AN OBSTRUCTION THEORY FOR THE EXISTENCE OF MAURER-CARTAN ELEMENTS IN CURVED $L_\infty$-ALGEBRAS AND AN APPLICATION IN INTRINSIC FORMALITY OF $P_\infty$-ALGEBRAS.

SILVAN SCHWARZ

ETH Zürich

Abstract. Let $g$ be a curved $L_\infty$-algebra endowed with a complete filtration $\mathfrak{F}g$. Suppose there exists an integer $r \in \mathbb{N}_0$ for which the curvature $\mu_0$ satisfies $\mu_0 \in \mathfrak{F}_{2r+1}g$ and the spectral sequence yields $E^{p,q}_{r+1} = 0$ for $p, q$ with $p+q = 2$. We prove that then a Maurer-Cartan element exists. In addition, we show, as a typical application, that for $P$ a possibly inhomogeneous Koszul operad with generating set in arities 1,2 (e.g. $P=\text{Com},\text{As},\text{BV},\text{Lie},\text{Ger}$), a $P_\infty$-algebra $A$ is intrinsically formal if its twisted deformation complex $\text{Def}(H(A) \xrightarrow{\text{id}} H(A))$ is acyclic in total degree 1.

Contents

1. Introduction 2
1.1. Overview of Results 3
2. Preliminaries 4
2.1. Curved $L_\infty$-algebra 4
2.2. Spectral sequences 5
2.3. Derivations and Differentials on operadic algebras 7
3. $GL_\infty$-equation as a power Series 9
4. Main Theorem 10
5. Applications 13
5.1. Koszul Operads 15
5.2. $OpA$-Category 15
5.3. $P_\infty$-algebra structures on co-homologies 16
5.4. $\infty$-(quasi-)isomorphisms and formality 16
5.5. (Intrinsic) Formality 17
5.6. Derivations of $P$-algebras revisited 17
5.7. Deformation Complex 20
5.8. A sufficient condition for intrinsic Formality 24
Appendix A. Proof of Equivalence of ($S$) $L_\infty$-algebra equations 28
References 30

E-mail address: silvan.schwarz@math.ethz.ch.
Date: October 3, 2022.
The author has been partially supported by the ERC starting grant 678156 GRAPHCPX.
Maurer-Cartan elements play an important role when it comes to finding morphisms which are compatible with some given structures, like the study of $\infty$-morphisms between $P_\infty$-algebras, the study of intrinsic formality of $P_\infty$-algebras or the existence of curved twisting morphisms between co-nilpotent curved co-properads and (not necessarily augmented) dg properads, to name a few.

We show in Theorem 1.1 that for a curved $L_\infty$-algebra $\g$ endowed with a descending bounded above and complete filtration $\g = \mathfrak{F}_1\g \supset \mathfrak{F}_2\g \supset \cdots$ there is the following implication: If there exists an $r \in \mathbb{N}_0$ for which the curvature satisfies $\mu_0 \in \mathfrak{F}_{2r+1}\g = 0$ and the $r + 1$st page of the spectral sequence of $\g$ vanishes $E_{r+1}^{p,q} = 0$ for all $p,q$ with $p + q = 2$, a Maurer-Cartan element always exists. Moreover, this Maurer-Cartan element lies in $\mathfrak{F}_{r+1}\g$.

In the second part of this paper we focus on an application of this result in the study of intrinsic formality and formulate a sufficient condition for e.g. $BV_{\infty}$-algebras or more general $P_\infty$-algebras for $P$ being a possibly inhomogeneous Koszul operad $^3$ generated in arities 1, 2, to be intrinsically formal.

To translate the search of intrinsic formality into the language of Maurer-Cartan elements, we recall the notion of deformation complex, an $L_\infty$-algebra $^4$ which has the property that its elements correspond to $U(P)$-co-algebra morphisms (where $U : \text{dgVect} \to \text{grVect}$ denotes the forgetful functor) and moreover Maurer-Cartan elements coincide with $\infty$-morphisms, i.e. $U(P)$-co-algebra-morphisms of the cofree $U(P)$-algebras $F_{U(P)}^c(A)$ and $F_{U(P)}^c(B)$ that in addition are also compatible with the differentials emerging from the relative bar-constructions.

However, since the study of intrinsic formality focuses on $\infty$-quasi-isomorphisms rather than $\infty$-morphisms, this falls short. Therefore, to establish the existence of an $\infty$-quasi-isomorphism between two $P_\infty$-algebras $A$ and $B$, we start with a quasi-isomorphism of the underlying dg vector spaces $f : (A,d_A) \to (B,d_B)$. This map induces a $U(P)$-co-algebra-morphism $F : F_{U(P)}^c(U(A)) \to F_{U(P)}^c(U(B))$ that by construction satisfies the ‘quasi-isomorphism-condition’ but fails to constitute an $\infty$-morphism.

Our approach is to look at the deformation complex twisted by $F$. Twisting an $L_\infty$-algebra means to look at the same underlying graded vector space but with brackets slightly altered. One of the main feature of twisted $L_\infty$-algebras is that $x$ is a Maurer-Cartan element of the $L_\infty$-algebra twisted by $F$ if and only if $x + F$ is a Maurer-Cartan element of the original non-twisted $L_\infty$-algebra. Since twisting an $L_\infty$-algebra by a non-Maurer-Cartan element leads to a curved $L_\infty$-algebra, that is an $L_\infty$-algebra with a non-vanishing 0-bracket, a twisted deformation complex does in general provide us merely with a curved $L_\infty$-algebra. We endow the twisted deformation complex with a descending bounded above and complete filtration compatible with the curved $L_\infty$-algebra structure such that elements of filtration degree \( \geq 2 \) do not alter the part relevant for the ‘quasi-isomorphism’ condition. If we can prove the existence of a Maurer-Cartan element on the twisted deformation complex that carries at least filtration degree 2, this yields an $\infty$-morphism that also is an $\infty$-quasi-isomorphism as it behaves as $F$ with regards to this question.

In this manner, the search of $\infty$-quasi-isomorphism can be conducted by finding

---

1. Introduction

For the operad $BV_{\infty}$ as defined in $^3$, Section 1.4.

2. That is to extend the Definition of being Koszul to operads that do not only involve quadratic but also linear relations, see $^6$. Appendices A, B. Sometimes, this is also referred to as linear quadratic Koszul operad.

3. It is a $S L_\infty$-algebra to be precise, that means there is an additional minor modification on the degrees.
Maurer-Cartan elements on curved $L_\infty$-algebras and as such gives a typical application of Theorem 1.1. Moreover, under the assumption of $P$ being a possibly inhomogeneous Koszul operad with the generating set in arities 1,2 the study of intrinsic formality of $F_\infty$-algebras provides a framework in which the choice of $f$ to be the identity-morphism (and hence in particular a quasi-isomorphism of dg vector spaces) satisfies all the requirements of Theorem 1.1, as we will explain in Theorem 5.2).

Theorem 1.1 extends previous work on obstruction theoretical approaches to the existence of Maurer-Cartan elements on differential graded Lie-algebras (see [6], Theorem 52) to curved $L_\infty$-algebras.

1.1. Overview of Results. In this paper we deal with curved $L_\infty$-algebras (introduced in Section 2). In short, a curved $L_\infty$-algebra differs from a ‘usual’ non-curved $L_\infty$-algebra by also allowing for a non-trivial zero-bracket $\mu_0$. The $L_\infty$-algebra equations it has to satisfy are the same as in the non-curved case but with the sums extended to start at zero (cf. Equation (2.1)).

For the scope of this paper we do focus exclusively on curved $L_\infty$-algebras which are endowed with a descending, bounded above and complete filtration that is compatible with the curved $L_\infty$-algebra structure. This means for a curved $L_\infty$-algebra $g$ being of such type it must satisfy $g = \mathfrak{F} \supset \mathfrak{F}_2 \supset \mathfrak{F}_3 \supset \ldots$, the degree of filtration must add under $L_\infty$-algebra brackets, that is to say $x_i \in \mathfrak{F}_k$, $g$ for $i = 1, \ldots, n$ implies $\mu_n(x_1, \ldots, x_n) \in \mathfrak{F}_{k_1+\ldots+k_n}$, and $g = \lim_i g/\mathfrak{F}_i g$ has to hold.

This additional requirement in turn allows us to speak about the Maurer-Cartan equation (notice that it starts at $n = 0$), that is

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \mu((x[1]))^{\otimes n} = 0 \]

and call elements satisfying this equation Maurer-Cartan elements.

In contrast to the setting of non-curved $L_\infty$-algebras, zero is not a Maurer-Cartan element any more and the natural question comes up if for a given curved $L_\infty$-algebra a Maurer-Cartan element exists.

Although $(g, \mu_1)$ is not a filtered complex (the ‘differential’ fails to square to zero), we can exploit $\mu_2^g(x) = -\mu_2^{(\mu_0, x)}$, i.e. we have that the differential squares to zero up to the filtration degree of $\mu_0$. More precisely, we apply the construction of a spectral sequence of a filtered complex on $(g, \mu_1)$ to construct $E_0, E_1, \ldots, E_{r+1}$, which we still call pages of the spec.seq. by abuse of notation, in the usual manner provided the curvature $\mu_0$ lies in $\mathfrak{F}_{2r+1} g$.

We show that under the sole premise of having a vanishing $r + 1$st page of spec.seq. for total degree $2$, i.e. $E^{p,q}_{r+1} = 0$ for $p + q = 2$, and the curvature satisfying $\mu_0 \in \mathfrak{F}_{2r+1} g$, a curved $L_\infty$-algebra gives rise to a Maurer-Cartan element.

**Theorem 1.1.** Let $g$ be a curved $L_\infty$-algebra equipped with descending, bounded above and complete filtration $g = \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \mathfrak{F}_3 \supset \ldots$ compatible with the curved $L_\infty$-algebra structures.

If there exists an integer $r \in \mathbb{N}_0$ for which the curvature $\mu_0$ of $g$ satisfies $\mu_0 \in \mathfrak{F}_{2r+1} g$ and the spectral sequence of $g$ vanishes at the $r + 1$st page for total degree $2$, i.e. $E^{p,q}_{r+1} = 0$ for all $p, q$ with $p + q = 2$, then there exists a Maurer-Cartan element $\alpha \in MC(g)$ that satisfies $\alpha \in \mathfrak{F}_{r+1} g$.

Since twisting of a curved $L_\infty$-algebra by a Maurer-Cartan element leads to a non-curved $L_\infty$-algebra, Theorem 1.1 equivalently shows the existence of a non-Abelian non-curved $L_\infty$-algebra structure on $g$ which is compatible with the filtration.
By applying the aforementioned results to the study of intrinsic formality of $P_\infty$-algebras, we show the following sufficient condition for a $P_\infty$-algebra $A$ to be intrinsically formal

**Theorem 1.2.** Let $A$ be a $P_\infty$-algebra for $P$ a possibly inhomogeneous Koszul operad generated in arities 1,2. If the twisted deformation complex $^3\text{Def}(H(A) \xrightarrow{\text{Id}} H(A))$ is acyclic in total degree 1 (i.e. $\delta_p H^q(\text{Def}(H(A) \xrightarrow{\text{Id}} H(A))) = 0$ for all $p,q$ with $p+q = 1$), then $A$ is intrinsically formal as a $P_\infty$-algebra.

As the Batalin-Vilkovisky-operad $BV$ is of such type, we get the following corollary.

**Corollary 1.3.** Let $A$ be a $BV_\infty$-algebra. If $A$ is subject to $\delta_p H^q(\text{Def}(H(A) \xrightarrow{\text{Id}} H(A))) = 0$ for all $p,q$ with $p+q = 1$, then $A$ is intrinsically formal as a $BV_\infty$-algebra.

In Section 2 we introduce some basic notation and recall the definition of spectral sequences. Section 3 is dedicated to the description of (curved) $L_\infty$-algebra as cohomological vector fields. In Section 4 we prove Theorem 1.1. Finally, Section 5 is used for introducing intrinsic formality and the deformation complex as well as proving Theorem 1.2. In Appendix A we provide proofs to the equivalence of the various descriptions of (curved) $L_\infty$-algebras.

**2. Preliminaries**

**2.1. Curved $L_\infty$-algebra.**

**Definition 2.1** (Curved $L_\infty$-algebra). A curved $L_\infty$-algebra structure on a graded vector space $g$ is a collection of linear maps $\mu_n : S^n(g[-1]) \rightarrow g[-1]$ of degree 1 that satisfy the so-called curved $L_\infty$-algebra relations

$$\sum_{i,j \in \mathbb{N}_0} \varepsilon(\sigma, x_1, \ldots, x_n) \mu_{i+1}(\mu_i(x_{\sigma(1)} \circ \ldots \circ x_{\sigma(i)}) \circ x_{\sigma(i+1)} \circ \ldots \circ x_{\sigma(n)}) = 0,$$

with $\varepsilon$ being the shifted Koszul sign (that is the usual Koszul sign w.r.t. elements viewed in the shifted graded vector space) and $\text{UnSh}(i,j)$ denoting the $(i,j)$-unshuffles.

It is immediate from the definition that a zero curvature (i.e. $\mu_0 = 0$) curved $L_\infty$-algebra coincides with a usual $L_\infty$-algebra (cf. [8], Equation (3.11)). To simplify the notation, we will use commata to separate the arguments of the $n$-brackets.

For the scope of this paper we mean by $L_\infty$-algebra only non-curved (a.k.a. flat) $L_\infty$-algebras and always write curved $L_\infty$-algebra when we also allow for non-zero curvature. The same of course also holds for the shifted counterparts (introduced below). We generally use the notation $g$ for both the $L_\infty$-algebra structure as well as the underlying vector space.

To avoid ubiquitous suspensions we work with shifted (curved) $L_\infty$-algebras instead.

**Definition 2.2** (Shifted (curved) $L_\infty$-algebra). A (curved) shifted $L_\infty$-algebra structure on a graded vector space $g$ is defined as a (curved) $SL_\infty$-algebra structure on the suspension $g[-1]$ and we will simply write (curved) $SL_\infty$-algebra for it.
Switching to shifted curved $L_{\infty}$-algebra does not make any significant difference but helps in keeping the notation as simple as possible. A more profound introduction may be found in [2].

In the theorems, we work with curved $\mathfrak{S}L_{\infty}$-algebras endowed with descending, bounded above and complete filtrations compatible with the $\mathfrak{S}L_{\infty}$-algebra structures. This means for a $\mathfrak{S}L_{\infty}$-algebra $\mathfrak{g}$ being of such type it must satisfy

\begin{equation}
\mathfrak{g} = \mathfrak{F}_1 \mathfrak{g} \supset \mathfrak{F}_2 \mathfrak{g} \supset \mathfrak{F}_3 \mathfrak{g} \supset \ldots,
\end{equation}

\begin{equation}
\mathfrak{g} = \lim \mathfrak{g}/\mathfrak{F}_p \mathfrak{g}
\end{equation}

and the degree of filtration must add under $\mathfrak{S}L_{\infty}$-algebra brackets.

