Dynamics of Fluctuations in the Open Quantum SSEP and Free Probability

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The quantum symmetric simple exclusion process (Q-SSEP) is a simple model of spinless fermions hopping on a one-dimensional chain to neighboring sites with random amplitudes. It was proposed as a toy model to study the fluctuations of coherent (i.e. purely quantum mechanical) effects in out-of-equilibrium many-body quantum systems. The aim is to describe these coherent effects at a mesoscopic scale and to establish a universal framework which might be applicable to a larger class of mesoscopic systems. Here we formulate the scaling limit of the time dynamics of fluctuating coherences in the open Q-SSEP, i.e. in the case when the one-dimensional chain is coupled to particle reservoirs on the boundaries to maintain it out-of-equilibrium. We show that the fluctuating coherences have a natural interpretation in terms of free cumulants, a concept from free probability theory which thus seems to be an appropriate framework to deal with fluctuating quantum many body systems out-of-equilibrium.

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I. INTRODUCTION

From the point of view of statistical physics, a system is completely characterised by the probability to observe any of its possible microscopic configurations. From this knowledge one can obtain the probability distribution of macroscopic (and in principle experimentally observable) variables such as the density or the current profile. In an equilibrium situation, where the probability distribution on microscopic configurations is well known since Boltzmann, macroscopic variables satisfy a large deviation principle: they are strongly peaked at their mean value around which their probability distribution decays exponentially with the number of particles in the system and proportional to the appropriate thermodynamic potential (e.g. the entropy or the free energy), see e.g. [1].

In contrast to this, out-of-equilibrium situations might depend on a large variety of system dependent details and there are no general formulas for the probability distribution on microscopic configurations. Nevertheless, over the course of the last 30 years a lot of progress has been made to understand the statistics of macroscopic variables in out-of-equilibrium systems, which show rich and new features. For example, the correlation of the density in non-equilibrium steady states extends to macroscopic distances, as had been experimentally observed by Dorf-
mann [2]. Furthermore, fluctuations relations that go beyond the linear response regime, have been shown to be generically applicable to out-of-equilibrium systems. [3, 4]. And the large deviation principle introduced above in the equilibrium context has been realised to hold also to the non-equilibrium setting – with the need to formulate appropriate out-of-equilibrium thermodynamic potentials, see e.g. [5].

In the case of classical systems with diffusive transport, which are maintained out-of-equilibrium by coupling the boundaries to reservoirs at different chemical potentials, this has been achieved by the so-called macroscopic fluctuation theory (MFT) [6, 7]. MFT provides a framework to specify the probability distribution of the density and current profile in a large deviation form starting from only two system-dependent quantities, the diffusion constant and the mobility. This framework has been developed through the study of microscopic toy models, more precisely, stochastic lattice gases, that revealed universal properties in the density and current fluctuations in the sense that they did not depend on the precise underlying microscopic dynamics. A very important role in this context has been played by the symmetric and asymmetric simple exclusion processes (SSEP and ASEP), since these toy models are exactly solvable [8–11] (for a review, see [5, 12]).

On the side of quantum many-body systems, the development of Generalized Hydrodynamics (GHD) has allowed to study out-of-equilibrium dynamics in integrable systems [13, 14] (see [15, 16] for a review). Since this theory treats the system as a composition of many thermodynamically large fluid cells that are locally in equilibrium with respect to a generalised Gibbs ensemble, it looses the statistical properties of quantum correlations between the fluid cells. An approach to restore these quantum correlations has recently been proposed in [17]. GHD is mostly concerned with ballistic transport due to the integrable nature of the systems to which it applies. However, in generic out-of-equilibrium quantum systems transport is rather diffusive than ballistic [18]. In the classical case this is well described by MFT.

The major question with which we are therefore concerned is whether MFT can be extended to a quantum setting [19]. Such a theory, which could be named the quantum mesoscopic fluctuation theory, should not only describe the statistical properties of the diffusive transport (i.e. density and current profiles) in out-of-equilibrium quantum systems, but also those of quantum coherent effects such as interference or entanglement. These properties have also been studied in the context of unitary random circuits [20, 21] (see [22, 23] for a review) and driven diffusive systems [24].

The picture we have in mind is that such a quantum extension of MFT should apply to mesoscopic systems in which the system size \( L \) is of the order of the coherence length \( \ell_0 \) of the dynamical degrees of freedom (e.g. electrons in a disordered metal of size \( \approx 1 \mu m \) [25]), see Fig. 1. Another length scale that enters the systems is the mean free path \( \ell \), below which transport is ballistic and the microscopic degrees of freedom change very rapidly with time. From the point of view of mesoscopic lengths and times, we expect that such a system has a description in terms of a noisy and unitary process, where the noise emerges from the coarse-graining over microscopic degrees of freedom inside ballistic cells of the size of the mean free paths \( \ell \). These ballistic cells should be assumed to be thermodynamically large, but nevertheless small compared to the system size, \( 1 \ll \ell \ll L \) (we identify the length scales with the number of lattice sites in a hypothetical one dimensional microscopic model by setting the lattice mesh to unity \( a_{uv} = 1 \)). At length above \( \ell \) we expect diffusive transport, and at length below \( \ell_0 \) we observe quantum mechanical interference.

A toy model that allows to study this regime is the quantum symmetric simple exclusion process (Q-SSEP) [26–31]. It describes the noisy and coherent hopping of spinless fermions on a one dimensional and discrete lattice chain which is maintained out-of-equilibrium by two particle reservoirs at different chemical potentials. While a simple average over the noise completely destroys coherent effects and reduces the model to the classical SSEP, the fluctuations of coherent effects survive in the non-equilibrium steady state [28], although they are sub-leading in the system size. It is in this sense that Q-SSEP describes mesoscopic transport. Furthermore, we note that the fluctuations of coherent effects satisfy a large deviation principle in analogy to the macroscopic variables in MFT. This can be seen as a first evidence that the fluctuations in Q-SSEP show universal properties that might apply to a larger class of mesoscopic systems.

In the equilibrium case, where one removes the boundary reservoirs and closes the chain periodically, it was shown in [31] that the dynamics of Q-SSEP has a well defined scaling limit where the number of sites goes to infinity and the lattice spacing to zero. Here we extend this result to the open case (with boundary reservoirs) and we show that physical correlation functions admit a diffusive scaling limit.

In view of the picture we outlined above, a single lattice site in Q-SSEP should correspond to a ballistic cell composed of many underlying microscopic degrees of freedom, such that in Q-SSEP the mean free path \( \ell \) has effectively been shrunk to the lattice spacing \( a_{uv} \). Though

![Fig. 1](image_url)
we are not yet able to show explicitly that Q-SSEP, in its scaling limit, is the effective description of a more detailed microscopic model after a coarse-graining to ballistic cells, we found hints in the mathematical structure that points towards such an interpretation. Namely, we found that correlation functions of so-called coherences (that characterise to what extend two spatially separated sites are entangled) are related to their connected parts in a similar fashion that moments are related to free cumulants in free probability theory [32–35]. Free probability theory deals with non-commutative random variables and an explicit realisation of its concepts can be provided by large random matrices [33, 36, 37]. In the picture of a mesoscopic quantum system referred to above, these large matrices of size scaling with \( \ell \) could emerge from the microscopic description of the ballistic cells. Free probability theory could therefore be the natural mathematical framework to characterise such systems.

Although there is only one degree of freedom per lattice site in Q-SSEP, which is hence not described by large random matrices, we nevertheless observe a mathematical structure in the Q-SSEP correlation functions that is reminiscent of free probability theory. We interpret this as a first evidence, one on hand, that Q-SSEP is indeed the emergent description of a more detailed physical model after a coarse-graining to ballistic cells and, on the other hand, that free probability tools promise to play a significant role in fluctuating out-of-equilibrium many-body quantum systems.

The structure of the paper is as follows. In section II we recall the main properties of Q-SSEP and explain the scaling limit. Sections III and IV are concerned with the formulation of the dynamical equations of the correlation functions of the open Q-SSEP in the scaling limit. Finally, section V explains why the connected correlation functions in Q-SSEP show the structure of free cumulants. Furthermore, we use this observation to construct the steady state solution recursively. Details and proofs are given in a few appendices.

II. REMINDER ON THE OPEN QUANTUM SSEP

The quantum symmetric simple exclusion process was first introduced in [26] (closed case) and [28] (open case) and was further elaborated in [27, 29, 31]. In this section we recall its definition and explain some of its properties.

a. Definition. The model describes a one-dimensional chain with \( L \) sites on which spinless fermions hop to neighbouring sites with random amplitudes. The boundary sites are coupled to particle reservoirs [38]. The bulk evolution of the system is stochastic and unitary. We describe it in terms of the systems density matrix \( \rho_t \) as

\[
\rho_t \rightarrow \rho_{t+dt} = e^{-i dH_t} \rho_t e^{i dH_t},
\]

where, by definition, the Hamiltonian increment is

\[
dH_t = \sum_{j=1}^{L-1} c_j^+ c_j dW^j_t + c_{j+1}^+ c_j dW^j_t,
\]

with \( c_j^+ (c_j) \) fermion creation (annihilation) operators at site \( j \) with the usual commutation relations \( \{c_j, c_k\} = \delta_{jk} \) and \( \{c_j^+, c_k\} = \{c_j, c_k\} = 0 \). Here \( dW^j_t \) is the increment of a complex (classical) Brownian motion that determines the random hopping amplitude along the edge \((j, j+1)\) at time \( t \). There is one complex Brownian motion per edge. This means that \( W^j_t \) at each instance \( t \) is a centred Gaussian random variable that has independent increments \( dW^j_t = W^j_{t+dt} - W^j_t \) with variance \( \mathbb{E}[dW^j_t dW^k_t] = J \delta^{j,k} dt \) if \( t = t' \), and zero otherwise. \( J \) is a rate parameter with units one over time and we set \( J = 1 \) in the following.

To the stochastic and unitary bulk evolution (1), we add a deterministic but dissipative Lindbladian evolution

\[
\mathcal{L}_{\text{bdry}} = \alpha_1 L_1^+ + \beta_1 L_1^- + \alpha_L L_L^+ + \beta_L L_L^-,
\]

that represents the interaction with the reservoirs. The operator \( L_j^+ (\bullet) = c_j^+ \bullet c_j - \frac{1}{2} \{ c_j^+ c_j, \bullet \} \) models particle injection and is multiplied by the injection rate \( \alpha_j \) while \( L_j^- (\bullet) = c_j \bullet c_j^+ - \frac{1}{2} \{ c_j c_j^+, \bullet \} \) models particle extraction with rate \( \beta_j \). For example, an isolated empty site \( \tau_i = \{0 \} \) that evolves according to \( \partial_t \tau_i = \alpha L^+_i (\tau_i) \) will be occupied after a time interval \( dt \) with probability \( \alpha dt \),

\[
|0\rangle \langle 0 | \rightarrow \alpha dt |1\rangle \langle 1 | + (1 - \alpha dt)|0\rangle \langle 0 |.
\]

The full evolution of the systems density matrix \( \rho_t \) can be expressed as a stochastic differential equation (SDE) in Itô convention (with Itô rules \( dW^j_t dW^k_t = \delta^{j,k} dt \)) by expanding (1) up to order \( dt \),

\[
d\rho_t = -i [dH_t, \rho_t] - \frac{1}{2} [dH_t, [dH_t, \rho_t]] + \mathcal{L}_{\text{bdry}} (\rho_t) dt.
\]

Occasionally we will refer to the “closed case”, by which we mean that there is no interaction with the reservoirs and the one-dimensional chain is closed periodically such that sites \( 1 \equiv L \) are identified.

b. Relation to the classical SSEP. The name of this model is inherited from the classical symmetric simple exclusion process (SSEP). The latter can be obtained from Q-SSEP in the special case where one is just interested in the mean density matrix \( \bar{\rho}_t := \mathbb{E}[\rho_t] \), where the expectation \( \mathbb{E}[\cdots] \) is taken with respect to the different realisations of the Brownian motions. This matrix evolves according to a Lindblad equation,

\[
\partial_t \bar{\rho}_t = \mathcal{L} (\bar{\rho}_t) = \sum_{j=1}^{L-1} \mathcal{L}^\text{edge}_j (\bar{\rho}_t) + \mathcal{L}_{\text{bdry}} (\bar{\rho}_t),
\]

with \( \mathcal{L}^\text{edge}_j (\bar{\rho}) = \bar{I}_j^+ \bar{\rho}_j^+ \bar{I}_j^- + \bar{I}_j^- \bar{\rho}_j \bar{I}_j^+ - \frac{1}{2} \{ \bar{I}_j^+ \bar{I}_j^- + \bar{I}_j^- \bar{I}_j^+, \bar{\rho} \} \) where \( \bar{I}_j^+ = c_{j+1} c_j \) and \( \bar{I}_j^- = c_j c_{j+1} \). Writing \( \bar{\rho}_t \) in the occupation number basis, this Lindbladian evolution preserves
its diagonal structure, while off-diagonal elements vanish exponentially in time. The diagonal elements of \( \rho_t \) provide the probability that the system is in one of the \( 2^L \) states with well defined occupation number \( \hat{n}_i = c_i^\dagger c_i \) on each site. This corresponds to a configuration of the classical SSEP, where one specifies the number of particles \( n_i = 0, 1 \) on each site. One can see easily that the Lindbladian evolution of the diagonal elements of \( \rho_t \) precisely corresponds to the Markov process of SSEP: During a time interval \( dt \) a particle in the bulk can jump to the left or right neighbouring site with probability \( \delta_t \) if the site is empty and particles get injected (extracted) at the boundaries with probability \( \alpha_i dt \) (\( \beta_i dt \)). This correspondence can also be formulated via the moment generating function,

\[
\langle e^{\sum_i a_i n_i} \rangle_{\text{SSEP}} = \text{Tr}(\rho e^{\sum_i a_i n_i}),
\]

and it illustrates the well known correspondence between Markov processes and Lindbladian evolutions. The relation between Q-SSEP and free cumulants we shall describe below implies a new relation between the classical SSEP and free probability [39].

c. Fluctuating coherences. If we go beyond the mean dynamics, then Q-SSEP has an additional structure, which describes purely quantum mechanical effects such as entanglement. In particular, we focus on the (so-called) coherence between site \( i \) and \( j \), defined as

\[
G_{ij}(t) = \text{Tr}(\rho_t c_i^\dagger c_j).
\]

In appendix A we outline a procedure how one could in principal experimentally measure \( G_{ij} \), following [24]. Since the density matrix \( \rho_t \) is a stochastic variable, so is the matrix of coherences \( G_{ij} \). It describes the entanglement between sites \( i \) and \( j \), in the sense that \( G_{ij} \) is zero if the complete state of the system \( \rho_t \) can be written as a product of the two parts that are formed by cutting the systems somewhere between site \( i \) and \( j \).

