Generalized (2 + 1)-dimensional BTZ black holes via Hojman symmetry

F. Darabi∗, M. Golmohammadi†, A. Rezaei-Aghdam‡

Department of Physics, Azarbaijan Shahid Madani University
53714-161, Tabriz, Iran

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Abstract

In this paper, we use the Hojman symmetry approach in the context of \( f(R) \) gravity to find new generalized (2 + 1)-dimensional BTZ black hole solutions as well as the associated symmetry vectors.

1 Introduction

The (2 + 1)-dimensional gravity has no black hole solutions for vanishing cosmological constant \[1\]. However, BTZ black hole solutions have been obtained for (2 + 1)-dimensional gravity with a negative cosmological constant, defined by the following action \[2\]

\[
I = \frac{1}{2} \int dx^3 \sqrt{-g} (R - 2\Lambda),
\]

where \( \Lambda = -l^{-2} \) is characterized by a length scale \( l \). The line element in \((t, r, \phi)\) coordinates is given by

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2,
\]

\[
f(r) = \left( -m + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right),
\]

where the mass \( m \) and angular momentum \( J \) are two parameters corresponding to time displacement and rotational symmetries associated with two Killing vectors \( \partial_t \) and \( \partial_\phi \), respectively. Unlike the asymptotically flat Schwarzschild and Kerr black hole solutions having curvature singularity at \( r = 0 \), the BTZ black hole is asymptotically anti-de-Sitter (AdS) having no curvature singularity at \( r = 0 \). A BTZ black hole with \( J \not= 0 \) describes a spacetime with a constant negative curvature having outer and inner horizons \( r_+ \) (event horizon) and \( r_- \) (Cauchy horizon) respectively, given by

\[
r_{\pm}^2 = \frac{l^2}{2} \left( m \pm \sqrt{m^2 - \frac{J^2}{l^2}} \right).
\]

A Noether symmetry approach has already been developed by the authors \[3\] to obtain (2 + 1) dimensional black hole solutions in \( f(R) \) gravity \[4\]. An alternative approach, so called Hojman symmetry approach, has recently been received attention by which one can find new exact solutions \[5\]-[10]. Unlike the Noether symmetry approach which needs Lagrangian and Hamiltonian functions, in the Hojman symmetry approach we just need the symmetry vectors and the corresponding conserved charges which are easily obtained by using the equations of motion. In the present paper, we intend to obtain (2 + 1) dimensional BTZ black hole solutions in the context of \( f(R) \) gravity, using Hojman symmetry approach, as an alternative to the Noether symmetry approach.

∗Corresponding author. e-mail: f.darabi@azaruniv.ac.ir
†author. e-mail: golmohammadi@azaruniv.ac.ir
‡author. e-mail: rezaei-a@azaruniv.ac.ir
2 (2 + 1)-dimensional $f(R)$ gravity

The action for (2 + 1)-dimensional $f(R)$ gravity is given by

$$I = \frac{1}{2} \int d^3x \sqrt{-g} f(R). \quad (5)$$

We consider the line element in the following form [2]

$$ds^2 = [-N^2(r) + r^2 M^2(r)] dt^2 + N^{-2}(r) dr^2 + 2r^2 M(r) dt d\phi + r^2 d\phi^2, \quad (6)$$

where the radial functions $N(r)$ and $M(r)$ are considered as degrees of freedom. The Ricci scalar is obtained

$$R = -\frac{1}{2r}(4r N'^2 + 4r N N'' - r^3 M'^2 + 8NN'), \quad (7)$$

where $'$ denotes the derivative with respect to $r$. Using the method of Lagrange multipliers to set $R$ as a constraint of the dynamics, generalizing the degrees of freedom and defining a canonical Lagrangian $\mathcal{L} = \mathcal{L}(N, M, R, N', M', R')$, the action (5) casts in the following form

$$S = \int d^3x \sqrt{-g} [f(R) - \lambda (R + \frac{1}{2r}(4r N'^2 + 4r N N'' - r^3 M'^2 + 8NN'))]. \quad (8)$$

After variation with respect to $R$, we find $\lambda = f_R \equiv df/dR$ and the action is rewritten as

$$S = \int d^3x \sqrt{-g} [f(R) - f_R(R + \frac{1}{2r}(4r N'^2 + 4r N N'' - r^3 M'^2 + 8NN'))]. \quad (9)$$

