WELL-POSEDNESS FOR A WHITHAM–BOUSSINESQ SYSTEM WITH SURFACE TENSION

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Abstract. We regard the Cauchy problem for a particular Whitham–Boussinesq system modelling surface waves of an inviscid incompressible fluid layer. The system can be seen as a weak nonlocal dispersive perturbation of the shallow water system. The proof of well-posedness relies on energy estimates. However, due to the symmetry lack of the nonlinear part, in order to close the a priori estimates one has to modify the traditional energy norm in use. Hamiltonian conservation provides with global well-posedness at least for small initial data in the one dimensional settings.

1. Introduction

Consideration is given to the following one-dimensional Whitham-type system

\[
\begin{align*}
\partial_t \eta &= -\partial_x v - i \tanh D(\eta v), \\
\partial_t v &= -i \tanh D(1 + \kappa D^2)\eta - i \tanh Dv^2/2
\end{align*}
\]

where \( D = -i\partial_x \) and \( \tanh D \) are Fourier multiplier operators in the space of tempered distributions \( S'(\mathbb{R}) \). The positive parameter \( \kappa \) stands for the surface tension here. The space variable is \( x \in \mathbb{R} \) and the time variable is \( t \in \mathbb{R} \). The unknowns \( \eta, v \) are real valued functions of these variables. We pick the initial values \( \eta(0), v(0) \) corresponding to the time moment \( t = 0 \) in Sobolev spaces as follows

\[
\eta_0 \in H^{s+1/2}(\mathbb{R}), \quad v_0 \in H^s(\mathbb{R})
\]

where \( s \geq 1/2 \). System (1.1) has the Hamiltonian structure

\[
\partial_t (\eta, v)^T = J \nabla H(\eta, v)
\]

with the skew-adjoint matrix

\[
J = \begin{pmatrix} 0 & -i \tanh D \\ -i \tanh D & 0 \end{pmatrix}
\]

and the energy functional

\[
H(\eta, v) = \frac{1}{2} \int_{\mathbb{R}} \left( \eta^2 + \kappa (\partial_x \eta)^2 + v \frac{D}{\tanh D} v + \eta v^2 \right) dx
\]

well defined on \( H^1 \times H^{1/2} \). The latter conserves on solutions together with momentum \( I(\eta, v) \) that has the same view as in the pure gravity case

\[
I(\eta, v) = \int_{\mathbb{R}} \eta \frac{D}{\tanh D} v dx.
\]

In case of the trivial surface tension \( \kappa = 0 \), System (1.1) was proposed in [6] as an approximate model for the study of water waves to provide a two-directional alternative to the well-known Whitham equation [22]. The latter was proved to be consistent with the KdV equation [18] in the long wave regime [19]. We also refer to [10] for another version of the fully-dispersive Boussinesq type. Importance of such models is supported by experiments [4]. The unknown \( \eta \) denotes the deflection of the free surface from its equilibrium position, corresponding to the vertical level \( z = 0 \). The bottom is assumed to be flat and located at the level \( z = -1 \). The variable \( v \) is associated with the free surface velocity as explained in [6].

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The initial value problem for Model (1.1) was studied in [5, 9] in the case of vanishing surface tension \( \kappa = 0 \). In the same framework existence of solitary waves was proved in [8]. A natural extension of the existing results is to consider the case of non-trivial capillarity \( \kappa > 0 \). Note that usually the term \( 1 + \kappa D^2 \) is applied to \( -v_2 \) in the first equation as it is done in [12], for example, to regularise the system regarded in [11]. However, the case regarded here is physically more relevant [7]. It turns out that surface tension spoils regularity. Indeed, the multiplication operator \( \eta \mapsto v \eta \) is not bounded in our problem. We have 1/2 loss of regularity here, which means that System (1.1) is quasilinear. As a result the proof of well-posedness demands a technique different from the one used in [9].

As to additional initial conditions, apart from inclusions given in (1.2), one has to impose a restriction essentially similar to the one used in [9], namely, smallness of the \( H^1 \times H^{1/2} \)-norm of \((\eta_0, v_0)\). This is important for the global-in-time existence. The meaning of this condition is that the total energy \( \mathcal{H}(\eta_0, v_0) \) should be positive and not too big. We point out that this condition cannot be significantly weakened even for the proof of the local result, which is also different from the non-capillarity situation. More precisely, for the local regular \( s \) is large enough in (1.2) well-posedness result it is enough to assume non-cavitation instead.

Definition 1. We say that elevation \( \eta \in C([0, T]; L^\infty(\mathbb{R})) \) satisfies the non-cavitation condition if there exist \( h, H > 0 \) such that \( H \geq \eta \geq h - 1 \) on \( \mathbb{R} \times [0, T] \). Analogously, one defines non-cavitation at a particular time moment.

The non-cavitation condition is a physical condition meaning that the elevation of the wave should not touch the bottom of the fluid for System (1.1) to be a relevant model. For convenience we have also included boundedness from above in this definition. We exploit the definition for providing with more general local existence formulation at high regularity level. However, in the low regularity case this condition cannot be controlled without imposing a stronger assumption, as we shall see below. We turn now to the formulation of the main result.

Theorem 1. Let \( s > 1/2 \). For any \( \eta_0 \in H^{s+1/2}(\mathbb{R}) \) and \( v_0 \in H^{s}(\mathbb{R}) \) having sufficiently small \( H^1 \times H^{1/2} \)-norm there exists a unique global solution \((\eta, v) \in C([0, \infty); H^{s+1/2}(\mathbb{R}) \times H^{s}(\mathbb{R})) \) of System (1.1) with the initial data \((\eta_0, v_0)\). Moreover, the solution depends continuously on the initial data on any finite time interval \([0, T]\).

Remark 1. Assuming instead smallness of \( H^1 \times H^{1/2} \)-norm the noncavitation condition for \( \eta_0 \) one obtains local well-posedness for \( s > 3/2 \).

The proof is essentially based on the energy method, that is natural to apply to quasilinear equations. The scaling \( H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}) \) is needed to rule out the linear terms. The main difficulty lies in the lack of symmetry of the nonlinearity. Indeed, a direct time differentiation of the norm \( \|\eta, v\|_{H^{s+1/2} \times H^s} \) leads to the term \( \int (J^{s-1/2} \partial_2 \eta) \eta J^{s+1/2} v \), where \( J^\sigma \) stands for the Bessel potential of order \( -\sigma \) (see the proof of Lemma [3] below). Note that this term cannot be handled by integration by parts or commutator estimates, and so cannot be estimated via the energy norm. To overcome this difficulty we modify the energy norm adding the cubic term \( \int \eta (J^{s-1/2} v)^2 \). The linear contribution of the derivative of this term will cancel out the mentioned inconvenient term. Meanwhile, the contribution coming from the nonlinear terms can easily be controlled. As we point out below a hint on the choice of the modifier comes from Hamiltonian (1.3). Note that after adding the cubic term the energy loses coercivity, and so one has to impose an additional condition. Either the noncavitation for big \( s \) or the smallness for small \( s \) of the initial data, both propagating through the flow of System (1.1), is enough to ensure that the modified energy is coercive.

Additionally, consideration is also given to a system posed on \( \mathbb{R}^{2+1} \) of the form

\[
\begin{cases}
\partial_t \eta + \nabla \cdot v = -K^2 \nabla \cdot (\eta v), \\
\partial_t v + K^2 \nabla (1 + \kappa |D|^2) \eta = -K^2 \nabla (|v|^2/2)
\end{cases}
\]

(1.4)
that is a direct two dimensional extension of Model (1.1). Here \( \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 \) is a curl free vector field, that is \( \nabla \times \mathbf{v} = 0 \), and

\[ K = \sqrt{\tanh |D|/|D|} \]

with \( D = -i\nabla \). So the corresponding symbol \( K(\xi) = \sqrt{\tanh(|\xi|/|\xi|} \). We complement (1.4) with the initial data

\[ \eta(0) = \eta_0 \in H^{s+1/2}(\mathbb{R}^2), \quad \mathbf{v}(0) = \mathbf{v}_0 \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2). \]  (1.5)

As above the variables \( \eta \) and \( \mathbf{v} \) stand for the surface elevation and the surface fluid velocity, respectively. The system enjoys the Hamiltonian structure

\[ \partial_t (\eta, \mathbf{v})^T = \mathcal{J} \nabla \mathcal{H}(\eta, \mathbf{v}) \]

with the skew-adjoint matrix

\[ \mathcal{J} = \begin{pmatrix} 0 & -K^2 \partial_{x_1} & -K^2 \partial_{x_2} \\ -K^2 \partial_{x_1} & 0 & 0 \\ -K^2 \partial_{x_2} & 0 & 0 \end{pmatrix}, \]

which in particular, guarantees conservation of the energy functional

\[ \mathcal{H}(\eta, \mathbf{v}) = \frac{1}{2} \int_{\mathbb{R}^2} \left( \eta^2 + \kappa |\nabla \eta|^2 + |K^{-1} \mathbf{v}|^2 + \eta |\mathbf{v}|^2 \right) \, dx. \] (1.6)

The noncavitation definition in the two dimensional problem has exactly the same view as in Definition 1 with the real line \( \mathbb{R} \) substituted by the plane \( \mathbb{R}^2 \).

