Oriented percolation with density close to one*

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Abstract

A unified treatment regarding the occupancy density of highly supercritical oriented site percolation with a perspective to applications for interacting particle systems is shown. The following refinement of a classical result is derived. The proportion of sites lying in the infinite cluster, along with a linearly growing interval which forms a cone of acute angle, exceeds any strictly less than one constant, eventually almost surely. Moreover, the corresponding exponentially decaying tail extension is obtained; this consolidates many special cases so far treated separately in the literature. We thus further provide exponential decay refinements of asymptotics known to play a central role in establishing existence of nontrivial stationary distributions for interacting particle systems.

0. Introduction

Interest in the oriented two-dimensional site percolation process with closed-site probability parameter in the proximity of zero is commonly known to emerge in connection with arguments involving renormalization (rescaling) of stochastic processes. This percolation process with parameter satisfying the proviso specified is typically employed in the field of interacting particle systems in particular for comparison with a (macroscopic) block construction that is embedded in another more complex process. The motivation for study of refinements of known from the literature results here originates in this perspective, since these may allow for more precise control in such constructions and furthermore, since these are generally developed in an ad-hoc way depending on the framework of the application under consideration.

In Theorem 1 we revisit properties concerning the empirical occupancy density of the infinite cluster of this process at fixed height levels, firstly appearing in [11].

*keywords: exponential decay; oriented percolation; convergence rates; stationary distributions of particle systems; renormalization; contact process; finite-range dependencies; shape theorem.
We thus derive a sharpened version of various other earlier results – a literature review in this regard, along with suggesting an alternative approach that simplifies the actual route in the verification of a criterion for the shape theorem of the super-critical contact process on finite-dimensional integer lattices, may be found in the concluding, discussion section. We should note here that the technique of proof of Theorem 1 relies on a series of deduction arguments that exploit various coupling connections in order to encompass an ensemble of preliminary results, most prominently a new consequence of the work of Durrett and Schonmann \cite{DS88}, derived to this end.

A very useful and powerful method for establishing existence of stationary distributions for interacting particle systems is developed in Section 4 of the lecture notes, Durrett \cite{D95}. This method relies on the renormalized site construction method of proof introduced by Bramson and Durrett \cite{BD88} for reproving the theorem of Gray and Griffeath \cite{GG82} regarding stability of attractive nearest-neighbor spin systems. In Theorem 2 and Corollary 3 below we provide with various improvements to the results typically invoked in conjunction with this general methodology.

1. Statement of Results

Two-dimensional oriented site percolation with closed-site probability $\epsilon$ is briefly described in order to state the result. A coherent treatment of general percolation theory was initiated with Broadbent and Hammersley \cite{BH57}, although much earlier references to such processes may be traced in the literature, for instance, Wood \cite{W1894}. The process considered here is a close variant of the extensively studied bond process, cf. Durrett \cite{D84}, which is the discrete-time analogue of the basic one-dimensional contact process, cf. Liggett \cite{L85}, Chpt. VI.

Consider the lattice graph $\mathbb{L}$ on the Cartesian plane with set of sites the points that are not below the vertical axis and have coordinates adding up to an even number, obtained by adding a bond from each such point $(y, n)$ to $(y - 1, n + 1)$ and to $(y + 1, n + 1)$. The first and second coordinates shall be thought of throughout here as space and time respectively. Consider a probability measure under which sites are designated open with probability $1 - \epsilon$ and closed (not open) otherwise. Designations at the same level for sites at distance smaller than or equal to a fixed constant need not be independent, while sites at time level zero receive no such assignment. Let $A_n^A$ denote the set of sites at time $n$ that are connected to some $(x, 0)$, $x \in A$, in the subgraph obtained by removing bonds starting from closed
sites from our lattice graph. Let also $\Omega_n^A$ be the event that $(A,0)$ percolates up to time-level $n$, that is, that $\{A_n^A \neq \emptyset\}$. We will omit superscripts for referring to $A = \{0\}$, for instance, we simply write $A_n$ instead of $A_n^{(0)}$. Let also $s_n^A$ denote the set of sites at time $n$ with spatial coordinate $y$, $|y| \leq an$, $a \in (0,1]$. We can now state the main result that concerns the density of $A_n$ over certain regions of the lattice graph, where the density of a set $S_1$ over another $S_2$ is defined to be $|S_1 \cap S_2|/|S_2|$ and $|\cdot|$ denotes cardinality as usual.

