Lp-theory for the exterior Stokes problem with Navier’s type slip-without-friction boundary conditions

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Abstract. In this paper, we consider the stationary Stokes equations in an exterior domain three-dimensional under a slip boundary condition without friction. We set the problem in weighted Sobolev spaces in order to control the behavior at infinity of the solutions. In this work, we try to investigate the existence and uniqueness of the weak solutions related to this problem in Lp-theory when p > 3. Our proof is based on obtaining Inf-Sup conditions that play a fundamental role.

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1. Introduction

In this paper, we are interested in the stationary Stokes equations, and this problem describes the flow of a viscous and incompressible fluid past an obstacle:

\[ -\Delta u + \nabla \pi = f \quad \text{in} \quad \Omega, \]
\[ \text{div} u = 0 \quad \text{in} \quad \Omega. \]

In (1.1), the flow domain \( \Omega \) is a three-dimensional external domain, i.e. the complement of a bounded domain which represents the obstacle, and the unknowns are the velocity of the fluid \( u \) and the pressure \( \pi \) and \( f \) the external forces acting on the fluid. For many years, the standard boundary conditions are the Dirichlet or no-slip boundary condition, which has been the most widely used, given its success in reproducing the standard velocity profiles for incompressible viscous fluid. These latter conditions prescribe that the fluid adheres to the boundary of the domain. More generally, in the case where the obstacles have an approximate limit, the standard conditions are no longer valid (see for example [30]). Therefore, another approach has been introduced, which assumes that due to the roughness of the boundary and the viscosity of the fluid, there is a stagnant fluid layer near the boundary, which allows the fluid to slip (see for example [23]). To overcome the complicated description of the problem, the Navier slip boundary conditions are commonly used.

These conditions, proposed by Navier [28], can be written as

\[ u \cdot n = 0, \quad (Du)n + \alpha u = 0 \quad \text{on} \quad \Gamma, \]

where

\[ Du = \frac{1}{2} (\nabla u + \nabla u^T) \]

denotes the rate-of-strain tensor field, \( \alpha \) is a scalar friction function, \( n \) is the unit normal vector to \( \Gamma \), and the notation \((\cdot)_\tau\) denotes the tangential component of a vector on \( \Gamma \). The first condition in (1.2) is...
the no-penetration condition, and the second condition expresses the fact that the tangential velocity is proportional to the tangential stress.

Now, let us recall some notations related to the boundary condition. First, for any vector field \( u \) on \( \Gamma \), we can write

\[
 u = u_\tau + (u \cdot n)n,
\]

where \( u_\tau \) is the projection of \( u \) on the tangent hyper-plan to \( \Gamma \). Next, for any point \( x \) on \( \Gamma \), one may choose an open neighborhood \( W \) of \( x \) in \( \Gamma \) small enough to allow the existence of two families of \( C^2 \) curves on \( W \) and where the lengths \( s_1 \) and \( s_2 \) along each family of curves are possible system of coordinates. Denoting by \( \tau_1, \tau_2 \) the unit tangent vectors to each family of curves, we have

\[
 u_\tau = (u \cdot \tau_1)\tau_1 + (u \cdot \tau_2)\tau_2.
\]

For a mathematical analysis of the Stokes system satisfying (1.2), the first pioneering paper is due to Solonnikov and Scadilov [31] with \( \alpha = 0 \). More recently, Beirao da Veiga [11] proved existence results for weak and strong solutions in the \( L^2 \) setting. However, we can prove that

\[
 2((Du)n)_\tau = \nabla_\tau(u \cdot n) + \left( \frac{\partial u}{\partial n} \right)_\tau - \sum_{k=1}^{2} \left( u_\tau \cdot \frac{\partial n}{\partial s_k} \right) \tau_k.
\]

On the other hand, we have the following relation:

\[
 \text{curl } u \times n = \nabla_\tau(u \cdot n) - \left( \frac{\partial u}{\partial n} \right)_\tau - \sum_{k=1}^{2} \left( u_\tau \cdot \frac{\partial n}{\partial s_k} \right) \tau_k.
\]

In the particular case \( u \cdot n = 0 \) on \( \Gamma \), which implies that

\[
 2((Du)n)_\tau = -\text{curl } u \times n - \sum_{k=1}^{2} \left( u_\tau \cdot \frac{\partial n}{\partial s_k} \right) \tau_k.
\]

Comparing with (1.2), the following boundary condition

\[
 u \cdot n = 0 \quad \text{and} \quad \text{curl } u \times n = 0 \quad \text{on} \quad \Gamma
\]

is in fact a slip-without-friction Navier boundary condition type for more details the interested reader can also refer to [12,17,27] and references therein.

Stokes problem with the boundary conditions (1.3) was studied by many authors. On the one hand, in the bounded domains, one can refer for instance to [8,13,15,16,26]. On the other hand, for the case for the exterior domains, we can just mention [4,25]. The exterior problem (1.1)–(1.2), where as far as know, we can mention [19,29,33]. However, the Stokes problem set in bounded domains with conditions (1.2) has been well studied by various authors (see for instance [6,10] or [7] for the case \( \alpha = 0 \) and [1,5] for the case \( \alpha \neq 0 \)).

The purpose of this work is to study the existence and the uniqueness of the weak solution for the problem (1.1)–(1.3). Since the flow domain is unbounded, we set the problem in weighted Sobolev spaces in order to control the behavior of functions at infinity. This functional framework allows us to find solutions with different behaviors to infinity (decay or polynomial growth). The study is based on a \( L^p \)-theory, \( p > 3 \). Our proof of solvability the problem (1.1)–(1.3) is based on a variational formulations and the Inf-Sup conditions.

The outline of this paper is as follows. In Sect. 2, we introduce the notations and the functional framework based on weighted Sobolev spaces. We shall precise in which sense, the Navier’s type slip-without-friction boundary conditions (1.3) are taken, we finish this section by giving a result concerning the Laplace problem with Neumann boundary conditions with data \( f \) belongs to \( L^p(\Omega) \), we got the existence of solutions in \( W^{-1,p}_0(\Omega) \). In Sect. 3, we recall one result about the vector potential problem (see [24]) and we prove the Inf-Sup conditions, which plays a crucial role in the existence and uniqueness of solutions.
Finally, in Sect. 4, we conclude with the main result of this paper related to well-posedness of the Stokes problem \((1.1)-(1.3)\). We prove the existence and the uniqueness of weak solutions, when \(p > 3\).

2. Notations and preliminaries

2.1. Notations

We recall the main notations and results, concerning the weighted Sobolev spaces, which we shall use later on. In what follows, \(p\) is a real number in the interval \([1, \infty[\). The dual exponent of \(p\) denoted by \(p'\) is given by the following relation:

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

We will use bold characters for vector and matrix fields. A point in \(\mathbb{R}^3\) is denoted by \(x = (x_1, x_2, x_3)\) and its distance to the origin by

\[
r = |x| = \left( x_1^2 + x_2^2 + x_3^2 \right)^{1/2}.
\]

For any multi-index \(\lambda \in \mathbb{N}^3\), we denote by \(\partial^\lambda\) the differential operator of order \(|\lambda|\),

\[
\partial^\lambda = \frac{\partial^{|\lambda|}}{\partial_1^{\lambda_1} \partial_2^{\lambda_2} \partial_3^{\lambda_3}}, \quad |\lambda| = \lambda_1 + \lambda_2 + \lambda_3.
\]

