RESTRICTED TESTING FOR THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

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ABSTRACT. We answer a special case of a question of T. Hytönen regarding the two weight norm inequality for the maximal function $M$ in the affirmative, namely that there is a constant $D > 1$, depending only on dimension $n$, such that the norm inequality

$$\int_{\mathbb{R}^n} M(f)^2 \, d\omega \leq C \int_{\mathbb{R}^n} f^2 \, d\sigma$$

holds for all $f \geq 0$ if and only if the $A_2$ condition holds, and the testing condition

$$\int_Q M(1_Q^2) \, d\omega \leq C |Q|_\sigma$$

holds for all cubes $Q$ satisfying $|2Q|_\sigma \leq D |Q|_\sigma$.

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1. INTRODUCTION

We begin with a brief history of ‘testing conditions’ as used in this paper. One of the earliest uses of testing conditions to characterize a weighted norm inequality occurs in the 1982 paper [Saw3] on the maximal function $M$ that showed

$$\int_{\mathbb{R}^n} M(f)^2 \, w(x) \, dx \leq C \int_{\mathbb{R}^n} f(x)^2 \, v(x) \, dx,$$

for all $f \geq 0$, if and only if the following testing condition holds:

$$\int_{\mathbb{R}^n} M(1_Q)^2 \, w(x) \, dx \leq C \int_Q v^{-1}(x) \, dx,$$  for all cubes $Q$ in $\mathbb{R}^n$.

Thus it suffices to test the weighted norm inequality over the simpler collection of test functions $f = 1_Q v^{-1}$ for cubes $Q$.

Two years later, David and Journé showed in their celebrated $T1$ theorem [DaJo], that the unweighted inequality

$$\int_{\mathbb{R}^n} T(f)^2 \, dx \leq C \int_{\mathbb{R}^n} f(x)^2 \, dx,$$

holds if and only if both a weak boundedness property and the following pair of dual testing conditions held:

$$\int_{\mathbb{R}^n} T(1_Q)^2 \, dx \leq C \int_Q dx$$

and

$$\int_{\mathbb{R}^n} T^*(1_Q)^2 \, dx \leq C \int_Q dx,$$  for all cubes $Q$ in $\mathbb{R}^n$.  

Here $T$ is a general Calderón-Zygmund singular integral on $\mathbb{R}^n$ and the testing functions are simply the indicators $1_Q$ for cubes $Q$. The following year David, Journé and Semmes extended the $T1$ theorem to a $Tb$ theorem \cite{DaJoSe} in which the testing conditions become $b1_Q$ and $b^*1_Q$ for appropriately accretive functions $b$ and $b^*$ on $\mathbb{R}^n$.

A couple of decades later, and motivated by the Painlevé problem of characterizing removable singularities for bounded analytic functions, Nazarov, Treil and Volberg solved in 2003 a particular one-weight formulation of the norm inequality for Riesz transforms $R$, including the Cauchy transform $Cg(z) \equiv \int_C \frac{1}{w-z}g(w)\,dw$ \cite{NTV},

$$\int_{\mathbb{R}^n} |R(f\mu)(x)|^2\,d\mu(x) \leq C \int_{\mathbb{R}^n} f(x)^2\,d\mu(x), \quad \text{for all} \ f \in L^2(\mathbb{R}^n;\mu),$$

if and only if a weak boundedness property and the following testing condition held:

$$\int_Q |R(1_{Q\mu})(x)|^2\,d\mu(x) \leq C \int_Q d\mu(x), \quad \text{for all cubes} \ Q \ in \ \mathbb{R}^n.$$

Here the testing functions are $f = 1_Q$. The Painlevé problem was solved in the same year by Tolsa \cite{Tol}, a culmination of an illustrious body of work by many mathematicians.

Finally, building on the work of Nazarov, Treil and Volberg in their 2004 paper \cite{NTV4} on the Hilbert transform, that in turn used the random dyadic grids of \cite{NTV} (that followed on those of Fefferman and Stein \cite{FeSe}, Garnett and Jones \cite{GaJo}, and Sawyer \cite{Saw3}), and the weighted Haar wavelets of \cite{NTV} (that followed on those of Coifman, Jones and Semmes \cite{CoJoSe}), the two weight norm inequality for the Hilbert transform was characterized in 2014 in the two-part paper Lacey, Sawyer, Shen and Uriarte-Tuero \cite{LaSaShUr3} - Lacey \cite{Lac} as follows:

$$\int_{\mathbb{R}^n} H(f\sigma)(x)^2\,d\omega(x) \leq C \int_{\mathbb{R}^n} f(x)^2\,d\sigma(x), \quad \text{for all} \ f \in L^2(\mathbb{R}^n),$$

if and only if both the strong Muckenhoupt $A_2$ condition

$$A_2(\sigma,\omega) \equiv \int_{\mathbb{R}} \frac{\ell(I)}{\ell(I) + |x-c_I|}^2\,d\omega(x) \cdot \int_{\mathbb{R}} \frac{\ell(I)}{\ell(I) + |x-c_I|}^2\,d\sigma(x) < \infty,$$

and the following dual testing conditions hold:

$$\int_Q H(1_{I\sigma})(x)^2\,d\omega(x) \leq A \int_Q d\sigma \text{ and } \int_Q H(f\omega)(x)^2\,d\sigma(x) \leq C \int_Q d\omega, \quad \text{for all intervals} \ I.$$

The extension to permitting common point masses in the measure pair $(\sigma,\omega)$ was added shortly after by Hytönen \cite{Hyt2}, where again the weighted norm inequality is tested over indicator functions $f = 1_I$ of intervals $I$. The two-weight inequality for the $g$ function was then characterized by testing conditions in \cite{LaLi}, and a further extension to a $Tb$ theorem for the Hilbert transform is in \cite{SaShUr3}.

**Point of departure:** The point of departure for the present paper begins with an observation of T. Hytönen, namely that in the one-weight formulation above of the norm inequality for Riesz transforms by Nazarov, Treil and Volberg, their testing condition is

$$\int_Q |R(1_{Q\mu})(x)|^2\,d\mu(x) \leq C \int_{2Q} d\mu(x), \quad \text{for all cubes} \ Q \ in \ \mathbb{R}^n,$$

where the double $2Q$ of the cube $Q$ appears on the right hand side. Moreover, one may restrict the testing functions to those functions $f = 1_Q$ for which $Q$ is a $\mu$-doubling cube for some appropriate positive constant $D$.

$$\int_{2Q} d\mu \leq D \int_Q d\mu.$$

This then motivated Hytönen to ask to what extent one can similarly restrict testing functions to doubling cubes for classical operators in other two-weight situations, including those discussed above.

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1. The moniker ‘$T1$ theorem’ refers to the equivalent formulation of the testing conditions as $T1 \in BMO$ and $T^*1 \in BMO$.

2. This philosophy was successfully carried out in the context of the one-weight $Tb$ theorem for nonhomogeneous square functions by Martikainen, Mourgoglou and Vuorinen in \cite{MaMoVu}.

3. Private communication with the first author circa November 2014.
An initial step in the two-weight setting was taken by the authors in [LiSa], where it was shown that such a restriction to doubling cubes is possible in the two weight norm inequality for fractional integrals. The maximal function $M$ was also considered in [LiSa], but only a much weaker result along these lines was obtained for $M$. The purpose of this paper is to prove the full result for $M$, namely that it suffices to restrict testing to doubling cubes in the two weight norm inequality for $M$.

**Motivation:** Besides the intrinsic interest in minimizing the functions over which an inequality must be tested in order to verify its validity, even a partial resolution of the question of restricted testing for singular integrals has the potential to characterize two weight norm inequalities for such operators - including Riesz transforms in higher dimensions, currently a very difficult open problem, see e.g. [SaShUr7], [LaWi] and [LiSaShUrWi]. Indeed, the non-doubling cubes have traditionally been viewed as the enemy in two weight inequalities for singular integrals, and (the techniques used in) the restriction of the testing conditions to just doubling cubes could help circumvent the difficulty that energy conditions fail to be necessary for two weight inequalities in higher dimensions [Saw3] - the point being that a similarly restricted energy condition could suffice.

Let $P = P^n$ be the collection of cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and side lengths $\ell (Q) \in \{2^j \}_{j \in \mathbb{Z}}$ equal to an integral power of 2. For $Q \in P$ and $\Gamma > 1$, let $\Gamma Q$ denote the cube concentric with $Q$ but having $\Gamma$ times the side length, $\ell (\Gamma Q) = \Gamma \ell (Q)$. As mentioned above, the purpose of this paper is to prove an answer to a question of T. Hytönen, in the context of the maximal function $M$. For a locally signed measure $\mu$ on $\mathbb{R}^n$ (meaning the total variation $|\mu|$ of $\mu$ is locally finite), we define the maximal function $M\mu$ of $\mu$ at $x \in \mathbb{R}^n$ by

$$M\mu (x) = \sup_{Q \in P^n : x \in Q} \frac{1}{|Q|} \int_Q |\mu|.$$

Given a pair $(\sigma, \omega)$ of weights (i.e. positive Borel measures) in $\mathbb{R}^n$ and $\Gamma > 1$, we say that $(\sigma, \omega)$ satisfies the $\Gamma$-testing condition for the maximal function $M$ if there is a constant $\mathfrak{F}_M (\Gamma) (\sigma, \omega)$ such that

$$\int_Q |M (1_Q \sigma)|^2 \, d\omega \leq \mathfrak{F}_M (\Gamma) (\sigma, \omega)^2 |\Gamma Q|_\sigma, \quad \text{for all } Q \in P^n,$$

and if so we denote by $\mathfrak{F}_M (\Gamma) (\sigma, \omega)$ the least such constant.

There is also the following weaker testing condition, in which one need only test the inequality over cubes that are 'doubling'. Given a pair $(\sigma, \omega)$ of weights in $\mathbb{R}^n$ and $D, \Gamma > 1$, we say that $(\sigma, \omega)$ satisfies the $D, \Gamma$-testing condition for the maximal function $M$ if there is a constant $\mathfrak{F}_M^D (\Gamma) (\sigma, \omega)$ such that

$$\int_Q |M 1_Q \sigma|^2 \, d\omega \leq \mathfrak{F}_M^D (\Gamma) (\sigma, \omega)^2 |Q|_\sigma, \quad \text{for all } Q \in P^n \text{ with } |\Gamma Q|_\sigma \leq D |Q|_\sigma,$$

and if so we denote by $\mathfrak{F}_M^D (\Gamma) (\sigma, \omega)$ the least such constant. Note that the $\Gamma$-testing condition implies the $D, \Gamma$-testing condition for all $D > 1$.

As shown in [LiSa], these restricted testing conditions are not by themselves sufficient for the norm inequality - the classical Muckenhoupt condition is needed as well:

$$A_2 (\sigma, \omega) \equiv \sup_{Q \in P^n} \left( \frac{1}{|Q|} \int_Q d\sigma \right) \left( \frac{1}{|Q|} \int_Q d\omega \right) < \infty.$$

Finally we let $\mathfrak{M}_M (\sigma, \omega)$ be the operator norm of $M$ as a mapping from $L^2 (\sigma) \to L^2 (\omega)$, i.e. the best constant $\mathfrak{M}_M (\sigma, \omega)$ in the inequality

$$\int_{\mathbb{R}^n} |M (f \sigma)|^2 \, d\omega \leq \mathfrak{M}_M (\sigma, \omega)^2 \int_{\mathbb{R}^n} |f|^2 \, d\sigma, \quad \text{for all } f \in L^2 (\sigma).$$

**Theorem 1.** Let $\Gamma > 1$. Then there is $D > 1$ depending only on $\Gamma$ and the dimension $n$ such that

$$\mathfrak{M}_M (\sigma, \omega) \approx \mathfrak{F}_M^D (\Gamma) (\sigma, \omega) + \sqrt{A_2 (\sigma, \omega)},$$

for all locally finite positive Borel measures $\sigma$ and $\omega$ on $\mathbb{R}^n$.\footnote{The supremum over $Q \in P^n$ used here is pointwise equivalent to the usual supremum over all cubes $Q$ with sides parallel to the coordinate axes.}
Remark 1. An inspection of the proof (Step 2 in Section 4) shows that the supremum over cubes \( Q \) in the testing constant \( \mathcal{T}_M^\Gamma(\sigma, \omega) \) in Theorem 1 may be further restricted to those cubes \( Q \) having null boundary, i.e. \( |\partial Q|_{\sigma+\omega} = 0 \) (cf. the one-weight theorem in [MaMoVu] where this type of reduction first appears).

The proof of this theorem splits neatly into two parts. In the first part of the proof, we adapt the argument in our previous paper [LiSa] to handle the difficulties arising when a triple cube spills outside a supercube - and this requires a careful application of a probabilistic argument of the type pioneered by Nazarov, Treil and Volberg ([NTV]). With this accomplished, the sufficiency of the stronger \( \Gamma \)-testing condition (1.1) is already proved. In the second part of the proof we use this interim result to establish an a priori bound on the operator norm \( \mathcal{M}_\Gamma(\sigma, \omega) \) in order to absorb additional terms arising from the absence of any testing condition at all in (1.2) when the cubes are not doubling - and this requires a reduction to mollifications of the measures \( \sigma \) and \( \omega \). As a consequence of this splitting, we will give the proof in two stages, beginning with the proof of the following weaker theorem, which requires probability, but not mollification, and which is then used to prove our main result Theorem 1. We emphasize that this paper is self-contained, and in particular does not rely on results from our earlier paper [LiSa].

Theorem 2. For \( \Gamma > 1 \) we have

\[
\mathcal{M}_\Gamma(\sigma, \omega) \approx \mathcal{T}_M(\Gamma)(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)},
\]

for all pairs \((\sigma, \omega)\) of locally finite positive Borel measures on \( \mathbb{R}^n \), and where the implicit constants of comparability depend on both \( \Gamma \) and dimension \( n \).

For convenience we will restrict our proof of Theorem 2 to the case \( \Gamma = 3 \), the general case of \( \Gamma \) large being an easy modification of this one.

2. Preliminaries

Here we introduce some standard tools we will use in the proof of Theorem 2.

2.1. Dyadic grids and conditional probability. In this subsection we introduce two parameterizations of grids, explain the conditional probability estimates we will need, and recall how the maximal function is controlled by an average over dyadic maximal functions.

To set notation we begin with the standard family of random dyadic grids \( G \) on \( \mathbb{R}^n \). Let

\[
D_0 := \{2^j([0,1]^n + k), j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.
\]

Then for \( \beta = \{\beta_j\}_{j=-\infty}^\infty \in \{\{0,1\}^n\}^\mathbb{Z} \), define

\[
D^\beta := \left\{ Q + \sum_{j: 2^{-j} \leq \ell(Q)} 2^{-j} \beta_j, Q \in D_0 \right\}.
\]

Denote by \( P_\Omega \) the natural probability measure on \( \Omega := (\{0,1\}^n)^\mathbb{Z} \), which we identify with the corresponding collection of grids \( G = \{D^\beta\}_{\beta \in \Omega} \), i.e. we write \( \Omega = G \). We will use grids in \( \Omega = \{D^\beta\}_{\beta \in (\{0,1\}^n)^\mathbb{Z}} \) to construct Whitney collections \( \mathcal{W}^\beta \) of cubes relative to a monotone family of open sets in Subsection 2.1 below. In probability calculations, we will use truncated versions of these grids. More precisely, given \( D = D^\beta \) with \( \beta \in \Omega \), and \( M, N \in \mathbb{Z} \) with \( N \leq M \), define the associated ‘truncated’ grid

\[
D_M^N := \{ Q \in D = D^\beta : 2^{-M} \leq \ell(Q) \leq 2^{-N} \}.
\]

Thus each \( D_M^N \) is a grid on the set of cubes \( \{Q \in D : 2^{-M} \leq \ell(Q) \leq 2^{-N}\} \). In particular, \( D_M^M = \{ Q \in D : \ell(Q) = 2^{-M} \} \) is the tiling of \( \mathbb{R}^n \) by the dyadic cubes in \( D \) of side length \( 2^{-M} \). If \( D, E \in \Omega \) (using our identification of \( \Omega \) with \( G \) above), then \( D \cap E \neq \emptyset \) if and only if \( D_M^M = E_M^M \) for some \( M \in \mathbb{Z} \), in which case \( D_M^{M'} = E_M^{M'} \) for all \( M' \in \mathbb{Z} \) with \( M' \geq M \). We will develop further properties of grids of the form \( D_M^N \) in the next subsection, including the fact that there are only finitely many (namely \( 2^{n(M-N)} \)) different grids of the form \( D_M^N \).
2.1.1. Parameterizations of a finite set of dyadic grids. Here we recall two constructions from [SaShUr10] of special collections of truncated grids of cubes - special because the origin is a vertex of any cube in which it is contained. We momentarily fix a large positive integer \( M \in \mathbb{N} \), and consider the tiling of \( \mathbb{R} \) by the family of intervals \( \mathbb{D}_M = \{ I^M_{\alpha} \}_{\alpha \in \mathbb{Z}} \) having side length \( 2^{-M} \) and given by \( I^M_{\alpha} = I_0^M + 2^{-M} \alpha \) where \( I_0^M = [0, 2^{-M}) \).

A dyadic grid \( \mathcal{D} \) built on \( \mathbb{D}_M \) is defined to be a family of intervals \( \mathcal{D} \) satisfying:

1. Each \( I \in \mathcal{D} \) has side length \( 2^{-\ell} \) for some \( \ell \in \mathbb{Z} \) with \( \ell \leq M \), and \( I \) is a union of \( 2^{M-\ell} \) intervals from the tiling \( \mathbb{D}_M \),
2. For \( \ell \leq M \), the collection \( \mathcal{D}_{\{0\}} \) of intervals in \( \mathcal{D} \) having side length \( 2^{-\ell} \) forms a pairwise disjoint decomposition of the space \( \mathbb{R} \),
3. Given \( I \in \mathcal{D}_{\{0\}} \) and \( J \in \mathcal{D}_{\{j\}} \) with \( j \leq i \leq M \), it is the case that either \( I \cap J = \emptyset \) or \( I \subset J \).

We denote the collection of all dyadic grids built on \( \mathbb{D}_M \) by \( \mathbb{A}_M \). We now also momentarily fix an integer \( N \in \mathbb{Z} \) with \( N \leq M \), and consider the collection \( \mathbb{A}^N_M \) of dyadic grids obtained by restricting the grids in \( \mathbb{A}_M \) to containing only intervals of side length at most \( 2^{-N} \). We refer to the dyadic grids in \( \mathbb{A}^N_M \) as (special truncated) dyadic grids built on \( \mathbb{D}_M \) of size \( 2^{-N} \).

**Notation 1.** We denote the collection of all intervals belonging to the grids in \( \mathbb{A}^N_M \) by \( \mathcal{S}^N_M \) (\( \mathcal{S} \) for special), and reserve \( \mathcal{P}^N_M \) (\( \mathcal{P} \) for parallel) for the collection of all intervals \( Q \in \mathcal{P} \) with \( 2^{-M} \leq \ell(Q) \leq 2^{-N} \).

There are now two traditional means of constructing probability measures on collections of such dyadic grids, namely parameterization by choice of parent, and parameterization by translation. We will typically use \( \mathcal{D} \) to denote one of these truncated grids when the underlying parameters \( M \) and \( N \) are understood. Here are the two constructions from [SaShUr10].

**Construction #1:** For any
\[
\beta = \{ \beta_i \}_{i \in \mathbb{Z}} \in \omega^N_M \equiv \{ 0, 1 \}^{\mathbb{Z} \times (0, 1)} \\cap \mathbb{Z},
\]
where \( \mathbb{Z} = \{ \ell \in \mathbb{Z} : N \leq \ell \leq M \} \), define the dyadic grid \( \mathcal{D}_\beta \) built on \( \mathbb{D} \) of size \( 2^{-N} \) by
\[
(2.2) \quad \mathcal{D}_\beta = \left\{ 2^{-\ell} \left( [0, 1) + k + \sum_{i: \ell \leq i \leq M} 2^{-i+\beta_i} \right) \right\}_{N \leq \ell \leq M, k \in \mathbb{Z}}.
\]
Place the uniform probability measure \( \rho^N_M \) on the finite index space \( \omega^N_M = \{ 0, 1 \}^{\mathbb{Z} \times (0, 1)} \), namely that which charges each \( \beta \in \omega^N_M \) equally.