**Theorem 2.** Let \( s > 1 \). Suppose that the initial data (1.5) has curl free velocity \( \nabla \times \mathbf{v}_0 = 0 \) and either has small enough \( H^1 \times H^{1/2} \times H^{1/2} \)-norm if \( s \leq 2 \) or satisfies the noncavitation condition if \( s > 2 \). Then there exist \( T > 0 \) depending only on the initial data and a unique solution \( (\eta, \mathbf{v}) \in C\left([0, T]; H^{s+1/2}(\mathbb{R}) \times (H^s(\mathbb{R}))^2\right) \) of System (1.4) associated with this initial data. Moreover, the solution depends continuously on the initial data.

Note that the theorem has the local character, in the opposite of the one dimensional case.

**Remark 2.** The same results hold in the periodic case as well. The proof is similar up to some small changes in the commutator estimates [15].

In the next section some important inequalities are recalled. In Section 3 we introduce the modified energy and obtain the corresponding energy estimate for System (1.1). In Section 4 we obtain the energy estimate for the difference of two solutions of System (1.1). In Section 5 the parabolic regularisation is studied and the corresponding energy estimate is deduced. In Section 6 a priori estimates are obtained. Finally, in Section 7 we comment on the last steps in the proof of Theorem 1, omitting only the thorough discussion of the initial data regularisation. In Section 8 we discuss some peculiarities of the two dimensional problem.

## 2. Preliminary estimates

We start this section by recalling all the necessary standard notations. For any positive numbers \( a \) and \( b \) we write \( a \preceq b \) if there exists a constant \( C \) independent on \( a, b \) such that \( a \leq Cb \). The Fourier transform is defined by the formula

\[ \hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx \]

on Schwartz functions. By the Fourier multiplier operator \( \varphi(D) \) with symbol \( \varphi \) we mean the line \( \mathcal{F}(\varphi(D)f) = \varphi(\xi) \hat{f}(\xi) \). In particular, \( D = -i\partial_x \) is the Fourier multiplier associated with the symbol \( \varphi(\xi) = \xi \). For any \( \alpha \in \mathbb{R} \) the Riesz potential of order \( -\alpha \) is the Fourier operator \( |D|^\alpha \) and the Bessel potential of order \( -\alpha \) is the Fourier operator \( J^\alpha = \langle D \rangle^\alpha \), where we exploit the notation
\( \langle \xi \rangle = \sqrt{1 + \xi^2} \). The \( L^2 \)-based Sobolev space \( H^\alpha \) is defined by the norm \( \| f \|_{H^\alpha} = \| J^\alpha f \|_{L^2} \), whereas the homogeneous Sobolev space \( \dot{H}^\alpha \) is defined by \( \| f \|_{\dot{H}^\alpha} = \| D^\alpha f \|_{L^2} \).

Introduce the operator
\[
K_\kappa = \sqrt{(1 + \kappa D^2) \tanh \frac{D}{\kappa}}
\]  

(2.1)

where \( \kappa \) is the surface tension. Note that \( \kappa > 0 \) is a fixed constant. We implement the notation \( K = K_0 = \sqrt{\tanh D^2} \) used in [9]. Its inverse \( K^{-1} \) and \( K_\kappa \) both have the domain \( H^{1/2}(\mathbb{R}) \) and are equivalent to the Bessel potential \( J^{1/2} \). Below we will need to compare \( J, |D| \) and \( K^{-2} \) and so we prove the following simple estimates.

**Lemma 1.** For any \( f \in L^2(\mathbb{R}) \) hold
\[
\| (J - K^{-2}) Df \|_{L^2} \leq \| (J - |D|) Df \|_{L^2} \leq \frac{1}{2} \| f \|_{L^2} .
\]

**Proof.** By the Plancherel identity it is enough to check the following inequalities
\[
\frac{\xi}{\tanh \xi} \leq \langle \xi \rangle - |\xi| \leq \frac{1}{2|\xi|}
\]

where the middle one is trivial. The rightmost inequality follows from
\[
\langle \xi \rangle - |\xi| = \frac{1}{(\langle \xi \rangle + |\xi|)} \leq \frac{1}{2|\xi|} .
\]

The leftmost one follows from the \( \tanh \)-definition via exponents and the obvious
\[
e^{2\xi} + e^{-2\xi} \geq 2 + 4\xi^2 .
\]

\( \square \)

Throughout the text we make an extensive use of the following bilinear estimates. Firstly, we state the Kato-Ponce commutator estimate [14].

**Lemma 2** (Kato-Ponce commutator estimate). Let \( s \geq 1, p, p_2, p_3 \in (1, \infty) \) and \( p_1, p_4 \in (1, \infty) \) be such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \). Then
\[
\| |J^s f| g|_{L^p} \lesssim \| \partial_x f \|_{L^{p_1}} \| J^{s-1} g \|_{L^{p_2}} + \| J^s f \|_{L^{p_3}} \| g \|_{L^{p_4}}
\]

(2.2)

for any \( f, g \) defined on \( \mathbb{R} \).

By the commutator \([A, B]f = ABf - BAf\) between operators \( A \) and \( B \) we mean the operator \([A, B]f = ABf - BAf\). Secondly, we state the fractional Leibniz rule proved in the appendix of [16].

**Lemma 3.** Let \( \sigma = \sigma_1 + \sigma_2 \in (0, 1) \) with \( \sigma_i \in (0, \sigma) \) and \( p, p_1, p_2 \in (1, \infty) \) satisfy \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then
\[
\| |D|^{\sigma} (fg) - f|D|^{\sigma} g - g|D|^{\sigma} f \|_{L^p} \lesssim \| |D|^{\sigma_1} f \|_{L^{p_1}} \| |D|^2 \sigma g \|_{L^{p_2}}
\]

(2.3)

for any \( f, g \) defined on \( \mathbb{R} \). Moreover, the case \( \sigma_2 = 0, p_2 = \infty \) is also allowed.

We also state an estimate, firstly appeared in [17] in a weaker form, and later sharpened in [21].

**Lemma 4.** Suppose \( a, b, c \in \mathbb{R} \). Then for any \( f \in H^a(\mathbb{R}), g \in H^b(\mathbb{R}) \) and \( h \in H^c(\mathbb{R}) \) hold
\[
\| fgh \|_{L^1} \lesssim \| f \|_{H^a} \| g \|_{H^b} \| h \|_{H^c}
\]

(2.4)

provided that
\[
a + b + c > \frac{1}{2},
\]
\[
a + b \geq 0, \quad a + c \geq 0, \quad b + c \geq 0 .
\]

Proving a global-in-time a priori estimate we will use the following limiting case of the Sobolev embedding theorem.
Lemma 5 (Brezis-Gallouet inequality). Suppose \( f \in H^s(\mathbb{R}^n) \) with \( s > n/2 \). Then
\[
\|f\|_{L^\infty} \leq C_{s,n} \left( 1 + \|f\|_{H^{n/2}} \sqrt{\log(2 + \|f\|_{H^s})} \right).
\] (2.5)

Inequality (2.5) was firstly put forward and proved for a domain in \( \mathbb{R}^n \) with \( n = 2 \) in the work by Brezis, Gallouet [2]. It was extended to the other Sobolev spaces in [3].

3. Modified energy

For \( s \geq 1/2 \) define the modified energy
\[
E^s(\eta, v) = \frac{\kappa}{2} \|\eta\|_{H^{s+1/2}}^2 + \frac{1}{2} \|v\|_{H^s}^2 + \frac{1}{2} \int \eta \left( J^s - \frac{1}{2} v \right)^2 d^i \quad (3.1)
\]
where the pair \( \eta(x, t), v(x, t) \) represents a possible solution of System (1.1). Note that in the limit case \( s = 1/2 \) this quantity almost coincides with the Hamiltonian
\[
E^{1/2}(\eta, v) = \mathcal{H}(\eta, v) + \frac{\kappa - 1}{2} \|\eta\|_{L^2}^2 + \frac{1}{2} \int v (J - K^{-2}) v.
\]

Lemma 6. Suppose \( s \geq 1/2 \) and \( \eta(t) \in H^{s+1/2}(\mathbb{R}), v(t) \in H^s(\mathbb{R}) \) solve System (1.1). Then
\[
\frac{d}{dt} E^s(\eta, v) \lesssim \|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2 + (\|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2)^2.
\]

Proof. Firstly, we regard the limit case \( s = 1/2 \). Taking into account energy conservation derive
\[
\frac{d}{dt} E^{1/2}(\eta, v) = (\kappa - 1) \int \eta \frac{dv}{dt} + \int v (J - K^{-2}) v_t.
\]
Substituting the right hand side of the first equation (1.1) to the first integral obtain
\[
\int \eta \frac{dv}{dt} = -\int \eta v_x - i \int \eta \tanh(D(\eta v)) \leq \|\partial_x \eta\|_{L^2} \|v\|_{L^2} + \|\eta\|_{L^2} \|\eta\|_{L^\infty} \|v\|_{L^2}
\]
following from Hölder’s inequality and boundedness of operator \( \tanh D \). Similarly, the second integral
\[
\int v (J - K^{-2}) v_t = -i \int v (J - K^{-2}) \tanh D \eta + i \kappa \int (D \tanh D \eta) (J - K^{-2}) D v
\]
\[
- \frac{i}{2} \int v (J - K^{-2}) \tanh D v^2 \leq \frac{1}{2} \|\eta\|_{L^2} \|v\|_{L^2} + \frac{\kappa}{2} \|\partial_x \eta\|_{L^2} \|v\|_{L^2} + \frac{1}{4} \|v\|_{L^2}^2 \|v\|_{L^4}^2
\]
is estimated using Hölder’s inequality and boundedness of operators \( \tanh D, (J - K^{-2}) D \) in \( L^2(\mathbb{R}) \). Applying standard Sobolev’s embeddings one proves the statement for \( s = 1/2 \).

