Modular framed vertex operator algebras

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Abstract

Framed vertex operator algebras over any algebraically closed field whose characteristic is different from 2 and 7 are studied. In particular, the rationality of framed vertex operator algebras is established. For a code vertex operator algebra, the irreducible modules are constructed and classified. Moreover, a \( \mathbb{Z}[\frac{1}{2}] \)-form for any framed vertex operator algebra over \( \mathbb{C} \) is constructed. As a result, one can obtain a modular framed vertex operator algebra from any framed vertex operator algebra over \( \mathbb{C} \).

1 Introduction

Based on the unitary representation theory of the Virasoro algebra with central charge 1/2, the framed vertex operator algebras over \( \mathbb{C} \) have been studied extensively, see [DMZ], [M1], [M2], [DGH], [LY2]. In this paper, we investigate framed vertex operator algebras over any algebraically closed field \( F \) whose characteristic is different from 2 and 7. Specifically, we determine the structure of a modular framed vertex operator algebra in terms of binary codes and establish the rationality. We also classify the irreducible modules for a modular code vertex operator algebra. In addition, we obtain a modular framed vertex operator algebra from any framed vertex operator algebra over \( \mathbb{C} \) by constructing a \( \mathbb{Z}[\frac{1}{2}] \)-form.

The study of framed vertex operator algebras over \( \mathbb{C} \) was initiated in [DMZ]. A systematic investigation of framed vertex operator algebras was given in [M2] and [DGH]. The moonshine vertex operator algebra \( V^\natural \) [FLM] which contains a vertex operator subalgebra \( L(\frac{1}{2},0)^{\otimes 48} \) is one of the most important examples of framed vertex operator algebras. Here \( L(\frac{1}{2},0) \) is the irreducible highest weight module for the Virasoro algebra with central charge \( \frac{1}{2} \). This fact leads to a better understanding of \( V^\natural \). In particular, \( V^\natural \) is holomorphic [D] and two weak versions of the Frenkel-Lepowsky-Meurman’s conjecture on uniqueness of the \( V^\natural \) have been given in [DGL] and [LY1]. A new construction of \( V^\natural \) has been obtained in [M3]. The theory of framed vertex operator algebra over \( \mathbb{C} \) also plays important
roles in the study of holomorphic vertex operator algebras with \( c = 24 \). Many holomorphic vertex operator algebras with \( c = 24 \) are framed vertex operator algebras \([\text{LS}]\).

The Virasoro vertex operator algebra \( L(\frac{1}{2}, 0) \) and its representation theory including the fusion rules \([\text{DMZ}]\), \([\text{W}]\) are the foundation of the framed vertex operator algebras. It is well known that \( L(\frac{1}{2}, 0) \) is rational and has exactly three irreducible modules \( L(\frac{1}{2}, h) \) with \( h = 0, \frac{1}{2}, \frac{1}{16} \). The fusion products also have the following simple forms:

\[
L\left(\frac{1}{2}, 0\right) \times L\left(\frac{1}{2}, h\right) = L\left(\frac{1}{2}, h\right), \quad L\left(\frac{1}{2}, \frac{1}{2}\right) \times L\left(\frac{1}{2}, \frac{1}{2}\right) = L\left(\frac{1}{2}, 0\right) \\
L\left(\frac{1}{2}, \frac{1}{2}\right) \times L\left(\frac{1}{2}, \frac{1}{16}\right) = L\left(\frac{1}{2}, \frac{1}{16}\right), \quad L\left(\frac{1}{2}, \frac{1}{16}\right) \times L\left(\frac{1}{2}, \frac{1}{16}\right) = L\left(\frac{1}{2}, 0\right) + L\left(\frac{1}{2}, \frac{1}{2}\right)
\]

A framed vertex operator algebra \( V \) contains a rational vertex operator subalgebra \( L(\frac{1}{2}, 0)^{\otimes r} \) where \( r/2 \) is the central charge of \( V \). The main idea in the theory of framed vertex operator algebra is to decompose the vertex operator algebra \( V \) into a direct sum of finitely many irreducible \( L(\frac{1}{2}, 0)^{\otimes r} \)-modules. One can then use the fusion rules for \( L(\frac{1}{2}, 0) \) to study this decomposition. It turns out that this decomposition is very powerful in understanding both structure and representation theory of \( V \).

Motivated by the theory of framed vertex operator algebra over \( \mathbb{C} \), we studied the vertex operator algebra \( L(\frac{1}{2}, 0)_F \) over any algebraically closed field \( F \) whose characteristic is different from 2, 7 \([\text{DR1}], [\text{DR2}]\). We proved that the representation theory of \( L(\frac{1}{2}, 0)_F \) is the same as before. That is, \( L(\frac{1}{2}, 0)_F \) is rational with the same irreducible modules \( L(\frac{1}{2}, h)_F \) for \( h = 0, \frac{1}{2}, \frac{1}{16} \), and the same fusion rules. This makes a theory of modular framed vertex operator algebra possible although the treatments are more complicated.

The structure of a framed vertex operator algebra \( V \) over \( F \) is similar to that over \( \mathbb{C} \). First, for every binary even code \( C \subset \mathbb{Z}_2^r \), there is a code vertex operator algebra

\[
M_C = \bigoplus_{x=(x_1, \ldots, x_r) \in C} L\left(\frac{1}{2}, \frac{x_1}{2}\right)_F \otimes \cdots \otimes L\left(\frac{1}{2}, \frac{x_r}{2}\right)_F
\]

associated to \( C \). Code vertex operator algebra \( M_C \) is a special class of framed vertex operator algebras which has no \( L(\frac{1}{2}, 16)_F \) involved. The representation theory of \( M_C \) over \( \mathbb{C} \) is well understood due to the work in \([\text{M1}], [\text{M2}], [\text{LY2}]\). We can associate two binary even codes \( C \) and \( D \) to \( V \) such that the code vertex operator algebra \( M_C \) is a subalgebra of \( V \) and

\[
V = \oplus_{d \in D} V^d
\]

where \( V^0 = M_C \) and each \( V^d \) is a simple current as \( M_C \)-module. Using this decomposition, we can show that \( V \) is rational. It is worthy to mention that the decomposition of \( V \) into irreducible \( M_C \)-modules is relatively easy over \( \mathbb{C} \) as the minimal weights of \( L(\frac{1}{2}, h)_C \) are obvious. But one needs extra effort to understand how to put \( L\left(\frac{1}{2}, \frac{x_1}{2}\right)_F \otimes \cdots \otimes L\left(\frac{1}{2}, \frac{x_r}{2}\right)_F \) in \( V \) whose \( \mathbb{Z} \)-gradation is not given by the weights anymore.

A classification of irreducible modules for an arbitrary framed vertex operator algebra seems difficult at this stage. This has not been carried out completely over \( \mathbb{C} \). As in \([\text{M1}]\) we can classify irreducible modules for any code vertex operator algebra \( M_C \). Although the main idea is similar to that given in \([\text{M1}], [\text{M2}]\), we adopt a different approach. The main tool we use in this paper is the vertex operator superalgebra \( V(H_F) \) associated to the
infinite dimensional Clifford algebra and its twisted modules $V(H_F, d)$ for any codeword $d \in \mathbb{Z}_2^r$ where $H_F$ is $r$-dimensional vector space with a nondegenerate bilinear form. Since $M_C$ is a vertex operator subalgebra of $V(H_F)$, we show that $V(H_F, d)$ is a completely reducible $M_C$-module if $d \in \mathbb{C}^*$ and any irreducible $M_C$-module is obtained in this way.

Constructing a $\mathbb{Z}^{[\frac{1}{2}]}$-form for a framed vertex operator algebra over $\mathbb{C}$ is the key to produce a modular framed vertex operator algebra from a framed vertex operator algebra over $\mathbb{C}$. It is easy to see that for any integral domain $\mathcal{D}$ and a free $\mathcal{D}$-module $H_D$ of rank $r$, $V(H_D)$ is a vertex operator superalgebra and $V(H_D, d)$ is its twisted module. If $\mathcal{D} = \mathbb{Z}^{[\frac{1}{2}]}$, we have natural $\mathbb{Z}^{[\frac{1}{2}]}$-forms $V(H_D)$ of $V(H_C)$ and $V(H_D, d)$ of $V(H_C, d)$. Consequently, we obtain a $\mathcal{D}$-form $(M_C)^{\mathcal{D}}$ of framed vertex operator algebra $M_C$ over $\mathbb{C}$ and a $\mathcal{D}$-form for any irreducible $M_C$-module which is a self-dual simple current. Although we could not prove in this paper that any irreducible $M_C$-module has a $\mathcal{D}$-form, the explicit construction of $\mathcal{D}$-form for any self-dual simple current for the $M_C$ over $\mathbb{C}$ is good enough for us to obtain a $\mathcal{D}$-form for any framed vertex operator algebra over $\mathbb{C}$.

We should mention that we use a lot of ideas and techniques developed in [M1], [M2], [DGH] and [LY2] for dealing framed vertex operator algebras over $\mathbb{C}$ in this paper.

The paper is organized as follows. In Section 2 we present basic materials on vertex operator superalgebras and their twisted modules over an integral domain. We also discuss the intertwining operators among twisted modules. We review the rational vertex operator algebra $L(\frac{1}{2}, 0)_F$ and its representation theory in Section 3. In Section 4, we investigate the structure of framed vertex operator algebras over any algebraically closed field $\mathbb{F}$. As in the case $\mathbb{F} = \mathbb{C}$, we can associate two even binary codes $C$ and $D$ to a framed vertex operator algebra $V$ over $\mathbb{F}$. These two codes play crucial roles in studying the structure and representation theory. We discuss the vertex operator superalgebra $V(H_D)$ and its twisted modules $V(H_D, d)$ over any integral domain $\mathcal{D}$ in Section 5. In the case $\mathcal{D} = \mathbb{F}$ we write down explicit decompositions of $V(H_F, d)$ into a direct sum of irreducible $L(\frac{1}{2}, 0)_F \otimes \cdots \otimes L(\frac{1}{2}, 0)_F$-modules $L(\frac{1}{2}, h_1)_F \otimes \cdots \otimes L(\frac{1}{2}, h_r)_F$. The code vertex operator algebra $M_C$ is studied in Section 6. We decompose each $V(H_F, d)$ into a direct sum of irreducible $M_C$-modules. We construct a $\mathcal{D}$-form for code vertex operator algebra $M_C$ over $\mathbb{C}$ and a $\mathcal{D}$-form for some irreducible $M_C$-module which is a simple current with $\mathcal{D} = \mathbb{Z}^{[\frac{1}{2}]}$ in Section 7. The main idea is to use the standard $\mathcal{D}$-form $V(H_D, d)$ of $V(H_C, d)$. In Section 8, we construct intertwining operators among irreducible $L(\frac{1}{2}, 0)$-modules over $\mathcal{D}$. More precisely, each $L(\frac{1}{2}, h)$ has a $\mathcal{D}$-form $L(\frac{1}{2}, h)_{\mathcal{D}}$. The restriction of the intertwining operator we constructed gives an intertwining operator among corresponding $\mathcal{D}$-forms. These results are used in Section 9 to construct a $\mathcal{D}$-form for any framed vertex operator algebra over $\mathbb{C}$.

2 Basics

We first recall from [B], [DR1], [DR2] the basics of vertex operator superalgebras and their twisted modules over an integral domain $\mathcal{D}$ with $\text{ch}\mathcal{D} \neq 2$ and $\frac{1}{2} \in \mathcal{D}$ (also see [DL], [DLM3], [FFR], [LL], [X], [DG], [Mc]).

A super $\mathcal{D}$-module is a $\mathbb{Z}_2$-graded free $\mathcal{D}$-module $V = V_0 \oplus V_1$ such that both $V_0$ and $V_1$ are free $\mathcal{D}$-submodules. As usual, we let $\bar{v}$ be 0 if $v \in V_0$, and 1 if $v \in V_1$. 

3
A vertex operator superalgebra $V = (V,Y,1,\omega)$ over $\mathbb{D}$ is a $\frac{1}{2}\mathbb{Z}$-graded super $\mathbb{D}$-module

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n = V_0 \oplus V_1$$

with $V_0 = \sum_{n \in \mathbb{Z}} V_n$ and $V_1 = \sum_{n \in \frac{1}{2}+\mathbb{Z}} V_n$ such that the axioms of vertex operator superalgebra over $\mathbb{C}$ hold and for $n \in \mathbb{Z}$, $s,t \in \frac{1}{2}\mathbb{Z}$, $u \in V_s,v \in V_t$, $u_n v \in V_{s+t-n-1}$. Here we assume that the central charge $c$ of the Virasoro algebra lies in $\mathbb{D}$. If $v \in V_s$, we will call $s$ the degree of $v$. We also have the notion of vertex operator superalgebra $V$ over $\mathbb{D}$ if $V_1 = 0$.