**Construction #2:** Momentarily fix a (truncated) dyadic grid \( \mathcal{D} \) built on \( \mathbb{D} \) of size \( 2^{-N} \). For any
\[
t \in \gamma^N_M \equiv \{ 2^{-m} \mathbb{Z} : |t| < 2^{-N} \},
\]
define the dyadic grid \( \mathcal{D}^t \) built on \( \mathbb{D} \) of size \( 2^{-N} \) by
\[
\mathcal{D}^t \equiv \mathcal{D} + t.
\]
Place the uniform probability measure \( \sigma^N_M \) on the finite index set \( \gamma^N_M \), namely that which charges each multiindex \( t \) in \( \gamma^N_M \) equally.

These constructions are then extended to Euclidean space \( \mathbb{R}^n \) by taking products in the usual way and using the product index spaces \( \Omega^N_M \equiv \omega^N_M \times \cdots \times \omega^N_M \) and \( \Gamma^N_M \equiv \gamma^N_M \times \cdots \times \gamma^N_M \), together with the uniform product probability measures \( \mu^N_M \equiv \rho^N_M \times \cdots \times \rho^N_M \) and \( \nu^N_M \equiv \sigma^N_M \times \cdots \times \sigma^N_M \), where there are \( n \) factors in each product above.

The two probability spaces \( \left( \{ \mathcal{D}_\beta \}_{\beta \in \Omega^N_M}, \mu^N_M \right) \) and \( \left( \{ \mathcal{D}^t \}_{t \in \Gamma^N_M}, \nu^N_M \right) \) are isomorphic since both collections \( \{ \mathcal{D}_\beta \}_{\beta \in \Omega^N_M} \) and \( \{ \mathcal{D}^t \}_{t \in \Gamma^N_M} \) describe the finite set \( \mathbb{A}^N_M \) of all (truncated) dyadic grids \( \mathcal{D} \) built on \( \mathbb{D} \) of size \( 2^{-N} \), and since both measures \( \mu^N_M \) and \( \nu^N_M \) are the uniform measure on this space. The first construction may be thought of as being parameterized by scales - each component \( \beta_i \in \beta = \{ \beta_i \}_{i \in \mathbb{Z}} \in \omega^N_M \) amounting to a choice of the \( 2^n \) possible tilings at level \( i \) that respect the choice of tiling at the level below - and since any grid in \( \mathbb{A}^N_M \) is determined by a choice of scales , we see that \( \{ \mathcal{D}_\beta \}_{\beta \in \Omega^N_M} = \mathbb{A}^N_M \). The second construction may be thought of as being parameterized by translation - each \( t \in \gamma^N_M \) amounting to a choice of translation of the grid \( \mathcal{D} \) fixed in construction #2 - and since any grid in \( \mathbb{A}^N_M \) is determined by any of the intervals at the top level, i.e. with side length \( 2^{-N} \), we see that \( \{ \mathcal{D}^t \}_{t \in \Gamma^N_M} = \mathbb{A}^N_M \) as well, since every interval at the top level in \( \mathbb{A}^N_M \) has the form \( Q + t \) for some \( t \in \Gamma^N_M \) and \( Q \in \mathcal{D} \) at the top level in \( \mathbb{A}^N_M \) (i.e. every cube at the
The top level in \( A^N_M \) is a union of small cubes in \( \mathbb{D}_M \times \cdots \times \mathbb{D}_M \), and so must be a translate of some \( Q \in \mathcal{D} \) by \( 2^{-M} \) times an element of \( \mathbb{Z}_2^n \). Note also that in all dimensions, \#\( \Omega^N_M \) = \#\( \Gamma^N_M \) = \( 2^{n(M-N)} \). We will use \( E_{A^N_M} \) to denote expectation with respect to this common probability measure on the finite set \( A^N_M \).

We will invoke these special collections of truncated grids in order to prove a conditional probability estimate \( \text{4.12} \) below. Then we will take limits as in Lemma \( \text{1} \) below to complete our argument. For this we will use the following observations.

Given a dyadic grid \( \mathcal{D} \in \mathcal{O} \), there is a unique \( s \in [0,2^{-M})^n \) such that

\[
(\mathcal{D} - s)^N_M = \mathcal{D}^N_M - s \in A^N_M,
\]

and so we have the decomposition

\[
\{\mathcal{D}^N_M : \mathcal{D} \in \mathcal{O}\} = \bigcup_{s \in [0,2^{-M})^n} A^N_M + s,
\]

that expresses the fact that the collection of truncations of arbitrary dyadic grids coincides with the collection of translations by a point in \( [0,2^{-M})^n \) of the special collection of truncated grids \( A^N_M \) constructed above.

2.1.2. Conditional probability. Here we consider the finite collection of grids \( A^N_M \) depending on a pair of integers \( M, N \) that was introduced in the previous subsubsection. Fix attention on a given cube \( I \in S^N_M \). Let \( P_{A^N_M} \) and \( E_{A^N_M} \) denote probability and expectation over the family \( A^N_M \) with respect to the measure \( \mu_{\Omega^N_M} \). When we wish to emphasize the variable grid \( \mathcal{D} \) being averaged in \( E_{A^N_M} \) we will include this as a superscript in the notation \( E^\mathcal{D}_{A^N_M} \) (in order to avoid confusion with any other variable grids \( \mathcal{G} \) that might be in consideration). Define the collection

\[
\left( A^N_M \right)_I \equiv \{ G \in A^N_M : I \in G \}
\]

of all dyadic grids \( \mathcal{G} \) in \( A^N_M \) that contain the cube \( I \). Then we claim that for \( D = \sum_{I \in \mathcal{D}} q(I) \) where \( q : S^N_M \rightarrow [0,\infty) \), we have the following identity by Fubini’s theorem:

\[
E_{A^N_M} p = \int_{A^N_M} p(\mathcal{D}) d\mu(\mathcal{D}) = \int_{A^N_M} \left( \sum_{I \in \mathcal{D}} q(I) \right) d\mu^N_M(\mathcal{D}) = \sum_{I \in S^N_M} q(I) \int_{(A^N_M)_I} d\mu^N_M(\mathcal{D}) = \sum_{I \in S^N_M} q(I) P_{A^N_M} \left( \left( A^N_M \right)_I \right).
\]

This identity can be rigorously proved simply by using the construction in the previous subsubsection and writing out explicitly the sums involved. Note however, that we make crucial use of the fact that counting measure on \( S^N_M \) is \( \sigma \)-finite, so that Fubini’s theorem applied\(^5\). Here are the details.

If we consider the parameterization of the family of grids \( \mathcal{D} \) in \( A^N_M \) by scale as above, then the expectation of a quantity \( p(\mathcal{D}) \), defined for all grids \( \mathcal{D} \in A^N_M \), is given by

\[
E_{\Omega} p = \frac{1}{\#\Omega^N_M} \sum_{\mathcal{D} \in \Omega^N_M} p(\mathcal{D}) = \frac{1}{\# A^N_M} \sum_{\mathcal{D} \in A^N_M} p(\mathcal{D}).
\]

A special case arises for a function \( q : S^N_M \rightarrow [0,\infty) \) defined on cubes in \( S^N_M \), if we set

\[
p(\mathcal{D}) = \sum_{I \in \mathcal{D} \cap S^N_M} q(I), \quad \text{for all} \ \mathcal{D} \in A^N_M.
\]

Then with the subset

\[
\Theta^N_M \equiv \{(I, \mathcal{D}) \in S^N_M \times A^N_M : I \in \mathcal{D} \}
\]

\(^5\)The analogous assertion that \( E_{A^N_M} p = \sum_{I \in P^N_M} q(I) P_{A^N_M} \left( \left( \Phi^N_M \right)_I \right) \), where \( \Phi^N_M \) is given by \( \text{4.40} \), fails because counting measure on \( P^N_M \) is not \( \sigma \)-finite, and this explains our ubiquitous use of the finite collections of grids \( \Omega^N_M \).
of the product $S_M^N \times A_M^N$, we can write

\[ E_{\Omega_M^N} p = \frac{1}{\# A_M^N} \sum_{D \in A_M^N} p(D) = \frac{1}{\# A_M^N} \sum_{D \in A_M^N} \sum_{I \in D} q(I) \]

\[ = \frac{1}{\# A_M^N} \sum_{(I,D) \in S_M^N \times A_M^N} 1_{\Theta_M^N}((I,D)) q(I) \]

\[ = \frac{1}{\# A_M^N} \sum_{I \in S_M^N} \sum_{D \in (A_M^N)_I} q(I) = \sum_{I \in S_M^N} q(I) \frac{\#(A_M^N)_I}{\# A_M^N} \].

Later, in the estimate (3.13) near the end of the paper, we will take a limit as $\varepsilon > 0$. Suppose that for some $\varepsilon > 0$ we have

\[ \sum_{I \in S_M^N} q(I) 1_B(I,D) \leq \varepsilon \sum_{I \in S_M^N} q(I). \]

We now illustrate, in a simple situation, the type of conditional estimate we will use in our proof below. For $I \in S_M^N$ let $P_{(A_M^N)_I}$ denote uniform probability on the finite set $(A_M^N)_I$. For $B \subset \Theta_M^N$ we have by definition

\[ P_{(A_M^N)_I} \left( \left\{ D \in (A_M^N)_I : (I,D) \in B \right\} \right) = \mu_{(A_M^N)_I} \left( B \cap (A_M^N)_I \right). \]

Suppose that for some $\varepsilon > 0$ we have

\[ P_{(A_M^N)_I} \left( \left\{ D \in (A_M^N)_I : (I,D) \in B \right\} \right) \leq \varepsilon \text{ for all } I \in S_M^N, \]

and furthermore suppose we are given a nonnegative quantity $q(I)$ that is defined for all cubes in $S_M^N$. Then we claim that

\[ E_{A_M^N} \sum_{I \in D} q(I) 1_B(I,D) \leq \varepsilon E_{A_M^N} \sum_{I \in D} q(I). \]

Indeed, to see this, we recall that our collections of truncated grids $\Omega_M^N$ are all finite, and write

\[ E_{A_M^N} \sum_{I \in D} q(I) 1_B(I,D) = \frac{1}{\# A_M^N} \sum_{D \in A_M^N} \sum_{I \in D: (I,D) \in B} q(I) \]

\[ = \frac{1}{\# A_M^N} \sum_{I \in S_M^N:} q(I) \# \left\{ D \in (A_M^N)_I : (I,D) \in B \right\} \]

\[ = \frac{1}{\# A_M^N} \sum_{I \in S_M^N:} q(I) \mu_{(A_M^N)_I} \left( B \cap (A_M^N)_I \right) \# \left\{ D \in (A_M^N)_I : (I,D) \in \Theta \right\} \]

\[ \leq \frac{1}{\# A_M^N} \sum_{I \in S_M^N:} q(I) \varepsilon \#(A_M^N)_I \]

\[ = \varepsilon \frac{1}{\# A_M^N} \sum_{D \in A_M^N} \sum_{I \in D} q(I) = \varepsilon E_{A_M^N} \sum_{I \in D} q(I). \]

A similar expectation argument, but complicated by a subtle point regarding Whitney grids, will be carried out in (3.12) below.

2.1.3. Control of the maximal function by dyadic operators. Recall the finite collections of dyadic grids $\Omega_M^N$ (equivalently parameterized by $\Gamma_M^N$) introduced in Subsubsection 2.1.1 and especially the decomposition (2.3). In particular, construction #2 in Subsubsection 2.1.1 shows that

\[ A_M^N = \left\{ (D_0)_M + t \right\}_{t \in \Omega_M^N} \]

where $D_0 := \left\{ 2^j([0,1) + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n \right\}$ is the standard dyadic grid in $\mathbb{R}^n$, $(D_0)_M$ consists of those cubes $Q$ in $D_0$ with side lengths between $2^{-M}$ and $2^{-N}$, and where $\Omega_M^N$ is the index set

\[ \Omega_M^N \equiv \left\{ t = (t_i)_{i=0}^\infty \in 2^{-M} \mathbb{Z}_+ : |t_i| < 2^{-N} \right\}. \]

Recall also that we denoted by $dP_{A_M^N}$ the uniform probability measure on the finite set $A_M^N$. 

Notation 2. We will now abuse notation by identifying the collection of dyadic grids \( A_M^N \) with its associated index set \( \Omega_N^M \). Thus we henceforth abandon the notation \( \Omega_N^M \) and write \( \Omega_N^M \) for the finite collection of dyadic grids built on \( D_M \times \cdots \times D_M \) of size \( 2^{-N} \).

We now denote the natural product probability measure on the (infinite) collection of truncated dyadic grids

\[
\Phi_N^M = \bigcup_{s \in \{0, 2^{-M}\}} (\Omega_N^M + s)
\]

by \( dP_{\Phi_N^M} \). More specifically, \( dP_{\Phi_N^M} \) is defined to be the product measure \( dP_{\Omega_N^M} (D_{\text{fin}}) \times 2^{-Mn} \mathbf{1}_{[0, 2^{-M}]^n} (s) \) \( ds \) on \( \Phi_N^M = \{ D_{\text{fin}} + s : D_{\text{fin}} \in \Omega_N^M \text{ and } s \in [0, 2^{-M}]^n \} \). Note that \( \Omega_N^M + s = \Omega_N^M + s' \) if \( s - s' \in 2^{-M} \mathbb{Z}^n \), and that we can then also write

\[
\Phi_N^M = \bigcup_{t \in \gamma_N^M} \bigcup_{s \in [0, 2^{-M}]^n} \{(D_0)_M^N + t + s\}.
\]

Notation 3. We are here using \( D_{\text{fin}} \) to denote an independent variable in the collection of finite dyadic grids \( \Phi_N^M \), so that - unlike the notation \( \Phi_N^M \), which depends on the choice \( D \) of an untruncated dyadic grid in \( \Omega \) - there is for \( D_{\text{fin}} \) no connection implied with an untruncated dyadic grid \( D \) in \( \Omega \). We will also use \( D_{\text{fin}} \) below to denote an independent variable in the larger collection of truncated dyadic grids \( \Phi_N^M \).

Then for each truncated dyadic grid \( D_{\text{fin}} \in \Phi_N^M \), we denote the natural probability measure on the collection of untruncated dyadic grids

\[
\mathcal{H}_{D_{\text{fin}}} = \{ D \in \Omega : D_M^N = D_{\text{fin}} \}
\]

by \( dP_{\mathcal{H}_{D_{\text{fin}}}} \). More specifically, if \( D_{\text{fin}} \in \Phi_N^M \) and \( D^\beta \in \Omega \) as in (2.1) is any fixed untruncated grid in \( \Omega \) such that \( (D_M^N)_{\text{fin}} = D_{\text{fin}} \), then the set \( \mathcal{H}_{D_{\text{fin}}} \) is given by

\[
\mathcal{H}_{D_{\text{fin}}} = \{ D^\gamma : D^\gamma \in \Omega : \gamma_j = \beta_j \text{ for all } j > N \},
\]

i.e. \( \mathcal{H}_{D_{\text{fin}}} \) consists of all grids \( D^\gamma \) whose tiling by cubes of side length \( 2^{-N} \) agrees with that of \( D^\beta \). The probability measure \( dP_{\mathcal{H}_{D_{\text{fin}}}} \) is that unique probability measure which assigns equal probability \( 2^{-nk} \) to each collection of grids indexed by the set \( S(\beta_{N-k}, \beta_{N-(k-1)}, \ldots, \beta_{N-1}) \) of indices

\[
S(\beta_{N-k}, \beta_{N-(k-1)}, \ldots, \beta_{N-1}) \equiv \{ \gamma \in \{(0,1)^{k}\}^n : \gamma_j = \beta_j \text{ for all } j > N \text{ and } \gamma_j = \beta_j \text{ for } N - k \leq j \leq N - 1 \},
\]

where \( (\beta_{N-k}, \beta_{N-(k-1)}, \ldots, \beta_{N-1}) \in \{(0,1)^{k}\}^n \) has length \( k \). These probability measures \( dP_{\mathcal{H}_{D_{\text{fin}}}} \) are translation invariant in the sense that

\[
dP_{\mathcal{H}_{D_{\text{fin}}} + s} = dP_{\mathcal{H}_{D_{\text{fin}}}} \text{ for } s \in [0, 2^{-M}]^n.
\]

For each choice of integers \( N < 0 < M \), we thus have

\[
\Omega = \bigcup_{D_{\text{fin}} \in \Phi_N^M} \mathcal{H}_{D_{\text{fin}}} = \bigcup_{D_{\text{fin}} \in \Omega_N^M} \mathcal{H}_{D_{\text{fin}}} = \bigcup_{t \in \gamma_M^N} \bigcup_{s \in [0, 2^{-M}]^n} \mathcal{H}_{(D_0)_M^N + t + s} = \bigcup_{t \in \gamma_M^N} \bigcup_{s \in [0, 2^{-M}]^n} \mathcal{H}_M^N + t + s ,
\]

where we have set \( \mathcal{H}_M^N = \mathcal{H}_{(D_0)_M^N} \), the set of dyadic grids \( D \in \Omega \) that agree with the standard grid \( D_0 \) at level \( N \), i.e. that share the same tiling of cubes with side length \( 2^{-N} \). For any quantity \( p(D) \) that is defined for all grids \( D \in \Omega \), and for each choice of integers \( N < 0 < M \), we thus have

\[
E_{\Omega} p = \int_{\Omega} p(D) \, dP_{\Omega}(D) = \int_{\Omega_M^N} \left[ \int_{[0, 2^{-M}]^n} \left( \int_{\mathcal{H}_M^N} p(D + t + s) \, dP_{\mathcal{H}_M^N}(D) \right) \, ds \right] \, \frac{dt}{2^{-Mn}} \, dP_{\Omega_M^N}(t)
\]

by Fubini’s theorem, since the measure \( dP_{\Omega} \) is the product measure \( dP_{\Omega_M^N} \times \mathbf{1}_{[0, 2^{-M}]^n} \, \frac{dt}{2^{-Mn}} \times dP_{\mathcal{H}_M^N} \) on \( \Omega_M^N \times [0, 2^{-M}]^n \times \mathcal{H}_M^N \), and where \( dP_{\Omega_M^N} \) is of course a finite convex sum of unit point masses. We also then
have

\[ E^{D}_{\Omega} p(D) = \int_{\mathcal{M}_{\Omega}^{N}} \left\{ \int_{\mathcal{H}_{\Omega}^{\text{fin}}} p(D) d\mathcal{P}_{\mathcal{H}^{\text{fin}}} (D) \right\} d\mathcal{P}_{\mathcal{M}_{\Omega}^{N}} (\mathcal{D}^{\text{fin}}). \]

Our main result in this subsection is the following lemma, which goes back to Fefferman and Stein [FeSt] page 112] and also [Saw3, Lemma 2]. For any dyadic grid \( D \in \Omega \), we denote the associated dyadic maximal operator by

\[ M^{D} f = \sup_{Q \in D} \frac{1}{|Q|} \int_{Q} |f|. \]

**Lemma 1.** For \( x \in \mathbb{R}^{n} \) and a positive Borel measure \( f \geq 0 \) on \( \mathbb{R}^{n} \) we have

\[ (2.9) \quad M^{D} f (x) \leq 2^{n+3} E^{D}_{\Omega} M^{D} f (x). \]

**Proof.** Fix \( x \in \mathbb{R}^{n} \), and let \( Q \in \mathcal{P} \) be such that \( x \in Q \) and

\[ \frac{1}{|Q|} \int_{Q} f > \frac{1}{2} M^{D} f (x). \]

