Spectral Analysis of the Schrodinger Operator with an Optical Potential

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Abstract

In this paper we give a complete description of the spectral analysis of the Schrodinger operator \( L(q) \) with the potential \( q(x) = 4 \cos^2 x + 4iV \sin 2x \) for all \( V > 1/2 \). First we consider the Bolch eigenvalues and spectrum of \( L(q) \). Then using it we investigate spectral singularities and essential spectral singularities (ESS). We prove that the operator \( L(q) \) has no ESS and has ESS respectively if and only if \( V \neq V_k \) and \( V = V_k \) for \( k \geq 2 \), where \( V_k \to \infty \) as \( k \to \infty \), \( V_2 \) is the second critical point and \( V_2 < V_3 < \cdots \). Using it we classify the spectral expansion in term of the critical points \( V_k \) for \( k \geq 2 \). Finally we discuss the critical points, formulate some conjectures and describe the changes of the spectrum of \( L(q) \) when \( V \) changes from \( 1/2 \) to \( \infty \).

Key Words: Optical potentials, spectrum, spectral singularities, spectral expansion.

AMS Mathematics Subject Classification: 34L05, 34L20.

1 Introduction

In this paper we consider in detail the spectrum, spectral singularities, ESS and spectral expansion of the Schrödinger operator

\[
L(q) = -\frac{d^2}{dx^2} + q
\]

with the optical potential

\[
q(x) = (1 + 2V)e^{i2x} + (1 - 2V)e^{-i2x}
\]

for all \( V > 1/2 \). The operator \( L(q) \) is defined in \( W^2_2(-\infty, \infty) \) by \( L(q)y = -y'' + qy \) and is a densely defined closed operator in \( L_2(-\infty, \infty) \). It is well-known that [1, 8, 11] the spectrum \( \sigma(L(q)) \) of \( L(q) \) is the union of the spectra \( \sigma(L_t(q)) \) of the operators \( L_t(q) \) for \( t \in (-1, 1] \) generated in \( L_2[0, \pi] \) by the expression \(-y'' + qy \) and the boundary conditions

\[
y(\pi) = e^{i\pi t}y(0), \quad y'(\pi) = e^{i\pi t}y'(0).
\]

The spectrum of \( L_t(q) \) consists of the eigenvalues \( \lambda_1(t), \lambda_2(t), \ldots \), called as the Bloch eigenvalues of \( L(q) \), that are the roots of the characteristic equation

\[
F(\lambda) = 2 \cos \pi t,
\]

where \( F(\lambda) := \varphi'(\pi, \lambda) + \theta(\pi, \lambda) \) is the Hill discriminant, \( \theta \) and \( \varphi \) are the solutions of

\[
-y''(x) + q(x)y(x) = \lambda y(x)
\]
satisfying the initial conditions \( \theta(0, \lambda) = \varphi'(0, \lambda) = 1, \theta'(0, \lambda) = \varphi(0, \lambda) = 0 \). Moreover, in [19] we proved that the eigenvalues \( \lambda_n(t) \) of \( L_1(q) \) can be numbered (counting the multiplicity) by elements of \( \mathbb{Z} \) and hence by elements of \( \mathbb{N} \) such that, for each \( n \) the function \( \lambda_n \) is continuous on \((-1, 1)\) and \( \lambda_n(t) \) uniformly with respect to \( t \in (-1, 1) \) approaches \( \infty \) as \( n \to \infty \). Thus the spectrum of \( L(q) \) is the union of the continuous curves

\[
\Gamma_n = \{ \lambda_n(t) : t \in (-1, 1) \} \tag{6}
\]

for \( n \in \mathbb{N} \) called as the \( n \)-th bands of the spectrum. By (3) and (4), \( \lambda_n(-t) = \lambda_n(t) \) and the end points of \( \Gamma_n \) are \( \lambda_n(0) \) and \( \lambda_n(1) \).

Some physically interesting results have been obtained by considering the potential (2). For the first time, the mathematical explanation of the nonreality of \( \sigma(L(q)) \) for \( V > 0.5 \) and finding the threshold 0.5 (first critical point \( V_1 \)) was done by Makris et al [6, 7]. Moreover, for \( V = 0.85 \) they sketch the real and imaginary parts of the first two bands by using the numerical methods. In [9] Midya et al reduce the operator \( L(q) \) with potential (2) to the Mathieu operator and using the tabular values establish that there is second critical point \( V_2 \approx 0.888437 \) after which no parts of the first and second bands remain real.

In [15] we proved that if \( ab = \tilde{a}\tilde{b} \), where \( a, b, \tilde{a}, \) and \( \tilde{b} \) are arbitrary complex numbers, then the operators \( L(q) \) and \( L(\tilde{q}) \) with potentials

\[
q(x) = ae^{-i2x} + be^{i2x} \tag{7}
\]

and \( \tilde{q}(x) = \tilde{a}e^{-i2x} + \tilde{b}e^{i2x} \) have the same Hill discriminants \( F(\lambda) \) and \( G(\lambda) \), Bloch eigenvalues and spectra. Thus for \( q(x) = (1 + 2V)e^{i2x} + (1 - 2V)e^{-i2x} \) and \( p(x) = 2ic \cos 2x \) we have

\[
F(\lambda) = G(\lambda), \quad \sigma(L(V)) = \sigma(H(c)), \quad \sigma(L_t(V)) = \sigma(H_t(c)) \tag{8}
\]

for \( t \in [0, 1] \) and \( c = \sqrt{4V^2 - 1} \), where for brevity of the notations, the operators \( L_t(q), L(q), L_t(p) \) and \( L(p) \) are denoted by \( L_t(V), L(V), H_t(c) \) and \( H(c) \) respectively. If \( V < 1/2 \) then \( ic \) is a real number and \( H(c) \) is the well-known self-adjoint Mathieu operator. The case \( V = 1/2 \) is also well-known (see [2, 5, 18] and references therein). Thus we need to consider the operator \( L(V) \) in the case \( V > 1/2 \) which is the consideration of the operator \( H(c) \) with pure imaginary potential \( 2ic \cos 2x \) for \( c > 0 \) due to (8).

In the paper [22] we have gave a complete description, provided with a mathematical proof, of the shape of \( \sigma(L(V)) \), when \( V \) changes from 1/2 to \( \sqrt{5}/2 \), that is, \( c \in (0, 2) \). We proved that the second critical point \( V_2 \) is a number between 0.888437005 and 0.8884370117. Moreover, it was proven that \( V_2 \) is the unique degeneration point for the first periodic eigenvalue, in the sense that the first periodic eigenvalue of the potential (2) is simple for all \( V \in (1/2, \sqrt{5}/2) \backslash \{ V_2 \} \) and is double for \( V = V_2 \). Besides, we gave an approach for finding the arbitrary close values of the second critical point \( V_2 \).

Let us briefly and visually explain those results of [22] about the spectrum of \( L(V) \), the eigenvalues of \( L_0(V) \) (periodic eigenvalues \( \lambda_1(0), \lambda_2(0), \ldots \)) and the eigenvalues of \( L_1(V) \) (antiperiodic eigenvalues \( \lambda_1(1), \lambda_2(1), \ldots \)). Antiperiodic eigenvalues are nonreal and simple for all \( V \in (1/2, \sqrt{5}/2) \). Moreover, they are pairwise conjugate numbers. To describe the periodic eigenvalues and spectrum of \( L(V) \) we consider the following cases.

**Case 1:** \( 1/2 < V < V_2 \), where \( V_2 \) is the second critical point. Then all periodic eigenvalues are real and simple. The bands \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \) are joined and they form together the connected subset \( \Omega_k =: \Gamma_{2k-1} \cup \Gamma_{2k} \) of the spectrum. The spectrum \( \sigma(L(V)) \) consists of the pairwise disjoint sets \( \Omega_1, \Omega_2, \ldots \) called as the components of the spectrum. Moreover, the \((2k-1)\)-th and \(2k\)-th bands \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \) become to be connected by their interior point \( \lambda_{2k-1}(t_k) = \lambda_{2k}(t_k) \), where \( t_k \in (0, \pi), \) which is the unique multiple and double Bloch eigenvalue lying in \( \Omega_k \). The real part of \( \Gamma_{2k-1} \cup \Gamma_{2k} \) is the closed interval \( I_k =: \)
In this section we introduce some notations, give the precise definitions of the spectral singularity and ESS and formulate some results of the papers [7, 11, 12, 17, 19-24] as
has a spectral singularity at infinity if there exist sequences such that $U$ is a spectral operator if and only if the projections of the operators $\lambda$ corresponding to the simple eigenvalues belong to the set $\mathcal{M}$ of boundary value problems.

As was noted in the papers [13, 14, 4], the spectral singularity of the operator $L(q)$ are the points $\lambda \in \sigma(L(q))$ for which the projections of $L(q)$ corresponding to the spectral arcs $\gamma \subset U(\lambda, \varepsilon)$ or the projections $e(\lambda_n(t))$ of $L(t)$ corresponding to the simple eigenvalues $\lambda_n(t) \in U(\lambda, \varepsilon)$ are not uniformly bounded for all $\varepsilon > 0$, where $U(\lambda, \varepsilon) = \{ z \in \mathbb{C} : |z - \lambda| < \varepsilon \}$ is the $\varepsilon$-neighborhood of $\lambda$. Therefore, by (10), we have the following definitions for the spectral singularities in term of $d_n$.

**Definition 1** A point $\lambda \in \sigma(L(q))$ is said to be a spectral singularity of $L(q)$ if there exist $n \in \mathbb{N}$ and sequence $\{t_k\} \subset (-1, 1)$ such that $\lambda_n(t_k) \to \lambda$ and $|d_n(t_k)| \to 0$ as $k \to \infty$, where $\lambda_n(t_k)$ is a simple eigenvalue of the operator $L(t_k)$. Similarly, we say that the operator $L(q)$ has a spectral singularity at infinity if there exist sequences $\{n_k\} \subset \mathbb{N}$ and $\{t_k\} \subset (-1, 1)$ such that $|\lambda_n(t_k)| \to \infty$ and $|d_n(t_k)| \to 0$ as $k \to \infty$, where $\lambda_n(t_k)$ is a simple eigenvalue.

In [19] using these definitions, (10) and the well-known result of McGarvey [8] that $L(q)$ is a spectral operator if and only if the projections of the operators $L(t)$ are bounded uniformly with respect to $t$ in $(-\pi, \pi]$ we obtained the following.

**Summary 2** The operator $L(q)$ is a spectral operator if and only if it has no spectral singularities in $\sigma(L(q))$ and at infinity. Moreover $L(q)$ is an asymptotically spectral operator if and only if it has no spectral singularity at infinity.

Thus the spectral singularities play the crucial rule for the investigations of the spectrality of $L$. Gesztesy and Tkachenko [4] proved two versions of a criterion for the Hill operator $L(q)$ to be a spectral operator, one analytic and one geometric. The analytic version was stated in term of the solutions of Hill’s equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems.

The problem of describing explicitly, for which potentials $q$ the Hill operators $L(q)$ are spectral operators appeared to have been open for about 60 years. In paper [16] we found the explicit conditions on the potential $q$ such that $L(q)$ is an asymptotically spectral operator. In [24] we find a criterion for asymptotic spectrality of the operator $L(q)$ with the potential (7) stated in term of $a$ and $b$. Moreover, in [24], we obtained the following result

**Summary 3** If $ab \in \mathbb{R}$, then the operator $L(q)$ with the potential (7) is a spectral operator if and only if it is self adjoint.

It readily implies the following result.
Proposition 1 The operator \( L(q) \) with potential (2) is a spectral operator if and only if \( V = 0 \), that is, \( q(x) = 2 \cos 2x \) and \( L(q) \) is a self-adjoint operator.

Thus Proposition 1 shows that the theory of spectral operators can not be used for the investigation of the non-self-adjoint operator \( L(q) \) with the optical potential (2).

Note that Definition 1 and Summary 2 show respectively that the boundlessness of \( \frac{1}{d_n} \) is the characterization of the spectral singularities and the spectrality of \( L(q) \). However, as was discovered in the papers [21, 23] to construct the spectral expansion we need to consider the integrability of \( \frac{1}{d_n} \) due to the following. In [3] it was proven that in the self-adjoint case the spectral expansion of \( L(q) \) has the following elegant form

\[
f = \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{(-1,1)} a_n(t) \Psi_{n,t} dt,
\]

where

\[
a_n(t) = \frac{1}{d_n(t)} \left( \int_{\mathbb{R}} f(x) \Psi_{n,t}^*(x) dx \right).
\]

In the non-self-adjoint case to obtain the spectral expansion, we need to consider the integrability of \( a_n(t) \Psi_{n,t} \) with respect to \( t \) over \((-1,1]\) which is connected with the integrability of \( \frac{1}{d_n} \) (see (12)). For this in [21, 23] we introduced a new notions essential spectral singularity (ESS) which is connected with the nonintegrability of \( \frac{1}{d_n} \), since it may have an integrable and nonintegrable bounlessness. Moreover we proved that it determines the form of the spectral expansion for \( L(q) \). That is why in this paper investigating the ESS of \( L(V) \) in term of \( V \) we classify the form of its spectral expansion also in term of \( V \). In [21] and [23] we defined the ESS as follows.

