Local perturbations of conservative $C^1$ diffeomorphisms

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Abstract

A number of techniques have been developed to perturb the dynamics of $C^1$-diffeomorphisms and to modify the properties of their periodic orbits. For instance, one can locally linearize the dynamics, change the tangent dynamics, or create local homoclinic orbits. These techniques have been crucial for the understanding of $C^1$ dynamics, but their most precise forms have mostly been shown in the dissipative setting. This work extends these results to volume-preserving and especially symplectic systems. These tools underlie our study of the entropy of $C^1$-diffeomorphisms in Buzzi et al (2016 (arXiv:1606.01765)). We also give an application to the approximation of transitive invariant sets without genericity assumptions.

Keywords: dominated splitting, conservative diffeomorphisms, perturbation

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1. Introduction

According to often-cited words of Poincaré, periodic and heteroclinic orbits provide a ‘breach into the fortress’ [34, p 2] that is differentiable dynamics. This key insight is still relevant today: indeed, the closing and connecting lemmas established [9, 26, 30, 36] in the $C^1$-topology lead to approximation by periodic orbits of chain-transitive sets and ergodic measures for $C^1$-generic systems (the $C^1$-topology offers flexibility while preserving the differentiable structure). Some key dynamical properties, such as the existence or lack of a dominated splitting

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on the tangent bundle, can then be detected on the periodic orbits. This text will focus on the perturbative approach.

1.1. The dissipative case

The first results about linearization and modification of the tangent dynamics were motivated by the $C^1$-stability conjecture: Franks [23] proved that one can perturb the tangent dynamics over a given finite set by a small $C^1$-perturbation. With this technique, the possible changes to the Lyapunov exponents and the angle between stable and unstable spaces were first studied by Pliss [33], Liao [29], and Mañé [30]. This very local analysis has been systematized in [15] and more recently in [12].

The investigation of robust transitivity [13, 22] led to a new approach where one is allowed to choose the periodic orbit supporting the perturbation. Taking advantage of homoclinic orbits, a notion of ‘transition’ (close to the specification property) was used to find periodic points exhibiting the properties necessary for the perturbations. This second technique is powerful but requires that the initial system already has homoclinic orbits. Moreover the perturbation is realized along a periodic orbit which may approximate a large set. Hence the perturbation is not local and its support is difficult to control.

Perturbations with controlled support creating rich dynamics have been recently built in [24, 38, 44]. These results yield homoclinic tangencies and transverse homoclinic orbits in an arbitrarily small neighborhood of a hyperbolic periodic orbit. Combining an improved Franks lemma with perturbative results of linear systems, Gourmelon in [24, 25] has shown how to perform various perturbations while preserving homoclinic relations. This is crucial in many applications, e.g. in order to work inside a given homoclinic class, see [10, 11, 35] among others.

For a survey of the dissipative case, we refer to [21].

1.2. The conservative case

For volume-preserving or symplectic systems, this topic has not been so systematically investigated. It has been shown that elliptic points of symplectic diffeomorphisms characterize a lack of hyperbolicity [4, 5, 19, 27, 32, 41]. We mention that some works in smooth ergodic theory (dealing with the tangent dynamics over Lebesgue-almost every point instead of over periodic orbits) have developed related arguments, see [7, 8]. Other interesting properties pertaining to the entropy have been studied [3, 17, 18, 20]. We note that these recent results mainly use the transition approach from [13] and do not provide local versions in the symplectic setting.

1.3. Results

Our goal is to systematically extend the perturbation tools to the conservative settings, trying as much as possible to follow the local approach of [12]. Let us list our results (deferring the precise statements):

- Franks’ lemma, linearization and preservation of homoclinic connections (theorems 3.1 and 3.3);
- perturbation of the spectrum of periodic orbits: achieving simplicity (proposition 4.1), realness (proposition 4.2) or equal modulus for the stable and for the unstable eigenvalues (theorem 4.9);
– further perturbations of the tangent dynamics above periodic orbits: making the angle between stable and unstable spaces arbitrarily small (theorem 4.5);
– birth of homoclinic tangencies for a hyperbolic periodic orbit without strong dominated splitting (theorem 5.1).

1.4. Consequences

This paper started during the preparation of [16] which studies the entropy of $C^1$-diffeomorphisms under a lack of domination. Hence, several applications of the present paper are explained there. We also note that the tools we present allow an easy and complete proof of such basic results as the necessity of a dominated splitting for robust transitivity (see the applications given in [6]). We close this introduction by giving an additional application.

An invariant compact set $\Lambda$ for a diffeomorphism $f$ on a boundaryless manifold $M$ has a dominated splitting if there exists a decomposition $TM|_\Lambda = E \oplus F$ of the tangent bundle of $M$ above $\Lambda$ in two invariant continuous subbundles and an integer $N \geq 1$ such that for all $x \in \Lambda$, all $n \geq N$ and all unit vectors $u \in E(x)$ and $v \in F(x)$ we have,

$$\|Df^nu\| \leq \|Df^nv\|/2.$$

**Theorem 1.1.** Let $\Lambda$ be a transitive invariant infinite compact set for a $C^1$-diffeomorphism $f$ on a manifold $M$. If $\Lambda$ has no dominated splitting, then there exists a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ having a horseshoe $K$ that is arbitrarily close to $\Lambda$ for the Hausdorff topology. If $f$ preserves a volume or a symplectic form, then one can choose $g$ to preserve it also.

We point out that this theorem makes no genericity assumption on the diffeomorphism $f$. In particular we do not suppose the existence of periodic orbits, so that the technique of transitions developed in [13] cannot be used here.

1.5. Comments and questions

We stress that the extension of the arguments of [12] and others to the conservative and especially to the symplectic setting is not direct. For instance, we were led to modify Gourmelon’s approach to theorem 5.1. The new argument is somewhat simpler, even in the dissipative setting.

Our techniques sometimes provide slightly weaker results in the symplectic case than the ones available in the dissipative case. For instance, consider a periodic cocycle with a large, given period and without strong dominated splitting. Can a small perturbation make: (1) all eigenvalues real? (2) All stable (all unstable) eigenvalues of equal moduli?

For (1), we need to assume the cocycle to be hyperbolic, see remark 4.3. For (2), we need to go to a possibly unbounded multiple of the period, see remark 4.10.

We also point out that although the perturbations are only small in the $C^1$-topology, the resulting diffeomorphism often has the same regularity as the unperturbed system. Our results could thus be used to provide examples and counter-examples with higher regularity in both the dissipative and conservative settings.

2. Preliminaries

In this section we review some properties of smooth dynamics and state a few fundamental perturbation properties that will be used throughout.
2.1. Space of diffeomorphisms

Let $M$ be a compact connected boundaryless Riemannian manifold with dimension $d_0$. The tangent bundle is endowed with a natural distance: considering the Levi-Civita connection, the distance between $u \in T_xM$ and $v \in T_yM$ is the infimum of $\|u - \Gamma_\gamma v\| + \text{Length}(\gamma)$ (where $\Gamma_\gamma$ denotes the parallel transport) over $C^1$-curves $\gamma$ between $x$ and $y$. Let $\text{Diff}^1(M)$ denote the space of $C^1$-diffeomorphisms of $M$ and let $d_{C^1}$ be the following usual distance defining the $C^1$-topology:

$$d_{C^1}(f, g) = \sup_{v \in T^1M} \max \{ d(Df(v), Dg(v)), d(Df^{-1}(v), Dg^{-1}(v)) \}.$$  

We say that $g$ is an $\varepsilon$-perturbation of $f$ when $d_{C^1}(g, f) < \varepsilon$.

2.2. Hyperbolic periodic points and homoclinic relations

Let $f \in \text{Diff}^r(M)$ and $p$ be a periodic point for $f$. We denote by $\pi(p)$ the (minimal) period of $p$ and by $O(p) = \{ f^n(p) : n = 0, \ldots, \pi(p) - 1 \}$ its orbit. A periodic point $p$ is hyperbolic if $Df^{\pi(p)}(p)$ has no eigenvalues on the unit circle. In this case there exists a stable manifold of $p$ denoted $W^s(p) = \{ y \in M : d(f^n(p)y, p) \to 0, n \to \infty \}$. It is a $C^r$ immersed submanifold tangent to the stable eigenspace at $p$. There is a similar definition for an unstable manifold. So we have $T_pM = E^s \oplus E^u$ where $E^s$ is the stable eigenspace for $p$ and $E^u$ is the unstable eigenspace for $p$. The periodic point is a saddle if the orbit $O(p)$ is hyperbolic and is neither a sink nor a source (so both $E^s$ and $E^u$ are nontrivial).

Two hyperbolic periodic orbits $O_1, O_2$ are homoclinically related if

- $W^s(O_1)$ has a transverse intersection point with $W^u(O_2)$, and
- $W^u(O_1)$ has a transverse intersection point with $W^s(O_2)$.

Equivalently, the orbits $O_1$ and $O_2$ are both contained in the same horseshoe (i.e. a topologically transitive, 0-dimensional compact invariant subset which is hyperbolic and locally maximal).

2.3. Conservative diffeomorphisms

If $\omega$ is a volume or a symplectic form on $M$, one denotes by $\text{Diff}^1_\omega(M)$ the subspace of diffeomorphisms which preserve $\omega$. The charts $\chi : U \to \mathbb{R}^{d_0}$ of $M$ that we will consider will always send $\omega$ on the standard Lebesgue volume or symplectic form of $\mathbb{R}^{d_0}$; it is well-known that any point admits a neighborhood with such a chart.

2.4. Symplectic linear algebra

We refer to [31]. For $d_0$ even, we write $d_0 = 2d$ and use the standard symplectic form,

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{(where } I = I_d \text{ is the } d \times d \text{-identity matrix),}$$

the Euclidean norm on $\mathbb{R}^{2d}$ and the operator norm on matrices. For $d \times d$-matrices $A, B, C, D$,

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5 We assume that $M$ and $\omega$ are $C^\infty$ smooth, see remark 2 in [6].
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{R}) \iff \begin{cases} A^T C = AC^T, \\ B^T D = D^T B, \text{ and} \\ A^T D - C^T B = I_d. \end{cases}
\]

The symplectic complement \(E^\omega\) of a subspace \(E \subset \mathbb{R}^{2d}\) is the linear subspace of vectors \(v\) such that \(\omega(v, u) = 0\) for all \(u \in E\).

A subspace \(E \subset \mathbb{R}^{2d}\) is symplectic if the restriction \(\omega|E \times E\) of the symplectic form is non-degenerate (i.e. symplectic). Two symplectic subspaces \(E, E'\) are \(\omega\)-orthogonal if for any \(u \in E, u' \in E'\) one has \(\omega(u, u') = 0\).

