CRITICAL IDEALS OF TREES

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Abstract. Critical ideals of a graph are the determinantal ideals of the generalized Laplacian matrix associated to it. In this paper we give an explicit description of a set of generators of the critical ideals of a tree $T$, that is, a simple connected graph without cycles. As a consequence we prove that the only simple graph with $n$ vertices and $n - 1$ trivial critical ideals is the path with $n$ vertices. Also, we prove that if $T$ has $n$ vertices, then the set of generators that we found is a Groebner basis for its $n - 1$-critical ideal. Moreover, we conjecture that is true for any of the critical ideals of $T$. Using the description that we get of the critical ideals of a tree we recovery some results obtained by Levine [5] and Toumpakari [9] about the critical group of a wired regular tree.

1. Introduction

Critical ideals of a graph were introduced in [4] as a generalization of the critical group and the characteristic polynomial of the adjacency and Laplacian matrices of a graph and have been studied in [2, 4]. Given a graph $G = (V, E)$, let $L(G, X_G)$ be the generalized Laplacian matrix of $G$, which is given by

$$L(G, X_G)_{u,v} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{if } u \neq v, \end{cases}$$

where $X_G = \{x_u | u \in V(G)\}$ is the set of indeterminates indexed by the vertices of $G$ and $m_{uv}$ is the number of edges between $u$ and $v$. For any $1 \leq j \leq n$, the $j$-critical ideal of $G$ is given by

$$I_j(G, X_G) = \langle j\text{-minors of } L(G, X_G) \rangle \subseteq \mathbb{Z}[X_G].$$

Critical ideals generalizes the critical group of a graph, see [4, Proposition 3.6]. The critical groups of a tree and wired trees have been studied by several authors, see for instance [5, 6, 9]. The critical ideals of a graph contain information about their critical group. For instance, if $\gamma(G) = \max\{j | I_j(G, X) = \mathbb{Z}[X_G]\}$, then rank$(K(G)) \leq n - 1 - \gamma(G)$. In contrast with the critical group of a graph, is more easy to relate some combinatorial invariants of the graph with its critical ideals. For instance, [4, Theorem 3.13] asserts that $\gamma(G) \leq \min(2n - \alpha(G), 2n - \omega(G) - 1)$, where $\alpha(G)$ and $\omega(G)$ are the stability and the clique numbers of $G$, respectively. In a similar way, the results obtained in [2] respect to the the characterization of the connected graphs with $\gamma(G) \leq 2$ and with two invariant factors of the critical group of a graph suggest a most evident role of the combinatorial structure of $G$ on critical ideals of a graph.

This paper focuses mainly to present an explicit description of a set of generators for the critical ideals of a tree when $\mathcal{P} = \mathbb{Z}$. Given a tree $T$, let $T^\ell$ be the (non-simple) graph that result by adding a loop at each vertex of $T$. Our description of the critical ideals of $T$ is based on the set of 2-matchings of $T^\ell$. 

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More precisely: To each 2-matching $M$ of $T^\ell$ we associate a minor of $L(T, X)$, denoted by $d(M, X)$, in such a way that: If $|M| = j$, then $d(M)$ is a $j$-minor. This association leads to our main result.

**Theorem 3.6.** Let $1 \leq j \leq n$, a tree with $n$ vertices $T$, and $V_2^j(T^\ell, j)$ be the set of minimal 2-matchings of $T^\ell$ of size $j$, see Definition 2.8. Then

$$I_j(T, X) = \left\{ d(M, X) \, | \, M \in V_2^j(T^\ell, j) \right\}.$$

An important invariant of a finite abelian group, in particular the critical group of a graph, is the number of non trivial invariant factors. The algebraic corank of a graph $G$, denoted by $\gamma(G)$, is given by $\gamma(G) = \max\{ j \, | \, I_j(G, X) = \mathbb{Z}[X_G] \}$. It is not difficult to see that $K(G)$ has at most $n - 1 - \gamma(G)$ non trivial invariant factors.

The invariant $\gamma$ have some nice features. For instance, Theorem 3.13 in [4] asserts that

$$\gamma(G) \leq \min(2(n - \alpha(G)), 2(n - \omega(G)) + 1),$$

where $\alpha(G)$ and $\omega(G)$ are the stability and the clique numbers of $G$, respectively. A remarkable consequence of Theorem 3.6 is the characterization of the algebraic corank of a tree in terms of its combinatorics.

If we set $\nu_2(G)$ as the maximum size of a 2-matching on $G$, then we get the following result:

**Theorem 3.8.** If $T$ is a tree, then $\gamma(T) = \nu_2(T)$.

This result led us to prove Conjecture 4.12 given in [4].

**Corollary 3.9.** If $G$ is a simple graph in $n$ vertices, then $\gamma(G) = n - 1$ if and only if $G = P_n$.

This paper is organized as follows: In Section 2, we introduce the concept of 2-matching and present some of their basic properties, which be useful to establish the main result of this paper. In Section 3 we establish the correspondence between 2-matchings on $T^\ell$ and minors of $L(T, X)$ and illustrate it with several examples. After doing this we focus in the algebraic relations between the minors associated to 2-matchings. In Section 4, we prove that the minors associated to the minimal 2-matching of $T^\ell$ form a reduced Gröbner Basis for the $n - 1$ critical ideals of $T$.

Finally, Section 5 is devoted to present three applications of the results obtained in the previous sections in the computation the critical ideals and critical groups of trees. Firstly we present some arithmetical trees associated to the reduction of elliptic curves of Kodaira type $I_n^\ell$. In the next subsection we study the critical ideals of the graph obtained from a regular tree by collapsing the leaves to a single vertex and reply some results obtained by Levine [5] and Toumpakary [9] about the critical groups of wired trees. Thirdly we describe critical ideals of all the trees with depth two.

## 2. 2-MATCHINGS OF TREES

In this section we introduce the concept of 2-matching of a graph, which play an important role through all the paper. After that, we present some of its properties when the graph is a tree, which will be very useful to give a description of its critical ideals.

**Definition 2.1.** Let $G$ be a graph (possibly with loops and multiple edges) and $M$ a set of edges of $G$. We say that $M$ is a 2-matching if every vertex of $G$ has at most two incident edges in $M$.

The set of all 2-matchings of a graph $G$, will be denoted by $V_2(G)$. Moreover, let $V_2(G, j)$ be the set of 2-matchings of $G$ of size $j$, that is, with $j$ edges. Also, the 2-matching number of $G$, denoted by $\nu_2(G)$, is the maximum number of edges of a 2-matching of $G$. A 2-matching of $G$ of size $\nu_2(G)$ is called maximum.

The concept of 2-matching apply for any class of graphs, however in this chapter we are primarily interested when $G$ is a tree. If $T$ is a tree, then is not difficult to see that its 2-matchings consist of a
union of paths. We recall that a vertex is a path of length zero. For instance, let $C$ be the tree given in Figure 1.a. If we take (see Figure 1b and 1c)

$$\mathcal{M}_1 = \{v_1v_2, v_2v_5, v_6v_7, v_6v_8\} \text{ and } \mathcal{M}_2 = \{v_1v_2, v_2v_3, v_2v_4, v_6v_8\}.$$ 

Then $\mathcal{M}_1 \in \mathcal{V}_2(C, 4)$ and $\mathcal{M}_2 \notin \mathcal{V}_2(C)$ because $\mathcal{M}_2$ has 3 incident edges on $v_2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A caterpillar tree $C$}
\end{figure}

It is important to note that a loop $vv$ is counted twice as an incident edge of $v$, however it is only one edge in the 2-matching. Now, we focus our attention in an special set of the 2-matchings, the maximal ones.

**Definition 2.2.** A 2-matching $M$ of a graph $G$ is called maximal if not exists a 2-matching $N$ such that $M \subseteq N$.

Note that a 2-matching with size $\nu_2(G)$ is immediately maximal, but a maximal 2-matching can have less than $\nu_2(G)$ edges. Even more, two maximal 2-matchings can have different size.

The 2-matching $\mathcal{M}_1$ (see Figure 1b) is maximal. The maximal 2-matchings will play an important role in the description of the critical ideals of a tree. In the follow we present the first property of the maximal 2-matching of a tree. Given two vertices $u, v$ of a tree $T$, let $P_{u,v}$ be the unique path in $T$ that join $u$ and $v$.

**Proposition 2.3.** If $\mathcal{M}$ is a maximal 2-matching of a tree $T$, then exists $u \neq v$ leaves of $T$ such that $E(P_{u,v}) \subseteq \mathcal{M}$.

**Proof.** The proof it follows by induction on the number of vertices of $T$. Is clear that the result is true for the trees with less or equal to three vertices.

Now, assume that the result is true for all the trees with $k$ or less vertices. Let $T$ be a tree with $k + 1$ vertices, $\mathcal{M}$ a maximal 2-matching of $T$, $a$ a leaf of $T$, and $e = ab$ the edge of $T$ incident with $a$. If $e \notin \mathcal{M}$, then $\mathcal{M}$ is a maximal 2-matching of $T \setminus a$ and the result it follows by the induction hypothesis. On the other hand, if $e \in \mathcal{M}$, then $\mathcal{M} \setminus e$ is a maximal 2-matching of $T \setminus a$. Now, by induction hypothesis, let $u \neq v$ leaves of $T \setminus a$ such that $E(P_{u,v}) \subseteq \mathcal{M} \setminus e$. If $b \neq u, v$, then the result it follows. Otherwise, if $b = u$, then $a$ and $v$ are leaves of $T$ such that $E(P_{a,v}) \subseteq \mathcal{M}$.

In the following we study the 2-matching number of a tree when we delete one of its edges. Before to present the result, we introduce some concepts. Given a tree $T$ and a vertex $v$, we say that $v$ is saturated if any maximum 2-matching of $T$ has two incident edges to it. In a similar way, we say that an edge $e$ of $T$ is saturated when it appears in all the maximum 2-matchings of $T$.

**Lemma 2.4.** If $T$ is tree without loops and $e = uv$ is an edge of $T$, then

$$\nu_2(T) - \nu_2(T \setminus e) = \begin{cases} 0 & \text{if and only if } u \text{ or } v \text{ are saturated in } T \setminus e, \\ 1 & \text{if and only if } e \text{ is saturated in } T. \end{cases}$$
Proof. Let $T_u$ and $T_v$ be the connected components of $T \setminus e$ that contain the vertex $u$ and $v$ respectively. Let $\mathcal{M}$ be a maximum 2-matching of $T$, $\mathcal{M}_u = \mathcal{M} \cap E(T_u)$ and $\mathcal{M}_v = \mathcal{M} \cap E(T_v)$. Note that $\mathcal{M}_u$ and $\mathcal{M}_v$ are not necessarily maximum 2-matchings of $T_u$ and $T_v$ respectively. However, we can assure that at least one of them it is and the other is almost of maximum size $(\nu_2(T_v) - 1)$. Since $|\mathcal{M}| = |\mathcal{M}_u| + |\mathcal{M}_u| + |\mathcal{M} \cap \{e\}|$ and $0 \leq |\mathcal{M} \cap \{e\}| \leq 1$, then $\nu_2(T) \leq \nu_2(T \setminus e) + 1$. In a similar way, taking maximum 2-matchings of $T_u$ and $T_v$ we get that $\nu_2(T \setminus e) \leq \nu_2(T)$ and therefore $\nu_2(T) - 1 \leq \nu_2(T \setminus e) \leq \nu_2(T)$.

