MOND as the weak field limit of an extended metric theory of gravity with torsion

E. Barrientos and S. Mendoza

Instituto de Astronomía, Universidad Nacional Autónoma de México, AP 70-264, Ciudad de México 04510, Mexico

Received: 22 March 2017 / Revised: 6 June 2017 Published online: 25 August 2017 – © Società Italiana di Fisica / Springer-Verlag 2017

Abstract. In this article we construct a relativistic extended metric theory of gravity, for which its weak field limit reduces to the non-relativistic MOdified Newtonian Dynamics regime of gravity. The theory is fully covariant and local. The way to achieve this is by introducing torsion in the description of gravity as well as with the addition of a particular function of the matter Lagrangian into the gravitational action.

1 Introduction

In 1983 Milgrom proposed MOND (MOdified Newtonian Dynamics), a theory that introduced a modification of Newton’s second law in order to explain the flattening of rotation curves in disc galaxies [1,2]. From this empirical proposal it is possible to recover the baryonic Tully-Fisher relation, for which the rotation velocity \( V \) scales as a power of the baryonic mass \( M \) (composed of stars and dust) in the following form: \( V \propto M^{1/4} \).

Although in principle, the Tully-Fisher relation was found for disc galaxies, recent surveys have proven that it holds in dwarf spheroidal galaxies, wide binaries and globular clusters [3–6]. Moreover, the astrophysical observations strongly suggest that MONDian gravity accurately describes pressure supported systems across 12 order of magnitude in mass [7].

Despite the success that MOND has at the phenomenological level, that formulation is non-relativistic. From a mathematical point of view, MOND should be conceived as the weak field limit of a relativistic proposal. Several attempts have been done towards building a relativistic version of MOND. Amongst the many proposals in this direction we can name the Tensor-Vector-Scalar theories [8–13], galileons [14], bimetric theories [15], non-local theories [16], modified energies [17] and field theories [18] to name a few.

Bernal et al. [19] built a relativistic proposal for MOND in the pure metric formalism. This theory is based on a dimensionally correct action for a \( f(\chi) \) function, where \( \chi \) is a dimensionless Ricci scalar defined as \( \chi = \frac{L^2}{M}R \), and \( L_M \) is a free coupling parameter of the theory with length dimensions, which is fixed by recovering MOND in the weak field limit. Taking this work as starting point Barrientos and Mendoza [20] analysed the previous proposal but in the Palatini framework. Both, metric and Palatini formalism yield \( \chi^{3/2} \) as the function that turns into MOND on its weak field limit. This value is coincident with the results obtained in some cosmological analysis [21,22].

The \( f(\chi) \) theory explains not only the flattening of rotation curves, but the correct bending angle of light for gravitational lensing in individual, groups and clusters of galaxies [23]. However, this proposal possesses a mathematical inconvenient since the coupling constant \( L_M \) has an explicit mass dependence and so, it makes this proposal non-local (there is however a mathematical way to deal with this caveat as explained by Carranza, Mendoza, and Torres [24], Mendoza [25]). As such, the \( f(\chi) \) action must not be seen as a complete theory but as a particular case of a more general idea.

In this article, we introduce a relativistic action, which in its weak field limit reduces to MOND, but unlike the \( f(\chi) \) theory, the coupling constants have exclusive dependence in pure physical constants: Newton’s gravitational constant \( G \), the speed of light \( c \) and Milgrom’s acceleration constant \( a_0 \), making the action entirely covariant and local. This theory has two departures with respect to general relativity. On the one hand, in the geometrical sector, we work with a \( f(R) \) theory with torsion. From the cosmological point of view, it has been proven that the torsion has interesting
implications in order to explain the accelerated expansion of the universe [26,27]. Our approach in this work is to find a MONDian behaviour in extended metric theories of gravity with torsion. On the other hand, based on the $f(\Sigma)$ and $f(L_m)$ theories [28–30], where $\Sigma$ is the trace of the energy momentum tensor $\Sigma_{\mu\nu}$ and $L_m$ is the matter Lagrangian, we also modify the matter sector with an action which, for this particular case, is only dependent on the derivatives of the matter Lagrangian.

The article is organised as follows. Section 2 introduces some of the theoretical background needed for torsion and for the weak field limit of a general metric theory of gravity. In sect. 3 we present our preliminary attempts which yield the correct MONDian proposal described in sect. 4. Finally, in sect. 5 we discuss our results.

2 Background information

Before dealing with our action proposals, we first introduce some of the mathematical concepts which we will use throughout our work. The reader is referred to the extensive reviews of Hehl [31], Hehl et al. [32] and the summaries of Capozziello et al. [26,27] for further information. As we are interested in a general scenario where there exists two fundamental variables, the metric $g_{\mu\nu}$, and a priori non-symmetric connection $\Gamma^\lambda_{\mu\nu}$, let us start defining the torsion tensor $S^\lambda_{\mu\nu}$ as

$$S^\lambda_{\mu\nu} := \frac{1}{2} (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}).$$

(1)

If we demand that this connection holds the metric compatibility $\nabla g_{\mu\nu} = 0$, then it is possible to relate it with the Levi-Civita connection $\{\}^\lambda_{\mu\nu}$ of the metric $g_{\mu\nu}$, through the following expression:

$$\Gamma^\lambda_{\mu\nu} = \{\} - K^\lambda_{\mu\nu},$$

(2)

where the contorsion tensor $K^\lambda_{\mu\nu}$ is given by [26]

$$K^\lambda_{\mu\nu} := -S^\lambda_{\mu\nu} + S^\lambda_{\mu\nu} - S^\lambda_{\nu\mu}.$$

(3)

The Riemann tensor is a geometric quantity defined entirely in terms of a general connection by

$$R^\alpha_{\epsilon\mu\nu} := \partial_{\epsilon} \Gamma^\alpha_{\mu\nu} - \partial_{\mu} \Gamma^\alpha_{\epsilon\nu} + \Gamma^\sigma_{\epsilon\nu} \Gamma^\alpha_{\sigma\mu} - \Gamma^\sigma_{\epsilon\mu} \Gamma^\alpha_{\sigma\nu}.$$

(4)