Pick \( N < 0 < 1 < M \) so that \( 2^{-M+100} \leq \ell(Q) \leq 2^{-N-100} \), which implies in particular that there exists \( s' \in [0, 2^{-M}) \) and \( \mathcal{D}^{\text{fin}} \in \Omega_{\mathcal{M}}^{N} + s' \) such that \( Q \subseteq \mathcal{D}^{\text{fin}} \). Thus the truncated grid \( \mathcal{D}^{\text{fin}} = \mathcal{D}^{\text{fin}} + s' \) belongs to the special collection \( \Omega_{\mathcal{M}}^{N} \) of truncated grids in Subsection 2.1.1 and \( Q - s' \) belongs to the corresponding special collection of cubes \( \mathcal{S}_{\mathcal{M}}^{N} \). Denote by \( \Delta_{s'} \) the set of indices \( t \in \Omega_{\mathcal{M}}^{N} \) such that the translated grid \( (\mathcal{D}_{0})_{\mathcal{M}}^{N} + t \) as given by Construction \#2 above has a cube \( K \) with side length twice that of \( Q \), and that contains \( Q \). For such a cube \( K \) we have

\[ \int_{|K|} f \geq \frac{1}{2^{n+3}} \int_{Q} f > \frac{1}{2^{n+3}} M^{D} f (x). \]

Moreover, the set \( \Delta_{s'} \) has probability \( \mu_{\mathcal{M}}^{N}(\Delta_{s'}) \geq \frac{1}{2} \). Thus we have

\[ \int_{\Omega_{\mathcal{M}}^{N}} M^{D_{\mathcal{M}, s'}} f (x) d\mu_{\mathcal{M}}^{N} (\beta) \geq \int_{\beta \in \Delta_{s'}} M^{D_{\mathcal{M}, s'}} f (x) d\mu_{\mathcal{M}}^{N} (\beta) \geq \int_{\beta \in \Delta_{s'}} \frac{1}{2^{n+1}} M^{D} f (x) d\mu_{\mathcal{M}}^{N} (\beta) = \frac{\mu_{\mathcal{M}}^{N}(\Delta_{s'})}{2^{n+1}} M f (x) \geq \frac{1}{2^{n+2}} M f (x). \]

Now using that \( 2^{-M+100} \leq \ell(Q) \leq 2^{-N-100} \), we easily see from the geometry of the cubes and grids that for every \( s \in [0, 2^{-M}) \) and any \( \mathcal{D}^{\text{fin}} \in \Omega_{\mathcal{M}}^{N} + s \), we have

\[ \int_{\Omega_{\mathcal{M}}^{N}} M^{D_{\mathcal{M}, s}} f (x) d\mu_{\mathcal{M}}^{N} (\beta) \geq \frac{\mu_{\mathcal{M}}^{N}(\Delta_{s})}{2^{n+1}} M f (x) \geq \frac{1}{2^{n+3}} M f (x), \]

upon using the crude estimate \( \mu_{\mathcal{M}}^{N}(\Delta_{s}) \geq \mu_{\mathcal{M}}^{N}(\Delta_{s'}) - \frac{1}{4} \geq \frac{1}{4} \). Taking the average over \( s \) in \([0, 2^{-M})\) and using (2.7) and (2.8) gives

\[ \int_{\Omega} M^{D} f (x) d\mathcal{P}_{\Omega} (D) = \int_{\Omega_{\mathcal{M}}^{N}} \left[ \int_{[0, 2^{-M})^{n}} \left( \int_{\mathcal{H}} M^{D_{\mathcal{M}, s}} f (x) d\mathcal{P}_{\mathcal{H}} (D) \right) \frac{ds}{2^{-Mn}} \right] d\mathcal{P}_{\mathcal{M}} (t) \]

\[ \geq \int_{[0, 2^{-M})^{n}} \left[ \int_{\Omega_{\mathcal{M}}^{N}} \left( \int_{\mathcal{H}} M^{D_{\mathcal{M}, s}} f (x) d\mathcal{P}_{\mathcal{H}} (D) \right) d\mathcal{P}_{\mathcal{M}} (t) \right] \frac{ds}{2^{-Mn}} \]

\[ \geq \int_{[0, 2^{-M})^{n}} \mu_{\mathcal{M}}^{N}(\Delta_{s}) M f (x) \frac{ds}{2^{-Mn}} \geq \frac{1}{2^{n+3}} M f (x), \]

which completes the proof of (2.9). \( \square \)
2.2. Whitney decompositions. Fix a finite measure \( \nu \) with compact support on \( \mathbb{R}^n \), and for \( k \in \mathbb{Z} \) let
\[
\Omega_k = \{ x \in \mathbb{R}^n : M \nu(x) > 2^k \}.
\]
Note that \( \Omega_k \neq \mathbb{R}^n \) is open for such \( \nu \). Fix a dyadic grid \( \mathcal{D} \in \Omega \) and an integer \( N \geq 5 \) (not to be confused with the different integer \( N \) in Subsubsection 2.1.1 above). We can choose \( R_W \geq 3 \) sufficiently large, depending only on the dimension and \( N \), such that there is a collection of \( \mathcal{D} \)-dyadic cubes \( \{ Q^k_j \}_j \) which satisfy the following properties for some positive constant \( C_W \):

\[
\begin{cases}
\text{(disjoint cover)} & \Omega_k = \bigcup_j Q^k_j \text{ and } Q^k_i \cap Q^k_j = \emptyset \text{ if } i \neq j \\
\text{(Whitney condition)} & R_W Q^k_j \subset \Omega_k \text{ and } 3R_W Q^k_j \cap \Omega^\ell_k \neq \emptyset \text{ for all } k, j \\
\text{(bounded overlap)} & \sum_{j} \chi_{Q^k_j} \leq C_W \chi_{\Omega_k} \text{ for all } k \\
\text{(crowd control)} & \# \{ Q^k_j : Q^k_j \cap NQ^k_i \neq \emptyset \} \leq C_W \text{ for all } k, j \\
\text{(side length comparability)} & \frac{1}{2} \leq \frac{\ell(Q^k_j)}{\ell(Q^k_i)} \leq 2 \text{ if } 3Q^k_j \cap 3Q^k_i \neq \emptyset \\
\text{(nested property)} & Q^k_i \subseteq Q^k_{i+1} \implies k > \ell
\end{cases}
\]

Indeed, one can choose the \( \{ Q^k_j \}_j \) from \( \mathcal{D} \) to satisfy an appropriate Whitney condition, and then show that the other properties hold. This Whitney decomposition and its use below are derived from work of C. Fefferman predating the two weight fractional integral argument of Sawyer [Saw2]. In particular, the properties above are as in [Saw2], with the exception of the side length comparability, which the reader can easily verify holds for \( R_W \) chosen sufficiently large.

3. Strong triple testing

Now we begin the proof of Theorem 2 which starts along the lines of the proof of the weaker result in [LiSa], but with the random grids of Nazarov, Treil and Volberg [NTV] used in place of the finite collection of grids constructed in the so-called ‘one third trick’ of Strömbärg.

Here is a brief description of the new features of the argument here as compared to that in [LiSa]. In [LiSa], we assumed the stronger hypothesis of \( D \)-parental tripling, which meant that the testing condition held for all cubes \( Q \) satisfying the property that for at least one of the \( 2^n \) possible dyadic parents \( P \) of \( Q \), we had \( |P|_\sigma \leq D |Q|_\sigma \). Thus the grids \( Q^k_{i,u} \) of nontripling cubes \( Q \) in a stopping cube \( Q^l_u \) (as defined in [LiSa]) were connected in the Whitney grid \( \mathcal{W} \), so that \( \pi_{\mathcal{W}} Q \subset Q^k_{i,u} \) and \( |Q|_\sigma < \frac{1}{4} |\pi_{\mathcal{W}} Q|_\sigma \), which could then be iterated and summed up to an acceptable Carleson estimate. In the analogous situation here, the tripled cube \( 3Q \) can spill outside the stopping cube \( Q^l_u \), which is then difficult to control because the averages of \( f \) outside the stopping cube are no longer controlled by the average of \( f \) over \( Q^l_u \). This spilling out then requires control of the ‘bad’ cubes \( Q \in \mathcal{W} \) whose triples are not contained in \( Q^l_u \). This control is effected by averaging over dyadic grids much as in [NTV], but is complicated by the fact that our cubes are contained in the subgrid of Whitney cubes, which necessitates some combinatoric arguments with finite grids.

We wish to prove the following estimate with \( \Gamma = 3 \) in the restricted testing condition,
\[
\mathcal{M}(\sigma, \omega) \lesssim \mathcal{F}(\sigma, \omega) + \sqrt{A_2}(\sigma, \omega).
\]
Fix \( f \) nonnegative and bounded with compact support, say \( \text{supp} f \subset Q(0, R) = [-R, R]^n \). Since \( M(f \sigma) \) is lower semicontinuous, the set \( \Omega_k \equiv \{ M(f \sigma) > 2^k \} \) is open and we can consider the standard Whitney decomposition of the open set \( \Omega_k \) into the union \( \bigcup_{j \in \mathbb{N}} Q^k_j \) of \( \mathcal{D}^\gamma \)-dyadic intervals \( Q^k_j \) with bounded overlap and packing properties as in (2.11). We denote the Whitney collection \( \{ Q^k_j \} \) by \( \mathcal{W}^\gamma \). We now use random grids to obtain from Lemma 1 in Subsubsection 2.1.3 that
\[
\mathcal{M}(f \sigma)(x) \lesssim E_D^D M^{D^\gamma} f(x), \quad x \in \mathbb{R}^n.
\]
Notice that if we replace \( \omega \) by \( \omega_N = \omega 1_{Q(0,N)} \) with \( N > R \), we have
\[
\int \mathcal{M}(f \sigma)^2 d\omega_N \leq \|f\|^2_{L^\infty} \int_{Q(0,N)} \mathcal{M}(\omega_N \sigma)^2 d\omega \leq \|f\|^2_{L^\infty} \mathcal{F}(\sigma, \omega) |3Q(0, N)|_\sigma < \infty,
\]
and therefore, without loss of generality, we can assume
\[
\int \mathcal{M}(f \sigma)^2 d\omega < \infty.
\]
We now have
\[ E_{\Omega} \int_{\mathbb{R}^n} \left\{ M^{D^\gamma} (f\sigma) (x) \right\}^2 \, d\omega (x) \leq E_{\Omega} C_n \sum_{k \in \mathbb{Z}} 2^{2(k+m)} \left| \left\{ M^{D^\gamma} (f\sigma) > 2^{k+m} \right\} \right|_{\omega} \]
\[ = E_{\Omega} C_n \sum_{k \in \mathbb{Z}, \, j \in \mathbb{N}} 2^{2(k+m)} \left| Q_j^k \cap \Omega_{k+m}^\gamma \right|_{\omega} \]
\[ \leq C_{n,m} E_{\Omega} \sum_{k \in \mathbb{Z}, \, j \in \mathbb{N}} 2^{2k} \left| E_{j,\gamma}^k \right|_{\omega} + 3^n C_n 2^{-2m_0} \int [M (f\sigma)]^2 \, d\omega , \]
where
\[ E_{j,\gamma}^k := Q_j^k \cap \Omega_{k+m}^\gamma \), \, \Omega_{k+m}^\gamma := \left\{ x : M^{D^\gamma} (f\sigma) > 2^{k+m} \right\} , \]
and we shall choose \( m_0 \) to be sufficiently large so that the second term can be absorbed (since it is finite).

So the goal is to prove
\[ E_{\Omega} \sum_{k \in \mathbb{Z}, \, j \in \mathbb{N}} 2^{2k} \left| E_{j,\gamma}^k \right|_{\omega} \lesssim \left( \mathcal{I}_M (3) (\sigma, \omega \right)^2 + A_2 (\sigma, \omega) \right) \| f\|_{L^2(\omega)}^2 . \]

Now fix \( \gamma \) and we will abbreviate \( E_{j,\gamma}^k \) by \( E_j^k \). As in [LiSa], we claim the maximum principle,
\[ 2^{k+m-1} < M^{D^\gamma} \left( 1_{Q_j^k} f\sigma \right) (x) , \quad x \in E_j^k . \]

Indeed, given \( x \in E_j^k \), there is \( Q \in D^\gamma \) with \( x \in Q \) and \( Q \cap (Q_j^k)^c \neq \emptyset \) (which implies that \( Q_j^k \subset Q \)), and also \( z \in \Omega_{k+m}^\gamma \), such that
\[ M^{D^\gamma} \left( 1_{(Q_j^k)^c} f\sigma \right) (x) \leq 2 \frac{1}{|Q|} \int_{Q \cap Q_j^k} f\sigma < 2 \frac{1}{|Q|} \int_{3 R W Q} f\sigma \]
\[ = \frac{2(3 R W)^n}{|3 R W Q|} \int_{3 R W Q} f\sigma \leq 2 (3 R W)^n M (f\sigma) (z) \leq 2^{k+m-1} \]
if we choose \( m > 1 \) large enough. Now we use \( 2^{k+m} < M^{D^\gamma} (f\sigma) (x) \) for \( x \in E_j^k \) to obtain
\[ 2^{k+m-1} < M^{D^\gamma} (f\sigma) (x) - M^{D^\gamma} \left( 1_{(Q_j^k)^c} f\sigma \right) (x) \leq M^{D^\gamma} \left( 1_{Q_j^k} f\sigma \right) (x) . \]

We now introduce some further notation which will play a crucial role below. Let
\[ H_j^k := \left\{ M^{D^\gamma} \left( 1_{Q_j^k} f\sigma \right) > 2^{k+m-1} \right\} , \]
\[ H_{j,\text{in}} := \left\{ M^{D^\gamma} \left( 1_{Q_j^k \cap \Omega_{k+m+m_0} f\sigma} \right) > 2^{k+m-2} \right\} , \]
\[ H_{j,\text{out}} := \left\{ M^{D^\gamma} \left( 1_{Q_j^k \setminus \Omega_{k+m+m_0} f\sigma} \right) > 2^{k+m-2} \right\} , \]
so that \( H_j^k \subset H_{j,\text{in}} \cup H_{j,\text{out}} \). We are here suppressing the dependence of \( H_j^k \) on \( \gamma \in \Omega \).

We will now follow the main lines of the argument for fractional integrals in [Saw2], but as in [LiSa], with two main changes:

(1) **Sublinearizations:** Since \( M \) is not linear, the duality arguments in [Saw2] require that we construct symmetric linearizations \( L \) that are dominated by \( M \), and
(2) **Tripling decompositions:** In order to exploit the triple testing conditions we introduce Whitney grids, and construct stopping times for tripling cubes, which entails some combinatorics. In particular, most of our effort is spent on decomposing and controlling the analogue of term IV from [Saw2] using good and bad cubes.

Now take \( 0 < \beta < 1 \) to be chosen later, and consider the following three exhaustive cases for \( Q_j^k \) and \( E_j^k \).

1. \( E_j^k \mid_\omega < \beta \mid 3 Q_j^k \mid_\omega \), in which case we say \((k, j) \in \Pi_1 \),
2. \( E_j^k \mid_\omega \geq \beta \mid 3 Q_j^k \mid_\omega \) and \( E_j^k \cap H_{j,\text{out}} \mid_\omega \geq \frac{1}{2} E_j^k \mid_\omega \), say \((k, j) \in \Pi_2 \),
3. \( E_j^k \mid_\omega \geq \beta \mid 3 Q_j^k \mid_\omega \) and \( E_j^k \cap H_{j,\text{in}} \mid_\omega \geq \frac{1}{2} E_j^k \mid_\omega \), say \((k, j) \in \Pi_3 \).
Here is a brief, and somewhat imprecise, schematic diagram of the decompositions, with bounds in used in this proof:

\[
\begin{align*}
\int_{\mathbb{R}^n} \left| \mathcal{M}(f\sigma)(x) \right|^2 \, d\omega(x) & \downarrow C_n, m \mathbf{E}_\Omega \sum_{k \in \mathbb{Z}, j \in \mathbb{N}} 2^{2k} |E^k_{j,\gamma}|_\omega + 3^n C_n 2^{-2m_0} \int_{\mathbb{R}^n} \left| \mathcal{M}(f\sigma) \right|^2 \, d\omega \\
& \downarrow \mathbf{E}_\Omega \sum_{(k,j) \in \Pi_1} 2^{2k} |E^k_{j,\gamma}|_\omega + \sup_\Omega \sum_{(k,j) \in \Pi_1} 2^{2k} |E^k_{j,\gamma}|_\omega + \sup_\Omega \sum_{(k,j) \in \Pi_2} 2^{2k} |E^k_{j,\gamma}|_\omega \\
& \downarrow \mathbf{E}_\Omega IV_{T - \text{good}} + \sup_\Omega \sum_{(t,u) \in \Gamma} V(t,u) + \mathbf{E}_\Omega III_{T - \text{bad}} \\
\left( \sum M(3)^2 + A_2 \right) \|f\|^2_{L^2(\sigma)} & \downarrow A_2 \|f\|^2_{L^2(\sigma)} \beta - \text{absorption} \quad r - \text{absorption}
\end{align*}
\]

where the notation is defined below. The expectation \( \mathbf{E}_\Omega \) is taken over dyadic grids \( D^\gamma \) in \( \Omega \), resulting in the absorption of the term \( \mathbf{E}_\Omega III_{T - \text{bad}} \) in the diagram, provided \( r \) is chosen sufficiently large. The term \( \sup_\Omega \sum_{(k,j) \in \Pi_1} 2^{2k} |E^k_{j,\gamma}|_\omega \) is absorbed by taking the parameter \( \beta > 0 \) sufficiently small, and the term \( 3^n C_n 2^{-2m_0} \int_{\mathbb{R}^n} \left| \mathcal{M}(f\sigma) \right|^2 \, d\omega \) is absorbed by taking the parameter \( m_0 \geq 1 \) sufficiently large.

3.1. The three cases. The first case is trivially handled, the second case is easy, and the third case consumes most of our effort.

Case (1): The treatment of case (1) is easy by absorption. Indeed,

\[
\sum_{(k,j) \in \Pi_1} 2^{2k} |E^k_{j,\gamma}|_\omega \lesssim \sum_{k \in \mathbb{Z}, j \in \mathbb{N}} 2^{2k} |3Q^k_j|_\omega \lesssim \beta \int \mathcal{M}(f\sigma)^2 \, d\omega,
\]

and then it suffices to take \( \beta \) sufficiently small at the end of the proof.

