Remarks on the Atick-Witten behavior and strings near black hole horizons

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Abstract

We present arguments pointing to a behavior of the string free energy in the presence of a black hole horizon similar to the Atick-Witten dependence on temperature beyond the Hagedorn transition. We give some evidence based on orbifold techniques applied to Rindler space and further support is found within a Hamiltonian approach. However, we argue that the interpretation in terms of a reduction of degrees of freedom is confronted by serious problems. Finally, we point out the problems concerning heuristic red-shift arguments and the local interpretation of thermodynamical quantities.
In attempts to better understand the nature of string theory, the behavior of strings at high energies \[1\] and high temperatures \[3\] has been investigated in recent years. In these regimes the fundamental degrees of freedom of the string are expected to show up and a recurrent feature in these investigations has been the fact that string theory at high energies seems to have a surprisingly low number of fundamental degrees of freedom, far fewer than expected in relativistic field theories.

There is still another scenario where the fundamental degrees of freedom of the string are expected to play a prominent role, i.e., the horizon of a black hole. Pioneering investigations by 't Hooft \[3\] showed that thermodynamical quantities like the entropy or the free energy of quantum fields diverge near the horizon. Since the presence of the horizon involves arbitrarily high frequencies, it is argued that a proper description must make reference to ultra-short distance physics where string theory is expected to play a fundamental role.

We will investigate whether the results of Atick and Witten \[2\], which showed that the high temperature behaviour of the free energy of the string presents a dependence on the temperature characteristic of two dimensional field theories, can be extended in the presence of a black hole horizon. Due to technical difficulties in quantizing the string in black hole backgrounds, the conclusions will not be completely air-tight, but different approaches will coincide in giving evidence of an Atick-Witten behavior. However, for reasons discussed below the statistical interpretation is not straightforward.

We start by briefly reviewing the analysis in Ref. \[2\]. The starting point is the closed bosonic string free energy \[2, 3\]

\[
F(\beta) = -\frac{V}{4\pi^2\alpha'} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \lambda(\tau) \sum_{m,n} \exp\left(-\frac{\beta^2|m\tau + n|^2}{4\pi\alpha'\tau_2}\right)
\]

where \(\lambda(\tau)\) is a modular invariant quantity given by

\[
\lambda(\tau) = \frac{|\eta(\tau)|^{-48}}{(4\pi^2\alpha'\tau_2)^{12}}
\]

and the integration region \(\mathcal{F}\) is the fundamental domain of the torus moduli space, \(-1/2 < \tau_1 < 1/2, |\tau| > 1, \tau_2 > 0\), which excludes the ultraviolet region \(\tau_2 \sim 0\). The term \(m = n = 0\) corresponding to the vacuum free energy is excluded from the sum.
The behavior for $\beta \to 0^+$ was analyzed in Ref. [2] by noting that, in that limit, the sums can be replaced by integrals,

$$\sum_{m,n} \to \beta^{-2} \int_{-\infty}^{\infty} dy_1 dy_2$$

with $y_1 = \beta m$ and $y_2 = \beta n$. One then easily finds

$$\beta F \simeq \frac{V}{\beta} 4\pi^2 \alpha' \Lambda$$

where $\Lambda$ is the one loop cosmological constant of the closed bosonic string. The latter is ill defined due to the presence of a tachyonic infrared divergence. A similar divergence is still present for the Type II superstring, showing that introduction of supersymmetry does not solve the problem. The origin of these divergences can be easily traced: the free energy Eq. (1) is ill defined beyond a certain value of $\beta$ due to the so-called Hagedorn singularity in the region $\tau_2 \to \infty$ [3, 2]. By examining the leading exponential behavior in this limit, we find an infrared divergence of the modular integral when $\beta \leq 4\pi\sqrt{\alpha'} \equiv \beta_c$ [3]. This is the Hagedorn instability.

