Research Article

Ulam–Hyers Stability of Caputo-Type Fractional Stochastic Differential Equations with Time Delays

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In this paper, we study a class of Caputo-type fractional stochastic differential equations (FSDEs) with time delays. Under some new criteria, we get the existence and uniqueness of solutions to FSDEs by Carathéodory approximation. Furthermore, with the help of Hölder’s inequality, Jensen’s inequality, Itô isometry, and Gronwall’s inequality, the Ulam–Hyers stability of the considered system is investigated by using Lipschitz condition and non-Lipschitz condition, respectively. As an application, we give two representative examples to show the validity of our theories.

1. Introduction

The fractional-order differential equations can better simulate many natural physical processes than integer-order differential equations, so it gradually becomes a powerful tool to analyze and solve problems in modern science and technology with the continuous development of natural science and production technology. It is mainly used in the fields of economy and insurance, the analysis of the quantitative structure of biological population, the control of diseases, and the research of genetic law, and we can see these monographs in [1–5]. For more notable achievements of this concept, the readers can also refer to [6–15].

As it is well known, stochastic disturbance is inevitable in practical systems, and it has an important influence on the stability of systems. In [16], \( du(t) = ku(t)dt \) was unstable when \( k > 0 \), but it increased the stochastic feedback control \( ru(t)dW(t) \) to become \( du(t) = ku(t)dt + ru(t)dW(t) \). Apparently, \( du(t) = ku(t)dt + ru(t)dW(t) \) was stable if and only if \( r^2 > 2k \). This fact indicated that the stochastic control \( ru(t)dW(t) \) can stabilize the unstable system \( du(t) = ku(t)dt \). Therefore, it is significant and challenging to study stochastic stabilization of deterministic systems. More relevant results can be found in [17–19].

The research on the existence and uniqueness of solutions to fractional differential equations is an important content of differential equations. At the same time, the existence and uniqueness have made rapid development in the field of applied mathematics. In [20], the authors studied the existence and uniqueness of positive solutions of some nonlinear fractional differential equations by using mixed monotone operators on cones. Under a number of new conditions and combined with the generalized Gronwall inequality, the uniqueness of solution for fractional \( \psi \)-Hilfer differential equation with time delays was investigated in [21]. In addition, for many other relevant conclusions, readers can refer to [22–25].

In 1940, S. M. Ulam proposed the stability to functional equations in a speech at the Wisconsin University [26].
Hyers [27] was the first to answer the question in 1941. From then, the Ulam–Hyers stability was produced. At the same time, more and more people were interested in exploring the Ulam–Hyers stability. In [28], by using fractional calculus, the properties of classical and generalized Mittag–Leffler functions and the Ulam–Hyers stability of linear fractional differential equations proved by utilizing the Laplace transform method. The authors investigated the Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability of impulsive integrodifferential equations with Riemann–Liouville boundary conditions in [29]. For more researched results, we can pay attention to [30–34].

Inspired by the abovementioned, in this article, we are concerned with the existence and Ulam–Hyers stability of Caputo-type FSDEs with time delays:

\[
\begin{cases}
\mathcal{C} D_0^\alpha X(t) = f(t, X(t), X(t - \tau)) + g(t, X(t), X(t - \tau)) \frac{dW(t)}{dt}, & t \in J = [0, T], \\
X(t) = \Phi(t), & t \in [-\tau, 0],
\end{cases}
\]

where \(X(0) = \Phi_{[0,t]}((1/2) < \alpha < 1, f: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \) and \(g: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}\) are measurable continuous functions. \(W(t)\) is an \(m\)-dimensional Brownian motion on a complete probability space \(\{\Omega, \mathcal{F}, \mathbb{P}\}\). \(\Phi(t): [-\tau, 0] \rightarrow \mathbb{R}^d\) is a continuous function, \(E|\Phi(t)|^2 < \infty, \) and \(E\) is the mathematical expectation.

Compared with the research results of [12, 20, 21, 24, 25, 28, 34], the major contributions of this paper include at least the following three points:

(1) In contrast to [20, 21, 28], the system we study is more generalized because it has not only the stochastic term but also the delay term.

(2) In the methods we investigate the existence and uniqueness of solutions to FSDEs are more novel than [24, 25]. In [24, 25], to explore the existence and uniqueness, Krasnoselskii’s fixed point theorem and Mönch’s fixed point theorem, respectively, were used. However, in this paper, we adopt the Caratheodory approximation to investigate the existence and uniqueness.

(3) In the study of various stability or existence and uniqueness of FSDEs, many literatures (see [12, 21, 34]) have used a stronger Lipschitz condition. However, in this paper, we used the weak non-Lipschitz condition to discuss the Ulam–Hyers stability of stochastic differential equations. This is a breakthrough in the exploration of the stability to FSDEs.

The structure of this article is arranged as follows. We present some basic definitions and necessary assumptions in Section 2. In Section 3, by Caratheodory approximation, a number of assumed conditions are established for existence and uniqueness of solutions. Section 4 is devoted to testify stability results for the FSDEs with time delays. Examples are given to certify the application of our findings in Section 5.

2. Preliminaries

In this section, we intend to recommend a few basic definitions, lemmas, and some necessary assumptions that will play a key role in the paper.
Hypothesis 1

and there exists a solution $X(t)$, $t \in [-\tau, 0)$, such that

$$X(t) = \begin{cases} 
\Phi_0 + \frac{1}{\Gamma(a)} \int_0^t (t - \nu)^{a-1} f(\nu, X(\nu), X(\nu - \tau))d\nu \\
\Phi(t), 
\end{cases}$$

where $E(\int_0^T \|X(t)\|^2 dt) < \infty$.

(4) For any other solution $\bar{X}(t)$, we obtain $P[X(t) = \bar{X}(t), -\tau \leq t \leq T] = 1$. 

Remark 1 (see [21]). A function $Z(t) \in ([0, T], \mathbb{R}^d)$ is a solution of equation (7) if and only if there exists a function $h(t) \in (0, T, \mathbb{R}^d)$, such that

(i) $E(\sup_{0 \leq t \leq T } \|h(t)\|_2 \leq \varepsilon$

(ii) $C_0 \cdot Z(t) = f(t, Z(t), Z(t - \tau)) + g(t, Z(t), Z(t - \tau)) \cdot \frac{dW(t)}{dt} + h(t)$

Hypothesis 1 (Lipschitz condition). As for any $f, g \in \mathbb{R}^d$, there is a constant $l > 0$ such that, for all $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^d, t \in [0, T],$

$$\|f(t, X_1, Y_1) - f(t, X_2, Y_2)\| \leq l(\|X_1 - X_2\| + \|Y_1 - Y_2\|),$$

where $f$ and $g$ are uniformly continuous functions and $\nu$ is defined as $Y_1 \vee Y_2 = \max\{Y_1, Y_2\}$. 

Hypothesis 2 (non-Lipschitz condition). There is a function $G(t, U, V), [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, such that

(1) For all $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^d$ and $0 \leq t \leq T,$

$$\|f(t, X_1, Y_1) - f(t, X_2, Y_2)\|^2 + \nu \|g(t, X_1, Y_1) - g(t, X_2, Y_2)\|^2 \leq G(t, \|X_1 - X_2\|^2, \|Y_1 - Y_2\|^2).$$

Definition 4 (see [36]). System (1) is Ulam–Hyers stable if there exists a real number $\delta > 0$ such that $\forall \varepsilon > 0$ and for each continuously differentiable function $Z(t) \in ([0, T], \mathbb{R}^d)$ satisfying

$$E(\sup_{0 \leq t \leq T } \|Z(t) - X(t)\|^2 ) < \varepsilon \delta.$$ 

(2) For every $t \in \mathbb{R}^+$ and any nonnegative function $Y(t)$ such that

$$Y(t) \leq m \int_0^t G(v, Y(v))dv,$$

where $m > 0$ is a constant and $G(t, U, V) = G(v, Y(v))$, we get $Y(t) \equiv 0$.

Hypothesis 3. There exist three functions $a(t), b(t),$ and $q(t),$ such that

$$G(t, U, V) \leq a(t) + b(t)U + q(t)V, \quad U, V > 0,$$

where $a(t)$, $b(t)$, and $q(t)$ are continuous as well as bounded functions, and for any fixed $t \geq 0, G(t, U, V)$ is monotone, nondecreasing, continuous, and concave function with $G(t, 0, 0) = 0$.