Let \(\Omega'\) be a bounded connected open set in \(\mathbb{R}^3\) with boundary \(\partial \Omega' = \Gamma\) of class \(C^{1,1}\) and let \(\Omega\) its complement \(\text{i.e.} \ \Omega = \mathbb{R}^3 \setminus \overline{\Omega'}\). In this work, we shall also denote by \(B_R\) the open ball of radius \(R > 0\) centred at the origin with boundary \(\partial B_R\). In particular, since \(\Omega'\) is bounded, we can find some \(R_0\), such that \(\Omega' \subset B_{R_0}\) and we introduce, for any \(R \geq R_0\), the set

\[
\Omega_R = \Omega \cap B_R.
\]

We denote by \(\mathcal{D}(\Omega)\) the space of \(C^\infty\) functions with compact support in \(\Omega\), \(\mathcal{D}(\overline{\Omega})\) the restriction to \(\Omega\) of functions belonging to \(\mathcal{D}(\mathbb{R}^3)\). We recall that \(\mathcal{D}'(\Omega)\) is the well-known space of distributions defined on \(\Omega\). We recall that \(L^p(\Omega)\) is the well-known Lebesgue real space and for \(m \geq 1\), we recall that \(W^{m,p}(\Omega)\) is the classical Sobolev space. We shall write \(u \in W^{m,p}_{\text{loc}}(\Omega)\) to mean that \(u \in W^{m,p}(\mathcal{O})\), for any bounded domain \(\mathcal{O} \subset \Omega\). We denote by \(|s|\) the integer part of \(s\). For any \(k \in \mathbb{Z}\), \(\mathcal{D}_k\) stands for the space of polynomials of degree less than or equal to \(k\) and \(\mathcal{D}_k^\Delta\) the harmonic polynomials of \(\mathcal{D}_k\). If \(k\) is a negative integer, we set by convention \(\mathcal{D}_k = \{0\}\).

Given a Banach space \(X\), with dual space \(X'\) and a closed subspace \(Y\) of \(X\), we denote by \(X' \perp Y\) the subspace of \(X'\) orthogonal to \(Y\), i.e.

\[
X' \perp Y = \{ f \in X'; \langle f, v \rangle = 0 \ \forall \ v \in Y \} = (X/Y)'.
\]

The space \(X' \perp Y\) is also called the polar space of \(Y\) in \(X'\). Finally, as usual, \(C > 0\) denotes a generic constant the value of which may change from line to line and even at the same line.

2.2. Weighted Sobolev spaces

In order to control the behavior at infinity of our functions and distributions we use for basic weights the quantity \(\rho(x) = (1 + r^2)^{1/2}\) which is equivalent to \(r\) at infinity, and to one on any bounded subset of \(\mathbb{R}^3\). For \(\alpha \in \mathbb{Z}\), we introduce

\[
W^{0,p}_\alpha(\Omega) = \left\{ u \in \mathcal{D}'(\Omega), \rho^\alpha u \in L^p(\Omega) \right\},
\]
which is a Banach space equipped with the norm:
\[ \|u\|_{W_0^\alpha} = \| \rho^\alpha u \|_{L^p(\Omega)}. \]
For any non-negative integers \( m \), real numbers \( p > 1 \) and \( \alpha \in \mathbb{Z} \). We define the weighted Sobolev space for \( 3/p + \alpha \notin \{1, \ldots, m\} \):
\[ W_\alpha^{m,p}(\Omega) = \left\{ u \in D'(\Omega); \forall \lambda \in \mathbb{N}^3 : 0 \leq |\lambda| \leq m, \rho^{\alpha - m + |\lambda|} \partial^\lambda u \in L^p(\Omega) \right\}. \]
It is a reflexive Banach space equipped with the norm:
\[ \|u\|_{W_\alpha^{m,p}(\Omega)} = \left( \sum_{0 \leq |\lambda| \leq m} \| \rho^{\alpha - m + |\lambda|} \partial^\lambda u \|_{L^p(\Omega)}^p \right)^{1/p}. \]
We define the semi-norm
\[ |u|_{W_\alpha^{m,p}(\Omega)} = \left( \sum_{|\lambda| = m} \| \rho^\alpha \partial^\lambda u \|_{L^p(\Omega)}^p \right)^{1/p}. \]
Let us give some examples of such space that will be often used in the remaining of the thesis.
1. For \( m = 1 \), we have
   \[ W_{\alpha+1}^{1,p}(\Omega) := \left\{ u \in D'(\Omega); \rho^{\alpha - 1} u \in L^p(\Omega), \rho^\alpha \nabla u \in L^p(\Omega) \right\} \]
2. For \( m = 2 \), we have
   \[ W_{\alpha+1}^{2,p}(\Omega) := \left\{ u \in W_{\alpha}^{1,p}(\Omega), \rho^{\alpha + 1} \nabla^2 u \in L^p(\Omega) \right\}. \]
Now, we present some basic properties on weighted Sobolev spaces. For more details, the reader can refer to [2,3,22].

**Properties 2.1.**
1. The space \( D(\Omega) \) is dense in \( W_\alpha^{m,p}(\Omega) \).
2. For any \( m \in \mathbb{N}^* \) and \( 3/p + \alpha \notin 1 \), we have the following continuous embedding:
   \[ W_\alpha^{m,p}(\Omega) \hookrightarrow W_{\alpha-1}^{m,p}(\Omega). \] (2.1)
3. For any \( \alpha, m \in \mathbb{Z} \) and for any \( \lambda \in \mathbb{N}^3 \), the mapping
   \[ u \in W_\alpha^{m,p}(\Omega) \rightarrow \partial^\lambda u \in W_\alpha^{m-|\lambda|,p}(\Omega) \] (2.2)
   is continuous.

The space \( W_\alpha^{m,p}(\Omega) \) sometimes contains some polynomial functions. Let \( j \) be defined as follow:
\[ j = \begin{cases} 
    m - (3/p + \alpha) & \text{if } 3/p + \alpha \notin \mathbb{Z}^- , \\
    m - 3/p - \alpha - 1 & \text{otherwise}. 
\end{cases} \] (2.3)
Then \( \mathcal{P}_j \) is the space of all polynomials included in \( W_\alpha^{m,p}(\Omega) \).
The norm of the quotient space \( W_\alpha^{m,p}(\Omega)/\mathcal{P}_j \) is given by:
\[ ||u||_{W_\alpha^{m,p}(\Omega)/\mathcal{P}_j} = \inf_{\mu \in \mathcal{P}_j} ||u + \mu||_{W_\alpha^{m,p}(\Omega)}. \]
All the local properties of \( W_\alpha^{m,p}(\Omega) \) coincide with those of the corresponding classical Sobolev spaces \( W^{m,p}(\Omega) \). Hence, it also satisfies the usual trace theorems on the boundary \( \Gamma \). Therefore, we can define the space
\[ \tilde{W}_\alpha^{m,p}(\Omega) = \{ u \in W_\alpha^{m,p}(\Omega), \gamma_0 u = 0, \gamma_1 u = 0, \cdots, \gamma_{m-1} u = 0 \}. \]
Note that when $\Omega = \mathbb{R}^3$, we have $\dot{W}_m^p(\mathbb{R}^3) = W_{m,p}^p(\mathbb{R}^3)$. The space $\mathcal{D}(\Omega)$ is dense in $\dot{W}_m^p(\Omega)$. Therefore, the dual space of $\dot{W}_m^p(\Omega)$, denoting by $\dot{W}_{-m,p}^p(\Omega)$, is a space of distributions with the norm

$$
\| u \|_{\dot{W}_{-m,p}^p(\Omega)} = \sup_{v \in \dot{W}_{-m,p}^p(\Omega)} \frac{\langle u, v \rangle_{\dot{W}_{-m,p}^p(\Omega) \times W_{m,p}^p(\Omega)}}{\| v \|_{W_{m,p}^p(\Omega)}}.
$$

### 2.3. Functional spaces and Trace Results

The purpose of this section is to introduce some weighted Sobolev spaces that are specific for the study of the Stokes problem (1.1) with the Navier boundary conditions (1.3). Let us first introduce some notations. For any vector fields $u$ and $v$ of $\mathbb{R}^3$, we define

$$
u \times \nu = (u_2v_3 - v_2u_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)^T$$

and

$$\text{curl } \nu = \nabla \times \nu.$$ 

We note that the vector-valued Laplace operator of a vector field $\nu$ is equivalently defined by

$$\Delta \nu = \nabla \text{div } \nu - \text{curl curl } \nu. \quad (2.4)$$

We start by introducing the following spaces for $\alpha \in \mathbb{Z}$ and $1 < p < \infty$.

