GLOBAL SOLVABILITY IN A TWO-DIMENSIONAL SELF-CONSISTENT CHEMOTAXIS-NAVIER-STOKES SYSTEM

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Abstract. In this paper we deal with the initial-boundary value problem for chemotaxis-fluid model involving more complicated nonlinear coupling term, precisely, the following self-consistent system

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi), & (x,t) \in \Omega \times (0,T), \\
  c_t + u \cdot \nabla c &= \Delta c - nc, & (x,t) \in \Omega \times (0,T), \\
  u_t + (u \cdot \nabla)u + \nabla P &= \Delta u - n \nabla \phi + n \nabla c, & (x,t) \in \Omega \times (0,T), \\
  \nabla \cdot u &= 0, & (x,t) \in \Omega \times (0,T),
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary.

The novelty here is that both the effect of gravity (potential force) on cells and the effect of the chemotactic force on fluid is considered, which leads to the stronger coupling than usual chemotaxis-fluid model studied in the most existing literatures. To the best of our knowledge, there is no global solvability result on this chemotaxis-Navier-Stokes system in the past works. It is proved in this paper that global weak solutions exist whenever \( m > 1 \) and the initial data is suitably regular. This extends a result by Di Francesco, Lorz and Markowich (Discrete Cont. Dyn. Syst. A 28 (2010)) which asserts global existence of weak solutions under the constraint \( m \in (\frac{3}{2}, 2] \) in the corresponding Stokes-type simplified system.

1. Introduction. Chemotaxis-fluid model. It has been observed experimentally that when bacteria of species bacillus subtilis are suspended in water, even in a simple setting, they may result in a complex spatio-temporal behavior. To describe such process of cells, Tuval et al. [25] proposed the following coupled chemotaxis-fluid system

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, \ t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, \ t > 0, \\
  u_t + \kappa (u \cdot \nabla)u + \nabla P &= \Delta u - n \nabla \phi, & x \in \Omega, \ t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, \ t > 0,
\end{align*}
\]

where the unknown \( n \) and \( c \) denote the bacterium density and the oxygen concentration, respectively, and \( u \) represents the velocity field of the fluid subject to an incompressible Navier-Stokes equation (\( \kappa \neq 0 \)) or Stokes equation (\( \kappa = 0 \)) with pressure \( P \) and a gravitational force \( \nabla \phi \).

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Since its introduction, several analytical results in the past several years have been established for corresponding initial-value problems of (1) in bounded or unbounded domains. In the bounded domain situation, when system (1) coupled to no-flux condition for \(c\) and \(n\) and zero Dirichlet for \(u\), well-posed problem for the two-dimensional case has been settles fairly complete since the establishment of local existence of weak solutions [17]: By establishing an energy inequality, Winkler showed in [34] that uniquely determined global-in-time classical solutions exist for all suitably regular initial data; the stabilization of the global solution was also investigated in [35], [41]. Very recently, a small-convection limit problem was investigated in [32]. For the three-dimensional case with Stokes-governed fluid, global weak solutions has been constructed in [34]. Recently, it was proved in [38] that global bounded weak solutions for a full chemotaxis-Navier-Stokes system exist. It is even known that these weak solutions become eventual smooth and classical after some relaxation time [39]. For the unbounded domain case, there are also lots of works addressing the problem of well-posedness and large time issues (see e.g. [5], [6], [8], [9], [16],[22],[43] and the recent survey [1]).

Since the diffusion of cells in a viscous fluid is more like movement in a porous medium (see the discussions in [2], [12], and [26], for instance), the authors of [7] extended the model (1) to one with a porous medium-type diffusion of cells as

\[
\begin{align*}
 n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi), \quad x \in \Omega, \ t > 0, \\
 c_t + u \cdot \nabla c &= \Delta c - nc, \quad x \in \Omega, \ t > 0, \\
 u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u - n \nabla \phi + n \nabla c, \quad x \in \Omega, \ t > 0, \\
 \nabla \cdot u &= 0, \quad x \in \Omega, \ t > 0.
\end{align*}
\] (2)

The global existence of weak solutions is asserted in [7] for bounded domains \(\Omega \subset \mathbb{R}^2\) when \(m \in (\frac{3}{2}, 2]\), \(\kappa = 0\). This global existence result was improved in [23] to cover the whole range \(m \in (1, \infty)\), and moreover it was proved there that all solutions evolving from sufficiently regular initial data even remain bounded in \(\Omega \times (0, \infty)\). Under the hypothesis of \(m \in (\frac{1+\sqrt{217}}{12}, 2]\), the global solutions were also obtained in the three-dimensional setting [7] and this was extended in [24] to the range of \(m > \frac{8}{7}\). In the Cauchy problem case, the paper [16] addresses the three-dimensional analogue, and establishes global existence of weak solutions for \(\kappa = 0\) and the precise value \(m = \frac{4}{3}\). The authors in [10] established the the global existence of weak solutions for all adiabatic exponents \(m \in [1, +\infty)\) for the three-dimensional chemotaxis-Stokes system and the two-dimensional chemotaxis-Navier-Stokes system, respectively. As for the case of three-dimensional chemotaxis-Navier-Stokes system, Zhang and Li [42] proved that \(m > \frac{2}{3}\) is enough to guarantee the existence of global weak solutions.

**Self-consistent chemotaxis-fluid model.** Although the model (1) has been used for numerical computations in the biophysical literature [25], it was pointed out in [7] that it could be more realistic to include both the effect of gravity (potential force) on cells and the effect of the chemotactic force on fluid. Based on this consideration, Di Francesco et al. [7] extended the model (2) to the following more complicated model system

\[
\begin{align*}
 n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi), \quad x \in \Omega, \ t > 0, \\
 c_t + u \cdot \nabla c &= \Delta c - nc, \quad x \in \Omega, \ t > 0, \\
 u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u - n \nabla \phi + n \nabla c, \quad x \in \Omega, \ t > 0, \\
 \nabla \cdot u &= 0, \quad x \in \Omega, \ t > 0.
\end{align*}
\]
Under the hypothesis of \( m \in (\frac{3}{4}, 2] \), they obtained the global existence of weak solutions to the Neumann boundary problem of this system with \( \kappa = 0 \) in the bounded domain \( \Omega \subset \mathbb{R}^2 \).

In this paper, we shall do some further study on this self-consistent model and consider the following initial-boundary value problem

\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi), & x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, t > 0, \\
    u_t + (u \cdot \nabla)u + \nabla P &= \Delta u - n \nabla \phi + n \nabla c, & x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, t > 0, \\
    \nabla n^m \cdot \nu &= \nabla c \cdot \nu = 0, & u = 0, \\
    u(x, 0) &= n_0(x), & x \in \Omega,
\end{aligned}
\]  

(3)

in a bounded domain \( \Omega \subset \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \), where \( \nu \) denotes the unit outward normal vector field on \( \partial \Omega \), the gravitational potential function \( \phi \) is supposed to be given parameter functions. It is easy to see that the coupling here is stronger and more nonlinear than that of (2) and the Stokes-fluid version considered in [7].

We shall assume throughout this paper that the initial data satisfy

\[
\begin{aligned}
    n_0 &\in C^0(\overline{\Omega}), & n_0 \geq 0 \text{ and } n_0 \neq 0 \text{ in } \Omega, \\
    c_0 &\in W^{1, \infty}(\Omega), & c_0 \geq 0 \text{ and } c_0 \neq 0 \text{ in } \Omega, \\
    u_0 &\in D(A^\alpha) \text{ for some } \alpha \in \left( \frac{1}{2}, 1 \right),
\end{aligned}
\]  

(4)

where \( A \) denotes the Stokes operator with domain \( D(A) := W^{2, 2}(\Omega) \cap W^{1, 2}(\Omega) \cap L^2_\sigma(\Omega) \) with \( L^2_\sigma(\Omega) := \{ \varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0 \} \).

As for the gravitational potential \( \phi \) in (3), we require that it is independent of time and satisfies

\[
\phi \in W^{2, \infty}(\Omega), \tag{5}
\]

and

\[
\frac{\partial \phi}{\partial \nu} = 0, \quad x \in \partial \Omega. \tag{6}
\]

Under these assumptions, we can establish the existence of global weak solutions to system problem (3). Precisely, we have the following global existence result.

**Theorem 1.1.** Suppose that \( m > 1 \) and (4)-(6) hold. System (3) admits a global weak solution \((n, c, u, P)\) in the sense of Definition 1.2.

Let us underline in which respects our result goes beyond what is known in the existing literature [7]. First, we deal with a fully Navier-Stokes coupled model rather than its Stokes version. Second, we obtain the global existence of this system whenever \( m > 1 \), and this extends the range \( m \in (\frac{3}{4}, 2] \) in [7].

We define the weak solution in the usual way as follows.

