Analysis of a particle antiparticle description of a soliton cellular automaton

Taichiro Takagi
Department of Applied Physics, National Defense Academy,
Kanagawa 239-8686, Japan

Abstract

We present a derivation of a formula that gives dynamics of an integrable cellular automaton associated with crystal bases. This automaton is related to type $D$ affine Lie algebra and contains usual box-ball systems as a special case. The dynamics is described by means of such objects as carriers, particles, and antiparticles. We derive it from an analysis of a recently obtained formula of the combinatorial $R$ (an intertwiner between tensor products of crystals) that was found in a study of geometric crystals.

I INTRODUCTION

The crystal basis theory [10, 11] has played an important role in studies of solvable lattice models and integrable systems since more than a decade. In this context Ref. [9] by Kang, Kashiwara and Misra has provided useful families of crystal bases associated with affine Lie algebras $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. We call the first algebra type $A$ and the fourth one type $D$. In Ref. [3] we have obtained a formula of the combinatorial $R$ (an intertwiner of the crystals) associated with the type $A$ algebra. From the viewpoint of integrable systems an intriguing fact in Ref. [3] is that this formula has been derived from a discrete soliton equation (the nonautonomous discrete KP equation) by a procedure known as the ultradiscretization. Since then there has been progress which goes beyond the type $A$ case. In a study of geometric crystals associated with the type $D$ algebra we have obtained an explicit formula of a tropical $R$, an intertwiner of geometric crystals [13]. The tropical $R$ is a birational map between totally positive rational functions, while the combinatorial $R$ is a bijective map between finite sets. Further analysis of this tropical $R$ and the combinatorial $R$ derived from it should be an important task in studies of integrable systems, since they are connected with discrete and ultradiscrete soliton equations of type $D$ Lie algebra symmetry [14].

The purpose of this paper is to investigate a piecewise linear formula of the above mentioned combinatorial $R$ for the type $D$ crystals in Ref. [13]. Our main result is the derivation of the limit of the formula that leads to the particle antiparticle description of an integrable cellular automaton (Theorem 13). This description was recently obtained [15] by using a factorization of the combinatorial $R$ into Weyl group operators, a property that had been found and proved in Ref. [7]. We emphasize that the result proves a non-trivial fact that the factorization of the combinatorial $R$ can be conducted in two different ways, via the Weyl operator description and via the piecewise linear formula. This point is new even in the type $A$ case (Theorem 3).
We briefly explain the background to our problem. There were studies on one dimensional cellular automata known as the box-ball systems [17, 18, 19, 20, 21]. It was found that dynamics in these automata was controlled by the combinatorial $R$ of the type $A$ crystals [2, 3]. Based on the families of crystals in Ref. [9] integrable cellular automata associated with crystals of the other types were also constructed [5, 6]. A question about such generalized automata arose as to whether we could give a description of their dynamics as box-ball like systems. To answer this question the particle antiparticle description was found [8, 15].

In Ref. [15] it was found that the automata associated with crystal bases of any types of affine Lie algebra in Ref. [9] can be embedded into the type $D$ case. Thus one can obtain the particle antiparticle description of these automata from that of the type $D$ case. This is the reason why we devote ourselves into this particular case.

The plan of this paper is as follows. In Sec. II the automaton associated with the type $A$ crystals is reviewed. The piecewise linear formula of the combinatorial $R$ is presented. The particle description of the automaton is proved in terms of the piecewise linear formula. In Sec. III we discuss the piecewise linear formula of the combinatorial $R$ of the type $D$ crystals. Its reduction to the type $A$ case is also shown. The automaton associated with the type $D$ crystals is explained in Sec. IV. The formula of a factorized dynamics of an inhomogeneous automaton (Theorem 13) is reviewed. This formula is proved in Sec. V using the piecewise linear formula of the combinatorial $R$. Proofs of several lemmas are given in the Appendix.

II $A_{n-1}^{(1)}$ CASE

A Combinatorial $R$

We begin with type $A$ case. Instead of $A_{n}^{(1)}$ we adopt $A_{n-1}^{(1)}$ crystals because it enables us to compare the results with those in the $D_{n}^{(1)}$ crystals. For the notation we use overlines to distinguish the symbols from those in the type $D$ case, writing $\overline{B}$ for a crystal, $\overline{R}$ for the combinatorial $R$ and so on.

As a set the $A_{n-1}^{(1)}$ crystal $\overline{B}_l$ ($l$ is any positive integer) is given by

$$\overline{B}_l = \left\{ (x_1, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \left| \sum_{i=1}^n x_i = l \right. \right\}.$$  \hspace{1cm} (1)

The other properties of this crystal are available in Ref. [9]. In this paper we use no other property of $\overline{B}_l$. We will write simply $\overline{B}$ or $\overline{B}'$ for $\overline{B}_l$ with arbitrary $l$.

**Definition 1.** Given a pair of variables $x = (x_1, \ldots, x_n) \in \overline{B}$, $y = (y_1, \ldots, y_n) \in \overline{B}'$, let $\overline{R} : (x, y) \mapsto (x', y')$ be the piecewise linear map defined by $x' = (x_1', \ldots, x_n')$, $y' = (y_1', \ldots, y_n')$ where

$$x'_i = y_i + P_{i+1} - P_i,$$

$$y'_i = x_i + P_i - P_{i+1}.$$  

Here $P_i$ is given by

$$P_i = \max_{1 \leq j \leq n} \left( \sum_{k=1}^{j-1} (y_{k+i-1} - x_{k+i-1}) + y_{j+i-1} \right).$$ \hspace{1cm} (2)

The indices herein involved are interpreted in modulo $n$. 


Except for the notation this formula is the same one as defined in Proposition 4.1 of Ref. [3]. The normalization of (2) is so chosen as the $P_1$ to take the same expression as the formula in Theorem 5.1 of Ref. [9]. The property of $R$ to intertwine the actions of Kashiwara operators in the crystal basis theory was essentially proved in Sec. 1 of Ref. [13]. It ensures that the $(x', y')$ falls into $\mathcal{B}' \times \mathcal{B}$.

We note that there exist other ways to present the combinatorial $R$ (e.g. Refs. [4] and [16]) but which are not used in this paper.

\section{Automaton}

Now we consider the automaton. In Ref. [3] the space of automaton extends infinitely towards both ends

$$\cdots \times \mathcal{B}_{l_{i-1}} \times \mathcal{B}_{l_i} \times \mathcal{B}_{l_{i+1}} \times \cdots.$$  

For our purposes it is sufficient to consider a finite size system like

$$\mathcal{B}_{l_1} \times \cdots \times \mathcal{B}_{l_N}. \quad (3)$$

Let $L(\gg \sum_{i=1}^{N} l_i)$ be an integer. We call a particular letter for the ground state of the automaton a \textit{vacuum}. We can use any letter in \{1, \ldots, n\} as a vacuum of the automaton [15]. Throughout this paper we adopt $n$ as the letter for the vacuum. As in Ref. [7] we define

$$\mathcal{B}_L[n] = \left\{ (x_1, \ldots, x_n) \in \mathcal{B}_L \mid x_n \gg x_a \text{ for any } a \neq n \right\}.$$ 

We write $\mathcal{B} \times \mathcal{B}' \simeq \mathcal{B}' \times \mathcal{B}$ for correspondence by $R$. Take any $x \in \mathcal{B}_L[n]$. Applying $R$ successively we have

$$\mathcal{B}_L[n] \times (\mathcal{B}_{l_1} \times \cdots \times \mathcal{B}_{l_N}) \simeq (\mathcal{B}_{l_1} \times \cdots \times \mathcal{B}_{l_N}) \times \mathcal{B}_L[n], \quad x \times Y \mapsto X' \times y',$$

that gives the following.

