A fast implementation of the Monster group
The Monster has been tamed
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Abstract
Let $M$ be the Monster group, which is the largest sporadic finite simple group, and has first been constructed in 1982 by Griess. In 1985 Conway has constructed a 196884-dimensional rational representation $\rho$ of $M$ with matrix entries in $\mathbb{Z}[\frac{1}{2}]$. We describe a new and very fast algorithm for performing the group operation in $M$.

For an odd integer $p > 1$ let $\rho_p$ be the representation $\rho$ with matrix entries taken modulo $p$. We use a generating set $\Gamma$ of $M$, such that the operation of a generator in $\Gamma$ on an element of $\rho_p$ can easily be computed.

We construct a triple $(v_1, v^+, v^-)$ of elements of the module $\rho_{15}$, such that an unknown $g \in M$ can be effectively computed as a word in $\Gamma$ from the images $(v_1 g, v^+ g, v^- g)$.

Our new algorithm based on this idea multiplies two random elements of $M$ in less than 30 milliseconds on a standard PC with an Intel i7-8750H CPU at 4 GHz. This is more than 100000 times faster than estimated by Wilson in 2013.

Key Words:
Monster group, finite simple groups, group representation, efficient implementation

MSC2020-Mathematics Subject Classification (2020): 20C34, 20D08, 20C11, 20–08

1 Introduction

Let $M$ be the Monster group, which is the largest sporadic finite simple group, and has first been constructed by Griess [7]. That construction has been simplified by Conway [2], leading to a rational representation $\rho$ of $M$ of dimension 196884 with matrix entries in $\mathbb{Z}[\frac{1}{2}]$. For a small odd integer $p > 1$ let $\rho_p$ be the representation $\rho$ with matrix entries taken modulo $p$. In this paper we deal with $\rho_{15}$ and, occasionally, with $\rho_3$ and $\rho_5$.

The first computer construction of the Monster group is due to Linton, Parker, Walsh and Wilson [11]. That construction is based on large 3-local subgroups. The reason for choosing 3-local subgroups (instead of the much larger 2-local subgroups) was that here computations can be done with scalars in $\mathbb{F}_2$. Holmes and Wilson [8] have presented a computer construction of the Monster based on 2-local subgroups with scalars in $\mathbb{F}_3$. The representation used in that construction resembles the representation $\rho_3$ mentioned above. Here the 3-local construction appears to be faster, but the 2-local construction allows general computations in a much larger maximal 2-local subgroup of $M$.

Any known faithful representation of $M$ has dimension at least 196882; so in practice we cannot store elements of $M$ as matrices acting on such a representation on a standard PC. We store elements of $M$ as words in a set $\Gamma$ of generators of $M$, where each generator corresponds to a sparse matrix acting on $\rho$. We write $\Gamma^*$ for the set of words in $\Gamma$. Inverting a word in $\Gamma^*$ is easy by construction of $\Gamma$. Group multiplication in $M$ is simply a concatenation
of words in $\Gamma^*$. But here the word shortening problem arises, since the length of a word may grow exponentially with the number of group operations. Wilson [19] has presented a general word shortening algorithm for the Monster group.

In this paper we present a construction of the Monster based on the representation $\rho$ mentioned above. We also give a new word shortening algorithm. We construct a triple $(v_1, v^+, v^-)$ of elements of $\rho_5$, such that for every $g \in \mathbb{M}$ we can effectively compute a word $g' \in \Gamma^*$ with $g' = g$ from the images $(v_1 g, v^+ g, v^- g)$, without using $g$. Obviously, the word $g'$ computed from these three images depends on the element $g$ of the Monster only, but not on the representation of $g$ as a word in $\Gamma^*$.

This means that we may effectively compute a unique reduced form of each element of the Monster as a word in $\Gamma^*$. Using standard data compression methods, we may store any such reduced form in less than 256 bits. So we may quickly find an element of the Monster in a list of millions of such elements. This was not possible before.

We have implemented the group operation in $\mathbb{M}$ using that new reduction algorithm in the software project [17]. For documentation of the project, see [18]. The run time for the group operation in that project is a bit less than 30 milliseconds on the author’s PC, which has an Intel i7-8750H CPU at about 4 GHz. So this is the first implementation of the group operation of the Monster that a user can run interactively on a computer.

We have also implemented the word shortening algorithm in [19]; here a group operation using that algorithm takes about 30 seconds. In 2013 the run time of this operation has been estimated to take 1–2 hours, see [19].

Dietrich, Lee, and Popiel [5] have used our new algorithm for solving the long-standing problem of finding all maximal subgroups of the Monster.

## 2 Overview of the new algorithm

In this section we give a brief overview of the new reduction algorithm. More details will be given in the following sections.

### 2.1 Construction of the Monster and of its representation

In this subsection we briefly recap Conway’s construction [2] of the Monster. A more detailed description of this construction given in Sections 3–5.

The Monster group contains two classes of involutions called 2A and 2B in the ATLAS [3]. The construction of the Monster $\mathbb{M}$ in [2] is based on a fixed 2B involution $x_{-1}$. The centralizer of $x_{-1}$ is a maximal subgroup $G_{x_0}$ of $\mathbb{M}$ of structure $2^{1+24}.Co_1$. We use the ATLAS notation for describing the structure of a group, see [3]. The normal subgroup $Q_{x_0}$ of structure $2^{1+24}_{+}$ of $G_{x_0}$ is an extraspecial 2 group of plus type. There is a natural homomorphism $\lambda$ from $Q_{x_0}$ to $\Lambda/2\Lambda$, i.e. to the Leech lattice $\Lambda$ modulo 2. The kernel of $\lambda$ is equal to the centre $\{1, x_{-1}\}$ of $Q_{x_0}$. The factor group $\text{Co}_1$ of $G_{x_0}$ is the automorphism group of $\Lambda/2\Lambda$. The automorphism group $\text{Co}_0$ of the Leech lattice $\Lambda$ has structure $2\cdot\text{Co}_1$ with centre of order 2. That centre is generated by the mapping $x \mapsto -x$, for $x \in \mathbb{R}^{24}$. So the operation of $\text{Co}_1$ (and hence also of $G_{x_0}$) on $\Lambda$ is defined up to sign.

The (unique) minimal faithful real representation of the Monster $\mathbb{M}$ has dimension 196883. We call that representation $196883_x$. The 196884-dimensional rational representation $\rho$ of $\mathbb{M}$ constructed in [2] has matrix entries in $\mathbb{Z}[\frac{1}{2}]$; and it is a direct sum of $196883_x$ and the trivial representation $1_x$. As a representation of $G_{x_0}$ the representation $\rho$ splits as follows:

$$1_x \oplus 196883_x = \rho = 300_x \oplus 98280_x \oplus (4096_x \otimes 24_x),$$  \hspace{1cm} (2.0.1)

where the numbers in the names of the representations indicate their dimensions. Here $24_x$ is the natural 24-dimensional representation of the automorphism group $\text{Co}_0$ of the Leech lattice, and $300_x$ is the symmetric tensor square of $24_x$. Representations and $4096_x$ and
98280 enters will be described in Sections 4.2 and 5. Ignoring signs, the basis vectors of monomial representation 98280 of G_{x_0} are in a one-to-one correspondence with the 2 \cdot 98280 shortest nonzero vectors of the Leech lattice \Lambda.

In this paper we focus on the part 300_{x_0} of \rho; and to some extent we also use 98280x. Since 300_{x_0} is the symmetric tensor square of 24x, a vector in 300_{x_0} has a natural interpretation as a symmetric matrix acting on the Euclidean space \mathbb{R}^{24} spanned by the Leech lattice. In the sequel we identify 300_{x_0} with the space of these symmetric matrices. The trivial part 1_{x_0} of representation \rho is the subspace of 300_{x_0} spanned by the unit matrix 1_{\rho}.

A four-group \{1, x_{-1}, x_\Omega, x_{-\Omega}\} of the Monster containing three commuting 2B involutions is also considered in [2]. The centralizer of that four-group is a maximal subgroup N_0 of \mathcal{M} of structure 2^{2+11+22}(M_{24} \times \text{Sym}_3). The factor 2^{2+11+22} describes a certain 2 group. The group M_{24} is the Mathieu group accounting as a permutation group on a set \Omega of 24 elements. Sym_3 is the symmetric permutation group of 3 elements. Sym_3 acts naturally on the set \{x_{-1}, x_\Omega, x_{-\Omega}\} of 2B involutions. The intersection N_{x_0} = N_0 \cap G_{x_0} has index 3 in N_0 and is a maximal subgroup of G_{x_0}. N_{x_0} and acts monomially on \rho. A (considerably more detailed) diagram of the relevant subgroups of the Monster is given in Figure 1.

A set of generators of N_{x_0} is given in [2], together with another generator \tau \in N_0 \setminus N_{x_0} of order 3 that cyclically exchanges x_{-1}, x_\Omega, and x_{-\Omega}. \tau is called the triality element. The operation of all these generators on \rho, and also sufficient information for effectively computing in N_0, is given in [2]. For obtaining a complete set of generators of \mathcal{M} we just need another element \xi of G_{x_0} \setminus N_{x_0}. Such a generator \xi, and its operation on \rho, has been constructed in [6]. Let \Gamma be the set of all these generators of \mathcal{M}; and let \Gamma^* be the set of all words in \Gamma.

### 2.2 Computing in the subgroup G_{x_0} of the Monster

According to the the Pacific Island model in [19] we may assume that computations in G_{x_0} are easy (or at least doable), while computations in \mathcal{M} outside G_{x_0} are difficult.

In Section 6 we will construct a vector \nu_1 \in \rho_{15} with the following properties:

- The only element of \mathcal{M} fixing \nu_1 is the neutral element.
- From \nu_1 G, with \nu \in \mathcal{M} unknown, we can effectively check if \nu is in G_{x_0} or not.
- From \nu_1 G, with \nu \in G_{x_0} unknown, we can effectively compute a word \nu' in \Gamma^* with \nu' = \nu.

By our construction of \nu_1 in Section 6 the first property of \nu_1 follows from a result in [11]. For testing our implementation this gives us the invaluable advantage, that equality of two words \nu_1 \nu \in \Gamma^* can be tested by comparing \nu_1 \nu_1 with \nu_1 \nu_2; here the correctness of this test follows from results that are independent of the new algorithm in this paper.

### 2.3 The basic idea for computing in the Monster

Our goal is to recognize an unknown element \nu of \mathcal{M} from the images \nu_{1} \nu of a few vectors in \nu_{1} in \rho. Here ‘recognizing’ means finding a word \nu in \Gamma^* that maps each image \nu_{1} \nu to its preimage \nu. If only the neutral element of \mathcal{M} fixes all vectors \nu_{1} then \nu = h^{-1}. So this leads to an algorithm for reducing an arbitrary word of generators of a \mathcal{M} to a standard form, and hence to an effective algorithm for the group operation in \mathcal{M}.

Here we give a brief overview of our reduction method based on this idea. More details will be given in Section 7.

We use a fixed pair (\beta^+, \beta^-) of 2A involutions in the subgroup Q_{x_0} of \mathcal{M} with \beta^+ \beta^- = x_{-1}. The centralizer of a 2A involution has structure 2.B (where B is the Baby Monster group); and it fixes a unique one-dimensional subspace of the representation 196883. In [2] a vector in 196883 fixed by the centralizer of a 2A involution \nu in \mathcal{M} is called an axis of \nu. For any 2A involution \nu in \mathcal{M} we will define a unique axis ax(\nu) in the representation.
\( \rho = 1_x \oplus 198883_x \). Then \( M \) is transitive on these axes; and the centralizer of a 2A involution \( t \) in \( M \) is equal to the centralizer of \( ax(t) \) in \( M \). We put \( v^+ = ax(\beta^+) \), \( v^- = ax(\beta^-) \); and we write \( H^+ \) for the centralizer of \( x_\beta \) (or of \( v^+ \)) in \( M \).

The key idea of the new algorithm is to track images of the axes \( v^+, v^- \) under the action of an element of the Monster. Given an image \( v^+ g \) of \( v^+ \) (with \( g \) unknown) we present an effective algorithm for computing a word \( h_1 \) in \( \Gamma^* \) with \( v^+ gh_1 = v^+ \). So we have \( gh_1 \in H^+ \).

This means that computation in the Monster can be reduced to computation in the group \( H^+ \) of structure 2A, which was not possible before.

In the next step, we deal with \( g' = gh_1 \). Given an image \( v^- g' \) of \( v^- \) (with \( g' \in H^+ \) unknown) we present an effective algorithm for computing a word \( h_2 \in H^+ \) (given as a word in \( \Gamma^* \)) with \( v^- g'h_2 = v^- \). Since \( H^+ \) fixes \( h_2 \), we have \( v^+ gh_1 h_2 = v^+ \) and \( v^- gh_1 h_2 = v^- \). So \( gh_1 h_2 \) centralizes both, \( x_\beta \) and \( x_{-\beta} \), and hence also the product \( x_{-1} = x_\beta x_{-\beta} \).

So we have \( gh_1 h_2 \in G_{x0} \). Using the idea in Section 2.2 we may compute a \( h_3 \in G_{x0} \) as a word in \( \Gamma^* \) with \( gh_1 h_3 h_3 = 1 \), provided that we know \( v_1 g \) (and hence also \( v_1 gh_1 h_2 \)).

So given a triple \( (v_1 g, v^+ g, v^- g) \) of vectors in \( \rho \), we can effectively compute the element \( g \) from that triple of vectors as a word in \( \Gamma^+ \). As we shall see in Section 6 et seq., it suffices if \( (v_1 g, v^+ g, v^- g) \) is given as a triple of vectors in \( \rho_{15} \).

### 2.4 Reducing an axis and the geometry of the Leech lattice

In this subsection we discuss the basic geometric idea used for finding an element of \( M \) that transforms an arbitrary image of axis \( v^+ \) to \( v^+ \). Here we generously assume that the relevant geometric computations in the Leech lattice are doable. Details will be given in Section 8.

According to [15] there are twelve orbits of \( G_{x0} \) on 2A axes. We may find representatives of all these orbits by multiplying axis \( v^+ \) with random elements of \( M \). This way we find out that the projections of all these representatives onto 300\( \times \) are nonzero positive semidefinite symmetric matrices. The elements of \( G_{x0} \) perform orthogonal transformations of the matrices in 300\( \times \). Hence the projections of all axes onto 300\( \times \) are nonzero and positive semidefinite. The multiset of the eigenvalues of such a matrix (counting each eigenvalue with the dimension of its eigenspace) is determined by the orbit of \( G_{x0} \) on the corresponding axis. It turns out that such a multiset of eigenvalues also characterizes the orbit of \( G_{x0} \) on an axis.

A positive semidefinite matrix \( A \in 300\times \) can be visualized as the (possibly degenerated) ellipsoid \( \{ x \in \mathbb{R}^{24} \mid x A x^T \leq 1 \} \) in the space spanned by the Leech lattice. The shape of such an ellipsoid is determined by the eigenvalues of \( A \); so it is a property of the orbit of \( G_{x0} \) on \( A \).

For any axis we may analyse the eigenspaces of its projection on 300\( \times \), leading to a wealth of geometric information related to the Leech lattice \( A \), on which \( G_{x0} \) operates naturally up to sign. It turns out that these eigenspaces are (usually) spanned by rather short vectors of the Leech lattice. This means that we have pretty good control over the geometry of an axis when operating inside \( G_{x0} \).

For mapping an axis to another orbit of \( G_{x0} \), we have to apply a power \( \tau^{\pm 1} \) of the triality element \( \tau \) to that axis. Note that this operation may change the shape of the ellipsoid corresponding to the projection of the axis onto 300\( \times \).

In Section 8 we will see that for a axis \( v \) the set \( \{ v^k G_{x0} \mid k = \pm 1 \} \) of orbits of \( G_{x0} \) depends on the orbit \( v N_{x0} \) of \( v \) only. In Section 1.2 we will see that \( N_{x0} \) can also be considered as the centralizer of the standard co-ordinate frame of the Leech lattice. The images of that frame correspond to the vectors of minimal norm 8 in \( \Lambda/2\Lambda \), i.e. in the Leech lattice mod 2.

