Nuclear clusters with Halo Effective Field Theory

RENA TO HIGA
Helmholtz-Institut für Strahlen- und Kernphysik (Theorie), Universität Bonn,
Nussallee 14-16, 53115 Bonn, Germany
higa@itkp.uni-bonn.de

After a brief discussion of effective field theory applied to nuclear clusters, I present
the aspect of Coulomb interactions, with applications to low-energy alpha-alpha and
nucleon-alpha scattering.

Keywords: Effective field theory, exotic nuclei, universality
21.45.-v, 21.60.Gx

1. Introduction

At low energies, few-body systems with large scattering length exhibit universal
features (universality) that are independent of the interaction details. Some conse-
quensequences are the existence of a shallow bound state and the Efimov effect in the
two- and three-body sectors, respectively. These universal properties have a wide
range of applications, from particle and nuclear to atomic and molecular physics.

Universality has been put on a different light in the language of effective field
theory (EFT). EFT is suitable for energies with an associated Compton wave-
length $\lambda \sim 1/M_{lo}$ that is much larger than the interaction radius $R \sim 1/M_{hi}$, where
$M_{lo}$ and $M_{hi}$ are respectively the characteristic low and high momentum scales.
The formalism allows for corrections of $O(M_{lo}/M_{hi})$, obtained in a systematic and
model-independent way. For nuclear systems with $A \leq 4$, it provides a convincing
explanation for some few-nucleon correlations, like the Phillips and Tjon lines,
as well as reliable error estimates for some astrophysical reactions like $n+p \rightarrow d+\gamma$.

Technical and numerical complications arise for nuclei with $A > 4$. There are,
however, interesting situations of halos and weakly bound nuclear clusters, where
large simplifications can be achieved. The typical momentum $M_{hi}$ required to excite
the core/clusters is much larger than the momentum $M_{lo}$ that binds the clusters
altogether. The degrees of freedom then become the clusters themselves, usually stable nuclei like alpha particles and nucleons. In the following I present
EFT studies for alpha-alpha ($\alpha\alpha$) and nucleon-alpha ($N\alpha$) interactions, which are
the basic ones in order to build more complex systems, like the $^6$He halo nuclei, the first $0_+$ excited state in $^{12}$C (Hoyle state), or the $\frac{1}{2}^+_1$ excited $^9$Be state.

2. EFT for nuclear clusters

In halo EFT each cluster is represented by an elementary field, with short-range forces represented by contact interactions. This is a good approximation for systems whose binding (or resonance) energy has a compton wavelength $\lambda_B \sim 1/M_{lo}$ much larger than the radius $r_c \sim 1/M_{hi}$ of the largest cluster. To simplify the discussion, let us consider a system of two identical bosons, with mass $m_\alpha$, represented by a field $\phi$ and an $S$-wave strong interaction. The latter is described by the Lagrangian

$$
\mathcal{L} = \phi^\dagger \left[ i\partial_0 + \frac{\vec{\nabla}^2}{2m_\alpha} \right] \phi - d^\dagger \left[ i\partial_0 + \frac{\vec{\nabla}^2}{4m_\alpha} - \Delta \right] d + g \left[ d^\dagger \phi \phi d + (\phi \phi)^\dagger d \right] + \cdots,
$$

where we introduce an auxiliary (dimeron) field $d$, with “residual mass” $\Delta$, carrying the quantum numbers of two bosons in $S$-wave and coupling with their fields through the constant $g$. The dots stand for higher order terms in a derivative expansion.

The form of the scattering amplitude depends on the magnitude of the parameters in the Lagrangian. Here we concentrate on the scale of $\Delta$, which determines the size of the scattering length $a_0$. A minimal fine-tuning on $\Delta \sim M_{hi}/M_{lo}$ provides an unnaturally large $a_0 \sim 1/M_{lo}$, corresponding to a system of strongly interacting bosons. This situation is analogous to the nucleon-nucleon ($NN$) case.

