Stress tensor correlators in the Schwinger–Keldysh formalism

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Abstract

We express stress tensor correlators using the Schwinger–Keldysh formalism. The absence of off-diagonal counterterms in this formalism ensures that the $+$-- and $+$-- correlators are free of primitive divergences. We use dimensional regularization in position space to explicitly check this at one loop order for a massless scalar on a flat space background. We use the same procedure to show that the $+$-- correlator contains the divergences first computed by 't Hooft and Veltman for the scalar contribution to the graviton self-energy.

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1. Introduction

Quantum fluctuations of the stress tensor operator play a role in at least three seemingly distinct physical phenomena: fluctuations of the Casimir force, radiation pressure fluctuations and passive fluctuations of the gravitational field. The Casimir force between a pair of material bodies is a mean force, which can be computed from the expectation value of the stress tensor. However, fluctuations around this mean value are expected, and have been discussed by several authors [1–4]. Unfortunately, these fluctuations seem to be too small to be observable at the present time.

Quantum fluctuations of the radiation pressure can also be interpreted as a manifestation of quantum stress tensor fluctuations [5]. Although this effect has not yet been observed, it is likely to be detected as part of the future development of laser interferometer detectors of gravity waves.

Just as the stress tensor describes forces on material bodies, it also acts as the source of the gravitational field in general relativity. The semiclassical theory assumes that this source is the expectation value of the stress tensor. This is an approximation which fails when the stress tensor fluctuations are significant, which can occur far from the Planck scale [6, 7].
this case, there are large fluctuations of the gravitational field around the mean value predicted by the semiclassical theory. Among the physical effects produced by such fluctuations are the angular blurring and luminosity fluctuations of a distant source [8]. Other effects of stress tensor fluctuations might play a role in the early universe, or near evaporating black holes [9–14].

These physical effects involve observables which are expressible as spacetime integrals of a stress tensor correlation function or correlator. This correlator (also known as the noise kernel) is a function of two spacetime points, \( x \) and \( x' \), which has a \((x - x')^{-8}\) singularity as \( x' \to x \) in four dimensions. As a result, the spacetime integrals are formally divergent and must be regularized. One approach which has been employed is an integration by parts procedure [5, 15], which is essentially differential regularization [16]. In many cases, this procedure leads to a finite result without the need for any renormalization.

In this paper, we will examine the ultraviolet singularities of the stress tensor correlators using dimensional regularization in position space and the Schwinger–Keldysh formalism. This will lead to insight as to when renormalization is required and when it is not. In section 2, we will review the Schwinger–Keldysh formalism. In section 3, the correlators for a massless scalar field will be constructed, and their ultraviolet singularities examined. Finally, in section 4, we discuss the results.

2. The Schwinger–Keldysh formalism

The Schwinger–Keldysh formalism is a technique that makes computing expectation values almost as simple as the Feynman rules do for computing in–out matrix elements [17–23]. To sketch the derivation, consider a real scalar field, \( \phi(x) \), whose Lagrangian (not Lagrangian density) at time \( t \) is \( L[\phi(t)] \). The well-known functional integral expression for the matrix element of an operator \( O_1[\phi] \) between states whose wavefunctionals are given at a starting time \( s \) and a last time \( \ell \) is

\[
\langle \Phi| T^* (O_1[\phi]) |\Psi \rangle = \int [d\phi] [\phi(\ell)] e^{i \int_s^\ell d\tau L[\phi(\tau)]} \Psi[\phi(s)].
\]

(1)

The \( T^* \)-ordering symbol in the matrix element indicates that the operator \( O_1[\phi] \) is time ordered, except that any derivatives are taken outside the time ordering. We can use (1) to obtain a similar expression for the matrix element of the \( \text{anti} \)-time-ordered product of some operator \( O_2[\phi] \) in the presence of the reversed states,

\[
\langle \Psi| \overline{T} (O_2[\phi]) |\Phi \rangle = \langle \Phi| T^* (O_2^*[\phi]) |\Psi \rangle^*,
\]

(2)

\[
= \int [d\phi] [\phi(\ell)] e^{-i \int_s^\ell d\tau L[\phi(\tau)]} \Phi^*[\phi(s)].
\]

(3)

