The density matrix recursion method: genuine multisite entanglement distinguishes odd from even quantum spin ladder states

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Abstract. We introduce an analytical iterative method, the density matrix recursion method, to generate arbitrary reduced density matrices of superpositions of short-range dimer coverings on periodic or non-periodic quantum spin-1/2 ladder lattices, with an arbitrary number of legs. The method can be used to calculate bipartite as well as multipartite physical properties, including bipartite and multi-partite entanglement. We apply this technique to distinguish between even- and odd-legged ladders. Specifically, we show that while genuine multi-partite entanglement decreases with increasing system size for the even-legged ladder states, it does the opposite for odd-legged ones.

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1. Introduction

The quantum spin ladder is an interesting platform to investigate quantum many-body systems in the intermediate sector between one-dimensional (1D) and two-dimensional (2D) lattice structures [1]. The possibility of relating doped even-legged quantum spin ladders to high-temperature superconductivity [2–4] makes such quantum systems extremely important. The characteristic pseudo 2D structure of ladders has generated considerable interest in several other areas of condensed matter physics [5, 6] and quantum information [7]. On the other hand, concepts such as entanglement [8] and other quantum correlations [9] have been applied to understand quantum critical phenomena in spin systems [10], including in spin ladders [7]. The properties of such systems have also been tested experimentally in several systems and proposals thereof have been presented, including in compounds and cold gases [11]. Ladder states formed by the superposition of short-range dimer coverings, also known as short-range resonating valence bond (RVB) states, can be simulated by using atoms in optical lattices [12] and, as shown recently, by using interacting photons [13]. These short-range RVB ladder states are believed to be possible ground states of certain undoped Heisenberg spin ladders, supported by various methods that include mean field theory [14], quantum Monte Carlo [4, 15] and Lanczos [5, 16].

A striking feature of the quantum spin ladder is that the interpolation from the 1D spin chain to the 2D square lattice by gradually increasing the number of legs is not straightforward. For example, the quantum characteristics of the Heisenberg ladder ensure that the odd- and the even-legged ladder ground states have different correlation properties. ‘Even ladders’ have a finite-gapped ground state excitation and exponential decay of two-site correlations, whereas ‘odd ladders’ are gapless and have power-law decay [6, 17]. Such effects also occur in quantum spin liquids, such as discussed in [18]. The results of Heisenberg ladders cannot, therefore, be directly extrapolated to the 2D regime. This difference in quantum characteristic of odd- and even-legged ladders may lead to interesting features in the entanglement properties of the system. However, calculating the bipartite or multi-partite entanglement in large-sized multi-legged quantum spin ladder states formed by the superposition of short-range dimer coverings is numerically challenging. Within a dimer-covering approach [6], it is possible to
derive energy density and spin-correlation functions for two-legged ladders by using generating functions [19] or state iterations [20]. Approximate solutions may also be obtained for the four-legged ladder [21]. For further work in this direction, including density matrix renormalization group calculations in multi-legged Heisenberg ladders, see [22].

In this paper, we introduce an analytical iterative technique, the ‘density matrix recursion method’ (DMRM), to obtain the reduced density matrix of an arbitrary number of sites of a state on a quantum spin-1/2 ladder with an arbitrary number of legs and with both open and periodic boundary conditions. We consider the spin-1/2 ladder state to be a superposition of short-range dimer coverings. Specifically, within a dimer covering approach for the state of the quantum spin ladders, we find separate iterative formalisms for even and odd ladders. These partial density matrices can be used to calculate and study the scaling and behavior of single-, two- and multi-site physical properties of the whole spin system, including two-site correlations and bipartite as well as multi-partite entanglement. The iterative method introduced here can also be a potential tool for studying the properties of reduced density matrices of general large superposed multipartite quantum states and hence can be used as an efficient method for investigating quantum correlations of multipartite quantum states.

We then apply our method to obtain the nature of genuine multi-partite entanglement, quantified by the generalized geometric measure (GGM) [23]. An understanding of the multipartite entanglement content is potentially advantageous, in assessing the importance of the corresponding state for future applications, over that of bipartite measures, as the former takes into account the distribution of information between the multi-site sub-regions of the entire system and the hidden correlations are not ignored due to tracing out [24]. We compare the odd- and even-legged ladder states by using genuine multisite entanglement, and find that the GGM of odd-legged ladder states increases with system size while it decreases in the even ones. We also find that the convergence of the GGM, for a given odd- or even-legged ladder, occurs for relatively small ladder lengths of the corresponding ladder.