A \(d\)-dimensional subspace \(E\) is Lagrangian if the restriction of the symplectic form vanishes, i.e. \(E^\omega = E\). Obviously, the stable (resp. unstable) space of a linear symplectic map is Lagrangian.

We now state and prove a few simple results we will need later.

**Lemma 2.1.** For any \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \((e_1, \ldots, e_d)\) is \(\delta\)-close to the first \(d\) vectors of the standard basis \((E_1, \ldots, E_{2d})\) of \(\mathbb{R}^{2d}\) and generates a Lagrangian space, then there exists \(A \in \text{Sp}(2d, \mathbb{R})\) that is \(\varepsilon\)-close to the identity such that \(A(e_i) = E_i\) for \(i = 1, \ldots, d\).

**Proof.** Let \(M = (m_j)_{1 \leq j \leq 2d, 1 \leq i \leq d}\) be the matrix defined such that \(e_j = \sum_{i=1}^{2d} m_j E_i\) for \(j = 1, \ldots, d\). Define a \(2d \times 2d\) matrix as:

\[
\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}
\]

such that \(\begin{pmatrix} A \\ C \end{pmatrix} = M\) and \(D = (A^T)^{-1}\).

This matrix sends \(E_i\) to \(e_i\) for \(1 \leq i \leq d\) and is symplectic since \((A^T C - C^T A)_y = \omega(e_i, e_j) = 0\). Hence its inverse has the claimed properties. \(\square\)

**Corollary 2.2.** There exists \(C > 0\) (only depending on \(d\)) such that for any Lagrangian space \(L \subset \mathbb{R}^{2d}\), there exists \(A \in \text{Sp}(2d, \mathbb{R})\) satisfying

\[
A(L) = \mathbb{R}^d \times \{0\}^d\text{ and }\|A\|, \|A^{-1}\| < C.
\]

**Proof.** It is well-known that the symplectic group acts transitively on the Lagrangian spaces (by a variation of the preceding proof). The claims follows from lemma 2.1 and the compactness of the set of Lagrangian spaces. \(\square\)

### 2.5. Perturbative tools

We will use the following definitions.

**Definition 2.3.** Consider \(f \in \text{Diff}^1(M)\), a finite set \(X \subset M\), a neighborhood \(V\) of \(X\), and \(\varepsilon > 0\). A diffeomorphism \(g\) is an \((\varepsilon, V, X)\)-perturbation of \(f\) if \(d_\text{C}^1(f, g) < \varepsilon\) and \(g(x) = f(x)\) for all \(x\) outside of \(V \setminus X\).

**Definition 2.4.** Given \(f \in \text{Diff}^1(M)\), a periodic point \(p\) of period \(\ell = \pi(p)\) and \(\varepsilon > 0\), an \(\varepsilon\)-path of linear perturbations at \(O(p)\) is a family of continuous \(\ell\) paths \((A_i(t))_{t \in [0,1], 1 \leq i, i \leq \ell}\) of linear maps \(A_i(t): T_{f^i(p)}M \to T_{f^{i+1}(p)}M\) satisfying:

- \(A_i(0) = Df(f^i(p))\),
- \(\sup_{t \in [0,1]} (\max ||Df(f^i(p)) - A_i(t)||, ||Df^{-1}(f^i(p)) - A_i^{-1}(t)||) < \varepsilon\).

The following folklore result modifies the image of one point.
Proposition 2.5. For any $C, \varepsilon > 0$, there is $\eta > 0$ with the following property. For any $f \in \text{Diff}^1(M)$ such that $Df, Df^{-1}$ are bounded by $C$, for any pair of points $x, y$ such that $r := d(x, y)$ is small enough, we can find $g$ with $d_{C}(g,f) < \varepsilon$ and $g = f$ outside of $B(x, r/\eta)$ such that $g(x) = f(y)$. Furthermore, if $f$ preserves a volume or a symplectic form, one can choose $g$ to preserve it.

2.6. Periodic cocycles

Let $G$ be a subgroup of $GL(d_{0}, \mathbb{R})$. A periodic cocycle with period $\ell$ is an integer $\ell \geq 1$ together with a sequence $(A_{i})_{i \in \mathbb{Z}}$ in $G$ such that $A_{i+\ell} = A_{i}$ for all $i \in \mathbb{Z}$. The eigenvalues (at the period) are the eigenvalues of $A := A_{\ell} \ldots A_{1}$. The cocycle is hyperbolic if $A$ has no eigenvalue on the unit circle. The cocycle is bounded by $C > 1$ if $\max(||A_{i}||, ||A_{i}^{-1}||) \leq C$ for $1 \leq i \leq \ell$. An $\varepsilon$-path of perturbations is a family of periodic cocycles $(A_{i}(t))_{t \in [0,1]}$ in $G$ such that $A_{0}(0) = A_{i}$ and $\max ||A_{i} - A_{i}(t)||, ||A_{i}^{-1} - A_{i}^{-1}(t)|| < \varepsilon$ for each $t \in [0,1]$.

3. Franks’ lemma

We will use two strengthening of the classical Franks’ lemma: we not only perturb the differential but also linearize on a neighborhood through a localized perturbation that keep the diffeomorphism conservative if it was so (theorem 3.1) and even keep a homoclinic orbit if the periodic orbit stays hyperbolic (theorem 3.3).

3.1. Linearization

Theorem 3.1 (Franks’ lemma with linearization). Consider $f \in \text{Diff}^1(M), \varepsilon > 0$ small, a finite set $X \subset M$ and a chart $\chi : V \rightarrow \mathbb{R}^{d_{0}}$ with $X \subset V$. For $x \in X$, let $A_{x} : T_{x}M \rightarrow T_{\chi(x)}M$ be a linear map such that

$$\max(||A_{x} - Df(x)||, ||A_{x}^{-1} - Df^{-1}(x)||) < \varepsilon/2.$$ 

Then there exists an $(\varepsilon, V, X)$-perturbation $g$ of $f$ such that for each $x \in X$ the map $\chi \circ g \circ \chi^{-1}$ is linear in a neighborhood of $\chi(x)$ and $Dg(x) = A_{x}$. Moreover if $f$ preserves a volume or a symplectic form, one can choose $g$ to preserve it also.

We note that the set $X$ need not be invariant, but in the applications this will typically be a periodic orbit. Also, a proof of another version of Franks’ Lemma appears in [2] where it is stated in the setting of Poisson diffeomorphisms.

The dissipative case follows from a variation on Franks’ original proof [23]. The symplectic case is obtained by a standard argument involving generating functions (see for instance [45]). The volume-preserving case requires an additional argument for which we will use two results from [6].

Proof in the volume-preserving case. We assume that $V$ is an open subset of $\mathbb{R}^{d_{0}}$ and suppose that $X = \{0\} \subset \mathbb{R}^{d_{0}}$ and $Df(0)$ is the identity. We leave the reduction of the general case to this situation to the diligent reader.

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We note that a proof of theorem 3.1 in the volume-preserving case has appeared [43] while we were finishing this text.
Regularization. For any open neighborhood $V'$ of $X$ with $\overline{V'} \subset V$, theorem 7 of [6] (applied with $K_0 = \emptyset$) gives a conservative, arbitrarily small $C^1$-perturbation with support in $V$ and $C^\infty$ on $V'$. The image $f(0)$ may have changed but can be restored by a composing with a conservative perturbation of the identity (e.g. built using again theorem 7 of [6]). Thus we can assume that $f$ is $C^\infty$ on $V'$.

Rectification. We denote by $B$, the open ball with center $0 \in \mathbb{R}^d$ and radius $r > 0$ and let

$$K := (B_2 \setminus B_1) \cup B_1 \subset U := (B_{1,1} \setminus B_{1,9}) \cup B_{1,1}.$$ 

Let $\eta := \varepsilon/4(\|Df\|_\infty + 1)$ where $\| \cdot \|_\infty$ is the supremum norm. Corollary 5 of [6] yields:

**Lemma 3.2.** There are an integer $k \geq 1$ and arbitrarily small number $\tau > 0$ with the following property for $h \in C^\infty(B_{3,1}, \mathbb{R}^d)$.

If $\|h - Id\|_{C^k} < \tau$ and $h|U$ is volume-preserving with $vol(h(B_2 \setminus B_1)) = vol(B_2 \setminus B_1)$, then there is a volume-preserving map $\tilde{h} \in C^\infty(B_{3,1}, \mathbb{R}^d)$ with $d_C(\tilde{h}, Id) < \eta$ and $\tilde{h} = h$ on $K$.

Linearization. Let us fix a map $a \in C^\infty([0,1])$ with $a(0) = 0$ if $r \leq 1.1$, $a(0) = 1$ if $r \geq 1.9$. For $\rho > 0$ small enough, $f_\rho(x) := \rho^{-1}f(\rho x)$ is well defined and arbitrarily $C^\infty$-close to the identity on $B_{3,1}$. Define $h : B_{3,1} \to \mathbb{R}^d$ by:

$$h(x) = (1 - a(\|x\|))x + a(\|x\|)f(x).$$

$h(B_2) = f_\rho(B_2)$, $h(B_1) = B_1$. Moreover, $\|h - Id\|_{C^k} \leq C(k)\|f_\rho - Id\|_{C^k}$ where $C(k)$ only depends on $k$ and $d_0$. Thus $\|h - Id\|_{C^k} < \tau$ for $\rho > 0$ small enough.

Lemma 3.2 yields a volume-preserving map $\tilde{h} : B_{3,1} \to \mathbb{R}^d$ with $d_C(\tilde{h}, Id) < \varepsilon/4$ that coincides with $f_\rho$ on $\overline{B_1} \setminus B_2$ and with the identity on $\overline{B_1}$. Replacing $f$ on $B_{3,1}$ by $\tilde{h}_{1/\rho} := \rho \tilde{h}(\cdot/\rho)$, we get a conservative $C^1$-diffeomorphism $\tilde{f}$ such that $\tilde{f} = Df(0)$ near 0 and, for $\rho > 0$ small enough,

$$d_C(\tilde{f}, f) \leq d_C(\tilde{h}, Id) + d_C(Id|_{B_{3,1,\rho}}) < \varepsilon/3.$$ 

This proves the theorem in the case $A_0 = Df(0) = Id$.