Assuming \( s > 1/2 \) calculate the derivative
\[
\frac{\kappa}{2} \frac{d}{dt} \|\eta\|_{H^{s+1/2}}^2 = -\kappa \int \left( J^{s+\frac{1}{2}} \eta \right) J^{s+\frac{1}{2}} \partial_x v - i \kappa \int \left( J^{s+\frac{1}{2}} \eta \right) J^{s+\frac{1}{2}} \tanh(D(\eta v))
\]
\[
= i \kappa \int (J^{s+1} D \eta) \ J^s v + i \kappa \int (J^{s+\frac{1}{2}} D \eta) \ J^{s+\frac{1}{2}} (\eta v) + I_1
\]
where the rest
\[
I_1 = i \kappa \int (J^{s+1} \tanh D \eta - J^s D \eta) \ J^s (\eta v)
\]
\[
= i \kappa \int J^s ((J - |D|) \tanh D \eta + D(|\tanh D| - 1) \eta) \ J^s (\eta v) \lesssim \|\eta\|_{H^s} \|v\|_{H^s}.
\]
The derivative of velocity norm
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{H^s} = -i \int (J^s v) J^s \tanh D \eta - i \varkappa \int (J^s v) J^s D^2 \tanh D \eta - \frac{i}{2} \int (J^s v) J^s \tanh D v^2
\]
where the first and the third integrals are put in \(I_2\). They can be estimated straightforwardly
\[
I_2 \lesssim \|\eta\|_{H^s} \|v\|_{H^s} + \|v\|_{H^s}^3
\]
and the rest
\[
I_3 = i \varkappa \int (J^s v) J^s (J - D \tanh D) D \eta
\]
\[
= i \varkappa \int (J^s v) J^s (J - |D| - |\tanh D| - 1) D \eta \lesssim \|\eta\|_{H^s} \|v\|_{H^s}.
\]
Summing up these derivatives obtain
\[
\frac{\varkappa}{2} \frac{d}{dt} \|\eta\|^2_{H^{s+1/2}} + \frac{1}{2} \frac{d}{dt} \|v\|^2_{H^s} = +i \varkappa \int \left(J^{s-\frac{1}{2}} D \eta\right) \left[J^{s+\frac{1}{2}} \eta \right] v + I_1 + I_2 + I_3 + I_4
\]
where the last part
\[
I_4 = i \varkappa \int \left(J^{s-\frac{1}{2}} D \eta\right) \left[J^{s+\frac{1}{2}} \eta \right] v \lesssim \|\eta\|_{H^{s+1/2}} \left(\|\partial_x \eta\|_{L^2} \|J^{s-\frac{1}{2}} v\|_{L^p} + \|J^{s+\frac{1}{2}} \eta\|_{L^2} \|v\|_{L^\infty}\right)
\]
is estimated applying H"older’s inequality and the Kato–Ponce commutator estimate. Taking
\[p_1(s) = \frac{1}{1-s}, \quad p_2(s) = \frac{2}{2s-1}\]
for \(s \in (\frac{1}{2}, 1)\) and \(p_1 = p_2 = 4\) in case \(s \geq 1\) one deduces to
\[
I_4 \lesssim \|\eta\|^2_{H^{s+1/2}} \|v\|_{H^s}
\]
after implementing the Sobolev embedding. Differentiation of the last summand of energy \(E^s\) gives
\[
\frac{1}{2} \frac{d}{dt} \int \eta \left(J^{s-\frac{1}{2}} v\right)^2 = -i \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} \tanh D \eta - i \varkappa \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} D^2 \tanh D \eta
\]
\[
- \frac{i}{2} \int \eta \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} \tanh D v^2 - \frac{1}{2} \int \partial_x v \left(J^{s-\frac{1}{2}} v\right)^2 - \frac{i}{2} \int \tanh D (\eta v) \left(J^{s-\frac{1}{2}} v\right)^2
\]
where the first summand, that we denote by \(I_5\), is estimated easily
\[
I_5 \lesssim \|\eta\|_{L^\infty} \|v\|_{H^{s-1/2}} \|\eta\|_{H^{s+1/2}}
\]
via H"older inequality. The third integral in \(3.2\), noted by \(I_6\), is estimated in a similar way
\[
I_6 \lesssim \frac{1}{2} \|\eta\|_{L^\infty} \|v\|_{H^{s-1/2}} \|v\|^2_{H^{s-1/2}} \lesssim \|\eta\|_{H^{s+1/2}} \|v\|^3_{H^s}
\]
where the last bound follows from the fact that \(H^s\) is an algebra. The fourth integral in \(3.2\) equals
\[
I_7 = -\frac{i}{2} \int \left(\text{sgn} D |D|^{\frac{1}{2}} v\right) |D|^{\frac{1}{2}} \left(J^{s-\frac{1}{2}} v\right)^2 - i \int \left(\text{sgn} D |D|^{\frac{1}{2}} v\right) \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v
\]
\[
- \frac{i}{2} \int \left(\text{sgn} D |D|^{\frac{1}{2}} v\right) \left[|D|^{\frac{1}{2}} \left(J^{s-\frac{1}{2}} v\right)^2 - 2 \left(J^{s-\frac{1}{2}} v\right) J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v\right]
\]
where the first integral can be treated with interpolation in Sobolev spaces and the second integral by the fractional Leibniz rule as follows
\[
I_7 \lesssim \|\text{sgn} D |D|^{\frac{1}{2}} v\|_{H^{s-1/2}} \|J^{s-\frac{1}{2}} v\|_{H^{1/2}} \|J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v\|_{L^2} + \|\text{sgn} D |D|^{\frac{1}{2}} v\|_{L^2} \|J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v\|_{L^2}^2 \lesssim \|v\|^3_{H^s}.
\]
The last integral in \(3.2\), that we denote by \(I_8\), is bounded by
\[
I_8 \lesssim \frac{1}{2} \|\eta\|_{L^\infty} \|v\|_{L^2} \|J^{s-\frac{1}{2}} v\|_{L^2}^2 \lesssim \|\eta\|_{H^{s+1/2}} \|v\|^3_{H^s}.
\]
It is left to calculate the second integral in (3.2). For this we approximate $D \tanh D$ by $J$ in exactly the same way as was done for estimating integral $I_3$ so that

$$-i\kappa \int \eta \left( J^{s-\frac{1}{2}} v \right) J^{s-\frac{1}{2}} D^2 \tanh D \eta = -i\kappa \int \eta \left( J^{s-\frac{1}{2}} v \right) J^{s+\frac{1}{2}} D \eta + I_9$$

$$= -i\kappa \int \eta \left( J^{s+\frac{1}{2}} v \right) J^{s-\frac{1}{2}} D \eta + I_9 + I_{10}$$

where the first rest

$$I_9 = i\kappa \int \eta \left( J^{s-\frac{1}{2}} v \right) J^{s-\frac{1}{2}} D (J - D \tanh D) \eta \lesssim \|\eta\|_{L^\infty} \|v\|_{H^{s-1/2}} \|\eta\|_{H^{s-1/2}}$$

follows from Hölder’s inequality together with boundedness of operator $D (J - D \tanh D)$ and the last part

$$I_{10} = -i\kappa \int \left( J^{s-\frac{1}{2}} D \eta \right) [J, \eta] J^{s-\frac{1}{2}} v \lesssim \|\eta\|_{H^{s+1/2}} (\|\partial_x \eta\|_{L^p} \|J^{s-\frac{1}{2}} v\|_{L^p} + \|\eta\|_{L^p} \|J^{s-\frac{1}{2}} v\|_{L^p})$$

follows from the Hölder and Kato–Ponce inequalities. Again taking $p_1 = p_3 = \frac{1}{1-s}$, $p_2 = p_4 = \frac{2}{3s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = p_3 = p_4 = 4$ for $s \geq 1$ one deduces

$$I_{10} \lesssim \|\eta\|_{H^{s+1/2}} \|v\|_{H^s}$$

after implementing the Sobolev embedding. Finally, summing Derivative (3.2) with the derivative of square of $H^{s+1/2} \times H^s$-norm according to Definition (3.1) obtain

$$\frac{d}{dt} E^s(\eta, v) = I_1 + \ldots + I_{10} \lesssim \|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2 + (\|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2)^2$$

which concludes the proof.