An automorphism $g$ of a vertex operator superalgebra $V$ is a $\mathbb{D}$-module automorphism of $V$ such that $gY(u,z)v = Y(gu,z)gv$ for all $u,v \in V$, $g1 = 1$, $g\omega = \omega$ and $gV_n = V_n$ for all $n \in \frac{1}{2}\mathbb{Z}$. It is clear that any automorphism preserves $V_0$ and $V_1$. There is a special automorphism $\sigma$ such that $\sigma|V_0 = 1$ and $\sigma|V_1 = -1$. We see that $\sigma$ commutes with any automorphism.

Fix $g \in \text{Aut}(V)$ of order $T < \infty$. We assume that $\frac{1}{T} \in \mathbb{D}$ and $\mathbb{D}$ contains a primitive $T$-th root of unity $\eta$. Then $V$ decomposes into the eigenspaces of $g$: $V = \oplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r$

where $V^r = \{v \in V \mid gv = \eta^r v\}$. Then we have the notion of weak, admissible $g$-twisted $V$-module $M = (M,Y_M)$ [DZ], [DR1] where $M = M_0 \oplus M_1$ is $\mathbb{Z}_2$-graded. If $\mathbb{D}$ is a field, vertex operator algebra $V$ is called $g$-rational if any admissible $g$-twisted $V$-module is completely reducible. $V$ is rational if $V$ is 1-rational.

**Theorem 2.1.** If $V$ is rational then there are only finitely many inequivalent irreducible admissible modules and the homogeneous subspaces of any irreducible admissible module are finite dimensional.

Theorem [2.1] in the case $\mathbb{F} = \mathbb{C}$ was obtained in [DZ], and in the case that $V$ is a vertex operator algebra over $\mathbb{F}$ was given in [DR1], [R]. The proof of Theorem 2.1 is similar to those given in [DZ], [DR1], [R].

The following result was obtained in [FHL] in the case $\mathbb{F} = \mathbb{C}$. The same proof works here.

**Lemma 2.2.** Let $V^i = (V^i,Y^i,1_i,\omega_i)$ be vertex operator superalgebra, $g_i$ be automorphism of $V^i$ of finite order, and $M^i = (M^i,Y_i)$ be admissible $g_i$-twisted $V^i$-module for $i = 1,...,n$.

1. $V = V^1 \otimes \cdots \otimes V^n$ is a vertex operator superalgebra, $g = g_1 \otimes \cdots \otimes g_n$ is an automorphism of $V$ of finite order, and $M = M^1 \otimes \cdots \otimes M^n$ is an admissible $g$-twisted $V$-module in an obvious way.

2. $M$ is irreducible if and only if each $M^i$ is irreducible. Moreover, every irreducible $V$-module is obtained in this way.

3. If $V^i$ is $g_i$-rational for all $i$ then $V$ is $g$-rational.

We need the contragredient module from [FHL], [X], [Y] and [DR1]. Let $g$ be an automorphism of $V$ of order $T < \infty$ and $M = \oplus_{n \geq 0} M(n)$ be an admissible $g$-twisted $V$-module. We define the graded dual $M'$ of $M$ as

$$M' = \oplus_{n \geq 0} M(n)^*$$
where $M(n)^* = \text{Hom}_F(M(n), F)$. We denote the natural pair from $M' \times M \to F$ by $(,)$. Also assume that the operators $L(n)^*$ make sense on $V$ for $n \geq 0$. Then $M' = (M', Y)$ is an admissible $g^{-1}$-twisted $V$-module such that
\[
(Y(v, z)w', w) = (u', Y(e^{zL(1)}(-1)^{\deg v + 2\deg v^2 + r}z^{-2\deg v}v, z^{-1})w)
\]
for any $v \in V$, $w = \tilde{v} \in \{0, 1\}$, $w' \in M'$ and $w \in M$. The reason for us to use $(-1)^{\deg v + 2\deg v^2 + r}$ instead of $(e^{\sqrt{-1}})^{\deg v}$ is that we do not assume the square root of $-1$ is contained in $F$. Moreover, if each homogeneous subspace of $M(n)$ is finite dimensional, then $M'$ is irreducible if and only if $M$ is irreducible.

Next we define intertwining operators and fusion rules among admissible $g_k$-twisted modules $(M_k, Y_k)$ for $k = 1, 2, 3$ where $g_k$ are commuting automorphisms of order $T_k$. In the case that $F = \mathbb{C}$, the definitions are the same as in [X] and [DLM0]. We now assume that $\text{chF} = p$ is a prime. Let $T$ be the least common multiple of $T_1, T_2, T_3$ and assume $T \neq 0$ in $F$ and $F$ contains a $T$-th primitive root of the unity $\eta$. In this case $V$ decomposes into the direct sum of common eigenspaces
\[
V = \bigoplus_{j_1, j_2} V^{(j_1, j_2)}
\]
where
\[
V^{(j_1, j_2)} = \{v \in V | g_k v = \eta^{j_k} v, k = 1, 2\}.
\]
Let $\mathbb{Q}(p) = \{m \mathbb{Z} | p \nmid n\}$. Assume that $L(0)|_{M(n)} = \lambda_i + n$ for all $n$ where $\lambda_i \in \mathbb{Q}(p)$ which is understood to be a number in $\mathbb{F}$.

An intertwining operator of type $\begin{pmatrix} M_3 \\ M_1 & M_2 \end{pmatrix}$ is a linear map
\[
I(\cdot, z) : M_1 \to \text{Hom}(M_2, M_3)\{z\}
\]
\[
u \mapsto I(u, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} u(n)z^{-n-1-\lambda_1-\lambda_2+\lambda_3}
\]
satisfying:
1. for any $u \in M_1$ and $v \in M_2$, $u(n)v = 0$ for $n$ sufficiently large;
2. $I(L(-1)v, z) = (\frac{d}{dz})I(v, z)$;
3. $u(m)M_2(n) \subset M_3(n - m - 1 + \deg u)$ for $m, n \in \frac{1}{T}\mathbb{Z}$;
4. for any $u \in V^{(j_1, j_2)}$, $v \in (M_1)_t$,
\[
z_0^{-1} \left(\frac{z_1 - z_2}{z_0}\right)^{j_1/T} \delta \left(\frac{z_1 - z_2}{z_0}\right) Y_3(u, z_1)I(v, z_2)
\]
\[-(-1)^{st}z_0^{-1} \left(\frac{z_2 - z_1}{z_0}\right)^{j_1/T} \delta \left(\frac{z_2 - z_1}{z_0}\right) I(v, z_2)Y_2(u, z_1)
\]
\[= z_2^{-1} \left(\frac{z_1 - z_0}{z_2}\right)^{-j_2/T} \delta \left(\frac{z_1 - z_0}{z_2}\right) I(Y_1(u, z_0)v, z_2).
\]
We denote the space of intertwining operators of type \( \begin{pmatrix} M_3 \\ M_1 & M_2 \end{pmatrix} \) by \( I \begin{pmatrix} M_3 \\ M_1 & M_2 \end{pmatrix} \). We call \( N_{M_1,M_2} = \dim I \begin{pmatrix} M_3 \\ M_1 & M_2 \end{pmatrix} \) the fusion rules. It is proved in \([DR1]\) that the fusion rules \( N_{M_1,M_2} \) are independent of choices of \( \lambda \). Note that if \( N_{M_1,M_2} > 0 \) then \( g_3 = g_1 g_2 \) (see \([X]\)). So we now assume that \( g_3 = g_1 g_2 \). This definition of intertwining operator here is the same as that given in \([DR2]\) when \( g_i = 1 \) for all \( i \).

We also remark that if \( F = \mathbb{C} \), we do not need \( \mathbb{Q}(p) \) in the definition, see \([X]\) and \([DLM0]\).

The following result is a generalization of Proposition 11.19 of \([DL]\) with the same proof by noting that the commutativity and associativity still hold in the current situation.

**Lemma 2.3.** Assume \( \mathbb{D} \) is a field. Let \( M_i \) and \( I \) be as before. We also assume that \( M_1, M_2 \) are irreducible and \( I(u,z)v = 0 \) for some nonzero \( u \in M_1, v \in M_2 \). Then \( I = 0 \).

The fusion rules have certain symmetry properties.

**Lemma 2.4.** Let \( M_k \) be as before and \( \mathbb{D} \) a field.

1. We have \( N_{M_2,M_1} = N_{M_1,M_2} \).

2. If each homogeneous subspaces of \( M_2 \) and \( M_3 \) are finite dimensional and \( \frac{L(1)^{n}}{m!} \) are well defined on \( V \) for \( n \geq 0 \), then \( N_{M_2,M_1}^{M_3} = N_{M_1,M_2}^{M_3} \).

Let \( V^i \) be rational vertex operator algebras and let \( M_j^i \) be irreducible admissible \( V^i \)-modules for \( i = 1, \ldots, n \) and \( j = 1, 2, 3 \). Also assume that \( N_{M_1,M_2}^{M_3} \) is finite for all \( i \). Using the exact proof given in \([DMZ]\) in the case \( F = \mathbb{C} \), we obtain the following result.

**Lemma 2.5.** The fusion rule \( N_{M_1^1 \otimes \cdots \otimes M_1^n, M_2^1 \otimes \cdots \otimes M_2^n}^{M_3} \) is equal to \( \prod_{i=1}^{n} N_{M_1^i,M_2^i}^{M_3} \).

We now formulate a notion of tensor product \( M_1 \boxtimes M_2 \) (which is also called the fusion product and denoted by \( M_1 \times M_2 \)) of \( M_1 \) and \( M_2 \); it is an admissible \( g_3 \)-twisted \( V \)-module defined by a universal mapping property \((L3, HL1, HL2)\): A tensor product for the ordered pair \((M_1, M_2)\) is a pair \((M, F)\) consisting of a weak \( g_3 \)-twisted \( V \)-module \( M \) and an intertwining operator \( F \) of type \( \begin{pmatrix} M \\ M_1 & M_2 \end{pmatrix} \) such that the following universal property holds: for any weak \( g_3 \)-twisted \( V \)-module \( W \) and any intertwining operator \( I(\cdot, z) \) of type \( \begin{pmatrix} W \\ M_1 & M_2 \end{pmatrix} \), there exists a unique \( V \)-homomorphism \( \psi \) from \( M \) to \( W \) such that \( I(\cdot, z) = \psi \circ F(\cdot, z) \). (Here \( \psi \) extends canonically to a linear map from \( M\{z\} \) to \( W\{z\} \).

It is easy to show that if \( V \) is \( g_k \)-rational for \( k = 1, 2, 3 \), then \( M_1 \boxtimes M_2 \) exists and is equal to \( \oplus_{M_3 \in \mathcal{M}(g_3)} N_{M_1,M_2}^{M_3} \mathcal{M}(g_3) \) where \( \mathcal{M}(g_3) \) is the set of inequivalent irreducible \( g_3 \)-twisted \( V \)-modules.

### 3 Vertex operator algebra \( L(\frac{1}{2}, 0)_F \)

In this section, we recall from \([DR2]\) the rational vertex operator algebra \( L(\frac{1}{2}, 0)_F \) associated to the Virasoro algebra with central charge \( c = \frac{1}{2} \) over any algebraically closed field \( F \) with \( \text{ch} F \neq 2, 7 \). (See also \([DMZ]\) and \([W]\).)
We begin with an integral domain \( \mathbb{D} \) such that \( 1/2 \in \mathbb{D} \). Set

\[
\text{Vir}_\mathbb{D} = \bigoplus_{n \in \mathbb{Z}} \mathbb{D} L_n \oplus \mathbb{D} C
\]

subject to the relation

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \quad [\text{Vir}_\mathbb{D}, C] = 0
\]

for \( m, n \in \mathbb{Z} \). Note that \( m^3 - m \) is divisible by 3 for any \( m \in \mathbb{Z} \), the commutators make sense. For any \( c, h \in \mathbb{D} \), we set

\[
V(c, h)_\mathbb{D} = U(\text{Vir}_\mathbb{D}) \otimes_{U(\text{Vir}^{>0}_\mathbb{D})} \mathbb{D}
\]

where \( \text{Vir}^{>0}_\mathbb{D} \) is the subalgebra generated by \( L_n \) for \( n \geq 0 \) and \( C \), and \( \mathbb{D} \) is a \( \text{Vir}^{>0}_\mathbb{F} \)-module such that \( L_n 1 = 0 \) for \( n > 0 \), \( L_0 1 = h \) and \( C 1 = c \). Then

\[
V(c, h)_\mathbb{D} = \bigoplus_{n \geq 0} V(c, h)_\mathbb{D}(n)
\]

is \( \mathbb{Z} \)-graded where \( V(c, h)_\mathbb{D}(n) \) has a basis

\[
\{ L_{-n_1} \cdots L_{-n_k} v_{c,h} \mid n_1 \geq \cdots \geq n_k \geq 1, \sum_i n_i = n \}
\]

where \( v_{c,h} = 1 \otimes 1 \). The \( V(c, h)_\mathbb{D} \) is again called the Verma module. It is easy to see that \( L_n V(c, h)_\mathbb{D}(m) \subset V(c, h)_\mathbb{D}(m - n) \) for all \( m, n \in \mathbb{Z} \). If \( \mathbb{D} = \mathbb{C} \), we will denote it by \( V(c, h)_\mathbb{C} \) instead of \( V(c, h)_\mathbb{D} \). The \( V(c, h)_\mathbb{F} \) has a unique maximal graded submodule \( W(c, h)_\mathbb{F} \) such that \( L(c, h)_\mathbb{F} = V(c, h)_\mathbb{F}/W(c, h)_\mathbb{F} \) is an irreducible highest weight \( \text{Vir}_F \)-module.

