Quantum Computing in the Presence of Spontaneous Emission By a Combined Dynamical Decoupling and Quantum Error Correction Strategy

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A new method for quantum computation in the presence of spontaneous emission is proposed. The method combines strong and fast (dynamical decoupling) pulses and a quantum error correcting code that encodes $n$ logical qubits into only $n+1$ physical qubits. Universal, fault-tolerant, quantum computation is shown to be possible in this scheme using Hamiltonians relevant to a range of promising proposals for the physical implementation of quantum computers.

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I. INTRODUCTION

Decoherence [1] remains the most daunting obstacle to the realization of quantum information processing, coherent control, and other applications requiring a high degree of quantum coherence. As quantum computation (QC) moves into the experimental realm it becomes increasingly important to design methods for overcoming this main obstacle to realization, that are tailored to particular systems and the resulting errors that afflict them. Here we show how to perform universal, fault-tolerant QC in the presence of decoherence due to spontaneous emission (SE). Since SE is a consequence of the inevitable coupling to the vacuum field, it cannot be "engineered away" and must eventually be dealt with, in all QC proposals. Several methods have been designed to this end, that may roughly be classified as "hardware" and "software": In the former category are proposals to construct quantum computers in materials where SE is strongly suppressed, e.g., placing atomic qubits in a photonic band-gap structure [2]. In the latter category are various error correction, avoidance, and suppression methods [4, 5, 6, 7, 8, 9, 10]. With the exception of the $2\pi$ pulsing method of [10], a unifying theme of these methods is to place the system under continuous observation. It is then well known that the Markovian quantum master equation can be unravelled into a set of quantum trajectories, consisting of a conditional evolution (governed by a non-Hermitian conditional Hamiltonian $H_c$, defined below), randomly interrupted by quantum jumps (wave-function collapse) into different observed decay channels [11, 12, 13, 14]. The time evolution conditional to a given set of time-ordered observations is called "a posteriori dynamics" [15], and is not Markovian. The continuous observation can lead to a Zeno-effect type suppression of decoherence, a fact that was exploited in [2], in conjunction with an encoding into a decoherence-free subspace (DFS) [16, 17], in order to resist SE. Quantum error correcting codes (QECCs) can correct both the conditional evolution and the jumps [3], but more efficient constructions are possible when one considers subspaces of the full system's Hilbert spaces that are invariant under the conditional evolution. It is then necessary to correct only the errors arising due to the quantum jumps [4, 5, 6, 7, 8]. The first proposal along these lines, [4], did not consider QC. A simple, but non fault-tolerant QC scheme, encoding a logical qubits into two physical qubits (four atomic levels), tailored to SE of phonons in trapped-ion QC, was subsequently presented in [5]. A QECC correcting one arbitrary single-qubit error and invariant under $H_c$ was given in [6], using an encoding of one logical qubits into 8 physical qubits. When one makes the assumption that the qubit undergoing the quantum jump can be identified ("detected-jump"), a more efficient encoding is possible. A family of such "detected-jump codes" (DJC) was first developed in [7], using a DFS to construct a subspace invariant under $H_c$. In [8] we showed how to perform fault-tolerant universal QC on a subclass of such codes encoding $n-1$ logical qubits into $2n$ physical qubits.

Here we present a new method for reducing and correcting SE errors. Rather than constructing a code subspace invariant under $H_c$, we dynamically eliminate $H_c$ by applying dynamical decoupling (or "bang-bang", BB) pulses [18, 19]. We then construct a QECC that deals with the remaining jump errors, under the detected-jump assumption. The advantage of this method compared to the previous methods using encoding is that it is significantly more economical in qubit and pulse timing resources: It uses a QECC in which $n$ logical qubits are encoded into only $n+1$ physical qubits; and, while in [10] the pulse interval has to satisfy the standard BB condition of being shorter than the inverse of the bath high-frequency cutoff [8, 10], in our case the requirement is that the pulses are faster than the average time between photon emission events, which can be orders of magnitude longer. Furthermore, our method is fully compatible with universal QC using Hamiltonians that are naturally available in a large variety of QC proposals [20], so unlike [8] does not rely on one specific architecture.

