The problem with the Brier score

Stephen Jewson*
RMS, London, United Kingdom

March 31, 2022

Abstract

The Brier score is frequently used by meteorologists to measure the skill of binary probabilistic forecasts. We show, however, that in simple idealised cases it gives counterintuitive results. We advocate the use of an alternative measure that has a more compelling intuitive justification.

1 Introduction

Users of meteorological forecasts need to be able to judge which forecasts are the best in order to decide which to use. We distinguish two cases. The first case is one in which the user plans to use the forecast for making a certain specific decision the details of which can be specified entirely in advance. The second is one in which the user plans to use the forecast for making one or more decisions which cannot be specified in detail in advance.

In the first case it may be possible to decide which forecast is the best by analysing the effect of using different forecasts on the quality of the final decisions made (for an example of this situation see Richardson (2000)). In the second case, however, the user cannot convert forecasts into decisions ahead of time because they do not know what decisions they are going to have to make. By the time they know what decision they are going to have to make, they do not have time to re-evaluate the available forecasts and potentially switch to a different forecast provider. In this second case forecasts have to be analysed and compared on their own merits, rather than on the merits of the decisions that can be based on them. In such a situation, the forecast user needs standard measures which can distinguish between forecasts at a general level. It is this second case that we will consider.

Forecasts can be divided into forecasts of the expectation of future outcomes and probabilistic forecasts that give probabilities of different outcomes. Probabilistic forecasts can then be divided into continuous and discrete probabilistic forecasts. A continuous probabilistic forecast gives a continuous density for the distribution of possible outcomes. We have discussed how to measure the skill of such forecasts in Jewson (2003b) and have applied the measures we propose to the calibration and the comparison of forecasts in a number of studies such as Jewson et al. (2003) and Jewson (2003a).

Discrete probabilistic forecasts give probabilities for a number of discrete events. Any number of events can be considered, but in this article we will restrict ourselves to the case of only two events, which we call a binary probabilistic forecast. We will address the question of how binary probabilistic forecasts can be compared.

One of the standard tools used by meteorologists to answer the question of which of two binary probabilistic forecasts is the better is the Brier score, first used over 50 years ago (Brier, 1950), and still in use today (see, for example, Vitart (2003), page 25). Nevertheless, we are going to argue that the Brier score is flawed. This is not something that can be proven mathematically, of course. Our arguments will be based on an appeal to intuition: we will present a simple case in which we believe it is intuitively clear which of two forecasts is the better, and we will show that the Brier score then comes to the opposite conclusion to our intuition i.e. it gives the wrong answer. We will then present an alternative score that overcomes this problem, that has a definition that accords more clearly with intuition, and that is also more firmly grounded in standard statistical theory.

*Correspondence address: RMS, 10 Eastcheap, London, EC3M 1AJ, UK. Email: x0stephenjewson.com

1A good example is the root mean square error, which is a general measure used for comparing forecasts for the expectation...
2 The Brier Score

The Brier score for a binary event is defined as:

\[ b = \langle (f - o)^2 \rangle \] (1)

where \( f \) is a forecast for the probability that an event \( X \) will happen, and \( o \) is an observation which takes the value 1 if the event happens and 0 otherwise. Lower values of the Brier score indicate better forecasts. A detailed discussion is given in Toth et al. (2003).

We can expand the Brier score as:

\[ b = \langle f^2 \rangle - 2 < fo > + \langle o^2 \rangle \] (2)

When we are comparing two forecasts on the same observed data set the difference in the Brier score is given by:

\[ b_2 - b_1 = \langle f_2^2 \rangle - 2 < f_2 o > - \langle f_1^2 \rangle + 2 < f_1 o > \] (3)

where the \( \langle o^2 \rangle \) term has cancelled because it is the same for both forecasts. If this difference is positive \( (b_2 > b_1) \) then we conclude that \( b_1 \) is the better forecast.

