Decay estimates for the linear damped wave equation on the Heisenberg group

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Abstract

This paper is devoted to the derivation of $L^2 - L^2$ decay estimates for the solution of the homogeneous linear damped wave equation on the Heisenberg group $\mathbb{H}_n$, for its time derivative and for its horizontal gradient. Moreover, we consider the improvement of these estimates when further $L^1(\mathbb{H}_n)$ regularity is required for the Cauchy data. Our approach will rely strongly on the group Fourier transform of $\mathbb{H}_n$ and on the properties of the Hermite functions that form a maximal orthonormal system for $L^2(\mathbb{R}^n)$ of eigenfunctions of the harmonic oscillator.

Keywords: damped wave equation, decay estimates, Heisenberg group, group Fourier transform, Hermite functions

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1. Introduction

In this work, we derive decay estimates for the solution of the damped wave equation on the Heisenberg group, for its time derivative and for its horizontal gradient. In other words, we estimate the $L^2(\mathbb{H}_n)$ - norm of the solution $u$ to the linear Cauchy problem

$$
\begin{align*}
\partial_t^2 u(t,\eta) - \Delta_{\mathbb{H}} u(t,\eta) + \partial_t u(t,\eta) &= 0, \quad \eta \in \mathbb{H}_n, \ t > 0, \\
u(0,\eta) &= u_0(\eta), \quad \eta \in \mathbb{H}_n, \\
\partial_t u(0,\eta) &= u_1(\eta), \quad \eta \in \mathbb{H}_n,
\end{align*}
$$

and its first order derivatives $\partial_t u$ and $\nabla_{\mathbb{H}} u$.

In the last decades, several papers have been devoted to the study of linear PDE in not-Euclidean structures. Let us recall some results from the literature in the case of Lie group for two equations, the heat equation and the wave equation, which are close, in some sense, to the equation that we will study in this paper, the damped wave equation. The properties of the solutions to the heat equation on the Heisenberg group or on more general H-type groups (and of the corresponding heat kernels) are investigated in [15, 2, 18, 4]. In the case of compact Lie group we refer to [9, 10] or to [16] for the special case $\mathbb{S}^{2n+1}$. For the description of the heat kernel in a nilpotent Lie group we refer to the pioneering work of Folland [12] and to [36]. On the other hand, for the wave equation on the Heisenberg group or on H-type groups we have to mention [24, 23, 1, 17, 22] while in the framework of compact group we cite [3, 14]. In the case of graded groups, we refer the recent works [33, 29, 31], where the well-posedness of the wave equation is studied in the case of time-dependent speed of propagation $a = a(t)$ in different function spaces, depending on the assumption on $a$.

The main idea in what follows is to generalize some well-know facts in the phase space analysis (WKB analysis) for the Euclidean case (cf. [8] for an overview on this topic) to the case of the Heisenberg group.

The Heisenberg group is the Lie group $\mathbb{H}_n = \mathbb{R}^{2n+1}$ equipped with the multiplication rule

$$(x, y, \tau) \circ (x', y', \tau') = (x + x', y + y', \tau + \tau' + \frac{1}{2}(x \cdot y' - x' \cdot y)).$$
where \( \cdot \) denotes the standard scalar product in \( \mathbb{R}^n \). A system of left-invariant vector fields that span the Lie algebra \( \mathfrak{h}_n \) is given by
\[
X_j = \partial x_j - \frac{y_j}{2} \partial t, \quad Y_j = \partial y_j + \frac{x_j}{2} \partial t, \quad T = \partial r,
\]
where \( 1 \leq j \leq n \). This system satisfies the commutation relations
\[
[X_j, Y_k] = \delta_{jk} T \quad \text{for} \quad 1 \leq j, k \leq n.
\]
Therefore, \( \mathfrak{h}_n \) admits the stratification \( \mathfrak{h}_n = V_1 \oplus V_2 \), where \( V_1 \approx \text{span}\{X_j, Y_j\}_{1 \leq j \leq n} \) and \( V_2 \approx \text{span}\{T\} \). This means that \( H_n \) is a 2 step stratified Lie group, whose homogeneous dimension is \( \mathcal{G} = 2n + 2 \). The sub-Laplacian on \( H_n \) is defined as
\[
\Delta_H \equiv \sum_{j=1}^{n} \left( X_j^2 + Y_j^2 \right) = \sum_{j=1}^{n} \left( \partial_{x_j}^2 + \partial_{y_j}^2 \right) + \frac{1}{4} \sum_{j=1}^{n} \left( x_j^2 + y_j^2 \right) \partial_{y_j}^2 + \sum_{j=1}^{n} \left( x_j \partial_{y_j} y_j - y_j \partial_{x_j} x_j \right).
\]

For a function \( v : H_n \to \mathbb{R} \), the horizontal gradient of \( v \)
\[
\nabla_H v = (X_1 v, \ldots, X_n v, Y_1 v, \ldots, Y_n v) \equiv \sum_{j=1}^{n} ((X_j v)X_j + (Y_j v)Y_j),
\]
where each fiber of the horizontal subbundle can be endowed with a scalar product in such a way that the horizontal vector fields \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) are orthonormal in each point \( \eta \in H_n \). For a function \( v \in L^2(H_n) \) we say that \( X_j v, Y_j v \in L^1_{\text{loc}}(H_n) \) exist in the sense of distributions, if the integral relations
\[
\int_{H_n} (X_j v)(\eta) \phi(\eta) \, d\eta = \int_{H_n} v(\eta)(X_j^\ast \phi)(\eta) \, d\eta \quad \text{and} \quad \int_{H_n} (Y_j v)(\eta) \phi(\eta) \, d\eta = \int_{H_n} v(\eta)(Y_j^\ast \phi)(\eta) \, d\eta
\]
are fulfilled for any \( \phi \in C_0^\infty(H_n) \), where \( X_j^\ast = -X_j \) and \( Y_j^\ast = -Y_j \) denote the formal adjoint operators of \( X_j \) and \( Y_j \), respectively. Therefore, in our framework, the Sobolev space \( H^1(H_n) \) is the set of all functions \( v \in L^2(H_n) \) such that \( X_j v, Y_j v \) exist in the sense of distributions and \( X_j v, Y_j v \in L^2(H_n) \) for any \( j = 1, \ldots, n \).

In [30], some ideas of the phase space analysis have been applied to the study of semilinear damped wave equation with mass on graded Lie groups \( \mathcal{G} \) for a positive Rockland operator \( \mathcal{R} \), namely, the semilinear Cauchy problem
\[
\begin{align*}
\partial_t^2 u(t, x) + \mathcal{R} u(t, x) + \partial_t u(t, x) + u(t, x) &= |u|^p, & x \in \mathcal{G}, \ t > 0, \\
 u(0, x) &= u_0(x), & x \in \mathcal{G}, \\
 \partial_t u(0, x) &= u_1(x), & x \in \mathcal{G}.
\end{align*}
\]

The main tool for this purpose is the group Fourier transform on \( \mathcal{G} \). In order to deal with this family of bounded operators \( \hat{u}(\pi) \) associated to irreducible unitary representations \( \pi \in \hat{\mathcal{G}} \) on some Hilbert spaces \( H_\pi \), the authors of [30] consider for each of these bounded operators its projection on a suitable basis of eigenfunctions for \( H_\pi \) that allows to “diagonalize” the operator given by the action of the infinitesimal representation \( d\sigma \) on \( \mathcal{R} \). As particular case, the Heisenberg group is considered with \( \mathcal{R} = -\Delta_H \). Due to the presence of a mass term in the linear part of the damped wave equation, an exponential decay rate is obtained for the \( L^2(\mathcal{G}) \)-norms of the solution of the corresponding homogeneous linear problem and its derivatives. Therefore, thanks to this extremely fast decay rate they can work with data just on \( L^2(\mathcal{G}) \) basis (and regularity related to the order of the derivative, of course). As a consequence, they need only the fact that \( d\sigma(\mathcal{R}) \) has discrete positive spectrum for any nontrivial unitary representation \( \pi \) of \( \mathcal{G} \) and they do not have to specifying neither the growth rate of the eigenvalues, nor the action of \( d\sigma \) on some of the generators of the Lie algebra of \( \mathcal{G} \) to estimates the \( L^2(\mathcal{G}) \)-norm of derivatives with respect to \( x \), nor an explicit description of Plancherel measure.
Nevertheless, if we do remove the mass term in (3), the situation changes drastically, since we cannot get exponential decay. In order to find still some decay rate, in this case of polynomial kind depending on the order and on the type of the derivative, we shall require further additional $L^1$ regularity for the Cauchy data. However, we will restrict our considerations to the case of the Heisenberg group with the sub-Laplacian, where the irreducible unitary representations are given, up to intertwining operators, by Schrödinger representations whose infinitesimal representations act on the generators of the first layer of $h_n$ in a known way, the Plancherel measure can be expressed through the parameter that parameterizes Schrödinger representations and the operator that has to be diagonalized is simply the harmonic oscillator (cf. Section 2).

