BOUNDARY FEEDBACK STABILIZATION FOR THE INTRINSIC GEOMETRICALLY EXACT BEAM MODEL

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ABSTRACT. The geometrically exact beam (GEB) model is a 1-D second-order nonlinear system of six equations which gives the position of a beam in $\mathbb{R}^3$. The beam may undergo large deflections and rotations, as well as shear deformation. A closely related model, the intrinsic formulation of GEB (IGEB), is a 1-D first-order semilinear hyperbolic system of twelve equations which has for states velocities and strains. Here, we consider a freely vibrating slender beam made of an isotropic linear-elastic material. Applying a feedback boundary control at one end of the beam, while the other end is clamped, we show that the steady state 0 of IGEB is locally exponentially stable for the $H^1$ and $H^2$ norms. The strategy employed is to choose the control so that the energy of the beam is nonincreasing and find appropriate quadratic Lyapunov functions, relying on the energy of the beam, the relationship between GEB and IGEB, and properties of the system’s coefficients.

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Date: December 10, 2019.
AMS subject classification. 35L50, 93D15.
Keywords. Geometrically exact beam, intrinsic beam, 1-D first-order semilinear hyperbolic systems, exponential stabilization, boundary feedback.
Funding: This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No.765579-ConFlex.
1. Introduction and main result

Beam models describe three-dimensional bodies which are much longer in one direction than in the other two. These models have found many applications in civil, mechanical and aerospace engineering. Depending on the assumptions made on the beam (material law, motion magnitude, shearing, etc.) there are various PDE models for flexible beams, famous ones being the Euler-Bernoulli, Rayleigh and Timoshenko beam equations. When deflections and rotations are not small compared to the overall dimensions of the body, a geometrically nonlinear model is needed for a better accuracy. Examples include robotic arms [6] as well as flexible aircraft wings [26] or wind turbine blades [33] designed to be lighter and slender to improve aerodynamic efficiency. The geometrically exact beam (GEB) model is such a geometrically nonlinear model giving the position of a beam in \( \mathbb{R}^3 \). It takes into account shearing and the beam may undergo large displacements of its centerline and large rotations of its cross sections. This 1-D second-order nonlinear system of 6 equations originates from the works of Reissner [28] and Simo [29] (see also its derivation in [13, eq. (78)-(79)] for instance). A closely related model is the intrinsic formulation of the GEB model (IGEB) which has for states velocities and strains; see [11, 24]. The adjective intrinsic stresses that the equations make no reference to displacements or rotations variables, but only involve velocities and strains (or forces, moments). Here we consider a slender beam made of an isotropic linear-elastic material, with constant geometrical and material parameters (density \( \rho > 0 \), cross section area \( a > 0 \), shear modulus \( G > 0 \), Young modulus \( E > 0 \), area moments of inertia \( I_2, I_3 > 0 \), shear correction factors \( k_2 > 0, k_3 > 0, \) and \( k_1 > 0 \) that corrects the polar moment of area), and with an additional geometrical assumption on the centerline and cross sections (see Section 2).

1.1. Description of the system. We now describe our problem of interest, the IGEB model, which is treated here mathematically, while the GEB model and the relationship between both models will be described in Section 2 and Section 3 respectively. The IGEB model is a 1-D first-order semilinear hyperbolic system of 12 equations \((d = 12)\) whose characteristic form, for a beam of length \( L \), is

\[
\begin{aligned}
\partial_t v + D \partial_x v + B(x)v &= g(v) \quad \text{in } (0, L) \times (0, T) \\
v_-(L, t) &= -v_+(L, t) \quad \text{for } t \in (0, T) \\
v_+(0, t) &= \kappa v_-(0, t) \quad \text{for } t \in (0, T) \\
v(x, 0) &= v^0(x) \quad \text{for } x \in (0, L),
\end{aligned}
\]

where \( v = v(x, t) \in \mathbb{R}^d, v^0 \in H^1(0, L; \mathbb{R}^d) \), and \( D, B, g, \kappa \) are known matrices and maps given below. The constant diagonal matrix \( D \in \mathbb{R}^{d \times d} \) has the form

\[
D = \text{diag}(-D_+, D_+),
\]

where \( D_+ = \text{diag}(D_{+1}, D_{+2}) \) is positive definite and

\[
D_{+1} = \rho^{-\frac{1}{2}} \text{diag}(\sqrt{E}, \sqrt{k_2 G}, \sqrt{k_3 G}), \quad D_{+2} = \rho^{-\frac{1}{2}} \text{diag}(\sqrt{G}, \sqrt{E}, \sqrt{E}).
\]
In line with the sign of the diagonal entries of $D$, for any $v \in \mathbb{R}^d$ we denote
\[ v = (v_-, v_+)^\top, \quad \text{where } v_-, v_+ \in \mathbb{R}^d. \]

**Remark 1.1.** We denote the diagonal entries of $D$ by \{\lambda_i\}_{i=1}^d. Note that they include two repeated values since $\lambda_1 = \lambda_5 = \lambda_6$ and $\lambda_i = -\lambda_{i-6}$ for $i > 6$. Moreover, $D_{+1} = \text{diag}(\lambda_7, \lambda_8, \lambda_9)$, $D_{+2} = \text{diag}(\lambda_{10}, \lambda_7, \lambda_7)$ with
\[ \lambda_7 = \sqrt{\rho^{-1}E}, \quad \lambda_8 = \sqrt{\rho^{-1}k_2G}, \quad \lambda_9 = \sqrt{\rho^{-1}k_3G}, \quad \lambda_{10} = \sqrt{\rho^{-1}G}. \]

We now turn to the lower order terms $B \in C^1([0, L]; \mathbb{R}^{d \times d})$ and $g \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$. The map $B$ is defined by
\[ B = \begin{bmatrix} -D_+ P + \tilde{J}^{-1} P^\top D_+ \tilde{J} & -D_+ P - \tilde{J}^{-1} P^\top D_+ \tilde{J} \\ D_+ P + \tilde{J}^{-1} P^\top D_+ \tilde{J} & D_+ P - \tilde{J}^{-1} P^\top D_+ \tilde{J} \end{bmatrix}, \]
where $P \in C^1([0, L]; \mathbb{R}^{6 \times 6})$ and the positive definite matrices $J \in \mathbb{R}^{3 \times 3}$ (inertia matrix) and $\tilde{J} \in \mathbb{R}^{6 \times 6}$ are defined by
\[ P = \begin{bmatrix} \hat{\Gamma}_c & (\hat{\Gamma}_c + e_1) \\ 0_3 & \hat{\Upsilon}_c \end{bmatrix}, \quad J = \text{diag}(k_1(I_2 + I_3), I_2, I_3), \quad \tilde{J} = \text{diag}(I_3, a^{-1}J), \]
and where the cross product $\xi \times \zeta$ is also denoted by $\hat{\xi} \hat{\zeta} = \xi \times \zeta$ (for $\xi, \zeta \in \mathbb{R}^3$), meaning that $\hat{\xi}$ is the skew-symmetric matrix
\[ \hat{\xi} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}. \] (1.2)

The map $P$ contains the strains of the beam before deformation $\Gamma_c, \Upsilon_c \in C^1([0, L]; \mathbb{R}^3)$ (see Eq. (3.3)), and $e_1 = (1, 0, 0)^\top$. Notice that $(B + B^T)(x)$ is indefinite for any $x \in [0, L]$, since its trace is equal to zero. Later, we will also refer to the diagonal entries of $J$ as $\{J_i\}_{i=1}^3$. The components $\{g_i\}_{i=1}^d \subset C^\infty(\mathbb{R}^d)$ of the nonlinear map $g$ are quadratic forms with respect to $v \in \mathbb{R}^d$.
\[ g_i(v) = v^\top G_i v, \] (1.3)
where $\{G_i\}_{i=1}^d \subset \mathbb{R}^{d \times d}$ are symmetric matrices defined in Appendix A. Notice that $(\text{Jac}_x g)(0) = 0$, where we denote by $\text{Jac}_x f$ the Jacobian matrix of any $f = f(x)$ such that $f \in C^1(\mathbb{R}^n; \mathbb{R}^m)$. Then one can see that $v = 0$ is a steady state of (1.1).

As we will see later, the boundary conditions correspond to a beam clamped at $x = L$ while a feedback control is applied at $x = 0$. In these boundary conditions appears the diagonal matrix $\kappa \in \mathbb{R}^{6 \times 6}$ which depends on two scalars $\mu_1, \mu_2 > 0$ (the feedback parameters) to be chosen. It is defined by
\[ \kappa = (\rho a \tilde{J}D_+ + \mu)^{-1}(\rho a \tilde{J}D_+ - \mu), \]
where $\mu \in \mathbb{R}^{6 \times 6}$ is the diagonal matrix
\[ \mu = \text{diag}(\mu_1, \mu_1, \mu_1, \mu_2, \mu_2, \mu_2). \]
The diagonal entries of $\kappa$ belong to $(-1, 1)$ since they have the form $(b - c)/(b + c)$ for $b > 0$ and $c > 0$.

1.2. Main result. The local existence and uniqueness of $C^0([0, T]; H^1(0, L; \mathbb{R}^d))$ solutions to general 1-D semilinear hyperbolic systems, as well as the local existence and uniqueness of $C^0([0, T]; H^2(0, L; \mathbb{R}^d))$ solutions to general 1-D quasilinear hyperbolic systems, have been addressed in [34, Lem. 2.3, Th. The local and semi-global existence and uniqueness of solutions to general 1-D semilinear hyperbolic systems, as well as the local existence and uniqueness of solutions to general 1-D quasilinear hyperbolic systems, have been addressed in [3] and [2, Ap. B]. Both results apply to Problem (1.1), assuming that $v^0$ fulfills the zero-order compatibility conditions (C0) and the first-order compatibility conditions (C1) respectively:

$$v^0(L) = -v^0_t(L), \quad v^0(0) = \kappa v^0(0). \quad (C0)$$

and

$$(-D(v^0)' - Bv^0 + g(v^0))_-(L) = -( -D(v^0)' - Bv^0 + g(v^0))_+(L)$$

$$(-D(v^0)' - Bv^0 + g(v^0))_+(0) = \kappa (-D(v^0)' - Bv^0 + g(v^0))_-(0). \quad (C1)$$

The local and semi-global existence and uniqueness of $C^1([0, L] \times [0, T]; \mathbb{R}^d)$ solutions to general 1-D quasilinear hyperbolic systems have been addressed in [34, Lem. 2.3, Th. 2.1] (which is an extension of [21, Lem. 2.3, Th. 2.5] to nonautonomous systems), and these results apply to (1.1) if $v^0$ fulfills (C0)-(C1).

Our aim is to prove Theorem 1.3 which, as explained in Section 2 and Section 3, may be read as follows: if a beam is freely vibrating (i.e. no applied external forces and moments), clamped at $x = L$, and if at $x = 0$ a feedback control of the form $h_1 = -\mu_1 \partial_k p$, $h_2 = -\mu_2 RW$ is applied (see in Section 2 the meaning of these variables), then the IGEB model writes as (1.1) and the steady state $v = 0$ of (1.1) is locally exponentially stable for the $H^1$ and $H^2$ norms, in the sense of the following definition.