Thus for an axis \( v \) the shapes of the ellipsoids corresponding to \( v \tau^{\pm 1} \) are determined by the position of the ellipsoid corresponding to \( v \) relative to the standard co-ordinate frame of the Leech lattice. Determining these shapes in all relevant cases is far from trivial; but at least we have a geometric idea how to proceed.

We say that two axes have distance \( l \) if any word in \( \Gamma^* \) transforming one axis into the other axis contains at least \( l \) powers of \( \tau \). So, geometrically, our task is to rotate an ellipsoid
in the Leech lattice corresponding to an axis into a 'good' position relative to the standard co-ordinate frame by applying a transformation in $G_{x0}$. Here a position of an ellipsoid is 'good' if applying one of the transformations $\tau^{\pm 1}$ decreases the distance of the corresponding axis to the standard axis $v^+$. This way we may decrease the distance of a given axis from the standard axis $v^+$ by a sequence of operations in $G_{x0}$ and in $\{\tau^{\pm 1}\}$. Finally, we obtain an axis that has distance 0 from axis $v^+$. Then an easy computation inside the group $G_{x0}$ will map that axis to $v^+$.

So we have to find 'good' co-ordinate frames in $\Lambda$ (or vectors of minimal norm 8 in $\Lambda/2\Lambda$), when an ellipsoid over the Leech lattice or, equivalently, a symmetric $24 \times 24$ matrix $A \in 300x$ is given. From a computational point of view we prefer to search for such 'good' vectors in $\Lambda/2\Lambda$ with respect to a given matrix $A \in 300x$. It turns out that the projection of an axis onto the subspace $98280x$ of $\rho$ gives us useful information about the $98280$ shortest nonzero vectors in $\Lambda/2\Lambda$, which we will also use for finding 'good' vectors in $\Lambda/2\Lambda$.

2.5 Dealing with a pair of axes

We call an axis *feasible* if it is equal to an image $v^-h$ of $v^-$ for some $h \in H^+$. So we have to find an element of $H^+$ that transforms a feasible axis to the axis $v^-$. For performing this task we use essentially the same methods as in the previous subsection.

Put $H = H^+ \cap G_{x0}$. By our construction of $\tau$ and $H^+$, the group $H^+$ is generated by $H$ and $\tau$. Since $H$ is a subgroup of $G_{x0}$, we'll have good control over the operation of $H$ on feasible axes. So we'll apply $\tau^{\pm 1}$ to a feasible axis whenever we may decrease the distance between a feasible axis and $v^-$, in a similar way as in the previous subsection. Here it is important to note that there are not too many orbits of $H$ on feasible axes. From [12] we will conclude that there are 10 such orbits.

2.6 Implementation

The implementation [17] contains highly optimized C programs for dealing with the structures involved in the construction of the representation $\rho$ in [2]. These structures include the binary Golay code and its cocode, the Parker loop, the Leech lattice $\Lambda$ (modulo 2 and 3), and also the automorphism groups of these structures, as discussed in Sections 3 and 4.

It suffices if all computations in representation $\rho$ described in this paper are done modulo 15, i.e. in $\rho_{15}$. In [17] we also have highly optimized functions for multiplying a vector in $\rho_3$ or $\rho_{15}$ with a generator in $\Gamma$. (We are a bit sloppy here, calling elements of $\rho_{15}$ also vectors, although 15 is composite.) Multiplication of a vector in $\rho_{15}$ with any generator in $\Gamma$ costs less than 160 microseconds on the author’s computer.

A reference implementation in Python for demonstrating the new reduction algorithm is presented in [18], Section *Demonstration code for the reduction algorithm*. The main function `reduce_monster_element` in that implementation reduces an element of the Monster.

3 The maximal subgroup $N_0$ of the Monster $\mathbb{M}$

The Monster has a maximal 2-local subgroup $N_0$ of structure $2^{2+11+22}(M_{24} \times \text{Sym}_3)$ that is used in the construction [2]. We briefly recap the structures given in [2] that are required for understanding the generators and relations defining $N_0$. For background we refer to [13][10].

3.1 The Golay code and its cocode

Let $\hat{\Omega}$ be a set of size 24 and construct the vector space $\mathbb{F}^{24}_2$ as $\prod_{\omega \in \hat{\Omega}} \mathbb{F}_2$. A *Golay code* $C$ is a 12-dimensional linear subspace of $\mathbb{F}^{24}_2$ whose smallest weight is 8. This characterizes the Golay code up to permutation. Golay code words have weight 0, 8, 12, 16, or 24. Code
words of weight 8 and 12 are called octads and dodecads, respectively. The automorphism group of $C$ is the Mathieu group $M_{24}$, which is quintuply transitive on the set $\Omega$.

We identify the power set of $\Omega$ with $\mathbb{F}_2^{24}$ by mapping each subset of $\Omega$ to its characteristic function, which is a vector in $\mathbb{F}_2^{24}$, as in [2]. So we may write $\Omega$ for the Golay code word containing 24 ones. For elements $d, e$ of $\mathbb{F}_2^{24}$ we write $d \cup e, d \cap e$ for their union and intersection, and $d + e$ for their symmetric difference. We write $|d|$ for the cardinality of a set $d$.

We use the specific Golay code constructed in [4], Ch. 11, which is also used in [16]. The implementation [17] fixes a basis of that Golay code for computational purposes; and it numbers the elements of the set $\Omega$ from 0 to 23. Occasionally we write a subset of $\Omega$ as in [2] representing an element of $\mathbb{F}_2^{24}$, with the obvious meaning.

Let $C^*$ be the cocode of $C$, with scalar product $\langle d, \delta \rangle$ in $\mathbb{F}_2$ for $d \in C, \delta \in C^*$. An element of $C^*$ has a unique representative of weight less than 4 in $\mathbb{F}_2^{24}$, or a set of six disjoint representatives of weight 4. Such a set of six representatives is called a sextet; the representatives in a sextet are called tetrads. For $\delta \in C^*$ let $|\delta|$ be the minimum weight of $\delta$; so we have $0 \leq |\delta| \leq 4$.

### 3.2 The Parker loop

The Parker loop $P$ is a non-associative Moufang loop which is a double cover of the Golay code written multiplicatively. For any $d \in P$ we write $\tilde{d}$ for the image of $d$ in the Golay code $C$ as in [2]. We write 1 for the neutral element in $P$ and $-1$ for the other preimage of the zero element of $C$ in $P$. Set-theoretic operations on elements of $P$ are interpreted as operations on subsets of $\mathbb{F}_2^{24}$ corresponding to the images of these elements in $C$, as in [2]. So the intersection of two elements of $P$ has a natural interpretation as an element of $\mathbb{F}_2^{24}$ or of $C^*$. For $i \in \Omega$ we abbreviate the cocode word $\{i\}$ of weight 1 to $i$. For $d, e, f \in P$ we have

$$d^2 = (-1)^{|d|/4} \cdot d \cdot e = (-1)^{|d \cap e|/2} \cdot e \cdot d, \quad (d \cdot e) \cdot f = (-1)^{|d \cap e \cap f|} \cdot d \cdot (e \cdot f).$$

For practical computations an element $d_i$ of the Parker loop $P$ is represented as a pair $(\tilde{d}_i, \mu_i)$ with $\tilde{d}_i \in C, \mu_i \in \mathbb{F}_2$. The implementation [17] defines a cocycle $\theta : C \times C \to \mathbb{F}_2$, so that the product in $P$ is given by:

$$(\tilde{d}_1, \mu_1) \cdot (\tilde{d}_2, \mu_2) = (\tilde{d}_1 + \tilde{d}_2, \mu_1 + \mu_2 + \theta(\tilde{d}_1, \tilde{d}_2)).$$

Cocycles in a loop like $P$ are discussed in [1], Chapter 4. The cocycle $\theta$ satisfies the conditions in [16], Section 3.3, so that we may use the results in [16] for computations. $\theta$ is quadratic in the first and linear in the second argument; so it can also be considered as a function from $C$ to $C^*$. An element $(\tilde{d}_i, \mu_i)$ of $P$ is called positive if $\mu_i = 0$ and negative otherwise. We write $\Omega$ for the positive preimage of the Golay code word $\Omega$ in $P$. The centre of $P$ is $\{\pm 1, \pm \Omega\}$.

Let $\text{Aut}_S P$ be the set of standard automorphisms of $P$, i.e. the set of automorphisms that map to an automorphism of $C$ in $M_{24}$ when factoring out $\{\pm 1\}$. Any $\delta \in C^*$ acts as a standard automorphism on $P$ given by $d \mapsto (-1)^{|d, \delta|} d$ for $d \in P$; we call $\delta$ a diagonal automorphism of $P$. An element $\pi$ of $\text{Aut}_S P$ is called even if it fixes $\Omega$ and odd if it negates $\Omega$; for $\delta \in C^* \subset \text{Aut}_S P$ this agrees with the parity of $\delta$ in $C^*$. The group $\text{Aut}_S P$ has structure $2^{12} \cdot M_{24}$; the extension does not split, and its normal subgroup of structure $2^{12}$ is isomorphic to the group $C^*$ of diagonal automorphisms.

We follow the conventions in [2] for denoting elements of $P, C$ and $C^*$:

- $d, e, f$ denote elements of $P$ or, loosely, of its homomorphic image $C$,
- $\delta, \epsilon, \iota$ denote elements of $C^*$,
- $i, j$ denote elements of $\Omega$, also considered as elements of $C^*$ of weight 1,
- $d \cap e$ denotes the subset $d \cap e$ of $\mathbb{F}_2^{24}$, usually considered as an element of $C^*$,
- $\pi, \pi', \pi''$ denote elements of $\text{Aut}_S P$. 

6
3.3 The subgroups $N_0$ and $N_{x_0}$ of the Monster

In [2] a subgroup $N_{x_0}$ of $M$ of structure $2^{1+24}.O_{11}.M_{24}$ is defined. This group has generators $x_\delta, x_d, y_d, x_\tau, \delta \in \Gamma, d \in P, \pi \in Aut_{St}P$. In this paper we use the generators of $N_{x_0}$ defined in [10], with the same names as in [2], but with slightly different sign conventions, leading to simpler relations in $N_{x_0}$. Generators $x_d$ and $y_d$ in [10] are equal to $x_d x_{-1}^{d/4}$ and $y_d y_{-1}^{d/4}$ in [2], respectively; the other generators are the same in both papers. This simplification has been proposed in [10], Ch. 2.7.

Theorem 3.1. In $N_{x_0}$ we have the following relations:

\[
\begin{align*}
\tau x_d \tau = x_{d^2}, & \quad \tau y_d \tau = y_{d^2}, \quad \tau x_\delta \tau = x_\delta, \\
\tau x_{d^2} \tau = x_{d^2}, & \quad \tau y_{d^2} \tau = y_{d^2}, \quad \tau x_\delta \tau = x_\delta, \\
\tau x_{d^2} \tau = x_{d^2}, & \quad \tau y_{d^2} \tau = y_{d^2}, \quad \tau x_\delta \tau = x_\delta, \\
\tau x_{d^2} \tau = x_{d^2}, & \quad \tau y_{d^2} \tau = y_{d^2}, \quad \tau x_\delta \tau = x_\delta,
\end{align*}
\]

for $d, e \in P; \delta, \epsilon \in \Gamma \subset Aut_{St}P; \pi, \pi', \pi'' \in Aut_{St}P, \pi$ even. We have to put $Z_{x,d} = y_{x,d}$ if $\delta$ is even, and $Z_{x,d} = (x_{x,d})^{-1}$ if $\delta$ is odd.

Theorem 3.1 follows from Theorem 1 and the definitions in §6 in [2], or from Theorem 5.1 in [10]. Here $[a, b]$ is the commutator $a^{-1}b^{-1}ab$; and $d \cap e$ is an element of $\Gamma$.

We obtain a larger group $N_0$ of structure $2^{2+11+22}.M_{24} \rtimes Sym_5$ with $N_0 : N_{x_0} = 3$ by adding another generator $\tau$ called the triality element in [2]. $\tau$ satisfies the relations:

\[
\tau^3 = 1, \quad \tau x_d \tau = \tau y_d, \quad \tau y_d \tau = \tau(x_d y_d)^{-1}, \quad [\tau, \pi] = 1, \quad \pi, \pi', \pi'' \in Aut_{St}P, \pi$ even, \ \pi', \pi'' odd.
\]

The group $N_0$ is a maximal subgroup of the Monster $M$. The relations (3.1.1) can be obtained from the discussion in [2], §6; or from [10], Theorem 5.1. Computation in the group $N_0$ is easy using the generators and relations given above. On the author's computer the group operation in $N_0$ costs about 2 microseconds.

4 The Leech lattice and the maximal subgroup $G_{x_0}$ of $M$

4.1 The Leech lattice and its relation to the subgroup $Q_{x_0}$ of $N_{x_0}$

Let $C$ be the Golay code in $F_2^{24}$ as in Section 3.3. Let $\{\eta_i, i \in \Omega\}$ be a basis of the Euclidean vector space $R^{24}$ so that the basis vectors of both, $F_2^{24}$ and $R^{24}$, are labelled by the elements of the same set $\Omega$. Then the Leech lattice $\Lambda$ is the set of vectors $u = \sum_{i \in \Omega} u_i \eta_i, u_i \in Z$, such that there is an $m \in \{0, 1\}$ and a $d \in C$ with

\[
\begin{align*}
\forall i \in \Omega : \quad u_i &= m + 2 \cdot (d, i) \pmod{4}, \\
\sum_{i \in \Omega} u_i &= 4m \pmod{8}.
\end{align*}
\]

Here we scale the basis vectors $\eta_i$ of $R^{24}$ so that they have length $\frac{1}{\sqrt{8}}$ and not 1. Then $\Lambda$ is the unique even unimodular lattice in $R^{24}$ such that the shortest nonzero vectors have squared norm 4, see e.g. [4], Ch. 4.11 for background. Thus for vectors $u, v \in \Lambda$ with coordinates $u_i, v_i$ the scalar product $\langle u, v \rangle$ is equal to $\frac{1}{2} \sum_{i \in \Omega} u_i v_i$. For $u \in \Lambda$ we define type($u$) = $\frac{1}{2} \langle u, u \rangle$; so a shortest nonzero vector in $\Lambda$ is of type 2.

The group $N_{x_0}$ has a normal subgroup $Q_{x_0}$ of structure $2^{1+24} \rtimes \Delta$, generated by $x_d, x_\delta, d \in P, \delta \in \Gamma$. So $Q_{x_0}$ is an extraspecial 2 group of order $2^{1+24}$ of plus type. The centre of $Q_{x_0}$ is $\{x_{\pm 1}\}$. Let $\Lambda/2\Lambda$ be the Leech lattice modulo 2. There is a homomorphism $\lambda$ from $Q_{x_0}$ onto $\Lambda/2\Lambda$, with kernel $\{x_{\pm 1}\}$, given by:

\[
x_d \mapsto \lambda_d = \frac{1}{2} \sum_{j \in d} \lambda_j, \quad x_i \mapsto \lambda_i, \quad \text{for } d \in P, i \in \tilde{\Omega}, \quad \text{where } \lambda_i = -4\eta_i + \sum_{j \in \Omega} \eta_j. \quad (4.0.1)
\]
We write
\[ x_r, x_s \quad \text{for general elements of } Q_{x0}, \]
\[ \lambda_r, \lambda_s \quad \text{for the elements } \lambda(x_r), \lambda(x_s) \text{ of } \Lambda/2\Lambda. \]

For \( x_r, x_s \in Q_{x0} \) we have
\[ [x_r, x_s] = x_r^{(\lambda_r, \lambda_s)}, \quad x_r^2 = x_r^{\text{type}(\lambda_r)}. \]

The mapping \( \lambda_r \mapsto \text{type}(\lambda_r) \) (mod 2) defines the natural non-singular quadratic form on \( \Lambda/2\Lambda \). For a proof of these statements see \[2\], Theorem 2, or \[10\], Theorem 6.1.

The type of a vector in \( \Lambda/2\Lambda \) is the type of its shortest preimage in \( \Lambda \). Every vector in \( \Lambda/2\Lambda \) has type 0, 2, 3, or 4. Vectors of type 2 are called short; there are 98280 short vectors in \( \Lambda/2\Lambda \). The automorphism groups of \( \Lambda \) and \( \Lambda/2\Lambda \) are called \( \text{Co}_0 \) and \( \text{Co}_1 \). The group \( \text{Co}_1 \) is simple; and \( \text{Co}_0 \) has structure 2 \[\text{Co}_1 \]. \( \text{Co}_0 \) and \( \text{Co}_1 \) are transitive on the sets of vectors of types 2, 3, or 4. A vector of type 2 or 3 in \( \Lambda/2\Lambda \) has two opposite preimages of the same type in \( \Lambda \).

