On two Thomae-type transformations for hypergeometric series with integral parameter differences

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Abstract

We obtain two new Thomae-type transformations for hypergeometric series with \( r \) pairs of numeratorial and denominatorial parameters differing by positive integers. This is achieved by application of the so-called Beta integral method developed by Krattenthaler and Rao [Symposium on Symmetries in Science (ed. B. Gruber), Kluwer (2004)] to two recently obtained Euler-type transformations. Some special cases are given.

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1. Introduction

The generalized hypergeometric function \( _pF_q(x) \) is defined for complex parameters and argument by the series

\[
_pF_q \left[ \begin{array}{c}
\{a_1, a_2, \ldots, a_p\} \\
\{b_1, b_2, \ldots, b_q\}
\end{array} ; x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}.
\]

When \( q \geq p \) this series converges for \( |x| < \infty \), but when \( q = p - 1 \) convergence occurs when \( |x| < 1 \) (unless the series terminates). In (1.1) the Pochhammer symbol or ascending factorial \((a)_n\) is given for integer \( n \) by

\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 
1 & (n = 0) \\
(a(a + 1) \cdots (a + n - 1) & (n \geq 1),
\end{cases}
\]

where \( \Gamma \) is the gamma function. In what follows we shall adopt the convention of writing the finite sequence of parameters \((a_1, a_2, \ldots, a_p)\) simply by \((a_p)\) and the product of \( p \) Pochhammer symbols by

\[
((a_p))_k \equiv (a_1)_k \cdots (a_p)_k,
\]

where an empty product \( p = 0 \) is interpreted as unity.

Recent work has been carried out on the extension of various summations theorems, such as those of Gauss, Kummer, Bailey and Watson [1, 6, 7], and also of Euler-type transformations to higher-order hypergeometric functions with \( r \) pairs of numeratorial and denominatorial parameters differing by positive integers [3, 4]. Our interest in this note is concerned with obtaining similar extensions of the two-term Thomae transformation [8, p. 52]

\[
_3F_2 \left[ \begin{array}{c}
a, b, c \\
d, e
\end{array} ; 1 \right] = \frac{\Gamma(d)\Gamma(e)\Gamma(\sigma)}{\Gamma(a)\Gamma(b + \sigma)\Gamma(c + \sigma)} \cdot \_3F_2 \left[ \begin{array}{c}
c - a, d - a, \sigma \\
b + \sigma, c + \sigma
\end{array} ; 1 \right]
\]

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for $\Re(\sigma) > 0$, $\Re(a) > 0$, where $\sigma = e + d - a - b - c$ is the parametric excess. Many other results of the above type, including three-term Thomae transformations, are given in [8, pp. 116-121]; see also [9].

The so-called Beta integral method introduced by Krathenthaler and Rao [2] generates new identities for hypergeometric series for some fixed value of the argument (usually 1) from known identities for hypergeometric series with a smaller number of parameters involving the argument $x$, $1 - x$ or a combination of their powers. The basic idea of this method is to multiply the known hypergeometric identity by the factor $x^{d-1}(1-x)^{e-d-1}$, where $e$ and $d$ are suitable parameters, integrate term by term over $[0,1]$ making use of the beta integral representation

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (\Re(a) > 0, \Re(b) > 0) \quad (1.2)$$

and finally to rewrite the result in terms of a new hypergeometric series. We apply this method to two Euler-type transformations recently obtained in [3, 4] to derive two two-term Thomae-type transformations for hypergeometric functions with $r$ pairs of numeratorial and denominatorial parameters differing by positive integers.

2. Extended Thomae-type transformations

Our starting point is the following Euler-type transformations for hypergeometric functions with $r$ pairs of numeratorial and denominatorial parameters differing by positive integers $(m_r)$.

**Theorem 1.** Let $(m_r)$ be a sequence of positive integers with $m := m_1 + \cdots + m_r$. Then we have the two Euler-type transformations [3, 4] for $|\arg(1-x)| < \pi$

$$r+2 F_{r+1} \left[ \begin{array}{c} a, b, (f_r + m_r) \\ c, \end{array} ; x \right] = (1-x)^{-a} m+2 F_{m+1} \left[ \begin{array}{c} a, c-b-m, (\xi_m + 1) \\ c, \end{array} ; x \right]$$

provided $b \neq f_j$ ($1 \leq j \leq r$), $(c-b-m)_m \neq 0$ and

$$r+2 F_{r+1} \left[ \begin{array}{c} a, b, (f_r + m_r) \\ c, \end{array} ; x \right] = (1-x)^{e-a-b} m+2 F_{m+1} \left[ \begin{array}{c} c-a-m, c-b-m, (\eta_m + 1) \\ c, \end{array} ; x \right]$$

provided $(c-a-m)_m \neq 0$, $(c-b-m)_m \neq 0$. The $(\xi_m)$ and $(\eta_m)$ are respectively the nonvanishing zeros of the associated parametric polynomials $Q_m(t)$ and $\hat{Q}_m(t)$ defined below.

