Aspects of Transplanckian Scattering

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Abstract. We attempt to address the well known problem of quantum gravity - the information paradox, through the study of transplanckian collisions, that may proceed with the production of intermediate black hole states. We apply the S-matrix approach to study $2 \rightarrow 2$ scattering of particles with the center of mass energy much greater than the Planck mass, and investigate the dependence of the scattering amplitude on the impact parameter. It is expected that the main properties of the S-matrix (unitarity, analyticity and crossing) may provide sufficiently strong constrain for the scattering amplitude to unravel non-trivial information about the underlying properties of quantum gravity. However, we argue that our present understanding is far from complete and additional novel ideas are required to resolve this puzzle.

1. Introduction

Although we understand how to construct consistent quantum field theory describing electromagnetic, strong and weak interactions, we still don’t know how to unite gravity with the quantum mechanics. One of the reasons is that in gravity, violation of unitarity occurs not only for short scales but also at long distances, not linked to the UV-behavior of gravity. Another major difference of gravity from the rest of the Standard Model interactions is that gravity generates an energy dependent scale – Schwarzschild radius. Although a lot of effort has been directed towards the resolution of this puzzle (see, e.g., [1] and references therein), there is no universally accepted understanding of even the simplest possible process involving $2 \rightarrow 2$ exclusive scattering that proceeds with the black hole production and its subsequent evaporation. Clearly, better understanding of some of these questions will allow us to address one of the fundamental problems of quantum gravity - the information paradox. This problem is one of the obstacles, preventing the unification of gravity with quantum mechanics.

In an attempt to reconcile gravity (GR) with quantum mechanics (QM) one faces multiple difficulties. Three of these problems are:

A) GR is not renormalizable;
B) unitarity violation (information paradox);
C) problem in defining gauge invariant observables in GR.

It seems, B) and C) suggest that we might want to abandon the notion of locality in order to find consistent theory of QG. Therefore, to study the problems of quantum gravity a framework is required that preserves causality without the need of demanding locality.

Although, string theory might seem to provide such a framework, it has been argued [2] that “stringy” effects are not very essential in the regime of interest relevant for the resolution of the
information paradox. On the other hand, it is expected that nonperturbative dual formulations such as AdS/CFT should resolve the problem [3].

Taking into account the problems with locality and renormalizability, along with the lack of knowledge of the exact theory of QG, the attempt has been made to understand this problem by employing the S-matrix formalism. Certainly, the existence of such a gravitational S-matrix is a very strong assumption that can only be tested a posteriori. It appears that the main properties of the S-matrix – unitarity, analyticity and crossing, are in no contradiction with non locality. The goal of this approach is to understand what the S-matrix description along with the expected properties of classical gravity can teach us about the underlying theory of quantum gravity. However, one should keep in mind that the S-matrix description has a lot of uncertainties. For instance, we don’t know what are the precise quantum numbers describing asymptotic states, and if these states are complete. Moreover, to avoid problems with IR divergences, the extension of the S-matrix to inclusive amplitudes has to be made, much in the same way as in QED [7].

In what follows, we will describe how to approach one of the simplest QG problems, involving the exclusive collision of two light particles at the center of mass energies much larger than the Planck scale. We consider scattering at different impact parameters. Studying the eikonal approximation, we argue that there exists a parameter, which is closely related to the average number of gravitons involved in the collision, and using it, explore the large-N limit of gravitational scattering. We then comment on the idea of fractionation, and think that this is a new constructive idea that is worth exploring. We also present the best available ansatz for the amplitude that describes the production of intermediate black hole state, and discuss its implications and limitations. Finally, we formulate the questions that are of immediate relevance, when one attempts to study the problem of transplanckian scattering.

2. Transplanckian Scattering
The relevant length scales involved in the two particle scattering (in $D = 4$) with the center of mass (c.o.m.) energy $\sqrt{s} \gg M_P$, where $M_P$ is the Planck mass, are:

i) $\ell_P \equiv \sqrt{\hbar G_N}$, quantum Planck’s length scale (since it contains $\hbar$);

ii) $R = 2G_N\sqrt{s}$, classical characteristic Schwarzschild radius;

iii) $b$ - impact parameter of scattering;

iv) $\ell_s = \sqrt{\hbar \alpha'}$, string length scale.

As usual, to classify different regimes of scattering, or to establish a perturbative expansion, it is instructive to define the dimensionless ratios made from these length scales, which are:

a. $(\ell_P/b)^2 = \hbar G_N/b^2 \equiv \alpha(b)$. We expect $b$ to be related to the typical (exchanged) graviton’s wave length, $\lambda$, in which case $\alpha$ will become the dimensionless graviton selfcoupling.

b. $(R/\ell_P)^2 = 4G_N s/\hbar \equiv N(s)$. For example, in [9], $N(s)$ itself plays the role of the dimensionless coupling. We will show below in what sense $N$ is related to the number of gravitons.

c. $(R/b)^2 = \alpha N$. While $\alpha$ and $N$ are quantum, $\alpha N$ is a classical parameter. For $b \leq R$ or $\alpha N \geq 1$, collision produces a trapped surface, therefore a black hole [8].

