Proper BRST quantization of relativistic particles.

Robert Marnelius

Institute of Theoretical Physics
Chalmers University of Technology
S-412 96 Göteborg, Sweden

Abstract

Recently derived general formal solutions of a BRST quantization on inner product spaces of irreducible Lie group gauge theories are applied to trivial models and relativistic particle models for particles with spin 0, 1/2 and 1. In the process general quantization rules are discovered which make the formal solutions exact. The treatment also gives evidence that the formal solutions are directly generalizable to theories with graded gauge symmetries. For relativistic particles reasonable results are obtained although there exists no completely Lorentz covariant quantization of the coordinate and momenta on inner product spaces. There are two inequivalent procedures depending on whether or not the time coordinate is quantized with positive or indefinite metric states. The latter is connected to propagators.
1 Introduction.

For a long time there has been a notorious difficulty to perform a unique and well defined BRST quantization of quantum mechanical systems like those describing relativistic particles, relativistic strings or any other field theory with nontrivial zero modes. These difficulties have their origin in the use of badly defined original state spaces. For instance, the solutions of a BRST quantization of particles and strings are usually presented in terms of momentum eigenstates which do not belong to an inner product space. Such a result is only possible if the projection to the BRST invariant subspace is obtained from a state space which also includes these momentum states. To achieve this one has to work with bilinear forms constructed from the chosen state space and its dual, which in the above case does not contain the momentum eigenstates. Such a construction has been used in a series of papers in which we have used state spaces which are analytic in the momenta while the dual state space is analytic in the coordinates $[1, 2, 3, 4]$. In this construction it was noted that BRST invariance of the dual state space selects states which make finite bilinear forms with the BRST invariant momentum states. New light was cast on this procedure in $[5, 6]$ where the allowed properties of general gauge fixing within a BRST scheme were determined. It was shown that the BRST invariant states may be further constrained by operators which form BRST doublets satisfying a closed algebra. In $[6]$ the relativistic particle and string were explicitly treated and it was shown exactly which constraints the dual BRST invariant states have to satisfy when one makes use of BRST invariant momentum states. These constraints are noncovariant and point to a difficulty with the above procedure involving the state space and its dual.

What one normally should require of the outcome of a BRST quantization is that the BRST cohomology should yield a positive definite state space (a Hilbert space). However, this causes a severe difficulty within the above construction. In order to satisfy this condition one must be able to define an inner product for the genuine physical states which belong to the BRST cohomology. In $[4]$ it was suggested that a
rather natural inner product may be constructed provided one makes use of dynamical Lagrange multipliers and antighosts. However, this proposal contained some confusing technical difficulties and was therefore not entirely correct. A suggestion how these difficulties could be overcome was given in [7]. There it was suggested that a correct treatment requires the use of spectral resolutions appropriate for an original state space being an inner product space. In [8, 9] it was then shown that a BRST quantization of general gauge models may always be performed and formally solved in a simple manner on inner product spaces and that the solutions modify the proposal in [4].

Presently it seems therefore as if the only way to get rid of all the previous difficulties and ambiguities is to require the original state space in a BRST quantization to be an inner product space, since in this case it looks as if we always can obtain a well defined BRST quantization. The purpose of the present paper is to further explore this possibility for models which usually are difficult to handle by a detailed treatment of formal solutions of the type given in [8, 9]. On our way we will then find general rules how the original state space should be chosen in order to have a well defined BRST quantization.

One point made in [7] was that Hermitian coordinate and momentum operators may be considered to act within an inner product space, and that we may make use of a complete set of coordinate and momentum eigenstates in a well defined manner although these states do not belong to the inner product space. How this works in the ordinary Hilbert space is well known. However, in the case when we have an indefinite metric state space it is not so well known that one may still make use of a complete set of eigenstates provided the eigenvalues are properly chosen. The hermitian coordinate and momentum operators must have imaginary eigenvalues when the corresponding state space is indefinite [10]. Real eigenvalues are only consistent with positive definite state spaces which is apparent from (assuming a positive measure $dx$)

$$\langle \phi | \phi \rangle = \int dx \langle \phi | x \rangle \langle x | \phi \rangle = \int dx |\phi(x)|^2 \geq 0 \quad (1.1)$$
(In \[7\] the possibility of using real eigenvalues and infinite normed state spaces regularized to antihermitian inner products was also considered.)

The implications of using an inner product space representation in a general BRST quantization of an arbitrary consistent model were summarized in section 5 of ref.\[11\] (extracted from \[1\]). There it was stated that in order for the genuine physical states to span a Hilbert space:

1. The number of constraints must be even in the chosen gauge theory.
2. The BRST charge $Q$ must be possible to write as
   \[ Q = \delta^\dagger + \delta \] (1.2)
   where
   \[ \delta^2 = \delta^\dagger \delta = [\delta, \delta^\dagger]_+ = 0 \] (1.3)
3. The genuine physical states may be chosen to have ghost number zero.
4. The genuine physical states may always be chosen to be determined by a generalized Gupta-Bleuler quantization.

(Points 1 and 3 are well known properties.)

These properties suggest that one in general should make use of dynamical Lagrange multipliers as well as antighosts since this will always imply that we have an even number of constraints and that the ghost number zero is contained in the original state space. Furthermore, they will in general allow for antiBRST invariance \[12\] which in turn allows for larger OSP invariances \[5\]. The presence of the antighosts makes sure that we have an even number of both bosonic and fermionic ghosts, the first of which allows for the possibility to have a Fock space representation and the second of which avoids the problem with an odd number of fermionic ghosts \[13\]. Indeed, in \[8, 9\] it was shown that condition 2 may be satisfied in several different ways, at least for any Lie group theory, and that the general solutions of the BRST
condition yields physical states which satisfy conditions 3 and 4 when dynamical Lagrange multipliers are introduced. (Decompositions like (1.2) with (1.3) have been used before. See e.g. [14].)

The paper is organized as follows: In section 2 the general properties found in [8, 9] is briefly reviewed with some additional remarks and in section 3 and 4 the main quantization rules are obtained from a detailed treatment of the abelian case. A detailed treatment of the spinless relativistic particle is then given in section 5 where the properties of state spaces of coordinate and momentum states are further clarified. The world-line supersymmetric spin-$\frac{1}{2}$ particle is treated in section 6, and the $O(2)$-extended world-line supersymmetric spin-1 particle is treated in section 7. In section 8 we give some final remarks. In an appendix the properties of the spinor states for the spin-$\frac{1}{2}$ particle are given.

2 The general case.

In [8] it was proved that the BRST charge of a general bosonic gauge model with finite number of degrees of freedom may be written in the form (1.2) with the properties (1.3) provided dynamical Lagrange multipliers and antighosts are introduced. The starting point was the BRST charge in the BFV form [15] ($a, b, c = 1, \ldots, m <$ $\infty$)

$$Q = \psi_a \eta^a - \frac{1}{2} i U_{bc}^a \mathcal{P}_a \eta^b \eta^c - \frac{1}{2} i U_{ab}^b \eta^a + \bar{\mathcal{P}}_a \pi_a$$ (2.1)

where the irreducible set of gauge generators $\psi_a$ satisfy

$$[\psi_a, \psi_b]_- = i U_{ab}^c \psi_c$$ (2.2)

and where $U_{ab}^c$ are constants. The anticommuting ghosts $\eta^a$, antighosts $\bar{\eta}_a$ and the commuting Lagrange multipliers $\nu^a$ satisfy the algebra (the nonzero part)

$$[\eta^a, \mathcal{P}_b]_+ = [\bar{\eta}^a, \bar{\mathcal{P}}_b]_+ = \delta^a_b, \quad [\nu^a, \pi_b]_- = i \delta^a_b$$ (2.3)
All these variables are assumed to be hermitian, which is no restriction. In [8, 9] it was shown that the charge (2.1) may be written in the form (1.2) where
\[ \delta = c^\dagger a \phi_a = \phi'_a c^\dagger a \]
where in turn the non-hermitian operators \( \phi_a \) or \( \phi'_a \) (= \( \psi_a + \ldots \)) satisfy the same Lie algebra as \( \psi_a \), i.e. (2.2). \( c^a \) is an expression in terms of ghosts, antighosts and Lagrange multipliers in [8]. In [9] the construction involves gauge fixing variables instead of Lagrange multipliers. By means of a bigrading which implies [11]

\[ Q|ph\rangle = 0 \iff \delta|ph\rangle = 0, \quad \delta^\dagger|ph\rangle = 0 \]

the BRST condition was then shown to be naturally solved by
\[ c^a|ph\rangle = 0, \quad \phi_a|ph\rangle = 0 \]

or
\[ c^\dagger a|ph\rangle = 0, \quad \phi'_a|ph\rangle = 0 \]
(or possibly a mixture of these conditions). Other solutions of (2.5) are in general zero norm states [9].

The formal solutions of (2.6) and (2.7) were shown to be
\[ |ph\rangle_\alpha = e^{\alpha[\rho,Q]}|\Phi\rangle \]

where \( |\Phi\rangle \) is a BRST invariant state and where \( \alpha \) is a strictly positive or a strictly negative real constant for (2.6) and (2.7) respectively. In [8] we have
\[ \rho \equiv P_a v^a. \]

and the state \( |\Phi\rangle \) satisfies the conditions
\[ \eta^a|\Phi\rangle = \pi_a|\Phi\rangle = 0 \]

which makes it BRST invariant. These conditions are trivially solved. In [9] we have the formal solution (2.8) with
\[ \rho \equiv \tilde{\eta}_a \chi^a. \]
where $\chi^a$ are gauge choices or gauge fixing variables for $\psi_a ( [\chi^a, \psi_b]_- \text{ must have an inverse}). The state $|\Phi\rangle$ satisfies here

\[
P_a |\Phi\rangle = \bar{P}_a |\Phi\rangle = (\psi_a + iU_{ab}^b)|\Phi\rangle = 0
\]

which makes it BRST invariant. The last conditions are just those of a Dirac quantization which are not trivial to solve in general. The norm of $|ph\rangle_\alpha$ should be independent of the value of $\alpha$ but can depend on the sign of $\alpha$ since the two signs requires different representations of the original state space $\Omega$.