**Definition 1.2.** For $k \in \{1, 2\}$, the steady state $v = 0$ of (1.1) is locally $H^k$-exponentially stable if there exist $\varepsilon > 0$, $\alpha > 0$ and $\eta \geq 1$ such that the following holds. Let the initial datum $v^0 \in H^k(0, L; \mathbb{R}^d)$ fulfill $\|v^0\|_{H^k(0, L; \mathbb{R}^d)} \leq \varepsilon$, as well as (C0) if $k = 1$ or (C0)-(C1) if $k = 2$. Then, Problem (1.1) admits a unique global (in time) solution $v \in C^0([0, +\infty); H^k(0, L; \mathbb{R}^d))$. Moreover, $v$ satisfies

$$\|v(\cdot, t)\|_{H^k(0, L; \mathbb{R}^d)} \leq \eta e^{-\alpha t}\|v^0\|_{H^k(0, L; \mathbb{R}^d)}, \quad \text{for any } t \in [0, +\infty).$$

We may now state our main result.

**Theorem 1.3.** The steady state $v = 0$ of (1.1) is locally exponentially stable for the $H^1$ and $H^2$ norms, in the sense of Definition 1.2.

**Remark 1.4.** Let $k \in \{1, 2\}$. Note that $C^0([0, +\infty); H^k(0, L; \mathbb{R}^d)) \subset C^{k-1}([0, L] \times [0, +\infty); \mathbb{R}^d)$ (for $k = 2$ see [2, Cor. B.2]), and there exists $\bar{\eta} > 0$ such that

$$\|v(\cdot, t)\|_{C^{k-1}([0, L] \times [0, +\infty); \mathbb{R}^d)} \leq \bar{\eta} e^{-\alpha t}\|v^0\|_{H^k(0, L; \mathbb{R}^d)}, \quad \text{for any } t \in [0, +\infty).$$
1.3. Notation. Let \( m, n \in \mathbb{N} \) and \( M \in \mathbb{R}^{n \times n} \). Here, the identity and null matrices are denoted by \( I_n \in \mathbb{R}^{n \times n} \) and \( O_{n,m} \in \mathbb{R}^{n \times m} \), and we use the abbreviation \( O_n = O_{n,n} \). The transpose, determinant and trace of \( M \), and the matrix with components \( |M_{i,j}| \) (for \( i, j \in \{1 \ldots n\} \)) are denoted by \( M^\top \), \( \det(M) \), \( \text{tr}(M) \) and \( |M| \) respectively. We use the notation \( \|M\| = \sup_{|\xi|=1} |M\xi| \), where \( |.| \) is the Euclidean norm. The symbol \( \text{diag}(\cdot, \ldots, \cdot) \) denotes a (block-)diagonal matrix composed of the arguments. We denote by \( \mathcal{D}^+(n) \) the set of positive definite diagonal matrices of size \( n \). Finally, we use the shortened notations \( L^2(0, L) = L^2(0, L; \mathbb{R}^d) \) and \( H^m(0, L) = H^m(0, L; \mathbb{R}^d) \).

1.4. Brief state of the art. Up to the best of our knowledge, global in time existence and uniqueness of \( C^1([0, L] \times [0, \infty); \mathbb{R}^d) \) solutions to \((1.1)\) is not provided by general results present in the literature, even though one may find such results for quasilinear and semilinear problems similar to \((1.1)\). For instance, in the case of initial value problems, [20, Ch. 4] assumes dissipativity of the lower order terms \((Bv \text{ and } g(v) \text{ here})\) and [4] gives a relaxation of this assumption, [32] considers \( C^0(\mathbb{R}; L^1(\mathbb{R}; \mathbb{R}^d)) \) solutions when there isn’t any linear lower order term \((Bv \text{ here})\) and the quadratic term satisfies certain constraints (which are satisfied by \( g \) here); while in the case of initial boundary value problems, [20, Ch. 5] assumes dissipativity of the boundary conditions and the absence of linear lower order terms, [23, 15] assume a specific sign and monotony of certain terms or a growth restriction on the lower order terms.

Stabilization of beam equations by means of feedback boundary controls goes back to [27] for the string, [14] for the Timoshenko beam; see also [10, 22, 35] and the references therein for other linear and nonlinear beam models. Among the methods usually used to study stability, we opt here for the Lyapunov approach. For 1-D first-order hyperbolic systems, such as \((1.1)\), several results of stabilization under boundary control are shown by means of quadratic Lyapunov functions in [2] and the references therein. There, when the system does not have any lower order term such as \( Bv \) and \( g(v) \) here (systems of conservation laws), the exponential stability may rely on the dissipativity of the boundary conditions alone. However, when lower order terms are present (systems of balance laws) the equations must also be taken into consideration. Some systems of nonlinear balance laws with a uniform steady state may be seen as systems of nonlinear conservation laws perturbed by the lower order terms: if the perturbation is small enough then the \( C^1 \)-exponential stability is preserved, see [2, Th. 6.1]. See also [8] for two by two quasilinear systems with small lower order terms. Problem \((1.1)\) does have dissipative boundary conditions, in the sense that \( \rho_\infty(K) < 1 \) where \( \rho_\infty(K) = \inf \{ \mathcal{R}_\infty(\Lambda K \Lambda^{-1}) : \Lambda \in \mathcal{D}^+(d) \} \) and

\[
K = \begin{bmatrix} O_6 & -I_6 \\ \kappa & O_6 \end{bmatrix}, \quad \mathcal{R}_\infty(M) = \max_{1 \leq i \leq d} \sum_{j=1}^d |M_{ij}|.
\]

Indeed, for \( \Lambda = \text{diag} ((1 + \varepsilon)|\kappa|, I_6) \), if \( \varepsilon > 0 \) is small enough then \( \mathcal{R}_\infty(\Lambda K \Lambda^{-1}) < 1 \). However, the perturbation is not small, in the sense that its derivative with respect to \( v \) evaluated at 0, which is equal to \( B \), cannot be assumed arbitrarily small. One may observe this, for example, for a beam that is straight, untwisted and with centerline \( x \mapsto x e_1 \).
before deformation (see Section 2), in which case \( \| B \| = \max\{ \lambda_8, \lambda_9, aI_2^{-1}\lambda_9, aI_3^{-1}\lambda_8 \} \).

Concerning general linear, semilinear and quasilinear systems, assumptions on both the boundary conditions and the structure of the governing equations are required in [2, Pr. 5.1], [3, Th. 10.2], [9] and [2, Th. 6.10] for \( L^2, H^1, C^1 \) and \( H^2 \)-exponential stability respectively. We also refer to [36] for semilinear systems with lower order terms of a specific form.

1.5. **Outline.** In Section 2 we present the GEB model. In Section 3 we explain the choice of the boundary feedback control and detail the transformations leading from GEB to Problem (1.1). In Section 4, we study the energy of the beam, thus gaining information of use in the next section. In Section 5, we prove the main result Theorem 1.3.

2. The GEB model

We now present the GEB model to which (1.1) is related by the transformations detailed in Section 3.2. Let \( \{e_i\}_{i=1}^3 = \{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\} \) be the canonical basis of \( \mathbb{R}^3 \), that we call the *global* basis. We consider three different configurations for a beam (see Fig. 1):

- \( \Omega_s \subset \mathbb{R}^3 \) is the beam in the *straight-reference configuration*: it is the straight, untwisted beam whose centerline’s position is \( x \mapsto xe_1 \) for \( x \in [0, L] \); it writes as \( \Omega_s = \bigcup_{x \in [0, L]} a(x) \) where \( a(x) \) is the cross section intersecting the centerline at \( xe_1 \) (i.e. all points of the beam sharing the same beam length coordinate \( x \) in \( \Omega_s \));
- \( \Omega_c \subset \mathbb{R}^3 \) is the beam in a *curved-reference configuration*: it is the beam before deformation; choosing \( \Omega_c \) different from \( \Omega_s \) permits to consider beams that are pre-curved and pre-twisted;
- \( \Omega_t \subset \mathbb{R}^3 \) is the beam in the *current (deformed) configuration*: it is the beam at time \( t \).

The states of GEB depend on \( x \in [0, L] \) and \( t \geq 0 \); they are the position \( p(x,t) \) of the centerline of \( \Omega_t \), and the rotation \( R(x,t) \) of the cross section \( a(x) \) from \( \Omega_s \) to \( \Omega_t \). The second unknown \( R(x,t) \) is a rotation matrix (i.e. unitary and of determinant equal to 1) that gives the orientation of \( a(x) \) at time \( t \), and whose columns \( \{R(x,t)e_i\}_{i=1}^3 \) form a *local* basis attached to the centerline of \( \Omega_t \) at \( p(x,t) \). For any \( X \in \Omega_s \) we denote:

\[
X = (x, X_2, X_3)^T.
\]

One recovers, from the two states, the position of \( X \in \Omega_s \) at time \( t \) as

\[
\dot{p}(X,t) = p(x,t) + R(x,t)(X_2e_2 + X_3e_3).
\]

For any skew-symmetric matrix \( M \in \mathbb{R}^{3 \times 3} \), we denote by \( \text{vec}(M) \in \mathbb{R}^3 \) the vector such that \( M = \text{vec}(M) \) (see (1.2)). We now introduce velocity and strain functions, defined from \( [0, L] \times [0, T] \) into \( \mathbb{R}^3 \). The velocity of the centerline \( V \) (or linear velocity) and the angular velocity \( W \) are

\[
V = R^T\partial_t p, \quad W = \text{vec}(R^T\partial_t R),
\]
while the translational strain $\Gamma$ and the curvature $\Upsilon$ (or rotational strain) are

$$\Gamma = R^T \partial_x p - R_c^T p_c', \quad \Upsilon = \text{vec}(R^T \partial_x R - R_c^T R_c'),$$

(2.2)

where $p_c \in C^2([0, L]; \mathbb{R}^3)$ and $R_c \in C^2([0, L]; \mathbb{R}^{3 \times 3})$ are defined analogously to $(p, R)$: $p_c(x)$ is the position of the centerline of $\Omega_c$ and $R_c(x)$ is the rotation of $a(x)$ from $\Omega_s$ to $\Omega_c$ ($R_c(x)$ is a rotation matrix). Correspondingly, the strain functions $\Gamma_c, \Upsilon_c \in C^1([0, L]; \mathbb{R}^3)$ of $\Omega_c$ are defined by

$$\Gamma_c = R_c^T p_c' - e_1, \quad \Upsilon_c = \text{vec}(R_c^T R_c'),$$

and the position of $X$ in $\Omega_c$ is

$$\bar{p}_c(X) = p_c(x) + R_c(x)(X_2 e_2 + X_3 e_3).$$

Remark 2.1. If the beam is straight and untwisted with centerline $p_c(x) = xe_1$ before deformation (i.e. $\Omega_c = \Omega_s$) then $R_c$ is the identity matrix and $p_c'(x) = e_1$, implying that $\Gamma_c = \Upsilon_c = 0$.