Also, the purpose of this paper is to develop a phase space analysis for the Cauchy problem (1). More precisely, our goal is to derive decay estimates on $L^2(H_n)$ - basis with possible additional $L^1(H_n)$ - regularity for the Cauchy data as it is stated in the next key theorem.

**Theorem 1.1.** Let $(u_0, u_1) \in H^1(H_n) \times L^2(H_n)$ and let $u \in \mathcal{C}([0, \infty), H^1(H_n)) \cap \mathcal{C}([0, \infty), L^2(H_n))$ solve the Cauchy problem (1). Then, the following estimates are satisfied

\[
\|u(t, \cdot)\|_{L^2(H_n)} \leq C \left( \|u_0\|_{L^2(H_n)} + \|u_1\|_{L^2(H_n)} \right) \tag{4}
\]

\[
\|\nabla H u(t, \cdot)\|_{L^2(H_n)} \leq C(1 + t)^{-\frac{3}{4}} \left( \|\nabla H u_0\|_{L^2(H_n)} + \|u_1\|_{L^2(H_n)} \right) \tag{5}
\]

\[
\|\partial_t u(t, \cdot)\|_{L^2(H_n)} \leq C(1 + t)^{-\frac{1}{8}} \left( \|\nabla H u_0\|_{L^2(H_n)} + \|u_1\|_{L^2(H_n)} \right) \tag{6}
\]

for any $t \geq 0$. Furthermore, if we assume additionally that $u_0, u_1 \in L^1(H_n)$, then, the following decay estimates are satisfied

\[
\|u(t, \cdot)\|_{L^2(H_n)} \leq C \left( 1 + t \right)^{-\frac{3}{4}} \left( \|u_0\|_{L^1(H_n)} + \|u_1\|_{L^1(H_n)} \right) \tag{7}
\]

\[
\|\nabla H u(t, \cdot)\|_{L^2(H_n)} \leq C \left( 1 + t \right)^{-\frac{3}{4}-\frac{1}{2}} \left( \|u_0\|_{L^1(H_n)} + \|\nabla H u_0\|_{L^2(H_n)} + \|u_1\|_{L^1(H_n)} + \|u_1\|_{L^2(H_n)} \right) \tag{8}
\]

\[
\|\partial_t u(t, \cdot)\|_{L^2(H_n)} \leq C \left( 1 + t \right)^{-\frac{1}{8}-\frac{1}{4}} \left( \|u_0\|_{L^1(H_n)} + \|\nabla H u_0\|_{L^2(H_n)} + \|u_1\|_{L^1(H_n)} + \|u_1\|_{L^2(H_n)} \right) \tag{9}
\]

for any $t \geq 0$. Here $C$ is a positive constant.

**Remark 1.** Let us point out that the estimates in the statement of Theorem 1.1 correspond exactly to the $L^2(\mathbb{R}^n)$ - $L^2(\mathbb{R}^n)$ estimates with eventual additional $L^1(\mathbb{R}^n)$ - regularity for the data in the Euclidean case proved in [21, Lemma 2], by replacing the dimension $n$ of $\mathbb{R}^n$ with the homogeneous dimension $Q = 2n + 2$ of $H_n$.

The paper is organized as follows: in Section 2 and Section 3 we will recall some properties of the group Fourier transform on the Heisenberg group and of the Hermite functions, which will play a fundamental role in the paper; then, in Section 4 Theorem 1.1 is proved.

2. Group Fourier transform on the Heisenberg group

In this section, we recall the properties of the group Fourier transform on $H_n$ that we will use in order to prove Theorem 1.1. In the following we follow the definitions and the notations of [11, Chapters 1 and 6]. For the general definition of group Fourier transform on locally compact group and its properties we refer to the classical works [6, 7, 34, 5, 13, 20] and references therein.

Let us recall the following equivalent realization of Schrödinger representations $\{\pi_\lambda\}_{\lambda \in \mathbb{R}^*}$ of $H_n$ on $L^2(\mathbb{R}^n)$. For any $\lambda \in \mathbb{R}^*$ the mapping $\pi_\lambda$ is a strongly continuous unitary representation defined by

\[
\pi_\lambda(x, y, \tau) \varphi(w) = e^{i\lambda(\tau + \frac{1}{2}x \cdot y)} e^{i\text{sign}(\lambda) \sqrt{|\lambda|} y \cdot w} \varphi(w + \sqrt{|\lambda|} x)
\]

for any $(x, y, \tau) \in H_n$, $\varphi \in L^2(\mathbb{R}^n)$ and $w \in \mathbb{R}^n$.

If $f \in L^1(H_n)$, the group Fourier transform of $f$ is the bounded operator on $L^2(\mathbb{R}^n)$ defined as

\[
\pi_\lambda(f) \equiv \hat{f}(\lambda) \equiv \int_{H_n} f(\eta) \pi_\lambda(\eta)^* d\eta
\]
for any $\lambda \in \mathbb{R}^*$, that is,

$$\langle \hat{f}(\lambda)\varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(\eta) \langle \pi_\lambda(\eta)^* \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^n)} \, d\eta$$

for any $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^n)$, where $\pi_\lambda(\eta)^* = \pi_\lambda(\eta)^{-1}$ denotes the adjoint operators of the unitary operator $\pi_\lambda(\eta)$. As Schrödinger representations are unitary, it follows immediately by the definition the inequality

$$\|\hat{f}(\lambda)\|_{\mathcal{F}(L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n))} \leq \|f\|_{L^1(\mathbb{H}_n)}$$

(10)

for any $\lambda \in \mathbb{R}^*$.

If $f \in L^2(\mathbb{H}_n)$, then, the following Plancherel formula holds

$$\|f\|_{L^2(\mathbb{H}_n)}^2 = c_n \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{\text{HS}[L^2(\mathbb{R}^n)]}^2 |\lambda|^n \, d\lambda,$$

(11)

where $c_n = (2\pi)^{-(\frac{3n}{2}+1)}$ and $\|\hat{f}(\lambda)\|_{\text{HS}[L^2(\mathbb{R}^n)]}$ denotes the Hilbert-Schmidt norm of $\hat{f}(\lambda)$ (cf. [11, Proposition 6.2.7]). Therefore, for an arbitrary maximal orthonormal system $\{\varphi_k\}_{k \in \mathbb{N}}$ of $L^2(\mathbb{R}^n)$ the definition of Hilbert-Schmidt norm

$$\|\hat{f}(\lambda)\|_{\text{HS}[L^2(\mathbb{R}^n)]}^2 = \text{Tr} \left( (\pi_\lambda(f))^* \pi_\lambda(f) \right) = \sum_{k \in \mathbb{N}} \|\hat{f}(\lambda)\varphi_k\|_{L^2(\mathbb{R}^n)}^2 = \sum_{k, \ell \in \mathbb{N}} |\langle \hat{f}(\lambda)\varphi_k, \varphi_\ell \rangle_{L^2(\mathbb{R}^n)}|^2$$

(12)

allows us to write (11) in more operative way, as follows:

$$\|f\|_{L^2(\mathbb{H}_n)}^2 = c_n \int_{\mathbb{R}^*} \sum_{k \in \mathbb{N}} \|\hat{f}(\lambda)\varphi_k\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n \, d\lambda = c_n \int_{\mathbb{R}^*} \sum_{k, \ell \in \mathbb{N}} |\langle \hat{f}(\lambda)\varphi_k, \varphi_\ell \rangle_{L^2(\mathbb{R}^n)}|^2 |\lambda|^n \, d\lambda.$$

In particular, the Plancherel measure $\mu$ on $\hat{\mathbb{H}}_n$ is supported on the equivalence classes of $\pi_\lambda$, $\lambda \in \mathbb{R}^*$ and $d\mu(\pi_\lambda) = c_n |\lambda|^n d\lambda$.

A further property related to the group Fourier transform, that we will apply extensively, is the action of the infinitesimal representation of $\pi_\lambda$ on the generators of the first layer of the Lie algebra $\mathfrak{h}_n$, namely,

$$d\pi_\lambda(X_j) = \sqrt{|\lambda|} \partial_{w_j} \quad \text{for} \quad j = 1, \ldots, n,$$

(13)

$$d\pi_\lambda(Y_j) = i \text{sign}(\lambda) \sqrt{|\lambda|} w_j \quad \text{for} \quad j = 1, \ldots, n.$$  

(14)

In particular, since the action of $d\pi_\lambda$ can be extended to the universal enveloping algebra of $\mathfrak{h}_n$, combining (13) and (14), it follows

$$d\pi_\lambda(\Delta_H) = |\lambda| \sum_{j=1}^n (\partial_{w_j}^2 - w_j^2) = -|\lambda| H_w,$$

(15)

where $H_w = -\Delta_w + |w|^2$ is the harmonic oscillator on $\mathbb{R}^n$.