Now, $e$ is not saturated in $T$ if and only if there exist $N_u$ and $N_v$ maximum 2-matchings of $T_u$ and $T_v$ respectively such that $N = N_u \cup N_v$ is a maximum 2-matching of $T$. Which happens if and only if $\nu_2(T) = \nu_2(T \setminus e)$. That is, $\nu_2(T) = \nu_2(T \setminus e) + 1$ if and only if $e$ is saturated in $T$.

Finally, $e$ is saturated in $T$ if and only if each maximum 2-matching $\mathcal{M}$ of $T$ satisfies that $e \in \mathcal{M}$ and $\mathcal{M} \setminus e$ is a maximum 2-matching of $T \setminus e$ if and only if $\deg_{T\setminus\mathcal{M}\setminus\{e\}}(u), \deg_{T\setminus\mathcal{M}\setminus\{e\}}(v) \leq 1$. Which happens if and only if $u$ and $v$ are not saturated in $T \setminus e$. That is, $e$ is saturated in $T$ if and only if $u$ and $v$ are not saturated in $T \setminus e$ or equivalently $\nu_2(T) = \nu_2(T \setminus e)$ if an only if $u$ or $v$ are saturated in $T \setminus e$. ⋄

Now, we present how it changes the 2-matching number of a tree when we delete one if its vertices. In the follow $N_T(v)$ denote the set of vertices on $T$ which are adjacent to $v$.

Lemma 2.5. Let $T$ be a tree without loops, $v$ a vertex of $T$, $N_T(v) = \{w_1, \ldots, w_s\}$, and $T_i$ the connected component of $T \setminus v$ that contains $w_i$. Then

$$\nu_2(T) - \nu_2(T \setminus v) = \begin{cases} 
2 & \text{if and only if } v \text{ is saturated in } T, \\
1 & \text{if and only if there exists } 1 \leq j \leq s \text{ such that } vw_j \text{ is saturated and } w_i \text{ is saturated in } T_i \text{ for all } i \neq j, \\
0 & \text{if and only if } w_i \text{ is saturated in } T_i \text{ for all } 1 \leq i \leq s.
\end{cases}$$

Proof. Given a maximum 2-matching $\mathcal{M}$ of $T$, let $\mathcal{M}_i = \mathcal{M} \cap E(T_i)$. Note that $\mathcal{M}_i$ is not necessarily a maximum 2-matchings of $T_i$. However, this is true in the following cases: (i) $vw_i \notin \mathcal{M}$ and (ii) $vw_i \in \mathcal{M}$ but $v$ is saturated in $T$. Is clear that if $\mathcal{M}_i$ is not a maximum 2-matching of $T_i$, then there exists a maximum 2-matchings $\mathcal{M}'_i$ of $T_i$ such that $|\mathcal{M}'_i| > |\mathcal{M}_i|$. Case (i). If $vw_i \notin \mathcal{M}$, then $\mathcal{M}' = (\mathcal{M} \setminus \mathcal{M}_i) \cup \mathcal{M}'_i$ is a 2-matching of $T$ with $|\mathcal{M}'| > |\mathcal{M}|$; a contradiction. Case (ii). If $vw_i \in \mathcal{M}$ and $v$ is saturated in $T$, then $\mathcal{M}' = (\mathcal{M} \setminus (\mathcal{M}_i \cup \{vw_i\})) \cup \mathcal{M}'_i$ is a 2-matching of $T$ with $|\mathcal{M}'| \geq |\mathcal{M}|$ and $\deg_{\mathcal{M}'(v)}(v) = 1$; a contradiction.

On the other hand, since $T \setminus v = T_1 \sqcup \cdots \sqcup T_s$, then $\nu_2(T \setminus v) = \nu_2(T_1) + \cdots + \nu_2(T_s)$ and

$$2 \geq |\mathcal{M} \cap \delta_T(v)| = |\mathcal{M} \setminus \bigcup_{i=1}^s \mathcal{M}_i| \geq \nu_2(T) - \sum_{i=1}^s \nu_2(T_i) = \nu_2(T) - \nu_2(T \setminus v) = \nu_2(T) - \nu_2(T \setminus v),$$

where $\delta_T(v) = \{vw \in E(T)\}$. That is $\nu_2(T) - \nu_2(T \setminus v) \leq 2$. Also, clearly $\nu_2(T \setminus v) \leq \nu_2(T)$ and therefore $\nu_2(T \setminus v) \leq \nu_2(T) + 2$.

Now, if $\nu_2(T) - \nu_2(T \setminus v) = 2$, then $|\mathcal{M} \cap \delta_T(v)| = 2$ and $v$ is saturated in $T$. Also, if $v$ is saturated in $T$, then the $\mathcal{M}_i$ are maximum 2-matchings in $T_i$ and

$$\nu_2(T) = |\mathcal{M}| = \sum_{i=1}^s |\mathcal{M}_i| + 2 = \sum_{i=1}^s \nu_2(T_i) + 2 = \nu_2(T \setminus v) + 2.$$

Also, it is not difficult to check that: $\nu_2(T) = \nu_2(T \setminus v)$ if and only if $w_i$ is saturated in $T_{w_i}$ for all $w_i \in N_T(v)$. Finally, if $\nu_2(T) = \nu_2(T \setminus v) + 1$, then there exist $1 \leq j \leq s$ such that $vw_j$ is saturated and
Proposition 2.6. Let \( T \) be a tree, \( u \in E(T) \), and \( T_u \) the connected component of \( T \setminus uv \) that contains \( u \). If \( u \) is saturated in \( T_u \), then \( u \) is saturated in \( T \).

Proof. Let \( T_u \) be the connected component of \( T \setminus uv \) that contains \( u \). Since \( u \) is saturated in \( T_u \) and \( T \setminus e = T_u \cup T_v \), then by Lemma 2.4, \( \nu_2(T) = \nu_2(T_u) + \nu_2(T_v) \). Thus

\[
\nu_2(T) - \nu_2(T \setminus u) = \nu_2(T_u) + \nu_2(T_v) = \nu_2(T_u) = \nu_2(T_u \setminus u) \leq \nu_2(T_v) = 0.
\]

Finally, by Lemma 2.5, \( u \) is saturated in \( T \). \( \square \)

Next result prove that if \( w_i \) is saturated in \( T_i \), then \( w_i \) is saturated in \( T \). Is not difficult to check that the converse is not true in general.

Proposition 2.7. If \( T \) is a tree with at least three vertices, then it has at least one saturated vertex.

Proof. It follows by Lemma 2.5 and Proposition 2.6. The only tree that not satisfies this theorem is the tree with only one edge.

Remark 2.9. Note that the definition of minimal 2-matching has sense only for graphs of the form \( G^\ell \).

Definition 2.8. A 2-matching \( M \) of \( G^\ell \) is called minimal if it does not exist a 2-matching \( M' \) of \( G^\ell \) such that \( \ell(M') \subseteq \ell(M) \) and \( |M| = |M'| \). The set of all minimal 2-matchings of \( G^\ell \) will be denoted \( \mathcal{V}_2^\ell(G^\ell) \). Moreover, let \( \mathcal{V}_j^\ell(G^\ell) = \mathcal{V}_2^\ell(G^\ell) \cap \mathcal{V}_2(G^\ell, j) \) for any \( 1 \leq j \leq n \).

Remark 2.9. Note that the definition of minimal 2-matching has sense only for graphs of the form \( G^\ell \).

If \( M \in \mathcal{V}_2(G^\ell, j) \setminus \mathcal{V}_2(G, j) \) and \( N \in \mathcal{V}_2(G, j) \), then \( |M| = |N| \) and \( \ell(N) = 0 \subseteq \ell(M) \). Thus \( \mathcal{V}_2(G^\ell, j) = \mathcal{V}_2(G, j) \) for all \( 1 \leq j \leq \nu_2(G) \). Moreover, next result shows that some maximal 2-matchings of \( T \) are part of a minimal 2-matching of \( T^\ell \).

Proposition 2.10. If \( M \) is a maximal 2-matching of \( T[N_T(M)] \), then

\[
\mathcal{N} = M \cup \{uu \mid u \notin V(M)\}
\]

is a minimal 2-matching of \( T^\ell \).

Proof. Assume that \( \mathcal{N} \) is a not minimal 2-matching of \( T^\ell \). Thus, there exists a 2-matching \( \mathcal{N}' \) of \( T^\ell \) such that \( |\mathcal{N}| = |\mathcal{N}'| \) and \( \ell(\mathcal{N}') \subseteq \ell(\mathcal{N}) \). That is, \( \mathcal{N}' \) has at least one more edge than \( \mathcal{N} \). Since \( M \) is maximal on \( N_T(M) \), \( \mathcal{N}' \) must have an edge with at least one end in \( V(T) \setminus V(M) \); a contradiction to the fact that \( \mathcal{N} \) has a loop in all the vertices of \( V(T) \setminus V(M) \). \( \square \)
Example 2.11. Let $C$ be the caterpillar tree consider in Figure 2a. Is not difficult to check that $\nu_2(C) = 4$. Thus, any minimal 2-matching of $C^\ell$ with at least one loop has at least size 5. The 2-matching $M_1 = \{v_1v_2, v_2v_5, v_5v_6, v_6v_9, v_3v_3\}$, see Figure 2a, is a minimal 2-matching of $C^\ell$ of size 5 with only one loop. Also, the 2-matching given in Figure 2b is a minimal 2-matching of $C^\ell$ of size 6.

![Figure 2. $C^\ell$ and some of its minimal 2-matchings.](image)

Let $M_3 = \{v_1v_1, v_3v_2, v_2v_4, v_5v_5, v_6v_6, v_7v_7, v_8v_8, v_9v_9\}$ be the 2-matching given in Figure 2c. Using Proposition 2.10 is not difficult to check that $M_3$ is a minimal 2-matching of size 8. Moreover, $M_4 = M_3 \setminus \{v_1v_1\}$ (see Figure 2d) is also a minimal 2-matching of size 7.

Now, we give a recursive description of all the minimal 2-matchings on $T^\ell$.

Proposition 2.12. If $T$ is a tree and $e = uv \in E(T)$, then

$$V_2^e(T^\ell) \subseteq \{V_2^e(T_u^\ell + e) \cup V_2^e(T_v^\ell + e)\} \cup \{V_2^e(T_u^\ell) \cup V_2^e(T_v^\ell)\},$$

where $T_x$ is the subtree of $T \setminus e$ that contains the vertex $x$ and $V_2^e(G)^e$ is the set of 2-matchings of $G$ that contain the edge $e$.

Proof. Let $M$ be a minimal 2-matching of $T^\ell$. First, assume that $e \in M$ and let $M_u = M \cap E(T_u^\ell + e)$ and $M_v = M \cap E(T_v^\ell + e)$. As $M = M_u \cup M_v$, it is enough to prove that $M_u$ is a minimal 2-matching of $T_u^\ell + e$. Assume that $M_u$ is not minimal. Thus there exists $M'_u \in V_2(T_u^\ell + e)$ such that $\ell(M'_u) \subseteq \ell(M_u)$ and $|M'_u| = |M_u|$. Note that $\ell(M'_u \cup M_v) \subseteq \ell(M)$ and

$$|M'_u \cup M_v| = \begin{cases} |M| + 1 & \text{if } e \notin M'_u, \\ |M| & \text{if } e \in M'_u. \end{cases}$$

Thus, since $M$ is minimal, $e \notin M'_u$ and $|M'_u \cup M_v| = |M| + 1$. If we remove one loop (or an edge different from $e$) of $M'_u$, then we get a 2-matching $M''_u$ of $T_u^\ell$ such that $|M''_u \cup M_v| = |M|$ and $\ell(M''_u \cup M_v) \subseteq \ell(M)$ which is also contradict the minimality of $M$. Thus, $M_u$ is minimal on $T_u^\ell + e$. As $e \in M_u$, $M_v$, $M \in V_2(T_u^\ell + e)^e \cup V_2(T_v^\ell + e)^e$.

