Superpotentials from variational derivatives rather than Lagrangians in relativistic theories of gravity

Joseph Katz and Gideon Livshits

1 The Racah Institute of Physics, The Hebrew University, Givat Ram, 91904 Jerusalem, Israel
2 Institute of Chemistry, The Hebrew University, Givat Ram, 91904 Jerusalem, Israel

E-mail: jkatz@phys.huji.ac.il and livshits.gideon@mail.huji.ac.il

Received 6 June 2008, in final form 13 July 2008
Published 19 August 2008
Online at stacks.iop.org/CQG/25/175024

Abstract

The prescription of Silva to derive superpotential equations from variational derivatives rather than from Lagrangian densities is applied to theories of gravity derived from Lovelock Lagrangians in the Palatini representation. Spacetimes are without torsion and isolated sources of gravity are minimally coupled. On a closed boundary of spacetime, the metric is given and the connection coefficients are those of Christoffel. We derive equations for the superpotentials in these conditions. The equations are easily integrated and we give the general expression for all superpotentials associated with Lovelock Lagrangians. We find, in particular, that in Einstein’s theory, in any number of dimensions, the superpotential, valid at spatial and at null infinity, is that of Katz, Bičák and Lynden-Bell, the KBL superpotential. We also give explicitly the superpotential for Gauss–Bonnet theories of gravity. Finally, we find a simple expression for the superpotential of Einstein–Gauss–Bonnet theories with an anti-de Sitter background: it is minus the KBL superpotential, confirming, as it should, the calculation of the total mass–energy of spacetime at spatial infinity by Deser and Tekin.

PACS numbers: 04.20.−q, 04.20.Cv, 04.20.Fy, 04.50.−h

1. Introduction

1.1. A view on superpotentials

In electromagnetism, one of Maxwell’s equations relates the stationary electric field \( \vec{E} \) to the density of charges \( \rho_e \): \( \nabla \cdot \vec{E} = 4\pi \rho_e \). From this follows that the total charge \( Q \) responsible for the field is equal to the flux of \( \vec{E} \) through a closed surface surrounding the sources. \( \vec{E} \) is the electric force acting on a unit test charge \( q \). In Newton’s theory of gravitation the gravitational field \( \vec{G} \) is related to the density of matter \( \rho_m \) in a similar way: \( \nabla \cdot \vec{G} = -4\pi \rho_m \). Thus the
flux of $-\vec{G}$ through a closed surface surrounding the source responsible for $\vec{G}$ is equal to the total mass $M$. $\vec{G}$ is the gravitational force acting on a unit test mass $m$. In Einstein’s theory of gravitation things get slightly more complicated. First there is a change of meaning: total mass is now total mass–energy $Mc^2$. Second, given a localized source of gravity, part of the total mass–energy is in the gravitational field itself though its density is not defined nor is it possible in general\(^3\) to even disentangle the total gravitational field energy from the total mass–energy of spacetime. Nevertheless, an isolated amount of matter together with its gravitational field appears from a great distance as a point-like source of gravitation, possibly spinning, and like in Newton’s theory and in electromagnetism its mass–energy is also equal to a flux across a closed surface at spatial infinity. However a surface element in spacetime is a 2-index anti-symmetric tensor so that instead of a vector ($\vec{E}$ or $\vec{G}$) we have a two index antisymmetric tensor whose flux across the surface at infinity equals mass–energy. The tensor is commonly called the superpotential. It is worth noting that the flux includes the energy of the sources [20].

Unfortunately the superpotential is not as well defined as in classical field theory. Given a Lagrangian from which Einstein’s equations are derived, there exists a unique superpotential associated with that Lagrangian\(^4\). The trouble is that the Lagrangian density itself is only defined up to a divergence and is thus not unique. That is the reason why efforts have been made to obtain the superpotential not from the Lagrangian but from the variational derivatives of the Lagrangian\(^5\). These are insensitive to additional divergences.

1.2. Superpotentials derived from variational derivatives

Silva [30] suggested a method to obtain superpotentials from variational derivatives. Previous works, in particular those of Rosen [29] and of Anderson and Torre [2] aimed at obtaining conserved quantities from the field equations. Here the emphasis is on superpotentials and there is no need for field equations. The fluxes we are after are in addition not necessarily conserved like, for instance, the Bondi mass [3] at null infinity. In the last decades relativists have become interested in spacetimes with more than four dimensions or which far from the sources are not flat; they become, in particular, anti-de Sitter\(^6\). The proper way to deal with this situation as well as in keeping the ability to calculate, for instance mass–energy, in asymptotically flat spacetimes in spherical coordinates, is by introducing a background metric\(^7\) [17]. Ferraris, Francaviglia and Raiteri [13], who generalized, to some extent, Silva’s idea were concerned with the variations of conserved quantities.

1.3. What is done in this paper

Silva’s Lagrangian approach is summarized in section 2. Julia and Silva [18] applied the method to Einstein’s relativistic theory of gravitation and found that given the asymptotic metric components at spatial or at null infinity, the unique answer is the KBL superpotential [20]. Silva’s prescription applies to variational derivatives that depend at most on first-order derivatives of the field components and whose Lie derivatives with respect to some arbitrary

---

\(^3\) See however [21, 22].

\(^4\) This is not shown here. It can be derived from the Belinfante-Rosenfeld identities, see [20], without invoking boundary conditions or field equations.

\(^5\) And not from field equations as is sometimes stated, see for instance [13].

\(^6\) The mass–energy in such spacetimes have been calculated in a series of fine works by Deser and collaborators [1, 8, 9] to which we shall come back later. See also the paper by Petrov [27].

\(^7\) Elsewhere backgrounds appear in the guise of ‘counter-terms’ like in [15] or a ‘regularization’ procedure as in [23].
vector field depend also at most on first-order derivatives of this vector field. These conditions are met in a $GL(D, \mathbb{R})$ formulation of Einstein’s theory. Ferraris, Francaviglia and Raiteri [13] extended Silva’s prescription to Lagrangians that contain higher order derivatives of the fields but, in the Lie derivatives, do not contain higher orders of derivatives of the vector field. Their work renders Silva’s prescription applicable to Einstein’s ordinary equations in terms of the metric components and the superpotential we derived that way is again the KBL one. In section 3 we extend Silva’s prescription to the case in which Lie derivatives contain second-order derivations of the vector field. This makes the method applicable to variational derivatives of Lovelock Lagrangians [24] (section 4) in the Palatini representation in which the metric and the symmetric connection coefficients are the independent fields. We find the equation of the superpotential and its solution. The superpotential of Einstein’s gravity theory in $N$ dimensions is, as expected, the KBL superpotential (section 5). The superpotential for Gauss–Bonnet variational derivatives, derived in section 6, is far more complicated than the KBL superpotential. The superpotential of Einstein–Gauss–Bonnet theories is a linear combination of both. In particular, the superpotential with an anti-de Sitter background is proportional to the KBL superpotential. The one associated with the spherically symmetric solution given in [1] is minus the KBL superpotential and on a sphere at spatial infinity this coincides with the superpotential found by Deser and Tekin [9].

1.4. A word about identities

It may be noted that nowhere field equations are used. From the beginning to the end we play with identities. That is not altogether surprising. Globally conserved or non-conserved quantities are constants or functions that appear in the boundary conditions. It is also interesting to note that the derivation of the superpotential from a Lagrangian via Noether identities, like it is done in [19] or [20], is very different from the derivation from variational derivatives. We have not yet tried to relate the two methods; the relation appears, to us at least, somewhat mysterious.

1.5. What has this to do with conservation laws?

This paper is far removed from conservation law considerations and is more of a mathematical character. It is therefore perhaps useful to reconnect superpotentials to conservation laws before we move on. The superpotential is an antisymmetric tensor density, linear in some arbitrary displacement vector say $\xi$, defined on the boundary of the domain, usually at spatial or null infinity where spacetime identifies with the background that does not have to be flat. The relation to ‘conservation laws’ is as follows, at least in Einstein’s theory of gravitation. The ordinary divergence of that antisymmetric tensor density is a divergenceless vector density, say $J^\rho$. If one uses Einstein’s field equations, and this is one of the rare occasions where they are used in this paper, it was found [10] [20] that $J^\rho = \sqrt{-g} T^\rho_\sigma \xi^\sigma$ plus terms which in the linear approximation are negligible. If, in particular, the background has a Killing field of time translations like in a flat spacetime, then one clearly sees that the total flux of the conserved current is related to mass–energy conservation. However, if the background possesses no Killing field then $J^\rho$ is a conserved vector with no obvious physical meaning. So much for conservation laws and the relevance of superpotentials.