3. Hermite functions and their properties

In the last section, we recalled some properties of Schrödinger representations. Because of (15) it will be helpful for what follows to recall the definition of the Hermite functions and to prove some elementary properties of them. These functions constitute a complete system of eigenfunctions for the essentially self adjoint operator $H_w$. Let us begin with the definition of $m$-th Hermite polynomial

$$H_m(x) \doteq e^{x^2} \left( \frac{d}{dx} \right)^m e^{-x^2}, \quad x \in \mathbb{R}.$$
It is well-known that Hermite polynomials satisfy the following recurrence properties
\begin{align}
    H_{m+1}(x) &= 2xH_m(x) - 2mH_{m-1}(x), \\
    H'_m(x) &= 2mH_{m-1}(x)
\end{align}
for any \( m \in \mathbb{N} \) and \( x \in \mathbb{R} \). Furthermore,
\[
    \int_{\mathbb{R}} H_m(x)H_\ell(x)e^{-x^2} \, dx = \sqrt{\pi}2^mm!\delta_{m,\ell} \quad \text{for any } m, \ell \in \mathbb{N}.
\]
The \( m \)-th Hermite function is
\[
    \psi_m(x) := a_m e^{-\frac{x^2}{2}} H_m(x)
\]
for any \( m \in \mathbb{N} \) and \( x \in \mathbb{R} \), where \( a_m := (\sqrt{\pi}2^m m!)^{-1/2} \). By using (16) and (17) it follows easily that \( \psi_m \) is an eigenfunction for the 1-d harmonic oscillator \(-\frac{\partial^2}{\partial x^2} + x^2\) relative to the eigenvalue \( 2m + 1 \) as
\[
    -\psi''_m(x) + x^2\psi_m(x) = (2m + 1)\psi_m(x).
\]

The multidimensional version of Hermite functions is given by
\[
e_k(w) := \prod_{j=1}^n \psi_{k_j}(w_j) = \prod_{j=1}^n a_{k_j}H_{k_j}(w_j)e^{-w_j^2/2}
\]
for any multi-index \( k = (k_j)_{1 \leq j \leq n} \in \mathbb{N}^n \) and any \( w = (w_j)_{1 \leq j \leq n} \in \mathbb{R}^n \). As the elements of \( \{e_k\}_{k \in \mathbb{N}^n} \) are functions with separate variables, by (18) we get that
\[
    H_w e_k(w) = (-\Delta_w + |w|^2)e_k(w) = \left( \sum_{j=1}^n (2k_j + 1) \right) e_k(w) = (2|k| + n)e_k(w)
\]
for any \( k \in \mathbb{N}^n \) and any \( w \in \mathbb{R}^n \).

Let us recall the following result (cf. [25, Theorem 2.2.3]).

**Proposition 3.1.** The system \( \{e_k\}_{k \in \mathbb{N}^n} \) is an orthonormal basis of \( L^2(\mathbb{R}^n) \).

Having in mind (13) and (14), we calculate now the action of multiplication and derivation operators on the elements of \( \{e_k\}_{k \in \mathbb{N}^n} \).

**Lemma 3.2.** Let \( k \in \mathbb{N}^n \) and \( 1 \leq j \leq n \). Then,
\begin{align}
    \partial_{w_j} e_k(w) &= \frac{1}{\sqrt{2}} \left( \sqrt{k \cdot \epsilon_j} e_{k-\epsilon_j}(w) - e_{k+\epsilon_j}(w) \right) \\
    w_j e_k(w) &= \frac{1}{\sqrt{2}} \left( \sqrt{k \cdot \epsilon_j} e_{k-\epsilon_j}(w) + e_{k+\epsilon_j}(w) \right)
\end{align}
for any \( w \in \mathbb{R}^n \), where \( \epsilon_j = (\delta_{j,h})_{1 \leq h \leq n} \) denotes the \( j \)-th element of the standard basis of \( \mathbb{R}^n \).

**Proof.** Let us begin with (21). By the definition (19) of multidimensional Hermite function it follows
\[
    \partial_{w_j} e_k(w) = \partial_{w_j} \left( a_{k_j}H_{k_j}(w_j)e^{-w_j^2/2} \right) \prod_{h=1, h \neq j}^n a_{k_h}H_{k_h}(w_h)e^{-w_h^2/2}.
\]
Applying (16) and (17), we find
\[
\begin{align*}
\partial_w \left( a_{k_j} H_{k_j} (w_j) e^{-w_j^2/2} \right) &= a_{k_j} \left( H'_{k_j} (w_j) - w_j H_{k_j} (w_j) \right) e^{-w_j^2/2} \\
&= a_{k_j} \left( k_j H_{k_j-1} (w_j) - \frac{1}{2} H_{k_j+1} (w_j) \right) e^{-w_j^2/2}.
\end{align*}
\]
Therefore,
\[
\begin{align*}
\partial_w \varepsilon_k (w) &= a_{k_j} \left( k_j H_{k_j-1} (w_j) - \frac{1}{2} H_{k_j+1} (w_j) \right) e^{-w_j^2/2} \prod_{h=1}^n a_{k_h} H_{k_h} (w_h) e^{-w_h^2/2} \\
&= \frac{a_{k_j} k_j}{a_{k_j-1}} \left( a_{k_j-1} H_{k_j-1} (w_j) e^{-w_j^2/2} \right) \prod_{h=1, h \neq j}^n a_{k_h} H_{k_h} (w_h) e^{-w_h^2/2} \\
&- \frac{2 a_{k_j+1}}{a_{k_j+1}} \left( a_{k_j+1} H_{k_j+1} (w_j) e^{-w_j^2/2} \right) \prod_{h=1, h \neq j}^n a_{k_h} H_{k_h} (w_h) e^{-w_h^2/2} = \frac{a_{k_j} k_j}{a_{k_j-1}} \varepsilon_{k-\varepsilon_j} (w) - \frac{a_{k_j}}{2 a_{k_j+1}} \varepsilon_{k+\varepsilon_j}.
\end{align*}
\]
Since \( a_{k_j} k_j / a_{k_j-1} = \sqrt{k_j / 2} \) and \( a_{k_j} / 2 a_{k_j+1} = 1 / \sqrt{2} \), the last equality yields (21). Similarly, by
\[
w_j H_{k_j} (w_j) e^{-w_j^2/2} = \left( k_j H_{k_j-1} (w_j) + \frac{1}{2} H_{k_j+1} (w_j) \right) e^{-w_j^2/2}
\]
we get (22). The proof is complete. \( \square \)

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. We split the proof in three parts for the estimate of the \( L^2 (\mathbf{H}_n) \) - norm of \( u(t, \cdot) \), \( \partial_t u(t, \cdot) \), and \( \nabla H u(t, \cdot) \), respectively.

The plan is to adapt the main ideas of WKB analysis from the pioneering work of Matsumura [21] to the case of the Heisenberg group. Since the group Fourier transform of a \( L^2 (\mathbf{H}_n) \) function \( f \) is no longer a function but a family of bounded linear operators \( \{ \hat{f}(\lambda) \}_{\lambda \in \mathbb{R}^n} \) on \( L^2 (\mathbb{R}^n) \), the trick is to project somehow these operators by using the basis \( \{ \varepsilon_k \}_{k \in \mathbb{N}^n} \), namely, by working with
\[
\{ (\hat{f}(\lambda) \varepsilon_k, \varepsilon_k)_{L^2 (\mathbb{R}^n)} \}_{k \in \mathbb{N}^n}.
\]

This approach has been introduced in [30]. However, differently from [30, Section 2], where the combined presence of a mass and a damping term produces an exponential decay for the \( L^2 (\mathbf{H}_n) \) - norm of the solution and all of its derivatives, in the massless case for each component we will split the corresponding integral term coming from Plancherel formula in two zones with respect to \( \lambda \). On the one hand, for \( |\lambda| \) “small” the analogous of Riemann-Lebesgue inequality in (10) is employed in order to get the decay rates of polynomial order as in (7), (8) and (9). On the other hand, for \( |\lambda| \) “large” by Plancherel formula we get exponential decay under suitable regularity assumptions for the Cauchy data, provided that suitable \( L^2 \) regularity is required for the Cauchy data. Let us stress that this splitting depends on the multi-index \( k \in \mathbb{N}^n \) and the corresponding eigenvalue \( \mu_k \) of \( \mathbf{H}_w \). In this way, we will show the validity of (7), (8) and (9) first. As byproduct of the above described procedure we get immediately estimates (4), (5) and (6) too, if we use Plancherel formula for “small” \( |\lambda| \) as well, rather than the Riemann-Lebesgue inequality.