Consider the regime, where the string length is the shortest of the scales (that is $\ell_s \ll \max\{R, b\}$), then for $b \gg R$, we have a small angle scattering (single graviton exchange dominates the amplitude). Elastic amplitude grows too fast with energy, and therefore, breaks unitarity. To maintain unitarity the full sum over ladder and cross-ladder diagrams is required (eikonal regime). Decreasing $b$ further, while keeping $b > R$, we will enter a regime, where intermediate graviton exchange diagrams among different legs of the ladder (H-diagrams) will start to contribute. Simplest H-diagram $\sim G_N s (R/b)^2$, will be described by $(R/b)^2$ classical expansion [13]. We have to add all H-type of diagrams, which in certain respect is a classical problem. For $b \sim R$, we have a large angle scattering. Finally, for $b \leq R$, collision produces a trapped surface, signalling the presence of a black hole (BH) [8].
3. \( N \) as the Number of Gravitons

For \( b \gg R, \ell_P \), the single graviton exchange dominates the scattering amplitude, \( A_0(s, t) \approx -8\pi G_N s^2 / t \). The latter is just the leading Born amplitude. When the impact parameter becomes smaller, but still much larger than \( R \), the scattering enters the eikonal regime, in which case [16]:

\[
iA_{\text{eik}}(s, t) = 2s \int d^2x_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} \left( e^{i\chi(x_\perp, s)} - 1 \right),
\]

\[
\chi(x_\perp, s) = \frac{1}{2s} \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot x_\perp} A_0(s, -k_\perp^2),
\]

where \( |x_\perp| = b \), and \( \chi \sim N(s) \ln(b\lambda_{IR}) \) is an eikonal phase with \( \lambda_{IR} \) being an IR cutoff. Soft graviton divergences can be eliminated in case the number of dimensions is \( D > 5 \), (see, e.g. [10]) in case of \( D > 5 \). However, this issue will not be relevant for our discussion.

The eikonal amplitude starts to dominate the Born amplitude, when \( \chi \geq 1 \). In \( D > 5 \) dimensions, \( \chi \sim G_D s / b^{D-4} \), therefore, there is a strong dependence on the impact parameter, while in \( D = 4 \), the dependence is only logarithmic. This suggests that in \( D = 4 \), the value of \( N(s) \) determines the approximation that we have to employ. In particular, for \( N \leq 1 \), the Born approximation can be sufficient, while in case \( N(s) \gg 1 \), eikonal approximation has to be used (if also \( R \ll b \)).

The eikonal approximation is described by the sum of Feynman ladder diagrams with multiple graviton exchanges. The sum of ladder and crossed-ladder diagrams at \( n \)-loop order can be written as follows [9]:

\[
iA_n(s, q) = \frac{2s}{(n + 1)!} \int \prod_{j=1}^{n+1} \frac{d^2k_j}{(2\pi)^2} \frac{iA_0(s, -k_j^2)}{2s} \left( 2\pi i \right)^{2d(n)} \left( \sum_j k_j - q_\perp \right). \]

It can be shown that for \( n \sim N(s) \), we have: \( A_{\text{eik}} = \sum_n A_n \approx A_N \). That is the eikonal sum is dominated by the \( N \)-ladder graviton diagram [9]. Indeed, the integrand in (1) has a phase function \( f(b) \equiv -iqb + i\chi(b, s) \), saddle point of which is at \( b \sim N(s) / q \). On the other hand, if we Taylor expand \( e^{i\chi} \) in the eikonal amplitude, the phase function would become: \( f_n(b) \approx -iqb + n \ln(\chi) \), saddle point of which is at \( b \sim n / q \).

Therefore, even for \( R \ll b \), the (ladder and cross-laddered) diagrams that give the dominant contribution to the scattering amplitude are characterized by \( N \) exchanged gravitons. It is also clear that in either the Born or eikonal regime, \( \alpha \ll 1 / N \).

3.1. Meaning of Imaginary \( N \)

The scattering amplitude of one particle, seen as a test body, in the background metric of the other particle, an Aichelburg-Sexl metric [17], was computed by ’t Hooft in [18]. The same result can be obtained applying eikonal approximation, see, e.g. [19].

