Strong Consistency of Nonparametric Bayesian Inferential Methods for Multivariate Max-Stable Distributions

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Predicting extreme events is important in many applications in risk analysis. The extreme-value theory suggests modelling extremes by max-stable distributions. The Bayesian approach provides a natural framework for statistical prediction. Marcon, Padoan and Antoniano [Electron. J. Stat. 10 (2016) 3310–3337] proposed a nonparametric Bayesian estimation method for bivariate max-stable distributions, representing the main (infinite dimensional) parametrizations of the dependence structure with polynomials in Bernstein form. In this article, we describe a similar inferential method, but which alternatively models the dependence structure by splines. Then, for both approaches we establish the strong consistency of the posterior distributions, under the main parametrizations of the dependence structure. Next, we describe an inferential framework that extends the Bernstein polynomials based approach to max-stable distributions in arbitrary dimensions (greater than two) and we derive the posterior consistency results also in this case. Initially, the consistency results are obtained assuming that the data follow a max-stable distribution with known margins. However, the latter only provides an asymptotic model for sufficiently large sample sizes and its margins are known, potentially, apart from some unknown parameters. Then, we extend the consistency results to the case where the data come from a distribution that is in a neighbourhood of a max-stable distribution and to the case where the margins of the max-stable distribution are heavy-tailed with unknown tail indices.

\textit{Keywords:} Bernstein polynomials, B-splines basis, Extreme-value copula, Multivariate max-stable distribution, Nonparametric estimation, Pickands dependence function, Posterior consistency.

1. Introduction

Predicting the extremes of multiple variables is important in many applied fields for risk management. For instance, when designing bridges in civil engineering it is crucial to quantify what forces they must sustain in the future, e.g. the maximum wind speed, maximum river level, etc. (e.g. Castillo et al., 2005, Ch. 9.3). In finance, the solubility of an investment is influenced by extreme changes in several assets in the financial market, such as
share prices, market indexes, currency values, etc. (e.g. Longin, 2016). The extreme-value theory develops several approaches for modelling multivariate extremes (e.g. Falk et al., 2011). In this paper we focus on the family of max-stable models which arises as a class of asymptotic distributions for suitably normalised componentwise maxima of random vectors (Falk et al., 2011, Ch. 4). Max-stable models have been successfully applied in several areas, e.g. in meteorological, environmental and insurance fields for analysing heavy rainfall, extreme temperatures, air pollution, clinical trials, insurance claims, etc. (e.g. Coles, 2001; Dey and Yan, 2016), in addition to those previously mentioned. In recent years, the popularity of some max-stable models is due to max-stable processes, which have been widely used in spatial applications (e.g. Blanchet and Davison, 2011; Davison et al., 2012; Asadi et al., 2015; Buhl and Klippelberg, 2016, to name a few).

The Bayesian approach provides a natural framework for statistical prediction. Establishing the consistency of the posterior distribution for the parameter of interest is informative for the robustness of the underlying Bayesian procedure (Ghosal and van der Vaart, 2017). The study of asymptotic properties in the nonparametric context can be challenging, however, in the last two decades several useful results for different interesting statistical problems have been obtained (among the most recent, see Ritov et al., 2014; Kleijn, 2017; Nickl, 2017; Nickl and Söhl, 2017; Ghosal and van der Vaart, 2017, and the references therein).

The Bayesian literature for univariate extremes is quite well developed (see e.g. Coles and Pericchi, 2003; Stephenson and Tawn, 2004; Beirlant et al., 2004; Stephenson and Tawn, 2016; Dey and Yan, 2016), while this is not the case for multidimensional extremes. There are two main reasons for the slow progress in the multivariate case.

The first motivation is that multivariate max-stable distributions define an infinite-dimensional (nonparametric) model class, since their (extreme-value) copula can not be fully characterised through a parametric class of copulas (e.g. Beirlant et al., 2004, Ch. 9.2). The extreme-value copula depends on an infinite-dimensional parameter, which is a function called the angular measure that permits an interpretation of the amount of dependence. A special mapping (reparametrization) of such a function yields the well-known Pickands dependence function, which is also commonly used to summarize the dependence level, as it is easy to interpret (e.g. Beranger and Padoan, 2015). To this extent, several semiparametric and nonparametric estimation methods based on polynomials and splines have been proposed for estimating the dependence structure under both parametrizations (e.g. Hall and Tajvidi, 2000; Klüppelberg and May, 2006; Cormier et al., 2014; Marcon et al., 2017, to name a few). In particular, Marcon et al. (2016) proposed a fully nonparametric Bayesian estimation method for bivariate max-stable distributions, where both dependence parametrizations are represented by means of polynomials in Bernstein form.

The second motivation is that the analytical expression of the likelihood function is complicated and computationally burdensome to calculate in practice (e.g. Dombry et al., 2017a). Accordingly, in high dimensions the statistical inference is often performed by the composite-likelihood approach (see Padoan et al., 2010; Ribatet et al., 2012) and hence the development of efficient inferential methods based on the full-likelihood still...
represents an active research area (e.g. Wadsworth and Tawn, 2014; Huser et al., 2016; Dombry et al., 2017a; Huser et al., 2019, to name a few). Assuming that the extreme-value copula belongs to a specific parametric model, Dombry et al. (2017b) have been able to derive a Bayesian inferential method based on the full-likelihood for fitting max-stable distributions to the data in arbitrary dimensions (greater than two).

In this paper we describe a nonparametric Bayesian estimation approach for bivariate max-stable distributions, where parametrizations of the dependence are represented through splines. For a certain order of the spline basis we derive the necessary and sufficient conditions to guarantee that the spline function provides a valid dependence structure of a max-stable distribution. Then, for this framework and the one introduced in Marcon et al. (2016), we establish the strong consistency of the posterior distribution under both parametrizations of the dependence, including predictive consistency. Next, we describe an inferential framework for max-stable distributions in arbitrary dimensions which extends that of Marcon et al. (2016). We derive the posterior consistency results also in this case. The latter findings extend, to some extent, the consistency result inferable from Dombry et al. (Section 3 2017b) to a more flexible nonparametric setup.

Initially, we derive our asymptotic results assuming that a dataset is sampled from a max-stable distribution with known margins, as in Dombry et al. (2017b). In practice, max-stable distributions are used to model a sequence of so-called block maxima, i.e. maxima are computed componentwise on a series of observations of a certain length (block), e.g. yearly maxima. Max-stable distributions provide an asymptotically justified model for block maxima, provided that the block size is sufficiently large. Hence, block maxima follow only approximately a max-stable distribution. Furthermore, the marginal distributions are known, potentially, apart from some unknown parameters. Accordingly, we extend the consistency results to the case where the data come from a distribution that is in a neighbourhood of a max-stable distribution, using suitable mathematical tools (Kleijn, 2017; Falk et al., 2019), and to the case where the margins of the max-stable distribution are heavy-tailed with unknown tail indices. Notice that our derivation of the strong consistency for the posterior distribution, under the mild condition that the componentwise maxima are obtained from observations whose distribution is in the variational domain of attraction of a max-stable model (Falk et al., 2019, and the references therein), provides a new original contribution to the extreme-value literature.

The paper is organised as follows. In Section 2 we briefly review the theory on max-stable distributions. After introducing some basic notation in Section 3.1, Section 3.2 describes some nonparametric Bayesian estimators of the dependence structure for bivariate max-stable distributions based on Bernstein polynomials and splines. Then, we establish strong consistency for the posterior distribution under different parametrizations of the dependence structure together with posterior predictive consistency. In Section 3.3 we describe an inferential framework for max-stable distributions in arbitrary dimensions and we derive similar consistency results. Finally, in Section 4, we extend the consistency results to more realistic sampling schemes. The proofs of the main theorems are postponed to the Appendix and those of Propositions 3.3–3.5, 3.10, Corollary 3.7 and Theorem 4.2 to the supplementary material.
2. Multivariate extremes

Let \( \xi = (\xi_1, \ldots, \xi_d) \) be a random vector (r.v.) with multivariate distribution \( F \). Let \( \xi_1, \xi_2, \ldots \) be independent and identically distributed (i.i.d.) copies of \( \xi \). Assume that \( F \) is in the maximum-domain of attraction of a multivariate max-stable class of distributions, say \( G \), shortly \( F \in D(G) \). This means that there are norming sequences \( a_n > 0 = (0, \ldots, 0) \) and \( b_n \in \mathbb{R}^d \), with \( n = 1, 2, \ldots \), such that

\[
\lim_{n \to \infty} F^n(a_n x + b_n) = G(x), \quad x \in \mathbb{R}^d, \tag{2.1}
\]

where the margins of \( G \) are nondegenerate (e.g. Falk et al., 2011, Ch. 4). The result in (2.1) states that the asymptotic distribution of the location-scale normalized block maxima \( M_{n,j} = \max_{1 \leq i \leq n} \xi_{i,j}, j = i, \ldots, d \), for a large sample size \( n \) (or block size), must be a max-stable distribution no matter what the underlying distribution \( F \) is, which is typically unknown in real applications. Precisely, \( G \) is called a max-stable distribution since it satisfies the max-stability properties \( G^k(\alpha_k x + \beta_k) = G(x) \) for \( k = 1, 2, \ldots \) and any \( x \in \mathbb{R}^d \), where \( \alpha_k > 0 \) and \( \beta_k \in \mathbb{R}^d \) are suitable norming constants (e.g. Falk et al., 2011, p. 143). It takes the form

\[
G(x) = C \{ G_{\gamma_1}(x_1), \ldots, G_{\gamma_d}(x_d) \}, \quad x \in \mathbb{R}^d, \tag{2.2}
\]

where \( G_{\gamma_j} \)’s are members of the univariate generalized extreme-value class of distributions, i.e.

\[
G_{\gamma_j}(x_j) = \begin{cases} 
\exp \left( - (1 + \gamma_j x_j)^{-1/\gamma_j} \right) & \text{if } \gamma_j \neq 0, \\
\exp(-\exp(-x_j)) & \text{if } \gamma_j = 0,
\end{cases} \tag{2.3}
\]

for \( j = 1, \ldots, d \), where \( \gamma_j \in \mathbb{R} \) is called the tail index and \( (x)_+ := \max(0, x) \) (e.g. Falk et al., 2011, p. 21 and Ch. 2). The second formula in (2.3) is obtained as the limit of the first one for \( \gamma_j \to 0 \). The support of \( G_{\gamma_j} \) is \( \text{supp}(G_{\gamma_j}) = \mathbb{R} \) if \( \gamma_j = 0 \), \( \text{supp}(G_{\gamma_j}) = (-1/\gamma_j, \infty) \) if \( \gamma_j > 0 \) and \( \text{supp}(G_{\gamma_j}) = (-\infty, -1/\gamma_j) \) if \( \gamma_j < 0 \). The function \( C \) is the extreme-value copula

\[
C(u) = \exp \left[ - L \{ (\ln u_1), \ldots, (\ln u_d) \} \right], \quad u \in (0,1]^d, \tag{2.4}
\]

where \( L : [0, \infty)^d \to [0, \infty) \) is a homogeneous function of order 1, named stable-tail dependence function (e.g. Falk et al., 2011, Ch. 4). By the homogeneity of \( L \) it follows that \( L(z) = (z_1 + \cdots + z_2) A(t) \) for all \( z \geq 0 \), with \( t_j = z_j/(z_1 + \cdots + z_d) \) for \( j = 1, \ldots, d-1 \) and \( t_d = 1 - t_1 - \cdots - t_{d-1} \), where \( A \) is the so-called Pickands dependence function. Precisely,

\[
A(t) = d \int_{S_d} \max(tw_1, \ldots, tw_d) H(dw), \tag{2.5}
\]

where \( H \) is a probability measure defined on the \( d \)-dimensional unit simplex \( S_d = \{ v \geq 0 : v_1 + \cdots + v_d = 1 \} \) subjected to the mean constraints \( \int_{S_d} w_j H(dw) = 1/d \) for all \( j = 1, \ldots, d \). For brevity we use \( H \) to denote both the probability measure and its
distribution function, the difference will be clear from the context, and we refer to them as the angular measure and distribution, respectively. In general, $H$ can place mass on all the $2^d - 1$ subspaces of $S_d$ which are of the form $S_{d,I} = \{v \in S_d : v_j > 0$ if $j \in I; v_j = 0$ if $j \notin I\}$, where $I$ is a non-empty subset of $\{1, \ldots, d\}$ (Beirlant et al., 2004, Ch. 7). In the sequel, for simplicity we focus on a subset of all the possible angular measures.