Perturbation of the differential. We again use lemma 3.2. As this procedure only effects small changes, one chooses a path $(A_t)_{t \in [0,1]}$ with

$$\max(\|A_t - A_0\|, \|A_t^{-1} - A_0^{-1}\|) < 2\varepsilon/3, \quad \text{for any } t \in [0,1].$$

Choosing $N$ large enough, one sets $A_i := A_{i/N}$ for $i = 0, \ldots, N$, so that:

$$\|A_i - A_{i+1}\| < \frac{\tau}{C(k)(\|Df\|_\infty + 1)}.$$ 

We are going to define $u_i \in D^\infty(M)$ such that $d_C(u_i, f) < \varepsilon$ and $u_i|_{B_{3-\rho}} = A_i$. We take $u_0 = \tilde{f}$. For $0 \leq i < N$, we build $u_{i+1}$ from $u_i$ by replacing it on $B_{3-\rho}$ by $A_{i+1} \circ \tilde{h}_{3-\rho}$ where $\tilde{h}$ given by the lemma starting from

$$h(x) := (1 - a(\|x\|))x + a(\|x\|)A_{i+1}^{-1}A_i(x)$$

on $B_{3,1}$. Indeed, $\|h - Id\|_{C^k} \leq C(k)\|A_{i+1}^{-1}A_i - Id\| \leq C(k)\|A_{i+1}^{-1}\| \cdot \|A_i - A_{i+1}\| < \tau$. Thus, for $\tau > 0$ small enough:

$$d_C(A_{i+1} \circ \tilde{h}, A_i) \leq \|A_{i+1}\| \cdot d_C(\tilde{h}, Id) + \|A_{i+1}\| \cdot \|A_{i+1}^{-1}\| \cdot \|A_i - A_{i+1}\| < (6/5)\|Df(0)\| \cdot \varepsilon + \frac{36/25}\|Df(0)\| + \|Df(0)\|^{-1} \cdot \tau < \varepsilon/3.$$
so \( d_C(u_{i+1}, f) \leq \max(d_C(u_i, \text{Id}), \varepsilon/3 + d_C(f|B, A_i)) < \varepsilon \), completing the induction. Thus \( g = u_N \) satisfies the claims of the theorem.

3.2. Homoclinic connections

The next result further strengthens the linearizing version of Franks’ lemma when the periodic orbit is kept hyperbolic: the perturbation preserves a given homoclinic relation. It has been proved in [25] in the dissipative case.

**Theorem 3.3 (Franks’ lemma with homoclinic connection).** Let \( f \in \text{Diff}^1(M) \) and \( \varepsilon > 0 \) small. Consider:

- a hyperbolic periodic point \( p \) of period \( \ell = \pi(p) \),
- a chart \( \chi : V \to \mathbb{R}^d \) with \( O(p) \subset V \),
- a hyperbolic periodic point \( q \) homoclinically related to \( O(p) \), and
- an \( \varepsilon/2 \)-path of linear perturbations \( (A_i(t))_{t \in [0,1]} \) \( 1 \leq i \leq \ell \), at \( O(p) \) such that the composition \( A \circ A_i(t) \) is hyperbolic for each \( t \in [0,1] \).

Then there exists an \((\varepsilon,V,O(p))\)-perturbation \( g \) of \( f \) such that, for each \( i \) the map \( \chi \circ g \circ \chi^{-1} \) is linear and coincides with \( A_i(1) \) near \( f^\ell(p) \), and \( O(p) \) is still homoclinically related to \( q \). Moreover if \( f \) and the linear maps \( A_i(t) \) preserve a volume or a symplectic form, one can choose \( g \) to preserve it also.

**Proof.** We need to extend [25] to the symplectic and volume-preserving cases, once the corresponding versions of Franks’ lemma (theorem 3.1) have been obtained. One can assume that the orbits of \( p \) and \( q \) are distinct since otherwise the statement follows from theorem 3.1.

In order to simplify the exposition, we assume that \( p \) is a fixed point, so that the linear perturbation is reduced to a single path \( A(t) = A_1(t) \). Let \( z \) be a transverse intersection point between \( W^s_{loc}(p) \) and \( W^u(q) \) and consider \( N \geq 1 \) large. We decompose the path \( \{A(t)\}_{t \in [0,1]} \) and consider the maps \( A(k/N), k = 0, \ldots, N \). Since \( q \notin O(p) \), there exists a small ball \( B_0 \) centered at \( p \) such that the backward orbit of \( z \) does not intersect \( B_0 \).

Working in the chart \( V \) and using inductively theorem 3.1, we build a first perturbation \( h \) and a family of small nested balls \( B_N \subset \cdots \subset B_1 \subset B_0 \) centered at \( p \) such that \( f(B_{i+1}) \) is contained in and much smaller than \( B_i \), \( h = f \) outside \( B_0 \), and such that the differential \( Dh \) is close to \( A(k/N) \) on the region \( B_{k-1} \setminus B_k \), coincides with \( A(k/N) \) on the boundary of \( B_k \), \( 1 \leq k \leq N \), and with \( A(1) \) inside \( B_N \).

To keep the homoclinic connection, we modify \( h \) as follows. Let \( f^n(z) \) be the first iterate of \( z \) in \( B_0 \). Since \( A(1/N) \) and \( Df(p) \) are \( C^1 \)-close, the stable manifolds of \( p \) for these two maps are \( C^1 \)-close. Using proposition 2.5, one can thus perturb \( h \) at a point of \( B_N \setminus B_1 \) such that the forward orbit of \( z \) under the new diffeomorphism meets the stable manifold of \( p \) for \( A(1/N) \) when it enters in \( B_1 \). Since \( A(1/N) \) and \( A(2/N) \) are \( C^1 \)-close, one can perturb \( h \) at a point of \( B_1 \setminus B_2 \) such that the forward orbit of \( z \) under the new diffeomorphism meets the stable manifold of \( p \) for \( A(2/N) \) when it enters in \( B_2 \). Perturbing inductively \( N \) times on disjoint domains, one ensures that some forward iterate of \( z \) belongs to the stable manifold of \( p \) for \( A(1) \) in \( B_N \). This preserves a transverse intersection at \( z \) between \( W^s(p) \) and \( W^u(q) \) for the new diffeomorphism.

One similarly maintains a transverse intersection between \( W^s_{loc}(p) \) and \( W^u(q) \) for the new diffeomorphism. Since the local stable and unstable manifolds of \( p \) are separated, this second
perturbation can be chosen with disjoint support so as to preserve the transverse intersection $\varepsilon$ between $W^u(p)$ and $W^s(p)$.

4. Perturbation of periodic linear cocycles

In this section we show how to modify eigenvalues and create small angles between eigenspaces through perturbations. The results in the dissipative case are essentially well-known. The perturbations preserve the Jacobian, hence the volume-preserving case also follows. The symplectic case, though, requires different arguments.

4.1. Simple spectrum

We first show how to obtain simple spectrum by arbitrarily small perturbations.

**Proposition 4.1 (Simple spectrum).** For any $d_0 \geq 1$ and $\varepsilon > 0$, any periodic cocycle in $GL(d_0, \mathbb{R})$, $SL(d_0, \mathbb{R})$ or $Sp(d_0, \mathbb{R})$ admits an $\varepsilon$-path of perturbations $(A_i(t))_{t \in [0,1]}$ with the same period $\ell$, such that $A(t) := A_{\ell}(t) \ldots A_1(t)$ satisfies:

- $A(1)$ has $d_0$ distinct eigenvalues; their arguments are in $\pi\mathbb{Q}$.

In $GL(d_0, \mathbb{R})$ and $SL(d_0, \mathbb{R})$ one can furthermore require that the moduli of the eigenvalues of $A(t)$ are constant in $t \in [0,1]$.

**Proof.** We first sketch the proof in the dissipative setting (which is essentially contained in the claim of the proof of [12, lemma 7.3]). It easily implies the volume-preserving case. Since the necessary perturbation is arbitrarily small and can therefore be performed at a single iterate, it is enough to consider the case $\ell = 1$.

We proceed by induction on the sum $\delta$ of the dimensions of the eigenspaces corresponding to eigenvalues with multiplicity or arguments outside $\pi\mathbb{Q}$. If $\delta = 0$, there is nothing to show. Otherwise, we are in one of the following three cases in each of which, we can decrease $\delta$ by 2. We note that the number of perturbations is at most $d_0/2$.

Case 1. $A$ is $R$-conjugate to

\[
\begin{pmatrix}
\lambda & B \\
0 & C
\end{pmatrix}
\]

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

One modifies $I$ as

\[
R_\theta = \begin{pmatrix}
\cos(\pi \theta) & \sin(\pi \theta) \\
-\sin(\pi \theta) & \cos(\pi \theta)
\end{pmatrix}.
\]

For $\theta > 0$ small, the eigenvalues of $R_\theta$ are non-real and of modulus 1 and their arguments can be made rational. The other eigenvalues are unchanged.

Case 2. $A$ is $R$-conjugate to

\[
\begin{pmatrix}
\lambda N & B \\
0 & C
\end{pmatrix}
\]

where $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
One modifies $N$ as

$$N_\varepsilon = \begin{pmatrix} 1 - \varepsilon & 1 \\ -\varepsilon & 1 \end{pmatrix}$$

and one concludes as in Case 1.

Case 3. $\mathcal{A}$ is $\mathbb{R}$-conjugate to

$$\begin{pmatrix} rR_\theta & B \\ 0 & C \end{pmatrix},$$

where $R_\theta$ is the rotation by angle $\pi \theta$. One perturbs $A$ by modifying the angle $\theta$: during the perturbation the eigenvalues along the invariant 2-plane $\mathbb{R}^2 \times \{0\}$ moves, but keep the same modulus. This concludes the proof in the dissipative case.

We now investigate the symplectic case (See also [39, section V] and [40].) Let us consider two integers $n, m \geq 1$ such that $2(n + m) = d_0$. For each matrix $R \in GL(n, \mathbb{R})$ and each matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m, \mathbb{R}),$$

the matrices:

$$\begin{pmatrix} R & RU & RS & RV \\ 0 & A & X & B \\ 0 & 0 & iR^{-1} & 0 \\ 0 & C & Z & D \end{pmatrix}$$

belong to $Sp(d_0, \mathbb{R})$, once $S, U, V, X, Z$ satisfy some relations independent from $R$. One can thus perturb $R$ while keeping the matrix in $Sp(d_0, \mathbb{R})$.

Case 1. $\mathcal{A}$ has a real eigenvalue $\lambda$.

One chooses an eigenvector $u$ and completes it as a symplectic basis. This defines a bounded change of coordinates after which $\mathcal{A}$ takes the form (2) above, with $n = 1$ and $R = \lambda$. One can change the eigenvalues $\lambda, \lambda^{-1}$ by perturbing $R$, without affecting the other eigenvalues.

Case 2. $\mathcal{A}$ has a complex eigenvalue $\sigma = \lambda e^{i\theta}$.

Thus there exists an invariant 2-plane $P$ where $\mathcal{A}$ induces the map $R := \lambda \text{Rot}_\theta$, where Rot is the rotation with angle $\theta$. If the symplectic form vanishes on $P$, one completes it as before to get a symplectic basis and a bounded change of coordinates. Now $\mathcal{A}$ takes the form (2) above. One can perturb $R$ to get four distinct eigenvalues $\sigma, \bar{\sigma}, \sigma^{-1}, \bar{\sigma}^{-1}$, all outside the unit circle with arguments rational multiples of $\pi$.

