On the Integrability of String Theory
in $AdS_5 \times S^5$

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Abstract

Integrability occupies an increasingly important role in direct tests of the AdS/CFT correspondence. Integrable structures have appeared in both planar $\mathcal{N} = 4$ super Yang-Mills theory and type IIB superstring theory on $AdS_5 \times S^5$. A generalized statement of the AdS/CFT conjecture has therefore emerged in which, in addition to string energies corresponding to gauge theory anomalous dimensions, an infinite tower of higher charges on each side of the duality should also be equated. Demonstrations of this larger equivalence have been successful in certain regimes. To test this correspondence in a more stringent setting, the bosonic sector of the fully quantized string theory on $AdS_5 \times S^5$ is expanded about the pp-wave limit to sextic order in fields, or to $O(1/J^2)$, where $J$ is the (large) angular momentum of string states boosted along an equatorial geodesic in the $S^5$ subspace. To avoid issues of renormalization, the analysis is restricted to zeroth order in the modified 't Hooft coupling where consistency conditions demand that integrability be realized. The string theory, however, fails to meet these conditions. This signals a potential problem with higher-order corrections in the large-$J$ expansion around the pp-wave limit.

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1 Introduction

Studies of the AdS/CFT correspondence have made impressive strides in recent years. The complications involved in quantizing type IIB superstring theory on $AdS_5 \times S^5$ in the presence of background Ramond-Ramond fields were circumvented to some extent by Metsaev, who was able to show that in a particular kinematic limit the string theory in this background becomes free $[1, 2]$. In this limit, states are boosted along a null geodesic in the $S^5$ subspace, and the geometry is reduced to a pp-wave $[3, 4, 5]$. On the gauge theory side a corresponding limit of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with $SU(N_c)$ gauge group was identified by Berenstein, Maldacena and Nastase, where the anomalous dimensions of a class of single-trace SYM operators with large $U(1)$ component of the $SU(4)$ $R$-charge were matched to the string theory energy spectrum on the pp-wave $[6]$.

The dimensions of $\mathcal{N} = 4$ SYM operators in the planar (large $N_c$) limit are perturbative in the 't Hooft coupling $\lambda = g^2_{YM} N_c$ and are analytic functions of the scalar $R$-charge, which is dual to the $S^5$ angular momentum $J$ of states in the string theory. Since string energies are exact in the string dual of the 't Hooft coupling (known as the modified 't Hooft coupling $\lambda' = g^2_{YM} N_c / J^2$), the resulting duality landscape is one in which agreement is obtained in the overlap between the large-$J$, small-$\lambda'$ limit of the string theory, and the large-$R$, small-$\lambda$ limit of the gauge theory. The correspondence can be probed at a much deeper level by including higher $\lambda$ loop corrections to the gauge theory and higher-order $1/J$ corrections to the string theory. This program has been pursued in a series of recent studies (see, eg. $[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]$).

Finite-$J$ corrections to the string spectrum can be interpreted as interaction perturbations to the worldsheet theory arising from finite-radius curvature corrections to the pp-wave geometry. This curvature expansion has thus far been carried out in the fully supersymmetric theory to $O(1/J)$ (or, in terms of the spacetime radius $\hat{R}$, to $O(1/\hat{R}^2)$) for two and three-impurity string states $[8, 9, 10]$. On the gauge theory side, the problem of computing anomalous dimensions of operators has undergone a striking simplification: in various subsectors of the theory the dilatation operator can be mapped to the Hamiltonian of an integrable spin-chain. The results have confirmed expectations of the AdS/CFT correspondence to $O(1/J)$ in the curvature expansion and $O(\lambda^2)$ in the gauge loop expansion. At the present time, however, there is a perplexing disagreement between each side of the correspondence at three loops in $\lambda$ $[8, 9, 10, 14, 17]$.

This disagreement aside, the emergence of integrable structures on each side of the duality has come to play a central role in explorations of the AdS/CFT mechanism. (For a review of current progress, see, eg. $[18, 19]$.) This direction of study was launched by Minahan and Zarembo in $[20]$, wherein it was shown that at one loop in $\lambda$ the anomalous dimension matrix of SYM operators in an $SU(2)$ subsector of the theory is precisely the Hamiltonian of the so-called XXX$_{1/2}$ integrable Heisenberg spin chain. Integrability in these systems can be proved by showing that the $R$ matrix associated with the dilatation operator satisfies the Yang-Baxter equation. Anomalous dimensions are then typically computed by means of the Bethe ansatz technique $[21, 22]$. (In $[23]$, this application was generalized to the full $PSU(2, 2|4)$ super spin chain at one loop in $\lambda$.) The integrable structure generates an infinite tower of higher commuting (local) charges, denoted by $\{Q_k\}$ (with $k = 0, \ldots, \infty$), where $Q_0$ is a cyclic shift in the trace, $Q_1$ is the anomalous dilatation operator and higher
$Q_k$ represent a series of hidden Abelian charges (most of which have not been linked to the known symmetries of $\mathcal{N} = 4$ SYM theory \cite{18}). In certain closed subsectors of the theory this structure has been shown to remain intact to the three-loop level \cite{11 15}, and an obvious implication is that the theory may in fact be integrable at all loops, generating a tower of hidden charges that are exact in $\lambda$. One may therefore promote the charges $\{Q_k\}$ to analytic functions of $\lambda$ where, in the notation of \cite{19},

$$Q_k(\lambda) = \sum_{j=0}^{\infty} \lambda^j Q_{k,2j} \, .$$  

It should also be noted that a similar direction of investigation has been pursued for a class of non-local (non-Abelian) charges generated by Yangian structures on each side of the correspondence \cite{24 25 26}. Successful contact was made at one loop between corresponding infinite-dimensional non-Abelian symmetry algebras on either side of the duality in \cite{27 28}. Here, however, we will primarily be concerned with the Abelian sector of the integrable structure.

The conjectured all-loop integrability is supported by the fact that certain integrable structures emerge from the classical string sigma model. Classical solutions corresponding to rigid strings moving in the $S^5$ subspace \cite{29 30 31 32} are characterized by Neumann integrable systems, and naturally give rise to an infinite tower of hidden commuting charges that are non-perturbative in $\lambda$ \cite{16 17 33}. At one loop, string energies in these semiclassical systems were matched to the $SU(2)$ Bethe ansatz results for the spin chain in \cite{14 34}. Beyond one loop, where the spin chain is characterized by non-nearest-neighbor interactions, the dilatation operator acquires long-range terms that are not immediately soluble in terms of the Bethe ansatz approach. This challenge was surmounted in \cite{17}, where the Inozemtsev long-range spin chain was employed to develop a long-range Bethe ansatz, and string predictions were matched to two loops. Moreover, the higher commuting charges in the gauge theory have been shown to match the corresponding classical string charges to two loops in $\lambda$ by comparing the long-range Bethe ansatz with the classical Bethe equation associated with the string sigma model \cite{18 35}. This matching can be seen as a consequence of the fact that, to two-loop order, the classical string action is identical to the effective two-dimensional action of the coherent-state vector field describing the corresponding spin chain system \cite{36}. (This analysis was extended to higher order in the semiclassical $1/J$ expansion in \cite{37}.) The conjectured exact equivalence of the full tower of commuting charges in the string and gauge theories (and the corresponding duality relationship in their respective couplings) can be taken as a generalized version of the AdS/CFT correspondence. In direct tests, however, this dramatic agreement begins to break down at three-loop order \cite{17 18}.

