Normal forms for parabolic Monge-Ampère equations

R. Alonso Blanco, G. Manno, F. Pugliese

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Abstract

We find normal forms for parabolic Monge-Ampère equations. Of these, the most general one holds for any equation admitting a complete integral. Moreover, we explicitly give the determining equation for such integrals; restricted to the analytic case, this equation is shown to have solutions. The other normal forms exhaust the different classes of parabolic Monge-Ampère equations with symmetry properties, namely, the existence of classical or nonholonomic intermediate integrals. Our approach is based on the equivalence between parabolic Monge-Ampère equations and particular distributions on a contact manifold, and involves a classification of vector fields lying in the contact structure. These are divided into three types and described in terms of the simplest ones (characteristic fields of 1st order PDE’s).

1 Introduction

In the present paper we give a contribution to the problem of classifying Monge-Ampère equations (MAE) up to contact transformations. MAE’s are second order equations of the form

\[ N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0, \]  

in the unknown function \( z = z(x, y) \), with coefficients \( A, B, C, D, N \) depending on \( x, y, z, z_x, z_y \). As is well known, any contact transformation maps a MAE into another one. Therefore, a major problem concerning equations (1) is their classification under the action of the contact pseudogroup. An aspect of this problem consists in finding normal forms, i.e. some particularly simple model equations, depending on functional parameters, such that any MAE is locally contact equivalent to one and only one of them, for a suitable choice of the parameters.

Below, we find normal forms of parabolic MAE’s, i.e. equations (1) satisfying \( B^2 - 4AC + 4ND = 0 \). Geometrically, this means that characteristic directions at any point of the 1-jet bundle \( J^1(\tau) = \{(x, y, z, z_x, z_y)\} \) of the trivial bundle \( \tau : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) define a 2-dimensional subdistribution \( D \) of the contact distribution \( C \):

\[ C = \{U = 0\}, \quad \text{with} \quad U = dz - z_x dx - z_y dy. \]  

As \( D \) is generally non integrable, it is necessary to consider also its derived flag

\[ D \subset D' = D + [D, D] \subset D'' = D' + [D', D'] \]  

whose properties allow to obtain important classification results on parabolic MAE’s in a simple and straightforward way. In fact, such a study is based, to a large extent, on the geometry of Cartan fields, i.e. sections of \( C \). Quite unexpectedly, generic Cartan fields are not contained in any integrable 2-dimensional subdistribution of \( C \). The degree of “genericity” of a Cartan field \( X \) is measured by a simple invariant, its type: the higher the type, the less symmetric \( X \) is with respect to \( C \). More precisely, \( X \) is of type 2, 3 or 4 if it is contained in many, one or no integrable 2-dimensional subdistribution of \( C \), respectively (the operative definition of type is given in section 3.2).

The main classification results in the present paper are summarized by the following two theorems.
Theorem 1.1 Let \( \mathcal{J} \) be a parabolic MAE with \( C^\infty \) coefficients on some domain of \( J^1(\tau) \). Then \( \mathcal{J} \) is locally contact equivalent to an equation of the form

\[
z_{yy} - 2az_{xy} + a^2z_{xx} = b,
\]

with \( a, b \in C^\infty(J^1(\tau)) \), if and only if it admits a complete integral.

Roughly speaking, a complete integral of \( \mathcal{J} \) is any 3-parametric family of solutions (see the more rigorous Definition 4.8). The existence of such a family does not seem to be a strong condition on \( \mathcal{J} \); in fact, in section 4.2.1, we provide a very large class of smooth parabolic MAE’s admitting a complete integral. Note that normal form (4) was proved to be true for every parabolic MAE with real analytic coefficients \((3)\). In that paper, the proof essentially consisted in showing the involutivity of a certain exterior differential system associated with \( \mathcal{J} \) and then applying Cartan-Kähler theorem to such a system; below (Theorem 4.12) we give an easier and more direct proof which makes use only of Cauchy-Kovalevsky existence theorem. An immediate corollary of this theorem and Theorem 1.1 is the existence of a complete integral for any real analytic parabolic MAE.

As it will be shown in section 4.2.2, the existence of a complete integral is equivalent to that of a generalized intermediate integral, i.e. a Cartan field of type less than 4 contained in \( \mathcal{D} \). This is a generalization of both the classical \((1)\) and the nonholonomic \((3)\) notion of intermediate integral; in fact, a classical intermediate integral \( f \in C^\infty(J^1(\tau)) \) of \( \mathcal{J} \) can be identified with a hamiltonian field \( X_f \) (a special kind of type 2 Cartan field, see Definition 3.3) belonging to \( \mathcal{D} \), while a nonholonomic intermediate integral is any Cartan field of type 2 in \( \mathcal{D} \).

The existence of intermediate integrals of equation \( \mathcal{J} \) is strictly linked to integrability properties of the derived flag \( \mathcal{F} \).

Theorem 1.2 Let \( \mathcal{J} \) be a parabolic MAE and \( \mathcal{D} \) be the corresponding characteristic distribution. If \( \dim \mathcal{D}'' < 5 \) then equation \( \mathcal{J} \) can always be locally reduced by a contactomorphism to one of the following forms:

1) \( z_{yy} = 0 \), when \( \mathcal{D} \) is integrable;

2) \( z_{yy} = b \), \( b \in C^\infty(J^1(\tau)) \), \( \partial_x(b) \neq 0 \), when \( \mathcal{D}'' \) is 4-dimensional and integrable;

3) \( z_{yy} - 2zz_{xy} + z^2z_{xx} = b \), \( b \in C^\infty(J^1(\tau)) \) with \( \partial_x(b) + z\partial_z(b) \neq 0 \), when \( \mathcal{D}'' \) is 4-dimensional and non integrable.

On the other hand, the three cases can be stated in terms of intermediate integrals of \( \mathcal{J} \), namely:

1’) There exist three (functionally independent) intermediate integrals;

2’) There exists just one intermediate integral;

3’) There are no (classical) intermediate integrals but there is exactly one nonholonomic intermediate integral.

The three normal forms of Theorem 1.2 were already known \((1, 8, 3)\) respectively. However, the alternative characterizations in terms of intermediate integrals are original. Moreover, the conditions given in \((3)\) for the validity of normal form 3) and the relative proof are completely different and, in our opinion, considerably more complicated and less transparent than ours: in fact, it must be emphasized that our conditions are easily computable for any given MAE.

The paper is structured as follows. In section 2, approximately following the approach of \((5, 6)\), the necessary preliminary notions on MAE’s in the framework of jet bundle formalism are given. Furthermore, the equivalence between parabolic MAE’s and lagrangian subdistributions of \( \mathcal{C} \) is explained.

Section 3 is devoted to the geometry of Cartan fields. We begin by studying the contact analogous of hamiltonian fields of symplectic geometry (in fact, they are the classical characteristic fields of first order PDE’s), along with several characterizations and properties. Then (section 3.2), we introduce the
type of a Cartan field $X$ as the rank of the system of Lie derivatives $U, X(U), X^2(U), \ldots$. Cartan fields of type 2 or 3 are characterized as linear combinations of involutive hamiltonian fields or, equivalently, as those belonging to integrable 2-dimensional subdistributions of $C$ (Theorem 3.15); according with this property, normal forms are derived (Theorems 3.17, 3.18).

In the final section, the main theorems are proved. In section 4.1 intermediate integrals, in the three senses explained above (classical, nonholonomic, generalized), are considered. In particular, we prove the aforementioned relations between existence of intermediate integrals and integrability properties of $D$ (Theorems 4.12 and Proposition 4.9). Using these results and normal forms of Cartan fields, Theorem 1.1 is proved (section 4.2). The following section contains the already mentioned example of a wide class of parabolic MAE’s admitting a complete integral, together with an explicit computation. Finally, results of section 3 on normal forms of Cartan fields allow to obtain normal forms for degenerate lagrangian distributions (Theorems 4.13, from which Theorem 1.2 immediately follows.

