Spatial localization of thermal structures

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Abstract. The Cauchy problem for the integro-differential equation of nonlinear heat conduction, which describes the evolution of a finite temperature field in a medium in the presence of volumetric heat absorption, is discussed in the paper. A feature of such one-dimensional and three-dimensional processes is the spatial localization of heat, when thermal perturbations evolve in a finite spatial domain.

1. Introduction
It was established during studying the heat conduction processes [1-3], that in parabolic equations describing such processes with nonlinearity, lead not only to quantitative, but also qualitative changes in the nature of such processes.

In particular, it was shown that effects of a finite velocity of heat waves propagation can be observed in nonlinear diffusion processes of thermal conductivity, and in the presence of volumetric heat absorption in the medium, the fronts of heat waves can penetrate into the medium only to a finite depth.

For example, if in the Cauchy problem for a nonlinear heat equation with a finite initial temperature distribution there is such a sphere of finite radius \( R = \text{const} < \infty \), that the temperature outside this sphere is zero at any time \( t \geq 0 \), then this generalized solution of the problem is called spatially localized decision [4].

It should be noted that the following conclusion was substantiated in [5]: nonlinear heat conduction effects can be observed even in the presence of nonlinearities only in the lower terms of the equation having the form

\[
\frac{\partial u}{\partial t} = a^2 \Delta u - p(u, t)
\]  

(1)

In the paper, we consider the Cauchy problem for such an equation, associating the lowest term of the equation with the volume heat absorption in the medium. More precisely, we will assume that at each point of the perturbation region \( \Omega \), where \( u > 0 \), the volume absorption depends on the average temperature of the thermal structure

\[
\bar{u}(t) = \frac{1}{V} \int_{\Omega} u(x, y, z, t) dV
\]

(2)

where \( V = \int_{\Omega} dV \) - volume of the thermal perturbation region at that moment of time.

2. Mathematical problem statement
Describing the process of heat conduction in a medium with volumetric heat absorption, we'll consider the Cauchy problem for equation (1) with the lowest term
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\[ p(u,t) = q(u,u) = \begin{cases} \frac{\partial}{\partial t}u - q(u,u) & \text{for } u > 0 \\ 0, u = 0 \end{cases} \]  

(3)

Here \( \alpha > 0 \) – some known constant characterizing the fraction of thermal energy, for example, released during ionization processes, radiation, etc.

It should be noted that in this formulation, equation (1) is a nonlinear integro-differential equation.

One-dimensional model \( x \in \mathbb{R}^1 \)

May a thermal structure be organized in the medium at the initial moment of time, the temperature distribution in which is given by the following finite function

\[ u_0(x) = \begin{cases} U \cos \frac{\pi x}{2l}, x \in (-\ell, +\ell) \\ 0, x \in \mathbb{R}^1 \setminus (-\ell, +\ell) \end{cases} \]  

(4)

Here \( \ell \) - characteristic size of the thermal structure at the initial time, and \( U > 0 \) – known maximum value of medium temperature at a point \( x = 0 \).

Let's show that for a certain value of \( \ell \), which we define below, a generalized solution of the Cauchy problem for the equation

\[ \frac{\partial u}{\partial t} = a^2 \Delta u - q(u,u) \]  

(5)

with the initial temperature distribution (4) has the following form

\[ u(t) = u_0(x) \exp(-\alpha t) \]  

(6)

Indeed, in this case in the region, where \( u > 0 \)

\[ \frac{\partial u}{\partial t} = -au_0(x) \exp(-\alpha t) = -\alpha U \cos \frac{\pi x}{\ell} \exp(-\alpha t) \]  

(7)

and

\[ a^2 \frac{\partial^2 u}{\partial x^2} = -\frac{a^2 \pi^2}{2l} U \cos \frac{\pi x}{2l} \exp(-\alpha t) \]  

(8)

the average temperature of such structure is

\[ \bar{u}(t) = \frac{U \exp(-\alpha t)}{2l} \int_{-\ell}^{\ell} \cos \frac{\pi x}{2l} dx = \frac{U}{2} \exp(-\alpha t) \]  

