1. Introduction and the main theorem

The main results of the present paper are the following:

**Theorem 1.1** (Nakano positive metric). Let $Z$ be a $n$-dimensional complex manifold, $X$ a 1-convex manifold, $S \subset X$ its exceptional set, $\pi : Z \to X$ a holomorphic submersion and $E \to Z$ a holomorphic vector bundle and $a : X \to Z$ a holomorphic section. Let $\varphi : X \to [0, \infty)$ be a plurisubharmonic exhaustion function, strictly plurisubharmonic on $X \setminus S$ and $\varphi^{-1}(0) = N(\varphi) = S$. Let $U = \varphi^{-1}([0, c])$ for some $c > 0$ be a given holomorphically convex set. Then there exist a neighbourhood $V$ of $a(U \setminus S)$ conic along $a(S)$ and a Nakano positive Hermitian metric on $E|_V$ with polynomial behaviour at the boundary and such that the curvature tensor $i\Theta(E)$ has at most polynomial poles and zeroes at the boundary.

The motivation for the present work was the paper [Pre] about h-principle on 1-convex spaces. In the proof we needed a way of linearizing small perturbations of a given section $a : X \to Z$. This is usually done by using holomorphic sprays generated by holomorphic vector fields that generate the vertical bundle $VT(Z) = \ker \pi$ on a neighbourhood in $Z$ of a given holomorphically convex compact set $a(K)$. In 1-convex case such vector fields can not exist on the whole neighbourhood but since we assume that the spray is only needed in $a(X \setminus S)$ we can work with vector fields with zeroes (of high order) on $a(S)$ and generating over $a(K)$ for $K$ satisfying $K \cap S = \emptyset$. Then their flows remain in conic neighbourhoods and thus generate the spray. These vector fields were obtained as extensions of the vector fields defined on $a(X)$ such that they were zero on $N(g) = g^{-1}(0)$, where $g : X \to \mathbb{C}$
is a holomorphic function, \( g(S) = 0 \), extended fiberwise constantly on \( Z \) and such that \( N(g) \cap K = \emptyset \). Such extensions exist but we were noted that we have not explained why

\[
Z
\]

Figure 1. Conic neighbourhoods of \( a(U \setminus N(g)) \) in \( Z \)

they can be chosen to go to zero when approaching \( N(g) \). This can be achieved by solving a suitable \( \partial \)-equation and the procedure is explained hereinafter.

**Theorem 1.2** (Vertical sprays on conic neighbourhoods). Let \( U = \varphi^{-1}[0, c] \), \( K \subset U, K \cap N(g) = \emptyset \). There exist a Stein neighbourhood \( V \) of \( a(U \setminus N(g)) \) conic along \( a(N(g)) \) and finitely many bounded holomorphic vector fields \( v_i \) generating \( VT(Z) \) over \( V \cap Z_K \) with zeroes on \( a(S) \) of arbitrary high order. Then there exist a \( \delta > 0 \) such that the integrals of \( v_i \)'s starting in a smaller conic neighbourhood \( \tilde{V} \subset V \) remain in \( V \) for times \( |t| < \delta \).

**Notation.** The notation from the main theorem will be fixed throughout the paper. The sets of the form \( \pi^{-1}(A) \) will sometimes be denoted by \( Z_A \). The local coordinate system on a neighbourhood \( U_Z \) of the a point from \( a(U) \) is \( (z, w) \), where \( z \) denotes the horizontal and \( w \) the vertical (or fibre) direction. For points in \( a(S) \) we have \( z = (z_1, z_2) \), where \( a(S) \cap U_Z = \{z_2 = 0\} \cap U_Z \). The function \( \varphi \) is extended to \( Z \) fiberwise and keeps the same notation throughout the paper. Its Levi form degenerates at most polynomially with respect to distance from \( S \). With the notation above this means that the smallest eigenvalue of the Levi form does not go to 0 faster than \( \|z\|^{2\alpha} \) for some \( \alpha \in \mathbb{N} \).

Recall that a complex space \( X \) is 1-convex if it possesses a plurisubharmonic exhaustion function which is strictly plurisubharmonic outside a compact set. It is known that there is a maximal nowhere discrete compact subset \( S \) of \( X \) called the exceptional set and that the Remmert reduction \( \rho : X \to \hat{X} \) maps a 1-convex space \( X \) to a Stein space \( \hat{X} \). Let \( \sigma : E \to Z \) be a holomorphic vector bundle of rank \( r \) equipped with some hermitian metric \( h \). If \( X \) were Stein the set \( a(U) \) would have a basis of Stein neighbourhoods in \( Z \) and a Nakano positive metric on \( E|_U \) would be given by \( he^{-\varphi} \) for some strictly plurisubharmonic function. If \( X \) is 1-convex then the set \( a(U) \) does not necessarily have a basis of 1-convex neighbourhoods and there are no strictly plurisubharmonic functions, since their Levi forms degenerate on exceptional sets.

2. **Basic theorems on \( \overline{\partial} \)-equation with values in a vector bundle**

We refer the reader to the Demailly’s book Complex analytic and algebraic geometry [Dem] and recall some theorems from it.
Let \((X, \omega)\) be a Kähler manifold, \(E \to X\) a vector bundle equipped with a hermitian metric \(h\). The matrix \(H\) that corresponds \(h\) in local coordinates is given by
\[
\langle u, v \rangle_h = \sum h_{ij} u_i v_j = u^T H v.
\]

Let \(i\Theta(E)\) be the Chern curvature tensor and \(\Lambda\) the adjoint of the operator \(u \to u \wedge \omega\) defined on \((p, q)\)-forms. Denote with \(L^2_{p,q}(X, E)\) the space of \((p, q)\)-forms with bounded \(L^2\)-norms with respect to the given metric \(h\). Define the hermitian operator \(A_{E,\omega}\) as the commutator
\[
A_{E,\omega} = [i\Theta(E), \Lambda].
\]

**Theorem 2.1** (Theorem VIII-4.5, [Dem]). If \((X, \omega)\) is complete and \(A_{E,\omega} > 0\) in bidegree \((p, q)\), then for any \(\overline{\partial}\)-closed form \(u \in L^2_{p,q}(X, E)\) with
\[
\int_X \langle A_{E,\omega}^{-1} u, u \rangle dV < \infty
\]
there exists \(v \in L^2_{p,q-1}(X, E)\) such that \(\overline{\partial} v = u\) and
\[
\|v\|^2 \leq \int_X \langle A_{E,\omega}^{-1} u, u \rangle dV.
\]

**Remark 2.2.** If \(v\) is replaced by the minimal \(L^2\)-norm solution and \(u\) is smooth, so is \(v\).

If the metric is locally represented by a matrix \(H\) then
\[
\Theta(E) = \overline{\partial}(H^{-1} \partial H).
\]
This is a matrix with \((1, 1)\)-forms as coefficients. If we denote the coefficient at \(dz_j \wedge d\overline{z}_k\) in the column \(\lambda\) and the row \(\mu\) by \(c_{jk\lambda \mu}\), then
\[
(2.1) \quad \Theta(E) = \sum c_{jk\lambda \mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu,
\]
where \(1 \leq j, k \leq \dim X\) and \(1 \leq \lambda, \mu \leq \text{rank } E\). Define the bilinear form \(\theta_E\) on \((TX \otimes E) \times (TX \otimes E)\) associated to \(\Theta(E)\) by
\[
(2.2) \quad \theta_E(\sum u_j \partial/\partial z_j \otimes e_\lambda, \sum v_{k\nu} \partial/\partial z_k \otimes e_\nu) = \sum c_{jk\lambda \mu} u_j v_{k\nu} (e_\mu, e_\nu)_h = \sum c_{jk\lambda \mu} u_j v_{k\nu} H_{\mu, \nu}.
\]
In orthonormal frame \(e_1, \ldots e_r\) the form can be written as
\[
(2.3) \quad \theta_E = \sum c_{jk\lambda \mu} (dz_j \otimes e_\lambda^*) \otimes (d\overline{z}_k \otimes e_\mu^*).
\]
The form \(2.3\) gives rise to several positivity concepts. The ‘lowest’ one is Griffiths positivity and that means that the form \(2.3\) is positive on the decomposable tensors \(\tau = \xi \otimes v, \xi \in TX, v \in E\) and then
\[
\theta_E(\tau, \tau) = \sum c_{jk\lambda \mu} \xi_j v_{k\nu} \epsilon_\lambda v_\mu.
\]
On the opposite side there is Nakano positivity, namely the form \(\theta\) must be positive on \(\tau = \sum t_{j\lambda} \partial/\partial z_j \otimes e_\lambda\),
\[
\theta_E(\tau, \tau) = \sum c_{jk\lambda \mu} t_{j\lambda} \epsilon_{k\mu}.
\]
In bidegree \((n, q)\) the positivity of the operator \(A_{E,\omega}\) is equivalent to Nakano positivity of \(E\).

