On the comoving distance as an arc-length in four dimensions

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1 INTRODUCTION

Liske (2000) recently showed a new variant for calculating the comoving distance \(\chi_{12}\) between two objects at cosmological distances, given their redshifts and angular separation. However, given two celestial positions (in right ascension and declination) \(\alpha_i, \delta_i, i = 1, 2\), before \(\chi_{12}\) can be calculated via equation (7) of Liske (2000), it is first necessary to calculate the angle \(\theta_{12}\) (i.e. \(\alpha\) in the notation of Liske 2000) between these two positions.

This angle can be calculated either by converting the angles to points in Euclidean 3-space, taking the inner product (dot product) of these two points treated as vectors, and inverting the cosine relation between the inner product and cosines.

However, the calculation of \(\chi_{12}\) is very closely analogous to the calculation of \(\theta_{12}\), apart from the addition of a dimension, a change in signature of the metric in the case of hyperbolic space (an ‘open’ universe), and multiplication by the curvature radius of the 3-sphere. The key to this is that just as a distance between two points ‘on’ the surface of the ordinary 2-sphere \(S^2\) is simply an arc-length (angle multiplied by radius) in ordinary Euclidean 3-space \(E^3\), the distance between two points ‘on’ a 3-sphere \(S^3\) (a 3-hyperboloid \(H^3\)) is simply an ‘arc-length’ in Euclidean 4-space \(E^4\) (Minkowski 4-space \(M^4\)), i.e. an ‘hyper-angle’ multiplied by the curvature radius of the 3-sphere (3-hyperboloid).

Key words: cosmology; observations — cosmology; theory

The use of the inner product to calculate \(\theta_{12}\) [denoted \(\alpha\) in the notation of Liske (2000)] in three dimensions is described in Sect. 3 and its generalisation to the calculation of \(\chi_{12}\) (in four dimensions) is derived in Sect. 4.

2 THE DISTANCE BETWEEN TWO POINTS IN \(S^2\) AS AN ARC-LENGTH IN \(E^3\)

Given two celestial positions in spherical polar (e.g. equatorial) coordinates \((\alpha_i, \delta_i), i = 1, 2\), these can be converted to Cartesian coordinates

\[
\begin{align*}
    x_i & = R \cos \delta_i \cos \alpha_i \\
    y_i & = R \cos \delta_i \sin \alpha_i \\
    w_i & = R \sin \delta_i
\end{align*}
\]

(1)

on the celestial sphere of arbitrary radius \(R\) (e.g. \(R = 1\)).

The standard Euclidean inner product on the two vectors \(a_i = (x_i, y_i, w_i), i = 1, 2\), can then be expressed either as

\[
(a_1, a_2) = x_1 x_2 + y_1 y_2 + w_1 w_2
\]

(2)

or as

\[
(a_1, a_1) = R^2 \cos \theta_{12}.
\]

(3)

Equations (1), (2) and (3) imply the value of \(\theta_{12}\). Hence, the length \(\chi_{12}\) of a geodesic in \(S^2\) between \(a_1\) and \(a_2\) is

\[
\chi_{12} = R \theta_{12} = R \cos^{-1} \left[ (a_1, a_2) / R^2 \right].
\]

(4)

In words, a distance in \(S^2\) is simply an arc-length in \(E^3\).
As long as two vectors are represented in Cartesian coordinates, the inner product and thereby the angle and distance can easily be calculated.

3 THE DISTANCE BETWEEN TWO POINTS IN $S^3 (H^3)$ AS AN ‘ARC-LENGTH’ IN $E^4 (M^4)$

How can this be generalised to one more dimension and to the hyperbolic (‘open’) case?

