We consider branched Josephson junction created by planar superconductors connected to each other through the Y-junction insulator. Assuming that the structure interacts with the external constant magnetic field, we study static sine-Gordon solitons in such system by modeling them in terms of the stationary sine-Gordon equation on metric graph. Exact analytical solution of the problem is obtained and their stability is analyzed.

Introduction. Low dimensional nanoscale materials are basic structures for many electronic devices. Optimization of their electronic properties and effective functioning of such devices require tuning the material properties and revealing most appropriate device architecture. This concerns also superconducting structures such as Josephson junctions. Remarkable feature of Josephson junctions is the fact that the phase difference at the junction is described in terms of the sine-Gordon equation (see, e.g. [1]-[7]). This makes them powerful testing ground for experimental realization of sine-Gordon solitons [8]-[15]. So far, different models have been proposed for the study of static and traveling solitons using Josephson junctions [16]-[22].

Among different realizations of Josephson junctions those having the discrete and branched structure is of special importance, as it allows to study soliton dynamics in discrete systems and networks. The early treatment of superconductor networks consisting of Josephson junctions meeting at one point dates back to [23]. An interesting realization of Josephson junction networks at tricrystal boundaries was discussed earlier in [24], which inspired later detailed study of the problem using the sine-Gordon equation on networks in [25-26]. Some versions of Josephson junction networks containing chain of the linear superconductors connected via the point-like insulators, have been studied on the basis of discrete sine-Gordon model [27]-[32]. From the viewpoint of practical applications, branched Josephson junction can be attractive for experimental realization of new version of the superconducting quantum interference devices (SQUID in networks), superconducting qubits in networks, as well as cold atom trapping in branched traps.

We note that the soliton dynamics in networks is becoming one of the hot topics in nonlinear and mathematical physics [25-27, 33-39]. Refs. [25-26] considered for the first time the sine-Gordon equation on branched domain for modeling Josephson junction at tricrystal surfaces. Integrable sine-Gordon equation on metric graphs is studied in [38-40]. Linear and nonlinear systems of PDE on metric graphs are considered in [47-49]. Despite the fact that different aspects of nonlinear wave equations on metric graphs are extensively studied, most of the publications are restricted by considering mathematical aspects of the problem, being far from the practical applications.

In this paper we address the problem of static solitons in branched Josephson junction containing planar superconductors connected to each other via the branched insulators having the shape of Y-junction. In such system the static solitons can be described in terms of the stationary sine-Gordon equation on metric graphs. For the boundary conditions at the branching point imposed as continuity of the derivative of wave function and local magnetic flux conservation, we obtain exact analytical solutions of the stationary sine-Gordon equation on metric graphs modeling branched Josephson junction. Particle and wave dynamics in branched Josephson junctions is more richer than that of in linear ones. From the viewpoint of practical applications, using branched Josephson junction instead of linear ones provides more effective tool for tuning of the functional properties of a device fabricated on the basis of such structures. This is caused by the fact that dynamics of quasiparticles and waves strongly depend on branching architecture and topology of a structure. The model we studied can be considered as branched version of its linear analog considered earlier in [20, 21]. Here we solve simplest, star-shaped branching. However, the approach we used can be utilized for arbitrary branching topology.

The paper is organized as follows. In the next section we give a formulation of the problem in terms of the sine-Gordon equation on metric graphs. Section 3 presents the derivation of exact analytical solutions for special cases and their stability analysis. Finally, Section IV presents some concluding remarks.

Modeling of branched Josephson junction in terms of metric graph. Consider the structure presented in Fig. 1a, which represents a Josephson junction consisting of three planar superconductors connected to each other via the branched insulator in the form of Y-junction. The whole system is assumed to interact with external constant magnetic field, \( H \) which is perpendicular to the plane of superconductors. Such structure can be con-
considered as the branched version of the Josephson junction considered in the Refs. [21, 22]. The structure can be modeled in terms of metric star graph having three branches, i.e., simple Y-junction (see, Fig. 1b). The phase difference on each branch \( \phi_j \), is given in terms of the stationary sine-Gordon equation on metric star graph [46]:

\[
\frac{d^2 \phi_j}{dx^2} = \frac{1}{\lambda_j^2} \sin(\phi_j), \quad 0 < x < L_j,
\]

where \( j = 1, 2, 3 \) is the bond (branch) number and the origin of coordinates is assumed at the branching point, \( O \). To solve this equation, one needs to impose boundary conditions at the branching point, \( O \). Such boundary conditions can be derived from the physical properties of the structure presented in Fig. 1a. Computing, at the branching point, the phase differences, \( \phi_2 = \theta_2 - \theta_1 \), \( \phi_3 = \theta_3 - \theta_2 \), \( \phi_3 = \theta_1 - \theta_3 \), where \( \theta_{1,2,3} \) are the phases on each superconductor, one can obtain first set of the vertex boundary conditions given by

\[
\phi_1|_{x=0} + \phi_2|_{x=0} + \phi_3|_{x=0} = 0.
\]