Remark 2.2. The functions $V, W, \Gamma$ and $\Upsilon$ are called local (or body-attached, or material) variables in the following sense. Let $y \in \mathbb{R}^3$, with components $\{y_i\}_{i=1}^3$. The local representation of $y$ is the vector $Y \in \mathbb{R}^3$ whose components $\{Y_i\}_{i=1}^3$ are the coordinates of $y$ relative to the local basis $\{Re_i\}_{i=1}^3$, meaning that $y = \sum_{i=1}^3 Y_i Re_i$. Observe that both vectors $(y = \sum_{i=1}^3 y_i e_i$ and $Y = \sum_{i=1}^3 Y_i e_i)$ are related by $y = RY$. Then, $y$ is called the global representation of $Y$. Here, the components of $V, W, \Gamma$ and $\Upsilon$ are coordinates relative to the local basis, and their global representations are consequently given by $RV, RW, R\Gamma$ and $R\Upsilon$.

In [31], a geometrically exact beam with linear elastic material law, and with the following geometrical and material restrictions is considered. Cross sections remain plane,
do not change of shape, rotate independently from the motion of the centerline (allowing shear deformation). The beam is thin, in the sense that the diameter of a is small compared to $L$ and $(\text{Jac}_X \tilde{p}_c)(X) \approx (\text{Jac}_X \tilde{p}_c)(xe_1)$. The parameters $(G, E, \rho)$ only vary along the centerline. In $\Omega_s$, the centerline connects the cross sections at their geometrical center which is assumed to be lying on the $e_1$-axis, and the principal axes of the cross sections are $e_2$ and $e_3$. In $\Omega_c$, the centerline is parameterized such that $e_1^T R_c p'_c = 1$. The material is isotropic. Strains are small in the sense that $|\tilde{\Gamma}(x, t)|$ and $|\tilde{\Upsilon}(x, t)(X_2 e_2 + X_3 e_3)|$ are small for $X \in \Omega_s$ and $t \geq 0$. The constitutive law are diagonal uncoupled: the internal forces and moments are $f_1 = R M_1 \Gamma$ and $f_2 = R M_2 \Upsilon$ respectively, where $M_1, M_2$ are the positive definite diagonal matrices:

$$M_1 = \rho a D^2_{+1}, \quad M_2 = \rho J D^2_{+2}.$$ 

For such a beam, the GEB model writes as

$$\begin{cases}
\rho \partial_t^2 \mathbf{p} = \partial_x (R M_1 \Gamma) + \tilde{f}_1 & \text{in } (0, L) \times (0, T) \\
\rho \partial_t (R J W) = \partial_x (R M_2 \Upsilon) + (\partial_x \mathbf{p}) \times (R M_1 \Gamma) + \tilde{f}_2 & \text{in } (0, L) \times (0, T) \\
\mathbf{p}(L, \cdot) = h^p, \quad R(L, \cdot) = h^R & \text{for } t \in (0, T) \\
-\mathbf{R}(0, \cdot) M_1 \Gamma(0, \cdot) = h_1, \quad -\mathbf{R}(0, \cdot) M_2 \Upsilon(0, \cdot) = h_2 & \text{for } t \in (0, T) \\
\mathbf{p}(\cdot, 0) = \mathbf{p}^0, \quad \partial_t \mathbf{p}(\cdot, 0) = v^0 & \text{for } x \in (0, L) \\
\mathbf{R}(\cdot, 0) = \mathbf{R}^0, \quad \mathbf{R}(\cdot, 0) W(\cdot, 0) = w^0 & \text{for } x \in (0, L).
\end{cases} \tag{2.3}
$$

The six governing equations are supplemented by Dirichlet boundary conditions at $x = L$ with boundary data $h^p(t) \in \mathbb{R}^3$, $h^R(t) \in \mathbb{R}^{3 \times 3}$, by Neumann boundary conditions at $x = 0$ with boundary data $h_1(t)$, $h_2(t) \in \mathbb{R}^3$, and by initial conditions with initial data $\mathbf{p}^0(x)$, $v^0(x)$, $w^0(x) \in \mathbb{R}^3$ and $\mathbf{R}^0(x) \in \mathbb{R}^{3 \times 3}$. 

Here, we consider model (2.3), additionally assuming that all geometrical and material parameters $(\rho, a, E, G, I_2, I_3)$ and $\{k_i\}_{i=1}^3$ are constant.

### 3. Boundary feedback and transformations

A part of this work relies on studying the energy $\mathcal{E}$ of the beam, which is by definition

$$\mathcal{E}(t) = \mathcal{K}(t) + \mathcal{V}(t),$$

where $\mathcal{K}$ is the kinetic energy and $\mathcal{V}$ the elastic energy (or strain energy) defined by

$$\mathcal{K} = \frac{1}{2} \int_{\Omega_s} \rho \partial_t \mathbf{p}^2 dX, \quad \mathcal{V} = \frac{1}{2} \int_{\Omega_s} \text{tr} (\mathbf{S}^T \mathbf{E}) \text{det}(\text{Jac}_X \tilde{p}_c) dX,$$

for $\mathbf{S}$ denoting the symmetric second Piola-Kirchhoff stress tensor (defined over $\Omega_s$), and $\mathbf{E}$ the Green-Lagrange strain tensor. However, for the GEB model described in Section 2, $\mathcal{K}$ and $\mathcal{V}$ also write as

$$\mathcal{K} = \frac{1}{2} \int_0^L (\rho a V^2 + \rho W^2 J W) dx, \quad \mathcal{V} = \frac{1}{2} \int_0^L (\Gamma^T M_1 \Gamma + \Upsilon^T M_2 \Upsilon) dx.$$

Here and in the following computations involving the energy, we drop the argument $t$ for clarity.
3.1. Design of the feedback control. We choose the feedback control at $x = 0$ by examining the derivative of $\mathcal{E}$. Denote $w = RW$. Useful identities are $\hat{R} = R \hat{\xi} R^\top$ for $\xi \in \mathbb{R}^3$ (from the invariance of the cross product in $\mathbb{R}^3$ under rotation), as well as

$$\partial_t R = R \hat{W} = \hat{w} R, \quad \partial_x R = R \hat{Y} + \hat{Y}_c. \quad (3.1)$$

If $p, R$ are regular solutions of (2.3) ($C^2$ in $[0, L] \times [0, T]$), then $t \mapsto \mathcal{E}(t)$ is nonincreasing on $[0, T]$ under the following assumption.

**Assumption 3.1.** Assume that the applied external forces are set to zero ($\bar{f}_1 = 0, \bar{f}_2 = 0$) and that the beam is clamped at $x = L$ (i.e. the Dirichlet boundary data $h^p$ and $h^R$ are independent of time). At $x = 0$, the following feedback control is applied:

$$h_1 = -\mu_1 \partial_t p, \quad h_2 = -\mu_2 RW,$$

with feedback parameters $\mu_1, \mu_2 > 0$ to be chosen.

Indeed, the derivatives of the kinetic and strain energies are

$$\frac{dK}{dt} = \int_0^L \left[ \rho a \partial_t^2 p \partial_t p + \rho \partial_t \hat{w} \hat{w} R \right] dx, \quad \frac{d\mathcal{V}}{dt} = \int_0^L \left[ \partial_t \Gamma \partial_t M_1 \Gamma + \partial_t \hat{Y} \partial_t M_2 \hat{Y} \right] dx.$$

Since $J$ is independent of $t$, we may rewrite $W^\top J \partial_t W = w^\top R \partial_t (JW)$. To make the term $\partial_t (RJW)$ of the governing equations of (2.3) appear, we write:

$$W^\top J \partial_t W = w^\top \partial_t (RJW) - w^\top \partial_t R JW.$$

By (3.1) and since $\hat{w}$ is skew-symmetric,

$$w^\top \partial_t R JW = w^\top \hat{w} R JW = - (\hat{w} w)^\top R JW = 0,$$

as $w \times w = 0$. Hence, $\rho W^\top J \partial_t W = w^\top \rho \partial_t (RJW)$ and the governing equations yield

$$\frac{dK}{dt} = \int_0^L \left[ \left( \partial_x (R M_1 \Gamma) + \bar{f}_1 \right)^\top \partial_t p + \left( \partial_x (R M_2 \hat{Y}) + \partial_x \hat{p} R M_1 \Gamma + \bar{f}_2 \right)^\top w \right] dx.$$

Consider now the derivative of $\mathcal{V}$. We compute the time derivative of $\Gamma = R^\top \partial_x p$ using (3.1) and $\hat{w}^\top = -\hat{w}$:

$$\partial_t \Gamma = \partial_t R^\top \partial_x p + R^\top \partial_t \partial_x p = R^\top (\partial_t \hat{p}) w + R^\top \partial_t \partial_x p.$$

Similarly, we compute the time derivative of $\hat{Y} = \text{vec}(R^\top \partial_x R)$ as follows. By (3.1) and $\hat{W}^\top = \hat{W}$,

$$\partial_t \hat{Y} = \partial_t R^\top \partial_x R + R^\top \partial_t \partial_x p = -\hat{W} \hat{Y} + \hat{Y}_c + R^\top \partial_t \partial_x R. \quad (3.2)$$

and one recognizes the term $R^\top \partial_x w = R^\top \partial_x \hat{w} R$ in (3.2), since

$$R^\top \partial_x w = R^\top \left[ (\partial_x \partial_t R) R^\top + \partial_t R \partial_x R^\top \right] R = R^\top \partial_x \partial_t R - \hat{W} \hat{Y} + \hat{Y}_c, \quad (3.3)$$
by (3.1) and the skew-symmetry of $\tilde{Y} + \tilde{Y}_t$. Consequently, $\partial_t \mathcal{Y} = R^T \partial_x w$. Then, the derivative of the elastic energy takes the form

$$\frac{d\mathcal{Y}}{dt} = \int_0^L \left[ -w^T \partial_x p R M_1 \Gamma + \partial_x \partial_x p^T R M_1 \Gamma + \partial_x w^T R M_2 \mathcal{Y} \right] dx.$$ 

