NORMAL ZETA FUNCTIONS OF THE HEISENBERG GROUPS
OVER NUMBER RINGS II - THE NON-SPLIT CASE

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Abstract. We compute explicitly the normal zeta functions of the Heisenberg groups $H(R)$, where $R$ is a compact discrete valuation ring of characteristic zero. These zeta functions occur as Euler factors of normal zeta functions of Heisenberg groups of the form $H(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an arbitrary number field $K$, which are indexed by the rational primes which are non-split in $K$. We show that these local zeta functions satisfy functional equations upon the inversion of the prime.

1. Introduction

Let $G$ be a finitely generated abstract or profinite group. For $m \in \mathbb{N}$, let $a^q_m(G)$ denote the number of (open) normal subgroups of $G$ of index $m$ in $G$. The normal zeta function of $G$ is the Dirichlet generating series

$$\zeta^q_G(s) = \sum_{m=1}^{\infty} a^q_m(G) m^{-s},$$

where $s$ is a complex variable. If $G$ is a finitely generated nilpotent group, then its normal zeta function converges on a complex half-plane and satisfies the Euler product

$$\zeta^q_G(s) = \prod_{p \text{ prime}} \zeta^q_{G,p}(s).$$

Here, for a prime $p$, the Euler factor $\zeta^q_{G,p}(s) = \sum_{k=0}^{\infty} a^q_{p^k}(G)p^{-ks}$ enumerates the normal subgroups of $G$ of $p$-power index in $G$. It may also be viewed as the normal zeta function of the pro-$p$ completion $\hat{G}^p$ of $G$. The Euler product reflects the facts that the normal zeta function of $G$ coincides with the normal zeta function of its profinite completion $\hat{G}$, and that $\hat{G} \cong \prod_{p \text{ prime}} \hat{G}^p$. In any case, the zeta functions $\zeta^q_{G,p}(s)$ are known to be rational functions in $p^{-s}$; cf. [2, Theorem 1].
Given a ring $\mathcal{R}$, the **Heisenberg group** $H(\mathcal{R})$ over $\mathcal{R}$ is the group of upper-triangular $3 \times 3$ matrices over $\mathcal{R}$:

$$H(\mathcal{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathcal{R} \right\}.$$ 

If $\mathcal{R}$ is a finitely generated torsion-free $\mathbb{Z}$-module of rank $n$, say, then $H(\mathcal{R})$ is a finitely generated torsion-free nilpotent group of nilpotency class 2 and Hirsch length $3n$. Given a prime $p$, the pro-$p$ completion of $H(\mathcal{R})$ is isomorphic to the $3n$-dimensional nilpotent $p$-adic analytic pro-$p$ group $H(\mathcal{R}_p)$, where $\mathcal{R}_p = \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and we have

$$\zeta_{H(\mathcal{R})}^\mathcal{R}(s) = \zeta_{H(\mathcal{R}_p)}^\mathcal{R}(s).$$

In this article we compute explicitly the normal zeta function of the Heisenberg group over an arbitrary compact discrete valuation ring $R$ of characteristic zero, i.e. a finite extension of the ring $\mathbb{Z}_p$ of $p$-adic integers. Let $m$ be the maximal ideal of the local ring $R$. The residue field $k = R/m$ is finite extension of the prime field $\mathbb{F}_p$. The degree $f = [k : \mathbb{F}_p]$ is called the inertia degree of $R$. The (absolute) ramification index $e$ of $R$ is given by $pR = m^e$. The ring $R$ is called unramified (over $\mathbb{Z}_p$) if $e = 1$ and totally ramified (over $\mathbb{Z}_p$) if $f = 1$. The degree of $R$ as an extension of $\mathbb{Z}_p$ is $n = ef$. It coincides with the rank of $R$ as $\mathbb{Z}_p$-module.

Normal zeta functions of Heisenberg groups of the form $H(R)$ occur as Euler factors of normal zeta functions of Heisenberg groups over number rings. Indeed, let $\mathcal{O}_K$ be the ring of integers of a number field $K$. Then $(\mathcal{O}_K)_p = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a local ring precisely if $p$ does not split in $K$, i.e. it decomposes in $K$ as $p\mathcal{O}_K = p^e$, where $p$ is a prime ideal of $\mathcal{O}_K$. We will call such primes non-split (in $K$). If $f = [\mathcal{O}_K/p : \mathbb{F}_p]$ denotes the inertia degree, then $ef = [K : \mathbb{Q}] = n$, say.

That normal zeta functions of groups of the form $H(R)$ are rational in $p^{-s}$ follows therefore from the general result [2, Theorem 1]. The more specific result [2, Theorem 3] asserts that the Euler factors of (normal) zeta functions of $H(\mathcal{O}_K)$ are rational in the two parameters $p^{-s}$ and $p$ on sets of rational primes with fixed decomposition type in $K$; cf. also [7] for details. There are, in particular, rational functions $W_{e,f}^\mathcal{O}(X,Y) \in \mathbb{Q}(X,Y)$ such that for all rational primes $p$ and $R$ as above, the following holds:

$$\zeta_{H(R)}^\mathcal{O}(s) = W_{e,f}^\mathcal{O}(p, p^{-s}).$$

In Theorem 3.9, our main result, we compute the rational functions $W_{e,f}^\mathcal{O}(X,Y)$ explicitly. Moreover, we prove the following functional equations in Corollary 3.13.

**Theorem 1.1.** Let $e, f \in \mathbb{N}$ with $ef = n$. Then

$$W_{e,f}^\mathcal{O}(X^{-1}, Y^{-1}) = (-1)^{3n} X^{\binom{n}{2}} Y^{5n-(e-1)f} W_{e,f}^\mathcal{O}(X, Y).$$

Note that $3n = \dim(H(R))$ and $5n = \dim(H(R)) + \dim(H(R)/H(R'))$, where $H(R')$ is the derived subgroup of $H(R)$. Here “dim” refers to the dimensions as $p$-adic analytic pro-$p$ groups. The term $(e-1)f$ describes the deviation from the “generic” symmetry factor in the functional equations for the local factors of normal zeta functions of finitely generated nilpotent groups of nilpotency class 2; cf. [10, Theorem C].
The results of the current paper complement those of [7], where we carry out analogous computations of the normal zeta functions of the groups $H((\mathcal{O}_K)_p)$ for primes $p$ which are unramified in $K$. There we also establish functional equations for these zeta functions that are comparable to those in Theorem 1.1 and coincide with them in the common special case of primes $p$ which are inert in $K$; see Theorem 3.1. In [7, Conjecture 1.4] we conjecture a functional equation for $\zeta_{H((\mathcal{O}_K)_p)}^\omega(s)$ for arbitrary (not necessarily unramified or non-split) rational primes, generalizing Theorem 1.1.

The methods used in the present paper are quite different from those of [7]. There the problem of computing the relevant zeta functions reduces to that of effectively enumerating subgroups of abelian $p$-groups varying in infinite, combinatorially described families. The precise shape these families take is determined by the decomposition type of the rational primes $p$ in the number field $K$. The simplest decomposition type is that of inert primes, namely the case where $p\mathcal{O}_K$ is a prime ideal. The present paper generalizes this case to non-split primes. We tackle this degeneration of the inert case using ideas introduced in [9] and [4], as we now explain.

