Hashimoto transform for stochastic Landau-Lifshitz-Gilbert equation

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1 Introduction

It is well known that the Heisenberg ferromagnet equation and the nonlinear Schrödinger equation are equivalent ([3], [4], [6] and references therein). In this paper we will show that Hashimoto transformation is applicable to the one dimensional stochastic Landau-Lifshitz-Gilbert (LLG) equation ([1], [2] and references therein) and transforms it to the stochastic nonlinear generalized heat equation with nonlocal (in space) interaction. We will start with the case of deterministic 1D LLG equation

$$u_t = \beta u \times u_{xx} - \alpha u \times (u \times u_{xx}), x \in \mathbb{R}, t \geq 0.$$  (1.1)

and prove in the Theorem 2.1 that Hashimoto transform $q = |u_x| e^{-i \int -\infty <u \times u_x, u_{xx}> dy}$ of a smooth solution satisfies to the following generalized heat equation

$$\partial_t q = \alpha [q_{xx} + \frac{q}{2} \int_{-\infty}^{x} (q_xq_x - q_xq_x) dy] + i \beta (q_{xx} + \frac{|q|^2}{2} - q_x \int_{-\infty}^{x} q_{xx} dy).$$  (1.2)

$$\equiv (\alpha + i \beta) [q_{xx} + \frac{1}{2} |q|^2] - \alpha q \int_{-\infty}^{x} q_{xx} dy.$$  (1.3)

Then we will consider the stochastic case and show that the following stochastic nonlinear heat equation

$$d_t q = \left( (\alpha + i \beta) q_{xx} + i \beta \frac{|q|^2}{2} + \frac{\alpha q}{2} \int_{a}^{x} (q_xq_x - q_xq_x) dy \right) dt$$

$$+ d\partial_t (W^1 + iW^2) - iq \int_{a}^{x} q^2 \circ d_t W^1 - q^1 \circ d_t W^2,$$  (1.4)

allows one to construct (assuming existence and certain smoothness of the solution) weak solution of stochastic LLG equation (Theorem 2.2)

$$d u = (\beta u \times u_{xx} - \alpha u \times (u \times u_{xx})) dt + u \times \circ d\tilde{W}_t,$$  (1.5)

where $(\tilde{W}_t)_{t \geq 0}$ is 3D space-time white noise iff we formally assume that $W^1, W^2$ are independent space-time white noises (remark 2.2). We will see that nonlinear stochastic heat equation (1.1) is the compatibility condition for the auxiliary system of linear equations (2.33)-(2.34) (where $p, C(q)$ are defined by identities (2.36)). The stochastic LLG equation is the consequence of the auxiliary system (2.33), (2.34), (2.35).
We can connect with Remark 2.1.

**Proof.**

Our proof will be divided in two steps. First, we will deduce equation for the pair \((\Theta, \eta)\) (we will omit argument \(u\) from now on). In the second step, we will deduce equation for \(q\).

**Step 1:**

We have by elementary calculations and identities

\[
\langle u_x, u \rangle = \frac{1}{2} \partial_x (|u|^2) = 0, |u_x|^2 = \partial_x (\langle u_z, u \rangle) = - \langle u_{xx}, u \rangle = - \langle u_{xx}, u \rangle,
\]

that

\[
\Theta' = \frac{\langle u_x, u_{xx} \rangle}{\Theta} = \frac{\beta \langle u_x, u_{xx} \rangle + \beta \langle u_x, u_{xx} \rangle + \alpha \langle u_{xx} \rangle^2 u_x + \alpha \Theta \partial_x (|u_x|^2), u_x \rangle}{\Theta} = \beta \langle u_x, u_{xx} \rangle + \alpha \langle u_{xx} \rangle^2 u_x + \alpha \Theta \partial_x(|u_x|^2), u_x \rangle
\]

\[
= \Theta
\]

\[
= \beta \langle u_x, u_{xx} \rangle + \alpha \langle u_{xx} \rangle^2 u_x,
\]

\[
= \alpha \beta \partial_x (\langle u_x, u_{xx} \rangle) + \alpha \partial_x (|u_{xx}|^2) - \Theta (|u_{xx}|^2) + |\langle u_{xx}, u \rangle|^2}
\]