**Definition 2.3** (Maurer-Cartan elements). In the category of curved $\mathfrak{S}L_{\infty}$-algebras, the Maurer-Cartan elements are defined by

\begin{equation}
\text{MC}(\mathfrak{g}) := \{ \alpha \in \mathfrak{g}^0 : M(\alpha) = 0 \},
\end{equation}

where

\begin{equation}
M(x) := \sum_{k=0}^{\infty} \frac{1}{k!} \mu_k(x, \ldots, x).
\end{equation}

**Definition 2.4** (Twisted curved $\mathfrak{S}L_{\infty}$-algebra). For a curved $\mathfrak{S}L_{\infty}$-algebra $\mathfrak{g} = (\mathfrak{g}, \mu_{\geq 0})$ and a degree 0 element $\beta \in \mathfrak{g}^0$, the twisted curved $\mathfrak{S}L_{\infty}$-algebra $\mathfrak{g}^\beta = (\mathfrak{g}, \mu^\beta_{\geq 0})$ is defined as being the same underlying vector space $\mathfrak{g}$ but with the curved $\mathfrak{S}L_{\infty}$-algebra brackets altered to

\begin{equation}
\mu^\beta_n(v_1, v_2, \ldots, v_n) := \sum_{k=0}^{\infty} \frac{1}{k!} \mu_{k+n}(\beta, \ldots, \beta, v_1, v_2, \ldots, v_n).
\end{equation}

In contrast to the non-curved setting, where we were only allowed to twist by Maurer-Cartan elements, in the curved case we may twist by any degree 0 element and obtain a curved $\mathfrak{S}L_{\infty}$-algebra out of it (see Lemma [4] for further details). Moreover, twisting a curved $\mathfrak{S}L_{\infty}$-algebra by a Maurer-Cartan element results in a non-curved $\mathfrak{S}L_{\infty}$-algebra. From this point of view it makes sense why in the subcategory of $\mathfrak{S}L_{\infty}$-algebras we only allow for twisting by Maurer-Cartan elements.

2.2. **Spectral sequences.**

**Definition 2.5** (Spectral Sequence up to page $k$). Let $R$ be a ring. By abuse of notation we define a spectral sequence up to page $k$ as a collection of differential bi-graded $R$-modules $\{E_{p,q}^r, d_r\}_{k \geq r \geq 0}$, for which $E_{p,q}^{r+1} \cong H^{p,q}(E_{p,*,}^r, d_r)$ for all $r \in \{0, \ldots, k-1\}$.

In the case of $k = \infty$ this definition coincides with the classical definition of a spectral sequence (cf. [14], Section 5).

Throughout this paper we use co-homological conventions, hence the differential is assumed to raise the degree by one. By abuse of notation we will still refer to a spectral sequence even if only the spectral pages up to some $r \in \mathbb{N}_0$ are defined. That being said, having an ‘up to page $r$ spectral sequence’ does a priori no longer allow to use spectral sequence arguments and one should consider them as filtered differential graded vector spaces with the additional attribute of the next page being isomorphic to the co-homology of the previous one.

There is a well-known construction for assigning a spectral sequence to a filtered complex $(V, d)$ (cf. [13], Section 5.4), by using the fact that the differential is
compatible with the filtration and hence satisfies \(d(\mathfrak{g}_p V) \subset \mathfrak{g}_p V\). In this construction one sets the zeroth page (also called associated graded) to
\[
E^p_0 := \mathfrak{g}_p V^{p+q}/\mathfrak{g}_{p+1} V^{p+q}
\]
with differential \(d^p_0 : E^p_0 \to E^{p+1}_0\) being induced by the original differential \(d\) and more general defines \(r\)th page as
\[
E^p_r := \{z \in \mathfrak{g}_p V^{p+q} : d(z) \in \mathfrak{g}_{p+r} V^{p+q+1}\} / \{d(y) \cap \mathfrak{g}_p V^{q+p} : y \in \mathfrak{g}_{p-r+1} V^{p+q-1}\}.
\]
There, the differential \(d^p_r : E^p_r \to E^{p+r,q-r+1}\) (note the shifts in the codomain) is given by the restriction of the original differential composed with the quotient map.

However, when we try to apply this construction in the setting of a curved \(\mathfrak{g}L_\infty\)-algebra \(\mathfrak{g}\) to \((\mathfrak{g}, \mu_1)\) we face the problem that the one bracket \(\mu_1\) does not square to zero but instead yields
\[
\mu_1^2(x) \equiv -\mu_2(\mu_0, x)
\]
and hence does not form a chain complex (or to put it in other words: \(-\mu_2(\mu_0, \cdot)\) measures the inability of \(\mu_1\) to square to zero).

Nevertheless, the construction intended for filtered complexes with \(\mu_1\) in the role of \(d\) remains applicable up to a certain (depending on the degree of filtration of the curvature) page, as we now demonstrate.

Let us assume that \(\mathfrak{g}\) is a curved \(\mathfrak{g}L_\infty\)-algebra, equipped with a descending, bounded above and complete filtration compatible with the curved \(\mathfrak{g}L_\infty\)-algebra structures. Moreover, let the curvature satisfy \(\mu_0 \in \mathfrak{g}_{2r+1}\g^1\) for some \(r\).

We start by noticing that as graded vector the \(s\)th page of the spectral sequence \(E_s\), for \(s \leq r + 1\) is a valid expression. The only non-trivial thing to show is that \(\{\mu_1(y) \cap \mathfrak{g}_p V^{p+q} : y \in \mathfrak{g}_{p-s+1}\g^{p+q-1}\} \subset \{z \in \mathfrak{g}_p V^{p+q} : \mu_1(z) \in \mathfrak{g}_{p+s}\g^{p+q+1}\}\).

So let \(x \in \{\mu_1(y) \cap \mathfrak{g}_p V^{p+q} : y \in \mathfrak{g}_{p-s+1}\g^{p+q-1}\}\). By virtue of \(\mu_0 \in \mathfrak{g}_{2r+1}\g^1\) and because of \(s \leq r + 1\) we have
\[
\mu_1(x) = \mu_1^2(y) \equiv -\mu_2\left(\mu_0 \begin{array}{c} x \\ \in \mathfrak{g}_{2r+1}\g^1 \end{array}, \begin{array}{c} \mu_0 \\ \in \mathfrak{g}_{2r+1}\g^1 \end{array}, \begin{array}{c} \mu_0 \begin{array}{c} \mathfrak{g}_{2r+1}\g^1 \end{array}, \mathfrak{g}_{p-s+1}\g^{p+q-1} \end{array}\right) \in \mathfrak{g}_{p-s+1+2r+1}\g^{p+q+1} \subset \mathfrak{g}_{p-s+1}\g^{p+q+1},
\]
which yields the inclusion.

Let us point out, that it is absolutely crucial for the curvature \(\mu_0\) to be subject to the condition \(\mu_0 \in \mathfrak{g}_{2r+1}\g^1\). It can directly be checked that for \(s = r + 1\) Equation (2.10) may only hold for \(\mu_0 \in \mathfrak{g}_{2r+1}\g^1\). As we will later see, this requirement is also needed in the proof of Theorem 11 itself.

Next, we check that the construction of \(E^{p,q}_r\) as described in Equation (2.8) and its respective differential as the map induced by the 1-bracket \(\mu_1\) forms a spectral sequence on the pages \(E_0, \ldots, E_{r+1}\), indeed.\(^5\)

For that we need to show that the ‘differential’ \(d_s\) (induced by the 1-bracket \(\mu_1\)) is well-defined and satisfies \(d^2_s = 0\) for \(s < r + 1\).

It being well-defined follows directly from the fact that
\[
\mu_1(\{x \in \mathfrak{g}_{p+1}\g^{p+q} : \mu_1(x) \in \mathfrak{g}_{p+r}\g^{p+q+1}\}) \subset \{\mu_1(y) \cap \mathfrak{g}_{p+r}\g^{p+q+1} : y \in \mathfrak{g}_{p+1}\g^{p+q}\}\)
\(^6\)

\(^5\)For now the object formally defined in Equation (2.3).
\(^6\)For \(E_{r+1}\) we may set the differential to be the zero map since the presented construction in general is not applicable to define \(d_{r+1}\).
and that for a \( z \in \{ \mu_1(y) \cap \mathfrak{F}_p g^{q+p} : y \in \mathfrak{F}_{p-s+1} g^{p+q-1} \} \) its image under \( \mu_1 \) satisfies the following two equations

\[
\mu_1(z) = \mu_1^2(y) = -\mu_2( \mu_0(\mu_0, y) ) \in \mathfrak{F}_{2r+1+p-s+1} g^{p+q+2} \subset \mathfrak{F}_{p-r+1} g^{p+q+1}
\]

and

\[
\mu_1^2(z) = -\mu_2( \mu_0(\mu_0, y) ) \in \mathfrak{F}_{p+2r+1+p-s+1} g^{p+q+2} \subset \mathfrak{F}_{p-r+1} g^{p+q+1}.
\]

This in turn proves

\[
\mu_1(\{ \mu_1(y) \cap \mathfrak{F}_p g^{q+p} : y \in \mathfrak{F}_{p-r+1} g^{p+q-1} \}) \subset \{ x \in \mathfrak{F}_{p+s+1} g^{p+q+1} : \mu_1(x) \in \mathfrak{F}_{p+s+s} g^{p+q+2} \}.
\]

It remains to show \( d_3^2 = 0 \), that is to say (written out to make the ‘filtration degree shift’ explicit)

\[
d_{s}^{p+s,q-s+1} d_{s}^{p,q} : E_{s}^{p,q} \to E_{s+1}^{p+2s,q-2s+2}
\]

vanishes. For \( [x] \in E_{s}^{p,q} \) we have \( x \in \mathfrak{F}_p g^{p+q} \) and \( \mu_1(x) \in \mathfrak{F}_{p+s+1} g^{p+q+1} \). Applying the differential two, respectively three times yields that

\[
\mu_1^2(z) = \mu_1^2(\mu_0(\mu_0, y)) = -\mu_2(\mu_0(\mu_0(\mu_0)), y) \in \mathfrak{F}_{2r+1+p-s+1} g^{p+q+2} \subset \mathfrak{F}_{p-r+1} g^{p+q+1}
\]

from which it clearly follows that \( \mu_1^2(z) \) vanishes under the projection to the quotient space \( E_{s+1}^{p+2s,q-2s+2} \).

The proof of \( E_{n+1}^{p,q} \simeq H^{p,q}(E_{n}^{\bullet, 
\bullet}, d_{r}) \) for \( n \leq r \) follows the usual path of the construction of a spectral sequence of filtered complexes (cf. [14], Sect.5.4).

2.3. Derivations and Differentials on operadic algebras. When it comes to (co-)derivations and (co-)differentials of (co-)operadic (co-)algebras we mainly follow the definitions of [2], Sect.2.2, but with co-homological conventions rather than homological. Moreover, we use the notation of [3], Vol.I, Sect.0.13, that is in case of \( \mathcal{C} \) being an enriched category over \( \mathcal{D} \) we distinguish between morphism sets \( \text{Mor}_{\mathcal{C}} \) and \( \text{Hom}_{\mathcal{D}} \) which denotes hom-objects with values in \( \mathcal{D} \).

Notice, that for closed symmetric monoidal categories \( M \) (as e.g. \( M = \text{grVect} \)) there is the internal hom-adjunction

\[
\text{Mor}_{\mathcal{M}}(A \otimes B, C) \simeq \text{Mor}(A, \text{Hom}_{\mathcal{M}}(B, C))
\]

for all \( A, B, C \in \text{Obj}(\mathcal{M}) \).

Therefore, for \( A, B \) being two graded vector spaces, \( f \in \text{Mor}_{\text{grVect}}(A, B) \) is a linear map that satisfies \( f(A_n) \subset B_n \) for all \( n \in \mathbb{Z} \). On the other hand (cf. [3], Vol.I, Sect.4.4), \( g \in \text{Hom}_{\text{grVect}}(A, B) \) is only required to be a linear map and hence in general is allowed to change the degree. By definition, \( \text{Hom}_{\text{grVect}}(A, B) \) carries a graded vector space structure given by \( \text{Hom}_{\text{grVect}}(A, B)^{(k)} = \{ f : A \to B \text{ linear : } f(A_n) \subset B_{n+k} \forall n \in \mathbb{Z} \} \).

Let \( U : \text{dgVect} \to \text{grVect} \) be the forgetful functor and let \( P \) be an operad in \( \text{dgVect} \) with differential \( d_p \).

In the following, we will introduce the three distinctive definitions of a differential on a \( U(P) \)-algebra, \( P \)-derivation on a \( P \)-algebra and differential-capable \( P \)-derivation.
**Definition 2.6 (Differential on $U(P)$-algebra).** Let $A$ be a $U(P)$-algebra (hence, $A$ as well the structure maps $\gamma : \mathcal{F}_U(P)(A)\to A$ lie in grVect). We say a graded vector space homomorphism $\xi : A\to A$ of degree 1 is a differential of $A$ if it squares to zero and makes the following diagram commutative:

\[
\begin{array}{c}
P(A) \\
\downarrow \gamma \\
A
\end{array} \xrightarrow{d_P \circ Id_A + Id_P \circ' \xi} \begin{array}{c}
P(A) \\
\downarrow \gamma \\
A
\end{array},
\]

where $f \circ g$ acts on $(\mu; a_1, \ldots, a_k)$ by $f \otimes S_n (g \otimes \ldots \otimes g)$ and $f \circ' g$ by $\sum_{j=1}^k f \otimes S_n (Id_A \otimes \ldots \otimes Id_A \otimes g \otimes Id_A \ldots \otimes Id_A)$. In particular, this requirement is equivalent to $(A, \xi)$ forming a $P$-algebra.

Let us point out, that for a differential graded vector space $(C, d_C)$, the map $d_{\mathcal{F}_U(P)(C)} := d_P \circ Id_C + Id_P \otimes d_C$ forms a differential on $\mathcal{F}_U(P)(C)$. The so constructed $P$-algebra is the free $P$-algebra $\mathcal{F}_P(C)$.

**Definition 2.7 (Derivation of a $P$-algebra).** Let $A$ be a $P$-algebra or a $U(P)$-algebra. We call a linear map $\tau : A\to A$ of (degree $k$) a derivation of $A$ (of degree $k$) or equivalently $P$-derivation (of degree $k$) when it makes the following diagram of graded vector space homomorphisms commutative:

\[
\begin{array}{c}
P(A) \\
\downarrow \gamma \\
A
\end{array} \xrightarrow{Id_P \circ' \tau} \begin{array}{c}
P(A) \\
\downarrow \gamma \\
A
\end{array}.
\]

Let us emphasise that in contrast to a differential on the $U(P)$-algebra $A$, in the definition of derivation on $A$ there is neither a requirement to square to zero nor to have degree +1. Moreover, the notion of derivation makes sense both in the category of graded vector spaces as well as differential graded vector spaces (also, in the latter case there is no requirement for the derivation to commute with the vector space differential $d_A$). It is immediate that if $P$ is an operad with zero-differential, the two definitions of differential on a $U(P)$-algebra and derivation on a $P$-algebra do coincide.

**Definition 2.8 (Differential-capable $P$-Derivation).** Let $A$ be a $P$-algebra with differential $d_A$. Then we say a $P$-derivation $\beta$ on $A$ is differential-capable relative to $d_A$, if it is of degree +1 and satisfies $(d_A + \beta)^2 = 0$.

Because of the requirement $(d_A + \beta)^2 = 0$, it is immediate that the sum of the differential on $A$ and a differential-capable $P$-derivation $\beta$ forms a differential on $A$ again.

Later in this paper, we will often work with quasi-free\footnote{Some authors (see \cite{7}) also call this almost-free.} algebras. Recall, that a $P$-algebra $B$ is quasi-free, if $B = \mathcal{F}_U(P)(V)$ as underlying graded vector space for some dg vector space $(V, d_V)$ but we allow $B$ to have a differential other than
More precisely, we allow $B$ to have differential $d_B = d_{F(P)} + \nu$ for $\nu$ being a differential-capable $P$-derivation on $F(U(P))(V)$. However, for several reasons (just think about e.g. the definition of an $\infty$-quasi-isomorphism) we do not want to have the differential (seen as a dg vector space differential) on the $V$-part of $B$ being changed. To this end, we impose on $P$-derivations $\nu$ of $A$ (and consequently on differential-capable $P$-derivations) the additional condition that the composition

$$V \xrightarrow{\cong} I(V) \xrightarrow{\eta \circ \Id} P(V) \xrightarrow{\nu} P(V) \xrightarrow{\eps} I(V) \xrightarrow{\cong} V$$

vanishes, where $\eta$ and $\eps$ denote the unit and the augmentation of the operad $P$.