The following theorem which is the foundation of the framed vertex operator algebra was obtained in [DR1] and [DR2].

**Theorem 3.1.** Let \( \mathbb{F} \) be an algebraically closed field with \( \text{ch} \mathbb{F} \neq 2, 7 \).

1. The \( L(\frac{1}{2}, 0)_\mathbb{F} \) is a rational vertex operator algebra which has exactly three irreducible modules \( L(\frac{1}{2}, \frac{1}{2})_\mathbb{F} \) with \( h = 0, \frac{1}{2}, \frac{1}{16} \).

2. The fusion product is given by

\[
L(\frac{1}{2}, 0)_\mathbb{F} \times L(\frac{1}{2}, h)_\mathbb{F} = L(\frac{1}{2}, h)_\mathbb{F}
\]

for \( h = 0, \frac{1}{2}, \frac{1}{16} \),

\[
L(\frac{1}{2}, \frac{1}{2})_\mathbb{F} \times L(\frac{1}{2}, \frac{1}{2})_\mathbb{F} = L(\frac{1}{2}, 0)_\mathbb{F},
\]

\[
L(\frac{1}{2}, \frac{1}{2})_\mathbb{F} \times L(\frac{1}{2}, \frac{1}{16})_\mathbb{F} = L(\frac{1}{2}, \frac{1}{16})_\mathbb{F},
\]

\[
L(\frac{1}{2}, \frac{1}{16})_\mathbb{F} \times L(\frac{1}{2}, \frac{1}{16})_\mathbb{F} = L(\frac{1}{2}, 0)_\mathbb{F} + L(\frac{1}{2}, \frac{1}{2})_\mathbb{F}.
\]

For the discussion of contragredient modules later, we give the following lemma.
Lemma 3.2. Let $M$ be a sum of highest weight modules for the Virasoro algebra $\text{Vir}_r$. Then $\frac{L_n}{n^2}$ is well defined on $M$ for $n \geq 0$.

Proof. We can assume that $M$ is a highest weight module with the highest weight $\lambda \in \mathbb{F}$ and the highest weight vector $v$. Then $M$ is spanned by $L_{-n_1} \cdots L_{-n_k} v$ for $n_1 \geq \cdots \geq n_k \geq 1$. Clearly, $\frac{L_n}{n} v = 0$. Assume that $u = L_m w$ and $\frac{L_n}{n} w$ is well defined. Then $\frac{L_n}{n} u = \sum_{i=0}^{n} \binom{1-m}{i} L_{m+i} \frac{L_{n-i}}{(n-i)} w$ is well defined for $n \geq 0$. □

4 Framed vertex operator algebras

We define framed vertex operator algebras and discuss properties of framed vertex operator algebras following [DGH].

A simple vertex operator superalgebra $V = \oplus_{n \geq 0} V_n$ is called a framed vertex operator superalgebra (FVOSA) if there exist $\omega_i \in V$ for $i = 1, \ldots, r$ such that (i) each $\omega_i$ generates a copy of the simple Virasoro vertex operator algebra of central charge $\frac{1}{2}$ and the component operators $L^i(n)$ of $Y(\omega_i, z) = \sum_{n \in \mathbb{Z}} L^i(n) z^{-n-2}$ satisfy $[L^i(m), L^j(n)] = (m-n)L^i(m+n) + \frac{m^3-m}{24} \delta_{m+n,0}$; (ii) the $r$ Virasoro algebras are mutually commutative; and (iii) $\omega = \omega_1 + \cdots + \omega_r$. The set $\{\omega_1, \ldots, \omega_r\}$ is called a Virasoro frame (VF). A framed vertex operator algebra is defined in an obvious way.

We remark that in the case $\mathbb{F} = \mathbb{C}$ the assumption $V = \oplus_{n \geq 0} V_n$ is unnecessary [DGH].

Let $T_r = L(\frac{1}{2}, 0)_\mathbb{F}$. Then the vertex operator subalgebra of $V$ generated by $\omega_1, \ldots, \omega_r$ is isomorphic to $T_r$ and is rational by Lemma 2.2 and Theorem 3.1. For $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$ with $i = 1, \ldots, r$, we set

$$L(h_1, \ldots, h_r) = L(\frac{1}{2}, h_1)_\mathbb{F} \otimes \cdots \otimes L(\frac{1}{2}, h_r)_\mathbb{F}.$$ 

Then $L(h_1, \ldots, h_r)_\mathbb{F}$ is an irreducible $T_r$-module and every irreducible $T_r$-module is given in this way. Using the rationality of $T_r$, we decompose $V$ into a direct sum of irreducible $T_r$-modules:

$$V = \bigoplus_{h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}} m_{h_1, \ldots, h_r} L(h_1, \ldots, h_r)_\mathbb{F},$$

where the nonnegative integer $m_{h_1, \ldots, h_r}$ is the multiplicity of $L(h_1, \ldots, h_r)_\mathbb{F}$ in $V$. If $\mathbb{F} = \mathbb{C}$, all the multiplicities are finite and $m_{h_1, \ldots, h_r}$ is at most 1 if all $h_i$ are different from $\frac{1}{16}$ as the gradation of $V$ is given by the eigenvalues of $L(0)$ [DMZ], [DGH]. If $\text{ch} \mathbb{F} = p$ is finite, the $L(0)$ has only $p$ different eigenvalues. Assume $M, W$ are two irreducible $T_r$-submodules of $V$ isomorphic to $L(h_1, \ldots, h_r)_\mathbb{F}$. Then there exist $s, t \geq 0$ such that $M = \oplus_{m \geq 0} M_{h+s+p+m}$ and $W = \oplus_{m \geq 0} W_{h+t+p+m}$ where $M_n = V_n \cap M$ and $h = \sum_i h_i$. We cannot prove that $s = t = 0$ from the definition. So it is not obvious that $m_{h_1, \ldots, h_r}$ is still finite.

For $d = (d_1, \ldots, d_r) \in \mathbb{Z}_+^r$, let $V^d$ be the sum of all irreducible submodules isomorphic to $L(h_1, \ldots, h_r)_\mathbb{F}$ such that $h_i = \frac{1}{16}$ if and only if $d_i = 1$. Let $D = D(V) = \{d \in \mathbb{Z}_+^r \mid V^d \neq 0\}$. Then

$$V = \bigoplus_{d \in D} V^d.$$

As in [DGH], we have the following lemma.
Lemma 4.1. Let $V$ be a FVOSA. Then

1. $D$ is a triply even linear binary code, i.e., $\text{wt}(\alpha) \equiv 0 \mod 8$ for any $\alpha \in D$;
2. for any $d \in D$, $V^d$ is a simple vertex operator superalgebra and each $V^d$ is an irreducible $V^0$-module.

The proof of Lemma 4.1 is similar to the same result in [DGH] by using Lemma 2.3 and Theorem 3.1.

For each $c = (c_1, \ldots, c_r) \in \mathbb{Z}_2^r$, let $V(c)$ be the sum of the irreducible submodules isomorphic to $L(\frac{1}{2}c_1, \ldots, \frac{1}{2}c_r)^F$. Then $V^0 = \bigoplus_{c \in \mathbb{Z}_2^r} V(c)$. Let $C = C(V) = \{c \in \mathbb{Z}_2^r \mid V(c) \neq 0\}$. The following result is an immediate consequence of Theorem 3.1 and Lemma 2.5.

**Lemma 4.2.** $C$ is a linear binary code. Moreover, $C$ is even if and only if $V^0$ is a vertex operator algebra.

Next result tells us that each $V(c)$ is an irreducible $T_r$-module.

**Lemma 4.3.** Let $V$ be a FVOA. For any $c \in C$, $V(c) = L(\frac{1}{2}c_1, \ldots, \frac{1}{2}c_r)^F$. In particular, the degree zero subspace $V_0$ of $V$ is one dimensional: $V_0 = \mathbb{F}1$.

**Proof.** Note that $V^0 = \bigoplus_{c \in C} V(c)$. Using a proof similar to that of Lemma 4.1 we know that $V(0)$ is a simple vertex operator algebra and each $V(c)$ is an irreducible $V(0)$-module. Let $U$ consist of vectors $v \in V$ such that $L'(n)v = 0$ for $i = 1, \ldots, r$ and $n \geq -1$. Then $U$ is the multiplicity of $T_r$ in $V$ and $U = \bigoplus_{m \geq 0} U_m$ is graded where $U_n = U \cap V_n$. Moreover, $V(0) = T_r \otimes U$. Clearly, $V_0 = U_0$. The key point is to prove that $U = \mathbb{F}1 = V_0$.

We claim that $U$ is a vertex algebra. This is clear by noting that $L'(n)u_s v = 0$ for $u, v \in U, s, n \in \mathbb{Z}$ with $n \geq -1$. Moreover, $U$ is a simple vertex algebra as $V(0)$ is. Also $U = \bigoplus_{i \in \mathbb{Z}} U_{ip}$ as $L(0)$ acts trivially on $U$.

Fix $0 \neq u \in U_{mp}, 0 \neq v \in U_{sp}$, and denote the irreducible $T_r$-modules generated by $u, v$ by $M$ and $N$, respectively. Then $(a_s b | a \in M, b \in N, s \in \mathbb{Z})$ is also an irreducible $T_r$-module $W$. It is obvious that $M, N$ isomorphic to $T_r$ as $T_r$-modules and $Y(a, z)b$ for $a \in M$ and $b \in N$ is an intertwining operator of type \( \begin{pmatrix} T_r & T_r \\ T_r & T_r \end{pmatrix} \). By Lemma 2.5 and Theorem 3.1, we know that $N_{T_r, T_r}^W = 1$.

From the definition, we know that $1_n = \delta_{n-1} Id_V$. Thus $Y(u, z)|_N = u_{-1+sp} z^{sp}$ for some $s \in \mathbb{Z}$. In particular, $Y(u, z)v = u_{-1+sp} vz^{sp}$ is nonzero. Take $w \in U_{ip}$. Then from the associativity [DR], there exists a positive number $q \in \mathbb{Z}$ such that

\[(z_0 + z_2)qY(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^qY(Y(u, z_0)v, z_2)w.\]

From the discussion above, $Y(u, z_0 + z_2)Y(v, z_2)w = u_{-1+jp} v_{-1+ip} w(z_0 + z_2)^{-jp} z_2^{ip}$ and $Y(Y(u, z_0)v, z_2)w = (u_{-1+sp} v_{-1+tp} z_0^{sp} z_2^{-sp})w$ for some integers $j, l, s, t$. As a result, we have

\[u_{-1+jp} v_{-1+ip} w(z_0 + z_2)^{-jp} z_2^{ip} = (u_{-1+sp} v_{-1+tp} z_0^{sp} z_2^{-sp}).\]

This forces $j = s = 0$ and $l = t$. Since $v \in U$ is arbitrary, we see that $Y(u, z) = u_{-1}$ on $U$ for any $u \in U$. This implies that

\[u_{-1} v_{-1} w = (u_{-1} v)_{-1} w.\]
That is, $U$ is an associative algebra over $\mathbb{F}$ with product $u \cdot v = u_{-1}v$.

Using the commutativity

$$(z_1 - z_2)^q Y(u, z_1)Y(v, z_2)w = (z_1 - z_2)^q Y(v, z_2)Y(u, z_1)w$$

for some $q \in \mathbb{Z}$, we see that $u_{-1}v_{-1}w = v_{-1}u_{-1}w$. In particular, if $w = 1$, we have $u_{-1}v = v_{-1}u$ and $U$ is a simple, commutative associative algebra. For $u \in U_{pm}, v \in U_{np}$, $u_{-1}v \in U_{p(m+n)}$. So $U = \oplus_{n \geq 0} U_{np}$ is a $\mathbb{Z}$-graded algebra and $I = \oplus_{n > 0} U_{np}$ is an ideal of $U$. From the simplicity of $U$, we conclude that $I = 0$ and $U = U_0$ is a finite dimensional simple commutative associative algebra over $\mathbb{F}$. This implies that $U$ is a finite field extension of $\mathbb{F}$. Since $\mathbb{F}$ is algebraically closed, $U = \mathbb{F} = \mathbb{F}1$.

Finally, each $V(c)$ is an irreducible $V(0)$-module. So $V(c) = L(\frac{1}{2}c_1, ..., \frac{1}{2}c_r)$. □

We remark that Lemma 4.3 was obtained in [DMZ] and [DGH] in the case $\mathbb{F} = \mathbb{C}$. But the proof here is much more complicated as we cannot use the unitarity of the modules for the Virasoro algebra with central $\frac{1}{2}$ in the current situation.

