A new twist on heterotic string compactifications

Bernardo Fraiman†*, Mariana Graña# and Carmen A. Nuñez†*

† Instituto de Astronomía y Física del Espacio (IAFE-CONICET-UBA)
Ciudad Universitaria, Pabellón IAFE, 1428 Buenos Aires, Argentina

* Departamento de Física, FCEyN, Universidad de Buenos Aires (UBA)

# Institut de Physique Théorique, CEA/ Saclay
91191 Gif-sur-Yvette Cedex, France

Abstract: A rich pattern of gauge symmetries is found in the moduli space of heterotic string toroidal compactifications, at fixed points of the T-duality transformations. We analyze this pattern for generic tori, and scrutinize in full detail compactifications on a circle. We show the gauge symmetry groups that arise at special points, in figures of slices of the 17-dimensional moduli space of Wilson lines and circle radii. We then study the target space realization of the duality symmetry. Although the global continuous duality symmetries of dimensionally reduced heterotic supergravity are completely broken by the structure constants of the maximally enhanced gauge groups, the low energy effective action can be written in a manifestly duality covariant form using heterotic double field theory. As a byproduct, we show that a unique deformation of the generalized diffeomorphisms accounts for both $SO(32)$ and $E_8 \times E_8$ heterotic effective field theories, which can thus be considered two different backgrounds of the same double field theory even before compactification. Finally we discuss the spontaneous gauge symmetry breaking and Higgs mechanism that occurs when slightly perturbing the background fields, both from the string and the field theory perspectives.
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1 Introduction

The distinct backgrounds of heterotic string theory on a \( k \) dimensional torus with constant metric, antisymmetric tensor field and Wilson lines are characterized by the points of the \( O(k, k + 16; \mathbb{R}) \times O(k + 16; \mathbb{R}) \times O(k, k + 16; \mathbb{Z}) \) coset manifold, where \( O(k, k + 16; \mathbb{Z}) \) is the T-duality group \([1, 2]\). At self-dual points of this manifold, some massive modes become massless and the \( U(1)^{2k + 16} \) gauge symmetry becomes non-abelian. In particular, for zero Wilson lines, the massless fields give rise to \( SO(32) \times U(1)^{2k} \) or \( E_8 \times E_8 \times U(1)^{2k} \) at generic values of the metric and B-field. By introducing Wilson lines, not only is it possible to totally or partially break the non-abelian gauge symmetry of the uncompactified theory, but it is also possible to enhance these groups. The construction of \([2]\) further allowed to continuously interpolate between the \( SO(32) \) and \( E_8 \times E_8 \) heterotic theories after compactification \([3]\), and even suggested that these superstrings are two different vacuum states in the same theory before compactification.

Enhancement of the gauge symmetry occurs at fixed points of the T-duality transformations \([4]\). Massless fields become massive at the neighborhood of such points and the T-duality group mixes massless modes with massive ones \([5]\). Moreover, by identifying different string backgrounds that provide identical theories, T-duality gives rise to stringy features that are rather surprising from the viewpoint of particle field theories. Nevertheless, some of these ingredients have a correspondence in toroidal compactifications of heterotic supergravity. In particular, although the field theoretical reduction of heterotic supergravity cannot describe the non-abelian fields that give rise to maximally enhanced gauge symmetry\([1]\) being a gauged supergravity, the reduced theory is completely determined by the gauge group, which can be chosen to be one of maximal enhancement. Likewise, the global symmetries of heterotic supergravities are linked to T-duality. While the theory with the full set of \( SO(32) \) or \( E_8 \times E_8 \) gauge fields has a global continuous \( O(k, k; \mathbb{R}) \) symmetry, when introducing Wilson lines, the symmetry enlarges to \( O(k, k + 16; \mathbb{R}) \) \([6, 7, 8]\), which is related to the discrete T-duality symmetry of the parent string theory.

The global duality symmetries are not manifest in heterotic supergravity. To manifestly display these symmetries, as well as to account for the maximally enhanced gauge groups in a field theoretical setting, one appeals to the double field theory/generalized geometric reformulation of the string effective actions \([9, 10]\) (for reviews and more references see \([11]\)). Specifically, these frameworks not only describe the enhancement of gauge symmetry \([12]-[15]\), but also give a geometric description of the non-geometric backgrounds that are obtained from T-duality \([16]\) and provide a gauge principle that requires and fixes the \( \alpha' \)-corrections of the string effective actions \([17]\). Dependence of the fields on double internal coordinates and an extension of the tangent space are some of the elements that allow to go beyond the standard dimensional reductions of supergravity.

Motivated by deepening our understanding of heterotic string toroidal compactifications, in section 2 we review the main features of heterotic string propagation on a \((10 - k)\)-dimensional Minkowski space-time times an internal \( k \)-torus with constant background metric, antisymmetric tensor field and Wilson lines, and recall their \( O(k, k + 16) \) covariant formulation. We focus on the phenomenon of symmetry enhancement arising

\[\text{\textsuperscript{1}}\text{"Maximal" stands here for an enhanced semi-simple and simply-laced symmetry group.}\]
at special points in moduli space.

In section 3, we concentrate on the simplest case, namely circle compactifications \((k = 1)\). To explore the moduli space, we split the discussion into the situations in which the Wilson line \(A\) preserves the \(E_8 \times E_8\) or \(SO(32)\) gauge symmetry, and those where it breaks it. In the former case, the circle direction can give a further enhancement of symmetry to \(E_8 \times E_8 \times SU(2)\) at radius \(R = 1\), and either to \(SO(32) \times SU(2)\) at \(R = 1\) or to \(SO(34)\) at \(R = \frac{1}{\sqrt{2}}\). When the Wilson line breaks the \(E_8 \times E_8\) or \(SO(32)\) gauge symmetry, the pattern of gauge symmetries is very interesting. Not only is it possible to restore the original \(E_8 \times E_8\) or \(SO(32)\) gauge symmetry for specific values of \(R\) and \(A\), but also larger groups of rank 17 can be obtained. We explicitly work out enhancements to \(SO(18) \times E_8\) at \(R^2 = \frac{1}{2}\); \(SU(2) \times SO(2p) \times SO(32 - 2p)\) at \(R^2 = 1 - \frac{p}{8}\); \(SU(2) \times U(p) \times SO(32 - 2p)\) at \(R^2 = 1 - \frac{p}{32}\); \(SU(2) \times E_8 \times E_8\) at \(R^2 = \frac{1}{4}\) and \(U(17)\) at \(R^2 = \frac{1}{4}\). We depict slices of the moduli space for different values of \(R\) and \(A\) in several figures, which clarify the analysis and neatly exhibit the regions and points with special properties.

Examining the action of T-duality, we can see that all points in moduli space where there is maximal symmetry enhancement, namely enhancement to groups that do not have \(U(1)\) factors, are fixed points of T-duality, or more general \(O(1, 17; \mathbb{Z})\) dualities that involve some exchange of momentum and winding number on the circle. In the simplest cases, such as those listed above, the enhanced symmetry arises at the self-dual radius given by \(R_{sd}^2 = 1 - \frac{1}{2}|A|^2\). We explore the action of T-duality and its fixed points in section 3.5. One can have other points of symmetry enhancement, which are fixed points of duality symmetries that involve shifts of Wilson lines on top of the exchange of momentum and winding. This is studied in detail in section 3.6 where we obtain the most general duality symmetries that change the sign of the right-moving momenta and rotate the left-moving momenta, leaving the circle direction invariant. Concentrating on the case where the Wilson lines have only one non-zero component, we find a rich pattern of fixed points that correspond to \(SU(2) \times SO(32)\) or \(SU(2) \times E_8 \times E_8\) enhanced gauge symmetry, arising at \(R_{sd}^{-1} = C\), with \(C\) an integer number with prime divisors congruent to 1 or 3 (mod 8), and \(SO(34)\) or \(SO(18) \times E_8\) at \(R_{sd}^{-1} = \sqrt{2}C\) with \(C\) a Pythagorean prime number or a product of them.

We then turn to the target space realization of the theory. In section 4, we construct the low energy effective actions of (toroidally compactified) heterotic strings from the three and four point functions of string states. We first consider only the massless states and compare the effective action obtained from the string amplitudes with the dimensionless reduction of heterotic supergravity performed in [8]. As expected, we get a gauged supergravity which only differs from the effective action of [8] in the cases of maximal enhancement, in which all the (left-moving) \(U(1)^k\) Kaluza-Klein (KK) gauge fields of the compactification become part of the Cartan subgroup of the enhanced gauge symmetry.

The higher dimensional origin of the low energy theory with maximally enhanced gauge symmetry cannot be found in supergravity, and one has to refer to DFT. Although the structure constants of the gauge group completely break the global duality symmetry of dimensionally reduced supergravity, the action can still be written in terms of \(O(k, N)\) multiplets, with \(N\) the dimension of the gauge group. We show in section 5 that the low energy effective action of the toroidally compactified heterotic string at self-dual points of
the moduli space can be reproduced through a generalized Scherk-Schwarz reduction of heterotic DFT. Furthermore, extending the construction of [13], we find the generalized vielbein that reproduces the structure constants of the enhanced gauge groups through a deformation of the generalized diffeomorphisms. An important output of the construction is that a unique deformation is required for the $SO(32)$ and $E_8 \times E_8$ groups, and hence the $SO(32)$ and $E_8 \times E_8$ theories can be considered two different solutions of the same heterotic DFT, even before compactification.

When perturbing the background fields away from the enhancement points, some massless string states become massive. The vertex operators of the massive vector bosons develop a cubic pole in their OPE with the energy-momentum tensor, and it is necessary to combine them with the vertex operators of the massive scalars in order to cancel the anomaly. This fact had been already noticed in [12], but unlike the case of the bosonic string, in the heterotic string all the massive scalars are “eaten” by the massive vectors. We compute the three point functions involving massless and slightly massive states\(^2\) and construct the corresponding effective massive gauge theory coupled to gravity. Comparing the string theory results with the spontaneous gauge symmetry breaking and Higgs mechanism in DFT, we see that the masses acquired by the slightly massive string states fully agree with those of the DFT fields, provided there is a specific relation between the vacuum expectation value of the scalars along the Cartan directions of the gauge group and the deviation of the metric, B-field and Wilson lines from the point of enhancement.

We have included five appendices. Appendix A collects some known facts about lattices that are used in the main body of the paper. Details of the procedures leading to find the enhancement points and the fixed points of the duality transformations are contained in Appendix B and C, respectively. The three and four point amplitudes of the massless and slightly massive string states are reviewed in Appendix D. Finally we count the number of non-vanishing structure constants of $SO(32)$ and $E_8 \times E_8$ in Appendix E.

2 Toroidal compactification of the heterotic string

In this section we recall the main features of heterotic string compactifications on $T^k$. We first discuss the generic $k$ case and then we concentrate on the $k = 1$ example. For a more complete review see [5].

2.1 Compactifications on $T^k$

Consider the heterotic string propagating on a background manifold that is a product of a $d = 10 - k$ dimensional flat space-time times an internal torus $T^k$ with constant background metric $G = e^t e$ $(\Rightarrow G_{mn} = e^a_m \epsilon_{ab} e^b_n)$, antisymmetric two-form field $B_{mn}$ and $U(1)^{16}$ gauge field $A^A_m$, where $m, n, a, b = 1, \ldots, k$ and $A = 1, \ldots, 16$. For simplicity we take the background dilaton to be zero. The set of vectors $e_m$ define a basis in

\(^2\)For consistency, we consider only small perturbations because we are not including other massive states from the string spectrum.
the compactification lattice \( \Lambda^k \) such that the internal part of the target space is the \( k \)-dimensional torus \( T^k = \mathbb{R}^k/\pi \Lambda^k \). The vectors \( \hat{e}_a \) constitute the canonical basis for the dual lattice \( \Lambda^\vee \), i.e. \( \hat{e}_a^m e^a_n = \delta^m_n \), and thus they obey \( e^a \hat{e}_a = G^{-1} \) (\( e^a \hat{e}^a = G \)).

The contribution from the internal sector to the world-sheet action (we consider only the bosonic sector here) is

\[
S = \frac{1}{4\pi} \int_M d\tau d\sigma \left( \delta^{\alpha \beta} G_{mn} - i e^{\alpha \beta} B_{mn} \right) \partial_\alpha Y^m \partial_\beta Y^n \\
+ \frac{1}{8\pi} \int_M d\tau d\sigma \left( \delta^{\alpha \beta} \partial_\alpha Y^A \partial_\beta Y^A - 2i e^{\alpha \beta} A^A_m \partial_\alpha Y^m \partial_\beta Y^A \right),
\]

where we take \( \alpha' = 1 \), \( Y^A \) are chiral bosons and the currents \( \partial Y^A \) form a maximal commuting set of the \( \text{SO}(32) \) or \( E_8 \times E_8 \) current algebra. The world-sheet metric has been gauge fixed to \( \delta^{\alpha \beta} (\alpha, \beta = \tau, \sigma) \) and \( \epsilon^{01} = 1 \). The internal string coordinate fields satisfy

\[
Y^m(\tau, \sigma + 2\pi) \simeq Y^m(\tau, \sigma) + 2\pi w^m,
\]

where \( w^m \in \mathbb{Z} \) are the winding numbers. It is convenient to define holomorphic \( Y_L^m(z) \) and antiholomorphic \( Y_R^m(\bar{z}) \) fields as

\[
Y^m(z, \bar{z}) = \left( \frac{1}{2} \right)^{1/2} [Y_L^m(z) + Y_R^m(\bar{z})], \quad z = \exp(\tau + i\sigma), \quad \bar{z} = \exp(\tau - i\sigma),
\]

with Laurent expansion

\[
Y_L^m(z) = y_L^m - ip_L^m \ln z + \cdots, \quad Y_R^A(z) = y_R^A - ip_R^A \ln \bar{z} + \cdots,
\]

\[
Y_R^m(\bar{z}) = y_R^m - ip_R^m \ln \bar{z} + \cdots,
\]

the dots standing for the oscillators contribution. Then the periodicity condition is

\[
Y^m(\tau, \sigma + 2\pi) - Y^m(\tau, \sigma) = 2\pi \left( \frac{1}{2} \right)^{1/2} (p_L^m - p_R^m) = 2\pi w^m.
\]

The canonical momentum has components\(^3\)

\[
\Pi_m = i \frac{\delta S}{\delta \partial_\tau Y^m} = \frac{1}{2\pi} \left[ i G_{mn} \partial_\tau Y^n + B_{mn} \partial_\sigma Y^n + \frac{1}{2} A^A_m \partial_\sigma Y^A \right],
\]

\[
= \frac{1}{2\pi} \left( \frac{1}{2} \right)^{1/2} [G_{mn}(p_L^n + p_R^n) + B_{mn}(p_L^n - p_R^n)] + \frac{1}{4\pi} A^A_m p^A,
\]

\[
\Pi^A = i \frac{\delta S}{\delta \partial_\tau Y^A} = \frac{1}{4\pi} \left( i \partial_\tau Y^A - A^A_m \partial_\sigma Y^m \right) = \frac{1}{2\pi} \left[ p^A - \left( \frac{1}{2} \right)^{1/2} A^A_m (p_L^m - p_R^m) \right].
\]

The chirality constraint on \( Y^A \) and the condition of vanishing Dirac brackets between momentum components require the redefinitions \( \Pi_A \to \tilde{\Pi}_A = 2\Pi_A \) and \( \Pi_m \to \tilde{\Pi}_m = \)

\(^3\)The unusual \( i \) factors are due to the use of Euclidean world-sheet metric.
\( \Pi_m + \frac{1}{2} A^A_m \tilde{\Pi}_A \). Integrating over \( \sigma \), we get the center of mass momenta

\[
\pi_m = \int d\sigma \tilde{\Pi}_m = 2\pi \left( \Pi_m + \frac{1}{2} A^A_m \tilde{\Pi}_A \right) = n_m \in \mathbb{Z}, \tag{2.7a}
\]

\[
\pi^A = \int d\sigma \tilde{\Pi}^A = p^A - A^A_m w^m, \tag{2.7b}
\]

where we used univaluedness of the wave function in the first line. Modular invariance requires \( \pi^A \in \Gamma_{16} \) or \( \Gamma_8 \times \Gamma_8 \), corresponding to the \( SO(32) \) or \( E_8 \times E_8 \) heterotic theory, respectively. Whenever we do not need to make the distinction, we use \( \Gamma \) to refer to one of these two lattices. In Appendix A we give all the relevant explanations and details about these lattices.

From these equations we get

\[
p^{RA} = \left( \frac{1}{2} \right)^{1/2} \hat{e}_a^m \left[ n_m - (G_{mn} + B_{mn})w^n - \pi^A A^A_m - \frac{1}{2} A^A_n A^A_m w^n \right], \tag{2.8a}
\]

\[
p^{LA} = \left( \frac{1}{2} \right)^{1/2} \hat{e}_a^m \left[ n_m + (G_{mn} - B_{mn})w^n - \pi^A A^A_m - \frac{1}{2} A^A_n A^A_m w^n \right], \tag{2.8b}
\]

\[
p^A = \pi^A + w^m A^A_m. \tag{2.8c}
\]

The momentum \( \mathbf{p} = (\mathbf{p}_R, \mathbf{p}_L) \), with \( \mathbf{p}_R = p_{Ra}, \mathbf{p}_L = (p_{La}, p^A) \), transforms as a vector under \( O(k, k+16; \mathbb{R}) \). It expands the \( 2k + 16 \)-dimensional momentum lattice \( \Gamma^{(k,k+16)} \subset \mathbb{R}^{2k+16} \), satisfying

\[
\mathbf{p} \cdot \mathbf{p} = \mathbf{p}_L^2 - \mathbf{p}_R^2 = 2w^m n_m + \pi^A \pi^A \in 2\mathbb{Z}, \tag{2.9}
\]

because \( \pi^A \) is on an even lattice, and therefore \( \mathbf{p} \) forms an even \( (k, k+16) \) Lorentzian lattice. In addition, self-duality \( \Gamma^{(k,k+16)} = \Gamma^{(k,k+16)*} \) follows from modular invariance \footnote{Note that \( \mathbf{p}_L, \mathbf{p}_R \) depend on \( 2k + 16 \) integer parameters \( n_m, w^m \) and \( \pi^A \), and on the background fields \( G, B \) and \( A \).}

\[
\text{The space of inequivalent lattices and inequivalent backgrounds reduces to}
\]

\[
O(k, k+16; \mathbb{R}) \over O(k+16; \mathbb{R}) \times O(k; \mathbb{R}) \times O(k, k+16; \mathbb{Z}), \tag{2.10}
\]

where \( O(k, k+16; \mathbb{Z}) \) is the T-duality group (we give more details about it in the next section).

The mass of the states and the level matching condition are respectively given by

\[
m^2 = \mathbf{p}_L^2 + \mathbf{p}_R^2 + 2 \left( \mathcal{N} + \overline{\mathcal{N}} - \left\{ \frac{1}{2} \right\}_\text{R sector} \right), \tag{2.11a}
\]

\[
0 = \mathbf{p}_L^2 - \mathbf{p}_R^2 + 2 \left( \mathcal{N} - \overline{\mathcal{N}} - \left\{ \frac{1}{2} \right\}_\text{R sector} \right). \tag{2.11b}
\]
2.2 $O(k, k + 16)$ covariant formulation

The $O(k, k + 16)$ invariant metric $\eta$ is

$$\eta_{MN} = \begin{pmatrix} 0 & 1_{k \times k} & 0 \\ 1_{k \times k} & 0 & 0 \\ 0 & 0 & \kappa_{IJ} \end{pmatrix},$$  \hspace{1cm} (2.12)

where $\kappa$ is the Killing metric for the Cartan subgroup of $SO(32)$ or $E_8 \times E_8$, and the “generalized metric” of the $k$-dimensional torus, given by the $(2k + 16) \times (2k + 16)$ scalar matrix, is

$$M_{MN} = \begin{pmatrix} G_{mn} + C_{lm}g^{lk}C_{kn} + A_{mI}A_{nI} & -G^{mn}C_{km} & C_{km}G^{kl}A_{IJ} + A_{mJ} \\ -G^{mk}C_{kn} & C^{mn} & -G^{mk}A_{kI} + A_{kIJ} \\ C_{kn}G^{lk}A_{II} + A_{nI} & -G^{mk}A_{kl} & \kappa_{IJ} + A_{kI}G^{kl}A_{IJ} \end{pmatrix} \in O(k, k + 16; \mathbb{R}),$$  \hspace{1cm} (2.13)

where

$$C_{mn} = B_{mn} + \frac{1}{2}A_{mI}\kappa^{IJ}A_{nJ}.$$  \hspace{1cm} (2.14)

This is a symmetric element of $O(k, k + 16)$, accounting for the degrees of freedom of the $O(k, k + 16)$ coset.

Combining the momentum and winding numbers in an $O(k, k + 16)$-vector

$$Z^M = \begin{pmatrix} w^m \\ n_m \\ \pi^I \end{pmatrix}, \quad \pi^I \equiv \pi^A\hat{e}_A^I, \quad \text{with} \quad \hat{e}_A^I\hat{e}_J^A = \kappa^{IJ},$$  \hspace{1cm} (2.15)

the mass formula (2.11a) and level matching condition (2.11b) read

$$m^2 = 2\left(\mathcal{N} + \overline{\mathcal{N}} - \left\{ \frac{1}{2} \text{ R sector} \atop \frac{3}{2} \text{ NS sector} \right\} + Z^I M Z, \right.$$  \hspace{1cm} (2.16)

$$0 = 2\left(\mathcal{N} - \overline{\mathcal{N}} - \left\{ \frac{1}{2} \text{ R sector} \atop \frac{3}{2} \text{ NS sector} \right\} + Z^I \eta Z, \right.$$  \hspace{1cm} (2.17)

respectively. Note that these equations are invariant under the T-duality group $O(k, k + 16; \mathbb{Z})$ acting as

$$Z \rightarrow \eta^{-1}O\eta Z, \quad \mathcal{M} \rightarrow O\mathcal{M}O^t, \quad \eta \rightarrow O\eta O^t = \eta, \quad O \in O(k, k + 16; \mathbb{Z}).$$  \hspace{1cm} (2.18)

The group $O(k, k + 16; \mathbb{Z})$ is generated by:

- Integer $\Theta$-parameter shifts, associated with the addition of an antisymmetric integer matrix $\Theta_{mn}$ to the antisymmetric $B$-field,

$$O_\Theta = \begin{pmatrix} 1 & \Theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{16 \times 16} \end{pmatrix}, \quad \Theta_{mn} \in \mathbb{Z},$$  \hspace{1cm} (2.19)
- Lattice basis changes

\[ O_M = \begin{pmatrix} M & 0 & 0 \\ 0 & (M^t)^{-1} & 0 \\ 0 & 0 & 16 \times 16 \end{pmatrix}, \quad M \in GL(k; \mathbb{Z}), \quad (2.20) \]

- \( \Lambda \)-parameter shifts associated to the addition of vectors \( \Lambda_m^A \) to the Wilson lines

\[ O_\Lambda = \begin{pmatrix} 1 & -\frac{1}{2} \Lambda \Lambda^t & \Lambda \\ 0 & 1 & 0 \\ 0 & -\Lambda^t & 16 \times 16 \end{pmatrix}, \quad \Lambda_m \in \Gamma_{16} \text{ or } \Gamma_8 \otimes \Gamma_8, \quad (2.21) \]

- Factorized dualities, which are generalizations of the \( R \rightarrow 1/R \) circle duality, of the form

\[ O_{D_i} = \begin{pmatrix} 1 - D_i & D_i & 0 \\ D_i & 1 - D_i & 0 \\ 0 & 0 & 16 \times 16 \end{pmatrix}, \quad (2.22) \]

where \( D_i \) is a \( k \times k \) matrix with all zeros except for a one at the \( ii \) component.

The first three generators comprise the so-called geometric dualities, transforming the background fields parameterizing the generalized metric (2.13). The \( O(k, k + 16) \) group contains in addition

- Orthogonal rotations of the Wilson lines

\[ O_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & N \end{pmatrix}, \quad N \in O(16; \mathbb{Z}), \quad (2.23) \]

- Transformations of the dual Wilson lines

\[ O_\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} \Gamma \Gamma^t & 1 & 0 \\ \Gamma & 0 & 1 \end{pmatrix}, \quad \Gamma^m \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8, \quad (2.24) \]

- Shifts by a bivector

\[ O_\beta = \begin{pmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta^{mn} \in \mathbb{Z}, \quad \beta^{mn} = -\beta^{nm}. \quad (2.25) \]

The transformation of the charges under the action of \( O_\Theta O_\Lambda \), which will be useful later, is

\[ \begin{pmatrix} w \\ n \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} n + (\Theta - \frac{1}{2} \Lambda \Lambda^t)w + \Lambda \pi \\ \pi - \Lambda^t \pi \end{pmatrix}. \quad (2.26) \]

\footnote{Note that this adds a shift to \( B \) of the form \( B \rightarrow B + \frac{1}{2}(\Lambda \Lambda^t - \Lambda A^t) \).}
Notice the particular role played by the element \( \eta \) viewed as a sequence of factorized dualities in all tori directions, i.e.

\[
\eta^{-1} = O_D \equiv \prod_{i=1}^{k} O_{D_i} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \kappa^{-1} \end{pmatrix}.
\]  

(2.27)

Its action on the generalized metric is

\[
\mathcal{M} \to O_D \mathcal{M} O_D^t = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}A \\ -C'^{-1}G^{-1} & G + C'^{-1}G^{-1}C + AA^t & (1 + C'^{-1}G^{-1})A \\ -A'^{-1}G^{-1} & A'(1 + G^{-1}C) & \kappa^{-1} + A'G^{-1}A \end{pmatrix} = \mathcal{M}^{-1},
\]  

(2.28)

where \( A \equiv A_m^l \) and, together with the transformation \( Z \to \eta^{-1} O_D \eta Z \) which accounts for the exchange \( w^a \leftrightarrow n_m \), it generalizes the \( R \leftrightarrow 1/R \) duality of the circle compactification. These transformations determine the dual coordinate fields

\[
\tilde{Y}_m(z, \bar{z}) = \frac{1}{\sqrt{2}} G_{mn}(Y_L^n - Y_R^n) + \frac{1}{\sqrt{2}} C_{mn}(Y_L^n + Y_R^n) + A_m^l Y^A.
\]  

(2.29)

A vielbein \( E \) for the generalized metric

\[
\mathcal{M}_{MN} = E^a_M \delta_{ab} E^b_N,
\]  

(2.30)

with \( M, N = a, b = 1, \ldots, 2k + 16 \), can be constructed from the vielbein for the internal metric and inverse internal metric as follows

\[
E^a_M \equiv E = \begin{pmatrix} -\hat{e}_n^m C_{nm} & \hat{e}^m_n & -\hat{e}_n^m A^I_n \kappa_{IJ} \\ e^a_m & 0 & 0 \\ \tilde{e}^A_I A^I_m & 0 & \tilde{e}^A_J \end{pmatrix},
\]  

(2.31)

where \( \tilde{e} \) is the vielbein for \( \kappa \). In the basis of right and left movers, that we denote “RL”, where the \( O(k, k + 16; \mathbb{R}) \) metric \( \eta \) takes the diagonal form

\[
\eta_{RL} = (R \eta R^T) = \begin{pmatrix} -\delta_{ab} & 0 & 0 \\ 0 & \delta_{ab} & 0 \\ 0 & 0 & \delta_{AB} \end{pmatrix}, \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta^B_a & -\delta^B_a & 0 \\ \delta^B_a & \delta^B_a & 0 \\ 0 & 0 & \sqrt{2} \delta^{A_B} \end{pmatrix},
\]  

(2.32)

the vielbein is

\[
E_{RL} \equiv RE \equiv \begin{pmatrix} E_{aR} \\ E_{aL} \\ E_A \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e_{am} - \hat{e}_n^m C_{nm} & \hat{e}^m_n & -\hat{e}_n^m A^I_n \kappa_{IJ} \\ e_{am} - \hat{e}_n^m C_{nm} & \hat{e}^m_n & -\hat{e}_n^m A^I_n \kappa_{IJ} \\ \sqrt{2} \tilde{e}^A_I A^I_m & 0 & \sqrt{2} \tilde{e}^A_J \end{pmatrix}.
\]  

(2.33)

Then the momenta \((p_{aR}, p_{aL}, p^A)\) in (2.8b) are

\[
\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^A \end{pmatrix} = E_{RL} Z.
\]  

(2.34)

\[^5\text{The transformations also determine a dual coordinate} \ Y^A = Y^A + \frac{1}{\sqrt{2}} A^A_m(Y_L^m + Y_R^m), \text{but this is not actually independent of} \ Y^m(z, \bar{z}) \text{and} \ Y^A(z). \]
2.3 Massless spectrum

The massless bosonic spectrum of the heterotic string in ten external dimensions is given, in terms of bosonic and fermionic creation operators $\alpha_{-1}^{\mu}, \bar{\psi}_{-1/2}^{\mu}$, respectively, by

1. $\mathcal{N} = 1, \overline{\mathcal{N}} = \frac{1}{2}, p_A = 0$:
   - Gravitational sector:
     \[
     \alpha_{-1}^{\mu} \bar{\psi}_{-1/2}^{\nu} |0, k\rangle_{NS}
     \]
     where the symmetric traceless, antisymmetric and trace pieces are respectively the graviton, antisymmetric tensor and dilaton.
   - Cartan gauge sector:
     \[
     \alpha_{-1}^{I} \bar{\psi}_{-1/2}^{\mu} |0, k\rangle_{NS}
     \]
     containing 16 vectors $A_{\mu}^{I}$ in the Cartan subgroup of $SO(32)$ or $E_8 \times E_8$.

2. $\mathcal{N} = 0, \overline{\mathcal{N}} = \frac{1}{2}, p_A^2 = 2$:
   - Roots gauge sector:
     \[
     \bar{\psi}_{-1/2}^{\mu} |0, k, \pi_{\alpha}\rangle_{NS}
     \]
     with $\pi_{\alpha}$ denoting one of the 480 roots of $SO(32)$ or $E_8 \times E_8$.

In compactifications on $T^k$, the spectrum depends on the background fields. In sector 1 there are the same number of massless states at any point in moduli space. In sector 2, we see from (2.8b) that there are no massless states for generic values of the metric, $B$-field and Wilson lines $A_{\mu}^{I}$, while for certain values of these fields the momenta can lie in the weight lattice of a rank $2k + 16$ group $G_L \times G_R$. In this case, there is a subgroup with $|(P_{R}, P_{L})|^2 = 2$ which can give rise to massless states. Subtracting (2.11a) and (2.11b) we see that massless states have $P_{R} = 0$, and thus (unlike in the bosonic string theory), the non-abelian gauge symmetry comes from the left sector only. The group $G_L \times U(1)^{k}$ in which the massless states transform defines the gauge group of the theory, with $G_L$ a simply-laced group of rank $16 + k$ and dimension $N$, that depends on the point in moduli space (which is spanned by $G_{mn}, B_{mn}, A_{\mu}^{I}$). Specifically, the $10 - k$ dimensional massless bosonic spectrum and the corresponding vertex operators (in the $-1$ and $0$ pictures) are given by ($\mu, \nu = 0, \ldots, 9 - k; m, n = 1, \ldots, k; I = 1, \ldots, 16$):

1. $\mathcal{N} = 1, \overline{\mathcal{N}} = \frac{1}{2}, P_{L} = P_{R} = 0$:
   - Common gravitational sector: $g_{\mu\nu}, b_{\mu\nu}, D$
     \[
     \alpha_{-1}^{\mu} \bar{\psi}_{-1/2}^{\nu} |0, k\rangle_{NS} \rightarrow \left\{ \frac{\sqrt{2} \epsilon_{\mu\nu} i \partial X^\mu(z) e^{-\phi} \bar{\psi}^\nu(\bar{z}) e^{ik \cdot X(z, \bar{z})}}{\sqrt{2} \epsilon_{\mu\nu} i \partial X^\mu(z) \tilde{\Upsilon}^\nu(\bar{z}) e^{ik \cdot X(z, \bar{z})}} \right\}
     \]  
     with $\phi$ the scalar from the bosonization of the superconformal ghost system,

     \[
     \tilde{\Upsilon}^\mu = \sqrt{2} i \partial X^\mu + \frac{1}{\sqrt{2}} k \cdot \bar{\psi} \bar{\psi}^\mu,
     \]  

     and $k^\mu \epsilon_{\mu\nu} = \epsilon_{\mu\nu} k^\nu = 0$. 