\[
= \Theta
\]

\[
= -\beta \partial_x (\Theta^2) + \alpha \varrho \Theta \left( \frac{\partial^2}{\partial x^2} - |u_{xx}|^2 + |\langle u_{xx}, u \rangle|^2 \right) \Theta
\]

\[
= -\beta \partial_x (\Theta^2) + \alpha \varrho \Theta \left( \frac{\partial^2}{\partial x^2} - |u_{xx}|^2 + |\langle u_{xx}, u \rangle|^2 \right) \Theta
\]

\[
= -\beta \varrho \Theta - 2\beta \varrho \Theta \eta + \alpha \Theta |u_{xx}|^2 - |\langle u_{xx}, u \rangle|^2 \Theta
\]

\[
= -\beta \varrho \Theta - 2\beta \varrho \Theta \eta + \alpha \Theta |u_{xx}|^2 - |\langle u_{xx}, u \rangle|^2 \Theta
\]

**Theorem 2.1.** Let \(u : [0, \infty) \times \mathbb{R} \to \mathbb{S}^2\) be a smooth solution of the Landau-Lifshitz-Gilbert equation

\[
u_t = \beta \nu \times \nu_{xx} - \alpha \nu \times (\nu \times \nu_{xx}), x \in \mathbb{R}, t \geq 0.
\]

Then its Hashimoto transform \(q = \mathcal{H}(u(t, \cdot)), t \in [0, \infty), q : [0, \infty) \times \mathbb{R} \to \mathbb{C}\) is a smooth solution of the following equation

\[
\partial_t q = \alpha \left[ q_{xx} + \frac{q}{2} \int_\mathbb{R} (q \cdot q_x^2 - q \cdot q_x) \, dy \right] + i \beta q_{xx} + \frac{|q|^2}{2} q
\]

\[
\equiv (\alpha + i \beta) \left[ q_{xx} + \frac{1}{2} |q|^2 \right] - \alpha q \int_\mathbb{R} q \cdot q_x \, dy.
\]
Consequently, by elementary identity $|a|^2 |b|^2 = |a \times b|^2 + |a, b|^2$, $a, b \in \mathbb{R}^3$ we have that

$$\Theta' = \alpha \Theta_{xx} + \alpha \frac{|\Theta_x|^2}{\Theta} - \alpha \frac{|u \times u_{xx}|^2}{\Theta} - \beta \eta_x \Theta - 2 \beta \Theta_x \eta.$$  \hspace{1cm} (2.7)

Denote $v := u \times u_x$. Now

$$\frac{|\Theta_x|^2 - |u \times u_{xx}|^2}{\Theta} = \frac{|\partial_x (\Theta^2)|^2}{4 |\Theta|^2} - |v_x|^2$$

$$= \frac{|\partial_x (|v|^2)|^2}{4 |v|^2} - |v_x|^2 = - \frac{|v \times v_x|^2}{|v|^3} = - |\eta^2 \Theta|.$$  \hspace{1cm} (2.8)

Hence by equation (2.7) and identity (2.7) we can deduce that

$$\Theta' = \alpha (\Theta_x^2 - \eta^2) \Theta - \beta \eta_x \Theta - 2 \beta \Theta_x \eta.$$  \hspace{1cm} (2.9)

It remains to deduce equation for $\eta$. We have

$$\eta' = -2 \alpha \frac{(\Theta_x^2 - \eta^2) \Theta - \beta \eta_x \Theta - 2 \beta \Theta_x \eta}{\Theta^3} \eta \Theta^2 + \frac{1}{\Theta^3} \frac{d}{dt} < u \times u_x, u_{xx} >$$

$$= -2 \alpha \frac{\Theta_{xx}}{\Theta} + 2 \alpha \eta^3 - \beta \eta_x - 2 \beta \Theta_x \eta + \frac{1}{\Theta^2} (\| u', u_x \times u_{xx} > + < u \times u'_x, u_{xx} > + < u \times u_x, u'_{xx} > )$$

$$= -2 \alpha \frac{\Theta_{xx}}{\Theta} + 2 \alpha \eta^3 - \beta \eta_x - 2 \beta \Theta_x \eta + \frac{1}{\Theta^2} (A + B + C)$$