While the coherences vanish in mean exponentially with time (we saw that the mean corresponds to the classical SSEP and there are no quantum correlations left), their fluctuations survive the long time limit, although they are sub-leading in the system size [28]. For example, at large system size \( L \to \infty \) with \( x = i/L \leq y = j/L \) fixed, the connected quadratic fluctuation (or 2nd cumulant) in the steady state is

\[
\mathbb{E}[G_{ij} G_{ji} ]^c \sim \frac{1}{L} (\Delta n)^2 x (1-y),
\]

where \( \Delta n = n_a - n_b \) is the difference in the particle density between the boundary reservoirs, see (10).

Let us also note that since the Hamiltonian is quadratic in fermionic creation and annihilation operators, the coherences \( G_{ij} \) completely characterise the state of the system due to Wick’s theorem. From (2) one finds that their time evolution is given by the SDE

\[
dG_{ij} = -i(G_{i,j-1} dW_t^{j-1} + G_{i,j+1} dW_t^j - G_{i-1,j} dW_t^{i-1} - G_{i+1,j} dW_t^i) + \delta_{ij} (G_{i+1,i+1} + G_{i-1,i-1}) dt - 2G_{ij} dt - \sum_{p \in \{1, L\}} \left( \frac{1}{2} (\delta_{ip} + \delta_{jp}) (\alpha_p + \beta_p - 1) G_{ij} - \delta_{ip} \delta_{jp} \alpha_p \right) dt.
\]

From this equation one can also see that Q-SSEP possesses a \( U(1)^L \) invariance: The transformation \( G_{ij} \to \tilde{G}_{ij} = e^{i\theta_i} G_{ij} e^{i\theta_j} \) (or in matrix form \( G \to e^{-i\theta} Ge^{i\theta} \) with \( \theta = \text{diag}(\theta_1, \ldots, \theta_L) \)) leaves (4) invariant if the Brownian motions are multiplied by a phase \( dW_t^{i} \to d\tilde{W}_t^{i} = e^{i(\theta_i + \theta_j)} dW_t^{j} \). Since \( dW_t^{i} \) and \( d\tilde{W}_t^{j} \) have the same distributions, the same is true for \( G \) and \( \tilde{G} \) if their initial conditions at \( t = 0 \) are diagonal (and therefore equal). Hence, the probability measure \( \mathbb{E}_t \) of \( G \) is \( U(1)^L \) invariant if the initial condition for \( G \) was diagonal [40].

d. Scaling limit. As has been shown in the closed case [31], Q-SSEP has a meaningful scaling limit of large system size \( L \to \infty \), if positions \( i_k \) and time \( t \) are scaled diffusively according to

\[
i_k \to x_k = i_k/L, \quad t \to \tau = t/L^2.
\]
are therefore the building blocks to understand the fluctuations of coherences in the scaling limit. They can be represented diagrammatically as

\[ g^* = \lim_{L \to \infty} L^{-n-1} \mathbb{E}_{L \tau} [G_{i_1 i_2 i_3 i_1}], \]

and as well as its connected part (or cumulant)

\[ g_{\tau}(x_1, \ldots, x_n) := \lim_{L \to \infty} L^{-n-1} \mathbb{E}_{L \tau} [G_{i_1 i_2 i_3 i_1} c], \]

with \( x_k = i_k/L \). If the arguments \((x_1, \ldots, x_n, \tau)\) are not explicitly specified, then we will denote these functions as \( g_{\tau} \) and \( g^* \). For simplicity, the scaled time variable \( \tau \) will also be denoted by \( t \) if the context is clear. We chose to denote the moments by the superscript “s”, since it turns out that \( g^* \) is singular if indices coincide. This is shown in appendix B for the example of \( n = 2 \).

III. BULK VS BOUNDARY THERMALISATION

When we casually talk about bulk and boundary thermalisation, one should keep in mind that the open Q-SSEP does not actually “thermalise”. At large times, the system approaches a steady state [41], in which observables do not depend on time any more longer. But they are not described by a thermal density matrix since there is a steady current, so that the system is out-of-equilibrium.

In this section, we show that the density of fermions in the open Q-SSEP approaches its steady state value much faster on the boundaries than in the bulk. In particular, the decay time scales with \( O(1) \) in the bulk and \( O(L^2) \) on the boundary. This information is then used in the next section to formulate the time evolution of coherent fluctuations in the scaling limit in the open Q-SSEP. Due to the correspondence (3) this property also hold true for the classical SSEP [42].

Under the evolution outlined above the mean fermion density \( n_i(t) := \mathbb{E}_t [\text{Tr}(\rho c_i^* c_i)] \) evolves according to

\[ \partial_t n_i(t) = \Delta n_i(t) + \sum_{p=1,L} \delta_{ip}(\alpha_p - (\alpha_p + \beta_p)n_p(t)), \]

where it is understood that the discrete Laplacian on the boundaries is truncated, i.e. \( \Delta n_1 := n_2 - n_1 \) and \( \Delta n_L := n_{L-1} - n_L \). Here we only discuss the special case, where the injection/extraction parameters are such that \( \alpha_1 + \beta_1 = 1 = \alpha_L + \beta_L \). The general case is discussed in Appendix C and works analogously with a little bit of help of numerical calculations. In the special case, (8) can be solved analytically

\[ n_j(t) = \sum_{k=1}^L \sin\left(\frac{jk\pi}{L+1}\right) \left(e^{-(2+2\cos(\frac{jk\pi}{L+1}))t}(c_k - b_k) + b_k\right), \]

where the coefficients \( c_k \) are determined by the initial condition and \( b_k := \frac{2}{L+1} \frac{\alpha_1 \sin(\frac{jk\pi}{L+1}) + \alpha_L \sin(\frac{Lk\pi}{L+1})}{2(1 - \cos(\frac{jk\pi}{L+1}))} \). The solution consists of two contributions: A time dependent term that decay exponentially in time and a constant term which provides the steady state value and can be simplified to \( n_j(\infty) = \frac{\alpha_1(L-j+1) + \alpha_L j}{L+1} \). We study the time scale with which the time dependent term decays on the boundary compared to its decay in the bulk.

A bulk site is characterised by \( j \sim aL \) with \( a \sim O(1) \). Due to the factor \( e^{-(2+2\cos(\frac{k\pi}{L+1}))t} \approx e^{-\pi^2 t^2/L^2} \) (for large \( L \)) the biggest contribution to the sum comes from the term with \( k = 1 \). Since we are in the bulk its amplitude is finite, \( \sin(j\pi/(L+1)) \sim \sin(a\pi) \). Therefore we find a time scale of \( t_{\text{decay}} \sim O(L^2) :\)

\[ n_{\text{bulk}}(t) \sim \text{const.} e^{-\pi^2 t^2/L^2} + n_{\text{bulk}}(\infty). \]

A site on the boundary is characterized by \( j = 1 \) or \( j = L \) and therefore the amplitude of the \( k = 1 \) term, \( \sin(j\pi/(L+1)) \), will be zero for large \( L \). A significant contribution therefore only comes from terms where \( k \sim bL \) with \( b \sim O(1) \), because then the amplitudes \( \sin(jk\pi/(L+1)) \sim \sin(b\pi) \) or \( \sim \sin(bL\pi) \) stays finite. The time scale with which these terms decay is of order one, \( t_{\text{decay}} \sim O(1) :\)

\[ n_{\text{bdry}}(t) \sim \text{const.} e^{-b^2 \pi^2 t^2} + n_{\text{bdry}}(\infty). \]

Note that the value of the density on the boundaries after a time of \( O(1) \) is that of the steady state. For general injection/extraction parameters these values are as in the classical SSEP

\[ n_1(\infty) = n_a := \frac{\alpha_1}{\alpha_1 + \beta_1}, \quad n_L(\infty) = n_b := \frac{\alpha_L}{\alpha_L + \beta_L}. \]

We conclude that there is a separation of time scales with which the boundary and bulk approach their steady state values. In the scaling limit (5) with rescaled time \( \tau \), this ensures that boundaries are equal to the steady state values at all times, while in the bulk it takes \( t_{\text{decay}} \sim O(1) \) of time.

IV. SCALING LIMIT WITH OPEN BOUNDARY CONDITIONS

First we discuss the scaling limit of the equation for the density (1-point function). The scaling limit for high-point functions then follow easily.
where $\Delta$ initial conditions. The agreement is very good. Injection rates are $\alpha = \alpha_L = \beta = 1$ and don’t fit the initial conditions. The agreement is very good.

A. One-point function

The aim is to find a partial differential equation in space and time for a continuous density $\rho(x, t)$ that approximates the $L$ coupled differential equations (8) for discrete density $n_i(t)$ in the scaling limit. This means that the discrete and continuous description are related by $\rho(x, t) \approx n_{Lx}(L^2t)$. In the last section we saw that the boundary conditions must be $\rho(0, t) = n_a, \rho(1, t) = n_b$. The bulk is just diffusion and therefore the continuous density satisfies

$$\partial_x \rho(x, t) = \partial_x^2 \rho(x, t).$$

This is confirmed numerically in Figure (2), where a solution for the discrete density $n_i(t)$ is compared to the solution for the continuous density $\rho(x, t)$. In particular the boundary values agree.

B. Higher-point functions

Higher-point functions in the Q-SSEP describe the fluctuations of coherences and (as pointed out in the end of section II) we should focus on the single loop correlations functions $g^s_i$. In the closed case, their time evolution was derived in [31],

$$(\partial_t - \Delta) g^s_i(x_1, \cdots , x_n)$$

$$= \sum_{i<j}^{n} 2\partial_x \partial_y (\delta(x_i, x_j) g^s_i(x_1, \cdots , x_{j-1}) g^s_i(x_j, \cdots , x_{i-1}))$$

where $\Delta \equiv \sum_{i=1}^{n} \Delta x_i, \partial_x \equiv \partial_x, \delta(x_i, x_j) = \delta(x_i - x_j)$ and indices $x_i$ are arranged in a closed loop where $x_n$ is followed by $x_1$. The equations reduces a loop of $n$ points to the product of two loops with less points by pinching the original loop along the nodes $x_i$ and $x_j$.

$$\begin{array}{c}
\text{FIG. 2. The discrete fermion density $n_{Lx}(L^2t)$ for system sizes $L = 24$ and $L = 48$ together with the scaling limit $\rho(x, t)$ at $t = 0.01$ as a function of space $x = i/L$. The extraction and injection rates are $\alpha = \alpha_L = \beta = 1$ and don’t fit the initial conditions. The agreement is very good.}
\end{array}$$

The claim we make here is that the exact same equation holds in the open case if we subject $g^s_i$ to boundary conditions that are equal to the steady state values. These are known from [28] for its connected part: $g^s_\infty(x_1, \cdots , x_n) = 0$ for all $n \geq 2$ and $(x_1, \cdots , x_n) \in \partial[0,1]^n$.