3.1 Friedmann-Lemaître-Robertson-Walker coordinates

Let us first write the standard Friedmann-Lemaître-Robertson-Walker metric as

$$ds^2 = -c^2 dt^2 + a^2(t)[dx^2 + \Sigma^2(\chi)(d\delta^2 + \cos^2 \delta d\alpha^2)]$$

where the dimensionless curvature is written

$$\Omega_\chi \equiv \Omega_m + \Omega_\Lambda - 1$$

$$k = \text{sign}(\Omega_\chi) = 0, \pm 1$$

and the proper distance $\chi$ [eq. (14.2.21), Weinberg (1972)] is

$$\chi = \frac{c}{H_0} \int_{1/(1+z)}^1 \frac{da}{a\sqrt{\Omega_m/a - \Omega_\chi + \Omega_\Lambda a^2}}$$

From here on, only spatial sections (hypersurfaces) at constant cosmological time are considered, i.e. $dt \equiv 0$ and $a(t) \equiv 1$.

Note that $\chi$ has a length dimension in this paper (e.g. h$^{-1}$ Mpc), whereas length units (e.g. h$^{-1}$ Mpc) are (presumably) included in the scale factor $a(t)$ by Liske (2000), in order that $\chi$ is dimensionless. This requires some (arbitrary) choice in the length scale of $a(t)$ by Liske (2000) for the flat case ($k = 0$), whereas here, there is no need to define $R$ for the flat case.

3.2 Cartesian coordinates, the metric and the inner product in 4-D

What is the meaning of $R$ for $k = \pm 1$?

Let us first define four-dimensional Cartesian coordinates so that the two points in comoving 3-space become two points $a_i = (x_i, y_i, z_i, w_i)$ in a four-dimensional space, via

$$x_i = \Sigma(\chi_i) \cos \delta_i \cos \alpha_i$$

$$y_i = \Sigma(\chi_i) \cos \delta_i \sin \alpha_i$$

$$z_i = \Sigma(\chi_i) \sin \delta_i$$

$$w_i = \begin{cases} R \cosh(\chi_i/R) & k = -1 \\ 0 & k = 0 \\ R \cosh(\chi_i/R) & k = +1 \end{cases}$$

for $i = 1, 2$, and similarly for any arbitrary point $(z, \alpha, \delta)$ [cf. eq. (12.4) of Peebles (1993)].

The metric of this four-dimensional space is

$$ds^2 = \begin{cases} k (dx^2 + dy^2 + dz^2) + dw^2 & k = \pm 1 \\ dx^2 + dy^2 + dz^2 & k = 0. \end{cases}$$

Why is this a useful choice of metric?

For $k = +1$, it is simply the obvious (by induction from $E^3$) metric defining Euclidean 4-space $E^4$, i.e.

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2.$$  \hspace{1cm} (12)

The meaning of $R$ for the case $k = +1$ then follows. A spatial hypersurface at constant cosmological time is a 3-plane $S^3$. The 2-plane $S^2$ is normally thought of as embedded in Euclidean 3-space $E^3$, since this is conceptually easy, although there is no mathematical necessity for the embedding. Similarly, $S^3$ can be thought of as embedded in $E^4$, in which case it has a centre located in $E^4$, but external to $S^3$ and a radius of size $R$.

The existence of the centre does not contradict the Copernican principle: all points in the physical space $S^3$ are equidistant from the centre [according to the metric of $E^4$, eq. (12), which itself is located in the mostly non-physical space $E^4$].

In the hyperbolic case, $k = -1$, the intuitive meaning of $R$ is less obvious. If the absolute value had not been used in eq. (8), then $R$ would have had an imaginary value. Peebles [discussion near eq. (12.4) of Peebles 1993] points out that substituting $R$ by $iR$ results in the required equations and relations for hyperbolic space. How can one imagine a ‘negatively curved’ sphere with imaginary radius $iR$? The relations

$$\cos \theta = \cosh \theta$$

$$\sin \theta = -i \sinh \theta.$$  \hspace{1cm} (13)

provide a clue. The fact that $\cos \theta$ is real but $\sin \theta$ is imaginary suggests that the symmetry between the four coordinates needs to broken.

This is why the metric [eq. (11)] shows that symmetry between the four coordinates is indeed broken. The metric is

$$ds^2 = -(dx^2 + dy^2 + dz^2) + dw^2$$  \hspace{1cm} (14)

and is of course just the metric defining Minkowski 4-space $M^4$, familiar from special relativity, in particular in the two-dimensional version, $S^2$.