Case (2): In case (2) we have

\[
\sum_{(k,j) \in \Pi_2} 2^{2k} |E^k_{j,\gamma}|_\omega \lesssim \sum_{(k,j) \in \Pi_2} 2^k \int_{E^k_j} L^k_j \left( 1_{Q^k_j \setminus \Omega_{k+m_0}} f\sigma \right) \, d\omega.
\]

Here the positive linear operator \( L^k_j \) given by

\[
L^k_j(h\sigma)(x) = \sum_{\ell=1}^{\infty} \frac{1}{|I^k_j(\ell)|} \int_{I^k_j(\ell)} h\sigma \mathbf{1}_{I^k_j(\ell)}(x),
\]
where \( I^k_j(\ell) \in \mathcal{D}^\nu \) are the maximal dyadic cubes in \( \mathcal{H}^k_{j,\text{in}} \), which implies that \( \mathcal{L}^k_j(I_{Q^k_j \cap \Omega_{k+m+m_0}} f \sigma) \approx 2^k 1_{\mathcal{H}^k_{j,\text{in}}} \). Now we can continue from (3.3) as follows:

\[
\sum_{(k,j) \in \Pi_2} 2^k \int_{E^k_j} \mathcal{L}^k_j \left( 1_{Q^k_j \setminus \Omega_{k+m+m_0}} f \sigma \right) d\omega \\
= \sum_{(k,j) \in \Pi_2} 2^k \int_{Q^k_j \setminus \Omega_{k+m+m_0}} \mathcal{L}^k_j \left( 1_{E^k_j \omega} \right) f d\sigma \\
\leq \sum_{(k,j) \in \Pi_2} 2^k \left( \int_{Q^k_j \setminus \Omega_{k+m+m_0}} \mathcal{L}^k_j \left( 1_{E^k_j \omega} \right)^2 d\sigma \right)^{\frac{1}{2}} \left( \int_{Q^k_j \setminus \Omega_{k+m+m_0}} f^2 d\sigma \right)^{\frac{1}{2}} \\
\leq \left( \sum_{(k,j) \in \Pi_2} 2^k \int_{Q^k_j \setminus \Omega_{k+m+m_0}} \mathcal{L}^k_j \left( 1_{Q^k_j \omega} \right)^2 d\sigma \right)^{\frac{1}{2}} \left( \sum_{(k,j) \in \Pi_2} \int_{Q^k_j \setminus \Omega_{k+m+m_0}} f^2 d\sigma \right)^{\frac{1}{2}} \\
\leq 2^k \sum_{(k,j) \in \Pi_2} 2^k \left( 1_{Q^k_j \omega} \right) \parallel f \parallel_{L^2(\sigma)} \\
\leq \beta^{-\frac{1}{2}} C_{m,m_0} A^2_k \left( \sum_{(k,j) \in \Pi_2} 2^k \left| E^k_j \omega \right| \right)^{\frac{1}{2}} \parallel f \parallel_{L^2(\sigma)},
\]

where we have used the following trivial estimate

\[
\int_{Q^k_j} \mathcal{L}^k \left( 1_{Q^k_j \omega} \right)^2 d\sigma \leq \sum_{\ell=1}^{\infty} \frac{|I^k_j(\ell)| \omega| I^k_j(\ell)| \sigma |I^k_j(\ell) \cap Q^k_j| \omega \leq A_2 |Q^k_j| \omega.}
\]

Then immediately we get

\[
\sum_{(k,j) \in \Pi_2} 2^k \left| E^k_j \omega \right| \leq \beta^{-1} C^2_{m,m_0} A_2 \parallel f \parallel_{L^2(\sigma)}^{2}.
\]

**Case (3):** For this case, we let \( \{I^k_j(\ell)\}_\ell \) be the collection of the maximal dyadic cubes in \( \mathcal{H}^k_{j,\text{in}} \) and define \( \mathcal{L}^k_j \) similarly. Then likewise, \( \mathcal{L}_j^k(1_{Q^k_j \cap \Omega_{k+m+m_0}} f \sigma) \approx 2^k 1_{\mathcal{H}^k_{j,\text{in}}} \) and therefore,

\[
\sum_{(k,j) \in \Pi_3} 2^k \left| E^k_j \omega \right| \leq \sum_{(k,j) \in \Pi_3} 2^k \int_{E^k_j} \mathcal{L}^k_j \left( 1_{Q^k_j \cap \Omega_{k+m+m_0}} f \sigma \right) d\omega \\
= \sum_{(k,j) \in \Pi_3} 2^k \int_{Q^k_j \cap \Omega_{k+m+m_0}} \mathcal{L}^k_j \left( 1_{E^k_j \omega} \right) f d\sigma \\
= \sum_{(k,j) \in \Pi_3} 2^k \sum_{i \in \mathbb{N}_+} \sum_{Q^k_{i+m+m_0} \subset Q^k_j} \int_{Q^k_{i+m+m_0}} \mathcal{L}^k_j \left( 1_{E^k_j \omega} \right) f d\sigma.
\]

Before moving on, let us make some observations. Since we only need to consider \( I^k_j(\ell) \) such that \( I^k_j(\ell) \cap E^k_j \neq \emptyset \), we have \( I^k_j(\ell) \not\subset \Omega_{k+m+m_0} \). Therefore, if we fix \( Q^k_{i+m+m_0} \), only those \( I^k_j(\ell) \) such that \( Q^k_{i+m+m_0} \subset I^k_j(\ell) \) contribute to \( \mathcal{L}^k_j \). In other words, \( \mathcal{L}^k_j \left( 1_{E^k_j \omega} \right) \) is constant on \( Q^k_{i+m+m_0} \). Set

\[
A^k_j = \frac{1}{|Q^k_j|} \int_{Q^k_j} f d\sigma.
\]
We have
\[
\sum_{(k,j) \in \Pi_3} 2^k |E^k_j|_\omega 
\lesssim \sum_{(k,j) \in \Pi_3} 2^k \sum_{i \in \mathbb{N}} Q_i^{k+m+m_0} \cdot \int_{Q_i^{k+m+m_0}} L^k_j \left( 1_{E^k_j} \right) \sigma 
= \lim_{N \to -\infty} \sum_{k \in \mathbb{Z}, k \geq N} 2^k \sum_{j \in \mathbb{N}, (k,j) \in \Pi_3} A^{k+m+m_0}_{i} \int_{Q_i^{k+m+m_0}} L^k_j \left( 1_{E^k_j} \right) \sigma.
\]

We make a convention that the summation over $k$ is understood as $k \equiv k_0 \mod (m + m_0)$ for some fixed $0 \leq k_0 \leq m + m_0 - 1$, and since we are summing over products with factor $|E^k_j|_\omega$, without loss of generality we only consider $Q^k_j$ for the largest $k$ if it is repeated, and define
\[
W^\gamma := \{ Q^k_j : k \equiv k_0 \mod (m + m_0), k \geq N \}.
\]

So in particular, there are no repeated cubes in $W^\gamma$, and $W^\gamma$ is in one-to-one correspondence with the set $W^\gamma_{\text{dis}}$ of distinguished pairs $(k,j)$ where $k$ is the largest $k$ among repeated cubes, and $k \equiv k_0 \mod (m + m_0)$.

**Notation 4.** We say that the cube $Q^k_j$ belongs to a set $\Lambda \subset W^\gamma_{\text{dis}}$ of distinguished pairs of indices when we have $(k,j) \in \Lambda$, i.e. we do not distinguish between the distinguished index $(k,j)$ and the corresponding cube $Q^k_j$ for $(k,j) \in W^\gamma_{\text{dis}}$. Thus if we write $Q \in \Lambda$, this means that $Q = Q^k_j$ for $(k,j) \in \Lambda$, and conversely we write $Q^k_j \in \Lambda$ if $(k,j) \in \Lambda$.

We now drop the superscript $\gamma$ when it does not matter. We have, using $|E^k_j|_\omega \approx |3Q^k_j|_\omega$ and $L^k_j(1_{Q^k_j \cap \Omega_{k+m+m_0} f \sigma}) \approx 2^k 1_{H^k_{\text{fin}}}$ for $(k,j) \in \Pi_3$ again, that
\[
\sum_{k \in \mathbb{Z}, k \geq N} 2^k \sum_{j \in \mathbb{N}, (k,j) \in \Pi_3} |E^k_j|_\omega \left[ \sum_{i \in \mathbb{N}} Q_i^{k+m+m_0} \cdot \int_{Q_i^{k+m+m_0}} L^k_j \left( 1_{E^k_j} \right) \sigma \right]^2 = III^*,
\]
where
\[
III^* = \sum_{k \in \mathbb{Z}, k \geq N} III^* (Q^k_j);
\]
\[
III^* (Q^k_j) = \frac{|E^k_j|_\omega}{|3Q^k_j|_\omega} \left[ \sum_{i \in \mathbb{N}} Q_i^{k+m+m_0} \cdot \int_{Q_i^{k+m+m_0}} L^k_j \left( 1_{E^k_j} \right) \sigma \right]^2.
\]

### 3.2. Control of bad cubes.

Now we encounter the main new argument needed for proving Theorem 2. Given a cube $Q$ in a grid $D = D^\gamma$ or $D_{\text{fin}}$, and a large positive integer $r$, we define $Q$ to be $r$-bad in $D$ if the level $r$ parent $\pi_D^{(r)} Q$ exists in the grid $D$, and the boundary of $\pi_D^{(r)} Q$ intersects the boundary of the tripled cube $3Q$ (equivalently, either $\partial Q$ ‘touches’ $\partial \pi_D^{(r)} Q$ or $3Q$ ‘touches’ $\partial \pi_D^{(r)} Q$). We write $D = D_{r\text{-bad}} \cup D_{r\text{-good}}$ where $D_{r\text{-bad}} = \{ Q \in D : Q \text{ is } r \text{-bad in } D \}$. Note that this definition of $r$ – bad is much more restrictive than the usual definition in [NTV], in that it requires actual ‘touching’ of the boundary of $Q$ or $3Q$ to that of the $r$-parent. With $\Omega$ and $P_\Omega$ as in the definition [24] of untruncated dyadic grids, it is well known that the set of grids $D^\gamma \in \Omega$ for which $Q \in D^\gamma$ and $Q$ is $r$-bad in $D^\gamma$ has conditional probability at most a multiple of $2^{-r}$, i.e.
\[
P_\Omega \left\{ D^\gamma \in \Omega : Q \text{ is } r \text{-bad in } D^\gamma \text{ conditioned on } Q \in D^\gamma \right\} \lesssim 2^{-r}.
\]
Indeed, this follows for example from the construction of $\Omega^N_M$ in Subsubsection 2.1.1 upon noticing that, given a cube $Q \in \mathcal{S}^N_M$ with $2^{-N} \leq \ell(Q) < 2^{-M-r}$, only $(2^r)^n - (2^r - 4)^n \approx (2^r)^{n-1}$ of the $2^{nr}$ possible level $r$ parents of the cube $Q$ have boundary that intersects that of $3Q$. This shows that the proportion of such $r$-bad cubes is $(2^r)^{n-1} \approx 2^{-r}$, which yields (3.3) after invoking the identities (2.3) and (2.8).

Now we observe that for $|E_j^k|_{\omega} \neq 0$, the quantity

$$\frac{|E_j^k|_{\omega}}{|3Q_j^k|_{\omega}} \left[ \sum_{i \in N: P(Q_j^k) = \mathcal{P}(Q_j^k)} A_{i}^{k+m+m_0} \int_{Q_j^k} \mathcal{L}_j^k \left( 1_{E_j^k \omega} \right) \sigma \right]^2 = q(Q)$$

depends only on the cube $Q = Q_j^k$ and not on the underlying grid $D^n$, since the operator $\mathcal{L}_j^k$ depends only on the dyadic grid structure within the cube $Q_j^k$. Before further decomposing the last sum $\mathcal{III}^*$ in (3.7) above into pieces $IV + V$, we will use probability to control the sum over $r$-bad cubes in (3.7),

$$III_{r-bad} = \sum_{Q \in W \cap D_{r-bad}} III^* (Q) = \sum_{Q \in W \cap D_{r-bad}} III^* (Q) .$$

At this point our grids $D^n$ are not truncated, and we are not yet working with the finite collection of truncated grids $\Omega^N_M$. When convenient, we also write $\mathcal{W}^D$ instead of $\mathcal{W}^\gamma$ when $D = D^n$, so that if $D$ is the underlying grid in the definition of $III_{r-bad}$, then

$$III_{r-bad} = \sum_{Q \in \mathcal{W}^D \cap D_{r-bad}} III^* (Q) .$$

A key point in what follows - already noted above - is that the quantity

$$q(Q) = III^* (Q_j^k) \quad \text{if } Q = Q_j^k \in \mathcal{W}^\gamma \text{ for some } (k,j) \in W_{\text{dis}} ,$$

which is defined for all $Q \in \widehat{\mathcal{W}} \equiv \bigcup_{\gamma \in \Omega} \mathcal{W}^\gamma$, depends only on the cube $Q$ and not on any of the untruncated grids $D^n$ for which $Q = Q_j^k \in \mathcal{W}^\gamma$, so that we have

$$q : \widehat{\mathcal{W}} \to [0, \infty) .$$

We would of course like to restrict matters to cubes with side length between $2^{-M}$ and $2^{-N}$ and use the conditional probability estimate (2.5) by simply extending the definition of our function $q : \widehat{\mathcal{W}} \to [0, \infty)$ to all of $(P^n)_M$ by setting $q(Q) = 0$ if $Q \in (P^n)_M \setminus \widehat{\mathcal{W}}$. However, a subtle point arises here that prevents such a simple application of (2.5). If $Q \in \mathcal{W}^{\gamma_1} \cap D^{r_2}$ for some $\gamma_1, \gamma_2 \in \Omega$, it need not be the case that $Q \in \mathcal{W}^{\gamma_2}$. However, if $Q \notin \mathcal{W}^{\gamma_2}$, then the $\gamma_2$-parent $\pi_{D^{r_2}} Q$ of $Q$ is in $\mathcal{W}^{\gamma_2}$, and this will prove to be a suitable substitute. We state and prove this in the following lemma.

**Lemma 2.** Suppose that $Q \in \mathcal{W}^{\gamma_1} \cap D^{r_2}$ for some $\gamma_1, \gamma_2 \in \Omega$. Then either $Q$ or $\pi_{D^{r_2}} Q$ belongs to $\mathcal{W}^{\gamma_2}$.

**Proof.** For this proof we use the notation,

$$e^{(\ell)} (Q) = \{ Q' : Q' \text{ is a dyadic subcube of } Q \text{ with } \ell(Q') = 2^{-\ell} \ell(Q) \} ,$$

and refer to a cube $Q' \in e^{(\ell)} (Q)$ as a level $\ell$ dyadic child of $Q$. Now pick a point $x \in Q$. Since $Q \in \mathcal{W}^{\gamma_1}$, there is an integer $k$ such that $Q = Q_j^k$ for some distinguished index $(k,j) \in W_{\text{dis}}$. Thus $x \in \Omega_k$, there is a unique cube $P \in \mathcal{W}^{r_2}$ such that $P = P_j^k$ for some index $(k,j')$ (not necessarily distinguished in the grid $\mathcal{W}^{r_2}$) and such that $P$ contains $x$. Clearly, $P$ cannot be a dyadic child of $Q$ at any level since then $P$ would be a strict subcube of $Q$ and hence not maximal in $D^{r_2}$ with respect to the property that $RWP \subset \Omega_k$. We now claim that $P \notin e^{(\ell)} (Q)$ for any $\ell \geq 2$. Indeed, if $P = e^{(\ell)} (Q)$ for some $\ell \geq 2$, then $RWP \subset \Omega_k$, and we now claim that $RWP \pi_{D^{r_2}} Q \subset \Omega_k$ as well. For this, consider the metric $d_{\infty} (x,y) = \max_{1 \leq j \leq n} |x_j - y_j|$ in $\mathbb{R}^n$, so that the ball $B_{d_{\infty}} (x, r)$ is the open cube centered $x$ with side length $2r$. Then if $e_{i_1}$ denotes the center of
the cube $I$ and $z \in R_W \pi_{D^1} Q$, we have for $R_W > \frac{3}{2}$ that
\[
d_\infty (z, c_P) \leq d_\infty (z, c_{\pi_{D^1} Q}) + d_\infty (c_{\pi_{D^1} Q}, c_P) \\
\leq R_W \left( \frac{\ell (\pi_{D^1} Q)}{2} + (2^\ell - 1) \ell (Q) \right) \\
= (R_W + 2^\ell - 1) \ell (Q) \\
< R_W 2^\ell - 1 \ell (Q) = R_W \frac{\ell (\pi_{D^1} Q)}{2},
\]
which shows that $z \in R_W \pi_{D^1} Q$. Here we have used that $R_W + 2^\ell - 1 < R_W 2^\ell - 1$ if and only if $2^\ell - 1 < R_W$, which holds for $R_W > \frac{3}{2}$ and $\ell \geq 2$. Thus we have
\[
R_W \pi_{D^1} Q \subset R_W \pi_{D^1} Q \subset \Omega_k,
\]
which contradicts the assumption that $Q$ is a maximal cube in $D^1$ with $R_W Q \subset \Omega_k$.

We now set up some definitions to deal with the subtle point discussed above. The quantity $q (Q^k)$ has the following upper bound where $D_K \equiv \{ Q \in D : Q \subset K \}$ is the grid of dyadic subcubes of $K$:
\[
q (Q^k) = \frac{|E^k|_\omega}{|3Q^k|_\omega} \left[ \sum_{i \in D} A_i^k \int_{Q^k_{i+m+m_0}} L^k (1_{E^k} \omega) \sigma \right]^2 \\
= \frac{|E^k|_\omega}{|3Q^k|_\omega} \left[ \sum_{i \in D} A_i^k \int_{E^k} L^k \left( \sum_{i \in D} A_i^k \right) \sigma \right]^2 \\
= \frac{|E^k|_\omega}{|3Q^k|_\omega} \left[ \int_{E^k} L^k \left( \sum_{i \in D} A_i^k \right) \sigma \right]^2 \\
\leq \frac{|E^k|_\omega}{|3Q^k|_\omega} \left[ \int_{E^k} \mathcal{M}^q (f \sigma) \omega \right]^2 \leq \left[ \frac{1}{|3Q^k|_\omega} \int_{Q^k} \mathcal{M} (f \sigma) \omega \right]^2,
\]
since for any cube $P \in D_Q$, we have
\[
\frac{1}{|P|} \int_P \left( \sum_{i \in D} A_i^k \int_{Q^k_{i+m+m_0}} 1_{Q_{i+m+m_0}} \sigma \right) \\
= \frac{1}{|P|} \int_P \left( \sum_{i \in D} A_i^k \int_{Q^k_{i+m+m_0}} 1_{Q_{i+m+m_0}} \int_{Q_{i+m+m_0}} |f| d\sigma \right) \sigma \\
= \frac{1}{|P|} \int_P \int_{Q^k_{i+m+m_0}} |f| d\sigma \leq \frac{1}{|P|} \int_P |f| d\sigma.
\]

Suppose that $Q \in \hat{W}$. If $Q \in W^D$ for some grid $D$, then $Q = Q^k$ for some distinguished index $(k, j) \in \mathcal{W}^D$. If we also have $Q \in W^{D'}$ for some grid $D'$, then $Q = Q^{k'}$ for some distinguished index $(k', j') \in \mathcal{W}^{D'}$. It is easy to see that $k' \geq k$, and then by symmetry that $k = k'$. Thus there is a unique integer $\kappa (Q) = k = k'$.
associated with \( Q \in \hat{W} \) that we refer to as the height of \( Q \). We now define

\[
q^* (Q) \equiv |E_Q|, \left[ \frac{1}{3|Q|} \int_Q \mathcal{M} (f \sigma) \right]^2 ;
\]

\[
E_Q \equiv Q \setminus \Omega_{k+m+m_0} \text{ where } k = \kappa (Q),
\]

for \( Q \in \hat{W} \), so that we have

\[
q (Q) \leq q^* (Q), \quad \text{for all } Q \in \hat{W},
\]

and finally we define

\[
q^{**} (Q) \equiv q^* (Q) + \sum_{Q' \in \mathcal{E} \cap \hat{W}} q^* (Q'), \quad \text{for all } Q \in \hat{W},
\]

\[
q^{**} (Q) \equiv q^{**} (Q) + \sum_{Q' \in \mathcal{E} \cap \hat{W}} q^{**} (Q'), \quad \text{for all } Q \in \hat{W},
\]

where if \( \mathcal{E} \cap \hat{W} = \emptyset \) in either line, the corresponding sum vanishes.

3.2.1. Truncated grids. Recall that we have already fixed a grid \( D \in \Omega \), and then by (2.3), the truncated grid \( D_M \) has the form \( D_{\text{fin}} + s \) for some \( s \in [0, 2^{-M}) \) and some \( D_{\text{fin}} \in \Omega_M \). We now also restrict the cubes \( Q = Q^k_j \) in our sums to belong to \( P_M \), i.e. to satisfy

\[
2^{-M} \leq \ell (Q) \leq 2^{-N}.
\]

Later, at the end of the proof, we will take the supremum of the estimates obtained over all \( N < 0 < M \).

