“Integrating in” and exact superpotentials in 4d

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We discuss integrating out matter fields and integrating in matter fields in four dimensional supersymmetric gauge theories. Highly nontrivial exact superpotentials can be easily obtained by starting from a known theory and integrating in matter.
1. Introduction

The Wilsonian effective superpotential of a four dimensional supersymmetric gauge theory is highly constrained and can often be obtained exactly [1,2]. In particular, symmetries constrain its functional form, dynamical considerations dictate its singularity structure, and holomorphy then provides the exact superpotential in all of field space by analytic continuation. The exact superpotentials so obtained can be highly non-trivial, reflecting interesting non-perturbative phenomena. By applying these techniques to a variety of examples, some conjectured general principles have emerged. These principles, discussed in [2], have to do with the linearity of the superpotentials in the coupling constants (see [3] for a related discussion). We discuss how and when it is possible to use these simple principles to easily obtain exact superpotentials. The technique, which we refer to as “integrating in matter”, can be easily applied even for highly non-trivial theories, where the more direct considerations of [2] would prove very difficult.

2. Integrating matter out and in

Consider two theories. The first, which we will refer to as the “downstairs” theory, is a supersymmetric gauge theory with gauge group $G = \prod_s G_s$ and matter chiral superfields $\phi_i$ transforming in representations $R_i$ of $G$. At the classical level there are flat directions in $\phi_i$ field space, with coordinates given by the gauge invariant polynomials $X_r$ in the fields $\phi_i$. The non-perturbative gauge dynamics generates an effective superpotential $W_d(X_r, \Lambda_{s,d}^{n_{s,d}})$ for the fields $X_r$. In this superpotential, $\Lambda_{s,d}$ are the scales of the $G_s$ gauge dynamics in the downstairs theory; they are related to the Wilsonian running coupling constants by the exact [4] 1-loop beta functions as $\Lambda_{s,d}^{n_{s,d}} = E^{n_{s,d}} e^{-8\pi^2/g_s^2(E)}$. Here $n_{s,d} = \frac{1}{2} (3G_s - \mu_s)$ with $G_s$ the index of the adjoint representation of gauge group $G_s$ and $\mu_s \equiv \sum_i \mu^i_s$, where $\mu^i_s$ is the index of the representation $R_i$ of matter field $\phi_i$ in the gauge group $G_s$.

The second theory, which we refer to as the “upstairs” theory, differs from the downstairs theory only in that it contains an additional matter field $\hat{\phi}$ in a representation $\hat{R}$ of $G$.

1 The $X_r$ are a finite basis of the basic gauge invariants; all gauge invariants can be reduced to sums and products of the $X_r$. 
and, if \( \hat{R} \) is not real, a conjugate field \( \hat{\phi}^c \) such that the “meson” \( \hat{M} = \hat{\phi}\hat{\phi}^c \) is gauge invariant. The gauge invariant polynomials of the upstairs theory are the \( X_r \) of the downstairs theory along with some additional polynomials \( X_{\hat{r}} \). The gauge dynamics generates an effective superpotential \( W_u(X_r, X_{\hat{r}}, \Lambda_s^{n_s}) \). The scales \( \Lambda_s \) of the gauge dynamics in the upstairs theory are related to the Wilsonian running coupling constants by 
\[
\Lambda_s^{n_s} = E_s^{n_s} \frac{e^{-8\pi^2/g_s^2(E)}}{g_s^2(E)},
\]
where \( n_s = n_{s,d} - \frac{1}{2} \hat{\mu}_s \) with \( \hat{\mu}_s \) the index of \( \hat{R} \) plus that of its conjugate (if it is not real) in \( G_s \).