Definition 2 A number \( \lambda_0 \in \sigma(L) \) is said to be an essential spectral singularity (ESS) of \( L(q) \) if there exist \( t_0 \in (-1,1] \) and \( n \in \mathbb{N} \) such that \( \lambda_0 = \lambda_n(t_0) \) and \( \frac{1}{d_n} \) is not integrable over \((t_0 - \delta, t_0 + \delta)\) for all \( \delta > 0 \). Similarly, we say that the operator \( L(q) \) has ESS at infinity if there exist sequence of integers \( n_\pi \) and sequence of closed subsets \( I(s) \) of \((-1,1] \) such that \( \lambda_{n_\pi}(t) \) for \( t \in I(s) \) are the simple eigenvalues and

\[
\lim_{s \to \infty} \int_{I(s)} |d_{n_\pi}(t)|^{-1} dt = \infty.
\]

To consider the spectral expansion of \( L(V) \) we use the following results of [23] formulated here as summary.

Summary 4 (a) The spectral expansion has the elegant form (11) if and only if \( L(q) \) has no ESS and ESS at infinity.

(b) If \( L(q) \) has no ESS at infinity, then it has at most finite number of ESS and its spectral expansion has the following asymptotically elegant form

\[
f(x) = \frac{1}{2\pi} \left( \int_{(-\pi,\pi]} \sum_{n \in \mathbb{N} \setminus \{0\}} a_n(t) \Psi_{n,t}(x) dt + \sum_{n \in \mathbb{N} \setminus \{0\}} \int_{(-\pi,\pi]} a_n(t) \Psi_{n,t}(x) dt \right),
\]

where \( N(\mathbb{E}) \) consists at most of a finite number of integers and is the set of the indices \( n \) for which \( \Gamma_n \) contains at least one ESS. Moreover

\[
\int_{(-\pi,\pi]} \sum_{n \in \mathbb{N} \setminus \{0\}} a_n(t) \Psi_{n,t}(x) dt = \lim_{\varepsilon \to 0} \sum_{n \in \mathbb{N} \setminus \{0\}} \int_{A(\varepsilon)} a_n(t) \Psi_{n,t}(x) dt,
\]
where $A(\varepsilon) = (-\pi + \varepsilon, -\varepsilon) \cup (\varepsilon, \pi - \varepsilon)$.

Thus to consider the spectral expansion it is necessary to investigate the ESS. In this paper we use the following results of [21, 24] about ESS formulated here as summary.

**Summary 5** (a) Let $E$, $S$, and $M$ be respectively the sets of ESS, spectral singularities and multiple eigenvalues of $L_t(q)$ for $t \in (-1, 1)$. Then $E \subset S \subset M$.

(b) If $t \in (0, 1)$ and $\lambda$ is a multiple eigenvalue of $L_t(q)$, then $\lambda$ is a spectral singularity of $L$ and not an ESS.

(c) If $\lambda$ is a multiple $2$-periodic eigenvalue with geometric multiplicity $1$, then it is an ESS, where the periodic and antiperiodic eigenvalues are called as the $2$-periodic eigenvalues.

(d) If $ab \neq 0$, then the operator $L(q)$ with potential (7) has no ESS at infinity.

The results (a)−(c) and (d) were proved in [21] and [24] respectively. By Summary 5(d) we have

**Corollary 1** If $V \neq \pm 1/2$, then $L(V)$ has no ESS at infinity.

To consider in detail the case $V \geq \sqrt{5}/2$, that is, the case $c \geq 2$ we use the following results of [17] about periodic and antiperiodic eigenvalues formulated here as summary.

**Summary 6** If $c \neq 0$, then the geometric multiplicity of the eigenvalues of the operators $H_0(c)$, $H_1(c)$, $D(c)$ and $N(c)$, called as periodic, antiperiodic, Dirichlet and Neumann eigenvalues respectively, is $1$ and the following equalities hold

$$
\sigma(D(c)) \cap \sigma(N(c)) = \emptyset, \quad \sigma(H_0(c)) \cup \sigma(H_1(c)) = \sigma(D(c)) \cup \sigma(N(c)),
$$

where $D(c)$ ($N(c)$) denotes the operator generated in $L_2[0, \pi]$ by (1) with potential $2ic \cos 2x$ and Dirichlet (Neumann) boundary conditions.

Moreover, we use the following obvious result and the results of the papers [11, 20] about the spectra of the operators $L(q)$ and $L_t(q)$ for $t \in [0, 1]$.

**Summary 7** (a) If $t_1 \in [0, 1]$, $t_2 \in [0, 1]$ and $t_1 \neq t_2$ then $\sigma(L_{t_1}(q)) \cap \sigma(L_{t_2}(q)) = \emptyset$. It readily follows from (4).

(b) The spectrum of $L(q)$ does not contain a closed curve (see [11]).

(c) A number $\lambda_n(t)$ is a multiple eigenvalue of $L_t(q)$ of multiplicity $p$ if and only if $p$ bands of the spectrum have common point $\lambda_n(t)$. If $n \neq m$, then the bands $\Gamma_n$ and $\Gamma_m$ may have only one common point (see [20]).

Finally, we use the following results of [7, 12, 20] about PT symmetric potentials.

**Summary 8** If $q(-x) = q(x)$, then the following statements hold.

(a) If two real numbers $c_1 < c_2$ belong to $\Gamma_n$, then $[c_1, c_2] \subset \Gamma_n$ (see [20]).

(b) If $\lambda \in \sigma(L_t(q))$, then $\overline{\lambda} \in \sigma(L_t(q))$ (see [7])

(c) If $\lambda \in \mathbb{R}$, then $F(\lambda) \in \mathbb{R}$, where $F$ is defined in (4) (see [12]).

3 **On the Bloch Eigenvalues and the Spectrum of $L(V)$**

First we consider the eigenvalues $\lambda_n(t)$ of $H_t(c)$ for all $t \in [0, 1]$, where $c \in (0, \infty)$. 
Theorem 1 All eigenvalues of the operator $H_t(c)$ lie on the union of the disks

$$D_{2c} \left( (2n + t)^2 \right) := \{ \lambda \in \mathbb{C} : |\lambda - (2n + t)^2| \leq 2c \} \quad (16)$$

for $n \in \mathbb{Z}$, where $t \in [0, 1]$.

Proof. Suppose to the contrary that there exists an eigenvalue $\lambda$ of $H_t(c)$ lying out of $D_{2c}((2n + t)^2)$ for all $n \in \mathbb{Z}$. Then the inequality $|\lambda - (2n + t)^2| > 2c$ holds for all $n \in \mathbb{Z}$. Using it, Parseval's equality for the orthonormal basis $\{ \pi^{-1/2} e^{i(2n + t)x} : n \in \mathbb{Z} \}$ in $L_2[0, \pi]$, and the following well-known relation

$$(\lambda - (2n + t)^2)(\Psi, e^{i(2n + t)x}) = ((2ic \cos 2x) \Psi, e^{i(2n + t)x}), \quad (17)$$

where $(\cdot, \cdot)$ is the inner product in $L_2[0, \pi]$ and $\Psi$ is a normalized eigenfunction of $H_t(c)$ corresponding to the eigenvalues $\lambda$ we get the following contradiction

$$\pi = \sum_{n \in \mathbb{Z}} |(\Psi, e^{i(2n + t)x})|^2 < \sum_{n \in \mathbb{Z}} \frac{|((2ic \cos 2x) \Psi, e^{i(2n + t)x})|^2}{(2c)^2} \leq \pi.$$ 

The theorem is proved. \(\blacksquare\)

Remark 1 It is well-known that the operator $H_t(c)$ is defined by the expression

$$-y'' + 2ity' + t^2y + (2ic \cos 2x)y \quad (18)$$

and the periodic boundary condition. Let us redenote the eigenvalues of $H_t(c)$ and $L_t(V)$ by $\lambda_n(t, c)$ and $\lambda_{n^*}(t, V)$ respectively, in order to stress their dependence on $c$ and $V$. Since $H_t(c)$ analytically depends on $t$ and $c$, if $t \in [0, 1]$ is fixed and $\lambda(t, c)$ $(\lambda(t, V))$ is a simple eigenvalue of $H_t(c)$ $(L_t(V))$, then there exists $\varepsilon > 0$ and an analytic function $\lambda(t, \cdot)$ $(\lambda(t, \cdot))$ in $\varepsilon$-neighborhood $U(c, \varepsilon)$ $(U(V, \varepsilon))$ of $c$ $(V)$ such that $\lambda(t, c)$ $(\lambda(t, V))$ is an eigenvalue of $H_t(c)$ $(L_t(V))$ for all $c \in U(c, \varepsilon)$ $(V \in U(V, \varepsilon))$. Moreover, for fixed $c$ $(V)$ the eigenvalue $\lambda_n(t, c)$ $(\lambda_{n^*}(t, V))$ analytically depend on $t$ if it is a simple eigenvalue. Recall that, as is noted in the introduction, for the fixed $c$ and $V$ the eigenvalues $\lambda_n(t, c)$ and $\lambda_{n^*}(t, V)$ of $H_t(c)$ and $L_t(V)$ are numerated so that they continuously depend on $t \in [0, 1]$.

Now using it and Summaries 7 and 8 we prove the following.

Theorem 2 (a) Let $c$ be a positive fixed number and $\lambda_n(0, c)$ be the $n$-th periodic eigenvalue of $H_0(c)$. If $\lambda_n(0, c)$ is a real number and $\lambda_n(t, c)$ are the simple eigenvalues for all $t \in [0, b)$, where $b \in [0, 1]$, then $\lambda_n(t, c)$ are the real numbers for all $t \in [0, b]$. Moreover, if $d_n(c) \in (0, 1)$ is the greatest positive number such that $\lambda_n(t, c)$ are the real eigenvalues for all $t \in [0, d_n(c)]$, then $\lambda_n(t, c)$ are the nonreal eigenvalues for all $t \in (d_n(c), 1]$.

(b) Let $t$ be a fixed number from $[0, 1]$ and $\lambda(t, c)$ be the simple eigenvalues of $H_t(c)$ for all $c \in [c_0(t), r(t))$ such that $\lambda(t, \cdot)$ is continuous on $[c_0(t), r(t))$, where $r(t) > c_0(t) > 0$. If $\lambda(t, c_0(t))$ is a real (nonreal) number, then $\lambda(t, c)$ are the real (nonreal) numbers for all $c \in [c_0(t), r(t))$.

(c) Let $t \in [0, 1]$ be a fixed number. If the eigenvalue $\lambda(t, c)$ of $H_t(c)$ continuously depends on $c$ and changes from real (nonreal) to nonreal (real) when $c$ moves from the left to the right of a point $c(t)$ then $\lambda(t, c(t))$ is a multiple eigenvalue.

Proof. (a) Since $c$ is fixed and $\lambda_n(0, c)$ is a simple eigenvalue, $\lambda_n(t, c)$ analytically depends on $t$ in some neighborhood of $0$ and there exist positive constants $\varepsilon$ and $\delta(0)$ such that the operator $H_0(c)$ has a unique eigenvalue $\lambda_n(t, c)$ in $U(\lambda_n(0, c), \varepsilon)$ whenever $|t| < \delta(0)$. 

(b) Let $t$ be a fixed number from $[0, 1]$ and $\lambda(t, c)$ be the simple eigenvalues of $H_t(c)$ for all $c \in [c_0(t), r(t))$ such that $\lambda(t, \cdot)$ is continuous on $[c_0(t), r(t))$, where $r(t) > c_0(t) > 0$. If $\lambda(t, c_0(t))$ is a real (nonreal) number, then $\lambda(t, c)$ are the real (nonreal) numbers for all $c \in [c_0(t), r(t))$.

