Time-dependent methods in inverse scattering problems for the Hartree-Fock equation

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February 1, 2019

Abstract

The inverse scattering theory for many-body systems in quantum mechanics is an important and difficult issue not only in physics—atomic physics, molecular physics and nuclear physics—but also mathematics. The major purpose in this paper is to establish a reconstruction procedure of two-body interactions from scattering solutions for a Hartree-Fock equation. More precisely, this paper gives a uniqueness theorem and proposes a new reconstruction procedure of the short-range and two-body interactions from a high-velocity limit of the scattering operator for the Hartree-Fock equation. Moreover, it will be found that the high-velocity limit of the scattering operator is equal to a small-amplitude limit of it. The main ingredients of mathematical analysis in this paper are based on the theory of integral equations of the first kind and a Strichartz type estimates on a solution to the free Schrödinger equation.

Keywords. Hartree-Fock equations, Non-linear Schrödinger equations, N-body systems, Inverse scattering problems.

2010 Mathematics Subject Classification. Primary: 35P25; Secondary: 81U40, 35R30.

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1 Introduction

1.1 Background

Inverse scattering problems in quantum many-body systems are important and difficult problems not only quantum physics but also mathematics. Some results on inverse scattering problems for $N$-body Schrödinger equations were investigated. A reconstruction problem of identifying the two-body interactions from the high-energy asymptotics was studied in Wang [17], Enss and Weder [2], Novikov [10] and Vasy [15]. Uhlmann and Vasy [14] studied a low-energy inverse scattering problem.

As is well known, the solution of the $N$-body Schrödinger equation on $\mathbb{R}^n$ is a high-dimensional complicated function on $\mathbb{R}^{nN}$, which usually causes exact or numerical calculations impractical. Therefore, methods of approximation in understanding the many-body problem in quantum mechanics have most often been proposed. A result on inverse scattering problems in nuclear physics by using the optical model, which is one of the method of approximation in the many-body problems, was reported by Isozaki-Nakazawa and Uhlmann [5].

The time-dependent Hartree-Fock approximation, which is one of the simplest approximate theories for solving the many-body Hamiltonian, has received much attention due to its effect of calculations and having a wide field of applications (see, e.g., Goeke and Reinhard [3], and Kramer and Saracero [6]). Time-dependent Hartree equations have also play an important role in the development of mathematical analysis due to its non-linear structure, which cause interesting behavior of the solutions (see, e.g., Cazenave [11]).

In this paper, we are interested in an inverse scattering problem for a time-dependent Hartree equation and a Hartree-Fock equation. Consider the $N$-body system of identical particles with the interaction potential $V_{\text{int}}$ consisting of sum of two-body force: $V_{\text{int}} = \sum_{i<j} V(x_i - x_j)$. Here we denoted the position of the $j$-th particle by $x_j$. The indistinguishability of identical particles permits that the interaction potential is symmetric: $V(x) = V(-x)$ for $x \in \mathbb{R}^n$. 
Put
\[ V_H(x, u_j)(x) = \left\{ \int_{\mathbb{R}^n} V(x - y) \sum_{k=1, k \neq j}^{N} |u_k(y)|^2 \, dy \right\} u_j(x), \]
\[ = \left( V \ast \sum_{k=1, k \neq j}^{N} |u_k|^2 \right) u_j(x), \quad u = \{ u_j \}_{1 \leq j \leq N} \]
and
\[ V_F(x, y) = -V(x - y) \sum_{k=1, k \neq j}^{N} \overline{u_k(y)} u_k(x). \]
then the Hartree equation (H equation) is written as
\[ i \frac{\partial u_j}{\partial t} = H_0 u_j + V_H(x, u) u_j \quad \text{for } 1 \leq j \leq N \quad (1.1) \]
and the Hartree-Fock equation (HF equation) is written as
\[ i \frac{\partial u_j}{\partial t} = \{ H_0 + V_H(x, u) \} u_j + \int_{\mathbb{R}^n} V_F(x, y) u_j(y) \, dy \quad \text{for } 1 \leq j \leq N, \quad (1.2) \]
where \( H_0 = -\frac{1}{2} \Delta \) and \( u_j = u_j(t, x) \) is an unknown function in \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \).
The terms \( V_H(x, u) u_j(x) \) and \( \int V_F(x, y) u_j(y) \, dy \) are called the Hartree term and the Fock term, respectively.

The problem considered in this paper is to reconstruct the interaction potential \( V(x) \) from the corresponding scattering operator defined below.

The H equation and the HF equation are non-linear Schrödinger equations with cubic convolution non-linearity. Thus, our inverse problems can be labeled as the inverse non-linear scattering problems of identifying the non-linearity from the scattering operator. As is well-known, inverse scattering problems are non-linear problems even if governing differential equations are linear equations. From an analytical point of view, inverse problems of non-linear differential equations are one of the most difficult problems in inverse problems.
Initial attempts for inverse non-linear scattering problems focused on identifying the coefficients of power type non-linearity from the small scattering data. The approach called small-amplitude method was developed by Strauss [13], Weder [24, 26, 25, 23, 27, 29, 30] and Angeles-Romo-Weder [11], which has been shown to be powerful to reconstruct coefficient functions of non-linear Schrödinger equations. However, the approach is valid only for small data. Reconstruction of coefficient functions from large scattering data requires an alternative approach. Recently, in [23], the author establishes the unique reconstruction of the power type non-linearity from the large scattering data by using the method of high-velocity limit developed by Enss and Weder [2].

On the inverse scattering problem for Hartree equations, most references we know are concerned with uniqueness results [12] and reconstructions for interactions with special form [18, 19, 20, 22]—all focused on identifying the interactions from the small scattering data.

With regard to the inverse scattering problem for Hartree-Fock equations, little work has been done even for a uniqueness problem. The only reference we know is the work [21] where the reconstruction formula is given for the special interactions of the form $V_j(x) = \lambda_j |x|^{-\sigma_j}, 1 \leq j \leq 3$ in the case of 3-body systems.

In this paper, we deal with the inverse scattering problem for both H equation (1.1) and HF equation (1.2). This paper presents a explicit reconstruction formula on recovering the two-body interaction $V(x)$ from the high energy asymptotics of the scattering solutions for the H equation (1.1) and the HF equation (1.2), respectively. A uniqueness theorem in the inverse scattering problem for the HF equation is also proved. As is mentioned above, the fundamental ingredients in mathematical analysis to investigate the non-linear inverse scattering problems are now two methods—the small-amplitude method and the high-velocity method. This paper will uncover the relation between the two methods.

1.2 Methods

Our method consists of the following analysis:

- Asymptotic expansion of the scattering operator acting on the function $\Phi_v(x) = e^{iv \cdot x} \varphi(x)$ as $|v| \to \infty$. 
• Derivation of a transformation of the Fourier transform of the interaction potential $\hat{V}(\xi)$ of the form $\int G(\xi, \lambda)\hat{V}(\xi)d\xi$ by using a scale transform $\varphi_\lambda(x) = \varphi((\lambda + 1)x)$.

• Invertibility of the transformation by using the Picard’s theorem for an equation of the first kind with a compact operator.

Previous researches [2, 23] explain that the high-velocity analysis of the scattering operator gives the Radon transform of the unknown coefficient functions. Due to the inversion formula of the Radon transform, the high-velocity analysis therefore provides a reconstruction formula for unknown coefficient functions.

With regard to Hartree equations, however, high-velocity analysis gives a transformation which is different from the Radon transform—the transformation has a complicated integral kernel due to those non-linearities. Invertibility of the transformation was unclear.

In order to overcome this difficulty, we employ the high-velocity analysis with scale transform, which leads an integral equation of the first kind for the unknown interaction $V$. It will be shown that the integral kernel is equicontinuous and equibounded in some function spaces. The Ascoli’s lemma therefore gives the compactness of the integral operator. Then, the Picard’s theorem for an equation with a compact operator can give an explicit solution of the integral equation. We also remark that the Picard’s theorem dose not imply uniqueness of the solution. Construction of the proper initial data for the free Schrödinger equation such that the non-linear interactions of the free solution can be localized at any fixed point in $\mathbb{R}^n$ lead us to prove a uniqueness theorem. This is done in Section 4.

It should be mentioned that the Picard’s theorem can be applicable on a Hilbert space. This paper finds the proper Hilbert space to apply the Picard’s theorem and consequently the proper function space to reconstruct the interaction $V$.

The fundamental ingredient in our proof is a time-space $L^4$ estimate on a solution to the free Schrödinger equations.

1.3 Results

We summarize our main results. Let $W^{k,l}(\mathbb{R}^n)$ be the usual Sobolev space in $L^l(\mathbb{R}^n)$. We abbreviate $W^{k,2}(\mathbb{R}^n)$ as $H^k(\mathbb{R}^n)$. 
We first state results for the restricted Hartree equation (RH equation), which is the case for $N = 2$ with $u_1 = u_2$ in equation (1.1). Proofs of theorems for H equation and HF equation are reduced to the proof of theorems on RH equation.

### 1.3.1 Restricted Hartree equation

Consider the RH equation:

$$i\partial_t u = H_0 u + V_H(x, u)u,$$

where $V_H(x, u)u = (V \ast |u|^2)u$. In order to formulate our inverse problem, let us state a result on the large date scattering problem. We denote solutions $u(t) := u(t, x)$ of the equation (1.3) with initial data $f$ as $U(t)f$ and the unitary group of the self-adjoint operator $H_0 = -\frac{1}{2}\Delta$ with a domain $H^1(\mathbb{R}^n)$ as $U_0(t)$.

**Theorem 1.1** (Nakanishi [9]). Let $n \geq 3$. Assume that $V(x)$ is a radial, non-negative and non-increasing function such that

$$V \in \mathcal{V} := \{V \in L^{p_1} + L^{p_2}; p_1, p_2 \geq 1, n/2 > p_1 \geq p_2 > n/4\}.$$

Then for any $f_- \in H^1(\mathbb{R}^n)$, there exists a unique pair of functions $f_+ \in H^1(\mathbb{R}^n)$ and $\varphi \in H^1(\mathbb{R}^n)$ such that

$$\|U(t)\varphi - U_0(t)f_\pm\|_{H^1} \to 0, \quad \text{as } t \to \pm \infty.$$

In addition, the scattering operator

$$S : H^1(\mathbb{R}^n) \ni f_- \rightarrow f_+ \in H^1(\mathbb{R}^n)$$

is homeomorphism in $H^1(\mathbb{R}^n)$.