4.1. Estimate of the \( L^2 (\mathbf{H}_n) \) - norm of \( u(t, \cdot) \)

Let us consider \( u \) solution to (1). By acting by the group Fourier transform on (1) we get a Cauchy problem related to a parameter dependent functional differential equation for \( \tilde{u}(t, \lambda) \)
\[
\begin{align*}
\begin{cases}
\partial_t \tilde{u}(t, \lambda) + \partial_{\lambda} \tilde{u}(t, \lambda) - \sigma_{\Delta u} (\lambda) \tilde{u}(t, \lambda) = 0, & \lambda \in \mathbb{R}^n, \ t > 0, \\
\tilde{u}(0, \lambda) = \tilde{u}_0 (\lambda), & \lambda \in \mathbb{R}^n, \\
\partial_t \tilde{u}(0, \lambda) = \tilde{u}_1 (\lambda), & \lambda \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]
where $\sigma_{\Delta_n}(\lambda)$ is the symbol of the sub-Laplacian. Due to (15), we have $\sigma_{\Delta_n}(\lambda) = -|\lambda|H_w$. Let us introduce the notation
\[
\hat{u}(t, \lambda)_{k,\ell} \doteq \left( \hat{u}(t, \lambda)e_k, e_\ell \right)_{L^2(\mathbb{R}^n)} \quad \text{for any } k, \ell \in \mathbb{N}^n,
\]
where $\{e_k\}_{k \in \mathbb{N}^n}$ is the system of Hermite functions. Since $H_w e_k = \mu_k e_k$, then, $\hat{u}(t, \lambda)_{k,\ell}$ solves an ordinary differential equation depending on parameters $\lambda \in \mathbb{R}^+$ and $k, \ell \in \mathbb{N}^n$
\[
\begin{aligned}
&\frac{\partial^2}{\partial t^2} \hat{u}(t, \lambda)_{k,\ell} + \partial_t \hat{u}(t, \lambda)_{k,\ell} + \mu_k|\lambda|\hat{u}(t, \lambda)_{k,\ell} = 0, \quad t > 0, \\
&\hat{u}(0, \lambda)_{k,\ell} = \tilde{u}_0(\lambda)_{k,\ell}, \\
&\partial_t \hat{u}(0, \lambda)_{k,\ell} = \tilde{u}_1(\lambda)_{k,\ell},
\end{aligned}
\]  \hspace{1cm} (25)
where
\[
\tilde{u}_h(\lambda)_{k,\ell} \doteq \left( \tilde{u}_h(\lambda)e_k, e_\ell \right)_{L^2(\mathbb{R}^n)} \quad \text{for any } h = 0, 1 \text{ and any } k, \ell \in \mathbb{N}^n.
\]

The roots of the characteristic equation $\tau^2 + \tau + \mu_k|\lambda| = 0$ are
\[
\tau_{\pm} = \left\{ \begin{array}{ll}
-\frac{1}{2} & \text{if } \mu_k|\lambda| > 1, \\
-\frac{1}{2} \pm i \sqrt{\mu_k|\lambda| - \frac{1}{4}} & \text{if } \mu_k|\lambda| = 1, \\
-\frac{1}{2} & \text{if } \mu_k|\lambda| < 1.
\end{array} \right.
\]

Elementary computations lead to the representation formula
\[
\hat{u}(t, \lambda)_{k,\ell} = \tilde{u}_0(\lambda)_{k,\ell} e^{-\frac{\tau_{\ell}}{2}} F(t, \lambda, k) + \left( \tilde{u}_0(\lambda)_{k,\ell} + \frac{1}{2}\tilde{u}_1(\lambda)_{k,\ell} \right) e^{-\frac{\tau_{\ell}^\ast}{2}} G(t, \lambda, k),
\]
where
\[
F(t, \lambda, k) = \begin{cases} 
\cos \left( \sqrt{\mu_k|\lambda| - \frac{1}{4}} t \right) & \text{if } \mu_k|\lambda| > 1, \\
1 & \text{if } \mu_k|\lambda| = 1, \\
\cosh \left( \sqrt{\frac{1}{4} - \mu_k|\lambda|} t \right) & \text{if } \mu_k|\lambda| < 1,
\end{cases}
\]
\[
G(t, \lambda, k) = \begin{cases} 
\sin \left( \sqrt{\mu_k|\lambda| - \frac{1}{4}} t \right) & \text{if } \mu_k|\lambda| > 1, \\
\sqrt{\mu_k|\lambda| - \frac{1}{4}} & \text{if } \mu_k|\lambda| = 1, \\
\sinh \left( \sqrt{\frac{1}{4} - \mu_k|\lambda|} t \right) & \text{if } \mu_k|\lambda| < 1.
\end{cases}
\]

Note that $F(t, \lambda, k) = \partial_t G(t, \lambda, k)$ for any $\lambda \in \mathbb{R}^+$. By using Plancherel formula and $\{e_k\}_{k \in \mathbb{N}^n}$ as orthonormal basis of $L^2(\mathbb{R}^n)$, we have
\[
\|u(t, \cdot)\|^2_{L^2(H_w)} = c_n \int_{\mathbb{R}^n} \|\tilde{u}(t, \lambda)\|^2_{H^1_0(L^2(\mathbb{R}^n))} |\lambda|^n \, d\lambda = c_n \sum_{k, \ell \in \mathbb{N}^n} \int_{\mathbb{R}^+} \left( \tilde{u}(t, \lambda)e_k, e_\ell \right)_{L^2(\mathbb{R}^n)}^2 |\lambda|^n \, d\lambda
\]
\[
= c_n \sum_{k, \ell \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{\sqrt{2} \mu_k}} \left( \hat{u}(t, \lambda)e_k, e_\ell \right)_{L^2(\mathbb{R}^n)}^2 |\lambda|^n \, d\lambda + \int_{|\lambda| > \frac{1}{\sqrt{2} \mu_k}} \left( \hat{u}(t, \lambda)e_k, e_\ell \right)_{L^2(\mathbb{R}^n)}^2 |\lambda|^n \, d\lambda
\]
\[
= f_{\text{low}} + f_{\text{high}}.
\]
We will estimate separately the terms $I^{\text{low}}$ and $I^{\text{high}}$. We start with
\[
I^{\text{high}} = c_n \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > \frac{1}{8n}} |\hat{u}(t, \lambda)_{k, \ell}|^2 |\lambda|^n d\lambda.
\]
For any $\lambda$ such that $|\lambda| > 1/(8\mu_k)$ we may estimate
\[
e^{-\frac{t}{2}|F(t, \lambda, k)|}, e^{-\frac{t}{2}|G(t, \lambda, k)|} \lesssim e^{-\frac{t}{4}}
\]
for some $\delta > 0$, where $\delta$ and the unexpressed multiplicative constant hereafter are independent of the time variable and of the parameters $\lambda$ and $k, \ell$ as well. Then, by the representation formula (26) we obtain
\[
|\hat{u}(t, \lambda)_{k, \ell}|^2 \lesssim e^{-\delta t} \left( |\hat{u}_0(\lambda)_{k, \ell}|^2 + |\hat{u}_1(\lambda)_{k, \ell}|^2 \right)
\]
for any $|\lambda| > 1/(8\mu_k)$. Therefore,
\[
I^{\text{high}} \lesssim e^{-\delta t} \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > \frac{1}{8n}} (|\hat{u}_0(\lambda)_{k, \ell}|^2 + |\hat{u}_1(\lambda)_{k, \ell}|^2) |\lambda|^n d\lambda
\]
\[
\lesssim e^{-\delta t} \int_{\mathbb{R}^n} \sum_{k, \ell \in \mathbb{N}^n} (|\hat{u}_0(\lambda)_{k, \ell}|^2 + |\hat{u}_1(\lambda)_{k, \ell}|^2) |\lambda|^n d\lambda
\]
\[
= e^{-\delta t} \int_{\mathbb{R}^n} (|\hat{u}_0(\lambda)|^2_{\mathcal{H}(\mathbb{R}^n)} + |\hat{u}_1(\lambda)|^2_{\mathcal{H}(\mathbb{R}^n)}) |\lambda|^n d\lambda \approx e^{-\delta t} \left( \|u_0\|_{L^2(\mathbb{R}^n)}^2 + \|u_1\|_{L^2(\mathbb{R}^n)}^2 \right),
\]
where in the last step we applied Plancherel formula to $u_0$ and $u_1$.

**Remark 2.** Let us point out that if we do not restrict to the range of $\lambda$ such that $|\lambda| > 1/(8\mu_k)$, then one has to replace (27) by
\[
|\hat{u}(t, \lambda)_{k, \ell}|^2 \lesssim (|\hat{u}_0(\lambda)_{k, \ell}|^2 + |\hat{u}_1(\lambda)_{k, \ell}|^2)
\]
since $|F(t, \lambda, k)|, |G(t, \lambda, k)| \to 1$ as $|\lambda| \to 0$. So, we would end up with the uniform estimate
\[
\|u(t, .)\|_{L^2(\mathbb{R}^n)}^2 \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}^2 + \|u_1\|_{L^2(\mathbb{R}^n)}^2
\]
with no time-dependent decay function on the right-hand side. Clearly, (29) implies immediately (4) in the statement of Theorem 1.1.