In order for (2.8) to be true solutions we must show that they belong to an inner product space. In other words although $|\Phi\rangle$ does not belong to an inner product space $|ph\rangle_\alpha$ must do. As a consequence

\[
|ph\rangle_\alpha = |\Phi\rangle + Q|\chi\rangle
\]

does not imply that we may through away $Q|\chi\rangle$!

The philosophy of the above approach is to start with a large inner product space $\Omega$ and project out a genuine physical state space $\Omega_{ph} \subset \Omega$ which then also is an inner product space. In a canonical theory (defined in [1]) the BRST condition (2.5) yields a unique set of solutions for a given original state space. If the BRST condition leads to several different conditions of the form (2.6) etc. with non-trivial solutions then there exist several different physical state spaces $\Omega_{ph}, \Omega'_{ph}, \cdots \subset \Omega$. If $\Omega$ really is an inner product space then $\Omega_{ph}, \Omega'_{ph}, \cdots$ may be combined into one physical state space which, however, means that the BRST cohomology contains ghost excitations which in turn implies that it cannot be a positive state space. Hence, a satisfactory theory must be a canonical one (see also section 7). Now the solutions (2.8) are obtained in a formal manner without using that $\Omega$ is an inner product space. Thus, in our procedure as described above we do not specify in advance how the ghosts and Lagrange multipliers are to be represented in $\Omega$. Instead we let the condition that $\Omega_{ph}$ must belong to an inner product space specify the appropriate representation in $\Omega$. Through the examples to be considered this condition leads to the proposal of the following general rules:
1. Lagrange multipliers must be quantized with opposite metric states to the unphysical variables in the matter space (the variables which $\psi_a$ eliminate).

2. Bosonic ghosts and antighosts must be quantized with opposite metric states. Only if these conditions are satisfied can $|\phi\rangle$ belong to an inner product space. However, in general we must also restrict the range of the Lagrange multipliers. It is the choice of representation of $\Omega$ which determines whether we have the solutions $|\phi\rangle_+$ or $|\phi\rangle_-$. We cannot have both since $+\langle\phi|\phi\rangle_- = \langle\Phi|\Phi\rangle$ is undefined. However, it is natural to require that $|\phi\rangle_+$ and $|\phi\rangle_-$ yield equivalent physics when both possibilities are allowed. We require therefore their norms to be exactly the same. In the following examples this condition is shown to partly specify the representation of $\Omega$ and sometimes to restrict the ranges of the Lagrange multipliers. The method can be used to determine a canonical theory in which case $\Omega$ is an inner product space for which the BRST condition yields a unique physical state space. In general this condition severely restricts the possible forms of $\Omega$ and the model.

3 The general abelian case.

Consider the BFV charge (2.1) in the abelian case when $U_{ab}c = 0$, i.e.

$$Q = \psi_a \eta^a + \bar{P}_a \pi^a$$

where $\psi_a$ satisfies $[\psi_a, \psi_b]_- = 0$. From the general formulas of [3] this charge is trivially written as (see also [14])

$$Q = c^a \phi_a + \phi_a^c c^a$$

where

$$\phi_a = \psi_a - i \pi_a, \quad c^a = \frac{1}{2} (\eta^a - i \bar{P}^a)$$

Hence, $Q$ is of the form (1.2) with

$$\delta = c^a \phi_a$$
Thus, if one imposes the bigrading leading to (2.7), the BRST condition \( Q|ph\rangle = 0 \) is either solved by
\[
\begin{align*}
c^a |ph\rangle &= 0, \quad \phi_a |ph\rangle &= 0 \quad (3.5) \\
c^{\dagger a} |ph\rangle &= 0, \quad \phi^{\dagger}_a |ph\rangle &= 0 \quad (3.6)
\end{align*}
\] (or a mixture of these cases). The solutions may be written
\[
|ph\rangle = e^{\pm [\rho,Q]}|\Phi\rangle
\]
where \( \rho = \mathcal{P}_a v^a \) and where \( |\Phi\rangle \) satisfies (2.10). \( |ph\rangle_+ \) solves (3.5) and \( |ph\rangle_- \) solves (3.6).

Now the decomposition (3.2) is not unique. Instead of (3.3) we may also choose
\[
\phi_a = \psi_a - i\pi_a \alpha, \quad c^a = \frac{1}{2}(\eta^a - i\alpha \bar{P}^a) \quad (3.8)
\]
for any real parameter \( \alpha \neq 0 \). The corresponding solutions are then
\[
|ph\rangle_{\pm \alpha} = e^{\pm \alpha [\rho,Q]}|\Phi\rangle \quad (3.9)
\]
where \( |\Phi\rangle \) still satisfies (2.10). In fact, (3.7) and (3.8) are connected by a unitary transformation
\[
\begin{align*}
(\pi_a, v^a) &\rightarrow \left(\frac{\pi_a}{\alpha}, \alpha v^a\right), \quad (\bar{P}^a, \bar{\eta}_a) &\rightarrow \left(\alpha \bar{P}^a, \frac{\bar{\eta}_a}{\alpha}\right) \quad (3.10)
\end{align*}
\]
We may therefore consider (3.9) for arbitrary real \( \alpha \neq 0 \). Consider, therefore, in particular \( |ph\rangle_{\pm \frac{1}{2}} \). Its inner product is given by
\[
\pm \frac{1}{2} \langle ph | ph \rangle_{\pm \frac{1}{2}} = \frac{1}{m!} (\pm i)^m \langle \Phi | (\mathcal{P}_a \bar{P}^a)^m e^{\pm \psi_a v^a} |\Phi\rangle \quad (3.11)
\]
which only is nonzero for those \( |\Phi\rangle \)-states which also satisfy
\[
\bar{\eta}_a |\Phi\rangle = 0 \quad (3.12)
\]
in addition to (2.10). Eqns (2.10) and (3.12) reduce \( |\Phi\rangle \) to
\[
|\Phi\rangle = |\phi\rangle |0\rangle_\pi |0\rangle_{\eta \bar{\eta}} \quad (3.13)
\]
where \( \pi_a |0\rangle = 0, \eta^a |0\rangle = \bar{\eta}_a |0\rangle = 0 \) and where \( |\phi\rangle \) only depends on the matter variables, which are involved in \( e.g. \psi_a \), but not on ghosts and Lagrange multipliers. Using the convention of \([13]\) we get

\[
\frac{1}{m!} (\pm i)^m \eta^a |0\rangle (P_a \bar{P}^a)^m |0\rangle = (\pm 1)^m C, \quad C = \pm 1
\]  

(3.14)

Hence, (3.11) reduces to

\[
(\pm \frac{1}{2}) \langle ph|ph\rangle_{\pm \frac{1}{2}} = (\pm 1)^m C \langle \phi| \pi \langle 0| e^{\pm \psi_a u^a} |0\rangle \pi |\phi\rangle
\]  

(3.15)

The middle term may be calculated by means of the spectral representation of the Lagrange multipliers \( v^a \). As was already mentioned in \([7]\) the only way to obtain a finite norm is to quantize \( v^a \) with opposite metric states to the variable which \( \psi_a \) eliminates. In particular, when \( |\phi\rangle \) belongs to a positive definite state space (a Hilbert space) all Lagrange multipliers \( v^a \) must be quantized with indefinite metric states for which the appropriate eigenvalues are imaginary \([10, 16, 17, 7]\). We have therefore the spectral representation

\[
v^a |iu\rangle = iu^a |iu\rangle
\]  

(3.16)

which satisfies

\[
\langle -iu|iu'\rangle = \delta^m (u' + u)
\]

\[
\langle -iu\rangle \equiv (|iu\rangle)^\dagger
\]  

(3.17)

Hence, we have

\[
\langle iu|iu'\rangle = \delta^m (u' - u)
\]

\[
\int d^n u |iu\rangle \langle iu| = 1 = \int d^n u | -iu\rangle \langle -iu| -iu|iu\rangle
\]  

(3.18)

(Notice that the range of \( u \) must be symmetric: \( -a \leq u \leq a \).) When this is inserted into (3.17) we obtain

\[
(\pm \frac{1}{2}) \langle ph|ph\rangle_{\pm \frac{1}{2}} = (\pm 1)^m C \langle \phi| \int d^n u e^{\pm \psi_a u^a} |\phi\rangle
\]

\[
= (\pm 1)^m C (2\pi)^m \int dx \delta^m (\psi_a) |\phi(x)|^2
\]  

(3.19)
where we also have introduced a complete set of states $|x\rangle$ in the matter space, *i.e.* $\int dx |x\rangle \langle x| = 1$. The last line in (3.19) is formal.

The sign difference between the norms of $|ph\rangle_+$ and $|ph\rangle_-$ when we have an odd number of constraints ($m$ odd) may also be directly understood from (3.6) and (3.6). Obviously the nontrivial solutions of (3.3) and (3.6) satisfy

$$c^a|ph\rangle_+ = k_a|ph\rangle_+ = 0$$
$$c^\dagger a|ph\rangle_- = k^\dagger a|ph\rangle_- = 0$$

(3.20)

respectively where

$$[c^a, k^\dagger_b]_+ = \delta^a_b$$

(3.21)

This means that (3.3) picks up the ghost vacuum

$$c^a|0\rangle = k_a|0\rangle = 0$$

(3.22)

while (3.6) picks up

$$c^\dagger a|\bar{0}\rangle = k^\dagger a|\bar{0}\rangle = 0$$

(3.23)

Both vacua have ghost number zero. In [13] (eqn (4.18)) it is shown that

$$\langle \bar{0}|\bar{0}\rangle = (-1)^m \langle 0|0\rangle$$

(3.24)

when $\langle 0|0\rangle = \pm 1$.