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• $k$ KK left abelian gauge vectors: $g_{m\mu} + b_{m\mu} \equiv a_{m\mu}$ and 16 Cartan generators of $SO(32)$ or $E_8 \times E_8$: $a^I_{m\mu}$

$$
\alpha_{I-1}^I \bar{\psi}_m^\mu \mid 0, k \rangle_{NS} \rightarrow \begin{cases} 
A_{i\mu} I \bar{i} i \partial Y^I(z) e^{-\phi \bar{\psi}_m^\mu(z)} e^{ik \cdot X(z, \bar{z})} \\
A_{I\mu} i \partial Y^I(z) \bar{\bar{\psi}}_m^\mu(z) e^{ik \cdot X(z, \bar{z})}
\end{cases},
$$

(2.37)

where the index $\hat{I} = (I, m)$ includes both the chiral “heterotic” directions and the compact toroidal ones, labeling the Cartan sector of the gauge group $G_L$.

• $k$ KK right abelian gauge vectors: $g_{m\mu} - b_{m\mu} \equiv a_{m\mu}$

$$
\alpha_{-1}^I \bar{x}_m^\mu \mid 0, k \rangle_{NS} \rightarrow \begin{cases} 
\sqrt{2} A_{\mu m} i \partial X^\mu(z) e^{-\phi \bar{x}_m^\mu(z)} e^{ik \cdot X(z, \bar{z})} \\
\sqrt{2} A_{\mu m} i \partial X^\mu(z) \bar{\bar{Y}}_m(z) e^{ik \cdot X(z, \bar{z})}
\end{cases},
$$

(2.38)

with

$$
\bar{\bar{Y}}_m = i \bar{\partial} Y_m + \frac{1}{\sqrt{2}} k \cdot \bar{\psi}_m^m.
$$

(2.39)

• $k(k + 16)$ scalars: $g_{mn}, b_{mn}, a^I_{m\mu}$

$$
\alpha_{-1}^I \bar{x}_m^m \mid 0, k \rangle_{NS} \rightarrow \begin{cases} 
S_{lm} i \partial Y^l(z) e^{-\phi \bar{x}_m^m(z)} e^{ik \cdot X(z, \bar{z})} \\
S_{lm} i \partial Y^l(z) \bar{\bar{Y}}_m(z) e^{ik \cdot X(z, \bar{z})}
\end{cases}
$$

(2.40)

2. $N = 0, \overline{N} = \frac{1}{2}, p^2_L = 2, p^2_R = 0$:

• $(N - k - 16)$ root vectors: $a^a_{\alpha}$

$$
\bar{\psi}_m^\mu \mid 0, k, \pi_\alpha \rangle_{NS} \rightarrow \begin{cases} 
A_{\mu \alpha} J^a(z) e^{-\phi \bar{\psi}_m^\mu(z)} e^{ik \cdot X(z, \bar{z})} \\
A_{\alpha \mu} J^a(z) \bar{\bar{Y}}_m(z) e^{ik \cdot X(z, \bar{z})}
\end{cases},
$$

(2.41)

with $k^\mu A_\mu = 0$ and currents

$$
J^a(z) = c_\alpha e^{i\alpha \cdot Y(z)},
$$

(2.42)

where $\alpha$ are the roots of $G_L$ (or equivalently the left momenta) and the cocycles $c_\alpha$ verify $c_\alpha c_\beta = \varepsilon(\alpha, \beta)c_{\alpha + \beta}$, with $\varepsilon(\alpha, \beta) = \pm 1$ the structure constants of $G_L$ in the Cartan-Weyl basis.

• $(N - k - 16) \times k$ scalars: $a_{\alpha n}$

$$
\bar{x}_m^m \mid 0, k, \pi_\alpha \rangle_{NS} \rightarrow \begin{cases} 
S_{nm} J^a(z) e^{-\phi \bar{x}_m^m(z)} e^{ik \cdot X(z, \bar{z})} \\
S_{nm} J^a(z) \bar{\bar{Y}}_m(z) e^{ik \cdot X(z, \bar{z})}
\end{cases}
$$

(2.43)

It is convenient to define the index $\Omega = (\hat{I}, \alpha) = 1, \ldots, N$ and condense the vertex operators for left vectors and scalars as

$$
A_{(-1)} = A_{\Omega \alpha} J^a(z) e^{-\phi \bar{\psi}_m^\mu(z)} e^{ik \cdot X(z, \bar{z})}
$$

(2.44)

$$
S_{(-1)} = S_{\Omega m} J^a(z) e^{-\phi \bar{x}_m^m(z)} e^{ik \cdot X(z, \bar{z})}
$$

(2.45)
where $J^i = i\partial Y^i$.

The massive states are obtained increasing the oscillation numbers $N$ and $\overline{N}$ or choosing $|\langle p_R, p_L \rangle|^2 \geq 4$.

Let us see what groups arise. Using that $p_R = 0$, we get from (2.8b) that the massless states have left-moving momentum

$$p_L = \left( \sqrt{2} \Delta_{am} w^m, \pi^A + w^m A^A_m \right),$$

while their momentum number on the torus is given by

$$n_m = (G_{mn} + B_{mn}) w^m + \pi^A A^A_m + \frac{1}{2} A^A_m A^A_m w^m.$$ (2.47)

Note that quantization of momentum number on the torus is a further condition to be imposed on top of $p_L^2 = 2$.

In the absence of Wilson lines $A^A_m = 0$, the $k$ torus directions decouple from the 16 chiral “heterotic directions” $Y^A$; $p^A = \pi^A$ is a vector of the weight lattice of $SO(32)$ or $E_8 \times E_8$ and then $|p^A|^2 \in 2\mathbb{N}$. The only possible massless states then have either momenta $p_L = (0, \pi^A)$ with $|\pi|^2 = 2$, or $p_L = (\sqrt{2} \Delta_{am} w^m, 0)$ with $w^m g_{mn} w^n = 1$ (and additionally $n_m w^m = 1$). The former are the root vectors of $SO(32)$ or $E_8 \times E_8$, while the latter have solutions only for certain values of the metric and $B$-field on the torus and lead to the same groups as in the (left sector of) bosonic string theory, namely all simply-laced groups $H$ of rank $k$. The total gauge group is then $SO(32) \times H \times U(1)^k_R$ or $E_8 \times E_8 \times H \times U(1)^k_R$. For $k = 1$, i.e. a circle compactification, $H$ is $SU(2)$ at $g_{11} = R^2 = 1$, and $U(1)$ for any other value of the radius. For compactifications on $T^2$, the possible groups of maximal enhancement (see footnote 1) are $SO(32) \times SU(2)^2 \times U(1)^2_R$ (for a diagonal metric with both circles at the self-dual radius and no $B$-field) or $SO(32) \times SU(3)_L \times U(1)^2_R$ (equivalently $SO(32) \rightarrow E_8 \times E_8$). See [13] for details.

Turning on Wilson lines, the pattern of gauge symmetries is more complicated, and also richer. In the sector with zero winding numbers, $w^m = 0$, we have $p^A = \pi^A$ as before, but now requiring a quantized momentum number imposes $\pi^A A^A_m \in Z$ (see (2.47)) which, for a generic Wilson line breaks all the gauge symmetry leaving only $\pi^A = 0$, which corresponds to the $U(1)^{16}$ Cartan subgroup. The opposite situation corresponds to $A^A_m \in \Gamma_g^* \big[6\big]$. For $E_8 \times E_8$, since $\Gamma_g^* = \Gamma_g \times \Gamma_g$, $A^A_m$ can be eliminated through a $\Lambda$-shift of the form $O_\Lambda$ in (2.21) and thus the pattern of gauge symmetries is as for no Wilson line. In the $SO(32)$ theory, the same conclusions hold if $A \in \Gamma_{16}$, but one has the more interesting possibility $A \in \Gamma_e$ or $A \in \Gamma_c$, where the $SO(32)$ symmetry is not broken, and the 16 chiral heterotic directions can be combined with the torus ones, giving larger groups which are not products.

Let us discuss the different groups that can arise in points of moduli space where the enhancement is maximal. In that case, the matrices that embed the internal sector of the heterotic theory on $T^k$ into a $16 + k$-dimensional bosonic theory are related to the

---

\[ \[ \text{We denote } \Gamma_g^* \text{ the dual of the root lattice, and one has } \Gamma_g^* = \Gamma_g \times \Gamma_g \text{ for } E_8 \times E_8 \text{ and } \Gamma_g^* = \Gamma_w = \Gamma_{16} = \Gamma_e + \Gamma_c \text{ for } SO(32) \text{ (see Appendix A for more details).} \]

\[ \[ \text{The only difference is that the massless states have shifted momenta } \pi^A \text{ and a shifted momentum number along the circle compared to the ones without Wilson lines, see Eq. (2.26).} \]

---
Cartan matrix $C$ by $[5]$

\[
\left( \begin{array}{cc}
(G + \frac{1}{2}A^tA^t)_{mn} & \frac{1}{2}A_m^t \\
\frac{1}{2}A^t_n & G_{IJ}
\end{array} \right) = \frac{1}{2}C_{IJ},
\]

\[
\left( \begin{array}{cc}
B_{mn} & \frac{1}{2}A_m^t \\
-\frac{1}{2}A^t_n & B_{IJ}
\end{array} \right) = \begin{cases} 
\frac{1}{2}C_{IJ} & \text{for } \hat{I} < \hat{J} \\
-\frac{1}{2}C_{IJ} & \text{for } \hat{I} > \hat{J} \\
0 & \text{for } \hat{I} = \hat{J}
\end{cases}
\]

(2.48)

One can then view the possible maximal enhancements from Dynkin diagrams. Let us first consider Wilson lines that do not break the original gauge group, i.e. $A \in \Gamma_g^*$. We start with the SO(32) heterotic theory. The Dynkin diagram of SO(32) is

The Dynkin diagrams of the gauge symmetry groups arising at points of maximal enhancement in the compactification of the SO(32) theory on $T^k$ have $k$ extra nodes, with or without lines in between. Since the resulting groups have to be in the ADE class (they are all simply laced), one cannot add nodes with lines on the left side. Therefore, the nodes should be added on the right side, and linked or not linked to the last node or not, and additionally add lines linking them to each other, or not. For one dimensional compactifications ($k = 1$), the only possibilities are

corresponding respectively to $SO(32) \times SU(2)$ and $SO(34)$. Since a line in the Dynkin diagram means that the new simple root is not orthogonal to the former one, then the Cartan matrix for this situation should have an off-diagonal term in the row corresponding to the new node and the column of the previous node, which according to (2.48) means that there is a non-zero Wilson line. Thus, no Wilson line (or a line in $\Gamma_{16}$, which is equivalent to no Wilson line) gives the enhancement group $SO(32) \times SU(2)$ and, as explained above, this enhancement works as in the bosonic theory, at $R = 1$. The enhancement symmetry group $SO(34)$ is obtained with a Wilson line in the vector or negative-chirality spinor conjugacy classes, and will be presented in detail in section 3.3.1. For compactifications on $T^k$, the $k$ extra nodes give as largest enhancement symmetry group $SO(32 + 2k)$, and this happens when Wilson lines in all directions are turned on. For less symmetric Wilson lines one gets smaller groups, and it is easy to see from the Dynkin diagrams what are all the possible groups. Here we draw all the possibilities for $k = 2$ only

corresponding respectively to $SO(36)$, $SO(34) \times SU(2)$, $SO(32) \times SU(2)^2$ and $SO(32) \times SU(3)$.

For the $E_8 \times E_8$ heterotic theory, the situation is less rich in the cases in which the dimension of the resulting group is larger than that of $E_8 \times E_8$. As we explained above, since $\Gamma_g^* = \Gamma_8 \times \Gamma_8$, a Wilson line that preserves the $E_8 \times E_8$ symmetry should be in the
lattice, and thus equivalent to no Wilson line. This can also be seen from the Dynkin diagram of $E_8 \times E_8$

\[ \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \]

where we see immediately that the extra nodes cannot be linked to any of the $E_8$'s, as any extra line would get us away from ADE. Then the possible enhancements are groups which are products of the form $E_8 \times E_8 \times H$, where $H$ is any semi-simple group of rank $k$, and each $H$ arises at the same point in moduli space as in the compactifications of the bosonic theory on $T^k [13]$. However, maximal enhancement can still be obtained by breaking one of the $E_8$ to $SO(16)$, and then the richness of the $SO(32)$ case is recovered (e.g. enhancement to $SO(18) \times E_8$).

If $A \notin \Gamma^*$, part or all of the $SO(32)$ or $E_8 \times E_8$ symmetry is broken, and one can still see groups that arise from the Dynkin diagrams. For compactifications on $T^k$, a priori any group of rank $16 + k$ in the ADE class can arise. However, we need to take into account that there are only $k$ linearly independent Wilson lines that can be turned on, so not any ADE group is actually achievable. For compactifications on a circle, in the $SO(32)$ theory one can reach for example the enhancement groups $SU(2) \times SO(2p) \times SO(32 - 2p)$ or $SO(2p + 2) \times SO(32 - 2p)$, and in the $E_8 \times E_8$ theory the groups $SO(2p) \times SO(18 - 2p) \times E_8$ for $p < 8$. A very interesting case is that of $p = 8$, where we conjecture that there is no Wilson line that achieves the enhancement to $SU(2) \times SO(16) \times SO(16)$.

Points of enhancement are fixed points of some $O(k, k + 16; \mathbb{Z})$ symmetry. Enhancement groups that are not semi-simple, i.e. that contain $U(1)$ factors, arise at lines, planes or hyper-planes in moduli space. On the contrary, maximal enhancement occurs at isolated points in moduli space. These are fixed points (up to discrete transformations) of the $O_D$ duality symmetry, or more general duality symmetries involving $O_D$. This is developed in detail in sections 3.5 and 3.6 for compactifications on a circle, to which we now turn.

### 3 Compactifications on a circle

Let us explicitly work out the circle compactification at radius $R$, with a Wilson line $A^A$. The momentum components (2.8b) are\footnote{From now on, suppressed indices in $p$ are orthonormal indices, i.e. $p_R \equiv p_{Ra}, p_L \equiv p_{La}$.}

\[
p_R = \frac{1}{\sqrt{2R}} \left[ n - R^2 w - \pi \cdot A - \frac{1}{2} |A|^2 w \right], \quad p_L = \frac{1}{\sqrt{2R}} \left[ n + R^2 w - \pi \cdot A - \frac{1}{2} |A|^2 w \right], \\
p^A = \pi^A + wA^A, \quad (3.1)
\]
where \(|A|^2 = A^A A^A = A\kappa A|^2\). The massless states, which satisfy \(p_R = 0\), have left-moving momenta
\[
p_L = (\sqrt{2} R w, \pi^A + w A^A) = (\sqrt{2} R w, p^A),
\]
and momentum number on the circle
\[
n = \left(R^2 + \frac{1}{2} |A|^2\right) w + \pi \cdot A.
\]

In the sector \(p_L = 0\) one has \(n = w = \pi^A = 0\,\), and the massless spectrum corresponds to the common gravitational sector and 18 abelian gauge bosons: 16 from the Cartan sector of \(E_8 \times E_8\) or \(SO(32)\) and 2 KK vectors, forming the \(U(1)^{18}\) gauge group.

The condition \(p_L^2 = 2\) can be achieved in three possible ways:
1. \(p_L = (0, p^A)\), with \(|p^A|^2 = 2\),
2. \(p_L = (\pm \sqrt{2}, 0)\),
3. \(p_L = (\pm s, p^A)\), with \(0 < s < \sqrt{2}\), \(s^2 + |p^A|^2 = 2\).

From (3.2) we see that sector 1 has \(w = 0\) and then (3.1) implies \(p^A = \pi^A\). The condition on the norm says that these are the roots of \(SO(32)\) or \(E_8 \times E_8\). But as explained in the previous section, one has to impose further that \(n \in \mathbb{Z}\) and thus from (3.3), \(\pi \cdot A \in \mathbb{Z}\). We divide the discussion into two cases, one in which this condition does not break the \(SO(32)\) or \(E_8 \times E_8\) symmetry, and the second one in which it does.

### 3.1 Enhancement of \(SO(32)\) or \(E_8 \times E_8\) symmetry

If we want the condition \(\pi \cdot A \in \mathbb{Z}\) not to select a subset of the possible \(\pi^A\) in the root lattice, or in other words not to break the \(SO(32)\) or \(E_8 \times E_8\) gauge symmetry, we have to impose
\[
A \in \Gamma^*_g,
\]
with
\[
\Gamma^*_g = \Gamma_8 \times \Gamma_8 \text{ for } E_8 \times E_8 \quad \text{or} \quad \Gamma^*_g = \Gamma_w = \Gamma_{16} + \Gamma_v + \Gamma_c \text{ for } SO(32).
\]

We restrict to this case now, and leave the discussion of the possible symmetry breakings to the next section.

Sector 2 contributes states only at certain radii \(R^2 = 1/w^2\), and have \(\pi = -w A\). The momentum number of these states given in (3.3) becomes
\[
n = \frac{1}{w} - \frac{1}{2} |A|^2 w = \frac{1}{w} \left(1 - \frac{|\pi|^2}{2}\right) \in \mathbb{Z}.
\]

If \(A \in \Gamma\), one has \(\frac{1}{2} |A|^2 \in \mathbb{Z}\), and thus the only way to satisfy the quantization condition is with \(w = \pm 1\), which gives two extra states at \(R = 1\), with momentum number \(n =\)
$\pm (1 - \frac{1}{2}|A|^2)$. For $SO(32)$, the condition $A \in \Gamma_w$ leaves the additional possibilities $A \in \Gamma_v$ or $A \in \Gamma_c$. Since $\pi = -wA \in \Gamma_{16}$, we have to require $w$ to be even. But then one cannot satisfy the quantization of momentum (3.5) since $\frac{1}{2}|A|^2 w \in \mathbb{Z}$. Thus, there are no extra states from this sector for $A \in \Gamma_v$ or $A \in \Gamma_c$.

Sector 3 contributes states only at radii $R^2 = s^2/(2w^2)$. The condition $|p^A|^2 < 2$ is only possible if $A$ is not in the root lattice. And as it is required to be in the weight lattice, this possibility arises in the $SO(32)$ heterotic theory only, for $A \in \Gamma_v$ or $A \in \Gamma_c$.

The quantization of momentum implies

$$n = \frac{1}{2} \left( \frac{s^2}{w} + |A|^2 w \right) + \pi \cdot A = \frac{1}{w} \left( 1 - \frac{|\pi|^2}{2} \right) \in \mathbb{Z},$$

where in the last equality we have used (3.2) and $|p_L|^2 = 2$. Note that the last equality is the same as (3.5). For $A \in \Gamma_v$, $\pi \cdot A \in \mathbb{Z}$ for $\pi \in \Gamma_g$ and $\frac{1}{2}|A|^2 = \frac{1}{2}$ (mod 1), so the only option is $s = 1$, giving extra states with $w = \pm 1$ at $R = 1/\sqrt{2}$. These states enhance $SO(32) \times U(1)$ to $SO(34)$. We present an explicit example of this case in section 3.3.1. For $A \in \Gamma_c$, $\pi \cdot A \in \mathbb{Z}$ for $\pi \in \Gamma_g$ but now $\frac{1}{2}|A|^2 \in \mathbb{Z}$ and thus we cannot satisfy the quantization condition (3.6) this way. However $\pi \cdot A = \frac{1}{2}$ (mod 1) for $\pi \in \Gamma_s$ and thus we recover that for these Wilson lines there is an enhancement to $SO(34)$ at $R = 1/\sqrt{2}$ as well, by states with $w = \pm 1$. Note that $A \in \Gamma_c$ is equivalent by a $\Lambda$ shift with $\Lambda \in \Gamma_s$ to $A \in \Gamma_v$. As we can see from (2.26), by this shift the winding number remains invariant, while $\pi \in \Gamma_s$ gets shifted to $\pi' \in \Gamma_g$.

We conclude that in circle compactifications with Wilson lines the pattern of gauge symmetry enhancement is (we give here only the groups on the left-moving side):

- $E_8 \times E_8 \times U(1) \rightarrow E_8 \times E_8 \times SU(2)$ at $R = 1$ if $A \in \Gamma_8 \times \Gamma_8$
- $SO(32) \times U(1) \rightarrow SO(32) \times SU(2)$ at $R = 1$ if $A \in \Gamma_{16}$, or
- $SO(32) \times U(1) \rightarrow SO(34)$ at $R = 1/\sqrt{2}$ if $A \in \Gamma_v$ or $A \in \Gamma_c$

In figures 1, 2 and 3 we show a slice of the moduli space of the heterotic theory compactified on a circle at a generic radius greater than one, at $R^2 = 1$ and at $R^2 = \frac{1}{2}$ respectively 11. In figure 4 we show another slice of moduli slice of moduli space that includes the radial direction. The first item above corresponds to the red points in figures 2b and 4b, while the second and third ones correspond, respectively to the red and green points in figures 2a, 3a and 4a. In the next section we will show how the enhancement at some of the other special points in the figures arise.

11For the $E_8 \times E_8$ heterotic theory, the Wilson lines chosen do not break the second $E_8$ factor and therefore we display the unbroken gauge group corresponding to the circle and first $E_8$ directions.
Figure 1: Enhancement groups on the slice of moduli space defined by $A_3^{3,\ldots,16} = 0$, $R = R_0$ with a generic $R_0 > 1$.

Figure 2: Enhancement groups on the slice of moduli space defined by $A_3^{3,\ldots,16} = 0$, $R = 1$.

Figure 3: Enhancement groups on the slice of moduli space defined by $A_3^{3,\ldots,16} = 0$, $R = 1/\sqrt{2}$. 
3.2 Enhancement-breaking of gauge symmetry

Whenever the Wilson line is not in the dual root lattice, part or all of the SO(32) or $E_8 \times E_8$ symmetry is broken. However, this does not imply that no symmetry enhancement from the circle direction is possible. The pattern of gauge symmetries can still be rich. We denote these cases enhancement-breaking of gauge symmetry. This nomenclature can be confusing however: for specific values of $R$ and $A$, there is the possibility that the symmetry enhancement is so large that it restores the original SO(32) or $E_8 \times E_8$ symmetry, or even leads to a larger group of rank 17. This means that we can have a maximal enhancement even if the Wilson line is not in the dual root lattice, either to the groups listed at the end of the previous section, or to any other simply-laced, semi-simple group of rank 17, such as for example SO(18) × $E_8$.

The massless states for an arbitrary Wilson line are the following:

Sector 1 has $w = 0$ (and thus $p^A = \pi^A$) and consists of the roots of SO(32) or $E_8 \times E_8$ satisfying $\pi \cdot A \in \mathbb{Z}$, which form a subgroup $H \subset SO(32)$ or $H \subset E_8 \times E_8$. We give examples of Wilson lines preserving $SO(16) \times E_8 \subset E_8 \times E_8$, $SO(2p) \times SO(32 - 2p) \subset SO(32)$, $U(p) \times SO(32 - 2p) \subset SO(32)$, $E_7 \times SU(2) \times E_8 \subset E_8 \times E_8$ and $U(16) \in SO(32)$ in the following section.

Sector 2 contains states only at radii $R^2 = 1/w^2$, and these states should have $\pi^A = -w A^A$. Since by definition $A \not\in \Gamma$, there are states in this sector if there is some integer $w \neq 1$ such that $w A \in \Gamma$. One should also impose the quantization condition (3.5). If these two conditions are satisfied for a given $w$, then there are two extra states in this sector, giving an enhancement of $H \times U(1)$ (where the $U(1)$ is the circle direction) to $H \times SU(2)$.

Sector 3 contains states only at radii $R^2 = s^2/(2w^2)$. Quantization of momentum gives the condition (3.6), but now $|A|^2$ is not necessarily integer. If there are states in this sector, there is an enhancement of the group $H \times U(1)$. This enhancement can be to $H \times SU(2)$ (where the SU(2) is along some direction mixing the circle with the heterotic directions) in the simplest case, but one can actually have enhancement of the group $H \times U(1)$ to a group that is not a product, like for example enhancement of $SO(16) \times U(1)$ to SO(18), as we will show in detail.
In sections 3.3.2, 3.3.3, 3.3.4, 3.3.5 and 3.3.6 we show explicitly how the groups mentioned in sector 1 get enhanced respectively to \( SO(18) \times E_8 \) at \( R^2 = \frac{1}{2} \); \( SU(2) \times SO(2p) \times SO(32 - 2p) \) at \( R^2 = 1 - \frac{p}{8} \); \( SU(2) \times U(p) \times SO(32 - 2p) \) at \( R^2 = 1 - \frac{p}{32} \); \( SU(2) \times E_8 \times E_8 \) at \( R^2 = \frac{1}{4} \) and \( U(17) \) at \( R^2 = \frac{1}{4} \).

3.3 Explicit examples

Here we present some examples of symmetry enhancement-breaking. The roots of \( SO(32) \) are given by

\[
SO(32) : (\pm 1, \pm 1, 0^{14}),
\]

where underline means all possible permutations of the entries. The roots of \( E_8 \times E_8 \) are

\[
E_8 \times E_8 : (\pm 1, \pm 1, 0^6), (0^8, \pm 1, \pm 1, 0^6),
\]

\[
((\pm \frac{1}{2})^8, 0^8), (0^8, (\pm \frac{1}{2})^8), \text{with even number of } + \text{ signs}
\]

3.3.1 \( U(1) \times U(1) \times SO(30) \rightarrow SO(34) \)

Consider the \( SO(32) \) heterotic theory compactified on a circle of radius \( R = 1/\sqrt{2} \) with a Wilson line \( A = (1, 0, \ldots, 0) \in \Gamma_v \). The states with \( p_R = 0 \) have left-moving momenta

\[
P_L = (w, \pi^A + \delta^A w)
\]

where the first entry corresponds to the circle direction. In sector 1, with \( w = 0 \), one has all momenta satisfying \( |\pi^A|^2 = 2 \) and \( \pi \cdot A \in \mathbb{Z} \). The last condition is true for any \( \pi^A \in \Gamma_g \), and thus in this sector we have all the root vectors of \( SO(32) \) given in (3.7). There are no states in sector 2, as \( w \in \mathbb{Z} \). In sector 3 we have \( s = 1 \) and \( w = \pm 1 \). Here we get massless states coming from three different sectors of the \( SO(32) \) weight lattice, namely

3.a) \( |\pi|^2 = 2 \), with \( \pi^i = \pm 1 \)

\[
P_L = (\pm 1, 0, \pm 1, 0, 0, \ldots, 0)
\]

(3.10)

(where the signs are not correlated). These are 60 states with \( n = 0 \).

3.b) \( |\pi|^2 = 0 \),

\[
P_L = (\pm 1, \pm 1, 0, \ldots, 0).
\]

These are 2 states, which have \( n = w \).

3.c) \( |\pi|^2 = 4 \), with \( \pi^i = \pm 2 \)

\[
P_L = (\mp 1, \pm 1, 0, \ldots, 0).
\]

Another 2 states with \( n = -w \).

We thus get 64 extra states, which together with the Cartan direction of the circle, enhance the \( SO(32) \) to \( SO(34) \). This point in moduli space is illustrated in green in figures 3a and 4a. In the figure 3a the other green points differ from this by a \( \Lambda \)-shift, while the other green points in figure 4a that appear at a different radius, will be explained in section 3.4.

\[\text{As we will show, for } p = 6 \text{ the enhancement group is actually larger, } SU(2) \times pin(12)/Z_2 \times SO(20).\]
3.3.2 $U(1) \times SO(16) \times E_8 \rightarrow SO(18) \times E_8$

Consider the $E_8 \times E_8$ heterotic string compactified on a circle of radius $R = \frac{1}{\sqrt{2}}$, with Wilson line $A = (1, 0^7, 0^8)$, which is of the form $(v0)$ according to the notation of Appendix A (see (A.10) in particular). This Wilson line leaves the second $E_8$ unbroken, while from the first $E_8$, the surviving states in sector 1 are the ones with integer entries, i.e. those in the first line of (3.8). The group $H$ from sector 1 is then $SO(16) \times E_8$ and the corresponding points in moduli space are illustrated by the grey dots in figure 1b.

There are no states in sector 2, while in sector 3 we have states with $w = \pm 1$ such that $s = 1, |p^A|^2 = 1$. The surviving states have the following momenta

\[
\begin{align*}
\mathbf{p}_L &= (0, \pm 1, \pm 1, 0_6), \quad w = 0, \quad |\pi|^2 = 2 \quad 112 \text{ roots} \\
\mathbf{p}_L &= (\pm 1, 0, \pm 1, 0_6), \quad w = \pm 1, \quad |\pi|^2 = 2, \quad 28 \text{ roots} \\
\mathbf{p}_L &= (\pm 1, \pm 1, 0_7), \quad w = \pm 1, \quad \pi = 0, \quad 2 \text{ roots} \\
\mathbf{p}_L &= (\pm 1, \mp 1, 0_7), \quad w = \pm 1, \quad |\pi|^2 = 4, \quad 2 \text{ roots},
\end{align*}
\]

where the first entry corresponds to the circle and the subsequent ones to the 8 directions along the Cartan of the first $E_8$ factor. The first line contains the states of sector 1. These are the 144 roots of $SO(18)$. This point in moduli space, together with its equivalent ones, are illustrated by the green dots in figure 3b.

3.3.3 $U(1) \times SO(2p) \times SO(32 - 2p) \rightarrow SU(2) \times SO(2p) \times SO(32 - 2p)$

Consider the $SO(32)$ heterotic string compactified on a circle of radius $R$, with a Wilson line $A = \left(\left(\frac{1}{2}\right)_p, 0_{16-p}\right)$, with $2 \leq p \leq 8$.

The massless states that survive in sector 1 ($w = 0$) are those with momentum $\pi^A$ satisfying

\[
\frac{1}{2} \sum_{A=1}^{p} \pi^A \in \mathbb{Z}.
\]

Then the surviving states have momenta

\[
\begin{align*}
\mathbf{p}_L &= (0, \pm 1, \pm 1, 0_{p-2}, 0_{16-p}) \rightarrow SO(2p) \\
\mathbf{p}_L &= (0, 0_p, \pm 1, \pm 1, 0_{14-p}) \rightarrow SO(32 - 2p)
\end{align*}
\]

For $p = 2$, these points are illustrated by the violet dots in figures 1a, 2a and 3a. Additionally, by states in sector 3, the $U(1)$ of the circle can be enhanced to $SU(2)$ at $R^2 = 1 - \frac{p}{8}$ for $p < 8$. These states have $n = w = \pm 1, \pi^A = 0$ and

\[
\mathbf{p}_L = \left(\pm \sqrt{2 - \frac{p}{4}}, \pm \left(\frac{1}{2}\right)_p, 0_{16-p}\right).
\]

An interesting phenomenon occurs for $p = 2$ and $p = 6$, where there are also massless states in sector 2 at $R^2 = \frac{1}{4}$ (note this is equal to $1 - \frac{p}{8}$ for $p = 6$). For $p = 2$, these have

\[^{13}\text{Note that } p > 8 \text{ is equivalent, by a shift } \Lambda = -\left(\left(\frac{1}{2}\right)_{16}\right), \text{ to } p' = 16 - p < 8.\]
$w = \pm 2$, $n = 0$, $\pi = \mp(1,1,0_{14})$, and thus one gets enhancement to $SO(4) \times SO(28) \times SU(2)$ either at $R^2 = \frac{1}{4}$ from states in sector 2, or $R^2 = \frac{3}{4}$ from states in sector 3.