The idea of using a hybrid BB-encoding approach to
suppress decoherence was first proposed in [21], where it was pointed out that BB is fully compatible with encoding into a QECC or DFS. In particular it was observed that one could use BB to suppress phase-flip errors, thus leaving the QECC with the need only to correct bit-flip errors. However, no method specifically tailored for SE errors was given. An experimental NMR implementation of a hybrid BB-QECC was presented in [22], where decoupling was used to remove coherent scalar coupling between protons (environment) and carbon qubits, together with QECC used to further correct for fast relaxation due to dipolar interactions modulated by random molecular motion.

Clearly, correcting for SE errors is only a part of a general procedure for offsetting decoherence, as additional decoherence sources will inevitably be present in any QC implementation. The methods we present here therefore will have to become part of this more general procedure, either as a first level of defense (in the case that SE is dominant), or at higher levels in a concatenated QECC scheme [23], after other, more dominant errors have been accounted for. The importance of the results presented here lies in the fact that SE is always present and therefore can never be ignored. A code that is optimized with respect to this type of error can potentially offer flexibility and significant savings in resources and overhead.

The structure of this paper is the following. In Section II we show how the conditional evolution during SE can be eliminated using a sequence of simple, global BB decoupling pulses. In Section III we construct a simple and economical QECC that corrects for the remaining quantum jump errors. We address fault tolerance and various imperfections in Section IV. We then show how to quantum compute in a universal and fault tolerant manner over our QECC, using a variety of model Hamiltonians pertinent to a wide class of promising quantum computing proposals. We conclude in Section VI.

II. ELIMINATING THE CONDITIONAL EVOLUTION OF SPONTANEOUS EMISSION WITH BB PULSES

Consider $N$ qubits that can each undergo SE, under the detected-jump assumption. This localizability of the SE events implies that the mean distance between qubits exceeds the wavelength of the emission. Note that this optical distinguishability between qubits does not limit our ability to couple the qubits via non-optical interactions, of the type we consider in Section V below.

The ground and excited states of each qubit are denoted by $|0\rangle$ and $|1\rangle$ respectively. Let $\sigma_i^+ = |0\rangle\langle 1|\sigma_i^+ |0\rangle\langle 1|$ denote the SE error generator acting on the $i$th qubit and let $\kappa_i$ denote the corresponding error rate. We use the quantum trajectories approach [11, 12, 13, 14] to describe the dynamics of the decohering system. The evolution is decomposed into two parts: a conditional non-Hermitian Hamiltonian $H_c$, interrupted at random times by occurrence of random jumps, each corresponding to an observation of decay channels in a quantum optical setting. For errors such as SE, where the jump can be detected by observation of the emission, the quantum trajectories approach also provides us with a way to combine QECCs and BB, in analogy to the way this was done for QECC and DFS in [15, 16]. The BB pulses take care of the conditional evolution, whereas the QECC deals with the random jumps. The conditional Hamiltonian is given in the SE case by $H_c = -\frac{1}{2} \sum_{i=1}^N \kappa_i \sigma_i^+ \sigma_i^-$, where $\sigma_i^+ \equiv (\sigma_i^-)^\dagger$. In [8] we assumed that the environment effectively does not distinguish among the qubits that undergo SEs ($\kappa_i = \kappa$) and the conditional Hamiltonian would then become: $-\frac{1}{2N} \sum_{i,j=1}^N |1\rangle\langle 1|$. This assumption is not necessary in the current work. From here on operators $X_i, Y_i, Z_i$ refer to the corresponding Pauli matrices acting on the $i$th qubit, and $I$ denotes the identity matrix. Now suppose that we apply a collective $X \equiv \otimes_{j=1}^N X_j$ pulse to the system, at intervals $T_c/2 \ll 1/\gamma$, where $\gamma$ is the SE rate. Under this condition, and using $X_i^\sigma_i^+ X_i = \sigma_i^+$ we can write the evolution after a full $T_c$ period as:

$$U = \exp(-i\frac{T_c}{2} H_c) X \exp(-i\frac{T_c}{2} H_c) X = \exp(-i\frac{T_c}{4} \sum_i \kappa_i |1\rangle\langle 1|) \exp(-i\frac{T_c}{4} \sum_i \kappa_i |0\rangle\langle 0|) = \exp(-i\frac{T_c}{4} \sum_i \kappa_i) I,$$

where $I$ is the identity operator. Therefore the decohering effect of the conditional Hamiltonian (that distinguishes states with different numbers of $1$’s) is removed and replaced by an overall shrinking norm. When the jumps are included in the dynamics, the state must be renormalized [11, 12, 13, 14], so this shrinking disappears. Note that we have not eliminated Markovian decoherence using BB pulses, since we have considered only a single trajectory. In fact, a comparison of the coherence $C = \text{Tr}(\rho^2)$ (where $\rho$ is the qubit density matrix) shows that if the results are ensemble-averaged over the a posteriori dynamics (recovering the Markovian master equation), and the jump errors are not corrected, then there is no advantage in using a BB sequence. More specifically, when comparing $C$ for the (1) free and (2) every $T_c/2$-pulsed evolution of a single qubit undergoing SE with rate $\gamma$, we find:

$$C_1 = 1 - \gamma T_c (\beta^2) + O(\gamma^2)$$
$$C_2 = 1 - \gamma T_c (\alpha^2 + \beta^2) + O(\gamma^2)$$

where the initial qubit state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ is normalized: $\alpha^2 + \beta^2 = 1$. Averaging over a random sample of initial states chosen from a uniform distribution (with $\alpha$ and $\beta$ subject to normalization), we have $\langle C_1 \rangle = \langle C_2 \rangle$, so as expected for purely Markovian dynamics, there is no improvement after using just BB pulses.
III. CORRECTING SPONTANEOUS EMISSION JUMPS WITH A QECC

We now introduce a very simple QECC that corrects the remaining part of the decoherence process, the random jumps. Since the error correction process by necessity takes place during the conditional evolution (the jump is instantaneous and the QECC takes time), we must ensure that the QECC keeps its error correcting properties under the conditional Hamiltonian and BB pulses. A minimal example of such a “decoupled-detected jump corrected” code is given by a subspaces of the $N = n + 1$ qubit Hilbert space $\mathcal{C}_n$, spanned by the codewords

$$|x\rangle_L \equiv |x_1, \ldots, x_n\rangle_L = \frac{|x_1, \ldots, x_n, 0\rangle + |x_1, \ldots, x_n, 1\rangle}{\sqrt{2}},$$

where $\{x_i\}$ is the binary representation of the $n$-qubit state $|x\rangle$ and $\overline{F}$ is an inverted $x$ in which 1 and 0 are interchanged. For example, for $n = 2$, the code $C_2$ is (up to normalization by factors of $\sqrt{2}$):

$$|00\rangle_L = |000\rangle + |111\rangle, \quad |01\rangle_L = |010\rangle + |101\rangle, \quad |10\rangle_L = |100\rangle + |011\rangle, \quad |11\rangle_L = |110\rangle + |001\rangle.$$

That $\mathcal{C}_n$ is a QECC against the jump errors follows from the fact that a spontaneous emission error at a given qubit position $i$ eliminates the component of the codeword with 0 in that position, which by construction results in a surjective mapping between the original codewords and the resulting states that are orthogonal to each other. More specifically, the sufficient condition that a QECC must satisfy is that orthogonal codewords must be mapped to orthogonal states after the occurrence of errors, so that the errors can be resolved and undone.