Integrating by parts lead to the Lagrangian

$$\mathcal{L} = r(f - R f_R) + \frac{r^3}{2} f_R M'^2 - 2f_R R' NN' + 2r f_R R' NN', \quad (10)$$

where $f_{RR} \equiv d^2f/dR^2$. The Euler-Lagrange equations for $N$, $M$ and $R$ are derived respectively as

$$N(f_{RR} R'^2 + f_{RR} R'') = 0, \quad (11)$$

$$(r^3 f_{RM}')' = 0, \quad (12)$$

$$-rR f_{RR} + \frac{r^3}{2} f_{RM} M'^2 - 4f_{RR} R NN' - 2r f_{RR} R N'^2 - 2r f_{RR} NN'' = 0. \quad (13)$$

3 Hojman symmetry approach

Consider a set of second-order ordinary differential equations

$$\ddot{q}^i = F^i(q^j, \dot{q}^j, t), \quad i, j = 1, 2, ... n \quad (14)$$

where $q^i$ and $F^i$ denote the generalized coordinates and forces, respectively, and each over dot denotes derivative with respect to time $t$. If there exists an associated symmetry vector $X^i = X^i(q^j, \dot{q}^j, t)$, then it should satisfy the differential equation [6, 7]

$$\frac{d^2X^i}{dt^2} - \frac{\partial F^i}{\partial q^j} X^j - \frac{\partial F^i}{\partial \dot{q}^j} \frac{dX^j}{dt} = 0, \quad (15)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial \dot{q}^i}. \quad (16)$$
The symmetry vector $X^i$ maps the solutions $q^i$ of Eq. (14) into the solutions $\hat{q}^i$ of the same equations (up to $\epsilon^2$ terms), under the infinitesimal transformation

$$\hat{q}^i = q^i + \epsilon X^i (q^j, \dot{q}^j, t).$$

Using this property, the Hojman conserved quantities are defined by the following theorem [10]:

**Theorem:**
1. Provided that $F^i$ satisfies

$$\frac{\partial F^i}{\partial q^i} = 0,$$

then

$$Q = \frac{\partial X^i}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left( \frac{dX^i}{dt} \right),$$

is a conserved quantity.

2. Provided that $F^i$ satisfies

$$\frac{\partial F^i}{\partial q^i} = -\frac{d}{dt} \ln \gamma,$$

then

$$Q = \frac{1}{\gamma} \frac{\partial (\gamma X^i)}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left( \frac{dX^i}{dt} \right),$$

is a conserved quantity, where $\gamma$ is merely a function of $q^i$.

4 (2+1)-dimensional black hole solutions via Hojman Symmetry

The equations of motion that are obtained in section 2, can be rewritten in accordance with Hojman symmetry approach as follows

$$R'' = -h(R)R'^2,$$

$$r^3 f_R M' = C,$$

$C$ being a constant, and

$$-r R f_{RR} + \frac{r^3}{2} f_{RR} M'^2 - 4f_{RR} N' - 2rf_{RR} N'^2 - 2rf_{RR} NN'' = 0,$$

where

$$h(R) = \frac{f_{RR}}{f_R}.$$

By comparing [21] and [14], we can recognize that $F(R, R') = -h(R)R'^2$. Also from Eq. (19), we obtain

$$\gamma(R) = \gamma_0 e^{\int 2h(R) dR},$$
where \( \gamma_0 \) is a constant. If there is no explicit dependence of \( X \) on \( r \), namely \( X = X(R, R') \), then we can rewrite Eqs. \([\ref{eq:26}]\) and \([\ref{eq:27}]\), respectively as

\[
\left( \frac{\partial^2 X}{\partial R^2} + h_R X + h(R) \frac{\partial X}{\partial R} \right) + R' h^2(R) \frac{\partial^2 X}{\partial R'\partial R} - R' \left(2h(R) \frac{\partial^2 X}{\partial R \partial R'} + h_R \frac{\partial X}{\partial R'} \right) = 0,
\]

and

\[
Q = \frac{1}{\gamma} \frac{\partial (\gamma X)}{\partial R} + \frac{\partial}{\partial R'} \left( \frac{dX}{dr} \right),
\]

where \( h_R \equiv \frac{dh}{dR} \). In general, solving the differential equation \([\ref{eq:26}]\) for vector \( X \) is difficult. In this regard, we will limit ourselves to some particular ansatz for vector \( X \), proposed in Ref. \([\ref{ref:6}]\), as follows.

## 4.1 \( X \sim X(R) \)

From Eqs. \([\ref{eq:26}]\) and \([\ref{eq:27}]\), we see that

\[
h(R)X + \frac{dX}{dR} = \frac{Q}{2},
\]

where \( Q \) is a conserved quantity.