Integrating by parts the second and third terms of the above right-hand side,

$$\frac{d\mathcal{Y}}{dt} = \int_0^L \left[ -w^T \left( \partial_x p R M_1 \Gamma + \partial_x (R M_2 \mathcal{Y}) \right) - \partial_t p^T \partial_x (R M_1 \Gamma) \right] dx$$

$$+ \left[ \partial_t p^T R M_1 \Gamma + W^T M_2 \mathcal{Y} \right]_0^L.$$ 

As a result, the derivative of the energy is given by

$$\frac{d\mathcal{E}}{dt} = \int_0^L \left[ \partial_t p^T \bar{f}_1 + w^T \bar{f}_2 \right] dx + \partial_t p^T R \mathcal{Y} + \partial_t (R \mathcal{Y} \mathcal{Y}) - \partial_t p^T (0, \cdot)^T R (0, \cdot) M_1 \Gamma (0, \cdot)$$

$$+ \left( R (0, \cdot) \mathcal{Y} \mathcal{Y} \right)^T \mathcal{Y} \mathcal{Y} - \partial_t R (0, \cdot) \mathcal{Y} \mathcal{Y} - \partial_t \mathcal{Y} \mathcal{Y}$$

$$- w (0, \cdot)^T h_1 + w (0, \cdot)^T h_2,$$

and under Assumption 3.1 we have

$$\frac{d\mathcal{E}}{dt} = -\mu_1 |\partial_t p|^2 - \mu_2 |w|^2 \leq 0.$$

### 3.2. Transformations.

Under Assumption 3.1, we apply the transformation

$$y = (V^T, W^T, \Gamma^T, \mathcal{Y}^T)^T$$

where $V, W, \Gamma, \mathcal{Y}$ are defined by (2.1)-(2.2). Then, the IGEB model is obtained:

$$\begin{cases}
\partial_t y + A \partial_x y + \tilde{B}(x) y = \tilde{g}(y) & \text{in } (0, L) \times (0, T) \\
\begin{bmatrix} I_6 & 0_6 \end{bmatrix} y(L, \cdot) = 0 & \text{for } t \in (0, T) \\
-\rho a \begin{bmatrix} 0_6 & J D_2 \end{bmatrix} y(0, \cdot) = -\mu \begin{bmatrix} I_6 & 0_6 \end{bmatrix} y(0, \cdot) & \text{for } t \in (0, T) \\
y(\cdot, 0) = y^0 & \text{for } x \in (0, L),
\end{cases}$$

where $y = y(x, t) \in \mathbb{R}^d$ and $d = 12$, setting

$$y^0 = \begin{bmatrix} (R^0)^T v_0^0 \\ (R^0)^T w_0^0 \\ (R^0)^T (p^0)' - R^0_p c' \end{bmatrix}.$$
The first six governing equations are derived from the governing equations of (2.3), while the last six come from the definition of $\Gamma$ and $\Upsilon$. The initial and boundary conditions are chosen so that they coincide with those of (2.3). The coefficients $A \in \mathbb{R}^{d \times d}$, $B \in C^1([0, L]; \mathbb{R}^{d \times d})$ and $\tilde{g} \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ are defined by

$$A = \begin{bmatrix} O_6 & -D_+^2 \\ -I_6 & O_6 \end{bmatrix}, \quad B = \begin{bmatrix} O_6 & \tilde{J}^{-1} P \tilde{J} D_+^2 \\ -P & O_6 \end{bmatrix}, \quad \tilde{g}(y) = y^\top \tilde{G}^i y,$$  

where $\{\tilde{g}_i\}_{i=1}^6 \subset C^\infty(\mathbb{R}^d)$ are the components of $\tilde{g}$, and $\{\tilde{G}^i\}_{i=1}^6 \subset \mathbb{R}^{d \times d}$ are symmetric matrices defined in Appendix A. This derived model is hyperbolic, in the sense that $A$ has real eigenvalues only, with $d$ independent associated left-eigenvectors. Hence, the characteristic form (1.1) of Problem (3.4) is obtained by using that $A = L^{-1} D L$, with the change of variable

$$v = Ly, \quad \text{where} \quad L = \begin{bmatrix} I_6 & D_+ \\ I_6 & -D_+ \end{bmatrix}, \quad L^{-1} = \frac{1}{2} \begin{bmatrix} I_6 & I_6 \\ D^{-1}_+ & -D^{-1}_+ \end{bmatrix},$$

and setting $v^0 = L y^0, B = L \tilde{B} L^{-1}$ and $g(v) = L \tilde{g}(L^{-1} v)$.

**Remark 3.2.** Assume that $v \in C^0([0, +\infty); \mathbb{H}^k(0, L))$ is the unique solution to (1.1) provided by Theorem 1.3, for $k \in \{1, 2\}$. If $y \in C^0([0, +\infty); \mathbb{H}^k(0, L))$ is defined by $y = L^{-1} v$, then it is the unique global solution to (3.4) with initial datum $y^0 = L^{-1} v^0$. Moreover, one has

$$\|y(\cdot, t)\|_{\mathbb{H}^k(0, L)} \leq \eta e^{-\alpha t} \|L\| \|L^{-1}\| \|v^0\|_{\mathbb{H}^k(0, L)},$$

where $\eta \geq 1$ and $\alpha > 0$ are the constants of Definition 1.2.

4. ENERGY OF THE BEAM

The boundary conditions of (1.1) have been chosen in such a way that the energy of the beam is nonincreasing. Moreover, the energy writes as

$$E(t) = \frac{1}{2} \int_0^L (V^\top, W^\top, \Gamma^\top, \Upsilon^\top) \tilde{Q}(V^\top, W^\top, \Gamma^\top, \Upsilon^\top)^\top dx,$$

for $\tilde{Q} = \rho a \text{diag}(\tilde{J}, D^2_+ \tilde{J})$ and for $V, W, \Gamma, \Upsilon$ defined by (2.1)-(2.2). Since the transformation from GEB to IGEB is $y = (V^\top, W^\top, \Gamma^\top, \Upsilon^\top)^\top$, and the change of variable $v = Ly$ leads to the characteristic form (1.1) of IGEB, we expect that the map $t \mapsto \mathcal{L}_0(t)$

$$\mathcal{L}_0(t) = \frac{1}{2} \int_0^L v(x, t)^\top \tilde{Q} v(x, t) dx,$$

where $\tilde{Q} = \rho a L^{-1} \text{diag}(\tilde{J}, D^2_+ \tilde{J}) L^{-1}$

is also nonincreasing if $v$ is solution to (1.1), as the definitions of $\mathcal{L}_0$ and $E$ coincide. Observe that $\tilde{Q}$ rewrites as:

$$\tilde{Q} = \frac{\rho a}{4} \begin{bmatrix} I_6 & D^{-1}_+ \\ I_6 & -D^{-1}_+ \end{bmatrix} \begin{bmatrix} \tilde{J} & 0 \\ 0 & \tilde{J} \end{bmatrix} \begin{bmatrix} I_6 & 0 \\ D^{-1}_+ & 0 \end{bmatrix} = \frac{\rho a}{2} \begin{bmatrix} \tilde{J} & 0 \\ 0 & \tilde{J} \end{bmatrix}.$$  

**Proposition 4.1.** Let $v$ be the unique solution to (1.1) in $C^1([0, L] \times [0, T]; \mathbb{R}^d)$. Then, the map $t \mapsto \mathcal{L}_0(t)$, defined by (4.1) is nonincreasing on $[0, T]$.  

Proof. Let \( v \) satisfy the assumptions of Proposition 4.1. The derivative of \( \mathcal{L}_0 \) is equal to
\[
\frac{d\mathcal{L}_0}{dt} = \int_0^L v^\top \tilde{Q} \partial_t v dx = - \int_0^L v^\top \tilde{Q} D \partial_x v dx + \int_0^L v^\top \tilde{Q} (-B v + g(v)) dx,
\]
where we used the governing equations. Integrating by parts the first integral,
\[
- \int_0^L v^\top \tilde{Q} D \partial_x v dx = \int_0^L \partial_x v^\top \tilde{Q} D v dx - \left[ v^\top \tilde{Q} D v \right]_0^L,
\]
and \(- \int_0^L v^\top \tilde{Q} D \partial_x v dx = - \frac{1}{2} \left[ v^\top \tilde{Q} D v \right]_0^L \) since \( \tilde{Q} \) and \( D \) commute. Hence,
\[
\frac{d\mathcal{L}_0}{dt} = - \frac{1}{2} \left[ v^\top \tilde{Q} D v \right]_0^L + \int_0^L v^\top \tilde{Q} (B v + g(v)) dx
\]

\[
= - \frac{\rho a}{4} ( - v_-(L,t) \top JD_+ v_-(L,t) + v_+(L,t) \top J D_+ v_+(L,t) )
\]

\[
+ \frac{\rho a}{4} ( - v_-(0,t) \top JD_+ v_-(0,t) + v_+(0,t) \top JD_+ v_+(0,t) ) + \int_0^L v^\top \tilde{Q} (B v + g(v)) dx.
\]

Using the boundary conditions,
\[
\frac{d\mathcal{L}_0}{dt} = \frac{\rho a}{4} v_-(0,t) \top J (\kappa^2 - \mathbf{1}_6) D_+ v_-(0,t) + \int_0^L v^\top \tilde{Q} (B v + g(v)) dx.
\]

In the above right-hand side, the first term is negative since the diagonal entries of \( \kappa \) belong to \((-1,1)\). It remain to see that the second term of above right-hand side is null. The product \( \tilde{Q} B \) is skew-symmetric since it writes as
\[
\tilde{Q} B = \frac{\rho a}{4} \begin{bmatrix}
-J D_+ P + (J D_+ P) \top & -J D_+ P - (J D_+ P) \top \\
J D_+ P + (J D_+ P) \top & -J D_+ P + (J D_+ P) \top
\end{bmatrix},
\]
hence \( v^\top \tilde{Q} B v = 0 \) for any \( v \in \mathbb{R}^d \). Finally, \( v^\top \tilde{Q} g(v) = 0 \) for any \( v \in \mathbb{R}^d \) is equivalent to \( y^\top \tilde{Q} g(y) = 0 \) for any \( y \in \mathbb{R}^d \) (by definition of \( \tilde{Q} \) and \( g \)), and showing that this quantity is indeed null does not involve the use of the governing equations but only basic computations and the definition of \( \tilde{Q} \) and \( \tilde{g} \).

Since \( \tilde{Q} \) is positive definite diagonal, there exists \( c > 0 \) such that
\[
c^{-1} \|v(\cdot,t)\|_{L^2(0,L)}^2 \leq \mathcal{L}_0(t) \leq c \|v(\cdot,t)\|_{L^2(0,L)}^2, \quad \text{for any } t \in [0,T],
\]
for example with \( c = \max\{\max_i \tilde{Q}_i, (\min_i \tilde{Q}_i)^{-1}\} \). Hence, Proposition 4.1 implies that the \( L^2(0,L) \) norm of the solution \( v \in C^1([0,L] \times [0,T]) \) is bounded on \([0,T]\):
\[
\|v(\cdot,t)\|_{L^2(0,L)} \leq c \|v(\cdot,t)\|_{L^2(0,L)} \quad \forall t \in [0,T].
\]

**Remark 4.2** (Structure of \( B \)). While the matrix \( B \) is neither skew-symmetric nor positive or negative semi-definite, one observes in the proof of Proposition 4.1 that
\[
2 \text{diag}(\tilde{J}, \tilde{J}) B = \begin{bmatrix}
-(\Phi - \Phi^\top) & -(\Phi + \Phi^\top) \\
\Phi + \Phi^\top & \Phi - \Phi^\top
\end{bmatrix}, \quad \text{for } \Phi = \tilde{J} D_+ P
\]
(in particular, the above product is skew-symmetric). In Section 5, we will use this information in the proof of the main result Theorem 1.3.
5. Proof of the main result

For any matrix $M \in D^+(d)$, we denote $M = \text{diag}(M_-, M_+)$, where $M_-, M_+ \in D^+(6)$.