Notice that the temperature dependence in Eq. (4) followed from simple formal power counting. Since a thermal dependence of this kind is what one would expect from a field theory in two dimensions, Atick and Witten have argued that, in spite of infrared singularities, this result can be interpreted to indicate a vast reduction of the fundamental degrees of freedom in string theory.

We now turn to study whether a similar behavior occurs in the presence of a black hole horizon. Our analysis will be carried out in Rindler space, which approximates the near vicinity of non-extremal black hole horizons, or equivalently, the geometry outside very large black holes (with vanishingly small curvature). The euclidean metric of Rindler space is

$$ds^2 = \xi^2 d\theta^2 + d\xi^2 + \sum_{i=1}^{D-2} dx_i^2.$$  

Here $\theta$ is the euclidean time to be periodically identified $\theta \sim \theta + \beta$, so that for $\beta \neq 2\pi$ a conical singularity is present. Sections $\xi = \text{const.}$ correspond to observers undergoing constant acceleration $a = 1/\xi$ or, in the black hole picture, at fixed distance above the
horizon, which is located at $\xi = 0$. The regularity requirement that $\theta \sim \theta + 2\pi$ corresponds to choosing the Hartle-Hawking thermal vacuum, which, in this approximation, is the Minkowski vacuum. It is the Hartle-Hawking state which corresponds to thermodynamic equilibrium with the (eternal) black hole. However, thermodynamical analysis requires considering general values $\beta \neq 2\pi$, as is clear, for instance, from the formula for the entropy

$$S = (\beta \frac{\partial}{\partial \beta} - 1)\beta F$$  \hspace{1cm} (6)

Now, quantization of string theory in the Rindler background with $\beta \neq 2\pi$ is far from straightforward. If we want to compute the free energy within an euclidean functional integral approach, we find that naive direct introduction of a conical singularity, e.g. by introducing a singular curvature term in the nonlinear sigma model, spoils conformal invariance. Consistent conical backgrounds for strings are obtained in the form of orbifolds \textsuperscript{[6]} and they have been applied to this problem in Refs. \textsuperscript{[7, 8]}.

The main distinguishing feature of the free energy in $Z_N$ orbifold string theory, as developed in Refs. \textsuperscript{[7, 8]}, is a double sum over twists in the nontrivial cycles of the torus corresponding to the fact that the winding number around the cone apex is conserved only modulo $N$, where $N = 2\pi/\beta$. Recall that the double sum over solitons in Eq. (4) was responsible for the high temperature dependence of the free energy. Remarkably, due to the double sum over twists (and after taking into account an additional factor $1/N$ coming from projection over $Z_N$-invariant states) one can expect a leading dependence of the one loop string amplitude like $A_N = -\beta F \sim N \sim \beta^{-1}$. In fact, in Ref. \textsuperscript{[7]} it has been noticed that, in the large $N$ limit, the one loop string amplitude $A_N = -\beta F$ behaves like $\sim N \log N$, i.e., $\sim \beta^{-1} \log \beta$, where the leading power factor comes from the double sum over twists. We want to point out the (previously unnoticed) similarity of this leading power dependence to the Atick-Witten behavior. It is likely that this dependence will hold when analytically continuing to non-integer $N$, but the discrete character of the sums makes it difficult to establish more precisely the relation and we will have to look for further evidence from other approaches.

We will follow now a more pedestrian Hamiltonian treatment. Since the current al-
gebra relations that determine the mass spectrum are invariant under field redefinitions, we expect the string spectrum in the Rindler background to be the same as in flat space. There could be a problem here if we wanted to regularize the horizon by cutting off a region $\xi < \xi_0$, because this introduces highly nontrivial complications (see Ref. [3] for a discussion). However, in the orbifold method one works in the whole conical space without introducing any cutoff. Therefore we will extend our discussion to distances arbitrarily close to the horizon $\xi = 0$, a way to proceed analogous to the limit $\beta \to 0^+$ considered in Ref. [2], and we will compute stringy quantities by summing over the already known string spectrum in flat space.