Notation and conventions. Throughout this paper, everything is supposed to be $C^\infty$ and local. For this reason, we do not lose in generality by working with jets of sections rather than of submanifolds. For simplicity, when $X$ is a vector field and $P$ is a distribution on the same manifold, we write $\{X \in P\}$ to mean that $X$ is a smooth (local) section of tangent subbundle $P$. We will use $X(T)$ to denote the Lie derivative of a tensor $T$ along $X$. Finally, first and second order jet coordinates will be indifferently denoted with $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ or $p, q, r, s, t$, respectively.

2 Preliminary notions

2.1 Jet bundles and contact distribution

Here we give the main definitions used in the present work. By $J^r(\tau)$ we denote the $r$-jet of the trivial bundle $\tau : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, i.e., the vector bundle of $r$-jets of smooth functions on $\mathbb{R}^2$. These are equivalence classes of smooth functions on $\mathbb{R}^2$ possessing the same partial derivatives up to $r$-th order at a given point. Jet bundles of different orders are linked by the obvious projections:

$$\cdots \longrightarrow J^2(\tau) \xrightarrow{\tau_{2,1}} J^1(\tau) \xrightarrow{\tau_{1,0}} \mathbb{R}^2 \times \mathbb{R} \xrightarrow{\tau} \mathbb{R}^2.$$  

For any $f \in C^\infty(\mathbb{R}^2)$, let $j_r f : \mathbb{R}^2 \ni p \mapsto [f]_p^r \in J^r(\tau)$, where $[f]_p^r$ is the $r$-jet of $f$ in $p$, be its $r$-th order prolongation. $R$-planes are the tangent planes to graphs of $r$-th order prolongations. For any $[f]_p^r \in J^r(\tau)$, $R$-planes passing through it biunivocally correspond to $(r + 1)$-jets projecting on $[f]_p^r$: namely, $[f]_p^{r+1}$ corresponds to the tangent plane $R_{[f]_p^r}$ to the graph of $j_r f$ at $[f]_p^r$ (2).

A chart $(x, y, z)$ on the bundle $\tau$ induces a natural chart on each $J^r(\tau)$. For instance $(x, y, z, p = z_x, q = z_y)$ are the induced coordinates on $J^1(\tau)$ and $(x, y, z, p, q, r = z_{xx}, s = z_{xy}, t = z_{yy})$ those on $J^2(\tau)$. The $R$-plane $R_\theta \subset T_{(x,y,z)} J^1(\tau)$ associated with point $\theta = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, \tilde{t}) \in J^2(\tau)$ is locally given by

$$R_\theta = <\tilde{\partial}_x|_{\theta} + \tilde{r} \partial_p|_{\theta} + \tilde{s} \partial_q|_{\theta}, \tilde{\partial}_y|_{\theta} + \tilde{t} \partial_a|_{\theta} > ,$$

where

$$\tilde{\partial}_x \overset{\text{def}}{=} \partial_x + p \partial_z, \quad \tilde{\partial}_y \overset{\text{def}}{=} \partial_y + q \partial_z.$$  

The contact space $C_\theta$ at $\theta \in J^1(\tau)$ is the span of $R$-planes at $\theta$. The distribution $\theta \mapsto C_\theta$ is then defined. From now on we shall focus on the case $r = 1$. On $J^1(\tau)$, $C$ is given by:

$$C = <\tilde{\partial}_x, \tilde{\partial}_y, \partial_a, \partial_q > .$$

Dually, $C$ is defined by $\{U = 0\}$ where

$$U = dz - p dx - q dy .$$  

(5)
Note that, as what really matters is the distribution $C$, one can substitute $U$ with any other multiple of it. $(J^1(\tau), C)$ is a contact manifold, i.e.

$$dU \wedge dU \wedge U \neq 0. \quad (6)$$

Hence, by $(\ref{5})$, $x, y, z, p, q$ are contact coordinates. As is well known, $(\ref{6})$ is equivalent to the fact that $C$ does not admit integral submanifolds of dimension greater than 2, or also to the non existence of infinitesimal symmetries $C$ belonging to it. Furthermore, condition $(\ref{6})$ is also equivalent to $(C, dU|_C)$ being a symplectic vector bundle.

Note that, for any $X \in C$, $X(U) = X| du$, i.e. one can express orthogonality in $C$ (with respect to $dU$) in terms of Lie derivatives. For example, the orthogonal complement of $X$ in $C$ is described by

$$X^\perp = \{ U = 0, \ X(U) = 0 \}.$$  

In particular, $X^\perp$ is 3-dimensional and contains $X$; moreover, any 3-dimensional subdistribution of $C$ is of this form. Analogously, if $D \subset C$ is a distribution spanned by vector fields $X, Y$ then its orthogonal complement is given by

$$D^\perp = \{ U = 0, \ X(U) = 0, \ Y(U) = 0 \}.$$  

In particular, $D$ is called a *lagrangian distribution* if $D = D^\perp$ (note that some other authors use the term *legendrian*).

### 2.2 Parabolic Monge-Ampère equations

Recall that a *scalar differential equation* (in two independent variables) $E$ of order $r$ is a hypersurface of $J^r(\tau)$. A solution of $E$ is a locally maximal integral manifold $\Sigma$ of the restriction to $E$ of the contact distribution on $J^r(\tau)$; when $\Sigma$ is the graph of $j_r f$, with $f$ smooth function on $\mathbb{R}^2$, then $\Sigma$ is a classical solution of $E$.

Let $\mathcal{I}(U) \subset \Lambda^*(J^1(\tau))$ be the differential ideal generated by $U$.

**Definition 2.1** Let $\omega \in \Lambda^2(J^1(\tau)) \setminus \mathcal{I}(U)$. Let us associate with $\omega$ the scalar second order equation

$$E_\omega \overset{\text{def}}{=} \{ \theta \in J^2(\tau) \ s.t. \ \omega|_{R_\theta} = 0 \}, \quad (7)$$

where $R_\theta \subset T_{r_2(\theta)}(J^1(\tau))$ is the $R$-plane associated with $\theta$. The equations of this form are called Monge-Ampère equations (see $(\ref{3})$ $(\ref{4})$).

In other words $E_\omega$ is the differential equation corresponding to the exterior differential system $\{ U = 0, \ \omega = 0 \}$.

**Coordinate expression.** Denote by $(x, y, z, p, q, r, s, t)$ a system of local contact coordinates on $J^2(\tau)$. In such a chart, a generic MAE takes the form

$$N(rt - s^2) + Ar + Bs + Ct + D = 0, \quad (8)$$

with $N, A, B, C, D \in C^\infty(J^1(\tau))$. The 2-forms $\omega$ on $J^1(\tau)$ such that $E_\omega$ is given by $(\ref{8})$ are

$$\omega = D \ dx \wedge dy + \left( \frac{B}{2} + b \right) dx \wedge dp + C \ dx \wedge dq - A \ dy \wedge dp$$

$$+ \left( -\frac{B}{2} + b \right) dy \wedge dq + N \ dp \wedge dq + \alpha \wedge U,$$

with arbitrary $b \in C^\infty(J^1(\tau)), \ alpha \in \Lambda^1(J^1(\tau))$.

It is clear from the above formula that the correspondence $\omega \mapsto E_\omega$ is not invertible. Let us consider in $\Lambda^2(J^1(\tau)) \setminus \mathcal{I}(U)$ the following equivalence relation:

$$\omega \sim \rho \iff \exists \mu \neq 0, \lambda \in C^\infty(J^1(\tau)) \ s.t. \ \rho|_C = \mu \omega|_C + \lambda (dU)|_C \quad (9)$$
(or \( \rho = \mu \omega + \lambda dU + \alpha \wedge U \) for some 1-form \( \alpha \)).

It can be proved (see [5]) that two 2-forms on \( J^1(\tau) \) are equivalent in the sense of (9) if and only if they define the same MAE.

