A Non-Hermitian two coupled Sachdev-Ye-Kitaev model

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We propose a non-Hermitian two coupled Sachdev-Ye-Kitaev model, but its thermodynamic structure is similar to Hermitian two coupled SYK model. The non-Hermiticity greatly modifies the properties of the ground states and the off-diagonal Green’s functions. Supersingly, the energy spectrum, the degree of entanglement of the ground states of the two SYK are not influenced by the non-Hermiticity. Meanwhile, the free energy as a function of temperature shows first order phase transition between gapped and gapless phases independent of the non-Hermiticity.

I. INTRODUCTION

Recently, non-Hermitian physics has gained widespread attention, such as non-Hermitian linear response theory [1], non-Hermitian topological systems [2, 3] and non-Hermitian holography [4, 5]. One of the primary reasons for those studies is due to the probability in nature effectively becomes non-conserving due to the presence of energy, particles, and information regarding the external degrees of freedom that are out of the Hilbert space. In a non-Hermitian system experiencing an exceptional point in the wave momentum, the corresponding eigenfrequencies change from real to complex numbers [6–10]. This strongly contradicts with Hermiticity, one of the key principles of quantum mechanics, ensuring the conservation of probability in an isolated quantum system and validating the expectation value of energy concerning a quantum state. However, seminal work by C. Bender and S. Boettcher demonstrates that in the physics of non-Hermitian systems, a huge class of non-conservative Hamiltonians can exhibit entirely real spectra as long as they commute with the parity-time (PT) operator [11]. Furthermore, it was found that a Hamiltonian with a real spectrum is pseudo-Hermitian. Moreover, all the PT symmetric Hamiltonians studied reported in the literature exhibited such property [12–14]. The similarity transformations can also enable one to construct non-Hermitian Hamiltonian with real spectrum [14–16].

In a quantum many-body system, the quantum two-level system can be simulated by coupling two copies of the Sachdev-Ye-Kitaev (SYK) model. The SYK mode, which is well-known as a disordered and strongly-coupled quantum system composed of Majorana fermions [17–19], has recently emerged as an exemplary model providing insight into the nature of non-Fermi liquids [20]. quantum chaos [21], holography [22, 23], strange metallic transport [24–25] and high temperature superconducting [26, 27]. SYK is closely related to two-dimensional dilaton gravity describing excitations above the near horizon external black hole [28, 29]. Therefore, an eternal traversable wormhole can be constructed by considering two copies of SYK models coupled by a simple interaction. This proposed model demonstrates that at low temperature, the coupling can drive phase transitions to a phase holographically dual to an eternal traversable wormhole with an AdS2 throat [30]. Conversely, at a higher temperature, the system reduces to two gapless black hole phases [30]. However, in non-Hermitian set up, one may expect the gapped-gapless physical picture drastically changed. We will prove that as long as the thermodynamical structure is concerned, non Hermiticity can nevertheless strengthen the wormhole-black hole physical picture.

Thus, considering the non-Hermiticity and two coupled SYK models, we developed novel non-Hermitian two-site SYK models. We first prove that the system yields real energy spectrum. Furthermore, we show that the degree of entanglement, the low energy effective action and the phase structure are non-Hermiticity independent. As illustrated in Fig.1 even though these two SYK sites are approaching a “ground state/excited state” picture in the regime of strong non-Hermitian limit, the thermodynamical phase structure indicated three distinct properties at different temperatures.

![FIG. 1: Sketched phase diagram for the physical picture. Left: We have a geometry connecting the two sides in the low temperature regime. The left Sachdev-Ye-Kitaev (SYK) and right SYK are in different states because of the non-Hermitian parameter. Middle: An unstable geometry connecting two SYK sites. Right: Two separated SYK sites at high temperature, representing the gapless two black hole phases. The non-Hermitian parameter can change the states of the left and right SYK sites, which are marked in orange and green colors.](image-url)
II. NON-HERMITIAN TWO COUPLED SYK MODEL

