Optomechanical position detection enhanced by de-amplification using intracavity squeezing

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It has been predicted and experimentally demonstrated that by injecting squeezed light into an optomechanical device it is possible to enhance the precision of a position measurement. Here, we present a fundamentally different approach where the squeezing is created directly inside the cavity by a nonlinear medium. Counterintuitively, the enhancement of the signal to noise ratio works by de-amplifying the quadrature that is sensitive to the mechanical motion. This enhancement works for systems with a weak optomechanical coupling and/or strong mechanical damping. This could allow for larger mechanical bandwidth of quantum limited detectors based on optomechanical devices. Our approach can be straightforwardly extended to Quantum Non Demolition (QND) qubit detection.

Recent progress in cavity optomechanics [1,2] has been so exceptional that the precision of a position measurement has been pushed until the limit set by the principles of quantum mechanics, the so-called Standard Quantum Limit (SQL) [3–5]. A measurement precision close to the SQL has been demonstrated in optomechanical devices with cavities both in the optical [6,7] and in the microwave [8] domain. Optomechanical position detection is not only of fundamental interest but finds also application in acceleration [9,10], magnetic field [11,12], and force detectors [13,14]. Thus, an important goal for the future is to develop new techniques to enhance its precision on different optomechanical platforms. The first step in this direction has been the proposal by Caves to enhance the precision of gravitational waves detection in an optomechanical interferometer by using externally generated squeezed light [15]. This technique has recently been demonstrated in the Laser Interferometer Gravitational Wave Observatory (LIGO) [16] and in a cavity optomechanics setup [17]. Externally generated squeezed light could also find application in QND qubit state detection [18,19]. Other techniques aiming to enhance the precision of a dispersive quantum measurement include operating close to the static bistability of a nonlinear cavity [20,22] and generating squeezing internally by the dissipative optomechanical interaction [24].

In this letter, we propose a new pathway to precision enhancement in optomechanical detection. In our approach, a nonlinear cavity is operated as a phase-sensitive parametric amplifier, as shown in Fig. 1. It amplifies a seed laser beam and its intensity fluctuations. Simultaneously, it de-amplifies the phase quadrature where the mechanical vibrations are imprinted. At first sight it might appear counter-intuitive that de-amplification can improve a (quantum) measurement. Here, we suggest that it might be worth to de-amplify a signal if the noise is suppressed by a larger factor thus obtaining a net enhancement of the signal to noise ratio. Indeed, our analysis shows that for optomechanical position detection a de-amplification of the phase quadrature induces only a limited suppression of the signal but simultaneously can strongly suppress the measurement noise. Our scheme could be implemented using a crystalline whispering gallery mode resonator [21]. Such devices offer a well established platform for optomechanics [1,2,25,26]. Resonators with an optical \(\chi^\text{(2)}\) nonlinearity can be operated as parametric amplifiers in the quantum regime [27,28]. The exciting perspective of an interplay of optical and optomechanical nonlinearities has already inspired a few theoretical investigations [29,32]. Alternative implementations of our scheme include optomechanical crystals [33] made out of a nonlinear medium [44] and a Josephson parametric amplifier [35] coupled to a mechanical membrane or a qubit.
We consider a degenerate parametric amplifier which is tuned to have a pair of modes with frequencies \( \omega_p \) and \( \omega_s \) (p pump, s signal) where \( \omega_p = 2 \omega_s \), and the pump mode is driven resonantly. In the following, we denote as \( \kappa_s \) and \( \kappa_p \) the decay rates of the corresponding cavity modes. We first describe our proposal by considering the standard description of an ideal degenerate parametric amplifier where the pump mode has been already adiabatically eliminated and we neglect intrinsic losses. In a frame rotating at frequency \( \omega_s \), the degenerate parametric amplifier is described by the standard linearized Hamiltonian \[ H_s = i \hbar \bar{n}_p^{1/2} \nu (\hat{a}_s^\dagger \hat{a}_s - \hat{a}_s^\dagger \hat{a}_s) / 2. \]

Here, \( \hat{a}_s \) is the ladder operator for the signal mode, \( \nu \) is the single-photon optical nonlinearity and \( \bar{n}_p \) is the number of photons circulating in the pump mode. The parametric amplifier is characterized by its pump parameter \( \sigma \),

\[ \sigma^2 = \bar{n}_p / \bar{n}_p^{(\text{thr})}, \quad \bar{n}_p^{(\text{thr})} = \left( \frac{\kappa_s}{2\nu} \right)^2, \quad (1) \]

The signal mode reaches the threshold of self-sustained (optical parametric) oscillations when the photon number circulating in the pump mode equals \( \bar{n}_p^{(\text{thr})} \), corresponding to the pump parameter \( \sigma = 1 \). Below threshold, the cavity behaves as a phase sensitive amplifier: An additional seed laser with the appropriate phase and frequency (the signal mode eigenfrequency \( \omega_s \)) is amplified with gain \( G = [(1+\sigma)/(1-\sigma)]^2 \), see sketch in Fig. 1. Notice that also the intensity fluctuations are amplified by a factor of \( G \) (inside the amplifier bandwidth \( (1-\sigma)\kappa_s \)) while the phase fluctuations are de-amplified by the same factor.