The kinetic term $i\partial_0 + \vec{\nabla}^2/4m_\alpha$ of the dimeron propagator is subleading compared to $\Delta$ and the amplitude has the form of the effective range expansion (ERE) formula, with the effective range $r_0$, shape parameter $P_0$, and higher order terms treated in perturbation theory. However, the power counting for nuclear clusters is often more complicated. In $\alpha\alpha$ scattering for example, an extra amount of fine-tuning ($\Delta \sim M_{hi}^2/m_\alpha$) is necessary to generate an even larger scattering length, $a_0 \sim M_{hi}/M_{lo}^2$. As a consequence, the dimeron’s kinetic and residual mass terms become of comparable order, which requires resummation of $r_0$ to all orders. This resummation is necessary to reproduce a narrow resonance at low energy.\(^\text{11}\)

![Graphic representation of $T_{CS}$, with multiple insertions of the bare dimeron propagator (double line) and the “bubble loop”. The latter contains Coulomb photons resummed to all orders (shaded ellipse).](image)

\(^a\) Even if these length scales are not quite well (but still) separated, corrections of the order $r_c/\lambda_B \sim M_{lo}/M_{hi}$ can be computed in a controlled and systematic way.

\(^b\) For simplicity, we set $g^2 \sim M_{lo}^2/m_\alpha$ to get an effective range with a natural size, $r_0 \sim 1/M_{hi}$.

\(^c\) A natural scale $\Delta \sim M_{hi}^2/m_\alpha$ generates $a_0 \sim 1/M_{hi}$ and a system of weakly interacting bosons.
Electromagnetic interactions are introduced in the usual way, among which Coulomb is the dominant one at low energies. The latter was formulated in the EFT framework by Kong and Ravndal for the two-protons system, and can be extended in a straightforward way to include resonances. The Coulomb-modified strong amplitude $T_{CS}$ is diagrammatically illustrated by Fig. 1 (see caption) and has the form of a geometric series. Like in the proton-proton case, the power counting for narrow resonances requires resummation. Up to next-to-leading order (NLO) one gets for our two-boson example, with reduced mass $\mu$ and charge $Z\alpha$,

$$T_{CS} = -\frac{2\pi}{\mu} C^2_\eta e^{2ig_0} \left[ -\frac{1}{a_0 + k^2 \frac{a_0}{4} - 2k_C H} + \frac{P_0}{4} \left( -\frac{1}{a_0 + k^2 \frac{a_0}{4} - 2k_C H} \right)^2 \right],$$

(2)

where $k_C = \mu Z^2_\alpha \alpha_{em}$ is the inverse of the Bohr radius, $\eta = k_C / k$, $C^2_\eta = 2\pi \eta / (e^{2\pi \eta} - 1)$, $\sigma_0 = \arg \Gamma(1 + i\eta)$, and $H(\eta) = \psi(i\eta) + (2i\eta)^{-1} - \ln(i\eta)$. There is a small complication in this formula, that around the resonance energy multiple “kinematical fine-tunings” are required. This is a technical rather than a conceptual problem, and can be handled via an expansion around the resonance pole.

3. Applications

3.1. $\alpha\alpha$ scattering

The $\alpha\alpha$ system is dominated by $S$-wave at low energies, with the presence of a very narrow resonance at $E_R \approx 92$ keV and width $\Gamma_R \approx 6$ MeV (the $^8\text{Be}$ ground state). The $\alpha\alpha$ scattering length $a_0 \sim 2000$ fm is nearly three orders of magnitude larger than the alpha matter radius, suggesting the large amount of fine-tuning discussed in the last section. The low-energy scale $M_{lo} \sim \sqrt{m_\alpha E_R} \approx 20$ MeV is roughly seven times smaller than a high momentum scale associated to either the pion mass or the excitation energy of the alpha particle, $M_{hi} \sim m_\pi \sim \sqrt{m_\alpha E^*_{\alpha}} \approx 140$ MeV. Within the power counting for resonances, one would expect convergent results for observables at laboratory energies up to 3 MeV.

Coulomb interactions are described in terms of the momentum scale $k_C$ which, due to the large $\alpha\alpha$ reduced mass, is numerically of $O(M_{hi})$. The low-energy amplitude is then obtained by expanding the function $H(\eta)$ for large $\eta$. However, it is interesting to discuss the other limit $k_C \to 0$, when Coulomb is turned off. In this case one has $2k_C H(\eta) \to i k \sim M_{lo}$, which is larger than the terms $1/a_0$ and $r_0 k^2/2 \sim M^2_{lo}/M_{hi}$. As a consequence, at leading order the denominator of the amplitude is given by the unitarity term $-ik$, and the $\alpha\alpha$ exhibits non-relativistic conformal invariance. Such unitary limit implies the existence of the $^8\text{Be}$ ground state right at threshold, and the corresponding three-body system ($^{12}\text{C}$) showing an exact Efimov spectrum. These results change for a physical value of $k_C$, due to the breaking of conformal invariance by Coulomb forces. Nevertheless, the fact that the ground state of $^8\text{Be}$ and the Hoyle state in $^{12}\text{C}$ remain very close to the threshold into $\alpha$-particles suggests that this conformal picture is not far from the real case.
The low-energy expansion of $H(\eta)$ up to NLO is given by $\frac{1}{12}\eta^2 + \frac{1}{120}\eta^4$, and resembles the usual effective range expansion. Defining $\tilde{r}_0 = r_0 - 1/3kC$, $\tilde{P}_0 = P_0 + 1/15k_3C$, and performing an expansion around the resonance pole, $T_{CS}$ reads