This paper is organized as follows. We introduce the model state in section 2. The genuine multipartite entanglement measure is defined in section 3. The main results are presented in sections 4–6. In particular, the DMRMs for even and odd ladders, respectively, are introduced in sections 4 and 5. We present the conclusion in section 7.

2. The model state

Within a short-range dimer-covering approach, the state of the quantum spin-1/2 ladder consists of an equal superposition of nearest neighbor (NN) directed dimer pairs on the spin-1/2 lattice, also called the RVB state. The ladder lattice under consideration can be divided into sublattices (see figure 1), A and B, in such a way that all NN sites of sublattice A belong to sublattice B, and vice versa. Such a lattice is called a bipartite lattice. Each bipartite lattice site is occupied by a spin-1/2 qubit. The multi-legged quantum spin-1/2 ladder consists of coupled parallel spin chains with \( N \) sites on each chain labeled from 1 to \( N \). The number of chains, referred to as legs, of the ladder are labeled from 1 to \( M \). The total number of spin-1/2 sites in the entire bipartite lattice is \( s \) (\( s = MN \)). The vertical chains, referred to as rungs, are labeled from 1 to \( n \), similar to the sites on the chain, and each rung contains \( M \) spins (see figure 1). The periodic ladder is conditionally defined by allowing dimer states to be formed between rungs 1 and \( n \) [19].
Figure 1. The $M$-legged, $N$-runged quantum spin-1/2 ladder in the form of a bipartite lattice. The solid circles are of sublattice $A$, whereas the hollow ones are of $B$. The red arrows (solid) show the NN dimer states ($| (a_i, b_j) \rangle$) from a site in sublattice $A$ to another in $B$. The arrows (dashed) at the boundary indicate that the lattice has a periodic boundary condition. The dashed lines indicate that the number of legs and rungs can be extended beyond the illustrated number of sites. The figure also shows the two-site ladder state, $|2\rangle$, and the one-site rung state, $|1\rangle$, which will be profusely used in the iteration method presented.

Of course as numbers, $n = N$. The (unnormalized) quantum state under study is therefore [25]

$$|\mathcal{N}\rangle = \sum_{k} \left[ \prod_{i,j} | (a_i, b_j) \rangle \right]_k,$$

where $| (a_i, b_j) \rangle$ refers to a dimer between sites $a_i$ and $b_j$ on the bipartite lattice. $(i, j)$ are NN sites with $i \neq j$. A product of all such dimers on the spin-1/2 lattice constitutes an RVB covering. The summation refers to the superposition of all such dimer product states that give us the superposed short-range dimer covering states. It is believed that these short-range dimer covered states are possible solutions of the 2D antiferromagnetic Heisenberg systems based on the understanding of the RVB theory and possible explanation of the results obtained by using numerical methods such as the density matrix renormalization group [20, 26].

3. Genuine multi-partite entanglement and the generalized geometric measure

Quantification of multi-partite entanglement plays a crucial role in quantum information processing. In the case of bipartite pure states, entanglement can be uniquely quantified as the von Neumann entropy of local density matrices. In comparison, the quantification of multi-partite entanglement involves characterizations and criteria of multipartite pure quantum states and cannot be uniquely defined [8]. For pure states, a genuinely multi-partite entangled state can be defined as a multipartite pure quantum state that cannot be expressed as a product across any bipartition. The ‘Greenberger–Horne–Zeilinger’ state of [27] and the ‘W state’ of [28] are common examples of genuine multi-partite entangled states. Genuine multi-partite entanglement is an important resource from the perspective of many-party quantum
communication [29] and is a potential resource for large-scale quantum computation [30]. Cluster states are multi-partite entangled states that form promising candidates in building a one-way quantum computer [31]. Multi-partite entanglement also has a fundamental application in understanding the role of entanglement in many-body physics [10].