The substitution of eq. (2) into the previous equation yields a relation between the general Riemann tensor $R^\alpha_{\epsilon\mu\nu}$ and the standard Riemann tensor built exclusively in terms of the Levi-Civita connection $R^\alpha_{\epsilon\mu\nu}(\{\})$:

$$R^\alpha_{\epsilon\mu\nu} = R^\alpha_{\epsilon\mu\nu}(\{\}) + \tilde{\nabla}_\epsilon K^\alpha_{\mu\nu} - \tilde{\nabla}_\mu K^\alpha_{\epsilon\nu} + K^\sigma_{\nu\epsilon} K^\alpha_{\sigma\mu} - K^\sigma_{\epsilon\mu} K^\alpha_{\sigma\nu}.$$

(5)

The Ricci tensor is defined by the contraction of the first and third index: $R_{\mu\nu} := R^\alpha_{\mu\alpha\nu}$. Performing this contraction in eq. (5), yields to

$$R_{\mu\nu} = R_{\mu\nu}(\{\}) + \tilde{\nabla}_\mu K^\alpha_{\alpha\nu} - \tilde{\nabla}_\nu K^\alpha_{\alpha\mu} + K^\sigma_{\nu\mu} K^\alpha_{\sigma\alpha} - K^\sigma_{\mu\alpha} K^\alpha_{\sigma\nu},$$

(6)

where $\tilde{\nabla}$ is the covariant derivative defined in terms of the Levi-Civita connection only.

Sometimes, instead of working with the torsion tensor, it is useful to express the results in terms of the torsion’s contraction. In order to simplify the notation we define the following tensor:

$$T^\lambda_{\mu\nu} := S^\lambda_{\mu\nu} + \delta^\lambda_{\mu} S_{\nu} - \delta^\lambda_{\nu} S_{\mu},$$

(7)

called the modified torsion tensor, and where $S_{\mu} := S^\lambda_{\mu\lambda}$.

In this work, we use a simplified torsion term which is only vectorial as described in the work of Capozziello, Lambiase, and Stornaioloi [33]. For this particular kind of torsion, the Ricci tensor is given by

$$R = R(\{\}) - 2\tilde{\nabla}_\alpha T^\alpha - \frac{2}{3} T^\alpha T_{\alpha}. $$

(8)

Since we are assuming the existence of torsion, there are two main differences when performing the variations and the use of Gauss’ theorem when integrating by parts, as compared to the purely metric formalism. Such differences are expressed in the following equations:

$$\delta R_{\mu\nu} = \tilde{\nabla}_\alpha \delta \Gamma^\alpha_{\nu\mu} - \tilde{\nabla}_\nu \delta \Gamma^\alpha_{\alpha\mu} + 2 S^\alpha_{\lambda\nu} \delta \Gamma^\lambda_{\alpha\mu},$$

(9)
and
\[ \nabla_\mu W^\mu = \partial_\mu W^\mu + 2 {\cal S}^\mu_{\ \mu \nu} W^\nu, \]  
(10)
where \( W^\mu \) is a density tensor of weight +1.

Also, since we are interested in the non-relativistic weak field limit for our proposals, the metric is expanded as a Minkowskian background plus a small perturbation. The perturbations are given in factor terms of order 1/c. For the purposes of this work, a second-order perturbation will be enough, since it is sufficient to explain the motion of matter and light particles at the non-relativistic level [34]. Taking as base the work of [35], the metric coefficients at second perturbation order are given by
\[ g_{00} = (0) g_{00} + (2) g_{00} = 1 + \frac{2\phi}{c^2}, \]
\[ g_{ij} = (0) g_{ij} + (2) g_{ij} = \delta_{ij} \left(-1 + \frac{2\phi}{c^2}\right), \]
\[ g_{0i} = 0. \]  
(11)

3 Warming up attempts

3.1 \( f(\{}\{, T)\)

Let us now make the assumption that the MONDian behavior of gravity is a physical effect due to the existence of torsion. The way to express this assumption is by the addition of torsion terms to the Hilbert action. Using eq. (8) as base, we propose the following action:
\[ S_2 = \frac{c^3}{16\pi G L_M} \int \sqrt{-g} \left[ R(\{}\{ + \kappa \left( \tilde{\nabla}_a T^a + T^a T_a \right)^b \right] \right] d^4x \]
\[ + \frac{1}{c} \int \sqrt{-g} L_m d^4x, \]  
(12)

where \( \kappa \) is a coupling constant. In this case the null variations are calculated with respect to the metric \( g_{\mu\nu} \) and the modified torsion tensor \( T^a \). The field equations derivated from the action (12) are
\[ R_{\mu\nu} \{ = \frac{1}{2} g_{\mu\nu} R \} - \frac{1}{2} g_{\mu\nu} \kappa (\tilde{\nabla}_a T^a + T^a T_a)^b \]
\[ + \kappa b (\tilde{\nabla}_a T^a + T^a T_a)^b - 1 T_\mu T_\nu \]
\[ - \kappa b T_\nu \tilde{\nabla}_\mu \left[ (\tilde{\nabla}_a T^a + T^a T_a)^b - 1 \right] \]
\[ = \frac{8\pi G}{c^4} \Sigma_{\mu\nu}, \]  
(13)

for the null variations with respect to the metric, and
\[ 2 T_\mu (\tilde{\nabla}_a T^a + T^a T_a)^b - 1 = \tilde{\nabla}_\mu \left[ (\tilde{\nabla}_a T^a + T^a T_a)^b - 1 \right], \]  
(14)

for the null variations with respect to the modified torsion. Equation (14) is a differential equation for the torsion \( T^a \) and it can be substituted into (13), yielding a single field equation:
\[ R_{\mu\nu} \{ = \frac{1}{2} g_{\mu\nu} R \} - \frac{1}{2} g_{\mu\nu} \kappa (\tilde{\nabla}_a T^a + T^a T_a)^b \]
\[ - \kappa b (\tilde{\nabla}_a T^a + T^a T_a)^b - 1 T_\mu T_\nu = \frac{8\pi G}{c^4} \Sigma_{\mu\nu}. \]  
(15)

From the latter equation we conclude that a relation between \( R(\{}\{ \) and \( \Sigma \) is not possible because eq. (14) is a differential equation involving only \( T \), and not \( L_m \).