The condition that $|ph\rangle_+$ and $|ph\rangle_-$ should yield equivalent physics requires from (3.17) and (3.19) that we either choose different ghost representations or different norms of the bosonic vacuum state in the two cases when $m$ is odd. (In the last line of (3.19) we have assumed that the bosonic vacuum has positive norm.) The last possibility is probably the most natural choice since $|ph\rangle_+$ and $|ph\rangle_-$ involve different vacuum states. It is probably natural that even these bosonic vacua should be related according to (3.24).
4 A trivial model.

In order to make the above description even more explicit we consider the simple abelian case when $\psi_a = p_a$ where $p_a$ are canonical momenta to some coordinates $x^a$ satisfying

$$[p_a, x^b] = i\delta_a^b$$

(4.1)

In this case we have either a spectral representation in which $x^a, p_a$ have real eigenvalues and $v^a, \pi_a$ imaginary ones, or vice versa, or a mixture. In all cases (3.15) reduces to

$$\pm\frac{1}{2}\langle ph|ph\rangle_{\pm\frac{1}{2}} = (\pm 1)^m C \int d^m p \int d^m ve^{\mp i p_a v^a} |\phi(p)|^2 =$$

$$= (\pm 1)^m C (2\pi)^m |\phi(0)|^2$$

(4.2)

after insertion of the completeness relations for the eigenstates of $p_a$ and $v^a$. Thus, the physical state space, which here only is a vacuum state, is either positive definite or negative definite and this sign depends on the choice of ghost representation of the original state space $\Omega$ and whether we have the solutions $|ph\rangle_+$ or $|ph\rangle_-$. (It does also depends on the vacuum normalization of the bosonic variables (4.1) which is assumed to be positive in (4.2).)

The meaning of the statement that $x^a$ and $v^a$ are quantized with positive or an indefinite metric refers to a choice of oscillator basis. For instance, if the indices are assumed to be raised or lowered with an Euclidean metric the oscillators

$$a^a = \frac{1}{\sqrt{2}}(x^a + ip^a), \quad b^a = \frac{1}{\sqrt{2}}(v^a + i\pi^a)$$

(4.3)

span a positive definite state space for $x^a$ and $p_a$, and an indefinite one for $v^a$ and $\pi_a$ while the oscillator basis spanned by

$$a^a = \frac{1}{\sqrt{2}}(x^a - ip^a), \quad b^a = \frac{1}{\sqrt{2}}(v^a - i\pi^a)$$

(4.4)

has the opposite properties. (This follows from (2.3) and (4.1).) Thus, the results (4.2) requires an original state space spanned by a Fock basis with equally many positive and indefinite oscillators.
The appropriate Fock space is here the state space spanned by the oscillators

\[ \phi_a = p_a - i\pi_a, \quad \xi_a = \frac{1}{2}(ix_a - v_a), \quad [\xi_a, \phi_b^\dagger] = \delta_{ab} \] (4.5)

since its vacuum state

\[ \phi_a|0\rangle = 0, \quad \xi_a|0\rangle = 0 \] (4.6)

will be picked up by the condition (3.3). The corresponding diagonal basis of (4.5) is spanned by e.g.

\[ a_a \equiv \frac{1}{2}(\xi_a + \phi_a), \quad b_a \equiv \frac{1}{2}(\xi_a - \phi_a) \] (4.7)

where \( a_a \) and \( b_a \) are positive and indefinite oscillators respectively. Thus, the oscillator basis (4.5) is equivalent to a basis with equally many positive as indefinite oscillators which was exactly the requirement for (4.2). Alternatively we may span the original state space by the oscillators (4.5) with \( \phi_a \) and \( \xi_a \) counted as creation operators. The vacuum state is then

\[ \phi_a^\dagger|\tilde{0}\rangle = 0, \quad \xi_a^\dagger|\tilde{0}\rangle = 0 \] (4.8)

which will be picked up by eqn (3.6). Thus, it is the choice of Fock space representation of the original state space which determines whether \(|ph\rangle_+ \) or \(|ph\rangle_- \) are non-trivial solutions. Notice also that the wave function representation of \(|0\rangle \) and \(|\tilde{0}\rangle \), \( \phi_0(p, v) \equiv \langle p, v|0\rangle \) and \( \phi_0(p, v) \equiv \langle p, v|\tilde{0}\rangle \), requires imaginary eigenvalues of either \( p_a \) or \( v^a \) if \(|0\rangle \) and \(|\tilde{0}\rangle \) are to be normalizable \( (\phi_0(p, v) \propto e^{pa^av^a}) \). The condition that \(|ph\rangle_+ \) and \(|ph\rangle_- \) should have the same norms is naturally satisfied if the normalization of the vacua (4.6) and (4.8) are related by

\[ \langle 0|0\rangle = (-1)^m \langle \tilde{0}|\tilde{0}\rangle \] (4.9)

(cf. (3.24)).

In order to get a first hint of what a graded gauge group requires within the framework of [8], we interchange the meaning of the variables in \( Q = p_a\eta^a + \pi_a\overline{P}^a \). Let therefore \( \eta^a \) be fermionic gauge generators and \( p_a \) bosonic ghosts, \( \overline{\eta}^a \) Lagrange
multipliers and $v^a$ antighosts. The BRST invariant solutions corresponding to (3.9) are then

$$|ph\rangle_{\pm\alpha} = e^{\pm\alpha[\bar{\rho},Q]}|\Phi\rangle$$

where $(\bar{\rho} = x^a\bar{\eta}_a)$

$$[\bar{\rho},Q]_+ = x^a\pi_a - i\bar{\eta}_a\eta^a$$

and where $|\Phi\rangle$ satisfies

$$\bar{P}^a|\Phi\rangle = p_a|\Phi\rangle = 0$$

(Actually, (4.10) satisfies the same conditions as (3.9) with $\alpha \rightarrow 1/\alpha$.) The inner product of (4.10) is (cf.(3.15))

$$\pm\frac{1}{2}\langle ph|ph\rangle_{\pm\frac{1}{2}} = (\pm 1)^m C\langle \phi'|P_0|e^{\pm x^a}\phi(0)|\pi\rangle$$

which may be calculated by means of the spectral representations of $x^a$ and $\pi_a$. The only way to get a constant as in (4.2) is then to impose the condition that bosonic ghosts and antighosts must be quantized with opposite metric states. With this rule (4.13) reduces to

$$\pm\frac{1}{2}\langle ph|ph\rangle_{\pm\frac{1}{2}} = (\pm 1)^m C(2\pi)^m |\phi'(0)|^2$$

which is exactly equivalent to (4.2) which we should have since we have used the same BRST charge and the same state representation. Notice that in this case (4.5) spans the conventional non-diagonal ghost state representation and the vacuum (4.6) is a ghost vacuum with zero ghost number.

From the point of view of the gauge fixing procedure of [9] the physical states (4.10) are solutions of the BRST condition with $p_a$ still counted as a constraint variable and $\eta^a$ as a ghost. $x^a$ in $\bar{\rho}$ is then a gauge fixing variable to $p_a$. Similarly (3.7) is a solution with $\eta^a$ as a constraint variable and $p_a$ as a ghost. $P_a$ in $\rho$ is then gauge fixing variable to $\eta^a$. 

13
5 The spinless relativistic particle.

A manifestly Lorentz covariant quantization of relativistic particles and strings requires an original state space for which there is a basis and a spectral representation which transform manifestly covariantly under Lorentz transformations. The problem is that these properties cannot be satisfied by an original state space which also is an inner product space \([7]\). In order to see this consider the manifestly Lorentz covariant and hermitian coordinate and momentum operators \(X^\mu\) and \(P^\mu\) satisfying

\[
[X^\mu, P^\nu] = i\eta^{\mu\nu}
\]  

(5.1)

where \(\eta^{\mu\nu}\) is a space-like Minkowski metric, \(i.e.\) \(\text{diag } \eta^{\mu\nu} = (-1, +1, +1, +1)\). The corresponding manifestly covariant oscillator representation

\[
a^\mu = \frac{1}{\sqrt{2}}(X^\mu + iP^\mu)
\]  

(5.2)

satisfies

\[
[a^\mu, a^{\nu\dagger}] = \eta^{\mu\nu}
\]  

(5.3)

which means that the space components yields positive metric states while the time components yields indefinite metric states. However, an inner product space built on this basis is not consistent with a spectral representation with real eigenvalues, \(i.e.\) states \(|x\rangle\) and \(|p\rangle\) satisfying

\[
X^\mu |x\rangle = x^\mu |x\rangle, \quad P^\mu |p\rangle = p^\mu |p\rangle
\]  

(5.4)

where \(x^\mu\) and \(p^\mu\) are real and with the completeness relations

\[
\int d^4 x |x\rangle \langle x| = 1, \quad \int d^4 p |p\rangle \langle p| = 1
\]  

(5.5)

Instead it turns out that a Hilbert space topology requires (5.4) with imaginary eigenvalues for \(X^0\) and \(P^0\) in conjunction with the oscillator basis (5.2)-(5.3), which means that the measures in (5.5) are not manifestly Lorentz invariant \([4]\). Thus, there are two possibilities to represent \(X^\mu\) and \(P^\mu\) on an inner product space: Either we choose the covariant oscillator basis (5.2)-(5.3) with indefinite metric states
and imaginary eigenvalues for $X^0$ and $P^0$, or we choose a non-covariant basis with positive metric states (interchange $a^0 \leftrightarrow a^0\dagger$) but with real eigenvalues for $X^0$ and $P^0$ and Lorentz invariant measures in (5.7). (In [7] an effort was made to combine a Lorentz covariant basis and real eigenvalues in (5.4) in an inner product space. However, the resulting inner products turned out to be non-hermitian.) In the following we shall perform a BRST quantization of the relativistic particle for the above two choices of an original inner product space.