While some authors use $\text{Der}^\ast$ and $\text{Diff}^\ast$ to distinguish $P$-derivations and differential-capable $P$-derivations subject to Equation (2.13) from those without, we dispense the use of an additional $\ast$, since we always use the ones where the additional requirement is invoked. Of course, all of the aforementioned definitions mutatis mutandis also hold for their dual counterparts.

We further elaborate on this topic in Section 5.6, when we introduce the notion of derivation w.r.t. a $U(P)$-algebra morphism. Moreover, by making use of extensions of scalars, we point out how freeness can be used in describing derivations of free $U(P)$-algebras.

3. \mathfrak{S}L_\infty-equation as a power series

In this section we introduce an equivalent description of (curved) $\mathfrak{S}L_\infty$-algebras in terms of power series since we will later use this language in the proof of Theorem 1.1. One advantage of this alternative definition is that one can circumvent the ubiquitous signs in the curved $(\mathfrak{S})L_\infty$-algebra Equation (2.1).

Let $g = (g, \{\mu_n\}_{n \geq 0})$ be a curved $\mathfrak{S}L_\infty$-algebra. Following ([5], Section 4 and [4], Section 1) we use the construction of the power series $M(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \mu_k(x, \ldots, x)$ as starting point.

So far nothing new, we already knew that e.g. solutions of the equation $M(x) = 0$ are some special elements, called the Maurer-Cartan elements.

Now comes the trick: We can also do the formal construction of $M(x)$ for some general n-ary degree 1 operations $\{\mu_n\}_{n \geq 0}$ which do not satisfy the $\mathfrak{S}L_\infty$-algebra equations. As it turns out, the question of these operations $\mu_n$ forming $\mathfrak{S}L_\infty$ brackets directly translates to whether the power-series $M(x)$ itself is a solution to a way simpler looking equation. Let us make the statement more concrete.

Let $g = (g, \{\mu_n\}_{n \geq 0})$ be a graded vector space equipped with n-linear graded vector space homomorphism $\mu_n$ of degree 1 for all $n \geq 0$, which we do refer to as n-brackets. Let $R$ be a nilpotent graded ring. By extension of scalars we may extend the n-brackets $\{\mu_n\}_{n \geq 0}$ to the completed tensor product $g \hat{\otimes} R$ and apply the construction of Equation (2.5) to it, i.e. we have

$$M^R(x \hat{\otimes} r) = \sum_{n=0}^{\infty} \frac{1}{n!} \pm \mu_n(x, \ldots, x) \otimes r \cdot \ldots \cdot r,$$

where the sign is due to the Koszul sign convention.

It turns out (see Appendix A for more details) that for the n-brackets $\{\mu_n\}_{n \geq 0}$ to satisfy the curved $\mathfrak{S}L_\infty$-algebra equation it is tantamount to

$$DM^R(x)[M^R(x)] = 0 \quad \forall x \in (g \hat{\otimes} R)^0,$$

where $D$ is the differential.
for all graded nilpotent Rings $R$ and for all degree 0 elements in $g \hat{\otimes} R$ with $D$ denoting the differential of the multilinear map and [...] being the point of evaluation.

This alternative definition is exactly the description of $(\mathfrak{g})L_{\infty}$-algebras as cohomological vector fields which is well established in physics (cf. [8], Sections 2.1 and 2.2).

If the $\mathfrak{g}L_{\infty}$-algebra can be fully described by $g = (g, MR)$, one may ask how to obtain an explicit expression for $\mu_n(x_1, \ldots, x_n)$ for some homogeneous elements $x_1, \ldots, x_n \in g$ out of this (in Equation (3.1) we only allowed for all the elements in the arguments being the same). The solution to this problems is obtained by graded polarisation and works as follows: Consider $\epsilon_i$ to be a formal variables of degree $-\text{deg}(x_i)$ and let $R$ be the graded Ring $R := \mathbb{K}[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1^2, \ldots, \epsilon_n^2)$. Then, $\mu_n(x_1, \ldots, x_n)$ is given by the $\epsilon_1, \ldots, \epsilon_n$ coefficient of the $\epsilon_1 \cdots \epsilon_n$ monomial of $MR(x_1 \otimes \epsilon_1, \ldots, x_n \otimes \epsilon_n)$.

There is yet another equivalent form (see Appendix A for more details) of the $\mathfrak{g}L_{\infty}$-algebra equation

$$M^{R \otimes R}(x \hat{\otimes} 1 + MR(x) \hat{\otimes} \epsilon) = MR(x) \hat{\otimes} 1 \quad \forall x \in (g \hat{\otimes} R)^0,$$

which we obtain by choosing $\epsilon$ to be a formal variable of degree $-1$ and $R'$ to be the graded algebra $R' = \mathbb{K}[\epsilon]/\epsilon^2$. This last version turns out to be useful when working with twisted curved $\mathfrak{g}L_{\infty}$-algebras, as it dramatically simplifies the proof for the twisted curved $\mathfrak{g}L_{\infty}$-algebra to be a curved $\mathfrak{g}L_{\infty}$-algebra as well.

4. Main Theorem

In this section we focus to the proof of Theorem 4.1.

Let us recall that we work in the setting of curved $\mathfrak{g}L_{\infty}$-algebras endowed with a descending, bounded above and complete filtration that is compatible with the curved $\mathfrak{g}L_{\infty}$-algebra structure. In particular, we always demand the filtration to start with $g = \mathfrak{g}1$ (also see Equations (2.2) and (2.3)), which together with completeness ensures convergence of the infinite sums as e.g. appearing in the Maurer-Cartan equations. By switching between the three equivalent descriptions of curved $\mathfrak{g}L_{\infty}$-algebra the main theorem can be proved almost in absence of any bigger calculation.

The following lemma shows, that in the curved $\mathfrak{g}L_{\infty}$-algebra setting we are not only allowed to twist by Maurer-Cartan elements but rather by any degree 0 element and still obtain a $\mathfrak{g}L_{\infty}$-algebra endowed with a ‘good’ filtration.

**Lemma 4.1.** Let $g = (g, \{\mu_n\}_{n \geq 0})$ be a curved $\mathfrak{g}L_{\infty}$-algebra endowed with a descending bounded above and complete filtration $g = \mathfrak{g}1 \supset \mathfrak{g}2 \supset \ldots$ compatible with the $\mathfrak{g}L_{\infty}$-algebra structure.

Let $\beta \in g^0$ be an arbitrary degree zero element.

Then $g^\beta = (g, \{\mu_n^\beta\}_{n \geq 0})$ with $\{\mu_n^\beta\}_{n \geq 0}$ as defined in Equation (2.4), also describes a curved $\mathfrak{g}L_{\infty}$-algebra. Moreover, the filtration maintains its qualities of being descending, bounded above, complete and compatible with the curved $\mathfrak{g}L_{\infty}$-algebra structure.

From a geometrical point of view we can interpret a curved $\mathfrak{g}L_{\infty}$-algebra $g$ as a formal vector field $Q(x)$ (which, when expanded as a power series, has $\mu_n$ as nth Taylor coefficient) that squares to zero. In this language twisting by $\beta$ corresponds to a displacement of the origin by $-\beta$. Clearly, this new vector field still squares to zero, hence still describes a curved $\mathfrak{g}L_{\infty}$-algebra equation. For non-curved $\mathfrak{g}L_{\infty}$-algebras however, there is a problem. A vanishing curvature translates to the vector field $Q(x)$ also having a root at the origin. When displacing the origin by some arbitrary $-\beta \in g^0$, this condition in general may no longer hold, hence making the
resulting $\mathfrak{S}L_\infty$-algebra to be curved.

Since this paper focuses on algebra, let us elaborate on a proof of this claim in the language of algebra.

**Proof of Lemma 4.1.** Let us denote the construction of Equation (3.1) by $M^R_\varnothing$ and $M^R_{\varnothing\varnothing}$, respectively, to emphasise which brackets were used.

Let $x \in (\varnothing \otimes R)^0$ and $\beta = \beta \otimes 1$ (by abuse of notation) for $R$ as in Section 3. Our first step is to verify

\begin{equation}
M^R_\varnothing(x) = M^R_\varnothing(x + \beta).
\end{equation}

On one hand we have

\[
M^R_\varnothing(x + \beta) = \sum_{n \geq 0} \frac{1}{n!} \mu_n(x + \beta, \ldots, x + \beta)
\]

\[
= \sum_{k \geq 0} \sum_{j \geq 0} \frac{(j+k)!}{j!k!} \mu_{j+k}(\beta, \ldots, \beta, x, \ldots, x)
\]

\[
= \sum_{k \geq 0} \sum_{j \geq 0} \frac{k!j!}{j!k!} \mu_{j+k}(\beta, \ldots, \beta, x, \ldots, x),
\]

where we used multilinearity, $|x| = 0$ and $|\beta| = 0$, relabelled the sums $n = j + k$ and made use of some elementary combinatorics, which yields that there are exactly $\binom{j+k}{j}$ combinations to have an $j + k$ bracket with $j$-many $\beta$ and $k$-many $x$.

On the other hand also

\[
M^R_\varnothing(x) = \sum_{n \geq 0} \frac{1}{n!} \mu_n^\varnothing(x, \ldots, x) = \sum_{n \geq 0} \sum_{j \geq 0} \frac{1}{n!j!} \mu_{n+j}(\beta, \ldots, \beta, x, \ldots, x)
\]

holds, which proves Equation (4.1).

In turn, Equation (1.1) allows for the following relation (with $R'$ as in Section 3)

\[
M^R_{\varnothing \varnothing}(x \otimes 1 + M^R_\varnothing(x) \otimes \epsilon) = M^{R \otimes R'}_{\varnothing}(x \otimes 1 + \beta \otimes 1 + M^R_\varnothing(x + \beta) \otimes \epsilon)
\]

\[
\overset{\varnothing = x + \beta}{\simeq} M^{R \otimes R'}_{\varnothing'(x)}(\tilde{x} \otimes 1 + M^R_\varnothing(\tilde{x}) \otimes \epsilon) = M^R_{\varnothing}(x) \otimes 1 = M^R_\varnothing(x + \beta) \otimes 1 = M^R_\varnothing(x) \otimes 1,
\]

which is just the $\mathfrak{S}L_\infty$-algebra equation for $\varnothing$ in the language of Equation (3.3).

Another technical lemma we need in the proof of the main theorem is the following:

**Lemma 4.2.** Let $\varnothing$ be curved $\mathfrak{S}L_\infty$-algebra equipped with a descending bounded above and complete filtration $\varnothing = \varnothing_1 \supset \varnothing_2 \supset \ldots$ compatible with the $\mathfrak{S}L_\infty$-algebra structure and let $r \in \mathbb{N}_0$ be an integer for which the curvature $\mu_0$ of $\varnothing$ satisfies $\mu_0 \in \varnothing_{2r+1}$ and the $r + 1$-th page of the spec.seq. vanishes, i.e. $E_{r+1}(\varnothing^\alpha) = 0$.

If $\alpha \in \varnothing_{r+1}$, then twisting of $\varnothing$ by $\alpha$ does preserve the condition of having a vanishing $r + 1$-th page of the spec.seq., i.e. $E_{r+1}(\varnothing^\alpha) = 0$ holds. This result also holds for each of the total degrees individually.

**Proof of Lemma 4.2.** Recalling Equation (2.8), $E_{r+1}^p = 0$ means that

\[
\begin{cases}
\forall x \in \varnothing_p \varnothing^{p+q} \text{ s.t. } \mu_1(x) \in \varnothing_{p+r+1} \varnothing^{p+q+1} \\
\exists y \in \varnothing_{p-r} \varnothing^{p+q-1} \text{ s.t. } \\
\mu_1(y) \in \varnothing_p \varnothing^{p+q} \text{ and } x - \mu_1(y) \in \varnothing_{p+1} \varnothing^{p+q}
\end{cases}
\]
Let $x \in \mathfrak{g}^{p+q}$ and $\alpha \in \mathfrak{g}^0$, then
\[
\mu_\alpha^\gamma(x) = \mu_1(x) + \mu_2(\underbrace{\alpha \cdot \cdots \cdot \alpha}_{\in \mathfrak{g}^p \mathfrak{g}^q}, x) + \frac{1}{2!} \mu_3(\alpha, \alpha, x) + \ldots = \mu_1(x) + O(\mathfrak{g}^{p+q+1})
\]
holds.
But on the other hand, for $y \in \mathfrak{g}^{p+q-1}$ we also have
\[
\mu_\alpha^\gamma(y) = \mu_1(y) + \mu_2(\underbrace{\alpha \cdot \cdots \cdot \alpha}_{\in \mathfrak{g}^p \mathfrak{g}^q}, y) + \frac{1}{2!} \mu_3(\alpha, \alpha, y) + \ldots = \mu_1(y) + O(\mathfrak{g}^{p+q}),
\]
allowing us to replace all the expressions in Equation (4.2) with its twisted counterparts (and vice versa), so particularly the statement $E_{r+1} = 0$ does not change under twisting.

One more technical lemma allows us in the setting of Theorem 4.1 to twist the curved $\mathfrak{g}L_\infty$-algebra in such a way that the filtration degree of the curvature of the twisted $\mathfrak{g}L_\infty$-algebras curvature is raised by one when compared to the curvature of the original untwisted $\mathfrak{g}L_\infty$-algebra.

**Lemma 4.3.** Let $\mathfrak{g}$ be a curved $\mathfrak{g}L_\infty$-algebra equipped with a descending, bounded above and complete filtration $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \ldots$ compatible with the curved $\mathfrak{g}L_\infty$-algebra structures. Let $r \in \mathbb{N}_0$ and $k \geq 2r + 1$ be some integer for which the curvature $\mu_0$ of $\mathfrak{g}$ satisfies $\mu_0 \in \mathfrak{g}_k$ and $E_{r+1}^{p+q} = 0$ holds for all $p, q$ with $p + q = 1$. Then there exists an $\alpha \in \mathfrak{g}_{k-r} \mathfrak{g}_0$ such that $\mu_0 - \mu_1(\alpha) \in \mathfrak{g}_{k+1} \mathfrak{g}_1$ and the curvature $\mu_0^\alpha$ of the twisted curved $\mathfrak{g}L_\infty$-algebra $\mathfrak{g}^{-\alpha}$ satisfies $\mu_0^{-\alpha} \in \mathfrak{g}_{k+1} \mathfrak{g}_1$.

**Proof of Lemma 4.3** A direct consequence of the curved $\mathfrak{g}L_\infty$-algebra equations is the curvature being closed $\mu_1(\mu_0) = 0$. This particularly means that $\mu_0$ fits into the scheme of Equation (4.2) (for $p = k$ and $q = 1 - k$) and we find
\[
\exists \alpha \in \mathfrak{g}_{k-r} \mathfrak{g}_0 \ \text{s.t.} \quad \mu_1(\alpha) \in \mathfrak{g}_k \mathfrak{g}_1 \text{ and } \mu_0 - \mu_1(\alpha) \in \mathfrak{g}_{k+1} \mathfrak{g}_1.
\]

Explicitly expanding the expression of the twisted curvature $\mu_0^{-\alpha}$ yields
\[
\mu_0^{-\alpha} = \underbrace{\mu_0 - \mu_1(\alpha)}_{\mathfrak{g}_{k+1} \mathfrak{g}_1} + \frac{1}{2!} \underbrace{\mu_2(\alpha, \alpha)}_{\mathfrak{g}_{k+1} \mathfrak{g}_1} + \ldots \in \mathfrak{g}_{k+1} \mathfrak{g}_1.
\]

Having stated these small technical lemmata, we can now pursue the proof of the main theorem of this paper.

