Abstract

In a recent preprint, Lai worked out the quotient of generating functions of weighted lozenge tilings of two “half hexagons with lateral dents” which differ only in width. Lai achieved this by using “graphical condensation” (i.e., application of a certain Pfaffian identity to the weighted enumeration of matchings).

The purpose of this note is to exhibit how this can be done by the Lindström–Gessel–Viennot method for nonintersecting lattice paths in a quite simple way. Basically the same observation, but restricted to mere enumeration (i.e., all weights of lozenge tilings are equal to 1), is contained in a recent preprint of Condon.

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Exposition: Lai’s observation for lozenge tilings

In a recent preprint, Lai [4] considers lozenge tilings of “half hexagons with lateral dents”. The literature on such tilings enumerations is abundant (see, for instance, [1]); for the experienced reader it certainly suffices to have a look at the left picture in Figure 1: A “half hexagon” is simply the upper half of some hexagon with a horizontal symmetry axis, drawn in the triangular lattice; and “lateral dents” are triangles of this “half hexagon” adjacent to its lateral sides which were removed from the “half hexagon”. All vertical lozenges of a tiling are labelled: This labelling is vertically constant and horizontally increasing by 1 from left to right, such that all vertical lozenges bisected by the vertical symmetry axis of the “half hexagon” have label 0 (see the left picture in Figure 1). Let $T$ be some lozenge tiling whose vertical lozenges are labelled $\{v_1, v_2, \ldots, v_m\}$, then the weight of $T$ is defined as

$$w(T) := \prod_{i=1}^{m} \frac{X q^{v_i} + Y q^{-v_i}}{2}.$$

Lai observed that if only the width $x$ (i.e., the length of the upper horizontal side) of such “half hexagon with lateral dents” is changed, then the corresponding generating function of all tilings (weighted as described above) changes by a factor which factors completely and does not contain the variables $X$ or $Y$. Lai provided a proof for this fact by “graphical condensation” (i.e., application of a certain Pfaffian identity to the enumeration of matchings).

The purpose of this note is to exhibit how this can be achieved in a simple way by the Lindström–Gessel–Viennot method [5, 3] of non–intersecting lattice paths.

Translation to non–intersecting lattice paths

The literature on the connection between lozenge tilings and non–intersecting lattice paths is abundant (see, for instance, [1, Section 5]); for the experienced reader it certainly suffices to have a look at the pictures in Figure 1: It is easy to see that there is a weight–preserving bijection between lozenge tilings
The left picture shows a “half hexagon” with side lengths 12, 7, 5, 7 in the triangular lattice: The lateral sides have “dents” (i.e., missing triangles; indicated in the picture by black colour), 4 on the left side and 3 on the right side. The triangle “on top” of this “half hexagon” shows the labelling of the vertical lozenges, which is constant vertically and increasing by 1 horizontally (from left to right). The picture also shows a lozenge tiling of this “half hexagon with dents”, where the three possible orientations of lozenges (left–tilted, right–tilted and vertical) are indicated by three different colours: This particular tiling has weight \( w_{-7} \cdot w_{-6} \cdot w_{-1} \cdot w_0 \cdot w_3 \cdot w_6 \), where \( w_i := \frac{X q^i + Y q^{-i}}{2} \). The evident non–intersecting lattice paths corresponding to this tiling are indicated by white lines in the left picture; the right picture shows a “reflected, rotated and tilted” version of these paths in the lattice \( \mathbb{Z} \times \mathbb{Z} \), where horizontal edges \((a, b) \to (a + 1, b)\) are labelled \( b - 2a \) (these labels are shown in the right picture only for the region of interest in our context, i.e., for \( 0 \leq y \leq x \)). Clearly, this bijection between lozenge tilings and non–intersecting lattice paths (introduced here “graphically”) is weight–preserving if we define the weight of some family \( P \) of non–intersecting lattice paths as the product of \( w_i \), where \( i \) runs over the labels of all horizontal edges belonging to paths in \( P \).