The vector \( \lambda_0 = \lambda(x_{10}) \) in \( \Lambda/2\Lambda \) is of type 4; its preimages of type 4 in \( \Lambda \) are the 48 vectors of type 4 proportional to the unit vectors in \( \mathbb{R}^{24} \). The images of the standard co-ordinate frame of \( \Lambda \) under \( \text{Co}_0 \) are in one-to-one correspondence with the vectors of type 4 in \( \Lambda/2\Lambda \), see \[4\], Ch. 10.3.3.

For \( \lambda_r \in \Lambda/2\Lambda \) put \( \Lambda(\lambda_r) = \{ v \in \Lambda \mid v = \lambda_r \text{ (mod 2)}, \text{type}(v) = \text{type}(\lambda_r) \} \). So \( \Lambda(\lambda_r) \) is the set of the shortest preimages of \( \lambda_r \) in \( \Lambda \). E.g. \( \Lambda(\lambda_{10}) \) is the set \( \{ \pm 8\eta_i \mid i \in \Omega \} \).

### 4.2 The maximal subgroup \( G_{x0} \) of the Monster

We construct a maximal subgroup \( G_{x0} \) of \( \text{M} \) with \( G_{x0} \cap N_0 = N_{x0} \) as follows. The extraspecial normal subgroup \( Q_{x0} \) of \( N_{x0} \) has a unique irreducible real representation \( 4096_x \) of dimension 4096, see \[2\][10]. That representation may be extended to a representation of a (unique) group of structure \( 2^{1+24} \).\( O_{24}^+(2) \), where \( O_{24}^+(2) \) is the orthogonal group on \( \mathbb{F}_2^{24} \) of plus type. Since \( \text{Co}_1 \subset O_{24}^+(2) \), we also obtain a representation \( 4096_x \) of a group \( G'_{x0} \) of structure \( 2^{1+24} \).\( \text{Co}_1 \).

If \( G_1, G_2 \) are groups with a common factor group \( H \) and homomorphisms \( \phi_i : G_i \to H \), \( i = 1, 2 \), then the fibre product \( G_1 \triangle_H G_2 \) is the subgroup of the direct product \( G_1 \times G_2 \) defined by:
\[ G_1 \triangle_H G_2 = \{(x, y) \in G_1 \times G_2 \mid \phi_1(x) = \phi_2(y)\}. \]

We define \( G_{x0} \) by
\[ G_{x0} = \frac{1}{2}(G'_{x0} \triangle \text{Co}_1 \text{Co}_0). \]

Here the factor \( \frac{1}{2} \) means that we identify the centres (of order 2) of the groups \( G'_{x0} \) and \( \text{Co}_0 \). We remark that \( G_{x0} \) and \( G'_{x0} \) are not isomorphic.

From \[2\] or from Theorem 8.3 we see that \( N_{x0} \) normalizes the four-group \( \{x_{\pm 1}, x_{\pm 10}\} \). By \[3\] the group \( 2^{11}.M_{24} \) is maximal in \( \text{Co}_1 \); so we conclude that \( N_{x0} \) is the normalizer of that four-group in \( G_{x0} \), and hence also the stabilizer of \( \lambda_0 = \lambda(x_{10}) \) in \( \Lambda/2\Lambda \) in \( G_{x0} \). So we may say that \( N_{x0} \) is the stabilizer of the standard co-ordinate frame of \( \Lambda \).

Computation in \( \text{Co}_0 \) is easy. Computation in \( 4096_x \) can be greatly accelerated by using the fact that a certain basis of the underlying vector space \( V = \mathbb{R}^{4096} \) has a natural structure as a 12-dimensional affine space over \( \mathbb{F}_2 \). Using such a basis, the standard basis of \( \text{hom}(V, V) \) inherits the structure of 24-dimensional affine space \( A \) over \( \mathbb{F}_2 \). It turns out that a matrix in \( \text{hom}(V, V) \) representing a group element has entries with a fixed absolute value in an affine subspace of \( A \), and entries 0 elsewhere. The signs of the nonzero entries of the matrix are essentially given by a quadratic form on that subspace. Using these ideas, the group operation in \( G_{x0} \) can be done in a bit more than 10 microseconds on the author’s computer. For details we refer to the documentation of the project \[17\].

Using the functions in \[17\] we may compute the character of any element of the subgroup \( G_{x0} \) of \( \text{M} \) in the 196883-dimensional real representation of \( \text{M} \) in a few ten milliseconds.
5  The 196884-dimensional representation of $\mathbb{M}$

Let $24_x$ be the representation of the group $\text{Co}_0$ on $\mathbb{R}^{24}$ as the automorphism group of the Leech lattice $\Lambda$, and let $4096_x$ be as in Section 4.2. Then the maximal subgroup $G_{x0}$ of $\mathbb{M}$ has a faithful real representation $4096_x \otimes 24_x$. A construction of $G_{x0}$ and of its representation $4096_x \otimes 24_x$ is given in [2], where the generators $x_8, x_9, y_9, x_\pi$ of the maximal subgroup $N_{x0}$ of $G_{x0}$ are given explicitly. In [16] we define another generator $\xi$ of order 3 in $G_{x0} \setminus N_{x0}$, and its action on $4096_x \otimes 24_x$. Using the basis of $\Lambda$ in $\mathbb{R}^{24}$ in Section 4.1, the generator $\xi$ acts on $\mathbb{R}^{24}$ by right multiplication with the matrix

$$
\begin{pmatrix}
AB \\
AB \\
AB \\
AB
\end{pmatrix}, \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{pmatrix}, \quad B = \begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix},
$$

(5.0.1)

up to sign. A suitable basis of $4096_x \otimes 24_x$ is given in [2] and also in [16], with slightly different sign conventions leading to a simpler construction of generator $\xi$. We omit the details, since we do not need them in this paper.

The group $G_{x0}$ operates on its normal subgroup $Q_{x0}$ by conjugation. Let $98280_x$ be a real vector space, with basis vectors $X_{d,\delta}, d \in \mathcal{P}, \delta \in \mathbb{C}^*$, such that $\text{type}(\lambda(x_d \cdot x_\delta)) = 2$. We identify $X_{-d,\delta}$ with $-X_{d,\delta}$. Then an element of $G_{x0}$ operates on $X_{d,\delta}$ in the same way as it operates on $x_d \cdot x_\delta$ by conjugation. With this operation the space $98280_x$ is a 98280-dimensional monomial representation of $G_{x0}$ with kernel $\{1, x_{-1}\}$.

For constructing the 196884-dimensional representation $\rho$ of $\mathbb{M}$ we also need the symmetric tensor square $300_x = 24_x \otimes_{\text{sym}} 24_x$. Note that $24_x$ is not a representation of $G_{x0}$, but $300_x$ is. Basis vectors of $24_x$ are $\eta_i, i \in \tilde{\Omega} = \{0, \ldots, 23\}$, as in Section 4.1. Then an element of $300_x$ has a natural interpretation as symmetric real $24 \times 24$ matrix. For $i, j \in \tilde{\Omega}$ we write $\eta_{i,j}$ for the symmetric matrix with an entry 1 in row $i$, column $j$, and also in row $j$, column $i$, and zeros elsewhere. So the elements of the set $\{\eta_{i,j} \mid i, j \in \tilde{\Omega}, i \leq j\}$ form a basis of $300_x$.

We define the representation $\rho$ of $G_{x0}$ by:

$$\rho = 300_x \oplus 98280_x \oplus (4096_x \otimes 24_x).$$

(5.0.2)

For extending $\rho$ to a representation of $\mathbb{M}$, Conway [2] defines the action of the generator $\tau \in \mathbb{M} \setminus G_{x0}$ of order 3 on $\rho$. For a proof that $\rho$ actually represents $\mathbb{M}$, he has to show that there is a certain algebra over $\mathbb{R}$ invariant under $\mathbb{M}$. Therefore he constructs an algebra visibly invariant under $G_{x0}$; and he shows that this algebra is also invariant under $\tau$. That algebra is called the Griess algebra; it has first been constructed by Griess [7].

Representation $300_x$ can also be decomposed as $300_x = 1_x \oplus 299_x$, where $1_x$ is the trivial representation of $G_{x0}$ corresponding to the subspace of $300_x$ spanned by the unit matrix $1_p$. Then $299_x$ is an irreducible representation of $G_{x0}$ containing the symmetric $24 \times 24$ matrices in $300_x$ with trace zero. Replacing $300_x$ by $299_x$ in (5.0.2) we obtain the smallest faithful irreducible representation 196883$_2$ of $\mathbb{M}$ in characteristic 0.

Representation $\rho$ preserves a positive-definite quadratic form. With respect to that form, basis vectors $\eta_{i,j}$ of $\rho$ have squared norm 2 for $i \neq j$, and all other basis vectors have norm 1. Any two basis vectors are orthogonal, unless equal or opposite.

The following set $\Gamma$ generates $N_{x0}$ and $G_{x0}$, and hence the Monster group $\mathbb{M}$:

$$\Gamma = \{x_\delta, x_d, y_e, x_\pi, \tau^\pm 1, \xi^\pm 1 \mid \delta \in \mathbb{C}^*; d, e \in \mathcal{P}; \pi \in \text{Aut}_{\text{St}}\mathcal{P}; \}.$$  

The operation of the generators in $\Gamma$ on $\rho$ (except for $\tau^\pm 1$ and $\xi^\pm 1$) is given in Table 1 in [2], or in Table 3 in [16]. The operation of $\tau^\pm 1$ is also given in Table 3 in [16]; in [2] this corresponds to a cyclic exchange of the three languages in the dictionary in Table 2. The operation of $\xi$ on $\rho$ is described in [16], Section 9. We write $\Gamma^*$ for the set of words in $\Gamma$. 

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The matrices in $\rho$ representing the generators in $\Gamma$ are sparse matrices. Any such matrix can be represented as a product of monomial matrices, and a sequence of at most six matrices of Hadamard type. Here a matrix of Hadamard type is a matrix with diagonal blocks and blocks of $2 \times 2$ matrices of shape $c(\frac{1}{2}, \frac{1}{2})$ for $c \in \{\frac{1}{2}, 1\}$. For any odd integer $p > 1$ we define the representation $\rho_p$ of the Monster as the representation $\rho$, with matrix entries taken modulo $p$. $\rho_p$ is well defined for odd $p$, since the denominators of the entries of the matrices in $\rho$ are powers of two.

6 Recognizing an element of $G_{x0}$ in $M$

In this section we will show that computing in the maximal subgroup $G_{x0}$ of $M$ is easy.

Given an element $g$ of $M$ as a word in $\Gamma^*$, we want to check if $g$ is in $G_{x0}$ or not. If this is the case then we also want to obtain a representation of $g$ as a word in the generators in $\Gamma$ that are in $G_{x0}$. That representation should depend on the value of $g$ only, and not on the given representation of $g$ as a word in $\Gamma^*$. In this section we construct a $\v_1 \in \rho_{15}$ such that every $g \in G_{x0}$ can effectively be reconstructed as a word in the generators of $G_{x0}$ from $\v_1 \cdot g$. The algorithm for reconstructing $g$ also detects if $g$ is in $G_{x0}$ or not.

An algorithm for solving that problem must certainly be able to detect if a word in $\Gamma^*$ is the identity or not. For this purpose we may use two nonzero vectors $\v_71$ and $\v_94$ in the representation $198883_3$ of $M$ that are fixed by an element of order 71 and negated by an element of order 94, respectively. In [11] it is shown that only the neutral element of $M$ fixes both vectors $\v_71$ and $\v_94$. So we may check if an element is the identity in the Monster. Here the corresponding calculations can be done modulo any odd prime; and we may even use different primes for the operation on $\v_71$ and on $\v_94$. Since we actually compute in the representation $\rho$ of $M$, we must make sure that the projections of the vectors $\v_71$ and $\v_94$ from $\rho$ onto the subspace $196883_3$ are not zero.

We generate $\v_71$ as a vector in the representation $\rho_3$ of the Monster in characteristic 3, and $\v_94$ as a vector in the representation $\rho_5$ of the Monster in characteristic 5. We combine these two vectors to a vector $\v_1$ in the representation $\rho_{15}$ via Chinese remaindering. Note that a computation in $\rho_{15}$ is faster than the combination of two similar computations in $\rho_3$. We will impose additional restrictions onto the vector $\v_3$, so that membership in $G_{x0}$ can be tested.

In Section 5 we have seen that an element of the subspace $300_2$ of $\rho$ corresponds to a symmetric matrix on the space $\mathbb{R}^{24}$ containing $\Lambda$. For $v \in \rho$ we write $M(v)$ for the symmetric matrix on $\mathbb{R}^{24}$ corresponding to the projection of $v$ onto $300_2$. When searching for a vector $\v_71 \in \rho_3$ fixing an element of order 71, we impose an additional condition on the matrix $M(\v_71)$. Modulo 3, we require that $M(\v_71)$ has corank 1, and that the kernel of $M(\v_71)$ contains a basis vector of $\mathbb{R}^{24}$. In Appendix B we estimate the cost of finding a suitable vector $\v_71$. On the author’s computer this takes less than a minute in average.

Given a fixed suitable $\v_71 \in \rho_3$ and a $g \in M$ as a word in $\Gamma^*$, we can compute the kernel of $M(\v_71 \cdot g)$ modulo 3. In case $g \in G_{x0}$ that kernel is spanned by the image of a basis vector of $\Lambda$ (modulo 3); and we can easily compute the image of that basis vector in $\Lambda$ under $g$ (modulo 3) from $M(\v_71 \cdot g)$ up to sign. So we can also compute the image $\lambda' = \lambda g_0$ of the standard co-ordinate frame $\lambda_0$ in $\Lambda/2\Lambda$. In Appendix A we show how to compute a $h_1 \in G_{x0}$ with $\lambda' h_1 = \lambda_0$, for any $\lambda' \in \Lambda/2\Lambda$ of type 4. So $gh_1$ fixes $\lambda_0$. Since $N_{x0}$ is the centralizer of the standard frame $\lambda_0$ in $G_{x0}$, we obtain $gh_1 \in N_{x0}$.

So we are left with the identification of an element $g$ of $N_{x0}$ from a vector $\v_1 g$, where $\v_1 \in \rho_{15}$ is fixed as above. The factor group $M_{24}$ of $N_{x0}$ acts on the matrix $M(\v_1)$ by permuting the rows and columns of $M(\v_1)$ as given by the natural permutation action of $M_{24}$, up to sign changes in the matrix. Thus $M_{24}$ permutes the multisets given by the absolute values of the entries of a row. For a random matrix $M(\v_1 \cdot g), g \in N_{x0}$, (with entries given modulo 15) we can almost certainly recover the permutation in $M_{24}$ corresponding to
g from the operation on these multisets. In the implementation we compute a hash function
on the 24 multisets given by the absolute values of the entries of the rows of M(v1); and we
precompute a new vector v1 if these 24 hash values are not mutually different. So we can
easily find a generator x, π ∈ AutS8P, such that gxπ−1 is in the kernel of structure 21+24+11
of the homomorphism Nx0 → M24.

An element g in the group 21+24+11 given above can be reduced to an element of Qx0 =
21+24 by a sequence of sign checks in the matrix M(v1 · g). In the unlikely case that M(v1)
has too many zero entries we just precompute another v1 ∈ ρ15.

Finally, an element of the extraspecial 2-group Qx0 can be recognized by a sequence of
sign checks in the parts of v1 that are not contained 300e. At the end we arrive at a relation
gh = 1 for a word h in the generators of Gx0; and we have to check that gh really fixes v1.

The algorithm sketched in this section succeeds if the input g is in Gx0 and fails otherwise.
For an implementation of that algorithm we refer to function reduce_Gx0 in [18], Section
Demonstration code for the reduction algorithm.

7 The strategy for reducing an element of Μ

In this section we explain the idea how to obtain an unknown element g of the Monster as
a word in Γ+ from a triple (v1, v g +, v − g), where v1, v +, and v − are fixed elements of ρ15.

