Cographs and 1-Sums

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Abstract

A graph that can be generated from $K_1$ using joins and 0-sums is called a cograph. We define a sesquicograph to be a graph that can be generated from $K_1$ using joins, 0-sums, and 1-sums. We show that, like cographs, sesquicographs are closed under induced minors. Cographs are precisely the graphs that do not have the 4-vertex path as an induced subgraph. We obtain an analogue of this result for sesquicographs, that is, we find those non-sesquicographs for which every proper induced subgraph is a sesquicograph.

Keywords Cographs · Join · 0-sum · 1-sum · Sesquicographs

Mathematics Subject Classification 05C40 · 05C75 · 05C83

1 Introduction

In this paper, we only consider finite and simple graphs. The notation and terminology follows [3] except where otherwise indicated. For graphs $G$ and $H$ having disjoint vertex sets, the 0-sum $G \oplus H$ of $G$ and $H$ is their disjoint union. A 1-sum $G \oplus_1 H$ of $G$ and $H$ is obtained by identifying a vertex of $G$ with a vertex of $H$. The join $G \join H$ of two disjoint graphs $G$ and $H$ is obtained from the 0-sum of $G$ and $H$ by joining every vertex of $G$ to every vertex of $H$. A cograph is a graph that can be generated from the single-vertex graph $K_1$ using the operations of join and 0-sum. We define a graph to be a sesquicograph if it can be generated from $K_1$ using the operations of join, 0-sum, and 1-sum. The class of cographs has been extensively studied over the last fifty years (see, for example, [2, 4, 9]). Many hard problems on graphs in general have been efficiently solved for cographs due to its simple structural decomposition. Due to the following characterization, cographs are also called $P_4$-free graphs [1].
Theorem 1.1 A graph $G$ is a cograph if and only if $G$ does not contain the path $P_4$ on four vertices as an induced subgraph.

Since we consider only simple graphs in this paper, when we write $G/e$ for an edge $e$ of a graph $G$, we mean the simple graph obtained from the multigraph that results from contracting the edge $e$ by deleting all but one edge from each class of parallel edges. An induced minor of a graph $G$ is a graph $H$ that can be obtained from $G$ by a sequence of operations each consisting of a vertex deletion or an edge contraction. In Section 2, we show that every induced minor of a sesquicograph is a sesquicograph. In addition, we provide an alternative definition of a sesquicograph in terms of the vertex connectivities of its induced subgraphs and their complements. The graph obtained from a 6-cycle by adding a chord to create two 4-cycles is called the dominograph. We let $C_6^+$ denote the domino; $P_5$ is the complement of a 5-vertex path. The next theorem is the main result of the paper.

Theorem 1.2 A graph $G$ is a sesquicograph if and only if $G$ does not contain any of the following graphs as an induced subgraph:

(i) Cycles of length exceeding four, and
(ii) $P_5, C_6^+, H_1, H_2, H_3, H_4,$ and $H_5$,

where the graphs in (ii) are shown in Fig. 1.

Its proof occupies most of Section 3. As a consequence of Theorem 1.2, we have the following characterization of sesquicographs in terms of forbidden induced minors.
Corollary 1.3 A graph $G$ is a sesquicograph if and only if $G$ has no induced minor isomorphic to a graph in \{$C_5$, $P_5$, $H_1$, $H_2$, $H_3$, $H_4$, $H_5$\}, where $C_5$ is the cycle of length five.

A graph $G$ is a 2-cograph if it can be generated from $K_1$ using the operations of complementation, 0-sum, and 1-sum. The class of 2-cographs has been studied in [7]. This paper has some similarities with [7] although the arguments for sesquicographs are not as complex as they are for 2-cographs. Since the class of sesquicographs is the smallest class of graphs that contains $K_1$ and is closed under the operations of join, 0-sum, and 1-sum, it is a proper subclass of 2-cographs and, thus, of the class of perfect graphs. Note the path $P_5$ on five vertices is a sesquicograph but its complement $P_5$ is not. It follows that the class of sesquicographs is not closed under complementation unlike the classes of cographs and 2-cographs.