**Condition 2.1.** Assume that the angular measure can place positive mass only on the vertices and interior $\{\{e_j\}, j = 1, \ldots, d, S_d\}$, where $S_d = \{v \in S_d : 0 < v_j < 1, j = 1, \ldots, d\}$. In particular, there are atoms $H(\{e_j\}) = p_j \in [0, 1/d]$ for $j = 1, \ldots, d$ and a Lebesgue integrable function $h(v) \geq 0$ such that

$$H(S_d) = \int_{S_d} h(w)dw = 1 - p_1 - \cdots - p_d.$$ 

We call $h$ the angular density.

The dependence level among the components of a max-stable r.v. can be described by means of a geometric interpretation of the angular measure. The more the mass of $H$ concentrates around $(1/d, \ldots, 1/d)$ (the center of the simplex) the more the variables are dependent on each other. On the contrary, the more the mass of $H$ accumulates close to the vertices of the simplex, the less dependent the variables are.

The Pickands dependence function $A$ also represents the level of dependence among extremes, indeed it satisfies $\max(t_1, \ldots, t_d) \leq A(t) \leq 1$, for all $t \in S_d$, with the lower and upper bounds representing the cases of complete dependence and independence, respectively.

Let $X = (X_1, \ldots, X_d)$ be a r.v. with max-stable distribution. Consider the transformation

$$y_j := U_j(x_j) = (1 + \gamma_j x_j)^{1/\gamma_j}, \quad j = 1, \ldots, d,$$

for $x_j \in \text{supp}(G_{x_j})$, then $Y = (Y_1, \ldots, Y_d)$ with $Y_j = U_j(X_j), j = 1, \ldots, d$, is a r.v. whose distribution is $G_\ast(y) = \exp(-V(y))$, where $V(y) = L(1/y), y > 0$. This is the so-called simple max-stable distribution, which is a max-stable distribution with common unit-Fréchet marginal distributions. The function $V(y) := \Lambda([0, y])$ is commonly called the exponent function, where $\Lambda$ is a Radon measure on $E := [0, \infty) \setminus \{0\}$, with $\infty = (\infty, \ldots, \infty)$, called the exponent measure (see also Appendix A.1.1). The density function of $G_\ast$ is given by the Faà di Bruno’s formula

$$g_\ast(y) = \sum_{P \in \mathcal{P}_d} g_\ast(y, P) = \sum_{P \in \mathcal{P}_d} G_\ast(y) \prod_{i=1}^m \Delta(I_i, y),$$

(2.6)

where $\mathcal{P}_d$ is the set of all the partitions $P = \{I_1, \ldots, I_m\}$ of $\{1, \ldots, d\}$, $m = |P|$ and, for fixed $i$, $\Delta(I_i, y) := \frac{\partial^{1/\gamma_j} V_\ast(y)}{\partial y_j} (\partial y_j, j \in I_i)$ denotes the partial derivative of the exponent function, obtained by differentiating the latter with respect to the variables $y_j$ with $j \in I_i$. Thus, the density function of $G$ is

$$g(x) = \prod_{j=1}^d U_j(x_j)^{1-\gamma_j} \sum_{P \in \mathcal{P}_d} G_\ast(U(x)) \prod_{i=1}^m \Delta(I_i, U(x))$$
where \( U(x_I) = (U_j(x_j), j \in I) \) with \( x_I = (x_j, j \in I) \) and \( \emptyset \neq I \subset \{1, \ldots, d\} \). Dombry et al. (2017a) showed that \( g^* \) exists if and only if \( \Lambda \) admits Lebesgue densities \( \lambda_I \) on \( E_I \subset E \), with \( E_I = \{ y \in E : y_j > 0 \text{ if } j \in I; y_j = 0 \text{ if } j \notin I \} \), i.e. for all Borel sets \( B \subset E \)

\[
\Lambda(B) = \sum_{\emptyset \neq I \subset \{1, \ldots, d\}} \int_{y_I; y \in B \cap E_I} \lambda_I(y_I) \nu_I(dy_I),
\]

where \( \nu_I \) is the Lebesgue measure on \( E_I \) (see also Dombry and Eyi-Minko, 2013). From this it follows that

\[
\Delta(I_i, y) = \sum_{I_i \subset I} \left( \int_{(0,y_{I_i}^c)} \lambda_I(z_I) \bigg|_{z_{I_i} = y_{I_i}} dz_I \right)
\]  

(2.7)

where \( y_{I_i} \) and \( y_{I_i}^c \) are the restrictions of \( y \) to \( I_i \) and \( I_i^c = \{1, \ldots, d\} \setminus I_i \). Under Condition 2.1, (2.7) reduces to

\[
\Delta(I_i, y) = \begin{cases} 
  d p_i y_{I_i}^{-2} + d \left( \int_{(0,y_{I_i}^c)} \|z\|_1^{-d-1} h(z/\|z\|_1) \bigg|_{z_{I_i} = y_{I_i}} dz_I \right), & \text{if } |I_i| = 1 \\
  d \left( \int_{(0,y_{I_i}^c)} \|z\|_1^{-d-1} h(z/\|z\|_1) \bigg|_{z_{I_i} = y_{I_i}} dz_I \right), & \text{otherwise}
\end{cases}
\]  

(2.8)

3. Bayesian nonparametric dependence modelling

In this section we assume for simplicity that the data follow a simple max-stable distribution (i.e. with known margins). In Section 4 we extend the consistency results obtained here to a more general setting.

We discuss the bivariate case apart from the higher dimensional case. The motivation is the following. In the bivariate case there is a simple explicit relationship between the angular distribution and density with the first and second derivatives of the Pickands dependence function. In particular, the map that transforms a Pickands to an angular density function is homeomorphic with respect to the uniform and the \( L_1 \) topologies, provided that the angular density is continuous. Modelling these functions with Bernstein polynomials or splines allows to capitalize such a relationship by means of a suitable relationship, in turn, among polynomials’ coefficients. As a result, consistency is readily obtained for both parametrizations of the dependence, with strong metrics. In the higher dimensional case, establishing a general explicit relationship between the angular distribution and the partial derivatives of the Pickands dependence function is a cumbersome task. Furthermore, the necessary and sufficient conditions for a function to be a valid Pickands dependence function are hard to verify, unlike the bivariate case. This means that only focusing on the modelling of the angular measure is more convenient.
3.1. Notation

Given $\mathcal{X} \subset \mathbb{R}^d$ and $d \in \mathbb{N}$, let $\ell^\infty(\mathcal{X})$ denote the spaces of bounded real-valued functions on $\mathcal{X}$. For $f : \mathcal{X} \rightarrow \mathbb{R}$, let $\|f\|_\infty = \sup_{x \in \mathcal{X}} |f(x)|$ and $\|f\|_1 = \int_\mathcal{X} |f(x)| \, dx$. In the case $d = 2$, we define the norms

$$
\|f\|_{1,\infty} := \|f\|_\infty + \|f^{(1)}\|_{\infty}, \quad \|f\|_{2,\infty} := \|f\|_{1,\infty} + \|f^{(2)}\|_{\infty},
$$

where $f^{(1)}$ and $f^{(2)}$ are the first and second derivative of $f$, and let $\mathcal{D}_{p,\infty}(f,g) := \|f - g\|_{p,\infty}$, $p = 1, 2$, be the associated metrics. For two probability density functions $f$ and $g$, we denote by $\mathcal{H}(f,g)$ and $\mathcal{D}_W(f,g)$ the Kullback-Leibler divergence and the Hellinger distance, respectively. We also denote by $\mathcal{D}_F$, $\mathcal{D}_\infty$ and $\mathcal{D}_W$ the total variation distance between probability measures, the uniform metric between functions or vectors and a metric on a space of probability measures that metrizes the topology of weak convergence, respectively (e.g. Ghosal and van der Vaart, 2017, Ch 2.3, 3.1 and 6.1).

The statistical setting that we consider here can be described in the following general terms. For $n \in \mathbb{N}$, let $X_1, \ldots, X_n$ be independent and identically distributed r.v.s on $\mathbb{R}^d$, with distribution $F_{\theta_0}(x)$, where $\theta_0$ is the true unknown parameter and belongs to a (possibly) infinite-dimensional space $\Theta$, endowed with metric $\mathcal{D}$. Let $\Pi(B) = \mathbb{P}(\theta \in B)$ be a prior distribution on the Borel sets $B$ of $(\Theta, \mathcal{D})$. The corresponding posterior distribution is the random measure

$$
\Pi_n(B) := \Pi(B | X_1, \ldots, X_n) = \frac{\int_B \prod_{i=1}^n F_{\theta}(X_i) \Pi(d\theta)}{\int_{\Theta} \prod_{i=1}^n F_{\theta}(X_i) \Pi(d\theta)}
$$

In this setting, almost sure convergence of the posterior distribution to the Dirac measure at $\theta_0$ is equivalent to the following: for all $\epsilon > 0$, $\lim_{n \to \infty} \Pi_n(\theta : \mathcal{D}(\theta, \theta_0) > \epsilon | X_1, \ldots, X_n) \to 0$, $F_{\theta_0}^\infty$-a.s.. When this happens we say that the posterior distribution is strongly consistent at $\theta_0$ (e.g. Ghosal and van der Vaart, 2017, Ch 6). As the Bayesian model conceives data as conditionally i.i.d. according to $F_{\theta}$, given a realization $\theta$ from $\Pi$, hereafter we replace the notation $F_{\theta}(\cdot)$ with $F(\cdot | \theta)$.

3.2. Bivariate case

Under Condition 2.1 the angular distribution is a function of the form $H(w) = p_0 + \int_0^w h(v) \, dv + p_1 1_{[0, w)}(1)$ for $w \in [0, 1]$, satisfying the mean constraint

$$(C1) \quad \int_{[0,1]} w H(dw) = \int_{[0,1]} (1-w) H(dw) = 1/2,$$

where $1_S(x)$ for $x \in \mathbb{R}$ and $S \subset \mathbb{R}$ is the indicator function. The Pickands dependence function is a function satisfying the convexity and boundary conditions

$$(C2) \quad A(at_1 + (1-a)t_2) \leq aA(t_1) + (1-a)A(t_2), \forall a, t_1, t_2 \in [0, 1],$$

$$(C3) \quad 1/2 \leq \max(t, 1-t) \leq A(t) \leq 1, \forall t \in [0, 1].$$
The functions $H$ and $A$ are related through the explicit expression $A(t) = 1 + 2 \int_0^t H(w)dw - t$, $t \in [0, 1]$, $A^{(1)}(t) = -1 + 2H(t)$ and $A^{(2)}(t) = 2h(t)$. Furthermore, $A^{(1)}(0) = 2p_0 - 1$ and $A^{(1)}(1) = 1 - 2p_1$, where $A^{(1)}$, with an abuse of notation, also denotes the continuous extension of the first derivative of $A$ on $[0, 1]$. We denote by $\mathcal{A}$ and $\mathcal{H}$ the sets of valid angular distribution and Pickands dependence functions, i.e. the sets of functions satisfying (C1) and (C2)-(C3), respectively.

Due to the one-to-one relationship between $A$ and $H$ we denote the simple max-stable distribution by $G_\ast(|\theta)$, where $\theta$ is an infinite-dimensional parameter, corresponding to one of those two equivalent representations of the dependence structure. We denote the density in (2.6) by $g_\ast(|\theta)$, to highlight its dependence on the parameter.