**Theorem 1.** For any $\beta$ and $\rho$ strictly less than one there exists sufficiently small $\epsilon > 0$ such that the probability that on $\Omega_n$ the density of $A_n$ over any subset $S$ of $s_n^\beta$ is less than $\rho$ admits an upper bound of the form $C(e^{-\gamma n} + e^{-\gamma|S|})$, for $C$ and $\gamma$ strictly positive, independent of the choice of $n$, finite constants.

The intuition behind Theorem 1 is that $A_n$ fills up a fraction $\rho$ of the number of available points within $\{-\beta n, \ldots, \beta n\}$ in consideration, outside of exponentially small probability in $n$ and in the number of points, where $\rho$ and $\beta$ tend to 1 as $\epsilon$ tends to 0.

The proof of Theorem 1 goes through coupling-based deductions for combining a series of preliminaries shown to this aim, mainly by exploiting a technique of Andjel [A93] for translating their counterparts for the supercritical bond process. These deductions build upon a new consequence of the usual large deviations behavior for the density of the upper invariant measure of this process due to Durrett and Schonmann [DS88], that actually dealt with the more difficult continuous-time analog case. Further, we point out that, by employing the celebrated, generalization of the latter mentioned technique from Liggett, Schonmann, and Stacey [LSS97], the results and overall approach taken here are resilient to any type of finite-range site designation dependencies.

The next result provides with exponential bounds for Theorem 4.2, or A.2, and Theorem A.3, or Lemma 6.1, from Durrett [D95] by means of an alternative method of proof relying on Theorem 1, as well as with an explicit expression of the lower bound that is identified here. To state it, let $A_n^{P^p}$ denote the process started with configuration which includes every point independently with probability $p$, $p > 0$, independently of other designations, and where we enhance the underlying probability space to support random starting states as usual. Let also $k = \{-2k, -2k + 2, \ldots, 2k - 2, 2k\}$, and $\Omega_k^\infty := \bigcap_n \Omega_n^k$. 




Theorem 2. We have that
\[
P \left( A_{2n}^{\Pi_p} \cap k \neq \emptyset \right) \geq \theta(\epsilon, k)(1 - Ce^{-\gamma n}),
\]
where \( \theta(\epsilon, k) = \mathbb{P}(\Omega_k^\infty) \). Furthermore,
\[
\theta(\epsilon, k) \geq 1 - C'e^{-\gamma'|k|},
\]
where \( C' \) and \( \gamma' \) depend only on \( \epsilon > 0 \).

Finally, from the last theorem above we derive the following refining of Theorem A.3 or Lemma 6.1 in Durrett [D95].

Corollary 3. \( \lim \inf_{n \to \infty} P \left( A_{2n}^{\Pi_p} \cap k \neq \emptyset \right) \to 1 \) as \( \epsilon \to 0 \), for all \( k \).

The remainder of this note comprises a section containing the proof of Theorem 1, followed by one with those of Theorem 2 and Corollary 3, along with a concluding, discussion section.

2. Proof of Theorem 1

The contents of this section are outlined as follows. The first statement below, Proposition 4, is a compilation of some preparatory results. The proof of Theorem 1 is carried out immediately after proving this statement. An alternative to this latter part route for deriving Theorem 1 on the grounds of the analogous statement for the (highly supercritical) bond process is also finally sketched.

Note that \( C \) and \( \gamma \) will represent positive and finite constants throughout here. Two-dimensional independent bond percolation with open bond probability parameter \( p \) is briefly defined as follows. Bonds of \( \mathbb{L}^2 \) are designated independently open with probability \( p \), and close otherwise. Let \( B_n^\infty \) denote the set of sites at time \( n \) that are connected to some \( (x, 0), x \in B \), in the subgraph obtained by removing bonds designated close, and further let \( p_c \) denote the usual critical value for percolation to occur with positive probability in this model, where \( p_c < 1 \). In what follows we denote the set of even integers by \( 2\mathbb{Z} \), the cardinality of a set \( S \) by \( |S| \), and the complement of an event \( E \) by \( E^c \). Points at a specified time level are said to be consecutive simply when at distance 2 from one another.