The purpose of the present study is to move beyond the semiclassical limit of the string theory and test integrability in the fully quantized theory at higher orders in the string background curvature expansion. We will focus on a particular conserved charge $Q_2(\lambda)$ and its string counterpart $Q_2^{\text{string}}(\lambda')$, restricted to two closed bosonic subsectors in each theory. In the CFT, these protected subsectors appear as $SL(2)$ and $SU(2)$ bosonic sectors that cannot mix with any other states in the theory, to all orders in $\lambda$. In the string theory these subsectors correspond to bosonic symmetric-traceless states that decouple in either the $SO(4)$ subspace descending from $AdS_5 (SO(4)_{AdS})$ or in the $SO(4)$ descending from the
$S^5 \ (SO(4)_{S^5})$.

In the gauge theory, the conserved charge $Q_2(\lambda)$ was studied to two loops in the closed $SU(2)$ bosonic subsector in [11], where it was shown that, in addition to commuting with the dilatation operator $Q_1(\lambda)$, $Q_2(\lambda)$ anticommutes with a parity operator $P$ which acts on single-trace operators by inverting the order of fields within the trace. In addition, $Q_2(\lambda)$ can also be shown to connect operators of opposite parity, so that the existence of $Q_2(\lambda)$ gives rise to degenerate pairs of operators (under $Q_1(\lambda)$) connected by $P$. Conversely, parity degeneracy in the spectrum of $Q_1(\lambda)$ at a given order in the loop expansion implies that $Q_2(\lambda)$ can be computed explicitly to that order. This degeneracy was originally used to fix the form of the three-loop dilatation operator $Q_{1,4}$ in the $SU(2)$ closed subsector which, without assuming integrability (and hence parity degeneracy), was only fixed up to two free coefficients [11, 13].

The status of $Q_{1,4}$ has since improved: it can be fixed by independent symmetry arguments [15] and, in accordance with the expectations of integrability, the theory in this subsector does indeed exhibit parity degeneracy to three loops. Beyond this, the requirements of BMN scaling and parity degeneracy fix the form of the dilatation operator to five loops. $Q_{1,6}$ and $Q_{1,8}$, however, have yet to be independently confirmed by symmetry arguments alone.

As demonstrated in [10], the string theory version of parity corresponds to exchanging left and right-moving modes on the worldsheet. Since the agreement between string energies at $O(1/J)$ and anomalous dimensions fails at three loops in $\lambda$, it may have been reasonable to expect the breakdown of parity degeneracy in the string theory at this order. According to the results in [10], however, this is in fact not the case: $Q_2^{\text{string}}(\lambda')$ commutes with the string Hamiltonian and parity degeneracy persists to $O(1/J)$ and to all loops in the gauge coupling. The disagreement between gauge and string theory at three loops in $\lambda$ is due to an overall shift in the string energy spectrum that preserves this particular facet of the integrable structure.

When treated as a constraint, integrability in the gauge theory is an extremely restrictive requirement [19]. Given the established disagreement with string theory at three loops, it may not be surprising to see a breakdown of integrability on the string side of the duality at some order in the $1/J$ expansion. At $O(1/J^2)$, the most basic question is of course whether there is any agreement between string energies and gauge theory anomalous dimensions. At this order, however, the string theory becomes subject to several renormalization issues, and these interesting yet complicated problems will be reserved for a subsequent paper. The subject of the present study will be a more immediate test on $Q_2^{\text{string}}(\lambda')$. In particular, we will expand the bosonic sector of the string theory to sextic order in fields and test whether $Q_2^{\text{string}}(\lambda')$ is still conserved at $O(1/J^2)$.

In section 2, curvature corrections to the Green-Schwarz superstring action in the pp-wave limit of $AdS_5 \times S^5$ will be reviewed. Employing the techniques described in [9], corrections to the action will be extended to $O(1/J^2)$ (or $O(1/R^4)$ in terms of the curvature radius) in the bosonic sector of the theory. In section 3, matrix elements of the resulting curvature

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1To be precise, one of these coefficients is fixed by demanding proper scaling in the BMN limit, and the other is fixed by parity degeneracy.

2It is necessary to consider at least three worldsheet impurities to study these aspects of parity in the string theory. The same statement in the gauge theory says that one requires at least three impurities in the trace to admit states with distinct parity in the same group representation [11].

3Note that the three-loop disagreement is by no means inextricably tied to the survival of integrability.
corrections will be computed for two classes of closed, three-impurity string states. It will be shown that the basic expectations from gauge theory integrability are not met at this order in the expansion. We conclude with a discussion of this problem and of future directions of study.

2 Higher curvature corrections to the Hamiltonian

As described in [8, 9], the pp-wave limit of $AdS_5 \times S^5$ can acquire finite-radius curvature corrections which lead to interaction perturbations to the free worldsheet theory. Here, corrections to the pp-wave Hamiltonian are arranged according to the expansion

$$H = \sum_{k=0}^{\infty} \frac{H^{(k)}}{R^{2k}},$$

where $H^{(0)}$ is the zeroth-order theory in the full Penrose limit of the geometry, and $\hat{R}$ is the spacetime radius. We will employ the same form of the $AdS_5 \times S^5$ metric used in [8, 9]:

$$ds^2 = \hat{R}^2 \left[ -\left(1 + \frac{1}{4}z^2\right)^2 \, dt^2 + \left(1 - \frac{1}{4}y^2\right)^2 \, d\phi^2 + \frac{dz_k dz_k}{(1 - \frac{1}{4}z^2)^2} + \frac{dy_k' dy_k'}{(1 + \frac{1}{4}y^2)^2} \right].$$

The time $t$ and $\phi$ directions will be combined to form lightcone coordinates $x^\pm$, while $x^A$ ($A = 1, \ldots, 8$) will label eight transverse directions which are broken into the two $SO(4)$ subspaces noted above. These are denoted by $z^2 = z_k z_k$, which span the $SO(4)_{AdS}$ (with $k = 1, \ldots, 4$), and $y^2 = y_{k'} y^{k'}$, which span the $SO(4)_{S^5}$ subgroup ($k' = 5, \ldots, 8$). The radius $\hat{R}$ is related to the gauge theory 't Hooft coupling by $\hat{R}^4 = \lambda (\alpha')^2$.

For reasons described in [9], we use a particular choice of lightcone coordinates given by

$$t = x^+, \quad \phi = x^+ + x^-.$$

In the Penrose limit, where states are boosted along an equator in the $S^5$ subspace, $\phi$ and the transverse coordinates $z_k$ and $y_{k'}$ are rescaled according to

$$\phi \rightarrow x^+ + \frac{x^-}{\hat{R}^2} \quad z_k \rightarrow \frac{z_k}{\hat{R}} \quad y_{k'} \rightarrow \frac{y_{k'}}{\hat{R}}.$$

This leads to the following curvature expansion of (2.2) in powers of $1/\hat{R}^2$ about the Penrose limit:

$$ds^2 = 2dx^+ dx^- - (x^A)^2 (dx^+)^2 + (dx^A)^2 + \frac{1}{\hat{R}^2} \left[ -2y^2 dx^+ dx^- + \frac{1}{2} (y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2} z^2 dz^2 - \frac{1}{2} y^2 dy^2 \right] + \frac{1}{\hat{R}^4} \left[ 16y^4 dx^+ dx^- - 3(x^A)^6 (dx^+)^2 - 16y^2 (dx^-)^2 + 3y^4 dy^2 + 3z^4 dz^2 \right] + O \left( \frac{1}{\hat{R}^6} \right).$$
The leading-order term is the lightcone metric of the pp-wave, and the \( O(1/\hat{R}^2) \) curvature correction leads to the first \( 1/J \) correction to the string spectrum, which is the subject of \([8, 9, 10]\).