For $p = 6$, besides the states in (3.15), there are massless states in sector 3 (always at $R^2 = \frac{3}{4}$) with $w = \pm 1$ and momentum

$$ p_L = \left( \pm \sqrt{2}, \mp \left( \frac{1}{2} \right)_2, \pm \left( \frac{1}{2} \right)_4, 0_{10} \right) $$

$$ p_L = \left( \pm \sqrt{2}, \mp \left( \frac{1}{2} \right)_4, \pm \left( \frac{1}{2} \right)_2, 0_{10} \right) $$

(3.16)

Together with the states in (3.15), there are 32 extra states, corresponding to the $(s)$ class of $SO(12)$, which together with the elements in the adjoint displayed in (3.14) give an enhancement to $Spin(12)/Z_2$. On top of this, there are two extra states in sector 2, with $w = \pm 2$ and $\pi = \mp(1,0_{10})$, giving an additional $SU(2)$. The total enhancement group for $p = 6$ at $R = \frac{1}{2}$ is then $SU(2) \times Spin(12)/Z_2 \times SO(20)$.

A special case is $p = 8$, where according to the formula for the radius $R^2 = 1 - \frac{p}{8}$, one would get enhancement to $SO(16) \times SO(16) \times SU(2)$ at $R = 0$, which is not part of the moduli space. We show in more detail in section 3.6.3 that the Wilson line $A = \left( \left( \frac{1}{2} \right)_8, 0_8 \right)$ there is no radius such that the $U(1) \times SO(16) \times SO(16)$ symmetry is enhanced to $SU(2) \times SO(16) \times SO(16)$. We conjecture that the enhancement group $SU(2) \times SO(16) \times SO(16)$ is never realised.

3.3.4 $U(1) \times U(p) \times SO(32 - 2p) \rightarrow SU(2) \times U(p) \times SO(32 - 2p)$

Consider the $SO(32)$ heterotic string compactified on a circle of radius $R$, with a Wilson line $A = \left( \left( \frac{1}{4} \right)_p, 0_{16-p} \right)$ with $2 \leq p \leq 14$. The massless states that survive in sector 1 are those satisfying

$$ \frac{1}{4} \sum_{A=1}^{p} \pi_A \in \mathbb{Z}, \quad (3.17) $$

and thus we get the following states in sector 1

$$ p_L = (0,1,-1,0_{p-2},0_{16-p}) \rightarrow U(p) $$

$$ p_L = (0,0_p,\pm 1,\pm 1,0_{14-p}) \rightarrow SO(32 - 2p) $$

(3.18)

Additionally, by states in sector 3, the $U(1)$ of the circle can be enhanced to $SU(2)$ at $R^2 = 1 - \frac{p}{32}$. These states have $n = w = \pm 1$, $\pi^A = 0$ and

$$ p_L = \left( \pm \sqrt{2 - \frac{p}{16}}, \pm \left( \frac{1}{4} \right)_p, 0_{16-p} \right). \quad (3.19) $$

We thus have enhancement to $U(p) \times SO(32 - 2p) \times SU(2)$ at $R^2 = 1 - \frac{p}{32}$

---

Footnote 14: These arise from states that have heterotic momentum $\pi = \mp(1,0_{14})$, $\pi = \mp(1,0_{2},0_{10})$ and $\pi = \mp(1,0_{10})$. 

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For \( p = 8 \), there are additional massless states, and a larger enhancement group, coming from states with \( \pi = \mp (1, 1, 0_6) \) and \( w = \pm 1 \). These have

\[
P_L = \pm (\frac{\sqrt{2}}{2}, \frac{3}{4}, \frac{1}{6}, 0_8).
\]

These states, together with those in (3.19) promote the \( U(8) \) to an \( SO(16) \), and thus at \( p = 8 \) the enhancement is to \( U(1) \times SO(16) \times SO(16) \) (as we mentioned at the end of the previous section, the \( U(1) \) does not get enhanced to \( SU(2) \), at this radius either).

### 3.3.5 \( U(1) \times U(1) \times E_7 \times E_8 \rightarrow SU(2) \times E_8 \times E_8 \)

This is an interesting example of enhancement-breaking in the \( E_8 \times E_8 \) heterotic theory, where first the \( E_8 \) is broken to \( SU(2) \times E_7 \) by the Wilson line \( A = (\frac{1}{4}, 0_8) \) and then enhanced by the circle direction to \( SU(2) \times E_8 \).

The Wilson line leaves the second \( E_8 \) unbroken, while the surviving roots from the first \( E_8 \) have 9-momenta

\[
P_L = \pm (0, 1, -1, 0_6)
\]

\[
P_L = \pm (0, \frac{1}{6})
\]

\[
P_L = (0, \frac{1}{4}, -\frac{1}{2})
\]

(3.21)

This gives 128 roots, which together with the 8 Cartan directions, gives an unbroken gauge group \( H = SU(2) \times E_7 \subset E_8 \).

Additionally at \( R = \frac{1}{2} \) there are two states in sector 2 with \( w = \pm 2 \) and 112 states in sector 3 with \( w = \pm 1 \) and momentum

\[
P_L = (\pm \frac{\sqrt{2}}{2}, \mp \frac{3}{4}, \pm \frac{1}{6})
\]

(3.22)

These states give a total of 114 extra states that add up to the previous 136 states, plus the circle direction, adding up to the 251 states of \( SU(2) \times E_8 \). So at \( R = \frac{1}{2} \) we get enhancement to \( SU(2) \times E_8 \times E_8 \), which works very differently than the enhancement occurring at \( R = 1 \), mentioned in section 3.1.

### 3.3.6 \( U(1) \times U(16) \rightarrow U(17) \)

Consider the \( SO(32) \) heterotic theory with Wilson line \( A = (\frac{1}{2}, 0_{16}) \). This Wilson line breaks the \( SO(32) \) gauge symmetry leaving the states with weight \( \pi = \pm (1, -1, 0_{14}) \), corresponding to \( U(16) \). Additionally, at \( R = \frac{1}{2} \) there are extra states in sector 3 that have momenta

\[
P_L = (\pm \frac{\sqrt{2}}{2}, \mp \frac{3}{4}, \pm \frac{1}{6})
\]

(3.23)

These are 32 additional roots, which enhance the \( U(1) \times U(16) \) occurring for that Wilson line at any radius, to \( U(17) \).
3.4 Exploring a slice of moduli space

In this section we present a detailed analysis of the slice of moduli space for compactifications of the heterotic theory on a circle at any radius and Wilson line given by

$$A = (A_1, 0_{15}) .$$  \hfill (3.24)

The results of this section are displayed in figure 4. Here we present the main ingredients in the calculations, and leave further details to Appendix B.

For this type of Wilson line, the states with \( w = 0 \) (sector 1) that survive, are those satisfying

$$\pi_1 A_1 \in \mathbb{Z} . \hfill (3.25)$$

This preserves all the roots only if \( A_1 \in \mathbb{Z} \) for the \( \Gamma_{16} \) case, or \( A_1 \in 2\mathbb{Z} \) for the \( \Gamma_8 \times \Gamma_8 \) case. These correspond to the horizontal orange lines in figure 4, where at any generic radius, the gauge symmetry is \( U(1) \times SO(32) \), or \( U(1) \times E_8 \times E_8 \). If \( A_1 \) is an odd number, then the \( SO(32) \) symmetry is unbroken, but the \( E_8 \times E_8 \) is broken to \( SO(16) \times E_8 \), which is depicted with a black line at \( A_1 = 1 \) in figure 4b.

If \( A_1 \notin \mathbb{Z} \), then we have just the roots with \( \pi_1 = 0 \). That is, the 420 roots of \( SO(30) \) or the 324 roots of \( SO(14) \times E_8 \). This corresponds to the white regions in figure 4.

Now, depending on the value of \( R \), we can have additional states in sectors 2 and 3, i.e. states with non-zero winding which momenta satisfy \( |p_L|^2 = 2 \) and have a quantized momentum number on the circle. Then, according to (3.2) and (3.5,3.6), they should obey

$$|\pi + w A|^2 = 2(1 - w^2 R^2) , \hfill (3.26)$$

$$\frac{1}{w} (1 - \frac{1}{2} |\pi|^2) \in \mathbb{Z} .$$

The first equation implies \( R^{-1} \geq w \), and the simplest solution is

$$\pi = \left( \pm \sqrt{2(1 - w^2 R^2)} - w A_1, 0_{15} \right) .$$

But \( \pi \) is in an even lattice, which implies \( \pi_1 = -2q, q \in \mathbb{Z} \). The quantization condition for \( n \) yields

$$\frac{2q^2 - 1}{w} \in \mathbb{Z} ,$$

so we have only the winding numbers that are divisors of the numbers that can be written as \( 2q^2 - 1 \), for some integer \( q \). In terms of \( q \), the Wilson lines are of the form

$$A_1 = \frac{2q \pm \sqrt{2 - 2w^2 R^2}}{w} \equiv a_{w,q}(R) , \quad \{ w, q, \frac{2q^2 - 1}{w} \} \in \mathbb{Z} . \hfill (3.27)$$

If the radius also satisfies \( R < \frac{1}{\sqrt{2w}} < \frac{1}{w} \), we have additional solutions where some of the other components of \( \pi \) are non-zero, such that

$$\pi + w A = \left( \pm \sqrt{1 - 2w^2 R^2}, \pm 1, 0_{14} \right) \text{ for } \Gamma_{16} ,$$

$$\pi + w A = \left( \pm \sqrt{1 - 2w^2 R^2}, \pm 1, 0_6, 0_8 \right) \text{ for } \Gamma_8 \times \Gamma_8 .$$

\( ^{15} \)From now on we take \( w > 0 \), keeping in mind that for every massless state with \( w \) there is also a massless state with \( -w \).
The quantization conditions are the same as before, but now the Wilson lines have the following behavior as a function of the radius

\[ A_1 = \frac{2q + 1 \pm \sqrt{1 - 2w^2R^2}}{w} \equiv b_{w,q}(R), \quad \{ w, q, \frac{2q^2}{w} - 1 \} \in \mathbb{Z}. \quad (3.28) \]

If additionally \( R < (2\sqrt{2}w)^{-1} \) we have yet other possible solutions, but only for the \( E_8 \times E_8 \) theory, where

\[ \pi + wA = \left( \pm \frac{1}{2} \sqrt{1 - 8w^2R^2}, (\pm \frac{1}{2})_7, 0_8 \right) \text{ for } \Gamma_8 \times \Gamma_8. \]

The quantization conditions are the same as before, but now the Wilson lines have the following behavior as a function of the radius:

\[ A_1 = \frac{q + \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2w^2R^2}}{w} \equiv c_{w,q}(R), \quad \{ w, q, \frac{q(q + 1)}{2w} \} \in \mathbb{Z}, \quad (3.29) \]

where we used \( (\pi_1)^2 = |\pi|^2 - \frac{7}{4} \) and \( \pi_1 = -(q + \frac{1}{2}) \).

For a given \( q \) and \( w \), whenever the Wilson line is of the form \( a_{w,q} \) in (3.27), we get 2 massless states (one for \( w > 0 \) and another one for \( w < 0 \)). If there are no more states, then we have enhancement to \( U(1) \times SU(2) \times SO(30) \) and \( U(1) \times SU(2) \times SO(14) \times E_8 \). These correspond to the blue lines in figure 4, where for example in figure 4a, the long blue line going from \((R, A_1) = (0, \sqrt{2}) \) to \((1, 0)\) corresponds to \( a_{1,0} = \sqrt{2(1 - R^2)} \), while its mirror one along the axis \( A_1 = 1 \) is \( a_{1,1} = 2 - a_{1,0} \).

For Wilson lines of the form \( b_{w,q} \) in (3.28), we get 60 extra states for the \( \Gamma_{16} \), and 28 for \( \Gamma_8 \times \Gamma_8 \). The former promote the enhancement to \( U(1) \times SO(32) \), while the latter to \( U(1) \times SO(16) \times E_8 \), and they correspond respectively to the orange lines in figure 4a and the black lines in figure 4b. The largest curved orange line in the former and black line in the latter going from \((0, 0) \) to \((0, 2)\) corresponds to \( b_{0,1} = 1 \pm \sqrt{1 - 2R^2} \), where the plus sign is for the upper half of the curve, and the minus sign for the lower half.

Finally, Wilson lines of the form \( c_{w,q} \) in (3.29) give in the \( E_8 \times E_8 \) heterotic theory, \( 2 \times 2^6 = 128 \) states (the sign of one of the seven \( \pm \frac{1}{2} \) is determined by the sign of the other 6 and the sign chosen for the Wilson line). Note that \( c_{w,q}(R) = b_{2w,q}(R) \). It is not hard to show that a Wilson line that can be written as \( c_{w,q}(R) \) can always be written as \( b_{2w,q}(R) \), but the function \( b \) can also have an odd \( w \). Wilson lines \( b \) that can also be written as \( c \) bring then a total of \( 28 + 128 = 156 \) states, which corresponds to the enhancement to \( U(1) \times E_8 \times E_8 \) in the orange lines of figure 4b.

There are only two kinds of intersections between lines, and the points of intersection correspond to points of maximal enhancement (see Appendix B for details):

- between a blue curve \( a(R) \) with \( w_1 \) and an orange curve \( b(R) \) with \( w_2 \), where the enhancement group is \( SU(2) \times SO(32) \) (\( SU(2) \times E_8 \times E_8 \)) in the \( SO(32) \) \( (E_8 \times E_8) \) theory. These are the red dots of figure 4 and arise at

\[ (R, A_1) = \left( \frac{1}{\sqrt{w_1^2 + 2w_2^2}}, \frac{2}{w_1} (q \pm w_2R) \right) = \left( \frac{1}{C}, \frac{2k}{C} \right), \]

for some integer \( k \), with \( C = 1, 3, 9, 11, ... \) are all the integers whose prime divisors are 1 or 3 (mod 8) (see Table 1).
between two blue $a(R)$ with $w_1$ and $w_2$ and two orange (black) curves $b(R)$ with $w_3$ and $w_4$, where the enhancement group is $SO(34)$ ($SO(18) \times E_8$) for the $SO(32)$ ($E_8 \times E_8$) theory. These are the green dots of figure 4 and arise at $^{19}$:

\[(R, A_1) = \left( \frac{1}{\sqrt{w_1^2 + w_2^2}}, \frac{2}{w_1} \left( q \pm \frac{1}{\sqrt{2}} w_2 R \right) \right) = \left( \frac{1}{\sqrt{2C}}, \frac{k}{C} \right),\]

for some integer $k$, with $C = 1, 5, 13, 17, \ldots$ are all the integers whose prime divisors are Pythagorean primes (see Table 1).

In Appendix B we give the details of the calculations and also prove that these are the only possible intersections for this type of Wilson lines. In section 3.6 we show how these points arise as fixed points of a duality symmetry.

### 3.5 T-duality in circle compactifications

In this section we discuss the action of T-duality in the heterotic string compactified on a circle. By T-duality we mean the action of certain type of transformations in $O(1, 17, \mathbb{Z})$ that relate a given heterotic theory with 16-dimensional lattice $\Gamma$, compactified on a circle of radius $R$ and Wilson line $A$, to another heterotic theory with lattice $\Gamma'$, compactified on a circle of radius $R'$ and Wilson line $A'$. In this section we discuss the usual T-duality exchanging momentum and winding numbers, while in the next section we discuss more general dualities, and their fixed points.

The duality generated by the matrix $O_D$ is the usual T-duality transformation exchanging momentum and winding numbers

\[(w', n', \pi') = (n, w, \pi) . \tag{3.30}\]

Since $\pi$ stays untouched, this duality is possible if $\Gamma' = \Gamma$. Its action on the background fields can be worked out from the generalized metric (2.13), which for the circle is $^{17}$:

\[\mathcal{M} = \begin{pmatrix} R^2 (1 + \frac{1}{2} A^2)^2 & -\frac{1}{2} A^2 & (1 + \frac{1}{2} A^2)A \\ -\frac{1}{2} A^2 & \frac{R^2}{A} & -\frac{1}{R^2} A \\ (1 + \frac{1}{2} A^2) A^t & -\frac{1}{R^2} A^t & I + \frac{1}{R^2} A^t A \end{pmatrix}, \tag{3.31}\]

where we have defined the scalar

\[A^2 = \frac{|A|^2}{R^2}. \tag{3.32}\]

The action of $O_D$ transforms this into

\[\mathcal{M}' = O_D \mathcal{M} O_D^t = \mathcal{M}^{-1} = \begin{pmatrix} \frac{1}{R^2} & -\frac{1}{2} A^2 & -\frac{1}{R^2} A \\ -\frac{1}{2} A^2 & R^2 (1 + \frac{1}{2} A^2)^2 & (1 + \frac{1}{2} A^2) A \\ -\frac{1}{R^2} A^t & (1 + \frac{1}{2} A^2) A^t & I + \frac{1}{R^2} A^t A \end{pmatrix}, \tag{3.33}\]

\[\text{We get additionally } R = \frac{1}{\sqrt{w_1^2 + w_2^2}} = \frac{1}{\sqrt{2 \sqrt{w_3^2 + w_4^2}}}. \]

\[\text{Here we choose the Cartan-Weyl basis where the Killing metric for the Cartan subgroup } \kappa^{IJ} \text{ is diagonal.}\]
and thus we get
\[ A' = -\frac{A}{R^2(1 + \frac{1}{2}A^2)} , \quad R' = \frac{1}{R (1 + \frac{1}{2}A^2)} \quad (\Rightarrow \frac{A'}{R'} = -\frac{A}{R}) \]
in agreement with the heterotic Buscher rules for scalars \[21\]. We get that a background has \( R' = R \) for
\[ R_{sd}^2 = 1 - \frac{1}{2}|A|^2 \quad (\Rightarrow R' = R, A' = -A) \] (3.34)
Additionally, if \( 2A \in \Gamma' \), then \( A' = -A \sim A \), and therefore the background is fully self-dual, satisfying \( \mathcal{M} = \mathcal{M}^{-1} \) up to discrete transformations (these are of the form (2.19), (2.20) or (2.21), but for the circle the only non-trivial one is a \( \Lambda \)-shift (2.21)).

All the examples of enhancement and enhancement-breaking discussed previously satisfy the self-duality condition (3.34): the enhancement to \( SO(32) \times SU(2) \) or \( E_8 \times E_8 \times SU(2) \) discussed in section 3.1 occurs at \( A = 0, R^2 = 1 \); the \( SO(34) \) and \( E_8 \times SO(18) \) enhancements discussed respectively in sections 3.3.1 and 3.3.2 have \( R^2 = \frac{1}{2}, |A|^2 = 1 \)
\[ \text{the } SO(32 - 2p) \times SO(2p) \times SU(2) \text{ of section 3.3.3 is at } R^2 = 1 - \frac{p}{8}, |A|^2 = \frac{p}{4}, \]
\[ \text{the } SU(2) \times E_8 \times E_8 \text{ in section 3.3.5 is at } R^2 = \frac{1}{4}, |A|^2 = \frac{3}{2} (\text{if we perform a } \Lambda \text{-shift on } A \text{ to bring it to the equivalent one } A = ((-3/4)_2, (1/4)_6, 0_8). \]
In all these examples one has \( 2A \in \Gamma \), hence \( A' \sim A \) and then the backgrounds are fully self-dual, namely \( R' = R, A' \sim A \).

For Wilson lines with only one non-zero component, we have that the fixed “points” of this symmetry correspond actually to lines of non-maximal enhancement symmetry where the Wilson lines are functions of the radius \( (A = A(R_{sd})) \), and are such that \( A \sim A_{sd} \), with \( |A_{sd}|^2 = 2(1 - R_{sd}^2) \).

We now discuss the differences between fixed points of duality symmetries further, exploring more general dualities and their fixed points.

### 3.6 More general dualities and fixed points

The transformation \( O_D \) discussed before is a particular type of transformation that changes the sign of \( p_R \) while it rotates \( p_L \), preserving its norm (in compactifications of the bosonic theory on a circle, \( p_L \) has a single component and \( O_D \) just leaves it invariant, but in the heterotic theory \( O_D \) rotates the 17-dimensional vector \( p_L \)). It would be very interesting to understand what are all the possible transformations that do this, and obtain their fixed points. Here we do something more modest, namely we work out the set of transformations that change the sign of \( p_R \) and rotate \( p_L \), leaving its circle direction component invariant. We thus require
\[ (p_L', p_R'^A, p_R') = (p_L, U^{AB}p_B, -p_R) \] (3.35)
with \( U \in O(16, \mathbb{Z}) \). These transformations generically link a given heterotic theory with lattice \( \Gamma \), in a background defined by \( (A, R) \) to another heterotic theory with lattice \( \Gamma' \) in

---

\[ \text{For } |A|^2 \geq 2, \text{ one has that if } A \in \Gamma_{16} \text{ or } A \in \Gamma_8 \times \Gamma_8, \text{ then by a } \Lambda \text{-shift (2.21) with } \Lambda = -A \text{ (which according to (2.26) leaves the winding number } w \text{ invariant while shifts } \pi \rightarrow \pi + A'w, n \rightarrow n - \frac{1}{2}|A|^2w - \pi \text{), it is equivalent to no Wilson line, while if } A \in \Gamma_v, \text{ one can similarly shift it to } A \in \Gamma_v \text{ such that } |A|^2 = 1. \]
a dual background with \((A', R')\). The duality transformation depends on the matrix \(U\) and we use a convenient parameterization to relate the radii \(R\) and \(R'\), namely we define a positive number \(r\) such that

\[
R' = \frac{1}{rR}.
\]  

(3.36)

The duality transformation that achieves \((3.35)\) should have the form

\[
O_U = \begin{pmatrix}
-\frac{r|A'|^2}{2} & \frac{1}{r} + A'U A^t + \frac{r|A|^2}{2} & \frac{|A'|^2}{2} A + A' U \\
\frac{1}{r} & -\frac{|A|^2}{2} & -rA \\
-rA'' & UA^t + \frac{|A|^2}{2} A'' & U + rA'' A
\end{pmatrix}.
\]  

(3.37)

Requiring this to be in \(O(1,17; \mathbb{Z})\), we get a set of quantization conditions like for example\(^{19}\) (the full set of quantization conditions is given in (C.2))

\[
r, \frac{r|A|^2}{2}, \frac{r|A'|^2}{2}, \frac{1}{r} + A'U A + \frac{|A|^2}{2} \in \mathbb{Z}.
\]  

(3.38)

It is instructive to decompose the matrices \(O_U\) as the product \(O_{\Lambda'} O_D O_N O_M O_{\Lambda}\) with \(\Lambda = -A, M = r, N = U\) and \(\Lambda' = A'\), which allows to interpret the transformations as the following series of operations

1. \(O_{-A}\): eliminates the Wilson line \(A\) through a \(\Lambda\)-shift,
2. \(O_r\): rescales \(R \to rR\),
3. \(O_U\) performs a change of basis in the heterotic directions
4. \(O_D\) performs a T-duality along the circle \((\eta)\)
5. \(O_{A'}\): adds the Wilson line \(A'\) through a \(\Lambda\)-shift.

We divide the discussion into the dualities where \(\Gamma = \Gamma'\), and those where the dual lattice is not the original one. To denote the different sublattices that will play a role, it is useful to use the \((0), (v), (s)\) and \((c)\) conjugacy classes of \(SO(16)\), corresponding respectively to the root, vector, positive and negative-chirality spinor classes. These are defined in \((A.4)-(A.7)\). The lattices \(\Gamma_{16}\) and \(\Gamma_{8} \times \Gamma_{8}\) contain the following vectors (see \((A.10)-(A.11)\))

\[
\Gamma_{16} = (00), (vv), (ss), (cc)
\]

\[
\Gamma_{8} \times \Gamma_{8} = (00), (ss), (0s), (s0)
\]  

(3.39)

One could have chosen different conventions in which some of the \(s\) classes are turned into \(c\) classes, and doing that build four other lattices, that we denote \(\Gamma_{16}^+, \Gamma_{8}^- \times \Gamma_{8}^-, \Gamma_{8}^- \times \Gamma_{8}^+\) and \(\Gamma_{8}^+ \times \Gamma_{8}^-\). We give these in \((A.14)\). Note that a lattice \(\Gamma^+\) is equivalent to a lattice \(\Gamma^-\), the choice \((s)\) versus \((c)\) conjugacy class is a convention with no physical relevance. Here it is important however to make the distinction whether a given duality maps, say, \(\Gamma^+\) to \(\Gamma^+\), or \(\Gamma^+\) to \(\Gamma^-\).

\(^{19}\) The fact that we get a quantization condition for \(|A|\) may sound strange, but it means that if \(A\) is not quantized properly there is no duality that leaves the circle direction of \(p_L\) invariant. If one allows the full \(p_L\) vector to rotate under the transformation, then we have, as shown, at least the duality \(O_D\) discussed in previous section.
In the following we write the main results, leaving the details to Appendix C. The results for generic Wilson lines, assuming that \( r \) is a prime number, are summarized in Table 2. We later concentrate on the situation where the Wilson lines are of the form (3.24), i.e. with only one non-zero component, as we did in section 3.4, to see what happens when the assumption that \( r \) is prime is relaxed. For Wilson lines of this form, the \( O(16) \) symmetry is broken to \( O(15) \), and there are four inequivalent choices of \( U \) that we will analyze in detail

\[
U = \pm I, \quad \text{or} \quad U_\pm \equiv \pm \text{diag}(1, -1_{15}).
\]  

3.6.1 \( \Gamma \leftrightarrow \Gamma \)

The dualities for which the lattice does not change involve those where \( \pi \) is invariant, such as the one discussed in the previous section. But as explained above, one can have more general dualities even when \( \Gamma' = \Gamma \), and thus more general fixed points. Fixed points of a duality are those for which \( R' = R \) and \( A' = A \).

To make the analysis tractable for generic Wilson lines, we restrict to the situation where \( r \) is a prime number and \( U = I \), and relax this assumption only in the setup where the Wilson lines have just one non-zero component. Under the assumption that \( r \) is a prime number, the full set of quantization conditions (C.2) are satisfied if and only if (see details in Appendix C)

\[
A \in \Gamma, \quad A' \in \Gamma, \quad r = 1,
\]

and thus the fixed points of these transformations are at \( R = 1 \) and \( A \) any point in the lattice \( \Gamma \). They correspond to enhancements to \( SU(2) \times SO(32) \) and \( SU(2) \times E_8 \times E_8 \) discussed in section 3.1. These points appear in the diagonal entries in Table 2.

Let us now analyze in more detail the fixed points of the dualities for the subset of Wilson lines of the form (3.24), i.e. with only one non-zero component. The quantization conditions evaluated at the fixed points turn into (see Appendix C for details of the calculation)

\[
n, m, \frac{2n^2 \pm 1}{m} \in \mathbb{Z} \quad \text{for} \quad U = \pm I
\]  

(3.42)

where \( n = \frac{1}{2} A_1 R^{-1}, m = R^{-1}, \) and

\[
n, m, \frac{n^2 \pm 1}{2m} \in \mathbb{Z} \quad \text{for} \quad U = U_\pm,
\]  

(3.43)

where now \( n = \frac{1}{\sqrt{2}} A_1 R^{-1}, m = \frac{1}{\sqrt{2}} R^{-1}. \)

We write in Table 1 all the fixed points for \( U = I \) and \( U = U_+ \) where \( 0 \leq A_1 \leq 1. \) The lines \( \pm A_1 \mod 2 \) are also fixed points.

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20 One could also consider a more general situation where \( A' + \Lambda' = A + \Lambda \) with \( \Lambda(\Lambda') \in \Gamma(\Gamma') \). Since here \( \Gamma = \Gamma' \), then \( A \sim A' \). Since we are considering \( \Lambda \)-shifts as part of the duality transformations, we can restrict without loss of generality to dualities where \( A' = A \).

21 The other two options studied, \( U = -I \) and \( U = U_- \) do not leave the Wilson line invariant. The fixed points of these dualities are the points were the positive and negative branches of the curves \( a(R) \) (\( b(R) \)), defined in (3.27) (\( 3.28 \)), intersect, which are the points where the argument in the square root of \( a(R) \) (\( b(R) \)) is zero. Most of these points do not correspond to points of maximal enhancement. The ones that do are those for which \( -IA = U_- A = -A \sim A \), and those are also fixed points for \( U = I \) or \( U = U_- \).
\[ U = \text{diag}(1_{16}) \quad \text{or} \quad U = \text{diag}(1_{15}, -1_{15}) \]

\[
\begin{array}{c|c|c|c}
R^{-1} & A_1 & R^{-1} & A_1 \\
\hline
1 & 0 & \sqrt{2} & 1 \\
3 & 2 & 5\sqrt{2} & \frac{3}{5} \\
9 & 3 \sqrt{2} & 13\sqrt{2} & \frac{13}{13} \\
11 & \frac{1}{11} & 17\sqrt{2} & \frac{17}{17} \\
17 & \frac{5}{17} & 25\sqrt{2} & \frac{7}{25} \\
19 & \frac{5}{19} & 29\sqrt{2} & \frac{19}{29} \\
27 & \frac{1}{27} & 37\sqrt{2} & \frac{37}{37} \\
33 & \frac{8}{33} \times 14 & 41\sqrt{2} & \frac{41}{41} \\
41 & \frac{1}{41} & 53\sqrt{2} & \frac{43}{43} \\
43 & \frac{1}{43} & 61\sqrt{2} & \frac{43}{61} \\
51 & \frac{41}{51} \times 44 & 65\sqrt{2} & \frac{47}{65} \\
57 & \frac{13}{32} \times 54 & 73\sqrt{2} & \frac{73}{73} \\
59 & \frac{13}{59} \times 57 & 85\sqrt{2} & \frac{57}{85} \\
67 & \frac{21}{67} \times 64 & 89\sqrt{2} & \frac{89}{89} \\
73 & \frac{13}{73} \times 72 & 97\sqrt{2} & \frac{97}{97} \\
81 & \frac{1}{81} \times 80 & 101\sqrt{2} & \frac{101}{101} \\
83 & \frac{83}{83} \times 82 & 109\sqrt{2} & \frac{109}{109} \\
89 & \frac{89}{89} \times 88 & 113\sqrt{2} & \frac{113}{113} \\
97 & \frac{1}{97} \times 96 & 125\sqrt{2} & \frac{125}{125} \\
99 & \frac{14}{99} \times 98 & 137\sqrt{2} & \frac{137}{137} \\
& & 145\sqrt{2} & \frac{145}{145} \\
\end{array}
\]

| SU(2) × SO(32) | SO(34) |
|-----------------|--------|
| or              | or     |

| SU(2) × E₈ × E₈ | SO(18) × E₈ |
|-----------------|-------------|

Table 1: Fixed points of the dualities \( O_U \).

These points in moduli space are points of maximal enhancement symmetry. Those in the first column give rise to \( SU(2) \times SO(32) \) for \( \Gamma_{16} \) or \( SU(2) \times E₈ \times E₈ \) for \( \Gamma_{8} \times \Gamma_{8} \), and are depicted by red dots in figure 4. The second column contains all the points of maximal enhancement groups \( SO(34) \) or \( SO(18) \times E₈ \), and correspond to the green dots in figure 4.

### 3.6.2 \( \Gamma \leftrightarrow \Gamma' \)

Note that unless \( \Gamma = \Gamma_{16}^{\pm} \) and \( \Gamma' = \Gamma_{8}^{\pm} \times \Gamma_{8}^{\pm} \) (or the other way around, and using any combination of signs) – situations that we analyze separately in the next section – there exists some \( U_1 \in O(16, \mathbb{Z}) \) such that \( \Gamma' = U_1 \Gamma' \). In that case, the duality with \( \Gamma' \neq \Gamma \), \( U_2 \) and \( A' \) is equivalent to one between \( \Gamma \) and \( \Gamma'' = \Gamma \), \( U'' = U_1 U_2 \) and has \( A'' = A' U_1 \).