\textbf{Definition 2.} The time evolution operator $\overline{T}$ of the automaton is given by

$$\overline{T} : Y \mapsto X'.$$

It means that we regard $Y$ and $X'$ in (4) as two automaton states before and after the time evolution. We note that the operator $\overline{T}$ actually depends on $x$.

\section{Particle description}

There is an interpretation of the automaton that we call a particle description. It is the description of the box-ball systems in Refs. [17, 18, 19, 20]. Suppose we have balls with index $a$ (1 \leq a \leq n - 1) that we call $a$-balls. For $x = (x_1, \ldots, x_n) \in \mathcal{B}_{l_i}$ we associate a box of capacity $l_i$ that has $x_a$ $a$-balls (1 \leq a \leq n - 1) in it. Then an element of $\mathcal{B}_{l_1} \times \cdots \times \mathcal{B}_{l_N}$ is regarded as a one dimensional array of boxes of capacities $l_1, \ldots, l_N$ with these balls. For any $a$ (1 \leq a \leq n - 1) we consider a carrier of $a$-balls that we call an $a$-carrier. We assume that the $a$-carrier has a sufficiently large capacity, so that it can carry arbitrary number of $a$-balls at a time.
First we suppose \( l_i = 1 \) for all \( i \). We call the associated automaton basic [15]. In this case the \( x \) represents a box with an \( a \)-ball if \( x_a = 1 \) for \( a \neq n \). It represents an empty box if \( x_n = 1 \). For any \( a \) we write \( a \) for \( x \) with \( x_a = 1 \). The carrier goes along the array of boxes. Then there are four actions in the loading-unloading process by the \( a \)-carrier:

1. If the carrier has at least one ball and meets an empty box, we unload a ball from the carrier and put it into the box.
2. If the carrier meets a box with an \( a \)-ball, we pick up the ball and load it into the carrier.
3. If the carrier meets a box with a \( b \)-ball (\( b \neq a \)), we do nothing.
4. If the carrier has no ball and meets an empty box, we do nothing.

These actions are depicted by the left four pictures in Fig. 1. For any \( a \) (\( 1 \leq a \leq n - 1 \)) let \( K_a \) be a particle motion operator that acts on the space of automaton (array of boxes) and does the actions in the loading-unloading process explained above. We assume that the \( K_a \) depends on \( x \) in (4) in such a way that the \( a \)-carrier has \( x_a \) balls in it at the beginning where \( x_a \) is the \( a \)-th element of the \( x \). Then for the basic automaton we have [2, 3]

\[
T = K_1K_2\ldots K_{n-1}.
\]

Now we consider a not necessarily basic case which we call inhomogeneous [15]. We denote by \( P \) the operator that reduces the automaton into a basic one, and by \( Q \) the operator that makes a rearrangement of balls [15]. To explain them we first let \( N = 1 \) in (4)

\[
\mathcal{B}_L[n] \times \mathcal{B}_L \simeq \mathcal{B}_1 \times \mathcal{B}_L[n].
\]

Then let

\[
P : \mathcal{B}_L \rightarrow \mathcal{B}_1 \times \mathcal{B}_1
\]

be the operator which sends \( y = (y_1, \ldots, y_n) \in \mathcal{B}_L \) into

\[
\begin{array}{cccc}
\square & \square & \cdots & \square \\
y_1 & y_2 & \cdots & y_n
\end{array}
\]

Its inverse \( P^{-1} \) can be defined only on such arrays in which the letters are arranged in decreasing order. Let

\[
Q : \mathcal{B}_1 \times \cdots \times \mathcal{B}_1 \rightarrow \mathcal{B}_1 \times \cdots \times \mathcal{B}_1
\]

be the operator which packs \( n \)'s into the left end. Next we consider the case \( N > 1 \). We insert walls between the \( \mathcal{B}_L \)'s in (3) to mark their positions. Then by \( P \) we denote the operator that applies the above \( P \) on each \( \mathcal{B}_L \), and by \( Q \) or \( P^{-1} \) those that applies the above \( Q \) or \( P^{-1} \) on each \( \mathcal{B}_1 \times \cdots \times \mathcal{B}_1 \) between the walls. Now we have [1, 15] the following.

**Theorem 3.** The time evolution operator of the inhomogeneous automaton is given by

\[
T = P^{-1}QK_1K_2\ldots K_{n-1}P.
\]

This theorem means that the time evolution of the inhomogeneous automaton can be reduced into that of the basic one only by inserting a simple rearrangement.
D Proof of the particle description

The reduction of an inhomogeneous automaton to a basic one (Theorem 3) was first presented by Fukuda [1]. We gave another proof of this theorem (and its generalization to type D case) in Ref. [15]. Here we show still another proof of this theorem that uses the piecewise linear formula in Definition 1. This is the proof that was inferred in Sec. II D of Ref. [3] but was not explicitly given there. Let \( p_i = \lim_{x \to \infty} P_i \). Then by (2) we have

\[
p_i = \max_{1 \leq j \leq n+1-i} \left( \sum_{k=1}^{j-1} (y_{k+i-1} - x_{k+i-1}) + y_{j+i-1} \right) .
\]

From Definition 1 and (6) we obtain

\[
p_n = y_n \quad \text{and} \quad p_i = \max\{y_i, y_i - x_i + p_{i+1}\},
\]

\[
x'_i = \min\{p_{i+1}, x_i\}.
\]

Note that the relation (7) is a descending recursion formula for \( p_i \)'s on \( i \). Let \( x \in B_L[n] \) and \( y \in B_l \) be a pair of variables. Let \( K_a \) be the particle motion operator introduced in Sec. II C. Then it is easy to see that the \( p_i \)'s (respectively \( x'_i \)'s) for \( 1 \leq i \leq n-1 \) obtained by (7) (respectively by (8) ) denote the number of empty boxes (respectively the number of boxes with balls with index \( i \)) in the automaton state \( K_i \ldots K_{n-1} P y \). We also see that the \( a \)-carrier finally has \( y'_a = x_a + y_a - x'_a \) balls in it. This proves the factorization of \( T \) in (5) for the case \( N = 1 \) in (4). The assertion of the theorem for the case \( N > 1 \) follows immediately by repeated use of this case, where we adopt the final states of the carriers for \( B_l_i \) in (3) as their initial states for \( B_{l+1} \).

III \( D^{(1)}_n \) COMBINATORIAL R

A Piecewise linear formula

As a set the \( D^{(1)}_n \) crystal \( B_l \) (\( l \) is any positive integer) is given by

\[
B_l = \left\{ (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1) \in \mathbb{Z}_{\geq 0}^n \left| x_n \overline{x}_n = 0, \sum_{i=1}^{n} (x_i + \overline{x}_i) = l \right. \right\} .
\]

The other properties of this crystal are available in a preprint version of Ref. [9] (Kyoto Univ., RIMS-887, 1992) or in, e.g., Ref. [12]. In this paper we use no other property of \( B_l \). We will write simply \( B \) or \( B' \) for \( B_l \) with arbitrary \( l \).