The paper [9] argues that the normal subgroup growth of a finitely generated nilpotent group $G$ of nilpotency class 2 is, to a large extent, determined by the geometry of its Pfaffian hypersurface. This is a projective hypersurface, defined explicitly by the Pfaffian of an antisymmetric matrix encoding the group’s structure constants with respect to a chosen (Mal’cev) basis. In the case that the Pfaffian hypersurface of $G$ is smooth and contains no lines (and $G$ satisfies some other mild hypotheses), [9, Theorem 3] gives an explicit formula for the Euler factors $\zeta_{G,p}^\omega(s)$ for almost all primes $p$, in terms of the numbers of $\mathbb{F}_p$-rational points on the Pfaffian hypersurface. In the special case $G = H((\mathcal{O}_K)_p)$, the results of [9] are not literally applicable. However, the ideas of the paper may be applied to the Euler factors $\zeta_{G,p}^\omega(s)$ for inert primes $p$, as in that case the Pfaffian hypersurface has no $\mathbb{F}_p$-rational points. In the setup of [9], this means that the set of solutions of a certain system of linear congruences has a particularly simple form. For non-split primes, ramification complicates this system slightly. The main idea of the current note is to control this complication using parabolic length functions on symmetric groups. These functions generalize the usual Coxeter length and were used to solve related enumeration problems in [4]. Theorem 3.9 expresses $\zeta_{H(R)}^\omega(s)$ in terms of parabolic length functions on the symmetric group $S_n$. Corollary 3.18 gives a formula in the totally ramified case in terms of parabolic length functions on $S_{n-1}$.

Prior to our work, the normal zeta functions $\zeta_{H(R)}^\omega(s)$ had been calculated for all cases occurring for $n \leq 3$; see [1, Theorems 2.2, 2.7, and 2.9].

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2. Preliminaries

By $p$ we denote a rational prime, and we fix the abbreviation $t = p^{-s}$. For an integer $m \geq 1$, we write $[m]$ for $\{1,2,\ldots,m\}$ and $[m]_0$ for $\{0,1,\ldots,m\}$. Given integers $a,b$ with $a \leq b$, we write $[a,b]$ for $\{a,a+1,\ldots,b\}$. Given a finite set $I$ of integers, we write $I = \{i_1,\ldots,i_t\}_{<}$ to indicate that $i_1 < \cdots < i_t$. 


2.1. Coxeter groups. The symmetric group $S_n$ of degree $n$ is a Coxeter group, with Coxeter generating set $S = \{s_1, \ldots, s_{n-1}\}$, where, for each $i \in [n-1]$, we denote by $s_i = (i \ i+1)$ the transposition of the letters $i$ and $i+1$ in the standard permutation representation of $S_n$. We will frequently identify elements of $S_n$ with permutations of $[n]$ in this way.

We write $\text{len} : S_n \to \binom{[n]}{2}_0$ for the usual Coxeter length function: for $w \in S_n$, $\text{len}(w)$ denotes the length of a shortest word representing $w$ as a product of elements of $S$.

Given $I \subseteq [n-1]$, we write $W_I = \langle s_i \mid i \in I \rangle$ for the parabolic subgroup of $S_n$ generated by the elements of $S$ indexed by elements of $I$. The restriction of $\text{len}$ to $W_I$ coincides with the standard length function on the Coxeter group $W_I$. Every element $w \in S_n$ can be factorized uniquely as $w = w^I w_I$, where $w_I \in W_I$ and $w^I$ is the unique element of shortest length in the coset $w W_I$, and that $\text{len}(w) = \text{len}(w_I) + \text{len}(w^I)$; cf. [3, Section 1.10]. We set $\text{len}^I(w) := \text{len}(w_I)$, and call $\text{len}^I$ the (right) parabolic length function associated to $I$; cf. [4, Definition 2.2].

The group $S_n$ has a unique longest element $w_0$ with respect to $\text{len}$, namely the inversion: $w_0(i) = n+1 - i$ for $i \in [n]$. Parabolic length functions are well-behaved with respect to (right) multiplication with $w_0$. Indeed, for every $I \subseteq [n-1]$ and $w \in S_n$,

\begin{equation}
\text{len}^I(w w_0) = \text{len}^I(w_0) - \text{len}^I(w);
\end{equation}

cf. [4, Lemma 2.3]. Clearly $\text{len} = \text{len}^\emptyset$. The other parabolic length function relevant for us is $\text{len}^{[n-2]}$. It is easy to check that $\text{len}^{[n-2]}(w) = n - \text{len}(w)$ for all $w \in S_n$, and in particular $\text{len}^{[n-2]}(w_0) = n - 1$.

The (right) descent set $\text{Des}(w)$ of an element $w \in S_n$ is defined as

$$\text{Des}(w) = \{i \in [n-1] \mid \text{len}(w s_i) < \text{len}(w)\}.$$ 

It is easily seen that $\text{Des}(w) = \{i \in [n-1] \mid w(i+1) < w(i)\}$ and

\begin{equation}
\text{Des}(w w_0) = [n-1] \setminus \text{Des}(w).
\end{equation}

Example 2.1. Let $n = 6$, $e = 2$, $f = 3$ and $i = 3$. Consider the element $w \in S_6$ corresponding to the permutation matrix

$$w = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}. $$

Here $\text{Des}(w) = \{3\}$, $\text{len}(w) = 3$ and $\text{len}^{[4]}(w) = 6 - w(6) = 2$.

For $I \subseteq [n-1]$ we have

\begin{equation}
\sum_{w \in S_n, \text{Des}(w) \subseteq I} Y^{\text{len}(w)} = \binom{n}{I}_Y
\end{equation}

where $\binom{n}{I}_Y \in \mathbb{Z}[Y]$ is the usual $Y$-multinomial coefficient; cf. [8, Section 1.7].
2.2. Grassmannians. Given an integer \( i \in [n] \), we denote by \( \text{Gr}(n, n - i) \) the Grassmannian of \((n - i)\)-dimensional subspaces of affine \( n \)-dimensional space. This \( i(n - i) \)-dimensional projective variety has a realization as follows. Fix a vector space basis for affine \( (n - i) \)-dimensional space. Subspaces of dimension \( n - i \) may be represented by \( \text{GL}_{n-i} \)-left cosets of matrices of size \( n \times (n - i) \) of full rank \( n - i \). A set of such matrices of the form

\[
\begin{pmatrix}
* & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \ldots & * \\
1 & 0 & \ldots & 0 \\
0 & * & \ldots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & * & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
\end{pmatrix}_{n \times (n - i)}
\]

where * stands for arbitrary field elements, is a set of unique coset representatives. More precisely, for any such matrix there is a subset \( J \subseteq [n] \) that the submatrix comprising rows labeled by elements of \( J \) is the \((n - i)\)-identity matrix. The matrix has zeroes in all entries below or to the right of a 1 in this submatrix, and arbitrary entries in the remaining positions. The set of cosets corresponding to such matrices for a fixed subset \( J = \{j_1, \ldots, j_{n-i}\} \subset \subseteq [n] \) may be identified with the cell \( \Omega_w \), where \( w \in S_n \) is the unique element in \( S_n \) whose descent set is contained in \( \{i\} \) and which satisfies \( w(i + m) = j_m \) for all \( m \in [n - i] \). This illustrates that each cell \( \Omega_w \) is an affine space of dimension \( i(n - i) - \text{len}(w) \), the number of * in the above matrix. Hence, given a prime \( p \), the number \( \# \text{Gr}(n, n - i; \mathbb{F}_p) \) of \( \mathbb{F}_p \)-rational points of \( \text{Gr}(n, n - i) \) is given by the formula (2.4)

\[
\# \text{Gr}(n, n - i; \mathbb{F}_p) = \sum_{w \in S_n, \text{Des}(w) \subseteq \{i\}} p^{\text{len}(w)} = \binom{n}{n - i} p^{i(n - i)} = \binom{n}{n - i} p^i ;
\]

cf. (2.3). We refer to [5, Section 3.2] for further information about Schubert cells.