5.1. Strategy. Applying Proposition 5.1 given below is sufficient to prove the main result Theorem 1.3, and is equivalent to finding a quadratic $H^k$-Lyapunov function for Problem (1.1) ($k \in \{1, 2\}$). This proposition is a special case of the general results [3, Th. 10.2] for 1-D first-order semilinear hyperbolic systems and [2, Th. 6.10] for 1-D first-order quasilinear hyperbolic systems.

**Proposition 5.1.** Assume that there exists $Q \in C^1([0, L]; D^+(d))$ such that:

$$\kappa^2 Q_+(0) - Q_-(0) \quad \text{and} \quad Q_-(L) - Q_+(L) \quad \text{are negative semi-definite;} \tag{5.1}$$

and, for any $x \in [0, L],$

$$Q'(x)D - Q(x)B(x) - B(x)^T Q(x) \quad \text{is negative definite.} \tag{5.2}$$

Then, the steady state $v = 0$ of (1.1) is locally exponentially stable for the $H^1$ and $H^2$ norms.

**Remark 5.2.** Note that (5.1) is a condition on the boundary feedback control, while (5.2) is a condition on the structure of the governing equations.

**Proof.** As we mentioned, this proposition is already proved in a more general case in [3, 2] for $k = 1$ and $k = 2$ respectively. Here, we only make two remarks to clarify the link between this theorem and the general result [3, 2].

Since $(\text{Jac}_v g)(0) = 0$, the derivative of $(-Bv + g(v))$ with respect to $v$, evaluated at $v = 0$, is equal to $B$. The matrix (10.19) in [3, Th. 10.2] (or (6.58) in [2, Th. 6.10]) has the form

$$- \begin{bmatrix} Q_-(0) & 0 \\ O_6 & Q_+(L)D_+ \\ \end{bmatrix},$$

in our particular case, which rewrites as the product

$$\begin{bmatrix} \kappa^2 Q_+(0) - Q_-(0) & 0 \\ 0 & Q_-(L) + Q_+(L) \end{bmatrix},$$

Hence, its is negative semi-definite if and only if (5.1) holds.

Then, our objective in what follows is to show the following theorem which, together with Proposition 5.1, would imply the main result Theorem 1.3.

**Theorem 5.3.** There exists $Q \in C^1([0, L]; D^+(d))$ such that

$$\kappa^2 Q_+(0) - Q_-(0) \quad \text{and} \quad Q_-(L) - Q_+(L) \quad \text{are negative semi-definite}$$

and, for any $x \in [0, L],$

$$Q'(x)D - Q(x)B(x) - B(x)^T Q(x) \quad \text{is negative definite.} \tag{5.3}$$
where \( \Phi = \text{hand} \), the product \( \Phi + \Phi^\top \) is null (implying that this matrix is indefinite): \( Q \) should be chosen so that (5.3) holds due to the presence of \( Q^\top D \). In the following subsections, we give three results (Prop. 5.4, Prop. 5.7 and Prop. 5.9) yielding the chain

\[ \text{Prop. 5.9 } \Rightarrow \text{Prop. 5.7 } \Rightarrow \text{Prop. 5.4 } \Rightarrow \text{Th. 5.3}. \]

Thus the proof of the main result Theorem 1.3 follows by proving Proposition 5.9.

5.2. Ansatz for \( Q \). Theorem 5.3 requires to find a map \( Q \in C^1([0, L]; D^+(d)) \) fulfilling three matrix inequalities. We now choose an ansatz for \( Q \) to simplify this task. Observing the beam energy (see Section 4), we have seen that the product \( \text{diag}(2\bar{J}, 2\bar{J})B \) has the specific form given in Remark 4.2. Consequently, we choose the following ansatz: for positive functions \( w_-, w_+ \in C^1([0, L]) \), that we call \textit{weights},

\[ Q(x) = W(x)\text{diag}(2\bar{J}, 2\bar{J}), \quad \text{where} \quad W = \text{diag} (w_- \mathbb{I}_6, w_+ \mathbb{I}_6). \quad (5.4) \]

Reformulating Theorem 5.3 with \( Q \) replaced by this ansatz, we obtain the Proposition 5.4 given below. Hence, if this proposition is proved, so is Theorem 5.3.

**Proposition 5.4 (Ansatz for \( Q \)).** For \( \{\kappa_i\}_{i=1}^6 \) denoting the diagonal entries of \( \kappa \), let \( C_\kappa \in (0, 1) \) be defined by

\[ C_\kappa = \max_{1 \leq i \leq 6} \kappa_i^2, \quad (5.5) \]

and let \( \Lambda \in \mathbb{R}^{d \times d} \) and \( \Theta \in C^1([0, L]; \mathbb{R}^{d \times d}) \) be defined by

\[ \Lambda = \text{diag}(\bar{J}D_+, \bar{J}D_+), \quad \Theta = -\left[ \begin{array}{cc} O_6 & \bar{J}D_+P + (\bar{J}D_+P)^\top \\
\bar{J}D_+P + (\bar{J}D_+P)^\top & O_6 \end{array} \right]. \quad (5.6) \]

There exist positive weights \( w_-, w_+ \in C^1([0, L]) \) such that

\[ w_+(0) \leq C^{-1}_\kappa w_-(0), \quad w_+(L) \leq w_-(L) \quad (5.7) \]

and, for any \( x \in [0, L] \),

\[ \Psi = \text{diag} (-w'_- \mathbb{I}_6, w'_+ \mathbb{I}_6) \Lambda + (w_+ - w_-)\Theta \quad \text{is negative definite}. \quad (5.8) \]

Indeed, let \( Q \) be defined by (5.4). Then, on the one hand

\[ \kappa^2 Q_+(0) - Q_-(0) = (w_+(0)\kappa^2 - w_-(0))\mathbb{I}_6)2\bar{J} \]

\[ Q_-(L) - Q_+(L) = (w_-(L) - w_+(L))2\bar{J}. \]

Both diagonal matrices are negative semi-definite if and only if (5.7) holds. On the other hand, the product \( QB \) and \( B^\top Q \) now write as (see Remark 4.2)

\[ QB = \begin{bmatrix} -w_-(\Phi - \Phi^\top) & -w_-(\Phi + \Phi^\top) \\
w_+(\Phi - \Phi^\top) & w_+(\Phi + \Phi^\top) \end{bmatrix}, \quad B^\top Q = \begin{bmatrix} w_-(\Phi - \Phi^\top) & w_+(\Phi + \Phi^\top) \\
-w_-(\Phi + \Phi^\top) & -w_+(\Phi - \Phi^\top) \end{bmatrix}, \]

where \( \Phi = \bar{J}D_+P \). Hence, the sum yields

\[ QB + B^\top Q = (w_+ - w_-) \begin{bmatrix} O_6 & \Phi + \Phi^\top \\
\Phi + \Phi^\top & O_6 \end{bmatrix}, \]
and (5.2) writes as (5.8).

5.3. Assumptions on the weights. Since $\Theta$ is indefinite, our strategy is to choose $w_-, w_+$ such that the first term in the right-hand side of (5.8) is negative definite and sufficiently large (in some sense), in comparison to the second term, for $\Psi$ to be negative definite. To make this first term negative definite, we add the following constraint on the weights:

$$ w'_- > 0, \quad w'_+ < 0, \quad \text{in } [0, L]. \quad (5.9) $$

**Remark 5.5** (Feedback parameters). If the weights satisfy (5.9) and (5.7), then necessarily $w_+ > w_-$ in $[0, L]$ and

$$ \frac{w_+(0)}{w_-(0)} \in (1, C_{\kappa}^{-1}], \quad (5.10) $$

for $C_{\kappa} \in (0, 1)$ defined by (5.5). Consequently, $\kappa$ determines how different from one another the weights are allowed to be at $x = 0$: if $\kappa$ is closer to the null matrix (in the sense that $C_{\kappa}$ is closer to zero), then the weights are less constrained. Since $\kappa$ depends on the feedback parameters $\mu_1, \mu_2 > 0$, the latter influence the constraint on the weights. From (5.10), it is indeed necessary that both feedback parameters be nonzero, as otherwise $C_{\kappa} = 1$ and the interval $(1, C_{\kappa}^{-1}]$ is empty. Moreover, one can show that, for fixed beam parameters $(a, \rho, E, G, I_2, I_3, \{k_i\}_{i=1}^3)$, the smallest $C_{\kappa}$ is obtained for

$$ \mu_1 = \sqrt{\left(\min_{1 \leq i \leq 3} \alpha_i \right) \left(\max_{1 \leq i \leq 3} \alpha_i \right)}, \quad \mu_2 = \sqrt{\left(\min_{1 \leq i \leq 3} \beta_i \right) \left(\max_{1 \leq i \leq 3} \beta_i \right)}, \quad (5.11) $$

where $\{\alpha_i\}_{i=1}^3$ and $\{\beta_i\}_{i=1}^3$ are the diagonal entries of $\rho a D_1$ and $\rho J D_2$ respectively.

By means of the following lemma, we give an explicit assumption on the weights and their derivatives that is sufficient for (5.8) to hold for any $x \in [0, L]$.

**Lemma 5.6.** Let $\sigma^2 \Theta(x)$ be the largest eigenvalue of the matrix $\Theta(x)$ defined in (5.6), let $\{\Gamma_{c_i}\}_{i=1}^3 \Gamma_{c_i}$ be the components of $\Gamma_c, \gamma_c$, and let $\{\theta_i\}_{i=1}^6 \in C^0([0, L])$ be the following positive functions:

$$ \begin{align*}
\theta_1 &= |1 - \lambda_8 \lambda_7^{-1}| |\gamma_{c_3}| + |1 - \lambda_9 \lambda_7^{-1}| |\gamma_{c_2}| + |\gamma_{c_3}| + |\gamma_{c_2}| \\
\theta_2 &= |1 - \lambda_7 \lambda_8^{-1}| |\gamma_{c_3}| + |1 - \lambda_9 \lambda_8^{-1}| |\gamma_{c_1}| + |\gamma_{c_3}| + |\gamma_{c_1} + 1| \\
\theta_3 &= |1 - \lambda_7 \lambda_9^{-1}| |\gamma_{c_2}| + |1 - \lambda_8 \lambda_9^{-1}| |\gamma_{c_1}| + |\gamma_{c_2}| + |\gamma_{c_1} + 1| \\
\theta_4 &= a \lambda_8 \lambda_7 J_1^2 \Gamma_{c_3} + a \lambda_9 (\lambda_7 J_1)^{-1} \Gamma_{c_2} + |1 - \lambda_7 J_1 | \gamma_{c_3} + |1 - \lambda_7 J_1 | \gamma_{c_2} \\
\theta_5 &= a J_2 \Gamma_{c_3} \gamma_{c_3} + a \lambda_9 \lambda_7 J_2 \gamma_{c_1} + |1 - \lambda_7 J_1 | \gamma_{c_3} + |1 - J_2 | \gamma_{c_1} \\
\theta_6 &= a J_3 \Gamma_{c_2} \gamma_{c_2} + a \lambda_8 \lambda_7 J_3 \gamma_{c_1} + |1 - \lambda_7 J_1 | \gamma_{c_2} + |1 - J_3 | \gamma_{c_1}.
\end{align*} $$
Define $q_1, q_2 \in C^0([0, L])$ by

$$q_1(x) = \max_{1 \leq i \leq 6} \theta_i(x), \quad q_2(x) = \sigma_d^{\Theta(x)} \left( \min_{1 \leq i \leq 6} \tilde{J}_i \lambda_i + 6 \right)^{-1}.$$  \hfill (5.12)

Let $\ell \in \{1, 2\}$. If there exist positive weights $w_-, w_+ \in C^1([0, L])$ fulfilling (5.9), as well as $w_+(L) \leq w_-(L)$ and

$$\min \{|w_-|, |w_+|\} > (w_+ - w_-)q_\ell \quad \text{in } [0, L],$$

then the matrix $\Psi(x)$ defined in (5.8) is negative definite for any $x \in [0, L]$.