A spinless relativistic particle with mass $m$ is e.g. given by the Lagrangian and Hamiltonian

$$L = \frac{1}{2}v\dot{x}^2 - \frac{1}{2}mv, \quad H = \frac{1}{2}(p^2 + m^2)v$$

(5.6)

where $v$ is the Lagrange multiplier (usually called the einbein variable). The corresponding BRST invariant formulation gives rise to the BRST charge operator

$$Q = \frac{1}{2}(P^2 + m^2)v + \pi\bar{P}$$

(5.7)

which may be written in the form (3.2) with e.g.

$$\phi_a = \phi = \frac{1}{2}(P^2 + m^2) - i\pi, \quad c^a = c = \frac{1}{2}(\eta - i\bar{P})$$

(5.8)

The solutions of (3.5) and (3.6) are from (3.7)

$$|ph\rangle_{\pm} = e^{\pm\frac{1}{2}(P^2 + m^2)v\pm i\bar{P}}|\phi\rangle$$

(5.9)

where $|\phi\rangle$ satisfies

$$\pi|\phi\rangle = \eta|\phi\rangle = 0$$

(5.10)

The inner product of (5.9) is

$$\pm\langle ph|ph\rangle_{\pm} = \pm2i\langle \phi|e^{\pm(P^2 + m^2)v\bar{P}}|\phi\rangle$$

(5.11)

where $|\phi\rangle$ now may be chosen to be in the form (3.13). We get then

$$\pm\langle ph|ph\rangle_{\pm} = \pm2\langle \phi|e^{\pm(P^2 + m^2)v}|\phi\rangle$$

(5.12)
We have now to choose between the two possibilities mentioned above. Consider first the case when $X^0$ and $P^0$ are represented by a positive definite state space. In this case the Lagrange multiplier $v$ must be quantized with indefinite metric states which means that we have to use a spectral representation with imaginary eigenvalues of $v$ which we choose to be $iu$. Inserting the appropriate completeness relations into (5.12) we get (choosing the plus sign by a choice of vacuum normalization)

$$\pm \langle ph|ph \rangle \pm = 2 \int d^4p \, du \, e^{\pm i(p^2 + m^2)u} |\phi(p)|^2 =$$

$$= 4\pi \int d^4p \, \delta(p^2 + m^2) |\phi(p)|^2$$

(5.13)

where we have chosen the vacuum normalizations so that the norms are positive. This is a correct inner product for a free spinless relativistic particle apart from the fact that it contains both positive and negative energy solutions. (Within the gauge fixing procedure of [9], the positive and negative energy solutions enter with opposite norms.)

If we instead make use of a manifestly covariant oscillator basis, then $X^0$ and $P^0$ are represented by indefinite metric states and their spectral representation requires imaginary eigenvalues. In this case (5.12) becomes (again choosing the plus sign by a choice of vacuum normalization)

$$\pm \langle ph|ph \rangle \pm = 2 \int d^4p \, dv \, e^{\pm (p^2 + m^2)v} \phi^*(-p^0, \mathbf{p})\phi(p^0, \mathbf{p})$$

(5.14)

where $p^0$ now is an Euclidean energy, i.e. $p^2 + m^2$ is positive definite and $d^4p$ is a Euclidean measure. In order for this expression to be finite, the range of $v$ has to be restricted: In $|ph\rangle_+$ $v$ has to have a finite maximal value, and in $|ph\rangle_-$ a finite minimal value. The condition that $|ph\rangle_\pm$ must yield equivalent results requires then

$$0 \leq v < \infty \text{ in } |ph\rangle_+$$

$$-\infty < v \leq 0 \text{ in } |ph\rangle_-$$

(5.15)

since only for these choices do we get the same inner products for $|ph\rangle_+$ and $|ph\rangle_-$ in (5.14), namely

$$\pm \langle ph|ph \rangle \pm = 2 \int d^4p \, \frac{\phi^*(-p^0, \mathbf{p})\phi(p^0, \mathbf{p})}{p^2 + m^2}$$

(5.16)
which is positive definite only if $\phi$ is an even function of $p^0$ (or an odd function if we choose a minus sign in (5.12)) In such a case this looks like a norm for a free Euclidean field. A Lorentz invariant way to make $\phi$ an even function of $p^0$ is to require the original state space to be invariant under strong reflection $p^\mu \rightarrow -p^\mu$.

We cannot have both solutions $|ph\rangle_+$ and $|ph\rangle_-$ simultaneously since their bilinear form is infinite. A simple way to make sure that only one of the above possibilities is allowed is to quantize the particle in such a way that the spectrum of $v$ is either in the range $0 \leq v < \infty$ or $-\infty < v \leq 0$. Such a model is the $OSp(4,2|2)$-invariant model considered in \[18\]. This model should therefore be a canonical model for the free spinless relativistic particle (cf. \[19\]).

In the Minkowski treatment leading to (5.13) there is no Lorentz invariant way to restrict the original state space to an inner product space such that the BRST condition yields a unique solution, \textit{i.e.} either $|ph\rangle_+$ or $|ph\rangle_-$. This follows since there is no Lorentz covariant basis in this case which means that there is no covariant canonical theory. The only possibility at our disposal is an analytic continuation of the Euclidean treatment above.

6 The massless relativistic spin-$\frac{1}{2}$ particle.

The standard worldline supersymmetric model for a massless spin-$\frac{1}{2}$ particle \[20\] may be described by the Lagrangian

$$L = \frac{1}{4v}(\dot{x} - i\lambda\gamma)^2 - i\frac{1}{2}\gamma \cdot \dot{\gamma}$$

(6.1)

where $\gamma^\mu$ is an odd Grassmann variable describing the spin degrees of freedom, and $v, \lambda$ are Lagrange multipliers. A BRST quantization of (6.1) with dynamical Lagrange multipliers and antighosts was considered in \[2, 4\]. Here we shall quantize this model on an inner product space using a graded generalization of the method of ref.\[8\]. The BRST charge is

$$Q = P^2 \eta + P \cdot \gamma c + \mathcal{P} c^2 + \pi \tilde{\mathcal{P}} + \kappa \tilde{k}$$

(6.2)
where the variables satisfy the following (anti-)commutation relations (the nonzero part):

\[ [\gamma^\mu, \gamma^\nu]_+ = -2\eta^\mu\nu, \quad [X^\mu, P^\nu]_- = i\eta^\mu\nu, \quad [\pi, v]_- = -i, \quad [\kappa, \lambda]_+ = 1, \]
\[ [\mathcal{P}, \eta]_+ = 1, \quad [\bar{\mathcal{P}}, \bar{\eta}]_+ = 1, \quad [k, c]_- = -i, \quad [\bar{k}, \bar{c}]_- = -i \]  

(6.3)

where \( k, c \) are bosonic ghosts and \( \bar{k}, \bar{c} \) the corresponding antighosts, \( \lambda \) is a fermionic Lagrange multiplier and \( \kappa \) its conjugate momentum. \( \eta^\mu\nu \) is a space-like Minkowski metric. Notice that

\[ [P \cdot \gamma, P \cdot \gamma]_+ = -2P^2 \]  

(6.4)

is the algebra of the world-line supersymmetry. In the matrix representation \( \gamma^\mu \) is turned into the ordinary Dirac gamma matrices as is shown in the appendix.

By means of a natural generalization of the method of ref.\[8\] it is easily shown that the BRST charge \((6.2)\) may be written in the form \((1.2)\). First one performs a unitary transformation to primed variables defined by (the nontrivial part)

\[ \bar{\mathcal{P}}' = \bar{\mathcal{P}} + \lambda c, \quad k' = k + i\lambda\bar{\eta}, \quad \kappa' = \kappa - \bar{\eta}c \]  

(6.5)

where hermiticity and ghost numbers are preserved. Then one introduces complex ghosts by

\[ \sigma \equiv \frac{1}{2}(\eta' - i\bar{\mathcal{P}}'), \quad \omega \equiv \mathcal{P}' - i\bar{\eta}' \]
\[ a \equiv \frac{1}{2}(c' - i\bar{k}'), \quad b \equiv ik' - \bar{c}' \]  

(6.6)

satisfying (the nonzero part)

\[ [a, b^\dagger]_- = 1, \quad [\sigma, \omega^\dagger]_+ = 1 \]  

(6.7)

In terms of these ghost variables \( \delta \) in \((1.2)\) may be written

\[ \delta \equiv [a^\dagger b + \sigma^\dagger \omega, Q]_- = a^\dagger D + \sigma^\dagger \phi \]  

(6.8)

where

\[ D = P \cdot \zeta - \lambda\pi - ik' + a^\dagger \omega + \omega^\dagger a, \quad \phi = P^2 - i\pi \]  

(6.9)
Notice that the ghost number operator is

\[ N = a^\dagger b + \sigma^\dagger \omega - b^\dagger a - \omega^\dagger \sigma \] (6.10)

