Representations attached to vector bundles on curves over finite and $p$-adic fields, a comparison

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1 The comparison

In [DW2] and [DW4] a partial analogue of the classical Narasimhan–Seshadri correspondence between vector bundles and representations of the fundamental group was developed. See also [F] for a $p$-adic theory of Higgs bundles. Let $\mathfrak{o}$ be the ring of integers in $\mathbb{C}_p = \hat{\mathbb{Q}}_p$ and let $k = \mathfrak{o}/\mathfrak{m} = \mathbb{F}_p$ be the common residue field of $\mathbb{Z}_p$ and $\mathfrak{o}$. Consider a smooth projective (connected) curve $X$ over $\mathbb{Q}_p$ and let $E$ be a vector bundle of degree zero on $X_{\mathbb{C}_p} = X \otimes \mathbb{C}_p$.

If $E$ has potentially strongly semistable reduction in the sense of [DW4] Definition 2, then for any $x \in X(\mathbb{C}_p)$ according to [DW4] Theorem 10 there is a continuous representation

$$\rho_{E,x} : \pi_1(X, x) \longrightarrow \text{GL}(E_x).$$

We now describe a special case of the theory where one can define the reduction of $\rho_{E,x} \mod \mathfrak{m}$. Assume that we are given the following data:

i) A model $\mathfrak{X}$ of $X$ i.e. a finitely presented proper flat $\mathbb{Z}_p$-scheme $\mathfrak{X}$ with $X = \mathfrak{X} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,

ii) A vector bundle $\mathcal{E}$ over $\mathfrak{X}_o = \mathfrak{X} \otimes_{\mathbb{Z}_p} \mathfrak{o}$ extending $E$.

Such models $\mathfrak{X}$ and $\mathcal{E}$ always exist. Consider the special fibre $\mathfrak{X}_k = \mathfrak{X} \otimes_{\mathbb{Z}_p} k = \mathfrak{X}_o \otimes_{\mathfrak{o}} k$ and set $\mathcal{E}_k = \mathcal{E} \otimes_{\mathfrak{o}} k$, a vector bundle on $\mathfrak{X}_k$. We assume that $\mathcal{E}_k$ restricted to $\mathfrak{X}_k^{\text{red}}$ is strongly semistable of degree zero in the sense of section 2 below.
In this case we say that $E$ has strongly semistable reduction of degree zero on $\mathfrak{X}_o$. Then [DW2] provides a continuous representation

\begin{equation}
\rho_{E,x_o} : \pi_1(X,x) \longrightarrow \text{GL} \left( \mathcal{E}_{x_o} \right),
\end{equation}

which induces $\pi$. Here $x_o \in \mathfrak{X}(o) = X(\mathbb{C}_p)$ is the section of $\mathfrak{X}$ corresponding to $x$ and $\mathcal{E}_{x_o} = \Gamma(\text{spec } o, x_o^* \mathcal{E})$ is an $o$-lattice in $\mathcal{E}_x$.

Denoting by $x_k \in \mathfrak{X}_k(k) = \mathfrak{X}_k^{\text{red}}(k)$ the reduction of $x_o$, we have $\mathcal{E}_{x_o} \otimes_o k = \mathcal{E}_x$ the fibre over $x_k$ of the vector bundle $\mathcal{E}_k$.

The aim of this note is to describe the reduction mod $m$ of $\rho_{E,x_o}$ i.e. the representation

\begin{equation}
\rho_{E,x_o} \otimes k : \pi_1(X,x) \longrightarrow \text{GL} \left( \mathcal{E}_x \right),
\end{equation}

using Nori’s fundamental group scheme [N].

Let us recall some of the relevant definitions. Consider a perfect field $k$ and a reduced complete and connected $k$-scheme $Z$ with a point $z \in Z(k)$. A vector bundle $H$ on $Z$ is \textit{essentially finite} if there is a torsor $\lambda : P \rightarrow Z$ under a finite group scheme over $k$ such that $\lambda^* H$ is a trivial bundle. Nori has defined a profinite algebraic group scheme $\pi(Z,z)$ over $k$ classifying the essentially finite bundles $H$ on $Z$. Every such bundle corresponds to an algebraic representation

\begin{equation}
\lambda_{H,z} : \pi(Z,z) \longrightarrow \text{GL}_H.
\end{equation}

The group scheme $\pi(Z,z)$ also classifies the pointed torsors under finite group schemes on $Z$. If $k$ is algebraically closed, it follows that the group of $k$-valued points of $\pi(Z,z)$ can be identified with Grothendieck’s fundamental group $\pi_1(Z,z)$. On $k$-valued points the representation $\lambda_{H,z}$ therefore becomes a continuous homomorphism

\begin{equation}
\lambda_{H,z} = \lambda_{H,z}(k) : \pi_1(Z,z) \longrightarrow \text{GL}_H(k).
\end{equation}

We will show the following result:

\textbf{Theorem 1} With notations as above, consider a vector bundle $E$ on $\mathfrak{X}_o$ with strongly semistable reduction of degree zero. Then $\mathcal{E}_k^{\text{red}}$, the bundle $\mathcal{E}_k$ restricted to $\mathfrak{X}_k^{\text{red}}$ is essentially finite. For the corresponding representation:

\begin{equation}
\lambda = \lambda_{\mathcal{E}_k^{\text{red}},x_k} : \pi_1(\mathfrak{X}_k^{\text{red}},x_k) \longrightarrow \text{GL}(\mathcal{E}_x),
\end{equation}

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the following diagram is commutative:

\[
\begin{array}{ccc}
\pi_1(X, x) & \xrightarrow{\rho_{E, x} \otimes k} & \text{GL}(E_{x_k}) \\
\downarrow & & \downarrow \\
\pi_1(X, x) & \xrightarrow{\lambda} & \text{GL}(E_{x_k})
\end{array}
\]

In particular, the reduction mod $m$ of $\rho_{E, x}$ factors over the specialization map $\pi_1(X, x) \to \pi_1(X_{\text{red}}^k, x_k)$. In general this is not true for $\rho_{E, x}$ itself according to Example 5.

