STANDARD COMPLEX FOR QUANTUM LIE ALGEBRAS

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ABSTRACT

For a quantum Lie algebra $\Gamma$, let $\Gamma^\wedge$ be its exterior extension (the algebra $\Gamma^\wedge$ is canonically defined). We introduce a differential on the exterior extension algebra $\Gamma^\wedge$ which provides the structure of a complex on $\Gamma^\wedge$. In the situation when $\Gamma$ is a usual Lie algebra this complex coincides with the "standard complex". The differential is realized as a commutator with a (BRST) operator $Q$ in a larger algebra $\Gamma^\wedge[\Omega]$, with extra generators canonically conjugated to the exterior generators of $\Gamma^\wedge$. A recurrent relation which defines uniquely the operator $Q$ is given.
1. A quantum Lie algebra \([1], [2], [3], [4]\) is defined by two tensors \(C^k_{ij}\) and \(\sigma^{mk}_{ij}\) (indices belong to some set \(N\), say, \(N = \{1, \ldots, N\}\)). By definition, the matrix \(\sigma^{mk}_{ij}\) has an eigenvalue 1; one demands that \((P_{(1)})^{mk}_{ij} C^m_{nk} = 0\), where \(P_{(1)}\) is a projector on the eigenspace of \(\sigma\) corresponding to the eigenvalue 1.

By definition, a quantum Lie algebra \(\Gamma\) is generated by elements \(\chi_i, i = 1, \ldots, N\), subjected to relations

\[
\chi_i \chi_j - \sigma^{mk}_{ij} \chi_m \chi_k = C^k_{ij} \chi_k .
\]

Here the structure constants \(C^k_{ij}\) obey

\[
C^p_{ni} C^l_{pj} = \sigma^{mk}_{ij} C^p_{nm} C^l_{pk} + C^p_{ij} C^l_{np} \quad \Leftrightarrow \quad C^{|1}\rangle = \sigma^{<4|}_{12} C^{|1>} C^{|4>} \rangle + C^{|13>} C^{|3>} \rangle , \quad (2)
\]

\[
C^k_{ni} \sigma^{pm}_{kj} = \sigma^{sj}_{iq} \sigma^{pk}_{ns} C^n_{kj} \quad \Leftrightarrow \quad C^{|1}\rangle = \sigma^{<4|}_{12} C^{|13>} + C^{|12>} C^{|3>} \rangle , \quad (3)
\]

\[
(\sigma^{pj}_{im} C^n_{qp} + \delta^n_j \sigma^j_{im}) \sigma^{ks}_{nj} = \sigma^{pq}_{ij} (\sigma^{ps}_{nm} C^k_{jp} + \delta^k_j \sigma^j_{nm}) \quad \Leftrightarrow \quad (\sigma^{<3|}_{12} + C^{|3>} \rangle ) \sigma^{13} = \sigma^{12} (\sigma^{<1|}_{12} + C^{|12>} \rangle) . \quad (4)
\]

The matrix \(\sigma^{mk}_{ij}\) satisfies the Yang-Baxter equation

\[
\sigma^{k1}_{j1} \sigma^{n2k3}_{j2} \sigma^{k1n2}_{j3} = \sigma^{k2}_{j1} \sigma^{n1k3}_{j2} \sigma^{k2n1}_{j3} \quad \Leftrightarrow \quad \sigma^{12} \sigma^{23} \sigma^{12} = \sigma^{23} \sigma^{12} \sigma^{23} . \quad (5)
\]

In the right hand side of (3)-(5) we use FRT matrix notations \([5]\); \(\{1, 2, 3, \ldots\}\) are the numbers of vector spaces, \(e.g., f_1 := f^{11}_1\) is a matrix which acts in the first vector space. Additionally, we use incoming and outcoming indices, \(e.g., \Omega^{<1|} := \Omega^{1i}\) and \(\gamma^{1|} := \gamma_j\), denote a covector with one outcoming index and a vector with one incoming index respectively. Thus, in this notation, the matrix \(f_1\) can be written as \(f_1 = f^{<1|}_{11}\).