Thus, in order to continue analysing this proposal, we need to make an extra assumption for the functional relation between \( T \) and \( \Sigma \) (see footnote\(^1\)). Let us assume the following:
\[ \tilde{\nabla}_a T^a = 0, \quad \text{and} \quad T_a = \kappa' \tilde{\nabla}_a \Sigma, \]  
(16)
\(^1\) By making this assumption, we are introducing additional information to the proposal, which makes it somewhat inviable, but it will give us a good idea on to the correct path to follow.
where $\kappa'$ is a constant of proportionality. At first view, it seems that these assumptions are very arbitrary, but the first one is for simplicity and the second one is based on an order of magnitude analysis that will recover MONDian acceleration as will be further discussed.

With eqs. (16), expression (15) takes the following form:

$$R_{\mu
u\lambda} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}g_{\mu\lambda}\kappa \kappa' (\partial_{\alpha} \Sigma \partial^{\alpha} \Sigma)^b + \kappa b \kappa' (\partial_{\alpha} \Sigma \partial^{\alpha} \Sigma)^{b-1} \partial_{\mu} \Sigma \partial_{\nu} \Sigma,$$

where we have changed $\tilde{\nabla}$ by $\partial_{\alpha}$ and dropped the Newtonian-like $4\pi G \rho$ term since we are only interested in the MONDian regime of gravity. Contracting the previous equation and substituting the trace of the energy-momentum for dust, we obtain:

$$-R = (b + 2) \kappa \kappa' c^{4b} (\partial_{\alpha} \rho \partial^{\alpha} \rho)^b.$$  

(17)

So far, we have not said anything about the constants $\kappa$ and $\kappa'$. Due to this freedom, we propose the following constraint:

$$\kappa \kappa' \approx \frac{1}{c^2}.$$  

(19)

This assumption implies that to second-order perturbation, the term in parenthesis in eq. (18) is a zeroth-order term. For the metric (11), the Ricci scalar at second perturbation order and the term involving the matter density are, respectively, given by

$$R = -\frac{2\nabla^2 \phi}{c^2} \quad \text{and} \quad \partial_{\alpha} \rho \partial^{\alpha} \rho = -\nabla \rho \cdot \nabla \rho.$$  

(20)

Thus, eq. (18) to second perturbation order is

$$-\nabla \cdot \mathbf{a} = (b + 2)(-1)^b \kappa \kappa' c^{2(2b+1)} (\nabla \rho \cdot \nabla \rho)^b,$$

for the acceleration $\mathbf{a} = -\nabla \phi$.

Up to the order of magnitude $\rho \approx M/r^3$ and $\nabla \approx 1/r$ and so on, the previous equation is

$$a \approx \kappa \kappa' c^{2(2b+1)} M^{2b} r^{-8b}.$$  

(22)

The MONDian acceleration has a $r^{-1}$ dependence. In order to obtain that, the parameter

$$b = \frac{1}{4}.$$  

(23)

With this value, the acceleration (22) is given by

$$a \approx \kappa \kappa' c^{1/2} M^{1/2} r^{-1}.$$  

(24)

The previous equation is important to our analysis. We have already obtained the correct dependence on $M$ and $r$ of the MONDian acceleration. Therefore, the constants $\kappa$ and $\kappa'$ depend exclusively on $c$, $a_0$ and $G$ in the following form:

$$\kappa \kappa' \approx \left(\frac{a_0 G}{c^3}\right)^{1/2}.$$  

(25)

This approach represents an entirely local and covariant relativistic formulation of MOND. However, it cannot be an option to obtain a correct relativistic formulation of MOND because the assumptions (16) have no physical or mathematical support. Despite this, the proposal gives us some clues towards the correct path to follow in order to enhance our theory.

### 3.2 $f(R\{\}, \tilde{\nabla}_\mu L_m)$

The next logical step in order to construct a relativistic formulation of MOND consist in substituting the assumptions (16) on the action (12). As such, we propose the following action:

$$S_3 = \frac{16\pi G}{c^3} \int \sqrt{-g} \left[ R + \lambda \tilde{\nabla}_\mu \left( L_m \tilde{\nabla}_\nu L_m \right) \right] d^4x,$$

(26)
where $\lambda$ is a coupling constant. This formulation, unlike the two previous ones, has only the metric as a dynamical variable. The field equations obtained from the null variations of the previous action with respect to the metric are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \lambda \left[ \tilde{\nabla}_a \left( \mathcal{L}_m \tilde{\nabla}^a \mathcal{L}_m \right) \right]^\gamma$$

$$- \gamma \lambda \left[ \tilde{\nabla}_a \left( \mathcal{L}_m \tilde{\nabla}^a \mathcal{L}_m \right) \right]^{\gamma - 1} \tilde{\nabla}_\mu \left( \mathcal{L}_m \tilde{\nabla}_\nu \mathcal{L}_m \right)$$

$$- \frac{\gamma}{2} \lambda \left( \mathcal{L}_m g_{\mu\nu} - \Sigma_{\mu\nu} \right) \mathcal{L}_m \tilde{\Delta} \left[ \left[ \tilde{\nabla}_a \left( \mathcal{L}_m \tilde{\nabla}^a \mathcal{L}_m \right) \right]^{\gamma - 1} \right],$$

(27)

where the Laplace-Beltrami operator $\Delta := \nabla_\mu \nabla^\mu$. Contracting the latter expression with the metric $g^{\mu\nu}$ yields

$$-R \{ \} = -\lambda (\gamma - 2) \left[ \tilde{\nabla}_a \left( \mathcal{L}_m \tilde{\nabla}^a \mathcal{L}_m \right) \right]^\gamma$$

$$- \frac{\gamma}{2} \lambda \left( 4 \mathcal{L}_m - \Sigma \right) \mathcal{L}_m \tilde{\Delta} \left[ \left[ \tilde{\nabla}_a \left( \mathcal{L}_m \tilde{\nabla}^a \mathcal{L}_m \right) \right]^{\gamma - 1} \right].$$

(28)

For the case of dust, the previous equation yields

$$R \{ \} = \lambda (\gamma - 2) e^{4\gamma} \left[ \tilde{\nabla}_a \left( \rho \tilde{\nabla}^a \rho \right) \right]^\gamma$$

$$+ \frac{3}{2} \gamma \lambda e^{4\gamma} \rho^2 \tilde{\Delta} \left[ \left[ \tilde{\nabla}_a \left( \rho \tilde{\nabla}^a \rho \right) \right]^{\gamma - 1} \right].$$

(29)