In the following obvious statement the non-cavitation condition plays a crucial role.

**Lemma 7** (Coercivity). Let $s \geq 1/2$ and $(\eta, v) \in C \left( [0, T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}) \right)$ be a solution of System (1.1) for some $T > 0$. If in addition $\eta$ satisfies the non-cavitation condition then $E^s(\eta, v) \sim \|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2$.

**Corollary 1** (Energy estimate). In the conditions of the previous lemma holds true

$$\frac{d}{dt} E^s(\eta, v) \lesssim E^s(\eta, v) + E^s(\eta, v)^2.$$

As we shall see below, the non-cavitation condition is convenient to work with only in the case of high regularity $s > 3/2$. Then the time interval on which the condition holds true can be easily estimated through the first equation in (1.1). Our goal is to study well-posedness in spaces of low regularity. So in case of $s \leq 3/2$ we will have to impose a stronger condition, instead of non-cavitation, namely smallness of the initial data norm to control it in time with the help of Hamiltonian conservation.

**Lemma 8.** There exists a constant $H > 0$ depending only on the surface tension $\kappa > 0$ such that for any $\epsilon \in (0, H]$ if a pair $u(t) = (\eta(t), v(t)) \in H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, having initial condition $\|u_0\|_{H^1 \times H^{1/2}} \leq \epsilon/2$, solves System (1.1) then $\|u(t)\|_{H^1 \times H^{1/2}} \leq \epsilon$ for any time $t$.

**Proof.** We use a continuity argument. Without loss of generality we prove the statement with a norm equivalent to $H^1 \times H^{1/2}$-norm instead. Regard the norm defined by

$$\|u\|^2 = \frac{\kappa}{2} \|\partial_x \eta\|_{L^2}^2 + \frac{1}{2} \|\eta\|_{L^2}^2 + \frac{1}{2} \|K^{-1} v\|_{L^2}^2.$$

Then there exists $C > 0$ such that

$$\|u\|^2 (1 - C\|u\|) \leq \mathcal{H}(u) \leq \|u\|^2 (1 + C\|u\|).$$
where \( u = u(t) \) is a solution of (1.1) defined on some interval. Take \( H = (2C)^{-1} \), any \( 0 < \epsilon \leq H \) and a solution with \( u_0 = u(0) \) having \( \|u_0\| \leq \epsilon / 2 \). By continuity \( \|u\| \leq \epsilon \) on some \([0, T_\epsilon]\) and so

\[
\|u\| \leq \sqrt{2H(u)} = \sqrt{2H(u_0)} \leq \sqrt{\frac{1 + C\epsilon^2}{2}} < \epsilon
\]

which means that the continuous function \( \|u(t)\| \) cannot touch the level \( \epsilon \) with time. \( \square \)

As a consequence of the lemma we can control \( \|\eta\|_{L^\infty} \) for any \( s \geq 1/2 \) admitting only small initial data. Paying this price we can guarantee non-cavitation, in particular.

4. Uniqueness Type Estimate

Suppose that we have two solution pairs \( \eta_1, v_1 \) and \( \eta_2, v_2 \) of System (1.1) on some time interval. Define functions \( \theta = \eta_1 - \eta_2, w = v_1 - v_2 \). Then \( \theta ) \) and \( w \) satisfy the following system

\[
\begin{align*}
\theta_t &= -w_x - i \tanh D(\theta v_2 + \eta_1 w), \\
w_t &= -i \tanh D(1 + \alpha D^2) \theta - i \tanh D((v_1 + v_2)w)/2. 
\end{align*}
\]

(4.1) (4.2)

We need an a priori estimate similar to one obtained in the previous section for the difference of solutions. For this purpose we introduce the difference energy

\[
E^r(\eta_1, v_1, \eta_2, v_2) = \frac{\kappa}{2} \|\theta\|_{\dot{H}^{r+1/2}}^2 + \frac{1}{2} \|w\|_{\dot{H}^r}^2 + \frac{1}{2} \int \eta_1 \left( J^{r-\frac{1}{2}} w \right)^2. 
\]

(4.3)

Note that \( E^*(\eta, v) = E^*(\eta, v, 0, 0) \) and so this new notation is in line with (3.1).

Lemma 9. Let \((\eta_1, v_1), (\eta_2, v_2) \in C([0, T]; \dot{H}^{r+1/2}(\mathbb{R}) \times H^s(\mathbb{R})) \) be solutions of System (1.1) for some \( T > 0 \) and \( s > 1/2 \). Their difference is denoted by \((\theta, w)\). Let \( 0 < r \leq s - 1/2 \). Then

\[
\frac{d}{dt} E^r(\eta_1, v_1, \eta_2, v_2) \leq \left(1 + \|\eta_1\|_{\dot{H}^{r+1/2}}^2 + \|v_1\|_{\dot{H}^r}^2 + \|v_2\|_{\dot{H}^r}^2 \right) \left( \|\theta\|_{\dot{H}^{r+1/2}}^2 + \|w\|_{\dot{H}^r}^2 \right).
\]

Proof. We follow the same arguments as in the proof of Lemma 6. The derivative of squared norm

\[
\frac{\kappa}{2} \frac{d}{dt} \|\theta\|_{\dot{H}^{r+1/2}}^2 + \frac{1}{2} \frac{d}{dt} \|w\|_{\dot{H}^r}^2 = -\kappa \int \left( J^{r+\frac{1}{2}} \theta \right) J^{r+\frac{1}{2}} \partial_x w - i\kappa \int \left( J^{r+\frac{1}{2}} \theta \right) J^{r+\frac{1}{2}} \tanh D(\theta v_2) \\
- i\kappa \int \left( J^{r+\frac{1}{2}} \theta \right) J^{r+\frac{1}{2}} \tanh D(\eta_1 w) - i \int (J^r w) J^r \tanh D \theta \\
- i\kappa \int (J^r w) J^r D^2 \tanh D \theta - \frac{i}{2} \int (J^r w) J^r \tanh D(v_1 + v_2)w
\]

\[
= I_1 + O \left( \|\theta\|_{\dot{H}^r} \|w\|_{\dot{H}^r} + \|v_1\|_{\dot{H}^{r+1/2}} + \|v_2\|_{\dot{H}^{r+1/2}} + \|\eta_1\|_{\dot{H}^r} \|\theta\|_{\dot{H}^r} \|w\|_{\dot{H}^r} + \|v_1 + v_2\|_{H^s} \|w\|_{H^r}^2 \right)
\]

where

\[
I_1 = i\kappa \int \left( J^{r-\frac{1}{2}} D \theta \right) J^{r+\frac{1}{2}} (\eta_1 w).
\]

In the case \( r \geq 1/2 \) we have the commutator estimate

\[
\left\| \left[ J^{r+\frac{1}{2}} \eta_1 \right] w \right\|_{L^2} \lesssim \|\partial_x \eta_1\|_{L^2} \left\| J^{r-\frac{1}{2}} w \right\|_{L^2} + \left\| J^{r+\frac{1}{2}} \eta_1 \right\|_{L^2} \left\| w \right\|_{L^2} \lesssim \|\eta_1\|_{\dot{H}^{r+1/2}} \|w\|_{\dot{H}^r}
\]

and so

\[
I_1 = i\kappa \int \left( J^{r-\frac{1}{2}} D \theta \right) \eta_1 J^{r+\frac{1}{2}} w + O \left( \|\eta_1\|_{\dot{H}^{r+1/2}} \|\theta\|_{\dot{H}^{r+1/2}} \|w\|_{\dot{H}^r} \right).
\]

(4.4)