Now note that summing over a complete set of states \( \Phi \) gives a delta functional,

\[
\sum_\Phi \Phi[\phi_-(\ell)] \Phi^*[\phi_+(\ell)] = \delta[\phi_-(\ell) - \phi_+(\ell)].
\]

(4)

Taking the product of (1) and (3), and using (4), we obtain a functional integral expression for the expectation value of any anti-time-ordered operator \( O_2 \) multiplied by any time-ordered operator \( O_1 \),

\[
\langle \Psi| \overline{T} (O_2[\phi]) T^* (O_1[\phi]) |\Psi \rangle = \int [d\phi_+][d\phi_-] \delta[\phi_-(\ell) - \phi_+(\ell)]
\]

\[
\times O_2[\phi_-] O_1^*[\phi_+](s) e^{i \int_s^\ell d\tau L[\phi_+(\tau)] - L[\phi_-(\tau)]} \Psi[\phi_+(s)].
\]

(5)
This is the fundamental relation between the canonical operator formalism and the functional integral formalism in the Schwinger–Keldysh formalism.

The Feynman rules follow from (5) in close analogy to those for in–out matrix elements. Because the same field is represented by two different dummy functional variables, $\varphi_\pm(x)$, the endpoints of lines carry a $\pm$ polarity. External lines associated with the operator $O_2[\varphi]$ have $-$ polarity whereas those associated with the operator $O_1[\varphi]$ have $+$ polarity. Interaction vertices are either all $+$ or all $-$. Vertices with $+$ polarity are the same as in the usual Feynman rules whereas vertices with the $-$ polarity have an additional minus sign. Propagators can be $++$, $-+$, $+-$, and $--$.

The four propagators can be read off from the fundamental relation (5) when the free Lagrangian is substituted for the full one. It is useful to denote canonical expectation values in the free theory with a subscript 0. With this convention we see that the $++$ propagator is just the ordinary Feynman propagator,

$$i\Delta_{++}(x; x') = \langle \Omega | T(\varphi(x)\varphi(x')) | \Omega \rangle_0 = i\Delta(x; x').$$

(6)

The other cases are simple to read off and to relate to the Feynman propagator,

$$i\Delta_{-+}(x; x') = \langle \Omega | \varphi(x)\varphi(x') | \Omega \rangle_0 = \theta(t - t')i\Delta(x; x') + \theta(t' - t)[i\Delta(x; x')]^*,$$

(7)

$$i\Delta_{+-}(x; x') = \langle \Omega | \varphi(x')\varphi(x) | \Omega \rangle_0 = \theta(t - t')i\Delta(x; x') + \theta(t' - t)i\Delta(x; x'),$$

(8)

$$i\Delta_{--}(x; x') = \langle \Omega | T(\varphi(x)\varphi(x')) | \Omega \rangle_0 = [i\Delta(x; x')]^*.$$ (9)

Therefore, we can get the four propagators of the Schwinger–Keldysh formalism from the Feynman propagator once that is known.

3. Massless scalar stress tensor correlators

The Lagrangian density for a massless, minimally coupled scalar $\varphi$ in the presence of an arbitrary spacelike metric $g_{\mu\nu}$ is

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{-g}.$$ (10)

Specializing its stress tensor to flat space $g_{\mu\nu} = \eta_{\mu\nu}$ gives

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \delta \mathcal{L}[\varphi, g] \bigg|_{g=\eta} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \varphi \partial_\rho \varphi.$$ (11)

From the discussion of the preceding section we see that there are four natural 2-point correlators of this operator in the Schwinger–Keldysh formalism,

$$\left[_{\mu\nu}^+ C^+_{\rho\sigma}\right](x; x') \equiv \langle \Omega | T(\mu_{\nu}(x)T_{\rho\sigma}(x')) | \Omega \rangle,$$

(12)

$$\left[_{\mu\nu}^- C^+_{\rho\sigma}\right](x; x') \equiv \langle \Omega | T(\mu_{\nu}(x))T(\rho_{\sigma}(x')) | \Omega \rangle = \langle \Omega | T_{\mu\nu}(x)T_{\rho\sigma}(x') | \Omega \rangle,$$

(13)

$$\left[_{\mu\nu}^+ C^-_{\rho\sigma}\right](x; x') \equiv \langle \Omega | T(\rho_{\sigma}(x'))T(\mu_{\nu}(x)) | \Omega \rangle = \langle \Omega | T_{\rho\sigma}(x')T_{\mu\nu}(x) | \Omega \rangle,$$

(14)

$$\left[_{\mu\nu}^- C^-_{\rho\sigma}\right](x; x') \equiv \langle \Omega | \bar{T}(\mu_{\nu}(x))\bar{T}(\rho_{\sigma}(x')) | \Omega \rangle.$$ (15)

These four quantities are closely related. For example, note that (12) and (15) are complex conjugates, as are (13) and (14). They have slightly different uses and physical interpretations.