Definition 4. Let \( x = (x_1, \ldots, \overline{x}_1) \in B \), \( y = (y_1, \ldots, \overline{y}_1) \in B' \) be a pair of variables. The involutive automorphisms \( * \), \( \sigma_1, \sigma_n \) on \( x, y \) are defined by

\[
* : x_i \leftrightarrow \overline{y}_i, \overline{x}_i \leftrightarrow y_i \quad (1 \leq i \leq n),
\]

\[
\sigma_1 : x_1 \leftrightarrow \overline{x}_1, y_1 \leftrightarrow \overline{y}_1 ,
\]

\[
\sigma_n : x_n \leftrightarrow \overline{x}_n, y_n \leftrightarrow \overline{y}_n .
\]

For any function \( F = F(x, y) \) we denote by \( F^a \) the function obtained from \( F \) by applying \( a = (*, \sigma_1, \sigma_n) \) to it. For \( x \in B_l \) we write \( \ell(x) \) for \( l \).
Definition 5. Given a pair of variables \( x = (x_1, \ldots, x_k) \in B, y = (y_1, \ldots, y_l) \in B' \), let \( R : (x, y) \mapsto (x', y') \) be the piecewise linear map defined by \( x' = (x'_1, \ldots, x'_l), y' = (y'_1, \ldots, y'_l) \) where

\[
\begin{align*}
x'_1 &= y_1 + V_0^{\sigma_0} - V_1, \\
x'_2 &= y_2 + V_1 - V_2, \\
x'_{i+1} &= y_{i+1} + V_i - V_{i+1} - V_i^{\sigma_i} - V_{i+1} \quad (2 \leq i \leq n - 1), \\
\overline{y}'_i &= y_i - V_i - V_i^{\sigma_i}, \\
\overline{x}'_i &= x_i - V_i^{\sigma_i} - V_i \\
\overline{y}'_n &= x_n + V_n^{\sigma_n} - V_n.
\end{align*}
\]

Here \( V_i \) and \( W_i \) are given by

\[
\begin{align*}
V_i &= \max_{1 \leq j \leq n-1} \{ \alpha_{i,j}, \alpha'_{i,j} \}, \\
W_i &= \max \{ V_i + V_i^{\sigma_i} - y_i, V_{i-1} + V_i^{\sigma_i} - x_i \} + \min(x_i, y_i), \quad (1 \leq i \leq n - 2), \\
W_{n-1} &= V_n + V_n^{\sigma_n}.
\end{align*}
\]

The functions \( \alpha_{i,j} = \alpha_{i,j}(x, y) \) and \( \alpha'_{i,j} = \alpha'_{i,j}(x, y) \) in (11) are given by

\[
\begin{align*}
\alpha_{i,j}(x, y) &= \max(\delta_{j,n-1} \beta_i, y_j - x_j) + \begin{cases} \ell(x) + \sum_{k=j+1}^i (y_k - z_k) & \text{for } j \leq i, \\
\ell(y) + \sum_{k=i+1}^j (x_k - y_k) & \text{for } j > i, \end{cases} \\
\alpha'_{i,j}(x, y) &= \max(\delta_{j,n-1} \beta'_i, x_j - y_j) + \ell(x) + \sum_{k=1}^i (y_k - z_k) + \sum_{k=1}^j (y_k - x_k),
\end{align*}
\]

where

\[
\beta_i = \begin{cases} x_n - y_n & \text{for } i \neq n - 1, n, \\
0 & \text{for } i = n - 1, \\
\overline{x}_n - y_n & \text{for } i = n, \end{cases} \quad \beta'_i = \begin{cases} \overline{y}_n - x_n & \text{for } i \neq n - 1, n, \\
\max(y_n - 2x_n, \overline{y}_n - 2x_n) & \text{for } i = n - 1, \\
y_n - \overline{x}_n & \text{for } i = n.
\end{cases}
\]

The map \( R \) is the combinatorial \( R \) for the \( B_n^{(1)} \) crystals. The property of \( R \) to intertwine the actions of Kashiwara operators in the crystal basis theory was proved in Theorem 4.28 of Ref. [13]. It ensures that the \((x', y')\) falls into \( B' \times B \).

Remark 6. We have changed the notation from Ref. [13] since our present formalism uses both \( x_n \) and \( \overline{x}_n \). The changes are listed in Table I.

According to the correspondence in Table I one of the formulas in Eq. (4.66) of Ref. [13] is now translated into

\[
x'_n = y_n + \max(V_n - \overline{y}_n, V_n^{\sigma_n} - y_n) - V_n^{\sigma_n}, \\
\overline{x}'_n = \overline{y}_n + \max(V_n - \overline{y}_n, V_n^{\sigma_n} - y_n) - V_n.
\]

In order to make them coincide with the relations in (10) we should define the \( V_{n-1} \) as \( \max(V_n - \overline{y}_n, V_n^{\sigma_n} - y_n) \). Actually the above definition of \( V_{n-1} \) is equivalent to this. In other words we have the following.
Table I: The correspondence of the notation between in this paper and in Ref. [13] for the piecewise linear formula of the $D_n^{(1)}$ combinatorial $R$. The $z$ in the first column denotes $x, y, x'$ or $y'$. It is assumed that $i \neq n-1$, $n$ in the last two columns.

| This paper | Ref. [13] | This paper | Ref. [13] | This paper | Ref. [13] |
|------------|------------|------------|------------|------------|------------|
| $z_n$      | $\max(z_n, 0)$ | $V_n$      | $V_{n-1}$  | $\alpha_{i,j}(\neq n-1)$ | $\max(\theta_{i,j}, \eta_{i,j})$ |
| $\overline{z}_n$ | $\max(-z_n, 0)$ | $V_n^{\sigma_n}$ | $V_{n-1}^*$ | $\alpha'_{i,j}(\neq n-1)$ | $\max(\theta'_{i,j}, \eta'_{i,j})$ |
| $z_{n-1}$  | $z_{n-1} + \min(z_n, 0)$ | $V_{n-1}, V_{n-1}^*$ | $-\sigma_n$ | $\alpha_{i,n-1}$ | $\max(\eta_{i,n-1}, \eta_{i,n})$ |
| $\overline{z}_{n-1}$ | $\overline{z}_{n-1} + \min(z_n, 0)$ | $*$ | $*$ $\sigma_n$ | $\alpha'_{i,n-1}$ | $\max(\eta'_{i,n-1}, \eta'_{i,n})$ |

Lemma 7. The following relation holds:

$$V_{n-1} = \max(V_n - \overline{y}_n, V_n^{\sigma_n} - y_n). \quad (16)$$

Proof. We have

$$\alpha_{n-1,j} = \max(\alpha_{n,j} - \overline{y}_n, \alpha_{n,j}^{\sigma_n} - y_n),$$

$$\alpha'_{n-1,j} = \max(\alpha'_{n,j} - \overline{y}_n, (\alpha'_{n,j})^{\sigma_n} - y_n),$$

for $1 \leq j \leq n - 1$. In order to check these relations we can use $\max(-x_n, -\overline{x}_n) = \max(-y_n, -\overline{y}_n) = 0$. The claim of the lemma follows immediately from these relations.

Remark 8. The transformation properties of the piecewise linear functions $V_i$ and $W_i$ under the automorphisms $\sigma_1, \sigma_n$ will be used afterwards, so we list them in Table II. It was quoted from Ref. [13] and adjusted by the correspondence in Table I.

Table II: The transformation of the piecewise linear functions $V_i$ and $W_i$ in Definition 5 by the automorphisms $\sigma_1, \sigma_n$. It was quoted from Ref. [13] and adjusted by the correspondence in Table I.

| $\sigma_1$ | $V_0^{\sigma_1}$ | $V_i (1 \leq i \leq n - 1)$ | $V_n$ | $W_i (1 \leq i \leq n - 1)$ |
|------------|-----------------|-----------------|--------|-----------------|
| $V_0$      | $V_i$           | $V_n$           | $W_i$  |
| $V_i^{\sigma_n}$ | $V_i^{\sigma_n}$ | $V_n$           | $W_i$  |

Some more relations on the piecewise linear functions will be used later. We give them at the beginning of the Appendix.