In order not to obtain a dependence on the speed of light at second perturbation order on the terms between parenthesis in the previous equation, it is required that

$$\lambda e^{4\gamma} \approx \frac{1}{c^2}. \quad (30)$$

At the same perturbation order, the terms involving $\rho$ are of the zeroth order. Using the metric (11), such terms are

$$\tilde{\nabla}_a \left( \rho \tilde{\nabla}^a \rho \right) = -\nabla \cdot (\rho \nabla \rho) \quad \text{and} \quad \tilde{\Delta} \psi = -\nabla^2 \psi. \quad (31)$$

The direct substitution of these last two expressions and relation (20) in eq. (29) yields

$$\frac{2 \nabla^2 \phi}{c^2} = -\lambda (\gamma - 2) e^{4\gamma} (-1)^\gamma \left[ \nabla \cdot (\rho \nabla \rho) \right]^\gamma$$

$$- \frac{3}{2} \gamma \lambda e^{4\gamma} (-1)^\gamma \rho^2 \nabla^2 \left[ \nabla \cdot (\rho \nabla \rho) \right]^{\gamma - 1}. \quad (32)$$

Based on the results of sect. 3.1, particularly on the ones in eqs. (23) and (25), we take the following values:

$$\gamma = \frac{1}{4} \quad \text{and} \quad \lambda = \zeta \frac{(a_0 G)^{1/2}}{c^4}, \quad (33)$$

in order to obtain the following formula for the acceleration (given by eq. (32)):

$$-\nabla \cdot a = \frac{(-1)^{1/4}}{4} \zeta (a_0 G)^{1/2} \left[ \frac{1}{2} \nabla^2 \rho^2 \right]^{1/4}$$

$$- \frac{3}{4} \rho^2 \nabla^2 \left[ \left( \frac{1}{2} \nabla^2 \rho^2 \right)^{-3/4} \right]. \quad (34)$$

An order-of-magnitude calculation of the previous equation yields

$$a \approx \frac{(a_0 GM)}{r}, \quad (35)$$

which is the right MONDian dependence for acceleration. For completeness, we must adjust the numerical value of $\zeta$. This is accomplished solving analytically eq. (34), but this expression is very complicated to handle and so, we will not solve eq. (34) directly. Instead in the following section we put together what we have learnt from sects. 3.1 and 3.2, in order to build a theory which in its weakest field limit yields a Poisson-like equation less complicated than the one of (34).
4 The final proposal

4.1 Field equations

With all the knowledge acquired from the previous attempts, let us start with the following action:

\[ S_4 = \omega \int \sqrt{-g} f(R) \, d^4x \]

\[ + \omega' \int \sqrt{-g} \left[ A \left( \nabla_\mu \mathcal{L}_m \nabla^\mu \mathcal{L}_m \right) \eta + B \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta \right] \, d^4x, \]

where \( \omega \) and \( \omega' \) are the action’s coupling constants. Since the action is a \( f(R) \) function, there are two variables again, the connection (via torsion) and the metric. The resulting field equations are

\[ \omega \left( f' R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f' \right) = \frac{1}{2} g_{\mu\nu} \omega' \left[ A \left( \nabla_\mu \mathcal{L}_m \nabla^\mu \mathcal{L}_m \right) \eta + B \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta \right] \]

\[ + A \omega' \eta \left( \mathcal{L}_m \nabla_\mu \mathcal{L}_m \right) \eta^{-1} \nabla_\mu \mathcal{L}_m \nabla_\nu \mathcal{L}_m \]

\[ + A \omega' \eta \left( \mathcal{L}_m g_{\mu\nu} - \Sigma_{\mu\nu} \right) \nabla_\mu \mathcal{L}_m \eta^{-1} \nabla_\nu \mathcal{L}_m \]

\[ = B \omega' \eta \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta^{-1} \left[ \mathcal{L}_m \nabla_\mu \mathcal{L}_m + \frac{1}{2} \left( \mathcal{L}_m g_{\mu\nu} - \Sigma_{\mu\nu} \right) \Delta \mathcal{L}_m \right] \]

\[ - B \omega' \eta \left( \mathcal{L}_m g_{\mu\nu} - \Sigma_{\mu\nu} \right) \Delta \left[ \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta^{-1} \mathcal{L}_m \right], \]

for the null variations with respect to the metric, and

\[ \partial_\lambda f' \left( \delta_\mu^\lambda \delta_\nu^\gamma - \delta_\nu^\lambda \delta_\mu^\gamma \right) + 2f'T^{\mu\tau\sigma} = 0, \]

for the null variations with respect to the connection and \( f' := \partial f/\partial R \). The corresponding traces of the previous equations are given by

\[ \omega \left( f' R - 2f \right) = \omega' \left( 2 - \eta \right) \left[ A \left( \nabla_\mu \mathcal{L}_m \nabla^\mu \mathcal{L}_m \right) \eta + B \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta \right] \]

\[ + A \omega' \eta \left( 4 \mathcal{L}_m - \Sigma \right) \nabla_\mu \left[ \left( \nabla_\alpha \mathcal{L}_m \nabla^\alpha \mathcal{L}_m \right) \eta^{-1} \nabla^\mu \mathcal{L}_m \right] \]

\[ - \frac{1}{2} B \omega' \eta \left( 4 \mathcal{L}_m - \Sigma \right) \left[ \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta^{-1} \Delta \mathcal{L}_m \Delta \left( \mathcal{L}_m \Delta \mathcal{L}_m \right) \eta^{-1} \mathcal{L}_m \right] \]

and

\[ \partial_\tau f' = \frac{2}{3} f'T_\tau. \]

For the dust case, eq. (40) remains the same, while eq. (39) turns into

\[ \omega \left( f' R - 2f \right) = \omega' \left( 2 - \eta \right) c^4 \left[ A \left( \nabla_\mu \rho \nabla^\mu \rho \right) \eta + B \left( \rho \Delta \rho \right) \eta \right] \]

\[ + 3A \omega' \eta c^4 \rho \nabla_\mu \left[ \nabla_\alpha \rho \nabla^\alpha \rho \right] \eta^{-1} \nabla^\mu \rho \]

\[ - \frac{3}{2} B \omega' \eta c^4 \rho \left[ \left( \rho \Delta \rho \right) \eta^{-1} \Delta \rho \right] \]

\[ + \Delta \left[ \left( \rho \Delta \rho \right) \eta^{-1} \rho \right]. \]