The parametric polynomials $Q_m(t)$ and $\hat{Q}_m(t)$, both of degree $m = m_1 + \cdots + m_r$, are given by

$$Q_m(t) = \frac{1}{(\lambda)^m} \sum_{k=0}^m (b)_k C_{k,r}(t)(\lambda-t)^{m-k}, \quad (2.3)$$

where $\lambda := b - a - m$, and

$$\hat{Q}_m(t) = \sum_{k=0}^m \frac{(-1)^k C_{k,r}(a)_k(b)_k(t)_k}{(c-a-m)_k(c-b-m)_k} G_{m,k}(t) \quad (2.4)$$

where

$$G_{m,k}(t) := \,_{3}F_{2} \left[ \begin{array}{c} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k+1 \end{array} ; 1 \right].$$

The coefficients $C_{k,r}$ are defined for $0 \leq k \leq m$ by

$$C_{k,r} = \frac{1}{\Lambda} \sum_{j=k}^m \sigma_j S_j^{(k)}, \quad \Lambda = (f_1)_{m_1} \cdots (f_r)_{m_r}, \quad (2.5)$$
with \(C_{0,r} = 1, C_{m,r} = 1/\Lambda\). The \(S^{(k)}_j\) denote the Stirling numbers of the second kind and the \(\sigma_j (0 \leq j \leq m)\) are generated by the relation

\[
(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_j x^j.
\]  

(2.6)

For \(0 \leq k \leq m\), the function \(G_{m,k}(t)\) is a polynomial in \(t\) of degree \(m - k\) and both \(Q_m(t)\) and \(\hat{Q}_m(t)\) are normalized so that \(Q_m(0) = \hat{Q}_m(0) = 1\).

**Remark 1.** In [5], an alternative representation for the coefficients \(C_{k,r}\) is given as the terminating hypergeometric series of unit argument

\[
C_{k,r} = \frac{(-1)^r}{k!} \binom{r}{1} F_r \left[ -k, \frac{(f_r + m_r)}{(f_r)} ; 1 \right].
\]

When \(r = 1\), with \(f_1 = f, m_1 = m\), Vandermonde’s summation theorem [8, p. 243] can be used to show that

\[
C_{k,1} = \frac{m - k}{(f_k)}. \tag{2.7}
\]

We first apply the Beta integral method [2] to the result in (2.2) to obtain a new hypergeometric identity. Multiplying both sides by \(x^{d-1}(1-x)^{e-d-1}\), where \(e, d\) are arbitrary parameters satisfying \(\Re(e - d) > 0, \Re(d) > 0\), we integrate over the interval \([0, 1]\). The left-hand side yields

\[
\int_0^1 x^{d-1}(1-x)^{e-d-1} r+2 F_r+1 \left[ a, b, \frac{(f_r + m_r)}{(f_r)} ; x \right] dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(a)k}{(c)k} \frac{(f_r + m_r)k}{(f_r)k} \int_0^1 x^{d+k-1}(1-x)^{e-d-1} dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(a)k}{(c)k} \frac{(f_r + m_r)k}{(f_r)k} \frac{\Gamma(d+k)\Gamma(e-d)}{\Gamma(e+k)}
\]

\[
= \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)} r+3 F_{r+2} \left[ a, b, d, \frac{(f_r + m_r)}{(f_r)} ; 1 \right]. \tag{2.8}
\]

upon evaluation of the integral by (1.2) and use of the definition (1.1) when it is supposed that \(\Re(s) > 0\), where \(s\) is the parametric excess given by

\[
s := c + e - a - b - d - m. \tag{2.9}
\]

Proceeding in a similar manner with the right-hand side of (2.2), we obtain

\[
\int_0^1 x^{d-1}(1-x)^{e-1} m+2 F_{m+1} \left[ c - a - m, c - b - m, \frac{(\eta_m + 1)}{(\eta_m)} ; x \right] dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(c-a-m)k(c-b-m)k}{(c)k} \frac{(\eta_m + 1)k}{(\eta_m)k} \int_0^1 x^{d+k-1}(1-x)^{e-1} dx
\]

\[
= \frac{\Gamma(d)\Gamma(s)}{\Gamma(c+e-a-b-m)} m+3 F_{m+2} \left[ c - a - m, c - b - m, d, \frac{(\eta_m + 1)}{(\eta_m)} ; 1 \right]. \tag{2.10}
\]