**Theorem 4.4** (Main Theorem). Let $\mathfrak{g}$ be a curved $L_\infty$-algebra equipped with a descending, bounded above and complete filtration $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \ldots$ compatible with the curved $L_\infty$-algebra structures. If there exists an integer $r \in \mathbb{N}_0$ for which the curvature $\mu_0$ of $\mathfrak{g}$ satisfies $\mu_0 \in \mathfrak{g}_{2r+1} \mathfrak{g}_2$ and the $r + 1$st page of the spec.seq. of $\mathfrak{g}$ vanishes in total degree 2, i.e. $E_{r+1}^{p+q} = 0$ for all $p, q$ with $p + q = 2$, then there exists a Maurer-Cartan element...
Proof of the Main Theorem. As before we do all our calculations in the language curved $\mathcal{S}L_\infty$-algebras, i.e. we work with $\mathfrak{g}[-1]$ instead of $\mathfrak{g}$ and curved $\mathcal{S}L_\infty$-algebra brackets to avoid having ubiquitous suspensions in the expressions.

We describe a procedure of repeated twisting such that in every step the curvature of the twisted $\mathcal{S}L_\infty$-algebra gets raised by one. Moreover, we explain why this construction converges and forms a non-curved $\mathcal{S}L_\infty$-algebra.

For $k \geq 2r + 1$, according to Lemma 4.3 there exists an $\alpha_1 \in \mathfrak{g}_{k-r}\mathfrak{g}^0$ such that the curvature $\mu_{\alpha_1}^0$ of the curved $\mathcal{S}L_\infty$-algebra $\mathfrak{g}^{\alpha_1}$ satisfies $\mu_{\alpha_1}^0 \in \mathfrak{g}_{k+1}^1$. Since $\alpha_1 \in \mathfrak{g}_{k-r}\mathfrak{g}^0$, by virtue of Lemma 4.2 $E_{r+1}^{p, q}(\mathfrak{g}^{\alpha_1}) = 0$ for $p + q = 1$ still holds.

As a consequence, we can repeat the very same process with $\mathfrak{g}$ replaced by $\mathfrak{g}^{\alpha_1}$ and $k + 1$ instead of $k$ and so eventually find an $\alpha_2 \in \mathfrak{g}_{k-r+1}\mathfrak{g}^0$ such that the curvature $\mu_{\alpha_1 + \alpha_2}^0$ of $\mathfrak{g}^{\alpha_1 + \alpha_2}$ satisfies $\mu_{\alpha_1 + \alpha_2}^0 \in \mathfrak{g}_{k+2}^1$. Of course this twisted curved $\mathcal{S}L_\infty$-algebra still preserves $E_{r+1}^{p, q}(\mathfrak{g}^{\alpha_1 + \alpha_2}) = 0$ for all $p, q$ with $p + q = 1$. But (see Equation (4.1)) $\mathfrak{g}^{\alpha_1 + \alpha_2}$ is the same as $\mathfrak{g}^{\alpha_1 + \alpha_2}$.

In other words we have a way of raising the degree of filtration of the twisted curved $\mathcal{S}L_\infty$-algebra by one without destroying the condition $E_{r+1}^{p, q} = 0$ for $p + q = 1$.

Repeated application of this twisting procedure and making use of the completeness of the filtration allows us to eventually obtain a non-trivial twisted curved $\mathcal{S}L_\infty$-algebra (still on the vector space $\mathfrak{g}$) with zero curvature. Also notice that by Lemma 4.3 we have $\alpha_i \in \mathfrak{g}_{k-r}\mathfrak{g}^0$, i.e. the element $\alpha_i$ we twist with also increases its degree of filtration in every step. Together with the completeness of the filtration this lets the sum $\alpha_1 + \alpha_2 + \ldots$ converge and therefore makes the construction well-defined.

By means of the equivalence between a curved $\mathcal{S}L_\infty$-algebra $\mathfrak{g}^\alpha$ being flat and $-\alpha$ forming a Maurer-Cartan element of $\mathfrak{g}$, this shows $\alpha = -\alpha_1 - \alpha_2 - \ldots$ to satisfy the requirements.

5. Applications

We start by presenting the general idea behind using Theorem 1.1 in the search of $\infty$-(quasi) isomorphisms.

So, let $P$ be a possibly inhomogeneous Koszul operad. A $P_\infty$-algebra, sometimes referred to as strong homotopy algebra, is an algebra over the co-bar-construction $\Omega P^!$ of the Koszul dual co-operad $P^!$. Notice, that even though in the definition of inhomogeneous Koszul operads we do not allow for operads with differentials other than the trivial one, the Koszul dual co-operad as well as its co-bar construction both do generally carry a non-trivial (co)-differential. Prominent examples of such $P_\infty$-algebras, apart from the $L_\infty$-algebras from before, are Com$_{\infty}$-algebras, Assoc$_{\infty}$-algebras and BV$_{\infty}$-algebras.

Remark 5.1. The main difference in the notion of $P_\infty$-algebras for $P$ being an inhomogeneous Koszul operad, as introduced in [4], Appendices A,B, compared to the ‘classical case’ of Koszul operads (with quadratic relations only) is that a $P_\infty$-algebra still is a $\Omega P^!$-algebra, but with the construction of $P^!$ being slightly more involved. More precisely, $P^!$ is defined as the ‘classical’ Koszul dual of $qP$

$^8\alpha \in MC(\mathfrak{g})$ means $M(\alpha) = 0$ and hence $M_R(\alpha \otimes R) = 0$. $\mathfrak{g}^\alpha$ being flat on the other hand is tantamount to $M_R^R(0) = 0$, as in $M_R^R(0)$, due to multilinearity of the higher ($n \geq 1$) brackets, only the curvature $\mu_{\alpha}^0$ survives. In Equation (4.1) we found $M_R^R(\mathfrak{g}^\alpha) = M_R(\mathfrak{g} + \alpha)$ which eventually shows the statement.
with \( q \) the map that removes the linear part of from the relations and some special co-differential is given to \( (qP) \). For most of the relevant properties, among others the ‘Rosetta Stone’ of equivalent descriptions of \( P_\infty \)-algebras (see \[4\], Theorem 1), the inhomogeneous Koszul case does not differ from the ‘classic’ quadratic case. Notice, however, that \( \Omega P^1 \), in general, does no longer form a minimal resolution of \( P \).

An \( \infty \)-morphism between two \( P_\infty \)-algebras \( F : A \rightsquigarrow B \) is defined as a \( P^1 \)-co-algebra (in \( \text{dgVect} \), hence compatible with the respective co-differentials) morphism between the corresponding relative (relative w.r.t. Koszul morphism \( i : P^1 \to \Omega P^1 = P_\infty \)) bar constructions \( F : B_s(A) \to B_s(B) \) (cf. \[4\], Section B.2).

Recall from \[4\], Section B.1 that the relative bar construction of the \( P_\infty \)-algebra \( A \) is a quasi-co-free co-algebra, hence an \( \infty \)-morphism \( F : A \rightsquigarrow B \) is completely determined by its projection\(^9\) to \( B \), that is a graded vector space morphism \( f : F_{U,p}^c(U(A)) \to B \).

It is crucial to notice that while every graded vector space morphism \( f : F_{U,p}^c(U(A)) \to U(B) \) induces a graded \( U(P^1) \)-co-algebra morphism \( f : F_{U,p}^c(U(A)) \to F_{U,p}^c(U(B)) \) (in the underlying category \( \text{grVect} \)), compatibility with the co-differentials does not hold in general. However, there is a way to work on the level of \( \text{Hom}_{\text{grVect}}(F_{U,p}^c(U(A)), U(B)) \) and analyse which elements induce co-free \( U(P^1) \)-co-algebra morphisms that are compatible with the respective co-differentials and as such form \( \infty \)-morphisms. To this end we define a formal vector field \( Q \) on \( \text{Mor}_{U(P^1)-\text{coalg}}(F_{U,p}^c(U(A)), F_{U,p}^c(U(B))) \) by the commutator with the co-differentials emerging from the bar-construction (that means maps compatible with the differential are the roots of \( Q \)) and pull this back along \( f \). This results in a \( \mathfrak{S}L_\infty \)-algebra structure on the graded vector space \( \text{Hom}_{\text{grVect}}(F_{U,p}^c(U(A)), U(B)) \) and we may call this \( \mathfrak{S}L_\infty \)-algebra deformation complex \( \text{Def}(A, B) \).

Maurer-Cartan elements on the Deformation Complex correspond to graded vector space morphisms whose induced \( U(P^1) \)-co-algebra morphisms commute with the differentials, that is to say they induce \( \infty \)-morphisms. Notice that the zero map also induces a \( U(P^1) \)-co-algebra morphism that is compatible with the differentials. Let us explain why the search of a Maurer-Cartan element of a curved \( \mathfrak{S}L_\infty \)-algebra is of relevance in the construction of \( \infty \)-(quasi-)isomorphism between \( P_\infty \)-algebras.

For the scope of this motivation let us assume \( P \) to be a binary generated (a requirement we will relax in the proper form of the theorem) possibly inhomogeneous Koszul operad and let \( A, B \) be two \( P_\infty \)-algebras. Consider the situation in which a (quasi-)isomorphism of dg vector spaces \( f_1 : A \to B \) is given. We may ask: Is there an \( \infty \)-morphism \( F : A \rightsquigarrow B \) which composed with the projection \( \pi_B : F_{U,p}(B) \to B \) yields \( f_1 \) (and as such by Definition \[5,6\] \( F \) forms an \( \infty \)-(quasi-)isomorphism). We address this question by inducing a descending, bounded above and complete filtration compatible with the \( \mathfrak{S}L_\infty \)-algebra structure on the deformation complex \( \text{Def}(A, B) \) such that elements of higher filtration degree have a vanishing \( U(P^1)(1) \otimes U(A) \to U(B) \) part. We then twist \( \text{Def}(A, B) \) by the graded vector space morphism \( \tilde{f}_1 : F_{U,p}^c(U(A)) \to U(B) \) that everywhere is zero except on

\[^9\text{For the projection } \pi_B : F_{U,p}^c(B) \to B \text{ defined by } F_{U,p}^c(B) \xrightarrow{\text{wlog}} I \otimes_{A^1} B \cong B, \]

\[^{10}\text{Since we later endow this space with a } \mathfrak{S}L_\infty \text{-structure, we do also want to allow for maps of degree other than zero and therefore work with } \text{Hom}_{\text{grVect}} \text{ instead of } \text{Mor}_{\text{grVect}}. \text{ Of course, only the degree 0 elements are the ones forming } U(P^1) \text{-co-algebra-morphisms.} \]
$U(P^i)(1) \otimes U(A)$, where it is $f_1$. Finding a Maurer-Cartan element $\alpha$ on the twisted deformation complex (for example by cleverly applying Theorem 4.3) is equivalent to $f_1 + \alpha$ being an $\infty$-morphism. Moreover, if the Maurer-Cartan element $\alpha$ carries a higher degree of filtration it does not alter the $U(P^i)(1) \otimes U(A) \rightarrow U(B)$ part from being $f_1$ and hence yields that $f_1 + \alpha$ is an $\infty$-(quasi-)isomorphism.

When discussing intrinsic formality we will follow this very rationale, start with the identity map as the initial dgVect quasi-isomorphism and search for Maurer-Cartan elements on the twisted deformation complex by e.g. applying Theorem 5.1 as these yield the $\infty$-quasi-isomorphism we looking for.

For the remainder of this paper $P$ is assumed to be a possibly inhomogeneous Koszul operad, sometimes (but always mentioned) with restriction on its generating set. Furthermore, we continue using $U$ to denote the forgetful functor $U : dgVect \rightarrow grVect$.

5.1. Koszul Operads.

**Definition 5.2** (Weight Grading on $P^i$, $\mathbb{F}_{U(P)}^c(U(A))$ and $\text{Hom}_{grVect}(\mathbb{F}_{U(P)}^c(U(A)), U(B))$).

Let $P$ be a possibly inhomogeneous Koszul operad. By definition ([R, Sect.A.2.]) the Koszul dual co-operad is a sub-cooperad of the co-free co-operad $\mathcal{J}(sE)$ (with $s$ denoting the suspension operator) for some $S$-module $E$, which we call generating set. Recalling from [R, Vol.I, Sect.C.1.], the definition of the co-free co-operad $\mathcal{J}(sE)$ in terms of trees with vertices decorated by $sE$ we may introduce a weight grading on $P^i$ by the number of vertices (cf. [R, Section C.1.) and write $P^{(k)}$ for the weight $k$-part of $P^i$ with respect to this weight grading.

Let $A$ be a $P_\infty$-algebra. We say $v \in P^i(m) \otimes_{S_m} A^{\otimes m} = \prod_{l \geq 1} P^l(m) \otimes_{S_m} A^{\otimes m}$ has weight $k$ if all the $P^l(m) \otimes_{S_m} A^{\otimes m}$ are zero except for $l = k$. Moreover, $w \in \mathbb{F}_{P^i}(A) = \bigoplus_{n \geq 1} P^i(n) \otimes_{S_n} A^{\otimes n}$ is said to be of weight $k$ if all the $v_i \in P^i(i) \otimes_{S_i} A^{\otimes i}$ in $w = v_1 + v_2 + \ldots$ are of weight $k$. Both definitions do also hold for their $grVect$ counterparts.

Let $A, B$ also be two $P_\infty$-algebras. We introduce a weight grading on $\text{Hom}_{grVect}(\mathbb{F}_{P^i}(U(A)), U(B))$ by setting the weight $k$-part to be the graded vector space homomorphisms that to vanish on all weights of $\mathbb{F}_{P^i}(U(A))$ except possibly for weight $k$.

5.2. $OpA$-Category. Next, let us present the category $OpA$ consisting of a tuple of an operad $P^i$ and a $P$-algebra.

**Definition 5.3** ($OpA$-category). Let $M$ denote a symmetric monoidal category (e.g. $M = dgVect, grVect$) and let $P$ be an operad in $M$. Furthermore, let $A$ be $P$-algebra.

We establish the category $OpA$ by defining

\begin{equation}
\text{Obj}(OpA) := (P, A)
\end{equation}

and setting the morphism to be

\begin{equation}
\text{Mor}_{OpA}((P, A), (Q, B)) = (f, g),
\end{equation}

where $f : P \rightarrow Q$ is an operad-morphism and $g : A \rightarrow f^*B$ is a $P$-algebra morphism. Recall, that the $P$-algebra $f^*B$ is $B$ with the initial $Q$-algebra structure $\gamma_B : Q \circ B \rightarrow B$ made into a $P$-algebra structure $\gamma : P \circ B \rightarrow B$ according to

\[ \gamma : P \circ B \xrightarrow{id_B} Q \circ B \xrightarrow{\gamma_B} B. \]

The composition of morphism works component-wise.
5.3. \(P_{(\infty)}\)-algebra structures on co-homologies.

**Lemma 5.4** (\(H(P)\)-algebra structure on \(H(A)\)). From \(\text{[3], Vol.I, Prop.3.1.1}\) it is immediate that every lax monoidal functor also forms a functor of \(\text{OpA}\) categories. According to the Künneth formula (see \(\text{[10], Chap.8 and [12], Sect.5.3}\), the co-homology forms a strong monoidal functor (and hence especially a lax monoidal functor) and so in e.g. \(M = \text{dgVect}\), the co-homology forms a functor

\[
H : \text{OpA}_{\text{dgVect}} \rightarrow \text{OpA}_{\text{grVect}}
\]

\[(P, A) \mapsto (H(P), H(A)).
\]

In particular, the co-homology \(H(A)\) of a \(P\)-algebra \(A\) forms a \(H(P)\)-algebra.