Note that the map $x \mapsto \sigma_d^{\Theta(x)}$ is continuous on $[0, L]$ since $x \mapsto \Theta(x)$ belongs to $C^0([0, L] ; \mathbb{R}^{d \times d})$, see [5, Coro. VI.1.6].

Proof. Let $x \in [0, L]$. First, we consider the case $\ell = 1$. By [12, Def. 6.1.9, Coro. 7.2.3], if a matrix is strictly diagonally dominant with negative diagonal entries, then it is negative definite. Since $\text{diag}(-w_-'(x)I_6, w_+'(x)I_6)\Lambda$ is negative definite and the diagonal entries of $(w_+(x) - w_-(x))\Theta(x)$ are null, we deduce that $\Psi(x)$ is negative definite if

$$(\text{diag}(|w_-'(x)|I_6, |w_+'(x)|I_6)\Lambda)_i > (w_+(x) - w_-(x)) \sum_{j=1}^d |\Theta_{ij}(x)|$$

holds for any $i \in \{1 \ldots d\}$. Below, even though $w_-, w_+, \Theta$ and $P$ depend on $x$, the argument $x$ is dropped for clarity. Due to the definition of $\Theta$ and $\Lambda$, (5.13) is equivalent to

$$\min \{|w_-|, |w_+|\} (\tilde{J}D_+)_i > (w_+ - w_-) \sum_{j=1}^6 (|\tilde{J}D_+P + (\tilde{J}D_+)\Lambda|)_i.$$

As $(\tilde{J}D_+P + (\tilde{J}D_+)\Lambda)_{ij} = (\tilde{J}D_+)_i \lambda_j + P_{ij} (\tilde{J}D_+)_j$, the above inequality holds if and only if

$$\min \{|w_-|, |w_+|\} > (w_+ - w_-)\theta_i,$$

where, for any $i \in \{1 \ldots d\}$, the map $\theta_i \in C^0([0, L])$ is defined by

$$\theta_i = \sum_{j=1}^6 \left| (P + \tilde{J}^{-1}D_+^{-1}P\tilde{J}D_+)_{i,j} \right|.$$

Using that $D_+ = \text{diag}(D_{+1}, D_{+2})$ and the definition of $P, \tilde{J}$ (see Section 1.1), as well as the skew-symmetry of $\tilde{\Gamma}_c$ and $\Gamma_c + e_1$, we compute

$$P + (\tilde{J}D_+)^{-1}P\tilde{J}D_+ = \begin{bmatrix} \tilde{\Gamma}_c - D_{+1}^{-1}\tilde{\Gamma}_cD_{+1} & \Gamma_c + e_1 \\ -aD_{+2}^{-1}J^{-1}\tilde{\Gamma}_c + e_1D_{+2} & \tilde{\Gamma}_c - D_{+2}^{-1}J^{-1}\tilde{\Gamma}_cD_{+2}J \end{bmatrix},$$
which also writes as

\[
\begin{pmatrix}
0 & (\frac{1}{\delta} - 1)\Gamma_{c3} & (1 - \frac{1}{\delta})\Gamma_{c1} & 0 & -\Gamma_{c3} & \Gamma_{c2} \\
(1 - \frac{1}{\delta})\Gamma_{c3} & 0 & (\frac{1}{\delta} - 1)\Gamma_{c1} & 0 & -\Gamma_{c3} & (\Gamma_{c1} + 1) \\
(\frac{1}{\delta} - 1)\Gamma_{c2} & (1 - \frac{1}{\delta})\Gamma_{c1} & 0 & (1 - \frac{1}{\delta})\Gamma_{c} & 0 & (\frac{1}{\delta} - 1)\Gamma_{c1} \\
0 & -\frac{1}{\delta}\Gamma_{c3} & -\frac{1}{\delta}\Gamma_{c2}(\Gamma_{c1} + 1) & 0 & (1 - \frac{1}{\delta})\Gamma_{c} & 0 \\
-\frac{1}{\delta}\Gamma_{c2} & -\frac{1}{\delta}\Gamma_{c3}(\Gamma_{c1} + 1) & 0 & (1 - \frac{1}{\delta})\Gamma_{c} & 0 & (\frac{1}{\delta} - 1)\Gamma_{c1} \\
\end{pmatrix}
\]

which finishes the proof for \( \ell = 1 \).

Finally, we consider the case \( \ell = 2 \). Denote by \( \{\sigma_i^M\}_{i=1}^n \) the eigenvalues of an Hermitian matrix \( M \in \mathbb{R}^{n \times n} \) in nondecreasing order (then the largest eigenvalue of \( M \) is \( \sigma_n^M \)). Weyl’s Theorem [12, Th. 4.3.1, Coro. 4.3.15] provides bounds on the eigenvalues of the sum of Hermitian matrices \( M_1, M_2 \in \mathbb{R}^{n \times n} \):

\[
\sigma_i^{M_1+M_2} \leq \sigma_i^{M_1} + \sigma_i^{M_2}.
\]

Hence, the eigenvalues of \( \Psi(x) \) necessarily satisfy

\[
\sigma_i^{\Psi(x)} \leq -\min \{ |w_-(x)|, |w_+(x)| \} \left( \min_{1 \leq i \leq 6} J_i \lambda_{i+6} \right) + (w_+(x) - w_-(x))\sigma_d^{\Theta(x)}.
\]

By Lemma 5.6, the following proposition implies Proposition 5.4.

**Proposition 5.7.** Let \( \ell \in \{1, 2\} \), let \( C_\kappa \in (0, 1) \) be defined by (5.5), and let \( q_\ell \in C^0([0, L]) \) be defined by (5.12). There exist positive weights \( w_-, w_+ \in C^1([0, L]) \) satisfying

\[
\begin{align*}
&w'_- > 0, \quad w'_+ < 0, \quad w_+ \geq w_-, \quad \text{in } [0, L], \quad (5.14) \\
&w_+(0) \leq C_\kappa^{-1} w_-(0), \quad (5.15) \\
&\min \{ |w'_-|, |w'_+| \} > (w_+ - w_-)q_\ell, \quad \text{in } [0, L]. \quad (5.16)
\end{align*}
\]
**Remark 5.8.** One can construct different weights satisfying (5.14), such as straight lines $w_- = ax + \beta$ and $w_+ = \alpha(2L - x) + \beta$ for $\alpha, \beta > 0$, or exponential functions (see Fig. 2)

$$w_- = e^{-\gamma(L-x)}, \quad w_+ = e^{\gamma(L-x)}$$

for $\gamma > 0$. For $\varepsilon, \delta > 0$, defining $f : [0, L] \rightarrow [\varepsilon, \frac{\pi}{4}]$ and $h : [0, L] \rightarrow [\varepsilon, \frac{\pi}{2}]$ by

$$f(x) = \pi x (4L)^{-1} + \varepsilon (1 - x L^{-1}), \quad h(x) = \pi x (2L)^{-1} + \varepsilon (1 - x L^{-1}),$$

one may also consider the weights (5.18) or (5.19) below (see Fig. 2)

$$w_- = \tan(f), \quad w_+ = \cot(f) \quad (5.18)$$

$$w_- = - \cot(h) + \cot(\varepsilon) + \delta, \quad w_+ = \cot(h) + \cot(\varepsilon) + \delta. \quad (5.19)$$

However, it is not straightforward to find weights also satisfying (5.15)-(5.16) for realistic beam parameters respecting the assumptions of the GEB model (such as the slenderness of the beam), especially as some of these parameters are linked to the others. For instance, $L, I_2, I_3$ and $a$ are related, and so are $E$ and $G$.

### 5.4. Existence of the weights

The following two results provide weights satisfying (5.14)-(5.15)-(5.16) without adding any constraint on the beam parameters, thus proving Proposition 5.7 and concluding the proof of the main result Theorem 1.3.

**Proposition 5.9.** Let $\ell \in \{1, 2\}$ and let $C_\kappa \in (0, 1)$ be defined by (5.5). For $q_\ell \in C^0([0, L])$ given in (5.12), define $C_{q_\ell} > 0$ by

$$C_{q_\ell} = \max_{x \in [0, L]} q_\ell(x). \quad (5.20)$$

If there exists $g \in C^1([0, L])$ such that

$$g > 0, \quad g' > 0, \quad g' > 2C_{q_\ell} (g(L) - g), \quad \text{in } [0, L], \quad (5.21)$$

then the weights $w_+, w_- \in C^1([0, L]; \mathbb{R})$ defined by

$$w_- = g, \quad w_+ = 2g(L) - g,$$

satisfy (5.14) and (5.16). Furthermore, (5.15) is also fulfilled if $g$ additionally satisfies

$$g(L) \in \left[ g(0), \frac{1 + C_\kappa^{-1} g(0)}{2} \right]. \quad (5.22)$$

**Proof.** Let $\ell \in \{1, 2\}$, and assume that there exists $g \in C^1([0, L])$ fulfilling (5.21). Then $g$ satisfies

$$g'(x) > 2q_\ell(x) (g(L) - g(x)), \quad \text{for } x \in [0, L]. \quad (5.23)$$

Both weights are positive since $w_- = g > 0$ and $w_+ > 2g(L) - g(0) > 0$. Moreover, $w'_- = g' > 0$ and $w'_+ = -g' < 0$. To finish, since for $x \in [0, L]$

$$\min \{ |w'_-(x)|, |w'_+(x)| \} = g'(x), \quad w_+(x) - w_-(x) = 2(g(L) - g(x)),$$

we deduce by (5.23) that $\min \{ |w'_-|, |w'_+| \} > q_\ell (w_+ - w_-)$. The last assertion of the proposition is the result of rewriting condition (5.15) by using that $w_-(0) = g(0)$ and $w_+(0) = 2g(L) - g(0).$ \qed
It remains to show that a function \( g \) as in Proposition 5.9 exists.