In the case $P$ is symplectic, $\lambda$ equals 1 and $R$ is a rotation. The symplectic complement $P^\perp$ of $P$ is invariant and $\mathcal{A}$ is the Cartesian product of the symplectic rotation $R$ with some map $Q \in Sp(d_0 - 2, \mathbb{R})$. One can again perturb $R$ to make the arguments of the eigenvalues $\sigma, \bar{\sigma}$, rational multiples of $\pi$ without modifying the rest of the spectrum. □

### 4.2. Real eigenvalues

We now show that for sufficiently long cocycles one can make all the eigenvalues real.

**Proposition 4.2 (Real eigenvalues).** For any $d_0 \geq 1$, $C > 1$, $\varepsilon > 0$, there exists $T \geq 1$ with the following property. Any periodic cocycle in $GL(d_0, \mathbb{R})$ or $SL(d_0, \mathbb{R})$ bounded by $C$ and with period $\ell \geq T$ admits an $\varepsilon$-path of perturbations $(A(t))_{t \in [0,1]}$ with the same period $\ell$ such that
– the eigenvalues of $A(t) := A_\ell(t) \ldots A_1(t)$ have moduli constant in $t \in [0, 1]$.
– $A(1)$ has only real eigenvalues.

The same result holds in $Sp(d_0, \mathbb{R})$ if the cocycle is hyperbolic.

A previous result for surfaces was obtained in [9, lemma 6.6]. The dissipative case was proved in [12, proposition 4.3] and the volume-preserving case is an immediate consequence. We extend it to the symplectic case.

**Remark 4.3.** We do not know if the same result holds for arbitrary symplectic cocycles (in particular for cocycles admitting an invariant symplectic 2-plane, when $d_0 \geq 4$).

**Proof in the symplectic case.** By corollary 2.2, there exists a change of coordinates by a bounded cocycle such that each map $A_i$ preserves the space $\mathbb{R}^d \times \{0\}$ and the moduli of the eigenvalues of $A$ along this space are smaller than 1. Consequently, one can assume each map $A_i$ to be a matrix of the form

$$
\begin{pmatrix}
B_i^T & C_i \\
0 & B_i^{-1}
\end{pmatrix}.
$$

(3)

The dissipative version of proposition 4.2 applied to the cocycle $(B_i)$ yields a path of perturbations $(U_i(t))_{t \in [0,1]}$ in $GL(d, \mathbb{R})$ such that the moduli of the eigenvalues of the cocycles $(B_i U_i(t))$ are constant in $t$ and the eigenvalues of the composition $(B_\ell U_\ell(1)) \ldots (B_1 U_1(1))$ are all real. To build the $\varepsilon$-path of perturbations, one can set $t \mapsto A_i(t) := A_i D_i(t)$, where $D_i(t)$ is given by the symplectic matrix

$$
D_i(t) := \begin{pmatrix} U(t)^T & 0 \\ 0 & U(t)^{-1} \end{pmatrix}.
$$

□

**Corollary 4.4 (Real simple eigenvalues).** For any $d_0 \geq 1$, $C > 1$, $\varepsilon > 0$, there exists $T \geq 1$ with the following property. Any periodic cocycle in $GL(d_0, \mathbb{R})$ or $SL(d_0, \mathbb{R})$ bounded by $C$ and with period $\ell \geq T$ admits an $\varepsilon$-path of perturbations $(A_i(t))_{t \in [0,1]}$ with the same period such that

– $A(t) := A_\ell(t) \ldots A_1(t)$ is hyperbolic for any $t > 0$,
– $A(1)$ has $d_0$ distinct real eigenvalues with modulus in $(0, 1/2) \cup (2, +\infty)$.

The same result holds in $Sp(d_0, \mathbb{R})$ if the cocycle is hyperbolic.

**Proof.** In the dissipative or volume-preserving cases, once all the eigenvalues are real, one can conjugate by a bounded cocycle (using Gram-Schmidt orthonormalization) to reduce to the case where all the $A_i$ are defined by triangular matrices. The result then follows easily by perturbing the diagonal coefficients.

In the symplectic case, the proof is the same: conjugacy by a bounded cocycle brings the cocycle $A$ to the form (3), where $B_i$ is a lower triangular matrix. □

**4.3. Small angle**

We use the lack of $N$-dominated splitting to find a perturbation making the angle between the stable and unstable bundles small.
Theorem 4.5 (Small angle). For any \( d_0 \geq 1 \), \( C > 1 \), \( \varepsilon > 0 \), there exist \( T, N \geq 1 \) with the following property. For any hyperbolic periodic cocycle \( (A_i)_{i \in \mathbb{Z}} \) in \( \text{GL}(d_0, \mathbb{R}) \), \( \text{SL}(d_0, \mathbb{R}) \) or \( \text{Sp}(d_0, \mathbb{R}) \) bounded by \( C \), with period \( \ell \geq T \) and such that the splitting defined by the stable and unstable eigenvalues is not an \( N \)-dominated splitting, there exists an \( \varepsilon \)-path of perturbations \( (A_i(t))_{t \in [0,1]} \) with period \( \ell \), such that:

- \( \mathcal{A}(t) := A_t(t) \ldots A_1(t) \) is hyperbolic for each \( t \in [0,1] \).
- for some \( j \in \{1, \ldots, \ell\} \), the angle between the stable and unstable spaces \( E^s_j, E^u_j \) of the cocycle \( (A_i(1)) \) is smaller than \( \varepsilon \).

This has been obtained in the dissipative (and volume-preserving) case in [25, proposition 4.7]. Previous results were obtained in [38, 44].

Proof of the symplectic case. Applying corollary 4.4, (and assuming that the period is large enough), one can perform a first \( \varepsilon/2 \)-perturbation and reduce to the case the cocycle has \( d_0 \) real eigenvalues with distinct moduli. One can assume that the angle between \( E^s, E^u \) is larger than \( \varepsilon \) since otherwise the theorem holds trivially.

Applying lemma 2.1, one can find a bounded symplectic change of coordinates such that each bundle \( \mathbb{R}^k \times \{0\}^{d-k+1}, 1 \leq k \leq d \), is invariant and contained in the stable bundle (where as before we let \( d_0 = 2d \)). The unstable bundle has the form \( E^u_j = \{ (L_t(u), u), u \in \mathbb{R}^d \} \). Since \( E^u_j \) is Lagrangian, the matrix

\[
\Delta_j := \begin{pmatrix} I_d & L_j \\ 0 & I_d \end{pmatrix}
\]

is symplectic. Moreover since the angle between \( E^s \) and \( E^u \) is bounded away from zero, the matrices \( \Delta_i \) and \( \Delta_i^{-1} \) are uniformly bounded. After conjugating by the cocycle \( (\Delta_i) \), we are thus reduced to the case the cocycle has the form

\[
\begin{pmatrix} B_i^T & 0 \\ 0 & B_i^{-1} \end{pmatrix}
\]

where \( B_i^T \) has been made upper triangular using as before Gram-Schmidt orthonormalization and has diagonal coefficients \( b_i(1), \ldots, b_i(d) \). Consequently \( B_i^{-1} \) is lower triangular and has diagonal coefficients

\[
b_i(d+1) = b_i(1)^{-1}, \ldots, b_i(d_0) = b_i(d)^{-1}.
\]

These coefficients define \( d_0 \) real cocycles.

4.3.1 The case \( d_0 = 2 \). The two-dimensional case now follows from the established argument in the dissipative case. We will use the following more precise statement for our proof in higher dimensions.

Lemma 4.6. For any \( C > 1 \), \( \varepsilon' > 0 \), there exist \( T, N' \geq 1 \) with the following property. For any hyperbolic periodic cocycle \( (D_i) \) of diagonal matrices in \( \text{SL}(2, \mathbb{R}) \) bounded by \( C \), with period \( \ell \geq T \) and no \( N' \)-dominated splitting, there exists an \( \varepsilon' \)-path of perturbations \( (D_i(t)) \) such that:

- the eigenvalues of \( D(t) := D_t(1) \ldots D_1(t) \) have moduli constant in \( t \);
- the matrices \( U_i(t) = D_i^{-1} \cdot D_i(t) \) are upper triangular;
– for some \( j \in \{1, ..., \ell\} \), the angle between the stable and the unstable spaces \( E^s_i, E^u_i \) of the cocycle \((D_i(1))\) is smaller than \( \varepsilon' \).

**Comment on the proof.** The proof follows the classical argument by Mañé, see [14, lemma 7.10]. Each matrix \( U_i(t) \) either has one of the following forms.

- \( \begin{pmatrix} 1 & \eta_j \\ 0 & 1 \end{pmatrix} \): it twists one of the bundles; or
- \( \begin{pmatrix} (1 + \eta_j) & 0 \\ 0 & (1 + \eta_j)^{-1} \end{pmatrix} \): it accentuates the contraction or the expansion along the invariant bundles.

Consequently the first bundle \( \mathbb{R} \times \{0\} \) is invariant. One can choose \( \prod_{i=1}^{\ell} (1 + \eta_j) = 1 \) so that the moduli of the eigenvalues are unchanged.

4.3.2. Choice of a symplectic plane. Let us come back to the general case. One chooses \( \varepsilon' > 0 \) small (see the condition later) and takes \( N' \) as given by the previous lemma. We now explain how to fix the integer \( N \) and how under the assumptions of theorem 4.5, one can find \( 1 \leq r \leq d \) such that \((b_i(r))_{1 \leq i \leq \ell}\) is not \( N' \)-dominated by the cocycle \((b_i(d + r))_{1 \leq i \leq \ell}\).

**Lemma 4.7.** For any \( N' \geq 1 \), there exists \( N \) with the following property. If for any \( j \in \{d + 1, \ldots, d_0\} \) and any \( k \in \{d + 1, \ldots, d_0\} \) the cocycle \((b_j(j))_{1 \leq i \leq \ell}\) is \( N' \)-dominated by the cocycle \((b_i(k))_{1 \leq i \leq \ell}\) then the bundle \( E^s \) is \( N \)-dominated by \( E^s \) for the cocycle \((A_i)_{1 \leq i \leq \ell}\).

**Proof.** Since the cocycle \((B_i^T)\) is upper triangular and uniformly bounded, there exists a uniform constant \( K > 0 \) such that for any \( n \geq 1 \), the norm of \( B^T_{d+n} \cdots B^T_{d+1} \) is bounded by \( K \max_{1 \leq r \leq d} |b_{d+n}(r) \cdots b_{d+1}(r)| \). Similarly, the co-norm (i.e. the minimal norm of the image of a unit vector) of \( B^{-1}_{d+n} \cdots B^{-1}_{d+1} \) is bounded from below by \( K^{-1} \min_{d+n \leq i \leq d_0} |b_{d+n}(r) \cdots b_{d+1}(r)| \).