We want to measure the displacement \( \hat{x} \) of a mechanical resonator with eigenfrequency \( \Omega \), effective mass \( m \), and decay rate \( \Gamma \), see Fig. 1. The mechanical resonance could be internal to the optical resonator (e.g. a breathing mode) or refer to the vibrations of an external nano-object coupled evanescently. A displacement \( \hat{x} \) induces a shift \(-G\hat{x}\) of the signal mode frequency, described by a Hamiltonian \( H_{OM} = -\hbar G \hat{a}_s \hat{x} \). We assume here that the effects of the coupling between the displacement and the pump mode are negligible (otherwise the additional information contained in the phase shift of the reflected pump beam would also have to be monitored). We measure the displacement \( \hat{x} \) by extracting the output signal phase of a seed drive injected at the bare cavity resonance \( \omega_s \), where it is most sensitive to the jittering of the optical resonance induced by the mechanical vibrations. When the seed laser injects a large number \( \bar{n}_s \) of circulating photons into the signal mode (below, we specify this condition more precisely), we can linearize the optomechanical interaction [1]. Then the mechanical vibrations couple to the optical field quadrature \( \hat{X} = (\hat{a}_s + \hat{a}_s^\dagger - 2\sqrt{\kappa_s}) / \sqrt{2} \) describing the amplitude fluctuations: \( \hat{H}_{OM} = -\hbar G \sqrt{2\kappa_s} \hat{X} \hat{x} \). We arrive at the Langevin equations for the optical signal mode quadratures \( \hat{X} \) (amplitude) and \( \hat{Y} \) (phase):

\[ \dot{\hat{X}} = -(1-\sigma)\kappa_s \hat{X} / 2 + \sqrt{\kappa_s} \hat{X}^{(\text{in})} \]
\[ \dot{\hat{Y}} = -(1+\sigma)\kappa_s \hat{Y} / 2 + \sqrt{2\kappa_s} \hat{G} \hat{x} + \sqrt{\kappa_s} \hat{Y}^{(\text{in})}. \quad (2) \]

Here, we have set all absorptive losses to zero (more on that later). We defined \( \hat{Y} = i(\hat{a}_s^\dagger - \hat{a}_s) / \sqrt{2} \), and \( \hat{X}^{(\text{in})} \) and \( \hat{Y}^{(\text{in})} \) are the standard vacuum input fields (the quantum fluctuations of the laser beam at the input) [36]. As seen in Eq. (2), the presence of the nonlinear medium and the pump drive manifests itself in the de-amplification of the phase quadrature and a corresponding amplification of the amplitude quadrature. In the limit \( \sigma \to 0 \), we recover the Langevin equations for a cavity measuring the mechanical displacement in the standard approach without squeezing.

We note that the de-amplification of the optical quadrature where the mechanical signal is imprinted is the key mechanism which will create an improvement of the measurement precision in our approach. We briefly compare to other schemes aiming at such an enhancement. In [20, 22], the amplification of this quadrature in an effectively detuned parametric amplifier built from a nonlinear cavity (near the static bistability) is considered, and the SQL is reached only away from the mechanical resonance (whereas for us it is reached precisely at resonance). Ref. [15] did not deal with amplification but rather with squeezing of the input noise (when squeezed light is injected from the outside). Finally, [23] proposed using an engineered dissipative interaction to the mechanical degree of freedom to feed squeezed noise to the cavity output, rather than de-amplifying a selected quadrature.