3.2.1. **Kulback-Leibler support of the prior**

Since the parametrizations of the dependence structure through $A$ or $H$ are equivalent, we focus on the prior distributions for the Pickands dependence function. We denote with $\Pi_A(\mathcal{B}) = \mathbb{P}(A \in \mathcal{B})$ a prior distribution on the Borel sets $\mathcal{B}$ of $(\mathcal{A}, \mathcal{D}_\infty)$. Observe that, under Condition 2.1, $\mathcal{A}$ is a set of continuously differentiable, convex functions. Thus, the uniform topology on such a class is equivalent to the topology induced by $\mathcal{D}_{1,\infty}$, see Proposition A.2(ii). Under Condition 2.1, the topological structure on $\mathcal{A}$ also allows to track angular distribution functions, as a result of the linear relationship between $A^{(1)}$ and $H$.

We assume that the data have been generated by a simple max-stable distribution $G_\ast(|\theta_0)$, where $\theta_0 = A_0$ is the true Pickands dependence function. A crucial condition for posterior consistency (see Section 3.2.2) is that the true probability density function is in the Kulback-Leibler support of the prior distribution. That is, for all $\epsilon > 0$

$$\Pi_A(A : \mathcal{K}(g_\ast(|A_0), g_\ast(|A)) < \epsilon) > 0. \quad (3.1)$$

When such a condition is satisfied we say that $g_\ast(|A_0)$ possesses the Kulback-Leibler property relative to $\Pi_A$. To derive the result in (3.1) we assume that the true angular density is continuous and either: has finite limits at the vertices or is bounded away from 0 and diverges at the vertices.

**Condition 3.1.** Let $A_0 \in \mathcal{A}_0$, where $\mathcal{A}_0 \subset \mathcal{A}$ is the class of Pickands dependence functions with corresponding angular density $h_0$ that satisfies one of the following conditions:

(i) $0 \leq \lim_{w \downarrow 0} h_0(w) < +\infty$ and $0 \leq \lim_{w \uparrow 1} h_0(w) < +\infty$;

(ii) $\inf_{w \in [0,1]} h_0(w) > 0$ and $\lim_{w \downarrow 0} h_0(w) = \lim_{w \uparrow 1} h_0(w) = +\infty$.

A nonparametric prior $\Pi_A$ can be constructed by modelling only a representative proper subset of $\mathcal{A}$, whose topological closure equals $\mathcal{A}$. Linear combinations of Bernstein polynomials and B-splines provide examples of such a $\mathcal{D}_\infty$-dense subset, as illustrated in Section 3.2.2. However, the uniform distance between Pickands dependence functions is insufficient to control the Kulback-Leibler divergence between the corresponding simple max-stable densities. Nonetheless, whenever $A \in \mathcal{A}$ can be approximated $\mathcal{D}_{2,\infty}$-sufficiently closely by a sequence $(A_k, k = 1, 2, \ldots)$ in the representative
subset, \( \log g_*(\cdot|A) - \log g_*(\cdot|A_k) \) can be controlled via \( D_{2,\infty}(A, A_k) \). If such an approximation is available for all \( A \) in a specific subclass \( A' \subset A_0 \), Kullback-Leibler property can be obtained over the entire class \( A_0 \), as asserted next.

**Theorem 3.2.** Let \( A' \) be the class of \( A \in A_0 \) satisfying Condition 3.1(i), with \( p_0, p_1 > 0 \) and \( \inf_{w \in (0,1)} h(w) > 0 \). Assume that for all \( A \in A' \) there exists a sequence \( A_k \in A \), \( k = 1, 2, \ldots \), satisfying

\[
D_{2,\infty}(A, A_k) = o(1), \quad k \to \infty,
\]

and that the prior distribution \( \Pi_A \) assigns positive mass to every \( D_{2,\infty} \)-ball centered at \( A_k \). Then, under Condition 3.1, \( \Pi_A \) satisfies (3.1).

### 3.2.2. Posterior consistency

A nonparametric Bayesian inferential method that uses polynomials in Bernstein form for representing both parametrizations \( H \) and \( A \) of the dependence structure has been proposed in Marcon et al. (2016). Here, we investigate a similar estimation approach, using a representation through piecewise polynomials, based on B-spline basis functions. For both approaches, we establish the strong consistency of the posterior distribution.

We recall that in the bivariate case the density function in (2.6) reduces to

\[
g_*(y|A) = G_*(y|A) \left[ \frac{A(t) - tA^{(1)}(t)}{(y_1y_2)^{2}} \right],
\]

where \( t = y_1/(y_1+y_2) \). Next, we briefly summarize the method introduced in Marcon et al. (2016), then we describe our proposal based on splines.

**Bernstein polynomial representation**

According to Marcon et al. (2016), the polynomial of degree \( k-1 \) in Bernstein form, for some \( k = 1, 2, \ldots, \), defined by

\[
H_{k-1}([0, w]) := \frac{1}{1} \sum_{j \leq k-1} \eta_j b_j(w; k-1) \quad \text{if} \quad w \in [0, 1)
\]

\[
H_{k-1}(w) := \frac{1}{1} \sum_{j \leq k-1} \eta_j b_j(w; k-1) \quad \text{if} \quad w = 1
\]

whose coefficients satisfy the restrictions:

(R1) \( 0 \leq \eta_0 \leq \eta_1 \leq \ldots \leq \eta_{k-1} = 1 - p_1 \leq 1; \)

(R2) \( \eta_0 + \cdots + \eta_{k-1} = k/2; \)

is a valid angular distribution, where \( b_j(w; k) = (k+1)^{-1} \text{Be}(w|j+1, k-j-1) \) for all \( w \in [0,1] \) and \( k \geq 1 \) and where \( \text{Be}(\cdot|a, b) \) is the beta density function with shape parameters \( a, b > 0 \). The polynomial of degree \( k \) in Bernstein form, for some \( k = 2, 3, \ldots, \), defined by

\[
A_k(t) := \sum_{j=0}^{k} \beta_j b_j(t; k), \quad t \in [0, 1],
\]

whose coefficients satisfy the restrictions:
(R3) \( \beta_0 = \beta_k = 1 \geq \beta_j, \text{ for all } j = 1, \ldots, k - 1; \)
(R4) \( \beta_1 = \frac{k - 1 + 2p_k}{k} \) and \( \beta_{k-1} = \frac{k - 1 + 2p_k}{k}; \)
(R5) \( \beta_{j+2} - 2\beta_{j+1} + \beta_j \geq 0, j = 0, \ldots, k - 2; \)
is a valid Pickands dependence function, see Marcon et al. (2016) for details. We call (3.3) and (3.4) angular distribution and Pickands dependence functions in Bernstein Polynomial (BP) form, respectively.

Let \( H_{k-1} = \{ w \mapsto H_{k-1}(w) = \sum_{j \leq k-1} \eta_j b_j(w; k - 1) : \eta_0, \ldots, \eta_{k-1} \in [0, 1] \} \) and (R1)-(R2) are satisfied

and \( A_k = \{ t \mapsto A_k(t) = \sum_{j \leq k} \beta_j b_j(t; k) : \beta_0, \ldots, \beta_k \in [0, 1] \text{ and (R3)-(R5) are satisfied} \}, \)
then \( \bigcup_{k=1}^\infty H_{k-1} \) and \( \bigcup_{k=1}^\infty A_k \) are dense subsets of the spaces \((\mathcal{H}, \mathcal{D}_\infty)\) and \((\mathcal{A}, \mathcal{D}_\infty)\), respectively, by Propositions 3.1-3.3 in Marcon et al. (2016). Furthermore, for each \( A_k \in A_k \) it is possible to derive a polynomial \( H_{k-1} \in H_{k-1} \) and vice versa, by means of precise relationships between the two polynomials’ coefficients (Marcon et al., 2016, Proposition 3.2).

**Piecewise polynomial representation with B-splines**

Piecewise polynomial representations of the angular distribution function and the Pickands dependence function on \([0, 1]\) as linear combinations of B-splines can be obtained as follows. For \( \kappa \geq 1 \) and \( m \geq 2 \) we define the interior knots as the sequence \( 0 < \tau_{m+1} < \cdots < \tau_{m+\kappa} < 1 \) and the exterior knots by \( \tau_1 = \cdots = \tau_m = 0 \) and \( \tau_{m+\kappa+1} = \cdots = \tau_{2m+\kappa} = 1 \).

Let the B-spline basis of order \( m \) be defined by the recursive formula (de Boor, 1978),

\[
\phi_{j,i}(t) = \frac{t - \tau_j}{\tau_{j+i-1} - \tau_j} \phi_{j,i-1}(t) + \frac{\tau_{j+i} - t}{\tau_{j+i} - \tau_{j+1}} \phi_{j+1,i-1}(t), \quad t \in [0, 1],
\]

for \( 1 < i \leq m \) and \( 1 \leq j \leq 2m + \kappa - i \), starting with

\[
\phi_{j,1}(t) = \begin{cases} 1, & \tau_j \leq t < \tau_{j+1}, \\ 0, & \text{otherwise,} \end{cases}
\]

for \( 1 \leq j \leq 2m + \kappa - 1 \). In general, given \( \kappa \) points in \((0, 1)\) a spline of order \( m \) can be expressed as the linear combination of \( m + \kappa \) piecewise polynomials of degree \( m - 1 \) (B-spline basis functions). Let

\[
H_{k-1}(w) = \begin{cases} \sum_{j=1}^{k-1} \eta_j \phi_{j,m-1}(w), & w \in [0, 1), \\ 1, & w = 1, \end{cases}
\] (3.5)
be a spline of order \( m - 1 \) whose basis consists of \( k - 1 = m + \kappa - 1 \) piecewise polynomials of degree \( m - 2 \). Notice that taking the first derivative of (3.5) with respect to \( w \) we obtain the following spline function of order \( m - 2 \)

\[
H_{k-1}^{(1)}(w) = \sum_{j=1}^{k-2} \frac{(m-2)(\eta_{j+1} - \eta_j)}{\tau_{j+m-1} - \tau_{j+1}} \phi_{j+1,m-2}(w), \quad w \in (0,1).
\] (3.6)

Similarly, let

\[
A_k(t) = \sum_{j=1}^{k} \beta_j \phi_{j,m}(t), \quad t \in [0,1],
\] (3.7)

be a spline of order \( m \) whose basis consists of \( k = m + \kappa \) piecewise polynomials of degree \( m - 1 \). Notice that the first two derivatives of (3.7) with respect to \( t \) gives the following spline functions of order \( m - s \) with \( s = 1,2 \),

\[
A_k^{(s)}(t) = \sum_{j=1}^{k-s} \beta_{j,s} \phi_{j+s,m-s}(t), \quad t \in (0,1),
\] (3.8)

where

\[
\beta_{j,s} = \frac{(m-s)(\beta_{j+1,s-1} - \beta_{j,s-1})}{\tau_{j+m+1-s} - \tau_{j+1}}, \quad s = 1,2,
\]

with \( \beta_{j,0} \equiv \beta_j \) for \( j = 1,2,\ldots,k-1 \).

The next result provides the necessary and sufficient conditions that the coefficients of the splines in (3.5) and (3.7) must satisfy so that the splines are valid angular distribution and Pickands dependence functions of order 2 and 3, respectively. The proof is provided in Section 2.4 of the supplementary material. We found that extending such a result to higher orders is less tractable, while nonessential for practical statistical purposes. Hereafter, for any given integer \( k > 3 \), we fix the sequence of internal knots \((i/(k-2), i = 1,\ldots,k-3)\). We refer to the functions (3.5) and (3.7), satisfying the restrictions of Proposition 3.3, as the angular distribution and Pickands dependence functions in B-Spline (BS) form.