Remark. Quantum Lie algebras defined by equations (1)-5 generalize the usual Lie (super-)algebras. Indeed in the non-deformed case, when

\[
\sigma^{mk}_{ij} = (-1)^{(m)(k)} \delta^{m}_{j} \delta^{k}_{i}
\]

is a super-permutation matrix (here \(\sigma^2 = 1\) and (3) is fulfilled; \((m) = 0, 1\) is the parity of a generator \(\chi_m\)), equations (1) and (2) coincide with the defining
relations and the Jacobi identities for Lie (super)-algebras. Equation (3) is then equivalent to the \( Z_2 \)-homogeneity condition \( C_{jk}^i = 0 \) for \( (i) \neq (j) + (k) \). Equation (4) follows from (3).

2. The exterior extension \( \Gamma^\wedge \) of the quantum algebra \( \Gamma \) (1) is obtained by adding new generators \( \gamma_i \), \( i = 1, \ldots, N \). The generators \( \gamma_i \) form a "generalized" wedge algebra. The definition of the wedge product of the elements \( \gamma_i \) is

\[
\gamma_1 \wedge \gamma_2 \cdots \wedge \gamma_n = A_{1 \to n} \gamma_1 \otimes \gamma_2 \cdots \otimes \gamma_n .
\]

(6)

Here the matrix operator \( A_{1 \to n} \) is an analog of the antisymmetrizer of \( n \)-spaces. This operator can be defined inductively (see e.g. [6])

\[
A_{1 \to n} = \left( 1 + \sum_{k=1}^{n-1} (-1)^{n-k} \sigma_{k \to n} \right) A_{1 \to n-1}
\]

(7)

where, for \( n > k \),

\[
\sigma_{k \to n} := \sigma_{kk+1} \sigma_{k+1k+2} \cdots \sigma_{n-1n} .
\]

Using the Yang-Baxter equation (5) one can rewrite (7) in the following three equivalent forms

\[
A_{1 \to n} = A_{1 \to n-1} \left( 1 + \sum_{k=1}^{n-1} (-1)^{k} \sigma_{k+1 \to 1} \right)
\]

\[
= \left( 1 + \sum_{k=1}^{n-1} (-1)^{k} \sigma_{k+1 \to 1} \right) A_{2 \to n} = A_{2 \to n} \left( 1 + \sum_{k=1}^{n-1} (-1)^{n-k} \sigma_{k \to n} \right) ,
\]

where

\[
\sigma_{n \to k} := \sigma_{n-1n} \cdots \sigma_{k+1k+2} \sigma_{kk+1}
\]

for \( n > k \).

If the sequence of operators \( A_{1 \to n} \) terminates at the step \( n = h + 1 \) (\( A_{1 \to h} \neq 0 \) and \( A_{1 \to n} = 0 \) for \( n > h \)) then the number \( h \) is called the height of the operator \( \sigma \).

The cross-commutation relations between the generators \( \gamma_i \) and \( \chi_j \) are:

\[
\gamma_1 \chi_2 = (\sigma_{12} \chi_1 + C_{12}^{<2|} \chi_2) \gamma_2 .
\]

(8)

The algebra \( \Gamma^\wedge \) is graded by the degree in the generators of \( \gamma_i \).
3. We further introduce a set of generators \( \{ \Omega^i \} \), \( i = 1, \ldots, N \), canonically conjugated to the generators \( \gamma_i \). The generators \( \Omega^i \) form a "wedge" algebra as well, with the wedge product defined by

\[
\Omega^{<r|} \wedge \Omega^{<r-1|} \wedge \ldots \wedge \Omega^{<1|} = \Omega^{<r|} \otimes \Omega^{<r-1|} \otimes \ldots \otimes \Omega^{<1|} A_{1 \rightarrow r}.
\] (9)

Here operators \( A_{1 \rightarrow n} \) are the same as in (7).

