THE HEIGHT OF AN nth-ORDER FUNDAMENTAL ROGUE WAVE FOR THE NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. The height of an nth-order fundamental rogue wave $q_{rw}^{[n]}$ for the nonlinear Schrödinger equation, namely $(2n+1)c$, is proved directly by a series of row operations on matrices appeared in the n-fold Darboux transformation. Here the positive constant $c$ denotes the height of the asymptotical plane of the rogue wave.

Keywords: Rogue wave, Nonlinear Schrödinger equation, Darboux transformation

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1. Introduction

The high-intensity light is of growing importance in the creation of few-cycle pulses at attosecond scale [1–3], which in general implies the vast boost of the high bit-rate optical fiber communication and the birth of the so called “attosecond physics” [4]. Parallel to above extensive researches on this featured light from the view of the optics, another kind of large amplitude optical wave, namely optical rogue wave (RW) for the nonlinear Schrödinger (NLS) equation, has been paid much attention since the experimental observation at year 2011 [5], almost 30 years later of the discovery of the solution [6]. This wave is described vividly as which appears from nowhere and disappears without a trace [7]. The first-order RW, which is also called Peregrine soliton [6], has a simple profile: two hollows allocated in the two sides of the center peak on a non-vanishing asymptotic plane, and maximum amplitude of the peak is three times of the height of the plane. However, higher-order RWs have several different interesting patterns [8–16]. For example, a fundamental pattern is consisted of a central main peak and several gradually decreasing peaks allocated in two sides on a non-vanishing asymptotical plane. There exists a conjecture [17–19]: the height of a nth-order RW under the fundamental pattern is $(2n+1)$ times of the height of the asymptotical plane, which has been confirmed for several RWs in both theory up to twelfth-order [20] and experiment results up to fifth-order [21]. Thus, higher-order RWs have larger amplitude (larger power), and hence can be more destructive in some disastrous events and be more useful to generate large-intensity short optical pulses. The higher-order RWs provide higher possibility for observation because of their large amplitudes, so that we can use or avoid them with more conveniences in real physical systems [22–25]. Therefore, it is physically important to pay more attention on the height of the higher-order RWs.

The first mathematical proof of the above expression for the maximal amplitude was given in [19]. Namely, the recurrence relations (7) and (8) of this work lead directly to the explicit formulae $(2n+1)$ for the expression of interest. Another work that gave the proof for the same

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expression for the maximal amplitude is [26]. The problem was also addressed in [27]. The purpose of this paper is to provide a direct proof of the above conjecture, which differs from above two works given in [19, 26]. The proof of the conjecture is a highly non-trivial work because the formula of the higher-order RW for the NLS is extremely cumbersome such that a fifth-order RW with eight parameters takes more than fourteen thousands pages [28]. The nonlinear Schrödinger equation [29,30] is the form of

$$i q_t + q_{xx} + 2|q|^2 q = 0. \quad (1)$$

Here $q = q(x,t)$ represents the envelop of electric field, $t$ is a normalized spatial variable and $x$ is a normalized time variable. In optics, the squared modulus $|q|^2$ usually denotes a measurable quantity optical power (or intensity). The NLS equation is a widely applicable integrable system [31] in physics, which is solved by several methods such as the inverse scattering method [31], the Hirota method [32] and the Darboux transformation (DT) [33]. Recently, the height of multi-breather of the NLS has been given in references [27,34], but which can not imply the height of the RWs because of the appearing of an indeterminate form $0^0$ under the degeneration of eigenvalues, namely $a_i \rightarrow \frac{1}{2}$ in reference [27] or $\nu_i \rightarrow 1$ in reference [34]. In general, this indeterminate form is unavoidable to construct RWs for many equations [35–55] by the DT. Thus it is worthwhile to provide a direct proof of the above conjecture based on the determinant representation of the DT for the NLS [13,33,56].

The rest of the paper is organized as follows. After a brief summary of the nth-order breather, a new formula of the nth-order RW $q_{rw}^{[n]}(x,t)$ is given by using a newly introduced function Cramer in section II. In section III, we provide a direct proof of the conjecture on the height of the $q_{rw}^{[n]}(x,t)$ by a series of row operations on the corresponding matrix appeared in the n-fold DT.