We are thus led to computing the free energy in a conical background. This can be conveniently performed by using heat kernel methods to write

$$F(\beta) = -\frac{1}{2\beta} \int_0^\infty \frac{ds}{s} \zeta(s) e^{-sm^2}$$

where $\zeta(s)$ is the heat kernel of the Laplacian in the Rindler geometry, Eq. (5). As argued above, we will obtain the string free energy by summing Eq. (7) over the string spectrum of oscillators. This is given by the mass formula

$$\frac{\alpha'}{2} m^2 = -2 + N + \tilde{N}$$

where $N$ and $\tilde{N}$ are left and right moving mode number operators. Additionally, the restriction $N = \tilde{N}$ must be imposed. The oscillator sum can be readily performed [4] and yields

$$F(\beta) = -\frac{1}{2\beta} \int_{-1/2}^{1/2} d\tau_1 \int_0^\infty \frac{ds}{s} \zeta(s) |\eta(\tau)|^{-48}$$

with $\tau = \tau_1 + i\tau_2$, $\tau_2 \equiv s/\pi\alpha'$. Here, the integral over $\tau_1$ is introduced as a means to implement the constraint $N = \tilde{N}$, and the dimension has been set to $D = 26$.

In the Rindler geometry of Eq. (5), the heat kernel $\zeta(s)$ factorizes into contributions from the $(D-2)$ dimensional transverse flat space and the two dimensional cone. The former yields the well known factor $V_{D-2}/(4\pi s)^{D/2-1}$, whereas the latter has been computed in Ref. [10] and is given by

$$\zeta_\beta(s) = \frac{\beta}{2\pi} \frac{A}{4\pi s} - \frac{1}{4\pi s} \int_0^\infty d\xi \xi \int_{-\infty}^{\infty} dw e^{-\xi^2 \cosh^2(w/2)/s} \cot \frac{\pi}{\beta}(\pi + iw)$$

\[10\]
The integrals can be done with the result that

\[ \zeta_\beta(s) = \frac{\beta}{2\pi} A - \frac{1}{12} \left( \frac{2\pi}{\beta} - \frac{\beta}{2\pi} \right) \]  

(11)

with \( A \) the area of the plane. At this point we must note that when computing the free energy, terms linear in \( \beta \) in the heat kernel correspond to zero-temperature contributions, \( F(\beta) = F^0 + O(\beta^{-1}) \) which, since they do not affect the thermal behavior, should be subtracted (they are automatically eliminated when computing the entropy using Eq. (6)). With these subtractions, the free energy is proportional to the transverse space ‘area’ and does not vanish for \( \beta = 2\pi \). Taking all this into account we eventually find the string free energy as

\[ \beta F(\beta) = -V \frac{\pi}{2\beta} \int_{E} \frac{d^2 \tau}{\tau_2^2} \lambda(\tau) \tau_2 \]  

(12)

where \( \lambda(\tau) \) is given by Eq. (2), and \( E \) is the strip \(-1/2 < \tau_1 < 1/2, \tau_2 > 0\).

The integrand is not modular invariant because of the factor \( \tau_2 \) but, following Ref. [5], we can formally perform a coset extension and rewrite the free energy as an integral over \( \mathcal{F} \) of modular invariant quantities,

\[ \beta F = -V \frac{\pi}{2\beta} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \lambda(\tau) \sum_{m,n} \frac{\tau_2}{m\tau + n} \]  

(13)

Remarkably, the \( \beta \) dependence in this expression is the same as in Eq. (4), and this raises some issues:

- It must be noted that \( \beta = 2\pi \) does not correspond to a high temperature, but this may not be a real problem since in any case the horizon involves fundamental aspects of strings. Note also that, since the local temperature \( T_\xi = (\xi \beta)^{-1} \) grows without limit as we approach the horizon, a local Atick-Witten behavior could be expected at distances very close to the horizon. However, the results above point to a different effect, namely, an Atick-Witten dependence on the global temperature. As discussed below, local interpretations are problematic.