The commutation relations between \( \Omega^i \) and \( \gamma_j \) are

\[
\gamma_j \Omega^i = -\Omega^p \sigma_{ji}^s \gamma_s + \delta_j^i \Rightarrow \gamma_{|2>} \Omega^{<2|} = -\Omega^{<1|} \sigma_{12}^{-1} \gamma_{|1>} + I_2.
\] (10)

Finally the commutation relations between \( \Omega^i \) and \( \chi_j \) are

\[
\chi_{|2>} \Omega^{<2|} = \Omega^{<1|} (\sigma_{12} \chi_{|1>} + C^{<2|}_{|12>}).
\] (11)

We denote the algebra generated by \( \{ \chi_i \} \), \( \{ \gamma_j \} \) and \( \{ \Omega^k \} \) by \( \Gamma^\wedge[\Omega] \). The algebra \( \Gamma^\wedge[\Omega] \) is graded by the rule: \( \text{deg} (\gamma_i) = 1 \) and \( \text{deg} (\Omega^i) = -1 \).

4. The main result of the present paper is a recursive formula for the BRST operator \( Q \) which satisfies \( Q^2 = 0 \).

Such an operator endows the algebra \( \Gamma^\wedge \) with the structure of the differential (chain) complex. To construct the differential (starting with the operator \( Q \)) one needs first to define the action of the algebra \( \Gamma^\wedge[\Omega] \) on the algebra \( \Gamma^\wedge \). The elements \( \chi_i \) and \( \gamma_j \) act on \( \Gamma^\wedge \) by the left multiplication. To define the action of generators \( \Omega^i \) on \( \Gamma^\wedge \) it suffices (due to relations (10) and (11)) to know \( \Omega^i(1) \), where 1 is the unit element of the algebra \( \Gamma^\wedge \). We set \( \Omega^i(1) = 0 \). The definition of the differential \( d \) is given by its action on an element \( \phi \) of the algebra \( \Gamma^\wedge \),

\[
d\phi = [Q, \phi]_\pm (1),
\] (13)

where \([,]_\pm \) is the graded commutator.
Now we are ready to formulate the main Proposition.

**Proposition.** The BRST operator $Q$ for the quantum algebra \([3]\) has the following form

$$Q = \Omega^i \chi_i + \sum_{r=1}^{h-1} Q_r ,$$

where $h$ is the height of the operator $A_{1 \to n} \,[3]$. Here the operators $Q_r$ are given by

$$Q_r = \Omega^{<r+1|} \Omega^{<r|} \cdots \Omega^{<1|} X^{<\tilde{1}, \ldots, \tilde{r}|}_{|1, \ldots, r+1>} \gamma_{[1]} \cdots \gamma_{[r]}$$

(the wedge product is implied); $X^{<\tilde{1}, \ldots, \tilde{r}|}_{|1, \ldots, r+1>}$ are tensors which satisfy the following recurrent relations

$$A_{1 \to r+1} X^{<\tilde{1}, \ldots, \tilde{r}|}_{|1, \ldots, r+1>} A_{1 \to r} = A_{1 \to r+1} ((-1)^r \sigma_{r+1-1} - 1) X^{<\tilde{2}, \ldots, \tilde{r}|}_{|2, \ldots, r+1>} A_{2 \to r}$$

with the initial condition $A_{12} X^{<0|}_{|12>} = -C^{<0|}_{|12>}.$

**Proof.** We have to verify the identity

$$Q^2 = (\Omega^{<2|} \chi_{|2>})^2 + [\Omega^{<2|} \chi_{|2>}, \sum_{r=1}^{h-1} Q_r] + (\sum_{r=1}^{h-1} Q_r)^2 = 0 .$$

Because of the lack of space we shall check a part of this identity which includes the linear in $\chi$ terms only.