**Definition 1.2.** A quadruple of functions \((n, c, u, P)\) is called a global weak solution of (3) if

\[
\begin{aligned}
    n &\in L^\infty((0, \infty); L^\infty(\Omega)), & \nabla n^m &\in L^2_{loc}((0, \infty); L^2(\Omega)) \\
    c &\in L^\infty((0, \infty); W^{1, \infty}(\Omega)), & \Delta u &\in L^2_{loc}((0, \infty); L^2(\Omega)) \\
    u &\in L^\infty((0, \infty); H^1(\Omega)), & \nabla \cdot u &\in L^2_{loc}((0, \infty); L^2(\Omega))
\end{aligned}
\]  

(7)

such that \( \nabla \cdot u = 0 \) in the distributional sense in \( \Omega \times (0, \infty) \).
and for all $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ and $\zeta \in C_0^\infty(\bar{\Omega} \times [0, \infty), \mathbb{R}^2)$ with $\nabla \cdot \zeta = 0$, the following integral equalities hold

$$
\int_0^\infty \int_{\Omega} n \varphi_t + \int_{\Omega} n_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\Omega} \nabla n \cdot \nabla \varphi - \int_0^\infty \int_{\Omega} n \nabla c \cdot \nabla \varphi \, dx \, dt + \int_0^\infty \int_{\Omega} n \varphi \cdot \nabla \varphi
$$

$$
- \int_0^\infty \int_{\Omega} n u \cdot \nabla \varphi,
$$

$$
- \int_0^\infty \int_{\Omega} c \varphi_t + \int_{\Omega} c_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\Omega} \nabla c \cdot \nabla \varphi + \int_0^\infty \int_{\Omega} n c \varphi - \int_0^\infty \int_{\Omega} c u \cdot \nabla \varphi,
$$

and

$$
\int_0^\infty \int_{\Omega} u \cdot \zeta_t \, dx \, dt + \int_{\Omega} u_0 \cdot \zeta(\cdot, 0) = - \int_0^\infty \int_{\Omega} u \cdot \Delta \zeta + \int_0^\infty \int_{\Omega} (u \cdot \nabla) u \cdot \zeta + \int_0^\infty \int_{\Omega} n \nabla \phi \cdot \zeta - \int_0^\infty \int_{\Omega} n \nabla c \cdot \zeta.
$$

Before going into details, let us mention that motivated by the particular experimental background [40] there are some recent works dealing with the well-posedness problem of chemotaxis-fluid model (1) with tensor-valued sensitivity. That is, the first equation is replaced by

$$
n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n S(x, n, c) \nabla c),
$$

where $S(x, n, c)$ is a tensor-valued function. See [3, 4, 28, 36] for the linear diffusion case and [11, 37] for the nonlinear diffusion case. More progress on the well-posedness problem of chemotaxis system with or without fluid coupling can be found in the recent survey [1].

We underline that the appearance of the effect of gravity on cells and the effect of the chemotactic force on fluid give rise to new mathematical challenges. Indeed, in the case of the chemotaxis-fluid system without including the chemotactic force on fluid, we can immediately establish the regularity for $u$ once we have the corresponding $L^p$ estimate of $n$ (see e.g. [15], [18], [19], [31], [29], [30], [37]). This is not valid in the present setting. Our approach underlying the derivation of Theorem 1.1 therefore consists at its core in an analysis of the functional

$$
y(t) := \int_{\Omega} n \varepsilon \ln n \varepsilon + \frac{1}{2} \int_{\Omega} \frac{|\nabla c \varepsilon|^2}{c \varepsilon} + 2K \int_{\Omega} |u \varepsilon|^2,
$$

which simultaneously involves all the components $n \varepsilon$, $c \varepsilon$, and $u \varepsilon$, for solutions of certain regularized versions of (3) (see Section 2). We shall establish an ODE for $y(t)$ and thus implying an upper bound for $y$ (see Lemma 3.4 and 3.5). Based on this upper bound, we can derive the $L^p$ estimate of $n \varepsilon$, $u \varepsilon$, and $\nabla c \varepsilon$ by some bootstrap arguments (see Section 4) and establish the existence of global classical solutions to regularized system. The global weak solutions can be constructed by some limitation process in Section 5.

2. Approximation by non-degenerate problems. To overcome the difficulties brought by the nonlinear diffusion, we shall first deal with some regularized approximate problems in this section.
We now introduce the following approximate problem
\[
\begin{cases}
n_{\varepsilon t} + u_{\varepsilon} \cdot \nabla n_{\varepsilon} = \Delta (n_{\varepsilon} + \varepsilon)^m - \nabla \cdot (n_{\varepsilon} \nabla c_{\varepsilon}) + \nabla \cdot (n_{\varepsilon} \nabla \phi), & x \in \Omega, t > 0, \\
c_{\varepsilon t} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} = \Delta c_{\varepsilon} - n_{\varepsilon} c_{\varepsilon}, & x \in \Omega, t > 0, \\
u_{\varepsilon t} + (u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + \nabla P_{\varepsilon} = \Delta u_{\varepsilon} - n_{\varepsilon} \nabla \phi + n_{\varepsilon} \nabla c_{\varepsilon}, & x \in \Omega, t > 0, \\
\nabla \cdot u_{\varepsilon} = 0, & x \in \Omega, t > 0, \\
\frac{\partial n_{\varepsilon}}{\partial \nu} = \frac{\partial c_{\varepsilon}}{\partial \nu} = 0, u_{\varepsilon} = 0, & x \in \partial \Omega, t > 0, \\
n_{\varepsilon}(x,0) = n_0(x), c_{\varepsilon}(x,0) = c_0(x), u_{\varepsilon}(x,0) = u_0(x), & x \in \Omega. \\
\end{cases}
\] (9)

By an adaptation of well-established fixed point arguments, one can readily establish the existence of a local-in-time smooth solution, nonnegativity in its first two components by the maximum principle, and the blow-up criteria (cf. Lemma 2.1 in [11], Lemma 2.1 in [34], for instance).

**Lemma 2.1.** Suppose that \( m > 1 \) and (4)-(6) hold. Then for each \( \varepsilon \in (0,1) \), there exist \( T_{\text{max},\varepsilon} \in (0,\infty) \) and a classical solution \((n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon}, P_{\varepsilon})\) to system (9) in \( \Omega \times (0,T_{\text{max},\varepsilon}) \) such that
\[
\begin{align*}
n_{\varepsilon} &\in C^0(\overline{\Omega} \times [0,T_{\text{max},\varepsilon}]) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\text{max},\varepsilon})), \\
c_{\varepsilon} &\in C^0(\overline{\Omega} \times [0,T_{\text{max},\varepsilon}]) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\text{max},\varepsilon})), \\
u_{\varepsilon} &\in C^0(\overline{\Omega} \times [0,T_{\text{max},\varepsilon}]) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\text{max},\varepsilon})), \\
P_{\varepsilon} &\in C^{1,0}(\overline{\Omega} \times (0,T_{\text{max},\varepsilon})).
\end{align*}
\]

Moreover, we have \( n_{\varepsilon} > 0 \) and \( c_{\varepsilon} > 0 \) in \( \Omega \times (0,T_{\text{max},\varepsilon}) \), and if \( T_{\text{max},\varepsilon} < \infty \), then
\[
\|n_{\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|c_{\varepsilon}(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u_{\varepsilon}(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \; t \to T_{\text{max},\varepsilon},
\] (10)

where \( \alpha \) is taken from (4).

The following lemma is very basic but important and will be frequently used in the sequel.

**Lemma 2.2.** For each \( \varepsilon \in (0,1) \), the solution of (9) satisfies
\[
\int_\Omega n_{\varepsilon}(\cdot,t) = \int_\Omega n_0 \quad \text{for all} \; t \in (0,T_{\text{max},\varepsilon})
\] (11)

and
\[
\|c_{\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \quad \text{for all} \; t \in (0,T_{\text{max},\varepsilon}).
\] (12)

**Proof.** The first conclusion directly results from an integration of the first equation in (9) over \( \Omega \). Here we need to use the fact \( \frac{\partial \phi}{\partial \nu} = 0 \) on the boundary of domain. Then (12) can be obtained immediately from the maximum principle due to the nonnegativity of \( n_{\varepsilon}c_{\varepsilon} \).

\( \square \)

3. **Energy-type inequality.** We now turn to the analysis of the coupled functional in (8). Here we first apply standard testing procedures to gain the inequalities in the following three lemmata.