For \( r \in (0, 1/2) \) we apply the Leibniz rule

\[
\left\| D^{r+\frac{1}{2}} (\eta_1 w) - w D^{r+\frac{1}{2}} \eta_1 - \eta_1 D^{r+\frac{1}{2}} w \right\|_{L^2} \lesssim \left\| D^{r+\frac{1}{2}} \eta_1 \right\|_{L^2} \|w\|_{L^2} \lesssim \|\eta_1\|_{\dot{H}^r} \|w\|_{\dot{H}^r}
\]

\[
\left\| D^{r+\frac{1}{2}} (\eta_1 w) - w D^{r+\frac{1}{2}} \eta_1 - \eta_1 D^{r+\frac{1}{2}} w \right\|_{L^2} \lesssim \left\| D^{r+\frac{1}{2}} \eta_1 \right\|_{L^2} \|w\|_{L^2} \lesssim \|\eta_1\|_{\dot{H}^r} \|w\|_{\dot{H}^r}
\]
where $p_2 > 2$ is such that $\sigma_2 = r - 1/2 + 1/p_2 > 0$. The last estimate is due to Sobolev’s embedding. Operator $J^{r+\frac{1}{2}} - |D|^{r+\frac{1}{2}}$ is bounded in $L^2$. Thus

$$I_1 = i\varepsilon \int \left( J^{r-\frac{1}{2}} D \theta \right) |D|^{r+\frac{1}{2}} (\eta, w) + O \left( \|\eta\|_{H^{s+1/2}} \|\theta\|_{H^{s+1/2}} \|w\|_{H^r} \right)$$

$$= i\varepsilon \int \left( J^{r-\frac{1}{2}} D \theta \right) w |D|^{r+\frac{1}{2}} \eta_1 + i\varepsilon \int \left( J^{r-\frac{1}{2}} D \theta \right) \eta_1 |D|^{r+\frac{1}{2}} w + O \left( \|\eta\|_{H^{s+1/2}} \|\theta\|_{H^{s+1/2}} \|w\|_{H^r} \right)$$

where the first integral can be estimated by interpolation in Sobolev spaces. In the second integral the fractional derivative $|D|^{r+\frac{1}{2}}$ can be approximated by $J^{r+\frac{1}{2}}$ to come again to (4.1) now for $0 < r < 1/2$.

Differentiation of the energy modifier gives

$$\frac{1}{2} \frac{d}{dt} \int \eta_1 \left( J^{r-\frac{1}{2}} w \right)^2 = -i \int \eta_1 \left( J^{r-\frac{1}{2}} w \right) J^{r-\frac{1}{2}} \tanh D \theta - i\varepsilon \int \eta_1 \left( J^{r-\frac{1}{2}} w \right) J^{r-\frac{1}{2}} D \tanh D \theta$$

$$- \frac{i}{2} \int \eta_1 \left( J^{r-\frac{1}{2}} w \right) J^{r-\frac{1}{2}} \tanh D (v_1 + v_2) - \frac{1}{2} \int \partial_x v_1 \left( J^{r-\frac{1}{2}} w \right)^2 - \frac{i}{2} \int \tanh D (\eta_1 v_1) \left( J^{r-\frac{1}{2}} w \right)^2$$

$$= I_2 + O \left( \|\eta_1\|_{H^s} \|\theta\|_{H^{s+1/2}} \|w\|_{H^{s+1/2}}^2 + 1 + \|\eta_1\|_{H^s} (\|v_1\|_{H^s} + \|v_2\|_{H^s}) \|w\|_{H^r}^2 \right)$$

where

$$I_2 = -i\varepsilon \int \left( J^{r-\frac{1}{2}} D \theta \right) J(\eta_1, J^{r-\frac{1}{2}} w) = -i\varepsilon \int \left( J^{r-\frac{1}{2}} D \theta \right) \eta_1 J^{r+\frac{1}{2}} w$$

$$+ \|\theta\|_{H^{s+1/2}} O \left( \|\partial_x \eta_1\|_{L^p} \|J^{r-\frac{1}{2}} w\|_{L^p} + \|\eta_1\|_{L^p} \|J^{r-\frac{1}{2}} w\|_{L^p} \right)$$

$$= -i\varepsilon \int \left( J^{r-\frac{1}{2}} D \theta \right) \eta_1 J^{r+\frac{1}{2}} w + O \left( \|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{s+1/2}} \|w\|_{H^r} \right)$$

following from the Kato–Ponce inequality with $p_1 = p_3 = \frac{1}{1-s}$, $p_2 = p_4 = \frac{2}{2s-1}$ for $s \in \left( \frac{1}{2}, 1 \right)$ and $p_1 = p_2 = p_3 = p_4 = 4$ for $s \geq 1$. Summing $I_2$ together with $I_1$ calculated in (4.1) we conclude the proof.

\[\square\]

**Corollary 2** (Energy estimate for difference). If in addition to the conditions of the previous we assume non-cavitation for $\eta_1$ then

$$\frac{d}{dt} E^r(\eta_1, v_1, \eta_2, v_2) \lesssim (1 + \|\eta_1\|_{H^{s+1/2}}^2 + \|v_1\|_{H^s}^2 + \|v_2\|_{H^s}^2) E^r(\eta_1, v_1, \eta_2, v_2).$$

**Proof.** Non-cavitation implies coercivity for $E^r$ and the rest is obvious. \[\square\]

**Remark 3.** The restriction $s > 1/2$ appeared in the lemma and its corollary is inconvenient. It comes from the loss of Hamiltonian structure of System (4.1)-(4.2). This results in the fact that we can obtain only a weak solution in case $s = 1/2$ and probably not unique.

## 5. PARABOLIC REGULARISATION

For application of the energy method we need to do a parabolic regularisation of the view

$$\begin{cases}
\eta_t + v_x + i \tanh D(\eta v) = -\mu |D|^p \eta, \\
v_t + i \tanh D(1 + \kappa D^2) \eta + i \tanh D v^2 / 2 = -\mu |D|^p v
\end{cases}$$

(5.1)

where $\mu \in (0, 1)$. We want to prove solution existence for (5.1) for any given $\mu$, by the contraction mapping principal and so $p$ should be big enough. However, we also do not want to spoil our energy estimates, and so $p$ should be small enough. As we shall see below, this bounds us to $p \in (1/2, 1]$. Here the left number comes from the following lemma.
Lemma 10. For any \( s \geq 1/2, \mu \in (0,1) \) and \( p > 1/2 \) there exists a finite positive bound \( C(T) \), tending to zero as \( T \to 0 \), such that

\[
\int_0^T \left\| e^{-\mu t |D|^p} (f(t)g(t)) \right\|_{H^r} \ dt \leq C(T) \| f \|_{C_T H^r} \| g \|_{C_T H^s}
\]

for any functions \( f, g \) defined on \([0,T]\). Here either \( r = s + 1/2 \) or \( r = s \).

Proof. In the case \( r = s + 1/2 \) the statement is obvious due to boundedness of \( \exp(-\mu t |D|^p) \) and the algebraic property \( \|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s} \). Hence \( C(T) = c_s T \) with some constant \( c_s \) depending only on \( s \).

Otherwise we use

\[
\left\| e^{-\mu t |D|^p} (fg) \right\|_{H^r} \leq \left\| e^{-\mu t |\xi|^p} (\xi)^{1/2} \right\|_{L_\infty} \|fg\|_{H^{r-1/2}}
\]

where in the case \( r = s = 1/2 \) by the Hölder inequality we have

\[
\|fg\|_{H^{r-1/2}} = \|fg\|_{L_2} \leq \|f\|_{L^4} \|g\|_{L^4} \lesssim \|f\|_{H^{1/4}} \|g\|_{H^{1/4}} \lesssim \|f\|_{H^s} \|g\|_{H^s}
\]

and in the case \( r = s + 1/2 \) we obviously have

\[
\|fg\|_{H^{r-1/2}} \lesssim \|f\|_{H^s} \|g\|_{H^s}.
\]

It is left to check that the \( L_\infty \)-norm above is locally integrable. Indeed, we can estimate the function at \( \xi \in [0,1] \) and at \( \xi \geq 1 \) separately

\[
e^{-\mu t |\xi|^p} (\xi)^{1/2} \leq \max \left\{ 2^{1/4} \sup_{\xi \geq 1} 2^{1/4} \xi^{1/2} e^{-\mu t |\xi|} \right\} \leq 2^{1/4} \max \left\{ 1, (2pe\mu t)^{1/4} \right\}
\]

that is an integrable function with respect to time over any bounded interval for \( p > 1/2 \). The integral of this function over \([0,T]\) defines the bound \( C(T) \). \( \square \)

With Lemma 10 in hand we can prove the local well-posedness in \( H^{s+1/2} \times H^s \) with \( s \geq 1/2 \) for System (5.1) by the fixed-point argument. Diagonalization has the matrix form

\[
\mathcal{S}(t) = \exp(-\mu t |D|^p) \mathcal{K} \begin{pmatrix} \exp(-it K_\varphi D) & 0 \\ 0 & \exp(it K_\varphi D) \end{pmatrix} \mathcal{K}^{-1}
\]

where

\[
\mathcal{K} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ K_\varphi & -K_\varphi \end{pmatrix}, \quad \mathcal{K}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & K_\varphi^{-1} \\ 1 & -K_\varphi^{-1} \end{pmatrix}
\]

with \( K_\varphi \) defined by (2.1). For any fixed \( u_0 = (\eta_0, v_0)^T \in X^s = H^{s+1/2} \times H^s \) function \( \mathcal{S}(t) u_0 \) solves the linear initial-value problem associated with (5.1). Let \( X^s_T = C([0,T];X^s) \) and regard a mapping \( \mathcal{A} : X^s_T \to X^s_T \) defined by

\[
\mathcal{A}(\eta, v; u_0)(t) = \mathcal{S}(t) u_0 + \int_0^t \mathcal{S}(t-t')(\eta \tanh D) \left( \frac{\eta'}{v'^2/2} \right) (t') dt'.
\]

Then the Cauchy problem for System (5.1) with the initial data \( u_0 \) may be rewritten equivalently as an equation in \( X^s_T \) of the form

\[
u = \mathcal{A}(u; u_0)
\]

where \( u = (\eta, v)^T \in X^s_T \).