We term solutions constructed in Theorem 1.1 scattering solutions.

**Remark 1.** The statement in [9, Theorem 1.1] assumes that $p_1, p_2 \geq 1$ satisfy $n/2 < p_1 \leq p_2 < n/4$. This is obviously an erratum. The correct condition is that $p_1, p_2 \geq 1$ satisfy $n/2 > p_1 \geq p_2 > n/4$.

We now formulate our inverse scattering problem.

**Inverse scattering problem:** Given the scattering operator $S$ with the domain $H^1(\mathbb{R}^n)$, determine the interaction potential $V$. 
This paper presents a reconstruction procedure of the interaction potential $V(x)$ from the scattering operator defined in Theorem 1.1.

We denote the multiplication operator with a fixed function $V(x)$ as $V$, the Schwartz class as $S$, the weighted $L^2$ space as $L^{2,\delta}$ and the set of compactly supported smooth functions as $C_0^\infty$. The Fourier transform of $f$ is denoted as $\hat{f}$ or $\mathcal{F}f$. Let $<\cdot,\cdot>_{L^2}$ be the inner product in $L^2(\mathbb{R}^n)$ and put

$$S_0 = \{ f \in S(\mathbb{R}^n); \hat{f} \in C_0^\infty(\mathbb{R}^n) \}.$$ 

**Theorem 1.2.** Let $3 \leq n \leq 6$ and $2\delta > n$. Assume that $V \in \mathcal{V} \cap L^{2,1+\delta}(\mathbb{R}^n)$ is a radial, non-negative and non-increasing function. In addition, suppose that $V$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,1+\delta}(\mathbb{R}^n)$. Then for any $\varphi \in S_0$, we have

$$\lim_{|v| \to \infty} \langle i(S-I)\Phi_v, \Phi_v \rangle_{L^2} = \int_{\mathbb{R}^n} \hat{V}(\xi)G(\xi) \, d\xi < \infty,$$

where $\Phi_v(x) = e^{iv \cdot x}\varphi(x)$, $v \in \mathbb{R}^n$ and

$$G(\xi) = \int_\mathbb{R} |\mathcal{F}(|U_0(t)\varphi|^2)(\xi)|^2 \, dt.$$

As is proved in [12], the identity

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \langle i(S-I)(\varepsilon \varphi), \varphi \rangle_{L^2} = \int_{\mathbb{R}^n} \hat{V}(\xi)G(\xi) \, d\xi$$

holds for any $\varphi \in H^1(\mathbb{R}^n)$. Hence we have

**Corollary 1.1.** Let $3 \leq n \leq 6$ and $2\delta > n$. Assume that $V \in \mathcal{V} \cap L^{2,1+\delta}(\mathbb{R}^n)$ is a radial, non-negative and non-increasing function. In addition, suppose that $V$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,1+\delta}(\mathbb{R}^n)$. Then for any $\varphi \in S_0$ and $\varepsilon > 0$, we have

$$\lim_{|v| \to \infty} \langle i(S-I)\Phi_v, \Phi_v \rangle_{L^2} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \langle i(S-I)(\varepsilon \varphi), \varphi \rangle_{L^2},$$

where $\Phi_v(x) = e^{iv \cdot x}\varphi(x)$, $v \in \mathbb{R}^n$.

This corollary shows that in the case of the RH equation, the high-velocity method on the inverse scattering problem is equivalent to the small-amplitude method on it.
Let $\varphi_\lambda(x) = \varphi((\lambda + 1)x)$ and put

$$G(\xi, \lambda) := \int_\mathbb{R} |\mathcal{F} (|U_0(t)\varphi_\lambda|^2)(\xi)|^2 \, dt$$

$$= \int_\mathbb{R} \left| e^{-it\xi^2} \int_{\mathbb{R}^n} e^{2it\xi \eta} \hat{\varphi}_\lambda(\xi - \eta) \hat{\varphi}_\lambda(\eta) \, d\eta \right|^2 \, dt$$

$$= \int_\mathbb{R} \left| \int_{\mathbb{R}^n} e^{2it\xi \eta} \left( \frac{1}{\lambda + 1} \right)^2 \hat{\varphi} \left( \frac{\xi - \eta}{\lambda + 1} \right) \hat{\varphi} \left( \frac{\eta}{\lambda + 1} \right) \, d\eta \right|^2 \, dt.$$  

Consider the integral equation of the first kind:

$$P(\lambda) = \int_{\mathbb{R}^n} \hat{V}(\xi) G(\xi, \lambda) \, d\xi. \quad (1.4)$$

**Theorem 1.3.** Let $\Gamma \subset \mathbb{R}$ be a compact set. Assume that $2 \leq n \leq 6$. Then for any $\varphi \in H^1(\mathbb{R}^n)$, the integral operator $T_G$:

$$(T_G f) = \int_{\mathbb{R}^n} f(\xi) G(\xi, \lambda) \, d\xi$$

is a compact operator from $H^k(\mathbb{R}^n)$ to $L^2(\Gamma)$ for $k > n/2$.

The Picard’s theorem for an equation of the first kind with a compact operator gives an explicit solution to the integral equation $(1.4)$ (see, e.g., Kress [7, Theorem 15.18]). To state our theorem on the reconstruction problem, we give a definition of the singular system of the compact operator.

**Definition 1.1.** Let $X$ and $Y$ be Hilbert space, $A : X \to Y$ be a compact linear operator, and $A^* : Y \to X$ be its adjoint. Singular values of $A$ is the non-negative square roots of the eigenvalue of non-negative self-adjoint compact operator $A^* A : X \to X$. The singular system of $A$ is the system $\{\mu_n, \varphi_n, g_n\}, n \in \mathbb{N}$, where $\varphi_n \in X$ and $g_n \in Y$ are orthonormal sequences such that $A\varphi_n = \mu_n g_n$ and $A^* g_n = \mu_n \varphi_n$ for all $n \in \mathbb{N}$.

We denote the null-space of the operator $T$ by $\mathcal{N}(T)$.

**Theorem 1.4.** Let $3 \leq n \leq 6$ and $2\delta > n$. Assume that $V \in \mathcal{V} \cap L^{2,1+\delta}(\mathbb{R}^n)$ is a radial, non-negative and non-increasing function. In addition, suppose that $V$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,1+\delta}(\mathbb{R}^n)$. Then for any $\varphi \in \mathcal{S}_0$, the function

$$P(\lambda) := \lim_{|v| \to \infty} \langle i(S - I)\Phi_v(\cdot, \lambda), \Phi_v(\cdot, \lambda) \rangle_{L^2}, \quad \Phi_v(x, \lambda) = e^{iv \cdot x} \varphi_\lambda(x)$$
is the $L^2$-function on a compact set $\Gamma \subset \mathbb{R}$. Moreover, letting $\{\mu_n, \varphi_n, g_n\}$, $n \in \mathbb{N}$ be a singular system of the compact operator $T_G$, the Fourier transform of the interaction potential is reconstructed by the formula:

$$\hat{V}(\xi) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \langle P, g_n \rangle_{L^2(\Gamma)} \varphi_n$$

if and only if $P \in \mathcal{N}(T_G^*)^\perp$ and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left| \langle P, g_n \rangle_{L^2(\Gamma)} \right|^2 < \infty.$$ 

**Remark 2.** Due to the fact that $\|f\|_{H^k} \leq C\|f\|_{L^2}$ for $k > 0$, the Picard’s theorem can be applied to the equation (1.4) for $V \in L^{2,1+\delta}(\mathbb{R}^n)$.

**Remark 3.** Uniqueness theorem on the inverse scattering problem of identifying $V(x)$ holds for a bounded continuous function $V(x)$ such that

$$|V(x)| \leq C|x|^{-\sigma}, \quad 2 \leq \sigma \leq 4, \sigma < n$$

for some $C > 0$ and $\hat{V}$ is the continuous function on $\mathbb{R}^n$ (see [12]).

### 1.3.2 Hartree equation

Consider the Hartree equation (1.1). The following theorem on the scattering problem is obtained easily from the proof of Theorem 1.1 because the Hartree term $V_H(x)u_j$ has the same structure as in the RH equation. We denote a vector-valued $H^1$-function $f = (f^{(j)})_{1 \leq j \leq N}$ with $f^{(j)} \in H^1(\mathbb{R}^n)$ by $f \in [H^1(\mathbb{R}^n)]^N$.

**Theorem 1.5.** Let $n \geq 3$. Assume that $V \in \mathcal{V}$ is a radial, non-negative and non-increasing function. Then for any $f_- \in [H^1(\mathbb{R}^n)]^N$, there exists a unique pair of functions $f_+ \in [H^1(\mathbb{R}^n)]^N$ and $\varphi \in [H^1(\mathbb{R}^n)]^N$ such that

$$\|U(t)\varphi - U_0(t)f_+\|_{H^1} \to 0, \quad \text{as } t \to \pm \infty.$$ 

In addition, the scattering operator

$$S : [H^1(\mathbb{R}^n)]^N \ni f_- \to f_+ \in [H^1(\mathbb{R}^n)]^N$$

is homeomorphism in $[H^1(\mathbb{R}^n)]^N$. 
Remark 4. The scattering operator is represented as
\[(S\varphi)^{(j)}(x) = \varphi^{(j)}(x) + \frac{1}{i} \int_{\mathbb{R}} e^{iH_0} V_H(x, u) u_j(x, t) \, dt,\]
where \(\varphi^{(j)}(x)\) is a radial, non-negative and non-increasing function. In addition, suppose that \(V\) is a compact operator from \(L^2(\mathbb{R}^n)\) to \(L^{2,1+\delta}(\mathbb{R}^n)\). Then for any \(\varphi^{(j)} \in \mathcal{S}_0, j = 1, 2, \ldots, N\), we have
\[\lim_{|v| \to \infty} \left\langle i((S-I)\Phi_v)^{(j)}, \Phi_v^{(j)} \right\rangle_{L^2} = \int_{\mathbb{R}^n} \hat{V}(\xi) H^{(j)}(\xi) \, d\xi,\]
where \(\Phi_v(x) = e^{iv \cdot x} \varphi(x), v \in \mathbb{R}^n\) and
\[H^{(j)}(\xi) = \sum_{k=1}^{N} \int_{\mathbb{R}} \mathcal{F} \left( |U_0(t)\varphi^{(k)}|^2 \right)(\xi) \mathcal{F} \left( |U_0(t)\varphi^{(j)}|^2 \right)(\xi) \, dt.\]

As is discussed in sub-subsection 1.3.1, the high-velocity limit of the scattering operator is equal to the small-amplitude limit of it.