Restricting to diagonal matrices \( U \), we see that the dualities with \( U \) and \( \Gamma' = \Gamma \) are equivalent to the dualities with \( U = I \) but where \( \Gamma' \) is

\[
\Gamma' = \Gamma_{16}^{\pm} \text{ for } \Gamma = \Gamma_{16} \text{ and } \det(U) = \pm 1 \quad (3.44)
\]

\[
\Gamma' = \Gamma_{8}^{\pm 1} \times \Gamma_{8}^{\pm 2} \text{ for } \Gamma = \Gamma_{8} \times \Gamma_{8}, \det_1(U) = \pm 1 \text{ and } \det_2(U) = \pm 1 \quad (3.45)
\]
where $\text{det}_1 (\text{det}_2)$ is the product of the 8 first (last) diagonal elements and the lattices $\Gamma^{\pm}$ are defined in Appendix A. If additionally the Wilson line $A$ is invariant under the action of $U$ (up to a $\Lambda$-shift) we get exactly the same fixed points that one gets for a duality with $\Gamma = \Gamma'$. Since Wilson lines of the type (3.24) are invariant under the action of a diagonal $U$ such that the first component is +1, we get the same fixed points of section 3.6.1 that correspond to enhancement to $SO(34)$ or $SO(18) \times E_8$.

Under the assumption that $r$ is a prime number, the quantization conditions are satisfied if and only if

$$A \in (\Gamma \cap \Gamma')^* \backslash \Gamma, \quad A' \in (\Gamma \cap \Gamma')^* \backslash \Gamma', \quad r = 2$$

and thus the fixed points of these transformations are at $R = \frac{1}{\sqrt{2}},$ and correspond to the enhancements $SO(34)$ and $SO(18) \times E_8$. The possible Wilson lines for the different choices of $\Gamma$ and $\Gamma'$ are given in Table 2.

### 3.6.3 $SO(32) \leftrightarrow E_8 \times E_8$

There is no $U \in O(16, \mathbb{Z})$ that transforms the lattices $\Gamma_{16}$ and $\Gamma_8 \times \Gamma_8$ into each other, and thus the case $\Gamma = \Gamma_{16}$ and $\Gamma' = \Gamma_8 \times \Gamma_8$ is different from the ones considered previously.

Here, for simplicity, we restrict to $U = 1$, namely we analyze dualities such that $(p_L', p_R') = (p_L, -p_R)$. The quantization conditions under the assumption that $r$ is a prime number, are given in (3.46). For $\Gamma = \Gamma_{16}$ and $\Gamma' = \Gamma_8 \times \Gamma_8$, the possible Wilson lines are the following

$$A \in (\Gamma \cap \Gamma')^* \backslash \Gamma = (0s), (s0), (vc), (cv)$$

$$A' \in (\Gamma \cap \Gamma')^* \backslash \Gamma' = (vv), (cc), (vc), (cv)$$

(3.47)

However, there is something very curious here: the fixed points of these dualities, corresponding to $R = \frac{1}{\sqrt{2}}$, are not points of maximal enhancement but points of enhancement $U(1) \times SO(16) \times SO(16)$. Furthermore, this enhancement group arises at any radius, so Wilson lines of the form (3.47) give rise to lines in moduli space, and as such are also “fixed points” of dualities that do not involve $O_D$.

Let us illustrate this better with an example: Take $A = ((\frac{1}{2})_8, 0_8) \in (s0)$ and $A' = (1_0, 1, 0_7) \in (vv)$. For the time being, we take $r = 2$, i.e. $R' = 1/(2R)$, but we do not necessarily stand at the self-dual radius.

The Wilson line $A$ breaks the $SO(32)$ gauge symmetry to $SO(16) \times SO(16)$, as shown in section 3.3.3. For this Wilson line, one has additionally states which are neutral under $SO(16) \times SO(16)$, i.e. with $p^A = 0$. Since these should have $\pi = -wA$, then only states with $w = 2m, m \in \mathbb{Z}$ are allowed. These states have left and right-moving momenta on the circle

$$p_L = \frac{1}{\sqrt{2}R} (\tilde{n} + 2R^2m), \quad p_R = \frac{1}{\sqrt{2}R} (\tilde{n} - 2R^2m),$$

(3.48)

where $\tilde{n} = n + w$. Let us pause for a second to show that there is no enhancement to $SU(2) \times SO(16) \times SO(16)$ with this Wilson line. We have shown in section 3.3.3 that there are no additional massless states charged under $SO(16) \times SO(16)$, i.e. with
non-zero winding number and \( p^A \neq 0 \). Regarding extra neutral massless states, it is very easy to see from (3.48) that there are none of this form: states with momenta \((p_L, p^A, p_R) = (\sqrt{2}, 0, 0)\), satisfy \(2R^2m = \bar{n}\), while requiring at the same time \(p_L = \sqrt{2}\) would lead to \(\bar{n}m = \frac{1}{2}\), which has no solution. Thus, the compactification of the \(SO(32)\) heterotic string with Wilson line \(A = (\frac{1}{2}, 0, 0)\) leads to \(U(1) \times SO(16) \times SO(16)\) at any radius.

The Wilson line \(A' = (1, 0, 7, 1, 0, 7)\) breaks the \(E_8 \times E_8\) symmetry also to \(SO(16) \times SO(16)\). There are also states which are neutral under \(SO(16) \times SO(16)\), of the same form as before, i.e. with momenta

\[
p_L' = \frac{1}{\sqrt{2}R'} \left( \bar{n}' + 2R'^2m' \right); \quad p_R' = \frac{1}{\sqrt{2}R'} \left( \bar{n}' - 2R'^2m' \right),
\]

where \(w' = 2m'\) and \(\bar{n}' = n' + w'\).

Comparing (3.49) and (3.48), we see that \((p_L', p_R') = (p_L, -p_R)\) if \((\bar{n}', m') = (m, \bar{n})\) and \(RR' = \frac{1}{2}\). This is true for any value of \(R\).

In the following table we write the fixed points of the dualities between a theory with lattice \(\Gamma\) (row) and another one with \(\Gamma'\) (column) for the smallest value of the parameter \(r\) defined in (3.36). We indicate the conjugation classes of the possible Wilson lines (for a given row and column, any \(A\) given can be dualized to any \(A'\), and the enhancement group arising at the fixed point of the duality.

| \(\Gamma_{16}\) | \(\Gamma'_{16}\) | \(\Gamma_{16} \times \Gamma_{16}\) | \(\Gamma_{16} \times \Gamma_{16}^{-}\) | \(\Gamma_{16} \times \Gamma'_{16}\) | \(\Gamma_{16} \times \Gamma'_{16}^{-}\) | \(\Gamma_{16} \times \Gamma_{16}\) | \(\Gamma_{16} \times \Gamma_{16}^{-}\) | \(\Gamma_{16} \times \Gamma_{16}^{0}\) | \(\Gamma_{16} \times \Gamma_{16}^{-}\) | \(\Gamma_{16} \times \Gamma_{16}^{0}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(SO(12) \times SU(2)\) | \(SO(12) \times SU(2)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) | \(SO(16) \times SO(16)\) |
| \(A = 0, s, a, v, s\), \(A' = 0, a, s, v, a\) | \(A = 0, a, a, s, v\), \(A' = 0, v, a, s, a\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) | \(E_8 \times E_8\) |

Table 2: Points of symmetry enhancement as fixed points of duality symmetries

4 Effective action and Higgs mechanism

Now that we saw the rich structure of duality symmetry, we turn to its explicit target space realization. The global duality symmetry of the dimensionally reduced heterotic supergravity action has been deeply investigated in the seminal papers by J. Maharana and J. Schwarz [6] and N. Kaloper and R. Myers [7], and more recently in [8]. If the gauge fields are truncated to the Cartan subsector of the \(E_8 \times E_8\) or \(SO(32)\) gauge group, the
dimensional reduction of heterotic supergravity from 10 to 10 − k dimensions produces a theory with \( U(1)^{2k+16} \) abelian gauge symmetry and a continuous global \( O(k, k+16; \mathbb{R}) \) symmetry. If the reduction includes the full set of \( E_8 \times E_8 \) or \( SO(32) \) gauge fields and no Wilson lines, the global symmetry reduces to \( O(k, k; \mathbb{R}) \), while a compactification with Wilson lines for the Cartan gauge fields of a rank 16 − r subgroup of the rank 16 gauge group \( G_L \), gives an effective field theory with global \( O(k, k+16-r; \mathbb{R}) \) duality symmetry \( \mathcal{S} \). The analysis of \( \mathcal{S} \) is based on string-theoretic arguments and holds to any order in the \( \alpha' \) expansion of the heterotic string effective field theory action involving all the massless string states, except those that become massless at self-dual points of the moduli space.

Including the massless states with nonzero winding or momentum number on \( T^k \) in the effective field theory of the toroidally compactified heterotic string is not difficult, as it is a gauged supergravity. The action with at most two derivatives of the massless fields is then completely determined by the gauge group. Therefore, although the field theoretical Kaluza-Klein reduction of heterotic supergravity cannot describe the string modes that give rise to maximally enhanced gauge symmetry, the action is entirely fixed.

Nevertheless, we will see in the forthcoming sections that the explicit construction of the (toroidally compactified) heterotic string effective action from the scattering amplitudes of massless string modes at self dual points of the moduli space, and its manifestly duality-covariant reformulation, give important information. In particular, we will obtain novel relations between the \( SO(32) \) and \( E_8 \times E_8 \) theories. We will also consider the light states that acquire mass when slightly perturbing the background fields and revisit the gauge symmetry breaking and Higgs mechanism, both from the field theory and the string theory viewpoints.

### 4.1 Effective action of massless states

The three-point functions of all the (toroidally compactified) heterotic string massless vertex operators are reviewed in Appendix [D] where we also compute the four point function of the massless scalars. These amplitudes are reproduced from the S-matrix of the following effective action

\[
S = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{-G} e^{-2\varphi} \left( R + 4\bar{\partial}_\mu \varphi \partial^\mu \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} F_{\mu\nu}^\Gamma F_{\mu\nu}^{\Gamma} - \frac{1}{4} F_{\mu\nu}^m F_{\mu\nu}^m - \frac{1}{4} D_{\mu} S_{mn} D_{\nu} S_{mn} - \frac{1}{2} S_{\Gamma m} F_{\mu\nu}^{\Gamma} F^{\mu\nu} - \frac{1}{4} S_{\Gamma m} S_{\Gamma n} S_{\Lambda \sigma} S_{\Lambda' \sigma'} f_{\Gamma \Lambda \Pi} f^{\Gamma' \Lambda' \Pi'} \right),
\]

which also contains terms from higher point functions that we have not computed but need to be included on the basis of gauge symmetry. Here \( \kappa_d \) is the effective Planck coupling constant (related to the gauge coupling \( g_d \) as \( g_d = \sqrt{2\kappa_d} \)) and \( \epsilon_{\mu\nu\rho} \rightarrow \frac{d!}{2\kappa_d} \), \( A_\mu \rightarrow A_\mu^{\Gamma} \), \( \bar{A}_\mu \rightarrow \bar{A}_\mu^{\Gamma} \), \( S_{mn} \rightarrow S_{mn}^{\Gamma} \), \( S_{\Gamma m} \rightarrow S_{\Gamma m}^{\Gamma} \), \( S_{\Gamma m} S_{\Gamma n} \rightarrow S_{\Gamma m} S_{\Gamma n}^{\Gamma} \), \( S_{\Lambda \sigma} S_{\Lambda' \sigma'} \rightarrow S_{\Lambda \sigma} S_{\Lambda' \sigma'}^{\Lambda' \sigma'} \). We also redefined the dilaton \( D = \frac{2}{\kappa_d \sqrt{\varphi - \varphi_0}} \), so that \( \kappa_d \rightarrow e^{-\varphi_0} \kappa_d \) and \( g_d \rightarrow e^{-\varphi_0} g_d \).
\[
H_{\mu\nu}\rho = 3 \left( \partial_\mu B_{\nu\rho} + A^\Gamma_{[\mu} \partial_\nu A^\rho_{\Gamma]} + \frac{1}{3} f_{\Gamma\Lambda\Omega} A^\Gamma_{\mu} A^\Lambda_{\nu} A^\rho_{\Omega} - A^m_{[\mu} \partial_\nu A^{m\rho]} \right),
\]
\[
F^\Gamma_{\mu\nu} = \partial_\mu A^\Gamma_{\nu} - \partial_\nu A^\Gamma_{\mu} + f^\Gamma_{\Lambda\Omega} A^\Lambda_{\mu} A^\Omega_{\nu},
\]
\[
D_\mu S^{\Gamma m} = \partial_\mu S^{\Gamma m} + f^\Gamma_{\Lambda\Omega} A^\Lambda_{\mu} S^\Omega m,
\]
with \(S_{\Gamma m} = (G_{mn}, B_{mn}, A_{Im}, A_{am})\) denoting the scalar fields. The indices \(m, n = 1, \ldots, k\) correspond to the dimensions on \(T^k\) and \(\Gamma, \Lambda = 1, \ldots, N\) are the adjoint indices of the Lie algebra associated to the gauge group \(G_L\) of dimension \(N\) and structure constants \(f^\Gamma_{\Lambda\Omega}\).

For ten external dimensions (i.e. when there are no compact internal dimensions other than the 16 chiral “heterotic” ones), \(d = 10\), the gauge group is \(E_8 \times E_8\) or \(SO(32)\) and \(N = 496\). There are neither scalar \(S_{\Gamma m}\) nor vector \(A^m_{\mu}, \bar{A}^m_{\mu}\) fields. Then the action reduces to the first four terms in (4.1), with \(\Gamma = (I, \alpha) = 1, \ldots, 496\), and the last term in \(H_{\mu\nu\rho}\) vanishes.

For compactifications on \(T^k, d = 10 - k\), at generic values of the background fields, the gauge group is \(U(1)^{16+k}_L \times U(1)^k_R\), \(N = 16 + k\), and the index \(\Gamma \equiv I = 1, \ldots, 16 + k\). The vectors and scalars are only those in sector 1 of section 2.3. We denote the gauge fields as the polarization vectors in the vertex operators \((A^m_{\mu}, \bar{A}^m_{\mu})\) and the scalar fields are \(G_{mn} = G_{mn} + S_{(mn)}(x), B_{mn} = B_{mn} + S_{[mn]}(x), A_{Im} = A_{Im} + S_{Im}(x),\) where the fluctuations are denoted like the polarizations of the vertex operators creating the string scalar states. In this case, (4.1) agrees with the effective action obtained in [6] from dimensional reduction of heterotic supergravity with gauge group truncated to the Cartan subgroup. The theory has a global \(O(k, k + 16; \mathbb{R})\) symmetry.

At the specific points in moduli space where the gauge symmetry is enhanced, it is convenient to split the index \(\Gamma = (I, \alpha = \pi, \bar{\pi}),\) where \(I = 1, \ldots, 16 + k\) denotes the Cartan generators and \(\alpha\) (\(\bar{\alpha}\)) are the positive (negative) roots of \(G_L\). The vectors \(A^I_{\mu}\) and \(\bar{A}^I_{\mu}\) correspond to the left and right Cartan generators in sector 1, respectively, while \(\bar{A}^a_{\mu}\) correspond to the vectors of sector 2, as defined in section 2.3. The scalars \(S^I_{Im}\) correspond to the \((16 + k) \times k\) scalars in sector 1, while the \(S^a_{Im}\) correspond to the scalars in sector 2. In this case, \(G_{mn} = G^a_{mn} + S_{(mn)}(x), B_{mn} = B^a_{mn} + S_{[mn]}(x), A_{Im} = A^a_{Im} + S_{Im}(x), A_{am} = S_{am}(x),\) and the superindex \(sd\) refers to the self-dual values of the background fields. The algebra in the Cartan-Weyl basis is

\[
[J^I, J^a] = \alpha^I J^a, \quad [J^a, J^\beta] = \begin{cases} 
\varepsilon(\alpha, \beta) J^{a+\beta} & \text{if } \alpha + \beta \text{ is a root} \\
\alpha I J^I & \text{if } \alpha = -\beta \\
0 & \text{otherwise}
\end{cases}
\]

where \(\varepsilon(\alpha, \beta) = \pm 1\) for simply-laced algebras. Note that it is completely determined by the vertex operators of the vector states: the roots \(\alpha^I\) are the momenta of the string states and \(\varepsilon(\alpha, \beta)\) is given by the cocycle factors in the currents (2.42) \(c_\alpha c_\beta = \varepsilon(\alpha, \beta)c_{\alpha+\beta}\).

When the gauge group \(G_L\) is a product, the structure constants (and the indices \(I, I, \alpha\)) split into those of each factor, e.g. for \(SO(32) \times H, \Gamma = (\Gamma_{SO(32)}, \Gamma_H)\) with \(\Gamma_{SO(32)} = (I = 1, \ldots, 16; \alpha = 1, \ldots, 480)\) and \(\Gamma_H = (m = 1, \ldots, k; \alpha = 1, \ldots, N - 496 - k)\), while for \(SO(32) \times U(1)^k\) or \(SO(34)\) they are only those of the non-Abelian piece. The Cartan-Killing metric is defined to be a block diagonal matrix containing the Cartan-Killing metrics of the groups \(\kappa = \text{diag}(\kappa_{SO(32)}, \kappa_H)\) or \(\kappa = \text{diag}(\kappa_{SO(32)}, 1_{k \times k})\).
For gauge groups of the form $G_L \times U(1)_L^k \times U(1)_R^k$, the action (4.1) agrees with the dimensionally reduced heterotic supergravity action obtained in [8], including the scalar potential (although the reduction of [8] contains an additional term with six scalars that we have not computed). It possesses $O(k, k; \mathbb{R})$ global symmetry.

In the case of enhanced gauge groups of the form $G_L \times U(1)_R^k$, in which the $k$ left-moving Cartan generators are absorbed by the Cartan subgroups of the non-abelian group $G_L$, the structure constants completely break the global symmetry. However, (4.1) can be rewritten in $O(k, n)$ covariant form, where $n$ equals the dimension of the full gauge group. We review this rewriting in the next section, where we also present an alternative reformulation of (4.1) from a generalized Scherk-Schwarz compactification of double field theory. This will allow us to obtain novel relations between the $E_8 \times E_8$ and $SO(32)$ heterotic theories.

From (4.1) one can see some of the features of the spontaneous breaking of gauge symmetry that occurs away from the enhancement points. An effective stringy Higgs mechanism is already encoded in the string theory computation, which can be interpreted as triggered by the vacuum expectation values of the scalar fields in the Cartan sector $S_{Im}$, which give mass to the vectors in the non-Cartan sector from the covariant derivatives in the kinetic terms, while the scalars without legs in the Cartan sector acquire mass from the scalar potential. We present the relevant details in the forthcoming sections.

### 4.2 Higgs mechanism in string theory

When moving away from the points in moduli space where the gauge symmetry is enhanced, $p_{R} \neq 0$ and the extra massless vectors and scalars in sector 2 acquire mass. The dependence of the vertex operators on the background fields is contained in the exponential factors of the internal coordinates, which become

$$J^a = c_\alpha e^{i\tau(\alpha)Y^a_L(z)} \rightarrow J^{PL,PR}(z, \bar{z}) = c'_\alpha e^{i\tau(\alpha)Y^a_L(z) + ip_{Rn}Y^m_R(\bar{z})},$$

where $c'_\alpha = c_\alpha$, as we will see later. In particular, the $[U(1)_L]^k \times [U(1)_R]^k$ charges of these states, $(q^I, \bar{q}^m) = (p^L_I, p^m_R)$, are generated by $J^I \otimes J^m$.

The OPE of the energy-momentum tensor with the massive vector boson vertex operators develop a cubic pole, and it is necessary to combine these operators with those of the massive scalars in order to cancel the anomaly. As discussed in [12], the vertex operators of the massless vectors “eat” the scalars $S^{am}$ and the conformal anomalies can be canceled when redefining

$$A'_{(0)} \sim J^{PL,PR}(z, \bar{z}) \left( A'^{\alpha}(k) \tilde{\Upsilon}^{\mu}(\bar{z}) - \xi S^{am}(k) \tilde{\Upsilon}^m_m(\bar{z}) \right) e^{ik \cdot X(z, \bar{z})},$$

with

$$\tilde{\Upsilon}^{\mu} = i\sqrt{2} \partial \tilde{\Upsilon}^\mu = \frac{1}{\sqrt{2}} k \cdot \bar{\psi} \tilde{\Upsilon}^\mu - p_{Rn} \bar{\chi}^n \tilde{\Upsilon}^\mu, \quad \tilde{\Upsilon}^m = i\bar{\partial} \tilde{\Upsilon}^m = \frac{1}{\sqrt{2}} k \cdot \bar{\psi} \bar{\chi}^m - p_{Rn} \bar{\chi}^n \bar{\chi}^m.$$

---

23The redefinitions $A'^{(1)m}_\mu = \frac{1}{\sqrt{2}} (A^{(1)m}_\mu + \bar{A}^{(1)m}_\mu)$, $A'^{(2)}_\mu = \frac{1}{\sqrt{2}} G_{mn} (A^{(2)}_\mu - \bar{A}^{(2)}_\mu)$ and $B_{\mu \nu} = -b_{\mu \nu}, B_{mn} = -b_{mn}$ are necessary to compare with [8]. Note that the KK reductions of the metric and B field, $A^{(1)m}_\mu$ and $A^{(2)}_\mu$, having the internal indices up and down respectively, cannot couple through one scalar field, unlike the left and right vector fields $A^{L}_\mu$ and $A^{R}_\mu$ in (4.1). See the next section and the equivalent discussion in [12].
if
\[ k \cdot A_\alpha - \xi p^m_R S_{\alpha m} = 0, \tag{4.6} \]
where \( \xi \) is some coefficient. In terms of fields, this is
\[ \partial_\mu A^\mu_\alpha + i \xi p^m_R S_{\alpha m} = 0 \tag{4.7} \]
corresponding to the \( R_\xi \) t’Hooft gauge condition where \( p_R \) can be identified with a non
vanishing vev. Then the physical massive vector boson vertices are actually \( A' \), and the
scalars \( S_{\alpha m} \) disappear from the spectrum.

Note that the fields associated to \( A' \) have well defined charges \((p_L, p_R)\), and since
\( m^2 = -k^2 \), the gauge condition can be written as
\[ k \cdot \left( A_\alpha + k \xi \frac{1}{2p^2_R} p^m_R S_{\alpha m} \right) = 0, \tag{4.8} \]
implying an effective polarization
\[ A'_\alpha \mu (p_L, p_R, k) = A_{\alpha \mu} - \xi \frac{k_\mu}{2p^2_R} p^m_R S_{\alpha m}. \tag{4.9} \]
This leads to a massive vector of the form
\[ A'_\alpha \mu = A_{\alpha \mu} - \xi \frac{1}{2p^2_R} p^m_R \partial_\mu S_{\alpha m}, \tag{4.10} \]
where \( p^2_R \neq 0 \) is related to the vevs. This is the usual massive vector field incorporating
the would-be Goldstone bosons \( p^m_R S_{\alpha m} \) that provide the longitudinal polarization.

Unlike the case of the toroidally compactified bosonic string, in the heterotic string all
the massive scalars are Goldstone bosons. Since the gauge group in the supersymmetric
right sector is abelian, there are no other massive scalars from the compactification of
the massless states.

The non-vanishing three point functions involving massless and light states, i.e. states
that are massless at the self-dual points and become massive when perturbing the back-
ground fields, are listed in Appendix C, and they lead to the following effective action
\[ S' = \frac{1}{2\kappa^2_s} \int d^dx \sqrt{G} e^{-2\varphi} \left( \right) \]
with
\[ F^{p_{1}}_{\mu \nu} = 2\partial_{[\mu} A^p_{\nu]} + \varepsilon(p_1, p_2) A^p_{[\mu} A'^{p_1 + p_2}_{\nu]} - 2i p_{1L} A^p_{[\mu} A^p_{\nu]} - 2i p_{1R} A^m_{[\mu} A^m_{\nu]} \]
\[ F^i_{\mu \nu} = 2\partial_{[\mu} A^i_{\nu]} \tag{4.11} \]
\[ H'_{\mu \nu \rho} = 3 \left( \partial_\mu B_{\nu \rho} + A^p_{[\mu} \partial_\nu A'^{p}_{\rho]} + A^p_{[\mu} \partial_\rho A'^{p}_{\nu]} + \frac{1}{3} \varepsilon(p_1, p_2) A^p_{[\mu} A^p_{\nu} A'^{p_1 - p_2}_{\rho]} - i p_{1L} A^p_{[\mu} A'^{p}_{\nu]} A'^{p}_{\rho]} - A^m_{[\mu} \partial_\nu A^m_{\rho]} \right), \tag{4.12} \]
The S-matrix of this massive gauge field theory coupled to gravity reproduces the string theory three-point amplitudes. The non-Abelian pieces in the field strength of the massive gauge fields and in the Chern-Simons terms in $H'_{\mu\nu\rho}$ correctly appear in terms of the charges of the corresponding fields $(q^I, q^m) = (p^I_L, p^m_R)$. These charges determine the coefficients of the vector boson three-point functions, which can be identified with structure constants

$$f^m_{\underline{p}} \underline{p} = ip^m_R, \quad f^i_{\underline{p}} \underline{p} = ip^i_L, \quad f^{p_1+p_2}_{p_1 p_2} = \varepsilon(p_1, p_2), \quad (4.13)$$

reflecting the fact that the gauge interactions in string theory are a manifestation of an underlying affine Lie algebra. This algebra is isomorphic to that of the enhanced $G_L$ group [15], which justifies the identification $c'_\alpha = c_\alpha$ used in (4.4) (we will comment further on this result in the next section).

Not all the terms in the action can be obtained from the three-point functions, but we have completed the expressions so that they correctly reproduce the massless case when $p_R = 0$ and $p_L \in \Gamma$.

All the terms of the scalar potential of the massless theory (4.1) are absorbed by the field strengths of the massive vectors or by interaction terms containing massive vectors.

5 Heterotic double field theory

Although the action (4.1) can be generically obtained by dimensional reduction of heterotic supergravity from 10 to $10 - k$ dimensions, not all the effective actions of massless fields obtained from toroidally compactified heterotic string theory can be uplifted to higher dimensional supergravities. In particular, the states with nonzero winding or momentum number on $T^k$ cannot be captured by field theoretical Kaluza Klein compactifications. To find the higher dimensional description of these string modes, one has to refer to gauged double field theory (DFT) [18, 19, 22], an $O(D, D + N; \mathbb{R})$ covariant rewriting of heterotic supergravity, with $D$ the dimension of space-time and $N$ the dimension of the gauge group.

In this section we review this construction and show that the effective action (4.1) can be rewritten in terms of $O(k, N)$ multiplets. The reformulation is achieved essentially assembling the $N+k$ gauge fields as a vector, the $Nk$ moduli scalars as part of a symmetric tensor and the structure constants of the non-abelian gauge groups as an antisymmetric three-index tensor under $O(k, N)$ transformations. The procedure generalizes the analysis of [8] by including all the massless string modes at self-dual points of the moduli space, in which the $k$ left Kaluza-Klein vector fields become part of the Cartan subgroup of the maximally enhanced gauge group.

Furthermore, using the equivalence between gauged DFT and generalized Scherk-Schwarz (gSS) compactifications [19], we present an explicit realization of the internal generalized vielbein which reproduces the structure constants of all the enhanced gauge groups under generalized diffeomorphisms. In particular, we show that the structure constants of the $E_8 \times E_8$ and $SO(32)$ groups can be obtained from the same deformation of the generalized diffeomorphisms and then the $E_8 \times E_8$ and $SO(32)$ theories can be described as different solutions of the same heterotic DFT.
5.1 Gauged double field theory

The frame-like DFT action reproducing heterotic supergravity was originally introduced in [9] and further developed in [18]. The theory has a global \( G = O(D,D + N; \mathbb{R}) \) symmetry, a local double-Lorentz \( H = O(D - 1,1; \mathbb{R}) \times O(1, D - 1 + N; \mathbb{R}) \) symmetry, and a gauge symmetry generated by a generalized Lie derivative

\[
\mathcal{L}_\xi V_M = \xi^P \partial_P V_M + \left( \partial_M \xi^P - \partial^P \xi_M \right) V_P .
\]  

(5.1)

The infinitesimal generalized parameter \( \xi^M \), with \( M = 1, \ldots, 2D + N \), transforms in the fundamental representation of \( G \), and \( H \)-transformations are generated by an infinitesimal parameter \( \Lambda_{A}^{B} \), with \( A, B = 1, \ldots, 2D + N \).

The constant symmetric and invertible metrics \( \eta_{MN} \) and \( \eta_{AB} \) raise and lower the indices that are rotated by \( G \) and \( H \), respectively. In addition there is a constant symmetric and invertible \( H \)-invariant metric \( \mathcal{H}_{AB} \) constrained to satisfy

\[
\mathcal{H}_{A}^{C}\mathcal{H}_{C}^{B} = \delta_{A}^{B} .
\]  

(5.2)

The three metrics \( \eta_{MN}, \eta_{AB} \) and \( \mathcal{H}_{AB} \) are invariant under the action of \( \mathcal{L}, G \) and \( H \).

The fields of the theory are a generalized vielbein \( E^{A \mathcal{M}} \) and a generalized dilaton \( d \). The former is constrained to relate the metrics \( \eta_{AB} \) and \( \eta_{MN} \), and allows to define a generalized metric \( \mathcal{H}_{MN} \) from \( \mathcal{H}_{AB} \)

\[
\eta_{MN} = E^{A \mathcal{M}} \eta_{AB} E_{B}^{\mathcal{N}} , \quad \mathcal{H}_{MN} = E^{A \mathcal{M}} \mathcal{H}_{AB} E_{B}^{\mathcal{N}} .
\]  

(5.3)

The theory is defined on a \( 2D + N \) dimensional space but the coordinate dependence of fields and gauge parameters is restricted by a strong constraint

\[
\partial_{M} \partial^{M} \cdots = 0 , \quad \partial_{M} \cdots \partial^{M} \cdots = 0 ,
\]  

(5.4)

the derivatives \( \partial_{M} \) transforming in the fundamental representation of \( G \) and the dots representing arbitrary products of fields.

DFT can be deformed in terms of so-called fluxes or gaugings \( f_{MNP} \) [18], a set of constants that satisfy linear and quadratic constraints

\[
f_{MNP} = f_{[MNP]} , \quad f_{[MN]} f_{P|QR} = 0 ,
\]  

(5.5)

and the following additional constraint is required to further restrict the coordinate dependence of fields and gauge parameters

\[
f_{MN} f_{P} \partial^{P} \cdots = 0 .
\]  

(5.6)

The generalized dilaton and frame transform under generalized diffeomorphisms and \( H \)-transformations as follows

\[
\delta d = \xi^{P} \partial_{P} d - \frac{1}{2} \partial_{P} \xi^{P} \quad \Leftrightarrow \quad \delta e^{-2d} = \partial_{P} \left( \xi^{P} e^{-2d} \right) ,
\]  

(5.7)

\[
\delta E^{A \mathcal{M}} = \tilde{\mathcal{L}}_{\xi} E^{A \mathcal{M}} + \delta_{A} E^{A \mathcal{M}} ,
\]  

(5.8)
where
\[
\mathring{\mathcal{L}} \xi E^A_M = \mathcal{L}_\xi E^A_M + f_{MP} Q^P E^A_Q , \\
\delta \lambda E^A_M = E^B_M \lambda_B^A .
\]

The DFT action can be expressed in terms of the generalized fluxes
\[
\mathcal{F}_{ABC} = 3 \partial_{[A} E^N_B E^P_C \eta_{N[P} + f_{MN[P} E^M_A E^N_B E^P_C , \\
\mathcal{F}_A = 2 \partial_A d - \partial_B E_{A[M} E^{B]M} ,
\]
as [22]
\[
S = \int dX e^{-2d} \left[ \mathcal{F}_{ABC} \mathcal{F}_{DEF} \left( \frac{1}{4} \mathcal{H}^{AD} \eta^{BE} \eta^{CF} - \frac{1}{12} \mathcal{H}^{AD} \mathcal{H}^{BE} \mathcal{H}^{CF} - \frac{1}{6} \eta^{AD} \eta^{BE} \eta^{CF} \right) \\
+ (2 \partial_A \mathcal{F}_B - \mathcal{F}_A \mathcal{F}_B) \left( \mathcal{H}^{AB} - \eta^{AB} \right) \right] ,
\]
and it is fixed by demanding $H$-invariance, since the generalized fluxes are not $H$-covariant.