2 Preliminaries

Let $G$ be a graph. A vertex $u$ of $G$ is a neighbour of a vertex $v$ of $G$ if $uv$ is an edge of $G$. The neighbourhood $N_G(v)$ of $v$ in $G$ is the set of all neighbours of $v$ in $G$. If $G$ is connected, a $t$-cut of $G$ is set $X_t$ of vertices of $G$ such that $|X_t| = t$ and $G - X_t$ is disconnected. A graph that has no $t$-cuts for all $t$ less than $k$ is $k$-connected. Viewing $G$ as a subgraph of $K_n$ where $n = |V(G)|$, we colour the edges of $G$ green while assigning the colour red to the non-edges of $G$. Similar to the terminology in [7], we use the terms green graph and red graph for $G$ and its complementary graph $\overline{G}$, respectively. An edge of $G$ is called a green edge while a red edge refers to an edge of $\overline{G}$. The green degree of a vertex $v$ of $G$ is the number of green neighbours of $v$, while the red degree of $v$ is its number of red neighbours.

We omit the straightforward proofs of the next three results.

Lemma 2.1 All graphs having at most four vertices are sesquicographs.

Lemma 2.2 A graph $G$ is a join of two graphs if and only if its complement $\overline{G}$ is disconnected.

Lemma 2.3 Let $G$ be a graph and let $uv$ be an edge $e$ of $G$. Then $\overline{G}/e$ is the graph obtained by adding a vertex $w$ with neighbourhood $N_{\overline{G}}(u) \cap N_{\overline{G}}(v)$ to the graph $\overline{G} - \{u, v\}$.

Lemma 2.4 Every induced subgraph of a sesquicograph is a sesquicograph.

Proof Let $G$ be a sesquicograph. It is enough to show that, for every vertex $v$ of $G$, the graph $G - v$ is a sesquicograph. Note that if $|V(G)| \leq 5$, then our result follows by Lemma 2.1. Let $|V(G)| = n$. We proceed via induction on $|V(G)|$ and assume that the result is true for all sesquicographs with order less than $n$. Since $G$ is a sesquicograph, $G$ is a 0-sum, a 1-sum, or a join of proper induced subgraphs $X$ and $Y$ of $G$. Observe that if $G$ is $X \oplus Y$ or $X \triangledown Y$, then $G - v$ equals $(X - v) \oplus Y$ or $(X - v) \triangledown Y$, and so the result follows by induction. Therefore we may assume that $G = X \oplus_1 Y$. Note that, in this case, $G - v$ is either $(X - v) \oplus (Y - v)$ or $(X - v) \oplus_1 Y$. Thus our result follows by induction. \qed

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The graph $K_1$ is called trivial. Cographs can also be characterized as the graphs in which every non-trivial connected induced subgraph has a disconnected complement. Similarly, a graph $G$ is a 2-cograph if $G$ has no non-trivial induced subgraph $H$ such that both $H$ and $\overline{H}$ are 2-connected. Next we show that sesquicographs can be characterized in a similar way.

**Proposition 2.5** A graph $G$ is a sesquicograph if and only if, for every non-trivial induced subgraph $H$ of $G$, the graph $H$ is not 2-connected or $\overline{H}$ is disconnected.

**Proof** Let $G$ be a sesquicograph and let $H$ be a non-trivial induced subgraph of $G$. By Lemma 2.4, $H$ is a sesquicograph. Since $H$ can be decomposed as a 0-sum, a 1-sum, or a join, it follows by Lemma 2.2, that $H$ is not 2-connected or $\overline{H}$ is disconnected.

Conversely, let $G$ be a graph such that, for every non-trivial induced subgraph $H$ of $G$, the graph $H$ is not 2-connected or $\overline{H}$ is disconnected. By Lemma 2.2, it follows that every non-trivial induced subgraph of $G$ can be written as a 0-sum, a 1-sum, or a join of smaller induced subgraphs of $G$. Therefore $G$ can be generated from $K_1$ using the operations of 0-sum, 1-sum, and join. Thus $G$ is a sesquicograph. $\Box$

A slight variation of the proof of the closure of 2-cographs under contractions [7, Proposition 2.8] shows that sesquicographs are also closed under contractions.

**Proposition 2.6** Let $G$ be a sesquicograph and $e$ be an edge of $G$. Then $G/e$ is a sesquicograph.

**Proof** Assume to the contrary that $G/e$ is not a sesquicograph. Then there is a non-trivial induced subgraph $H$ of $G/e$ such that $H$ is 2-connected and $\overline{H}$ is connected. Let $e = uv$ and let $w$ denote the vertex in $G/e$ obtained by identifying $u$ and $v$. We may assume that $w$ is a vertex of $H$, otherwise $H$ is an induced subgraph of $G$, a contradiction. We assert that the subgraph $H'$ of $G$ induced on the vertex set $(V(H) \cup \{u, v\}) - \{w\}$ is 2-connected and its complement $\overline{H'}$ is connected. To see this, note that, since $H$ is 2-connected, $H'$ is 2-connected unless one of $u$ and $v$, say $u$, is a leaf of $H'$. In the exceptional case, we have $H' - u \cong H$, so $G$ has a 2-connected induced subgraph for which its complement is connected, a contradiction. We deduce that $H'$ is 2-connected.