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2 sss-bundles on curves over finite fields

In this section we collect a number of definitions and results related to Nori’s fundamental group [N]. The case of curves over finite fields presents some special features.

Consider a reduced complete and connected scheme $Z$ over a perfect field $k$ with a rational point $z \in Z(k)$. According to [N] the $\otimes$-category of essentially finite vector bundles $H$ on $Z$ with the fibre functor $H \mapsto H_z$ is a neutral Tannakian category over $k$. By Tannakian duality it is equivalent to the category of algebraic representations of an affine group scheme $\pi(Z, z)$ over $k$ which turns out to be a projective limit of finite group schemes.

Let $f : Z \to Z'$ be a morphism of reduced complete and connected $k$-schemes. The pullback of vector bundles induces a tensor functor between the categories of essentially finite bundles on $Z'$ and $Z$ which is compatible with the fibre functors in $f(z)$ and $z$. By Tannakian functoriality we obtain a morphism
\(f_* : \pi(Z, z) \to \pi(Z', f(z))\) of group schemes over \(k\). If \(k\) is algebraically closed the induced map on \(k\)-valued points

\[
\pi_1(Z, z) = \pi(Z, z)(k) \to \pi(Z', f(z))(k) = \pi_1(Z', f(z))
\]
is the usual map \(f_*\) between the Grothendieck fundamental groups.

We will next describe the homomorphism

\[\lambda_{H,z} = \lambda_{H,z}(k) : \pi_1(Z, z) = \pi(Z, z)(k) \to \text{GL}(H_z)\]
in case \(H\) is trivialized by a finite étale covering. Consider a scheme \(S\) with a geometric point \(s \in S(\Omega)\). We view \(\pi_1(S, s)\) as the automorphism group of the fibre functor \(F_s\) which maps any finite étale covering \(\pi : S' \to S\) to the set of points \(s' \in S'(\Omega)\) with \(\pi(s') = s\).

**Proposition 2** Let \(Z\) be a reduced complete and connected scheme over the algebraically closed field \(k\) with a point \(z \in Z(k)\). Consider a vector bundle \(H\) on \(Z\) for which there exists a connected finite étale covering \(\pi : Y \to Z\) such that \(\pi^*H\) is a trivial bundle. Then \(H\) is essentially finite and the map \(\lambda_{H,z} : \pi_1(Z, z) \to \text{GL}(H_z)\) in (5) has the following description. Choose a point \(y \in Y(k)\) with \(\pi(y) = z\). Then for every \(\gamma \in \pi_1(Z, z)\) there is a commutative diagram:

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(6) \(\begin{array}{ccc}
\left(\pi^*H\right)_y & \xrightarrow{y^*} & \Gamma(Y, \pi^* H) \\
\downarrow & & \downarrow \\
H_z & \xrightarrow{\lambda_{H,z}(\gamma)} & H_z
\end{array}\)
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**Proof** The covering \(\pi : Y \to Z\) can be dominated by a finite étale Galois covering \(\pi' : Y' \to Z\). Let \(y' \in Y'(k)\) be a point above \(y\). If the diagram (6) with \(\pi, Y, y\) replaced by \(\pi', Y', y'\) is commutative, then (6) itself commutes. Hence we may assume that \(\pi : Y \to Z\) is Galois with group \(G\). In particular \(H\) is essentially finite. Consider the surjective homomorphism \(\pi_1(Z, z) \to G\) mapping \(\gamma\) to the unique \(\sigma \in G\) with \(\gamma y = y\sigma\). The right action of \(G\) on \(Y\) induces a left action on \(\Gamma(Y, \pi^* H)\) by pullback and it follows from the definitions that \(\lambda_{H,z}\) is the composition

\[
\lambda_{H,z} : \pi_1(Z, z) \to G \to \text{GL}(\Gamma(Y, \pi^* H)) \xrightarrow{\sim y^*} \text{GL}(H_z) .
\]
Now the equations
\[(\gamma y)^* \circ (y^*)^{-1} = (y^* \sigma) \circ (y^*)^{-1} = (\sigma \circ y)^* \circ (y^*)^{-1} = y^* \sigma^* \circ (y^*)^{-1} = \lambda_H(k)(\gamma)\]
imply the assertion. Here \(\sigma^*\) is the automorphism of \(\Gamma(Y, \pi^* H)\) induced by \(\sigma\).

