Contact Equivalence Problem for Linear Parabolic Equations

Oleg I. Morozov
Department of Mathematics, Snezhinsk Physical and Technical Academy,
Snezhinsk, 456776, Russia
morozov@sfti.snz.ru

Abstract. The moving coframe method is applied to solve the local equivalence problem for the class of linear parabolic equations in two independent variables under an action of the pseudo-group of contact transformations. The structure equations and the complete sets of differential invariants for symmetry groups are found. The solution of the equivalence problem is given in terms of these invariants.

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Introduction

In this article we consider a local equivalence problem for the class of linear second order parabolic equations

\[ u_{xx} = T(t, x) u_t + X(t, x) u_x + U(t, x) u \]  

under a contact transformation pseudo-group. Two equations are said to be equivalent if there exists a contact transformation mapping one equation to the other. Élie Cartan developed a general method for solving equivalence problems, [1] - [5]. The method provides an effective means of computing complete systems of differential invariants and associated invariant differential operators. The necessary and sufficient condition for equivalence of two submanifolds under an action of a Lie pseudo-group is formulated in terms of the differential invariants. The invariants parameterize the classifying manifold associated with given submanifolds. Cartan’s solution to the equivalence problem states that two submanifolds are (locally) equivalent if and only if their classifying manifolds (locally) overlap. The symmetry classification problem for classes of differential equations is closely related to the problem of local equivalence: symmetry groups and their Lie algebras of two equations are necessarily isomorphic if these equations are equivalent, while the converse statement is not true in general. The symmetry analysis of linear second order parabolic equations [1] is done by Sophus Lie, [1] Vol. 3, pp 492-523. In [1] § 9, Ovsiannikov gives the finite defining equation for the equivalence pseudo-group and the symmetry classification in terms of a normal form \( u_{xx} = u_t + H(t, x) u \) for equations [1]. In [9], the Laplace type semi-invariant, i.e., the
function remaining unchanged under a transformation $\tau = \sigma(t,x)u$ for every $\sigma(t,x)$, is
found for the class (1). This function

$$K = (2TX_x - X^2 T_x + 2T_x X_x + 2T^2 X_t - 2T X_x x + 4 T U_x - 4 U T_x) / (2T^4)$$

is not invariant under the full symmetry group of equation (1). In [10], it is shown that
equation (1) is reducible to the heat equation $u_{xx} = u_t$ under some contact transforma-
tion if and only if $\lambda = 0$, where

$$\lambda = 8T^8 K_{xx} + 20T^7 T_x K_x + 12 T^7 T_{xx} K + 288 T^2 T_x T_{xx}^2 + 220 T^2 T_x T_{xxx}$$

$$- 64T^3 T_{xx} T_{xxx} - 40 T^3 T_x T_{xxxx} + 4 T^4 T_{xxxxx} + 4 T^6 T_{txx} - 8 T^5 T_{txx} + 405 T_x^5$$

$$- 810 T T_x^3 T_{xx} + 4 T^4 T_x T_{t}^2 + 4 T^5 T_x T_{t}^2 + 80 T^2 T_t T_x^3 - 4 T^5 T_t T_t x - 80 T^3 T_x^2 T_t x$$

$$+ 28 T^4 T_{tx} T_{xx} + 36 T^4 T_x T_{txx} + 8 T^4 T_t T_{xxx} - 64 T^3 T_t T_x T_{xx}^2) / T^{10},$$

and $K$ is defined by (2).

In the present paper, we apply Cartan’s equivalence method, [1] - [5], [8], [15], in its form developed by Fels and Olver, [3] [7], to find all differential invariants of
symmetry groups for equations (1) and to solve the local contact equivalence problem
for equations from the class (1) in terms of their coefficients. Examples of computing
structure for symmetry pseudo-groups of partial differential equations via the method
of [3] [7] are given in [13]. Unlike Lie’s infinitesimal method, Cartan’s approach allows
us to find differential invariants and invariant differential operators without analysing
over-determined systems of PDEs at all, and requires differentiation and linear algebra
operations only.

The paper is organized as follows. In Section 1, we begin with some notation,
and use Cartan’s equivalence method to find the invariant 1-forms and the structure
equations for the pseudo-group of contact transformations on the bundle of second-order
jets. In Section 2, we briefly describe the approach to computing symmetry groups of
differential equations via the moving coframe method of Fels and Olver. In Section 3,
the method is applied to the class of parabolic equations (1). Finally, we make some
concluding remarks.

1. Pseudo-group of contact transformations

In this paper, all considerations are of local nature, and all mappings are real analytic.
Suppose $\mathcal{E} = \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a trivial bundle with the local base coordinates $(x^1, \ldots, x^n)$
and the local fibre coordinate $u$; then by $J^2(\mathcal{E})$ denote the bundle of the second-order jets
of sections of $\mathcal{E}$, with the local coordinates $(x^i, u, p_i, p_{ij})$, $i, j \in \{1, \ldots, n\}, i \leq j$. For every
local section $(x^i, f(x))$ of $\mathcal{E}$, the corresponding 2-jet $(x^i, f(x), \partial f(x)/\partial x^i, \partial^2 f(x)/\partial x^i \partial x^j)$
is denoted by $j_2(f)$. A differential 1-form $\vartheta$ on $J^2(\mathcal{E})$ is called a contact form, if it is
annihilated by all 2-jets of local sections: $j_2(f)^* \vartheta = 0$. In the local coordinates every
contact 1-form is a linear combination of the forms $\vartheta_0 = du - p_t \, dx^i$, $\vartheta_i = dp_t - p_{ij} \, dx^j,$
\(i, j \in \{1, ..., n\}, \quad p_{ji} = p_{ij}\) (here and later we use the Einstein summation convention, so \(p_i dx^i = \sum_{i=1}^n p_i dx^i\), etc.) A local diffeomorphism

\[
\Delta : J^2(\mathcal{E}) \rightarrow J^2(\mathcal{E}), \quad \Delta : (x^i, u, p_i, p_{ij}) \mapsto (\tilde{x}^i, \tilde{u}, \tilde{p}, \tilde{p}_{ij}),
\]

is called a contact transformation, if for every contact 1-form \(\vartheta\), the form \(\Delta^* \tilde{\vartheta}\) is also contact. To obtain a collection of invariant 1-forms for the pseudo-group of contact transformations on \(J^2(\mathcal{E})\), we apply Cartan’s method of equivalence, [5,15]. For this, take the coframe \(\{\vartheta_0, \vartheta_i, dx^i, dp_{ij} \mid i, j \in \{1, ..., n\}, i \leq j\}\) on \(J^2(\mathcal{E})\). A contact transformation \([4]\) acts on this coframe in the following manner:

\[
\Delta^* \begin{pmatrix} \tilde{\vartheta}_0 \\ \tilde{\vartheta}_i \\ d\tilde{x}^i \\ dp_{ij} \end{pmatrix} = S \begin{pmatrix} \vartheta_0 \\ \vartheta_k \\ dx^k \\ dp_{kl} \end{pmatrix},
\]

where \(S\) is an analytic function on \(J^2(\mathcal{E})\), taking values in the Lie group \(G\) of non-degenerate block matrices of the form

\[
\begin{pmatrix}
1 & -\tilde{a}^k & 0 & 0 \\
\tilde{g}_i & h^k_i & 0 & 0 \\
\tilde{c}_i & \tilde{f}^{ik} & b^l_k & \tilde{r}^{ikl} \\
\tilde{s}_{ij} & \tilde{w}^k_{ij} & \tilde{z}_{ijk} & \tilde{q}^{kl}_{ij} \\
\end{pmatrix}.
\]

In this matrix, \(i, j, k, l \in \{1, ..., n\}\), \(\tilde{r}^{ikl}\) are defined for \(k \leq l\), \(\tilde{s}_{ij}, \tilde{w}^k_{ij}\), and \(\tilde{z}_{ijk}\) are defined for \(i \leq j\), and \(\tilde{q}^{kl}_{ij}\) are defined for \(i \leq j, \ l \leq k\).

Let us show that \(\tilde{a}^k = 0\). Indeed, the exterior (non-closed!) ideal \(\mathcal{I} = \text{span}\{\vartheta_0, \vartheta_i\}\) has the derived ideal \(\delta \mathcal{I} = \{\omega \in \mathcal{I} \mid d\omega \in \mathcal{I}\} = \text{span}\{\vartheta_0\}\). Since \(\Delta^* \tilde{\mathcal{I}} \subset \mathcal{I}\), it follows that \(\Delta^* \tilde{\mathcal{I}}_0 = \vartheta_0\).

For convenience in the following computations, we denote by \((B^i_j)\) the inverse matrix for \((b^j_i)\), so \(b^j_i B^j_k = \delta^j_k\), by \((H^i_j)\) denote the inverse matrix for \((h^i_j)\), so \(h^i_j H^k_j = \delta^k_i\), define \(Q^{kl}_{ij}\) by \(Q^{kl}_{ij} \vartheta_{ij} = \delta^k_i \delta^l_j\), and change the variables on \(G\) such that

\[
g_i = \tilde{g}^i a^{-1}, \quad f^{ij} = \tilde{f}^{ik} H^j_k, \quad c^i = \tilde{c}^i a^{-1} - f^{ik} g_k, \quad s_{ij} = \tilde{s}_{ij} a^{-1} - \tilde{w}^k_{ij} H^m_k g_m - \tilde{z}_{ijm} B^m_k c^k, \quad w^k_{ij} = \tilde{w}^m_{ij} H^k_m - \tilde{z}_{ijm} B^m_k f^{ik} z_{ijk} = \tilde{z}_{ijm} B^m_k, \quad \text{and} \quad d^k_{ij} = \tilde{q}^{kl}_{ij} - \tilde{z}_{ijm} B^m_k r_{ijkl}. \]

In accordance with Cartan’s method of equivalence, we take the lifted coframe

\[
\begin{pmatrix}
\Theta_0 \\
\Theta_i \\
\Xi^i \\
\Sigma_{ij} \\
\end{pmatrix} = S \begin{pmatrix}
\vartheta_0 \\
\vartheta_k \\
dx^k \\
dp_{kl} \\
\end{pmatrix} = \begin{pmatrix}
\vartheta_0 \\
\vartheta_k \\
dx^k \\
dp_{kl} \\
\end{pmatrix}
\]

on \(J^2(\mathcal{E}) \times G\). Expressing \(du, dx^k, dp_k\), and \(dp_{kl}\) from \([4]\) and substituting them to \(d\Theta_0\), we have

\[
d\Theta_0 = da \wedge \vartheta_0 + a d\vartheta_0 = da a^{-1} \wedge \Theta_0 + a dx^i \wedge dp_i = da a^{-1} \wedge \Theta_0 + a dx^i \wedge \vartheta_i
\]

\[
= \Phi^0 \wedge \Theta_0 + a B^i_k H^m_i \Xi^k \wedge \Theta_m + a H^m_i R^{ikl} \Sigma_{kl} \wedge \Theta_m
\]

\[
+ a H^m_i \left( B^i_j f^{ij} + R^{ikl} w^j_{kl} \right) \Theta_j \wedge \Theta_m,
\]
where
\[
\Phi_0^0 = da a^{-1} + a H_i^{m'} \left( B_k^i \left( c^k + R_{ikl} s_{kl} \right) \Theta_{m'} - g_{m'} B_k^i \left( \Xi^k - c^k \Theta_0 - f^{kj} \Theta_j \right) \right) \\
- g_{m'} R_{ikl} \left( \Sigma_{kl} - s_{kl} \Theta_0 - w_{kl}^m \Theta_m - z_{klm} \Xi^m \right)
\]
and \( R_{ikl} = -\gamma_{ikl'} B_{i}^{l'} Q_{kll'} \).