Finally, if assume that $e \notin M$ and $M_u = M \cap T_u$ and $M_v \cap T_v$, then the minimality of $M_u$ and $M_v$ can be deduced in a similar way. $\square$

3. Critical Ideals of Trees

This section is devoted to establish a relationship between the generators of the critical ideals of a tree and its 2-matchings. This relationship allows to give a complete and compact combinatorial description of the critical ideals of a tree. Moreover, we prove that the critical ideals are generated from an special set of the 2-matchings of the tree.

Since the $j$ critical ideal of a graph $G$ is generated by the $j$-minors of their generalized Laplacian matrix, then it only depends of the non-vanishing $j$-minors of $L(G, X_G)$. Therefore, we begin by giving a description of the non-vanishing $j$-minors of the generalized Laplacian matrix of a tree.
3.1. The non-vanishing minors of $L(T, X_T)$. In this section we prove that the non-vanishing $j$-minors of $L(T, X_T)$ correspond to the 2-matchings of $T$. We begin by introducing some notation.

Given a 2-matching $\mathcal{M}$ of $T^\ell$, we associate the sets $t(\mathcal{M}), h(\mathcal{M}) \subset V(T)$ as follows: First, if $\mathcal{M} = \{v_j, v_j, v_j, \ldots, v_{j_m}, v_{j_m+1}\}$ is a path, then

$$h(\mathcal{M}) = \{v_{j_2}, \ldots, v_{j_m+1}\} \text{ and } t(\mathcal{M}) = \{v_{j_1}, \ldots, v_{j_m}\}.$$ That is, if $\overrightarrow{\mathcal{M}}$ is the oriented path obtained from $\mathcal{M}$ and the order on the vertices of $T$, then $h(\mathcal{M})$ and $t(\mathcal{M})$ are the heads and tails of their arcs, respectively. Moreover, if $\mathcal{M}$ is non-connected and $\{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ are their connected components, then $h(\mathcal{M}) = \bigcup_{i=1}^k h(\mathcal{M}_i)$ and $t(\mathcal{M}) = \bigcup_{i=1}^k t(\mathcal{M}_i)$.

On the other hand, let $L(T, X)[t(\mathcal{M}), h(\mathcal{M})]$ be the square submatrix of $L(T, X)$ of size $|\mathcal{M}| = |t(\mathcal{M})| = |h(\mathcal{M})|$, $a_\mathcal{M}$ the leading coefficient of $\det (L(T, X)[t(\mathcal{M}), h(\mathcal{M})])$, and

$$d(\mathcal{M}, X) = \left\{ \begin{array}{ll} \det (L(T, X)[h(\mathcal{M}), t(\mathcal{M})]) & \text{if } a_\mathcal{M} > 0, \\
-\det (L(T, X)[h(\mathcal{M}), t(\mathcal{M})]) & \text{if } a_\mathcal{M} < 0. \end{array} \right.$$ Thus $d(\mathcal{M}, X)$ is a generator of the $|\mathcal{M}|$-critical ideal of $T$. As we will see in Lemma 3.2, $d(\mathcal{M}, X)$ does not depend on the order of the vertices of $T$. That is, the correspondence $\mathcal{M} \mapsto d(\mathcal{M}, X)$ between $\mathcal{V}_2(T^\ell)$ and $\mathbb{Z}[X_C]$ is well defined. Next example illustrate this correspondence between 2-matchings and generators of the critical ideals of $T$.

**Example 3.1.** Let $T$ be the tree given in Figure 1. It is not difficult to see that

$$\mathcal{M} = \{v_1v_3, v_2v_5, v_7v_6, v_6v_8, v_9v_9\}$$ is a 2-matching of $T^\ell$. Moreover, the paths $P_1 = v_3v_2v_5$ and $P_2 = v_7v_6v_8$ and the loops $L_1 = v_1v_1$ and $L_2 = v_9v_9$ are the connected components of $\mathcal{M}$. Since $h(P_1) = \{v_2, v_5\}$, $t(P_1) = \{v_3, v_2\}$, $h(P_2) = \{v_6, v_8\}$, $t(P_2) = \{v_7, v_6\}$, $h(L_1) = \{v_1\} = t(L_1)$, and $h(L_2) = \{v_1\} = t(L_2)$, then $h(\mathcal{M}) = \{v_1, v_2, v_3, v_6, v_7, v_8, v_9\}$ and $t(\mathcal{M}) = \{v_1, v_2, v_3, v_6, v_7, v_9\}$. Thus

$$L(T, X)[h(\mathcal{M}), t(\mathcal{M})] = \begin{pmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ -1 & x_2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & x_6 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & x_9 \end{pmatrix}$$

and $d(\mathcal{M}, X) = x_1x_9$.

**Lemma 3.2.** If $T$ is a tree and $\mathcal{M}$ a 2-matching of $T^\ell$, then

$$d(\mathcal{M}, X) = x_{\ell(\mathcal{M})} + \text{ “terms of lower degree”}.$$ 

**Proof.** First, is not difficult to prove that if $\mathcal{P}$ is a disjoint union of paths in a tree, then

$$L(T, X)[h(\mathcal{P}), t(\mathcal{P})] \sim \begin{pmatrix} 1 & * & * \\ \vdots & \ddots & * \\ 0 & 0 & 1 \end{pmatrix}.$$ Thus

$$L(T, X)[h(\mathcal{M}), t(\mathcal{M})] \sim \begin{pmatrix} L(T, X)[\ell(\mathcal{M}), \ell(\mathcal{M})] & * & * \\ 0 & 1 & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
where $\ell(M) = \{u_1, \ldots, u_r\}$ is the set of loops of $M$. Since

$$L(T, X)[\ell(M), \ell(M)] = \begin{pmatrix} x_{u_1} & * & * \\ * & \ddots & * \\ * & * & x_{u_r} \end{pmatrix}$$

and $\det(L(T, X)[h(M), t(M)]) = \det(L(T, X)[l(M), l(M)])$, then the result is clear.

The next lemma is a partial converse of the previous result.

**Lemma 3.3.** If $f(X)$ is a non-vanishing minor of $L(T, X)$ with positive leader coefficient, then there exists $M \in V_2(T^d)$ such that $f(X) = d(M, X)$.

**Proof.** Let $I, J \subseteq V(T)$ such that $|I| = |J| \neq 0$ and $f(X) = \det(L(T, X)[I, J])$. Since $f(X)$ is non-zero, we can assume that all the entries in the main diagonal of $L(T, X)[I, J]$ are different from zero. Now, let $N = \{v_i v_j, \ldots, v_i v_{j_r}\}$, where $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_t\}$. Since $i_r \neq i_s$ and $j_r \neq j_s$ for all $r \neq s$, then $N$ is a 2-matching of $T^d$ with $|N| \leq t = |I|$. If $|N| < t$, then exists $1 \leq r < s \leq t$ such that $v_r v_{j_r} = v_i v_{j_s}$. Since $i_r \neq i_s$, $i_r = j_s$, $j_r = i_s$ and $N' = N \cup \{v_{i_r} v_{j_r}, v_i v_{j_s}\}$ \{\{v_{i_r} v_{j_r}, v_i v_{j_s}\} is a 2-matching $N'$ of $T^d$ with $|N'| = |N| + 1$. We can repeat this process until to get a 2-matching $M$ of size $t$ such that $I = t(M)$ and $J = h(M)$.

By Lemma 3.3, $I_j(T, X) = \{d(M, X) \mid M \in V_2(T^d)\}$ with $|M| = j$ for any tree $T$ with $n$ vertices and $1 \leq j \leq n$. However, this description is not minimal. For instance, is not difficult to find a tree $T$ and 2-matchings $M \neq N$ of $T^d$ with $\ell(M) = \ell(N)$. That is, $d(M, X) = d(N, X)$ (Lemma 3.2) and therefore the previous description of $I_j(T, X)$ contains repeated elements. Moreover, the minors of $L(T, X)$ are related by more complex algebraic identities.

In the next we exploit the combinatorial structure of $T$ in order to develop some identities between the minors of $L(T, X)$ which allows to find a better description of the critical ideals of a tree. Before to present the first of this identities, we fix some notation. For any graph $G$, let $d(G, X) = \det(L(G, X))$.

**Lemma 3.4.** If $T$ is a tree and $S \subseteq E(T)$, then

$$d(T \setminus S, X) = \sum_{\mu \in \mathcal{V}_1(S)} d(T \setminus V(\mu), X),$$

where $\mathcal{V}_1(S)$ is the set of matchings of the subgraph of $T$ induced by $S$.

**Proof.** We use induction on $|S|$. First, let $S = \{uv\}$. Since

$$\mathcal{V}_1(T) = \mathcal{V}_1(T \setminus uv) \cup \{\{uv\} \cup \mu \mid \mu \in \mathcal{V}_1(T \setminus \{u, v\})\}$$

and $d(T, X) = \sum_{\mu \in \mathcal{V}_1(T)} (-1)^{|\mu|} \prod_{v \in V(\mu)} x_v$, ([4] Lemma 4.4) we get the result, that is, $d(T \setminus uv, X) = d(T, X) + d(T \setminus \{u, v\}, X)$.

Now, let $S = \{uv\} \cup S'$ with $|S'| > 0$. If $T' = T \setminus S'$, then by induction hypothesis

$$d(T \setminus S, X) = d(T' \setminus uv, X) = d(T', X) + d(T' \setminus \{u, v\}, X)$$

and $d(T', X) = d(T \setminus S', X) = \sum_{\mu \in \mathcal{V}_1(S')} d(T \setminus V(\mu), X)$. On the other hand, since $T' \setminus \{u, v\} = T_{u,v} \setminus S''$, where $T_{u,v} = T \setminus \{u, v\}$ and $S'' = \{e \in S' \mid u, v \notin V(e)\}$,

$$d(T' \setminus \{u, v\}, X) = d(T_{u,v} \setminus S'', X) = \sum_{\mu \in \mathcal{V}_1(S'')} d(T_{u,v} \setminus V(\mu), X).$$
Moreover, since $\mathcal{V}_1(S) = \mathcal{V}_1(S') \cup \{\{uv\} \cup \mu | \mu \in \mathcal{V}(S')\}$,
\[
d(T \setminus S, X) = \sum_{\mu \in \mathcal{V}_1(S')} d(T \setminus \mathcal{V}(\mu), X) + \sum_{\mu \in \mathcal{V}_1(S')} d(T_{\mu,v} \setminus \mathcal{V}(\mu), X) = \sum_{\mu \in \mathcal{V}_1(S)} d(T \setminus \mathcal{V}(\mu), X).
\]

This lemma is a fundamental result on this chapter, in fact that almost all the identities between the generators of the critical ideals of a tree are derived from it. For instance we have the next corollary:

**Corollary 3.5.** If $\mathcal{M}$ is a 2-matching of $T$ and $w$ a vertex such that $ww \notin \ell(\mathcal{M})$, then
\[
x_w d(\mathcal{M}, X) = d(\mathcal{N}, X) + \sum_{v \in U} d(\mathcal{M} \setminus \{vv\}, X),
\]
where $\mathcal{N} = \{uv | uv \in \mathcal{M} \land w \neq u, v\} \cup \{ww\}$ and $U = \{v \in V(T) | vv \in \ell(\mathcal{M}), vw \in E(T)\}$.