Recall that here we are assuming that we know the location of the error, after recording the position of the SE. Hence we need only compare orthogonal codewords after the action of an error in a known location $i$:

$$L\langle y|\sigma^+_i\sigma^-_i|x\rangle_L = \left\{ \begin{array}{ll} \delta_{xy}/2 & \text{if } y_i = x_i \\ 0 & \text{if } y_i \neq x_i \end{array} \right.$$  

The second line is explained in the following way: If $x_i \neq y_i$, then either $x_i$ or $y_i$ is 0. Suppose $y_i$ is zero, then the component of the $|y\rangle_L$ codeword that remains after the SE is $\sigma^-_i$, and the component of $|x\rangle_L$ that remains is $\sigma^+_i|x, 0\rangle$ which is always orthogonal to each other. Thus the QECC condition is satisfied. To see that recovery from the errors is indeed possible, we describe a simple (non fault-tolerant) scheme. The recovery operation after the detection of an error in position $i$, is given by $U = \prod_{n \neq i} X_n \prod_{n \neq i} \text{CNOT}_{in} H_i$, where $H_i$ is a Hadamard operation on qubit $i$ and $\text{CNOT}_{ij}$ is a CNOT gate with qubit $i \leftrightarrow j$. That unitary operation fixes the SE error can be seen as follows by considering the transformation of the codewords after the error and recovery for the two cases: $x_i = 1$ and $x_i = 0$. If $x_i = 1$ ($x_i = 0$) then the codeword after the error becomes $|x_1 \ldots 0 \ldots x_{n+1}\rangle$ ($|x_1 \ldots 0 \ldots x_{n+1}\rangle$). It is easy to verify that applying $U$ to this state indeed returns the original logical codeword $|x\rangle_L$. To illustrate this we discuss in detail the case of $C_2$. The conditional evolution, under the collective BB pulse $X = X_1 X_2 X_3$, has the sole effect of shrinking the norm of all codewords in Eq. 2 

IV. FAULT TOLERANT PREPARATION, MEASUREMENT, AND RECOVERY

So far we have assumed perfect error detection, recovery, and gates. Of course, in reality these assumptions must be relaxed. Here we discuss the implications of imperfections.

In general, a procedure is said to be fault tolerant if the occurrence of an error in one location does not lead (via the applied procedure) to the catastrophic multiplication of errors in other locations, an event that the code cannot correct.

Let us first discuss preparation of the encoded qubits. The code word $|0\rangle_L$ is prepared by cooling all physical qubits in their ground state ($|0\rangle$), which can be done, e.g., via cooling, a strong polarizing field, or repeated strong measurements of all qubits, followed by a Hadamard on the $n + 1$-th qubit, and a collective CNOT from the $n + 1$-th qubit to all the other physical qubits 1 to $n$. Once $|0\rangle_L$ has been prepared computation proceeds using the fault-tolerant logical operations given in Section V below, so any other state can be reached fault-tolerantly. Readout is also simple: First apply the same collective CNOT and then measure the first $n$ physical qubits. The measurement procedure must be tailored to the specific implementation, but our only assumption is that single-qubit measurements are possible, and that these measurements do not couple qubits. The measurement procedure is then fault-tolerant. If means of applying fault-tolerant CNOT are available then both preparation and readout are fault tolerant.

Next consider recovery. The code $C_n$ is an especially simple example of CSS stabilizer codes, with stabilizer generated by the single element $\otimes_{j=1}^n X_j$. It is well known how in general to perform fault-tolerant recovery from this class of codes (see also [24]), so we will not repeat the general construction here, which in-
volves preparing and measuring encoded ancilla qubits (note that this typically doubles the number of physical qubits required, even before concatenation).

Finally consider detection of SE events. Above we assumed that it is possible to perfectly identify the position of a qubit that underwent SE. Note that this measurement is in itself fault-tolerant, in the sense that observing an SE event on a specific qubit cannot cause errors to multiply. Clearly, detecting which qubit emitted a photon is very demanding experimentally, and can in practice only be done to some finite precision (though there is no fundamental limit, provided the distance between the qubits is larger than the wavelengths of emitted photons), and at the cost of introducing potentially cumbersome detection apparatus. The same difficulty is shared by previous “detected-jump” schemes 7, 8, 9.