Consider now modifying the upstairs theory by turning on a tree level superpotential 
\[
W_{\text{tree}} = \sum_{\hat{r}} g_{\hat{r}} X_{\hat{r}} \text{ for the macroscopic variables containing the field } \hat{\phi} \text{ or its conjugate.}
\]
We will now assume the following

- **Principle of Linearity**: the full superpotential \( W_f \) is then simply
  \[
  W_f(X_r, X_{\hat{r}}, \Lambda_s^{n_s}, g_{\hat{r}}) = W_u(X_r, X_{\hat{r}}, \Lambda_s^{n_s}) + \sum_{\hat{r}} g_{\hat{r}} X_{\hat{r}}.
  \]

At this stage we could consider integrating out the field \( \hat{\phi} \) and, correspondingly, the fields \( X_{\hat{r}} \). Solving for the fields \( X_{\hat{r}} \) in (2.1) using their equations of motion \( \frac{\partial W_f}{\partial X_{\hat{r}}} = 0 \) will yield a new superpotential

\[
W_f(X_r, \langle X_{\hat{r}} \rangle, \Lambda_s^{n_s}, g_{\hat{r}}) \equiv W_l(X_r, \Lambda_s^{n_s}, g_{\hat{r}}) = W_d(X_r, \Lambda_s^{n_s,d}) + W_I(X_r, \Lambda_s^{n_s}, g_{\hat{r}}),
\]

where \( W_d \) is the dynamically generated superpotential of the downstairs theory and \( W_I \) is an additional superpotential, which is irrelevant in the renormalization group sense. In particular, \( W_I \to 0 \) for \( \hat{m} \to \infty \) where \( \hat{m} \) is the \( g_{\hat{r}} \) for \( \hat{M} \) in \( W_{\text{tree}} \), the mass of the field \( \phi \). Also, \( W_I = 0 \) when \( g_{\hat{r}} = 0 \) for all \( g_{\hat{r}} \) besides \( \hat{m} \).

We now introduce a second conjectured principle

- **Simple Thresholds**: The scales \( \Lambda_{s,d} \) of the downstairs theory in (2.2) are exactly related to the scales \( \Lambda_s \) of the upstairs theory by matching the running couplings \( g_{s}^2(E) \) at the scale \( E = \hat{m} \), independent of all other couplings:
  \[
  \Lambda_{s,d}^{n_{s,d}} = \Lambda_s^{n_s} \hat{m}^{\hat{\mu}_s/2}.
  \]

This conjecture is a generalization of the conjecture in [2] stating that the superpotential with the massive fields \( S_s \) integrated in is linear in the couplings \( \log \Lambda_s^{n_s} \). It is possible
to prove these conjectures in examples on a case by case basis but no general proof is known. To summarize, if the superpotential $W_u$ is known, we can “flow” down to the superpotential of the downstairs theory.

Although we have integrated out the variables $X_r$ in obtaining (2.2), we actually have not lost any information. This is very different from the usual idea of coarse graining and is related to the linearity principle. In fact, $W_l$ is simply a Legendre transform of $W_u$; it is possible to obtain $W_u$ from $W_l$ by an inverse Legendre transform. In particular, consider a theory with the superpotential

$$W_n(X_r, Y_{\bar{r}}, \Lambda_s^{n_s}, g_{\bar{r}}) = W_l(X_r, \Lambda_s^{n_s}, g_{\bar{r}}) - \sum_{\bar{r}} g_{\bar{r}} Y_{\bar{r}},$$

(2.4)

where $Y_{\bar{r}}$ are some additional gauge singlets which, as in (2.1), can be added to the theory without affecting $W_l$. Now suppose that we consider the couplings $g_{\bar{r}}$ in the above superpotentials to be fields. If we integrate out the $g_{\bar{r}}$ from (2.4) by solving for them using their equations of motion $\frac{\partial W}{\partial g_{\bar{r}}} = 0$, we will obtain

$$W_n(X_r, Y_{\bar{r}}, \Lambda_s^{n_s}, \langle g_{\bar{r}} \rangle) = W_u(X_r, X_{\bar{r}} = Y_{\bar{r}}, \Lambda_s^{n_s}).$$

(2.5)

The equality in (2.5) follows because we could have added the singlets $Y_{\bar{r}}$ and their contribution in (2.4) to the original theory (2.1) to obtain the superpotential

$$W = W_u(X_r, X_{\bar{r}}, \Lambda_s^{n_s}) + \sum_{\bar{r}} g_{\bar{r}}(X_{\bar{r}} - Y_{\bar{r}}).$$

(2.6)