Then by (2.8) and (2.10) we obtain the two-term Thomae-type hypergeometric identity given in the following theorem, where the restriction \(\Re(d) > 0\) can be removed by appeal to analytic continuation:

**Theorem 2.** Let \((m_r)\) be a sequence of positive integers with \(m := m_1 + \cdots + m_r\). Then

\[
r+3 F_{r+2} \left[ a, b, d, \frac{(f_r + m_r)}{(f_r)} ; 1 \right]
\]
Theorem 3. Let \( m_r \) be a sequence of positive integers with \( m := m_1 + \cdots + m_r \). Then, for non-negative integer \( n \)
\[
r + 3F_{r+2} \left[ \begin{array}{c}
-n, b, d, \, (f_r + m_r) \\
c, e, \, (f_r)
\end{array} ; 1 \right] = \frac{(e - d)_n}{(e)_n} \, m + 3F_{m+2} \left[ \begin{array}{c}
-n, c - b - m, \, (\xi_m + 1) \\
c, 1 - e + d - n, \, (\xi_m) \end{array} ; 1 \right]
\] (2.13)
provided \( b \neq f_j \) (1 \( \leq j \leq r) \), \( (c - b - m)_m \neq 0 \) and \( \Re(e - d) > 0 \).

3. Examples

When \( r = 0 \) (with \( m = 0 \)) we recover from (2.11) and (2.13) the known results [9]
\[
3F_2 \left[ \begin{array}{c}
a, b, d \\
c, e
\end{array} ; 1 \right] = \frac{\Gamma(e)\Gamma(c + e - a - b - d)}{\Gamma(e - d)\Gamma(c + e - a - b)} \, 3F_2 \left[ \begin{array}{c}
c - a, c - b, d \\
c, c + e - a - b
\end{array} ; 1 \right]
\]
for \( \Re(e - d) > 0, \Re(e + c - a - b - d) > 0 \) and
\[
3F_2 \left[ \begin{array}{c}
n, b, d \\
c, e
\end{array} ; 1 \right] = \frac{(e - d)_n}{(e)_n} \, 3F_2 \left[ \begin{array}{c}
n, c - b, d \\
c, 1 - e + d - n
\end{array} ; 1 \right]
\]
for \( \Re(e - d) > 0 \) with \( n \) a non-negative integer.

In the particular case \( r = 1, \; m_1 = m = 1, \; f_1 = f \), we have the parametric polynomial from (2.3)
\[
Q_1(t) = 1 + \frac{(b - f)t}{(c - b - 1)f}
\]
with the nonvanishing zero \( \xi_1 = \xi \) (provided \( b \neq f, \; c - b - 1 \neq 0 \)) given by
\[
\xi = \frac{(c - b - 1)f}{f - b},
\] (3.1)
and from (2.4)
\[
\hat{Q}_1(t) = 1 - \frac{(c - a - b - 1)f + ab)t}{(c - a - 1)(c - b - 1)f}
\]
with the nonvanishing zero $\eta_1 = \eta$ (provided $c - a - 1 \neq 0$, $c - b - 1 \neq 0$) given by

$$\eta = \frac{(c - a - 1)(c - b - 1)f}{ab + (c - a - b - 1)f}.$$  

(3.2)

Then we have from (2.11) and (2.13) the transformations

$$4F_3 \left[ \begin{array}{c} a, b, d, f + 1 \\ c, e, f + 1 \\ \end{array} ; 1 \right] = \frac{\Gamma(e)\Gamma(s)}{\Gamma(e - d)\Gamma(s + d)} 4F_3 \left[ \begin{array}{c} c - a - 1, c - b - 1, d, \eta + 1 \\ c, s + d, \eta + 1 \\ \end{array} ; 1 \right]$$

provided $c - a - 1 \neq 0$, $c - b - 1 \neq 0$, $\Re(e - d) > 0$ and $\Re(s) > 0$, where $s$ is defined by (2.9) with $m = 1$, and

$$4F_3 \left[ \begin{array}{c} -n, b, d, f + 1 \\ c, e, f + 1 \\ \end{array} ; 1 \right] = \frac{(e - d)_n}{(e)_n} 4F_3 \left[ \begin{array}{c} -n, c - b - 1, d, \xi + 1 \\ c, 1 - e + d - n, \xi + 1 \\ \end{array} ; 1 \right]$$

for non-negative integer $n$ and $\Re(e - d) > 0$.