B Reduction to the $A_{n-1}^{(1)}$ case

We realize that the piecewise linear formula in type $D$ case has a rather bulky expression in contrast with its type $A$ counterpart: See Definitions 1 and 5. In order to understand its structure it is worth trying to study some special limits of the formula. Here we consider a reduction to the type $A$ case. We observe that the piecewise linear map $R$ in Definition 5 for the $D_n^{(1)}$ crystals becomes the intertwiner of the $A_{n-1}^{(1)}$ crystals under the reduction.
Theorem 9. Set
\[ \pi_i = \overline{y}_i = 0 \quad \text{for} \quad 1 \leq i \leq n. \] (17)

Then the map \( R : (x, y) \mapsto (x', y') \) in Definition 5 reduces to
\[
\begin{align*}
  x'_i &= y_i + P_{i+1} - P_i, \\
  y'_i &= x_i + P_i - P_{i+1}, \\
  \pi'_i &= 0, \\
  \overline{y}'_i &= 0,
\end{align*}
\]
where \( P_i \) was defined by (2).

This theorem follows from Lemma 10 below.

Lemma 10. Under the specialization (17) the following relations hold
\[
\begin{align*}
  V_i &= \ell(x) + P_1, \quad V_i^* = \ell(x) + P_{i+1} \quad (0 \leq i \leq n), \\
  V_0^{\sigma_1} &= \ell(x) + P_2, \quad V_n^{\sigma_n} = \ell(x) + P_n, \\
  W_i &= 2\ell(x) + P_1 + P_{i+1} \quad (1 \leq i \leq n-1).
\end{align*}
\]

We shall give a proof of this lemma in Appendix.

We note that the reduction from type \( D \) to type \( A \) (Theorem 9) itself can also be obtained from the description of the combinatorial \( R \) in Ref. [4] since the insertion algorithms for types \( A \) and \( D \) in Ref. [4] coincide under the condition (17).

IV \( D_n^{(1)} \) AUTOMATON

A Definition

We now present a brief definition of the \( D_n^{(1)} \) automaton using the crystals and the combinatorial \( R \). For a more complete definition, see Refs. [5] and [6]. We consider a finite size system like
\[ B_{l_1} \times \cdots \times B_{l_N}. \]

Let \( L(\gg \sum_{i=1}^N l_i) \) be an integer. We define
\[ B_L[n] = \left\{ (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1) \in B_L \left| x_n \gg x_a \text{ for any } a \neq n \right. \right\}. \] (18)

Take any \( x \in B_L[n] \). Applying the combinatorial \( R \) successively we have
\[ B_L[n] \times (B_{l_1} \times \cdots \times B_{l_N}) \simeq (B_{l_1} \times \cdots \times B_{l_N}) \times B_L[n], \]
\[ x \times Y \mapsto X' \times y', \] (19)
that gives the following.

Definition 11. The time evolution operator \( T \) of the automaton is given by
\[ T : Y \mapsto X'. \]

Remarks similar to those after Definition 2 also apply here.
B Particle antiparticle description

We consider a particle antiparticle description of this automaton [8, 15]. This is a generalization of the particle description in Sec. III C. Suppose we have balls with index $a$ and $\overline{a}$, $(1 \leq a \leq n - 1)$ that we call an $a$-ball and an $\overline{a}$-ball respectively. The $a$-ball and the $\overline{a}$-ball are regarded as a particle and an antiparticle one another. We introduce a pair annihilation process in which a pair of particle and antiparticle makes a bound state, and a pair creation process where the bound state breaks up into a pair of particle and antiparticle of another kind. To each $x = (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1) \in B_l$ we associate a box of capacity $l_i$ that has $x_a$ $a$-balls, $\overline{x}_a$ $\overline{a}$-balls $(1 \leq a \leq n - 1)$, and $\overline{x}_n$ bound states in it. Then any element of $B_l \times \cdots \times B_l$ can be regarded as a one dimensional array of boxes of capacities $l_1, \ldots, l_N$ with the balls and the bound states. For any $a (1 \leq a \leq n - 1)$ we introduce the notion of an $a$-carrier as in Sec. III C, and that of a carrier for $\overline{a}$-balls that we call an $\overline{a}$-carrier. Assume that their capacities are sufficiently large.

First we consider a basic case, i.e. we suppose $l_i = 1$ for all $i$. In this case the $x$ represents a box with an $a$-ball if $x_a = 1$ and a box with an $\overline{a}$-ball if $x_a = 1$ for $a \neq n$. It represents an empty box if $x_n = 1$ and a box with a bound state if $\overline{x}_n = 1$. We write $[a]$ for $x$ with $x_a = 1$ and write $[\overline{a}]$ for $x$ with $\overline{x}_a = 1$.

Remark 12. In what follows we write $\overline{a}$ also for a number with an overline as well as that without an overline. We interpret $\overline{x}_a = x_a$ and $x_{\overline{a}} = \overline{x}_a$. We call $x_a$ the $a$-th element of $x$.

Besides the four actions in the loading-unloading process in Sec. III C we have three additional actions by the $a$-carrier:

5. If the carrier has at least one ball and meets a box with an $\overline{a}$-ball, we unload a ball from the carrier and make a bound state in the box.

6. If the carrier meets a box with a bound state, we extract an $a$-ball from the bound state, load it into the carrier, and leave an $\overline{a}$-ball in the box.

7. If the carrier has no ball and meets a box with an $\overline{a}$-ball, we do nothing.

These actions are depicted by the right three pictures in Figure 1.

For any $a \in \{1, \ldots, n - 1\} \cup \{n - 1, \ldots, 1\}$ (see Remark 12) let $K_a$ be a particle motion operator [8, 15] that acts on the space of automaton and does the actions in the loading-unloading process explained above. We assume that the $K_a$ depends on $x$ in (19) in such a way that the $a$-carrier has $x_a$ balls in it at the beginning, where $x_a$ is the $a$-th element of $x$. Then for the basic automaton we have [8]

$$T = K_{\overline{n-1}} \cdots K_{\overline{2}} K_2 K_1 \cdots K_{n-1}. $$

Now we consider an inhomogeneous case. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ as in Sec. III C but modify them to be suitable for the type $D$ case [15]. To define them we first set $N = 1$ in (19)

$$B_L[n] \times B_l \simeq B_l \times B_L[n]. $$

Then let

$$\mathcal{P} : B_l \to \underbrace{B_l \times \cdots \times B_l}_l$$

9
be the operator which sends \( y = (y_1, \ldots, y_n; \overline{y}_n, \ldots, \overline{y}_1) \in B_l \) into
\[
\begin{array}{cccccccc}
1 \times \cdots \times 1 & \times \cdots \times 1 \\
\overline{y}_l & \pi & \times \cdots \times \pi & \times \cdots \times \pi & \pi & \times \cdots \times \pi & \pi & 1 \times \cdots \times 1 \\
y_l & y_n & \cdots & y_n & y_n & \cdots & y_n & y_l
\end{array}
\]  
(21)

Its inverse \( P^{-1} \) can be defined only on such arrays in which the letters are arranged as in (21). Let
\[
Q : B_l \times \cdots \times B_l \to B_l \times \cdots \times B_l
\]
be the operator which packs \( \overline{n} \)'s into the left end and \( \pi \)'s into the right end. For the case \( N > 1 \) we generalize the definitions of these operators in the same way as in Sec. III C. Now we have the following.