We test this claim on the level of the two-point function, for which the above equation simplifies to

$$(\partial_t - \Delta) g^s_i(x, y) = 2\delta(x - y) \partial_x g_i(x) g_i(y),$$

and we identified the one-point function with the density, $g_i(x) = \rho(x, t)$. We saw that the steady state values are very simple for the connected part of $g^s_i(x,y)$ which according to its definition (7) is

$$g_i(x, y) = \lim_{L \to \infty} L(\mathbb{E}_{L^2}[G_{ij}G_{ji}] - \delta_{ij} \mathbb{E}[G_{ii}]^2)$$

$$= g^s_i(x, y) - \delta(x - y) g_i(x) g_i(y).$$

From (14) its time evolution equation with appropriate boundary conditions is found to be

$$(\partial_t - \Delta) g_i(x, y) = 2\delta(x - y) \partial_x g_i(x) \partial_y g_i(y),$$

$$g_i(x, y) = 0 \text{ if } (x, y) \in \partial[0,1]^2,$$

which is a again heat equation, but with a new source term.

The analytic solution of this equation (which is constructed in appendix D) is compared to a numerical solution of the discrete evolution equations for $L(\mathbb{E}_{L^2}[G_{ij}G_{ji}] - \delta_{ij} \mathbb{E}[G_{ii}]^2)$ for given $L$ – which can be derived from (4). Figures 4 and 3 show the result of this comparison for different system sizes $L$. The agreement is excellent.

Notice that the injection and extraction rates enter (15) only through the source term, since the density $g^s_i(x)$ depends on these rates through its own boundary conditions. Apart from that, the equation for the 2-point function $g_i(x, y)$ (and for all higher point functions) never explicitly depend on the injection and extraction rates. As a consequence, without loss of generality, we use the convention

$$n_a = 0, \quad n_b = 1,$$

in the following, through out the paper. In particular, the density (one-point function) in the steady state $g^s_\infty(x) = n_a + x(n_b - n_a)$ reduces to $g^s_\infty(x) = x$ in this convention.
As explained in the introduction, the Q-SSEP is expected to be the effective description of more detailed models after a coarse-graining on the microscopic scale. Though we are far from showing this explicitly, there is a hint for this interpretation in the mathematical structure of Q-SSEP which can be phrased in terms of so-called free cumulants. Free cumulants appear in the context of free probability theory, which deals with non-commutative random variables where one introduces the concept of “freeness” in analogy to “independence” of commuting random variables. The fact that single sites in Q-SSEP show a structure that usually is associated with non-commuting variables is what we have in mind, when we say that Q-SSEP might be an effective description after a coarse-graining on the microscopic scale.

After a small introduction on standard and free cumulants in subsections VA and VB, we return to the open Q-SSEP in the scaling limit. Motivated by the need for a time evolution equation of the connected correlation function \( g_t \) (which is continuous at coinciding indices in contrast to \( g^s_t \)), we find in subsection VC a relation between \( g^s_t \) and \( g_t \) that resembles that between moments and free cumulants. In the next subsection VE, this relation allows us to find the time evolution equation for \( g_t \) – the derivation of which is explained in detail in Appendix E2. In subsection VE, we go one step further and interpret \( g_t \) (without the delta functions) as the free cumulant of a new measure \( \nu_t \) with its own time evolution. The derivation of this evolution equation is given in Appendix E3. Finally, in subsection VF we show that steady state solution of \( \nu_t \) is particularly simple, which allows us to recursively formulate explicit solutions for the steady connected correlations \( g_\infty \). The relation between connected correlations in the steady regime and free cumulants was first discovered by Ph. Biane in [43] by extending the combinatorial solution of the steady measure described in [29]. Our approach gives an alternative proof and extends the validity of this relation from the steady state to the entire time evolution.

V. Q-SSEP AND FREE PROBABILITY THEORY

A. Standard cumulants and partitions

a. Moments and cumulants. Let \( \{X_1, \ldots, X_n\} \) be a family of classical random variables with moment-generating-function

\[
Z[a, u] := E[e^{u \sum i a_i X_i}] = \sum_{q \geq 0} \frac{u^q}{q!} \sum_{i_1 \cdots i_q} a_{i_1} \cdots a_{i_q} \mathbb{E}[X_{i_1} \cdots X_{i_q}],
\]

where \( u \) is an (optional) parameter that counts the order of the joint moment (or correlation function) \( \mathbb{E}[X_{i_1} \cdots X_{i_q}] \). The joint cumulant (or connected correlation function) \( \mathbb{E}[X_{i_1} \cdots X_{i_q}]^c \) is defined as the term proportional to \( a_{i_1} \cdots a_{i_q} \) in the expansion of the cumulant generating function \( W[a, u] := \log Z[a, u] \).

\[
W[a, u] = \sum_{q \geq 0} \frac{u^q}{q!} \sum_{i_1 \cdots i_q} a_{i_1} \cdots a_{i_q} \mathbb{E}[X_{i_1}, \cdots, X_{i_q}]^c.
\]

In fact, cumulants and moments are related by a combinatorial formula. Expanding \( Z[a, u] \) in terms of the cumulants and grouping together terms with the same power of \( u \) one can derive that a moment \( \mathbb{E}[X_{i_1} \cdots X_{i_q}] \) can be expressed as a sum of products of cumulants arranged according to partitions \( \pi \) of the set \( \{i_1, \ldots, i_q\} \),

\[
\mathbb{E}[X_{i_1} \cdots X_{i_q}] = \sum_{\pi \in P\{(i_1, \ldots, i_q)\}} \mathbb{E}_\pi[X_{i_1}, \cdots, X_{i_q}]^c, \tag{16}
\]

where \( \mathbb{E}_\pi[X_{b_1}, \cdots, X_{b_q}]^c := \prod_{b \in \pi} \mathbb{E}[X_{b_1} X_{b_2} \cdots] \) and \( b = \{b_1, b_2, \ldots\} \) denote the elements of a block of the partition \( \pi \). The number of partitions of a set of \( n \) elements
is called the Bell number $B_n$, with recursion relation $B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k$ and $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$ and $B_5 = 52$, etc.

Let us give an example for $n = 4$ and the set $\{1, 2, 3, 4\}$. A possible partition is $\pi = \{(1, 2), \{3, 4\}\}$ and the corresponding term $E_{\pi}[X_1, \ldots, X_4]$ can be represented by a diagram: We place four nodes on a circle and connect nodes belonging to the same block of the partition. This yields

![Diagram](image)

The expansion of the moment $E[X_1X_2X_3X_4]$ in terms of its cumulants, represented by diagrams, then reads

![Diagrams](image)

where the subscript $\circ k$ tells us that by cyclic permutation there are in total $k$ such diagram. Note the fact, that the last diagram (in a dotted box) corresponds to a crossing partition $\pi = \{\{1, 3\}, \{2, 4\}\}$. In free probability theory these diagrams do not appear as we will see below.

b. **Standard cumulants of a single variable.** The moment-cumulant relation (16) allows us to express the cumulants recursively through the moments. In the case of a single variable $X = X_1 = \cdots = X_n$ we illustrate how this can be done up to order four. Let us denote by $m_k = E[X^k]$ and $c_k = E[X^k]$, the moments and cumulants of this variable, then

$$
\begin{align*}
m_1 &= c_1, \\
m_2 &= c_2 + c_1^2, \\
m_3 &= c_3 + 3c_2c_1 + c_1^3, \\
m_4 &= c_4 + 4c_3c_1 + 3c_2^2 + 6c_1^2c_2 + c_1^4.
\end{align*}
$$

This can be solved recursively for $c_k$,

$$
\begin{align*}
c_1 &= m_1, \\
c_2 &= m_2 - m_1^2, \\
c_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\
c_4 &= m_4 - 4m_1m_3 + 12m_1^2m_2 - 3m_2^2 - 6m_1^4.
\end{align*}
$$

The generating function of cumulants is the logarithm of that of moments, as above. Below we will see how this formula differs for the free cumulants

b. **Free cumulants and non-crossing partitions**

a. **Moments and free cumulants.** In classical probability, two variables are independent if their moments factorise at all orders, $E[X^mY^n] = E[X^m]E[Y^n]$ for all $n, m \in \mathbb{N}$. As a consequence, the (standard) cumulants add in the sense that $c_n(X + Y) = c_n(X) + c_n(Y)$. In free probability theory one deals with non-commuting variables $\{a_1, \ldots, a_n\}$ that live in some algebra $\mathcal{A}$ and a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ that can be seen as a generalisation of the expectation value $E$ for commuting variables. There is a natural extension of the notion of independence, called "freeness", due to Voiculescu [32]. For example, if $a \in \mathcal{A}$ and $b \in \mathcal{A}$ are free, then $\varphi(ab^\circ) = \varphi(a)\varphi(b^\circ)$. However, additional structure occurs if one interchanges the order such that free variables are no longer grouped together. For example $\varphi(ab^\circ) \neq \varphi(a^\circ)b^\circ$. The precise definition of freeness can be found in [32–34]. It allows to recursively compute the multipole point expectation values in terms of the moments of the free variables therein. In this setting it turns out that there exists a natural extension of the notion of cumulants, the so-called "free cumulants" $\kappa_\pi(a)$. They are such that if $a$ and $b$ are free then their free cumulants are additive, in the sense that $\kappa_n(a + b) = \kappa_n(a) + \kappa_n(b)$, see [44].

Free cumulants $\kappa$ are implicitly defined [34] through the moments $\varphi(a_{i_1} \cdots a_{i_k})$ as a sum over non-crossing partitions of the set $\{i_1, \ldots, i_k\}$,

$$
\varphi(a_{i_1} \cdots a_{i_k}) := \sum_{\pi \in NC(\{i_1, \ldots, i_k\})} \kappa_\pi(a_{i_1}, \ldots, a_{i_k}),
$$

where $\kappa_\pi(a_{i_1}, \ldots, a_{i_k}) := \prod_{b \in \pi} \kappa(a_{b_1}, a_{b_2}, \ldots)$ and the set of non-crossing partitions is denoted by $NC(\cdots)$. Recall that, if we draw the ordered set $\{i_1, \ldots, i_k\}$ on a circle and picture each block of a given partition by polygons whose vertices are the points of the blocks, the non-crossing partitions are those for which these polygons do not intersect. In the example for $n = 4$ above we would recover all diagrams except for the last diagram in the dotted box, if we would expand $\varphi(a_1a_2a_3a_4)$ in terms of free cumulants $\kappa_\pi$. The number of non-crossing partitions of a set of $n$ elements is the Catalan number $C_n = \frac{1}{n+1}(2n)!$, with $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$ and $C_5 = 42$, etc. The formula (20) is triangular and can be inverted to express the free cumulants in terms of the moments [34].

b. **Free cumulants of a single variable.** In the case of a single variable $a = a_1 = \cdots = a_n$, we denote $\kappa_k := \kappa(a, \ldots, a)$ the free cumulant at order $k$ and $m_k := \varphi(a^k)$ the moment at order $k$, then we have

$$
\begin{align*}
m_1 &= \kappa_1, \\
m_2 &= \kappa_2 + \kappa_1^2, \\
m_3 &= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3, \\
m_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 2\kappa_2^2 + 6\kappa_1^2\kappa_2 + \kappa_1^4.
\end{align*}
$$
The equations can be solved for $\kappa_k$ recursively,

\begin{align}
\kappa_1 &= m_1, \\
\kappa_2 &= m_2 - m_1^2, \\
\kappa_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\
\kappa_4 &= m_4 - 4m_1m_3 + 10m_1^2m_2 - 2m_1^2 - 5m_4^2.
\end{align}

Note, that the difference between standard and free cumulants only shows up at order 4 since here a crossing-partition become possible for the first time. Also notice the correspondence between the diagrams in (17) and the equation for $m_4$. For a single variable, the relation between the moments and the free cumulants can be found by inverting the resolvent associated to the distribution, see \cite{34, 35}.

Below we will see that the Q-SSEP correlation functions also reveal an expansion over non-crossing partitions and therefore have an interpretation as free cumulants.

c. Free cumulants in Random Matrix Theory. It has been known since the 90’s that large random matrices provide an explicit realisation of the concepts of free probability theory. See \cite{32, 33, 37} for a simple introductory review on this topics.

For instance, a random variable whose non-zero free cumulants are only the first two $\kappa_1$ and $\kappa_2$ is distributed according Wigner’s semi-circle law of random matrix theory for the Gaussian unitary ensemble (GUE), so that large GUE random matrices are analogous of Gaussian variables but from a free probability point of view.

Furthermore, as proved by Voiculescu \cite{36}, $N \times N$ random matrices of the form $X_N = U_N A_N U_N^\dagger$, where the $U_N$’s are chosen independently from the Haar distribution over the unitary group and $A_N$ are deterministic matrices with well defined spectral density $\mu_A$ in the limit of large matrix size (i.e. $m_k := \lim_{N \to \infty} \frac{1}{N} \text{Tr}(A_N^k) = \int \lambda^k d\mu_A(\lambda)$ exists), are free non-commuting variables with $\varphi := \frac{1}{N} \text{Tr}$ in the limit $N \to \infty$. In this limit, it is furthermore known that the (classical) cumulants of such matrices $X_N$ w.r.t. the Haar measure can be expressed as free cumulants w.r.t. to its spectral density $\mu_A$. That is,

\[
\mathbb{E}[e^{N \text{Tr}(B_N X_N)}] \simeq_{N \to \infty} e^{N \sum_{k=1}^{9} \frac{1}{k!} \kappa_k \text{Tr}(B_k)},
\]

where $B_N$ is a sequence of matrices with fixed rank (such that $\text{Tr}(B_N^k)$ does not scale with $N$), for instance a rank one projector, and $\kappa_k$ are the free cumulants of the spectral density $\mu_A$ (defined recursively via the moments $m_k$ as in (21) and (22)).