The multipliers of \( \Xi^k \wedge \Theta_m, \Sigma_{kl} \wedge \Theta_m, \) and \( \Theta_j \wedge \Theta_m \) in (6) are essential torsion coefficients. We normalize them by setting \( a B_k^i H_i^{m'} = \delta_k^m, R_{ikl} = 0, \) and \( f^{kj} = f^{jk} \).

Therefore the first normalization is
\[
h_i^k = a B_i^k, \quad r_{ikl} = 0, \quad f^{kj} = f^{jk}. \tag{7}
\]

Analyzing \( d\Theta_i, d\Xi^i, \) and \( d\Sigma_{ij} \) in the same way, we obtain the following normalizations:
\[
q_{ij}^{kl} = a B_i^k B_j^l, \quad s_{ij} = s_{ji}, \quad w_{ij}^k = w_{ji}^k, \quad z_{ij} = z_{ji} = z_{ikj}. \tag{8}
\]

After these reductions the structure equations for the lifted coframe have the form
\[
d\Theta_0 = \Phi_0^0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i, \\
d\Theta_i = \Phi_i^0 \wedge \Theta_0 + \Phi_k^i \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik}, \\
d\Xi^i = \Phi_0^0 \wedge \Xi^i \wedge \Phi_i^k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k, \\
d\Sigma_{ij} = \Phi_i^k \wedge \Sigma_{ki} - \Phi_0^0 \wedge \Sigma_{ij} + \Gamma_{ij}^0 \wedge \Theta_0 + \Sigma_{ij}^0 \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k,
\]

where the forms \( \Phi_0^0, \Phi_i^0, \Phi_i^k, \Psi^{i0}, \Psi^{ij}, \Theta_0^0, \Theta_k^0, \) and \( \Lambda_{ijk} \) are defined by the following equations:
\[
\Phi_0^0 = da a^{-1} - g_k \Xi^k + (c^k + f^{km} g_m) \Theta_k, \\
\Phi_i^0 = dg_i + g_k db_k^i B_i^j - (g_i g_k + s_{ik} + c^j z_{ijk}) \Xi^k + c^k \Sigma_{ik} \\
+ (g_i c^k + g_i g_m f^{mk} - c^j w_{ij}^k + f^{mk} s_{im}) \Theta_k, \\
\Phi_i^k = \delta_i^k da a^{-1} - db_k^i B_i^j + (g_i \delta_j^k - w_{ij}^k - f^{km} z_{ijm}) \Xi^j + f^{km} \Sigma_{im} + f^{jm} w_{ij}^k \Theta_m, \\
\Psi^{i0} = dc^i + f^{ij} \Phi^0_j + c^k \Phi_i^k + (c^j f^{mj} g_m - c^k f^{mj} w_{ij}^k) \Theta_j - c^k f^{ij} \Sigma_{kj} \\
+ c^k (f^{im} z_{kjm} + w_{ij}^k - g_k \delta_j^i - g_j \delta_i^k) \Xi^j, \\
\Psi^{ij} = df^{ij} + (f^{ik} \delta_j^m + f^{jk} \delta_i^m) \Phi^0_m + (c^j \delta_k^i + c^i \delta_k^j - f^{ij} g_k + f^{im} f^{jm} z_{klm}) \Xi^k \\
+ f^{ij} (c^k + f^{km} g_m) \Theta_k - f^{ik} f^{jm} \Sigma_{km}, \\
\Gamma_{ij}^0 = ds_{ij} - s_{ij} da a^{-1} + s_{kj} db_k^i B_i^m + s_{ik} db_k^i B_i^j + s_{ij} \Phi^0_0 + w_{ij}^k \Phi^0_i + z_{ijk} \Psi_{kj}^0, \\
\Gamma_{ij}^k = dw_{ij}^k - w_{ij}^k da a^{-1} + (w_{ij}^k s_{ij} + w_{ij}^k s_{ij}^m) db_i^l B_i^{mj} + (s_{ij} \delta_k^m + z_{ijkl} f^{mk} w_{ij}^l B_i^m) \Xi^m \\
+ w_{ij}^m \delta_k^m + f^{ik} (w_{ij}^m s_{ij} + w_{ij}^m s_{ij}^m) \Sigma_{m}^m - (c^k + f^{mk} g_m) \Sigma_{ij}, \\
\Lambda_{ijk} = dz_{ijk} - 2 z_{ijk} da a^{-1} + z_{ijl} db_l^i B_i^m + z_{ilk} db_l^i B_i^j + z_{ijk} db_l^i B_i^m + z_{ijk} \Phi^0_0.
\]
Let $H$ be the subgroup of $G$ defined by (7) and (8). We shall prove that the restriction of the lifted coframe $[5]$ to $J^2 (E) \times H$ satisfies Cartan’s test of involutivity, [15, def 11.7]. The structure equations remain unchanged under the following transformation of the modified Maurer-Cartan forms $\Phi_0^0, \Phi_0^i, \Phi_0^k, \Psi^{ij}, \Psi^{ik}, \Psi^{ij}, \Psi^{ij},$ and $\Lambda_{ijk}$:

\[
\begin{align*}
\Phi_0^0 & \rightarrow \Phi_0^0 + K \Theta_0, \\
\Phi_i^k & \rightarrow \Phi_i^k + L_i^{kl} \Theta_l + M_i^k \Theta_0, \\
\Phi_0^i & \rightarrow \Phi_0^i + M_i^k \Theta_k + N_i \Theta_0, \\
\Psi^{ij} & \rightarrow \Psi^{ij} + P^{ij} \Theta_0 + S^{ijk} \Theta_k - L_i^{ij} \Xi^k, \\
\Psi^{i0} & \rightarrow \Psi^{i0} + P^{ij} \Theta_j + T^i \Theta_0 + K \Xi^i - M_i^k \Xi^k, \\
\Psi^{00} & \rightarrow \Psi^{00} + \Psi^{i0} \Theta_i + V^{ij} \Theta_k + W_{ijk} \Xi^k + K \Sigma_{ij} + M_i^k \Sigma_{kj}, \\
\Psi^{ij} & \rightarrow \Psi^{ij} + X_{ij}^k \Theta_k + V_{ij} \Theta_0 + Y_{ij}^i \Xi^l + L_i \Sigma_{ij}, \\
\Lambda_{ijk} & \rightarrow \Lambda_{ijk} + Z_{ijkl} \Xi^l + Y_{ijk}^i \Theta_l + W_{ijk} \Theta_0,
\end{align*}
\]

where $K, L_i^{kl}, M_i^k, N_i, P^{ij}, S^{ijk}, T^i, U_{ij}, V_{ij}^k, W_{ijk}, X_{ij}^k, X_{ijkl}^1, X_{ijkl}^2, Z_{ijkl}$ are arbitrary constants satisfying the following symmetry conditions: $L_i^{kl} = L_k^{li}$, $P^{ij} = P^{ji}$, $S^{ijk} = S^{kji} = S^{ikj}$, $U_{ij} = U_{ji}$, $V_{ij}^k = V_{ji}^k$, $W_{ijk} = W_{ikj}$, $X_{ij}^k = X_{ji}^k$, $Y_{ij}^k = Y_{ji}^k$, and $Z_{ijkl} = Z_{jikl} = Z_{ijk}$. The number of such constants

\[
r^{(1)} = 1 + \frac{n^2 (n + 1)}{2} + n^2 + n + \frac{n (n + 1)}{2} + \frac{n (n + 1) (n + 2)}{6} + \frac{n (n + 1)}{2} + \frac{n (n + 1) (n + 2)}{6} + \frac{n^2 (n + 1) (n + 2)}{4} + \frac{n^2 (n + 1) (n + 2)}{6} + \frac{n (n + 1) (n + 2)}{24} = \frac{1}{24} (n + 1) (n + 2) (11 n^2 + 29 n + 12)
\]

is the degree of indeterminacy of the lifted coframe, [15, def 11.2]. The reduced characters of this coframe, [15, def 11.4], are easily found:

\[
\begin{align*}
s_i' &= \frac{(n + 1) (n + 4)}{2} - i, \quad i \in \{1, \ldots, n + 1\}, \\
s_{n+1}' &= \frac{(n + 1 - j) (n + 2 - j)}{2}, \quad j \in \{1, \ldots, n\}.
\end{align*}
\]

A simple calculation shows that

\[
r^{(1)} = s_1' + 2 s_2' + 3 s_3' + \ldots + (2 n + 1) s_{2n+1}'.
\]

So the Cartan test is satisfied, and the lifted coframe is involutive.
It is easy to directly verify that a transformation $\hat{\Delta} : J^2(\mathcal{E}) \times \mathcal{H} \to J^2(\mathcal{E}) \times \mathcal{H}$ satisfies the conditions

$$\hat{\Delta}^* \Theta_0 = \Theta_0, \quad \hat{\Delta}^* \Theta_i = \Theta_i, \quad \hat{\Delta}^* \Xi^i = \Xi^i, \quad \hat{\Delta}^* \Sigma_{ij} = \Sigma_{ij}$$  \hspace{1cm} (9)

if and only if it is projectable on $J^2(\mathcal{E})$, and its projection $\Delta : J^2(\mathcal{E}) \to J^2(\mathcal{E})$ is a contact transformation.