2. The nth-order Rogue Waves of the NLS generated by the n-fold DT

We recall briefly to construct higher-order RWs from higher-order breathers of the NLS by the DT [13]. In order to construct the DT, it is necessary to introduce a proper “seed” as follows:

$$q^{[0]} = ce^{i\rho}, \quad (2)$$

with $\rho = ax + (2c^2 - a^2)t, a \in \mathbb{R}, c > 0, i^2 = -1$. Corresponding to this “seed” solution, the eigenfunction of the Lax equation associated with $\lambda$ is expressed by

$$\phi(\lambda) = \begin{pmatrix} e^{i\frac{\lambda}{2}} & 0 \\ 0 & e^{-i\frac{\lambda}{2}} \end{pmatrix} \varphi(\lambda), \quad (3)$$

$$\varphi(\lambda) = \begin{pmatrix} \varphi_1(\lambda) \\ \varphi_2(\lambda) \end{pmatrix} = \begin{pmatrix} ce^{id(\lambda)} + i(\lambda + \frac{a}{2} + h(\lambda))e^{-id(\lambda)} \\ ce^{-id(\lambda)} + i(\lambda + \frac{a}{2} + h(\lambda))e^{id(\lambda)} \end{pmatrix}, \quad (4)$$

in which $h(\lambda) = \sqrt{(\lambda + \frac{a}{2})^2 + c^2}, d(\lambda) = (x + (2\lambda - a)t)h(\lambda)$. In order to construct nth-fold DT of the NLS, introduce following 2n eigenfunctions

$$f_{2k-1} \triangleq \phi(\lambda_{2k-1}) = \begin{pmatrix} f_{2k-1,1} \\ f_{2k-1,2} \end{pmatrix} \text{ for } \lambda_{2k-1}, \quad (5)$$

and

$$f_{2k} \triangleq \begin{pmatrix} -f_{2k-1,2}^* \\ f_{2k-1,1}^* \end{pmatrix} = \begin{pmatrix} f_{2k,1} \\ f_{2k,2} \end{pmatrix} \text{ for } \lambda_{2k} = \lambda_{2k-1}^*, \quad (6)$$

$$f_{2k}$$

$$f_{2k}$$
and \( k = 1, 2, 3, \ldots, n \). Here the asterisk denotes the complex conjugation. Using above “seed” solution and the eigenfunctions, the \( n \)-fold DT generates an \( n \)-th-order breather of the NLS \([13]\)

\[ q^{[n]} = q^{[0]} - 2i \frac{|\Delta_1^{[n]}|}{|\Delta_2^{[n]}|}. \]  

(7)

Here, \( |\cdot| \) denotes the determinant of a matrix, and two matrices in \( q^{[n]} \) are

\[ \Delta_1^{[n]} = (\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}, \alpha_{2n+1}), \]
\[ \Delta_2^{[n]} = (\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}, \alpha_{2n}), \]

with

\[ \alpha_{2k-1} = \begin{pmatrix} \lambda_1^{k-1} f_{1,1} \\ \lambda_2^{k-1} f_{2,1} \\ \vdots \\ \lambda_{2n}^{k-1} f_{n,1} \end{pmatrix}, \]
\[ \alpha_{2k} = \begin{pmatrix} \lambda_1^{k-1} \varphi_{1,1} \\ \lambda_2^{k-1} \varphi_{1,2} \\ \vdots \\ \lambda_{2n}^{k-1} \varphi_{n,1} \end{pmatrix} e^{i\frac{\rho}{2}}, \quad (k = 1, 2, \cdots, n + 1), \]

(9)

and

\[ \alpha_{2k} = \begin{pmatrix} \lambda_1^{k-1} f_{1,2} \\ \lambda_2^{k-1} f_{2,2} \\ \vdots \\ \lambda_{2n}^{k-1} f_{n,2} \end{pmatrix}, \]
\[ \alpha_{2k} = \begin{pmatrix} \lambda_1^{k-1} \varphi_{1,2} \\ \lambda_2^{k-1} \varphi_{1,2} \\ \vdots \\ \lambda_{2n}^{k-1} \varphi_{n,2} \end{pmatrix} e^{-i\frac{\rho}{2}}, \quad (k = 1, 2, \cdots, n). \]

(10)

It can be seen that the \( n \)-th-order breather \( q^{[n]} \) has two variables \( x \) and \( t \), two real parameters \( a \) and \( c \), and \( n \) unique complex spectrum parameters \( \lambda_k \) corresponding eigenfunctions \( f_k(k = 1, 3, 5, \cdots, 2n - 1) \).