Notice also that

\[ [D, D]_+ = -2\phi, \quad [D, \sigma]_+ = a, \quad [D, \sigma^\dagger]_+ = a^\dagger \] (6.11)

imply \( \delta^2 = 0 \) and \( [\delta, \delta^\dagger]_+ = 0 \), and that \( D \) and \( \phi \) satisfy the original worldline supersymmetry (5.4). By means of a bigrading the BRST invariant states must satisfy

\[ \delta |ph\rangle = \delta^\dagger |ph\rangle = 0 \] (6.12)

whose non-trivial solutions we find to be determined by

\[ D|ph\rangle = \phi|ph\rangle = a|ph\rangle = \sigma|ph\rangle = 0 \] (6.13)

or

\[ D^\dagger|ph\rangle = \phi^\dagger|ph\rangle = a^\dagger|ph\rangle = \sigma^\dagger|ph\rangle = 0 \] (6.14)

All other solutions of (6.12) are decoupled zero norm states. In order to solve these conditions we introduce the transformations (\( x \) is any operator)\[8\]

\[ x \rightarrow e^A x e^{-A} \] (6.15)

where

\[ A \equiv [\rho, Q]_+ = P^2 v + iP \cdot \gamma \lambda + k\bar{k} + i\mathcal{P}\mathcal{P} + 2i\lambda \mathcal{P} c, \quad \rho \equiv \mathcal{P} v + k\lambda \] (6.16)

We find then

\[ e^A \eta e^{-A} = \eta - 2i\lambda c - i\mathcal{P} - \lambda\bar{k} = 2\sigma - 2i\lambda a, \quad e^A ce^{-A} = c - i\bar{k} = 2a \]

\[ e^A(-i\pi)e^{-A} = \phi, \quad e^A(-\lambda\pi - ik)e^{-A} = D + 2\omega a \] (6.17)

These expressions imply then that (6.13) is solved by

\[ |ph\rangle_+ = e^{[\rho, Q]} |\Phi\rangle \] (6.18)
where \( |\phi\rangle \) satisfies
\[
c|\Phi\rangle = \eta|\Phi\rangle = \pi|\Phi\rangle = \kappa|\Phi\rangle = 0
\] (6.19)
which are trivially solved. Since the hermitian conjugation of (6.17) yields
\[
e^{-A}\eta e^{A} = 2\sigma^{\dagger} - 2i\lambda a^{\dagger}, \quad e^{-A}\eta e^{A} = 2a^{\dagger}
e^{-A}\eta e^{A} = 2\sigma^{\dagger} - 2i\lambda a^{\dagger}, \quad e^{-A}\eta e^{A} = 2\sigma^{\dagger} - 2i\lambda a^{\dagger},
\] (6.20)
eqn (6.17) is solved by
\[
|ph\rangle = e^{-[\rho,Q]}|\Phi\rangle
\] (6.21)
where \( |\Phi\rangle \) satisfies (6.19).

The inner products of \( |ph\rangle_{\pm} \) are
\[
\pm \langle ph|ph\rangle_{\pm} = \langle \Phi|e^{\pm2(P^{2}v+iP \cdot \gamma \lambda + kP + \bar{P}^{\dagger} - 2i\lambda P\sigma)}|\Phi\rangle = 
= \langle \Phi|e^{\pm2P^{2}v}e^{\pm2iP \cdot \gamma \lambda}e^{\pm2kk}e^{\pm2iP\gamma e^{4i\lambda P\sigma}e^{-4k\lambda P}}|\Phi\rangle = 
= -8\langle \Phi|e^{\pm2P^{2}v}e^{\pm2kk}P \cdot \gamma \lambda \bar{P}|\Phi\rangle \mp 8i\langle \Phi|e^{\pm2P^{2}v}e^{\pm2kk}k\lambda \bar{P}|\Phi\rangle
\] (6.22)
The integration over the bosonic ghosts is simple if one uses the rule that \( c, k \) and \( \bar{c}, \bar{k} \) must be quantized with opposite choices of state spaces, \( i.e. \) one with positive and the other with indefinite metric states. The integration over \( k \) yields then in the first term \( \delta(\bar{k}) \) and in the second \( \delta(\bar{k})\bar{k} = 0 \). Hence, we arrive at (\( C \) is a positive constant)
\[
\pm \langle ph|ph\rangle_{\pm} = -C'\langle \Phi|e^{\pm2P^{2}v}\bar{P} \cdot \gamma \lambda \bar{P}|\Phi\rangle'
\] (6.23)
where \( |\Phi\rangle' \) is equal to \( |\Phi\rangle \) without the bosonic ghost part. We may choose
\[
|\Phi\rangle' = \sum_{\alpha=1}^{4} |\psi_{\alpha}\rangle|\alpha\rangle|0\rangle_{\kappa}|0\rangle_{\eta\bar{\eta}} \]
(6.24)
where \( |\alpha\rangle \) are spinor states built from the operators \( \gamma^{\mu} \) (see appendix). Eqn (6.23) becomes then
\[
\pm \langle ph|ph\rangle_{\pm} = 
= -C \sum_{\alpha,\beta=1}^{4} \langle \psi_{\alpha}|e^{\pm2P^{2}v}\bar{P} \cdot \gamma^{\mu} \bar{P}|\Phi\rangle|\alpha\rangle|\gamma^{\mu} \lambda |\beta\rangle|0\rangle_{\kappa}|\gamma^{\mu} \lambda \bar{P}|\beta\rangle|0\rangle_{\eta\bar{\eta}}|\psi_{\beta}\rangle
\] (6.25)
In the appendix it is shown that

\[ \kappa\langle 0 | \alpha | \gamma^\mu \lambda | \beta \rangle |0\rangle_\kappa = i(\gamma^0 \gamma^\mu \gamma^5)_{\alpha\beta} \]  

(6.26)

where \( \gamma^0 \gamma^\mu \) are Dirac’s gamma matrices. Furthermore we may choose

\[ \eta\bar{\eta}\langle 0 | \mathcal{P} \bar{\mathcal{P}} |0\rangle_{\eta\bar{\eta}} = i. \]  

(6.27)

Thus, if we quantize \( P^0 \) with positive metric states and \( v \) with indefinite ones then (6.25) becomes

\[ \pm \langle ph|ph \rangle_\pm \propto \int d^4p \delta(p^2) \bar{\psi}(p) \hat{\mathbf{p}} \gamma^5 \psi(p) \]  

(6.28)

which is a Lorentz invariant expression. However, it is not a positive norm. (The left-handed positive energy part and the right-handed negative energy part have opposite norms to the right-handed positive energy part and the left-handed negative energy part.) It should be emphasized that \( \gamma^5 \) in (6.28) is forced on us as soon as odd Grassmann numbers are introduced into the quantum theory. (If odd Grassmann numbers are banned from the quantum theory then (6.28) without \( \gamma^5 \) should be an allowed possibility. However, this would e.g. exclude pseudoclassical path integrals.)

Even within the gauge fixing procedure of [9] the norm (6.28) is obtained.

If we instead quantize \( P^0 \) with indefinite metric states and \( v \) with positive ones, then the argument leading to (5.16) applies. Hence, we get

\[ \pm \langle ph|ph \rangle_\pm \propto \int d^4p \left( -p^0, \mathbf{p} \right) \frac{\hat{\mathbf{p}} \gamma^5}{p^2} \psi(p^0, \mathbf{p}) \]  

(6.29)

where \( p^0 \) is real and Euclidean, i.e. \( p^2 > 0 \), and \( \mathbf{p} \equiv ip^0 \gamma^0 + \mathbf{p} \cdot \gamma \). Also this norm is not positive definite.

If we instead of \( \gamma^\mu \) in (6.2)-(6.3) had used fermionic operators \( \zeta^\mu \) satisfying

\[ [\zeta^\mu, \zeta^\nu]_+ = 2\eta^{\mu\nu} \]  

(6.30)

then we would not obtain any factor \( \gamma^5 \) in (6.28) and (6.29). The reason for this is obvious since we may make the identification (see appendix)

\[ \zeta^\mu = \gamma^\mu \gamma^5 \]  

(6.31)
This is a consistent possibility only for massless spin-1/2 particles. (However, this does not make (6.28) positive since the positive and negative energy parts now will have opposite norms.)

As in the spinless case a canonical theory of the spin-1/2 particle requires a quantization with an Euclidean spectrum for the four-momentum and the ranges \(0 \leq v < \infty\) or \(-\infty < v \leq 0\) for the Lagrange multipliers.

Contrary to the spinless case there is no natural manifestly covariant way to make the norms (6.28) and (6.29) positive. The replacement \(\gamma^\mu \rightarrow \zeta^\mu\) does not improve the situation.

### 7 The massless relativistic spin-one particle.

The following generalization of the Lagrangian (6.1) was given in [21]:

\[
L = \frac{1}{4v_1}(\dot{x} - i \sum_{k=1}^{2} \lambda_k \gamma_k)^2 - i \frac{1}{2} \sum_{k=1}^{2} \gamma_k \cdot \dot{\gamma}_k - iv_2 \gamma_1 \cdot \gamma_2
\]

(7.1)

Its gauge invariance is an \(O(2)\)-extended world-line supersymmetry. In distinction to (6.1) we have here introduced two odd Grassmann variables, \(\gamma_1^\mu\) and \(\gamma_2^\mu\), describing the spin degrees of freedom. There are also two odd Lagrange multipliers, \(\lambda_1\) and \(\lambda_2\). An additional even Lagrange multiplier \(v_2\) is introduced as well. That this Lagrangian describes a spin-one particle was first proposed in [22]. It was further treated in [23, 3]. A particular interesting feature of this Lagrangian is that there exists no gauge fixing to the constraint \(\gamma_1 \cdot \gamma_2\) which means that there exists no corresponding regular Lagrangian. A generalized BRST quantization of (7.1) was performed in [3] on a particular state space. Here we shall perform a standard BRST quantization on an inner product space by means of the procedure of [8, 9]. The BRST charge is [3]

\[
Q = P^2 \eta_1 + c_1 \phi_1 + c_2 \phi_2 + \eta_2 B + \mathcal{P}_1 (c_1^2 + c_2^2) + 2\eta_2 (k_1 c_2 - k_2 c_1) + \\
+ \kappa_1 \tilde{k}_1 + \kappa_2 \tilde{k}_2 + \pi_1 \tilde{P}_1 + \pi_2 \tilde{P}_2
\]