Since (9) imply $\hat{\Delta}^* d\Theta_0 = d\Theta_0$, $\hat{\Delta}^* d\Theta_i = d\Theta_i$, $\hat{\Delta}^* d\Xi^i = d\Xi^i$, and $\hat{\Delta}^* d\Sigma_{ij} = d\Sigma_{ij}$, we have

$$\hat{\Delta}^* (\Theta_0^i \land \Theta_0 \land \Xi^i \land \Theta_i) = (\hat{\Delta}^* \Theta_0^i) \land \Theta_0 + \Xi^i \land \Theta_i = \Phi_0^i \land \Theta_0 + \Xi^i \land \Theta_i,$$

$$\hat{\Delta}^* (\Theta_i^k \land \Theta_0 + \Xi^i \land \Theta_i) = \hat{\Delta}^* (\Theta_i^k) \land \Theta_0 + \hat{\Delta}^* (\Xi^i) \land \Theta_i \land \Theta_k + \Xi^k \land \Sigma_{ik},$$

$$\hat{\Delta}^* (\Xi^i \land \Xi^i \land \Xi^k) = \hat{\Delta}^* (\Xi^i) \land \Xi^i \land \Xi^k \land \Xi^i \land \Xi^k \land \Xi^k \land \Xi^i \land \Xi^k,$$

$$\hat{\Delta}^* (\Xi^i \land \Xi^i \land \Xi^k) = \hat{\Delta}^* (\Xi^i) \land \Xi^i \land \Xi^k \land \Xi^i \land \Xi^k \land \Xi^k \land \Xi^i \land \Xi^k,$$

Therefore,

$$\hat{\Delta}^* (\Phi_0^0) = \Phi_0^0 + K \Theta_0,$$
$$\hat{\Delta}^* (\Phi_i^k) = \Phi_i^k + L_i^j \Theta_j + M_i^k \Theta_0,$$
$$\hat{\Delta}^* (\Psi_{ij}) = \Psi_{ij} + P_{ij} \Theta_0 + S_{ijk} \Theta_k - L_{ij}^k \Xi^k,$$
$$\hat{\Delta}^* (\Phi_0^0) = \Phi_0^0 + M_i^k \Theta_k + N_i \Theta_0,$$
$$\hat{\Delta}^* (\Phi_i^k) = \Phi_i^k + M_i^k \Theta_k + N_i \Theta_0,$$

$$\hat{\Delta}^* (\Xi^i) = \Xi^i + \Psi_{ij} \Theta_0 + S_{ijk} \Theta_k - L_{ij}^k \Xi^k,$$
$$\hat{\Delta}^* (\Theta_0^i) = \Theta_0^i + \Theta_0^i \Theta_j + T_i \Theta_0 + K \Xi^i - M_i^j \Xi^k,$$
$$\hat{\Delta}^* (\Theta_i^k) = \Theta_i^k + X_{ij}^k \Theta_j + V_{ij}^k \Theta_0 + W_{ijk} \Xi^k + K \Sigma_{ij} + M_i^k \Sigma_{kj},$$
$$\hat{\Delta}^* (\Xi^i) = \Xi^i + Y_{ijkl} \Xi^j + X_{ij}^k \Theta_j + V_{ij}^k \Theta_0 + Y_{ijkl} \Xi^j + L_i \Sigma_{ij},$$

$$\hat{\Delta}^* (\Xi^i) = \Xi^i + Z_{ijkl} \Xi^j + Y_{ijkl} \Xi^j + L_i \Sigma_{ij},$$

with some functions $K$, $L_i^j$, $M_i^k$, $N_i$, $P_{ij}$, $S_{ijk}$, $T_i$, $U_{ij}$, $V_{ij}^k$, $W_{ijk}$, $X_{ij}^k$, $Y_{ijkl}$, and $Z_{ijkl}$ on $J^2(\mathcal{E}) \times \mathcal{H}$.
2. Contact symmetries of differential equations

Suppose \( \mathcal{R} \) is a second-order differential equation in one dependent and \( n \) independent variables. We consider \( \mathcal{R} \) as a sub-bundle in \( J^2(\mathcal{E}) \). Let \( \text{Cont}(\mathcal{R}) \) be the group of contact symmetries for \( \mathcal{R} \). It consists of all the contact transformations on \( J^2(\mathcal{E}) \) mapping \( \mathcal{R} \) to itself. The moving coframe method, \([6, 7]\), is applicable to find invariant 1-forms characterizing \( \text{Cont}(\mathcal{R}) \) is the same way, as the restriction of the lifted coframe \( \mathcal{E} \) to \( J^2(\mathcal{E}) \times \mathcal{H} \) characterizes \( \text{Cont}(J^2(\mathcal{E})) \). We briefly outline this approach.

Let \( \iota : \mathcal{R} \to J^2(\mathcal{E}) \) be an embedding. The invariant 1-forms of \( \text{Cont}(\mathcal{R}) \) are restrictions of the coframe \([3, 4, 10]\) to \( \mathcal{R} \): \( \theta_0 = \iota^*\Theta_0 \), \( \theta_i = \iota^*\Theta_i \), \( \xi^i = \iota^*\Xi^i \), and \( \sigma_{ij} = \iota^*\Sigma_{ij} \) (for brevity we identify the map \( \iota \times \text{id} : \mathcal{R} \times \mathcal{H} \to J^2(\mathcal{E}) \times \mathcal{H} \) with \( \iota : \mathcal{R} \to J^2(\mathcal{E}) \)). The forms \( \theta_0 \), \( \theta_i \), \( \xi^i \), and \( \sigma_{ij} \) have some linear dependencies, i.e., there exists a non-trivial set of functions \( E^0, E^i, F_i, \) and \( G^i_j \) on \( \mathcal{R} \times \mathcal{H} \) such that \( E^0 \theta_0 + E^i \theta_i + F_i \xi^i + G^i_j \sigma_{ij} \equiv 0 \). These functions are lifted invariants of \( \text{Cont}(\mathcal{R}) \). Setting them equal to some constants allows us to specify some parameters \( a, b^k_i, c_i, f^{ij}, s_{ij}, w^k_{ij} \), and \( z_{ijk} \) of the group \( \mathcal{H} \) as functions of the coordinates on \( \mathcal{R} \) and the other group parameters.

After these normalizations, some restrictions of the modified Maurer - Cartan forms \( \phi^0_i = \iota^*\Phi^0_i \), \( \phi^k_i = \iota^*\Phi^k_i \), \( \psi^0_i = \iota^*\Psi^0_i \), \( \psi^{ij} = \iota^*\Psi^{ij} \), \( \psi^{0i} = \iota^*\Psi^{0i} \), \( \psi_{ij}^0 = \iota^*\Psi_{ij}^0 \), \( \psi_{ij}^k = \iota^*\Psi_{ij}^k \), and \( \lambda_{ijk} = \iota^*\Lambda_{ijk} \), or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of \( \mathcal{H} \). From \([10]\), we have the following statements: (i) if \( \phi^0_i \) is semi-basic, then its coefficients at \( \theta_k \), \( \xi^k \), and \( \sigma_{kl} \) are lifted invariants of \( \text{Cont}(\mathcal{R}) \); (ii) if \( \phi^0_i \) or \( \phi^k_i \) are semi-basic, then their coefficients at \( \xi^k \) and \( \sigma_{kl} \) are lifted invariants of \( \text{Cont}(\mathcal{R}) \); (iii) if \( \psi^{0i} \), \( \psi^{ij} \), or \( \lambda_{ijk} \) are semi-basic, then their coefficients at \( \sigma_{kl} \) are lifted invariants of \( \text{Cont}(\mathcal{R}) \). Setting these invariants equal to some constants, we get specifications of some more parameters of \( \mathcal{H} \) as functions of the coordinates on \( \mathcal{R} \) and the other group parameters.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

\[
\begin{align*}
d\theta_0 &= \phi^0_0 \land \theta_0 + \xi^i \land \theta_i \\
&
d\theta_i &= \phi^k_i \land \theta_0 + \phi^0_i \land \theta_k + \xi^k \land \sigma_{ik} \\
&
d\xi^i &= \phi^0_i \land \xi^i - \phi^k_i \land \xi^k + \psi^{0i} \land \theta_0 + \psi^{ij} \land \theta_k \\
&
d\sigma_{ij} &= \phi^k_i \land \sigma_{ki} - \phi^0_0 \land \sigma_{ij} + \psi^0_{ij} \land \theta_0 + \psi^k_{ij} \land \theta_k + \lambda_{ijk} \land \xi^k.
\end{align*}
\]

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of defining forms for \( \text{Cont}(\mathcal{R}) \). In the second case, when the reduced lifted coframe does not satisfy Cartan’s test, we should use the procedure of prolongation, \([15, \text{Ch 12}]\).
3. Structure and invariants of symmetry groups for linear parabolic equations

We apply the method described in the previous section to the class of linear parabolic equations (1). Denote \( x^1 = t, \) \( x^2 = x, \) \( p_1 = u_t, \) \( p_2 = u_x, \) \( p_{11} = u_{tt}, \) \( p_{12} = u_{tx}, \) and \( p_{22} = u_{xx}. \) The coordinates on \( \mathcal{R} \) are \( \{t, x, u, u_t, u_x, u_{tt}, u_{tx}\}, \) and the embedding \( \iota : \mathcal{R} \to J(\mathcal{E}) \) is defined by (1). Computing the linear dependence conditions for the reduced forms \( \theta_0, \theta_i, \xi_i, \) and \( \sigma_{ij} \) by means of MAPLE, we express the group parameters \( b_1^1, z_{122}, z_{222}, w_{12}^1, w_{22}^0, \) and \( s_{22} \) as functions of the coordinates on \( \mathcal{R} \) and the other parameters of the group \( \mathcal{H}. \) Particularly, since

\[
\sigma_{22} \equiv - (b_1^2)^2 (b_2^2)^{-2} \sigma_{11} - 2b_1^1 (b_2^2)^{-1} \sigma_{12} \pmod{\theta_0, \theta_1, \theta_2, \xi_1, \xi_2},
\]

and without loss of generality \( b_1^1 \neq 0, b_2^2 \neq 0, \) we take \( b_1^2 = 0. \) After that, we have

\[
\sigma_{22} \equiv \left( z_{22}^1 + a \left( b_2^2 (T u_{tt} + X u_{tx} + (U + T) u_t + X u_x + U u) - b_1^1 \left( T u_{tx} + (T X + T_x) u_t + (X^2 + U + X_x) u_x + (U_x + X U) u \right) \right) \right) (b_1^1)^{-1}(b_2^2)^{-3} \xi^1
\]

\[
+ \left( z_{22}^2 + a \left( T u_{tx} + (T X + T_x) u_t + (X^2 + U + X_x) u_x + (U_x + X U) u \right) \right) (b_2^2)^3 \xi^2
\]

\( \pmod{\theta_0, \theta_1, \theta_2}. \)

Then we take

\[
z_{22}^1 = -a \left( b_2^2 (T u_{tt} + X u_{tx} + (U + T) u_t + X u_x + U u) - b_1^1 \left( T u_{tx} + (T X + T_x) u_t + (X^2 + U + X_x) u_x + (U_x + X U) u \right) \right) (b_1^1)^{-1}(b_2^2)^{-3},
\]

\[
z_{22}^2 = -a \left( T u_{tx} + (T X + T_x) u_t + (X^2 + U + X_x) u_x + (U_x + X U) u \right) (b_2^2)^3.
\]

After that, setting the coefficients of \( \sigma_{22} \) at \( \theta_1, \theta_2, \) and \( \theta_0 \) equal to 0, we find \( w_{12}^1, w_{22}^0, \) and \( s_{22} \) as the functions of the coordinates on \( \mathcal{R} \) and the other parameters of \( \mathcal{H}. \) These expressions are too long to be written out in full here.