Proposition 4. (a) If \( r_n = \sup A_n \) and \( l_n = \inf A_n \), then \( A_n = A_n^{2\mathbb{Z}} \cap [l_n, r_n] \) on \( \Omega_n \).
(b) If \( \bar{r}_n = \sup A_n^{[\cdots-2,0]} \), then \( \bar{r}_n = r_n \) on \( \Omega_n \).

(c) Let \( \Omega_\infty := \bigcap_n \Omega_n \). There exists \( \epsilon > 0 \) such that \( \mathbb{P}(\Omega_n \cap \Omega_\infty) \leq C e^{-\gamma n} \).

(d) Let \( \bar{r}^b_n = \sup B_n^{[\cdots-2,0]} \). For all \( p > p_c \), there is \( \alpha := \alpha(p) \) such that if \( a > \alpha \), then \( \mathbb{P}(\bar{r}^b_n < an) \leq Ce^{-\gamma n} \); further \( \alpha(p) \to 1 \), whenever \( p \to 1 \).

(e) For all \( p < 1 \) there is \( \epsilon > 0 \) such that \( B_n^B \subset_{st} A_n^A \), whenever \( B \subset A \).

(f) Let \( p > p_c \). If \( p' < p \), then \( \mathbb{P}(|B_n^{2Z} \cap S| < p'|S|) \leq Ce^{-\gamma|S|} \), for any finite set of consecutive points \( S \) at time \( n \).

Proof of Proposition 4. Parts (a) and (b) are shown by coupling as for the bond process, Durrett \[D84\]. Part (c) follows from the corresponding result for the bond case and part (e) may be proved by a simple coupling argument, see the respective background section in Liggett \[L99\]. The proof of (d) may be found in Durrett \[D84\], where for the second part one employs continuity of \( \alpha(p) \) shown there.

Part (f) is a corollary of the discrete-time analogue of Theorem 1 of Durrett and Schonmann \[DS88\]. To derive (f) from the part of this statement proved there, one first uses monotonicity in distribution of the initial set and then relies on the (distributional) invariance property to get the uniformity of the bound in time.

\[\Box\]

Proof. From \([c]\) the result reduces to that on \( \Omega_\infty \). By \([a]\) then it suffices to that: 1) For any \( \beta < 1 \), there is \( \epsilon > 0 \) such that \( \mathbb{P}(l_n \geq -\beta n, r_n \leq \beta n, \Omega_\infty) \leq C e^{-\gamma n} \); and that, 2) For any \( \rho < 1 \), there is \( \epsilon > 0 \) such that \( \mathbb{P}(|A_{nZ}^{2Z} \cap s_{n^\beta}| < \rho|s_{n^\beta}|) \leq Ce^{-\gamma|s_{n^\beta}|} \).

By \([b]\), some reflection and set theory basics show that 1) follows from: 1') For any \( \beta < 1 \) there is \( \epsilon > 0 \), such that the \( \mathbb{P}(\bar{r}_n \leq \beta n) \leq C e^{-\gamma n} \). Appropriate applications of \([e]\) imply 1') from \([d]\), and 2) from \([f]\) by choosing \( p' \) there close to 1.

\[\Box\]

Remark on alternative Proof. Owing to the event in consideration the result cannot be deduced from the analogue of Theorem 1 for \( B_n^{(0)} \) with a direct application of \([e]\) a proof by means of this and for example the restart argument in \[T11-2\], Proposition 3.5, for embedding \( B_n^{(0)} \) may however also be done.