Next we deal with the multiplicities $m_{h_1, ..., h_r}$ in general. The same result was given in Proposition 2.5 [DGH] when $\mathbb{F} = \mathbb{C}$. The proof in [DGH] works here.

**Lemma 4.4.** Let $V$ be a FVOA. Let $d \in D$ and suppose that $(h_1, ..., h_r)$ and $(h'_1, ..., h'_r)$ are $r$-tuples with $h_i, h'_i \in \{0, \frac{1}{2}, \frac{1}{10}\}$ such that $h_i = \frac{1}{10}$ (resp. $h'_i = \frac{1}{10}$) if and only if $d_i = 1$. If both $m_{h_1, ..., h_r}$ and $m_{h'_1, ..., h'_r}$ are nonzero then $m_{h_1, ..., h_r} = m_{h'_1, ..., h'_r}$. That is, all irreducible modules inside $V^d$ for $T_r$ have the same multiplicities.

Recall the codes $\mathcal{C}$ and $\mathcal{D}$. The main result in this section is the following.

**Theorem 4.5.** Any FVOA $V$ is rational. Moreover,

$$\mathcal{C} \subseteq D^+ = \{x = (x_1, ..., x_r) \in \mathbb{Z}_2^r \mid x \cdot d = 0 \text{ for any } d \in D\}.$$ 

The proof of Theorem 2.12 of [DGH] in the case $\mathbb{F} = \mathbb{C}$ is valid here.

## 5 Clifford algebras and vertex operator algebras

In this section, we study the vertex operator superalgebra $V(H_\mathbb{F})$ and its twisted modules. For the purpose of later discussion on integral forms and modular vertex operator algebra, we consider vertex algebras and their twisted modules over an integral domain $\mathbb{D}$.

Fix a positive integer $r$ and a codeword $d = (d_1, ..., d_r) \in \{0, 1\}^r$. Define the support of $d$ as $\text{supp}(d) = \{i|d_i = 1\}$ and the length $|d|$ of $d$ as the cardinality of $\text{supp}(d)$. Let $H_\mathbb{D} = \sum_{i=1}^r \mathbb{D}a_i$ be a free $\mathbb{D}$-module of rank $r$ with a nondegenerate symmetric bilinear form $(.,.)$ such that $\{a_i|i = 1, 2, ... r\}$ is an orthonormal basis of $H_\mathbb{D}$. The Clifford algebra $A(H_\mathbb{D}, d)$ is an associative algebra over $\mathbb{D}$ generated by $\{a_i(n_i) \mid 1 \leq i \leq r, n_i \in \mathbb{Z} + \frac{1}{2}(d_i + 1)\}$ subject to the relation

$$[a(n), b(m)]_+ = (a, b)\delta_{m+n, 0}$$

for $a, b \in H_\mathbb{D}$. Let $A^+(H_\mathbb{D}, d)$ be the subalgebra generated by $\{a_i(n_i)|1 \leq i \leq r, n_i \in \mathbb{Z} + \frac{1}{2}(1 + d_i), n_i > 0\}$, and make $\mathbb{D}$ a 1-dimensional $A^+(H_\mathbb{D}, d)$-module so that $a_i(n)1 = 0$ for $n > 0$. We have the induced module

$$V(H_\mathbb{D}, d) = A(H_\mathbb{D}, d) \otimes A^+(H_\mathbb{D}, d) \mathbb{D}$$

$$\cong \wedge_{\mathbb{D}}[a_i(-n_i)|n_i \geq 0, i = 1, 2, ... r](\text{linearly})$$

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so that the action of $a_i(n)$ is given by $\frac{\partial}{\partial n_i(-n)}$ if $n$ is positive and by multiplication by $a_i(n)$ if $n$ is nonpositive. The $V(H_d,d)$ is naturally graded by $\frac{1}{2}Z$ with

$$V(H_d,d)_{n+m} = \left\langle a_{i_1}(-n_1)a_{i_2}(-n_2)\cdots a_{i_k}(-n_k)|n_1 + n_2 + \cdots + n_k = n\right\rangle.$$ 

Note that $V(H_d,d) = V(H_d,d)_{0} \oplus V(H_d,d)_{1}$ where $V(H_d,d)_{i}$ is the span of monomials whose length is congruent to $i$ modulo $2$.

For short, we let $V(H_d) = V(H_d, (0,\ldots,0))$. Then $V(H_d) = V(\mathbb{D}a_1) \wedge \cdots \wedge V(\mathbb{D}a_r))$. Set $1 = 1$ and $\omega = \frac{1}{2} \sum_{i=1}^{r} a_i(-\frac{1}{2})a_i(-\frac{3}{2})$ lie in $V(H_d)$. From [FFR, [KW] and [L1], we have

**Theorem 5.1.** (1) $(V(H_d), Y, 1, \omega)$ is a vertex operator superalgebra over $\mathbb{D}$ generated by $a(-1/2)$ for $a \in H_d$ and with $Y(a(-1/2), z) = a(z) = \sum_{n \in \frac{1}{2} + Z} a(n)z^{-n-1/2}$. Moreover, if $\mathbb{D}$ is a field, $V(H_d)$ is a simple vertex operator superalgebra.

(2) $V(H_d)_{0}$ is a vertex operator algebra. In the case $\mathbb{D}$ is a field, both $V(H_d)_{0}$ and $V(H_d)_{1}$ are irreducible $V(H_d)_{0}$-modules. In particular, $V(H_d)_{0}$ is simple.

To see how the vertex operator $Y(v, z)$ is defined for $v = b_1(-n_1 - \frac{1}{2})\cdots b_k(-n_k - \frac{1}{2}) \in V(H_d)$, we need a normal ordering:

$$: b_1(n_1)\cdots b_k(n_k) := (-1)^{[\sigma]}b_{i_1}(n_{i_1})\cdots b_{i_k}(n_{i_k})$$

such that $n_{i_1} \leq \cdots \leq n_{i_k}$ where $\sigma$ is the permutation of $\{1,\ldots,k\}$ by sending $j$ to $i_j$. It is easy to see that

$$Y(v, z) := (\partial_{n_1} b_1(z)) \cdots (\partial_{n_k} b_k(z)) :$$

where $\partial_n = \frac{1}{n!}(\frac{d}{dz})^n$. Note that for any $m \in Z$ the constant $\binom{m}{n}$ for $n \geq 0$ is an integer. So the component operators of $Y(v, z)$ are well defined linear operators on $V(H_d)$.

We now define a $\mathbb{D}$-valued bilinear form $\langle \cdot, \cdot \rangle$ on $V(H_d)$ such that the monomials form an orthonormal basis. It is clear that the form is symmetric and nondegenerate.

**Proposition 5.2.** Let $a \in H_d, u, v \in V(H_d)$, $w \in V(H_d)_{i}$, $m \in \frac{1}{2} + Z$ and $n \in Z$. Then

(1) $(a(m)u, v) = (u, a(-m)v)$,

(2) $(L(n)u, v) = (u, L(-n)v)$,

(3) $\frac{L(1)^n}{n!}$ is well defined if $n \geq 0$,

(4) The form is invariant:

$$\langle Y(w, z)u, v \rangle = (u, Y(e^zL(1)(-1)^{\deg w + 2(\deg w)^2} + rz^{-2z} \deg w)w, z^{-1})v).$$

**Proof.** (1) It is good enough to take $a = a_i$ for some $i$. The result follows immediately from the definition of form.

(2) follows from (1) by noting that

$$L(n) = \frac{1}{2} \sum_{i=1}^{r} \sum_{j \in \frac{1}{2} + Z} j : a_i(-j)a_i(j + n) :.$$ 

(3) Since $\frac{L(1)^n}{n!}$ is well defined for $n \geq 0$, using the invariant property we see that

$$\left(\frac{L(1)^n}{n!} u, v \right) = \left(u, \frac{L(1)^n}{n!} v \right)$$
and \( \frac{L(1)^n}{n!} \) is well defined.

(4) From the definition of the bilinear form, we know the invariant property holds for \( w = a(-\frac{1}{2}) \). Note that \( V(H_\mathcal{D}) \) is generated by \( a_i(-\frac{1}{2}) \) for \( i = 1, \ldots, r \). It follows from the proofs of Proposition 2.11 of [DLin] and Proposition 2.5 of [AL] that the invariant property holds for any \( w \). □

We now assume that \( \mathcal{D} = \mathcal{F} \) is an algebraically closed field with \( \text{ch}\mathcal{F} \neq 2, 7 \). Notice that \( V(\mathcal{F}a_i)_0 \cong L(1, 0)_\mathcal{F} \) and \( V(\mathcal{F}a_i)_1 \cong L(1, 0)_\mathcal{F} \) as modules for the Virasoro algebra generated by the component operators of \( \omega_i \) [KR], [DR2]. As a result we have the decomposition

\[
\begin{align*}
V(H_\mathcal{F}) &\cong \bigoplus_{h_i \in \{0, \frac{1}{2}\}} L(h_1, \ldots, h_r)_\mathcal{F} \\
V(H_\mathcal{F})_0 &\cong \bigoplus_{h_i \in \{0, \frac{1}{2}\}, \sum_i h_i \in \mathbb{Z}} L(h_1, \ldots, h_r)_\mathcal{F} \\
V(H_\mathcal{F})_1 &\cong \bigoplus_{h_i \in \{0, \frac{1}{2}\}, \sum_i h_i \in \mathbb{Z}+\frac{1}{2}} L(h_1, \ldots, h_r)_\mathcal{F}.
\end{align*}
\]

as modules for \( T_r = \langle \omega_i | i = 1, \ldots, r \rangle \).

**Proposition 5.3.** (1) \( (V(H_\mathcal{F}), Y, 1, \omega) \) is a holomorphic vertex operator superalgebra in the sense that \( V(H_\mathcal{F}) \) is rational and \( V(H_\mathcal{F}) \) is the only irreducible module for itself.

(2) Set \( \omega_i = \frac{1}{2} a_i(-\frac{3}{2}) a_i(-\frac{1}{2}) \) for \( i = 1, \ldots, d \). Then each \( \omega_i \) generates a vertex operator subalgebra \( \langle \omega_i \rangle \) isomorphic to \( L(1, 0)_\mathcal{F} \) and \( \{\omega_1, \ldots, \omega_r\} \) form a Virasoro framed. In fact, \( V(H_\mathcal{F}) \) is a framed vertex operator superalgebra.

**Proof.** (2) follows from Theorem 4.3 of [DR2]. The rationality of \( V(H_\mathcal{F}) \) follows immediately from Theorem 2.1. To prove that \( V(H_\mathcal{F}) \) is holomorphic, let \( W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}+} W(n) \) be an irreducible \( V(H_\mathcal{F}) \)-module with \( W(0) \neq 0 \). Let \( Y_W(a(\frac{1}{2}), z) = \sum_{m \in \mathbb{Z}} a(\frac{1}{2}) z^{-m-1} \) for \( a \in H_\mathcal{F} \). Then \( W \) is an irreducible \( A(H_\mathcal{F}, 0) \)-module such that \( a(\frac{1}{2}+s) \) acts as \( a(\frac{1}{2})-1+s \) for \( s \in \mathbb{Z} \). Clearly, \( a(-\frac{1}{2}+s) W(0) = 0 \) for \( a \in H_\mathcal{F} \) and \( s > 0 \). This implies \( W \) is isomorphic to \( V(H_\mathcal{F}) \) as \( A(H_\mathcal{F}, 0) \)-modules. Since \( V(H_\mathcal{F}) \) is generated by \( a(-\frac{1}{2}) \) for \( a \in H_\mathcal{F} \), \( W \) is isomorphic to \( V(H_\mathcal{F}) \) as \( V(H_\mathcal{F}) \)-modules. □

Write \( Y(\omega_i, z) = \sum_{n \in \mathbb{Z}} L_i(n) z^{-n-2} \) for \( i = 1, \ldots, r \). Set \( \tau_i = (-1)^{2L_i(0)}. \) Then \( \tau_i \) is an automorphism of \( V(H_\mathcal{F}) \) from the fusion rules of \( L(1, 0)_\mathcal{F} \)-modules [DR1] and Lemma 2.5 and is the Miyamoto involution associated to \( \omega_i \) [MM1], [MM2]. Recall the codeword \( d \) and \( V(H_\mathcal{F}, d) \). Let \( \tau(d) = \prod_{d_i=1} \tau_i \). Then \( \tau(d) \) is an automorphism of \( V(H_\mathcal{F}) \).

**Proposition 5.4.** Let \( d \) be as before. Then

(1) \( V(H_\mathcal{F}) \) has a unique \( \tau(d) \)-twisted module \( V(H_\mathcal{F})(\tau(d)) \) if \( |d| \) is even and has two inequivalent \( \tau(d) \)-twisted modules \( V(H_\mathcal{F})(\tau(d))^i \) for \( i = 1, 2 \) if \( |d| \) is odd.