This connection between Haar distributed random matrices and free cumulants is adapted to the closed Q-SSEP in the steady state. Indeed, since in the closed case the Q-SSEP dynamics is unitary and ergodic over the unitary group, its steady state distribution is the one induced by the Haar measure on the orbit of the initial matrix of coherences $G_0$ \cite{27}. That is, the matrix of coherence $G$ is distributed as $UG_0U^\dagger$ with $U$ a Haar distributed unitary $L \times L$ matrix. As a consequence of (23) above, the large deviation function for coherence fluctuations in the closed Q-SSEP is the generating function of the free cumulants of the spectral measure of the initial matrix of coherences $G_0$. The aim of the following section is to show that a connection with free cumulants persists in the open case.

c. Expansion into connected correlation functions

After this little excursion to free cumulants in general, we now return to the open Q-SSEP in the scaling limit. Our goal is to express the correlation functions (moments) $g_t = L^{n-1} \mathbb{E}[G_{i_1 i_2} \cdots G_{i_n i_1}]$ in terms of their connected parts (cumulants) $g_t = L^{n-1} \mathbb{E}[\text{Connected}(G_{i_1 i_2} \cdots G_{i_n i_1})]^c$, since this is the first step to find the time evolution of the connected correlation functions. To do so, we use the moment–cumulant relation in (16) and illustrate our argument on the example $n = 4$.

a. Example $n = 4$. The correlation function $\mathbb{E}[G_{i_1 j_1} G_{j_2 k_1} G_{k_2 l_1}]$ can be expanded into cumulants precisely as in (17) with the identification $X_1 = G_{i_1}, X_2 = G_{j_2}, X_3 = G_{k_2}$ and $X_4 = G_{l_1}$. However, the $U$ invariance of the measure forces us to add delta functions such that all left and right indices of the appearing variables $G_{ij}$ are paired. This yields (with $[\cdots] = \mathbb{E}[\cdots]$ to simplify notation),

\[
[G_{ij} G_{jk} G_{kl} G_{li}] = [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{ij} [G_{ij}] [G_{jk} G_{kl} G_{li}]^c + \delta_{jk} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{lk} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{il} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{jl} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{lj} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{kl} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{il} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{jl} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \delta_{lk} [G_{ij} G_{jk} G_{kl} G_{li}]^c + \text{cyclic perm.}
\]

In our diagrammatic notation we choose to label the nodes (rather than the edges) by the indices $\{i, j, k, l\}$ such that $G_{ij}$ lives on an edge and we are dealing with a partition of edges. We include delta functions such as $\delta_{ij}$ into our diagrammatic notation by adding a dashed line between node $i$ and $j$. For a non-crossing partition $\pi$ of the edges, the dashed lines form a non-crossing partition of the nodes, which we refer to as the dual partition $\pi^*$.
(also called the Kreweras complement in the mathematical literature). Then, in full analogy to (17),

\begin{equation}
\left[ G_{ij}G_{jk}G_{kl}G_{il} \right] = k
\end{equation}

We analyse the scaling with \( L \): all terms scale with \( L^{-3} \) (using the rule \( \delta_{ij} \rightarrow L^{-1}\delta(x_1 - x_2) \) and \( \mathbb{E}[G_{i_1i_2}\cdots G_{i_ni_1}]^c \sim L^{-n+1} \) except the last term (or crossing diagram) which scales with \( L^{-5} \) (\( \mathbb{E}[G_{i_1j_1}]^c \) is the product of two loops and must scale at least with \( L^{-2} \)). Therefore the crossing diagram does not contribute in the scaling limit.

Note that this expansion into non-crossing partitions is very generic in the sense that we only used the \( U(1)^L \) invariance of the measure and the scaling property \( \mathbb{E}[G_{i_1i_2}\cdots G_{i_ni_1}]^c \sim L^{-n+1} \).

b. Generalisation. This result generalises to the \( n \)-point correlation functions \( g^n \) as follows: Denote by \( NC([n]) \) all non-crossing partitions \( \pi \) of the set \( [n] := \{ x_1, ..., x_n \} \). In order to obtain its dual, we label (by convention) a node and an edge that follows this node in anti-clockwise direction by the same name. Then we think of \( \pi \) as a non-crossing partition of edges that uniquely defines non-crossing partition of nodes \( \pi^* \). For example, \( \pi = \{ \{ x_1, x_2 \}, \{ x_3, x_4 \} \} \) gives rise to \( \pi^* = \{ \{ x_1, x_3 \}, \{ x_2, x_4 \} \} \), as can be read off the diagram

\begin{equation}
\begin{array}{c}
D_{\pi} = x_3 \rightarrow x_1 \cdot
\end{array}
\end{equation}

For better readability, we usually only show the labels of the nodes. With this convention we have

**Theorem 1.**

\begin{equation}
g^n([n]) = \sum_{\pi \in NC([n])} D_{\pi}
\end{equation}

where the free cumulants \( D_{\pi} \) are defined by

\begin{equation}
D_{\pi} := \prod_{d \in \pi^*} \delta(d) \prod_{b \in \pi} g_l(b),
\end{equation}

and diagrammatically represented as in (26).

Here we adopted the notation to represent \( \delta \)-functions as \( \delta(d) = \delta(d_1,d_2,\cdots) := \delta(d_1 - d_2)\delta(d_2 - d_3)\cdots \) where \( d = \{ d_1,d_2,\cdots \} \in \pi^* \) and by convention \( \delta(d_1) = 1 \) if the block \( d = \{ d_1 \} \) has only one element.

The proof follows the combinatorial scaling argument outline above the four point function and is given in appendix E 1.

**D. Time evolution of connected correlation functions**

The relation (27) allows to find the time evolution of the connected correlation functions \( g_t \).

**Theorem 2.**

\begin{equation}
(\partial_t - \Delta)g_t(x_1,\cdots,x_n) = \sum_{i<j} 2 \delta(x_i,x_j)\partial_t g_t(x_i,\cdots,x_{j-1})\partial_j g_t(x_j,\cdots,x_{i-1}).
\end{equation}

This equation differs from (12) by the relative position of the derivatives and the delta function.

It is derived from (12) by induction over \( n \) as is shown appendix (E 2). Here we illustrate our method for the easiest case \( n = 2 \). The time evolution of the two point function \( g^n \) is

\begin{equation}
(\partial_t - \Delta)g^n_t(x_1,x_2) = 2\partial_t \partial_x(\delta(x_1,x_2)g_t(x_1)g_t(x_2)).
\end{equation}

The partitions of \( \{ x_1, x_2 \} \) are \( \{ \{ x_1 \}, \{ x_2 \} \} \) and \( \{ \{ x_1, x_2 \} \} \). We therefore have

\begin{equation}
g^n_t(x_1,x_2) = g_t(x_1,x_2) + \delta(x_1,x_2)g_t(x_1)g_t(x_2).
\end{equation}

We also need the action of \( (\partial_t - \Delta) \) on these terms. Computing \( (\partial_t - \Delta) (\delta(x_1,x_2)g_t(x_1)g_t(x_2)) \), we get
Recall that we use a convention such that $\Delta = 0$.

For the second equal sign, use $(\partial_1 + \partial_2)\delta(x_1, x_2) = 0$ from which follows that $\Delta\delta(x_1, x_2) = -2\partial_1\partial_2\delta(x_1, x_2)$. Comparing to (29), the first term cancels and we are left with the result,

$$ (\partial_1 - \Delta)g_1(x_1, x_2) = 2\delta(x_1, x_2)\partial_1g_1(x_1)\partial_2g_1(x_2). $$

For higher orders, there is a similar cancellation of the terms involved. However, one usually needs to reorder the sums over non-crossing partitions to see this cancellation. We would like to stress that this reordering is only possible due to the non-crossing nature of the partitions. For more details, see appendix (E.2).

E. Connected correlations as free cumulants of a new measure $\hat{\varphi}$

Having realised that $g^n$ has a natural interpretation as the moment of free cumulants $D_n$ we turn the logic around and ask if the connected correlation functions without the delta functions, namely

$$ g_{t,\pi} := \prod_{b \in \pi} g_t(b), $$

for any non-crossing partition $\pi$, could have an interpretation as free cumulants of a new (time-dependent) measure $\hat{\varphi}_t$ on a family of (possibly non-commuting) random variables $\{a_x\}_{x \in [0,1]}$. The moments of $\hat{\varphi}$

$$ \hat{\varphi}_t(x_1, \ldots, x_n) := \varphi(a_{x_1}a_{x_2} \cdots a_{x_n}) $$

should then be related to the $g_{t,\pi}$ by

$$ \hat{\varphi}_t(x_1, \ldots, x_n) := \sum_{\pi \in NC((x_1, \ldots, x_n))} g_{t,\pi}. \quad (30) $$

Recall that we use a convention such that $n_a = 0$ and $n_b = 1$ so that the one point function satisfies the boundary conditions $g_t(0) = 0$ and $g_t(1) = 1$, and the steady density reads $g_\infty(x) = x$.

**Theorem 3.** The moments $\varphi_t(x_1, \ldots, x_n)$ satisfy the same time evolution equation as the corresponding free cumulants $g_{t,\pi}(x_1, \ldots, x_n)$,

$$ (\partial_t - \Delta)\varphi_t(x_1, \ldots, x_n) = \sum_{i < j} 2\delta(x_i, x_j)\partial_i\varphi_t(x_1, \ldots, x_{j-1})\partial_j\varphi_t(x_j, \ldots, x_{i-1}). \quad (31) $$

**F. Steady state solution as free cumulants**

The boundary conditions (32) are fulfilled by $\varphi_\infty(x_1, x_2) = \min(x_1, x_2)$ for $(x_1, x_2) \in \partial[0,1]^n$. The theorem below shows, that this is even a complete solution of (31) in the steady state.

**Theorem 4.** $\varphi_\infty(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$ is a steady state solution of (31).

The proof is given in Appendix E.4. As a consequence, we can use (30) to recursively reconstruct the connected correlations functions $g_t$ in the steady state. This works similarly to (21) and (22). Denoting $\min(x_1, \ldots, x_n) =: x_1 \wedge \cdots \wedge x_n$, we have

$$ g_\infty(x_1) = x_1 $$

$$ g_\infty(x_1, x_2) = x_1 \wedge x_2 - x_1x_2 $$

$$ g_\infty(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge x_3 - x_1(x_2 \wedge x_3) \wedge x_3 + 2x_1x_2x_3, $$

while the four point function $g_\infty(x_1, x_2, x_3, x_4)$ reads

$$ x_1 \wedge x_2 \wedge x_3 \wedge x_4 - x_1(x_2 \wedge x_3 \wedge x_4) \wedge x_4 \wedge x_3(x_2 \wedge x_4) \wedge x_4 \wedge x_2(x_3 \wedge x_4) \wedge x_4 $$

$$ + 2x_1x_2x_3x_4, $$

where $\cdots \wedge y$ denotes the sum of all terms obtained by $q$ successive cyclic permutations of the arguments of the term in question. Note that due to the absence of crossing partitions from 4-points onward, $g_\infty(x_1, \ldots, x_n)$ is no longer invariant under the permutation of its arguments for $n \geq 4$. In the example above it is the term $(x_1 \wedge x_2)(x_3 \wedge x_4) \wedge 2$ which prevents $g_\infty(x_1, \ldots, x_4)$ from being invariant under the exchange of $x_2$ and $x_3$. However with different boundary conditions. Indeed, given $\{x_1, \ldots, x_n\} \in \partial[0,1]^n$ there will be at least one $x_i = y$ with $y = 0, 1$. Then

$$ \hat{\varphi}_t(x_1, \ldots, x_i = y, \ldots, x_n) = y \hat{\varphi}_t(x_1, \ldots, \hat{x}_i, \ldots, x_n) \quad (32) $$

where the hat on $\hat{x}_i$ indicates that $x_i$ is missing from the set $\{x_1, \ldots, x_n\}$.
Indeed, ordering the variables as $0 < x_1 < x_2 < x_3 < x_4 < 1$, we get $g_\infty(x_1, x_2) = x_1(1 - x_2)$ and $g_\infty(x_1, x_2, x_3) = x_1(1 - 2x_2)(1 - x_3)$ and

$$g_\infty(x_1, x_2, x_3, x_4) = x_1(1 - 3x_2 - 2x_3 + 5x_2x_3)(1 - x_4),$$

$$g_\infty(x_1, x_3, x_2, x_4) = x_1(1 - 4x_2 - 2x_3 + 5x_2x_3)(1 - x_4),$$

whereas $g_\infty(x_1, x_3, x_4, x_2) = g_\infty(x_1, x_2, x_3, x_4)$, in agreement with [28].