To improve a measurement by de-amplification might not sound promising. The measurement noise will be de-amplified but one could reasonably expect that this effect will be offset by the de-amplification of the signal. Indeed, it is true that the response of the cavity to both the vacuum noise and the mechanical vibrations is decreased by the same factor. From Eq. (2), the intracavity phase quadrature in frequency space is

\[ \hat{Y}[\omega] = \chi_Y(\omega) \left( \sqrt{2\kappa_s} \hat{G} [\omega] + \sqrt{\kappa_s} \hat{Y}^{(\text{in})} [\omega] \right) \quad (3) \]

with the intracavity susceptibility

\[ \chi_Y = -i\omega + (1 + \sigma) \kappa_s / 2 \quad [1]. \]

We note in passing that the largest possible suppression, a factor of 2, occurs in the limit \( \omega \to 0 \) and \( \sigma \to 1 \). This is the well known 3dB limit of intracavity squeezing [37]. However, the suppression of the background noise and of the mechanical signal is different outside the cavity. From the input/output relation \( \hat{Y}^{(\text{out})} = \hat{Y}^{(\text{in})} - \sqrt{\kappa_s} \hat{Y} \).
Output phase noise $\bar{S}_{YY}(\omega)$ as a function of frequency. Comparison between the phase noise in presence and in absence of the pump drive for the same number of circulating photons $\bar{n}_s$. In presence of the pump laser (pump parameter $\sigma = 0.6$), the background noise inside the amplifier bandwidth is squeezed below the shot noise level by more than 3dB. The signal amplitude is also reduced, but in this case the reduction is bounded by the 3dB limit. The number of circulating photons $\bar{n}_s$ is chosen to yield the minimum added noise allowed by the SQL for $\sigma = 0.6$. Thus, the imprint noise and the backaction noise (shown in the zoom) have the same intensity at the mechanical resonator eigenfrequency $\Omega$. The remaining parameters are: $\Omega = 0.2\kappa_s$, $\Gamma = 10^{-3}\kappa_s$, $k_B T/h \Omega = 1$.

we find

$$
\hat{Y}^{(\text{out})}[\omega] = [1 - \kappa_s\chi_Y(\omega)]\hat{Y}^{(\text{in})}[\omega] - \sqrt{2\kappa_s\bar{n}_s}G\chi_Y(\omega)\hat{\varepsilon}[\omega]
$$

(4)

From this formula we see that the response of the phase quadrature of the transmitted signal to the mechanical vibrations is still governed by the intracavity susceptibility and is thus subject to the 3dB limit of squeezing. In contrast, the output phase noise is squeezed below the 3dB limit by the destructive interference between the reflected input noise and the response to the cavity that noise. Indeed, it is well-known that the output noise squeezing can be arbitrarily large [28]. Thus, we expect an overall enhancement of the measurement precision due to the de-amplification, which we quantify below. In a homodyne setup, see Fig. 1, it is possible to directly measure the symmetrized noise spectral density of the output phase quadrature

$$
\bar{S}_{YY}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2} e^{i\omega t} \langle \{\hat{Y}^{(\text{out})}(t), \hat{Y}^{(\text{out})}(0)\} \rangle.
$$

(5)

The typical behavior of the output noise phase in presence of phase squeezing is shown in Fig. 2. In order to quantify the net enhancement of the measurement precision, it is convenient to define the measured noise referred back to the input $\bar{S}_{xx}^{(\text{meas})} = \bar{S}_{YY}/(2\kappa_s\bar{n}_s)G^2|\chi_Y|^2$. Then, from Eq. (4) the measured noise takes the form $\bar{S}_{xx}^{(\text{meas})}(\omega) = \bar{S}_{xx}(\omega) + \bar{S}_{xx}^{(\text{add})}$ where $\bar{S}_{xx}(\omega)$ describes the symmetrized mechanical noise in absence of optomechanical backaction, whereas $\bar{S}_{xx}^{(\text{add})}$ is the noise added during the measurement. We are interested in the noise at frequency $\Omega$ where the mechanical spectrum is peaked. Since there is a typical number of circulating photons (specific of the device) that can be tolerated without inducing strong heating effects, we use as a figure of merit of our measurement scheme the added noise $\bar{S}_{xx}^{(\text{add})}(\Omega)$ for a fixed total number of circulating photons $\bar{n} = \bar{n}_s + \bar{n}_p$.

Below, we show that internally generated optical squeezing can strongly enhance the precision for optomechanical couplings that are small compared to the optical nonlinearities, when

$$
C_{\text{thr}} \equiv \frac{g^2\bar{n}_s}{\Gamma\Omega^2} = \frac{4g^2\bar{n}_p^{(\text{thr})}}{\Gamma \bar{n}_s} \ll 1.
$$

(6)

Here, $g_0 = G/\sqrt{2}\Gamma^F$ is the single-photon optomechanical coupling (the optical frequency shift by a single phonon) [1]. The parameter $\bar{C}_{\text{thr}}$, which we refer to as threshold cooperativity, is the optomechanical cooperativity if $\bar{n}_s = \bar{n}_p^{(\text{thr})}$ photons were in the signal mode. It quantifies the ratio of optomechanical and nonlinear coupling. Notice that in the absence of squeezing the SQL is reached for the optomechanical cooperativity $C = 1/4$ [1]. Thus, if $C_{\text{th}} \ll 1$ it is not possible to achieve a precision close to the SQL by injecting all available photons $\bar{n}_s \sim \bar{n}_p^{(\text{thr})}$ directly into the signal mode. Instead, one can enhance the measurement precision by injecting part of the photons into the pump mode to generate squeezing, as shown below.