Theorem 1.6. Let \(3 \leq n \leq 6\) and \(2\delta > n\). Assume that \(V \in \mathcal{V} \cap L^{2,1+\delta}(\mathbb{R}^n)\) is a radial, non-negative and non-increasing function. In addition, suppose that \(V\) is a compact operator from \(L^2(\mathbb{R}^n)\) to \(L^{2,1+\delta}(\mathbb{R}^n)\). Then for any \(\varphi^{(j)} \in \mathcal{S}_0, j = 1, 2, \ldots, N\) and \(\varepsilon > 0\), we have
\[\lim_{|v| \to \infty} \left\langle i((S-I)\Phi_v)^{(j)}, \Phi_v^{(j)} \right\rangle_{L^2} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \left\langle i((S-I)(\varepsilon \varphi))^{(j)}, \varphi^{(j)} \right\rangle_{L^2},\]
where \(\Phi_v(x) = e^{iv \cdot x} \varphi(x), v \in \mathbb{R}^n\).

Let \(\varphi_\lambda(x) = \varphi((\lambda + 1)x)\) and put
\[H^{(j)}(\xi, \lambda) = \sum_{k=1}^{N} \int_{\mathbb{R}} \mathcal{F} \left( |U_0(t)\varphi_\lambda^{(k)}|^2 \right)(\xi) \mathcal{F} \left( |U_0(t)\varphi^{(j)}\lambda|^2 \right)(\xi) \, dt.\]

Consider the integral equation of the first kind:
\[P^{(j)}(\lambda) = (T_H\hat{V})^{(j)}(\lambda) := \int_{\mathbb{R}^n} \hat{V}(\xi) H^{(j)}(\xi, \lambda) \, d\xi. \quad (1.5)\]
Theorem 1.7. Let \( \Gamma \subset \mathbb{R} \) be a compact set. Assume that \( 2 \leq n \leq 6 \). Then for any \( \varphi \in [H^1(\mathbb{R}^n)]^N \), the integral operator \( T_H \) is a compact operator from \( H^k(\mathbb{R}^n) \) to \( L^2(\Gamma) \) for \( k > n/2 \).

The same argument as in sub-subsection 1.3.1 leads us a reconstruction formula.

Theorem 1.8. Let \( 3 \leq n \leq 6 \) and \( 2\delta > n \). Assume that \( V \in \mathcal{V} \cap L^{2,1+\delta}(\mathbb{R}^n) \) is a radial, non-negative and non-increasing function. In addition, suppose that \( V \) is a compact operator from \( L^2(\mathbb{R}^n) \) to \( L^{2,1+\delta}(\mathbb{R}^n) \). Then for any \( \varphi^{(j)} \in S_0 \), \( j = 1, 2, \cdots, N \), the function

\[
P^{(j)}(\lambda) := \lim_{|v| \to \infty} \left\langle (i(S - I)\Phi_v)^{(j)}, \Phi^{(j)}_v \right\rangle_{L^2}, \quad \Phi_v(x, \lambda) = e^{iv \cdot x}\varphi_\lambda(x)
\]

is the \( L^2 \)-function on a compact set \( \Gamma \subset \mathbb{R} \). Moreover, letting \( \{\mu_n, \varphi_n, g_n\} \), \( n \in \mathbb{N} \) be a singular system of the compact operator \( T_H \), the Fourier transform of the interaction potential is reconstructed by the formula:

\[
\hat{V}(\xi) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \left\langle P^{(j)}, g_n \right\rangle_{L^2(\Gamma)} \varphi_n
\]

if and only if \( P^{(j)} \in \mathcal{N}(T^{*}_H)^\perp \) and satisfies

\[
\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left| \left\langle P^{(j)}, g_n \right\rangle_{L^2(\Gamma)} \right|^2 < \infty.
\]

1.3.3 Hartree-Fock equation

Consider the HF equation (1.2). In contrast to H equation, the large data scattering for HF equation in \( H^1(\mathbb{R}^n) \) space does not follow in the same way as in the proof of Theorem 1.1. It still remains to be a poorly understood problem, although basic results—the global existence and the \( L^2 \)-conservation law of solutions—was obtained by Isozaki [4]. We here state results on the small data scattering in the space \( H^1(\mathbb{R}^n) \) and the large data scattering in a weighted space because of a difference for assumptions on \( V \).

The following theorem on the small data scattering follows from the result in Mochizuki [8].
Theorem 1.9. Assume that \( V(x) \) satisfies
\[
|V(x)| \leq C_V|x|^{-\sigma}, \quad 2 \leq \sigma \leq n \text{ and } \sigma < n
\]
for some \( C_V > 0 \). Then for any \( f_\pm \in [H^1(\mathbb{R}^n)]^N \), there exists a unique pair of functions \( f_+ \in [H^1(\mathbb{R}^n)]^N \) and \( \varphi \in [H^1(\mathbb{R}^n)]^N \) such that
\[
\|U(t)\varphi - U_0(t)f_\pm\|_{H^1} \to 0, \quad \text{as } t \to \pm\infty.
\]
In addition, the scattering operator \( S : f_- \to f_+ \) is defined on \( [H^1(\mathbb{R}^n)]^N = \{\varphi \in [H^1(\mathbb{R}^n)]^N; \|\varphi^{(j)}\|_{H^1} < \varepsilon, j = 1, \cdots, N\} \) for small \( \varepsilon > 0 \) depending on \( C_V \).

The large data scattering is stated as follows:

Theorem 1.10 (Wada [16]). Let \( \ell, m \in \mathbb{N} \). Assume that \( V(x) \) satisfies
\[
|V(x)| \leq C|x|^{-\sigma}, \quad \frac{4}{3} < \sigma < \min(4, n)
\]
for some \( C > 0 \). Suppose that \( m \geq 2 \) if \( \sigma \leq \sqrt{2} \). Then for any \( f_- \in \sum^{\ell, m} \), there exists a unique pair of functions \( f_+ \in \sum^{\ell, m} \) and \( \varphi \in \sum^{\ell, m} \) such that
\[
\|U(t)\varphi - U_0(t)f_\pm\|_{\sum^{\ell, m}} \to 0, \quad \text{as } t \to \pm\infty.
\]
In addition, the scattering operator \( S : f_- \to f_+ \) is defined on
\[
\sum^{\ell, m} = \{\varphi \in [L^2(\mathbb{R}^n)]^N; \|\varphi\|^2_{\sum^{\ell, m}} = \sum_{|\alpha| \leq \ell} \|\nabla^\alpha \varphi\|^2_{L^2} + \sum_{|\beta| \leq m} \|x^\beta \varphi\|^2_{L^2} < \infty\}.
\]

Remark 5. Due to the fact that in each case, the scattering operator is represented as
\[
(S\varphi)^{(j)}(x) = \varphi^{(j)}(x) + \frac{1}{i} \int_{\mathbb{R}} e^{itH_0}(F_{HF}(u))^{(j)} dt,
\]
\[
(F_{HF}(u))^{(j)} = V_H(x)u_j(x, t) + \int_{\mathbb{R}^n} V_F(x, y)u_j(y, t) dy,
\]
\( j = 1, 2, \cdots, N \), our reconstruction formula given below is valid for both of the scattering although assumptions on \( V \) are different.
We consider now a inverse scattering problem of identifying the interaction potential $V(x)$ from the scattering operator $S$ with the domain $[H^1_ε(\mathbb{R}^n)]^N$ or $\sum t^m$.

**Theorem 1.11.** Let $2 \leq n \leq 6$. Assume that $V$ satisfies the assumption in Theorem 1.9 or Theorem 1.10. In addition, suppose that $V$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,1+\delta}(\mathbb{R}^n)$ with $2\delta > n$. Then for any $\varphi(j) \in \mathcal{S}_0$, $j = 1, 2, \cdots, N$, we have

$$
\lim_{|v| \to \infty} \left\langle \hat{S}(S-I)\Phi_v^{(j)}, \Phi_v^{(j)} \right\rangle_{L^2} = \int_{\mathbb{R}^n} \hat{V}(\xi)H_{HF}^{(j)}(\xi) \, d\xi,
$$

where $\Phi_v^{(j)}(x) = e^{iv \cdot x} \varphi(j)(x), v \in \mathbb{R}^n$ and

$$
H_{HF}^{(j)}(\xi) = \sum_{k=1}^N \int_{\mathbb{R}} \mathcal{F} \left( \frac{|U_0(t)\varphi(k)|^2}{(S-I)(\xi)} \mathcal{F}(U_0(t)\varphi(j))^2 \right)(\xi) \, dt
$$

$$
- \sum_{k=1}^N \int_{\mathbb{R}} \left| \mathcal{F} \left( \frac{(U_0(t)\varphi(j))}{(U_0(t)\varphi(k))} \right)(\xi) \right|^2 dt.
$$

Similarly to the RH equation and H equation, the high-velocity limit of the scattering operator is equal to the small-amplitude limit of it.

**Corollary 1.3.** Let $2 \leq n \leq 6$. Assume that $V$ satisfies the assumption in Theorem 1.9 or Theorem 1.10. In addition, suppose that $V$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,1+\delta}(\mathbb{R}^n)$ with $2\delta > n$. Then for any $\varphi(j) \in \mathcal{S}_0$, $j = 1, 2, \cdots, N$ and $\varepsilon > 0$, we have

$$
\lim_{|v| \to \infty} \left\langle (S-I)\Phi_v^{(j)}, \Phi_v^{(j)} \right\rangle_{L^2} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \left\langle (S-I)(\varepsilon \varphi)^{(j)}, \varphi^{(j)} \right\rangle_{L^2},
$$

where $\Phi_v^{(j)}(x) = e^{iv \cdot x} \varphi(j)(x), v \in \mathbb{R}^n$.