Note that, by Lemma 2.3, $\overline{H}$ is obtained from $\overline{H'}$ by adding a vertex $w$ with neighbourhood $N_{\overline{H'}}(u) \cap N_{\overline{H'}}(v)$ to the graph $\overline{H'} - \{u, v\}$. Since $\overline{H}$ is connected, it follows that $\overline{H'}$ is connected, a contradiction. $\Box$

It now follows that the class of sesquicographs is closed under taking induced minors. For an input graph $G$ and, for $t$ in \{0, 1\}, the algorithm in Fig. 2 attempts to decompose $G$ as joins of $t$-sums of graphs having at most four vertices. Note that such graphs are sesquicographs by Lemma 2.1. Since we can compute the components and blocks of a graph in polynomial time [10, 4.1.23], the algorithm in Fig. 2 recognizes sesquicographs in polynomial time.

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Fig. 2 Algorithm for recognizing a sesquicograph

3 Induced-subgraph-minimal non-sesquicographs

We noted in Section 2 that sesquicographs are closed under induced subgraphs. In this section, we consider those non-sesquicographs for which every proper induced subgraph is a sesquicograph. We call these minimal non-sesquicographs induced-subgraph-minimal non-sesquicographs. The goal of this section is to characterize such graphs. We begin by showing that all cycles of length exceeding four are examples of such graphs.

Lemma 3.1 Let $G$ be a cycle of length exceeding four. Then $G$ is an induced-subgraph-minimal non-sesquicograph.

Proof Note that both $G$ and $\overline{G}$ are 2-connected and so, by Proposition 2.5, $G$ is not a sesquicograph. For any vertex $v$ of $G$, the graph $G - v$ is a path and so is a sesquicograph. \hfill \Box

The next result can be easily checked by examining the graphs.

Lemma 3.2 The graphs $\overline{P}_5$, $C_6^+$, $H_1$, $H_2$, $H_3$, $H_4$, and $H_5$ are induced-subgraph-minimal non-sesquicographs.

Lemma 3.3 Let $G$ be an induced-subgraph-minimal non-sesquicograph. Then $G$ is 2-connected and $\overline{G}$ is connected.

Proof Assume the contrary. Then for some proper induced subgraphs $X$ and $Y$ of $G$, we can decompose $G$ as $X \oplus Y$, as $X \oplus_1 Y$, or, by Lemma 2.2, as $X \triangledown Y$. Since $G$ is an induced-subgraph-minimal non-sesquicograph, both $X$ and $Y$ are sesquicographs. It now follows that $G$ is a sesquicograph, a contradiction. \hfill \Box
A 2-connected graph $H$ is critically 2-connected if, for each vertex $v$ of $H$, the graph $H - v$ is not 2-connected.

**Lemma 3.4** Let $G$ be an induced-subgraph-minimal non-sesquicograph. Then $G$ is critically 2-connected, or $G$ has connectivity two and $\overline{G}$ has connectivity one.

**Proof** Note that, by Lemma 3.3, $G$ is 2-connected and $\overline{G}$ is connected, and, by Proposition 2.5, for each vertex $v$ of $G$, the graph $G - v$ is not 2-connected or $\overline{G} - v$ is disconnected. Observe that $G$ has a vertex $v$ such that $\overline{G} - v$ is connected and so $G - v$ is not 2-connected. Therefore $G$ has connectivity two. Suppose that $G$ is not critically 2-connected. Then there is a vertex $w$ of $G$ such that $G - w$ is 2-connected and so $\overline{G} - w$ is disconnected. Therefore the connectivity of $\overline{G}$ is one. \(\square\)

Next we find those induced-subgraph-minimal non-sesquicographs $G$ such that $G$ is critically 2-connected. We will use the following result of Nebesky [6].

**Lemma 3.5** Let $G$ be a critically 2-connected graph such that $|V(G)| \geq 6$. Then $G$ has at least two distinct paths of length exceeding two such that the internal vertices of these paths have degree two in $G$.