As mentioned above, a genuine multisite entangled pure quantum state is one which is entangled across every partition, into two subsets, of the set of the observers involved. A natural definition of the amount of genuine multi-partite entanglement of a pure multipartite quantum state is therefore the minimum distance of that state from the set of all states that are not genuinely multipartite entangled. A widely used measure of distance is the fidelity subtracted from unity [32]. The GGM of an \(N\)-party pure quantum state \(|\phi_N\rangle\) is defined as the minimum of this ‘fidelity-based distance’ of \(|\phi_N\rangle\) from the set of all states that are not genuinely multipartite entangled [23, 33]. In other words, the GGM of \(|\phi_N\rangle\), denoted by \(\mathcal{E}(|\phi_N\rangle)\), is given by

\[
\mathcal{E}(|\phi_N\rangle) = 1 - \Lambda_{\text{max}}^2(|\phi_N\rangle),
\]

where \(\Lambda_{\text{max}}(|\phi_N\rangle) = \max \{|\langle\chi|\phi_N\rangle|, \text{ with the maximization being over all } N\text{-party quantum states } |\chi\rangle \text{ that are not genuinely multipartite entangled.}

It was shown in [23, 33] that the GGM of a multipartite pure state can be effectively calculated by using

\[
\mathcal{E}(|\phi_N\rangle) = 1 - \max\{\lambda_{A:B}^2|A \cup B = \{1, 2, \ldots, N\}, A \cap B = \emptyset\},
\]

where \(\lambda_{A:B}\) is the maximal Schmidt coefficient in the bipartite split \(A:B\) of \(|\phi_N\rangle\). The GGM can, therefore, be calculated once the reduced density matrices of the pure quantum state are obtained. We use the GGM to quantify the amount of genuine multisite entanglement present in multi-legged ladder states of different sizes.

4. The density matrix recursion method: the even ladder

We will now describe the iterative methods for generating arbitrary local density matrices for the even and odd ladders. The recursion relation for the non-periodic (i.e. with open boundary conditions) even-legged ladder can be written as

\[
|\mathcal{N} + 2\rangle = |\mathcal{N} + 1\rangle|1\rangle_{n+2} + |\mathcal{N}\rangle|2\rangle_{n+1,n+2}
\]

\[
= |\mathcal{N}\rangle|2\rangle_{n+1,n+2} + |\mathcal{N} - 1\rangle|2\rangle_{n,n+1}|1\rangle_{n+2},
\]

where \(|\mathcal{N}\rangle\) is an even-legged ladder containing \(\mathcal{N}\) sites in each chain and corresponds to the rungs numbered from 1 to \(n\). On the other hand, e.g. \(|\mathcal{N}\rangle_{2,n+1}\) represents an even-legged ladder containing \(\mathcal{N}\) sites in each chain, and corresponds to the rungs numbered from 2 to \(n + 1\). \(|1\rangle\) is a vertical chain containing \(M\) spins representing a single rung of the ladder with the dimer coverings between adjacent sites. \(|2\rangle_{n+1,n+2}\) is a two-rung even-legged ladder of rungs \(n + 1\) and \(n + 2\). Also

\[
|2\rangle_{n+1,n+2} = |2\rangle_{n+1,n+2} - |1\rangle_{n+1}|1\rangle_{n+2}.
\]

The term \(|2\rangle_{n+1,n+2}\) contains all the coverings of a two-rung even leg ladder \(|2\rangle_{n+1,n+2}\), apart from the coverings formed by the product of the two rungs \(|1\rangle_{n+1}|1\rangle_{n+2}\). The subtraction removes possible repetition of states between the two terms in the recursion for \(|\mathcal{N} + 2\rangle\) in relation (3).
We further note that if no site numbers are mentioned in the subscript of $|\mathcal{N}\rangle$, it is to be considered from 1 to $n = \mathcal{N}$.

The periodic boundary condition entails that the ladder also forms dimer states between the rungs 1 and $n$. Accounting for the additional states in the ladder system, due to the periodicity, the recursion relation can be modified into

$$|\mathcal{N} + 2\rangle_p = |\mathcal{N} + 2\rangle_{1,n+2} + |\mathcal{N}\rangle_{2,n+1}\hat{2}_{n+2,1},$$  \hspace{1cm} (4)$$

where the subscript $P$ stands for a periodic ladder state. Throughout the paper, a state without a subscript $P$ will imply a non-periodic state. The density matrix for the periodic ladder state can be calculated by using the recursion relation in equation (4):

$$\rho_p^{(\mathcal{N}+2)} = |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2|_p $$
$$= |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2| + |\mathcal{N} + 2\rangle\langle \mathcal{N}|_{2,n+1}\hat{2}_{n+2,1} + |\mathcal{N}\rangle_{2,n+1}\hat{2}_{1,n+2}\langle \mathcal{N} + 2| $$
$$+ |\mathcal{N}\rangle\langle \mathcal{N} |_{(2,n+1)}\hat{2}_{(1,n+2)}.$$  \hspace{1cm} (5)$$