We consider non-Hermitian two coupled SYK model with the Hamiltonians

\[ H = -\sum_{ijkl} J_{ijkl} \sum_{A=R,L} (c_A C_i C_j A C_k C_l A) + c_2 C_i C_j C_k (c_A C_l A) + H_{int}, \]

\[ H_{int} = i\mu \sum_i (e^{-2\alpha} C_i L C_i R - e^{2\alpha} C_i R C_i L), \]

where the coupling \( J_{ijkl} \) are random real numbers and \( c_1 \) and \( c_2 \) are two constants. \( A = L, R \) refers to the “left” and “right” side of the two identical copies. We choose \( c_1 = 2 \) and \( c_2 = 4 \) in what follows. The random real numbers obey the Gaussian distribution and satisfy \( J_{ijkl} = -J_{jilk} = -J_{ilkj} \) with \( < J_{ijkl} >= 0, \quad < J_{ijkl}^2 >= \frac{J^2}{8N}. \) The parameter \( \alpha \) is a real number controlling the strength of non-Hermiticity, which is introduced by a non-Hermitian particle-hole similarity transformation. See appendix A for details. By analytically continued to an imaginary value \( \alpha \rightarrow i\alpha \), one can recover the Hermitian Hamiltonian. This Hamiltonian can be inspired by performing a non-Hermitian particle-hole similarity transformation on the original MQ model and dropping all the nonphysical terms. In the absence of the interacting term, the Hamiltonian describes two complex SYK (cSYK) models at zero chemical potential with the two identical copies of Dirac fermions \( L, \ R \) referring to the “left” and “right” side of the system, respectively. Also without \( H_{int} \), the gravity dual of this Hamiltonian describes a two sided AdS\(_2\) space.

III. ENERGY SPECTRUM AND DEGREE OF ENTANGLEMENT

Energy spectrum is an important feature of the non-Hermitian quantum system. We compute the energy spectrum by using exact diagonalization techniques. The energy spectrum is real and independent of the non-Hermitian parameter \( \alpha \), as demonstrated in Fig. 2.

For non-Hermitian systems, we need construct the ground state by introducing a biorthogonal set \{\( |\psi^l_n\rangle, |\psi^r_n\rangle \}\}. The right/left eigenstates are defined as

\[ H|\psi^l_n\rangle = E_n|\psi^l_n\rangle, \quad H^\dagger|\psi^r_m\rangle = E_m^*|\psi^r_m\rangle. \]

The eigenstates satisfy the properties as follow

\[ \sum_n |\psi^l_n\rangle \langle \psi^r_n| = I, \quad \langle \psi^l_n| \psi^r_m\rangle = \delta_{nm}. \]

We impose the constraints the system yields the ground state energy as those of \[37\] by taking \( H^\dagger_{int}|\psi^l_0\rangle = -\mu N|\psi^l_0\rangle, H_{int}|\psi^r_0\rangle = -\mu N|\psi^r_0\rangle \) and \( \langle \psi^l_0|\psi^r_0\rangle = 1 \). Without loss of generality, the generated ground states are proposed as

\[ |\psi^l_0\rangle = \prod_j \frac{1}{\sqrt{2}} (\tilde{A}|1\rangle_{L,j}|0\rangle_{R,j} + i\tilde{B}|0\rangle_{L,j}|1\rangle_{R,j}), \quad \langle \psi^l_0| = \prod_j \frac{1}{\sqrt{2}} (\tilde{C}|1\rangle_{L,j}|0\rangle_{R,j} + i\tilde{D}|0\rangle_{L,j}|1\rangle_{R,j}), \]

where the coefficients \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) satisfy the relation \( \tilde{B} = \tilde{A}e^{2\alpha} \) and \( \tilde{D} = \tilde{C}e^{-2\alpha} \). The constraint \( \langle \psi^l_0|\psi^r_0\rangle = 1 \) further leads to \( \tilde{A}\tilde{C} = B\tilde{D} = \frac{1}{2} \). The ground states can return to the ground state of Hermitian case \( \alpha = 0 \) consistently if setting

\[ \tilde{A} = \frac{1}{\sqrt{2}}, \quad \tilde{B} = \frac{e^{2\alpha}}{\sqrt{2}}, \quad \tilde{C} = \frac{1}{\sqrt{2}}, \quad \tilde{D} = \frac{e^{-2\alpha}}{\sqrt{2}}. \]