**Proposition 2.2** For any \( \omega \in \Lambda^2(J^1(\tau)) \setminus \mathcal{I}(U) \), there are at most two 2-forms equivalent, up to a factor, to \( \omega \) in the sense of (9) and such that their restriction to \( C \) is degenerate (so that they are decomposable).

**Proof.** The restriction to \( C \) of a 2-form equivalent to \( \omega \) is, up to a factor, always of the form
\[
(\omega + \lambda dU)|_C = (\alpha + 2k\lambda + \lambda^2)(dU \wedge dU)|_C = 0,
\]
with
\[
(\omega \wedge \omega)|_C = \alpha(dU \wedge dU)|_C \quad \text{and} \quad (\omega \wedge dU)|_C = k(dU \wedge dU)|_C.
\]
As (10) is quadratic in \( \lambda \), the proposition is proved.

Note that the sign of the discriminant \( k^2 - \alpha \) in (10) is the same of the expression \( \Delta = B^2 - 4AC + 4ND \).

Let us recall the following basic notion.

**Definition 2.3** Let \( \mathcal{E} \subset J^2(\tau) \) be a second order scalar differential equation, and let \( \theta \in J^1(\tau) \). A line \( r \subset C_\theta \) is said to be characteristic for \( \mathcal{E} \) in \( \theta \) if it belongs to more than one \( R \)-plane \( R_\theta \), with \( \hat{\theta} \in E \cap \tau^{-1}_2(\theta) \).

Characteristic directions are those belonging to more than one integral manifold of \( \mathcal{E} \); a curve \( \gamma \subset J^1(\tau) \) (locally) determines the integral surface passing through it if and only if the tangent lines to \( \gamma \) are not characteristic. In the case of MAE’s, it is not difficult to check that a line is characteristic for \( \mathcal{E} \) if and only if it belongs to the radical of a degenerate 2-form \( \omega \lambda \) equivalent to \( \omega \). According to the previous proposition, there are three possibilities:

1) if \( \Delta > 0 \), there are two distinct \( \lambda \)'s such that \( \text{Rad} \ \omega \lambda \neq 0 \); hence, there exist two distinct families of characteristic lines (hyperbolic case);

2) if \( \Delta = 0 \), there is just one \( \lambda \) for which \( \omega \lambda \) is degenerate; in this case there is only one family of characteristics (parabolic case);

3) if \( \Delta < 0 \), \( \omega \lambda \) is always non degenerate, so that there are no characteristics (elliptic case).

**Warning.** As the paper is devoted to the parabolic case, from now on, when writing \( \mathcal{E}_\omega \), we mean that \( \omega \) is (up to a factor) the only degenerate representative of the equation.

**Definition 2.4** The 2-dimensional distribution \( D = \text{Rad} \ \omega|_C \) is called characteristic distribution of the parabolic MAE \( \mathcal{E}_\omega \).

**Proposition 2.5** Let \( \mathcal{E}_\omega \) be a parabolic MAE. Then its characteristic distribution \( D \) is lagrangian. Conversely, any lagrangian distribution is characteristic for one and only one parabolic MAE.

**Proof.** We must prove that \( dU|_D = 0 \), which is equivalent to \( \omega \wedge dU|_C = 0 \). But, by equation (10) and the assumptions made, one has \( \alpha = k = 0 \), and the proposition follows.

Note that, by the above proposition, a parabolic MAE can be specified by assigning a lagrangian subdistribution of \( C \). In fact, let \( D = \langle X, Y \rangle \). Then the corresponding MAE is \( \mathcal{E}_\omega \), with
\[
\omega = X(U) \wedge Y(U).
\]
If the generators of $D$ are locally expressed by

$$
X = \hat{\partial}_x + R\partial_p + S\partial_q, \quad Y = \hat{\partial}_y + S\partial_p + T\partial_q, \tag{11}
$$

with $R, S, T \in C^\infty(J^1(\tau))$, then

$$
X(U) = dp - Rdx - Sdy, \quad Y(U) = dq - Sdx - Tdy,
$$

from which it follows that equation $E_{X(U)\wedge Y(U)} \subset J^2(\tau)$ is

$$(s - S)^2 - (r - R)(t - T) = 0. \tag{12}
$$

Recall that the Legendre transformation maps $\hat{\partial}_x, \hat{\partial}_y, \partial_p, \partial_q$ into $\partial_p, \partial_q, -\hat{\partial}_x, -\hat{\partial}_y$, respectively. A partial Legendre transformation just exchanges $\hat{\partial}_x$ with $\partial_p$ or $\hat{\partial}_y$ with $\partial_q$ (up to a sign). Thus expression (11) is the most general coordinate representation of $D$, up to contact transformations.

### 3 Geometry of Cartan fields

As the contact distribution $C$ is completely non integrable, the flow of any Cartan field $X \in C$ deforms it; the sequence of iterated Lie derivatives

$$
U, X(U), X^2(U), X^3(U)
$$

gives a measure of this deformation (as $J^1(\tau)$ is 5-dimensional and all the forms $X^j(U)$ vanish on $X$, there is no need to consider the remaining derivatives).

**Definition 3.1** Let $X \in C$. The type of $X$ is the rank of system (12).

The following cases are possible:

1) Fields of type 2: $X^2(U)$ depends on $U$ and $X(U)$ (which is equivalent to $X$ being characteristic for $X^\perp = \{U = X(U) = 0\}$);

2) Fields of type 3: $U, X(U), X^2(U)$ are independent but $X^3(U)$ depends on them (which is equivalent to $X$ being characteristic for distribution $\{U = X(U) = X^2(U) = 0\}$);

3) Fields of type 4: $U, X(U), X^2(U), X^3(U)$ are independent.

Note that, due to the complete non integrability of the contact distribution, it can not be $X(U) = \lambda U$, for $X \in C\setminus\{0\}$ (“type 1”). Note also that the above three cases are well defined, i.e. they do not depend on the choice of $U$ nor on the length of $X$ (in other words, what we are dealing with are line distributions, rather than vector fields). As one can realize from the definition, the higher is the type, the more complicated is the structure of Cartan fields.

In the rest of the section we will study the main properties of different types of Cartan fields, starting from the simplest and the most basic ones: hamiltonian vector fields.

#### 3.1 Hamiltonian fields and integrable distributions

The map

$$
\chi : C \longrightarrow \Lambda^1(J^1(\tau))/<U>, \quad X \longmapsto X(U) \mod <U>, \tag{13}
$$

is a $C^\infty(J^1(\tau))$-module isomorphism: it associates with each Cartan field $X$ the restriction of $X(U)$ to $C$. Note that, although $\chi$ depends on the choice of $U$ (by substituting it with a multiple $U = \lambda U$ one gets $\chi(X) = X^\perp$ does not change. By inverting $\chi$, with each $\sigma \in \Lambda^1(J^1(\tau))$ one associates a Cartan vector field

$$
X_\sigma \overset{\text{def}}{=} \chi^{-1}(\sigma),
$$
where \([\sigma]\) is the equivalence class of \(\sigma\) in \(\Lambda^1(J^1(\tau))/\langle U \rangle\); in other words, \(X_\sigma \in C\) is determined by the relation
\[
X_\sigma(U) = X_\sigma \cdot dU = \sigma + \lambda U
\]
for some \(\lambda \in C^\infty(J^1(\tau))\) (in fact, if \(U\) is given by (3), then \(\lambda = -\sigma(\partial_z))\).

**Proposition 3.2** \(X_\sigma^\perp = \{U = 0, \sigma = 0\}\). Furthermore, \(X_\sigma\) is characteristic for \(X_\sigma^\perp\) if and only if it is of type 2.

**Proof.** It follows from (14) that \(\sigma(X_\sigma) = 0\). But then
\[
X_\sigma(\sigma) = X^2_\sigma(U) - X_\sigma(\lambda)U - \lambda X_\sigma(U),
\]
hence, \(X_\sigma\) is characteristic for \(X_\sigma^\perp\) if and only if \(X^2_\sigma(U)\) linearly depends on \(U\) and \(X_\sigma(U)\). ■

In the case \(\sigma\) is exact, \(\sigma = df\), we simply write \(X_f\) instead of \(X_{df}\). Due to the apparent analogy with the case of symplectic geometry, we give the following

**Definition 3.3** Let \(f \in C^\infty(J^1(\tau))\), then the vector field \(X_f \in C\) is called the (contact-)hamiltonian vector field associated with \(f\).