The supersymmetric Green-Schwarz action describing type IIB string theory on this background is constructed from the Cartan one-forms and superconnections on the coset space \( G/H = [SO(4, 2) \times SO(6)]/[SO(4, 1) \times SO(5)] \). Here we intend to focus on the bosonic sector of the theory: the salient points in this study can be made without confronting the complications present in the fermionic sector (specifically, these arise from the existence of second-class constraints on the fermionic degrees of freedom). Without fermions, the full \( AdS_5 \times S^5 \) Lagrangian takes the form

\[
\mathcal{L} = -\frac{1}{2} h^{ab} L^a_{\mu} L^\mu_b ,
\]

where the Cartan one-forms \( L^\mu_a \) are given simply by

\[
L^\mu_a = e^{\mu}_\nu \partial_a x^\nu .
\]

The indices \( a, b = 0, 1 \) are used to denote the worldsheet coordinates \( \tau \) \((a, b = 0) \) and \( \sigma \) \((a, b = 1) \).

The general lightcone gauge-fixing procedure is to eliminate unphysical degrees of freedom by imposing the gauge condition \( x^+ = p_+ \tau \) and enforcing both the \( x^- \) equations of motion and the conformal gauge constraints in the action. In the present setting, these operations can be achieved order-by-order in the large-\( \hat{R} \) expansion. The only complication is that, with the lightcone coordinates chosen in (2.3), the worldsheet metric must acquire curvature corrections to remain consistent with the equations of motion. To organize the calculation, the worldsheet metric is taken to be flat at leading order and higher order curvature corrections are arranged according to the following expansion:

\[
\begin{align*}
h^{00} &= -1 + \frac{h^{00}_0}{\hat{R}^2} + \frac{h^{00}_4}{\hat{R}^4} + O(\hat{R}^{-6}) , \\
h^{11} &= 1 + \frac{h^{11}_0}{\hat{R}^2} + \frac{h^{11}_4}{\hat{R}^4} + O(\hat{R}^{-6}) , \\
h^{01} &= \frac{h^{01}_0}{\hat{R}^2} + \frac{h^{01}_4}{\hat{R}^4} + O(\hat{R}^{-6}) .
\end{align*}
\]

As described in \([8, 9]\), this simply rewrites \( h^{ab} \) and does not (at this stage) amount to a particular gauge choice. Using a similar notation, the worldsheet derivatives of \( x^- \) are expanded as

\[
\dot{x}^- = \dot{x}^-_{(0)} + \frac{\dot{x}^-_{(2)}}{\hat{R}^2} + \frac{\dot{x}^-_{(4)}}{\hat{R}^4} + O(\hat{R}^{-6}) , \quad x'^- = x'^-_{(0)} + \frac{x'^-_{(2)}}{\hat{R}^2} + \frac{x'^-_{(4)}}{\hat{R}^4} + O(\hat{R}^{-6}) .
\]

To proceed, we construct the basic combinations of Cartan one-forms appearing in the
higher-order corrections to the sheet metric $h_{\alpha\beta}$.

The Virasoro constraints are obtained by varying the Lagrangian with respect to the world-sheet metric $h_{\alpha\beta}$:

$$T_{ab} = L_b^\mu L_\mu^a - \frac{1}{2} h_{\alpha\beta} h^{\alpha\delta} L_\delta^\mu L_\mu^a = 0 .$$

These constraints will be used in conjunction with the $x^-$ equations of motion to solve for higher-order corrections to $h_{\alpha\beta}$. The $x^-$ equations of motion can be satisfied by setting the following variations to zero:

$$\frac{\delta L}{\delta \dot{x}^-} = p_+ + \frac{1}{R^2} \left[ \dot{x}^- - p_- (y^2 - h^{00}_{(2)}) \right] + \frac{1}{R^4} \left[ p_- (h^{00}_{(2)} y^2 - h^{00}_{(4)} + \frac{1}{2} y^2) - \dot{x}^- (h^{00}_{(2)} - y^2) - h^{01}_{(2)} x^- \right] + O(\hat{R}^{-6}) ,$$

$$\frac{\delta L}{\delta x'^-} = -\frac{1}{R^2} \left[ x'^- + p_- h^{01}_{(2)} \right] + \frac{1}{R^4} \left[ p_- (h^{01}_{(2)} y^2 - h^{01}_{(4)}) - h^{01}_{(2)} \dot{x}^- + x'^- (y^2 - h^{11}_{(2)}) \right] + O(\hat{R}^{-6}) .$$

In terms of the transverse $SO(4)_{AdS} \times SO(4)_S$ coordinates, the expansion of $x^-$ to $O(\hat{R}^{-2})$ is

$$\dot{x}^-_{(0)} = \frac{p_+}{2} (x^A)^2 - \frac{1}{2p_-} \left[ (\dot{x}^A)^2 + (x'^A)^2 \right] , \quad \dot{x}^-_{(1)} = -\frac{1}{p_-} x^A x'^A ,$$

$$\dot{x}^-_{(2)} = \frac{1}{8p_-^2} \left\{ -4(\dot{x}^A a'^A)^2 - \left[ (\dot{x}^A)^2 - 3(x'^A)^2 \right] \left[ (\dot{x}^A)^2 + (x'^A)^2 \right] + 2 \left[ y^2 (2y^2 + z^2 - \dot{z}^2) + z^2 \left( y^2 - y'^2 - 2z'^2 \right) \right] + p_+ \left[ (x'^A)^2 \right]^2 \right\} ,$$

$$x'^-_{(2)} = \frac{1}{2p_-^2} \left\{ p_- \left[ (\dot{x}^A y^A y_{k'} - y^2 \dot{z}^A z'_k) - (x^A x'^A) \right] \left[ (\dot{x}^A)^2 - (x'^A)^2 \right] \right\} .$$
At $O(\hat{R}^{-4})$ we only need to solve for $\dot{x}^-$ (ie. $x^-_{(4)}$) to eliminate all instances of $x^-$ from the Lagrangian. We find
\[
\dot{x}^-_{(4)} = -\frac{1}{32p_-^5} \left\{ 2 \left[ \dot{y}^6 - \dot{y}^4(y'^2 - 3\dot{z}^2) + 4(\dot{y}_k y'_{k'})^2(\dot{z}^2 - 3y'^2) \right] \\
+ (\dot{z}^2 + y'^2)(5y'^4 - 2y'^2\dot{z}^2 + \dot{z}^4) + \dot{y}^2(4(\dot{y}_k y'_{k'})^2 + 3y'^4 - 2y'^2\dot{z}^2 + 3\dot{z}^4) \right\} \\
+ (16\dot{y}_k y'_{k'}\dot{z}_k z'_k + 8(\dot{z}_k z'_k)^2)((\dot{x}^A)^2 - 3y'^2) - 2\dot{z}^2 \left[ \dot{y}^4 - 6\dot{y}^2y'^2 - 15y'^4 + 2y'^2\dot{z}^2 \\
- 6y'^2\dot{z}^2 + 3\dot{z}^4 + 12(\dot{x}^A x'^A)^2 \right] + 6\dot{z}^4((\dot{x}^A)^2 + 5y'^2) + 10\dot{z}^6 - 2p^2 \left\{ y^2 \left[ \dot{y}^4 - 6\dot{y}^2y'^2 \\
- 15y'^4 - 4y'^2\dot{z}^2 - 3\dot{z}^4 + 4(\dot{x}^A x'^A)(3\dot{y}_k y'_{k'} + \dot{z}_k z'_k) - 2\dot{z}^2(2\dot{y}^2 + 12y'^2 + \dot{z}^2) - 9\dot{z}^4 \right] \\
+ z^2 \left[ 3\dot{y}^4 + 4(\dot{y}_k y'_{k'})^2 - 2\dot{y}^2y'^2 + 3\dot{y}^4 + 4\dot{y}^2\dot{z}^2 + \dot{z}^4 - 4(\dot{z}_k z'_k)^2 \right] \\
+ 2\dot{z}^2(6\dot{y}^2 + \dot{z}^2) + 9\dot{z}^4 \right\} \right\} + p^4 \left\{ y^4 \left[ \dot{y}^2 + 9y'^2 + 2(\dot{z}^2 + \dot{z}'^2) \right] - 8y^2\dot{z}^2(x'^A)^2 \\
+ z^4 \left[ 2y^2 + 2y'^2 + \dot{z}^2 + 9\dot{z}^2 \right] \right\} - p_\perp^4 (x^A)^2 (y^4 + y'^2\dot{z}^2 + \dot{z}^4) \right\} .
\]

