On the Kähler cone of irreducible symplectic orbifolds

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Abstract

We generalize the Hodge version of the global Torelli theorem in the framework of irreducible symplectic orbifolds. We also propose a generalization of several results related to the Kähler cone and the notion of wall divisors introduced in the smooth case by Mongardi in [Mon15]. As an application we propose a definition of the mirror symmetry for irreducible symplectic orbifolds.

1 Introduction

A compact Kähler analytic space is called holomorphically symplectic if it admits a holomorphic 2-form which is non-degenerate on its smooth locus. In the last years there have been many developments regarding these objects with the objective of generalizing parts of the theory existing for smooth compact hyperkähler manifolds. These generalizations respond to two necessities. The first one is to get around the lack of examples in the smooth case and the second is to find a framework more adapted to the minimal model program, where certain singularities naturally arise. The two most important developments in this area during the last years concern the Beauville–Bogomolov decomposition theorem [GKP16], [HP19] and the global Torelli theorem [BL18], [Men20]. This paper continues the quest in the framework of irreducible symplectic orbifolds.

A complex analytic space is called an orbifold if it is locally isomorphic to a quotient of an open subset of $\mathbb{C}^n$ by a finite automorphism group (Definition 2.1). An orbifold $X$ is called irreducible (holomorphically) symplectic if $X \setminus \text{Sing} X$ is simply connected, admits a unique (up to a scalar multiple), nondegenerate holomorphic 2-form, and $\text{Codim Sing} X \geq 4$ (Definition 2.4). Irreducible symplectic orbifolds have several key properties, making them a particularly interesting class of geometrical objects. First, they are elementary bricks in a Bogomolov decomposition theorem ([Cam04, Theorem 6.4]). Second, they are well adapted to the minimal model program since their singularities are $\mathbb{Q}$-factorial and terminal ([Nam01, Corollary 1]). Third, the construction of twistor spaces ([Men20, Theorem 5.4]) generalizes to these objects which is a powerful tool when generalizing results from the smooth case. Finally, many examples have already been constructed in this framework ([Fuj83, Section 13]).

After the generalization of the global Torelli theorem in the moduli space setting ([Men20, Theorem 1.1]), it appears natural to generalize its Hodge version (see [Mar11, Theorem 1.3] in the smooth case).

Theorem 1.1. Let $X$ and $X'$ be irreducible symplectic orbifolds.

(i) $X$ and $X'$ are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ which is an isometry of integral Hodge structures.

(ii) Let $f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ be a parallel transport operator, which is an isometry of integral Hodge structures. There exists an isomorphism $\tilde{f} : X \to X'$ such that $f = \tilde{f}_*$ if and only if $f$ maps some Kähler class on $X$ to a Kähler class on $X'$.

As in the smooth case, the second cohomology group of an irreducible symplectic orbifold is endowed with the Beauville–Bogomolov form (see [Men20, Theorem 3.17]). See Definition 2.13 for the definition of parallel transport operator.
The proof of Theorem 1.1 uses a generalization of Huybrechts’ Theorem on bimeromorphic hyperkähler manifolds [Huy03, Theorem 2.5] to the orbifold setting (see Proposition 3.5).

Theorem 1.1 shows the importance of understanding the Kähler classes on irreducible symplectic orbifolds, which is the subject of the next part of this paper. First, we generalize the Boucksom criterion which characterizes the Kähler classes by their intersections with rational curves (Theorem 4.1).

A central part of this paper is to develop the theory of wall divisors for irreducible symplectic orbifolds, thus generalizing the theory developed independently by Mongardi [Mon15] and Amerik–Verbitsky [AV15]. By definition, the wall divisors provide orthogonal hyperplanes which (even after applying Hodge preserving monodromy operators) intersect trivially the birational Kähler chambers (see Definition 4.3). We prove that wall divisors are invariant under arbitrary parallel transport and that extremal rays of the cone of effective curves give rise to wall divisors. More precisely, we have the following result:

**Theorem 1.2.** (i) (compare Theorem 4.6) Let $X$ and $Y$ be two irreducible symplectic orbifolds such that there exists a parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$. Let $D$ be a wall divisor of $X$ such that $f(D) \in \text{Pic}(Y)$. Then $f(D)$ is a wall divisor on $Y$.

(ii) (compare Proposition 4.12) Let $R$ be an extremal ray of the cone of effective curves on $X$ with negative self intersection with respect to the Beauville–Bogomolov form. Suppose that either $X$ is projective or $b_2(X) \geq 5$. Then the hyperplane $R^2 \subseteq H^2(X, \mathbb{Z})$ is the orthogonal complement of a wall divisor.

These results have many applications. One of them is to provide information on the singularities of the irreducible symplectic orbifolds (see Section 3.2). In particular, we obtain a positive answer for the Fu–Namikawa conjecture ([FN04, Conjecture 2]) in the following context: two crepant resolutions of a given symplectic space will lead to deformation equivalent irreducible symplectic orbifolds (see Corollary 3.8).

On the other hand, the global Torelli theorem (Theorem 1.1) and our results on wall divisors allow to generalize the notion of lattice mirror symmetry to the orbifold setting in Section 5. In [Huy04, Section 6], Huybrechts proposes a definition of mirror symmetry at the level of the following period domain:

$$G_\Lambda := \{ (\alpha, \beta + ix) \in (\Lambda \otimes \mathbb{C})^2 \mid \alpha^2 = 0, \alpha \cdot \beta > 0, \alpha \cdot x = 0, x^2 > 0 \}.$$ 

In Theorem 5.4, using our results on the Kähler cone and the global Torelli theorem, we are able to provide an isomorphism between a sub-period domain of $G_\Lambda$ and the moduli space $\tilde{M}_X := \{(X, \varphi, \omega_X, \sigma_X, \beta) \}\) which parametrizes deformation equivalent marked irreducible symplectic orbifolds of Beauville–Bogomolov lattice $\Lambda$ endowed with a Kähler class $\omega_X$, a holomorphic 2-form $\sigma_X$ and a class $\beta \in H^2(X, \mathbb{R})$. Theorem 5.4 makes possible to define the mirror symmetry as an involution on a sub-moduli space of $\tilde{M}_\Lambda$. Therefore, this allows to define the symmetric mirror of a precise irreducible symplectic orbifold. This is a new approach compared to what was done in [Del95] and [Cam10] where the mirror symmetry acts on a set of moduli spaces. This is an improvement of results in [FJM19, Section 4] and it is also new in the setting of irreducible symplectic manifolds.

In the literature (see [Fuj83, Section 13, table I.2], [MT07, Corollary 5.7], [Men15, Section 3.2]), an important example of irreducible symplectic orbifolds is the orbifold denoted by $M'$, which is constructed as follows. We consider $X$ a manifold deformation equivalent to an Hilbert scheme of two points on a K3 surface endowed with a symplectic involution $\iota$; then $M'$ is given by a crepant resolution in codimension two of the quotient $X/\iota$ (see Example 3.5). In particular, we prove that this orbifold is deformation equivalent to a Fujiki orbifold (see Proposition 3.10) and to the Markushevich–Tikhomirov orbifold (see Proposition 3.12).

In a sequel of this paper, we will provide a full description of the wall divisors on $M'$. This is an application of the results from Section 4 and will make it possible to determine the Kähler cone of irreducible symplectic orbifolds of this deformation type explicitly.

The paper is organized as follows. In Section 2, we provide some reminders for important results on irreducible symplectic orbifolds. Section 3.1 is devoted to the proof of Theorem 1.1 with applications to the singularities in Section 3.2. In Section 3.3, we provide an example of non-separated...
points in a moduli space of marked irreducible symplectic or orbifolds. In Section 4, we describe the Kähler cone providing the Boucksom criterion and the results on wall divisors. Finally, in Section 5, we apply our results in order to define mirror symmetry for irreducible symplectic orbifolds.

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2 Reminder on holomorphically symplectic orbifolds

2.1 Definition

First we briefly recall the definition of symplectic orbifold used in this paper.

**Definition 2.1.** A $n$-dimensional orbifold is a connected paracompact complex space $X$ such that for every point $x \in X$, there exists an open neighborhood $U$ and a triple $(V,G,\pi)$ such that $V$ is an open subset of $\mathbb{C}^n$, $G$ is a finite automorphisms group of $V$ and $\pi: V \to U$ is the composition of the quotient map $V \to V/G$ and an isomorphism $V/G \simeq U$. Such a quadrupole $(U,V,G,\pi)$ is called a local uniformizing system of $x$.

**Remark 2.2.** Note that an orbifold is normal (see for instance [Car57, Théorème 4]).

Let $X$ be an orbifold. A smooth differential $k$-form $\omega$ on $X$ is a $C^\infty$ differential $k$-form on $X_{\text{reg}} := X \setminus \text{Sing} X$ such that for all local uniformizing systems $(U,V,G,\pi)$, $\pi^*(\omega|_{U_{\text{reg}}})$ extends to a $C^\infty$-differential $k$-form on $V$ (see [Men20, Section 2] and [Bla96, Section 1 and 2] for more details).

A smooth differential form $\omega$ on $X$ is called Kähler if it is a Kähler form on $X_{\text{reg}}$ such that for all local uniformizing systems $(U,V,G,\pi)$, the pullback $\pi^*(\omega|_{U_{\text{reg}}})$ extends to a Kähler form on $V$. A Kähler orbifold is an orbifold which admits a Kähler form.

**Remark 2.3.** An orbifold is Kähler if and only if it is Kähler as a complex space (cf. [Fuj79, Definition 1.2, 4.1 and Remark 4.2] for the definition of a Kähler complex space and [CFZ07, pages 793-795] for the proof of the equivalence).

We denote by $K_X$ the Kähler cone of $X$ which is the set of all De Rham classes of Kähler forms on $X$.

**Definition 2.4 ([Men20], Definition 3.1).** A compact Kähler orbifold $X$ is called primitively symplectic if:

1. $X_{\text{reg}} := X \setminus \text{Sing} X$ is endowed with a non-degenerated holomorphic 2-form which is unique up to scalar.

2. $\text{Codim} \text{Sing} X \geq 4$.

If moreover $X_{\text{reg}}$ is simply connected, $X$ is said an irreducible symplectic orbifold.

We refer to [Cam04, Section 6], [Men20, Section 3.1] and [FM01, Section 3.1] for discussions about this definition. In particular, the condition (2) is not restrictive since quotient singularities in codimension 2 can always be solved (see [BCHM10]).

**Example 2.5 ([Men20, Section 3.2]).** Let $X$ be a hyperkähler manifold deformation equivalent to a Hilbert scheme of 2 points on a K3 surfaces and $\iota$ a symplectic involution on $X$. By [Mon12, Theorem 4.1], $\iota$ has 28 fixed points and a fixed K3 surface $\Sigma$. We denote by $M'$ the blow-up of $X/\iota$ in the image of $\Sigma$. The orbifold $M'$ is irreducible symplectic (see [Men20, Proposition 3.8]).

**Example 2.6.** A similar construction of an irreducible symplectic orbifold (starting from a generalized Kummer variety) can also be found in [Men20, Proposition 3.8]; we denote this orbifold by $K'$. See [Fuj83, Section 13] and [FM01, Section 5] for further examples of primitively symplectic orbifolds.
Section 3.5], this set can be endowed with a complex structure such that the orbifold \(X\) (marked irreducible symplectic orbifolds \(\beta\)). Let \(2.3\) Period map can shrink \(U\) and non-degenerated (see [Men20, Theorem 3.17]), we can find \(\beta\) defined as follows. By [Men20, Corollary 2.7] and since the Beauville–Bogomolov form is integral (\(\beta\)), we will write \(\sigma\) is simply connected. We denote \(\Lambda\) is a local isomorphism with \(D\). When \(\Lambda\) is clear, we will write \(\mathcal{D}\) instead of \(\mathcal{D}_\Lambda\).

\[H^2(M',\mathbb{Z}) \cong U(2)^3 \oplus E_8(-1) \oplus (-2)^2 \text{ and } H^2(K',\mathbb{Z}) \cong U(3)^3 \oplus \begin{pmatrix} -5 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\]

Remark 2.8. Let \(\beta \in H^{2n-1,2n-1}(X,\mathbb{Q})\). We can associate to \(\beta\) its dual class \(\beta' \in H^{1,1}(X,\mathbb{Q})\) defined as follows. By [Men20, Corollary 2.7] and since the Beauville–Bogomolov form is integral and non-degenerated (see [Men20 Theorem 3.17]), we can find \(\beta' \in H^2(X,\mathbb{Q})\) such that for all \(\alpha \in H^2(X,\mathbb{C})\):

\[(\alpha,\beta')_q = \alpha \cdot \beta,\]

where the dot on the right hand side is the cup product. Since \((\beta',\sigma_X)_q = \beta \cdot \sigma_X = 0\), we have \(\beta' \in H^{1,1}(X,\mathbb{Q})\).

2.3 Period map

Let \(\Lambda\) be an abstract lattice of signature \((3,\text{rk}\Lambda - 3)\). A marking of a primitively symplectic orbifold \(X\) is an isometry \(\varphi : H^2(X,\mathbb{Z}) \to \Lambda\). We denote by \(\mathcal{M}_\Lambda\) the set of isomorphism classes of marked irreducible symplectic orbifolds \((X,\varphi)\) with \(\varphi : H^2(X,\mathbb{Z}) \to \Lambda\). As explained in [Men20 Section 3.5], this set can be endowed with a complex structure such that the period map:

\[\mathcal{P} : \mathcal{M}_\Lambda \longrightarrow \mathcal{D}_\Lambda\]

is a local isomorphism with \(\mathcal{D}_\Lambda := \{\alpha \in \mathbb{P}(\Lambda_\mathbb{C}) \mid \alpha^2 = 0, \alpha \cdot \sigma > 0\}\). The complex manifold \(\mathcal{M}_\Lambda\) is called the moduli space of marked primitively symplectic orbifolds of Beauville–Bogomolov lattice \(\Lambda\). When \(\Lambda\) is clear, we will write \(\mathcal{D}\) instead of \(\mathcal{D}_\Lambda\).

Let \(\mathcal{X} \xrightarrow{f} S\) be a deformation of \(X\) by primitively symplectic orbifolds, where the base \(S\) is simply connected. We denote \(\sigma := f(X)\). By [Men20 Theorem 3.17], we can see \(S\) as an analytic subset of \(\mathcal{M}_\Lambda\). Moreover, by [Men20 Corollary 3.12], there exists a natural isomorphism \(u_\varphi : H^*(\mathcal{X},\mathbb{C}) \to H^*(X,\mathbb{C})\) which respects the cup product. Let \(\varphi\) be a mark of \(X\). The period map restricted to \(S\) has the following expression:

\[\mathcal{P} : S \longrightarrow \mathcal{D},\]

For instance, let \(f : \mathcal{X} \to \text{Def}(X)\) be the Kuranishi deformation of \(X\) (see for instance [Fuj83 Remark 3.4]). By [Men20 Proposition 3.16], there exists \(U\) an open neighborhood of \(o \in \text{Def}(X)\) such that for all \(t \in U\), the fiber \(\mathcal{X}_t := f^{-1}(t)\) is a primitively symplectic manifold. Moreover, we can shrink \(U\) to a smaller open neighborhood of \(o\) such that \(\mathcal{P} : U \to \mathcal{P}(U)\) is an isomorphism.