The following class of vector bundles contains the essentially finite ones. A vector bundle \(H\) on a reduced connected and complete \(k\)-scheme \(Z\) is called strongly semistable of degree zero (\(sss\)) if for all \(k\)-morphisms \(f : C \to Z\) from a smooth connected projective curve \(C\) over \(k\) the pullback bundle \(f^*(H)\) is semistable of degree zero, c.f. \([DM]\) (2.34). It follows from \([N]\) Lemma (3.6) that the \(sss\)-bundles form an abelian category. Moreover a result of Gieseker shows that it is a tensor category, c.f. \([Gi]\). If \(Z\) is purely one-dimensional, a bundle \(H\) is \(sss\) if and only if the pullback of \(H\) to the normalization \(\tilde{C}_i\) of each irreducible component \(C_i\) of \(Z\) is strongly semistable of degree zero in the usual sense on the smooth projective curve \(\tilde{C}_i\) over \(k\), see e.g. \([DW3]\) Proposition 4.

Generalizing results of Lange–Stuhler and Subramanian slightly we have the following fact, where \(F_q\) denotes the field with \(q = p^r\) elements.

**Theorem 3** Let \(Z\) be a reduced complete and connected purely one-dimensional scheme over \(F_q\). Then the following three conditions are equivalent for a vector bundle \(H\) on \(Z\).

1. \(H\) is strongly semistable of degree zero.
2. There is a finite surjective morphism \(\varphi : Y \to Z\) with \(Y\) a complete and purely one-dimensional scheme over \(F_q\) such that \(\varphi^* H\) is a trivial bundle.
3. There are a finite étale covering \(\pi : Y \to Z\) and some \(s \geq 0\) such that for the composition \(\varphi : Y \xrightarrow{\pi} Y \xrightarrow{\pi} Z\) the pullback \(\varphi^* H\) is a trivial bundle. Here \(F = Fr_q = Fr_p^r\) is the \(q\)-linear Frobenius morphism on \(Y\).

If \(Z\) has an \(F_q\)-rational point, these conditions are equivalent to

4. \(H\) is essentially finite.

**Remark** If \(Z(F_q) \neq \emptyset\), then according to 4 the trivializing morphism \(\varphi : Y \to Z\) in 2 can be chosen to be a \(G\)-torsor under a finite group scheme \(G/Fq\).

**Proof** The equivalence of 1 to 3 is shown in \([DW2]\) Theorem 18 by slightly generalizing a result of Lange and Stuhler. It is clear that 4 implies 2. Over
a smooth projective curve $Z/F_q$ the equivalence of 1 and 4 was shown by Subramanian in [S], Theorem (3.2) with ideas from [MS] and [BPS]. His proof works also over our more general bases $Z$ and shows that 1 implies 4. Roughly the argument goes as follows: Using the fibre functor in a point $z \in Z(F_q)$ the abelian tensor category $\mathcal{T}_Z$ of sss-bundles on $Z$ becomes a neutral Tannakian category over $F_q$. Note by the way that the characterization 2 of sss-bundles shows without appealing to [Gi] that $\mathcal{T}_Z$ is stable under the tensor product. Consider the Tannakian subcategory generated by $H$. Its Tannakian dual is called the monodromy group scheme $M_H$ in [BPS]. Let $n$ be the rank of $H$. The $\text{GL}_n$-torsor associated to $H$ allows a reduction of structure group to $M_H$. Hence we obtain an $M_H$-torsor $\alpha: P \to Z$ such that $\alpha^*H$ is a trivial bundle. We have $\text{Fr}_{q^s}H = \text{Fr}_{q^t}H$ for some $s > t \geq 0$ because there are only finitely many isomorphism classes of semistable vector bundles of degree zero on a smooth projective curve over a finite field. See [DW2] Proof of Theorem 18 for more details. A short argument as in [S] now implies that $M_H$ is a finite group scheme and we are done. 

Later on we will need the following fact:

**Proposition 4** Let $S_0$ be a scheme over $F_q$ and let $F = \text{Fr}_q$ be the $q$-linear Frobenius morphism on $S_0$. Set $k = \bar{F}_q$ and let $\bar{F} = F \otimes_{F_q} k$ be the base extension of $F$ to a morphism of $S = S_0 \otimes_{F_q} k$. Then for any geometric point $s \in S(\Omega)$ the induced map $\bar{F}_*: \pi_1(S, s) \to \pi_1(S, \bar{F}(s))$ is an isomorphism.

**Proof** Let $F_k$ be the automorphism of $k$ with $F_k(x) = x^q$ for all $x \in k$. Then $\psi = \text{id}_{S_0} \otimes F_k$ is an automorphism of the scheme $S$ and hence it induces isomorphisms on fundamental groups. It suffices therefore to show that

$$(\psi \circ \bar{F})_*: \pi_1(S, s) \to \pi_1(S, \psi(\bar{F}(s)))$$

is an isomorphism. The morphism $\psi \circ \bar{F}$ is the $q$-linear Frobenius morphism $\text{Fr}_q$ on $S$. For any finite étale covering $\pi: T \to S$ the relative Frobenius morphism is known to be an isomorphism and hence the commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\text{Fr}_q} & T \\
\pi \downarrow & & \downarrow \pi \\
S & \xrightarrow{\text{Fr}_q} & S
\end{array}
\]

is cartesian. It follows that $\text{Fr}_{q^s} = (\psi \circ \bar{F})_*$ is an isomorphism on fundamental groups. \qed
3 Proof of theorem 1

For the proof of theorem 1 we first give a description of the representation \(\rho_{\mathcal{E},x_0} \otimes k\) which follows immediately from the construction of \(\rho_{\mathcal{E},x_0}\) in [DW2] section 3.

