Supersymmetry representation of Bose-Einstein condensation of fermion pairs

Alexander Olemsko† and Irina Shuda‡
Institute of Applied Physics, Nat. Acad. Sci. of Ukraine, 58, Petropavlovskaya St., 40030 Sumy, Ukraine
Sumy State University, 2, Rimskii-Korsakov St., 40007 Sumy, Ukraine
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We consider supersymmetry field theory with supercomponents being the square root of the Bose condensate density, the amplitude of its fluctuations and Grassmannian fields related to the Fermi particles density. The fermion number is demonstrated to be conserved in degenerated Fermi-Bose mixtures with unbroken supersymmetry when the system is invariant with respect to inversion of the time arrow. We show the supersymmetry breaking allows one to derive field equations describing behavior of real Bose-Fermi mixtures. Solution of related field equations reveals the cooling of homogeneously distributed fermions gives first spontaneous rise to strong inhomogeneous fluctuations, while the Bose condensate appears at a lower temperature dependent of the fermion density.

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I. INTRODUCTION

Supersymmetry is one of the most beautiful and productive conception of contemporary physics which has been proposed for description of microworld (see Ref. 1 to review supersymmetry applications to the quantum mechanics, the survey 2 deals with disorder metals, and book 3 relates to the theory of superstrings). Basing on idea proposed in the work 4 and developed in Refs. 5 and 6, it has been shown that the supersymmetry can be used effectively to present fluctuative fields determining a picture of phase transition with symmetry breaking in non-equilibrium condensed matter 7 and random heteropolymers 8. Along this line, the Martin-Siggia-Rose method 9 has been used as a basis permitting to combine stochastic fields into supersymmetry construction 10. One can perceive the supersymmetry approach is developed along two main lines: the former is based on the quantum operator representation 1, 2, while the latter considers evolution of supersymmetric field 3, 4, 5, 6, 8, 9, 10.

Initially, the supersymmetry had been proposed as a symmetry that relates bosons and fermions in elementary particle physics 11, 12, 13. Along this line, the supersymmetric string theory is a unique theory expected to give a unified description of all interactions in nature 3, however none of the superpartners of any known elementary particles has been found in experiments so far. Therefore, it is very important to study the supersymmetry breaking. Such an opportunity is opened with experimental progress in atomic mixtures of ultracold Bose and Fermi atoms 14. Theoretically, an ultracold superstring model was constructed 15, as well as an exactly solvable model of one-dimensional Bose-Fermi mixture was investigated 16. According to Ref. 17, the supersymmetry is always broken either spontaneously or by a chemical potential difference between bosons and fermions. This article is devoted to consideration of the Bose-Fermi mixture within field-theoretical supersymmetry approach 10.

The outline of the paper is as follows. In section II we adduce main field-theoretical statements based on the generating functional method introduced by Martin, Siggia and Rose. Making use of this method allows us to write down both supersymmetric Lagrangian of the problem and Euler equations related. Section III is devoted to derivation of the field equations for the superfield components being the most probable values of the square root of the Bose condensate density, the amplitude of its fluctuations and the most probable Grassmannian fields giving the Fermi particles density. Combining above equations, we show the fermion number is conserved in supersymmetrically degenerated systems whose superspace is invariant with respect to a rotation. According to section IV the supersymmetry invariance is broken by means of a rotation that results in loss of the Grassmannian components invariance with respect to the time inversion due to the Bose-Einstein condensation. We show the supersymmetry breaking allows one to derive field equations describing behavior of real Bose-Fermi mixtures. Section V is devoted to discussion of obtained results and Appendix contains details of calculations at derivation of the field equations.