**Lemma 3.1.** Suppose that \( m > 1 \) and (4)-(6) hold. There exists \( C > 0 \) such that for all \( \varepsilon \in (0,1) \) we have
\[
\frac{d}{dt} \int_\Omega n_{\varepsilon} \ln n_{\varepsilon} + \frac{2}{m} \int_\Omega |\nabla n_{\varepsilon}|^2 \leq \int_\Omega \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} + C \quad \text{for all} \; t \in (0,T_{\text{max},\varepsilon}).
\] (13)
Proof. Testing the first equation of (9) by $\ln n_\varepsilon$, we have
\[
\frac{d}{dt} \int_\Omega n_\varepsilon \ln n_\varepsilon + m \int_\Omega \frac{(n_\varepsilon + \varepsilon)^{m-1}}{n_\varepsilon} |\nabla n_\varepsilon|^2 = \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega \nabla n_\varepsilon \cdot \nabla \phi
\]
for all $t \in (0, T_{\text{max},\varepsilon})$, which shows
\[
\frac{d}{dt} \int_\Omega n_\varepsilon \ln n_\varepsilon + m \int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 \leq \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega \nabla n_\varepsilon \cdot \nabla \phi \quad (14)
\]
for all $t \in (0, T_{\text{max},\varepsilon})$. Using Young’s inequality and Hölder’s inequality, we find that
\[
\int_\Omega \nabla n_\varepsilon \cdot \nabla \phi = \int_\Omega n_\varepsilon^{m-1} \nabla n_\varepsilon \cdot n_\varepsilon^{1-\frac{m}{2}} \nabla \phi
\]
\[
\leq \frac{m}{2} \int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 + \frac{||\nabla \phi||^2_{L^\infty(\Omega)}}{2m} \int_\Omega n_\varepsilon^{2-m} \leq \frac{m}{2} \int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 + \frac{||\nabla \phi||^2_{L^\infty(\Omega)} |\Omega|^{m-1}}{2m} ||n_0||^{2-m}_{L^1(\Omega)},
\]
where we used the fact $2-m < 1$ and $\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0$. Combining this inequality with (14) and noticing the identity $\int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 = \frac{4}{m^2} \int_\Omega |\nabla n_\varepsilon|^2$, we can immediately draw our conclusion. \hfill \Box

Lemma 3.2. Suppose that $m > 1$ and (4)-(6) hold. There exist positive constants $k, K, C$ such that for all $\varepsilon \in (0,1)$ we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + k \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} + k \int_\Omega |\nabla c_\varepsilon|^4 \leq - \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + K \int_\Omega |\nabla u_\varepsilon|^2 + C \quad (15)
\]
on $(0, T_{\text{max},\varepsilon})$.

Proof. From the second equation in (9) we see that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} = \int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla c_{\varepsilon t}}{c_\varepsilon} - \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2 \cdot c_{\varepsilon t}}{c_\varepsilon^2} = - \int_\Omega \frac{\Delta c_\varepsilon \cdot c_{\varepsilon t}}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^2 \cdot c_{\varepsilon t}}{c_\varepsilon^2} - \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2 \cdot c_{\varepsilon t}}{c_\varepsilon^2} = - \int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{\Delta c_\varepsilon (u_\varepsilon \cdot \nabla c_\varepsilon)}{c_\varepsilon} - \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2 \cdot \Delta c_\varepsilon}{c_\varepsilon^2} - \frac{1}{2} \int_\Omega \frac{n_\varepsilon |\nabla c_\varepsilon|^2}{c_\varepsilon} - \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} \quad (16)
\]
for all $t \in (0, T_{\text{max},\varepsilon})$. We know from Lemma 2.7 vi) of [13] there exists $\varepsilon$–independent positive constants $k_1, k_2$ such that
\[
- \int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2 \cdot \Delta c_\varepsilon}{c_\varepsilon^2} \leq -k_1 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 - k_1 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} + k_2 \int_\Omega c_\varepsilon. \quad (17)
\]
On the other hand, \[
\int_\Omega \frac{\Delta c_\varepsilon (u_\varepsilon \cdot \nabla c_\varepsilon)}{c_\varepsilon} = -\int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{(u_\varepsilon \cdot D^2 c_\varepsilon) \nabla c_\varepsilon}{c_\varepsilon} + \int_\Omega \frac{(u_\varepsilon \cdot \nabla c_\varepsilon) |\nabla c_\varepsilon|^2}{c_\varepsilon} \]
and \[
\frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2 u_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon^2} = -\frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2 u_\varepsilon \cdot \nabla \frac{1}{c_\varepsilon} = \int_\Omega \frac{(u_\varepsilon \cdot D^2 c_\varepsilon) \cdot \nabla c_\varepsilon}{c_\varepsilon} \]
implies \[
\int_\Omega \frac{\Delta c_\varepsilon (u_\varepsilon \cdot \nabla c_\varepsilon)}{c_\varepsilon} - \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2 u_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon^2} = -\int_\Omega \frac{1}{c_\varepsilon} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon). \quad (18)
\]
Noticing that \(\|c\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}\) and the nonnegativity of \(n_\varepsilon, c_\varepsilon\), we conclude from (16)-(18) that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + k_1 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} \leq -\int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega \frac{1}{c_\varepsilon} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) + k_2 \|c_0\|_{L^\infty(\Omega)}
\leq -\int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \left( \frac{k_1}{4} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} + \frac{1}{k_1} \int_\Omega c_\varepsilon |\nabla u_\varepsilon|^2 + k_2 \|c_0\|_{L^\infty(\Omega)} \right). \quad (19)
\]
Since \(\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} \geq \frac{1}{\|c_0\|_{L^\infty(\Omega)}^4} \int_\Omega |\nabla c_\varepsilon|^4\), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + k_1 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} + \frac{k_1}{2\|c_0\|_{L^\infty(\Omega)}^3} \int_\Omega |\nabla c_\varepsilon|^4 \leq -\int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \frac{\|c_0\|_{L^\infty(\Omega)}}{k_1} \int_\Omega |\nabla u_\varepsilon|^2 + k_2 \|c_0\|_{L^\infty(\Omega)}. \quad (20)
\]

The derivation of the time evolution for \(\int_\Omega |u_\varepsilon|^2\) is more involved due to the appearance of external coupling term \(n \nabla c\).

**Lemma 3.3.** Suppose that \(m > 1\) and (4)-(6) hold. Let \(k, K\) be the constants in (15). Then there exists constant \(C > 0\) such that for any \(\varepsilon \in (0, 1)\)
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq \frac{1}{2mK} \int_\Omega |\nabla n_\varepsilon|^2 + \frac{k}{2K} \int_\Omega |\nabla c_\varepsilon|^4 + C \quad (21)
\]
for all \(t \in (0, T_{\text{max},\varepsilon})\).

**Proof.** Multiplying \(u_\varepsilon\) in the third equation of (9), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = -\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla c_\varepsilon \quad (22)
\]
for all \(t \in (0, T_{\text{max},\varepsilon})\). Recalling the Poincaré inequality we can find constant \(K_p > 0\) fulfilling
\[
\|\phi\|_{L^2(\Omega)}^2 \leq K_p \|\nabla \phi\|_{L^2(\Omega)}^2 \quad \text{for all } \phi \in W^{1,2}_0(\Omega).
\]
Then the Young inequality along with the assumed boundedness of \( \phi \) yields
\[
- \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \leq \frac{1}{4K_p} \int_\Omega |u_\varepsilon|^2 + K_p \|\nabla \phi\|_{L^\infty(\Omega)}^2 \int n_\varepsilon^2 \\
\leq \frac{1}{4} \int_\Omega |\nabla u_\varepsilon|^2 + K_p \|\nabla \phi\|_{L^\infty(\Omega)}^2 \int n_\varepsilon^2.
\]

(23)

Noticing the fact \( W^{1,2}(\Omega) \to L^q(\Omega) \) for any \( q < +\infty \) in the 2D case and making use of H"older inequality and Young's inequality we can estimate the second term in the right hand of (22) as
\[
\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla c_\varepsilon \leq \|u_\varepsilon\|_{L^{\frac{4(1+s)}{3}}(\Omega)} \cdot \|n_\varepsilon \cdot \nabla c_\varepsilon\|_{L^{1+s}(\Omega)} \\
\leq \frac{1}{4} \int_\Omega |\nabla u_\varepsilon|^2 + k_1 \|n_\varepsilon \cdot \nabla c_\varepsilon\|_{L^{1+s}(\Omega)}^2
\]
for some positive constant \( k_1 \) independent of \( \varepsilon \) and arbitrarily small \( \delta > 0 \). We use H"older inequality and Young's inequality again to find that for any \( \eta_1 > 0 \), there exists \( k(\eta_1) > 0 \) such that
\[
k_1 \|n_\varepsilon \cdot \nabla c_\varepsilon\|_{L^{1+s}(\Omega)} \leq k_1 \|n_\varepsilon\|_{L^{\frac{4(1+s)}{3}}(\Omega)} \cdot \|\nabla c_\varepsilon\|_{L^s(\Omega)} \\
\leq \eta_1 \|\nabla c_\varepsilon\|_{L^s(\Omega)}^4 + k(\eta_1) \|n_\varepsilon\|_{L^{\frac{4(1+s)}{3}}(\Omega)}^4.
\]