**Theorem 13.** The time evolution operator of the inhomogeneous automaton is given by
\[
T = P^{-1}K_1 \cdots K_nQK_1K_2 \cdots K_{n-1}P.
\]  
(22)

In Ref. [15] a proof of this theorem was given by means of the factorization of the combinatorial \( R \) in Ref. [7]. In the remaining part of this paper we give another proof of this theorem that uses the piecewise linear formula of the combinatorial \( R \) in Definition 5.

**V PROOF OF THE PARTICLE ANTIPARTICLE DESCRIPTION**

**A Limit of the piecewise linear formula**

We study a limit of the piecewise linear formula of the map \( R \) in Definition 5. Let \( F = F(x, y) \) be any function of \( (x, y) \in B \times B' \). The limit we consider here is to adopt the \( B_l[n] \) in (18) as the \( B \). We introduce the following normalized limits:
\[
\begin{align*}
\lim_\ast F &= \lim_{x \to \infty, x_n \to 0} (F(x, y) - \ell(x)), \\
\lim^\ast_\ast F &= \lim_{x \to \infty, x_n \to 0} (F(x, y) - 2\ell(x)).
\end{align*}
\]  
(23)

(24)

First we consider the limit (23) of \( V_i \). For the sake of notational simplicity we denote \( \lim_\ast V_i \) by \( v_i \). For \( a = \sigma_1, \sigma_n \) or * we denote \( \lim^\ast_\ast V_i^a \) by \( v_i^a \). Note that if \( a = \sigma_n \) or * the \( v_i^a \) is not necessarily equal to the function that is obtained from \( v_i \) by applying \( a \) to it, whereas if \( a = \sigma_1 \) it is. Next we consider the limit (24) of \( W_i \). We shall denote \( \lim^\ast_\ast W_i \) by \( w_i \).

The relations in Lemma 23 in the Appendix become recursion relations in the limit (23). We let \( (x)_+ \) denote \( \max(x, 0) \).

**Lemma 14.** For \( F = V_i, V_i^\ast, V_0^\ast \) or \( V_n^\ast \) the limit \( \lim_\ast F \) exists. Moreover the following relations hold
\[
\begin{align*}
v_i^\ast &= v_i^{n-1} = y_n - \overline{y}_n, \\
v_i - 1 &= y_i - x_i + \max\{v_i^\ast, (x_i - \overline{y}_i)_+\}, \quad (1 \leq i \leq n - 1) \\
v_i &= \max\{\overline{y}_i - \pi_i + v_i - 1, (\overline{y}_i - x_i)_+\}, \quad (1 \leq i \leq n - 1) \\
v_n &= \overline{y}_n - y_n + \max\{y_n + \overline{y}_{n-1} - \pi_{n-1} + v_n - 2, (y_n + \overline{y}_{n-1} - x_{n-1})_+\}.
\end{align*}
\]  
(25)

(26)

(27)

(28)
We consider the limit (24) of the defining relations of $W_i$ ((12) and (13)).

**Lemma 15.** The following relation holds

$$w_i = v_i + v_i^* - \min\{v_i^* - v_{i-1} + y_i, v_i - v_{i-1} + \overline{y}_i\} + \min\{x_i, \overline{y}_i\} \quad (1 \leq i \leq n - 1).$$  

(29)

These lemmas will be used in proofs of Lemmas 20 and 21. We shall give their proofs in the Appendix.

**B Analysis of the particle antiparticle description**

We now consider a recursion formula satisfied by the numbers of items in the particle antiparticle description. For this purpose we introduce the following.

**Definition 16.** For any non-negative integers $A, B, C, D, E$, define the piecewise linear map

$$\gamma: (A, B, C, D, E) \mapsto (F, G, H, I, J)$$

by

$$F = \min(A, E),$$
$$G = B + (A - E)_+, $$
$$H = \min(C, B + (E - A)_+),$$
$$I = D + (C - B - (E - A)_+),$$
$$J = D + (B - C + (E - A)_+).$$

The identities $F + G = A + B$, $H + I = C + D$, and $F + H + J = B + D + E$ can be checked easily and will be used afterwards. We give an interpretation of the map $\gamma$ in the particle antiparticle description that is illustrated in Figure 2. Recall the seven actions in the loading-unloading process by the $a$-carrier (Figure 1) that were explained in Secs. II C and IV B. We write act-$i$ for the action with number $i$ in the lists. Note that the $a$ can represent an overlined number in Figure 1. In Figure 2 the boxes with $b$-balls ($b \neq a, \overline{a}$) have been omitted because of act-3. In the upper picture of Figure 2 we are applying act-5 (or act-7 if $E = 0$), act-6, act-1 (or act-4 if $B + (E - A)_+ = 0$), and act-2 from left to right. In the lower picture we are applying act-1 (or act-4 if $E = 0$), act-2, act-5 (or act-7 if $B + (E - A)_+ = 0$), and act-6.

In what follows we always assume

$$x_n \gg 0 \quad \text{and} \quad \overline{x}_n = 0.$$  

(30)

Let $B$ and $B'$ be the $D_n^{(1)}$ crystals.

**Definition 17.** For each pair of variables $x = (x_1, \ldots, x_{2n}) \in B, y = (y_1, \ldots, y_{2n}) \in B'$, define the set of variables $z(i), \overline{z}(i) (0 \leq i \leq 2n - 2), y_i, \overline{y}_i (1 \leq i \leq n - 1), x'_i, \overline{x}'_i, y'_i, \overline{y}'_i (1 \leq i \leq n)$ as follows.

1. Set

$$z(0) = \overline{y}_n, \quad z(0) = y_n.$$  

(31)
2. Define \( z(n-i), y_i^O, y_i' (1 \leq i \leq n-1) \) as
\[
\gamma(\varphi_y, z(n-1-i), y_i, x_i) = (\varphi_y, y_i^O, y_i', z(n-i), y_i')
\]
by descending recursion on \( i \). Here the function \( \gamma \) is given by Definition 16.

3. Define \( z(n-1+i), x_i', x_i', y_i' (1 \leq i \leq n-1) \) as
\[
\gamma(\varphi_x, z(n-1+i), x_i', x_i', y_i') = (x_i', \varphi_x, x_i', y_i', y_i')
\]
by recursion on \( i \).

4. Set
\[
x'_n = z(2n-2), \quad \varphi_n' = \varphi(2n-2), \quad y'_n = \ell(x) - \sum_{i=1}^{n-1}(y'_i + y'_i), \quad \varphi_n = 0.
\]

**Remark 18.** Let us consider the case when \( x \in B_L[n] \) and \( y \in B_l \). Then the numbers represented by the variables \( z(i) \) etc. in Definition 17 are equal to the numbers of items in the particle antiparticle description in Sec. IV B. More precisely these items appear within the time evolution process by \( T \) in Theorem 13 for the case \( N = 1 \). See Table III. In the table \( t_i \) and \( t_i' \) are defined as follows: For \( 1 \leq i \leq n-1 \) we let \( t_i \) (respectively \( t_i' \)) denote the time just after the \( i \)-carrier (respectively \( i' \)-carrier) has passed, where the automaton state is given by \( K_i \ldots K_{n-1}P_{iy} \) (respectively \( K_{i'} \ldots K_{1}Q_{K_{i'}} \ldots K_{n-1}P_{iy} \)).

| Variables | Items at time \( t_i \) | Variables | Items at time \( t_i' \) |
|-----------|----------------------|-----------|----------------------|
| \( z(n-i) \) | empty boxes | \( z(n-1+i) \) | empty boxes |
| \( \varphi_y(n-i) \) | boxes with bound states | \( \varphi_y(n-1+i) \) | boxes with bound states |
| \( y_i^O \) | boxes with \( i \)-balls | \( x_i' \) | boxes with \( i \)-balls |
| \( \overline{y}_i^O \) | boxes with \( \overline{i} \)-balls | \( \overline{y}_i \) | boxes with \( \overline{i} \)-balls |
| \( y_i' \) | balls in the \( i \)-carrier | \( y_i' \) | balls in the \( i' \)-carrier |

**C Proof**

We now give the proof of Theorem 13 that we have promised at the end of Sec. IV. It is obtained from the following.