2.4 Global Torelli theorem

The moduli space \(\mathcal{M}_\Lambda\) introduced in the previous section is a non-separated manifold, however by [Men20 Corollary 3.25], there exists a Hausdorff reduction \(\overline{\mathcal{M}}_\Lambda\) of \(\mathcal{M}_\Lambda\) such that the period map \(\mathcal{P}\) factorizes through \(\overline{\mathcal{M}}_\Lambda\):

\[\mathcal{M}_\Lambda \longrightarrow \overline{\mathcal{M}}_\Lambda \longrightarrow \mathcal{D}.\]
Moreover, two points in \( \mathcal{M}_\Lambda \) map to the same point in \( \overline{\mathcal{M}}_\Lambda \) if and only if they are non-separated in \( \mathcal{M}_\Lambda \). Finally, we can recall the global Torelli theorem.

**Theorem 2.9** ([Men20], Theorem 1.1). Let \( \Lambda \) be a lattice of signature \((3, b - 3)\), with \( b \geq 3 \). Assume that \( \mathcal{M}_\Lambda \neq \emptyset \) and let \( \mathcal{M}^\Lambda_\Lambda \) be a connected component of \( \mathcal{M}_\Lambda \). Then the period map:
\[
\mathcal{P} : \overline{\mathcal{M}}^\Lambda_\Lambda \to \mathcal{D}
\]
is an isomorphism.

### 2.5 Positive cone

We denote by \( C_X \) the positive cone of \( X \), it is the connected component of
\[
\{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid q(\alpha) > 0 \}
\]
that contains the Kähler cone (see Section 2.1). We recall a first result, about elements of the positive cone, which will be used several times in this paper.

**Proposition 2.10** ([Men20], Proposition 5.5). Let \( \mathcal{X} \) be an irreducible symplectic orbifold and let \( \alpha \in C_X \) be a very general element of the positive cone, i.e. \( \alpha \) is contained in the complement of countably many nowhere dense closed subsets. Then there exist two smooth proper families \( \mathcal{X} \to S \) and \( \mathcal{X}' \to S' \) of irreducible symplectic orbifolds over a one-dimensional disk \( S \) and a bimeromorphism \( F : X' \to X \) compatible with the projection to \( S \), such that \( F \) induces an isomorphism \( \mathcal{X}'_{|S \setminus \{0\}} \simeq \mathcal{X}_{|S \setminus \{0\}} \); the special fiber \( \mathcal{X}_0 \) is isomorphic to \( X \), and \( F^* \alpha \) is a Kähler class on \( \mathcal{X}_0 \).

In this proposition by \( F^* \alpha \) we refer to the following. If \( \mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{X}' \) resolves the bimeromorphic map \( F \), we can consider \( \Gamma = \text{Im} \ Z_0 \to X \times X' \) and \( p : \Gamma \to X \), \( p' : \Gamma \to X' \) the projections. Then, we set \( F^* \alpha(\gamma) := p'_* (\Gamma \cdot p^* (\alpha)) \).

**Remark 2.11.** In Section 4.2 we will introduce the concept of wall divisors. This will make it possible to replace the condition that \( \alpha \) is very general by a much more explicit condition, which can be verified for individual elements \( \alpha \in C_X \) (compare Corollary 4.19).

### 2.6 Twistor space

Let \( \Lambda \) be a lattice of signature \((3, \text{rk} \Lambda - 3)\). We denote by "\( \cdot \)" its bilinear form. A **positive three-space** is a subspace \( W \subset \Lambda \otimes \mathbb{R} \) such that \( ^\perp W \) is positive definite.

**Definition 2.12.** For any positive three-space \( W \), we define the associated twistor line \( T_W \) by:
\[
T_W := \mathcal{D} \cap \mathbb{P}(W \otimes \mathbb{C}).
\]

A twistor line is called *generic* if \( W^\perp \cap \Lambda = 0 \). A point of \( \alpha \in \mathcal{D} \) is called very general if \( \alpha^\perp \cap \Lambda = 0 \).

**Theorem 2.13** ([Men20], Theorem 5.4). Let \((X, \varphi)\) be a marked irreducible symplectic orbifold with \( \varphi : H^2(X, \mathbb{Z}) \to \Lambda \). Let \( \alpha \) be a Kähler class on \( X \), and \( W_\alpha := \text{Vect}_\mathbb{R}(\varphi(\alpha), \varphi(\text{Re} \sigma_X), \varphi(\text{Im} \sigma_X)) \).

Then:

(i) There exists a metric \( g \) and three complex structures (see [Men20, Section 5.1] for the definition) \( I \), \( J \) and \( K \) in quaternionic relation on \( X \) such that:
\[
\alpha = [g(\cdot, I \cdot)] \quad \text{and} \quad g(\cdot, J \cdot) + ig(\cdot, K \cdot) \in H^{0,2}(X).
\]

(ii) There exists a deformation of \( X \):
\[
\mathcal{X} \to T(\alpha) \simeq \mathbb{P}^1,
\]
such that the period map \( \mathcal{P} : T(\alpha) \to T_W^\alpha \) provides an isomorphism. Moreover, for each \( s = (a, b, c) \in \mathbb{P}^1 \), the associated fiber \( \mathcal{X}_s \) is an orbifold diffeomorphic to \( X \) endowed with the complex structure \( aI + bJ + cK \) (identifying \( \mathbb{P}^1 \) with \( S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\} \)).
Remark 2.14. Note that if the irreducible symplectic orbifold $X$ of the previous theorem is endowed with a marking then all the fibers of $\mathcal{X} \to T(\alpha)$ are naturally endowed with a marking.

For the statement of the next lemma, we recall the definition of a parallel transport operator.

**Definition 2.15.** Let $X_1$ and $X_2$ be two irreducible symplectic orbifolds. An isometry $f : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is called a parallel transport operator if there exists a deformation $s : X \to B$, two points $b_1 \in B$, two isomorphisms $\psi_i : X_i \to X_{b_i}$, $i = 1, 2$ and a continuous path $\gamma : [0, 1] \to B$ with $\gamma(0) = b_1$, $\gamma(1) = b_2$ and such that the parallel transport in the local system $R_s, \mathbb{Z}$ along $\gamma$ induces the morphism $\psi_2 \circ f \circ \psi_1$.

**Remark 2.16.** Note that a deformation of irreducible symplectic orbifolds is always locally trivial (see [Men20 Proposition 3.10]).

**Lemma 2.17.** Two marked irreducible symplectic orbifolds $(X, \varphi)$, $(Y, \psi)$ in the $\mathcal{M}_2$ are connected by twistor spaces. That is: There exists a sequence of generic twistor spaces $f_i : \mathcal{X}_i \to \mathbb{P}_1^i$ with $(x_i, x_i') \in P_1^i \times P_1^i$, $i \in \{0, \ldots, k\}$, $k \in \mathbb{N}$ such that

$$f_0^{-1}(x_0) \simeq (X, \varphi), \quad f_k^{-1}(x_k) \simeq f_{i+1}^{-1}(x_{i+1}) \quad \text{and} \quad f_k^{-1}(x_k) \simeq (Y, \psi),$$

for all $0 \leq i \leq k - 1$, which induces $\psi^* \circ \varphi$ as parallel transport operator.

**Proof.** We split the proof in two steps.

**First case:** We assume that $(X, \varphi)$ and $(Y, \psi)$ are very general (that is $\text{Pic} X = 0$ and $\text{Pic} Y = 0$). By [Huy12 Proposition 3.7] the period domain $\mathcal{D}_A$ is connected by generic twistor lines. Note that the proof of [Huy12 Proposition 3.7] in fact shows that the twistor lines can be chosen in a such a way that they intersect in very general points of $\mathcal{D}_A$. In particular, we can connect $\mathcal{P}(Y, \psi)$ and $\mathcal{P}(X, \varphi)$ by such generic twistor lines in $\mathcal{D}_A$. Since for a very general element $(\tilde{X}, \tilde{\varphi})$ of $\mathcal{M}_A$ we know $\mathcal{K}_\tilde{X} = \mathcal{C}_\tilde{X}$ by [Men20 Corollary 5.6], Theorem 2.13 shows that all these twistor lines can be lifted to twistor spaces. By [Men20 Proposition 3.22] the period map $\mathcal{P}$ is injective on the set of very general points in $\mathcal{D}_A$. Therefore, all these twistor spaces intersect and connect $(X, \varphi)$ to $(Y, \psi)$.

**Second case:** If $(X, \varphi)$ is not very, we consider a very general Kähler class $\alpha$. Then the associated twistor space $\mathcal{X} \to T(\alpha)$ have a fiber which is a very general marked irreducible symplectic orbifold. Hence we are back to the first case. \hfill\Box

### 3 Hodge version of the global Torelli theorem

**3.1 Proof of Theorem 1.1**

**Lemma 3.1.** Let $X$ be an orbifold and $f : \tilde{X} \to X$ the blow-up of $X$ in a smooth subvariety $Y$. Let $E$ be the exceptional divisor of $f$. Then:

$$H^2(\tilde{X}, \mathbb{R}) \simeq H^2(X, \mathbb{R}) \oplus \mathbb{R}[E].$$

**Proof.** The map $f$ is the blow-up of $X$ in an analytic subset $Y$. Let $U := X \setminus Y$. Since $\text{Codim} Y \geq 2$, we have $H^2(X, \mathbb{R}) \simeq H^2(U, \mathbb{R})$ (see for instance [Fuj83 Lemma 1.6]). Then the commutative diagram

$$\begin{array}{ccc}
H^2(X, \mathbb{R}) & \xrightarrow{f^*} & H^2(\tilde{X}, \mathbb{R}) \\
\downarrow & & \downarrow \\
H^2(U, \mathbb{R}) & \xrightarrow{f^*} & H^2(\tilde{X}, \mathbb{R})
\end{array}$$

shows that $f^* : H^2(X, \mathbb{R}) \to H^2(\tilde{X}, \mathbb{R})$ is an injection. We have $f^*(H^2(X, \mathbb{R})) \oplus \mathbb{R}[E] \subset H^2(\tilde{X}, \mathbb{R})$. Let $V := \tilde{X} \setminus \text{Sing} \tilde{X}$. Still because of [Fuj83 Lemma 1.6], $H^2(\tilde{X}, \mathbb{R}) \simeq H^2(V, \mathbb{R})$. Then we conclude

$$H^2(\tilde{X}, \mathbb{R}) \simeq H^2(X, \mathbb{R}) \oplus \mathbb{R}[E].$$
with the following exact sequence:

\[
\begin{array}{c}
\text{H}^2(\text{X}, \mathbb{R}) \\
\text{H}^2(\text{V}, \text{V} \setminus \text{E}, \mathbb{R}) \longrightarrow \text{H}^2(\text{V}, \mathbb{R}) \longrightarrow \text{H}^2(\text{V} \setminus \text{E}, \mathbb{R}) \\
\text{H}^0(\text{E} \cap \text{V}, \mathbb{R}),
\end{array}
\]

where the isomorphisms are given respectively by Thom’s isomorphism and [Fuj83, Lemma 1.6].

**Lemma 3.2.** Let \( f : \text{X} \rightarrow \text{X}' \) be a bimeromorphic map between two primitively symplectic orbifolds. Then \( f \) is an isomorphism in codimension 1.

**Proof.** Since \( \text{X} \) is normal (Remark 2.2), \( f \) is well defined on \( U_0 \subset \text{X} \) with \( \text{Codim} \text{X} \setminus U_0 \geq 2 \). Moreover, we can look at \( U := U_0 \setminus \text{Sing} U_0 \). We also have \( \text{Codim} \text{X} \setminus U \geq 2 \). Let \( U' \) be the smooth locus of \( \text{X}' \). First, observe that \( f(U) \) is contained in \( U' \). Indeed, if this was not the case, \( f|_U \) would be a crepant resolution of some singularities of \( \text{X}' \) which is impossible since \( \text{X}' \) has only terminal singularities (see [Nam01, Corollary 1]).

Consider the subset \( \text{Y} \subset U \) consisting of all points \( x \) such that \( f^{-1}(f(x)) \neq \{x\} \). Let \( \sigma' \) be the unique non-degenerate holomorphic 2-form on \( U' \), then \( f^*(\sigma') \) is also the unique non-degenerate holomorphic 2-form on \( U \) (up to a scalar) because \( f^*(\sigma') \) extends to all \( \text{X} \) (see for instance [Fuj83, Lemma 2.1]). Since \( f^*(\sigma') \) is non-degenerate, \( Y \) cannot be a divisor, otherwise there would need to be fibers of dimension bigger than 0, and they would be in the set of degeneration of \( f^*(\sigma') \) which is impossible. Finally, we set \( V := U \cap f^{-1}(U') \setminus Y \) which verifies \( \text{Codim} \text{X} \setminus V \geq 2 \) and \( f : V \rightarrow \text{X}' \) is an injective bimeromorphic morphism. With the same argument, there exists \( V' \subset \text{X}' \) such that \( \text{Codim} \text{X}' \setminus V' \geq 2 \) and \( f^{-1} : V \rightarrow \text{X} \) is an injective bimeromorphic morphism. Then \( f : V \cap f^{-1}(V') \rightarrow V' \cap f(V) \) is an isomorphism and using the same arguments as above, \( \text{Codim} \text{X} \setminus V' \cap f(V) \geq 2 \); \( \text{Codim} \text{X} \setminus V' \cap f(V) \geq 2 \).

**Proposition 3.3.** Let \( f : \text{X}' \rightarrow \text{X} \) be a bimeromorphic map between two primitively symplectic orbifolds. If \( \alpha \in \text{H}^2(\text{X}, \mathbb{R}) \) is a class such that \( \alpha \cdot [\text{C}] > 0 \) and \( f^*(\alpha) \cdot [\text{C}'] > 0 \) for all rational curves \( \text{C} \subset \text{X} \) and \( \text{C}' \subset \text{X}' \), then \( f \) extends to an isomorphism.