We now calculate the added noise $\bar{S}_{xx}^{(\text{add})}$. We distinguish between two different contributions [3–5]: the so-called imprint noise $\bar{S}_{xx}^{(\text{imp})}(\omega)$ and the backaction noise $\bar{S}_{xx}^{(\text{back})}(\omega)$. The former is due to the shot noise phase fluctuations. The latter is the additional mechanical noise induced by the backaction of the light onto the mechanics. It can be expressed as $\bar{S}_{xx}^{(\text{back})}(\omega) = [\chi_M(\omega)]^2\bar{S}_{FF}(\omega)$ in terms of the mechanical susceptibility $\chi_M(\omega) = m^{-1}(\omega^2 - \omega^2 + i\omega\Gamma)^{-1}$ and the noise spectrum $\bar{S}_{FF}$ of the radiation pressure force $F = \sqrt{2\bar{n}_s}hG\hat{X}$. We note in passing that our measurement scheme could also find application in the detection of any degree of freedom coupled dispersively to the cavity, e. g. a qubit [4]. From Eq. (2) we can readily derive the identity $\bar{S}_{xx}^{(\text{imp})}(\omega)\bar{S}_{FF}(\omega) = \hbar^2/4$ valid for all values of $\sigma$. It is well-known that when this equality holds both the position detection of resonant vibrations and the QND qubit state detection are quantum limited [3–5]. We compute the overall added noise $\bar{S}_{xx}^{(\text{add})} = \bar{S}_{xx}^{(\text{imp})} + \bar{S}_{xx}^{(\text{back})}$ from Eq. (2):

$$
\bar{S}_{xx}^{(\text{imp})} \approx \bar{S}_{xx}^{(\text{SQL})} \left( 1 - \sigma^2 \right) + 4\Omega^2/\kappa_s^2\bar{S}_{xx}^{(\text{back})}\left( \bar{n}/\bar{n}_p^{(\text{thr})} - \sigma^2 \right),
$$

(7)

Here, we have introduced the minimum added noise allowed by the SQL $\bar{S}_{xx}^{(\text{SQL})} = \hbar/m\Omega$ [3–5]. Fig. 3(a) shows the added noise Eq. (7) as a function of the circulating photon number $\bar{n}$ and the pump parameter $\sigma$. For $\sigma = 0$ (no circulating photons in the pump mode), we recover
the result for standard optomechanical detection. The SQL is reached for $S^{(add)}_{xx} = 2S^{(imp)}_{xx} = 2S^{(back)}_{xx} = S^{SQL}_{xx}$, see also the zoom in Fig. 3. From Eq. (7), we find the required photon number

$$\bar{n}^{SQL}(\sigma) = \left(1 - \sigma^2 + 4\Omega^2/\kappa_s^2\right) + \sigma^2. \quad (8)$$

It is shown as a yellow solid line in Fig. 3(a). By minimizing $\bar{n}^{SQL}(\sigma)$ as a function of $\sigma$, we find the minimal number of circulating photons $\bar{n}^*$ necessary to reach the SQL and the corresponding optimal pump parameter $\sigma^*$,

$$\bar{n}^* = \bar{n}^{SQL}(\sigma^*), \quad \sigma^* = (1 + 4\kappa_{thr})^{-1}. \quad (9)$$

Compared to the standard scheme, where the SQL is reached for $\bar{n}_{standard} = \bar{n}^{SQL}(\sigma = 0)$ circulating photons, the required number of photons is suppressed by a factor of

$$\frac{\bar{n}^{SQL}}{\bar{n}^*} = \frac{1 + 4\Omega^2/\kappa_s^2}{1 - (4\kappa_{thr} + 1)^{-1} + 4\Omega^2/\kappa_s^2}. \quad (10)$$