Let us make the following assumption:

\[ f(R) = R^d. \]
With this explicit relation, the traces (eqs. (41) and (40)) are given by

\[ \omega(d-2)R^d = \omega'(2-\eta)c^4 \left[ A \left( \hat{\nabla}_\mu \rho \hat{\nabla}^\mu \rho \right)^\eta + B \left( \rho \hat{\Delta} \rho \right)^\eta \right] \]
\[ + 3A\omega\eta c^4 \hat{\nabla}_\nu c \left[ \left( \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho \right)^{\eta-1} \hat{\nabla}^\nu \rho \right] \]
\[ - \frac{3}{2} B\omega'\eta c^4 \rho \left[ \left( \rho \hat{\Delta} \rho \right)^{\eta-1} \hat{\Delta} \rho \right] \]
\[ + \hat{\Delta} \left[ \left( \rho \hat{\Delta} \rho \right)^{\eta-1} \right] \]  

(43)

and

\[ T_\sigma = \frac{3}{2} (d-1) \frac{\partial_\sigma R}{R} . \]  

(44)

### 4.2 MOND

Based on the results of sects. 3.1 and 3.2, we choose the following values:

\[ d = 4, \quad \eta = 1. \]  

(45)

The direct substitution of these values into eqs. (43) and (44) yields

\[ 2\omega R^4 = \omega' c^4 \left[ A \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho + (3A - 2B)\rho \hat{\Delta} \rho \right], \]  

(46)

and

\[ T_\sigma = \frac{9}{2} \frac{\partial_\sigma R}{R} . \]  

(47)

Let us analyse in more detail these expressions. From eq. (46) we obtain a relation \( R = R(\rho) \) and the substitution of this into eq. (47) yields \( T = T(\rho) \). Thus, for a vectorial torsion (8) we find a relation \( R(\cdot) = R(\cdot) \rho \). The end result of performing these substitutions yields a complicated expression and so, instead we perform an analogous procedure to the one followed by Barrientos and Mendoza [20] and write eq. (8) as

\[ R = R(\cdot) + H(R), \]  

(48)

in which we have used eq. (47) which allows us to express \( T_\mu = T_\mu(R) \). By performing Taylor expansion for \( H(R) \), and keeping only terms up to the linear term in \( R \), it follows that

\[ H(R) = \vartheta R + O(R^2), \]  

(49)

where \( \vartheta \) is a constant. The substitution of this result in eq. (8) gives

\[ R(\cdot) = \vartheta R, \quad \text{where} \quad \vartheta' := 1 - \vartheta. \]  

(50)

The direct substitution of this equation into eq. (46) yields

\[ R(\cdot) = \vartheta' c^4 \left[ \frac{\omega'}{2\omega} \right]^{1/4} \left[ A \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho + (3A - 2B)\rho \hat{\Delta} \rho \right]^{1/4}. \]  

(51)

Since we are only interested in second-order terms of \( 1/c \), we require that the coupling constants \( \omega \) and \( \omega' \) must satisfy the following constraint:

\[ \left[ \frac{\omega'}{\omega} \right]^{1/4} \propto \frac{1}{c^7}. \]  

(52)

From this restriction and using eqs. (20) and (31), the acceleration derived from eq. (51) to second perturbation order is given by

\[ \nabla \cdot a = \vartheta' \frac{c^3}{25/4} \left[ \frac{\omega'}{\omega} \right]^{1/4} \left[ A \nabla \rho \cdot \nabla \rho + (3A - 2B)\rho \nabla^2 \rho \right]^{1/4}, \]  

(53)
which, to order of magnitude yields
\[ a \approx \left( \frac{\omega'}{\omega} \right)^{1/4} \frac{c^3 M^{1/2}}{r}. \] (54)

In order to recover a MONDian acceleration, the coupling constants \( \omega \) and \( \omega' \) must satisfy the following condition:
\[ \left( \frac{\omega'}{\omega} \right)^{1/4} \propto \left( \frac{a_0 G}{c^3} \right)^{1/2}. \] (55)

Using Buckingham’s theorem of dimensional analysis (see, e.g., \[ 36 \]) with \( a_0, G \) and \( c \) as the independent variables, it follows that
\[ \omega = \Lambda \frac{c^{15}}{a_0 G}, \]
\[ \omega' = \Lambda' \frac{c^3 G}{a_0^2}, \] (56)

which satisfy the requirement (55). Defining \( \Lambda'/\Lambda := \Xi \) and using the previous expression for the coupling constants, eq. (53) is
\[ \nabla \cdot a = \vartheta' \left( \frac{-\Xi}{2^{3/4} \left( Ga_0 \right)^{1/2}} \left[ A \nabla \rho \cdot \nabla \rho + (3A - 2B) \rho \nabla^2 \rho \right] \right)^{1/4}. \] (57)

Since we are looking for a Poisson-like equation as simply as possible, we choose \( A = 1 \) and \( B = 3/2 \), so that eq. (57) turns into
\[ \nabla \cdot a = \vartheta' \left( \frac{-\Xi}{2^{3/4} \left( Ga_0 \right)^{1/2}} \left[ \nabla \rho \cdot \nabla \rho \right] \right)^{1/4}. \] (58)

Solving analytically the last relation (see appendix A), the following value of \( \Xi \) is found:
\[ \Xi = -\frac{128\pi^2}{9\vartheta'^2}. \] (59)

### 4.3 PPN consistency

In this analysis, we expand the metric \( g_{\mu\nu} \) as
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \] (60)
where \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the Minkowskian metric and \( h_{\mu\nu} \) is a small perturbation. To first order in \( h_{\mu\nu} \) (second order in \( 1/c^2 \)), the components of the Ricci tensor are given by
\[ (2) R_{00} \{} = \frac{1}{2} \nabla^2 h_{00}, \quad \text{and} \quad (2) R_{ij} \{} = \frac{1}{2} \nabla^2 h_{ij}, \] (61)

for the PPN gauge (see, e.g., \[ 34 \]).