Now, consider small angle scattering (when \( \theta^2 \sim -t / s \ll 1 \)) for which the important contribution comes from \( b \approx R / \theta \gg R \). In this case, the small-angle amplitude can be computed by just extending the previous result up to \( b = 0 \). As a result:

\[
iA_{\text{eik}}(s, t) = \frac{2\pi s N}{-t} \left( \frac{\pi(1 - iN/4)}{\Gamma(1 + iN/4)} \right) \left( \frac{4\lambda^2 R}{-t} \right)^{-iN/4}, \]

where the first fraction is just the Born amplitude. (We also assumed \( m \ll s \), where \( m \) is the mass of the scattered particle.) The complex \( N \)-poles of this amplitude are analogous to those in the Coulomb scattering, due to the \( 1/r \) potential. Therefore, ’t Hooft poles originate from the singularity at \( b = 0 \), that we attempted to ignore, without consistently treating the \( b < R \) region. More about the nature of these poles can be found in [19].
3.2. Fractionation

It is claimed in [9] that the transplanckian scattering in gravity probes long-distance physics. This suggests that the short-distance singularities in the loop amplitudes are not crucial in studying such collisions, however, unitarity is. Also, in transplanckian scattering the exchanged gravitons seems to redistribute democratically the momentum transfer. This idea, is called fractionation, and is partly motivated by the saddle point approximation made above.

However, we think that even thought the idea of fractionation is plausible, some of the details must be clarified. In particular, since the eikonal approximation is essentially a random walk in the scattering angle, it can be shown that the average value of the momentum transfer in a single graviton, for a dominant diagram with \( N \) ladders, must be \( q/\sqrt{N} \), instead of \( q/N \), expected in [9]. This suggests that the average graviton wavelength in the scattering process must be: \( \lambda \sim \frac{b}{N/\sqrt{q}} \geq \sqrt{N}\ell_{P} \). Since \( b = \ell_{P}/\sqrt{\alpha} \), when \( R \sim b \), we have \( \alpha \sim 1/N \), therefore, \( b \sim \sqrt{N}\ell_{P} \). Taking into account that \( b \sim \sqrt{N}/q \), we can verify that when \( R \sim b \), we have \( q \sim M_{P} \). In other words, in the regime of black hole formation we expect \( s \sim M_{P}^{2} \) and \(-t/s \sim \alpha N \sim 1\).

Combining ideas of classicalization and fractionation one can deduce that in any transplanckian scattering the gravitons, can not have wavelengths shorter than \( \ell_{P} \), as a result they are getting multiplied in quantity and redistribute the large momentum transfer. This principle can be used to calculate the entropy of the collision process, when \( N \gg 1 \). In this case, the entropy would be related to the number of distributions of momenta \( k_{i} \) among \( N \) gravitons, such that \( k_{i} \leq M_{P} \) and \( \sum_{i=1}^{N} k_{i} = q \).

When \( b \) approaches \( R \), the diagrams with intermediate graviton exchanges among gravitons themselves in the ladder (\( H \)-diagrams) may become important. Moreover, one can wonder about the possible Reggeization of the gravitons by considering vertical ladder diagrams. However, due to fractionation, one expects two vertical gravitons in the \( H \)-diagram to be soft, making the contribution of the vertical ladder suppressed. Therefore, one would expect that the Regge regime will be irrelevant [9], unless \( b \sim R \).

3.3. The Black Hole Ansatz in the Large \( N \) limit

Assuming that the S-matrix obeys unitarity condition from the beginning, the partial-wave expansion of the scattering amplitude (in \( D \) dimensions) can be written as follows [10,14]:

\[
A(s,t) = \psi_{h} s^{2-D/2} \sum_{\ell=0}^{\infty} (\ell + h) C_{\ell}^{h}(1 + 2t/s)f_{\ell}(s),
\]

\[
f_{\ell}(s) = \frac{1}{2i} \left( e^{2i\delta_{\ell}(s) - 2\beta_{\ell}(s)} - 1 \right),
\]

where \( h = (D-3)/2 \), \( \psi_{h} \equiv 2^{4h+3} \pi^{h} \Gamma(h) \), \( \delta_{\ell} \) is the phase shift, and \( \beta_{\ell} \) is an absorption parameter (\( C_{\ell}^{h}(x) \) are Gegenbauer polynomials that in case of \( D = 4 \) coincide with the Legendre polynomials). As one can check, partial-wave amplitudes satisfy unitarity condition:

\[
\text{Im} f_{\ell} \geq |f_{\ell}|^{2},
\]

where \( \ell \geq 0 \) and is an integer. Also, notice that in terms of our dimensionless parameters the angular momentum is: \( \ell \sim \sqrt{s} b \sim N/\sqrt{\alpha N} \).