The suppression factor increases monotonically with increasing optical nonlinearity (decreasing threshold cooperativity $\kappa_{thr}$) and reaches the asymptotic value $\kappa_s^2/4\Omega^2$ for large optical nonlinearities ($\kappa_{thr} \ll 1$, in the bad cavity limit $\Omega \ll \kappa_s$). Our method is still useful even when it is not possible to reach the maximum precision allowed by the SQL because the typical number of circulating photons tolerated in the device is too small (smaller than $\bar{n}^*$). In this case, the added noise remains larger than $S^{SQL}_{xx}$, yet it can still be decreased by the squeezing. By minimizing $S^{(add)}_{xx}$ in Eq. (7) as a function of $\sigma$ for a fixed $\bar{n}$ (smaller than $\bar{n}^*$), we find the optimal pump parameter

$$\sigma^{(opt)} = \frac{B}{2} - \left(\frac{B^2}{4} - \frac{\bar{n}}{\bar{n}_{p}^{(thr)}}\right)^{1/2}, \quad B = 1 + \frac{\bar{n}}{\bar{n}_{p}^{(thr)}} + 4\Omega^2/\kappa_s^2. \quad (11)$$

It increases monotonically with the number of circulating photons and reaches the value $\sigma = \sigma^*$ for $\bar{n} = \bar{n}^*$, see the white dashed line in Fig. 3(a).

Next we go beyond the ideal description of a parametric amplifier by considering the effects of losses. Those are potentially deleterious as they decrease the amount of achievable squeezing. In a parametric amplifier there are two main loss channels: (i) photon absorption and (ii) photon up-conversion, where photons which are up-converted by the $\chi^{(2)}$ interaction $i\hbar\nu(\hat{a}_s^2\hat{a}_p - h.c.)/2$ decay via the pump mode. This process is enhanced in the presence of a large number $\bar{n}_s$ of signal mode photons. Thus, the overall loss rate takes the form (see also Appendix B for a full derivation)

$$\kappa_s^{(loss)} = \kappa_s^{(abs)} + 4\nu^2\bar{n}_s/\kappa_p. \quad (12)$$

In order to suppress the losses via the pump mode it is therefore important to have a large pump decay rate $\kappa_p$. The effect of losses is investigated in Fig. 3(b). It shows that the measurement precision can still be noticeably enhanced by the squeezing in a broad range of threshold cooperativities $\kappa_{thr}$. Indeed, an explicit calculation shows that the added noise is dramatically increased only for $\kappa_s^{(loss)} \approx \Omega^2/\kappa_s$, see Appendix [3]. A large pump decay rate is helpful to suppress losses. For a small pump decay rate $\kappa_p$, one could suppress the upconversion losses by introducing a detuning between the pump laser and the pump mode, while keeping the (pump) seed laser at (twice) the effective signal mode eigenfrequency. Alternatively, one could improve the measurement precision by monitoring also the light scattered by the pump mode.

The inevitable enhancement of the intensity fluctuations in the proposed measurement scheme represents a potential contradiction of the assumption of small fluctuations, inherent to the linearized Langevin equations [2]. However, it can be shown that the enhanced fluctuations remain compatible with the linearization, provided that the single-photon nonlinearity $\nu$ is not too large, $\nu \ll \Omega$, see Appendix [A].

The regime of small threshold cooperativities $\kappa_{thr} \ll 1$ is realized in state-of-the-art lithium-niobate microdisks [27, 28, 39]. These devices have breathing modes with eigenfrequencies $\Omega$ in the MHz range. Typical single-photon optomechanical couplings are in the sub-Hz.
range, whereas single-photon optical nonlinearities $\nu$ are in the kHz range. Thus, the regime $C_{thr} \ll 1$ is compatible with the bad cavity limit even for disks with large mechanical quality factors. Moreover, the nonlinear corrections to the Langevin equations will be small.

In conclusion, we have shown that the precision of optomechanical position detection can be strongly enhanced by de-amplification. Our method could pave the way to the quantum limited position detection of mechanical resonators with larger decay rates. This would allow faster detection of forces yielding an increase of the bandwidth of quantum limited detectors based on optomechanical devices. A natural extension of our scheme could find application in QND qubit state detection.