Consequently, \( E^s \) is \( N \)-dominated by \( E^s \) provided for all \( n \geq N \)

\[
K^2 \max_{1 \leq r \leq d} |b_{d+n}(r) \cdots b_{d+1}(r)| \leq \frac{1}{2} \min_{d+n \leq i \leq d_0} |b_{d+n}(r) \cdots b_{d+1}(r)|.
\]

One thus chooses \( m \geq 1 \) such that \( K^2 < 2^{-m-1} \) and sets \( N' = mN \).

Let us assume that \((b_i(j))\) is not \( N' \)-dominated by the cocycle \((b_j(j))\) for some \( 1 \leq j \leq d < k \leq d_0 \). There exists \( i \in \mathbb{Z} \) and \( n \geq N' \) such that

\[
|b_i(j) \cdots b_{i+n-1}(j)| > 1/2 |b_i(j) \cdots b_{i+n-1}(j)|.
\]

Since \( b_i(d + s) = b_i(s)^{-1} \) for each \( 1 \leq s \leq d \), one also gets

\[
|b_i(k - d) \cdots b_{i+n-1}(k - d)| > 1/2 |b_i(j + d) \cdots b_{i+n-1}(j + d)|.
\]

Let us assume that \((b_i(k - d))\) is \( N' \)-dominated by \((b_i(k))\). This gives

\[
|b_i(k) \cdots b_{i+n-1}(k)| \geq 2 |b_i(k - d) \cdots b_{i+n-1}(k - d)|.
\]

Combining these inequalities, one deduces

\[
|b_i(j) \cdots b_{i+n-1}(j)| > 1/2 |b_i(j + d) \cdots b_{i+n-1}(j + d)|,
\]

and \((b_i(j))\) is not \( N' \)-dominated by \((b_i(j + d))\). We have thus shown:
Lemma 4.8. If there are $1 \leq j \leq d < k \leq d_0$ such that the cocycle $(b_i(j))_{1 \leq i \leq \ell}$ is not $N'$-dominated by the cocycle $(b_i(k))_{1 \leq i \leq \ell}$, then there exists $1 \leq r \leq d$ such that $(b_i(r))_{1 \leq i \leq \ell}$ is not $N'$-dominated by the cocycle $(b_i(d + r))_{1 \leq i \leq \ell}$.

4.3.3. The perturbation. Let us assume now that $(b_i(r))_{1 \leq i \leq \ell}$ is not $N'$-dominated by $(b_i(d + r))_{1 \leq i \leq \ell}$. One can decompose the stable and unstable bundles as follows

$$\mathbb{R}^d \times \{0\}^d = \mathbb{R}^{r-1} \times \mathbb{R} \times \mathbb{R}^{d-r} \times \{0\}^d = E_1^s \oplus E_2^s \oplus E_3^s,$$

$$\{0\}^d \times \mathbb{R}^d = \{0\}^d \times \mathbb{R}^{r-1} \times \mathbb{R} \times \mathbb{R}^{d-r} = E_1^u \oplus E_2^u \oplus E_3^u.$$

The spaces $E_* := E_*^s \oplus E_*^u$ for $* = 1, 2$ or 3 are symplectic. In the coordinates $E_1 \oplus E_2 \oplus E_3$, the cocycle takes the form

$$A_i = \begin{pmatrix} D_{i,1} & D_{i,2} & D_{i,3} \\ 0 & D_{i,4} & D_{i,5} \\ 0 & 0 & D_{i,6} \end{pmatrix}.$$

Each matrix

$$D_{i,4} = \begin{pmatrix} b_i(r) & 0 \\ 0 & b_i(d + r) \end{pmatrix}$$

is two-dimensional, symplectic, and diagonal; by our assumptions it has no $N'$-dominated splitting and the composition $D_{5,4} \ldots D_{5,1}$ is hyperbolic. The matrices $D_2$ have the form

$$D_{2,1} = \begin{pmatrix} V_i & 0 \\ 0 & V_i' \end{pmatrix},$$

where $V_i$ and $V_i'$ are $1 \times (r - 1)$ matrices.

Let us apply lemma 4.6 to the cocycle $(D_{5,4})$: this gives an $\varepsilon'$-path of perturbations $(D_{5,4}(t))$. We set $U_i(t) = D_{5,1}^{-1} \cdot D_{5,1}(t)$ and define the symplectic matrices

$$A_i(t) = A \cdot \begin{pmatrix} I_{r-1} & 0 & 0 \\ 0 & U_i(t) & 0 \\ 0 & 0 & I_{d-r} \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \cdot U_i(t) & D_3 \\ 0 & D_{5,1}(t) & D_6 \\ 0 & 0 & D_9 \end{pmatrix}.$$

If $\varepsilon'$ has been chosen small enough, this is a $\varepsilon$-path of perturbations. By construction, the modulus of the eigenvalues are unchanged, hence the cocycle is hyperbolic for each $t \in [0, 1]$.

By construction, there exists some $1 \leq j \leq \ell$ such that the stable and unstable spaces at index $j$ of the cocycle $(D_{5,1}(1))$ contain nonzero vectors $w^s, w^u \in E_1^s \oplus E_2^u$ which satisfy $\angle(w^s, w^u) \leq \varepsilon'$.

Lemma 4.6 keeps $E_1^s \oplus E_2^u$ and therefore $w^s$ in the stable space of $(A_i(1))$. As $E_1 \oplus E_2$ remains invariant for $(A_i(1))$, the unstable space of $(D_{5,1}(1))$ lifts inside $E_1^s \oplus E_2^s$ to that of $(A_i(1))$. Thus we get an unstable vector for $(A_i(1))$ of the form $w^s + v_1^s + v_1^u$ with $v_1^s \in E_1^s$.

As $E_2^u$ contains the unstable space of $(A_i(1))$, the vector $\tilde{w}^u := (w^s + v_1^s + v_1^u) - v_1^u$ is in the unstable space for $(A_i(1))$. As $E_2^u$ is an invariant stable subspace for $(A_i(1))$, the vector $\tilde{w}^s := w^s + v_1^s$ is in the stable space.

To conclude the proof of the theorem, observe that $\tilde{w}^u - \tilde{w}^s = w^u - w^s$ and $\|\tilde{w}^u\| \geq \|w^u\|$ for $*=u,s$ so $\angle(\tilde{w}^u, \tilde{w}^s) \leq \angle(w^u, w^s) \leq \varepsilon'$. 

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4.4. Mixing the exponents

The lack of strong domination leads to further perturbations making all stable (resp. unstable) eigenvalues to have equal modulus. The next statement has been proved in the dissipative (and volume-preserving) setting in [12, theorem 4.1].

**Theorem 4.9 (Mixing the exponents).** For any \( d_0 \geq 1 \), \( C > 1 \), \( \varepsilon > 0 \), there exists \( N' \geq 1 \) with the following property. For any hyperbolic periodic cocycle \( (A_i)_{i \in \mathbb{Z}} \) in \( \text{GL}(d_0, \mathbb{R}) \), \( \text{SL}(d_0, \mathbb{R}) \) or \( \text{Sp}(d_0, \mathbb{R}) \) bounded by \( C \), with period \( \pi \) and no \( N \)-dominated splitting, there exists an \( \varepsilon \)-path of perturbations \( (A_i(t))_{i \in [0,1]} \) with period \( \ell \), a multiple of \( \pi \), such that:

- \( \mathcal{A}(t) := A_0(t) \ldots A_1(t) \) is hyperbolic for any \( t \in [0,1] \),
- \( t \mapsto |\det(\mathcal{A}(t)|_{E^u})| \) and \( t \mapsto |\det(\mathcal{A}(t)|_{E^s})| \) are constant in \( t \in [0,1] \),
- the stable (resp. unstable) eigenvalues of \( \mathcal{A}(1) \) have the same moduli.

**Remark 4.10.** In fact, a stronger statement is proven in [12] for \( \text{GL}(d_0, \mathbb{R}) \) and \( \text{SL}(d_0, \mathbb{R}) \). Namely, there exists \( T \geq 1 \) which only depends on \( d_0, C, \varepsilon \), and \( N \), such that any such that \( \ell \geq T \) of \( \pi \) satisfies the conclusion of theorem 4.9. In particular, if the period \( \pi \) is larger than \( \ell \), one can choose \( \ell = \pi \). We do not know if this uniformity holds in \( \text{Sp}(d_0, \mathbb{R}) \). Note also that [12] allows to realize other spectra for non-dominated cocycles in \( \text{GL}(d_0, \mathbb{R}) \). We do not know to what extend this generalizes to the symplectic case.

It remains to prove the symplectic case of theorem 4.9. The proof is by reduction to the dissipative case. We will see that, in the symplectic category, any hyperbolic cocycle without strong dominated splitting admits a perturbation (maybe with a larger period) whose restriction to its stable subbundle is also without any strong dominated splitting. The dissipative case of the theorem can then be applied.

Let us fix numbers \( C, \varepsilon > 0 \). In this section we always consider hyperbolic cocycles \( A = (A_i)_{i \in \mathbb{Z}} \) in \( \text{Sp}(d_0, \mathbb{R}) \) bounded by \( 2C \). Choosing \( \varepsilon > 0 \) small enough, any \( \varepsilon \)-perturbation of a cocycle bounded by \( C \) is still bounded by \( 2C \).

**Lemma 4.11.** There exists \( N_0 \geq 1 \) such that if the stable bundle of \( A = (A_i) \) has no \( N_0' \)-dominated splitting, then there exists an \( \varepsilon/2 \)-path of perturbations of \( A \) (possibly with larger period) satisfying the conclusion of theorem 4.9 in \( \text{Sp}(d_0, \mathbb{R}) \).

**Proof.** As before, one uses corollary 2.2 to reduce to the case of cocycles of the form

\[
\begin{pmatrix}
B_\ell C_1 \\
0 & B_\ell^{-1}
\end{pmatrix},
\]

where \( \mathbb{R}^d \times \{0\}^d \) is the stable space. Since the cocycle \( (B_\ell) \) has no strong dominated splitting, the version of theorem 4.9 for \( \text{GL}(d, \mathbb{R}) \) provides a path of perturbations \( (B_\ell(t)) \) (with possibly larger period \( \ell \)), whose composition \( B(t) = B_{\ell}(t) \ldots B_1(t) \) has constant Jacobian, only eigenvalues with modulus smaller than 1, and such that \( B(1) \) has all its eigenvalues with the same modulus. One concludes as in section 4.2. \( \square \)

The key to this reduction is the next proposition which analyzes the dominated decompositions \( E^s = E^u \oplus E^s \) of the stable spaces. The domination is not quantified and only requires that the eigenvalues along \( E^u \) have smaller moduli than along \( E^s \).