**Proof.** Let $T' = T[\ell(\mathcal{N})]$ and $S$ be the set of edges in $T'$ that contains $w$. Since $\mathcal{V}_1(S) = \emptyset \cup \{vw | vv \in \ell(\mathcal{M}), vw \in E(T)\}$, then applying Lemma 3.4 to $T'$ and $S$ we get that
\[
d(T' \setminus S, X) = \sum_{\mu \in \mathcal{V}_1(S)} d(T' \setminus \mathcal{V}(\mu), X) = d(T', X) + \sum_{v \in U} d(T' \setminus \{w, v\}, X).
\]
Since $w$ and $T' \setminus S$ are not connected, then $d(T' \setminus S, X) = x_w d(T' \setminus w, X) = x_w d(\mathcal{M}, X)$. On the other hand, by Lemma 3.2 $d(T', X) = d(\mathcal{N}, X)$ and $d(T' \setminus \{w, v\}, X) = d(\mathcal{M} \setminus \{vv\}, X)$ for all $v \in U$. Combining these identities we get the result. $\square$

This corollary allows us to prove one of the most important results in this chapter.

**Theorem 3.6.** If $T$ is a tree with $n$ vertices, then
\[
I_j(T, X) = \left\{d(\mathcal{M}, X) | \mathcal{M} \in \mathcal{V}_2^n(T^\ell, j)\right\} \text{ for all } 1 \leq j \leq n.
\]

**Proof.** By Lemma 3.3, $I_j(T, X) \subseteq \{d(\mathcal{M}, X) | \mathcal{M} \in \mathcal{V}_2^n(T^\ell, j)\}$. Thus, we only need to prove that the minor of a not minimal 2-matching can be expressed in term of minors associated to minimals 2-matchings of the same size.

Let $\mathcal{M}$ be a not minimal 2-matching of size $j$. Then, there exists $\mathcal{N} \in \mathcal{V}_2(T, j)$ and $w \in V(T)$ such that $\ell(\mathcal{M}) = \ell(\mathcal{N}) \cup \{ww\}$. Applying Proposition 3.5 to $\mathcal{N}$ we get that
\[
d(\mathcal{M}, X) = x_w d(\mathcal{N}, X) - \sum_{v \in U} d(\mathcal{N} \setminus \{vv\}, X),
\]
where $U = \{v \in V(T) | vv \in \ell(\mathcal{M}), vw \in E(T)\}$.

For all $v \in U$, let $\mathcal{N}_{vw} = (\mathcal{N} \setminus \{vv\}) \cup \{vw\}$. Since $vw \notin \mathcal{N} \setminus \{vv\}$, clearly $|\mathcal{N}_{vw}| = |\mathcal{N} \setminus \{vv\}| + 1$ and therefore $\mathcal{N}_{vw}$ is 2-matching of $T^\ell$ of size $j$. On the other hand, since $\ell(\mathcal{N}_{vw}) = \ell(\mathcal{N}) \setminus \{vv\}$, by Lemma 3.2 we get that $d(\mathcal{N} \setminus \{vv\}, X) = d(\mathcal{N}_{vw}, X)$. Therefore
\[
d(\mathcal{M}, X) = x_w d(\mathcal{N}, X) - \sum_{v \in U} d(\mathcal{N}_{vw}, X).
\]
Since $\ell(\mathcal{N}) \subseteq \ell(\mathcal{M})$ and $\ell(\mathcal{N}_{vw}) \subseteq \ell(\mathcal{M})$ for all $v \in U$, we can repeat this process until we get an expression of $d(\mathcal{M}, X)$ as an algebraic combination on the minors associated to some minimal 2-matchings of $T^\ell$ of size $j$. $\square$

Next example illustrate how it works the previous theorem.
Example 3.7. Let \( C \) be the tree given in Figure 7 and
\[
\mathcal{M} = \{ v_1v_1, v_2v_2, v_3v_3, v_4v_4, v_5v_5, v_6v_6 \}
\]
be a 2-matching of \( C \) of size 6. Since \( \mathcal{M}_1 = \{ v_1v_1, v_2v_2, v_3v_3, v_4v_4, v_5v_5, v_6v_9 \} \) is a 2-matching of size 6 with \( \ell(\mathcal{M}) = \ell(\mathcal{M}_1) \cup \{ v_6v_9 \} \), then \( \mathcal{M} \) is non-minimal. Thus \( d(\mathcal{M}, X) = x_9d(\mathcal{M}_1, X) - d(\mathcal{M}_2, X) \), where \( \mathcal{M}_2 = \{ v_1v_1, v_2v_2, v_3v_3, v_4v_4, v_5v_6, v_6v_9 \} \).

In a similar way, since \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are not minimal, then \( d(\mathcal{M}_1, X) = x_9d(\mathcal{M}_2, X) - d(\mathcal{M}_3, X) \) where \( \mathcal{M}_3 = \{ v_1v_1, v_2v_5, v_3v_3, v_4v_4, v_5v_6, v_6v_9 \} \) and \( d(\mathcal{M}_2, X) = x_9d(\mathcal{M}_3, X) - d(\mathcal{M}_4, X) - d(\mathcal{M}_5, X) - d(\mathcal{M}_6, X) \) with \( \mathcal{M}_4 = \{ v_1v_2, v_3v_3, v_4v_4 \} \cup \mathcal{P}, \mathcal{M}_5 = \{ v_1v_1, v_2v_3, v_4v_4 \} \cup \mathcal{P}, \mathcal{M}_6 = \{ v_1v_1, v_2v_4, v_3v_3 \} \cup \mathcal{P}, \) and \( \mathcal{P} = \{ v_2v_5, v_5v_6, v_6v_9 \} \).

Finally, since \( \mathcal{M}_4, \mathcal{M}_5 \) and \( \mathcal{M}_6 \) are minimal 2-matchings and \( d(\mathcal{M}_3, X) = x_1d(\mathcal{M}_4, X) \), then
\[
d(\mathcal{M}, X) = (x_1 \cdot p_{2,5,6} - p_{5,6}) \cdot d(\mathcal{M}_4, X) - p_{5,6} \cdot d(\mathcal{M}_5, X) - p_{5,6} \cdot d(\mathcal{M}_6, X),
\]
where \( p_{2,5,6} = x_2x_5x_6 - x_2 - x_6 \) and \( p_{5,6} = x_5x_6 - 1 \). In a similar way, we can get that
\[
d(\mathcal{M}, X) = (x_1 \cdot p_{2,5,6} - p_{5,6}) \cdot d(\mathcal{M}_6, X) - p_{5,6} \cdot d(\mathcal{M}_4, X) - p_{5,6} \cdot d(\mathcal{M}_5, X),
\]
which gives us an expressing of \( d(\mathcal{M}, X) \) in terms of minors associated to some minimal 2-matchings of \( C \) of size 6.

Next result is fundamental identity in the study of the critical ideals of trees, which prove that the first non trivial critical ideal of a tree is the \( \nu_2(T) \)-critical ideal.

Theorem 3.8. If \( T \) is a tree, then \( \gamma(T) = \nu_2(T) \).

Proof. Let \( \mathcal{M} \) be a maximum 2-matching of \( T \). By 3.2, \( d(\mathcal{M}, X) = 1 \) and since \( d(\mathcal{M}, X) \in I_{\nu_2(T)}(T, X) \) then \( I_{\nu_2(T)}(T, X) \) is trivial. Thus, we only need to prove that \( I_{\nu_2(T)+1}(T) \) is non-trivial. We will use induction in the number of vertices of the tree. It is not difficult to check that the result is true for all the trees with less or equal to four vertices, therefore we can assume that \( |V(T)| \geq 5 \). Let \( k = \nu_2(T) + 1 \) and \( v \in V(T) \). By 3.2 Claim 3.12
\[
I_k(T, X) \subseteq (x_v, I_{k-1}(T \setminus v, X), I_{k-2}(T \setminus v, X), I_k(T \setminus v, X)).
\]
Moreover, since \( I_k(T \setminus v, X) \subseteq I_{k-1}(T \setminus v, X) \subseteq I_{k-2}(T \setminus v, X) \), then \( I_k(T, X) \subseteq (x_v, I_{k-2}(T \setminus v, X)) \). By induction hypothesis \( \gamma(T \setminus v) = \nu_2(T \setminus v) \) for all \( v \in V(T) \). If we assume that \( I_k(T, X) \) is trivial, then \( I_{k-2}(T \setminus v, X) \) is trivial and therefore
\[
\nu_2(T) - 1 = k - 2 \leq \gamma(T \setminus v) = \nu_2(T \setminus v) \text{ for all } v \in V(T);
\]
a contradiction to Lemma 2.3.

As a consequence we get that \( P_n \) is the only one simple graph with \( n \) vertices and \( \gamma(G) = n - 1 \).

Corollary 3.9. If \( G \) is a simple graph in \( n \) vertices, then \( \gamma(G) = n - 1 \) if and only if \( G = P_n \).

Proof. \((\Rightarrow)\) If \( G = P_n \), by Theorem 3.8, \( \gamma(G) = \nu_2(G) = \nu_2(P_n) = n - 1 \). \((\Leftarrow)\) Let \( G \) be a graph with \( n \) vertices and \( \gamma(G) = n - 1 \). Since \( I_{\nu_2(G)}(G, X) = \{ 1 \} \), by 3.7 Proposition 3.7 the critical group of \( G \) must be trivial. Then by the Kirchhoff’s Matrix Tree Theorem [3 Theorem 6.2], \( G \) is a tree. By Theorem 3.8 we get that \( \nu_2(G) = n - 1 \). Thus, exist a 2-matching \( \mathcal{M} \) of \( T \) with size \( n - 1 \). Let \( P_{n_1}, \ldots, P_{n_s} \) be paths on \( G \) such that \( \mathcal{M} = E(P_{n_1}) \cup \cdots \cup E(P_{n_s}) \). Since \( G \) has \( n \) vertices,
\[
n \geq |V(G)| = |V(P_{n_1})| + \cdots + |V(P_{n_s})| = |E(P_{n_1})| + \cdots + |E(P_{n_s})| + s = |\mathcal{M}| + s = n - 1 + s,
\]
hence \( s = 1 \) and then \( n = |\mathcal{M}| + 1 = |E(P_{n_1})| + 1 = n_1 \). Thus, since \( G \) contains a path with \( n \) vertices, \( G = P_n \).
4. Gröbner basis of critical ideals

Usually the theory of Gröbner basis deals with ideals in a polynomial ring over a field. However, in this section we deal with ideals in a polynomial ring over the integers. There exists a theory of Gröbner basis over almost any kind of rings.