**Definition 5.5** (\(P_{(\infty)}\)-algebra structures on \(H(A)\)). Let \(P\) be a possibly inhomogeneous Koszul operad and \(A\) be a \(P_{(\infty)}\)-algebra. Recall the notion of an operad being possibly inhomogeneous Koszul (cf. \(\text{[6], Sect.A.3}\), which among others yields the cobar construction of the Koszul dual co-operad to provide a quasi-free resolution of \(P\).

\[
\Omega P^A_{\text{dgOp quasi-iso}} P.
\]

But because our definition of Koszul operads does not allow for operads with non-trivial differentials, Equation (5.3) also shows

\[
H(\Omega P^A) \cong \text{End}_{H(A)} P.
\]

From Lemma 5.4 applied to \(\Omega P^A\) we know that \(H(A)\) is endowed with a \(H(\Omega P^A)\)-algebra structure. Subsequently, the composition of the structure map \(H(\Omega P^A) \rightarrow \text{End}_{H(A)}\) with the morphism of Equation 5.4 yields a \(P\)-algebra structure on \(H(A)\).

Furthermore, using the fact every \(P\)-algebra \(B\) also is a \(P_{(\infty)}\)-algebra simply by composition of the structure map with the projection

\[
P_{(\infty)} = \Omega P^B \cong P \rightarrow \text{End}_B,
\]

we can also speak of a \(P_{(\infty)}\)-algebra structure on \(H(A)\).

Apart from the just found \(P\)-algebra (and \(P_{(\infty)}\)-algebra structure, respectively) on \(H(A)\), there is yet another construction for a \(P_{(\infty)}\)-algebra structure on \(H(A)\).

More precisely, by virtue of the homotopy transfer theorem (see \(\text{[6], Lemma 48}\), we can endow the space \(H(A)\) also with a proper \(P_{(\infty)}\)-algebra structure, which we denote by \(H(A)_{\text{HTT}}\) to avoid confusion with the structures from above (even though the underlying graded vector spaces are the same). Among others, \(H(A)_{\text{HTT}}\) has the special property that it extends the \(P\)-algebra structure from above, that is to say the corresponding \(\mathbb{S}\)-module morphism \(\rho_{H(A)_{\text{HTT}}} : P^A \rightarrow \text{End}_{H(A)_{\text{HTT}}}\) coincides with \(\rho_{H(A)}\) in \(P^{(1)}\).

Moreover, the homotopy transfer theorem allows us to extend the inclusion and projection, respectively, to \(\infty\)-quasi-isomorphisms (see Definition 5.6) \(\iota : H(A)_{\text{HTT}} \rightarrow P\) and \(p : A \rightarrow H(A)_{\text{HTT}}\).

5.4. \(\infty\)-(quasi-)isomorphisms and formality.

**Definition 5.6** (\(\infty\)-(quasi-)isomorphism). An \(\infty\)-algebra morphism \(f : A \rightarrow B\) between two \(P_{(\infty)}\)-algebras \(A, B\) for a possibly inhomogeneous Koszul operad \(P\) is by definition a co-algebra morphism between the co-free \(U(P)\)-co-algebras \(F : \text{F}_U(P^A)(U(A)) \rightarrow \text{F}_U(P^B)(U(B))\) which in addition is compatible with the respective co-differentials emerging from the relative bar-constructions (see Equations 5.20-5.28). Due to co-freeness such a map \(f\) is fully described by a graded vector space morphism \(\tilde{f} : \text{F}_U(P^A)(U(A)) \rightarrow U(B)\) and hence using the internal hom adjunction.
we translate the notion of a differential of $U$ directly utilise freeness when analysing derivations of free morphisms. By describing derivations as $U$-morphisms, we can equivalently be fully characterised as an $S$-module morphism $\xi \in \text{Mor}_{\text{Mod}}(U(P^*), \text{End}_{U(B)})$, where
\[
\text{End}_{U(B)} = \{\text{Hom}(U(A)^{\otimes n}, U(B))\}_{n \in \mathbb{Z}_{\geq 1}}.
\]
For an $\infty$-morphism the image of the weight 0-part under $\xi$, that is $\xi(P^{(0)}) : A \to B$, is a chain map.
So it makes sense for an $\infty$-morphism $f : A \rightsquigarrow B$ to be called $\infty$-isomorphism if $\xi(P^{(0)})$ is an isomorphism of dg vector spaces.
Similarly, we say an $\infty$-morphism $f : A \rightsquigarrow B$ is an $\infty$-quasi-isomorphism if $\xi(P^{(0)})$ is a quasi-isomorphism of dg vector spaces, that is, it induces an isomorphism in the respective co-homologies.

$\infty$-quasi-isomorphisms of $P_\infty$-algebras are of great relevance as they do form the weak equivalences in a model category structure on $P_\infty$-algebras with $\infty$-morphisms (cf. [13], Sect.4.1).

5.5. (Intrinsic) Formality.

**Definition 5.7** (Formality of $P_\infty$-algebras). Let $A$ be a $P_\infty$-algebra for some possible inhomogeneous Koszul operad $P$. We say that $A$ is formal if there exists an $\infty$-quasi-isomorphism
\[
H(A) \xrightarrow{\sim} A.
\]
Besides the notion of formality, there is also the stronger concept of intrinsic formality, meaning that a $P_\infty$-algebra is called intrinsically formal if every $P_\infty$-algebra $B$ with co-homology $H(B)$ (in the sense of Definition 5.5) that as a $P$-algebra is isomorphic to $H(A)$, is formal itself.

**Definition 5.8** (Intrinsic Formality of $P_\infty$-algebras). Let $A$ be a $P_\infty$-algebra for $P$ a possibly inhomogeneous Koszul operad. We say $A$ is intrinsically formal, if for every $P_\infty$-algebra $B$ there is the implication\(^{11}\)
\[
H(B) \stackrel{p-\text{alg.-iso.}}{\simeq} H(A) \quad \implies \quad B \underset{\text{\text{-iso.}}}{\sim} H(B).
\]
Before introducing the deformation complex, let us first further elaborate on the notion of derivations of $P$-algebras (and its dual counterparts, respectively), which we have introduced at the end of Section 2.

5.6. Derivations of $P$-algebras revisited. We start, by using extension of scalars (cf. [11], Section 3.10), to provide an alternative but equivalent definition of derivations of $P$-algebras as $U(P) \otimes R$-algebra morphisms for some graded ring $R$. Furthermore, we extend this definition to derivations with respect to $U(P)$-algebra morphisms. By describing derivations as $U(P) \otimes R$-algebra morphisms we can directly utilise freeness when analysing derivations of free $U(P)$-algebras. Eventually, we translate the notion of a differential of $U(P)$-algebra and a derivation w.r.t. a $U(P)$-algebra morphism to the $OpA$ language.

\(^{11}\)To be precise it is merely an almost model category structure as even though it admits finite products and pullback of fibrations, in general the category of $P_\infty$-algebras with $\infty$-morphisms fails to admit finite limits and co-limits (see [13], Section 4.1 and [2], Sect.B.6.3).

\(^{12}\)The implication can equivalently (using the fact that isomorphic $P$-algebras are in particular also $\infty$-isomorphic and hence $\infty$-quasi-isomorphic as well) be formulated as
\[
H(B) \stackrel{p-\text{alg.-iso.}}{\simeq} H(A) \quad \implies \quad B \underset{\text{\text{-iso.}}}{\sim} H(A).
\]
Moreover, $H(B) \stackrel{p-\text{alg.-iso.}}{\simeq} H(A) \quad \implies \quad B \underset{\text{\text{-iso.}}}{\sim} A$ is another equivalent definition of intrinsic formality, justifying to speak of intrinsic formality being a property of $A$. 

17
Definition 5.9 (Derivation with respect to $U(P)$-algebra morphism, equivalent description of $P$-algebra derivations). Let $P$ be an operad, $A$, $B$ two $P$-algebras and $f : U(A) \to U(B)$ a graded $U(P)$-algebra morphism. We say a linear map $\xi : U(A) \to U(B)$ is a derivation of degree $k$ with respect to $f$, if

\begin{equation}
(f \otimes \text{Id}_R + \xi \otimes \epsilon) : U(A) \otimes R \to U(B) \otimes R
\end{equation}

forms a $U(P) \otimes R$-algebra morphism for $R$ the ring $R := \mathbb{K}[\epsilon]/\epsilon^2$ with $\epsilon$ a formal variable of degree $-k$.

That is to say, the following diagram commutes

\begin{equation}
\begin{tikzcd}
(U(P) \otimes R)(U(A) \otimes R) \ar{r}{(U(P) \otimes R)(f \otimes \text{Id}_R + \xi \otimes \epsilon)} \ar{d}{\gamma} & (U(P) \otimes R)(U(B) \otimes R) \ar{d}{\gamma} \\
U(A) \otimes R \ar{r}{f \otimes \text{Id}_R + \xi \otimes \epsilon} & U(B) \otimes R
\end{tikzcd}
\end{equation}

Recall, that $(U(P) \otimes R)(f \otimes \text{Id}_R + \xi \otimes \epsilon)$ operates on an element in $(U(P) \otimes R)(U(A) \otimes R)$ by leaving the $U(P) \otimes R$-part untouched but simultaneously acting with $f \otimes \text{Id}_R + \xi \otimes \epsilon$ on all the $U(A) \otimes R$ decorations. Due to the definition of $R$, only the terms where one or none $\xi \otimes \epsilon$ operation occurs, do survive. From $f$ being a graded $U(P)$-algebra morphism it is clear, that

\begin{equation}
\begin{tikzcd}
(U(P) \otimes R)(U(A) \otimes R) \ar{r}{(U(P) \otimes R)(f \otimes \text{Id}_R)} \ar{d}{\gamma} & (U(P) \otimes R)(U(B) \otimes R) \ar{d}{\gamma} \\
U(A) \otimes R \ar{r}{f \otimes \text{Id}_R} & U(B) \otimes R
\end{tikzcd}
\end{equation}

commutes and hence commutativity of the diagram of Equation (5.7) is tantamount to commutativity of

\begin{equation}
\sum_j \text{Id}_{U(P)} \otimes ((f \otimes \text{Id}_R) \otimes \ldots (f \otimes \text{Id}_R) \otimes (\xi \otimes \epsilon) \otimes (f \otimes \text{Id}_R) \otimes \ldots (f \otimes \epsilon)) \otimes \text{Id}_{U(P)}
\end{equation}

\begin{equation}
\begin{tikzcd}
(U(P) \otimes R)(U(A) \otimes R) \ar{r}{(U(P) \otimes R)(f \otimes \text{Id}_R)} \ar{d}{\gamma} & (U(P) \otimes R)(U(B) \otimes R) \ar{d}{\gamma} \\
U(A) \otimes R \ar{r}{\xi \otimes \epsilon} & U(B) \otimes R
\end{tikzcd}
\end{equation}

When comparing the diagrams of Equation (5.9) and Equation (2.12), one realises that the latter is just the special case of $A = B$ and $f$ being the identity map. Hence, it makes sense to consider the definition of a derivation with respect to an algebra morphism as a generalisation of the ‘classical’ definition of $P$-derivations from Section 2 and under this new notion ‘classical’ $P$-derivations correspond to derivations w.r.t. the identity morphism.

One of the main advantages of this generalised definition is that $\text{Hom}_{gr\text{Vect}}(A, \mathbb{F}_{U(P)}(A)) \cong \text{Der}(\mathbb{F}_{U(P)}(A))$ follows directly from freeness applied to the corresponding $U(P) \otimes R$-algebra morphism.
Lemma 5.10 (Derivations w.r.t. \( U(P) \)-algebra morphisms and differentials in the \( \text{Op}_A \) language). Let \( P \) be an operad and let \( A \) be a \( P \)-algebra. Consider the space

\[
\text{Der}(P, A) := \{ \xi = (\xi_1, \xi_2) : \xi_1 \in \text{Hom}_{\text{grVect}}(P, P), \xi_2 \in \text{Hom}_{\text{grVect}}(A, A), \quad Id + \epsilon \xi \in \text{Mor}_{\text{Op}_A}((P, A) \otimes R, (P, A) \otimes R) \},
\]

where \( R = \mathbb{K}[e]/e^2 \) with \( e \) a formal variable of degree 0. We have the following equivalent characterisation of \( P \)-derivation and differential of a \( P \)-algebra \( A \) in terms of elements of \( \text{Der}(P, A) \).

- A graded linear map \( \beta : A \rightarrow A \) describes a \( P \)-derivation of \( A \) if
  \[
  (0, \beta) \in \text{Der}(U(P), U(A)).
  \]

- A graded linear map \( \beta : A \rightarrow A \) of degree 1, subject to \( \beta^2 = 0 \) forms a differential on \( U(A) \) if
  \[
  (d_P, \beta) \in \text{Der}(U(P), U(A))^{(1)}
  \]

Furthermore, also derivations w.r.t. a \( U(P) \)-algebra morphism can be characterised in this manner. More precisely, let \( f : A \rightarrow B \) be a \( U(P) \)-algebra morphism. Then, a graded linear map \( \xi : A \rightarrow B \) forms a derivation w.r.t. \( f \), if

\[
(Id_P, f) + \epsilon(0, \xi) \in \text{Mor}_{\text{Op}_A}((U(P), U(A)) \otimes R, (U(P), U(B)) \otimes R)
\]

Proof of Lemma 5.10. For \((Id_P, f) + \epsilon(0, \xi)\) to form an \( \text{Op}_A \) morphism the following diagram has to commute

\[
\begin{array}{ccc}
(U(P) \otimes R)(U(A) \otimes R) & \xrightarrow{(Id_{U(P)} \otimes Id_{R}) \circ (f + \epsilon \xi)} & (U(P) \otimes R)(Id_{U(P)} \otimes Id_{R})^*(U(B) \otimes R) \\
\gamma_A & \circ & \gamma_B \\
U(A) \otimes R & \xrightarrow{f + \epsilon \xi} & U(B) \otimes R
\end{array}
\]

But this is the very same diagram as in Equation (5.7), proving equivalency of the two definitions. Moreover, we notice that Equation (5.11) is just the special case \( f = Id_A \) of Equation (5.13). But, as discussed in Definition 5.3, a derivation w.r.t. identity morphism is nothing else than a ‘classical \( P \)-derivation’.

It remains to show that Equation (5.12) coincides with the definition of differential of a \( U(P) \)-algebra from Definition 2.6. \((d_P, \beta) \in \text{Der}(U(P), U(A))^{(1)}\) means we have the following commutative diagram

\[
\begin{array}{ccc}
(U(P) \otimes R)(U(A) \otimes R) & \xrightarrow{(Id_{U(P)} \otimes Id_{R}) \circ (Id + \epsilon \beta)} & (U(P) \otimes R)(Id_{U(P)} \otimes Id_{R})^*(U(A) \otimes R) \\
\gamma_A & \circ & \gamma_A \\
U(A) \otimes R & \xrightarrow{Id_A + \epsilon \beta} & U(A) \otimes R
\end{array}
\]

Commutativity of the non-\( \epsilon \)-part is equivalent to the identity map forming a \( U(P) \)-algebra morphism and hence is trivially satisfied.

