Optimal bounds for Neuman-Sándor mean in terms of the convex combination of the logarithmic and the second Seiffert means

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Abstract
In the article, we prove that the double inequality

\[ \alpha L(a, b) + (1 - \alpha)T(a, b) < NS(a, b) < \beta L(a, b) + (1 - \beta)T(a, b) \]

holds for \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \geq 1/4 \) and \( \beta \leq 1 - \pi/[4 \log(1 + \sqrt{2})] \), where \( NS(a, b), L(a, b) \) and \( T(a, b) \) denote the Neuman-Sándor, logarithmic and second Seiffert means of two positive numbers \( a \) and \( b \), respectively.

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1 Introduction
For \( a, b > 0 \) with \( a \neq b \), the Neuman-Sándor mean \( NS(a, b) \) [1], the second Seiffert mean \( T(a, b) \) [2], and the logarithmic mean \( L(a, b) \) [1] are defined by

\[ NS(a, b) = \frac{a - b}{2 \sinh^{-1}[(a - b)/(a + b)]}, \]
\[ T(a, b) = \frac{a - b}{2 \tan^{-1}[(a - b)/(a + b)]}, \]
\[ L(a, b) = \frac{a - b}{\log a - \log b}, \]

respectively. It can be observed that the logarithmic mean \( L(a, b) \) can be rewritten as (see as [1])

\[ L(a, b) = \frac{a - b}{2 \tanh^{-1}[(a - b)/(a + b)]}, \]

where \( \sinh^{-1}(x) = \log(x + \sqrt{1 + x^2}) \), \( \tanh^{-1}(x) = \log \sqrt{(1 + x)/(1 - x)} \) and \( \tan^{-1}(x) = \arctan(x) \), are the inverse hyperbolic sine, inverse hyperbolic tangent, and inverse tangent, respectively.
Recently, the means $NS$, $T$, $L$ and other means have been the subject of extensive research. In particular, many remarkable inequalities for the Neuman-Sándor, second Seiffert and logarithmic means can be found in the literature [2–16].

Let $P(a, b) = (a - b)/(2\sin^{-1}[(a - b)/(a + b)])$, $S(a, b) = \sqrt{(a^2 + b^2)/2}$, $A(a, b) = (a + b)/2$, $I(a, b) = 1/e(b^\alpha/a^\beta)^{1/(\beta - \alpha)}$, $G(a, b) = \sqrt{ab}$, and $H(a, b) = 2ab/(a + b)$ denote the first Seiffert, root-square, arithmetic, identric, geometric, and the harmonic means of two positive numbers $a$ and $b$ with $a \neq b$, respectively. Then it is well known that the inequality

$$S(a, b) > T(a, b) > NS(a, b) > A(a, b) > I(a, b) > P(a, b) > G(a, b) > H(a, b)$$

holds for $a, b > 0$ with $a \neq b$.

In [17] and [18], the authors proved that the double inequalities

$$S(a, b)^{1-\alpha_3}A^{\alpha_3}(a, b) < NS(a, b) < S(a, b)^{\beta_3}A^{1-\beta_3}(a, b),$$

$$\alpha_3 S(a, b) + (1 - \alpha_3)G(a, b) < NS(a, b) < \beta_3 S(a, b) + (1 - \beta_3)G(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/3$, $2(\log(2 + \sqrt{2}) - \log 3)/\log 2 \leq \beta_3 \leq 1$, $\alpha_4 \leq 2/3$ and $\beta_4 \geq 1/[\sqrt{2}\log(1 + \sqrt{2})]$.

In [19], it was shown that the inequality

$$P^{\alpha_2}(a, b)T^{1-\alpha_2}(a, b) < NS(a, b) < P^{\beta_2}(a, b)T^{1-\beta_2}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 > 1/3$ and

$$\beta_2 \leq \log\left(\frac{4\log(1 + \sqrt{2})}{\pi}\right)/\log 2 = 0.1663 \ldots$$

Let $L_p(a, b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ be the Lehmer mean of two positive numbers $a$ and $b$ with $a \neq b$. In [10], the authors proved the double inequality

$$L_{\alpha_1}(a, b) < NS(a, b) < L_{\beta_1}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 = 1.8435 \ldots$ is the unique solution of the equation $(p + 1)^{1/p} = 2\log(1 + \sqrt{2})$, and $\beta_1 = 2$.