---

8 The method is equally applicable to more general higher order derivative Lagrangians.
9 For a definition of notations see section 2 below.
10 When spacetime is flat far from the sources.
2. The Silva prescription

2.1. Elements

In what follows we are given a set of tensors on a curved spacetime with components \( y^A(x^\lambda) \) and a Lagrangian density\(^{11}\)

\[
\mathcal{L} = \mathcal{L}(y^A, D_\sigma y^A).
\]  

(2.1)

The particularity of \( \mathcal{L} \) is that variational derivatives,

\[
\mathcal{L}^A \equiv \partial \mathcal{L} \partial y^A - D_\sigma \partial \mathcal{L} \partial (D_\sigma y^A),
\]  

(2.2)

contain only first-order derivatives of \( y^A \). Note that partial derivatives with respect to \( y^A \) are constant and vice versa. The variation of \( \mathcal{L} \) with respect to \( y^A \) can thus be written in this form

\[
\delta \mathcal{L} = \mathcal{L}^A \delta y^A + D_\sigma \left[ \frac{\partial \mathcal{L}}{\partial (D_\sigma y^A)} \delta y^A \right] = \mathcal{L}^A \delta y^A + \partial_\sigma \left[ \frac{\partial \mathcal{L}}{\partial (D_\sigma y^A)} \delta y^A \right].
\]  

(2.3)

The spacetime has here the passive role of a background so that \( \delta D = D \delta \). The Lie derivative of a Lagrangian density\(^{12}\),

\[
\mathcal{L}^A(D_\xi y^A) = \partial_\sigma \left( \frac{\partial \mathcal{L}^A}{\partial (D_\sigma y^A)} D_\xi y^A - \mathcal{L}^A \xi^\sigma \right) = 0.
\]  

(2.4)

Thus, if \( \delta = \mathcal{L}^A(D_\xi y^A) \) minus (2.4) must be equal to zero

\[
\mathcal{L}^A(\xi y^A) = \mathcal{L}^A \xi^A + \partial_\sigma \left[ \frac{\partial \mathcal{L}^A}{\partial (D_\sigma y^A)} \xi^A - \mathcal{L}^A \xi^\sigma \right] = 0.
\]  

(2.5)

Equation (2.5) is commonly referred to as Noether’s identity. Next we assume that \( \mathcal{L}^A y^A \) has this form

\[
\mathcal{L}^A y^A = \Lambda_{A \lambda} \xi^\lambda + \Lambda_{A \rho}^\sigma D_\rho \xi^\lambda.
\]  

(2.6)

This is the case for tensor fields. The \( \Lambda \)'s depend on \( y^A \) and \( D_\rho y^A \) and are tensors. Inserting (2.6) into \( \mathcal{L}^A(\xi y^A) \), we see from (2.5) that this expression, which we designate by \( X \), is necessarily of the following form:

\[
X \equiv \mathcal{L}^A(\xi y^A) = \mathcal{L}^A \Lambda_{A \lambda} \xi^\lambda + \mathcal{L}^A \Lambda_{A \rho}^\sigma D_\rho \xi^\lambda = \partial_\sigma \left( J^\sigma_\xi \xi^\lambda + U^\sigma_\rho \partial_\rho \xi^\lambda \right).
\]  

(2.7)

\( J \)'s and \( U \)'s are components of tensor densities. A derivation by part of the second term after the first equality sign leads to what has been called since the late 1940's generalized Bianchi identities because this is what they are if \( y^A \) is the gravitational field

\[
\mathcal{L}^A \Lambda_{A \lambda} - \partial_\rho \left( \mathcal{L}^A \Lambda_{A \rho}^\sigma \right) = 0.
\]  

(2.8)

All this has been derived in innumerable papers but has been re-derived here to clarify our notations.

\(^{11}\) In Einstein’s theory of gravity which is in four dimensions, Greek indices go from 0 to 3. Latin indices from 1 to 3. The signature of the metric \( g_{\mu \nu} \) is \(-2\) and \( g \) is its determinant. Covariant derivatives are indicated by a \( D \); partial derivatives by a \( \partial \). The permutation symbol in four dimensions is \( \epsilon_{\mu \nu \rho \sigma} \) with \( \epsilon_{0123} = 1 \) and in three dimensions by \( \epsilon_{klm} \) with \( \epsilon_{123} = 1 \). The 4-volume element \( dV_4 = \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \), a surface element in four dimensions \( dS_4 = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \) and in three dimensions \( dS_3 = \frac{1}{2} \epsilon_{klm} dx^k \wedge dx^l \wedge dx^m \). The sections on Einstein’s gravity and Gauss–Bonnet gravity are in \( N \) dimensions to which these definitions extend in an obvious way. In those sections, Greek indices run from 0 to \( N - 1 \) and the signature of the metric is \(-\langle N - 2 \rangle\).

\(^{12}\) One should pay attention to the fact that we deal with vector densities and that ordinary divergences of such vectors as well as of anti-symmetric tensor densities are scalar or vector densities.
2.2. The ‘cascade identities’ of Julia and Silva

In their paper of 1998 [17], $\xi^\lambda$ is replaced by

$$\xi^\lambda = \xi^\lambda_0 \epsilon,$$

(2.9)

where $\xi^\lambda_0(x^\mu)$ is regarded as ‘fixed’ for a moment and $\epsilon(x^\mu)$ is an arbitrary scalar function. Inserting (2.9) into (2.7) and expanding in terms of $\epsilon, \partial_\rho \epsilon$ and $D_\rho \epsilon$, one obtains for the left-hand side of (2.7), an expression of this form

$$X = X_0 \epsilon + W_0^\rho \partial_\rho \epsilon$$

where

$$W_0^\rho = L^A K_{\lambda A}^0 \xi^\lambda_0.$$

(2.10)

$X_0$ is $X$ in which $\xi^\lambda_0$ has been replaced by $\xi^\lambda_0$. In the right-hand side of (2.7) we have a divergence of

$$J_0^\rho \xi^\lambda + U_0^{\rho\sigma} D_\rho \xi^\lambda = J_0^\rho \epsilon + U_0^{\rho\sigma} \partial_\rho \epsilon$$

with

$$J_0^\rho = J_0^\rho \xi^\lambda_0 + U_0^{\rho\sigma} D_\rho \xi^\lambda_0$$

and

$$U_0^{\rho\sigma} = U_0^{\rho\sigma} \xi^\lambda_0.$$

(2.11)

But following (2.7), (2.10) is equal to the divergence of (2.11) and since $\xi^\lambda_0$ is arbitrary the identity also holds if we replace $\xi^\lambda_0$ by any $\xi^\lambda$ (we remove then the indice 0). Thus,

$$X = \partial_\rho J^\rho, \quad W^\rho = J^\rho + \partial_\sigma U^{\rho\sigma}$$

and $U^{\rho\sigma} = 0$. (2.13)

The interesting point is that $U^{\rho\sigma}$ is anti-symmetrical, $U^{\rho\sigma} = U^{[\rho\sigma]}$, and the identities may be rewritten

$$X = \partial_\rho J^\rho, \quad J^\rho = W^\rho + \partial_\sigma U^{\rho\sigma}.$$

(2.14)

If $L^A = 0$, then $X = 0$, $W^\rho = 0$ and the vector density $J^\rho$ is the divergence of an antisymmetric tensor density

$$J^\rho = \partial_\sigma U^{\rho\sigma}, \quad \text{(only if } L^A = 0).$$

(2.15) will not be used later.

2.3. Functional equation for a superpotential

Now first for the motivation. Consider the integral, over a volume $V$ with boundary $S$, of $J^\rho$ as given by (2.14)

$$\int_V J^\rho \, dV_\rho = \int_V W^\rho \, dV_\rho + \oint_S U^{\rho\sigma} \, dS_{\rho\sigma}. $$

(2.16)

In particular, let the volume be a spacelike hypersurface, say $x^0 = 0$. Equation (2.16) can then be written as

$$\int_V J^0 \, d^3x = \int_V W^0 \, d^3x + \oint_S U^{0k} \, dS_k.$$

(2.17)

Let further $\xi^\lambda$ be a timelike vector which on the boundary, at spatial or null infinity, is associated with time translations in a flat background. The timelike field exists in the whole spacetime (no black holes). In coordinates in which $\xi^\lambda = \{1, 0, 0, 0\}$ the volume integral becomes

$$\int_V J^0 \, d^3x = \int_V L^A K_{\lambda A}^0 \, d^3x + \oint_S U^{0k} \, dS_k.$$

(2.18)

In general relativity, (2.18) is the Hamiltonian

$$\int_V J^0 \, d^3x = \int_V s (G^0_0 - \kappa T^0_0) \, d^3x + \oint_S U^{0k} \, dS_k \quad \text{with} \quad G^0_0 = R^0_0 - \frac{1}{2} \delta^0_0 R.$$

(2.19)
\( \mathbf{R}_{\rho\sigma} \) is the Ricci tensor and \( R \) the scalar curvature while \( T^{\rho}_{\sigma} \) is the energy momentum of the source of gravity and \( \kappa \) is the usual coupling constant\(^{13}\). When the field equations are satisfied, the Hamiltonian is equal to the total mass–energy

\[
M c^2 = \int_S U^{\rho\sigma} dS_{\rho\sigma} = \int_S U^{0k} dS_k, \quad \text{(only if } \mathcal{L}^A = 0). \tag{2.20}
\]

\( U^{\rho\sigma} \) is thus the superpotential by definition.