If, for example, we trace out all but two rungs (say $n + 1$ and $n + 2$) from the state, we will obtain a two-rung (mixed state) ladder containing $2M$ spins, where $M$ is the number of legs of the ladder. Note that $M$ is even here. These reduced states can then be used to calculate the multisite entanglement properties of the ladder system. In particular,

$$\rho_p^{(2)}_{P(n+1,n+2)} = \text{tr}_{1,...,n}(|\mathcal{N} + 2\rangle \langle \mathcal{N} + 2|_p $$

is the two-rung state of the periodic ladder. Now, using relation (3), we can obtain the trace for the non-periodic part to have

$$\text{tr}_{1,...,n}(|\mathcal{N} + 2\rangle \langle \mathcal{N} + 2|) = \text{tr}_{1,...,n}(|\mathcal{N}\rangle \langle \mathcal{N}| \otimes |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2|_{n+1,n+2} + |\mathcal{N} - 1\rangle \langle \mathcal{N} - 1| \otimes |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2|_{n+1,n+2} + |\mathcal{N}\rangle\langle \mathcal{N}|\otimes |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2|_{n+1,n+2} $$
$$+ |\mathcal{N}\rangle\langle \mathcal{N}|\otimes |\mathcal{N} - 1\rangle \langle \mathcal{N} - 1| \otimes |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2|_{n+1,n+2} + |\mathcal{N}\rangle\langle \mathcal{N}|\otimes |\mathcal{N} + 2\rangle\langle \mathcal{N} + 2|_{n+1,n+2}).$$  \hspace{1cm} (7)$$

Extending the trace to the periodic ladder state, we obtain

$$\rho_{P(n+1,n+2)}^{(2)} = \rho_{n+1,n+2}^{(2)} + \text{tr}_{1,...,n}(|\mathcal{N}\rangle \langle \mathcal{N}|_{(2,n+1)}\hat{2}_{(1,n+2)} $$
$$+ (|\mathcal{N}\rangle_{2,n+1}\hat{2}_{1,n+2}\langle \mathcal{N} + 2| + \text{h.c.}) $$
$$|\mathcal{N}\rangle_{(2,n+1)}\hat{2}_{(1,n+2)}\langle \mathcal{N} + 2| + \text{h.c.}) $$
$$= \rho_{n+1,n+2}^{(2)} + \beta_{2(n+1,n+2)}^{2} + ||\xi_{\mathcal{N} - 1}\rangle_{n+1} \otimes \tilde{\rho}_{n+2} + \text{h.c.} $$
$$\text{h.c.}.$$  \hspace{1cm} (9)$$

Here,

$$\rho_{n+1,n+2}^{(2)} = Z_{\mathcal{N} - 1}|\mathcal{N} - 1\rangle \otimes \tilde{\rho}_{n+2} + Z_{\mathcal{N} - 2}\tilde{\rho}_{n+1} \otimes \tilde{\rho}_{n+2} + ||\xi_{\mathcal{N} - 1}\rangle_{n+1} \otimes \tilde{\rho}_{n+2} + \text{h.c.} $$

with

$$\langle \xi_{\mathcal{N} - 1}|_{n+1} = \tilde{\xi}_{n+1} \langle \mathcal{N} - 2|_{2,...,n-1}\mathcal{N} - 1)_{2,...,n}.$$

$$\langle \xi_{\mathcal{N} - 1}|_{n+1} = \tilde{\xi}_{n+1} \langle \mathcal{N} - 2|_{2,...,n-1}\mathcal{N} - 1)_{2,...,n}.$$  \hspace{1cm} (10)$$
Further,
\[
\rho_{2(n+1,n+2)}^2 = |2\rangle_{n+1,n+2} \langle 1|_{n+1} \langle \xi_{n+1} |_{n+2} + |2\rangle_{n+1,n+2} \langle \phi_{n+1} |_{n+1,n+2} + \bar{\rho}_{n+1} \otimes |1\rangle_{n+2} \langle \xi_{N-1} |_{n+2}
\]
\[
+ \frac{1}{A} |\xi_{n+1} |_{n+2} \langle 1|_{n+1} \langle \eta_{N-1} |_{n+2},
\]
where
\[
\langle \phi_{n+1} |_{n+1,n+2} = \langle \tilde{2}|_{n+2} \langle \tilde{2}|_{n+1} \langle N-2 |_{2,n-1} N, \]
\[
\langle \eta_{N-1} |_{n+2} = \langle \tilde{2}|_{n+2} \langle N-1 |_{2,n} N-1 |_{n-1} 1 |_{n} \]
and \( A = \langle 1|1 \rangle = \langle \bar{1}| \bar{1} \rangle \).