When examining commutativity of the maps containing \( \epsilon \), we can restrict to those
with exactly one $\epsilon$ because of $\epsilon^2 = 0$, i.e. we find

$$((U(P) \otimes R)(U(A) \otimes R) \xrightarrow{(d \circ \text{Id}_{(A) \otimes R} + \text{Id}_{(P) \otimes R} \circ (\epsilon \beta))} (U(P) \otimes R)(U(A) \otimes R)),$$

(5.16)

$$U(A) \otimes R \xrightarrow{\epsilon \beta} U(A) \otimes R,$$

where the $(d \circ \text{Id}_{A \otimes R})$ term is due to $(d_{P \otimes R} + d_{P} \otimes \text{Id}_{R})^*$. But this coincides with the diagram from Equation (2.11) and as such implies equivalency of Equation (5.12) and Definition 2.6. □

All the claims of Section 5.6 mutatis mutandis also hold for their dual counterparts.

**Lemma 5.11** (Commutator with co-differential). Let $C$ be a co-operad with co-operadic co-differential $d_C$, $A$, $B$ two $C$-co-algebras with co-differentials $d_A$ and $d_B$, respectively. Moreover, let $f : U(A) \to U(B)$ be a $U(C)$-co-algebra morphism (i.e. compatible with the co-algebraic co-multiplication, but not necessarily with the respective co-differentials).

Then $d_B f - f d_A$ is a co-derivation with respect to $f$.

**Proof of Lemma 5.11** Let $R$ denote the Ring $R = \mathbb{K}[\epsilon]/\epsilon^2$ for $\epsilon$ a formal variable of degree 0. By assumption, $d_A$ is a differential of a $U(C)$-co-algebra and as such by means of Equation (5.12) $(d_C, d_A) \in \text{coDer}(U(C), U(A))$, which in turn yields $(\text{Id}_C + \epsilon d_C, \text{Id}_A + \epsilon d_A) \in \text{Mor}_{U(C)}((U(C), U(A)) \otimes R, (U(C), U(A)) \otimes R)$. Analogously, $(\text{Id}_C - \epsilon d_C, \text{Id}_B - \epsilon d_B)$ forms a $U(C)$-co-algebra morphism as well. We further notice that because $f$ is a $U(C)$-co-algebra morphism, also $(\text{Id}_C, f)$ forms a $U(C)$-co-algebra morphism.

Consequently, as a composition of $U(C)$-co-algebra morphisms

$$((\text{Id}_C + \epsilon d_C, \text{Id}_A + \epsilon d_A) \circ (\text{Id}_C, f) \circ (\text{Id}_C - \epsilon d_C, \text{Id}_B - \epsilon d_B)),$$

(5.17)

describes a $U(C)$-co-algebra morphism, too.

However, a short calculation yields

$$(\text{Id}_C + \epsilon d_C, \text{Id}_A + \epsilon d_A) \circ (\text{Id}_C, f) \circ (\text{Id}_C - \epsilon d_C, \text{Id}_B - \epsilon d_B)$$

$$= ((\text{Id}_C + \epsilon d_C) \circ (\text{Id}_C) \circ (\text{Id}_C - \epsilon d_C), (\text{Id}_A + \epsilon d_A) \circ (f) \circ (\text{Id}_B - \epsilon d_B))$$

$$= (\text{Id}_C, f + \epsilon(d_A \circ f - f \circ d_B))$$

$$= (\text{Id}_C, f + \epsilon(0, d_A f - f d_B)).$$

Recalling Equation (5.13) we know that in order for $(d_A f - f d_B)$ to be a co-derivation w.r.t. $f$ the expression $(\text{Id}_C, f + \epsilon(0, d_A f - f d_B))$ has to form a $U(C)$-co-algebra morphism. Though, as a composition of $U(C)$-co-algebra morphism, this certainly is the case. □

5.7. Deformation Complex. Deformation complex is an often used expression to denote an $L_\infty$-algebra whose Maurer-Cartan elements form some structures or denote morphisms of a certain kind (e.g. see [9], Sect.12.2 and [1], Sect.2.1, where the authors call it convolution $\mathcal{G}L_\infty$-algebra).

For the scope of this paper we follow [1] and use the notion deformation complex to describe the $\mathcal{G}L_\infty$-algebra whose Maurer-Cartan elements are $\infty$-morphisms (see Proposition 5.13).
Therefore, the remaining step is to investigate $F_B\cdot A$ endowed with a descending, bounded above and complete filtration compatible with the $\Def$.

Then the Maurer-Cartan elements of the deformation complex $\gr Vect(P^{\infty}(U(A)), U(B))$ with $\gr\Hom(\gr Vect(P^{\infty}(U(A)), U(B)))$ are defined as vector space morphisms $P^{\infty}(U(A)) \to U(P^{\infty}(U(A)))$, where $\gr Vect(U(P^{\infty}(U(A))))$ (as described in [11], Equation (3.64)), $d_B$ the internal differential of $B$ (seen as dgVect) and $\psi_B$ is the $P^{\infty}$-algebra structure describing graded vector space homomorphism $\psi_B : F^{\infty}_U(U(B)) \to U(B)$.

Moreover, $d_B(A)$ denotes the differential on $F^{\infty}_U(U(B))$ emerging from the relative bar construction of $A$ (see Equation (5.20)). We denote this $\mathcal{S}L^{\infty}$-algebra by $\Def(A, B)$ and call it deformation complex.

Next, we show that Maurer-Cartan elements of the deformation complex are in 1-to-1 correspondence with $\infty$-morphisms.

**Proposition 5.13** ($\infty$-morphisms are Maurer-Cartan elements of the deformation complex). Let $P$ be a possibly inhomogeneous Koszul operad and $A, B$ be two $P^{\infty}$-algebras. Moreover, we assume the deformation complex $\Def(A, B)$ to be endowed with a descending, bounded above and complete filtration compatible with the $\mathcal{S}L^{\infty}$-algebra structure.

Then the Maurer-Cartan elements of the deformation complex $\Def(A, B)$ correspond to $\infty$-morphisms between the two $P^{\infty}$-algebras $A$ and $B$.

**Proof of Proposition 5.13.** First, we recall that $\infty$-morphisms $A \hookrightarrow B$ are defined as vector space morphisms $F^{\infty}_U(U(A)) \to F^{\infty}_U(U(B))$ that are compatible with the $U(P^{\infty})$-co-algebra co-multiplications and commute with the respective co-differentials emerging from the relative bar-constructions.

For a given map $f \in \gr\Hom(\gr\Vect(F^{\infty}_U(U(A)), U(B)))$, due to co-freeness, $F := U(P^{\infty})(f)\Delta$ describes a $U(P^{\infty})$-co-algebra morphism, that is a linear map $F : F^{\infty}_U(U(A)) \to F^{\infty}_U(U(B))$ compatible with the $U(P^{\infty})$-co-algebra co-multiplication.

Hence it comes down to proving that $F$ does also commute with the co-differentials $d_B(A), d_B(B)$ on $F^{\infty}_U(U(A))$ and $F^{\infty}_U(U(B))$ provided $f$ is a Maurer-Cartan element.

From Lemma 5.11 we know that the commutator of $F$ with the respective co-differentials forms a co-derivation w.r.t. $F$. As discussed in Definition 5.4 a co-derivation w.r.t. $F$ is a $(U(P^{\infty}) \otimes R)$-co-algebra morphism: $F^{\infty}_U(U(P^{\infty}) \otimes R)(U(A) \otimes R) \to F^{\infty}_U(U(P^{\infty}) \otimes R)(U(B) \otimes R)$. From the target being co-free such a co-algebra morphism, and consequently also the co-derivation, is completely described by its composition with the projection to $B$. Moreover, this means that $F$ commutes with the respective co-differentials if the projection to $U(B) \otimes R$ of the commutator vanishes.

Therefore, the remaining step is to investigate $\pi_B(d_B(A)F -Fd_B(A))$. Expanding...

---

13See [11], Appendix A for a proof why this forms a $\mathcal{S}L^{\infty}$-algebra, indeed.

14By means of [7], Prop.2.15, a $P^{\infty}$-algebra structure on a dg vector space $B$ may equivalently be described by a co-differential-capable $P^{\infty}$-co-derivation on $F^{\infty}_P(B)$ and hence can be interpreted (see Definition 5.10) as a $U(P^{\infty}) \otimes R$-co-algebra morphism $(U(P^{\infty}) \otimes R)(U(A) \otimes R) \to (U(P^{\infty}) \otimes R)(U(B) \otimes R)$ for $R = K[\epsilon]/\epsilon^2$, where $\epsilon$ is a formal variable in degree $-1$. But because of co-freeness, this may also be described by the graded vector space homomorphism $\psi_B : F^{\infty}_U(U(B)) \to U(B)$.

15The requirement of such a filtration is mainly to ensure convergence of the Maurer-Cartan equation.
$F$ in terms of $f$ and explicitly writing out $d_{B,(A)}$ (cf. Equations (5.26)–(5.28)) shows that $\pi_B(d_{B,(A)} F - F d_{B,(A)}) = 0$ and hence $f$ corresponding to an $\infty$-morphism, is tantamount to $f$ satisfying the Maurer-Cartan equation on the deformation complex.

\[\square\]

**Lemma 5.14** (Weight induced filtration deformation complex $\text{Def}(A,B)$). Let $P$ be a possibly inhomogeneous Koszul operad generated in arities 1, 2, that is to say the generating set of $P$ (by definition a Koszul operad is a quotient operad of the free-operad $F(E)$ for some $\mathbb{S}$-module $E$, which we call generating set) is non-zero in arities 1 and 2 only. Further, let $A,B$ be two $P_\infty$-algebras.

Recall Definition 5.3 where we introduced weight filtrations on $P^i$, $P^r_{U(P)}(A)$ and $\text{Hom}_{P\text{-alg}}(P^c_{U(P)}(U(A)), U(B))$, respectively. We say an element $f \in \text{Def}(A,B)$ is of filtration degree $p$ if it does not contain non-trivial components in weights strictly smaller than $p - 1$, i.e. we set

\[
\mathfrak{F}_p \text{Def}(A,B) := \{ f \in \text{Def}(A,B) : f|_{U(P) \langle 1 \rangle (n) \otimes_{\mathbb{S}} U(A) \langle 1 \rangle n} = 0 \ \forall k < p - 1, \forall n \in \mathbb{N}_0 \}.
\]

This endows $\text{Def}(A,B)$ with a filtration

\[
\text{Def}(A,B) = \mathfrak{F}_1 \text{Def}(A,B) \supseteq \mathfrak{F}_2 \text{Def}(A,B) \supseteq \ldots
\]

that is descending, bounded above, complete and compatible with the $\mathfrak{S}L_\infty$-algebra structure on the deformation complex.

**Proof of Lemma 5.14** By restricting to generating sets of arities 1, 2, we can ensure the arity $k$ part of $U(P^i)$ to carry at least weight $k - 1$ (using $P^i$ is as a sub-co-operad of the co-free co-operad $F(s^{-1}E)$).

Notice, that the co-operadic co-multiplication of the co-free co-operad conserves the total (i.e. the sum amongst all the factors in the tensor product) weight and so does the co-free co-algebra co-multiplication on $P^c_{U(P)}(U(A))$, cf. [H], Equation (3.64).

Let $f_1, \ldots, f_i$ be such that $f_s, \in \mathfrak{F}_p \text{Def}(A,B)$ for $s = 1, \ldots, i$. Next, we analyse what weight $v \in P^c_{U(P)}(U(A))$ at least must carry for $\{f_1, \ldots, f_i\}(v)$ not to trivially vanish.

Recall that $v \in P^c_{U(P)}(U(A))$ in general is a combination of elements of the form

\[v_{l_1} \cdots v_{l_m} = U(P^i)(t)(k) \otimes_{\mathbb{S}} U(A)^{\otimes k}, \]

where $t \geq 1$ and $l_1 + \ldots + l_m = k$. Then, by making use of the associator (monoidal category), the $U(P^i)(l_s)$ are gathered with the $U(A)^{\otimes l_s}$.

As explained above, because the generating set of $P$ is concentrated in arities 1, 2, as an arity $i$ element $U(P^i)(i)$ must carry weight $i - 1$ or bigger.

Moreover, each of the $U(P^i)(l_s)$ is required to carry weight $\geq p_s - 1$ as otherwise application of $f_x$ on $U(P^i)(l_s) \otimes_{\mathbb{S}} U(A)^{\otimes l_s}$ would trivially vanish because of $f_x \in \mathfrak{F}_p \text{Def}(A,B)$.

Due to the aforementioned conservation of the total weight by the co-free co-algebra co-multiplication, we can proceed by simply adding the minimal requirements for the individual terms we have just found:

\[t \geq (i - 1) + (p_1 - 1) + \ldots + (p_i - 1) = (p_1 + \ldots + p_i) - 1.\]
In other words, $v_{t,k}$ must at least have weight $(p_1 + \ldots + p_i) - 1$, otherwise it would trivially vanish under $\{f_1, \ldots, f_i\}$. But according to Equation (5.21) this means $\{f_1, \ldots, f_i\} \in \mathfrak{g}(p_1 + \ldots + p_i)\text{Def}(A, B)$, hence compatibility of the filtration with the $\mathfrak{S}L_\infty$-algebra structure holds.

Eventually, $\mathfrak{g}_1\text{Def}(A, B) = \text{Def}(A, B)$ follows directly from Equation (5.21), thus the filtration is guaranteed to be bounded above. □

Notice that the filtration used in Lemma 5.14 differs from the one of [1], Sect. 2.1.1, in the sense that we focus on the operadic weight rather than the arity (for binary generated operads in fact both definitions coincide). The reason for doing so is that if we have a $P_\infty$-algebra structure on $A$, described by $\rho_\infty \in \text{Mor}\mathfrak{S}(U(P^i), \text{End}_{U(A)})$, that does extend a $P$-algebra structure $\rho \in \text{Mor}\mathfrak{S}(U(P^i), \text{End}_{U(A)})$, they do differ in weight $\geq 2$ only. Hence, using the weight for filtration is a natural choice, in particular as by this we can get rid of $\rho_\infty$-exclusive terms when going to higher pages of the spectral sequence.

Now we have a 'good' filtration at hand, hence both the Maurer-Cartan equation and the twisting-procedure are well-defined.

For the scope of this section we will use the following special notation to denote the twisted deformation complex.

**Definition 5.15 (Twisted Deformation Complex).** Let $P$ be a possibly inhomogeneous Koszul operad generated in arities 1, 2 and let $A, B$ be two $P_\infty$-algebras.

Notice, that a graded vector space homomorphism $f: U(A) \to U(B)$ corresponds to an element in $F \in \text{Def}(A, B)$, namely the graded vector space homomorphism

(5.22)

$$F: \bigoplus_{n \in \mathbb{N}_0} U(P^i)(n) \otimes \mathfrak{S}_n U(A)^{\otimes n} \to U(P^i)(1) \otimes U(A) \xrightarrow{\epsilon \circ \text{id}_{U(A)}} K \otimes U(A) \cong U(A) \xrightarrow{f} U(B),$$

where $\epsilon$ denotes the co-operadic co-unit of $P^i$.

Hence, for a degree zero graded vector space homomorphism $f: U(A) \to U(B)$ it makes sense to twist the deformation complex $\text{Def}(A, B)$ by the corresponding $F \in \text{Def}(A, B)$.