Let $\varphi_\lambda(x) = \varphi((\lambda + 1)x)$ and

$$
H_{HF}^{(j)}(\xi, \lambda) = \sum_{k=1}^N \int_{\mathbb{R}} \mathcal{F} \left( \frac{|U_0(t)\varphi^{(k)}_\lambda|^2}{(S-I)(\xi)} \mathcal{F}(U_0(t)\varphi^{(j)}_\lambda)^2 \right)(\xi) \, dt
$$

$$
- \sum_{k=1}^N \int_{\mathbb{R}} \left| \mathcal{F} \left( \frac{(U_0(t)\varphi^{(j)}_\lambda)(U_0(t)\varphi^{(k)}_\lambda)}{(S-I)(\xi)} \right) \right|^2 dt.
$$
Consider the integral equation of the first kind:
\[
P^{(j)}(\lambda) = (T_{HF} \hat{V})^{(j)}(\lambda) := \int_{\mathbb{R}^n} \hat{V}(\xi) H_{HF}^{(j)}(\xi, \lambda) \, d\xi. \tag{1.6}
\]

**Theorem 1.12.** Let $\Gamma \subset \mathbb{R}$ be a compact set. Assume that $2 \leq n \leq 6$. Then for any $\varphi^{(j)} \in H^1(\mathbb{R}^n)$, $j = 1, \cdots, N$, the integral operator $T_{HF}$ is a compact operator from $H^k(\mathbb{R}^n)$ to $L^2(\Gamma)$ for $k > n/2$.

Similarly to the RH equation and the H equation, the Picard’s theorem allows us to obtain a reconstruction formula of $\hat{V}$ in terms of the singular system of $T_{HF}$.

**Theorem 1.13.** Let $2 \leq n \leq 6$. Assume that $V$ satisfies the assumption in Theorem 1.12 or Theorem 1.10. In addition, suppose that $V$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,1+\delta}(\mathbb{R}^n)$ with $2\delta > n$. Then for any $\varphi \in [S_0]^N$, the function
\[
P^{(j)}(\lambda) := \lim_{|v| \to \infty} \left\langle i ((S - I) \Phi_v(\cdot, \lambda))^{(j)}, \Phi_v^{(j)}(\cdot, \lambda) \right\rangle_{L^2}, \quad j = 1, \cdots, N,
\]
\[
\Phi_v(x, \lambda) = e^{iv \cdot x} \varphi((\lambda + 1)x)
\]
is the $L^2$-function on a compact set $\Gamma \subset \mathbb{R}$. Moreover, letting $\{\mu_n, \varphi_n, g_n\}$, $n \in \mathbb{N}$ be a singular system of $T_{HG}$, the Fourier transform of the interaction potential is reconstructed by the formula:
\[
\hat{V}(\xi) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \left\langle P^{(j)}, g_n \right\rangle_{L^2(\Gamma)} \varphi_n
\]
if and only if $P^{(j)} \in \mathcal{N}(T_{HF}^\perp)$ and satisfies
\[
\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left| \left\langle P^{(j)}, g_n \right\rangle_{L^2(\Gamma)} \right|^2 < \infty.
\]

**Theorem 1.14.** Let $2 \leq n \leq 6$. Assume that $V_1, \sharp = 1, 2$ satisfy the assumption in Theorem 1.12. Let $S_1$ are the scattering operator for the HF equation (1.2) with interaction potential $V_1$. If $S_1 = S_2$, then we have $V_1 = V_2$.

The structure of this paper is as follows. Section 2 is devoted to an analysis of high-velocity analysis of the scattering operator. A time-space estimate on $V_H(x, u)U_0(t)\Phi_v$ plays an important role. We give proofs of Theorem 1.3, Theorem 1.7 and Theorem 1.12 in Section 3. It will be shown that the set of functions $\{T_{GF}\}$, $\{T_{HF}\}$ and $\{T_{HF,F}\}$ are equicontinuous and equibounded in the set of continuous functions $C(\Gamma)$. Section 4 gives a proof of Theorem 1.14.
2 High velocity limit of the scattering operator

In this section, we analyze the asymptotic behavior of the scattering operator for the RH equation, the H equation and the HF equations. Due to the similarity of the proof, we give a proof in detail only for the case of the RH equation.

Consider the RH equation \((1.3)\). Let \(S\) be the scattering operator for \((1.3)\) defined in Theorem 1.1. Our goal in this section is to prove the following theorem.

**Theorem 2.1.** Let \(3 \leq n \leq 6\) and \(2\delta > n\). Assume that \(V \in V \cap L^{2,1+\delta}(\mathbb{R}^n)\) is a radial, non-negative and non-increasing function. In addition, suppose that \(V\) is a compact operator from \(L^2(\mathbb{R}^n)\) to \(L^{2,1+\delta}(\mathbb{R}^n)\). Then for any \(\varphi \in S_0\), we have

\[
\langle i(S - I)\Phi_v, \Phi_v \rangle_{L^2} = \int_{\mathbb{R}^n} \hat{V}(\xi)G(\xi) d\xi + R(v),
\]

where \(\Phi_v(x) = e^{iv \cdot x} \varphi(x), v \in \mathbb{R}^n\) and

\[
G(\xi) = \int_{\mathbb{R}} |F(|U_0(t)\varphi|^2)(\xi)|^2 dt,
\]

\[
R(v) = O(|v|^{-2}), \quad |v| \to \infty.
\]

2.1 Preliminary lemmas

In order to prove Theorem 2.1, we need some lemmas.

**Lemma 2.1.** Let \(n \geq 2\) and \(s > 1\). Assume that \(q\) is a compact operator from \(L^2(\mathbb{R}^n)\) to \(L^{2,s}(\mathbb{R}^n)\). Then for any \(\varphi \in S_0\), there exist a positive constant \(C\) such that

\[
\int_{-\infty}^{\infty} \|q U_0(t) e^{iv \cdot x} \varphi\|_{L^2} dt \leq \frac{C}{|v|}
\]

for \(|v|\) large enough.

**Proof.** The proof will be found in [2, Lemma 2.2] and its proof. \(\Box\)

We establish a similar estimate for the RH equation.

**Lemma 2.2.** Let \(n \geq 3\) and \(s > 1\). Assume that \(V \in V\) is a radial, non-negative and non-increasing function. Suppose that \(V\) is a compact operator
from $L^2(\mathbb{R}^n)$ to $L^2_{\text{sc}}(\mathbb{R}^n)$. Then for any $\varphi \in S_0$, there exist a positive constant $C$ such that

$$\int_{-\infty}^{\infty} \|V_H U_0(t) \Phi_v\|_{L^2} dt \leq \frac{C}{|v|}$$

for $|v|$ large enough, where $V_H(x, u) = (V * |u(t)|^2)(x)$ and $u(t) = u(x, t)$ is the scattering solution for the RH equation (1.3).

Proof. Due to the $L^2$ boundedness of the scattering solution to the RH equation (1.3), we have

$$|\mathcal{F}(|u|^2)(t, \xi)| \leq \|u(t)\|_{L^1} \leq \|f_-\|_{L^2}^2.$$ 

From this inequality and the identity $\hat{V_H} = \hat{V} \mathcal{F}(|u|^2)$, one has

$$\|V_H(\cdot, u) U_0(t) \Phi_v\|_{L^2} = \|\hat{V} |u|^2 \ast \mathcal{F}(U_0(t) \Phi_v)\|_{L^2} \leq \sup_{t, \xi} |\hat{|u|^2(t, \xi)}| \|\hat{V} \ast \mathcal{F}(U_0(t) \Phi_v)\|_{L^2} \leq C \|V U_0(t) \Phi_v\|_{L^2}.$$ 

Then, applying Lemma 2.1 to the right hand side of the above inequality achieves the desired estimate. \hfill \Box

Let $u$ be the scattering solution to (1.3). Consider the wave operator $\Omega_- : H^1 \ni f_- \rightarrow u(0) \in H^1$. 

**Lemma 2.3.** Let $n \geq 3$ and $\Phi_v = e^{iv \cdot x} \varphi$. Assume that $V(x)$ satisfies the same condition as in Lemma 2.2. Then for any $\varphi \in S_0$, we have

$$\|\left(\Omega_- - I\right) U_0(t) \Phi_v\|_{L^2} = O(|v|^{-1})$$

as $|v| \rightarrow \infty$ uniformly in $t \in \mathbb{R}$.

Proof. In view of the representation of the wave operator, one has

$$\langle \Omega_- f, g \rangle_{L^2} - \langle f, g \rangle_{L^2} = i \int_{-\infty}^{0} \langle V_H(x, u) u(s), U_0(s) g \rangle_{L^2} ds.$$ 

Then, Lemma 2.2 and the duality argument enables us to obtain

$$\|\left(\Omega_- - I\right) U_0(t) \Phi_v\|_{L^2} \leq \int_{-\infty}^{\infty} \|u(s)\|_{L^2} \|V_H(\cdot, u) U_0(s) g\|_{L^2} ds \leq \frac{C}{|v|},$$

Here $C$ is a positive constant independent of $t$. This completes the proof. \hfill \Box
Lemma 2.4. Let \( n \geq 2, \delta > 0 \) and \( \Phi_v = e^{iv \cdot x}, v \in \mathbb{R}^n \). Assume that \( V \) is a compact operator from \( L^2(\mathbb{R}^n) \) to \( L^{2,1+\delta}(\mathbb{R}^n) \). Then for any \( \varphi \in S_0 \) and \( y \in \mathbb{R}^n \), there exist positive constants \( C_1, C_2 \) and \( C_3 \) such that

\[
\|V(\cdot - y)U_0(t)\Phi_v\|_{L^2} \leq C_1 \left( 1 + \frac{|vt|}{4} \right)^{-3} + C_2 (1 + |vt|)^{-3/2} + C_3 \left\{ 1 + \left( \frac{3}{8} |vt| - |y| \right)^2 \right\}^{-(1+\delta)/2}.
\]

Proof. The proof of this lemma is almost the same as in [2]. \qed

We denote by \( L^p(\mathbb{R}; L^q) \) the set of \( L^q \)-valued \( L^p \) functions.