This calculation provides an alternative proof of Biane’s formula (6.1) in [43] that relates the steady state connected correlations of the open Q-SSEP to free cumulants of the measure $\varphi_\infty$. This measure can be realized as the Lebesgue measure on the interval $[0, 1]$ of the indicator function $I_x := 1_{[0,x]}$, e.g. $\varphi_\infty(I_x I_y) := \int_0^1 1_{[0,x]}(z)1_{[0,y]}(z)dz = \min(x,y)$. It is surprising that $\varphi_\infty$ has a realization in terms of commuting variables, since free cumulants usually appear in a setting of non-commuting variables. Furthermore (31) suggests that $\varphi_t$ might not be invariant under a permutation of its arguments at finite time.

VI. CONCLUSION

In this paper we presented two main points. One the one hand, we have shown that the evolution equation of correlation functions in the open Q-SSEP has a diffusive scaling limit, where the only difference to the closed case is that we subject the equations to boundary conditions that are equal to the steady state values. The reason for this is that the density approaches its steady state value much faster on the boundary than in the bulk. In the scaling limit (where time is rescaled by $t \rightarrow \tau = t/L^2$) the boundary conditions can therefore directly be identified with the steady state values and this property proliferates to higher order correlation functions.

One the other hand, we have transformed this evolution equation into an equation for the connected correlation functions and thereby discovered a link with free probability theory. More precisely, we found that the expansion of correlations functions into their connected parts is a sum over non-crossing partitions. This expansion is very generic in the sense that we only used the $U(1)\ell^2$ invariance of the measure and the scaling property of the correlation functions $E[O_{i_1}O_{i_2}\cdots O_{i_n}] \sim L^{-n+1}$. Furthermore, we showed that the connected correlations can be interpreted as the free cumulants of a new measure for whose moments we derived a time evolution equation. The steady state solution of this measure was shown to agree with the formula found in [43] and allows to recursively find the connected correlations in the steady state.

However, it remains to construct the stochastic process associated to the time dependent measure $\varphi_t$ which would then fully characterize Q-SSEP in its scaling limit.

Furthermore, we don’t yet know how to interpret the fact that this new measure becomes a measure of commuting variables at large time, despite the fact that at intermediate times this seems not to be the case.

We believe that the appearance of tools from free probability theory is a generic feature of the type of mesoscopic systems that we described in the introduction. Intuitively, at a mesoscopic scale, an observable has to be invariant under relabelling of the large number of internal degrees of freedom within the ballistic cells of size $\ell$. As a consequence, these observables inherit properties from large random matrices and hence from free probability. Let us elaborate on this idea. Since information about the local degrees of freedom within the ballistic cells are lost after a coarse-graining, an observable $X_x$ that probes the system on the scale of a ballistic cell in the neighbourhood of a point $x$ should be invariant under conjugation by a unitary transformation of the microscopic degrees of freedom within this cell. From a maximal local entropy principle, we expect that this observable should be similar to $X_x \sim U O_a U^\dagger$ with $U$ Haar distributed over the unitary group of the ballistic cell and where $O_a$ is a specific choice of this observable that depends only on the microscopic degrees of freedom $a$ inside the cell at $x$. The flat distribution of $U$ over the ballistic cell unitary group can be thought of as arising from the dynamics of the microscopic degrees of freedom providing the latter is ergodic within the ballistic cells on ultra-fast time scale. As a consequence, the cumulants of the observable $X_x$ scale with the ballistic length $\ell$ as Haar distributed random matrices do. In the mesoscopic picture, although small compared to the decoherence length, this length $\ell$ is large compared to the ultra-violet microscopic length (say the mesh of the lattice). And thus, by arguments similar to those presented in this text, only non-crossing cumulants of these mesoscopic observables are expected to survive. However, it remains a challenge to make this naive argument more concrete and to construct a quantum mesoscopic fluctuation theory describing fluctuations of quantum coherences in generic diffusive many-body systems at coarse-grained mesoscopic scales.

Acknowledgements. This paper has been submitted simultaneously with ”Eigenstate Thermalization Hypothesis and Free Probability” by S. Pappalardi, L. Foini and J. Kurchan [45], where the relation between these two frameworks is discussed. The occurrence of free probability in both problems has a similar origin: the coarse-graining at microscopic either spatial or energy scales, and the unitary invariance at these microscopic scales. Thus the use of free probability tools promises to be ubiquitous in chaotic or noisy many-body quantum systems. We thank Fabian Essler and Adam Nahum for numerous discussions on this topic. This work was in part supported by CNRS, by the ENS and by the ANR project “ESQuisses”, contract number ANR-20-CE47-0014-01.
Appendix A: Measuring coherences

To gain a better understanding of the coherences $G_{ij} = \text{Tr}(p_j \rho_p c_i)$ we outline an experiment to measure them that was proposed in [24]. The setup is shown in Figure (5). The idea is to probe the system at spatially separated places and let the two signals interfere before each output is measured separately. Let us outline the steps of the measurement protocols in detail.

![Figure 5](image_url)

Two wires are attached to the system at sites $i$ and $j$ such that only one fermion can enter at a time. First the fermions in the wire are allowed to interact via the beam splitter $S$. Then their occupation number $n_L$ and $n_R$ is measured on each side. In the first measurement (a) one uses a symmetric beam splitter, which allows to measure the imaginary part of $G_{ij}$. In the second measurement (b), one needs to use a beam splitter where the fermion that is transmitted from $R$ to $L$ accumulates a phase $\pi$, while it does not accumulate this phase when being transmitted in the other direction. In this way one can measure the real part of $G_{ij}$.

- The total state of system, left and right wire is described by a state in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_L \otimes \mathcal{H}_R$. Let us assume that the system is in a pure state and that initially the wires are empty and not yet coupled to the system,

$$|\psi(0)\rangle = |\psi_S\rangle|0,0\rangle.$$

- Now we couple the two wires to the system. A very simple description of this coupling could be given by the unitary evolution with $U_{\text{int}} = e^{-i\lambda(c_L^d c_i + c_R^d c_j + \text{h.c.})}$, where $\lambda$ is the product of coupling strength and the time during which we allow the wires to couple to the system, and $c_L$, $c_R$ are fermionic operators on the left (right) wire. If we tune the coupling strength and duration such that $\lambda \ll 1$ is small, we can neglect $\mathcal{O}(\lambda^2)$ terms and find the complete state to be

$$|\psi(1)\rangle := U_{\text{int}}|\psi_S\rangle = |\psi_S\rangle|0,0\rangle - i\lambda (c_j|\psi_S\rangle|1,0\rangle + c_j|\psi_S\rangle|0,1\rangle) + \mathcal{O}(\lambda^2).$$

- Next, the fermions in the left and right wire interfere in a beam splitter. Written in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the beam splitter can in general be described by the scattering matrix

$$S = \begin{pmatrix} 1 & r' & t' & r r' - t t' \end{pmatrix},$$

where $r$ and $t$ ($r'$ and $t'$) are the reflection and transmission amplitudes for a fermion incident from the left (right) side. Though not important in our case, since the state $|1,1\rangle$ where there is a fermion in each wire is suppressed by $\lambda^2$, lets give a short explanation for how to to obtain last entry $(1,1) \rightarrow (r r' - t t')|1,1\rangle$. One has take into account that the wave function is antisymmetric, $|1,1\rangle = |\phi_L\rangle \otimes |\phi_R\rangle - |\phi_R\rangle \otimes |\phi_L\rangle$. Here the position in the tensor product labels the fermion (say they are called 1 and 2), whereas $|\phi_L\rangle$ and $|\phi_R\rangle$ are single fermion states in the left and right wire. Then

$$|1,1\rangle \rightarrow (r|\phi_L\rangle + t|\phi_R\rangle) \otimes (r'|\phi_R\rangle + t'|\phi_L\rangle) - (r'|\phi_R\rangle + t'|\phi_L\rangle) \otimes (r|\phi_L\rangle + t|\phi_R\rangle)$$

leads to the entry $(r r' - t t')$. Unitarity demands $|r|^2 + |t|^2 = 1$, $|r'|^2 + |t'|^2 = 1$ and $r t' + t r' = 0$ (the condition $|r r' - t t'|^2 = 1$ is then automatically fulfilled).

- (a): Choosing a symmetric beam splitter, $r = r'$ and $t = t'$, allowed to measure the imaginary part of $G_{ij}$. Note that the unitary constraints can now be expressed as $r = \sin \theta e^{i\varphi}$ and $t = i \cos \theta e^{i\varphi}$ and we set the overall phase $\varphi = 0$ since this will not change the result. The state evolves to

$$|\psi(2,a)\rangle = S(a)|\psi(1)\rangle = |\psi_S\rangle|0,0\rangle - i \lambda c_i |\psi_S\rangle (\sin \theta |1,0\rangle + i \cos \theta |0,1\rangle) - i \lambda c_j |\psi_S\rangle (\sin \theta |0,1\rangle + i \cos \theta |1,0\rangle).$$
- (b): If the beam splitter is symmetric, except that transmitted fermions incident from the right will accumulate an additional phase \(\pi\), i.e. \(r = r'\) and \(t = -t'\), this allows to measure the real part of \(G_{ij}\). Note that the unitary constraints result it \(r = \sin \theta e^{i\varphi}\) and \(t = \cos \theta e^{i\varphi}\) and again we set \(\varphi = 0\). The state evolves to

\[
|\psi^{(2,b)}\rangle = \mathcal{S}^{(b)}|\psi^{(1)}\rangle \\
= |\psi_S\rangle|0,0\rangle - i\lambda c_{j}|\psi_S\rangle (\sin \theta|1,0\rangle + \cos \theta|0,1\rangle) - i\lambda c_{j}|\psi_S\rangle (\sin \theta|0,1\rangle - \cos \theta|1,0\rangle).
\]

- Finally, we measure the particle number \(n_L = c_R^\dagger c_L^\dagger\) and \(n_R = c_R^\dagger c_R\) in the left and right wire. Denoting averages w.r.t the system \(|\psi_S\rangle\) by \(\langle \ldots \rangle_S\) as in \(G_{ij} = \langle c_R^\dagger c_S^\dagger \rangle_S\) we find for case (a)

\[
\langle n_L \rangle^{(a)} = \lambda^2 (\sin^2 \theta |n_i\rangle_S + \cos^2 \theta |n_j\rangle_S - 2 \sin \theta \cos \theta \mathcal{G}(G_{ij})) \\
\langle n_R \rangle^{(a)} = \lambda^2 (\cos^2 \theta |n_i\rangle_S + \sin \theta |n_j\rangle_S + 2 \sin \theta \cos \theta \mathcal{G}(G_{ij})).
\]

Choosing an angle \(\theta = \pi/4\) gives the imaginary part of \(G_{ij}\),

\[
2 \lambda^2 \mathcal{G}(G_{ij}) = \langle n_R \rangle^{(a)} - \langle n_L \rangle^{(a)}.
\]

For the case (b), one gets

\[
\langle n_L \rangle^{(a)} = \lambda^2 (\sin^2 \theta |n_i\rangle_S + \cos^2 \theta |n_j\rangle_S - 2 \sin \theta \cos \theta \mathcal{R}(G_{ij})) \\
\langle n_R \rangle^{(a)} = \lambda^2 (\cos^2 \theta |n_i\rangle_S + \sin \theta |n_j\rangle_S + 2 \sin \theta \cos \theta \mathcal{R}(G_{ij})).
\]

Choosing the same angle \(\theta = \pi/4\) gives the real part of \(G_{ij}\),

\[
2 \lambda^2 \mathcal{R}(G_{ij}) = \langle n_R \rangle^{(a)} - \langle n_L \rangle^{(a)}.
\]

Hence, both the real and imaginary parts of the coherence, as well as their statistical distribution, are experimentally measurable (at least in principle).

**Appendix B: Singular behaviour of non-connected correlation functions**

For the example of \(n = 2\), we will show here why (12) produces solutions \(g^e_i\) that are singular at coinciding points, whereas solutions \(g_t\) of (28) are continuous (but not differentiable). The corresponding equations for \(n = 2\) are (14) and (15). Namely,

\[
(\partial_t - \Delta) g^e_i(x,y) = 2\partial_x \partial_y (\delta(x-y) g_t(x) g_t(y)),
\]

and

\[
(\partial_t - \Delta) g_t(x,y) = 2\delta(x-y) \partial_x g_t(x) \partial_y g_t(y).
\]

Integrating these equations across the diagonal line \(\{x = y\}\) reveals that the van Neumann boundary conditions will be singular for \(g^e_i\) and regular for \(g_t\).