Lemma 1 (see [37]). Suppose Hypothesis 2 and Hypothesis 3 are fulfilled. Then, there exists constant $c > 0$ such that, for any $t, X, Y \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\|f(t, X, Y)\|^2 + \nu \|g(t, X, Y)\|^2 \leq c(1 + \|X\|^2 + \|Y\|^2).$$

Proof. Applying Jensen’s inequality and Hypothesis 2 and Hypothesis 3, we have
\[ \| f(t, X, Y) \|^2 = \| f(t, X, Y) - f(t, 0, 0) + f(t, 0, 0) \|^2 \leq 2\| f(t, X, Y) - f(t, 0, 0) \|^2 + 2\| f(t, 0, 0) \|^2 \leq 2G(t, \|X\|^2, \|Y\|^2) + 2\sup_{0\leq t \leq T} \| f(t, 0, 0) \|^2 \]
\[
\leq 2\alpha(t) + 2b(t)\|X\|^2 + 2q(t)\|Y\|^2 + 2\sup_{0\leq t \leq T} \| f(t, 0, 0) \|^2 \leq 2 \sup_{0\leq t \leq T} \alpha(t) + 2 \sup_{0\leq t \leq T} b(t)\|X\|^2 + 2 \sup_{0\leq t \leq T} q(t)\|Y\|^2 \leq k_1 \left( 1 + \|X\|^2 + \|Y\|^2 \right),
\]

where \( k_1 = \max\left\{ 2 \sup_{0\leq t \leq T} \alpha(t) + 2 \sup_{0\leq t \leq T} \| f(t, 0, 0) \|^2, 2 \sup_{0\leq t \leq T} b(t), 2 \sup_{0\leq t \leq T} q(t) \right\} < \infty \). In a similar way, we obtain

\[ \|g(t, X, Y)\|^2 \leq k_1 \left( 1 + \|X\|^2 + \|Y\|^2 \right). \]  

Let us set \( c = \max(k_1, k_2) \). The proof is therefore complete.

3. Existence and Uniqueness

Utilizing Carathéodory approximation \([35, 38]\), the existence and uniqueness of solutions to SFDEs can be obtained. So, let us define the Carathéodory approximation as follows. For any integer \( n \geq 1, 0 \leq t \leq T \) define

\[
X_n(t) = \Phi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \nu)^{\alpha - 1} f\left( \nu, X_n(\nu - \frac{1}{n}), X_n(\nu - \frac{1}{n}) \right) d\nu + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \nu)^{\alpha - 1} g\left( \nu, X_n(\nu - \frac{1}{n}), X_n(\nu - \frac{1}{n}) \right) dW(\nu),
\]

and \( X_n(t) = X(0) = \Phi_0 \) for all \(-1 + \tau \leq t \leq 0\).

**Theorem 1.** Suppose that Hypothesis 2 and Hypothesis 3 hold and \( 3^{(3/2)cT^{2a-1}} < (\alpha - (3/4)^{1/2} F^2(\alpha)), (3/4) < \alpha < 1 \); then, system (1) has a unique solution \( X(t), t \in [-\tau, T] \).

**Proof.** The proof will be divided into three steps, when \( t \in [0, T] \).

**Step 1.** The boundedness of the sequence \( \{X_n(t), n \geq 1\} \).

By (16) and Jensen’s inequality, we have

\[
E \left( \sup_{0 \leq t \leq T} \|X_n(t)\|^2 \right) \leq 3E\|\Phi_0\|^2 + \frac{3}{\Gamma^2(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (s - \nu)^{\alpha - 1} f\left( \nu, X_n(\nu - \frac{1}{n}), X_n(\nu - \frac{1}{n}) \right) d\nu \right\|^2 \right) + \frac{3}{\Gamma^2(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (s - \nu)^{\alpha - 1} g\left( \nu, X_n(\nu - \frac{1}{n}), X_n(\nu - \frac{1}{n}) \right) dW(\nu) \right\|^2 \right).
\]
According to Itô isometry, Cauchy–Schwarz inequality, and Lemma 1, it is easy to obtain

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} \| X_n(s) \|^2 \right) \leq 3\mathbb{E}\| \Phi_0 \|^2 + \frac{3}{\Gamma^2(\alpha)} \int_0^t \left( t - v \right)^{\alpha - 1} \mathbb{E} \left[ \int_0^v \left( v, X_n\left( v - \frac{1}{n} \right), X_n\left( v - \frac{1}{n} \right) \right) \, dv \right]^2 \, dv \\
+ \frac{3}{\Gamma^2(\alpha)} \int_0^t (t - v)^{2\alpha - 2} \mathbb{E} \left[ g\left( v, X_n\left( v - \frac{1}{n} \right), X_n\left( v - \frac{1}{n} \right) \right) \right]^2 \, dv \\
\leq 3\mathbb{E}\| \Phi_0 \|^2 + \frac{3T^{2\alpha - 1}}{(2\alpha - 1)\Gamma^2(\alpha)} \int_0^t (t - v)^{2\alpha - 2} \left( 1 + \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v \leq v} \| X_n(v) \|^2 \right] + \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v \leq v} \| X_n(v - \frac{1}{n}) \|^2 \right] \right) \, dv \\
+ \frac{3}{\Gamma^2(\alpha)} \int_0^t (t - v)^{2\alpha - 2} \left( 1 + \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v \leq v} \| X_n(v) \|^2 \right] + \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v \leq v} \| X_n(v - \frac{1}{n}) \|^2 \right] \right) \, dv \\
= J_1 + J_2 + J_3. 
\]

Letting \( v_1 = v - \frac{1}{n} \) and \( v_2 = v - \tau - \frac{1}{n} \). It is obvious that \( v_1 \in [-1, v], v_2 \in [-(1 + \tau), v] \), and

\[
J_2 \leq \frac{3cT^{2\alpha - 1}}{(2\alpha - 1)\Gamma^2(\alpha)} \int_0^t \left( 1 + \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v_1 \leq v} \| X_n(v_1) \|^2 \right] + \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v_2 \leq v} \| X_n(v_2) \|^2 \right] \right) \, dv \\
\leq \frac{3cT^{2\alpha - 1}}{(2\alpha - 1)\Gamma^2(\alpha)} \int_0^t \left( 1 + 2\mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v_2 \leq v} \| X_n(v_2) \|^2 \right] \right) \, dv \\
\leq c_1 \left[ T + 2 \int_0^t \max\left( \mathbb{E}\left[ \sup_{-\frac{1}{n} \leq v_2 \leq \frac{1}{n}} \| X_n(v_2) \|^2 \right], \mathbb{E}\left[ \sup_{0 \leq v_2 \leq v} \| X_n(v_2) \|^2 \right] \right) \, dv \right] \\
= c_1 T + 2c_1 \int_0^t \max\left( \mathbb{E}\| \Phi_0 \|^2, \mathbb{E}\left[ \sup_{0 \leq v_2 \leq v} \| X_n(v_2) \|^2 \right] \right) \, dv,
\]
where \( c_1 = (3cT^{2α-1}/(2α - 1)!)^2 (a)) \). Using Hölders inequality and Jensen’s inequality, we obtain

\[
J_3 \leq \frac{3c}{Γ^2(α)} \left( \int_0^t (t-v)^{4α-4} \, dv \right)^{(1/2)}
\]

\[
\mathbb{E}\left( \int_0^t \left( 1 + \|X_n(v - \frac{1}{n})\|^4 + 2\|X_n(v - \frac{1}{n})\|^2 \right) \, dv \right)^{(1/2)}
\]

\[
\leq \frac{3cT^{2α-3/2}}{(4α - 3)^{(1/2)}Γ^2(α)} \mathbb{E}\left( \int_0^t \left( \frac{3}{4} \|X_n(v - \frac{1}{n})\|^4 + \frac{3}{4} \|X_n(v - \frac{1}{n})\|^2 \right) \, dv \right)^{(1/2)}
\]

where \( 3 + 3\|X_n(v - (1/n))\|^4 + 3\|X_n((v - τ) - (1/n))\|^4 \) is continuous function on \([0, t]\). By the mean value theorem of integrals, there exists \( y \in [0, t] \) such that

\[
\int_0^t \left( \frac{3}{4} \|X_n(v - \frac{1}{n})\|^4 + \frac{3}{4} \|X_n(v - \frac{1}{n})\|^2 \right) \, dv
\]

\[
= t \left( \frac{3}{4} \|X_n(y - \frac{1}{n})\|^4 + \frac{3}{4} \|X_n((y - τ) - \frac{1}{n})\|^2 \right)
\]