Lemma 2.5. Let \( 2 \leq n \leq 6 \). Then for any \( \varphi \in H^1(\mathbb{R}^n) \),

\[
\|U_0(\cdot)\varphi\|_{L^4(\mathbb{R}; L^4)} \leq C \|\varphi\|_{H^1}, \quad C > 0.
\]

Proof. The proof will be found in [12, Proposition 6]. \qed

2.2 Proof of Theorem 2.1

We are now in a position to prove Theorem 2.1. Let \( F_{RH}(u) = (V * |u(t)|^2)u(t) \). We break the scattering operator in four parts:

\[
\langle i(S - I)\Phi_v, \Phi_v \rangle_{L^2} = L(v) + R_1(v) + R_2(v) + R_3(v),
\]

where

\[
L(v) = \int_{-\infty}^{\infty} \langle F_{RH}(U_0(s)\Phi_v), U_0(s)\Phi_v \rangle_{L^2} ds,
\]

\[
R_1(v) = \int_{-\infty}^{\infty} \langle \left[ V * \{ u(s) - U_0(s)\Phi_v \} \right] U_0(s)\Phi_v, U_0(s)\Phi_v \rangle_{L^2} ds,
\]

\[
R_2(v) = \int_{-\infty}^{\infty} \langle \left[ V * u(s) \left\{ u(s) - U_0(s)\Phi_v \right\} \right] U_0(s)\Phi_v, U_0(s)\Phi_v \rangle_{L^2} ds,
\]

\[
R_3(v) = \int_{-\infty}^{\infty} \langle \left\{ V * |u(s)|^2 \right\} \{ u(s) - U_0(s)\Phi_v \}, U_0(s)\Phi_v \rangle_{L^2} ds.
\]

Here \( u(t) \) is the scattering solution to (1.3).

To calculate the leading term \( L(v) \), we first observe that the identity

\[
\langle F_{RH}(U_0(t)\Phi_v), U_0(t)\Phi_v \rangle_{L^2} = \langle F_{RH}(U_0(t)\varphi), U_0(t)\varphi \rangle_{L^2}
\]
holds. In fact, by using the identity

\[(U_0(s)\Phi_v)(x) = e^{i(v \cdot x - |v| s^2/2)}(U_0(s)\varphi)(x - vs),\]

and the change of variables \(s = t/|v|, x' = x - vt\) and \(y' = y - vt\), we obtain

\[
\langle F_{RH}(U_0(t)\Phi_v), U_0(t)\Phi_v \rangle_{L^2} = \int_{\mathbb{R}^n} \left| e^{i(v \cdot x - |v| t^2/2)}(U_0(t)\varphi)(x - vt) \right|^2 (V \ast |(U_0(t)\varphi)(\cdot - vt)|^2)(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left| (U_0(t)\varphi)(x') \right|^2 \left( \int_{\mathbb{R}^n} V(x' + vt - (y' + vt))(U_0(t)\varphi(y'))^2 \, dy' \right) \, dx
\]

\[
= \int_{\mathbb{R}^n} V_H(x, U_0(t)\varphi)(U_0(t)\varphi)(\overline{U_0(t)\varphi})(x') \, dx'
\]

\[
= \langle F_{RH}(U_0(t)\varphi), U_0(t)\varphi \rangle_{L^2}.
\]

The Plancherel’s theorem implies that

\[
\langle F_{RH}(U_0\varphi), U_0(t)\varphi \rangle_{L^2} = \left\langle \hat{V}\mathcal{F}(|U_0(t)\varphi|^2), \mathcal{F}(|U_0(t)\varphi|^2) \right\rangle_{L^2}.
\]

Then, the Fubini’s theorem yields the expression of the leading term

\[
L(v) = \int_{\mathbb{R}^n} \hat{V}(\xi) \left( \int_{\mathbb{R}} |\mathcal{F}(|U_0(t)\varphi|^2)|^2 \, dt \right) \, d\xi.
\]

Here we note that \(L(v)\) is bounded. In fact, thanks to Lemma 2.3 one gets

\[
|L(v)| \leq \|\hat{V}\|_{L^\infty} \int_{\mathbb{R}^n} \left| \mathcal{F}(|U_0(t)\varphi|^2) \right|^2 \, d\xi \, dt
\]

\[
\leq C\|V\|_{L^1} \int_{\mathbb{R}} \|\mathcal{F}(|U_0(t)\varphi|^2)\|^2_{L^2} \, dt
\]

\[
= C\|U_0(t)\varphi\|_{L^4(\mathbb{R};L^4)}^4
\]

\[
\leq C\|\varphi\|_{H^1}^4.
\]

Next, we will show that \(R_3(v) = O(|v|^{-2})\) as \(|v| \to \infty\). Thanks to Lemma
and Lemma 2.3, one has
\[ |R_3(v)| \leq \int_{-\infty}^{\infty} |\langle (\Omega_1 - I)(U_0(s)\Phi_v), V_H(\cdot, u)U_0(s)\Phi_v \rangle| L^2 \, ds \]
\[ \leq \int_{\mathbb{R}} \| (\Omega_1 - I)(U_0(s)\Phi_v) \| L^2 \| V_H(\cdot, u)(U_0(s)\Phi_v) \| L^2 \, ds \]
\[ \leq \frac{C}{|v|} \int_{\mathbb{R}} \| V_H(\cdot, u)(U_0(s)\Phi_v) \| L^2 \, ds \]
\[ \leq \frac{C}{|v|^2}, \]
due to the fact that \( u(t) = \Omega_1(U_0(t)\phi) \).

We will claim that \( R_j(v) = O(|v|^{-2}) \), \( j = 1, 2 \) as \( |v| \to \infty \). Due to the fact that
\[ \| (V * ab)c \|_{L^2}^2 \leq \| a \|_{L^2}^2 \int_{\mathbb{R}^n} |b(y)|^2 \left( \int_{\mathbb{R}^n} |V(x - y)c(x)|^2 \, dx \right) dy, \]
we obtain
\[ |R_1(v)| \leq \int_{\mathbb{R}} \| V * \{(\Omega_1 - I)(U_0(s)\Phi_v)\}U_0(s)\Phi_v \| L^2 \| U_0(s)\Phi_v \| L^2 \, dt \]
\[ \leq \| \varphi \|_{L^2} \int_{\mathbb{R}} \| V * \{(\Omega_1 - I)(U_0(s)\Phi_v)\}U_0(s)\Phi_v \| L^2 \, dt \]
\[ \leq \| \varphi \|_{L^2} \int_{\mathbb{R}} \| (\Omega_1 - I)U_0(s)\Phi_v \| L^2 \]
\[ \left\{ \int_{\mathbb{R}^n} |U_0(s)\Phi_v(y)|^2 \left( \int_{\mathbb{R}^n} |V(x - y)(U_0(s)\Phi_v)(x)|^2 \, dx \right) dy \right\}^{1/2} dt. \]

Thanks to Lemma 2.3 and Lemma 2.4, one has
\[ |R_1(v)| \leq \frac{C}{|v|} \left( R_1^{(1)}(v) + R_1^{(2)}(v) \right), \]
where
\[ R_1^{(1)}(v) = \int_{\mathbb{R}} \left\{ \left( 1 + \frac{|vt|}{4} \right)^{-3} + (1 + |vt|)^{-3/2} \right\} \| U_0(s)\Phi_v \| L^2 \, dt \]
\[ R_1^{(2)}(v) = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^n} \left( 1 + \left( \frac{3}{8} |vt| - |y| \right)^2 \right)^{-1+\delta} \| U_0(t)\Phi_v \|^2 \, dy \right\}^{1/2} dt. \]
Let us prove that $R_1^{(1)}(v) \leq C|v|^{-1}$ for some $C > 0$. Thanks to Lemma 2.3, it is easy to verify that

$$R_1^{(1)}(v) = \|\varphi\|_{L^2} \int_\mathbb{R} \left\{ \left(1 + \frac{|vt|}{4}\right)^{-3} + (1 + |vt|)^{-3/2} \right\} \, dt \leq \frac{C}{|v|}.$$  

We will show that $R_1^{(2)}(v) \leq C|v|^{-1}$ for some $C > 0$. By using Lemma 2.4 and estimate $\|U_0(t)\Phi_v\|_{L^\infty} \leq \|\hat{\varphi}\|_{L^1}$, one gets

$$R_1^{(2)}(v) \leq C\|\hat{\varphi}\|_{L^1} \int_\mathbb{R} \left\| \left(1 + \left(\frac{3}{8}|vt| - |y|\right)^2\right)^{-\frac{(1+\delta)}{2}} \right\|_{L^2} \, dt.$$  

The polar coordinate yields that

$$\left\| \left(1 + \left(\frac{3}{8}|vt| - |y|\right)^2\right)^{-\frac{(1+\delta)}{2}} \right\|_{L^2}^2 = |\mathbb{S}^{n-1}| \int_0^\infty \frac{r^{n-1}}{(1 + (3|vt|/8 - r)^2)^{(1+\delta)}} \, dr,$$

where $|\mathbb{S}^{n-1}|$ is the surface area of the unit $(n-1)$-sphere in $n$-dimensional Euclidean space. It is easy to verify that

$$\int_0^{3|vt|/16} \frac{r^{n-1}}{(1 + (3|vt|/8 - r)^2)^{(1+\delta)}} \, dr \leq \int_0^{3|vt|/16} \frac{r^{n-1}}{(1 + (3|vt|/16)^2/4)^{(1+\delta)}} \, dr = \frac{1}{n (1 + (3|vt|/16)^n)^{(1+\delta)}},$$

and

$$\int_0^{\infty} \frac{r^{n-1}}{(1 + (3|vt|/8 - r)^2)^{(1+\delta)}} \, dr \leq \int_0^{3|vt|/16} \frac{(1 + r)^{n-1}}{(1 + r)^{2(1+\delta)}} \, dr = \frac{2}{2(1 + \delta) - n} \left(1 + \frac{3}{16}|vt|\right)^{n-2(1+\delta)}.$$
for $n < 2(1 + \delta)$. Then we have

$$
\left\| \left(1 + \left(\frac{3}{8}|vt| - |y|\right)^2\right)^{-1/(1+\delta)/2} \right\|^2_{L^2} \leq \frac{C_1 (3|vt|/16)^n}{(1 + (3|vt|/8)^2/4)^{(1+\delta)}}
$$

$$
+ \frac{C_2}{2(1+\delta) - n} \left(1 + \frac{3}{16}|vt|\right)^{n-2(1+\delta)}
$$

$$
\leq C_3 \left(1 + \frac{3}{16}|vt|\right)^{n-2(1+\delta)}
$$

for $n < 2(1 + \delta)$, where $C_3$ is a positive constant depending only on $n$ and $\delta$. Therefore, one has

$$
R_1^{(2)}(v) \leq \frac{C}{|v|} \int_{\mathbb{R}} \left(1 + s/2\right)^{(n-2(1+\delta))/2} dx = \frac{C}{|v|}
$$

for $n/2 < \delta$, due to the change of variables $3|vt|/8 = s$.