Let us do this explicitly for (14). We rotate the variables \((x,y)\) by \(\pi/4\) clockwise, \((v,u) = (\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}})\). Then the derivative \(\partial_v = \frac{\partial_x - \partial_x}{\sqrt{2}}\) is orthonormal to the line \(\{x = y\}\) and \(\partial_u = \frac{\partial_x + \partial_x}{\sqrt{2}}\) is parallel to the line. In these variables the equation reads

\[
(\partial_t - (\partial^2_v + \partial^2_u)) g^e_i\left(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}}\right) = (\partial^2_u - \partial^2_v) (\delta(\sqrt{2}) v) \rho_t\left(\frac{u+v}{\sqrt{2}}\right) \rho_t\left(\frac{u-v}{\sqrt{2}}\right).
\]

Integrating \(\int_{-\varepsilon}^{\varepsilon} dv\) this equation and keeping only terms of \(\mathcal{O}(1)\),

\[
\partial_v g^e_i\left(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}}\right) \bigg|_{-\varepsilon}^{\varepsilon} = \partial_v (\delta(\sqrt{2}) v) \rho_t\left(\frac{u+v}{\sqrt{2}}\right) \rho_t\left(\frac{u-v}{\sqrt{2}}\right) \bigg|_{-\varepsilon}^{\varepsilon}.
\]
The amplitude,\( \lambda \)

\[
2\partial_v g^\dagger \left( \frac{u + v}{\sqrt{2}}, \frac{u - v}{\sqrt{2}} \right) \bigg|_{v = \epsilon} = 2\sqrt{2} \delta'(\sqrt{2}v) \rho(\frac{u + v}{\sqrt{2}}) \rho(\frac{u - v}{\sqrt{2}}) \bigg|_{v = \epsilon}
\]

Finally we take \( \epsilon \to 0^+ \) and find the van Neumann boundary condition in both sectors \( \{x > y\} \) and \( \{x < y\} \),

\[
\partial_v g^\dagger \bigg|_{x = y^+} = -\partial_v g^\dagger \bigg|_{x = y^-} = \sqrt{2} \delta'(0^+) \rho(x)^2.
\]

The delta function will make this derivative blow up and therefore \( g^\dagger_i(x, y) \) is indeed singular at \( x = y \). The same procedure for (15) yields

\[
\partial_v g_i \bigg|_{x = y^+} = -\partial_v g_i \bigg|_{x = y^-} = -\frac{1}{\sqrt{2}} \rho_i(x)^2,
\]

which is finite.

**Appendix C: Solution for the discrete density**

The time evolution equation for the discrete density (8) can be rewritten as \( \partial_t n = An + n, \) with \( n = (n_1, ..., n_L)^T, \)

\[
b = (\alpha_1, ..., \alpha_L)^T
\]

and

\[
A = \begin{pmatrix}
-1 - (\alpha_1 + \beta_1) & 1 & \\
1 & -2 & \\
... & ... & ...
\end{pmatrix}
\]

If \( A \) is diagonalisable, \( A = SDS^{-1} \), a general solution for an initial condition \( n(0) = u \) is given by

\[
n(t) = \exp(At)u + (\exp(At) - 1)A^{-1}b
\]

\[
= S \exp(Dt)(S^{-1}u + D^{-1}S^{-1}b) + A^{-1}b \quad (C1)
\]

If \( \alpha_1 + \beta_1 = 1 = \alpha_L + \beta_L, A \) has is a so-called Töpitz matrix and has eigenvalues \( \lambda_k \) and normalised eigenvectors \( v_k \),

\[
\lambda_k = -2 + 2\cos(k)
\]

\[
(v_k)_j = \sqrt{2/(L + 1)} \sin(jk)
\]

with \( k = \frac{\pi}{L+1}, \frac{2\pi}{L+1}, ..., \frac{L\pi}{L+1} \), from which (9) can be derived. For general injection/extraction rates we make an ansatz

\[
\lambda_k = -2 + 2\cos(k)
\]

\[
(v_k)_j = e^{jk} + s_k e^{-jk}
\]

where \( s_k \) and \( k \) are determined by the eigenvalue equations. Denoting \( z := e^{ik} \) this leads to

\[
0 = (z^{L+1} - z^{-(L+1)}) - (A + B)(z^L - z^{-L}) + AB(z^{L-1} - z^{-(L-1)})
\]

\[
s_k = -e^{2ik} \frac{A - e^{ik}}{A - e^{-ik}}.
\]

where \( A = 1 - (\alpha_1 + \beta_1) \) and \( B = 1 - (\alpha_L + \beta_L) \). Note that even though this equation has \( 2L + 2 \) solutions for \( z \), only \( L \) on them give rise to eigenvalues with linearly independent eigenvectors: If \( k \) is a solution then also \(-k\) is a solution, but the corresponding eigenvectors are linearly dependent, \( v_{-k} = s_k v_k \). Furthermore \( k = 0 \) is a solution with zero eigenvector. Numerical solutions of C2 show that for \( A, B \in [1, -1] \) all solutions of \( z \) lie on the unit circle and therefore (taking into account that \( k \sim -k \) and \( k \neq 0 \) we have \( k \in (0, \pi) \).

A site in the bulk, \( j \sim aL \), approaches the steady value \( (A^{-1}b)_j \) according to the time dependent part of (C1). Again, since \( \lambda_k < 0 \), there is an exponential decay. Only the smallest solution for \( k \), denoted by \( k^*, \) will contribute. In the limit of large system-size \( L \) one can check that \( k^* \approx \pi/L \) is the smallest positive solution for \( k \). Note that the amplitude \( (v_{k^*})_j \approx e^{i\alpha_k} - e^{-i\alpha_k} \) of this term stays finite \( (s_{k^*} \approx -1) \). Then, the time scale with which the bulk approaches the steady value is, as before, \( t_{\text{decay}} \sim O(L^2) \).

At the boundary, the amplitude of the term corresponding to \( k^* \) will be zero. E.g. for \( j = 1, (v_{k^*})_1 = e^{ik^*} + s_{k^*} e^{-ik^*} = 0 \). To have a non-zero amplitude, one has to consider terms where \( k \sim bL \) with \( b \sim O(1) \), which leads to \( t_{\text{decay}} \sim O(1) \).
Appendix D: Analytic solution of the connected 2-point function

Here we outline how to solve the connected 2-point function analytically, which was needed for the comparison with the solution of the discrete equations in the scaling limit in Fig. (3) and (4).

a. **Solution for the density.** First we construct an analytic solution of the density (11) with domain wall initial condition \( \rho(x,0) = \Theta(1/2-x) \) and boundary conditions \( \rho(0,t) = n_a, \rho(1, t) = n_b \). We simplify the boundary conditions by subtracting the stationary solution, \( \rho_\infty(x) = n_a + x(n_b - n_a) \). Then we solve for \( \hat{\rho}(x,t) = \rho(x,t) - \rho_\infty(x) \), which has easier boundary conditions.

\[
\begin{align*}
(\partial_t - \partial_x^2)\hat{\rho} &= 0 \\
\hat{\rho}(x,0) &= \Theta(1/2 - x) - n_a - x(n_b - n_a) \\
\hat{\rho}(0,t) &= \hat{\rho}(1,t) = 0
\end{align*}
\]

We find the solution by an expansion in \( \{\sin(n\pi x)\}_{n=1}^\infty \). These functions satisfy the correct boundary conditions and are orthogonal in the sense that \( \int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \frac{1}{2} \delta_{nm} \). Importantly, they form a complete basis of \( L^2([0,1]) \) which justifies the expansion.

Taking into account the initial condition, this leads to

\[
\hat{\rho}(x,t) = \sum_{n=1}^\infty c_n \sin(n\pi x) e^{-n^2\pi^2 t}
\]

\[
c_n = \frac{2}{n\pi} (1 - (-1)^n(n_a - n_b) - 2n_a \delta_{n,odd} - (-1)^{n/2} \delta_{n,even}).
\]

In the special case where the boundary conditions match the initial conditions \( (n_a = 1, n_b = 0) \) one finds

\[
\rho(x,t) = 1 - x - \sum_{k=1}^\infty \frac{(-1)^k}{k\pi} \sin(2\pi kx) e^{-4\pi^2 k^2 t}.
\]

b. **Solution for the connected two point function.** In appendix B we saw that the connected two point function \( g_t(x,y) \) satisfies the van-Neumann boundary conditions (B1) on the diagonal \( \{x = y\} \). Here we construct a solution of (15) on the lower triangle \( T^+ = \{(x,y) \in [0,1]^2: x \geq y\} \): First we identify a function that satisfies the boundary conditions,

\[
w(x,y,t) = y(1-x)\partial_x \rho(x,t)\partial_y \rho(y,t).
\]

Note that for \( t \to \infty \), \( w \) becomes the correct stationary solution on \( T^+ \). Then we solve for \( f(x,y,t) := g_t(x,y) - w(x,y,t) \), which satisfies an inhomogeneous heat equation with homogeneous boundary conditions

\[
\begin{align*}
(\partial_t - \Delta) f(x,y,t) &= S(x,y,t) := 2(1-x)\partial_x \rho(x,t)\partial_x^2 \rho(y,t) - 2y\partial_y \rho(x,t)\partial_y \rho(y,t) \\
f(x,y,0) &= -y(1-x)\delta(1/2 - x)\delta(1/2 - y) \\
f(x,0,t) &= 0 = g(1,0,t) \text{ (Dirichlet condition)} \\
\partial_v f|_{x=y} &= 0 \text{ (Neumann condition)}
\end{align*}
\]

with \( \partial_v := (\partial_x - \partial_y)/\sqrt{2} \) as in appendix B. This is solved, as before, by the method of eigenfunction expansion. Note that

\[
\psi_{nm}(x,y) := \sin(n\pi x) \sin(m\pi y) + \sin(m\pi x) \sin(n\pi y)
\]

is a complete basis of \( L^2(T^+) \) that satisfies the correct Dirichlet and Neumann boundary conditions. It satisfies

\[
\int_{T^+} \psi_{nm}(x,y) \psi_{kl}(x,y) dx dy = \frac{\delta_{nk}\delta_{ml} + \delta_{nl}\delta_{mk}}{4}.
\]

We write \( S(x,y,t) = \sum_{n \geq m \geq 1} \hat{S}_{nm}(t) \psi_{nm}(x,y) \) and \( f(x,y,t) = \sum_{n \geq m \geq 1} \hat{f}_{nm}(t) \psi_{nm}(x,y) \), where the coefficients are given by

\[
\hat{S}_{nm}(t) = \begin{cases} 
\frac{4}{t} \int_{T^+} S \psi_{nm} & \text{if } n > m \\
\frac{2}{t} \int_{T^+} S \psi_{nm} & \text{if } n = m.
\end{cases}
\]
This leads to the \( \partial_t \hat{f}_{nm} + \pi^2 (n^2 + m^2) \hat{f}_{nm} = \hat{S}_{nm} \) which is solved by

\[
\hat{f}_{nm}(t) = \hat{f}_{nm}(0) e^{-\pi^2 (n^2 + m^2) t} + \int_0^t e^{\pi^2 (n^2 + m^2) (\tau - t)} \hat{S}_{nm}(\tau) d\tau.
\]

We get a solution \( \hat{f}_{nm} = \sum_{n \geq m \geq 1} \hat{f}_{nm}(t) \psi_{nm} \) of the homogeneous equation that satisfies the initial condition and a solution \( f_{part} = \sum_{n \geq m \geq 1} f_{nm}(t) \psi_{nm} \). One finds

\[
\hat{f}_{hom}(x, y, t) = \sum_{k, l \geq 0} (-1)^{k+l+1} e^{-\pi^2 ((2k+1)^2 + (2l+1)^2) t} \sin((2k + 1)\pi x) \sin((2l + 1)\pi y) \\
= -\frac{1}{4} \partial_x(\pi x, e^{-\pi^2 t}) \partial_y(\pi y, e^{-\pi^2 t}).
\]

The precise expression for \( \hat{S}_{nm} \) is rather complicated, and it is easier to solve for \( f_{part} \) using the Mathematica NDSolve function. The complete solution is then \( g_t(x, y) = w(x, y, t) + \hat{f}_{hom}(x, y, t) + f_{part}(x, y, t) \). By symmetry in \( x \) and \( y \), i.e. \( g_t(x, y) = g_t(y, x) \), this also determines a solution on \( T^- = \{(x, y) \in [0, 1]^2 : x \geq y\} \).

### Appendix E: Proofs

1. Correlation functions as sums over non-crossing partitions (Theorem 1)

The proof of (27) which relates the correlation function \( g_t^c = [G_{i_1, i_2} \cdots G_{i_n, i_1}]^c \) to its connected part \( g_t = [G_{i_1, i_2} \cdots G_{i_n, i_1}] \) is based on only two assumptions that hold in the scaling limit:

- The measure \( E_t \) is \( U(1) \) invariant (we will often denote the measure by \( [ \cdot ] \)).
- Connected correlations on single loops of \( n \) points scale as \( [n]^c \sim \frac{L}{\pi^2} \) and all other connected correlations with equal number of points are sub-leading.

Using the moment-cumulant formula (16), the essential point to show is that cumulants \( E_t [G_{i_1, i_2} \cdots G_{i_n, i_1}]^c \) corresponding to crossing partitions \( \pi \) will be sub-leading compared to non-crossing partitions \( \pi \) in the scaling limit.

a. Scaling of non-crossing partitions. We start the proof by showing that if \( \pi = \{b^{(1)}, \ldots, b^{(m)}\} \) is a non-crossing partition on the edges of a loop with \( n \) nodes then \( [\pi]^c := E_t [G_{i_1, i_2} \cdots G_{i_n, i_1}]^c \) scales as \( L^{-n+1} \) independently of the number of blocks \( m \) – and therefore behaves in the same way as the single loop \( E_t [G_{i_1, i_2} \cdots G_{i_n, i_1}]^c \) with \( n \) points.