II. MAIN FIELD-THEORETICAL STATEMENTS

We consider attractive Fermi system characterised by the pair of conjugate wave functions \( \psi(\mathbf{r}, t) \) and \( \overline{\psi}(\mathbf{r}, t) \) and Bose condensate with the density \( n(\mathbf{r}, t) \) where \( \mathbf{r}, t \) are coordinate and time, respectively. Behavior of the Bose subsystem is presented by the fluctuating order parameter

\[
x(\mathbf{r}, t) := \sqrt{n(\mathbf{r}, t)} e^{i\phi(\mathbf{r}, t)}
\]
with \( \phi(r, t) \) being a condensate phase. Within framework of the standard field-theoretical scheme \([10]\), the system evolution is described by the Langevin equation

\[
\dot{x}(r, t) - D \nabla^2 x = -\gamma \frac{\partial F}{\partial x} + \zeta(r, t). \tag{2}
\]

Here, \( \dot{\cdot} \) stands for the time derivative, \( \nabla \equiv \partial / \partial r \), \( D \) and \( \gamma \) are diffusion and kinetic coefficients, respectively, \( F(x) \) is specific free energy of condensate, \( \zeta(r, t) \) is stochastic addition defined by the white noise conditions

\[
\langle \zeta(r, t) \rangle = 0, \quad \langle \zeta(r, t) \zeta(0, 0) \rangle = \gamma T \delta(r) \delta(t) \tag{3}
\]

where angle brackets notice averaging over system states scattered with variance \( T \) being the temperature measured in energy units. With introduction of the scales \( t_s = (\gamma T)^2 / D^3, \quad r_s = \gamma T / D, \quad F_s = D^3 / \gamma^2 T^2, \quad \zeta_s = D^3 / (\gamma T)^2 \) for time \( t \), coordinate \( r \), specific free energy \( F \) and stochastic force \( \zeta \), respectively, the equation of motion (2) takes the simplest form

\[
\dot{x}(r, t) = -\frac{\delta F}{\delta x} + \zeta(r, t) \tag{4}
\]

where short denotations of the variational derivative

\[
\frac{\delta F}{\delta x} \equiv \frac{\delta F(x, \{r(t)\})}{\delta x} = \frac{\partial F(x)}{\partial x} - \nabla^2 x \tag{5}
\]

is used for the Ginzburg-Landau model

\[
F\{x\} \equiv \int \left[ F(x) + \frac{1}{2} (\nabla x)^2 \right] \, dr. \tag{6}
\]

Along the standard line \([10]\), our approach is stated on the generating functional

\[
Z\{u(r, t)\} = \int Z\{x\} \exp \left( \int u x \, dr \, dt \right) \, Dx \tag{7}
\]

being determined by the partition functional

\[
Z\{x(r, t)\} := \left\langle \prod_{i(t)} \delta \left( \dot{x} + \frac{\delta F}{\delta x} - \zeta \right) \det \left[ \frac{\delta \zeta}{\delta x} \right] \right\rangle \tag{8}
\]

where argument of the \( \delta \)-functional takes into account the equation of motion (4) and determinant is the Jacobian of the transition from the noise field \( \zeta(r, t) \) to the order parameter \( x(r, t) \) over whose distribution the continuous integration in the definition (7) is carried out.

Within the simplest case of the Itô calculus, the Jacobian determinant equals one and the expression (8) arrives at the pair of the Bose field only \([8]\). Much more interesting situation is generated by the Stratonovich calculus when the Jacobian

\[
\det \left[ \frac{\delta \zeta}{\delta x} \right] = \int \exp \left( \psi \frac{\delta \zeta}{\delta x} \right) d^2 \psi, \quad d^2 \psi = d\psi \, d\overline{\psi} \tag{9}
\]

is presented by Grassmannian conjugate fields \( \psi(r, t) \) and \( \overline{\psi}(r, t) \) which subject to the following conditions:

\[
\overline{\psi}\psi + \psi\overline{\psi} = 0, \quad \int d\psi = 0; \quad \int \psi \, d\psi = 1, \quad \int d\overline{\psi} = 0, \quad \int \psi \, d\overline{\psi} = 1. \tag{10}
\]