Local magnetic field in terms of \( \phi_j \) can be written as

\[
h_j(x) = \frac{\partial \phi_j}{\partial x}.
\]

The current density on each branch of the junction is given as [21, 52, 53]

\[
j_j(x) = \frac{1}{4\lambda_j^2} \sin \phi_j(x),
\]

while for the total current flowing though the whole junction we have

\[
J = \sum_{j=1}^{3} J_j,
\]

where \( J_j = \int_{0}^{L_j} j_j(x) dx \) or [52]

\[
J_j = \frac{1}{4} \left( \left. \frac{d\phi_j}{dx} \right|_{x=L_j} - \left. \frac{d\phi_j}{dx} \right|_{x=0} \right).
\]

Using continuity of the local magnetic field \( h_j(x) \) at the branching point \( (h_1(0) = h_2(0) = h_3(0)) \) we get the second set of vertex boundary conditions:

\[
\left. \frac{d\phi_1}{dx} \right|_{x=0} = \left. \frac{d\phi_2}{dx} \right|_{x=0} = \left. \frac{d\phi_3}{dx} \right|_{x=0}.
\]

Finally, the relation between the external magnetic field and the phase difference given by [21]

\[
H = \frac{1}{4} \sum_{j=1}^{3} \left. \frac{d\phi_j}{dx} \right|_{x=L_j} + \frac{1}{12} \sum_{j=1}^{3} \left. \frac{d\phi_j}{dx} \right|_{x=0},
\]

leads to the (Dirichlet) boundary conditions at the branch ends:

\[
\left. \frac{d\phi_1}{dx} \right|_{x=L_1} = H - J + 4J_1,
\]

\[
\left. \frac{d\phi_2}{dx} \right|_{x=L_2} = H - J + 4J_2,
\]

\[
\left. \frac{d\phi_3}{dx} \right|_{x=L_3} = H + J - 4J_3,
\]

The problem given by Eqs. (1), (2), (7) and (9) determines completely the problem of sine-Gordon equation on metric star graph, which is the model for the static solitons branched Josephson junction presented in Fig. 1a.

Exact solutions of Eq. (1) for the boundary conditions providing the absence of current-carrying states \( (J = 0) \), have been obtained in [46], where the stability of such solutions also was analyzed. Here we consider current-carrying states \( (J \neq 0) \) in the branched Josephson junction described in terms of Eqs. (1), for which the boundary conditions (2), (7) and (9) are imposed. Although the above formulation deals with simplest basic star graph, it can be extended for the star graph with arbitrary number of branches, as well as for the graph having arbitrary branching topology.

**Static soliton and their stability.** The problem given by Eqs. (1), (2), (7) and (9) have different types of solutions. However, only the stable solutions of this problem can be considered as the physical one. These latter describe the phase difference in branched Josephson junction in Fig. 1a. Therefore, following the Refs. [21, 22], we provide prescription for stability analysis for the solutions.
of Eq. (1). Starting point for such analysis is the Gibbs free-energy functional which can be written as [20, 21]

$$\Omega_G = \sum_{j=1}^{3} \Omega_G^{(j)} \left[ \phi_j, \frac{d\phi_j}{dx}; H, J_j \right],$$

where

$$\Omega_G^{(j)} \left[ \phi_j, \frac{d\phi_j}{dx}; H, J_j \right] = \frac{H^2 L_j}{2\pi^2} - (H \mp J \pm 4J_j) \times$$

$$\times \phi_j(L_j) + (H \mp J \pm 4J_j) \phi_j(0) +$$

$$+ \int_{0}^{L_j} \left[ \frac{1}{\lambda_j^2} (1 - \cos \phi_j(x)) + \frac{1}{2} \left( \frac{d\phi_j(x)}{dx} \right)^2 \right] dx.$$

Eq. (10) together with the boundary conditions (7)-(9) follows from the condition

$$\delta \Omega_G = 0.$$  (12)