Let

$$M_p(a, b) = \begin{cases} 
\left(\frac{a^{p+1} + b^{p+1}}{2}\right)^{1/p}, & p \neq 0, \\
\sqrt{ab}, & p = 0,
\end{cases}$$

be the $p$th power means of two positive numbers $a$ and $b$ with $a \neq b$. In [20], the authors proved the sharp double inequality

$$M_{\log 2/(\log \pi - \log 2)}(a, b) < T(a, b) < M_{5/3}(a, b)$$

holds.
Gao [21] proved the optimal double inequality

\[ I(a, b) < T(a, b) < \frac{2e}{\pi} I(a, b) \]

holds for all \( a, b > 0 \) with \( a \neq b \).

Yang [22] proved the inequality

\[ A^{1/3}(a, b) G^{1-1/3}(a, b) < L(a, b) < A^{1/3}(a, b) G^{1-1/3}(a, b) \]

holds for all \( a, b > 0 \) if and only if \( p \geq 1/\sqrt{3} \) and \( 0 < q \leq 1/3 \). And the inequality

\[ M_0(a, b) < L(a, b) < M_{1/3}(a, b) \]

was proved by Lin in [23].

In [24], the authors present bounds for \( L \) in terms of \( G \) and \( A \)

\[ G^{2/3}(a, b) A^{1/3}(a, b) < L(a, b) < \frac{2}{3} G(a, b) + \frac{1}{3} A(a, b) \]

for all \( a, b > 0 \) with \( a \neq b \).

The purpose of this paper is to answer the following questions: What are the least value \( \alpha \) and the greatest value \( \beta \) such that

\[ \alpha L(a, b) + (1 - \alpha) T(a, b) < NS(a, b) < \beta L(a, b) + (1 - \beta) T(a, b) \]

holds for all \( a, b > 0 \) with \( a \neq b \) ?

2 Lemmas

It is well known that, for \( x \in (0, 1) \),

\[ \tanh^{-1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}, \quad (2.1) \]

\[ \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \quad (2.2) \]

To establish our main result, we need several lemmas as follows.

**Lemma 2.1** ([25]) Let

\[ H(x) = \frac{1}{\sinh^{-1} x} - \frac{x}{\sqrt{1 + x^2(\sinh^{-1} x)^2}}. \]

Then \( H(x) \) is strictly increasing on \((0, 1)\). Moreover, the inequality

\[ H(x) < \frac{x}{3} - \frac{x^3}{9} \quad (2.3) \]
holds for any \( x \in (0,0.76) \) and the inequality

\[
H(x) > \frac{x}{3} - \frac{17x^3}{90}
\]  

(2.4)

holds for any \( x \in (0,1) \).

**Lemma 2.2** Let \( S(x) = 1/\tanh^{-1}x - x/[(1 - x^2)(\tanh^{-1}x)^2] \). Then

\[
S(x) < -\frac{2}{3} x - \frac{1}{3} x^3 - \frac{1}{3} x^5
\]  

(2.5)

for any \( x \in (0,1) \) and

\[
S(x) > -\frac{2}{3} x - x^3 - \frac{x^5}{4}
\]  

(2.6)

for any \( x \in (0,0.76) \).

**Proof** Let

\[
G(x) = (1 - x^2)(\tanh^{-1}x)^2 \left[ S(x) + \frac{2}{3} x + \frac{1}{3} x^3 + \frac{1}{3} x^5 \right].
\]

Then direct computation leads to

\[
G(0) = 0,
\]  

(2.7)

\[
G'(x) = \frac{1}{3} g(x) \tanh^{-1}x,
\]  

(2.8)

where \( g(x) = (2 - 7x^6 - 3x^2)\tanh^{-1}x + 2x^5 + 2x^3 - 2x \). It follows that

\[
g'(x) = \frac{1}{1 - x^2} g_1(x),
\]  

(2.9)

where \( g_1(x) = (-42x^5 - 6x)(1 - x^2)\tanh^{-1}x - 17x^6 + 4x^4 + 5x^2 \). Considering (2.1), we have

\[
g_1(x) < (-42x^5 - 6x)(1 - x^2) \left( x + \frac{x^3}{3} + \frac{x^5}{5} \right) - 17x^6 + 4x^4 + 5x^2
\]

\[
= \frac{1}{5} \left( 42x^{12} + 28x^{10} + 146x^8 - 291x^6 + 40x^4 - 5x^2 \right)
\]

\[
< x^2(216x^6 - 291x^4 + 40x^2 - 5) < 0,
\]  

(2.10)

for \( x \in (0,1) \). Thus, (2.9) and (2.10) as well as \( g(0) = 0 \) imply \( g(x) < 0 \) for \( x \in (0,1) \). Therefore, combining (2.7) and (2.8), we get \( G(x) < 0 \) for \( x \in (0,1) \). It means inequality (2.5) holds.