(24)

Combining (22)-(25), we arrive at
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq 2K_p \|\nabla \phi\|_{L^\infty(\Omega)}^2 \int n_\varepsilon^2 + 2k(\eta_1) \|n_\varepsilon\|_{L^{\frac{4(1+s)}{3}}(\Omega)}^4 + 2\eta_1 \|\nabla c_\varepsilon\|_{L^s(\Omega)}^4.
\]

(26)

The Gagliardo-Nirenberg inequality, the mass conservation (11) and the Young inequality tell that for any \( \eta_2 > 0 \) there exists positive constant \( k_i (i = 2, 3, 4) \) satisfying
\[
\int n_\varepsilon^2 = \|n_\varepsilon\|_{L^{\frac{4}{3-s}}(\Omega)}^4 \leq k_2 \left( \|\nabla n_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{3}} \cdot \|n_\varepsilon\|_{L^{\infty}(\Omega)}^{\frac{4}{3}} + \|n_\varepsilon\|_{L^{\infty}(\Omega)}^{\frac{4}{3}} \right) \\
\leq k_3 \|\nabla n_\varepsilon\|_{L^2(\Omega)}^4 + k_3 \\
\leq \eta_2 \|\nabla n_\varepsilon\|_{L^2(\Omega)}^4 + k_4.
\]

(27)

Here we used the fact \( m > 1 \). Similarly, there exists positive constant \( k_5, k_6 \) such that
\[
\|n_\varepsilon\|^4_{L^{\frac{4(1+s)}{3-s}}(\Omega)} = \|n_\varepsilon\|_{L^{\frac{4(1+s)}{3-s}}(\Omega)}^4 \leq k_5 \left( \|\nabla n_\varepsilon\|_{L^2(\Omega)}^{\frac{2(1+s)}{3}} \cdot \|n_\varepsilon\|_{L^{\infty}(\Omega)}^{\frac{2(3-s)}{3}} + \|n_\varepsilon\|_{L^{\infty}(\Omega)}^{\frac{2(3-s)}{3}} \right) \\
\leq k_6 \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + k_6.
\]

(28)

Without loss of generality, we may assume \( m < 5 \). As \( m > 1 \), we can choose \( \delta = \frac{m-1}{2(5-m)} \), which implies
\[
\frac{2(1+5\delta)}{m(1+\delta)} < 2.
\]
Then the Young inequality guarantees the existence of \( k_7 > 0 \) such that for any \( \eta_3 > 0 \)
\[
\|n_\varepsilon\|_{L^{\frac{4(1+\varepsilon)}{4}}(\Omega)}^4 \leq \eta_3 \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + k_7.
\] (29)

Substituting (27) and (29) into (26), we obtain that
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq \left(2\eta_2 K_p \|\nabla \phi\|^2_{L^\infty(\Omega)} + 2\eta_3 k(\eta_1)\right) \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + 2\eta_1 \|\nabla c_\varepsilon\|^4_{L^4(\Omega)} + k_4 + k_7.
\] (30)

Let \( \eta_1 = \frac{k}{4\varepsilon}, \eta_2 = \frac{1}{8mK_p \|\nabla \phi\|^2_{L^\infty(\Omega)}}, \eta_3 = \frac{1}{8mK_k(\eta_1)} \). Then we can draw our conclusion (21) with \( C = k_4 + k_7 \).

Collecting the result of Lemma 3.1 to Lemma 3.3, we thus establish the following energy-type inequality which simultaneously involves all the components \( n_\varepsilon, c_\varepsilon \) and \( u_\varepsilon \).

**Lemma 3.4.** Suppose that \( m > 1 \) and (4)-(6) hold. There exists \( C > 0 \) such that for all \( t \in (0,T_{\text{max},\varepsilon}) \)
\[
\frac{d}{dt} \left\{ \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2K \int_\Omega |u_\varepsilon|^2 \right\} + \frac{1}{m} \int_\Omega |\nabla n_\varepsilon|^2 \leq C + \int_\Omega |\nabla u_\varepsilon|^2 \leq C
\] (31)
whenever \( \varepsilon \in (0,1) \).

Thereupon, we can state the following boundedness consequence.

**Lemma 3.5.** Suppose that \( m > 1 \) and (4)-(6) hold. There exists \( C > 0 \) such that
\[
\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^2 \leq C \quad \text{for all} \quad t \in (0,T_{\text{max},\varepsilon}),
\] (32)
\[
\int_0^T \int_\Omega |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 + \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C(T+1) \quad \text{for all} \quad T \in (0,T_{\text{max},\varepsilon})
\] (33)
whenever \( \varepsilon \in (0,1) \).

**Proof.** We first fix \( p \in (1,2) \) and observe that
\[
\xi \ln \xi \leq \frac{1}{p(p-1)} \xi^p \quad \text{for all} \quad \xi > 0.
\]
Consequently, an application of the Gagliardo-Nirenberg inequality yields \( k_1 > 0, k_2 > 0 \) such that
\[
\int_\Omega n_\varepsilon \ln n_\varepsilon \leq \frac{1}{p(p-1)} \int_\Omega n_\varepsilon^p
\]
\[
= \frac{1}{p(p-1)} \|n_\varepsilon^\frac{1}{p}\|_{L^\frac{p}{p-1}(\Omega)}^p
\]
\[
\leq k_1 \left( \|\nabla n_\varepsilon\|_{L^2(\Omega)} + \|n_\varepsilon\|^\frac{1}{p}\|_{L^\frac{p}{p-1}(\Omega)} + \|n_\varepsilon\|_{L^\infty(\Omega)} \right)^{\frac{1}{p}}
\]
\[
\leq k_2 \int_\Omega |\nabla n_\varepsilon|^2 + k_2
\] (34)
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Here we used the mass conservation (11) and the fact \( \frac{2(p-1)}{m} < 2 \). On the other hand, making use of the boundedness of \( \|c_\varepsilon\|_{L^\infty(\Omega)} \) (12) and the Young inequality, we know there exists \( k_3 > 0 \) fulfilling
\[
\int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon}^2 \leq k \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon}^4 + \frac{1}{k} \int_\Omega c_\varepsilon \leq k \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon}^4 + k_3 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}). \tag{35}
\]
At last, it is easy to see from Pincaré’s inequality that
\[
\int_\Omega |u_\varepsilon|^2 \leq K_p \int_\Omega |\nabla u_\varepsilon|^2, \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{36}
\]
where \( K_p \) is the constant as in Lemma 3.3.

Substituting (34)-(36) into (31), we achieve that there exist positive constants \( k_4 \) and \( k_5 \) such that
\[
\frac{d}{dt} \left( \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon}^2 + 2K \int_\Omega |u_\varepsilon|^2 \right) + k_4 \left( \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon}^2 + 2K \int_\Omega |u_\varepsilon|^2 \right) \leq k_5. \tag{37}
\]
As \( \int_\Omega |\nabla c_\varepsilon|^2 = \int_\Omega |\nabla c_\varepsilon|^2 \cdot c_\varepsilon \leq \|c_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |\nabla c_\varepsilon|^2 \), an integration of (37) immediately yields estimates (32). And then (33) can be obtained by an integration of (31).

4. Existence of global classical solutions to the approximate systems.

With the above regularization properties of each component \( n_\varepsilon, c_\varepsilon, u_\varepsilon \) at hand, we are now in the position to make sure that all approximate problems (9) are in fact globally solvable. To this end, we shall first making use of Lemma 3.5 to derive some \( L_p \) estimates for \( n_\varepsilon, u_\varepsilon \) and \( \nabla c_\varepsilon \).

**Lemma 4.1.** Suppose that \( m > 1 \) and (4)-(6) hold. For all \( T_0 > 0 \) and any \( p > 1 \) there exists \( C > 0 \) satisfying for all \( \varepsilon \in (0, 1) \),
\[
\int_\Omega n_\varepsilon^p(x,t)dx \leq C \quad \text{for all } t \in (0, T), \tag{38}
\]
with \( T := \min\{T_0, T_{\text{max}, \varepsilon}\} \).