We assume that we are in the situation of theorem 1. By assumption \(E_{\text{red}}^k\) is strongly semistable of degree zero on \(X_{\text{red}}^k\). According to [DW2] theorem 17 there is a proper morphism \(\pi : Z \to X\) with the following properties:

a. The generic fibre \(Z = Z \otimes \mathbb{Q}_p\) is a smooth projective connected \(\mathbb{Q}_p\)-curve.

b. The induced morphism \(\pi : Z \to X\) is finite and for an open dense subscheme \(U \subset X\) the restriction \(\pi : \pi^{-1}(U) = W \to U\) is étale. Moreover we have \(x \in U(\mathbb{C}_p)\) for the chosen base point \(x \in X(\mathbb{C}_p)\).

c. The scheme \(Z\) is a model of \(Z\) over \(\mathbb{Z}_p\) whose special fibre \(Z_k\) is reduced. In particular \(Z/\mathbb{Z}_p\) is cohomologically flat in degree zero.

d. The pullback \(\pi^*_k \mathcal{E}_k\) is a trivial vector bundle on \(Z_k\).

The following construction gives a representation of \(\pi_1(U,x)\) on \(\mathcal{E}_{x_k}\). For \(\gamma \in \pi_1(U,x) = \text{Aut}(F_x)\) choose a point \(z \in W(\mathbb{C}_p)\) with \(\pi(z) = x\). Then \(\gamma z \in W(\mathbb{C}_p)\) is another point over \(x\). From \(z\) and \(\gamma z\) in \(W(\mathbb{C}_p) \subset Z(\mathbb{C}_p)\) we obtain points \(z_k\) and \((\gamma z)_k\) in \(Z_k(k)\) as in the introduction. Consider the diagram

\[
\begin{align*}
\mathcal{E}_{x_k} = (\pi^*_k \mathcal{E}_k)_{z_k} \xleftarrow{z_k^*} \Gamma(Z_k, \pi^*_k \mathcal{E}_k) \xrightarrow{(\gamma z)_k^*} (\pi^*_k \mathcal{E}_k)_{(\gamma z)_k} = \mathcal{E}_{x_k}.
\end{align*}
\]

Here the pullback morphisms along \(z_k : \text{spec } k \to Z_k\) and \((\gamma z)_k : \text{spec } k \to Z_k\) are isomorphisms because \(\pi^*_k \mathcal{E}_k\) is a trivial bundle and \(Z/\mathbb{Z}_p\) is cohomologically flat in degree zero.

It turns out that the map

\[
\rho : \pi_1(U,x) \to \text{GL}(\mathcal{E}_{x_k}) \quad \text{defined by } \rho(\gamma) = (\gamma z)_k^* \circ (z_k^*)^{-1}
\]

is a homomorphism of groups which (by construction) factors over a finite quotient of \(\pi_1(U,x)\). Thus \(\rho\) is continuous if \(\text{GL}(\mathcal{E}_{x_k})\) is given the discrete topology. Moreover \(\rho\) does not depend on either the choice of the point \(z\) above \(x\) nor on the choice of morphism \(\pi : Z \to X\) satisfying a–d. It follows from [DW2] Theorem 17 and Proposition 35 that \(\rho\) factors over \(\pi_1(X,x)\). The resulting representation \(\rho : \pi_1(X,x) \to \text{GL}(\mathcal{E}_{x_k})\) agrees with \(\rho_{\mathcal{E},x_0} \otimes k\).
In order to prove theorem 17 we will now construct given $\mathcal{E}_k$ a suitable morphism $\mathcal{Z} \to \mathcal{X}$. We use a modification of the method from the proof of theorem 17 in [DW2]. In that proof the singularities were resolved at the level of $\mathcal{Y}$ which is too late for our present purposes because it creates an extension of $\mathcal{Y}_k$ which is hard to control discussing the Nori fundamental group. Instead, we will resolve the singularities of a model of $X$. Then $\mathcal{Y}$ does not have to be changed later.

We proceed with the details:

Choose a finite extension $K/\mathbb{Q}_p$ with ring of integers $\mathfrak{o}_K$ and residue field $k$ such that $(\mathcal{X}, \mathcal{E}_k, x_k)$ descends to $(\mathcal{X}_{\mathfrak{o}_K}, \mathcal{E}_0, x_0)$. Here $\mathcal{X}_{\mathfrak{o}_K}$ is a proper and flat $\mathfrak{o}_K$-scheme with $\mathcal{X}_{\mathfrak{o}_K} \otimes_{\mathfrak{o}_K} \mathbb{Z}_p = \mathcal{X}$ and $\mathcal{E}_0$ a vector bundle on $\mathcal{X}_0 = \mathcal{X}_{\mathfrak{o}_K} \otimes k$ with $\mathcal{E}_0 \otimes_k k = \mathcal{E}_k$. Since $\mathcal{E}_k^{\text{red}}$ is an sss-bundle on $\mathcal{X}_k^{\text{red}}$ the restriction $\mathcal{E}_0^{\text{red}}$ of $\mathcal{E}_0$ to $\mathcal{X}_0^{\text{red}}$ is an sss-bundle as well. Finally $x_0 \in \mathcal{X}_0(k)$ is a point which induces $x_k$ after base change to $k$. Theorem 3 implies that $\mathcal{E}_0^{\text{red}}$ is essentially finite and hence $\mathcal{E}_k^{\text{red}}$ is essentially finite as well.