We now conclude that $|R_1(v)| \leq C/|v|^2$. Similarly, the remainder term $R_2(v)$ is estimated as

$$
|R_2(v)| \leq \frac{C}{|v|^2}, \quad \text{for } n/2 < \delta.
$$

Consequently, letting $R(v) = R_1(v) + R_2(v) + R_3(v)$, we obtain

$$
|R(v)| \leq \frac{C_1}{|v|^2} + \frac{C_2}{|v|^2}
$$

which proves the theorem.

### 3 Integral equations

In this section, we will show that the integral operators $T_G$, $T_H$ and $T_{HF}$ are compact operators. After giving a proof in detail for $T_G$ in subsection 3.1, we give a sketch of proofs for $T_H$ and $T_{HF}$ in subsection 3.2 and 3.3 respectively.
3.1 Integral operator $T_G$

Consider the integral operator $T_G$:

$$(T_G f)(\lambda) = \int_{\mathbb{R}^n} f(\xi) G(\xi, \lambda) \, d\xi,$$

where

$$G(\xi, \lambda) = \int_{\mathbb{R}} |F(|U_0(t)\varphi_\lambda|^2)(\xi)|^2 \, dt$$

and $\varphi_\lambda(x) = \varphi((\lambda + 1)x)$.

Due to the Sobolev embedding theorem $H^k(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $2k > n$, in order to prove Theorem 1.3, it suffices to verify that $T_G$ is a compact operator from $L^\infty(\mathbb{R}^n)$ to $C(\Gamma)$.

**Theorem 3.1.** Let $\Gamma \subset \mathbb{R}$ be a compact set. Assume that $2 \leq n \leq 6$. Then for any $\varphi \in H^1(\mathbb{R}^n)$ the integral operator $T_G$ is a compact operator from $L^\infty(\mathbb{R}^n)$ to $C(\Gamma)$.

*Proof.* Due to the fact that $||u|^2 - |v|^2| \leq |u - v|^2 + 2(|u| + |v|)|u - v|$, we have

$$|(T_G f)(\lambda) - (T_G f)(\lambda')| \leq T_G^{(1)}(\lambda, \lambda') + 2T_G^{(2)}(\lambda, \lambda'),$$

where

$$T_G^{(1)}(\lambda, \lambda') = \int_{\mathbb{R}^n} |f(\xi)| \left( \int_{\mathbb{R}} |F(|U_0(t)\varphi_\lambda|^2)(\xi)| - |F(|U_0(t)\varphi_{\lambda'}|^2)(\xi)| \right)^2 \, d\xi,$$

$$T_G^{(2)}(\lambda, \lambda') = \int_{\mathbb{R}^n} |f(\xi)| \left( \int_{\mathbb{R}} \left| |F(|U_0(t)\varphi_\lambda|^2)(\xi)| - |F(|U_0(t)\varphi_{\lambda'}|^2)(\xi)| \right| \right)^2 \, d\xi.$$

Thanks to Lemma 2.3, one has

$$T_G^{(1)}(\lambda, \lambda') \leq ||f||_{L^\infty} \int_{\mathbb{R}} \left( |F(|U_0(t)\varphi_\lambda|^2 - |U_0(t)\varphi_{\lambda'}|^2)| \right)^2 \, dt$$

$$\leq ||f||_{L^\infty} \int_{\mathbb{R}} \left( ||U_0(t)\varphi_\lambda| + |U_0(t)\varphi_{\lambda'}|)(|U_0(t)\varphi_\lambda| - |U_0(t)\varphi_{\lambda'}|) \right)^2 \, dt$$

$$\leq ||f||_{L^\infty} \int_{\mathbb{R}} ||U_0(t)\varphi_\lambda| + |U_0(t)\varphi_{\lambda'}| ||^2 \, dt$$

$$\leq ||f||_{L^\infty} \left( ||U_0(t)\varphi_\lambda||^2_{L^2(\mathbb{R};L^4)} + ||U_0(t)\varphi_{\lambda'}||^2_{L^2(\mathbb{R};L^4)} \right)$$

$$\leq C||f||_{L^\infty} \left( ||\varphi_\lambda||^2_{H^1} + ||\varphi_{\lambda'}||^2_{H^1} \right)$$

$$\leq C||f||_{L^\infty} \lambda - \lambda'^2.$$
Similarly, for the function $T^{(2)}_G$, one gets

$$T^{(2)}_G(\lambda, \lambda') \leq \|f\|_{L^\infty} \int_\mathbb{R} \left\| \mathcal{F}(|U_0(t)\varphi_\lambda|^2) \right\|_{L^2} \left\| \mathcal{F}(|U_0(t)\varphi_\lambda|^2 - |U_0(t)\varphi_{\lambda'}|^2) \right\|_{L^2} dt$$

$$+ \|f\|_{L^\infty} \int_\mathbb{R} \left\| \mathcal{F}(|U_0(t)\varphi_{\lambda'}|^2) \right\|_{L^2} \left\| \mathcal{F}(|U_0(t)\varphi_{\lambda'}|^2 - |U_0(t)\varphi_\lambda|^2) \right\|_{L^2} dt$$

$$\leq \|f\|_{L^\infty} \int_\mathbb{R} \left( \|U_0(t)\varphi_\lambda\|_{L^4}^2 + \|U_0(t)\varphi_{\lambda'}\|_{L^4}^2 \right)$$

$$\|U_0(t)\varphi_\lambda| + |U_0(t)\varphi_{\lambda'}| \|_{L^4} \|U_0(t)\varphi_\lambda| - |U_0(t)\varphi_{\lambda'}| \|_{L^4} dt$$

$$\leq \|f\|_{L^\infty} \left( \|U_0(t)\varphi_\lambda\|_{L^4}^3 + \|U_0(t)\varphi_{\lambda'}\|_{L^4}^3 \right) \|U_0(t)(\varphi_\lambda - \varphi_{\lambda'})\|_{L^4}$$

$$\leq C\|f\|_{L^\infty} \left( \|\varphi_\lambda\|_{H^1}^3 + \|\varphi_{\lambda'}\|_{H^1}^3 \right) \|\varphi_\lambda - \varphi_{\lambda'}\|_{H^1}$$

Thus we obtain

$$|(T_G)(f)(\lambda) - (T_G)(f)(\lambda')| \leq C\|f\|_{L^\infty}|\lambda - \lambda'|$$

for $\lambda \in \Gamma$. This implies that $\{T_Gf\}$ is equicontinuous and equibounded for $\|f\|_{L^\infty}$. It therefore follows from the theorem Ascoli that $\{T_Gf\}$ contains a Cauchy subsequence in $C(\Gamma)$, which implies the integral operator $T_G$ is a compact operator from $L^\infty(\mathbb{R}^n)$ to $C(\Gamma)$. The proof is complete.

$\square$

### 3.2 Integral operator $T_H$

Consider the integral operator $T_H$:

$$(T_Hf)(\lambda) := \int_{\mathbb{R}^n} f(\xi)H(\xi, \lambda) \, d\xi.$$  \hspace{1cm} (3.1)

where

$$H(\xi, \lambda) = \sum_{k=1}^N \int_{\mathbb{R}} \mathcal{F} \left( \left| U_0(t)\varphi_\lambda^{(k)} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi_\lambda^{(j)} \right|^2 \right)(\xi) \, dt$$

and $\varphi_\lambda^{(j)}(x) = \varphi^{(j)}((\lambda + 1)x)$. 
Theorem 3.2. Let $\Gamma \subset \mathbb{R}$ be a compact set. Assume that $2 \leq n \leq 6$. Then for any $\varphi_j \in H^1(\mathbb{R}^n)$, $j = 1, \cdots, N$ the integral operator $T_H$ is a compact operator from $L^\infty(\mathbb{R}^n)$ to $C(\Gamma)$.