$$T_{CS} = -\frac{2\pi}{\mu} C^2_\eta e^{2i\sigma_0} \left[ \frac{1}{\eta} \frac{\tilde{r}_0^2 (k^2 - k_R^2) - ikC^2_\eta}{\frac{r_0^2}{2} (k^2 - k_R^2) - ikC^2_\eta} \right] + \frac{\tilde{P}_0}{4} \left( \frac{k^4 - k_R^4}{\frac{r_0^2}{2} (k^2 - k_R^2) - ikC^2_\eta} \right)^2,$$

where $k_R$ is the momentum of the resonance. The scattering length is obtained from

$$a_0^{-1} = \tilde{r}_0 k_R^2/2 - \tilde{P}_0 k_R^4/4.$$

An intriguing puzzle arises at this point. In the $NN$ case, the fine-tuning necessary to generate the low-energy bound and virtual states reflects a sensitivity to QCD parameters. For the present case, this sensitivity seems to be much larger, as indicated by two orders of magnitude of $\Delta$ away from a natural size. Apart from that, one observes an extra fine-tuning generated by a roughly 90% cancellation between strong and electromagnetic contributions in the parameter $\tilde{r}_0$, now of $O(M_{lo}/M_{hi}^2)$. That leads to a scattering length of $O(M_{hi}^2/M_{lo}^3)$, an order of magnitude larger than its purely strong part. It is remarkable, that if the strong forces generated an $r_0$ 11% larger the $^8$Be ground state would be bound, with drastic consequences in the formation of elements in the universe (see also Ref. 14).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{EFT results at LO (dotted) and NLO (solid), compared against the scattering data. Left panel: phase shift $\delta_0$. Right panel: $K(\eta) \equiv C^2_\eta (\cot \delta_0 - i)/2\eta + H(\eta)$.}
\end{figure}

Despite the large fine-tunings and cancellations of strong and electromagnetic contributions, we obtain a successful description of $\alpha\alpha$ scattering at low energies, as shown in Fig. 2. S-wave phase shifts and ERE parameters can be found in a review by Afzal et al. [15] and references therein. The latest extraction of ERE parameters before our work was done in Ref. [16] with numerical values given in Table 1. We used the available scattering data, combined with the most recent measurement of the resonance properties [17], and we were able to extract the ERE parameters with smaller errorbars. The disagreement between Ref. [16] and our results for $a_0$ is likely due to an approximation made in the latter, as discussed in details in Ref. [11].
Nuclear clusters with Halo Effective Field Theory

Table 1. S-wave effective range parameters.

|       | $a_0$ (10^3 fm) | $r_0$ (fm) | $P_0$ (fm^3) |
|-------|-----------------|------------|--------------|
| LO    | −1.80           | 1.083      | —            |
| NLO   | −1.92 ± 0.09    | 1.098 ± 0.005 | −1.46 ± 0.08 |
| ERE (our fit) | −1.92 ± 0.09 | 1.099 ± 0.005 | −1.62 ± 0.08 |
| Ref[10] | −1.65 ± 0.17    | 1.084 ± 0.011 | −1.76 ± 0.22 |

Our combined fit, albeit showing a convergence pattern, still has a relative large $\chi^2$/datum. If one uses only the scattering data, the fit becomes much better but the resonance width is well underpredicted. This happens regardless if one uses EFT or the conventional ERE. We take this as indication that the measurement of the resonance properties and the (rather old) scattering data are incompatible with each order or, at least, one of them has overestimated quoted errors. Reanalysis or even new measurements of scattering data seem necessary to resolve this discrepancy.