### 4.1.1 \( X = R \)

As a first step, if we simply consider \( X = R \) then from equation \([\ref{eq:28}]\) we find

\[
h(R) = \frac{Q}{2R} - \frac{1}{R},
\]

and from equations \([\ref{eq:21}]\) and \([\ref{eq:24}]\) we have

\[
f(R) = C_1 + C_2 R + C_3 R^{\frac{Q}{2} + 1},
\]

and

\[
R(r) = \frac{1}{2r^2} [Q (C_5 + C_4 r)]^{\frac{2}{Q}}.
\]

Finally, from equations \([\ref{eq:26}]\) and \([\ref{eq:28}]\) for \( M(r) \) and setting \( C_5 = 0 \) for simplicity, we obtain

\[
M(r) = -\frac{CC_0^2 C_4^2 \ln(C_2 + C_6 C_4 r)}{C_2^2} + \frac{CC_0^2 C_4^2 \ln(r)}{C_2^3} + \frac{CC_0 C_4}{C_2^2} - \frac{1}{2} \frac{C}{C_2 r^2} + C_7,
\]

where \( C_6 = \frac{Q}{2} C_3 (\frac{Q}{2} Q + 1) \). Using \([\ref{eq:30}]\), \([\ref{eq:31}]\) and \([\ref{eq:32}]\) in the equation of motion \([\ref{eq:23}]\) we obtain

\[
N^2(r) = -2C_9 + \frac{C^2}{4 C_2^2 r^2} + 2C_8 \frac{1}{r} + \frac{3}{2} \frac{C^2 C_6 C_4}{C_2^2} \frac{1}{r} - \frac{1}{4 C_2^2 r^2} \left[2C^2 C_6 C_4 r \ln \left(-\frac{C_2}{C_6 C_4 r} - 1 \right)(3C_6 C_4 r + 2) - \frac{Q}{16(Q + 1)(3Q + 2)} (41 - \frac{Q + 1}{Q} + 24Q) \left(\frac{C_4 Q}{2}\right)^{\frac{2}{Q}} r^{2 + \frac{2}{Q}} + \frac{Q}{3Q + 2} \left(\frac{C_4 Q}{2}\right)^{\frac{2}{Q}} r^{2 + \frac{2}{Q}} \right],
\]

where \( C_8 \) and \( C_9 \) are constants of integration. Now \([\ref{eq:32}]\) and \([\ref{eq:33}]\) determine the spherically symmetric solutions for the metric \([\ref{eq:30}]\) subject to the Ricci scalar \([\ref{eq:31}]\). These solutions are obtained by imposing Hojman symmetry. Note that since \( f(R) \) is not appeared in the field equations \([\ref{eq:26}]\) and \([\ref{eq:28}]\), the constant \( C_1 \) is not appeared in the solutions for \( M(r) \) and \( N(r) \) as a direct consequence of imposing the Hojman symmetry along \( X = R \).

In order to investigate the black hole property of these solutions, we write the metric \([\ref{eq:30}]\) in the following form

\[
ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2 [M^2(r)dt + d\phi]^2.
\]
For the given constants in the shift function, we may set $N^2(r) = 0$ to find the horizons of the black hole metric (34). Therefore, the spherical solutions (32) and (33) are capable of being as a black hole solution for $f(R)$ gravity (30) subject to the Ricci scalar (31).

In order to compare these solutions with those of obtained in [3], by imposing Noether symmetry, we may consider the solution (51) in [3] and set $C_6 = 0$ (or $C_3 = 0$) here to recover $f(R) = C_1 + C_2 R$ which is linear in terms of $R$ similar to $f(R) = R + D_3$, in [3]. This provides us with the following solution

$$M(r) = -\frac{1}{2} \frac{C}{C_2 r^2} + C_7, \quad (35)$$

$$N^2(r) = \frac{C^2}{4 C_2^2} \frac{1}{r^2} + 2 \frac{C_8}{r} + \frac{1}{4} \frac{C^2 C_4^2}{C_2 (3Q + 1)(3Q + 2)} r^{2+Q} - 2C_9. \quad (36)$$

A comparison between the solutions (43) and (51) in [3] and the solutions (35) and (36) here, shows that both solutions are the same, up to some identifications between the constants in each solution.