First of all we find (see \([3]\))

$$(\Omega^{<2|} \chi_{|2>})^2 = \Omega^{<2|} (\Omega^{<1|} (\sigma_{12} \chi_{|1>}) + C^{<2|}_{|12>}) \chi_{|2>}

= \Omega^{<2|} \otimes \Omega^{<1|} \sigma_{12} C^{<2|}_{|12>} \chi_{|2>} + \Omega^{<2|} \otimes \Omega^{<1|} (1 - \sigma)_{12} C^{<2|}_{|12>} \chi_{|2>}

= \Omega^{<2|} \otimes \Omega^{<1|} C^{<2|}_{|12>} \chi_{|2>} .$$

Consider then the anticommutator $[\Omega^{<2|} \chi_{|2>}, Q_r]$ in which we commute all $\chi_i$ to the right and extract only the terms which are linear in the generators $\chi_i$:

$$[Q_r, \Omega^{<r|} \chi_{|r>} + \Omega^{<r+1|} \chi_{|r+1>} Q_r] + Q_r \Omega^{<r|} \chi_{|r>}$$
(dots denote the terms independent of \( \chi_r \)). Here eqs. (11), (8) and (12) have been used.

Equations (19) and (18) give the whole contribution to the \( \chi \)-linear terms in \( Q^2 \) since \((\sum_{r=1}^{h-1} Q_{(r)})^2\) is independent of \( \chi_i \).

The substitution of (18) and (19) produces the initial data

\[
A_{12} X_{[12]}^{<0|} = -C_{[12]}^{<0|} \quad \text{and recurrent relations}
\]

\[
A_{1\rightarrow r+1} X_{[1\ldots r+1]}^{<1\ldots r|} \left( \sum_{k=1}^{r} (-1)^{r-k} \sigma_{r-k}^{-1} \right) A_{1\rightarrow r-1} = -A_{1\rightarrow r+1} \left( \sigma_{r+1-1} + (-1)^{r-1} 1 \right) X_{[2\ldots r+1]}^{<2\ldots r|} \sigma_{r-1}^{-1} A_{1\rightarrow r-1}
\]

where the matrix operator \( A_{1\rightarrow r} \) is defined in (7). These relations express coefficients \( X_{[1\ldots r]}^{<1\ldots r|} \) via \( X_{[1\ldots r]}^{<1\ldots r-1|} \).

Using an identity \( \sigma_{r-1}^{-1} A_{1\rightarrow r-1} = A_{2\rightarrow r} \sigma_{r-1}^{-1} \) and inductive relations (7) for the projectors \( A_{1\rightarrow r} \) one can rewrite (20) in the form (16).

5. Comments.

\[i.\] For general \( \sigma^{ij}_{kl} \) and \( C_{jk}^{ii} \) it is rather difficult to solve equations (16) explicitly. However for the case \( \sigma^2 = 1 \) the main equations (16) become simpler and the general solution for \( Q \) can be found. Indeed the relation (16) for \( r = 2 \) gives

\[
A_{1\rightarrow 3} X_{[123]}^{<12|} (1 - \sigma_{12}) = A_{1\rightarrow 3} (\sigma_{23} \sigma_{12} - 1) X_{[23]}^{<2|}.
\]

For \( \sigma^2 = 1 \) we have \( A_{1\rightarrow 3} (\sigma_{23} \sigma_{12} - 1) = 0 \) and therefore \( Q_{(r)} = 0 \) for \( r \geq 2 \). Thus the BRST operator (14) has the familiar form

\[
Q = \Omega^{<1|} \chi_{[1]} > - \Omega^{<2|} \otimes \Omega^{<1|} C_{[12]}^{<1|} \gamma_{[1]} >.
\]
In the case when the matrix $\sigma$ is the (super)-permutation matrix the algebra $\Gamma^\wedge$ with the differential (13) becomes the standard complex for the Lie (super)-algebra $\Gamma$ (see e.g. [4]).

In general, for $\sigma^2 \neq 1$, the sum in (14) will be limited only by the height $h$ of the operator $\sigma$.

Below we present an explicit form for $Q$ for the standard quantum deformation $\Gamma = U_q(gl(N))$ of the universal enveloping algebra of the Lie algebra $gl(N)$ ($\sigma^2 \neq 1$ in this case).

ii. When the algebra (1) is a Hopf algebra, the algebraic structure (6), (8) – (11) is related to the differential calculus on quantum groups (see [2], [8], [9]). The BRST operator $Q$ given by (14) generates the differential $d$ (introduced in [2]) on the algebra dual to $\Gamma^\wedge$.