**Theorem 19.** Let \( x = (x_1, \ldots, \varphi_1) \in B, y = (y_1, \ldots, \varphi_1) \in B' \) be a pair of variables, and suppose the condition (30) on \( x \). Let \( v_i, v_i', v_i^{0}, v_i^{n} \) be the functions defined in Sec. V A, and let \( x_i', \varphi_i, y_i', \varphi_i (1 \leq i \leq n-1) \) as
\[
\gamma(\varphi_y, z(n-1-i), y_i, x_i) = (\varphi_y, y_i^O, y_i', z(n-i), y_i')
\]
by descending recursion on \( i \). Here the function \( \gamma \) is given by Definition 16.
i ≤ n) be the variables given by Definition 17. Then the following relations hold

\[
\begin{align*}
x'_1 &= y_1 + v''_{01} - v_1, \\
x'_j &= y_j + v_{i-1} - v_i + w_i - w_{i-1} \quad (2 ≤ i ≤ n - 1), \\
x'_n &= y_n + v_{n-1} - v''_{n1}; \\
\overline{y}'_i &= \overline{y}_i + v_{i-1} - v_i \quad (1 ≤ i ≤ n), \\
y'_i &= x_i + v''_{i-1} - v'_i \quad (1 ≤ i ≤ n), \\
\overline{y}'_1 &= \overline{y}_1 + v''_{01} - v'_1, \\
\overline{y}'_i &= \overline{y}_i + v''_{i-1} - v'_i \quad (2 ≤ i ≤ n - 1), \\
\overline{y}'_n &= \overline{y}_n + v''_{n1} - v''_{n1}.
\end{align*}
\]

(35)

A proof of Theorem 19 will be given after the following two lemmas.

**Lemma 20.** Let \( A = \overline{y}_i, B = \min(x_{i+1}, \overline{y}_{i+1}), C = v'_i + \min(x_{i+1}, \overline{y}_{i+1}), D = y_i \) and \( E = x_i \) in \( \gamma : (A, B, C, D, E) \mapsto (F, G, H, I, J) \). Then

\[
\begin{align*}
F &= \min(x_i, \overline{y}_i), \\
G &= \overline{y}_i - \min(x_i, \overline{y}_i) + \min(x_{i+1}, \overline{y}_{i+1}), \\
H &= - \min(x_i, \overline{y}_i) + \min(x_{i+1}, \overline{y}_{i+1}) + y_i + v'_i - v''_{i-1}, \\
I &= v''_{i-1} + \min(x_i, \overline{y}_i), \\
J &= x_i + v''_{i-1} - v'_i,
\end{align*}
\]

for \( 1 ≤ i ≤ n - 1 \).

In what follows we set \( w_0 = 2v_0 \). Note that we have \( v'_0 = v_0 \) from Table II and \( w_1 = v_0 + v''_{01} \) from Lemma 22 in the Appendix.

**Lemma 21.** Let \( A = v_{i-1} + \min(x_i, \overline{y}_i), B = \overline{y}_i - \min(x_i, \overline{y}_i) + \min(x_{i+1}, \overline{y}_{i+1}), C = - \min(x_i, \overline{y}_i) + \min(x_{i+1}, \overline{y}_{i+1}) + y_i + v'_i - v''_{i-1}, D = \min(x_i, \overline{y}_i) + v_{i-1} + v''_{i-1} - w_{i-1}, \) and \( E = \overline{x}_i \) in \( \gamma : (A, B, C, D, E) \mapsto (F, G, H, I, J) \). Then

\[
\begin{align*}
F &= \overline{y}_i + v_{i-1} - v_i, \\
G &= v_i + \min(x_{i+1}, \overline{y}_{i+1}), \\
H &= \min(x_{i+1}, \overline{y}_{i+1}) + v_i + v'_i - w_i, \\
I &= y_i + w_i - w_{i-1} - v_i + v_{i-1}, \\
J &= \overline{x}_i + w_i - w_{i-1} - v''_{i-1} + v''_{i-1},
\end{align*}
\]

for \( 1 ≤ i ≤ n - 1 \).

We shall give proofs of these lemmas in the Appendix.

**Proof of Theorem 19.** Suppose \( i = n - 1 \) in Lemma 20. Then we have \( B = \overline{y}_n = z(0) \) and \( C = \overline{y}_n = z(0) \) because of (25), (30), and (31). Then by comparing (32) with Lemma 20 we see that \( \overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4, \overline{z}_5 \) should be equal to \( F, G, H, I, \) and \( J \). Thus the expression for \( y'_{n-1} \) was obtained. The expressions for \( y'_{i} \) \( 1 ≤ i ≤ n - 2 \), as well as those for \( \overline{z}_1, \overline{y}_i, z(i) \) will be obtained by descending recursion on \( i \), where one uses \( F \) and \( I \) as \( B \) and \( C \) in the next step.
Since $B$ and $C$ in Lemma 21 are equal to $G$ and $H$ in Lemma 20, we have $B = y_i^C$ and $C = y_i^C$ in Lemma 21. We also see that when $i = 1$ we have $A = z^{(n-1)}$ and $D = \min(x_1, y_1) = z^{(n-1)}$ in Lemma 21, from the result obtained in the preceding paragraph. Then by comparing (33) with Lemma 21 we see that $\bar{y}_1, z^{(n)}, \bar{x}_1$, and $y_1'$ in (33) are equal to $F, G, H, I$, and $J$ in Lemma 21 if $i = 1$. Thus the expressions for $x_1', \bar{x}_1, y_1'$ were obtained. The expressions for $x_i', \bar{x}_i, y_i'$ ($2 \leq i \leq n-2$) will be obtained by recursion on $i$, where one uses $G$ and $H$ as $A$ and $D$ in the next step.

Then from (34) we can obtain the expressions for $x_n'$ and $\bar{x}_n'$ in (35) since we have (30) and (25). It is clear that the relation $\bar{y}_n = 0 = \bar{x}_n + v_{n-1} - v_n^{\sigma_0}$ holds. Then the expression for $y_n'$ in (35) is obtained from the condition $\ell(y') = \ell(x)$. The proof is completed. \hfill \Box

Finally we give the proof of Theorem 13.

**Proof of Theorem 13.** If we impose the condition (30) on the defining relations (10) in Definition 5 then their right hand sides become those of (35) because of the existence of the limiting functions defined in Sec. V A. Then Theorem 19 tells that the numbers represented by $x_i', \bar{x}_i, y_i', \bar{y}_i'$ in Definition 5 are equal to those by the same symbols in Definition 17 under this condition. Then according to Remark 18 we see that the time evolution $T$ by Definition 11 is identical to the $T$ in Theorem 13 for the case $N = 1$. The assertion for the case $N > 1$ follows immediately by repeated use of this case, where we adopt the final states of the carriers for $B_l_1$ in (19) as their initial states for $B_{l_{i+1}}$.

\hfill \Box

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**APPENDIX : PROOFS OF THE LEMMAS**

Before giving the proofs we present some relations between piecewise linear functions. They are used in the main text and in Appendix. These relations have been essentially obtained in Ref. [13].