If \( \alpha \neq 0 \), \( \langle \psi^l_0|\psi^r_0\rangle \neq 1 \) and \( \langle \psi^r_0|\psi^l_0\rangle \neq 1 \). In our non-Hermitian model, the degree of entanglement turns out to be

\[ P_E = -tr(\rho_A \log \rho_A) = 1, \]

which can be derived from the von-Neumann entropy of the reduced density matrix

\[ \rho_{LR} = |\psi^r_0\rangle \langle \psi^l_0| = \tilde{A}\tilde{C}|10\rangle\langle 01| + i\tilde{B}\tilde{C}|01\rangle\langle 01| - i\tilde{A}\tilde{D}|10\rangle\langle 10| + B\tilde{D}|01\rangle\langle 01|. \]

\[ \rho_L = tr_R(\rho_{LR}) = \begin{pmatrix} \tilde{A}\tilde{C} & 0 \\ 0 & \tilde{B}\tilde{D} \end{pmatrix}. \]

Because of \( P_E = 1 \), which is independent of \( \alpha \), the ground states are maximally entangled between two systems. Note that for large \( \alpha \), the ground states \[4\] and \[5\] will approach the pure state, but the degree of entanglement...
remains unchanged. That is to say
\[ |\psi_0^R\rangle \rightarrow \prod_{j=1}^{N} \frac{1}{\sqrt{2}} |1\rangle_{L,J} |0\rangle_{R,J}, \quad |\psi_0^L\rangle \rightarrow \prod_{j=1}^{N} \frac{i e^{-2\alpha}}{\sqrt{2}} |0\rangle_{L,J} |1\rangle_{R,J} \]
as \( \alpha \to -\infty \) and
\[ |\psi_0^L\rangle \rightarrow \prod_{j=1}^{N} \frac{i}{\sqrt{2}} |0\rangle_{L,J} |1\rangle_{R,J}, \quad |\psi_0^R\rangle \rightarrow \prod_{j=1}^{N} \frac{i e^{2\alpha}}{\sqrt{2}} |0\rangle_{L,J} |1\rangle_{R,J} \]
as \( \alpha \to +\infty \).

Therefore, the left and right SYK sites are no longer symmetric as illustrated in Fig[1].

**IV. LOW ENERGY EFFECTIVE ACTION**

In the low energy limit, the model simplifies due to the emergence of a conformal symmetry. We demonstrate that the low energies physics of our non-Hermitian Hamiltonian is also independent of the non-Hermitian parameter \( \alpha \). The retarded Green function of non-Hermitian system is defined as
\[ G_{AB}(\tau_1, \tau_2) = \frac{1}{N} \sum_n \langle \psi_n^A | T C_{\tau_1}^A \{ \phi(\tau_1) \} C_{\tau_2}^B (\tau_2) | \psi_n^B \rangle, \quad (11) \]
where \( A, B = L, R \). The saddle-point equations are invariant under the time reparametrization \( \tau \rightarrow h(\tau) \) and the \( U(1) \) symmetry, which is the same as the complex SYK model in [38,40].

\[ \tilde{G}_{AB}(\tau_1, \tau_2) = \langle h_A(\tau_1) h_B(\tau_2) \rangle G_{AB}(h(\tau_1), h(\tau_2)) e^{i \phi_A(\tau_1) - i \phi_B(\tau_2)}, \]
\[ \tilde{\Sigma}_{AB}(\tau_1, \tau_2) = [h_A(\tau_1) h_B(\tau_2)]^{1-\Delta} \Sigma_{AB}(h(\tau_1), h(\tau_2)) e^{i \phi_B(\tau_2) - i \phi_A(\tau_1)}. \]