(2) \( V(H_\mathcal{F}, d) \) is a \( \tau(d) \)-twisted \( V(H_\mathcal{F}) \)-module such that

\[
Y(a_i(-1/2), z) = \sum_{n \in \frac{1}{2}(d_i+1)+\mathbb{Z}} a_i(n) z^{-n-1/2}
\]

for \( i = 1, \ldots, r \).
(3) We have the decomposition
\[
V(H_F, d) \cong \begin{cases} 
2^{d/2}V(H_F)(\tau(d)) & |d| \in 2\mathbb{Z} \\
2^{d/2-1}V(H_F)(\tau(d))^2 & |d| \in 2\mathbb{Z} + 1.
\end{cases}
\]

Proof: (1) and (2) follow from Propositions 4.3 of \textbf{[L2]} (also see \textbf{[DZ]}). To prove (3), consider the subalgebra \(cl(d)\) of \(A(H_F, d)\) generated by \(a_i(0)\) where \(i \in \text{supp}(d)\). Then \(cl(d)\) is a Clifford algebra of dimension \(2^{|d|}\) and is a semisimple module for itself. It is well known that \(cl(d)\) is a simple algebra with the unique irreducible module of dimension \(2^{|d|/2}\) if \(|d|\) is even, and is a sum of two simple algebras with two inequivalent simple modules of dimension \(2^{(|d|-1)/2}\) if \(|d|\) is odd. It is clear that any irreducible \(cl(d)\)-submodule \(W\) of \(cl(d)\) generates an irreducible \(\tau(d)\)-twisted module \(A(H_F, d)W\). The proof is complete. \(\Box\)

We also have the following decomposition of \(V(H, d)\) as a module for \(T_r\):
\[
V(H, d) = \bigoplus_{h_i \in \{0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}, h_i=\frac{1}{2} \Leftrightarrow d_i=1} 2^{|d|} L(h_1, \ldots, h_r).
\]

6 Code vertex operator algebras

We study the code vertex operator algebra \(MC\) for any binary even code \(C\) in this section following \textbf{[M1]}, \textbf{[M2]}, and \textbf{[LY2]}.

Fix an even linear code \(C \subset \mathbb{Z}_2^r\). Let \(c = (c_1, \ldots, c_r) \in C\). For short, we set \(L(c) = L(h_1, \ldots, h_r)\) where \(h_i = 0\) if \(c_i = 0\) and \(h_i = \frac{1}{2}\) if \(c_i = 1\). Regard each \(L(c)\) as a subspace of \(V(H_F)_0\). Then
\[
MC = \bigoplus_{c \in C} L(c)
\]
is a vertex operator algebra by Theorem 3.1 and Lemma 2.5 with \(C(M_C) = C\) \textbf{[M2]}. Moreover, any code vertex operator algebra \(V\) with \(C(V) = C\) is isomorphic to \(MC\) by the uniqueness of the simple current extension \textbf{[DM]}.

Clearly, \(\tau(d)\) is also an automorphism of \(MC\) and \(\tau(d)|_{MC} = 1\) if and only if \(d \in C^\perp\) which consists of all codewords in \(\mathbb{Z}_2^r\) orthogonal to \(C\) with respect to the standard bilinear form on \(\mathbb{Z}_2^r\). As a result, \(V(H_F, d)\) is a \(\tau(d)\)-twisted \(MC\)-module and \(V(H_F, d)\) is a \(MC\)-module if and only if \(d \in C^\perp\). We next decompose \(V(H_F, d)\) into a direct sum of irreducible \(MC\)-modules for \(d \in C^\perp\). For this purpose, we consider the decomposition
\[
V(\mathbb{F}a_i, d_i) = V(\mathbb{F}a_i, d_i)_0 \oplus V(\mathbb{F}a_i, d_i)_1
\]
where
\[
V(\mathbb{F}a_i, d_i)_s = \langle a_i(-n_1)a_i(-n_2)\cdots a_i(-n_k)|k \equiv s(\mod 2), n_i \geq 0, n_i \in \frac{1}{2}(1 + d_i) + \mathbb{Z}\rangle
\]
for \(s = 0, 1\). Then \(V(\mathbb{F}a_i, d_i)_s \cong L\left(\frac{1}{2}, \frac{1}{16}\right)\) if \(d_i = 1\) for \(s = 0, 1\) (cf. \textbf{[KR]}, \textbf{[DR2]}). Then
\[
V(H_F, d) = \bigoplus_{s=(s_1, \ldots, s_r) \in \mathbb{Z}_2^r} V(\mathbb{F}a_1, d_1)_{s_1} \wedge \cdots \wedge V(\mathbb{F}a_r, d_r)_{s_r}.
\]
For each coset \((x_1, \ldots, x_r) + C \in \mathbb{Z}_2^r/C\), we set
\[
V(H_F, d, C + (x_1, \ldots, x_r)) = \bigoplus_{s=(s_1, \ldots, s_r) \in (x_1, \ldots, x_r) + C} V(\mathbb{F}a_1, d_1)_{s_1} \wedge \cdots \wedge V(\mathbb{F}a_r, d_r)_{s_r}.
\]
Let \( cl(d) \) be a subalgebra of \( A(H_\mathbb{F}, d) \) generated by \( a_i(0) \) with \( d_i = 1 \). Then \( cl(d) \) is a finite dimensional semisimple associative algebra which has one simple module if \( d \) is even and two inequivalent simple modules if \( d \) is odd. Let \( K(d) = \{ c \in C | \text{supp}(c) = \{i_1, \ldots, i_k\} \subset \text{supp}(d) \} \) and \( E(d) \) a maximal subcode of \( K(d) \) such that \( E(d) \subset E(d)^\perp \). For \( c \in K(d) \), we set \( e^c = 2^{k/2}a_{i_1}(0) \cdots a_{i_k}(0) \in cl(d) \). Then \( G(d) = \{ \pm e^c | c \in K(d) \} \) is a finite group and the group algebra \( \mathbb{F}[G(d)] = \bigoplus_{e \in K(d)} e^c \) is a semisimple subalgebra of \( cl(d) \). Moreover, \( A(d) = \{ \pm e^c | c \in E(d) \} \) is a maximal abelian subgroup of \( G(d) \).

Theorem 6.1. Let \( d \in C^\perp \) and \( (x_1, \ldots, x_r) + C \in \mathbb{Z}_2/C \).

(1) \( V(H_\mathbb{F}, d, C + (x_1, \ldots, x_r)) \) is a direct sum of \( |E(d)| \) irreducible \( M_C \)-modules and each irreducible submodule is a direct sum of \( |C/E(d)| \) irreducible \( T_r \)-modules. Moreover, each irreducible \( M_C \)-submodule is determined by a character of \( A(d) \).

(2) Every irreducible \( M_C \)-module is obtained in this way.

Proof. (1) Clearly, \( V(H_\mathbb{F}, d, C + (x_1, \ldots, x_r)) \) is a \( M_C \)-module. Let \( x = (x_1, \ldots, x_r) \) and set

\[
W = \bigoplus_{t=(t_1, \ldots, t_r) \in x+E(d)} V(\mathbb{F}a_1, d)_{t_1} \wedge \cdots \wedge V(\mathbb{F}a_r, d)_{t_r}.
\]

Note that if \( d_i = 0 \) then \( t_i = x_i \). Also, \( V(\mathbb{F}a_i, 0)_{t_i} \) is isomorphic to \( L(\frac{1}{2}, \frac{-1}{2})_{\mathbb{F}} \), and if \( d_i = 1 \), \( V(\mathbb{F}a_i, 1)_{t_i} \) is isomorphic to \( L(\frac{1}{2}, \frac{1}{2})_{\mathbb{F}} \). This gives

\[
W \cong |E(d)|L(h_1, \ldots, h_r)_{\mathbb{F}}
\]

as \( T_r \)-module where \( h_i = \frac{1}{16} \) if \( i \in \text{supp}(d) \), \( h_i = x_i \) if \( i \notin \text{supp}(d) \).

It is evident that the top level \( T(W) \) of \( W \) has a basis

\[
e^c \prod_{d_i=1,x_i=1} a_i(0) \prod_{d_i=0,x_i=1} a_i(-1/2)
\]

for \( c \in E(d) \). Moreover, \( T(W) \) is isomorphic to \( \mathbb{F}[A(d)]/(-e^0 + 1) \) as \( A(d) \)-modules where \( (-e^0 + 1) \) is the ideal of \( \mathbb{F}[A(d)] \) generated by \( -e^0 + 1 \). Since \( A(d) \) is an abelian group, \( T(W) = \oplus_{\lambda} T(W)_\lambda \) is a sum of irreducible \( A(d) \)-modules \( T(W)_\lambda \) with a character \( \lambda \) such that \( \lambda(-e^0) = -1 \). Note that the module for the code vertex operator algebra \( M(d)_{E(d)} \) generated by \( T(W)_\lambda \) is an irreducible \( T_r \)-module isomorphic to \( L(h_1, \ldots, h_r)_{\mathbb{F}} \). We denote this \( M_{E(d)} \)-module by \( M_{E(d)}(x, \lambda) \).

Now consider the induced module \( \text{Ind}_{A(d), T(W)_\lambda}^{G(d)}(W) \) which is an irreducible \( G(d) \)-module by [FLM] and whose dimension is \( |K(d) : E(d)| \). Then the \( M_K(d) \)-module generated by \( \text{Ind}_{A(d), T(W)_\lambda}^{G(d)}(W) \) is an irreducible \( M_K(d) \)-module as \( M_K(d) \) is rational [DR1]. Clearly, this irreducible is isomorphic to \( [K(d) : E(d)]L(h_1, \ldots, h_r)_{\mathbb{F}} \) as \( T_r \)-module. We denote this irreducible \( M_K(d) \)-module by \( M_K(d)(x, \lambda) \).

Let \( M_C(d, x, \lambda) \) be the \( M_C \)-module generated by \( T(W)_\lambda \). Then

\[
M_C(d, x, \lambda) = \sum_{s+K(d) \in C/K(d)} M_{K(d) + s} \cdot M_{K(d)}(x, \lambda)
\]
where $M_{K(d)+s} \cdot M_{K(d)}(x,\lambda)$ is spanned by $u_{n}M_{K(d)}(x,\lambda)$ for $u \in M_{K(d)+s}$ and $n \in \mathbb{Z}$. Clearly, $M_{K(d)+s} \cdot M_{K(d)}(x,\lambda)$ is also the fusion product of $M_{K(d)+s}$ with $M_{K(d)}(x,\lambda)$ as $M_{K(d)+s}$ is a simple current. Moreover,

$$M_{K(d)+s} \cdot M_{K(d)}(x,\lambda) \cong M_{K(d)}(s+x,\lambda)$$

is an irreducible $M_{K(d)}$-module. Note that if $s+K(d) \neq t+K(d)$ for $s,t \in C$ then each irreducible $T_{r}$-module in $M_{K(d)}(s+x,\lambda)$ and $M_{K(d)}(t+x,\lambda)$ has different $\frac{1}{2}$ positions. This shows that $M_{C}(d,\lambda)$ is an irreducible $M_{C}$-module.

From the definitions of $V(H_{\mathcal{F}},d,C+x)$ and $W$, we see that $V(H_{\mathcal{F}},d,C+x) = M_{C} \cdot W$. This leads to the decomposition

$$V(H,d,C+x) = \oplus_{\lambda} M_{C}(d,\lambda). \quad (6.2)$$

The proof of (2) is similar to that given in [M1], [M2], [LY2] when $\mathcal{F} = \mathbb{C}$. \hfill \square

7 $\mathbb{D}$-forms

In this section, we construct a $\mathbb{D}$-form for any framed vertex operator algebra over $\mathbb{C}$ where $\mathbb{D} = \mathbb{Z}[\frac{1}{2}]$. The main idea is to investigate the $\mathbb{D}$-forms of code vertex operator algebras and their certain irreducible modules.

**Definition 7.1.** A $\mathbb{D}$-form of a vertex superalgebra $V$ over $\mathbb{C}$ is a $\mathbb{D}$-submodule $I$ of $V$ such that $(I,Y,1,\omega)$ is a vertex operator superalgebra over $\mathbb{D}$ and for each $n$, $I_{n} = I \cap V_{n}$ is a $\mathbb{D}$ form of $V_{n}$ for all $n$ in the sense that $I_{n}$ is a free $\mathbb{D}$-module whose rank is equal to the dimension of $V_{n}$.

Unlike [DG], we do not assume that there is a nondegenerate symmetric bilinear form on $V$. Also note that $I \cap V_{0}$ is a vertex operator algebra $\mathbb{D}$-form for $V_{0}$.

**Definition 7.2.** Let $V$ be a vertex operator superalgebra and $I$ a $\mathbb{D}$-form of $V$. Assume that $M = \oplus_{h \in \mathbb{C}} M_{h}$ is an ordinary $V$-module [DLM2]. A $\mathbb{D}$-form $S = \oplus_{h \in \mathbb{C}} S_{h}$ with $S_{h} = S \cap M_{h}$ of $M$ over $I$ is a module for vertex operator algebra $I$ and each $S_{h}$ is a $\mathbb{D}$-form of $M_{h}$.