The local density matrices can be calculated by using the iterations, derived below, of the complete and partial inner products defined above. For \( M < 6 \), using (3), the normalization of the non-periodic ladder is given by
\[
Z_N = \langle N | N \rangle
\]
\[
= \bar{A} Z_{N-1} + B Z_{N-2} + 2C \mathcal{Y}^1_{N-2} + 2D \mathcal{Y}^2_{N-1},
\]
where \( \bar{A} = \langle 1|\bar{1} \rangle \) and \( B = \langle \tilde{2}|\tilde{2} \rangle, \ C, \ D, \tilde{C} \) and \( \tilde{D} \) are given by \( \langle 1|\bar{2} \rangle = C \langle 1|D|\bar{1} \rangle \) and \( \langle \bar{1}|\tilde{2} \rangle = \tilde{C} \langle 1|\tilde{D}|1 \rangle \), where \( |1 \rangle \) is a vertical rung with a periodic dimer covering between the topmost and lowermost sites of that rung. The other terms in the normalization can be calculated as follows:
\[
\mathcal{Y}^1_N = \langle N | N-1 | 1 \rangle = \bar{A} Z_{N-1} + C \mathcal{Y}^1_{N-1} + D \mathcal{Y}^2_{N-1},
\]
\[
\mathcal{Y}^2_N = \langle N | N-1 | \bar{1} \rangle = \bar{A} Z_{N-1} + \tilde{C} \mathcal{Y}^1_{N-1} + \tilde{D} \mathcal{Y}^2_{N-1}.
\]

The other term in the expression for \( \rho_{2(n+1,n+2)}^{(2)} \) that is to be calculated using iterations is
\[
\langle \xi_{N} | = \langle \tilde{2}|_{n+1} \langle N-1 | N \rangle
\]
\[
= \langle 1| (CA^1_N + \tilde{C} A^2_N) + \langle \bar{1}| (D A^1_N + \tilde{D} A^2_N),
\]
where the iteration variables are given by
\[
A^1_N = \sum_{i=1}^{N} Z_{N-i} g_{i-1},
\]
\[
A^2_N = \sum_{i=1}^{N} Z_{N-i} h_{i-1}
\]
with
\[
g_{i+1} = C g_i + \tilde{C} h_i,
\]
\[
h_{i+1} = D g_i + \tilde{D} h_i.
\]
Here, \( g_0 = 1 \) and \( h_0 = 0 \).

The extra iterative terms in the periodic two-rung density matrix, \( \rho_{P(n+1,n+2)}^{(2)} \), can be expressed as
\[
\langle \phi_{n+1,n+2} = \langle \tilde{2}|_{n+2} \langle \tilde{2}|_{n+1} \langle N-2 |_{2,n-1} N
\]
\[
= \langle \tilde{2}|_{n+2} \langle \tilde{2}|_{n+1} \langle N \rangle + \mathcal{X}^N_{1N} |\bar{1}\rangle_1 |_1 \rangle_n + \mathcal{X}^N_{2N} |\bar{1}\rangle_1 |\bar{1}\rangle_1 \rangle_n + \mathcal{X}^N_{3N} |\bar{1}\rangle_1 |\bar{1}\rangle_1 \rangle_n
\]
\[
+ \langle \tilde{2}|_{n-2,n-1} \ldots \langle \tilde{2}|_{3} \rangle_{12} \ldots |\tilde{2}\rangle_{n-1,n} \).
\]
Here, the iterative variables, $X_i^N$, are defined by using $A_1^N$ and $A_2^N$ as

$$X_1^N = \sum g_{2i} (A_1^N - 1 - 2i + CA_1^N - 2 - 2i) + \bar{g}_{2i} DA_1^N - 2 - 2i,$$

$$X_2^N = \sum g_{2i} (A_2^N - 1 - 2i + CA_2^N - 2 - 2i) + \bar{g}_{2i} DA_2^N - 2 - 2i,$$

$$X_3^N = \sum h_{2i} (A_1^N - 1 - 2i + CA_1^N - 2 - 2i) + \bar{h}_{2i} DA_1^N - 2 - 2i,$$

$$X_4^N = \sum h_{2i} (A_2^N - 1 - 2i + CA_2^N - 2 - 2i) + \bar{h}_{2i} DA_2^N - 2 - 2i,$$