Likewise, the curvature corrections to the worldsheet metric are found at $O(\hat{R}^{-2})$ to be
\[
h^{00}_{(2)} = \frac{1}{2}(\ddot{z}^2 - y'^2) - \frac{1}{2p_-^2} [(\dot{x}^A)^2 + (x'^A)^2] \quad h^{01}_{(2)} = \frac{1}{p_-} \dot{x}^A x'^A .
\]

Continuing to $O(\hat{R}^{-4})$ in the expansion, we find
\[
h^{00}_{(4)} = -\frac{1}{8p_-^4} \left\{ -4(\dot{x}^A x'^A)^2 + \left[ 3(\dot{x}^A)^2 - (x'^A)^2 \right] \left[ (\dot{x}^A)^2 + (x'^A)^2 \right] \\
- 2p_-^2 \left[ \dot{y}^2(2\dot{y}^2 - \dot{z}^2 + z'^2) + z^2(3\dot{y}^2 + y'^2 + 2\dot{z}^2) \right] + p_\perp^4 (y^2 - \dot{z}^2)^2 \right\} ,
\]
\[
h^{01}_{(4)} = \frac{1}{2p_-^4} \left\{ \dot{x}^A x'^A \left[ (\dot{x}^A)^2 - (x'^A)^2 \right] - p_-^2 \left[ y^2\dot{z}_k z'_k + \dot{y}_k y'_{k'}(2\dot{y}^2 + \dot{z}^2) \right] \right\} .
\]

The remaining metric components are fixed by setting $\det h = -1$.

At this stage we have all the information necessary to compute the Hamiltonian as the generator of time translations on the worldsheet:
\[
H = \frac{\delta L}{\delta \dot{x}^+} .
\]

As a final step we must quantize the theory by converting all fields to conjugate coordinates and momenta. The momenta of the transverse $SO(8)$ coordinates $p_A = \delta L/\delta \dot{x}^A$ can be computed at each order in the expansion. The $p_A$ are then substituted into the Hamiltonian order-by-order, so that all coordinates $\dot{x}^A$ are replaced by $p_A$ plus higher-order corrections. For completeness, we record these corrections for the transverse $SO(4)_{AdS}$ and $SO(4)_{S^5}$.
moments:

\[
(p_z)_k = \dot{z}_k + \frac{1}{2p^2 R^2} \left\{-2z_k' (\dot{x}^A x'^A) + \dot{z}_k ((\dot{x}^A)^2 + (x'^A)^2) + p^2 \dot{z}_k y^2 \right\} \\
+ \frac{1}{16p^4 R^4} \left\{ 8z_k' \left[ -(\dot{x}^A x'^A)((\dot{x}^A)^2 - (x'^A)^2) + p^2 (2y^2 y' + (y^2 - z^2) \dot{z} \cdot z') \right] \\
+ \dot{z}_k \left[ 2 \left(-4(\dot{x}^A x'^A)^2 + (3(\dot{x}^A)^2 - (x'^A)^2)((\dot{x}^A)^2 + (x'^A)^2) \right) \\
- 4p^2 \left[ y^2 (2y^2 - z^2 + z'^2) + z^2 (2y^2 + \dot{z}^2 - z'^2) \right] + p^4 (2y^4 + z^4) \right] \right\} + O(\hat{R}^{-6}) ,
\]

\( (2.21) \)

\[
(p_y)_{k'} = \dot{y}_{k'} + \frac{1}{2p^2 R^2} \left\{-2y_{k'}' (\dot{x}^A x'^A) + \dot{y}_{k'} ((\dot{x}^A)^2 + (x'^A)^2) - p^2 \dot{y}_{k'} z^2 \right\} \\
+ \frac{1}{16p^4 R^4} \left\{ 2p^2 \left[ 4y_{k'}' (\dot{x}^A x'^A)((\dot{x}^A)^2 - (x'^A)^2) + \dot{y}_{k'} \left[ -4(\dot{x}^A x'^A)^2 \\
+ (3(\dot{x}^A)^2 - (x'^A)^2)((\dot{x}^A)^2 + (x'^A)^2) \right] \right] - 2p^2 y_{k'} \left[ 3y^2 y' y + 2y^2 \dot{z} \cdot z' + z^2 \dot{y} \cdot y' \right] \\
+ p^4 \dot{y}_{k'} (y^4 + 2z^4) \right\} + O(\hat{R}^{-6}) .
\]

\( (2.22) \)

The final result for the Hamiltonian, computed to \( O(\hat{R}^{-4}) \) in the expansion, is

\[
H^{(0)}_{pp-\text{wave}} = \frac{1}{2} \left[ (x^A)^2 + (p_A)^2 + (x'^A)^2 \right] , \tag{2.23}
\]

\[
H^{(2)} = \frac{1}{4} \left[ z^2 \left( p_y^2 + y'^2 + 2z'^2 \right) - y^2 \left( p_z^2 + z'^2 + 2y^2 \right) \right] + \frac{1}{8} \left[ (x^A)^2 \right]^2 \\\n- \frac{1}{8} \left\{ [(p_A)^2]^2 + 2(p_A)^2 (x'^A)^2 + [(x'^A)^2]^2 \right\} + \frac{1}{2} \left( x'^A p_A \right)^2 , \tag{2.24}
\]
\[ H^{(4)} = \frac{1}{32p_-} \left\{ 2(p_y^2 + p_z^2) - 8p_z^2(p_z \cdot z')^2 + 6p_z^4y^2 - 8y^2(p_z \cdot z')^2 + 6p_z^2y^4 + 2y^6 + 6p_z^4y^2 \right. \\
\left. - 24p_z^2y^2(p_z \cdot z')^2 + 16p_z^2y^2p_z^2 + 10p_z^2y^2y^4 + 2p_z^4y^4 + 9p_z^4y^2y^4 + p_z^6y^6 \\
+ 6p_z^4z^2 - 8z^2(p_z \cdot z')^2 + 12p_z^2y^2z^2 + 6y^4z^2 + 12p_z^2p_y^2y^2z^2 \\
+ 16p_z^2y^2y^2z^2 + 2p_z^4y^8z^2 + 6p_z^2z^4 + 6y^2z^4 + 6p_z^2y^2z^4 + 2z^6 + 2p_y^4 \left[ 3p_z^2 + 3(x' A)^2 \right. \\
\left. + p_z^2(y^2 - z^2) \right] - \left[ 8(p_y y')^2 + 16(p_y y')(p_z \cdot z') \right] \left[ p_z^2 + (x' A)^2 + p_z^2(3y^2 - z^2) \right] \\
+ 2p_z^2z^2 \left[ p_z^4 + 4(p_z \cdot z')^2 - y^4 - 4p_z^2y^2y^2 + p_z^4y^4 - 3y^2 - z^4 \right] \\
+ p_z^4z^4(2y^2 - p_z^2 + 2p_z^2y^2 + 9z^2^2) + p_z^6z^6 + p_y^4 \left[ 6p_z^4 - 8(p_A x' A)^2 + 6y^4 + 12p_z^2y^2y^2 \\
- p_z^4y^4 + 12y^2z^2 + 8p_z^2y^2z^2 + 6z^4 + 4p_z^2 \left( 3y^2 + 2p_z^2y^2 + 3z^2 \right) \\
- 4p_z^2z^2(y^2 + 2z^2) + 2p_z^4z^4 \right\} \right\} . \] (2.25)

As expected, the leading-order system is exactly the quadratic pp-wave Hamiltonian originally reported by Metsaev in [1]. Similarly, \( H^{(2)} \) agrees with the bosonic sector of the \( O(1/\hat{R}^2) \) quartic Hamiltonian computed in [8, 9].