In the absence of the interacting term, the Schwarzian effective action of the left or right copy turns out to be
\[ S_A = -N \alpha_S \int d\tau \{ \tanh \frac{h_A(\tau)}{2}, \tau \} + \frac{N K}{2} \int d\tau \left( \phi_A'(\tau) + i \varepsilon_A h_A'(\tau) \right)^2. \quad (12) \]
where \( \varepsilon_A \) is related to the charge \( Q_A \) with \( A = L, R \) and \( \alpha_S \) is determined by four-point calculation of the SYK model. Note that
\[ \{ h, \tau \} = \frac{h''(\tau)}{h'(\tau)} - \frac{3}{2} \left( \frac{h''(\tau)}{h'(\tau)} \right)^2. \]
The effective action of the coupled part is written as
\[ S_{int} = \frac{N \mu}{2} \int d\tau \left[ \frac{bh_L'(\tau) h_R'(\tau)}{\cosh^2 \left( \frac{h_L(\tau) - h_R(\tau)}{2} \right)} \right]^\Delta \cos(h_L(\tau) - h_R(\tau)) \left[ e^{i(\phi_L - \phi_R) - 2\alpha} + e^{-i(\phi_L - \phi_R) + 2\alpha} \right]. \quad (13) \]
The action has the global \( SL(2) \times U(1) \) symmetry generated by
\[ \delta h_L = e^0 + e^+ e^{ih_L} + e^- e^{-ih_L}, \]
\[ \delta h_R = e^0 - e^+ e^{ih_R} - e^- e^{-ih_R}, \]
\[ \delta \phi_A = -i \varepsilon \delta h_A + \epsilon. \quad (14) \]
The total action could be simplified to
\[ S \frac{1}{N} = -2 \alpha_S \int d\tau \left\{ \frac{h(\tau)}{2}, \tau \right\} + K \int d\tau \left( \phi'(\tau)^2 - e^{2\phi} \right)^{2\Delta}, \quad (15) \]
with the solution
\[ h_L = h_R = h, \quad \phi_L = \phi_R - 2i \alpha = \text{const}. \quad (16) \]

We derive the \( SL(2) \) Noether charge in appendix B. The \( SL(2) \) Noether charge vanishes with the simple solution \( h(\tau) = t' \tau \), so it can be treated as gauge symmetry. The solutions of equation (16) lead to \( Q = 0 \), and
\[ Q_0/N = 2 e^{-\phi} - e^{2\phi} + \Delta \mu e^{2\phi} = 0, \quad (17) \]
by introducing \( \phi = \log h' \). We can derive the equations of motion from the action of a non-relativistic particle in a potential
\[ S = N \int du \left( \phi'(u)^2 - (e^{2\phi} - \mu e^{\phi/2}) \right). \quad (18) \]
The effective potential is independent of the non-Hermitian parameter \( \alpha \), but same as that of MQ model [39]. One can therefore conclude that there is an \( \alpha \)-independent energy gap at low energies. We can thus add a boundary interaction to the bulk action
\[ S_{int} = g \sum_{i=1}^{N} \int du \left( e^{-2\alpha} O_L^i(u) O_R^i(u) - e^{2\alpha} O_L^i(u) O_R^i(u) \right), \quad (19) \]
where \( O \) is a set of \( N \) operators with dimension \( \Delta \) and \( g \) is proportional to the coupling \( \mu \). When \( \alpha \) and \( g \) is small, the interacting \( (19) \) corresponds to the interaction term of the low energy effective action in Eq. [13]. The coupling of left and right black holes \( (1 - 2\alpha) O_L^i O_R^i, (1 + 2\alpha) O_L^i O_R^i \) are not symmetric. The two sides of \( AdS_2 \) is directly coupled by the double trace deformation. Since \( e^{2\alpha} \) or \( e^{-2\alpha} \) is always positive, the double trace interaction generates negative null energy in the bulk without violating causality same as that of [11]. Therefore quantum entangled states at left and right boundaries are connected. So a pair of infalling particles traverse the wormhole from one side of the horizon to the other side.
V. THERMODYNAMIC PHASE STRUCTURE BEYOND THE LOW ENERGY LIMIT

At finite temperature, the retarded Green function receives great contribution from the non-Hermitian parameter $\alpha$. But the whole thermodynamic phase structure unchanged by the non-Hermitian parameter $\alpha$.

The effective action can be obtained as,

$$S_{\text{eff}} = -\log \det(\sigma_{AB} - \Sigma_{AB}) - \int d\tau_1 d\tau_2 \left( \Sigma_{BA}(\tau_2, \tau_1) \right),$$

where

$$\sigma_{AB} = \begin{pmatrix} \partial_\tau & i \mu e^{-2\alpha} \\ -i \mu e^{2\alpha} & \partial_\tau \end{pmatrix}. \quad (21)$$