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Appendix A: Derivation of the linearized Hamiltonian for the optomechanical degenerate parametric amplifier

We start from the standard input-output formalism for a degenerated parametric amplifier (DPMA) coupled via radiation pressure to an underdamped mechanical oscillator. Both optical modes are driven resonantly. In a frame where they are rotating at their respective eigenfrequencies, $\omega_p$ and $\omega_s = \omega_p/2$ the Langevin equation reads

\[
\dot{\hat{a}}_p = -\kappa_p \hat{a}_p - \nu \hat{a}_s^2/2 + \sqrt{\kappa_p} \langle \hat{a}_{in} \rangle \hat{a}_p + \sqrt{\kappa_p} \langle \hat{a}_{abs} \rangle \hat{a}_p
\]

\[
\dot{\hat{a}}_s = i g_0 \left( \hat{b} + \hat{b}^\dagger \right) \hat{a}_s - \kappa_s \hat{a}_s/2 + \nu \hat{a}_s^2 + \sqrt{\kappa_s} \langle \hat{a}_{in} \rangle \hat{a}_s + \sqrt{\kappa_s} \langle \hat{a}_{abs} \rangle \hat{a}_s
\]

\[
\dot{\hat{b}} = (-i\Omega - \Gamma/2) \hat{b} + i g_0 \hat{a}_s^\dagger \hat{a}_s + \sqrt{\Gamma} \hat{b}_{in}
\]

(A1)

Moreover, we denote by $\hat{b}$ and $\hat{a}_{p/s}$ the phonon and photon annihilation operators, respectively; $\kappa_{p/s}$ and $\kappa_{p/s}^{(abs)}$ are the photon outcoupling and absorption rates, respectively; Their sums give the overall decay rates $\kappa_{p/s} = \kappa_{p/s}^{(in)} + \kappa_{p/s}^{(abs)}$; The pump and seed laser have amplitudes $\langle \hat{a}_{p/s}^{(in)} \rangle$.

We use the standard procedure to linearize the Langevin equations: we divide the ladder operators $\hat{b}$ and $\hat{a}_{p/s}$ into the sum of their average stationary amplitudes and the corresponding fluctuating fields, $\hat{b} = \langle \hat{b} \rangle + \delta \hat{b}$, $\hat{a}_{p/s} = \langle \hat{a}_{p/s} \rangle + \delta \hat{a}_{p/s} = \alpha_{p/s} + \delta \hat{a}_{p/s}$; By plugging this decomposition into the full nonlinear Langevin equation (A1) and neglecting all the correlations between the fluctuating fields $\delta \hat{b}$ and $\delta \hat{a}_{p/s}$, we arrive to two set of equations for the stationary average fields and the noise operators. The classical equations for the stationary amplitudes,

\[
-\left(\kappa_p \alpha_p + \nu \alpha_s^2\right)/2 + \sqrt{\kappa_p} \alpha_p^{(in)} = 0,
\]

\[
i 2 g_0^2 n_s \alpha_s/\Omega - (\kappa_s \alpha_s - 2 \nu \alpha_s^* \alpha_p)/2 + \sqrt{\kappa_s} \alpha_s^{(in)} = 0,
\]

\[
-\alpha \Omega \beta + ig_0 n_s = 0,
\]

yield the laser light amplitudes $\alpha_p^{(in)}$ and $\alpha_s^{(in)}$ required to generate a specific combination of intracavity average light amplitudes $\alpha_s$ and $\alpha_p$. In our investigation, we focus on the parameter regime where the precision of the optomechanical position detection could be enhanced, $\alpha_p = \sigma \kappa_s/2\nu$ with $0 \leq \sigma < 1$ and $\alpha_s = \bar{n}_s^{1/2}$ (thus $\alpha_s$ and $\alpha_p$ are positive and real). The noise operators dynamics is governed by the linearized Langevin equations,

\[
\delta \dot{\hat{a}}_p = -\frac{\kappa_p}{2} \delta \hat{a}_p - \nu \sqrt{\kappa_s} \delta \hat{a}_s + \sqrt{\kappa_p} \delta \hat{a}_p^{(in)}
\]

\[
+ \sqrt{\kappa_p} \langle \hat{a}_{abs} \rangle \hat{a}_p^{(abs)},
\]

\[
\delta \dot{\hat{a}}_s = ig_0 \sqrt{\kappa_s} \left( \delta \hat{b} + \delta \hat{b}^\dagger \right) + i 2 g_0^2 n_s \delta \hat{a}_s/\Omega - \frac{\kappa_s}{2} \delta \hat{a}_s
\]

\[
+ \sigma \kappa_s \delta \hat{a}_s + \sqrt{\kappa_s} \langle \hat{a}_{in} \rangle \delta \hat{a}_s^{(in)} + \sqrt{\kappa_s} \langle \hat{a}_{abs} \rangle \hat{a}_s^{(abs)}
\]

\[
\delta \dot{\hat{b}} = \left( -i \Omega - \frac{\Gamma}{2} \right) \delta \hat{b} + ig_0 \sqrt{\kappa_s} \left( \delta \hat{a}_s^{\dagger} + \delta \hat{a}_s \right) + \sqrt{\Gamma} \delta \hat{b}_{in}\]

The above equations describe the linearized dynamics beyond the ideal description of Eq. (2). Compared to the ideal description, they include also intrinsic losses, the coupling between the signal and pump mode fluctuations, and a radiation pressure induced effective detuning of the signal mode (by $2 g_0^2 n_s/\Omega$).