**Definition 2.2.** The space $H^p_\alpha(\text{curl}, \Omega)$ is defined by:

$$H^p_\alpha(\text{curl}, \Omega) = \left\{ \nu \in W^{0,p}_\alpha(\Omega); \text{curl } \nu \in W^{0,p}_{\alpha+1}(\Omega) \right\},$$

and is provided with the norm:

$$\| \nu \|_{H^p_\alpha(\text{curl}, \Omega)} = \left( \| \nu \|_{W^{0,p}_\alpha(\Omega)}^p + \| \text{curl } \nu \|_{W^{0,p}_{\alpha+1}(\Omega)}^p \right)^{\frac{1}{p}}.$$

The space $H^p_\alpha(\text{div}, \Omega)$ is defined by:

$$H^p_\alpha(\text{div}, \Omega) = \left\{ \nu \in W^{0,p}_\alpha(\Omega); \text{div } \nu \in W^{0,p}_{\alpha+1}(\Omega) \right\},$$

and is provided with the norm:

$$\| \nu \|_{H^p_\alpha(\text{div}, \Omega)} = \left( \| \nu \|_{W^{0,p}_\alpha(\Omega)}^p + \| \text{div } \nu \|_{W^{0,p}_{\alpha+1}(\Omega)}^p \right)^{\frac{1}{p}}.$$

Finally, we set

$$X^p_\alpha(\Omega) = H^p_\alpha(\text{curl}, \Omega) \cap H^p_\alpha(\text{div}, \Omega).$$

It is provided with the norm

$$X^p_\alpha(\Omega) = \left( \| \nu \|_{W^{0,p}_\alpha(\Omega)}^p + \| \text{div } \nu \|_{W^{0,p}_{\alpha+1}(\Omega)}^p + \| \text{curl } \nu \|_{W^{0,p}_{\alpha+1}(\Omega)}^p \right)^{\frac{1}{p}}.$$

These definitions will be also used when $\Omega$ is replaced by $\mathbb{R}^3$.

Note that $\mathcal{D}(\overline{\Omega})$ is dense in $H^p_\alpha(\text{div}, \Omega)$ and in $H^p_\alpha(\text{curl}, \Omega)$ and so in $X^p_\alpha(\Omega)$. For the proof, one can use the same arguments than for the proof of the density of $\mathcal{D}(\overline{\Omega})$ in $W_{m,p}^p(\Omega)$ (see [21, 22]). If $\nu$ belongs to $H^p_{\alpha}(\text{div}, \Omega)$, then $\nu$ has normal trace $\nu \cdot n$ in $W^{-1/p,p}(\Gamma)$, where $W^{-1/p,p}(\Gamma)$ denotes the dual space of $W^{1/p,p'}(\Gamma)$. By the same way, if $\nu$ belongs to $H^p_\alpha(\text{curl}, \Omega)$, then $\nu$ has a tangential trace $\nu \times n$ that
exists a constant $C > 0$, such that
\[ \forall v \in H^p_\alpha(\div, \Omega), \quad ||v \cdot n||_{W^{-1/p},p(\Gamma)} \leq C||v||_{H^p_\alpha(\div, \Omega)}, \] (2.5)
\[ \forall v \in H^p_\alpha(\curl, \Omega), \quad ||v \times n||_{W^{-1/p},p(\Gamma)} \leq C||v||_{H^p_\alpha(\curl, \Omega)}, \] (2.6)
and the following Green’s formulas holds: For any $v \in H^p_\alpha(\div, \Omega)$ and $\varphi \in W^{-1/p}_\alpha(\Omega)$, we have
\[ \langle v \cdot n, \varphi \rangle_\Gamma = \int_\Omega v \cdot \nabla \varphi \, dx + \int_\Omega \varphi \, \div v \, dx. \] (2.7)
For any $v \in H^p_\alpha(\curl, \Omega)$ and $\varphi \in W^{-1/p}_\alpha(\Omega)$, we have
\[ \langle v \times n, \varphi \rangle_\Gamma = \int_\Omega v \cdot \curl \varphi \, dx - \int_\Omega \curl v \cdot \varphi \, dx, \] (2.8)
where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between $W^{-1/p,p}(\Gamma)$ and $W^{1/p,p}(\Gamma)$.

The closures of $\mathcal{S}(\Omega)$ in $H^p_\alpha(\div, \Omega)$ and in $H^p_\alpha(\curl, \Omega)$ are denoted, respectively, by $\hat{H}^p_\alpha(\curl, \Omega)$ and $\hat{H}^p_\alpha(\div, \Omega)$ and can be characterized, respectively, by:
\[ \hat{H}^p_\alpha(\curl, \Omega) = \{ v \in H^p_\alpha(\curl, \Omega); \ v \times n = 0 \text{ on } \Gamma \}, \]
\[ \hat{H}^p_\alpha(\div, \Omega) = \{ v \in H^p_\alpha(\div, \Omega); \ v \cdot n = 0 \text{ on } \Gamma \}. \]

For $1 < p < \infty$ and $\alpha \in \mathbb{Z}$, we denote by $[\hat{H}^p_\alpha(\div, \Omega)]'$ and $[\hat{H}^p_\alpha(\curl, \Omega)]'$ the dual spaces of $\hat{H}^p_\alpha(\curl, \Omega)$ and $\hat{H}^p_\alpha(\div, \Omega)$, respectively. We can characterize these spaces as it is stated in the following proposition.

**Proposition 2.3.**

1. A distribution $f$ belongs to $[\hat{H}^p_\alpha(\div, \Omega)]'$ if and only if there exist functions $\psi \in W^{0,p'}_{-\alpha}(\Omega)$ and $\chi \in W^{0,p'}_{-\alpha-1}(\Omega)$, such that $f = \psi + \nabla \chi$. Moreover
\[ ||\psi||_{W^{0,p'}_{-\alpha}(\Omega)} + ||\chi||_{W^{0,p'}_{-\alpha-1}(\Omega)} \leq C||f||_{[\hat{H}^p_\alpha(\div, \Omega)]'}. \] (2.9)

2. A distribution $f$ belongs to $[\hat{H}^p_\alpha(\curl, \Omega)]'$ if and only if there exist functions $\psi \in W^{0,p'}_{-\alpha}(\Omega)$ and $\chi \in W^{0,p'}_{-\alpha-1}(\Omega)$, such that $f = \psi + \curl \chi$. Moreover
\[ ||\psi||_{W^{0,p'}_{-\alpha}(\Omega)} + ||\chi||_{W^{0,p'}_{-\alpha-1}(\Omega)} \leq C||f||_{[\hat{H}^p_\alpha(\curl, \Omega)]'}. \] (2.10)

The proof of Proposition 2.3 can be found in [18, Proposition II.2]. As a consequence of Proposition 2.3 and the imbedding (2.1) we have, for any $\alpha \in \mathbb{Z}$ and $1 < p < \infty$, the following imbeddings
\[ [\hat{H}^p_\alpha(\div, \Omega)]' \subset W^{-1/p'}_{-\alpha-1}(\Omega) \] (2.11)
and
\[ [\hat{H}^p_\alpha(\curl, \Omega)]' \subset W^{-1/p'}_{-\alpha-1}(\Omega). \] (2.12)

Let us consider the following space:
\[ E^p(\Omega) = \{ v \in W^{1,p}_0(\Omega); \ \Delta v \in [\hat{H}^{p'}_{-1}(\div, \Omega)]' \}, \]
equipped with the following norm
\[ ||v||_{E^p(\Omega)} = ||v||_{W^{1,p}_0(\Omega)} + ||\Delta v||_{[\hat{H}^{p'}_{-1}(\div, \Omega)]'}. \]

We have the following preliminary result.