Now let us calculate $A, B$ and $C$. We will deal separately with the terms proportional to $\alpha$ and $\beta$. We have $A = A_\alpha + A_\beta$, where

$$A_\alpha = \alpha < u_{xx} > |u_x|^2 u_x \times u_{xx} > = \alpha \eta \Theta^4,$$

$$A_\beta = \beta < u \times u_{xx}, u_x \times u_{xx} >$$

$$= - \beta < u, u_{xx} > < u_x, u_{xx} > = \beta \Theta^2 \frac{\partial_x (\Theta^2)}{2} = \beta \Theta^3 \Theta_x.$$  \hspace{1cm} (2.11)

We have $B = B_\alpha + B_\beta$, where

$$B_\alpha = - \alpha < u_{xxx}, u \times u_{xx} > - \alpha |u_x|^2 < u_x, u \times u_{xx} >$$

$$= \alpha \eta \Theta^4 - \alpha < u_{xxx}, u \times u_{xx} >.$$  \hspace{1cm} (2.12)

To calculate $B_\beta$ we will need the following auxiliary identity.

$$|u_{xx}|^2 = \Theta^4 + |\Theta_x|^2 + \eta^2 \Theta^2.$$  \hspace{1cm} (2.13)

Indeed, expanding $|u_{xx}|^2$ in the orthonormal basis $\{u, \frac{u_x}{|u_x|}, \frac{u_{xx}}{|u_{xx}|}\}$ we get

$$|u_{xx}|^2 = | < u_{xx}, u > |^2 + | < u_{xx}, \frac{u_x}{|u_x|} > |^2 + | < u_{xx}, \frac{u \times u_x}{|u_x|} > |^2$$

$$= \Theta^4 + \frac{1}{\Theta^2} \frac{\partial_x (\Theta^2)}{2} + \eta^2 \Theta^2$$

$$= \Theta^4 + |\Theta_x|^2 + \eta^2 \Theta^2.$$
Thus, we get

\[ B_\beta = \beta \langle u \times (u \times u_{xxx} + u_x \times u_{xxx}), u_{xxx} \rangle \]

(2.14)

\[ = \beta \langle u \times u_{xxx} > - u_{xxx} + u_x < u, u_{xxx} >, u_{xxx} \rangle \]

= \beta < u, u_{xxx} > - \beta < u_{xxx}, u_{xxx} > + \beta < u, u_{xxx} > < u_x, u_{xxx} >

= \beta < u, u_{xxx} > \partial_x (< u, u_{xxx} >) - < u_x, u_{xxx} > = - \beta \Theta^2 \partial_x (-\Theta^2) - \frac{1}{2} \partial_x (|u_{xxx}|^2)

= 2 \beta \Theta^4 \Theta_x - \frac{1}{2} \partial_x (\Theta^4 + |\Theta_x|^2 + \eta^2 \Theta^2)

= - \beta \Theta_x \Theta_{xxx} - \beta \eta \Theta \eta^2 - \beta \Theta_x \eta^2.

Hence by elementary calculations and identity (2.13) we get

\[ C = C_\alpha + C_\beta, \]

where

\[ C_\alpha = \alpha < u \times u_x, u_{xxx} + |u_x|^2 u_{xxx} = \alpha \eta \Theta^4 + \alpha < u \times u_x, u_{xxx} >. \]  (2.15)

\[ C_\beta = \beta < u \times u_x, u \times u_{xxx} + 2u_x \times u_{xxx} > \]

(2.16)

= \beta < u_x, u_{xxx} > - 2 \beta \Theta^2 < u, u_{xxx} >

Now we can notice that

\[ < u, u_{xxx} > \partial_x (< u, u_{xxx} >) - < u_x, u_{xxx} > = \partial_x (-\Theta^2) - \partial_x (\Theta^2) = -3 \Theta \Theta_x. \]