In the case of holomorphic vector bundles the Griffiths curvature decreases in subbundles and increases in quotient bundles. But this is not the case with Nakano positive bundles.
Curvature in the sense of Nakano decreases in subbundles but does not increase in quotient bundles. And thus the dual of Nakano negative bundle is not necessarily Nakano positive.

In bidegree \((n, q)\) we have a theorem that provides the estimates in possibly noncomplete Kähler metric provided that the manifold possesses a complete one.

**Theorem 2.3** (Theorem VIII-6.1, \[Dem\]). Let \((X, \hat{\omega})\) be a complete \(n\)-dimensional Kähler manifold, \(\omega\) another Kähler metric, possibly non complete, and \(E \to X\) a Nakano semi-positive vector bundle. Let \(u \in L^2_{n,q}(X, E)\), \(q \geq 1\), be a closed form satisfying

\[
\int_X \langle A_{E,\omega}^{-1}u, u \rangle dV_\omega < \infty.
\]

Then there exists \(v \in L^2_{p,q-1}(X, E)\) such that \(\bar{\partial}v = u\) and

\[
\|v\|^2 \leq \int_X \langle A_{E,\omega}^{-1}u, u \rangle dV_\omega.
\]

**Theorem 2.4** (Theorem VII-8.1, \[Dem\]). If \(E >_{\text{Grif}} 0\) then \(E \otimes (\det E) >_{\text{Nak}} 0\).

Let \(E, F\) be holomorphic vector bundles equipped with metrics and let \(D_E, D_F\) be the corresponding metric connections respectively. Let \(e_1, \ldots, e_r\) a base of \(E\) at \(z_0\). The following formulas hold (sect 4. chap. V, sect 8., chap. VII, \[Dem\]):

\[
\Theta(D_E \otimes D_F) = \Theta(D_E) \otimes \text{Id}_F + \text{Id}_E \otimes \Theta(D_F),
\]

\[
\Theta(D_{\Lambda^*E})(e_1 \wedge \ldots \wedge e_s) = \sum_j e_1 \wedge \ldots \wedge \Theta(E)e_j \ldots \wedge e_s,
\]

\[
\Theta(D_{\det E})(e_1 \wedge \ldots \wedge e_r) = \sum_{1 \leq j \leq r} \langle \Theta(E)e_j, e_j \rangle_h(e_1 \wedge \ldots \wedge e_r)
\]

\[
\Theta(E \otimes (\det E)) = \Theta(E) + \text{Tr}_E(\Theta(E)) \otimes \text{Id}_E.
\]

Let \(H\) be a matrix defining the metric \(h\) on \(E\), \(H(z_0) = I\). Then

\[
\theta_{E \otimes (\det E)} = \theta_E + \text{Tr}_E(\theta_E) \otimes h,
\]

where

\[
\text{Tr}_E(\theta_E)(\xi, \xi) = \sum_{1 \leq \lambda \leq r} \theta_E(\xi \otimes e_\lambda, \xi \otimes e_\lambda), \xi \in TX
\]

and

\[
(2.4) \quad \theta_{E \otimes (\det E)}(\tau, \tau) = \sum_{j,k} c_{jk\lambda\mu} \tau_{j\lambda} \overline{\tau}_{k\mu} + \sum_{j,k} c_{jk\lambda\lambda} \tau_{j\mu} \overline{\tau}_{k\mu}.
\]

The last sum comes from the induced metric on \(\det E\), the form \(\bar{\partial} \bar{\partial} \log \det H\). In matrix form it is represented as \((\bar{\partial} \bar{\partial} \log \det H)I\) and the curvature of the tensor product is

\[
\bar{\partial} \bar{\partial}(\bar{\partial}^{-1} \bar{\partial}(\bar{\partial} \log \det H)) + (\bar{\partial} \bar{\partial} \log \det H)I.
\]

If we calculate the curvature form of \(\det E\) for \(\tau^{\mu\lambda} := (\sum j\mu) \otimes e_\lambda\) we get

\[
(2.5) \quad \theta_{\det E}(\tau^{\mu\lambda}, \tau^{\mu\lambda}) = \sum_{j,k} c_{jk\lambda\lambda} \tau_{j\mu} \overline{\tau}_{k\mu}
\]

and this means that if \(E\) is Griffiths positive then \(\det E\) is positive.
3. Nakano positive metric and 1-convex sets

In this section we prove the main theorem. Nakano positive Hermitian metric will be obtained from the induced metric on the quotient space of the trivial bundle. We will first construct an almost Griffiths positive metric, correct it to Griffiths positive one and then simulate the tensor product by the determinant bundle \( \det E \) using a suitable weight to obtain almost Nakano positive metric and in the last step correct this metric with another weight to make it Nakano positive. In order to do this we have to have finitely many sections of \( E \) generating it on \( V_z \).

If we were given a metric that would have been Nakano positive on a neighbourhood of \( a(S) \) then this procedure would not be needed because we could have achieved positivity by using a weight of the form \( e^{-\Phi} \), where \( \Phi \) is strictly plurisubharmonic on a neighbourhood, conic along \( a(S) \). In general we do not have such a metric.

**Proposition 3.1** (Almost holomorphic global sections). *Notation as above, \( E \to Z \) a holomorphic vector bundle. For any \( l \in \mathbb{N}_0 \) there exist \( k_0 \in \mathbb{N} \) such that for each \( k \geq k_0 \) there are finitely many smooth sections \( F_i \) of \( E \) generating \( E \) on some open neighbourhood \( U_Z \) of \( a(U) \) in \( Z \) except on \( Z|_{\mathcal{S}} \), holomorphic in the vertical directions. There exists \( c, C > 0 \) such that for \( z \in a(S) \) and small \( \|w\| \) we have \( F_i(z, w) \leq C\|z_2\|^k \), \( \overline{\partial}F_i(z, w) \cong \|w\|^l\|z_2\|^k \) and if \( z \not\in a(S) \) we have \( \overline{\partial}F_i(z, w) \cong \|w\|^{l+1} \).

**Remark 3.2.** Note that given \( l \) the number \( k \) can be chosen arbitrarily large.

**Proof.** Let and \( \mathcal{E} \) be a coherent sheaf of sections of a holomorphic vector bundle \( E \). Denote by \( \mathcal{Q} = \mathcal{J}(a(X)) \) the ideal in \( \mathcal{O}_Z \) generated by (the analytic set) \( a(X) \) and recall that by the theorem \( A \) for relatively compact 1-convex sets there are finitely many sections \( f_1, \ldots, f_m \) of the quotient sheaf \( \mathcal{J}(a(S))^k \mathcal{E}/\mathcal{Q}^{l+1} \) generating it on a neighbourhood \( a(U_1) \) of \( a(U) \) in \( a(X) \). The sheaf is a finite dimensional vector bundle with coefficients in \( \mathcal{J}(a(S))^k \) and it is supported on \( a(X) \). This means that the sections represent Taylor series in the \( w \)-variable up to order \( l \) with coefficients in \( \mathcal{J}(a(S))^k \).

Let \( f \) be one of these sections and \( z \in a(S) \). Then there exist a local holomorphic section \( f_z \) of \( \mathcal{J}(a(S))^k \mathcal{E} \) (a local holomorphic lift) defined on a neighbourhood \( U_z \) of \( z \) in \( Z \) that represents the section \( f \) on \( U_z \cap a(X) \). Any other such lift coincides with this one up to order \( l \) in \( w \). If \( z \) is not in \( a(S) \) then we (may and will) assume that the neighbourhood \( U_z \) does not intersect \( a(S) \). Each \( f_i \) thus defines an open covering of \( a(U) \) in \( Z \) and has a locally finite subcovering.