Ultradiscretization [18, 20, 21] is a procedure to derive an equation of piecewise linear functions from an equation of totally positive (i.e. having no minus sign) rational functions. It is realized as a transformation that replaces $+, \times$ and $/$ by $\max(\min)$, $+$ and $-$, respectively. As the ultradiscretization of Lemma 4.12 of Ref. [13] we have the following.

**Lemma 22.** The following relation holds

$$W_1 = V_0 + V_0^{\sigma_1}.$$  \hfill (A 1)

This lemma is used just above Lemma 21. The next lemma is obtained from the formulas (4.23)*,(4.23), and (4.24) of Ref. [13] by the ultradiscretization.

**Lemma 23.** The following relations hold

$$\max\{V_i, \ell(x), \ell(x) + x_i - \bar{y}_i\} = \max\{x_i - y_i + V_{i-1}, \ell(y), \ell(y) + x_i - \bar{y}_i\},$$  \hfill (A 2)

$$\max\{V_i, \ell(y), \ell(y) + \bar{y}_i - x_i\} = \max\{\bar{y}_i - \bar{x}_i + V_{i-1}, \ell(x), \ell(x) + \bar{y}_i - x_i\},$$  \hfill (A 3)

$$\max\{V_n, \ell(y) + X\} = \max\{\bar{y}_{n-1} + \bar{y}_n - \bar{x}_{n-1} - \bar{x}_n + V_{n-2}, \ell(x) + X\},$$  \hfill (A 4)

where $1 \leq i \leq n-2$ and $X = \bar{y}_n - y_n + (\bar{y}_{n-1} + y_{n-1} - \bar{x}_{n-1})$.

Here we write $(x)_+$ for $\max(x, 0)$. This lemma will be used in the proof of Lemma 14.
Proof of Lemma 10

Proof. We derive the expression for $V_i$. First we suppose $i \neq n - 1, n$. Then under the specialization (17) we have $V_i = \max_{1 \leq j \leq n-1} \{\alpha_{i,j}, \alpha_{i,j}'\}$ with

$$
\alpha_{i,j} = x_n \delta_{j,n-1} + \begin{cases} 
\ell(x) & \text{for } j \leq i, \\
\ell(y) & \text{for } j > i,
\end{cases}
\alpha_{i,j}' = \ell(x) + \sum_{k=1}^{j-1} (y_k - x_k) + y_j.
$$

Since $\alpha_{i,j(\leq i)} \leq \alpha_{i,1}'$ and $\alpha_{i,j(> i)} \leq \alpha_{i,n-1}$, we can drop off $\alpha_{i,j(\neq n-1)}$ in the max. Thus we obtain the desired result from (2) with $i = 1$. Now we suppose $i = n - 1$ or $n$. Then we have

$$
\alpha_{i,j} = \ell(x) + \begin{cases} 
0 & \text{for } i = n - 1, \\
\max(-y_n, -x_{n-1}) & \text{for } i = n,
\end{cases}
\alpha_{i,j}' = \ell(x) + \sum_{k=1}^{j-1} (y_k - x_k) + y_j + \delta_{j,n-1} \max(y_n - x_{n-1}, 0).
$$

Since $\alpha_{i,j} \leq \alpha_{i,1}'$, we can drop off $\alpha_{i,j}$ in the max and obtain the desired result.

We derive the expression for $V_i^*$. Note that if $i = 0, n$ it has already been proved since $V_i^* = V_i$ for $i = 0, n$. Suppose $i \neq 0, n$. Then under the specialization (17) we have $V_i^* = \max_{1 \leq j \leq n-1} \{\alpha_{i,j}', (\alpha_{i,j}')^*\}$ with

$$
\alpha_{i,j}^* = x_j + \begin{cases} 
\ell(y) + \sum_{k=j+1}^{i} (x_k - y_k) & \text{for } j \leq i, \\
\ell(x) + \sum_{k=i+1}^{j} (y_k - x_k) & \text{for } j > i,
\end{cases}
(\alpha_{i,j}')^* = \delta_{j,n-1} x_n + \ell(y) + \sum_{k=1}^{i} (x_k - y_k).
$$

Since $(\alpha_{i,j(\neq n-1)})^* \leq \alpha_{i,1}'$, we can drop off $(\alpha_{i,j(\neq n-1)})^*$ in the max. The remaining candidates are

$$
\begin{align*}
\alpha_{i,i+1}^* &= \ell(x) + y_{i+1}, \\
\alpha_{i,i+2}^* &= \alpha_{i,i+1}^* - x_{i+1} + y_{i+2}, \\
& \quad \vdots \\
\alpha_{i,n-1}^* &= \alpha_{i,n-2}^* - x_{n-2} + y_{n-1}, \\
(\alpha_{i,n-1}')^* &= \alpha_{i,n-1}^* - x_{n-1} + y_n, \\
\alpha_{i,1}^* &= (\alpha_{i,n-1}')^* - x_n + y_1, \\
\alpha_{i,2}^* &= \alpha_{i,1}^* - x_1 + y_2, \\
& \quad \vdots \\
\alpha_{i,i}^* &= \alpha_{i,i-1}^* - x_{i-1} + y_i.
\end{align*}
$$

Thus we obtain the desired result.
We derive the expression for $V_0^{\sigma_1}$. Under the specialization (17) we have $V_0^{\sigma_1} = \max_{1 \leq j \leq n-1} \{ \alpha_{0,j}^{\sigma_1}, (\alpha_{0,j}')^{\sigma_1} \}$ with
\[
\alpha_{0,j}^{\sigma_1} = \ell(y) + x_1 - (1 - \delta_{j,1}) y_1 + \delta_{j,n-1} x_n,
\]
\[
(\alpha_{0,j}')^{\sigma_1} = \ell(x) + \sum_{k=2}^{j} (y_k - x_k) + (1 - \delta_{j,1}) x_j.
\]

Since $\alpha_{0,j}(\neq n-1) \leq \alpha_{0,1}^{\sigma_1}$ and $(\alpha_{0,1}')^{\sigma_1} \leq (\alpha_{0,2}')^{\sigma_1}$, we can drop off $\alpha_{0,j}(\neq n-1)$ and $(\alpha_{0,1}')^{\sigma_1}$ in the max and obtain the desired result.

We derive the expression for $V_n^{\sigma_n}$. Under the specialization (17) we have $V_n^{\sigma_n} = \max_{1 \leq j \leq n-1} \{ \alpha_{n,j}^{\sigma_n}, (\alpha_{n,j}')^{\sigma_n} \}$ with
\[
\alpha_{n,j}^{\sigma_n} = \ell(x) + y_n - (1 - \delta_{j,n-1}) x_n,
\]
\[
(\alpha_{n,j}')^{\sigma_n} = \ell(x) + y_n - x_n + \sum_{k=1}^{j-1} (y_k - x_k) + y_j.
\]

Since $\alpha_{n,j}(\neq n-1) \leq (\alpha_{n,1}')^{\sigma_n}$, we can drop off $\alpha_{n,j}(\neq n-1)$ in the max and obtain the desired result.

The expression for $W_{n-1}$ is derived from (13), and that for $W_{i(\neq n-1)}$ is from (12) and the following lemma. 