After performing a Fourier transformation

$$f(\tau) = \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} f(\omega_n), \quad f(\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} f(\tau),$$

the saddle-point equations can be written as,

$$\Sigma_{AB}(\tau_1, \tau_2) = -36J^2 G_{AB}^{2}(\tau_1, \tau_2) G_{BA}^{2}(\tau_2, \tau_1),$$

$$G_{LL}(i\omega_n, \alpha) = \frac{-i\omega_n - \Sigma_{LL}(i\omega_n, \alpha)}{D(i\omega_n, \alpha)},$$

$$G_{RR}(i\omega_n, \alpha) = \frac{-i\omega_n - \Sigma_{RR}(i\omega_n, \alpha)}{D(i\omega_n, \alpha)},$$

$$G_{LR}(i\omega_n, \alpha) = \frac{i \mu e^{-2\alpha} + \Sigma_{LR}(i\omega_n, \alpha)}{D(i\omega_n, \alpha)},$$

$$D(i\omega_n, \alpha) = \left( -i\omega_n - \Sigma_{LL} \right) \left( -i\omega_n - \Sigma_{RR} \right) + \left( i \mu e^{-2\alpha} - \Sigma_{LR} \right) \left( i \mu e^{2\alpha} + \Sigma_{RL} \right), \quad (22)$$

with the Matsubara frequency $\omega_n = 2\pi(n + \frac{1}{2})/\beta$. The numerical results in Fig. 3(a) show that $G_{RR}(i\omega_n, \alpha) = G_{LL}(i\omega_n, \alpha)$, $G_{LR}(i\omega_n, \alpha) = -G_{RL}(i\omega_n, -\alpha)$. When $\alpha = 0$, the model recovers the pseudo-complex SYK model at zero chemical potential [37].

Green function decays exponentially no matter at low temperature $T = 0.005$ or high temperature $T = 0.05$, and $E_{\text{gap}}$ decreases as temperature increases from $T = 0.005$ to $T = 0.05$(see Fig. 3(c) and 3(d)). According to the approximate behavior of the saddle-point equations Eq. (22), as $\alpha \to +\infty$, the off-diagonal Green function $G_{RL} \sim -\frac{1}{\Sigma_{LR}}$ dominates while as $\alpha \to -\infty$, the term $G_{LR} \sim -\frac{1}{\Sigma_{RL}}$ dominates. The approximate solutions are indicative of decoupled SYK behavior in the IR limit ($G \sim -\frac{1}{\Sigma}$). The results with $\alpha = 10$ supports this statement numerically in Fig. 3(c) and 3(d).

We evaluate the free energy of this non-Hermitian coupled model in this section. Substituting the saddle-point solutions into the action in Eq. (20) as the method in [12], we obtain the free energy

$$F = -T \log \frac{Z}{N} = T S_{\text{eff}}$$

$$N = -T \log \frac{Z}{N} = T S_{\text{eff}}$$

$$= -T \left[ 2 \log 2 + \sum_{\omega_n} \log \frac{D(i\omega_n, \alpha)}{(i\omega_n)^2} + \sum_{\omega_n} \left( \frac{3}{4} \Sigma_{LL}(i\omega_n, \alpha) \right) \right]$$

$$G_{LL}(i\omega_n, \alpha) + \frac{3}{4} \Sigma_{RR}(i\omega_n, \alpha) G_{RL}(i\omega_n, \alpha) + \frac{3}{4} \Sigma_{LR}(i\omega_n, \alpha) G_{RL}(i\omega_n, \alpha) \right]. \quad (23)$$

In turn, we calculate the free energy from the high temperature to the low temperature, and increase back to the high temperature with the non-zero $|\alpha|$. The free energy as a function of temperature are plotted in Fig. 4. The free energy obtained in Fig. 4 is analogous to
the free energy of the pseudo-complex SYK model with the Hermitian coupled term in Ref. [37, 43, 44]. Noticeably, our numerical result shows the free energy is non-Hermitian parameter $\alpha$-independent. As mentioned previously, the term $G_{LR} \sim -\frac{1}{\Sigma_{LR}}$ does not vanish when $\alpha \to -\infty$, and $G_{RL} \sim \frac{1}{\Sigma_{RL}}$ doesn’t vanish when $\alpha \to +\infty$. Now Green function and the self-energy satisfy $\Sigma_{LR}(i\omega_n, \alpha) = -\Sigma_{RL}(i\omega_n, -\alpha)$. Obviously, the free energy does not change by $\alpha = \pm \infty$. However, for a non-zero $\alpha$, the coupling of left and right black holes is not symmetric. The first-order phase transition from the low temperature traversable wormhole phase to the high temperature two black hole phase ending at a second-order critical point ($T_c = 0.25, \mu_c = 0.7$), which is not in deduced by the non-Hermitian parameter. The almost zero slopes of free energy is a signature of the gapped wormhole phase, and the constant negative slope is a signature of the black hole phase [44].