(c) Let $t \in [0, 1]$ be a fixed number. If the eigenvalue $\lambda(t, c)$ of $H_t(c)$ continuously depends on $c$ and changes from real (nonreal) to nonreal (real) when $c$ moves from the left to the right of a point $c(t)$ then $\lambda(t, c(t))$ is a multiple eigenvalue.
If $\lambda_n(t, c)$ is a nonreal number then by Summary 8(b), $\lambda_n(t, c)$ is also eigenvalue of $H_t(c)$ lying in $U(\lambda_n(0, c), \varepsilon)$ which contradicts the uniqueness of $\lambda_n(t, c)$. Thus $\lambda_n(t, c)$ is a real number for all $t \in [0, \delta(0))$. Let $d > \delta$ be greatest number such that $\lambda_n(t, c)$ is real for $t \in [0, d)$ and $d < b$. Then by the condition of the theorem and by continuity of $\lambda_n(t, c)$, there exists $t_1 \in (d_n(c), d_n(c) + \varepsilon)$ such that $\lambda_n(t_1, c)$ is a nonreal number.

Now to prove the second statement of (a) suppose to the contrary that $\lambda_n(t_0, c)$ is a nonreal eigenvalue for some $t_0 = (d_n(c) + \delta) \in [0, 1]$, where $\delta > 0$. By the definition of $d_n(c)$ for each $0 < \varepsilon < \delta$ there exists $t_1 \in (d_n(c), d_n(c) + \varepsilon)$ such that $\lambda_n(t_1, c)$ is a nonreal number. Then using Summary 8(b) and taking into account that $\lambda_n(t, c)$ continuously depend on $t$, one can readily see that the spectrum of $H_t(c)$ contains a closed curve

$$\{\lambda_n(t, c) : t \in [d_n(c), t_0]\} \cup \{\lambda_n(t, c) : t \in [d_n(c), t_0]\}.$$ 

It contradicts to the Summary 7(b).

(b) Instead of the fixed $c$ and variable $t$ using respectively the fixed $t$ and variable $c$ and repeating the proof of the first statement of (a) we obtain the proof of (b) when $\lambda(t, c_0)$ is a real number. If $\lambda(t, c_0)$ is a nonreal number and simple eigenvalue, then $\lambda(t, c)$ is a nonreal number for $c \in [c_0, c_0 + \delta]$ for some $\delta > 0$. Let $d > \delta$ be greatest number such that $\lambda(t, c)$ is nonreal for $c \in [c_0, d)$ and $d < r(t)$. Then $\lambda(t, d)$ is a real number. It implies that the operator $H_t(c)$ has a unique eigenvalue $\lambda$ in $\varepsilon$-neighborhood of $\lambda(t, d)$, whenever $c \in (d - \gamma(t), d)$ for some $\gamma(t) > 0$. Then by the definition of $d$, $\lambda$ is a nonreal eigenvalue and by Summary 8(b), $\lambda$ is also an eigenvalues of $H_t(c)$ lying in $\varepsilon$-neighborhood of $\lambda(t, d)$, which contradicts the above uniqueness.

(c) Suppose to the contrary that $\lambda(t, c(t))$ is a simple eigenvalue. Then by (b) the eigenvalue $\lambda(t, c)$ does not change from real (nonreal) to nonreal (real) when $c$ moves from the left to the right of the point $c(t)$. The theorem is proved.

Now we consider the periodic and antiperiodic eigenvalues for $c \geq 2$. For this first let us formulate the corresponding results of [22] for $c < 2$ as the next summary.

**Summary 9** If $c < 2$, then the operator $H_1(c)$ (or $H_0(c)$) have 2 eigenvalues in the disks $D_1((2n - 1)^2)$ for $n = 1, 2, \ldots$ (in $D_1((2n)^2)$ for $n = 2, 3, \ldots$), defined by (16) These eigenvalues are simple. One of those antiperiodic (periodic) eigenvalues is AD (PD) and the other is AN (PN) eigenvalue, where AD, PD, AN and PN eigenvalues are respectively the eigenvalues belonging to $\sigma(H_0(c)) \cap \sigma(D(c)), \sigma(H_0(c)) \cap \sigma(D(c)), \sigma(H_1(c)) \cap \sigma(N(c))$ and $\sigma(H_0(c)) \cap \sigma(N(c))$ (see (15)). Moreover, in the outsides of the disks $D_1((2n)^2)$ for $n = 2, 3, \ldots$ there exist two PN eigenvalues $\lambda_1(0)$ and $\lambda_2(0)$ and one PD eigenvalue $\lambda_3(0)$.

**Theorem 3** (a) Let $n_1 = \lfloor (c + 1)/2 \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of the positive number $x$. For each $n > n_1$ the number of the eigenvalues (counting multiplicity) of $H_0(c)$ lying in $D_2c((2n)^2)$ is 2. They are the simple eigenvalues and the real numbers. One of them is the PD and the other is the PN eigenvalue. The number of eigenvalues (counting multiplicity) of $H_0(c)$ lying outside the disks $D_2c((2n)^2)$ for $n > n_1$ is $2n_1 + 1$. They lie in the rectangle

$$\left\{ \lambda \in \mathbb{C} : \text{Im} \lambda \leq 2c, -2c \leq \text{Re} \lambda \leq (2n_1)^2 + 2c \right\}. \quad (19)$$

(b) Let $n_2 = \lfloor c/2 \rfloor$. For each $n > n_2$ the number of the eigenvalues (counting multiplicity) of $H_1(c)$ lying in $D_2c((2n + 1)^2)$ is 2. They are the simple eigenvalues and the nonreal conjugate numbers. One of them is the AD and the other is the AN eigenvalue. The number
of the eigenvalues of $H_1(c)$ lying outside of the disks $D_{2c}((2n+1)^2)$ for $n > n_2$ is $2n_2 + 2$. They lie in the rectangle

$$\left\{ \lambda \in \mathbb{C} : |\Im \lambda| \leq 2c, \ -2c \leq \Re \lambda \leq (2n_2 + 1)^2 + 2c \right\}.$$  \hspace{1cm} (20)

**Proof.** (a) If $n > n_1$, then $n > (c + 1)/2$ and the disk $D_{2c}((2n)^2)$ has no common points with the other disks $D_{2c}((2m)^2)$, because $|(2m)^2 - (2n)^2| > 4c$ for $m \neq n$. Hence, it follows from Theorem 1 that the boundary of $D_{2c+\varepsilon}((2n)^2)$ lies in the resolvent sets of the operators $H_0(\alpha)$ for all $\alpha \in [0, c]$ if $\varepsilon$ is a sufficiently small positive number. Therefore the projection of $H_0(\alpha)$ defined by contour integration over the boundary of $D_{2c+\varepsilon}((2n)^2)$ depends continuously on $\alpha$. It implies that the number of eigenvalues (counting the multiplicity) of $H_0(\alpha)$ lying in $D_{2c+\varepsilon}((2n)^2)$ are the same for all $\alpha \in [0, c]$. Since $H_0(0)$ has two eigenvalues in $D_{2c+\varepsilon}((2n)^2)$, the operator $H_0(c)$ has also 2 eigenvalues. Letting $\varepsilon$ tend to zero we obtain that $H_0(c)$ has two eigenvalues (counting the multiplicity) in $D_{2c}((2n)^2)$.

Instead of the operator $H_0(c)$ using the operators $N(c)$ and $D(c)$, taking into account that $N(0)$ and $D(0)$ have one eigenvalue in $D_{2c}((2n)^2)$ and repeating the above arguments we obtain that the operators $N(c)$ and $D(c)$, have one eigenvalue in $D_{2c}((2n)^2)$. It with (15) implies that the periodic eigenvalues lying in $D_{2c}((2n)^2)$ are different numbers and hence they are simple eigenvalues. Now using Summary 1 and Theorem 2(b) we obtain that they are the real numbers.

The disks $D_{2c}((2n)^2)$ for $n = 0, 1, \ldots, n_1$ are contained in the rectangle (19). Moreover the rectangle (19) has no common points with the disks $D_{2c}((2m)^2)$ for $n > (c + 1)/2$. On the other hand, the rectangle (19) contains $2n_1 + 1$ eigenvalues of $H_0(0)$. Therefore instead of $D_{2c}((2n)^2)$ using (19), and arguing as above we complete the proof of (a).

(b) Instead of $D_{2c}((2n)^2)$ and the rectangle (19) using $D_{2c}((2n+1)^2)$ and the rectangle (20) respectively, and arguing as above we get the proof of (b). \hfill \blacksquare

The following theorem plays the crucial roles in the study of the spectrum of $H(c)$.

**Theorem 4** Let $n_3 = \lceil (2c + 1)/2 \rceil + 1$ and $t \in (0, 1)$. Then for any $b \geq (2n_3)^2 - 2c$ the following statements hold.

(a) The number of the eigenvalues of $H_t(c)$ lying in $(b, b + 4)$ is less than 3.

(b) The multiplicity of any eigenvalue of $H_t(c)$ lying in $(b, \infty)$ is less than 3.

**Proof.** First, let us consider the case $t \in (0, 1/2]$. If $n \in \mathbb{N}$ and $n \geq n_3 > (2c + 1)/2$, then the rectangle

$$R_n(t) := \{ \lambda \in \mathbb{C} : |\Im \lambda| \leq 2c, \ (2n - t)^2 - 2c \leq \Re \lambda \leq (2n + t)^2 + 2c \}.$$

has no common points with the other rectangles $R_m(t)$ for $m \neq n$. Therefore arguing as in the proof of Theorem 3(a) we obtain that $R_n(t)$ contains two eigenvalue (counting the multiplicity). Moreover, using Theorem 1 and taking into account that the rectangle $R_n(t)$ contains the disks $D_{2c}((2n - t)^2)$ and $D_{2c}((2n + t)^2)$ we obtain that the eigenvalues lying in the half-plane $\{ \lambda \in \mathbb{C} : \Re \lambda > (2n_3)^2 - 2c \}$ belong to the rectangle $R_n(t)$ for some $n \geq n_3$. On the other hand, the distance between $R_n(t)$ and $R_{n+1}(t)$ for $n \geq n_3$ is greater than 4. Therefore for any $b \geq (2n_3)^2 - 2c$ the interval $(b, b + 4)$ may have common points only with one of the rectangles $R_n(t)$ for $n \geq n_3$. Thus this interval may contain at most two eigenvalues of the operator $H_t(c)$.

Instead of the rectangle $R_n(t)$ using the rectangle

$$\{ \lambda \in \mathbb{C} : |\Im \lambda| \leq 2c, \ (2n + t)^2 - 2c \leq \Re \lambda \leq (2n + 2 - t)^2 + 2c \}$$

and repeating the proof of the case $t \in (0, 1/2]$ we get the proof of (a) in the case $t \in (1/2, 1)$.
The proof of (b) immediately follows from (a). □

Now using the above theorems, the following notations and some well-known properties of the Hill discriminant $F(\lambda)$ defined in (4) we consider the spectrum of $H(c)$.

**Notation 1** We denote the periodic eigenvalues of $H(c)$ lying in rectangle (19) and in the disks $D_{2c}((2n)^2)$ for $n > n_1$ by $\lambda_1(0), \lambda_2(0), \ldots, \lambda_{2n-1}(0)$ and $\lambda_{2n}(0), \lambda_{2n+1}(0)$ respectively, where $n_1$ is defined in Theorem 3(a). By Theorem 3(a), $\lambda_{2n}(0)$ and $\lambda_{2n+1}(0)$ for $n > n_1$ are the distinct real numbers and simple eigenvalues. We denote they in increasing order $\lambda_{2n}(0) < \lambda_{2n+1}(0)$ and hence

$$
\lambda_{2n+2}(0) < \lambda_{2n+3}(0) < \lambda_{2n+4}(0) < \lambda_{2n+5}(0) < \ldots
$$

Thus from now on the periodic eigenvalues are numerated in the described manner, the eigenvalues of $H_t(c)$ for $t \in (0, 1]$, as is noted in Remark 1, are numerated so that $\lambda_n$ is a continuous function on $[0, 1]$ and the $n$-th band $\Gamma_n$ is defined by (6).

By (4) the eigenvalues of $H_0(c)$ and $H_1(c)$ are respectively the roots of $F(\lambda) = 2$ and $F(\lambda) = -2$ and $\sigma(H(c)) = \{ \lambda \in \mathbb{C} : -2 \leq F(\lambda) \leq 2 \}$. Thus by Summary 8(c) we have

$$
\lambda \in \text{Re}(\sigma(H(c))) \iff \lambda \in \mathbb{R}, \ (\lambda, F(\lambda)) \in S(2),
$$

where $S(2) = \{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \}$. Moreover, it is well known [1] that

$$
\lim_{\lambda \to -\infty} F(\lambda) = \infty.
$$

We also use the following well-known [1] asymptotic formulas for the eigenvalues $\lambda_{2n}(0)$ and $\lambda_{2n+1}(0)$ lying in the disks $D_{2c}((2n)^2)$

$$
\lambda_{2n}(0) = (2n)^2 + O(n^{-1}), \ \lambda_{2n+1}(0) = (2n)^2 + O(n^{-1}).
$$

**Theorem 5** (a) If $\mu$ is a smallest real periodic eigenvalue, then the part of the spectrum of $H(c)$ lying in the half-plane $\{ \lambda \in \mathbb{C} : \text{Re} \lambda < \mu \}$ is nonreal.