The Hamiltonian field equations are obtained by applying the variational principle to the Hamiltonian. Regge and Teitelboim [28] brought attention to the fact that if boundary conditions are given to begin with, the variation of the Hamiltonian should have no boundary terms like in classical mechanics. Hamiltonians have no time derivatives. Silva’s prescription is a covariant expression of that remark. So here is how it goes.

Consider the variation of (2.16) due to arbitrary variations of \( y_A \); it can be written as

\[
\delta \left( \int_V J^\rho dV_\rho \right) = \int_V \frac{\delta W^\rho}{\delta y_A} \delta y_A dV_\rho + \oint_S (W^{\rho\sigma} + \delta U^{\rho\sigma}) dS_{\rho\sigma} \quad \text{where}
\]

\[
W^{\rho\sigma} \equiv \frac{\partial W^\rho}{\partial (D_\rho y_A)} \delta y_A. \tag{2.21}
\]

The condition that no boundary term appears in (2.21) provides a functional differential equation for the superpotential

\[
\delta U^{\rho\sigma} = -W^{\rho\sigma} \quad \text{so that } \quad \delta \left( \int_V J^\rho dV_\rho \right) = \int_V \frac{\delta W^\rho}{\delta y_A} \delta y_A dV_\rho. \tag{2.22}
\]

Note that the surface integral in (2.21) and equation (2.22) are only correct if \( W(\rho\sigma) = 0 \). That this is indeed true can be seen as follows. Consider the variational derivatives of \( X \), which according to (2.13) on the left, must be equal to zero

\[
\frac{\delta X}{\delta y_A} = \frac{\partial X}{\partial y_A} - D_\rho \left( \frac{\partial X}{\partial (D_\rho y_A)} \right) = 0 \quad \text{where } \quad X = \mathcal{L}^A A_{A\lambda} \xi^\lambda + \mathcal{L}^A A^A_{\rho} D_\rho \xi^\lambda. \tag{2.23}
\]

Now in (2.23) replace \( \xi^\lambda \) by \( \xi_0^\lambda \), expand as a polynomial in \( \epsilon, \partial_\rho \epsilon \) and \( D_\rho \epsilon \) and equate the factors of \( \epsilon, \partial_\rho \epsilon \) and \( D_\rho \epsilon \) to zero. Then remove the indice 0 because \( \xi_0^\lambda \) is arbitrary. The factor of \( D_\rho \epsilon \) is \( W^{\rho\sigma} \). Thus \( W^{\rho\sigma} \) is indeed antisymmetrical.

Next suppose that \( U^{\rho\sigma}_0 \) is a solution of equation (2.22) on the left. Then, \( U^{\rho\sigma}_0 + C^{\rho\sigma} \), where \( C^{\rho\sigma}(x^\lambda) \) is an arbitrary function independent of \( y_A \), is also a solution of (2.22). When the superpotential relates, in particular to mass–energy, \( C^{\rho\sigma} \) defines its ‘zero’ point. As it was done in [20] and also in [18], we take \( C^{\rho\sigma} = -U^{\rho\sigma}_0 \) which is the superpotential of the background. The superpotential of the background is obtained from \( U^{\rho\sigma}_0 \) by equating to zero the source terms\(^{14}\). The final solution is then given by \( U^{\rho\sigma} = U^{\rho\sigma}_0 - U^{\rho\sigma}_0 \). If, however, we use a background from the very beginning, as we have done here and shall do in our examples, then \( U^{\rho\sigma}_0 \equiv 0 \) and \( U^{\rho\sigma} = U^{\rho\sigma}_0 \). So much about the works of Silva and that of Julia and Silva.

For completeness let us mention a work of Fatibene, Ferraris and Francaviglia [12] which deals with relative conservation laws.

\(^{13}\) About \( \kappa \) see (4.7) below and the corresponding footnote.

\(^{14}\) For instance, in the Schwarzschild spacetime we take \( m = 0 \) in whichever coordinates the metric is written. Note that if spacetimes are asymptotically flat, \( U^{\rho\sigma}_0 dS_{\rho\sigma} = 0 \) while if they are anti-de Sitter, \( U^{\rho\sigma}_0 dS_{\rho\sigma} = \infty \) but in both cases \( U^{\rho\sigma}_0 dS_{\rho\sigma} \) is bounded as can be seen in the example considered at the end of the last section.
3. Higher order derivatives of $\xi$

The method described so far is not applicable to the Palatini formulation of general relativity because the Lie derivatives of the connection coefficients contain second-order derivatives of $\xi^\lambda$. So, we first generalize the prescription of Silva to the case where Lie derivatives contain second-order derivatives of $\xi^\lambda$.

3.1. Basic elements

Equations (2.1) to (2.5) remain valid here but instead of (2.6) we assume that

$$\mathcal{L}_\xi \gamma_A = \Lambda_{\lambda A} \xi^\lambda + \Lambda_{\rho A} \partial_\rho \xi^A + \Lambda_{\rho \sigma A} \partial_{\rho \sigma} \xi^A \Rightarrow \Lambda_{\lambda A} = \Lambda^\sigma_{\lambda A}.$$  \hspace{1cm} (3.1)

Equation (2.7) has now this form

$$X = \mathcal{L}^A \xi \gamma_A = \partial_\rho (\mathcal{J}_\rho^A \xi^A + \tilde{U}_{\rho}^\sigma \partial_\rho \xi^\sigma + V_{\rho \sigma}^\mu \partial_\rho \xi^\mu).$$ \hspace{1cm} (3.2)

Set again, like in (2.9), $\xi^A = \xi^A_0$. The left-hand side of (3.2) can then be written like this, thanks to (3.1),

$$X = X_0 + W_0^\rho \partial_\rho \epsilon + Y_0^\rho \partial_\rho \epsilon$$

and

$$Y_0^\rho_0 \equiv \mathcal{L}^A \Lambda_{\lambda A} \xi^A_0.$$ \hspace{1cm} (3.3)

As before, the indice 0 indicates that $\xi^A_0$ is there instead of $\xi^A$. Thus (3.2) takes the following form:

$$X_0 + W_0^\rho \partial_\rho \epsilon + Y_0^\rho \partial_\rho \epsilon = \partial_\sigma (\mathcal{J}_\sigma^A \epsilon + \tilde{U}_0^\sigma \partial_\sigma \epsilon + V_0^\rho \partial_\sigma \epsilon)$$

where $V_0^\rho \partial_\rho \epsilon = 0$.\hspace{1cm} (3.4)

3.2. Cascade equations

In local Minkowski coordinates, $D_{\rho \sigma} \epsilon = \partial_{\rho \sigma} \epsilon$ and its factor, $V(\rho \sigma)$ in the right-hand side of (3.5), must be zero because there is no similar factor on the left-hand side and $\epsilon$ is arbitrary. Thus, we must have both

$$V_{\rho \sigma}^\mu = V_{\rho \sigma}^\mu$$

and

$$V_{\rho \sigma}^\mu = \frac{1}{2} (V_{\rho \sigma}^\mu + V_{\rho \sigma}^\mu + V_{\rho \sigma}^\mu) = 0.$$ \hspace{1cm} (3.6)

With (3.6), one finds that (3.7) can equally be written as follows:

$$V_{\rho \sigma}^\mu D_{\rho \sigma} \epsilon = -\frac{2}{3} V_{\mu \nu \lambda}^\rho R_{\rho \mu \nu} \partial_\rho \epsilon,$$ \hspace{1cm} (3.8)

and with (3.8), we can now rewrite (3.5) like this

$$X_0 + W_0^\rho \partial_\rho \epsilon + Y_0^\rho \partial_\rho \epsilon = (\partial_\rho J^\rho \epsilon + (J^\rho + D_\rho U^{\rho \sigma}) \partial_\rho \epsilon$$

$$+ (\tilde{U}^{(\rho \sigma)} + D_\rho V^{\rho \sigma}) D_{\rho \sigma} \epsilon + V^{\rho \sigma} D_{\rho \sigma} \epsilon.$$ \hspace{1cm} (3.9)

This identity holds for any $\epsilon$. Thus the factors of $\epsilon, \partial_\rho \epsilon$ and $D_{\rho \sigma} \epsilon$ from both sides of the equality must be identical, i.e.

$$X = \partial_\rho J^\rho, \hspace{0.5cm} W^\rho = J^\rho + D_\rho U^{\rho \sigma} - \frac{2}{3} V_{\mu \nu \lambda}^\rho R_{\mu \nu \lambda} \partial_\rho \epsilon \hspace{0.5cm} \text{and} \hspace{0.5cm} Y_{\rho \sigma} = \tilde{U}^{(\rho \sigma)} + D_\rho V^{\rho \sigma}.$$ \hspace{1cm} (3.10)

These cascade equations are similar to (2.13). We shall now reduce this set to one that looks exactly like (2.13).
3.3. An equivalent set of identities

We start from \( W^\rho \) as given in (3.10) in which we replace \( \tilde{U}^{\sigma\rho} \) by \( \left( -\tilde{U}^{\rho\sigma} + \tilde{U}^{(\rho\sigma)} \right) \)

\[
W^\rho = J^\rho - \partial_\sigma \tilde{U}^{[\rho\sigma]} + D_\sigma \tilde{U}^{(\rho\sigma)} - \frac{2}{3} V_{\mu\nu\lambda} R^{\rho\mu\nu\lambda}. \tag{3.11}
\]