We estimate now the other term
\[
I^{\text{low}} = c_n \sum_{k, \ell \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{8n}} |\hat{u}(t, \lambda)_{k, \ell}|^2 |\lambda|^n d\lambda.
\]
For $\lambda$ such that $|\lambda| < 1/(8\mu_k)$ it holds
\[
e^{-\frac{t}{2}} |F(t, \lambda, k)| = e^{\frac{t}{4}} \cosh \left( \sqrt{\frac{1}{4} - \mu_k |\lambda|} t \right) \lesssim e^{\left( -\frac{t}{4} + \frac{t}{8} \sqrt{1 - 4\mu_k |\lambda|} \right) t},
\]
\[
e^{-\frac{t}{2}} |G(t, \lambda, k)| = \frac{e^{-\frac{t}{2}} \sinh \left( \sqrt{\frac{1}{4} - \mu_k |\lambda|} t \right)}{\sqrt{\frac{1}{4} - \mu_k |\lambda|}} \lesssim e^{\left( -\frac{t}{4} + \frac{t}{8} \sqrt{1 - 4\mu_k |\lambda|} \right) t}.
\]
Using the inequality
\[-4z \leq -1 + \sqrt{1 - 4z} \leq -2z \quad \text{for any } z \geq 0,\]
(30)
it follows that
\[
|\tilde{u}(t,\lambda)_{k,\ell}|^2 \lesssim e^{-t+\sqrt{1-4\mu_k|\lambda|}} \left( |\tilde{u}_0(\lambda)_{k,\ell}|^2 + |\tilde{u}_1(\lambda)_{k,\ell}|^2 \right)
\lesssim e^{-2\mu_k|\lambda|} \left( |\tilde{u}_0(\lambda)_{k,\ell}|^2 + |\tilde{u}_1(\lambda)_{k,\ell}|^2 \right),
\]
for any \(\lambda\) such that \(|\lambda| < 1/(8\mu_k)\). Applying the last estimate, we find
\[
I_{\text{low}} \lesssim \sum_{k,\ell \in \mathbb{N}^n} \int_{0<|\lambda|<8\mu_k} \frac{e^{-2\mu_k|\lambda|} \left( \sum_{\ell \in \mathbb{N}^n} \left( |\tilde{u}_0(\lambda)e_k, e_\ell|_{L^2(\mathbb{R}^n)} \right)^2 + \left( |\tilde{u}_1(\lambda)e_k, e_\ell|_{L^2(\mathbb{R}^n)} \right)^2 \right) |\lambda|^n d\lambda}
= \sum_{k \in \mathbb{N}^n} \int_{0<|\lambda|<8\mu_k} \frac{e^{-2\mu_k|\lambda|} \left( \sum_{\ell \in \mathbb{N}^n} \left( |\tilde{u}_0(\lambda)e_k|_{L^2(\mathbb{R}^n)} \right)^2 \right) |\lambda|^n d\lambda}
\lesssim \sum_{k \in \mathbb{N}^n} \int_{0}^{\mu_k} e^{-2\mu_k|\lambda|\lambda^n} d\lambda \left( \|u_0\|_{L^1(\mathbb{H}_n)}^2 + \|u_1\|_{L^1(\mathbb{H}_n)}^2 \right),
\]
where in the third step we employed Parseval’s identity
\[
\|\tilde{u}_h(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 = \sum_{\ell \in \mathbb{N}^n} \left( |\tilde{u}_0(\lambda)e_k, e_\ell|_{L^2(\mathbb{R}^n)} \right)^2 \quad \text{for } h = 0, 1
\]
and in the last step we used (10) and \(\|e_k\|_{L^2(\mathbb{R}^n)} = 1\) to get the inequality
\[
\|\tilde{u}_h(\lambda)e_k\|_{L^2(\mathbb{R}^n)} \leq \|\tilde{u}_h(\lambda)\|_{L^2(\mathbb{R}^n)} \|e_k\|_{L^2(\mathbb{R}^n)} \leq \|u_h\|_{L^1(\mathbb{H}_n)} \quad \text{for } h = 0, 1.
\]
Carrying out the change of variables \(\theta = 2\mu_k\lambda t\) in the last integral, we have
\[
I_{\text{low}} \lesssim \sum_{k \in \mathbb{N}^n} (2\mu_k t)^{-(n+1)} \int_{0}^{t} e^{-2\theta\lambda^n} d\theta \left( \|u_0\|_{L^1(\mathbb{H}_n)}^2 + \|u_1\|_{L^1(\mathbb{H}_n)}^2 \right)
\lesssim \Gamma(n+1) \frac{t^{-(n+1)}}{\sum_{k \in \mathbb{N}^n} \mu_k^{-(n+1)}} \left( \|u_0\|_{L^1(\mathbb{H}_n)}^2 + \|u_1\|_{L^1(\mathbb{H}_n)}^2 \right)
\lesssim t^{-\frac{n}{2}} \left( \|u_0\|_{L^1(\mathbb{H}_n)}^2 + \|u_1\|_{L^1(\mathbb{H}_n)}^2 \right),
\]
where \(\Gamma\) denotes Euler integral of the second kind and in the last estimate we used that the series
\[
\sum_{k \in \mathbb{N}^n} \mu_k^{-(n+1)} = \sum_{k \in \mathbb{N}^n} (2|k| + n)^{-(n+1)}
\]
is convergent. Combining (28), (33) and (29) (which allows us to exclude a singular behavior of \(I_{\text{low}}\) as \(t \to 0^+\)), we get finally (7).

4.2. Estimate of the \(L^2(\mathbb{H}_n)\) norm of \(\partial_t u(t, \cdot)\)

In this section we estimate the \(L^2(\mathbb{H}_n)\) norm of the time derivative of \(u\). As in the case of the \(\|u(t, \cdot)\|_{L^2(\mathbb{H}_n)}\), we will split the estimate in two parts. By Plancherel formula we have
\[
\|\partial_t u(t, \cdot)\|_{L^2(\mathbb{H}_n)}^2 = c_n \int_{\mathbb{R}^n} \|\partial_t \tilde{u}(t, \lambda)\|_{H^0[\mathbb{R}^n]}^2 |\lambda|^n d\lambda
\]
\[
= c_n \sum_{k,\ell \in \mathbb{N}^n} \left( \int_{0<|\lambda|<\frac{1}{8\mu_k}} \|\partial_t \tilde{u}(t, \lambda)e_k, e_\ell\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n d\lambda + \int_{|\lambda|>\frac{1}{8\mu_k}} \|\partial_t \tilde{u}(t, \lambda)e_k, e_\ell\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n d\lambda \right)
\]
\[
= c_n \sum_{k,\ell \in \mathbb{N}^n} \left( \int_{0<|\lambda|<\frac{1}{8\mu_k}} \|\partial_t \tilde{u}(t, \lambda)e_k, e_\ell\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n d\lambda + \int_{|\lambda|>\frac{1}{8\mu_k}} \|\partial_t \tilde{u}(t, \lambda)e_k, e_\ell\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n d\lambda \right) \doteq J_{\text{low}} + J_{\text{high}}.
\]
We estimate $J_{\text{low}}$ first. For $|\lambda| < 1/(8\mu_k)$ by (26) we find

$$
\partial_t \mathcal{u}(t, \lambda)_{k, \ell} = -\frac{e^{-\frac{\lambda}{2}} \sinh \left( \sqrt{\frac{1}{4} - \mu_k |\lambda|} t \right)}{\sqrt{\frac{1}{4} - \mu_k |\lambda|}} \mu_k |\lambda| \mathcal{u}_0(\lambda)_{k, \ell} \\
+ e^{-\frac{\lambda}{2}} \left( 1 - \frac{1}{2 \sqrt{\frac{1}{4} - \mu_k |\lambda|}} \right) e^{\frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left[ \mathcal{u}_1(\lambda)_{k, \ell} \right]
$$

and, rewriting the factor that multiplies $\mathcal{u}_1(\lambda)_{k, \ell}$, we arrive at

$$
\partial_t \mathcal{u}(t, \lambda)_{k, \ell} = -\frac{e^{-\frac{\lambda}{2}} \sinh \left( \sqrt{\frac{1}{4} - \mu_k |\lambda|} t \right)}{\sqrt{\frac{1}{4} - \mu_k |\lambda|}} \mu_k |\lambda| \mathcal{u}_0(\lambda)_{k, \ell} \\
+ e^{-\frac{\lambda}{2}} \left( 1 - \frac{1}{2 \sqrt{\frac{1}{4} - \mu_k |\lambda|}} \right) e^{\frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \mathcal{u}_1(\lambda)_{k, \ell}.
$$

Combining the last expression and (30), we may estimate for $|\lambda| < 1/(8\mu_k)$

$$
|\partial_t \mathcal{u}(t, \lambda)_{k, \ell}| \lesssim e^{-\frac{\lambda}{2} + \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \mathcal{u}_0(\lambda)_{k, \ell} \left( |\mathcal{u}_0(\lambda)_{k, \ell}| + |\mathcal{u}_1(\lambda)_{k, \ell}| \right) + \left( 1 + e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \right) |\mathcal{u}_1(\lambda)_{k, \ell}|.
$$