(b) Let $n = n_1 + 2$, where $n_1$ is defined in Theorem 3(a). Then the part of $\sigma(H(c)) \cap \mathbb{R}$ lying in the half-space $P(n) = \{ \lambda : \text{Re} \lambda > \lambda_{2n}(0) \}$ consist of the intervals

$$
I_{n+1} = [\lambda_{2n+1}(0), \lambda_{2n+2}(0)], \ I_{n+2} = [\lambda_{2n+3}(0), \lambda_{2n+4}(0)], \ldots
$$

The gaps in the real part of the spectrum lying in $P(n)$ are the intervals

$$
(\lambda_{2n}(0), \lambda_{2n+1}(0)), \ (\lambda_{2n+2}(0), \lambda_{2n+3}(0)), \ldots
$$

**Proof.** (a) It readily follows from (23) that the leftmost point $(\lambda, F(\lambda))$ of the intersection of the graph $G(F) := \{(\lambda, F(\lambda)) : \lambda \in \mathbb{R} \}$ with the strip $S(2)$ lies in the line $y = 2$, that is, $\lambda$ is a smallest real periodic eigenvalue $\mu$. Therefore by (22) $\mu$ is the smallest point of $\text{Re}(\sigma(H(c)))$. It means that the half-plane $\{ \lambda \in \mathbb{C} : \text{Re} \lambda < \mu \}$ may contain only the nonreal part of $\sigma(H(c))$.

(b) Since $\lambda_{2n}(0), \lambda_{2n+1}(0), \ldots$ are real and simple (see Notation 1) the intersection points of $G(F)$ and the line $y = 2$ lying the half-plane $\overline{P(n)} := \{ \lambda : \text{Re} \lambda \geq \lambda_{2n}(0) \}$ are

$$
(\lambda_{2n}(0), 2), \ (\lambda_{2n+1}(0), 2), \ (\lambda_{2n+2}(0), 2), \ (\lambda_{2n+3}(0), 2), \ (\lambda_{2n+4}(0), 2), \ldots
$$

On the other hand, it follows from Theorem 3(b) that all antiperiodic eigenvalues lying in the half-plane $\overline{P(n)}$ are nonreal numbers due to the following. Since $\lambda_{2n}(0)$ is the eigenvalue lying in the disk $D_{2c}((2n)^2)$, it readily follows from the definition of $n$ and $n_1$ that $2n > c + 3$
and $\lambda_{2n}(0) > c^2 + 4c + 9$. If the antiperiodic eigenvalue $\lambda(1)$ is a real, then by Theorem 3(b) it belong to the rectangle (20) and hence $\lambda(1) \leq (2n_2 + 1)^2 + 2c \leq (c + 1)^2 + 2c = c^2 + 4c + 1 < \lambda_{2n}(0)$. That is why the antiperiodic eigenvalues lying in $\mathcal{P}(n)$ are nonreal. It means that $G(F)$ does not intersect the line $y = -2$ in the later half-plane. Thus we have

$$F(\lambda) > -2, \forall \lambda \in \{\lambda \in \mathbb{R} : \lambda \geq \lambda_{2n}(0)\}.$$  \hspace{1cm} (28)

These arguments imply that in $\mathcal{P}(n)$ the graph $G(H)$ may get in and out of the strip $S(2)$ at the points (27) called respectively the points of entry and exit. It is clear that if $(\lambda_m(0), 2)$ is a point of entry (point of exit) then $(\lambda_{m+1}(0), 2)$ is the point of exit (point of entry). Thus the points of entry and exit are alternating points. It implies that if $[\lambda_m(0), \lambda_{m+1}(0)]$ is the interval of $\sigma(H(c)) \cap \mathbb{R}$, then $(\lambda_{m+1}(0), \lambda_{m+2}(0))$ is the gap in $\sigma(H(c)) \cap \mathbb{R}$. On the other hand, by Theorem 3(a) of [20], in the neighborhood of infinity the lengths of the intervals of $\sigma(H(c)) \cap \mathbb{R}$ approach infinity. Thus, taking into account that the lengths of the intervals in (25) and (26) respectively approach infinity and zero (see (24)), we get the proof of (b) \Box

Now using the graph of $F$ we obtain the following result.

**Proposition 2** Suppose that $\lambda_m(t)$ is a simple eigenvalue for all $t \in (0, t_m)$ and $\lambda_m(0)$ is a real number. If $\lambda_m(0)$ is a point of entry (point of exit) then $\lambda_m(t_m) > \lambda_m(0)$ ($\lambda_m(t_m) < \lambda_m(0)$) and $[\lambda_m(0), \lambda_m(t_m)] \subset \Gamma_m$.

**Proof.** By Theorem 2(a) $\lambda_m(t)$ is a real eigenvalue for all $t \in [0, t_m]$. Since $\lambda_m(0)$ is a point of entry we have $\lambda_m(t_m) > \lambda_m(0)$ and by Summary 8(a), $[\lambda_m(0), \lambda_m(t_m)] \subset \Gamma_m$. In the same way we prove the statement for the point of exit. The proposition is proved. \Box

The following remarks which readily follows from (23) will be used in the next theorems.

**Remark 2** Let $(A, 2)$ and $(B, 2)$ be respectively the neighboring point of entry and point of exit of the graph $G(F)$ defined in the proof of Theorem 5(a) such that $A$ and $B$ are the simple periodic eigenvalues and $A < B$. Then there exists $\varepsilon > 0$ such that $F'(\lambda) < 0$ and $F'(\lambda) > 0$ respectively for $\lambda \in [A - \varepsilon, A + \varepsilon]$ and $\lambda \in [B - \varepsilon, B + \varepsilon]$. Assume that $F''(\lambda) \neq 0$ for all $\lambda \in [A - \varepsilon, B + \varepsilon]$. Let $\mu_1$ and $\mu_2$ be respectively the largest and smallest points of the interval $[A + \varepsilon, B - \varepsilon]$ such that $F'(\lambda) < 0$ and $F'(\lambda) > 0$ for $\lambda \in [\mu_1, \mu_2]$. It is clear that both $\mu_1$ and $\mu_2$ are the local minimum points. There are two cases:

**Case 1.** $\mu_1 < \mu_2$. Then $F$ has at least two local minimum points $\mu_1$ and $\mu_2$.

**Case 2.** $\mu_1 = \mu_2$. Then $F$ has only one local minimum point $\mu_1$ and it decreases and increases on $(A, \mu_1)$ and $(\mu_1, B)$ respectively.

**Remark 3** Since the differential equation

$$-y''(x) + (2ic \cos 2x)y(x) = \lambda y(x)$$

analytically depend on the real parameters $c$ and $\lambda$, its solutions $\theta$ and $\varphi$ and hence the Hill discriminant $F(\lambda, c) := \varphi'(\pi, \lambda, c) + \theta(\pi, \lambda, c)$ analytically depend on $\lambda$ and $c$. Moreover $F(\lambda, c)$ and its derivatives $F'(\lambda, c)$ and $F''(\lambda, c)$ with respect to $\lambda$ continuously depend on the pair $(\lambda, c)$. If $F'(\lambda, c_0) = 0$ and $F''(\lambda, c_0) \neq 0$ then by implicit function theorem there exists $\varepsilon > 0$ and differentiable function $\lambda(\cdot)$ such that $F'(\lambda(c), c) = 0$ for all $c \in U(c_0, \varepsilon)$.

Now we are ready to prove the main result about the spectrum of $H(c)$ $(L(V))$ for $c \in [2, \infty)$ $(V \geq \sqrt{3}/2)$. Note that the case $c \in (0, 2)$ was considered in detail in Theorems 6-8 of [22]. Therefore we assume that $c \geq 2$.

**Theorem 6** For each $k > n_3$ the following statements about the bands $\Gamma_{2k-1}$ and $\Gamma_{2k}$ of the spectrum of $H(c)$ hold, where $n_3$ defined in Theorem 4.
(a) There exists unique \( t_k \in (0, 1) \) such that \( \lambda_{2k-1}(t_k) \) is a multiple real eigenvalue lying in \( I_k = [\lambda_{2k-1}(0), \lambda_{2k}(0)] \). Its multiplicity is 2 and \( \lambda_{2k-1}(t_k) = \lambda_{2k}(t_k) \).

(b) The real parts of the bands \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \) are respectively the intervals \( \{\lambda_{2k-1}(t) : t \in [0, t_k]\} = [\lambda_{2k-1}(0), \lambda_{2k-1}(t_k)] \) and \( \{\lambda_{2k}(t) : t \in [0, t_k]\} = [\lambda_{2k}(t_k), \lambda_{2k}(0)] \).

The nonreal parts of \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \) are respectively the curves \( \{\lambda_{2k-1}(t) : t \in (t_k, 1]\} \) and \( \{\lambda_{2k}(t) : t \in (t_k, 1]\} \).

**Proof.** (a) First we prove that there exists multiple real eigenvalue lying in the interval \( I_k \). Since the endpoints of the interval \( I_k \) are the periodic and antiperiodic eigenvalues, we have \( F(\lambda_{2k-1}(0)) = F(\lambda_{2k}(0)) = 2 \). Therefore by the Roll’s Theorem there exists \( \lambda \in (\lambda_{2k-1}(0), \lambda_{2k}(0)) \) such that \( F'(\lambda) = 0 \). Moreover by Theorem 5 and Theorem 3 \( (\lambda_{2k-1}(0), \lambda_{2k}(0)) \) is the interval of the real part of the spectrum and does not contain the periodic and antiperiodic eigenvalues. Note that for \( c \geq 2 \) we have \( n_3 \geq n_1 + 2 \) and \( n_3 > n_2 \). That is why we can use Theorem 5 and Theorem 3. Therefore we have \( F(\lambda) = 2 \cos \pi t_k \) for some \( t_k \in (0, 1) \). Hence \( \lambda \) is a multiple eigenvalue of \( H_{\alpha} \) lying in the interior of \( I_k \). Moreover by Theorem 4(b) \( \lambda \) is a double eigenvalue and hence \( F''(\lambda, c_0) \neq 0 \). Thus we have proved that there exist double eigenvalue lying in \( I_k \).

Now we prove the uniqueness of the double eigenvalue lying in \( I_k \). Suppose that it does not hold for some \( c \geq 2 \), that is, there exist at least two points \( \mu_1 \) and \( \mu_2 \) in \( I_k \) such that \( F'(\mu_1, c) \) and \( F'(\mu_2, c) \) lie in the interval \( (-2, 2) \) and \( F'(\mu_1, c) = 0 \) and \( F'(\mu_2, c) = 0 \). By Theorem 4(b) we have \( F''(\mu_1, c) \neq 0 \) and \( F''(\mu_2, c) \neq 0 \). Therefore by implicit function theorem the uniqueness does not hold in some open neighborhood of \( c \). It implies that the uniqueness holds in the closed set. Let \( c_0 \) be the largest number such that the uniqueness holds for \( 0 < c \leq c_0 \). If \( |c_0 + 1/2| + 2 = k \) then the uniqueness is proved. Otherwise, there exists \( \alpha > 0 \) such that for \( c \in (c_0, c_0 + \alpha) \) the operator \( H(c) \) has at least two double Bloch eigenvalues \( \mu_1(c) \) and \( \mu_2(c) \). Since \( F'(\lambda, c_0) \) is a continuous function and it has unique zero \( \mu_0 \) in the interval \( [A, B] \), where \( A = \lambda_{2k-1}(0) - \beta, B = \lambda_{2k}(0) + \beta \) and \( \beta \) is chosen so that \( F'(\lambda, c_0) \neq 0 \) for \( \lambda \in [A, \lambda_{2k-1}(0)] \cup [\lambda_{2k}(0), B] \), there exists \( \varepsilon > 0 \) such that \( |F'(\lambda, c_0)| > \varepsilon \) whenever \( \lambda \) belongs to the compact \( [A, B] \setminus (\mu_0 - 1, \mu_0 + 1) \). Then using the uniform continuity of \( F' \) with respect to the pair \( (\lambda, c) \) we obtain that there exists \( \gamma > 0 \) such that \( |F'(\lambda, c)| > \varepsilon/2 \) whenever \( c \) and \( \lambda \) belongs to \( [c_0, c_0 + \gamma] \) and \( [A, B] \setminus (\mu_0 - 1, \mu_0 + 1) \) respectively. Here \( \gamma \) can be chosen so that \( [\lambda_{2k-1}(0), \lambda_{2k}(0)] \subset [A, B] \) for all \( c \in [c_0, c_0 + \gamma] \), since the periodic eigenvalues continuously depend on \( c \). Therefore \( H(c) \) has at least two double Bloch eigenvalue lying in \( (\mu_0 - 1, \mu_0 + 1) \). It means that Case 1 of Remark 2 holds. Thus \( F \) has two local minimum points \( \mu_1 \) and \( \mu_2 \) in \( (\mu_0 - 1, \mu_0 + 1) \). Without loss of generality it can be assumed that \( F(\mu_1) \geq F(\mu_2) \). Since, \( \mu_1 \) is a local minimum point there exists \( \mu_3 \in (\mu_1, \mu_2) \) such that \( F(\mu_3) > F(\mu_1) \). The last two inequalities with the intermediate value theorem for the continuous function \( F \) imply that there exists \( \mu_4 \in [\mu_2, \mu_3] \) such that \( F(\mu_4) = F(\mu_1) \) (see the graph of \( F \) between the first points of entry and exit). Then the points \( (\mu_1, F(\mu_1)) \) and \( (\mu_4, F(\mu_4)) \) of the graph of \( F \) belong to the horizontal line \( y = F(\mu_1) = 2 \cos t_1 \) for some \( t_1 \in (0, 1) \). It implies that \( \mu_1 \) and \( \mu_4 \) are the eigenvalues of the operator \( H_{t_1} \). Moreover, by definition \( \mu_1 \) is a double eigenvalue. Thus \( H_{t_1} \) has at least three eigenvalue (counting the multiplicity) on \( [\mu_1, \mu_2] \). It contradicts Theorem 4, since \( [\mu_1, \mu_2] \) is a subinterval of the interval \( (\mu_0 - 1, \mu_0 + 1) \) of the length 2. Thus we have proved that there exist multiple eigenvalue lying in \( I_k \) and it is a double eigenvalue.