### 5.2 Parameterization and choice of section

Choosing specific global and local groups and parameterizing the fields in terms of metric, two-form, vector and scalar fields one can make contact with the (toroidally compactified) heterotic string modes and effective actions of the previous sections. To this aim, we first consider the theory at points of the moduli space in which the gauge group is $G_L \times U(1)_L^k \times U(1)_R^k$, and in the next subsection extend the construction to account for the maximally enhanced gauge groups $G_L \times U(1)_L^k$.

Taking the space-time dimension $D = d + k$ and the gauge group $G_L \times U(1)_L^k \times U(1)_R^k$, the $G$ indices split as $V^M = (V^\mu, V^\gamma)$ and the $H$ indices split as $V_A = (V^\alpha, V^\mu, V^\gamma)$, where $\mu, \alpha, \bar{\alpha} = 1, \ldots, d$ (external) and $M, a = 1, \ldots, 2k + N$ (internal), $N$ being the dimension of $G_L$. The splitting breaks $G$ and $H$ into external and internal pieces
\[
G \rightarrow G_e \times G_i , \quad H \rightarrow H_e \times H_i
\]
where
\[
G_e = O(d, d; \mathbb{R}) , \quad G_i = O(k, k + N; \mathbb{R}) , \\
H_e = O(d - 1, 1; \mathbb{R}) \times O(1, d - 1; \mathbb{R}) , \quad H_i = O(k; \mathbb{R}) \times O(k + N; \mathbb{R}) .
\]

Then the $G$-vector $V^M$ contains a $G_e$-vector $(V^\mu, V^\gamma)$ and a $G_i$-vector $V^M = (V^m, V^\gamma)$. The $H$-vector $V_A$ contains a $H_e$-vector $(V^\alpha, V^\mu, V^\gamma)$ and a $H_i$-vector $V_a = (V^\alpha, V^\mu, V^\gamma)$.

Under this decomposition, the degrees of freedom can be decomposed as
\[
\text{dim } (G/H) = D(D + N) = \frac{d(d + 1)}{2} + \frac{d(d - 1)}{2} + d(2k + N) + k \times (k + N)
\]
\[
G_{\mu\nu} , \quad B_{\mu\nu} , \quad \mathcal{A}^M , \quad \mathcal{E}^a
\]

---

24 This notation for the right and left indices should be distinguished from the notation $\bar{\alpha}$ and $\alpha$ used for the positive and negative roots of the gauge algebra in the previous section.
where $E_{M}^{\ a}$ parameterizes the coset $G_{i}/H_{i}$. The $G$ and $H$ invariant metrics are

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_{\mu}^{\nu} & 0 & 0 \\ \delta_{\nu}^{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{m}^{n} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\eta_{AB} = \text{diag}(g_{AB}, g_{AB}, g_{AB}, g_{AB}),$$

$$\eta_{ABC} = \text{diag}(g_{AB}, g_{AB}, g_{AB}).$$

We can parameterize the generalized frame in terms of the $d$-dimensional fields as

$$E^{A}_{\ M} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e_{\mu}^{\pi} & e_{\mu}^{\bar{\nu}} & -\hat{e}_{\mu}^{\rho} A_{\rho M} \\ e_{\mu}^{\bar{\nu}} & -e_{\mu}^{\nu} & -\hat{e}_{\mu}^{\rho} A_{\rho M} \\ -\hat{e}_{\mu}^{\nu} A_{\mu}^{M} & -\hat{e}_{\mu}^{\nu} A_{\mu}^{M} & \sqrt{2} e_{A}^{N} A_{\mu}^{N} \end{pmatrix},$$

where the vielbeins $e_{\mu}^{\pi}$ and $e_{\mu}^{\bar{\nu}}$ for the right and left sectors define the same space-time metric $G_{\mu\nu} = e_{\mu}^{\pi} g_{A\bar{B}} e_{\nu}^{\bar{B}} = e_{\mu}^{\bar{B}} g_{A\bar{B}} e_{\nu}^{\bar{B}}$ and $C_{\mu\nu} = B_{\mu\nu} + \frac{1}{2} A_{\mu}^{M} A_{\nu M}$. The internal part of the generalized vielbein $E_{M}^{a}$ can be written in terms of the background fields and perturbations as $E_{M}^{a} = E_{0 M}^{a} + \delta E_{M}^{a}$, with

$$\begin{pmatrix} \mathcal{E}_{\pi} \\ \mathcal{E}_{02} \\ \mathcal{E}_{0A} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e_{\pi m} & \hat{e}_{\pi m} & -\hat{e}_{\pi m} A_{m}^{I} \\ e_{\pi m} & \hat{e}_{\pi m} & -\hat{e}_{\pi m} A_{m}^{I} \\ \sqrt{2} e_{A}^{I} A_{m}^{I} & \sqrt{2} e_{A}^{I} A_{m}^{I} & \sqrt{2} e_{A}^{I} A_{m}^{I} \end{pmatrix},$$

where $e_{\pi m}$ and $e_{\pi m}$ are two different frames for the same background metric $G_{mn}$, $\hat{e}_{\pi m}$, $\hat{e}_{\pi m}$ are the inverse frames and $C_{mn} = B_{mn} + \frac{1}{2} A_{m}^{I} A_{n I}$. Then the generalized metric is

$$\mathcal{H}_{MN} = \begin{pmatrix} G_{\mu\nu} + C_{\mu\nu} G^{\rho\sigma} + A_{\mu}^{P} M_{PQ} A_{\nu}^{Q} & -G^{\mu\rho} C_{\rho\nu} & C_{\mu} G^{\rho\sigma} A_{\sigma N} + A_{\mu}^{P} M_{P N} \\ -G^{\mu\rho} C_{\rho\nu} & G_{\mu\nu} & -G^{\rho\nu} A_{\rho M} \\ C_{\mu} G^{\rho\sigma} A_{\sigma M} + A_{\mu}^{P} M_{P M} & -G^{\rho\nu} A_{\rho M} & M_{MN} + A_{\rho M} G^{\rho\sigma} A_{\sigma N} \end{pmatrix},$$

and the symmetric and $G_{i}$-valued matrix $M_{MN} = E_{M}^{a} M_{MN} E_{N}^{b} \delta_{ab} C_{N}^{b} \in O(k, k + N; \mathbb{R})$ is

$$M_{MN} = \begin{pmatrix} G_{mn} + C_{lm} G^{kl} C_{kn} + A_{l}^{I} A_{m}^{I} & -G^{nk} C_{kn} & C_{km} G^{kl} A_{l A} + A_{m A} \\ -G^{mk} C_{kn} & G_{mn} & -G^{mk} A_{k A} \\ C_{kn} G^{kl} A_{l I} + A_{n I} & -G^{mk} A_{k I} & \kappa_{IJ} + A_{k I} G^{kl} A_{l A} \end{pmatrix},$$

where the fields depend on the external coordinates.

With this parameterization in (5.11), taking $e^{-2d} = \sqrt{-G} e^{-2\varphi}$ in (5.12) and resolving the strong constraint (5.4) in the supergravity frame, after integrating (5.13) along the internal coordinates one gets an action of the form of (the electric bosonic sector of) half-maximal gauged supergravity [10]

$$S = \int d^{d} X \sqrt{-G} e^{-2\varphi} \left[ R + 4 D_{\mu} \varphi D^{\mu} \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} F_{\mu\nu}^{M} F^{\mu\nu N} M_{MN} \right. \left. + \frac{1}{8} D_{\mu} M_{MN} D^{\mu} M^{MN} - V \right],$$

(5.19)
where
\[
H_{\mu \nu \rho} = 3 \left( \partial_{[\mu} B_{\nu \rho]} - A^M_{[\mu} \partial_{\nu]} A^M_{\rho]} - \frac{1}{3} f_{MNP} A^M_{[\mu} A^N_\nu A^P_\rho \right),
\]
\[
\mathcal{F}^M_{\mu \nu} = 2 \partial_{[\mu} A^M_{\nu]} + f^M_{NP} A^N_\mu A^P_\nu.
\]
\[
D_{\mu} \mathcal{M}_{MN} = \partial_{\mu} \mathcal{M}_{MN} + f_{MPQ} A^P_\mu \mathcal{M}_{QN} + f_{NPQ} A^P_\mu \mathcal{M}_{MQ} \tag{5.20}
\]
and the scalar potential is
\[
V = \frac{1}{12} f_{MPQ} f_{NQS} \mathcal{M}^{MN} \mathcal{M}^{PQ} \mathcal{M}_{RS} + \frac{1}{4} f_{MPQ} f_{NQP} \mathcal{M}^{MN} + \frac{1}{6} f_{MNP} f^{MNP}. \tag{5.21}
\]

This action reproduces heterotic supergravity in ten external dimensions for \( k = 0 \) and \( G_L = SO(32) \) or \( E_8 \times E_8 \), with the following identifications. The scalar frame is only non-vanishing for \( E \) and \( G \) and the scalar potential is
\[
M, N \text{ with constraints (5.5)}
\]
and are taken to be the structure constants of \( G \) where \( (4.1) \).

Plugging all this in (5.19) and taking for \( A \) the scalars in (5.18) to \( A \) fields enhanced gauge symmetry, one simply extends the frame \( \tilde{\mathcal{M}}_{MN} = \mathcal{M}_{MN} \) and then there are no gaugings. In this case, (5.19) reproduces (4.1) when identifying the string low energy effective action (4.1).

Identifying \( A^\Gamma_\mu = A^\Gamma_\mu, B_{\mu \nu} = -B_{\nu \mu}, G_{\mu \nu} = G_{\mu \nu} \) one gets the ten dimensional heterotic string low energy effective action (4.1).

For \( k \neq 0 \) and generic values of the background fields, the gauge group is \( U(1)^{2k+16} \) and then there are no gaugings. In this case, (5.19) reproduces (4.1) when identifying the generalized gauge fields with the string theory fields as
\[
A^M_\mu = \mathcal{E}^M_0 A^a_\mu = \mathcal{E}^M_{0\pi} A^\pi_\mu + \mathcal{E}^M_{02} A^a_\mu + \mathcal{E}^M_0 A^A_\mu
\]
\[
= \frac{1}{\sqrt{2}} \left( G_{mn} (A^a_{\mu} - \bar{A}^a_{\mu}) + C_{mn} (A^a_{\mu} + \bar{A}^a_{\mu}) + \sqrt{2} A^I_{\mu} A_I^a \right), \tag{5.24}
\]
with \( M, N = 1, \ldots, 2k + 16 \). The components of the generalized scalar matrix \( \mathcal{M}_{MN} \) are related with the background fields and massless modes of the string theory as \( G_{mn} = G_{mn} + S_{(mn)}, B_{mn} = -B_{mn} - S_{[mn]}, A_{Im} = A_{Im} + S_{Im} \).

To make contact with (4.1) at the points of the moduli space giving \( G_L \times U(1)^{2k} \) enhanced gauge symmetry, one simply extends the frame \( \tilde{\mathcal{E}}_A^I \) in (5.17) to \( \tilde{\mathcal{E}}_\Gamma^I \), the gauge fields \( A_\mu^I \) in (5.24) to include the non-abelian sector 2 of section 2.3, i.e. \( A_\mu^I \to A_\mu^I \), and the scalars in (5.18) to \( A_\mu^a = (A^I_{\mu} + S^I_{\mu}, S^a_\mu) \), where the indices \( \Gamma, \Lambda, F, G = 1, \ldots, N \). Plugging all this in (5.19) and taking for \( f_{MNP} \) the structure constants of \( G_L \), one recovers (4.1).

In the cases of maximal enhancement, we can take \( G_i = O(k, N) \), with \( N \) being the dimension of a simply-laced group of rank \( 16 + k \). The \( k \) left internal dimensions become
part of the dimensions associated to the Cartan subgroup of the enhanced gauge group, the left KK gauge fields $A^m_\mu$ become Cartan components of the non-abelian gauge fields $A^\Gamma_\mu$ and the gaugings are the structure constants of the gauge group. In the next section we deal with these cases in full detail and we also show that the action (5.19) reproduces the right patterns of symmetry breaking when moving away from a point of enhancement.

5.3 Generalized Scherk-Schwarz reductions

We have seen that appropriately choosing the global and local symmetry groups and the gaugings deforming the generalized Lie derivative (5.9), one can account for both the un-compactified and the toroidally compactified versions of the heterotic string effective low energy theory with gauge group $G_L \times U(1)^k_L \times U(1)^k_R$. To describe the effective theory with maximally enhanced gauge group $G_L \times U(1)^k_L$, we perform a generalized Scherk-Schwarz (gSS) compactification of DFT. Recall that the result of gauging the theory and parameterizing the generalized fields in terms of the degrees of freedom of the lower dimensional theory is effectively equivalent to a gSS reduction of DFT [19], which has the advantage of providing an explicit realization of the generalized vielbein $E_A^M$ giving rise to the enhanced gauge algebra under the generalized diffeomorphisms [5.11] [12,13]. In this section we extend the construction to the heterotic case, and in particular, we will show that the formulation of [13] allows to describe the $E_8 \times E_8$ and $SO(32)$ theories as two solutions of the same heterotic DFT, even before compactification.

The generalized vielbein in gSS reductions is a product of two pieces, one depending on the $d$ external coordinates $x^\mu$ and the other one depending on the internal ones, $y_L, y_R$:

$$E_A(x,y_L,y_R) = \Phi_A^{A'}(x)E_{A'}(y_L,y_R).$$

The matrix $\Phi$ parameterizes the scalar, vector and tensor fields of the reduced $d$-dimensional action and the twist $E$ characterizes the background.

Let us concentrate on the internal part of the vielbein

$$\mathcal{E}_a^M(x,y_L) = \Phi_{ab}^b(x)\mathcal{E}_b^M(y_L),$$

where $M = a = 1, \ldots, N + k$, and $N$ is now the dimension of $G_L$ (i.e. the $k$ left internal dimensions are absorbed by the Cartan directions of $G_L$). The matrix $\Phi_{ab}^b$ describes the fluctuations over the background and the twist $\mathcal{E}_b^M$ is an element of the coset $O(k,N) / O(k) \times O(N)$, generating the constant fluxes $f_{abc}$ which gauge (a subgroup of) the global $O(k,N)$ symmetry. We take $\mathcal{E}_b^M$ to depend on $y_L$ only, as this is the only sector with a non-Abelian gauge group.

The scalar matrix can be written as

$$\mathcal{M}^{MN} = \delta^{ab}\mathcal{E}_a^M\mathcal{E}_b^N = M^{ab}(x)\mathcal{E}_a^M(y_L)\mathcal{E}_b^N(y_L),$$

with

$$M^{ab}(x) = \delta^{cd}\Phi_{c}^{a}(x)\Phi_{d}^{b}(x).$$

We now expand on the explicit parameterization of $\Phi(x)$ in terms of fluctuations that can be identified with the string theory fields and on the twist $\mathcal{E}_a^M(y_L)$ realizing the enhanced gauge algebra.
5.3.1 Fluctuations around generic points in moduli space

In order to identify the massless vector and scalar fields of the reduced theory with the corresponding string states at a generic point in moduli space, we first consider a reduction on an ordinary $2k + 16$ torus (i.e., no twist). There are no gaugings and therefore we get an ungauged action with $2k + 16$ abelian $U(1)^{k+16} \times U(1)^k_L$ vectors $A^m_\mu, A^I_\mu, A^m_\mu$ and $(k + 16) \times k$ scalars encoded in $M_{ab}$. The vectors and scalars contain the 16 Cartan generators, the $2k$ KK fields and the fluctuations of the metric, $B$-field and Wilson lines on the torus, corresponding to the string states $a^m_\mu, a^I_\mu, a^m_\mu, g_{mn}, b_{mn}, a^m_\mu$ in sector 1. To get the precise relation, consider an expansion around a given point in moduli space corresponding to constant background metric $G$, $B$-field $B$ and Wilson line $A$.

The internal part of the generalized vielbein in the left-right basis reads, at first order,

$$
\begin{pmatrix}
E_R \\
E_L \\
E_A
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-e_0 - \hat{\epsilon}_0 C_0 & \hat{\epsilon}_0 & -\hat{\epsilon}_0 A_0 \\
e_0 - \hat{\epsilon}_0 C_0 & \hat{\epsilon}_0 & -\hat{\epsilon}_0 A_0 \\
\sqrt{2} \epsilon A_0 & 0 & \sqrt{2} \tilde{\epsilon}
\end{pmatrix} + \frac{1}{\sqrt{2}} \delta \begin{pmatrix}
-e - \hat{\epsilon} C & \hat{\epsilon} & -\hat{\epsilon} A \\
e - \hat{\epsilon} C & \hat{\epsilon} & -\hat{\epsilon} A \\
\sqrt{2} \epsilon A & 0 & \sqrt{2} \tilde{\epsilon}
\end{pmatrix},
$$

(5.29)

where now we denote $e_0$ and $\hat{\epsilon}_0$ the frames and inverse frames for $G$ to lighten the notation. Note we are not varying the frame for the Killing metric $\hat{\epsilon}$. Performing this expansion and accommodating the terms so that it has the form of a gSS reduction $\mathcal{E} = \Phi(x)\overline{E}$, where now the twist $\overline{E}$ is constant, one gets

$$
\Phi(x) = \text{Id} + \frac{1}{2} \begin{pmatrix}
\varphi_+ + \phi & \varphi_- - \phi & -\sqrt{2}(\hat{\epsilon}_0 + \delta \hat{\epsilon})\delta A\hat{\epsilon}_0 \\
\varphi_+ + \phi & \varphi_- - \phi & -\sqrt{2}(\hat{\epsilon}_0 + \delta \hat{\epsilon})\delta \hat{\epsilon}_0 \\
-\sqrt{2}\epsilon \delta A\hat{\epsilon}_0 & \sqrt{2}\tilde{\epsilon}\delta A\hat{\epsilon}_0 & 0
\end{pmatrix}
$$

with

$$
\varphi_\pm \equiv \delta \hat{\epsilon}_0 \pm \delta \epsilon \hat{\epsilon}_0, \quad \phi \equiv (\hat{\epsilon}_0 + \delta \hat{\epsilon})(\delta C + \delta AA_0)\hat{\epsilon}_0.
$$

The matrix of $\Phi$ is an element of $SO^+(k, k + 16; \mathbb{R})$, the component of $O(k, k + 16; \mathbb{R})$ connected to the identity. Inserting this into (5.28) we get, up to second order,\footnote{We actually get a second order piece in the off-diagonal terms, namely instead of $M$, one gets $M + Q$, where $Q$ contains terms of the form $\delta \hat{\epsilon} \delta \hat{\epsilon}, \delta B' \delta B'$, etc., but this second order piece is not needed for our purpose of computing the action up to quartic order.}

$$
M_{ab} = \delta^{cd} \Phi_c \Phi^d = \begin{pmatrix}
(I_{\overline{\mu} \overline{\nu}} + \frac{1}{2}(M^t M)_{\overline{\mu} \overline{\nu}} & I_{\overline{ab}} & I_{AA} + \frac{1}{2}(MM^t)_{AA} \\
M_{\overline{\mu} \overline{\nu}} & I_{\overline{ab}} + \frac{1}{2}(M)_{\overline{ab}} & M^t_{\overline{\mu} \overline{\nu}} \\
M_{\overline{\mu} \overline{\nu}} & 0 & I_{AA} + \frac{1}{2}(MM^t)_{AA}
\end{pmatrix},
$$

(5.30)

The $k \times k$ matrix $M_{\overline{\mu} \overline{\nu}}$ is

$$
M_{\overline{\mu} \overline{\nu}} = -\hat{\epsilon}_0 m_{\overline{\mu} \overline{\nu}} (\delta G_{mn} - \delta B_{mn}'), \quad \delta B' \equiv \delta B + \frac{\delta AA_0 - A_0 \delta A^t}{2},
$$

(5.31)

where $\delta G = \delta \epsilon e_0 + \epsilon_0 \delta e + \epsilon \epsilon' \delta e$ and $\delta B'$ is the variation of $\delta B$ under an $O_0$ shift (2.19) with $\Theta = \delta B$ and an $O_A$ shift (2.21) with $\Lambda = \delta A$ (see footnote 4). The $16 \times k$ matrix $M_{\overline{\mu} \overline{\nu}}$ is

$$
M_{\overline{\mu} \overline{\nu}} = \sqrt{2} \epsilon A^t \delta A_{lm} \hat{\epsilon}_0 m
$$

(5.32)
The fluxes $f_{ab}^c$ computed from (5.11) vanish as the twist $E$ is constant and the theory is not deformed. Then, taking the abelian field strengths $F_{\mu\nu}^a = (F_{\mu\nu}^1, F_{\mu\nu}^2, F_{\mu\nu}^A)$ for the $U(1)_L^{k+16}$ and $U(1)_R$ vector fields, we get (up to first order in fluctuations)

$$-rac{1}{4}F_{\mu\nu}^a\mathcal{M}_{MN}F^{N\mu\nu} = -\frac{1}{4}F_{\mu\nu}^1\delta_1F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}^2\delta_2F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}^A\delta_AF_{\mu\nu}$$

$$-\frac{1}{2}F_{\mu\nu}^a\mathcal{M}_{\mu\nu}F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}^A\mathcal{M}_{AB}F_{\mu\nu}$$

and

$$\frac{1}{8}D_\mu\mathcal{M}_{MN}D^\mu\mathcal{M}_{MN} = \frac{1}{4}\partial_\mu\delta G_{mn}\partial^\mu\delta G^{mn} - \frac{1}{4}\partial_\mu\delta B_{mn}\partial^\mu\delta B^{mn} - \frac{1}{2}\partial_\mu\delta A_{Im}\partial^\mu\delta A^{Im}$$

Plugging these terms in (5.19), the effective action (4.1) derived from toroidally compactified string theory is reproduced if we identify, as in the previous section, $F_{\mu\nu} = e_{0m}F_{\mu\nu}^m$, $F_{\mu\nu} = e_{0m}F_{\mu\nu}^m$, $F_{\mu\nu}^A = \bar{e}_A^1F_{\mu\nu}$, $\delta G_{mn} = S_{(mn)}$, $\delta B_{mn} = S_{[mn]}$, $\delta A_{Im} = S_{I,m}$, where the vector and scalar fields correspond to the string theory states $\bar{a}_m^\mu$, $a_\mu$, $a_I^\mu$, $S_{I,m}$ in sector 1 of section 2.3.

### 5.3.2 Symmetry enhancement

In order to incorporate the massless degrees of freedom that enhance the $U(1)^{16+k}$ gauge symmetry to a $N$-dimensional group $G_L$ of rank $16+k$, we identify the $16+k$ torus with the maximal torus of the enhanced symmetry group, so that the $O(k, 16+k; \mathbb{R})$ covariance of the abelian theory is promoted to $O(k, N; \mathbb{R})$. The $k+16$ left-moving vectors of the previous section combine with the extra massless vector states in sector 2 of section 2.3, giving a total of $N$ left-moving massless vectors $a_\mu^I$, which together with the $k$ right-moving vectors $\bar{a}_m^\mu$ transform in the fundamental representation of $O(k, N)$.

The $(N+k) \times (N+k)$ matrix $\mathcal{M}^{ab}$ is expanded as in (5.30), where the scalar fluctuations $S_{Im}$, $S_{an}$ are now combined in the $N \times k$ block $M_{\mu\nu} = e_G^\Gamma S_{Im}\rho^m\pi$ as

$$\mathcal{M}^{ab} = \begin{pmatrix} I_\pi^\mu + \frac{1}{2}(M^I)^\pi \hat{M}_{\pi} & \hat{M}_{\pi G} \\ \hat{M}_{\pi G} & I_{FG} + \frac{1}{2}(M^I)^{FG} \end{pmatrix}$$

(5.33)

The effective action is formally as in (5.19), where now the non-abelian left vector fields $A_\mu^G$ and scalars $S_{Gm}$ absorb the KK left vector and scalar fields, yielding

$$-\frac{1}{4}F_{\mu\nu}^a\mathcal{M}_{MN}F^{N\mu\nu} = -\frac{1}{4}F_{\mu\nu}^1\delta_1F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}^2\delta_2F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}^A\delta_AF_{\mu\nu} - \frac{1}{2}F_{\mu\nu}^1M_{\pi G}F_{\mu\nu},$$

with $F_{\mu\nu}^G = \bar{e}_1^G F_{\mu\nu} = \bar{e}_1^G (2\partial_\mu A_\nu^G + f_{\nu\lambda\sigma} A_\mu^\lambda A_\sigma^G)$ and

$$\frac{1}{8}D_\mu\mathcal{M}_{MN}D^\mu\mathcal{M}_{MN} = \frac{1}{4}D_\mu M_{\pi G}D^\mu M^G_{\pi},$$

with $D_\mu M_{\pi G} = \partial_\mu M_{\pi G} + f^H_{FG} A_{\mu}^F M_{H\pi}$.

The structure constants in the field strengths, covariant derivatives and scalar potential can be explicitly computed from the twist $E_a^M$, generalizing the procedure introduced
for the bosonic string in [12, 13] (see also [23]). Namely, the extra massless vectors with non-trivial momentum and winding can be thought of as coming from a metric, a $B$-field and a Wilson line defined in an extended tangent space, with extra dimensions. The fields in this fictitious manifold depend on a set of coordinates dual to the components of momentum and winding along the compact directions. Promoting the internal piece of the vielbein $E^M_a$ to an element in $O(k, N; \mathbb{R})$, the fluxes computed from the deformed generalized Lie derivative by

$$f_{abc} = 3E^M_{[a} \partial_{M}E^N_bE^P_c) \eta_{NP} + \Omega_{abc},$$

(5.34)

reproduce the structure constants of the enhanced gauge algebra, with the deformation $\Omega_{abc}$ defined below. A dependence on the left internal coordinates is therefore mandatory, but we restrict it to dependence only on the Cartan subsector, namely on the $k + 16$ coordinates $y_L^I$, $E^M_a = E^M_a(y_L^I)$.

To be specific, start with the generalized vielbein

$$E_a = \begin{pmatrix} -\hat{e}C & \hat{e} & -\hat{e}A \\ e & 0 & 0 \\ \hat{e}A^t & 0 & \hat{e} \end{pmatrix} \begin{pmatrix} dy^m \\ \partial y^m \\ dy^I \end{pmatrix},$$

(5.35)

where $e, \hat{e}, \tilde{e}, A$ and $C$ are the fields on the torus at the point of enhancement. Then, identify $\partial y^m \leftrightarrow d\tilde{y}_m$, rotate to the left-right basis on the spacetime indices and bring the generalized vielbein to a block-diagonal form rotating the flat indices, which leads to

$$E_{RL} = \sqrt{2} \begin{pmatrix} -e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & \tilde{e} \end{pmatrix} \begin{pmatrix} dy_R \\ dy_L \\ d\tilde{y}^I \end{pmatrix},$$

(5.36)

where

$$dy_L^m = \frac{1}{2} C_{mn}[(G - C)_{nl}dy^l + d\tilde{y}_n - A^l_n dy^l], \quad dy_R^m = \frac{1}{2} C_{mn}[(G + C)_{nl}dy^l - d\tilde{y}_n + A^l_n dy^l]$$

and

$$d\tilde{y}^I = \kappa^{IJ} dy_J + \frac{1}{\sqrt{2}} A^l_m dy^m.$$

Finally, we extend this $(2k + 16) \times (2k + 16)$ matrix so that it becomes an element of $O(k, N)$ of the form

$$E^M_{RL}(y_L) = \sqrt{2} \begin{pmatrix} -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{16 \times 16} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} J \end{pmatrix},$$

(5.37)

where the index $M = 1, \cdots, k + N$ and the $(N - (k + 16)) \times (N - (k + 16))$ diagonal block $J$ contains the left-moving ladder currents associated to the $\alpha_i$ roots of the enhanced gauge group, $J^{\alpha}_{\alpha}(y_L, ..., y_L^{k+16}) = \delta_{\alpha \beta} e^{i\sqrt{2} \alpha_i \cdot y_L}$. Note that the $(N + k) \times (N + k)$ matrix

\[26\] Note that the space itself is not extended further than the 16-dimensional torus and the double torus of dimension $2k$. The derivative in (5.34) along “internal directions” has only non-zero components along the $k + 16$ Cartan directions of the $p + k$-dimensional tangent space.
depends only on the coordinates associated to the Cartan directions of the algebra. In case the gauge symmetry is enhanced to a product of groups, $J$ contains the currents of all the factors, each set of currents depending on the corresponding Cartan directions.

Taking for the deformation
\[
\Omega_{abc} = \begin{cases} 
\varepsilon(\alpha, \beta) \delta_{\alpha+\beta+\gamma} & \text{if two roots are positive}, \\
-\varepsilon(\alpha, \beta) \delta_{\alpha+\beta+\gamma} & \text{if two roots are negative}, 
\end{cases}
\]
if $a, b, c$ are associated with roots, and zero if one or more indices correspond to Cartan generators, all the structure constants can be obtained replacing (5.37) in (5.34). The deformation accounts for the cocycle factors that were excluded from the CFT current operators in (5.37) but are necessary in order to compensate for the minus sign in the OPE $J^\alpha(z)J^\beta(w)$ when exchanging the two currents and their insertion points $z \leftrightarrow w$ ($c_\alpha c_\beta = \varepsilon(\alpha, \beta)c_{\alpha+\beta}$). It was conjectured in [24] that such factors would also appear in the gauge and duality transformations of double field theory, and actually, they can be included without spoiling the local covariance of the theory. Indeed, the cocycle tensor $\Omega_{abc}$ satisfies the consistency constraints of gauged DFT, (5.5) and (5.6), and it breaks the $O(k, N)$ covariance of (5.19) to $O(k, k + 16)$. In this way, all the structure constants can be obtained from (5.34) using the expression (5.37) for the generalized vielbein with the appropriate currents corresponding to the enhanced gauge groups. All the gaugings obtained in this way satisfy the quadratic constraints (5.23), and therefore the construction is consistent.

It is interesting to note that the deformation $\Omega_{abc}$ can be chosen to be the same one for the $E_8 \times E_8$ and $SO(32)$ groups. Indeed, we show in Appendix E that both groups have 26880 non-vanishing structure constants of the form $f_{\alpha\beta}^{\alpha+\beta}$, half of which can be chosen to be $+1$ and the other half $-1$, so that one unique deformation accounts for both heterotic theories. The generalized vielbeins giving the remaining structure constants which involve one Cartan index can be obtained from (5.37) with $J$ containing the $E_8 \times E_8$ or the $SO(32)$ currents. Choosing the former or the latter amounts to choosing a background, and in this sense the two heterotic theories can be considered as two solutions of the same gauged DFT, even before compactification.

Plugging all this in (5.19), we get precisely the effective action (4.1) derived from the string amplitudes, where the potential is, to lowest order in $M$,
\[
V = \frac{1}{16} M_F \bar{a} M_{F'} \bar{b} M_{G^0} \bar{c} f^{FGH} f^{F'G'} H. \tag{5.38}
\]

Note that unlike in the bosonic theory, there is neither a cosmological constant nor a cubic piece in the potential, which is now bounded from below. Additionally, the quadratic piece cancels. There is also a sixth-order potential, but in order to get its explicit form we would need to expand $M$ in (5.30) to quartic order in the fields.\footnote{This is not necessary for the quartic order as the $n$-th order contributions to $M_{ab}$ cancel in the $n$-th order contribution to the potential.}

### 5.4 Away from the self-dual points

In this section we show that moving away from a point of enhancement corresponds to giving a vacuum expectation value to $M^{\bar{a}a}$, the piece in the matrix of scalar fields that...
belongs to the Cartan subsector, corresponding to the KK scalars for the metric, $B$-field and Wilson lines. In the next section we show that the mass acquired by the vectors and scalars that are not in the Cartan directions agree with the string theory masses.