(21)

where the summation is from $i = 0$ to $N$. The variables $\bar{g}_i$ and $\bar{h}_i$ mimic the same relations as $g_i$, $h_i$, with initial conditions $\bar{g}_0 = 0$, $\bar{h}_0 = 1$. Similarly, we can derive

$$\langle \eta_{N-1}|_{n+1} \rangle = \langle \tilde{2}|_{1,n+1} \langle N - 1|_{2,n} N - 1 \rangle_{1,n-1}|1 \rangle_n$$

$$= \langle AA_1^{N-1} + \tilde{A} A_2^{N-1} \rangle \langle C \rangle + \mathcal{D} \langle \tilde{1} \rangle + \langle \tilde{2}|_{n+2,1} \langle X_1^{N-1} \rangle |_{2,1} |_{n} + \langle X_2^{N-1} \rangle |_{2,1} |_{n}$$

$$+ X_3^{N-1} |_{2,1} |_{n} + X_4^{N-1} |_{2,1} |_{n},$$

(22)

where the iterative variables can be derived using (21). These iterations can be used to obtain the partial density matrices of a periodic even-legged ladder, and hence its bipartite as well as genuine multi-partite entanglement and other single- and multi-site physical quantities, provided that the values of $A$, $\bar{A}$, $B$, $C$, $\bar{C}$, $\mathcal{D}$, $\mathcal{Z}$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are exactly calculated. The size of the reduced density matrices depends on the type of ladder considered. For example, a two-rung reduced density matrix for an $M$-legged ladder is a $\tau = 2M$-spin matrix. Other reduced density matrices of spins smaller than $\tau$, required, e.g., for calculating the GGM, can be obtained from the $\tau$-spin matrix by partial tracings.

For $M \geq 6$, the iterations involve algebra that is slightly more complicated. $\langle 1|2 \rangle = \sum_{k=1}^{k} \alpha_k^1 |\alpha_k^1 \rangle$, where $|\alpha_k^1 \rangle = |1 \rangle$ and $|\alpha_k^2 \rangle = |1 \rangle$. $|\alpha_k \rangle$ ($k \neq 1, 2$) are other singlet combinations of an $M$-legged single vertical rung. For $M = 4$ (in the previous derivation), $\alpha_k^1 (k \neq 1, 2) = 0$. $\alpha_1^1$ and $\alpha_2^1$ are $\bar{C}$ and $\mathcal{D}$, respectively. Hence, for $M \geq 6$, we have the following relation: $\langle \alpha_k \rangle_n |_{n+1} = (-1)^{n-1} \sum_j \alpha_j^k |\alpha_j \rangle$. The normalization terms then work out to be

$$\mathcal{Z}_N = \langle N|N \rangle_n$$

$$= AZ_{N-1} + BZ_{N-2} + 2(-1)^{n-1} \sum_j \alpha_j^k \mathcal{Y}_{N-1}^j,$$

(23)

$$\mathcal{Y}_N^j = n \langle \alpha_j \rangle |_{1,n} \langle N - 1|N \rangle_{1,n}$$

$$= A_{ij} \mathcal{Z}_{N-1} + (-1)^{n-1} \sum_k \alpha_k^j \mathcal{Y}_{N-1}^k,$$

(24)

where $A_{ij} = \langle \alpha_i |\alpha_j \rangle$. The rest of the iterations can be derived using these terms in a similar fashion to the derivation done for $M < 6$. The terms $\alpha_j^i (i, j = 1$ to $k)$, $A$, $B$ and $A_{ij}$ need to be exactly calculated.

One of the main motivations for this work was that spin ladders of the type discussed in this paper can be implemented in the laboratory. It is, therefore, important to find out whether the obtained effects are resilient to noise effects, such as small admixtures of higher multiplets.
or of singlets that are not of NNs. Since the genuine multipartite entanglement that we use in the paper is a continuous function of the state parameters, such small admixtures of noise will not substantially alter the amount of the measure present in the multipartite state. This feature is valid independent of whether we are considering even- or odd-legged ladders.

