Characterizing the second smallest eigenvalue of the normalized Laplacian of a tree

Israel Rocha

May 7, 2014

Abstract

In this paper we show a monotonicity theorem for the harmonic eigenfunction of \( \lambda_1 \) of the normalized Laplacian over the points of articulation of a graph. We introduce the definition of Perron component for the normalized Laplacian matrix of a graph and show how its second smallest eigenvalue can be characterized using this definition.

1 Main Concepts

As usual, in this paper a graph is a pair of sets \( G = (V, E) \), where the elements of \( E \) are subsets of two elements of \( V \). The elements of \( V \) are vertices of the graph and the elements of \( E \) are its edges.

Given a graph \( G = (V, E) \) on \( n \) vertices, the normalized Laplacian matrix of \( G \) is the matrix of order \( n \) \( \mathcal{L}(G) \) given by

\[
\mathcal{L}(v_i, v_j) = \begin{cases} 
1, & v_i = v_j \text{ and } d_{v_i} \neq 0; \\
-\frac{1}{\sqrt{d_{v_i}d_{v_j}}}, & \text{whenever } v_i \text{ and } v_j \text{ are adjacent}; \\
0, & \text{otherwise}.
\end{cases}
\]

Also the Laplacian matrix of \( G \) is the matrix of order \( n \) given by

\[
L(v_i, v_j) = \begin{cases} 
\sqrt{d(v_i)}, & v_i = v_j; \\
-1, & \text{whenever } v_i \text{ and } v_j \text{ are adjacent}; \\
0, & \text{otherwise}.
\end{cases}
\]

In the survey [6], some known results about on Laplacian matrix are exhibit. Fiedler in [2], has shown that a graph is connected if and only if the second smallest Laplacian eigenvalue is positive. This eigenvalue is called algebraic connectivity and plays a fundamental role in the field of Spectral Graph Theory.

Throughout this paper, \( G \) does not have isolated vertices. In that case \( D \) is invertible and \( \mathcal{L} \) and \( L \) are related by the formula

\[
\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}.
\]
We consider the normalized laplacian matrix $L$ of a tree and, following the notation of [1], we denote the eigenvalues of $L$ by $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$. Let $g$ denote an function which assigns to each vertex $v$ of $G$ a real value $g(v)$. We can view $g$ as a column vector and whenever $Lg = \lambda g$ we call $g$ an eigenfunction of $L$. We define the harmonic eigenfunction of $\lambda$ as $f = D^{-\frac{1}{2}}g$.

The following result, that we can find at [4], concerns the harmonic eigenfunction of $\lambda_1$.

Theorem 1. Let $G$ be a connected graph and $L(G)$ be its normalized Laplacian matrix. Let $f$ be a harmonic eigenfunction corresponding to $\lambda_1$ and $v$ be a cut vertex of $G$, let $G_0, G_1, \ldots, G_r$ be all connected components of the graph $G\setminus v$. Then:

1. If $f(v) > 0$ then exactly one of the components $G_i$ contains a vertex negatively valuated by $f$. For all vertices $u$ in the remaining components $f(u) > f(v)$.

2. If $f(v) = 0$ and there exists a component $G_i$ containing both positively and negatively valuated vertices, then there is exactly one such component, all remaining components being zero valuated.

3. If $f(v) = 0$ and no component contains both positively and negatively valuated vertices then each component contains either only positively valuated, or negatively valuated, or zero valuated vertices.

We notice that this result is similar to the result of Fiedler [2] where the eigenvector associated with the algebraic connectivity was considered. In this paper we show a property of the harmonic eigenfunction of $\lambda_1$ over the points of articulation which, likewise in [2], enables us to classify every graph in two distinct families. Also, we introduce the definition of Perron component for the normalized Laplacian matrix of a graph, and using this we can provide a characterization for $\lambda_1$ in terms of this definition. Moreover, we introduce the notion of normalized bottleneck matrix of a branch of a tree which allow us to easily describe $\lambda_1$. Furthermore, we shall perform a more careful analysis on the structure of normalized bottleneck matrices in order to understand how $\lambda_1$ behaves when we change the structure of trees.

2 Monotonicity Theorem

In this section we show an interesting property of the harmonic eigenfunction of $\lambda_1$ over the points of articulation of a graph. We shall provide a monotonicity theorem for such harmonic eigenfunction. This enable us to classify every graph in two distinct families.