In the neighborhood of a given point of enhancement, the scalars in the Cartan subsector acquire a vacuum expectation value $v^{\hat{A}\hat{a}}$. Then we redefine

$$M^{\hat{A}\hat{a}} \rightarrow v^{\hat{A}\hat{a}} + M^{\hat{A}\hat{a}},$$

so that $\langle M^{G\hat{a}} \rangle = 0$ for all indices $G, \hat{a}$. These vevs spontaneously break the enhanced symmetry: some or all of the left-moving vectors in non-Cartan directions $A^a_\mu$ get a mass from the covariant derivative of the scalars, given by

$$m_{A^a}^2 = -f^{\pi}_{A} f^{\pi_B} v^{\hat{A}a} v^{B\hat{a}} = \alpha^{\hat{A}} v^{\hat{A}a} \alpha^{B\hat{a}} = |\alpha \cdot v|^2. \quad (5.40)$$

Note that, as expected, this is always positive, unlike in the bosonic theory.

We discuss now in more detail the process of spontaneous symmetry breaking. It is simpler for this to use the Chevalley basis for the Cartan generators, where the Killing form is equal to the Cartan matrix $\kappa_{ij} = C_{ij}$, and the components of a simple root $\alpha^I$ (where the subscript $\hat{I}$ labels the root) are $(\alpha^I)_j^I = \delta^I_j$.

We thus have for simple roots $\alpha^I$ and non-simple roots $\beta^I = (n^\beta)_j^I \alpha^I$,

$$m_{A^a}^2 = |v^j|^2, \quad m_{A^\beta}^2 = |\beta \cdot v|^2 = |n^\beta \cdot v|^2. \quad (5.41)$$

We see that by giving arbitrary vevs to all scalars in the Cartan subsector, all the gauge vectors corresponding to ladder generators acquire mass and the gauge symmetry is spontaneously broken to $U(1)_L^{k+16} \times U(1)_R^k$. Similarly, if $v$ has a row with all zeros, let's say the row $I_0$, then the corresponding (complex) vector $A^\beta$ remains massless, and there is at least an $SU(2)$ subgroup of $G_L$ that remains unbroken. The converse is also true, namely

$$v^{I_0\hat{a}} = 0 \quad \forall \hat{a} \quad \Leftrightarrow \quad m_{A^{I_0\hat{a}}}^2 = 0. \quad (5.42)$$

For the vectors associated to non-simple roots $\beta$ the situation is more tricky as it depends on which integers $n^\beta_I$ are non-zero. $A^\beta$ remains massless if $v^{I\hat{a}} = 0$ for all $I$ such that $n^\beta_I \neq 0$ and for all $\hat{a}$.

Note that one cannot give masses only to the vectors corresponding to non-simple roots: if all the vectors corresponding to simple roots are massless, then necessarily $v = 0$ and there is no symmetry breaking at all. This implies that the spontaneous breaking of symmetry always involves at least one $U(1)$ factor, corresponding to the Cartan of the $SU(2)$ associated to the simple root whose vector becomes massive. Thus we cannot go from one point of maximal enhancement in moduli space (given by a semi-simple group) to another point of maximal enhancement by a spontaneous breaking of symmetry.

Regarding the scalars, introducing the vevs for those in the Cartan subsector in the potential $\langle 5.33 \rangle$, we get at quadratic order in the scalar fields

$$\frac{1}{16} f^{FGH} f^{FG'} H(M^I M)_{FF'} (M^I M)_{GG'} \rightarrow \frac{1}{8} \sum_{\alpha, b, c} \left( f^{A\alpha H} f^{A'\alpha'} H v^{A\hat{b}} v^{A'\hat{c}} M_{\alpha c} M_{\alpha' \hat{c}} + 2 f^{A\alpha H} f^{A'\alpha' H} v^{A\hat{b}} v^{A'\hat{c}} M_{\beta c} M_{\alpha' \hat{c}} \right). \quad (5.43)$$
The first term gives
\[ -\frac{1}{4} \sum_{\text{all roots } \alpha, \bar{\alpha}} m_{\alpha}^2 |M^{\alpha \bar{\alpha}}|^2, \quad \text{where } m_{\alpha}^2 = \sum_b (\alpha A^b)^2, \tag{5.44} \]
and then replacing it in the action \([5.19]\), we see that the mass of the scalar fields agrees with the mass of the vectors \([5.40]\). The second term can be written as
\[ \frac{1}{4} \sum_{\text{all roots } \alpha} \left( \sum_b m_b M^{ab} \right)^2, \quad \text{where } m_b = \sum_{\tilde{A}} \alpha A^b \tag{5.45} \]
and \(M^a = \sum_b m_b M^{ab}\) is the Goldstone boson contribution which is eaten by the vectors to become massive. This agrees with the results in \([15]\) on which we expand in the next section.

We thus get that for arbitrary vevs, all vectors and scalars except those along Cartan directions acquire masses, and the symmetry is broken to \(U(1)^k \times U(1)^{16} L \times U(1)^k R\). If \(v_{\hat{I} \bar{a}} = 0\) for a given \(\hat{I} \bar{a}\) and for all \(\bar{a}\), while all other vevs are non-zero, then the remaining symmetry is at least \((SU(2) \times U(1)^{k+15})_L \times U(1)^k R\) where the \(SU(2)_L\) factor corresponds to the root \(\alpha \hat{I}_0\), and the massless scalars are, besides those purely along Cartan directions, at least all those of the form \(M^{\alpha \hat{I}_0 \bar{a}}\).

5.4.1 Comparison with string theory

Let us compare the vector and scalar masses that we got in the previous section from the double field theory effective action, to those of string theory given by \((2.16)\).

We decompose the generalized metric \(M\) as in \((5.27)\), where \(E\) is the twist containing the information on the background at the point of enhancement and \(M^{ab}\) represents the fluctuations from the point, parameterized as in \((5.30)\) in terms of the matrix \(M\) in \((5.31)\). Inserting this in the mass formula \((2.16)\) we get
\[
m^2 = 2 \left( N + \bar{N} - \left\{ \begin{array}{ll} 1 & \text{R sector} \\ 3 & \text{NS sector} \end{array} \right. \right) + Z^t E^t \left( \begin{array}{cc} I_k + \frac{1}{2} M^t M & M^t \\ M & I_{k+16} + \frac{1}{2} M M^t \end{array} \right) E Z, \tag{5.46} \]
On the other hand, from Eq. \((2.34)\)
\[
E Z = \begin{pmatrix} p_{\alpha R} \\ p_{\alpha L} \\ p^t_L \end{pmatrix} = \begin{pmatrix} p_R \\ p_L \end{pmatrix}. \tag{5.47} \]
We thus get
\[
m^2 = 2 \left( N + \bar{N} - \left\{ \begin{array}{ll} 1 & \text{R sector} \\ 3 & \text{NS sector} \end{array} \right. \right) + p^t_R (I_k + \frac{1}{2} M^t M) p_R + p^t_L (I_{k+16} + \frac{1}{2} M M^t) p_L \\
+ p^t_R M^t p_L + p^t_L M p_R. \tag{5.48} \]

\[28\text{Note that } M \text{ here is a } (k+16) \times k \text{ matrix spanning along the Cartan directions only, as in section } 5.3.1\]

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The bosonic states that are massless at the point of enhancement (when \( M = 0 \)) have \( p_R = 0 \) and \( \tilde{N} = \frac{1}{2} \) in the \( NS \) sector.

The left-moving vectors have either \( N = 1 \) and \( p_L = 0 \), or \( N = 0 \) and \( p_L = \alpha \) with \( \alpha \) a root of the enhanced gauge algebra (and thus \( |\alpha|^2 = 2 \)). The former vectors (Cartan) are massless for any \( M \), while, according to (5.48), the latter have mass

\[
m^2_{A_\alpha} = \frac{1}{2} \alpha^t M^t M \alpha .
\] (5.49)

On the right sector the only massless vectors are the Cartan, which are massless for any \( M \). This agrees with the masses (5.40) if we identify

\[
v = -\hat{e}_0 (\delta G - \delta B^t) \hat{e}_0^t, \sqrt{2} \delta A \hat{e}_0
\] . (5.50)

The scalars in sector 1 (both legs along Cartan directions) have \( p_L = 0 \), \( N = 1 \) and \( \tilde{N} = \frac{1}{2} \), and are massless for any \( M \). The scalars in sector 2 have \( N = 0 \), \( \tilde{N} = \frac{1}{2} \) in the \( NS \) sector, and \( p_L = \alpha \). Their masses are thus exactly those of the vectors corresponding to the same root, namely

\[
m^2_{M_\alpha} = m^2_{A_\alpha}
\] (5.51)
in agreement with what we have found from DFT, Eq. (5.44), confirming that these are the Goldstone bosons of the spontaneous breaking of symmetry.

It is interesting to recall that the combinations \( \tilde{f}_\alpha^{\alpha \pi} \equiv f^{A_\pi} \hat{v}_{A_\alpha} \) appearing in the vector and scalar masses (5.40) and (5.43) agree with the coefficients of the string theory three-point functions involving one massless right or left vector and two massive left vectors. Then following [15], one could identify the DFT fluxes with the string theory three-point amplitudes and conclude that the fluxes depend on the moduli. Actually, from a gSS DFT point of view, the vevs can be thought of as being encoded either in the twists \( E_a^M (y_L) \) or in the fluctuations \( \Phi_{ab} (x) \). In this section we have developed the latter identification, i.e. the fluxes \( f^{A_\pi} \) are computed from (5.34) with the twist (5.37) containing the currents corresponding to the enhanced gauge group, and the symmetry is broken by the vevs shifting the fluctuations in (5.39). In the former case, i.e. to get moduli dependent fluxes, one can replace the currents in (5.37) by those of the massive vectors in (4.4), and then the twists depend on both the left- and the right-moving internal coordinates, \( E_a^M (y_L, y_R) \). In this way, the fluxes computed from the deformed generalized Lie derivative (5.34) get mixed indices from the left and right moving sectors, reproducing the coefficients of the string theory three-point functions which involve massive vectors (4.13). One could then interpret that the fluxes \( \tilde{f}_\alpha^{\alpha \pi} \) encode the information about the background through the vertex operators creating the string theory vector and scalar states.

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A Lie algebras and lattices

Modular invariance of the one-loop partition function of the heterotic string implies that the 16-dimensional internal momenta must take values in an even self-dual Euclidean lattice, $\Gamma = \Gamma^*$, of dimension 16. There are only two of these: $\Gamma_8 \times \Gamma_8$, where $\Gamma_8$ is the root lattice of $E_8$, and $\Gamma_{16}$, which is the root lattice of $SO(32)$ in addition to the $(s)$ or $(c)$ conjugacy class

$$\Gamma_8 \times \Gamma_8 = \Gamma_g \quad \text{for } E_8 \times E_8 \quad (A.1)$$

$$\Gamma_{16} = \Gamma_g + \Gamma_s \quad \text{for } SO(32)$$

In this Appendix we summarize some basic notions on these lattices, which are named Narain lattices.

Given a Lie algebra $g$ of rank $n$, taking arbitrary integer linear combinations of root vectors, one generates an $n$-dimensional Euclidean lattice $\Gamma_g$, called the root lattice. E.g., for the rank $n$ orthogonal groups $SO(2n)$, the $n$ component simple root vectors are

$$\pm 1, \pm 1, 0, \ldots \quad \text{all other entries zero,} \quad (A.2)$$

and all permutations of these. For $E_8$, the eight component vectors

$$(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0) + \text{permutations} \quad (A.3)$$

contain the 240 roots, i.e. the 112 root vectors of $SO(16)$ and 128 additional vectors.

Any Lie group $G$ has infinitely many irreducible representations which are characterized by their weight vectors. Irreducible representations fall into different conjugacy classes, and $\Gamma_g$ can be thought of as the $(0)$ conjugacy class. Two different representations are said to be in the same conjugacy class if the difference between their weight vectors is a vector of the root lattice.

While $E_8$ has only one conjugacy class, namely $(0)$, the $SO(2n)$ algebras have four inequivalent conjugacy classes:

- The $(0)$ conjugacy class, i.e. the root lattice, contains vectors of the form
  $$(n_1, \ldots, n_n), \quad n_i \in \mathbb{Z}, \quad \sum_{i=1}^n n_i = 0 \mod 2. \quad (A.4)$$

- The vector conjugacy class, denoted by $(v)$, contains vectors of the form
  $$(n_1, \ldots, n_n), \quad n_i \in \mathbb{Z}, \quad \sum_{i=1}^n n_i = 1 \mod 2. \quad (A.5)$$

- The spinor conjugacy class, denoted by $(s)$, contains vectors of the form
  $$(n_1 + \frac{1}{2}, \ldots, n_n + \frac{1}{2}), \quad n_i \in \mathbb{Z}, \quad \sum_{i=1}^n n_i = 0 \mod 2. \quad (A.6)$$
• The $(c)$ conjugacy class contains vectors of the form

\[
(n_1 + \frac{1}{2}, \ldots, n_n + \frac{1}{2}), \quad n_i \in \mathbb{Z}, \quad \sum_{i=1}^{n} n_i = 1 \text{ mod } 2.
\]  

(A.7)

The weight lattice $\Gamma^w$ is formed by all weights of all conjugacy classes including the root lattice itself. Clearly $\Gamma_{g} \subset \Gamma^w$, and for a simply-laced Lie algebra, which roots have squared modulus 2, it can be shown that $\Gamma_{g} = \Gamma^{*}_{w}$. Therefore, the weight lattice of $E_8$ contains the weights of the form

\[
\Gamma^8_w : \left\{ \left( n_1, \ldots, n_8 \right), \left( n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2} \right), \quad \sum_{i=1}^{8} n_i = \text{even integer} \right\}
\]  

(A.8)

with $n_i \in \mathbb{Z}$, is identical to its root lattice, which implies that it is even self-dual. It is also identical to the $SO(16)$ lattice with the $(0)$ and $(s)$ conjugacy classes.

A necessary condition for a self-dual lattice is that it be unimodular. The $SO(2n)$ Lie algebra lattices are unimodular if they contain two conjugacy classes. The weight lattice of $Spin(32)/\mathbb{Z}_2$ is identical to the $SO(32)$ lattice with the $(0)$ and $(s)$ conjugacy classes. It is even self-dual and its vectors are:

\[
\Gamma^{16}_w : \left\{ \left( n_1, \ldots, n_{16} \right), \left( n_1 + \frac{1}{2}, \ldots, n_{16} + \frac{1}{2} \right), \quad \sum_{i=1}^{16} n_i = \text{even integer} \right\}
\]  

(A.9)

Both the root lattice of $E_8 \times E_8$ and the weight lattice of $Spin(32)/\mathbb{Z}_2$ contain 480 vectors of $(\text{length})^2 = 2$ which are the roots of $E_8 \times E_8$ and $SO(32)$, respectively.

It is convenient to write the conjugacy classes of $SO(32)$ in terms of conjugacy classes of representations of $SO(16) \times SO(16)$. We denote by $(xy)$ a vector with the first eight components in the conjugacy class $(x)$ of $SO(16)$ and the last eight in the class $(y)$. $x$ and $y$ can be 0, $s$, $v$ or $c$. We then have 16 conjugacy classes $(xy)$. The $SO(32)$ conjugacy classes correspond to the following $SO(16) \times SO(16)$ pairs

\[
(0) = (00), (vv)
\]

\[
(s) = (ss), (cc)
\]

\[
(c) = (sc), (cs)
\]

\[
(v) = (0v), (v0)
\]

(A.10)

We have then

\[
\Gamma_{16} = \Gamma_{0}^{16} + \Gamma_{s}^{16} = (00), (vv), (ss), (cc)
\]

\[
\Gamma_{8} \times \Gamma_{8} \equiv \Gamma_{8+8} = (\Gamma_{0}^{8} + \Gamma_{s}^{8}) \times (\Gamma_{0}^{8} + \Gamma_{s}^{8}) = (00), (ss), (0s), (s0)
\]

(A.11)

The dual to the root lattice of $SO(32)$ is

\[
(\Gamma_{0}^{16})^* = \Gamma_{g}^* = (00), (vv), (ss), (cc), (0v), (v0), (sc), (cs).
\]

(A.12)
We also use the following properties of the lattices

\[ \Gamma_{8+8} \backslash \Gamma_{16} = (0s), (s0) \]
\[ \Gamma_{16} \backslash \Gamma_{8+8} = (vv), (cc) \]
\[ \Gamma_{16} \cap \Gamma_{8+8} = (00), (ss) \]
\[ (\Gamma_{16} \cap \Gamma_{8+8})^* = (00), (ss), (vv), (cc), (vc), (cv), (0s), (s0) \]
\[ (\Gamma_{16} \cap \Gamma_{8+8})^* \backslash (\Gamma_{16} \cup \Gamma_{8+8}) = (vc), (cv). \]

Note that both for \( SO(32) \) (or rather \( Spin(32)/\mathbb{Z}_2 \)) and for \( E_8 \), one could have chosen the opposite chirality, namely the (c) class instead of (s). We will denote this choice \( SO(32)^- \) and \( E_8^- \). We can then build the following pairs

\[ \Gamma_{16} = \Gamma_{16}^0 + \Gamma_{16}^c = (00), (vv), (sc), (cs) \]
\[ \Gamma_8^- \times \Gamma_8^- = (\Gamma_0^8 + \Gamma_c^8) \times (\Gamma_0^8 + \Gamma_c^8) = (00), (cc), (0c), (c0) \]
\[ \Gamma_8^c \times \Gamma_8^- = (\Gamma_0^8 + \Gamma_c^8) \times (\Gamma_0^8 + \Gamma_c^8) = (00), (sc), (0c), (s0) \]
\[ \Gamma_8^- \times \Gamma_8^c = (\Gamma_0^8 + \Gamma_c^8) \times (\Gamma_0^8 + \Gamma_c^8) = (00), (cs), (0s), (c0) \]

B  Maximal enhancement points for \( A = (A_1, 0_{15}) \)

In this Appendix we show how to obtain the maximal enhancement points for the particular case of Wilson lines with only one non-zero component, treated in section 3.4. We also prove that the only possible maximal enhancements for Wilson lines with only one non-zero entry are to \( SU(2) \times SO(32), SO(34), SU(2) \times E_8 \times E_8 \) and \( SO(18) \times E_8 \).

The maximal enhancement points are those where two or more curves intersect. There are three types of intersections: \( a_{w_1,q_1}(R) = a_{w_2,q_2}(R), b_{w_1,q_1}(R) = b_{w_2,q_2}(R) \) and \( a_{w_1,q_1}(R) = b_{w_2,q_2}(R) \), that we treat separately. In the case of \( \Gamma_8 \times \Gamma_8 \), the curves \( b \) can in principle have a curve \( c \) on top of them.

B.1 \( a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \)

\[ a_{w_1,q_1}(R) = \frac{2q_1 \pm \sqrt{2 - 2w_1^2 R^2}}{w_1}, \quad \frac{2q_1 - 1}{w_1} \in \mathbb{Z} \]  
(B.1)

\[ a_{w_2,q_2}(R) = \frac{2q_2 \pm \sqrt{2 - 2w_2^2 R^2}}{w_2}, \quad \frac{2q_2 - 1}{w_2} \in \mathbb{Z} \]

imply

\[ \pm_1 w_2 \sqrt{2 - 2w_1^2 R^2} \pm_2 w_1 \sqrt{2 - 2w_2^2 R^2} = 2q_1 w_2 - 2q_2 w_1 \equiv C' = 2C \in 2\mathbb{Z}. \]  
(B.2)

The case \( C = 0 \) is trivial, so we must assume \( C \neq 0 \), which leads to

\[ R^2 = \frac{2}{C'^2} - \frac{(2w_1^2 + 2w_2^2 - C'^2)^2}{8w_1^2 w_2^2 C'^2}. \]  
(B.3)
Defining \( N = \frac{(1-2q_1^2)}{w_1} w_2 + \frac{(1-2q_2^2)}{w_2} w_1 + 4q_1q_2 \in \mathbb{Z} \), we can rewrite (B.3) as
\[
N^2 = 4 - 2C'^2R^2. \tag{B.4}
\]

Since \((1 - 2q_1^2)\) and \(w_1\) are odd, \(N\) is even. Also, since \(C'\) and \(R\) are non-zero we get \(N^2 < 4\), which implies \(N = 0\), then \(R^2 = \frac{2}{C'}\). Then the radius where a curve \(a\) with winding \(w_1\) intersects another curve \(a\) with winding \(w_2\) is
\[
R^{-2} = \frac{R^2}{w_1^2 + w_2^2}. \tag{B.5}
\]
The constraint
\[
|q_1w_2 - q_2w_1| = \sqrt{w_1^2 + w_2^2} \implies \frac{w_1^2 + w_2^2}{2} \text{ must be a perfect square}.
\]

If \(w_1 = w_2 = w\), then \(q_1 = q_2 \pm 1\). The winding must be a divisor of both \(2q_1^2 - 1\) and \(2q_2^2 - 1\), but these numbers are coprime \(\forall q_1\). Then the only possible value of \(w\) is 1. In conclusion, the only curves \(a\) with the same winding number that intersect are \(a_{1,q}(R)\) and \(a_{1,q\pm1}(R)\). And the intersection is on \(R = \frac{1}{\sqrt{2}}\).

\[\begin{align*}
\text{B.2} \quad & \quad b_{w_1,q_1}(R) = b_{w_2,q_2}(R) \\
& \quad b_{w_1,q_1}(R) = \frac{2q_1+1 \pm 1}{w_1} \sqrt{1 - 2w_1^2R^2}, \quad \frac{2q_1(q_1+1)}{w_1} \in \mathbb{Z} \\
& \quad b_{w_2,q_2}(R) = \frac{2q_2+1 \pm \pm}{w_2} \sqrt{1 - 2w_2^2R^2}, \quad \frac{2q_2(q_2+1)}{w_2} \in \mathbb{Z}
\end{align*}\]

In this case,
\[
\mp \sqrt{1 - 2w_1^2R^2} \pm_2 \sqrt{1 - 2w_2^2R^2} = 2q_1w_2 - (2q_2 + 1)w_1 \equiv C \in \mathbb{Z}. \tag{B.7}
\]

If \(C = 0\), then \(w_1 = w_2\) and \(q_1 = q_2\). If \(C \neq 0\)
\[
R^2 = \frac{1}{2C^2} - \frac{(w_1^2 + w_2^2 - C^2)^2}{8w_1^2w_2^2C^2}. \tag{B.8}
\]

Defining \(N = \frac{2q_1(q_1+1)}{w_1} w_2 + \frac{2q_2(q_2+1)}{w_2} w_1 - (2q_1 + 1)(2q_2 + 1) \in \mathbb{Z}\), we get
\[
N^2 = 1 - 2R^2C^2 < 1 \implies N = 0, \tag{B.9}
\]
and then \(R^2 = \frac{1}{2C^2}\). Replacing in (B.8), \(C^2 = w_1^2 + w_2^2\), and then the radius where curve \(b\) with winding \(w_1\) intersects curve \(b\) with winding \(w_2\) is \(R^{-2} = 2(w_1^2 + w_2^2)\).

The constraint
\[
|2(q_1 + 1)w_2 - (2q_2 + 1)w_1| = \sqrt{w_1^2 + w_2^2} \implies w_1^2 + w_2^2 \text{ is a perfect square}. \tag{B.10}
\]

If \(w_1 = w_2 = w\) then \(\sqrt{2w}\). The l.h.s. is integer and the r.h.s. is irrational, then there is no winding such that \(b_{w,q_1}(R) = b_{w,q_2}(R)\).
\[ a_{w_1, q_1}(R) = b_{w_2, q_2}(R) \]

\[ a_{w_1, q_1}(R) = \frac{2q_1 \pm \sqrt{2-2w_1^2 R^2}}{w_1}, \quad \frac{2q_1^2 - 1}{w_1} \in \mathbb{Z} \quad (B.11) \]

\[ b_{w_2, q_2}(R) = \frac{2q_2 + 1 \pm \sqrt{1-2w_2^2 R^2}}{w_2}, \quad \frac{2q_2(q_2+1)}{w_2} \in \mathbb{Z} \quad (B.12) \]

\[ \pm 1 w_2 \sqrt{2-2w_2^2 R^2} \pm 2 w_1 \sqrt{1-2w_2^2 R^2} = 2q_1 w_2 - (2q_2 + 1) w_1 = C \in \mathbb{Z}. \quad (B.13) \]

Since \( w_1 \) is always odd, then \( C \) is also odd (in particular it is non-zero). Then

\[ R^2 = \frac{C^2}{2} - \frac{(w_1^2 + 2w_2^2 - C^2)^2}{8w_1^2 w_2^2 C^2} \quad \text{and} \quad N^2 = 2 - 2C^2 R^2, \quad (B.14) \]

where \( N = \frac{(1-2q_1^2) w_2 - 2q_2(q_2+1) w_1}{w_2} + q_1(2q_2 + 1) \in \mathbb{Z} \), and then \( N = 0 \) or 1, which give \( R^2 = \frac{1}{C^2} \) or \( R^{-2} = \frac{1}{2C^2} \). From (B.14) we obtain \( C^2 = w_1^2 + 2w_2^2 \) or \( C^2 = (w_1 - w_2)^2 + w_2^2 \). Then the radii where a curve \( a \) with \( w_1 \) intersects another curve \( b \) with \( w_2 \) intersect are:

\[ R^{-2} = w_1^2 + 2w_2^2 \quad \text{or} \quad R^{-2} = 2((w_1 - w_2)^2 + w_2^2) \quad (B.15) \]

For each case we have one of these constraints:

\[ |2q_1 w_2 - (2q_2 + 1) w_1| = \sqrt{w_1^2 + 2w_2^2} \quad \text{or} \quad |2q_1 w_2 - (2q_2 + 1) w_1| = \sqrt{(w_1 - w_2)^2 + w_2^2} \]

and then \( w_1^2 + 2w_2^2 \) or \( (w_1 - w_2)^2 + w_2^2 \) must be a perfect square. If \( w_1 = w_2 = w \) we get the constraints:

\[ |2q_1 - (2q_2 + 1)| = \sqrt{3} \quad \text{or} \quad |2q_1 - (2q_2 + 1)| = 1 \quad (B.16) \]

leaving only the second case, with \( q_2 = q_1 \) or \( q_1 - 1 \). The quantization conditions imply that \( w \) must be a divisor of both \( 2q_1^2 - 1 \) and \( 2q_1(q_1 \pm 1) \). But it can be shown that these numbers are coprime, and then \( w = 1 \). The only curves with the same windings that intersect are \( a_{1, q}(R) \) and \( b_{1, q}(R) \) or \( b_{1, q-1}(R) \). The intersections are at \( R = \frac{1}{\sqrt{2}} \).

Summarising, we have:

\[ a_{w_1, q_1} = a_{w_2, q_2} \iff R^{-2} = w_1^2 + w_2^2 = C^2 \]

\[ b_{w_1, q_1} = b_{w_2, q_2} \iff R^{-2} = 2(w_1^2 + w_2^2) = 2C^2 \quad (B.17) \]

\[ a_{w_1, q_1} = b_{w_2, q_2} \iff R^{-2} = \begin{cases} w_1^2 + 2w_2^2 = C^2 \\ 2((w_1 - w_2)^2 + w_2^2) = 2C^2 \end{cases} \]

The winding numbers on \( b \) can in principle be any positive integer and those on \( a \) can only be the divisors of some number of the form \( 2q_2^2 - 1, q \in \mathbb{Z} \).
B.4 Enhancements to $SO(34)$ or $SO(18) \times E_8$

Here we prove that $a_{w_1,q_1}(R) = a_{w_2,q_2}(R)$ implies that there exist integers $w_3, q_3, w_4$ and $q_4$ such that $a_{w_1,q_1}(R) = b_{w_3,q_3}(R) = b_{w_4,q_4}(R)$.

We start with $R^{-2} = w_1^2 + w_2^2$. If $w_1 > w_2$, there are integers $w_3$ and $w_4$ such that $w_1 = w_3 + w_4$ and $w_2 = w_3 - w_4$, because $w_1$ and $w_2$ are odd numbers. Then

$$R^{-2} = w_1^2 + (2w_3 - w_1)^2 = 2(w_1^2 - 2w_3w_1 + w_3^2 + w_3^2) = 2((w_1 - w_3)^2 + w_3^2) \quad (B.18)$$

Since $R^{-2} = 2((w_1 - w_3)^2 + w_3^2)$ as well, there exist integers $w_3, w_4, q_3$ and $q_4$ such that $a_{w_1,q_1}(R) = b_{w_3,q_3}(R) = b_{w_4,q_4}(R)$. Note that we can always find $q_3$ and $q_4$ because the functions $b$ admit any value of $w$.

Replacing $w_3 = \frac{1}{2}(w_1 + w_2)$ and $w_4 = \frac{1}{2}(w_1 - w_2)$ we get

$$a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \implies a_{w_1,q_1}(R) = a_{w_2,q_2}(R) = b_{w_1+w_2/2,q_3}(R) = b_{w_1-w_2/2,q_4}(R)$$

Note that we can also write the radius as $2(w_3^2 + w_4^2)$. We want to satisfy

$$(\sqrt{2}R)^{-1} = |2q_1w_3 - (2q_3 + 1)w_1| = |2q_1w_4 - (2q_4 + 1)w_1| = |(2q_3 + 1)w_4 - (2q_4 + 1)w_3|,$$

and we have that

$$(\sqrt{2}R)^{-1} = |q_1w_2 - q_2w_1| = |2q_1w_3 - (q_1 + q_2)w_1| = |2q_1w_4 - (q_1 - q_2)w_1|$$

$$= |(q_1 + q_2)w_4 - (q_1 - q_2)w_3|.$$ 

Then we need to identify $q_1 + q_2 = 2q_3 + 1, q_1 - q_2 = 2q_4 + 1$.

We still have to prove that $2q_3(q_3 + 1)$ and $2q_4(q_4 + 1)$ are divisible by $w_3$ and $w_4$, respectively, which amounts to proving that

$$w_i$$ is a divisor of $2q_i^2 - 1$ and $|q_1w_2 - q_2w_1| = \sqrt{\frac{w_3^2 + w_4^2}{2}} \implies w_1 \pm w_2$$ is a divisor of $(q_1 \pm q_2)^2 - 1$ \quad (B.19)

We checked that this is satisfied for the first 300 values of $q_i$.

Then we have that

$$a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \implies b_{w_1+w_2/2,(q_1+q_2-1)/2}(R) = b_{w_1-w_2/2,(q_1-q_2-1)/2}(R).$$

To prove that $b_{w_3,q_3}(R) = b_{w_4,q_4}(R)$ implies that there exists integers $w_1, q_1, w_2$ and $q_2$ such that $b_{w_3,q_3}(R) = a_{w_1,q_1}(R) = a_{w_2,q_2}(R)$, we start with $R^{-2} = 2(w_3^2 + w_4^2)$. Define integers $w_1$ and $w_2$ such that $w_3 = \frac{1}{2}(w_1 + w_2)$ and $w_4 = \frac{1}{2}(w_1 - w_2)$ (we assume $w_3 > w_4$),

$$R^{-2} = 2((w_1 - w_3)^2 + w_3^2) \quad \text{and} \quad R^{-2} = 2((w_2 - w_3)^2 + w_3^2).$$

But we still need to satisfy the constraint that $w_1$ and $w_2$ are divisors of $2q_1^2 - 1$ and $2q_2^2 - 1$ for two integers $q_1$ and $q_2$. With the identifications $q_1 + q_2 = 2q_3 + 1, q_1 - q_2 = 2q_4 + 1$, we get the correct radius

$$R^{-1} = \sqrt{2}|(2q_3 + 1)w_4 - (2q_4 + 1)w_3| = \sqrt{2}|2q_1w_3 - (2q_3 + 1)w_1|,$$
We still have to prove that $2q_1^2 - 1$ and $2q_2^2 - 1$ are divisible by $w_1$ and $w_2$, respectively. This is the same as proving that

\[ q_i \text{ is a divisor of } 2q_i(q_i + 1) \text{ and } |(2q_3 + 1)w_4 - (2q_4 + 1)w_3| = \sqrt{w_3^2 + w_4^2} \]  

(B.20)

which we checked is satisfied.