Lemma 11. Let \( s \geq 1/2, p > 1/2, \mu \in (0,1) \) and \( u_0 = (\eta_0, v_0)^T \in X^s \). Then there is \( T = T(s, p, \mu, \|u_0\|_{X^s}) > 0 \), decreasing to zero with increase of the norm of \( u_0 \), such that there exists a unique solution \( u = (\eta, v)^T \in X^s_T \) of Problem (5.3).

Moreover, for any \( R > 0 \) there exists \( T = T(s, p, \mu, R) > 0 \) such that the flow map associated with Equation (5.3) is a real analytic mapping of the open ball \( B_R(0) \subset X^s \) to \( X^s_T \).
Lemma 12. Suppose $\eta(t) \in H^{s+1/2}(\mathbb{R})$, $v(t) \in H^s(\mathbb{R})$ solve System (5.1) with $\mu \in (0, 1)$ and $p \in (1/2, 1]$. Then

$$\frac{d}{dt} E^s(\eta, v) \lesssim \|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2 + \left(\|\eta\|_{H^{s+1/2}}^2 + \|v\|_{H^s}^2\right)^2.$$

In other words, the parabolic regularisation (5.1) does not spoil the energy estimate.

Proof. Following the proof of Lemma 6 one arrives at

$$\frac{d}{dt} E^s(\eta, v) \lesssim \tilde{I}_1 + \tilde{I}_2 + \ldots$$

where

$$\tilde{I}_1 = -\mu \varkappa \left\|D^{p/2} \eta\right\|_{H^{s+1/2}}^2 - \mu \left\|D^{p/2} v\right\|_{H^s}^2 \leq 0,$$

$$\tilde{I}_2 = -\frac{\mu}{2} \int \left(\tilde{J}^{s-1/2} v\right)^2 |D|^p \eta - \mu \int \eta \left(\tilde{J}^{s-1/2} v\right) \tilde{J}^{s-1/2} |D|^p v \lesssim \|\eta\|_{H^{s+1/2}} \|v\|_{H^s}^2$$

for $p \leq 1$ and the rest is the same as in Lemma 6. \(\square\)

As was noticed at the end of Section 3 one has to make sure that the modified energy is coercive. An effective way to do it at the low level of regularity is to control $\|\eta\|_{L^\infty}$ via the energy conservation. One can get the same controllability for the regularised problem via the energy dissipation due to the following result.

Lemma 13. Suppose $\eta(t) \in H^1(\mathbb{R})$, $v(t) \in H^{1/2}(\mathbb{R})$ solve System (5.1) with $\mu \in (0, 1)$ and $p \in (1/2, 1]$. Then there exists $\delta > 0$ independent on the viscosity $\mu$ such that $\mathcal{H}(\eta, v)$ is a non-increasing function of time $t$ provided $\|\eta(t)\|_{H^1} + \|v(t)\|_{H^{1/2}} \leq \delta$ holds for any $t$.

Proof. Hamiltonian (4.3) has the derivative

$$\frac{1}{\mu} \frac{d}{dt} \mathcal{H}(\eta, v) = -\|\eta\|_{H^{p/2}}^2 - \varkappa \|\eta\|_{H^{p/2+1}}^2 - \left\|K^{-1} v\right\|_{H^{p/2}}^2 - I_1 - I_2,$$

where the rest integrals

$$I_1 = \int \eta v |D|^p v, \quad I_2 = \frac{1}{2} \int v^2 |D|^p \eta$$

are of no definite sign. One has to check that $I_1$, $I_2$ are absorbed by the first three norms. The main difficulty arising here is that two different homogeneous Sobolev spaces cannot be compared for inclusion, however, there is interpolation between them.
Using the Hölder inequality, the fractional Leibniz rule for $|D|^{p/2}$, the Sobolev embeddings $H^{1/4} \hookrightarrow L^4$, $H^1 \hookrightarrow L^\infty$ and $H^{1/2} \hookrightarrow H^{1/4} \hookrightarrow L^2$ obtain

$$|I_1| = \left| \int |D|^{p/2}(\eta v)D|^{p/2}v \right| \leq \left| \left| |D|^{p/2}(\eta v) - v|D|^{p/2}\eta - \eta|D|^{p/2}v \right| \right|_{L^2} \left| |D|^{p/2}v \right|_{L^2}$$

$$+ \left| \left| v|D|^{p/2}\eta \right| \right|_{L^2} \left| |D|^{p/2}v \right|_{L^2} + \left| \left| \eta|D|^{p/2}v \right| \right|_{L^2} \left| |D|^{p/2}v \right|_{L^2}$$

$$\lesssim \left| \eta \right|_{L^\infty} \left| \left| |D|^{p/2}v \right| \right|_{L^2}^2 + \left| \left| D|^{p/2}v \right| \right|_{L^4} \left| \left| v \right| \right|_{L^4} \left| \left| |D|^{p/2}v \right| \right|_{L^2}$$

$$\lesssim \left| \eta \right|_{H^1} \left| K^{-1}v \right|_{H^{p/2}}^2 + \left| \left| v \right| \right|_{H^{1/2}} \left| \left| \eta \right| \right|_{H^{p/2+1/4}} \left| K^{-1}v \right|_{\dot{H}^{p/2}}$$

where $\left| \eta \right|_{\dot{H}^{p/2+1/4}}$ can be interpolated between $\dot{H}^{p/2}$-norm and $\dot{H}^{p/2+1}$-norm. Hence $I_1$ can be absorbed provided $H^1 \times H^{1/2}$-norm of the solution is small.

The second integral $I_2$ can be treated similarly for $p = 1$ exploiting $|D| = D \text{sgn} D$. Indeed,

$$I_2 = -\int |D|^{1/2}(v \text{sgn} D\eta)|D|^{1/2} \text{sgn} Dv$$

and so it can be estimated by the same chain of inequalities since $\text{sgn} D$ preserves Sobolev norms. For $p \in (1/2, 1)$ we have

$$2|I_2| \leq \left| \left| D|^{p/2} \eta \right| \right|_{H^{1-p/4}} \left| \left| v \right| \right|_{L^{4/p}} \left| \left| v \right| \right|_{L^{4/p}} \lesssim \left| \eta \right|_{H^{1-p/4}} \left| \left| v \right| \right|_{H^{1-p/4}} \left| \left| v \right| \right|_{\dot{H}^{p/2}}$$

by the Hölder inequality and the Sobolev embeddings $H^{1-p/4} \hookrightarrow L^{4/p}$, $\dot{H}^{p/2} \hookrightarrow L^{4/p}$. Again we interpolate the norm of $\eta$ between $\dot{H}^{p/2}$-norm and $\dot{H}^{p/2+1}$-norm. Estimate $\left| \left| \eta \right| \right|_{H^{1-p/4}} \leq \left| \left| \eta \right| \right|_{H^{1}}$ and $\left| \left| \left| v \right| \right|_{\dot{H}^{p/2}} \leq \left| \left| K^{-1}v \right| \right|_{\dot{H}^{p/2}}$. Eventually we obtain

$$|I_1| + |I_2| \lesssim \left( \left| \left| \eta \right| \right|_{H^{p/2}}^2 + \left| \left| \eta \right| \right|_{H^{p/2+1}}^2 + \left| \left| K^{-1}v \right| \right|_{\dot{H}^{p/2}}^2 \right) \max \left\{ \left| \left| \eta \right| \right|_{H^1}, \left| \left| v \right| \right|_{H^{1/2}} \right\}$$

that concludes the proof.

As a simple corollary with the proof similar to that of Lemma 8 one obtains the following.

**Corollary 3.** There exists a constant $\delta > 0$ depending only on the surface tension $\kappa > 0$ and the parabolic regularisation power $p$ such that if a pair $u(t) = (\eta(t), v(t)) \in H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, having initial condition $\|u_0\|_{H^1 \times H^{1/2}} \leq \delta/2$, solves System (5) then $\|u(t)\|_{H^1 \times H^{1/2}} \leq \delta$ for any time $t$.

The dependence of $\delta$ on the parabolic regularisation power $p$ is unimportant since below we stick only to the case $p = 1$.

6. A priori estimate

We have an a priori global bound for solutions of both systems (1.1) and (5.1) in $H^1 \times H^{1/2}$ due to Lemma 8 and Corollary 3 respectively. Our aim is to obtain estimates in $H^{s+1/2} \times H^s$ with $s > 1/2$.