3.2. $N\alpha$ scattering

At low energies $N\alpha$ scattering is dominated by the waves $S_{1/2}$ (or the notation $0^+$), $P_{3/2}$ (1+), and $P_{1/2}$ (1−). A narrow, low-energy resonance is seen in the 1+ channel, and a very broad one, at the 1− channel. The latter is a perturbative effect that appears only beyond the NLO that we are working on[9,18]. Therefore, for EFT up to this order, only 0+ and 1+ are relevant.

The power counting for a narrow $P$-wave resonance was developed in Refs.[9,10] with an application to neutron-alpha ($n\alpha$) scattering. Recently we incorporated the expansion around the resonance pole to this process, obtaining similar to (but showing better convergence than) the previous works[18].

![Fig. 3. EFT results at LO (dotted) and NLO (solid), compared against the partial wave analysis results from Arndt et al. (circles).](image-url)

In proton-alpha ($p\alpha$) scattering, Coulomb interactions are included by extending the formalism from $S$-wave[11] to $P$-wave strong interactions. One important difference is that in the latter not only the “scattering length” $a_{1+}$, but also the “effective range” $r_{1+}$, are renormalized by Coulomb loops. Fig. shows our prelim-
inary results for $\alpha\alpha$ cross-section at $\theta = 140^\circ$ laboratory angle, compared against results of Ref.\textsuperscript{[19]} using the ERE. Clearly, a good agreement is achieved already at LO.

4. Outlook

I presented the EFT formalism for cluster resonances in the presence of Coulomb interactions. As applications, low-energy $\alpha\alpha$ and $N\alpha$ scattering were successfully described. These two interactions are the basic ones before considering more complicated clusters of $\alpha$ and nucleons. The Hoyle state in $^{12}$C is particularly interesting due to its relevance in the formation of heavy elements. A model-study with this state in mind was developed in Ref.\textsuperscript{[20]}, where a perturbative treatment of the Coulomb interaction was proposed. This idea could be useful to handle the technical difficulties involving three charged particles.

Acknowledgments

I would like to thank Hans-Werner Hammer, Bira van Kolck, and Carlos Bertulani for stimulating collaboration, and the organizers of the Asia-Pacific Few-Body 2008 for the invitation and enjoyable conference. This work was partially support by the BMBF under contract number 06BN411.

References

1. V. Efimov, Phys. Lett. \textbf{33B}, 563 (1970).
2. E. Braaten and H.-W. Hammer, Phys. Rept. \textbf{428}, 259 (2006).
3. L. Tomio, this proceedings.
4. P.F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. \textbf{52}, 339 (2002).
5. P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Nucl. Phys. \textbf{A676}, 357 (2000).
6. L. Platter, Phys. Rev. C \textbf{74}, 037001 (2006).
7. L. Platter, H.-W. Hammer, and U.-G. Meißner, Phys. Lett. \textbf{B607}, 254 (2005).
8. G. Rupak, Nucl. Phys. A \textbf{678}, 405 (2000).
9. C. A. Bertulani, H.-W. Hammer, and U. van Kolck, Nucl. Phys. \textbf{A712}, 37 (2002).
10. P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Lett. \textbf{B569}, 159 (2003).
11. R. Higa, H.-W. Hammer, and U. van Kolck, Nucl. Phys. \textbf{A809}, 171 (2008).
12. D. L. Canham and H.-W. Hammer, \texttt{arXiv:0807.3258v1} [nucl-th].
13. X. Kong and F. Ravndal, Phys. Lett. \textbf{B450} 320 (1999); Nucl. Phys. \textbf{A665}, 137 (2000).
14. H. Oberhummer, A. Csótó, and H. Schlattl, Science \textbf{289}, 88 (2000).
15. S.A. Afzal, A.A.Z. Ahmad, and S. Ali, Rev. Mod. Phys. \textbf{41}, 247 (1969).
16. G. Rasche, Nucl. Phys. \textbf{A94}, 301 (1967).
17. S. Wüstenbecker, H.W. Becker, H. Ebbing, W.H. Schulte, M. Berheide, M. Buschmann, C. Rolfs, G.E. Mitchell, and J.S. Schweitzer, Z. Phys. A \textbf{344}, 205 (1992).
18. C.A. Bertulani, R. Higa, and U. van Kolck, in progress.
19. R.A. Arndt, L.D. Roper, and R.L. Shotwell, Phys. Rev. C \textbf{3}, 2100 (1971).
20. H.-W. Hammer and R. Higa, Eur. Phys. J. A \textbf{37}, 193 (2008).