5. The density matrix recursion method: the odd ladder

The periodic recursion for the odd ladder is rather different from that of the even ladder. This is due to the fact that there exists no analogous state for $|1\rangle$ in the odd ladder. The non-periodic $|\mathcal{N}\rangle$ can be written as a series of $\mathcal{N} = 2$ odd-legged ladders. $|\mathcal{N}\rangle = |2\rangle_{1,2} |2\rangle_{3,4} \cdots |2\rangle_{n-1,n}$. The periodic recursion for the odd $M$-legged $\mathcal{N}$-spin-per-chain ladder is then given by

$$|\mathcal{N}+2\rangle_P = |\mathcal{N}\rangle_{1,n} |2\rangle_{n+1,n+2} + |\mathcal{N}\rangle_{2,n+1} |2\rangle_{n+2,1},$$

where $|2\rangle_{n+1,n+2}$ is a two-rung odd-legged ladder at rungs $n+1, n+2$. We observe that there is no repetition of terms in $|\mathcal{N}+2\rangle_P$ and hence no subtraction of states is required, as is needed for the even ladder. The repetition is avoided by using the periodic condition in the initial recursion. This is possible due to the absence of $|1\rangle$ state in the odd ladder.

The density matrix can now be calculated as before, using (25): $\rho^{\mathcal{N}+2}_P = |\mathcal{N}+2\rangle \langle \mathcal{N}+2|_P$. The reduced density matrices can be obtained by tracing out the requisite number of spins. In particular, for obtaining a two-rung reduced density matrix, we trace out the rungs ranging from 1 to $n$:

$$\rho^{(2)}_{P(n+1,n+2)} = \text{tr}_{1,...,n} [|\mathcal{N}+2\rangle \langle \mathcal{N}+2|_P]$$

$$= \text{tr}_{1,...,n} [|\mathcal{N}\rangle \langle \mathcal{N}|_{1,n} |2\rangle_{2,n+1,n+2} + |\mathcal{N}\rangle \langle \mathcal{N}|_{2,n+1} |2\rangle_{2,n+2,1}$$

$$+ (|\mathcal{N}\rangle_{1,n} |2\rangle_{n+1,n+2} \langle \mathcal{N}|_{2,n+1} |2\rangle_{n+2,1} + \text{h.c.}).$$

After simplification, the above equation reads

$$\rho^{(2)}_{P(n+1,n+2)} = Z_N |2\rangle \langle 2|_{n+1,n+2} + Z_N^{-2} \tilde{\rho}_{n+1} \otimes \tilde{\rho}_{n+2} + (|2\rangle_{n+1,n+2} \langle \Omega_N|_{n+1,n+2} + \text{h.c.}),$$

where

$$(\Omega_N)_{n+1,n+2} = \langle \mathcal{N}|_{1,n} \langle 2|_{n+1,n+2} + \langle \mathcal{N}|_{2,n+1} \langle 2|_{n+2,1} + \text{h.c.})$$

and the normalization is $Z_N = \langle \mathcal{N}|_N \rangle = Z_2^{N/2}$. Now, for an $M$-legged ladder, $Z_2$ corresponds to a two-legged $M$-site-per-chain ladder, and can be calculated from the previous section. The recursion for $|\Omega_N\rangle$ is $|\Omega_N|_{n+1,n+2} = |\Omega_N|_{n+2} \langle 2|_{n+1,n+2} \langle \mathcal{N}|_{1,n} = \langle 2|_{1,n+2} \langle \mathcal{N}|_{2,n+1} \langle 2|_{n+2,1} + \text{h.c.})$. Hence the computation involves writing an algorithm to iterate the step $|2|_{i+2} \langle 2|_{i+1,n+2} \langle \mathcal{N}|_{1,n}$. Once the reduced density matrices are obtained by the iterative method, we can again use them to obtain the different single- and multi-site physical quantities of the system.

6. Even versus odd

To illustrate the effectiveness of the DMRM, we apply it to obtain a multisite entanglement of multi-legged ladders with both even and odd numbers of legs. In particular, we consider two- and four-legged ladders among even ladders, and three- and five-legged ladders among odd

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Genuine multisite entanglement decreases with system size for even ladders. We perform the iterations for $M = 2$ and 4. The iterations are carried out up to 18 rungs, i.e. 36 and 72 sites, respectively, for $M = 2$ and 4. However, in both cases, the GGM converges much before those sizes. The vertical axis represents the GGM, while the horizontal one represents the number of rungs. Both axes are dimensionless.