**Lemma 14 (A priori estimate).** Suppose $s > 1/2$. Let $(\eta, v) \in C([0, T^*); H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}))$ be a solution of System (1.1) (or of the regularised system 5.1) with $\mu \in (0, 1)$ and $p = 1$ defined on its maximal time of existence and satisfying the blow-up alternative

$$T^* < +\infty \text{ implies } \lim_{t \to T^*} \|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} = +\infty. \quad (6.1)$$

Suppose that its initial data 1.2 either satisfies the non-cavitation condition for $s > 3/2$ or has small enough $H^1 \times H^{1/2}$-norm for $s \leq 3/2$. Then there exists $T_0 = T_0(\|\eta_0, v_0\|_{H^{s+1/2} \times H^s}) < T^*$ such that

$$\sup_{t \in [0, T_0]} \|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} \leq C \|\eta_0, v_0\|_{H^{s+1/2} \times H^s} \quad (6.2)$$

for some $C > 0$ independent on $\mu$. 

Proof. We closely follow the arguments in \cite{12} since we have essentially the same energy estimates. The main difference lies in the control of coercivity of the modified energy (3.1) for small \( s \). Let \( h_0, H_0 \) define non-cavitation of \( \eta_0 \) according to Definition 1. \( h_0 = h_0/2 \) and \( H = H_0 + h_0/2 \). If the wave \( \eta \) satisfies the non-cavitation condition associated with \( h, H \) then there exist positive constants \( c_0(h), C_0(H) \) such that

\[
c_0 \| \eta, v \|^2_{H^{s+1/2} \times H^s} \leq E^s(\eta, v) \leq C_0 \| \eta, v \|^2_{H^{s+1/2} \times H^s}
\]

by coercivity of the energy. These constants depend only on \( h_0, H_0 \). They are used to define the time set

\[
\mathcal{T} = \left\{ T \in (0, T^*) : \sup_{t \in [0, T]} \| \eta(t), v(t) \|_{H^{s+1/2} \times H^s} \leq 3\sqrt{C_0/c_0} \| \eta_0, v_0 \|_{H^{s+1/2} \times H^s} \right\}
\]

that is non-empty and closed in \( (0, T^*) \) by the solution continuity. Moreover, for \( \widetilde{T} = \sup \mathcal{T} \) we have either \( \widetilde{T} < T^* \) and so \( \widetilde{T} \in \mathcal{T} \) or \( \widetilde{T} = T^* = +\infty \) by the blow-up alternative (6.1). Introduce \( T_0 = \min \{ T_1, T_2 \} \) with

\[
T_1 = \frac{1}{C_1} \log \left( 1 + \frac{1}{1 + C_1 C_0 \| \eta_0, v_0 \|^2_{H^{s+1/2} \times H^s}} \right),
\]

\[
T_2 = \frac{h_0}{C_2 \left( \| \eta_0, v_0 \|^2_{H^{s+1/2} \times H^s} + \| \eta_0, v_0 \|^2_{H^{s+1/2} \times H^s} \right)} \quad \text{for } s > 3/2
\]

1 otherwise

where \( C_1, C_2 \) are two big positive constants to be fixed below in the proof. The idea is to show that these constants can be chosen, independently on the initial data, in such a way that \( T_0 \in \mathcal{T} \) or equivalently \( T_0 \leq \widetilde{T} \).

Assume the opposite \( \widetilde{T} < T_0 \). Firstly, we will check that the non-cavitation condition holds on \([0, \tilde{T}]\). Indeed, in the low regularity case \( s \in (1/2, 3/2] \) it is assumed smallness of the initial data and so \( H^1 \times H^{1/2} \)-norm of the solution stays small with time evolution by Lemma 8 and Corollary 3. In particular, the wave satisfies the non-cavitation condition. For \( s > 3/2 \) one can estimate \( \eta \) using the first equation in System (1.1) (or in System (5.1)) as follows

\[
\eta(x, t) = \eta_0(x) + \int_0^t \partial_t \eta(x, t') dt'
\]

where

\[
\| \partial_t \eta \|_{L^\infty} \lesssim \| \partial_x \eta \|_{L^\infty} + \| \tanh D(\eta) \|_{L^\infty} + \mu \| D[\eta] \|_{L^\infty} \lesssim \| \eta \|_{H^s} + \| v \|_{H^s} + \| \eta \|_{H^s} \| v \|_{H^s}
\]

with the implicit constant independent on \( \mu \in (0, 1) \), obviously. Hence

\[
\| \partial_t \eta \|_{L^\infty} \lesssim \| \eta_0, v_0 \|_{H^{s+1/2} \times H^s} + \| \eta_0, v_0 \|^2_{H^{s+1/2} \times H^s}
\]

uniformly on \( (0, \tilde{T}) \subset \mathcal{T} \). Thus we have

\[
\left\| \int_0^t \partial_t \eta(x, t') dt' \right\|_{L^\infty} \leq \tilde{T} \sup_{t \in (0, \tilde{T})} \| \partial_t \eta(t) \|_{L^\infty} \leq \frac{h_0}{2}
\]

for big enough \( C_2 \) since \( \tilde{T} < T_2 \). As a result the non-cavitation

\[
h - 1 = h_0/2 - 1 \leq \eta \leq H_0 + h_0/2 = H
\]

holds on \( \mathbb{R} \times [0, \tilde{T}] \). Without loss of generality one can assume that for \( s \leq 3/2 \) the non-cavitation of \( \eta \) is governed by the same \( h, H \).
Let \( E(t) = E^s(\eta, v)(t) \) be the energy defined by (3.1) and \( E_0 = E(0) \). For System (1.1) (or for System (5.1)) we have the a priori energy estimate given in its differential form by Corollary 1. A straightforward integration gives

\[
E(t) \left(1 - \frac{E_0}{1 + E_0} e^{ct}\right) \leq \frac{E_0}{1 + E_0} e^{ct}
\]

for any \( t \in [0, \bar{T}] \) with \( c \) depending only on \( h \). Note that

\[
e^{ct} \leq 1 + \frac{1}{1 + C_1 E_0}
\]

for any \( C_1 \geq c \) and \( 0 \leq t \leq \bar{T} < T_1 \). In particular,

\[
\frac{E_0}{1 + E_0} e^{ct} \leq \frac{1/C_1 + E_0}{1 + E_0} < 1
\]

if in addition \( C_1 > 1 \). Thus

\[
E(t) \leq \left(\frac{E_0}{1 + E_0} e^{ct}\right)^{-1} \leq E_0 \frac{2 + C_1 E_0}{1 + (C_1 - 1)E_0} \leq 2E_0
\]

if in addition \( C_1 \geq 2 \). As a result we have

\[
\|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} \leq \sqrt{2C_0/c_0} \|\eta_0, v_0\|_{H^{s+1/2} \times H^s}
\]

for all \( t \in [0, \bar{T}] \). Taking into account \( \bar{T} < T^* \) and continuity of the solution one can find \( \bar{T} < T' < T^*, T_0 \) such that on \([0, T']\) holds

\[
\|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} \leq 2\sqrt{C_0/c_0} \|\eta_0, v_0\|_{H^{s+1/2} \times H^s}
\]

which contradicts the definition of \( \bar{T} \). Therefore, we showed that \( T_0 \leq \bar{T} \) concluding the proof. \( \square \)

**Lemma 15.** Suppose \( s > 1/2 \) and a pair \( \eta(t) \in H^{s+1/2}, v(t) \in H^s \) solves System (1.1) (or the regularised system (5.1) with \( \mu \in (0, 1) \) and \( p = 1 \)). Then if \( s < 1 \) the following holds true

\[
\frac{d}{dt} E^s(\eta, v) \lesssim \left(1 + \|v\|_{L^\infty} + \|\eta, v\|_{H^{1/2} \times H^{1/2}}^2\right) \|\eta, v\|^2_{H^{s+1/2} \times H^s},
\]

and if \( s \geq 1 \) then

\[
\frac{d}{dt} E^s(\eta, v) \lesssim \left(1 + \|\eta, v\|^2_{H^{s+1/2} \times H^{s+1/2}}\right) \|\eta, v\|^2_{H^{s+1/2} \times H^s}.
\]

Moreover, the implicit constants do not depend on \( \mu \).

**Proof.** The estimates obtained while proving Lemmas 3, 12 need to be refined for \( s > 1/2 \) as follows. We stick to the notations used in the corresponding proofs. Note that by Lemma 4 we have

\[
\bar{I}_2 \lesssim \left\| J^{s-1/2}v \right\|_{L^2} \left\| J^{s-1/2}v \right\|_{H^{1/2}} \left\| D\eta \right\|_{H^{s-1/2}} + \left\| \eta \right\|_{H^s} \left\| J^{s-1/2}v \right\|_{H^{1/2}} \left\| Dv \right\|_{H^{s-1/2}} \lesssim \left\| \eta, v \right\|_{H^{s+1/2} \times H^s} \left\| \eta, v \right\|^2_{H^{s+1/2} \times H^s}.
\]

Since \( H^s \cap L^\infty \) is an algebra under the point-wise product one obtains

\[
I_2 \lesssim \left\| \eta \right\|_{H^s} \left\| v \right\|_{H^s} + \left\| v \right\|_{L^\infty} \left\| v \right\|_{H^s}^2 \lesssim (1 + \left\| v \right\|_{L^\infty}) \left\| \eta, v \right\|^2_{H^{s+1/2} \times H^s}.
\]

The integral \( I_4 \) is essentially estimated already as

\[
I_4 \lesssim \left\| \eta \right\|^2_{H^{s+1/2}} \left\{ \left\| v \right\|_{H^{s+1/2}} + \left\| v \right\|_{L^\infty} \quad \text{for } s \in (1/2, 1)
\]

\[
\left\| v \right\|_{H^{s-1/2}} \quad \text{for } s \geq 1
\].