(7.2)
where
\[ \phi_i \equiv P \cdot \gamma_i, \quad B \equiv i \gamma_1 \cdot \gamma_2 \] (7.3)
and where \( P, \eta \) and \( k, c \) are fermionic and bosonic ghosts respectively, and \( \bar{P}, \bar{\eta} \) and \( \bar{k}, \bar{c} \) the corresponding antighosts. \( \pi, v \) and \( \kappa, \lambda \) are the bosonic and fermionic Lagrange multipliers. All variables satisfy the following (anti-)commutation relations (the non-zero part)

\[
\left[ \gamma_i^\mu, \gamma_j^\nu \right]_+ = -2 \delta_{ij} \eta^{\mu\nu}, \quad \left[ X^\mu, P^\nu \right]_- = i \eta^{\mu\nu}, \\
\left[ \pi_k, v_l \right]_- = -i \delta_{kl}, \quad \left[ \kappa_i, \lambda_j \right]_+ = \delta_{ij}, \quad \left[ P_i, \eta_j \right]_+ = \delta_{ij} \\
\left[ \bar{P}_i, \bar{\eta}_j \right]_+ = \delta_{ij}, \quad \left[ k_i, c_j \right]_- = -i \delta_{ij}, \quad \left[ \bar{k}_i, \bar{c}_j \right]_- = -i \delta_{ij}
\] (7.4)

\( \phi, B \) and \( P^2 \) are generators of the \( O(2) \)-extended world-line supersymmetry. Their algebra is

\[
\left[ \phi_i, \phi_j \right]_+ = -2 \delta_{ij} P, \quad \left[ B, \phi_1 \right]_+ = 2i \phi_2, \quad \left[ B, \phi_2 \right]_+ = -2i \phi_1
\] (7.5)

From the treatment of [8, 9] we expect the general formal solutions of \( \delta |ph\rangle = \delta^\dagger |ph\rangle = 0 \) to be

\[
|ph\rangle_\pm = e^{\pm [\rho, Q]} |\Phi\rangle
\] (7.6)

We shall consider the case when \( \rho = P_i v_i + k_i \kappa_i \) for which case the state \( |\Phi\rangle \) must satisfy

\[
c_i |\Phi\rangle = \eta_i |\Phi\rangle = \pi_i |\Phi\rangle = \kappa_i |\Phi\rangle = 0
\] (7.7)

In order to calculate the norms

\[
\langle ph |ph\rangle_\pm = \langle \Phi |e^{\pm [\rho, Q]} |\Phi\rangle
\] (7.8)

we have to simplify \( e^{\pm [\rho, Q]} \). We have explicitly

\[
[\rho, Q] = P^2 v_1 - i \lambda_i \phi_i + v_2 B + i P_i \bar{P}_i + k_i \bar{k}_i + \\
+ 2i P_1 (\lambda_1 c_1 + \lambda_2 c_2) + 2i \eta_2 (k_1 \lambda_2 - k_2 \lambda_1) + 2v_2 (k_1 c_2 - k_2 c_1)
\] (7.9)
Thus,
\[ e^{\pm \rho, Q} = e^{\pm P^2 v_1} e^{\pm B} e^{\pm R} \] (7.10)

where
\[ \bar{B} \equiv -i \lambda_i \phi_i + v_2 B; \quad R \equiv i \mathcal{P}_i \bar{\mathcal{P}}_i + k_i \bar{k}_i + \\
2i \mathcal{P}_1 (\lambda_1 c_1 + \lambda_2 c_2) + 2i \eta_2 (k_1 \lambda_2 - k_2 \lambda_1) + 2v_2 (k_1 c_2 - k_2 c_1) \] (7.11)

The last two exponentials in (7.10) may be further simplified. For \( e^{\pm B} \) we find
\[ e^{\pm B} = \left( 1 \pm \alpha(v_2) a + \beta(v_2) b + \frac{\beta(v_2)}{2v_2} a^2 \pm \frac{1}{2v_2} (\alpha(v_2) - 1) ab \right) e^{\pm v_2 R} \] (7.12)

where
\[ a \equiv -i \lambda_i \phi_i, \quad b \equiv \lambda_1 \phi_2 - \lambda_2 \phi_1, \quad a^2 = 2 \lambda_1 \lambda_2 \phi_1 \phi_2, \quad ab = 2i \lambda_1 \lambda_2 P^2, \]
\[ \alpha(v_2) \equiv \frac{\sinh 2v_2}{2v_2}, \quad \beta(v_2) \equiv \frac{1}{2v_2} (\cosh 2v_2 - 1) \] (7.13)

and
\[ e^{\pm R} = e^{\pm i \mathcal{P}_1 \bar{\mathcal{P}}_1 e^{\pm k_i \bar{k}_i} G_\pm} \] (7.14)

where
\[ G_\pm \equiv e^{\pm c + d}, \]
\[ c \equiv 2i \mathcal{P}_1 (\lambda_1 c_1 + \lambda_2 c_2) + 2i \eta_2 (k_1 \lambda_2 - k_2 \lambda_1) + 2v_2 (k_1 c_2 - k_2 c_1), \]
\[ d \equiv iv_2 (k_1 \bar{k}_2 - k_2 \bar{k}_1) - \mathcal{P}_1 (\lambda_1 \bar{k}_1 + \lambda_2 \bar{k}_2) + \bar{\mathcal{P}}_1 (k_2 \lambda_1 - k_1 \lambda_2) \] (7.15)

When we now insert these expressions into the right-hand-side of (7.8) we realize that the terms in (7.15) involving \( \mathcal{P}_1, \bar{\mathcal{P}}_2 \) and \( \eta_2 \) will not contribute to the norm.
We may therefore effectively set
\[ c = 2v_2 (k_1 c_2 - k_2 c_1), \quad d = iv_2 (k_1 \bar{k}_2 - k_2 \bar{k}_1) \] (7.16)
in (7.13). We find then
\[ G_\pm = e^{\frac{1}{2}i(k_1 \bar{k}_2 - k_2 \bar{k}_1) \sinh 2v_2} e^{\pm k_i \bar{k}_i} e^{\cosh 2v_2} e^{\pm c} \] (7.17)
Inserting (7.10) with (7.12), (7.14), and (7.17) into (7.8) we find

\[ \pm \langle ph|ph \rangle_\pm = \langle \Phi| e^{\pm P_2 v_1} e^{\pm B} e^{\pm i P_1^\dagger P_1} e^{\pm k_i \bar{k}_i} G_\pm |\Phi \rangle = \]

\[ = \langle \Phi| e^{\pm P_2 v_1} e^{\pm B} P_1 P_2 \bar{P}_1 \bar{P}_2 e^{\frac{i}{2} [(k_1 \bar{k}_2 - k_2 \bar{k}_1)] \sinh 2v_2 e^{\pm k_i \bar{k}_i (1 + \frac{1}{2} \cosh 2v_2)} |\Phi \rangle \]  

(7.18)

A problem with this \( O(2) \)-model is that the gauge generator \( B \) cannot be said to eliminate any dynamical variable since there exists no gauge fixing to \( B \). As a consequence it is unclear how the quantization rule in section 2 should be interpreted, \( i.e. \) whether or not the spectrum of \( v_2 \) should be chosen to be real or imaginary. In the following we shall therefore consider both cases and derive under what conditions they lead to finite results. First we notice that in order for (7.18) to have a chance to be finite the eigenvalues of \( k_i \bar{k}_i \) must be imaginary which requires the bosonic ghosts and antighosts to be quantized with opposite metric states in accordance with the second rule in section 2. However, in addition also \( i (k_1 \bar{k}_2 - k_2 \bar{k}_1) \sinh 2v_2 \) must be imaginary. This yields the following two possibilities:

1. if the spectrum of \( v_2 \) is real then \( k_1 \) and \( \bar{k}_2 \) must be quantized with the same metric but opposite to those for \( k_2 \) and \( \bar{k}_1 \).

2. if the spectrum of \( v_2 \) is imaginary then \( k_1 \) and \( k_2 \) must be quantized with the same metric but opposite to those of \( \bar{k}_1 \) and \( \bar{k}_2 \).

Both these possibilities are consistent with the quantization rules of section 2.