**Proof.** The proof of this proposition is an adaptation of the Huybrechts’ proof [Huy03, Proposition 2.1]. By Hironaka’s theorem (see for instance [BM97]) there exists a sequence of blow-ups in smooth loci \( \pi : Z \rightarrow \text{X} \) resolving \( f \). In particular, we can find a combination \( \sum n_i E_i \) of exceptional divisors with \( n_i \in \mathbb{N}^+ \) such that \( -\sum n_i E_i \) is \( \pi \)-ample. We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Z} & \xrightarrow{\pi} & \text{X} \\
\downarrow{\pi'} & \searrow{f} & \swarrow{=} \\
\text{X}' & \xrightarrow{=} & \text{X}.
\end{array}
\]

By Lemma 3.2, any exceptional divisors \( E_i \) for \( \pi \) is also an exceptional divisor for \( \pi' \). Let \( \beta \in \text{H}^2(\text{X}, \mathbb{R}) \). Using Lemma 3.1 there exists \( \gamma \in \text{H}^2(\text{X}', \mathbb{R}) \) such that:

\[
\pi^* \beta = \pi'^* \gamma + \sum a_i [E_i].
\]

Let \( W \subset \text{X} \) and \( W' \subset \text{X}' \) such that \( \text{Codim} \text{X} \setminus W \geq 2 \), \( \text{Codim} \text{X}' \setminus W' \geq 2 \) and the restriction \( f : W \rightarrow W' \) is an isomorphism. We have a commutative diagram:

\[
\begin{array}{ccc}
\pi^{-1}(W) & \xrightarrow{\pi} & \text{W}' \\
\downarrow{\pi} & \searrow{f} & \swarrow{=} \\
W & \xrightarrow{=} & \text{W}'.
\end{array}
\]
The commutativity of this diagram and \([Fuj83]\) Lemma 1.6 imply that \(\gamma = f^*\beta\). That is:

\[
p^*\beta = p^*(f^*(\beta)) + \sum_i a_i [E_i].
\]  

Taking \(\alpha \in H^2(X, \mathbb{R})\) as in the statement of the proposition, exactly as Huybrechts in \([Huy03]\) Proof of Proposition 2.1, we can see that the \(a_i\) vanish in this case:

\[
p^*\alpha = p^*(f^*(\alpha)).
\]

We conclude as Huybrechts proving that a rational curve \(C\) is contracted by \(\pi\) if and only if it is contracted by \(\pi'\) regarding the intersections \(\pi^*\alpha \cdot C\) and \(\pi'^*(f^*(\alpha)) \cdot C\). Since all the exceptional divisors are covered by rational curves (our resolution is a sequence of blow-ups), all contractions \(\pi_i\) and \(\pi'_i\) coincide. So \(f\) extends to an isomorphism. \(\Box\)

**Lemma 3.4.** Let \(f : X' \dashrightarrow X\) be a bimeromorphic map between two primitively symplectic orbifolds. By Lemma 3.3, there exist two open sets \(j : W \hookrightarrow X\) and \(j' : W' \hookrightarrow X'\) such that \(\text{Codim } X \setminus W \geq 2\), \(\text{Codim } X' \setminus W' \geq 2\) and the restriction \(f : W' \to W\) is an isomorphism. Then the map induced by \([Fuj83]\) Lemma 1.6:

\[
f^* : H^2(X, \mathbb{R}) \to H^2(W, \mathbb{R}) \simeq H^2(W', \mathbb{R}) \to H^2(X', \mathbb{R}),
\]

is an isometry with respect to the Beauville–Bogomolov form.

**Proof.** Let \(Z\) resolving \(f\) as in the proof of Proposition 3.3:

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi'} & X' \\
\cap & \pi & \downarrow \\
X' & \xrightarrow{\sim} & \to X.
\end{array}
\]

Pulling back the Beauville–Bogomolov form to \(Z\), the result is a direct consequence of (1) and the projection formula \([Men20]\) Remark 2.8]. \(\Box\)

**Proposition 3.5.** Let \(X\) and \(X'\) be irreducible symplectic orbifolds and \(f : X' \dashrightarrow X\) a bimeromorphism. Then there exist smooth proper families \(\mathcal{X} \to S\) and \(\mathcal{X}' \to S\) over a one-dimensional disk \(S\) with the following properties:

(i) The special fibers are \(X_0 \simeq X\) and \(X'_0 \simeq X'\).

(ii) There exists a bimeromorphism \(F : \mathcal{X}' \dashrightarrow \mathcal{X}\) which is an isomorphism over \(S \setminus \{0\}\), i.e. \(F : \mathcal{X}'|_{S \setminus \{0\}} \simeq \mathcal{X}|_{S \setminus \{0\}}\), and which coincides with \(f\) on the special fiber, i.e. \(F_0 = f\).

**Proof.** Using Lemmas 3.1 and 3.2 and Propositions 3.3 and 2.10, this result can be proved exactly as Huybrechts did in \([Huy03]\) proof of Theorem 2.5]. \(\Box\)

**Proof of Theorem 2.29.** Statement (i) is given by Proposition 3.3 and \([Men20]\) Corollary 5.11. In the following, we show statement (ii). Clearly, if there exists \(f : X \to X'\) with, then \(\tilde{f}_s\) maps some Kähler class to a Kähler class. Hence, we only need to prove the reverse implication.

Let \(f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})\) be a parallel transport which is a Hodge isometry. By Theorem 2.29 and Proposition 3.3, there exist smooth proper families \(\mathcal{X} \to S\) and \(\mathcal{X}' \to S\) over a one-dimensional disk \(S\) such that \(X_0 \simeq X\) and \(X'_0 \simeq X'\) and \(F : \mathcal{X}' \dashrightarrow \mathcal{X}\) which is an isomorphism over \(S \setminus \{0\}\). Let \(\mathcal{X} \leftarrow Z \to X'\) be a resolution of \(\tilde{F}\) obtained by a sequence of blow-ups in smooth loci (Hironaka’s theorem). Let \(Z_0\) be the fiber of \(Z \to S\) over \(0\), we consider the cycle \(\Gamma := \text{Im}(Z_0 \to X \times X')\) which decomposes as \(\Gamma = S + \sum Y_i\). As explained in \([Men20]\) Remark 3.24, we know that \(Z\) is the graph of a bimeromorphism between \(X\) and \(X'\). Moreover \([\Gamma], (x) := \pi'_i([\Gamma] \cdot \pi^*(x))\) for all \(x \in H^2(X, \mathbb{R})\). Assume that there exists \(\alpha \in H^{1,1}(X, \mathbb{R})\) a Kähler class such that \(\beta = f^*(\alpha) = [\Gamma]_* (\alpha)\) is a Kähler class too. We are now in the same situation as the one in the proof of \([Huy03]\) Theorem 2.5 and with the same argument, we prove that \(\pi'(Y_i)\) has codimension at least \(2\) for all \(i\). Hence \(f = [\Gamma]_* = [Z]_*\). Finally, we conclude the proof using Proposition 3.3. \(\Box\)
3.2 Applications to the singularities of an irreducible symplectic orbifold

Two bimeromorphic irreducible symplectic orbifolds have the same singularities.

We note the following corollary of Proposition 3.5.

Corollary 3.6. Let X and X' be two bimeromorphic irreducible symplectic orbifolds. Then, X and X' have the same singularities.

Proof. We have seen with Proposition 3.5 that X and X' are deformation equivalent. However, quotient singularities of codimension \( \geq 3 \) are rigid under deformation (see [Fuj83] Lemma 3.3).

Uniqueness of the crepant resolution in codimension 2

The following corollary of Proposition 3.5 can be seen as a partial answer to [LN03] Conjecture 2. By [BCHM10], we know that the symplectic singularities in codimension 2 always have a crepant resolution. Is this crepant resolution unique?

Definition 3.7. Let X be an orbifold. Let \( \text{Sing}_2 X \) be the union of the irreducible components of \( \text{Sing} X \) of codimension 2. A partial resolution in codimension 2, \( r : \tilde{X} \to X \) of X, is a proper bimeromorphic map such that:

(i) the restriction \( r : r^{-1}(X \setminus \text{Sing}_2 X) \to X \setminus \text{Sing}_2 X \) is an isomorphism;

(ii) \( \text{Codim Sing} \tilde{X} \geq 3 \).

Corollary 3.8. Let X and X' be two irreducible symplectic orbifolds. We assume that X and X' are two partial resolutions in codimension 2 of the same orbifold Y. Then X and X' are deformation equivalent.

Proof. Indeed, X and X' are bimeromorphic. Then Proposition 3.5 concludes the proof.

This corollary can help to classify the irreducible symplectic orbifolds in dimension 4. We provide an example here.

Example 3.9. In [Fuj83] Section 13], Fujiki constructs the following symplectic orbifold. He considers S a K3 surface and \( i \) a symplectic involution on S. Let \( s_2 \) be the reflection on \( S \times S \) acting by \( s_2(x,y) = (y,x) \). Let \( G = \langle s_2, i \rangle \) be the automorphism group on \( S \times S \) with \( i(x,y) = (i(x),i(y)) \). Fujiki defined \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) a crepant resolution in codimension 2 of \( S \times S/G \). Fujiki shows that \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) is a primitively symplectic orbifold.

We recall that \( M' \) is defined in Example 3.5.

Proposition 3.10. The orbifold \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) is symplectic irreducible and deformation equivalent to \( M' \).

Proof. It is enough to show that \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) is symplectic irreducible. Indeed \( M' \) and \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) are both crepant resolutions in codimension 2 of \( S \times S/G \) and so Corollary 3.8 applies.

Necessarily, \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) and \( M' \) are bimeromorphic. Indeed, let \( \Delta := \{ \{x,x\} \in S \times S/G \mid x \in S \} \) and \( \Sigma := \{ \{x,i(x)\} \in S \times S/G \mid x \in S \} \). Let \( U := S \times S/G \setminus (\Delta \cup \Sigma) \). The set U can be seen as an open set in \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) and in \( M' \). This gives rise to a natural bimeromorphism \( f : Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \to M' \) (extending \( \text{id}_U \)).

By Lemma 3.2, \( f \) extends to an isomorphism from an open set \( U' \subset Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) to an open set \( V' \subset M' \) such that \( \text{Codim} Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \setminus U' \geq 2 \) and \( \text{Codim} M' \setminus V' \geq 2 \) with \( U \subset U' \) and \( U \subset V' \). Moreover, \( U \) contains all the singularities which are given by \( \{x_i,x_j\} \mid i < j \), where \( (x_i)_{i \in \{1,\ldots,8\}} \) are the fixed points of \( i \) on S. It follows that \( \pi_1(Y_{K3}(\mathbb{Z}/2\mathbb{Z})_{\text{reg}}) = \pi_1(U'_{\text{reg}}) = \pi_1(V'_{\text{reg}}) = 0 \), where the index \text{reg} means the regular subset.

Remark 3.11. Note that this proposition is in contradiction with the Euler characteristic computed in [MT07] Proposition 5.1, Corollary 5.7 and the one suggested in [Fuj83] Remark 13.2 (4)].

However, we can prove that the Euler characteristic of \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) and \( M' \) is actually 212 using two different methods. Indeed, it is proved in [Men15] Proposition 2.40] using the link between the
invariant cohomology and the cohomology of the quotient that \( \chi(M') = 212 \). Moreover, in [FM01]
Proposition 3.6, using Blaschke–Riemann–Roch theorem [Bla96] Theorem 3.5 and 3.17, we provide
an expression such that the Euler characteristic of a 4-dimensional primitively symplectic orbifold
\( X \) depends only on the second Betti number, the third Betti number and the singularities:

\[
b_4(X) + b_3(X) - 10b_2(X) = 46 + s,
\]

where \( s \) only depends on the singularities. That is:

\[
\chi(X) = (48 + s) + 12b_2(X) - 3b_3(X).
\]

(2)

More precisely

\[
s = \sum_{x \in \text{Sing } X} s_x,
\]

where \( s_x \) only depends of the analytic type of the singularity \( x \). It was shown in [FM01] (16), that
\( s_x = -1 \) if \( x \) is a singularity analytically equivalent to \( \mathbb{C}^4 / \text{id} \). By
Men15 Propostions 2.8 and 2.40, [MT07], Corollary 5.7, [Fuj83] Theorem 13.1, Table 1.12, \( b_3(M') = b_2(Y_{K3}(\mathbb{Z}/2\mathbb{Z})) = 16 \),
\( b_2(M') = b_2(Y_{K3}(\mathbb{Z}/2\mathbb{Z})) = 0 \) and \( M' \), \( Y_{K3}(\mathbb{Z}/2\mathbb{Z}) \) have 28 singularities analytically equivalent
to \( \mathbb{C}^4 / \text{id} \). Hence by (2):

\[
\chi(M') = \chi(Y_{K3}(\mathbb{Z}/2\mathbb{Z})) = 212.
\]

3.3 Application: example of non-separated orbifolds in their moduli space

In [MT07], Markushevich and Tikhomirov provide a primitively symplectic orbifold related by a
Mukai flop to \( M' \) (see Example 2.3 for the definition of \( M' \)). The construction starts with a K3
surface \( S \) endowed with an anti-symplectic involution \( i \) such that the quotient \( X/i \) is a del Pezzo
surface of degree 2. We endow \( S \) with a polarization \( H \) which is a pull-back of the anti-canonical
bundle of \( X/i \). In particular, the divisor \( H \) is very ample and provides an embedding of \( S \) in \( \mathbb{P}^3 \)
(see [MT07] Lemma 1.1). Then we consider \( \mathcal{M} = M_S^{H,s_2}(0, H, -2) \), the moduli space of \( H \)-semi-
stable sheaves on \( S \) with Mukai vector \( (0, H, -2) \). Then the Markushevich–Tikhomirov orbifold
\( \mathcal{P} \) is constructed as a connected component of the fixed locus of the involution \( i^* \circ \tau \) where \( \tau \) is a
generalization of the dual map adapted to \( \mathcal{M} \) (see [MT07] Section 3) for more details).

Restricting to the case, where \( S \rightarrow \mathbb{P}^3 \) does not contain any lines, one can consider the Beauville
involution \( \iota \) on \( S^{[2]} \) (see [Bea83a, p.21]) which is an antisymplectic involution. The composition
\( j^{[2]} \circ \iota \) is therefore symplectic, and one can associate the orbifold \( M' \) as the partial resolution in
codimension 2 of \( S^{[2]} / j^{[2]} \circ \iota \).

Proposition 3.12. The varieties \( \mathcal{P} \) and \( M' \) are irreducible symplectic orbifolds of dimension 4
with only 28 singular points analytically equivalent to \( (\mathbb{C}^4 / \{ \pm 1 \}, 0) \). Moreover they are related by a
Mukai flop. In particular they are deformation equivalent.

Proof. By Example 2.3 we know the properties of \( M' \). By [MT07] Theorem 3.4 and Corollary 5.7
we know that \( \mathcal{P} \) is a primitively symplectic orbifold related to \( M' \) by a Mukai flop. As explained in
the proof of [MT07] Lemma 5.3, the indeterminacy locus of this Mukai flop does not contain any
singularities of \( \mathcal{P} \). It follows that \( \pi_1(\mathcal{P}_{\text{reg}}) = \pi_1(M'_{\text{reg}}) = 0 \). That is \( \mathcal{P} \) is an irreducible symplectic
orbifold. We conclude with Proposition 5.5 that \( M' \) and \( \mathcal{P} \) are deformation equivalent.

Assume that \((S,i)\) is a very general K3 surface as in the previous paragraph, i.e. satisfying
the additional condition \( \text{Pic } S = H^2(S, \mathbb{Z})^* \). Denote by \( j : H^2(S, \mathbb{Z}) \rightarrow H^2(S^{[2]}, \mathbb{Z}) \) the natural
injection constructed in [Bea83a] Proposition 6]. We have the following description of the action of
the Beauville involution.

Proposition 3.13 ([Og03], Proposition 4.1). The involution \( \iota^* \) restricted to \( \text{Pic } S^{[2]} \) is the
reflection in the span of \( \theta := j(H) - \delta \), where \( \delta \) is half the class of the diagonal.