Unlike the case of special relativity, the full space here does not have physical meaning. On the contrary, just as the physical space $S^3$ is only a subset of the mostly non-physical $E^4$, there is a particular subset of $M^4$ which does have physical meaning. This subset can be referred to as a 3-hyperboloid $H^3$, and defined in an equation uniting $S^3$ and $H^3$ by first defining the inner product

$$(a_1, a_2) \equiv \begin{cases} k (x_1x_2 + y_1y_2 + z_1z_2) + w_1w_2 & k = \pm 1 \\ x_1x_2 + y_1y_2 + z_1z_2 & k = 0. \end{cases}$$

This clearly satisfies (in $E^4$ and in the domain of $M^4$ restricted to $w > \sqrt{x^2 + y^2 + z^2}$) the properties required of an inner product that

$$\forall a_1, a_2, a_3 \in E^4 \text{(resp. } M^4) \text{, } \forall \epsilon \in \mathbb{R}$$
although all the points of the space \( H \) (convention here) or space-like (negative here).

The invariant interval may be either time-like (positive by the relativistic terminology is adopted here (keeping in mind the domain \( \langle a_1, a_2 \rangle \equiv \sqrt{x_1^2 + y_1^2 + z_1^2} \) for the \( k = -1 \) case is required in order to satisfy the standard definitions of inner product, norm and metric as always non-negative. In special relativistic terminology, where \( w \) is the time variable, these definitions are extended so that the invariant interval may be either time-like (positive by the convention here) or space-like (negative here).

In the present case, an extension is also needed, because although all the points of the space \( H^3 \) lie in the ‘upper cone’ and represent ‘time-like’ vectors, the small difference vectors which are of interest in integrating along this surface are in the complementary domain \( w < \sqrt{x_2^2 + y_2^2 + z_2^2} \). If relativistic terminology is adopted here (keeping in mind that the ‘time’ variable \( w \) here is purely artificial and has no physical meaning), then it can be said that although the invariant intervals (vector lengths) from the origin to the points on \( H^3 \) are all time-like, the difference element vectors tangent to the surface of \( H^3 \) are all space-like.

Hence, for \( k = \pm 1 \), let us replace equation (11) by the more useful ‘space-like’ metric

\[
\begin{align*}
\text{ds}^2_{\text{sp}} &= k[k(dx^2 + dy^2 + dz^2) + dw^2] \\
&= dx^2 + dy^2 + dz^2 + k dw^2
\end{align*}
\]

which can be integrated along \( H^3 \) (and, of course, also along \( S^3 \)).

3.3 The arc-length formula in four dimensions

Does the definition in eq. (15) lead to the following equation, which would appear to be the generalisation of eq. (12)?

\[
\chi_{12} = \begin{cases} 
R \cosh^{-1} \left[ \langle a_1, a_2 \rangle / R^2 \right] & k = -1 \\
\sqrt{\langle a_1 - a_2, a_1 - a_2 \rangle / R^2} & k = 0 \\
R \cos^{-1} \left[ \langle a_1, a_2 \rangle / R^2 \right] & k = +1.
\end{cases}
\]

3.3.1 Flat case

The case \( k = 0 \) is simply the Euclidean 3-distance in the 3-plane \( w = 0 \).

3.3.2 Curved cases

The cases \( k = \pm 1 \) can be established by applying an appropriate ‘rotation’ (isometry) and integrating the metric \( ds_{\text{sp}} \) between the ‘rotated’ positions \( a_1 \) and \( a_2 \).

As pointed out by Liske [just before eq. (6) of Liske 2000], ‘since the curvature of the three-dimensional space under consideration is constant, one can generate all totally
geodesic hypersurfaces from any given one by mere translations and rotations', where the word ‘rotations’ is interpreted loosely to include isometries of the 3-hyperboloid, \( \mathcal{H}^3 \), as well as rotations of the 3-sphere, \( S^3 \).