Now, we investigate on the possibility of recovering the well known BTZ black hole solution

$$M(r) = -\frac{J}{2r^2}, \quad (37)$$

$$N^2(r) = -m + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad (38)$$

where $m$ and $J$ are the mass and angular momentum of the black hole, respectively. In this regard, we may use the following initial identifications

$$C = J, \quad C_1 = 2/l^2, \quad C_2 = 1, \quad C_7 = C_8 = 0, \quad C_9 = m/2. \quad (39)$$

At this stage, it is left to identify the third term in (30) with the term $r^2/l^2$. To this end, we have no choice but to consider a relation $Q \sim l$, assuming $l \to \infty$, which gives us $r^2/l^2$ as the the asymptotic behavior of the third term in (30). Therefore one may conclude that the regular BTZ black hole is a solution corresponding to $f(R) = C_1 + R$ equipped with a Hojman symmetry vector $X = R$ whose freedom in choosing the constant term $C_1$ is fixed as $C_1 = 2l^{-2} \sim Q^{-2}$. Moreover, it turns out that similar to $m$ and $J$ which are the conserved charges corresponding to the time translation and rotation symmetries, respectively, we may consider $Q$ as the conserved charge corresponding to the symmetry of solutions under the infinitesimal displacement of $R$ by a cosmological constant. This is an important result accounting for the fact that the cosmological constant is nothing but the manifestation of Hojman symmetry.

By assuming $C_7 = 0, C_8 \neq 0$, we find

$$M(r) = -\frac{J}{2r^2}, \quad (40)$$

$$N^2(r) = -m + \frac{r^2}{l^2} + 2 \frac{C_8}{r} + \frac{J^2}{4r^2}, \quad (41)$$

which is the same generalized BTZ black hole solution which was obtained in [3] using Noether symmetry.

4.1.2 $X = \lambda \tan(R), \quad \lambda = \text{const}$

According to the previous subsection, by fixing $Q = 2\lambda$, we find

$$f(R) = C_1 + C_2 R + C_3 \cos R, \quad (42)$$

1We mean the solution (43) for which $D_1 = 0, D_2 = 1$ are imposed.
and

\[ R(r) = \arcsin(C_4 r + C_5). \]  (43)

For simplicity, we fix \( C_5 = 0 \), then we have

\[ M(r) = -\frac{C C_3 C_4^2}{C_5^3} \ln(-C_2 + C_3 C_4 r) + \frac{C C_3^2 C_4^2}{C_5^3} \ln r - \frac{C}{2C_2} \frac{1}{r^2} - \frac{C C_3 C_4}{C_5} \frac{1}{r} + C_6. \]  (44)

Using (42), (43) and (44) in the equation of motion (23), we obtain

\[ N^2(r) = -2C_7 + \frac{C^2}{4C_2^2 r^2} + 2C_8 \frac{1}{r} + \frac{3C^2 C_3 C_4}{2C_2} \frac{1}{r} - \sqrt{1 - \frac{C^2}{C_4^2 r^2}} \frac{1}{36C_4^2 r^2} (5C_4^2 r^2 - 8)
+ \frac{1}{4C_2^2 r^2} \left[ 2C^2 C_4 C_3 r \ln(C_3 - \frac{C_2}{C_4 r}) \right] (-3C_3 C_4 r + 2C_2)
- \frac{1}{12C_4^2} (2C_4^2 r^2 - 3) \arcsin(C_4 r), \]  (45)

where \( C_7 \) and \( C_8 \) are integration constants. For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric (44). Therefore, the spherical solutions (44) and (51) are capable of being as a black hole solution for \( f(R) \) gravity subject to the Ricci scalar (42).

For the special case \( C_3 = 0 \), namely \( f(R) = C_1 + C_2 R \), we obtain

\[ M(r) = -\frac{C}{2C_2} \frac{1}{r^2} + C_6, \]  (46)

\[ N^2(r) = \frac{1}{12C_4^2} (2C_4^2 r^2 - 3) \arcsin(C_4 r) - \frac{1}{36C_4^3} \frac{8 + 5C_4^2 r^2}{r} \sqrt{1 - \frac{C^2}{C_4^2 r^2}} + \frac{C^2}{4C_2^2 r^2} \frac{1}{r} + 2C_8 \frac{1}{r} - 2C_7. \]  (47)

Unlike the previous solutions (35) and (36), the solutions (46) and (47) are new in comparison to the solutions (43) and (51) obtained for the same \( f(R) = C_1 + C_2 R \) gravity in (3).