Notation 3.14. We denote by \( \mathcal{M} \) the moduli space of marked irreducible symplectic orbifolds
equivalent by deformation to \( \mathcal{P} \).
Theorem 3.15. Let \( \mathcal{P} \) be a very general Markushevich–Tikhomirov variety (constructed from a very general 2-elementary K3 surface). Let \( \rho : M' \longrightarrow \mathcal{P} \) be the Makai flop of Proposition 3.14. Let \( \varphi \) be a mark for \( \mathcal{P} \), then \( (\mathcal{P}, \varphi) \) and \( (M', \rho \circ \varphi) \) are distinct non-separated points in \( \mathcal{M} \). More precisely, the orbifolds \( M' \) and \( \mathcal{P} \) are not isomorphic.

Proof. Because of Propositions 3.12 and 3.13, we know that \( (\mathcal{P}, \varphi) \) and \( (M', \rho \circ \varphi) \) are non-separated points in \( \mathcal{M} \). Now, we are going to prove that \( M' \) and \( \mathcal{P} \) are not isomorphic. We will assume that there exists an isomorphism \( \mathcal{P} \cong M' \) and we will find a contradiction. The composition \( \psi : M' \longrightarrow \mathcal{P} \cong M' \) induces a bimeromorphism of \( M' \) to itself. We will show that this bimeromorphism is necessarily an isomorphism; since \( \rho \) is not an isomorphism it will be a contradiction. 

Let \( \pi : S^{[2]} \longrightarrow M := S^{[2]}/i^{[2]} \circ \iota \) be the quotient map. Let \( \Sigma \) be the surface fixed by the involution \( i^{[2]} \circ \iota \) and \( \Sigma' \) the exceptional divisor of the blow-up \( r : M' \longrightarrow M \). We have a commutative diagram:

\[
\begin{array}{ccc}
M' & \xrightarrow{\psi} & M' \\
\downarrow r & & \downarrow r \\
M & \xrightarrow{\psi_0} & M.
\end{array}
\]  

Because of Proposition 3.13:

\[
\text{Pic} \, M \otimes \mathbb{Q} = \mathbb{Q} \, \pi_*(\theta) \quad \text{and} \quad \text{Pic} \, M' \otimes \mathbb{Q} = \mathbb{Q} \, r^* \pi_*(\theta) \oplus \mathbb{Q} \, \Sigma'.
\]  

Let \( \mathcal{L} \) be the line bundle on \( M \) associated to \( \pi_*(\theta) \). By Lemma 3.2, \( \psi \) is an isomorphism in codimension 1. It follows that \( \psi_0 \) is also an isomorphism in codimension 1. Hence by Banica and Stanasila theorem [BS76, Corollary II 3.15]:

\[
H^0(M, \mathcal{L}^n) = H^0(M, \psi_0^*(\mathcal{L})^n),
\]  

for all \( n \geq 0 \).

By construction \( M \) is projective and by [Men20], \( \mathcal{L} \) is an ample line bundle. It follows from [Men20], that:

\[
\psi_0^*(\pi_*(\theta)) = \pi_*(\theta).
\]

Therefore, by commutativity of the diagram [Men20], we have:

\[
\psi^*(r^*(\pi_*(\theta))) = r^*(\pi_*(\theta)).
\]

We recall the developed form of the Fujiki formula (see [Men20] Theorem 3.17]):

\[
\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 = \frac{c_M}{24} \sum_{s \in S_4} B_{M'}(\alpha_{s(1)}, \alpha_{s(2)}) B_{M'}(\alpha_{s(3)}, \alpha_{s(4)}),
\]

with \( B_{M'} \) the Beauville–Bogomolov form and \( c_M \) the Fujiki constant. Hence:

\[
r^*(\pi_*(\theta))^3 \cdot \tilde{\Sigma} = c_M B_{M'}(\pi_*(\theta), \pi_*(\theta)) B_{M'}(\pi_*(\theta), \Sigma').
\]

By the projection formula ([Men20, Remark 2.8]):

\[
r^*(\pi_*(\theta))^3 \cdot \tilde{\Sigma} = 0.
\]

Moreover by [Men18, Proposition 7.10]:

\[
B_{M'}(r^*(\pi_*(\theta)), r^*(\pi_*(\theta))) \neq 0.
\]

It follows:

\[
B_{M'}(r^*(\pi_*(\theta)), \Sigma') = 0.
\]

Moreover, by Lemma 3.4, \( \psi^* \) is an isometry on \( (\text{Pic} \, M', B_{M'}) \). Therefore by [Men20] and [Men18], we have \( \psi^*(\Sigma') = \pm \Sigma' \). We conclude as before using Banica and Stanasila theorem that:

\[
\psi^*(\Sigma') = \Sigma'.
\]

It follows that \( \psi^* \) induces the identity on \( \text{Pic} \, M' \). So \( \psi^* \) necessarily sends an ample class to an ample class. Hence \( \psi \) extends to an isomorphism.
4 The Kähler cone

4.1 Generalization of Boucksom criterion

This section is devoted to the proof of the following theorem. We recall that the positive cone $\mathcal{C}_X$ is defined in Section 2.3.

**Theorem 4.1.** Let $X$ be an irreducible symplectic orbifold. Let $\alpha \in \mathcal{C}_X$ such that $C \cdot \alpha > 0$ for all rational curves $C \subset X$. Then $\alpha$ is a Kähler class.

**Lemma 4.2.** Let $X$ be an irreducible symplectic orbifold. Let $\alpha \in \overline{\mathcal{C}_X}$ such that $C \cdot \alpha \geq 0$ for all rational curves $C \subset X$. Then $\alpha \in \overline{\mathcal{C}_X}$.

**Proof.** Following the same argument as [Bou01, Proposition 3.2], this lemma is a consequence of Propositions 2.10 and 2.11.

We will adapt the proof of [Bou01, Theorem 4.1] to obtain Theorem 4.1 from the previous lemma.

Let $X$ be an irreducible symplectic orbifold endowed with a marking $\varphi: H^2(X,\mathbb{Z}) \to \Lambda$ and let $\alpha \in \mathcal{C}_X$ as in the statement of Theorem 4.1. Let $f: \mathcal{X} \to \text{Def}(X)$ be the Kuranishi deformation of $X$. We denote $o := f(X)$. As explained in Section 2.3, there exists $U$ an open neighborhood of $o \in \text{Def}(X)$ such that the period map induces an isomorphism $\mathcal{P}: U \to \mathcal{P}(U)$. We denote:

$$W_\alpha := \text{Vect}_\mathbb{C}(\varphi(\alpha), \varphi(\text{Re }\sigma_X), \varphi(\text{Im }\sigma_X)) \text{ and } T(\alpha)|_U := \mathcal{P}^{-1}(\mathcal{P}(U) \cap \mathbb{P}(W_\alpha)).$$

By shrinking $U$ if necessary, we may assume that $U$ and $\mathcal{P}(U) \cap \mathbb{P}(W_\alpha)$ are simply connected.

Then we consider the one dimensional deformation of $X$:

$$f: f^{-1}(T(\alpha)|_U) \to T(\alpha)|_U$$

(9)

For simplicity of the notation, we still denote $\mathcal{X} := f^{-1}(T(\alpha)|_U)$. The deformation $f: \mathcal{X} \to T(\alpha)|_U$ can be seen as a local twistor space.

For all $t \in T(\alpha)|_U$, we denote by $\varphi_t := \varphi \circ u_t$ the markings, with $u_t : H^*(\mathcal{X}_t,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ the parallel transport operator induced by the deformation $f$. We denote by $\sigma_t$ the holomorphic 2-form on $\mathcal{X}_t$ and choose $\alpha_t$ such that $\mathbb{R}_{>0} \alpha_t = \varphi_t^{-1}(W_\alpha) \cap H^{1,1}(\mathcal{X}_t,\mathbb{R}) \subset \mathcal{C}_{\mathcal{X}_t}$.

**Lemma 4.3.** There exists $t \in T(\alpha)|_U$ such that $\alpha_t$ is a Kähler class on $\mathcal{X}_t$.

**Proof.** If there exists $t \in T(\alpha)|_U$ such that $\mathcal{X}_t$ does not contain any rational curve, by Lemma 2.12 we have $\mathcal{C}_{\mathcal{X}_t} = \mathcal{K}_{\mathcal{X}_t}$. Therefore, we assume that for all $t \in T(\alpha)|_U$, the orbifold $\mathcal{X}_t$ contains a rational curve and we will find a contradiction. Let $\beta \in H^{4n-2}(X,\mathbb{Z})$, we consider the set $S_\beta$ of $t \in U$ such that $u_t^{-1}(\beta)$ is a cohomology class of type $(2n-1,2n-1)$. As explained in the beginning of [Men20, Section 4.2], the $S_\beta$ are analytic subsets of $U$. By [Men20, Proposition 2.12] the class of a rational curve is of type $(2n-1,2n-1)$, hence $T(\alpha)|_U \subset \cup_{\beta \in H^{4n-2}(X,\mathbb{Z})} S_\beta$. Since $H^{4n-2}(X,\mathbb{Z})$ is countable, necessarily there exists $\beta \in H^{4n-2}(X,\mathbb{Z})$ and $\mathcal{W}$ the complement of a countable number of points in $T(\alpha)|_U$ such that $u_t^{-1}(\beta)$ is the class of a rational curve for all $t \in \mathcal{W}$. Let $\beta$ be such a class and $\mathcal{W}$ such a set. Then by [Nab84, Theorem 1.2], $\beta$ is the class of an effective combination of rational curves in $X$. However, $\sigma_t \cdot u_t^{-1}(\beta) = 0$ for all $t \in \mathcal{W}$, that is $u_t(\sigma_t) \cdot \beta = 0$ for all $t \in \mathcal{W}$. This implies that $w \cdot \beta = 0$ for all $w \in W_\alpha$, and in particular $\alpha \cdot \beta = 0$ which is a contradiction.

**Proof of Theorem 4.4.** Let $t \in T(\alpha)|_U$, as in Lemma 4.3. We consider:

$$W_t := \text{Vect}_\mathbb{C}(\varphi_t(\alpha_t), \varphi_t(\text{Re }\sigma_t), \varphi_t(\text{Im }\sigma_t)) \text{ and } T_W_t := \mathbb{P}(W_t) \cap \mathcal{D}.$$ 

By Lemma 4.3, $\alpha_t$ is a Kähler class, and therefore we can apply Theorem 2.13 to consider $f' : \mathcal{X}' \to T(\alpha_t)$ the twistor space associated to $\mathcal{X}_t$ and $\alpha_t$ (the period map provides an isomorphism $\mathcal{P}: T(\alpha_t) \to T_W_t$).

However, by construction $W_t = W_\alpha$. Hence, we have $\varphi(\sigma_t) \in T_W_t$. Let $o_t \in T(\alpha_t)$ be such that $\mathcal{P}(o_t) = \varphi(\sigma_t) \in T_W_t$ and $X' := f'^{-1}(o_t)$. We denote by $\varphi'$ the mark on $X'$ which arises by parallel transport from $\varphi$. We consider $\mathcal{C}_{X'} \cap \varphi'^{-1}(W_t) = \mathbb{R}_{>0} \alpha'$. Since $f'$ is a twistor space,
\(\alpha'\) is a Kähler class on \(X'\). Moreover by construction \(\varphi(\alpha) = \varphi'(\alpha')\). The two marked irreducible symplectic orbifolds \((X, \varphi)\) and \((X', \varphi')\) have the same image by the period map, hence by Theorem 2.9 they are non-separating points in their moduli space.

So by \([\text{Men20}]\) Remark 3.24 there exists a \(2n\)-dimensional analytic cycle \(\Gamma = Z + \sum Y_k \subset X \times X'\) such that:

(i) \(Z\) induces a bimeromorphism:

\[
\pi \xleftarrow{\varphi} Z \quad \pi' \xrightarrow{g} X' ;
\]

(ii) the components \(Y_k\) dominate neither \(X\) nor \(X'\);

(iii) \([\Gamma]_*(\alpha) = \varphi'^{-1} \circ \varphi(\alpha) = \alpha'\), where \([\Gamma]_*(\alpha) := \pi'_*([\Gamma] \cdot \pi^*\alpha)\).

Using Lemmas 3.1 and 3.2 for the same reason as explained in \([\text{Huy03}]\) proof of Theorem 2.5, the maps \([Y_k]_* : H^2(X, \mathbb{C}) \to H^2(X', \mathbb{C})\) are trivial. Then by Proposition 3.3 the bimeromorphism induced by \(Z\) extends to an isomorphism.

\[\square\]

### 4.2 Wall divisors

In this section we will define wall divisors in the same way as Mongardi in \([\text{Mon15}]\) Definition 1.2]. The proofs of \([\text{Mon15}]\) carry over in the same way to the setting of irreducible symplectic orbifolds. However, we streamline the exposition and some arguments, so we find it worthwhile to include the proofs here.

**Definition 4.4.** Let \(X\) be an irreducible symplectic orbifold of dimension \(2n\).

(i) We denote by \(BK_X\) the birational Kähler cone which is the union \(\cup f^*(K_X)\), for \(f\) running through all bimeromorphic map between \(X\) and any irreducible symplectic orbifold \(X'\). This is well-defined by Lemma 3.4.

(ii) We call the Mori cone the cone in \(H^{2n-1, 2n-1}(X, \mathbb{R})\) generated by the classes of effective irreducible curves contained in \(X\).

(iii) A Kähler-type chamber of the positive cone \(C_X\) is a subset of the form \(g[f^*(K_X)]\), where \(g \in Mon_{\mathbb{H}^{2,0}}^n(X)\), and \(f : X \dashrightarrow X'\) is a bimeromorphic maps to an irreducible symplectic orbifold \(X'\).

**Definition 4.5.** Let \(X\) be an irreducible symplectic orbifold and let \(D \in \text{Pic}(X)\). Then \(D\) is called a wall divisor if \(D^2 < 0\) and \(g(D^\perp) \cap BK_X = \emptyset\), for every parallel transport Hodge isometry \(g \in Mon_{\mathbb{H}^{2,0}}^n(X)\).

Note that an equivalent formulation for this definition is that \(D^\perp\) does not intersect any Kähler-type chamber.

One of our main result on wall divisor is Theorem 1.2) that we recall here.

**Theorem 4.6.** Let \((X, \varphi)\) and \((Y, \psi)\) be two marked irreducible symplectic orbifolds in the same connected component of their moduli space. Let \(D \subset X\) be a wall divisor of \(X\) such that \((\psi^{-1} \circ \varphi)(D) \in \text{Pic}(Y)\). Then \((\psi^{-1} \circ \varphi)(D)\) is a wall divisor on \(Y\).

**Proof.** We adapt the proof of \([\text{Mon15}]\) Theorem 1.3]. Let \(M^\circ_A\) be a connected component of the moduli spaces \(M_A\) which contains \((X, \varphi)\) and \((Y, \psi)\). Let \(D \subset \text{Pic}(X)\) be a divisor on \(X\) with \(D^2 < 0\) such that \(D_Y := (\psi^{-1} \circ \varphi)(D) \in \text{Pic}(Y)\). Assume that \(D_Y\) is not a wall divisor on \(Y\). We need to deduce that \(D\) cannot be a wall divisor on \(X\). Note that we may assume that \(D_Y\) is primitive.