**Proposition 3.3.** For fixed \( k \), the spline function \( H_{k-1} \) in (3.5) with \( m = 3 \) is a valid angular distribution if and only if the coefficients \( \eta_1,\ldots,\eta_{k-1} \) satisfy the restrictions:

(R6) \( 0 \leq p_0 = \eta_1 \leq \eta_2 \leq \cdots \leq \eta_{k-1} = 1 - p_1 \leq 1 \);
(R7) \( \eta_1 + 2(\eta_2 + \cdots + \eta_{k-2}) + \eta_{k-1} = (k - 2) \);

where \( 0 \leq p_0, p_1 \leq 1/2 \). Likewise, the spline function \( A_k \) in (3.7) with \( m = 3 \) is a valid Pickands dependence function if and only if the coefficients \( \beta_1,\ldots,\beta_k \) satisfy the restrictions:

(R8) \( \beta_1 = \beta_k = 1 \geq \beta_j, \text{ for all } j = 2,\ldots,k-1 \);
(R9) \( \beta_2 = 1 + (p_0 - 1/2)/(k - 2) \) and \( \beta_{k-1} = 1 + (p_1 - 1/2)/(k - 2) \);
(R10) \( \beta_3 - 2\beta_2 + 2\beta_1 \geq 0, 2\beta_k - 3\beta_{k-1} + \beta_{k-2} \geq 0 \) and \( \beta_j - 2\beta_{j-1} + \beta_{j-2} \geq 0, j = 4,\ldots,k-1 \).
Moreover, when \( m = 3 \), the class of valid spline angular distribution functions is related to the class of valid spline Pickands dependence functions through a precise relationship that link their coefficients, as described in the following proposition. See Section 2.5 of the supplementary material for its proof.

**Proposition 3.4.** For fixed \( k \), let \( H_{k-1} \) be defined in (3.5) with \( m = 3 \). Define the coefficients \( \eta_1, \ldots, \eta_{k-1} \) by

\[
\eta_j = \frac{1}{2} + \frac{\beta_{j+1} - \beta_j}{\tau_{j+2} - \tau_j}, \quad j = 1, \ldots, k-1,
\]

where the coefficients \( \beta_1, \ldots, \beta_k \) satisfy the restrictions (R8)-(R10) and \( (\tau_j, j = 1, \ldots, k+1) = (0, 0, 1/(k-2), \ldots, (k-3)/(k-2), 1, 1) \). Then, \( H_{k-1} \) is a valid angular distribution.

Let \( A_k \) be defined in (3.7) with \( m = 3 \). Define

\[
\beta_1 = 1, \quad \beta_j = \sum_{i=1}^{j-1} (\eta_i - 1/2)(\tau_{i+3} - \tau_{i+1}) + 1, \quad j = 2, \ldots, k.
\]

where the coefficients \( \eta_1, \ldots, \eta_{k-1} \) satisfy the restrictions (R6)-(R7) and \( (\tau_j, j = 1, \ldots, k+3) = (0, 0, 0, 1/(k-2), \ldots, (k-3)/(k-2), 1, 1, 1) \). Then, \( A_k \) is a valid Pickands dependence function.

Finally, we show that the spaces of angular distributions and Pickands dependence functions, \( \mathcal{H} \) and \( \mathcal{A} \), are well approximated by the spaces of angular distributions and Pickands dependence functions in BS form, respectively. For the proof, see Section 2.6 of the supplementary material.

**Proposition 3.5.** Let

\[
\mathcal{H}_{k-1} = \{ w \mapsto H_{k-1}(w) = \sum_{j=1}^{k-1} \eta_j \phi_{j,m-1}(w) : \eta_0, \ldots, \eta_{k-1} \in [0,1] \text{ and } (R6)-(R7) \text{ are satisfied} \}
\]

and

\[
\mathcal{A}_k = \{ t \mapsto A_k(t) = \sum_{j=1}^{k} \beta_j \phi_{j,m}(t) : \beta_0, \ldots, \beta_k \in [0,1] \text{ and } (R8)-(R10) \text{ are satisfied} \},
\]

then \( \cup_{k=4}^\infty \mathcal{H}_{k-1} \) and \( \cup_{k=4}^\infty \mathcal{A}_k \) are dense subsets of the spaces \( (\mathcal{H}, \mathcal{D}_\infty) \) and \( (\mathcal{A}, \mathcal{D}_\infty) \).

The spline in (3.5), with \( m = 3 \), is a piecewise linear function. However, it may be desirable to have a smoothed representation of the angular distribution function. By following a method similar to that in Guillotte et al. (2011), we propose to smooth the function in (3.5) as described next.
The method consists of two steps. In the first step we construct the following step function. For the points $y_1 = 0$, $y_j = \tau_{j+1}$ with $j = 2, \ldots, k-2$ and $y_{k-1} = 1$ we have $H_{k-1}(y_1) = \eta_1$, $H_{k-1}(y_j) = \eta_j$ with $j = 2, \ldots, k-2$ and $H_{k-1}(y_{k-1}) = \eta_{k-1}$. Then, we define the step function by

$$
\overline{H}_{k-1}(w) = (\eta_j + \eta_{j+1})/2, \quad w \in [y_j, y_{j+1}], \quad j = 1, \ldots, k-2,
$$

and $\overline{H}_{k-1}(1) = 1$. Next, we define the interpolating functions $\overline{H}_{k-1}(w) = S(t; y, b_{\pm})$, where $y = (y_1, \ldots, y_{k-1})^T$,

$$
b_{-} = (\overline{H}_{k-1}(y_1), \overline{H}_{k-1}(y_2), \ldots, \overline{H}_{k-1}(y_{k-1})), \quad (3.9)
$$

$$
b_{+} = (\overline{H}_{k-1}(y_1), \overline{H}_{k-1}(y_2), \ldots, \overline{H}_{k-1}(y_{k-1})), \quad (3.10)
$$

$\overline{H}_{k-1}(1) = 1$ and $S$ is the following monotone nondecreasing cubic spline interpolant (Fritsch and Butland, 1984),

$$
S(t; y, b_{\pm}) = \frac{\pi_j + \pi_{j+1} - \Pi_j}{1/(k-2)^2} (t - y_j)^3 + \frac{3\Pi_j - 2\pi_j - j - \pi_{j+1}}{1/(k-2)} (t - y_j)^2 + \pi_j (t - y_j) + b_{j}^{\pm}
$$

for every $t \in [y_j, y_{j+1}]$, where $\Pi_j = (k-2)(b_{j+1}^{\pm} - b_{j}^{\pm})$ and

$$
\pi_j = \begin{cases} 
2\frac{\Pi_j - \Pi_{j-1}}{\Pi_j + \Pi_{j-1}}, & \text{if } \Pi_{j-1}\Pi_j > 0, \\
0, & \text{otherwise,}
\end{cases}
$$

for $j = 1, \ldots, k-1$. By construction we have $\overline{H}_{k-1}(w) \leq \overline{H}_{k-1}(w) \leq \overline{H}_{k-1}(w)$ for all $w \in [0, 1]$. Define $\tilde{H}_{k-1}(w) := \omega_k \overline{H}_{k-1}(w) + (1 - \omega_k) \overline{H}_{k-1}(w)$, $w \in [0, 1]$, where $\omega_k \in (0, 1)$ is such that

$$
\omega_k \int_0^1 w \overline{H}_{k-1}(w) dw + (1 - \omega_k) \int_0^1 w \overline{H}_{k-1}(w) dw = 1/2.
$$

Then, $\tilde{H}_{k-1}$ defines a valid angular distribution function with point masses $(\eta_1 + \eta_2)/2$ and $1 - (\eta_{k-2} + \eta_{k-1})/2$ at 0 and 1, respectively. $\tilde{H}_{k-1}$ provides a smooth version of $H_{k-1}$.

Reexpressing $H_{k-1}$ as a piecewise linear spline $H_{k'} - 1$, with $k' \gg k$, we also obtain a step function $\overline{H}_{k'-1}$ on a denser grid of points, that accurately approximates $H_{k-1}$. Consequently, its smooth version $\tilde{H}_{k'-1}$ does the same. Then, from Proposition 3.5 it follows that the class of smoothed splines just introduced also defines a dense subset of $(\mathcal{H}, \mathcal{D}\infty)$.

**Main results**

The next theorem establishes the strong consistency of the posterior distribution under the parametrization of the dependence structure through the Pickands dependence function. The result is obtained for both the inferential approach based on Bernstein polynomials and the one based on splines.
Theorem 3.6. Let \( Y_1, \ldots, Y_n \) be i.i.d. r.v. with distribution \( G_\ast(\cdot|A_0) \), where \( A_0 \) satisfies Condition 3.1. Let \( \Pi_A \) be the prior distribution on \( A \) induced by a prior distribution \( \Pi \) defined on \( \bigcup_{k \geq k'} \{ (k) \times \mathcal{B}_k \} \), for some \( k' \in \mathbb{N} \), satisfying:

(i) \( \Pi(\{k\}) > 0 \), \( \Pi(B|k) > 0 \) for all Borel sets \( B \subset \mathcal{B}_k \) and \( k \geq k' \);
(ii) \( \Pi(\{k, k+1, k+2, \ldots\}) \lesssim e^{-qk} \), for some \( q > 0 \);

where \( \mathcal{B}_k = \{ (\beta_0, \ldots, \beta_k) : (R3)-(R5) \) are satisfied \} or \( \mathcal{B}_k = \{ (\beta_0, \ldots, \beta_k) : (R8)-(R10) \) are satisfied \}, for each \( k = 1, 2, \ldots \). Then, for all \( \epsilon > 0 \) we have

\[
\lim_{n \to \infty} \Pi_n(A \in A : D_H(g_\ast(\cdot|A), g_\ast(\cdot|A_0)) > \epsilon) = 0, \quad G_\ast^\infty(\cdot|\theta_0) - \text{a.s.}, \\
\lim_{n \to \infty} \Pi_n(A \in A : D_1(\cdot, A_0) > \epsilon) = 0, \quad G_\ast^\infty(\cdot|\theta_0) - \text{a.s.}
\]

where \( \Pi_n(\cdot) := \Pi_A(\cdot|Y_1, \ldots, Y_n) \) is the posterior distribution induced by the prior distribution \( \Pi_A \).

The first result of Theorem 3.6 asserts that the posterior distribution induced on the space of simple max-stable densities is strongly consistent with respect to \( D_H \). As a result, the Bayesian estimator of \( g_\ast(\cdot|A_0) \), i.e. the predictive density \( \hat{\rho}_{n+1}(z) := \int_A g_\ast(z|A)\Pi_n(dA) \), is strongly consistent too.

Given the one-to-one relationship that exists among the coefficients of the Pickands dependence and angular distribution functions, in both BP and BS form, a prior distribution defined on the class of Pickands dependence functions induces a prior distribution on the class of angular distribution functions. Furthermore, a similar implication is also valid for the smooth spline angular distributions obtained by the construction on pages 12-13. Thus, one expects that the posterior distributions corresponding to such prior distributions, denoted by the symbol \( \Pi_H \), are also consistent. The following corollary corroborates this conjecture. The proof is provided in Section 2.7 of the supplementary material.

Corollary 3.7. Under the conditions of Theorem 3.6, for all \( \epsilon > 0 \) we have

\[
\lim_{n \to \infty} \Pi_n(H \in H : \|H - H_0\|_\infty > \epsilon) = 0, \quad G_\ast^\infty(\cdot|\theta_0) - \text{a.s.}, \\
\lim_{n \to \infty} \Pi_n(H \in H : \|h - h_0\|_1 > \epsilon) = 0, \quad G_\ast^\infty(\cdot|\theta_0) - \text{a.s.}
\]

where \( \Pi_n(\cdot) := \Pi_H(\cdot|Y_1, \ldots, Y_n) \) is the posterior distribution induced by the prior distribution \( \Pi_H \).