Now, we investigate on the possibility of recovering the well known BTZ black hole solution from the solutions (46) and (47). It turns out that the existence of first and second terms in (47), subject to \( C_4 \neq 0 \), does not allow for such recovery even if we set \( C_6 = C_8 = 0 \). Therefore, the Hojman symmetry provides us with the new generalized BTZ black hole solution, in comparison to the solution which has already been obtained by Noether symmetry in (3).

4.1.3 \( X = \lambda \sinh(R), \quad \lambda = \text{const} \)

By this ansatz, again from the procedure mentioned at the previous subsections and setting \( Q = 2\lambda \), we have

\[ f(R) = C_1 + C_2 R + C_3 \left[ -\ln(\tanh(\frac{1}{2} R) - 1) - \ln(\tanh(\frac{1}{2} R) + 1) \right], \]  (48)

and

\[ R(r) = -2 \arctanh(C_4 r + C_5). \]  (49)

Finally, by fixing \( C_5 = 0 \) for simplicity, we obtain

\[ M(r) = -\frac{C C_3 C_4}{C_5^3} \ln(-C_2 + C_3 C_4 r) + \frac{C C_3^2 C_4^2}{C_5^3} \ln r - \frac{C}{2C_2} \frac{1}{r^2} - \frac{C C_3 C_4}{C_5} \frac{1}{r} + C_6, \]  (50)

Using (48), (49) and (50) in the equation of motion (23), we obtain

\[ N^2(r) = -\frac{C^2 C_4 C_3}{2C_2^4} (3C_3 C_4 r - 2C_2) \ln(C_3 - \frac{C_4}{C_4 r}) \frac{1}{r} - \frac{1}{36C_4^3} \ln \left( \frac{C_4^2 r^2 - 1}{r} \right) \frac{1}{r} + \frac{r^2}{3} \tanh(C_4 r) - \frac{1}{C_4^2} \arccoth(C_4 r) + \frac{2}{3C_4} r + \frac{C^2}{2C_2^2} \frac{1}{r^2} - \frac{3C^2 C_3 C_4}{2C_2^2} \frac{1}{r} + 2C_8 \frac{1}{r} + 2C_7, \]  (51)
where $C_7$ and $C_8$ are integration constants. For the given constants in the shift function, we may set $N^2(r) = 0$ to find the horizons of the black hole metric (21). Therefore, the spherical solutions (50) and (51) are capable of being as a black hole solution for $f(R)$ gravity (15) subject to the Ricci scalar (49). For the special case $C_3 = 0$, namely $f(R) = C_1 + C_2 R$, we obtain

$$M(r) = -\frac{C}{2C_2 r^2} + C_6,$$

$$N^2(r) = \frac{1}{3C_4} \ln \left( \frac{C_4^2 r^2 - 1}{r} \right) + \frac{1}{2C_4} \arcoth(C_4 r) + \frac{1}{3} r^2 \arctanh(C_4 r) + \frac{2}{3} C_4 + \frac{C^2}{4C_2^2 r^2} + 2C_8 \frac{1}{r} - 2C_7. \quad (53)$$

These solutions are also new for $f(R) = C_1 + C_2 R$ gravity, because of the “ln”, “arctanh” and “arcoth” terms, in comparison to the solutions obtained by Noether symmetry in [3]. Similar to the previous case, the existence of first three terms in (53), subject to $C_4 \neq 0$, does not allow for recovering the BTZ black hole, even if we set $C_6 = C_8 = 0$. Therefore, the Hojman symmetry provides us with the new generalized BTZ black hole solution, in comparison to the solution which has already been obtained by Noether symmetry in [3].

4.2 $X = X(R')$

For the choice $X = X(R')$, the equation of symmetry vector $X$, (26), reads as Euler equation

$$h_R X + R'^2 h^2(R) \frac{d^2 X}{dR'^2} - R' h_R \frac{dX}{dR'} = 0, \quad (54)$$

from which $X$ and $h(R)$ are obtained respectively as follows

$$X = A_1 R' + A_2 R'^n, \quad (55)$$

$$h(R) = -\frac{1}{nR + h_0}, \quad (56)$$

where $A_1, A_2, h_0$ and $n$ are constant parameters, and the Hojman conserved quantity reads as

$$Q = 2h(R) R'^n - h(R) n(n + 1) R'^n. \quad (57)$$

We can easily verify that the symmetry vectors $X \sim R'$ and $X \sim R'^2$ give rise to vanishing conserved charge, namely $Q = 0$. Therefore, we may discard $n = 1, -2$ cases. By using eqs. (24) and (57) we have

$$f(R) = C_1 + C_2 \left( R + \frac{h_0}{n} \right) + C_3 \left( R + \frac{h_0}{n} \right)^{\frac{n-1}{n}} \quad (58)$$