Now the $g_{\bar{r}}$ are simply Lagrange multipliers and, upon integrating them out and setting $Y_{\bar{r}}$ to $X_{\bar{r}}$, we have done nothing. We thus see that the superpotential $W_u(X_r, X_{\bar{r}}, \Lambda_s^{n_s})$ of the upstairs theory can be obtained from

$$W_n = W_l(X_r, \Lambda_s^{n_s}, g_{\bar{r}}) - \sum_{\bar{r}} g_{\bar{r}} X_{\bar{r}},$$

(2.7)

by integrating out the $g_{\bar{r}}$. The key point is that the downstairs theory can be much simpler than the upstairs one and hence it is often much easier to obtain $W_l$ by direct methods than it would be to directly analyze the upstairs theory. Nevertheless, once $W_l$ is known, we can “flow” up to the superpotential $W_u$ of the upstairs theory by using (2.7).
3. Theories with only quadratic gauge invariants

When the gauge invariants \( X^r \) are only quadratic, the tree level superpotential added in (2.1) simply gives the field \( \hat{\phi} \) and its conjugate a mass \( \hat{m} \); the superpotential \( W_I = 0 \) in (2.2). In theories with only quadratic gauge invariants, it is sensible to choose pure glue Yang Mills as the downstairs theory, with all matter to be integrated in using (2.7).

Consider, for example, \( SU(N_c) \) QCD with \( N_f < N_c \) flavors as the “upstairs” theory and \( SU(N_c) \) Yang-Mills as the “downstairs” theory. The \( X^r \) are the mesons \( M = Q\tilde{Q} \) in the \((N_f, N_f)\) of the global \( SU(N_f)_L \times SU(N_f)_R \) flavor symmetry. We can decouple these fields by turning on the tree level mass terms \( W_{\text{tree}} = \text{Tr} \ m M \), where the mass matrix \( m \) is in the \((\tilde{N}_f, \tilde{N}_f)\) of the flavor symmetry, to obtain pure glue \( SU(N_c) \) Yang-Mills theory.

\( SU(N_c) \) Yang Mills theory is known to have gluino condensation\(^2\) which is conveniently described by the effective superpotential for the (massive) glueball superfield \( S = -W^2 \)

\[
W_d = S \left[ \log \left( \frac{\Lambda_d^{3N_c}}{S^{N_c}} \right) + N_c \right].
\] (3.1)

The scale \( \Lambda_d \) is related to the scale \( \Lambda \) of the upstairs theory by the matching condition on the running coupling at the scales where the fields decouple; as in (2.3), \( \Lambda_d^{3N_c} = \Lambda^{3N_c-N_f} \det m \).

Using (2.7), the superpotential of the upstairs theory is obtained from

\[
W_n = S \left[ \log \left( \frac{\Lambda_d^{3N_c-N_f} \det m}{S^{N_c}} \right) + N_c \right] - \text{Tr} \ m M
\] (3.2)

by integrating out the “field” \( m \). Setting \( \frac{\partial W_n}{\partial m} = 0 \) gives \( \langle m \rangle = SM^{-1} \) and (3.2) becomes

\[
W_u = S \left[ \log \left( \frac{\Lambda_d^{3N_c-N_f}}{S^{N_c-N_f} \det Q\tilde{Q}} \right) + N_c - N_f \right].
\] (3.3)

Because \( S \) is always massive, it should be integrated out. Upon doing so, (3.3) indeed gives the correct low energy effective superpotential of the upstairs theory [6,7].

As another example, consider \( SU(2) \) gauge theory with \( 2N_f \) doublets, \( Q_{i\alpha} \) with \( i = 1, \ldots, 2N_f \) a \( SU(2N_f) \) flavor index (with \( N_f < 6 \) for asymptotic freedom) and \( \alpha \) a \( SU(2) \)