The resulting curved $\mathfrak{S}L_\infty$-algebra we denote by

(5.23)

$$\text{Def}(A \xrightarrow{f} B) := \text{Def}(A, B)^F.$$

Notice, that this is not the most general notation of a twisted deformation complex since, according to Definition 5.14 we could twist by any degree zero element $F \in \text{Def}(A, B)$ rather than those induced by degree zero graded vector space homomorphisms $f: U(A) \to U(B)$. However, we motivate this more specific notation by the fact that in Theorem 1.2 we start with a degree zero graded vector space morphism $f: A \to B$ that is a quasi-isomorphism of dg vector spaces, but whose induced co-free $U(P^i)$-co-algebra morphism fails to commute with the co-differentials from the relative bar constructions and twist the therefore non-flat twisted deformation complex $\text{Def}(A \xrightarrow{f} B)$ by an element $\alpha \in \mathfrak{g}_2\text{Def}(A \xrightarrow{f} B)$ resulting in a flat $\mathfrak{S}L_\infty$-algebra. As explained in a remark in Theorem 1.3 this is equivalent to $F + \alpha$ forming a Maurer-Cartan element. From $\alpha$ being of filtration degree 2 the twist by $\alpha$ does not change the behaviour on weight 0. But since the weight 0 part is the one responsible for the property of the $\infty$-morphism being an $\infty$-(quasi)-isomorphism, the final element behaves in this regard according to $f$. By explicitly writing $\text{Def}(A \xrightarrow{f} B)$, we highlight this aspect.
5.8. A sufficient condition for intrinsic Formality. With all these definitions and formalism at hand, we are finally ready to work on giving a sufficient condition for intrinsic formality of $P_\infty$-algebras. To this end, we will proceed with some lemmata that will later be used in the proof of Theorem 1.2.

**Lemma 5.16** (Intrinsic Formality in Maurer-Cartan-language). Let $A$ be a $P_\infty$-algebra for some possibly inhomogeneous Koszul operad $P$ that is generated in arities 1 and 2. Then $A$ is intrinsically formal.

\[(5.24)\]
\[\text{MC}(\mathfrak{g}_2(\text{Def}(H(B)_{\text{HTT}} \xrightarrow{\text{id}} H(B)))) \neq \emptyset,\]

for all $P_\infty$-algebras $B$ satisfying $H(B)_{P_{\text{-alg}}-\text{iso}} H(A)$.

**Proof of Lemma 5.16.** We start by observing that $\alpha \in \text{MC}(\text{Def}(H(B)_{\text{HTT}} \xrightarrow{\text{id}} H(B)))$ is equivalent to $\alpha + \text{Id}$ being a Maurer-Cartan element of $\text{Def}(H(B)_{\text{HTT}}, H(B))$ and by Proposition 5.13 $\alpha + \text{Id}$ forming an $\infty$-morphism $\alpha + \text{Id} : H(B)_{\text{HTT}} \to H(B)$. Moreover, if $\alpha$ is subject to $\alpha \in \mathfrak{g}_2(\text{Def}(H(B)_{\text{HTT}}, H(B)))$ (the underlying vector space and the filtration do not change under twisting), $\alpha$ vanishes in weight zero. This leads to the conclusion that only the Id-part is responsible for the weight zero part of the just found $\infty$-morphism.

Since, according to Definition 5.6 the property of being an $\infty$-(quasi-)isomorphism does only involve the weight zero part, which here is $\alpha : H(B)_{\text{HTT}} \to H(A)$, $\alpha + \text{Id}$ forms an $\infty$-isomorphism, indeed.

Composing this $\infty$-isomorphism with the $\infty$-quasi-isomorphism (by the homotopy transfer theorem, see [6], Theorem 49) $p : B \xrightarrow{\sim} \text{ quasi-iso.} H(B)$ results in an $\infty$-quasi-isomorphism $B \xrightarrow{\sim} \text{ quasi-iso.} H(B)$. \hfill $\square$

**Lemma 5.17.** Let $B$ be a $P_\infty$-algebra for a possibly inhomogeneous Koszul operad $P$ that is generated in arities 1 and 2. Then, there is an isomorphism of graded vector spaces

\[(5.25)\]
\[E_1(\text{Def}(H(B)_{\text{HTT}} \xrightarrow{\text{id}} H(B)))_{\text{grVec}} \cong \text{Def}(H(B)_{\text{id}} \xrightarrow{\text{id}} H(B)),\]

where $E_1$ denotes the first page of the spectral sequence\[^{16}\].

**Proof of Lemma 5.17.** First, we demonstrate that the 1-bracket $\{\}_1^\text{id}$ on the deformation complex raises the degree of filtration by one. To this end let us take $f \in \mathfrak{g}_p \text{Def}(H(B)_{\text{HTT}} \xrightarrow{\text{id}} H(B))$. We examine which weight $v \in \mathcal{F}_{\text{U}(P)}(H(B)_{\text{HTT}})$ must carry to not trivially vanish under $\{f\}_1^\text{id}$.

We notice that all terms originating from the twist by Id can be ignored, since these terms do all involve at least one Id term that satisfies $\text{Id} \in \mathfrak{g}_1 \text{Def}(H(B)_{\text{HTT}}, H(B))$ and hence, due to compatibility of the filtration with the $\mathfrak{g}L_\infty$-algebra brackets, guarantees the expressions of the form $\{\text{Id}, \ldots, \text{Id}, f\}$ to lie in $\mathfrak{g}_{p+1} \text{Def}(H(B)_{\text{HTT}} \xrightarrow{\text{id}} H(B))$.

Let $v$ be of weight $k$ and of the form $P^{i(k)}(m) \otimes_{\mathbb{Z}_m} H(B)_{\text{HTT}} \otimes^m$. We continue by analysing the threshold on $k$ for producing a non-vanishing contribution on each of the three terms of $\{f\}_1^\text{id}(v) = d_{H(B)} f(v) + (-1)^{|f|} f(d_{B_{\text{H}(B)_{\text{HTT}}}})(\Delta_1(v)) + \psi_{H(B)}(Id_{\text{U}(P)} \otimes f)(\Delta_1(v))$.

\[^{16}\]Here, we used the notion from Definition 5.6 to emphasise which $P_{(\infty)}$-algebra structure on the dg vector $H(B)$ we consider. By $\mathfrak{g}_p$ we denote the filtration introduced in Lemma 5.13.

\[^{17}\]In Lemma 5.19 we prove the curvature $\mu_{0, \text{ twisted}}$ of the twisted deformation complex on the l.h.s. of Equation (5.22) to satisfy $\mu_{0, \text{ twisted}} \in \mathfrak{g}_1 \text{Def}(H(B)_{\text{HTT}} \xrightarrow{\text{id}} H(B))$, hence we can, as explained in Section 2.2, apply the construction of a spectral sequence on a filtered chain complex up to page 2. Moreover, this ensures the differential on the first page to be a proper differential.
individually.

Due to \( d_{H[B]} = 0 \), the term \( d_{H[B]} f \) vanishes.

For the third term, \( \psi H[B](1 \otimes f)(\Delta_1(v)) \), we know that
\begin{equation}
\Delta_1(v) \in \sum_{k_1, k_2} (P)^{(k_1)} \otimes (P)^{(k_2)} \otimes H(B)_{HTT} \otimes m.
\end{equation}

But the \( P_{\infty} \)-algebra structure describing graded \( \mathbb{S} \)-module morphism \( \rho H(B) : U(P) \to \text{End}(H(B)) \) vanishes in weight 0 (see \( [9] \), Sect. 10.1.8) and so does \( \psi H[B] \in \text{Hom}_{\text{grVect}}(\mathbb{F}_U(P) (H(B)), H(B)) \), due to the internal hom adjunction. Therefore, \( \psi H[B](1 \otimes f) \) applied to \( \Delta_1(v) \) trivially vanishes unless \( k_1 \geq 1 \). Furthermore, because of \( f \in \mathfrak{F}_p \text{Def}(H(B)_{HTT} \xrightarrow{id} H(B)) \), the expression does also vanish for \( k_2 < p - 1 \). As a direct consequence of these two observations \( k \geq p \) is a necessity for \( \psi H[B](1 \otimes f)(\Delta_1(v)) \) not being trivial and therefore we have \( \psi H[B](1 \otimes f) \Delta_1 \in \mathfrak{F}_{p+1} \text{Def}(H(B)_{HTT} \xrightarrow{id} H(B)) \).

Eventually, there remains the term \( f(d_{B_i}(H(B)_{HTT})) v \), where (cf. \( [9] \), Sect.11.2.2)
\begin{equation}
d_{B_i}(H(B)_{HTT}) = d_1 + d_2
\end{equation}
with
\begin{equation}
d_1 = d_{P^1} \circ \text{Id}_{H(B)_{HTT}} + \sum_{i=0} d_{P^1} \circ' \text{Id}_{H(B)_{HTT}}
\end{equation}
and \( d_2 \) is the differential-capable \( P^1 \)-co-derivation induced by the \( P_{\infty} \)-structure describing homomorphism \( \psi H(B)_{HTT} : \mathbb{F}_U(P) (H(B)_{HTT}) \to H(B)_{HTT} \), that is (see \( [11] \), Vol.II, Prop.3.83)
\begin{equation}
d_2 = \sum_{n \in \mathbb{Z}_{\geq 1}} \left( \sum_{j=0}^{n-1} \text{Id}_{U(P^1)} \otimes (\pi H(B)_{HTT} \otimes \psi H(B)_{HTT} \otimes \pi H(B)_{HTT} \otimes \pi H(B)_{HTT} \otimes \pi H(B)_{HTT} \otimes n-j-1) \right) \Delta_n,
\end{equation}
where \( \pi H(B)_{HTT} : \mathbb{F}_U(P) (H(B)_{HTT}) \to H(B)_{HTT} \) denotes the projection described in the introduction of Section \( [\] \). Due to \( d_{H(B)_{HTT}} = 0 \), the \( \text{Id}_{U(P^1)} \) \( \circ' \text{Id}_{H(B)_{HTT}} \) term vanishes. Moreover, \( d_1 \) reduces the weight by one since the co-operadic co-differential of a Koszul dual co-operad lowers the weight by one (see \( [9] \), Sect.C.1.). Further, as \( \psi H(B)_{HTT} \) is non-vanishing in weights \( \geq 1 \) only and since the co-algebra co-multiplication \( \Delta \) leaves the weight unaltered, it follows that also \( d_2 \) does lower the weight by one.

Therefore, we find \( f(d_{B_i}(H(B)_{HTT})) \in \mathfrak{F}_{p+1} \text{Def}(H(B)_{HTT} \xrightarrow{id} (H(B))) \) and the 1-bracket raises the degree of filtration by one, indeed.

A direct consequence is that as graded vector spaces (but not as dg vector spaces) the zeroth and the first page of the spectral sequence of \( \text{Def}(H(B)_{HTT} \xrightarrow{id} H(B)) \), \{\( \cdot \)\}_1 \) are isomorphic.

Furthermore, by definition of the zeroth page of the spectral sequence (see Equation \( (23) \)), an element \( g \in E_0^{p,q}(\text{Def}(H(B)_{HTT} \xrightarrow{id} H(B))) \) can be represented by \( g : \mathbb{F}_U(P) (H(B)_{HTT}) \to H(B) \) that is only non-zero in weight \( p-1 \). Notice, that as filtered (with the filtration from Lemma \( (5.13) \)) graded vector spaces \( \text{Def}(H(B)_{HTT} \xrightarrow{id} H(B)) \) and \( \text{Def}(H(B) \xrightarrow{id} H(B)) \) do not differ and the same also holds for the respective zeroth pages of the spec.seq.

Consequently, among others using
\begin{equation}
\prod_{p \geq 0} \mathfrak{F}_p \mathfrak{g}/\mathfrak{F}_{p+1} \mathfrak{g}_{\text{grVect}} \mathfrak{g},
\end{equation}
\textcopyright 25
we find
\begin{equation}
\Phi : \prod_{p \geq 0} E^{p,q}_1(\Def(H(B))_{\text{HTT}} \xrightarrow{id} H(B))_{\text{grVect}} \prod_{p \geq 0} E^{p,q}_0(\Def(H(B))_{\text{HTT}} \xrightarrow{id} H(B))_{\text{grVect}} \Def(H(B) \xrightarrow{id} H(B)) : \Phi^{-1},
\end{equation}
where the graded vector space isomorphism $\Phi$ from left to right is mapping a representative $f \in \Def(H(B))_{\text{HTT}} \xrightarrow{id} H(B))$ to its part that is non-vanishing in weight $p - 2$ only and in the other direction we have that $\Phi^{-1}$ is decomposing $\Def(H(B)) \xrightarrow{id} H(B))$ into its weight components (see Definition 5.2) and then take the weight $p - 2$, degree $q$ part and map it to $E^{p,q}_1(\Def(H(B))_{\text{HTT}} \xrightarrow{id} H(B))$ by means of the quotient map (since the $\{\}$-id does raise the degree of filtration by at least one, this is well-defined).

\textbf{Lemma 5.18.} The graded vector space isomorphism of Equation \eqref{Equation:GradedVectorSpaceEquation} is an isomorphism of filtered dg vector spaces.

\textbf{Proof of Lemma 5.18.} In order for $\Phi$ to form an isomorphism of dg vector spaces we need
\begin{equation}
\Phi(d_E(\Def(H(B))_{\text{HTT}} \xrightarrow{id} H(B)) \{()\}) = \{\Phi(\{()\})\}_{\text{Def}(H(B)) \xrightarrow{id} H(B)}^{-1},
\end{equation}
or equivalently
\begin{equation}
d_E(\Def(H(B))_{\text{HTT}} \xrightarrow{id} H(B)) \{()\} = \Phi^{-1}\{\Phi(\{()\})\}_{\text{Def}(H(B)) \xrightarrow{id} H(B)}^{-1}
\end{equation}
for $\Phi$ and $\Phi^{-1}$ the isomorphisms from Equation \eqref{Equation:GradedVectorSpaceEquation}. Let us emphasise that on the left-hand side of Equation \eqref{Equation:DecompositionEquation} the differential is on the first page, i.e. it is given by $\{\}$-id seen as a map $E^{p,q}_1 \rightarrow E^{p+1,q}_1$, so in particular there is a shift in the filtration degree of the domain.

Let $f : \text{End}_{U(P)}(H(B))_{\text{HTT}} \rightarrow H(B)$ be a graded vector space homomorphism such that it vanishes in all weights except $p - 1$. We now go through all the terms in $\{\} \_{\text{Def}(H(B)) \xrightarrow{id} H(B)}$ and investigate which of them survive after we apply the quotient map to it. Moreover, we also analyse where the result differs from its counterpart $\{\Phi(f)\}_{\text{Def}(H(B)) \xrightarrow{id} H(B)}$ where there is no quotient map involved.

We first deal with the terms, emerging from the twisting by $\text{Id}$. In general, this twisting leads to additional terms of the form $\{\text{Id}, \ldots, \text{Id}, f\}_{\geq 2}$. Recall that the higher brackets, as defined in Equation \eqref{Equation:HigherBracketDefinition}, include the graded vector space homomorphism $\psi_H(B) : \text{End}_{U(P)}(H(B)) \rightarrow H(B)$ that describes the $P_{\infty}$-algebra structure. However, $H(B)$ merely being a $P$-algebra, $\psi_H(B)$ is non-zero in weight 1 only and since the generating set is concentrated in arities 1, 2, this implies the bracket of highest arity to be the 2-bracket. Therefore, the only additional contribution due to twisting is $\{\text{Id}, f\}_{\geq 2}$ and this is non-vanishing in weight $p$ only and as such the term $\{\text{Id}, f\}_{\geq 2}$ remains unchanged when passing to the quotient space.

Nonetheless, the very same term $\{\text{Id}, f\}_{\geq 2}$ also appears on the right-hand side of Equation \eqref{Equation:DecompositionEquation} as the only contribution due to the twisting on the right-hand side. We continue the analysis of the composition of $f$ with the differentials by writing them down explicitly using Equation \eqref{Equation:DecompositionEquation}.