Let \( \mathcal{P} (Q) \) denote the collection of \( 2^n \) dyadic parents of the cube \( Q \). For a given cube \( I \in \hat{W} \) satisfying \( 2^{-M} \leq \ell (I) \leq 2^{-1-N} \), and a parent \( J \in \mathcal{P} (I) \), we decompose the grids \( D_{\text{fin}} \in (\Omega_M) \) according to \( J = \pi_{\text{fin}} I \) and also to whether or not \( I \in W^D_{\text{fin}} \):

\[
(\Omega_M) = \bigcup_{J \in \mathcal{P} (I)} \left\{ \{ D_{\text{fin}} \in (\Omega_M) : I \in W^D_{\text{fin}} \} \cup \{ D_{\text{fin}} \in (\Omega_M) : I \notin W^D_{\text{fin}} \} \right\}.
\]

Thus the collection \( (\Omega_M)^{\text{fin}}_{\text{fin}} \) consists of all grids \( D_{\text{fin}} \in (\Omega_M) \) with the property that \( I \in W^D_{\text{fin}} \), while \( (\Omega_M)^{\text{fin}}_{\text{fin}} \) consists of all grids \( D_{\text{fin}} \in (\Omega_M) \) with the property that \( I \notin W^D_{\text{fin}} \). However, by Lemma 2 above, it will be the case that \( J \in W^D_{\text{fin}} \) if \( I \notin W^D_{\text{fin}} \). Thus for every grid \( D_{\text{fin}} \in (\Omega_M) \) and every dyadic child \( I \) of \( J \) which satisfies \( I \in \hat{W} \), it is the case that either \( J \) or its child \( I \) belongs to \( W^D_{\text{fin}} \). As we will see below, this is the reason we defined the quantities \( q^{**} (Q) \) and \( q^{***} (Q) \) above. Finally we note that

\[
\# (\Omega_M) = \sum_{J \in \mathcal{P} (I)} \left( \# (\Omega_M) \in W^D_{\text{fin}} + \# (\Omega_M) \notin W^D_{\text{fin}} \right).
\]

Now let \( I_{\text{r-bad}} \) denote the collection of grids \( D_{\text{fin}} \in (\Omega_M) \) such that \( I \) is \( r \) bad in \( D_{\text{fin}} \),

i.e. \( I_{\text{r-bad}} = \{ D_{\text{fin}} \in (\Omega_M) : I \text{ is r-bad in } D_{\text{fin}} \} \),

so that

\[
I_{\text{r-bad}} \subset (\Omega_M) \text{ and } \# I_{\text{r-bad}} \leq C^{2-r} \# (\Omega_M) < 2^{-r-N}.
\]

We restrict the side length of \( I \) to \( \ell (I) < 2^{r-N} \) in order that the level \( r \) parent \( \pi (r) I \) of \( I \) in the grid \( D_{\text{fin}} \) belongs to \( (\Omega_M) \). Recall also that one should think of \( M \) near \( \infty \) and \( N \) near \( -\infty \). Define the collection

\[
(\Omega_M)^{\text{fin}} = \{ D_{\text{fin}} \in (\Omega_M) : I \in W^D_{\text{fin}} \}
\]

to consist of those grids \( D_{\text{fin}} \) that not only contain \( I \), but satisfy
\( I \in \mathcal{W}^{D_{\text{fin}}} \). Fix \( I \in \hat{\mathcal{W}} \) and let \( D_{\text{fin}} \in (\Omega_M^N)_I \). Then \( D_{\text{fin}} \in (\Omega_M^N)_J \) for some \( J \in \Psi(I) \), and Lemma 2 implies that either \( D_{\text{fin}} \in (\Omega_M^N)_J \) or \( D_{\text{fin}} \in (\Omega_M^N)_J \). Thus we have \( (\Omega_M^N)_I = (\Omega_M^N)_J \cup \bigcup_{J \in \Psi(I)} (\Omega_M^N)_J \) and so

\[
I_{r-\text{bad}} \subseteq \left[ (\Omega_M^N)_J \cap I_{r-\text{bad}} \right] \cup \bigcup_{J \in \Psi(I)} \left[ (\Omega_M^N)_J \cap J_{(r-1)-\text{bad}} \right],
\]

since \( D_{\text{fin}} \in (\Omega_M^N)_J \cap I_{r-\text{bad}} \) for a dyadic child \( I \in \mathcal{C}(J) \) implies that \( J \) itself is \((r-1)-\text{bad}\) in \( D_{\text{fin}} \).

We have

\[
\#I_{r-\text{bad}} \leq \# \left[ (\Omega_M^N)_J \cap I_{r-\text{bad}} \right] + \sum_{J \in \Psi(I)} \# \left[ (\Omega_M^N)_J \cap J_{(r-1)-\text{bad}} \right]
\leq \sum_{J \in \Psi(I)} \left( \# \left[ (\Omega_M^N)_J \cap (\Omega_M^N)_J \cap I_{r-\text{bad}} \right] + \# \left[ (\Omega_M^N)_J \cap J_{(r-1)-\text{bad}} \right] \right),
\]

since \( (\Omega_M^N)_J \) is the pairwise disjoint union of the sets \( \{(\Omega_M^N)_J \} \). We now deviate slightly from the treatment of conditional probability in (2.5) above by setting

\[
\Theta_M^N \equiv \left\{ (I, D_{\text{fin}}) \in P_M^N \times \Omega_M^N : I \in \mathcal{W}^{D_{\text{fin}}} \right\},
\]
\[
B_M^N \equiv \left\{ (I, D_{\text{fin}}) \in \Theta_M^N : I \text{ is } r \text{-bad in } D_{\text{fin}} \right\}.
\]

By (3.9) we have \( q(I) \leq q^*(I) \leq q^{**}(I) \) for all cubes \( I \in \hat{\mathcal{W}} \). We now denote by \( III_{r-\text{bad}}(M, N + r + 1) \) the term \( III_{r-\text{bad}} \) but with cubes \( Q \) restricted to satisfying

\[
(3.11) \quad 2^{-M} \leq \ell(Q) \leq 2^{-(N+r+1)}.
\]

Now fix \( s \in [0, 2^{-M}]^n \). We have

\[
(3.12) \quad \mathbf{E}_{\Omega_M^N + s} \left( \sum_{I \in \mathcal{W}^{D_{\text{fin}}} \cap P_M^{N+r+1}} q^*(I) \mathbf{1}_{B_M^N}(I, D_{\text{fin}}) \right)
\]
\[
= \frac{1}{\#\Omega_M^N} \sum_{D \in \Omega} \sum_{I \in \mathcal{W}^{D \cap P_M^{N+r+1}}} q^*(I) \leq \frac{1}{\#\Omega_M^N} \sum_{I \in \mathcal{W} \cap P_M^{N+r+1}} q^*(I) \#I_{r-\text{bad}}
\]
\[
= \frac{1}{\#\Omega_M^N} \sum_{I \in \mathcal{W} \cap P_M^{N+r+1}} q^*(I) \left( \# \left[ (\Omega_M^N)_J \cap I_{r-\text{bad}} \right] + \sum_{J \in \Psi(I) \cap \hat{\mathcal{W}}} \# \left[ (\Omega_M^N)_J \cap J_{(r-1)-\text{bad}} \right] \right)
\]
\[
\leq \frac{1}{\#\Omega_M^N} \sum_{I \in \mathcal{W} \cap P_M^{N+r+1}} q^*(I) \# \left[ (\Omega_M^N)_J \cap I_{r-\text{bad}} \right] + \frac{2^n}{\#\Omega_M^N} \sum_{J \in \Psi(I) \cap \hat{\mathcal{W}}} \# \left[ (\Omega_M^N)_J \cap J_{(r-1)-\text{bad}} \right]
\]
\[
\leq C \frac{1}{\#\Omega_M^N} \sum_{I \in \mathcal{W} \cap P_M^{N+r+1}} q^{**}(J) \# \left[ (\Omega_M^N)_J \cap J_{(r-1)-\text{bad}} \right] \leq C2^{-r} \frac{1}{\#\Omega_M^N} \sum_{J \in \mathcal{W} \cap P_M^{N+r+1}} q^{**}(J) \# \left( \Omega_M^N \right)_J,
\]
by the probability estimate \(\text{Lemma 2}\) with \(r - 1\) in place of \(r\). Now for \(J \in \hat{\mathcal{W}}\) we have \((\Omega_M^N)_j = (\Omega_M^N)_j \cup \bigcup_{K \in \mathfrak{W}(j)} (\Omega_M^N)_K\) by Lemma 2 and so we can continue \(\text{Lemma 2}\) with

\[
C_2^{-r} \frac{1}{\#\Omega_M^N} \sum_{J \in \hat{\mathcal{W}} \cap P_{M-1}^{N+r}} q^{**}(J) \#(\Omega_M^N)_j
\]

\[
\leq C_2^{-r} \frac{1}{\#\Omega_M^N} \sum_{J \in \hat{\mathcal{W}} \cap P_{M-1}^{N+r}} q^{**}(J) \#(\Omega_M^N)_j + C_2^{-r} \frac{1}{\#\Omega_M^N} \sum_{K \in \mathfrak{W}(j)} q^{**}(K) \#(\Omega_M^N)_K
\]

\[
\leq C_2^{-r} \frac{1}{\#\Omega_M^N} \sum_{K \in \mathfrak{W}(j)} q^{**}(K) \#(\Omega_M^N)_K
\]

\[
= C_2^{-r} \frac{1}{\#\Omega_M^N} \sum_{D_{\text{fin}} \in \Omega_M^N} \sum_{K \in \mathfrak{W}(j) \cap P_{M-1}^{N+r}} q^{**}(K) \leq C_2^{-r} \mathcal{E}_{\Omega_M^{N+r}} \left( \sum_{K \in \mathfrak{W}(j) \cap P_{M-1}^{N+r}} q^{**}(K) \right),
\]

where we have used \(\text{Lemma 2}\) above once more. Now take an average over \(s \in [0, 2^{-M}]\) in the above inequality

\[
\mathcal{E}_{\Omega_M^{N+r}} \left( III_{-\text{bad}}(M, N + r + 1) \right) \leq C_2^{-r} \mathcal{E}_{\Phi_M^N} \left( \sum_{K \in \mathfrak{W}(j) \cap P_{M-1}^{N+r}} q^{**}(K) \right),
\]

where \(\Phi_M^N\) is defined in Subsubsection 2.1.3 as a union of translates of the grids in \(\Omega_M^N\), and we remind the reader that the sum in \(III_{-\text{bad}}(M - 2, N + r)\) on the left hand side satisfies \(\text{Lemma 2}\) while the sum on the right hand side satisfies \(\text{Lemma 2}\).

Now we estimate the sums \(\sum_{K \in \mathfrak{W}(j) \cap P_{M-1}^{N+r}} q^{**}(K)\) uniformly over grids \(D_{\text{fin}}\) as follows. Fix a grid \(D_{\text{fin}}\) for the moment. If \(\mathcal{M}_{D_{\text{fin}}}^\omega\) denotes the \(D_{\text{fin}}\)-dyadic maximal operator with respect to the measure \(\omega\), i.e.

\[
\mathcal{M}_{D_{\text{fin}}}^\omega h(x) \equiv \sup_{Q \in D_{\text{fin}}; x \in Q} \frac{1}{|Q|} \int_Q |h| \, d\omega,
\]

then for \(Q_{k}^{j} \in \mathcal{W}_{D_{\text{fin}}}\), we have

\[
q^{**}(Q_{k}^{j}) = |E_{k}^{j}|_\omega \left[ \frac{1}{|3Q_{k}^{j}|_\omega} \int_{Q_{k}^{j}} \mathcal{M}(f \sigma) \omega \right]^2 + \sum_{Q' \in \mathcal{E}(Q_{k}^{j})} |E_{Q'}^{j}|_\omega \left[ \frac{1}{|3Q'_{k}|_\omega} \int_{Q'} \mathcal{M}(f \sigma) \omega \right]^2 + \sum_{Q'' \in \mathcal{E}(Q_{k}^{j})} |E_{Q''}^{j}|_\omega \left[ \frac{1}{|3Q''_{k}|_\omega} \int_{Q''} \mathcal{M}(f \sigma) \omega \right]^2
\]

\[
\leq \int_{E_{k}^{j}} \mathcal{M}_{D_{\text{fin}}}^\omega (\mathcal{M}(f \sigma))^2 \, d\omega + \sum_{Q' \in \mathcal{E}(Q_{k}^{j})} \int_{E(Q')} \mathcal{M}_{D_{\text{fin}}}^\omega (\mathcal{M}(f \sigma))^2 \, d\omega
\]

\[
\leq I_{k}^{j} + II_{k}^{j} + III_{k}^{j},
\]
where \( E(Q') = Q' \setminus \Omega_{k(Q')} + m + m_0 \) and \( E(Q'') = Q'' \setminus \Omega_{k(Q'')} + m + m_0 \). Now we trivially have
\[
\sum_{k,j: Q_j^k \in W_{\text{fin}}} T_j^k \leq \sum_{k,j: Q_j^k \in W_{\text{fin}}} \int_{E_j^k} M_{\omega}^D fin (M(f\sigma))^2 \, d\omega
\]
\[
\leq \int_{\mathbb{R}^n} M_{\omega}^D fin (M(f\sigma))^2 \, d\omega \leq C \int_{\mathbb{R}^n} (M(f\sigma))^2 \, d\omega,
\]

since the collection of sets \( \{E_j^k\}_{(k,j) \in W_{\text{fin}}} \) is pairwise disjoint in \( \mathbb{R}^n \).

To estimate the sum of the terms \( T_j^k \), we will require a bounded overlap constant for the collection of sets \( \{E(Q') : Q' \in \mathcal{C}(Q_j^k) \cap \hat{W}\} \). Recall that the cubes \( Q_j^k \) all belong to the Whitney grid \( W_{\text{fin}} \).

To obtain such a bounded overlap constant, suppose that \( T = \{Q_{\ell}\}_{\ell=1}^L \) is a strictly increasing consecutive tower of cubes \( Q_{\ell} \supseteq Q_{\ell+1} \) with \( Q_{\ell} \in \hat{W} \) (by consecutive we mean that every cube \( Q \) in \( \hat{W} \) that satisfies \( Q_1 \subset Q \subset Q_L \) is included in the tower \( T \)). By Lemma 2, we see that if \( Q_\ell \notin W_{\text{fin}}, \) then \( \pi Q_\ell \in W_{\text{fin}} \subset \hat{W} \) which shows that \( Q_{\ell+1} = \pi Q_\ell \notin W_{\text{fin}} \). Thus we see that at least half of the cubes in the tower belong to \( W_{\text{fin}} \). Now focus attention on the subtower \( S = \{Q_{\ell}\}_{\ell=1}^I \) is a strictly increasing consecutive of cubes which belong to \( W_{\text{fin}} \). It thus suffices to establish a bounded overlap constant for the subtower \( S \). However, there are clearly at most \( m + m_0 \) cubes in the tower \( S \) since \( E(Q_{\ell}) = Q_{\ell} \setminus \Omega_{k(Q_{\ell}) + m + m_0} \) where \( k(Q_{\ell}) \) is strictly decreasing in \( i \) because all the cubes \( Q_{\ell} \) belong to a common grid, namely \( D_{\text{fin}} \). Thus \( \#S = I \leq m + m_0 \) and \( \#T \leq 2m + 2m_0 \). It follows in particular that the collection of sets \( \{E(Q') : Q' \in \mathcal{C}(Q_j^k) \cap \hat{W}\} \) has bounded overlap at most \( 2m + 2m_0 \), and we conclude that
\[
\sum_{k,j: Q_j^k \in W_{\text{fin}}} T_j^k \leq \sum_{k,j: Q_j^k \in W_{\text{fin}}} \sum_{Q' \in \mathcal{C}(Q_j^k) \cap \hat{W}} \int_{E_j^k} M_{\omega}^D fin (M(f\sigma))^2 \, d\omega
\]
\[
\leq (2m + 2m_0) \int_{\mathbb{R}^n} M_{\omega}^D fin (M(f\sigma))^2 \, d\omega \leq C \int_{\mathbb{R}^n} (M(f\sigma))^2 \, d\omega.
\]

The sum of the terms \( III_j^k \) satisfies a similar estimate. Indeed, we have already shown above that the tower \( T = \{Q_{\ell}\}_{\ell=1}^L \) satisfies \( \#T \leq 2m + 2m_0 \), and it follows in particular that \( \{E(Q'') : Q'' \in \mathcal{C}^{(2)}(Q_j^k) \cap \hat{W}\} \) also has bounded overlap at most \( 2m + 2m_0 \). Altogether then we have
\[
\sum_{K \in W_{\text{fin}}} q^{***}(K) = \sum_{k,j: Q_j^k \in W_{\text{fin}}} q^{***}(Q_j^k) \leq 3 \sum_{k,j: Q_j^k \in W_{\text{fin}}} (T_j^k + II_j^k + III_j^k)
\]
\[
\leq C \int_{\mathbb{R}^n} M_{\omega}^D fin (M(f\sigma))^2 \, d\omega \leq C \int_{\mathbb{R}^n} (M(f\sigma))^2 \, d\omega,
\]

for all \( D_{\text{fin}} \in G^N_M \). Finally, we average over \( H_{D_{\text{fin}}} \) for each \( D_{\text{fin}} \in \Phi^N_M \) and use (2.8) to obtain
\[
(3.13) \quad E_\Omega^D (III^{*}_{r-bad} (M, N + r + 1)) \leq C 2^{-r} E_{\Phi^N_M}^D E_{H_{D_{\text{fin}}}}^D \left( \sum_{K \in W_D \cap D_M^N} q^{***}(K) \right)
\]
\[
\leq C 2^{-r} E_\Omega^D \left( C \int_{\mathbb{R}^n} (M(f\sigma))^2 \, d\omega \right) = C 2^{-r} \int (|M\sigma|)^2 \, d\omega,
\]

where the sum over cubes in \( III^{*}_{r-bad} (M, N + r + 1) \) on the left hand side satisfies (3.11). This estimate will be applied at the end of the proof in order to estimate \( III^{*}_{r-bad} \) by taking a supremum over cubes \( Q \) satisfying (3.11), i.e. \( 2^{-M} \leq \ell(Q) \leq 2^{-(N+r+1)} \).

3.3. Principal cube decomposition. Recall our convention regarding distinguished index pairs \((k,j)\): namely that \( k \equiv k_0 \mod (m + m_0) \) for some fixed \( 0 \leq k_0 \leq m + m_0 - 1 \), and that \( k \) is maximal among equal cubes \( Q_j^k \). Fix an integer \( L \in \mathbb{Z} \) (thought of as near \(-\infty\)) such that \( L \equiv k_0 \mod (m + m_0) \), and let \( G_0 \) consist of the \( D^\gamma \)-maximal cubes in \( \Omega_L \). With the grid \( W = W^\gamma \) in hand, we now introduce principal cubes as in [MuWh] page 804 (note that we are suppressing the dependence of \( W \) on \( \gamma \) for reduction of
notation). If \(G_n\) has been defined, let \(G_{n+1}\) consist of those indices \((k,j)\) for which \(Q^k_j \in \mathcal{W}\), there is an index \((t,u) \in G_n\) with \(k \geq t\) and \(Q^k_j \subset Q^t_u\), and

(i) \(A^k_t > \eta A^k_u\),

(ii) \(A^k_t \leq \eta A^k_u\) whenever \(Q^k_j \not\subset Q^t_u \subset Q^u_u\).