Then we replace \( D_\sigma \tilde{U}^{(\rho\sigma)} \) by its value deduced from \( Y^{\rho\sigma} \), given in the right-hand side of (3.10), and we get for (3.11)

\[
W^\rho = J^\rho - \partial_\sigma \tilde{U}^{[\rho\sigma]} + D_\sigma Y^{\rho\sigma} - \left( D_{\sigma\lambda} V^{\lambda\rho\sigma} + \frac{2}{3} V_{\mu\nu\lambda} R^{\rho\mu\nu\lambda} \right). \tag{3.12}
\]

However, thanks to the symmetries of \( V^{\lambda\rho\sigma} \) we readily find that

\[
D_{\sigma\lambda} V^{\lambda\rho\sigma} + \frac{2}{3} V_{\mu\nu\lambda} R^{\rho\mu\nu\lambda} = -\frac{2}{3} \partial_\sigma (D_{\lambda} V^{[\rho\sigma]}). \tag{3.13}
\]

Thus instead of (3.12) we may also write

\[
W^\rho = J^\rho - \partial_\sigma U^{\rho\sigma} + D_\sigma Y^{\rho\sigma}\] where \( U^{\rho\sigma} \equiv \tilde{U}^{[\rho\sigma]} - \frac{2}{3} D_{\lambda} V_{\mu\nu\lambda}^{\rho\sigma} \). \tag{3.14}

The tensor \( U^{\rho\sigma} \) is anti-symmetrical. Thus (3.14) provides identities similar to (2.13)

\[
X = \partial_\rho J^\rho, \quad J^\rho = \ast W^\rho + \partial_\rho U^{\rho\sigma}, \quad \text{where} \quad \ast W^\rho \equiv W^\rho - D_\sigma Y^{\rho\sigma}. \tag{3.15}
\]

and the corresponding generalized Bianchi identities are

\[
\mathcal{L}^\lambda \Lambda_{\alpha\lambda} - D_\rho \left( \mathcal{L}^\lambda \Lambda_{\rho\alpha\lambda} \right) + D_{\rho\sigma} \left( \mathcal{L}^\lambda \Lambda_{\rho\sigma\lambda} \right) = 0. \tag{3.16}
\]

3.4. More identities

Consider now (3.3) which we rewrite like this

\[
X = X_0 + W_0^\rho \partial_\rho \epsilon + Y_0^{\rho\sigma} D_{\rho\sigma} \epsilon = X_0 + W_0^\rho \partial_\rho \epsilon + Y_0^{\rho\sigma} \partial_{\rho\sigma} \epsilon \quad \text{where} \quad \tilde{W}^\rho = W^\rho - Y^{\mu\nu\rho} \epsilon_{\mu\nu}. \tag{3.17}
\]

Since \( X \) is a divergence we have

\[
\frac{\delta X}{\delta y_A} = \frac{\partial X}{\partial y_A} - D_\sigma \left( \frac{\partial X}{\partial (D_\sigma y_A)} \right) = 0. \tag{3.18}
\]

Inserting (3.17) into (3.18), identifying the coefficients of \( \epsilon, \partial_\rho \epsilon, \partial_{\rho\sigma} \epsilon \) and \( \partial_{\rho\lambda\sigma} \epsilon \) to zero, removing the indice 0, because the resulting identities are valid for any \( \xi^\lambda \), one obtains four identities, two of which do not involve \( X \)

\[
\frac{\delta Y^{\rho\sigma}}{\delta y_A} = \tilde{W}^{(\rho\sigma)} \Rightarrow \tilde{W}^{(\rho\sigma)} \equiv \frac{\partial \tilde{W}^\rho}{\partial (D_\sigma y_A)} \delta y_A \quad \text{and} \quad Y^{(\lambda\rho\sigma)} = 0 \Rightarrow Y^{(\lambda\rho\sigma)} \equiv \frac{\partial Y^{\rho\sigma}}{\partial (D_\lambda y_A)} \delta y_A. \tag{3.19}
\]

We note that \( Y^{(\lambda\rho\sigma)} \) has the same symmetries as \( Y^{\lambda\rho\sigma} \). Therefore (3.13) holds for \( Y^{\lambda\rho\sigma} \) except that in (3.17) we have ordinary derivatives as if we were in a flat space and therefore

\[
\partial_\sigma (Y^{\lambda\rho\sigma}) = -\frac{2}{3} \partial_\sigma (\partial_\lambda Y^{[\rho\sigma]}). \tag{3.20}
\]
3.5. The superpotential equation

We now consider the variation of $J^\rho$ as given in (3.15), using $\tilde{W}^\rho$ as defined in (3.17):

$$\delta J^\rho = \delta (W^\rho - D_\sigma Y^{\rho\sigma}) + \partial_\sigma \delta U^{\rho\sigma} = \delta \tilde{W}^\rho - \partial_\sigma \delta Y^{\rho\sigma} + \partial_\sigma \delta U^{\rho\sigma}. \quad (3.21)$$

Note that $\tilde{W}^\rho$ like $Y^{\rho\sigma}$ contains at most first-order derivatives of $y_A$. Therefore, we may write (3.21), making use of (3.17) as follows:

$$\delta J^\rho = \delta(\tilde{W}^\rho \delta y_A) + \partial_\sigma [\tilde{W}^{\rho\sigma} \delta y_A] = \delta \tilde{W}^\rho - \partial_\sigma \delta Y^{\rho\sigma} + \partial_\sigma \delta U^{\rho\sigma}. \quad (3.22)$$

With (3.19), the sum of the first two terms in parenthesis may be written

$$\tilde{W}^{\rho\sigma} = \tilde{W}^{\rho\sigma} - \partial_\sigma (D_\lambda y_A) \delta y_A,$$

and therefore $\delta J^\rho$ can also be written

$$\delta J^\rho = \delta \left( \frac{\partial Y^{\rho\sigma}}{\partial (D_\lambda y_A)} \right) \delta y_A. \quad (3.23)$$

But using (3.20) and the definition of $\tilde{W}^\rho$ in (3.17) we find that

$$\partial_\sigma \tilde{W}^{\rho\sigma} = \partial_\sigma \left( Y^{\rho\sigma} + \frac{2}{3} D_\lambda (Y^{\rho\sigma} \lambda) \right). \quad (3.24)$$

With (3.19), the sum of the first two terms in parenthesis may be written

$$\tilde{W}^{\rho\sigma} = \tilde{W}^{\rho\sigma} - \partial_\sigma (D_\lambda y_A) \delta y_A,$$

and therefore $\delta J^\rho$ can also be written

$$\delta J^\rho = \delta \left( \frac{\partial Y^{\rho\sigma}}{\partial (D_\lambda y_A)} \right) \delta y_A. \quad (3.25)$$

Thus the differential equation for the superpotential is now

$$\delta U^{\rho\sigma} = - W^{\rho\sigma} + \frac{2}{3} D_\lambda (Y^{\rho\sigma} \lambda)$$

and

$$\delta \left( \int_V J^\rho \, dV^\rho \right) = \int_V \frac{\partial W^\rho}{\partial y_A} \delta y_A \, dV^\rho. \quad (3.26)$$

Putting together the various elements defined along the way, we have an equation for the superpotential—the first equality in (3.28)—that looks like this

$$\delta U^{\rho\sigma} = - \left( \frac{\partial W^{\rho\sigma}}{\partial (D_\lambda y_A)} \right) \delta y_A + \frac{2}{3} D_\lambda \left( \frac{\partial Y^{\rho\sigma}}{\partial (D_\lambda y_A)} \delta y_A \right). \quad (3.27)$$

in which

$$W^\rho = A^A A^A_{\lambda} \lambda + 2 A^A A^\rho_{\lambda} D_\lambda \xi^\lambda \quad \text{and} \quad Y^{\rho\sigma} = A^A A^{\rho\sigma} \xi^\lambda. \quad (3.28)$$

Equations (3.29) and (3.30), together with appropriate boundary conditions, are all we need to calculate the superpotential in the following example.