We remark that if $I$ is a $\mathbb{D}$-form of $V$, then $I \cap V_{1}$ is a $\mathbb{D}$-form of $V_{1}$ over the $\mathbb{D}$-form $I \cap V_{0}$.

Fix an even linear binary code $C \subset \mathbb{Z}_{2}^{r}$. Let $\mathbb{D} = \mathbb{Z}[\frac{1}{2}]$. Set $(M_{C+x})_{\mathbb{D}} = M_{C+x} \cap V(H_{\mathbb{D}})$ for any $x \in \mathbb{Z}_{2}^{r}$. It is clear that $(M_{C})_{\mathbb{D}}$ is vertex operator algebra over $\mathbb{D}$ and $(M_{C+x})_{\mathbb{D}}$ is an $\mathbb{M}_{C}$-$\mathbb{D}$-module. Clearly, $(M_{C+x})_{\mathbb{D}}$ is $\mathbb{D}$-form of $M_{C+x}$ over $(M_{C})_{\mathbb{D}}$. Recall the invariant bilinear from $(\cdot,\cdot)$ on $V(H_{\mathbb{D}})$. We also denote the restriction of the form to $(M_{C+x})_{\mathbb{D}}$ by $(\cdot,\cdot)$. We call

$$(M_{C+x})_{\mathbb{D}}^{*} = \{ u \in M_{C+x}| (u,(M_{C+x})_{\mathbb{D}}) \subset \mathbb{D} \}$$

the dual of $(M_{C+x})_{\mathbb{D}}$ in $M_{C+x}$ with respect to the form.

**Lemma 7.3.** The form on $(M_{C+x})_{\mathbb{D}}$ is self dual. That is, $(M_{C+x})_{\mathbb{D}}^{*} = (M_{C+x})_{\mathbb{D}}$. 


Proof. From the definition we see that

\[(M_{C+x})_D = \bigoplus_{c=(c_1,\ldots, c_r) \in C+x} V(Da_1)_{c_1} \wedge \cdots \wedge V(Da_r)_{c_r}\]

and \((M_{C+x})_D\) has a \(D\)-base consisting of monomials in \(M_{C+x}\). Since these monomials form an orthonormal base of \((M_{C+x})_D\), the result follows. \(\square\)

Next we study the \(D\)-form of \(M_C\)-module with the \(\frac{1}{16}\) position code \(d \in C^\perp\). Recall the irreducible \(M_C\)-module \(M_C(d, x, \lambda)\) over \(\mathbb{C}\) and the code \(E(d)\) from Section 6. For our purpose, we only deal with the case when \(M_C(d, x, \lambda)\) is a self-dual simple current. The following result comes from \([LY2]\) Corollary 4 and Proposition 5].

**Lemma 7.4.** Let \(K(d)\) and \(E(d)\) be as in Section 6. An \(M_C\)-module \(M_C(d, x, \lambda)\) is a simple current if and only if \(E(d)\) is a self-dual subcode of \(K(d)\). Furthermore, if \(M_C(d, x, \lambda)\) is self-dual, then we can choose \(E(d)\) to be doubly even.

Here we present a useful fact about elementary abelian 2-groups.

**Lemma 7.5.** Let \(G\) be an elementary abelian 2-group, i.e., \(o(a) = 2\) for all \(a \in G \setminus \{e\}\). Let \(k = o(G)\). Then \(\mathbb{C}[G]\) has a basis \(\{u_1, \ldots, u_k\} \subset \mathbb{D}[G]\) such that each \(\mathbb{D}u_i\) is a \(\mathbb{D}[G]\)-module.

**Proof.** We proceed by induction on \(k\). If \(k = 2\) then \(G = \{e, a\}\). We can take \(u_1 = e + a\), and \(u_2 = e - a\). Assume the result for \(k\), we now prove the result for \(2k\). Let \(G\) be generated by \(\{a_1, \ldots, a_m\}\) such that \(2^{m-1} = k\). Let \(G_1\) be the subgroup of \(G\) generated by \(\{a_1, \ldots, a_{m-1}\}\). Then \(\mathbb{C}[G] = \mathbb{C}[G_1](\mathbb{C}e + \mathbb{C}a_m)\). Let \(v_1, \ldots, v_k\) be a basis of \(\mathbb{C}[G_1]\) such that \(\mathbb{D}v_i\) is a \(\mathbb{D}[G_1]\)-module. Set \(u_i = v_i(e + a_m)\) for \(i = 1, \ldots, k\) and \(u_i = v_i(e - a_m)\) for \(i = k + 1, \ldots, 2k\). It is clear \(\mathbb{D}u_i\) is a \(\mathbb{D}[G]\)-module. \(\square\)

We remark that only numbers 0, 1, −1 are used in the proof of Lemma 7.5, so Lemma 7.5 is valid with \(\mathbb{D}\) replaced by \(\mathbb{Z}\).

We can now give our key Lemma.

**Lemma 7.6.** If \(M_C(d, x, \lambda)\) is a self-dual simple current, then \(M_C(d, x, \lambda)\) has a \(D\)-form \(M_C(d, x, \lambda)\) such that \(M_C(d, x, \lambda)_D\) is an \((M_C)_D\)-module.

**Proof.** From the definition, we notice that \(V(H_D, d)\) is a \(D\)-submodule of \(V(H_C, d)\). In fact, the monomials form a \(D\)-base of \(V(H_D, d)\). We claim, in fact, that \(V(H_D, d)\) is a \(D\)-form of \(V(H_C, d)\) over \(V(H_D)\). It is good enough to show that for any \(u \in V(H_D)\) and \(m \in \mathbb{Z}\), \(u_mV(H_D, d) \subset V(H_D, d)\). Let \(u = a_i(-n_1 - 1/2)\cdots a_i(-n_k - 1/2)\) with \(n_j \in \mathbb{Z}_+\), and we prove the claim by induction on \(k\). If \(k = 1\) then \(Y(a_i(-n - 1/2), z) = \partial_n a_i(z)\) where

\[a_i(z) = \sum_{n \in \frac{1}{2}(d_i+1) + \mathbb{Z}} a_i(n)z^{-n-1/2}.\]

It follows immediately that \(Y(a_i(-n - 1/2), z)V(H_D, d) \subset V(H_D, d)[[z^{1/2}, z^{-1/2}]]\).

Now assume that the claim is true for \(k\). Using the twisted Jacobi identity, we see that for any \(w \in V(H_D, d)\),

\[(a_{i_1}(-n_1 - 1/2)\cdots a_{i_{k+1}}(-n_{k+1} - 1/2))_mw = \sum_j z_j a_{i_1}(p_j)v_{q_j}w\]

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for some $z_j \in \mathbb{D}$ where $v = (a_{i_2}(-n_2 - 1/2) \cdots a_{i_{k+1}}(-n_{k+1} - 1/2)$. So the claim is true for $k + 1$. One can also use the explicit expression of $Y(u, z)$ from [FFR] to see that each component $u_m$ is a $\mathbb{D}$ linear combination of operators $a_{i_k}(m) \cdots a_{i_1}(m_k)$.

Recall that $M_C(d, x, \lambda)$ is a $M_C$-submodule of $V(H_C, d)$. Let

$$M_C(d, x, \lambda)_{\mathbb{D}} = M_C(d, x, \lambda) \cap V(H_{\mathbb{D}}, d).$$

Clearly, $M_C(d, x, \lambda)_{\mathbb{D}}$ is an $(M_C)_{\mathbb{D}}$-module. It remains to show that $M_C(d, x, \lambda)_{\mathbb{D}}$ is nonzero. By Lemma 7.4, $E(d)$ is a doubly even self dual subcode of $K(d)$. So the length of each codeword of $E(d)$ is a multiple of 4. Since $a_i(0)a_j(0) + a_j(0)a_i(0) = \delta_{i,j}$, $A(d)$ is an abelian group such that each element has order less than or equal to 2. Moreover $G = \{ e^c | c \in E(d) \}$ is a subgroup of $A(d)$ of index 2 such that any irreducible $A(d)$-module such that $-e^0$ acts as $-1$ is an irreducible $G$-module.

We need to recall how $M_C(d, x, \lambda)$ is constructed in the proof of Theorem 6.1. In particular, the top level $T(W)$ of $W$ has a basis $e^c\prod_{d_i=1,x_i=1} a_i(0) \prod_{d_i=0,x_i=1} a_i(-1/2)$ for $c \in E(d)$. Clearly, $T(W)$ is isomorphic to $\mathbb{C}[G]$ as $G$-module by identifying $e^c$ with $e^c\prod_{d_i=1,x_i=1} a_i(0) \prod_{d_i=0,x_i=1} a_i(-1/2)$ for $c \in E(d)$. From the definition of $V(H_{\mathbb{D}}, d)$, it is evident that $e^c\prod_{d_i=1,x_i=1} a_i(0) \prod_{d_i=0,x_i=1} a_i(-1/2)$ lies in $V(H_{\mathbb{D}}, d)$ for $c \in E(d)$. By Lemma 7.5 there is basis

$$\{ u_1, \ldots, u_k \} \subset \mathbb{D}[G] \prod_{i,d_i=0,x_i=1} a_i(-1/2)$$

of $T(W)$ such that $Cu_i$ is an irreducible $G$-module. Let $T(W)_\lambda = Cu_i$ for some $i$. Then $u_i \in M_C(d, x, \lambda)_{\mathbb{D}}$. We conclude that $M_C(d, x, \lambda)_{\mathbb{D}}$ is a $\mathbb{D}$-form over $(M_C)_{\mathbb{D}}$. □

**Remark 7.7.** From the proof of 7.6, we see that $Cu_i$ is an irreducible $G$-module. Let $\lambda_i$ be the corresponding characters. Then

$$V(H, d, C + x) = \oplus_{i=1}^k M_C(d, x, \lambda_i)$$

and

$$V(H_{\mathbb{D}}, d, C + x) = \oplus_{i=1}^k M_C(d, x, \lambda_i)_{\mathbb{D}}.$$

To prove any framed vertex operator algebra over $\mathbb{C}$ has a $\mathbb{D}$-form, we need an invariant bilinear form on $V(H_C, d)$ and $M_C(d, x, \lambda)$. Recall that $V(H_{\mathbb{D}}, d)$ has a monomial basis. We define a bilinear form $(\cdot, \cdot)$ on $V(H_C, d)$ such that the monomials form an orthogonal basis. More precisely, let $u = a_{i_1}(0) \cdots a_{i_k}(0)v$ where $v$ is a monomial without any $a_i(0)$ and $i_p \neq i_q$ if $p \neq q$. We define the square length of $u$ is $\frac{1}{2^n}$. It is trivial to show that for any $u, v \in V(H_C, d), i = 1, \ldots, n$ and $r \in \frac{1}{2} \mathbb{Z}$, $(a_i(n)u, v) = (u, a_i(-n)v)$. The next result is an analogue of Proposition 5.2.

**Lemma 7.8.** Let $u, v \in V(H_{\mathbb{D}}, d), w \in V(H_{\mathbb{D}})_i, m \in \frac{1}{2} + \mathbb{Z}$ and $n \in \mathbb{Z}$. Then

1. The $\mathbb{D}$-bilinear $\mathbb{D}$-valued form $(\cdot, \cdot)$ on $V(H_{\mathbb{D}}, d)$ is positive definite.
2. $V(H_{\mathbb{D}}, d)$ is self dual in $V(H_C, d)$ in the sense that

$$\{ w \in V(H_C, d) | (w, V(H_{\mathbb{D}}, d)) \subset \mathbb{D} \} = V(H_{\mathbb{D}}, d).$$

3. $(L(n)u, v) = (u, L(-n)v)$,
4. $\frac{L(1)^n}{n!}$ is well defined on $V(H_{\mathbb{D}}, d)$ if $n \geq 0$,
5. The form is invariant:

$$(Y(w, z)u, v) = (u, Y(e^{zL(1)}(-1)^{\deg w + 2(\deg w)^2 + r} z^{-2\deg w} w, z^{-1})v).$$
Proof. (1) and (2) are clear from the definition of the form. 
(3) We know from [KoR] that
\[ L(0) = \frac{|d|}{16} + \frac{1}{2} \sum_{d_i=0} \sum_{j \in \frac{1}{2} + z} j : a_i(-j)a_i(j) : + \frac{1}{2} \sum_{d_i=1} \sum_{j \in z} j : a_i(-j)a_i(j) : \]
and
\[ L(n) = \frac{1}{2} \sum_{d_i=0} \sum_{j \in \frac{1}{2} + z} j : a_i(-j)a_i(j+n) : + \frac{1}{2} \sum_{d_i=1} \sum_{j \in z} j : a_i(-j)a_i(j+n) : \]
if \( n \neq 0 \) where the normal order is defined as in (5.1). Using the explicit expression of
\( L(n) \) gives \( (L(n)u,v) = (u,L(-n)v) \).
(4) A special case of the commutator formula
\[ [L(m), a_i(s)] = -(s + \frac{m}{2})a_i(m + s) \]
gives \( [L(1), a_i(s)] = -(s + \frac{1}{2})a_i(1 + s) \). So
\[ \frac{L(1)^n}{n!} a_i(-s) = \frac{(-1)^n(-s+1/2)\cdots(-s+n-1/2)}{n!} a_i(-s+n) \}
for \( s > 0 \). Clearly, \( \frac{L(1)^n}{n!} a_i(-s) \) lies in \( V(H_D, d) \). Now assume that \( \frac{L(1)^n}{n!} v \) is well defined for \( v \in V(H_D, d) \) and \( u = a_i(-s)v \). Then
\[ \frac{L(1)^n}{n!} u = \sum_{j=0}^n \frac{(ad L(1))^j}{j!} a_i(-s) \frac{L(1)^{n-j}}{(n-j)!} v \]
as \( \frac{(ad L(1))^j}{j!} a_i(-s) = \frac{(-1)^j(-s+1/2)\cdots(-s+j-1/2)}{j!} a_i(-s+j) \) and \( \frac{(-1)^j(-s+1/2)\cdots(-s+j-1/2)}{j!} \in D \).
Since \( a_j(-s+j) \) preserves \( V(H_D, d) \), \( \frac{L(1)^n}{n!} v \) is an element of \( V(H_D, d) \).
(5) The proof is similar to that given in Proposition 6.2 (4). □

Corollary 7.9. Assume that \( M_C(d, x, \lambda) \) is a self-dual simple current. There is a positive definite \( D \)-valued, invariant bilinear form on \( (M_C(d, x, \lambda))_D \). Moreover, the dual of \( M_C(d, x, \lambda)_D \) in \( M_C(d, x, \lambda) \) is itself.