VI. CONCLUSION AND DISCUSSION

We have constructed a novel non-Hermitian two coupled SYK model yielding a real energy spectrum. This is a pseudo-Hermitian Hamiltonian, where the non-Hermiticity is reflected in the coupling between two copies of the pseudo-complex SYK model. The ground states of the total Hamiltonian receive contribution from the non-Hermitian parameter. In the strong non-Hermitian limit, the wave function $|\psi_i^T\rangle$ approaches the “ground states” on the left and “excited states” on the right or vice versa. The effective action and the free energy are $\alpha$-independence, although the left and right side states are actually $\alpha$-dependent. Low energy analysis further reveals an $\alpha$-independent energy gap. However, a key observation is that the off-diagonal Green’s functions $G_{LR}$ and $G_{RL}$ are no longer symmetric and strongly $\alpha$-dependent. Note that the transmission amplitude of particles across the traversable wormhole is proportional to the retarded Green’s function. Thus, this may elucidate the observable aspects of the dynamics.

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A. non-Hermitian similarity transformation

In this appendix, we provide more details about a similarity transformation on the left hand-side and the right hand-side Majorana fermion. One can generalize the phase $\phi$ from a real number to an imaginary number $\phi = -i\alpha$ and the imaginary phase transformation becomes a non-Hermitian particle-hole similarity transformation $C \to e^{-\alpha}C$ and $C^\dagger \to e^{i\alpha}C^\dagger$. The Majorana fermion can be decomposed as $\psi = C + C^\dagger$, so that one can perform a similarity transformation on the left hand-side and the right hand-side Majorana fermion. It can be written as

$$\psi_{2i-1}^L = e^{\alpha}C_{i}^L + e^{-\alpha}C_{i}^{L\dagger}, \quad \psi_{2i}^L = i(e^{\alpha}C_{i}^L - e^{-\alpha}C_{i}^{L\dagger})$$

$$\psi_{2i-1}^R = e^{-\alpha}C_{i}^R + e^{\alpha}C_{i}^{R\dagger}, \quad \psi_{2i}^R = i(e^{-\alpha}C_{i}^R - e^{\alpha}C_{i}^{R\dagger})$$

Here we keep the coupling of $\psi_{2i-1}$ only. After the self-similarity transformation, the Majorana fermions become non-Hermitian $(\psi^L)^\dagger \neq (\psi^L), (\psi^R)^\dagger \neq \psi^R$ as long as $\alpha \neq 0$ in which the Hermitian operators $C^L, C^R, C^{L\dagger}, C^{R\dagger}$ satisfy the anti-commutation relations as follow

$$\{C_{i}^{A\dagger}, C_{j}^B\} = \delta_{ij}\delta_{AB}, \quad \{C_{i}^{A\dagger}, C_{j}^{B\dagger}\} = \{C_{i}^{A\dagger}, C_{j}^B\} = 0.$$

Then, $\alpha$ could be introduced by performing the non-Hermitian particle-hole similarity transformation on the original Maldacena and Qi’s model

$$H = -\sum_{ijkl} J_{ijkl} (\psi^L_i \psi^L_j \psi^L_k \psi^L_l + \psi^R_i \psi^R_j \psi^R_k \psi^R_l) + i\mu \sum_i \psi^L_i \psi^R_i.$$

(24)

Note that all the non-physical terms should be neglected after the similarity transformation.