Now the form \( \phi_2^1 \) is semi-basic. We have

\[
\phi_2^1 \equiv f^{11} \sigma_{12} + b_1^1 T (b_2^2)^{-2} \xi^2 \pmod{\theta_0, \theta_1, \theta_2, \xi_1, \sigma_{11}},
\]

therefore we take \( f^{11} = 0, b_1^1 = (b_2^2)^2 T^{-1}. \) After that, setting the coefficient of \( \phi_2^1 \) at \( \xi_1 \) equal to 0, we find \( w_{12}^1. \)

Then the linear combination \( 2 \phi_2^2 - \phi_1^1 - \phi_0^0 \) becomes semi-basic. Since

\[
2 \phi_2^2 - \phi_1^1 - \phi_0^0 \equiv f^{12} \sigma_{12} + \left( 4 g_2 + \left( 2 T b_1^2 + (2 T X - T_x) b_2^2 \right) (b_2^2)^{-2} T^{-1} \right) \xi^2
\]

\( \pmod{\theta_0, \theta_1, \theta_2, \xi_1, \sigma_{11}}, \)

we take \( f^{12} = 0, g_2 = -(2 T b_1^2 + (2 T X - T_x) b_2^2)/(4 (b_2^2)^2 T). \) Setting the coefficient of \( 2 \phi_2^2 - \phi_1^1 - \phi_0^0 \) at \( \xi_1 \) equal to 0, we find \( w_{12}^2. \)

Since for the semi-basic linear combination \( 2 \phi_0^2 - \phi_1^2 \) we have \( 2 \phi_0^2 - \phi_1^2 \equiv (2 c^1 - f^{22}) \sigma_{12} \pmod{\theta_0, \theta_1, \theta_2, \xi_1, \xi_2, \sigma_{11}}, \) the normalization \( c^1 = f^{22}/2 \) is possible. Setting the coefficient of \( 2 \phi_0^2 - \phi_1^2 \) at \( \xi_1 \) and \( \xi_2 \) equal to 0, we find \( s_{12} \) and \( g_1 \).
After that, we obtain the following reduced structure equations

\[ d\theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \]
\[ d\theta_1 = \alpha_1 \wedge \theta_1 - \alpha_2 \wedge \theta_1 + 2 \alpha_3 \wedge \theta_2 + \alpha_4 \wedge \theta_0 + \xi^1 \wedge \sigma_{12} + \xi^2 \wedge \sigma_1 + \frac{1}{3} f^{22} \theta_1 \wedge \theta_2, \]
\[ d\theta_2 = \alpha_1 \wedge \theta_2 - \frac{1}{2} \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_0 + \xi^1 \wedge \sigma_{12} + \xi^2 \wedge \theta_1, \]
\[ d\xi^1 = \alpha_2 \wedge \xi^1 + \alpha_4 \wedge \theta_0 + \frac{1}{2} f^{22} \xi^2 \wedge \theta_2, \]

where \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \) are 1-forms on \( J^2(\mathcal{E}) \times \mathcal{H} \) depending on differentials of the parameters of \( \mathcal{H} \). We normalize the essential torsion coefficient \( f^{22} \) in these equations by setting \( f^{22} = 0 \). Then, there are the following structure equations

\[ d\theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \]
\[ d\theta_1 = \alpha_1 \wedge \theta_1 - \alpha_2 \wedge \theta_1 + 2 \alpha_3 \wedge \theta_2 + \alpha_4 \wedge \theta_0 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12} - c^2 \theta_1 \wedge \theta_2, \]
\[ d\theta_2 = \alpha_1 \wedge \theta_2 - \frac{1}{2} \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_0 + \xi^1 \wedge \sigma_{12} + \xi^2 \wedge \theta_1, \]
\[ d\xi^1 = \alpha_2 \wedge \xi^1 \]

(the forms \( \alpha_i \) can change after the normalizations). The structure equations have the essential torsion coefficient \( c^2 \), therefore we normalize \( c^2 = 0 \). After that, we set the coefficients of \( d\sigma_{12} \) at \( \theta_0 \wedge \xi^2 \) and \( \theta_2 \wedge \xi^2 \) equal to 0 and express \( w_{11}^2 \) and \( s_{11} \) as functions of the coordinates on \( \mathcal{R} \) and the remaining parameters of \( \mathcal{H} \). The formulas for \( w_{11}^2 \) and \( s_{11} \) are too long to be written out in full here. Then we get

\[ d\sigma_{11} = \alpha_1 \wedge \sigma_{11} - 2 \alpha_2 \wedge \sigma_{11} + 4 \alpha_3 \wedge \sigma_{12} + 6 \alpha_4 \wedge \theta_1 + \alpha_5 \wedge \xi^2 + \alpha_6 \wedge \xi^1 \]
\[ + \lambda\left(b_2^2\right)^{-5} \theta_0 \wedge \xi^2, \]
\[ d\tau_{12} = \alpha_1 \wedge \sigma_{12} - \frac{3}{2} \alpha_2 \wedge \sigma_{11} + 3 \alpha_3 \wedge \theta_1 + 3 \alpha_4 \wedge \theta_2 + \alpha_5 \wedge \xi^1 + \xi^2 \wedge \sigma_{11}, \]

where \( \lambda = \frac{1}{45} \lambda T_5 \), \( \lambda \) is given by (3), and all the essential torsion coefficients in the other structure equations are constants.

There are two possibilities now: \( I = 0 \) or \( I \neq 0 \). By \( \mathcal{P}_1 \) we denote the subclass of all equations \( (\Pi) \) such that \( I \neq 0 \). For an equation from \( \mathcal{P}_1 \) all the essential torsion coefficients in the reduced structure equations are constants, but the lifted coframe \( \theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \) and \( \sigma_{12} \) is not involutive yet. Therefore we use the procedure of prolongation, [15 Ch 12], and obtain the structure equations

\[ d\theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \]
\[ d\theta_1 = \alpha_1 \wedge \theta_1 - \alpha_2 \wedge \theta_1 + 2 \alpha_3 \wedge \theta_2 + \alpha_4 \wedge \theta_0 + \xi^2 \wedge \sigma_{12} + \xi^1 \wedge \sigma_{11}, \]
\[ d\theta_2 = \alpha_1 \wedge \theta_2 - \frac{1}{2} \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_0 + \xi^1 \wedge \sigma_{12} - \theta_1 \wedge \xi^2, \]
\[ d\xi^1 = \alpha_2 \wedge \xi^1, \]
\[ d\xi^2 = -2 \alpha_3 \wedge \xi^1 + \frac{1}{2} \alpha_2 \wedge \xi^2, \]
\[ d\sigma_{11} = \alpha_1 \wedge \sigma_{11} - 2 \alpha_2 \wedge \sigma_{11} + 4 \alpha_3 \wedge \sigma_{12} + 6 \alpha_4 \wedge \theta_1 + \alpha_5 \wedge \xi^2 + \alpha_6 \wedge \xi^1, \]
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\[ d\sigma_{12} = \alpha_1 \wedge \sigma_{12} - \frac{3}{2} \alpha_2 \wedge \sigma_{12} + 3 \alpha_3 \wedge \theta_1 + 3 \alpha_4 \wedge \theta_2 + \alpha_5 \wedge \xi + \xi^2 \wedge \sigma_{11}, \]
\[ d\alpha_1 = -\alpha_3 \wedge \xi^2 - \alpha_4 \wedge \xi^1, \]
\[ d\alpha_2 = 4 \alpha_4 \wedge \xi^1, \]
\[ d\alpha_3 = -\alpha_4 \wedge \xi^2 - \frac{1}{2} \alpha_2 \wedge \alpha_3, \]
\[ d\alpha_4 = -\alpha_2 \wedge \alpha_4, \]
\[ d\alpha_5 = \pi_1 \wedge \xi^1 + \alpha_1 \wedge \alpha_5 - \frac{5}{2} \alpha_2 \wedge \alpha_5 - 5 \alpha_3 \wedge \alpha_1 - 10 \alpha_4 \wedge \sigma_{12} - \alpha_6 \wedge \xi^2, \]
\[ d\alpha_6 = \pi_1 \wedge \xi^2 + \pi_2 \wedge \xi^1 + \alpha_1 \wedge \alpha_6 - 3 \alpha_2 \wedge \alpha_6 + 6 \alpha_3 \wedge \alpha_5 - 15 \alpha_4 \wedge \sigma_{11}, \]

where \( \alpha_1, ..., \alpha_6, \pi_1, \) and \( \pi_2 \) are 1-forms on \( R \times H \) (they are too long to be written explicitly). From these structure equations, it follows that the only non-zero reduced character of the lifted coframe \( \theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{12}, \alpha_1, \alpha_2, ..., \alpha_6 \) is \( s^i_1 = 2 \), while the degree of indeterminacy is \( r^{(1)} = 2 \). So the Cartan test is satisfied, and the lifted coframe is involutive.

The same calculations show that the symmetry group of the linear heat equation

\[ u_{xx} = u_t \tag{11} \]

has the identical structure equations, but with the different lifted coframe. All the essential torsion coefficients in the structure equations are constants. Thus, applying Theorem 15.12 of [15], we have

**Theorem 1.** ([10], Theorem 3.2) The linear parabolic equation \( \square \) is equivalent to the linear heat equation \( \square \) under a contact transformation if and only if it belongs to \( \mathcal{P}_1 \), i.e., \( \iff I = 0 \).