Proof. It is clear that

$$|H(\xi, \lambda) - H(\xi, \lambda')| \leq \sum_{k=1 \atop k \neq j}^N \left| \int_{\mathbb{R}} \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda'} \right|^2 \right)(\xi) \right| - \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda'} \right|^2 \right)(\xi) dt \right|$$

$$\leq \sum_{k=1 \atop k \neq j}^N (H^{(1)}(\lambda, \lambda') + H^{(2)}(\lambda, \lambda')),$$

where

$$H^{(1)}(\xi, \lambda, \lambda') = \int_{\mathbb{R}} \left| \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda} \right|^2 \right)(\xi) \right| \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda'} \right|^2 \right)(\xi) dt$$

$$H^{(2)}(\xi, \lambda, \lambda') = \int_{\mathbb{R}} \left| \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda} \right|^2 \right)(\xi) \right| \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda'} \right|^2 \right)(\xi) dt.$$

The same technique as in the estimate on $T^{(2)}_G(\lambda, \lambda')$ finds that

$$\int_{\mathbb{R}^n} |f(\xi)||H(\xi, \lambda) - H(\xi, \lambda')| d\xi$$

$$\leq \|f\|_{L^\infty} \sum_{k=1, k \neq j}^N \left\{ c_j \|\varphi^{(k)}_{\lambda} - \varphi^{(k)}_{\lambda'}\|_{H^1} + c_k \|\varphi^{(j)}_{\lambda} - \varphi^{(j)}_{\lambda'}\|_{H^1} \right\}$$

for some $c_j, c_k > 0$. This estimate implies that

$$|(T_H)(f)(\lambda) - (T_H f)(\lambda')| \leq C\|f\|_{L^\infty} |\lambda - \lambda'|$$
for $\lambda \in \Gamma$. Due to the same argument as in the subsection 3.1, the integral operator $T_H$ is the compact operator from $L^\infty(\mathbb{R}^n)$ to $C(\Gamma)$. The proof is complete. \hfill \Box

## 3.3 Integral operator $T_H F$

Consider the integral operator $T_H F$:

$$(T_H F f)(\lambda) := \int_{\mathbb{R}^n} f(\xi) H_{HF}(\xi, \lambda) \, d\xi.$$  \hfill (3.2)

where

$$H_{HF}(\xi, \lambda) = \sum_{k=1}^{N} \int_{\mathbb{R}} \mathcal{F}
\left(
\left|U_0(t) \varphi_{(k)}^{(j)}(\lambda)\right|^2
\right)(\xi) \mathcal{F}
\left(
\left|U_0(t) \varphi_{(j)}^{(k)}(\lambda)\right|^2
\right)(\xi) \, dt$$

$$- \sum_{k=1}^{N} \int_{\mathbb{R}} \mathcal{F}
\left(
\left|U_0(t) \varphi_{(j)}^{(k)}(\lambda)\right|^2
\right)(\xi) \, dt$$

and $\varphi_{\lambda}(x) = \varphi_{(j)}((\lambda + 1)x)$.

**Theorem 3.3.** Let $\Gamma \subset \mathbb{R}$ be a compact set. Assume that $2 \leq n \leq 6$. Then for any $\varphi_j \in H^1(\mathbb{R}^n)$, $j = 1, \cdots, N$ the integral operator $T_H F$ is a compact operator from $L^\infty(\mathbb{R}^n)$ to $C(\Gamma)$.

**Proof.** We write

$$H_{HF}(\xi, \lambda) - H_{HF}(\xi, \lambda') = \sum_{k=1}^{N} \left(H_{HF}^{(1)}(\xi, \lambda, \lambda') - H_{HF}^{(2)}(\xi, \lambda, \lambda')\right),$$
where
\[
H_{HF}^{(1)}(\xi, \lambda, \lambda') = \int_{\mathbb{R}} \left\{ \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda} \right|^2 \right)(\xi) - \mathcal{F} \left( \left| U_0(t)\varphi^{(k)}_{\lambda'} \right|^2 \right)(\xi) \mathcal{F} \left( \left| U_0(t)\varphi^{(j)}_{\lambda'} \right|^2 \right)(\xi) \right\} \, dt,
\]
\[
H_{HF}^{(2)}(\xi, \lambda, \lambda') = \int_{\mathbb{R}} \left\{ \left| \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda}} \right) (\xi) \right|^2 - \left| \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda'} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda'}} \right) (\xi) \right|^2 \right\} \, dt.
\]

Due to the fact that \( |H_{HF}^{(1)}| \leq H^{(1)} + H^{(2)} \), where \( H^{(1)} \) and \( H^{(2)} \) are defined in the proof of Theorem 3.2, we have
\[
\int_{\mathbb{R}^n} |H_{HF}^{(1)}(\xi, \lambda, \lambda')| \, d\xi \leq C |\lambda - \lambda'|
\]
for some \( C > 0 \) and for \( \lambda \in \Gamma \).

For \( H_{HF}^{(2)} \), thanks to the inequality \(|g|^2 - |f|^2 \leq |g - f|^2 + 2|f||g - f|\), we have
\[
|H_{HF}^{(2)}(\xi, \lambda, \lambda')| \leq \int_{\mathbb{R}} \left| \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda}} \right) (\xi) - \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda'} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda'}} \right) (\xi) \right|^2 \, dt
\]
\[
+ 2 \int_{\mathbb{R}} \left| \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda}} \right) (\xi) \right| \times
\]
\[
\times \left| \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda'} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda'}} \right) (\xi) - \mathcal{F} \left( \left( U_0(t)\varphi^{(j)}_{\lambda'} \right) \overline{U_0(t)\varphi^{(k)}_{\lambda'}} \right) (\xi) \right| \, dt.
\]
Thanks to Lemma 2.5, one gets
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}} \left| \mathcal{F} \left( \left( U_0(t) \varphi_{\lambda}^{(j)} \right) \left( U_0(t) \varphi_{\lambda'}^{(k)} \right) \right) - \mathcal{F} \left( \left( U_0(t) \varphi_{\lambda'}^{(j)} \right) \left( U_0(t) \varphi_{\lambda'}^{(k)} \right) \right) \right|^2 \, dt \, d\xi \\
\leq \int_{\mathbb{R}} \left\| \left( U_0(t) \varphi_{\lambda}^{(j)} \right) \left( U_0(t) \varphi_{\lambda'}^{(k)} \right) - \left( U_0(t) \varphi_{\lambda'}^{(j)} \right) \left( U_0(t) \varphi_{\lambda'}^{(k)} \right) \right\|^2_{L^2} \, dt \\
\leq \int_{\mathbb{R}} \left\| \left( U_0(t) \varphi_{\lambda}^{(k)} \right) \left\{ \left( U_0(t) \varphi_{\lambda}^{(j)} \right) - \left( U_0(t) \varphi_{\lambda'}^{(j)} \right) \right\} \right\|^2_{L^2} \, dt \\
+ \int_{\mathbb{R}} \left\| \left( U_0(t) \varphi_{\lambda'}^{(j)} \right) \left\{ \left( U_0(t) \varphi_{\lambda}^{(k)} \right) - \left( U_0(t) \varphi_{\lambda'}^{(k)} \right) \right\} \right\|^2_{L^2} \, dt \\
\leq C_1 \| \varphi_{\lambda'}^{(k)} \|^2_{H^1} \| \varphi_{\lambda}^{(j)} \|^2_{H^1} + C_2 \| \varphi_{\lambda'}^{(j)} \|^2_{H^1} \| \varphi_{\lambda}^{(k)} \|^2_{H^1} - \| \varphi_{\lambda'}^{(k)} \|^2_{H^1} \\
\leq C | \lambda - \lambda'| 
\] 
for some $C > 0$ and $\lambda \in \Gamma$. In the same way as in the proof of Theorem 3.1.
we also obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}} \mathcal{F} \left( \left( U_0(t) \varphi^{(j)}_\lambda \right) \left( U_0(t) \varphi^{(k)}_{\lambda'} \right) \right) \times \\
\mathcal{F} \left( \left( U_0(t) \varphi^{(j)}_\lambda \right) \left( U_0(t) \varphi^{(k)}_{\lambda'} \right) \right) - \mathcal{F} \left( \left( U_0(t) \varphi^{(j)}_\lambda \right) \left( U_0(t) \varphi^{(k)}_{\lambda'} \right) \right) \ dt \, d\xi
\]
\[
\leq \int_{\mathbb{R}} \left\| \left( U_0(t) \varphi^{(j)}_\lambda \right) \left( U_0(t) \varphi^{(k)}_{\lambda'} \right) \right\|_{L^2} \times \\
\left\| \left( U_0(t) \varphi^{(j)}_\lambda \right) \left( U_0(t) \varphi^{(k)}_{\lambda'} \right) - \left( U_0(t) \varphi^{(j)}_\lambda \right) \left( U_0(t) \varphi^{(k)}_{\lambda'} \right) \right\|_{L^2} \ dt
\]
\[
\leq \int_{\mathbb{R}} \left\{ \left\| U_0(t) \varphi^{(j)}_\lambda \right\|^2_{L^2} + \left\| U_0(t) \varphi^{(k)}_{\lambda'} \right\|^2_{L^2} \right\} \times \\
\left\{ \left\| U_0(t) \varphi^{(j)}_\lambda \right\|_{L^4} \left\| U_0(t) \varphi^{(k)}_{\lambda'} - U_0(t) \varphi^{(k)}_{\lambda'} \right\|_{L^4} \right. \\
+ \left\| U_0(t) \varphi^{(k)}_{\lambda'} \right\|_{L^4} \left\| U_0(t) \varphi^{(j)}_\lambda - U_0(t) \varphi^{(j)}_{\lambda'} \right\|_{L^4} \right\} \ dt
\]
\[
\leq \int_{\mathbb{R}} \left( \left\| U_0(t) \varphi^{(j)}_\lambda \right\|_{L^4} + \left\| U_0(t) \varphi^{(k)}_{\lambda'} \right\|_{L^4} \right)^3 \times \\
\left( \left\| U_0(t) \varphi^{(k)}_{\lambda} - U_0(t) \varphi^{(k)}_{\lambda'} \right\|_{L^4} + \left\| U_0(t) \varphi^{(j)}_\lambda - U_0(t) \varphi^{(j)}_{\lambda'} \right\|_{L^4} \right) \ dt
\]
\[
\leq \left( \left\| U_0(t) \varphi^{(j)}_\lambda \right\|_{L^4}^3 + \left\| U_0(t) \varphi^{(k)}_{\lambda'} \right\|_{L^4}^3 \right) \times \\
\left( \left\| U_0(t) \varphi^{(k)}_{\lambda} - U_0(t) \varphi^{(k)}_{\lambda'} \right\|_{L^4} + \left\| U_0(t) \varphi^{(j)}_\lambda - U_0(t) \varphi^{(j)}_{\lambda'} \right\|_{L^4} \right) \right)
\]
\[
\leq C \left( \left\| \varphi^{(j)}_\lambda \right\|_{H^1}^3 + \left\| \varphi^{(k)}_{\lambda'} \right\|_{H^1}^3 \right) \left( \left\| \varphi^{(k)}_{\lambda} - \varphi^{(k)}_{\lambda'} \right\|_{H^1} + \left\| \varphi^{(j)}_\lambda - \varphi^{(j)}_{\lambda'} \right\|_{H^1} \right)
\]
\[
\leq C |\lambda - \lambda'|
\]
for some $C > 0$ and $\lambda \in \Gamma$. 