**Lemma 2.4.** The space $\mathcal{D}(\Omega)$ is dense in $E^p(\Omega)$.
Proof. The proof is quite similar to the proof when \( p=2 \) (see [4]). Let \( P \) be a continuous linear mapping from \( W^{1,p}_0(\Omega) \) to \( W^{1,p}(\mathbb{R}^3) \), such that \( P\mathbf{v}|_{\Omega} = \mathbf{v} \) and let \( \ell \in (E^p(\Omega))^l \), such that for any \( \mathbf{v} \in \mathcal{D}(\Omega) \), we have \( \langle \ell, \mathbf{v} \rangle = 0 \). We want to prove that \( \ell = 0 \) on \( E^p(\Omega) \). Then there exists \((f, g) \in W^{-1}((\mathbb{R}^3) \times \dot{H}^1_p(\text{div}, \Omega)) \) such that: for any \( \mathbf{v} \in E^p(\Omega) \),

\[
\langle \ell, \mathbf{v} \rangle = \langle f, P\mathbf{v} \rangle_{W^{-1}((\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3))} + \langle \Delta \mathbf{v}, g \rangle_{\dot{H}^1_p(\text{div}, \Omega)' \times \dot{H}^1_p(\text{div}, \Omega)}.
\]

Observe that we can easily extend by zero the function \( g \) in such a way that \( \tilde{g} \in H^1_p(\text{div}, \mathbb{R}^3) \). Now we take \( \varphi \in \mathcal{D}(\mathbb{R}^3) \). Then we have by assumption that:

\[
\langle f, \varphi \rangle_{W^{-1}((\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3))} + \int_{\mathbb{R}^3} \tilde{g} \cdot \Delta \varphi dx = 0,
\]

because \( \langle f, \varphi \rangle = \langle f, P\mathbf{v} \rangle \) where \( \mathbf{v} = \varphi|_{\Omega} \). Thus we have \( f + \Delta \tilde{g} = 0 \) in \( \mathcal{D}'(\mathbb{R}^3) \). Then we can deduce that \( \Delta \tilde{g} = -f \in W^{-1,p}_0(\mathbb{R}^3) \) and due to [3, Theorem 1.3], there exists a unique \( \lambda \in W^{1,p}_0(\mathbb{R}^3) \) such that \( \Delta \lambda = \Delta \tilde{g} \). Thus the harmonic function \( \lambda - \tilde{g} \) belonging to \( W^{0,p}_0(\mathbb{R}^3) \) is necessarily equal to zero. Since \( g \in W^{1,p}_0(\Omega) \) and \( \tilde{g} \in W^{1,p}_0(\mathbb{R}^3) \), we deduce that \( g \in W^{1,p}_0(\Omega) \). As \( \mathcal{D}(\Omega) \) is dense in \( W^{1,p}_0(\Omega) \), there exists a sequence \( g_k \in \mathcal{D}(\Omega) \) such that \( g_k \rightarrow g \) in \( W^{1,p}_0(\Omega) \), when \( k \rightarrow \infty \). Then \( \nabla \cdot g_k \rightarrow \nabla \cdot g \) in \( L^p(\Omega) \). Since \( W^{1,p}_0(\Omega) \) is imbedded in \( W^{-1,p}_0(\Omega) \), we deduce that \( g_k \rightarrow g \) in \( H^{-1}_p(\text{div}, \Omega) \). Now, we consider \( \mathbf{v} \in E^p(\Omega) \) and we want to prove that \( \langle \ell, \mathbf{v} \rangle = 0 \). Observe that:

\[
\langle \ell, \mathbf{v} \rangle = \langle \Delta \tilde{g}, P\mathbf{v} \rangle_{W^{-1}((\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3))} + \langle \Delta \mathbf{v}, g \rangle_{\dot{H}^1_p(\text{div}, \Omega)' \times \dot{H}^1_p(\text{div}, \Omega)}
\]

\[
= \lim_{k \rightarrow \infty} \langle \Delta g_k \cdot \mathbf{v} \rangle_{W^{-1}((\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3))} + \langle \Delta \mathbf{v}, g_k \rangle_{\dot{H}^1_p(\text{div}, \Omega)' \times \dot{H}^1_p(\text{div}, \Omega)}
\]

\[
= \lim_{k \rightarrow \infty} \langle \Delta g_k \cdot \mathbf{v} \rangle_{W^{-1}((\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3))} + \langle \mathbf{v} \cdot \Delta g_k \rangle_{\dot{H}^1_p(\text{div}, \Omega)} = 0.
\]

\[\square\]

Next, we introduce the following space:

\[
V^p_{0,T}(\Omega) = \left\{ \mathbf{z} \in X^{p}_{0,T}(\Omega) ; \text{div} \mathbf{z} = 0 \text{ in } \Omega \right\},
\]

where

\[
X^{p}_{0,T}(\Omega) = \{ \mathbf{z} \in X^{p}_{0}(\Omega) ; \mathbf{z} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}.
\]

As a consequence, we have the following result.

**Lemma 2.5.** The linear mapping \( \gamma : \mathbf{v} \rightarrow \text{curl} \mathbf{v} \big|_{\Gamma} \times \mathbf{n} \) defined on \( \mathcal{D}(\overline{\Omega}) \) can be extended to a linear continuous mapping

\[
\gamma : E^p(\Omega) \rightarrow W^{-1/p,p}(\Gamma).
\]

Moreover, we have the Green formula: for any \( \mathbf{v} \in E^p(\Omega) \) and any \( \varphi \in V^{p'}_{0,T}(\Omega) \),

\[
\langle -\Delta \mathbf{v}, \varphi \rangle_{\Omega} = \int_{\Omega} \text{curl} \mathbf{v} \cdot \text{curl} \varphi \, dx + \langle \text{curl} \mathbf{v} \times \mathbf{n}, \varphi \rangle_{\Gamma}
\]

(2.13)

Where \( \langle ., . \rangle_{\Gamma} \) denotes the duality between \( W^{-1/p,p}(\Gamma) \) and \( W^{1/p,p'}(\Gamma) \) and \( \langle ., . \rangle_{\Omega} \) denotes the duality between \( \dot{H}^1_p(\text{div}, \Omega)' \) and \( \dot{H}^1_p(\text{div}, \Omega) \).
Proof. Let \( v \in \mathcal{D}(\overline{\Omega}) \). Observe that if \( \varphi \in W^{1,p'}_0(\Omega) \) such that \( \varphi \cdot n = 0 \) on \( \Gamma \) we deduce that \( \varphi \in X_{p'}(\Omega) \), then (2.13) holds for such \( \varphi \).