Note that, although \(X_f\) depends on the particular choice of \(U\), its direction, which by the previous proposition is orthogonal to \(\{U = 0, df = 0\}\), only depends on \(C\) (and \(f\), of course). Furthermore, as in the symplectic case, \(f\) is a first integral of the corresponding field: \(X_f(f) = df(X_f) = 0\), from which it easily follows that \(X_f\) is of type 2. By the previous proposition, \(X_f\) is characteristic for \(X_f^\perp\): in other words, \(X_f\) coincides with the classical characteristic vector field of the first order equation \(f = 0\). Its local expression in a contact coordinate system \((x, y, z, p, q)\) on \(J^1(\tau)\) is
\[
X_f = \partial_p(f) \hat{\partial}_x + \partial_q(f) \hat{\partial}_y - \hat{\partial}_z(f) \partial_p - \hat{\partial}_y(f) \partial_q.
\]
In particular:
\[
X_x = -\partial_p, \quad X_y = -\partial_q, \quad X_z = -p\partial_p - q\partial_q, \quad X_p = \hat{\partial}_x, \quad X_q = \hat{\partial}_y.
\]

**Example 3.4** Let \(X \in C\), and \(f\) be a first integral of \(X\) then \(X_f \in X^\perp:\)
\[
dU(X_f, X) = X_f(U)(X) = (df + \lambda U)(X) = 0.
\]
Hence, if \(f, g, h\) are three first integrals such that \(df, dg, dh, U\) are independent, then
\[
X^\perp = \langle X_f, X_g, X_h \rangle.
\]

**Theorem 3.5** Let \(f, g \in C^\infty(J^1(\tau))\). Then the following properties are equivalent:

1) the distribution \(\langle X_f, X_g \rangle\) is integrable;
2) \(X_f\) and \(X_g\) are orthogonal with respect to \(dU\);
3) \(X_f(g) = X_g(f) = 0\);

Furthermore, if \(f, g\) are functionally independent, then the following two properties can be added to the above list of equivalences:

4) there exists a third function \(h \in C^\infty(J^1(\tau))\) such that \(U\) linearly depends on \(df, dg, dh\);
5) there exists a system of contact coordinates \((x, y, z, p, q)\) in which \(x = f, y = g\);
Proof. 1) implies 2). It follows from
\[ dU(X_f, X_g) = -U([X_f, X_g]) \]
and from the fact that \([X_f, X_g]\) depends on \(X_f\) and \(X_g\). Also, 2) implies 1). It follows from
\[ dU(X_f, [X_f, X_g]) = df([X_f, X_g]) = X_f(X_g(f)) - X_g(X_f(f)) = 0 \]
and the analogous relation for \(X_g\), keeping in mind that \(<X_f, X_g>^\perp = <X_f, X_g>\).

The equivalence of 2) and 3) is an immediate consequence of (1) applied to the cases \(\sigma = df\) and \(\sigma = dg\), respectively.

Let us now assume the functional independence of \(f\) and \(g\). If 1) holds, then by 2) \(df\) and \(dg\) vanish on \(<X_f, X_g>\), so that there exists a third function \(h\), independent from \(f\) and \(g\), such that \(<X_f, X_g> = \{df = 0, dg = 0, dh = 0\}\). As \(U\) vanishes on \(X_f, X_g\) it linearly depends on \(df, dg, dh\).

Let now 4) hold, then:
\[ \lambda U = dh - adf - bdg \tag{15} \]
for some functions \(\lambda, a, b \in C^\infty(J^1(\tau))\) (note that, as \(U\) is completely non integrable, in (15) all the three differentials must appear). But, then
\[ x = f, \ y = g, \ h = z, \ p = a, \ q = b \]
are contact coordinates on \(J^1(\tau)\), which proves 5). Finally, let 5) hold. Then
\[ X_f(g) = X_x(y) = -\partial_p(y) = 0, \]
which implies 3). ■

We note that the previous theorem is a special case of a more general result, essentially due to Jacobi (the statement and proof can be found in [7]).

**Definition 3.6** Two functions \(f, g \in C^\infty(J^1(\tau))\) are in involution when they satisfy any of the equivalent properties 1), 2), 3) of the previous theorem.

**Theorem 3.7 (structure of integrable distributions)** Let \(D\) be a 2-dimensional distribution in \(C\). Then \(D\) is integrable if and only if it is spanned by two hamiltonian fields \(X_f, X_g\), with \(f\) and \(g\) independent and in involution.

**Proof.** One of the two implications has already been proved in the previous theorem. As to the converse implication, let \(D \subset C\) be 2-dimensional and integrable. Then \(D = \{df = dg = dh = 0\}\) for some independent functions \(f, g, h\). But, as \(U\) vanishes on \(D\), it linearly depends on \(df, dg, dh\), i.e. is of the form (15), so that, by the same argument used there, \(f\) and \(g\) are in involution (and, obviously, \(D\) contains \(X_f\) and \(X_g\)). ■

As a consequence of Theorems 3.7 and 5) of Theorem 3.5 one has that every integrable, 2-dimensional distribution in \(C\) can be reduced to the form
\[ D = <\partial_p, \partial_q> = <X_x, X_y> \]
in a suitable contact chart; a partial or total Legendre map gives the alternative representations
\[ D = <\partial_x, \partial_y> \text{ or } D = <\partial_x, \partial_q> \text{ or } D = <\partial_y, \partial_p>. \]

The following proposition, together with Proposition 3.2, completes the discussion of integrability of subdistributions in \(C\).

**Proposition 3.8** Let \(P \subset C\) be a 3-dimensional distribution. Then its derived distribution \(P'\) is not contained in \(C\); in particular, \(P\) is not integrable.
Proof. Assume, by contradiction, that \( \mathcal{P}' \subset \mathcal{C} \). Then, for any couple of fields \( X_1, X_2 \in \mathcal{P} \) it would hold \( dU(X_1, X_2) = -U([X_1, X_2]) = 0 \), i.e. \( (dU)|_\mathcal{C} \) would identically vanish on \( \mathcal{P} \).  

Below we will need the following general lemma on derived distributions. The proof is straightforward.

 Lemma 3.9 Let \( \mathcal{P} \) be a \( k \)-dimensional distribution on a smooth manifold \( M^n \) and let \( I_\mathcal{P} \) be the corresponding Pfaffian system. Then the Pfaffian system associated with the derived distribution \( \mathcal{P}' \) is:

\[
I'_\mathcal{P} = \{ \omega \in I_\mathcal{P} \text{ s.t. } X(\omega) \in I_\mathcal{P} \forall X \in \mathcal{P} \}.
\]

The next proposition characterizes hamiltonian fields by integrability properties of their orthogonal complements.

 Proposition 3.10 Let \( X \in \mathcal{C} \). Then \( X \) is a multiple of a hamiltonian field \( X_f \) if and only if \( (X^\perp)' \) is 4-dimensional and integrable.

Proof. Assume \( X = X_f \) (or a multiple of it), then

\[
X^\perp = X_f^\perp = \{ U = X_f(U) = 0 \} = \{ U = df = 0 \}.
\]

Furthermore

\[
X_f(df) = X_g(df) = X_h(df) = 0
\]

i.e., by the previous lemma, \( df \) belongs to the derived system of \( \{ U, df \} \). Hence,

\[
(X_f^\perp)' = \{ df = 0 \}
\]

which is 4-dimensional and integrable.

Viceversa, let \( (X^\perp)' \) be 4-dimensional and integrable, then there exists a function \( f \) such that \( (X^\perp)' = \{ df = 0 \} \); therefore

\[
X^\perp = (X^\perp)' \cap \mathcal{C} = \{ U = df = 0 \} = \{ U = X_f(U) = 0 \} = X_f^\perp
\]

which entails the parallelism between \( X \) and \( X_f \).  