Since by assumption \(D_Y\) is not a wall divisor on \(Y\), there exists another marking \(\psi_2\) on \(Y\) such that \(\psi_2^{-1} \circ \psi\) is a parallel transport Hodge isometry which satisfies \((\psi_2^{-1} \circ \psi)(D_Y^\perp) \cap BK_Y = (\psi_2^{-1} \circ \varphi)(D^\perp) \cap BK_Y \neq \emptyset\). Therefore, up to replacing \(\psi\) by \(\psi_2\), we can assume \(D^\perp \cap BK_Y = \emptyset\).
By the definition of the birational Kähler cone, this implies that there exists a bimeromorphic map \( f : Y \to Y' \) between irreducible symplectic orbifolds and such that \( D_Y^+ \cap f^+ (K_{Y'}) \neq \emptyset \). However, by Proposition 3.5, we know that \( (Y, \psi) \) and \( (Y', \psi \circ f) \) are deformation equivalent. Thus, up to replacing \( Y \) by \( Y' \) and \( D_Y \) by \( f^{-1}D_Y \), we can assume that \( D_Y^+ \cap K_Y \neq \emptyset \).

Choose a Kähler class \( \omega \in D_Y^+ \cap K_Y \) such that \( \omega \) is orthogonal to \( \omega_0 \). This is possible, since \( D_Y^+ \cap K_Y \) is a non-empty open subset. Consider the twistor family \( \mathcal{P}^0 \to \mathbb{P}^1 \) associated to \( \omega_0 \) (see Theorem 2.13). By construction of the twistor space, the period map identifies this \( \mathbb{P}^1 \) with the twistor line \( T_{W} \) for the positive 3-space \( W := \text{Vect}_{\mathbb{R}}(\psi(\omega), \psi(\text{Re } \sigma_Y), \psi(\text{Im } \sigma_Y)) \). The choice of \( \omega \) implies that \( W^+ \cap \psi(H^{1,1}(Y,\mathbb{Z})) = \mathbb{Z} \psi(D_Y) \). Therefore, a very general fiber \( (Z, \eta) \) of the twistor space satisfies that \( \text{Pic}(Z) = D_Z \cdot Z \), where \( D_Z := \eta^{-1} \circ \psi(D_Y) \). Note that \( D_Z \) cannot be a wall divisor on \( Z \), since by the construction of the twistor family \( Z \) is equipped with a Kähler form \( \omega_Z \in W \subseteq D_Z \).

We claim that this implies \( K_Z = C_Z \). Indeed, \( Z \) cannot contain an effective curve, since \( H^{2n-1,2n-1}(Z,\mathbb{Z}) \) consists exactly of the ray dual to \( D_Z \), which pairs trivially with the Kähler class \( \omega_Z \). Theorem 4.1 therefore implies that \( K_Z = C_Z \) as claimed.

Note that this has the following consequence. Let \( W' \subseteq \Lambda_{\mathbb{R}} \) be any positive 3-space such that \( \mathcal{P}(Z, \eta) \in T_{W'} \) (i.e., \( W' \subseteq \text{Vect}_{\mathbb{R}}(\text{Re}(\sigma_Z), \text{Im}(\sigma_Z)) \)). Then there exists a twistor family \( \mathcal{X}_1 \to \mathbb{P}^1 \) such that \( \mathbb{P}^1 \) is identified with \( T_{W'} \), via the period map. Indeed, \( W' \cap \text{Vect}_{\mathbb{R}}(\text{Re}(\sigma_Z), \text{Im}(\sigma_Z))^{+} \) is a one-dimensional positive subspace, which therefore contains an element \( \omega_{W'} \in C_Z = K_Z \) and \( \mathcal{X}_1 \) is obtained as the twistor family associated to \( \omega_{W'} \).

We now connect \( \mathcal{P}(Z, \eta) \) back to \( \mathcal{P}(X, \varphi) \) via convenient twistor spaces. Let \( N \) be the sublattice of \( \Lambda \) defined by \( N := \varphi(D)_{+} = \psi(D_Y)^+ = \eta(D_Z)^+. \) Since \( D^2 < 0 \), the lattice \( N \) has signature \((3, rk\Lambda - 3)\). Therefore by [Huy12] Proposition 3.7, the period domain

\[
D_N := \mathbb{P}(\{ \sigma \in N \otimes \mathbb{C} | \sigma^2 = 0, (\sigma + \overline{\sigma})^2 > 0 \}) \subseteq \mathbb{P}(N \otimes \mathbb{C}) \subseteq \mathbb{P}^1(\Lambda \otimes \mathbb{C})
\]

is connected by generic twistor lines. Note that the proof of [Huy12] Proposition 3.7 in fact shows that the twistor lines can be chosen in such a way that they intersect in very general points of \( D_N \). In particular, we can connect \( \mathcal{P}(Z, \eta) \) and \( \mathcal{P}(X, \varphi) \) by generic twistor lines in \( D_N \) intersecting in very general points.

As we have seen above, any generic twistor line \( T_{W'} \) through \( \mathcal{P}(Z, \eta) \) lifts to a twistor family. Repeating the same arguments as above, we observe that the fibre \( (Z', \eta') \) over a very general point of this family satisfies \( \text{Pic}(Z') = \mathbb{Z} D_{Z'} \), with \( D_{Z'} := \eta'^{-1} \varphi(D) \) which cannot be a wall divisor, since it is orthogonal to a Kähler class. As we observed above, this implies that \( C_{Z'} = K_{Z'} \), which allows us to repeat the same arguments.

In this way, we obtain an irreducible symplectic orbifold \((X', \varphi') \) with \( \mathcal{P}(X', \varphi') = \mathcal{P}(X, \varphi) \), which satisfies \((\varphi'^{-1} \circ \varphi(D))^+ \cap \mathcal{K}_{X'} \neq \emptyset \) (since it comes with a Kähler class orthogonal to \( \varphi'^{-1} \circ \varphi(D) \) by construction). Since we assumed that \((X, \varphi) \) and \((Y, \psi)\) are deformation equivalent, \( \psi^{-1} \circ \varphi \) is a parallel transport operator. On the other hand \( \varphi'^{-1} \circ \psi \) is a parallel transport operator by construction. Therefore, \( \varphi'^{-1} \circ \varphi \) is a parallel transport operator, which is a Hodge isometry since \( \mathcal{P}(X', \varphi') = \mathcal{P}(X, \varphi) \). By [Men20] Theorem 1.1 and Proposition 3.22, there is a bimeromorphism \( f : X \dashrightarrow X' \). Proposition 3.5 shows that \( f^* \) (and thus \( f^* \circ \varphi'^{-1} \circ \varphi \)) is a parallel transport Hodge isometry. By construction, we have \( \emptyset \neq (f^* \circ \varphi'^{-1} \circ \varphi(D))^+ \cap f^*(K_{X'}) \subseteq (f^* \circ \varphi'^{-1} \circ \varphi(D))^+ \cap \mathcal{K}_{X'} \). Thus \( D \) is not a wall divisor, which is what we wanted to show.

**Definition 4.7.** Let \( \mathcal{M}_\Lambda \) be the moduli space of marked irreducible symplectic orbifolds with Beauville–Bogomolov lattice \( \Lambda \) (see Section 2.3).

For a given irreducible symplectic orbifold, we denote by \( \mathcal{W}_X \subseteq \text{Pic}(X) \) the set of all primitive wall divisors. Furthermore, after choosing a connected component \( \mathcal{M}_\Lambda^0 \), let \( \mathcal{W}_X \subseteq \Lambda \) be the set of all classes \( \alpha \in \Lambda \) such that \( \psi^{-1}(\alpha) \in \mathcal{W}_Y \) is a primitive wall divisor for some \((Y, \psi) \in \mathcal{M}_\Lambda^0 \).

As an immediate consequence of Theorem 4.6 we obtain

**Corollary 4.8.** Let \( \Lambda \) be a lattice of signature \((3, rk\Lambda - 3)\) and \( \mathcal{M}_\Lambda^0 \) a connected component of the associated moduli space of marked irreducible symplectic orbifolds. Then for any \((X, \varphi) \in \mathcal{M}_\Lambda^0 \), the set \( \varphi^{-1}(\mathcal{W}_X) \cap \text{Pic}(X) \) consists of the (primitive) wall divisors on \( X \).
Example 4.9 ([Mon15, Proposition 2.12]). If $\mathcal{M}_\Lambda$ is a connected component of the moduli space of marked K3 surface, then:

$$\mathcal{M}_\Lambda = \{ D \in \Lambda \mid D^2 = -2 \}.$$ 

If $\mathcal{M}_\Lambda$ is a connected component of the moduli space of marked irreducible symplectic manifolds deformation equivalent to a Hilbert scheme of 2 points on a K3 surface, then:

$$\mathcal{M}_\Lambda = \{ D \in \Lambda \mid D^2 = -2 \} \cup \{ D \in \Lambda \mid D^2 = -10 \text{ and } D \cdot \Lambda \subset 2\mathbb{Z} \}.$$ 

Proposition 4.10. Let $X$ be an irreducible symplectic orbifold with $b_2(X) \geq 5$. Then the square of primitive wall divisors on $X$ is bounded, i.e. there exists a natural number $N \in \mathbb{N}$ such that for every primitive wall divisor $D$ the Beauville–Bogomolov square of $D$ satisfies $-N < q(D) < 0$.

Proof. Let us prove the statement under the stronger assumption that $b_2(X) > 5$. Under this assumption the statement is the equivalent of [AV17, Theorem 5.3]. This makes it easier to present the analogy in detail. It is possible to extend the statement to the case $b_2(X) = 5$ by a similar but slightly more technical analogy to the proof of [AV16 Corollary 1.4].

The concept of MBM-classes in [AV17] and the concept of wall divisors in [Mon15] which we introduced here, correspond to each other, but we will stick to using wall divisors here.

By definition (Definition [Mon15, Lemma 2.38]) the group $\text{Mon}^2_{\text{Hdg}}(X)$ preserves the set $\mathcal{W}_X$ of primitive wall divisors. Bakker and Lehn show in [BL18, Theorem 1.1] that $\text{Mon}^2_{\text{Hdg}}(X) \subseteq \text{Aut}(H^2(X,\mathbb{Z}))$ is a subgroup of finite index (since $b_2(X) \geq 5$). In particular this implies that $\text{Mon}^2_{\text{Hdg}}(X)$ restricts to an arithmetic subgroup of $\text{SO}(\text{Pic}(X),q)$.

In a first step, suppose that $X$ is an irreducible symplectic orbifold with maximal Picard rank $\rho(X) := \text{rkPic}(X) = b_2(X) - 2 > 3$. Under this assumption Pic($X$) is a lattice of signature $(1,b_2(X) - 3)$ containing $\mathcal{W}_X$. Therefore, [AV17, Theorem 1.3] implies that $\text{Mon}^2_{\text{Hdg}}(X)$ acts on $\mathcal{W}_X$ with finitely many orbits (since the set of orthogonal hyperplanes to elements in $\mathcal{W}_X$ cannot be dense in $C_X$). In particular, since elements in $\text{Mon}^2_{\text{Hdg}}(X)$ preserve the Beauville–Bogomolov square, the set $\{ q(D) \mid D \in \mathcal{W}_X \}$ is finite, which proves the claim.

The general case for $X$ with $b_2(X) > 5$ follows by deforming to an irreducible symplectic orbifold with maximal Picard group which contains the parallel transport of Pic($X$). This can be achieved using the surjectivity of the period map (see [Men20, Proposition 5.8]) and the deformation equivalence of wall divisors (see Theorem [Ari09]).

Lemma 4.11. Let $X$ be an irreducible symplectic orbifold such that $b_2(X) \geq 5$. Then the chamber structure in $C_X$ cut out by wall divisors is locally finite in the following sense: Let $\Pi \subset C_X$ be a rational polyhedral cone. Then the set $\{ D \in \mathcal{W}_X \mid D^+ \cap \Pi \neq \emptyset \}$ is finite.

Proof. Using the boundedness of the Beauville–Bogomolov squares from Proposition [AV17, Proposition 2.2] this is an immediate consequence of [MY15, Proposition 2.2].

We recall that the dual $\beta^\vee$ of a class $\beta \in H^{2n-1,2n-1}(X,\mathbb{Q})$ is defined in Remark [Mon15, Proposition 2.2].

Proposition 4.12. Let $X$ be an irreducible symplectic orbifold such that either $X$ is projective or such that $b_2(X) \geq 5$. Let $R$ be an extremal ray of the Mori cone of $X$ of negative self intersection. Then any divisor $D \in Q R^\vee$ is a wall divisor.

Proof. Let $D$ be the divisor in the statement, i.e. a primitive divisor with negative square, dual to an extremal ray $R$. We claim that this implies that $D^+$ cuts out a wall of $K_X$, i.e. $D^+ \cap \varnothing K_X \subseteq D^+$ contains an open subset $V$ of $D^+$.

Assuming this, choose $\alpha \in V$ generic (i.e. such that the only elements in $H^{1,1}(X,\mathbb{Z})$ which are orthogonal to $\alpha$ are collinear to $D$). Choose a marking $\varphi: H^2(X,\mathbb{Z}) \to \Lambda$ on $X$. Consider the 3-space $W := \text{Vect}_\mathbb{R}(\varphi(\alpha), \varphi(\text{Re} \sigma), \varphi(\text{Im} \sigma))$. By choice of $\alpha$ the space $W^\perp \cap \varnothing (H^2(X,\mathbb{Z})) = \mathbb{Z} \varphi(D)$. Therefore, $W$ gives rise to a generic twistor line $T_W$ in $D_X$, where $N := \varphi(D)^\perp$ as before. A priori $T_W$ is not coming from a twistor family, since $\alpha$ is not a Kähler class.

However, as in [Ari09, locally around $(X,\varphi)$, we can consider a family over some open simply by using the local Torelli theorem ([Men20, Theorem 3.17]). Let $(Y,\psi)$ be a very general element of this family (i.e. an element with $\text{Pic}(Y) = \mathbb{Z} \psi^{-1} \circ \varphi(D)$.)
Assume for contradiction that $D$ is not a wall divisor. Then use Theorem 4.10 to deduce that 
$\psi^{-1} \circ \varphi(D)$ is not a wall divisor either. Therefore, as in the proof of Theorem 4.10, $C_Y = K_Y$. Note 
that $W \cap \psi(\text{Re } \sigma_Y) \perp \psi(\text{Im } \sigma_Y) \subseteq \psi(H^1(Y, \mathbb{R}))$ is a positive 1-space and thus contains a Kähler 
form $\omega_Y$. By construction, $\text{Vect}_G(\varphi(\omega_Y), \varphi(\text{Re } \sigma_Y), \varphi(\text{Im } \sigma_Y)) = W$.

Therefore, the twistor family $\mathcal{Y}$ associated to $\omega_Y$ surjects to the twistor line $T_W$ associated to $W$. In particular the fibre $(\mathcal{Y}_0, \psi_0)$ over $\mathcal{D}(X, \varphi)$ is an irreducible symplectic orbifold, with Kähler class $\omega_{\mathcal{Y}_0}$, which satisfies

$$
\psi_0(\omega_{\mathcal{Y}_0}) \in W \cap \psi_0(\text{Re } \sigma_{\mathcal{Y}_0}) \perp \psi_0(\text{Im } \sigma_{\mathcal{Y}_0}) = W \cap \varphi(\text{Re } \sigma) \perp \varphi(\text{Im } \sigma) = \varphi(\alpha) \cdot \mathbb{R} \subseteq \varphi(\partial K_X),
$$

which is absurd.