**Lemma 24.** Let $P_i$ be the function that was defined by (2). Then
\[
P_{i+1} \geq P_i - y_i.
\]

**Proof.** It is easy to see that
\[
P_i - y_i = \max_{1 \leq j \leq n} \{ A_j \}, \quad P_{i+1} = \max_{2 \leq j \leq n+1} \{ x_i + A_j \},
\]
where
\[
A_j = \sum_{k=1}^{j-1} (y_{i+k} - x_{i+k-1}).
\]

The claim of the lemma holds since we have $x_i \geq 0$ and $x_i + A_2 = y_{i+1} \geq 0 = A_1$. 

**Proof of Lemma 14**

**Proof.** By definition we have $V_n^{\sigma_n} = \max_{1 \leq j \leq n-1} \{ \alpha_{n,j}^{\sigma_n}, (\alpha_{n,j}')^{\sigma_n} \}$, where
\[
(\alpha_{n,j})^{\sigma_n} = \max(\delta_{j,n-1}(x_n - \overline{y}_n), \overline{y}_j - x_j)
\]
\[
\quad + \ell(x) + \sum_{k=j+1}^{n-1} (\overline{y}_k - \overline{y}_j) + y_n - x_n,
\]
\[
(\alpha_{n,j}')^{\sigma_n} = \max(\delta_{j,n-1}(\overline{y}_n - x_n), x_j - \overline{y}_j)
\]
\[
\quad + \ell(x) + \sum_{k=1}^{j-1} (\overline{y}_k - \overline{y}_j) + \sum_{k=1}^{j} (y_k - x_k) + y_n - x_n.
\]
In the limit \( \lim_n \), the only element that survives in the max is \( (\alpha_{n,n-1})^\sigma_n \), which yields \( v_n^{\sigma_n} = y_n - \overline{y}_n \). In the same way the relation

\[
v_n^{\sigma} = y_n - x_n - \overline{y}_n + \left( x_n - \overline{y}_n + y_n \right)_+
\]  

(A 5)
can be obtained by a direct calculation. Then from (A 2) we see that the \( v_i^\sigma \)s exist and the relation (26) holds for \( 1 \leq i \leq n - 2 \) by descending induction on \( i \). Since \( V_0 = V_0^\sigma \) (Table II) we have \( v_0 = v_0^\sigma \). Then from (A 3) and (A 4) we see that \( v_i \)s exist for \( i \neq n - 1 \), and that the relations (27) for \( 1 \leq i \leq n - 2 \) and (28) hold, by induction on \( i \). Since \( v_n \) and \( v_n^{\sigma_n} \) exist, we see by (16) that the function \( v_n \) also exists and equals to \( \max \{ v_n - \overline{y}_n, - \overline{y}_n \} \). Substituting (28) into \( v_n = \max \{ v_n - \overline{y}_n, - \overline{y}_n \} \) we obtain (27) for \( i = n - 1 \). From \( \sigma \) of (16) we obtain \( v_n^\sigma = y_n - \overline{y}_n \). Then from (A 5) and \( \max \{ -y_n, - \overline{y}_n \} = 0 \) we obtain (26) for \( i = n - 1 \). From \( \sigma_1 \) of (A 2) the existence of \( v_0^{\sigma_1} \) can be verified. The proof is completed.

**Proof of Lemma 15**

**Proof.** For \( 1 \leq i \leq n - 2 \) the relations follow immediately from (12). We consider the case \( i = n - 1 \). We obtain \( w_{n-1} = v_n + v_n^{\sigma_n} \) from (13). First we suppose \( x_{n-1} \leq \overline{y}_{n-1} \). Then from Lemma 14 we obtain \( v_{n-1} = \overline{y}_{n-1} + \max \{ -\overline{x}_{n-1} + v_{n-2}, -x_{n-1} \} \). It yields the relations \( v_n + v_n^{\sigma_n} = v_{n-1} + y_n \) and

\[
\text{RHS of (29)} = v_{n-1} + y_n - \overline{y}_n + x_{n-1} - \min \{ x_{n-1} - \overline{y}_n, \overline{x}_{n-1} - v_{n-2} - v_{n-1} \}
\]

\[
= v_{n-1} + y_n - \min \{ 0, y_n + (\overline{y}_{n-1} - x_{n-1}) + (\overline{x}_{n-1} - x_{n-1} - v_{n-2})_+ \}
\]

\[
= v_{n-1} + y_n.
\]

Thus the assertion of the lemma was proved in this case. Now we suppose \( x_{n-1} > \overline{y}_{n-1} \). Then

\[
\text{RHS of (29)} = v_{n-1} + y_n - \overline{y}_n + \overline{y}_{n-1}
\]

\[
- \min \{ x_{n-1} - (x_{n-1} - \overline{y}_{n-1} - y_n)_+ - \overline{x}_{n-1} - v_{n-2} - v_{n-1} \}
\]

\[
= \max \{ (y_n + \overline{y}_{n-1} - x_{n-1})_+ + v_{n-1}, v_{n-2} + \overline{y}_{n-1} - \overline{x}_{n-1} + y_n - \overline{y}_n \}.
\]

From Lemma 14 we have \( v_{n-1} = (\overline{y}_{n-1} - \overline{x}_{n-1} + v_{n-2})_+ \). Therefore

\[
\text{RHS of (29)} = \max \left\{ (y_n + \overline{y}_{n-1} - x_{n-1})_+, \overline{y}_{n-1} - \overline{x}_{n-1} + y_n + v_{n-2} + \max \{ -y_n, - \overline{y}_n, \overline{y}_{n-1} - x_{n-1} \} \right\}.
\]

The last expression gives \( v_n + v_n^{\sigma_n} \) since the inner max vanishes. The proof is completed.

**Proof of Lemma 20**

**Proof.** The expressions for \( F \) and \( G \) are given by definition. The expression for \( I \) is given by \( I = C + D - H \), and that for \( J \) is by \( J = I + B - C + E - F \). Thus it suffices to prove \( H \). We have

\[
H = \min(C, B - E - F)
\]

\[
= \min(x_{i+1}, \overline{y}_{i+1}) + \min \{ v_i^\sigma, -\min(x_i, \overline{y}_i) + x_i \}
\]

\[
= \min(x_{i+1}, \overline{y}_{i+1}) + v_i^\sigma - \min(x_i, \overline{y}_i) + x_i - \max \{ v_i^\sigma, -\min(x_i, \overline{y}_i) + x_i \}.
\]

Then by (26) we obtain the desired result. \( \square \)
Proof of Lemma 21

Proof. The expression for $G$ is given by $G = A + B - F$, that for $I$ is by $I = C + D - H$, and that for $J$ is by $J = I + B - C + E - F$. Thus it suffices to prove $F$ and $H$. For $F$ we have

$$F = \min(A, E)$$
$$= \overline{y}_i + v_{i-1} + \min\{\min(x_i, \overline{y}_i) - \overline{y}_i, \overline{x}_i - \overline{y}_i - v_{i-1}\}$$
$$= \overline{y}_i + v_{i-1} - \max\{-\min(x_i, \overline{y}_i) + \overline{y}_i, -\overline{x}_i + \overline{y}_i + v_{i-1}\}.$$ 

Then by (27) we obtain the desired result. For $H$ we have

$$H = \min(C, B + E - F)$$
$$= \min(x_{i+1}, \overline{y}_{i+1}) - \min(x_i, \overline{y}_i) + \min\{v_{i}^* - v_{i-1}^* + y_i, v_i - v_{i-1} + \overline{x}_i\}.$$ 

Then by (29) we obtain the desired result. \qed

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Figure 1: The loading and unloading process by the carrier of balls with index $a$. A gray ball represents the bound state of a particle and an antiparticle. It is assumed that $b \neq a, \bar{a}$. 
Figure 2: The meaning of the map $\gamma$ in Definition 16 by the particle antiparticle description of the automaton.