Therefore, combining the last inequality and (31), it results

$$
J_{\text{low}} \leq c_n \sum_{k, \ell \in \mathbb{N}_0^n} \int_{0 < |\lambda| < \frac{\lambda}{16 \pi} \sqrt{\mu_k}} \left( |\mathcal{u}_0(\lambda)_{k, \ell}|^2 + |\mathcal{u}_1(\lambda)_{k, \ell}|^2 \right) \frac{|\lambda|^{n+2} d\lambda}{|\lambda|^{n+2}} \\
+ e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left( \sum_{k, \ell \in \mathbb{N}_0^n} \left( |\mathcal{u}_0(\lambda)_{k, \ell}|^2 + |\mathcal{u}_1(\lambda)_{k, \ell}|^2 \right) \right) \frac{|\lambda|^{n+2} d\lambda}{|\lambda|^{n+2}} \\
+ e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left( \sum_{k, \ell \in \mathbb{N}_0^n} \left( |\mathcal{u}_0(\lambda)_{k, \ell}|^2 + |\mathcal{u}_1(\lambda)_{k, \ell}|^2 \right) \right) \frac{|\lambda|^{n+2} d\lambda}{|\lambda|^{n+2}}
$$

So, applying the inequality in (32) and using the fact that the series (34) converges, it follows

$$
J_{\text{low}} \lesssim \sum_{k \in \mathbb{N}_0^n} \frac{\mu_k^{n+2}}{10} |\lambda|^{n+2} d\lambda \left( |\mathcal{u}_0|^2_{L^1(H_n)} + |\mathcal{u}_1|^2_{L^1(H_n)} \right) \\
+ e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left( |\mathcal{u}_0|^2_{L^1(H_n)} + |\mathcal{u}_1|^2_{L^1(H_n)} \right) + e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left( |\mathcal{u}_1|^2_{L^1(H_n)} \right)
$$

$$
\lesssim t^{-n+3} \sum_{k \in \mathbb{N}_0^n} \frac{\mu_k^{n+1}}{10} |\lambda|^{n} d\lambda \left( |\mathcal{u}_0|^2_{L^1(H_n)} + |\mathcal{u}_1|^2_{L^1(H_n)} \right) \\
+ e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left( |\mathcal{u}_0|^2_{L^1(H_n)} + |\mathcal{u}_1|^2_{L^1(H_n)} \right) + e^{-\frac{\lambda}{2} - \frac{1}{2} \sqrt{1 - 4 \mu_k |\lambda|} t} \left( |\mathcal{u}_1|^2_{L^1(H_n)} \right),
$$

(37)
where in second inequality we performed the change of variables $\theta = 2\mu_k \lambda$ in the first integral. Now we will prove that $J^{\text{high}}$ decay with exponential speed with respect to $t$. For $\lambda$ such that $1/(8\mu_k) < |\lambda| < 1/(4\mu_k)$ by (35) we get

$$|\partial_t \tilde{u}(t, \lambda)_{k,\ell}| \lesssim e^{-\frac{\delta}{4} t} (|\tilde{u}_0(\lambda)_{k,\ell}| + |\tilde{u}_1(\lambda)_{k,\ell}|)$$

(38)

for some $\delta > 0$, that may change from line to line in the following. When $|\lambda| = 1/(4\mu_k)$, by (26) we have

$$\partial_t \tilde{u}(t, \lambda)_{k,\ell} = e^{-\frac{\pi}{2} t} \tilde{u}_1(\lambda)_{k,\ell} - \frac{1}{2} e^{-\frac{\pi}{2} t} (\tilde{u}_1(\lambda)_{k,\ell} + \frac{1}{2} \tilde{u}_0(\lambda)_{k,\ell})$$

(39)

while for $\lambda$ such that $1/(4\mu_k) < |\lambda| < 1/(2\mu_k)$

$$\partial_t \tilde{u}(t, \lambda)_{k,\ell} = -\frac{1}{\sqrt{2\mu_k |\lambda| - \frac{4}{\pi} t}} \mu_k |\lambda| \tilde{u}_0(\lambda)_{k,\ell}$$

$$+ e^{-\frac{\pi}{2} t} \left( \cos \left( \sqrt{\mu_k |\lambda| - \frac{4}{\pi} t} \right) - \frac{\sin \left( \sqrt{\mu_k |\lambda| - \frac{4}{\pi} t} \right)}{2} \right) \mu_k |\lambda| \tilde{u}_1(\lambda)_{k,\ell}$$

(40)

In the case $1/(4\mu_k) \leq |\lambda| < 1/(2\mu_k)$ from (39) and (40) we find again the estimate (38). Finally, for $|\lambda| > 1/(2\mu_k)$ we have to estimate

$$|\partial_t \tilde{u}(t, \lambda)_{k,\ell}| \lesssim e^{-\frac{\delta}{4} t} (\sqrt{\mu_k |\lambda|} |\tilde{u}_0(\lambda)_{k,\ell}| + |\tilde{u}_1(\lambda)_{k,\ell}|).$$

Hence, applying the last estimate and (38) to the definition of $J^{\text{high}}$, we arrive at

$$J^{\text{high}} \lesssim \sum_{k,\ell \in \mathbb{N}} \int_{|\lambda| > 1/(2\mu_k)} |\partial_t \tilde{u}(t, \lambda)_{k,\ell}|^2 |\lambda|^n \, d\lambda$$

$$\lesssim e^{-\delta t} \sum_{k,\ell \in \mathbb{N}} \int_{|\lambda| < 1/(2\mu_k)} \left( \mu_k |\lambda| \left| \tilde{u}_0(\lambda)_{k,\ell} \right|^2 + \left| \tilde{u}_1(\lambda)_{k,\ell} \right|^2 \right) |\lambda|^n \, d\lambda$$

$$\lesssim e^{-\delta t} \sum_{k,\ell \in \mathbb{N}} \int_{|\lambda| < 1/(2\mu_k)} \left( \mu_k |\lambda| \left( \| \tilde{u}_0(\lambda)_{e_k, e_\ell} \|_{L^2(\mathbb{R}^n)} \right)^2 + \left( \| \tilde{u}_1(\lambda)_{e_k, e_\ell} \|_{L^2(\mathbb{R}^n)} \right)^2 \right) |\lambda|^n \, d\lambda$$

$$\lesssim e^{-\delta t} \int_{\mathbb{R}^n} \sum_{k,\ell \in \mathbb{N}} \left( \| \nabla \tilde{u}_0 \|^{\lambda}_{H^1(\mathbb{R}^n)} + \| \tilde{u}_1(\lambda) \|_{H^1(\mathbb{R}^n)} \right) |\lambda|^n \, d\lambda$$

$$\approx e^{-\delta t} \left( \| \nabla \tilde{u}_0 \|_{L^2(\mathbb{H}_a)}^2 + \| u_1 \|_{L^2(\mathbb{H}_a)}^2 \right),$$

(41)

where we employed

$$\sum_{k,\ell \in \mathbb{N}} \| \tilde{u}_0(\lambda)_{e_k, e_\ell} \|_{L^2(\mathbb{R}^n)}^2 \approx \int_{\mathbb{R}^n} \left( \| \nabla \tilde{u}_0 \|^{\lambda}_{H^1(\mathbb{R}^n)} + \| \tilde{u}_1(\lambda) \|_{H^1(\mathbb{R}^n)} \right)$$

in the second last step and Plancherel formula in the last step.

**Remark 3.** Also in this case, if we had used only $L^2$ regularity for any $\lambda \in \mathbb{R}^n$, we would have found the estimate

$$\| \partial_t u(t, \cdot) \|_{L^2(\mathbb{H}_a)} \lesssim \left( \| \nabla \tilde{u}_0 \|_{L^2(\mathbb{H}_a)}^2 + \| u_1 \|_{L^2(\mathbb{H}_a)}^2 \right)$$

which excludes the possibility of $J^{\text{low}}$ to be estimate by a singular term as $t \to 0^+$. 

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Due Remark 3, by (37) and (41) we get eventually (9).

**Remark 4.** Besides the uniform estimate in Remark 3, which is necessary to exclude a singular behavior of \( \|\partial_t u(t, \cdot)\|_{L^2(H_\mu)} \) as \( t \to 0^+ \), by using just \( L^2 \) regularity in the estimate for \( \partial_t u(t, \cdot) \) and making more sharp the intermediate steps, we may get some decay rate as well (of course, weaker than the one we got by assuming additional \( L^1 \) regularity). Indeed, keeping the estimate of \( J_{\text{low}} \) in (36) and applying (11) in spite of (10), we have

\[
J_{\text{low}} \lesssim \sum_{k \in \mathbb{N}^n} \mu_k^2 \int_{0 < |\lambda| < \frac{1}{8 \pi^2}} e^{-2 \mu_k |\lambda| t} \left( \|\tilde{u}_0(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 + \|\tilde{u}_1(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 \right) |\lambda|^{n+2} \, d\lambda \\
+ e^{-t} \sum_{k \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{8 \pi^2}} \|\tilde{u}_1(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n \, d\lambda \\
\lesssim t^{-2} \sum_{k \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{8 \pi^2}} \left( \|\tilde{u}_0(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 + \|\tilde{u}_1(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 \right) |\lambda|^n \, d\lambda \\
+ e^{-t} \sum_{k \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{8 \pi^2}} \|\tilde{u}_1(\lambda)e_k\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n \, d\lambda \\
\lesssim t^{-2} \left( \|u_0\|_{L^2(H_\mu)}^2 + \|u_1\|_{L^2(H_\mu)}^2 \right) + e^{-t} \|u_1\|_{L^2(H_\mu)}^2,
\]

where in the second step we used the inequality \( |\lambda|^n e^{-2 \mu_k |\lambda| t} \lesssim (\mu_k t)^{-2} \). Combining the last estimate and (41) we get (6).