For example, $-i4\pi G$ times (12) gives the scalar contribution to the graviton self-energy whose divergent part was computed by ’t Hooft and Veltman [24]. The stress tensor fluctuations...
whose effect upon focusing has been studied recently [8] are given by $+\frac{1}{2}$ times the sum of (13) and (14).

Each of the four Schwinger–Keldysh scalar propagators takes the same form in $D$-dimensional flat space,

$$i\Delta_{\pm\pm}(x; x') = \frac{\Gamma(\frac{D}{2} - 1)}{4\pi^\frac{D}{2}} \left( \frac{1}{\Delta x_{\pm\pm}} \right)^{\frac{D}{2} - 1}.$$  \hspace{1cm} (16)

The four $\pm$ variations only affect what we mean by the invariant interval,

$$\Delta x^2_{\pm\pm} = ||\vec{x} - \vec{x}'||^2 - (t - t' - i\delta)^2,$$
$$\Delta x^2_{\pm\pm} = ||\vec{x} - \vec{x}'||^2 - (t - t' + i\delta)^2,$$
$$\Delta x^2_{\pm\pm} = ||\vec{x} - \vec{x}'||^2 - (t - t' - i\delta)^2.$$  \hspace{1cm} (17)

Because the $++$ and $--$ intervals involve $t - t'$, rather than $|t - t'|$, second derivatives of these propagators are straightforward,

$$\partial^2_x \delta_x i\Delta_{-+}(x; x') = \frac{\Gamma(\frac{D}{2})}{2\pi^\frac{D}{2}} \left[ 1 - D\frac{\Delta x_+ \Delta x_+}{\Delta x_{++}^{D+2}} \right],$$  \hspace{1cm} (18)

$$\partial^2_x \delta_x i\Delta_{+-}(x; x') = \frac{\Gamma(\frac{D}{2})}{2\pi^\frac{D}{2}} \left[ 1 - D\frac{\Delta x_- \Delta x_-}{\Delta x_{--}^{D-2}} \right] - i\delta_x^0 \delta_x^0 \delta^D(x - x').$$  \hspace{1cm} (19)

Second derivatives of the $++$ and $--$ propagators involve another term owing to the absolute value [25, 26],

$$\partial^2_x \delta_x i\Delta_{++}(x; x') = \frac{\Gamma(\frac{D}{2})}{2\pi^\frac{D}{2}} \left[ 1 - D\frac{\Delta x_+ \Delta x_+}{\Delta x_{++}^{D+2}} \right] + i\delta_x^0 \delta_x^0 \delta^D(x - x'),$$  \hspace{1cm} (20)

$$\partial^2_x \delta_x i\Delta_{--}(x; x') = \frac{\Gamma(\frac{D}{2})}{2\pi^\frac{D}{2}} \left[ 1 - D\frac{\Delta x_- \Delta x_-}{\Delta x_{--}^{D-2}} \right] - i\delta_x^0 \delta_x^0 \delta^D(x - x').$$  \hspace{1cm} (21)

However, note that this extra term goes away when the derivatives act inside the time-ordering (or anti-time-ordering) symbol,

$$\langle \Omega | T(\partial_\mu \phi(x) \partial^\mu \phi(x')) | \Omega \rangle = \frac{\Gamma(\frac{D}{2})}{2\pi^\frac{D}{2}} \left[ 1 - D\frac{\Delta x_+ \Delta x_+}{\Delta x_{++}^{D+2}} \right].$$  \hspace{1cm} (22)

Note also that the coincidence limits of such quantities vanish in dimensional regularization [25, 26],

$$\langle \Omega | T(\partial_\mu \phi(x) \partial_\rho \phi(x)) | \Omega \rangle = 0 = \langle \Omega | \overline{T}(\partial_\mu \phi(x) \partial_\rho \phi(x)) | \Omega \rangle.$$  \hspace{1cm} (23)