In the eikonal regime, and when \( D = 4 \), one has: \( \delta_{\ell}^{\text{eik}}(s) \sim N(s) \) and \( \beta_{\ell}^{\text{eik}}(s) = 0 \). The contribution from the graviton bremsstrahlung of soft gravitons can be estimated to give the following correction to the absorptive part: \( \beta_{\ell}^{\text{soft}} \sim \alpha N^{2} \).

According to Giddings and Srednicki [10], the entropy of the produced BH is \( S_{BH}(s,b) \sim G_{N}s/h \sim N(s) \), in which case, for \( R \leq b \) or \( \ell \leq N \), they obtained:

\[
\delta_{\ell}^{GS}(s) \sim \pi S_{BH}, \quad \beta_{\ell}^{GS} \sim S_{BH}.
\]
As a result, the elastic and absorptive cross sections can be shown to be:

\[ \sigma_{el} \approx \sigma_{abs} \sim \sum_{\ell=0}^{N} (\ell + h) \sim N^2. \]  

(9)

To my knowledge this result (with later modifications in [1]) seems to be the best guess for such an amplitude. However, I think, a lot of work has to be done to obtain a more realistic result. In particular, the following questions must be addressed.

4. Reformulating Old Questions  

What is the number of BH states? Consider multiparticle collision with the c.o.m. energy \( E \) that results to the production of BH states. Assume that the initial state is \( |E, a\rangle_{in} \), where \( a \) run through the number of states that can be produced from the collapse of matter of energy \( E \), therefore, \( a \) takes \( \exp\{E^{3/2}\} \) integer values. Indeed, in \( D \)-dimensions, \( E \sim R^{D-1}T^D \), \( S \sim (TR)^{D-1} \) and we need \( R > R_S \), where \( T \) is the temperature, \( R \) is the size of the sphere inside of which matter is confined, and \( R_S = (G_pmE)^{1/(D-3)} \). Since not all of the energy goes into the production of BH, the produced state will be a mixture of BH state and radiation. The numbers of states corresponding to BH of mass \( M \) and radiation of energy \( E - M \) are: \( \exp\{M^2\} \) and \( \exp\{E^2\} \), respectively. This is so, since the typical radiation quanta have energies \( \sim 1/R \), then given the radiation energy \( E - M \), the entropy would be of order \( \sim R(E - M) \sim E^2 \). Therefore, the number of states from the collapse of matter of energy \( E \) is smaller than the number of states of the BH, suggesting that not all BH states can be accessed by the collision, since \( E/2 < M < E \). This raises an important question: What is the number of BH states accessed by the collision? Recently, similar issues were discussed in [15].

Can BH be resonances of quantum gravity? It may seem that the answer is positive, since the width of the BH state is \( \Gamma \sim 1/R \), and moreover, \( \Gamma/M \sim 1/S_{BH} \ll 1 \), suggesting that the resonances might even be sharp. On the other hand, no matter how narrow the resonances are, if the distance between the subsequent resonances is smaller than their widths, resonances can not be distinguished. In this case, the application of the Levinson’s theorem, that led to the result in [10] might be under question. Since we don’t have a solid ground to address this problem, we can discuss this question in the frameworks of certain models. As an example, consider a model of the BH, where it is assumed that the area is quantized [20]. In this case, it was shown that BH states indeed can be considered as resonances. However, I think that this assumption is not universal if not incorrect. The reason for such a suspicion becomes clear when we try to extend this idea to AdS BH, with subsequent application of the AdS/CFT correspondence. Indeed, the area quantization of the AdS BH in 5D implies \( M_n \sim n^{3/2} \) and \( \Delta M_n \sim \sqrt{n} \), where \( n \) is a large natural number. This suggests that the dual CFT side, should contain such widely separated states with huge degeneracy \( \sim \exp\{\#n\} \). Neither this spectral pattern nor this degeneracy is expected to be present in CFT.

Should we modify the unitarity condition? If we only consider channels that (classically) are doomed to undergo gravitational collapse, then there is no reason to assume that the S-matrix has a non-scattering term. This assumption leads to a modification of the optical theorem. We proposed a model [21] without such a non-scattering term in the S-matrix. This model is assumed to describe interactions of particle and BH states. It is based on the appropriate generalization of the well-known Breit-Wigner formula for resonant scattering. The other main ingredient is that couplings between BH and other states are governed by random complex numbers obeying certain sum-rules. These assumptions lead to the emergence of the Hawking-like thermal properties.

Finally, why is it justified to consider only processes that proceed with intermediate BH formation, and ignore the infinite number of diagrams that do not have intermediate BH states?
It may be that the Hawking’s original arguments that lead to the information paradox should still work, if we ignore diagrams (with trivial topology) that do not contain intermediate BH states.

Needless to say, there are many more questions that are left to be answered, however, to our view, the questions above require immediate attention, if one attempts to understand simple aspects of quantum gravity.

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