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\(^2\) In [3,4], gaugino condensation was proven to occur by starting with the calculated [3] instanton induced superpotential for \( N_f = N_c - 1 \) and integrating out matter. We are here taking the opposite route as an illustration of integrating in matter.
color index, as the upstairs theory. Decoupling the doublets with the mass terms \( W_{\text{tree}} = \frac{1}{2} m^{ij} V_{ij} \), where \( m \) is the skew-symmetric mass matrix, \( V_{ij} = Q_{ie} Q_{jd} \epsilon^{cd} \) are the gauge invariant objects, and the \( \frac{1}{2} \) corrects for a double counting, gives \( SU(2) \) Yang-Mills as the downstairs theory. The scales are related as in (2.3) by \( \Lambda_{\frac{3}{2} G_s} = \Lambda_{\frac{3}{2} G_s} - N_f \text{Pf } m = \Lambda_{\frac{3}{2} G_s}^0 \). Using (3.1) and (2.7), the superpotential of the original theory is thus obtained from

\[
W_n = S \left[ \log \left( \frac{\Lambda_{\frac{3}{2} G_s}^0 \text{Pf } m}{S^2} \right) + 2 \right] - \frac{1}{2} m^{ij} V_{ij} \quad (3.4)
\]

by integrating out the \( m \). Enforcing \( \partial W_n / \partial m = 0 \) gives \( \langle m \rangle = SV^{-1} \) and (3.4) becomes

\[
W_u = S \left[ \log \left( \frac{\Lambda_{\frac{3}{2} G_s}^0 \text{Pf } V}{S^2 - N_f} \right) + 2 - N_f \right]. \quad (3.5)
\]

Integrating out \( S \), these \( W_u \) are indeed correct, giving results explained in [8].

For the general case of a theory with gauge group \( G = \prod_s G_s \) and matter fields \( \phi_i \) such that all gauge invariants \( X_r \) are quadratic in the \( \phi_i \), we can take for the downstairs theory the different decoupled pure glue \( G_s \) Yang Mills theories. The superpotential \( W_l \) in (2.7) is then simply a sum over the decoupled gaugino condensation superpotentials, i.e. (3.1) with \( N_c \) generalized to \( \frac{1}{2} G_s \), of each \( G_s \) Yang-Mills theory.

It is now possible to integrate in the matter fields. To take any non-abelian flavor symmetries into account, label the \( \phi_i \) as \( \phi_{r,a} \), where \( a \) is a flavor symmetry index. The gauge invariant objects are the mesons \( (M_r)_a^b = \phi_{r,a} \tilde{\phi}_{r,b} \) and the the mass terms are \( \sum_r m_r M_r \), with the appropriate sum over the flavor indices implicit. The conjecture (2.3) gives for the matching of the scales

\[
\Lambda_{s,d}^{\frac{3}{2} G_s} = \Lambda_{s}^{\frac{3}{2} G_s - \mu_s} \prod_i m_i^{\frac{1}{2}}, \quad (3.6)
\]

where \( m_i \) are the masses where the matter fields \( \phi_i \) decouple. When there are non-abelian flavor symmetries (as in the above examples) under which the masses transform, the \( m_i \) in (3.6) are to be understood as the eigenvalues of the mass matrices. The products of eigenvalues \( \prod_i m_i^{\frac{1}{2}} \) are invariant under the non-abelian symmetries and will thus be given by products of \( \text{det } m_r \) (or Pf \( m_r \) for pseudo-real representations). Using (2.7), the superpotential of the theory with matter is obtained from

\[
W_n = \sum_s S_s \left[ \log \left( \frac{\Lambda_{s}^{\frac{3}{2} G_s - \mu_s} \prod_i m_i^{\frac{1}{2}}}{S_s^{G_s/2}} \right) + \frac{1}{2} G_s \right] - \sum_r m_r M_r \quad (3.7)
\]
by integrating out the $m_r$ and the $S_s$.