4. Application to variational derivatives of Lovelock Lagrangians in the Palatini representation

4.1. Lagrangian, variational derivatives and boundary conditions

In the Palatini formulation we take the inverse metric components $g^{\mu\nu}$ and the connection coefficients $\Gamma^\lambda_{\rho\sigma} = \Gamma^\lambda_{\sigma\rho}$ as independent field components. The curvature tensor does not
have all the symmetries of the Riemannian one. To avoid any confusion we shall denote
the curvature tensor, like in Eisenhart \cite{10}, by $B^\lambda_{\nu\rho\sigma}$ and reserve the usual notation of
the curvature tensor $R^\lambda_{\nu\rho\sigma}$ for when the connection coefficients are Christoffel symbols. $B^\lambda_{\nu\rho\sigma}$
is antisymmetrical in the last two indices only, $B^\lambda_{\nu\rho\sigma} = -B^\lambda_{\rho\sigma\nu}$. Additional symmetry
properties, similar to those of the Riemann curvature tensor, are:

$$B^\lambda_{\nu\rho\sigma} = \frac{1}{2} \left( \partial\!_{\rho} \Gamma^\lambda_{\sigma\nu} \right)_{\nu}^{} + \Gamma^\lambda_{\eta\rho} \left( \Gamma^\eta_{\sigma\nu} \right)_{\nu}^{}.$$  \hfill (4.1)

Here a semi-column means covariant differentiation. The curvature tensor itself is given by

$$B^\lambda_{\nu\rho\sigma} = \frac{1}{2} \left( \partial\!_{\rho} \Delta^\lambda_{\sigma\nu} \right)_{\nu}^{} + \Delta^\lambda_{\eta\rho} \left( \Delta^\eta_{\sigma\nu} \right)_{\nu}^{} + \overline{B}^\lambda_{\nu\rho\sigma}.$$  \hfill (4.2)

In accordance with \cite{20}, we introduce a background whose metric components are $\overline{g}_{\mu\nu}$. Instead
of $\Gamma$’s we introduce

$$\Delta^\lambda_{\rho\sigma} \equiv \Gamma^\lambda_{\rho\sigma} - \frac{1}{2} g^{\lambda\mu} \left( \partial\!_{\rho} g_{\mu\sigma} + \partial\!_{\sigma} g_{\mu\rho} - \partial\!_{\mu} g_{\rho\sigma} \right).$$  \hfill (4.3)

The bars like on $\overline{D}$’s refer to the background. $\Delta^\lambda_{\rho\sigma}$ is a tensor. The curvature tensor can be
written in terms of these $\Delta$’s

$$B^\lambda_{\nu\rho\sigma} = \frac{1}{2} \left( \partial\!_{\rho} \Delta^\lambda_{\sigma\nu} \right)_{\nu}^{} + \Delta^\lambda_{\eta\rho} \left( \Delta^\eta_{\sigma\nu} \right)_{\nu}^{} + \overline{B}^\lambda_{\nu\rho\sigma}.$$  \hfill (4.4)

As boundary conditions we want to impose the value of the metric components on the boundary $S$, i.e.

$$g_{\mu\nu}\big|_S = \overline{g}_{\mu\nu}.$$  \hfill (4.5)

We shall also demand that on the boundary the $\Gamma$’s be Christoffel symbols, i.e. that

$$D\!_{\lambda} \overline{g}^{\mu\nu}\big|_S = 0 \quad \text{and thus} \quad B^\lambda_{\nu\rho\sigma}\big|_S = R^\lambda_{\nu\rho\sigma}.$$  \hfill (4.6)

The action of the gravitational field is of the form

$$A = \frac{1}{2\kappa} \int L\left( g_{\mu\nu}, B^\lambda_{\nu\rho\sigma} \right) d^N x \quad \text{where} \quad \kappa = \frac{2S_{N-2}}{\mathcal{N}} G_N c^4,$$  \hfill (4.7)

where $\mathcal{L}$ is a Lovelock Lagrangian \cite{24}, $N$ is the dimension of spacetime. The coupling
constant $\kappa$ is normalized like in \cite{1}, except for a factor 2: $S_{N-2}$ is the surface of a sphere of
dimension $N - 2$ and $G_N$ the gravitational coupling constant\footnote{For $N = 4$, $S_2 = 4\pi$ and $\kappa = \frac{8\pi G}{c^4}$ in which $G$ is Newton’s gravitational constant.}.

Contrary to our assumption in sections 2 and 3 that spacetime has a passive role here it is part of the game: the metric components and the connection coefficients are the field components. Nonetheless, section 3 is applicable to variational derivatives of Lovelock Lagrangians because: (1) the connection coefficients enter into tensorial combinations, (2) the variational derivatives as well as Lie derivatives of the connection coefficients are tensors as well and (3) on the boundary the connection coefficients are Christoffel symbols.

The variation of the action has this form

$$\delta A = \int \left( \mathcal{L}_{\mu\nu} \delta g^{\mu\nu} + \mathcal{L}_\alpha^{\rho\sigma} \delta \Delta^\alpha_{\rho\sigma} \right) d^N x + \oint_S D\!^\mu dS\!_{\mu}.$$  \hfill (4.8)

$\mathcal{L}_{\mu\nu}$ and $\mathcal{L}_\alpha^{\rho\sigma}$ are variational derivatives. In applying the variational principle to the action of
the gravitational field plus its sources, ($\delta A + \delta A_{\text{sources}} = 0$). $\mathcal{L}_{\mu\nu}$ is defined by the sources
but for minimally coupled matter, $\mathcal{L}_\alpha^{\rho\sigma} = 0$. This last equation is linear and homogeneous in
$D\!_{\lambda} g^{\mu\nu}$ and contains no derivatives of $B^\lambda_{\nu\rho\sigma}$. This is the particularity of Lovelock Lagrangians.
As a consequence, and as shown by Exirifart and Sheikh-Jabbari \cite{11}, $D\!_{\lambda} g^{\mu\nu} = 0$ will always
be a solution of this equation. We shall, of course, not use that solution except to note that it
is in accordance with the boundary condition (4.6). In addition to $\mathcal{L}^{\rho\sigma} = 0$, the variational principle would impose some boundary condition, i.e., for isolated sources,

$$D^\mu dS_\mu = 0.$$  (4.9)

Adding a divergence to the Lagrangian may change this condition but, contrary to what happens in the Hamiltonian formalism, there is no way to get rid of a condition like (4.9) by adding a divergence to $\mathcal{L}$. The condition (4.9) may be satisfied if (4.5), (4.6) hold. Otherwise (4.9) must at least be compatible with (4.5), (4.6).

4.2. The superpotential

The Lie derivatives $\mathcal{L}_\xi y_A$ are as follows, the second equality is a repeat in terms of $\Lambda$’s like in (3.1) where indices $A$ are surrounded by curly brackets

$$\mathcal{L}_\xi g^{\mu\nu} = D_\lambda g^{\mu\nu} \xi^\lambda - 2 g^{\eta(\mu} D_\eta g^{\nu)} = \Lambda_\lambda^{[(\mu]} \xi^\lambda + \Lambda_\lambda^{(\mu|\nu)]} D_\eta \xi^\lambda,$$  (4.10)

$$\mathcal{L}_\xi \Gamma^\eta_{\rho\sigma} = - R^\eta_{(\rho|\sigma)} \xi^\lambda + D_{(\rho|\sigma)} \xi^\eta = \Lambda_\lambda^{[\eta]} (\rho|\sigma) \xi^\lambda + \Lambda_\lambda^{(\eta|\rho)} D_{(\rho|\sigma)} \xi^\lambda.$$  (4.11)

The only nonzero $\Lambda$’s appearing in $W^\rho$ and $Y^{\rho\sigma}$, see (3.30), are

$$\Lambda_\lambda^{[\nu\eta]} = -2 g^{\eta(\mu} D_\eta \xi^{\rho)}$$ and $$\Lambda_\lambda^{(\eta|\rho]} = \frac{1}{2} (\delta^\eta_\mu \delta^\rho_\nu + \delta^\rho_\mu \delta^\eta_\nu) \delta^\lambda_\tau.$$  (4.12)

Introducing these $\Lambda$’s into $W^\rho$ and $Y^{\rho\sigma}$, we obtain

$$W^\rho = 2 (-\mathcal{L}_\xi \xi^\lambda + \mathcal{L}_\xi \rho^{\rho\sigma} D_\sigma \xi^\lambda)$$ and $$Y^{\rho\sigma} = \mathcal{L}_\xi \rho^{\rho\sigma} \xi^\lambda.$$  (4.13)

With this, the solution of equation (3.29) is absolutely straightforward. Since $g_{\mu\nu}|_S = \bar{g}_{\mu\nu}$, we have $\delta g_{\mu\nu} = 0$ in the equation and if (4.6) and (4.7) hold, the following equalities must also hold on the boundary $S$:

$$D_\lambda \delta g^{\mu\nu}|_S = -2 \bar{g}^{\xi^{(\mu} \delta \Delta^{\nu)}}_{\tau \eta}$$ and $$\delta (D_\lambda \xi^\lambda)|_S = \xi^\tau \delta \Delta^{\lambda}_{\mu\nu}.$$  (4.14)

The equation for $U^{\rho\sigma}$ reduces thus to a linear differential form in $\delta \Delta^{\lambda}_{\mu\nu}$. With (4.14) and $\delta g_{\mu\nu} = 0$, the equation for the superpotential (3.29) may be written as

$$\delta U^{\rho\sigma} = -4 \left[ \frac{\partial \mathcal{L}_\xi \rho^{\rho\sigma}}{\partial B^{\mu\nu}_{\mu\nu |\rho\sigma}} + \frac{1}{3} \frac{\partial}{\partial (D_\sigma g^{\mu\nu})} g^{\eta\rho} \right] \delta \Delta^{\lambda}_{\mu\nu}.$$  (4.15)

In this expression it is understood that the square brackets $[\rho \sigma]$ mean that the expression is anti-symmetrized in $\rho \sigma$. Since the factor of $\delta \Delta^{\lambda}_{\mu\nu}$ contains only the metric and the curvature tensor which on the boundary are given thanks to (4.5) and (4.6), provided $\bar{\Gamma}$’s are derivable at least once, we may write that

$$U^{\rho\sigma} = -4 \left[ \frac{\partial \mathcal{L}_\xi \rho^{\rho\sigma}}{\partial B^{\mu\nu}_{\mu\nu |\rho\sigma}} + \frac{1}{3} \frac{\partial}{\partial (D_\sigma g^{\mu\nu})} g^{\eta\rho} \right] \Delta^{\lambda}_{\mu\nu}.$$  (4.16)

Note that $\bar{U}^{\rho\sigma} = 0$ because $\bar{\Delta}^{\lambda}_{\rho\sigma} = 0$. $U^{\rho\sigma}$ is the superpotential for theories of gravity derived from a Lovelock Lagrangian.