A particularly simple case is where the forecast probabilities have constant values, giving:

\[ b_2 - b_1 = f_2^2 - 2 f_2 < o > - f_1^2 + 2 f_1 < o > \] (4)

A further simplification is possible if the event occurs with a constant probability \( p \), in which case \( < o > = p \) and

\[ b_2 - b_1 = f_2^2 - 2 f_2 p - f_1^2 + 2 f_1 p \] (5)

3 A simple example

We now consider a very simple example, with constant event probability and constant forecast probabilities. We set \( p = \frac{1}{10} \), and consider the forecasts \( f_1 = 0 \) and \( f_2 = \frac{1}{4} \).

In this case the difference between the Brier scores is given by:

\[ b_2 - b_1 = f_2^2 - 2 f_2 p - f_1^2 + 2 f_1 p \] (6)

\[ = \left( \frac{1}{4} \right)^2 - 2 \frac{1}{4} \frac{1}{10} \]

\[ = \frac{1}{16} - \frac{1}{20} \]

\[ = \frac{1}{80} \]

The Brier score leads us to conclude that forecast \( f_1 \) is the better forecast. However, this does not agree with our intuition. Forecast \( f_1 \) is a disaster: it predicts a zero probability (a very strong statement!) for something that happens not infrequently. Forecast \( f_1 \) is completely invalidated whenever event \( X \) actually occurs (on average, 1 in every 10 trials). Forecast \( f_2 \), on the other hand, is not so bad. It gives a lowish probability for something that does indeed occur with a low probability. Its only fault is that the probability is not exactly correct.

The reason that the Brier score makes this mistake is that it does not penalise forecasts that predict a zero probability strongly enough when they are wrong, even though our intuition tells us that they should be heavily penalised. More generally, the Brier score does not penalise forecasts that give very small probabilities when they should be giving larger probabilities to the same extent that we penalise such forecasts with our intuition. This is because the Brier score is based on a straight difference between \( f \) and \( o \). Our intuition, on the other hand, considers the difference between probabilities of 0% and 10% to be very different from the difference between probabilities of 40% and 50%. Intuition apparently uses fractional or logarithmic rather than absolute differences in probability.

One can easily construct other examples that illustrate this point. The more extreme the events considered, the more striking is the problem with the Brier score. Consider, for example, \( p = \frac{1}{100} \), \( f_1 = 0 \) and \( f_2 = \frac{1}{400} \). Again the Brier score prefers \( f_1 \), while our intuition considers \( f_1 \) to be a failure, and \( f_2 \) to be a reasonably good attempt at estimating a very small probability.

We conclude that the Brier score cannot be trusted to make the right decision about which of two forecasts is better. It should also not be used to calibrate forecasts or evaluate forecasting systems since it will over-encourage prediction of very small or zero probabilities. We need a different measure.
4 The likelihood score

The standard measure used in classical statistics for testing which of two distributions gives the best fit to data is the likelihood \( L \) defined as the probability (or probability density) of the observations given the model and the parameters of the model [Fisher, 1922]. In our case this becomes the probability of the observations given the forecast.

We advocate the likelihood as the best metric for calibrating and comparing continuous probabilistic forecasts (see the previous citations) mainly on the basis that it is very intuitively reasonable: the forecast that gives the highest probability for the observations is the better forecast. We also advocate the likelihood as the best metric for calibrating and comparing binary forecasts. In this case the likelihood is given by:

\[
L = p(x|f)
\]

where \( x \) is the full set of observations and \( f \) is the full set of forecasts. If we assume that the forecast errors are independent in time then this becomes:

\[
L = \Pi_{i=1}^{n} p(x_i|f_i) = \Pi_{i=1}^{n} o_i f_i + (1 - o_i)(1 - f_i)
\]

We can also use the log-likelihood, which gives a more compressed range of values, and is given by:

\[
l = \ln L = \ln \left[ \Pi_{i=1}^{n} o_i f_i + (1 - o_i)(1 - f_i) \right] = \sum_{i=1}^{n} \ln \left[ o_i f_i + (1 - o_i)(1 - f_i) \right]
\]

If we put all cases of event \( X \) occurring into set \( A \), and all cases of event \( X \) not occurring into set \( B \) then:

\[
L = \Pi_{A} f_i \Pi_{B}(1 - f_i)
\]

and

\[
l = \sum_{A} f_i + \sum_{B} (1 - f_i)
\]

If we now consider the special case in which \( f \) is constant then:

\[
L = f_i^a (1 - f_i)^b
\]

and

\[
l = a \ln f_i + b \ln (1 - f_i)
\]

where \( a \) is the number of occurences of \( X \), \( b \) is the number of occurences of not \( X \), and \( b = n - a \).