Furthermore,

\[ < u, u_{xxx} > = \partial_x (< u_x, u_{xxx} >) - < u_x, u_{xxx} > \]

\[ = \partial_x (\partial_x (< u_x, u_{xxx} >) - |u_{xxx}|^2) - \frac{1}{2} \partial_x (|u_{xxx}|^2) \]

\[ = - \frac{3}{2} \partial_x (|u_{xxx}|^2) + \frac{1}{2} \partial_x (\Theta^4 + |\Theta_x|^2 + \eta^2 \Theta^2) + \frac{1}{2} \partial_x (\Theta^2). \]

Hence,

\[ C_\beta = \beta (\Theta \Theta_{xxx} - 3 \eta \Theta \Theta_x^2 - 3 \Theta \Theta_x \eta^2). \]

Thus

\[ A_\alpha + B_\alpha + C_\alpha = 3 \alpha \eta \Theta^4 + \alpha (\langle u \times u_x, u_{xxx} > + < u \times u_x, u_{xxx} > ), \]  (2.17)

and

\[ A_\beta + B_\beta + C_\beta = \beta (\Theta \Theta_{xxx} - \Theta_x \Theta_{xxx} + \Theta^3 \Theta_x - 4 \eta \Theta \Theta_x^2 + 4 \Theta \Theta_x \eta^2). \]  (2.18)

Since

\[ \partial_x (\eta \Theta^2) = \partial_x (< u \times u_x, u_{xxx} >) = \langle u \times u_x, u_{xxx} > + < u \times u_x, u_{xxx} > \rangle, \]  (2.19)

we deduce that

\[ A_\alpha + B_\alpha + C_\alpha = 3 \alpha \eta \Theta^4 + \alpha \partial_x (\langle u \times u_x, u_{xxx} > + 2 \alpha < u \times u_x, u_{xxx} >. \]  (2.20)

Thus, it remains to calculate \( u_{xxx} \times u, u_{xxx} \times u \). Expanding this quantity in orthonormal basis \( \{ u, \frac{u_x}{|u_x|}, \frac{u_{xxx}}{|u_{xxx}|} \} \) we get

\[ < u_{xxx} \times u, u_{xxx} > = < u_{xxx} \times u, u_x > < u_{xxx} > + < u_{xxx} \times u, \frac{u_x}{|u_x|} > < u_{xxx} > \]

(2.21)

\[ + < u_{xxx} \times u, \frac{u_x}{|u_x|} > < u_{xxx} > \]

\[ = 0 + \eta < u_{xxx} > < u_{xxx} > - \frac{u_x, u_{xxx}}{|u_x|^2} < u \times u_x, u_{xxx} > \]

\[ = \eta < u, u_{xxx} > - \frac{u_x, u_{xxx}}{|u_x|^2} \partial_x (\eta \Theta^2) \]

\[ = \eta [\partial_x (< u_x, u_{xxx} >) - |u_{xxx}|^2] \frac{1}{\Theta^2} \partial_x (\eta \Theta^2) \frac{\Theta^2}{\Theta} \]

\[ = \eta \partial_x \left( \frac{\Theta^2}{\Theta} \right) = \eta |u_{xxx}|^2 - \frac{\Theta_x}{\Theta} (\Theta^2 \eta + 2 \eta \Theta \Theta_x). \]
Combining \((2.21)\) and \((2.13)\) we deduce that
\[
\langle u_{xx} \times u, u_{xxx} \rangle = \eta \Theta_{xx} + \eta \Theta_{x}^{2} - \eta \Theta^{4} - \eta \Theta_{x}^{2} - \Theta \Theta_{x} \eta_{x} - 2 \eta \eta \Theta_{x}^{2}.
\] (2.22)

Hence, combining \((2.20)\) and \((2.22)\) we conclude that
\[
A_{\alpha} + B_{\alpha} + C_{\alpha} = \alpha \eta \Theta_{x}^{4} + \alpha \eta_{x} \Theta_{x}^{2} + 2 \alpha \eta \eta \Theta_{x} - 2 \alpha \eta \Theta_{x}^{2} + 2 \alpha \eta \Theta_{xx} - 2 \alpha \eta^{3} \Theta^{2}.
\] (2.23)