We can now evaluate the various correlators quite simply. Consider first the $++$ case,

$$\langle \Omega | T(T_{\mu\nu}(x) T_\rho\sigma(x')) | \Omega \rangle = \frac{\Gamma(\frac{D}{2})}{2\pi^\frac{D}{2}} \left[ 1 - D\frac{\Delta x_+ \Delta x_+}{\Delta x_{++}^{D+2}} \right] \delta^D(x - x').$$  \hspace{1cm} (24)

$$\times 2 \langle \Omega | T(\partial_\mu \phi^\dagger(\nu) \partial_\rho \phi^\dagger(\sigma)) | \Omega \rangle \langle \Omega | T(\partial_\mu \phi^\dagger(\nu) \partial_\rho \phi^\dagger(\sigma)) | \Omega \rangle.$$  \hspace{1cm} (25)

$$= \frac{\Gamma^2(\frac{D}{2})}{2\pi^D} \left[ \frac{\eta_{\mu\nu} \eta_{\rho\sigma}^*}{\Delta x_{++}^{D+2}} - 2D \frac{\Delta x_+ \eta_{\mu\nu} \eta_{\rho\sigma}}{\Delta x_{++}^{D+2}} + D^2 \frac{\Delta x_+ \Delta x_+ \Delta x_\sigma \Delta x_\nu}{\Delta x_{++}^{D+2}} \right] \delta^D(x - x').$$  \hspace{1cm} (26)
Although the intermediate steps are different, the result takes the same form for all four correlators,

\[
\left[ C_{\mu}^{\pm \pm} \right](x; x') = \frac{\Gamma^2(\frac{D}{2})}{2\pi^D} \left\{ \frac{\eta_{\mu\nu} \eta_{\rho\sigma}}{\Delta x_{\pm \pm}^{2D}} - 2D \frac{\Delta x_{\mu} \eta_{\nu(\gamma \rho \sigma)}}{\Delta x_{\pm \pm}^{2D+2}} + D^2 \frac{\Delta x_{\mu} \Delta x_{\mu} \Delta x_{\rho} \Delta x_{\sigma}}{\Delta x_{\pm \pm}^{2D+4}} \right\} + \frac{1}{2} (D - 2) D \left\{ \frac{\eta_{\mu\nu} \Delta x_{\rho} \eta_{\rho\sigma}}{\Delta x_{\pm \pm}^{2D+2}} + \frac{1}{4} (D^2 - 4) \frac{\eta_{\mu\nu} \eta_{\rho\sigma}}{\Delta x_{\pm \pm}^{2D}} \right\}.
\]

(27)

The next step is to partially integrate up to the logarithmically divergent power \(1/\Delta x^{2D-4}\). This is facilitated by the following identities [25–27]:

\[
\frac{1}{\Delta x_{\pm \pm}^{2D}} = \left\{ \frac{\partial^4}{4(D - 2)^2(D - 1)D} \right\} \frac{1}{\Delta x_{\pm \pm}^{2D-4}},
\]

(28)

\[
\frac{\Delta x_{\mu} \Delta x_{\nu} \Delta x_{\rho} \Delta x_{\sigma}}{\Delta x_{\pm \pm}^{2D+4}} = \left\{ \frac{[\eta_{\mu\nu} \eta_{\rho\sigma} + 2\eta_{\mu(\gamma \rho \sigma)}] \partial^4}{16(D - 2)^2(D - 1)D^2(D + 1)} \right\} \frac{1}{\Delta x_{\pm \pm}^{2D-4}}.
\]

(29)

\[
\frac{D^2 \Delta x_{\mu} \Delta x_{\mu} \Delta x_{\rho} \Delta x_{\sigma}}{\Delta x_{\pm \pm}^{2D+4}} = \left\{ \frac{[\eta_{\mu(\rho \eta \sigma)} + 2\eta_{\mu(\gamma \rho \sigma)}] \partial^4}{(D - 2)^2(D - 1)(D + 1)} \right\} \frac{1}{\Delta x_{\pm \pm}^{2D-4}}.
\]

(30)

At this stage the result takes the form

\[
\left[ C_{\mu}^{\pm \pm} \right](x; x') = \frac{\Gamma^2(\frac{D}{2})}{16\pi^D} \left\{ \frac{(D^2 - 2D - 2)[\eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu}] \eta_{\rho(\gamma \rho \sigma)} \partial^2 - \partial_{\rho} \partial_{\sigma}]}{2(D - 2)^2(D - 1)(D + 1)} \right\} \frac{1}{\Delta x_{\pm \pm}^{2D-4}}.
\]

(31)

Note the manifest transversality of (31) which is a consequence of stress–energy conservation.