We recall some basic concepts on Gröbner basis, for more details see [1]. First, let \( \mathcal{P} \) be a principal ideal domain. A monomial order or order term in the polynomial ring \( R = \mathcal{P}[x_1, \ldots, x_n] \) is a total order \( \prec \) in the set of monomials of \( R \) such that

(i): \( 1 \prec x^\alpha \) for all \( \mathbf{0} \neq \alpha \in \mathbb{N}^n \), and

(ii): if \( x^\alpha \prec x^\beta \), then \( x^{\alpha+\gamma} \prec x^{\beta+\gamma} \) for all \( \gamma \in \mathbb{N}^n \),

where \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

Now, given a monomial order \( \prec \) and \( p \in R \), let \( \text{lt}(p) \), \( \text{lp}(p) \), and \( \text{lc}(p) \) be the leading term, the leading power, and the leading coefficient of \( p \), respectively. Given a subset \( S \) of \( R \) its leading term ideal is

\[ \text{Lt}(S) = \langle \text{lt}(s) \mid s \in S \rangle. \]

A finite set of nonzero polynomials \( B = \{b_1, \ldots, b_s\} \) of an ideal \( I \) is called a Gröbner basis of \( I \) with respect to an order term \( \prec \) if \( \text{Lt}(B) = \text{Lt}(f) \). Moreover, it is called reduced if \( \text{lc}(b_i) = 1 \) for all \( 1 \leq i \leq s \) and no nonzero term in \( b_i \) is divisible by any \( \text{lp}(b_j) \) for all \( 1 \leq i \neq j \leq s \).

A good characterization of Gröbner basis is given in terms of the so called S-polynomials.

**Definition 4.1.** Let \( f, f' \) be polynomials in \( \mathcal{P}[X] \) and \( B \) a set of polynomials in \( \mathcal{P}[X] \). We say that \( f \) reduces strongly to \( f' \) modulo \( B \) if

- \( \text{lt}(f') \prec \text{lt}(f) \), and
- there exist \( b \in B \) and \( h \in \mathcal{P}[X] \) such that \( f' = f - hb \).

Moreover, if \( f^* \in \mathcal{P}[X] \) can be obtained from \( f \) in a finite number of reductions, we write \( f \rightarrow_B f^* \).

That is, if \( f = \sum_{j=1}^l p_i b_i f^* \) with \( p_i \in \mathcal{P}[X] \) and \( \text{lt}(p_i b_i) \neq \text{lt}(p_j b_k) \) for all \( j \neq k \), then \( f \rightarrow_B f^* \).

Now, given \( f \) and \( g \) polynomials in \( \mathcal{P}[X] \), their \( S \)-polynomial, denoted by \( S(f, g) \), is given by

\[ S(f, g) = \frac{c}{c_f X_f} X g - \frac{c}{c_g X_g} f, \]

where \( X_f = \text{lt}(f) \), \( c_f = \text{lc}(f) \), \( X_g = \text{lt}(g) \), \( c_g = \text{lc}(g) \), \( X = \text{lcm}(X_f, X_g) \), and \( c = \text{lcm}(c_f, c_g) \).

The next lemma, known as Buchberger’s criterion, gives us an useful criterion for checking whether a set of generators of an ideal is a Gröbner basis.

**Lemma 4.2.** Let \( I \) be an ideal of polynomials over a PID and \( B \) be a generating set of \( I \). Then \( B \) is a Gröbner basis for \( I \) if and only if \( S(f, g) \rightarrow_B 0 \) for all \( f \neq g \in B \).

In this paper we only work with the so called degree lexicographic order.

**Definition 4.3.** Let \( x^\alpha \) and \( x^\beta \) be two monomials on \( \mathcal{P}[x_1, \ldots, x_n] \), then \( x^\alpha \prec x^\beta \) whenever

- \( \alpha_1 + \cdots + \alpha_n < \beta_1 + \cdots + \beta_n \),

- or \( \alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n \) and exist \( i = 1, \ldots, n \) such that
  \[ \alpha_1 = \beta_1, \ \alpha_2 = \beta_2, \ldots, \ \alpha_{i-1} = \beta_{i-1} \ \text{and} \ \alpha_i < \beta_i. \]

In this section we prove that if \( T \) is a tree with \( n \) vertices, then \( \{d(M, X) \mid M \in \mathcal{V}_2^t(T^d, n - 1)\} \) is not only a generating set this is also a Gröbner basis for \( I_{n-1}(T) \). First we prove that for any \( 1 \leq j \leq n \) strong reduction by \( \mathcal{V}_2(T^d, j) \) is equivalent with strong reduction by \( \mathcal{V}_2^t(T^d, j) \).
Lemma 4.5. Thus, if

Proof. Suppose that \( d(M, X) \in V_2(T^t, j) \) and \( h(x) \in \mathbb{Z}[X_T] \) are such that \( g(x) = f(x) - h(x)d(M, X) \) and \( x_g \prec x_f \). If \( M \) is minimal, then there is nothing left to prove. On the other hand, if \( M \) is not minimal, then according to Theorem 3.6, there exists \( N_1, \ldots, N_s \in V_2^s(T^t, j) \) and \( t_1(x), \ldots, t_s(x) \in \mathbb{Z}[X_T] \) such that \( d(M, X) = t_1(x)d(N_1, X) + \cdots + t_s(x)d(N_s, X) \). Thus

\[
g(x) = f(x) - \sum_{i=1}^{s} t_i(x)p(N_i, X)h(x).
\]

Following the proof of Theorem 3.6, we can ensure that for each \( i = 1, \ldots, s - 1 \)

\[
\text{lt}(t_i(x)p(N_i, X)) < \text{lt}(t_{i+1}(x)p(N_{i+1}, X)).
\]

Thus, if

\[
\begin{align*}
f_1(x) &= f(x) - t_1(x)p(N_1, X)h(x), \\
f_2(x) &= f_1(x) - t_2(x)p(N_2, X)h(x), \\
&\vdots \\
f_s(x) &= f_{s-1}(x) - t_s(x)p(N_s, X)h(x),
\end{align*}
\]

then \( x_{f_s} < \cdots < x_{f_1} < x_f \). Therefore \( f(x) \rightarrow V_2^s(T^t) f_1(x) \rightarrow V_2^s(T^t) \cdots \rightarrow V_2^s(T^t) f_s(x) = g(x). \)

Now, before to proceed to deal with the reduction of S-polynomials, we begin with the reduction of a monomial and a minor of size \( n - 1 \). In the next, if \( e_1, e_2 \) are two different edges in \( T \), then \( P(e_1, e_2) \) is the unique path in \( T \) that join \( e_1 \) and \( e_2 \).

Lemma 4.5. If \( T \) is a tree and \( P \) is a not empty path of \( T \), then

\[
x_P d(T \setminus P, X) = d(T, X) + \sum_{e \in E(N_T(P))} d(T \setminus V(e), X) + \sum_{(e_1, e_2) \in \Lambda} \frac{x_P(e_1, e_2)}{x_{e_1}x_{e_2}} d(T \setminus P(e_1, e_2), X),
\]

where \( \Lambda = \{(e_1, e_2) \in V_1(N_T(P)) | e_1, e_2 \in E(N_T(P)) \} \).

Proof. Let \( S = E(N_T(P)) \). Clearly \( V(P) \) is a free set of \( T \setminus S \). Thus, by Lemma 3.4

\[
x_P d(T \setminus P, X) = \sum_{\mu \in V_1(S), |\mu| \leq 1} d(T \setminus V(\mu), X) + \sum_{\mu \in V_1(S), |\mu| \geq 2} d(T \setminus V(\mu), X).
\]

Each \( \mu \in V_1(S) \) with \( |\mu| = 2 \) is a member in \( \Lambda \). If \( E_\mu \) is the neighborhood of \( V(P(\mu))/V(\mu) \) in \( T[V(P(\mu))/V(\mu)] \), then \( \{\mu \in V_1(N_T(P)) | |\mu| \geq 2\} = \bigcup_{\mu \in \Lambda} \{\mu \cup \rho | \rho \in V_1(E_\mu)\} \). This relation allow us to write

\[
\sum_{\mu \in V_1(S), |\mu| \geq 2} d(T \setminus V(\mu), X) = \sum_{\mu \in \Lambda} \sum_{\rho \in V_1(E_\mu)} d(T \setminus V(\mu \cup \rho), X).
\]

For each \( \mu \in \Lambda \) we apply Lemma 3.4 in \( T \setminus V(\mu) \) and \( E_\mu \) to get that

\[
\sum_{\rho \in V_1(E_\mu)} d(T \setminus V(\mu \cup \rho), X) = \frac{x_P(\mu)}{x_\mu} d(T \setminus P(\mu), X).
\]

\[\square\]
Remark 4.6. Note that a 2-matching $M$ has size $n - 1$ if and only if $T \setminus \ell(M)$ is a path (possibly with size 0). Thus, $T[M] = T \setminus P$ for some path $P$ and $d(M, X) = d(T \setminus P, X)$. Conversely, for each path $P$, $T \setminus P = T[M]$ for some $M \in V_2(T', n - 1)$.

Now we deal with other case of the product of a monomial and a minor of size $n - 1$. Suppose that $P$ and $Q$ are non empty paths of $T$ with $Q \subset P$. Then $P \setminus Q$ is composed of one or two paths, which we call $P_1$ and $P_r$ ($P_r$ could be empty). Let $L = E(N_{T \setminus Q}(P_1))$ and $R = E(N_{T \setminus Q}(P_r))$.

Proposition 4.7. Let $P$ be a path in a tree $T$ and $Q$ a not empty subpath of $P$. If $L$ and $R$ are defined as above, then

$$\frac{x_P}{x_Q} d(T \setminus P, X) = d(T \setminus Q, X) + \sum_{e \in L} \frac{x_{P(e, Q)}}{x_e x_Q} d(T \setminus P(e, Q), X)$$

$$+ \sum_{e \in R} \frac{x_{P(e, Q)}}{x_{Q} x_e} d(T \setminus P(e, Q), X) + \sum_{e_l \in L, e_r \in R} \frac{x_{P(e_l, e_r)}}{x_{e_l} x_{Q} x_{e_r}} d(T \setminus P(e_l, Q, e_r), X).$$

Proof. Set $T' = T \setminus Q$. As $L \cup R$ is the set of edges of $N_T(V(P) \setminus V(Q))$ and $V(P) \setminus V(Q)$ is free in $T' \setminus S = T \setminus P$, by Lemma 3.4

$$\frac{x_P}{x_Q} d(T \setminus P, X) = \sum_{\nu \in V_1(L \cup R)} d(T' \setminus V(\nu), X).$$

For each $e \in L$ let $P(e, Q)$ be the path in $T$ that join the vertices in $e$ and $Q$ and set $V_{e, Q} = V(P(e, Q)) \setminus (V(e) \cup V(Q))$. If we set $S_{e, Q} = \{uv \in E(T) | u, v \in V_{e, Q}\}$, then $S_{e, Q}$ is a set of edges on $T_{e, Q} = T \setminus (V(e) \cup V(Q))$. Thus by Lemma 3.4

$$d(T_{e, Q} \setminus S_{e, Q}, X) = \sum_{\nu \in V_1(S_{e, Q})} d(T_{e, Q} \setminus V(\nu), X).$$

Since, $T_{e, Q} \setminus S_{e, Q} = T \setminus P(e, Q) \cup V_{e, Q}$ and $V_1(L) = \{\emptyset\} \cup_{e \in L} \{\{e\} \cup V_1(S_{e, Q})\}$ thus

$$\sum_{\nu \in V_1(L) \setminus \{\emptyset\}} d(T' \setminus V(\nu), X) = \sum_{e \in L} \sum_{\nu \in V_1(S_{e, Q})} d(T' \setminus V(\{e\} \cup \nu), X)$$

$$= \sum_{e \in L} \sum_{\nu \in V_1(S_e)} d(T_{e, Q} \setminus V(\nu), X) = \sum_{e \in L} x_{V_{e, Q}} d(T \setminus P(e, Q), X).$$

In the same way we get the expression that involve $V_1(R)$.