There exists a (product) covering \( \{W_j \cong U_j \times B^m\} \) of \( a(U) \) in \( Z \) by product neighbourhoods with respect to the submersion \( Z \to X \) finer than any of the above subcoverings. Let \( \{\chi_j\} \) be a partition of unity subordinate to the product covering that only depends on the base direction \( z \). Summing up the local lifts \( f_{i,j} \) of \( f_{i} \) using this partition of unity we obtain sections \( F_i(z, w) = \sum f_{i,j}(z, w)\chi_j(z) \) on an open neighbourhood \( U_Z \) of \( a(U) \) in \( Z \) that are holomorphic in the vertical direction and their nonholomorphicity is of the order \( \|w\|^{l+1} \) as we see by expanding \( f_{i,j} \) in the Taylor series with respect to the vertical direction \( w \). The terms in expansion coincide up to order \( l \) and therefore we have \( f_{i,j} = f_{i,l}(z, w) + f_{i,j,l} \), where
\( f_{i,j,l} \) are of order \(|w|^{l+1}\). Then

\[
\overline{J} F_i(z, w) = \sum f_{i,j}(z, w) \partial \chi_j(z) = f_i^l(z, w) \partial \sum \chi_j(z) + \sum f_{i,j,l} \partial \chi(z)
\]

We still have to prove that the sections generate \( E \) on tubular neighbourhood of \( a(U) \) except on \( Z_S \). Since the statement is local, we may assume that \( E \) is trivial. Problem arises near points from \( a(S) \) where \( F_i \)-s degenerate. We claim that they become linearly dependent on \( a(S) \) because their coefficients go to zero and not because some direct ions would vanish.

By definition of \( f_i \)-s for any monomial in \( J(a(S))^k \), \( z_2^\alpha \) at the point \( z_0 = ((0, z_2), 0) \in a(S) \) where \( \alpha \) is a multiindex of order \( k \), there exist finitely many sections \( f_i \) and coefficients \( g_i \) in the stalk \( O(a(X))_{z_0} \) such that \( f_\alpha := \sum g_i f_i = z_2^\alpha e_1 \). We may consider \( g_i \)-s to be functions on a neighbourhood of \( z_0 \) obtained by representing first the germs by a functions on a neighbourhood of \( z_0 \) in \( a(X) \) and then fiberwisely extending them to functions \( g_i(z) \) depending only on \( z \). We have therefore a canonical way of seeing functions on the section \( a(X) \) as functions on \( Z \) and that is important in the sequel. Let \( e_1, \ldots e_r \) denote the local base. Assume that the (local) sections \( f_i \) of the sheaf are represented by sections of \( E \) as above and denoted by the same letters. Then by definition of \( f_i \)-s we have

\[
f_\alpha(z, w) = \sum g_i(z) f_i(z, w) = z_2^\alpha e_1 + O(|w|^{l+1}|z_2|^k)
\]

and the same holds for the corresponding extensions \( F_i \), because they coincide with \( f_i \)-s to order \( l \) in \( |w| \),

\[
F_\alpha(z, w) = \sum g_i(z) F_i(z, w) = z_2^\alpha e_1 + O(|w|^{l+1}|z_2|^k).
\]

If the sections \( F_\alpha \) for all possible \( \alpha \) would be linearly dependent on any neighbourhood \( U_{z_0} \) of \( z_0 \) except on fibres over \( S \) their first coordinates \( F_{\alpha,1}(z, w) \) would have had common zeroes on \( Z \setminus Z_S \) arbitrarily close to \( Z_S \). We can estimate \(|F_{\alpha,1}(z, w)| \geq |z_2^\alpha| - c_\alpha |w|^{l+1} |z_2|^k \) and so we have

\[
\sum_\alpha |F_{\alpha,1}(z, w)| \geq \sum_\alpha |z_2^\alpha| - (\sum_\alpha c_\alpha) |w|^{l+1} |z_2|^k.
\]

Because \( \sum_\alpha |z_2^\alpha| \geq c_1 (\sum |z_{2,i}|)^k \geq c |z_2|^k \), \( c, c_1 > 0 \) we conclude that

\[
\sum_\alpha |F_{\alpha,1}(z, w)| \geq |z_2|^k (c - (\sum_\alpha c_\alpha) |w|^{l+1})
\]

and is nonzero for all small \( w \). This means that the vector fields \( F_i \) generate \( E \) on a tubular neighbourhood of \( a(U) \) except on \( Z_S \).

\[\square\]

**Remark 3.3.** Choose local coordinates in \( E \) near \( z_0 \in a(S) \) and define the matrix \( A = [\ldots F_i, \ldots] \). It has full rank on a tubular neighbourhood of \( a(U) \) except on \( Z_S \) and therefore all the zeroes of the determinant \( \text{det}(AA^*)|_{U_{z_0}} \) are on \( Z_S \). The determinant decreases polynomially with respect to \(|z_2|^k\) on this tubular neighbourhood.
3.1. Construction of a polynomially degenerating strictly plurisubharmonic function and the Kähler metric. Here we describe the construction of a function $\Phi$, strictly plurisubharmonic on a neighbourhood of $a(U \setminus S)$, conic along $S$. Its Levi form is decreasing polynomially and the rate of decreasing does not depend on the sharpness of the cone.

With exactly the same construction as in the proposition 3.1 (i.e. we may take a trivial line bundle) we produce a finite number of functions $\varphi_{1,i}$ defined on an open neighbourhood of $a(U)$ obtained from lifts of the sections of the sheaf $\mathcal{F}(a(S))^k; \mathcal{F}(a(U'))/\mathcal{F}^{l_1+1}(a(U'))$, $U \Subset U'$. The sections are 0 on $a(U')$, holomorphic to order $l_1$ in the $w$-direction and such that off $\pi^{-1}(S)$ their derivatives generate the cotangent bundle on some cone. Let’s see why.

Near a point form $a(S)$ the functions are of the form

$$\varphi_{1,i}(z, w) = \sum_{j,|\beta|=k_1} c_{ij\beta} w_j z_2^\beta + O(\|w\| z_2^k).$$

Then

$$\partial \varphi_{1,i}(z, w) = \sum_{j,|\beta|=k_1} c_{ij\beta} (dw_j z_2^\beta + w_j \partial z_2^\beta) + O(\|w\| z_2^k + \|w\|^2 z_2^{k-1}).$$

We conclude that the forms $\partial \varphi_{1,i}$ will generate the vertical cotangent bundle and degenerate as $\|z_2\|^k$. For points on $a(U \setminus S)$ with $\|z_2\| > \delta$ we have a uniform estimate.

Define $\varphi_1 = \sum |\varphi_{1,i}|^2$. Its Levi form

$$\bar{\partial} \varphi_1 = \sum \bar{\partial} \varphi_{1,i} \wedge \varphi_{1,i} + \sum \bar{\partial} \varphi_{1,i} \wedge \bar{\partial} \varphi_{1,i} + \sum \varphi_{1,i} \bar{\partial} \varphi_{1,i} + \sum \bar{\partial} \varphi_{1,i} \varphi_{1,i}$$

has positive first two terms and all possible negative terms are in the last two. Since they involve at least one $\bar{\partial} \varphi_{1,i}$ they go to zero at least as $\|w\|^{l_1-1}$. The Levi form of $\Phi = \varphi + \varphi_1$ in coordinates $(z, w)$ does not decrease faster than

$$\begin{bmatrix}
\|z_2\|^{2\alpha} + \|w\|^2 \|z_2\|^{2k_1-2} + \|w\|^{2l_1+2} \|z_2\|^{2k_1-2} & \|w\| \|z_2\|^{2k_1-1} \\
\|w\| \|z_2\|^{2k_1-1} & \|z_2\|^{2k_1}
\end{bmatrix}\begin{bmatrix}
\|w\|^{l_1+2} \|z_2\|^{2k_1-1} & \|w\|^{l_1} \|z_2\|^{2k_1} \\
\|w\|^{l_1} \|z_2\|^{2k_1} & 0
\end{bmatrix},$$

where the first matrix consists of the bound $\|z_2\|^{2\alpha}$ for the lowest eigenvalue of Levi form of $\varphi$ and the first two terms of the above sum and is therefore positive and the second consists of the last two terms and might be negative. It is clear that this form is positive on a neighbourhood of points from $a(U \setminus S)$. If we assume, say, that $\|w\| \leq \|z_2\|^{\alpha+2}$ then the sum of such matrices is a positive definite matrix, since the diagonal block

$$\begin{bmatrix}
\|z_2\|^{2\alpha} & 0 \\
0 & \|z_2\|^{2k_1}
\end{bmatrix}$$

dominates. Instead of that we may assume that $l_1 > 2\alpha$ and take the cone $\|w\| \leq \|z_2\|^2$. In any case the Levi $L\Phi$ is positive on a conic neighbourhood of $a(U \setminus S)$ and the form