After replacing $K$ by a finite extension and performing a base extension to the new $K$ we can find a semistable model $\mathcal{X}'_{\mathfrak{o}_K}$ of the smooth projective curve $X_K = \mathcal{X}_{\mathfrak{o}_K} \otimes K$ together with a morphism $\alpha_{\mathfrak{o}_K} : \mathcal{X}'_{\mathfrak{o}_K} \to \mathcal{X}_{\mathfrak{o}_K}$ extending the identity on the generic fibre $X_K$. This is possible by the semistable reduction theorem, c.f. [A] for a comprehensive account. By Lipman’s desingularization theorem we may assume that $\mathcal{X}'_{\mathfrak{o}_K}$ besides being semistable is also regular, c.f. [Li] 10.3.25 and 10.3.26. The irreducible regular surface $\mathcal{X}'_{\mathfrak{o}_K}$ is proper and flat over $\mathfrak{o}_K$.

Let $\mathcal{E}'_0$ be the pullback of $\mathcal{E}_0$ along the morphism $\alpha_0 : \mathcal{X}'_0 = \mathcal{X}' \otimes k \to \mathcal{X}_0$. Since $\mathcal{X}'_0$ is reduced the map factors as $\alpha_0 : \mathcal{X}'_0 \to \mathcal{X}_0^{\text{red}} \subset \mathcal{X}_0$ and $\mathcal{E}'_0$ is also the pullback of the sss-bundle $\mathcal{E}_0^{\text{red}}$. Hence $\mathcal{E}'_0$ is an sss-bundle as well.

Using theorem 3 we find a finite étale covering $\pi_0 : \mathcal{Y}_0 \to \mathcal{X}'_0$ by a complete and one-dimensional $\kappa$-scheme $\mathcal{Y}_0$ and an integer $s \geq 0$ such that under the composed map $\varphi : \mathcal{Y}_0 \xrightarrow{F^s} \mathcal{Y}_0 \xrightarrow{\pi_0} \mathcal{X}'_0$ the pullback $\varphi^* \mathcal{E}'_0$ is a trivial bundle. Here $F = \text{Fr}_q$ is the $q = |\kappa|\cdot \text{linear Frobenius}$ morphism on $\mathcal{Y}_0$. Let $\tilde{\kappa}$ be a finite extension of $\kappa$ such that all connected components of $\mathcal{Y}_0 \otimes_\kappa \tilde{\kappa}$ are geometrically connected. Let $\tilde{K}/K$ be the unramified extension with residue field $\tilde{\kappa}$. We replace $\mathcal{X}_{\mathfrak{o}_K}, \mathcal{X}_{\mathfrak{o}_K}^\prime$ and $\mathcal{E}_0, \mathcal{E}_0^\prime$ by their base extensions with $\mathfrak{o}_{\tilde{K}}$ resp. $\tilde{\kappa}$ and $F$ by the $|\tilde{\kappa}|$-linear Frobenius morphism. We also replace $\mathcal{Y}_0$ be a connected component of $\mathcal{Y}_0 \otimes_\kappa \tilde{\kappa}$ and $\pi_0$ by the induced morphism. Then the new $\mathcal{X}_{\mathfrak{o}_K}, \mathcal{X}_{\mathfrak{o}_K}^\prime, \varphi, \ldots$ keep the previous properties and $\mathcal{Y}_0$ is now geometrically connected. Using
IX Théorème 1.10 we may lift \( \pi_0 : Y_0 \rightarrow X'_0 \) to a finite étale morphism \( \pi_{o_K} : Y_{o_K} \rightarrow X'_{o_K} \). The proper flat \( o_K \)-scheme \( Y_{o_K} \) is regular with geometrically reduced fibres over \( o_K \) because \( X'_{o_K} \) has these properties. In particular, the morphism \( Y_{o_K} \rightarrow \text{spec} \ o_K \) is cohomologically flat in degree zero. Since the special fibre \( Y_0 \) is geometrically connected and reduced it follows that the generic fibre \( Y_K \) of \( Y_{o_K} \) is geometrically connected and hence a smooth projective geometrically irreducible curve over \( K \). In particular \( Y_{o_K} \) is irreducible in addition to being regular and proper flat over \( o_K \). By a theorem of Lichtenbaum [Li] there is thus a closed immersion \( \tau_K : Y_{o_K} \hookrightarrow \mathbb{P}^N_{\kappa} \) for some \( N \). Composing with a suitable automorphism of \( \mathbb{P}^N_{o_K} \) we may assume that \( \tau_K^{-1}(G_{m,K}) \subset Y_K \) contains all points in \( Y_K(\kappa) \) over \( x \in X_K(\kappa) = X(\kappa) \). In particular, \( \tau_K^{-1}(G_{m,K}) \) is open and dense in \( Y_K \) with a finite complement. Thus there is an open subscheme \( U_K \subset X_K \) with \( x \in U_K(\kappa) \) and such that \( V_K = \tau_K^{-1}(U_K) \) is contained in \( \tau_K^{-1}(G_{m,K}) \).

Consider the finite morphism \( F_{o_K} : \mathbb{P}^N_{o_K} \rightarrow \mathbb{P}^N_{o_K} \) given on \( A \)-valued points where \( A \) is any \( o_K \)-algebra, by sending \([x_0 : \ldots : x_N] \) to \([x_0^q : \ldots : x_N^q] \). The reduction of \( F_{o_K} \) is the \( q \)-linear Frobenius morphism on \( \mathbb{P}^N_{\kappa} \).