For every \( \mu \in W^{1/p,p'}(\Gamma) \), then there exists \( \varphi \in W^{1,p'}_0(\Omega) \) such that \( \varphi = \mu \) on \( \Gamma \) and that \( \text{div} \varphi = 0 \) with

\[
\| \varphi \|_{W^{1,p'}_0(\Omega)} \leq C\| \mu \|_{W^{1/p,p'}(\Gamma)} \leq C\| \mu \|_{W^{1/p,p'}(\Gamma)}.
\]

(2.14)

As a consequence, using (2.13) we have

\[
\| \text{curl} v \times n, \mu \|_{\Gamma} \leq C\| v \|_{E^p(\Omega)} \| \mu \|_{W^{1/p,p'}(\Gamma)}.
\]

Thus,

\[
\| \text{curl} v \times n \|_{W^{-1/p,p}(\Gamma)} \leq C\| v \|_{E^p(\Omega)}.
\]

We deduce that the linear mapping \( \gamma \) is continuous for the norm \( E^p(\Omega) \). Since \( \mathcal{D}(\overline{\Omega}) \) is dense in \( E^p(\Omega) \), \( \gamma \) can be extended to by continuity to \( \gamma \in \mathcal{L}(E^p(\Omega), W^{-1/p,p}(\Gamma)) \) and formula (2.13) holds for all \( v \in E^p(\Omega) \) and \( \varphi \in W^{1,p'}_0(\Omega) \) such that \( \text{div} \varphi = 0 \) in \( \Omega \) and \( \varphi \cdot n = 0 \) on \( \Gamma \).

\[\square\]

2.4. The Laplace problem

This section is devoted to recall the solution of the Laplace equations in \( \mathbb{R}^3 \), and we give a result concerning the Laplace problem with Neumann boundary conditions. The result that we state below was proved by Amrouche et all in [2].

**Theorem 2.6.** Assume that \( 3/p + \alpha \neq 1 \) and \( 3/p' - \alpha \neq 1 \), then for any integer \( m \geq 1 \), the following Laplace operator is an isomorphism:

\[
\Delta : W^{1+m,p}_{\alpha+m}([\mathbb{R}^3]/P_{[1-3/p-\alpha]}) \rightarrow W^{-1+m,p}_{\alpha+m}([\mathbb{R}^3] \perp P_{[1-3/p'+\alpha]})
\]

(2.15)

Now, we are interested into the following Neumann problem:

\[
-\Delta u = f \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma.
\]

(2.16)

J. Giroire in [21] studied the problem (2.16) in the Hilbert framework. In [24], the authors investigated the harmonic Neumann problem in \( L^p \) theory, for the exterior domain with boundary of class \( C^{1,1} \), note that these results have been also proved by Specovius-Neugebauer [32] with boundary of class at least \( C^2 \). C. Amrouche, V. Girault and J. Giroire in [3] studied the problem (2.16) with data \( f \) belongs to \( W^{-1,p}_0(\Omega) \cap L^p(\Omega) \), they got the existence of solutions in \( W^{1,p}_0(\Omega) \). In this work, we give a result concerning the Laplace problem with Neumann boundary conditions with data \( f \) belongs to \( L^p(\Omega) \), we got the existence of solutions in \( W^{1,p}_0(\Omega) \).

Let us first introduce the kernel of the Laplace operator with Neumann boundary condition. For any integer \( \alpha \in \mathbb{Z} \) and \( 1 < p < \infty \)

\[
\mathcal{N}_{p,\alpha} = \left\{ w \in W^{1,p}_\alpha(\Omega); \Delta w = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma \right\}.
\]

The next theorem states an existence, uniqueness and regularity result for problem (2.16).

**Theorem 2.7.** Let \( 1 < p < \infty \) be any real number and assume that \( \Gamma \) is of class \( C^{1,1} \) if \( p \neq 2 \). For any \( f \) in \( L^p(\Omega) \) and \( g \) in \( W^{-1/p,p}(\Gamma) \). Then, the problem (2.16) has a solution \( u \in W^{1,p}_0(\Omega) \) unique up to element of \( \mathcal{N}_{p,\alpha}(\Omega) \) and we have the following estimate:

\[
\| u \|_{W^{1,p}_0(\Omega)/\mathcal{N}_{p,\alpha}(\Omega)} \leq C(\| f \|_{L^p(\Omega)} + \| g \|_{W^{-1/p,p}(\Gamma)}).
\]

(2.17)
If in addition, \( f \in W_1^{1,p}(\Omega) \) and \( g \in W^{1/p',p}(\Gamma) \), the solution \( u \) of problem (2.16) belongs to \( W_0^{2,p}(\Omega) \) and satisfies

\[
\|u\|_{W_0^{2,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1/p',p}(\Omega)}) \tag{2.18}
\]

Proof. Let us extend \( f \) by zero in \( \Omega' \) and let \( \tilde{f} \) denote the extended function. Then \( \tilde{f} \) belongs to \( L^p(\mathbb{R}^3) \).

Using Theorem 2.6, there exists a unique function \( \tilde{v} \in W_0^{2,p}(\mathbb{R}^3) / \mathcal{P}_{(2-3/p)} \) such that

\[
-\Delta \tilde{v} = \tilde{f} \quad \text{in} \quad \mathbb{R}^3. \tag{2.19}
\]

Then \( \nabla \tilde{v} \cdot n \) belongs to \( W^{-1/2-p,p}(\Gamma) \). It follows from [24, Theorem 3.12], that the following problem:

\[
\Delta w = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \nabla w \cdot n = g - \nabla \tilde{v} \cdot n \quad \text{on} \quad \Gamma, \tag{2.20}
\]

has a solution \( w \in W_0^{1,p}(\Omega) \) unique up to element of \( \mathcal{N}_{p-1}(\Omega) \). Thus \( u = \tilde{v}|_{\Omega} + w \in W_0^{1,p}(\overline{\Omega}) \) is the required solution of (2.16). The uniqueness follows immediately from [24, Proposition 3.11].

Now, suppose that \( g \) belongs to \( W^{1/p',p}(\Gamma) \). The aim is to prove that \( u \) belongs to \( W_0^{2,p}(\Omega) \). To that end, let us introduce the following partition of unity:

\[
\varphi, \psi \in \mathcal{C}^\infty(\mathbb{R}^3), \quad 0 \leq \varphi, \psi \leq 1, \quad \varphi + \psi = 1 \quad \text{in} \quad \mathbb{R}^3,
\]

\[
\varphi = 1 \quad \text{in} \quad B_R, \quad \text{supp} \varphi \subset B_{R+1}.
\]

Let \( P \) be a continuous linear mapping from \( W_0^{1,p}(\Omega) \) to \( W_0^{1,p}(\mathbb{R}^3) \), such that \( Pu = \tilde{u} \). Then \( \tilde{u} \) belongs to \( W_0^{1,p}(\mathbb{R}^3) \) and can be written as:

\[
\tilde{u} = \varphi \tilde{u} + \psi \tilde{u}.
\]

Next, one can easily observe that \( \tilde{u} \) satisfies the following problem:

\[
-\Delta \psi \tilde{u} = f_1 \quad \text{in} \quad \mathbb{R}^3, \tag{2.21}
\]

with

\[
f_1 = \tilde{f} \psi - (2\nabla \tilde{u} \nabla \psi + \tilde{u} \Delta \psi).
\]