3.2 Cartan fields of type 2

The following result generalizes Proposition 3.10 and gives a characterization of type 2 Cartan fields.

 Proposition 3.11 Let \( X \in \mathcal{C} \). Then \( X \) is of type 2 if and only if the derived distribution \( (X^\perp)' \) has dimension 4.

Proof. Let \( \dim(X^\perp)' = 4 \). Then, by Lemma 3.9 applied to the case \( \mathcal{P} = X^\perp \), \( (X^\perp)' \) is described by equation \( \sigma = 0 \), with \( \sigma \) linear combination of \( U \) and \( X(U) \)

\[
\sigma = \alpha U + X(U)
\]

(by Proposition 3.8 \( \sigma \) is not a multiple of \( U \)) and such that, for any \( W \in X^\perp \), \( W(\sigma) \) linearly depends on \( U \) and \( X(U) \). In particular,

\[
X^2(U) \equiv X(\sigma) \equiv 0 \mod \langle U, X(U) \rangle.
\]

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Viceversa, let $X$ be of type 2. To prove our statement we must find an $\alpha$ in $\mathbb{C}$ such that $X^\perp$ is described by equation $\sigma = 0$. To this end, let $\{X, Y, Z\}$ be a basis of $X^\perp$, then $X(\sigma), Y(\sigma)$ and $Z(\sigma)$ must vanish on $X^\perp$. By assumption it holds

$$X(\sigma) = X^2(U) + X(\alpha)U + \alpha X(U) \equiv 0 \mod <U, X(U)>$$

and, therefore, $X(\sigma)$ vanishes on $X^\perp$ for any choice of $\alpha$. As to $Y(\sigma)$, relations

$$Y(\sigma)(X) = -X(\sigma)(Y) = 0, \quad Y(\sigma)(Y) = d\sigma(Y, Y) = 0$$

hold true for any $\alpha$, whereas equation

$$0 = Y(\sigma)(Z) = dX(U)(Y, Z) + \alpha dU(Y, Z)$$

determines $\alpha$. Therefore, by choosing $\alpha$ in this way, one has that $Y(\sigma)$ vanishes on $X^\perp$; the same holds for $Z(\sigma)$, due to the symmetry of roles of $Y$ and $Z$.

**Proposition 3.12** Let $D \subset C$ be a lagrangian, non integrable distribution. Then, it contains at most one field of type 2; if such a field exists, it spans $(D')^\perp$.

**Proof.** Let $X \in D$ be of type 2. Then, if $D = <X, Y>$, it holds $dU(X, X) = dU(X, Y) = 0$ and

$$dU(X, [X, Y]) = X(U)([X, Y]) = X(X(U)(Y)) - X^2(Y)(Y) = 0.$$

**Proposition 3.13** Let $X \in C$ be of type 2. For any first integral $f$ of $X$ the distribution $<X, X_f>$ is integrable. Conversely, every 2-dimensional integrable distribution in $C$ which contains $X$ is of this form.

**Proof.** Let $f \in C^\infty(J^1(\tau))$ be a first integral of $X$, then the lagrangian distribution $D = <X, X_f>$ is integrable. In fact, $[X, X_f] \in D$ if and only if it is orthogonal to both $X$ and $X_f$. But

$$dU(X_f, [X, X_f]) = (df - f U)([X, X_f]) = df([X, X_f]) = X(X_f(f)) - X_f(X(f)) = 0$$

(this holds for any $X \in C$ having $f$ as a first integral) and also

$$dU(X, [X, X_f]) = X(dU(X, X_f)) - dX(U)(X, X_f) = 0 - X^2(X_f) = 0$$

because $X^2(U)$ depends on $U$ and $X(U)$.

Viceversa, let $D \subset C$ be a 2-dimensional integrable distribution. Then, by Theorem 3.9, $D = <X_f, X_g>$ with $f$ and $g$ in involution. Therefore, if $X \in D$, then $f$ and $g$ are first integrals of $X$.

### 3.3 Normal forms of Cartan fields

In this section normal forms for Cartan fields are given. The following proposition gives us the simplest possible form valid for any Cartan field. For fields of type less than 4, more precise normal forms can be obtained. These are a consequence of next theorem, which characterizes non generic Cartan fields in terms of involutive hamiltonian fields.

**Proposition 3.14** For any field $X \in C$ there exists a contact coordinate system in which $X$ takes the form

$$X = a\partial_x + b\partial_y + c\partial_q, \quad a, b, c \in C^\infty(J^1(\tau)).$$

**Proof.** Let $f$ be a first integral of $X$ (equivalently, $X_f$ be orthogonal to $X$), then one may assume, according to Theorem 3.5, that in a certain contact chart is $f = y$ and consequently $X_f = \partial_y$, from which the statement follows, because $\partial^\perp_q$ is spanned by $\partial_x, \partial_p, \partial_q$.
Theorem 3.15 Let \( X \in \mathcal{C} \), then the following equivalences hold:

1) \( X \) is of type 2 or 3;

2) \( X = aX_f + bX_g \) with \( f \) and \( g \) in involution and \( a, b \in C^\infty(J^1(\tau)) \);

3) \( X = a\partial_p + b\partial_q \) in an appropriate contact chart \((x,y,z,p,q)\), and \( a, b \in C^\infty(J^1(\tau)) \);

4) \( X \) admits two independent first integrals in involution;

5) \( X \) belongs to at least one 2-dimensional integrable subdistribution of \( \mathcal{C} \).

Proof. 1) implies 2). In fact, if \( X \) is of type 2 then the statement follows from Proposition 3.13. If, instead, \( X \) is of type 3, then it is characteristic for the distribution \( D_X = \{ U = X(U) = X^2(U) = 0 \} = \langle X, Y \rangle \), for some \( Y \in X^\perp \). Hence, \( D_X \) is integrable (because it contains \([X,Y]\)) and, consequently, it is spanned by two vector fields in involution (Theorem 3.7). Also, 2) implies 1). In fact, if we put \( X_0 = \text{id} \), in this case the following relations hold:

\[
X^j(U) \equiv X^j - 1(a)df + X^j - 1(b)dg \mod \langle U, \ldots, X^j - 1(U) \rangle, \quad 1 \leq j \leq 3.
\]

from which the linear dependence of \( U, X_0(U), X^2(U), X^3(U) \) follows.

Equivalence between 2) and 3) immediately follows from 4) of Theorem 3.5. Equivalence between 2) and 5) is just Theorem 3.7. 4) trivially follows from 2). Now, assuming 4) to hold, let \( f \) and \( g \) be the two (independent) involutive first integrals, then: \( X(f) = X(g) = 0 \), \( X_f(g) = 0 \), or also, in terms of orthogonality, \( X \in \langle X_f, X_g \rangle^\perp = \langle X_f, X_g \rangle \). #

Remark 3.16 We have already proved (Proposition 3.13) that, if \( X \in \mathcal{C} \) is of type 2, then it is contained in a family of 2-dimensional integrable subdistributions of \( \mathcal{C} \) (one for each first integral). On the other hand, if \( X \) is of type 3, it is contained in just one 2-dimensional integrable subdistribution of \( \mathcal{C} \), namely the distribution \( D_X \) defined in the proof of the above theorem.

We have seen in Theorem 3.15 that, modulo a contact transformation, a field \( X \in \mathcal{C} \) of type less than 4 takes the form \( X = \partial_p + b\partial_q \) (as the type of a field depends only on its direction, we have chosen \( a = 1 \) in point 3) of above theorem). Then \( X(U) = -dx - bdy \) and \( X^2(U) = -X(b)dy \) from which it follows that \( X^2(U) \) depends on \( U \) and \( X(U) \) if and only if \( b \) is a first integral of \( X \). Therefore, one gets the following

Theorem 3.17 Let \( X \in \mathcal{C} \). Then

1) \( X \) is of type 2 if and only if, in a suitable contact chart, it takes the form

\[
X = a\partial_p + b\partial_q, \quad \text{with} \ X(b/a) = 0;
\]

2) \( X \) is of type 3 if and only if \((17)\) holds, with \( X(b/a) \neq 0 \).