First, a block of a graph is a maximal induced connected subgraph not containing a point of articulation. A path is said to be pure if it contains at most two points of articulation of each block.

Theorem 2. Let $G$ be a connected graph and let $f$ be the harmonic eigenfunction for $\lambda_1$. Then only one of the following cases can occur:
Case 1 There is no mixed block. In this case, there is a unique point of articulation \( z \) having \( f(z) = 0 \) and a nonzero neighbor. Each block (with the exception of the vertex \( z \)) is either a positive block, or a negative block, or a zero block. Let \( P \) be a pure path which starts at \( z \). Then the \( f \) at the points of articulation (with the exception of \( z \)) form either an increasing, or decreasing, or a zero sequence. Every path containing both positive and negative vertices passes through \( z \).

Case 2 There is a unique block \( B_0 \) which is mixed. In this case, each remaining block is positive, negative or null. Moreover, each pure path \( P \) starting in \( B_0 \) and containing only one vertex \( v \in B_0 \) has the property that \( f \) at the points of articulation contained in \( P \) form either an increasing, or decreasing, or a zero sequence according to whether \( f(v) > 0 \), \( f(v) < 0 \) or \( f(v) = 0 \). In the last case \( f \equiv 0 \) along the path.

Proof. First, for case 1, if no block is mixed, since \( \sum d_v f(v) = 0 \), there is a path containing both positive and negative vertices. We claim that \( P \) has a vertex \( z \) with \( f(z) = 0 \) and a nonzero neighbor. Indeed, the intersection of blocks has only articulation points and no block is mixed, it follows that exists such vertex. Thus, it follows from Theorem 1 that part (3) must occurs. Therefore, there is no other vertex \( v \neq z \) having \( f(v) = 0 \) and a nonzero neighbor. This shows the first part of case 1.

Now, if \( P \) contains another vertex \( v \) with \( f(v) = 0 \), part (3) of Theorem 1 ensures that \( f \equiv 0 \) over the vertex of \( P \). On the other hand, if \( P \) has a vertex \( v \) with \( f(v) \neq 0 \) then part (1) of Theorem 1, we obtain that \( f \) does not change sign neither vanish over \( P \). Denote by \( z = v_0, v_1, \ldots, v_s \) the points of articulation at \( P \) in the order they appear. If \( f(v) > 0 \), then by part (1) of Theorem 1 we obtain \( f(v_i) < f(v_{i+1}) \), \( i = 0, \ldots, s - 1 \). If \( f(v) < 0 \), then the same argument applied to the eigenfunction \( -f \), shows that this form a decreasing sequence.

Now we proceed proving case 2. If \( G \) has only one block, then we are done. Otherwise, denote by \( B_1 \) some other block different of \( B_0 \). In this case, there is a articulation point \( v \) separating them. Let \( G_0, G_1, \ldots, G_r \) be the connected components of \( G \setminus v \), where \( G_0 \) contains \( B_0 \) and \( G_1 \) contains \( B_1 \). If \( f(v) > 0 \) (or \( f(v) < 0 \)), then by part (1) of Theorem 1, we obtain that \( f \) has the same sign over \( G_1 \). If \( f(v) = 0 \), then using part (2) of Theorem 1, we obtain that \( f \equiv 0 \) over \( G_1 \). This completes the first part of case 2.

Finally, denote by \( v = v_0, v_1, \ldots, v_s \) the points of articulation at \( P \) in the order they appear. If \( f(v) > 0 \), then by part (1) of Theorem 1 we obtain \( f(v_i) < f(v_{i+1}) \), \( i = 0, \ldots, s - 1 \). If \( f(v) < 0 \), then the same argument applied to the eigenfunction \( -f \), shows that this form a decreasing sequence. If \( f(v) = 0 \), then using part (2) of Theorem 1, we obtain that \( f \equiv 0 \) over the vertices of \( P \). This concludes the proof.

Remark 3. Since \( \text{sign}(f(v)) = \text{sign}(g(v)) \) for each vertex \( v \) at \( G \), we can provide the following result which is straightforward from Theorem 4.