2.3. Lattices. The paper [9] uses some notation to parameterize sublattices \( \Lambda \leq \mathbb{Z}_p^n \), which we recall here for the reader’s convenience. A sublattice \( \Lambda \leq \mathbb{Z}_p^n \) of finite index in
$\mathbb{Z}_p^n$ is maximal in $\mathbb{Z}_p^n$ if $p^{-1} \Lambda \nsubseteq \mathbb{Z}_p^n$. Such a lattice is called of type $\nu(\Lambda) = (I, r_I)$, where $I = \{i_1, \ldots, i_\ell\} \subseteq [n-1]$ and $r_I = (r_{i_1}, \ldots, r_{i_\ell}) \in \mathbb{N}_I^\ell$, if $\Lambda$ has elementary divisors

$$p^{\nu} := \left( 1, \ldots, p^{\nu_{i_1}}, \ldots, p^{\nu_{i_\ell}}, \ldots, p^{\sum_{\iota \in I} r_\iota}, \ldots, p^{\sum_{\iota \in I} r_\iota} \right)$$

with respect to $\mathbb{Z}_p^n$. (Note that this ordering differs from the one used in [10, Section 3.1].)

The group $\Gamma = \text{SL}_n(\mathbb{Z}_p)$ acts transitively on the finite set of maximal sublattices of $\mathbb{Z}_p^n$ of given type $\nu = (I, r_I)$. We denote by $\Gamma_{(I, r_I)}$ the stabilizer in $\Gamma$ of the diagonal lattice diag($p^{\nu}$), and sometimes identify a given maximal lattice with a coset $\alpha \Gamma_{(I, r_I)}$, where $\alpha \in \Gamma$. The number of maximal lattices of type $(I, r_I)$ inside $\mathbb{Z}_p^n$ is given by

$$|\Gamma : \Gamma_{(I, r_I)}| = \binom{n}{I} p^{-1} \prod_{\iota \in I} r_\iota (n-\iota). \tag{2.5}$$

2.4. Linearization. The problem of counting finite-index normal subgroups of $H(R)$ turns out to be equivalent to the problem of counting finite-index ideals in a certain Lie ring, which we now introduce. Given a ring $R$, the Heisenberg Lie ring $L(R)$ over $R$ is defined as

$$L(R) = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in R \right\},$$

equipped with the Lie bracket induced from $\mathfrak{gl}_3(R)$. The derived subring $L(R)'$ of $L(R)$ is equal to its center and consists of those matrices for which $a = b = 0$. Let $L(R) = L(R)/L(R)'$ be the abelianization.

If $R$ is an $A$-module of finite rank, for some commutative ring $A$, then so is $L(R)$. In this case, $L(R)$ has only finitely many $A$-ideals of each finite index. The ideal zeta function of $L(R)$ is then defined as the Dirichlet generating function

$$\zeta^\infty_{L(R)}(s) = \sum_{n=1}^{\infty} a_n^\infty(L(R)) n^{-s}, \tag{2.6}$$

where $a_n^\infty(L(R))$ denotes the number of $A$-ideals of index $n$ in $L(R)$. In the cases considered in this paper, $A = \mathbb{Z}_p$.

3. Computation of the functions $W_{e,f}^a(X,Y)$

3.1. The set-up. Let $R$ be a compact discrete valuation ring of characteristic zero, with maximal ideal $m$ and finite residue field $k = R/m$. Fix a uniformizer $\pi \in m$, and let $\text{val}$ be the discrete valuation on $R$, normalized so that $\text{val}(\pi) = 1$. Let $p$ be the characteristic of $k$, and $f = [k : \mathbb{F}_p]$ the inertia degree. Denote by $e$ the ramification index of $R$, which satisfies $pR = m^e$. Note that there is a natural ring embedding of $\mathbb{Z}_p$ into $R$, endowing $R$ with a $\mathbb{Z}_p$-module structure.

Let $(\overline{\beta}_1, \ldots, \overline{\beta}_f)$ be an ordered $\mathbb{F}_p$-basis of $k$. For each $i \in [f]$, we fix a lift $\beta_i \in R$ of $\overline{\beta}_i$. Then $R$ is a free $\mathbb{Z}_p$-module of rank $n = ef$, and the set

$$\mathcal{B} = \{ \beta_i \pi^j \mid i \in [f], j \in [e-1][a] \}$$
is a \( \mathbb{Z}_p \)-basis; cf. [6, Proposition II.6.8]. We order it as follows: \( \mathcal{B} = (d_1, \ldots, d_n) \), where 
\( d_{i+j} = \beta_i \pi^j \). Setting 
\[
 a_i = \begin{pmatrix} 0 & d_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{n+i} = \begin{pmatrix} 0 & 0 & d_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_i \\ 0 & 0 & 0 \end{pmatrix}
\]
for each \( i \in [n] \), we obtain the following presentation of the Heisenberg Lie ring \( L(R) \) defined in Section 2.4:
\[
(3.1) \quad L(R) = \langle a_1, \ldots, a_{2n}, c_1, \ldots, c_n | [a_i, a_j] = M(c)_{ij}, i, j \in [2n] \rangle.
\]
Here \( M(Y) \in \text{Mat}_{2n}(\mathbb{Z}_p[Y_1, \ldots, Y_n]) \) is a matrix whose entries are linear forms in the variables \( Y_1, \ldots, Y_n \), and \( M(c) \) is the matrix obtained after making the substitution \( Y_i = c_i \) for all \( i \in [n] \). More precisely, \( M(Y) \) has the form
\[
(3.2) \quad M(Y) = \begin{pmatrix} 0 & B(Y) & -B(Y) \\ -B(Y) & 0 & 0 \end{pmatrix},
\]
where \( B(Y) \) is an \( n \times n \) matrix whose entries are given by 
\( B(Y)_{ij} = \sum_{k=1}^{n} \gamma_{ij}^k Y_k \), and the “structure constants” \( \gamma_{ij}^k \in \mathbb{Z}_p \) are defined by the relations 
\( d_i d_j = \sum_{k=1}^{n} \gamma_{ij}^k d_k \). Note that (3.1) differs from the presentation appearing in [7, Section 2.1] by a reordering of the generators \( a_i \).

By the remark after [2, Lemma 4.9], we have that
\[
(3.3) \quad \zeta^\mathbb{C}_{H(R)} = \zeta^\mathbb{C}_{L(R)},
\]
where the ideal zeta function on the right hand side was defined in (2.6); see the discussion in [7, Section 1.3] for more details.

It is well known that, for all \( d \in \mathbb{N} \), the (normal) zeta function of the free abelian pro-\( p \) group \( \mathbb{Z}_p^d \) of rank \( d \) is given by
\[
(3.4) \quad \zeta^{\mathbb{Z}_p^d}(s) = \prod_{i=0}^{d-1} \zeta_p(s - i),
\]
where \( \zeta_p(s) = (1 - p^{-s})^{-1} \) is the Euler factor of the Riemann zeta function \( \zeta(s) \) at the prime \( p \); see, for instance, [2, Proposition 1.1].

3.2. The unramified case. First suppose that \( e = 1 \), covering the case of finite unramified extensions \( R \) of \( \mathbb{Z}_p \).

**Theorem 3.1.** Let \( p \) be a prime and \( R \) a finite unramified extension of \( \mathbb{Z}_p \). Then
\[
(3.5) \quad \zeta_{H(R)}^\mathbb{C}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s) \frac{1}{1 - x_0} \sum_{I \subseteq [n-1]} \binom{n}{I} \prod_{i \in I}^{p-1} \frac{x_i}{1 - x_i},
\]
with numerical data \( x_i = p^{(2n+i)(n-i)-3(n-i)s} \) for \( i \in [n-1]_0 \).

**Proof.** In order to keep the notation of this paper compatible with [9], we have labeled the numerical data in reverse order to that of [7]. By [7, Corollary 3.7], we have
\[
(3.6) \quad \zeta_{H(R)}^\mathbb{C}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s) \frac{1}{1 - x_0} \sum_{I \subseteq [n-1]} \binom{n}{I} \prod_{i \in I}^{p-1} \frac{x_{n-i}}{1 - x_{n-i}}.
\]
Define \( n - I \subseteq [n - 1] \) to be the set \( \{ n - i \mid i \in I \} \). Our claim follows by the identity
\[
\binom{n}{n - I}_{p^{-1}} = \binom{n}{I}_{p^{-1}}; \]
\( \square \)

cf. [7, Remark 2.13].