7.1 Two 2A involutions and their centralizers H+, H− in Μ

The Monster Μ has two classes of involutions, which are called 2A and 2B in the ATLAS [3].
The centralizer of a 2A involution is a group of structure 2.B, where B is the Baby Monster.
An element t of the subgroup Qx0 of Μ is a 2A involution in Μ if and only if λ(t) is of type 2
in Λ/2Λ, see [12][10]. In the remainder of this paper let β be the fixed element of the Golay
cocode C* given by the subset (2, 3) of Ω = {0, ..., 23}, using the basis of C in [17]. Then
xβ ∈ Qx0. Put x− = x−1xβ, where x−1 is the central involution in Qx0 as in Section 4.1.

Then xβ and xβ− are 2A involutions in Μ; and λβ := λ(xβ) = λ(xβ−) is of type 2 in
Λ/2Λ. Let H+ and H− be the centralizers of xβ and xβ−, respectively; and let H = H+ ∩ H−.
Since H centralizes x−1 = xβ · xβ−, we have H ⊂ Gx0. xβ centralizes ξ and τ, see [16].

One can show that H = H+ ∩ Gx0, and that H has structure 22+22,C02, where C02 is the
(simple) subgroup of C01 fixing the type-2 vector λβ in Λ/2Λ.

A few words for justifying our choice of β ∈ C* are appropriate. We require that xβ is a
2A involution in Μ. For simplifying computations in the centralizer H+ of xβ we also want
to have ξ, ξ ∊ H+ for the generators ξ, ξ. This restricts β to an element of C* of weight 2
which is coloured in the terminology of [16]. The coloured cocode words correspond to the
elements of the cocode of the hexacode, which is a 3-dimensional linear code in F4 with
Hamming distance 4, as defined in [3], Ch. 11.2. Here our Golay cocode word β corresponds
to the element of the cocode of the hexacode given by (1, 0, ..., 0) ∈ F4, which is a fairly
natural choice.

The subgroups of Μ relevant for our purposes (together with their structure) are shown in
Figure 11. There arrows mean inclusion of groups. If an arrow is labelled with one or
more generators then all generators of that shape must be added to the subgroup in order to
obtain the group to which the arrow points. If an arrow is labelled with a set of generators
preceded by a \( 'C' \) symbol then some, but not all generators of that shape must be added
instead.

Note that M22 is the Mathieu group acting on the set Ω \ {2, 3} of size 22.

7.2 Using axes of 2A involutions

If the centralizer ΩM(t) of an element t of Μ fixes a unique one-dimensional subspace of the
representation 196883x of Μ then a nonzero vector in that subspace is called an axis of t
The total number of axes in $\eta$ with $300$ an additive multiple of the element $1$ cannot expect this task to be easy. The group $G$ in $[2]$. Since $G_{\text{v}}$ axis $gh$ in $[2]$. For any $2A$ involution $gh$ we want to find an element $v_\beta$, $v^-\parallel$ and $\xi_{\beta}$ respectively, see $[2]$. The axes $v^\pm$ have squared norm $\|v^\pm\|^2 = 8$ and scalar product $\langle v^\pm, 1_\rho \rangle = 2$. Also, $v^\pm \ast v^\pm$ is a positive scalar multiple of $v^\pm$, where the operation $\ast : \rho \times \rho \rightarrow \rho$ denotes the Griess algebra defined in $[2]$. For any $2A$ involution $t \in \mathcal{M}$ there is a unique axis satisfying these three conditions, and we write $ax(t)$ for that specific axis. In this paper an axis is always equal to $ax(t)$ for a $2A$ involution $t \in \mathcal{M}$. For practical computations it suffices to know the co-ordinates of an axis modulo $15$, i.e. we may assume $ax(t) \in \rho_{15}$.

Let $v_1$ be as in Section $[6]$. Given a triple $(v^+g, v^-g, v_1g)$ of vectors in $\rho_{15}$ for some unknown element $g$ of $\mathcal{M}$ we want to find an element $h$ of $\mathcal{M}$ (given as a word in $\Gamma^*$) with $gh = 1$.

In Section $[8]$ we construct a $h_1 \in \mathcal{M}$ that maps $v^+ \cdot g$ to $v^+$. Then $gh_1 \in H^+$. We call an axis in $\rho$ feasible if it is in the set $\{v^-h \mid h \in H^+\}$. Since $gh_1 \in H^+$, the axis $v^-gh_1$ is feasible. In Section $[9]$ we construct a $h_2 \in H^+$ that maps the feasible axis $v^-gh_1$ to $v^-$. Note that $h_2$ fixes $v^+$. Therefore $gh_1h_2$ fixes both, $v^+$ and $v^-$; hence $gh_1h_2 \in H^+ \cap H^- \subset G_{x_0}$. Using $v_1g$ and the method in Section $[4]$ we may find a $h_3 \in G_{x_0}$ with $gh_1h_2h_3 = 1$.

8 Reducing an axis of a $2A$ involution in the Monster

In this section we show how to transform an arbitrary axis $v$ in $\rho$ to the standard axis $v^+$. The total number of axes in $\rho$ (or of $2A$ involutions in $\mathcal{M}$) in is a bit less than $10^{20}$, so we cannot expect this task to be easy. The group $G_{x_0}$ has $12$ orbits on the set of $2A$ involutions,
see [15]; and the co-ordinates of an axis of a 2A involution encode quite a bit of geometric information about that orbit. We will use this information to find an \( h \in M \) with \( vh = v^+ \).

For an implementation demonstrating that transformation of an axis we refer to function \texttt{reduce_axis} in [18]. Section \textit{Demonstration code for the reduction algorithm}.

### 8.1 Using the orbits of \( G_{x_0} \) on the axes for reducing an axis

For \( v \in \rho \) let \( M(v) \) be the real symmetric \( 24 \times 24 \) matrix corresponding to the projection of \( v \) onto \( 300_\text{z} \), as in the Section 2. An element \( g \) of \( G_{x_0} \) acts as an orthogonal transformation matrix \( M_g \) on the Leech lattice, which is defined up to sign. Then \( g \) maps \( M(v) \) to \( M_g \cdot \cdot \cdot M(v) \cdot M_g \). Hence the eigenvalues of \( M(v) \) are actually properties of the orbit \( vG_{x_0} \); so they can be used to identify the orbit \( vG_{x_0} \).

A transformation in \( M \) that maps an axis \( v \) to \( v^+ \) has a representation as a word

\[
g_1 \cdot \tau_1 \cdot g_2 \cdot \tau_2 \cdot \ldots \cdot \tau_{n-1} \cdot g_n ,
\]

with \( g_r \in G_{x_0} \) and \( \tau_r \) a power of \( \tau \). Let \( v \) be an axis. Then \( \tau_v \) may change the orbit \( vG_{x_0} \), but \( g_r \) does not. Let \( l(v) \) be the minimum number of occurrences of a power of \( \tau \) in a word \( h' \) such that \( vh' = v^+ \). Actually, \( l(v) \) depends on the orbit \( vG_{x_0} \) only. So given an axis \( v \) with \( l(v) > 0 \), we essentially have to find a \( h \in G_{x_0} \) such that \( l(v \cdot h \cdot \tau^k) < l(v) \) for a \( k = \pm 1 \).

For an axis \( v \), how does a power of \( \tau \) change the orbit \( vG_{x_0} \)? From (3.1.1) we conclude

\[
N_{x_0} \tau^k \subset \tau^k N_{x_0} \cup \tau^{-k} N_{x_0} , \quad \text{for } k = \pm 1 . 
\]

Thus the set \( \{ v\tau^k G_{x_0} \mid k = \pm 1 \} \) of orbits depends on the orbit \( vN_{x_0} \) of \( v \) only. So given an axis \( v \) we may first identify its orbit \( vG_{x_0} \), and then search for an orbit \( vhN_{x_0} \), \( h \in G_{x_0} \), such that \( l(vhN_{x_0}) < l(v) \) for a \( k = 1 \).

The group \( N_{x_0} \) is the stabilizer of the standard frame \( \lambda_0 \) in \( \Lambda/2\Lambda \) with respect to the action of \( G_{x_0} \). Thus searching for a suitable \( hN_{x_0} \) in \( G_{x_0} \) amounts to searching for a type-4 vector \( \lambda_r \) in \( \Lambda/2\Lambda \) such that \( \lambda_r h = \lambda_0 \). In Appendix [A] we will show how to compute a representative of the coset \( hN_{x_0} \) in \( G_{x_0} \) from a vector \( \lambda_r \in \Lambda/2\Lambda \), such that we have \( \lambda_r h = \lambda_0 \). Note that both, \( \lambda_0 \) and \( \lambda_r \) are type-4 vectors in \( \Lambda/2\Lambda \). So given an axis \( v \) with \( l(v) > 0 \), our task is to find a type-4 vector \( \lambda_r \in \Lambda/2\Lambda \), such that for the coset \( hN_{x_0} \) with \( \lambda_r h = \lambda_0 \) we have \( l(vhN_{x_0} \tau^k) \) for a \( k = 1 \) that minimizes \( l(vh \tau^k) \).

For each axis \( v \) with \( l(v) > 0 \) we will specify a reasonably small set \( U_4(v) \) of type-4 vectors in \( \Lambda/2\Lambda \) that can be used for decrementing the value \( l(v) \) for an axis \( v \) as described above. Here we have to treat the different orbits of \( G_{x_0} \) on the axes separately. Details are given in Section 8.3. Lemma 8.1 deals with axes \( v \) satisfying \( l(v) = 0 \).

### 8.2 Enumeration of the orbits of \( G_{x_0} \) on the axes

Let \( v^+ = ax(x_\beta) \) be the axis of the 2A involution \( x_\beta \) as defined in Section 7.2. In this section we assume that a given vector \( v \) in \( \rho \) is the axis \( ax(t) \) of an (unknown) 2A involution \( t \).

Norton [15] has enumerated the twelve orbits of \( G_{x_0} \) on the 2A involutions in \( M \). For a 2A involution \( t \) the orbit of \( t \) under the action of \( G_{x_0} \) is characterized by the of class \( tx_{-1} \) in \( M \), where \( x_{-1} \) is the central involution in \( G_{x_0} \). The classes of \( tx_{-1} \) (in ATLAS notation) are given in column 1 of Table [4]. The powers of these classes are given in column 2. The structure of the centralizer of \( tx_{-1} \) in \( G_{x_0} \) is given in column 3, as in [15].

We have computed representatives of the 12 orbits of \( G_{x_0} \) on 2A involutions with the software package [17]. Here the essential information required from [15] is that there are no more than 12 such orbits. Then we may easily find such representatives by computing images.
$v^+ h$ for random elements $h$ of $\mathbb{M}$, and watermark them e.g. with the set of eigenvalues of the matrix $M(ax(t)) = M(v^+ h)$, where $t = h^{-1} x_\beta h$. These eigenvalues are given in column 4 of Table 1. Column 5 contains the squared norm $\| M(ax(t)) \|$ of the matrix (modulo 15), which we define as the sum of the squares of the eigenvalues or, equivalently, as the sum of the squares of the entries of the symmetric matrix.

The package [17] contains a function for computing the real character of the representation $196883_x$ of an element of $\mathbb{M}$ that powers up to a 2B involution. Thus for any 2A involution $t$ we may check the class containing $tx_{-1}$ against column 1 of Table 1.

| $tx_{-1}$ | Powers of $tx_{-1}$ | $C_{G_{x_0}}(tx_{-1})$ | Eigenvalues of $256 M(ax(t))$ | $\| M(ax(t)) \|$ mod 15 |
|-----------|----------------------|--------------------------|-------------------------------|--------------------------|
| 2A        | $2^{2+22}, C_{02}$   | 512$^4$, 0$^{23}$        | 4                             |                          |
| 2B        | $2^{2+8+16} O^+_8(2)$ | 64$^8$, 0$^{16}$         | 8                             |                          |
| 4A 2B     | $2^{1+22}, M_{23}$   | 144$^4$, 16$^{23}$       | 14                            |                          |
| 4B 2A     | $(2^7 \times 2^{1+8}), S_6(2)$ | 72$^1$, 24$^{16}$, 8$^7$ | 13                            |                          |
| 4C 2B     | $2^{1+14+5}, A_8$   | 32$^8$, 16$^{16}$        | 3                             |                          |
| 6A 2A     | $2^2 U_6(2), 2$     | 90$^1$, 26$^1$, 18$^{22}$ | 4                             |                          |
| 6C 2B     | $2^{2+8} (3 \times U(4)_2), 2$ | 34$^6$, 18$^{16}$, 11$^2$ | 5                             |                          |
| 6F 2B     | $2^{1+8}, A_9$      | 24$^{16}$, 16$^{8}$      | 14                            |                          |
| 8B 4A.2B  | $2.2^{10}. M_{11}$  | 48$^1$, 24$^{12}$, 16$^{11}$ | 2                             |                          |
| 10A 2A    | $2. HS_2$           | 60$^1$, 20$^{22}$, 12$^1$ | 4                             |                          |
| 10B 2B    | $2^{1+8}, (A_5 \times A_5), 2$ | $(24 \pm 4 \sqrt{5})^1$, 20$^{16}$ | 8                             |                          |
| 12C 6A.4B.2A | $2 \times S_6(2)$ | 42$^1$, 26$^{7}$, 18$^{16}$ | 10                            |                          |

Table 1: Orbits of $G_{x_0}$ on 2A involutions in $\mathbb{M}$

For labeling an orbit $ax(t) G_{x_0}$ on the set of the axes we use the name of the corresponding class $tx_{-1}$ in column 1 of Table 1 and we put this name in single quotes. In Section 8.4 we will show how to identify the orbit $v G_{x_0}$ of a given axis $v$.

Let $l(v)$ be as Section 8.3. In case $l(v) = 0$ the axis $v$ is in the orbit $v^+ G_{x_0}$ with name ‘2A’. From the definition of $v^+ = ax(x_\beta)$ in Section 7.1 we see that matrix $M(v^+)$ has rank 1 and an eigenvalue 2, and that $M(v^+)$ is proportional to $\lambda_\beta \otimes \lambda_\beta$. So we obtain:

**Lemma 8.1.** The orbit ‘2A’ of $G_{x_0}$ on the axes consists of the elements $x_\nu$ of $G_{x_0}$ such that $\lambda_r = \lambda(x_\nu)$ is of type 2. For any such $x_\nu$, the kernel $\ker(M(ax(x_\nu)) - 2 \cdot 1_\rho)$ is one-dimensional and spanned by the set $\Lambda(\lambda_r)$ of the shortest preimages of $\lambda_r$ in the Leech lattice $\Lambda$.

### 8.3 Mapping an axis to a different orbit of $G_{x_0}$

For any axis $v$ not in the orbit ‘2A’ we will show how to compute a non-empty set $U_4 = U_4(v)$ of type-4 vectors in $\Lambda/2\Lambda$ with:

$$\forall \lambda_r \in U_4(v) \\forall h \in G_{x_0} : \lambda_r h = \lambda_\Omega \implies \min \{ l(vh\tau^k) \mid k = \pm 1 \} < l(v). \quad (8.1.1)$$

In Section 8.4 we will construct a set $U = U(v) \subseteq \Lambda/2\Lambda$ such that the mapping $v \mapsto U(v)$ visibly commutes with the action of $G_{x_0}$ and we define $U_4(v)$ to be the set of type-4 vectors in $U(v)$. So it suffices to check Equation 8.1.1 for one representative $v$ of each orbit of $G_{x_0}$ on the axes. By 8.3.1 it suffices to check each case of $v$ and each $\lambda_r \in U_4(v)$ for one $h \in G_{x_0}$ with $\lambda_r h = \lambda_\Omega$. This check can be done (and has been done) computationally.

By a sequence of such reduction steps we may reduce $l(v)$ to 0 as shown in Figure 2.

In the remainder of this subsection we define certain subsets of $\Lambda/2\Lambda$ depending on an axis $v$ that will be used for constructing $U(v)$.