Theorem 4. Let \( G \) be a connected graph and \( L(G) \) be its normalized Laplacian matrix. Let \( g \) be a eigenfunction corresponding to \( \lambda_1 \) and \( v \) be a cut vertex of
Let \( G, G_0, G_1, ..., G_r \) be all connected components of the graph \( G \setminus v \). Then:

1. If \( g(v) > 0 \) then exactly one of the components \( G_i \) contains a vertex negatively valuated by \( g \).
2. If \( g(v) = 0 \) and there exists a component \( G_i \) containing both positively and negatively valuated vertices, then there is exactly one such component, all remaining components being zero valuated.
3. If \( g(v) = 0 \) and no component contains both positively and negatively valuated vertices then each component contains either only positively valuated, or negatively valuated, or zero valuated vertices.

We notice that in part (1) of Theorem 4 unlike Theorem 1, we can not guarantee that \( g(u) > g(v) \), since degree \( d_v \) could be much larger than \( d_u \). Remark 3 and Theorem 2 give us the following result.

**Theorem 5.** Let \( G \) be a connected graph and let \( g \) be the eigenfunction for \( \lambda_1 \). Then only one of the following cases can occur:

**Case 1** There is no mixed block. In this case, there is a unique point of articulation \( z \) having \( g(z) = 0 \) and a nonzero neighbor. Each block (with the exception of the vertex \( z \)) is either a positive block, or a negative block, or a zero block.

**Case 2** There is a unique block \( B_0 \) which is mixed. In this case, each remaining block is positive, negative or null.

Henceforth, we use Theorem 5 as it describes directly the valuation of an eigenvector at the vertices of \( G \).

### 3 Characterizing the Second Smallest Eigenvalue

Despite of giving classification of graphs and a good insight about the behavior of the harmonic eigenfunction, Theorems 2 and 5 do not give us information about \( \lambda_1 \) itself. However, we can provide an alternative characterization for cases 1 and 2 such that information about \( \lambda_1 \) arises.

More precisely, in this section we are interested in describe \( \lambda_1 \) in terms of the Perron value of special matrices. This results were inspired by [5].

Consider the normalized Laplacian matrix \( \mathcal{L}(G) \) for a graph \( G \). The relation between the matrix \( L(G) \) and \( \mathcal{L}(G) \) is well-known, and it is given by

\[
\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}},
\]

where \( D \) is the degree matrix.

We shall denote by \( M_k \), the principal submatrix of \( M \) formed by removing the \( k \)-th row and column of \( M \). Now, consider the matrix \( \mathcal{L}_k \). We call normalized bottleneck matrix of a component \( C \) at \( k \), the corresponding block at \( \mathcal{L}^{-1}_k \).

If we call \( N = \mathcal{L}^{-1}_k(C) \) the normalized bottleneck matrix of the component \( C \), we say that it is a Perron component if it has largest \( \rho(N) \) among all components.

Let \( T \) be a tree. We call branch of \( T \) at \( k \) some of the connected components of \( T - k \) obtained from \( T \) by deleting the vertex \( k \) and its edges. If \( T \) satisfies
Let $\text{Theorem 5}$ then we say $T$ is a Type 1 tree. If $T$ satisfies case 2 of $\text{Theorem 5}$ then we say $T$ is a Type 2 tree.

If $T$ is a Type 1 tree, then the only null vertex adjacent to a non-null vertex (see $\text{Theorem 5}$) is said to be the characteristic vertex of $T$.

If $T$ is a Type 2 tree, by $\text{Theorem 5}$ the only mixed block is formed by only two adjacent vertices. For a Type 2 tree, we say that two vertices $i$ and $j$ are characteristic vertices if and only if they are adjacent and satisfies $\text{sign}(g(i)) \neq \text{sign}(g(j))$.

**Theorem 6.** Let $T$ be a tree and $g$ a eigenfunction of $\lambda_1$. Then $T$ is a Type 1 tree with characteristic vertex $v$ if and only if there are at least two Perron branches at $v$. In this case, $\lambda_1 = \frac{1}{\rho(L(C)^{-1})}$ for each Perron branch $C$ at $v$.

**Proof.** Suppose that $T$ is a Type 1 tree and $v$ is its characteristic vertex. Let $C_0, C_1, \ldots, C_r$ be the branches of $T \setminus v$ and assume the normalized Laplacian matrix is in the form

$$L = \begin{bmatrix}
    L(C_0) & 0 & \cdots & 0 & c_0 \\
    0 & L(C_1) & 0 & c_1 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & L(C_r) & c_r \\
    (c_0)^T & (c_1)^T & \cdots & (c_r)^T & d_v
\end{bmatrix}, \quad (3.1)
$$

where $L(C_i)$ corresponds to vertices of the connected component $C_i$, for $i = 0, 1, \ldots, r$ and $c_i$ is a 0,1 vector that accounts for the edges between the vertex $v$ and the connected component $C_i$. For convenience, we assume that the last rows and columns of $L$ represent the vertex $v$.