3.3. High dimensional case

In dimensions greater than two \((d > 2)\) the Pickands dependence function is a function that still needs to satisfy the convexity and boundary constraints

\[ A(at_1 + (1-a)t_2) \leq aA(t_1) + (1-a)A(t_2), \quad a \in [0, 1], \forall t_1, t_2 \in S_d, \]
(C5) \(1/d \leq \max(t_1, \ldots, t_d) \leq A(t) \leq 1, \forall t \in \mathcal{S},\)

however these are necessary but not sufficient conditions to characterise the class of valid multivariate Pickands dependence functions, see e.g., Beirlant et al. (2004, p. 257) for a counter example. In the multivariate case the class of valid stable dependence functions is fully characterised through the conditions stated in Ressel (2013). However to define a suitable prior distribution on such a class is very challenging. A more viable approach is to work with the class of valid angular measures, consisting of the probability measures on \(\mathcal{S}_d\) that satisfy the simpler conditions

(C6) \(\int_{\mathcal{S}_d} w_j H(d\mathbf{w}) = 1/d, \text{ for all } j = 1, \ldots, d.\)

Therefore, hereafter we focus on the class \(\mathcal{H}\) of valid angular measures satisfying Condition 2.1.

3.3.1. Kulback-Leibler support of the prior

Again, we assume that the data have been generated by a simple max-stable distribution with true angular distribution \(H_0\). In the sequel we use the notation \(G_\ast(\cdot|H)\) and \(g_\ast(\cdot|H)\) to explicitly link the angular measure \(H\) to the corresponding simple max-stable distribution and probability density. Recall that \(g_\ast(\cdot|H)\) is as in (2.6). We denote by \(\Pi_\mathcal{H}\) a prior distribution on \(\mathcal{H}\). Analogously to the bivariate case, we provide sufficient conditions for \(g_\ast(\cdot|H_0)\) to possess the Kulback-Leibler property relative to \(\Pi_\mathcal{H}\), that is

\[
\Pi_\mathcal{H}(H : \mathcal{K}(g_\ast(\cdot|H_0), g_\ast(\cdot|H)) < \epsilon) > 0, \tag{3.11}
\]

for all \(\epsilon > 0\). The property in (3.11) is derived assuming that the true angular measure satisfies the following condition.

**Condition 3.8.** \(H_0 \in \mathcal{H}_0 \subset \mathcal{H},\) where \(\mathcal{H}_0\) is the class of angular measures whose density functions are continuous (on \(\mathcal{S}_d\)) and admit a continuous extension to \(\mathcal{S}_d\).

Note that the class of angular measures that satisfy Condition 3.8 is less flexible than that considered in the bivariate case, whose members satisfy Condition 3.1(i) or 3.1(ii). The latter condition is less tractable in higher dimensions than two, while nonessential for practical statistical purposes.

**Theorem 3.9.** Let \(\mathcal{H}'\) be the subset of \(H \in \mathcal{H}_0\) whose angular density and point massess satisfy \(\inf_{w \in \mathcal{S}_d} h(w) > 0\) and \(p_j > 0, j = 1, \ldots, d,\) respectively. Assume that for all \(H \in \mathcal{H}'\) there exists a sequence \(H_k \in \mathcal{H}, k = 1, 2, \ldots,\) satisfying

\[
\mathcal{D}_\infty(h, h_k) = o(1), \quad k \to \infty, \tag{3.12}
\]

and that the prior distribution \(\Pi_\mathcal{H}\) assigns positive mass to the sets \(\{H' \in \mathcal{H} : D_\infty(h, h') \leq \delta\}\), for all \(k\) and \(\delta > 0\). Then, under Condition 3.8, \(\Pi_\mathcal{H}\) satisfies (3.11).
3.3.2. Posterior consistency

For positive integer $k > d$, let $\Gamma_k$ be the set of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in \{0,1,\ldots,k\}^{d-1}$ such that $\alpha_1 + \cdots + \alpha_{d-1} \leq k$. The cardinality of $\Gamma_k$ is equal to the number of multi-indices $\alpha \in \{0,1,\ldots,k\}^d$ such that $\alpha_1 + \cdots + \alpha_d = k$; just set $\alpha_d = k - \alpha_1 - \cdots - \alpha_{d-1}$. As pointed out in Marcon et al. (2017) the cardinality of $\Gamma_k$ is

$$|\Gamma_k| = \binom{k + d - 1}{d - 1}.$$  \hfill (3.13)

Define the Bernstein basis polynomial $b_\alpha(\cdot; k)$ on $S_d$ of degree $k$ by

$$b_\alpha(w; k) = \binom{k}{\alpha}w_\alpha, \quad w \in S_d,$$  \hfill (3.14)

where

$$\binom{k}{\alpha} = \frac{k!}{\alpha_1! \cdots \alpha_d!}, \quad w_\alpha = w_1^{\alpha_1} \cdots w_d^{\alpha_d}.$$  

In particular, the Bernstein basis polynomial of index $\alpha$ and degree $k-d$ can be rewritten as $b_\alpha(w; k) = \text{Dir}(w; \alpha + 1)/(k + d - 1) \cdots (k + 1)$, where $\text{Dir}(w; \alpha + 1)$ denotes the Dirichlet probability density with parameters $\alpha + 1$. We exploit this relationship to model the angular density through the Bernstein polynomial. Accordingly, let $\Gamma_k$ be the set of multi-indices $\alpha \in \{1, \ldots, k - d + 1\}^d$ such that $\alpha_1 + \cdots + \alpha_d - 1 \leq k$. Thus the cardinality of $\Gamma_k$ is as in (3.13) but with $k$ replaced with $k - d$.

For positive integer $k$ and fixed dimension $d > 2$, the $(k-d)$-th degree Bernstein polynomial representation associated to the angular density is given by

$$h_{k-d}(w) = \sum_{\alpha \in \Gamma_k} \psi_\alpha \text{Dir}(w; \alpha), \quad w \in \hat{S}_d.$$  \hfill (3.15)

Let $\kappa_j = ke_j$ with $e_j$, $j = 1, \ldots, d$, that is the canonical unit vector. According to Hanson et al. (2017) the function in (3.15) is a valid angular density if and only if the non-negative coefficients $(\psi_{\kappa_j}, j = 1, \ldots, d, \psi_\alpha, \alpha \in \Gamma_k)$ satisfy the restrictions:

(R11) $\sum_{\alpha \in \Gamma_k} \psi_\alpha + \psi_{\kappa_1} + \cdots + \psi_{\kappa_d} = 1$;

(R12) $\psi_{\kappa_j} = \frac{1}{k} - \sum_{l=1}^{k-1} \frac{1}{l} \sum_{\alpha \in \Gamma_k; \alpha_j = l} \psi_\alpha$, for $j = 1, \ldots, d$.

For all Borel sets $B \subset S_d$, the measure defined by

$$H_k(B) := \sum_{1 \leq j \leq d} 1_B(e_j)\psi_{\kappa_j} + \int_{S_d \cap B} h_{k-d}(w)dw,$$  \hfill (3.16)

is a valid angular measure. For each integer $k > d$, define $\Psi_k := \{\psi_{\kappa_j}, j = 1, \ldots, d, \psi_\alpha, \alpha \in \Gamma_k : (R11)-(R12) \text{ are satisfied}\}$ and

$$\mathcal{H}_k := \{B \mapsto H_k(B) = \sum_{1 \leq i \leq d} 1_B(e_j)\psi_{\kappa_j} + \int_{S_d \cap B} h_{k-d}(w)dw : (\psi_{\kappa_j}, j = 1, \ldots, d, \psi_\alpha, \alpha \in \Gamma_k) \in \Psi_k\}.$$
Consistency of Bayesian Inference for Max-stable Distributions

It is already known from Boldi and Davison (2007) that $\cup_{k=1}^{\infty} H_k$ is a dense subset of $(\mathcal{H}, \mathcal{D}_W)$. We establish an analogue result for the stronger topology induced by $\mathcal{D}_T$, see Section 2.9 of the supplementary material for a concise proof.

Proposition 3.10. For fixed dimension $d > 2$, $\cup_{k=1}^{\infty} H_k$ is a dense subset of $(\mathcal{H}, \mathcal{D}_T)$, where $H \subset H$ is the class of angular measures with continuous angular density on $\mathcal{S}_d$.

The following results establish the posterior consistency of our proposed inferential procedure in arbitrary dimensions.

Theorem 3.11. Let $Y_1, \ldots, Y_n$ be iid rv with distribution $G_\star(\cdot|H_0)$, where $H_0$ satisfies Condition 3.8. Let $\Pi_H$ be the prior distribution on $H$ induced by a prior distribution $\Pi$ defined on $\cup_{k \geq k'}(\{k\} \times \Psi_k)$, for some $k' \in \mathbb{N} \setminus \{1, \ldots, d\}$, satisfying:

(i) $\Pi(\{k\}) > 0$, $\Pi(B|k) > 0$ for all Borel sets $B \subset \Psi_k$ and $k \geq k'$;
(ii) $\Pi(\{k, k + 1, k + 2, \ldots\}) \lesssim e^{-qk}$, for some $q > 0$.

Then, for all $\epsilon > 0$ we have

$$
\lim_{n \to \infty} \Pi_H\{H \in H : \mathcal{D}_H(g_\star(\cdot|H), g_\star(\cdot|H_0)) > \epsilon\} = 0, \quad G_\infty^\star(\cdot|\theta_0) - a.s.,
$$

$$
\lim_{n \to \infty} \Pi_H\{H \in H : \mathcal{D}_\infty(A, A_0) > \epsilon\} = 0, \quad G_\infty^\star(\cdot|\theta_0) - a.s.,
$$

$$
\lim_{n \to \infty} \Pi_H\{H \in H : \mathcal{D}_W(H, H_0) > \epsilon\} = 0, \quad G_\infty^\star(\cdot|\theta_0) - a.s.,
$$

where $A$ and $A_0$ are the Pickands dependence functions associated with $H$ and $H_0$, and $\Pi_H(\cdot) := \Pi_H(\cdot|Y_1, \ldots, Y_n)$ is the posterior distribution corresponding to $\Pi_H$.

Observe that consistency of the posterior on the angular distribution is now obtained with a weaker metric than in the bivariate case. The lack of an explicit relation between the gradient $\nabla A$ and the angular distribution $H$ does not allow to turn the second consistency result of Theorem 3.11 into consistency on the space $H$ with Kolmogorov-Smirnov distance.

4. Extensions

Here we extend the consistency results discussed in Section 3 weakening two of the assumptions considered therein. First, we assume that the margins of max-stable distributions are members of the Fréchet family with unknown shape parameters. Second, we assume to work with block maxima instead of data that exactly follow a max-stable distribution. The maxima are derived from a block of (normalised) observations, whose distribution is assumed to be in the domain of attraction of a simple max-stable distribution. Hence, the distribution of the block maxima is only approximately simple max-stable, provided that the block size is large enough.
4.1. Fréchet marginal distributions with unknown shape parameters

Consider the generalized extreme-value distribution in (2.3) with \( \gamma_j \neq 0, \ j = 1, \ldots, d \). For a tail index \( \gamma_j > 0 \), set \( a_j = 1/\gamma_j \) and \( G_{a_j}(x_j) = G_{\gamma_j}((x_j - 1)/\gamma_j) \), then \( G_{a_j}(x_j) = \exp(-x^{-\gamma_j}) \) for \( x_j > 0 \) is the so-called \( a_j \)-Fréchet distribution (e.g. Falk et al., 2011, p. 35).

Here we assume to deal with a multivariate max-stable distribution with \( a_j \)-Fréchet margins, where the shape parameters \( a_j \) can be different from each other. For given angular measure \( H \in \mathcal{H} \) and shape parameters \( a = (a_1, \ldots, a_d) \), the density function is now of the form

\[
g(x|H,a) = \prod_{j=1}^{d} x_{i_j}^{a_j-1} g_*(x_{a_1}, \ldots, x_{a_d}|H), \quad x \geq 0,
\]

with \( g_*(\cdot|H) \) as in (2.6). We denote by \( G(x|H,a) \) the corresponding max-stable distribution function. We consider independent prior distributions for \( H \) and \( a \). A prior distribution on \( H \) is specified via a Bernstein polynomials representation as in Sections 3.2.2 and 3.3.2, for the cases \( d = 2 \) and \( d > 2 \), respectively. For brevity, we herein refer to such prior distribution as Bernstein-polynomials prior distribution on \( H \). A sequence of prior distributions for \( a \) is specified as follows.