**Lemma 3.6** Let $G$ be an induced-subgraph-minimal non-sesquicograph such that $G$ is not isomorphic to a cycle and let $wxyz$ be a path $P$ of $G$ such that both $x$ and $y$ have degree two in $G$. Then $w$ and $z$ are adjacent.

**Proof** Assume that $w$ and $z$ are not adjacent. By Lemma 3.3, $G$ is 2-connected, so there is a path $P'$ joining $w$ and $z$ such that $P$ and $P'$ are internally disjoint. We may assume that $P'$ is a shortest such path. It now follows that $G$ has a cycle $C$ of length exceeding four as an induced subgraph. Since a cycle of length exceeding four is not a sesquicograph, $G = C$, a contradiction. \(\square\)

**Proposition 3.7** Let $G$ be an induced-subgraph-minimal non-sesquicograph such that $G$ is critically 2-connected. Then $G$ is isomorphic to a cycle of length exceeding four or to the domino.

**Proof** We may assume that $G$ is not isomorphic to a cycle exceeding four otherwise we have our result. Note that, by Lemma 2.1, $|V(G)| \geq 5$. Since the cycle of length five is the only critically 2-connected graph on five vertices, we may assume that $|V(G)| \geq 6$. By Lemma 3.5, $G$ has two distinct paths $P_1 = abcd$ and $P_2 = wxyz$ of length three such that their internal vertices have degree two. By Lemma 3.6, $a$ and $d$ are adjacent, and $w$ and $z$ are adjacent. Consider the graph $G' = G - \{b, c\}$. Note that $G'$ is 2-connected and so, by Lemma 2.5, $\overline{G}$ is disconnected. It is now easy to check that $|V(G')| = 4$ and so $G$ is isomorphic to the domino. \(\square\)

**Proof of Theorem 1.2** We may assume that $G$ is not critically 2-connected; otherwise we are done by Proposition 3.7. By Lemma 3.3, $G$ has connectivity two and $\overline{G}$ has connectivity one. We first show the following.

3.7.1 $\overline{G}$ has at most three cut vertices.
Set FinalList ← ∅, i ← 0
Generate all two connected graphs of order 6 using nauty geng [5] and
store in an iterator L
for g in L such that connectivity of g is 2 and g is 1 do
    for v in V(g) do
        h = g\v
        if connectivity of h < 2 or connectivity of \( \overline{h} < 1 \) then
            i ← i + 1
        if i equals |V(g)| then
            Add g to FinalList

Fig. 3 Algorithm to find induced-subgraph-minimal non-sesiquicographs of order six

Let \( \{u, v\} \) be a 2-cut of \( G \) and let the components of \( G - \{u, v\} \) be partitioned into subgraphs \( A \) and \( B \) such that \( |V(A)| \geq |V(B)| \) and \(|V(A)| - |V(B)|\) is a minimum. Observe that \( \overline{G} - x \) is connected for a vertex \( x \) in \( V(G) \) unless \( x \) is the only red neighbour of \( u \) or the only red neighbour of \( v \), or \(|V(B)| = 1\) and \( x \) is in \( V(B) \). Thus 3.7.1 holds.

We show next that the number of vertices of \( G \) can be bounded.

3.7.2 \(|V(G)| \leq 6\).

Assume that \(|V(G)| > 6\). By 3.7.1, \( \overline{G} \) has at most three cut vertices. First suppose that \( \overline{G} \) has one cut vertex \( x \). Let the components of \( \overline{G} - x \) be partitioned into subgraphs \( R_1 \) and \( R_2 \) such that \(|V(R_1)| \geq |V(R_2)|\) and \(|V(R_1)| - |V(R_2)|\) is a minimum. Since \(|V(G)| \geq 7\), we have \(|V(R_1)| \geq 3\). Observe that, if \(|V(R_2)| \geq 2\), then there exists a vertex \( r \) in \( R_1 \) such that \( x \) has two green neighbours in \( G - r \). Note that every edge joining a vertex in \( R_1 \) to a vertex in \( R_2 \) is a green edge and so \( G - r \) is connected. Since every vertex in \( V(G) - x \) is in a green 2-cut, this is a contradiction. Therefore \(|V(R_2)| = 1\) and so \(|V(R_1)| \geq 5\). Let \( R_2 = \{\alpha\} \). Note that \( G - x \) is 2-connected since \( G \) is not critically 2-connected. It is now clear that \( G - \{x, \alpha\} \) is connected.