More specifically, in reality there is a finite probability that the emitted photon will (i) Go undetected; (ii) Be attributed to the wrong atom (misidentification). The latter possibility applies also to other qubit measurements; (iii) In case (ii), there is the additional possibility of an error by applying the correction step to the wrong qubit. In general, fault-tolerance results again come to the rescue: provided that the probability of an undetected photon and/or misidentification can be kept sufficiently low, concatenated QECC guarantees that the procedure will remain robust 24, 25, 28. However, several additional comments are in order. First, we note that the performance of DJC codes in the presence of imperfections such as detection inefficiencies and time delay between error detection and recovery operations, has been analyzed in 29, with favorable conclusions regarding fidelity degradation. We expect similar conclusions for our current method. Second, unlike the case of DJC codes 7, 8, 29, we do not require equal error rates κi. Hence our qubits need not be identical: qubits can be tuned to different cavity modes and therefore emit distinguishable photons. This should enable a significant reduction in the misidentification error rate. Third, we can take advantage of the fact that after any SE event, each codeword is transformed to a state which is orthogonal to all original codewords. Thus, we can perform an extra measurement (of the stabilizer X1 . . . Xn+1) that determines whether an error has occurred at all. This is done by adding one more ancilla qubit a (initialized into the 0-state) that functions as a syndrome-measurement bit. Now repeated apply \( \prod_{i=1}^{n+1} H_i \text{CNOT}_a H_i \) and periodically check the qubit a to see if it has changed to 1. In such a case if the position of the SE is undetected the computation has to be restarted and the ancilla qubit has to be reset; otherwise the recovery procedure may still be applied. Fault tolerant procedures are known for syndrome measurement as well 24, 25, 28. There is also the possibility of SE on the ancilla qubit, but this can only be caused by two successive spontaneous emissions (one on the code qubits and one on the parity qubit), which has a lower probability \( p^2 \), where \( p \) is the probability of two SE errors occurring during the same observation period, before the first one is detected. Note that the parity bit also helps preventing the error of applying a correction step without an SE event having taken place.

V. FAULT TOLERANT COMPUTATION

So far we have described a fault tolerant implementation of quantum memory in the presence of SE. Now we describe how to perform universal quantum computation fault tolerantly on our code. Formally, one can use the formalism of normalizer group operations, together with a non-normalizer element such as the \( \pi/8 \) or Toffoli gate 28. However, here we are interested in how to carry this out from the perspective of the naturally available interactions for a given physical system. Similar questions have been raised recently under the heading of “encoded universality”: the ability to quantum compute universally directly in terms of a given and limited set of Hamiltonians, possibly by use of encoded qubits (see, e.g., 30, 31 and references therein). The problem then translates into finding sets of Hamiltonians that generate a universal set of logic gates on the code. There are many options, depending on the set of naturally available interactions. Nevertheless, all encoded universality constructions rely on showing that the well-known universal set of all single-qubit operations and a single entangling gate can be generated, on the encoded qubits. Underlying this are a few elementary identities. Let us define conjugation as:

\[
A \xrightarrow{B, \phi} e^{-i\phi B} A e^{i\phi B}.
\]

Then for any three \( su(2) \) generators \( \{J_x, J_y, J_z\} \):

\[
J_z \xrightarrow{J_x, \phi} J_x \cos \phi + J_y \sin \phi.
\]

This can be lifted to unitary evolutions using

\[
U e^{A U^\dagger} = e^{U A U^\dagger},
\]

valid for any unitary \( U \). Hence where convenient we present our arguments in terms of transformed Hamiltonians. Eqs. 2, 3 show that given two \( su(2) \) generators one can generate a unitary evolution about any axis. This is also the basis for the well-known Euler angle construction, used to argue that all single qubit operations can be generated from \( \sigma^x \) and \( \sigma^z \) Hamiltonians: an arbitrary rotation by an angle \( \omega \) around the unit vector \( \mathbf{n} \) is given by three successive rotations around the \( z \) and \( x \) axes:

\[
e^{-i\omega \mathbf{n} \cdot \sigma} = e^{-i3\beta^z \sigma^z} e^{-i3\alpha^x \sigma^x} e^{-i\omega \mathbf{n} \cdot \sigma}.
\]