and

$$R(r) = \frac{Q_0^{\frac{1}{n-1}}}{n} \left[(1 - n)(-r + C_4)\right]^{\frac{n}{n-1}} - \frac{h_0}{n}, \quad (59)$$

which $Q_0 = -\frac{n}{2-n(n+1)}$. Finally from equations (62) and (28) we have

$$M(r) = -\frac{CC_7^2 C_8}{C_2^2} \ln(C_2 + C_3 C_5 r) + \frac{CC_7^2 C_8}{C_2^4} \ln(r) - \frac{C}{2C_2} \frac{1}{r^2} + \frac{CC_3 C_5}{C_2} \frac{1}{r} + C_6, \quad (60)$$

where $C_5 = \frac{2n-1}{n} Q_0^{\frac{1}{n}} (n-1)$ and $C_4 = 0$ for simplicity. Again, using equations (58), (59) and (23) we obtain

$$N^2(r) = \frac{C^2 C_3 C_5}{2C_2^2} \left[ (3 C_3 C_5 r + 2 C_2) \ln((n-1)r) - \ln(C_2 + C_3 C_5 r) \right] \frac{1}{r} - \frac{Q_0^{\frac{1}{n-1}}}{n(3n-2)(4n-3)} ((n-1)r)^\frac{n-2}{n-1} +$$

$$+ \frac{1}{4C_2^2 r^2} + \left( 2C_7 + \frac{C_3 C_5}{2C_2^3} \right) \frac{1}{r} - \frac{3C_3 C_5^{2} C_8}{2C_2^4} - 2C_8. \quad (61)$$
For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric (34). Therefore, the spherical solutions (60) and (61) are capable of being as a black hole solution for \( f(R) \) gravity (58) subject to the Ricci scalar (59).

For \( C_3 = 0 \), namely \( f(R) = C_1 + C_2(R + \frac{h_n}{n}) \), we obtain

\[
M(r) = -\frac{C}{2C_2 r^2} + C_6,
\]

\[
N^2(r) = -\frac{3}{32} \frac{C_2(-8C_2C_2^2 r^2 + 8C_7C_2^2 r)C_2}{C_2^8} \left[ -\ln((n - 1)r) + \ln \left( -\frac{1}{4} \frac{(-8C_2C_2^2 r^2 + 8C_7C_2^2 r)C_2}{C_2^2 r} \right) \right].
\]

The \( f(R) = C_1 + C_2(R + \frac{h_n}{n}) \) gravity is the same as \( f(R) = C_1 + C_2 R \) gravity, so the above solutions are also new, because of the “\( \ln \)” terms, in comparison to the solutions obtained by Noether symmetry in \([3]\). Similar to the previous cases, the existence of “\( \ln \)” terms in (63) does not allow for recovering the BTZ black hole, even if we set \( C_6 = 0 \). Therefore, the Hojman symmetry provides us with the new generalized BTZ black hole solution, in comparison to the solution which has already been obtained by Noether symmetry in \([3]\).

### 4.3 \( X(R, R') \sim R'g(R) \)

We consider another ansatz \( X(R, R') \sim R'g(R) \), where \( g(R) \) is an arbitrary function. By this assumption, in order for \( X \) to be the symmetry vector, we obtain

\[
h(R) = \frac{g_{RR}}{g_R}
\]

Also, the Hojman conserved quantity is obtained as

\[
Q_0 = R'g_R.
\]

Now, by considering some ansatzs for \( g(R) \), we find some exact solutions in the following.

#### 4.3.1 \( g(R) = \lambda e^{\alpha R} \)

By this choice and using equation (24), (34) and (35), we obtain

\[
f(R) = C_1 + C_2 R + C_3 e^{\alpha R},
\]

and

\[
R(r) = \frac{1}{\alpha} \ln \left( \frac{Q_0(r + C_4)}{\lambda} \right).
\]