Here \(\eta\) is any constant larger than 1, for example \(\eta = 4\) works fine. Now define \(\Gamma \equiv \bigcup_{n=0}^{\infty} G_n\) and for each index \((k,j)\) define \(P(Q^k_j)\) to be the smallest dyadic cube \(Q^k_{\diamond}\) containing \(Q^k_j\) and with \((t,u) \in \Gamma\). Then we have

\[
(i) \quad P(Q^k_j) = Q^t_u \quad \implies \quad A^k_t \leq \eta A^k_u, \\
(ii) \quad Q^k_j \not\subset Q^t_u \quad \text{with} \quad (k,j),(t,u) \in \Gamma \quad \implies \quad A^k_t > \eta A^k_u.
\]

Now we return to the sum of \(III^* (Q^k_j)\) over \((k,j)\) satisfying \(k \geq L\), and decompose the sum over \(i\) inside \(III^* (Q^k_j)\) according to whether \((k+m+m_0,i) \in \Gamma\) or the predecessor \(P(Q^k_{i+m+m_0})\) of \(Q^k_{i+m+m_0}\) in the grid \(\Gamma\) coincides with the predecessor \(P(Q^k_j)\) of \(Q^k_j\):

\[
\sum_{k \in \mathbb{Z}, k \geq L} \sum_{j \in \mathbb{N}} III^* (Q^k_j) \lesssim \sum_{k \in \mathbb{Z}, k \geq L} \sum_{j \in \mathbb{N}} \frac{|E^k_j|}{|3Q^k_j|^2} \left[ \sum_{i \in \mathbb{N}: P(Q^k_{i+m+m_0}) = P(Q^k_j)} A^{k+m+m_0}_i \int_{Q^k_{i+m+m_0}} L^k_j (1_{E^k_j}) d\sigma \right]^2 \\
+ \sum_{k \in \mathbb{Z}, k \geq L} \sum_{j \in \mathbb{N}} \frac{|E^k_j|}{|3Q^k_j|^2} \left[ \sum_{i \in \mathbb{N}: (k+m+m_0,i) \in \Gamma} A^{k+m+m_0}_i \int_{Q^k_{i+m+m_0}} L^k_j (1_{E^k_j}) d\sigma \right]^2 \\
=: IV + V.
\]

It is relatively easy to estimate term \(V\) by the Cauchy-Schwarz inequality and (3.14),

\[
(3.15) \quad V = \sum_{k \in \mathbb{Z}, k \geq L} \sum_{j \in \mathbb{N}} \frac{|E^k_j|}{|3Q^k_j|^2} \left[ \sum_{i \in \mathbb{N}: (k+m+m_0,i) \in \Gamma} A^{k+m+m_0}_i \int_{Q^k_{i+m+m_0}} L^k_j (1_{E^k_j}) d\sigma \right]^2 \\
\leq \sum_{k \in \mathbb{Z}, k \geq L} \sum_{j \in \mathbb{N}} \frac{|E^k_j|}{|3Q^k_j|^2} \left[ \sum_{i \in \mathbb{N}: (k+m+m_0,i) \in \Gamma} |Q^k_{i+m+m_0}|_{\sigma} (A^{k+m+m_0}_i)^2 \right] \\
\times \left[ \sum_{i \in \mathbb{N}: (k+m+m_0,i) \in \Gamma} \left( \int_{Q^k_{i+m+m_0}} L^k_j (1_{E^k_j}) d\sigma \right)^2 |Q^k_{i+m+m_0}|_{\sigma}^{-1} \right] \\
\leq \sum_{k \in \mathbb{Z}, k \geq L} \sum_{j \in \mathbb{N}} \frac{|E^k_j|}{|3Q^k_j|^2} \left[ \sum_{i \in \mathbb{N}: (k+m+m_0,i) \in \Gamma} |Q^k_{i+m+m_0}|_{\sigma} (A^{k+m+m_0}_i)^2 \right] \int_{Q^k_j} L^k_j (1_{Q^k_j})^2 d\sigma \\
\lesssim A_2 \sum_{(t,u) \in \Gamma} (A^k_u)^2 |Q^k_{\diamond}|_{\sigma} \lesssim A_2 \|f\|_{L^2(\sigma)}^2.
\]
Thus we are left to estimate term $IV$ which we decompose as

$$IV = \sum_{k \in \mathbb{Z}, k \geq L, j \in \mathbb{N} \text{ and } Q^j_i \in \mathcal{D}_{r - \text{good}}} \frac{|E^j_i|_\omega}{|3Q^j_i|^2} \left[ \sum_{i \in \mathbb{N}: P(Q^i_k + m + m_0) = P(Q^j_i)} A^{k+m+m_0} \int_{Q^i_k + m + m_0} L^j_i \left(1_{E^j_i} \right) \sigma \right]^2$$

$$+ \sum_{k \in \mathbb{Z}, k \geq L, j \in \mathbb{N} \text{ and } Q^j_i \in \mathcal{D}_{r - \text{bad}}} \frac{|E^j_i|_\omega}{|3Q^j_i|^2} \left[ \sum_{i \in \mathbb{N}: P(Q^i_k + m + m_0) = P(Q^j_i)} A^{k+m+m_0} \int_{Q^i_k + m + m_0} L^j_i \left(1_{E^j_i} \right) \sigma \right]^2$$

$$= IV_{r - \text{good}} + IV_{r - \text{bad}}.$$

Fix $(t,u)$, and consider the sum

$$IV_{r - \text{good}} = \sum_{Q^j_i \in \mathcal{W} \cap \mathcal{D}_{r - \text{good}}: Q^j_i \subset Q^t_u} \frac{|E^j_i|_\omega}{|3Q^j_i|^2} \left[ \sum_{i \in \mathbb{N}: P(Q^i_k + m + m_0) = P(Q^j_i)} A^{k+m+m_0} \int_{Q^i_k + m + m_0} L^j_i \left(1_{E^j_i} \right) \sigma \right]^2$$

$$\leq C \left( \sum_{Q^j_i \in \mathcal{W} \cap \mathcal{D}_{r - \text{good}}: Q^j_i \subset Q^t_u} q(Q) \right) (A^t_u)^2 = CS^t_u \left(A^t_u\right)^2,$$

where

$$S^t_u = \sum_{Q \in \mathcal{W} \cap \mathcal{D}_{r - \text{good}}: Q \subset Q^t_u} q(Q).$$

It is here in estimating $S^t_u$, that the only quantitative use of the triple testing condition occurs.

**Lemma 3.** We claim that

$$S^t_u \leq C \left( (\mathcal{I}_M(3))^2 + A_2 \right) |Q^t_u|_\sigma.$$

**Proof.** Let $\{K_i\}_{i \in I}$ be the collection of maximal $D$-cubes $K_i$ satisfying $5K_i \subset Q^t_u$. Then for all cubes $K_i$ we have

$$\sum_{Q \in \mathcal{W}: Q \subset K_i} q(Q) \leq \sum_{Q^j_i \in \mathcal{W}, Q^j_i \subset K_i} |E^j_i|_\omega \left[ \frac{1}{|Q^j_i|_\omega} \int_{Q^j_i} 1_{K_i} \mathcal{M}(1_{K_i} \sigma) d\omega \right]^2$$

$$\leq C \int_{K_i} \left[ \mathcal{M}^D \left(1_{K_i} \mathcal{M}(1_{K_i} \sigma)\right)\right]^2 d\omega$$

$$\leq C \int_{K_i} \mathcal{M}(1_{K_i} \sigma)^2 d\omega \lesssim (\mathcal{I}_M(3))^2 |3K_i|_\sigma.$$

Thus we have

$$\sum_{i \in I} \sum_{Q \in \mathcal{W}: Q \subset K_i} q(Q) \leq \sum_{i \in I} \left( \mathcal{I}_M(3))^2 |3K_i|_\sigma \leq C_{\text{bound}} (\mathcal{I}_M(3))^2 |Q^t_u|_\sigma,$$

where $C_{\text{bound}}$ is a constant such that $\sum_{i \in I} 1_{3K_i} \leq C_{\text{bound}} 1_{Q^t_u}$. We also have

$$\sum_{Q \in \mathcal{W}: Q \subset Q^t_u \text{ and } \ell(Q) \geq 2^{-r}(Q^t_u)} q(Q) \leq C 2^{n_{\text{bound}} A_2} |Q^t_u|_\sigma.$$  

Finally, we note that if a cube $Q \in \mathcal{W}$ is contained in $Q^t_u$ and satisfies $\ell(Q) < 2^{-r}(Q^t_u)$, but is not contained in any $K_i$, then $Q$ is $r$-bad. Indeed, if we consider the tiling of $Q^t_u$ by dyadic subcubes $Q$ of side length $\ell(Q) = 2^{-m} \ell(Q^t_u)$ for some fixed $m > r$, then the only cubes $Q$ in this tiling that do not satisfy $5Q \subset Q^t_u$ are those for which $\overline{3Q} \cap \overline{Q^t_u} \neq \emptyset$. This completes the proof of (3.16) and hence that of Lemma 3. \hfill \square
Then summing over \((t, u) \in \Gamma\) we obtain
\[
(3.17) \quad IV = IV_{r-\text{good}} + IV_{r-\text{bad}} = \sum_{(t, u) \in \Gamma} (A_t^u)^2 IV_{r-\text{good}} + IV_{r-\text{bad}}
\]
which combined with (3.15) gives
\[
\leq \left((\mathcal{I}_M(3))^2 + A_2\right) \sum_{(t, u) \in \Gamma} |Q_{tu}| \sigma (A_t^u)^2 + III_{r-\text{bad}}
\]
\[
\leq \left((\mathcal{I}_M(3))^2 + A_2\right) \|f\|_{L^2(\sigma)}^2 + III_{r-\text{bad}},
\]
which by the estimates (3.2), (3.5) and (3.18), together with (3.17), then gives
\[
(3.19) \quad \int_{\mathbb{R}^n} |\mathcal{M}(f\sigma)(x)|^2 d\omega(x) \leq E^P_{\Omega} \int_{\mathbb{R}^n} |\mathcal{M}^P(f\sigma)(x)|^2 d\omega(x)
\]
\[
\leq E^P_{\Omega} \left(\sum_{(k, j) \in \Pi_1} 2^{2k} + 3^n C_n 2^{-2m_0} \int |\mathcal{M}(f\sigma)|^2 d\omega\right)
\]
\[
\leq E^P_{\Omega} \left(\sum_{(k, j) \in \Pi_2} 2^{2k} + \sum_{(k, j) \in \Pi_3} 2^{2k} + 2^{2k}\right) + 3^n C_n 2^{-2m_0} \int |\mathcal{M}(f\sigma)|^2 d\omega,
\]
which by the estimates (3.2), (3.5) and (3.18), together with (3.17), then gives
\[
(3.20) \quad \int_{\mathbb{R}^n} |\mathcal{M}(f\sigma)(x)|^2 d\omega(x)
\]
\[
\leq (\beta + 2^{-2m_0}) \int \mathcal{M}(f\sigma)^2 d\omega + \beta^{-1} C_{m+m_0} A_2 \|f\|_{L^2(\sigma)}^2 + \left((\mathcal{I}_M(3))^2 + A_2\right) \|f\|_{L^2(\sigma)}^2 + \sup_{N < 0 < M} E^P_{\Omega} (III_{r-\text{bad}}(M, N + r + 1))
\]
\[
\leq (\beta + 2^{-2m_0} + 2^{-r}) \int \mathcal{M}(f\sigma)^2 d\omega + \beta^{-1} C_{m+m_0} A_2 \|f\|_{L^2(\sigma)}^2 + \left((\mathcal{I}_M(3))^2 + A_2\right) \|f\|_{L^2(\sigma)}^2.
\]
Now we can absorb the first term on the right hand side by choosing \(\beta > 0\) sufficiently small and \(m_0\) and \(r\)
sufficiently large since the integral \(\int \mathcal{M}(f\sigma)^2 d\omega\) is finite. Then we take the supremum over \(f \in L^2(\sigma)\) with \(\|f\|_{L^2(\sigma)} = 1\) to obtain
\[
\mathcal{R}_M \leq C \left(\mathcal{I}_M(3) + \sqrt{A_2}\right).
\]
As the opposite inequality is trivial, this completes the proof of Theorem 2.

4. WEAK TRIPLE TESTING

Now we adapt the previous arguments to prove our main result, Theorem 3. Recall that given a pair \((\sigma, \omega)\) of weights (i.e. locally finite positive Borel measures) in \(\mathbb{R}^n\) and \(D, \Gamma > 1\), we say that \((\sigma, \omega)\) satisfies the \(D\)-\(\Gamma\)-testing condition for the maximal function \(\mathcal{M}\) if there is a constant \(\mathcal{I}_M^D(\Gamma)(\sigma, \omega)\) such that
\[
\int_Q |\mathcal{M}Q\sigma|^2 d\omega \leq \mathcal{I}_M^D(\Gamma)(\sigma, \omega)^2 |Q|_{\sigma}, \quad \text{for all } Q \in \mathcal{P}^n \text{ with } |\Gamma Q|_{\sigma} \leq D |Q|_{\sigma},
\]
and if so we denote by \(\mathcal{I}_M^D(\Gamma)(\sigma, \omega)\) the least such constant. Here again is the main result of this paper.

**Theorem 3.** Let \(\Gamma > 1\). Then there is \(D > 1\) depending only on \(\Gamma\) and the dimension \(n\) such that
\[
(4.1) \quad \mathcal{R}_M(\sigma, \omega) \approx \mathcal{I}_M^D(\Gamma)(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)},
\]
for all locally finite positive Borel measures \(\sigma\) and \(\omega\) on \(\mathbb{R}^n\).
To begin the proof, we point out the well known fact that for locally finite positive Borel measures $\sigma$ and $\omega$,  

$$P_\Omega (\{ D \in \Omega : |\partial Q|_\sigma + |\partial Q|_\omega > 0 \text{ for some } Q \in D \}) = 0.$$  

Indeed, for $0 \leq k \leq n - 1$, there are at most countably many $k$-planes parallel to the coordinate $k$-planes that are charged by $\sigma + \omega$. Now note that with probability zero, a random grid $D \in \Omega$ includes a cube $Q \in D$ whose boundary $\partial Q$ contains one of these countably many $k$-planes. More precisely, consider the subcase of hyperplanes $(k = n - 1)$ parallel to the hyperplane  

$$P_n \equiv \{(x_1, ..., x_{n-1}, 0) : (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\}$$  

that passes through the origin and is perpendicular to the $x_n$-axis. Let $F \equiv \{z \in \mathbb{R} : |P_n + (0, 0, ..., 0, z)|_{\sigma + \omega} > 0\}$. If $B(0, j) = \{x \in \mathbb{R}^n : |x| < j\}$ is the ball of radius $j$, then the sets  

$$F_j \equiv \{z \in \mathbb{R} : |B(0, j) \cap (P_n + (0, 0, ..., 0, z)|_{\sigma + \omega} > \frac{1}{j}\}$$  

are clearly finite for each $j$ since the measure $\sigma + \omega$ is locally finite, i.e. $|B(0, j)|_{\sigma + \omega} < \infty$, and it follows that $F$ is at most countable. Now if $D \in \Omega$ is any grid, and if $\partial_n D$ denotes the collection of all hyperplanes $P$ that are parallel to $P_n$ and contain a face of a dyadic cube from $D$, then $\partial_n D$ is countable. Thus with $D_0$ equal to the standard dyadic grid on $\mathbb{R}^n$, this shows that the set of $t \in \mathbb{R}^n$ such that $\partial_n (D_0 + t) \cap F \neq \emptyset$ has Lebesgue measure zero, and thus that  

$$P_\Omega (\{ D \in \Omega : \partial_n D \cap F \neq \emptyset \}) = 0.$$  

Now we repeat this calculation for hyperplanes parallel to $P_1$, where $P_1$ is the hyperplane perpendicular to the $x_i$-axis. And then we perform similar calculations for $k$-planes with $0 \leq k \leq n - 2$. The case $k = 0$ is particularly easy since the set of points in $\mathbb{R}^n$ that are charged by $\sigma + \omega$ is clearly countable.

We say that a random grid $D \in \Omega$ has null boundaries if $|\partial Q|_{\sigma + \omega} = 0$ for all cubes $Q \in D$, and set $\Omega_{null} = \{ D \in \Omega : D \text{ has null boundaries} \}$, $P_{null} = \bigcup_{D \in \Omega_{null}} D$ and for any positive Borel measure $f$ on $\mathbb{R}^n$,  

$$M_{null} f (x) \equiv \sup_{Q \in P_{null} : x \in Q} \frac{1}{|Q|} \int_Q f.$$  

Then, using that $P_\Omega (\Omega \setminus \Omega_{null}) = 0$, equivalently $P_\Omega (\Omega_{null}) = 1$, together with (2.9) in Lemma 1 we have  

$$M f (x) \leq 2^{n+3} E_\Omega \sup_{D \in P_{null}} M_D f (x) = 2^{n+3} E_{\Omega_{null}} M_D f (x)$$  

$$\leq 2^{n+3} \sup_{D \in P_{null}} M_D f (x) \leq 2^{n+3} \sup_{Q \in P_{null} : x \in Q} \frac{1}{|Q|} \int_Q f = M_{null} f (x) \leq 2^{n+3} M f (x)$$  

for all positive Borel measures $f$ on $\mathbb{R}^n$. Thus we have the pointwise equivalence  

$$M_{null} f (x) \approx \sup_{D \in P_{null}} M_D f (x) \approx M f (x),$$  

and in particular, we conclude that (4.1) is equivalent to  

$$\mathfrak{M}_{null} (\sigma, \omega) \approx E_M (\Gamma) (\sigma, \omega) + \sqrt{A_2 (\sigma, \omega)}.$$  

We complete the proof of (4.4), and hence of Theorem 1 by modifying the proof of Theorem 2 in the following seven steps.

**Step 1:** There were only two places in the proof of Theorem 2 where the hypothesis of triple testing was used:

1. qualitatively, at the beginning of the argument, in order to assume without loss of generality that  
   $$\int M (f \sigma)^2 d\omega < \infty,$$
2. and quantitatively, near the end of the argument, in the proof of Lemma 3
   