Then, after generalized Laplace transform of the \( \delta \)-functional in Eq.(8) we obtain the supersymmetry Lagrangian

\[
\mathcal{L}(x, p, \psi, \overline{\psi}) = \left( px - \frac{p^2}{2} + \frac{\delta F}{\delta x} \right) \overline{\psi} - \psi \left( \frac{\partial}{\partial t} + \frac{\delta^2 F}{\delta x^2} \right) \psi \tag{11}
\]

with a ghost field \( p(r, t) \). Introducing the four-component superfield

\[
\Phi := x + \overline{\psi} \psi + \psi \theta + \theta \overline{\psi}, \tag{12}
\]

easily to convince that expression (11) can be written in the canonical supersymmetric form

\[
\mathcal{L} = \int \Lambda d^2 \theta, \quad d^2 \theta \equiv d\theta d\overline{\theta}, \quad \Lambda(\Phi) \equiv \frac{1}{2} (\overline{\Phi}(\overline{D}) (D \Phi) + F(\Phi) \tag{13}
\]

where \( \theta, \overline{\theta} \) are Grassmannian conjugate coordinates defined by the properties

\[
\overline{\theta} \theta + \theta \overline{\theta} = 0, \quad \int d\theta = 0; \quad \int \theta d\theta = 1 \tag{14}
\]

being similar to Eqs.(10). Supersymmetry generators in Eq.(13) are as follows:

\[
D := \frac{\partial}{\partial \theta} - 2\theta \frac{\partial}{\partial t}, \quad \overline{D} := \frac{\partial}{\partial \overline{\theta}}. \tag{15}
\]

**III. SUPERSYMMETRY FIELD EQUATIONS**

Variation of the action related to the supersymmetric Lagrangian \([13]\) over the superfield \([12]\) derives the supersymmetry Euler equation

\[
\frac{1}{2} \left\langle \overline{D}, D \right\rangle \Phi = \frac{\delta F}{\delta \Phi} \tag{16}
\]

where the square brackets notice commutation. As is shown in Appendix, projection of Eq.(16) onto the superspace axes \( \theta, \overline{\theta}, \theta, \overline{\theta} \) and \( \overline{\theta} \overline{\theta} \) arrives at the explicit form of the equations of motion:

\[
\dot{\eta} - \nabla^2 \eta = -\frac{\partial F}{\partial \eta} + \varphi, \tag{17}
\]


\[ \dot{\psi} + \nabla^2 \varphi = \frac{\partial^2 F}{\partial \eta^2} \varphi - \frac{\partial^3 F}{\partial \eta^3} \nabla \Psi, \quad (18) \]

\[ \dot{\Psi} - \nabla^2 \Psi = - \frac{\partial^2 F}{\partial \eta^2} \Psi, \quad (19) \]

\[ - \ddot{\Psi} - \nabla^2 \ddot{\Psi} = - \frac{\partial^2 F}{\partial \eta^2} \ddot{\Psi}. \quad (20) \]

Since minimal action relates to the most probable realizations of the superfield (12), solutions of equations (17) – (20) determine the most probable components \( x^{(max)} \equiv \eta \) [19], \( \psi^{(max)} \equiv \varphi \) and \( \ddot{\Psi}^{(max)} \equiv \ddot{\Psi} \). The first of these equations takes the form of the Langevin equation (4) that shows a ghost field \( \varphi \equiv \tilde{p}^{(max)} \) represents the most probable realization of the fluctuation amplitude \( \zeta \). Specific peculiarity of the field \( \varphi \) is that gradient term in the governing equation (18) has inverse sign, so that inhomogeneity in the space distribution of the most probable fluctuation increases in the course of the time until non-linearity stabilizes its amplitude.

Another feature consists in the Grassmannian conjugation of the equations (19) and (20) which coincide if the time arrow is inverted in one of them. Thus, one can conclude that the pair of conjugate fields \( \Psi \) and \( \Psi \) describes evolution of the Fermi particle and antiparticle for which the time runs in opposite directions. Combining Eqs. (19) and (20) arrives at the continuity equation

\[ \dot{\rho} + \nabla j = 0 \quad (21) \]

for the fermion density

\[ \rho := \nabla \dot{\Psi}, \quad (22) \]

and the current related

\[ j := \nabla \dot{\Psi} - \nabla \nabla \Psi. \quad (23) \]

The equation (21) expresses the conservation law of the Fermi particles number in supersymmetric system whose state space spanned onto axes 1, \( \tilde{\theta} \), \( \theta \) and \( \overline{\theta} \) is invariant with respect to a direction choice.