First notice, that if the blocks of \( \pi \) are not nested into each other (e.g. \( \pi = \{12, 23\}, \{34, 45, 56, 61\}\) then, by \( U(1) \) invariance, we have to connect starting and endpoints of each block by a \( \delta \)-function. If we arrive at the last block, this conditions is already satisfied due to the other delta functions. We therefore need \( m - 1 \) delta functions,

\[
[\pi]^c = \delta \cdots \delta [b^{(1)}]^c \cdots [b^{(m)}]^c \sim L^{-m+1} L^{-\sum_{i=1}^m |b^{(i)}| + m} \sim L^{-n+1},
\]

where \( \delta \sim L^{-1} \) in the scaling limit and the sum over the size \( |b^{(i)}| \) of each block is equal to the total number of elements \( m \).

Next, assume that \( \pi \) has some nested blocks (e.g. \( \pi = \{12, 23\}, \{45\}, \{34, 56, 61\}\)), then treat each collection of nested blocks as a big block \( B \) (e.g. \( B = \{45\}, \{34, 56, 61\}\)), such that the argument above applies to the non-nested blocks and the big blocks. Now we can iterate the argument for each big block \( B \) and possible collections of nested sub-blocks therein. In the end we are left with \( m - 1 \) delta functions needed to close each block in \( \pi \) to form a loop. This shows that \( [\pi]^c \sim L^{-n+1} \) if \( \pi \) is non-crossing.

b. Scaling of crossing partitions. Now assume that \( \pi \) consists of a collection \( B \) of non-crossing blocks with \( |B| \) elements in total and a collection \( C \) of crossing blocks that cannot be disentangled from another with \( |C| \) elements in total (e.g. \( \pi = \{12, 34\}, \{23, 45\}, \{56\}, \{67\}\), \( B = \{56\}, \{67\}\) and \( C = \{12, 34\}, \{23, 45\}\)). Then we have \( |B| + |C| = n \). If we treat \( B \) as an independent partition of the cyclic set formed by all its elements, then by the argument above, \( |B|^c \sim L^{-|B|+1} \). Alternatively we can view \( B \) as the partition of the full loop from which we removed
all edges that belong to the collection \( C \) of crossing blocks, leaving us with a smaller loop of \( |B| \) edges. If we assume that \( [C]^c \sim L^{-|C|} \) (as will be shown below) then

\[
[\pi]^c = \delta \ [B]^c [C]^c \sim L^{-1} L^{-|B|+1} L^{-|C|} \sim L^{-n},
\]

where only a single delta function is needed to connect the collection \( B \) and \( C \) (because all the other delta functions are already included in \([B]^c\) and \([C]^c\). If \( \pi \) has more than one collection \( C \) of crossing blocks that cannot be disentangled, then it will scale with even higher negative power of \( L \). This shows that crossing partitions are sub-leading in the scaling limit compared to non-crossing partitions.

It remains to show that \( [C]^c \sim L^{-|C|} \) (probably it is even true that \( [C]^c \sim L^{-|C|-1} \). The idea is to produce a collection of crossing blocks starting with non-crossing blocks and permuting its elements. Assume that \{\( b^{(1)}, b^{(2)} \)\} is a collection of two non-crossing blocks

\[
b^{(1)} = \{i_1 i_2, \ldots, i_k i_{k+1}\} \quad \text{and} \quad b^{(2)} = \{i_{k+1} i_{k+2}, \ldots, i_n i_1\}
\]

which would mean that there is only one delta function \( \delta(i_1, i_{k+1}) \) necessary for the product of these blocks to be non-zero. Then construct a first crossing by inserting an element \( i_l i_{l+1} \) from the "middle" of \( b^{(1)} \) (and not from the boundary) into \( b^{(2)} \),

\[
\begin{align*}
b^{(1)} &\to b^{(1)} = \{i_1 i_2, \ldots, i_{l-1} i_l, i_l i_{l+1}, \ldots, i_k i_{k+1}\} \quad \text{and} \quad b^{(2)} &\to b^{(2)} = \{i_{k+1} i_{k+2}, \ldots, i_n i_1\},
\end{align*}
\]

which makes it necessary to insert a second delta function, \( \delta(i_l, i_{l+1}) \). In this way one can continue to permute elements between the two blocks, create more crossings and obtain \( C = \{b^{(1)}, b^{(2)}\} \) of sizes \( |b^{(1)}| = k' \) and \( |b^{(2)}| = n - k' \). Each new crossing necessitates a new delta function. Note, however, that a new crossing is only created, if one permutes an element that (a) is still in its original block, (b) whose neighbours have not yet been permuted (we sort the elements \( i_l i_{l+1} \) in ascending order with respect to the index \( l \)) and (c) is not taken from the boundary of the block. Since no correlation function is more dominant than single loops, the two blocks, after an arbitrary permutation between them, will scale at most as \( [b^{(1)}] \sim L^{-k'+1} \) and \( [b^{(2)}] \sim L^{-n-k'+1} \). We therefore have

\[
[C]^c = \sum_{\#\text{crossings}+1} \delta \ [b^{(1)}]^c [b^{(2)}]^c \sim L^{-n+1-\#\text{crossings}},
\]

As a consequence, in the case where \( C \) consists of two crossing blocks, this shows that it scales at most with \([C]^c \sim L^{-|C|}\) where \( |C| = n \) in our example.

If \( C \) is a collection of \( m \) crossing blocks it is probably still true that the number of delta functions is equal to \#crossings + 1. However, all we need here is that \( m \) crossing blocks will cause at least \( m \) delta functions. Then the scaling is

\[
[C]^c = \sum_{m} \delta \ [b^{(1)}]^c [b^{(m)}]^c \sim L^{-|C|},
\]

is sufficient. We saw that 2 crossing blocks cause at least 2 delta functions. But each originally non-crossing block that we add to the collection of crossing blocks already comes with one delta function, even before we permute its elements with the other blocks. For example, if we have the crossing blocks \{\{12, 34\}, \{23, 41\}\} which need 2 delta functions \( \delta(2, 3) \delta(1, 4) \) and we add a block, we have \{\{12, 34\}, \{23, 45\}, \{56, 61\}\}, which needs 3 delta functions, \( \delta(2, 3) \delta(1, 4) \delta(4, 5) \). If we now cross the new block with the others this cannot reduce the number of delta functions. Therefore, this argument shows that \( m \) crossing blocks indeed cause \( m \) delta functions and concludes the proof.

2. Time evolution of connected correlations (Theorem 2)

We will show how (28) follows from (12) by induction over \( n \). We start by writing the sum \( 2 \sum_{i<j} = \sum_{ij \neq i} \) in these equations as a sum over all ordered, non-crossing sets \( r = \{r_1, r_2, \ldots\} \) and \( s = \{s_1, s_2, \ldots\} \) with \( i = r_1 \) and \( j = s_1 \) such that \( r \sqcup s = [n] \equiv \{x_1, \ldots, x_n\} \). Throughout this paper we the symbol \( \sqcup \) to denote the union of ordered,
non-crossing subsets. Furthermore, instead of $g_t(x_1, x_2, ..., x_{j-1})$ we simply write $g(r)$ if $r = \{x_1, x_2, ..., x_{j-1}\}$. Then the two equations become,

$$ (\partial_k - \Delta)g_t^k([n]) = \sum_{r \cup s = [n]} \partial_k \partial_s (\delta(r_1, s_1)g_t^r(r)g_t^s(s)), \quad (E1) $$

and

$$ (\partial_k - \Delta)g_t([n]) = \sum_{r \cup s = [n]} \delta(r_1, s_1)\partial_k g_t(r)\partial_s g_t(s). \quad (E2) $$

For $n = 1$ the two equations are identical. Let’s assume that (E2) holds for all $k \leq n - 1$ for some $n \in \mathbb{N}$. We will use (E1) to show that it holds also for $k = n$.

Recall from (27) that the $g_t^k$ can be expanded into $g_t$ as a sum over non-crossing partitions $\pi$,

$$ g_t^k([n]) = \sum_{\pi \in NC([n])} \prod_{d \in \pi^*} \delta(d) \prod_{b \in \pi} g_t(b). \quad (E3) $$

Then we can evaluate the right hand side of (E1)

$$ rhs = \sum_{r \cup s = [n]} \partial_k \partial_s \sum_{\rho \in NC(r)} D_{\rho} \sum_{\sigma \in NC(s)} D_{\sigma} \sum_{\pi \in NC([n])\setminus[n]} \prod_{d \in \pi^*} \partial_x \partial_y D_{\pi}, \quad (E4) $$

where $(x, y) \in d$ denotes all tuples with $x \neq y$. To see the second equal sign, one has to establish the bijection

$$ \{\delta(r_1, s_1)D_{\rho}D_{\sigma} \mid \rho \in NC(r), \sigma \in NC(s), s \cup r = [n]\} $$

$$ \longleftrightarrow $$

$$ \{(D_{\pi}, x, y) \mid \pi \in NC([n])\setminus[n], (x, y) \in d \text{ st. } d \in \pi^*\}. $$

“→”: One draws the two diagrams $D_{\rho}$ and $D_{\sigma}$ inside a single big loop with nodes $\{r_1, r_2, ..., s_1, s_2, ...\} = [n]$ and connects $r_1$ and $s_1$ by a dotted line to represent the delta function $\delta(r_1, s_1)$. This is equal to $D_{\pi}$ with two marked nodes $x = r_1$ and $y = s_1$, where $\pi = \rho \cup \sigma$ is a partition on the big circle with at least two elements — therefore excluding the partition $\pi = \{[n]\}$.

“←”: One starts with $(D_{\pi}, x, y)$. Since the two marked nodes $(x, y) \in d \in \pi^*$ belong to a block of the dual partition, one can cut the diagram at these two nodes without breaking any block $b \in \pi$. The cut therefore defines independent non-crossing partitions $\rho$ and $\sigma$ on the two non-crossing subsets $r \cup s = [n]$ where $r_1 = x$ and $s_1 = y$.

![Diagram](image)

With the help of (E4) and (E3) we can rewrite (E1) and isolate the connected correlation function $g([n])$, which is the term we are aiming for,

$$ (\partial_k - \Delta)g_t([n]) = \sum_{\pi \in NC([n])\setminus[n]} \left( -(\partial_k - \Delta)D_{\pi} + \sum_{d \in \pi^* \setminus \{(x, y) \in d\}} \partial_x \partial_y D_{\pi} \right). \quad (E6) $$
The only information which is missing is the action of \((\partial_t - \Delta)\) on \(D_{\pi}\). We claim that

\[
(\partial_t - \Delta)D_{\pi} = \sum_{(x,y) \in d \in \pi^*} (\partial_x \partial_y - \partial_x^2 \partial_y)D_{\pi} + \sum_{(x,y) \in b \in \pi} \partial_x^2 \partial_y^2 D_{\pi \backslash (b \cup b_1(x,y) \cup b_2(x,y))},
\] (E7)

the derivation of which comes at the end of this section. Let us explain the arising terms:

- The first term is a sum over all blocks \(d\) of the dual partition \(\pi^*\), from which we choose all possible tuples \((x,y)\) with \(x \neq y\). The symbol \(\partial_x^2 \partial_y\) means that the derivative only acts on the \(g\)'s and not the delta functions that appear in \(D_{\pi}\), while \(\partial_x\) acts on both.

- The second term is a sum over all blocks \(b \in \pi\), from which we choose all possible tuples of edges \((x,y)\) with \(x \neq y\). By the convention (26), we denote the neighbouring nodes by the same name. This allows us to cut the block \(b\) of edges along the nodes \((x,y)\). The two resulting blocks are denoted by \(b_1(x,y)\) and \(b_2(x,y)\). The partition \(\pi \backslash b \cup b_1(x,y) \cup b_2(x,y)\) is the one where \(b\) was removed and replaced by the two blocks \(b_1(x,y)\) and \(b_2(x,y)\).