Owing to the support of \( \psi \), \( f_1 \) has the same regularity as \( f \) and so belongs to \( L^p(\mathbb{R}^3) \). It follows from Theorem 2.6 that there exists \( z \in W_0^{2,p}(\mathbb{R}^3) \) such that \( -\Delta z = f_1 \) in \( \mathbb{R}^3 \). This implies that \( \psi \tilde{f} - z \) is a harmonic tempered distribution and therefore a harmonic polynomial that belongs to \( \mathcal{P}_{(2-3/p)} \). The fact that \( \mathcal{P}_{(2-3/p)} \subset W_0^{2,p}(\mathbb{R}^3) \) yields that \( \psi \tilde{u} \) belongs to \( W_0^{2,p}(\mathbb{R}^3) \). In particular, we have \( \psi \tilde{u} = u \) outside \( B_{R+1} \), so the restriction of \( u \) to \( \partial B_{R+1} \) belongs to \( W^{2-1/p,p}(\partial B_{R+1}) \). Therefore, \( \varphi \tilde{u} \in W^{1,p}(\Omega_{R+1}) \) satisfies the following problem:

\[
\begin{cases}
-\Delta \varphi u = f_2 & \text{in} \quad \Omega_{R+1}, \\
\nabla \varphi u \cdot n = g & \text{on} \quad \Gamma, \\
\varphi u = \psi \tilde{u} & \text{on} \quad \partial B_{R+1},
\end{cases} \tag{2.22}
\]

where \( f_2 \in L^p(\Omega_{R+1}) \) have similar expression as \( f_1 \) with \( \psi \) replaced by \( \varphi \). According [Remark 3.2, [3]], \( \varphi u \in W^{2,p}(\Omega_{R+1}) \), which in turn shows that \( \varphi \tilde{u} \) also belongs to \( W^{2,p}(\Omega_{R+1}) \). This implies that \( u \in W_0^{2,p}(\Omega) \). \(\square\)
3. Potential vector and Inf-Sup condition

In this section, we recall a result concerning the existence a vector potential, and we give the proof of the Inf-Sup condition, which plays a crucial role in the existence and uniqueness of solutions. Let us recall the abstract setting of Babuška-Brezzi’s Theorem (see Babuška [9] and Brezzi [14]).

**Theorem 3.1.** Let $X$ and $M$ be two reflexive Banach spaces and $X'$ and $M'$ their dual spaces. Let $a$ be the continuous bilinear form defined on $X \times M$, let $A \in \mathcal{L}(X; M')$ and $A' \in \mathcal{L}(M; X')$ be the operators defined by

$$\forall v \in X, \forall w \in M, a(v, w) = < A v, w > = < v, A' w >$$

and $V = \text{Ker } A$. The following statements are equivalent:

i) There exists $\beta > 0$ such that

$$\inf_{w \in M \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{a(v, w)}{\|v\|_X \|w\|_M} \geq \beta.$$  \hspace{1cm} (3.1)

ii) The operator $A : X/V \mapsto M'$ is an isomorphism and $1/\beta$ is the continuity constant of $A^{-1}$.

iii) The operator $A' : M \mapsto X' \perp V$ is an isomorphism and $1/\beta$ is the continuity constant of $(A')^{-1}$.

**Remark 3.2.** As consequence, if the Inf-Sup condition (3.1) is satisfied, then we have the following properties:

i) If $V = \{0\}$, then for any $f \in X'$, there exists a unique $w \in M$ such that

$$\forall v \in X, a(v, w) = < f, v > \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta} \|f\|_{X'}.$$  \hspace{1cm} (3.2)

ii) If $V \neq \{0\}$, then for any $f \in X'$, satisfying the compatibility condition: $\forall v \in V, < f, v > = 0$, there exists a unique $w \in M$ such that (3.2).

iii) For any $g \in M'$, $\exists v \in X$, unique up an additive element of $V$, such that:

$$\forall w \in M, a(v, w) = < g, w > \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta} \|g\|_{M'}.$$  

Now, we recall a result concerning the potential vector provided in [24].

**Theorem 3.3.** Assume that $1 < p < 3/2$. Let $z$ in $H^p_0(\text{div}, \Omega)$ satisfying

$$\text{div } z = 0 \quad \text{in } \Omega \quad \text{and} \quad (z \cdot n, 1)_\Gamma = 0.$$  

Then there exists a unique vector potential $\psi$ in $W^{1,p}_0(\Omega)$ such that:

$$z = \text{curl } \psi, \quad \text{div } \psi = 0 \quad \text{in } \Omega \quad \text{and} \quad \psi \cdot n = 0 \quad \text{on } \Gamma.$$  \hspace{1cm} (3.3)

In addition we have the following estimate:

$$\|\psi\|_{W^{1,p}_0(\Omega)} \leq C \|z\|_{L^p(\Omega)}. \hspace{1cm} (3.4)$$

We introduce the following space:

$$Z^p_\Gamma(\Omega) = \left\{ v \in W^{1,p}_\Gamma(\Omega) : \text{div } v \in L^p(\Omega), \text{ curl } v \in L^p(\Omega), \quad v \cdot n = 0 \quad \text{on } \Gamma \right\}.$$  

**Theorem 3.4.** Let $1 < p < 3/2$. There exists a constant $C > 0$ such that for all $\varphi \in Z^p_\Gamma(\Omega)$, we have

$$\|\varphi\|_{W^{1,p}_0(\Omega)} \leq C \left( \|\text{div } \varphi\|_{L^p(\Omega)} + \|\text{curl } \varphi\|_{L^p(\Omega)} \right) \hspace{1cm} (3.5)$$
Proof. Let $\varphi$ in $Z^p_0(\Omega)$. It follows from [24, Theorem 5.1] that $Z^p_0(\Omega)$ is continuously imbedded in $W^{1,p}_0(\Omega)$ and so $\varphi$ belongs to $W^{1,p}_0(\Omega)$. Observe that for any $\chi \in W^{1,p}_0(\Omega)$, the following Green formula holds:

$$\langle \text{curl} \varphi \cdot n, \chi \rangle_{\Gamma} = \int_\Omega \text{curl} \varphi \cdot \nabla \chi \, dx. \tag{3.6}$$

Because $1 < p < 3/2$, the constants belong to $W^{1,p}_0(\Omega)$, then (3.6) is still valid for $\chi = 1$ and thus we have:

$$\langle \text{curl} \varphi \cdot n, 1 \rangle_{\Gamma} = 0.$$

Using Theorem 3.3, we prove that there exists a unique vector potential $\psi \in W^{1,p}_0(\Omega)$ such that:

$$\text{curl} \varphi = \text{curl} \psi, \quad \text{div} \psi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \psi \cdot n = 0 \quad \text{on} \quad \Gamma.$$

In addition, we have

$$\|\psi\|_{W^{1,p}_0(\Omega)} \leq C \|\text{curl} \varphi\|_{L^p(\Omega)}.$$

On the other hand, let us solve the following problem,

$$\Delta w = \text{div} \varphi \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma. \tag{3.7}$$

Thanks to Theorem 2.7, problem (3.7) has a solution $w \in W^{2,p}_0(\Omega)$ that satisfies

$$\|\nabla w\|_{W^{1,p}_0(\Omega)} \leq C \|\text{div} \varphi\|_{L^p(\Omega)} \tag{3.8}$$

Setting $z = \varphi - \nabla w$. It is clear that $z$ belongs to $W^{1,p}_0(\Omega)$, it is divergence-free, $z \cdot n = 0$ on $\Gamma$ and $\text{curl} \varphi = \text{curl} z$ in $\Omega$. The uniqueness of $\psi$ implies that $z = \psi$ and thus we obtain

$$\|z\|_{W^{1,p}_0(\Omega)} \leq C \|\text{curl} \varphi\|_{L^p(\Omega)}. \tag{3.9}$$

Finally, the inequality (3.5) follows immediately from (3.9) and (3.8).

Remark 3.5. As a consequence of Theorem 3.4, the seminorm in the right-hand side of (3.5) is an norm on $Z^p_0(\Omega)$ equivalent to the norm $\|\varphi\|_{W^{1,p}_0(\Omega)}$.