(b) By (a) for \( t \in [0, 1] \setminus \{t_k\} \) the eigenvalues of the operator \( H_t \) lying in \( [\lambda_{2k-1}(0), \lambda_{2k}(0)] \) are simple and \( H_{t_k} \) has a double eigenvalue \( \lambda(t_k) \) lying in that interval. Now using it Theorem 2(a) and Summary 8(a) we conclude that the intervals \( [\lambda_{2k-1}(0), \lambda_{2k-1}(t_k)] \) and \( [\lambda_{2k}(t_k), \lambda_{2k}(0)] \) belong respectively to the bands \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \).
Now we prove that the sets \( \{ \lambda_{2k-1}(t) : t \in (t_k, \pi) \} \) and \( \{ \lambda_{2k}(t) : t \in (t_k, \pi) \} \) has no intersections points with the real line. If \( \lambda_{2k-1}(t) \in \mathbb{R} \) for some \( t \in (t_k, \pi) \), then by Summary 8(a) the interval \( [\lambda_{2k-1}(t_k), \lambda_{2k-1}(t)] \) has common subinterval either with \( [\lambda_{2k-1}(0), \lambda_{2k-1}(t_k)] \) or \( [\lambda_{2k}(t_k), \lambda_{2k}(0)] \). It contradict either Summary 7(a) or Summary 7(c). □

Now we prove some theorems which will be used in the next sections.

**Theorem 7** For each \( V \neq \pm 1/2 \) the geometric multiplicity of the periodic and antiperiodic eigenvalues of \( L(V) \) is 1.

**Proof.** Suppose two the contrary that the geometric multiplicity of the periodic eigenvalue \( \lambda_n(0) \) of \( L(V) \) is 2. Then both solution \( \varphi(x, \lambda_n(a)) \) or \( \theta(x, \lambda_n(a)) \) of the equation

\[
- y''(x) + (1 + 2V)e^{izx} + (1 - 2V)e^{-izx}y(x) = \lambda y(x)
\]

is a periodic function. On the other hand, one easily verify that

\[
(1 + 2V)e^{izx} + (1 - 2V)e^{-izx} = 2ic\cos(2x + a),
\]

where \( c = \sqrt{4V^2 - 1} \) and \( \alpha = -\frac{\pi}{2} + i \ln \frac{2V-1}{2V+1} \). Note that formula (31) was found in [9], where instead of the function \( \tanh^{-1} \) was used. Moreover, the solutions \( \varphi(z, \lambda_n(0)) \) and \( \theta(z, \lambda_n(0)) \) of the equation obtained from (30) by replacing the real variable \( x \) with the complex variable \( z \) analytically depend on \( z \), since they are the entire function of the potential \( q(z) = (1 + 2V)e^{izx} + (1 - 2V)e^{-izx} \) and \( q \) is the entire function of \( z \). Using these arguments we conclude that \( \varphi(x - \alpha, \lambda_n(0)) \) or \( \theta(x - \alpha, \lambda_n(0)) \) are the linearly independent periodic solutions of \(- y''(x) + (2ic\cos 2x)y(x) = \lambda y(x). \) It contradicts Summary 6. Thus the theorem is proved for \( \lambda_n(0) \). In the same way we prove it for \( \lambda_n(1) \). □

In [22] we proved that (see Summary 1) if \( c \) is a small number, then all periodic eigenvalues are real numbers and simple eigenvalues and hence by Notation 1 can be numbered in increasing order \( \lambda_1(0, c) < \lambda_2(0, c) < \ldots \) and they satisfy the formulas

\[
\lambda_1(0, c) = O(c), \quad \lambda_{2n}(0, c) = (2n)^2 + O(c), \quad \lambda_{2n+1}(0, c) = (2n)^2 + O(c)
\]

(32) as \( c \to 0 \) for \( n = 1, 2, \ldots \). Moreover, we proved that \( \lambda_1(0, c) \) and \( \lambda_2(0, c) \) are the PN and \( \lambda_3(0, c) \) is the PD eigenvalue (see Summary 9). It means that PN eigenvalues lying in \( O(c) \) neighborhood of \( 4 \) is less than the PD eigenvalues lying in \( O(c) \) neighborhood of \( 4 \). Now we proof that for each \( k = 1, 2, \ldots, \) the eigenvalues \( \lambda_{4k+1}(0, c) \) and \( \lambda_{4k+2}(0, c) \) are the PN and \( \lambda_{4k}(0, c) \) and \( \lambda_{4k+3}(0, c) \) are the PD eigenvalues (see Theorem 8). These results well be used very much in the Section 5. To consider the PD and PN eigenvalues \( \lambda \) and \( \mu \) lying in \( O(a) \) neighborhood of \((2n)^2\) we use the formulas

\[
(\lambda - 4)b_1 = ab_2, \quad (\lambda - (2k)^2)b_k = ab_{k-1} + ab_{k+1},
\]

(33)

(34)

\[
(\mu - 4 - \frac{2a^2}{\mu})a_1 = aa_2, \quad (\mu - (2k)^2)a_k = aa_{k-1} + aa_{k+1},
\]

(35)

(36)

where \( a = ic, c > 0 \) and without loss of generality, it can be assumed that \( b_n = a_n = 1 \).

**Theorem 8** If \( c \) is a small number and \( k \in \mathbb{N} \), then \( \lambda_{4k}(0, c) \) and \( \lambda_{4k+3}(0, c) \) are the PD and \( \lambda_{4k+1}(0, c) \) and \( \lambda_{4k+2}(0, c) \) are the PN eigenvalues.
Proof. To consider the PD eigenvalue $\lambda$ lying in $O(c)$ neighborhood of $(2n)^2$ for $n \geq 2$ we iterate $(2n-1)$-times formula

$$(\lambda - (2n)^2)b_n = ab_{n-1} + ab_{n+1}, \quad (37)$$

which is (34) for $k = n$ by using the formulas

$$b_k = \frac{ab_{k-1} + ab_{k+1}}{\lambda - (2k)^2}, \quad b_1 = \frac{ab_2}{\lambda - 4} \quad (38)$$

obtained respectively from (34) and (33) as follows. In (37) we use (38) for $b_{n-1}$ and $b_{n+1}$. Then in the obtained formula we isolate the terms with multiplicand $b_k$, and do not change they and use (38) for the terms with multiplicand $b_k$ when $k \neq n$. Continuing these process, $2n-1$ times we obtain

$$\lambda = (2n)^2 + G_n(\lambda) + S_n(\lambda) + S_{n-1}(\lambda) + R_n(\lambda) \quad (39)$$

where $G_n(\lambda), S_n-1(\lambda), S_n(\lambda)$ and $R_n(\lambda)$ are defined as follows. One can readily see that the $(2n-1)$-th usages (38) in (37) give the terms with multiplicands $b_n$ and $b_{n+k}$, where $|k| \geq 2$. In (39) $R_n(\lambda)$ denotes the sum of the terms with multiplicand $b_{n+k}$ for $|k| \geq 2$. Since these terms are obtained after $(2n-1)$-th usage (38) in (37) they contain the multiplicand $a^{2n}$.

Moreover, it follows from (38) that $b_{n+k} = O(a^2)$ for $|k| \geq 2$. Thus $R_n(\lambda) = O(a^{2n+2})$ and (39) has the form

$$\lambda = (2n)^2 + G_n(\lambda) + S_n(\lambda) + S_{n-1}(\lambda) + O(a^{2n+2}), \quad (40)$$

where $G_n(\lambda), S_n(\lambda)$ and $S_{n-1}(\lambda)$ contains the multiplicand $b_n$ and without loss of generality, it can be assumed that $b_n = 1$. In (40) $G_n(\lambda)$ denotes the sum of the terms whose denominators does not contain the multiplicand $\lambda - 4$. It is clear that the terms with multiplicand $\lambda - 4$ may be obtained as a result of $(2n-3)$-th and $(2n-1)$-th usage of the second formula of (38). In (40) $S_{n-1}(\lambda)$ and $S_n(\lambda)$ are the sums of the terms whose denominators contain the multiplicand $\lambda - 4$ and are obtained in $(2n-3)$-th and $(2n-1)$-th usages the second formula of (38) respectively. It is clear that $S_{n-1}$ is obtained by using the formulas in (38) in the following order $k = n-2, n-1, n-2, ..., 2, 1, 2, ..., n-1$. Therefore it has the form

$$S_{n-1} = \frac{a^{2n-2}}{2}\prod_{k=2}^{n-1} (\lambda - (2k)^2)^{-2}. \quad (41)$$

Similarly iterating the formulas (36) $2n-1$ times and arguing as above we get

$$\mu = (2n)^2 + G_n(\mu) + \tilde{S}_{n-1}(\mu) + \tilde{S}_n(\mu) + O(a^{2n+2}). \quad (42)$$

Here $\tilde{S}_{n-1}(\lambda)$ and $\tilde{S}_n(\lambda)$ are obtained respectively from $S_{n-1}(\lambda)$ and $S_n(\lambda)$ by replacing the $\lambda - 4$ with $\lambda - 4 - \frac{2a^2}{\lambda}$. Therefore, using (32) and taking into account that $S_n(\lambda)$ and $\tilde{S}_n(\lambda)$ are obtained in $(2n-1)$-th usages the second formula of (38) and (35) respectively, and hence they contains the multiplicand $a^{2n}$ we get

$$\tilde{S}_n(\lambda) = S_n(\lambda) + O(a^{2n+2}). \quad (43)$$

In (41) replacing $\lambda - 4$ with $\lambda - 4 - \frac{2a^2}{\lambda}$ we get

$$\tilde{S}_{n-1}(\lambda) = S_{n-1}(\lambda) + \frac{2a^{2n}}{(\lambda - 4)^2} \prod_{k=2}^{n-1} (\lambda - (2k)^2)^{-2} + O(a^{2n+2}). \quad (44)$$
Now in the second term of the right-hand side of (44) instead of \( \lambda \) writing \( \lambda = (2n)^2 + O(a^2) \) (it readily follows from (40)) we obtain

\[
\bar{S}_{n-1}(\lambda) = S_{n-1}(\lambda) + \frac{2a^{2n}}{(2n)^2 - 4} \prod_{k=2}^{n-1} \left( (2n)^2 - (2k)^2 \right)^{-2} + O(a^{2n+2}). \tag{45}
\]

Now subtracting equality (42) from (40) and using (43) and (45) we obtain

\[
\lambda - \mu = f(\lambda) - f(\mu) - \frac{2a^{2n}}{(2n)^2 - 4} \prod_{k=2}^{n-1} \left( (2n)^2 - (2k)^2 \right)^{-2} + O(a^{2n+2}), \tag{46}
\]

where \( f(\lambda) = S_n(\lambda) + S_{n-1}(\lambda) \). It is clear that \( f'(\lambda) = O(a^2) \) for \( \lambda = (2n)^2 + O(a^2) \). Therefore by the mean value theorem we have \( f(\lambda) - f(\mu) = (\lambda - \mu)(1 + O(a^2)) \). Now using it in (46) and taking into account that \( a = ic, c > 0 \) we get

\[
\lambda - \mu = \frac{-2(-1)^n c^{2n}}{(2n)^2 - 4} \prod_{k=2}^{n-1} \left( (2n)^2 - (2k)^2 \right)^{-2} + O(a^{2n+2}). \tag{47}
\]

It is clear that the multiplicands \( (2n)^2 - (2k)^2 \) for \( k = 2, 3, ..., (n-1) \) are the positive number. It follows from (47) that if \( n = 2k \), then the PD eigenvalue \( \lambda \) lying in the neighborhood of \( (2n)^2 \) is less than the PN eigenvalues \( \mu \) lying in the neighborhood of \( (2n)^2 \). On the other hand by Notation 1 the periodic eigenvalues lying in \( (2n)^2 \) are denoted as \( \lambda_{2n}(0, c) < \lambda_{2n+1}(0, c) \). Thus \( \lambda_{2k}(0, c) \) and \( \lambda_{2k+1}(0, c) \) are the PD and PN eigenvalues respectively. Instead of \( n = 2k \) using \( n = 2k + 1 \) and repeating the above proof we obtain that \( \lambda_{4k+2}(0, c) \) and \( \lambda_{4k+3}(0, c) \) are the PN and PD eigenvalues respectively.