The object of this section is to give a second proof of Theorem 3.1, based on the ideas in [9]. This will prepare the way for arguments in the general case in the remainder of the article.

Note that, since \( e = 1 \), we have \( \mathcal{B} = (\beta_1, \ldots, \beta_n) \). Hence \( \mathcal{B} \) reduces modulo \( \mathfrak{m} = pR \) to an \( \mathbb{F}_p \)-basis of the residue field \( k \).

As in [9], we consider the Pfaffian hypersurface \( \mathfrak{P}_{H(R)} \subseteq \mathbb{F}^{n-1} \) defined by the equation \( \det(B(\mathbf{Y})) = 0 \).

**Lemma 3.2.** Let \( q = p^n \), and let \( T : \mathbb{F}_q \to \mathbb{F}_p \) be a non-zero \( \mathbb{F}_p \)-linear map. Let \( \{x_1, \ldots, x_n\} \) be an \( \mathbb{F}_p \)-basis of \( \mathbb{F}_q \). Then the matrix \( A_T = (T(x_ix_j))_{ij} \in \text{Mat}_n(\mathbb{F}_p) \) is nonsingular.

**Proof.** Given \( x \in \mathbb{F}_q \), consider the \( \mathbb{F}_p \)-linear map \( U_{T,x} : \mathbb{F}_q \to \mathbb{F}_p \) given by \( U_{T,x}(y) = T(xy) \). Let \( \mathbb{F}_q^\vee = \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_q, \mathbb{F}_p) \) be the dual space of \( \mathbb{F}_q \). Observe that the map
\[
\mathbb{F}_q \to \mathbb{F}_q^\vee, \quad x \mapsto U_{T,x}
\]
is \( \mathbb{F}_p \)-linear and injective and therefore is an isomorphism of \( \mathbb{F}_p \)-vector spaces. The matrix \( A_T \) is just the matrix of this map with respect to the \( \mathbb{F}_p \)-basis \( \{x_1, \ldots, x_n\} \) and its dual basis \( \{x_1^\vee, \ldots, x_n^\vee\} \), so the claim follows. \( \square \)

**Lemma 3.3.** The Pfaffian hypersurface \( \mathfrak{P}_{H(R)} \) has no \( \mathbb{F}_p \)-rational points.

**Proof.** Let \( \mathbf{v} = (v_1, \ldots, v_n)^t \in \mathbb{Z}^n \) be a column vector, and set \( q = p^n \). Let \( \overline{\mathbf{v}} \in \mathbb{F}_p \) be the reduction modulo \( p \) of \( v_i \in \mathbb{Z} \). Choose an isomorphism \( k \simeq \mathbb{F}_q \) and use it to identify these two fields. Now let \( \overline{\beta_i} \in \mathbb{F}_q \) be the reduction modulo \( p \) of \( \beta_i \in R \), for \( i \in [n] \). Recall that \( \{ \overline{\beta_1}, \ldots, \overline{\beta_n} \} \) is an \( \mathbb{F}_p \)-basis of \( \mathbb{F}_q \) and consider the \( \mathbb{F}_p \)-linear map \( T_{\mathbf{v}} : \mathbb{F}_q \to \mathbb{F}_p \) given by \( T_{\mathbf{v}}(\overline{\beta_i}) = \overline{v_i} \). Observe that the reduction modulo \( p \) of the matrix \( B(\mathbf{v}) \) is just the matrix \( A_{T_{\mathbf{v}}} \) defined in the statement of Lemma 3.2. The conclusion of that lemma then implies that \( \det(B(\overline{\mathbf{v}})) = 0 \) only if \( \overline{\mathbf{v}} = 0 \). \( \square \)

In the notation of [9], Lemma 3.3 states that \( n_{\mathfrak{P}_{H(R)}} = 0 \) if \( p \) is inert in \( K \). Furthermore, it implies that the Pfaffian hypersurface has no points defined over \( \mathbb{Q} \). Therefore it is vacuously smooth and has no lines. Hence, in the notation of [9, Theorem 3], \( c_{H(R),p}^2(s) = W_0(p, p^{-s}) \), where \( W_0(p, p^{-s}) \) is implicitly computed in [9, Section 4.2.1]. It is easily seen to match the formula given in (3.5). This concludes the second proof of Theorem 3.1.

### 3.3. The general case

We start off by studying the structure of the matrix \( B(\mathbf{Y}) \) in more detail. To do this, it will be convenient to relabel the variables. Write \( Y_{i,j} = Y_{i+ej} \) for all \( i \in [f] \) and \( j \in [e - 1] \). We write \( \mathbb{Z}_p[\mathbf{Y}] \) for the ring \( \mathbb{Z}_p[Y_1, \ldots, Y_n] \). Now, for all \( r \in [f] \) and all \( s \in \mathbb{N} \), define a linear form \( Y_{r,s} \in \mathbb{Z}_p[\mathbf{Y}] \) as follows:
\[
Y_{r,s} = \sum_{m=1}^{n} \delta_{r,s}^m c_m d_i.
\]
where the \( \delta_{r,s}^m \in \mathbb{Z}_p \) are such that \( \beta_i \pi^s = \sum_{m=1}^{n} \delta_{r,s}^m d_i \).
Remark 3.4. Note that the \( Y_{r,s} \) thus defined are consistent with our earlier notation if \( s \leq e - 1 \). If \( s \geq ge \) for some \( g \in \mathbb{N} \), then \( c_{r,s} \in p^{g} \mathbb{Z}_{p}[Y] \).

Given an element \( F = \sum_{i=1}^{f} \sum_{j=0}^{e-1} \gamma_{ij} Y_{i,j} \in \mathbb{Z}_{p}[Y] \) and a natural number \( m \geq 0 \), define \( F^{(m)} = \sum_{i=1}^{f} \sum_{j=0}^{e-1} \gamma_{ij} Y_{i,j+m} \). Similarly, if \( M \in \text{Mat}_{n}(\mathbb{Z}_{p}[Y]) \) is a matrix, then we write \( M^{(m)} \) for the matrix in which each entry \( M_{i,j} \) has been replaced by \( M_{i,j}^{(m)} \).

Let \( A(Y) \in \text{Mat}_{f}(\mathbb{Z}_{p}[Y]) \) be the matrix given by \( A(Y)_{i,j} = B(Y)_{i,j} \) for \( i,j \in [f] \); in other words, \( A(Y) \) is just the \( f \times f \) block in the upper left corner of \( B(Y) \).

Given a real number \( x \), we denote by \( \lceil x \rceil \) the smallest integer greater than or equal to \( x \), and by \( \lfloor x \rfloor \) the largest integer less than or equal to \( x \).

Definition 3.6. Let \( \alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{Z}_{p}^{n} \setminus p\mathbb{Z}_{p}^{n} \) be a vector. Put \( \mu(\alpha) = \max\{i \in [n] \mid \text{val}(\alpha_{i}) = 0\} \) and define \( [\alpha] = [\mu(\alpha)] \).

Lemma 3.7. Let \( m \in [e] \), and let \( \alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{Z}_{p}^{n} \setminus p\mathbb{Z}_{p}^{n} \) be a vector such that \( [\alpha] = m \). Then:

1. \( A^{(m-1)}(\alpha) \in \text{GL}_{f}(\mathbb{Z}_{p}) \).
2. \( p^{-1} A^{(e+m-1)}(\alpha) \in \text{GL}_{f}(\mathbb{Z}_{p}) \).