$$\omega = \bar{\partial} \partial \Phi$$

defines the Kähler metric we are going to use.
Proof of the theorem 1.1. By proposition 3.1 there exist finitely many smooth vector fields \( f_1, \ldots, f_m \) on an open neighbourhood \( U_z \), of \( a(U) \), holomorphic to order \( l \) in the vertical direction, defining a surjective vector bundle homomorphism \( f : U_Z \times \mathbb{C}^m \to E|_{U_Z} \), where \( U_Z = U_{Z_1} \setminus \pi^{-1}(S) \). Thus the bundle \( E|_{U_Z} \) can be given the metric of the orthogonal complement of \( \ker f \). Consider the mapping \( f \) in some local chart, denote by \( r \) the rank of the bundle and let \( (z, w) \) be the local coordinates as usual. Then the mapping \( f \) can be represented as a \( r \times m \) matrix \( A \) with coefficients \( f_{i,j} \) that are holomorphic up to order \( l \) in the vertical direction and therefore \( \partial A \approx ||w||^l \). The linear mapping given by \( A \) has an inverse \( A^{-1} : E|_{U_Z \setminus a(S)} \to \ker f^\perp \). Then for \( u, v \in E|_{U_Z \setminus a(S)} \) we have
\[
\langle u, v \rangle_h := \langle A^{-1}u, A^{-1}v \rangle,
\]
where the right scalar product is the usual one on \( \mathbb{C}^m \). By definition the matrix \( H = \{ h_{ij} \} \) associated with the \((1,1)\)-form that defines the scalar product is
\[
\langle u, v \rangle_h = \sum h_{i,j} u_i \overline{v_j} = u^\top H v = u^\top A^{-1} \overline{A^{-1} v}
\]
and has poles on \( Z_S \). So
\[
H = \overline{A^{-1}} A^{-1}.
\]
The Nakano curvature tensor can be calculated by the formula
\[
\Theta(E) = \overline{\partial (\overline{\partial}^{-1} \partial \overline{\partial})}.
\]
Before continuing let’s express \( \overline{\partial}^{-1} \) with the matrix \( A \). Since off \( S \) the matrix \( A \) has full rank it has a singular value decomposition
\[
A = V \Sigma U^*,
\]
where \( V, U \) are unitary matrices and \( \Sigma \) is a \( r \times m \) matrix with all entries equal 0 except those on the diagonal, \( d_1, \ldots, d_r \), which are square roots of eigenvalues of \( AA^* \). The partial inverse \( A^{-1} \) is then given by \( U \Sigma^{-1} V^* \), where \( \Sigma^{-1} \) is \( m \times r \) matrix with only diagonal elements \( d_1^{-1} \geq \ldots \geq d_r^{-1} > 0 \) nonzero. We have
\[
A^{-1} = V \Sigma^{-1} U^* U \Sigma^{-1} V^* = V D^{-2} V^*,
\]
where \( D \) is a diagonal matrix with diagonal \( d_1, \ldots, d_r \). By construction we have
\[
AA^* = V \Sigma U^* U \Sigma V = V D^2 V^*
\]
and so
\[
(AA^*)^{-1} = V D^{-2} V^* = A^{-1} A^{-1}.
\]
This means that
\[
\overline{\partial} = (AA^*)^{-1}.
\]
For any invertible matrix \( B \) we have \( \partial B^{-1} = -B^{-1} \partial B B^{-1} \). The curvature is
\[
\overline{\partial}(\overline{\partial}^{-1} \partial \overline{\partial}) = -\overline{\partial}(AA^*)^{-1} \partial (AA^*)^{-1} (AA^*)^{-1}
\]
\[
= -\overline{\partial}(AA^*)^{-1} (AA^*)^{-1}
\]
\[
= -\overline{\partial} (AA^*)^{-1} \partial (AA^*)^{-1} + \partial (AA^*) \wedge \overline{\partial} (AA^*)^{-1}
\]
\[
= -\overline{\partial} (AA^*)^{-1} \partial (AA^*)^{-1} - \partial (AA^*) (AA^*)^{-1} \wedge \overline{\partial} (AA^*) (AA^*)^{-1}.
\]
We are interested in calculating the curvature tensor some point $z_0$. Let’s make a change of coordinates such that the diagonal $D(z_0) = I$. Then $AA^*(z_0) = I$ and the above expression simplifies to

$$-\overline{\partial}(AA^*)(AA^*)^{-1} - \partial(AA^*)(AA^*)^{-1}\overline{\partial}(AA^*)(AA^*)^{-1} = -\overline{\partial}(AA^*) - \partial(AA^*)\overline{\partial}(AA^*).$$

Let’s calculate each of the terms separately. The first one is

$$\overline{\partial}(AA^*) = \overline{\partial}((\partial A)A^* + A(\overline{\partial}A)^*) = (\overline{\partial}(\partial A)A^* - \partial A \wedge (\partial A)^* + \overline{\partial}A \wedge (\overline{\partial}A)^* + A(\overline{\partial}A)^*),$$

and the second one is

$$\partial(AA^*) \wedge \overline{\partial}(AA^*) = ((\partial A)A^* + A(\overline{\partial}A)^*) \wedge ((\overline{\partial}A)A^* + A(\partial A)^*).$$

All of the terms containing $\overline{\partial}A$ are small when close to the given section $a(U)$. If $z_0 \in a(U \setminus S)$ then they are 0. We divide the curvature form into two forms: the one without the $\overline{\partial}A$ expressions is denoted by $\Theta_1$ and the remaining part by $\Theta_2$. Then

$$\Theta_1 = -(-\partial A \wedge (\partial A)^*) - \partial AA^* \wedge A(\partial A)^* = \partial A \wedge (\partial A)^* - \partial AA^* \wedge (\partial A)^*.$$  

Denote by $A_s$ the $s$-th column of $A$. Since we have chosen $D(z_0) = I$ we have $A^*A = \text{pr}_{C^r}$ and this means that

$$\Theta_1(\xi \otimes v, \xi \otimes v) = \sum_{s=1}^m |\langle (\partial A_s(\xi), v) \rangle|^2 - \sum_{s=1}^r |\langle (\partial A_s(\xi), v) \rangle|^2 \geq 0$$

is nonnegative on some tubular neighbourhood of $a(U)$ except on the fibres over $S$, therefore the bundle has nonnegative Griffiths curvature at least on $a(U)$, where $\Theta_2 = 0$.

If we multiply our initial trivial metric by $e^{-\Phi}$ the curvature tensor gets an additional term $L\Phi$, where $L$ denotes the Levi form of $\Phi$. Wherever $\Phi$ is strictly plurisubharmonic we are adding a strictly positive $(1,1)$-form. The bad news is that $\Phi$ is such only on a conic neighbourhood and its Levi form decreases polynomially as we approach $a(S)$. But if we manage to show that the form $\Theta_2$ goes to 0 even faster, then we can make Griffiths curvature positive on a conic neighbourhood. In order to find the rate of decreasing we must work in ambient coordinates (and hence can not assume that $D = I$ at a point of $a(S)$). The form $\Theta_2$ is therefore equal to

$$\Theta_2 = (-\overline{\partial}AA^* - A(\overline{\partial}A)^*)(AA^*)^{-1} + (\partial AA^* + A(\overline{\partial}A)^*)(AA^*)^{-1}\overline{\partial}(AA^*)(AA^*)^{-1} + A(\overline{\partial}A)^*(AA^*)^{-1}A(\overline{\partial}A)^*(AA^*)^{-1}.$$  

By construction the $\det(AA^*) = 0$ only on fibres above $S$ and goes to 0 polynomially with respect to distance from the $\pi^{-1}(S)$. If $z = (z_1, z_2)$ denotes the base directions and $S = \{z_2 = 0\}$, we can write that $\det(AA^*) \geq \|z_2\|^{n_2}$ for some constant $n_2$. Because of noninvertibility of $AA^*$ the form $\Theta_2$ has poles and they are hidden in the determinant $\det(AA^*)$. Each term involving $(AA^*)^{-1}$ also involves a term of the form $\overline{\partial}A \approx \|w\|^{l+1}\|z_2\|^k$. So if $\|w\| \leq c\|z_2\|^{n_2+n_3}$ for some $n_3 \in \mathbb{N}$ all the terms will go to 0 at least as $\|z_2\|^{n_3}$ inside this cone as we approach the set $a(S)$. If $n_3$ is large enough the possible negativity of $\Theta_2$ will be compensated by the Levi form $L\Phi$. Since we only have Griffiths nonnegative curvature it can be made strict by adding another factor $e^{-\Phi}$. The new hermitian metric on $E$ is denoted by

$$h_1 = he^{-2\Phi}.$$
Remark. Note that in the associated bilinear form $\theta$ besides the coefficients of $\Theta$ there are also the coefficients of the matrix $H$ if the coordinates of $E$ are not orthonormal at a given point and that is the case when comparing the size of $\Theta_2$ and $L\Phi$. This coefficients in the worst case give multiplication by the terms of the form $\|z_2\|^{-n_3}$ and if $n_3$ is large then the coefficients in the associated form $\theta_3$ still go to 0 as approaching $a(S)$. The coefficients of the form $\theta_{\Phi \theta}$ associated to $L\Phi$ decrease in the worst case as $\|z_2\|^{2k-n_2}$ and this is still slower that $\|z_2\|^{n_3}$ if $n_3$ is large.