**Proposition 5.10.** Let \( c > 0 \). There exists \( g \in C^1([0, L]) \) such that
\[
g(x) > 0, \quad g'(x) > 0, \quad g'(x) > 2c(g(L) - g(x)), \quad \text{for } x \in [0, L], \tag{5.24}
\]
and \( 0 < g(0) < g(L) \) may be chosen arbitrarily.

**Proof.** Notice that (5.24) is equivalent to
\[
0 < g(0) < g(L), \quad g'(x) > 2c(g(L) - g(x)), \quad \text{for } x \in [0, L]. \tag{5.25}
\]
Let \( \alpha = 2c \). The inequality \( g' > 2c(g(L) - g) \) is equivalent to
\[
e^{\alpha x}(g'(x) + \alpha(g(x) - g(L))) > 0, \quad \text{for } x \in [0, L]. \tag{5.26}
\]
In the above left-hand side, one recognizes the derivative of \( x \mapsto e^{\alpha x}(g(x) - g(L)) \), hence (5.26) holds if and only if there exists \( \varepsilon > 0 \) such that
\[
\frac{d}{dx}(e^{\alpha x}(g(x) - g(L))) \geq \varepsilon.
\]
Integrating over \([0, x]\) and isolating the term \( g(x) \) on one side, this is equivalent to
\[
g(x) \geq g(L) - e^{-\alpha x}(g(L) - g(0) - \varepsilon x).
\]
We choose as a candidate the function \( g \) defined by
\[
g(x) = g(L) - e^{-\alpha x}(g(L) - g(0) - \varepsilon x).
\]
Equality at \( x = L \) is true if and only if \( \varepsilon = \frac{g(L) - g(0)}{L} \), which is positive if and only if \( g(L) > g(0) \). Then, \( g \) writes as
\[
g(x) = g(L) - e^{-\alpha x}(1 - xL^{-1})(g(L) - g(0)), \tag{5.27}
\]
and fulfills \( g' > 2c(g(L) - g) \) by construction. Assuming that \( g(0) > 0 \), the function \( g \) defined by (5.27) now satisfies (5.25), and this concludes the proof. \( \square \)

**Remark 5.11.** Let \( \alpha > 0 \) and \( c = C_q \ell \) for \( \ell \in \{1, 2\} \). The function \( g \) fulfilling (5.21)- (5.22) can be chosen as
\[
g(x) = \beta - e^{-2c\ell\left(1 - \frac{x}{L}\right)}(\beta - \alpha), \quad \text{with } \beta \in \left[\alpha, \frac{1 + C^{-1}_{\kappa}}{2}\alpha\right].
\]
Then \( g(0) = \alpha \) and \( g(L) = \beta \). In this case, the weights have the form (see Fig. 3)
\[
w_-(x) = \beta - e^{-2c\ell\left(1 - \frac{x}{L}\right)}(\beta - \alpha)
\]
\[
w_+(x) = \beta + e^{-2c\ell\left(1 - \frac{x}{L}\right)}(\beta - \alpha). \tag{5.28}
\]

**Remark 5.12.** In the particular case \( \Omega_c = \Omega_s \), we may compute the constants \( C_{q1} \) and \( C_{q2} \) defined by (5.20) as follows. Both \( \{\theta_i\}_{i=1}^4 \) and \( \Theta \) are independent of \( x \). On the one hand,
\[
\theta_1 = \theta_4 = 0, \quad \theta_2 = \theta_3 = 1, \quad \theta_5 = a\sqrt{k_3G(I_2\sqrt{E})}^{-1}, \quad \theta_6 = a\sqrt{k_2G(I_3\sqrt{E})}^{-1},
\]
Figure 3. Weights defined by (5.28) for different values of $c$ and $\beta - \alpha = 1$ (left), for different values of $\beta - \alpha$ and $c = 1$ (right). Here, $L = \alpha = 1$. The upper decreasing curve is $w_+$ and the lower increasing curve is $w_-$. 

implying that 

$$C_{q_1} = \max \left\{1, \frac{a\sqrt{k_3}G}{I_2\sqrt{E}}, \frac{a\sqrt{k_2}G}{I_3\sqrt{E}} \right\}.$$ 

On the other hand, $\Theta$ has the form 

$$\Theta = -\begin{bmatrix} O_3 & \bar{\Theta} \\ \Theta & O_3 \end{bmatrix}, \quad \bar{\Theta} = \begin{bmatrix} O_3 & M \\ M^T & O_3 \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda_8 \end{bmatrix}.$$ 

Since $\det(\lambda\mathbf{I}_d - \Theta) = \det(\lambda^2 \mathbf{I}_6 - \bar{\Theta}^2)$, where 

$$\bar{\Theta}^2 = \begin{bmatrix} MM^T & O_3 \\ O_3 & M^TM \end{bmatrix}, \quad MM^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_8^2 & 0 \\ 0 & 0 & \lambda_9^2 \end{bmatrix}, \quad \text{ and } M^TM = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$ 

$\Theta$ has for eigenvalues $-\lambda_8, \lambda_8, -\lambda_9$ and $\lambda_9$ with multiplicity 2, and 0 with multiplicity 4. Its largest eigenvalue is $\sigma_\Theta^{\bar{\Theta}} = \max\{\lambda_8, \lambda_9\}$, hence 

$$C_{q_2} = \min\{\lambda_7, \lambda_8, \lambda_9, a^{-1}k_1(I_2 + I_3), a^{-1}I_2\lambda_7, a^{-1}I_3\lambda_7\}.$$ 

Note that $C_{q_1} \geq 1$ and $C_{q_2} \geq 1$.

6. Conclusion and perspectives

In this work, we studied a freely vibrating beam made of an isotropic linear-elastic material, that may undergo large deflections and rotations as well as shear deformation. We assumed that the beam is clamped at one end. Applying a feedback control at the other end, we addressed the local $H^1$ and $H^2$-exponential stability of the steady state $v = 0$ of Problem (1.1), which is the intrinsic formulation of the geometrically exact beam model (or IGEB model). The strategy involved the study of the energy of the beam, of the relationship between the geometrically exact beam model and its intrinsic
formulation and of the structure of the system’s coefficients, in order to find appropriate feedback boundary controls and quadratic Lyapunov functions.

Beams may also be studied as part of networks to describe flexible structures: see the modelling done in [18] and the simulations of networks of Cosserat elastic rods carried out in [30]. Different control problems for networks of linear and nonlinear Timoshenko beams have been treated for instance in [7, 16, 17, 19]. Our next interest related to this work is the exponential stabilization for a network of IGEB by applying feedback controls at the nodes.

One may want to consider a more general IGEB model. For example, if the size of the cross sections varies along the centerline then \((a, I_2, I_3, k_2, k_3)\) depend on \(x\), implying that the IGEB equations contain additional lower order terms, \(g\) depends on \(x\), and \((A, D, L)\) also depend on \(x\) due to the presence of \(k_2\) and \(k_3\). Similarly, the equations are changed by considering material parameters \(\rho, E, G\) (and eventually \(k_2, k_3\)) varying along the centerline. Another perspective is to assume that the applied external forces are nonzero functions of \(x\) and/or \((p, R)\), such as gravity [1, eq. (4)] or aerodynamic forces [25, eq. (12)]. Then, a term of the form \(q = \frac{1}{\rho a} L(F^T_1 F^T_2, O_{1,6})\) appears in the right-hand side of the governing equations of (1.1), where \(F_1, F_2\) are local applied external forces and moments (see Remark 2.2 for the meaning of local). Consequently, IGEB might have a steady state depending on \(x\).

In the case of weights defined as in Proposition 5.9, it is also valuable to understand how the choice of \(g\) and of the feedback parameters \(\mu_1, \mu_2\) influence the exponential decay of the solution. Here, we have seen that both parameters must be positive and that they constrain the choice of the weights (hence also the choice of \(g\)), with the least constraining choice of \(\mu_1, \mu_2\) being (5.11) (see Remark 5.5). Besides, we observe in Appendix B that the exponential decay has the form

\[
\alpha = 2^{-1}C_Q(-C_g - 4C_Q C_g \delta)
\]

where \(Q = 2\text{diag}(g\bar{J}, (2g(L) - g)\bar{J})\), and \(C_g < 0\) is the maximum over \([0, L]\) of the largest eigenvalue of \(\Psi = -g\Lambda + 2(g(L) - g)\Theta\) (the positive definite diagonal matrix \(\Lambda\) and the indefinite matrix \(\Theta\) are defined in (5.6)), the size of the initial datum is constrained by \(\delta > 0\), and \(C_g, C_Q > 0\) are constants depending on \(g, Q\) defined by (B.1).

**Appendix A. Complements on the nonlinear term**

Here we define the symmetric matrices \(\{G^i\}^d_{i=1} \subset \mathbb{R}^{d \times d}\) and \(\{\bar{G}^i\}^d_{i=1} \subset \mathbb{R}^{d \times d}\) which are involved in the definitions of \(g\) and \(\bar{g}\), in (1.3) and (3.5). Denote by \(S(M) = \frac{1}{2}(M + M^T)\) the symmetric part of \(M \in \mathbb{R}^{n \times n}\), and let \(\{\bar{e}_i\}^3_{i=1} \subset \mathbb{R}^{3 \times 3}\) be defined using (1.2). In the remainder of Appendix A we write 0 instead of \(O_3\) in order to lighten the notation. For \(\{\lambda_i\}^d_{i=1}\) and \(L\) defined in (3.6) and Remark 1.1,

\[
G^i = \begin{cases} 
L^{-1}(\bar{G}^i + \lambda_{i+6} \bar{G}^{i+6})L^T & \text{if } i \leq 6 \\
L^{-1}(\bar{G}^{i-6} - \lambda_i \bar{G}^i)L^T & \text{if } i > 6
\end{cases}
\]
and \( \{\tilde{G}^i\}^d_{i=1} \) are defined by:

\[
\tilde{G}^1 = \frac{1}{2} \begin{bmatrix}
0 & -\tilde{e}_1 & 0 & 0 \\
-\tilde{e}_1 & 0 & 0 & 0 \\
0 & 0 & D_{+1}^2 \tilde{e}_1 & 0 \\
0 & 0 & -\tilde{e}_1 D_{+1}^2 & 0 
\end{bmatrix}, \quad \tilde{G}^2 = \frac{1}{2} \begin{bmatrix}
0 & -\tilde{e}_2 & 0 & 0 \\
\tilde{e}_2 & 0 & 0 & 0 \\
0 & 0 & D_{+1}^2 \tilde{e}_2 & 0 \\
0 & 0 & -\tilde{e}_2 D_{+1}^2 & 0 
\end{bmatrix},
\]

\[
\tilde{G}^3 = \begin{bmatrix}
0 & -\tilde{e}_3 & 0 & 0 \\
\tilde{e}_3 & 0 & 0 & 0 \\
0 & 0 & D_{+1}^2 \tilde{e}_3 & 0 \\
0 & 0 & -\tilde{e}_3 D_{+1}^2 & 0 
\end{bmatrix}, \quad \tilde{G}^4 = \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix},
\]

\[
\tilde{G}^5 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}, \quad \tilde{G}^6 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix},
\]

On a side note, for \( y = (u_1^T, u_2^T, u_3^T, u_4^T)^T \), where \( \{u_i\}^4_{i=1} \subset \mathbb{R}^3 \), the map \( \tilde{g} \) equivalently writes as

\[
\tilde{g}(y) = \begin{bmatrix}
u_1 \times u_2 + (\rho a)^{-1} u_4 \times (M_1 u_3) \\
u_4 \times u_1 + u_3 \times u_2 + (\rho J)^{-1}(u_4 \times (M_2 u_4))\end{bmatrix}.
\]

where \( J, M_1, M_2 \) are defined in Section 1.1 and Section 2.