Since we are only interested in the closed bosonic subsectors that are restricted to each of the \( SO(4)_{AdS} \) and \( SO(4)_{S^5} \) subspaces, the results at \( O(1/\hat{R}^4) \) can be dramatically simplified by projecting \( H^{(4)} \) onto these subspaces:

\[ H^{(4)}_{AdS} = \frac{1}{32p_-} \left\{ 2(p_z^2 + z^2)(p_z^2 - 2p_z \cdot z' + z^2)(p_z^2 + 2p_z \cdot z' + z^2) \right. \\
\left. + 2p_-^2z^2 \left[ 4(p_z \cdot z')^2 + (p_z^2 - 3z^2)(p_z^2 + z^2) \right] - p_-^2z^4(p_z^2 - 9z^2) + p_-^6z^6 \right\} , \] (2.26)

\[ H^{(4)}_{S^5} = \frac{1}{32p_-} \left\{ 2(p_y^2 + y^2)(p_y^2 - 2p_y \cdot y' + y^2)(p_y^2 + 2p_y \cdot y' + y^2) \right. \\
\left. + 2p_-^2y^2 \left[ -12(p_y y')^2 + (p_y^2 + y^2)(p_y^2 + 5y^2) \right] - p_-^4y^4(p_y^2 - 9y^2) + p_-^6y^6 \right\} . \] (2.27)

At this point the Hamiltonian can be expanded in terms of raising and lowering operators, and matrix elements can be computed between states lying in the closed bosonic subsectors of the theory.

### 3 Matrix elements and integrability

The unperturbed string eigenstates are the exact eigenstates of the pp-wave background, and the ground state \( | J \rangle \) (of the bosonic theory) is that of the eight transverse string oscillators satisfying

\[ \hat{x}^A - x'^A + p_-^2x^A = 0 \]  

(3.1)
and carrying angular momentum \( J \) on the \( S^5 \) subspace. This is solved by the usual Fourier expansion

\[
x^A(\sigma, \tau) = \sum_{n=-\infty}^{\infty} x_n^A(\tau) e^{-ik_n\sigma},
\]

\[
x_n^A(\tau) = \frac{i}{\sqrt{2\omega_n}} \left( a_n^A e^{-i\omega_n\tau} - a_n^{\dagger A} e^{i\omega_n\tau} \right), \tag{3.2}
\]

with integer mode numbers \( k_n = n \) and \( \omega_n \) defined by \( \omega_n \equiv \sqrt{p_n^2 + k_n^2} \). The creation and annihilation operators obey the standard relation \( [a_m^A, a_n^{\dagger B}] = \delta_{mn}\delta^{AB} \), in terms of which the pp-wave Hamiltonian \( (2.23) \) takes the form

\[
H^{(0)}_{\text{pp-wave}} = \frac{1}{p} \sum_{n=-\infty}^{\infty} \omega_n \left( a_n^{\dagger A} a_n^A + 4 \right). \tag{3.3}
\]

The zero-point term is canceled when fermions are included.

As noted above, the string theory version of the SYM parity operator, which inverts the ordering of fields within single-trace operators, is one that invokes an overall sign change on the worldsheet mode indices of the unperturbed eigenstates:

\[
P(a_{q_1}^{A_1} a_{q_2}^{A_2} \ldots | J) = a_{q_1}^{A_1}\bar{a}_{-q_2}^{A_2} \ldots | J). \tag{3.4}
\]

In the language of spin chains, \( P \) acts in a similar fashion on pseudoparticle states by applying a sign switch to the lattice momenta of each state. At least three worldsheet (spin-chain) impurities with non-zero momenta are required to admit string (pseudoparticle) states that are distinct under the action of \( P \).

As described in [10], the bosonic three-impurity string Fock space consists of the 512-dimensional space spanned by the states

\[
a_q^{A}\bar{a}_r^{B}\bar{a}_s^C | J \rangle,
\]

subject to the level-matching constraint \( q + r + s = 0 \). The upper-case indices \( A, B, C, \ldots = 1, \ldots, 8 \) span the transverse \( SO(8) \), and the lower-case notation \( a, b, c = 1, \ldots, 4 \) and \( a', b', c' = 5, \ldots, 8 \) will be used to indicate vectors in the \( SO(4)_{\text{AdS}} \) and \( SO(4)_{S^5} \) subspaces, respectively. Within the bosonic sector of the theory there are two subsectors which decouple at all orders in \( \lambda' \). These subsectors consist of symmetrized, traceless bosonic impurities restricted to lie in either \( SO(4)_{\text{AdS}} \) or \( SO(4)_{S^5} \):

\[
a_q^{(a} a_r^{b)} a_s^{c)} | J \rangle, \quad a_q^{(a'} a_r^{b')} a_s^{c')} | J \rangle.
\]

(Here, tracelessness implies \( a \neq b \neq c \) and \( a' \neq b' \neq c' \).) By restricting to these protected subsectors, we can compute matrix elements that do not mix with any other sectors of the theory and are exact in \( \lambda' \). The three-impurity block-diagonalization of the Hamiltonian on these closed bosonic subsectors was demonstrated in more detail in [10].

To \( O(\hat{R}^{-2}) \), the matrix elements in these sectors were reported in [10]. At this order, the Hamiltonian is quartic in oscillators and matrix elements taken between three-impurity
string states always involve a single forward-scattering contraction. This fact breaks the state space into two distinct classes, where the mode indices \((q, r, s)\) are either completely inequivalent \((q \neq r \neq s)\), or two mode indices are taken to be equal \((q = r = n, s = -2n)\). At \(O(\hat{R}^{-6})\), this complication does not arise: the Hamiltonian is sixth-order in fields and matrix elements taken in the three-impurity regime do not involve contractions taken directly between the external unperturbed eigenstates.

Using the AdS/CFT relation

\[
\hat{R}^2 \Rightarrow p_- J ,
\]

all instances of the radius \(\hat{R}\) will henceforth be replaced with the \(S^5\) angular momentum \(J\). In the \(SO(4)_{\text{AdS}}\) subspace, to \(O(1/J^2)\), the Hamiltonian in the closed symmetric-traceless subsector exhibits the following matrix elements

\[
\langle J | a_q^{(a} a_r^{b} a_s^{c)} (H_{\text{AdS}}^{(4)}) a_s^{(a} a_r^{b} a_q^{c)} | J \rangle = \frac{1}{4J^2 \lambda^{3/2} \omega_q \omega_r \omega_s} \left\{ 15 + \lambda' \left[ 9rs + 16s^2 \\
-3q(r + s)(-3 + 2rs\lambda') + 2r^2(8 + 5s^2\lambda') + 2q^2(8 - 3rs\lambda' + 5s^2\lambda') \\
+ r^2\lambda'(5 + 12s^2\lambda') \right] + \lambda' \omega_r \omega_s \left[ 1 + 2\lambda'(-rs + q(r + s) + q^2(1 - 4rs\lambda')) \right] \\
+ \lambda' \omega_q \left[ (1 + 2s(r + s)\lambda' - 2q\lambda'(r - s + 4rs^2\lambda')) \omega_r + \omega_s \left[ 1 + 2r(r + s)\lambda' \\
- 2q\lambda'(s + r(-1 + 4rs\lambda')) \right] \right] \right\} .
\]