B. Noether charge

In this appendix, we discuss the Noether charges associated with the $SL(2)$ symmetry in the low energy effective action. The action is given by the combination of [12] and [13]. The general form of the Noether charge is
described by

\[ Q = \sum_{k \geq 1} \left( \frac{d}{d\tau} \right)^{k-1} \left[ Y \sum_{m \geq k} \binom{m}{k} \left( -\frac{d}{d\tau} \right)^{m-k} \frac{\partial L}{\partial h(m)} \right], \]

where the infinitesimal transformation is given by \( \tau \rightarrow \tau + aY \) with a small parameter \( a \). Since the effective Lagrangian contains up to third-derivative terms, the Noether charge can be explicitly written as

\[ Q = Y \left[ \frac{\partial L}{\partial h'} - \left( \frac{\partial L}{\partial h''} \right)' \right] + \frac{dY}{d\tau} \left[ \frac{\partial L}{\partial h'} - 3 \left( \frac{\partial L}{\partial h''} \right)'' + \frac{d^2 Y}{d\tau^2} \left( \frac{\partial L}{\partial h''} \right) \right]. \]

Considering the SL(2) transformation generated by (14), the corresponding charges are given by

\[ \frac{Q_0}{N} = Q_0[h_L, \phi_L] + Q_0[h_R, \phi_R] + \left( \frac{1}{h'^2_L} + \frac{1}{h'^2_R} \right) I_c, \]

\[ \frac{Q_+}{N} = Q_+[h_L, \phi_L] - Q_+[h_R, \phi_R] + \left( e^{ih_L} h'^2_L - e^{ih_R} h'^2_R \right) I_c, \]

\[ \frac{Q_-}{N} = Q_[-h_L, \phi_L] - Q_-[-h_R, \phi_R] + \left( e^{-ih_L} h'^2_L - e^{-ih_R} h'^2_R \right) I_c, \]

where

\[ Q_0[h, \phi] = i\epsilon K (\phi' + i\epsilon h') - \alpha_s \left( h' + \frac{h''}{h'^2} - h'^2 \frac{h'^2}{h'^3} \right), \]

\[ Q_+[h, \phi] = e^{i\epsilon K (\phi' + i\epsilon h')} - \alpha_s \left( h' + \frac{h''}{h'^2} - h'^2 \frac{h'^2}{h'^3} \right), \]

\[ Q_-[-h, \phi] = e^{-i\epsilon K (\phi' + i\epsilon h')} - \alpha_s \left( h' + \frac{h''}{h'^2} - h'^2 \frac{h'^2}{h'^3} \right), \]

\[ I_c = \mu \Delta \left[ \frac{bh_L h_R}{\cosh^2 \frac{h_R - h_L}{2}} \right] \cosh(\epsilon L - \epsilon h_R) \]

\[ \times \cosh(2\alpha - i\phi_L + i\phi_R). \]

FIG. 5: The energy gap extracted from the exponential decay at low temperature \( T = 0.001 \) with \( \alpha = 10 \). The red line represents the power law behavior \( E_{\text{gap}} \sim \mu^{2/3} \).

and

\[ 0 = \delta \cosh(2\alpha - i\phi_L + i\phi_R) \]

\[ = \sinh(2\alpha - i\phi_L + i\phi_R)\delta(\phi_L - \phi_R), \]

where we have assumed \( h_L = h_R \). These conditions are satisfied if \( \phi_L = \phi_R - 2i\alpha \) and \( \delta \phi_L = -\delta \phi_R = \epsilon(\tau) \), where \( \phi_L \) and \( \phi_R \) are constants and \( \epsilon(\tau) \) is an arbitrary infinitesimal function. To be self-consistency, the Noether charge associated with the variation of \( \delta \phi_L = -\delta \phi_R = \epsilon(\tau) \) is given by

\[ Q_\phi/N = K \left[ (\phi'_L - \phi'_R) + i\epsilon(h'_L - h'_R) \right], \]

which vanishes if \( h_L = h_R \) and \( \phi_L = \phi_R - 2i\alpha \).

**C. energy gap**

As mentioned in the previous sections, Green function decays exponentially \( G_{ab}(\tau) \sim e^{-E_{\text{gap}}\tau} \). In Fig. 4 we show that the energy gap extracted from the exponential decay of \( G_{AB}(\tau) \) in which the gap scaling \( E_{\text{gap}} \sim \mu^{2/3} \), same as that of the Hermitian two coupled SYK model in [30]. The gap scaling in Fig. 5 indicate that our non-Hermitian model return to the Hermitian model consistently if setting \( \alpha = 0 \).

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