**APPENDIX B. THE EXPONENTIAL DECAY**

Here, we precise the form of the constants \( \varepsilon > 0, \eta \geq 1, \alpha > 0 \) of the main theorem in the case of \( H^1 \)-exponential stabilization (see Definition 1.2 with \( k = 1 \)), by following the proof of [3, Th. 10.2] for the special case of Problem (1.1) while making the constants explicit.
For $Q \in C^1([0,L]; D^+(d))$ as in Proposition 5.1, denote $\Psi = Q^T D - QB - B^T Q$ and let the constants $C_Q \geq 1$, $C_g > 0$, $C_B > 0$, $C_{\text{IB}}(r) > 0$ (for any $r > 0$) and $C_\Psi < 0$ be defined by:

$$C_Q = \max \left\{ \max_{x \in [0,L]} \min_{1 \leq i \leq d} Q_i(x), \left( \min_{x \in [0,L]} \max_{1 \leq i \leq d} Q_i(x) \right)^{-1} \right\}, \quad C_g = \sqrt{\sum_{i=1}^d \|G^2\|^2}, \quad (B.1)$$

$$C_B = \max_{x \in [0,L]} \|B(x)\|, \quad C_{\text{IB}}(r) = \max \left\{ \|D\|, \|D\|^{-1}, (C_B + C_g r), \frac{C_B + C_g r}{\|D\|} \right\},$$

$$C_\Psi = \max_{x \in [0,L]} \sigma_{\Psi(x)}^d.$$

Here $\sigma_{\Psi(x)}^d$ denotes the largest eigenvalue of $\Psi(x)$. Let $C_1 > 0$ be the constant from the standard Sobolev inequality

$$\|\varphi\|_{C^0([0,L]; \mathbb{R}^d)} \leq C_1 \|\varphi\|_{H^1(0,L)}, \quad \text{for } \varphi \in H^1(0,L).$$

By [3, Th. 10.1], there exists $\delta_0 > 0$ and $T_{\text{max}} > 0$ such that if $v^0 \in H^1(0,L)$ satisfies (C0) and $\|v^0\|_{H^1(0,L)} \leq \delta_0$, then Problem (1.1) admits a unique maximal solution $v \in C^0([0,T_{\text{max}}]; H^1(0,L))$. Moreover, $T_{\text{max}} = +\infty$ if

$$\|v\|_{C^0([0,T_{\text{max}}]; H^1(0,L))} \leq \delta_0.$$ 

Then, we will see below that $\varepsilon > 0$, $\eta \geq 1$ and $\alpha > 0$ can be chosen as follows: for any constant $\delta > 0$ fulfilling $-C_\Psi - 4C_Q C_g \delta > 0$, we may choose:

$$\alpha = \frac{C_Q}{2} \left( -C_\Psi - 4C_Q C_g \delta \right), \quad \eta = C_Q (2C_{\text{IB}}(\delta) + 1), \quad \varepsilon = \min \left\{ \frac{\delta}{2C_1 \eta}, \frac{\delta_0}{\eta} \right\}. \quad (B.2)$$

We now proceed with showing this by following the proof of [3, Th. 10.2]. Consider the quadratic $H^1$-Lyapunov function $\mathcal{L}$ defined on $[0,T]$ by

$$\mathcal{L} = \int_0^T (v^T Q v + \partial_t v^T Q \partial_t v) dx,$$

for $v: [0,L] \times [0,T] \rightarrow \mathbb{R}^d$. Note that the nonlinear term in (1.1) also writes as $g(v) = M(v)v$ for $v \in \mathbb{R}^d$, with $M(v) \in \mathbb{R}^{d \times d}$ defined by

$$M(v) = [G^1 v, G^2 v, \ldots, G^d v]^T.$$

Moreover, $\|M(v)\| \leq C_g |v|$ for any $v \in \mathbb{R}^d$.

**Lemma B.1.** Let $T > 0$. There exists $\delta > 0$ such that the following holds: if $v$ is the unique $C^0([0,T]; H^1(0,L))$ solution to (1.1) and if $\|v\|_{C^0([0,L] \times [0,T]; \mathbb{R}^d)} \leq \delta$ then there exists $\eta \geq 1$ and $\alpha > 0$ for which

$$\frac{1}{\eta} \|v(\cdot, t)\|_{H^1(0,L)}^2 \leq \mathcal{L}(t) \leq \eta \|v(\cdot, t)\|_{H^1(0,L)}^2, \quad \text{for any } t \in [0,T], \quad (B.3)$$

$$\mathcal{L}(t) \leq e^{-2\alpha t} \mathcal{L}(0), \quad \text{for any } t \in [0,T]. \quad (B.4)$$
Note that if \( v \in C^0([0,T]; \mathbf{H}^1(0,L)) \) is the solution to (1.1), then \( \partial_t v \) belongs to \( C^0([0,T]; \mathbf{L}^2(0,L)) \) and \( \mathcal{L}(t) < \infty \) for any \( t \in [0,T] \).

**Proof of Lemma B.1.** For \( \delta > 0 \) to be chosen, let \( v \) satisfy the assumptions of Lemma B.1 and assume additionally that \( v \in C^2([0,L] \times [0,T]; \mathbb{R}^d) \). Using the governing equations, integration by parts (for the space partial derivative) and the boundary conditions, one deduces that

\[
\frac{d\mathcal{L}}{dt} = \int_0^L \left[v^\top (\Psi + 2QM(v))v + \partial_tv^\top (\Psi + 4QM(v))\partial_tv\right] dx \\
+ v_+(L,t)^\top D_+ (Q_-(L) - Q_+(L)) v_+(L,t) + v_-(0,t)^\top D_+ (\kappa^2 Q_+(0) - Q_-(0)) v_-(0,t) \\
+ \partial_t v_+(L,t)^\top D_+ (Q_-(L) - Q_+(L)) \partial_t v_+(L,t) \\
+ \partial_t v_-(0,t)^\top D_+ (\kappa^2 Q_+(0) - Q_-(0)) \partial_t v_-(0,t).
\]

From the assumptions on \( Q \),

\[
\frac{d\mathcal{L}}{dt} \leq \int_0^L \left( C_\Psi + 4C_Q C_\delta \right) (|v|^2 + |\partial_tv|^2) dx \leq \left( C_\Psi + 4C_Q C_\delta \right) C_\Psi \mathcal{L}.
\]

Gronwall’s inequality yields (B.4) for \( \alpha \) defined by (B.2) and \( \delta \) small enough to have \( \alpha > 0 \). Since \( v \) is solution to Problem (1.1), the governing equations yield that

\[
|\partial_tv| = | -D_2 v - Bv + M(v)v| \leq \|D_2||\partial_tv| + (C_B + C_\delta)|v|,
\]

\[
|\partial_tv| = |D_2 (-\partial_tv - Bv + M(v)v)| \leq \|D_2^{-1}||v| + (C_B + C_\delta)|v|).
\]

Hence, for \( C_{IB} = C_{IB}(\delta) \), we have \( |\partial_tv| \leq C_{IB}(|v| + |\partial_tv|) \) and \( |\partial_tv| \leq C_{IB}(|v| + |\partial_tv|) \).

Setting \( \eta = C_Q(2C_{IB}^2 + 1) \), we obtain (B.3) by using

\[
\frac{1}{2C_{IB}^2 + 1} (|v|^2 + |\partial_tv|^2) \leq (|v|^2 + |\partial_tv|^2) \leq (2C_{IB}^2 + 1) (|v|^2 + |\partial_tv|^2)
\]

and

\[
\frac{1}{C_Q} \int_0^L (|v|^2 + |\partial_tv|^2) dx \leq \mathcal{L}(t) \leq C_Q \int_0^L (|v|^2 + |\partial_tv|^2) dx.
\]

The selection of \( \eta, \alpha \) depends on the \( C^0([0,L] \times [0,T]; \mathbb{R}^d) \) norm of the solution. By a density argument similar to [2, Comment 4.6], the estimates (B.3)-(B.4) remain valid for \( v \in C^0([0,T]; \mathbf{H}^1(0,L)) \). \( \square \)

Then, the proof of local \( H^1 \)-exponential stability unfolds as follows. Let \( \delta, \eta, \alpha > 0 \) be the constants from Lemma B.1. We set \( \varepsilon \) as in (B.2). Assume that the initial datum satisfies \( \|v^0\|_{\mathbf{H}^1(0,L)} \leq \varepsilon \leq \delta_0 \) (since \( \eta \geq 1 \)), and let \( v \) be the associated maximal solution. Then \( \|v^0\|_{C^0([0,L]; \mathbb{R}^d)} \leq C_1 \varepsilon \), and by definition of \( \varepsilon \) we have \( \|v^0\|_{C^0([0,L]; \mathbb{R}^d)} < \delta \). Hence \( \mathcal{L}(0) \leq \eta \varepsilon^2 \) by (B.3). Since \( v \) is continuous in \([0,L] \times [0,T] \) and \( \|v^0\|_{C^0([0,L]; \mathbb{R}^d)} < \delta \), there exists \( \tau > 0 \) such that \( \|v(\cdot,t)\|_{C^0([0,L]; \mathbb{R}^d)} \leq \delta \) for \( t \in [0,\tau] \). Hence by (B.4), \( \mathcal{L}(t) \leq \eta \varepsilon^2 \) for \( t \in [0,\tau] \), and by (B.3)

\[
\|v(\cdot,t)\|_{C^0([0,L]; \mathbb{R}^d)} \leq C_1 \|v(\cdot,t)\|_{\mathbf{H}^1(0,L)} \leq C_1 (\eta \mathcal{L}(t))^\frac{1}{2} \leq C_1 \eta \varepsilon < \delta, \quad \text{for} \quad t \in [0,\tau].
\]
Using this reasoning, we deduce that on $[0, T_{\max})$, 
\[ \| v(\cdot, t) \|_{C^0(\{0, L\}; \mathbb{R}^d)} \leq \delta, \quad \text{and} \quad \| v(\cdot, t) \|_{H^1(0, L)} \leq \delta_0. \]
The second above inequality implies that $T_{\max} = +\infty$, while the first inequality yields, by (B.3)-(B.4), that 
\[ \| v(\cdot, t) \|_{H^1(0, L)} \leq \eta e^{-\alpha t} \| v^0 \|_{H^1(0, L)}, \quad \text{for } t \in [0, +\infty). \]

ACKNOWLEDGMENTS

The authors are grateful to Martin Gugat for the helpful discussions and advice.

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