**Proposition 4.12.** Given \( N_0, \varepsilon' > 0 \), \( C > 1 \), and \( j \in \{1, \ldots, d-1\} \), there exists an integer \( N' = N'(N_0, j, \varepsilon', C) \geq 1 \) with the following property. Let \( A = (A_i) \) be any hyperbolic cocycle in \( \text{Sp}(d_0, \mathbb{R}) \) bounded by \( 2C \) with period \( \pi \) such that its composition \( A_\pi \ldots A_1 \) has \( d_0 \) eigenval-
ues with pairwise distinct moduli. If the stable space of $A$ admits a splitting $E^s = E^s_a \oplus E^s_w$ with $\dim(E^s) = j$, then

- either $A$ admits a $N'$-dominated splitting,
- or there exists an $\varepsilon'$-path of perturbations $A(t) = (A_i(t))$ of $A$ with possibly larger period $t$ such that

1. each $A(t) := A(t) \ldots A(t)$ is hyperbolic,
2. $A(t)$ preserves $E^s$ and coincides with $A$ on $E^s_w$, for all $0 \leq t \leq 1$,
3. $A(t)$ has a dominated splitting $E'(t) = E'' \oplus E^s(t)$ for all $0 \leq t \leq 1$,
4. $t \rightarrow |\det(A(t))_{E''}|$ and $t \rightarrow |\det(A(t))_{E^s(t)}|$ are constant in $t$,
5. the splitting $E'(1) = E'' \oplus E^s(1)$ for $A(1)$ is not $N_0$-dominated.

Proof of theorem 4.9 from proposition 4.12. Lemma 4.11 gives an integer $N_0$, depending on $\varepsilon$. We will apply the proposition for each possible $E''$-dimension $1 \leq j \leq d - 1$ to remove any $N_0$-dominated splitting in the stable subbundle by an $\varepsilon/2$-perturbation. The $\varepsilon/2$-perturbation given by lemma 4.11 will finish the proof. More precisely, we pick $0 < \varepsilon_{d-1} < \varepsilon/2d$ and $N_{d-1} \geq N'(N_0, d - 1, \varepsilon_{d-1}, C)$ and then, inductively, select $N_j, \varepsilon_j$ for $1 \leq j < d - 1$, given $N_{j+1}$, by:

- $0 < \varepsilon_j < \varepsilon/2d$ so small that non $2N_{j+1}$-dominated splitting for some cocycle implies non $N_{j+1}$-dominated splitting for any $\varepsilon_j$-perturbation; and
- $N_j \geq N'(N_0, j, \varepsilon_j, C)$ with $N_j \geq 2N_{j+1}$.

Now, given a cocycle $A$ without $N_1$-dominated splitting, we inductively get cocycles $A^{(1)}, \ldots, A^{(d)}$ by setting $A^{(1)} := A$ and taking, for $2 \leq j \leq d$, $A^{(j)}$ to be an $\varepsilon_{j-1}$-perturbation of $A^{(j-1)}$ satisfying:

$A^{(j)}$ has no $N_0$-dominated splitting of index strictly less than $j$ inside $E''$ and, if $j < d$, no $N_j$-dominated splitting in $\mathbb{R}^{2d}$.

In particular, the distance from $A$ to $A^{(d)}$ is less than $\varepsilon_1 + \cdots + \varepsilon_{d-1} < \varepsilon/2$ and the restriction $A^{(d)}|E'$ has no $N_0$-dominated splitting. Theorem 4.9 now follows from lemma 4.11.

It remains to prove proposition 4.12. Note that the symmetry of the spectrum of a symplectic cocycle implies that there also exists a dominated splitting $E'' = E''_a \oplus E''_w$ with $\dim E''_a = \dim E''_w$. We prove three preliminary lemmas.

Lemma 4.13. For any $\varepsilon > 0$ and any integer $N_0$, there exists $\eta > 0$ with the following property. If $(A_i)$ has a dominated splitting $E'' \oplus E''_a \oplus E''_w \oplus E''_w$ with $\dim E'' = j$ such that $E'' = E''_a \oplus E''_w$ is $N_0$-dominated, then for any $k \in \mathbb{Z}$ and any space $E''_{k+1} \subseteq E''_{k+1} \oplus E''_{k+1}$ that is $\eta$-close to $E''_{k+1}$ and with equal dimension, there exists an $\varepsilon'$-path of perturbations $(A_i(t))$ in $\text{Sp}(2d, \mathbb{R})$ such that

- $A_k(1).E''_w = \hat{E}'_{k+1}$, and
- the maps $A_i(t)$ restricted to $E''_a$ are constant in $t$.

Proof. Since $E'' = E''_a \oplus E''_w$ is $N_0$-dominated, the angle between $E''_a$ and $E''_w$ is uniform. Arguing as in corollary 2.2, one finds $C' > 0$, depending only on $C, d, N_0$, such that, after conjugation by a cocycle in $\text{Sp}(2d, \mathbb{R})$ and bounded by $C'$, the subspace $E''_a$ becomes $\mathbb{R}^j \times \{0\}^{2d-j}$ and the subspace $E''_w$ becomes $\{0\}^j \times \mathbb{R}^{d-j} \times \{0\}^d$. In these new coordinates, there exists a
linear map $B: \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathbb{R}^d \times \{0\}^{d-j} \subset \ker(B)$ and $E^x_{k} = \{(u, B(u)) : u \in E^x_i\}$.

Note that $B$ is (uniformly) close to 0 when $\eta$ is small. One builds the path of perturbations by composing $A_{k-1}$ with the symplectic matrices, given (in the new coordinates) by:

$$U(t) = \begin{pmatrix} I & 0 \\ -tB & 1 \end{pmatrix}.$$

We denote the partial compositions $A_{k+n-1} \cdots A_i$ by $A^\eta_{ij}$.

**Lemma 4.14.** For any $\eta > 0$ and any integer $N_0$, there is $\tilde{N}$ with the following property. Let $(A_i)$ be a cocycle with a dominated splitting $E^u \oplus E^s \oplus E^c \oplus E^{cu}$. If $E^s = E^c \oplus E^{cu}$ is $N_0$-dominated and $E^s \oplus (E^c \oplus E^{cu})$ is not $\tilde{N}$-dominated, then there exist integers $i_0 \in \mathbb{Z}$, $m_0 \geq N_0$ and unit vectors $u^s \in E^u_{i_0} \oplus E^c_{i_0}$, $u^{cu} \in E^{cu}_{i_0}$ such that

1. the line $A^m_{i_0}(\mathbb{R}u^s)$ is $\eta$-close to $E^s_{i_0+m_0}$ and
2. $\|A^m_{i_0}(u^{cu})\| > \frac{1}{2}\|A^m_{i_0}(u^s)\|$.

**Proof.** Let $a = (C - 2m_0)/10$ and let $m_0$ be an integer larger than $N_0|\log(\eta a)|$. Since $E^s \oplus E^c$ is $N_0$-dominated, for any $i \in \mathbb{Z}$ and any unit vectors $u^s \in E^s_i$ and $u^{cu} \in E^{cu}_i$ we have

$$\|A^m_{i_0}(u^s)\| > \frac{3}{2\eta a}\|A^m_{i_0}(u^{cu})\|.$$  \hspace{1cm} (4)

Pick an integer $\tilde{N} > \max(N_0, 2m_0^2(\log C + 2)/\log\frac{1}{2})$. The lack of the $\tilde{N}$-dominated splitting of $E^s \oplus (E^c \oplus E^{cu})$ yields $j_0 \in \mathbb{Z}$, $n \geq \tilde{N}$ and some unit vectors $u^s_{j_0} \in E^s_{j_0}$ and $u^{cu}_{j_0} \in E^{cu}_{j_0}$ such that

$$\|A^m_{j_0}(u^s_{j_0})\| > \frac{1}{2}\|A^m_{j_0}(u^{cu}_{j_0})\|.$$  \hspace{1cm}

Thus the positive numbers

$$a_i := \frac{\|A^m_{j_0+j_0+1}(u^s_{j_0})\|}{\|A^m_{j_0+j_0}(u^s_{j_0})\|} \times \frac{\|A^m_{j_0+j_0+1}(u^{cu}_{j_0})\|}{\|A^m_{j_0+j_0}(u^{cu}_{j_0})\|} \leq C^2 \quad (i = j_0, \ldots, j_0 + n - 1)$$

satisfy $a_{j_0} \cdots a_{j_0+n-1} > 1/2$. From $n \geq \tilde{N}$ and the choice of $\tilde{N}$, one deduces that there is an integer $i_0$ with $j_0 \leq i_0 \leq j_0 + n - m_0 - 1$ such that $a_{i_0} \cdots a_{i_0+N_0+1} > 2/3$ and $a_{i_0} \cdots a_{i_0+m_0+1} > 2/3$. Thus, the unit vectors

$$u^s = \frac{A^m_{j_0+j_0}(u^s_{j_0})}{\|A^m_{j_0+j_0}(u^s_{j_0})\|} \in E^s_{j_0}, \quad \tilde{u}^{cu} = \frac{A^m_{j_0+j_0}(u^{cu}_{j_0})}{\|A^m_{j_0+j_0}(u^{cu}_{j_0})\|} \in E^{cu}_{j_0} \oplus E^{cu}_{i_0},$$

satisfy

$$\|A^m_{j_0}(u^s)\| > \frac{2}{3}\|A^m_{j_0}(\tilde{u}^{cu})\|,$$  \hspace{1cm} (5)

$$\|A^m_{j_0}(u^{cu})\| > \frac{2}{3}\|A^m_{j_0}(\tilde{u}^s)\|.$$  \hspace{1cm} (6)
Let $u' \in E^{\alpha^j}_{u_0}$ be any unit vector. Define
$$u^c = (i^c + a.u^c')/\|u^c + a.u^c'\|.$$ 
Since the cocycle $(A_i)$ and its inverse are bounded by $C$, equation (5) yields
$$\|A^N_{u_0}(a'')\| > \frac{2}{3}(1-a)\|A^N_{u_0}(u'')\| - \frac{2}{3}a\|A^N_{u_0}(u'')\| > \frac{2}{3}(1-a)\|A^N_{u_0}(u'')\| - \frac{2}{3}a\|C2^{N}\|\|A^N_{u_0}(a'')\|$$
which gives item (b). Note that $A^N_{u_0}(u')$ decomposes as
$$\|i^c + a.u^c'\|^{-1} \left( A^N_{u_0}(i^c) + a.A^N_{u_0}(u') \right).$$
From (6) and (4), one gets
$$\|A^N_{u_0}(u')\| < \frac{3}{2}\|A^N_{u_0}(u'')\| < \eta a\|A^N_{u_0}(u^c)\|.$$ 
This implies item (a). 