Finally, by identities \((2.10)\), \((2.18)\) and \((2.23)\) we can conclude that
\[
\eta' = \alpha \eta_{xx} + 2 \alpha \left[ \eta \Theta_{x}^{2} + \beta \left( \Theta_{xx} - \eta^{2} \right) \right].
\] (2.24)

Hence, we conclude that from LLG equation \((2.4)\) follows that
\[
\Theta' = \alpha \left( \partial_{xx}^{2} \Theta - \eta^{2} \right) \Theta - \beta \left( \eta \Theta_{x} + 2 \Theta_{x} \eta \right),
\] (2.25)

\[
\eta' = \alpha \eta_{xx} + 2 \alpha \left[ \eta \Theta_{x}^{2} + \beta \left( \Theta_{xx} - \eta^{2} \right) \right].
\] (2.26)

where \(\Theta = |u_{x}|, \eta = \frac{\langle u \times u_{x}, u_{xxx} \rangle}{|u_{x}|^{2}}\).

Step 2:
By definition of \(q\) we have,
\[
\partial_{t} q = q \left( \frac{\Theta'}{\Theta} + i \int_{-\infty}^{x} \eta' dy \right).
\]

By identities \((2.24)\) we deduce that
\[
\partial_{t} q = \alpha q \left( \frac{\Theta_{xx}}{\Theta} - \eta^{2} + i \left( \eta_{x} + 2 \eta \Theta_{x} \Theta \right) + i \int_{-\infty}^{x} \eta \Theta^{2} dy \right) + \beta q \left( - \left( \eta_{x} + 2 \Theta_{x} \eta \right) + i \left( \Theta_{xx} - \eta^{2} + \Theta^{2} \right) \right)
\] (2.27)

Now we will calculate these four terms:
\[
R = i q \int_{-\infty}^{x} \eta \Theta^{2} dy = \frac{q}{2} \int_{-\infty}^{x} \left( q_{x} \eta - q \eta_{x} \right) dy
\]
\[
= \frac{1}{2} q |q|^{2} - q \int_{-\infty}^{x} q d\eta
\] (2.28)

To calculate terms \(P\) and \(Q\) we will need following auxiliary identities. They can be deduced immediately by elementary calculations.

\[
\Theta = |q|, \quad \eta = i \frac{q \eta_{x} - q_{x} \eta}{2 |q|^{2}}
\] (2.29)

\[
\Theta_{x} = \frac{q_{x} \eta + q \eta_{x}}{2 |q|}
\]

\[
\Theta_{xx} = \frac{(q_{x} \eta + q \eta_{x})^{2}}{4 |q|^{3}} + \frac{q_{xx} \eta + q \eta_{xx} + 2 q_{x} \eta_{x}}{2 |q|}.
\]

\[
\eta_{x} = i \frac{|q|^{2} q \eta_{xx} - q_{x} \eta \eta_{x}}{2 |q|^{2}} - \frac{q_{x}^{2} \eta_{x}^{2} + q_{x}^{2} \eta_{x}^{2}}{|q|^{4}}.
\]

Consequently,
\[
P = q \left( \Theta_{xx} - \eta^{2} \right) = \frac{q_{x} x^{2} + \eta_{xx} q_{x}^{2}}{2 |q|^{2}}.
\] (2.30)
\[ Q = iq \eta x + 2iq \eta \Theta(t) = \frac{q_{xx}}{2} - \frac{q^2}{2|q|^2}. \] (2.31)

Consequently,

\[ S = i(q_{xx} + \frac{|q|^2}{2}q). \] (2.32)

Thus, by identities (2.26), (2.30), (2.31), (2.28) and (2.32) we can conclude that the equation (2.25) holds.