For $d_{H(B)}f$ there is nothing to do, as this vanishes from $d_{H(B)}$ being zero, anyway. The next term we have to consider is $\psi_{H(B)}(\text{Id}_P \otimes f)(\Delta_1(v))$. This term does appear on both sides of Equation \eqref{Equation:DecompositionEquation}, however we still have to make sure that we do not lose any terms due to passing to the quotient space on the left-hand side.

From $H(B)$ merely carrying a $P$-algebra structure, its $P_{\infty}$-algebra structure characterising map $\rho_H(B) : P \rightarrow \text{End}_{H(A)}$ and $\psi_{H(B)} \in \text{Hom}_{\text{grVect}}(\text{End}_{U(P)}(H(B)), H(B))$
by the internal hom-adjunction, respectively, are concentrated in weight 1. Hence, \( \psi_{H(B)}(1 \otimes f)\Delta_1 \) is potentially non-zero only for \( v \in F^c_{U(P)}(H(B)) \) being of weight \( p \), as this is the only case in which \( \Delta_1 \) can allocate a weight 1 term in the first and a weight \( p - 1 \) term in the second co-operadic factor. Consequently \( \psi_{H(B)}(1 \otimes f)\Delta_1 \) cannot be cancelled out by a term in \( \mathfrak{F}_p \text{Def}(H(B)) \) as such a term by definition must vanish on all weights less than or equal to \( p \).

By the same arguments that showed \( \{\} \) to raise the degree of filtration by one it follows that for \( f \in \mathfrak{F}_p \text{Def}(H(B)) \) being concentrated in weight \( p - 1 \), the expression \( f \Delta_1 \) is possibly non zero only on weight \( p \). Hence, the contribution due to \( \Delta_1 \) remains unchanged when passing to the quotient space on the left-hand side of Equation (6.30). For the right-hand side, there is no quotient space and \( H(B) \) instead of \( H(B) \) the exact same terms emerge from \( f \Delta_1 \).

We proceed by analysing \( f \). By virtue of the homotopy transfer theorem, the \( P_\infty \)-algebra structure on \( H(B) \) extends the \( P \)-algebra structure on \( H(B) \), that is, to say on \( P^{(1)} \) the \( P_\infty \)-algebra-structure describing morphisms \( \phi_{H(B)} \in \text{Mod}(U(P)), \text{End}_U(H(B)) \) and \( \phi_{H(B)} \in \text{Mod}(U(P)), \text{End}_U(H(B)) \) coincide. By means of the internal hom-adjunction this also implies \( \psi_{H(B)} \in \text{Hom}_{grV ect}(\mathfrak{F}_c^{U(P)}(H(B)), H(B)) \) and \( \psi_{H(B)} \in \text{Hom}_{grV ect}(\mathfrak{F}_c^{U(P)}(H(B)), H(B)) \)

We write \( H(B) \) and \( H(B) \) for clarity even though as dg vector spaces they coincide) to be equal in weight 1.

After noticing that, according to Equation (6.29), the term \( \psi_{H(B)} \) appears in \( f \) exactly once, we continue by decomposing \( \psi_{H(B)} \) in its weight 1 part \( \psi_{H(B)}^{(1)} \), which is just \( \psi_{H(B)} \), and its weight \( \geq 2 \) part that we will denote by \( \psi_{H(B)}^{(2)} \) and call their \( d_2 \) contributions \( d_2^{(1)} \) and \( d_2^{(2)} \), respectively. A short investigation shows that \( f \Delta_1 \) is non-zero in weight \( p \) only and as such does not get lost when passing to the quotient space. On the other hand the appearance of \( \psi_{H(B)}^{(2)} \) in \( f \) requires at least an additional weight 2, making it a total of at least \( p + 1 \). Hence \( f \) becomes zero under the quotient map.

But this is in line with the right-hand side of Equation (5.30) and consequently validates Equation (5.30).

By carefully going through all the previous steps one can verify that \( \Phi \) even is an isomorphism of filtered chain complexes.

\[ \mu_{0,\text{twisted}} \in \mathfrak{F}_3 \text{Def}(H(B)) \]

**Lemma 5.19 (Filtration of Curvature).** Let \( B \) be a \( P_\infty \)-algebra for a possibly inhomogeneous Koszul operad \( P \) that is generated in arities 1 and 2. Then, the curvature \( \mu_{0,\text{twisted}} \) of the twisted deformation complex \( \text{Def}(H(B))(H(B)) \) satisfies

\[ \mu_{0,\text{twisted}} \in \mathfrak{F}_3 \text{Def}(H(B))(H(B)). \]

**Proof of Lemma 5.19.** By unravelling the definitions of the twisted deformation complex and the filtration thereon, we see that for \( \mu_{0,\text{twisted}} \) to satisfy the requirement of Equation (5.31) is tantamount to the graded vector space homomorphism \( F^c_{U(P)}(H(B)) \to H(B) \), given by \( M(\text{Id}) \), to vanish in weights 1 and 2.

To this end, we notice that \( \text{Id} \) induces an \( \infty \)-morphism \( \text{Id} : H(B) \to H(B) \) for \( H(B) \) the \( P \)-algebra from the first part of Definition 5.5 but seen as a \( P_\infty \)-algebra, and as such constitutes for a Maurer-Cartan element of \( \text{Def}(H(B) \to H(B)) \). Consequently the curvature \( \mu_{0,\text{twisted}} \) (we added an extra \( \sim \) to our notation to distinguish
it from the curvature of \( \text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B)) \) of the twisted deformation complex \( \text{Def}(H(B) \xrightarrow{id} H(B)) \) is zero.

As previously discussed in Lemmata 5.17 and 5.18, \( \text{Def}(H(B), H(B)) \) and \( \text{Def}(H(B)_{\text{HTT}}, H(B)) \) along with the twisted counterparts, respectively, do only differ by the latter having some additional terms in its brackets due to the occurrence of a \( P_{\infty} \)-algebra structure rather than a \( P \)-algebra structure. This remains valid also for the curvatures \( \mu_{0, \text{twisted}} \) and \( \tilde{\mu}_{0, \text{twisted}} \) of the twisted deformation complexes. That being said, due to \( \tilde{\mu}_{0, \text{twisted}} \) being zero, we may neglect all the terms of \( \mu_{0, \text{twisted}} \) not involving weight \( \geq 2 \) \( P_{\infty} \)-algebra (and hence \( H(B)_{\text{HTT}} \) specific) terms.

The remaining terms are \( \text{Id} \ d_2^{(\geq 3)} \), i.e. those from \( d_2 \) that do involve \( \psi_{H(B)_{\text{HTT}}} : \mathcal{F}_{U(M)}(H(B))_{\text{HTT}} \rightarrow H(B) \) in weight \( \geq 2 \) only and therefore clearly need weight \( \geq 2 \) to attribute a non-zero contribution with.

This, however, directly validates the statement \( \mu_{0, \text{twisted}} \in \mathfrak{F}_3 \text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B)) \).

**Theorem 5.20.** Let \( A \) be a \( P_{\infty} \)-algebra for a possibly inhomogeneous Koszul operad \( P \) that is generated in arities 1 and 2.

If the twisted deformation complex \( \text{Def}(H(A) \xrightarrow{id} H(A)) \) is acyclic in total degree 1 (i.e. \( H^p(\mathfrak{F}_3 \text{Def}(H(A)) \xrightarrow{id} H(A)) = 0 \) for all \( p,q \) with \( p + q = 1 \)), then \( A \) is intrinsically formal as a \( P_{\infty} \)-algebra.

**Proof of Theorem 5.20.** By means of Lemma 5.16 it suffices to show that all \( P_{\infty} \)-algebras \( B \) with co-homologies isomorphic to \( H(A) \) are subject to \( \text{MC}(\mathfrak{F}_2 \text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B)))) \neq 0 \).

Because of the co-homologies being isomorphic, also

\[
\text{Def}(H(B) \xrightarrow{id} H(B))_{\text{filtered}} \xrightarrow{\sim} \text{Def}(H(A) \xrightarrow{id} H(A))
\]

has to hold.

Moreover, we can apply Lemma 5.18 on \( B \) and, after invoking Equation (5.32), find

\[
E_1(\text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B)))_{\text{filtered}} \xrightarrow{\sim} \text{Def}(H(A) \xrightarrow{id} H(A)).
\]

Taking co-homologies and using both, the initial assumption of acyclicity \( \text{Def}(H(A) \xrightarrow{id} H(A)) \) in total degree 1 and the fact that the second page of a spectral sequence is isomorphic to the co-homology of the first, we obtain

\[
E_2^{p,q}(\text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B))) = 0
\]

for all \( p,q \) with \( p + q = 1 \).

Because we also know from Lemma 5.19 that the filtration degree of the curvature of \( \text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B)) \) is at least 3, this puts us in a situation in which all assumptions of Theorem 4.3 (for \( r = 1 \)) are satisfied. Consequently, a Maurer-Cartan element \( \alpha \in \text{MC}(\mathfrak{F}_2 \text{Def}(H(B)_{\text{HTT}} \xrightarrow{id} H(B))) \) must exist. \( \square \)

**Appendix A. Proof of Equivalence of \((\mathfrak{S})L_{\infty}\)-algebra Equations**

**Lemma A.1** (Equivalence of curved \((\mathfrak{S})L_{\infty}\)-algebra Equations 1). Let \( g \) be a curved \( \mathfrak{S}L_{\infty} \)-algebra endowed with a filtration that is descending, bounded above, complete and compatible with the \( \mathfrak{S}L_{\infty} \)-algebra structure. Then the curved \( \mathfrak{S}L_{\infty} \)-algebra Equation (2.7)\(^{18}\) can equivalently be encoded in the form of Equation (2.3).

\(^{18}\)To be precise, Equation (2.3) describes the curved \( L_{\infty} \)-algebra equation. But as explained in Definition (2.7) curved \( \mathfrak{S}L_{\infty} \)-algebra brackets do share the very same condition but with the brackets being maps \( \mu_n : S^n(g) \rightarrow g \) instead of \( \mu_n : S^n(g[-1]) \rightarrow g[-1] \).
Proof of Lemma A.1. The implication (3.2) ⇒ (2.1) follows directly from graded polariation (see Section 3).

For the other direction we first recall from Equation (3.1) that we can extend the $GL_{\infty}$-algebra structure to $\mathfrak{g}\otimes R$ for $R$ being a nilpotent graded ring. We may rewrite Equation (2.1) for some fixed $n$ in the case of all $x_i$ being set to some arbitrary $x \in (\mathfrak{g}\otimes R)^0$ and get

$$\sum_{j,k=0}^{n} j+k=n \frac{1}{j!k!} \mu_{j+1}(\mu_k(x, \ldots, x), x, \ldots, x) = 0,$$

since $x$ is of degree 0 and therefore we can reorder without picking up additional signs. 

Summing over $n$, adding an additional factor $\frac{1}{n!}$ (for each fixed $n$ the summand vanishes separately) and using $\binom{n+k}{k} = \frac{n^n}{k!n!}$ leads to

$$\sum_{n \geq 0} \sum_{j,k=0}^{n} \frac{1}{j!k!} \mu_{j+1}(\mu_k(x, \ldots, x), x, \ldots, x) = \sum_{j \geq 0} \frac{1}{j!} \mu_{j+1}(\mu(x, \ldots, x), x, \ldots, x) = \sum_{l \geq 1} \frac{1}{(l-1)!} \mu_l(M(x), x, \ldots, x) = \sum_{l \geq 1} \frac{l}{l!} \mu_l(M(x), x, \ldots, x) [A.2] = 0.$$  

□

Lemma A.2 (Equivalence of curved ($\mathcal{G}$) $L_{\infty}$-algebra Equations II). Let $\mathfrak{g}$ be a curved $\mathcal{G}L_{\infty}$-algebra endowed with a filtration that is descending, bounded above, complete and compatible with the $\mathcal{G}L_{\infty}$-algebra structure. Then the curved $\mathcal{G}L_{\infty}$-algebra Equation (A.2) can equivalently be encoded in the form of Equation (3.3).

Proof of Lemma A.2. Let $x$ be an element of $\mathfrak{g}\otimes R)^0$ for $R := \mathbb{K}[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1^2, \ldots, \epsilon_n^2)$ where $\epsilon_i$ is a formal variable of degree $-|x_i|$. Using the definition of the derivative of a multilinear map combined with the fact that for $|x| = 0$ graded symmetry allows us to change the order of arguments freely without picking up additional signs, we may find

$$DM^R(x)[M^R(x)] = \sum_{n \geq 1} \frac{1}{n!} \sum_{j=1}^{n} \mu_n(x, \ldots, x, M^R(x), x, \ldots, x) = \sum_{n \geq 1} \frac{n}{n!} \mu_n(M^R(x), x, \ldots, x).$$

But on the other hand

$$M^{R@R}(x@1 + M^R(x)@\epsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_n(x@1 + M^R(x)@\epsilon, \ldots, x@1 + M^R(x)@\epsilon) = \sum_{n \geq 1} \frac{1}{n!} \sum_{j=1}^{n} \mu_n(x@1, \ldots, x@1, M^R(x)@\epsilon, x@1, \ldots, x@1) = M^R(x@1 + \sum_{n \geq 1} \frac{n}{n!} \mu_n(M^R(x)@\epsilon, x@1, \ldots, x@1).$$
also holds, and so it is immediate that $M^{R \otimes R'}(x \otimes 1 + M(x) \otimes 1) = M^{R}(x) \otimes 1$ is tantamount to $DM^{R}(x)[M^{R}(x)] = 0$. □

References

[1] Vasily A. Dolgushev, Alexander E. Hoffnung, and Christopher L. Rogers. What do homotopy algebras form? arXiv e-prints, page arXiv:1406.1751, June 2014.
[2] Vasily A. Dolgushev and Christopher L. Rogers. On an Enhancement of the Category of Shifted L-infinity-Algebras. Applied Categorical Structures, 25(4):489–503, Aug 2017.
[3] Benoit Fresse. Homotopy of operads and Grothendieck-Teichmüller groups. Mathematical surveys and monographs vol. 217. American Mathematical Society, Providence R.I, 2017.
[4] Benoit Fresse, Victor Turchin, and Thomas Willwacher. The rational homotopy of mapping spaces of $E_n$ operads. arXiv e-prints, page arXiv:1703.06123, March 2017.
[5] Benoit Fresse and Thomas Willwacher. Mapping Spaces for DG Hopf Cooperads and Homotopy Automorphisms of the Rationalization of $E_n$-operads. arXiv e-prints, page arXiv:2003.02939, March 2020.
[6] Imma Gálvez-Carrillo, Andrew Tonks, and Bruno Valette. Homotopy Batalin–Vilkovisky algebras. Journal of Noncommutative Geometry, 6(3):539–602, 2012.
[7] Ezra Getzler and J. D. S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. arXiv e-prints, pages hep-th/9403055, March 1994.
[8] Olaf Hohm and Barton Zwiebach. $L_{\infty}$-algebras and field theory. Fortschritte der Physik, 65(3-4):1700014, March 2017.
[9] Jean-Louis Loday and Bruno Vallette. Algebraic operads, volume 346 of Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 2012.
[10] Saunders Mac Lane. Homology. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete Bd. 114. Springer-Verlag, Berlin, 1st ed. 1963, 4th pr. edition, 1994 - 1963.
[11] M. Markl, S. Shnider, and J.D. Stasheff. Operads in Algebra, Topology and Physics. Mathematical surveys and monographs. American Mathematical Society, 2002.
[12] Edwin Henry Spanier. Algebraic topology. Springer-Verlag, New York, 1st corr. springer ed. edition, 1989 - 1989.
[13] Bruno Vallette. Homotopy theory of homotopy algebras. arXiv e-prints, page arXiv:1411.5533, November 2014.
[14] Charles A. Weibel. An Introduction to Homological Algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.