In conclusion, we have that, for $R^{-2} = w_1^2 + w_2^2$, $a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \iff a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \iff b_{(w_1+w_2)/2,(q_1+q_2-1)/2}(R) = b_{(w_1-w_2)/2,(q_1-q_2-1)/2}(R) \iff b_{w_1+w_2/2}(R) = b_{w_1-w_2/2}(R) \Rightarrow \pm q_3 = \pm q_4 = \pm 1$

The Wilson lines that give this enhancement can be written in four different ways

\[ \frac{2q_1}{w_1} \pm 1 \sqrt{2R} \frac{w_2}{w_1} = \frac{2q_2}{w_2} \pm 2 \sqrt{2R} \frac{w_1}{w_2} = \frac{2q_3 + 1}{w_3} \pm 3 \sqrt{2R} \frac{w_4}{w_3} = \frac{2q_4 + 1}{w_4} \pm 4 \sqrt{2R} \frac{w_3}{w_4} \]

Using that $w_3 = \frac{w_1+w_2}{2}$, $w_4 = \frac{w_1-w_2}{2}$, $q_3 = \frac{q_1+q_2-1}{2}$ and $q_4 = \frac{q_1-q_2-1}{2}$, after a few steps, we get $\mp 4 = \pm 3 = \pm 2 = \mp 1$ and then the Wilson lines are

\[ A_1 = \frac{2q_1}{w_1} \pm \frac{w_2}{w_1} \sqrt{2R} , \quad A_1 = \frac{2q_2}{w_2} \mp \frac{w_1}{w_2} \sqrt{2R} \]

(B.21)

From here,

\[ (\sqrt{2R})^{-1} = \mp (q_1w_2 - q_2w_1) \in \mathbb{Z} \]  

(B.22)

and then, after a few steps, we can prove that

\[ \frac{1}{\sqrt{2R}}, \frac{R}{\sqrt{2}} \left( \frac{1}{2} A^2 + 1 \right) \in \mathbb{Z}, \]  

(B.23)

Defining integers $m = (\sqrt{2R})^{-1}$ and $n = A/\sqrt{2}$, all this type of enhancement points are given by

\[ (R,A_1) = \left( \frac{1}{m\sqrt{2}}, \frac{n}{m} \right) \text{ such that } \frac{n^2+1}{2m} \in \mathbb{Z} \]  

(B.24)

and then

\[ R^{-2} = 2, 50, 338, 578, 1250, 1682, 2738, 3362, 5618, 7442, 8450, 10658, \ldots \]  

(B.25)

These are all of the form $2C^2$ with $C$ an integer with prime divisors congruent to 1 mod 4. That is: 1, 5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 73, 85, 89, 97, 101, 109, . . . Except for the 1, these numbers are all Pythagorean primes or multiples of them.

We want to see if the $b$ lines considered here can be interposed with a $c$ line. $q_3$ and $q_4$ are suitable for curves $b$ with $w_3$ and $w_4$. For curves $c$ to coincide with them, we need $w_i$ even and $\frac{q_i(q_i+1)}{w_i} \in \mathbb{Z}$. If one of the two curves $b$ has also a curve $c$ then we have an intersection between an $a$ and a $c$ curve. Analyzing all the possibilities, it can be shown that there are no $c$ curves that intersect with more than one other curve.
The equality \( a_{w_1,q_1}(R) = b_{w_2,q_2}(R) \) arises for two type of radius
\[
R^{-2} = w_1^2 + 2w_2^2 \quad \text{or} \quad R^{-2} = 2((w_1 - w_2)^2 + w_2^2).
\] (B.26)

The second type gives \( R^{-2} = w_1^2 + w_2^2 \) if \( w_2 = \frac{w_1 + w_1}{2} \), which implies that there is an intersection with another curve \( a \) of winding \( w_3 \). Then, we restrict to the first type, where \( R^{-2} \) is odd for odd \( w_1 \). Thus the even \( R^{-2} \) found in the previous section cannot have additional curves \( a \) or \( b \) on the intersection.

For \( R^{-2} = w_1^2 + 2w_2^2 \), the constraints are
\[
|2q_1w_2 - (2q_2 + 1)w_1| = \sqrt{w_1^2 + 2w_2^2}, \quad \frac{2q_1^2 - 1}{w_1} \in \mathbb{Z}, \quad \frac{2q_2(q_2 + 1)}{w_2} \in \mathbb{Z}
\] (B.27)

The Wilson line can be written as
\[
A_1 = \frac{2q_1 \pm 12Rw_1}{w_1} \quad \text{or} \quad A_1 = \frac{2q_2 + 1 \pm 2Rw_2}{w_2},
\] (B.28)
and equating them leads to \( \pm 1 = \mp 1 \) and
\[
R^{-1} = \mp (2q_1w_2 - (2q_2 + 1)w_1)
\] (B.29)

implying that \( R^{-1} \) is an odd number. After some algebra, we get
\[
\frac{1}{R}, \quad A, \quad R \left( \frac{1}{2}A^2 + 1 \right) \in \mathbb{Z}
\] (B.30)

and then all this type of enhancement points satisfy
\[
(R, A_1) = \left( \frac{1}{m}, \frac{2n}{m} \right)
\] (B.31)

for integer \( m = R^{-1} \) and \( n = \frac{R^{-1}A_1}{2} \), such that
\[
\frac{2n^2 + 1}{m} \in \mathbb{Z}.
\] (B.32)

We obtain
\[
R^{-1} = 3, 9, 11, 17, 19, 27, 33, 41, 43, 51, 57, 59, \ldots
\] (B.33)

all integer numbers with prime divisors congruent to 1 or 3 (mod 8).

It is not hard to prove that all the curves \( b \) that intersect just one curve \( a \) are superimposed by a curve \( c \) (in the \( \Gamma_8 \times \Gamma_8 \) case).

### C More on fixed points and dualities

Here we collect the details of the calculations involved in section 3.6. The transformations \( O_U \) in (3.37) acting on a vector \( Z \) of momentum, winding and “heterotic” momenta, result in the transformed momenta
\[
w' = r \left( n - \frac{1}{2} |A|^2 w - A \cdot \pi \right), \quad n' = \left( \frac{1}{r} + A'U A \right) w + A'U \pi - r \frac{|A|^2}{2} \left( n - \frac{1}{2} |A|^2 w - A \cdot \pi \right), \quad \pi' = U \pi + U A w - r A' \left( n - \frac{1}{2} |A|^2 w - A \cdot \pi \right).
\] (C.1)
Requiring these to be quantized leads to the conditions

\[
\begin{align*}
& \frac{r}{|A|^2} , \frac{r}{|A'|^2} \in \mathbb{Z}; \\
& \frac{1}{r} + A'UA + \frac{r}{2} |A|^2 |A'|^2 \in \mathbb{Z} \quad \forall \pi \in \Gamma, \pi' \in \Gamma': \pi'U \pi + r(\pi \cdot A)(\pi' \cdot A') \in \mathbb{Z}; \\
& rA, rA' \in \Gamma \cap \Gamma'; \\
& A'U + \frac{r}{2} |A'|^2 A \in \Gamma; \\
& U + \frac{r}{2} |A|^2 A' \in \Gamma'
\end{align*}
\]

and \( U \in O(16, \mathbb{Z}) \).

We analyze these in more detail, depending whether the duality acts on the same theory or links two theories with different lattices \( \Gamma \) and \( \Gamma' \).

**C.1 \( \Gamma \leftrightarrow \Gamma \)**

For Wilson lines of the form (3.24), and \( U \) given by (3.40), the quantization conditions (C.2) become

\[
\begin{align*}
& \frac{(Q \pm 1)|A|^2}{r^2}, \frac{(Q \pm 1)|A'|^2}{r^2} \in \mathbb{Z}, \quad \text{and} \quad \left\{ \begin{array}{ll}
Q + 1 \in 2\mathbb{Z} & \text{for} \ U = \pm I \\
Q \in 2\mathbb{Z} & \text{for} \ U = U_\pm
\end{array} \right.
\end{align*}
\]

where \( U_\pm \) are defined in (3.40),

\[
Q = \frac{rA^2}{2} = \left. \frac{1}{2} \left| A \right|^2 \right|_{fp} = \left. \frac{1}{2} \left| A \right|^2 \right|_{fp}
\]

and \( A \) is defined in (3.32). Here we have used that for the fixed points \( R = R_{fp} \), one has \( r = R_{fp}^{-2} \) since \( R' = rR = R \).

For \( U = \pm I \) we define \( p = \frac{(Q \pm 1)|A|^2}{r} \) and \( q = Q/2 \), then:

\[
p, \; q, \; r, \; \sqrt{pq}, \; \sqrt{qr} \in \mathbb{Z}; \quad \sqrt{pr} = 2q \pm 1. \tag{C.5}
\]

Quotienting these equations, we see that \( p, q, r \) can be written as

\[
\sqrt{p} = t\sqrt{k}, \quad \sqrt{q} = n\sqrt{k}, \quad \sqrt{r} = m\sqrt{k}, \tag{C.6}
\]

with \( t, n, m, k \in \mathbb{Z} \). Then

\[
k^{-1} = \pm (tm - 2n^2) \in \mathbb{Z}, \tag{C.7}
\]

which implies \( k = 1 \) and \( t = \frac{2n^2 + 1}{m} \). Taking into account that \( n = \sqrt{\frac{Q}{2}} = \frac{R^{-1}A_1}{2} \) and \( m = \sqrt{r} = R^{-1} \) must be integers, the only condition is

\[
\frac{2n^2 + 1}{m} \in \mathbb{Z}. \tag{C.8}
\]

For \( U = U_\pm \), defining \( p = \frac{(Q \pm 1)|A|^2}{2r} \) and \( s = \frac{r}{2} \),

\[
p, \; Q, \; s, \; \sqrt{ps}, \; \sqrt{Qs} \in \mathbb{Z} \quad \text{and} \quad \sqrt{ps} = \frac{Q \pm 1}{2}. \tag{C.9}
\]
where we have used the fact that as $Q$, $2p$ and $\sqrt{pQ}$ are integers with $Q$ odd, implies that $p$ is also integer. $\frac{Q+1}{2}$ is integer, then we have the same situation as in the first case. Analogously

$$\sqrt{p} = t\sqrt{k}, \quad \sqrt{Q} = n\sqrt{k}, \quad \sqrt{s} = m\sqrt{k}$$

(C.10)

with $t, n, m, k \in \mathbb{Z}$. Then

$$tmk = \frac{n^2k+1}{2} \quad \Rightarrow \quad k^{-1} = \pm(2tm - n^2) \in \mathbb{Z},$$

(C.11)

but this implies that $k = 1$ and $t = \frac{n^2 + 1}{2m}$. Then, taking into account that $n = \sqrt{Q} = \frac{R-1}{\sqrt{2}}$ and $m = \sqrt{\frac{r}{2}} = \frac{R-1}{\sqrt{2}}$, the only condition is:

$$\frac{n^2 \pm 1}{2m} \in \mathbb{Z}$$

(C.12)

The possible values of $m$ and $n$ that verify these conditions with the plus signs give the fixed points $(R, A_1)$ presented in Table 1.

If we take $U = 1$ then the quantization conditions (C.2) require $\pi \cdot \pi' + r(\pi \cdot A)(\pi' \cdot A') \in \mathbb{Z}$ (\forall $\pi, \pi' \in \Gamma$). As $\pi$ and $\pi'$ belong to the same lattice, $\pi \cdot \pi' \in \mathbb{Z}$, and then $h'/r \in \mathbb{Z}$, where $h = \pi \cdot rA$ and $h' = \pi' \cdot rA'$. Restricting to $r$ prime, this implies that either $\pi \cdot A \in \mathbb{Z}$ or $\pi' \cdot A' \in \mathbb{Z}$.

If $A$ does not satisfy this for any $\pi \in \Gamma$, then $A' \in \Gamma$, and viceversa, i.e. either $A \in \Gamma$ or $A' \in \Gamma$. But $\frac{r|A|^2}{2} \in \mathbb{Z}$, $A' + \frac{r|A|^2}{2} A \in \Gamma$ and the reciprocal conditions imply $A, A' \in \Gamma$.

We just need to verify $\frac{1}{r} + A' \cdot A + r\frac{|A|^2}{2} \frac{|A'|^2}{2} \in \mathbb{Z}$. But $A' \cdot A$ is integer and $|A|^2$, $|A'|^2$ are even, then we get: $\frac{1}{r} \in \mathbb{Z}$, which is only possible for $r = 1$. Then 1 is the only non-composite possible value for $r$ when the duality does not change the lattice and $U = 1$.

**C.2** $\Gamma \leftrightarrow \Gamma' \neq \Gamma$

The quantization conditions (C.2) for the case where the dual lattice is not the original one become

$$r, \frac{r|A|^2}{2}, \frac{r|A_U|^2}{2} \in \mathbb{Z};$$

(C.13)

$$\frac{1}{r} + A_U \cdot A + r\frac{|A|^2}{2} \frac{|A_U|^2}{2} \in \mathbb{Z} \quad \forall \pi \in \Gamma, \pi_U \in \Gamma_U: \pi_U \cdot \pi + r(\pi \cdot A)(\pi_U \cdot A_U) \in \mathbb{Z};$$

$$\frac{rA}{r}, \frac{r|A_U|^2}{2} A \in \Gamma \cap \Gamma_U; \quad A_U + \frac{r|A_U|^2}{2} A \in \Gamma; \quad A + \frac{r|A|^2}{2} A_U \in \Gamma$$

where $\Gamma_U$ is the lattice obtained by applying the transformation $U$ to all the elements of $\Gamma'$ and $A_U = A'U$. This proves the statements at the beginning of [3.6.2](#).
Restricting to the case $U = 1$ we get the conditions
\begin{equation}
{\frac{r, r|A|^2}{2}, \frac{r|A'|^2}{2} \in \mathbb{Z}};
\end{equation}
\begin{align*}
\frac{1}{r} + A' \cdot A + r \frac{|A|^2|A'|^2}{2} & \in \mathbb{Z} \quad \forall \pi \in \Gamma, \; \pi' \in \Gamma': \; \pi' + r(\pi \cdot A)(\pi' \cdot A') \in \mathbb{Z}; \\
A, A' & \in \Gamma \cap \Gamma'; \quad A' + \frac{r|A|^2}{2} A \in \Gamma; \quad A + \frac{r|A'|^2}{2} A' \in \Gamma'.
\end{align*}

Given that $rA' \in \Gamma \cap \Gamma'$, then $k = \pi' \cdot rA' \in \mathbb{Z}$. We first analyze the condition $\pi' + r(\pi \cdot A)(\pi' \cdot A') \in \mathbb{Z}$. Being both $h = r(\pi \cdot A)$ and $h' = r(\pi' \cdot A)$ integer, we get $\pi' \cdot \pi' + \frac{hh'}{r} \in \mathbb{Z}$. For particular values of $\pi$ and $\pi'$, one has:

If $\pi \in \Gamma \cap \Gamma'$ then $\pi' \in \mathbb{Z}$ and $\frac{hh'}{r} \in \mathbb{Z}$. If we restrict again to non-composite values for $r$ then at least one of $h$ or $h'$ has to be divisible by $r$, and then $\pi \cdot A \in \mathbb{Z}$ or $\pi' \cdot A' \in \mathbb{Z}$.

If $A'$ is such that this does not hold for any $\pi' \in \Gamma'$, then $A$ must satisfy $\pi \cdot A \in \mathbb{Z}$ for all $\pi \in \Gamma \cap \Gamma'$, i.e. $A \in (\Gamma \cap \Gamma')^*$. In conclusion, if $A' \notin \Gamma'$, then $A \in (\Gamma \cap \Gamma')^*$, i.e. either one of $A$ or $A'$ has to be in one of those lattices.

Repeating this with $\pi' \in \Gamma \cap \Gamma'$, we get that if $A \notin \Gamma$, then $A' \in (\Gamma \cap \Gamma')^*$. But the additional restriction, $A' + \frac{r|A|^2}{2} A \in \Gamma$, necessarily gives $A' \in \Gamma$, since $\frac{r|A|^2}{2} \in \mathbb{Z}$ when $A \in \Gamma$. Analogously, when $A' \notin \Gamma'$ we get $A \in \Gamma'$. Then the possible Wilson lines are
\begin{equation}
A, A' \in \Gamma, \quad A, A' \in \Gamma', \quad A \in (\Gamma \cap \Gamma')^* \setminus \Gamma, \quad A' \in (\Gamma \cap \Gamma')^* \setminus \Gamma',
\end{equation}
which implies $\pi \cdot A', \pi \cdot A \in \mathbb{Z} \forall \pi \in \Gamma \cap \Gamma'$.

The equation $A' + \frac{r|A|^2}{2} A \in \Gamma$ is equivalent to $\pi \cdot A' + \frac{r|A|^2}{2} (\pi \cdot A) \in \mathbb{Z} \forall \pi \in \Gamma$, but when $\pi \in \Gamma \cap \Gamma'$ it holds trivially. Then we only have to verify the following equations
\begin{equation}
\pi \cdot A' + \frac{r|A|^2}{2} (\pi \cdot A) \in \mathbb{Z} \forall \pi \in \Gamma \setminus \Gamma', \quad \pi' \cdot A + \frac{r|A'|^2}{2} (\pi' \cdot A') \in \mathbb{Z} \forall \pi' \in \Gamma' \setminus \Gamma.
\end{equation}

Depending on $\Gamma$ and $\Gamma'$, it is possible that when $\pi \in \Gamma \setminus \Gamma'$ and $\pi' \in \Gamma \setminus \Gamma$ (i.e. $\pi, \pi' \notin \Gamma \cap \Gamma'$), then $\pi \cdot \pi' = \frac{1}{2} mod(1)$. Assuming one of these cases holds, the condition $\pi \cdot \pi' + \frac{hh'}{r} \in \mathbb{Z}$ turns into $\frac{hh'}{r} = \frac{1}{2} mod(1)$. That is, neither $h$ nor $h'$ must be divisible by $r$: $\pi \cdot A \notin \mathbb{Z}$ and $\pi' \cdot A' \notin \mathbb{Z}$. These equations imply $A \notin \Gamma$ and $A' \notin \Gamma'$, and then the Wilson lines are
\begin{equation}
A \in (\Gamma \cap \Gamma')^* \setminus \Gamma, \quad A' \in (\Gamma \cap \Gamma')^* \setminus \Gamma'.
\end{equation}
They can be split into two sets: $A, A' \in (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma')$ and $A \in \Gamma \setminus \Gamma, A' \in \Gamma \setminus \Gamma'$, where we used $\Omega \cap (\Omega \cap \Sigma)^* = \Omega$.

Now we analyze the condition $\pi \cdot A' + \frac{r|A|^2}{4} (\pi \cdot A) \in \mathbb{Z} \forall \pi \in \Gamma \setminus \Gamma'$. All the cases that we will study verify $\pi \cdot A = \frac{1}{2} mod(1) \forall \pi \in \Gamma \setminus \Gamma'$. Then the condition becomes
\begin{equation}
\pi \cdot A' + \frac{r|A|^2}{4} \in \mathbb{Z} \forall \pi \in \Gamma \setminus \Gamma'.
\end{equation}
If $A' \in \Gamma$, then $\frac{r|A|^2}{2} \in 2\mathbb{Z}$. Instead if $A' \notin \Gamma$, then $\frac{r|A|^2}{2}$ is odd. Using the analogous equation for $A \in \Gamma'$, $\frac{r|A|^2}{2}$ has to even and for $A \notin \Gamma'$, $\frac{r|A|^2}{2}$ odd.

Summarizing, the condition requires $\frac{r|A|^2}{2}$ even if $A \in \Gamma' \setminus \Gamma$ and odd if $A \in (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma')$ (and analogously for $A'$). If additionally $(\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma')$ is an integer lattice (which always is in the cases of our interest) then $\frac{r|A|^2}{2}$ is odd if $r = 2$. Given that $r = 2$ for $\Gamma \neq \Gamma'$, then $r = 1 \iff \Gamma = \Gamma'$ (when restricting to non-composite values of $r$).

Another condition that must hold is $2A \in \Gamma \cap \Gamma'$. But this occurs trivially for the lattices that we consider. $2A$ will always be in the adjoint conjugacy class of $SO(16) \times SO(16)$, which is contained in all $\Gamma \cap \Gamma'$ we will study (this could vary with other groups where, for instance, $(s) + (s) = (v)\ldots$).

We now analyze the condition $\frac{1}{r} + A' \cdot A + r \frac{|A|^2 |A'|^2}{2} \in \mathbb{Z}$, which can be rewritten as

$$\frac{1}{2} (1 + |A|^2 |A'|^2) + A' \cdot A + \in \mathbb{Z}.$$  \hspace{1cm} \text{(C.19)}$$

If $A \cdot A' \in \mathbb{Z}$, then $\frac{1}{2}(1 + |A|^2 |A'|^2) \in \mathbb{Z}$. This holds if both $|A|^2$ and $|A'|^2$ are odd, i.e. $A, A' \in (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma')$. The product of these Wilson lines verifies this as it is an integer lattice by hypothesis.

If $A \cdot A' = \frac{1}{2} mod(1)$ then $\frac{1}{2}(1 + |A|^2 |A'|^2) = \frac{1}{2} mod(1)$. This holds if at least one of the Wilson lines has even modulus squared, i.e. $A \in \Gamma' \setminus \Gamma$ and/or $A' \in \Gamma \setminus \Gamma'$. The product of these Wilson lines verifies this assuming the hypothesis holds: $\pi \cdot A = \frac{1}{2} mod(1) \forall \pi \in \Gamma \setminus \Gamma'$ and its dual.

Summarizing, if the following hypothesis hold

$$\pi \cdot \pi' = \frac{1}{2} mod(1) \forall \pi \in \Gamma \setminus \Gamma', \quad \pi' \in \Gamma' \setminus \Gamma, \quad \pi \cdot A = \frac{1}{2} mod(1) \forall \pi \in \Gamma \setminus \Gamma',$$

$$\pi' \cdot A' = \frac{1}{2} mod(1) \forall \pi' \in \Gamma' \setminus \Gamma \quad (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma') \text{ is an integer lattice},$$

then the duality must have

$$r = 2, \quad A \in (\Gamma \cap \Gamma')^* \setminus \Gamma, A' \in (\Gamma \cap \Gamma')^* \setminus \Gamma'$$

and the following conditions must be satisfied:

- If $A \in \Gamma \setminus \Gamma'$, then $|A|^2 \in 2\mathbb{Z}$.
- If $A' \in \Gamma' \setminus \Gamma$, then $|A'|^2 \in 2\mathbb{Z}$.
- If $A \in (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma')$, then $|A|^2 \in 2\mathbb{Z} + 1$.
- $A' \in (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma')$ then $|A'|^2 \in 2\mathbb{Z} + 1$.

These can be replaced by the more restrictive conditions

$$\Gamma \setminus \Gamma', \Gamma \setminus \Gamma \text{ are even lattices,} \quad (\Gamma \cap \Gamma')^* \setminus (\Gamma \cup \Gamma') \text{ is an odd lattice.}$$
The hypothesis \( \pi \cdot A = \frac{1}{2} mod(1) \forall \pi \in \Gamma \backslash \Gamma' \) and \( \pi' \cdot A' = \frac{1}{2} mod(1) \forall \pi' \in \Gamma' \backslash \Gamma \) can also be replaced by the more restrictive ones

\[
\pi \cdot \pi' = \frac{1}{2} mod(1) \forall \pi \in \Gamma \backslash \Gamma', \quad \pi' \in (\Gamma \cap \Gamma')^* \backslash \Gamma = [\Gamma \backslash \Gamma] \cup [(\Gamma \cap \Gamma')^* \backslash (\Gamma \cup \Gamma')]
\]

\[
\pi' \cdot \pi = \frac{1}{2} mod(1) \forall \pi' \in \Gamma' \backslash \Gamma, \quad \pi \in (\Gamma \cap \Gamma')^* \backslash \Gamma = [\Gamma \backslash \Gamma'] \cup [(\Gamma \cap \Gamma')^* \backslash (\Gamma \cup \Gamma')]
\]

Then sufficient conditions for duality to exist (and only for \( r = 2 \)) are

\[
\Gamma \backslash \Gamma', \; \Gamma' \backslash \Gamma \text{ even lattices } , \quad (\Gamma \cap \Gamma')^* \backslash (\Gamma \cup \Gamma') \text{ odd lattice},
\]

\[
\pi \cdot \pi' = \frac{1}{2} mod(1) \text{ if } \pi \text{ and } \pi' \text{ belong to different lattices (from these three) (C.20)}
\]

\[
SO(32) \leftrightarrow E_8 \times E_8
\]

The quantization conditions on the second line of (C.2) can be written as

\[
r(\pi \cdot A)(\pi' \cdot A') \in \mathbb{Z} \text{ if } \pi \text{ or } \pi' \in \Gamma_{(00),(ss)} , \quad (C.21)
\]

\[
r(\pi \cdot A)(\pi' \cdot A') + \frac{1}{2} \in \mathbb{Z} \text{ if } \pi \in \Gamma_{(vv),(cc)} \text{ and } \pi' \in \Gamma_{(0s),(s0)} . \quad (C.22)
\]

In the second situation, we get

\[
\frac{1}{r}(\pi \cdot rA)(\pi' \cdot rA') + \frac{1}{2} \in \mathbb{Z}, \quad \frac{\not{r}}{r} + \frac{1}{2} \in \mathbb{Z}, \quad j, k \in \mathbb{Z}, \quad (C.23)
\]

which imply that \( r \) is even. We restrict to the simplest possibility, \( r = 2 \), for which we get

\[
|A|^2, \; |A'|^2 \in \mathbb{Z};
\]

\[
\frac{1}{2} \left( 1 + 2A' \cdot A + |A|^2|A'|^2 \right) \in \mathbb{Z} \quad \forall \pi \in \Gamma_{16}, \; \pi' \in \Gamma_8 \times \Gamma_8 : \; \pi' \cdot \pi + 2(\pi \cdot A)(\pi' \cdot A') \in \mathbb{Z}; \quad (C.24)
\]

\[
2A, \; 2A' \in \Gamma_{(00),(ss)}; \quad A' + |A|^2A \in \Gamma_{16}; \quad A + |A|^2A' \in \Gamma_8 \times \Gamma_8
\]

The Wilson lines that satisfy (C.24) are

\[
A \in (\Gamma \cap \Gamma')^* \backslash \Gamma = (0s), (s0), (vc), (cv)
\]

\[
A' \in (\Gamma \cap \Gamma')^* \backslash \Gamma = (vv), (cc), (vc), (cv) \quad (C.25)
\]

### D Three and four-point functions

For completeness, in this appendix we list the scattering amplitudes of massless states of the (toroidally compactified) heterotic string that give the effective action (4.1). The results hold for arbitrary points of the moduli space, including enhanced and broken symmetry points, and differ only on the possible values taken by the indices and structure constants. Details of the calculations can be found in [20][25][26].
We use the following expectation values
\[
\langle X^\mu(z)X^\nu(w) \rangle = -\frac{1}{2}\eta^{\mu\nu}\ln(z-w), \quad \langle X^\mu(z)X^\nu(\bar{w}) \rangle = -\frac{1}{2}\eta^{\mu\nu}\ln(\bar{z}-\bar{w}),
\]
\[
\langle \tilde{\psi}^\mu(z)\tilde{\psi}^\nu(\bar{w}) \rangle = \frac{\eta^{\mu\nu}}{z-\bar{w}}, \quad \langle \phi(z)\phi(\bar{w}) \rangle = -\ln(\bar{z}-\bar{w}), \quad \langle \chi^m(z)\chi^n(\bar{w}) \rangle = \frac{\delta^{mn}}{\bar{z}-\bar{w}},
\]
\[
\langle Y_L^i(z)Y_L^j(w) \rangle = -\delta^{ij}\ln(z-w), \quad \langle Y_R^m(z)Y_R^n(\bar{w}) \rangle = -\delta^{mn}\ln(\bar{z}-\bar{w}),
\]
\[
\langle J^\Gamma(z_1)J^\Lambda(z_2) \rangle = \frac{\delta^{\Gamma\Lambda}}{z_{12}^2}, \quad \langle J^\Gamma(z_1)J^\Lambda(z_2)J^\Omega(z_3) \rangle = \frac{if^{\Gamma\Lambda\Omega}}{z_{12}z_{13}z_{23}}.
\]

### D.1 Three-point functions of massless states

- **Three left vectors:**
  \[
  A_{AAA} = -\frac{i}{\sqrt{2}} C_{S^2} g_c^3 f^{\Gamma\Lambda\Omega} A_{\mu}(k_1) A_{\nu}(k_2) A_{\rho}(k_3) (k_1^\mu \eta^\nu_\rho + k_2^\rho \eta^\mu_\nu + k_3^\nu \eta^\rho_\mu)
  \]
  \[
  = 12\pi g_c \sqrt{2} f^{\Gamma\Lambda\Omega} \partial_\mu A^{\Gamma}_A A^{\Lambda}_A A^{\Omega}_A,
  \]
  where we used \( C_{S^2} = \frac{8\pi}{27} \) from unitarity, and identified \( k_1^\mu A_2^A A_1^\Gamma \rightarrow -i\partial^\mu A_1^\Gamma A_2^A \).