As an example, consider $\prod_{s=1}^4 SU(2)_s$ gauge theory with matter content in the representations $\phi_{1,a} = (2, 1, 1, 1)$ for $a = 1 \ldots 4$, $\phi_2 = (2, 2, 1, 1)$, $\phi_3 = (1, 2, 2, 1)$ and $\phi_4 = (1, 1, 2, 2)$. The gauge invariants are $(M_1)_{ab} = \phi_{1,a} \phi_{1,b}$, in the 6 of the $SU(4)$ global flavor symmetry, $M_2 = \phi_2^2$, $M_3 = \phi_3^2$, and $M_4 = \phi_4^2$. The superpotential (3.7) is

$$W_n = S_1 \left[ \log \left( \frac{A_1^3(\text{Pf } m_1) m_2}{S_1^2} \right) + 2 \right] + S_2 \left[ \log \left( \frac{A_2^4 m_2 m_3}{S_2^2} \right) + 2 \right] - \frac{1}{2} \text{Tr } m_1 M_1$$

$$+ S_3 \left[ \log \left( \frac{A_3^4 m_3 m_4}{S_3^2} \right) + 2 \right] + S_4 \left[ \log \left( \frac{A_4^5 m_4}{S_4^2} \right) + 2 \right] - \sum_{r=2}^4 m_r M_r.$$  

Integrating out the $S_s$ and $m_r$ by their equations of motion gives a superpotential $W_u = S_4 - S_1$, where $S_1$ and $S_4$ are obtained by solving

$$\frac{A_1^3(S_1 + S_2)}{\text{Pf } M_1 M_2} = \frac{A_2^4(S_1 + S_2)(S_2 + S_3)}{S_2^2 M_2 M_3} = \frac{A_3^4(S_2 + S_3)(S_3 + S_4)}{S_3^2 M_3 M_4} = \frac{A_4^5(S_3 + S_4)}{S_4^2 M_4} = 1.$$  

Although this gives a superpotential which is very complicated, reflecting some of the complicated non-perturbative dynamics, it was obtained simply from gaugino condensation in the decoupled downstairs Yang-Mills theories along with the simple matching relations (3.6). All of the superpotentials discussed in [2] can be easily obtained using this technique.

4. Theories with non-quadratic gauge invariants

Theories with non-quadratic gauge invariants are more complicated because the superpotentials $W_I$ in (2.2) are nonzero. In the simplest case we would have $W_I = W_{\text{tree},d}$, the superpotential obtained from integrating $\hat{\phi}$ and its conjugate out from the $W_{\text{tree}}$ in (2.1) by their equations of motion. Writing $W_I = W_{\text{tree},d} + W_\Delta$, it is sometimes possible to use the symmetries, along with the requirement that $W_\Delta \to 0$ in the $\hat{m} \to \infty$ limit (where only $W_d$ remains) and also in the limit when the $\Lambda_s \to 0$ (where only $W_{\text{tree},d}$ remains), to argue that $W_\Delta = 0$. When this is the case, the “integrating in” procedure is still useful. Even in this case, once a nonzero tree level superpotential is generated, integrating out the remaining matter would result in a terrible mess. So we will be unable to obtain as nice of an expression as (3.7). Nevertheless, the technique of integrating in matter can be used to easily obtain superpotentials on a case by case basis.
As an example, consider $SU(N_c)$ QCD with $N_c$ flavors as the upstairs theory and take $SU(N_c)$ QCD with $N_c - 1$ flavors as the downstairs theory. By a flavor rotation we can add the mass term $m Q_N \tilde{Q}_N = m M_{NN}$ only for the $N$-th flavor. In addition we should add couplings for $B = \det Q$ and $\tilde{B} = \det \tilde{Q}$ since they involve the fields $Q_N$ and $\tilde{Q}_N$: $W_{\text{tree}} = m M_{NN} + b B + \tilde{b} \tilde{B}$. The superpotential (2.7) is $W_n = W_d + W_{\text{tree},d} + W_\Delta - W_{\text{tree}}$. The dynamically generated superpotential of the downstairs theory is $W_d = \frac{\Lambda^{2N_c+1}_d}{\det M_d}$, where $M_d$ are the mesons involving the $N_c - 1$ flavors of the downstairs theory. Integrating out the fields $Q_N$ and $\tilde{Q}_N$ from $W_{\text{tree}}$ gives $W_{\text{tree},d} = -\frac{b \tilde{b}}{m} \det M_d$. The symmetries determine $W_\Delta = \frac{\Lambda^{2N_c+1}_d}{bb \det M_d}$ where, because the gauge group is completely broken by $\det M_d$, $f(u) = \sum_{n=0}^{\infty} a_n u^n$. Adjusting the relative strength of the limits $m \to \infty$ and $\Lambda_d \to 0$, where it is known that $W_\Delta \to 0$, shows that $W_\Delta = 0$ everywhere. The matching condition (2.3) on the scales is $\Lambda^{2N_c+1}_d = m \Lambda^{2N}$, independent of $b$ and $\tilde{b}$. It is possible to prove this: again, the symmetries would allow the equality to be modified by a function $g \left( \frac{\Lambda^{2N_c+1}_d}{bb \det M_d} \right)$. We know $g \to 1$ for $m \to \infty$ and also for $\det M_d \to \infty$, where the theory is very weakly coupled. Adjusting the relative strength of these limits gives $g=1$ identically.