Criteria for the stability of the problem (1), (2), (7) and (9), can be obtained from the second variation of $\Omega_G$ (which should be zero for stable solutions) that leads to the problem for finding the lowest eigenvalue, $\mu = \mu_0$, of the following Sturm-Liouville problem [20, 21, 23]:

$$-\frac{d^2 \psi_j}{dx^2} + \frac{1}{\lambda_j^2} \cos \phi_j(x) \psi_j = \mu \psi_j, \quad x \in (0; L_j),$$

$$\psi_j|_{x=0} + \psi_j|_{x=L_j} = 0,$$

$$\frac{d\psi_j}{dx}|_{x=0} = \frac{d\psi_j}{dx}|_{x=L_j} = 0, \quad j = 1, 2, 3.$$  (13)

Having found $\mu_0$ the criterion for stability of the solution can be formulated as follows. If $\mu_0 < 0$, the solution $\phi_j(y)$ corresponds to a saddle point of Eq. (11) which implies that the solution is absolutely unstable and unphysical. Stable (physical) solutions correspond to the case, when $\mu_0 > 0$, ($\delta^2 \Omega_G < 0$). The boundaries of the stability regions for these solutions is determined by the condition $\mu_0 = 0$ ($\delta^2 \Omega_G = 0$). This boundary is given by the following Sturm-Liouville problem:

$$-\frac{d^2 \tilde{\psi}_j}{dx^2} + \frac{1}{\lambda_j^2} \cos \phi_j(x) \tilde{\psi}_j = 0, \quad x \in (0; L_j),$$

$$\tilde{\psi}_j|_{x=0} + \tilde{\psi}_j|_{x=L_j} = 0, \quad j = 1, 2, 3.$$  (17)

Using Eqs. (14)-(17) one can determine explicitly the stability boundary for each type of solution of the problem given by Eqs. (1), (2), (7) and (9).

General solution of Eq. (11) can be obtained from the following first integral [20, 21]:

$$\frac{1}{2} \left( \frac{d\phi_j}{dx} \right)^2 = \cos \phi_j = C_j - 1 \leq C_j < \infty,$$  (18)

with $C_j$ being the integration constant. Depending on the value of $C_j$ this general solution can be determined as type I and II. Namely, for $C_j \in (-1, 1)$ we have solution of type I, while solution of type II corresponds to the values, $C_j \in [1, \infty)$. Both solutions for $H \neq 0$, and $J = 0$ have been found in [44] where it was shown that only the special case of the solution of type II is stable. Following the Refs. [20, 21], instead of $C_j$ we introduce new parametrization constant, $k_j$, which is defined, for the solution of type I as

$$k_j^2 = \frac{1 + C_j}{2}, -1 < k_j < 1,$$
FIG. 4: (Color online) The dependence $J_c = J_c(L)$ for $H = 0$ (solid line). The stability region is shaded, parameters are the same as in Fig. 2.

and

$$k_j^2 = \frac{2}{1 + C_j}, \quad -1 < k_j < 1,$$

for solution of type II. General (type I) solution of Eq. (1) can be written as [20, 21, 46]

$$\phi_j(x) = (2n_j + 1)\pi + 2 \arcsin \left \{ k_j \sin \left [ \frac{x}{\lambda} - x_0, k_j \right ] \right \} \quad (19)$$

where sn is Jacobian elliptic function, and $x_0$ are integration constants which obey the constraints given by the following inequality:

$$- \min K(k_j) < x_0 < \min K(k_j), \quad j = 1, 2, 3.$$

Fulfilling by Eq. (19) the vertex boundary conditions given by Eqs. (7) and (2) leads to the constraints in Eqs. 4 and (21). Stable solutions and the border between stability and unstable regions can be determined similarly to that for solution type I.

From Eqs. (6) and (8) we get the expressions for currents:

$$J_j^{(s)} = -\frac{k_c}{2\lambda} \left ( \text{cn} \left [ \frac{L_j}{\lambda} - x_0, k_c \right ] - \text{cn} \left [ x_0, k_c \right ] \right ), \quad (23)$$

$$H = -\frac{k_c}{2\lambda} \sum_{j=1}^{n} \text{cn} \left [ \frac{L_j}{\lambda} - x_0, k_c \right ] - \frac{k_c}{2\lambda} \text{cn} \left [ x_0, k_c \right ]. \quad (24)$$

Fig. 2 presents plot of $k_c$ as a function of the parameter, $L$ determined from $L_1 = L, L_2 = 2L, L_3 = 3L$. The left (colored) area of each plot corresponds to the stability region. Since $k_c$ appears as the value of $k$ at which the Sturm-Liouville (stability) problem has zero ($\mu_0 = 0$) eigenvalue, it is important to check, at which values of $x_0$ this is possible. Fig. 3 presents plot of $k_c$ as a function of $x_0$, i.e., the stability region of $\phi$ in the parametric plane. Colored area corresponds to the stability region.