**Corollary 2** If \( c \) is a small number, then for each \( k \in \mathbb{N} \), the \( (2k-1) \)-th entry and exit points of the graph \( G(F) = \{ (\lambda, F(\lambda)) : \lambda \in \mathbb{R} \} \) to the strip \( S(2) \) are the PN eigenvalues and the \( 2k \)-th entry and exit points are the PD eigenvalues.

Now we consider the antiperiodic eigenvalues for the small value of \( c \) which also will be used in Section 5. By Summary 1 all antiperiodic eigenvalues for small \( c \) are nonreal numbers. Moreover, by Theorem 3(b) for the small value of \( c \) the disk \( D_{2c}((2n - 1)^2) \) contains one AD and one AN eigenvalues \( \lambda(c) \) and \( \mu(c) \) and

\[
\lambda(c) = (2n - 1)^2 + O(c), \quad \mu(c) = \overline{\lambda(c)}. \tag{48}
\]

**Theorem 9** For the AD eigenvalue \( \lambda(c) \) lying in \( D_{2c}((2n - 1)^2) \) the formula

\[
\text{Im} (\lambda(c)) = C_n a^{2n-1} + O(c^{2n}) \tag{49}
\]

holds as \( c \to 0 \), where \( a = ic \) and \( C_n \) is a positive number.

**Proof.** Formula (49) for \( n = 1 \) follows from formula (24) of [22]. To prove it for \( n > 1 \) we iterate \((2n - 2)\)-times the formula

\[
(\lambda(c) - (2n - 1)^2) c_n = ac_{n-1} + ac_{n+1} \tag{50}
\]

(see (18) of [22] for \( k = n \)) as follows. Each time isolate the terms with multiplicand \( c_n \) (we call they as isolated terms) and use the formulas

\[
c_1 = \frac{ac_2}{\lambda(c) - 1 - a}, \quad c_k = \frac{ac_{k-1} + ac_{k+1}}{\lambda(c) - (2k - 1)^2} \tag{51}
\]
where \( k = 2, 3, \ldots \), for the terms with multiplicand \( c_k \) when \( k \neq n \). After \((2n-2)\) times usages of (51) in (50) we obtain

\[
\lambda(c) = (2n+1)^2 + G_n(\lambda(c)) + S_n(\lambda(c)) + O(a^{2n}),
\]

(52)

where \( S_n \) is obtained by using the formulas (51) for \( c_k \) in the following order \( k = n-1, n-2, \ldots, 2, 1, 2, \ldots, n-1 \) and hence has the form

\[
S_n = \left( (\lambda(c) - 1 - a)^{-1} \prod_{k=2}^{n-1} (\lambda(c) - (2k-1))^{-2} \right) a^{2n-2}.
\]

(53)

The sum of other isolated terms is denoted by \( G_n(\lambda(c)) \). Thus \( G_n(\lambda(c)) \) is the sum of fractions whose numerators are \( a^{2k} \) for \( k = 1, 2, \ldots, (n-1) \), denominators are the products of \( \lambda(c) - (2s-1)^2 \) for \( s \neq n \) and hence

\[
\overline{G_n(a, \lambda(c))} = \overline{G_n(a, \lambda(c))} - G_n(a, \lambda(c)) = \overline{(\lambda(c) - \lambda(c))} O(a^3).
\]

(54)

Using (48) we obtain \( (\lambda(c)-1-a)^{-1} = \gamma(1+a\gamma)^{-1} = \gamma(1+a+O(a^2)) \), where \( \gamma = (\lambda(c)-1)^{-1} \). It with (53) implies that

\[
\text{Im} S_n = \left( (2n-1)^2 - 1 \right)^{-1} \prod_{k=2}^{n-1} ((2n-1)^2 - (2k-1))^{-2} a^{2n-1} + O(a^{2n})
\]

(55)

Now using (54) and (55) in (52) we get the proof of (49). ■

4 Spectral Singularities, ESS, and Spectral Expansion

First using the results of Section 3 we consider spectral singularities and ESS.

**Theorem 10** If \( V > 1/2 \) then the operator \( L(V) \) has infinitely many spectral singularities and spectral singularity at infinity.

**Proof.** By Theorem 6(a) for each \( k > n_3 = \lfloor (2c + 1)/2 \rfloor + 1 \) there exists \( t_k \in (0, \pi) \) such that \( \lambda_{2k-1}(t_k) = \lambda_{2k}(t_k) \) is a double eigenvalue. It is spectral singularity due to Summary 5(b). Thus all bands \( \Gamma_k \) for \( k > n_3 \) contains a spectral singularity. On the other hand, it readily follows from Definition 1 that if \( L(V) \) has a sequence of spectral singularities converging to infinity then it has the spectral singularity at infinity. ■

To consider the ESS we use the following theorem.

**Theorem 11** A number \( \lambda \) is an ESS of \( L(V) \) if and only if it is either a multiple periodic or a multiple antiperiodic eigenvalue.

**Proof.** Let \( \lambda \in S(L(V)) \) be an ESS. Then by Summary 5(a) it is multiple Bloch eigenvalue. Since, by Summary 5(b), the Bloch eigenvalues \( \lambda_n(t) \) for \( t \in (0, \pi) \) and \( n \in \mathbb{N} \) are not ESS, \( \lambda \) is either periodic or antiperiodic eigenvalue. Now suppose that \( \lambda \in S(L(V)) \) is a multiple periodic or antiperiodic eigenvalue. Then by Theorem 7 the geometric multiplicity of \( \lambda \) is 1 and by Summary 5(c) it is an ESS. ■

**Remark 4** Note that by (8) the operators \( H(c) \) and \( L(V) \) have the same Hill discriminants \( F(\lambda) \) and \( G(\lambda) \) if \( c = \sqrt{4V^2 - 1} \). Therefore if \( \lambda \) is a multiple eigenvalue of \( H(c) \), then it is also a multiple eigenvalue of \( L(V) \) with the same multiplicity. Note that if \( V < 1/2 \), then \( ic \in (-\infty, 0) \) and the well known self-adjoint Mathieu-Hill operator \( H(c) \) has no double
Bloch eigenvalues. Therefore, by Summary 5(a) for \( V < 1/2 \) the operator \( L(V) \) has no spectral singularities and ESS. If \( V = 1/2 \) then all 2-periodic eigenvalues of \( L(V) \) except 0 are double eigenvalue with geometric multiplicities 1 (see [5]) and hence are ESS (see Summary 5(c)).

Now we consider the ESS of \( L(V) \) for \( V > 1/2 \).

**Theorem 12** Let \( 1/2 < V < \sqrt{5}/2 \). (a) If \( V \neq V_2 \), then the operator \( L(V) \) has no ESS. 
(b) If \( V = V_2 \), then \( L(V) \) has a unique ESS and it is \( \lambda_1(0) = \lambda_2(0) \).

**Proof.** (a) By Summary 1 all periodic and antiperiodic eigenvalues are simple. Therefore by Theorem 11 the operator \( L(V) \) has no ESS.
(b) By Summary 1 \( \lambda_1(0) = \lambda_2(0) \) is a unique double periodic eigenvalue and all antiperiodic eigenvalues are simple. Therefore by Theorem 11 \( \lambda_1(0) \) is a unique ESS. ■

**Theorem 13** There exists a sequence \( 0 < c_2 < c_3 < \cdots \) such that \( c_k \to \infty \) as \( k \to \infty \) and \( H(c) \) has no ESS and has ESS respectively if and only if \( c \neq c_k \) for all \( k \) and \( c = c_k \) for some \( k \), where \( c_2 = \sqrt{4V^2 - 1} \) and \( V_2 \) is the second critical point. Moreover, the number of ESS of \( H(c_k) \) is not greater than \( c_k + 2 \). The number of the bands of \( H(c_k) \) containing at least one ESS is not greater than \( 2c_k + 3 \) respectively.

**Proof.** Let \( c < 2n - 1 \). Then using Theorem 3(a) and (b) we conclude that the total number of multiple periodic and antiperiodic eigenvalues (counting multiplicity) is not greater than \( 4n - 1 \). It means that there exist at most \( s \) different numbers denoted by \( \rho_1(c), \rho_2(c), \ldots, \rho_s(c) \) which are the multiple periodic and antiperiodic eigenvalues, where \( s \leq 2n - 1 \). Since the operators \( H_0(c) \) and \( H_1(c) \) analytically depend on \( c \), by the well-known perturbation theory if \( \rho_k(c_0) \) is a multiple periodic eigenvalue then there exists \( \epsilon_k \) such that the eigenvalues of the operators \( H_0(c) \) and \( H_1(c) \) for \( c \in U(c_0, \epsilon_k) \) lying in the small neighborhood of \( \rho_k(c_0) \) are simple. Therefore there exist at most finite number \( c_1, c_2, \ldots, c_m \) from \( (0, 2n - 1) \) such that \( H_0(c) \) and \( H_1(c) \) may have multiple eigenvalue.

Let \( n_k \) be the smallest integer such that \( c_k < 2n_k - 1 \). Instead of \( c \) and \( n \) using \( c_k \) and \( n_k \) respectively and repeating the above argument we obtain that the total number of multiple 2-periodic (periodic and antiperiodic) eigenvalues without counting multiplicity and counting multiplicity are not greater than \( 2n_k - 1 \) and \( 4n_k - 1 \) respectively. By the definition of \( n_k \) we have \( 2n_k - 1 \leq c_k + 2 \). Therefore by Theorem 11 and Summary 7(c) the number of the ESS of the operator \( H(c_k) \) and the number of the bands of \( H(c_k) \) containing at least one ESS are not greater than \( c_k + 2 \) and \( 2c_k + 3 \) respectively. ■

**Remark 5** It was suitable to formulate Theorem 13 in term of the parameter \( c \), since we have used the notation of Theorem 3. By (8) and Theorem 13 the operator \( L(V) \) for \( V > 1/2 \) has no ESS and has ESS respectively if and only if \( V \neq V_k \) for all \( k \) and \( V = V_k \) for some \( k \), where \( V_k = 1/2 \sqrt{c_k^2 + 1} \) is said to be the \( k \)-th critical point. Moreover the number of ESS of \( L(V_k) \) and the number of the bands of \( L(V_k) \) containing at least one ESS is not greater than \( \sqrt{4V_k^2 - 1} + 2 \) and \( 2\sqrt{4V_k^2 - 1} + 3 \) respectively. Denote by \( \Gamma_1, \Gamma_2, \ldots, \Gamma_{m(k)} \) the bands containing at least one ESS of \( L(V_k) \), where \( m(k) \leq 2\sqrt{4V_k^2 - 1} + 3 \). Then the bands \( \Gamma_{m(k)+1}, \Gamma_{m(k)+2}, \ldots \), don’t contain an ESS.

Now using Summary 4, Corollary 1 and Theorem 12 we get the following results about the spectral expansions of \( L(V) \) for \( 1/2 < V < \sqrt{5}/2 \).