Set $L \cup R$ as the (non-empty) matchings on $L \cup R$ that intersect both $L$ and $R$. For each $e_l \in L$ and $e_r \in R$ let $P(e_l, e_r)$ be the only path that join $e_l$ and $e_r$. If we set $S_{e_l, e_r} = V(P(e_l, e_r)) \setminus (V(e_l) \cup V(Q) \cup V(e_r))$ and $S_{e_l, e_r} = \{uv \in E(T) | u, v \in V_{e_l, e_r}\}$, then $S_{e_l, e_r}$ is a set of edges on $T_{e_l, e_r} = T \setminus (V(e_l) \cup Q \cup V(e_r))$. By Lemma 3.4

$$d(T_{e_l, e_r} \setminus S_{e_l, e_r}, X) = \sum_{\nu \in V_1(S_{e_l, e_r})} d(T_{e_l, e_r} \setminus V(\nu), X).$$

If we notice that $T_{e_l, e_r} \setminus S_{e_l, e_r} = T \setminus P(e_l, e_r) + V_{e_l, e_r}, L \cup R = \bigcup_{e_l \in L} \bigcup_{e_r \in R} \{\{e_l, e_r\} \cup V_1(S_{e_l, e_r})\}$ and that for each $\nu \in V_1(S_{e_l, e_r})$, $T' \setminus V(\{e_l, e_r\} \cup \nu) = T_{e_l, e_r} \setminus V(\nu)$, then we get

$$\sum_{\nu \in L \cup R} d(T' \setminus V(\nu), X) = \sum_{e_l \in L} \sum_{e_r \in R} \sum_{\nu \in V_1(S_{e_l, e_r})} d(T_{e_l, e_r} \setminus V(\nu), X)$$

$$= \sum_{e_l \in L} \sum_{e_r \in R} d(T_{e_l, e_r} \setminus S_{e_l, e_r}, X) = \sum_{e \in L \cup R} x_{V_{e, Q}} d(T \setminus P(e, Q), X).$$

This complete the proof of the theorem as $V_1(L \cup R) = V_1(L) \cup V_1(R) \cup L \cup R$. 

$\Box$
By Lemma 4.5, is a reduced Groebner basis for \( B_{n-1} = \{ d(M, X) \mid M \in V_2^\ast(T^\ell, n-1) \} \) is a reduced Groebner basis for \( I_{n-1}(T, X) \) with respect to the degree lexicographic order.

**Proof.** By Proposition 4.2 and Theorem 3.6, we only need to prove that \( S(f, g) \to B_{n-1} \) 0 for all \( f, g \in B_{n-1} \). If \( M_1, M_2 \in V_2^\ast(T^\ell, n-1) \), then there are two paths \( P_1 \) and \( P_2 \) in \( T \) such that \( d(M_i, X) = d(T \setminus P_i, X) \).

We can suppose that either \( P_1 \) or \( P_2 \) is empty and that \( P_1 \neq P_2 \), thus

\[
S(d(M_1, X), d(M_2, X)) = x_{P_2 \setminus P_1} d(T \setminus P_1, X) - x_{P_1 \setminus P_2} d(T \setminus P_2, X),
\]

where \( P_1^c = V(T) \setminus V(P_1) \). If \( P_1 \cap P_2 = \emptyset \), then

\[
S(d(M_1, X), d(M_2, X)) = x_{P_1} d(T \setminus P_1, X) - x_{P_2} d(T \setminus P_2, X).
\]

By Lemma 4.5, \( S(d(M_1, X), d(M_2, X)) \to_G 0 \).

If \( P_1 \cap P_2 \neq \emptyset \), then this must be a path. If we set \( Q = P_1 \cap P_2 \), then

\[
S(d(M_1, X), d(M_2, X)) = \frac{x_{P_1}}{x_Q} \left( d(T \setminus P_1, X) - \frac{x_{P_2}}{x_Q} d(T \setminus P_2, X) \right),
\]

and by Proposition 4.7, \( S(d(M_1, X), d(M_2, X)) \to_G 0 \). \( \square \)

Next result gives us an alternative more compact description of minimal 2-matchings of \( T^\ell \) of size \( n-1 \).

**Proposition 4.9.** If \( P_{u,v} \) is the unique path on \( T \) that joins the vertices \( u \) and \( v \), then

\[
V_2^\ast(T^\ell, n-1) = \{ P_{u,v} \cup \{ u w \mid w \notin V(P_{u,v}) \} \mid u \text{ and } v \text{ are leaves of } T \}.
\]

**Proof.** If \( P \) is any path on \( T \), then by Proposition 2.10, \( M = P \cup \{ u w \mid w \notin V(P) \} \) is a minimal 2-matching of size \( n-1 \) of \( T^\ell \). Therefore, we need to prove that if \( M \in V_2^\ast(T^\ell, n-1) \), then \( M = P_{u,v} \cup \{ u w \mid w \notin V(P_{u,v}) \} \) for some \( u, v \) leaves of \( T \).

Let \( M \in V_2^\ast(T^\ell, n-1) \). If \( M \) has no edges, that is, \( M \) has \( n-1 \) loops, then let \( u \in V(T) \) such that \( u \notin V(M) \) and \( v \in V(T) \) such that \( u v \in E(G) \). Since \( M' = \{ u v \} \cup (M \setminus \{ u v \}) \) has size \( n-1 \) and that \( \ell(M') \subseteq \ell(M) \), then \( M \) is not minimal. Thus, \( M \) contains at least one path. Furthermore, since \( T \) has \( n \) vertices, \( M \) has exactly a path. Let \( P = M \setminus \ell(M) \). If \( P' \) is a path on \( T \) such that \( P \subseteq P' \), then \( N = P' \cup \{ u w \mid w \notin V(P') \} \) is a 2-matching of size \( n-1 \) and \( \ell(N) \subseteq \ell(M) \); a contradiction to the minimality of \( M \). Therefore, \( P \) is maximal in the sense that \( P \) is equal to \( P_{u,v} \) for some leaves \( u, v \) and \( M = P_{u,v} \cup \{ u w \mid w \notin V(P_{u,v}) \} \). \( \square \)

**Remark 4.10.** If \( T \) is a tree with \( n \) vertices and \( m \) leaves, then \( B_{n-1} \) contains \( \binom{m}{2} \) polynomials.

We finish this section with a conjecture about the minimality of the generating sets found in Theorem 3.6. In the next section we present several examples of the validity of this conjecture.

**Conjecture 4.11.** If \( T \) is a tree and \( 1 \leq j \leq n \), then

\[
B_j = \{ d(M, X) \mid M \in V_2^\ast(T^\ell, j) \}
\]

is a reduced Groebner basis for \( I_j(T, X) \) with respect to the degree lexicographic order.

We finish this section with an example of how to get the \( n-1 \) critical ideal of a tree with \( n \) vertices.
Example 4.12. Let $n_1, n_2, n_3 \geq 2$ and $J(n_1, n_2, n_3)$ be the tree with a vertex $r$ as root and three paths $P_{n_1}, P_{n_2}$, and $P_{n_3}$ from it of lengths $n_1, n_2$ and $n_3$, see Figure 3.

If $n = n_1 + n_2 + n_3 - 2 = |J(n_1, n_2, n_3)|$, then by Theorem 4.8

$$I_{n-1}(J(n_1, n_2, n_3), X) = \langle \det(P_{n_1} \setminus r, X), \det(P_{n_2} \setminus r, X), \det(P_{n_3} \setminus r, X) \rangle$$

In particular $I_9(J(5, 4, 3), X) = \langle x_1x_2x_3x_4 - x_1x_2 - x_3x_4 - x_1x_4 + 1, x_6x_7x_8 - x_6 - x_8, x_9x_{10} - 1 \rangle$.

5. Applications on critical group

Although the critical group of a tree is always trivial, critical ideals of trees can be used to obtain information about the structure of critical groups associated to a large class of interesting graphs. This section is devoted to present tree applications of the results of sections 3 and 4.

5.1. Trees of depth one and two. We begin with the trees of depth one or stars, which are (along with the paths) the simplest trees. Note the star with two leaves is $P_3$.

Theorem 5.1. Let $S(m)$ be the star with root $r$ and leaves $\{1, 2, \ldots, m\}$. If $m \geq 3$, then $\gamma(S(m)) = 2$, for each $1 \leq k \leq m - 2$

$$I_{2+k}(S(m), X) = \left\{ \prod_{s=1}^{k} x_{j_s}, \ 1 \leq j_1 < \cdots < j_k \leq m \right\},$$

and $I_{m+1} = \langle x_r x_1 \cdots x_m - x_1 \cdots x_{m-1} - x_1 \cdots x_{m-2} x_m - \cdots - x_2 \cdots x_m \rangle$.

Proof. Is pretty clear that $\nu_2(S(m)) = \gamma(S(m)) = 2$ and since $|V(S(m))| = m + 1$, there are $m - 1$ non-trivial critical ideals of $S(m)$. Moreover, it is not difficult to see that for each $1 \leq k \leq m - 2$,

$$\mathcal{V}_2^*(S(m)^k, k + 2) = \left\{ \{j_s j_s\}_{s=1}^{k} \cup \{p_1 r, p_2\}, \ 1 \leq p_1 < p_2 \leq m \text{ and for all } 1 \leq s \leq k, j_s \in \{1, \ldots, m\} \setminus \{p_1, p_2\} \right\}.$$

Thus, a straightforward application of Theorem 3.6 gives the result about $I_{2+k}(S(m), X)$. Finally, for $I_{m+1}(S(m), X) = \langle \det(S(m), X) \rangle$ we use Theorem 3.4 with $S = E(S(m))$. \hfill $\Box$

Now, we continue with the trees with depth two. Let $s \geq 2$ and $T = T_2(m_1, \ldots, m_s)$ be the tree of depth two with $r$ as the root and $s$ branches with $m_i$ leaves each, see Figure 4. Note that $T_2(\emptyset)$ consists only of the root. If $m_i \geq 2$ for all $1 \leq i \leq s$, then it is not difficult to see that $\nu_2(T) = 2s$. Since $n = |V(T)| = 1 + s + \sum_{i=1}^{s} m_i$, then $T$ has $n - 2s = \sum_{i=1}^{s} m_i - s + 1$ non-trivial critical ideals.