Numerous examples of equations \( \square \) reducible to the linear heat equation are given in [10], [16].

Now we return to the case \( I \neq 0 \). Then we can take \( b^2_1 = I \). Setting the essential torsion coefficient in the structure equation for \( d\theta_2 \) at \( \theta_2 \wedge \xi^2 \) equal to 0 and expressing \( u^1_{11} \), we get the following structure equations

\[ d\theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \]
\[ d\theta_1 = \alpha_1 \wedge \theta_1 + 2 \alpha_2 \wedge \theta_2 - \frac{1}{2} J_1 \alpha_2 \wedge \theta_0 + Z \theta_0 \wedge \xi^1 - \left(b^2_1 J_{1x} - I J_1 \right)/\left(4 I^3 \right) \theta_0 \wedge \xi^1 \]
\[ -J_1 \xi^2 \wedge \theta_1 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12}, \]
\[ d\theta_2 = \alpha_1 \wedge \theta_2 + \alpha_2 \wedge \theta_0 + \xi^2 \wedge \theta_1 - \frac{1}{2} J_1 \xi^2 \wedge \theta_2 + \xi^1 \wedge \sigma_{12}, \]
\[ d\xi^1 = -J_1 \xi^1 \wedge \xi^1, \]
\[ d\xi^2 = -2 \alpha_2 \wedge \xi^1, \]

where

\[ J_1 = (2 T I_x - I T_x) T^{-1} I^{-2}, \]
and $Z$ is a function of $T$, $X$, $U$, $I$, $J_1$, their derivatives w.r.t. $t$, $x$, and $b_1^2$. Recall that the forms $\alpha_1$, $\alpha_2$ are not necessary the same as in the previous structure equations.

Consider the subclass $\mathcal{P}_2$ of all equations $\square$ such that $I \neq 0$, $J_{1x} \neq 0$. This subclass is not empty, e.g., the equation $u_{xx} = u + x^4 u$ belongs to $\mathcal{P}_2$. For an equation from $\mathcal{P}_2$, we normalize the coefficient in the structure equation for $d\theta_1$ at $\theta_0 \wedge \xi^1$ by setting $b_1^2 = -I J_{1t} J_{1x}^{-1}$. Then, after a prolongation, we obtain the following structure equations

\[
d\theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,
\]

\[
d\theta_1 = \alpha_1 \wedge \theta_1 + J_3 \theta_0 \wedge \xi^1 - \frac{1}{4} J_1 J_2 \theta_0 \wedge \xi^2 + J_1 \theta_1 \wedge \xi^2 + J_4 \theta_2 \wedge \xi^1 + J_2 \theta_2 \wedge \xi^2 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12},
\]

\[
d\theta_2 = \alpha_1 \wedge \theta_2 + \frac{1}{2} J_4 \theta_0 \wedge \xi^1 + \frac{1}{2} J_2 \theta_0 \wedge \xi^2 + \frac{1}{2} J_1 \theta_2 \wedge \xi^2 + \xi^1 \wedge \sigma_{12} - \theta_1 \wedge \xi^2,
\]

\[
d\xi^1 = -J_1 \xi^1 \wedge \xi^2,
\]

\[
d\xi^2 = -J_2 \xi^1 \wedge \xi^2,
\]

\[
d\sigma_{11} = \alpha_1 \wedge \sigma_{11} + \alpha_2 \wedge \xi^1 + \alpha_3 \wedge \xi^2,
\]

\[
d\sigma_{12} = \alpha_1 \wedge \sigma_{12} + \alpha_3 \wedge \theta_0 \wedge \xi^1 + \frac{3}{2} (J_4 + J_1 J_2) \theta_1 \wedge \xi^1 + \frac{3}{2} J_2 \theta_1 \wedge \xi^2 + 3 J_3 \theta_2 \wedge \xi^1 - \frac{3}{2} J_1 \xi^2 \wedge \sigma_{11} + 2 J_2 \xi^1 \wedge \sigma_{12} + \xi^2 \wedge \sigma_{11} - \frac{3}{2} J_1 \xi^2 \wedge \sigma_{12},
\]

\[
d\alpha_1 = \frac{1}{4} (2 J_4 + J_1 J_2) \xi^1 \wedge \xi^2,
\]

\[
d\alpha_2 = \pi_1 \wedge \xi^1 + \pi_2 \wedge \xi^2 + \alpha_1 \wedge \alpha_2 + J_1 \alpha_2 \wedge \xi^2 + J_2 \alpha_3 \wedge \xi^2 - \frac{1}{4} (2 J_4 + J_1 J_2) \xi^2 \wedge \sigma_{11}
\]

\[
d\alpha_3 = \frac{3}{2} (2 D_2(J_4) + 2 J_1 D_2(J_2) + 2 J_2 D_2(J_1) + 6 J_2^2 + 3 J_4 + 1 J_1^2 + 3 J_2 + 4 J_3) - 2 D_1(J_2) \theta_1 \wedge \xi^2 + \frac{1}{2} J_1 J_2^2 + 6 J_3 J_1 - 1 + \frac{3}{4} J_1 D_1(J_2) \theta_2 \wedge \xi^2 + \frac{3}{2} J_2 + 2 D_2(J_1) + 2 (D_2(J_2) - J_1 J_2 - 2 J_4) \xi^2 \wedge \sigma_{12},
\]

where $\alpha_1$, $\alpha_2$, $\alpha_3$, $\pi_1$, and $\pi_2$ are 1-forms on $\mathcal{R} \times \mathcal{H}$. The functions $J_2$, $J_3$, and $J_4$ are defined as follows:

\[
J_2 = \frac{1}{2} \left( 2 T I J_{1x} J_{1tx} - 2 T I J_{1t} J_{1xx} + T I^2 J_1 J_{1t} J_{1x} + I T J_{1x} J_{1t} J_{1x} \right)
\]

\[
J_3 = \frac{1}{32} \left( -135 I^2 T^4 J_{1x}^2 J_{1xx} - 32 T^6 I^2 J_1^2 + 16 J_1^2 T^3 I^2 T_{xx} T_t + 16 J_1^2 T^5 I^2 X_{tx} + 216 I^2 T^2 T_{xx} J_{1x}^2 + 32 T^4 I^2 U_{xx} J_{1x}^2 + 16 T^3 J_1^2 T_{xx} J_{1x}^2 + 8 T^3 I^2 T_{xxx} J_{1x}^2 - 32 I^2 U T^3 J_{1x} J_{1xx}^2 - 36 T^3 I^2 T_{xx} J_{1x}^2 - 16 T^4 I^2 T_{xxx} J_{1x}^2 + 16 J_1^2 T^6 I J_{1t} + 16 T^4 I^2 X_x J_{1x}^2 + 8 T^6 I J_{1x} J_{1xx} + 40 I^2 T^2 T^2 X_x J_{1x}^2 + 8 T^6 I^3 J_{1t} J_{1xx} J_{1x} \right)
\]
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$$-8 J_t^2 T^6 J_{1x} I^3 - 8 J_{1x} T^6 I^3 J_{1tt} J_1 - 8 T^5 I^3 T_t J_{1x} J_{1t} J_1 - 8 T^3 I^2 X^2 T_{xx} J_{1x}^2$$

$$+8 J_t^4 T^5 I^6 J_1 J_{1x} J_2 - 4 J_t^2 T^6 I^4 J_1^2 + 16 T^4 I^2 X_{xx} X_{1x} + 40 T^3 I^2 T_x T_{xx} J_{1x}^2$$

$$+20 T^2 T_x^2 T^2 X^2 J_{1x}^2 - 8 T^4 I^2 X_t T_x J_{1x}^2 - 40 T^2 T^3 X X_x J_{1x}^2 - 40 T^2 T_x^2 T_x J_{1x}^2$$

$$-56 T^2 I^2 T_{xxx} T_x J_{1x}^2 - 16 T^4 I^2 X_{xxx} J_{1x}^2 + 16 T^5 I T_t J_1 J_1 J_t I_t + 40 T^2 T^3 X X_x T_x J_{1x}^2$$

$$+80 T^2 T_x U T^2 J_{1x}^2 - 80 T^2 T^3 U_x T_x J_{1x}^2) ) T^{-4} J_{1x}^2 I^{-6},$$

$$J_4 = 1/8 (-8 J_t^2 T^6 I^3 - 32 T^6 t I^2 J_{1x} - 135 T^2 T_x J_{1x} + 16 T^5 I T_t J_{1x} I_t$$

$$-8 T^4 I^2 X_t T_x J_{1x} + 20 T^2 T_x^2 T^2 X^2 J_{1x} + 40 T^2 T^3 X_x T_x J_{1x} - 8 I^3 J_t T^4 X_{xx} J_1$$

$$-80 T^2 T^3 U_x T_x J_{1x} - 16 T^4 I^2 X_{xxx} J_{1x} - 40 T^2 T^3 X X_x T_x J_{1x} - 8 T^3 I^2 X^2 T_{xx} J_{1x}$$

$$+40 T^3 I^2 T_x T_x J_{1x} - 40 T^2 T^2 T_x T_x J_{1x} + 16 T^2 T^4 X X_x J_{1x} + 16 T^4 I^2 X_x J_{1x}$$

$$+80 T^2 U T^2 J_{1x} - 40 T^2 T^2 T_x J_{1x} - 56 T^2 T_x T_x J_{1x} - 32 T^4 J_{1x} I_0 J_0$$

$$+8 I^3 J_{1x} T^5 X_t J_1 + 216 I^2 T_x^2 T_x J_{1x} + 8 T^3 I^2 T_{xxx} J_{1x} + 8 I^3 J_{1x} T^4 X X_x J_1$$

$$-36 T^2 I^2 T_x J_{1x} + 16 J_{1x} T^6 I_{tt} + 16 J_{1x} T^3 I^2 T_x T_t - 8 J_{1x} T^4 T_x I^3 J_1$$

$$-4 I^3 J_{1x} T_x T^3 X^2 J_1 + 15 I^3 J_{1x} T^3 T_x J_1 + 4 I^3 J_{1x} T^3 T_{xxx} J_1 - 18 I^3 J_{1x} T^2 X_x T_{xx} J_1$$

$$-16 I^3 J_{1x} T_x U T^3 J_1 + 16 T^3 I^2 X_x T_{xxx} J_{1x} - 16 T^4 I^2 T_{xxx} J_{1x} + 8 I^3 J_{1x} T_x T^3 X_x J_{1x}$$

$$-32 I^2 U T^3 T_{xxx} J_{1x} + 16 I^3 J_{1x} T^4 U_x J_1 + 32 T^4 I^2 U_x J_{1x} + 8 J_{1x} T_x T^3 T_t I^3 J_1$$

$$+16 J_{1x} T^5 I^2 X_x J_{1x}) J_{1x} J_{1x} T^{-4} I^{-6} J_{1x}^{-1}.$$

They are invariant of the symmetry pseudo-group for equation \(\Upsilon\) from \(\mathcal{P}_2\). The invariant differential operators are

$$\mathcal{D}_1 = \frac{\partial}{\partial t} = T I^{-1} D_t - J_t I^{-2} J_{1x} D_x, \quad \mathcal{D}_2 = \frac{\partial}{\partial x} = I^{-1} D_x,$$

where \(D_t\) and \(D_x\) are the operators of total differentiation w.r.t. \(t\) and \(x\). These \(\mathcal{D}_1\) and \(\mathcal{D}_2\) are found without any integration. Indeed, they satisfy \(dF = \mathcal{D}_1(F) \xi_1 + \mathcal{D}_2(F) \xi_2\) for an arbitrary function \(F = F(t, x)\). Since \(\xi_1 = I^2 T^{-1} dt\) and \(\xi_2 = I J_{1x} J_{1x} dt + I dx\), we have \(\mathcal{D}_2\).