**Theorem 14** Let \( 1/2 < V < \sqrt{5}/2 \).
(a) If \( V \neq V_2 \), then the spectral expansion for \( L(V) \) has the elegant form \((11)\).
(b) If \( V = V_2 \), then the spectral expansion for \( L(V) \) has the following form

\[
 f(x) = \frac{1}{2\pi} \left( \int_{(-\pi,\pi]} [a_1(t)\Psi_{1,t}(x) + a_2(t)\Psi_{2,t}(x)] dt + \sum_{n=3}^{\infty} \int_{(-\pi,\pi]} a_n(t)\Psi_{n,t}(x) dt \right). \tag{56}
\]

Moreover

\[
 \int_{(-\pi,\pi]} a_1(t)\Psi_{1,t} + a_2(t)\Psi_{2,t} dt = \lim_{\varepsilon \to 0} \left( \int_{(-\pi,\pi]} a_1(t)\Psi_{1,t} + \int_{(-\pi,\pi)\setminus(-\varepsilon,\varepsilon)} a_2(t)\Psi_{2,t} dt \right)
\]

**Proof.** (a) By Theorem 12(a) and Corollary 1 the operator \( L(V) \) has no ESS and ESS at infinity. Therefore the proof follows from Summary 4(a).

(b) By Theorem 12(b), \( \lambda_1(0) = \lambda_2(0) \) is a unique ESS of \( L(V) \). Therefore the set \( \mathcal{N}(\mathcal{E}) \) defined in Summary 4(b) is \{1, 2\}. Moreover, by Corollary 1 the operator \( L(V) \) has no ESS at infinity. Therefore (56) follows from (14).

Now changing the variable to \( \lambda \) in (11) and (56) as was done in [19] and using the relations

\[
 \Gamma_n = \lim_{\varepsilon \to 0} \Gamma_n(\varepsilon), \quad \Omega_1(\varepsilon) = \Gamma_1(\varepsilon) \cup \Gamma_2(\varepsilon),
\]

where \( \Gamma_n(\varepsilon) = \{\lambda = \lambda_n(t) : t \in [\varepsilon, \pi - \varepsilon]\} \) we obtain

**Theorem 15** Let \( 1/2 < V < \sqrt{3}/2 \).

(a) If \( V \neq V_2 \), then the spectral expansion for \( L(V) \) has the form

\[
 f(x) = \frac{1}{\pi} \sum_{k \in \mathbb{N}} \left( \int_{\Omega_k} \left( \phi(x, \lambda) \right) \frac{1}{p(\lambda)} d\lambda \right), \tag{57}
\]

where

\[
 \phi(x, \lambda) = \theta' h(\lambda) \varphi(x, \lambda) + \frac{1}{2} (\theta - \varphi')(h(\lambda)\theta(x, \lambda) + g(\lambda)\varphi(x, \lambda) - \varphi g(\lambda)\theta(x, \lambda),
\]

\[
 h(\lambda) = \int_{-\infty}^{\infty} \varphi(x, \lambda)f(x) dx, \quad g(\lambda) = \int_{-\infty}^{\infty} \theta(x, \lambda)f(x) dx, \quad p(\lambda) = \sqrt{1 - F^2(\lambda)},
\]

\[
 \varphi = \varphi(\pi, \lambda), \quad \varphi' = \varphi'(\pi, \lambda), \quad \theta = \theta(\pi, \lambda) \text{ and } \theta' = \theta'(\pi, \lambda).
\]

(b) If \( V = V_2 \), then the spectral expansion for \( L(V) \) has the following form

\[
 f(x) = \frac{1}{\pi} \text{p.v.} \left( \int_{\Omega_1} \left( \phi(x, \lambda) \right) \frac{1}{p(\lambda)} d\lambda \right) + \frac{1}{\pi} \sum_{k=3}^{\infty} \left( \int_{\Omega_k} \left( \phi(x, \lambda) \right) \frac{1}{p(\lambda)} d\lambda \right), \tag{58}
\]

where p.v. integral over \( \Omega_1 \) is the limit of the integral over \( \Omega_1(\varepsilon) \) as \( \varepsilon \to 0 \).

Now instead of Theorem 12 using Theorem 13 and Remark 5 and repeating the proof of Theorems 14 and 15 we get the following spectral expansion of \( L(V) \) for all \( V > 1/2 \).

**Theorem 16** Suppose that \( V > 1/2 \).

(a) If \( V \neq V_k \) for \( k \geq 2 \), then the spectral expansion for \( L(V) \) has the form (11). Moreover, the spectral expansion for \( L(V) \) in term of \( \lambda \) has the form (57).
Remark 5 the operator differences of these cases and the properties of the critical case. By Theorems 12 and 13 and of case

conclusions

1 cases, since the unbounded functions the unbounded function.

L cases the operator

of the nonintegrability of 1 spectral operator. Note that by Definition 1 the spectral singularities is connected by the

V and the critical points . Moreover, one can obtain a spectral expansion without parenthesis and 2 the spectral decomposition of the operator

k > 1/2 and the critical points 2 < 3 < ... are defined in Remark 5. Then we illustrate the changes of the spectrum and spectral expansion of \( L(V) \) when \( V \) changes from 1/2 to \( \infty \) and give some conjectures.

Similarities: By Theorem 10 in the both cases the operator \( L(V) \) has infinitely many spectral singularities and spectral singularity at infinity. Hence by Summary 2 in the both cases the operator \( L(V) \) is not an asymptotically spectral operator and hence is not a spectral operator. Note that by Definition 1 the spectral singularities is connected by the boundlessness of the function \( \frac{1}{m} \) and in the both (critical and noncritical) cases they are the unbounded function.

Differences: The detailed investigation of the spectral singularities, namely, the study of the nonintegrability of \( \frac{1}{m} \) helps us to see the differences between the critical and noncritical cases, since the unbounded functions \( \frac{1}{m} \) may become as integrable as well as nonintegrable. It was the reason to introduce the new types of spectral singularities called as ESS (see Definition 2) which is defined by the nonintegrability of the functions \( \frac{1}{m} \). Thus the study only the boundlessness of \( \frac{1}{m} \) or equivalently of the projections \( e(\lambda_m(t)) \) does not explain the

differences of these cases and the properties of the critical case. By Theorems 12 and 13 and Remark 5 the operator \( L(V) \) in the critical case \( V = V_k \) has ESS, while in the noncritical case \( V \neq V_k \) for \( k = 2, 3, \ldots \) has no ESS. Moreover in the critical case \( V = V_k \) the existence of the ESS does not allow to be the spectral decomposition of the operator \( L(V_k) \) in the elegant form (11) (see Theorems 14 and 16), while the spectral expansion of the operator \( L(V) \) for \( V \neq V_k \) has an elegant form. Note that in the critical case \( V = V_2 \) the spectral decomposition of the operator \( L(V) \) has no elegant form, because the functions \( a_1(t)\Psi_{1,t} \) and \( a_1(t)\Psi_{1,t} \) have nonintegrable singularities at \( t = 0 \). Similarly in the critical cases \( V = V_k \) for \( k > 2 \) the spectral decomposition of the operator \( L(V_k) \) has no elegant form, because the functions \( a_n(t)\Psi_{n,t} \) for \( n = 1, 2, \ldots, m \) has nonintegrable singularities either at \( t = 0 \) or at \( t = \pi \). However their sum is integrable over \((-\pi, \pi]\). It means that, in the spectral expansions (56) and (59) the bracket comprising the functions with indices \( n = 1, 2 \)

and \( n = 1, 2, \ldots, m \) respectively is necessary. Note also that, if we consider the spectral expansion in term of \( \lambda \), then we need to use the p.v. integral about ESS, since the integrals about this point do not exist. We do not need the p.v. integral if and only if \( V \neq V_k \) for \( k = 2, 3, \ldots \) . Moreover, one can obtain a spectral expansion without parenthesis and p.v. integrals if and only if \( V \neq V_k \) for \( k = 2, 3, \ldots \), that is, only in the noncritical case. It is a principle difference between the noncritical and critical cases.

Now we consider the changes of the spectrum and spectral expansion of \( L(V) \) when \( V \) changes from 1/2 to \( \infty \). First we assume that the following two principles formulated as conjectures hold.

(b) If \( V = V_k \), then the operator \( L(V) \) has the spectral expansion

\[
f(x) = \frac{1}{2\pi} \left( \int_{[-\pi,\pi]} \sum_{n=1}^{m(k)} a_n(t)\Psi_{n,t}(x) \right) dt + \sum_{n=m(k)+1}^{\infty} \int_{[-\pi,\pi]} a_n(t)\Psi_{n,t}(x) dt ,
\]

where \( m(k) \) is defined in Remark 5. Moreover, the spectral expansion for \( L(V_k) \) in term of \( \lambda \) can be obtained from (58) by replacing \( \Omega_1 \) and \( \sum_{k=3}^{\infty} \) with \( \bigcup_{n=1}^{m(k)} \Gamma_n \) and \( \sum_{n=m(k)+1}^{\infty} \) respectively.
Conjecture 1 (Leaving principle.) If \( c \) increases from 0 to \( \infty \), then all Bloch eigenvalues of the operator \( H(c) \) with the pure imaginary potential \( 2ic\cos 2x \) leave the real line. In other words, if \( V \) increases from \( 1/2 \) to \( \infty \), then all Bloch eigenvalues of \( L(V) \) leave the real line.

Conjecture 2 (Irreversibility principle.) The Bloch eigenvalues never came back to the real line after leaving it.

The discussion of the Leaving principle. Conjecture 1 was proved in [22] for the Bloch eigenvalues \( \lambda_1(t, V) \) and \( \lambda_2(t, V) \). Moreover, in [22] we have proved that if \( V \) moves from \( 1/2 \) to \( V_2 \), where \( V_2 \) is the second critical point, then the all antiperiodic eigenvalues \( \lambda_n(1, V) \) leave the real line while the periodic eigenvalues \( \lambda_n(0, V) \) moves over real line (see Summary 1). Since \( \lambda_n(t, V) \) continuously depends on \( t \) after antiperiodic eigenvalues \( \lambda_n(1, V) \) the Bloch eigenvalues \( \lambda_n(t, V) \) for \( t \) close to 1 leave the real line.

The periodic eigenvalues do not leave the real line for \( V \in (1/2, V_2) \) due to the following. By Theorem 2(c) if the eigenvalue \( \lambda(t, V) \) changes from real to nonreal when \( V \) moves from the left to the right of the constant \( V(t) > 1/2 \), then \( \lambda(t, V(t)) \) is a multiple eigenvalue, where \( t \) is a fixed number from \([0, 1]\). The first periodic eigenvalues is simple for \( V \in (1/2, V_2) \) and hence it can not leave the real line until being the multiple eigenvalue. As was proven in [22] (see Case 1 and Summary 1 in the introduction), if \( V \) moves from the left to the right of \( V_2 \) then the first and the second eigenvalues get close to each other for \( V < V_2 \), the equality \( \lambda_1(0, V_2) = \lambda_2(0, V_2) \) holds and they leave the real line for \( V > V_2 \). Moreover the last equality is possible since both \( \lambda_1(0, V_2) \) and \( \lambda_2(0, V_2) \) are the PN eigenvalues (see Summary 9 and (15)).

Now we show that the eigenvalue \( \lambda_3(0, V) \) moving over real line may become the double eigenvalues if and only if \( \lambda_3(0, V) = \lambda_4(0, V) \). If \( V \in (V_2, \sqrt{5}/2) \), then on the left side of \( \lambda_3(0, V) \) there is not a real periodic eigenvalue, since \( \lambda_1(0, V) \) and \( \lambda_2(0, V) \) are nonreal eigenvalues (see Summary 1). Moreover by Summary 9 and Theorem 8, \( \lambda_3(0, a) \) and \( \lambda_4(0, a) \) are PD eigenvalues while \( \lambda_5(0, a) \) is a PN eigenvalue and lies on the right of \( \lambda_4(0, a) \). Thus by moving over real line the eigenvalue \( \lambda_3(0, a) \) have no possibility to coincide with the eigenvalues lying on the left of \( \lambda_3(0, a) \) and \( \lambda_4(0, a) \) have no possibility to coincide with the eigenvalues lying on the right of \( \lambda_4(0, a) \), since PN and PD eigenvalue are different (see (15)). Using these arguments and taking into account that all Bloch eigenvalues sooner or later leave the real line we conclude that there exists \( V_3 \) such that \( \lambda_3(0, V_3) = \lambda_4(0, V_3) \). Moreover by Summary 1 we have \( V_3 > \frac{\sqrt{5}}{2} > V_2 \). In the same way we conclude that there exist a number \( V_k \) such that if \( 1/2 < V < V_k \), then \( 2(k-3) \)-th and \( 2(k-2) \)-th periodic eigenvalues \( \lambda_{2k-3}(0, V) \) and \( \lambda_{2k-2}(0, V) \) are real numbers, \( \lambda_{2k-3}(0, V) = \lambda_{2k-2}(0, V) \in \mathbb{R} \), and both of them are nonreal respectively. Moreover Theorem 3(a) implies that \( V_k \to \infty \) as \( k \to \infty \). Thus relying on these arguments, Theorem 3(a) and Theorem 13 we believe that the following conjecture holds.