6. Example. The BRST operator $Q$ for the quantum algebra $\Gamma = U_q(gl(N))$.

The quantum algebra $U_q(gl(N))$ is defined (as a Hopf algebra) by the relations (5)

$$\hat{R} L_1^\pm L_1^\pm = L_1^\pm L_1^\pm \hat{R}, \quad \hat{R} L_2^\pm L_2^\mp = L_2^\mp L_2^\pm \hat{R},$$

$$\Delta(L^\pm) = L^\pm \otimes L^\pm, \quad \varepsilon(L^\pm) = 1, \quad S(L^\pm) = (L^\pm)^{-1},$$

where elements of the $N \times N$ matrices $(L^\pm)_i^j$ are generators of $U_q(gl(N))$; the matrices $L^+$ and $L^-$ are respectively upper and lower triangular, their diagonal elements are related by $(L^+_i)_i^j = 1$ for all $i$. The matrix $\hat{R}$ is defined as $\hat{R} := R_{12} = P_{12} R_{12}$ (the permutation matrix); The algebra $R_{12}$ is the standard Drinfeld-Jimbo $R$-matrix for $GL_q(N)$,

$$R_{12} = R_{12}^{j_1,i_2,j_2} = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} (1 + (q - 1) \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}) + (q - q^{-1}) \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \Theta_{i_1 i_2},$$

where

$$\Theta_{ij} = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i \leq j. \end{cases}$$

This $R$-matrix satisfies the Hecke condition $\hat{R}^2 = \lambda \hat{R} + 1$, where $\lambda = (q - q^{-1})$ and $q$ is a parameter of deformation.

The generators of the algebra $\Gamma$ are defined by the formula (10), (11), (3)

$$\chi_k^i = \frac{1}{\chi} \left[ (D^{-1})^i_k - (D^{-1})_i^j J_{kj} \right].$$

7
Here \( f^i_{jk} = L^{-1}_k S (L^+_j) \) and the numerical matrix \( D \) can be found by means of relations

\[
Tr_2 \mathring{R}_{12} \Psi_{23} = P_{13} = Tr_2 \Psi_{12} \mathring{R}_{23} , \quad D_1 := Tr_2 \Psi_{12} \Rightarrow Tr_1 (D_1^{-1} \mathring{R}^{-1}) = 1_2 ,
\]

where \( Tr_1 \) and \( Tr_2 \) denote the traces over first and second spaces.

It is convenient to write down the complete set of commutation relations for the exterior algebra \( \Gamma^\wedge [\Omega] \) in terms of generators

\[
L_i^j = (L^+_i)^k S ((L^-)_j^k) = \delta^i_j - \lambda S^{-1}(\chi^k_j) D^j_k ,
\]

\[
J_i^j = - S^{-1}(f^{ik}_j) \gamma^j_k D^i_k , \quad \omega^i_j = \Omega_i^m f_{kj}^m .
\]
The indices now are pairs of indices; the roles of the elements \( \chi^i_j, \gamma^i_j \) and \( \Omega^k_j \) are played by the generators \( \chi_j^i, \gamma_j^i \) and \( \Omega_j^i \) respectively.

The commutation relations are \([12], [11], [9]\):

\[
\omega_2 \mathring{R} \omega_1 = \mathring{R}^{-1} \omega_2 \mathring{R}^{-1} \omega_2 , \quad \omega_2 \mathring{R} L_2 \mathring{R} = \mathring{R} L_2 \mathring{R} \omega_2 , \quad \omega_2 \mathring{R} L_2 \mathring{R} = \mathring{R} L_2 \mathring{R} \omega_2 , \quad (24)
\]

\[
L_2 \mathring{R} L_2 \mathring{R} = \mathring{R} L_2 \mathring{R} L_2 , \quad J_2 \mathring{R} L_2 \mathring{R} = \mathring{R} L_2 \mathring{R} J_2 , \quad J_2 \mathring{R} J_2 \mathring{R} = - \mathring{R}^{-1} J_2 \mathring{R} J_2 . \quad (26)
\]