To construct all the other invariants of the pseudo-group, we apply \(\mathcal{D}_1\) and \(\mathcal{D}_2\) to \(J_i\) in an arbitrary order: \(\mathcal{D}_1^{k_1} \mathcal{D}_2^{k_2} ... \mathcal{D}_1^{k_{n-1}} \mathcal{D}_2^{k_n} J_i\). The commutator identity

$$[\mathcal{D}_1, \mathcal{D}_2] = J_1 \mathcal{D}_1 + J_2 \mathcal{D}_2$$

allows us to permute the coframe derivatives, so we need only to deal with the derived invariants \(J_{i,kl} = \mathcal{D}_1^k \mathcal{D}_2^l(J_i), i \in \{1, ..., 4\}, k \geq 0, \text{ and } l \geq 0\). For \(s \geq 0\) define the \(s\)th order classifying manifold associated with the coframe \(\theta = \{\theta_0, \theta_1, \theta_2, \xi_1, \xi_2, \sigma_{11}, \sigma_{12}, \alpha_1, \alpha_2, \alpha_3\}\) and an open subset \(U \subset \mathbb{R}^2\) as

$$\mathcal{C}^{(s)}(\theta, U) = \{(J_{i,kl}(t, x)) \mid i \in \{1, ..., 4\}, \ k + l \leq s, \ (t, x) \in U\}$$
Contact Equivalence Problem for Linear Parabolic Equations

Since all the functions $J_{i,kl}$ depend on two variables $t$ and $x$, it follows that $\rho_s = \dim \mathcal{C}^s(\theta, U) \leq 2$ for all $s \geq 0$. Let $r = \min\{s \mid \rho_s = \rho_{s+1} = \rho_{s+2} = \ldots\}$ be the order of the coframe $\theta$. Since $J_{1x} \neq 0$, we have $1 \leq \rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots \leq 2$. In any case, $r + 1 \leq 2$. Hence from Theorem 15.12 of [13] we see that two linear parabolic equations \(1\) from the subclass $\mathcal{P}_2$ are locally equivalent under a contact transformation if and only if their second order classifying manifolds \(14\) locally overlap.

**Remark** A Lie pseudo-group is called structurally intransitive, [12], if it is not isomorphic to any transitive Lie pseudo-group. In [4], Cartan proved that a Lie pseudo-group is structurally intransitive whenever it has essential invariants. An invariant of a Lie pseudo-group with the structure equations

$$d\omega^i = A_{jk}^i \pi^j \wedge \omega^k + T_{jk}^i \omega^j \wedge \omega^k$$

is called *essential*, if it is a first integral of the systatic system $A_{jk}^i \omega^k$. From the structure equations \(12\), it follows that the systatic system for the symmetry pseudo-group for an equation from $\mathcal{P}_2$ is generated by the forms $\xi^1$ and $\xi^2$. First integrals of these forms are arbitrary functions of $t$ and $x$. Therefore all the invariants $J_1, \ldots, J_4$, and all the derived invariants are essential. Thus the symmetry pseudo-group of equation \(1\) from the subclass $\mathcal{P}_2$ is structurally intransitive.

Now we return to the case $J_{1x} = 0$. Then the structure equations have the form

$$d\theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,$$

$$d\theta_1 = \alpha_1 \wedge \theta_1 + 2 \alpha_2 \wedge \theta_2 - \frac{1}{2} J_1 \alpha_2 \wedge \theta_0 - \frac{1}{2} T^2 I^-4 J_{1t} (b_1^2 - L_0) \theta_0 \wedge \xi^1 + \frac{1}{4} T J_{1t} I^-2 \theta_0 \wedge \xi^2 + J_1 \theta_1 \wedge \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12},$$

$$d\theta_2 = \alpha_1 \wedge \theta_2 + \alpha_2 \wedge \theta_0 - \theta_1 \wedge \xi^2 + \frac{1}{2} J_1 \theta_2 \wedge \xi^1 + \xi^1 \wedge \sigma_{12},$$

$$d\xi^1 = -J_1 \xi^1 \wedge \xi^2,$$

$$d\xi^2 = -2 \alpha_2 \wedge \xi^1,$$

$$d\sigma_{11} = \alpha_1 \wedge \sigma_{11} + 4 \alpha_2 \wedge \sigma_{12} - 3 J_1 \alpha_2 \wedge \theta_1 + \alpha_3 \wedge \xi^1 + \alpha_4 \wedge \xi^2,$$

$$d\sigma_{12} = \alpha_1 \wedge \sigma_{12} + 3 \alpha_2 \wedge \theta_1 - \frac{3}{2} J_1 \alpha_2 \wedge \theta_2 + \alpha_4 \wedge \xi^1 - \theta_0 \wedge \xi^1 - \frac{3}{4} T I^-2 J_{1t}^2 (2 \theta_1 \wedge \xi^1 - \theta_2 \wedge \xi^2) + 2 J_1 \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{11} - \frac{3}{2} J_1 \xi^2 \wedge \sigma_{12},$$

where

$$L_0 = -1/16 (135 T_x^4 I^2 + 16 J_1 T^3 I^3 T_x U - 16 T^5 T^2 X_{tx} + 16 T^4 I^2 X_{xxx} - 16 J_1 T^4 I^3 U_x$$

$$- 8 J_1 T^{3} T_{x} T_{t} + 40 T^2 I^2 T_{t} T_{x}^2 - 15 J_1 T I^3 T_{x}^3 - 4 J_1 T^3 I^3 T_x X^2 - 20 T^2 I^2 T_x^2 X^2$$

$$- 80 T^2 I^2 T_{x}^2 U + 8 J_1 T^4 I^3 X_{tx} - 16 T^4 I^2 X_{xx} X_{xx} + 32 T^3 I^2 U T_{xx} - 216 T I^2 T_{x}^2 T_{xx}$$

$$+ 18 J_1 T^2 I^3 T_{x} T_{xx} + 8 T^3 I^2 X^2 T_{xx} - 40 T^3 I^2 X_{xx} T_{x} - 16 T^4 I^2 X_{x}^2 - 16 T^3 I^2 T_{t} T_{xx}.$$
+56 \ T^2 I^2 T_{xxx} T_x + 80 \ T^3 I^2 U_x T_x - 4 \ J_1 T^3 I^3 T_{xxx} + 32 \ T^6 I_t^2 + 36 \ T^2 I^2 T_{xx}^2 \\
-8 \ J_1 T^4 I^3 X X_x - 8 \ J_1 T^3 I^3 T_x X_x + 40 \ T^2 I^2 T_x^2 X_x - 16 \ T^5 I T_t I_t + 40 \ T^3 I^2 X X_x T_x \\
-16 \ T^3 I^2 X_x T_{xx} - 16 \ T^6 I I_t - 8 \ J_1 T^5 I X_t + 8 \ T^4 I^2 X T_x - 32 \ T^4 I^2 U_x \\
+8 \ J_1 T^4 I^3 T_{tx} - 40 \ T^3 I^2 T_{tx} T_x + 16 \ T^4 I^2 T_{txx} - 8 \ T^3 I^2 T_{xxxx} \ T^{-6} I^{-2} J_{tt}^{-1}.

Consider the subclass \( \mathcal{P}_3 \) of all equations \( \text{II} \) such that \( I \neq 0, \ J_{t_x} = 0, \) and \( J_{tt} \neq 0. \) This subclass is not empty, since the equation \( u_{xx} = u_t + Q(t) x^{-2} u \) with \( Q'(t) \neq 0 \) belongs to \( \mathcal{P}_3. \) For an equation from \( \mathcal{P}_3, \) we normalize the coefficient in the structure equation for \( d\theta_1 \) at \( \theta_0 \land \xi^1 \) by setting \( b_1^2 = L_0. \) Then we prolong the structure equations and obtain

\[
d\theta_0 = \alpha_1 \land \theta_0 + \xi^2 \land \theta_2 + \xi^1 \land \theta_1,
\]

\[
d\theta_1 = \alpha_1 \land \theta_1 - \frac{1}{4} J_1 L_2 \theta_0 \land \xi^1 + \frac{1}{4} (D_1(J_1) - J_1 L_1) \theta_0 \land \xi^2 + J_1 \theta_1 \land \xi^2 + L_1 \theta_2 \land \xi^2 \\
+ \xi^2 \land \sigma_{12} + \xi^1 \land \sigma_{11} + L_2 \theta_2 \land \xi^1,
\]

\[
d\theta_2 = \alpha_1 \land \theta_2 + \frac{1}{2} L_2 \theta_0 \land \xi^1 + \frac{1}{2} L_1 \theta_0 \land \xi^2 - \theta_1 \land \xi^2 + \frac{1}{2} J_1 \theta_2 \land \xi^2 + \xi^1 \land \sigma_{12},
\]

\[
d\xi^1 = - J_1 \xi^1 \land \xi^2,
\]

\[
d\xi^2 = - L_1 \xi^1 \land \xi^2,
\]