**Proof.** Without loss of generality, we assume \( p > 2(m - 1) \). We multiply the first equation in (9) by \( n_\varepsilon^{p-1} \) to obtain
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega n_\varepsilon^p + m(p-1) \int_\Omega n_\varepsilon^{p+m-3} |\nabla n_\varepsilon|^2 \\
\leq (p-1) \int_\Omega n_\varepsilon^{p-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon - (p-1) \int_\Omega n_\varepsilon^{p-1} \nabla n_\varepsilon \cdot \nabla \phi
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \), which leads to
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega n_\varepsilon^p + \frac{m(p-1)}{2} \int_\Omega n_\varepsilon^{p+m-3} |\nabla n_\varepsilon|^2 \\
\leq \frac{p-1}{2m} \int_\Omega n_\varepsilon^{p-m+1} |\nabla c_\varepsilon|^2 - (p-1) \int_\Omega n_\varepsilon^{p-1} \nabla n_\varepsilon \cdot \nabla \phi \tag{39}
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \), from Young’s inequality. Now we estimate the two integrals on the right hand.
As $\phi \in W^{2,\infty}(\Omega)$, it is more simple to first estimate the second one as

$$(p - 1) \int_{\Omega} n_{\varepsilon}^{p-1} \nabla n_{\varepsilon} \cdot \nabla \phi = (p - 1) \int_{\Omega} n_{\varepsilon}^{\frac{p+1}{p} - 1} \nabla n_{\varepsilon} \cdot n_{\varepsilon}^{\frac{p-1}{p+1}} \nabla \phi$$

$$\leq \frac{m(p - 1)}{16} \int_{\Omega} n_{\varepsilon}^{p+m-3} |\nabla n_{\varepsilon}|^2 + \frac{4(p - 1)}{m} \|\nabla \phi\|_{L^{\infty}(\Omega)} \int_{\Omega} n_{\varepsilon}^{p-m+1}. \quad (40)$$

By using the Gagliardo-Nirenberg inequality, we can find there exists $k_1 > 0$ such that for all $t \in (0, T_{\text{max}, \varepsilon})$,

$$\int_{\Omega} n_{\varepsilon}^{p-m+1} \leq \frac{m(p - 1)}{16} \int_{\Omega} n_{\varepsilon}^{p+m-3} |\nabla n_{\varepsilon}|^2 + k_2 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}). \quad (41)$$

In summary, we have

$$(p - 1) \int_{\Omega} n_{\varepsilon}^{p-1} \nabla n_{\varepsilon} \cdot \nabla \phi \leq \frac{m(p - 1)}{8} \int_{\Omega} n_{\varepsilon}^{p+m-3} |\nabla n_{\varepsilon}|^2 + k_2 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (42)$$

for all $\varepsilon \in (0, 1)$. For another integral on the right hand of (39), we can first use the Hölder inequality to derive

$$\int_{\Omega} n_{\varepsilon}^{p-m+1} |\nabla c_{\varepsilon}|^2 \leq \left( \int_{\Omega} n_{\varepsilon}^{2(p-m+1)} \right)^{\frac{1}{2}} \cdot \left( \int_{\Omega} |\nabla c_{\varepsilon}|^4 \right)^{\frac{1}{2}}.$$  

Then the Gagliardo-Nirenberg again yields $k_3 > 0$ satisfying

$$\left( \int_{\Omega} n_{\varepsilon}^{2(p-m+1)} \right)^{\frac{1}{2}} = \left\| n_{\varepsilon}^{\frac{p+m-1}{2}} \right\|_{L^{(p+m-1)(p+m-1)}} \leq k_3 \left( \left\| \nabla n_{\varepsilon}^{\frac{p+m-1}{2}} \right\|_{L^{2(\Omega)}} \cdot \left\| n_{\varepsilon}^{\frac{p+m-1}{2}} \right\|_{L^{\frac{p+m}{p+m-1}}} \cdot \left\| n_{\varepsilon}^{\frac{p+m-1}{2}} \right\|_{L^{\frac{p+m}{p+m-1}}} \right),$$

where $a = \frac{p-2m+2}{2(p-m+1)} \in (0, 1)$ due to the fact $p > 2(m - 1)$. Therefore, we find the existence of $k_4 > 0$ such that

$$\left( \int_{\Omega} n_{\varepsilon}^{2(p-m+1)} \right)^{\frac{1}{2}} \leq k_4 \left( \int_{\Omega} n_{\varepsilon}^{p+m-3} |\nabla n_{\varepsilon}|^2 \right)^{\frac{p-2(m-1)}{4(p+m-1)}} \cdot \left( \int_{\Omega} n_{\varepsilon}^p \right)^{\frac{1}{2}} + k_4.$$
As $\frac{p-2(m-1)}{2(p+m-1)} < \frac{1}{2}$, by using Young’s inequality, we can pick $k_3 > 0$ such that
\[
\frac{(p-1)}{2m} \int_{\Omega} n_{\epsilon}^{p-2m+1} |\nabla c_{\epsilon}|^2 \leq m(p-1) \int_{\Omega} n_{\epsilon}^{p+m-3} |\nabla n_{\epsilon}|^2 + k_3 \left( \int_{\Omega} |\nabla c_{\epsilon}|^4 \right)^{\frac{1}{4}} \cdot \left( \int_{\Omega} n_{\epsilon}^p + 1 \right) .
\]

Collecting (39), (42) and (43), we obtain that $y(t) := \int_{\Omega} n_{\epsilon}^p(x,t)dx$, $t \in [0,T)$, satisfies the ODI
\[
y'(t) \leq k_6 \left( \int_{\Omega} |\nabla c_{\epsilon}|^4 + 1 \right) (y(t) + 1) \quad \text{for all } t \in (0,T)
\]
with some $k_6 > 0$. On integration we infer that
\[
y(t) + 1 \leq (y(0) + 1) \cdot e^{k_6 \int_0^T \int_{\Omega} |\nabla c_{\epsilon}|^4} \quad \text{for all } t \in (0,T),
\]
whereupon an application of Lemma 3.5 completes the proof. \hfill \Box

The conclusion of above Lemma together with the boundedness of $\int_0^T \int_{\Omega} |\nabla c_{\epsilon}|^4$ obtained in Lemma 3.5 imply the following regular estimate for $u_{\epsilon}$.

**Lemma 4.2.** Suppose that $m > 1$ and (4)-(6) hold. For all $p > 1$, $T_0 > 0$ there exists $C > 0$ such that for each $\epsilon \in (0,1)$
\[
\|u_{\epsilon}(\cdot,t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0,T)
\]  
with $T := \min\{T_0, T_{\max,\epsilon}\}$.

**Proof.** Let $\tau = \frac{T}{6}$. We can first obtain from Lemma 4.1 and assumption on $\phi$ that there exists $k_1 > 0$ such that
\[
\frac{1}{\tau} \int_{t}^{t+\tau} \int_{\Omega} |n_{\epsilon} \nabla \phi|^2 \leq k_1 , \quad \text{for all } t \in (0,T-\tau).
\]
Next we also have $k_2 > 0$ fulfilling
\[
\frac{1}{\tau} \int_{t}^{t+\tau} \int_{\Omega} |n_{\epsilon} \nabla c_{\epsilon}|^2 \leq k_2 , \quad \text{for all } t \in (0,T-\tau).
\]
Indeed,
\[
\frac{1}{\tau} \int_{t}^{t+\tau} \int_{\Omega} |n_{\epsilon} \nabla c_{\epsilon}|^2 \leq \frac{1}{2\tau} \int_{t}^{t+\tau} \int_{\Omega} |n_{\epsilon}|^4 + \frac{1}{2\tau} \int_{t}^{t+\tau} \int_{\Omega} |\nabla c_{\epsilon}|^4 \leq k_2
\]
from the results of Lemma 4.1 and Lemma 3.5. Then we can infer from Lemma 3.3 of [31] that
\[
\int_{\Omega} |\nabla u_{\epsilon}|^2 \leq k_3 \quad \text{for all } t \in (0,T)
\]  
with some $k_3 > 0$. Since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, the $L^p$ estimate of $u_{\epsilon}$ is an immediate consequence of Poincaré’s inequality and (47). \hfill \Box

**Lemma 4.3.** Suppose that $m > 1$ and (4)-(6) hold. Let $q < 4$. For all $T_0 > 0$ there exists $C > 0$ satisfying
\[
\|\nabla c_{\epsilon}(\cdot,t)\|_{L^q(\Omega)} \leq C \quad \text{for all } t \in (0,T)
\]  
with $T := \min\{T_0, T_{\max,\epsilon}\}$. 

The variation-of-constants formula and well-known smoothing estimates for the Neumann heat semigroup ([33], Lemma 1.3) yield the estimate
\[ \|\nabla c_\varepsilon\|_{L^2(\Omega)} \leq k_1\|\nabla c_0\|_{L^\infty(\Omega)} + k_1\int_0^t (t-s)^{-\frac{1}{2}}\left(\frac{3}{2} - \frac{1}{2}\right)\|n_\varepsilon c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^\frac{4}{3}(\Omega)} ds, \] (49)
for all \( t \in (0, T) \). It is easy to find \( k_2 > 0 \) such that
\[ \|n_\varepsilon c_\varepsilon\|_{L^\frac{4}{3}(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^\frac{4}{3}(\Omega)} \leq k_2 \quad \text{for all} \quad t \in (0, T) \] (50)
from Lemma 4.1. We use the Hölder inequality to conclude from (32) and (44) that there exists \( k_3 > 0 \) satisfying
\[ \|u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^\frac{4}{3}(\Omega)} \leq \|u_\varepsilon\|_{L^4(\Omega)} \cdot \|\nabla c_\varepsilon\|_{L^2(\Omega)} \leq k_3 \quad \text{for all} \quad t \in (0, T). \] (51)
From (49)-(51), we know that for any \( q < 4 \),
\[ \|\nabla c_\varepsilon\|_{L^q(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T) \] (52)
with some \( C > 0 \).