**Condition 4.1.** Let \( \Pi_{a_n} \) be a sequence of prior distribution on \( a \) supported on \( \mathcal{A}_n = (1/2, n^{1/d})^d \). Assume that \( \Pi_{a_n} \) admits a positive and continuous Lebesgue density on \( \mathcal{A}_n \).

Such assumption helps to control the metric complexity of the support of the joint prior distribution on \( (H,a) \). Details on this point are provided in the proof of the following result, in Section 2.10 of the supplementary material.

**Theorem 4.2.** Let \( X_1, \ldots, X_n \) be iid rv with distribution \( G(\cdot|H_0, a_0) \), where \( H_0 \) satisfies Condition 3.8 and \( a_0 \in (1/2, \infty)^d \). Let \( \Pi_{H \times a_n} := \Pi_H \times \Pi_{a_n} \), where \( \Pi_{a_n} \) satisfies Condition 4.1 and \( \Pi_H \) is a Bernstein-polynomials prior distribution on \( H \), satisfying the assumptions of Corollary 3.7 and Theorem 3.11 for \( d = 2 \) and \( d > 2 \), respectively. Then, for \( \gamma(H_0)|H_0, a_0| - a.s.

\[
\lim_{n \to \infty} \Pi_n \{ H \in \mathcal{H}, \ a \in (1/2, \infty)^d : \mathcal{D}_H(g(\cdot|H,a), g(\cdot|H_0,a_0)) > \epsilon \} = 0,
\]

\[
\lim_{n \to \infty} \Pi_n \{ a \in (1/2, \infty)^d : \|a - a_0\|_1 > \epsilon \} = 0,
\]

where \( \Pi_n(\cdot) = \Pi_{H \times a_n}(\cdot|X_1, \ldots, X_n) \) is the posterior distribution corresponding to \( \Pi_{H \times a_n} \).

Notice that for every \( H, H' \in \mathcal{H} \) and \( a, a' \in (1/2, \infty)^d \)

\[
\mathcal{D}_H(g_*(\cdot|H), g_*(\cdot|H')) \leq \mathcal{D}_H(g(\cdot|H,a), g(\cdot|H',a')).
\]
Consequently, Theorem 4.2 and Propositions A.1-A.2 guarantee that all the consistency results established in Theorems 3.6, 3.11 and Corollary 3.7 also obtain in the present setting, with prior $\Pi_{H\times A_n}$. Finally, notice also that for technical convenience we focused on the case of $a_j > 1/2$, for $j = 1, \ldots, d$. Our results are still valid assuming $a_j > \varepsilon$, with $\varepsilon > 0$ arbitrarily small. Though, this is nonessential for statistical purposes: with $1/2 < a_j \leq 1$ distributions with very heavy tails are considered, while with $a_j > 1$ still heavy-tailed distributions are taken into account, but with lighter tails than those in the previous case. This is already a quite rich class of distributions for practical applications. Also, consistency is preserved if $n^{1/d}$ is replaced with another positive power, in the definition of $A_n$.

### 4.2. Arrays of componentwise maxima

The consistency results introduced in Section 3 are obtained exploiting Theorem 6.23 in Ghosal and van der Vaart (2017). In particular, for an appropriate map $\varphi$ and a distance $D$, strong consistency is attained from the exponential bounds

$$G_n^\varphi(\Pi_n \{ \varphi(\theta), \varphi(\theta_0) \}) > \epsilon \} > 1/\epsilon \}
\varepsilon \} \gtrsim e^{-c_n n}.$$

and Borel-Cantelli lemma. In the above display, $\epsilon'$, $\epsilon$ and $c_\epsilon$ are positive constants, $\Pi_n(\cdot) \equiv \Pi_\Theta(\cdot|Z_1,\ldots,Z_n)$ is the posterior distribution corresponding to the prior distribution $\Pi_\Theta$, $(Z_1,\ldots,Z_n)$ follows the distribution $G^\ast(\cdot|\theta_0)$ and $E_0$ denotes the expectation with respect to $G^\ast(\cdot|\theta_0)$.

We now assume that the data sample, $M_{m_1,n},\ldots,M_{m_n,n}$, consists of $n$ i.i.d. block maxima with distribution $F_{0,m_1}^{m_n}(m_n)$, weakly converging to a simple max-stable distribution $G_\ast(\cdot|\theta_0)$. Precisely, we recall that each $M_{m_i,n}$, $i = 1,\ldots,n$, is a r.v. of normalised componentwise maxima obtained from a block of $m_i$ i.i.d. r.v.s whose distribution is $F_0$. We assume $m_n \to \infty$ as $n \to \infty$. This simplifies the asymptotic analysis, avoiding considerations about double limits. For brevity we denote with $Q_{m_n}$ the probability measure pertaining $F_{0,m_n}^{m_n}(m_n)$, and by $Q_{m_n}$ the corresponding $n$-fold product measure. We are interested in establishing $Q_{m_n}^\ast$-almost sure consistency of the pseudo-posterior distribution defined via

$$\tilde{\Pi}_n(B) := \frac{\int_B \prod_{i=1}^n g_\ast(M_{m_i,n}|\theta)\Pi_\Theta(d\theta)}{\int_{\Theta} \prod_{i=1}^n g_\ast(M_{m_i,n}|\theta)\Pi_\Theta(d\theta)},$$

for all $\Pi_\Theta$-measurable sets $B$.

To accomplish the above objective, we resort to a specific form of remote-contiguity. Recently, Kleijn (2017) has introduced a weaker notion of contiguity than the classical one. The latter can be successfully exploited to establish weak consistency (i.e. in probability) of the pseudo-posterior distribution with parametric limiting models (e.g. de Boor and Fix, 1973), while is unsuitable to obtain strong consistency and less accessible for nonparametric models.
Definition 4.3. Consider two sequences \( \rho_n, \tau_n > 0 \) such that as \( n \to \infty \), \( \rho_n, \tau_n \to 0 \). As \( n \to \infty \), \( Q_{m_n}^n \) is said to be:

- \( i \): contiguous with respect to \( G_n^*(\cdot|\theta_0) \) if, for a sequence of measurable events \( E_n \),
  \[ G_n^*(E_n|\theta_0) = o(1) \implies Q_{m_n}^n(E_n) = o(1); \]
- \( ii \): \( \rho_n \)-remotely contiguous with respect to \( G_n^*(\cdot|\theta_0) \), if \( G_n^*(E_n|\theta_0) = o(\rho_n) \implies Q_{m_n}^n(E_n) = o(1); \)
- \( iii \): \( \rho_n \)-to-\( \tau_n \)-remotely contiguous with respect to \( G_n^*(\cdot|\theta_0) \), if \( G_n^*(E_n|\theta_0) = o(\rho_n) \implies Q_{m_n}^n(E_n) = o(\tau_n), \)

where \( \Longrightarrow \) denotes the usual implication symbol.

See Kleijn (2017, Section 3) for a comprehensive account. Now, we establish the following result.

**Theorem 4.4.** Assume \( F_0 \in D(G_\lambda^*(\cdot|\theta_0)) \). Also, assume that the stable tail-dependence function corresponding to \( G_\lambda^*(\cdot|\theta_0) \) has partial derivatives of order \( d \) and that the copula corresponding to \( F_0 \) satisfies condition (13) in Falk et al. (2019). Then, \( Q_{m_n}^n \) is \( e^{-cn} \)-to-\( e^{-cn'} \)-remotely contiguous with respect to \( G_n^*(\cdot|\theta_0) \), for all \( 0 < c' < c \).

The interest in \( e^{-cn} \)-to-\( e^{-cn'} \)-remote contiguity is motivated by (4.2). Together with Theorem 4.4 and Borel-Cantelli Lemma, the latter entails that for all \( \epsilon > 0 \)

\[
\lim_{n \to \infty} \Pi_n \{ D(\varphi(\theta), \varphi(\theta_0)) > \epsilon \} = 0, \quad Q_{k_n}^n - a.s.
\]

provided that \( \theta_0 \) and \( \Pi_\Theta \) satisfy suitable conditions, as discussed in Section 3. This extends the strong consistency results presented therein to the present setting.

**Appendix A: Proofs**

### A.1. Metric structure of multivariate simple max-stable models

In this section we provide some notations and auxiliary results that will be useful in Appendices A.2-A.4. In the sequel we consider distributions of arbitrary dimension \( d \geq 2 \), unless otherwise specified.

**A.1.1. Notation**

We recall that by the spectral representation (e.g. Falk et al., 2011, Ch. 4-5), any max-stable r.v. can be written as \( Y = \max(\xi_i, i = 1, 2, \ldots) \), where \( \xi_i, i = 1, 2, \ldots \) are the points of a Poisson process on \( E \) with intensity measure \( \Lambda \). In particular, under Condition 2.1, the latter has Lebesgue densities \( \lambda_{(j)}(z) = z_j^{-2} H((e_j)) \) on \( E_{(j)}, j = 1, \ldots, d \), and \( \lambda_{(1,\ldots,d)}(z) = \|z\|^{d-1} h(z/\|z\|) \) on \( E_{(1,\ldots,d)} \). We recall that the definition of \( E \) and \( E_I \) is given in Section 2. For \( j = 1, \ldots, d \), define the random index \( i^*_j \) by

\[
i^*_j = \max_{i=1,2,\ldots} \xi_{i,j}.
\]
Then, the set \{i^*_1, \ldots, i^*_d\} induces a random partition of \{1, \ldots, d\}. For a given angular measure \(H \in \mathcal{H}\), we denote the joint probability density of a simple max-stable r.v. and the corresponding random partition by

\[
f(y, \mathcal{P}|H) = G_*(y|H) \prod_{i=1}^{m} \Delta(I_i, y), \quad \mathcal{P} = \{I_1, \ldots, I_m\} \in \mathcal{P}_d, \quad y \in (0, \infty),
\]

see e.g. (Domby and Eyi-Minko, 2013, pp. 4816-4817) for details. For \(\delta > 0\), let \(B_{\delta, \infty}(H) := \{H' \in \mathcal{H} : \mathcal{D}_\infty(h, h') \leq \delta\}\) and \(N(\delta, B, \mathcal{D})\) be the \(\delta\)-covering number of a set \(B\) with respect to the metric \(\mathcal{D}\), see e.g. Ghosal and van der Vaart (2017, Appendix C).

A.1.2. Metric and divergence results

The results presented in Section 3 involve several types of distance or divergence for simple max-stable distributions and the associated dependence functions. The following propositions describe the relationships existing among those.

**Proposition A.1.** Let \(A\) and \(A'\) be the Pickands dependence functions corresponding to the angular measures \(H, H' \in \mathcal{H}\), respectively. The following results hold true:

(i) \(e^{-1}\mathcal{D}_\infty(A, A') \leq \mathcal{D}_H(g_*(\cdot|H), g_*(\cdot|H')) \leq \|f(\cdot, \cdot|H) - f(\cdot, \cdot|H')\|_1^{1/2}\);

(ii) \(\mathcal{D}_\infty(A, A') \leq \min\{2\mathcal{D}_\infty(h, h')/\Gamma(d), 2d\|h - h'\|_1\};

(iii) \(\forall \epsilon > 0, \exists \eta > 0\) such that, if \(\mathcal{D}_\infty(G_*(\cdot|H), G_*(\cdot|H')) < \eta\), then \(\mathcal{D}_W(H, H') < \epsilon\);

where \(\Gamma(\cdot)\) is the gamma function.

For the proof, see Section 2.1 in the supplementary material.

**Proposition A.2.** In the specific case of \(d = 2\), the following results hold true for any \(H', H'' \in \mathcal{H}\), with corresponding Pickands \(A', A''\):

(i) \(\mathcal{D}_H^\top(g_*(\cdot|H'), g_*(\cdot|H'')) \leq c\|h' - h''\|_1\), for a positive global constant \(c\);

(ii) \(\forall \epsilon > 0, \exists \eta > 0\) such that, if \(\mathcal{D}_\infty(A', A'') < \eta\), then \(\mathcal{D}_{1, \infty}(A', A'') \leq \epsilon\); in particular, \(\mathcal{D}_\infty(H', H'') \leq \epsilon\);

(iii) further assuming that \(H' \in \mathcal{H}\), then \(\forall \epsilon > 0, \exists \eta > 0\) such that, if \(\mathcal{D}_\infty(H', H'') < \eta\), then \(\|h' - h''\|_1 \leq \epsilon\);

where \(\mathcal{H}\) is as in Proposition 3.10.