Now the construction of the BRST operator \( Q \) is in order. To begin we find the first term in the sum \([14]\):

\[
\Omega^k_m \chi^m_k = \frac{1}{\lambda} Tr_q (\omega (L - 1)) , \quad (27)
\]

where we have introduced the quantum trace \( Tr_q (X) := Tr (D^{-1} X) \). Then one can resolve the chain of the recurrent relations \([13]\) where we have to substitute the expressions for the structure constants

\[
\sigma^r_{pq} = R^{ij}_p (R^{-1})^m_{kr} (D^{-1})^j_o R^{no}_{ut} D^i_l (R^{-1})^r_l ,
\]

\[
C^{<q>_r}_{ij} = \delta^q_j \delta^i_p - \sigma^{<q>_r}_{ij} ,
\]

and find the set of coefficients \( X^{<1...r>}_{1...r+1} \). After straightforward but tiresome calculations one can obtain the following result:

\[
Q = Tr_q \left( \omega (L - 1)/\lambda - \omega L (\omega J) + \lambda \omega L (\omega J)^2 - \lambda^2 \omega L (\omega J)^3 + \ldots \right)
\]
\[ \text{Tr}_q \left( \omega (L - 1)/\lambda - \omega L (\omega J) (1 + \lambda \omega J)^{-1} \right) = -\frac{1}{\lambda} \text{Tr}_q(\omega) + \frac{1}{\lambda} \text{Tr}_q(W), \] 

(28)

where \( W = \omega L (1 + \lambda \omega J)^{-1} \) and the sum in the first line of (28) is limited by the requirement that monomials of \( \omega \)'s of the order \( N^2 + 1 \) are equal to zero.

One can check directly that the operator \( Q \) given by (28) satisfies:

\[ Q^2 = 0, \quad [Q, L] = 0, \quad [Q, J] = \frac{1}{\lambda} (1 - L). \]

To obtain these relations one has to use identities

\[ \text{Tr}_q(X) \text{1}_2 = \text{Tr}_q(\hat{R} \hat{X} \hat{R}^{-1}) \]

and relations

\[ \hat{R} \hat{W}_2 \hat{R}^{-1} \omega_2 = -\omega_2 \hat{R}^{-1} W_2 \hat{R}, \]

\[ \hat{R} \hat{W}_2 \hat{R}^{-1} W_2 = -W_2 \hat{R}^{-1} W_2 \hat{R}^{-1}, \quad \hat{R}^{-1} W_2 \hat{R} L_2 = L_2 \hat{R} W_2 \hat{R}^{-1}, \]

\[ J_2 \hat{R} \hat{W}_2 \hat{R}^{-1} + \hat{R}^{-1} W_2 \hat{R} J_2 = -L_2 (1 + \lambda \omega J)_2 \hat{R}^{-1} (1 + \lambda \omega J)_2, \]

which follow from (24)-(26).

Remark. The operator \( Q \) given by (28) has the correct classical limit for \( q \to 1 \) (\( \lambda \to 0, L \to 1 + \lambda \tilde{\chi}, \omega \to \tilde{\omega}, J \to \tilde{\gamma} \))

\[ Q \to Q_{cl} = Tr(\tilde{\omega} \tilde{\chi} + \tilde{\omega}^2 \tilde{\gamma}) = Tr(\tilde{\omega} X - \tilde{\omega} \tilde{\gamma} \tilde{\omega}), \]

where \( X := \tilde{\chi} + \tilde{\omega} \tilde{\gamma} + \tilde{\gamma} \tilde{\omega} \) and the classical algebra is

\[ [\hat{\omega}_2, \hat{\gamma}_1]_+ = P_{12}, \quad [\hat{\omega}_2, \hat{\omega}_1]_+ = 0 = [\hat{\gamma}_2, \hat{\gamma}_1]_+, \]

\[ [X_2, X_1] = P_{12}(X_2 - X_1), \quad [X_2, \hat{\omega}_1] = 0 = [X_2, \hat{\gamma}_1]. \]
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