Let \( \rho_{o_K} : Y'_{o_K} \rightarrow Y_{o_K} \) be the base change of \( F_{o_K} \) via \( \tau_K \). It is finite and its generic fibre \( \rho_K : Y'_K \rightarrow Y_K \) is étale over \( V_K \). Now we look at the reductions and we define a morphism \( i : Y_0 \rightarrow Y'_0 \) over \( \kappa \) by the commutative diagram

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{i} & Y'_0 \\
\downarrow{\tau_0} & & \downarrow{\rho_0} \\
\mathbb{P}^N_{\kappa} & \xrightarrow{F^s} & \mathbb{P}^N_{\kappa}
\end{array}
\]

In [DW2] Lemma 19 it is shown that \( i \) induces an isomorphism \( i : Y_0 \isom Y'_0 \). Here the index 0 always refers to the special fibre over \( \text{spec} \ k \).

Taking the normalization of \( Y_{o_K} \) in the function field of an irreducible component of \( Y'_K \) we get a proper, flat \( o_K \)-scheme \( Y''_{o_K} \) which is finite over \( Y'_{o_K} \). Its generic fibre \( Y''_K \) is a smooth projective connected curve over \( K \) (maybe not
geometrically connected). The following diagram summarizes the situation

```
\begin{array}{c}
Y''_0 \rightarrow Y''_K \leftarrow Y''_K \\
\downarrow \quad \downarrow \quad \downarrow \\
Y'_0 \rightarrow Y'_K \leftarrow Y'_K \\
\downarrow \quad \downarrow \quad \downarrow \\
Y_0 \rightarrow Y_K \leftarrow Y_K \\
\downarrow \quad \downarrow \quad \downarrow \\
X_0 \rightarrow X_K \leftarrow X_K \\
\downarrow \quad \downarrow \quad \downarrow \\
X_0^{\text{red}} \rightarrow X_K^{\text{red}} \leftarrow X_K^{\text{red}}
\end{array}
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For a suitable finite extension $\tilde{K}/K$ all connected components of $Y''_K \otimes_K \tilde{K}$ will be geometrically connected. Let $Y'''_K$ be one of them and let $Y'''_o \otimes \tilde{K}$ be its closure with the reduced scheme structure in $Y''_o \otimes \tilde{K}$. By the semistable reduction theorem there are a finite extension $L/\tilde{K}$ and a semistable model $Z_{o_L}$ of $Y'''_K \otimes \tilde{K}$ over $Y'''_o \otimes \tilde{K}$. Base extending $X_{o_K}$, $Y_{o_K}$, $X_{o_K}$, and $Z_{o_L}$ over $o_K$ and $o_L$ we get a commutative diagram, where $\delta$ is the composition $\delta: Z \rightarrow Y''' \rightarrow Y'' \rightarrow Y' \rightarrow Y$,

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\begin{array}{c}
Z_k \rightarrow Z \leftarrow Z \\
\downarrow \quad \downarrow \quad \downarrow \\
Y_k \rightarrow Y \leftarrow Y \\
\downarrow \quad \downarrow \quad \downarrow \\
X'_L \rightarrow X \leftarrow X \\
\downarrow \quad \downarrow \quad \downarrow \\
X_k^{\text{red}} \rightarrow X_k \leftarrow X_k
\end{array}
```

Here the morphism $\beta_k: Z_k \rightarrow Y_k$ comes about as follows: Since $Z_k$ is reduced, the composition $Z_k \rightarrow Y'''_K \rightarrow Y''_K \rightarrow Y'_K$ factors over $Y'_K \rightarrow Y_K$ and this
defines $\beta_k$. By construction, the map $\pi_{Q_p} \circ \delta_{Q_p} : Z \to X$ is finite and such that its restriction to a map $W = (\pi_{Q_p} \circ \delta_{Q_p})^{-1}(U) \to U$ is finite and étale. By construction the bundle $E_k' = \alpha_k^*E_k = E_0' \otimes \kappa$ is trivialized by pullback along $\pi_k \circ (F \otimes \kappa)$ and hence also along $(\pi \circ \delta)_k = \pi_k \circ (F \otimes \kappa) \circ \beta_k$. For later purposes note that we have a commutative diagram

$$(10)$$

\[
\begin{array}{ccc}
\mathcal{Y}_k & \overset{F \otimes \kappa}{\longrightarrow} & \mathcal{Y}_k \\
\pi_k \downarrow & & \downarrow \pi_k \\
\mathcal{X}_k' & \overset{F \otimes \kappa}{\longrightarrow} & \mathcal{X}_k'
\end{array}
\]

obtained by base changing the corresponding diagram over $\kappa$:

\[
\begin{array}{ccc}
\mathcal{Y}_0 & \overset{F}{\longrightarrow} & \mathcal{Y}_0 \\
\pi_0 \downarrow & & \downarrow \pi_0 \\
\mathcal{X}_0' & \overset{F}{\longrightarrow} & \mathcal{X}_0'
\end{array}
\]

The inclusion $\mathfrak{X}_k \to \mathfrak{X}$ induces a natural isomorphism $\pi_1(\mathfrak{X}_k, x_k) \sim \pi_1(\mathfrak{X}, x_k)$. This follows from [SGA1] Exp. X, Théorème 2.1 together with an argument to reduce the finitely presented case to a Noetherian one as in the proof of [SGA1], Exp. IX, Théorème 6.1, p. 254 above.