Conjecture 3 There exists a sequence \( V_2 < V_3 < \ldots \) of real numbers, called the critical points, approaching infinity such that the following hold. If \( V_k < V < V_{k+1} \), then \( \lambda_1(0, V), \lambda_2(0, V), \ldots, \lambda_{2k-2}(0, V) \) are the nonreal eigenvalues and \( \lambda_{2k-1}(0, V), \lambda_{2k}(0, V), \lambda_{2k+1}(0, V), \lambda_{2k+2}(0, V) \) are the real simple eigenvalues. If \( V = V_{k+1} \), then \( \lambda_1(0, V), \lambda_2(0, V), \ldots, \lambda_{2k-2}(0, V) \) are the nonreal eigenvalues, \( \lambda_{2k-1}(0, V) = \lambda_{2k}(0, V) \) is a real double eigenvalue and \( \lambda_{2k+1}(0, V), \lambda_{2k+2}(0, V) \) are the real simple eigenvalues.

These conjecture for \( k = 2 \) was proved in [22] and is explained in the introduction. Let us give some explanation for this conjecture. The perturbation \( 2ic_2 \cos 2x \) of norm \( 2c_2 \), where \( c_2 = \sqrt{4V_2^2 - 1} \) is enough to move the first and second eigenvalues 0 and 4 of the unperturbed operator \( H_0(0) \) so that \( \lambda_1(0, c_2) \) and \( \lambda_2(0, c_2) \) coincide, since \( 2c_2 > 2 \). However this perturbation is not enough to move the third and forth eigenvalue 4 and 16 of \( H_0(c) \).
so that the equality \( \lambda_3(0, c) = \lambda_4(0, c) \) holds for some \( c \in (0, 2) \) since the distance between 4 and 16 is 12. Therefore \( V_3 > V_2 \). The same explanation show that \( V_{k+1} > V_k \). Moreover, by Theorem 3(a) and Notation 1 \( \lambda_{2k}(0, c) \) is a real eigenvalue if \( c < k - 1 \). It implies that \( c_k \geq k - 1 \), where \( c_k \) is \( \sqrt{4V_k^2 - 1} \), and hence \( V_k \to \infty \) as \( k \to \infty \).

Now let us explain why the Bloch eigenvalues \( \lambda_n(t, V) \) leave the real line for all \( t \in (0, 1) \). In [22] it was proved if \( 1/2 < V < V_2 \), then \( \text{Re}(\Gamma_{2k-1} \cup \Gamma_{2k}) = [\lambda_{2k-1}(0), \lambda_{2k}(0)] \) (see Case 1 in introduction). It implies that \( \lambda_{2k-1}(t) \) and \( \lambda_{2k}(t) \) lie between \( \lambda_{2k-1}(0) \) and \( \lambda_{2k}(0) \) if they are real numbers. On the other hand, by Conjecture 3 the eigenvalues \( \lambda_{2k-1}(0) \) and \( \lambda_{2k}(0) \) leave the real line when \( V \) moves to the right of \( V_{k+1} \). Therefore the eigenvalues \( \lambda_{2k-1}(t) \) and \( \lambda_{2k}(t) \) leave the real line for all \( t \in [0, 1] \) and hence as a result no parts of \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \) remain real when \( V \) moves to the right of \( V_{k+1} \). Moreover, it is clear that the eigenvalues \( \lambda_n(t, V) \) leave the real line with decreasing order of \( t \) from 1 to 0. These arguments and Theorem 6 imply the following conclusions about the spectrum of \( L(V) \).

**Conclusion 1** Suppose that conjectures 1-3 hold and \( V > 1/2 \). Then 
(a) For each \( k \geq 2 \) the spectrum of \( L(V_k) \) has the following properties: 

**Separation:** The bands \( \Gamma_{2s-3} \) and \( \Gamma_{2s-2} \) for \( s < k \) are nonreal separated curves symmetric with respect to the real line.

**Connection by the end point:** The bands \( \Gamma_{2s-3} \) and \( \Gamma_{2s-2} \) are connected by \( \lambda_{2s-3}(0) \), that is, \( \Gamma_{2s-3} \cap \Gamma_{2s-2} = \{ \lambda_{2s-3}(0) \} \).

**Connection by the interior point:** The bands \( \Gamma_{2s-3} \) and \( \Gamma_{2s-2} \) for \( s > k \) are connected by double eigenvalue \( \lambda_{2s-3}(t) = \lambda_{2s-2}(t) \), where \( t \in (0, \pi) \), that is, \( \Gamma_{2s-3} \cap \Gamma_{2s-2} = \{ \lambda_{2s-3}(t) \} \).

(b) If \( V_k < V < V_{k+1} \), then the **Separation** and **Connection by the interior point** \( \lambda_{2s-3}(t) = \lambda_{2s-2}(t) \) occurs for \( s \leq k \) and \( s > k \) respectively. In this case **Connection by the end points** does not occurs.

(c) The number \( V_k \) is a point after which no parts of the \( (2k - 3) \)-th and \( (2k - 2) \)-th bands are real. That is why it is natural to call it as the \( k \)-th critical point.

**Explanation of the conjectures by using the graph of the Hill discriminant** \( F \). First let us explain Summary 1 in term of the graph. In the case \( V < V_2 \) the first point of entry and the first point of exit of the graph \( G(F) \) defined in the proof of the Theorem 5 are \( (\lambda_1(0), 2) \) and \( (\lambda_2(0), 2) \) respectively. The part \( \{(\lambda, F'() : \lambda \in [\lambda_1(0), \lambda_2(0)] \} \) of the graph \( G(F) \), called as the first part, consists of the points \( (\lambda_1(t), 2 \cos t) \) and \( (\lambda_2(t), 2 \cos t) \) for real \( \lambda_1(t) \) and \( \lambda_2(t) \). If \( V \) approaches \( V_2 \) from the left then the first and second periodic eigenvalues get close to each other and the real eigenvalues \( \lambda_1(t) \) and \( \lambda_2(t) \) become nonreal numbers (see Case 1 of Case 3 in introduction). It means that the first part of the graph of \( F \) is leaving the strip \( S(2) \). In other word the graph of \( F \) is rising up. As \( V \) reaches \( V_2 \) we get the equality \( \lambda_1(0) = \lambda_2(0) \) which means that the first point of entry \( (\lambda_1(0), 2) \) and the first point of exit \( (\lambda_2(0), 2) \) coincides and the line \( y = 2 \) becomes tangent line to the curve \( y = F(\lambda) \) at the point \( (\lambda_1(0), 2) \). If \( V \) moves to the right of \( V_2 \), then the eigenvalues \( \lambda_1(0) \) and \( \lambda_2(0) \) get off the real line. Therefore the first part of the graph of \( F \) completely get out of the strip \( S(2) \).

In general Corollary 2 shows that for each \( k \) both the first coordinates of the \( k \)-th points of entry and exit are either PN or PD eigenvalues. Therefore they may get close to each other coincides and get out of the strip as \( V \) increases and moves from the left to the right of \( V_{k+1} \). It means that the \( k \)-th part of the graph of \( F \) get out of the strip \( S(2) \) and by Theorem 6(b) no parts of the \( (2k - 1) \)-th and \( 2k \)-th bands remain real.

Let us stress also the following. As is noted in above first the antiperiodic eigenvalues \( \lambda(1, V) \) leave real line and after antiperiodic eigenvalues the Bloch eigenvalues \( \lambda(t, V) \) for \( t \) close to 1 leave the real line. On the other hand, in the end of the proof of Theorem 6(a) we noted that the graph of \( F \) between the \( k \)-th point of entry and the \( k \)-th point of exit
has the form as it is sketched in the picture of Section 3 between the second points of entry and exit. It implies that when the points of entry and exit get close to each other, then the eigenvalues \( \lambda(t, V) \) leave the real line with decreasing order of \( t \) from 1 to 0. Finally the periodic eigenvalues live the real line. Moreover, Theorem 5 implies that if all periodic eigenvalues lying in the left of some number \( \mu \) leave the real number then all spectrum of \( L(V) \) on the left of \( \mu \) left the real line.

Now we discuss the spectral expansion by assuming that the following conjecture holds.

**Conjecture 4 (Simplicity principle.)** The nonreal periodic and antiperiodic eigenvalues are the simple eigenvalues.

Let us give some explanation for this conjecture. The nonreal periodic (antiperiodic) eigenvalues \( \lambda_{2k-1}(0) \) and \( \lambda_{2k}(0) \) (\( \lambda_{2k-1}(1) \) and \( \lambda_{2k}(1) \)) are complex conjugate numbers and by Conjecture 2 they do not came back to real line. That is why they can not coincide. On the other hand, Theorem 8 shows that if \( \lambda_{2k-1}(0) \) is PN (PD) eigenvalues lying in upper (lower) half-plane then the neighboring nonreal complex periodic eigenvalues \( \lambda_{2k-3}(0) \) and \( \lambda_{2k+1}(0) \) lying in the same half-plane are PD (PN) eigenvalues. Therefore \( \lambda_{2k-1}(0) \) can not be multiple eigenvalue due to the neighboring eigenvalues. Similarly it follows from Theorem 9 and Summary 1 that if \( \lambda_{2k-1}(1) \) is AD (AN) eigenvalues lying in upper (lower) half-plane then the neighboring nonreal antiperiodic eigenvalues \( \lambda_{2k-3}(0) \) and \( \lambda_{2k+1}(0) \) lying the same half-plane are AN (AD) eigenvalues. Therefore \( \lambda_{2k-1}(1) \) also can not be multiple eigenvalue due to the neighboring eigenvalues. On the other hand the overlapping of the unneibored eigenvalues \( \lambda_{2k-1}(1) \) and \( \lambda_{2k+3}(1) \) is the unlikely event, since they must pass through the \((2k+1)-th\) band \( \Gamma_{2k+1}(1) \) which is the continuous curve lying between these eigenvalues. That is why Conjecture 4 holds with high probability.

By Summary 1 and Conjectures 2 and 4 for \( V > 1/2 \) the operator \( L(V) \) has no multiple antiperiodic eigenvalues. Moreover by Summary 1 and Conjectures 3 and 4 if \( V \neq V_k \) for all \( k \geq 2 \) and \( V = V_{k+1} \) then \( L(V) \) has no multiple periodic eigenvalues and has only one double periodic eigenvalue \( \lambda_{2k-1}(0) = \lambda_{2k}(0) \) respectively. Therefore by Summary 4 and Theorems 11 and 16 the spectral expansion of \( L(V) \) has the form

**Theorem 17** Suppose that Conjectures 1-4 hold and \( V > 1/2 \). Then for each \( k \geq 2 \) the spectral expansion for the operator \( L(V_{k+1}) \) has the following form

\[
f = \frac{1}{2\pi} \left( \int_{(-\pi,\pi]} [a_{2k-1}(t)\Psi_{2k-1,t} + a_{2k}(t)\Psi_{2k,t}] dt + \sum_{n \in \mathbb{N} \setminus \{2k-1,2k\}} \int_{(-\pi,\pi]} a_n(t)\Psi_{n,t} dt \right)
\]

and

\[
f(x) = \frac{1}{\pi} \text{p.v.} \left( \int_{\Omega_k} \left( \phi(x,\lambda) \frac{1}{p(\lambda)} d\lambda \right) \right) + \frac{1}{\pi} \sum_{n \in \mathbb{N} \setminus \{2k-1,2k\}} \left( \int_{\Omega_k} \left( \phi(x,\lambda) \frac{1}{p(\lambda)} d\lambda \right) \right).
\]

If \( V \neq V_k \) for all \( k \geq 2 \), then the spectral expansion for \( L(V) \) has the elegant forms (11) and (57).

Thus if the conjectures hold, then the spectral decomposition of the operator \( L(V_{k+1}) \) has no elegant form, because the functions \( a_{2k-1}(t)\Psi_{2k-1,t} \) and \( a_{2k}(t)\Psi_{2k,t} \) has nonintegrable singularities at \( t = 0 \). It means that, in the spectral expansion (60) the bracket comprising the functions with indices \( 2k-1 \) and \( 2k \) corresponding to ESS \( \lambda_{2k-1}(0) = \lambda_{2k}(0) \) is necessary. Besides, if we consider the spectral expansion in term of \( \lambda \), then it is necessary to use the p.v. integral about ESS \( \lambda_{2k}(0) \), since the integrals about this point do not exist. That is
why only for the integral over the component \( \Omega_k = \Gamma_{2k-1} \cup \Gamma_{2k} \) containing the ESS \( \lambda_2(0) \) we use the p.v. integral (see (61)).

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