**Lemma 4.4.** Suppose that \( m > 1 \) and (4)-(6) hold. Then for each \( \varepsilon \in (0, 1) \), the classical solution to system (9) is global; that is, \( T_{\text{max}, \varepsilon} = \infty \).

**Proof.** Let \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)\) denote the classical solution of (9). We now make sure that \( T_{\text{max}, \varepsilon} = \infty \) for any \( \varepsilon \in (0, 1) \). To this end, we assume on the contrary that \( T_{\text{max}, \varepsilon} < \infty \) for some \( \varepsilon > 0 \), and proceed to derive a contradiction to (10). We first assert that if \( T_{\text{max}, \varepsilon} < \infty \),
\[ \sup_{t \in (0, T_{\text{max}})} \|A^\alpha u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} < \infty \] (53)

For this purpose, we first apply the Helmholtz projection \( \mathcal{P} \) to both sides of the third equation of (9) and multiply the resulting identity by \( Au_\varepsilon \). Using the Young inequality and the projection property \( \|\mathcal{P}\varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)} \) for \( \varphi \in L^2(\Omega) \), we thereby obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |A^\frac{1}{2} u_\varepsilon|^2 + \int_\Omega |Au_\varepsilon|^2
= \int_\Omega \mathcal{P}((u_\varepsilon \cdot \nabla)u_\varepsilon)Au_\varepsilon - \int_\Omega \mathcal{P}(n_\varepsilon \nabla \phi)Au_\varepsilon + \int_\Omega \mathcal{P}(n_\varepsilon \nabla c_\varepsilon)Au_\varepsilon
\leq \frac{1}{2} \int_\Omega |Au_\varepsilon|^2 + \int_\Omega |(u_\varepsilon \cdot \nabla)u_\varepsilon|^2 + 2 \int_\Omega |n_\varepsilon \nabla \phi|^2 + 2 \int_\Omega |n_\varepsilon \nabla c_\varepsilon|^2
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). (54)
As \( \phi \in W^{2,\infty}(\Omega) \), remembering Lemma 4.1, 4.3, we can easily find \( k_1 > 0, k_2 > 0 \) such that
\[
2 \int_\Omega |n_\varepsilon \nabla \phi|^2 + 2 \int_\Omega |n_\varepsilon \nabla c_\varepsilon|^2
\leq 2\|\nabla \phi\|_{L^\infty(\Omega)}^2 n_\varepsilon^2 + 2 \left( \int_\Omega n_\varepsilon^2 \right)^\frac{1}{2} \left( \int_\Omega |\nabla c_\varepsilon|^3 \right)^\frac{2}{3}
\leq k_1 + k_2
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). We can thereupon obtain that
\[
\frac{d}{dt} \int_{\Omega} |A^2 u_\varepsilon|^2 + \int_{\Omega} |Au_\varepsilon|^2 \leq 2 \int_{\Omega} |(u_\varepsilon \cdot \nabla)u_\varepsilon|^2 + k_3 \quad \text{for all} \ t \in (0, T_{\text{max}, \varepsilon}) \tag{55}
\]
with some \( k_3 = 2k_1 + 2k_2 \). Now by the Gagliardo-Nirenberg inequality, (47) and Lemma 4.2 we can pick \( k_4 > 0, k_5 > 0 \) and \( k_6 > 0 \) such that
\[
\int_{\Omega} |(u_\varepsilon \cdot \nabla)u_\varepsilon|^2 \leq \|u_\varepsilon\|^2_{L^\infty(\Omega)} \cdot \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}
\leq k_4 \|u_\varepsilon\|^2_{W^{2,2}(\Omega)} \cdot \|u_\varepsilon\|_{L^2(\Omega)} \cdot \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}
\leq k_5 \|u_\varepsilon\|^2_{W^{2,2}(\Omega)}
\leq \frac{1}{4} \|Au_\varepsilon\|^2_{L^2(\Omega)} + k_6, \quad \text{for all} \ t \in (0, T_{\text{max}, \varepsilon})
\]
where we have employed Young’s inequality and the fact that \( \|A(\cdot)\|_{L^2(\Omega)} \) defines a norm equivalent to \( \|\cdot\|_{W^{2,2}(\Omega)} \) on \( D(A) \) ([21], Theorem 2.1.1).

Therefore, recalling that \( A = -\mathcal{P} \Delta \) and hence \( \|A^2 u_\varepsilon\|_{L^2(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega)} \), we see that (55) provides \( k_7 > 0 \) satisfying
\[
\frac{d}{dt} \int_{\Omega} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \int_{\Omega} \|\Delta u_\varepsilon\|^2 \leq k_7 \quad \text{for all} \ t \in (0, T_{\text{max}, \varepsilon}). \tag{56}
\]
In conjunction with the boundedness of \( \|\nabla u_\varepsilon\|_{L^2(\Omega)} \) this means that there exists \( k_8 > 0 \) such that
\[
\int_0^{T_{\text{max}, \varepsilon}} \int_{\Omega} \|\Delta u_\varepsilon\|^2 \leq k_8. \tag{57}
\]
From the variation-of-constants formula for \( u \) and the contractivity of the Stokes semigroup in \( L^2(\Omega) \) we know that
\[
\|A^\alpha u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}
\leq \|A^\alpha u_0\|_{L^2(\Omega)} + \int_0^t \|A^\alpha e^{-(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s) \nabla \phi)\|_{L^2(\Omega)} ds
+ \int_0^t \|A^\alpha e^{-(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s) \nabla c_\varepsilon)\|_{L^2(\Omega)} ds
+ \int_0^t \|A^\alpha e^{-(t-s)A} \mathcal{P}(u_\varepsilon(\cdot, s) \nabla u_\varepsilon(\cdot, s))\|_{L^2(\Omega)} ds \quad \text{for all} \ t \in (0, T_{\text{max}, \varepsilon}), \tag{58}
\]
where thanks to Lemma 4.1, Lemma 4.3 and Hölder’s inequality there exist \( k_i > 0(i = 9, 10, 11) \) such that whenever \( 0 < s < t < T_{\text{max}, \varepsilon} \),
\[
\|A^\alpha e^{-(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s) \nabla \phi)\|_{L^2(\Omega)} \leq k_9 (t - s)^{-\alpha} \tag{59}
\]
and
\[
\|A^\alpha e^{-(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s) \nabla c_\varepsilon)\|_{L^2(\Omega)} \leq k_{10} (t - s)^{-\alpha} \|n_\varepsilon(\cdot, s)\|_{L^5(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \leq k_{11} (t - s)^{-\alpha}. \tag{60}
\]
Moreover, since $\alpha < 1$ we can find $p > 2$ large such that $p' := \frac{p}{p-1}$ satisfies $p'\alpha < 1$, and use the Hölder inequality to estimate

$$
\int_0^t \|A^\alpha e^{-(t-s)A}P(u_\varepsilon \cdot \nabla)u_\varepsilon(\cdot, s)\|_{L^2(\Omega)} ds
\leq k_{12} \int_0^t (t-s)^{-\alpha} \|(u_\varepsilon \cdot \nabla)u_\varepsilon(\cdot, s)\|_{L^2(\Omega)} ds
\leq k_{12} \left( \int_0^t (t-s)^{-p'\alpha} \left( \int_0^t \|(u_\varepsilon \cdot \nabla)u_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^p ds \right)^{\frac{1}{p'}} \right) \frac{1}{\alpha}
$$

for all $t \in (0, T_{\text{max}, \varepsilon})$ with some $k_{12} > 0$. Recalling (44), (36) and (57), we can use the Hölder and the Gagliardo-Nirenberg inequality to find positive $k_{13}, k_{14}$ and $k_{15}$ fulfilling

$$
\int_0^{T_{\text{max}, \varepsilon}} \|u_\varepsilon\|_{L^p(\Omega)} ds
\leq \int_0^{T_{\text{max}, \varepsilon}} \|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^p \cdot \|\nabla u_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^{2p} ds
\leq k_{13} \int_0^{T_{\text{max}, \varepsilon}} \|\nabla u_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^p ds
\leq k_{14} \int_0^{T_{\text{max}, \varepsilon}} \|\Delta u_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \cdot \|\nabla u_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^{p-2} ds
\leq k_{15}
$$

Thereupon, collecting (58)-(61) we can obtain (53). As a consequence, we have

$$
\|u_\varepsilon\|_{L^\infty(\Omega)} \leq k_{16} \quad \text{in } \Omega \times (0, T_{\text{max}, \varepsilon})
$$

(62) for some $k_{16} > 0$ since $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ due to the fact $\alpha \geq \frac{1}{2}$ [21].