At this point, we pause to note that no delta functions emerge from the derivatives in (28)–(30) because the only power that can give them in dimensional regularization is \(1/\Delta x^{D-2}\). This happens for the ++ and −− cases [25–27],

\[
\partial^2 \left( \frac{1}{\Delta x_{++}^{D-2}} \right) = \frac{\Gamma^2(\frac{D}{2})}{\Gamma(\frac{D}{2} - 1)} \partial^4(x - x') = -\partial^2 \left( \frac{1}{\Delta x_{--}^{D-2}} \right).
\]

(32)

It does not happen for the ++ and −− cases [25–27],

\[
\partial^2 \left( \frac{1}{\Delta x_{++}^{D-2}} \right) = 0 = \partial^2 \left( \frac{1}{\Delta x_{--}^{D-2}} \right).
\]

(33)

The point of partially integrating, as we did to reach (31), is to write the result as a derivative operator with respect to \(x^\mu\), acting upon a function of \(x'^\mu\) which is integrable in \(D = 4\). We have not quite achieved this in (31) but the next partial integration does,

\[
\frac{1}{\Delta x_{\pm \pm}^{2D-4}} = \frac{\partial^2}{2(D - 3)(D - 4)} \left( \frac{1}{\Delta x_{\pm \pm}^{2D-6}} \right).
\]

(34)

Except for the explicit factor of \(1/(D - 4)\) we could take \(D = 4\) in this expression.
The next step—and the first at which we must distinguish between the four \( \pm \) variations—is to transfer the divergence to a local term by adding zero in the form of the identities (32) and (33). For the ++ term this gives

\[
\frac{1}{\Delta x_{++}^{2D-4}} = \frac{\partial^2}{2(D-3)(D-4)} \left( \frac{1}{\Delta x_{++}^{2D-6}} - \frac{\mu^{D-4}}{\Delta x_{++}^{D-2}} \right) + \frac{i4\pi^2 \mu^{D-4}}{\Gamma\left(\frac{D}{2}-1\right)} \frac{\delta^D(x-x')}{2(D-3)(D-4)}. \tag{35}
\]

Note the dimensional regularization mass scale \( \mu \). The expression on the first line of (35) is both integrable and finite so we can take \( D = 4 \),

\[
\frac{\partial^2}{2(D-3)(D-4)} \left( \frac{1}{\Delta x_{++}^{2D-6}} - \frac{\mu^{D-4}}{\Delta x_{++}^{D-2}} \right) \rightarrow -\frac{\partial^2}{4} \left( \frac{\ln (\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} \right). \tag{36}
\]

The result for each of the four \( \pm \) variations is

\[
\frac{1}{\Delta x_{++}^{2D-4}} \rightarrow -\frac{\partial^2}{4} \left( \frac{\ln (\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} \right) + \frac{i4\pi^2 \mu^{D-4}}{\Gamma\left(\frac{D}{2}-1\right)} \frac{\delta^D(x-x')}{2(D-3)(D-4)}. \tag{37}
\]

\[
\frac{1}{\Delta x_{+}^{2D-4}} \rightarrow -\frac{\partial^2}{4} \left( \frac{\ln (\mu^2 \Delta x_{+}^2)}{\Delta x_{+}^2} \right). \tag{38}
\]

\[
\frac{1}{\Delta x_{-}^{2D-4}} \rightarrow -\frac{\partial^2}{4} \left( \frac{\ln (\mu^2 \Delta x_{-}^2)}{\Delta x_{-}^2} \right). \tag{39}
\]

\[
\frac{1}{\Delta x_{--}^{2D-4}} \rightarrow -\frac{\partial^2}{4} \left( \frac{\ln (\mu^2 \Delta x_{--}^2)}{\Delta x_{--}^2} \right) - \frac{i4\pi^2 \mu^{D-4}}{\Gamma\left(\frac{D}{2}-1\right)} \frac{\delta^D(x-x')}{2(D-3)(D-4)}. \tag{40}
\]