Proof. Let \( \bar{A} \) be the reduction modulo \( p \) of \( A^{(m-1)}(\alpha) \). To prove the first part of the claim, we need to verify that \( \bar{A} \in \text{GL}_{f}(\mathbb{F}_{p}) \). From Remark 3.4 and our assumption on \( \alpha \), it follows that \( \bar{A}_{i,j} = \sum_{\ell=1}^{f} \bar{\gamma}^{\ell}_{j} \bar{\alpha}_{(m-1)\ell+i} \), where the overline denotes reduction modulo \( p \), and the \( \gamma^{\ell}_{j} \) are as defined immediately following (3.2). Since \( \bar{\beta}_{j} = \sum_{\ell=1}^{f} \bar{\gamma}^{\ell}_{j} \beta_{\ell} \) for all \( i,j \in [f] \), we find that \( \bar{A} = A_f \), in the notation of Lemma 3.2, where \( T : k \to \mathbb{F}_{p} \) is the non-zero \( \mathbb{F}_{p} \)-linear operator given by \( T(\bar{\alpha}) = \bar{\alpha}_{(m-1)\ell+i} \) for all \( i \in [f] \). The first statement of our claim now follows from Lemma 3.2.

Observe that \( p^{-1} A^{(e+m-1)}(\alpha) \in \text{Mat}_{f}(\mathbb{Z}_{p}) \) by Remark 3.4. Let \( \bar{A}' \) be the reduction modulo \( p \) of this matrix. Let \( (\bar{\alpha}_{1}, \ldots, \bar{\alpha}_{f}) = \psi(\bar{\alpha}_{(m-1)f+1}, \ldots, \bar{\alpha}_{mf}) \), where \( \psi : k \to k \) is the \( \mathbb{F}_{p} \)-linear isomorphism corresponding to multiplication by \( \psi(\bar{\alpha}) = \bar{\psi}(\bar{\alpha}) \) for all \( i \in [f] \). Thus we again have \( \bar{A}' \in \text{GL}_{f}(\mathbb{F}_{p}) \) by Lemma 3.2.

Remark 3.8. It follows from the proof that if \( [\alpha] < m \), then \( A^{(m-1)}(\alpha) \in p \text{Mat}_{f}(\mathbb{Z}_{p}) \).

Write \( I_{f} \) for the \( f \times f \) identity matrix. If \( m \in [f] \), we define

\[
J_{m} = \begin{pmatrix}
0 & I_{mf} \\
p^{-1}I_{(e-m)f} & 0 \\
\end{pmatrix} \in \text{Mat}_{n}(\mathbb{Q}_{p}).
\]

Lemma 3.9. Let \( m \in [e] \), and let \( \alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{Z}_{p}^{n} \setminus p\mathbb{Z}_{p}^{n} \) be a vector such that \( [\alpha] = m \). Then \( B(\alpha) J_{m} \in \text{GL}_{n}(\mathbb{Z}_{p}) \).

Proof. It follows from the definitions that \( B(Y) \) has the following form, where the entries of the matrix below are \( f \times f \) blocks and we write \( A^{(\ell)} \) for \( A^{(\ell)}(Y) \) for \( \ell \in [2e-2] \) and
Theorem 3.9. \( \text{Let } R \text{ be a finite extension of } \mathbb{Z}_p \text{ with inertia degree } f \text{ and ramification index } e. \text{ Set } n = ef. \text{ Then} \\
\zeta_H^q(s) = \zeta_{\mathbb{Z}^n_p}(s) \left( \frac{\sum_{w \in S_n} \left( w \right)^{-2f} \left[ \frac{\text{len}(w)^{n-2}}{f} \right]^s \prod_{j \in \text{Des}(w)} x_j}{\prod_{i=0}^{n-1} (1-x_i)} \right), \\
\text{with numerical data } x_i = p^{(2n+i)(n-i)-(3n-i)s} \text{ for } i \in [n-1]. \)

Proof. We saw in (3.3) that \( \zeta_H^q(s) = \zeta_{L(R)}^q(s) \), where \( L(R) \) is the Heisenberg Lie ring; cf. Section 2.4. The abelianization \( L(R) \) and the derived subring \( L(R)' \) are free \( \mathbb{Z}_p \)-modules of rank \( 2n \) and \( n \), respectively. By [9, Lemma 1], which is essentially [2, Lemma 6.1], we have

\[ \zeta^q_L(L(R))(s) = \zeta_{\mathbb{Z}^n_p}(s) \zeta_p(3ns - 2n^2) \sum_{\Lambda' \leq L(R)' \text{ maximal}} |L(R)' : \Lambda'|^{2n-s} |L(R) : X(\Lambda')|^{-s}, \]

where for every finite-index sublattice \( \Lambda' \leq L(R)' \), we define \( X(\Lambda') \) to be the sublattice of \( L(R) \) such that \( X(\Lambda')/\Lambda' \) is the center of \( L(R)/\Lambda' \). Note that \( \zeta_p(3ns - 2n^2) = \frac{1}{1-x_0} \).

Let \( \Lambda' \leq L(R)' \) be a maximal sublattice of finite index, of type \( \nu(\Lambda') = (I, r_I) \), where \( I = \{i_1, \ldots, i_t\} \subseteq [n-1] \) and \( r_I = (r_{i_1}, \ldots, r_{i_t}) \in \mathbb{N}^I \); cf. Section 2.3. We write \( i \) for \( i_I \). As in Section 2.3, we identify \( \Lambda' \) with a coset \( \alpha \Gamma_{(I, r_I)} \), where \( \alpha \in \text{SL}_n(\mathbb{Z}_p) \). For \( j \in [n] \), let \( \alpha^j \) denote the \( j \)-th column vector of the matrix \( \alpha \). We define

\[ \kappa(\Lambda') := e - \max \{|\alpha^j| \mid n - i < j \leq n\} \in [e - 1]. \]

An informal description of \( \kappa(\Lambda') \) is as follows. Consider the reduction modulo \( p \) of the \( n \times (n-i) \) matrix composed of the last \( n-i \) columns of \( \alpha \). Then \( \kappa(\Lambda') = \kappa \) if and only
if the last $\kappa f$ rows of this matrix are zero, but the $(\kappa + 1)$-st block of $f$ rows from the bottom contains a nonzero element.

The most mysterious ingredient of (3.7) is the quantity $|L(R) : X(\Lambda^\prime)|$, which we will now compute.

**Lemma 3.10.** Let $\Lambda^\prime \leq L(R)^\prime$ be a maximal sublattice. Let $\nu(\Lambda^\prime) = (I, r_I)$. Then

$$|L(R) : X(\Lambda^\prime)| = p^{2(n\sum_{i \in I} r_i - \kappa(\Lambda^\prime)f)}.$$

**Proof.** By [9, Theorem 6], $|L(R) : X(\Lambda^\prime)|$ is equal to the index in $\overline{L(R)} \cong \mathbb{Z}_p^{2n}$ of the sublattice of solutions to the following system of linear congruences:

$$(3.9) \quad \overline{g} M(\alpha^j) \equiv 0 \mod (p^r)_{\j J} \text{ for } j \in [n]$$

in variables $\overline{g} = (g_1, \ldots, g_{2n})$. Here $M(\alpha^j)$ is the commutator matrix $M(Y)$ of (3.2) evaluated at $\alpha^j$.