To be more convenient to formulate the \( n \)-th-order breather, introduce Cramer(\( \cdot \)) function,

\[ \text{Cramer} (\Delta) = \frac{|\Delta_1|}{|\Delta_2|}. \]

(11)

Here, \( \Delta = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{2n}|\bar{a}_{2n+1}) \) is an augmented matrix of \( 2n \)-dimensional column vectors \( \bar{a}_j(j = 1, 2, 3, \cdots, 2n + 1), \) \( \Delta_1 = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{2n-1}, \bar{a}_{2n+1}) \) and \( \Delta_2 = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{2n-1}, \bar{a}_{2n}) \) are two sub-matrices of \( \Delta \). Setting

\[ \Delta^{[n]} = (\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}, \alpha_{2n} | \alpha_{2n+1}) = \Delta^{[n]}(f_{j_1}, f_{j_2}; \lambda_j), \]

(12)

in which the second equality just means that \( \Delta^{[n]} \) is generated by functions \( f_j(j = 1, 3, 5, \cdots, 2n - 1) \). Note that \( f_{2k} \) and \( \lambda_{2k}(k = 1, 2, 3, \cdots, n) \) are given by \( (6) \). Using newly introduced Cramer function, then \( n \)-th-order breather \( (7) \) can be re-written as

\[ q^{[n]} = q^{[0]} - 2i \text{Cramer} (\Delta^{[n]}), \]

(13)

which will be used to study the height of the \( n \)-th-order RWs.

Simplify determinants in numerator and denominator of Cramer \( (\Delta^{[n]}) \) simultaneously by using \( (9) \) and \( (10) \), then

\[ \text{Cramer} (\Delta^{[n]}(f_{j_1}, f_{j_2}; \lambda_j)) = \text{Cramer} (\Delta^{[n]}(\varphi_{j_1}, \varphi_{j_2}; \lambda_j)) e^{i\rho}. \]

(14)

Equation \( (14) \) shows that Cramer \( (\Delta^{[n]}) \) is generated equivalently by functions \( \varphi_j(j = 1, 3, 5, \cdots, 2n - 1) \) except of a phase \( \rho \). Moreover, the Cramer function has an important property.
Lemma 1 Set $P$ is an invertible $2n \times 2n$ matrix, then
\[
\text{Cramer} \left( P \cdot \Delta \right) = \text{Cramer} \left( \Delta \right).
\]  
(15)

Proof:
\[
\text{Cramer} \left( P \cdot \Delta \right) = \left| P \Delta_1 \right| \left| P \Delta_2 \right| = \left| P \right| \left| \Delta_1 \right| \left| \Delta_2 \right| = \text{Cramer} \left( \Delta \right).
\]

This lemma shows that the ratio of the two determinants in $\text{Cramer} \left( P \cdot \Delta \right)$ is invariant under elementary row operations, and will be used repeatedly to do row operations on the matrices appeared in the $n$th-order RW.

It is known that an $n$th-order RW $q_{rw}^{[n]}$ of the NLS is obtained from an $n$th-order breather $q_{0}^{[n]}$ (13) by higher-order Taylor expansion of an indeterminate form $\frac{a}{0}$ which is appeared from the double degeneration $\lambda_j \rightarrow \lambda_1 \rightarrow \lambda_0 = -\frac{a}{2} + ic(j = 1, 3, 5, \cdots, 2n - 1)$ [13]. According to (13) and (14), the $n$th-order RW of the NLS becomes
\[
q_{rw}^{[n]} = q_{rw}^{[n]}(x, t) = q^{[0]} - 2i \text{Cramer} \left( \Delta_{rw}^{[n]} \right) \cdot e^{i\rho}.
\]  
(16)

Here, matrix $\Delta_{rw}^{[n]}$ is defined by following elements
\[
\left( \Delta_{rw}^{[n]} \right)_{j,k} = \frac{1}{(j+1)!} \frac{\partial^{(j+1)}}{\partial \epsilon^{j+1}} \left( \Delta^{[n]}(\varphi_{11}, \varphi_{12}; \lambda_0 + \epsilon) \right)_{j,k|\epsilon=0},
\]  
(17)

because of
\[
\frac{1}{(j+1)!} \frac{\partial^{(j+1)}}{\partial \epsilon^{j+1}} \left( \Delta^{[n]}(\varphi_{j1}, \varphi_{j2}; \lambda_0 + \epsilon) \right)_{j,k|\epsilon=0} = \frac{1}{(j+1)!} \frac{\partial^{(j+1)}}{\partial \epsilon^{j+1}} \left( \Delta^{[n]}(\varphi_{11}, \varphi_{12}; \lambda_0 + \epsilon) \right)_{j,k|\epsilon=0},
\]  
(18)