We can define functions $g^{(i)}$ over each branch $C_i$ as $g^{(i)}(x) = g(x)$ where $x \in C_i$. From the relation $Lg = \lambda_1 g$, we have

$$L(C_i)g^{(i)} = \lambda_1 g^{(i)},$$

since $g(v) = 0$. From the fact that $\sum \sqrt{d_i}g(x) = 0$, we know $g$ assumes positive and negative values. Applying case 1 of $\text{Theorem 5}$ we notice that there are functions $g^{(r)} > 0$ e $g^{(s)} < 0$. Using Perron-Frobenius theorem, the only positive eigenfunction are the Perron vector. Hence, $g^{(r)}$ and $g^{(s)}$ are Perron vectors for $L(C_i)^{-1}$ and $L(C_s)^{-1}$, respectively. If $g^{(i)}(x) = 0$ for some $x \in C_i$, by case 1 of $\text{Theorem 5}$, $g^{(i)} \equiv 0$ and it is not a Perron branch. Therefore, for each non-null branch we have the relation for the Perron vector $g^{(i)}$

$$L(C_i)^{-1}g^{(i)} = \frac{1}{\lambda_1(G)}g^{(i)}.$$

It remains to show that $C_r$ and $C_s$ are Perron branch at $v$. Suppose, by contradiction, that it is not true. Then it would exist another component, say $C_3$, such that the Perron value is larger than $1/\lambda_1$. We call $z$ the Perron vector of $L(C_3)^{-1}$, normalized so that $1^T D_3^\frac{1}{2} z = 1/2$, where $D_3$ is the diagonal degree
matrix of $C_3$. Also, we define $u = g^{(r)}/\sqrt{2}D_1 g^{(r)}$, where $D_1$ is the diagonal degree matrix of $C_r$. Thereof, we consider the vector
\[ w = [u, 0, \ldots, 0, -z, 0, \ldots, 0]^T, \]
which is obviously orthogonal to $D_1$, where $D_1$ is the diagonal degree matrix of $T$, and also $\|w\| = 1$. Since
\[ w^T L w = \lambda_1 u^T u + \frac{1}{\rho(L(C_3)^{-1})} z^T z < \lambda_1 u^T u + \lambda_1 z^T z = \lambda_1 w w^T \]
we obtain a contradiction with the fact that
\[ \lambda_1(T) = \min_{\|x\| = 1} x^T L x. \]
Thus, we obtain that $C_r$ and $C_s$ are indeed the Perron branches at $v$. This concludes the first part.

Conversely, assume that there are at least two Perron branches at vertex $v$, let us say $C_i$ and $C_j$ are two of them. Let $y$ and $z$ be the Perron vectors of $L(C_i)^{-1}$ and $L(C_j)^{-1}$, respectively. Taking into account (3.1), we can make $y$ and $z$ normalized such that $c_{ij}^T x - c_{ij}^T y = 0$. Now, we define the function $g$ as
\[
\begin{cases}
  g(u) = y(u) & u \in C_i; \\
  g(u) = -z(u) & u \in C_j; \\
  0 & \text{otherwise}.
\end{cases}
\]
Hence, we have the relation $Lg = \frac{1}{\rho(L(C_i)^{-1})} g$. It is easy to see that if $\lambda_1(T) = \frac{1}{\rho(L(C_i)^{-1})}$, then $g$ is an eigenfunction that makes $T$ a Type 1 tree with characteristic vertex $v$, since $g(v) = 0$.

In order too see that $\lambda_1(T) = \frac{1}{\rho(L(C_i)^{-1})}$, consider the submatrix $M$ of $L$ obtained by deleting the column and row corresponding to vertex $v$. It is easy to see that the eigenvalues of $M$ are the union of eigenvalues of all matrices $L(C_t)$, for $t = 0, 1, \ldots, r$. Since $\rho(L(C_i)^{-1})$ is the largest among all branches, we can say that $\frac{1}{\rho(L(C_i)^{-1})}$ is the smallest eigenvalue of $M$. In fact, we have at least two eigenvalues equal to $\frac{1}{\rho(L(C_i)^{-1})}$. Therefore, we by the interlacing property of eigenvalues for principal submatrices, we obtain
\[
\frac{1}{\rho(L(C_i)^{-1})} \leq \lambda_1(T) \leq \frac{1}{\rho(L(C_i)^{-1})}.
\]
This shows the theorem.