- Although the orbifold calculation and the Hamiltonian method presented above are both based on conical backgrounds, the origin of the Atick-Witten dependence...
seems to be different in both cases. Actually, the $\beta$ dependence in Eq. (13) could be objected on the grounds that it directly follows from the conical heat kernel, Eq. (11), after subtraction of terms linear in $\beta$. Apparently, there are no stringy features here, in contrast to the stringy peculiarities of the orbifold technique. This leads us to a digression on the distinction between particles and strings as regards to the physics near horizons. The plausibility of the argument will rely more on physical arguments rather than on technical ones.

We start by recalling that the temperature dependence expected for a particle field theory is $\beta F \propto \beta^{1-D}$. Apparently, the bidimensional character of the $\beta$ dependence comes from the factorization of Rindler space into (cone) $\times$ (transverse space), so that the latter factor cannot not influence the $\beta$ dependence. However, as argued in Ref. [11], the correct $\beta$ dependence for particles is recovered after imposing a cutoff $\xi_0$ near the horizon, which alters the scaling properties of the heat kernel in the global space. The introduction of this cutoff is related to the expectation that the field theory description has to be radically altered near the horizon [3]. However, when considering strings we do not expect a more fundamental description to be necessary; instead, we need a more complete (non perturbative) understanding of the phase structure of string theory. This is needed here because the divergences in Eq. (13) appear in the infrared region $\tau_2 \to \infty$, where they are related to the Hagedorn instability. It is widely accepted that infrared problems, as opposed to ultraviolet ones, do not point to new underlying degrees of freedom, but rather to changes in the qualitative behavior of the system. If we had tried to perform the analysis above (i.e., by not introducing a horizon regularization) for particles, the divergences would be present in the ultraviolet region $s \to 0$. In this respect, we just want to point out that the remarks made in the analysis of Atick and Witten can be applied here, the small $\beta$ limit in Ref. [2] being replaced by the limit $\xi_0 \to 0$. In both cases we need non-perturbative knowledge to solve the problem of singularities. Notice that, according to the arguments above, the association of the heat kernel on the whole cone to the thermodynamics of Rindler space would
be physically sensible only for a putatively fundamental theory like string theory. That this yields a dependence on temperature that may be physically relevant is a remarkable fact.

- We now come to the problem of the interpretation in the black hole context of this global Atick-Witten dependence. Note that, since we have integrated all distances up to the horizon, the free energy that we have computed not only contains the contribution of high local-temperature degrees of freedom. This suggests that Eq. (13) does not measure the degrees of freedom in the usual sense of local quantum field theory and the interpretation of the $\beta$ dependence is not clear at all. This is somewhat reminiscent of the well-known difficulties in understanding in a statistical mechanical way the Bekenstein-Hawking entropy associated to black holes —notice that we could have equally rephrased Eqs. (11) and (13) in terms of the entropy, which is perhaps a more meaningful quantity. Although it may be that the nontrivial common features that we have found point to some universal property —perhaps in a spirit similar to the extension of thermodynamics to include black holes—, it is difficult to imagine a gedanken experiment to test in which sense the Atick-Witten dependence that we are discussing represents a low number of degrees of freedom of the string.

At this point we would like to compare the result of Eq. (13) with another approach recently applied to string theory in Rindler space [12, 9, 13]. This makes use of a heuristic ‘red-shift argument’ and involves two basic steps:

(i) Obtain local thermodynamical densities by dividing by the volume extensive quantities in flat space like the free energy or the entropy.

(ii) The basic assumption of the red-shift argument: local densities in Rindler space are equal to the ones in thermal flat space by an adequate red-shift of the local temperature, $\beta \rightarrow \sqrt{g_{00}} \beta = \xi \beta$.