Using the inequality \( \sqrt{L_1 + L_2 + L_3} \leq \|L_1\| + \|L_2\| + \|L_3\| \), we derive

\[
J_3 \leq \frac{3cT^{2α-3/2}}{(4α - 3)^{(1/2)}Γ^2(α)} \mathbb{E}\left( \int_0^t \left( \frac{3}{4} \|X_n(y - \frac{1}{n})\|^4 + \frac{3}{4} \|X_n((y - τ) - \frac{1}{n})\|^2 \right) \, dv \right)^{(1/2)}
\]

Let us set \( ν_3 = y - (1/n) \), \( ν_4 = y - τ - (1/n) \), \( ν_3 \in [-1, t] \), and \( ν_4 \in [-1 + τ, t] \), and have

\[
J_3 \leq \frac{3^{(3/2)}cT^{2α-3/2}}{(4α - 3)^{(1/2)}Γ^2(α)} \left( \sup_{-1 ≤ ν_3 ≤ t} \|X_n(ν_3)\|^2 + \mathbb{E}\left( \sup_{-1 + τ ≤ ν_4 ≤ t} \|X_n(ν_4)\|^2 \right) \right)
\]

\[
\leq \frac{3^{(3/2)}cT^{2α-3/2}}{(4α - 3)^{(1/2)}Γ^2(α)} + \frac{3^{(3/2)}cT^{2α-3/2}}{(4α - 3)^{(1/2)}Γ^2(α)} \sup_{-1 + τ ≤ ν_4 ≤ t} \|X_n(ν_4)\|^2
\]

\[
= c_2 + 2c_2 \max\left( \mathbb{E}\|Φ_0\|^2, \mathbb{E}\left( \sup_{0 ≤ ν_4 ≤ t} \|X_n(ν_4)\|^2 \right) \right)
\]
where \( c_2 = \left(3^{(3/2)}cT^{2\alpha-1}/(4\alpha - 3)^{(1/2)}\right)^2(a) \).

If \( \max(\sup_{0 \leq s \leq t} \|X_n(v)\|, \sup_{0 \leq s \leq v} \|X_n(v)\|^2) \), we obtain
\[
E \left( \sup_{0 \leq s \leq t} \|X_n(s)\|^2 \right) 
\leq I_1 + I_2 + I_3
\leq 3E\|\Phi_0\|^2 + c_1T + 2c_1 \int_0^t E\|\Phi_v\| \, dv + c_2E\|\Phi_0\|^2
\leq (3 + 2c_1T + 2c_2)E\|\Phi_0\|^2 + c_1T + c_2 = \beta_1.
\] (25)

If \( \max(E\|\Phi_0\|^2, E(\sup_{0 \leq s \leq v} \|X_n(v)\|^2)) = E(\sup_{0 \leq s \leq v} \|X_n(v)\|^2) \), we obtain
\[
E \left( \sup_{0 \leq s \leq t} \|X_n(s)\|^2 \right) 
\leq I_1 + I_2 + I_3
\leq 3E\|\Phi_0\|^2 + c_1T + 2c_1 \int_0^t E\left( \sup_{0 \leq v \leq s} \|X_n(v)\|^2 \right) \, dv + c_2E\|\Phi_0\|^2
\]
\[
= 3E\|\Phi_0\|^2 + c_1T + 2c_1 \int_0^t E\left( \sup_{0 \leq v \leq s} \|X_n(v)\|^2 \right) \, dv + 2c_2E\left( \sup_{0 \leq s \leq t} \|X_n(s)\|^2 \right).
\] (26)

and then,
\[
E \left( \sup_{0 \leq s \leq t} \|X_n(s)\|^2 \right) 
\leq \frac{3}{1 - 2c_2}E\|\Phi_0\|^2 + \frac{c_1T + c_2}{1 - 2c_2} \int_0^t E \left( \sup_{0 \leq v \leq s} \|X_n(v)\|^2 \right) \, dv.
\] (27)

By Gronwall’s inequality, we can conclude that
\[
E \left( \sup_{0 \leq s \leq t} \|X_n(s)\|^2 \right) \leq \left( \frac{3}{1 - 2c_2}E\|\Phi_0\|^2 + \frac{c_1T + c_2}{1 - 2c_2} \right) e^{(2c_1T/1 - 2c_2)}
\] (28)

Letting \( \beta = \max(\beta_1, \beta_2) \), we obtain
\[
E \left( \sup_{0 \leq s \leq t} \|X_n(s)\|^2 \right) \leq \beta,
\] (29)
where \( \beta \) is a positive constant. So, we have proved that the sequence \( \{X_n(t), n \geq 1\} \) is bounded.

**Step 2.** For \( 0 \leq s < t \leq T \) and any integer \( n \geq 1 \), we obtain by (16)

\[
X_n(t) - X_n(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} f(\nu, X_n(\nu - \frac{1}{n}), X_n((\nu - \nu) - \frac{1}{n})) \, d\nu
+ \frac{1}{\Gamma(\alpha)} \int_s^t (t - \nu)^{\alpha-1} f(\nu, X_n(\nu - \frac{1}{n}), X_n((\nu - \nu) - \frac{1}{n})) \, d\nu
+ \frac{1}{\Gamma(\alpha)} \int_0^s (t - \nu)^{\alpha-1} g(\nu, X_n(\nu - \frac{1}{n}), X_n((\nu - \nu) - \frac{1}{n})) \, dW(\nu)
+ \frac{1}{\Gamma(\alpha)} \int_s^t (t - \nu)^{\alpha-1} g(\nu, X_n(\nu - \frac{1}{n}), X_n((\nu - \nu) - \frac{1}{n})) \, dW(\nu).
\] (30)
By using Jensen’s inequality, Cauchy–Schwarz inequality, and Itô isometry, we conclude

\[
\mathbb{E}\|X_n(t) - X_n(s)\|^2 \\
\leq \frac{4}{\Gamma^2(\alpha)} \mathbb{E} \int_0^T \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) f \left( \nu, X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) d\nu \\
+ \frac{4}{\Gamma^2(\alpha)} \mathbb{E} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) g \left( \nu, X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) dW(\nu) \\
+ \frac{4}{\Gamma^2(\alpha)} \mathbb{E} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) d \left( X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) \\
+ \frac{4}{\Gamma^2(\alpha)} \mathbb{E} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) g \left( \nu, X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) dW(\nu) \\
\leq \frac{4T}{\Gamma^2(\alpha)} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) f \left( \nu, X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) d\nu \\
+ \frac{4T}{\Gamma^2(\alpha)} \mathbb{E} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) g \left( \nu, X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) d\nu \\
+ \frac{4}{\Gamma^2(\alpha)} \mathbb{E} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) d \left( X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) \\
+ \frac{4}{\Gamma^2(\alpha)} \mathbb{E} \int_s^t \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) g \left( \nu, X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) d\nu.
\]

Recalling Lemma 1, we obtain

\[
\mathbb{E}\|X_n(t) - X_n(s)\|^2 \\
\leq \frac{4c(1 + T)}{\Gamma^2(\alpha)} \int_0^T \left( (t - \nu)^{\alpha-1} - (s - \nu)^{\alpha-1} \right) \left( 1 + \left\| X_n \left( \nu - \frac{1}{n} \right) \right\|^2 + \left\| X_n \left( \nu - \frac{1}{n} \right) \right\|^2 \right) d\nu \\
+ \frac{4c(1 + T)}{\Gamma^2(\alpha)} \int_s^t \left( (t - \nu)^{2\alpha-2} \right) d \left( X_n \left( \nu - \frac{1}{n} \right), X_n \left( \nu - \frac{1}{n} \right) \right) \\
= c_3 \left( J_4 + J_5 \right),
\]

where \( c_3 = (4c(1 + T)/\Gamma^2(\alpha)) \).

Applying Hölder’s inequality and Step 1, we obtain
\[ J_4 \leq \left[ \int_0^t (t-v)^a - (s-v)^a \right]^{1/2} \left[ \int_0^s \left( 1 + \mathbb{E} \left( X_n \left( v - \frac{1}{n} \right) \right)^2 + \mathbb{E} \left( X_n \left( v - \frac{(1-n)}{n} \right) \right)^2 \right) dv \right]^{1/2} \]

\[ \leq \left[ \int_0^t (t-v)^a - (s-v)^a \right]^{1/2} \left[ \int_0^s \left( 1 + \mathbb{E} \left( \sup_{0 \leq u \leq v} \left| X_n \left( u_1 - \frac{1}{n} \right) \right|^2 \right) \right) dv \right]^{1/2} \]

\[ \leq \left[ \int_0^t (t-v)^a - (s-v)^a \right]^{1/2} \left[ \int_0^s \left( 1 + 2 \beta \right)^2 dv \right]^{1/2} \]

\[ \leq (1 + 2 \beta)^r \int_0^t \left( 2(t-v)\alpha^4 + 2(s-v)\alpha^4 + 4(s-v)\alpha^4 - 8(t-v)\alpha^4 \right) dv \]

\[ = (1 + 2 \beta) \int_0^t \left( 6(s-v)\alpha^4 - 6(t-v)\alpha^4 \right) dv \]

\[ = (1 + 2 \beta)(6T)^{(1/2)} \left( t - s \right)^{2 \alpha - 1} \]

\[ = c_4 \left( t - s \right)^{2 \alpha - 1}, \]

where \( c_4 = (1 + 2 \beta)(6T)^{(1/2)}/(4 \alpha - 3)^{(1/2)} \).