Now we are going to show that a solution of the nonlinear stochastic heat equation can be used to construct the weak solution of stochastic Landau-Lifshitz-Gilbert equation. The main idea of the construction is to inverse (in certain sense explained below) Hashimoto transform. We will need the following auxiliary system:

**Definition 2.2.** Let \( \mathbf{q} = q^1 + iq^2 \in L^\infty([0, \infty), L^2(\Omega \times S^1, \mathbb{C}) \), and \( \mathbf{p} = p^1 + ip^2 \in L^2([0, \infty), L^2(\Omega \times S^1, \mathbb{C}) \). \]

\[
\begin{align*}
\frac{du}{dt} & = \begin{pmatrix} u & e \\ e \times u & e \end{pmatrix} = \begin{pmatrix} 0 & p^1 & p^2 \\ -p^1 & 0 & C(q) \\ -p^2 & C(-q) & 0 \end{pmatrix} \begin{pmatrix} u & e \\ e \times u & e \end{pmatrix} dt + \begin{pmatrix} 0 & dW^1 & dW^2 \\ -dW^1 & 0 & d\Psi \\ -dW^2 & -d\Psi & 0 \end{pmatrix} \begin{pmatrix} u & e \\ e \times u & e \end{pmatrix}, \\
& \quad \text{where } W^1, W^2 \text{ are independent Wiener processes given by} \\
& W^i(t, x) = \sum_{i=1}^\infty \sum_{i=1}^\infty \{\beta_i^1\}^\infty_{i=1} \text{ i.i.d. 1D Brownian motions, } \{\sigma_i^1\}^\infty_{i=1} \text{ orthonormal basis in } L^2(S^1, \mathbb{R}), i \in \mathbb{N} \}
\end{align*}
\] (2.33)

where \( W^1, W^2 \) are independent Wiener processes given by

\[
W^i(t, x) = \sum_{i=1}^\infty \sum_{i=1}^\infty \{\beta_i^1\}^\infty_{i=1} \text{ i.i.d. 1D Brownian motions, } \{\sigma_i^1\}^\infty_{i=1} \text{ orthonormal basis in } L^2(S^1, \mathbb{R}), i \in \mathbb{N} \}
\] (2.35)

\[
p = (\alpha + i\beta)q_x, C(q) = -\frac{1}{2} |q|^2 + \frac{i\alpha}{2} \int_a^x (q_x \bar{q} - \bar{q}_x q) dy, \Psi(t, x) = \int_0^t \int_a^x q^2 \circ dW^1 - q^1 \circ dW^2, \] (2.36)

and stochastic integrals above are understood in Stratonovich sense.

**Theorem 2.2.** Assume that we are given solution \( \mathbf{q} \in L^2(\Omega, C([0, \infty), W^{2,2}(S^1, \mathbb{C})) \) of the following equation

\[
\begin{align*}
\frac{d\mathbf{q}}{dt} & = \left( (\alpha + i\beta)q_{xx} + i\beta |q|^2 + \frac{\alpha q}{2} \int_a^x (q_x \bar{q} - \bar{q}_x q) dy \right) dt \\
& \quad + d\partial_x(W^1 + iW^2) - iq^1 \int_a^x q^2 \circ d\partial_x W^1 - q^1 \circ d\partial_x W^2,
\end{align*}
\] (2.37)

and \( \mathbf{m} \in S^2, \mathbf{e}_0 \in T_{\mathbf{m}} S^2, |\mathbf{e}_0| = 1 \). Then system (2.28), (2.31), (2.32), (2.26) has a solution \( (\mathbf{u}, \mathbf{e}) \in L^\infty([0, \infty), L^2(\Omega, H^1(S^1, TS^2))) \) such that \( \mathbf{u}(0, \mathbf{a}) = \mathbf{m}, \mathbf{e}(0, \mathbf{a}) = \mathbf{e}_0 \). Furthermore, \( \mathbf{u} \) solves Landau-Lifshitz-Gilbert equation

\[
d\mathbf{u} = (\beta \mathbf{u} \times \mathbf{u}_{xx} - \alpha \mathbf{u} \times (\mathbf{u} \times \mathbf{u}_{xx})) dt + \mathbf{u} \times d\overline{\mathbf{W}}_t \] (2.38)

where

\[
\overline{\mathbf{W}}_t = \int_0^t \mathbf{e}(s) dW^2(s) + \mathbf{e} \times dW^1(s) + \mathbf{u}(s) dW^3(s),
\]
gaussian process with zero mean and quadratic covariation given by formula

\[ \mathbb{E} \left[ \langle \mathbf{W}_t, \phi \rangle_{L^2(S^1, \mathbb{R}^3)} \langle \mathbf{W}_t, \psi \rangle_{L^2(S^1, \mathbb{R}^3)} \right] \]