The previous theorem is a natural application of the same method used in [5] where, in the context of Laplacian matrix, it was characterized the algebraic connectivity for Type I trees. However, if we want to find some characterization for Type 2 trees using the normalized Laplacian matrix, we must perform a different calculation in order to obtain matrices that characterize $\lambda_1$. As the next theorem show us, these matrices are more complicated than those in [5].
Theorem 7. Let $T$ be a tree on $n$ vertices with normalized Laplacian matrix $\mathcal{L}$ and let $i$ and $j$ be adjacent vertices of $T$. For $i$ and $j$ be characteristic vertices of $T$ it is necessary and sufficient that there exists a $\gamma \in (0, 1)$ such that

$$\rho(M_1 - \gamma \frac{1}{\sqrt{d_i d_j}} e_k e_k^T) = \rho(M_2 - (1 - \gamma) \frac{1}{\sqrt{d_i d_j}} e_k e_k^T) = \frac{1}{\lambda_1},$$

where $M_1$ is the normalized bottleneck matrix for the branch at $j$ containing $i$ and $D_1$ is the degree matrix of this branch; $M_2$ is the normalized bottleneck matrix for the branch at $i$ containing $j$ and $D_2$ is the degree matrix of this branch.

Proof. We can put the normalized Laplacian matrix of $T$ in the following format

$$\mathcal{L} = \begin{bmatrix} M_1^{-1} & -\frac{1}{\sqrt{d_i d_j}} e_k e_k^T \\ -\frac{1}{\sqrt{d_i d_j}} e_k e_k^T & M_2^{-1} \end{bmatrix},$$

where the last row of $M_1^{-1}$ represents the vertex $i$ and the first row of $M_2^{-1}$ represents the vertex $j$.

First, we suppose that $i$ and $j$ are characteristic vertices of $T$. By using part (1) of the Theorem 4, we have that both branchs at $i$ and at $j$ have the same sign each. Moreover, the theorem ensures that we can write the eigenvector associated with $\lambda_1$ as $v = [-v_1, v_2]^T$, where $v_1$ and $v_2$ are both positive vectors. Since $1^T D_2^\frac{1}{2} v = 0$, we have $1^T D_1^\frac{1}{2} v_1 = 1^T D_2^\frac{1}{2} v_1$.

From the equation $\mathcal{L} v = \lambda_1 v$, if we set $\alpha = e_1^T v_2$ and $\beta = e_k^T v_1$, we find that

$$-M_1^{-1} v_1 - \frac{\alpha}{\sqrt{d_i d_j}} e_k = -\lambda_1 v_1,$$

which we can rewrite as

$$\frac{v_1}{\lambda_1} = M_1 v_1 - \frac{\alpha}{\lambda_1 \sqrt{d_i d_j}} M_1 e_k.$$

Using Lemma 9 we conclude that $M_1 e_k = \sqrt{d_i} D_1^\frac{1}{2} 1$, because $|P_{a,i,j}| = 1$ for any vertex $a$ in the branch at $j$ containing $i$. Hence, we have

$$\frac{v_1}{\lambda_1} = M_1 v_1 - \frac{\alpha}{\lambda_1 \sqrt{d_i d_j}} D_1^\frac{1}{2} 1. \quad (3.2)$$