Let

\[
Q(x) = (1 - x^2)(\tanh^{-1}x)^2 \left[ S(x) + \frac{2}{3} x + x^3 + \frac{x^5}{4} \right].
\]
Direct computation leads to

\[ Q(0) = 0, \quad (2.11) \]
\[ Q'(x) = \frac{1}{12} q_1(x) \tanh^{-1} x, \quad (2.12) \]

where

\[ q_1(x) = 6x^5 + 24x^3 - 8x + (8 + 12x^2 - 45x^4 - 21x^6) \tanh^{-1} x. \]

When \( x \in (0, 0.7] \), considering (2.1) and the fact \( 8 + 12x^2 - 45x^4 - 21x^6 = (3 - 21x^4) + (5 + 12x^2 - 45x^4) > 0 \), we can get

\[ q_1(x) > 6x^5 + 24x^3 - 8x + (8 + 12x^2 - 45x^4 - 21x^6) \left( x + \frac{x^3}{3} + \frac{x^5}{5} \right) \]
\[ = -\frac{21}{5} x^{11} - 16x^9 - \frac{168}{5} x^7 - \frac{167}{5} x^5 + \frac{116}{3} x^3 \]
\[ > x^3 \left( -\frac{269}{5} x^4 - \frac{167}{5} x^2 + \frac{116}{3} \right) > 0. \]

When \( x \in (0.7, 0.76) \), direct computation leads to

\[ q_1(0.76) = 1.8639 \ldots > 0, \quad (2.13) \]
\[ q_1(x) = q_2(x)/(1 - x^2), \quad (2.14) \]

where \( q_2(x) = 92x^2 - 87x^4 - 51x^6 + (126x^7 + 54x^5 - 204x^3 + 24x) \tanh^{-1} x. \) Considering (2.1) and the fact \( 126x^7 + 54x^5 - 204x^3 + 24 < 12x(15x^4 - 17x^2 + 2) < 0 \), we can get

\[ q_2(x) < 92x^2 - 87x^4 - 51x^6 + (126x^7 + 54x^5 - 204x^3 + 24x) \left( x + \frac{x^3}{3} + \frac{x^5}{5} \right) \]
\[ = \frac{126}{5} x^{12} + \frac{264}{5} x^{10} + \frac{516}{5} x^8 - \frac{301}{5} x^6 - 283x^4 + 116x^2 \]
\[ < 2x^4(91x^4 - 30x^2 - 20) + x^2(116 - 243x^2) < 0. \quad (2.15) \]

Thus, (2.13)-(2.15) imply that

\[ q_1(x) > 0 \quad (2.16) \]

holds for any \( x \in (0.7, 0.76) \).

Therefore, \( Q(x) > 0 \) for \( x \in (0, 0.76) \) follows from (2.11), (2.12) and (2.16). That means inequality (2.6) holds. \( \square \)

**Lemma 2.3** Let \( T(x) = 1/\tan^{-1}x - x/[(1 + x^2)(\tan^{-1}x)^2] \). Then

\[ T(x) < \frac{2}{3}x - \frac{1}{3}x^3 + \frac{2}{7}x^5 \quad (2.17) \]
for any $x \in (0, 1)$ and
\[ T(x) > \frac{2}{3} x - \frac{2}{5} x^3 + \frac{x^5}{7} \]  \quad (2.18)
for any $x \in (0, 0.76)$.

**Proof** Let
\[
M(x) = \left[ T(x) - \frac{2}{3} x + \frac{x^3}{3} - \frac{2}{7} x^5 \right] (1 + x^2) (\tan^{-1} x)^2.
\]
Differentiating $M(x)$, we have $M'(x) = [t(x) \tan^{-1} x]/21$, where
\[
t(x) = 14x + 14x^3 - 12x^5 + (-42x^6 + 5x^4 - 21x^2 - 14) \tan^{-1} x.
\]
For $x \in (0, 1)$, we have $-42x^6 + 5x^4 - 21x^2 - 14 < -42x^6 - 16x^2 - 14 < 0$. Thus from (2.2), we can get
\[
t(x) < 14x + 14x^3 - 12x^5 + (-42x^6 + 5x^4 - 21x^2 - 14) \left( x - \frac{x^3}{3} \right)
\]
\[
= 14x^9 - \frac{131}{3} x^7 - \frac{7}{3} x^3
\]
\[
< -\frac{89}{3} x^7 - \frac{7}{3} x^3 < 0.
\]
Therefore $M'(x) < 0$ for $x \in (0, 1)$. Considering the fact $M(0) = 0$, we get $M(x) < 0$ for $x \in (0, 1)$. So the inequality (2.17) holds.