Eqs. 2, 3 show that this is true also for “encoded Hamiltonians”, which we define as Hamiltonians that have the same effect on encoded states as do regular Hamiltonians on “bare” (uncoded) qubits. We denote encoded Hamiltonians by a bar. For the code states 10 these are given by:

\[
\bar{Z}_i = Z_i Z_{n+1}, \quad \bar{X}_i = X_i,
\]
and generate $su(2)$. Therefore controllable $Z_iZ_{n+1}$ and $X_i$ Hamiltonians suffice to generate arbitrary single encoded-qubit transformations. To complete the set of universal logic gates we require some non-trivial (entangling) gate \[8\] such as controlled-phase: $CP = \text{diag}(1,1,1,-1)$, in the computational basis. $CP$ can be generated from the Ising interaction $Z_iZ_j$ as follows: $CP_{ij} = e^{-i\pi(Z_i+Z_j)}e^{-\frac{i\pi}{2}Z_iZ_j}$. An entangling gate can also be generated from the Hamiltonian $X_iX_j$ (one way to see this is to note that it can be rotated to $Z_iZ_j$ using $Y_i$ and $Y_j$ in Eqs. 2, 3). Encoded $CP$ can thus be generated from the encoded Hamiltonians $Z_iZ_j = Z_iZ_j$ or $X_iX_j = X_iX_j$. Note that in both cases the physical interaction is also the corresponding encoded Hamiltonian. Thus the sets of controllable Hamiltonians $\{X_i, Z_i\}$ or $\{X_i, Z_iZ_{n+1}, X_iX_j\}$ suffice for encoded universal QC on our code. Importantly, these sets moreover exhibit “natural fault-tolerance” \[17\]: they preserve the code subspace and hence will not expose the code to uncorrectable errors. An accuracy error in the time over which the Hamiltonians are turned on can be dealt with using the technique of concatenated QECCs \[23\]. The question now is how to generate these sets, or an equivalent fault tolerant universal set, from the given, naturally available interactions. We will consider here the most important cases, extending methods developed in \[20, 24, 25\]. Note that the decoupling procedure requires us to assume in any case the ability to apply a global (non-selective) $X$ pulse, and the recovery procedure requires the ability to apply a CNOT gate. We comment on these requirements in each of the cases we next analyze.

A. Case 1: Natural $\{Z_i, X_i, X_iX_j\}$

The Hamiltonians $Z_i, X_iX_j$ are naturally available, e.g., in the Sørensen-Mølmer scheme for trapped-ion QC \[30\], and in proposals using Josephson charge qubits \[32\]. However these do not form a universal set for our code and hence we must assume the ability to turn on spin-selective $X_i$ Hamiltonians. This will also be sufficient for producing the encoded $Z_iZ_j$ coupling.