**Lemma 4.15.** For any integer $\tilde{N} \geq 1$, there exists an integer $N' \geq 1$ such that if the splitting $E^s \oplus (E^s \oplus E^u)$ is $\tilde{N}$-dominated, then the cocycle $(A_i)$ has an $N'$-dominated splitting.

**Proof.** By $\tilde{N}$-domination, the angle between $E^s$ and $E^s = E^s \oplus E^u$ is lower bounded and a variant of corollary 2.2 yields a conjugacy by a bounded, symplectic cocycle which sends $E^s$ and $E^s = E^s \oplus E^u$ to the constant bundles $\mathbb{R} \times \{0\}^{d_h-1}$ and $\{0\}^j \times \mathbb{R}^{d_h-2j} \times \{0\}^j$. The cocycle $(A_i)$ is bounded by $C'$ and has the form:

$$A_i = \begin{pmatrix} B_i^T & 0 & D_i \\ 0 & C_i & 0 \\ 0 & 0 & B_i^{-1} \end{pmatrix},$$

where $B_i, D_i$ are $j \times j$ matrices and $C_i \in Sp(d_h - 2j, \mathbb{R})$.

In the following we denote by $\|A^{-1}\|$ the co-norm of a matrix. Since the change of coordinates is bounded, the $\tilde{N}$-dominated splitting gives uniform numbers $a > 0$ and $b \in (0, 1)$ such that for any $n \geq 0$ and any $i \in \mathbb{Z}$,
$$\|B_i^1 \ldots B_i^{n+1}\| \leq ab^n m(C_{i+n} \ldots C_{i+1}).$$

Now, there is a constant $c$ (depending only on $d_h$) such that
$$\|(B_i^{-1} \ldots B_i^{-1})^{-1}\| \leq c\|(B_i^{-1} \ldots B_i^{-1})^{-1}\| = c\|B_i^{-1} \ldots B_i^{-1}\|.$$ 
Hence,
$$\|(B_i^{-1} \ldots B_i^{-1})^{-1}\| \leq c\|C_{i+n} \ldots C_{i+1}\|^{-1}.$$ 
As each $C_i$ is symplectic, $JC_i^{-1} = C_i^T J$, so we have $\|(C_{i+n} \ldots C_{i+1})^{-1}\| = \|C_{i+n} \ldots C_{i+1}\|$ and
$$\|C_{i+n} \ldots C_{i+1}\| \leq ab^n m(B_i^{-1} \ldots B_i^{-1}).$$

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This also gives
\[ \| B_{1:n}^1 \cdots B_{1:n}^t \| \leq c a^2 b^{2m} (B_{1:n}^{t-1} \cdots B_{1:n}^1). \]

Since the matrices \( A_t \) are uniformly bounded, the cone-field criterion is satisfied and gives a uniform dominated splitting between the bundle \( \mathbb{R}^d \times \{ 0 \} \) and a transverse bundle with dimension \( j \).

The dominated splitting for the initial cocycle \( (A_t) \) is obtained by pulling back by the bounded conjugacy. This shows that the splitting \( (E^u \oplus E^s) \oplus E^0 \) is \( N' \)-dominated for some uniform integer \( N' \).

**Proof of the proposition 4.12.** Given \( N_0, \varepsilon' \), the lemmas above give \( \eta, \tilde{N}, N' \). Consider a cocycle \( (A_t) \) as in the statement of the proposition. One may assume that the splitting \( E^0 = E^s \oplus E^u \) is \( N_0 \)-dominated (otherwise there is nothing to show) and that \( E^u \oplus (E^s \oplus E^0) \) is not \( N \)-dominated (otherwise lemma 4.15 provides a \( N' \)-dominated splitting). Hence, lemma 4.14 gives integers \( i_0 \in \mathbb{Z}, m_0 \geq N_0 \) and some unit vectors \( u' \in E^u_{i_0} \oplus E^s_{i_0}, u'' \in E^u_{i_0} \).

One chooses a subspace \( E^u_{i_0+m_0} \oplus E^s_{i_0+m_0} \) that is \( \eta \)-close to \( E^u_{i_0+m_0} \) (with equal dimension) and contains \( A^m_{i_0} (u') \). Since \( E^u \oplus E^s \) is (not necessarily) dominated, there exists an index \( k < i_0 \) such that
\[ \hat{E}^u_{k+1} := (A^0_{k+1} \cdots A^k_0 - 1)(E^u_{i_0+m_0}) \]
is \( \eta \)-close to \( E^u_{k+1} \). Applying lemma 4.13 twice, one builds two \( \varepsilon' \)-paths of perturbations \( (A_t (t)) \) and \( (A_{i_0+m_0} (t)) \) of \( A_k \) and \( A_{i_0+m_0} \) respectively such that

- the restrictions to \( E^s \) of \( A_k(t) \) and \( A_{i_0+m_0} \) are constant in \( t \), and
- \( A_k(E^u_t) = \hat{E}^u_{k+1} \) and \( A_{i_0+m_0} (E^s_{i_0+m_0}) = E^s_{i_0+m+1} \).

One then chooses some large even multiple \( \ell \) of the period of the initial cocycle and sets \( A_{i_0+\ell} = A_{i_0} \) for each \( s \in \mathbb{Z} \) and \( i \in \{ -\ell/2, \ldots, \ell/2 - 1 \} \). Since \( \ell \) is large, the Oseledets splitting \( E^u \oplus E^s \oplus E^0 \oplus E^0 \) for the \( \ell \)-periodic cocycle \( (A_t(t)) \) can be followed continuously with \( t \). The items 1–3 hold.

Item 5 follows from the lemma 4.14 (b), since \( u'' \in E^s_{i_0}, u' \in E^u_{i_0} \) for the cocycle \( (A_t(1)) \) and since \( A_{i_0+n} (1) = A_{i_0+n} \) for \( n = 0, \ldots, N_0 - 1 \).

Note also that the determinant along \( E^s \) of
\[ A_{t/2-1} \circ A_{t/2} \circ A_{t/2-1} \circ A_{t/2} (t) \]
changes only by a factor bounded independently from \( \ell \) (the maps and the spaces of the bundle \( E^s \) are unchanged for indices not in \( \{ k, \ldots, i_0 + m_0 \} \)). Since the spaces of the Oseledets splitting are \( \omega \)-orthogonal, one can compose by symplectic maps which act as the identity along \( E^0 \) and as homotheties along \( E^s \) for some indices \( j = \ell/4, \ldots, \ell/2 - 1 \). For these indices \( E^s \) coincides with the initial bundle \( E^s \), hence is uniformly far from \( E^u \). Lemma 2.1 controls the size of this perturbation. One thus obtains \( \varepsilon' \)-paths of perturbations \( (A_{t/4} (t)), (A_{t/2-1} (t)) \) such that the determinant of \( A_{t/2-1} (t) \circ A_{t/2} \circ A_{t/2-1} \circ A_{t/2} (t) \) along the continuation of the bundle \( E^s \) is constant in \( t \). This implies the item 4 for the stable bundle. The determinant along the stable and unstable bundles being inverse of each other, item 4 also holds for the unstable bundle.

**5. Homoclinic tangencies**

This section extends to the conservative setting the following theorem of Gourmelon [24, 25] on the creation of homoclinic tangencies from a lack of dominated splitting. We will follow
the main steps of [25] with the exception of the induction on the dimension. This induction is quite technical in the dissipative setting and difficult to adapt in the conservative setting. Avoiding it results in a simplification of Gourmelon’s proof.

**Theorem 5.1 (Homoclinic tangency).** For any \( d_0 \geq 1, \ C > 1, \ \varepsilon > 0, \) there exist \( N, T \geq 1 \) with the following property. Consider

- a diffeomorphism \( f \in \text{Diff}^1(M) \) of a \( d_0 \)-dimensional manifold \( M \) such that the norms of \( Df \) and \( Df^{-1} \) are bounded by \( C \),
- a periodic saddle \( \mathcal{O} \) with period larger than \( T \) such that the splitting defined by the stable and unstable bundles is not an \( N \)-dominated splitting, and
- a neighborhood \( V \) of \( \mathcal{O} \).

Then there exists an \((\varepsilon, V, \mathcal{O})\)-perturbation \( g \) of \( f \) and \( p \in \mathcal{O} \) such that

- \( W^s(p), W^u(p) \) have a tangency \( z \) whose orbit is contained in \( V \), and
- the differential of \( f \) and \( g \) coincide along \( \mathcal{O} \).

Moreover if \( \mathcal{O} \) is homoclinically related to a periodic point \( q \) for \( f \), then the perturbation \( g \) can be chosen to still have this property. If \( f \) preserves a volume or a symplectic form, one can choose \( g \) to preserve it also.

**Remark 5.2.** As a consequence, one can also obtain a transverse intersection \( z' \) between \( W^s(\mathcal{O}) \) and \( W^u(\mathcal{O}) \), whose orbit is contained in \( V \). This implies that \( \mathcal{O} \) belongs to a horseshoe of \( g \) which is contained in \( V \).

**Proof of theorem 5.1.** As noted, the above theorem was proved in the dissipative setting by Gourmelon in [24, theorem 3.1] and [25, theorem 8]. We consider the symplectic setting. One fixes \( \eta > 0 \) given by proposition 2.5 and \( \chi > 0 \) much smaller than \( \eta \).

**Step 1. Reduction.** After a first perturbation, using the linear statements (proposition 4.2, theorem 4.5 and corollary 4.4) together with the Franks’ lemmas (theorems 3.1 and 3.3) one can assume the following properties:

- a neighborhood of \( \mathcal{O} \) admits a chart \( \psi \) such that \( \psi \circ f \circ \psi^{-1} \) is linear near each point of \( \mathcal{O} \),
- \( \mathcal{O} \) has \( d_0 \) distinct real eigenvalues, with moduli in \((0, 1/2) \cup (2, +\infty)\),
- the angle between stable and unstable spaces at some iterate \( p \in \mathcal{O} \) is smaller than \( \chi^{d_0} \).

When \( \mathcal{O} \) is homoclinically related to a periodic point \( q \), one fixes two transverse intersections \( z^s \in W^s(\mathcal{O}) \cap W^u(q) \) and \( z^u \in W^s(q) \cap W^u(\mathcal{O}) \) and choose the above perturbation to preserve these.