In case (1) (7.18) will involve the integral (\( v_2 \) denotes here the eigenvalue of the operator \( v_2 \))

\[ \int dv_2 d^2 k d^2 \bar{k} f(v_2, \ldots) e^{\frac{i}{2} [(k_1 \bar{k}_2 - k_2 \bar{k}_1)] \sinh 2v_2 e^{\pm i k_i \bar{k}_i (1 + \frac{1}{2} \cosh 2v_2)} \]  

(7.19)

where the function \( f(v_2, \ldots) \) does not involve \( k_i \) and \( \bar{k}_i \). The integration over \( k_i \) and \( \bar{k}_i \) yields here

\[ (2\pi)^2 \int dv_2 f(v_2, \ldots) \frac{1}{\frac{5}{4} + \cosh 2v_2} \]  

(7.20)
In case (2) (7.18) will instead involve the integral

\[ \int du_2 d\mathbf{k} \mathcal{K} f(iu_2, \ldots) e^{i(k_1 \mathbf{k}_2 - k_2 \mathbf{k}_1) \sin 2u_2} e^{\pm ik_i(1 + \frac{1}{4} \cos 2u_2)} = \]

\[ = (2\pi)^2 \int du_2 f(iu_2, \ldots) \frac{1}{1 + \cos 2u_2 + \frac{1}{4} \cos 4u_2} \quad (7.21) \]

where \( iu_2 \) is the eigenvalue of the operator \( v_2 \). Inserting (7.12) into (7.18) we find

\[ \pm \langle \phi h | \phi h \rangle_\pm = (2\pi)^2' \langle \Phi | \pi_1 \langle 0 | e^{\pm p^2 v_1} | 0 \rangle_{\pi_1} \times \]

\[ \times \int dv_2 \left( -i \frac{\beta(v_2)}{v_2} \phi_1 \phi_2 \pm \frac{1}{v_2} (\alpha(v_2) - 1) \right) \frac{e^{\pm u_2 B}}{\frac{1}{4} + \cosh 2v_2} |\Phi\rangle' \quad (7.22) \]

in case (1), and

\[ \pm \langle \phi h | \phi h \rangle_\pm = (2\pi)^2' \langle \Phi | \pi_1 \langle 0 | e^{\pm p^2 v_1} | 0 \rangle_{\pi_1} \times \]

\[ \times \int du_2 \left( \frac{\beta(iu_2)}{u_2} \phi_1 \phi_2 \pm \frac{1}{iu_2} (\alpha(iu_2) - 1) \right) \times \]

\[ \times \frac{e^{\pm iu_2 B}}{1 + \cos 2u_2 + \frac{1}{4} \cos 4u_2} |\Phi\rangle' \quad (7.23) \]

in case (2). We have used the fact that

\[ |\Phi\rangle = |\Phi\rangle' |0\rangle_{\kappa} |0\rangle_{\pi} |0\rangle_{\eta} |0\rangle_{\bar{\eta}} \]

\[ \kappa, \eta, \bar{\eta} \langle 0 | \lambda_1 \lambda_2 \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_1 \mathcal{P}_2 | 0 \rangle_{\kappa, \eta, \bar{\eta}} = -i \quad (7.25) \]

In order to proceed we must calculate \( \phi_1 \phi_2 \) and \( e^{\pm v_2 B} \) and to do that we have to specify the state \( |\Phi\rangle' \). In fact, \( |\Phi\rangle' \) may be expanded in terms of eigenvalues of \( B \) as follows

\[ |\Phi\rangle' = \sum_{n=-2}^{2} |\Phi, n\rangle, \quad B |\Phi, n\rangle = 2n |\Phi, n\rangle \quad (7.26) \]

where \( n \) is an integer. These eigenstates are easily constructed [3]: Define \( b^\mu \) by

\[ b^\mu \equiv \frac{1}{2} (\gamma_1^\mu - i\gamma_2^\mu) \quad (7.27) \]

We have then

\[ \gamma_1^\mu = (b^\mu + b^{\mu \dagger}), \quad \gamma_2^\mu = i(b^\mu - b^{\mu \dagger}) \quad (7.28) \]
and

$$[b^\mu, b^{\nu\dagger}]_+ = -\eta^{\mu\nu} \quad (7.29)$$

Hence,

$$B \equiv i\gamma_1 \cdot \gamma_2 = -2b^{\dagger} \cdot b - 4 \quad (7.30)$$

The possible matter states are then

$$|\Phi, -2\rangle = |A\rangle |0\rangle, \quad |\Phi, -1\rangle = |A_\mu\rangle b^{\dagger\mu}|0\rangle, \quad |\Phi, 0\rangle = |A_\mu_\nu\rangle b^{\dagger\mu} b^{\dagger\nu}|0\rangle$$

$$|\Phi, 1\rangle = |A_{\mu\nu\rho}\rangle b^{\dagger\mu} b^{\dagger\nu} b^{\dagger\rho}|0\rangle, \quad |\Phi, 2\rangle = |A_{\mu\nu\rho\lambda}\rangle b^{\dagger\mu} b^{\dagger\nu} b^{\dagger\rho} b^{\dagger\lambda}|0\rangle \quad (7.31)$$

where the vacuum state $|0\rangle$ satisfies $b^{\mu}|0\rangle = 0$. Since $\phi_1\phi_2$ and $B$ commute $\phi_1\phi_2$ does not change the eigenvalues of $|\Phi, n\rangle$. This is also easily realized since the operator $\phi_1\phi_2$ has the following form in terms of $b^{\mu}$

$$\phi_1\phi_2 = i (2P \cdot b^{\dagger} P \cdot b + P^2) \quad (7.32)$$

When inserting (7.26) into (7.22) and (7.23) we find that $\pm \langle ph|ph\rangle\pm$ can only be finite if the range of the spectrum of $v_2$ is finite. If we choose the range in (7.22) to be $-L \leq v_2 \leq L$ we find

$$\pm \langle ph|ph\rangle\pm = (2\pi)^2 \sum_{n=-2}^2 \langle n, \Phi|\pi_1, 0|e^{\pm P^2 v_1}|0\rangle \pi_1 \times$$

$$\times \left( a_n (2P \cdot b^{\dagger} P \cdot b + P^2) + b_n P^2 \right) |\Phi, n\rangle \quad (7.33)$$

where

$$a_n \equiv \int_{-L}^{L} dv_2 \left( \frac{\cosh 2v_2 - 1}{2v_2^2} \right) \frac{\cosh nv_2}{\frac{4}{9} + \cosh 2v_2} = a_{-n} > 0$$

$$b_n \equiv \int_{-L}^{L} dv_2 \left( \frac{\sinh 2v_2 - 2v_2}{2v_2^2} \right) \frac{\sinh nv_2}{\frac{4}{9} + \cosh 2v_2} = -b_{-n} \quad (7.34)$$

In (7.23) the range of $u_2$ must be of the form $-L \leq u_2 \leq L$ since $u_2$ represents an imaginary eigenvalue of an hermitean operator. We find here (7.33) but with

$$a_n \equiv \int_{-L}^{L} du_2 \left( \frac{1 - \cos 2u_2}{2u_2^2} \right) \frac{\cos nu_2}{(1 + \cos 2u_2 + \frac{1}{4} \cos 4u_2)} = a_{-n} > 0$$

$$b_n \equiv \int_{-L}^{L} du_2 \left( \frac{\sin 2u_2 - 2u_2}{2u_2^2} \right) \frac{\sin nu_2}{(1 + \cos 2u_2 + \frac{1}{4} \cos 4u_2)} = -b_{-n} \quad (7.35)$$
In the case when \( P^\mu \) is chosen to have a real spectrum and \( v_1 \) an imaginary one we find for the above two cases:

\[
\pm \langle \text{ph} | \text{ph} \rangle_\pm = 2(2\pi)^3 \int d^4p \delta(p^2) \left( -a_{-1}p_\mu A^{*\mu}(p)p_\nu A^\nu(p) + 4a_0 p_\mu A^{*\mu\nu}(p)p^\rho A_{\rho\nu}(p) - 18a_1 p_\mu A^{*\mu\nu\rho}(p)p^\lambda A_{\lambda\nu\rho}(p) \right) \tag{7.36}
\]

For the case when \( P^0, X^0 \) are chosen to have an indefinite metric basis and imaginary eigenvalues, and \( v_1 \) a real spectrum with the ranges \((0, \infty)\) or \((-\infty, 0)\) we find

\[
\pm \langle \text{ph} | \text{ph} \rangle_\pm = (2\pi)^2 \int d^4p \left\{ (a_{-2} + b_{-2})A^*(\tilde{p}^*)A(\tilde{p}) + 2a_{-1} \left( \frac{\tilde{p}_\mu A^{*\mu}(\tilde{p}^*)\tilde{p}_\nu A^\nu(\tilde{p})}{\tilde{p}^2} \right) - (a_{-1} + b_{-1})A^{*\mu}(\tilde{p}^*)A_\mu(\tilde{p}) + 2a_0 \left( \frac{-4\tilde{p}_\mu A^{*\mu\nu}(\tilde{p}^*)\tilde{p}_{\rho\nu} A_{\rho\nu}(\tilde{p})}{\tilde{p}^2} + A^{*\mu\nu}(\tilde{p}^*)A_{\mu\nu}(\tilde{p}) \right) + 36a_1 \left( \frac{\tilde{p}_\mu A^{*\mu\nu\rho}(\tilde{p}^*)p^\lambda A_{\lambda\nu\rho}(\tilde{p})}{\tilde{p}^2} \right) - 6(a_1 + b_1)A^{*\mu\nu\rho}(\tilde{p}^*)A_{\mu\nu\rho}(\tilde{p}) - 24(a_1 - b_1)A^{*\mu\nu\rho\lambda}(\tilde{p}^*)A_{\mu\nu\rho\lambda}(\tilde{p}) \right\} \tag{7.37}
\]

where \( \tilde{p} \equiv (ip^0, \mathbf{p}) \) and \( p^2 \equiv (p^0)^2 + \mathbf{p}^2 \). Thus, the measure \( d^4p \) is Euclidean here.

The above expressions involve too many degrees of freedom. In order to describe only a spin one particle we need further projections. An obvious condition to impose is

\[ B | \Phi \rangle = 0 \tag{7.38} \]

which may be written in a manifestly allowed form \[ \overline{B} | \Phi \rangle = 0 \] where \( \overline{B} \equiv [P_2, Q]_+ \).