IV. SUPERSYMMETRY BREAKING

To break above invariance we shall follow to the Bogolyubov method of quasi-averages, according to which taking off a system degeneration is provided by switching an infinitesimal source type of slight magnetic field in magnets [18]. In our case, the role of such a field is played by the conjugate Grassmannian components \( \nabla \Psi \) and \( \nabla \theta \) the superfield (12) which relate to forward and backward directions of the time arrow. Formally, we should replace the superfield (12) by the transformed field

\[ \tilde{\Phi} := e^{-\nabla \Psi} \Phi e^{\nabla \Psi}. \quad (24) \]

Writing this superfield in the explicit form

\[ \tilde{\Phi} = x + (1 - x) \nabla \Psi + (1 + x) \nabla \theta + (p + x\rho) \overline{\theta}, \quad (25) \]

easily to see that transformation (24) squeezes the axis \( \nabla \Psi \) and stretches the axis \( \nabla \theta \) by the same value \( x \) being the order parameter of the Bose condensate, while the axis \( \overline{\theta} \) is stretched by the value \( x\rho \) proportional to both order parameter and density of the Fermi particles \( \rho = \nabla \Psi \). In any case, above transformation breaks the supersymmetry, so that the Euler superequation (16) is reduced to the following components (see Appendix):

\[ \frac{\partial \eta}{\partial t} - \nabla^2 \eta = \frac{\partial F}{\partial \eta} + \varphi + \eta \rho, \quad (26) \]

\[ \frac{\partial}{\partial t} (\varphi + \eta \rho) + \nabla^2 \varphi = \frac{\partial^2 F}{\partial \eta^2} (\varphi + \eta \rho) - \frac{\partial^3 F}{\partial \eta^3} (1 - \eta^2) \rho, \quad (27) \]

\[ \frac{\partial}{\partial t} \ln [(1 - \eta) \Psi] - \nabla^2 \Psi = - \frac{\partial^2 F}{\partial \eta^2} \Psi, \quad (28) \]

\[ - \frac{\partial}{\partial t} \ln [(1 + \eta) \overline{\Psi}] - \nabla^2 \overline{\Psi} = - \frac{\partial^2 F}{\partial \eta^2} \overline{\Psi}. \quad (29) \]

In contrast to Eqs. (19) and (20), the pair of equations (25) and (26) becomes non-invariant with respect to the time inversion due to the Bose-Einstein condensate appearance \( \eta \neq 0 \). Combining the equations (25) and (26), one obtains

\[ \rho \frac{\partial}{\partial t} \ln [(1 - \eta^2) \rho] + \nabla j = 0. \quad (30) \]

At steady-state condensation \( j = \text{const} \), Eq. (30) arrives at the relation

\[ n = 1 - \frac{\rho_c}{\rho} \quad (31) \]

where integration constant \( \rho_c \) plays the role of a critical density of fermions and one takes into account the definition (1) according to which \( \eta^2 = n \). The dependence (31) means the density \( n \) of the Bose condensate increases steadily from \( n = 0 \) to \( n = 1 \) with growth of the density \( \rho \) of Fermi particles above a critical value \( \rho_c \).

V. DISCUSSION

Characteristic feature of our consideration consists in making use of the supersymmetry field theory that is based on principle of the minimal superaction

\[ S\{\Phi(r, t)\} := \int L(\Phi(r, t)) dr dt \]
whose values relate to maximal probability

\[ P\{\Phi(r,t)\} \propto \exp(-S\{\Phi(r,t)\}) \]

in the system distribution over the superfields \( \{\Phi\} \). As a result, the governing equations (26) - (29) determine the most probable Bose components \( \eta, \varphi \) and Fermi ones \( \Psi, \bar{\Psi} \). Such a description is differs crucially from the standard picture, within whose framework observable values are determined in terms of averages over sets of quantum states. It is worthwhile to stress though the most probable fields are determined without averaging over quantum fluctuations, however they take into account scattering over statistical states (see related averaging in Eqs. (3)).