It remains to prove the claim. Since the Kähler cone is cut out by effective curves (see e.g. Theorem 4.1) it suffices to prove that the extremal ray $R$ cannot be a limit of other extremal rays.

Let us first prove the claim under the condition that $b_2(X) \geq 5$. Suppose for contradiction that $R$ is the limit of extremal rays $R_i$ for $i \to \infty$. Observe that, since all $R_i$ are extremal rays, there exist elements $0 \neq \alpha_i \in \partial K_X \cap R_i^+$. We can choose these $\alpha_i$ such that they converge toward an element $0 \neq \alpha \in \partial K_X \cap R^+$. We will show that for each $\alpha_i$, there exists a wall divisor $D_i \in M_X$ with $D_i \perp \alpha_i$. This contradicts the locally finite structure of the walls from Lemma 4.11. Suppose for contradiction, that there exists $i$ such that $\alpha_i$ is not orthogonal to any wall divisor. Consider $W := \text{Vect}_G(\varphi(\alpha_i), \varphi(\text{Re } \sigma), \varphi(\text{Im } \sigma))$ and repeat the first part of this proof to reach a contradiction (this is applicable since by construction $\varphi^{-1}(W^\perp)$ does not contain any wall divisors).

In the case where $X$ is projective, the fact that $R$ cannot be a limit of other extremal rays follows from application of the cone theorem (see e.g. [Deb01, Theorem 7.38]): Choosing $L \in \text{Pic}(X) \cap C_X$, which satisfies $L \cdot R < 0$, the pair $(X, \epsilon L)$ is klt for small $\epsilon$, and $R$ is $\epsilon L$-negative by the choice of $L$.

This provides another criterion for Kähler classes.

**Definition 4.13.** Given an irreducible symplectic orbifold $X$ endowed with a Kähler class $\omega$. Define $\mathcal{W}^+_X := \{D \in \mathcal{M}_X | (D, \omega)_q > 0\}$, i.e. for every wall divisor, we choose the primitive representative in its line, which pairs positively with the Kähler cone.

**Corollary 4.14.** Let $X$ be an irreducible symplectic orbifold such that either $X$ is projective or $b_2(X) \geq 5$. Then

$$
\mathcal{K}_X = \{\alpha \in C_X | (\alpha, D)_q > 0 \ \forall D \in \mathcal{W}^+_X\}.
$$

**Proof.** Let $\alpha \in C_X$ satisfy $(\alpha, D)_q > 0$ for all $D \in \mathcal{W}^+_X$. We need to show that $\alpha \in \mathcal{K}_X$. The other inclusion follows immediately from the definitions. By Theorem 4.11 it is enough to show that $\alpha \cdot R > 0$ for all extremal rays $R$.

If $q(R^\vee) < 0$ this is true by Proposition 4.14.

If on the other hand $q(R^\vee) \geq 0$, then $R^\vee \in \mathcal{C}_X$ (since effective curves pair positively with Kähler classes), and therefore $\alpha \in C_X$ implies that $0 < (R^\vee, \alpha)_q = R \cdot \alpha$.

The following proposition will be useful in Section 5.

**Proposition 4.15.** Every very general class $\alpha \in C_X$ belongs to some Kähler-type chamber.

**Proof.** We adapt the proof of [Mar11, Lemma 5.1]. By Proposition 2.10 and [Men20, Remark 3.24], there exists $X'$ an irreducible symplectic orbifold and a cycle $\Gamma := Z + \sum_i Y_i$ in $X \times X'$ such that $Z$ defines a binumeromorphism under the following commutative diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi'} & \ \\ \ X' \\
\pi & \downarrow & \ \ \ \ f \\
X' & \rightarrow & \ \ \ \ X,
\end{array}
$$

where $\pi : X \times X' \to X$ and $\pi' : X \times X' \to X'$ are the projection. Moreover the map $[\Gamma]_* : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ defined for all $\beta \in H^2(X', \mathbb{Z})$ by $[\Gamma]_* (\beta) = \pi'_*([\Gamma] : \pi'^*(\beta))$ is a parallel transport operator and $[\Gamma]_* (\alpha)$ is a Kähler class of $X'$. We set $g := f^* \circ [\Gamma]_*$. By Proposition 5.5 $f^* \in \text{Mon}^2_{Hdg}(X)$ and so $g \in \text{Mon}^2_{Hdg}(X)$. Hence $g^{-1}(f^*(\mathcal{K}_X))$ is a Kähler-type chamber. Since $\alpha \in g^{-1}(f^*(\mathcal{K}_X))$, this concludes the proof.
Using the results on wall divisors, we can deduce the following more explicit form of this proposition.

**Corollary 4.16.** Let $X$ be an irreducible symplectic orbifold such that either $X$ is projective or $b_2(X) \geq 5$, and let $\alpha \in \mathcal{C}_X$ be such that $(\alpha, D)_q \neq 0$ for all $D \in \mathcal{W}_X$. Then $\alpha$ belongs to a Kähler-type chamber.

**Proof.** First observe that there exists an open neighborhood $U$ of $\alpha$ such that $U \cap D^\perp = \emptyset$ for all wall divisors $D \in \mathcal{W}_X$: Assume for contradiction that $\alpha$ would be the limit of points $\alpha_i$ satisfying $\alpha_i \in D^i_+$ for some wall divisor $D_i$. Then one can find a subsequence of these hyperplanes, which is converging, resulting in a hyperplane $W \ni \alpha$. This implies that $W$ does not intersect any Kähler chamber, since the $D^i_+$ do not. Therefore, $W$ is by definition the orthogonal complement of some wall divisor $D$. However, this gives the desired contradiction to the choice of $\alpha$.

Applying Proposition 4.15, we deduce that there exists $\alpha' \in U$ such that $\alpha'$ is in a Kähler type chamber, i.e. there exists $g \in \text{Mon}_\text{Hdg}(X)$ and $f: X \to X'$ such that $\alpha' \in g^{-1}f^*(K_Y)$. We consider the set $\mathcal{W}_\alpha^+ := \{D \in \mathcal{W}_X | (\alpha, D)_q > 0\}$ of wall divisors pairing positively with $\alpha$. Note that for every wall divisor $D \in \mathcal{W}_X$ either $D$ or $-D$ is in this set (since we assumed that $(\alpha, D)_q \neq 0$).

Furthermore, by the choice of $\alpha'$, the conditions $(\alpha, D)_q > 0$ and $(\alpha', D)_q > 0$ are equivalent for every $D \in \mathcal{W}_X$. Therefore, $g^{-1}f^*(K_Y) = \{\beta \in \mathcal{C}_X | (\beta, D)_q > 0 \ \forall D \in \mathcal{W}_\alpha^+\}$ by Corollary 4.14. Since this set obviously contains $\alpha$, this concludes the proof.

We also provide a criterion for the birational Kähler cone. Using Lemma 3.3 and Proposition 2.10, the following proposition can be proved exactly as [Huy03, Proposition 4.2].

**Proposition 4.17.** Let $X$ be an irreducible symplectic orbifold. Then $\alpha \in H^{1,1}(X, \mathbb{R})$ is in the closure $\overline{\mathcal{B}K}_X$ of the birational Kähler cone $\mathcal{B}K_X$ if and only if $\alpha \in \mathcal{C}_X$ and $(\alpha, [D])_q \geq 0$ for all uniruled divisors $D \subset X$.

## 5 Application to mirror symmetry

### 5.1 Motivation and some previous results

In this section, we propose a definition of the mirror symmetry for an irreducible symplectic orbifold endowed with a Kähler class. The main idea of mirror symmetry is to exchange the holomorphic 2-form and the Kähler metric. Several definitions of mirror symmetry are possible. We follow the definition of Huybrechts which is given by an involution on a period subdomain. When Huybrechts proposed his definition, the knowledge on mirror symmetry obtained by combining Theorem 5.4 and Huybrechts’ definition was not yet developed sufficiently to deduce the mirror symmetry between moduli spaces of lattice polarized K3 surfaces (see Definition 4.1.2), we are able to make this missing step in the smooth and or bifold setting. More precisely, via the Kähler cone for irreducible symplectic manifolds was not yet developed sufficiently to deduce that Huybrechts’ definition coincides with the one of Dolgachev and Camere.

In addition, our work is a generalization of [FJM19, Section 4.1 and 4.2], where a weaker version of Theorem 5.3 was obtained for the smooth case.

Section 4.2 is devoted to the proof of Theorem 5.3 using our previous results on the global Torelli theorem (Theorem 3.1) and the Kähler cone (Section 4.3). In Section 5.3, we discuss the definition of mirror symmetry obtained by combining Theorem 5.3 and Huybrechts’ definition [Huy03, Section 6.4].

For simplicity of the equations in this section, we denote the Beauville–Bogomolov form simply by the dot ".".
5.2 Global Torelli theorem for marked irreducible symplectic orbifolds endowed with a Kähler class

Let \( \Lambda \) be a lattice of \( \text{rk} \Lambda \geq 5 \) and signature \((3, \text{rk} \Lambda - 3)\). For a marked irreducible symplectic orbifold \((X, \varphi)\), we denote by \([X, \varphi]\) its class of isomorphism. Fix a connected component \(M_\Lambda^x\) of the moduli space of marked irreducible symplectic orbifolds of Beauville–Bogomolov lattice \(\Lambda\).

In this section, we consider quadruplets \((X, \varphi, \sigma_X, \omega_X, \beta)\), where \((X, \varphi)\) is a marked irreducible symplectic orbifold, \(\sigma_X \in H^{2,0}(X), \omega_X \in \mathcal{K}_X\) and \(\beta \in H^2(X, \mathbb{R})\). The class \(\beta\) is called a \(B\)-field.

We denote

\[
\tilde{M}_\Lambda := \{ (X, \varphi, \sigma_X, \omega_X, \beta) \mid [X, \varphi] \in M_\Lambda^x, 0 \neq \sigma_X \in H^{2,0}(X), \omega_X \in \mathcal{K}_X, \beta \in H^2(X, \mathbb{R}) \}/\sim,
\]

where \(\sim\) is the identification of isomorphic objects, i.e. \((X, \varphi, \sigma_X, \omega_X, \beta) \sim (X', \varphi', \sigma_X', \omega_X', \beta')\) if and only if there exists an isomorphism \(f : X \rightarrow X'\) such that \(f^* = \varphi^{-1} \circ \varphi', f^*(\sigma_X) = \sigma_X, f^*(\omega_X) = \omega_X,\) and \(f^*(\beta') = \beta\). We denote by \(\mathcal{W}_\Lambda \subset \Lambda\) the set from Definition 4.7 which gives the wall divisors of orbifolds in \(\tilde{M}_\Lambda\) via the marking.

Remark 5.1. The set \(\tilde{M}_\Lambda\) is endowed with a structure of differential manifold obtained as an open submanifold of

\[
\tilde{\mathcal{M}}_\Lambda := \{ (X, \varphi, \sigma_X, \omega_X, \beta) \mid [X, \varphi] \in M_\Lambda^x, 0 \neq \sigma_X \in H^{2,0}(X), \omega_X \in H^{1,1}(X, \mathbb{R}), \beta \in H^2(X, \mathbb{R}) \}/\sim,
\]

where \(\sim\) is identification of isomorphic objects, as before. The structure of differential manifold of \(\tilde{\mathcal{M}}_\Lambda\) is given by the period map which is locally a bijection by the local Torelli theorem [Men20, Theorem 3.17]:

\[
\tilde{\mathcal{P}} : \quad \tilde{\mathcal{M}}_\Lambda \rightarrow \mathcal{G}_\Lambda \subseteq (\Lambda \otimes \mathbb{C})^2
\]

\[
(X, \varphi, \sigma_X, \omega_X, \beta) \mapsto (\varphi(\sigma_X), \varphi(\beta + i\omega_X)),
\]

with

\[
\mathcal{G}_\Lambda := \{ (\alpha, \beta + ix) \in (\Lambda \otimes \mathbb{C})^2 \mid \alpha^2 = 0, \alpha \cdot \overline{\alpha} > 0, \alpha \cdot x = 0, x^2 > 0 \}.
\]

In [Huy04, Section 4.4], the space considered is slightly different:

\[
\text{Gr}_2^\mathbb{C}(\mathbb{A}_\mathbb{R}) \times \mathbb{A}_\mathbb{R} := \{ (\alpha, \beta + ix) \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \times \Lambda \otimes \mathbb{C} \mid \alpha^2 = 0, \alpha \cdot \overline{\alpha} > 0, \alpha \cdot x = 0, x^2 > 0 \};
\]

it can be obtained as a quotient of \(\mathcal{G}_\Lambda\).

Remark 5.2. If we choose to restrict to the case, where \(\beta \in H^{1,1}(X, \mathbb{R})\), then

\[
\widetilde{M}_\Lambda^{1,1} := \{ (X, \varphi, \sigma_X, \omega_X, \beta) \mid [X, \varphi] \in M_\Lambda^x, 0 \neq \sigma_X \in H^{2,0}(X), \omega_X \in H^{1,1}(X, \mathbb{R}), \beta \in H^{1,1}(X, \mathbb{R}) \}/\sim
\]

can be endowed with the structure of a complex manifold, inherited from the period map (compare [Mag12, Chapter 1 Section 2]).

We can generalize the global Torelli theorem [Men20, Theorem 1.1] for \(\tilde{M}_\Lambda\).

Definition 5.3. Define the following generalized period domain:

\[
\tilde{D}_\Lambda := \{ (\alpha, \beta + ix) \in (\Lambda \otimes \mathbb{C})^2 \mid \alpha^2 = 0, \alpha \cdot \overline{\alpha} > 0, x^2 > 0, \alpha \cdot x = 0, \alpha \perp x \perp \mathcal{W}_\Lambda = \emptyset \} \subseteq \mathcal{G}_\Lambda.
\]

Theorem 5.4. Assume that \(\Lambda\) satisfies \(\text{rk}\Lambda \geq 5\) and that \(M_\Lambda^x\) is non-empty. The period domain \(\tilde{D}_\Lambda\) has two connected components \(\tilde{D}_1\) and \(\tilde{D}_2\), and there exists \(i \in \{1, 2\}\) such that the period map

\[
\tilde{\mathcal{P}} : \quad \tilde{M}_\Lambda \rightarrow \tilde{D}_i
\]

\[
(X, \varphi, \sigma_X, \omega_X, \beta) \mapsto (\varphi(\sigma_X), \varphi(\beta + i\omega_X)),
\]

is an isomorphism.