Next we note that there is a canonical isomorphism

$$\pi_1(\mathfrak{X}, x_k) = \text{Aut}(F_{x_k}) = \text{Aut}F_x = \pi_1(\mathfrak{X}, x).$$

Namely, for a finite étale covering $\mathcal{Y} \to \mathfrak{X}$, by the infinitesimal lifting property, any point $y_k \in \mathcal{Y}_k(k)$ over $x_k$ determines a unique section $y_\phi \in \mathcal{Y}(\phi)$ over $x_\phi \in \mathfrak{X}(\phi)$ and hence a point $y \in \mathcal{Y}(C_p)$ over $x \in \mathfrak{X}(C_p)$. In this way one obtains a bijection between the points $y_k$ over $x_k$ and the points $y$ over $x$. Thus the fibre functors $F_{x_k}$ and $F_x$ are canonically isomorphic.

Finally, by [SGA1], Exp. IX, Proposition 1.7, the inclusion $\mathfrak{X}_k^{\text{red}} \hookrightarrow \mathfrak{X}_k$ induces an isomorphism $\pi_1(\mathfrak{X}_k^{\text{red}}, x_k) \sim \pi_1(\mathfrak{X}_k, x_k)$. Thus we get an isomorphism

$$\pi_1(\mathfrak{X}_k^{\text{red}}, x_k) \sim \pi_1(\mathfrak{X}_k, x_k) = \pi_1(\mathfrak{X}, x_k) = \pi_1(\mathfrak{X}, x)$$
and hence a commutative diagram

(11)

\[
\begin{array}{ccc}
\pi_1(X', x) & \xrightarrow{\alpha_*} & \pi_1(X'_k, x'_k) \\
\downarrow & & \downarrow \\
\pi_1(X, x) & \xrightarrow{\alpha_k*} & \pi_1(X_{\text{red}}^k, x_k).
\end{array}
\]

For $\gamma \in \pi_1(X, x)$ choose an element $\gamma \in \pi_1(U, x)$ which maps to $\overline{\gamma}$ and let $\overline{\gamma}_k$ be the image of $\overline{\gamma}$ in $\pi_1(X'_k, x'_k)$. Fix a point $z \in W(C_p)$ which maps to $x \in U(C_p)$ in diagram (9). As explained at the beginning of this section the automorphism $\rho_{E, x}(\gamma) \otimes k$ of $E_{x_k}$ is given by the formula

(12)

\[
\rho_{E, x}(\gamma) \otimes k = (\gamma z)^* \circ (z_k^*)^{-1}.
\]

Here the isomorphisms $z_k^*$ and $(\gamma z)_k^*$ are the ones in the upper row of the following commutative diagram, where we have set $F_k = (\pi_k \circ (F^s \otimes \kappa))^* E'_k$, so that $(\alpha \circ \pi \circ \delta)^* E_k = \beta_k^* F_k$. Moreover $\overline{y}_1 := \beta_k(z_k)$ and $\overline{y}_2 := \beta_k((\gamma z)_k)$ in $Y_k(k)$,

(13)

\[
\begin{array}{ccc}
E_{x_k} \xrightarrow{(\beta_k^* F_k)_{z_k}} & \Gamma(Z_k, \beta_k^* F_k) \xrightarrow{(\gamma z)_k^*} & E_{x_k} \\
\downarrow & \approx & \downarrow \\
E_{x_k} \xrightarrow{(F_k)_{\overline{y}_1}} & \Gamma(Y_k, F_k) \xrightarrow{(\gamma z)_k} & E_{x_k}.
\end{array}
\]

Note here that $F_k$ is already a trivial bundle and that $Y_k$ and $Z_k$ are both reduced and connected. It follows that all maps in this diagram are isomorphisms. Using (12) we therefore get the formula:

(14)

\[
\rho_{E, x}(\gamma) \otimes k = \overline{y}_2 \circ (\overline{y}_1^{-1}).
\]

The point $y = \delta_{\overline{y}_2}(z)$ in $V(C_p) \subset Y(C_p)$ lies above $x$ and we have $\gamma y = \delta_{\overline{y}_2}(\gamma z)$. Moreover the relations

(15)

\[(F^s \otimes \kappa)k(\overline{y}_1) = y_k \quad \text{and} \quad (F^s \otimes \kappa)k(\overline{y}_2) = (\gamma y)_k = \overline{\gamma}_k(y_k)\]

hold because $\gamma y = \overline{\gamma}y$ implies that $(\gamma y)_k = (\overline{\gamma}y)_k = \overline{\gamma}_k(y_k)$. Setting $G_k = (F^s \otimes \kappa)^* E'_k$, a bundle on $X'_k$, we have $F_k = \pi_k^* G_k$.  

12
Next we look at representations of Nori’s fundamental group. For the point \( \pi_1 = \pi_k(\pi_1) \) in \( X'_k(k) \) we have \((F^s \otimes k)(\pi_1) = x'_k\).