With these premises, one can construct a free energy density in Rindler space out of the one in flat space, and then obtain the global free energy by integrating over Rindler coordinates. A free energy of this kind was first obtained in Ref. [9], and it can be readily
checked that when the $\xi$ coordinate is integrated from 0 to $\infty$ one finds the same result as Eq. (13).

However, there are problems concerning the validity of points (i) and (ii) above. The definition of local densities for strings is not as easy as point (i) assumes, since localization of strings presents serious obstacles [14]: confining strings in a box either yields an infinite result by breaking conformal invariance or introduces highly non-local behavior in the form of solitons wrapping around the box.

Additional problems concern point (ii). We want to stress that (ii) is just a heuristic assumption, and it is important to notice that this assumption on the local behavior of densities does not follow from the global thermal character of the Hartle-Hawking state. What we actually know is that Green's functions in Rindler space at temperature $1/2\pi$ are the same as the corresponding Green's functions of the Minkowski observer at $T = 0$. By no means it is immediate to conclude from here the local behavior of densities that the red-shift argument assumes.

These problems concerning the interpretation in terms of local quantities can be conveniently illustrated by noting that the red-shift arguments lead to a determination of the position of the local Hagedorn instability different from the one obtained from the conical approach above.

In order to find the position of the local singularity predicted by the conical approach we will consider the representation of the free energy in terms of the integral over the strip $E$, Eq. (9). It is well known [4, 5] that in the $E$ representation of the free energy the Hagedorn singularity appears in the region $\tau_2 \to 0$ (i.e., $s \to 0$). A close examination of this limit in the $\xi$-dependent integrand of the heat kernel in Eq. (10), together with the asymptotics $|\eta(\tau)|^{-48} \sim \exp(4\pi/\tau_2)$ would place the singularity at $\xi_{Hag} = 2\pi\sqrt{\alpha'} = \beta_c/2$. This is the value previously found in Ref. [15] by using a different approach.

In contrast, the heuristic red-shift arguments would place the Hagedorn transition at the point where the local inverse temperature $\beta \xi$ equals the critical value $\beta_c$, i.e., at $\xi'_{Hag} = \beta_c/\beta$ [8, 13], clearly different from the previous value.

Therefore, although the red-shift arguments may eventually lead to correct global
results, the underlying assumptions on the local behavior are quite dubious. An attempt to give a formal basis for the red-shift results can be found in Ref. [9] where, using a WKB approximation, a derivation of the red-shifted free energy is given. In any case, due to the extended character of the strings it is probably meaningless to refer to any local behavior, a feature strongly suggested by the ‘stretching effect’ of strings near the horizon [16]. We stress that neither the orbifold nor the Hamiltonian approach presented in this paper rely on local properties.

There are a number of points that have been left out throughout this work: (i) Interactions have been neglected. Their importance to the problem of Hawking radiation has been recently stressed in Ref. [17]. It remains to be seen how this influences the calculation of thermodynamical quantities. (ii) In Ref. [4] it has been argued that the Hagedorn singularity in thermal flat space at genus one means that we have not treated correctly the genus zero contribution. However, an effective field theory argument has been given in Ref. [8] that such genus zero condensate vanishes (to first order) in the Rindler background when $\beta = 2\pi$.

In this paper we have given some tentative arguments pointing to a behavior of the free energy of strings in the presence of black hole horizons similar to the high-temperature phase of string theory in flat space. In both cases, the resolution of infrared singularities seems to require non-perturbative knowledge of string theory. However, although independent orbifold and Hamiltonian approaches point to a similar behavior, they can not be considered as rigorous derivations and a sounder basis (e.g., by using an (unknown) proper off-shell formulation) would be required. Also, it seems difficult to interpret these results in terms of a counting of degrees of freedom, a situation somewhat reminiscent of well-known problems regarding the statistical interpretation of the Bekenstein-Hawking entropy. Finally, we have pointed out the problems of using arguments based on local properties to study thermodynamics in the presence of horizons.

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