We see that the ++ and -- correlators are completely finite

\[
\left\{ \mu C_{\rho\sigma}^- \right\}(x; x') \rightarrow -\frac{\partial^2}{1280\pi^4} \left\{ \eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu} \right\} \ln \left( \frac{\mu^2 \Delta x_{--}^2}{\Delta x_{--}^2} \right). \tag{41}
\]

\[
\left\{ \mu C_{\rho\sigma}^+ \right\}(x; x') \rightarrow -\frac{\partial^2}{1280\pi^4} \left\{ \eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu} \right\} \ln \left( \frac{\mu^2 \Delta x_{++}^2}{\Delta x_{++}^2} \right). \tag{42}
\]

Since they are complex conjugates, their average is also real. The ++ and -- terms have similar finite parts but they also harbour ultraviolet divergences,

\[
\left\{ \mu \tilde{C}_{\rho\sigma}^+ \right\}(x; x') \rightarrow -\frac{\partial^2}{1280\pi^4} \left\{ \eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu} \right\} \ln \left( \frac{\mu^2 \Delta x_{--}^2}{\Delta x_{++}^2} \right) + \frac{i\Gamma\left(\frac{D}{2}\right) \mu^{D-4}}{16\pi^2} \left\{ \frac{(D^2 - 2D - 2)\eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu}}{2(D-4)(D-3)(D-2)(D-1)(D+1)} \right\} \delta^D(x-x'). \tag{43}
\]
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\[ \left[ \eta_{\mu\nu} C_{\rho\sigma} \right](x; x') \rightarrow -\frac{\partial^2}{1280\pi^4} \left\{ \left[ \eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right] \left[ \eta_{\rho\sigma} \partial^2 - \partial_\rho \partial_\sigma \right] \right. \\
+ \frac{1}{3} \left[ \eta_{\mu(\rho} \eta_{\sigma)\nu} \right] \partial^2 - 2 \partial_\mu \eta_{(\rho} \partial_\sigma \partial_{\nu)} \partial^2 + \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \right\} \left( \ln(\mu^2 \Delta x^2_{\rightarrow}) \right) \Delta x^2_{\rightarrow} \\
- \frac{i\Gamma(\frac{2}{3})}{16\pi^2} \left\{ (D^2 - 2D - 2) \left[ \eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right] \left[ \eta_{\rho\sigma} \partial^2 - \partial_\rho \partial_\sigma \right] \right. \\
+ \left[ \eta_{\mu(\rho} \eta_{\sigma)\nu} \right] \partial^2 - 2 \partial_\mu \eta_{(\rho} \partial_\sigma \partial_{\nu)} \partial^2 + \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \right\} \frac{1}{(D - 4)(D - 3)(D - 2)(D - 1)(D + 1)} \delta^D(x - x'). \] (44)

These four correlators have been previously evaluated at one loop order, also using dimensional regularization but in momentum space, by Campos and Verdaguer [28] and by Martin and Verdaguer [29].

4. Discussion

We have expressed the stress tensor correlators of previous studies [8] using the Schwinger–Keldysh formalism [17–23]. In this language it is the average of the ++ and −− correlators (41), (42) which has been the object of earlier study. However, the four ± variations are so closely related that a unified treatment was simple. The reduction procedure is by now familiar from analogous computations in a locally de Sitter background [25–27, 30].

It might seem curious that the ++ and −− correlators (41), (42) are ultraviolet finite. That this must be so derives from the relation between stress tensor correlators and the graviton self-energy. To see this relation, define the graviton field \( h_{\mu\nu}(x) \) by perturbing the full metric about flat space,

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \quad \text{where} \quad \kappa^2 \equiv 16\pi G. \] (45)

As usual in perturbative quantum gravity we raise and lower indices with the background metric, \( \eta_{\mu\nu} \). The 3-point interaction between gravitons and scalars can be read off from (10),

\[ g^{\mu\nu} \sqrt{-g} = \eta^{\mu\nu} - \kappa \left( h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) + O(\kappa^2) \quad \Rightarrow \quad \mathcal{L}^{(3)} = \frac{\kappa}{2} h^{\mu\nu} T_{\mu\nu}. \] (46)