Now we multiply $e_k^T$ by (3.2), to obtain

$$\frac{e_k^T v_1}{\lambda_1} = e_k^T \left( M_1 v_1 - \frac{\alpha}{\lambda_1 \sqrt{d_i d_j}} D_1^\frac{1}{2} 1 \right) = \sqrt{d_i} 1^T D_1^\frac{1}{2} v_1 - \frac{\alpha \sqrt{d_i}}{\lambda_1 \sqrt{d_j}}.$$
Hence, we obtain
\[
\frac{\beta}{\lambda_1} = \sqrt{d_i} 1^T D^\frac{1}{2}_1 v_1 - \frac{\alpha \sqrt{d_i}}{\lambda_1 \sqrt{d_j}}
\]
which can be rewritten as
\[
\frac{\beta \sqrt{d_i d_j}}{\beta \sqrt{d_j + \alpha \sqrt{d_i}}} 1^T D^\frac{1}{2}_1 v_1 = \frac{1}{\lambda_1}.
\]
Now, we replace \( \frac{1}{\lambda_1} \) in (5.2), to obtain
\[
\frac{v_1}{\lambda_1} = M_1 v_1 - \frac{\alpha \sqrt{d_i d_j}}{\sqrt{d_j} (\beta \sqrt{d_j} + \alpha \sqrt{d_i})} 1^T D^\frac{1}{2}_1 v_1 D^\frac{1}{2}_1 1
\]
\[
= M_1 v_1 - \frac{\alpha \sqrt{d_i}}{\beta \sqrt{d_j} + \alpha \sqrt{d_i}} D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 v_1
\]
Therefore, we have
\[
\frac{v_1}{\lambda_1} = \left( M_1 - \frac{\alpha \sqrt{d_i}}{\beta \sqrt{d_j} + \alpha \sqrt{d_i}} D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 \right) v_1.
\]
The same calculation for the matrix \( M_2 \), gives us the relation
\[
\frac{v_2}{\lambda_1} = \left( M_2 - \frac{\beta \sqrt{d_j}}{\beta \sqrt{d_j} + \alpha \sqrt{d_i}} D^\frac{1}{2}_2 11^T D^\frac{1}{2}_2 \right) v_1.
\]
Now, from Lemma (9), we conclude that \( |P_{a,b,i}| \geq 1 \) and \( |P_{a,b,j}| \geq 1 \), since the edge between \( i \) and \( j \) is in any set of that form. Hence, we have \( M_1 \geq D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 \) and \( M_2 \geq D^\frac{1}{2}_2 11^T D^\frac{1}{2}_2 \). Besides, if we define \( \gamma = \frac{\beta \sqrt{d_j}}{\beta \sqrt{d_j} + \alpha \sqrt{d_i}} \) and notice that \( \gamma \in (0, 1) \), we conclude that \( v_1 \) is a positive eigenvector of the positive matrix \( M_1 - \gamma D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 \) and that \( v_2 \) is a positive eigenvector for the matrix \( M_2 - (1 - \gamma) D^\frac{1}{2}_2 11^T D^\frac{1}{2}_2 \). Therefore, from the Perron-Frobenius theory, we have
\[
\rho(M_1 - \gamma D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 ) = \rho(M_2 - (1 - \gamma) D^\frac{1}{2}_2 11^T D^\frac{1}{2}_2 ) = \frac{1}{\lambda_1},
\]
as required.

Reciprocally, assume that there is a \( \gamma \in (0, 1) \) that satisfies \( \rho(M_1 - \gamma D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 ) = \rho(M_2 - (1 - \gamma) D^\frac{1}{2}_2 11^T D^\frac{1}{2}_2 ) = \frac{1}{\lambda_1} \), where \( v_1 \) and \( v_2 \) are the Perron vectors of \( M_1 - \gamma D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 \) and \( M_2 - (1 - \gamma) D^\frac{1}{2}_2 11^T D^\frac{1}{2}_2 \), respectively. Then we can compute
\[
\frac{e_k^T v_1}{\lambda_1} = e_k^T \left( M_1 - \gamma D^\frac{1}{2}_1 11^T D^\frac{1}{2}_1 \right) v_1
\]
\[
= \left( \sqrt{d_i} 1^T D^\frac{1}{2}_1 - \gamma \sqrt{d_i} 1^T D^\frac{1}{2}_1 \right) v_1
\]
\[
= (1 - \gamma) \sqrt{d_i} 1^T D^\frac{1}{2}_1 v_1.
\]
Also, we can choose the eigenvectors $v_1$ and $v_2$ normalized such that $1^T D_1^{\frac{1}{2}} v_1 = 1^T D_2^{\frac{1}{2}} v_1$, and then we can write

$$\frac{e^T v_1}{\lambda_1} = (1 - \gamma) \sqrt{d_1} 1^T D_2^{\frac{1}{2}} v_2. \quad (3.3)$$

Similarly, using the same procedure, we can compute

$$e_1^T \left( M_2 - (1 - \gamma) D_2^{\frac{1}{2}} 11^T D_2^{\frac{1}{2}} \right) v_2$$

to obtain the relation

$$\frac{e^T v_2}{\lambda_1} = \gamma \sqrt{d_j} 1^T D_1^{\frac{1}{2}} v_1. \quad (3.4)$$