The generic upper indices \(a, b, c\) are taken to be any indices in the \(SO(4)_{\text{AdS}}\) subspace \((a, b, c = 1, \ldots, 4)\) and, because the state is traceless, \(a \neq b \neq c\). We can perform a small-\(\lambda'\) expansion to obtain

\[
\langle J | a_q^{(a'} a_r^{b'} a_s^{c')} (H_{\text{AdS}}^{(4)}) a_s^{(a} a_r^{b} a_q^{c)} | J \rangle = \frac{9}{2J^2} + \frac{9\lambda'}{4J^2}(q^2 + qr + r^2) + \frac{11\lambda'^2}{16J^2}(q^2 + qr + r^2)^2 \\
- \frac{3\lambda'^3}{64J^2} \left[ 14q^6 + 42q^5r - 51q^4r^2 - 172q^3r^3 - 51q^2r^4 + 42qr^5 + 14r^6 \right] + O(\lambda'^4) .
\]

Here we have made the substitution \(s = -q - r\) (using the level-matching constraint) to simplify the resulting expression. In the \(SO(4)_{S^5}\) subspace, we find

\[
\langle J | a_q^{(a' a_r^{b'}} a_s^{c')} (H_{S^5}^{(4)}) a_s^{(a} a_r^{b} a_q^{c)} | J \rangle = \frac{1}{4J^2 \lambda^{3/2} \omega_q \omega_r \omega_s} \left\{ 15 + \lambda' \left[ 9rs + 16s^2 \\
+ q(r + s)(9 + 10rs\lambda') + 2r^2(8 + 9s^2\lambda') + 2q^2(8 + 5rs\lambda' + 9s^2\lambda' + 3r^2\lambda'(3 + 4s^2\lambda')) \right] \\
- \omega_q \omega_r \left[ -1 + 6r\lambda' - 2s\lambda' + 2q\lambda'(3s + r(5 + 4s^2\lambda')) \right] - \omega_s \lambda' \left[ \omega_q \omega_r \left[ 1 - 2r(r - 3s)\lambda' \\
+ 2q\lambda'(5s + r(3 + 4rs)) \right] + \omega_r \left[ 1 + 2\lambda'(5rs + 3q(r + s) + q^2(-1 + 4rs\lambda')) \right] \right] \right\} \\
= \frac{9}{2J^2} + \frac{33}{4J^2}(q^2 + qr + r^2)\lambda' + \frac{11}{16J^2}(q^2 + qr + r^2)^2\lambda'^2 \\
- \frac{\lambda'^3}{64J^2} \left[ 74q^6 + 222q^5r + 39q^4r^2 - 292q^3r^3 + 39q^2r^4 + 222qr^5 + 74r^6 \right] + O(\lambda'^4) .
\]
Again, we have expanded in small $\lambda'$, setting $s = -q - r$ in the end. The indices $a', b', c' = 5, \ldots, 8$ lie in $SO(4)_{S^5}$, and are again taken to be inequivalent.

To find the full energy shifts of these states to $O(1/J^2)$, we would need to compute the full contribution from $H^{(2)}$ in second-order perturbation theory. As noted above, such calculations require knowledge of how the theory is renormalized at this order, and these issues will be dealt with elsewhere. We can avoid these complications in the present setting by restricting to zeroth order in the small-$\lambda'$ expansion: the salient points regarding integrability can still be made at this level.

To $O(\lambda'^0)$, all energy levels are degenerate, and a trivial consequence of this fact is that, to any order in $1/J$,

$$[H_{AdS}, P] = 0 + O(\lambda') , \quad [H_{S^5}, P] = 0 + O(\lambda') .$$

(3.9)

To test integrability, we aim to determine whether the Hamiltonian also commutes with $Q_{\text{string}}^{\lambda'}$ to this order. Since $Q_{\text{string}}^{\lambda'}$ must anticommute with $P$ and connect degenerate parity pairs (by definition), and because the Hamiltonian commutes with $P$ to $O(\lambda'^0)$, $Q_{\text{string}}^{\lambda'}$ can only commute with the Hamiltonian if the Hamiltonian itself does not connect states of opposite parity at zeroth order in $\lambda'$. To test whether $Q_{\text{string}}^{\lambda'}$ is truly a conserved charge in the theory, we can therefore compute matrix elements that connect string states of opposite parity. Starting with the first-order contributions from $H^{(4)}$ on the $SO(4)_{AdS}$ side, we find

$$\langle J | a_{-q} \tilde{a}_{-s} (H^{(4)}_{AdS}) a^\dagger_{s+b} | J \rangle = \frac{1}{4J^2 \lambda'^{3/2}} \frac{1}{\omega_q \omega_r \omega_s} \left\{ 15 + \lambda' \left[ -9rs + 5s^2 ight. \\
-3q(r + s)(3 + 2rs\lambda') + r^2(5 + 6s^2\lambda') + q^2(5 + 6(r^2 - rs + s^2)\lambda') \right. \\
+ \lambda' \omega_q \omega_s \left[ 1 + 2(q^2 - 3rs + q(r + s))\lambda' \right] + \lambda' \omega_q \omega_r \left[ (1 + 2q(s - 3r)\lambda' + 2s(r + s)\lambda') \\
+ \omega_s(1 + 2q(r - 3s)\lambda' + 2r(r + s)\lambda') \right] \right\} \\
= \frac{9}{2J^2} + \frac{9}{4J^2} (q^2 + qr + r^2)\lambda' + \frac{11}{16J^2} (q^2 + qr + r^2)^2 \lambda'^2 \\
- \frac{3\lambda'^3}{64J^2} \left[ 14q^6 + 42q^5r + 165q^4r^2 + 260q^3r^3 + 165q^2r^4 + 42qr^5 + 14r^6 \right] + O(\lambda'^4) .$$

(3.10)
In the $SO(4)_{S^5}$ subspace, we have

$$
\langle J | a_{-q}^{(a')} a_{-r}^{(a')} (H_{SO}^{(4)}) a_s^{(a') \tilde{w}'} a_q^{(c')} | J \rangle = \frac{1}{4 J^2 \chi^{3/2} \omega_r \omega_s} \left\{ 15 + \chi' \left[ -9 r s + 5 s^2 
+ q(r + s)(-9 + 10 r s \lambda') + r^2(5 + 14 s^2 \lambda') + q^2(5 + 2(7r^2 + 5rs + 7s^2) \lambda') \right] 
+ \chi' \omega_r \omega_s \left[ 1 + 2(q^2 - 7rs - 3q(r + s)) \lambda' \right] + \chi' \omega_r \left[ (1 + 2(-7qr - 3(q + r)s + s^2) \lambda') \omega_r 
+ \left[ 1 + 2r(r - 3s) \lambda' - 2q(3r + 7s) \lambda' \right] \omega_s \right\} 
= \frac{9}{2 J^2} + \frac{33}{4 J^2} (q^2 + qr + r^2) \lambda' + \frac{11}{16 J^2} (q^2 + qr + r^2)^2 \lambda'^2 
- \frac{\chi'^3}{64 J^2} \left[ 74q^6 + 222q^5 r + 687q^4 r^2 + 1004q^3 r^3 + 687q^2 r^4 + 222qr^5 + 74r^6 \right] + O(\lambda'^4). \tag{3.11}
$$