The “Inf-Sup” condition is given by the following lemma:

Lemma 3.6. Assume that $p > 3$. The following Inf-Sup condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\varphi \in V^{p'}_0(\Omega)} \sup_{\psi \neq 0} \frac{\int_\Omega \text{curl} \psi \cdot \text{curl} \varphi \, dx}{\|\psi\|_{Z^p_0(\Omega)} \|\varphi\|_{Z^{p'}_0(\Omega)}} \geq \beta. \tag{3.10}$$

Proof. Let $g \in L^p(\Omega)$ and let us introduce the following Dirichlet problem:

$$-\Delta \chi = \text{div} g \quad \text{in} \quad \Omega, \quad \chi = 0 \quad \text{on} \quad \Gamma.$$

It is shown in Theorem 3.7 of [24], that this problem has a solution $\chi \in W^{1,p}_0(\Omega)$ and we have

$$\|\nabla \chi\|_{L^p(\Omega)} \leq C \|g\|_{L^p(\Omega)}.$$

Set $z = g - \nabla \chi$. Then we have $z \in L^p(\Omega)$, $\text{div} z = 0$ and we have

$$\|z\|_{L^p(\Omega)} \leq C \|g\|_{L^p(\Omega)}. \tag{3.11}$$
Let \( \varphi \) any function of \( V^{p'}_{0,T}(\Omega) \), by Theorem 3.4 we have \( \varphi \in Z^{p'}_{T}(\Omega) \hookrightarrow W^{1,p'}_{0}(\Omega) \). Then due to (3.5) we can write
\[
\|\varphi\|_{W^{1,p'}_{0}(\Omega)} \leq C \|\text{curl } \varphi\|_{L^{p'}(\Omega)} = C \sup_{g \in L^{p'}(\Omega), g \neq 0} \frac{\|\int_{\Omega} \text{curl } \varphi \cdot g \, dx\|}{\|g\|_{L^{p'}(\Omega)}}.
\] (3.12)

Using the fact that \( \text{curl } \varphi \in H^{p'}_{0}(\text{div}, \Omega) \) and applying (2.7), we obtain
\[
\int_{\Omega} \text{curl } \varphi \cdot \nabla \chi \, dx = 0.
\] (3.13)

Now, let \( \lambda \in W^{2,p}_{1}(\Omega) \cap W^{2,2}_{1}(\Omega) \) the solution of the following problem:
\[
\Delta \lambda = 0 \quad \text{in } \Omega \quad \text{and} \quad \lambda = 1 \quad \text{on } \Gamma.
\]
It follows from Lemma 3.11 of [20] that
\[
\left\langle \frac{\partial \lambda}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma} = \int_{\Gamma} \frac{\partial \lambda}{\partial \mathbf{n}} \, d\sigma = C_{1} > 0.
\]
Now, setting
\[
\tilde{z} = z - \frac{1}{C_{1}} \langle z \cdot \mathbf{n}, 1 \rangle_{\Gamma} \nabla \lambda.
\]

It is clear that \( \tilde{z} \in L^{p}(\Omega), \text{div } \tilde{z} = 0 \) in \( \Omega \) and that \( \langle \tilde{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = 0 \). Due to Theorem 3.3, there exists a potential vector \( \psi \in W^{1,p}_{0}(\Omega) \) such that
\[
\tilde{z} = \text{curl } \psi, \quad \text{div } \psi = 0 \quad \text{in } \Omega \quad \text{and} \quad \psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.
\] (3.14)

In addition, we have the estimate
\[
\|\psi\|_{W^{1,p}_{0}(\Omega)} \leq C \|	ilde{z}\|_{L^{p}(\Omega)} \leq C \|z\|_{L^{p}(\Omega)}.
\] (3.15)

Then, we obtain that \( \psi \) belongs to \( V^{p'}_{0,T}(\Omega) \). Since \( \varphi \) is \( W^{1,p'}_{\text{loc}}(\Omega) \) in a neighborhood of \( \Gamma \), then \( \varphi \) has a \( W^{1,p'}_{\text{loc}} \) extension in \( \Omega' \) denoted by \( \tilde{\varphi} \). Applying Green’s formula in \( \Omega' \), we obtain
\[
0 = \int_{\Omega'} \text{div}(\text{curl } \tilde{\varphi}) \, dx = \langle \text{curl } \tilde{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = \langle \text{curl } \varphi \cdot \mathbf{n}, 1 \rangle_{\Gamma}.
\]

Using the fact that \( \text{curl } \varphi \) in \( H^{p'}_{0}(\text{div}, \Omega) \) and \( \lambda \in W^{2,p}_{1}(\Omega) \hookrightarrow W^{1,p}_{0}(\Omega) \) and applying (2.7), we obtain
\[
0 = \langle \text{curl } \varphi \cdot \mathbf{n}, 1 \rangle_{\Gamma} = \langle \text{curl } \varphi \cdot \mathbf{n}, \lambda \rangle_{\Gamma} = \int_{\Omega} \text{curl } \varphi \cdot \nabla \lambda \, dx.
\] (3.16)

Using (3.13) and (3.16), we deduce that
\[
\int_{\Omega} \text{curl } \varphi \cdot g \, dx = \int_{\Omega} \text{curl } \varphi \cdot z \, dx = \int_{\Omega} \text{curl } \varphi \cdot \tilde{z} \, dx.
\] (3.17)

From (3.11), (3.15) and (3.17), we deduce that
\[
\frac{\|\int_{\Omega} \text{curl } \varphi \cdot g \, dx\|}{\|g\|_{L^{p'}(\Omega)}} \leq C \frac{\|\int_{\Omega} \text{curl } \varphi \cdot \tilde{z} \, dx\|}{\|\tilde{z}\|_{L^{p}(\Omega)}} = C \frac{\|\int_{\Omega} \text{curl } \varphi \cdot \text{curl } \psi \, dx\|}{\|\text{curl } \psi\|_{L^{p}(\Omega)}}.
\]

Applying again (3.5), we obtain
\[
\frac{\|\int_{\Omega} \text{curl } \varphi \cdot g \, dx\|}{\|g\|_{W^{1,p}(\Omega)}} \leq C \frac{\|\int_{\Omega} \text{curl } \varphi \cdot \text{curl } \psi \, dx\|}{\|\psi\|_{W^{1,p}_{0}(\Omega)}}.
\]
and the Inf-Sup Condition (3.10) follows immediately from (3.12). □

4. Existence of weak solution for $p > 3$

We prove in this sequel the existence and the uniqueness of weak solutions for the following Problem:

$$\mathcal{S}_T \begin{cases} -\Delta u + \nabla \pi = f & \text{in } \Omega, \\ u \cdot n = g & \text{on } \Gamma. \end{cases}$$

The following result concerns the existence and uniqueness of the weak solution of the problem ($\mathcal{S}_T$) where $\chi = g = 0$.

**Theorem 4.1.** Assume that $p > 3$, suppose that $\chi = g = 0$. Then, for any $f \in [\dot{H}^{-1/p}_1(\text{div, } \Omega)]'$ and $h \in W^{-1/p,p}(\Gamma)$, the problem ($\mathcal{S}_T$) has a unique solution $(u, \pi) \in (W^{1,p}_0(\Omega) \times L^p(\Omega))$. In addition, we have the following estimate:

$$\|u\|_{W^{1,p}_0(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\|f\|_{[\dot{H}^{-1/p}_1(\text{div, } \Omega)]'} + \|h\|_{W^{-1/p,p}(\Gamma)}).$$

**Proof.** On the first hand, observe that Problem ($\mathcal{S}_T$) is reduced to the following variational problem:

$$\begin{cases} \text{Find } u \in V_{0,T}^p(\Omega) \text{ such that,} \\ \int_{\Omega} \text{curl } u \cdot \text{curl } \phi \, dx = (f, \phi)_\Omega + (h, \phi)_\Gamma, \forall \phi \in V_{0,T}^p(\Omega) \end{cases}$$

(4.2)