For the proof, see Section 2.2 in the supplementary material.

**Proposition A.3.** Let \(H \in \mathcal{H}'\), with \(\mathcal{H}'\) as in Theorem 3.9. Then, for every \(\epsilon > 0\) there exists \(\delta > 0\) such that, for all \(H' \in B_{\delta, \infty}(H)\) and \(H'' \in \mathcal{H}\):

\[
\int_{(0, \infty)} \sum_{\mathcal{P} \in \mathcal{P}_d} \log \frac{f(y, \mathcal{P}|H)}{f(y, \mathcal{P}|H')} f(y, \mathcal{P}|H'') dy \leq \epsilon + \log(1 + \epsilon). \tag{A.1}
\]

In particular, \(\mathcal{X}(f(\cdot, \cdot|H), f(\cdot, \cdot|H')) \leq \epsilon + \log(1 + \epsilon)\).
Proof. Define \( \bar{\epsilon} \) via \( (1 + \bar{\epsilon}) = (1 + \epsilon)^{1/d} \) and \( h_{\inf} := \inf_{w \in S_d} h(w) \). Let

\[
0 < \delta < \min \left\{ \frac{\epsilon}{2cd}, \frac{\bar{\epsilon}}{1 + \epsilon} h_{\inf}, \frac{\bar{\epsilon} \cdot d}{1 + \epsilon} \cdot \min_{j=1,\ldots,d} p_j \right\},
\]

(A.2)

with \( c = 1/\Gamma(\delta) \). Fix \( H' \in B_{\delta,\infty}(H) \). Then, the left hand-side of (A.1) equals

\[
T_1 + T_2 \equiv \int_{(0,\infty)} \sum_{p \in \mathcal{P}_d} [V'(y) - V(y)] f(y, P|H'')dy
+ \int_{(0,\infty)} \sum_{p \in \mathcal{P}_d} \log \prod_{i=1}^m \frac{\Delta(I_i, y)}{\Delta'(I_i, y)} f(y, P|H'')dy.
\]

where \( V' \) and \( \Delta' \) are the exponent function and its partial derivatives (see (2.6)) corresponding to the angular measure \( H' \).

By Proposition A.1(ii) and the bound in (A.2) we have that

\[
|T_1| \leq \int_{(0,\infty)} \sum_{p \in \mathcal{P}_d} \left| A' \left( \frac{1/y}{\|1/y\|_1} \right) - A \left( \frac{1/y}{\|1/y\|_1} \right) \right| \|1/y\|_1 f(y, P|H'')dy
\]

\[
\leq d \mathcal{D}_\infty(A, A') < \epsilon.
\]

Also, it holds that \( T_2 \leq \log(1 + \epsilon) \), since for all \( i \in \{1, \ldots, m\}, P \in \mathcal{P}_d, y \in (0, \infty) \) we have

\[
\frac{\Delta(I_i, y)}{\Delta'(I_i, y)} \leq 1 + \bar{\epsilon}.
\]

The latter inequality easily follows from (2.8) and these facts:

(i) since \( \mathcal{D}_\infty(h, h') < \delta \), then \( h - h' < \delta = \frac{\delta}{h_{\inf} - \bar{\epsilon}} (h_{\inf} - \delta) < \bar{\epsilon} h' \) and

\[
\int_{(0,y_{\delta,c})} \frac{d \left[ h(z/\|z\|_1) - h'(z/\|z\|_1) \right]}{\|z\|_1^{d+1}} \left| z_{\|z\|_1} = y_{i_1} \right| dz_{\|z\|_1} \leq \epsilon \int_{(0,y_{\delta,c})} \frac{dh'(z/\|z\|_1)}{\|z\|_1^{d+1}} \left| z_{\|z\|_1} = y_{i_1} \right| dz_{\|z\|_1};
\]

(ii) since

\[
(p_j - p'_j)/p'_j = \int_{S_d} w_j [h'(w) - h(w)] dw_1 \ldots dw_d \leq \frac{\delta c d^{-1} - 1}{p_j - \delta c d^{-1}} < \bar{\epsilon},
\]

thus \( p_j y_{i_j}^{-2} \leq (1 + \bar{\epsilon}) p'_j y_{i_j}^{-2} \), whenever \( I_i = \{j\} \).

The proof is now complete. \( \square \)

A.2. Kullback-Leibler property

In this appendix, we prove Theorems 3.2 and 3.9. We highlight the differences between the bivariate case and the higher dimensional case.
A.2.1. Proof of Theorem 3.2

We recall that in the bivariate case we have that \( A(t) = 1 + 2 \int_0^t H(w)dw - t \), \( A^{(1)}(t) = -1 + 2H(t) \) and \( A^{(2)}(t) = 2h(t) \), for all \( A \in \mathcal{A} \). In particular, such relations entail that, for \( A \in \mathcal{A}' \) and a sequence \( \{A_k, k = 1, \ldots, \infty \} \subset \mathcal{A} \),

\[
\lim_{k \to \infty} \mathcal{D}_{2, \infty}(A, A_k) = 0 \iff \lim_{k \to \infty} \mathcal{D}_{\infty}(h, h_k) = 0, \tag{A.3}
\]

where “\( \iff \)” denotes the usual “if and only if” symbol. Consequently, when \( A_0 \) satisfies Condition 3.1(i), which is the bivariate analogue of Condition 3.8, then the result follows directly from Theorem 3.9. Instead, the case \( A_0 \) satisfying Condition 3.1(ii) is not covered in the higher dimensional case, yet a similar proof scheme can be exploited. For brevity, here we only sketch the main changes.

For small constants \( \epsilon_1, \epsilon_2 \), there exist positive bounded functions \( \gamma_{\epsilon_1}, \gamma_{\epsilon_2} \) which are continuous on \([0, 1] \), satisfy \( \gamma_{\epsilon_1} \leq h_0(t), \forall t \in (0, \epsilon_1) \), \( \gamma_{\epsilon_2} \leq h_0(t), \forall t \in (1 - \epsilon_2, 1) \), and are such that the function

\[
h_{i_0}(t) := \begin{cases} 
\gamma_{\epsilon_1}(t), & t \in (0, \epsilon_1) \\
h_0(t), & t \in (\epsilon_1, 1 - \epsilon_2) \\
\gamma_{\epsilon_2}(t), & t \in (1 - \epsilon_2, 1) 
\end{cases}
\]

is continuous on \([0, 1] \) and bounded from below by \( \inf_{w \in (0, 1)} h_0(w) > 0 \). Furthermore, set \( p_{i_0,0} = 1/2 - \int_0^1 (1 - t)h_{i_0}(t)dt \) and \( p_{i_0,1} = 1/2 - \int_0^1 t h_{i_0}(t)dt \). Then, the corresponding angular measure \( H_{i_0} \) is an element of \( \mathcal{H}' \), with \( \mathcal{H}' \) as in Theorem 3.9 (equivalently, the associated Pickands \( A_{i_0} \) is in \( \mathcal{A}' \)). Thus, we can exploit (A.5), setting \( p = 0 \), and conclude by showing that \( \mathcal{A}(f(\cdot, \cdot | H_0), f(\cdot, \cdot | H_{i_0})) \) can be made arbitrarily small if \( \epsilon_1 \) and \( \epsilon_2 \) are chosen small enough. See Section 2.3 of the supplementary material for details.

A.2.2. Proof of Theorem 3.9

It is sufficient to show that

\[
\Pi_{\mathcal{H}}(H : \mathcal{A}(f(\cdot, \cdot | H_0), f(\cdot, \cdot | H)) < \epsilon) > 0, \quad \forall \epsilon > 0 \tag{A.4}
\]

since, for any pair \( H, H' \in \mathcal{H} \), we have that

\[
\mathcal{A}(g_*(\cdot | H), g_*(\cdot | H')) \leq \mathcal{A}(f(\cdot, \cdot | H), f(\cdot, \cdot | H')).
\]

We do so by exploiting the following argument. For suitable \( p \in \mathbb{N} \) and indexes \( \{i_0, \ldots, i_p\} \) to be specified later, we introduce a set of angular measures \( \{H_{i_0}, H_{i_1}, \ldots, H_{i_p}\} \subset \mathcal{H} \), with \( H_{i_k} \in \mathcal{H}' \). By hypothesis, there exists a sequence \( \{H_k, k = 1, \ldots, \infty\} \subset \mathcal{H} \) satisfying
(3.12), with \( h \) replaced by \( h_{i_j} \). Let \( H' \in B_{\delta,\infty}(H_k) \), then we have that
\[
\mathcal{K}(f(\cdot,\cdot|H_0), f(\cdot,\cdot|H')) = \mathcal{K}(f(\cdot,\cdot|H_0), f(\cdot,\cdot|H_{i_0}))
\]
\[
+ \sum_{j=0}^{p-1} \int_{(0,\infty)} \sum_{P \in \mathcal{P}_d} \log \frac{f(y, P|H_i)}{f(y, P|H_{i+1})} f(y, P|H_0) dy
\]
\[
+ \int_{(0,\infty)} \sum_{P \in \mathcal{P}_d} \log \frac{f(y, P|H_{i_0})}{f(y, P|H_k)} f(y, P|H_0) dy \tag{A.5}
\]
and, by Proposition A.3, the third and fourth terms on the right hand-side can be made arbitrarily small by choosing \( k \) sufficiently large and \( \delta \) sufficiently small. If \( H_{i_j}'s \) are chosen to make the first two terms sufficiently small, the expression on right-hand side can be bounded above by \( \epsilon \). Then, the conclusion follows from the assumption: \( \Pi(B_{\delta,\infty}(H_k)) > 0 \), for every \( k \) and \( \delta \).

We now construct suitable angular measure \( H_{i_j}'s \), considering different types of \( H_0 \) separately.

**Case 1:** \( \inf_{w \in \mathcal{S}_d} h_0(w) > 0 \) and \( p_{0,j} > 0, \ j = 1,\ldots, d \).
In this case we have \( H_0 \in \mathcal{H} \) and therefore there exists a sequence \( (H_k, k = 1,\ldots, \infty) \) satisfying (3.12) with \( h \) replaced by \( h_0 \). Consequently, no intermediate angular measure \( H_{i_j} \) is needed, i.e. in the right-hand side of (A.5) it is sufficient to keep the first and last term, substituting \( H_{i_0} \) with \( H_k \).

**Case 2:** \( \inf_{w \in \mathcal{S}_d} h_0(w) > 0 \) and \( p_{0,j} = 0 \) for some \( j \in \{1,\ldots, d\} \).
In this case, we set \( p = 0 \), i.e. a single intermediate angular measure is needed. Fix arbitrarily small \( \epsilon' < 2cM_0 \), where \( M_0 := \|h_0\|_\infty \) and \( c = 1/\Gamma(d) \). Set \( \epsilon' \) via \( (1 + \epsilon') = (1 + \epsilon')^{1/d} \) and
\[
0 < c_0 < \min \left\{ \frac{\epsilon'}{2dcM_0}, \epsilon' \right\},
\]
then define \( h_{i_0} = h_0/(1 + c_0) \) and
\[
p_{0,j} = d^{-1} - \int_{\mathcal{S}_d} w_j h_{i_0}(w) dw_1 \cdots dw_d, \quad j = 1,\ldots, d.
\]
Denote by \( H_{i_0} \) the associated angular measure. Proceeding as in the proof of Proposition A.3 and using results similar to (i)-(ii) therein, we can show \( \mathcal{K}(f(\cdot,\cdot|H_0), f(\cdot,\cdot|H_{i_0})) \leq \epsilon' + \log(1 + \epsilon') \).