Let
\[
N(x) = \left[ T(x) - \frac{2}{3} x + \frac{2}{5} x^3 - \frac{x^5}{7} \right] (1 + x^2) (\tan^{-1} x)^2.
\]
Differentiating $N(x)$, we have $N'(x) = n(x) \tan^{-1} x$, where
\[
n(x) = \frac{2}{3} x + \frac{4}{5} x^3 - \frac{2}{7} x^5 - \left( x^6 - \frac{9}{7} x^4 + \frac{4}{5} x^2 + \frac{2}{3} \right) \tan^{-1} x.
\]
Because of
\[
\left( \frac{4}{5} x^2 - \frac{9}{7} x^4 \right) + x^6 + \frac{2}{3} > 0
\]
for $x \in (0, 0.76)$, it follows that
\[
n(x) > \frac{2}{3} x + \frac{4}{5} x^3 - \frac{2}{7} x^5 - \left( x^6 - \frac{9}{7} x^4 + \frac{4}{5} x^2 + \frac{2}{3} \right) x
\]
\[
= x^5 - x^7 > 0.
\]
Considering the fact $N(0) = 0$, the inequality (2.18) holds. \qed
Lemma 2.4 The function \( f(x) = \lambda S(x) + (1 - \lambda)T(x) - H(x) \) is strictly decreasing on \((0.76, 1)\), where \( \lambda = 1 - \pi / [4 \log(1 + \sqrt{2})] = 0.1089 \ldots \) and \( H(x), S(x) \) and \( T(x) \) are defined as in Lemmas 2.1, 2.2 and 2.3, respectively.

Proof Direct computation leads to
\[
S'(x) = 2 \frac{x - \tanh^{-1} x}{(1 - x^2)(1 + \tanh^{-1} x)^2},
\]
\[
S''(x) = 2 \frac{\psi(x)}{(1 - x^2)(1 + \tanh^{-1} x)^4},
\]
where \( \psi(x) = 3(1 + x^2)\tanh^{-1} x - 3x - 4x(\tanh^{-1} x)^2 \). It follows that
\[
\psi'(x) = R(x) \frac{1}{1 - x^2},
\]
where \( R(x) = -4(1 - x^2)(\tanh^{-1} x)^2 - (6x^3 + 2x) \tanh^{-1} x + 6x^2 \). From (2.3), we can get
\[
R(x) < -4(1 - x^2) \left( x + \frac{x^3}{3} \right)^2 - (6x^3 + 2x) \left( x + \frac{x^3}{3} \right) + 6x^2 = \frac{4}{9}x^4 + \frac{2}{9}x^6 - \frac{16}{3}x^4 < 0.
\]
Thus \( \psi(x) \) is strictly decreasing on \((0.76, 1)\). Considering the fact \( \psi(0.76) = -0.5821 \ldots < 0 \), we have \( \psi(x) < 0 \) for any \( x \in (0.76, 1) \). In other words, \( S'(x) \) is strictly decreasing on \((0.7, 1)\). Let \( \phi(x) = \lambda S(x) + (1 - \lambda)T(x) \). It was proved that \( T'(x) \) is strictly decreasing on \((0.7, 1)\) in Lemma 5 of [26]. Thus, from the monotonicity of \( S'(x) \) and \( T''(x) \), we have
\[
\phi'(x) < \lambda S'(0.76) + (1 - \lambda)T''(0.76) = -0.0043 \ldots < 0
\]
for any \( x \in (0.76, 1) \). That is to say, \( \phi(x) \) is strictly decreasing on \((0.76, 1)\). Considering the monotonicity of \( H(x) \) in Lemma 2.1, the proof is completed.

\[\square\]

Lemma 2.5 We have
\[
\frac{4 - 11\lambda}{28}x^4 - \frac{27\lambda + 13}{45}x^2 + \frac{1 - 4\lambda}{3} > 0
\]
for \( x \in (0, 0.76) \), where \( \lambda = 1 - \pi / [4 \log(1 + \sqrt{2})] = 0.1089 \ldots \).