For any $2 < q < 4$, by an application of the variation-of-constants formula and the smoothing estimates for the Neumann heat semigroup we can estimate

$$
\|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}
\leq k_{17}\|\nabla c_0\|_{L^\infty(\Omega)} + k_{17} \left( \int_0^t (t-s)^{-\frac{3}{2} - \frac{1}{q}} \|(n_\varepsilon c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot, s)\|_{L^q(\Omega)} ds \right)
\leq k_{18} + k_{18} \int_0^t (t-s)^{-\frac{3}{2} - \frac{1}{q}} \|(n_\varepsilon c_\varepsilon\|_{L^q(\Omega)} + \|\nabla c_\varepsilon\|_{L^q(\Omega)} ds
\leq k_{19} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon})
$$

(63) for some positive $k_{17}, k_{18}$ and $k_{19}$, where we used the boundedness of $\|c_\varepsilon\|_{L^\infty(\Omega)}$, (48), (38) and (62). Finally, by a straightforward iteration procedure of Moser-type as in [23] or [37], we arrive at the $L^\infty$ estimate for $n$, that is,

$$
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq k_{20} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon})
$$

(64) with some positive constant $k_{20}$. In consequence, we infer from (53), (63) and (64) that (10) does not hold, which is absurd. We therefore conclude that $T_{\text{max}, \varepsilon} = +\infty$ whenever $\varepsilon \in (0, 1)$, that is, $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is global in time.
5. **Passing to the limit. Proof of Theorem 1.1.** In this section, we shall use an approximate procedure to prove that the global solutions to system (9) converge to the weak solution of problem (3). For this purpose, we first recall that Lemma 4.4 shows that if \( m > 1 \) the regularized system (9) possesses a global classical solution \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)\) for any \( \varepsilon \in (0, 1) \), and for each \( T > 0 \), we can find \( \varepsilon \)-independent \( K(T) > 0 \) satisfies

\[
\|n_\varepsilon(t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} + \|u_\varepsilon(t)\|_{L^\infty(\Omega)} + \|\nabla u_\varepsilon(t)\|_{L^2(\Omega)} \\
\leq K(T) \quad \text{for all } t \in (0, T),
\]

\[
\int_0^T \|\Delta u_\varepsilon(s)\|_{L^2(\Omega)}^2 ds \leq K(T).
\]

To achieve the convergence result, we need the following further regularity estimates.

**Lemma 5.1.** Let \( T > 0 \) and let \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)\) be a solution of (9) on \((0, T)\). Then

\[
\|\nabla(n_\varepsilon + \varepsilon)^\gamma\|_{L^2((0,T);L^2(\Omega))} \leq C_1, \quad \text{for any } \gamma > \max\{m - 1, 1\},
\]

\[
\|(n_\varepsilon + \varepsilon)^\theta\|_{L^1((0,T);W^{2,\infty}_0(\Omega))} \leq C_2 \quad \text{for any } \theta > \max\{m, 2\},
\]

**Proof.** We first prove (67). Let \( p > \max\{m - 1, 1\} \). We test the first equation of (9) by \( p(n_\varepsilon + \varepsilon)^{p-1} \) to see that

\[
\frac{d}{dt}\|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \frac{2mp(p - 1)}{(p + m - 1)^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 \\
\leq \frac{p(p - 1)}{m} \int_\Omega \varepsilon^{p-m+1} |\nabla c_\varepsilon|^2 + \frac{p(p - 1)}{m} \int_\Omega n_\varepsilon^{p-m+1} |\nabla \phi|^2.
\]

Noticing that \( p - m + 1 > 0 \), we have from the \( L^\infty \)-estimate for \( n_\varepsilon \), \( L^\infty \)-estimate for \( c_\varepsilon \) in (65) and the boundedness of \( \nabla \phi \) that there exists some \( k_1 > 0 \)

\[
\frac{d}{dt}\|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \frac{2mp(p - 1)}{(p + m - 1)^2} \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 \leq k_1.
\]

Integrating this inequality with respect to \( t \in (0, T) \) and replacing \( \frac{p+m-1}{2} = \gamma \), we can obtain (67).

Next, let \( \varphi \in W^{2,2}_0(\Omega) \) and \( \theta > \max\{m, 2\} \), we test the first equation of (9) by \( \theta(n_\varepsilon + \varepsilon)^{\theta-1}\varphi \) to obtain that

\[
\int_\Omega \left( \frac{\partial}{\partial t}(n_\varepsilon + \varepsilon)^\theta \right) \varphi \\
= -\theta \int_\Omega \nabla(n_\varepsilon + \varepsilon) \cdot \nabla((n_\varepsilon + \varepsilon)^{\theta-1}\varphi) + \theta \int_\Omega n_\varepsilon \nabla c_\varepsilon \cdot \nabla((n_\varepsilon + \varepsilon)^{\theta-1}\varphi) \\
- \theta \int_\Omega n_\varepsilon \nabla \varphi \cdot \nabla((n_\varepsilon + \varepsilon)^{\theta-1}\varphi) + \int_\Omega (n_\varepsilon + \varepsilon)^\theta u_\varepsilon \cdot \nabla \varphi \\
=: I_1 + I_2 + I_3 + I_4.
\]

We can first estimate \( I_1 \) as follows.

\[
I_1 = -m\theta(\theta - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m+\theta-3} |\nabla n_\varepsilon|^2 \varphi - m\theta \int_\Omega (n_\varepsilon + \varepsilon)^{m+\theta-2} \nabla n_\varepsilon \cdot \nabla \varphi,
\]
where
\[
|m\theta(\theta - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m+\theta-3} |\nabla n_\varepsilon|^2 \varphi| \\
\leq \frac{4m\theta(\theta - 1)}{(m + \theta - 1)^2} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 \cdot |\varphi| \\
\leq \frac{4m\theta(\theta - 1)}{(m + \theta - 1)^2} \|\varphi\|_{L^\infty(\Omega)} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2
\]
and
\[
|m\theta \int_\Omega (n_\varepsilon + \varepsilon)^{m+\theta-2} \nabla n_\varepsilon \cdot \nabla \varphi| \\
= \frac{m\theta}{(m + \theta - 1)} \int_\Omega \nabla (n_\varepsilon + \varepsilon)^{m+\theta-1} \cdot \nabla \varphi \\
\leq \frac{m\theta}{(m + \theta - 1)} \|\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \\
\leq \frac{m\theta}{2(m + \theta - 1)} \left( \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 + 1 \right) \|\nabla \varphi\|_{L^2(\Omega)}.
\]

Therefore, there exists \(k_1 > 0\) such that
\[
|I_1| \leq k_1 \left( \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{m+\theta-1}{2}} + \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 + 1 \right) \|\varphi\|_{W_0^{2,2}(\Omega)}. \tag{70}
\]
Here we used the fact \(W_0^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)\). Similarly,
\[
I_2 = \theta(\theta - 1) \int_\Omega n_\varepsilon (n_\varepsilon + \varepsilon)^{\theta-2} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \cdot \varphi + \theta \int_\Omega n_\varepsilon (n_\varepsilon + \varepsilon)^{\theta-1} \nabla c_\varepsilon \cdot \nabla \varphi.
\]
As
\[
\left| \theta(\theta - 1) \int_\Omega n_\varepsilon (n_\varepsilon + \varepsilon)^{\theta-2} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \cdot \varphi \right| \\
\leq (\theta - 1) \|\varphi\|_{L^\infty(\Omega)} \|\nabla c_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)|^\theta
\]
and
\[
\left| \theta \int_\Omega n_\varepsilon (n_\varepsilon + \varepsilon)^{\theta-1} \nabla c_\varepsilon \cdot \nabla \varphi \right| \leq \theta \int_\Omega (n_\varepsilon + \varepsilon)^{\theta} \|\nabla c_\varepsilon\| \|\nabla \varphi\|,
\]
we know from the estimate (65) that there exists \(k_2 > 0\) such that
\[
|I_2| \leq k_2 \left( \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{\theta}|^2 + 1 \right) \|\varphi\|_{W_0^{2,2}(\Omega)}. \tag{71}
\]
By quite a similar way, we can obtain that there exists \(k_3 > 0\) fulfilling
\[
|I_3| \leq k_3 \left( \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{\theta}|^2 + 1 \right) \|\varphi\|_{W_0^{2,2}(\Omega)}. \tag{72}
\]
Finally, recalling the boundedness of \(n_\varepsilon\) and \(u_\varepsilon\) in (65), we can easily see there exists \(k_4 > 0\) such that
\[
|I_4| \leq k_4 \|\varphi\|_{W_0^{2,2}(\Omega)}. \tag{73}
\]
Collecting (70)-(73), we can find \(k_5 > 0\) satisfying
\[
\left| \int_\Omega \left( \frac{\partial}{\partial t} (n_\varepsilon + \varepsilon)^\theta \right) \varphi \right|
\leq k_5 \left( \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 + \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 + \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^\theta|^2 \right) \|\varphi\|_{W_0^{2,2}(\Omega)}
\]
Integrating from \(t\) to \(T\), we arrive at
\[
\int_0^T \left\| \int_\Omega \left( \frac{\partial}{\partial t} (n_\varepsilon + \varepsilon)^\theta \right) \right\|_{(W_0^{2,2}(\Omega))^*}^2 \leq k_5 \left( \int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 + \int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+\theta-1}|^2 
+ \int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^\theta|^2 + T \right) \|\varphi\|_{W_0^{2,2}(\Omega)}.
\]
Then we can use (67) to derive our conclusion.