So, either in the \( \mathcal{L}^4 \) representation of \( S^3 \), or in the \( \mathcal{M}^4 \) representation of \( \mathcal{H}^3 \), an isometry \( f \) can be chosen such that \((0, 0, 0, 0)\) is kept as a fixed point and \( a_1 \) is shifted to

\[
a'_1 = f(a_1) = (0, 0, 0, R). \tag{21}
\]

After applying \( f \), let us apply another isometry, a 3-rotation \( g \) in the \( x - y - z \) 3-plane, about \((0, 0, 0, 0)\), which leaves \( w \) values unchanged and shifts \( a_2 \) to the point

\[
a'_2 = g[f(a_2)] = (x'_2, 0, 0, w'_2). \tag{22}
\]

This leaves \( a'_1 \) unchanged, i.e. \( g(a'_1) = a'_1 \).

Both \( a'_1 \) and \( a'_2 \) still lie on the surface defined by eq. \((12)\), i.e. the relation

\[\sqrt{x^2 + w^2} = R, \tag{23}\]

holds on this surface.

To calculate the distance \( \chi_{12} \), let us integrate the metric \( ds|_{sp} \) [eq. \((19)\)] along the geodesic from \( a'_1 \) to \( a'_2 \) (which lies in the \( X' \)-\( W' \) plane), see Figs \(1, 2\), i.e.

\[
\chi_{12} = \int_{a'_1}^{a'_2} ds|_{sp} \tag{24}
\]

and parametrise \( x \) and \( w \) in terms of a parameter \( \theta \) via

\[
\theta \equiv \begin{cases} \cosh^{-1}(w/R) & k = -1 \\ \cos^{-1}(w/R) & k = +1 \end{cases} \tag{25}
\]

so that

\[
x = \begin{cases} R \sinh \theta & k = -1 \\ R \sin \theta & k = +1 \end{cases}
\]

\[
w = \begin{cases} R \cosh \theta & k = -1 \\ R \cos \theta & k = +1 \end{cases}
\]

\[dx^2 = w^2 d\theta^2 \]

\[dw^2 = x^2 d\theta^2. \tag{26}\]

The endpoints of the integral \( a'_1, a'_2 \), then become

\[
\theta_1 = 0
\]

and

\[
\theta_2 = \begin{cases} \cosh^{-1}(w'_2/R) & k = -1 \\ \cos^{-1}(w'_2/R) & k = +1 \end{cases}, \tag{28}
\]

using eqs \((15), (21), (22)\).

The integral [eq. \((24)\)] is then

\[
\chi_{12} = \int_{\theta_1}^{\theta_2} ds|_{sp} \]

\[= \int_0^{\theta_2} \int_0^{\theta_2} \sqrt{dx^2 + k dw^2} \]

[using eq. \((19)\)]

\[= \int_0^{\theta_2} \int_0^{\theta_2} \sqrt{w^2 + k x^2} \]

[using eq. \((26)\)]

\[= \int_0^{\theta_2} \sqrt{R d\theta} \]

[using eq. \((25)\)]

\[= \int_0^{\theta_2} \int_0^{\theta_2} \sqrt{R^2 - k w^2} \]

[using eqs \((15), (21), (22)\)]

\[= \int_0^{\theta_2} \int_0^{\theta_2} \sqrt{R^2 - k w^2} \]

[since \( f \) and \( g \) are isometries]

\[= \int_0^{\theta_2} \int_0^{\theta_2} \sqrt{R^2 - k w^2} \]

(29)

Thus, eq. \((21)\) is the correct generalisation of eq. \((4)\).

3.3.3 The flat case as a limit of the curved cases

Why is the inner product used differently in the curved and flat cases in eq. \((20)\)?

The reason can easily be seen by taking the limit as \( R \to \infty \), after isometries \( f \) and \( g \) have been applied as above.