under the double degeneration of eigenvalues $\lambda_j \rightarrow \lambda_1 = \lambda_0 + \epsilon$. It can be seen clearly that $\Delta_{rw}^{[n]}$ is just generated by one eigenfunction $f_1$, or equivalently by two components $\varphi_{11}$ and $\varphi_{12}$ of $\varphi_1$. By the simplification of the formula of $\Delta_{rw}^{[n]}$, then
\[
\Delta_{rw}^{[n]} = \Delta_{rw}^{[n]}(\varphi_{11}, \varphi_{12}),
\]  
(19)

\[
\left( \Delta_{rw}^{[n]}(\varphi_{11}, \varphi_{12}) \right)_{j,k} = \begin{cases} 
\frac{1}{(j+1)!} \frac{\partial^{(j+1)}}{\partial \epsilon^{j+1}} ((\lambda_0 + \epsilon)^{\frac{k}{2}} \varphi_{11})|_{\epsilon=0}, & j \text{ is Odd, } k \text{ is Odd}, \\
\frac{1}{(j+1)!} \frac{\partial^{(j+1)}}{\partial \epsilon^{j+1}} ((\lambda_0 + \epsilon)^{\frac{k-2}{2}} \varphi_{12})|_{\epsilon=0}, & j \text{ is Odd, } k \text{ is Even}, \\
\frac{1}{(k)!} \frac{\partial^{(k)}}{\partial \epsilon^{k}} ((\lambda_0 + \epsilon)^{\frac{k-2}{2}} \varphi_{12})^*|_{\epsilon=0}, & j \text{ is Even, } k \text{ is Odd}, \\
\frac{1}{(k)!} \frac{\partial^{(k)}}{\partial \epsilon^{k}} ((\lambda_0 + \epsilon)^{\frac{k}{2}} \varphi_{11})^*|_{\epsilon=0}, & j \text{ is Even, } k \text{ is Even}.
\end{cases}
\]  
(20)

It it clear that the coefficient of $\epsilon^l$ in above formula has zero contribution when $l$ is an integer. Formula (16) of the $n$th-order RW $q_{rw}^{[n]}(x, t)$ is crucial to study the height in this paper, because it is convenient to introduce row operations in order to simplify determinants in numerator and denominator.

3. The Height of an $n$th-order Fundamental RW

The $q_{rw}^{[n]}(x, t)$ in (16) is an $n$th-order fundamental rogue wave of the NLS [13], and the height of its asymptotical plane is $c$. Because one can always move the central peak to origin of the coordinate on the $(x, t)$-plane, we shall set $x = 0$ and $t = 0$ in $q_{rw}^{[n]}$ in following theorem to study the height without the loss of the generality.
The height of an nth-order fundamental RW is $|q_{rw}^{[n]}|_{\text{height}} = (2n + 1)c$. Here $n$ is a positive integer and $c$ is the height of the asymptotical plane.

**Proof:** We shall prove the theorem by four steps from $q_{rw}^{[n]}(x, t)$ in (16). The main idea of the calculation of Cramer $(\Delta_{rw}^{[n]})$ is to utilize row operations according to Lemma 1, such that the matrix $\Delta_{rw}^{[n]}$ becomes a strict upper triangular matrix.

**Step 1:** Simplify the formula of $q_{rw}^{[n]}(0, 0)$. It is known by double degeneration of eigenvalues that there is only one eigenfunction $\varphi_1$ in $q_{rw}^{[n]}(x, t)$ associated with eigenvalue $\lambda_1 = \lambda_0 + \epsilon$, then

$$h = h(\lambda_1) = \sqrt{(ic + \epsilon)^2 + \epsilon^2} = \sqrt{2ic\epsilon + \epsilon^2} = h_0\sqrt{\epsilon}\sqrt{2ic},$$

(21)

here,

$$h_0 = \sqrt{1 + \frac{\epsilon}{2ic}} = \sum_{k=0}^{\infty} c_k \epsilon^k = c_0 + c_1 \epsilon + c_2 \epsilon^2 + \cdots + c_k \epsilon^k + \cdots,$$