\[\square\]

**Lemma 5.2.** Let \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)\) be a global classical solution of (9). Then there exist triple of functions \((n, c, u)\) satisfying
\[
n \in L^\infty(0, \infty; L^\infty(\Omega)) \quad \text{with} \quad \nabla n^m \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)),
\]
\[
c \in L^\infty(0, \infty; L^\infty(\Omega)) \quad \text{with} \quad \nabla c \in L^\infty(0, \infty; L^\infty(\Omega)),
\]
\[
u \in L^\infty(0, \infty; W^{1,2}(\Omega)) \quad \text{with} \quad \Delta u \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)),
\]
such that \(\nabla \cdot u = 0\) in the distributional sense in \(\Omega \times (0, \infty)\)
and a subsequence \(\{\varepsilon_j\}_{j=1}^\infty\) such that
\[
n_{\varepsilon_j} \to n \quad \text{weakly* in} \quad L^\infty(0, \infty; L^\infty(\Omega)); \quad (74)
\]
\[
\nabla n_{\varepsilon_j}^m \to \nabla n^m \quad \text{in} \quad L^2_{\text{loc}}([0, \infty); L^2(\Omega)); \quad (75)
\]
\[
c_{\varepsilon_j} \to c \quad \text{weakly* in} \quad L^\infty(0, \infty; L^\infty(\Omega)); \quad (76)
\]
\[
\nabla c_{\varepsilon_j} \to \nabla c \quad \text{weakly* in} \quad L^\infty(0, \infty; L^\infty(\Omega)); \quad (77)
\]
\[
u_{\varepsilon_j} \to \nu \quad \text{in} \quad L^\infty(0, \infty; H^1(\Omega)); \quad (78)
\]
\[
\Delta u_{\varepsilon_j} \to \Delta u \quad \text{in} \quad L^2_{\text{loc}}([0, \infty); L^2(\Omega)); \quad (79)
\]
\[
n_{\varepsilon_j} \to n \quad \text{in} \quad L^2(0, \infty; L^2(\Omega)); \quad (80)
\]
as \(j \to \infty\).

**Proof.** From the estimate (65)-(67) we know there exists a subsequence \(\{\varepsilon_j\}_{j=1}^\infty\)
and non-negative functions \(n \in L^\infty(0, \infty; L^\infty(\Omega))\) with \(\nabla n^m \in L^2_{\text{loc}}([0, \infty); L^2(\Omega))\),
\(c \in L^\infty(0, \infty; L^\infty(\Omega))\) with \(\nabla c \in L^\infty(0, \infty; L^\infty(\Omega))\) and \(u \in L^\infty(0, \infty; W^{1,\infty}(\Omega))\)
with \(\Delta u \in L^2_{\text{loc}}([0, \infty); L^2(\Omega))\) satisfying (74)-(78). As \(\nabla \cdot u = 0\), we have \(\nabla \cdot u = 0\)
in the distributional sense in \(\Omega \times (0, \infty)\).

On the other hand, we fix \(\gamma > \max\{m, 2\}\), the estimate (67) and (69) shows that
\[
(n_\varepsilon + \varepsilon)^\gamma_{\varepsilon \in (0, 1)} \in L^2((0, T); W^{1,2}(\Omega)),
\]
\[
\left( \frac{\partial}{\partial t} (n_\varepsilon + \varepsilon)^\gamma \right)_{\varepsilon \in (0, 1)} \in L^1((0, T); (W_0^{2,2}(\Omega))^*).
\]
Thereupon, an application of the Aubin-Lions lemma shows that \((n_\varepsilon + \varepsilon)^\gamma_\varepsilon \in (0,1)\) is relatively compact in \(L^2((0,T);L^2(\Omega))\). By using the Egorov’s theorem, we can find a subsequence (still denoted by \(\{\varepsilon_j\}_{j=1}^\infty\)) fulfilling (80).

**Proof of Theorem 1.1.** Let \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\) be the global classical solution of (9). Lemma 5.2 asserts that there exists the limit function \((n, c, u)\). We now prove that \((n, c, u)\) satisfies (3) in the weak formulation of Definition 1.2. Indeed, taking \(L\) relatively compact in \(\Omega\),

\[
\int_0^\infty \int_\Omega u_\varepsilon \cdot \zeta_t \,dx \,dt + \int_\Omega u_0 \cdot \zeta(\cdot,0) = - \int_0^\infty \int_\Omega u_\varepsilon \cdot \Delta \zeta + \int_0^\infty \int_\Omega (u_\varepsilon \cdot \nabla)u_\varepsilon \cdot \zeta + \int_0^\infty \int_\Omega n_\varepsilon \nabla \phi \cdot \zeta - \int_0^\infty \int_\Omega n_\varepsilon \nabla c_\varepsilon \cdot \zeta.
\]

The convergence (74), (76)-(80) and \(\phi \in W^{1,\infty}(\Omega)\) yields that there exists a subsequence \(\{\varepsilon_j\} (\varepsilon_j \to 0 \text{ as } j \to \infty)\) such that

\[
\int_0^\infty \int_\Omega u_{\varepsilon_j} \cdot \zeta_t \,dx \,dt \to \int_0^\infty \int_\Omega u \cdot \zeta_t \,dx \,dt,
\]

\[
\int_0^\infty \int_\Omega u_{\varepsilon_j} \cdot \Delta \zeta \to \int_0^\infty \int_\Omega u \cdot \Delta \zeta,
\]

\[
\int_0^\infty \int_\Omega n_{\varepsilon_j} \nabla \phi \cdot \zeta \to \int_0^\infty \int_\Omega n \nabla \phi \cdot \zeta,
\]

\[
\int_0^\infty \int_\Omega n_{\varepsilon_j} \nabla c_{\varepsilon_j} \cdot \zeta \to \int_0^\infty \int_\Omega n \nabla c \cdot \zeta
\]

as \(j \to \infty\).

We now consider the second term on the right hand of (81). From the maximal Sobolev regularity in [20] (Theorem 1.2), we know

\[
\|u_{\varepsilon_j}\|_{L^2((0,T);L^2(\Omega))} + \|\Delta u_{\varepsilon_j}\|_{L^2((0,T);L^2(\Omega))} \\
\leq \|u_0\|_{W^{1,2}(\Omega)} + \|(u_{\varepsilon_j} \cdot \nabla)u_\varepsilon + n_\varepsilon \nabla \phi - n_\varepsilon \nabla c_\varepsilon\|_{L^2((0,T);L^2(\Omega))}.
\]

Since \(n_\varepsilon \in L^\infty\), \(\nabla c_\varepsilon \in L^\infty\), \(u_{\varepsilon_j} \in L^\infty\), \(H^1(\Omega)\), and \(u_\varepsilon \in L^2\), we obtain that the right-hand side of above inequality is bounded. Then, boundedness of \(\|(u_{\varepsilon_j})_t\|_{L^2((0,T);L^2(\Omega))}\) and \(\|(u_{\varepsilon_j})^t\|_{L^2((0,T);W^{1,2}(\Omega))}\) yield

\[
u_{\varepsilon_j} \to u \text{ in } L^2((0,T);L^2(\Omega))
\]

as \(j \to \infty\), which together with the convergence (78) gives

\[
\int_0^\infty \int_\Omega (u_{\varepsilon_j} \cdot \nabla)u_{\varepsilon_j} \cdot \zeta \to \int_0^\infty \int_\Omega (u \cdot \nabla)u \cdot \zeta \quad \text{as } j \to \infty.
\]

We therefore proved that the limit function satisfies the weak formulation of the third equation. As to the first and second equations, we refer [34] (Section 5.3) and [14] (Section 3.2, 3.3). Hence, \((n, c, u, P)\) is a global weak solution of (3).
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