5. Application to general relativity in $N$ dimensional spacetimes

5.1. The variational derivatives

Let

$$B = g^{\mu\nu} B_{\mu\nu} = b^{\mu\nu\rho\sigma} B_{\mu\nu\rho\sigma},$$  (5.1)
where
\[ b_{\mu\nu\rho\sigma} = g^{[\rho} g^{\sigma]}_{\nu} \]  \hspace{1cm} (5.2)
Then, if a hat like in \( \hat{g}_{\mu\nu} \) means multiplication by \( \sqrt{-g} \), the ‘relative’ Einstein–Hilbert action is
\[ A_1 = \int \mathcal{L}_1 \, d^N x \equiv \frac{1}{2\kappa} \int (\hat{B} - B) \, d^N x. \]  \hspace{1cm} (5.3)
The variation of the action,
\[ \delta A_1 = \int \left( \mathcal{L}_{\mu\nu} \delta g_{\mu\nu} + \mathcal{L}_{\lambda}^{\rho\sigma} \delta \Delta^\lambda_{\rho\sigma} \right) \, d^N x - \frac{1}{\kappa} \oint_S \hat{b}_{\mu\rho\sigma} \lambda \delta \Delta^\lambda_{\rho\sigma} \, dS_{\mu}, \]  \hspace{1cm} (5.4)
The variational derivatives,
\[ L_{\mu\nu} = \frac{\delta \mathcal{L}_1}{\delta g_{\mu\nu}}, \quad \text{and} \quad \mathcal{L}_{\lambda}^{\rho\sigma} = \frac{\delta \mathcal{L}_1}{\delta \Delta^\lambda_{\rho\sigma}} = \frac{1}{2\kappa} \left( \delta_{\lambda}^{\rho} D_{\eta} \hat{g}_{\sigma}^{\eta\rho} - D_{\lambda} \hat{g}_{\rho\sigma}^{\eta} \right). \]  \hspace{1cm} (5.5)
We may and have indeed used \( B_{\mu\nu} \) in terms of background derivatives and in terms of \( \Delta \)'s so as to keep every term covariant. The boundary conditions that would follow from the action principle would be \( \delta b_{\mu\rho\sigma} \lambda = 0 \). Since neither \( \delta b_{\mu\rho\sigma} \lambda = 0 \) nor \( \delta \Delta^\lambda_{\rho\sigma} \, dS_{\mu} = 0 \) are, in general, compatible with (4.5) and (4.6), we shall add a divergence and take the following new action\(^{17}\):
\[ A_1' = A_1 + \frac{1}{\kappa} \oint_S \hat{b}_{\mu\rho\sigma} \lambda \delta \Delta^\lambda_{\rho\sigma} \, dS_{\mu}. \]  \hspace{1cm} (5.6)
The variation of this action is
\[ \delta A_1' = \int \left( \mathcal{L}_{\mu\nu} \delta g_{\mu\nu} + \mathcal{L}_{\lambda}^{\rho\sigma} \delta \Delta^\lambda_{\rho\sigma} \right) \, d^N x + \frac{1}{\kappa} \oint_S \delta \left( \hat{b}_{\mu\rho\sigma} \lambda \right) \Delta^\lambda_{\rho\sigma} \, dS_{\mu}. \]  \hspace{1cm} (5.7)
The boundary conditions may now be \( \delta \left( \hat{b}_{\mu\rho\sigma} \lambda \right) = 0 \) or \( b_{\mu\nu\rho\sigma} \big|_S = \hat{g}_\rho^{[\nu} g^{\sigma]}_{\mu} \) and these conditions are compatible with (4.5).

5.2. The superpotential

The superpotential (4.16) in this case is simple to calculate
\[ U_1^{\rho\sigma} = \frac{3}{\kappa} \xi^{[\mu} b_{\rho\sigma] \mu} \bigg|_S \Delta^\lambda_{\mu\nu} = \frac{3}{\kappa} \xi^{[\lambda} (b_{\rho\sigma] \nu) \bigg|_S \mathcal{D}_{\mu} g_{\rho\lambda}. \]  \hspace{1cm} (5.8)
The last equality follows from (4.3). \( U_1^{\rho\sigma} \) is, as expected, the KBL superpotential. This may not be apparent because the KBL superpotential is more often written locally like this [20]
\[ U_1^{\rho\sigma} = \frac{1}{\kappa} \left( D[\rho \xi^{\sigma} \eta] - D[\rho \xi^{\sigma} \eta] \right) + \frac{1}{\kappa} \xi^{[\rho} k_{\sigma]} \quad \text{where} \quad k^\sigma \equiv \hat{g}^{\sigma\mu} \Delta^\mu_{\rho\nu} - \hat{g}^{\rho\mu} \Delta^\mu_{\sigma\nu}. \]  \hspace{1cm} (5.9)
To see that (5.9) is the same as (5.8) we note that \( k^\sigma \) can be written as follows:
\[ k^\sigma = 2 \hat{g}^{\sigma\rho \lambda \mu} \Delta^\lambda_{\mu\nu} \quad \text{and} \quad D[\rho \xi^{\sigma}] - D[\rho \xi^{\sigma}], \xi^{[\rho} = (\hat{g}^{\mu\rho} - \hat{g}^{\mu[\rho} \mathcal{D}_{\mu} \xi^{\sigma]} + \hat{g}^{\mu[\rho} \Delta_{\mu\nu}^{\sigma]} \xi^{\nu}. \]  \hspace{1cm} (5.10)

\(^{16}\) The 1/2\( \kappa \) factor is for convenience.

\(^{17}\) This is not the only surface term possible which leads to acceptable boundary conditions. We might also add \( \frac{1}{\kappa} \oint_S \hat{b}_{\mu\rho\sigma} \lambda \delta \Delta^\lambda_{\rho\sigma} dS_{\mu} \) instead to \( A_1 \).
But since $g_{\mu\nu}S = \bar{g}_{\mu\nu}$, the first term on the right-hand side is zero. So, taking (5.10) into account and using the expression for the tensor $b$ defined in (5.2), one easily finds that (5.9) is the same as (5.8).

One may also write (5.8) like this

$$U_1^{\rho\sigma} = \frac{1}{k} \xi_{\mu} \bar{D}_{\nu} \left[ (\hat{g}^{\rho\sigma} b_{\nu\mu}) - \bar{g}^{\rho\sigma} b_{\nu\mu} \right].$$

(5.11)

It appears in this form, on a sphere at spatial infinity, in [1, 9] with $\hat{g}^{\mu\nu}$ to leading order in $1/r$. The physical properties of the KBL superpotential in general relativity are summarized in [18]. It is worth noting that the superpotential is valid on any $S$, whether at spatial or at null infinity.

6. Application to Gauss–Bonnet theories in the Palatini representation

6.1. Lagrangian, variational derivatives and boundary conditions

Gauss–Bonnet theories of gravity, in a Palatini formulation apply to spacetimes with more than four dimensions. Another difference with Einstein’s theory of gravitation lies in the relation between the metric and the connection coefficient. In Einstein’s theory $L_{\lambda\rho\sigma} = 0$ implies $D_{\lambda} g^{\mu\nu} = 0$. In Gauss–Bonnet theories $D_{\lambda} g^{\mu\nu} = 0$ is always a solution of $L_{\lambda\rho\sigma} = 0$.