We write $\text{rank}_3(v, k)$ for the rank of the matrix $M(v) - k \cdot 1_\rho$, considered as a matrix in $\mathbb{F}_3^{n \times n}$ by taking the entries of $M$ modulo 3. This is well defined since $M(v)$ has entries in
$l(v) = \begin{array}{cccc}
3 & 2 & 1 & 0 \\
'6F' & '4C' & & \\
'10B' & '4B' & '2B' & \\
'12C' & '6A' & '2A' & \\
'10A' & '6C' & '4A' & \\
'8B' & & & 
\end{array}$

Figure 2: Reduction of an axis $v$ depending on its orbit $vG_{x0}$

For an endomorphism $\phi$ of $\mathbb{R}^{24}$, with $\Lambda \subset \mathbb{R}^{24}$, we put

$$\ker_{3,\Lambda}(\phi) = (\phi^{-1}(3\Lambda) \cap \Lambda) + 3\Lambda.$$  

This defines the kernel of $\phi$ with respect to the Leech lattice modulo 3; and this kernel is a subspace of the space $\Lambda/3\Lambda$ considered as a vector space over $\mathbb{F}_3^4$.

Matrix $M(v)$ acts naturally as an endomorphism of $\mathbb{R}^{24}$, with $\Lambda \subset \mathbb{R}^{24}$. For a 2A axis $v$ and an integer $k$ we define $\ker_{3}(v, k)$ by

$$\ker_{3}(v, k) = \{ w \in \ker_{3,\Lambda}(M(v) - k \cdot 1_\Lambda) | 2 \leq \text{type}(w) \leq 4 \} + 2\Lambda \subset \Lambda/2\Lambda.$$  

The term $2\Lambda$ in that formula implies $\ker_{3}(v, k) \subset \Lambda/2\Lambda$. We need $\ker_{3}(v, k)$ in case $\text{rank}_{3}(v, k) = 23$ only. In this case we have $| \ker_{3}(v, k) | \leq 1$; and $\ker_{3}(v, k)$ can easily be computed from matrix $M(v)$ with entries taken modulo 3. For this computation we may use the information in Table 4 about the 'shapes' of vectors in $\Lambda$ of type $\leq 4$.

By construction, $\text{rank}(v, 3)$ is a property of the orbit $vG_{x0}$; and the mapping $v \mapsto \ker_{3}(v, k)$ commutes with the action of $G_{x0}$ on 300 and $\Lambda$.

Apart from analysing matrix $M(v)$ we also consider the component of the axis $v$ in the subspace $98280_x$ of $\rho$. $G_{x0}$ acts monomially on $98280_x$, without changing the number of co-ordinates with the same absolute value. For $v \in \rho_5$ we let $S_\lambda(v)$ be the set of type-2 vectors $\lambda_r$ in $\Lambda/2\Lambda$ such that the co-ordinate of $v$ with respect to the basis vector of $98280_x$ corresponding to $\lambda_r$ has absolute value $k$ (mod 15).

If $S$ is a subset of $\Lambda/2\Lambda$ then we write $\text{span}(S)$ for the linear subspace of $\Lambda/2\Lambda$ generated by $S$, and we write $S^\perp$ for the orthogonal complement of $S$ (with respect to the natural scalar product) in $\Lambda/2\Lambda$. Let $\text{rad}(S) = \text{span}(S) \cap \text{span}(S)^\perp$ be the radical of $S$; so the radical of a linear subspace $S$ of $\Lambda/2\Lambda$ is the intersection of $S$ with its orthogonal complement.

For any axis $v$ we construct $U(v)$ as a sum of certain sets defined in this subsection, depending on the orbit $vG_{x0}$. We write $U' + U''$ for the sum of two subsets $U', U''$ of $\Lambda/2\Lambda$.

For identifying the orbit $vG_{x0}$ of $v$ we sometimes evaluate $M(v)$ at a type-2 vector $\lambda_r \in \Lambda/2\Lambda$. Define $M(v, \lambda_r) = w \cdot M(v) \cdot w^\top$, for a $w \in \Lambda(\lambda_r)$. This is well defined, since $\Lambda(\lambda_r)$ is unique up to sign if $\lambda_r$ is of type 2.

8.4 Reducing an axis inside an orbit of $G_{x0}$

In this subsection we assume that an axis $v$ is given. For each of the 12 orbits of $G_{x0}$ on the axes we will show how to check that axis $v$ is in that orbit. For each orbit different from orbit '2A' we will also show how to compute a set $U = U(v)$ such that the (non-empty) set $U_4(v)$ of type-4 vectors in $U(v)$ satisfies (8.1.1).
Case '2A'  
We check $\|M(v)\| = 4$ (mod 15), rank$_3(v, 2) = 23$; and we check that $\ker_3(v, 2)$ is a singleton $\{\lambda_r\}$ with $\lambda_r$ of type 2 and $M(v, \lambda_r) = 4$ (mod 15). We put $U = \{\}$ in this case.

By Lemma 5.1 we have $v = ax(x_{\pm r})$ for an $x_r$ with $\lambda(x_r) = \lambda_r$. We can find a $h \in G_{x_0}$ that maps $\lambda_r$ to $\lambda_{\beta}$, as explained in Appendix C. Then $vh = ax(x_{\pm \beta}) \in \{v^+, v^\right\}$. In case $vh = v^+$ we are done. In case $vh = v^\right$ we replace $vh$ by $vhx_d$ for a $d \in P$ with $\langle d, \beta \rangle = 1$.

Case '2B'  
We check $\|M(v)\| = 8$ (mod 15) and rank$_3(v, 0) = 7$. We have $|S_4(v)| = 120$. Put $U = \text{span}(S_4(v))$. Then $|U| = 256$; and $U$ contains 135 type-4 vectors. If $h \in G_{x_0}$ maps a type-4 vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '2A' of $G_{x_0}$.

Case '4A'  
We check $\|M(v)\| = 14$ (mod 15) and rank$_3(v, 0) = 23$. Then $U = \ker_3(v, 0)$. Then $U$ is a singleton $\{\lambda_r\}$ with $\lambda_r$ of type 4. If $h \in G_{x_0}$ maps $\lambda_r$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '4A' of $G_{x_0}$.

Case '4B'  
We check $\|M(v)\| = 13$ (mod 15). We have $|S_4(v)| = 512$. Put $U = \text{rad}(S_4(v))$. Then $|U| = 128$; and $U$ contains 63 type-4 vectors. If $h \in G_{x_0}$ maps a type-4 vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '2B' of $G_{x_0}$.

Case '4C'  
We check $\|M(v)\| = 3$ (mod 15). We have $|S_4(v)| = 16$. Put $U = \text{rad}(S_4(v))$. Then $|U| = 32$; and $U$ contains 15 type-4 vectors. If $h \in G_{x_0}$ maps a type-4 vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '4E' of $G_{x_0}$.

Case '6A'  
We check $\|M(v)\| = 4$ (mod 15), rank$_3(v, 2) = 23$; and we check that $\ker_3(v, 2)$ is a singleton $\{\lambda_r\}$ with $\lambda_r$ of type 2 and $M(v, \lambda_r) = 7$ (mod 15). Put $U = \{\lambda_r\} + S_4(v)$. Then $|U| = 891$; and all vectors in $U$ are of type 4. If $h \in G_{x_0}$ maps one vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '4A' of $G_{x_0}$.

Case '6C'  
We check $\|M(v)\| = 5$ (mod 15). We have $|S_4(v)| = 36$; Put $U = \text{span}(S_4(v))$ Then $|U| = 64$; and $U$ contains 27 type-4 vectors. If $h \in G_{x_0}$ maps one type-4 vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '4A' of $G_{x_0}$.

Case '8B'  
We check $\|M(v)\| = 2$ (mod 15), rank$_3(v, 0) = 8$. We have $|S_7(v)| = 144$. Put $U = S_7(v)$. Then $|U| = 134$; and $U$ contains one type-4 vector $\lambda_r$. If $h \in G_{x_0}$ maps $\lambda_r$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '4A' of $G_{x_0}$. Note that $\lambda_r \in \{\lambda_s\} + S_1(v)$ holds for all $\lambda_s \in S_1(v)$.

Case '10A'  
We check $\|M(v)\| = 4$ (mod 15) and rank$_3(v, 2) = 2$. Then $|S_3(v)| = 1$ and $|S_1(v)| = 100$. Put $U = S_3(v) + S_1(v)$. Then all the 100 vectors in $U$ are of type 4. If $h \in G_{x_0}$ maps one vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in orbit '6A' of $G_{x_0}$.

Case '10B'  
We check $\|M(v)\| = 8$ (mod 15) and rank$_3(v, 0) = 24$. Then $|S_4(v)| = 680$. Put $U = \text{rad}(S_4(v))$. Then $|U| = 256$; and $U$ contains 135 type-4 vectors. If $h \in G_{x_0}$ maps one type-4 vector in $U$ to $\lambda_\Omega$ then one of the axes $vh\tau^{\pm 1}$ is in the orbit '4B' or '4C' of $G_{x_0}$. An axis can be mapped to orbit '4B' (or '4C') for 60 (or 75) of the 135 type-4 vectors, respectively.
Case '12C

We check ||M(v)|| = 10 (mod 15). We have |S_7(v)| = 56. Put U = rad(S_7(v)). Then |U| = 256; and U contains 135 type-4 vectors. If h ∈ G_{x_0} maps one type-4 vector in U to \lambda_1 then one of the axes vhτ^±1 is in the orbit '4B' or '6A' of G_{x_0}. An axis can be mapped to orbit '4B' (or '6A') for 63 (or 72) of the 135 type-4 vectors, respectively.

Remark

It has not escaped our attention that the sets U of vectors in \Lambda/2\Lambda discussed in the different cases for v = ax(t) provide a wealth of geometric information about the centralizer of the 2A involution t ∈ M in G_{x_0}. According to Table 1 in Case '10A' the Higman-Sims group HS is involved in that centralizer. The 100 short vectors in \Lambda/2\Lambda mentioned in that case correspond to the 100 short vectors in \Lambda discussed in the analysis of the Higman-Sims group in [3], Ch. 10.3.5. In Case '10B' the sum of the two 4-dimensional eigenspaces of matrix M(ax(t)) contains a sublattice \sqrt{2}E_8 of \Lambda; and this can be interpreted in terms of the icosians, see [4], Ch. 8.2.1. Most of the other cases have simpler geometric interpretations. As the objective of this paper is computational, we omit further details.

9 Reducing a feasible axis in the group \( H^+ \)

In this section we will define the term feasible axis; and we will show how to transform an arbitrary feasible axis \( v \) in \( \rho \) to the standard feasible axis \( v^- \) using a computation in the group \( H^+ \). For an implementation demonstrating that transformation of an feasible axis we refer to function reduce_feasible_axis in [15]. Section Demonstration code for the reduction algorithm.

9.1 Enumeration of the orbits of \( H \) on feasible axes

Let \( v^+ \), \( v^- \) be the axes of the 2A involutions \( x_\beta \), \( x_\beta^- \), and let \( H^+ \) be the centralizer of \( v^+ \) (or of \( x_\beta \)) as in Section 4. We call an axis in \( \rho \) feasible if it is in the set \( \{ v^- h \mid h \in H^+ \} \), as in Section 7.1. Since \( H^+ \) fixes \( v^+ \), and \( v^+ \) is orthogonal to \( v^- \) with respect to the scalar product given by the natural norm in \( \rho \), any feasible axis is also orthogonal to \( v^+ \). For two 2A involutions \( t_1, t_2 \in M \) their axes \( ax(t_1) \) and \( ax(t_2) \) are orthogonal if and only if the product \( t_1 \cdot t_2 \) is a 2B involution in \( \mathbb{M} \); and \( \mathbb{M} \) is transitive on the pairs of orthogonal axes; see [2] or [10].

Given a feasible axis \( v \), we construct a \( g \in H^+ \) with \( vg = v^- \) in this section. Therefore we have to compute in the group \( H^+ \). The group \( H^+ \cap N_{x_0} \) is generated by

\[
\Gamma_0^\rho = \{ x_\delta, x_d, y_d, x_\pi \in \Gamma \mid \delta \in C^+, d \in P, \langle d, \beta \rangle = 0, \pi \in \text{Aut}_W P, \beta^\pi = \beta \};
\]

so computation in \( H^+ \cap N_{x_0} \) is easy. The group \( H = H^+ \cap G_{x_0} \) is generated by \( \Gamma_0^\rho \) and \( \xi \); the group \( H^+ \) is generated by \( H \) and \( \tau \). Put \( \Gamma_f = \Gamma_0^\rho \cup \{ \xi^{\pm1}, \tau^{\pm1} \} \), so that \( \Gamma_f \) generates \( H^+ \). Let \( \Gamma_f \) be the set of words in \( \Gamma_f \).

Our strategy for finding a \( g \in H^+ \) that maps a feasible axis to \( v^- \) is similar to the strategy in Section 8 for finding an element of \( M \) that maps an arbitrary axis to the standard axis \( v^+ \). Here we will replace the rôles of the groups \( \mathbb{M}, G_{x_0}, N_{x_0} \) by their intersections with the group \( H^+ \), as indicated in Table 2.

We present an element \( g \) of \( H^+ \) as a word

\[ g_1 \cdot \tau_1 \cdot g_2 \cdot \tau_2 \cdot \ldots \cdot \tau_{n-1} \cdot g_n, \]

with \( g_\nu \in H \) and \( \tau_\nu \) a power of \( \tau \). Let \( v \) be a feasible axis. Then \( \tau_\nu \) may change the orbit \( vH \), but \( g_\nu \) does not. Let \( l(v) \) be the minimum number of occurrences of a power of \( \tau \) in a word \( h' \) in \( \Gamma_f \) such that \( vh' = v^- \). Actually, \( l(v) \) depends on the orbit \( vH \) only. So given
Table 2: Analogies between reducing axes to $v^+$ and feasible axes to $v^-$

| Operating in group | Reduce axis to $v^+$ | Reduce feasible axis to $v^-$ |
|--------------------|----------------------|-------------------------------|
| $\Gamma$          | $\Gamma_f$           | $H$                           |
| $G_{x_0}$          | $H$                  |                               |
| 12, see [15]       | 10, see [12]         |                               |
| see Figure 2       | see Figure 9         |                               |
| type($\lambda_e$) = 4 | type($\lambda_e$) = 2, type($\lambda_\Omega + \lambda_\beta$) = 4 |
| $\lambda_\Omega$ | $\lambda_\Omega + \lambda_\beta$ |
| $N_{x_0}$          | $H \cap N_{x_0}$    |                               |
| Group fixing that image |                       |                               |

Thus the set $\{v\tau^k H \mid k = \pm 1\}$ of orbits depends on the orbit $v(H \cap N_{x_0})$ of $v$ only. So given an axis $v$ we may first identify its orbit $vH$, and then search for an orbit $vh(H \cap N_{x_0})$, $h \in H$, such that $\min\{l(vh(H \cap N_{x_0})\tau^k) \mid k = \pm 1\} < l(v)$.

We call a vector $\lambda_e$ in $\Lambda/2\Lambda$ feasible if type($\lambda_e$) = 2 and type($\lambda_e + \lambda_\beta$) = 4. This is equivalent to claiming that any two shortest preimages of $\lambda_e$ and $\lambda_\beta$ in the Leech lattice $\Lambda$ are of type 2 and perpendicular. The group $H$ fixes $\beta$; and it is transitive on the set of feasible vectors in $\Lambda/2\Lambda$. The group $H \cap N_{x_0}$ fixes $\lambda_\beta$ and $\lambda_\Omega$, and hence also the feasible type-2 vector $\lambda_\Omega + \lambda_\beta$. From [3] and the structures of $H$ and of $H \cap N_{x_0}$ we see that $H \cap N_{x_0}$ is a maximal subgroup of $H$. So $H \cap N_{x_0}$ is the stabilizer of the feasible vector $\lambda_\Omega + \lambda_\beta$ in $H$.

Thus searching for a suitable coset $h(H \cap N_{x_0})$ in $H$ amounts to searching for a feasible vector $\lambda_e$ in $\Lambda/2\Lambda$ such that $\lambda_e h = \lambda_\Omega + \lambda_\beta$. In Appendix D we will show how to compute a representative of the coset $h(H \cap N_{x_0})$ in $H$ from a feasible vector $\lambda_e$ satisfying $\lambda_e h = \lambda_\Omega + \lambda_\beta$.