Choose some local coordinates in $E$ and let $H_1$ be the matrix representing $h_1$. Then the determinant bundle has a metric given by $\tau_1 = \text{det}(h_{1,\lambda\mu})$ and since the curvature of $\text{det} E$ is positive by Proposition 3.1 we have

$$-\partial \overline{\partial} \log \tau_1 = \partial \overline{\partial} \log \tau_1^{-1} > 0.$$ 

Consider the induced metric on the dual bundle $E^*$. Let $e_1, \ldots, e_r$ be local coordinates and $e_1^*, \ldots, e_r^*$ the dual coordinates. Each $e_i^*$ can be represented as scalar product by vector $f_i$ satisfying the equation $(e_i, f_i)_{h_1} = \delta_{ij}$ or $H \overline{F} = I$ where $F = [f_1, \ldots, f_r]$. Then the induced scalar product is given by the matrix $F^T H_1 \overline{F} = F^T = (\overline{H_1}^{-1})^T = H_1^{-1}$. The induced metric $\text{det}(h_1)^*$ on $\text{det} E^*$ in the dual coordinates is thus represented by $\tau_1^{-1}$. Let $v_1^*, \ldots, v_k^*$ be almost holomorphic sections of the $(\text{det} E)^*$ given by proposition 3.1. They generate the bundle on a neighbourhood of $a(U)$ in $Z$ except over the fibres over $a(S)$. Then we can multiply the metric $h_1$ by the weight $e^{-\log \Phi_1} = \Phi_1^{-1},$

$$\Phi_1 = \sum_i \langle v_i^*, v_i^* \rangle_{\text{det}(h_1)^*},$$

i.e. we have the metric

$$h_2 = h_1 e^{-\log \Phi_1}.$$ 

In local coordinates $e_1, \ldots, e_r$ of $E$ we have with $e^* := (e_1 \wedge \ldots \wedge e_r)^*$ the norm

$$\langle e^*, e^* \rangle_{\text{det}(h_1)^*} = \tau_1^{-1}$$ 

and since $v_i^* = \alpha_i e^*$ for some almost holomorphic functions $\alpha_i$ we have

$$\langle v_i^*, v_i^* \rangle_{\text{det}(h_1)^*} = \tau_1^{-1} |\alpha_i|^2$$

and so the weight equals

$$\Phi_1 = \sum_i (\tau_1^{-1} |\alpha_i|^2) = \tau_1^{-1} \sum |\alpha_i|^2.$$ 

The metric is

$$h_2 = h_1 \tau_1 \frac{1}{\sum |\alpha_i|^2}$$

and has again polynomial poles only on $Z_S$ if restricted to some small tubular neighbourhood of $a(U)$ in $Z$. The curvature tensor is then

$$i \partial \overline{\partial} (\overline{H_1}^{-1} \partial H_1) + i \partial \overline{\partial} \log \tau_1^{-1} + i \partial \overline{\partial} \log \sum |\alpha_i|^2$$

and has polynomial poles on $Z_S$. Note that taking other local coordinates multiplies $v_i$-s by a fixed holomorphic function $\tilde{\alpha}$ and thus adds an additional factor $|\tilde{\alpha}|^2$ to the last term. Since $\partial \overline{\partial} \log |\tilde{\alpha}|^2 = 0$, the choice of the local coordinates does change the last term.
The first two terms represent the curvature tensor of $E \otimes \det E$ and it is Nakano positive by (2.4) wherever $E$ is Griffiths positive. The last term would have been positive if $\alpha_i$ were holomorphic. Since they are only almost holomorphic there might be some negative terms hidden in the last term of the curvature tensor. But all the negative terms are multiplied by terms of the form $\overline{\partial} \alpha_i$ and should only add terms that are bounded (and go to zero) on some conic neighbourhood:

\[
\overline{\partial} \log \sum \alpha_i \overline{\alpha_i} = \frac{1}{(\sum |\alpha_i|^2)^2} \left( \sum |\alpha_i|^2 \sum \partial \alpha_i \wedge \overline{\partial} \alpha_i - \sum \partial \alpha_j \overline{\partial} \alpha_j \wedge \sum \alpha_i \partial \overline{\alpha_i} \right) +
\frac{1}{(\sum |\alpha_i|^2)^2} \left( \sum \alpha_j \overline{\partial} \alpha_j \wedge (\alpha_i \overline{\partial} \alpha_i + \overline{\alpha_i} \partial \alpha_i) + \overline{\alpha_i} \partial \alpha_j \wedge \overline{\partial} \alpha_i \right) +
\frac{1}{\sum |\alpha_i|^2} \left( \sum \alpha_i L \overline{\alpha_i} + L \alpha_i \overline{\alpha_i} + \overline{\partial} \alpha_i \wedge \overline{\partial} \alpha_i \right).
\]

First line is positive by Lagrange identity and the rest is potentially negative. Take a point $(z, 0) \in a(U \setminus S)$. There we have $\partial \alpha_i = 0$ and $\overline{\partial} \alpha_i(z, w) \approx \|w\|^{l_2}$ for some $l_2 > 2$ otherwise. On a neighbourhood of $(z, 0) \in a(S)$ we have for some $k_2 > 2$

\[
\sum |\alpha_i|^2 \approx \|z_2\|^{2k_2},
\overline{\partial} \alpha_i(z, w) \approx \|w\|^{l_2} \|z_2\|^{k_2},
L \alpha_i(z, w) \approx \|w\|^{l_2-1} \|z_2\|^{k_2-1} (\|z_2\| + \|w\|).
\]

The second and the third line of the Levi form are thus of the form

\[
C_1 \frac{\|w\|^{l_2}}{\|z_2\|} + C_2 \|w\|^{2l_2} + C_3 \|w\|^{l_2} + C_4 \frac{\|w\|^{l_2-1}}{\|z_2\|}
\]

and decrease polynomially in conic neighbourhoods of the form $\|w\| < \|z_2\|^{k_3}$ for $k_3$ large enough. Therefore in some thin enough (with respect to $\|w\|$) and (along $a(S)$) sharp enough conic neighbourhood the negativity of these two terms can be compensated by the weight $e^{-C\Phi}$ for some positive constant $C$ as before. Since $S$ is compact there exist $C$ that works for all $(z, 0) \in a(S)$. The positive part grows at most polynomially when approaching $a(S)$.

The desired metric is therefore

\[
h_3 = h_2 e^{-(C+1)\Phi} = he^{-(C+3)\Phi + \log \Phi_1}, C > 0
\]

and has polynomial poles on $a(S)$ with respect to $\|z_2\|$. The curvature tensor of $h_2 e^{-C\Phi}$ is strictly positive and by adding $e^{-\Phi}$ we conclude that the smallest eigenvalue decreases at most polynomially with respect to $\|z_2\|$. The inverse of the curvature tensor $A_E^{-1}$ then has at most polynomial poles on $a(S)$ and the rate is controlled by the Levi form of $\Phi$ (i.e. by $\|z_2\|^{-2 \min(a, k)}$).

\[\square\]

**Remark 3.4.** Note that choosing $k_2$ large produces a large pole on $a(S)$ in the weight.

4. $\overline{\partial}$ Equation in Bidegree $(n, q)$

We can now solve the $\overline{\partial}$ problem for $(n, q)$-forms. The Nakano curvature equals

\[
i \Theta_1 = i \Theta + i \overline{\partial}((C + 3)\Phi + \log \Phi_1)
\]
and therefore the curvature form \( A_{E,\omega} \) is strictly positive on the neighbourhood of \( a(U \setminus S) \), conic along \( a(S) \). Given \( g : X \rightarrow \mathbb{C}, \ N(g) := g^{-1}(0) \supset S \) there exist by \([Pr]\) an arbitrarily thin Stein neighbourhood \( V \) of \( a(U \setminus N(g)) \), conic along \( N(g) \) and it possesses a complete Kähler metric. As a result the theorem \([2.3]\) yields the following

**Theorem 4.1.** Let \( u \) be a closed smooth \((n, q)\)-form on \( V \) with

\[
\int_V \langle A_{E,\omega}^{-1}, u \rangle_{h_3} e^{-M \log |g|} dV_\omega < \infty
\]

for some \( M \geq 0 \). Then there exist a smooth \((n, q - 1)\)-form \( v \) solving \( \overline{\partial} v = u \) with

\[
\|v\|^2 = \int_V \langle v, v \rangle_{h_3} e^{-M \log |g|} dV_\omega \leq \int_V \langle A_{E,\omega}^{-1}, u \rangle_{h_3} e^{-M \log |g|} dV_\omega.
\]

By multiplying the metric with \( e^{-M \log |g|} \) for large \( M \) we do not change the curvature, since \( \log |g| \) is plurisubharmonic.