And, by fixing \( C_4 = 0 \) for simplicity, we have

\[
M(r) = -\frac{CC_3^2\alpha^2 Q_0^2}{\lambda^2 C_2^3} \ln \left( \frac{C_2 \lambda}{r} + C_3 \alpha Q_0 \right) - \frac{1}{2} \frac{C}{C_2 r^2} + \frac{CC_3 \alpha Q_0}{\lambda C_2} \frac{1}{r} + C_5,
\]

and

\[
N^2(r) = -\frac{C_2^2 \alpha Q_0 C_3}{2 \lambda^2 C_2^2} (3C_3 \alpha Q_0 r + 2C_2 \lambda) \ln \left( C_2 + \frac{C_3 \alpha Q_0}{\lambda} r \right) \frac{1}{r} + \frac{1}{6 \lambda^2 C_2^4 \alpha} (-\lambda^2 C_2^4 r^3 + 9Q_0^2 C_2^2 C_3^5 r + 6Q_0 C_2^2 C_3 \lambda^2 r) \ln \left( \frac{Q_0 r}{\lambda} \right) \frac{1}{r} + \frac{5}{36 \alpha} r^2 + 3Q_0 C_2^2 C_3 \alpha \frac{1}{2C_2^2 r} + \frac{C_2}{4C_2^2 r^2} + 2C_7 \frac{1}{r} - 2C_6.
\]
For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric \( \text{(34)} \). Therefore, the spherical solutions \( \text{(68)} \) and \( \text{(69)} \) are capable of being as a black hole solution for \( f(R) \) gravity \( \text{(66)} \) subject to the Ricci scalar \( \text{(67)} \).

For \( C_3 = 0 \), namely namely \( f(R) = C_1 + C_2 R \), we obtain

\[
M(r) = -\frac{1}{2} C_2 \frac{1}{r^2} + C_5,
\]

\[
N^2(r) = -\frac{1}{6\alpha} r^2 \ln(\frac{Q_0 r}{\lambda}) + \frac{C^2}{4C_2^3} \frac{1}{r^2} + 2 C_8 \frac{1}{r} + \frac{5}{36\alpha} r^2 - 2 C_7.
\]

The above solutions are new, because of the extra first term \( r^2 \ln(\frac{Q_0 r}{\lambda}) \), in comparison to the solutions obtained by Noether symmetry in \( \text{(3)} \), for \( f(R) = C_1 + C_2 R \) gravity. Similar to the previous cases, the existence of “ln” term in \( \text{(77)} \) does not allow for recovering the BTZ black hole, even if we set \( C_5 = C_8 = 0 \). Therefore, the Hojman symmetry provides us with the new generalized BTZ black hole solution, in comparison to the solution which has already been obtained by Noether symmetry in \( \text{(3)} \).

### 4.3.2 \( g(R) = \frac{(g_0 + R)^{1+\alpha}}{1+\alpha} \)

Here \( g_0 \) and \( \alpha \) are constants parameters. Again, from Eqs. \( \text{(64)} \), \( \text{(65)} \), \( \text{(62)} \) and \( \text{(23)} \) we find respectively \( f(R) \), \( R(r) \), \( M(r) \) and \( N^2(r) \) for the case \( C_4 = 0 \), as follows

\[
f(R) = C_1 + C_2 (g_0 + R) + C_3 (g_0 + R)^{\alpha+2},
\]

\[
R(r) = [Q_0 (1 + \alpha) r]^{\frac{1}{1+\alpha}} - g_0,
\]

\[
M(r) = -\frac{C_3 C_5}{C_2^3} \ln(\frac{C_2}{r} + C_3 C_5) - \frac{1}{2} C \frac{1}{C_2} \frac{1}{r^2} + \frac{C C_3 C_5}{C_2^2} \frac{1}{r} + C_5
\]

and

\[
N^2(r) = \frac{C^2 C_3 C_5}{2C_2^4} (3C_3 C_5 r + 2C_2) \left[ \ln(\frac{Q_0 (\alpha + 1) r}{(\alpha + 1) r}) - \ln\left(\frac{C_2 + C_3 C_5 r}{C_2}ight) \right] \frac{1}{r} - \frac{(\alpha + 1)^2}{2(\alpha + 3)(3\alpha + 4)} \left( \frac{Q_0 (\alpha + 1) r}{(\alpha + 1) r} \right)^{\frac{1}{1+\alpha}} + \frac{g_0}{6} r^2 + \frac{C^2}{C_2} \frac{1}{r^2} + \left( \frac{C^2 C_3 C_5}{C_2^3} \right) \frac{1}{r} - 2 C_6 - \frac{3C^2 C_3 C_5^2}{2C_2^4},
\]

where \( C_5 = (\alpha + 1)(\alpha + 2) Q_0 \). For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric \( \text{(34)} \). Therefore, the spherical solutions \( \text{(74)} \) and \( \text{(75)} \) are capable of being as a black hole solution for \( f(R) \) gravity \( \text{(72)} \) subject to the Ricci scalar \( \text{(73)} \).