Proof. It is clear from Lemma 7.8 that the restriction of the form to \( M_C(d, x, \lambda)_D \) is positive definite and invariant. In order to prove that \( M_C(d, x, \lambda)_D \) is self dual in \( M_C(d, x, \lambda) \), we recall the decomposition (6.1). Since the dual of \( V(H_D, d) \) in \( V(H_C, d) \) is itself by Lemma 7.8 (2), and each \( V(H_C, d, C+x) \) has a basis consisting of monomial basis, we see that \( V(H_D, d, C+x) \) is self dual in \( V(H_C, d, C+x) \). By (6.2) and the proof of Lemma 7.6
\[ V(H_D, d, C+x) = \oplus_\lambda (M_C(d, x, \lambda))_D. \]
This immediately implies that \( M_C(d, x, \lambda)_D \) is self dual in \( M_C(d, x, \lambda) \). □
8 Intertwining operators over \( \mathbb{D} \).

We construct various intertwining operators associated to the vertex operator algebra \( V(Ca) \) and \( V(H_D) \) and \( (M_C)_D \) over an integral domain \( \mathbb{D} \) in this section. These intertwining operators will be used later to construct a \( \mathbb{D} \)-form for a framed vertex operator algebra over \( \mathbb{C} \).

Let \( V = V(Ca) \oplus V(Ca, 1) \) and define a nondegenerate symmetric bilinear form \( (\cdot, \cdot) \) on \( V \) such that the restriction of the form to \( V(Ca) \) and \( V(Ca, 1) \) is as before, and \( (V(Ca), V(Ca, 1)) = 0 \). We also set \( V_D = V(\mathbb{D}a) \oplus V(\mathbb{D}a, 1) \).

Define a linear map
\[
Y : V \to (\text{End} V)[[z^{1/8}, z^{-1/8}]]
\]
such that \( Y(v, z) \) for \( v \in V(Ca) \) is exactly the vertex operator which defines the vertex operator subalgebra structure on \( V(Ca) \) and the \( \tau \)-twisted \( V(Ca) \)-module structure on \( V(Ca, 1) \) where \( \tau = (-1)^{2L(0)} \) is the canonical automorphism of \( V(Ca) \). For \( u \in V(Ca, 1) \) and \( v \in V(Ca, \tau) \), we define \( Y(u, z)v = (-1)^{s} e^{L(1)z} L(0) - 1/16 z^{-2L(0)} u, z^{-1} w) \) for any \( w \in V(Ca, \tau) \). Using the notation from Section 5, we also denote \( V(Ca) \) by \( V(Ca, 0) \).

**Lemma 8.1.** Let \( d_1, d_2, s, t, 0 = 1 \) and \( u \in V(Ca, d_1) \), \( v \in V(Ca, d_2) \). Then \( Y(u, z)v \in V(Ca, d_1 + d_2)_{s \tau t}[z^{1/8}, z^{-1/8}] \).

**Proof.** If \( d_1 = 0 \), the conclusion is clear from the construction of \( \tau^{d_2} V(Ca) \)-twisted module \( V(Ca, d_2) \). For the case \( d_1 = 1 \), we first notice that \( (V(Ca, d_1), V(Ca, d_2)) = 0 \) for any \( d = 0, 1 \) if \( s_1 \neq t_1 \). The result follows from the definition of \( Y(u, z)v \) and the result with \( d_1 = 0 \). \( \square \)

**Lemma 8.2.** The restriction of \( Y \) to \( V_D \) defines a linear map
\[
V_D \to (\text{End} V_D)[[z^{1/8}, z^{-1/8}]].
\]

**Proof.** From the proof of Lemma 7.6, we know that \( Y(u, z)v \in V_D[[z^{1/8}, z^{-1/8}]] \) for \( u \in V(\mathbb{D}a) \) and \( v \in V_D \). Since \( \frac{1}{\pi} L(-1)^n \) preserves \( V_D \) for \( n \geq 0 \), we see immediately from the definition that \( Y(u, z)v \in V_D[[z^{1/8}, z^{-1/8}]] \) for \( u \in V(\mathbb{D}a, 1) \) and \( v \in V(\mathbb{D}a) \).

Finally we assume that \( u \in V(\mathbb{D}a, 1) \) and \( v \in V(\mathbb{D}a, 1) \). Let \( w \in V(\mathbb{D}a) \). Then
\[
(Y(u, z)v, w) = (-1)^{st}(v, Y(e^{L(1)z} L(0) - 1/16 z^{-2L(0)} u, z^{-1} w)) \in \mathbb{D}[[z^{1/8}, z^{-1/8}]].
\]

Since \( V(\mathbb{D}a) \) is selfdual in \( V(Ca) \), we conclude that \( Y(u, z)v \in V_D[[z^{1/8}, z^{-1/8}]] \), as desired. \( \square \)

**Lemma 8.3.** (1) For \( u \in V(Ca, 1) \) and \( v \in V(Ca) \), \( Y(u, z)v \) is an intertwining operator of type \( \left( \begin{array}{c} V(Ca, 1) \\ V(Ca) \end{array} \right) \).

(2) For \( u \in V(Ca, 1) \) and \( v \in V(Ca, 1) \), \( Y(u, z)v \) is an intertwining operator of type \( \left( \begin{array}{c} V(Ca) \\ V(Ca, 1) \end{array} \right) \).
Proof. (1) The same arguments from [FHL] and [X] show that $Y(L(-1)u, z)v = \frac{d}{dz} Y(u, z)v$ for $u \in V(Ca, 1)$ and $v \in V(Ca)$. It remains to prove the following Jacobi identity for $u \in V(Ca)_r$, $v \in V(Ca)_s$ and $w \in V(Ca)_t$:

$$z_0^{-1} \left( \frac{z_1 - z_2}{z_0} \right)^{r/2} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) w$$

$$= (-1)^{rs} z_0^{-1} \left( \frac{z_2 - z_1}{-z_0} \right)^{r/2} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y(v, z_2) Y(u, z_1) w$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2) w.$$

From the definition, it is equivalent to show that

$$(-1)^{rt} z_0^{-1} \left( \frac{z_1 - z_2}{z_0} \right)^{r/2} \delta \left( \frac{z_1 - z_2}{z_0} \right) e^{z_2 L(-1)} Y(u, z_1 - z_2) Y(w, -z_2) v$$

$$= (-1)^{rs+s(r+t)} z_0^{-1} \left( \frac{z_2 - z_1}{z_0} \right)^{r/2} \delta \left( \frac{z_2 - z_1}{-z_0} \right) e^{z_2 L(-1)} Y(Y(u, z_1)w, -z_2) v$$

or

$$z_0^{-1} \left( \frac{z_1 - z_2}{z_0} \right)^{r/2} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1 - z_2) Y(w, -z_2) v$$

$$= (-1)^{rt} z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(w, -z_2) Y(u, z_0) v$$

$$= z_0^{-1} \left( \frac{z_2 - z_1}{-z_0} \right)^{r/2} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y(Y(u, z_1)w, -z_2) v.$$

Note that $z^{r/2} Y(u, z)$ only involves integral powers of $z$. So

$$z_0^{-1} \left( \frac{z_1 - z_2}{z_0} \right)^{r/2} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1 - z_2) = z_1^{-1} \delta \left( \frac{z_0 + z_2}{z_1} \right) Y(u, z_0).$$

Also,

$$z_0^{-1} \left( \frac{z_2 - z_1}{-z_0} \right)^{r/2} \delta \left( \frac{z_2 - z_1}{-z_0} \right) = (-z_2)^{-1} \left( \frac{z_0 - z_1}{-z_2} \right)^{-r/2} \delta \left( \frac{z_0 - z_1}{-z_2} \right).$$

Thus we need to show that

$$z_1^{-1} \delta \left( \frac{z_0 + z_2}{z_1} \right) Y(u, z_0) Y(w, -z_2) v$$

$$= (-1)^{rt} z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(w, -z_2) Y(u, z_0) v,$$

$$= (-z_2)^{-1} \left( \frac{z_0 - z_1}{-z_2} \right)^{-r/2} \delta \left( \frac{z_0 - z_1}{-z_2} \right) Y(Y(u, z_1)w, -z_2) v$$

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which is the Jacobi identity in the definition of $\tau$-twisted module.

(2) Now let $u \in V(\mathbb{C}a)_r$, $v \in V(\mathbb{C}a, 1)_s$ and $w \in V(\mathbb{C}a, 1)$. It follows from the proof of Theorem 4.4 of [X] that the following Jacobi identity holds:

$$z_0^{-1} \left( \frac{z_1 - z_2}{z_0} \right)^{r/2} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2)w$$

$$= (-1)^{rs}z_0^{-1} \left( \frac{z_2 - z_1}{z_0} \right)^{r/2} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(v, z_2)Y(u, z_1)w$$

$$= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/2} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2)w.$$ 

Also see [X] for the $L(-1)$-derivation property. □

Recall from Section 5 that $H = \bigoplus_{i=1}^r \mathbb{C}a_i$ is vector space with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ such that $\{a_i | i = 1, ..., r\}$ form an orthonormal basis. Let $d = (d_1, ..., d_r)$, $e = (e_1, ..., e_r) \in \mathbb{Z}_r$. Let $\tau(d)$ and $\tau(e)$ be the associated automorphism of vertex operator superalgebra $V(H)$. Then $V(H, d)$ is a $\tau(d)$-twisted $V(H)$-module.

We now define a linear map:

$$Y : V(H, d) \rightarrow (\text{Hom}(V(H, e), V(H, d + e))[\{z^{1/8}, z^{-1/8}\}])$$

such that $Y(u^1 \wedge \cdots \wedge u^r, z) = Y(u^1, z) \cdots Y(u^r, z)$ where $u^i \in V(\mathbb{C}a_i, d_i)$ for $i = 1, ..., r$ and $Y(u^i, z)$ is defined as before. The following result is an immediate consequence of Lemma 8.2

**Lemma 8.4.** The restriction of $Y$ to $V(H_D, d)$ defines a linear map

$$V(H_D, d) \rightarrow (\text{Hom}(V(H_D, e), V(H_D, d + e))[\{z^{1/8}, z^{-1/8}\}]).$$

The following result is an extension of Lemma 8.3

**Lemma 8.5.** The linear map

$$Y : V(H, d) \rightarrow (\text{Hom}(V(H, e), V(H, d + e))[\{z^{1/8}, z^{-1/8}\}])$$

defines an intertwining operator of type $\begin{pmatrix} V(H, d + e) \\ V(H, d) \end{pmatrix}$. 

**Proof.** We only prove the following Jacobi identity for any $v \in V(H)^{(j_1,j_2)}, u \in V(H, d)$:

$$z_0^{-1} \left( \frac{z_1 - z_2}{z_0} \right)^{j_1/2} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(v, z_1)Y(u, z_2)$$

$$= (-1)^{s+t} z_0^{-1} \left( \frac{z_2 - z_1}{z_0} \right)^{j_1/2} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(u, z_2)Y(v, z_1)$$

$$= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-j_2/2} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(v, z_0)u, z_2).$$

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where $V(H)^{(ji,jk)}$ is the common eigenspace for $\tau(d), \tau(e)$ with eigenvalues $(-1)^{ji}, (-1)^{jk}$ respectively.