**Corollary 4.2.** Notation as above. Assume that the \((n, 1)\)-form \( u \) has at most polynomial growth when approaching the boundary with respect to some ambient Hermitian metric on \( Z \). Then \( v \) has at most polynomial growth at the boundary. If \( \|u\|_\infty \) is bounded and \( M \) is large enough, then \( \lim_{z \to z_0} v(z) = 0 \), \( z_0 \in N(g) \).

This follows from Bochner-Martinielli-Koppelman (BMK) formula. Let \( v \) be a \((p, 0)\)-form, \( v(z) = \sum_{|P|=p} a_P(z) dz_P \), and define \( |v(z)|_\infty := \max_P |a_P(z)| \). If we verbatim repeat the proof in \([FL]\), Lemma 3.2. for \((p, 0)\)-forms, we obtain

**Lemma 4.3.** Let \( v \) be a \((p, 0)\)-form with coefficients in \( C^1(\varepsilon B^n(0, 1)) \), where \( B^n(0, 1) \) is the unit ball in \( \mathbb{C}^n \). Then we have the estimate

\[
|v(0)|_\infty \leq C(\varepsilon^{-n} \|v\|_{L^2(\varepsilon B^n(0, 1))} + \varepsilon \|\overline{\partial} v\|_{L^\infty(\varepsilon B^n(0, 1))}).
\]

The constant \( C \) depends on \( n \) only.

**Proof.** Let \( \chi \) be a cut-off function on \( B = B^n(0, 1) \), \( \chi = 1 \) on \( \frac{1}{2} B \). Fix a multiindex \( P \) and estimate \( v(\zeta)_P = a(\zeta)_P d\zeta_P \). The BMK kernel is

\[
B(z, \zeta) = \frac{(n-1)!}{(2\pi)^n} \sum_{|j|=n} (-1)^j \langle \zeta^{-j} \rangle \wedge d(\zeta - z)[j] \wedge d(\zeta - z),
\]

where \( dz = dz_1 \wedge \ldots \wedge dz_n \) and \( dz[j] \) is \((n-1)\)-form obtained from \( dz \) by omitting \( dz_j \) and the form \( B_0^0(z, \zeta) \) is the part of \( B \) that is \((p, 0)\) in the \( z \) variable and let \( B_0^{p,P} \) be the term in \( B_\varepsilon^\mu \) that is \( dz_P \) in the \( z \) variable.

The BMK formula gives

\[
(-1)^p v(0)_P = \int_{\partial B} v(\zeta) \chi(\varepsilon^{-1} \zeta) \wedge B_0^{p,P}(0, \zeta) - \int_{\varepsilon B} \overline{\partial} v(\zeta) \chi(\varepsilon^{-1} \zeta) \wedge B_0^{p,P}(0, \zeta)
\]

\[
- \int_{\varepsilon B} \overline{\partial} v(\zeta) \wedge \chi(\varepsilon^{-1} \zeta) B_0^{p,P}(0, \zeta) - \int_{\varepsilon B} v(\zeta) \wedge \overline{\partial} \chi(\varepsilon^{-1} \zeta) \wedge B_0^{p,P}(0, \zeta).
\]

In the second integral the form \( \overline{\partial} \chi(\varepsilon^{-1} \zeta) \wedge B_0^{p,P}(0, \zeta) \) has support on \( \varepsilon/2 < |\zeta| < \varepsilon \) and is \( C^\infty \), \( B_0^0 \) has coefficients bounded by \( \|\varepsilon\|^{-2n+1}, \overline{\partial} \chi(\varepsilon^{-1} \zeta) = \overline{\partial} \chi(z)|_{z=\varepsilon^{-1} \zeta \varepsilon^{-1}} \) and by Cauchy-Schwarz the integral can be estimated by \( \varepsilon^{-n} C_1 \|v\|_{L^2(\varepsilon B)} \). By Hölder inequality the first
integral is bounded by \( \varepsilon C_2 \| \nabla v \|_{L^\infty} \).

\[ \square \]

**Proof of 4.2** Since the weight in the integral and the metric have polynomial behaviour at \( a(N(g)) \), so does the \( L^2 \) norm with respect to some ambient Hermitian metric \( h_Z \) on \( Z \) and hence the form \( v \) has at most polynomial growth by the above lemma. If \( M > 0 \), then for small ball of radius \( \delta/2 \) and center \( z_1 \) at the distance \( \delta \) from \( N(g) \) we have the estimate

\[
\| v \|_2 \geq \| v \|_{h_3, B(z_1, \delta/2)}^2 \geq \frac{1}{\inf_{B(z_1, \delta/2)} |g(z)|^{M}} \| v \|_{h_3, B(z_1, \delta/2)}. 
\]

This implies that the \( L^2 \)-norm on \( \delta/2 \)-balls goes to 0 at least as \( \| g \|^M \). With respect to the metric \( h_3 \) the rate of the decreasing close to \( a(S) \) is even larger since the metric \( h_3 \) has poles on \( a(S) \). The distance from \( N(g) \) depends polynomially on \( |g| \) and so together with the above lemma we conclude that for \( M \) large enough the values of \( v \) go to 0 when approaching \( N(g) \).

\[ \square \]

**Corollary 4.4** (Extensions). Notation as above. Let \( v \in \Lambda^{n,0} T^* Z \otimes E|_{a(U')} \) be a holomorphic \((n,0)\)-form with coefficients in \( J(S)^k \). There exist a Stein neighbourhood \( V \) of \( a(U \setminus N(g)) \) conic along \( N(g) \cap a(U) \) and a \((n,0)\)-form \( \tilde{v} \) with holomorphic coefficients and values in \( E \) extending it that has polynomial growth at the boundary.

Before proceeding to the proof let’s state an observation in the following

**Lemma 4.5.** Let \( f \) be a holomorphic section of a holomorphic bundle \( E \) over \( Z \) defined on \( a(U) \) and \( \{ W_j \} \) the product covering. Then the sum \( F \) given in local coordinates \( (z, w) \) by \( F(z, w) = \sum f_j(z, w) \chi_j(z) \) where \( f_j \)-s are local fibrewise constant extensions of \( f \), is holomorphic in the fibre directions,

\[
f_j(z, w) = f(z) + \sum w^\kappa c_{j,\kappa}(z) + \ldots
\]

\[
F(z, w) = f(z) + \sum_{|\kappa| \geq 1, j} w^\kappa c_{j,\kappa}(z) \chi_j(z) \quad \text{and}
\]

\[
\bar{\partial} F(z, w) = \sum_{|\kappa| \geq 1, j} w^\kappa c_{j,\kappa}(z) \bar{\partial} \chi_j(z) = \mathcal{O}(|w|).
\]

If \( f \) is zero up to order \( k \) on \( a(S) \) so is \( F \).

**Proof of 4.4**. Let \( n \) be the fibre dimension. Since \( V \) is Kähler and complete and \( a(X) \) in \( V \) is given as a zero set of finitely many global functions, the Kähler manifold \( V \setminus a(X) \) is also complete (see [Dem]). The function \( \Phi_2 = \varphi + \varphi_1 + \log(\varphi_1) \) is strictly plurisubharmonic on some conic neighbourhood \( \| w \| \leq \| z_2 \|^k \) and has a logarithmic pole on the given section \( a(U) \). This follows immediately from the estimates derived in the proof of the theorem 2.2.