It is clear from (3.2) that the index of the solution sublattice of (3.9) in $\overline{L(R)}$ is the square of the index in $\mathbb{Z}_p^n$ of the solution sublattice of the system

$$(3.10) \quad h B(\alpha^j) \equiv 0 \mod (p^r)_{\j J} \text{ for } j \in [n],$$

where $h = (h_1, \ldots, h_n) \in \mathbb{Z}_p^n$. By Lemma 3.8, each matrix $B(\alpha^j) \in \text{Mat}_{2n}(\mathbb{Z}_p)$ becomes invertible after all the entries in its last $(e - \lfloor \alpha^j \rfloor)f$ columns have been divided by $p$. Therefore, $h = (h_r) \in \mathbb{Z}_p^n$ is a solution to (3.10) if and only if, for all $j \in [n]$,

$$h_r \equiv 0 \mod (p^r)_{\j J} \text{ if } r \leq \lfloor \alpha^j \rfloor f,$$

$$ph_r \equiv 0 \mod (p^r)_{\j J} \text{ if } r > \lfloor \alpha^j \rfloor f.$$

It follows that the congruences where $(p^r)_{\j J}$ is maximal, namely those with $i < j \leq n$, dominate all the others.

By the considerations above, $h \in \mathbb{Z}_p^n$ is a solution to (3.10) if and only if

$$h_r \equiv 0 \mod p^{\sum_{i \in I} r_i} \quad \text{if } r \leq (e - \kappa(\Lambda^\prime))f,$$

$$h_r \equiv 0 \mod p^{(\sum_{i \in I} r_i) - 1} \quad \text{if } r > (e - \kappa(\Lambda^\prime))f.$$

Recalling that $n = ef$, it follows that the index in $\overline{L(R)}$ of the sublattice of simultaneous solutions to the congruences (3.9) is the quantity in the statement of the lemma. $\Box$

It is obvious from the definition of the type of a lattice that, if $\Lambda^\prime \leq L(R)^\prime$ is a sublattice of type $(I, r_I)$, then

$$|L(R)^\prime : \Lambda^\prime| = p^{\sum_{i \in I} (n-i)r_i}.$$

Given $\kappa \in [e - 1]_0$ and a type $(I, r_I)$, define

$$N^\kappa_{(I, r_I)} = \{\Lambda^\prime \mid \nu(\Lambda^\prime) = (I, r_I), \kappa(\Lambda^\prime) = \kappa\}.$$

It follows from (3.7) and Lemma 3.10 that

$$(3.11) \quad \zeta^\kappa_{L(R)}(s) = \frac{\zeta_{\mathbb{Z}_p^{2n}}(s)}{1 - x_0} \sum_{I \subseteq [n-1]} \sum_{r_I \in \mathbb{N}^I} \sum_{\kappa = 0}^{e-1} N^\kappa_{(I, r_I)} p^{(\sum_{i \in I} (n-i)r_i)(2n-s)-2s(n\sum_{i \in I} r_i - \kappa f)}}.$$
As preparation for computing the numbers $N^\nu_{(I,\nu)}$, we fix a type $(I, r_I)$ as above and consider the surjective map
\[
\varphi_{(I, r_I)} : \{ \Lambda' \mid \nu'(\Lambda') = (I, r_I) \} \to \text{Gr}(n, n - i; \mathbb{F}_p).
\]
\[
\alpha \Gamma_{(I, r_I)} \mapsto (\alpha \gamma_i \mid i < j \leq n)_{\mathbb{F}_p}.
\]
As before, we identify lattices of type $(I, r_I)$ with cosets $\alpha \Gamma_{(I, r_I)}$ for $\alpha \in \Gamma$. Informally, $\varphi(\alpha \Gamma_{(I, r_I)})$ is the subspace of $\mathbb{F}_p^n$ spanned by the reduction modulo $p$ of the last $n - i$ columns of the matrix $\alpha \in \Gamma$.

**Lemma 3.11.** The fibres of $\varphi_{(I, r_I)}$ all have the same cardinality
\[
(3.12) \quad p^{-(n-i)i} \binom{n-i}{I} p^{\sum_i r_i(n-i)i}.
\]

**Proof.** The map $\varphi_{(I, r_I)}$ is just the natural surjection
\[
\Gamma / \Gamma_{(I, r_I)} \to \Gamma / \Gamma_{(I_{(1)}, 1)}, \quad \alpha \Gamma_{(I, r_I)} \mapsto \alpha \Gamma_{(I_{(1)}, 1)}.
\]
Of course $|\Gamma : \Gamma_{(I, r_I)}| = |\Gamma : \Gamma_{(I_{(1)}, 1)}||\Gamma_{(I_{(1)}, 1)} : \Gamma_{(I, r_I)}|$.

By (2.5) we have
\[
|\Gamma : \Gamma_{(I, r_I)}| = \binom{n}{I} p^{\sum_i r_i(n-i)i} = \binom{n}{n-i} p^{r_i(n-i)i} \cdot \binom{n-i}{I} p^{\sum_i r_i(n-i)i}.
\]

Formula (3.12) for the index $|\Gamma_{(I_{(1)}, 1)} : \Gamma_{(I, r_I)}|$ follows now from (2.4), as
\[
|\Gamma : \Gamma_{(I_{(1)}, 1)}| = \# \text{Gr}(n, n - i; \mathbb{F}_p) = \binom{n}{n-i} p^{i(n-i)}, \quad \Box
\]

Consider the following filtration on $\text{Gr}(n, n - i; \mathbb{F}_p)$. Let $(e_1, \ldots, e_n)$ denote the standard $\mathbb{F}_p$-basis of $V = \mathbb{F}_p^n$, and consider the flag $(V_d)_{d=0}^e = ((e_1, \ldots, e_{jd})_{\mathbb{F}_p})_{d=0}^e$. Define
\[
\psi : \text{Gr}(n, n - i; \mathbb{F}_p) \to [e], \quad W \mapsto \min\{d \mid W \subseteq V_d\}.
\]
One verifies easily that the fibres of $\psi$ are unions of cells. Indeed, if $\lambda \in [e]$ and $W \in \text{Gr}(n, n - i; \mathbb{F}_p)$, then $\psi(W) = \lambda$ if and only if the bottom $(e - \lambda)f$ rows of the matrix of $W$ (cf. Section 2.2) consist of zeroes, whereas the previous block of $f$ rows does contain a non-zero matrix element. It is clear from the discussion in Section 2.2 that the lowest non-zero element in the matrix of $W$ is a 1 in the $(w(n), n - i)$ position, where $w \in S_n$ is such that $W \in \Omega_w$. In other words, $\psi(W) = \lambda$ if and only if $w(n) \in [(\lambda - 1)f + 1, \lambda f]$. Recalling that $\text{len}^{n-2}(w) = n - w(n)$, this condition is clearly equivalent to $\left\lfloor \frac{\text{len}^{n-2}(w)}{f} \right\rfloor = e - \lambda$. Therefore, for every $\lambda \in [e]$,

\[
(3.13) \quad \psi^{-1}(\lambda) = \bigcup_{\substack{w \in S_n, \text{len}(w) \leq (1) \\text{len}^{n-2}(w) \geq \left\lfloor \frac{e - \lambda}{f} \right\rfloor}} \Omega_w.
\]

Now consider the composition
\[
\psi \circ \varphi_{(I, r_I)} : \{ \Lambda' \mid \nu'(\Lambda') = (I, r_I) \} \to [e], \quad \alpha \Gamma_{\nu} \mapsto \psi(\alpha \gamma_i \mid i < j \leq n)_{\mathbb{F}_p}).
\]
From the definition of \( \kappa(\Lambda') \) in (3.8), it is evident for all \( \kappa \in [e - 1]_0 \) that \( \kappa(\Lambda') = \kappa \) if and only if \( \psi \circ \varphi(I,r_I)(\Lambda') = e - \kappa \). Thus, by (3.12), (3.13), and the fact that \( |\Omega_w| = p^{(n-i)-\text{len}(w)} \) for all \( w \in S_n \) (cf. Section 2.2), we obtain

\[
(3.14) \quad N_{(I,r_I)}^p = \# \{ \Lambda' \mid \nu(\Lambda') = (I,r_I), \kappa(\Lambda') = \kappa \} = \# \{ \Lambda' \mid \nu(\Lambda') = (I,r_I), \psi(\varphi(I,r_I)(\Lambda')) = e - \kappa \} = \left( \sum_{w \in S_n, \text{Des}(w) \subseteq \{ I \}} p^{-\text{len}(w)} \left\lfloor \frac{\text{len}(w) - 2(n - \kappa)}{n} \right\rfloor = \kappa \right) \left( n - i \right) \left( I \right)^{p \sum_{\imath \in I} r_{\imath}(n-i) \imath} \alpha_I^p \prod_{\imath \in I} p^{r_{\imath}(n-i) \imath},
\]

where we set

\[
\alpha_I^p = \sum_{w \in S_n, \text{Des}(w) \subseteq \{ I \}} p^{-\text{len}(w)} \left\lfloor \frac{\text{len}(w) - 2(n - \kappa)}{n} \right\rfloor = \kappa
\]