Müller [12] has enumerated the ten orbits of $2^{1+22}.Co_2$ on the representation of the Baby Monster $B$ acting on the cosets of $2^{1+22}.Co_2$ as a permutation group. Thus the group $H^+$ of structure 2.B has ten orbits of $H$ (of structure $2^{2+22}.Co_2$) acting on the cosets of $H$. The centralizer of the feasible axis $v^-$ in $H^+$ is $H^+ \cap H^- = H$. Since $H^+$ is transitive on the set of feasible axes, we conclude that there are ten orbits of $H$ on the feasible axes.

We have computed representatives of the 10 orbits of $H$ on the feasible axes with the software package [17]. Here the essential information required from [12] is that there are no more than 10 such orbits. Then we may easily find such representatives by computing images $v^+ h$ for random elements $h$ of $H^+$. Here we watermark the feasible axes in the same way as the axes in Section 8 and we augment our watermark of an axis $v$ by the value $M(v, \lambda_\beta)$, as defined in Section 8.3. $M(v, \lambda_\beta)$ is invariant under the action of $H$.

For labelling an orbit of $H$ on the set of the feasible axes we use the name of the orbit of $G_{x_0}$ on the set of all axes, which contains that orbit of $H$. These names are defined as in Section 8.2. We append a ‘0’ digit to that name if $M(v, \lambda_\beta) = 0$; otherwise we append a ‘1’ digit. This labelling is sufficient to distinguish between the 10 orbits. Thus for identifying the orbit of $H$ containing the feasible axis $v$ we first have to compute the orbit $vG_{x_0}$ on the axes as described Section 8.4. If there are several orbits $vH$ on the feasible axes contained in the same orbit $vG_{x_0}$ then we also have to compute $M(v, \lambda_\beta)$. It suffices to compute $M(v, \lambda_\beta)$ modulo 15. We obtain the 10 orbits of $H$ on the feasible axes shown in Figure 3.
9.2 Reducing a feasible axis

In this subsection we assume that a feasible axis $v$ is given. Let $U(v)$ be as in Section 8.3. We define the set $U_f(v)$ of feasible type-2 vectors in $\Lambda/2\Lambda$ by:

$$U_f(v) = \{ \lambda_r + \lambda_\beta \mid \lambda_r \in U(v), \text{ type}(\lambda_r) = 4, \text{ type}(\lambda_r + \lambda_\beta) = 2 \},$$

(9.0.2)

If $v$ is in one of the orbits of $H$ on the feasible axes shown in Figure 3 (except for orbits '2A0' and '2A1'), then $U_f(v)$ is non-empty, and we can show computationally:

$$\forall \lambda_r \in U_f(v) \forall h \in H : \lambda_r h = \lambda_\Omega + \lambda_\beta \implies \min\{ l(\nu h \tau^k) \mid k = \pm 1 \} < l(v)$$

Therefore we use a similar method as described in Section 8.3.

We obtain the following cases of orbits of $H$ on feasible axes.

Case '2A1'

Since $v^- = ax(x_\beta)$ is in this orbit of $H$, and $H$ fixes $\lambda_\beta$, the axis $v^-$ is the only feasible axis in this orbit, so that we are done.

Case '2A0'

By Lemma 8.1 there is an $x_\tau$ with $v = ax(x_\tau)$, and $x_\tau$ can be effectively computed from $v$. Since $v$ is feasible and $M(v, x_\beta) = 0$, the vectors in $\Lambda(\lambda_r)$ and $\Lambda(\lambda_\beta)$ in the Leech lattice are perpendicular, and hence $\text{type}(\lambda_\Omega + \lambda_r) = 4$, i.e. $\lambda_r$ is feasible.

So we can find a $h \in H$ that maps $\lambda_r$ to $\lambda_\Omega + \lambda_\beta$ without changing $\lambda_\beta$, as explained in Appendix D. Then $\nu h = ax(x_\Omega x_\beta)$. Note that the element $\tau$ of $H^+$ cyclically permutes the axes $ax(x_\beta)$, $ax(x_\Omega x_\beta)$, and $ax(x_\Omega x_\beta)$. Thus there is a $k = \pm 1$ with $\nu h \tau^k = ax(x_- \beta) = v^-$; i.e. $\nu h \tau^k$ is the unique axis in the orbit '2A1'.

The remaining cases of orbits of $H$ on feasible axes

The sizes of the sets $U_f(v)$ in these cases are given in Table 3. For any $\lambda_\nu \in U_f(v)$ we can find a $h \in H$ that maps $\lambda_\nu$ to $\lambda_\Omega + \lambda_\beta$, as explained in Appendix D. Then one of the axes $\nu h \tau^{\pm 1}$ is in an orbit of $H$ as shown in Table 3.

In case '4B1' the set $U_f(v)$ has size 31; for 30 elements of $U_f(v)$ one of the axes $\nu h \tau^{\pm 1}$ is in the orbit '2B0'; for the remaining element of $U_f(v)$ one of these axes is in the orbit '2B1.'

| Orbit of $H$ | 2B0 | 2B1 | 4A1 | 4B1 | 4C1 | 6A1 | 6C1 | 10A1 |
|--------------|-----|-----|-----|-----|-----|-----|-----|------|
| is mapped to orbit | 2A0 | 2A0 | 2A0 | 2B0 | 2B1 | 2B0 | 4A1 | 4A1 | 6A1 | 10A1 |
| Size of $U_f$ | 15 | 63 | 1 | 30 | 1 | 15 | 891 | 27 | 100 |

Table 3: Reduction of feasible axes
10 Conclusion

We have shown how to reconstruct an element $g$ of the Monster $\mathbb{M}$ as a word in the generators of $\mathbb{M}$ from the images of three fixed vectors in the representation $\rho$ under the action of $g$. It suffices if the co-ordinates of these three images are known modulo 15. This leads to an extremely fast word shortening algorithm. In [17] we have implemented the group operation of the Monster $\mathbb{M}$ based on this word shortening algorithm.

By construction, the new algorithm computes a unique representation of an element of $\mathbb{M}$. This representation can be compressed to a bit string of less than 256 bit length. So we may quickly find an element of $\mathbb{M}$ in an array of millions of elements of $\mathbb{M}$.

The package [17] can also perform the following computations efficiently. Given two arbitrary conjugate involutions $t_1, t_2 \in \mathbb{M}$, we can compute a $g \in \mathbb{M}$ with $t_1^g = t_2$. In the smallest faithful real representation $196883_\mathbb{x}$ of $\mathbb{M}$ we can compute the character of an element of $\mathbb{M}$ centralizing a known 2B involution. Therefore we also use a strategy called 'changing post’ in [19].

The ATLAS [3] describes a well-known homomorphism from a certain Coxeter group $Y_{555}$ to a group of structure $(\mathbb{M} \times \mathbb{M}).2$, which is called the Bimonster. For background, see [9], Section 8. The software package [17] also contains the first efficient implementation of this homomorphism, based on the ideas in [6,14]. During the construction of that homomorphism in [17] we have verified the presentation of the Monster given in [14].

The new algorithm has been used to solve the long-standing problem of finding all maximal subgroups of the Monster, see [5].

In principle, the new algorithm can also compute unique representatives of the cosets in a chain of subgroups

$$\ldots \subset 2^{2+22}.2^{10}.M_{22}.2 \subset 2^{2+22}.\text{Co}_2 \subset 2.B \subset \mathbb{M}.$$ 

So using the methods in [13], we may be able to enumerate rather big orbits of some very large subgroups of the Monster $\mathbb{M}$.

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### Notation

| Symbol | Description |
|--------|-------------|
| Aut$_{SL}$P | The group of standard automorphisms of the Parker loop $\mathcal{P}$. |
| axis | Vector in representation $\rho$, corresponding to a 2A involution in $\mathbb{M}$. |
| aX(t) | The axis of a 2A involution $t \in \mathbb{M}$ in the representation $\rho$. |
| B | The Baby Monster group. $\mathbb{M}$ has a subgroup of structure 2.B. |
| $\beta$ | A fixed element of $C^*$ used for constructing $x_\beta, x_{-\beta}$, and $\lambda_\beta$. |
| $C, C^*$ | $C$ is the 12-dimensional Golay code in $\mathbb{F}_2^{12}$; $C^*$ is its cocode $\mathbb{F}_2^{24}/C$. |
| Co$_{0}, $Co$_{1}$ | Automorphism groups of $\Lambda$ and of $\Lambda/2\Lambda$. Co$_{0}$ has structure 2.Co$_{1}$. |
| d, e, f | Elements of the Parker loop $\mathcal{P}$ or of the Golay code $C$. |
| $\delta, \epsilon$ | Elements of the Golay cocode $C^*$. |
| $\eta_i$ | $i$-th unit vector of the space $\mathbb{R}^{24}$ containing $\Lambda$, for $i \in \tilde{\Omega}$. |
| $\eta_{ij}$ | Matrix in $300_x$ with entry 1 in row $i$, column $j$, and in row $j$, col. $i$. |
| feasible | Property of an axis or a vector in $\Lambda/2\Lambda$, related to involutions $x_{\pm\beta}$. |
| $G_{20}$ | Maximal subgroup of $\mathbb{M}$ of structure $2_+^{1+24}.Co_1$ with centre $\{1, x_{-1}\}$. |
| $\Gamma, \Gamma^*$ | $\Gamma$ is a set of elements generating $\mathbb{M}$. $\Gamma^*$ is the set of words in $\Gamma$. |
| H | Group of structure $2^{2+22}.Co_2$, with $H = H^+ \cap H^- = G_{20} \cap H^+$. |
| $H^+, H^-$ | Centralizers of the involutions $x_\beta, x_{-\beta}$ in $\mathbb{M}$, both of structure 2.B. |
| i, j | Elements of $\tilde{\Omega}$, also considered as elements of $C^*$ of weight 1. |
| $\Lambda, \Lambda/2\Lambda$ | The 24-dimensional Leech lattice, and the Leech lattice mod 2. |
| $\Lambda(\lambda_r)$ | For $\lambda_r \in \Lambda/2\Lambda$ this is the set of shortest preimages of $\lambda_r$ in $\Lambda$. |
| $\lambda$ | Homomorphism from $Q_{x_0}$ onto $\Lambda/2\Lambda$ with kernel $\{1, x_{-1}\}$. |
| $\lambda_\beta$ | Fixed vector in $\Lambda/2\Lambda$ of type 2, equal to $\lambda(x_{\beta})$. |
| $\lambda_{01}$ | Fixed vector in $\Lambda/2\Lambda$ of type 4; $\Lambda(\lambda_{01})$ is the co-ordinate frame of $\Lambda$. |
| $\mathbb{M}$ | The Monster group, i.e. the largest sporadic simple group. |
| $M_{24}$ | Mathieu group, acts on $\tilde{\Omega}$ as the automorphism group of $\tilde{C}$. |
| M(v) | Projection of a vector $v \in \rho$ onto the subspace $300_x$ of $\rho$; also considered as a symmetric $24 \times 24$ matrix, with $\Lambda \subset \mathbb{R}^{24}$. |
| $M(v, \lambda_r)$ | Equal to $uM(v)u^{-1}$ for a shortest preimage $u \in \Lambda$ of $\lambda_r \in \Lambda/2\Lambda$. |
| $N_0$ | A maximal subgroup of $\mathbb{M}$ of structure $2^{2+11+22}.(M_{24} \times S_3)$. |
| $N_{20}$ | Subgroup of structure $2_+^{1+24}.2^{24}.M_{24}$ of $\mathbb{M}$, with $G_{20} \cap N_0 = N_{20}$. |
| $\Omega$ | The element $(\tilde{\Omega}, 0)$ of the Parker loop $\mathcal{P}$. |
| $\tilde{\Omega}$ | The set $\{0, \ldots, 23\}$ used for labelling the basis vectors of $\mathbb{F}_2^{24}$ and $\mathbb{R}^{24}$. The power set $\tilde{\Omega}$ is identified with $\mathbb{F}_2^{24}$, and we have $\mathbb{C} \subset \mathbb{F}_2^{24}$. |
| $\mathcal{P}$ | The Parker loop, any $d \in \mathcal{P}$ has the form $(d, \mu)$, $d \in C, \mu \in \mathbb{F}_2$. |
| $\pi, \pi', \pi''$ | Standard automorphisms of the Parker Loop $\mathcal{P}$ in Aut$_{SL}$P. |
| $Q_{x_0}$ | A normal subgroup of structure $2_+^{1+24}$ of the group $G_{x_0}$. |
| $\rho$ | 196884-dimensional representation of $\mathbb{M}$ with matrix entries in $\mathbb{Z}[\frac{1}{2}]$. We have $\rho = 1_x \oplus 196883_x = 300_x \oplus 98280_x \oplus (24_x \otimes 4096_x)$. |
| $\rho_p$ | Representation $\rho$ of $\mathbb{M}$ with matrix entries taken modulo $p$, $p$ odd. |
| $S_{k}(v)$ | Set of short vectors in $\Lambda/2\Lambda$ depending on $v \in \rho_{15}$ and integer $k$. |
| type($\lambda_r$) | Type of a vector $\lambda_r$ in $\Lambda$. This is half the squared length of $\lambda_r$. |
| $\theta$ | For $\lambda_r \in \Lambda/2\Lambda$ this is the type of the shortest preimage of $\lambda_r$ in $\Lambda$. |
| $\tau$ | Cocycle of the Parker loop $\mathcal{P}$, with $(\tilde{d}, 0) \cdot (\tilde{e}, 0) = (\tilde{d} + \tilde{e}, \theta(\tilde{d}, \tilde{e}))$. |
| $U(v)$ | Triality element, a generator of $\tilde{\mathbb{M}}$ in $\Gamma$, which is in $N_0 \setminus N_{20}$. |
| $U_4(v)$ | Subset of $U(\mathcal{V})$ containing the vectors in $U(v)$ of type 4. |
| $v_{1}$ | A vector in $\rho_{15}$ fixed by the neutral element of $\mathbb{M}$ only. Any unknown $g \in G_{x_0}$ can be constructed from $v_{1}$ as a word in $\Gamma^*$. |
| $v^+, v^-$ | Axes of the 2A involutions $x_\beta, x_{-\beta}$ in the representation $\rho$. |
which are orthogonal except when equal or opposite; see e.g. \([4]\), \([9]\).

where \(\Lambda\)

we may assume

vector. E.g. a vector of shape (3

vectors of type 2, 3, and 4 in \(\Lambda\). The table at Lemma 4.6.1 in \([9]\) assigns a name and one

/ co-ordinates with absolute value 1.

also assign a subtype (which is a 2-digit number) to each orbit. The first digit of the subtype specifies the type of the vectors in that orbit and the second digit is used to distinguish between orbits of the same type. Hints for memorizing the second digit are given in Table \([5]\).

A Mapping a type-4 vector in \(\Lambda/2\Lambda\) to the standard frame

In this appendix we show how to find an element \(g\) of the group \(Gx_0\) defined in Section \(4.2\) that maps an arbitrary type-4 vector \(\lambda_r\) in \(\Lambda/2\Lambda\) to the standard frame \(\lambda_0\) in \(\Lambda/2\Lambda\). The normal subgroup \(Qx_0\) of \(Gx_0\) of structure \(2^{1+24}\) operates trivially on \(\Lambda/2\Lambda\). Hence the group \(Gx_0\) of structure \(2^{1+24}\) acts on \(\Lambda/2\Lambda\) in the same way as \(Co_1\) acts as an automorphism group on \(\Lambda/2\Lambda\). So we may work in the factor group \(Co_1\) of \(Gx_0\); and we may assume that \(Co_1\) is generated by \(y_d, x_\tau, \xi\), as defined in Section \(3.3\). Taking these generators modulo \(Qx_0\), we may assume \(d \in C/\{1, \Omega\}, \pi \in M_{24}\). So given a \(\lambda_r \in \Lambda/2\Lambda\) of type 4, it suffices to find an element of \(Co_1\) (represented as a word in the generators \(y_d, x_\tau,\) and \(\xi\)) that maps \(\lambda_r\) to \(\lambda_0\).

Modulo \(Qx_0\), the group generated by \(y_d, x_\tau\) is a maximal subgroup \(Nx_0 = N_{x_0}/Q_{x_0}\) of \(Co_1\) of structure \(2^{1+24}\). From the discussion in Section \(1.2\) we see that \(Nx_0\) is the stabilizer of \(\lambda_0\) in \(Co_1\). The action of \(Co_1\) (or of \(N_{x_0}\)) on the Leech lattice \(\Lambda\) is defined up to sign.