Substituting the value \( \eta = 1 \), the functional form \( f(R) = R^4 \), the definition of \( \Xi \) and eqs. (56) into the full field eqs. (37), the following equation is obtained:
\[ 4R^3 R_{\mu\nu} - \frac{1}{2} R^4 = \Xi \left( \frac{(Ga_0)^2}{c^{12}} \right) \frac{1}{2} g_{\mu\nu} \left[ \nabla_\alpha \mathcal{L}_m \nabla^\alpha \mathcal{L}_m \\
+ \frac{3}{2} \mathcal{L}_m \Delta \mathcal{L}_m \right] - \nabla_\mu \mathcal{L}_m \nabla_\nu \mathcal{L}_m - \frac{3}{2} \mathcal{L}_m \Delta \nabla_\mu \nabla_\nu \mathcal{L}_m \\
- \frac{1}{2} \left( \mathcal{L}_m g_{\mu\nu} - \Sigma_{\mu\nu} \right) \Delta \mathcal{L}_m, \] (62)

with a trace given by
\[ 2R^4 = \Xi \left( \frac{(Ga_0)^2}{c^{12}} \right) \left[ \nabla_\alpha \mathcal{L}_m \nabla^\alpha \mathcal{L}_m + \frac{3}{2} \mathcal{L}_m \Delta \mathcal{L}_m \\
- \frac{1}{2} \left( 4\mathcal{L}_m - \Sigma \right) \Delta \mathcal{L}_m \right]. \] (63)
Using this relation in eq. (62), the field equations are
\[
4R_{\mu\nu} = \left( \frac{8(\alpha_0)^2}{c^2} \right)^{1/4} \left[ \frac{3}{4} g_{\mu\nu} \left( \hat{\nabla}_\alpha \mathcal{L}_m \hat{\nabla}^\alpha \mathcal{L}_m \right) \\
+ \frac{3}{2} \mathcal{L}_m \Delta \mathcal{L}_m \right] - \hat{\nabla}_\mu \mathcal{L}_m \hat{\nabla}_\nu \mathcal{L}_m - \frac{3}{2} \mathcal{L}_m \hat{\nabla}_\mu \hat{\nabla}_\nu \mathcal{L}_m \\
- \frac{1}{2} \left( \mathcal{L}_m g_{\mu\nu} - \Sigma_{\mu\nu} \right) \Delta \mathcal{L}_m - \frac{8}{8} g_{\mu\nu} \left( 4 \mathcal{L}_m - \Sigma \right) \Delta \mathcal{L}_m \right] \\
\times \left[ \hat{\nabla}_\alpha \mathcal{L}_m \hat{\nabla}^\alpha \mathcal{L}_m + \frac{3}{2} \mathcal{L}_m \Delta \mathcal{L}_m - \frac{1}{2} \left( 4 \mathcal{L}_m - \Sigma \right) \Delta \mathcal{L}_m \right]^{-3/4}. \tag{64}
\]

Based on eq. (48), we can express \( R_{\mu\nu} \) as
\[
R_{\mu\nu} = R_{\mu\nu} \{ \} + H_{\mu\nu}(R), \tag{65}
\]
From eq. (49), we conclude
\[
H_{\mu\nu} = \vartheta_{\mu\nu} R, \tag{66}
\]
where \( \vartheta_{\mu\nu} \) is a second-rank tensor.

Using eq. (65) for dust, the 00 component of eq. (64) at second order of approximation is given by
\[
(2) R_{00} \{ \} + (2) H_{00} = \frac{3(\xi)^{1/4} \left( \alpha_0 G \right)^{1/2}}{2^{1/4} c^2} \left[ \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho + \rho \hat{\Delta} \rho \right] \\
\times \left[ \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho \right]^{-3/4}, \tag{67}
\]
where we have used the fact that the derivatives with respect to the coordinate \( x^0 \) are of order \( 1/c \). Comparing this latter equation with eq. (58), we find the following relation:
\[
\frac{1}{2} \nabla^2 h_{00} + (2) H_{00} = \frac{3}{4} \nabla^2 \phi + G(\phi), \tag{68}
\]
where we have already substituted eqs. (61), (20) and (31), and we define \( G(\phi) \) as
\[
G(\phi) = \frac{3(-\xi)^{1/4} \left( \alpha_0 G \right)^{1/2}}{2^{1/4} c^2} \rho \nabla^2 \rho \left[ \nabla \rho \cdot \nabla \rho \right]^{-3/4}. \tag{69}
\]

The explicit dependence in \( \phi \) is given for the solution \( \rho = \rho(\phi) \) obtained by solving eq. (58).

In order to be in agreement with the metric (11) employed in our exploration examples, the following relation must hold: \( h_{00} = 2\phi/c^2 \), and so
\[
(2) H_{00} = -\nabla^2 \phi \left( \frac{3}{4} \frac{1}{\rho} + 1 \right) + G(\phi). \tag{70}
\]

Using eqs. (61) and (65), the spatial components of eq. (64) for dust are
\[
\frac{1}{2} \nabla^2 h_{ij} + (2) H_{ij} = \frac{\xi^{1/4} \left( \alpha_0 G \right)^{1/2}}{2^{5/4} c^2} \left[ \frac{1}{4} g_{ij} \left( 3 \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho \\
+ \rho \hat{\Delta} \rho \right) \right] \\
\times \left[ \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho \right]^{-3/4}, \tag{71}
\]
To handle this equation in a better way, we contract it with \( \eta^{ij} \). Defining \( H_3 := \eta^{ij} H_{ij} \) and \( h_3 := \eta^{ij} h_{ij} \), eq. (71) turns into
\[
\frac{1}{2} \nabla^2 h_3 + (2) H_3 = \frac{\xi^{1/4} \left( \alpha_0 G \right)^{1/2}}{2^{5/4} c^2} \left[ \frac{3}{4} \left( 3 \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho \\
+ \rho \hat{\Delta} \rho \right) \right] \\
\times \left[ \hat{\nabla}_\alpha \rho \hat{\nabla}^\alpha \rho \right]^{-3/4}. \tag{72}
\]
Using eqs. (20), (31) and (69) and comparing with eq. (57), the latter expression can be expressed as

\[
\frac{1}{2} \nabla^2 \phi + (2) H_3 = -\frac{5}{4} \frac{\nabla^2 \phi}{c^2} - G(\phi). \tag{73}
\]

Since we are looking for \( H_{ij} \) in order to have \( h_{ij} = (2\phi/c^2) \delta_{ij} \), therefore

\[
(2) H_3 = \frac{\nabla^2 \phi}{c^2} \left( 3 - \frac{5}{4}\vartheta' \right) - G(\phi), \tag{74}
\]

and because we are working in an isotropic frame, we conclude that

\[
(2) H_{ij} = -\frac{\nabla^2 \phi}{c^2} \left( 1 - \frac{5}{12}\vartheta' \right) \delta_{ij} + \frac{1}{3} G(\phi) \delta_{ij}. \tag{75}
\]

In order to keep the contribution of \( H_{\mu\nu} \) as small as possible, we choose the following values:

\[
\vartheta' = \frac{5}{12} \quad \text{and} \quad \vartheta = \frac{7}{12}, \tag{76}
\]

which guarantee a sufficiently small value of \( (2) H_{ij} \) given by the second term on the right-hand side of eq. (75).