Equations (4.1)–(4.6) and (4.10)–(4.16) hold in Gauss–Bonnet theories. The variational derivatives are derived from a Lovelock Lagrangian of order two which we took from Jacobson and Myers [16]. Define

$$B = g^{\mu\rho} g^{\sigma\sigma} \left( B^{\mu\rho\sigma\sigma} - B^{\eta}_{\mu\rho\sigma} B_{\eta\sigma\sigma} - 2 B^{\lambda}_{\mu\rho\sigma} B_{\rho\lambda\sigma} + B_{\mu\rho\sigma} B_{\rho\mu\sigma} - B_{\rho\mu\sigma} B_{\rho\mu\sigma} \right).$$

(6.1)

Then, the relative action with respect to the background,

$$A_2 = \int L_2 d^N x = \frac{1}{4\kappa} \int (\hat{B} - \bar{B}) d^N x.$$ 

(6.2)

The variation of the action is of this form

$$\delta A_2 = \int \left( L_{\mu\nu} \delta g^{\mu\nu} + L_{\lambda\rho\sigma} \delta D_{\lambda} \right) d^N x - \frac{1}{k} \oint_S \hat{P}_{\mu\rho\sigma} \delta D_{\lambda} dS_{\mu\rho\sigma};$$

(6.3)

in this the variational derivatives with respect to the metric components are as follows:

$$L_{\mu\nu} = \frac{\delta L_2}{\delta g^{\mu\nu}} = -\frac{1}{2} L_2 \hat{g}_{\mu\nu} + \sqrt{-g} \hat{L}_{\mu\nu},$$

(6.4)

in which

$$2\kappa \hat{L}_{\mu\nu} = g^{\rho\sigma} \left( B^{\rho\sigma}_{\mu\rho\sigma} B_{\rho\sigma\sigma} + B^{\mu\rho\sigma}_{\mu\rho\sigma} B_{\rho\sigma\sigma} + B_{\rho\sigma\sigma} B_{\rho\sigma\sigma} \right) + B^{\lambda}_{\rho\sigma\sigma} B_{\rho\lambda\sigma}$$

while variational derivatives with respect to the connection coefficients are of the form $D(\mathcal{O}(g)B)$ where $\mathcal{O}$ depends on the metric components only, the Bianchi identities eliminate the derivatives of $B$’s and the result is of the form $D(\mathcal{O}(g)B)$; explicitly

$$2\kappa \hat{L}_{\lambda}^{\rho\sigma} = 2\kappa \frac{\delta L_2}{\delta \Gamma^{\rho\sigma}_{\lambda\rho\sigma}} = \frac{3}{2} D_\rho \left[ (\hat{g}^{\mu\rho} \delta^\sigma_\mu g^{\nu\nu}) + (\hat{g}^{\mu\sigma} \delta^\rho_\mu g^{\nu\nu}) \right] B^\rho_{\mu\nu\tau} + 3 D_\rho \left( \hat{g}^{\mu\rho} \delta^\sigma_\mu g^{\nu\nu} B^\rho_{\mu\nu\tau} + \hat{g}^{\mu\rho} \delta^\rho_\mu B_{\mu\nu}\right).$$

(6.5)

18 Gauss–Bonnet theories of gravity are rarely considered alone. What is usually used is Einstein’s theory in $N$ dimensions to which a Gauss–Bonnet term is added. This is then called an Einstein–Gauss–Bonnet theory which we consider briefly below.
The boundary term introduces a tensor denoted by Davis [5] as

\[ P_{\mu\nu\rho\sigma} | S \equiv (R_{\mu\nu\rho\sigma} - 2g_{\mu\rho}R_{\nu}^{\sigma} + 2g_{\mu\sigma}R_{\nu}^{\rho} + Rg_{\mu\rho}g_{\nu}^{\sigma}) \].

\[ (6.7) \]

\( P_{\mu\nu\rho\sigma} \), which has manifestly the symmetries of the curvature tensor like \( b_{\mu\nu\rho\sigma} \), is also divergenceless: \( D_{\mu} P_{\mu\nu\rho\sigma} = 0 \).

We may take a new action which is indeed compatible with (4.5) and (4.6) by adding a divergence to \( L_{2} \)

\[ A' = \int L_{2} d^{N}x + \frac{1}{\kappa} \oint_{S} \hat{P}_{\mu\rho\sigma\lambda} \Delta_{\rho\sigma} dS_{\mu} \],

so that

\[ \delta A' = \int (\mathcal{L}_{2} \delta \hat{g}_{\mu\nu} + \mathcal{L}_{2} \delta \hat{g}_{\rho\sigma} \hat{\Delta}_{\rho\sigma}) d^{N}x + \frac{1}{\kappa} \oint_{S} \delta \hat{P}_{\mu\rho\sigma\lambda} \Delta_{\rho\sigma} dS_{\mu} \].

(6.8)

The boundary condition that follows from the variational principle is now \( \delta \hat{P}_{\mu\rho\sigma\lambda} = 0 \) which is equivalent to (6.7). This condition will automatically hold if (4.5) and (4.6) are satisfied and the \( \Gamma \)'s are derivable at least once.

6.2. The superpotential

The calculations of the superpotential \( U_{2}^{\rho\sigma} \) given by equation (4.16) is somewhat more complicated than for Einstein’s theory in \( N \) dimensions. MathTensor and Mathematica were valuable tools to check our calculations. A condensed but readable formula for \( U_{2}^{\rho\sigma} \) is perhaps this one

\[ U_{2}^{\rho\sigma} = 3 \left[ \hat{g}^{\mu\sigma} R^{\rho\sigma} \right] \Delta_{\mu\nu}^{\hat{\lambda}} + (\hat{g}^{\mu\sigma} G_{\mu}^{\rho} \Delta_{\nu}^{\sigma} - \hat{g}^{\mu\sigma} G_{\nu}^{\rho} \Delta_{\sigma}^{\mu}) \hat{\xi}^{\eta} \]

\[ + \frac{2}{5} (\hat{g}^{\mu\sigma} R^{\rho\sigma} \hat{\xi}^{\eta} + \hat{R}^{\mu\rho\sigma} \hat{\xi}^{\eta} - \hat{R}^{\mu\sigma\rho} \hat{\xi}^{\eta}) + (\hat{g}^{\mu\sigma} G_{\mu}^{\rho} \Delta_{\nu}^{\sigma} - \hat{g}^{\mu\sigma} G_{\nu}^{\rho} \Delta_{\sigma}^{\mu}) \hat{\xi}^{\eta} \]

\[ - \frac{2}{5} (\hat{g}^{\mu\sigma} G_{\mu}^{\rho} \Delta_{\nu}^{\sigma} - \hat{g}^{\mu\sigma} G_{\nu}^{\rho} \Delta_{\sigma}^{\mu}) \hat{\xi}^{\eta} \] \[ + \hat{R}^{\mu\rho\sigma} \hat{\xi}^{\eta} - \hat{R}^{\mu\sigma\rho} \hat{\xi}^{\eta} \]

\[ (6.10) \]

the factors of \( \Delta \)'s are, of course, to be evaluated on the boundary, that is, in terms of the background geometry which is generally much simpler than that of the spacetime itself.

Deser and Tekin [8, 9] have calculated the mass–energy in generic higher curvature gravity theories, in particular on anti-de Sitter backgrounds, motivated by the role of these backgrounds in string theory.

Because of its importance in string theory and as an opportunity to relate our results with previous calculations, we consider now what becomes of \( U_{2}^{\rho\sigma} \) if the curvature tensor of the background is of the form

\[ R_{\mu\nu\rho\sigma} = \frac{1}{l^{2}} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) = \frac{2}{l^{2}} P_{\mu\nu\rho\sigma} \].

(6.11)

With such a background (6.10) becomes rather simple:19

\[ U_{2}^{\rho\sigma} = \frac{1}{l^{2}} (N - 3)(N - 4)U_{1}^{\rho\sigma} \].

(6.12)

We now consider the Einstein–Gauss–Bonnet theory of gravity and follow Deser and Tekin [9]. Our action is written like this

\[ A = \int (\mathcal{L}_{1} + a \mathcal{L}_{2}) d^{N}x + \frac{1}{\kappa} \int 2\Lambda_{0}(\sqrt{-g} - \sqrt{-g}) d^{N}x \]

(6.13)

19 Formulae (6.10) as well as (6.12) are at variance with the superpotential suggested in [7].
where $a$ is a coupling constant and $\Lambda_0$ contributes to the overall effective cosmological constant $\Lambda$ which is related to $l^2$ as follows:

$$l^2 = -\frac{(N-1)(N-2)}{2\Lambda}, \quad \Lambda < 0. \quad (6.14)$$

The superpotential in this case is of the form

$$U^{\rho\sigma} = U^{\rho\sigma}_1 + aU^{\rho\sigma}_2 = \left(1 + a \frac{(N-3)(N-4)}{l^2}\right) U^{\rho\sigma}_1. \quad (6.15)$$

In the case considered by Deser and Tekin,

$$a = -2\kappa \gamma = \frac{-2l^2}{(N-3)(N-4)}. \quad (6.16)$$

Thus,

$$U^{\rho\sigma} = -U^{\rho\sigma}_1. \quad (6.17)$$

This is indeed their superpotential on a sphere at spatial infinity, as can be figured out from their formula (31) in which we must set $\alpha = \beta = 0$.

7. A brief summary and some comments

7.1. First the summary

The superpotential $U^{\rho\sigma}_n$ of a Lovelock Lagrangian of order $n$, $L_n$, $n = 1, 2, 3, \ldots$, or the superpotential $U^{\rho\sigma} = \sum a^n U^{\rho\sigma}_n$ of a linear combination of such Lagrangians $L = \sum a^n L_n$, with coupling constants $a^n$, is given by equation (4.16). The structure of $U^{\rho\sigma}_n$ is rather obvious. $U^{\rho\sigma}_n$ is homogeneous of order $(n-1)$ in the curvature tensor of the background and is linear homogeneous in $\Delta^{\rho\sigma} = \Gamma^{\rho\sigma} - \Gamma^{\rho\sigma}_\xi$.