Proof. The end of this section is devoted to the proof of this theorem.
Step 1: The set $\mathcal{D}_\Lambda$ has at least two connected components

Consider $(\alpha, \beta + ix) \in \mathcal{D}_\Lambda$. By the surjectivity of the usual period map $\mathcal{P}$ (see [Men20 Proposition 5.8]), there exists a marked irreducible symplectic orbifold $(X_\alpha, \varphi_\alpha) \in \mathcal{M}_\alpha^1$ such that $\alpha \in \varphi_\alpha(H^{2,0}(X_\alpha))$. Since by assumption $x^2 > 0$, this implies that either $\varphi_\alpha^{-1}(x) \in \mathcal{C}_{x_\alpha}$ (i.e. $x \cdot \varphi_\alpha(K_{x_\alpha}) > 0$, or $-\varphi_\alpha^{-1}(x) \in \mathcal{C}_{x_\alpha}$. By continuity, this splits $\mathcal{D}_\Lambda$ into two disjoint open sets:

$$\mathcal{D}_1 := \{ (\alpha, \beta + ix) | \varphi_\alpha^{-1}(x) \in \mathcal{C}_{x_\alpha} \}, \quad \text{and} \quad \mathcal{D}_2 := \{ (\alpha, \beta + ix) | -\varphi_\alpha^{-1}(x) \in \mathcal{C}_{x_\alpha} \}. $$

Note that by exchanging $x$ and $-x$ one verifies that both sets are non-empty.

Step 2: The moduli space $\mathcal{M}_\Lambda$ is connected

Let $(X, \varphi, \sigma_X, \omega_X, \beta)$ and $(Y, \psi, \sigma_Y, \omega_Y, \gamma)$ be two elements in $\mathcal{M}_\Lambda$. By Lemma 2.17 we can connect $(X, \varphi)$ and $(Y, \psi)$ by twistor spaces. The first twistor line is given by the positive 3-space $W := \text{Vect}(\varphi(\text{Re} \sigma_X), \varphi(\text{Im} \sigma_X), \omega_X)$ for some $\omega_X \in K_X$. Since any fibre $X_1$ of the twistor family is endowed with an induced marking $\varphi_1$ and a canonical Kähler class $\omega_1$, one can connect $(X, \varphi, \sigma_X, \omega_X, \beta)$ to $(X_1, \varphi_1, \sigma_1, \omega_1, \varphi^{-1} \circ \varphi(\beta))$ for some $\sigma_1 \in H^{2,0}(X_1)$. Use that $K_X$ is connected to observe that this is actually connected to the original $(X, \varphi, \sigma_X, \omega_X, \beta)$. Repeating this process for the other two twistors, we can find $\sigma'_X$ and $\omega'_X$ such that $(X, \varphi, \sigma_X, \omega_X, \beta)$ and $(Y, \psi, \sigma_Y, \omega_Y, \varphi^{-1} \circ \varphi(\beta))$ are connected. Using again that the Kähler cone, the space of non-zero holomorphic 2-forms, and the second cohomology group with real coefficient are connected, it follows that $(X, \varphi, \sigma_X, \omega_X, \beta)$ and $(Y, \psi, \sigma_Y, \omega_Y, \gamma)$ can be connected in $\mathcal{M}_\Lambda$.

We prove now that $\mathcal{P} : \mathcal{M}_\Lambda \to \mathcal{D}_1$ is an isomorphism.

Step 3: The map $\mathcal{P}$ is injective

Indeed, choose $(X, \varphi, \sigma_X, \omega_X, \beta)$ and $(X', \varphi', \sigma_X', \omega_X', \beta')$ in $\mathcal{M}_\Lambda$. Since $(X, \varphi)$ and $(X', \varphi') \in \mathcal{M}_\Lambda^0$ are deformation equivalent

$$\varphi^{-1} \circ \varphi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

defines a parallel transport operator. Assume that:

$$\mathcal{P}(X, \varphi, \sigma_X, \omega_X, \beta) = \mathcal{P}(X', \varphi', \sigma_X', \omega_X', \beta').$$

Then $\varphi^{-1} \circ \varphi$ is a Hodge isometry and which sends a Kähler class to a Kähler class. Hence by Theorem 1.3 (ii), there exists an isomorphism $f : X' \to X$ such that $f^* = \varphi^{-1} \circ \varphi$. That is $(X, \varphi, \sigma_X, \omega_X, \beta) \simeq (X', \varphi', \sigma_X', \omega_X', \beta')$ are isomorphic.

Step 4: The map $\mathcal{P}$ is surjective

By the convention from Step 1, $\mathcal{P}(\mathcal{M}_\Lambda) \subseteq \mathcal{D}_1$. Let $(\alpha, \beta + ix) \in \mathcal{D}_1$. Theorem 2.4 implies that there exists $(X, \varphi) \in \mathcal{M}_\Lambda^0$ such that $\varphi^{-1}(\alpha) \in H^{2,0}(X)$. Then $\varphi^{-1}(\beta) \in H^2(X, \mathbb{R})$. It remains to study $\varphi^{-1}(x)$, which is an element of $C_X$ by definition of $\mathcal{D}_1$. By Corollary 1.8 the wall divisors on $X$ are given by $\varphi^{-1}(\mathcal{W}_\Lambda) \cap \text{Pic}(X)$, and therefore $\varphi^{-1}x \cdot D \neq 0$ for all wall divisors $D \in \mathcal{W}_X$ by the definition of $\mathcal{D}_\Lambda$.

Hence by Corollary 1.10 there exists $g \in \text{Mon}_{\text{bim}}(\mathcal{H}^0(X))$ and $f : X \to Y$ a bimeromorphic map such that $\varphi^{-1}(x) \in g(f^* (K_Y))$. We consider $\psi := \varphi \circ g \circ f^*$ (note that by Proposition 3.5 $f^*$ is a parallel transport operator). It follows that $\psi^{-1}(x) \in K_Y$ and $\psi^{-1}(\alpha) \in H^{2,0}(Y)$.

Then $(Y, \psi, \psi^{-1}(\alpha), \psi^{-1}(x), \psi^{-1}(\beta)) \in \mathcal{M}_\Lambda$ and $\mathcal{P}(Y, \psi, \psi^{-1}(\alpha), \psi^{-1}(x), \psi^{-1}(\beta)) = (\alpha, \beta + ix)$. \qed

Remark 5.5. Of course, the previous theorem remains true if we remove the data of the B-field $\beta$ in $\mathcal{M}_\Lambda$ and in $\mathcal{D}_1$. This data will be useful only in the framework of mirror symmetry.

Remark 5.6. Theorem 5.4 also shows that $\mathcal{M}_\Lambda$ is separated.
Remark 5.7. There is a natural isomorphism between \( \tilde{\mathcal{D}}_1 \) and \( \tilde{\mathcal{D}}_2 \) given by \(-id\):
\[
\begin{array}{c}
\tilde{\mathcal{D}}_1 \\
\alpha, \beta + ix
\end{array} 
\begin{array}{c}
\tilde{\mathcal{D}}_2 \\
(-\alpha, -\beta - ix)
\end{array}
\]
Moreover if \( \tilde{\mathcal{M}}_\Lambda \rightarrow \tilde{\mathcal{D}}_1 \) is an isomorphism then we can consider
\[
\tilde{\mathcal{M}}_\Lambda := \left\{ (X, \varphi, \sigma_X, \omega_X, \beta) \mid (X, -\varphi, \sigma_X, \omega_X, \beta) \in \tilde{\mathcal{M}}_\Lambda \right\}
\]
and the induced map \( \tilde{\mathcal{M}}_\Lambda \rightarrow \tilde{\mathcal{D}}_2 \) is an isomorphism.

Remark 5.8. An alternative way to determine the connected component \( \tilde{\mathcal{D}}_1 \) of an element \((\alpha, \beta + ix) \in \tilde{\mathcal{D}}_\Lambda \) is the following. Fix a positive definite 3-space \( W \subseteq \Lambda_R \) with a determinant form \( \det_W \).

Let \( \pi_W : \Lambda_R \rightarrow W \) be the orthogonal projection to \( W \). Note that \( \text{Vect}_R(\text{Re} \alpha, \text{Im} \alpha, x) \subseteq \Lambda_R \) is also a positive definite threespace.

Moreover, if \( \Lambda \) is primitive and \( \pi_W : \Lambda \rightarrow W \) is an isomorphism, and therefore \( \det_W(\pi_W(\text{Re} \alpha), \pi_W(\text{Im} \alpha), \pi_W(x)) \neq 0 \). Therefore, by continuity the sign of \( \det_W(\pi_W(\text{Re} \alpha), \pi_W(\text{Im} \alpha), \pi_W(x)) \) determines the connected component of \((\alpha, \beta + ix)\), since both signs are achieved by elements of \( \tilde{\mathcal{D}}_\Lambda \) possible (consider e.g. \((\alpha, -\beta - ix)\)).

5.3 Definition of the mirror symmetry

Using Theorem 5.4 we can define mirror symmetry algebraically at the level of a subset of the period domain \( \mathcal{D}_\Lambda \) and it will induce a symmetry on a subset of the moduli space \( \mathcal{M}_\Lambda \).

Let \( \Lambda \) be a lattice of rk \( \Lambda \geq 5 \) and signature \((3, \text{rk} \Lambda - 3)\). For any field \( K \), we denote:
\[
\Lambda_K := \Lambda \otimes K.
\]
For \( n \in \mathbb{N}^* \) let \( U(n) \) be the lattice of rank 2 with basis \((v, v^*)\) such that \( v^2 = v^*v = 0 \) and \( v \cdot v^* = n \). Assume that there exists a primitive embedding \( j : U(n) \hookrightarrow \Lambda \), such that the sublattice \( j(U(n)) \) is a direct summand, i.e. \( \Lambda = \Lambda' \oplus \Lambda'' \).

For simplicity of the notation, we also denote by \((v, v^*)\) the basis of \( j(U(n)) \) and when there is no ambiguity, we simply write \( U(n) \) for \( j(U(n)) \).

Let \( U'(n) \) be another hyperbolic lattice isometric to \( U(n) \). We denote by \( \xi \) the isometry of \( O(\Lambda \oplus U'(n)) \) which fixes \( \Lambda' \) and exchanges \( U(n) \) and \( U'(n) \). As explained in [Huy04 Section 6], the mirror map is well defined on the following period domain:
\[
\text{Gr}_{2,2}^R(\Lambda_R \oplus U'(n)_R) := \left\{ \omega_1^2 = \omega_2^2 = \alpha_1 \cdot \alpha_2 = 0, \right. \
\left. \alpha_1 \cdot \overline{\alpha_1} > 0, \alpha_2 \cdot \overline{\alpha_2} > 0 \right\}.
\]
The notation \( \text{Gr}_{2,2}^R(\Lambda_R \oplus U'(n)_R) \) was chosen, since this space can be identified with the Grassmannian which parametrizes pairs of orthogonal positive 2-planes in \( \Lambda_R \oplus U'(n)_R \). On \( \text{Gr}_{2,2}^R(\Lambda_R \oplus U'(n)_R) \) the mirror map is given by:
\[
\overline{\mathcal{M}}_\xi := \iota \circ \xi,
\]
where \( \iota(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) \).

Remark 5.9. The general concept of mirror symmetry is to exchange the holomorphic 2-form and the metric. This operation will roughly be given by \( \iota \). We refer to [AM97] for the physical meaning of the composition by \( \xi \).

Remark 5.10. In [Huy04 Section 6] he considers the case \( n = 1 \). However, this condition is too restrictive in the framework of orbifolds (see Example 5.7). This is why we allow any \( n \in \mathbb{N}^* \). This does not have any consequence on the definition since \( U(n)_R = U_R \).

In the following, we consider a lift of \( \overline{\mathcal{M}}_\xi \) to a subset of \( \overline{\mathcal{D}}_\Lambda \). We recall that \( G_\Lambda \) is defined in [10]. It is shown in [Huy04 Section 4] that \( G_\Lambda \) which contains \( \overline{\mathcal{D}}_\Lambda \) admits a natural map to \( \text{Gr}_{2,2}^R(\Lambda_R \oplus U'(n)_R) \) obtained as the composition of the quotient map \( Q : G_\Lambda \rightarrow \text{Gr}_{2,2}^R(\Lambda_R) \times \Lambda_R \) and the following natural embedding:
\( \overline{m} : \text{Gr}_{2,1}^p(\Lambda R) \times \Lambda R \to \text{Gr}_{2,2}^p(\Lambda R \oplus U'(n)_R) \)
\[
(\alpha, \beta + ix) \mapsto \left( \sqrt{n\alpha} - (\alpha \cdot \beta)w, \sqrt{n\beta} + \frac{i}{2}(x^2 - \beta^2)w + w^* + i(\sqrt{n}x - (x \cdot \beta)w) \right),
\]
where \( (w, w^*) \) is a basis of \( U'(n)_R \); we set \( h := \overline{m} \circ Q \). Unfortunately, \( h(\mathcal{G}_\Lambda) \) is not fixed by \( \overline{m}_j \). We will therefore consider a subspace of \( \mathcal{G}_\Lambda \) such that its image under \( h \) is fixed by \( \overline{m}_j \):
\[
\text{Dom}(m_j)^\bigtriangleup := \{ (\alpha, \beta + ix) \in \mathcal{G}_\Lambda \mid \text{Im}(\alpha) \cdot v = 0, \text{Re}(\alpha) \cdot v \neq 0, x \cdot v = \beta \cdot v = 0 \}.
\]
Let \( pr : \Lambda \to j(U(n))^\perp \) be the projection. Then similar to [Huy04, Proposition 6.8], we can define the following action \( m_j \) on \( \text{Dom}(m_j)^\bigtriangleup \):
\[
m_j : \text{Dom}(m_j)^\bigtriangleup \to \text{Dom}(m_j)^\bigtriangleup
\]
\[
(\alpha, \beta + ix) \mapsto \left( \frac{\sqrt{n}pr(\beta + ix) - \frac{i}{2}(\beta + ix)^2v + v^*}{\text{Re}(\alpha) \cdot v}, \frac{\sqrt{n}pr(\alpha) - (\alpha \cdot \beta)w}{\text{Re}(\alpha) \cdot v} \right).
\]

**Proposition 5.11.** The map \( m_j \) is an involution, which satisfies \( h \circ m_j = \overline{m}_j \circ h \).

**Proof.** The claimed compatibility with \( \overline{m}_j \) is an immediate consequence of [Huy04, Proposition 6.8]. Note that there are only two differences between the map \( m_j \) and the map which is considered in [Huy04]: One difference is that in loc. cit. the domain of \( m_j \) is a quotient space of \( \mathcal{G}_\Lambda \). The second difference is, that we performed a change of variables, replacing \( v \) by \( \sqrt{n}v \) (and similarly for \( v^*, w, \) and \( w^* \)) in order to obtain the same intersection values. We can verify that \( m_j \) is an involution by a direct computation. We denote \( (\alpha^\vee, \beta^\vee + ix^\vee) := m_j \circ m_j(\alpha, \beta + ix) \). We are going to check that \( \beta^\vee + ix^\vee = \beta + ix \); the verification that \( \alpha^\vee = \alpha \) is very similar and is left to the reader. By definition of \( m_j \), we have:
\[
\beta^\vee + ix^\vee = \frac{n pr(\beta + ix) - \frac{i}{2}(\beta + ix)^2v + v^*}{\text{Re}(\alpha) \cdot v}, \quad \frac{\sqrt{n}pr(\alpha) - (\alpha \cdot \beta)w}{\text{Re}(\alpha) \cdot v}.
\]
It remains to compute \( pr(\beta + ix) \cdot pr(\text{Re}(\alpha)) \):
\[
pr(\beta + ix) \cdot pr(\text{Re}(\alpha)) = \left( \frac{\beta + ix}{n} \right) \cdot \left( \frac{\text{Re}(\alpha)}{n} \cdot v - \frac{\text{Re}(\alpha) \cdot v^*}{n} \right)
\]
\[
= \beta \cdot \text{Re}(\alpha) - \frac{\beta + ix}{n} \cdot (v \cdot \text{Re}(\alpha)).
\]
Combined with the previous equation, we obtain:
\[
\beta^\vee + ix^\vee = pr(\beta + ix) + \frac{\beta + ix}{n} \cdot v = \beta + ix.
\]
Finally, \( m_j \) will be well defined on the following period subdomain of \( \overline{D}_\Lambda \): For \( (\alpha, \beta + ix) \in \text{Dom}(m_j)^\bigtriangleup \), choose the notation \( (\alpha^\vee, \beta^\vee, + ix^\vee) := m_j(\alpha, \beta + ix) \). Then set
\[
\text{Dom}(m_j) := \left\{ (\alpha, \beta + ix) \in \overline{D}_\Lambda \mid \text{Im}(\alpha) \cdot v = 0, \text{Re}(\alpha) \cdot v \neq 0, x \cdot v = \beta \cdot v = 0, \alpha^\perp \cap x^\perp \cap \mathcal{W}_\Lambda = \emptyset \right\}.
\]
Moreover, we denote:
\[
\text{Dom}(m_j)_1 := \text{Dom}(m_j) \cap \overline{D}_1 \quad \text{and} \quad \text{Dom}(m_j)_2 \equiv \text{Dom}(m_j) \cap \overline{D}_2.
\]
We obtain the following proposition.
Proposition 5.12. The mirror involution \( m_j \) exchanges \( \text{Dom}(m_j)_1 \) and \( \text{Dom}(m_j)_2 \).