Proof Let
\[
\eta(x) = \frac{4 - 11\lambda}{28}x^4 - \frac{27\lambda + 13}{45}x^2 + \frac{1 - 4\lambda}{3}.
\]
Then it is easy to verify that \( \eta(x) \) is decreasing on \((0, \mu)\), where
\[
\mu = \sqrt{\frac{14}{15}} \times \sqrt{\frac{160 \log(1 + \sqrt{2}) - 27\pi}{11\pi - 28 \log(1 + \sqrt{2})}} = 1.3303 \ldots
\]
Considering \( \eta(0.76) = 0.01693 \ldots > 0 \), we have \( \eta(x) > 0 \) for \( x \in (0, 0.76) \).

\[\square\]
3 Main results

Theorem 3.1  The double inequality

\[ \alpha L(a, b) + (1 - \alpha)T(a, b) < NS(a, b) < \beta L(a, b) + (1 - \beta)T(a, b) \]

holds for any \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \geq 1/4 \) and

\[ \beta \leq 1 - \frac{\pi}{4 \log(1 + \sqrt{2})} = 0.1089 \ldots. \]

Proof  Because \( NS(a, b), L(a, b), T(a, b) \) are symmetric and homogeneous of degree 1, without loss of generality, we can assume that \( a > b \) and \( x := (a - b)/(a + b) \in (0, 1) \). Let \( p \in (0, 1) \) and \( \lambda = 1 - \pi /[4 \log(1 + \sqrt{2})] = 0.1089 \ldots \). Then by (1.1)-(1.3), direct computation leads to

\[
\begin{align*}
\frac{NS(a, b)}{A(a, b)} &= \frac{x}{\sinh^{-1} x}, \\
\frac{L(a, b)}{A(a, b)} &= \frac{x}{\tanh^{-1} x}, \\
\frac{T(a, b)}{A(a, b)} &= \frac{x}{\tan^{-1} x}.
\end{align*}
\]

Let

\[ F_t(x) = \frac{tL(a, b) + (1 - t)T(a, b) - M(a, b)}{A(a, b)} \]

\[ = t - \frac{x}{\tanh^{-1} x} + (1 - t) \frac{x}{\tan^{-1} x} - \frac{x}{\sinh^{-1} x}. \] \hspace{1cm} (3.1)

Then it follows that

\[ F^1_{\lambda}(0^+) = 0, \] \hspace{1cm} (3.2)

\[ F^2_{\lambda}(0^+) = F^2_{\lambda}(1^-) = 0. \] \hspace{1cm} (3.3)

Differentiating \( F_t(x) \), we have

\[ F_t'(x) = t \left[ \frac{1}{\tanh^{-1} x} - \frac{x}{1 - x^2} \left( \tanh^{-1} x \right)^2 \right] + (1 - t) \left[ \frac{1}{\tan^{-1} x} - \frac{x}{1 + x^2} \left( \tan^{-1} x \right)^2 \right] \]

\[ - \left[ \frac{1}{\sinh^{-1} x} - \frac{x}{\sqrt{1 + x^2}} \left( \sinh^{-1} x \right)^2 \right] \]

\[ := tS(x) + (1 - t)T(x) - H(x), \]

where \( H(x), S(x) \) and \( T(x) \) are defined as in Lemmas 2.1-2.3, respectively.
On one hand, from inequalities (2.4), (2.5) and (2.16), we clearly see that

\[ F_{\frac{4}{3}}(x) = -S(x) + \frac{3}{4}T(x) - H(x) \]
\[ < \frac{1}{4} \left( -\frac{2}{3}x^2 + \frac{1}{3}x^3 - \frac{1}{3}x^5 \right) + \frac{3}{4} \left( \frac{2}{3}x^2 - \frac{2}{5}x^3 + \frac{x^5}{7} \right) - \left( \frac{x}{3} - \frac{17}{90}x^3 \right) \]
\[ = -\frac{13}{90}x^3 + \frac{11}{84}x^5 < 0 \]

for any \( x \in (0, 1) \). It leads to

\[ F_{\frac{4}{3}}(x) < F_{\frac{4}{3}}(0) = 0 \>

(3.4)

for any \( x \in (0, 1) \). Thus, from (3.1) it follows that

\[ NS(a, b) > \frac{1}{4}L(a, b) + \frac{3}{4}T(a, b) \]

for all \( a, b > 0 \) with \( a \neq b \). Considering \( L(a, b) < NS(a, b) < T(a, b) \), we can get

\[ NS(a, b) > \alpha L(a, b) + (1 - \alpha)T(a, b) \>

(3.5)

for all \( \alpha \geq 1/4 \) and \( a, b > 0 \) with \( a \neq b \).