(9)

Substituting the found values of the derivatives and average value into equation (5), we'll get

\[ \alpha \cos \frac{\pi x}{2l} = \frac{a^2 \pi^2}{2l} \cos \frac{\pi x}{2l} + \frac{\alpha}{2}, -\ell < x < +\ell \]  

(10)

It can be verified that if the size (half width) of the structure \( \ell \) is chosen as

\[ l = L = \frac{a \pi}{\sqrt{\alpha}} \]  

(11)

then relation (10) accomplishes identically. This means that in this case the function (6) is the desired generalized solution of the Cauchy problem. It describes the non-stationary temperature distribution in an environment with a volume heat absorption.

The evolution of the found thermal structure in time occurs without changing the width of the thermal perturbation region (Figure 1). Such self-isolation of a spatially localized thermal structure is based on the process of volumetric heat absorption, which suppresses the diffusion mechanism of medium heating. It should be noted that at the boundaries of the self-isolated structure \( x = \pm L \) physical conditions of temperature and heat flux continuity are satisfied at any moment of time.
Figure 1. Evolution of the thermal perturbation region (hatched) in time.

3D model \( r \in \mathbb{R}^3 \)

Let the initial temperature distribution in the medium be described by a spherically symmetric finite function

\[
u_0(r) = \begin{cases} 1, & 0 \leq r < r_0 \\ 0, & r > r_0 \end{cases}
\]

(12)

Here \( \psi(\xi) = \xi^{-1} \sin \xi \), moreover, the parameter \( \nu = 4.493 \) corresponds to the value of the first positive root of the equation \( \psi'(\xi) = 0 \).

Such a non-negative generalized function \( \nu(r) \), whose carrier is a sphere of \( r_0 \), radius, describes the thermal structure of a single spherically symmetric thermal pulse of finite spatial size equal to \( r_0 \), with a maximum temperature value at the center \( r = 0 \) equal to \( U \). At the front of the heat pulse \( r = r_0 \) the temperature becomes equal to zero, and the outlet to the zero unperturbed background occurs with a zero gradient, providing the heat flux at the front equal to zero.

It can be convinced by the direct verification that if the thermal structure (12) has a well-defined size equal to

\[
r_0 = R = \nu \alpha^{-\frac{1}{2}}
\]

(13)

Then the following function is the generalized solution to the Cauchy problem for equation (1), written in a spherical coordinate system,

\[
u(r,t) = \nu_0(r) \exp(-\alpha t),
\]

(14)

which describes the thermal structure evolution in time. It should be noted that, as in the one-dimensional case, during the thermal structure evolution (14), the size of the thermal perturbation region, where \( \nu > 0 \), does not change in time (Figure 2).

Figure 2. The evolution of thermal perturbation in time.
It should be noted that the size $R$ of a localized structure depends on the value of the coefficient $\alpha$, which characterizes the fraction of the structure thermal energy that dissipates turning in other energy.

3. Results
Thus, in a medium with volumetric heat absorption, the evolution of a finite thermal structure in time can proceed in the mode of spatial localization of thermal perturbation. In this case, the process of heat propagation is described by the nonlinear heat conduction equation.

It should be noted that thermal perturbations are spatially localized in the processes described by solutions (6) and (14), and the solution carrier does not change in time in the stable localization mode.

The spatial localization of the examined structures can be explained physically as follows. The initial temperature profile of the heat pulse is such that the heat flux at the pulse front is zero. It is necessary to change its steepness in order for such a front to be set in motion. But the diffusion mechanism of heating the environment near the thermal front, which could have done this, is suppressed by the volumetric heat absorption. Therefore, the sloping front remains motionless. In real environments, the processes of volumetric heat absorption may be conditioned by the processes of medium ionization or radiation processes of heated media.

In the considered cases, we did not take into account the dependence of the thermophysical properties of the medium on temperature. It should be noted that taking into account the dependence of the thermal conductivity on temperature expands the possibilities of implementing the modes of thermal structures spatial localization [3, 4].

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