Let us introduce the space

$$\mathcal{D}_\sigma(\Omega) = \left\{ \phi \in \mathcal{D}(\Omega), \text{div } \phi = 0 \right\}.$$

Indeed, every solution of ($\mathcal{S}_T$) also solves (4.2). Conversely, let us take $\phi \in \mathcal{D}_\sigma(\Omega)$ as the test function in (4.2), then we get

$$\langle -\Delta u - f, \phi \rangle_{\mathcal{D}_\sigma(\Omega) \times \mathcal{D}_\sigma(\Omega)} = 0.$$  (4.3)

By De Rham theorem, there exists a function $\pi \in L^p(\Omega)$ such that

$$-\Delta u + \nabla \pi = f \quad \text{in } \Omega.$$

Moreover, by the fact that $u$ belongs to the space $V_{0,T}^p(\Omega)$, we have $\text{div } u = 0$ in $\Omega$, $u \cdot n = 0$ on $\Gamma$. The remainder boundary condition $\text{curl } u \times n = h$ on $\Gamma$ is implicitly contained in (4.2). Observe that since $\nabla \pi$ is elements of $[\dot{H}^{-1/p}_1(\text{div, } \Omega)]'$, it is the same for $\Delta u$. Since $\mathcal{D}(\Omega)$ is dense in $\dot{H}^{-1/p}_1(\text{div, } \Omega)$, it is clear then that for any $\phi \in \dot{H}^{-1/p}_1(\text{div, } \Omega)$

$$\langle \nabla \pi, \phi \rangle_\Omega = -\int_{\Omega} \pi \text{div } \phi \, dx, \quad \forall \phi \in \dot{H}^{-1/p}_1(\text{div, } \Omega).$$

Particularly

$$\langle \nabla \pi, \phi \rangle_\Omega = 0 \quad \text{if } \phi \in V_{0,\sigma}^p(\Omega).$$

Moreover, if $\phi \in V_{0,T}^p(\Omega)$, using the Green formula (2.13) we have

$$\langle -\Delta u, \phi \rangle_{[\dot{H}^{-1/p}_1(\text{div, } \Omega)]' \times \dot{H}^{-1/p}_1(\text{div, } \Omega)} = \int_{\Omega} \text{curl } u \cdot \text{curl } \phi \, dx + \langle \text{curl } u \times n, \phi \rangle_{\Gamma}$$

(4.4)

Therefore, from (4.2) and (4.4) we deduce that for all $\phi \in V_{0,T}^p(\Omega)$

$$\langle \text{curl } u \times n, \phi \rangle_{\Gamma} = \langle H, \phi \rangle_{\Gamma}.$$
Let now \( \mu \) any element of the space \( W^{1-1/p',p'}(\Gamma) \). So, there exists an element \( \varphi \in W_0^{1,p}(\Omega) \) such that \( \text{div} \varphi = 0 \) in \( \Omega \) and \( \varphi = \mu_\tau \) on \( \Gamma \). Then, we have
\[
\langle \text{curl} \ u \times n, \mu \rangle_\Gamma - \langle H, \mu \rangle_\Gamma = \langle \text{curl} \ u \times n, \mu_\tau \rangle_\Gamma - \langle H, \mu_\tau \rangle_\Gamma = \langle \text{curl} \ u \times n, \varphi \rangle_\Gamma - \langle H, \varphi \rangle_\Gamma
\]

This implies that \( \text{curl} \ u \times n = H \) on \( \Gamma \).

This implies that problem \( (\mathcal{F}_T) \) and problem \( (4.2) \) are equivalent. Now, to solve Problem \( (4.2) \), we use the Inf-Sup condition \( (3.10) \). We consider the bilinear form \( a : V_{0,T}^p(\Omega) \times V_{0,T}^{p'}(\Omega) \rightarrow \mathbb{R} \) such that
\[
a(u, \varphi) = \int_\Omega \text{curl} \ u \cdot \text{curl} \varphi \, dx.
\]

Let consider the following mapping \( \ell : V_{0,T}^{p'}(\Omega) \rightarrow \mathbb{R} \) such that \( \ell(\varphi) = \langle f, \varphi \rangle_\Omega + \langle h, \varphi \rangle_\Gamma \). It is clear that \( \ell \) belongs to \( (V_{0,T}^{p'}(\Omega))^* \) and according to Remark 3.2, there exists a unique solution \( u \in V_{0,T}^p(\Omega) \) of Problem \( (4.2) \). Due to Theorem 3.4, we prove that this solution \( u \) belongs to \( W_0^{1,p}(\Omega) \). It follows from Remark 3.2 i) that
\[
\|u\|_{W_0^{1,p}(\Omega)} \leq C (\|f\|_{\tilde{H}^{-1}_0(\text{div},\Omega)})' + \|h\|_{W^{-1/p,p}(\Gamma)}.
\]
(4.5)

Since \( \nabla \pi = f + \Delta u \), then we have the following estimate
\[
\|\pi\|_{L^p(\Omega)} \leq C \|\nabla \pi\|_{W_0^{1,p}(\Omega)} \leq C (\|f\|_{\tilde{H}^{-1}_0(\text{div},\Omega)})' + \|u\|_{W_0^{1,p}(\Omega)}.
\]
(4.6)

The estimate \( (4.1) \) follows from \( (4.5) \) and \( (4.6) \). \( \square \)

We can also solve the Stokes problem when the divergence operator does not vanish and \( g \neq 0 \).

Corollary 4.2. Assume that \( p > 3 \). Let \( f, \chi, g, h \) such that
\[
f \in [\tilde{H}^{-1}_0(\text{div}, \Omega)]', \chi \in L^p(\Omega), g \in W^{1-1/p,p}(\Gamma) \text{ and } h \in W^{-1/p,p}(\Gamma).
\]

Then, the Stokes problem \( (\mathcal{F}_T) \) has a unique solution \( (u, \pi) \in W_0^{1,p}(\Omega) \times L^p(\Omega) \) and we have:
\[
\|u\|_{W_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C (\|f\|_{\tilde{H}^{-1}_0(\text{div},\Omega)})' + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|h\|_{W^{-1/p,p}(\Gamma)}.
\]
(4.7)

Proof. Let \( \chi \in L^p(\Omega) \) and \( g \in W^{1-1/p,p}(\Gamma) \). We solve the following Neumann problem in \( \Omega \):
\[
-\Delta \theta = \chi \quad \text{in } \Omega, \quad \frac{\partial \theta}{\partial n} = g \quad \text{on } \Gamma.
\]
(4.8)

It follows from Theorem 2.7 that the Problem \( (4.8) \) has a solution \( \theta \) in \( W_0^{2,p}(\Omega) \) and we have:
\[
\|\theta\|_{W_0^{2,p}(\Omega)} \leq C (\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)}).
\]
(4.9)

Setting \( z = u - \nabla \theta \), then Problem \( (\mathcal{F}_T) \) becomes: Find \( (z, \pi) \in W_0^{1,p}(\Omega) \times L^p(\Omega) \) such that
\[
\begin{cases}
-\Delta z + \nabla \pi = f + \nabla \chi & \text{in } \Omega, \\
z \cdot n = 0 & \text{and } \text{curl} z \times n = h \quad \text{on } \Gamma,
\end{cases}
\]
(4.10)

Observe that \( f + \nabla \chi \) belongs to \( [\tilde{H}^{-1}_0(\text{div}, \Omega)]' \). According to Theorem 4.1, this problem has a unique solution \( (z, \pi) \in W_0^{1,p}(\Omega) \times L^p(\Omega) \). Thus \( u = z + \nabla \theta \) belongs to \( W_0^{1,p}(\Omega) \) and estimate \( (4.7) \) follows from \( (4.1) \) and \( (4.9) \). \( \square \)

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