(2.39)

\[ = \sum_{l=1}^{\infty} c_l^2 \int_0^t \mathbb{E} \left[ \int_S \int_S \phi \mathbf{e} \cdot \mathbf{e} \sigma^i dx \int_S \int_S \psi \mathbf{e} \cdot \mathbf{e} \sigma^i dx + \int_S \int_S \phi \mathbf{u} \cdot \mathbf{e} \sigma^i dx \int_S \int_S \psi \mathbf{u} \cdot \mathbf{e} \sigma^i dx \right] ds, \phi, \psi \in L^2(S^1, \mathbb{R}^3). \]

Moreover, \( u(t) \) and \( q(t) \), \( t \geq 0 \) are connected with each other through Hashimoto transform introduced in the previous theorem.

**Remark 2.2.** Note that if \( c_l = 1, l \in \mathbb{N} \) then from formula (2.39) follows that

\[ \mathbb{E} \left[ \langle \mathbf{W}_t, \phi \rangle_{L^2(S^1, \mathbb{R}^3)} \langle \mathbf{W}_t, \psi \rangle_{L^2(S^1, \mathbb{R}^3)} \right] = t < \phi, \psi >_{L^2(S^1, \mathbb{R}^3)}, \phi, \psi \in L^2(S^1, \mathbb{R}^3), \]

i.e. if \( W^i, i = 1, 2 \) are two independent real-valued space-time white noises then \( \mathbf{W} \) is an \( \mathbb{R}^3 \) valued space-time white noise.

**Proof.** The system (2.33), (2.34), (2.35), (2.36) has a solution iff and only if compatibility conditions

\[ \begin{align*}
  dt \partial_x u &= \partial_x dt \mathbf{u}, \quad (4.0) \\
  dt \partial_x e &= \partial_x dt \mathbf{e}, \quad (4.1)
\end{align*} \]

are satisfied. First, we will look at the condition (4.0). We have by elementary calculations and equality (2.31) that

\[ \begin{align*}
  dt \partial_x u &= (dt q^1 - q^2 C(q) - d\Psi) \mathbf{e} - (p^1 q^1 + p^2 q^2 + q^1 dW^1 + q^2 dW^2) \mathbf{u} \quad (4.2) \\
  + (dt q^2 + q^1 C(q) + d\Psi) \mathbf{u} \times \mathbf{e},
\end{align*} \]

\[ \begin{align*}
  \partial_x dt \mathbf{u} &= (\partial_x p^1 + dt \partial_x W^1) \mathbf{e} - (p^1 q^1 + p^2 q^2 + q^1 dW^1 + q^2 dW^2) \mathbf{u} \quad (4.3) \\
  + (\partial_x p^2 + dt \partial_x W^2) \mathbf{u} \times \mathbf{e}.
\end{align*} \]

Equating coefficients in (4.2) and (2.33) we can deduce that

\[ dt \mathbf{q} = [-i q C(q) + \partial_x p] dt - i q d\Psi + d\partial_x (W^1 + i W^2). \]

(4.4)

Now let us look at the second compatibility condition (4.1). We have by elementary calculations and equality (2.34) that

\[ dt \partial_x e = -dt q^1 \mathbf{u} - (q^1 p^1 dt + dW^1) \mathbf{e} - (q^1 p^2 + q^1 dW^2) \mathbf{u} \times \mathbf{e}, \]

(4.5)

and

\[ \begin{align*}
  \partial_x dt \mathbf{e} &= ([-\partial_x p^1 - q^2 C(q)] dt - dt \partial_x W^1 - q^2 d\Psi) \mathbf{u} - (p^1 q^1 + q^1 dW^1) \mathbf{e} \quad (4.6) \\
  + ([-p^1 q^2 + \partial_x C(q)] dt - q^2 dW^1 + d\partial_x \Psi) \mathbf{u} \times \mathbf{e}.
\end{align*} \]