We are solving the \( \bar{\partial} \)-equation with the scalar product \( h_4 = h_3 e^{-r\Phi_2}, r = \dim V T(Z) \). Take an extension of the form \( v \) in the vertical direction obtained by patching together local holomorphic lifts, denote it again by \( v \) and let \( u = \bar{\partial} v \). Since \( u(z,0) = 0 \), the coefficients of \( u \) are bounded by \( C \| w \| \| z_2 \|^k \) close to \( a(S) \) and by \( C \| w \|^2 \) off \( a(S) \). By construction we have \( \varphi_1 \geq \| w \|^2 \| z_2 \|^{2k_1} \). The inverse \( A_{E,\omega} \) has a polynomial pole on \( a(S) \) and the induced scalar product \( h_3 \) has a polynomial pole there, so we have a polynomial pole in the scalar product. Let the whole term be bounded from below by \( \| z_2 \|^{-2k_3} \).
The integrand of 
\[
\int_V \langle A_{E,\omega}^{-1} u, u \rangle_{h,3} e^{-r\Phi_2} dV_\omega
\]
at a point from \(a(S)\) is of the form 
\[
(\|z_2\|^{-2k_3})(\|w\|^2 \|z_2\|^{2k})(\|w\|^{-2r} \|z_2\|^{-2rk_1})(\|w\|^{2r-1} \|z_2\|^{2k_5}) = \|w\| \|z_2\|^{2(k-k_3-rk_1+2k_5)}.
\]
The terms in the last bracket come from the volume form if we introduce the polar coordinates in the base and fibre directions and take the form \(\omega\) into account. The integral on some neighbourhood of this point is reduced to
\[
c_1 \int_0^\delta d\|z_1\| \int_0^\delta d\|z_2\| \int \|z_2\|^{k_4} \|w\| \|z_2\|^{2(k-k_3-rk_1+2k_5)} d\|w\| =
\]
\[
= c_2 \int_0^\delta \|z_2\|^{2(k-k_3-rk_1+2k_5+k_4)} d\|z_2\|
\]
and it converges if either the cone is sharp enough (i.e. \(k_4\) large) or the form has a zero of high enough order (\(k\) large).

Off the set \(a(S)\) the integral is approximately of the type \(\|w\|\) and is therefore finite because the set \(V\) is relatively compact.

The integrability condition
\[
\|v\|^2_V = \int_V \langle v, v \rangle_{h,3} e^{-r\Phi_2} dV_\omega < \infty
\]
then by theorem VIII-7.1, [Dem] forces the solution of the \(\overline{\partial}\)-equation to be zero on the given section.

\[\square\]

**Remark 4.6.** Similar ideas work for jet interpolation at one point (not in \(a(S)\)), since we have a local holomorphic extension. The weight is defined as \(M \log(\|z\|^2 + \|w\|^2)\) on a neighbourhood of the given point and continued as a constant outside. The negativity of such weight can be compensated by \(e^{-c\Phi}\), since we are away from \(a(S)\).

5. \(\overline{\partial}\)-equation in bidegree \((0, q)\).

In this section we prove the analogous theorem for \((0, q)\)-forms. In this case the positivity of the curvature tensor is no longer ensured by the positivity of the bundle curvature. Therefore we view a \((0, q)\)-form as a \((n, q)\)-form with values in a different vector bundle.

Notation as above. Let \(u \in \Lambda^{0,q} T^*Z \otimes E|_{V'}\), where \(V' \subset Z\) possesses a complete Kähler metric \(\omega_1\) and let \(\omega\) be another Kähler metric. The canonical pairing locally gives a decomposition \(1 = v \otimes v^*\), where \(v \in \Lambda^{n,0} T^*Z\) and \(v^* \in \Lambda^{n,0} TZ\). Thus \(u\) can be viewed as a \((n, q)\)-form \(\tilde{u}\) with values in \(\tilde{E} = \Lambda^{n,0} TZ \otimes E\). This adds an additional term to the curvature tensor, namely the curvature of \(\Lambda^{n,0} TZ\) with respect to the given metric \(\omega\). The curvature is the Ricci curvature and so the curvature tensor equals
\[
A_{\tilde{E},\omega} = A_{E,\omega} + \text{Ricci}(\omega).
\]
Let’s check that we gave a well defined section in this new bundle. In local coordinates \( z \) we have
\[
  u = u_z = \sum u_{Q, \lambda}^z d\bar{z}^Q \otimes e_\lambda \equiv \\
  = \sum u_{Q, \lambda}^z d\bar{z}^Q \wedge dz_1 \ldots \wedge dz_n \otimes (\partial/\partial z_1) \wedge \ldots \wedge (\partial/\partial z_n) \otimes e_\lambda = \tilde{u}_z.
\]
for multiindices \(|Q| = q\). Let \( \xi \) be the new coordinates in \( Z \), \((\partial/\partial z) = A(\partial/\partial \xi)\) then \( dz = (A^{-1})^T d\xi \) and in the new coordinates the form \( u \) equals
\[
  u = u^\xi = \sum u_{Q, \lambda}^\xi d\bar{\xi}_Q \otimes e_\lambda = \sum u_{Q, \lambda}^\xi (A^{-1} d\bar{\xi}_Q) \otimes e_\lambda
\]
and induces the section
\[
  \tilde{u}^\xi = \sum u_{Q, \lambda}^\xi d\bar{\xi}_Q \wedge d\xi_1 \ldots \wedge d\xi_n \otimes (\partial/\partial \xi_1) \ldots \wedge (\partial/\partial d\xi_n) \otimes e_\lambda.
\]
On the other hand the section \( \tilde{u}^z \) in coordinates \( \xi \) equals
\[
  (\tilde{u}^z)^\xi = \sum u_{Q, \lambda}^z \overline{A^{-1} d\bar{\xi}_Q} \wedge \det A^{-1} d\bar{\xi}_1 \ldots \wedge d\xi_n \otimes \det A(\partial/\partial \xi_1) \ldots \wedge (\partial/\partial d\xi_n) \otimes e_\lambda = \\
  = \sum u_{Q, \lambda}^z d\bar{z}_Q \wedge d\xi_1 \ldots \wedge d\xi_n \otimes (\partial/\partial \xi_1) \ldots \wedge (\partial/\partial d\xi_n) \otimes e_\lambda
\]
and this coincides with the previously defined form because the terms \( \det A \) and \( \det A^{-1} \) cancel out. Therefore \( \tilde{u} \) is a section and if \( H_Z \) is a matrix representing metric on \( TZ \) and \( H_E \) represents metric on \( E \) then
\[
  |\tilde{u}|^2(z) = \sum u_{Q, \lambda}^z \overline{u_{Q', \chi}^z(z)} \langle d\bar{z}_Q, d\bar{z}_{Q'} \rangle_{H_Z} \cdot \|dz_1 \ldots \wedge dz_n\|_{H_Z}^2 \\
  \|\langle \partial/\partial z_1 \ldots \wedge (\partial/\partial d\xi_n)\rangle_{H_Z}^H \rangle_{H_Z} = \det(H_Z^{-1}) \quad \text{and} \quad \|\langle \partial/\partial z_1 \ldots \wedge (\partial/\partial d\xi_n)\rangle_{H_Z}^H \rangle_{H_Z} = \det H_Z \quad \text{the norm is equal to the norm of} \ u.
\]
Because \( \|dz_1 \ldots \wedge dz_n\|_{H_Z}^2 = \det(H_Z^{-1}) \) and \( \|\langle \partial/\partial z_1 \ldots \wedge (\partial/\partial d\xi_n)\rangle_{H_Z}^H \rangle_{H_Z} = \det H_Z \) the norm is equal to the norm of \( u \).

We would like to find a weight that would remove the Ricci curvature. By proposition 3.1 with \( E = \det Z \) there exist finitely many almost holomorphic sections \( v_i \), holomorphic to order \( l_3 \) in \( w \) with zeroes of order \( k_3 \) on \( a(S) \) that generate the det \( Z \) off \( Z_S \). The metric on the determinant bundle \( h_{\det Z} \) defines the squares of the norms
\[
  f_i(z, w) = \langle v_i(z, w), v_i(z, w) \rangle_{h_{\det Z}}.
\]
The function
\[
  \varphi_2(z, w) = \sum \langle v_i(z, w), v_i(z, w) \rangle_{h_{K}}
\]
defined on \( a(U_1) \) is zero over \( a(S) \) and has locally polynomial zeroes over \( a(S) \) (the metric itself is polynomially degenerate and the vector fields are polynomially degenerate).

Let \( v \) be a nonzero section of the determinant bundle defined on a neighbourhood of a point \((z, 0) \in a(S)\). Then the metric \( h_{\det Z} \) is can be represented as multiplication by the function \( f(z, w) = \langle v(z, w), v(z, w) \rangle_{h_{\det Z}} \) and the Ricci curvature equals \(-i \partial \bar{\partial} \log f\).

By construction we have \( v_i = \alpha_i v \) for some functions \( \alpha_i \), holomorphic in the fibre direction, holomorphic to the degree \( l_3 \) with zeroes of order \( k_3 \) on the fibres over \( a(S) \). This implies that
\[
  \varphi_2 = \sum \langle v_i, v_i \rangle_h = \sum \alpha_i \overline{\alpha_i} \langle v, v \rangle_h = (\sum |\alpha_i|^2) f = \|\alpha\|^2 f,
\]
where \( \alpha \) is a vector with components \( \alpha_i \) equipped with Euclidean norm. The function \( \sum |\alpha_i|^2 \) has zeroes only (on the fibres) over \( a(S) \) so we have an estimate \( \sum |\alpha_i|^2 \geq \|z\|^{2k_3} \).