Using Hölder’s inequality and Step 1 again,

\[ J_5 \leq \left[ \int_0^t (t-v)^{4 \alpha - 4} dv \right]^{1/2} \left[ \int_0^t (1 + 2 \beta)^3 dv \right]^{1/2} \]

\[ = (t - s)^{(1/2)} \left( 1 + 2 \beta \right) \left[ \int_0^t (t-v)^{4 \alpha - 4} dv \right]^{1/2} \]

\[ = \left( t - s \right)^{2 \alpha - 1} \left( 1 + 2 \beta \right) \left[ \int_0^t (t-v)^{4 \alpha - 4} dv \right]^{1/2} \]

\[ = c_5 \left( t - s \right)^{2 \alpha - 1}, \]

where \( c_5 = ((1 + 2 \beta)/4 \alpha - 3)^{(1/2)} \).

Then,

\[ \mathbb{E} \left\| X_n(t) - X_n(s) \right\|^2 \leq c_3 (J_4 + J_5) \]

\[ \leq c_3 c_4 \left( t - s \right)^{2 \alpha - (3/2)} + c_3 c_5 \left( t - s \right)^{2 \alpha - 1} \]

\[ = r_1 \left( t - s \right)^{2 \alpha - (3/2)} + r_2 \left( t - s \right)^{2 \alpha - 1}, \]

where \( r_1 = c_3 c_4 \) and \( r_2 = c_3 c_5 \).

Step 3. We claim that \( \{ X_n(t), n \geq 1 \} \) is a Cauchy sequence.

For integer \( m > n \geq 1 \), one can obtain
\[ X_m(s) - X_n(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s - v)^{\alpha - 1} \left[ f(v, X_m(v - \frac{1}{m}), X_m((v - \tau) - \frac{1}{m})) - f(v, X_n(v - \frac{1}{n}), X_n((v - \tau) - \frac{1}{n})) \right] dv \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^s (s - v)^{\alpha - 1} \left[ g(v, X_m(v - \frac{1}{m}), X_m((v - \tau) - \frac{1}{m})) - g(v, X_n(v - \frac{1}{n}), X_n((v - \tau) - \frac{1}{n})) \right] dW(v). \]

(36)

By Jensen’s inequality, Hölder’s inequality, and Itô isometry, we can obtain

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} \left\| X_m(s) - X_n(s) \right\|^2 \right) \\
\leq \frac{2}{\Gamma^2(\alpha)} \mathbb{E}\left( \int_0^t (t - \tau)^{2\alpha - 2} \left[ f(v, X_m(v - \frac{1}{m}), X_m((v - \tau) - \frac{1}{m})) - f(v, X_n(v - \frac{1}{n}), X_n((v - \tau) - \frac{1}{n})) \right] dv \right)^2 \\
+ \frac{2}{\Gamma^2(\alpha)} \mathbb{E}\left( \int_0^t (t - \tau)^{2\alpha - 2} \left[ g(v, X_m(v - \frac{1}{m}), X_m((v - \tau) - \frac{1}{m})) - g(v, X_n(v - \frac{1}{n}), X_n((v - \tau) - \frac{1}{n})) \right] dW(v) \right)^2 \\
\leq \frac{2}{\Gamma(\alpha)} \int_0^t (t - \tau)^{2\alpha - 2} dv \int_0^t \mathbb{E}\left( \left\| f(v, X_m(v - \frac{1}{m}), X_m((v - \tau) - \frac{1}{m})) - f(v, X_n(v - \frac{1}{n}), X_n((v - \tau) - \frac{1}{n})) \right\|^2 dv \\
+ \frac{2}{\Gamma(\alpha)} \int_0^t (t - \tau)^{2\alpha - 2} \mathbb{E}\left( \left\| g(v, X_m(v - \frac{1}{m}), X_m((v - \tau) - \frac{1}{m})) - g(v, X_n(v - \frac{1}{n}), X_n((v - \tau) - \frac{1}{n})) \right\|^2 dv \\
= J_7 + J_8. 
\]
By applying Jensen’s inequality and Hypothesis 2, we obtain

\[ J_7 \leq \frac{2T^{2\alpha - 1}}{(2\alpha - 1)!^2 (\alpha)} \int_0^t \mathbb{E} \left[ f \left( v, X_m \left( \frac{v - 1}{m} \right), X_m \left( \left( v - \tau - \frac{1}{m} \right) \right) \right) \\
- f \left( v, X_n \left( \frac{v - 1}{n} \right), X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right) \\
+ f \left( v, X_n \left( \frac{v - 1}{n} \right), X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right) \right] \text{dv} \\
\leq 2c_6 \int_0^t \mathbb{E} \left[ f \left( v, X_m \left( \frac{v - 1}{m} \right), X_m \left( \left( v - \tau - \frac{1}{m} \right) \right) \right) \\
- f \left( v, X_n \left( \frac{v - 1}{n} \right), X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right) \right] \text{dv}, \\
+ 2c_6 \int_0^t \mathbb{E} \left[ f \left( v, X_n \left( \frac{v - 1}{n} \right), X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right) \right] \text{dv} \\
- f \left( v, X_n \left( \frac{v - 1}{n} \right), X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right) \right] \text{dv} \\
\leq 2c_6 \int_0^t \mathbb{E} \left[ G \left( v, \mathbb{E} \left[ X_m \left( \frac{v - 1}{m} \right) - X_n \left( \frac{v - 1}{n} \right) \right] \right)^2 \right] \text{dv} \\
\mathbb{E} \left[ \left( X_m \left( \left( v - \tau - \frac{1}{m} \right) \right) - X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right)^2 \right] \text{dv} \\
+ 2c_6 \int_0^t \mathbb{E} \left[ G \left( v, \mathbb{E} \left[ X_n \left( \frac{v - 1}{n} \right) - X_n \left( \frac{v - 1}{n} \right) \right] \right)^2 \right] \text{dv} \\
\mathbb{E} \left[ \left( X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) - X_n \left( \left( v - \tau - \frac{1}{n} \right) \right) \right)^2 \right] \text{dv} \]
where $c_6 = (2T^{2\alpha - 1} / (2\alpha - 1) \Gamma^2 (\alpha))$.

Using Jensen’s inequality and Hypothesis 2 again, we acquire

$$I_8 \leq \frac{2}{\Gamma^2 (\alpha)} \int_0^t \left( \sup_{0 \leq v \leq t} (t - v)^{2\alpha - 2} \right) \mathbb{E} \left\| g \left( v, X_m \left( v - \frac{1}{m} \right), X_m \left( (v - \tau) - \frac{1}{m} \right) \right) \right\|^2 dv$$

$$- g \left( v, X_n \left( v - \frac{1}{m} \right), X_n \left( (v - \tau) - \frac{1}{m} \right) \right)$$

$$+ g \left( v, X_n \left( v - \frac{1}{m} \right), X_n \left( (v - \tau) - \frac{1}{m} \right) \right)$$

$$- g \left( v, X_n \left( v - \frac{1}{n} \right), X_n \left( (v - \tau) - \frac{1}{n} \right) \right) \|^2 dv$$

$$\leq \frac{4 \sup_{0 \leq v \leq t} (t - v)^{2\alpha - 2}}{\Gamma^2 (\alpha)} \int_0^t \mathbb{E} \left\| g \left( v, X_m \left( v - \frac{1}{m} \right), X_m \left( (v - \tau) - \frac{1}{m} \right) \right) \right\|^2 dv$$

$$- g \left( v, X_n \left( v - \frac{1}{m} \right), X_n \left( (v - \tau) - \frac{1}{m} \right) \right) \|^2 dv$$

$$+ \frac{4 \sup_{0 \leq v \leq t} (t - v)^{2\alpha - 2}}{\Gamma^2 (\alpha)} \int_0^t \mathbb{E} \left\| g \left( v, X_n \left( v - \frac{1}{m} \right), X_n \left( (v - \tau) - \frac{1}{m} \right) \right) \right\|^2 dv$$