To summarize, we have obtained for the superpotential (2.7)

$$W_n = \frac{m \Lambda^{2N_c}}{\det M_d} - \frac{b \tilde{b}}{m} \det M_d - m M_{NN} - b B - \tilde{b} \tilde{B}. \quad (4.1)$$

The superpotential of the upstairs theory is obtained from (4.1) by integrating out $m$, $b$, and $\tilde{b}$. Doing so yields

$$W_u = 0 \quad \text{with} \quad \det M - B \tilde{B} = \Lambda^{2N}, \quad (4.2)$$

where we substituted the flavor invariant quantity $\det M$ for the quantity $M_{NN} \det M_d$, obtained because of our particular choice of integrating out the $N$-th flavor. This is indeed the quantum deformed moduli space of vacua obtained (by similar reasoning) in [1].

As another example, consider $SU(2)_L \times SU(2)_R$ gauge theory with matter in the representations $Q = (2,2)$, $L_\pm = (2,1)$ and $R_\pm = (1,2)$. Without the field $Q$, this would be the two decoupled $SU(2)_L$ and $SU(2)_R$ gauge theories, each with a single flavor; we will take this as the downstairs theory: $W_d = \frac{\Lambda^{2}_{L,d}}{X_L} + \frac{\Lambda^{2}_{R,d}}{X_R}$, where $X_L = L_+ L_-$ and $X_R = R_+ R_-$. To get from the upstairs theory to this downstairs theory we would add...
the tree level superpotential $W_{\text{tree}} = m_Q X_Q + \vec{\lambda} \cdot \vec{Z}$ where $X_Q = Q^2$ and $\vec{Z} = LQR$, in the $(2,2)$ representation of the global $SU(2) \times SU(2)$ flavor symmetry. Integrating out the field $Q$ at tree level gives $W_{\text{tree},d} = -\frac{\bar{\lambda}^2}{4m_Q}X_LX_R$. The superpotential (2.7) is given by $W_n = W_d + W_{\text{tree},d} + W_{\Delta} - W_{\text{tree}}$. The symmetries can be used [1,2] to show $W_{\Delta} = \frac{\bar{\lambda}^2}{m_Q}X_LX_Rf(\frac{m_Q\Lambda^5_{L,d}}{\lambda^2X_L^2X_R} - \frac{\Lambda^5_{R,d}X_L}{\lambda X_LX_R})$. Because the gauge group is completely broken for nonzero $X_L$ and $X_R$, $f(u,v) = \sum_{n=0}^{\infty} \sum_{m\leq n} a_{n,m}u^n v^m$. Further, $W_{\Delta} \to 0$ in the limits $m_Q \to \infty$ or $\Lambda_L, \Lambda_R \to 0$. Adjusting the relative strength of these limits gives $W_{\Delta} = 0$. The matching (2.3) of the scales is $\Lambda^5_{L,d} = \Lambda^4_{L}m_Q$ and $\Lambda^5_{R,d} = \Lambda^4_{R}m_Q$, independent of $\bar{\lambda}$. Again, it is here possible to prove this using the symmetries and the behavior in different limits. Using (2.7), the effective superpotential of the upstairs theory is thus obtained from

$$W_n = \frac{\Lambda^4_{L}m_Q}{X_L} + \frac{\Lambda^4_{R}m_Q}{X_R} - \frac{\bar{\lambda}^2}{4m_Q}X_LX_R - m_QX_Q - \vec{\lambda} \cdot \vec{Z}, \tag{4.3}$$

by integrating out $m_Q$ and $\vec{\lambda}$. Doing so yields

$$W_u = 0 \quad \text{with} \quad X_QX_LX_R - \vec{Z}^2 = X_L\Lambda^4_{R} + X_R\Lambda^4_{L}. \tag{4.4}$$