Solution of type II can be treated similarly to that of type I, by considering two cases. The case $H > 0, J_j < 0$ has been studied in detail in the Ref. [46]. Therefore we drop this part. Here we will focus on the case $H > 0, J_j > 0$. General (type II) solution for this case can be written as

$$\phi_j(x) = \pi(2n_j + 1) + 2 \text{am} \left ( \frac{x}{\lambda k_j} - x_0, k_j \right ). \quad (25)$$

Fulfilling the boundary conditions given by Eqs. (7) and (2) leads to the constraints in Eqs. 4 and (21). Stable solutions and the border between stability and unstable regions can be determined similarly to that for solution type I.

From Eqs. (6) and (8) we get the expressions for cur-
FIG. 6: (Color online) Tree-like branched Josephson junction

rent and magnetic field

\[ J_j^{(c)} = \frac{1}{2\lambda_k} \left( \text{dn} \left[ \frac{L_j}{\lambda_k} - x_0, k \right] - \text{dn} [x_0] \right), \]

\[ H = \frac{1}{2\lambda_k} \sum_{j=1}^{3} \text{dn} \left[ \frac{L_j}{\lambda_k} - x_0, k \right] + \frac{1}{2\lambda_k} \text{dn} [x_0, k], \]

\[ x_0 \in [0; x_0, \gamma]. \]

In Fig. 4, dependence of current on the branch length, \( L_j \) is plotted. Colored (lower) parts corresponds to the the stability area. Fig.5 presents the plot of current, \( J_j \) as a function of the magnetic field. The colored area in each plot corresponds to the stability region, i.e., the stability region of \( \phi \) in the physical plane \( (J, H) \).

It is meaningful to compare the above results with those for their linear (unbranched) counterpart considered in the Ref.\([21]\). Comparing dependence of \( k_c \) on \( L \) in Fig.2 with corresponding plot from for linear case (corresponds to Fig.2 in \([21]\)) one can conclude that they are exactly the same. However, differences between linear and branched cases appear in the plots of \( J_j(L) \) and \( J_j(H) \) presented in Figs. 4 and 5, respectively. Comparing Fig. 3 with corresponding plot of Fig.3 from the Ref.\([21]\) one can find that the shape of the stability region border is the same with the linear case for the first branch, while shapes of second and third branches are completely different. However, the total area of stability regions for all three branches equal to that for linear case. Completely different (than that from the linear case) behavior of \( J_j \) can be observed by comparing Fig.5 with the corresponding plots for linear case. Cyclic dependence can be observed for branched case, while for linear case \( J \) decays by increasing of \( H \) (see, Fig.6 in the Ref.\([21]\)). The total area of the stability region for this case is much smaller than that for the linear case. All this shows that branched Josephson junction is more attractive from the viewpoint of tuning the device properties. Especially, this should be important for the case of more complicated branching architecture, e.g., junction with tree-like branching presented in Fig. 6. Static solitons in this structure can be modeled in terms of the sine-Gordon equation with the boundary conditions given in metric tree graph.

Conclusions. We have studied the current carrying states in branched Josephson junction interacting with external magnetic field. The structure is assumed to be constructed, from three planer superconductors connected to each other via the insulating (or normal metal) Y-junction. The system is modeled in terms of the stationary sine-Gordon equation on metric star graph, whose solutions describe the phase different in the junction. The boundary conditions for sine-Gordon equation at the branching points are derived. Explicit analytical solutions are obtained. The stability regions are determined in terms of the integration constant using the Gibbs functional based approach. Physical observable values of the current described in terms of the stable solutions are derived explicitly as a function magnetic field. Finally, we note that although we considered very simple branching having the form of Y-junction, the approach we used can be can be extended for modeling static solitons in more general branching architectures of the junction, such as tree, loop, triangle, etc. This can be done similarly to that in \([10]\), where sine-Gordon equation on metric graphs is solved for \( J = 0 \). Considering such complicated branching architectures is of importance form the viewpoint of the device tuning and optimization in such problems as SQUID, superconducting qubit, cold atom trapping and Majorana wire networks.

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