In order to describe the non-trivial critical ideals we need characterize the minimal 2-matchings of $T^f$. Before to do this, we introduce some notation: let $1, \ldots, s$ be the children of the root $r$ of $T$, for each $1 \leq t \leq s$ let $t_1, \ldots, t_m$ be the children of $t$, $S_t$ the $t$ branch of $T$ ie. the star induced by the vertices $\{t, t_1, \ldots, t_m\}$ (see Figure 3) and $V_i$ will denote a subset of $\{t_1, \ldots, t_m\}$. We use $P(u, v)$ to denote the edges of the path joining the vertices $u$ and $v$ of $T$.
Lemma 5.2. If $T = T_2(m_1, \ldots, m_s)$ and $\mathcal{M} \in \mathcal{V}_2(T^\ell, 2s + l)$ with $l \geq 1$, then

$$
\mathcal{M} = \begin{cases} 
P(i_{1j_1}, i_{2j_2}) \cup \bigcup_{p=3}^{s} P(i_{p_{jp}, i_{p_{kp}}}) \cup V_i^\ell \cup \cdots \cup V_i^\ell & \text{where for each } 1 \leq p \leq s \\
1 \leq j_p, k_p \leq m_i \text{ and } j_p \neq k_p, \\
P(i_{1j_1}, r) \cup \bigcup_{p=2}^{s} P(i_{p_{jp}, i_{p_{kp}}}) \cup V_i^\ell \cup \cdots \cup V_i^\ell & \text{where for each } 1 \leq p \leq s \\
1 \leq j_p, k_p \leq m_i \text{ and } j_p \neq k_p, \\
\bigcup_{p=1}^{s} P(i_{p_{jp}, i_{p_{kp}}}) \cup V_i^\ell \cup \cdots \cup V_i^\ell & \text{where for each } 1 \leq p \leq s \\
1 \leq j_p, k_p \leq m_i \text{ and } j_p \neq k_p, \\
\bigcup_{p=1}^{q} P(i_{p_{jp}, i_{p_{kp}}}) \cup \{rr\} \cup V_i^\ell \cup \cdots \cup V_i^\ell & \text{where } 0 \leq q < s \text{ and for each } 1 \leq p \leq q \\
1 \leq j_p, k_p \leq m_i \text{ and } j_p \neq k_p,
\end{cases}
$$

Proof. Let $I = V(\ell(\mathcal{M})) \cap \{1, \ldots, s\}$. If $I = \emptyset$, then the minimality of $\mathcal{M}$ implies that the degree on each of the vertices $1, \ldots, s$ is 2. Thus, $E(\mathcal{M})$ is a maximum 2-matching of $T$ so there are $1 \leq j_p, k_p \leq m_i$ with $j_p \neq k_p$ such that

$$
E(\mathcal{M}) = P(i_{1j_1}, i_{2j_2}) \cup \bigcup_{p=3}^{s} P(i_{p_{jp}, i_{p_{kp}}}) \cup P(i_{1j_1}, r) \cup \bigcup_{p=2}^{s} P(i_{p_{jp}, i_{p_{kp}}}) \cup \bigcup_{p=1}^{s} P(i_{p_{jp}, i_{p_{kp}}}).
$$

In the first two cases $\ell(\mathcal{M}) \subseteq V_i^\ell \cup \cdots \cup V_s^\ell$ and in the third one $\ell(\mathcal{M}) \subseteq \{rr\} \cup V_i^\ell \cup \cdots \cup V_s^\ell$.

If $I \neq \emptyset$ the minimality of $\mathcal{M}$ implies that $\mathcal{M}$ has degree degree 2 in $r$. Thus, if $rr \not\in \ell(\mathcal{M})$, then exists $i_1, i_2 \in \{1, \ldots, n\}$ such that $i_1r, i_2r \in E(\mathcal{M})$, and since $\mathcal{M}$ is minimal, the degree on each of the vertices $1, \ldots, s$ must also be 2. This ensure that exist $2 \leq q < s$ such that

$$
E(\mathcal{M}) = P(i_{1j_{1}}, i_{2j_{2}}) \cup \bigcup_{p=3}^{q} P(i_{p_{jp}, i_{p_{kp}}}),
$$

and thus $V(\ell(\mathcal{M})) = V_i^\ell \cup \cdots \cup V_i^\ell \cup S_{i_{1}}^\ell \cup \cdots \cup S_{i_{s}}^\ell$.

Finally, if $rr \in \ell(\mathcal{M})$ similar arguments show that $E(\mathcal{M}) = \bigcup_{p=1}^{q} P(i_{p_{jp}, i_{p_{kp}}})$ for some $0 \leq q < s$. Thus, $\ell(\mathcal{M}) = V_i^\ell \cup \cdots \cup V_i^\ell \cup S_{i_{1}}^\ell \cup \cdots \cup S_{i_{s}}^\ell$. \hfill \Box

Lemma 5.2 give us a complete description of the critical ideals of $T = T_2(m_1, \ldots, m_s)$. For example, if $\mathcal{M} \in \mathcal{V}_2^s(T, 2s + 1)$, then $|E(\mathcal{M})| = 2s$ and $\ell(\mathcal{M}) \in V_i^\ell$ for some $i$, $1 \leq i \leq s$. Thus,

$$
I_{2s+1}(T, X) = \langle x_v | v \text{ is a leaf of } T \rangle
$$

In the next we give a description of the critical ideals of some trees of depth two with tree branches.
According to Proposition 5.2, the critical ideals of $T_2(m_1, m_2, m_3)$ has two types of generators: monomials and products of a monomial with the determinant of a tree of depth one. Thus, for each $I \subseteq \{1, 2, 3\}$, let $Q_I = \det(L|_{i \in I} S_i, X)$. Also, let

$$P_{r,s,t}^i = \left\{ \prod_{l=1}^{r_1} x_{1i_l}, \prod_{l=1}^{s_1} x_{2i_l}, \prod_{l=1}^{t_1} x_{3i_l} \mid 1 \leq i_1 < \cdots < i_r \leq m_1, 1 \leq j_1 < \cdots < j_s \leq m_2, 1 \leq k_1 < \cdots < k_t \leq m_3, r_1 \leq r, s_1 \leq s, t_1 \leq t, r_1 + s_1 + t_1 = i. \right\}$$

for all $r, s, t \geq 0$. Moreover, by convention $P_{r,s,t}^i = \emptyset$ when either $i$, $r$, $s$, or $t$ is negative.

**Example 5.3.** Let $T = T_2(3, 4, 5)$ be the tree with tree branches, the first one with tree leaves, the second one with four leaves and a third one with five leaves. Since $n = |V(T)| = 16$ and $v_2(T) = 6$, then $T$ has 10 non-trivial critical ideals. Furthermore, by Theorem 3.6 and Proposition 5.2,

$$I_{6+i}(T, X) = \begin{cases} \langle x_r \cdot P_{1,2,3}^{i-1}, P_{2,3,4}^i, P_{0,2,3}^{i-3} \cdot Q_1, P_{1,0,3}^{i-4} \cdot Q_2, P_{1,2,0}^{i-5} \cdot Q_3, P_{0,0,3}^{i-6} \cdot Q_{1,2}, P_{0,2,0}^{i-7} \cdot Q_{1,3} \rangle & \text{if } 1 \leq i \leq 7, \\ \langle P_{2,3,4}^8, P_{0,2,3}^5 \cdot Q_1, P_{1,0,3}^4 \cdot Q_2, P_{1,2,0}^3 \cdot Q_3, P_{0,0,3}^2 \cdot Q_{1,2}, P_{0,2,0}^1 \cdot Q_{1,3}, P_{1,2,3}^1 \rangle & \text{if } i = 8, \\ \langle P_{0,0,3}^9 \cdot Q_{1,2}, P_{0,2,0}^2 \cdot Q_{1,3}, P_{1,0,0}^1 \cdot Q_{2,3} \rangle & \text{if } i = 9. \end{cases}$$

Also, let $T = T_2(2, 2, m)$ be the tree of depth two with three branches, the first two with 2 leaves and the third one with $m$ leaves. Since $n = |V(T)| = m + 8$ and $v_2(T) = 6$, then $T$ has $m + 2$ non-trivial critical ideals. By Theorem 3.6 and Proposition 5.2,

$$I_{6+i}(T, X) = \begin{cases} \langle x_r \cdot P_{0,0,3}^{i-1}, P_{1,1,m-1}^i, P_{0,0,2}^{i-2} \cdot \{Q_1, Q_2\}, P_{0,0,1,3}^{i-3} \cdot Q_{1,2} \rangle & \text{if } 1 \leq i \leq m - 1, \\ \langle P_{0,0,m-2}^m \cdot Q_1, P_{0,0,m-2}^{m-2} \cdot \{Q_1, Q_2\}, P_{0,0,m-3}^{m-3} \cdot Q_{1,2}, Q_3 \rangle & \text{if } i = m, \\ \langle P_{0,0,m-2}^m \cdot Q_{1,2}, P_{1,1,0}^1 \cdot Q_3 \rangle & \text{if } i = m + 1. \end{cases}$$

**5.2. Wired d-regular trees.** A wired tree is a graph obtained from a tree by collapsing its leaves to a singular vertex. The critical group of a wired regular tree (obtained from a regular tree) and some variants of them have been recently study on [3] [9] [8].

For $d \geq 3$, let $T(d, h)$ be the $d$-regular tree of depth $h$ and $T'(d, h)$ the tree obtained from $T(d, h)$ by deleting one of his principal branches. In other words, $T'(d, h)$ is a tree of depth $h$ in which the root has degree $d - 1$, and all the other vertices, except the leaves, has degree $d$. We begin by calculating the 2-matching numbers of $T'(d, h)$ and $T(d, h)$.
Lemma 5.4. If $h \geq 2$, then

$$\nu_2(T'(d, h)) = \begin{cases} 2(d-1)^{h+1} - (d-1) & \text{if } h \text{ is even}, \\ \frac{2(d-1)^{h+1} - 1}{(d-1)^2 - 1} & \text{if } h \text{ is odd}. \end{cases}$$

Proof. We will prove that if $\mathcal{M}$ is a maximum 2-matchings of $T'(d, h)$, then

$$|\mathcal{M}| = \begin{cases} (d-1)\nu_2(T'(d, h-1)) & \text{if } h \text{ is even}, \\ (d-1)\nu_2(T'(d, h-1)) + 2 & \text{if } h \text{ is odd}. \end{cases} \tag{5.1}$$

Moreover, if $r$ is the root of $T(d, h)$ and $h$ is odd, then $\deg\mathcal{M}(r) = 2$, that is, $r$ is saturated.

Since $|\mathcal{M}| = \nu_2(T'(d, h))$, the relations above leads to the fact that $\nu_2(T'(d, h+2)) = (d-1)^2\nu_2(T'(d, h)) + 2(d-1)$ for even $h$. This difference equations and the initial value $\nu_2(T'(d, 2)) = 2(d-1)$ leads to the formulae for odd $h$. The formulae for odd $h$ is a consequence of the even case.

To prove 5.1 we use induction on $h$. The base step, i.e. $T'(d, 2)$ and $T'(d, 3)$ can be done by inspection. Suppose that $\mathcal{M}$ is a maximum 2-matchings of $T'(d, h+1)$. Note that $T'(d, h+1)$ has $d-1$ copies of $T'(d, h)$ so $|\mathcal{M}| \geq (d-1)\nu_2(T'(d, h))$.

Suppose that $h+1$ is even. If $\mathcal{M}$ has just one edge on the root of $T'(d, h+1)$ then, let $w$ be the vertex connected to $r$ by an edge of $\mathcal{M}$. Let $T'_w(d, h)$ be the copy of $T'(d, h)$ inside $T'(d, h+1)$ rooted at $w$. As $\mathcal{M}_w = \mathcal{M} \cap T'_w(d, h)$ has degree at most one on $w$ and $h$ is odd, by the induction hypothesis it follows that $\mathcal{M}_w$ is not maximum on $T'_w(d, h)$. Thus,

$$|\mathcal{M}| < \nu_2(T'(d, h)) + 1 + (d-2)\nu_2(T'(d, h)) = (d-1)\nu_2(T'(d, h)) + 1,$$

which implies that $|\mathcal{M}| = (d-1)\nu_2(T'(d, h))$. The case in which $\mathcal{M}$ has degree 2 on $r$ can be treated in the same way.