Equating coefficients in (4.5) and (4.6) we can deduce that

\[ \begin{align*}
  d^1 q^2 - p^2 q^1 - \partial_x C(q) \right) dt + q^2 dW^1 - q^1 dW^2 - d\partial_x \Psi &= 0. \quad (4.7)
\end{align*} \]

Now we can notice that compatibility conditions (4.1) and (4.7) together with (2.36) give us equation (2.37). It remains to show that equation (2.38) holds. We have by equation (2.33) that

\[ dt \mathbf{u} = (p^1 \mathbf{e} + p^2 \mathbf{u} \times \mathbf{e}) dt + dW^1 \mathbf{e} + dW^2 \mathbf{u} \times \mathbf{e}. \]

(4.8)
Moreover, we can deduce by elementary calculation and equation (2.34) that
\[ \partial^2_{xx} u = q^1_x e + q^2_x u \times e - |q|^2 u, \]
and, consequently,
\[ \beta u \times \partial^2_{xx} u - \alpha u \times (u \times \partial^2_{xx} u) = (\alpha q^1_x - \beta q^2_x)e + (\alpha q^2_x + \beta q^1_x)u \times e. \]
Hence identity $p = (\alpha + i \beta)q_x$ implies that
\[ \beta u \times \partial^2_{xx} u - \alpha u \times (u \times \partial^2_{xx} u) = p^1 e + p^2 u \times e. \] (2.49)
Combining (2.48) and (2.49) we deduce equation (2.38). It remains to show that $u$ and $q$ are connected through Hashimoto transform. We will deduce it from the equation (2.34). First, by antisymmetry of the matrices in the equations (2.34) and (2.33) we can see that quantities $|u|_{R^3}$, $|e|_{R^3}$, $<u, e>_{R^3}$ are constants both in space and time. Consequently, by conditions on $u_0$ and $e_0$ we have that $|u|_{R^3} = |e|_{R^3} = 1$, $<u, e>_{R^3} = 0$. We have system
\[ \partial_t u = q^1_x e + q^2_x u \times e, \] (2.50)
\[ \partial_t e = -q^1_x u. \] (2.51)
Equation (2.50) immediately implies that $|q| = |\partial_t u|$. Consequently, we have
\[ q^1 = |\partial_t u| \cos \omega, \quad q^2 = |\partial_t u| \sin \omega. \] (2.52)
Moreover, elementary calculations allow us to deduce from equations (2.50)-(2.51) that
\[ e = \frac{q^1 \partial_t u + q^2 u \times \partial_t u}{|\partial_t u|^2}, \] (2.53)
\[ = \frac{\cos \omega \partial_t u + \sin \omega u \times \partial_t u}{|\partial_t u|}. \] (2.54)
Furthermore, equation (2.51) implies that
\[ <\partial_t e, u \times e>_{R^3} = 0. \] (2.55)
Inserting in the equation (2.55) representation (2.54) of $e$ we deduce that
\[ \omega = \frac{<u \times \partial_t u, \partial^2_{xx} u>}{|\partial_t u|^2}. \] (2.56)
Equation (2.50) together with representation (2.52) implies the result. \qed

**Remark 2.3.** The equation (2.37) is a zero-curvature representation (although no dependence on spectral parameter here) of the system (2.33), (2.34), (2.35), (2.36). Consequently, natural question is if there exist soliton solutions for LLG and SLLG equations?

**Remark 2.4.** The existence of the strong solution of the equation (2.37) is a subject of further research.

**Remark 2.5.** The existence and regularity assumptions on the solution of the equation (2.37) in the theorem 2.2 can be circumvented by consideration of proper space discretisation of the system (2.33) - (2.34).

**Remark 2.6.** Here we consider only the case of exchange energy $E(u) := \int |u_s|^2 dx$ in the SLLG equation. The general case can be obtained in the same fashion by properly modifying definitions of $p$ and $C(q)$ in (2.36).

**Remark 2.7.** System (2.33), (2.34), (2.36) in the case of absence of noise ($W^1 = W^2 = \Psi = 0$) and zero viscosity coefficient has been considered in [6].

**Remark 2.8.** It would be of interest to see if in the case of absence of the noise term the equation (2.37) is equivalent to the complex Ginzburg-Landau equation deduced in [5].
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