Consequently, we obtain
\[ \int_{\mathbb{R}^n} \left| H^{(2)}(\xi, \lambda, \lambda') \right| d\xi \leq C|\lambda - \lambda'| \]
for some \( C > 0 \) and \( \lambda \in \Gamma \). Hence one gets
\[ |(T_{HF}f)(\lambda) - (T_{HF}f)(\lambda')| \leq \|f\|_{L^\infty} \int_{\mathbb{R}^n} \left| H_{HF}(\xi, \lambda) - H_{HF}(\xi, \lambda') \right| d\xi \]
\[ \leq \|f\|_{L^\infty} \sum_{k=1}^{N} \int_{\mathbb{R}^n} \left| H^{(1)}_{HF}(\xi, \lambda, \lambda') \right| + \left| H^{(2)}_{HF}(\xi, \lambda, \lambda') \right| d\xi \]
\[ \leq C\|f\|_{L^\infty}|\lambda - \lambda'| \]
for some \( C > 0 \) and \( \lambda \in \Gamma \). This implies that the operator \( T_{HF} \) is a compact operator from \( L^\infty(\mathbb{R}^n) \) to \( C(\Gamma) \), due to the same argument as in the proof of Theorem 3.1. The proof is complete. □

4 Uniqueness

In this section, we prove the following uniqueness theorem in the inverse scattering problem for the HF equation (1.2).

**Theorem 4.1.** Let \( 2 \leq n \leq 6 \). Assume that \( V_{\sharp} \), \( \sharp = 1, 2 \) satisfy the assumption in Theorem 1.11. Let \( S_\sharp \) are the scattering operator for the HF equation (1.2) with interaction potential \( V_\sharp \). If \( S_1 = S_2 \), then we have \( V_1 = V_2 \).

4.1 Lemmas

In order to prove theorem 4.1, we need some lemmas. Set
\[ (G_1\varphi^{(k)})(t, \xi) = \mathcal{F} \left( |e^{-itH_0\varphi^{(k)}}|^2 \right)(\xi), \]
\[ (G_2[\varphi^{(j)}, \varphi^{(k)}])(t, \xi) = \mathcal{F} \left( e^{-itH_0\varphi^{(j)}}e^{-itH_0\varphi^{(k)}} \right)(\xi) \]
and put \( B_R(a) = \{ x \in \mathbb{R}^n ; |x - a| < R \} \).
Lemma 4.1. Let $\varepsilon > 0$ and $p_k \in \mathbb{R}^n$ such that $\cap_{k=1}^N B_\varepsilon(p_k) = \emptyset$. Then for any $\varphi^{(k)} \in S_0$ with $\text{supp}_{\xi} \hat{\varphi}^{(k)}(\xi) \subset B_\varepsilon(p_k)$, $k = 1, 2, \cdots N$, we have

$$
(G_1 \varphi^{(k)})(t, \xi)(G_1 \varphi^{(j)})(t, \xi) = 0, \quad k \neq j,
$$
on $(t, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Proof. Due to the fact that $|\hat{f}|^2 = \hat{f} \ast \hat{f}$, we have

$$
(G_1 \varphi^{(k)})(t, \xi) = e^{-it\xi^2} \hat{\varphi}^{(k)}(\xi) = e^{it\xi^2} \hat{\varphi}^{(k)}(\xi) = \int_{\mathbb{R}^n} e^{-it(\xi - \eta)^2} \hat{\varphi}^{(k)}(\xi - \eta) e^{it\eta^2} \hat{\varphi}^{(k)}(\eta) d\eta
$$

$$
= \int_{\mathbb{R}^n} e^{-it\xi^2} e^{2it\xi \cdot \eta} \hat{\varphi}^{(k)}(\xi - \eta) \hat{\varphi}^{(k)}(\eta) d\eta. \quad (4.1)
$$

Note that if $\text{supp}_\eta \hat{\varphi}^{(k)}(\eta) \subset B_\varepsilon(p_k)$, then $\text{supp}_\xi \hat{\varphi}^{(k)}(\xi - \eta) \hat{\varphi}^{(k)}(\eta) \subset B_{2\varepsilon}(2p_k)$ for each $\eta \in B_\varepsilon(p_k)$. It is also clear that if $\cap_{k=1}^N B_\varepsilon(p_k) = \emptyset$, then $\cap_{k=1}^N B_{2\varepsilon}(2p_k) = \emptyset$. Thus, one has

$$
\cap_{k=1}^N \text{supp}_{\xi} \left( \hat{\varphi}^{(k)}(\xi - \eta) \hat{\varphi}^{(k)}(\eta) \right) = \emptyset
$$

for $\eta \in B_\varepsilon(p_k)$, which implies that $\text{supp}_{(t, \xi)} G_1 \varphi^{(k)} \cap \text{supp}_{(t, \xi)} G_1 \varphi^{(j)} = \emptyset$, due to the identity (4.1), we complete the proof. \qed

Lemma 4.2. Let $\varepsilon > 0$ and $p_\ell \in \mathbb{R}^n$, $\ell = 1, 2, \cdots, N$ such that $\cap_{\ell=1}^N B_\varepsilon(p_\ell) = \emptyset$. Then for any $\varphi^{(\ell)} \in S_0$ with $\text{supp}_{\xi} \hat{\varphi}^{(\ell)}(\xi) \subset B_\varepsilon(p_\ell)$, we have

$$
\text{supp}(G_2[\varphi^{(j)}, \varphi^{(k)}])_{(t, \xi)} \subset \mathbb{R} \times B_{2\varepsilon}(p_j + p_k), \quad 1 \leq j \neq k \leq N.
$$

Proof. It is clear that

$$
(G_2[\varphi^{(j)}, \varphi^{(k)}])(t, \xi) = e^{-it\xi^2} \int_{\mathbb{R}^n} e^{2it\xi \cdot \eta} \hat{\varphi}^{(j)}(\xi - \eta) \hat{\varphi}^{(k)}(\eta) d\eta.
$$

In view of the assumption $\text{supp}_{\xi} \hat{\varphi}^{(b)}(\xi) \subset B_\varepsilon(p_b)$, $b = j, k$, we have

$$
\text{supp}_{\xi} \hat{\varphi}^{(j)}(\xi - \eta) \hat{\varphi}^{(k)}(\eta) \subset B_{2\varepsilon}(p_j + p_k),
$$

which proves Lemma 4.2. \qed
4.2 Proof of Theorem 4.1

We are now in a position to prove Theorem 4.1. Let

\[ w(\xi) = \hat{V}_1(\xi) - \hat{V}_2(\xi). \]

By virtue of Theorem 1.11 and assumption \( S_1 = S_2 \), one has

\[
0 = \int_{\mathbb{R}^n} w(\xi) H^{(j)}(\xi) d\xi \\
= \int_{\mathbb{R}^n} w(\xi) \sum_{k=1}^{N} \int_{\mathbb{R}} (G_1\varphi^{(k)})(t, \xi)(G_1\varphi^{(j)})(t, \xi) dt \\
- \int_{\mathbb{R}^n} w(\xi) \sum_{k=1}^{N} \int_{\mathbb{R}} |(G_2[\varphi^{(k)}, \varphi^{(j)}])(t, \xi)|^2 dt
\]

for any \( \varphi^{(\ell)} \in S_0, \ell = 1, 2, \ldots, N \). Taking \( \varphi^{(\ell)} \in S_0 \) with \( \text{supp}_{\xi} \varphi^{(\ell)}(\xi) \subset B_\varepsilon(p_\ell) \) as in Lemma 4.1, we have

\[
\int_{\mathbb{R}^n} w(\xi) \sum_{k=1}^{N} \int_{\mathbb{R}} (G_1\varphi^{(k)})(t, \xi)(\overline{G_1\varphi^{(j)}})(t, \xi) dt = 0.
\]

This implies that for such functions \( \varphi^{(b)}, b = j, k \)

\[
\int_{\mathbb{R}^n} \text{Re}(w(\xi)) \sum_{k=1}^{N} \int_{\mathbb{R}} |(G_2[\varphi^{(k)}, \varphi^{(j)}])(t, \xi)|^2 dt = 0, \quad (4.2)
\]

\[
\int_{\mathbb{R}^n} \text{Im}(w(\xi)) \sum_{k=1}^{N} \int_{\mathbb{R}} |(G_2[\varphi^{(k)}, \varphi^{(j)}])(t, \xi)|^2 dt = 0,
\]

where \( \text{Re}(w(\xi)) \) and \( \text{Im}(w(\xi)) \) denote the real part and the imaginary part of the complex-valued function \( w(\xi) \), respectively.

Assume that for \( m > n/2, H^m(\mathbb{R}^n) \ni w \neq 0 \) on \( \mathbb{R}^n \). Due to the Sobolev embedding theorem, we find \( w \in C^{m-[n/2]-1,\gamma}(\mathbb{R}^n) \) for some \( 0 < \gamma < 1 \). Here \( C^{m,\gamma}(\mathbb{R}^n) \) denotes the set of H"older continuous functions with exponent \( \gamma \). Therefore, there exist \( \delta > 0 \) and \( p \in \mathbb{R}^n \) such that \( \text{Re}(w(\xi)) > 0 \) on \( B_\delta(p) \).

For \( \varepsilon > 0 \) sufficiently small and fixed \( p_j \in \mathbb{R}^n \), we define \( p_\ell, \ell = 1, 2, \ldots, N \) so that \( p_\ell \neq p_j, 1 \leq \ell, j \leq N \) and

\[
\bigcup_{\ell=1,\ell\neq j}^{N} B_{2\varepsilon}(p_\ell + p_j) \subset B_\delta(p).
\]
Then, thanks to Lemma 4.2, there exist $\varphi^{(b)} \in S_{0}$, $b = j, k$ such that the function $\sum_{k=1}^{N} (\int |G_{2}[\varphi^{(k)}, \varphi^{(j)}]|^2 dt)(\xi)$ is a non-negative function with the support in $B_{3}(p)$. This and the positivity $\text{Re}(w(\xi)) > 0$ on $B_{3}(p)$ show that the integral of the left hand side in the identity (4.2) never vanishes for such functions $\varphi^{(b)}$. This contradicts the identity (4.2). Thus, we conclude that $w \equiv 0$ on $\mathbb{R}^{n}$, which proves Theorem 4.1.

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