If any of the predictions \( f \) are 0 or 1 (i.e. are completely certain) then \( L = 0 \) and \( l = -\infty \). If not, then \( L > 0 \) and \( l > -\infty \). We see that use of the likelihood penalises the use of probability forecasts with values of 0 or 1 very heavily. Such forecasts get the worst possible score, as they should (since one can never be completely certain).

In our simple example the difference in likelihoods for the two forecasts is:

\[
L_2 - L_1 = f_2^a (1 - f_2)^b - f_1^a (1 - f_1)^b
\]

Since this is positive for all samples we see that the likelihood concludes that forecast 2 is better, in line with our intuition.

5 Summary

Meteorologists have used the Brier score to compare binary probabilistic forecasts for over 50 years. However, we find that in simple cases it makes the wrong decision as to which is the better of two forecasts (where we define wrong in terms of our intuition). We reach this conclusion independently of any detailed analysis of the preferences of the user of the forecast.

We advocate scores based on the likelihood as a replacement for the Brier score. On the one hand the likelihood is conceptually simpler than the Brier score: it decides which forecast is better simply according
to which forecast gives the higher probability for the observed data, which seems immediately reasonable. On the other hand the likelihood accords with our intuition in the simple example that we present, and punishes forecasts that give probabilities of 0 and 1 appropriately. We conclude that use of the Brier score should be discontinued, and should be replaced by a score based on the likelihood.

6 Acknowledgements

The author would like to thank Rie Kondo and Christine Ziehmann for useful discussions, and Christine for reading the manuscript and making some helpful comments.

7 Legal statement

The author was employed by RMS at the time that this article was written. However, neither the research behind this article nor the writing of this article were in the course of his employment, (where ‘in the course of his employment’ is within the meaning of the Copyright, Designs and Patents Act 1988, Section 11), nor were they in the course of his normal duties, or in the course of duties falling outside his normal duties but specifically assigned to him (where ‘in the course of his normal duties’ and ‘in the course of duties falling outside his normal duties’ are within the meanings of the Patents Act 1977, Section 39). Furthermore the article does not contain any proprietary information or trade secrets of RMS. As a result, the author is the owner of all the intellectual property rights (including, but not limited to, copyright, moral rights, design rights and rights to inventions) associated with and arising from this article. The author reserves all these rights. No-one may reproduce, store or transmit, in any form or by any means, any part of this article without the author’s prior written permission. The moral rights of the author have been asserted.

References

G Brier. Verification of forecasts expressed in terms of probabilities. *Monthly Weather Review*, 78:1–3, 1950.

R Fisher. On the mathematical foundations of statistics. *Philosophical Transactions of the Royal Society, A*, 222:309–368, 1922.

S Jewson. Moment based methods for ensemble assessment and calibration. *arXiv:physics/0309042*, 2003a. Technical report.

S Jewson. Use of the likelihood for measuring the skill of probabilistic forecasts. *arXiv:physics/0308046*, 2003b. Technical report.

S Jewson, A Brix, and C Ziehmann. A new framework for the assessment and calibration of ensemble temperature forecasts. *Atmospheric Science Letters*, 2003. Submitted.

D Richardson. Skill and relative economic value of the ECMWF ensemble prediction system. *Q. J. R. Meteorol. Soc.*, pages 649 – 668, 2000.

Z Toth, O Talagrand, G Candille, and Y Zhu. Probability and ensemble forecasts. In *Forecast verification*, chapter 7, pages 137–162. Wiley, 2003.

F Vitart. Monthly forecasting system. *ECMWF Research Department*, 10 2003.