This result can be refined in the case of a field of type 2.

Theorem 3.18 A vector field \( X \in \mathcal{C} \) is of type 2 if and only if, in some contact chart, it takes one of the forms

\[
X = \partial_p \quad \text{or} \quad X = \partial_p + z\partial_q.
\]
Proof. Let \((X^\perp)'\) be locally described by equation \(\sigma = 0\) (see also Proposition 3.11). By Darboux theorem, one can choose independent functions \(f, g, h, k, l\) in such a way that, up to a factor, one of the following three expressions holds: either

\[
\sigma = df \tag{18}
\]

or

\[
\sigma = df - gh \tag{19}
\]

or

\[
\sigma = df - gh - kd. \tag{20}
\]

Expression (20) can be excluded because, otherwise, \(\{\sigma = 0\}\) would be a contact structure containing a 3-dimensional distribution, \(X^\perp\), such that \((X^\perp)' = \{\sigma = 0\}\), which is impossible by Proposition 3.8.

If (18) holds, \(X\) is a multiple of \(X_f\) (Proposition 3.10); on the other hand, by Theorem 3.5, there exists a contact transformation sending \(f\) into coordinate \(x\), so that, modulo a factor,

\[
X = \partial_p. \tag{21}
\]

Finally, in case (19) one has

\[
X = X_\sigma = X_f - gX_h. \tag{21}
\]

Hence,

\[
X(U) = df - gh - (f_z - gh_z)U, \quad X^2(U) = -X_h(f)dg + X_g(f - gh)dh. \tag{22}
\]

But, being \(X\) of type 2, one gets

\[
- X_h(f)dg + X_g(f - gh)dh = \lambda U + \mu (df - gh) \tag{22}
\]

for some \(\lambda, \mu \in C^\infty(J^1(\tau))\). As the contact form \(U\) is determined up to a factor, one may assume that \(\lambda\) does not vanish. Hence, it follows from (22) that

\[
U = -\frac{X_h(f)}{\lambda} \left( df + \frac{\mu}{X_h(f)} X_g(gh - f) - \frac{\mu}{X_h(f)} dh \right). \tag{22}
\]

Hence the functions

\[
x = f, \quad y = h, \quad z = -g, \quad p = \frac{\mu}{X_h(f)}, \quad q = \frac{X_g(gh - f) - \mu}{X_h(f)}
\]

form a contact chart. Consequently, \(X\) of (21) assumes the form

\[
X = X_x + zX_y = \partial_p + z\partial_q. \tag{21}
\]

As a remarkable application of normal form (17), we prove the following proposition.

**Proposition 3.19** Let \(D \subset \mathbb{C}\) be a non integrable lagrangian distribution, and let \((D')^\perp\) be spanned by vector field \(X\). Then \(X\) is not of type 3.

**Proof.** Assume the type of \(X\) less than 4. Then it is 2 or 3. By Theorem 3.17 in some contact coordinates \(X\) takes the form

\[
X = \partial_p + a\partial_q, \quad a \in C^\infty(J^1(\tau))
\]

(as the type only depends on the direction of \(X\), the coefficient of \(\partial_p\) in (17) can be assumed equal to 1). Let \(D = \langle X, Y \rangle\), then \(Y \in X^\perp\) and, hence, is of the form

\[
Y = \partial_x - \frac{1}{a}\partial_y + b\partial_p + c\partial_q,
\]

for some functions \(b, c \in C^\infty(J^1(\tau))\). Let us now impose the orthogonality between \(X\) and \([X, Y] \in D'\). As \(X_dU = -dx - ady\), one gets:

\[
0 = dU(X, Y) = -(dx + ady)([X, Y]) = -[X, Y](x) - a[X, Y](y) = \frac{X(a)}{a},
\]

so that \(X(a) = 0\), i.e., by Theorem 3.18 \(X\) is of type 2. ■
4 Normal forms of parabolic Monge-Ampère equations

In this section Theorems 1.1 and 1.2 are eventually proved. Normal forms of parabolic MAE’s are derived by the corresponding normal forms of the associated characteristic distributions. The relation between each normal form and the existence of intermediate integrals is shown. Furthermore, the existence of a complete integral for the general analytic parabolic MAE’s is proved.

4.1 Intermediate integrals and their generalization

Definition 4.1 Let $E$ be a second order PDE. An intermediate integral of $E$ is a function $f \in C^\infty(J^1(\tau))$ such that solutions of the equations $f = k$, $k \in \mathbb{R}$, are also solutions of $E$.

In the case of MAE’s, the following theorem provides a practical method for finding intermediate integrals.

Theorem 4.2 (1) Let $\rho \in \Lambda^2(J^1(\tau))$ and $\mathcal{E}_\rho$ be the corresponding MAE. Then, $f \in C^\infty(J^1(\tau))$ is an intermediate integral of $\mathcal{E}_\rho$ if and only if

$$U \wedge df \wedge (X_f|\rho) = 0.$$  

(23)

Coming back to the parabolic case, the following proposition holds.

Proposition 4.3 A function $f \in C^\infty(J^1(\tau))$ is an intermediate integral of $\mathcal{E}_\omega$ if and only if $X_f \in \mathcal{D}$, i.e. $X_f$ is characteristic for the equation. Furthermore, as $\mathcal{D}$ is lagrangian, $f$ is a first integral of any characteristic field of $\mathcal{E}_\omega$.

Proof. If $\omega = X(U) \wedge Y(U)$, with $X, Y$ generating the characteristic distribution of $\mathcal{E}_\omega$, then, by taking $\rho = \omega$ in (23), one gets

$$U \wedge df \wedge W(U) = 0$$  

(24)

with $W = Y(f)X - X(f)Y$. But from (24) follows $W(U) = \alpha df + \beta U$ and, by dividing by $\alpha$, we obtain

$$\frac{1}{\alpha}W(U) = df + \frac{\beta}{\alpha}U.$$  

(25)

On the other hand $X_f(U) = df - f_2 U$, so that, subtracting (25) from it, one gets

$$\left(X_f - \frac{1}{\alpha}W\right)(U) = \lambda U$$

from which follows that $X_f - \frac{1}{\alpha}W = 0$ (otherwise, it would be a non-trivial characteristic field of $\mathcal{C}$), and the proposition follows.

Theorem 4.4 Let $\mathcal{D} \subset \mathcal{C}$ be the characteristic distribution associated with $\mathcal{E}_\omega$. Then, such equation admits intermediate integrals if and only if: 1) $\mathcal{D}$ is integrable or 2) $\mathcal{D}''$ is 4-dimensional and integrable. In the first case, intermediate integrals are all and only the functions of the form $f = \phi(f_1, f_2, f_3)$ with $\phi$ arbitrary function of three real variables and $f_1, f_2, f_3$ independent first integrals of $\mathcal{D}$; in the second case, there exists (up to functional dependence) only one intermediate integral, given by the function $f$ such that $\mathcal{D}' = \{df = 0\}$.

Proof. According to Proposition 4.3, $f$ is an intermediate integral if and only if $X_f \in \mathcal{D}$. If $\mathcal{D}$ is integrable, then $\mathcal{D} = \langle X_{f_1}, X_{f_2} \rangle$ with $f_1$ and $f_2$ in involution. Hence $X_{f_1}(f) = X_{f_2}(f) = 0$ which proves the statement in case 1).