## 5 Discussion

As mentioned in the introduction, many proposals of extended theories of gravity have been constructed. Recently, a new approach by Verlinde [37] yields an estimate of the excess gravity in terms of the baryonic mass distribution and the Hubble parameter. In a first astrophysical test, this approach has been able to account reasonably well for the expected lens signal of low redshift galaxies [38]. Despite this, it is not very clear from the theoretical developments of the theory how to apply such results to an extended system such as a cluster of galaxies.

From the very early stages in the introduction of torsion onto gravitational phenomena, it has never been thought as to which effect it can produce. Furthermore, it has never become clear how it can affect standard gravitational interactions. In this work, we have shown that if we want to understand MONDian phenomenology in the relativistic regime, we require to extend gravity in such a way that the functional action \( f(R, \mathcal{L}_m) \) has the following form — see eq. (36):

\[
f(R, \mathcal{L}_m) = \omega R^4 - \omega' \left[ \nabla_{\mu} \mathcal{L}_m \nabla^{\mu} \mathcal{L}_m \right] + \frac{3}{2} \left( \Delta \mathcal{L}_m \right), \tag{77}
\]

where

\[
\omega = \frac{5^{4/15}}{2^{11/15} a_0^5 G} \approx 0.02 e^{15}, \quad \text{and} \quad \omega' = \frac{9\pi^2 e^3 G}{a_0^4}. \tag{78}
\]

This formalism is fully covariant and local and so, unlike many of the previous attempts built to generalise MOND to a relativistic regime, it can be tested in many astrophysical systems, such as weak and strong lensing of individual, groups and clusters of galaxies. It can also be applied to a Friedmann-Lemaître-Robertson-Walker universe and test the behaviour of the large-scale universe at the present epoch. We intend to deal with all these problems elsewhere.

The departures introduced in the matter sector of the action (77) with respect to the classical matter action \( \mathcal{L}_m \) brings with them some theoretical concerns since it is not clear that such a choice would lead, e.g., to geodesic trajectories, but this is a much broader subject to discuss in the present article. However, the motivation of choosing this particular action comes from the field equations at the non-relativistic level, since at this level of approximation, the field equations can be expressed as

\[
(\nabla^2 \phi)^4 \approx (\nabla \rho)^2. \tag{79}
\]

In terms of the mass \( M \), the radial coordinate \( r \) and the acceleration \( a \), at order of magnitude, the previous equation can be written as

\[
\left( \frac{a}{r^2} \right)^4 \approx \left( \frac{M}{r^2} \right)^2. \tag{80}
\]

This last expression yields the correct mass and radial dependence for the MONDian acceleration. Therefore, our choice (77) was made in order to recover the dependence (80). From the above simple calculation, this choice is not unique and others actions can be built in order to achieve (80). Such actions may in principle contain the theoretical issues that the approach introduced in this work presents.
The fact that the matter Lagrangian appears inside the gravitational action contradicts the precise measurements performed on Earth and on the solar system with respect to this fact. As has been noted throughout the article, the MONDian behaviour of gravity occurs at mass to length ratios quite different from the characteristic ones associated to the solar system. In this respect, the proposal constructed in this article cannot be applied to any mass to length ratio system similar to those of the solar system. It can only be applied to systems where that ratio is much less than one, in which essentially the equivalent Newtonian gravitational acceleration is \( \lesssim a_0 \). It is precisely on these systems where the matter Lagrangian will appear inside the gravitational action.

The main conclusion that we can derive from this work is that in order to recover a MONDian acceleration from a \( F(R) \) theory, derivatives of the matter Lagrangian must be present in the field equations. The proposal of a matter Lagrangian function appearing on the gravitational action is not new and has been studied previously [39,40]. The possibility of building similar field equations from a gravitational action, that does not involve derivatives of the matter Lagrangian and satisfies standard conservation laws, is beyond the scope of this work, but will be studied by us in future research.

The authors acknowledge the referee for useful suggestions that were added to the final version of the article. This work was supported by DGAPA-UNAM (IN112616) and CONACyT (CB-2014-01 No. 240512) grants. EB and SM acknowledge economic support from CONACyT (517586 and 26344).

**Appendix A. Evaluation of the constant \( \Xi \)**

Since we are making the assumption that acceleration has only a radial component, the spherical coordinate system is the most suitable one. As such

\[
a = a r^\tau \hat{r},
\]

with divergence

\[
\nabla \cdot a = \alpha (\tau + 2) r^{\tau - 1}.
\]

Also, the gradient of \( \rho \) in these coordinates is given by

\[
\nabla \rho = \frac{d\rho}{dr} \hat{r}.
\]

Squaring eq. (58) and substituting on it eqs. (A.2) and (A.3) yields

\[
\alpha^2 (\tau + 2)^{2} r^{2(\tau - 1)} = \vartheta^2 \left( -\frac{\Xi}{2^5} \right)^{1/2} G a_0 \frac{d\rho}{dr},
\]

Integrating over \( r \) the latter expression gives the following result:

\[
\frac{\alpha^2 (\tau + 2)^{2}}{2\tau - 1} r^{2\tau - 1} = \vartheta^2 \left( -\frac{\Xi}{2^5} \right)^{1/2} G a_0 \rho,
\]

with \( \tau \neq 1/2 \). For a point mass source, the matter density is \( \rho = M \delta(r)/4\pi r^2 \). Using this expression and integrating over \( r \), we obtain

\[
\frac{\alpha^2 (\tau + 2)^2}{2\tau (2\tau - 1)} r^{2\tau} \bigg|_{r=0}^{r=\infty} = \vartheta^2 \left( -\frac{\Xi}{2^5} \right)^{1/2} \frac{GMa_0}{4\pi} \frac{1}{r^2} \bigg|_{r=0},
\]

with the additional condition \( \tau \neq 0 \). Since \( \Xi \) is just a constant, it does not depend on \( r \) and so, in order that eq. (A.6) has a meaning, it is necessary that \( \tau = 1 \). This value was expected because we built our theory with the requirement that \( a \approx r^{-1} \). Using all this, eq. (A.6) can be written as

\[
-\frac{\alpha^2}{6} = \vartheta^2 \left( -\frac{\Xi}{2^5} \right)^{1/2} \frac{GMa_0}{4\pi}.
\]