Although there is a 4-point interaction, it makes no contribution to the graviton self-energy because the coincident massless scalar propagator vanishes in dimensional regularization. We can therefore write the four Schwinger–Keldysh self-energies in terms of the four correlators,

\[ -i\left[ i_{\mu\nu} \Sigma^\pm_{\mu\nu} \right](x; x') = \left( \pm \frac{i\kappa}{2} \right) \left[ i_{\mu\nu} C^\pm_{\mu\nu} \right](x; x'). \] (47)

The reason that the ++ and −− correlators are finite becomes clear when we consider how the various correlators enter the Schwinger–Keldysh effective action [22, 26–28],

\[ \Gamma[g_+, g_-] = S[g_+] - S[g_-] + i \frac{\kappa^2}{8} \int d^4 x \int d^4 x' \times \left\{ h^\mu_+ (x) \left[ i_{\mu\nu} C^+_{\rho\sigma} \right](x; x') h^{\rho\sigma}_+(x') - h^\mu_-(x) \left[ i_{\mu\nu} C^-_{\rho\sigma} \right](x; x') h^{\rho\sigma}_-(x') \right. \\
- h^\mu_-(x) \left[ i_{\mu\nu} C^+_{\rho\sigma} \right](x; x') h^{\rho\sigma}_+(x') + h^\mu_+(x) \left[ i_{\mu\nu} C^-_{\rho\sigma} \right](x; x') h^{\rho\sigma}_-(x') \right\} + O(\kappa^3). \] (48)

Counterterms derive from the ‘classical’ actions, \( S[g_+] \) and \( -S[g_-] \), hence they can involve only all + or all − fields. It follows that the mixed correlators cannot harbour primitive
divergences at any order. At one loop order the only divergences are primitive, so these correlators are simply finite.

The ++ and −− correlators do harbour divergences. It is illuminating to express the ++ pole term in the invariant form

\[
\Gamma_{++}^{\infty}[g] = -\frac{\kappa^2}{1280\pi^2} \frac{1}{D-4} \int d^4x \left\{ \left[ h^{\mu \nu} - h^{\mu \nu} \right]^2 \right. \\
+ \frac{1}{3} \left[ h^{\rho \sigma \mu} h_{\rho \sigma}^{\quad \nu} - 2 h^{\rho \sigma \mu} h^{\nu}_{\rho, \mu, \nu} + h^{\rho \sigma \mu} h^{\nu}_{\rho, \mu} \right] + O(\kappa^3),
\]

\[
= -\frac{1}{960\pi^2} \frac{1}{D-4} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R^2 + R_{\mu \nu} R_{\mu \nu} \right\}.
\]

This is exactly \( -\frac{1}{2} \) of the counterterm—their equation (3.34)—that ‘t Hooft and Veltman long ago computed for removing the one loop divergences induced by a complex scalar [24]. Because a complex scalar contributes as two real scalars, our result is in perfect agreement, as it is with subsequent studies [28, 29].

We can also now understand why the integration by parts procedure used [5, 8, 15] leads to a finite result with no infinite subtraction required. These papers were concerned solely with the +− and −+ correlators, which we have shown to be finite in dimensional regularization. Even if dimensional regularization is not used explicitly, integration by parts will produce a finite result.

Although the focus of this paper has been understanding the singularity structure of stress tensor correlators, the motivation for the exercise is the interesting physics associated with these objects. In flat space, the physical effects arising from quantum stress tensor fluctuations are associated with the finite, state-dependent parts of the correlator. Examples include radiation pressure fluctuations for an electromagnetic field in a coherent state [5] and the angular blurring and luminosity fluctuations of the image of a distant source produced by Ricci tensor fluctuations, which can in turn arise from stress tensor fluctuations of a matter field in a thermal state [8]. In curved space, the nontrivial geometry can give rise to interesting effects. Much work has been done in the homogeneous and isotropic geometry of cosmology and in black-hole geometries [9–14]. Moving beyond the stress tensor, it will be seen that the same considerations apply to the correlators of any field. Recent examples of correlators in a locally de Sitter background include the vacuum polarization of scalar QED [26], the fermion self-energy of Yukawa theory [27] and the self-mass-squared of a self-interacting scalar [30].

The analysis of this paper allows one to be clear about the singular parts so that the calculation of the finite parts may proceed unambiguously.

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