According to Eqs. (19) and (20), Fermi-Bose mixtures with unbroken supersymmetry are invariant with respect to inversion of the time arrow. To break this symmetry we transform the superfield (12) to the form (24) whose explicit appearance (25) reveals breaking of above invariance. Then, solution of Eqs. (32) – (34) gives stationary order parameter

\[ T \]

with parameter \( \varepsilon \) determining a moving off a characteristic temperature \( T_c \). Then, solution of Eqs. (32) – (34) arrives at the stationary order parameter

\[ \eta_0 \approx \sqrt{\frac{T_c - T}{2T_{c_0}}} \]

in the \( T_c - T \ll T_{c_0} \) vicinity of the critical temperature

\[ T_c \equiv \left(1 - \sqrt{6\rho_c}\right)T_{c_0}. \] (39)

Here, Eqs. (32) – (34) give the stationary amplitude of fluctuations

\[ \varphi_0 \approx -\left(\rho_c + \sqrt{6\rho_c}\right)\sqrt{\frac{T_c - T}{2T_{c_0}}} \] (40)

and related density of fermions

\[ \rho_0 \approx \rho_c \left(1 + \frac{T_c - T}{2T_{c_0}}\right). \] (41)

According Eqs. (38) and (41), with the temperature decrease near the critical temperature (38) the densities of both fermions and bosons grow linearly, while the fluctuation amplitude (40) takes negative magnitudes varying in square root manner.

Let us consider finally inhomogeneous steady-state system that is described by the relation (34) together with the equations

\[ \nabla^2 \eta = -\left(\varepsilon - \eta^2\right)\eta - (\varphi + \eta \rho), \]

\[ \nabla^2 \varphi = -\left(\varepsilon - 3\eta^2\right)(\varphi + \eta \rho) - 6\left(1 - \eta^2\right)\eta \rho, \] (43)

following from Eqs. (26) and (27) where the Landau free energy (35) is used. Linearization of Eqs. (12) and (24) over \( \eta \) and \( \varphi \) shows the \( \varepsilon \) increase arrives initially (at \( \varepsilon = 0 \)) at loss of the homogeneity in space distribution of the fluctuation amplitude and then (at \( \varepsilon = \varepsilon_c \)) the system becomes unstable with respect to the Bose-Einstein condensation. What about the fermion distribution, it is supposed to be homogeneous due to the equilibrium condition \( j = 0 \) in the continuity equation (30) (consideration of more general steady-state condition \( j = \text{const} \neq 0 \) is out of the scope of our study). From physical point of view, above means that with cooling of the Bose-Fermi mixture characterized by a homogeneous distribution of fermions strong inhomogeneous fluctuations appear spontaneously at a characteristic temperature \( T_{c_0} \), while following temperature decrease arrives at the Bose-Einstein condensation in the critical point \( T_c \) determined by Eq. (38).

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Appendix

With accounting definitions (12) and (15), one obtains l.h.s. of Eq. (16):

\[ \frac{1}{2} [\bar{D}, D] \Phi = (\varphi - \dot{\eta}) - \bar{\theta} \dot{\psi} + \bar{\theta} \theta \dot{\varphi}. \]  \hspace{1cm} (A.1)

Being supersymmetry variational derivative, r.h.s. of this equation is written as the generalization of the expression (5):

\[ \delta F = F' (\eta + \bar{\theta} \Psi + \bar{\psi} \theta + \theta \Psi \dot{\varphi}) - \left[ \nabla^2 \eta + \bar{\theta} (\nabla^2 \Psi) + (\nabla^2 \bar{\Psi}) \theta + \bar{\theta} \theta (\nabla^2 \dot{\varphi}) \right] \]  \hspace{1cm} (A.2)

where the prime denotes the differentiation over related argument. According to the first rule (14), expansion in powers of the addition \( \bar{\theta} \Psi + \bar{\psi} \theta + \theta \Psi \dot{\varphi} \) gives

\[ F' (\Psi) = F' (\eta) \]  \hspace{1cm} (A.3)

Comparison of multipliers standing before \( 1, \bar{\theta}, \theta \) and \( \bar{\theta} \theta \) arrives at the system of equations (17) – (20).

In more complicated case of the transformed superfield (25), the expressions (A.1) and (A.3) take the forms:

\[ \frac{1}{2} [\bar{D}, D] \tilde{\Phi} = [(\varphi - \dot{\eta}) + \eta \rho] \]  \hspace{1cm} (A.4)

\[ \frac{\partial}{\partial t} \left( \bar{\psi} + \bar{\theta} \Psi \right) \]  \hspace{1cm} (A.5)

Comparison of related supersymmetry terms derives to the system (26) – (29).

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