If, instead, $\mathcal{D}$ is not integrable and $X_f \in \mathcal{D}$, then $\mathcal{D}' = X_f^\perp$ (see Proposition 3.12). It is easily checked that $\mathcal{D}'' = \{df = 0\}$; in fact, two vector fields are orthogonal to $X_f$ if and only if both have $f$ as a first integral, so that their commutator vanishes on $df$.  ■
It follows from the previous theorem that there exist parabolic MAE’s without intermediate integrals: in fact, as we shall see later, these are the majority. For this reason, it is interesting to consider possible extensions of the classical notion of intermediate integral. Note that a field $X$ is a multiple of an $X_f$, with $f$ intermediate integral of $\mathcal{E}_\omega$, if and only if $X$ is a field of type 2 in $\mathcal{D}$ such that $(X^\perp)'$ is integrable. If one checks the last condition out, one obtains nonholonomic intermediate integrals in the sense of [5].

**Definition 4.5** Let $\mathcal{D} \subset \mathcal{C}$ be the characteristic distribution associated with $\mathcal{E}_\omega$. A nonholonomic intermediate integral of $\mathcal{E}_\omega$ is a type 2 vector field contained in $\mathcal{D}$.

**Theorem 4.6** If $\mathcal{D}''$ is 4-dimensional, then $\mathcal{E}_\omega$ admits exactly one nonholonomic intermediate integral $X \in \mathcal{D}$ which spans $(\mathcal{D}')^\perp$. Such an integral is classical if $\mathcal{D}''$ is integrable and genuinely nonholonomic otherwise.

**Proof.** It is an easy corollary of Propositions 3.11, 3.12 and Theorem 4.4.

Below we propose a further generalization.

**Definition 4.7** A generalized intermediate integral of a parabolic MAE $\mathcal{E}_\omega$ is a field $X \in \mathcal{D}$ of type less than 4.

Note that an intermediate integral of $\mathcal{E}_\omega$ is a 4-dimensional foliation of $J^1(\tau)$ whose leaves (which are first order scalar differential equations) are such that their solutions are also solutions of $\mathcal{E}_\omega$. By applying the method of Lagrange-Charpit one obtains a complete integral (2 functional parameters) of each leaf ($\infty^1$ leaves), so that one obtains a family of $\infty^3$ solutions of $\mathcal{E}_\omega$.

**Definition 4.8** A complete integral of $\mathcal{E}_\omega$ is a 2-dimensional foliation of $J^1(\tau)$ whose leaves are solutions or, equivalently, a 2-dimensional integrable distribution $\hat{\mathcal{D}} \subset \mathcal{C}$ such that $\omega|_{\hat{\mathcal{D}}} = 0$ for any $\theta \in J^1(\tau)$.

Let us now show the (almost) equivalence of the two above definitions.

**Proposition 4.9** Starting from a generalized intermediate integral, one can construct a complete integral, and vice versa.

**Proof.** If $X \in \mathcal{D}$ is of type 2 or 3, then it belongs to at least one lagrangian integrable distribution $\hat{\mathcal{D}}$ (Theorem 3.15). Conversely, a complete integral $\hat{\mathcal{D}}$, whose fields are all of type 2 or 3, has a non trivial intersection with $\mathcal{D}$: any non zero vector field in $\mathcal{D} \cap \hat{\mathcal{D}}$ is a generalized intermediate integral.

Note that the correspondence between intermediate integrals and complete integrals is not biunivocal. Namely, when $X$ is of type 2 it belongs to a family of integrable distributions, whereas, when it is of type 3 the distribution is unique. Conversely, if $\dim \mathcal{D} \cap \hat{\mathcal{D}} = 2$, i.e. $\mathcal{D}$ is integrable, then every field in $\mathcal{D}$ is an intermediate integral; if, instead, $\dim \mathcal{D} \cap \hat{\mathcal{D}} = 1$, then the intermediate integral is unique (up to a multiple). As we shall see in the next section, the latter is the generic case.

**4.2 The general case: proof of Theorem 1.1**

Let us assume that there exists a complete integral of $\mathcal{E}_\omega$. Then, by Proposition 4.9 there exists a generalized intermediate integral $Z \in \mathcal{D}$. As $Z$ is of type less than 4, by Theorem 3.15 one has that, up to contactomorphisms and a factor,

$$Z = \partial_p + a\partial_q,$$

Therefore, $\mathcal{D}$ is spanned by $Z$ and a vector field orthogonal to it,

$$W = \partial_q - a\partial_x + b\partial_q,$$
so that, up to a factor, is \( \omega = Z(U) \wedge W(U) \), i.e.

\[
\omega = -(dx + ady) \wedge (dq - adp - bdy)
\]

whose associated equation \( \mathcal{E}_\omega \) is (4), i.e.

\[
z_{yy} - 2a z_{xy} + a^2 z_{xx} = b.
\]

(26)

Viceversa, an equation of the above form admits the characteristic field \( Z = \partial_p + a \partial_q \) which belongs to the integrable distribution \( \hat{D} = \langle \partial_p, \partial_q \rangle \). This completes the proof of Theorem 1.1.

The condition of the existence of a complete integral seems to be not very restrictive in the \( C^\infty \) category, as we shall see in section 4.2.1. Furthermore we shall prove in section 4.2.2 that, in the analytic case, this condition is not a restriction at all.

4.2.1 Does a complete integral always exist?

Here we shall see how a large class of \((C^\infty)\) parabolic MAE’s admits a complete integral and, hence, is reducible to normal form (4). Let us consider the parabolic MAE:

\[
z_{xy}^2 - z_{xx} z_{yy} + T z_{xx} - 2Sz_{xy} + Rz_{yy} + S^2 - RT = 0
\]

which is associated with the distribution \( \mathcal{D} \) spanned by vector fields

\[
X = \partial_x + R \partial_p + S \partial_q, \quad Y = \partial_y + S \partial_p + T \partial_q
\]

(27)

(see the end of section 2.2). Assume either \( R \) to be independent of \( q \) or \( T \) to be independent of \( p \). Then \( \mathcal{D} \) contains a vector field of type 2 or 3. In fact, if \( \partial_q(R) = 0 \), then \( [X, \partial_q] = -\partial_q(S) \partial_q \), so that the distribution \( \langle X, \partial_q \rangle \) is integrable and the assertion follows from Theorem 3.15. In the second case \( (\partial_p(T) = 0) \) \( Y \) belongs to the integrable distribution \( \langle Y, \partial_p \rangle \).

As an example, in order to give completely explicit computations, we assume \( R = 1 \). The distribution \( \langle X, \partial_q \rangle \) is integrable and spanned by three common first integrals of the generators, namely:

\[
\langle X, \partial_q \rangle = \{ dy = d\alpha = d\beta = 0 \}, \quad \alpha = z - \frac{p^2}{2}, \quad \beta = x - p.
\]

Then \( \{ y = k_1, \alpha = k_2, \beta = k_3 \}, k_i \in \mathbb{R} \), turns out to be a complete integral of the MAE under consideration. A direct computation shows that \( U = d\alpha - p \, d\beta - q \, dy \). Therefore, functions

\[
\overline{x} = \beta = x - p, \quad \overline{y} = y, \quad \overline{z} = \alpha = z - \frac{p^2}{2}, \quad \overline{p} = p, \quad \overline{q} = q
\]

are contact coordinates, with respect to which \( X \) and \( Y \) are given by

\[
X = \partial_{\overline{x}} + S \partial_{\overline{y}}, \quad Y = \partial_{\overline{z}} - S \partial_{\overline{x}} + (T - S^2) \partial_{\overline{p}}
\]

Since \( \partial_{\overline{y}} = X - S \partial_{\overline{x}} \), \( \mathcal{D} \) is spanned by

\[
X = \partial_{\overline{x}} + S \partial_{\overline{y}}, \quad Y' = \partial_{\overline{z}} - S \partial_{\overline{x}} + (T - S^2) \partial_{\overline{p}}
\]

and the associated equation becomes

\[
\overline{z}_{yy} - 2S \overline{z}_{xy} + S^2 \overline{z}_{xx} - (T - S^2) = 0.
\]
4.2.2 The analytic case

In [3] it is proved that every parabolic MAE with real analytic coefficients can be reduced to form (4) by means of Cartan-Kähler theorem. In this section we give an alternative proof based only on the Cauchy-Kovalevsky theorem.