For the case \( C_3 = 0 \), namely namely \( f(R) = C_1 + C_2 R \), we obtain

\[
M(r) = -\frac{1}{2} C \frac{1}{C_2} \frac{1}{r^2} + C_5,
\]

\[
N^2(r) = -\frac{Q_0^{\frac{1}{1+\alpha}} [(\alpha + 1) r]^{\frac{2\alpha+3}{1+\alpha}}}{(2\alpha + 3)(3\alpha + 4)} + \frac{C^2}{4C_2^2} \frac{1}{r^2} + 2 C_8 \frac{1}{r} + \frac{g_0}{6} r^2 - 2 C_7.
\]

These solutions are considered as new, because of the first term, in comparison to the solutions obtained by Noether symmetry in \( \text{(3)} \) for \( f(R) = C_1 + C_2 R \) gravity. Similar to the previous cases, the existence of first term in \( \text{(77)} \) does not allow for recovering the BTZ black hole, even if we set \( C_5 = C_8 = 0 \), unless we set \( \alpha = -5/4 \). Therefore, the Hojman symmetry provides us with the new generalized BTZ black hole solution, in comparison to the solution which has already been obtained by Noether symmetry in \( \text{(3)} \).
4.3.3 $g(R) = \lambda \ln R$

For this ansatz we have

$$f(R) = C_1 + C_2 R + C_3 R \ln R,$$

$$R(r) = C_4 e^{\frac{Q_0 r}{\lambda R}},$$

$$M(r) = -\frac{CC_3 Q_0^2}{\lambda^2 C_5^3} \ln \left( \frac{C_5 r}{Q_0} \right) - \frac{C}{2C_5} \frac{1}{r^2} + \frac{CC_3 Q_0}{\lambda C_5^2} \frac{1}{r} + C_6$$

(78)

(79)

(80)

where $C_5 = C_2 + C_3(1 + \ln C_4)$, and

$$N^2(r) = -\frac{Q_0 C^2 C_3}{2C_5^4 \lambda^2} (3C_3 Q_0 r + 2C_5 \lambda) \left[ \ln \left( \frac{C_5 \lambda}{Q_0} \right) + C_3 \right] \frac{1}{r} - \frac{C_4 \lambda^2 e^{\frac{Q_0 r}{\lambda R}}}{Q_0^3} \left( \frac{Q_0}{r} - \frac{2 \lambda}{r^2} \right) -$$

$$\frac{C^2}{4C_5^2 r^2} + \frac{3Q_0 C^2 C_3}{2C_5^3} \frac{1}{r} + 2C_7 \frac{1}{r} - 2C_8.$$  

(81)

For the given constants in the shift function, we may set $N^2(r) = 0$ to find the horizons of the black hole metric (34). Therefore, the spherical solutions (80) and (81) are capable of being as a black hole solution for $f(R)$ gravity (78) subject to the Ricci scalar (79).

For the case $C_3 = 0$, namely namely $f(R) = C_1 + C_2 R$, we obtain

$$M(r) = -\frac{C}{2C_5} \frac{1}{r^2} + C_6,$$

$$N^2(r) = -\frac{C_4 \lambda^2 (-2 \lambda + Q_0 r) e^{\frac{Q_0 r}{\lambda R}}}{Q_0^3} + \frac{C^2}{4C_5^2} \frac{1}{r^2} + 2C_8 \frac{1}{r} - 2C_7.$$  

(82)

(83)

These solutions are considered as new, because of the first term $e^{\frac{Q_0 r}{\lambda R}}$, in comparison to the solutions obtained by Noether symmetry in [3] for $f(R) = C_1 + C_2 R$ gravity. Similar to the previous cases, the existence of first term in (83) does not allow for recovering the BTZ black hole, even if we set $C_6 = C_8 = 0$. Therefore, the Hojman symmetry provides us with the new generalized BTZ black hole solution, in comparison to the solution which has already been obtained by Noether symmetry in [3].

5 Conclusions

We have obtained $(2 + 1)$-dimensional spherically symmetric solutions in the context of $(2 + 1)$-dimensional $f(R)$ gravity by using the Hojman symmetry. These solutions have capability of being $(2 + 1)$-dimensional black holes. In some special cases, these solutions cast in the form of $(2 + 1)$ dimensional BTZ black hole and generalized $(2 + 1)$ dimensional BTZ black hole which has already been obtained by Noether symmetry in [3]. In the case of Hojman symmetry along $X = R$, leading to BTZ black hole, we have shown that the cosmological constant is nothing but the manifestation of Hojman symmetry.

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