Before adding more technical details to the discussion of the main text regarding the effects of losses and of the pump-signal coupling we have to explain why we have omitted the effective detuning of the signal mode in the main text. The effective detuning is negligible when it is much smaller than the cavity bandwidth $\kappa_s$, $\Omega \kappa_s \gg g_0^2 n_s$. Keeping in mind that the Standard Quantum Limit (SQL) in a linear cavity is reached for $\Gamma \kappa_s = 16 g_0^2 n_s$, we can conclude that the shift is negligible if the circulating photon number is optimized to reach the best possible precision and, at the same time, the mechanical quality factor is large (this is typically the case in WGMs). Even in the case of mechanical oscillators with not too large quality factors, it could still be possible to eliminate the radiation pressure induced detuning by tuning appropriately the cavity spectrum and the laser frequencies.

In the absence of detuning, it is most convenient to
write the Langevin equations in terms of quadratures,
\[ \dot{Y}_p = -\frac{k_p}{2} Y_p - \nu \sqrt{n_s} \dot{Y} + \sqrt{k_p^{(in)}} \dot{Y}_p^{(in)} + \sqrt{k_p^{(abs)}} \dot{Y}_p^{(abs)} \]
\[ \dot{X}_p = -\frac{k_p}{2} X_p - \nu \sqrt{n_s} \dot{X} + \sqrt{k_p^{(in)}} \dot{X}_p^{(in)} + \sqrt{k_p^{(abs)}} \dot{X}_p^{(abs)} \]
\[ \dot{Y} = -(1 + \sigma) \kappa_s Y/2 + \nu \sqrt{n_s} \dot{Y}_p + \sqrt{n_s} G x \]
\[ + \sqrt{\kappa_s^{(in)}} \dot{Y}^{(in)} + \sqrt{\kappa_s^{(abs)}} \dot{Y}^{(abs)} \]
\[ \dot{X} = -(1 - \sigma) \kappa_s X/2 + \nu \sqrt{n_s} \dot{X}_p + \sqrt{n_s} G x \]
\[ + \sqrt{\kappa_s^{(in)}} \dot{X}^{(in)} + \sqrt{\kappa_s^{(abs)}} \dot{X}^{(abs)} \]
\[ \dot{x} = \frac{\hat{p}}{m} - \Gamma \dot{x} + \sqrt{\Gamma} x^{(in)} \]
\[ \dot{p} = -m \Omega^2 \dot{x} - \frac{\Gamma}{2} \dot{p} + \sqrt{2n_s h G \dot{X}} + \sqrt{\Gamma} \dot{p}^{(in)} \]

(A2)

Here, \( \dot{X} = (\delta a_s + \delta a_s^\dagger)/\sqrt{2} \) and \( \dot{Y} = -(i \delta a_s + \delta a_s^\dagger)/\sqrt{2} \) describe the signal mode intensity and phase fluctuations. Analogous definitions apply to the pump quadratures \( \dot{X}_p \) and \( \dot{Y}_p \) and the noise operators \( \dot{X}^{(in)} \), \( \dot{X}^{(abs)} \), \( \dot{Y}^{(in)} \), \( \dot{Y}^{(abs)} \), \( \dot{X}_p^{(in)} \), \( \dot{X}_p^{(abs)} \), \( \dot{Y}_p^{(in)} \), and \( \dot{Y}_p^{(abs)} \). Moreover, \( \dot{p} = (m \Omega^2/2)^{1/2}(\delta b + \delta b^\dagger) \) is the oscillator momentum and \( \dot{x} = (\hbar/2m \Omega)^{1/2}(\delta b + \delta b^\dagger) \) is the displacement counted off from the stationary position, \( hGn_s/m \Omega^2 \). We note in passing that the form of the mechanical damping, yielding a dissipation of the mechanical energy which is independent from the phase of the vibrations, is consistent with the assumption of underdamped vibrations decaying over many cycles, \( \Gamma \ll \Omega \).