On the other hand, from inequalities (2.3), (2.6) and (2.17), we have

\[ F_{\frac{3}{4}}(x) > -\lambda \left( \frac{2}{3}x^2 + \frac{x^5}{4} \right) + (1 - \lambda) \left( \frac{2}{5}x^2 - \frac{2}{5}x^3 + \frac{x^5}{7} \right) - \left( \frac{x}{3} - \frac{x^3}{9} \right) \]
\[ = x \left[ \frac{4 - 11\lambda}{28}x^4 - \frac{27\lambda + 13}{45}x^2 + \frac{1 - 4\lambda}{3} \right] \]

for \( x \in (0, 0.76) \). According to Lemma 2.5, we have

\[ F_{\frac{3}{4}}(x) > 0 \>

(3.6)

for \( x \in (0, 0.76) \). Lemma 2.4 shows that \( F_{\frac{3}{4}}(x) \) is strictly decreasing on \((0.76, 1)\). This fact and \( F_{\frac{3}{4}}(0.76) = 0.0713... > 0 \) together with \( F_{\frac{3}{4}}(1^-) = -\infty \) imply that there exists \( x_0 \in (0.76, 1) \) such that \( F_{\frac{3}{4}}(x) \) is strictly increasing on \((0, x_0] \) and strictly decreasing on \([x_0, 1)\).

Equations (3.1) and (3.3) together with the piecewise monotonicity of \( F_{\frac{3}{4}}(x) \) lead to the conclusion that

\[ NS(a, b) < \lambda L(a, b) + (1 - \lambda)T(a, b) \]

for all \( a, b > 0 \) with \( a \neq b \). Considering \( L(a, b) < M(a, b) < T(a, b) \), we can get

\[ NS(a, b) < \beta L(a, b) + (1 - \beta)T(a, b) \>

(3.7)

holds for \( \beta \leq \lambda \) and all \( a, b > 0 \) with \( a \neq b \).
Finally, we prove that \(L(a,b)/4 + 3T(a,b)/4 \) and \(\lambda L(a,b) + (1-\lambda)T(a,b)\) are the best possible lower and upper mean bound for the Neuman-Sándor mean \(M(a,b)\).

For any \(\epsilon_1, \epsilon_2 > 0\), let \(t_1 = 1/4 - \epsilon_1, t_2 = \lambda + \epsilon_2\). Then one can get

\[
F_1(x) = \left(\frac{1}{4} - \epsilon_1\right) \frac{x}{\tanh^{-1} x} + \left(\frac{3}{4} + \epsilon_1\right) \frac{x}{\tan^{-1} x} - \frac{x}{\sinh^{-1} x},
\]

\[
F_2(x) = \left(\lambda + \epsilon_2\right) \frac{x}{\tanh^{-1} x} + (1-\lambda - \epsilon_2) \frac{x}{\tan^{-1} x} - \frac{x}{\sinh^{-1} x}.
\]

Let \(x_1 \to 0^+\) and \(x_2 \to 1\), then the Taylor expansion leads to

\[
F_1(x_1) = \frac{2}{3} \epsilon_1 x_1^3 + O(x_1^4),
\]

\[
F_2(x_2) = -4 \epsilon_2 / \pi + O(x_2 - 1).
\]

Equations (3.8) and (3.10) imply that if \(\alpha < 1/4\), then, for any \(\epsilon_1 > 0\), there exists \(\sigma_1 \in (0,1)\) such that \(NS(a,b) < (1/4 - \epsilon_1)L(a,b) + (3/4 - \epsilon_1)T(a,b)\) for all \(a, b\) with \((a-b)/(a+b) \in (0, \sigma_1)\).

Equations (3.9) and (3.11) imply that if \(\beta > \lambda\), then, for any \(\epsilon_2 > 0\), there exists \(\sigma_2 \in (0,1)\) such that \(NS(a,b) > (\lambda + \epsilon_2)L(a,b) + (1-\lambda - \epsilon_2)T(a,b)\) for all \(a, b\) with \((a-b)/(a+b) \in (1 - \sigma_2, 1)\).

\[\square\]

4 Conclusion

In the article, we give the sharp upper and lower bounds for Neuman-Sándor mean in terms of the linear convex combination of the logarithmic and second Seiffert means.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All the authors worked jointly. All the authors read and approved the final manuscript.

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