As another derivation of (4.4), take the theory without the fields $L_\pm$ as the downstairs one. The dynamically generated superpotential of this downstairs theory can be determined using (3.7) and was discussed in detail in [2]: $W_d = \Lambda^5_{L,d}X_R/(X_QX_R - \Lambda^4_{R,d})$. In addition, there is a constraint in the downstairs theory $X_QX_R - (QR_+)(QR_-) = \Lambda^4_{R,d}$. This downstairs theory is obtained from our upstairs one by adding $W_{\text{tree}} = m_LX_L + \vec{\lambda} \cdot \vec{Z}$ and integrating out the fields $L_\pm$. Integrating out $L_\pm$ at tree level gives $W_{\text{tree},d} = -\frac{\bar{\lambda}^2}{4m_L}(QR_+)(QR_-) = -\frac{\bar{\lambda}^2}{4m_L}(X_QX_R - \Lambda^4_{d,R})$, where we used the mentioned constraint. Again, the symmetries and the limiting behaviors determine $W_{\Delta} = 0$. The scales match as $\Lambda^5_{L,d} = m_L\Lambda^4_{L}$ and $\Lambda^4_{R,d} = \Lambda^4_{R}$. Using (2.7), $W_u$ can be obtained from

$$W_n = \frac{m_L\Lambda^4_{L}X_R}{X_QX_R - \Lambda^4_{R}} - \frac{\bar{\lambda}^2}{4m_L}(X_QX_R - \Lambda^4_{R}) - m_LX_L - \vec{\lambda} \cdot \vec{Z}, \tag{4.5}$$

by integrating out $m_L$ and $\vec{\lambda}$. Doing so indeed reproduces the same result (4.4).

Although in these examples we could argue that $W_{\Delta} = 0$, this is not always the case. In fact, whenever $\mu_s > G_s$ for some $G_s$, the symmetries and limiting behaviors can allow a nonzero $W_{\Delta}$. A more direct and involved analysis would then be necessary.
5. Conclusions and limitations

We have discussed how matter can be integrated in, as well as integrated out. For theories with only quadratic gauge invariants, this allowed us to obtain the general expression (3.7). For theories with non-quadratic gauge invariants, the technique of integrating in matter is useful when it is possible to determine the obstructing superpotential $W_\Delta$.

An obvious general limitation of the technique of integrating in matter is that only non-chiral matter can be so integrated in. This is unfortunate since chiral theories are quite important: they can dynamically break supersymmetry. For these theories, at present, it is necessary to conduct the detailed analysis discussed in [2] on a case by case basis.

We should mention an important place where integrating in seemingly breaks down. Consider, for example, $SU(2)$ gauge theory with matter $\phi$ in the adjoint as the upstairs theory. Adding a mass term $m\phi^2$ to the upstairs theory and integrating out $\phi$ gives $SU(2)$ Yang-Mills as the downstairs theory, with scale matching $\Lambda_d^6 = \Lambda^4 m^2$. Applying our integrating in procedure then suggests that the upstairs theory is described by the superpotential $W_u = 0$ with a constraint $\phi^2 = \pm 2\Lambda^2$ – this is incorrect: $\phi^2$ is only so constrained for $m \neq 0$ [9]. This appears to contradict principle (2.1) but that is only because there is more to the story. As discovered in [9], there are additional matter fields, magnetic monopoles, which must be taken into account in order to properly describe this upstairs theory when $m=0$. The moral is that it is important to take care to correctly identify the spectrum of light fields involved in the low energy effective theory.

Finally, our analysis was based on assuming the principles (2.1) and (2.3). It would be useful to better understand their veracity.

Acknowledgements

I would like to thank R. Leigh and N. Seiberg for stimulating discussions and helpful comments. This work was supported in part by DOE grant #DE-FG05-90ER40559.
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