### 4.3. Estimate of the \( L^2(H_\mu) \) - norm of \( \nabla H u(t, \cdot) \)

In this section, we estimate the \( L^2(H_\mu) \) - norm of the horizontal gradient of \( u(t, \cdot) \), that is,

\[
\|\nabla H u(t, \cdot)\|_{L^2(H_\mu)}^2 = \sum_{j=1}^{n} \left( \|X_j u(t, \cdot)\|_{L^2(H_\mu)}^2 + \|Y_j u(t, \cdot)\|_{L^2(H_\mu)}^2 \right).
\]

Let us fix \( 1 \leq j \leq n \). We start with the estimate of the \( L^2(H_\mu) \) - norm of \( X_j u(t, \cdot) \). Combining (13) and (21), we get

\[
(X_j u)^\wedge (t, \lambda)e_k, e_\ell)_{L^2(\mathbb{R}^n)} = \sqrt{|\lambda|} \left( \sqrt{\frac{|\lambda|}{\lambda}} \langle \widehat{u}(t, \lambda)e_k, e_\ell \rangle_{L^2(\mathbb{R}^n)} - \sqrt{\frac{|\lambda|}{\lambda}} \langle \widehat{u}(t, \lambda)e_{k+e_j}, e_\ell \rangle_{L^2(\mathbb{R}^n)} \right) = \sqrt{|\lambda|} \left( \sqrt{k \cdot \epsilon_j} \widehat{u}(t, \lambda)k_{-\epsilon_j}, \ell - \widehat{u}(t, \lambda)k_{+e_j}, \ell \right)
\]

for any \( k, \ell \in \mathbb{N}^n \). Consequently, by Plancherel formula we have

\[
\|X_j u(t, \cdot)\|_{L^2(H_\mu)}^2 = c_n \int_{\mathbb{R}^n} \| (X_j u)^\wedge (t, \lambda) \|_{H^1[L^2(\mathbb{R}^n)]}^2 |\lambda|^n \, d\lambda = \frac{c_n}{2} \sum_{k, \ell \in \mathbb{N}^n} \left( \int_{0 < |\lambda| < \frac{1}{8 \pi^2}} + \int_{|\lambda| > \frac{1}{8 \pi^2}} \right) \sqrt{k \cdot \epsilon_j} \widehat{u}(t, \lambda)k_{-\epsilon_j}, \ell - \widehat{u}(t, \lambda)k_{+e_j}, \ell \right| \left. 2|\lambda|^{n+1} \, d\lambda \right)
\]

\[
\lesssim K_{\text{low}} + K_{\text{high}}.
\]

Note that we slightly changed the partition of the domain of integration with respect to the previous sections. We estimate \( K_{\text{high}} \) first. Clearly,

\[
K_{\text{high}} \lesssim \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > \frac{1}{8 \pi^2}} \left( k \cdot \epsilon_j \left| \widehat{u}(t, \lambda)k_{-\epsilon_j}, \ell \right|^2 + \left| \widehat{u}(t, \lambda)k_{+e_j}, \ell \right|^2 \right) |\lambda|^{n+1} \, d\lambda.
\]

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From the representation formula (26), for \( \lambda \) such that \(|\lambda| > 1/(8\mu_{k-\varepsilon_j}) > 1/(8\mu_{k+\varepsilon_j})\) we get the estimates

\[
|\tilde{u}(t, \lambda)_{k-\varepsilon_j, \ell}| \lesssim e^{-\delta t} (|\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}| + (\mu_{k-\varepsilon_j}|\lambda|)^{-1/2} |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|),
\]
\[
|\tilde{u}(t, \lambda)_{k+\varepsilon_j, \ell}| \lesssim e^{-\delta t} (|\tilde{u}_0(\lambda)_{k+\varepsilon_j, \ell}| + (\mu_{k+\varepsilon_j}|\lambda|)^{-1/2} |\tilde{u}_1(\lambda)_{k+\varepsilon_j, \ell}|),
\]

where \( \delta > 0 \) is a suitable constant. Hence,

\[
K^{\text{high}} \lesssim e^{-\delta t} \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > |\mu_{k-\varepsilon_j}|} k \cdot \varepsilon_j (|\lambda| |\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + (\mu_{k-\varepsilon_j}|\lambda|)^{-1} |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2) |\lambda|^n \, d\lambda
\]
\[
+ e^{-\delta t} \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > |\mu_{k+\varepsilon_j}|} (|\lambda| |\tilde{u}_0(\lambda)_{k+\varepsilon_j, \ell}|^2 + (\mu_{k+\varepsilon_j}|\lambda|)^{-1} |\tilde{u}_1(\lambda)_{k+\varepsilon_j, \ell}|^2) |\lambda|^n \, d\lambda
\]
\[
\lesssim e^{-\delta t} \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > |\mu_{k-\varepsilon_j}|} (|\lambda|^{1/2} |\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2) |\lambda|^n \, d\lambda
\]
\[
+ e^{-\delta t} \sum_{k, \ell \in \mathbb{N}^n} \int_{|\lambda| > |\mu_{k+\varepsilon_j}|} (|\lambda|^{1/2} |\tilde{u}_0(\lambda)_{k+\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k+\varepsilon_j, \ell}|^2) |\lambda|^n \, d\lambda,
\]

where we used the inequalities \((k \cdot \varepsilon_j)/(\mu_{k-\varepsilon_j}) \lesssim 1\) and \(\mu_{k+\varepsilon_j} \gtrsim 1\) in the second step. Also,

\[
K^{\text{high}} \lesssim e^{-\delta t} \int_{\mathbb{R}^3} \sum_{k, \ell \in \mathbb{N}^n} (|\lambda|^{1/2} |\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2) |\lambda|^n \, d\lambda
\]
\[
+ e^{-\delta t} \int_{\mathbb{R}^3} \sum_{k, \ell \in \mathbb{N}^n} (|\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2) |\lambda|^n \, d\lambda
\]
\[
\lesssim e^{-\delta t} \left( \|X_j + iY_j\|_{L^2(H_n)}^2 + \|X_j - iY_j\|_{L^2(H_n)}^2 + \|u_1\|_{L^2(H_n)}^2 \right)
\]
\[
\lesssim e^{-\delta t} \left( \|X_ju_0\|_{L^2(H_n)}^2 + \|Y_ju_0\|_{L^2(H_n)}^2 + \|u_1\|_{L^2(H_n)}^2 \right).
\]

Note that in the previous chain of inequalities, according to (13) and (14), we used

\[
\sum_{k, \ell \in \mathbb{N}^n} (|\lambda|^{1/2} |\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2)
\]
\[
= 2 \sum_{k, \ell \in \mathbb{N}^n} \left( \|\lambda\|_{L^2(H_n)}^2 + \|\lambda\|_{L^2(H_n)}^2 \right)
\]
\[
= 2 \left( \|X_j + iY_j\|_{H^1}^2 + \|X_j - iY_j\|_{H^1}^2 \right).
\]