In the case $r = 1$, $m_1 = 2$, $f_1 = f$, we have $C_{0,r} = 1$, $C_{1,r} = 2/f$ and $C_{2,r} = 1/(f)^2$ by (2.7). From (2.3) and (2.4) we obtain after a little algebra the quadratic parametric polynomials $Q_2(t)$ (with zeros $\xi_1$ and $\xi_2$) and $\hat{Q}_2(t)$ (with zeros $\eta_1$ and $\eta_2$) given by

$$Q_2(t) = 1 - \frac{2(f - b)t}{(c - b - 2)f} + \frac{(f - b)_2 t(t + 1)}{(c - b - 2)_2 f_2}$$

and

$$\hat{Q}_2(t) = 1 - \frac{2Bt}{(c - a - 2)(c - b - 2)} + \frac{Ct(1 + t)}{(c - a - 2)(c - b - 2)} ,$$

where

$$B := \sigma' + \frac{ab}{f}, \quad C := \sigma' \sigma' + 2 \frac{ab \sigma'}{f} + \frac{(a)_2 (b)_2}{(f)_2} , \quad \sigma' := c - a - b - 2 .$$

For example, if $a = \frac{1}{4}$, $b = \frac{5}{2}$, $c = \frac{3}{2}$ and $f = \frac{1}{2}$ we have

$$Q_2(t) = 1 - \frac{8}{3} t + \frac{4}{3} t(1 + t), \quad \hat{Q}_2(t) = 1 + \frac{16}{9} t - \frac{68}{27} t(1 + t) ,$$

whence $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{9}{3}$ and $\eta_1 = \frac{1}{7}$, $\eta_2 = -\frac{27}{33}$. The transformations in (2.11) and (2.13) then yield

$$4F_3 \left[ \begin{array}{c} \frac{1}{3}, \frac{5}{2}, d, \frac{5}{2} \\ \frac{1}{2}, e, \frac{1}{2} \\ \end{array} ; 1 \right] = \frac{\Gamma(e)\Gamma(e - d - \frac{11}{14})}{\Gamma(e - d)\Gamma(e - \frac{11}{14})} 4F_3 \left[ \begin{array}{c} -\frac{3}{4}, -3, d, \frac{7}{2} \\ e - \frac{11}{14}, \frac{1}{2}, -\frac{33}{2} \\ \end{array} ; 1 \right]$$

(3.3)

providing $\Re(e - d) > \frac{14}{17}$, and

$$4F_3 \left[ \begin{array}{c} -n, \frac{5}{2}, d, \frac{5}{2} \\ \frac{3}{2}, e, \frac{1}{2} \\ \end{array} ; 1 \right] = \frac{(e - d)_n}{(e)_n} 4F_3 \left[ \begin{array}{c} -n, -3, d, \frac{11}{2} \\ 1 - e + d - n, \frac{1}{2}, \frac{9}{2} \\ \end{array} ; 1 \right]$$

(3.4)

for non-negative integer $n$. We remark that a contraction of the order of the hypergeometric functions on the right-hand sides of (3.3) and (3.4) has been possible since $c = \xi_1 + 1 = \eta_1 + 1 = \frac{5}{2}$.

In addition, both series on the right-hand sides terminate: the first with summation index $k = 3$ and the second with index $k = \min\{n, 3\}$. A final point to mention is that for real parameters $a$, $b$, $c$ and $f$ it is possible (when $m \geq 2$) to have complex zeros.

4. Concluding remarks

We have employed the Beta Integral method of Krattenthaler and Rao [2] applied to two recently obtained Euler-type transformations for hypergeometric functions with $r$ pairs of numeratorial
and denominatorial parameters differing by positive integers \((m_r)\). By this means, we have established two Thomae-type transformations given in Theorems 2 and 3.

In order to write the hypergeometric series in (2.11) and (2.13) we require the zeros \(\eta_m\) and \(\xi_m\) of the parametric polynomials \(Q_m(t)\) and \(Q_m(t)\) respectively. However, to evaluate the series on the right-hand sides of (2.11) and (2.13), it is not necessary to evaluate these zeros. This observation can be understood by reference to the hypergeometric series

\[
F \equiv \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} (1 + \frac{k}{\xi_1}) \cdots (1 + \frac{k}{\xi_m})
\]

upon use of the fact that \((a + 1)k/(a)_k = 1 + (k/a)\). Since the parametric polynomial \(Q_m(t)\) in (2.3) can be written as \(Q_m(t) = \prod_{r=1}^{m} \{1 - (t/\xi_r)\}\) it follows that

\[
F = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} Q_m(-k).
\]

Consequently it is sufficient to know only the parametric polynomial \(Q_m(t)\). A similar remark applies to the series involving the zeros \(\eta_m\) with the parametric polynomial \(Q_m(-k)\) replaced by \(\hat{Q}_m(-k)\).

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