$$- g \left( v, X_n \left( v - \frac{1}{n} \right), X_n \left( (v - \tau) - \frac{1}{n} \right) \right) \|^2 dv$$

$$\leq \frac{4 \sup_{0 \leq v \leq t} (t - v)^{2\alpha - 2}}{\Gamma^2 (\alpha)} \int_0^t G \left( v, \mathbb{E} \left\| X_m \left( v - \frac{1}{m} \right) - X_n \left( v - \frac{1}{m} \right) \right\|^2 \right) dv$$

$$+ \frac{4 \sup_{0 \leq v \leq t} (t - v)^{2\alpha - 2}}{\Gamma^2 (\alpha)} \int_0^t G \left( v, \mathbb{E} \left\| X_n \left( v - \frac{1}{m} \right) - X_n \left( v - \frac{1}{n} \right) \right\|^2 \right) dv$$

$$+ \frac{4 \sup_{0 \leq v \leq t} (t - v)^{2\alpha - 2}}{\Gamma^2 (\alpha)} \int_0^t G \left( v, \mathbb{E} \left\| X_n \left( (v - \tau) - \frac{1}{m} \right) - X_n \left( (v - \tau) - \frac{1}{n} \right) \right\|^2 \right) dv.$$
In terms of Step 2, we can conclude that

\[
E \left( \sup_{0 \leq s \leq t} \left\| X_m(s) - X_n(s) \right\|^2 \right) \leq f_7 + f_8
\]

\[
\leq \left( 2c_6 + 4 \sup_{0 \leq s < t} \frac{(t - v)^{2a-2}}{\Gamma^2(\alpha)} \right) \int_0^t \left( \left\| X_m(v) - X_n(v) \right\|^2 \right) dv
\]

\[
+ \left( 2c_6 + 4 \sup_{0 \leq s < t} \frac{(t - v)^{2a-2}}{\Gamma^2(\alpha)} \right) \int_0^t \left( \left\| X_m(v) - X_n(v) \right\|^2 \right) dv
\]

\[
\leq \left( 2c_6 + 4 \sup_{0 \leq s < t} \frac{(t - v)^{2a-2}}{\Gamma^2(\alpha)} \right) \int_0^t \left( \left\| X_m(v) - X_n(v) \right\|^2 \right) dv
\]

\[
\leq \left( 2c_6 + 4 \sup_{0 \leq s < t} \frac{(t - v)^{2a-2}}{\Gamma^2(\alpha)} \right) \int_0^t \left( \left\| X_m(v) - X_n(v) \right\|^2 \right) dv
\]

\[
= \left( 2c_6 + 4 \sup_{0 \leq s < t} \frac{(t - v)^{2a-2}}{\Gamma^2(\alpha)} \right) \int_0^t \left( \left\| X_m(v) - X_n(v) \right\|^2 \right) dv
\]

Thus, by Hypothesis 2, we have

\[
Y(t) = \lim_{m,n \to \infty} E \left( \sup_{0 \leq s \leq t} \left\| X_m(s) - X_n(s) \right\|^2 \right) = 0.
\]

indicating that \( \{X_n(t), n \geq 1\} \) is a Cauchy sequence. The Borel-Cantelli lemma makes clear, as \( n \to \infty \), \( X_n(t) \to X(t) \), \( t \in [0, T] \) holds uniformly. So, if we take the limit of both sides of (16), we get that \( X(t) \) is a solution to (1), with the property

\[
Y(t) \leq \left( 2c_6 + 4 \sup_{0 \leq s < t} \frac{(t - v)^{2a-2}}{\Gamma^2(\alpha)} \right) \int_0^t G(v,Y(v)) dv.
\]
or \( \tau \in \mathbb{R} \). We are going to research the solution \( X(t), t \in [0, T] \) of system (1) is Ulam–Hyers stable and prove the stability theory of solutions to FSDEs (1) with Lipschitz and non-Lipschitz coefficients in this section.

### 4. Ulam–Hyers Stability Analysis of FSDEs

We are going to research the solution \( X(t), t \in [0, T] \) of system (1) is Ulam–Hyers stable and prove the stability theory of solutions to FSDEs (1) with Lipschitz and non-Lipschitz coefficients in this section.

Theorem 2. Assume that Hypothesis 1 holds and \( 12^{(2)}T^{2\alpha - (1/2)} < (4\alpha - 3)^{(1/2)} I^{2}(\alpha), (3/4) < \alpha < 1 \). The FSDE (1) is Ulam–Hyers which is stable at \([0, T]\).

**Proof.** From Definition 3 and Remark 1, we know

\[
Z(t) = \Phi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(v, Z(v), Z(v - \tau))d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(v, Z(v), Z(v - \tau))dW(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} h(v)dv.
\]  

(45)

According to Definition 3 and equation (45), we have

\[
E \left( \sup_{0 \leq t \leq T} \|Z(t) - X(t)\|^2 \right) \leq \frac{3}{\Gamma^2(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (t - \tau)^{\alpha - 1} (f(v, Z(v), Z(v - \tau)) - f(v, X(v), X(v - \tau))d\tau \right. \right.
\]

\[
- f(v, X(v), X(v - \tau))d\tau \right) + \frac{3}{\Gamma^2(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (t - \tau)^{\alpha - 1} (g(v, Z(v), Z(v - \tau)) - g(v, Z(v), Z(v - \tau)))dW(\tau) \right. \right.
\]

\[
- g(v, Z(v), Z(v - \tau)))dW(\tau) \right) + \frac{3}{\Gamma^2(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (t - \tau)^{\alpha - 1} h(v)dv \right. \right.
\]

\[
= I_1 + I_2 + I_3.
\]  

(47)
Now, we use Hölder’s inequality and Hypothesis 1, and one can obtain

\[
I_1 \leq \frac{3}{\Gamma^2(\alpha)} \left( \sup_{0 \leq \tau \leq T} \int_0^T (t - v)^{2\alpha - 1} dv \right)
\]

\[
\cdot \mathbb{E} \left[ \int_0^T \| f(v, Z(v), Z(v) - \tau) - f(v, X(v), X(v) - \tau) \|^2 dv \right]
\]

\[
\leq \frac{3T^{2\alpha - 1}}{(2\alpha - 1)\Gamma^2(\alpha)} \mathbb{E} \left[ \int_0^T (\|Z(v) - X(v)\| + \|Z(v) - X(v)\|)^2 dv \right]
\]

\[
\leq \frac{6T^{2\alpha - 1}}{(2\alpha - 1)\Gamma^2(\alpha)} \int_0^T \left( \mathbb{E}[Z(v) - X(v)]^2 + \mathbb{E}[Z(v) - X(v)]^2 \right) dv
\]

\[
= b_1 T^{2\alpha - 1} \int_0^T \mathbb{E}[Z(v) - X(v)]^2 dv
\]

\[
+ b_1 T^{2\alpha - 1} \int_0^T \mathbb{E}[Z(v) - X(v)]^2 dv,
\]

where \( b_1 = (6T^2/(2\alpha - 1)\Gamma^2(\alpha)) \).

Then, by Itô isometry and Hölder’s inequality, we obtain

\[
I_2 \leq \frac{3}{\Gamma^2(\alpha)} \mathbb{E} \left[ \int_0^T (T - v)^{2\alpha - 1} (g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau)) dW(v) \right]^2
\]

\[
= \frac{3}{\Gamma^2(\alpha)} \mathbb{E} \left( \left( \int_0^T (T - v)^{2\alpha - 1} (g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau)) dv \right)^2 \right)
\]

\[
\leq \frac{3}{\Gamma^2(\alpha)} \left( \int_0^T (T - v)^{4\alpha - 4} dv \right)^{(1/2)}
\]

\[
\cdot \mathbb{E} \left( \left( \int_0^T \| g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau) \|^2 dv \right)^{(1/2)} \right)
\]

\[
\leq \frac{3T^{2\alpha - (3/2)}}{(4\alpha - 3)\Gamma^2(\alpha)} \mathbb{E} \left( \int_0^T \| g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau) \|^4 dv \right)^{(1/2)}
\]

\[
= b_2 T^{2\alpha - (3/2)} \mathbb{E} \left( \int_0^T \| g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau) \|^4 dv \right)^{(1/2)},
\]

where \( b_2 = (3/(4\alpha - 3)\Gamma^2(\alpha)) \). Since \( \| g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau) \|^4 \) is a continuous function on [0, T], according to the mean value theorem of integrals, there exists \( \bar{y} \in [0, T] \), such that

\[
\int_0^T \| g(v, Z(v), Z(v) - \tau) - g(v, X(v), X(v) - \tau) \|^4 dv
\]

\[
= T \| g(\bar{y}, Z(\bar{y}), Z(\bar{y} - \tau)) - g(\bar{y}, X(\bar{y}), X(\bar{y} - \tau)) \|^4.
\]
By Hypothesis 1 and Jensen’s inequality, we obtain

\[ I_2 \leq b_1 T^{2\alpha-(1/2)} \mathbb{E} \| g((\tilde{y}, Z(\tilde{y}), Z(\tilde{y} - r)) - g(\tilde{y}, X(\tilde{y}), X(\tilde{y} - r)) \|^2 \]

\[ \leq 2T^2 b_1 T^{2\alpha-(1/2)} \left( \mathbb{E} \| Z(\tilde{y}) - X(\tilde{y}) \|^2 + \mathbb{E} \| Z(\tilde{y} - r) - X(\tilde{y} - r) \|^2 \right). \]  

\[ (51) \]