MONDian acceleration sets the value \( \alpha = -(GMa_0)^{1/2} \) and so

\[
-\frac{1}{6} = \vartheta^2 \left( -\frac{\Xi}{2^5} \right)^{1/2} \frac{1}{4\pi}.
\]

The algebraic manipulation of this expression yields eq. (59).
References

1. M. Milgrom, Astrophys. J. **270**, 365 (1983).
2. M. Milgrom, Astrophys. J. **270**, 371 (1983).
3. X. Hernandez, S. Mendoza, T. Suarez, T. Bernal, Astron. Astrophys. **514**, A101 (2010) arXiv:0904.1434 [astro-ph.GA].
4. X. Hernandez, M.A. Jiménez, Astrophys. J. **750**, 9 (2012) arXiv:1108.4021.
5. X. Hernandez, M.A. Jiménez, C. Allen, Mon. Not. R. Acad. Sci. **428**, 3196 (2013) arXiv:1206.5024 [astro-ph.GA].
6. X. Hernandez, M.A. Jiménez, E. C. Allen, Phys. J. C **72**, 1884 (2012) arXiv:1105.1873 [astro-ph.GA].
7. X. Hernandez, R. Durazo, B. Cervantes-Sodi, H.J. Ibarra-Medel, O. Lopez-Cruz, arXiv:1506.04099 (2015).
8. J.D. Bekenstein, R.H. Sanders, Mon. Not. R. Acad. Sci. **421**, L59 (2012) arXiv:1110.5048 [astro-ph.CO].
9. J.D. Bekenstein, Phys. Rev. D **70**, 083509 (2004) astro-ph/0403694.
10. T.G. Zlosnik, P.G. Ferreira, G.D. Starkman, Phys. Rev. D **74**, 040337 (2006) gr-qc/0606039.
11. R.H. Sanders, Astrophys. J. **480**, 492 (1997) astro-ph/9612099.
12. R.H. Sanders, Mon. Not. R. Acad. Sci. **363**, 459 (2005) astro-ph/0502222.
13. C. Skordis, Class. Quantum Grav. **26**, 143001 (2009) arXiv:0903.3602 [astro-ph.CO].
14. E. Babichev, C. Deffayet, G. Esposito-Farèse, Phys. Rev. D **84**, 061502 (2011) arXiv:1106.2538 [gr-qc].
15. M. Milgrom, Phys. Rev. D **80**, 123536 (2009) arXiv:0912.0790 [gr-qc].
16. C. Deffayet, G. Esposito-Farèse, R.P. Woodard, Phys. Rev. D **84**, 124054 (2011) arXiv:1106.4984 [gr-qc].
17. D.A. Demir, C.N. Karahan, Eur. Phys. J. C **74**, 1 (2014).
18. J.-P. Bruneton, G. Esposito-Farèse, Phys. Rev. D **76**, 124012 (2007).
19. T. Bernal, S. Capozziello, J.C. Hidalgo, S. Mendoza, Eur. Phys. J. C **71**, 1794 (2011) arXiv:1108.5588 [astro-ph.CO].
20. E. Barrientos, S. Mendoza, arXiv:1602.05644 [gr-qc] (2016).
21. S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002) gr-qc/0201033.
22. S. Capozziello, A. DeFelice, J. Cosmol. Astropart. Phys. **8**, 016 (2008) arXiv:0804.2163 [gr-qc].
23. S. Mendoza, T. Bernal, X. Hernandez, J.C. Hidalgo, L.A. Torres, Mon. Not. R. Acad. Sci. **433**, 1802 (2013) arXiv:1208.6241 [astro-ph.CO].
24. D.A. Carranza, S. Mendoza, L.A. Torres, Eur. Phys. J. C **73**, 2282 (2013) arXiv:1208.2502.
25. S. Mendoza, Can. J. Phys. **93**, 217 (2015).
26. S. Capozziello, R. Cianci, C. Stornaiolo, S. Vignolo, Phys. Scr. **78**, 065010 (2008) arXiv:0810.2549 [gr-qc].
27. S. Capozziello, R. Cianci, C. Stornaiolo, S. Vignolo, Class. Quantum Grav. **24**, 6417 (2007) arXiv:0708.3938 [gr-qc].
28. T. Harko, F.S.N. Lobo, S. Nojiri, S.D. Odintsov, Phys. Rev. D **84**, 024020 (2011) arXiv:1104.2669 [gr-qc].
29. Z. Haghani, T. Harko, F.S.N. Lobo, H.R. Sepangi, S. Shahidi, Phys. Rev. D **88**, 044023 (2013) arXiv:1304.5957 [gr-qc].
30. F.S.N. Lobo, T. Harko, arXiv:1211.0426 [gr-qc] (2012).
31. F.W. Hehl, Gen. Relativ. Gravit. **4**, 333 (1973).
32. F.W. Hehl, P. von der Heyde, G.D. Kerlick, J.M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
33. S. Capozziello, G. Lambiase, C. Stornaiolo, Ann. Phys. **513**, 713 (2001) gr-qc/0101038.
34. C.M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, UK, 1993) ISBN: 0521439736.
35. S. Mendoza, G.J. Olmo, arXiv:1401.5104 [gr-qc] (2014).
36. L.I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic Press, New York, 1959).
37. E.P. Verkhó, arXiv:1611.02269 [hep-th] (2016).
38. M.M. Brouwer, M.R. Visser, A. Dvornik, H. Hoekstra, K. Kuijken, E.A. Valentijn, M. Blickei, C. Blake, S. Brough, H. Baudelmeijer T. Erben, C. Heymans, H. Hildebrandt, B.W. Holwerda, A.M. Hopkins, D. Klaes, J. Liske, J. Loveday, J. McFarland, R. Nakajima, C. Sifón, E.N. Taylor et al., arXiv:1612.03034 (2016).
39. T. Harko, F.S.N. Lobo, E.N. Saridakis, arXiv:1405.7019 [gr-qc] (2014).
40. P. Pani, T.P. Sotiriou, D. Vernieri, Phys. Rev. D **88**, 121502 (2013) arXiv:1306.1835 [gr-qc].