Using the relation (3.3) in the equation $\left( M_2 - (1 - \gamma) D_2^{\frac{1}{2}} 11^T D_2^{\frac{1}{2}} \right) v_2 = \frac{1}{\lambda_1} v_2$, we obtain

$$\frac{1}{\lambda_1} v_2 = M_2 v_2 - (1 - \gamma) D_2^{\frac{1}{2}} 11^T D_2^{\frac{1}{2}} v_2$$

$$= M_2 v_2 - \frac{1}{\lambda_1 \sqrt{d_i}} D_2^{\frac{1}{2}} 1 e_k^T v_1.$$ 

By applying Lemma 9, we use the relation $M_2 e_1 = \sqrt{d_j} D_2^{\frac{1}{2}} 1$ to get

$$\frac{1}{\lambda_1} v_2 = M_2 v_2 - \frac{1}{\lambda_1 \sqrt{d_i d_j}} M_2 e_1 e_k^T v_1,$$

which is equivalent to

$$\lambda_1 v = M_2^{-1} v_2 + \frac{1}{\sqrt{d_i d_j}} e_k e_1^T v_1. \quad (3.5)$$

In the same way, we can use the relation (3.3) and then rewrite the equation

$$\left( M_1 - \gamma D_1^{\frac{1}{2}} 11^T D_1^{\frac{1}{2}} \right) v_1 = \frac{1}{\lambda_1} v_1$$

as follows

$$- \lambda_1 v_1 = - M_1^{-1} v_1 + \frac{1}{\sqrt{d_i d_j}} e_k e_1^T v_2. \quad (3.6)$$

Therefore, equation (3.5) and (3.6) show that the vector $v = [-v_1 \mid v_2]^T$ satisfies $L v = \lambda_1 v$. This proofs the result. \qed
4 Normalized Bottleneck Matrix

The previous section pointed us to the bottleneck matrices in order to characterize $\lambda_1$ of trees. Hence, in this section we shall perform a more careful analysis on the structure of these matrices with the expectation of giving prolific results about $\lambda_1$. In fact, it allows us to extremize the $\lambda_1$ over the set of trees.

First, we define the set $P_{i,j,k}$ as the set of edges of $T$ which are on both the path from vertex $i$ to vertex $k$ and the path from the vertex $j$ to vertex $k$. The following lemma was obtained by Kirkland in [5], where it was investigated Perron components of trees using the Laplacian matrix.

Lemma 8. Consider a tree $T$ at $n$ vertex. Denote by $L_k$ the principal submatrix of the Laplacian matrix $L(T)$ obtained by deleting the $k$–th column and the $k$–the row from $L(T)$. Then the entry $(i, j)$ of $L_k^{-1}$ equals to the number of edges at $P_{i,j,k}$.

The following lemma concerns the normalized Laplacian, and also we can describe the entries of $L_k^{-1}$.

Lemma 9. Consider a tree $T$ with $n$ vertex. Then $(i, j)$ entry of $L_k^{-1}$ is equal to $\sqrt{d_i d_j} |P_{i,j,k}|$.

Proof. We observe that, since $D$ is a diagonal matrix, then

$$L_k = \left( D^{-\frac{1}{2}} L D^{-\frac{1}{2}} \right)_k = D_k^{-\frac{1}{2}} L_k D_k^{-\frac{1}{2}}.$$

Thus, it is straightforward to obtain $L_k^{-1} = D_k^\frac{1}{2} L_k^{-1} D_k^\frac{1}{2}$. By applying Lemma 8 we obtain that the $(i, j)$ entry of $L_k^{-1}$ is equal to

$$\left( D_k^\frac{1}{2} L_k^{-1} D_k^\frac{1}{2} \right)_{i,j} = \left( D_k^\frac{1}{2} \right)_{i,i} |P_{i,j,k}| \left( D_k^\frac{1}{2} \right)_{j,j} = \sqrt{d_i d_j} |P_{i,j,k}|.$$

The next result describes Perron branches of trees in a similar fashion to [5].

Lemma 10. $T$ is a Type 2 tree with characteristic $i$ and $j$ if and only if $i$ and $j$ are adjacent and the branch at $i$ containing vertex $j$ is the unique Perron branch at $i$, while the branch at $j$ containing $i$ is the unique Perron branch at $j$.