Finally, we use Cauchy–Schwarz inequality and Remark 1 to yield

\[ I_3 \leq \frac{3}{T^2(\alpha)} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_0^T (t - \nu) e^{-2\alpha - 2} \mathrm{d} \nu \cdot \int_0^T \| h(\nu) \|^2 \mathrm{d} \nu \right) \right] \]

\[ \leq \frac{3 T^{2\alpha - 1}}{(2\alpha - 1) \| T \|^2(\alpha)} \left( \int_0^T \mathbb{E} \left( \sup_{0 \leq \tau \leq \nu} \| h(\nu) \|^2 \right) \mathrm{d} \nu \right) \]

\[ \leq \frac{3 T^{2\alpha} \varepsilon}{(2\alpha - 1) \| T \|^2(\alpha)} \]

\[ = b_3 T^{2\alpha} \varepsilon, \]

\[ (52) \]

\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} \| Z(t) - X(t) \|^2 \right) \]

\[ \leq b_1 T^{2\alpha - 1} \left( \int_0^T \mathbb{E} \| (Z(\nu) - X(\nu)) \|^2 \mathrm{d} \nu + \int_0^T \mathbb{E} \| (Z(\nu - r) - X(\nu - r)) \|^2 \mathrm{d} \nu \right) \]

\[ + 2T b_2 T^{2\alpha - (1/2)} \left( \mathbb{E} \| Z(\tilde{y}) - X(\tilde{y}) \|^2 + \mathbb{E} \| Z(\tilde{y} - r) - X(\tilde{y} - r) \|^2 \right) + b_3 T^{2\alpha} \varepsilon \]

\[ \leq b_1 T^{2\alpha - 1} \left( \int_0^T \mathbb{E} \left( \sup_{0 \leq \nu \leq \nu} \| Z(s_1) - X(s_1) \|^2 \right) \mathrm{d} \nu \right) \]

\[ + b_1 T^{2\alpha - 1} \left( \int_0^T \mathbb{E} \left( \sup_{0 \leq \nu \leq \nu} \| Z(s_1 - \tau) - X(s_1 - \tau) \|^2 \right) \mathrm{d} \nu \right) \]

\[ + 2T b_2 T^{2\alpha - (1/2)} \mathbb{E} \left( \sup_{0 \leq \nu \leq \nu} \| Z(\tilde{y}_1) - X(\tilde{y}_1) \|^2 \right) \]

\[ + 2T b_2 T^{2\alpha - (1/2)} \mathbb{E} \left( \sup_{0 \leq \nu \leq \nu} \| Z(\tilde{y}_1 - \tau) - X(\tilde{y}_1 - \tau) \|^2 \right) + b_3 T^{2\alpha} \varepsilon, \]

\[ (53) \]

Different from the approach of dealing with the delay in [18, 39, 40], we obtain

\[ V(T) = \mathbb{E} \left( \sup_{0 \leq t \leq T} \| Z(t) - X(t) \|^2 \right), \]

\[ \mathbb{E} \left( \sup_{-\tau \leq t \leq 0} \| Z(t) - X(t) \|^2 \right) = 0, \]

\[ (54) \]

and then, we acquire

\[ \mathbb{E} \left( \sup_{0 \leq \nu \leq \nu} \| Z(s_1 - \tau) - X(s_1 - \tau) \|^2 \right) = V(\nu - \tau). \]

\[ (55) \]

Hence,

\[ V(T) \leq b_1 T^{2\alpha - 1} \left( \int_0^T V(\nu) \mathrm{d} \nu + \int_0^T V(\nu - T) \mathrm{d} \nu \right) \]

\[ + 2T b_2 T^{2\alpha - (1/2)} (V(\tilde{y}) + V(\tilde{y} - \tau)) + b_3 T^{2\alpha} \varepsilon. \]

\[ (56) \]

Let us set \( U(T) = \sup_{0 \leq r \leq T} V(\theta) \), then \( V(\nu) \leq U(\nu) \), and \( V(\nu - \tau) \leq U(\nu) \). Thus, \( \varepsilon \leq T \)

\[ V(T) \leq 2b_1 T^{2\alpha - 1} \int_0^T U(\nu) \mathrm{d} \nu + 4T b_2 T^{2\alpha - (1/2)} U(\tilde{y}) + b_3 T^{2\alpha} \varepsilon. \]

\[ (57) \]
For \( \forall \theta \in [0, T] \), we obtain

\[
V(\theta) \leq 2b_1 \theta^{2\alpha-1} \int_0^\theta U(v)dv + 4t^2 b_2 \theta^{2\alpha-(1/2)} U(\bar{y}) + b_3 \theta^{2\alpha} \\
\leq 2b_1 T^{2\alpha-1} \int_0^T U(v)dv + 4t^2 b_2 T^{2\alpha-(1/2)} U(\bar{y}) + b_3 T^{2\alpha}.
\]

(58)

Then, we can obtain

\[
U(T) = \sup_{\theta \in [-r, T]} V(\theta) \\
\leq \max \left\{ \sup_{\theta \in [-r, 0]} V(\theta), \sup_{\theta \in [0, T]} V(\theta) \right\} \\
\leq 2b_1 T^{2\alpha-1} \int_0^T U(v)dv + 4t^2 b_2 T^{2\alpha-(1/2)} U(T) + b_3 T^{2\alpha}.
\]

(59)

Then,

\[
U(T) \leq \frac{2b_1 T^{2\alpha-1}}{1 - 4t^2 b_2 T^{2\alpha-(1/2)}} \int_0^T U(v)dv + \frac{b_3 T^{2\alpha}}{1 - 4t^2 b_2 T^{2\alpha-(1/2)}}.
\]

(60)

Using Gronwall’s inequality, we obtain

\[
U(T) \leq \frac{b_3 T^{2\alpha}}{1 - 4t^2 b_2 T^{2\alpha-(1/2)}} e^{(2b_1 T^{2\alpha-1} - 4t^2 b_2 T^{2\alpha-(1/2)})T}.
\]

(61)

Therefore,

\[
E \left( \sup_{0 \leq t \leq T} \|Z(t) - X(t)\|^2 \right) \leq \frac{b_3 T^{2\alpha}}{1 - 4t^2 b_2 T^{2\alpha-(1/2)}} e^{(2b_1 T^{2\alpha-1} - 4t^2 b_2 T^{2\alpha-(1/2)})T}.
\]

(62)

consequently, \( \forall \varepsilon > 0 \), there exists \( \delta = (b_3 T^{2\alpha}/1 - 4t^2 b_2 T^{2\alpha-(1/2)}) e^{(2b_1 T^{2\alpha-1} - 4t^2 b_2 T^{2\alpha-(1/2)})T} \), such that

\[
E \left( \sup_{0 \leq t \leq T} \|Z(t) - X(t)\|^2 \right) \leq \varepsilon \delta.
\]

(63)

Therefore, this theorem is proved.

**Theorem 3.** Assume that Hypothesis 2 and Hypothesis 3 hold, 
\( 6\bar{k}T^{2\alpha-(1/2)} < (4\alpha - 3)^{(1/2)}T^2 (\alpha) \), 
\( \bar{k} = \max\{\sup_{0 \leq t \leq T} b(t), \sup_{0 \leq t \leq T} q(t)\} \), and there exists a constant \( \gamma \) satisfying 
\( (3(4\alpha - 3)^{(1/2)}T^{2\alpha} + 3(2\alpha - 1)T^{2\alpha-(1/2)}/(2\alpha - 1) (4\alpha - 3)^{(1/2)}T^2 (\alpha) - 6(2\alpha - 1)\bar{k}T^{2\alpha-(1/2)}\) \( \sup_{0 \leq t \leq T} \gamma(t) \leq \gamma \), \( (3/4) < \alpha < 1 \). The FSDE (1) is Ulam–Hyers which is stable at \( [0, T] \).