\[
d\sigma_{11} = \alpha_1 \land \sigma_{11} + \alpha_2 \land \xi^1 + \alpha_3 \land \xi^2,
\]

\[
d\sigma_{12} = \alpha_1 \land \sigma_{12} + \alpha_3 \land \xi^1 - \theta_0 \land \xi^1 - \frac{3}{2} (D_1(J_1) - J_1 L_1 - L_2) \theta_1 \land \xi^1 \\
+ \frac{3}{2} L_1 \theta_1 \land \xi^2 - \frac{3}{4} J_1 L_2 \theta_2 \land \xi^1 + \frac{3}{4} (D_1(J_1) - J_1 L_1) \theta_2 \land \xi^2 + 2 J_1 \xi^1 \land \sigma_{11} \\
+ 2 L_1 \xi^1 \land \sigma_{12} + \xi^2 \land \sigma_{11} - \frac{3}{2} J_1 \xi^2 \land \sigma_{12},
\]

\[
d\alpha_1 = \frac{1}{4} (2 L_2 - D_1(J_1) + J_1 L_1) \xi^1 \land \xi^2,
\]

\[
d\alpha_2 = \pi_1 \land \xi^1 + \pi_2 \land \xi^2 + \alpha_1 \land \alpha_2 - J_1 \alpha_2 \land \xi^2 + L_1 \alpha_3 \land \xi^2 \\
- \frac{1}{4} (2 L_2 - D_1(J_1) + J_1 L_1) \xi^2 \land \sigma_{11},
\]

\[
d\alpha_3 = \pi_2 \land \xi^1 + \alpha_1 \land \alpha_3 - \alpha_2 \land \xi^2 + \frac{9}{4} J_1 \alpha_3 \land \xi^2 + \left(\frac{5}{8} D_1(J_1)^2 - \frac{3}{4} D_1(J_1) J_1 L_1 \\
+ \frac{3}{8} L_1^2 L_1^2 - \frac{5}{2} J_1 \right) \theta_0 \land \xi^2 + \frac{3}{2} (D_2(D_1(J_1)) - 6 D_2(L_2) - 6 J_1 D_2(L_1) + J_1 D_1(J_1) \\
- 6 J_1 L_2 - J_1^2 L_1 - 2 L_1^2 + 6 D_1(L_1)) \theta_1 \land \xi^2 + \left( \frac{9}{4} L_1 D_1(J_1) - \frac{15}{4} J_1 L_1^2 - \frac{3}{2} J_1^2 L_2 \right. \\
- 1 + \frac{3}{4} J_1 D_1(L_1) - \frac{3}{4} J_1 D_2(L_2) - \frac{3}{4} D_2(J_1) \right) \theta_2 \land \xi^2 + \left( \frac{9}{2} L_1 + 5 J_1^2 \right) \xi^2 \land \sigma_{11} \\
- 2 \left( L_2 - 2 D_1(J_1) + J_1 L_1 - D_2(L_1) \right) \xi^2 \land \sigma_{12},
\]

where

\[
L_1 = T \ I^{-3} (L_{0x} - I_t),
\]
\[ L_2 = -1/8 (8 I^2 T^3 T_{xx} - 8 I^2 T_x T^2 X_x - 15 I^2 T_x^3 + 4 I^2 T_x T^2 X^2 + 16 I^2 T_x U T^2 - 8 T^4 I^2 X_t - 8 I^2 T_x T^2 T_t - 8 IT^5 L_0 t + 18 T I^2 T_x T_{xx} - 8 L_0 T^4 I T_t - 4 L_0^2 T^5 I J_1 + 8 T^4 L_0 L_1 I^3 + 16 T^5 I_t L_0 + 8 I^2 T^3 X_{xx} - 8 I^2 T^3 X X_x - 4 T^2 I^2 T_{xxx} - 16 I^2 T^3 U_x) I^{-5} T^{-3}, \]

and the invariant differential operators are defined by
\[ \mathcal{D}_1 = \frac{\partial}{\partial \xi^1} = T I^{-2} D_t - T I^{-3} L_0 D_x, \quad \mathcal{D}_2 = \frac{\partial}{\partial \xi^2} = I^{-1} D_x. \]

The commutator relation for invariant differentiations is
\[ [\mathcal{D}_1, \mathcal{D}_2] = J_1 \mathcal{D}_1 + L_1 \mathcal{D}_2. \]

The \( s^{th} \) order classifying manifold associated with the involutive coframe \( \theta = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{12}, \alpha_1, \alpha_2, \alpha_3\} \) and an open subset \( U \subset \mathbb{R}^2 \) is
\[ \mathcal{C}(\theta, U) = \left\{ (D_{1}^{k}D_{2}^{l}(J_1(t, x)), D_{1}^{k}D_{2}^{l}(L_1(t, x))) \mid i \in \{1, 2\}, k + l \leq s, (t, x) \in U \right\} \quad (15) \]

Thus two linear parabolic equations (14) from the subclass \( \mathcal{P}_3 \) are locally equivalent under a contact transformation if and only if their second order classifying manifolds (15) locally overlap.

Since all the invariants of the symmetry pseudo-group for an equation from \( \mathcal{P}_3 \) are first integrals of the systatic system \( \xi^1, \xi^2 \), this pseudo-group is structurally intransitive.

Now we return to the case \( J_{1x} = J_{1t} = 0 \). We denote \( J_1 = N = \text{const} \), then the structure equations have the form
\[
\begin{align*}
  d\theta_0 &= \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \\
  d\theta_1 &= \alpha_1 \wedge \theta_1 - \frac{1}{2} N \alpha_2 \wedge \theta_0 + 2 \alpha_2 \wedge \theta_2 + M_1 \theta_0 \wedge \xi^1 + N \theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12}, \\
  d\theta_2 &= \alpha_1 \wedge \theta_2 + \alpha_2 \wedge \theta_0 - \theta_1 \wedge \xi^2 + \frac{1}{2} N \theta_2 \wedge \xi^2 + \xi^1 \wedge \sigma_{12}, \\
  d\xi^1 &= -N \xi^1 \wedge \xi^2, \\
  d\xi^2 &= -2 \alpha_2 \wedge \xi^1, \\
  d\sigma_{11} &= \alpha_1 \wedge \sigma_{11} - 3 N \alpha_2 \wedge \theta_1 + 4 \alpha_2 \wedge \sigma_{12} + 2 \alpha_3 \wedge \xi^1 + \alpha_4 \wedge \xi^2, \\
  d\sigma_{12} &= \alpha_1 \wedge \sigma_{12} + 3 \alpha_2 \wedge \theta_1 - \frac{3}{2} N \alpha_2 \wedge \theta_2 + \alpha_4 \wedge \xi^1 - \theta_0 \wedge \xi^1 + 3 M_1 \theta_2 \wedge \xi^1 + 2 N \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{11} - \frac{3}{2} N \xi^2 \wedge \sigma_{12},
\end{align*}
\]

where
\[
\begin{align*}
  M_1 &= 1/32 (-40 I^2 T^3 X X_x T_x + 32 I^2 T^4 U_{xx} - 32 T^6 I_t^2 + 16 N I^3 T^4 U_x + 8 I^2 T^3 T_{xxxx} + 16 T^4 X X_{xx} - 8 N I^3 T^4 X_x X_x + 8 N I^3 T^4 X X_x + 8 N I^3 T^5 X_t + 16 I^2 T^6 X_{tx} - 8 N I^3 T^4 T_x + 16 T^5 I T_t I_t + 80 I^2 T^2 T_x^2 U + 20 I^2 T^2 T_x^2 X^2 - 16 N I^3 T^3 T_x U + 15 N I^3 T T_x^3 - 56 I^2 T^2 T_{xxx} T_x + 216 I^2 T T_x^2 T_{xx} + 40 I^2 T^3 X_{xx} T_x - 18 N I^3 T^2 T_x T_{xx} + 16 I^2 T^4 X_x^2 + 40 I^2 T^3 T_{tx} T_x - 135 I^2 T_x^4 + 8 N I^3 T^3 T_x X_x
\end{align*}
\]
All the essential torsion coefficients now are independent of the group parameters, but

\[-8 I^2 T^4 X_t T_x - 40 I^2 T^2 T_x^2 X_x - 80 I^2 T^3 U_x T_x - 8 I^2 T^3 X^2 T_x x - 32 I^2 T^3 U T_x x\]

\[+16 I^2 T^3 X_t T_{xx} + 4 N I^3 T^3 T_{xx} + 16 I^2 T^3 T_t T_x + 8 N I^3 T^3 T_x T_t - 40 I^2 T^2 T_t T_x^2\]

\[-4 N I^3 T^3 T_x X^2 - 36 I^2 T^2 T_{xx}^2 - 16 I^2 T^4 X_{xxx} - 16 I^2 T^4 T_{xxx} + 16 IT^6 T_{tt} I^{-6} T^{-4}.\]

All the essential torsion coefficients now are independent of the group parameters, but

\[d M_1 = \left( \left( \frac{3}{2} N M_1 + 1 \right) b_1^2 + M_{1t} \right) T I^{-2} \xi^1 - \left( \frac{3}{2} N M_1 N - 1 \right) \xi^2.\]