We know from [DL] that the Jacobi identity is equivalent to the commutativity

$$(z_1 - z_2)^{ji/2+n} Y(v, z_1) Y(u, z_2) = (-z_2 + z_1)^{ji/2+n} Y(u, z_1) Y(v, z_2)$$

for some positive integer $n$, and the associativity

$$(z_0 + z_2)^{ji/2+m} Y(v, z_0 + z_2) Y(u, z_2) w = (z_2 + z_0)^{ji/2+m} Y(Y(v, z_0) u, z_2) w$$

where $w \in V(H, e)$ and $m$ is a nonnegative integer which depends on $v$ and $w$ only. Note that

$$Y(u^1 \wedge \cdots \wedge u^r, z) = Y(u^1, z) \cdots Y(u^r, z)$$

for $u^i \in V(\mathbb{C}a_i, x_i)$ and $x_i = 0, 1$ and that $Y(u^i, z)Y(u^j, z) = (-1)^{st} Y(u^j, z)Y(u^i, z)$ if $i \neq j$ and $u^i \in V(\mathbb{C}a_i, x_i)$ and $u^j \in V(\mathbb{C}a_j, x_j)$. The Jacobi identity obtained in the proof of Lemma 8.3 gives the commutativity and associativity. The proof is complete. \qed

We now fix an even binary code $C \subset \mathbb{Z}_2^n$ and assume that $e, d \in C^\perp$.

**Lemma 8.6.** Let $x, y \in \mathbb{Z}_2^n$. Then the restriction of $Y$ to $V(H, d, C + x) \subset V(H, d)$ gives an intertwining operator of type $\begin{pmatrix} V(H, d + e, C + x + y) \\ V(H, d, C + x) \end{pmatrix}$ for vertex operator algebra $M_C$. Moreover, the restriction of $Y$ to $V(H_D, d, C + x)$ gives a linear map

$$Y : V(H_D, d, C + x) \to \text{Hom}(V(H_D, e, C + y), V(H_D, d + e, C + x + y))[[z^{1/8}, z^{-1/8}]].$$

**Proof.** The result is an immediate consequence of Lemmas 8.1, 8.4 and 8.5. \qed

We now assume that $M_C(d, x, \lambda), M_C(e, y, \mu)$ and $M_C(d + e, x + y, \nu)$ are self-dual simple currents.

**Corollary 8.7.** The restriction of the intertwining operator $Y$ in Lemma 8.6 to $M_C(d, x, \lambda)$ is an intertwining operator of type $\begin{pmatrix} M_C(d + e, x + y, \nu) \\ M_C(d, x, \lambda) \end{pmatrix}$. Moreover, for any $u \in M_C(d, x, \lambda)_D$ and $v \in M_C(e, y, \mu)_D$, $n \in \mathbb{Z}$, we have $u_nv \in M_C(d + e, x + y, \nu)_D$.

**Proof.** Let $W_1$ be an irreducible $M_C$-module occurring in $V(H, d, C + x)$ given in Remark 7.7. Similarly we pick an irreducible $M_C$-module $W_2$ in $V(H, e, C + y)$ given in Remark 7.7. Then the span of $u_nv$ for $u \in W_1$ and $v \in W_2$, $n \in \mathbb{Z}$ is an irreducible $M_C$-module inside $V(H, d + e, C + x + y)$. There is an irreducible $M_C$-module $W_3$ in $V(H, d + e, C + x + y)$ given in Remark 7.7 such that the projection of $W$ to $W_3$ is an isomorphism. Clearly, this produces an intertwining operator of type $\begin{pmatrix} M_C(d + e, x + y, \nu) \\ M_C(d, x, \lambda) \end{pmatrix}$. Using Remark 7.7 and Lemma 8.6, we immediately see that if $u \in M_C(d, x, \lambda)_D$ and $v \in M_C(e, y, \mu)_D$, $n \in \mathbb{Z}$ then $u_nv \in M_C(d + e, x + y, \nu)_D$. \qed
9 The $\mathcal{D}$-forms of FVOAS

In this section, we present the main result of this paper. Namely, any framed vertex operator algebra over $\mathbb{C}$ has a $\mathcal{D}$-form. As a result, we can obtain a framed vertex operator algebra over any algebraically closed field whose characteristic is different from 2,7 from any framed vertex operator algebra over $\mathbb{C}$.

Let $V$ be a framed vertex operator algebra over $\mathbb{C}$. It follows from [DGH] that

$$V = \oplus_{d \in \mathcal{D}} V^d$$

and $V^0 = M_C$ is a code vertex operator algebra for some binary even codes $C, D \subset \mathbb{Z}_2^r$ with $D \subset C^\perp$. It follows from [DGH], [LY2], [DJX] that each $V^d$ is a self-dual simple current for the vertex operator algebra $M_C$. According to [LY2], we know that for each $d \in D$, $V^d$ is isomorphic to some $M_C(d, x(d), \lambda(d))$ for some $x(d) = (x(d)_1, \ldots, x(d)_r) \in \mathbb{Z}_2^r$ and $\lambda(d)$. We will use $\mathcal{Y}$ to denote the intertwining operators defined in Corollary 8.7.

**Lemma 9.1.** If $V$ is a framed vertex operator algebra over $\mathbb{C}$ such that $D(V)$ is isomorphic to $\mathbb{Z}_2$, then $V$ has a $\mathcal{D}$-form $V_\mathcal{D}$.

**Proof.** Assume $V = M_C \oplus M_C(d, x, \lambda)$ for some $d \in C^\perp$, $x \in \mathbb{Z}_2^r$ and $\lambda$. It follows from Corollary 8.7 that $\mathcal{Y}(u, z)v \in (M_C)_D \oplus M_C(d, x, \lambda)_D[[z, z^{-1}]]$ for $u, v \in (M_C)_D \oplus M_C(d, x, \lambda)_D$. On the surface, $(M_C)_D \oplus M_C(d, x, \lambda)_D$ is a perfect candidate for the $\mathcal{D}$-form of $V$. The problem is that we do not know the $\mathcal{Y}$ which defines the vertex operator algebra structure on $V$ is equal to $\mathcal{Y}$. From the definition of $\mathcal{Y}$ and the skew symmetry, we know that $Y(u, z)v = \mathcal{Y}(u, z)v$ for $u \in M_C$, or $u \in M_C(d, x, \lambda)$ and $v \in M_C$. Now we assume that $u, v \in M_C(d, x, \lambda)$. Since $M_C(d, x, \lambda)$ is a simple current, there exists a nonzero complex number $a$ such that $Y(u, z)v = a\mathcal{Y}(u, z)v$ for any $u, v$. We now set $V_\mathcal{D} = (M_C)_D \oplus a^{-1/2}M_C(d, x, \lambda)_D$. It is clear that $V_\mathcal{D}$ is a free module over $\mathcal{D}$. To prove that $V_\mathcal{D}$ is a $\mathcal{D}$-form, it is good enough to show that $Y(u, z)v \in V_\mathcal{D}[[z, z^{-1}]]$ for $u, v \in V_\mathcal{D}$. From the discussion before, this is clear if $u \in (M_C)_D$, or $u \in M_C(d, x, \lambda)_D$ and $v \in (M_C)_D$. Now we let $u = a^{-1/2}u^1, v = a^{-1/2}v^1 \in a^{-1/2}M_C(d, x, \lambda)_D$. Then $Y(u, z)v = \mathcal{Y}(u^1, z)v^1 \in V_\mathcal{D}$. □

We certainly believe that the constant $a$ in the proof of Lemma 7.3 is 1. But we cannot prove it in this paper.

We now can deal with the general case.

**Theorem 9.2.** Any framed vertex operator algebra $V$ over $\mathbb{C}$ has a $\mathcal{D}$-form $V_\mathcal{D}$. Moreover, for any algebraically closed field $\mathbb{F}$ whose characteristic is different from 2,7, $V_\mathcal{F} = \mathbb{F} \otimes_\mathcal{D} V_\mathcal{D}$ is a framed vertex operator algebra over $\mathbb{F}$.

**Proof.** Let $D = D(V)$. For any $d, e \in D$, $u \in M_C(d, x(d), \lambda(d))$, $v \in M_C(e, x(e), \lambda(e))$ and $n \in \mathbb{Z}$,

$$Y(u, z)v \in M_C(d + e, x(d + e), \lambda(d + e))[[z, z^{-1}]].$$

By Lemmas 8.1 and 8.5

$$\mathcal{Y}(u, z)v \in V(H, d + e, x(d) + x(e) + C)[[z, z^{-1}]].$$
Since both $M_C(d, x(d), \lambda(d))$ and $M_C(e, x(e), \lambda(e))$ are simple currents, this implies that $M_C(d + e, x(d + e), \lambda(d + e))$ is an irreducible $M_C$-submodule of $V(H, d + e, x(d + e) + C)$ and $x(d) + x(e) + C = x(d + e) + C$. So $V(H, d + e, x(d + e) + C) = V(H, d + e, x(d + e) + C)$.

Now let $\{d^1, \ldots, d^k\}$ be a least generating set of $D$. Then

$$V = \bigoplus_{1 \leq i_1 < \cdots < i_s \leq k} V^s \sum_{p=1}^s d^p = \bigoplus_{1 \leq i_1 < \cdots < i_s \leq k} M_C(\sum_{p=1}^s d^p, x(\sum_{p=1}^s d^p), \lambda(\sum_{p=1}^s d^p)).$$

Using the commutativity and associativity of vertex operators (see Propositions 4.5.7 and 4.5.8 of [1]), we see that $M_C(\sum_{p=1}^s d^p, x(\sum_{p=1}^s d^p), \lambda(\sum_{p=1}^s d^p))$ is spanned by $u_{1n_1} \cdots u_{sn_s} 1$ for $u^v \in M(d^v, x(d^v), \lambda(d^v))$ and $n_p \in \mathbb{Z}$.

According to Lemma 4.11 for each $d^j$, there exists a nonzero constant $a_j$ such that $(M_C)_d + a_j M_C(d^j, x(d^j), \lambda(d^j))_d$ is a $d$-form of $M_C + M_C(d^j, x(d^j), \lambda(d^j))$. For short, we set $V^0_d = (M_C)_d$ and $V^d_d = a_j M_C(d^j, x(d^j), \lambda(d^j))$. For any subset $X, Y$ of $V$, we denote the $d$-span of $x_n$ for $x \in X$ and $n \in \mathbb{Z}$ by $X \cdot Y$. For any $d \in D$, there exists

$$V_d = \bigoplus_{d \in D} V^d_d.$$

Clearly, $V = C \otimes D V_d$. We claim $V_d$ is the $d$-form of $V$.

First we prove that each $V^d_d$ is a free $d$-module. We need a general result that for $d, e \in D$, $0 \neq a, b \in C$, the $d$-span of $u_n v$ for $u \in a M_C(d, x(d), \lambda(d))$, $v \in b M_C(e, x(e), \lambda(e))$ and $n \in \mathbb{Z}$ is contained in $c M_C(d + e, x(d + e), \lambda(d + e))$ for some $0 \neq c \in C$. Since $Y(u, z) = f Y(u, z) v$ for some nonzero constant $f$ which depends on $d$ and $e$ only, and

$$Y(u, z) v \in ab M_C(d + e, x(d + e), \lambda(d + e)) \mathbb{Z}[z, z^{-1}],$$

we see that $u_n v \in f a b M_C(d + e, x(d + e), \lambda(d + e))$. Now let $d = d^i + \cdots + d^s$. Then there exists a nonzero constant $\alpha \in C$ such that $V^d_d$ is a $d$-submodule of $\alpha M_C(d, x(d))$. In particular, $V^d_d$ is a free $d$-module.

We finally prove that $V_d$ is a vertex operator algebra over $D$. Observe that for $u, v, w \in V$, $m, n \in \mathbb{Z},$

$$Y(u_m v, z) = \text{Res}\{(z_1 - z)^m Y(u, z_1) Y(v, z) - (z - z_1)^m Y(v, z) Y(u, z_1)\},$$

$$u_m v_n w = \sum_i \alpha_i(u, v)_i t_i u_w$$

for some integers $\alpha_i, \beta_j$.

Now let $d = d^i + \cdots + d^s$, $e = d^j + \cdots + d^t \in D$. Then $V^d_d = V^{d^i} d \cdots V^{d^s} d \cdot V^d_d$ and $V^e_d = V^{d^j} d \cdots V^{d^t} d \cdot V^d_d$. Using the three relations above we see that

$$V^d_d \cdot V^e_d \subset V^{d^i} d \cdots V^{d^s} d \cdot V^{d^j} d \cdots V^{d^t} d \cdot V^d_d.$$

Note that for any $j$, $V^0_d \cdot V^d_d \subset V^d_d$ and $V^d_d \cdot V^0_d \subset V^0_d$, we immediately see that $V^d_d \cdot V^e_d$ is contained in $V^{d^i} d \cdots V^{d^s} d \cdot V^{d^j} d \cdots V^{d^t} d$. This shows that $V^d_d$ is a vertex operator algebra over $D$, as desired. □
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