If \( G - \{x, \alpha\} \) has a vertex \( r \) such that \( G - \{x, \alpha, r\} \) is connected and contains two green neighbours of \( x \), then \( G - \alpha \) is 2-connected, a contradiction. It now follows that \( G - \{x, \alpha\} \) is a path and its leaves are the only green neighbours of \( x \). Note that \( G - \alpha \) is a cycle of length exceeding four, a contradiction.

Next suppose that \( \overline{G} \) has two cut vertices \( x_1 \) and \( x_2 \). For \{i, j\} = \{1, 2\}, let \( R_i \) be the disjoint union of the components of \( \overline{G} - x_i \) that do not contain \( x_j \). Let \( R_3 \) be the subgraph induced on \( V(G) - (V(R_1) \cup V(R_2) \cup \{x_1, x_2\}) \). We first consider the case when \( V(R_3) \) is empty. We may assume that \(|V(R_1)| \geq |V(R_2)|\) and so \(|V(R_1)| \geq 3\). Note that if \(|V(R_2)| \geq 2\), then there is a vertex \( r \) in \( R_1 \) such that \( G - r \) is 2-connected, a contradiction. Therefore \(|V(R_2)| = 1\) and so \(|V(R_1)| \geq 4\). Let \( \beta \) be a green neighbour of \( x_1 \) in \( R_1 \). Note that \( G - r \) is 2-connected for every vertex \( r \) in \( V(R_1) - \beta \), a contradiction. Therefore \( V(R_3) \) is non-empty. Observe that, if both \( R_1 \) and \( R_2 \) have at least two vertices, then \( G - r \) is 2-connected for any vertex \( r \) in \( R_3 \), a contradiction. Therefore we may assume that \(|V(R_1)| = 1\). We show that neither \( R_2 \) nor \( R_3 \) has more than two vertices. Assume that \( R_i \) has more than two vertices for some \( i \) in \{2, 3\}. Then there exists a vertex \( r \) in \( V(R_i) \) such that both \( x_1 \) and \( x_2 \) have at least two green neighbours in \( G - r \). Note that \( G - r \) is 2-connected, a contradiction.
Therefore \(|V(R_2)| = |V(R_3)| = 2\). Observe that there is a vertex \(r\) in \(R_3\) such that both \(x_1\) and \(x_2\) have green degree at least two in \(G - r\). It follows that \(G - r\) is 2-connected, a contradiction. Thus \(G\) has three cut vertices.

Let \(X = \{x_1, x_2, x_3\}\) be the set of cut vertices of \(\overline{G}\). We may assume that for the cut vertex \(x_1\) of \(\overline{G}\), the components of \(\overline{G} - x_1\) can be partitioned into subgraphs \(P\) and \(Q\) such that \(x_2\) is in \(P\) and \(x_3\) is in \(Q\), and \(|V(P)| \geq |V(Q)| \geq 2\). Note that all vertices in \(P\) are green neighbours of \(x_3\) and all vertices in \(Q\) are green neighbours of \(x_2\). If \(|V(P)| \geq 4\), then there is a vertex \(r\) in \(P\) such that all vertices in \(X\) have at least two green neighbours in \(G - r\) and so \(G - r\) is 2-connected, a contradiction. Therefore \(|V(P)| = |V(Q)| = 3\). Note that there is a vertex \(r\) in \(P \cup Q\) such that all vertices in \(X\) have at least two green neighbours in \(G - r\) and so \(G - r\) is 2-connected, a contradiction. Thus 3.7.2 holds.

By Lemma 2.1, it is clear that \(|V(G)| \geq 5\) and so \(|V(G)|\) is either 5 or 6. Suppose \(|V(G)| = 5\). Since \(P_5\) is the only graph on five vertices that is not critically 2-connected, has connectivity two, and whose complement has connectivity one, by Lemma 3.2, we have \(G \cong P_5\). Next suppose that \(|V(G)| = 6\). Implementing the algorithm in Fig. 3 in Sagemath [8], it can be easily checked that \(G\) is isomorphic to one of the graphs in \(\{H_1, H_2, H_3, H_4, H_5\}\). This completes the proof. \(\square\)

**Proof of Corollary 1.3** Note that every cycle of length exceeding five has the cycle of length five as an induced minor. Also, the domino graph \(C_{6}^+\) contains \(P_5\) as an induced minor. The result now follows by Theorem 1.2.

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