**Proof.** Using Remark 5.8, we can fix a convenient positive definite three-space \( W \subseteq \Lambda_R \) with a volume form \( \beta \), and we only need to compare the signs of \( \det_W(\pi_W(\text{Re} \alpha), \pi_W(\text{Im} \alpha), \pi_W(x)) \) and \( \det_W(\pi_W(\text{Re} \alpha'), \pi_W(\text{Im} \alpha'), \pi_W(x')) \).

Let us choose \( W = \text{Vect}_\mathbb{R}(\frac{1}{2}(v + v^*), \text{pr}(\text{Im} \alpha), \text{pr}(x)) \) with \( \det_W(\frac{1}{2}(v + v^*), \text{pr}(\text{Im} \alpha), \text{pr}(x)) = 1 \).

We start by determining \( \det_W(\pi_W(\text{Re} \alpha), \pi_W(\text{Im} \alpha), \pi_W(x)) \). Notice that

\[
\pi_W(\text{Im} \alpha) = \pi_W(\text{pr} \text{Im} \alpha) + \frac{1}{n}(\text{Im} \alpha \cdot v^*),
\]

and similarly \( \pi_W(\text{Re} \alpha) = 0 + \frac{1}{n}(\text{Re} \alpha \cdot v^*) \frac{1}{2}(v + v^*) \), and \( \pi_W(x) = \text{pr} x + \frac{1}{n}(x \cdot v^*) \frac{1}{2}(v + v^*) \).

Therefore,

\[
\det_W(\pi_W(\text{Re} \alpha), \pi_W(\text{Im} \alpha), \pi_W(x)) = \det \begin{pmatrix}
\frac{1}{n}(\text{Re} \alpha \cdot (v + v^*)) & \frac{1}{n}(\text{Im} \alpha \cdot v^*) & \frac{1}{2}(x \cdot v^*) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \frac{1}{n} \text{Re} \alpha \cdot (v + v^*) .
\]

Since \( \text{Re} \alpha^2 > 0 \), \( (\text{Re} \alpha) \cdot v \) and \( (\text{Re} \alpha) \cdot v^* \) have the same sign. In particular the sign of \( \frac{1}{n} \text{Re} \alpha \cdot (v + v^*) \) is the same as the sign of \( \text{Re} \alpha \cdot v \).

As a second step, we need to determine the sign of \( \det_W(\pi_W(\text{Re} \alpha'), \pi_W(\text{Im} \alpha'), \pi_W(x')) \). Note, that by replacing \( \beta \) with 0, one obtains an element in the same connected component of \( \text{Dom}(m_j) \), so it is sufficient to treat the special case \( \beta = 0 \). Using (11) and the same reasoning as above, we need to determine the determinant of the following matrix:

\[
M := \frac{1}{\text{Re} \alpha \cdot v} \begin{pmatrix}
\frac{1}{2}x^2 + 1 & 0 & 0 \\
0 & 0 & \sqrt{n} \\
0 & \sqrt{n} & 0
\end{pmatrix} .
\]

Observe that \( \det(M) = -\frac{1}{(\text{Re} \alpha \cdot v)}(\frac{1}{2}x^2 + 1) \). Since \( x^2 > 0 \), this has the same sign as \( -\text{Re} \alpha \cdot v \).

Therefore \( (\alpha, \beta + ix) \) and \( m_j(\alpha, \beta + ix) = (\alpha', \beta' + ix') \) lie in different connected components of \( \tilde{D}_\Lambda \), which proves that \( m_j \) interchanges \( \text{Dom}(m_j)_1 \) and \( \text{Dom}(m_j)_2 \).

**Remark 5.13.** Proposition 5.12 can colloquially be rephrased by saying that the mirror map changes the orientation of positive three-spaces, which happens to match our experience with real life mirrors.

To be able to have a better idea in practice of the elements contained in \( \text{Dom}(m_j) \), we provide the following lemma which is a straight forward consequence of [FJM19 Proposition 4.6].

**Lemma 5.14.** Let \( (\alpha, \beta + ix) \in \tilde{D}_\Lambda \) such that:

- \( (\alpha, \beta + ix) \in \text{Dom}(m_j)^\mathbb{R} \),
- \( x^2 \in \mathbb{R} \setminus \mathbb{Q} \),
- For each \( D \in \mathcal{W}_\Lambda \) one of the following two implications is satisfied:
  - Either \( \text{pr}(\text{Re} \alpha) \cdot D \neq 0 \) \( \Rightarrow \) \( \text{pr}(\beta) \cdot D \neq 0 \) or \( \alpha \cdot D \neq 0 \) \( \Rightarrow \) \( \text{Im} \alpha \cdot D \neq 0 \).

Then there exists a dense uncountable subset \( \Pi \in \mathbb{R}^* \) such that for all \( \lambda \in \Pi \), we have \( (\alpha, \lambda \beta + ix) \in \text{Dom}(m_j) \).

We are now ready to state the definition of the mirror symmetry of a marked irreducible symplectic orbifold endowed with a Kähler class and a B-field.

**Definition 5.15.** Let \( \bar{\mathcal{T}} : \tilde{M}_\Lambda \rightarrow \tilde{D}_1 \) be the isomorphism provided by Theorem 5.4 and consider the associated natural map \( \bar{\mathcal{T}} : \tilde{M}_\Lambda^\mathbb{R} \rightarrow \tilde{D}_2 \) from Remark 5.7. Consider the moduli subspaces

\[
\tilde{M}_\Lambda^j := \bar{\mathcal{T}}^{-1}(\text{Dom}(m_j)_1) \text{ and } \tilde{M}_\Lambda^{-j} := \bar{\mathcal{T}}^{-1}(\text{Dom}(m_j)_2).
\]
The mirror symmetry \( \tilde{m}_j \) is an involution defined on the moduli space \( \tilde{M}_\Lambda^j \cup \tilde{M}_\Lambda^{-j} \) by:

\[
\tilde{m}_j := \tilde{\Theta}^{-1} \circ m_j \circ \tilde{\Theta}.
\]

Since \( m_j \) interchanges \( \tilde{D}_1 \) and \( \tilde{D}_1 \), the map \( \tilde{m}_j \) exchanges \( \tilde{M}_\Lambda^j \) and \( \tilde{M}_\Lambda^{-j} \).

Actually, a mirror can be defined for all elements of the following dense set of \( M_\Lambda^0 \): (Note that the constructed mirror depends on the choice of \( \sigma_X \) and \( \beta \) for the element \( (X, \varphi, \sigma_X, \omega_X, \beta) \in \tilde{M}_\Lambda^j \))

**Proposition 5.16.** Let \( (X, \varphi) \in M_\Lambda^0 \). Assume that the Kähler cone \( K_X \) satisfies that \( K_X \cap v^\perp \neq \emptyset \) then there exists \( \sigma_X \in H^{2,0}(X), \omega_X \in K_X \) and \( \beta \in H^2(X, \mathbb{R}) \) such that:

\[
(X, \varphi, \sigma_X, \omega_X, \beta) \in \tilde{M}_\Lambda^j.
\]

**Proof.** Let \( (X, \varphi) \) be as in the statement of the proposition. Let \( \sigma'_X \in H^{2,0}(X) \). For any \( (a + ib) \in \mathbb{C}^* \), we have:

\[
\text{Im} [(a + ib)\sigma'_X] = a \text{Im} \sigma'_X + b \text{Re} \sigma'_X.
\]

Choose \( (a + ib) \in \mathbb{C}^* \) such that:

\[
a(\text{Im} \sigma'_X \cdot v) + b(\text{Re} \sigma'_X \cdot v) = 0.
\]

Then we set \( \sigma_X := (a + ib)\sigma'_X \). Since by assumption \( K_X \cap v^\perp \neq \emptyset \), we can choose \( \omega_X \in K_X \) such that \( \omega_X \cdot v = 0 \). By rescaling \( \omega_X \), we can furthermore assume that \( \omega_X \in \mathbb{R} \setminus \mathbb{Q} \). Finally, by Lemma 5.14 we can choose \( \beta \in H^2(X, \mathbb{R}) \) such that \( (X, \varphi, \sigma_X, \omega_X, \beta) \in \tilde{M}_\Lambda^j \).

Now, we show that our definition of mirror symmetry coincides with the one of Dolgachev [Dol96] and Camere [Cam18] for lattice polarized irreducible symplectic manifolds.

**Definition 5.17.** Let \( \Lambda \) be a lattice of signature \((3, \text{rk}\Lambda - 3)\). Let \( \nu : N \hookrightarrow \Lambda \) be a primitive embedding of a sublattice \( N \) of signature \((1, \text{rk}N - 1)\). An \((N, \nu)\)-polarized irreducible symplectic orbifold is a couple \((X, \varphi) \in M_\Lambda \) such that:

\[
\nu(N) \subset \varphi(\text{Pic} X).
\]

We say that \((X, \varphi)\) is strictly \((N, \nu)\)-polarized if:

\[
\nu(N) = \varphi(\text{Pic} X).
\]

**Proposition 5.18.** Let \( \nu : N \hookrightarrow \Lambda \) be a primitive embedding with \( N \) of signature \((1, \text{rk}N - 1)\) such that \( j : U(n) \hookrightarrow \nu(N)^\perp \). We set \( N^\nu := U(n)^\perp \cap \nu(N)^\perp \) and \( \nu^\vee : N^\nu \hookrightarrow \Lambda \) the natural embedding. Let \((X, \varphi) \in M_\Lambda^0 \) be a strictly \((N, \nu)\)-polarized irreducible symplectic orbifold. Then there exists \( \sigma_X \in H^{2,0}(X), \omega_X \in K_X \) and \( \beta \in H^2(X, \mathbb{R}) \) such that:

\[
(X, \varphi, \sigma_X, \omega_X, \beta) \in \tilde{M}_\Lambda^j,
\]

and \((X^\vee, \varphi^\vee)\) is \((N^\vee, \nu^\vee)\)-polarized, where \((X^\vee, \varphi^\vee, \sigma_{X^\vee}, \omega_{X^\vee}, \beta^\vee) := m_j(X, \varphi, \sigma_X, \omega_X, \beta).

**Proof.** For simplicity of the notation, we denote \( \nu(N) \) by \( N \). Note that \( \text{Pic}(X) = \varphi^{-1}(N) \) has signature \((1, \text{rk}N - 1)\) implies by the projectivity criterion ([Men20 Theorem 1.2]) that \( X \) is ample, and therefore the ample cone \( K_X \cap \varphi^{-1}(N) \) is non-empty. Choose \( \omega_X \in K_X \cap \varphi^{-1}(N) \) with \( \omega_X^2 \in \mathbb{R} \setminus \mathbb{Q} \). Furthermore, fix \( \beta \in \varphi^{-1}(N) \) such that for all \( D \in U \setminus \varphi^{-1}(N^\perp \cap U) \):

\[
\text{pr}(\beta) \cdot D \neq 0.
\]

In particular, the third condition of Lemma 5.14 is satisfied for all \( D \in U \setminus \varphi^{-1}(N^\perp \cap U) \).

Let \( \sigma'_X \in H^{2,0}(X) \) and \( (a + ib) \in \mathbb{C}^* \). Again, we have:

\[
\text{Im} [(a + ib)\sigma'_X] = a \text{Im} \sigma'_X + b \text{Re} \sigma'_X.
\]
and we can choose \((a + ib) \in \mathbb{C}^*\) such that:

\[
\text{Im}((a + ib)\sigma_X) \cdot v = a(\text{Im}\sigma_X \cdot v) + b(\text{Re}\sigma_X \cdot v) = 0.
\]  

(12)

Set \(\sigma_X := (a + ib)\sigma_X^\vee\). To use Lemma 5.14 we need to check the third condition for \(D\). To do this, we will show that

\[
\text{Im}(\sigma_X) \cdot D = a(\text{Im}\sigma_X \cdot D) + b(\text{Re}\sigma_X \cdot D) \neq 0,
\]  

(13)

for all \(D \in \nu^{-1}(\mathcal{W}_A \cap N^\perp)\). Assume for contradiction that (13) does not hold, i.e. \(\text{Im}(\sigma) \cdot D = 0\). For any real number \(c \in \mathbb{R}\), observe that in this case \(\sigma_X \cdot (D - cv) = \text{Re}(\sigma_X) \cdot (D - cv)\). Note that \(\text{Re}(\sigma_X) \cdot v = \sigma_X \cdot v \neq 0\), since \(v \notin \text{Pic}(X)\). Hence by setting \(c = \frac{\text{Re}(\sigma_X) \cdot D}{\text{Re}(\sigma_X) \cdot v}\), one can achieve that \(\sigma_X \cdot (D - cv) = 0\). This implies that \(D - cv \in \text{Pic}(X)\). However, we have chosen \(D \in \text{Pic} X^\perp\) and \(v \in \text{Pic} X^\perp\). Since the Beauville-Bogomolov form is non-degenerate, we have \(D - cv = 0\). This is impossible because \(D^2 < 0\) and \(v^2 = 0\).

Therefore, we can apply Lemma 5.14 to see that there exists \(\lambda \in \mathbb{R}\) such that \((X, \varphi, \sigma_X, \omega_X, \lambda \beta) \in \tilde{\mathcal{M}}_A^J\). Finally, since by hypothesis \(\nu^\vee(\nu^\perp) = U(n) \oplus \nu(N)\), we obtain from (11) that \(\sigma_X^\vee \in \nu^\vee(\nu^\perp)^\perp\). That is \((X^\vee, \varphi^\vee)\) is \((\nu^\vee, \nu^\perp)\)-polarized. \(\square\)

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