Consider the commutative diagram:

\[
\begin{array}{ccc}
\pi_1(X'_k, \pi_1) & \xrightarrow{\lambda_{G_k, \pi_1}} & \text{GL}((G_k)_{\pi_1}) \\
\downarrow \quad (F^s \otimes k)_* & & \downarrow \\
\pi_1(X'_k, x'_k) & \xrightarrow{\lambda_{E'_k, x'_k}} & \text{GL}((E'_k)_{x'_k}) \\
\downarrow \quad \alpha_k & & \downarrow \\
\pi_1(X^\text{red}_k, x_k) & \xrightarrow{\lambda_{E^\text{red}_k, x_k}} & \text{GL}((E_k)_{x_k}) 
\end{array}
\]

It is obtained by passing to the groups of \( k \)-valued points in the corresponding diagram for representations of Nori’s fundamental group schemes. Recall that as observed above \( E_k^\text{red} \) is an essentially finite bundle on \( X_k^\text{red} \). The fact that \((F^s \otimes k)_* \) is an isomorphism on fundamental groups was shown in Proposition 4. Let \( \tilde{\gamma}_k \in \pi_1(X'_k, \pi_1) \) be the element with \((F^s \otimes k)_* (\tilde{\gamma}_k) = \pi_k\). Using the diagrams (11) and (16), theorem 4 will follow once we have shown the equation

\[
(\rho_{E, x_k}(\gamma) \otimes k = \lambda_{G_k, x_k}(\tilde{\gamma}_k) \quad \text{in} \quad \text{GL}((E_k)_{x_k}) \).
\]

We now use the description of \( \rho_{E, x_k} \otimes k \) in formula (14) and the one of \( \lambda_{G_k, x_k} \) in Proposition 2 applied to the finite étale covering \( \pi_k : Y_k \to X'_k \) which trivializes \( G_k \). It follows that (17) is equivalent to the following diagram being commutative where we recall that \( F_k = \pi_k^* G_k \):

\[
\begin{array}{ccc}
E_k & \xrightarrow{(\pi_k)_!} & \Gamma(Y_k, F_k) \\
\downarrow \quad \overline{\pi}_1 & & \downarrow \\
E_k & \xrightarrow{(\pi_k)_!} & \Gamma(Y_k, F_k) \\
\downarrow \quad \overline{\pi}_2 & & \downarrow \\
E_k & \xrightarrow{(\pi_k)_!} & \Gamma(Y_k, F_k)
\end{array}
\]

But this is trivial since we have \( \overline{\pi}_2 = \tilde{\gamma}_k(\overline{\pi}_1) \). Namely (15) implies the equations:

\[
(F^s \otimes k)(\overline{\pi}_2) = \tilde{\gamma}_k(\overline{\pi}_1) = \tilde{\gamma}_k((F^s \otimes k)(\overline{\pi}_1)) = (F^s \otimes k)(\tilde{\gamma}_k(\overline{\pi}_1))
\]

and \( F^s \otimes k \) is injective on \( k \)-valued points because \( F \) is universally injective.
Example 5 The following example shows that in general the representation
\[ \rho_{E,x} : \pi_1(X, x) \to \text{GL}(E_x) \]
in theorem [1] does not factor over the specialization map \( \pi_1(X, x) \to \pi_1(\mathfrak{X}_k, x_k) \).
Let \( \mathfrak{X} \) be an elliptic curve over \( \mathbb{Z}_p \) whose reduction \( \mathfrak{X}_k \) is supersingular. Then we have \( \mathfrak{X}_k^{\text{red}} = \mathfrak{X}_k \) and \( \pi_1(\mathfrak{X}_k, 0)(p) = 0 \). The exact functor \( E \mapsto \rho_{E,0} \) of [DW2] or [DW4] induces a homomorphism
\[ \rho_* : \text{Ext}^1_{X_{\mathbb{C}_p}}(\mathcal{O}, \mathcal{O}) \to \text{Ext}^1_{\pi_1(X, 0)}(\mathbb{C}_p, \mathbb{C}_p) = \text{Hom}(\pi_1(X, 0), \mathbb{C}_p). \]
Here the second Ext-group refers to the category of finite dimensional \( \mathbb{C}_p \)-vector spaces with a continuous \( \pi_1(X, 0) \)-operation. Moreover, Hom refers to continuous homomorphisms. In [DW1] Corollary 1, by comparing with Hodge–Tate theory it is shown that \( \rho_* \) is injective. For an extension of vector bundles \( 0 \to \mathcal{O} \to E \to \mathcal{O} \to 0 \) on \( X_{\mathbb{C}_p} \) the corresponding representation \( \rho_{E,0} \) of \( \pi_1(X, 0) \) on \( \text{GL}(E_0) \) is unipotent of rank 2 and described by the additive character
\[ \rho_*( [E] ) \in \text{Hom}(\pi_1(X, 0), \mathbb{C}_p) = \text{Hom}(\pi_1(X, 0)(p), \mathbb{C}_p). \]
In particular \( \rho_{E,0} \) factors over \( \pi_1(X, 0)(p) \) and \( \rho_{E,0} \) is trivial if and only if \( [E] = 0 \). Thus any extension \( [\mathcal{E}] \) in \( H^1(\mathfrak{X}, \mathcal{O}) \) whose restriction to \( H^1(X, \mathcal{O}) \) is non-trivial has a non-trivial associated representation
\[ \rho_{\mathcal{E},0} : \pi_1(X, 0) \to \text{GL}(\mathcal{E}_0). \]
Since \( \rho_{\mathcal{E},0} \) factors over \( \pi_1(X, 0)(p) \) it cannot factor over \( \pi_1(\mathfrak{X}_k, 0) \) because then it would factor over \( \pi_1(\mathfrak{X}_k, 0)(p) = 0 \).

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