The iterative variables can be evaluated by using their explicitly calculated initial set of values. The iterations provide the reduced density matrices of the system, which are thereafter utilized to obtain the GGM.

Specifically, for $M = 2$, the relevant initial parameters are

\[
\begin{align*}
Z_0 &= 1, & Z_1 &= 2, \\
A &= 2, & \bar{A} &= 2, \\
C &= 1, & D &= 0, & \bar{C} &= 0, & \bar{D} &= 0, \\
Y^1_1 &= 2, & Y^2_1 &= 0.
\end{align*}
\]

For $M = 4$, the initial parameters are

\[
\begin{align*}
Z_0 &= 1, & Z_1 &= 4, \\
A &= 4, & \bar{A} &= 2, \\
C &= 5, & D &= 1, & \bar{C} &= 2, & \bar{D} &= 3, \\
Y^1_1 &= 4, & Y^2_1 &= 2.
\end{align*}
\]

A similar analysis can also be done for higher even-legged ladders.

In the case of odd ladders, the initial parameter required in the recursion for $M = 3$ is $Z_2 = 44$, and for $M = 5$ is $Z_2 = 804$.

We are now ready to compare the multisite entanglements for the even- and odd-legged ladders. The GGMs obtained by the iterative methods clearly capture the characteristic complementary nature of even and odd ladders. We show that the GGM decreases with an increase of system size in the case of even ladders (figure 2). The opposite is true for odd ladders—the GGM increases with system size (figure 3).
Figure 3. Genuine multisite entanglement increases with system size for odd ladders. We perform the iterations for $M = 3$ and $5$. The iterations are up to 18 rungs, i.e. 54 and 90 sites, respectively, for $M = 3$ and $5$. Again in both cases, the GGM converges much before 18 rungs. The vertical axis represents the GGM, while the horizontal one represents the number of rungs. Both the axes are dimensionless.

Although the comparison is made by taking $M = 3$ and $5$ among odd ladders, we have actually performed the computations also for $M = 1$, which again shows the characteristic increasing GGM of odd ladders. In all cases, the GGM is calculated by considering reduced density matrices up to $2M$ spins: numerical exact diagonalizations corroborate that considering up to 4 spins is already enough.

We have performed the iterative algorithms up to 18 rungs in all cases (which, e.g., is equivalent to 72 spins for the four-legged and 90 spins for the five-legged ladder). As seen in figures 2 and 3, the GGM has already converged much before the maximum number of rungs that we have considered.

The entanglement properties of the quantum spin ladder states can be generalized and scaled to study other important aspects of quantum spin systems. The two-site reduced density matrix of the superposed dimer-covered state is a Werner state [34] and the bipartite entanglement of the state is a monotone of the Werner parameter. The Werner parameter can also be used to study the ground state energy and spin correlation length (see, e.g., [19–21] and references therein). There is also a correspondence between the entropic properties of spin ladder states and entanglement [35].

7. Conclusion

We have introduced an analytical iterative technique, the DMRM, which can be efficiently used to obtain arbitrary reduced density matrices of the states of spin-1/2 quantum spin ladders with an arbitrary number of legs, formed by the superposition of dimer coverings. This technique immediately allows us to obtain single- and multi-site physical properties of the system. In particular, we use the method to obtain the scaling of genuine multisite entanglement in the states of both odd- and even-legged ladders. We find that the genuine multisite entanglement
can capture the disparity between the even and odd ladder states. These entanglement properties may prove insightful to researchers interested in studying other physical aspects of spin systems. It would be interesting to find the extrapolation of the results obtained to the case of broad rungs.

The behavior of multi-partite entanglement of such systems has attracted a lot of interest owing to recent experimental developments. Simulating large qubit systems, using optical superlattices [12] and interacting photon states [13], to generate dimer-covered superposed states in the laboratory points to the future applications of multi-partite entanglement properties of these states. These multipartite entangled states can potentially find applications in building cluster states for large-scale quantum computation [31] and in quantum metrology [36].

Our analytical method enables investigation of the bipartite and multi-partite entanglement properties in these superposed dimer-covered systems with relative control and ease even for systems containing a considerable number of quantum spins. The results obtained may prove useful in predicting the prospects of the entanglement properties of experimentally generated states for future applications.

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