B. Case 2: $\{Z_i, X_i, XY \text{ Model}\}$

Members of a relatively large class of promising QC proposals (quantum dots \[32, 35\], atoms in a cavity \[40\], quantum Hall qubits \[11\], subradiant dimers in a solid host \[12\], capacitively coupled superconducting qubits \[13\]) have a controllable Hamiltonian of the XY form: $H_{ij}^{XY} = J_{ij}(X_iX_j + Y_iY_j)$. Let $T_{ij} = \frac{1}{2}(X_iX_j + Y_iY_j)$. Then $|01\rangle \overset{T_{ij}}\rightarrow |10\rangle$, and annihilates $|00\rangle, |11\rangle$. i.e., the XY Hamiltonian cannot change the total number of 1’s in a computational basis state \[34, 37\]. Therefore by itself, or even if supplemented with $Z_i$ Hamiltonians, it cannot generate $su(2)$ on our code. This conclusion is unchanged even if one considers conjugating $H_{ij}^{XY}$ with $H_{ik}^{XY}$: then $\{T_{12}, T_{13}, -Z_1Z_2T_{23}\}$ close as $su(2)$, and still preserve the total number of 1’s. Therefore in this case we must assume the ability to tune $X_i$ Hamiltonians as well, to obtain universality. Now, $X_iX_j(T_{jk})X_iX_j = \frac{1}{2}(X_jX_k - Y_jY_k)$, which commutes with $T_{jk}$. Therefore, using Eq. 9, we have $X_iX_j e^{-\frac{i\pi}{2}T_{jk}} X_iX_j e^{-\frac{i\pi}{2}T_{jk}} = e^{-i\theta}X_iX_k$, showing that the Hamiltonian $X_iX_k$ can be generated in four steps. At this point we have the same set of Hamiltonians as in Case 1, so that universal encoded computation is possible, as are the global $X$ pulse and recovery.

C. Case 3: $\{Z_i, X_i, Heisenberg interaction\}$

Next we consider the case of single-qubit $X-Z$ control together with the Heisenberg interaction $H_{ij}^{\text{Heis}} = J_{ij}(X_iX_j + Y_iY_j + Z_iZ_j)$. Heisenberg interactions prevail in QC proposals using spin-coupled quantum dots \[12, 16\] and donor atoms in Si \[17, 18\]. This case is similar to that of the XY model, since $H_{ij}^{\text{Heis}}$ also preserves the total number of 1’s in a computational basis state. Therefore, as in the XY case, we must assume the ability to generate an $X_iX_j$ pulse. Then, $X_iX_j(H_{ij}^{\text{Heis}})X_iX_j = J_{jk}(X_jX_k - Y_jY_k - Z_jZ_k)$, which commutes with $H_{ik}^{\text{Heis}}$, so that $X_iX_j e^{-itH_{ij}^{\text{Heis}}} X_iX_j e^{-itH_{ij}^{\text{Heis}}} = e^{-2itJ_{jk}}X_jX_k$, and we are back to Case 1. There is now another option for generating an entangling gate: we can generate a pure ZZ interaction using $ZI(H_{ij}^{\text{Heis}})ZI = -XX -YY -ZZ$, which commutes with $H_{ij}^{\text{Heis}}$, so that $e^{-itH_{ij}^{\text{Heis}}} e^{-i\frac{\pi}{2}ZI} e^{-itH_{ij}^{\text{Heis}}} e^{-i\frac{\pi}{2}ZI} = e^{-2itJJZ}$. This is a four-step, naturally fault tolerant procedure. The decoupling pulse and recovery are now the same as in Case 1.

Finally, there remains the issue of compatibility between the encoded logic operations and the decoupling pulses that are being constantly applied to the system. All three interaction Hamiltonians we have considered commute with the global $X$ BB-pulse, so are fully compatible with the BB operations. Furthermore, the logical single-qubit terms also commute with the $X$ pulse. Thus whenever use of a single body $Z_i$ Hamiltonian is required, it must be synchronized to be applied only after an even number of collective BB pulses, to ensure the compatibility of quantum manipulation and dynamical decoupling.

VI. CONCLUSIONS

We have proposed a new method for performing universal, fault tolerant quantum computation in the presence of spontaneous emission. The method combines dynamic decoupling pulses with a particularly simple and efficient quantum error correcting code, encoding $n$ logical qubits into $n+1$ physical qubits. Computation is performed by controlling single-qubit $\sigma^x$ and $\sigma^z$ terms...
together with any of three major examples of qubit-qubit interaction Hamiltonians, applicable to a wide range of quantum computing proposals. The proposed method offers an improvement over previous schemes for protecting quantum information against spontaneous emission in that the code is at least twice as efficient in terms of qubit resources, and the method is fully compatible with computation using physically reasonable resources and interactions.

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