By \( P_4 \) we denote the subclass of all equations (11) such that \( I \neq 0, J_1 = N = \text{const}, \) and \( 3 N M_1 \neq -2. \) This subclass contains, e.g., the equation \( u_{xx} = u_t + (\kappa x^{-2} + \nu x) u \) with \( \kappa \neq 0, \nu \neq 0. \) For an equation from \( P_4, \) we set \( b_1^2 = -2 M_{1t} (3 N M_1 + 2)^{-1}. \) After this normalization, we prolong the structure equations and obtain

\[d \theta_0 = \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\]

\[d \theta_1 = \alpha_1 \wedge \theta_1 + (M_1 - N M_3) \theta_0 \wedge \xi^1 - \frac{1}{4} N M_2 \theta_0 \wedge \xi^2 + N \theta_1 \wedge \xi^2 + 4 M_3 \xi_2 \wedge \xi^1 + M_2 \theta_2 \wedge \xi^2 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12},\]

\[d \theta_2 = \alpha_1 \wedge \theta_2 + 2 M_3 \theta_0 \wedge \xi^1 + \frac{1}{4} M_2 \theta_0 \wedge \xi^2 - \theta_1 \wedge \xi^2 + \frac{1}{2} N \theta_2 \wedge \xi^2 + \xi^1 \wedge \sigma_{12},\]

\[d \xi_1 = -N \xi^1 \wedge \xi^2,\]

\[d \xi_2 = -M_2 \xi^1 \wedge \xi^2,\]

\[d \sigma_{11} = \alpha_1 \wedge \sigma_{11} + \alpha_2 \wedge \xi^1 + \alpha_3 \wedge \xi^2,\]

\[d \sigma_{12} = \alpha_1 \wedge \sigma_{12} + \alpha_3 \wedge \xi^1 - \theta_0 \wedge \xi^1 + (6 M_3 + \frac{3}{2} N M_2) \theta_1 \wedge \xi^1 + \frac{3}{2} M_2 \theta_1 \wedge \xi^2 + 3 (M_1 - N M_3) \theta_2 \wedge \xi^1 - \frac{3}{4} N M_2 \theta_2 \wedge \xi^2 + 2 N \xi^1 \wedge \sigma_{11} + 2 M_2 \xi^1 \wedge \sigma_{12},\]

\[+ \xi^2 \wedge \sigma_{11} - \frac{3}{4} N \xi^2 \wedge \sigma_{12},\]

\[d \alpha_1 = \left( 2 M_3 + \frac{1}{4} N M_2 \right) \xi^1 \wedge \xi^2,\]

\[d \alpha_2 = \pi_1 \wedge \xi^1 + \pi_2 \wedge \xi^2 + \alpha_1 \wedge \alpha_2 + N \alpha_2 \wedge \xi^2 + M_2 \alpha_3 \wedge \xi^2 - (2 M_3 + \frac{1}{4} N M_2) \xi^2 \wedge \sigma_{11},\]

\[d \alpha_3 = \pi_2 \wedge \xi^1 + \alpha_1 \wedge \alpha_2 - \frac{9}{4} N \alpha_2 \wedge \xi^2 + \left( \frac{3}{8} N^2 M_2^2 - \frac{5}{2} N \right) \theta_0 \wedge \xi^2 + \left( 3 M_1 - \frac{3}{2} D_1(M_2) + 6 N M_3 + \frac{9}{4} N^2 M_2 + \frac{9}{2} M_2^2 + \frac{3}{2} N D_2(M_2) \right) \theta_1 \wedge \xi^2 + \left( 6 N M_1 - \frac{15}{4} N M_2^2 - 6 N^2 M_3 - 1 + 3 D_2(M_1) \right) \theta_{2} \wedge \xi^2 + \left( \frac{9}{2} M_2 + 5 N^2 \right) \xi^2 \wedge \sigma_{11} + \left( 8 M_3 + 2 N M_2 - 2 D_2(M_2) \right) \xi^2 \wedge \sigma_{12},\]

where

\[M_2 = T (M_{0x} - I_t) I^{-3}, \quad M_0 = -2 M_{1t} (3 N M_1 + 2)^{-2};\]

\[M_3 = -1/32 (-8 IT^5 M_{0t} - 8 T^4 M_0 M_2 I^3 + 16 T^5 I_t M_0 - 4 M_0^2 T^3 I N - 16 I^2 T^3 U_x)\]
+8 I^2 T^3 X_{xx} + 8 I^2 T^3 T_{tx} - 4 T^2 I^2 T_{xxx} + 16 I^2 T_x U T^2 - 8 M_0 T^4 I T_t \\
-8 I^2 T^3 X X_j + 18 I^2 T_x T_{xx} + 4 I^2 T_x T^2 X^2 - 8 I^2 T_x T^2 T_t - 8 I^2 T_x T^2 X_x \\
-15 I^2 T_x ^3 - 8 T^4 I^2 X_t) I^{-5} T^{-3}.

The invariant differential operators

\[ D_1 = T I^{-2} D_t - T M_0 I^{-3} D_x, \quad D_2 = I^{-1} D_x, \]

satisfy the commutator relation

\[ [D_1, D_2] = N D_1 + M_2 D_2. \]

The \( s^{th} \) order classifying manifold associated with the involutive coframe \( \theta = \{ \theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{12}, \alpha_1, \alpha_2, \alpha_3 \} \) and an open subset \( U \subset \mathbb{R}^2 \) is

\[ C^s(\theta, U) = \{ (D^k \theta^l (M_i(t, x))) | i \in \{1, 2, 3\}, k + l \leq s, (t, x) \in U \}. \] (16)

So two linear parabolic equations \( (\mathcal{P}) \) from the subclass \( \mathcal{P}_4 \) are locally equivalent under a contact transformation if and only if their second order classifying manifolds \( (16) \) locally overlap.

The systatic system for the symmetry pseudo-group of equation \( (\mathcal{P}) \) from the subclass \( \mathcal{P}_4 \) is generated by \( \xi^1 \) and \( \xi^2 \) again, and, as all the differential invariants are essential, this pseudo-group is structurally intransitive.

Finally, consider the subclass \( \mathcal{P}_5 \) of all equations \( (\mathcal{P}) \) such that \( I \neq 0, J_1 = N = const, \) and \( M_1 = -2/(3N) \). For an equation from \( \mathcal{P}_5 \), after a prolongation, the structure equations have the form

\[
\begin{align*}
d\theta_0 &= \alpha_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2, \\
d\theta_1 &= \alpha_1 \wedge \theta_1 - \frac{N}{2} \alpha_2 \wedge \theta_0 + 2 \alpha_2 \wedge \theta_2 - \frac{2}{3N} \theta_0 \wedge \xi^1 + N \theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12}, \\
d\theta_2 &= \alpha_1 \wedge \theta_2 + \alpha_2 \wedge \theta_0 - \theta_1 \wedge \xi^2 + \frac{N}{2} \theta_2 \wedge \xi^2 + \xi^1 \wedge \sigma_{12}, \\
d\xi^1 &= -N \xi^1 \wedge \xi^2, \\
d\xi^2 &= -2 \alpha_2 \wedge \xi^1, \\
d\sigma_{11} &= \alpha_1 \wedge \sigma_{11} - 3N \alpha_2 \wedge \theta_1 + 4 \sigma_{12} + \alpha_3 \wedge \xi^1 + \alpha_4 \wedge \xi^2, \\
d\sigma_{12} &= \alpha_1 \wedge \sigma_{12} - \frac{3N}{2} \alpha_2 \wedge \theta_2 + 3 \alpha_2 \wedge \theta_1 + \alpha_4 \wedge \xi^1 - \theta_0 \wedge \xi^1 - \frac{2}{N} \theta_2 \wedge \xi^1 + 2N \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{11} - \frac{3N}{2} \xi^2 \wedge \sigma_{12}, \\
d\alpha_1 &= \frac{N}{2} \alpha_2 \wedge \xi^1 - \alpha_2 \wedge \xi^2, \\
d\alpha_2 &= N \alpha_2 \wedge \xi^2 - \frac{2}{3N} \xi^1 \wedge \xi^2, \\
d\alpha_3 &= \pi_1 \wedge \xi^1 + \pi_2 \wedge \xi^2 + \alpha_1 \wedge \alpha_3 + 6 \alpha_2 \wedge \alpha_4 - 2 \alpha_2 \wedge \theta_0 - \frac{8}{N} \alpha_2 \wedge \theta_2 - \frac{2N}{3} \alpha_2 \wedge \sigma_{11} \\
&\quad + N \alpha_3 \wedge \xi^2 + 2 \theta_1 \wedge \xi^2 + \frac{8}{3N} \xi^2 \wedge \sigma_{12},
\end{align*}
\]
\[ d\alpha_4 = \pi_2 \wedge \xi^1 + \alpha_1 \wedge \alpha_4 - 6 N^2 \alpha_2 \wedge \theta_1 - 5 \alpha_2 \wedge \sigma_{11} + 13 N \alpha_2 \wedge \sigma_{12} - \alpha_3 \wedge \xi^2 \]
\[ + \frac{9N}{2} \alpha_4 \wedge \xi^2 - \frac{5N}{2} \theta_0 \wedge \xi^2 - \frac{4}{N} \theta_1 \wedge \xi^2 - 4 \theta_2 \wedge \xi^2 + 5 N^2 \xi^2 \wedge \sigma_{11}. \]

From these structure equations, it follows that the classifying manifold is a point, and that two equations from the subclass \( P_5 \) are equivalent under a contact transformation iff they have the same value of the constant \( N \). Repeating the calculations for the equation
\[ u_{xx} = u_t + \tilde{N} x^{-2} u, \quad (17) \]
we see that its symmetry pseudo-group has the same structure equations whenever \( \tilde{N} = -4/(3 N^5) \). Thus the linear parabolic equation (11) is equivalent to an equation of the form (17) under a contact transformation if and only if it belongs to the subclass \( P_5 \).

The results of the calculations are summarized in the following statement:

**Theorem 2** The class of linear parabolic equations (1) is divided into the five subclasses \( P_1, P_2, ..., P_5 \) invariant under an action of the pseudo-group of contact transformations:

- \( P_1 \) consists of all equations (1) such that \( I = 0 \);
- \( P_2 \) consists of all equations (1) such that \( I \neq 0 \) and \( J_{1x} \neq 0 \);
- \( P_3 \) consists of all equations (1) such that \( I \neq 0, J_{1x} = 0, \) and \( J_{1t} \neq 0 \);
- \( P_4 \) consists of all equations (1) such that \( I \neq 0, J_1 = N = \text{const}, \) and \( 3 N M_1 \neq -2 \);
- \( P_5 \) consists of all equations (1) such that \( I \neq 0, J_1 = N = \text{const}, \) and \( 3 N M_1 = -2 \).

Every equation from the subclass \( P_1 \) is equivalent to the linear heat equation (11). Two equations from one of the subclasses \( P_2, P_3, \) or \( P_4 \) are locally equivalent to each other if and only if the classifying manifolds (14), (15), or (16) for these equations locally overlap.

Every equation from the subclass \( P_5 \) is locally equivalent to the equation (17) whenever \( \tilde{N} = -4/(3 N^5) \).

**Conclusion**

In this paper, the moving coframe method of [6] is applied to the local equivalence problem for the class of linear second-order parabolic equations in two independent variables under an action of the pseudo-group of contact transformations. The class is divided into the five invariant subclasses. We have found the structure equations and the complete sets of differential equations for all the subclasses. The solution of the equivalence problem is given in terms of the differential invariants. It is shown that the moving coframe method is applicable to structurally intransitive symmetry pseudo-groups. The moving coframe method allows us to find invariant 1-forms, structure equations, differential invariants, and operators of invariant differentiation for symmetry pseudo-groups of differential equations without analyzing over-determined systems of partial differential equation or using procedures of integration.
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