3. Proofs of Theorem 2 and Corollary 3

Proof of Theorem 2. We will actually prove the following, sharpened from \([1]\), result

\[\mathbb{P} \left( |A_{2n}^{1p} \cap k| \geq \rho p(|2(k + \beta n)|_2 + 1) \right) \geq \theta(\epsilon,k)(1 - Ce^{-\gamma n}), \quad (3)\]
where \( \lfloor r \rfloor \) denotes the biggest even number smaller than \( r \). This suffices because (2) follows, as pointed out in the corresponding background section of Liggett \[L99\], from an application of (e) in Proposition (4) by the corresponding statement for the bond case, which is proved in section 10 of Durrett \[D84\].

By a path property of \( (A_n) \) known as duality (see Durrett \[D84\] or \[D95\]), we have that \( |A_{2n}^k \cap \Pi| \) is equal in distribution to \( |A_{2n}^k \cap \Pi_p| \). Thus, by an application of Bayes’ formula, because \( P(\Omega_n^k) \geq \theta(\epsilon, k) \), for all \( n \), we have that it suffices to show that

\[
P(|A_{2n}^k \cap \Pi_p| \leq \rho p(2(k + \beta n)]_2 + 1), \Omega_n^k) \leq Ce^{-\gamma n},
\]

We prove the last inequality which completes the proof. Let \( \Pi_{2n}^p = \Pi_p \cap [-2\beta n - 2k, 2\beta n + 2k] \), and note that \( E \Pi_{2n}^p = p(2(k + \beta n)]_2 + 1) \). Elementarily, since, by the Chernoff bound, \( P(\Pi_{2n}^p < E \Pi_{2n}^p) \) decays exponentially in \( n \), we have that it suffices to prove (3) on \( \Pi_{2n}^p \geq E \Pi_{2n}^p \). However, this follows easily by an application of the law of total probability, the analog of Theorem 1 for \( A_{2n}^k \), and use of independence of \( \Pi_p \) from \( (A_{2n}^k) \).

Proof of Corollary 3. Because, by monotonicity of the process, for all \( k \), \( \theta(\epsilon, \{0\}) \geq \theta(\epsilon, k) \), the statement follows from Theorem 2 since we have that as \( \epsilon \to 0 \), \( P(\Omega_\infty) \to 1 \). This last claim may be derived from the respective result for the bond process by applying Proposition (4) (e) This result for the bond process in turn follows by (7) in section 5 of Durrett \[D84\], along with the statement in Proposition (4) (d).

4. Discussion

Regarding other known from the literature results related to Theorem 1, we note that the renormalization construction arguments of Mountford and Sweet \[MS00\], also employed in Andjel \textit{et.al}. \[AMPV10\], are founded upon a weaker statement, Lemma 3 in \[MS00\], and that Theorem 1 may thus be used for throughout refining the exposition of these works. Theorem 1 also improves the corresponding result from Durrett and Neuhauser \[DN91\], pp.204-205. We further note that it may also be used for shortening a part of the actual argument from Durrett \[D91\] for the verification of criterion (7) of the Theorem of Durrett and Griffeath \[DG82\], known as the shape theorem, for the basic contact processes –for more recent applications on this estimate, see also e.g. Proposition 5, (11), or (45) in Garet and Marchand \[GM12\]. To see this simplification one utilizes the following idea. First observe that Theorem 1 implies that the cardinality of the intersection at time \( n \) of two surviving
i.i.d. copies of \( (A_n) \) started from sites at distance \( an \), for any \( a < 1 \), is not of linear order in \( n \) occurs outside of probability that decays exponentially in \( n \). Following then the construction and restart arguments in p.16-17 from \[D91\] and applying this result twice (one for the construction embedded in the contact process going forward in time, and one for the dual process going backwards in time there), one may derive (5.8) there without going through the modifications of those arguments in the last paragraph of the proof there. We also note that this last consequence of Theorem 1 pointed out gives the corresponding exponential estimates for the intersection of two independent copies of \( (A_n) \) from Bramson, Wan-Ding and Durrett \[BDD91\]. We also refer to \[Txx\] for another application, regarding a simple and direct proof of the asymptotically coupled region for a class of contact processes, Theorem 1 finds.

Finally, we remark that, although the part of the large deviations behaviour for the density of the invariant measure of the site process corresponding to that shown in Durrett and Schonmann \[DS88\] for the bond one is not known, Theorem 1 suffices for the purposes and scopes elaborated in the introduction.

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