   The qualitative use of the triple testing condition is easily handled using $D$-triple testing as follows. If we replace $\omega$ by $\omega_N = \omega 1_{B(0, N)}$ with $N > R$ where $\omega$ is supported in $B(0, R)$, then the $D$-triple testing condition and $A_2$ condition still hold, and with constants no larger than before. Moreover, the testing
condition for the cube $Q_m = [-3^m N, 3^m N]$ must hold for some $m \geq 0$, since otherwise iteration of the inequality $|Q_m|_\sigma \leq \frac{1}{D} |Q_{m+1}|_\sigma$ eventually violates the $A_2$ condition,
\[
A_2 (\sigma, \omega) \geq \frac{|Q_m|_\sigma |Q_m|_\omega}{|Q_m|^2} \geq \frac{D^m |Q_0|_\sigma |Q_0|_\omega}{2^{mn} |Q_0|^2} = \left( \frac{D}{2^{2n}} \right)^m \frac{|Q_0|_\sigma |Q_0|_\omega}{|Q_0|^2},
\]
if $D$ is chosen greater than $2^{2n+1}$. Thus if the testing condition holds for the cube $Q_m$ we have
\[
\int \mathcal{M} (f \sigma)^2 \, d\omega_N \leq \|f\|_{L^\infty} \int_{B(0,N)} \mathcal{M}(1_{Q_m})^2 \, d\omega < \infty,
\]
and therefore, without loss of generality, we can assume \( \int \mathcal{M}(f \sigma)_\omega \, d\omega < \infty \).
For the quantitative use of the triple testing condition, recall that Lemma 3 asserted
\[
S_{r_{\sigma}}^{t,u} = \sum_{Q \in \mathcal{W}: \mathcal{W}_{\sigma}^{Q} \subseteq Q} q(Q) \leq C \left( (\mathfrak{T}_{\mathcal{M}}(3))^2 + A_2 \right) |Q_u|_\sigma,
\]
where $\mathcal{W}$ was the Whitney grid associated with a given dyadic grid $D \in \Omega$, and $\mathcal{W}_{\sigma}^{Q}$ was the associated subgrid of $r_{\sigma}$ -- good cubes. The triple testing condition was used only in the inequality
\[
(4.5) \qquad \sum_{Q \in \mathcal{W}: \mathcal{W}_{\sigma}^{Q} \subseteq K_i} q(Q) \leq C \int_{K_i} \mathcal{M}(1K_i)^2 \, d\omega \leq C (\mathfrak{T}_{\mathcal{M}}(3))^2 |3K_i|_\sigma.
\]
However, if we only have the testing condition over $D$-tripling cubes, then we have
\[
\int_{K_i} \mathcal{M}(1K_i)^2 \, d\omega \leq (\mathfrak{T}_{\mathcal{M}}(3))^2 |K_i|_\sigma \leq (\mathfrak{T}_{\mathcal{M}}(3;D))^2 |3K_i|_\sigma, \quad \text{if } |3K_i|_\sigma \leq D |K_i|_\sigma,
\]
where in the testing condition
\[
\mathfrak{T}_{\mathcal{M}}(3;D) \equiv \sup_{K \in \mathcal{D}} \sqrt{\int_{K_i} \mathcal{M}(1K)^2 \, d\omega},
\]
the supremum is taken over only $D$-dyadic cubes $K$ satisfying $|3K|_\sigma \leq D |K|_\sigma$. On the other hand, for the $D$-nontripling cubes $K_i$, we can only use the inequality
\[
\int_{K_i} \mathcal{M}(1K_i)^2 \, d\omega \leq (\mathfrak{T}_{\mathcal{M}})^2 |K_i|_\sigma \leq \frac{1}{D} (\mathfrak{T}_{\mathcal{M}})^2 |3K_i|_\sigma, \quad \text{if } |3K_i|_\sigma > D |K_i|_\sigma,
\]
where
\[
\mathfrak{T}_{\mathcal{M}} \equiv \sup_{K \in \mathcal{P}} \sqrt{\int_{K_i} \mathcal{M}(1K)^2 \, d\omega},
\]
and this gives
\[
\int_{K_i} \mathcal{M}(1K_i)^2 \, d\omega \leq \frac{1}{D} (\mathfrak{T}_{\mathcal{M}})^2 |3K_i|_\sigma, \quad \text{if } |3K_i|_\sigma > D |K_i|_\sigma.
\]
Altogether then we obtain
\[
\sum_{Q \in \mathcal{W}: \mathcal{W}_{\sigma}^{Q} \subseteq K_i} q(Q) \leq C \left[ (\mathfrak{T}_{\mathcal{M}})^2 (\mathfrak{T}_{\mathcal{M}}(3; D))^2 + \frac{1}{D} (\mathfrak{T}_{\mathcal{M}})^2 \right] |3K_i|_\sigma,
\]
and the only difference from (4.5) is that $(\mathfrak{T}_{\mathcal{M}}(3))^2$ has been replaced with $(\mathfrak{T}_{\mathcal{M}}(3; D))^2 + \frac{1}{D} (\mathfrak{T}_{\mathcal{M}})^2$. As a consequence, the previous inequalities (4.19) and (4.20), together with the fact that $P_{\Omega} (\Omega \setminus \Omega^{\text{null}}) = 0$, can be modified to yield the inequalities (where $\Omega$ gets replaced by $\Omega^{\text{null}}$),
\[
\int_{\mathbb{R}^n} [\mathcal{M}(f \sigma)(x)]^2 \, d\omega(x) \lesssim E_{\Omega^{\text{null}}}^{D} \int_{\mathbb{R}^n} [\mathcal{M}^D(f \sigma)(x)]^2 \, d\omega(x)
\]
\[
\lesssim E_{\Omega^{\text{null}}}^{D} \left( \sum_{(k,j) \in \Pi_1} 2^{2k} + \sum_{(k,j) \in \Pi_2} 2^{2k} + \sum_{(k,j) \in \Pi_3} 2^{2k} \right) + 3^n C_n 2^{-2m_0} \int [\mathcal{M}(f \sigma)]^2 \, d\omega,
\]
and
\[ \int_{\mathbb{R}^n} |M(f\sigma)(x)|^2 \, d\omega(x) \]
\[ \lesssim (\beta + 2^{-2m_0}) \int_{\mathbb{R}^n} |M(f\sigma)|^2 \, d\omega + \beta^{-1} C_{m+m_0}^2 A_2 \|f\|_{L^2(\sigma)}^2 + \left( (\Sigma_M(3; \mathcal{P}^{\text{null}}))^2 + A_2 \right) \|f\|_{L^2(\sigma)}^2 \]
\[ + \sup_{N<0<\mathcal{M}} E_{\Omega^{\text{null}}}^N (III_{r-\text{bad}}(M, N + r + 1)) \]
\[ \lesssim (\beta + 2^{-2m_0} + 2^{-r}) \int_{\mathbb{R}^n} |M(f\sigma)|^2 \, d\omega + \beta^{-1} C_{m+m_0}^2 A_2 \|f\|_{L^2(\sigma)}^2 \]
\[ + \left( (\Sigma_M(3; \mathcal{P}^{\text{null}}))^2 + A_2 + \frac{1}{\sqrt{D}} (\Sigma_M)^2 \right) \|f\|_{L^2(\sigma)}^2, \]

which, after absorption of the first term on the right hand side, give the conclusion that
\[ \mathcal{N}_M(\sigma, \omega) \leq C \left( \Sigma_M^D(3; \mathcal{P}^{\text{null}})(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)} + \frac{1}{\sqrt{D}} \Sigma_M(\sigma, \omega) \right), \]

where
\[ \Sigma_M^D(3; \mathcal{P}^{\text{null}}) \equiv \sup_{Q \in \mathcal{P}^{\text{null}}} \frac{1}{|Q|_\sigma} \int_{Q} M(1_K \sigma)^2 \, d\omega \approx \sup_{Q \in \mathcal{P}^{\text{null}}} \frac{1}{|Q|_\sigma} \int_{Q} M^{\text{null}}(1_K \sigma)^2 \, d\omega. \]

Thus in the testing constant $\Sigma_M^D(3; \mathcal{P}^{\text{null}})$, the supremum over cubes $Q$ is restricted to those cubes $Q$ satisfying both $|\partial Q|_\sigma + |\partial Q|_\omega = 0$ and $|Q|_\sigma \leq D |Q|_\sigma$.

**Step 2:** If $\Sigma_M(\sigma, \omega) < \infty$, then Step 1 shows that $\mathcal{N}_M(\sigma, \omega) < \infty$, and since we trivially have $\Sigma_M(\sigma, \omega) \leq \mathcal{N}_M(\sigma, \omega)$, we can then absorb the term $\frac{1}{\sqrt{D}} \Sigma_M(\sigma, \omega)$ into the left hand side of the inequality to obtain the apriori inequality
\[ \mathcal{N}_M(\sigma, \omega) \leq C \left( \Sigma_M^D(3; \mathcal{P}^{\text{null}})(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)} \right), \]

whenever $\Sigma_M(\sigma, \omega) < \infty$.

However, noting that all of the above holds with $\Gamma' > 1$ in place of $3$ and a corresponding $D' = D'(\Gamma', n) > 1$ in place of $D$, we obtain
\[ \mathcal{N}_M(\sigma, \omega) \leq C \left( \Sigma_M^{D'}(\Gamma'; \mathcal{P}^{\text{null}})(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)} \right), \]

for a fixed $D' = D'(\Gamma', n) > 1$ depending only on $\Gamma'$ and dimension $n$, and where the cubes $Q$ used to define the testing condition $\Sigma_M^{D'}(\Gamma'; \mathcal{P}^{\text{null}})(\sigma, \omega)$ are restricted to those $Q$ satisfying both $|\partial Q|_\sigma + |\partial Q|_\omega = 0$ and $|Q|_\sigma \leq D' |Q|_\sigma$.

### 4.1. Approximation by mollified weights

It remains to appropriately approximate the measure pair $(\sigma, \omega)$ by a family of measure pairs $(\sigma_\varepsilon, \omega_\varepsilon)$ for which $\mathcal{N}_M(\sigma_\varepsilon, \omega_\varepsilon) < \infty$. A standard mollification will serve this purpose.

**Step 3:** Suppose that $\omega$ is supported in the compact cube $K = Q(0, R) \equiv [-R, R]^n$. Fix $\varphi : (-1, 1)^n \to [0, 1]$ smooth and compactly supported in $(-1, 1)^n$ with $\varphi \geq 2^{-n}$ on $(-\frac{5}{6}, \frac{5}{6})^n$ and $\int \varphi = 1$. For $0 < \varepsilon < 1$ define $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi \left( \frac{x}{\varepsilon} \right)$ and
\[ \sigma_\varepsilon \equiv \sigma * \varphi_\varepsilon \text{ and } \omega_\varepsilon' \equiv \omega * \varphi_\varepsilon', \quad 0 < \varepsilon, \varepsilon' < 1. \]

We claim that
\[ \Sigma_M(\sigma_\varepsilon, \omega_\varepsilon') < \infty, \quad \text{for } 0 < \varepsilon, \varepsilon' < \frac{1}{4}. \]
Indeed, $d\sigma(x) = s(x) \, dx$ and $dw(x) = w(x) \, dx$ where $s(x)$ and $w(x)$ are smooth functions, and thus if $Q \in \mathcal{P}$ we have

$$\int_Q \mathcal{M}(1_Q\sigma)(x)^2 \, dx \leq \|w\|_\infty \int_Q \mathcal{M}(1_Q\sigma)(x)^2 \, dx \leq C_{\text{class}}^2 \|w\|_\infty \int_Q \sigma(x)^2 \, dx \leq C_{\text{class}}^2 \|w\|_\infty \|1_Q\sigma\|_\infty \int_Q \sigma(x) \, dx,$$

where $C_{\text{class}}$ is the classical bound for $M$ on (unweighted) $L^2$. Now $\|w\|_\infty < \infty$ since $\omega$ is compactly supported, and by the same token, $\sup_{Q \subseteq 3K} \|1_Q\sigma\|_\infty \leq \|1_{3K}\sigma\|_\infty < \infty$, yielding

$$\int_Q \mathcal{M}(1_Q\sigma)(x)^2 \, dx \leq C_{\text{class}}^2 \int_Q \sigma(x) \, dx, \quad \text{for } Q \subset 3K.$$

So it only remains to consider a cube $Q$ with $Q \cap B_K(\varepsilon') \neq \emptyset$ and $Q \cap [\mathbb{R}^n \setminus 3K] \neq \emptyset$, where $B_K(\varepsilon') = \cup_{\varepsilon \in K} Q(x, \varepsilon')$. We may assume $K$ is big enough, e.g., $\ell(K) = 2R > 100$. Then $B_K(\varepsilon') \subset \frac{51}{50} K$ and we can write

$$\int_Q \mathcal{M}(1_Q\sigma)(x)^2 \, dx \leq \int_{Q \cap B_K(\varepsilon')} \mathcal{M}(1_Q\sigma)(x)^2 \, dx + \int_{Q \cap B_{\varepsilon'}(\varepsilon')} \mathcal{M}(1_Q\sigma)(x)^2 \, dx,$$

where the first term is handled using the estimates in the above, i.e.

$$\int_{Q \cap B_K(\varepsilon')} \mathcal{M}(1_Q\sigma)(x)^2 \, dx \leq C_{\text{class}}^2 \|w\|_\infty \|1_{3K}\sigma\|_\infty \int_Q \sigma(x) \, dx \leq C_{\text{class}}^2 \int_Q \sigma(x) \, dx,$$

and the second term satisfies

$$\int_{Q \cap B_{\varepsilon'}(\varepsilon')} \mathcal{M}(1_Q\sigma)(x)^2 \, dx \leq C \left[ \frac{51}{50} K \right]_{\varepsilon'}^\omega \mathcal{M}(1_Q\sigma)(cK)^2 \leq C \left| \partial Q \right| \frac{1}{3^m |K|} \left[ \int_{Q \cap (3^m K \setminus 3K)} d\sigma(x) \right]^2 \leq C \left[ \sup_{\ell \geq 2} \frac{|3^\ell K|}{3^{m \ell} |K|^2} \right] \left| \partial Q \right| \mathcal{M}(1_Q\sigma)^2 \leq C A_2(\varepsilon, \omega) \|Q\|_{\sigma_e} \leq 3^{\ell(K)} |\varepsilon| < \infty,$$

where we have used the fact that $|3^\ell K|_{\sigma_e} \leq |B(3^\ell K, \varepsilon)|_{\sigma_e} \leq \frac{51}{50} 3^{\ell(K)} |\varepsilon|$ and similar estimates for $|3^\ell K|_{\omega.e}$.

**Step 4**: Combining Steps 2 and 3, we obtain

$$(4.6) \quad \mathfrak{R}_M(\varepsilon, \omega) \leq C \left( \mathfrak{X}_M^D(\Gamma', \mathfrak{b}^\text{null}) \left( \varepsilon, \omega \right) + \sqrt{A_2(\varepsilon, \omega)} \right), \quad \text{for } 0 < \varepsilon, \varepsilon' < 1,$$

for a fixed $D' = D'(\Gamma', n) > 1$ depending only on $\Gamma'$ and dimension $n$, and where the cubes $Q$ used to define the testing condition $\mathfrak{X}_M^D(\Gamma', \mathfrak{b}^\text{null}) \left( \varepsilon, \omega \right)$ are restricted to those $Q$ satisfying both $|\partial Q|_{\sigma} + |\partial Q|_{\omega} = 0$ and $|\Gamma'Q|_{\sigma} \leq D' |\Gamma'Q|_{\omega}$.

We will now prove the general statement in Theorem $\text{H}$ for $\Gamma > 1$, namely that given $\Gamma > 1$, there is $D = D(\Gamma, n) > 1$ such that (4.3) holds. We will do this by choosing any fixed $\Gamma' > \Gamma$, e.g. $\Gamma' = \Gamma + 1$ works just fine, and then proving that (4.4) holds with the constant $D$ given by

$$D = 2^n D'(\Gamma', n),$$

where $D'(\Gamma', n)$ is the constant in (4.6).

From this point on we will consider only pairs $(\varepsilon, \varepsilon')$ with $\frac{\varepsilon}{\varepsilon'} = 8$, and so we will replace the pair $(\varepsilon, \varepsilon')$ with $(8\varepsilon, \varepsilon)$. We claim that for $\Gamma'$ and $D'$ chosen as above, namely $\Gamma' > \Gamma$ and $D' > 1$ so that (4.6) holds,
then we have

$$\mathcal{N}_{\text{null}} (\sigma, \omega) \leq \liminf_{\varepsilon \searrow 0} \mathcal{N}_M (\sigma_{8\varepsilon}, \omega_\varepsilon)$$

$$\leq \liminf_{\varepsilon \searrow 0} \mathcal{Y}_M (\Gamma') (\sigma_{8\varepsilon}, \omega_\varepsilon) + \liminf_{\varepsilon \searrow 0} \sqrt{A_2 (\sigma_{8\varepsilon}, \omega_\varepsilon)}$$

$$\leq \mathcal{Y}_M (\Gamma) (\sigma, \omega) + \sqrt{A_2 (\sigma, \omega)},$$

which, once established, will complete the proof of (4.4), and hence that of Theorem 1. We will prove (4.7) by proving three assertions, namely

$$\mathcal{N}_{\text{null}} (\sigma, \omega) \leq \liminf_{\varepsilon \searrow 0} \mathcal{N}_M (\sigma_{8\varepsilon}, \omega_\varepsilon),$$

$$\sup_{0 < \varepsilon < \frac{\delta}{2}} \sqrt{A_2 (\sigma_{8\varepsilon}, \omega_\varepsilon)} \leq \sqrt{A_2 (\sigma, \omega)},$$

$$\sup_{0 < \varepsilon < \frac{\delta}{2}} \mathcal{Y}_M (\Gamma') (\sigma_{8\varepsilon}, \omega_\varepsilon) \leq \mathcal{Y}_M (\Gamma) (\sigma, \omega) + \sqrt{A_2 (\sigma, \omega)}.$$

**Step 5**: We begin with the first line in (4.8), and prove that

$$\mathcal{N}_{\text{null}} (\sigma, \omega) \leq \mathcal{I}_{\text{null}} (\mathcal{P}^{\text{null}}) (\sigma, \omega) \leq \liminf_{\varepsilon \searrow 0} \mathcal{N}_M (\sigma_{8\varepsilon}, \omega_\varepsilon),$$

where

$$\mathcal{I}_{\text{null}} (\mathcal{P}^{\text{null}}) (\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}^{\text{null}}} \sqrt{\frac{1}{|Q|_\sigma} \int \mathcal{M} (1_Q \sigma)^2 d\omega}.$$  

The first inequality in (4.9) follows from (4.3) and $\mathcal{N}_M (\sigma, \omega) \leq \mathcal{I}_{\text{null}} (\mathcal{P}^{\text{null}}) (\sigma, \omega)$, which in turn follows from the observation that, with probability one, the grids $D \in \Omega$ used in (2.3) of Lemma 1 above have null boundaries, and so contain only cubes belonging to the collection $\mathcal{P}^{\text{null}}$. Thus Theorem 2 yields the even stronger conclusion that

$$\mathcal{N}_M (\sigma, \omega) \leq \mathcal{I}_M (3; \mathcal{P}^{\text{null}}) (\sigma, \omega);$$

where

$$\mathcal{I}_M (3; \mathcal{P}^{\text{null}}) (\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}^{\text{null}}} \sqrt{\frac{1}{|3Q|_\sigma} \int \mathcal{M} (1_Q \sigma)^2 d\omega}.$$  

Now we turn to proving the second inequality in (4.9). Fix $Q \in \mathcal{P}^{\text{null}}$. We begin by noting that

$$\left| \left( 1 - \frac{2\varepsilon}{\ell(Q)} \right) Q \right|_\sigma \leq |Q|_\sigma \leq |B_Q(\varepsilon)| \leq \left| \left( 1 + \frac{2\varepsilon}{\ell(Q)} \right) Q \right|_\sigma,$$

so that by the regularity of locally finite positive Borel measures on $\mathbb{R}^n$, together with $|\partial Q|_\sigma = 0$, we have

$$\limsup_{\varepsilon \searrow 0} |Q|_{\sigma, \varepsilon} \leq |Q|_\sigma \leq \liminf_{\varepsilon \searrow 0} |Q|_{\sigma, \varepsilon},$$

$$\Rightarrow |Q|_\sigma = \lim_{\varepsilon \searrow 0} |Q|_{\sigma, \varepsilon}.$$  

A similar argument shows that $|R|_\sigma = \lim_{\varepsilon \searrow 0} |R|_{\sigma, \varepsilon}$ for any rectangle $R = Q \cap K$ with $Q, K \in \mathcal{P}^{\text{null}}$. Moreover, we also have $|R|_\omega = \lim_{\varepsilon \searrow 0} |R|_{\omega, \varepsilon}$ for any rectangle $R = Q \cap K$ with $Q, K \in \mathcal{P}^{\text{null}}$.

Next, still supposing that $Q \in \mathcal{P}^{\text{null}}$, we claim that

$$\mathcal{M}^{\text{null}} (1_Q \sigma) (x) \leq \liminf_{\varepsilon \searrow 0} \mathcal{M}^{\text{null}} (1_{Q \sigma, \varepsilon}) (x), \quad x \in \mathbb{R}^n.$$

Indeed, given $\delta > 0$, there is a cube $K \in \mathcal{P}^{\text{null}}$ such that $x \in K$ and

$$\mathcal{M}^{\text{null}} (1_Q \sigma) (x) - \delta < \frac{|Q \cap K|_\sigma}{|K|}.$$  

---

6One can also obtain the first inequality in (4.9) by observing that, with probability one, the grids used in Lemma 2 of [Saw3] contain only cubes in $\mathcal{P}^{\text{null}}$. 
Then since $|Q \cap K|_\sigma = \lim_{\varepsilon \to 0} |Q \cap K|_{\sigma_\varepsilon}$ for the rectangle $R = Q \cap K$, we have

$$M^{null}(1_Q\sigma)(x) - \delta < \frac{|Q \cap K|_{\sigma_\varepsilon}}{|K|} = \lim_{\varepsilon \to 0} \frac{|Q \cap K|_{\sigma_\varepsilon}}{|K|} \leq \liminf_{\varepsilon \to 0} M(1_Q\sigma_\varepsilon)(x),$$

which proves (4.11) upon letting $\delta \searrow 0$. An application of Fatou’s lemma then gives for $Q \in \mathcal{P}^{null}$ that

$$\frac{1}{|Q|_\sigma} \int_Q M^{null}(1_Q\sigma)^2 \, d\omega \leq \frac{1}{|Q|_\sigma} \int_Q \liminf_{\varepsilon \to 0} M(1_Q\sigma_\varepsilon)^2 \, d\omega \leq \frac{1}{|Q|_\sigma} \liminf_{\varepsilon \to 0} \int_Q M(1_Q\sigma_\varepsilon)^2 \, d\omega.$$

We next observe that the following oscillation inequality holds for any cube $Q \in \mathcal{P}$:

$$M(1_Q\sigma_\varepsilon)(x) \leq CM(1_Q\sigma_{8\varepsilon})(x + h), \quad x, h \in \mathbb{R}^n \text{ with } |h| < \varepsilon < \frac{1}{8}.