Finally, we have all the ingredients necessary to compute \( \zeta_{H(R)}^q(s) \) and finish the proof of Theorem 3.9. Indeed, a simple calculation using (3.14) shows that

\[
\sum_{r_I \in [n]^{\{ I \}}} N_{(I,r_I)}^p (\sum_{\imath \in I} r_{\imath}) (2n - s - 2s(n \sum_{\imath \in I} r_{\imath}, - \kappa f) = \alpha_I^p p^{2s\kappa f} \sum_{r_I \in [n]^{\{ I \}}} \prod_{\imath \in I} \left( p^{(2n+\kappa)(n-i)-(3n-i)s} \right) r_i = \alpha_I^p p^{2s\kappa f} \prod_{\imath \in I} \frac{x_{\imath}}{1-x_{\imath}},
\]

where \( x_i = p^{(2n+\kappa)(n-i)-(3n-i)s} \) for \( i \in I \subseteq [n-1] \) as in the statement of Theorem 3.9. By (3.11) this implies

\[
\zeta_{H(R)}^q(s) \frac{1-x_0}{\zeta_{p^{2s\kappa f}}(s)} = \sum_{I \subseteq [n-1]} \sum_{\kappa=0}^{e-1} \alpha_I^p p^{2s\kappa f} \prod_{\imath \in I} \frac{x_{\imath}}{1-x_{\imath}}.
\]

Bringing the right hand side to a common denominator, we get

\[
\frac{\sum_{\kappa=0}^{e-1} \sum_{I \subseteq [n-1]} \beta_I^p p^{2s\kappa f} \prod_{\imath \in I} x_{\imath}}{\prod_{i=1}^{n-1} (1-x_{i})},
\]

where

\[
\beta_I^p = \sum_{I \subseteq [n-1]} (-1)^{|I|-|J|} \alpha_J^p = \sum_{w \in S_n, \text{Des}(w) \subseteq J} p^{-\text{len}(w)},
\]
the last equality following from a simple inclusion-exclusion argument; cf. [8, (1.34)]. Hence,

$$\zeta_H^c(R)(s) \frac{\prod_{i=0}^{n-1} (1 - x_i)}{\zeta^c_{\mathbb{Z}_p}(s)} = \sum_{I \subseteq [n-1]} \sum_{\kappa=0}^{e-1} \sum_{p^{2n}\mathfrak{f}} \left( \sum_{w \in \mathcal{S}_n, \text{Des}(w) = I} \frac{p^{-\text{len}(w)}}{\left\lfloor \frac{\text{len}[n-2](w)}{f} \right\rfloor} \right) \prod_{i \in I} x_i$$

$$= \sum_{w \in \mathcal{S}_n} p^{-\text{len}(w)} \prod_{j \in \text{Des}(w)} x_j,$$

as claimed.

**Remark 3.12.** Theorem 3.1 is indeed a special case of Theorem 3.9. If \( e = 1 \) and \( f = n \), then \( \text{len}^{[n-2]}(w) \leq n - 1 < f \) for all \( w \in \mathcal{S}_n \), and hence \( \left\lfloor \frac{\text{len}^{[n-2]}(w)}{f} \right\rfloor = 0 \) for all \( w \in \mathcal{S}_n \). One verifies easily that

$$\frac{\sum_{w \in \mathcal{S}_n} p^{-\text{len}(w)} \prod_{j \in \text{Des}(w)} x_j}{\prod_{i=0}^{n-1} (1 - x_i)} = \frac{1}{1 - x_0} \sum_{I \subseteq [n-1]} \left( \frac{n}{I} \right) \prod_{i \in I} \frac{x_i}{1 - x_i},$$

by bringing the right hand side to a common denominator as in [9, Section 4.1] and using (2.3).

**Corollary 3.13.** Let \( R \) be a finite extension of \( \mathbb{Z}_p \) with inertia degree \( f \) and ramification index \( e \). Set \( n = ef \). Then \( \zeta_H^c(s) \) satisfies the following functional equation:

$$\zeta_H^c(R)(s) \big|_{p \to p^{-1}} = (-1)^{3n} p^{(3n)-(5n+2f(e-1))s} \zeta_H^c(R)(s).$$

**Proof.** Recall from (2.1) that, for all \( w \in \mathcal{S}_n \), we have \( \text{len}^{[n-2]}(ww_0) = \text{len}^{[n-2]}(w_0) - \text{len}^{[n-2]}(w) \), where \( w_0 \) is longest element. The key observation is that the fractional part of \( \frac{\text{len}^{[n-2]}(ww_0)}{f} = \frac{\text{len}^{[n-2]}(w_0)}{f} - \text{len}^{[n-2]}(w) \) is the largest possible. Hence, for all \( w \in \mathcal{S}_n \),

$$\left\lfloor \frac{\text{len}^{[n-2]}(ww_0)}{f} \right\rfloor = \left\lfloor \frac{\text{len}^{[n-2]}(w_0)}{f} \right\rfloor - \left\lfloor \frac{\text{len}^{[n-2]}(w)}{f} \right\rfloor = (e - 1) - \left\lfloor \frac{\text{len}^{[n-2]}(w)}{f} \right\rfloor.$$

Using (3.4), (2.2), and (2.3) it then follows that

$$\zeta_H^c(R)(s) \big|_{p \to p^{-1}} = \zeta^c_{\mathbb{Z}_p}(s) \big|_{p \to p^{-1}} \frac{\sum_{w \in \mathcal{S}_n} p^{-\text{len}(w)} \prod_{j \in \text{Des}(w)} x_j}{\prod_{i=0}^{n-1} (1 - x_i)^{-1}}$$

$$= (-1)^{3n} p^{(3n)-(5n+2f(e-1))s} \zeta^c_{\mathbb{Z}_p}(s) x_0 \frac{\sum_{w \in \mathcal{S}_n} p^{-\text{len}(w)} \prod_{j \in \text{Des}(w)} x_j}{\prod_{i=0}^{n-1} (1 - x_i^{-1})}$$

$$= (-1)^{3n} p^{(3n)-(5n+2f(e-1))s} \zeta^c_{\mathbb{Z}_p}(s) x_0 \frac{\sum_{w \in \mathcal{S}_n} p^{-\text{len}(ww_0) + 2f(e-1)s} \prod_{j \in \text{Des}(ww_0)} x_j}{\prod_{i=0}^{n-1} (1 - x_i^{-1})}$$

$$= (-1)^{3n} p^{(3n)-(5n+2f(e-1))s} \zeta^c_{\mathbb{Z}_p}(s) x_0 \frac{\sum_{w \in \mathcal{S}_n} p^{-\text{len}(ww_0) + 2f(e-1)s} \prod_{j \in \text{Des}(ww_0)} x_j}{\prod_{i=0}^{n-1} (1 - x_i^{-1})}$$

as claimed.
3.4. An alternative formulation of the main result. We now prove an alternative formula for $\zeta_{H(R)}^{\omega}(s)$ to that of Theorem 3.9 by showing that, in general, the fraction on the right hand side of (3.6) admits some cancellation.