(22)

with Taylor expansion coefficients

$$c_0 = 1, c_1 = \frac{1}{2 \cdot (2ic)}, c_k = \frac{(-1)^{k-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2k - 3)}{2^k \cdot (2ic)^k \cdot k!}, (k = 2, 3, \cdots).$$

(23)

Setting $x = 0$ and $t = 0$ in $q_{rw}^{[n]}$ (16), yields

$$\begin{cases} 
\varphi_{11} = i(\epsilon + h), \\
\varphi_{12} = i(\epsilon + h), 
\end{cases}$$

(24)

and then

$$q_{rw}^{[n]}(0, 0) = c - 2iCramer (\Delta_{rw}^{[n]}(i(\epsilon + h), i(\epsilon + h))).$$

(25)

Because coefficient of $\epsilon^l$ in the calculation of $\Delta_{rw}^{[n]}$ through (20) has zero contribution when $l$ is an integer, the formula of the nth-order RW is further simplified as

$$q_{rw}^{[n]}(0, 0) = c - 2iCramer (\Delta_{rw}^{[n]}(ih, ih)).$$

(26)

**Step 2:** Remove the common factors in each row of $\Delta_{rw}^{[n]}$. According to (15) in Lemma 1, the elementary row operations $P_0$ for $\Delta_{rw}^{[n]}(ih, ih)$ remove a nonzero common factor $i\sqrt{2ic}$ in odd rows while $-i\sqrt{-2ic}$ in even rows, and the transformed matrix $\Delta_{rw}^{[n]}(h_0\sqrt{\epsilon}, h_0\sqrt{\epsilon})$ is given in (37). This implies

$$\text{Cramer} (\Delta_{rw}^{[n]}(ih, ih)) = \text{Cramer} (P_0\Delta_{rw}^{[n]}(ih, ih)) = \text{Cramer} (\Delta_{rw}^{[n]}(h_0\sqrt{\epsilon}, h_0\sqrt{\epsilon})), $$

(27)

and the second equality can be seen from (21). Here

$$P_0 = \text{diag} (r_1, r_2, r_1, r_2, \cdots, r_1, r_2),$$

$$r_1 = (i\sqrt{2ic})^{-1} \text{ and } r_2 = (-i\sqrt{-2ic})^{-1}.$$ Therefore the nth-order RW becomes

$$q_{rw}^{[n]}(0, 0) = c - 2iCramer (\Delta_{rw}^{[n]}(h_0\sqrt{\epsilon}, h_0\sqrt{\epsilon})).$$

(28)
Step 3: Transform $\Delta^{[n]}_{rw}$ to be a block upper triangular matrix. We further construct a series of row operation matrices $P_1, P_2, \ldots, P_{n-1}$ (see appendix) acting on $\Delta^{[n]}_{rw}(h_0\sqrt{\epsilon}, h_0\sqrt{\epsilon})$ such that the transformed matrix

$$\Delta^{[n]}_{rw}(\sqrt{\epsilon}, \sqrt{\epsilon}) = P_{n-1} \cdots P_2 P_1 \Delta^{[n]}_{rw}(h_0\sqrt{\epsilon}, h_0\sqrt{\epsilon})$$

is given by (38). According to Lemma 1,

$$\text{Cramer} \left( \Delta^{[n]}_{rw}(h_0\sqrt{\epsilon}, h_0\sqrt{\epsilon}) \right) = \text{Cramer} \left( \Delta^{[n]}_{rw}(\sqrt{\epsilon}, \sqrt{\epsilon}) \right),$$

which leads to a new formula of the $n$th-order RW

$$q^{[n]}_{rw}(0, 0) = c - 2i \text{Cramer} \left( \Delta^{[n]}_{rw}(\sqrt{\epsilon}, \sqrt{\epsilon}) \right).$$

(29)

**Step 4: Transform $\Delta^{[n]}_{rw}$ to be a strict upper triangular matrix.** We are now in a position to do final row operations on the matrix in Cramer, i.e. $P_n \Delta^{[n]}_{rw}(\sqrt{\epsilon}, \sqrt{\epsilon})$, such that matrix given by (38) becomes a strict upper triangular matrix, and thus the Cramer of the transformed matrix is nic. Here the $2n \times 2n$ block diagonal matrix $P_n$ is given by

$$P_n = \text{diag} \left( J, J, \ldots, J \right),$$

and

$$J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$  

(30)

(31)