**Proof.** From inequality (48), we obtain

\[
\begin{align*}
E \left( \sup_{0 \leq t \leq T} \|Z(t) - X(t)\|^2 \right) & \leq \frac{3}{\Gamma'(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (t - v)^{\alpha - 1} \left( f(v, Z(v), Z(v - r)) - f(v, X(v), X(v - r)) \right) dv \right\|^2 \right) \\
& + \frac{3}{\Gamma'(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (t - v)^{\alpha - 1} \left( g(v, Z(v), Z(v - r)) - g(v, X(v), X(v - r)) \right) dW(v) \right\|^2 \right) \\
& + \frac{3}{\Gamma'(\alpha)} E \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (t - v)^{\alpha - 1} h(v) dv \right\|^2 \right) \\
& = I_1 + I_2 + I_3.
\end{align*}
\]

(64)

Taking inequality (48) and Hypothesis 2 into account to achieve,

\[
I_1 \leq \frac{3T^{2\alpha-1}}{(2\alpha - 1)T^2 (\alpha)} E \left( \int_0^T \left\| f(v, Z(v), Z(v - r)) - f(v, X(v), X(v - r)) \right\|^2 dv \right)
\]

(65)

\[
\leq b_4 T^{2\alpha-1} \int_0^T E \left( \|Z(v) - X(v)\|^2, \|Z(v - r) - X(v - r)\|^2 \right) dv,
\]

where \( b_4 = (3/(2\alpha - 1)T^2 (\alpha)) \).
By inequalities (50)–(52) and Hypothesis 2, one can obtain

\[
I_2 \leq b_2 T^{2\alpha - (1/2)} \mathbb{E}\|g(\bar{y}, Z(\bar{y})), Z(\tilde{y} - t)) - g(\bar{y}, X(\bar{y}), X(\tilde{y} - t))\|^2
\leq b_2 T^{2\alpha - (1/2)} \mathbb{E}\|Z(\bar{y}) - X(\bar{y})\|^2. \tag{66}
\]

Using Hypothesis 3, it is immediate to obtain

\[
I_1 + I_2 \\
\leq b_4 T^{2\alpha - 1} \int_0^T \mathbb{E}\|g(\bar{y}, Z(\bar{y})), Z(\tilde{y} - t)) - g(\bar{y}, X(\bar{y}), X(\tilde{y} - t))\|^2 \, dt
\]
\[
+ b_2 T^{2\alpha - (1/2)} \int_0^T \mathbb{E}\|Z(\bar{y}) - X(\bar{y})\|^2 \, dt \leq b_2 T^{2\alpha - (1/2)} \int_0^T \mathbb{E}\|Z(\bar{y}) - X(\bar{y})\|^2 \, dt.
\]

Now, through inequality (52), we have

\[
I_3 \leq b_2 T^{2\alpha} \epsilon. \tag{68}
\]

Then,

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \|Z(t) - X(t)\|^2 \right)
\leq \left( b_4 T^{2\alpha} + b_2 T^{2\alpha - (1/2)} \right) \sup_{0 \leq s \leq T} a(v)
+ b_4 \tilde{K} T^{2\alpha - 1} \int_0^T \mathbb{E}\left( \sup_{0 \leq s \leq t} \|Z(s) - X(s)\|^2 \right) \, dt
+ b_2 \tilde{K} T^{2\alpha - (1/2)} \mathbb{E}\left( \sup_{0 \leq s \leq t} \|Z(s) - X(s)\|^2 \right)
+ b_4 \tilde{K} T^{2\alpha - (1/2)} \mathbb{E}\left( \sup_{0 \leq s \leq T} \|Z(s) - X(s)\|^2 \right)
+ b_3 T^{2\alpha} \epsilon. \tag{69}
\]

Indeed, we can conclude that

\[
V(T) \leq b_4 \tilde{K} T^{2\alpha - 1} \int_0^T \mathbb{E}\|Z(t) - X(t)\|^2 \, dt
+ b_2 \tilde{K} T^{2\alpha - (1/2)} \mathbb{E}\|Z(\tilde{y}) - X(\tilde{y})\|^2
+ b_4 T^{2\alpha} + b_3 T^{2\alpha} \epsilon. \tag{71}
\]

Letting \( U(T) = \sup_{0 \leq t \leq T} V(t) \), then \( V(t) \leq U(t) \) for \( V(\tau) \leq U(\tau) \), and \( V(t) \leq U(t) \). Thus,

\[
V(T) \leq 2b_4 \tilde{K} T^{2\alpha - 1} \int_0^T U(t) \, dt + 2b_4 \tilde{K} T^{2\alpha - (1/2)} U(\tilde{y})
+ b_4 T^{2\alpha} + b_4 T^{2\alpha - (1/2)} \sup_{0 \leq s \leq T} a(v) + b_3 T^{2\alpha} \epsilon. \tag{72}
\]

For all \( \forall \theta \in [0, T] \), we get that

\[
V(\theta) \leq 2b_4 \tilde{K} T^{2\alpha - (1/2)} U(\tilde{y})
+ b_4 T^{2\alpha} + b_4 T^{2\alpha - (1/2)} \sup_{0 \leq s \leq T} a(v) + b_3 T^{2\alpha} \epsilon.
\]

Let us set \( V(T) = \mathbb{E}\left( \sup_{0 \leq t \leq T} \|Z(t) - X(t)\|^2 \right) \) and \( E\left( \sup_{-\tau \leq t \leq 0} \|Z(t) - X(t)\|^2 \right) = 0 \), we can obtain

\[
\mathbb{E}\left( \sup_{-\tau \leq t \leq 0} \|Z(t) - X(t)\|^2 \right) = V(\tau) - V(\tilde{y}) = V(\tau) - V(\bar{y}). \tag{70}
\]
Moreover, we have
\[
U(T) = \sup_{\theta \in [-\tau, T]} V(\theta) \\
\leq \max \left\{ \sup_{\theta \in [-\tau, 0]} V(\theta), \sup_{0 \leq t \leq T} V(\theta) \right\} \\
\leq 2b_2 k T^{2\alpha - 1} \int_0^T U(v) dv + 2b_2 k T^{2\alpha - (1/2)} U(T) \\
+ (b_4 T^2 + b_5 T^{2\alpha - (1/2)}) \sup_{0 \leq t \leq T} a(v) + b_5 T^2 \epsilon.
\] (74)

Furthermore,
\[
U(T) \leq \frac{2b_4 k T^{2\alpha - 1}}{1 - 2b_2 k T^{2\alpha - (1/2)}} \int_0^T U(v) dv \\
+ b_4 T^2 + b_5 T^{2\alpha - (1/2)} \sup_{0 \leq t \leq T} a(v) + \frac{b_5 T^2 \epsilon}{1 - 2b_2 k T^{2\alpha - (1/2)}}.
\] (75)

In view of Gronwall’s inequality, we get that
\[
U(T) \leq \left( \frac{b_4 T^2 + b_5 T^{2\alpha - (1/2)}}{1 - 2b_2 k T^{2\alpha - (1/2)}} \sup_{0 \leq t \leq T} a(v) + \frac{b_5 T^2 \epsilon}{1 - 2b_2 k T^{2\alpha - (1/2)}} \right) e^{(2b_2 k T^{2\alpha - (1/2)})}
\] (76)

Therefore,
\[
E \left( \sup_{0 \leq t \leq T} \| Z(t) - X(t) \|^2 \right) \\
\leq \left( \frac{b_4 T^2 + b_5 T^{2\alpha - (1/2)}}{1 - 2b_2 k T^{2\alpha - (1/2)}} \sup_{0 \leq t \leq T} a(v) + \frac{b_5 T^2 \epsilon}{1 - 2b_2 k T^{2\alpha - (1/2)}} \right) e^{(2b_2 k T^{2\alpha - (1/2)})}
\] (77)

which implies that there exists \( \delta = (y + (b_4 T^{2\alpha} / 1 - 2b_2 k T^{2\alpha - (1/2)}) e^{(2b_2 k T^{2\alpha - (1/2)})} > 0, \forall \epsilon > 0, \) satisfying
\[
E \left( \sup_{0 \leq t \leq T} \| Z(t) - X(t) \|^2 \right) \leq \epsilon \delta.
\] (78)

This completes the proof.