In the sequel we will describe the orbits of \(N_{x_0}\) on \(\Lambda/2\Lambda\).

The orbits of the group \(N_{x_0}\) on the vectors of type 2, 3, and 4 on \(\Lambda\) have been described in \([7]\), Lemma 4.4.1. \(N_{x_0}\) acts monomially on the Leech lattice \(\Lambda\). Thus an orbit \(\lambda_r N_{x_0}\) of \(N_{x_0}\) on \(\Lambda/2\Lambda\) can be described by the \(\text{shapes}\) of the vectors in \(\Lambda(\lambda_r)\) for any \(\lambda_r \in \lambda_r N_{x_0}\), where \(\Lambda(\lambda_r)\) is the set of the shortest preimages of \(\lambda_r\) in the Leech lattice as in Section \(4.1\). Here the shape is the multiset of the absolute values of the co-ordinates of the vector. E.g. a vector of shape \((3^51^9)\) has 5 co-ordinates with absolute value 3 and 19 co-ordinates with absolute value 1.

A vector of type 2 or 3 in \(\Lambda/2\Lambda\) has two opposite preimages of the same type in \(\Lambda\); so its shape is uniquely defined. A vector of type 4 in \(\Lambda/2\Lambda\) has 2 \(\cdot 24\) preimages of type 4 in \(\Lambda\) which are orthogonal except when equal or opposite; see e.g. \([3]\), \([9]\).

The table at Lemma 4.4.1 in \([7]\) assigns a name and a shape to each orbit of \(N_{x_0}\) on the vectors of type 2, 3, and 4 in \(\Lambda\). The table at Lemma 4.6.1 in \([9]\) assigns a name and one or more shapes to each orbit of \(N_{x_0}\) on the vectors of type 4 in \(\Lambda/2\Lambda\). We reproduce this information for the orbits of \(N_{x_0}\) on \(\Lambda/2\Lambda\) in Table \([5]\). In column \(\text{Subtype}\) of that table we also assign a subtype (which is a 2-digit number) to each orbit. The first digit of the subtype specifies the type of the vectors in that orbit and the second digit is used to distinguish between orbits of the same type. Hints for memorizing the second digit are given in Table \([5]\).

### Table: Symbols and Descriptions

| Symbol | Description | Section |
|--------|-------------|---------|
| \(x_{-1}\) | The unique central involution in group \(Qx_0\) and in group \(Gx_0\). | \(4.1\) |
| \(x_\beta, x_{-\beta}\) | Fixed involutions in \(Qx_0\), both in class \(2A\) in \(M\); \(\lambda(x_{\pm\beta}) = \lambda_\beta\). | \(7.1\) |
| \(x_d, x_\tau, x_\Omega\) | Generators of \(N_{x_0}\) in \(\Gamma\), for \(d \in \mathcal{P}, \delta \in C^*, \pi \in \text{Aut}_\mathbb{R}\mathcal{P}\). | \(3.5\) |
| \(x_Q, x_{-\Omega}\) | Fixed involutions in the group \(Qx_0\), with \(\lambda(x_{\pm\Omega}) = \lambda_\Omega\). | \(3.4\) |
| \(\xi\) | A generator of \(M\) in \(\Gamma\), which is in \(G_{x_0} \setminus N_{x_0}\). | \(5\) |
| \(y_d\) | A generator of \(N_{x_0}\) in \(\Gamma\), for \(d \in \mathcal{P}\). | \(3.3\) |
| \(1_\rho\) | The unit matrix in the representation \(300_2\); we have \(1_\rho = 1_x\). | \(6\) |
| \(1_x\) | Trivial representation of \(M\) or of \(G_{x_0}\); we have \(1_x \subset 300_x\). | \(6\) |
| \(24_\tau\) | Natural representation of \(Co_1\) as the automorphism group of \(\Lambda\). | \(5\) |
| \(300_2\) | Rational representation of \(G_{x_0}\), subspace of \(\rho\), isomorphic to the space of real symmetric \(24 \times 24\) matrices; we have \(\Lambda \subset \mathbb{R}^{24}\). | \(5\) |
| \(4096_x\) | A representation such that \(24_\tau \otimes 4096_\xi\) is a representation of \(G_{x_0}\). | \(1.3\) |
| \(9820_\tau\) | A monomial rational representation of \(G_{x_0}\), subspace of \(\rho\). | \(6\) |
| \(196883_x\) | Minimal real faithful representation of \(M\), with \(\rho = 1_\tau \oplus 196883_\xi\). | \(5\) |
| \(\tilde{d}\) | Image of \(d \in \mathcal{P}\) in \(C\) under the natural homomorphism \(\mathcal{P} \to C\). | \(8.2\) |
| \(|S|, |\delta|\) | Cardinality of a finite set \(S\); minimum weight of cocode word \(\delta\). | \(9.1\) |
| \(\|M\|\) | Sum of the squares of the entries of a symmetric matrix \(M\). | \(5.5\) |
| \(\langle \ldots \rangle\) | The scalar product, e.g. on \(C \times C^*, \Lambda \times \Lambda\), or on \(\rho \times \rho\). | \(3.4\) |
| Subtype | Name in [9] | Shape | $|d|$ | $|\delta|$ | $\langle d, \delta \rangle$ | Remark |
|---------|-------------|-------|------|---------|----------------|--------|
| 00      | $A_2^2$     | $(0^01^02^0)$ | 0    | 0       | 0              |        |
| 20      | $A_2^3$     | $(4^01^01^02^02^0)$ | 0,24 | 2       | 0              |        |
| 21      | $A_2^3$     | $(3^01^01^02^02^0)$ | 0    | 1       | $|d|/4$         |        |
| 22      | $A_2^5$     | $(2^01^01^02^02^0)$ | 0,16 | even    | 0              | 1.     |
| 31      | $A_2^3$     | $(4^01^01^02^02^0)$ | 0    | 1       | $|d|/4+1$       |        |
| 33      | $A_2^3$     | $(3^01^01^02^02^0)$ | 0    | 3       | $|d|/4$         |        |
| 34      | $A_2^5$     | $(4^01^01^02^02^0)$ | 0,16 | even    | 1              |        |
| 36      | $A_2^3$     | $(2^01^01^02^02^0)$ | 0    | 1       |                 |        |
| 40      | $A_2^4$     | $(4^01^01^02^02^0)$ | 0,24 | 4       | 0              |        |
| 42      | $A_2^4$     | $(2^01^01^02^02^0)$ | 0,16 | even    | 0              | 2.     |
| 43      | $A_2^3$     | $(5^01^01^02^02^0)$ | 0    | 3       | $|d|/4+1$       |        |
| 44      | $A_2^4$     | $(4^01^01^02^02^0)$ | 0,16 | even    | 0              | 3.     |
| 46      | $A_2^4$     | $(3^01^01^02^02^0)$ | 0    | 1       |                 |        |
| 48      | $A_2^4$     | $(8^01^01^02^02^0)$ | 0    | 0       |                 |        |

Remarks:

1. $|\delta|/2 = 1 + |d|/8 \pmod 2$, $\delta' \subset d\Omega^{1+|d|/8}$ for a suitable representative $\delta'$ of the cocode element $\delta$ in $\mathbb{F}_2$.  
2. $|\delta|/2 = |d|/8 \pmod 2$, $\delta' \subset d\Omega^{1+|d|/8}$ for a suitable representative $\delta'$ of the cocode element $\delta$ in $\mathbb{F}_2$.
3. None of the conditions stated in Remarks 1 and 2 holds.

Table 4: Orbits of $N_{x_0}$ of $\Lambda/2\Lambda$

For $d \in \mathcal{P}, \delta \in \mathbb{C}^*$ let $x_d, x_\delta \in Q_{x_0}$ as in Section 3.3. Put $\lambda_d = \lambda(x_d) \in \Lambda/2\Lambda$, $\lambda_\delta = \lambda(x_\delta) \in \Lambda/2\Lambda$, as in Section 4.1. Then $\lambda_d$ is well defined also for $d \in \mathbb{C}$, and each vector $\lambda_\delta \in \Lambda/2\Lambda$ has a unique decomposition $\lambda_\delta = \lambda_d + \lambda_\delta, d \in \mathbb{C}, \delta \in \mathbb{C}^*$. The subtype of a vector $\lambda_d + \lambda_\delta \in \Lambda/2\Lambda$ can be computed from $d$ and $\delta$, as indicated in the Table 4.

Columns Name and Shape in Table 4 list the names and the shapes of the orbits as given in [9], Lemma 4.1.1 and 4.6.1. Columns $|d|$ and $|\delta|$ list conditions on the weight of a Golay code word $d$ and of (a shortest representative of) the Golay cocode element $\delta$, respectively. Column $\langle d, \delta \rangle$ lists conditions on the scalar product of $d$ and $\delta$. The information in the four rightmost columns of Table 4 can be derived by using (4.0.1). Conditions for vectors of type 2 are also given in [2].

Table 4 provides the information required for effectively computing the subtype of an element $\lambda_d + \lambda_\delta$ from $d$ and $\delta$.

From the generators $y_d, x_\delta$, and $\xi$ of $\text{Co}_1$ mentioned above, only the generators $\xi^{\pm 1}$ may change the subtype of a vector in $\Lambda/2\Lambda$. We say that a vector $v \in \mathbb{Z}^n$ is of shape $(m^\alpha 0^{n-\alpha} \pmod 2m)$ if $v$ has $\alpha$ co-ordinates equal to $m \pmod 2m$, and $n-\alpha$ co-ordinates equal to $0 \pmod 2m$. For a vector $v = (v_0, \ldots, v_{23}) \in \mathbb{R}^{24}$ define $(v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3}) \in \mathbb{R}^4$ to be the $i$-th column of $v$. This definition is related to the Miracle Octad Generator (MOG) used for the description of the Golay code, see [4], Ch. 11. The co-ordinates of a vector in the Leech lattice $\Lambda$ may be labelled with the entries of the MOG. Using this MOG labelling, we will depict the co-ordinates of a vector $v \in \Lambda \subset \mathbb{Z}^{24}_2$ as follows:

$$v = (v_0, \ldots, v_{23}) = \begin{pmatrix} v_0 & v_4 & v_8 & v_{12} & v_{16} & v_{20} \\ v_1 & v_5 & v_9 & v_{13} & v_{17} & v_{21} \\ v_2 & v_6 & v_{10} & v_{14} & v_{18} & v_{22} \\ v_3 & v_7 & v_{11} & v_{15} & v_{19} & v_{23} \end{pmatrix}.$$
| Subtype | Meaning |
|---------|---------|
| x0      | Orbit contains a vector $\lambda_d + \lambda_3$ with $|d| = 0$; $|\delta|$ even. |
| x1      | Vectors $\lambda_d + \lambda_3$ with $|\delta| = 1$. |
| x2      | Vectors $\lambda_d + \lambda_3$ with $|d| = 8, 16$; $|\delta|$ even; and Remark 1 or 2 in Table 4 applies. |
| x3      | Vectors $\lambda_d + \lambda_3$ with $|\delta| = 3$. |
| x4      | Vectors $\lambda_d + \lambda_3$ with $|d| = 8, 16$; $|\delta|$ even; and none of the Remarks 1 or 2 in Table 4 apply. |
| x6      | Vectors $\lambda_d + \lambda_3$ with $|d| = 12$; $|\delta|$ even. |
| x8      | Reserved for the standard frame $\lambda_0$. |

Table 5: Subtypes of orbits of $N_{x_0}$ of $\Lambda/2\Lambda$

The following lemma provides some more information about the operation of $\xi^k$ on $\Lambda$.

**Lemma A.1.** Let $v \in \Lambda \subset \mathbb{Z}^{24}$, and let $w$ be a column of $v$ in the MOG. Let $w^{(k)}$ be the corresponding column of $v \cdot \xi^k$. Then $w^{(k)}$ depends only on $w$, and the squared sums of the entries of $w$ and $w^{(k)}$ are equal. If $w$ has shape $(m^4 \text{ mod } 2m)$ then there is a unique $k \in \{\pm 1\}$ such that $w^{(k)}$ has shape $(0^4 \text{ mod } 2m)$. If $w$ has shape $(m^20^2 \text{ mod } 2m)$ then $w^{(k)}$ has shape $(m^20^2 \text{ mod } 2m)$ for $k = \pm 1$.

**Sketch proof**

In Equation (5.0.1) the operation of $\xi$ on a column of the MOG is given as a product of two orthogonal symmetric $4 \times 4$ matrices $A$ and $B$. Thus $A$ and $B$ are involutions, and $\xi^{-1}$ acts as multiplication by $BA$ on a column. Note that multiplication of a column with $-A$ is equivalent to subtracting the halved sum of the column from each entry, and multiplication with $B$ means negating the first entry. So it is easy to show the stated properties of the action of $\xi^\pm 1$ on columns of shape $(m^4 \text{ mod } 2m)$ and $(m^20^2 \text{ mod } 2m)$. $\square$

We apply Lemma A.1 to vectors $v \in \Lambda$ of type 4. Here the entries of the MOG columns of $v$ are either all even or all odd; and they are usually quite small. So the shape of a column of $v$ imposes rather severe restrictions on the shapes of the corresponding column of $v\xi^\pm$. In the cases discussed below this allows to infer the subtype of $v\xi$ or $v\xi^{-1}$ from the shapes of the columns of $v$.

In the sequel we apply an operation $x_\pi\xi^k$ to a vector $\lambda_r$ in $\Lambda/2\Lambda$ of a given subtype 4$\chi$, so that the subtype is changed as in Figure 4. The graph in that figure is a subgraph of the Leech graph in [9], Section 4.7. By a sequence of such operations we may eventually map an arbitrary vector in $\Lambda/2\Lambda$ to the (unique) vector $\lambda_0$ of subtype 48. For the following discussion of the individual subtypes we assume that the reader is familiar with the Mathieu group $M_{24}$, the Golay Code $C$, and the MOG, as presented in [4], Ch. 11.

![Figure 4: Reduction of the orbits of $N_{x_0}$ on type-4 vectors in $\Lambda/2\Lambda$](image)

In the following discussion the phrase *up to MOG permutations* means up to permutations of the columns of the MOG and arbitrary permutations inside a column of the MOG.

**From subtype 46 to subtype 44**

Let $\lambda_r = \lambda_d + \lambda_3$ be of subtype 46. Then $d$ is a dodecad. We first assume that $d$ contains...
one column of the MOG.

From the discussion of the MOG in [11] we see that the union of two different columns of
the MOG is an octad, and that the intersections of \( d \) with all columns must have the same
size (modulo 2). Since a dodecad may not contain an octad, \( d \) must intersect four columns
of the MOG in a set of size 2; and its intersection with the remaining column is empty.

Note that each vector in \( \Lambda(\lambda_r) \) has exactly one co-ordinate \( \pm 4 \); and such a co-ordinate
may occur at all 24 positions, when considering all 48 vectors in \( \Lambda(\lambda_r) \).

Thus up to signs and MOG permutations there is a vector \( v \in \Lambda(\lambda_r) \) with co-ordinates

\[
\begin{array}{ccccccc}
2 & 0 & 2 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 \\
2 & 0 & 4 & 2 & 2 & 2 \\
2 & 0 & 0 & 2 & 2 & 2 \\
\end{array}
\]

By Lemma [A.1] there is a \( k = \pm 1 \) such that column 0 of the vector \( w = v \cdot \xi^k \) has shape
\((4^02^00^0)\). The other columns of \( w \) have the same shape as the corresponding columns of \( v \). Thus
\( w \) has shape \((4^12^20^1)\). So from Table [3] we see that \( w + 2\Lambda \) has subtype 44.

If dodecad \( d \) does not contain a column of the MOG then we select a permutation in \( M_{24} \)
that maps four entries of dodecad \( d \) to the subset \( \{0, 1, 2, 3\} \) of \( \tilde{\Omega} \), i.e. to the first column of
the MOG. This is possible, since \( M_{24} \) is quintuply transitive. Then we proceed as above.

From subtype 43 to subtype 42

Let \( \lambda_r = \lambda_d + \lambda_\delta \) be of subtype 43. Then \( |\delta| = 3 \). The three elements of (the shortest
representative of) \( \delta \) can lie in one, two, or three different columns of the MOG.