2. Analysis of the limit of validity of the nonlinear corrections

We have also verified that the linearization of the Langevin equations is a good approximation for the parameter regime compatible with state of the art devices. This is true even for small threshold cooperativities \( C_{thr} \ll 1 \) where the amplifier is operated close to threshold (\( \sigma \approx 1 \)) and the intensity fluctuations are strongly enhanced, \( \langle X^2 \rangle = (1 - \sigma)^{-1}/2 \). We consider the case where the minimum added noise allowed by the SQL is realized in presence of the smallest possible number of circulating photons \( \bar{n}_s \) for the optimal pump parameter \( \alpha^* \), see Eqs. (7,8) and Fig. 3(a) of the main text. We require that the average number of additional photons due to the fluctuations \( \langle X^2 + \dot{Y}^2 \rangle/2 \) is comparatively small, \( \langle X^2 + \dot{Y}^2 \rangle/2 \ll \bar{n}_s \). From Eqs. (7,8) we find that in the limit \( C_{thr} \ll (\Omega/\kappa_s)^2 \), where this constraint is more stringent, it amounts to an upper bound for the optical nonlinearity, \( \nu \ll \Omega \). Thus, the linearized Langevin equations accurately describe state of the art lithium-niobate microdisks [39] where typical optical nonlinearities are in the kHz range and typical frequencies of vibrational breathing modes are in the MHz range.

Appendix B: Technical details regarding the calculation of the added noise in presence of losses

In Fig. 3(b), we have computed directly from the system of equations (A2) the added noise referred back to the input and minimized over the pump parameter \( \sigma \). We found that a rather large pump parameter was necessary to keep the added noise small. In the regime of large pump decay rates, \( \kappa_p \gg \kappa_s, \nu \sqrt{n_s} \), it is possible to eliminate adiabatically the pump mode and we arrive to the Langevin equations
\[ \dot{Y} = -(1 + \sigma) \kappa_s Y/2 + \sqrt{2n_s G \dot{x}} + \sqrt{\kappa_s^{(in)}} \dot{Y}^{(in)} \]
\[ + \sqrt{\kappa_s^{(loss)}} Y^{(loss)} , \]
\[ \dot{X} = -(1 - \sigma) \kappa_s X/2 + \sqrt{\kappa_s^{(in)}} \dot{X}^{(in)} + \sqrt{\kappa_s^{(loss)}} \dot{X}^{(loss)} , \]
\[ \dot{x} = \frac{\hat{p}}{m} - \Gamma/2 \dot{x} + \sqrt{\Gamma} x^{(in)} , \]
\[ \dot{p} = -m \Omega^2 \dot{x} - \frac{\Gamma}{2} \dot{p} + \sqrt{2n_s h G \dot{X}} + \sqrt{\Gamma} \dot{p}^{(in)} . \]

In this regime, the coupling to the pump fluctuations merely increases the signal mode losses, \( \kappa_s^{(loss)} = \kappa_s^{(abs)} + 4\nu^2 n_s/\kappa_p \) \((\kappa_s = \kappa_s^{(in)} + \kappa_s^{(loss)})\). Physically any upconverted photon leaks out of the cavity before it can be down-converted again. In the adiabatic approximation, the imprecision and backaction noise become
\[ \frac{\bar{n}_s^{(imp)}}{\bar{n}_s^{SQL}} = \frac{(1 - \sigma) \kappa_s - \kappa_s^{(loss)} \kappa_s^{(in)}}{8C_{thr}(\bar{n}^{(thr)}_s \kappa_p - \sigma^2) \kappa_s^2} + \frac{4}{\kappa_s^{(in)}} \frac{\kappa_s^{(loss)}}{\kappa_s} + \Omega^2 . \]
$$\frac{\bar{S}_{xx}^{(\text{back})}}{\bar{S}_{xx}^{\text{SQL}}} = \frac{2\tilde{C}_{hr}(\bar{n}_{n}^{(\text{thr})} - \sigma^2)}{(1 - \sigma)^2 + 4(\Omega/\kappa_{s})^2}. \quad (B1)$$

For the parameters of Fig. 3b, the minimal added noise calculated from these expressions fits well (it can not be distinguished with the bare eye) with the result obtained directly from Eqs. (A2). From the simple analytical formula in Eq. (B1), we see that the imprecision noise is very sensitive to the losses. Even in the case where the signal mode is overcoupled $\kappa_{s}^{(in)} \gg \kappa_{s}^{(loss)}$, the increase can be substantial if $\kappa_{s}^{(loss)} \gg \Omega^2$. We roughly estimate the loss rate $\kappa_{s}^{(loss)}$ that can be tolerated without strongly affecting the measurement precision in our scheme to be $\kappa_{s}^{(loss)} \lesssim \Omega^2/\kappa_{s}$.

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