5. Examples

Example 1. Consider the Ulam–Hyers stability and existence and uniqueness of the solution to the following equation:

\[
C D_t^{(2/5)} X(t) = \frac{1}{15} \sin X(t) + \frac{2}{23} \cos X(t - \tau) \\
+ \left[ \frac{\Gamma (1/2)}{24\pi} \cos^2 X(t) + \frac{\ln (3/2)}{\pi e^4 X(t - \tau)} - X(t - \tau) \right] \frac{dW(t)}{dt},
\] (79)

where \( \alpha = (4/5), \ t \in [0, 8], \) and uniformly continuous functions.
\[ f(t, X(t), X(t - \tau)) = \frac{1}{15} \sin X(t) + \frac{2}{23} \cos X(t - \tau), \]

\[ g(t, X(t), X(t - \tau)) = \frac{\Gamma(1/2)}{24\pi} \cos^2 X(t) + \frac{\ln(3/2)}{\pi e^4} X(t - \tau). \]  

(80)

Due to

\[
\| f(t, Z(t), Z(t - \tau)) - f(t, X(t), X(t - \tau)) \|
\leq \frac{1}{15} \| \sin Z(t) - \sin X(t) \| + \frac{2}{23} \| \cos Z(t - \tau) - \cos X(t - \tau) \|
= \frac{1}{15} \| 2 \cos Z(t) + \frac{Z(t) + X(t)}{2} \| + \frac{2}{23} \| 2 \sin Z(t - \tau) + \frac{Z(t - \tau) + X(t - \tau)}{2} \|
\leq \frac{1}{15} \| 2 \sin Z(t) - X(t) \| + \frac{2}{23} \| 2 \sin Z(t - \tau) - X(t - \tau) \|
\leq \frac{1}{15} \| Z(t) - X(t) \| + \frac{2}{23} \| Z(t - \tau) - X(t - \tau) \|
\leq \frac{2}{23} \left( \| Z(t) - X(t) \| + \| Z(t - \tau) - X(t - \tau) \| \right),
\]

(81)

\[
\| g(t, Z(t), Z(t - \tau)) - g(t, X(t), X(t - \tau)) \|
\leq \frac{\Gamma(1/2)}{24\pi} \cos^2 Z(t) + \frac{\ln(3/2)}{\pi e^4} Z(t - \tau) - \frac{\Gamma(1/2)}{24\pi} \cos^2 X(t) - \frac{\ln(3/2)}{\pi e^4} X(t - \tau)
\leq \frac{\Gamma(1/2)}{24\pi} \| \cos^2 Z(t) - \cos^2 X(t) \| + \frac{\ln(3/2)}{\pi e^4} \| Z(t - \tau) - X(t - \tau) \|
\leq \frac{\Gamma(1/2)}{12\pi} \| \cos Z(t) - \cos X(t) \| + \frac{\ln(3/2)}{\pi e^4} \| Z(t - \tau) - X(t - \tau) \|
\leq \frac{2}{23} \left( \| Z(t) - X(t) \| + \| Z(t - \tau) - X(t - \tau) \| \right),
\]

Now, a numerical simulation will be carried out to find the solution of (79) is Ulam–Hyers stable, and we can see it in Figure 1.

Example 2. Consider the Ulam–Hyers stability of the following especial FSDEs with time delays:

\[ f(t, X(t), X(t - \tau)) \text{ and } g(t, X(t), X(t - \tau) \text{ satisfy Hypothesis 1, } \frac{3^{(3/2)}\Gamma^{2 \alpha - 1}}{2 \times (2/23)^{2} \times (11/10)_{1} \times 0.30} = 0.09 < (\alpha - (3/4))^{(1/2)} \Gamma^{2} (\alpha) = \sqrt{1(20/2)} \times \Gamma^{2} (3/4) = 0.30 \text{ and } 12^{(3/2)}\Gamma^{2 \alpha - (1/2)} = 12 \times (2/23)^{2} \times (5)^{(11/10)}_{1} \times 0.54 < (4\alpha - 3)^{(1/2)} \Gamma^{2} (\alpha) = \sqrt{1(7/5)} \times 0.60. \text{ Therefore, according to Remark 2 and Theorem 2, we can see that there is a unique solution to equation (80) and the solution is Ulam–Hyers stable.} \]
\[ C_{D_{\alpha}}^{(5/6)} X(t) = \frac{1}{n} X(t) + \frac{\bar{\varepsilon}}{\sqrt{2}} X(t - \tau) + \left[ \frac{1}{n^2} X(t) + \frac{\bar{\varepsilon}^2}{4} \sin \frac{1}{X(t - \tau)} \right] \frac{dW(t)}{dt}, \]  

(82)

where \( \alpha = (5/6), t \in [0, 2], n \geq 10, f(t, X(t), X(t - \tau)) = (1/n)X(t) + \bar{\varepsilon}X(t - \tau), \) and \( g(t, X(t), X(t - \tau)) = (1/n^2)X(t) + (\bar{\varepsilon}^2/4)\sin(1/X(t - \tau)) \) are measurable continuous functions, \( \bar{\varepsilon} \) is an arbitrary number, and \( 0 < \bar{\varepsilon} < (\sqrt{2}/8) \).
Because

\[
\|f(t, Z(t), Z(t) - r) - f(t, X(t), X(t) - r)\|^2 \\
+ \|g(t, Z(t), Z(t) - r) - g(t, X(t), X(t) - r)\|^2 \\
= \left\| \frac{1}{n} Z(t) + \frac{\varepsilon}{\sqrt{2}} Z(t) - r - \frac{1}{n} X(t) - \frac{\varepsilon}{\sqrt{2}} X(t) - r \right\|^2 \\
+ \left\| \frac{1}{n^2} Z(t) + \frac{\varepsilon^2}{4} \sin \frac{1}{Z(t) - r} - \frac{1}{n^2} X(t) - \frac{\varepsilon^2}{4} \sin \frac{1}{X(t) - r} \right\|^2 \\
\leq \frac{4}{n} \|Z(t) - X(t)\|^2 + \frac{\varepsilon^2}{2} \|Z(t) - X(t)\|^2 + \frac{\varepsilon^2}{8} \||Z(t) - r| - \frac{1}{\sin (1/2)} \| \frac{1}{X(t) - r} - \frac{1}{\sin (1/2)} \| \|^2 \\
= \frac{4}{n} \|Z(t) - X(t)\|^2 + \frac{\varepsilon^2}{2} \|Z(t) - X(t)\|^2 + \frac{\varepsilon^2}{8} \|Z(t) - X(t)\|^2 + \frac{\varepsilon^2}{2} \|Z(t) - X(t)\|^2 \\
= G(t, \|Z(t) - X(t)\|^2, \|Z(t) - X(t) - r\|^2)
\]

obviously, \(G(t, \|Z(t) - X(t)\|^2, \|Z(t) - X(t) - r\|^2)\) is nondecreasing, continuous, and concave function and \(\bar{k} = \max\{\sup_{t \in [0, T]} b(t), \sup_{t \in [0, T]} q(t)\} = \max \{(4/n^2), \varepsilon^2\} = (2/50), \bar{k} = (2^2 - 1)(1/2) \approx 0.54 < (4a - 3)^{(1/2)} \Gamma^2 (\alpha) \approx 0.57. \) From the arbitrariness of \(\varepsilon\), we know \(G(t, 0, 0) = 0\). And, \(\forall \varepsilon > 0, \exists \gamma = \left(1/5 \varepsilon\right) > 0\), such that \((3/4)T a - (3)T a = 3(2a - 1) T a - (1/2) (2a - 1) T a - (1/2) (2a - 1) T a - (1/2) - \sup_{t \in [0, T]} a(t) < 0.20 = (1/5 \varepsilon) = \gamma \varepsilon. \) That satisfies all the conditions of Theorem 2. Therefore, we can conclude that system (82) is Ulam–Hyers stable on \([0, 2]\).

Next, we will use a numerical simulation to verify the solution of (82) is Ulam–Hyers stable, and we can see it in Figure 2.

6. Conclusion

In this work, the objective is to research the existence and uniqueness of FSDEs with time delays using the novel Caratheodory approximation and the weaker non-Lipschitz condition. Furthermore, different assumptions are used to prove the Ulam–Hyers stability of the solutions. Finally, we present two examples to test the validity of the proposed theory. Our future work will focus on exploring Ulam–Hyers stability of various types of fractional differential equations with weaker conditions, and the explored conditions can be applied to a wider range of differential equations.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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