The Hamiltonian is a priori $2 \times 2$ block-diagonal in this three-impurity basis of degenerate parity pairs. The theory, however, does not conserve impurity number, and the unperturbed eigenstates above do not constitute a complete basis. To properly compute these matrix elements to the order of interest, one must include higher perturbative corrections to the zeroth-order three-impurity basis states that involve different numbers of excitations in the intermediate channels. Denoting, for example, the zeroth-order $SO(4)_{S^5}$ eigenstates above as

$$
|+^{(0)}\rangle = a_s^{(a') a_{-r}^{(a')} a_{-q}^{(c')}} |J\rangle \quad |-(0)\rangle = a_{-s}^{(a') a_{-r}^{(a')} a_{-q}^{(c')}} |J\rangle, \tag{3.12}
$$

the full matrix element in eqns. (3.10,3.11), to $O(1/J^2)$, takes the form

$$
\langle -(0) | H | +^{(0)} \rangle = \langle -(0) | H^{(4)} | +^{(0)} \rangle + \sum_{\psi \neq \pm} \frac{\langle -(0) | H^{(2)} \rangle \langle \psi^{(0)} \rangle \langle \psi^{(0)} | H^{(2)} | +^{(0)} \rangle}{E^{(0)}_\psi - E^{(0)}_{\psi \pm}}. \tag{3.13}
$$

The intermediate state sum, mediated by $H^{(2)}$, can in principle involve transitions that mix the three-impurity states $|+^{(0)}\rangle$ and $|-(0)\rangle$ with any one-, three-, five- and seven-impurity states in the theory. In addition to the purely bosonic sector $H^{(2)}_{BB}$, these sums can also involve the pure-fermi sector $H^{(2)}_{FF}$ and the bose-fermi mixing sector $H^{(2)}_{BF}$ (see [8, 9] for details). Some of these summations are excluded by simple arguments, however.

Starting with the pure-boson sector $H^{(2)}_{BB}$, intermediate channels involving one excitation vanish by direct calculation in both the $SO(4)_{AdS}$ and $SO(4)_{S^5}$ subspaces:

$$
\langle J | a_{-p_1}^{A_1} (H^{(2)}_{BB}) a_s^{(a') a_{-r}^{(a')} a_{-q}^{(c')}} | J \rangle = 0 \quad \langle J | a_{-p_1}^{A_3} (H^{(2)}_{BB}) a_s^{(a') a_{-r}^{(a')} a_{-q}^{(c')}} | J \rangle = 0. \tag{3.14}
$$

The three-to-three impurity channel has no contributions at $O(\lambda^6)$:

$$
\langle J | a_{-p_1}^{A_1} a_{-p_2}^{A_2} a_{-p_3}^{A_3} (H^{(2)}_{BB}) a_s^{(a' a_{-r}^{(a')} a_{-q}^{(c')}} | J \rangle = O(\lambda') \quad \langle J | a_{-p_1}^{A_1} a_{-p_2}^{A_2} a_{-p_3}^{A_3} (H^{(2)}_{BB}) a_s^{(a' a_{-r}^{(a')} a_{-q}^{(c')}} | J \rangle = O(\lambda') \tag{3.15}
$$
It should be noted that, since the propagator in this channel is nonzero at \(O(\lambda^{-1})\), contributions from eqn. (3.15) at \(O(\lambda^{1/2})\) could potentially affect the final result. The matrix elements in (3.15) vanish to \(O(\lambda')\), however, and this is not a concern. The zeroth-order three-to-five impurity contributions to eqn. (3.13) from the \(SO(4)_{AdS}\) and \(SO(4)_{S^5}\) sectors are

\[
- \frac{1}{2} \sum_{p_1, A_1} \left[ \langle J | a_r^{a} a_s^{b} a_s^{-c} (H_{BB}^{a}) a_p^{A_1} a_p^{A_2} a_p^{A_3} a_p^{A_4} a_p^{A_5} | J \rangle \right]
\]

\[
\times \langle J | a_p^{A_5} a_p^{A_4} a_p^{A_3} a_p^{A_2} a_p^{A_1} (H_{BB}^{a}) a_s^{a r} a_q^{a} | J \rangle \right] = - \frac{675}{J^2} + O(\lambda')
\]

\[
- \frac{1}{2} \sum_{p_1, A_1} \left[ \langle J | a_r^{a'} a_s^{b'} a_s^{-c'} (H_{BB}^{a}) a_p^{A_1} a_p^{A_2} a_p^{A_3} a_p^{A_4} a_p^{A_5} | J \rangle \right]
\]

\[
\times \langle J | a_p^{A_5} a_p^{A_4} a_p^{A_3} a_p^{A_2} a_p^{A_1} (H_{BB}^{a}) a_s^{a' r} a_q^{b'} a_q^{c'} | J \rangle \right] = - \frac{1755}{J^2} + O(\lambda') .
\] (3.16)

Finally, three-to-seven impurity contributions from \(H_{BB}^{a}\) vanish to this order:

\[
- \frac{1}{4} \sum_{p_1, A_1} \left[ \langle J | a_r^{a} a_s^{b} a_s^{-c} (H_{BB}^{a}) a_p^{A_1} a_p^{A_2} a_p^{A_3} a_p^{A_4} a_p^{A_5} a_p^{A_6} a_p^{A_7} | J \rangle \right]
\]

\[
\times \langle J | a_p^{A_7} a_p^{A_6} a_p^{A_5} a_p^{A_4} a_p^{A_3} a_p^{A_2} a_p^{A_1} (H_{BB}^{a}) a_s^{a r} a_q^{a} | J \rangle \right] = O(\lambda')
\]

\[
- \frac{1}{4} \sum_{p_1, A_1} \left[ \langle J | a_r^{a'} a_s^{b'} a_s^{-c'} (H_{BB}^{a}) a_p^{A_1} a_p^{A_2} a_p^{A_3} a_p^{A_4} a_p^{A_5} a_p^{A_6} a_p^{A_7} | J \rangle \right]
\]

\[
\times \langle J | a_p^{A_7} a_p^{A_6} a_p^{A_5} a_p^{A_4} a_p^{A_3} a_p^{A_2} a_p^{A_1} (H_{BB}^{a}) a_s^{a' r} a_q^{b'} a_q^{c'} | J \rangle \right] = O(\lambda') .
\] (3.17)

By inspection, the bose-fermi sector \(H_{BF}^{a}\) cannot mediate mixing between three- and one-impurity string states:

\[
\langle J | a_q^{a b} a_s^{a} (H_{BF}^{a}) | J \rangle = \langle J | a_q^{a b} a_s^{a} (H_{BF}^{a}) | J \rangle = 0
\]

\[
\langle J | a_q^{a'} a_s^{a'} (H_{BF}^{a}) | J \rangle = \langle J | a_q^{a'} a_s^{a'} (H_{BF}^{a}) | J \rangle = 0 .
\] (3.18)

This sector can mix three bosonic impurities with spacetime boson states comprised of a single bosonic excitation and two fermionic excitations. At \(O(\lambda^0)\), however, there are no three-to-three impurity matrix elements of \(H_{BF}^{a}\) in this channel:

\[
\langle J | a_q^{a b} a_s^{a} (H_{BF}^{a}) b_p^{a \alpha} | J \rangle = O(\lambda')
\]

\[
\langle J | a_q^{a'} a_s^{a'} (H_{BF}^{a}) b_p^{a' \alpha} | J \rangle = O(\lambda') .
\] (3.19)
The three-to-five impurity interaction gives equal contributions from the $AdS_5$ and $S^5$ sides:

$$-\frac{1}{2} \sum_{p_i,A_i,\alpha_i} \left[ \langle J | a_{-q,a_r,a_s} b_{-r} (H^{(2)}_{\text{BF}}) b_{p_1} b_{p_2} a_{p_3} a_{p_4} a_{p_5} | J \rangle \right. $$

$$ \times \left. \langle J | a_{p_5} a_{p_4} a_{p_3} a_{p_4} a_{p_5} b_{p_1} (H^{(2)}_{\text{BF}}) a_{s} a_{r} a_{q} | J \rangle \right] = -\frac{10}{9J^2} + O(\lambda')$$