According to Lemma 1, $\text{Cramer} \left( \Delta^{[n]}_{rw}(\sqrt{\epsilon}, \sqrt{\epsilon}) \right) = \text{nic}$. Substitute it back into formula (29), then

$$q^{[n]}_{rw}(0, 0) = c - 2i(\text{nic}) = (2n + 1)c.$$ 

(32)

Therefore, the height of an $n$th-order fundamental RW of the NLS is

$$\left| q^{[n]}_{rw} \right|_{\text{height}} = \left| q^{[n]}_{rw}(0, 0) \right| = (2n + 1)c.$$ 

(33)

This is the end of the proof. $\square$

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Appendix

A series of row operation matrices in Step 3:

\[
P_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-c_1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -c_1^* & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-c_3 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & -c_3^* & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-c_{m-1} & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & -c_{m-1}^* & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\] \hspace{1cm} (34)

\[
P_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -c_1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & -c_1^* & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & -c_3 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & -c_3^* & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -c_{m-2} & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & -c_{m-2}^* & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\] \hspace{1cm} (35)

\[
P_{n-1} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & -c_1 & 0 \\
0 & 0 & \cdots & 0 & 0 & -c_1^* & 0 \\
\end{pmatrix}
\] \hspace{1cm} (36)
The transformed matrix after Step 2:

\[
\begin{pmatrix}
1 & 1 & \lambda_0 & \lambda_0 & \ldots & \lambda_0^{n-1} & \lambda_0^{n-1} \\
-1 & 1 & -\lambda_0^* & \lambda_0^* & \ldots & -\lambda_0^{n-1} & \lambda_0^{n-1} \\
c_1 & c_1 & \lambda_0 c_1 + 1 & \lambda_0 c_1 + 1 & \ldots & \lambda_0^{n-1} c_1 + \binom{n-1}{1} \lambda_0^{n-2} & \lambda_0^{n-1} c_1 + \binom{n-1}{1} \lambda_0^{n-2} \\
-c_1^* & c_1^* & -\lambda_0^* c_1^* - 1 & \lambda_0^* c_1^* + 1 & \ldots & -\lambda_0^{n-1} c_1^* - \binom{n-1}{1} \lambda_0^{n-2} & \lambda_0^{n-1} c_1^* + \binom{n-1}{1} \lambda_0^{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{n-1} & c_{n-1} & \lambda_0 c_{n-1} + c_{n-2} & \lambda_0 c_{n-1} + c_{n-2} & \ldots & \sum_{j=0}^{n-1} \binom{n-1}{j} \lambda_0^{n-j-1} c_{n-j-1} & \sum_{j=0}^{n-1} \binom{n-1}{j} \lambda_0^{n-j-1} c_{n-j-1} \\
-c_{n-1}^* & c_{n-1}^* & -\lambda_0^* c_{n-1}^* - c_{n-2} & \lambda_0^* c_{n-1}^* + c_{n-2} & \ldots & -\sum_{j=0}^{n-1} \binom{n-1}{j} \lambda_0^{n-j-1} c_{n-j-1}^* & -\sum_{j=0}^{n-1} \binom{n-1}{j} \lambda_0^{n-j-1} c_{n-j-1}^* \\
\end{pmatrix}
\]

The transformed matrix after Step 3:

\[
\begin{pmatrix}
1 & 1 & \lambda_0 & \lambda_0^2 & \lambda_0^2 & \ldots & \lambda_0^{n-1} & \lambda_0^{n-1} \\
-1 & 1 & -\lambda_0^* & \lambda_0^* & \ldots & -\lambda_0^{n-1} & \lambda_0^{n-1} \\
0 & 0 & 1 & 1 & 2 \lambda_0 & 2 \lambda_0 & \ldots & \binom{n-2}{0} \lambda_0^{n-2} & \binom{n-2}{0} \lambda_0^{n-2} \\
0 & 0 & -1 & 1 & -2 \lambda_0^* & 2 \lambda_0^* & \ldots & -\binom{n-2}{0} \lambda_0^{n-2} & -\binom{n-2}{0} \lambda_0^{n-2} \\
0 & 0 & 0 & 0 & 1 & 1 & \ldots & \binom{n-3}{0} \lambda_0^{n-3} & \binom{n-3}{0} \lambda_0^{n-3} \\
0 & 0 & 0 & 0 & -1 & 1 & \ldots & -\binom{n-3}{0} \lambda_0^{n-3} & -\binom{n-3}{0} \lambda_0^{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & -1 & 1 \\
\end{pmatrix}
\]

(37)

(38)