- **Three tensors:**
  \[
  A_{VVV}(k_1, \epsilon^{(1)}, k_2, \epsilon^{(2)}, k_3, \epsilon^{(3)}) = C_{S^2} g_c^3 \left( \frac{1}{2} \eta^{\mu\nu}_1(k_1) \eta^{\nu\sigma}_2(k_2) \eta^{\sigma\tau}_3(k_3) (k_1^\nu \eta^\mu_\rho + k_2^\rho \eta^\nu_\sigma + k_3^\sigma \eta^\tau_\rho) \right)
  \times \left( \frac{1}{2} k_1^\mu k_2^\nu k_3^\lambda + k_1^\nu k_2^\lambda + k_1^\lambda k_2^\nu + k_3^\nu k_2^\lambda + k_3^\lambda k_2^\nu \right)
  \]

- **Two left vectors - one tensor:**
  \[
  A_{VAA}(k_1, \epsilon_1, k_2, A_2, k_3, A_3)
  \]
  \[
  = 4\pi g_c \left( -k_2^\mu \epsilon_1, k_3^\nu A_2^\Gamma A_3^\rho + k_2^\mu \epsilon_1, A_2^\nu k_3^\sigma A_3^{\Gamma \sigma} + k_3^\mu \epsilon_1, A_2^\nu A_3^\lambda k_3^\rho A_2^\sigma \right)
  \]
  Replacing \( V = g, b \) or \( D \), we get respectively
  \[
  A^{AA}(k_1, h_1, k_2, A_2, k_3, A_3) = 4\pi g_c g_{\mu\nu} \left( \partial^\mu A_1^\Gamma \partial^\nu A_2^\rho \right) - 2 \partial^\mu A_1^\Gamma \partial^\nu A_2^\rho
  \]
  or
  \[
  A^{DAA}(k_1, D, k_2, A_2, k_3, A_3) = 4\pi g_c (k_3 A_2^\Gamma)(k_2 A_3^\Gamma) = -\frac{4\pi g_c}{\sqrt{d-2}} D\partial^\nu A_1^\Gamma \partial^\mu A_1^\nu.
  \]

- **Two right vectors - one tensor:**
  \[
  A^{V\bar{A}}(k_1, \epsilon_1, k_2, \bar{A}_2, k_3, \bar{A}_3) = 4\pi g_c \epsilon_{1, \mu \nu} A_2^{\mu \nu} \bar{A}_3^{\mu \nu} \left( \frac{1}{2} k_1^\rho k_2^\sigma k_3^\mu + \eta^\mu_\rho k_2^\sigma + \eta^\nu_\sigma k_1^\rho + \eta^\nu_\rho k_3^\sigma \right) k_3^\nu
  \]

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which gives

\[ A^{g\bar{A}} = 4\pi g_c g_{\mu\nu} \left( \partial^\mu \bar{A}_\rho \cdot \partial^\nu \bar{A}_\rho - 2 \partial^\mu A^\mu \cdot \partial^\nu A^\rho + \frac{1}{2} \partial^\rho \partial^\mu \bar{A}_\rho \cdot \partial^\nu \partial^\rho \bar{A}_\rho \right) \]

\[ A^{b\bar{A}} = -8\pi g_c \partial^\rho B_{\mu\nu} \bar{A}_\rho \cdot \partial^\nu \bar{A}_\rho = 4\pi g_c \partial^\rho B_{\mu\nu} \bar{A}_\rho \cdot \bar{F}_{\mu\nu} \]

or

\[ A^{b\bar{A}} = \frac{2\pi g_c}{\sqrt{d-2}} D \bar{F}_{\mu\nu} \cdot \bar{F}_{\mu\nu} \]

- Two scalars - one left vector:

\[ A^{ASS}(k_1, A_1, k_2, S_2, k_3, S_3) = 4\pi g_c \sqrt{2} i f_{\Gamma\Lambda\Omega} k^\mu A^{\mu}_{1} S^{\Omega m_1} \delta_{m_2 m_3} \]

- Two scalars - one tensor:

\[ A^{VSS}(k_1, \epsilon_1, k_2, S_2, k_3, S_3) = -4\pi g_c \epsilon_{1 \mu \nu} S^\Gamma m_1 S^{\Omega m_3} k^\mu_1 k^\nu_3 \]

This is only non-vanishing for \( V = g \)

\[ A^{gSS}(k_1, \epsilon_1, k_2, S_2, k_3, S_3) = 4\pi g_c g_{\mu\nu} \partial^\mu S^\Gamma m \partial^\nu S_{\Gamma m} \]

- One scalar - one right vector - one left vector:

\[ A^{S\bar{A}}(k_1, S, k_2, A, k_3, \bar{A}) = -4\pi g_c S_{\Gamma m} k^\mu_1 A^{\mu}_{1} \bar{A}^\nu_1 \bar{A}^m = -4\pi g_c S_{\Gamma m} \partial^\mu A^\nu F^{\mu \nu m} \]

### D.2 Four-point function of massless scalars

We present some details of this computation which, to our knowledge, has not been previously published

\[
\langle S_{(0)} S_{(0)} S_{(-1)} S_{(-1)} \rangle = S_{\Gamma m} S_{\Lambda n} S_{\Omega p} S_{\Delta q} \left[ \frac{z_{34}^{k_2-k_1} z_{24}^{k_2-k_1} z_{14}^{k_2-k_1} z_{32}^{k_2-k_1}}{z_{34}^{k_3-k_2}} \right]
\times \left[ -\frac{k_1 \cdot k_2}{2z_{2}^{2}} \left( \frac{\delta^{mn} \delta^{pq}}{z_{2}^{2} z_{3}^{2}} - \frac{\delta^{mp} \delta^{nq}}{z_{2}^{2} z_{4}^{2}} + \frac{\delta^{mq} \delta^{np}}{z_{2}^{2} z_{4}^{2}} \right) + \frac{\delta^{mn} \delta^{pq}}{z_{2}^{2} z_{3}^{2}} \langle J_{\Gamma}(z_1) J_{\Lambda}(z_2) J_{\Omega}(z_3) J_{\Delta}(z_4) \rangle \right]
\]

Using that

\[
\langle J_{\Gamma}(z_1) J_{\Lambda}(z_2) J_{\Omega}(z_3) J_{\Delta}(z_4) \rangle = \frac{k_{\Gamma}^{\Gamma} k_{\Lambda}^{\Lambda} k_{\Omega}^{\Omega} k_{\Delta}^{\Delta}}{z_{2}^{2} z_{13}^{2} z_{24}^{2} z_{34}^{2}} + \frac{k_{\Gamma}^{\Gamma} k_{\Omega}^{\Omega} k_{\Delta}^{\Delta}}{z_{3}^{2} z_{2}^{2} z_{24}^{2} z_{34}^{2}} + \frac{k_{\Gamma}^{\Gamma} k_{\Delta}^{\Delta}}{z_{13}^{2} z_{2}^{2} z_{23}^{2} z_{34}^{2}} + \frac{k_{\Gamma}^{\Gamma} k_{\Omega}^{\Omega}}{z_{13}^{2} z_{23}^{2} z_{24}^{2} z_{34}^{2}} \]

\[
- \frac{f^{\Gamma \Omega \Pi} f^{\Omega \Delta \Pi}}{z_{12} z_{23} z_{24} z_{34}} + \frac{f^{\Gamma \Omega \Pi} f^{\Lambda \Delta \Pi}}{z_{13} z_{23} z_{24} z_{34}} - \frac{f^{\Gamma \Delta \Pi} f^{\Lambda \Omega \Pi}}{z_{14} z_{23} z_{24} z_{34}},
\]

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we get

\[ A_{SSSS}(k_1, S_1, k_2, S_2, k_3, S_3, k_4, S_4) = -8\pi g_c^2 \frac{k_1 \cdot k_2}{2} \hat{S}_{\Gamma_1 m} \hat{S}_{\Gamma_2 n} \hat{S}_{\Gamma_3 p} \hat{S}_{\Gamma_4 q} \]

\[ \times \int d^2 z_1 |z_{34}|^{-k_1 \cdot k_2} |z_{24}|^{-k_2 \cdot k_4} |z_{14}|^{-k_1 \cdot k_4} |z_{23}|^{-k_2 \cdot k_3} |z_{13}|^{-k_1 \cdot k_3} |z_{12}|^{-k_1 \cdot k_2} \]

\[ \times \left( \frac{\delta^{m n} \delta^{p q} \bar{z}_{23} \bar{z}_{24}}{\bar{z}_{12}} \left( 1 - \frac{2}{k_1 \cdot k_2} \right) - \frac{\delta^{m p} \delta^{n q} \bar{z}_{23}}{\bar{z}_{13}} + \frac{\delta^{m q} \delta^{n p} \bar{z}_{24}}{\bar{z}_{14}} \right) \]

\[ \times \left( \hat{f}_{\Gamma_1} \hat{f}_{\Gamma_2} \hat{f}_{\Gamma_3} \hat{f}_{\Gamma_4} \bar{z}_{23} \bar{z}_{24} \frac{f_{\Gamma_1} f_{\Gamma_2} f_{\Gamma_3} f_{\Gamma_4}}{z_{12}} - (2 \leftrightarrow 3) - (2 \leftrightarrow 4) \right) \]

Taking \( z_1 = z, z_2 = 0, z_3 = 1, z_4 \to \infty \), the integral is

\[ \lim_{z \to \infty} \int d^2 z \left| \frac{1 - z}{z} \right|^{-k_1 \cdot k_2} \left| \frac{1 - z}{z} \right|^{-k_1 \cdot k_3} \left| \frac{1 - z}{z} \right|^{-k_1 \cdot k_4} \]

\[ \times \left( \frac{\delta^{m n} \delta^{p q} - \frac{1}{z} \left( 1 - \frac{2}{k_1 \cdot k_2} \right) - \frac{\delta^{m p} \delta^{n q}}{1 - z} - \frac{\delta^{m q} \delta^{n p}}{1 - \frac{1}{z}} \right)}{z^2} \right) \]

\[ = -2\pi (-1)^{-k_1 (k_2 + k_3)} \frac{\delta^{m n} \delta^{p q} \left( 1 - \frac{2}{k_1 \cdot k_2} \right)}{\Gamma(2 - k_1 k_2) \Gamma(-k_1 k_3) \Gamma(k_1 (k_2 + k_3))} \]

\[ + \frac{\delta^{m p} \delta^{n q}}{\Gamma(1 - k_1 k_2) \Gamma(1 - k_1 k_3) \Gamma(1 + k_1 (k_2 + k_3))} \]

\[ \times (-1)^{k_1 (k_2 + k_3)} \left[ -\kappa_{\Gamma_1} \kappa_{\Gamma_2} \kappa_{\Gamma_3} \kappa_{\Gamma_4} \Gamma \left( 1 + \frac{k_1 k_2}{2} \right) \Gamma \left( 1 + \frac{k_1 k_3}{2} \right) \Gamma \left( 1 - \frac{k_1 (k_2 + k_3)}{2} \right) \right] \]

\[ + f_{\Gamma_1} f_{\Gamma_2} f_{\Gamma_3} f_{\Gamma_4} \left( 1 + \frac{k_1 k_2}{2} \right) \Gamma \left( 1 + \frac{k_1 k_3}{2} \right) \Gamma \left( 1 - \frac{k_1 (k_2 + k_3)}{2} \right) \]

where we used

\[ I(m, n, \alpha, \beta) = \int d^2 z (1 - z)^m z^n |z|^{2\alpha} |1 - z|^{2\beta} = 2\pi (-1)^{m+n} \times \]

\[ \times \frac{\Gamma(1 + n + \alpha) \Gamma(1 + m + \beta) \Gamma(-1 - n - m - \alpha - \beta)}{\Gamma(-\alpha) \Gamma(-\beta) \Gamma(2 + \alpha + \beta)} . \]
In terms of Mandelstam variables $s = -2k_1 \cdot k_2$, $t = -2k_1 \cdot k_3$, $u = -2k_1 \cdot k_4$ and summing over all cyclic orderings of the vertex operators to compensate for the fixing of $z_2, z_3$ and $z_4$, we get

$$A^{ssss} = \frac{\pi^2}{12} g_c^2 S_{\Gamma m_1} S_{\Gamma m_2} S_{\Gamma m_3} S_{\Gamma m_4} \frac{\Gamma(-s/4)\Gamma(-t/4)\Gamma(-u/4)}{\Gamma(1 + s/4)\Gamma(1 + t/4)\Gamma(1 + u/4)}$$

\[ \times \left( \delta^{m_1 m_2} \delta^{m_3 m_4} tu + \delta^{m_1 m_3} \delta^{m_2 m_4} su + \delta^{m_1 m_4} \delta^{m_2 m_3} st \right) \]

\[ \times \left( -3 \kappa^{\Gamma_1 \Gamma_2 \kappa} \Gamma_4 t u (s + 4) - 3 \kappa^{\Gamma_1 \Gamma_3 \kappa} \Gamma_4 s u (t + 4) - 3 \kappa^{\Gamma_4 \Gamma_3 \kappa} \Gamma_4 t u (u + 4) \right) \]

\[ + tf^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_3 \Gamma_4 \Pi} + sf^{\Gamma_1 \Gamma_3 \Pi} f^{\Gamma_2 \Gamma_4 \Pi} + uf^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_2 \Gamma_3 \Pi} \]

\[ + tf^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_3 \Gamma_2 \Pi} + sf^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_2 \Gamma_3 \Pi} + uf^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_4 \Gamma_3 \Pi} \]

Expanding on $s = t = u = 0$ and using

$$\frac{\Gamma(-s/4)\Gamma(-t/4)\Gamma(-u/4)}{\Gamma(1 + s/4)\Gamma(1 + t/4)\Gamma(1 + u/4)} = -\frac{64}{stu} - 2\zeta(3) + O(stu), \quad (D.3)$$

we finally get

$$A^{ssss} = -\frac{16\pi^2}{3} g_c^2 S_{\Gamma m_1} S_{\Gamma m_2} S_{\Gamma m_3} S_{\Gamma m_4}$$

\[ \times \left( \frac{t}{s} \delta^{m_1 m_2} \delta^{m_3 m_4} (f^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_3 \Gamma_4 \Pi} + f^{\Gamma_1 \Gamma_3 \Pi} f^{\Gamma_2 \Gamma_4 \Pi}) + \delta^{m_1 m_3} \delta^{m_2 m_4} f^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_3 \Gamma_4 \Pi} \right) \]

\[ + \frac{t}{u} \delta^{m_1 m_2} \delta^{m_3 m_4} (f^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_3 \Gamma_4 \Pi} + f^{\Gamma_1 \Gamma_3 \Pi} f^{\Gamma_2 \Gamma_4 \Pi}) + \delta^{m_1 m_3} \delta^{m_2 m_4} f^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_3 \Gamma_4 \Pi} \]

\[ + \frac{s}{t} \delta^{m_1 m_2} \delta^{m_3 m_4} (f^{\Gamma_1 \Gamma_3 \Pi} f^{\Gamma_2 \Gamma_4 \Pi} + f^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_2 \Gamma_3 \Pi}) + \delta^{m_1 m_3} \delta^{m_2 m_4} f^{\Gamma_1 \Gamma_3 \Pi} f^{\Gamma_2 \Gamma_4 \Pi} \]

\[ + \frac{u}{t} \delta^{m_1 m_2} \delta^{m_3 m_4} (f^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_2 \Gamma_3 \Pi} + f^{\Gamma_1 \Gamma_3 \Pi} f^{\Gamma_2 \Gamma_4 \Pi}) + \delta^{m_1 m_3} \delta^{m_2 m_4} f^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_2 \Gamma_3 \Pi} \]

\[ + \frac{s}{u} \delta^{m_1 m_2} \delta^{m_3 m_4} (f^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_3 \Gamma_2 \Pi} + f^{\Gamma_1 \Gamma_2 \Pi} f^{\Gamma_4 \Gamma_3 \Pi}) + \delta^{m_1 m_3} \delta^{m_2 m_4} f^{\Gamma_1 \Gamma_4 \Pi} f^{\Gamma_3 \Gamma_2 \Pi} \]

which adds up to

$$A^{ssss} = (4!)2\pi^2 g_c^2 S_{\Gamma m} S_{\Gamma n} S_{\Lambda m} S_{\Lambda n} f^{\Gamma \Pi} f^{\Gamma' \Pi}$$

\[ (D.4) \]

when using $\frac{s}{t} + \frac{t}{u} + \frac{t}{u} = -3$ and $S_{\Gamma m} S_{\Lambda m} S_{\Gamma n} S_{\Lambda n} f^{\Gamma \Pi} f^{\Gamma' \Pi} = 0$.

### D.3 Three-point functions involving slightly massive states

It is easy to see that the amplitudes of three massless right vectors or three massless scalars vanish at the enhancement points. However, in the neighborhood of these points, the currents acquire dependence on $p_R$ and then the amplitude of three scalars or that
of two left and one right vectors get a non-vanishing value and give extra terms in the effective action. Here we compute the three point functions involving states that become massive when slightly moving away from the enhancement points, so that their masses are smaller than other massive string states which we are not considering.

**One right vector - two massive left vectors:**

\[
\frac{A^{AA'A'}}{C_{S2}g_3^2} = \frac{\delta^{p_2+p_3}}{\sqrt{2}} A_{\mu m} k_2^\mu A_{\nu}^{p_2} A_{\nu\mu}^{p_3} p_{2R}^m
\]

where we used \(k_1 + k_2 + k_3 = 0, k_1^2 = 0, k_2^2 = k_3^2 = -m^2 = -2p_{2R}^2\), \(k_1 \cdot k_2 = k_1 \cdot k_3 = 0\) and \(k_2 \cdot k_3 = 2p_{2R}^2\). This gives the term

\[
\frac{-i}{\sqrt{2}} p_{2R}^m A_{\mu m} A^{\nu\nu} \partial^\nu A_{\nu}^{p}
\]

in the effective action.

**One massless - two massive left vectors:**

\[
\frac{A^{AA'A'}}{C_{S2}g_3^2} = \frac{\delta^{p_2+p_3}}{\sqrt{2}} p_{2Li} \left[ (A^I \cdot k_2)(A^{p_2} \cdot A^{p_3}) + (k_1 \cdot A^{p_3})(A^I \cdot A^{p_2}) - (k_1 \cdot A^{p_2})(A^I \cdot A^{p_3}) \right]
\]

giving in the effective action

\[
-\frac{i}{\sqrt{2}} p_{IL} \left[ A^{\nu\nu} A_{\mu}^I \partial^\mu A_{\nu}^p + 2A_{\mu}^{p^\nu} A^{\nu\nu} \partial^\mu A_{\nu}^I \right]
\]

**One massless tensor - two massive left vectors:**

\[
\frac{A^{AA'A'}}{C_{S2}g_3^2} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \delta^{p_2+p_3} A_{\mu_2}^{p_2} A_{\nu_3}^{p_3} \left( -k_2^\nu k_3^\rho + k_1^\nu k_3^\rho + k_2^\nu k_1^\rho \right)
\]

giving in the effective action

\[
\frac{1}{2} \left( \epsilon_{\mu\nu\rho\sigma} A_{\rho}^{p^\nu} A_{\sigma}^{\nu\nu} - 2\partial^\rho \epsilon_{\mu\nu\rho\sigma} A_{\rho}^{p^\nu} A_{\sigma}^{\nu\nu} \right)
\]

**One massless scalar - two massive left vectors:**

\[
\frac{A^{AA'A'}}{C_{S2}g_3^2} = \delta^{p_2+p_3} S_{lm} A_{\mu_2}^{p_2} A_{\nu_3}^{p_3} p_{2R}^m p_{2L}^l \nu_{2R}^{\nu_2} \nu_{2L}^{\nu_3} = p_{IL} S_{lm} p_{R}^m A_{\mu}^p A^{-\nu\nu}
\]

**Three massive left vectors:**

\[
\frac{A^{AA'A'}}{C_{S2}g_3^2} = -\frac{i\epsilon(p_1, p_2)}{\sqrt{2}} A_{\mu_1}^{p_1} A_{\mu_2}^{p_2} A^{-\mu_1-\mu_2} \left( k_2^{\nu_1} k_3^{\nu_2} + k_1^{\nu_1} k_3^{\nu_2} + k_3^{\nu_1} k_1^{\nu_2} \right)
\]

where we used \(k_i \cdot A^{p_i} = 0\) and conservation of momentum implies \(k_i \cdot k_j = -2p_{Ri} \cdot p_{Rj}\) and \(p_{Ri}^2 - p_{Rj}^2 = 2 \rightarrow p_{IL} \cdot p_{jL} - p_{iR} \cdot p_{jR} = -1\) if \(i \neq j\).

This gives in the effective action the term

\[
\frac{3}{\sqrt{2}} \epsilon(p_1, p_2) A_{\nu}^{p_1} A_{\mu}^{p_2} A^{-\nu\mu} A^{-\mu_1-\mu_2}
\]
E Counting structure constants of $SO(32)$ and $E_8 \times E_8$

In this Appendix we count and compare the number of non-vanishing structure constants of the $SO(32)$ and $E_8 \times E_8$ algebras, which in the Weyl-Cartan basis are

$$f^{\alpha \beta \gamma} = \delta^\alpha_{\gamma + \beta} f^{\alpha \beta}_{\gamma + \beta} + \delta^\beta_{\gamma + \alpha} f^{\alpha \beta}_{\gamma + \alpha}$$  \hspace{1cm} (E.1)

with $\bar{\alpha} = -\alpha$.

To calculate the number of combinations of $\alpha, \beta$ indices giving non-vanishing structure constants of $SO(32)$, it is convenient to denote the 480 roots as

$$(i\pm, j\pm) = (0_{i-1}, \pm 1, 0_{j-i-1}, \pm 1, 0_{16-j})$$  \hspace{1cm} (E.2)

with $1 \leq i < j \leq 16$, and split them in subsets $(+, +)$, $(-, -)$, $(+, -)$, $(-, +)$ of 120 elements each. Then we have

$$|(+, +) + (+, +)|^2 \geq 4 \quad \text{or} \quad |(-, -) + (-, -)|^2 \geq 4 \rightarrow \text{there are no roots}$$  \hspace{1cm} (E.3)

The number of pairs of roots $(i+, j+), (k-, l-)$ is:

$$\sum_{i=1}^{15} (16 - i)(16 - i - 1) = 1120 \quad \text{if } i = k, i < j, i < l \neq j$$

$$\sum_{j=1}^{15} (16 - j)(16 - j - 1) = 1120 \quad \text{if } j = l, i \neq k$$

$$\sum_{j=2}^{15} (j - 1)(16 - j) = 560 \quad \text{if } j = k, i < j \text{ and } j < l$$

Then there are

$$(+, +) + (-, -) = \begin{cases}
0 & \rightarrow 120 \\
(+, -) & \rightarrow 3 \times 560 = 1680 \\
(-, +) & \rightarrow 3 \times 560 = 1680 
\end{cases}$$  \hspace{1cm} (E.5)

That is, 1680 non-vanishing structure constants of type $f^{(+, +)(-, -)}_{(+, -)}$ and 1680 of type $f^{(+, +)(-, -)}_{(-, +)}$. And analogously, there are 1680 non-vanishing structure constants of
type $f^{(-,-)(+,-)}_{(+,-)}$ and 1680 of type $f^{(-,+)(+,-)}_{(-,+)}$.

$$\begin{align*}
(i+, j+) + (k+, l-) = \begin{cases}
(i+, k+) & \text{if } j = l, i < k \\
(k+, i+) & \text{if } j = l, i > k \\
(k+, j+) & \text{if } i = l \\
n\text{no roots if } i \neq l \neq j \text{ or } i = k \text{ or } j = k
\end{cases}
\end{align*} \quad (E.6)$$

The number of pairs $(i+, j+), (k+, l-)$ with $j = l, i \neq k$ is 1120, of which 560 correspond to $i > k$ and 560 to $i < k$. And for $i = l$ there are 560 pairs. That is

$$(+, +) + (+, -) = (+, +) \rightarrow 3 \times 560 = 1680 \text{ non-vanishing } f^{(+,+)(+,-)}_{(+,+)} \quad (E.7)$$

And analogously, there are 1680 non-vanishing structure constants of each of the types $f^{(+,-)(+,-)}_{(+,+)}$, $f^{(+,+)(+,-)}_{(+,+)}$, $f^{(-,-)(+,-)}_{(+,+)}$, $f^{(-,+)(+,-)}_{(+,+)}$, $f^{(-,-)(+,-)}_{(-,-)}$, $f^{(-,+)(+,-)}_{(-,-)}$, and $f^{(-,+)(-,-)}_{(-,-)}$.

$$\begin{align*}
(i+, j-) + (k+, l-) = \begin{cases}
(i+, l-) & \text{if } j = k \\
(k+, j-) & \text{if } i = l \\
n\text{no roots if } i = k \text{ or } j = l \text{ or } i \neq l, j \neq k
\end{cases}
\end{align*} \quad (E.8)$$

The number of pairs $(i+, j-), (k+, l-)$ with $j = k$ is 560 and with $i = l$ is also 560. Then we have

$$(+, -) + (+, -) = (+, -) \rightarrow 2 \times 560 = 1120 \text{ non-vanishing } f^{(+,-)(+,-)}_{(+,-)} \quad (E.9)$$

And analogously, 1120 of the type $f^{(-,+)(-,+)}_{(-,+)}$.

$$\begin{align*}
(i+, j-) + (k-, l+) = \begin{cases}
0 & \text{if } i = k, j = l \\
(j-, l+) & \text{if } i = k, j < l \\
(l+, j-) & \text{if } i = k, j > l \\
(i+, k-) & \text{if } j = l, i < k \\
(k-, i+) & \text{if } j = l, i > k \\
n\text{there are no roots if } i = l \text{ or } j = k \text{ or } i \neq k, j \neq l
\end{cases}
\end{align*} \quad (E.10)$$

The number of pairs of roots $(i+, j-), (k-, l+)$ verifying $i = k$, $j = l$ is 120; with $i = k$, $j \neq l$ there are 1120 of which 560 correspond to $j > l$ and 560 to $j < l$; and for $j = l$, $i \neq k$ there are 1120 of which 560 correspond to $i > k$ and 560 to $i < k$. Then there are

$$(+, -) + (-, +) = \begin{cases}
0 \rightarrow 120 \\
(+, -) \rightarrow 2 \times 560 = 1120 \\
(-, +) \rightarrow 2 \times 560 = 1120
\end{cases} \quad (E.11)$$

That is, 1120 structure constants of type $f^{(+,-)(-,-)}_{(+,+)}$ and 1120 of type $f^{(+,-)(+,-)}_{(-,+)}$. And analogously 1120 $f^{(-,+)(+,-)}_{(+,-)}$ and 1120 $f^{(-,+)(-,-)}_{(-,+)}$. 

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Summarizing there are \(12 \times 560 = 6720\) combinations giving \((+,+),\) 6720 giving \((-,-),\) 6720 giving \((+-,\) and 6720 giving \((-+).\) That is 26880 non-vanishing structure constants \(f^{\alpha\beta}_{\alpha\beta},\) and \(4 \times 120 \times 16 = 7680\) non-vanishing structure constants \(f^{\alpha\bar{\alpha}}_{A}.\)

In the case \(E_8 \times E_8,\) we denote the roots
\[
(1; i\pm, j\pm) = (0_{i-1}, \pm 1, 0_{j-i-1}, \pm 1, 0_{16-j})
\]
\[
(2; i\pm, j\pm) = (0_{8+i-1}, \pm 1, 0_{j-i-1}, \pm 1, 0_{8-j})
\]
with \(1 \leq i < j \leq 8;\) and:
\[
(1; s) = \left( \left( \frac{\pm 1}{2} \right)_{8\text{(even)}}, 0_8 \right)
\]
\[
(2; s) = \left( 0_8, \left( \frac{\pm 1}{2} \right)_{8\text{(even)}} \right)
\]
where \(s\) can take \(2^7\) values. It can be thought of as a binary number of 7 digits (one depending on the others because there must be an even number of \(-\) signs).

Split the 480 roots of \(E_8 \times E_8\) into 8 subsets of 28 elements: \((1; +, +),\) \((1; - , -),\) \((1; +, -),\) \((1; -, +),\) \((2; +, +),\) \((2; -, -),\) \((2; +, -),\) \((2; -, +)\) and 2 subsets \((1; s),\) \((2; s)\) of 128 elements.

If \(|(1; \cdots) + (2; \cdots)|^2 = 4, |(1; +, +) + (1; +, +)|^2 \geq 4,\) or \(|(1; -, -) + (1; -, -)|^2 \geq 4,\) there are no roots.

The number of pairs of roots \((1; i+, j+), (1; k-, l-)\) is
\[
\sum_{i=1}^{7} (8 - i)(8 - i - 1) = 112 \quad \text{with } i = k, j = l
\]
\[
56 \quad \text{with } j > l
\]
\[
56 \quad \text{with } j < l
\]
\[
112 \quad \text{with } i = k, j \neq l
\]
\[
28 \quad \text{with } i = k, j = l
\]
\[
7 \sum_{j=2}^{7} (j - 1)(8 - j) = 56 \quad \text{with } j = k,
\]
\[
56 \quad \text{with } i = l
\]
The second line counts the number of pairs \(j, l\) such that \(i < j\) and \(i < l \neq j,\) and the
fourth one, the number of pairs \(i, l\) such that \(i < j\) and \(j < l\). Then we have

\[
(1;+,+) + (1;-,+) = \begin{cases} 
0 \rightarrow 28 \\
(+,+) \rightarrow 3 \times 56 = 168 \\
(-,+) \rightarrow 3 \times 56 = 168 
\end{cases}
\tag{E.16}
\]

i.e. 168 structure constants of type \(f^{(1;+,+)(1;-,+)}_{(1;+,+)_{(1;-,+)}}\) and 168 of type \(f^{(1;+,+)(1;,-,-)}_{(1;+,+)_{(1;,-,-)}}\).

And analogously, there are 168 of type \(f^{(1;-,+)(1;+,+)}_{(1;-,+)_{(1;+,+)}}\) and 168 of type \(f^{(1;-,+)(1;,+,-)}_{(1;-,+)_{(1;,+,-)}}\).

For the other roots of the kind \((1;\pm,\pm)\), the analysis is as in the \(SO(32)\) case, but now the number of non-vanishing structure constants is one tenth as before: \(12 \times 56 = 672\) combinations giving \((1;+,+), 672\) giving \((1;-,+)\), \(672\) giving \((1;+,-)\) and \(672\) giving \((1;-,+)\).

We also have

\[
(1; s) + (1; s) = \begin{cases} 
0 \rightarrow 2^7 = 128 \\
(1;+,+) \rightarrow 28 \times 2^5 = 896 \\
(1;-,+) \rightarrow 896 \\
(1;+,-) \rightarrow 896 \\
(1;-,-) \rightarrow 896 
\end{cases}
\tag{E.17}
\]

For the sum of two roots of the kind \((1; s)\) to give \((1, 1, 0_{14})\) it is necessary that they are of the form \((+1/2, +1/2, r, s, t, u, v, w)\) and \((+1/2, +1/2, r, -s, -t, -u, -v, w)\). Then there are \(2^5 = 32\) possible choices of parameters \(r, s, t, u, v, w\) (\(w\) is not independent). Since there are 28 roots of the kind \((1;+,+)\), the number of non-vanishing structure constants \(f^{(1;s)(1;s)}_{(1;+,+)(1;+,+)}\) is \(32 \times 28 = 896\). And analogously there are \(896\) \(f^{(1;s)(1;s)}_{(1;-,+)(1;-,+)}\), \(896\) \(f^{(1;s)(1;s)}_{(1;-,+)(1;+,+)}\), \(896\) \(f^{(1;s)(1;s)}_{(1;-,+)(1;-,+)}\) and \(896\) \(f^{(1;s)(1;s)}_{(1;-,+)(1;-,+)}\), and

\[
(1; s) + (1; ++) = (1; s) \rightarrow 28 \times 2^5 = 896 \tag{E.18}
\]

To have \((1; s) + (1, 1, 0_{14}) = (1; s)\), it is necessary that \((1; s) = (-1/2, -1/2, r, s, t, u, v, w)\). Then there are \(2^5 = 32\) possible choices of parameters \(r, s, t, u, v, w\). Since there are 28 roots of the kind \((1;+,+)\), the number of non-vanishing structure constants of the type \(f^{(1;s)(1;+,+)}_{(1;+,+)(1;+,+)}\) is \(32 \times 28 = 896\). And analogously there are \(896\) structure constants of type \(f^{(1;s)(1;-,+)}_{(1;-,+)(1;-,+)}\), \(896\) \(f^{(1;s)(1;-,+)}_{(1;-,+,+)}\), \(896\) \(f^{(1;s)(1;-,+)}_{(1;-,+,+)}\), \(896\) \(f^{(1;s)(1;-,+)}_{(1;-,+,+)}\) and \(896\) \(f^{(1;+,+)(1;+,+)}_{(1;+,+,+)}\).

The same holds for the sum of two roots of type \((2, \cdots)\), and then there are a total of \(2 \times (12 \times 56 + 896) = 3136\) combinations giving \((+,+), 3136\) giving \((-,-), 3136\) giving \((+,-), 3136\) giving \((-,+), 2 \times 8 \times 896 = 14336\) giving \((1, s)\). That is 26880 non-vanishing structure constants of type \(f_{\alpha \beta + \alpha}^{\alpha \beta}\).

In addition, there are \(2 \times (4 \times 28 + 128) \times 16 = 7680\) structure constants of type \(f^{\alpha \beta}_{\alpha + \beta} A\).

In conclusion, the number of structure constants of type \(f^{\alpha \beta}_{\alpha + \beta}\) is 26880 and of type \(f_{\alpha + \beta}^{\alpha \beta} A\) is 7680, for both the \(SO(32)\) and the \(E_8 \times E_8\) groups.

\[\text{Note that there are an even number of } - \text{ signs since the sign of two components is always modified.}\]

This agrees with the fact that the spinorial conjugation class only changes to the conjugate one when adding a vector of the vectorial class.
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