**Step 2. Choice of iterates.** Let \( TM|_{\mathcal{O}} = E_1^s \oplus \cdots \oplus E_k^s \oplus E_1^u \oplus \cdots \oplus E_{d_0-k}^u \) be the invariant decomposition into one-dimensional bundles. Since the angle between \( E^s(p), E^u(p) \) is smaller than \( \chi^{d_0} \), one can find \( 1 \leq i \leq k \) and \( 1 \leq j \leq d_0 - k \) such that

- the angle \( \theta \) between \( E := E_1^s \oplus \cdots \oplus E_i^s \oplus F := E_1^u \oplus \cdots \oplus E_j^u \) is in \((0, \chi^{d_0})\);
- the angles between \( E \) and \( F' := E_1^s \oplus \cdots \oplus E_{i-1}^s \) and between \( E' := E_1^u \oplus \cdots \oplus E_{j-1}^u \) and \( F \) are larger than \( \theta/\chi \) (by convention the angle of the zero subspace with any other subspace is infinite).

Indeed, setting \( \theta_{ij} := \angle(E_1^s \oplus \cdots \oplus E_i^s, E_1^u \oplus \cdots \oplus E_j^u) \), one builds inductively a sequence \( S \) of pairs \((i,j)\) such that the angle \( \theta_{ij} \) satisfies (a). The initial pair is \((k, d_0 - k)\). If \((i,j) \in S \) and
if $\theta_{ij}$ does not satisfies the condition (b), then the new pair in the sequence is either $(i − 1, j)$ or $(i, j − 1)$: one of these two pairs has to satisfy (a). The last pair obtained during this construction satisfies (b) as required.

For $\rho > 0$ small, one can choose $u \in E$ and $v \in F$ with norm $\rho$ such that $\|u − v\| < \theta \cdot \rho$. Since the angle between $v$ and $E'$ is larger than $\theta / \chi$ (and since $\chi$ is small), the orthogonal projection of $u$ on $E \cap (E')^\perp$ has norm larger than $\theta \rho / (2 \chi)$.

Since $\eta \gg \chi$, the quantity $\theta \rho / \eta$ is much smaller than $\theta \rho / (2 \chi)$. Consequently, the orthogonal projection on $E \cap (E')^\perp$ of the ball $B$ centered at $u$ and with radius $\|u − v\| / \eta$ has diameter much smaller than the projection of $u$. Since $E'$ and $E$ are preserved by $Dg(\pi(O))$, since $E'/E$ is one-dimensional, and since the eigenvalue of $Dg(\pi(O))$ on the quotient $E'/E'$ belongs to $(-1/2, 1/2)$, one deduces that the forward orbit of $u$ does not intersect the ball $B$. Similarly the backward iterate of $v$ does not intersect the ball $B$. Using the linearity of $f$ in the chart near $O$, this also holds for the $f$-orbits of $u$ and $v$ as points in $M$.

Step 3. The homoclinic tangency. Proposition 2.5 gives an $\varepsilon$-perturbation $g$ of $f$ such that $f^{-1} \circ g$ is supported in $B$ and such that $g(v) = f(u)$. One deduces that the orbit of $v$ for $g$ is homoclinic to $O$ (and contained in a small neighborhood of $O$). Since the angle between $E$ and $F$ is small, with theorem 3.1, one can ensure that $Dg(v)(F)$ is tangent to $E$ along the line directed by $u$. One has obtained a homoclinic tangency.

Assume now that $O$ is homoclinically related to some point $q$ through the orbits of $z'$, $z''$. By construction, the orthogonal projection of $B \cap E$ on $E \cap (E')^\perp$ has a small diameter in comparison to the distance to the origin. Since the eigenvalue of $Dg(\pi(O))$ on the quotient $E'/E'$ belongs to $(-1/2, 1/2)$, one can choose $B$ (i.e. the norm $\rho$) so that $B$ is disjoint from the orbit of $z'$. The same can be done for $z''$. This proves that the homoclinic connection with $q$ is preserved.

Finally, one can apply theorem 3.1 so that the differential of $f$ and of the perturbed system coincide along the orbit of $O$ while keeping the homoclinic connection.

6. Proof of theorem 1.1

The proof can be summarized as follows: Let $\Lambda$ be an infinite, transitive compact set for the diffeomorphism $f$. It will be approximated by a periodic orbit created by Pugh’s closing lemma. This periodic orbit cannot have strong domination. Theorem 5.1 (more precisely, remark 5.2) will yield a horseshoe close to the periodic orbit, provided this orbit is a saddle.

To reduce to this case, we will use a different argument in the conservative and dissipative cases. In the conservative setting, we will perturb the differential and linearize to build saddles (with higher periods). In the dissipative setting (where, generically, there are weak sinks or sources accumulating on $\Lambda$), we will rely on the following result, see [33]:

**Proposition 6.1 (Pliss).** For any $d_0 \geq 2$, $C \geq 1$ and $\varepsilon > 0$, there exists $N, T \geq 1$ such that, if $g$ is a diffeomorphism with $Dg, Dg^{-1}$ bounded by $C$ and if $O$ is an attracting periodic orbit with period $\ell$ larger than $T$, then

- either $\prod_{x \in O} \|Dg^N(x)\| \leq 2^{-\ell}$,
- or there exists a diffeomorphism $C^1$-close to $g$ which preserves $O$ and such that $O$ is a hyperbolic saddle (it has both stable and unstable eigendirections).
Proof of theorem 1.1. We fix $\varepsilon > 0$ small and $C > 0$ which bounds the norms of $Dg$ and $Dg^{-1}$ for any diffeomorphism that is $\varepsilon$-close to $f$ for the $C^1$-topology. The dimension of $M$ is $d_0 \geq 2$. Theorem 5.1 provides integers $N, T \geq 1$ given $d_0, C, \varepsilon/2$.

We first perturb $f$ to create a periodic orbit approximating $\Lambda$.

Lemma 6.2. For any $\varepsilon_1 > 0$, there exists $f_1$ with $d_{C^1}(f, f_1) < \varepsilon_1$ and a periodic orbit $\mathcal{O}$ of $f_1$ that is $\varepsilon_1$-close to $\Lambda$ for the Hausdorff distance.

Proof. Since $\Lambda$ is transitive, there exists a point $x \in \Lambda$ whose orbit is dense in $\Lambda$. Since $\Lambda$ is infinite, $x$ is not periodic. From Pugh’s closing lemma [36], there exists $N \geq 1$ with the following property: for any neighborhood $U$ of $x$, there exists a diffeomorphism $f_1$ having a periodic orbit $O$ which intersects $U$ such that:

\begin{align*}
- & d_{C^1}(f, f_1) < \varepsilon_1, \\
- & f_1 = f \text{ on } M \setminus (U \cup \cdots \cup f^{N-1}(U)), \\
- & O \subset O(x) \cup U \cup \cdots \cup f^{N-1}(U).
\end{align*}

When $f$ is conservative, $f_1$ can still be chosen conservative (the closing lemma is still valid for conservative systems [37]).

Since the orbit of $x$ is dense in $\Lambda$, if the diameter of $U$ is small enough, the orbit $O$ (which intersects $U$) intersects the $\varepsilon_1$-neighborhood of any point of $\Lambda$. From the third item above, it is contained in the $\varepsilon_1$-neighborhood of $\Lambda$. Hence $O$ and $\Lambda$ are $\varepsilon_1$-close.

We now turn the periodic orbit $O$ into a hyperbolic saddle.

Lemma 6.3. For any $\varepsilon_2 > 0$, there exists $f_2$ with $d_{C^1}(f, f_2) < \varepsilon_2$ and a hyperbolic orbit $\tilde{O}$ of $f_2$ that is $\varepsilon_2$-close to $\Lambda$ for the Hausdorff distance and has saddle type (i.e. both stable and unstable eigenvalues).

Proof. Since $\Lambda$ is non-periodic, the period of the orbit $O$ given by the previous lemma is arbitrarily large, provided $\varepsilon_1$ has been chosen small enough. The proof then differs in the conservative and in the dissipative cases.

6.1. In the dissipative case

Proposition 4.1 and Franks’ lemma (e.g. theorem 3.1) provide a perturbation of $f_1$ so that $O$ becomes a hyperbolic periodic orbit (see also Kupka-Smale theorem [28, 42]), which is a saddle, a sink or a source. Let $N_2, T_2 \geq 1$ be integers defined by proposition 6.1 given $\frac{1}{2} \varepsilon_2 > 0$. Since the period of $O$ can be chosen arbitrarily large, proposition 6.1 will apply whenever $O$ is attracting. If, after a $\frac{1}{2} \varepsilon_2$-perturbation, one gets a saddle periodic orbit $\tilde{O} = O$, we are done. Otherwise, we can assume that all such perturbations have a sink (the case of a source is left to the reader). Proposition 6.1 then yields that the average of $\log \|Df_{N_2}\|$ for the measure $\mu_O$ defined by $O$ is smaller than $-\log 2$. Since $f_1$ can be chosen arbitrarily close to $f$ and $O$ to $\Lambda$, one can take an accumulation point of the measures $\mu_O$ and an ergodic component $\mu$. It will be carried on $\Lambda$ with $\int \log \|Df_{N_2}\| d\mu \leq -\log 2$. In particular $\mu$-almost every point has a stable manifold of dimension $d_0$ (see for instance [1, theorem 3.11]) and $\mu$ is a sink. This is a contradiction since $\Lambda$ is transitive and infinite.
6.2. In the conservative case

Proposition 4.1 provides a perturbation $f_1$ such that the eigenvalues of $O$ are simple and that the eigenvalues with modulus 1 are roots of the unity. Theorem 3.1 linearizes the dynamics in a neighborhood of $O$ by a further perturbation. This implies that there exists a periodic orbit $\hat{O}$ contained in an arbitrarily small neighborhood of $O$ and whose period $\hat{\ell}$ is a multiple of $\ell$, such that $Df_{\hat{O}}^{\hat{\ell}}$ is the product of a hyperbolic linear map and the identity. Since the identity can be turned into a hyperbolic linear map by a small perturbation in the symplectic group, a further small perturbation provided by theorems 3.1 ensures that $\hat{O}$ is hyperbolic as required.

The existence of a $N$-dominated splitting passes to the limit when one considers sequences of diffeomorphisms and of invariant compact sets. Consequently, if $\epsilon_2 \in (0, \epsilon/2)$ is chosen small enough, the previous lemma provides a diffeomorphism $f_2$ with a periodic orbit $O$ whose decomposition of the tangent bundle into stable and unstable spaces is not $N$-dominated and whose period is larger than $T$.

Theorem 5.1 and remark 5.2 then build a diffeomorphism $g$ that is $\epsilon/2$-close to $f_2$ such that $O$ has a transverse homoclinic orbit in an arbitrarily small neighborhood of $O$. In particular, this gives a horseshoe in an arbitrarily small neighborhood of $\Lambda$, as required.

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