**Case 3:** \( \inf_{w \in \mathcal{S}_d} h_0(w) = 0 \).
Set \( p = 1 \) and fix an arbitrarily small \( \epsilon' < 1 \). Let \( \epsilon', c_0 \) and \( H_{i_0} \) be defined as above. Define
\[
0 < c_1 < \min \left\{ \frac{\epsilon'}{2dc}, \frac{1}{c_0}, \frac{\epsilon'}{1 + \epsilon'} \right\}.
\]
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\[ h_{i_1} := h_{i_0} + c_1 \] and

\[ p_{i_1,j} := d^{-1} - \int_{S_d} w_j h_{i_1}(w) dw_1 \ldots dw_d, \quad j = 1, \ldots, d. \]

Again, with a few adaptations to the proof of Proposition A.3, we obtain

\[ \mathcal{K}(f(\cdot, \cdot \mid H_0), f(\cdot, \cdot \mid H_{i_0})) < \epsilon' + \log(1 + \epsilon') \]

and

\[ \int_{(0, \infty)} \sum_{P \in \mathcal{D}} \log \frac{f(y, P \mid H_{i_0})}{f(y, P \mid H_{i_1})} f(y, P) dy < \epsilon' + \log(1 + \epsilon'). \]

The proof is now complete.

A.3. Proof of Theorem 3.6

To prove the first statement, we resort to Theorem 6.23 in Ghosal and van der Vaart (2017) and verify that the conditions therein are satisfied. In particular, an application of Theorem 3.2 allows to conclude that \( g_\ast(\cdot \mid A_0) \) is in the Kulback-Leibler support of the prior distribution. The existence of suitable approximating sequences required by the latter follows from Theorem 6.3.2 in Davis (1975), when the extremal dependence is represented through Bernstein polynomials, and by Theorem 2.1 in de Boor and Fix (1973) and Proposition 3.3, when the extremal dependence is represented through splines.

Now, let \( A_k \) and \( \tilde{A}_k \) be two Pickands dependence functions in BP form, with degree \( k \). Let \( H_{k-1} \) and \( \tilde{H}_{k-1} \) be the corresponding angular distributions in BP form obtained by applying Proposition 3.2(i) in Marcon et al. (2016). The associated angular densities satisfy

\[ \| h_{k-2} - \tilde{h}_{k-2} \|_1 = \left| \int \sum_{j=0}^{k-2} (\eta_{j+1} - \eta_j - \tilde{\eta}_{j+1} + \tilde{\eta}_j) \text{Be}(w|j+1, k-j-2) \right| dw \]

\[ \leq \sum_{j=0}^{k-2} |\eta_{j+1} - \eta_j - \tilde{\eta}_{j+1} + \tilde{\eta}_j|. \]

Analogously, let \( A_k \) and \( \tilde{A}_k \) be two valid Pickands dependence spline functions of order 3 and \( H_{k-1} \) and \( \tilde{H}_{k-1} \) be the corresponding angular distribution spline functions, obtained by applying Proposition 3.4. By (3.6) with \( m = 3 \), their angular densities satisfy

\[ \| h_{k-2} - \tilde{h}_{k-2} \|_1 = \left| \int \sum_{j=1}^{k-2} \left( \frac{\eta_{j+1} - \eta_j - \tilde{\eta}_{j+1} + \tilde{\eta}_j}{\tau_{j+2} - \tau_{j+1}} \right) 1_{[\tau_{j+1}, \tau_{j+2}]}(w) \right| dw \]

\[ \leq \sum_{j=1}^{k-2} |\eta_{j+1} - \eta_j - \tilde{\eta}_{j+1} + \tilde{\eta}_j|. \]
By the above inequalities, Proposition A.2(i) and Ghosal and van der Vaart (2017, Proposition C.2) we have that, for any $\tilde{A}_k \subset A_k$ and set of the form $G^{(k)} = \{ g_\ast(\cdot | A_k) : A_k \in \tilde{A}_k \}$, 

$$N(\epsilon, G^{(k)}, \mathcal{D}_H) \leq N(\epsilon' \epsilon^2, \{ x \in \mathbb{R}^{k-1} : \| x \|_1 \leq 1 \}, L_1) \leq (3/\epsilon' \epsilon^2)^{k-1},$$

for some $\epsilon' > 0$, where, without loss of generality, we assume $\epsilon' \epsilon^2 < 1$. Therefore, Conditions i. and ii. in Ghosal and van der Vaart (2017, Theorem 6.23) can be verified by using similar arguments to those in Appendix A.4.

The second statement follows from the first one together with Proposition A.1(i) and Proposition A.2(ii).

### A.4. Proof of Theorem 3.11

Firstly, we verify the conditions of Theorem 6.23 in Ghosal and van der Vaart (2017) to establish the first of the three results. By assumption, the angular measure $H_0$ satisfies Condition 3.8. Therefore, combining Lemma 2.1 in the supplementary material, assumption (i) of Theorem 3.11 and Theorem 3.9, we can conclude that $g_\ast(\cdot | H_0)$ is in the Kulback-Leibler support of the prior. Next, observe that $\mathcal{D}_H$ is a metric that generates convex balls and define 

$$G^{(k)} := \{ g_\ast(\cdot | H_k) : H_k \in \mathcal{H}_k; \mathcal{D}_H(g_\ast(\cdot | H_k), g_\ast(\cdot | H_0)) > 4\epsilon \},$$

$$G_{n,1} := \bigcup_{k=d+1}^{\nu_n} G^{(k)},$$

$$G_{n,2} := \bigcup_{k=\nu_n+1}^{\infty} G^{(k)},$$

where $\nu_n$ is a sequence of positive integers. Then, by Lemma 2.5 in the supplementary material and Ghosal and van der Vaart (2017, Proposition C.2) we have 

$$N(2\epsilon, G^{(k)}, \mathcal{D}_H) \leq N \left( c' \epsilon^2, \{ x \in \mathbb{R}^{k-1} : \| x \|_1 \leq 1 \}, L_1 \right) \leq (3/c' \epsilon^2)^{\left( \frac{k-1}{d} \right)},$$

where $c'$ is a positive global constant and, without loss of generality, we assume $c' \epsilon^2 < 1$. As a consequence, we also have that 

$$N(2\epsilon, G_{n,1}, \mathcal{D}_H) \leq \sum_{k=d+1}^{\nu_n} \left( 3/c' \epsilon^2 \right)^{\left( \frac{k-1}{d} \right)} \leq \nu_n \left( 3/c' \epsilon^2 \right)^{\nu_n^d} \quad (A.6)$$

and by choosing $\nu_n \sim (nc^2)^{1/d} (\log(3/c' \epsilon^2))^{-1/d}$ it follows that 

$$\log N(2\epsilon, G_{n,1}, \mathcal{D}_H) \leq \log \nu_n + \nu_n^d \log(3/c' \epsilon^2) \lesssim n\epsilon^2. \quad (A.7)$$

The first condition in (Ghosal and van der Vaart, 2017, Theorem 6.23) is therefore satisfied. The second condition therein is satisfied in light of assumption (ii) in Theorem 3.11. From all this, the first result in the statement of Theorem 3.11 now follows. The second result is a direct consequence of the first one and Proposition A.1(i). The third result is a direct consequence of the first one and Proposition A.1(iii).
A.5. Proof of Theorem 4.4

Preliminarily observe that, denoting by \( q_{m_n} \) the Lebesgue density of \( Q_{m_n} \), both \( q_{m_n}(z) > 0 \) and \( g_*(z|\theta_0) > 0 \), for all \( z \in (0, \infty)^d \). Moreover, under the considered assumptions, Proposition 3.1 in Falk et al. (2019) implies that

\[
\mathcal{D}_r(Q_{m_n}, G_*(\cdot|\theta_0)) \to 0, \quad n \to \infty. \tag{A.8}
\]

Without loss of generality, let \( \tau > 0 \) be a constant such that, defining \( \delta = 2e^{-\tau/2}/(1 - e^{-\tau/2})^2 \), it holds that \( \delta > 1 \), e.g. \( \tau = 1/2 \). Let \( Z_i, i = 1, \ldots, n \), be \( n \) i.i.d. r.v.s distributed according to \( G_*(\cdot|\theta_0) \) and define

\[
\Phi_{m_n}(Z_i) = \log \left( \frac{q_{m_n}(Z_i)}{g_*(Z_i|\theta_0)} \right),
\]

\[
\tilde{\Phi}_{m_n}(Z_i) = \begin{cases} \Phi_{m_n}(Z_i), & \text{if } \Phi_{m_n}(Z_i) > \tau; \\ -\tau, & \text{otherwise}. \end{cases}
\]

Let \( E_n, n = 1, 2, \ldots \) denote a sequence of measurable sets such that \( G^n_*(E_n|\theta_0) = o(e^{-cn}) \), as \( n \to \infty \), for some positive constant \( c > 0 \). Observe that \( Q_{m_n}(E_n) \) equals

\[
\int_{E_n} \exp \left( \sum_{i=1}^{n} \Phi_{m_n}(z_i) \right) \prod_{i=1}^{n} g_*(z_i|\theta_0) \, dz_1 \cdots dz_n \\
\leq \int_{E_n} \exp \left( \sum_{i=1}^{n} \tilde{\Phi}_{m_n}(z_i) \right) \prod_{i=1}^{n} g_*(z_i|\theta_0) \, dz_1 \cdots dz_n \\
= \int_{E_n} \exp \left[ n^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^{n} \tilde{\Phi}_{m_n}(z_i) - \mathbb{E}_0 \tilde{\Phi}_{m_n}(Z_1) \right\} + n\mathbb{E}_0 \tilde{\Phi}_{m_n}(Z_1) \right] G^n_*(dz_1, \ldots, dz_n|\theta_0) \\
=: I_n.
\]

Consider two arbitrary positive constants \( p_1, p_2 \), satisfying \( p_1 + p_2 < 1 \). From (A.8) and the inequality \( \mathcal{D}_r^2(q_{m_n}, g_*(\cdot|\theta_0)) \leq 2\mathcal{D}_r(Q_{m_n}, G_*(\cdot|\theta_0)) \), it follows that for \( n \) sufficiently large

\[
\mathbb{E}_0 \tilde{\Phi}_{m_n}(Z_1) \leq -(1-\delta)\mathcal{D}_r^2(q_{m_n}, g_*(\cdot|\theta_0)) \leq p_1 c.
\]

Moreover, letting \( t_n = n^{1/2}p_2 c \), Wong et al. (1995, Lemma 6) entails that

\[
P \left( n^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{\Phi}_{m_n}(Z_i) - \mathbb{E}_0 \tilde{\Phi}_{m_n}(Z_1) \right) > t_n \right) \\
< \exp \left[ -t_n^2 \left\{ 8(c_r \mathcal{D}_r^2(q_{m_n}, g_*(\cdot|\theta_0)) + p_2 c) \right\}^{-1} \right],
\]

where \( c_r \) is a positive constant. As \( n \to \infty \), we have that

\[
n \exp \left[ -t_n^2 \left\{ 8(c_r \mathcal{D}_r^2(q_{m_n}, g_*(\cdot|\theta_0)) + c) \right\}^{-1} \right] \sim \exp \{ \log n - nc' \},
\]
where $c'$ is a positive constant and the term on the right-hand side converges to 0. Therefore, Borel-Cantelli Lemma implies that for large $n$

$$n^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{\Phi}_{m_n}(Z_i) - \mathbb{E}_0 \tilde{\Phi}_{m_n}(Z_1) \right) \leq n^{1/2} p_2 c, \quad G^*_n(\cdot|\theta_0) - \text{a.s.}$$

From all the above arguments it now follows that, as $n \to \infty$

$$I_n \leq \int_{E_n} \exp\{np_1c + np_2c\} G^n_n(dz_1, \ldots, dz_n|\theta_0)$$

$$= e^{n(p_1+p_2)c} G^n_n(E_n|\theta_0)$$

and

$$e^{(1-p_1-p_2)c_n} Q^n_{m_n}(E_n) \leq e^{cn} G^n_n(E_n|\theta_0) = o(1),$$

which is the result.

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