In order to refine \( I_7 \) we need to estimate

\[
\left\| (\text{sgn } D |D|^{1/2}v) J^{s-1/2}v \right\|_{L^2} \lesssim \left\{ (\text{sgn } D |D|^{1/2}v) \right\|_{L^p_1} \left\| J^{s-1/2}v \right\|_{L^p_2}
\]
following from Hölder’s inequality with \( p_1(s) = \frac{1}{1-s}, \) \( p_2(s) = \frac{2}{2s-1} \) for \( s \in (\frac{1}{2}, 1) \) and \( p_1 = p_2 = 4 \) in case \( s \geq 1 \). Implementing the Sobolev embedding and gathering the rest of \( I_7 \) obtain

\[
I_7 \lesssim \|v\|_{H^s}^2 \left\{ \begin{array}{ll}
\|v\|_{H^{1/2}} & \text{for } s \in (1/2, 1) \\
\|v\|_{H^{s-1/4}} & \text{for } s \geq 1
\end{array} \right.
\]

The integral \( I_{10} \) is also estimated already as

\[
I_{10} \lesssim \|\eta\|_{H^{s+1/2}}^2 \left\{ \begin{array}{ll}
\|\eta\|_{H^{1/2}} & \text{for } s \in (1/2, 1) \\
\|\eta\|_{H^{s-1/4}} & \text{for } s \geq 1
\end{array} \right.
\]

Thus gathering all the parts obtain

\[
\bar{I}_1 + \bar{I}_2 + I_1 + \ldots + I_{10} \lesssim \|\eta, v\|_{H^{s+1/2} \times H^s}^2 \left\{ \begin{array}{ll}
1 + \|v\|_{L^\infty} + \|\eta, v\|_{H^{1/2} \times H^{1/2}}^2 & \text{for } s \in (1/2, 1) \\
1 + \|\eta, v\|_{H^{s+1/4} \times H^{s-1/4}}^2 & \text{for } s \geq 1
\end{array} \right.
\]

which are the desired estimates. \( \square \)

Knowing coercivity of the energy \( E^s \), controlled either by the smallness or by the non-cavitation of the initial data, one can deduce from the lemma that the time of existence depends only on \( \|\eta_0, v_0\|_{H^{s+1/2} \times H^{s'}} \) where \( 1/2 < s' < s \). Taking into account the boundedness of \( \|\eta, v\|_{H^{1/2} \times H^{1/2}} \), holding true at least for small initial data, one can get a stronger result thanks to the Brezis-Gallouet limiting embedding (2.5). In order to exploit it we need the following Gronwall inequality.

**Lemma 16** (Gronwall inequality). Let \( y \) be an absolutely continuous positive function defined on some interval \([0, T]\). Suppose that almost everywhere

\[
y' \leq Ay \log y
\]

where \( A > 0 \) is constant. Then there exists \( C > 0 \) independent on \( T \) such that

\[
y(t) \leq \exp (Ce^{At})
\]

Proof. Denote the right hand side by \( z(t) = \exp (Ce^{At}) \), where we take \( C > 0 \) such that \( z(0) > y(0) \). Regard the derivative

\[
\left( \frac{y}{z} \right)' = \frac{y'z - yz'}{z^2} \leq A\frac{y}{z} \log \frac{y}{z}
\]

where the latter is less than zero at least for \( t = 0 \). So the fraction \( y/z \) decreases and stays always below the unity. \( \square \)

**Corollary 4** (Persistence of regularity). In the conditions of the a priori estimate lemma \( \text{14} \) the following holds true

\[
\|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} \lesssim \exp (Ce^{At})
\]

provided \( s < 1 \), and if \( s \geq 1 \) then

\[
\|\eta(t), v(t)\|_{H^{s+1/2} \times H^s} \lesssim \|\eta_0, v_0\|_{H^{s+1/2} \times H^s} \exp \left( Ct + C \int_0^t \|\eta, v\|_{H^{s+1/4} \times H^{s-1/4}} \right)
\]

where the constant \( C > 0 \) does not depend on \( \mu \). In particular, the maximal time of existence \( T^* = +\infty \) provided \( \|\eta_0, v_0\|_{H^{1/2} \times H^{1/2}} \) is small enough.

Proof. The statement is obvious for \( s \geq 1 \). Suppose \( s \in (1/2, 1) \). By Lemma \( \text{8} \) and Corollary \( \text{3} \) the norm \( \|\eta(t), v(t)\|_{H^{1/2} \times H^{1/2}} \) stays bounded with time. Hence from the Brezis-Gallouet inequality (2.5) one deduces

\[
\|v(t)\|_{L^\infty} \lesssim 1 + \log (2 + \|v(t)\|_{H^s})
\]

Thus applying Lemma \( \text{15} \) and taking into account that \( E^s \) is coercive obtain

\[
\frac{d}{dt} E^s \lesssim (1 + \log (2 + E^s)) E^s.
\]
As a result, after the application of the previous lemma with \( y = 2 + E^s \), we have the estimate
\[
E^s \leq \exp \left( C e^{Ct} \right),
\]
which again due to coercivity of \( E^s \) leads to the first inequality of the corollary after renaming the constant.

\[\square\]

7. Proof of Theorem 1

With the a priori estimate (6.2) in hand we can reapply the local existence lemma 11 for the regularised problem (5.1) with \( \mu \in (0, 1) \) and \( p = 1 \) in order to obtain solution \( u^\mu = (\eta^\mu, v^\mu) \) on the time interval \([0, T_0]\) defined by Lemma 14. Convergence of \( u^\mu \) as \( \mu \to 0 \) follows from an adaptation of Lemma 9 to the difference energy (4.3) with \( \eta_j = \eta^\mu_j, v_j = v^\mu_j \) (\( j = 1, 2 \)) and \( 0 < \mu_2 < \mu_1 < 1 \). The proof repeats the arguments of Lemma 9 and Lemma 13. Moreover, using the Gagliardo–Nirenberg interpolation one can obtain that \( u^\mu \) converges to some \( u \) in \( C([0, T_0]; H^{s+1/2} \times H^s) \) as \( \mu \to 0 \) for any \( 0 < r < s \). This \( u \) is a solution of (1.1) in the distributional sense. Furthermore, to prove persistence \( u \in C([0, T_0]; H^{s+1/2} \times H^s) \), justify all the previous steps and obtain continuity of the flow map one has to regularize the initial data (1.2) as \( u^0 = (\eta^0 * \rho_\epsilon, v^0 * \rho_\epsilon) \), where \( \rho_\epsilon \) is an approximation of the identity parametrised by \( 0 < \epsilon < 1 \) [11, 13]. An application of the Bona–Smith argument in a straightforward standard way [1, 15, 20] results in the persistence and continuous dependence. We omit further details.

8. The two-dimensional problem

In this section we comment briefly on adaptation of the proof for the two dimensional case. Firstly, we define the modified energy
\[
E^s(\eta, v) = \frac{\kappa}{2} \| \eta \|_{H^{s+1/2}}^2 + \frac{1}{2} \| v \|_{H^s \times H^s}^2 + \frac{1}{2} \int \eta \left| J^{s-1/2} v \right|^2 \tag{8.1}
\]
and notice that it is coercive provided the wave \( \eta \) satisfies the noncavitation condition or has small \( H^1 \)-norm. Note that the latter does not imply the first one, since now we do not have embedding of \( H^1 \) to \( L^{\infty} \). The smallness of \( H^1 \times H^{1/2} \times H^{1/2} \)-norm can be controlled by the energy conservation. Indeed, by Hölder’s inequality and the Sobolev embedding the cubic part of Hamiltonian (1.6) is estimated as
\[
\int_{\mathbb{R}^2} \eta|v|^2 dx \lesssim \| \eta \|_{L^2} \| v \|_{H^{1/2} \times H^{1/2}}^2
\]
and so repeating the arguments given in the proof of Lemma 8 we arrive at the conclusion that the small enough initial data stays small through the flow. For \( s > 2 \) the noncavitation preserves locally-in-time due to the first equation in (1.4). The energy estimates and the rest of the proof of Theorem 2 can be done in exactly the same manner as in the one dimensional case, and so we omit further details.

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