Suppose that $h+1$ be odd. As $T'(d, h+1) \setminus r$ consist of $d-1$ copies of $T'(d, h)$ it follows that $\nu_2(T'(d, h+1)) \geq (d-1)\nu_2(T'(d, h)) + 2$. Even more, in each one of this copies there exist a 2-matchings of size $\nu_2(T'(d, h))$ with degree 0 on their root. The union of this 2-matchings is a 2-matchings of $T'(d, h+1)$ with degree 0 on $r$ and on their pendants vertices. Thus, $\nu_2(T'(d, h+1)) \geq (d-1)\nu_2(T'(d, h)) + 2$ and $\mathcal{M}$ must be composed of 2 edges at $r$ and a maximum 2-matchings in each copy of $T'(d, h)$.

Corollary 5.5. If $h \geq 3$, then

$$\nu_2(T(d, h)) = \frac{2(d-1)^h - 1}{d-2}$$

Proof. Since $\nu_2(T(d, h)) = \nu_2(T'(d, h)) + \nu_2(T'(d, h-1))$, the result it follows from Lemma 5.4.

The proof of Lemma 5.3 can be refined to describe the first non trivial critical ideal of $T'(d, h)$ and $T(d, h)$. Since the proofs for $T'(d, h)$ and $T(d, h)$ are very similar, we only present the proof for $T'(d, h)$.

Corollary 5.6. If $h \geq 3$ and $d \geq 4$, then

$$I_k(T'(d, h), X) = \langle \{x_v \mid v \text{ is a leaf of } T'(d, h)\}, I_{k'}(T'_v(d, h-2), X) \rangle,$$

where $k = \nu_2(T'(d, h)) + 1$ and $k' = \nu_2(T'(d, h-2)) + 1$. \qed
Proof. Each minimal 2-matching of \( T(d, h)^{t} \) of size \( \nu_{2}(T'(d, h)) + 1 \) consist of a maximum 2-matching \( \mathcal{M} \) of \( T(d, h) \) and a loop on one of the vertices not covered by \( \mathcal{M} \). Let \( j = 1 \) when \( h \) is odd and \( j = 0 \) when \( h \) is even. From the inductive description of the maximum 2-matching given in the proof of Lemma 5.4, it follows that the leaves of \( T^r(d, h - 1), \ldots, T^r(j + 1) \) are saturated. Then, if \( \mathcal{M} \in V_{2}^{r}(T'(d, h)^{t}, k) \) the loop can only be on the leaves of \( T^r(d, h), T^r(d, h - 2), \ldots, T^r(d, j) \). Furthermore, each leaf of \( T'(d, h), T^r(d, h - 2), \ldots, T^r(d, j) \) is free for some maximum 2-matching of \( T'(d, h) \). \( \square \)

Corollary 5.7. If \( h \geq 3 \) and \( d \geq 4 \), then

\[
I_{k}(T(d, h), X) = \langle \{x_{v} \mid v \text{ is a leaf of } T(d, h)\}, I_{k'}(T_{v}(d, h - 2), X) \rangle,
\]

where \( k = \nu_{2}(T(d, h)) + 1 \) and \( k' = \nu_{2}(T(d, h - 2)) + 1 \).

Let \( T(d, h) \) be the wired tree obtained from \( T(d, h) \). In [9], Toumparaky calculate the rank, the exponent and the order of the critical group \( K(T(d, h)) \) of \( T(d, h) \). It is not difficult to see that \( T(d, h) \) is equal to the multigraph obtained from adding a new vertex \( v \) to \( T(d, h - 1) \) and \( d - 1 \) edges between each leaf of \( T(d, h - 1) \) and \( v \). Thus, using Corollary 5.6 we can calculate easily the rank of \( K(T(d, h)) \).

Proposition 5.8. If \( d \geq 4 \), then the critical group of \( T(d, h) \) has rank \( (d - 1)^{h} \) and its first non trivial invariant factor is equal to \( d \).

Proof. Let \( n = |T(d, h - 1)| \) and \( f_{1}, \ldots, f_{n} \) be the invariant factors of \( K(T(d, h)) \). By [4] Proposition 3.7, for each \( i = 1, \ldots, n \), \( f_{1} \cdots f_{i} \) can be determined by evaluating the set of generators for \( I_{i}(T(d, h), X) \) and after that calculating its greatest common divisor. Since \( k = \nu_{2}(T(d, h - 1)) \), clearly \( f_{1} = \cdots = f_{k} = 1 \) and \( K(T(d, h - 1)) \) has rank \( n - k = (d - 1)^{h} \). Furthermore, since \( \deg_{T(d, h)}(v) = d \) for any leave of \( T'(d, h) \), then using Corollary 5.6 and induction on \( h \) we get that \( f_{k+1} = d \). \( \square \)

In [5], Levine describe the critical group of the multigraph \( T_{h} \) obtained from \( T'(d, h) \) by collapsing all the leaves to a single vertex \( v \) and adding an edge between \( v \) and the root of \( T'(d, h) \), which is very similar to \( T(d, h) \). Note that, at difference to \( T(d, h) \), the multigraph \( T_{h} \) is \( d \)-regular. In a similar way that in Proposition 5.8, we can use Corollary 5.6 to calculate the rank of \( K(T_{h}) \) by evaluating the first non trivial critical ideal of \( T'(d, h - 1) \).

Proposition 5.9. If \( d \geq 4 \), then the critical group of \( T_{h} \) has rank \( (d - 1)^{h} \). Furthermore, the first non trivial invariant factor is equal to \( d \).

Proof. Let \( v \) the vertex of \( T_{h} \) and \( T(d, h - 1) \) obtained by collapsing the leaves of \( T'(d, h) \). Since \( T_{h} \setminus v = T'(d, h - 1) \setminus v \), the result following similar arguments that those given in Proposition 5.8. \( \square \)

Unfortunately, the technique described by Levine in [5] can not be used to calculate the critical group of \( T(d, h) \). This technique consists on apply a recursive relation among the critical group of \( T_{h} \) and the critical groups of its principal branches [5] Proposition 4.3]. We found certain similarity between this recursive relation and the description of the first critical ideal of \( T'(d, h) \) given in Proposition 5.6. Thus, we not doubt that a deeper study of the critical ideals of \( T'(d, h) \) allow us to replicate all the results obtained by Levine [5] and Toumpakary [9].

5.3. Arithmetical trees. An arithmetical graph is a triplet \((G, d, r)\) given by a graph \( G \) and \( d, r \in \mathbb{Z}^{\nu(G)}_{+} \) such that \((\text{Diag}(d) - A)r = 0\), where \( A \) is the adjacency matrix of \( G \). Any graph \( G \) belongs to an arithmetical graph in a natural way, just taking \( d \) as its degree vector and \( r = (1, \ldots, 1)^{t} \). The matrix \( M = \text{Diag}(d) - A \) arise in algebraic geometry as an intersection matrix of degenerating curves, see [3] [7] and the references contained there for more details.
Given an arithmetical graph \((G,d,r)\), we define its critical group \(K(G,d,r)\) (also called the group of components) as the torsion part of \(\mathbb{Z}^{|V(G)|}/\text{Im}(M)\). In \([6]\), Lorenzini proved that the \(\mathbb{Z}\)-rank of \(K(G,d,r)\) is equal to \(n-1\). Furthermore, if the Smith Normal Form of \(M\) is \(\text{diag}(f_1, \ldots, f_{n-1}, 0)\), then \(K(G,d,r) = \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_{n-1}\). Since for each \(1 \leq j \leq n-1\), \(\prod_{i=1}^{j} f_i\) is the greatest common divisor of the \(j\)-minors of \(M\) and \(M = L(G,d)\), it follows that \(\langle \prod_{i=1}^{j} f_i \rangle\) is the generator of \(I_j(G,d)\), the \(j\)-critical ideal of \(G\) evaluated in \(d\).

Thus, the invariant factors of \(K(G,d,r)\) can be found as follows: First, find a set of generators of the critical ideals of \(G\). After that, we evaluate them on \(d\) and compute its greatest common divisor. For instance, consider the family of arithmetical graphs associated to the reduction of elliptic curves of Kodaira type \(I_5\). For any \(m \in \mathbb{N}\), let \(C_{5,m}\) the tree obtained by identifying the center of a star with two leaves with each leaf of the path \(P_{m+1}\), see Figure 5.

![Figure 5. The tree \(C_{5,m}\) and the two types of 2-matchings of size \(m+3\).](image)

Now, we will describe the critical ideals of \(C_{5,m}\). First, since \(V(C_{5,m}) \setminus \{v_1, v_3\}\) induces a path isomorphic to \(P_{m+3}\), it follows that \(\nu_2(C_{5,m}) \geq m+2\). Moreover, it is not difficult to check that \(\nu_2(C_{5,m}) = m+2\). Thus, by Theorem 3.8 \(\gamma(C_{5,m}) = m+2\) and \(C_{5,m}\) has only 3 non trivial critical ideals. The \(m+2\)-critical ideal is the determinant of the generalized Laplacian matrix. For simplicity, we will assume that \(m \geq 5\).

By Proposition 4.5 we get that
\[
I_{m+4}(C_{5,m},X) = \langle x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, P_{1,2} P_{7,8} - x_1 x_2 P_{9,8}, P_{3,4} P_{7,8} - x_3 x_4 P_{7,10} \rangle,
\]
where \(P_{i,j} = \det(P(v_i, v_j))\) and \(P(v_i, v_j)\) is the unique path in \(C_{5,m}\) that join the vertices \(v_i\) and \(v_j\). Note that, \(\det(C_{5,m} \setminus P(v_3,v_4)) = \mathcal{P}_{1,2} P_{7,8} - x_1 x_2 P_{9,8}\) and similarly in the case of \(\det(C_{5,m} \setminus P(v_1,v_2))\).

Finally, in Figure 5 are sketched the two types of minimal 2-matchings of \(C_{5,m}\) of size \(m+3\). Thus
\[
I_{m+3}(C_{5,m},X) = \langle x_1, x_2, x_3, x_4, P_{7,8} \rangle.
\]

Now, taking \(d_{5,m} = (2, \ldots, 2)\) and \(r_{5,m} = (1, 1, 1, 1, 2, \ldots, 2)^t\) we get that \((C_{5,m},d_{5,m},r_{5,m})\) is an arithmetical graph. Since \(\gamma(C_{5,m}) = m+2\), \(f_i = 1\) for all \(1 \leq i \leq m+2\). On the other hand, using Corollary 4.5 we get that the polynomial \(P_{i,j}\) evaluated in \(d = (2, \ldots, 2)\) is odd if and only if the path \(P(v_i, v_j)\) has an even number of vertices and \(P_{1,2}\) and \(P_{3,4}\) evaluated in \((2,2,2)\) are equal to 4. Thus, \(f_{m+3} = 1\) when \(m\) is odd, \(f_{m+3} = 2\) when \(m\) is even. Finally, since \(f_{m+3} = I_{m+4}(C_{5,m},(2, \ldots, 2)) = 4\), then
\[
K((C_{5,m},d_{5,m},r_{5,m})) = \begin{cases} 
\mathbb{Z}_2^2 & \text{if } m \text{ is even}, \\
\mathbb{Z}_4 & \text{if } m \text{ is odd}.
\end{cases}
\]

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