The three-to-seven $H^{(2)}_{\text{BF}}$ channel yields

$$-\frac{1}{2} \sum_{p_i,A_i,\alpha_i} \left[ \langle J | a_{-q,a_r,a_s} b_{-r} (H^{(2)}_{\text{BF}}) b_{p_1} b_{p_2} a_{p_3} a_{p_4} a_{p_5} a_{p_6} a_{p_7} a_{s} | J \rangle \right. $$

$$ \times \left. \langle J | a_{p_7} a_{p_6} a_{p_5} a_{p_4} a_{p_5} a_{p_6} a_{s} (H^{(2)}_{\text{BF}}) a_{s} a_{r} a_{q} | J \rangle \right] = O(\lambda')$$

The only interaction permitted in the pure-fermi sector $H^{(2)}_{\text{FF}}$ is the three-to-seven impurity transition with intermediate states composed of three bosonic excitations and four fermionic excitations. These contributions vanish in both the $SO(4)_{\text{AdS}}$ and $SO(4)_{S^5}$ sectors:

$$-\frac{1}{4} \sum_{p_i,A_i,\alpha_i} \left[ \langle J | a_{-q,a_r,a_s} b_{-r} (H^{(2)}_{\text{FF}}) b_{p_1} b_{p_2} b_{p_3} a_{p_4} a_{p_5} a_{p_6} a_{p_7} a_{s} | J \rangle \right. $$

$$ \times \left. \langle J | a_{p_7} a_{p_6} a_{p_5} a_{p_4} a_{p_5} a_{p_6} a_{s} (H^{(2)}_{\text{FF}}) a_{s} a_{r} a_{q} | J \rangle \right] = O(\lambda')$$

No other contributions can arise from $H^{(2)}_{\text{FF}}$ (apart from those involving normal-ordering terms, which are excluded by supersymmetry). In the end we find that the matrix elements mixing bosonic parity pairs are nonzero at $O(1/J^2)$:

$$\langle J | a^{(a'} b' c')_{-q,a_r,a_s} (H_{S^5}) a^{{\dagger}(a} b c)_{s} | J \rangle \neq 0$$

$$\langle J | a^{(a'} b' c')_{-q,a_r,a_s} (H_{AdS}) a^{{\dagger}(a} b c)_{s} | J \rangle \neq 0 .$$
This is the first order in the $1/J$ expansion where this sort of mixing can possibly be observed: at lower orders the Hamiltonian is either quartic or quadratic in fields and therefore cannot mix distinct three-impurity states connected by parity.

To interpret this mixing in terms of the comparison with gauge theory dynamics, we first note that operators of definite and distinct parity cannot mix in $\mathcal{N} = 4$ SYM. When string eigenstates are arranged into states of definite parity there are no off-diagonal matrix elements of the Hamiltonian that connect states of opposite parity. This aspect of the string theory is therefore in agreement with gauge theory predictions. We have also established that to $O(1/J^2)$ and $O(\lambda^0)$ the string Hamiltonian commutes with the parity operator $P$. As noted above, since $Q_2^{\text{string}}(\lambda')$ must anticommute with $P$ and connect degenerate parity pairs, it can only commute with the Hamiltonian if the Hamiltonian itself does not connect states that are themselves connected by parity (in other words, string eigenstates of definite parity are non-degenerate). The results in eqn. (3.23) show that $Q_2^{\text{string}}(\lambda')$ therefore fails to commute with the Hamiltonian at this order:

$$[H_{\text{AdS}}, Q_2^{\text{string}}(\lambda')] \neq 0, \quad [H_{S^5}, Q_2^{\text{string}}(\lambda')] \neq 0.$$ (3.24)

There is no charge $Q_2^{\text{string}}(\lambda')$ in the string theory that satisfies all of the requirements set forth by the gauge theory. This of course indicates the breakdown of integrability at $O(1/J^2)$ in the curvature expansion.

This result, however, can also be seen to indicate a larger inconsistency within the string analysis. At zeroth order in $\lambda'$ the eigenstates $a_s^{\dagger}(a'_{i}a'_{j}a'_{k}|J\rangle$ and $a_s^{\dagger}(a_{i}a_{j}a_{k}|J\rangle$ can be reinterpreted as zero-mode string states and the computation above amounts to an energy eigenvalue calculation for the superparticle. From the analysis put forth in [9], the superparticle spectrum cannot acquire any corrections associated with the curvature of the target space geometry. In this light, eqn. (3.23) states that the string theory fails to meet a fairly basic constraint.

4 Discussion and conclusions

Previous tests of the AdS/CFT correspondence near the BMN limit have indicated that, at $O(1/J)$, the string theory begins to disagree with gauge theory predictions at three loops in the gauge coupling. Originally, this disagreement seemed to signal a failure of string theory integrability at that order, since the $SU(2)$ SYM dilatation operator is uniquely fixed by assuming integrability (and proper BMN scaling) [11, 13]. Following the study in [10], however, it became apparent that the string Hamiltonian at $O(1/J)$ preserves parity degeneracy among three-impurity string states and commutes with $Q_2^{\text{string}}(\lambda')$. While it may have been promising that this sector of the integrable structure is preserved at $O(1/J)$, it now appears that, if integrability is to survive at $O(1/J^2)$, some additional ingredient is needed.

It has been suggested that higher-order disagreements with gauge theory may be due to an order-of-limits issue [17]. Specifically, we assume that there is some expression for a given charge on either side of the duality that is exact in $\lambda$ (or $\lambda$ and $R$): $Q_k^{\text{string}}(\lambda', J) =$

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4I thank N. Beisert for clarification on this point.
$Q_k(\lambda, R)$. On the string side, the charges $Q_k^{\text{string}}(\lambda', J)$ are first expanded in powers of $1/J$, followed by an expansion in $\lambda'$ for comparison with the gauge theory. Conversely, the gauge theory (spin chain) charges $Q_k(\lambda, R)$ are derived perturbatively near $\lambda = 0$, and a subsequent expansion in the $R$-charge (or spin chain length $L$) is performed for comparison with the string side. In [17] it was shown that the order in which these limits are taken can lead to an erroneous thermodynamic limit ($L \to \infty$) in certain spin chain systems (a limit that is naturally well-defined in the string theory). While this particular problem is resolved in [19], it provides a concrete example of how these issues can lead to superficially discordant results between both sides of the duality. A more recent suggestion involving spin-chain wrapping interactions was given in [19]. Wrapping terms, characterized as having an interaction range greater than the length of the spin chain, should naturally affect corrections to the BMN limit that specifically incorporate finite-length effects. In the gauge theory, the loop expansion followed by the thermodynamic limit is expected to drop wrapping terms, while the inverse operation on the string side is expected to include these effects. At present these considerations have not been realized in any quantitative fashion. To include these effects in the spin chain analysis, one would have to sum all perturbative loop corrections prior to taking the thermodynamic limit. This is a daunting proposal but, in light of the recent developments in [19], such a computation may soon be within reach.

At this stage we do not have a precise algorithm for rescuing integrability in the string theory or interpreting the failure thereof in the context of the AdS/CFT correspondence. If the string spectrum is to remain internally consistent at zeroth-order in $\lambda'$, however, it must also align with integrability expectations at that order. One possibility is that the small-$\lambda'$ expansion should not be executed before computing corrections associated with second-order intermediate state sums mediated by $H^{(2)}$. With our current computing capabilities, however, this has not yet been possible. When this problem is solved and higher-order $\lambda'$ corrections to the string spectrum are successfully computed at $O(1/J^2)$, the methods developed here will provide a simple and concrete test of integrability and of any mechanism that hopes to resolve the standing mismatch between string and gauge theory at three-loop order in $\lambda'$.

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