If \( k = \pm 1 \), but \( R \gg \max\{x, y, z\} \), i.e. \( R \gg x'_2 \), then

\[
\chi_{12} = \begin{cases} R \cosh^{-1}\left[(a'_1, a'_2)/R^2\right] & k = -1 \\ R \cos^{-1}\left[(a'_1, a'_2)/R^2\right] & k = +1 \end{cases}
\]

\[= \begin{cases} R \sinh^{-1}(w'_2/R) & k = -1 \\ R \cos^{-1}(w'_2/R) & k = +1 \end{cases}
\]

\[\approx R \left(x'_2/R\right) \]

\[= x'_2, \tag{30}\]

i.e.

\[
\lim_{R \to \infty, k = \pm 1} \chi_{12} = x'_2. \tag{31}\]

For \( k = 0 \),

\[
\chi_{12} = \sqrt{(x'_2 - 0)^2 + 0^2 + 0^2} = x'_2 = \lim_{R \to \infty, k = \pm 1} \chi_{12}. \tag{32}\]

Thus, the flat case is a limit of the curved cases as expected.

4 Conclusion

So, just as a distance in \( S^2 \) is an arc-length in \( \mathcal{R}^3 \), the distance between two objects at cosmological distances in a curved universe can be thought of as the arc-length corresponding to an ‘hyper-angle’ in four-dimensional Euclidean or Minkowski space and is thus obtained directly from the inner product.

This is algebraically equivalent to the solutions of Osmer (1981), Peebles (1993), Peacock (1999) and Liske (2000).
but is expressed completely via the definitions and equations (6), (7), (8), (9), (10), (15) and (20).

That is, the complete formulae for calculating an FLRW comoving distance between two objects at cosmological distances, given $\Omega_m$, $\Omega_\Lambda$, $H_0$, $(z'_i, \alpha_i, \delta_i)$, $i = 1, 2$, are the following:

$$\Sigma(\chi) \equiv \begin{cases} R \sinh(\chi/R) & k = -1 \\ \chi & k = 0 \\ R \sin(\chi/R) & k = +1 \end{cases}$$

$$R \equiv \begin{cases} (c/H_0)(\Omega_\kappa)^{-0.5} & k = \pm 1 \\ \text{undefined} & k = 0 \end{cases}$$

$$\chi = \frac{c}{H_0} \int_{1/(1+z')}^1 \frac{da}{a \sqrt{\Omega_m/a - \Omega_\kappa + \Omega_\Lambda a^2}}$$

$$\Omega_\kappa \equiv \Omega_m + \Omega_\Lambda - 1$$

$$k = \text{sign}(\Omega_\kappa) = 0, \pm 1$$

$$x_i = \Sigma(\chi_i) \cos \delta_i \cos \alpha_i$$

$$y_i = \Sigma(\chi_i) \cos \delta_i \sin \alpha_i$$

$$z_i = \Sigma(\chi_i) \sin \delta_i$$

$$w_i = \begin{cases} R \cosh(\chi_i/R) & k = -1 \\ 0 & k = 0 \\ R \cos(\chi_i/R) & k = +1 \end{cases}$$

$$\langle a_1, a_2 \rangle = \begin{cases} k \left( x_1 x_2 + y_1 y_2 + z_1 z_2 \right) + w_1 w_2 & k = \pm 1 \\ x_1 x_2 + y_1 y_2 + z_1 z_2 & k = 0. \end{cases}$$

$$\chi_{12} = \begin{cases} R \cosh^{-1} \left[ \langle a_1, a_2 \rangle / R^2 \right] & k = -1 \\ \sqrt{\langle a_1 - a_2, a_1 - a_2 \rangle} & k = 0 \\ R \cos^{-1} \left[ \langle a_1, a_2 \rangle / R^2 \right] & k = +1. \end{cases}$$

(33)

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References

Liske J., 2000, MNRAS, 319, 557 [arXiv:astro-ph/0007341]

Osmer P. S., 1981, ApJ, 247, 762

Peacock J. A., 1999, Cosmological Physics, Cambridge: Cambridge University Press

Peebles, P.J.E., 1993, Principles of Physical Cosmology, Princeton, U.S.A.: Princeton Univ. Press

Weinberg S., 1972, Gravitation and Cosmology, New York: Wiley