We calculated $U^{\rho\sigma}_1$ which is well known and has been obtained by various authors with different methods. It was, however, useful to show the reader that we recovered at least well known results. One practical novelty, besides equation (4.16), is an explicit expression for $U^{\rho\sigma}_2$ on arbitrary backgrounds, formula (6.10). Another one is an expression for $U^{\rho\sigma}_2$, see (6.12), on anti-de Sitter backgrounds. This later expression provides the same superpotential as that found by Deser and Tekin at spatial infinity, which they obtained using a very different method. The result gave us further confidence that the procedure of Silva, extended in section 3 to Palatini’s representation of gravity fields, did indeed work and that equation (4.16) was correct. One might have calculated $U^{\rho\sigma}_3$ and $U^{\rho\sigma}_4$ using $L_3$ and $L_4$ explicitly written in an appendix of a paper by Wheeler [32]. This would have been hard work with little direct prospect of applicability. The method of section 3 may of course be applied to variational derivatives of higher order that are not Lovelock Lagrangians, the type of Lagrangians that interest string theorists.

7.2. Now some comments

(a) Regarding $U^{\rho\sigma}_1$ as given by equation (5.8). On a flat background, in Minkowski coordinates and with $\xi$ the Killing vectors of spacetime translations, $U^{\rho\sigma}_1$ is exactly the superpotential that Von Freud [31] found, almost 70 years ago, to calculate mass–energy and total linear momentum. One wonders why Freud did not calculate the angular momentum on the same occasion.

---

20 In [9], $l^2$ is defined with the opposite sign so that their $l^2$ for AdS spacetimes is negative.
21 See also Müller-Hoissen [25] who gives $L_3$. 

---
We emphasized several times that the superpotential holds at null as well as at spatial infinity and that $U^{\rho\sigma}$ at null infinity gives the Bondi mass. We do not know if anybody became interested in radiating fields for Einstein–Gauss–Bonnet theories. $U^{\rho\sigma}_2$ is, of course, the superpotential appropriate for calculating the Bondi mass.

We imposed Dirichlet boundary conditions. Julia and Silva showed that imposing Neumann boundary conditions lead to Komar’s superpotential. Neumann boundary conditions have been considered in recent works. See for instance a paper by Kofinas and Ola [23] on Lovelock anti-de Sitter gravity. Our superpotential $U^{\rho\sigma}_2$ does not apply to such spacetimes.

Jacobson and Myers used the first law of thermodynamics to define mass–energy of Lovelock black holes. Mass–energy has, however, little to do with thermodynamics because asymptotic spacetimes ignore the source of gravity. A direct calculation of the mass–energy of Lovelock black holes, independent of thermodynamic considerations, is naturally provided by the superpotential $U^{\rho\sigma}_1$. An approach, similar to that of Jacobson and Myers, was used by Gibbons et al [14] to find the mass of Kerr–anti-de Sitter black holes. Needless to say the same mass–energy is obtained with the KBL superpotential $U^{\rho\sigma}_1$, see [6].

Finally, we are still wondering about the following problem. Noether’s identity provides a conserved current from a given Lagrangian. The conserved current is the divergence of a non-unique superpotential. The form of that current (see section 1) tells us what physical meaning to attribute to closed surface integrals of the superpotential assuming it has been properly chosen. With Silva’s prescription, things work the other way round. One calculates a unique superpotential but one must look at its divergence to figure out the physical meaning of its total flux. Both methods are sound and well defined but what is the formal connexion?

Acknowledgments

We thank Nathalie Deruelle for very useful discussions.

References

[1] Abbot J F and Deser S 1982 Stability of gravity with a cosmological constant Nucl. Phys. B 195 76
[2] Anderson I M and Torre C G 1996 Asymptotic conservation laws in field theory Phys. Rev. Lett. 77 4109 (Preprint hep-th/9608008)
[3] Bondi H 1960 Gravitational waves in general relativity Nature 186 535
[4] Cai R-G 2002 Gauss–Bonnet black holes in AdS spaces Phys. Rev. D 65 084014 (Preprint hep-th/0109133)
[5] Davis S C 2003 Generalized Israel junction conditions for a Gauss–Bonnet brane world Phys. Rev. D 67 024030 (Preprint hep-th/0208205)
[6] Deruelle N and Katz J 2005 On the mass of a Kerr–anti-de Sitter spacetime in D Dimensions Class. Quantum Grav. 22 421 (Preprint arXiv:gr-qc/0410135 v1)
[7] Deruelle N, Katz J and Ogushi S 2004 Conserved charges in Einstein Gauss–Bonnet theory Class. Quantum Grav. 21 1971 (Preprint hep-th/gr-qc/0310098)
[8] Deser S and Tekin B 2002 Gravitational energy in quadratic curvature gravities Phys. Rev. Lett. 89 101101 (Preprint hep-th/0205318)
[9] Deser S and Tekin B 2003 Energy in generic higher curvature gravity theories Phys. Rev. D 67 084009 (Preprint hep-th/0212292)

A generalization of Komar’s superpotential, that does not need a background, to any type of boundary conditions, can be found in several papers of Obukhov and Rubilar’s; see for instance [26]. The role of backgrounds is here replaced by another ingredient, ‘generalized’ Lie derivatives.
[10] Eisenhart L P 1927 Non-Riemannian Geometry (AMS Colloquium Publications vol 8) (New York: Dover) pp 184 (reprinted in 2005)

[11] Exirifard Q and Sheikh-Jabbari M M 2008 Lovelock gravity at the crossroads of Palatini and metric formulations Phys. Lett. B 661 158 (Preprint arXiv:0705.1879)

[12] Fatibene L, Ferraris M and Francaviglia M 2005 Augmented variational principles and relative conservation laws in classical field theory Int. J. Geom. Meth. Mod. Phys. 2 373 (Preprint math-ph/0411029)

[13] Ferraris M, Francaviglia M and Raitere M 2003 Conserved quantities from the equations of motion: with applications to natural and gauge natural theories of gravitation Class. Quantum Grav. 29 4043 (Preprint gr-qc/0305047)

[14] Gibbons G W, Perry M J and Pope C N 2005 The first law of thermodynamics for Kerr-Anti-de Sitter black holes Class. Quantum Grav. 22 1503 (Preprint arXiv:0705.1879)

[15] Hollands S, Ishibashi S and Marolf D 2005 Counter-term charges generate bulk symmetries Phys. Rev. D 72 104025 (Preprint hep-th/0503105)

[16] Jacobson T and Myers R C 1993 Entropy of Lovelock black holes Phys. Rev. Lett. 70 3684 (Preprint hep-th/9305016)

[17] Julia B and Silva S 1998 Currents and superpotentials in classical gauge theories: I. Local results with applications to perfect fluids and general relativity Class. Quantum Grav. 15 2173 (Preprint gr-qc/9804029)

[18] Julia B and Silva S 2000 Currents and superpotentials in classical gauge theories: II. Global aspects and the example of affine gravity Class. Quantum Grav. 17 4733 (Preprint gr-qc/0008127)

[19] Katz J 1985 A note on Komar’s anomalous factor Class. Quantum Grav. 2 423

[20] Katz J, Bičák J and Lynden-Bell D 1997 Relativistic conservation laws and integral constraints for large cosmological perturbations Phys. Rev. D 55 5957 (Preprint gr-qc/0504041)

[21] Katz J, Lynden-Bell D and Bičák J 2006 Gravitational energy in stationary spacetimes Class. Quantum Grav. 23 7111 (Preprint gr-qc/0610052)

[22] Katz J, Lynden-Bell D and Bičák J 2007 Energy and angular momentum densities of stationary gravity fields Phys. Rev. D 75 024040

[23] Kofinas G and Olea R 2007 Universal regularization prescription for Lovelock AdS gravity J. High Energy Phys. JHEP11(2007)069 (Preprint arXiv:0708.0782)

[24] Lovelock D 1971 The Einstein tensor and its generalizations J. Math. Phys. 12 498

[25] Mielke-Goossens F 1985 Spontaneous compactification with quadratic and cubic curvature terms Phys. Lett. B 163 106

[26] Obukhov Y N and Rubilar G F 2006 Invariant conserved currents in gravity theories with local Lorentz and diffeomorphism symmetry Phys. Rev. D 74 064002 (Preprint arXiv:gr-qc/0608064)

[27] Petrov A N 2005 A note on the Deser–Teitelboim charges Class. Quantum Grav. 22 L83 (Preprint gr-qc/0504058)

[28] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity Ann. Phys. 88 286

[29] Rosen J 1974 Generalized Noether’s theorem. I. Theory Ann. Phys. 82 54

[30] Silva S 1999 On superpotentials and charge algebras of gauge theories Nucl. Phys. B 558 391 (Preprint hep-th/9909109)

[31] von Freud P 1939 Über die Ausdrücke der Gesamtenergie und des Gesamtimpulses eines Materiellen Systems in der Allgemeinen Relativitätstheorie Ann. Math. J. 40 417

[32] Wheeler J T 1986 Symmetric solutions to the Gauss–Bonnet extended Einstein equations Nucl. Phys. B 268 737