If all entries of \( \delta \) lie in the same column of the MOG then up to signs and MOG permuta-
tions, there is a vector \( v \in \Lambda(\lambda_r) \) with co-ordinates

\[
\begin{array}{ccccccc}
5 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

By Lemma [A.1] there is a \( k = \pm 1 \) such that one column of \( w = v \cdot \xi^k \) (and hence all
columns) have even entries. Column 0 of \( w \) has squared norm 44 = \( 6^2 + 2^2 + 2^2 + 0^2 \). That
decomposition of 44 into a sum of four even squares is unique, so column 0 has shape \((6^12^20^3)\).
The other columns have shape \((2^10^3)\). So \( w \) is of shape \((6^12^20^115^2)\); and hence \( \lambda_r \xi^k = w + 2\Lambda \)
is of subtype 42.

If the entries of \( \delta \) lie in two or three different columns of the MOG then we apply a
permutation in \( M_{24} \) that maps \( \delta \) to \( \{0, 1, 2\} \), and proceed as above.

From subtype 44 to subtype 40

Let \( \lambda_r = \lambda_d + \lambda_\delta \) be of subtype 44. Then \( d \) is an octad or a complement of an octad. We
call that octad \( o \). If the cocode word \( \delta \) is a duad (i.e. \( |\delta| = 2 \)) then we have \( \delta = \{c_0, c_1\} \)
and \( o \cap \{c_0, c_1\} = \{\} \). Otherwise, \( \delta \) corresponds to a sextet containing a tetrad \( c \) intersecting
with \( o \) in two points; in that case we let \( c_0, c_1 \) be the two elements of \( c \) not contained in \( o \).

Assume first that \( o \) is a union of two columns of the MOG, and that the points \( c_0, c_1 \) are
in the same column of the MOG.

Then up to signs and MOG permutations there is a vector \( v \in \Lambda(\lambda_r) \) with co-ordinates

\[
\begin{array}{ccccccc}
2 & 2 & 4 & 0 & 0 & 0 \\
2 & 2 & 4 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 \\
\end{array}
\]

Similar to the previous cases, we can show that there is a \( k = \pm 1 \) such that \( w = v \cdot \xi^k \)
has shape \((4^40^20^9)\). So \( w \) is of subtype 40.
Assume now that \( o \) is any octad and that none of the two points \( c_0, c_1 \in \tilde{\Omega} \) is in \( o \). In the sequel we will construct a permutation \( \pi \in M_{24} \) that maps \( o \) to a union of two columns of the MOG and \( (c_0, c_1) \) to \( (2,3) \). Then we may proceed with \( \pi(o) \) as above.

For constructing \( \pi \) we select three elements \( o_0, o_1, o_3 \) from \( o \); and we compute the syndrome \( \sigma \) of the set \( \{c_0, c_1, o_0, o_1, o_2\} \), which is a cocode word of length 3. \( \sigma \) intersects with \( o \) in exactly one point \( o_3 \). Then we construct a permutation in \( M_{24} \) that maps the tuple \( (c_0, c_1, o_0, o_1, o_2, o_3) \) to the tuple \( (2,3,4,5,6,7) \). Such a permutation exists, since both tuples are subsets of an octad, \( M_{24} \) is transitive on octads, and the stabilizer of an octad in \( M_{24} \) acts as the alternating permutation group on that octad. See [1] for proofs of these statements. This permutation maps \( \{c_0, c_1\} \) to \( (2,3) \); and it maps \( o \) to an octad containing the column \( \{4,5,6,7\} \) of the MOG. Any octad containing one column of the MOG consists of two columns of the MOG.

From subtype 42 to subtype 40

Let \( \lambda_r = \lambda_d + \lambda_\delta \) be of subtype 42. Then \( d \) is an octad or a complement of an octad. We call that octad \( o \).

Assume first that \( o \) is a union of two columns of the MOG. Then up to signs and MOG permutations there is a vector \( v \in \Lambda(\lambda_r) \) with co-ordinates

\[
\begin{array}{cccccc}
6 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Similar to the previous cases, we can use Lemma \( \Lambda.1 \) to show that there is a \( k = \pm 1 \) such that \( w = v \cdot \xi^k \) has shape \( (4^40^02^0) \). So \( w \) is of subtype 40.

Otherwise we first map the octad \( o \) to the octad \( \{0,1,2,3,4,5,6,7\} \), using a permutation in \( M_{24} \).

From subtype 40 to subtype 48

Let \( \lambda_r \) be of subtype 40. Then \( \lambda_r = \alpha \lambda_\Omega + \lambda_\delta \), \( \alpha = 0,1 \), for some \( \delta \in \mathbb{C}^\ast \), \( |\delta| = 4 \).

If \( \delta \) is equal to the standard tetrad represented by \( \{0,1,2,3\} \), then a column of a vector \( v \in \Lambda(\lambda_r) \) in the MOG has shape \( (4^4) \). By Lemma \( \Lambda.1 \) there is a \( k = \pm 1 \) such that \( \xi^k \) maps this column to a column of shape \( (8^30^02^0) \). Thus \( \lambda_r \xi^k \) is of subtype 48 and hence equal to \( \lambda_\Omega \).

Otherwise we first apply a permutation in \( M_{24} \) that maps a tetrad in \( \delta \) to \( \{0,1,2,3\} \).

B The probability that the kernel of a random symmetric matrix on \( \Lambda/3\Lambda \) is spanned by a type-4 vector

We first calculate the probability \( p_{n,k}^{(q)} \) that a symmetric \( n \times n \) matrix with random entries in \( \mathbb{F}_q \) has corank \( k \). For any symmetric \( n \times n \) matrix \( Q_n \) of corank \( k \) we can find a basis of the underlying vector space such that \( Q_n \) has shape

\[
Q_n = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \quad \text{where } A \text{ is a non-singular symmetric } (n-k) \times (n-k) \text{ matrix.}
\]

Let \( Q_{n+1} \) be the symmetric \( (n+1) \times (n+1) \) matrix obtained from \( Q_n \) by adding one row and one column:

\[
Q_{n+1} = \begin{pmatrix} 0 & 0 & b_0^\top \\ 0 & A & b_1^\top \\ b_0 & b_1 & c \end{pmatrix}, \quad \text{with } b_0 \in \mathbb{F}_q^k, \ b_1 \in \mathbb{F}_q^{n-k}, \ c \in \mathbb{F}_q.
\]
Matrix $Q_{n+1}$ has corank $k - 1$ if $b_0 \neq 0$. In case $b_0 = 0$ it has corank $k + 1$ if $c = b_1 A^{-1} b_1^T$ and corank $k$ otherwise. Assuming that $b_0, b_1, c$ are selected at random this implies:

$$\text{corank}(Q_{n+1}) = \begin{cases} k - 1 & \text{if } b_0 \neq 0 \\ k & \text{if } b_0 = 0 \end{cases} \quad \text{with probability} \quad \begin{cases} 1 - q^{-k} & \text{if } b_0 \neq 0 \\ q^{-k} - q^{-(k+1)} & \text{if } b_0 = 0 \end{cases} \quad \text{for } k > 0.$$

$q^{-k}$ is spanned by a type-4 vector with probability about 1/3. Generating a vector $k$ a call that octad $\lambda$ from subtype 22 to subtype 20 fixing an element of order 71, we may also check $\ker M$ and it is easy to see that $\lambda$ has corank $k$ and $\beta$ has shape (3,4,1,0). Hence for fixed $q$ the distribution of $p^{(q)}_{n,k}$, $0 \leq k \leq n$, can be modelled as a Markov chain with discrete time parameter $n$ and start value $p^{(q)}_{0,0} = 1$. So we easily obtain $p^{(q)}_{22,0} \approx 0.31950$.

There are 398034000 vectors of type 4 in the Leech lattice $\Lambda$, see e.g. [4] Ch.4.11, and $2^{24} - 1$ nonzero vectors in $\Lambda/3\Lambda$, so a random one-dimensional subspace of $\Lambda/3\Lambda$ is spanned by a type-4 vector with probability about 1/709.56. Thus the kernel of a random symmetric $24 \times 24$ matrix over $\mathbb{F}_3$ (acting on $\Lambda/3\Lambda$) is spanned by a type-4 vector with probability about 1/2221.

Let $M(\cdot)$ be as in Section 6. For $v \in \rho_3$ let $M_3(v)$ be the matrix $M(v)$ with entries taken modulo 3. Generating a vector $v$ in a vector of order 71 costs 70 group operations on $\rho_3$. So in average we need about 155000 group operations on $\rho_3$ to find a such vector $v \in \rho_3$ with the additional property that $M_3(v)$ is spanned by a type-4 vector (modulo 3). This is certainly doable, and has been done. Once having found a vector $v$ fixing an element of order 71, we may also check $\ker M_3(v \pm 1_\rho)$.

So we can find a vector $v$ satisfying the properties mentioned above, such that $\ker M_3(v)$ is spanned by $w + 3\Lambda$, for a type-4 vector $w \in \Lambda$. Given the co-ordinates of $w$ (modulo 3), we can easily compute the corresponding type-4 vector in $\Lambda$, and hence also the corresponding type-4 vector $\lambda_r$ in $\Lambda/2\Lambda$. We put $v_1 = v_g$, where $g$ is an element of $G_{x_0}$ that maps the type-4 vector $\lambda_r$ to the standard frame $\Lambda_0$. We use the method in Appendix A for computing a suitable $g \in G_{x_0}$. Then $\ker M_3(v_1)$ is spanned by $\eta_j + 3\Lambda$ for some basis vector $\eta_j$ of $\mathbb{R}^{24}$.

## C Mapping a type-2 vector in $\Lambda/2\Lambda$ to the standard type-2 vector $\lambda_\beta$

In this appendix we show how to find an element $g$ of $G_{x_0}$ that maps an arbitrary type-2 vector $\lambda_r$ in $\Lambda/2\Lambda$ to the standard type-2 vector $\lambda_\beta$ in $\Lambda/2\Lambda$.

We use the same notation as in Appendix A. We apply an operation $x_{\pi}^k$ to a vector in $\Lambda/2\Lambda$ of a given subtype 2X, so that the subtype is changed as in Figure 5. Finally, we apply an operation in $N_{x_0}$ to a vector of subtype 20 in order to map that vector to $\lambda_\beta$.

![Diagram of orbits reduction](image)

**Figure 5:** Reduction of the orbits of $\tilde{N}_{x_0}$ on type-2 vectors in $\Lambda/2\Lambda$.

### From subtype 21 to subtype 22

Let $\lambda_r$ be of subtype 21. Then $\Lambda(\lambda_r)$ is of shape $(31123)$. By Lemma A.1 there is an element $\xi^k$ such that $\Lambda(\lambda_r)\xi^k$ has even co-ordinates. Then $\lambda_r\xi^k$ cannot be of subtype 21; and it is easy to see that $\lambda_r\xi^k$ is of subtype 22.

### From subtype 22 to subtype 20

Let $\lambda_r = \lambda_d + \lambda_\beta$ be of subtype 22. Then $d$ is an octad or a complement of an octad. We call that octad $o$. Then $\Lambda(\lambda_r)$ has shape $(280180)$. Assume first that $o$ is a union of two columns of the MOG. Then by Lemma A.1 there is a $k = \pm 1$ such that $\Lambda(\lambda_r)\xi^k$ has shape $(42022)$. So $\lambda_r\xi^k$ is of subtype 40.
Otherwise we first map the octad \( o \) to the octad \{0, 1, 2, 3, 4, 5, 6, 7\}, using a permutation in \( M_{24} \).

**From subtype 20 to the vector \( \alpha \beta \)**

Let \( \lambda_r \) be of subtype 20. Then \( \lambda_r = \alpha \lambda_\Omega + \lambda_\delta \), \( \alpha \in \mathbb{F}_2 \), for some \( \delta \in C^* \), \( |\delta| = 2 \). So there is an \( x_\pi \) that maps \( \delta \) to \( \beta \); i.e. \( \lambda_r x_\pi = \alpha' \lambda_\Omega + \lambda_\beta \). In case \( \alpha' = 0 \) we are done. Otherwise we apply a transformation \( y_d \) with \( (d, \beta) = 1 \). So we have \( \lambda_r x_\pi y_d = \lambda_\beta \).

**D  Mapping a feasible type-2 vector in \( \Lambda/2\Lambda \) to the type-2 vector \( \lambda_\Omega + \lambda_\beta \), fixing the standard type-2 vector \( \lambda_\beta \)**

In this appendix we show how to find an element \( g \) of \( H \) that maps an arbitrary feasible type-2 vector \( \lambda_r \) in \( \Lambda/2\Lambda \) to the standard feasible vector \( \lambda_\Omega + \lambda_\beta \) in \( \Lambda/2\Lambda \). Here the term *feasible* is defined as in Section 9.1.

We use the same notation as in Appendix A. We apply an operation \( x_\pi \xi^k \), with \( \pi \in H \cap Aut_{S_4} \mathcal{P} \) to a feasible vector in \( \Lambda/2\Lambda \) of a given subtype 2\( X \), so that the subtype is changed as in Figure 5. Note that \( H \cap Aut_{S_4} \mathcal{P} \) operates on set \( \Omega = \{0, \ldots, 23\} \) as the subgroup of the permutation group \( M_{24} \) fixing the set \( \beta = \{2, 3\} \). Finally, we apply an operation in \( H \) to a feasible vector of subtype 20 in order to map that vector to \( \lambda_\Omega + \lambda_\beta \).

**From subtype 21 to subtype 22**

The same argument as in the corresponding case in Appendix C shows that there is a \( k = \pm 1 \) such that \( \lambda_r \xi^k \) is of subtype 22. Since \( \xi \in H \), the vector \( \lambda_r \xi^k \) is feasible.

**From subtype 22 to subtype 20**

Let \( \lambda_r = \lambda_d + \lambda_\delta \) be of subtype 22. Then \( d \) is an octad or a complement of an octad. We call that octad \( o \). Then \( \Lambda(\lambda_r) \) has shape \((2^301^4)\). Since \( \lambda_r \) is feasible, the set \( \{2, 3\} \) is either contained in or disjoint to the octad \( o \); otherwise we would have type(\( \lambda_r + \lambda_\beta \)) \( \neq 4 \).

Assume first that \( o \) is a union of two columns of the MOG. Then by Lemma A.1 there is a \( k = \pm 1 \) such that \( \Lambda(\lambda_r) \cdot \xi^k \) has shape \((4^202^2)\). So \( \lambda_r \xi^k \) is of subtype 20.

Otherwise, if the set \( \{2, 3\} \) is contained in \( o \) then we first map the octad \( o \) to the octad \( \{0, 1, 2, 3, 4, 5, 6, 7\} \), using a permutation in \( M_{24} \) that fixes the set \( \{2, 3\} \). Then we proceed as above.

If the set \( \{2, 3\} \) is disjoint from \( o \) then we construct a permutation \( \pi \in M_{24} \) that maps octad \( o \) to a union of two columns of the MOG, and that fixes the tuple \( (2, 3) \). Therefore we use the method in Appendix A in the case 'From subtype 44 to subtype 40'. Then we proceed as above.

**From subtype 20 to the vector \( \lambda_\Omega + \lambda_\beta \)**

Let \( \lambda_r \) be feasible and of subtype 20. Then \( \lambda_r = \alpha \lambda_\Omega + \lambda_\delta \), \( \alpha = 0, 1 \), for some \( \delta \in C^* \), \( |\delta| = 2 \). In case \( \delta = \beta \) we have \( \lambda_r = \lambda_\Omega + \lambda_\beta \), so that we are done.

Otherwise both elements of \( \delta \) are different from 2 and 3. In case \( \delta = \{0, 1\} \) we have \( (-1)^{\alpha+1} \cdot 4\eta_0 + 4\eta_1 \in \Lambda(\lambda_r) \); and a direct calculation using (5.0.1) shows \( \lambda_r \xi^{2-\alpha} = \lambda_\Omega + \lambda_\beta \).

In case \( \delta \neq \{0, 1\} \) we first apply a permutation in \( M_{24} \) that maps \( \delta \) to \( \{0, 1\} \) and fixes \( \beta = \{2, 3\} \). Then we proceed as in case \( \delta = \{0, 1\} \).
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