Indeed, we have

$$M(1_Q\sigma_\varepsilon)(x) \leq \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} \geq \varepsilon} \frac{|Q \cap K|_{\sigma_\varepsilon}}{|K|} + \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} < \varepsilon} \frac{|Q \cap K|_{\sigma_\varepsilon}}{|K|},$$

and using the inequality $\varphi_\varepsilon(z) \leq C\varphi_{8\varepsilon}(z + h)$ for $|h| < \varepsilon$, we obtain that for any $|h| < \varepsilon$,

$$\sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} \geq \varepsilon} \frac{|Q \cap K|_{\sigma_\varepsilon}}{|K|} = \frac{1}{|K|} \int_{Q \cap K} \left( \int \varphi_\varepsilon(x - y) \, d\sigma(y) \right) \, dx

\leq \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} \geq \varepsilon} \frac{1}{|K|} \int_{Q \cap K} \left( \int C\varphi_{8\varepsilon}(x + h - y) \, d\sigma(y) \right) \, dx

= C \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} \geq \varepsilon} \frac{1}{|K|} \int_{Q \cap K} \sigma_{8\varepsilon}(x + h) \, dx

\leq C \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} \geq \varepsilon} \frac{1}{|K|} \int_{Q \cap K \setminus (B_k(\varepsilon))} \sigma_{8\varepsilon}(x) \, dx

\leq CM(1_Q\sigma_{8\varepsilon})(x + h).$$

We also have

$$\sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} < \varepsilon} \frac{|Q \cap K|_{\sigma_\varepsilon}}{|K|} \leq \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} < \varepsilon} \|1_{Q \cap K}\sigma_\varepsilon\|_\infty = \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} < \varepsilon} \sup_{z \in Q \cap K} \int \varphi_\varepsilon(z - y) \, d\sigma(y)

\leq \sup_{K \in \mathcal{P}: \frac{z}{\theta(K)} < \varepsilon} \sup_{z \in Q \cap K} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(z)} d\sigma(y)

\leq \frac{1}{|B_\varepsilon|} \int_{B_{2z}(x)} d\sigma(y) \leq C\sigma_{8\varepsilon}(x) \leq CM(1_Q\sigma_{8\varepsilon})(x + h),$$

for $|h| < \varepsilon$, where $B_\delta(x)$ denote the cube of side length $2\delta$ centered at $x$ (if $x$ is the origin then we simply denote it by $B_1$). This completes the proof of the oscillation inequality (4.12). With (4.12), it follows immediately that for any cube $Q \in \mathcal{P}$ that

$$\int_Q M(1_Q\sigma_\varepsilon)^2 \, d\omega \leq \int_Q \int \varphi_\varepsilon(h) \{CM(1_Q\sigma_{8\varepsilon})(x + h)\}^2 \, dh \, d\omega(x)

= \int_Q \{CM(1_Q\sigma_{8\varepsilon})(x)\}^2 \left\{ \int \varphi_\varepsilon(h) \, d\omega(x - h) \, dh \right\}

= C^2 \int_Q M(1_Q\sigma_{8\varepsilon})^2 \, d\omega_\varepsilon.$$
Now, restricting to cubes \( Q \in \mathcal{P}^{null} \), and using \(|Q|_\sigma = \lim_{\varepsilon \searrow 0} |Q|_{\sigma\varepsilon} \), we have

\[
(4.13) \quad \frac{1}{|Q|_\sigma} \int_Q M^{null}(1_Q\sigma)^2 \, d\omega \leq \liminf_{\varepsilon \searrow 0} \frac{1}{|Q|_\sigma} \int_Q M(1_Q\sigma\varepsilon)^2 \, d\omega \\
\leq C^2 \liminf_{\varepsilon \searrow 0} \frac{1}{|Q|_{\sigma\varepsilon}} \int_Q M(1_Q\sigma_{8\varepsilon})^2 \, d\omega \varepsilon \\
= C^2 \liminf_{\varepsilon \searrow 0} \frac{1}{|Q|_{\sigma\varepsilon}} \int_Q M(1_Q\sigma_{8\varepsilon})^2 \, d\omega \\
\leq C^2 \liminf_{\varepsilon \searrow 0} \mathfrak{N}_M(\sigma_{8\varepsilon}, \omega\varepsilon),
\]

which is a bound independent of the cube \( Q \). If we now take the supremum over all cubes \( Q \in \mathcal{P}^{null} \) we obtain

\[
\mathfrak{X}_{M^{null}}(\mathcal{P}^{null})(\sigma, \omega) = \sup_{Q \in \mathcal{P}^{null}} \frac{1}{|Q|_\sigma} \int_Q M^{null}(1_Q\sigma)^2 \, d\omega \\
\leq \sup_{Q \in \mathcal{P}^{null}} C^2 \liminf_{\varepsilon \searrow 0} \mathfrak{N}_M(\sigma_{8\varepsilon}, \omega\varepsilon) = C^2 \liminf_{\varepsilon \searrow 0} \mathfrak{N}_M(\sigma_{8\varepsilon}, \omega\varepsilon),
\]

which completes the proof of the second line in (4.19), and hence the first line in (4.8).

**Step 6:** Now we turn to the second line in (4.8), and prove that

\[
(4.14) \quad \sup_{0 < \varepsilon < \frac{1}{9}} A_2(\sigma_{8\varepsilon}, \omega\varepsilon) \leq \sup_{0 < \varepsilon < \frac{1}{9}} \sup_{Q \in \mathcal{P} : \ell(Q) \geq \varepsilon} \frac{|Q|_{\sigma\varepsilon}}{|Q|^2} \left| \int Q \varphi_{8\varepsilon}(x-y) \, dx \right| + \sup_{0 < \varepsilon < \frac{1}{9}} \sup_{Q \in \mathcal{P} : \ell(Q) < \varepsilon} \frac{|Q|_{\sigma\varepsilon}}{|Q|^2} \left| \int Q \varphi_{8\varepsilon}(x-y) \, dx \right| \leq C A_2(\sigma, \omega).
\]

Indeed, to see this, we bound the first summand in (4.14) by

\[
\sup_{0 < \varepsilon < \frac{1}{9}} \sup_{Q \in \mathcal{P} : \ell(Q) \geq \varepsilon} \frac{|Q|_{\sigma\varepsilon}}{|Q|^2} \left| \int Q \varphi_{8\varepsilon}(x-y) \, dx \right| \\
\leq C \sup_{0 < \varepsilon < \frac{1}{9}} \sup_{Q \in \mathcal{P} : \ell(Q) \geq \varepsilon} \left( 1 + \frac{16\varepsilon}{\ell(Q)} \right)^n \left( 1 + \frac{2\varepsilon}{\ell(Q)} \right)^n \left| \frac{1}{1 + \frac{16\varepsilon}{\ell(Q)}} \right| \left| \frac{1}{1 + \frac{2\varepsilon}{\ell(Q)}} \right| \left| \int Q \varphi_{8\varepsilon}(x-y) \, dx \right|
\]

Then using

\[
|Q|_{\sigma\varepsilon} = \int_Q \left( \int \varphi_{8\varepsilon}(x-y) \, d\sigma(y) \right) \, dx = \int \left( \int_Q \varphi_{8\varepsilon}(x-y) \, dx \right) \, d\sigma(y) \leq C \frac{1}{|B_{8\varepsilon}|} \int_{B_Q(8\varepsilon)} d\sigma
\]

and similarly \(|Q|_{\omega\varepsilon} \leq C \frac{1}{|B_{8\varepsilon}|} \int_{B_Q(\varepsilon)} d\omega\), we bound the second summand in (4.14) by

\[
(4.15) \quad \sup_{0 < \varepsilon < \frac{1}{9}} \sup_{Q \in \mathcal{P} : \ell(Q) < \varepsilon} \frac{|Q|_{\sigma\varepsilon}}{|Q|^2} \left| \int Q \varphi_{8\varepsilon}(x-y) \, dx \right| \\
\leq C \sup_{0 < \varepsilon < \frac{1}{9}} \sup_{Q \in \mathcal{P} : \ell(Q) < \varepsilon} \left( \frac{1}{|B_{8\varepsilon}|} \int_{B_Q(8\varepsilon)} d\sigma \right) \left( \frac{1}{|B_{\varepsilon}|} \int_{B_Q(\varepsilon)} d\omega \right)
\]

This completes the proof of (4.14).

**Step 7:** In order to complete the proof of (4.7), it remains to prove the third line in (4.8), namely that for \( \Gamma' > \Gamma \) and \( D = D' \) where \( D' = D'(\Gamma', n) \) is such that (4.9) holds, we have

\[
(4.16) \quad \sup_{0 < \varepsilon < \frac{1}{9}} \mathfrak{X}^{D'}_{M}(\Gamma')(\sigma_{8\varepsilon}, \omega\varepsilon) \leq C \left[ \mathfrak{X}^{D}_{M}(\Gamma)(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)} \right].
\]
Suppose first that $Q \in \mathcal{P}$ satisfies $|\Gamma'Q|^\sigma_{8\varepsilon} \leq D' |Q|^\sigma_{8\varepsilon'}$ and $\varepsilon \leq \ell(Q)$. Recall that $B_\delta$ is the cube of side length $\delta$ centered at the origin. Fix $x \in Q$ and $\delta > 0$, and choose $K \in \mathcal{P}$ such that $x \in K$ and

$$\mathcal{M} (1_Q \sigma_{8\varepsilon})(x) - \delta \leq \frac{1}{|K|} \int_{K \cap Q} \sigma_{8\varepsilon}(z) \, dz = \frac{1}{|K|} \int_{K \cap Q} \left\{ \int_{\mathbb{R}^n} \varphi_{8\varepsilon}(z - y) \, d\sigma(y) \right\} \, dz$$

$$= \frac{1}{|K|} \int_{B_{K \cap Q}(\varepsilon)} \left\{ \int_{K \cap Q} \varphi_{8\varepsilon}(z - y) \, d\sigma(y) \right\} \, d\sigma(y) \leq C \frac{1}{|K|} \int_{B_{K \cap Q}(\varepsilon)} \left\{ \left[ B_{8\varepsilon} \right] \int_{K \cap Q} 1_{B_{8\varepsilon}}(z - y) \, d\sigma(y) \right\} \, d\sigma(y)$$

$$= C \int_{B_{K \cap Q}(\varepsilon)} \left\{ \left[ K \cap Q \cap (B_{8\varepsilon} + y) \right] \right\} \frac{|B_{8\varepsilon}|}{|K|} \, d\sigma(y).$$

There are now two subcases, $\varepsilon \leq \ell(K)$ and $\ell(K) < \varepsilon$. In the first case $\varepsilon \leq \ell(K)$ we continue with

$$\int_{B_{K \cap Q}(\varepsilon)} \left\{ \frac{|K \cap Q \cap (B_{8\varepsilon} + y)|}{|K| \left| B_{8\varepsilon} \right|} \right\} \, d\sigma(y) \leq C \frac{1}{|B_{K}(8\varepsilon)|} \int_{B_{K}(8\varepsilon)} 1_{B_{K}(8\varepsilon)}(y) \, d\sigma(y) \leq C \mathcal{M} (1_{B_{K}(8\varepsilon)} \sigma)(x),$$

while in the second case $\ell(K) < \varepsilon$ we continue with

$$\int_{B_{K \cap Q}(\varepsilon)} \left\{ \frac{|K \cap Q \cap (B_{8\varepsilon} + y)|}{|K| \left| B_{8\varepsilon} \right|} \right\} \, d\sigma(y) \leq C \frac{1}{|B_{8\varepsilon}|} \int_{B_{K \cap Q}(\varepsilon)} \, d\sigma(y) \leq C \mathcal{M} (1_{B_{8\varepsilon}} \sigma)(x).$$

Thus altogether we have $\mathcal{M} (1_Q \sigma_{8\varepsilon})(x) - \delta < C \mathcal{M} (1_{B_{Q}(8\varepsilon)} \sigma)(x)$ for all $\delta > 0$, which yields $\mathcal{M} (1_Q \sigma_{8\varepsilon})(x) \leq C \mathcal{M} (1_{B_{Q}(8\varepsilon)} \sigma)(x)$, when $Q \in \mathcal{P}$ and $\varepsilon \leq \ell(Q)$.

Hence, using (4.12) and the restricted testing constant $\mathfrak{T}^\sigma_\mathcal{M}(\Gamma)$, we will have

$$\int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})^2 \, d\omega = \int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})(x)^2 \left\{ \int \varphi_{\varepsilon}(x - z) \, d\omega(z) \right\} \, dx$$

$$= \int_Q \int_{\mathbb{R}^n} \mathcal{M} (1_Q \sigma_{8\varepsilon})(x)^2 \varphi_{\varepsilon}(x - z) \, dx \, d\omega(z) = \int_Q \int_{\mathbb{R}^n} \mathcal{M} (1_Q \sigma_{8\varepsilon})(z + h) \varphi_{\varepsilon}(h) \, dh \, d\omega(z)$$

$$\leq C^2 \int_Q \int_{\mathbb{R}^n} \mathcal{M} (1_Q \sigma_{64\varepsilon})(z)^2 \varphi_{\varepsilon}(h) \, dh \, d\omega(z) \leq C^2 \int_{B_Q(\varepsilon)} \int_{B_Q(\varepsilon)} \mathcal{M} (1_Q \sigma_{64\varepsilon})(z)^2 \varphi_{\varepsilon}(h) \, dh \, d\omega(z)$$

$$\leq C^2 \left( \mathfrak{T}^\sigma_\mathcal{M}(\Gamma)(\sigma, \omega) \right)^2 |B_{Q}(64\varepsilon)|_\sigma,$$

provided that $|\Gamma B_{Q}(64\varepsilon)|_\sigma \leq D |B_{Q}(64\varepsilon)|_\sigma$. But we claim this latter inequality will hold for

$$0 < \varepsilon \leq \min \{ \frac{T - 1}{288}, \frac{1}{32} \} \ell(Q) = \alpha \ell(Q), \quad \text{where} \quad \alpha = \min \{ \frac{T - 1}{288}, \frac{1}{32} \} > 0.$$

Indeed, since $B_{Q}(64\varepsilon) = \left( 1 + 128 \frac{\ell(Q)}{\ell(Q)} \right) Q$, we have

$$|\Gamma B_{Q}(64\varepsilon)|_\sigma \leq \left| \Gamma \left( 1 + \frac{128 \ell(Q)}{\ell(Q)} \right) \frac{1}{1 - \frac{128 \ell(Q)}{\ell(Q)}} Q \right| \leq |\Gamma'Q|^\sigma_{\sigma_{8\varepsilon}} \leq D' |Q|^\sigma_{\sigma_{8\varepsilon}} \leq D' |B_{Q}(8\varepsilon)|_\sigma \leq D |B_{Q}(64\varepsilon)|_\sigma.$$

The first inequality is by (4.10). Then the second inequality is by using $\varepsilon \leq \alpha$. The third inequality in (4.18) follows from our starting assumption that $|\Gamma'Q|^\sigma_{\sigma_{8\varepsilon}} \leq D' |Q|^\sigma_{\sigma_{8\varepsilon}}$, and the fourth inequality is trivial. Thus
\[ (4.18) \] shows that \( (4.17) \) holds for \( 0 < \varepsilon \leq \alpha \ell (Q) \). Now we note an additional consequence of \( (4.18) \), namely that for \( 0 < \varepsilon \leq \alpha \ell (Q) \) we have
\begin{equation}
(4.19) \quad |B_Q (64\varepsilon)|_\sigma \leq |\Gamma B_Q (64\varepsilon)|_\sigma \leq D' |Q|_{\sigma_{8\varepsilon}}.
\end{equation}
Thus when \( Q \in \mathcal{P} \) is a \( D' \Gamma' \sigma_{8\varepsilon} \)-cube and \( 0 < \varepsilon \leq \alpha \ell (Q) \), we have from \( (4.17) \) and \( (4.19) \) that
\[ \int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})^2 \, d\omega \leq C^2 (\Xi_M^D (\Gamma) (\sigma, \omega))^2 \big| B_Q (64\varepsilon) \big|_\sigma \]
\[ \leq C^2 (\Xi_M^D (\Gamma) (\sigma, \omega))^2 D' |Q|_{\sigma_{8\varepsilon}} \]
\[ = C^2 D' (\Xi_M^D (\Gamma) (\sigma, \omega))^2 |Q|_{\sigma_{8\varepsilon}}, \]
or in other words,
\[ \frac{1}{|Q|_{\sigma_{8\varepsilon}}} \int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})^2 \, d\omega \leq C^2 D' (\Xi_M^D (\Gamma) (\sigma, \omega))^2, \quad \text{for } 0 < \varepsilon \leq \alpha \ell (Q). \]

If on the other hand, we have \( \ell (Q) < \frac{1}{\alpha} \varepsilon \), then
\[ \| 1_Q \omega \|_{\infty} = \sup_{x \in Q} \int \varphi_x (x-z) \omega (z) \, dz \leq \sup_{x \in Q} \frac{2^n}{|B_z|} \int_{B_z + x} \omega (z) \, dz \leq 2^n \frac{|B_Q (\varepsilon)|_\omega}{|B_z|}, \]
and similarly \( \| 1_Q \sigma_{8\varepsilon} \|_{\infty} \leq 2^n \frac{|B_Q (8\varepsilon)|_\sigma}{|B_z|} \),
and so
\[ \int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})^2 \, d\omega \leq \| 1_Q \omega \|_{\infty} \int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})^2 \, dx \]
\[ \leq \| 1_Q \omega \|_{\infty} C_{\text{class}}^2 \int_Q (\sigma_{8\varepsilon})^2 \leq C_{\text{class}}^2 \| 1_Q \omega \|_{\infty} \| 1_Q \sigma_{8\varepsilon} \|_{\infty} \int_Q \sigma_{8\varepsilon} \]
\[ \leq C_{\text{class}}^2 4^n \frac{|B_Q (\varepsilon)|_\omega |B_Q (8\varepsilon)|_\sigma}{|B_z| |B_{8\varepsilon}|} |Q|_{\sigma_{8\varepsilon}} \]
\[ \leq C_{\text{class}}^2 A_2 (\sigma, \omega) |Q|_{\sigma_{8\varepsilon}}. \]

Thus altogether we have
\[ \frac{1}{|Q|_{\sigma_{8\varepsilon}}} \int_Q \mathcal{M} (1_Q \sigma_{8\varepsilon})^2 \, d\omega \leq C^2 D' (\Xi_M^D (\Gamma) (\sigma, \omega))^2 + C A_2 (\sigma, \omega), \]
provided \( Q \in \mathcal{P} \) and \( |\Gamma' Q|_{\sigma_{8\varepsilon}} \leq D' |Q|_{\sigma_{8\varepsilon}} \).

Upon taking the supremum over all \( Q \in \mathcal{P} \) satisfying \( |\Gamma' Q|_{\sigma_{8\varepsilon}} \leq D' |Q|_{\sigma_{8\varepsilon}} \), it follows that
\[ \Xi_M^D (\Gamma) (\sigma_{8\varepsilon}, \omega) \leq C \left( (\Xi_M^D (\Gamma) (\sigma, \omega)) + \sqrt{A_2 (\sigma, \omega)} \right). \]
This proves \( (4.16) \), and so we obtain \( (4.7) \) and hence \( (4.4) \), which completes the proof of Theorem II.

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