Note that the term with \(\partial_x \partial_y\) is cancelled once we plug (E7) into (E6). This equation simplifies further if we do the sum over all \(\pi \in NC([n]) \backslash [n]\),

\[
\sum_{\pi \in NC([n]) \backslash [n]} \left( - \sum_{(x,y) \in d \in \pi^*} \partial_x^2 \partial_y^2 D_{\pi} + \sum_{(x,y) \in b \in \pi} \partial_x^2 \partial_y^2 D_{\pi \backslash (b \cup b_1(x,y) \cup b_2(x,y))} \right) = - \sum_{\tau \cup s = [n]} \partial_1^2 \partial_2^2 D_{(r,s)}.
\] (E8)

Let us explain this: Since \(\pi \neq \{[n]\}\) one can always find a block \(d \in \pi^*\) that consists of at least two nodes \((x,y)\). We can join the two blocks \(b(x) \in \pi\) and \(b(y) \in \pi\) to which the corresponding edges \(x\) and \(y\) belong. This forms a block \(b\) of a new partition \(\pi'\) which differs from \(\pi\) only by this block, \(\pi' \backslash b \cup b(x) \cup b(y) = \pi\). Note, that if \(\pi\) consists of only two blocks, then \(\pi'\) will consist of a single block, so \(\pi' = \{[n]\}\). There is hence a bijection

\[
\{(D_{\pi}, x, y) \mid \pi \in NC([n]) \backslash [n], (x, y) \in d \text{ st. } d \in \pi^*\} 
\leftrightarrow 
\{\pi' \backslash b \cup b_1(x, y) \cup b_2(x, y) \mid \pi' \in NC[n], (x, y) \in b \in \pi'\}.
\]

But \(\pi' = [n]\) is not available in the above sum, therefore leaving all those terms

\[- \sum_{(x,y) \in d \in \pi^*} \partial_x^2 \partial_y^2 D_{\pi}\]

uncancelled, where \(\pi\) consists of only two blocks. Then we can write \(\pi = \{r, s\} \text{ where } r \cup s = [n]\), which establishes the above equality.

Equation (E6) then simplifies to

\[
(\partial_t - \Delta)g_t([n]) = \sum_{\tau \cup s = [n]} \partial_{r1}^2 \partial_{s2}^2 D_{(r,s)} = \sum_{\tau \cup s = [n]} \delta(r_1, s_1) \partial_{r1} g_t(r) \partial_{s1} g_t(s),
\] (E9)

which is what we wanted to show.

**Proof of (E7).** Recall that our notation \((x,y) \in b\), for \(b\) any block of a partition, assumes that \(x \neq y\). Two identities about \(\delta\)-functions of several variables that we will need are

\[
\sum_{x \in [n]} \partial_x \delta([n]) = 0
\] (E10)

\[
\Delta_{[n]} \delta([n]) = - \sum_{(x,y) \in [n]} \sum_{x \neq y} \partial_x \partial_y \delta([n])
\] (E11)
Using the subscript of the Laplacian denotes the set on which it acts. For example, \( \Delta_b = \sum_{x \in b} \partial_x^2 \) for some set \( b \subseteq [n] \). For \( \pi \in NC([n]) \setminus \odot \) one finds,

\[
(\partial_t - \Delta) D_\pi = -\left( \Delta \prod_{d \in \pi^*} \delta(d) \prod_{b \in \pi} g_t(b) - 2 \sum_{x \in [n]} \left( \partial_x \prod_{d \in \pi^*} \delta(d) \right) \left( \partial_x \prod_{b \in \pi} g_t(b) \right) \right) \tag{I}
+ \prod_{d \in \pi^*} \delta(d)(\partial_t - \Delta) \prod_{b \in \pi} g_t(b). \tag{II}
\]

Using \( \sum_{x \in [n]} = \sum_{e \in \pi^*} \sum_{x \in e} \),

\[
(\partial_t - \Delta) D_\pi = -\sum_{e \in \pi^*} \prod_{d \in \pi^* \setminus e} \delta(d) \left( \Delta_t \delta(e) \prod_{b \in \pi} g_t(b) + 2 \sum_{x \in e} \partial_x \delta(e) \partial_x \prod_{b \in \pi} g_t(b) \right). \tag{I}
\]

With the help of the \( \delta \)-function identities (E10) and (E11),

\[
(\partial_t - \Delta) D_\pi = \sum_{e \in \pi^*} \prod_{d \in \pi^* \setminus e} \delta(d) \left( \sum_{(x,y) \in e} \left( \partial_x \delta(y) \right) \prod_{b \in \pi} g_t(b) + 2 \sum_{(x,y) \in e} \partial_y \delta(e) \partial_x \prod_{b \in \pi} g_t(b) \right), \tag{II}
\]

which, rewritten in terms of \( D_\pi \), is the first term that appears in (E7),

\[
(\partial_t - \Delta) D_\pi = \sum_{e \in \pi^*} \sum_{(x,y) \in e} (\partial_x \delta(y) - \partial_x \partial_y \delta(e)) D_\pi. \tag{I}
\]

For the second term we need to use (E2), which by assumption holds for all \( k \leq n - 1 \). Since \( \pi \in NC([n]) \setminus \odot \) the assumption applies,

\[
(\partial_t - \Delta) D_\pi = \prod_{d \in \pi^*} \delta(d) \sum_{e \in \pi} (\partial_t - \Delta_e) g_t(c) \prod_{b \in \pi \setminus c} \prod_{b \in \pi} g_t(b) \tag{II}
= \prod_{d \in \pi^*} \delta(d) \sum_{e \in \pi} \sum_{r \cup s = c} \delta(r_1, s_1) \partial_{r_1} g_t(r) \partial_{s_1} g_t(s) \prod_{b \in \pi \setminus c} g_t(b).
\]

Instead of summing over \( r \cup s = c \), we can also sum over \( (x,y) \in c \) (where as usual \( x \neq y \)). In this case \( r \) and \( s \) are the blocks \( b_1(x,y) \) and \( b_2(x,y) \) that result from cutting \( c \) along the nodes \( (x,y) \),

\[
(\partial_t - \Delta) D_\pi = \prod_{d \in \pi^*} \delta(d) \sum_{e \in \pi} \sum_{(x,y) \in e} \delta(x,y) \partial_x g_t(b_1(x,y)) \partial_y g_t(b_2(x,y)) \prod_{b \in \pi \setminus c} g_t(b). \tag{II}
\]

The expression can therefore be rewritten in terms of the partition \( \pi \setminus c \cup b_1(x,y) \cup b_2(x,y) \) and provides the second term in (E7),

\[
(\partial_t - \Delta) D_\pi = \sum_{e \in \pi} \sum_{(x,y) \in e} \partial_x \partial_y D_\pi \prod_{c \cup b_1(x,y) \cup b_2(x,y)} g_t(b). \tag{II}
\]

This concludes the derivation of the time evolution equation of connected correlation functions.
3. Time evolution of the new measure (Theorem 3)

Here, we present the proof of the evolution equation (31) of the new measure $\varphi_t$. We follow the explanation before (E2) to rewrite the evolution of the connected correlations $g_t$ as

$$(\partial_t - \Delta)g_t([n]) = \sum_{r \sqcup s = [n]} \delta(r_1, s_1) \partial_{r_1} g_t(r) \partial_{s_1} g_t(s),$$  \hspace{1cm} (E12)$$

where $r \sqcup s = [n]$ denotes the union of non-crossing subsets $r = \{r_1, r_2, \cdots\}$ and $s = \{s_1, s_2, \cdots\}$ of the set $[n] = \{x_1, \cdots, x_n\}$. The proof of (31) is by induction over $n$. For $n = 1$, $\varphi_t$ and $g_t$ are identical and therefore satisfy the same equation. Assume the formula holds for all $n \leq k - 1$ for some $k \in \mathbb{N}$.

We start by evaluating its left hand side making use of (E12) (for better readability we suppress write the time argument)

$$(\partial_t - \Delta)\varphi([n]) = \sum_{\pi \in NC([n])} \sum_{c \in \pi} (\partial_t - \Delta_c)g(c)g_{\pi \setminus c}$$

$$(E13) = \sum_{\pi \in NC([n])} \sum_{c \sqcup a' = c} \delta(a_1, a'_1)\partial_{a_1} g_t(a)\partial_{a'_1} g_t(a')g_{\pi \setminus c}.$$  \hspace{1cm} (E14)$$

Here $\Delta_c = \sum_{x \in c} \Delta x$ and $\pi \setminus c$ is the partition $\pi$ without the block $c$.

We continue with the right hand side of (E12),

$$\sum_{r \sqcup s = [n]} \delta(r_1, s_1) \partial_{r_1} \varphi(r) \partial_{s_1} \varphi(s) = \sum_{r \sqcup s = [n]} \delta(r_1, s_1) \left( \partial_{r_1} \sum_{\rho \in NC(r)} g_\rho \partial_{s_1} \sum_{\sigma \in NC(s)} g_\sigma \right)$$

$$(E14) = \sum_{\rho \in NC(r)} \sum_{\sigma \in NC(s)} \delta(r_1, s_1) \partial_{r_1} g_t(b(r_1)) \partial_{s_1} g_t(b(s_1)) g_\rho \cap b(r_1) \cap g_\sigma \cap b(s_1).$$

where the blocks $b(r_1) \in \rho$ and $b(s_1) \in \sigma$ are uniquely defined by the fact that $r_1 \in b(r_1)$ and $s_1 \in b(s_1)$.

Note that terms appearing in the sum in (E14) and (E14) agree iff $a = b(r_1)$, $a' = b(s_1)$ and $\pi \setminus c = \rho \sqcup b(r_1) \sqcup \sigma \sqcup b(s_1)$. To show equivalence between the two sums, we should show that there is a bijection between the sets

$$\{(a, a', \pi) \mid \pi \in NC([n]), a \sqcup a' = c \in \pi\}$$

$$\longleftrightarrow \{(r_1, s_1, \rho, \sigma) \mid \rho \in NC(r), \sigma \in NC(s), r \sqcup s = [n]\}.$$  \hspace{1cm} (E15)$$

"\rightarrow": Given $\pi \in NC([n]), a \sqcup a' = c \in \pi$, this completely specifies a terms in the sum in (E12). Define $r_1 = a_1$ and $s_1 = a'_1$. By the condition that $r \sqcup s = [n]$ is non-crossing, this uniquely defines the sets $r$ and $s$. Since $\pi$ is non-crossing, once we take away the “connecting block” $c$, $\pi \setminus c$ factorizes in a unique way into two non-crossing partitions $\hat{\rho}$ of $r \setminus a$ and $\hat{\sigma}$ of $s \setminus a'$. That is $\pi \setminus c = \hat{\rho} \sqcup \hat{\sigma}$. From these we can define $\rho = \hat{\rho} \cup \{a\}$ and $\sigma = \hat{\sigma} \cup \{a'\}$ such that indeed, $\pi \setminus c = \rho \setminus a \sqcup \sigma \setminus a'$. This produces a corresponding term in the sum of (E14), which is completely specified by the data $(r_1, s_1, \rho, \sigma)$.

"\leftarrow": Given $\rho \in NC(r), \sigma \in NC(s), r \sqcup s = [n]$, we define $a = b(r_1)$, $a' = b(s_1)$ and $c = a \sqcup a'$. It remains to construct $\pi \in NC([n])$ such that $\pi \setminus c = \rho \setminus a \sqcup \sigma \setminus a'$. This is achieved by defining $\pi := \rho \sqcup \sigma \sqcup \{c\}$.

4. Steady measure (Theorem 4)

Here, we prove that $\min(x_1, ..., x_n)$ is a steady solution.

We represent the minimum by a sum of Heaviside-functions $\Theta(\text{condition})$ which are one if the condition is true and zero otherwise,

$$\min(x_1, ..., x_n) = \sum_{i=1}^{n} x_i \Theta(x < \{x_1, \cdots, \hat{x}_i, \cdots, x_n\}).$$
The hat on $\hat{x}_i$ suggests that $x_i$ is missing from the set $\{x_1, \ldots, x_n\}$. We have $\partial_i \min(x_1, \ldots, x_n) = \Theta(x_i < \{x_1, \ldots, \hat{x}_i, \ldots, x_n\})$. Furthermore, derivatives of the Heaviside function evaluate to

$$\partial_i \Theta(x_i < \{x_1, \ldots, \hat{x}_i, \ldots, x_n\}) = -\sum_{j \neq i} \delta(x_i, x_j) \Theta(x_i < \{x_1, \ldots, \hat{x}_i, \hat{x}_j, \ldots, x_n\}),$$

$$\partial_j \Theta(x_i < \{x_1, \ldots, \hat{x}_i, \ldots, x_n\}) = \delta(x_i, x_j) \Theta(x_j < \{x_1, \ldots, \hat{x}_i, \hat{x}_j, \ldots, x_n\}).$$

With these formulas, it is easy to check that

$$-\Delta \min(x_1, \ldots, x_n) = \sum_{i,j} \delta(x_i, x_j) \Theta(x_j < \{x_1, \ldots, \hat{x}_i, \hat{x}_j, \ldots, x_n\}).$$

Furthermore, we have

$$\delta(x_i, x_j) \Theta(x_j < \{x_1, \ldots, \hat{x}_i, \hat{x}_j, \ldots, x_n\}) = \delta(x_i, x_j) \Theta(x_i < \{x_{i+1}, \ldots, \hat{x}_j\}) \Theta(x_j < \{x_{j+1}, \ldots, \hat{x}_{i-1}\}).$$

As a consequence,

$$-\Delta \min(x_1, \ldots, x_n) = \sum_{i,j} \delta(x_i, x_j) \partial_i \min(x_1, \ldots, x_{j-1}) \partial_j \min(x_j, \ldots, x_{i-1}),$$

This is all we needed to show the claim.

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