As we already explained, all that we have to do is to find a complete integral. As a first step, we give some equivalent formulations of this problem without yet assuming the analyticity condition.

**Lemma 4.10** A vector field $Z \in \mathcal{C}$ is of type less than 4 if and only if it admits a first integral $f$ satisfying the equation

$$dU(Z, [Z, X_f]) = 0, \text{ with } X_f \neq 0.$$  \hspace{1cm} (28)

**Proof.** If $Z$ is a multiple of $X_f$ for some $f$, then both of them are of type 2. So, we can assume that they are independent. It is easy to prove that if $Z(f) = 0$ then $dU(X_f, [Z, X_f]) = 0$. Assume that the first integral $f$ is a solution of (28); then $[Z, X_f]$ is orthogonal to the lagrangian distribution spanned by $Z$ and $X_f$ and, hence, belongs to it; but this implies that such distribution is integrable. By applying Theorem 3.15 one obtains that $Z$ is of type 2 or 3.

Conversely, if $Z$ is of type 2 or 3 then, again by Theorem 3.15, $Z$ linearly depends on two fields $X_f, X_g$ with $f$ and $g$ in involution: obviously, both functions are solutions of (28). \hfill \blacksquare

**Theorem 4.11** Let $\mathcal{D} = \left< X, Y \right>$ be the lagrangian distribution associated with equation $\mathcal{E}_\omega$. Then, the following equivalences hold:

1) There exists a complete integral of $\mathcal{E}_\omega$;

2) There exists a generalized intermediate integral;

3) There exists a field $Z \in \mathcal{D}$ such that type $Z < 4$;

4) There exists a field $Z \in \mathcal{D}$ which is also contained in an integrable lagrangian distribution $\hat{\mathcal{D}}$;

5) There exists an integrable lagrangian distribution $\hat{\mathcal{D}}$ such that the graph of the corresponding section $J^1(\tau) \to J^2(\tau)$ is contained in $\mathcal{E}_\omega$.

6) There exists a function $f \in C^\infty(J^1(\tau))$ such that the field $Z_f = Y(f)X - X(f)Y$ satisfies the equation

$$dU(Z_f, [Z_f, X_f]) = 0.$$  \hspace{1cm} (29)

**Proof.** The equivalence of properties 1), 2), 3), 4), 5) has been already proved. Let us focus on the equivalence between 4) and 6). First, 4) implies 6). In fact, let us suppose $Z \in \hat{\mathcal{D}}$. Since $\hat{\mathcal{D}}$ is integrable, there exists a function $f$ such that $X_f \in \hat{\mathcal{D}}$ (see Theorem 3.15) and $Z(f) = 0$, which implies that $Z$ is a multiple of $Z_f$. So $Z_f, [Z_f, X_f] \in \hat{\mathcal{D}}$, that is lagrangian, and (29) follows. Second, 6) implies 4). If $Z_f = 0$, then $X(f) = Y(f) = 0$, which implies $X_f \in \mathcal{D}$. Then we can choose $Z = X_f$. If $Z_f \neq 0$, then it is sufficient to apply previous lemma with $Z = Z_f$. \hfill \blacksquare

The determining equation (29) provides a tool for proving the existence of a complete integral in the real analytic case.

**Theorem 4.12** Any parabolic analytic MAE admits a complete integral. In particular, it can be reduced to form (4).

**Proof.** Equation (29) can be written in the equivalent form:

$$Y(f)^2dU(X, [X, X_f]) - 2X(f)Y(f)dU(X, [Y, X_f]) + X(f)^2dU(Y, [Y, X_f]) = 0.$$  \hspace{1cm} (30)
It is straightforward to check that this equation, in a contact chart where $X$ and $Y$ assume the form (27), takes the form
\[ \sum_{i,j=1}^{5} A_{ij} f_{x_i x_j} + B = 0, \] (31)
where we have denoted by $(x^1, x^2, x^3, x^4, x^5)$ the chart $(x, y, z, p, q)$, and $A_{ij}$ and $B$ are analytic functions of $x^1, \ldots, x^5, f_{x^1}, \ldots, f_{x^5}$. Hence, by applying Cauchy-Kovalevsky theorem to equation (31), the existence of a complete integral in a neighborhood of an arbitrary analytic hypersurface of $J^1(\tau)$ is proved. □

4.3 The non generic case: proof of Theorem 1.2

In the previous section (proof of Theorem 1.1) we derived the normal form (26) of a parabolic MAE admitting a complete integral from that of the associated characteristic distribution:

\[ D = \{ \partial_p + a\partial_q, \partial_y - a\partial_x + b\partial_q \} \]

As we have already seen, such canonical form holds for all analytic parabolic MAE's and for a large class of $C^\infty$ ones (indeed, we strongly suspect, for all). In particular, one can reduce to form (26) all non generic parabolic MAE's, i.e. those for which $D''$ has dimension less than 5. However, for such equations more precise normal forms can be obtained.

**Theorem 4.13** Let $D \subset \mathcal{C}$ be a non generic lagrangian distribution. Then, there exist contact local coordinates on $J^1(\tau)$ in which $D$ takes one the following normal forms:

- $D = \{ \partial_p + a\partial_q, \partial_y - a\partial_x + b\partial_q \}$
- $D = \{ \partial_p + z\partial_q, \partial_y - z\partial_x + b\partial_q \}$, $b \in C^\infty(J^1(\tau))$, $\partial_p(b) + z\partial_z(b) \neq 0$.

**Proof.** According to the “integrability degree” of $D$, one can distinguish the following cases:

1) $D = D'$, i.e. $D$ is integrable;

2) $D \neq D'$, i.e. $D$ is non integrable: in this case $\dim D' = 3$ and $D' \subset \mathcal{C}$ (the latter property is due to the fact that $D$ is lagrangian); by Proposition 3.8 $D'$ is non integrable.

Case 2) splits into the following subcases:

2-1) $D \neq D' \neq D''$ and $\dim D'' = 4$; in this case there are two possibilities:

- 2-1-1) $D''$ integrable;
- 2-1-2) $D''$ non integrable;

2-2) the generic case: $D \neq D' \neq D''$ and $\dim D'' = 5$.

- In case 1), in view of Theorem 5.7 in a suitable contact chart $D$ takes the form

\[ D = \{ \partial_p, \partial_q \} \]

and, by a Legendre transformation, we obtain normal form a).

- In case 2), $D'$ is determined by a generator $X$ of its orthogonal complement. Let us examine, first, case 2-1). From Theorem 1.6 and in view of Theorem 5.18 one obtains the normal form for the field $X \in (D')^\perp$:

- the case 2-1-1) corresponds to the normal form $X = \partial_p$, so that we obtain normal form b).
• the case 2-1-2) corresponds to the normal form $X = \partial_p + z\partial_q$, so that we obtain normal form $c$).

• The case 2-2) is excluded by hypothesis.

Note that it is possible to distinguish the various types of parabolic MAE's according to the number and kind of their intermediate integrals, namely:

- in case 1) there are three intermediate integrals, up to functional dependence, and according to Theorem 3.15 $\mathcal{D}$ contains only vector fields of type less than 4;
- in case 2-1-1) there exists only one intermediate integral and, in view of Proposition 3.12, only one vector field of type 2 which turns out to be hamiltonian;
- in case 2-1-2) there are no classical intermediate integrals, but there exists a nonholonomic one in the sense of [5], which is also, up to a factor, the only vector field of type 2 (Proposition 3.12);
- in case 2-2) there is not even a nonholonomic integral. For what said in the previous section, there exists a generalized intermediate integral (fields of type 3) in the real analytic case, while we don’t know in the $C^\infty$ case.

In order to obtain normal of Theorem 1.2 by using the results of previous theorem, it is sufficient to compute $\mathcal{E}_\omega$ where $\omega = X(U) \wedge Y(U)$ with $\mathcal{D} = \langle X, Y \rangle$ (see also the reasoning in the end of section 2.2). This completes the proof of Theorem 1.2.

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