Next, we estimate \(K^{\text{low}}\). Combining

\[
K^{\text{low}} \lesssim \sum_{k, \ell \in \mathbb{N}^n} \int_{0 < |\lambda| < |\mu_{k-\varepsilon_j}|} (k \cdot \varepsilon_j |\tilde{u}(t, \lambda)_{k-\varepsilon_j, \ell}|^2 + |\tilde{u}(t, \lambda)_{k+\varepsilon_j, \ell}|^2) |\lambda|^{n+1} \, d\lambda
\]
and

\[
|\tilde{u}(t, \lambda)_{k-\varepsilon_j, \ell}|^2 \lesssim e^{-2\mu_{k-\varepsilon_j}|\lambda| t} (|\tilde{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2)
\]
\[
|\tilde{u}(t, \lambda)_{k+\varepsilon_j, \ell}|^2 \lesssim e^{-2\mu_{k+\varepsilon_j}|\lambda| t} (|\tilde{u}_0(\lambda)_{k+\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k+\varepsilon_j, \ell}|^2)
\]
\[
\lesssim e^{-2\mu_{k-\varepsilon_j}|\lambda| t} (|\tilde{u}_0(\lambda)_{k+\varepsilon_j, \ell}|^2 + |\tilde{u}_1(\lambda)_{k+\varepsilon_j, \ell}|^2)
\]
for $|\lambda| < 1/(8\mu_{k-\varepsilon_j})$, where we used again (30) and $\mu_{k+\varepsilon_j} > \mu_{k-\varepsilon_j}$, we obtain

$$K^{low} \lesssim \sum_{k, \ell \in \mathbb{N}^n} k \cdot \varepsilon_j \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} e^{-2\mu_{k-\varepsilon_j}|\lambda|t} \left( |\widehat{u}_0(\lambda)_{k-\varepsilon_j, \ell}|^2 + |\widehat{u}_1(\lambda)_{k-\varepsilon_j, \ell}|^2 \right) |\lambda|^{n+1} d\lambda$$

$$+ \sum_{k, \ell \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} e^{-2\mu_{k-\varepsilon_j}|\lambda|t} \left( |\widehat{u}_0(\lambda)_{k+\varepsilon_j, \ell}|^2 + |\widehat{u}_1(\lambda)_{k+\varepsilon_j, \ell}|^2 \right) |\lambda|^{n+1} d\lambda$$

$$\lesssim \sum_{k, \ell \in \mathbb{N}^n} k \cdot \varepsilon_j \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} e^{-2\mu_{k-\varepsilon_j}|\lambda|t} \left( \|\widehat{u}_0(\lambda) e_{k-\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 + \|\widehat{u}_1(\lambda) e_{k-\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 \right) |\lambda|^{n+1} d\lambda$$

$$+ \sum_{k, \ell \in \mathbb{N}^n} \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} e^{-2\mu_{k-\varepsilon_j}|\lambda|t} \left( \|\widehat{u}_0(\lambda) e_{k+\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 + \|\widehat{u}_1(\lambda) e_{k+\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 \right) |\lambda|^{n+1} d\lambda, \quad (43)$$

where in the second inequality we used Parseval’s identity. Therefore, by (32) it results

$$K^{low} \lesssim \sum_{k \in \mathbb{N}^n} (k \cdot \varepsilon_j + 1) \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} e^{-2\mu_{k-\varepsilon_j}|\lambda|t} |\lambda|^{n+1} d\lambda \left( \|u_0\|_{L^1(H_{\lambda})}^2 + \|u_1\|_{L^1(H_{\lambda})}^2 \right)$$

$$\lesssim \frac{n}{2} \sum_{k \in \mathbb{N}^n} (k \cdot \varepsilon_j + 1) \mu_{k-\varepsilon_j}^{-1} \int_0^{\frac{1}{2}} e^{-\theta|\lambda|t} \frac{1}{\theta^{n+1}} d\theta \left( \|u_0\|_{L^1(H_{\lambda})}^2 + \|u_1\|_{L^1(H_{\lambda})}^2 \right)$$

$$\lesssim t^{-\frac{n}{2} - 1} \sum_{k \in \mathbb{N}^n} (k \cdot \varepsilon_j + 1)(|k| - 1 + n)^{-(n+2)} \left( \|u_0\|_{L^1(H_{\lambda})}^2 + \|u_1\|_{L^1(H_{\lambda})}^2 \right)$$

$$\lesssim t^{-\frac{n}{2} - 1} \left( \|u_0\|_{L^1(H_{\lambda})}^2 + \|u_1\|_{L^1(H_{\lambda})}^2 \right). \quad (44)$$

where we performed the change of variables $\theta = 2\mu_{k-\varepsilon_j}\lambda t$ in the second line and we employed the convergence of the series

$$\sum_{k \in \mathbb{N}^n} (k \cdot \varepsilon_j + 1)(|k| - 1 + n)^{-(n+2)} < \infty$$

in the last one. Similarly as in Remarks 2 and 3, we can exclude a singular coefficients with respect to $t$ as $t \to 0^+$ for $K^{low}$. Summarizing, we proved

$$\|X_j u(t, \cdot)\|_{L^2(H_{\lambda})}^2 \lesssim (1 + t)^{-\frac{n}{4} - 1} \left( \|u_0\|_{L^1(H_{\lambda})}^2 + \|X_j u_0\|_{L^2(H_{\lambda})}^2 + \|X_j u_0\|_{L^2(H_{\lambda})}^2 + \|u_1\|_{L^1(H_{\lambda})}^2 + \|u_1\|_{L^1(H_{\lambda})}^2 \right).$$

In a completely analogous way it is possible to prove the same kind of estimate for $\|Y_j^T u(t, \cdot)\|_{L^2(H_{\lambda})}^2$. So, this complete the proof of (8).

**Remark 5.** If we continue the estimate (43) for $K^{low}$ by using (11) in place of (10), we arrive at

$$K^{low} \lesssim t^{-1} \sum_{k \in \mathbb{N}^n} (k \cdot \varepsilon_j) \mu_{k-\varepsilon_j}^{-1} \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} \left( \|\widehat{u}_0(\lambda) e_{k-\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 + \|\widehat{u}_1(\lambda) e_{k-\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 \right) |\lambda|^{n} d\lambda$$

$$+ t^{-1} \sum_{k, \ell \in \mathbb{N}^n} \mu_{k-\varepsilon_j}^{-1} \int_{0 < |\lambda| < \frac{1}{\mu_{k-\varepsilon_j}}} \left( \|\widehat{u}_0(\lambda) e_{k+\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 + \|\widehat{u}_1(\lambda) e_{k+\varepsilon_j} \|_{L^2(\mathbb{R}^n)}^2 \right) |\lambda|^{n} d\lambda$$

$$\lesssim t^{-1} \left( \|u_0\|_{L^2(H_{\lambda})}^2 + \|u_1\|_{L^2(H_{\lambda})}^2 \right)$$

where in the second step we applied the inequality

$$(k \cdot \varepsilon_j) \mu_{k-\varepsilon_j}^{-1} \lesssim \frac{|k|}{2(|k| - 1) + n} \lesssim 1$$
and the fact that the set of the eigenvalues of the harmonic oscillator is bounded from below by a positive constant. Thus, combining the previous estimate for $K_{\text{low}}$ and (42), we obtain (5).

5. Final remarks and future applications

In Theorem 1.1, we restricted our considerations to the $L^2(H_n)$ estimate for the horizontal gradient of $u(t, \cdot)$. This choice seems to be quite natural due to the differential operator on the left-hand side of (1). Nevertheless, one might wonder what happens if we apply the machinery developed in Section 4 to the derivative $Tu(t, \cdot)$ (as in the introduction, $T$ denotes here the generator of the second layer of the Lie algebra $h_n$). The infinitesimal generator of $\pi_\lambda$ acts on $T$ as follows:

$$d\pi_\lambda(T) = i\lambda \text{Id}.$$  

As in Section 4, if we employ Plancherel formula, we split the integral of containing the Hilbert-Schmidt norm of $(Tu)^\wedge(t, \lambda) = i\lambda \hat{u}(t, \lambda)$ in two regions and we use $L^1(H_n)$ regularity for the data in the low $\lambda$ region and $L^2(H_n)$ regularity for the data in the high $\lambda$ region, then, we end up with the estimate

$$\|Tu(t, \cdot)\|_{L^2(H_n)} \lesssim (1 + t)^{-Q_4^{-1}} \left(\|u_0\|_{L^1(H_n)} + \|Tu_0\|_{L^2(H_n)} + \|u_1\|_{L^1(H_n)} + \|T^{1/2}u_1\|_{L^2(H_n)} + \|u_1\|_{L^2(H_n)}\right),$$  

(45)

where

$$\|T^{1/2}u_1\|_{L^2(H_n)} \doteq c_n \int_{R^*} |\hat{u}_1(\lambda)|^2 \|\lambda\|_{H^2(R^n)} |\lambda|^{n+1} d\lambda.$$  

So, in (45) we get the same decay rate as in (9), which is faster than the one for the horizontal gradient of $u(t, \cdot)$ in (8), but we are forced somehow to require more regularity for $u_1$ on $L^2$ level.

Concerning future applications for the tools derived in this work, in the forthcoming paper [28], the decay estimates proved in Theorem 1.1 are going to be used for proving a global existence result for small data solutions for the semilinear damped wave equation with power nonlinearity on the Heisenberg group, namely, for the Cauchy problem

$$\begin{cases}
\partial_t^2 u(t, \eta) - \Delta_{H} u(t, \eta) + \partial_t u(t, \eta) = |u|^p, & \eta \in H_n, \ t > 0, \\
u(0, \eta) = u_0(\eta), & \eta \in H_n, \\
\partial_t u(0, \eta) = u_1(\eta), & \eta \in H_n,
\end{cases}$$  

(46)

requiring a suitable lower bound for $p > 1$. By following the main ideas from [35, 19] in the Euclidean case, an exponential weight function will be introduced. Hence, the global existence of small data solutions for exponents $p$ in the super-Fujita case will be shown in the corresponding weighted energy space. Note that this will provide a critical exponent of Fujita-type as for the corresponding semilinear heat equation on the Heisenberg group or on more general nilpotent Lie groups (cf. [32, 26, 27]).

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