Figure 1: Pictures corresponding to Figures 1.2.a and 2.1.a in Lai’s preprint: The length of the upper horizontal side of the “half hexagon” in the left picture is Lai’s “width parameter” \( x \) (so \( x = 5 \) in this picture).
and families of non–intersecting lattice paths in the lattice $\mathbb{Z} \times \mathbb{Z}$ with steps to the right and downwards, where steps to the right from $(a, b)$ to $(a + 1, b)$ are labelled $a - 2b$ and thus have weight

$$\frac{Xq^{a-2b} + Yq^{2a-b}}{2}$$

(and all downward steps have weight 1). As usual, the weight of a lattice path is the product of all the weights of steps it consists of.

It is easy to see that the generating function of all lattice paths from initial point $(a, b)$ to terminal point $(c, d)$ is zero for $a > c$ or $b < d$, otherwise it is equal to:

$$\text{gf}(a, b, c, d) = \prod_{j=1}^{c-a} \left( \frac{Xq^{j-1-2b+a} + Yq^{-j+1+2d-a}}{2 (1 - q^{2j})} \right) \left(1 - q^{2(b-d)+2j}\right). \quad (1)$$

This follows immediately by showing that (1) fulfils the recursion

$$\text{gf}(a, b, a, d) = 1,$$

$$\text{gf}(a, b, c, b) = \prod_{i=a-2b}^{c-2b-1} \frac{Xq^{i} + Yq^{-i}}{2},$$

$$\text{gf}(a, b, c, d) = \frac{Xq^{a-2b} + Yq^{2b-a}}{2} \text{gf}(a + 1, b, c, d) + \text{gf}(a, b - 1, c, d)$$

for $a \leq c$ and $b \geq d$.

We have to specialize this to our situation, i.e., to initial point $(a, a)$ and terminal point $(c, 0)$: The generating function of all lattice paths from $(a, a)$ to $(c, 0)$ is zero for $c < a$, and for $c \geq a$ it is equal to

$$\text{gf}(a, c) = 2^{a-c} q^{(a-c)a} \prod_{j=1}^{c-a} \left( \frac{Xq^{j-1} + Yq^{1-j}}{1 - q^{2j}} \right) \left(1 - q^{2a+2j}\right). \quad (2)$$

Note that increasing the width of the “half hexagon with lateral dents” by some $k \in \mathbb{N}$ corresponds bijectively to shifting all initial and terminal
points of the corresponding non–intersecting lattice paths (i.e., \((a, a) \rightarrow (a + k, a + k)\) and \((c, 0) \rightarrow (c + k, 0)\)), and from (2) we immediately obtain

\[
\text{gf}(a + k, c + k) = \text{gf}(a, c) \cdot \prod_{j=1}^{c-a} \frac{1 - q^{2a+2k+2j}}{q^x \left(1 - q^{2a+2j}\right)},
\]

which, by using standard \(q\)–Pochhammer notation

\[
(q; a)_n := \prod_{j=0}^{n-1} (1 - a \cdot q^j),
\]

we may rewrite as

\[
\text{gf}(a + k, c + k) = \text{gf}(a, c) \cdot \frac{(q^2; q^2)_{c+1} \cdot q^{xa} \cdot (q^2; q^2)_a}{(q^x; q^2)_{a+1}}.
\]

(3)

By the well–known Lindström–Gessel–Viennot argument [5, 3], the generating function of all families of non–intersecting lattice paths can be written as a determinant, and by the multilinearity of the determinant, we get for all \(n \in \mathbb{N}\) and all \(n\)–tuples \((a_1 < a_2 < \cdots < a_n)\) and \((c_1 < c_2 < \cdots < c_n)\) with \(a_i \leq c_i, 1 \leq i \leq n:\)

\[
\frac{\text{det} (\text{gf}(a_i + k, c_j + k))_{i,j=1}^{n}}{\text{det} (\text{gf}(a_i, c_j))_{i,j=1}^{n}} = \prod_{l=1}^{n} \left( \frac{(q^{2k}; q^2)_{c+l} \cdot q^{xa_l} \cdot (q^2; q^2)_{a_l}}{(q^{x}; q^2)_{a+l+1}} \right).
\]

(4)

By the weight–preserving bijection between lozenge tilings and non–intersecting lattice paths, this is equivalent to Lai’s observation [4, Theorem 1.1]. (Basically the same simple approach, but restricted to mere enumeration, is contained in a recent preprint of Condon [2].)

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