Proof. By Theorem 7 for $T$ be a Type 2 it is necessary and sufficient that there exists a $\gamma \in (0, 1)$ such that

$$\rho(M_1 - \gamma D_1^\frac{1}{2} 11^T D_1^\frac{1}{2}) = \rho(M_2 - (1 - \gamma) D_2^\frac{1}{2} 11^T D_2^\frac{1}{2}) = \frac{1}{\lambda_1}.$$

We consider the values $\rho(M_1 - xD 11^T D_1^\frac{1}{2})$ and $\rho(M_2 - (1 - x) D_2^\frac{1}{2} 11^T D_2^\frac{1}{2})$ as functions of $x$, and notice that they are decreasing and increasing, respectively. Also, they are both continuous functions, hence the existence of such $\gamma \in (0, 1)$ is equivalent to $\rho(M_1 - D 11^T D_1^\frac{1}{2}) < \rho(M_2)$ and $\rho(M_1) > \rho(M_2 - D_2^\frac{1}{2} 11^T D_2^\frac{1}{2})$. 

Using the description of the entries of $M_1$ and $M_2$ given by Lemma 9, it is easy to see that the matrix $M_2 - D_2^\frac{1}{2}11^TD_2^\frac{1}{2}$ is similar to the bottleneck matrix of the branches at $j$ which do not contain $i$. Also, we the matrix $M_1 - D_1^T D_1^\frac{1}{2}$ is similar to the bottleneck matrix of the branches at $i$ which do not contain $j$. Therefore, the inequalities $\rho(M_1 - D_1^T D_1^\frac{1}{2}) < \rho(M_2 - D_2^\frac{1}{2}11^TD_2^\frac{1}{2})$ holds if and only if the branch at $i$ containing vertex $j$ is the unique Perron branch at $i$, while the branch at $j$ containing $i$ is the unique Perron branch at $j$. 

The following result provides a simple way to characterize Type 1 and Type 2 trees.

**Theorem 11.** Let $T$ be a tree. $T$ is a Type 1 tree if and only if there is only one vertex such that there are at least two Perron branches. $T$ is a Type 2 tree if and only if at each vertex there is a unique Perron branch.

**Proof.** First, assume that there is only one vertex such that there are at least two Perron branches. Then by Theorem 6, $T$ is a Type 1 tree. Conversely, assume that $T$ is a Type 1 tree with characteristic vertex $v$. Take any branch at some vertex $u \neq v$. Let $P$ be the branch at $u$ containing $v$ and $Q$ be any other branch at $u$. Let $C$ be the component at $v$ that contains $u$. In light of Lemma 9, we can see that $L(Q)^{-1} \leq L(C)^{-1} \leq L(P)^{-1}$ with the strict inequality in at least one entry. Hence we conclude that $\rho(L(Q)^{-1}) < \rho(L(P)^{-1})$ and that there is only one Perron component at $u$.

If $T$ is a Type 2 tree, then by Lemma 10 there are a pair of adjacent vertex $i$ and $j$ such there is a unique Perron branch at each one. If we consider a vertex different from $i$ and $j$, then we can use the same argument of the previous part to conclude that there is only one Perron branch at this vertex. Finally, assume that at each vertex there is a unique Perron branch. If $T$ is not a Type 2 tree, then we have a contradiction with Theorem 6. This completes the theorem. 

**Corollary 12.** Let $T$ be a tree and $u$ a vertex which not be its characteristic vertex. Then the unique Perron branch at $u$ is the branch containing the characteristic vertex or vertices of $T$.

**References**

[1] Chung, Fan R.K. Spectral graph theory, Issue 92, 1997.

[2] Fiedler, M. Algebraic connectivity of graphs, Czechoslovak Mathematical Journal, Vol. 23 (1973), 298-305.

[3] Fiedler, M. A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 4, 619-633
[4] Hong-Hai Li, Jiong-Sheng Li & Yi-Zheng Fan. The effect on the second smallest eigenvalue of the normalized Laplacian of a graph by grafting edges, Linear and Multilinear Algebra, 56:6, 627-638, 2008.

[5] S. Kirkland, M. Neumann and B. Shader, Characteristic Vertices of Weighted Trees Via Perron Values, Linear and Multilinear Algebra 40 (1996), 311-325.

[6] Merris, R. Laplacian Matrices of Graphs: A Survey, Linear Algebra and its Applications, Vol. 197 (1994), 198:143-176.