Consider the $n$-cycle $c = (1 \ 2 \ \cdots \ n) \in S_n$.

**Lemma 3.14.** Let $w \in S_n$, and let $m \in [n - 1]_0$ be such that $w(1) \leq n - m$. Then the following holds:

$$p^{-\text{len}(c^m w) + 2 \text{len}^m(c^m w) s} \prod_{j \in \text{Des}(c^m w)} x_j = (p^{2n-3} m p^{-\text{len}(w) + 2 \text{len}^m(w) s}) \prod_{j \in \text{Des}(w)} x_j.$$

**Proof.** It suffices to prove the statement for $m = 1$; the general case clearly follows from iterated application of this result. So let $w \in S_n$ and set $j = w^{-1}(n)$. Recall that $\text{len}^m(w) = n - w(n)$ and observe that $\text{len}(cw) - \text{len}(w) = 2j - n - 1$. Moreover, we observe that $\text{Des}(cw) = (\text{Des}(w) \cup \{j - 1\}) \setminus \{j\}$.

If $j < n$, then $w(n) < n$ and so $w(n) - cw(n) = -1$. Hence

$$p^{-\text{len}(c^m w) + 2 \text{len}^m(c^m w) s} \prod_{j \in \text{Des}(c^m w)} x_j = p^{-2j+1+2(w(n) - cw(n)) s} \prod_{j \in \text{Des}(w)} x_j = p^{n+2j-1-2s} \cdot p^{n+2j-1-s} = p^{2n-3s}.$$

If $j = n$, then $w(n) - cw(n) = n - 1$ and the analogous computation gives

$$p^{-\text{len}(c^m w) + 2 \text{len}^m(c^m w) s} \prod_{j \in \text{Des}(c^m w)} x_j = p^{n+2n-1-s} \prod_{j \in \text{Des}(w)} x_j = p^{2n-3s} \cdot p^{(3n-1)-(2n+1)s} = p^{2n-3s}. \square$$

**Lemma 3.15.** Suppose that $w \in S_n$ and $f \in \mathbb{N}$ satisfies $1 \leq w(1) \leq f$. Then for any $m \leq \left\lfloor \frac{n-f}{f} \right\rfloor$ the following holds:

$$p^{-\text{len}(c^m w) + 2f \left\lfloor \frac{\text{len}^m(c^m w)}{f} \right\rfloor s} \prod_{j \in \text{Des}(c^m w)} x_j = (p^{2n-3} m p^{-\text{len}(w) + 2f \left\lfloor \frac{\text{len}^m(w)}{f} \right\rfloor s}) \prod_{j \in \text{Des}(w)} x_j.$$

**Proof.** Let $w$ and $m$ be as in the lemma. Then $\text{len}(c^m w) = \text{len}^m(w) - mf$. Thus

$$f \left\lfloor \frac{\text{len}^m(w)}{f} \right\rfloor - \text{len}^m(w) = f \left\lfloor \frac{\text{len}^m(w)}{f} \right\rfloor - \text{len}^m(w).$$

The lemma follows immediately from this and Lemma 3.14. \square

**Theorem 3.16.** Let $R$ be a finite extension of $\mathbb{Z}_p$ with inertia degree $f$ and ramification index $e$. Set $n = ef$ and $S_n^{(f)} = \{w \in S_n \mid w(1) \leq f\}$. Then

$$\zeta_{H(R)}^{\omega}(s) = \zeta_{\mathbb{Z}_p}(s) \frac{\sum_{w \in S_n^{(f)} p^{-\text{len}(w) + 2f \left\lfloor \frac{\text{len}^m(w)}{f} \right\rfloor s}} \prod_{j \in \text{Des}(w)} x_j}{(1 - p^f(2n-3s)) \prod_{i=1}^{n-1} (1 - x_i)}.$$
Proof. This follows from Theorem 3.9 and Lemma 3.15, and the observations that

\[ 1 - x_0 = 1 - (p^{2n-3s})^n = (1 - (p^f(2n-3s))) \sum_{m=0}^{e-1} p^{mf(2n-3s)} \]

and that every element of $S_n$ can be written uniquely in the form $c^{mf}w$, where $m \in [e-1]$ and $w \in S_n^{(f)}$.

\[ \square \]

Remark 3.17. An interesting question is whether the fraction in Theorem 3.16 is always in lowest terms and admits no more cancellation. T. Bauer has verified this for all pairs $(e,f)$ with $n = ef \leq 10$.

In the case that $e = 1$, Theorem 3.16 is exactly Theorem 3.1. In the other extreme, the case $f = 1$, we obtain an interesting corollary. We identify the stabilizer in $S_n$ of the letter 1, viz. the parabolic subgroup $(S_n)_{s_i | 2 \leq i \leq n-1}$, with $S_{n-1}$.

Corollary 3.18. Let $R$ be a totally ramified extension of $\mathbb{Z}_p$ of degree $n$. Then

\[ \zeta_{H(R)}^\varnothing(s) = \zeta_{\mathbb{Z}_p^\varnothing}(s) \sum_{w \in S_{n-1}} \frac{P^{-\text{len}(w)+2\text{len}^{(0)}(w)s}}{(1-p^{2n-3s}) \prod_{i=1}^{n-1} (1-x_i)} \prod_{j \in \text{Des}(w)} x_j^{j+1}, \]

with numerical data $x_i = p^{(2n+i)(n-i)-(3n-i)s}$ for $i \in [n-1]$.

Example 3.19. We illustrate our results in the case $e = 3$, $f = 1$. Thus let $R$ be a totally ramified cubic extension of $\mathbb{Z}_p$. It is shown in [2, Proposition 8.15] that

\[ \zeta_{H(R)}^\varnothing(s) = \frac{1 + p^{7-5s}}{(\prod_{i=0}^{5}(1-p^{i-s}))(1-p^{6-3s})(1-p^{8-7s})(1-p^{14-8s})}. \]

Theorem 3.9 presents this zeta function as

\[ \zeta_{H(R)}^\varnothing(s) = \zeta_{\mathbb{Z}_p^\varnothing}(s) \sum_{w \in S_3} p^{-\text{len}(w)+2\text{len}^{(1)}(w)s} \prod_{j \in \text{Des}(w)} x_j, \]

with numerical data

\[ x_0 = p^{18-9s}, \quad x_1 = p^{14-8s}, \quad x_2 = p^{8-7s}. \]

The Coxeter group $S_3$ is generated by the involutions $s_1$ and $s_2$. We tabulate the values of the functions Des, len, and len$^{(1)}$ on $S_3$.

| $w \in S_3$ | Des($w$) | len($w$) | len$^{(1)}(w)$ |
|--------------|----------|----------|---------------|
| 1            | \emptyset| 0        | 0             |
| $s_1$        | \{1\}   | 1        | 0             |
| $s_2$        | \{2\}   | 1        | 1             |
| $s_2s_1$     | \{1\}   | 2        | 1             |
| $s_1s_2$     | \{2\}   | 2        | 2             |
| ($s_2s_1s_2 =) s_1s_2s_1$ | \{1, 2\} | 3 | 2 |
We deduce that
\[
\sum_{w \in S_3} p^{-\text{len}(w)+2\text{len}(1)(w)} x_j \\
= 1 + p^{13-8s} + p^{7-5s} + p^{12-6s} + p^{6-3s} + p^{19-11s} = \frac{(1 - p^{18-9s})(1 + p^{7-5s})}{1 - p^{6-3s}},
\]
showing that (3.16) accords with (3.17). Formula (3.16) also illustrates (3.15), as \(\langle s_2 \rangle \cong S_2\) and
\[
1 + p^{7-5s} = 1 + p^{-1+2s}x_2 = \sum_{w \in S_2} p^{-\text{len}(w)+2\text{len}(w)} x_{j+1}.
\]

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