If we further restrict \( A^{\mu\nu} \) to be a real field we find from (7.38) (we set \( F^{\mu\nu} \equiv A^{\mu\nu} \) where \( F^{\mu\nu} \) is real and give three equivalent forms)

\[
\pm \langle \text{ph} | \text{ph} \rangle_\pm \propto \int d^4p \delta(p^2)p_\mu F^{\mu\nu\rho\nu} F_{\rho\mu} \equiv \frac{1}{2} \int d^4p \delta(p^2) F^{\mu\nu} (p_\mu p_\rho F_{\rho\nu} - p_\nu p_\rho F_{\rho\mu}) \equiv -\frac{1}{6} \int d^4p \delta(p^2) G^{\mu\nu\rho} G_{\mu\nu\rho} \tag{7.39}
\]

where \( G_{\mu\nu\rho} \) is a totally antisymmetric field given by

\[
G_{\mu\nu\rho} \equiv p_\mu F_{\nu\rho} + p_\nu F_{\rho\mu} + p_\rho F_{\mu\nu} \tag{7.40}
\]
Eqns (5.13), (6.28) and the second form of (7.39) suggest the general form

\[ \langle p|ph \rangle \propto \int d^4p A^*(p) A_{ph}(p) \]  

(7.41)

where \( A(p) \) is the off-shell field and \( A_{ph}(p) \) is an on shell field, \textit{i.e.} a solution of the equations of motion. We have the following table for massless particles:

| Particle | \( A^*(p) \) | \( A_{ph}(p) \) |
|----------|---------------|-----------------|
| spin 0   | \( \phi(p) \) | \( \delta(p^2)\phi(p) \) |
| spin 1/2 | \( \psi(p) \) | \( \delta(p^2)\gamma^5\psi(p) \) |
| spin 1   | \( F_{\mu\nu}(p) \) | \( \delta(p^2)(p_\mu p^\rho F_{\rho\nu}(p) - p_\nu p^\rho F_{\rho\mu}(p)) \) |

For the spin one case we have

\[ F^{ph}_{\mu\nu}(p) \equiv \delta(p^2)(p_\mu A^T_{\nu}(p) - p_\nu A^T_{\mu}(p)) \]  

(7.42)

where

\[ A^T_{\mu}(p) \equiv p^\rho F_{\rho\mu}(p) \]  

(7.43)

is a transverse vector field \( (p_\mu A^T_{\mu}(p) \equiv 0) \). The first form in (7.39) may therefore be written as

\[ \langle p|ph \rangle \propto \int d^4p A^T_{\mu}(p) A^{T\mu}(p) > 0 \]  

(7.44)

For the Euclidean treatment leading to (7.37) the condition (7.38) and reality of the field \( (A_{\mu\nu}(\tilde{p}) \rightarrow F_{\mu\nu}(\tilde{p})) \) implies

\[ \langle p|ph \rangle = 2a_0 \int d^4p \left( -4\tilde{p}_\mu F^{\mu\nu}(\tilde{p})^* F_{\rho\nu}(\tilde{p}) + F^{\mu\nu}(\tilde{p})^* F_{\mu\nu}(\tilde{p}) \right) \]  

(7.45)

When this expression is analytically continued to Minkowski space it agrees with the path integral results of [24]. (The condition (7.38) is imposed there as well.)

8 Final remarks.

We have performed a detailed analysis of the general formal solutions of a BRST quantization on inner product spaces found in [8, 9]. This analysis has shown that
these formal solutions can be made exact for all models we have considered and in 
the process we have unraveled the appropriate quantum structure of the state spaces 
for such a BRST quantization. We have e.g. found that Lagrange multipliers and 
the unphysical gauge degrees of freedom as well as the bosonic ghosts and antighosts 
must be quantized with opposite metric state spaces. We have not concentrated on 
the physical implications of the treated models. However, we notice that

1. it is in general possible to calculate the physical norms in a BRST quantization 
of a relativistic particle model,

2. one may use positivity and the choice of a canonical theory as criteria to select 
models. Hopefully a further analysis of such criteria will make them useful for 
a deeper understanding of second quantization.

After the above analysis was completed we have investigated the implications for 
the path integral quantization [25]. The results are that the above quantum rules 
in general make the solutions of [8, 9] consistent with the conventional path integral 
expressions [15]. It is always the case if the Lagrange multipliers are quantized with 
indefinite metric states. Our Euclidean treatment with positive metric states for 
the Lagrange multiplier belonging to the mass shell condition corresponds to the 
derivation of propagators when they are analytically continued to Minkowski space. 
In the path integral formulation this derivation requires instead the introduction of 
a convergence factor in the conventional expressions [26, 24].
Appendix

Properties of the spinor states considered in section 5.

In section 5 in connection with the spin-$\frac{1}{2}$ particle we introduced the fermionic hermitian operator $\gamma^\mu$ which also is a Lorentz vector. It satisfies the anticommutation relations

$$[\gamma^\mu, \gamma^\nu]_+ = -2\eta^{\mu\nu} \quad (A.1)$$

where $\eta^{\mu\nu}$ is a space-like Minkowski metric. These $\gamma$-operators may be written in terms of standard canonical operators, e.g. as follows:

$$\gamma^0 = \theta + \mathcal{P}\theta, \quad \gamma^1 = \xi + \xi^\dagger$$

$$\gamma^2 = i(\xi - \xi^\dagger), \quad \gamma^3 = \mathcal{P}\theta - \theta \quad (A.2)$$

where $\xi, \xi^\dagger$ are fermionic oscillators satisfying

$$[\xi, \xi^\dagger]_+ = -1 \quad (A.3)$$

and where $\theta$ and $\mathcal{P}\theta$ are hermitian fermionic coordinate and momentum operators satisfying

$$[\theta, \mathcal{P}]_+ = 1 \quad (A.4)$$

The compatible state space of $\xi, \theta$ and $\mathcal{P}\theta$ is spanned by four independent states. We may choose

$$|1\rangle \equiv -\xi^\dagger|0\rangle_\theta$$

$$|2\rangle \equiv |0\rangle\mathcal{P}|0\rangle_\theta$$

$$|3\rangle \equiv \xi^\dagger|0\rangle\mathcal{P}|0\rangle_\theta$$

$$|4\rangle \equiv |0\rangle|0\rangle_\theta \quad (A.5)$$

where $|0\rangle$ is the vacuum state for $\xi$ ($\langle 0|0\rangle = 0$) and $|0\rangle_\theta$ the vacuum state for $\theta$ satisfying $\theta|0\rangle_\theta = 0$. Their normalization may be chosen to be

$$\langle 0|0\rangle = 1, \quad \theta\langle 0|0\rangle_\theta = 0, \quad \theta\langle 0|\mathcal{P}|0\rangle_\theta = 1 \quad (A.6)$$
which implies
\[ \langle \alpha | \beta \rangle = g_{\alpha \beta} \]  \hspace{1cm} (A.7)

where \( g \) is the block matrix
\[
g \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  \hspace{1cm} (A.8)

Obviously \( g^2 = 1 \). There is a metric operator \( \eta \) satisfying
\[ \langle \alpha | \eta | \beta \rangle = \delta_{\alpha \beta} \]  \hspace{1cm} (A.9)

and \( \eta^\dagger = \eta \) and \( \eta^2 = 1 \). In fact we have
\[ \eta = \gamma^0 \]  \hspace{1cm} (A.10)

If we define the matrix representation of \( \gamma^\mu \) by
\[ \gamma^\mu_{\alpha \beta} \equiv ' \langle \alpha | \gamma^\mu | \beta \rangle , \quad ' \langle \alpha | \equiv \langle \alpha | \gamma^0 \]  \hspace{1cm} (A.11)

then \( (\gamma^\mu_{\alpha \beta}) \) are Dirac’s gamma matrices in the following representation
\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \]  \hspace{1cm} (A.12)

where \( \sigma^i \) are the Pauli matrices. We may also define a \( \gamma^5 \)-operator by
\[ \gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \]  \hspace{1cm} (A.13)

Notice that \( (\gamma^5)^2 = 1 \) and \( \gamma^{5\dagger} = -\gamma^5 \). Its matrix representation is
\[ \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  \hspace{1cm} (A.14)

If one requires the existence of eigenstates to \( \gamma^\mu \) then the eigenvalues must be odd Grassmann numbers. As soon as odd Grassmann numbers are introduced one must prescribe the Grassmann parity of all states and operators. In this analysis was carefully performed and the resulting possibilities given. In the above calculations
we have chosen the Grassmann parity of the $\xi$-vacuum in (A.5) to be even which is a natural choice. However, notice that the $\theta$-vacuum $|0\rangle_{\theta}$ is neither even nor odd due to the property (A.6). In fact it has a mixed parity, $|0\rangle_{\theta} = |0\rangle^+_{\theta} + |0\rangle^-_{\theta}$. Using the properties given in [13] we find

$$\kappa \langle 0|\langle \alpha|\gamma^\mu \lambda|\beta \rangle|0\rangle = i(\gamma^0 \gamma^\mu \gamma^5)_{\alpha\beta} \quad (A.15)$$

This follows since

$$\lambda|\beta\rangle = \gamma^5|\beta\rangle \lambda \quad (A.16)$$

where $|0\rangle_{\theta}$ is replaced by $|0\rangle_{\theta} \equiv |0\rangle^+_{\theta} - |0\rangle^-_{\theta}$ in $|\beta\rangle$, and since [13]

$$\theta\langle 0|\mathcal{P}_{\theta}|0\rangle_{\theta} = i \quad (A.17)$$

If we had started from the hermitian fermionic operators $\zeta^\mu$ satisfying

$$[\zeta^\mu, \zeta^\nu]_+ = 2\eta^{\mu\nu} \quad (A.18)$$

instead of (A.1), then we had obtained the matrix representation

$$(\gamma^\mu \gamma^5)_{\alpha\beta} \equiv ^t\langle \alpha|\zeta^\mu|\beta \rangle \quad (A.19)$$

This follows directly from the above calculations with the identification $\zeta^\mu = \gamma^\mu \gamma^5$. With $\gamma^\mu$ replaced by $\zeta^\mu$ we would then obtain

$$\kappa \langle 0|\langle \alpha|\zeta^\mu \lambda|\beta \rangle|0\rangle = -i(\gamma^0 \gamma^\mu)_{\alpha\beta} \quad (A.20)$$

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