Let’s multiply the metric with the weight

\[ e^{-\log \varphi_2}. \]

The weight adds the term \( i\partial \bar{\partial} \log f + i\partial \bar{\partial} \log \|\alpha\|^2 \) to the curvature thus killing the Ricci curvature and adding a term that has bounded negative part in a conic neighbourhood (calculation is the same as in the section on Nakano curvature). As before we can compensate the negativity of curvature by multiplying the metric with the weight \( e^{-c\Phi} \) and at the same time achieve that the lowest eigenvalue decreases at most polynomially.

As a result for some large constant \( c \) the curvature tensor

\[ A_{E,\omega} = A_{E,\omega} + \text{Ricci}(\omega) + L(\log \varphi_2) + cL(\Phi) \]

is positive and this enables us to solve the \( \partial \)-equation with at most polynomial growth at the boundary and with zeroes on \( N(g) \). As a corrolary of the theorem 4.1 we have a

**Theorem 5.1.** Let \( u \) be a closed \((0,q)\)-form on \( V \) with

\[ \int_V \langle A_{E,\omega}^{-1} u, u \rangle_h e^{-M \log |g|} dV_\omega \]

for some \( M \geq 0 \). Then there exist a smooth \((0,q-1)\)-form \( v \) solving \( \partial \overline{\partial} v = u \) with

\[ \|v\|^2 = \int_V \langle v, v \rangle_h e^{-M \log |g|} dV_\omega \leq \int_V \langle A_{E,\omega}^{-1} u, u \rangle_h e^{-M \log |g|} dV_\omega. \]

**Remark 5.2.** Note that now the sign of the Ricci curvature does not play any role since we are removing the Ricci curvature by the weight as a contrast to the previous theorem where we needed the positivity of the induced curvature on the determinant bundle in order to compensate the possible negativity of the Hermitian metric.

6. **Vertical sprays on conic neighbourhoods**

**Proof of the theorem 5.2.** Consider the set \( U \). We are looking for sections defined on a conic neighbourhood of a given compact set \( a(U) \) and such that they generate \( VT(Z) = \ker \pi \) on an open neighbourhood of \( a(K) \) and to avoid too many notations we use the letter \( U \) for such a neighbourhood and will shrink \( U \) if necessary. Let \( VT(Z) \) denote the sheaf of sections of the vertical tangent space \( \ker \pi = VT(Z) \). Let \( v_i \) be almost holomorphic sections of \( VT(Z) \), holomorphic to the degree \( l_4 \) in \( w \) and with zeroes of order \( k_4 \) given by proposition 3.1. Define \( u_i = \partial \overline{\partial} v_i \) and view it as a \((n,1)\)-form as in the previous section. We have to show that over a suitable conic neighbourhood the integral

\[ I = \int_V \langle A_{E,\omega}^{-1} u_i, u_i \rangle_h e^{-r_1 \Phi_2} dV_\omega \]

is convergent for \( r_1 \geq r \). As in the \((n,1)\)-case the integrability is problematic only at the points on \( a(S) \). The terms in the integrand are of the following form: \( A_{E,\omega}^{-1} \) has in the worst case a polynomial pole in \( \|z_2\| \), say \( \|z_2\|^{-m} \). The form \( u \) is of the type \( \|w\|^{l_4+1} \|z_2\|^{k_4} \), \( \text{Exp}(-r \log \Phi_2) \) is of the type \((\|w\|^2 \|z_2\|^{2k})^{-r} \) and \( dV_\omega \) of the type \((\|z_2\|^{2(\alpha(n-r)+rk)})dV_{h_2} \). After
introducing the polar coordinates in $w$ and $z_2$ direction (the direction $z_1$ is not problematic) the integral near $a(S)$ takes the form

\[
I = \int_0^\delta \left( \int_0^{\|z_2\|^{k_5}} \|w\|^{2l_4+2-2r_1+2r-1}d\|w\| \right) \cdot \|z_2\|^{k_5} \cdot \int_0^{\|z_2\|} \|w\|^{2l_4+2-2r_1+2r-1}d\|w\|,
\]

where $\|w\| \leq \|z_2\|^{k_5}$ describes the type of the cone. Put $r_1 = r_0$. Then if either $k_4$ is large, meaning that the initial vector fields have zeroes of high order on $a(S)$ or the cone is sharp enough, for example $k_5 > n_1$, or the vector fields are holomorphic to a very high order ($l_4$ large) the integral converges. Even if we start with any vector field with zeroes of large order on $a(S)$ and construct an extension $v$ by lemma 4.5 the integral converges. In this case we have $l_4 = 0$. By the theorem 4.1 and corollary 4.2 we get the $(n,0)$-forms $\tilde{v}_i$ with values in $\Lambda^n TZ \otimes VT(Z)$ of polynomial growth that are zero on our section and after the pairing we get vector fields that still have zeroes on $a(U)$. Vector fields $v_i - \tilde{v}_i$ still generate the $VT(Z)$ on a neighbourhood of $a(U \setminus S)$. In particular, they generate the bundle on a neighbourhood of $a(K)$ in $Z$.

If $r_1 >> r_0$ then again the integral converges $l_4$ is large. As a result the integral of the solution of the $\overline{\partial}$-equation

\[
\|\tilde{v}_i\| = \int_V \langle v_i, v_i \rangle_{h_0} e^{-r_1 \Phi_2} < \infty
\]

converges by the theorem 4.1 the sections $\tilde{v}_i$ are smooth and have zeroes on $a(U)$ and because the coefficients in the integrals have large poles on $a(U) \cup Z_S$ the $L^2$-norms with respect to $h_Z$ on small balls close to $\|z_2\| = 0$ go to 0 faster than $\|z_2\|^{2r_1 k}$. The estimate from the Bochner-Martelli-Koppelman formula 4.3 tells us that in a smaller conic neighbourhood $\tilde{V}$ we have the estimate

\[
\|\tilde{v}_i(z, w)\| \leq \delta^{-n} \|\tilde{v}_i\|_{B((z,w),\delta)} + \delta \|\tilde{v}_i\|_{B((z,w),\delta),\infty},
\]

where $\delta$ is the distance from $(z, w) \in \tilde{V}$ of to the boundary of $\tilde{V}$ and it depends polynomially on $\|z_2\|$, $\delta \approx \|z_2\|^m$. Then as $\|\tilde{v}_i\|_{B((z,w),\delta),h_4} \geq C \|z_2\|^{-2r_1 k} \|v_i\|_{B((z,w),\delta),h_Z}$ we have

\[
\|\tilde{v}_i(z, w)\| \leq C^{-1} \|z_2\|^{2r_1 k} \|z_2\|^{2r_1 k} + \|z_2\|^m \|z_2\|^{k_4}
\]

and this means that on $\tilde{V}$ the resulting sections are bounded and go to 0 polynomially as approaching $a(S)$. They are 0 on $a(U)$ this implies that close to $a(S)$ they are of the form $\|z_2\|^{k_4} \|w\|$. In case $\|w\| \leq \|z_2\|^{k_4+k_5}$ they decrease faster than the vector fields $v_i$ and since the latter were generating the bundle so do the vector fields $v_i - \tilde{v}_i$.

If we take a slightly thinner and sharper conic (with respect to the fibre directions) and shrink $U$ a little they will be bounded when away from $N(g)$. Denote this conic neighbourhood by $\tilde{V}$. But then the vector fields $g^k(v_i - \tilde{v}_i)$ for sufficiently large $k$ still generate the bundle wherever $v_i - \tilde{v}_i$ did and approach 0 near $g^{-1}(0)$ as fast as we want. In particular they are (at least) continuous on the closure of $V$ and can be extended to global continuous vector fields.

Their flows $\varphi_{i,t_i}(z)$ remain in $V$ for $z$ in thinner and sharper conic neighbourhood $V'$ (see figure 11) for small times $t < \varepsilon$ and so generate a continuous vertical spray $s := \varphi_{1,t_1} \circ \ldots \circ \varphi_{m,t_m} : V_Z \times \Delta(0,\varepsilon) \to Z$ for sufficiently small neighbourhood $V_Z$ of $a(U)$ in $Z$. 17
Over $V'$ the spray is holomorphic and dominating over a given compact set $a(K)$ for $K \subset a(U \setminus N(g))$ (provided $V$ is thin enough). The restriction of $s$ to $a(U) \times \Delta(0, \varepsilon)^m$ is smooth and holomorphic on $a(U \setminus N(g)) \times \Delta(0, \varepsilon)^m$ and is therefore holomorphic on $a(U) \times \Delta(0, \varepsilon)^m$ since $N(g)$ is analytic of codimension 1. This completes the proof of the main theorem in [Pre] in the case of manifolds.

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