On gauge fields - strings duality as an integrable system.

M. Zyskin *

Abstract

It was suggested in [1], that semiclassically, a partition function of a string theory in the 5 dimensional constant negative curvature space with a boundary condition at the absolute satisfy the loop equation with respect to varying the boundary condition, and thus the partition function of the string gives the expectation value of a Wilson loop in the 4 dimensional QCD. In the paper, we present the geometrical framework, which reveals that the equations of motion of the string are integrable, in the sense that they can be written via a Lax pair with a spectral parameter. We also show, that the issue of the loop equation rests solely on the properly posing the boundary condition.

*BRIMS, Hewlett Packard Labs, Filton Road, Stoke Gifford, Bristol, BS34 8QZ, UK; maxim@hplb.hpl.hp.com
1 Introduction and summary.

According to [1], [2], there is a semiclassical evidence, that the expectation value of a QCD Wilson loop in dimension 4 is given by a partition function of a string theory with the 5 dimensional constant negative curvature target space and with a boundary condition on the loop at the absolute, as the partition function satisfy the loop equation with respect to variation of the boundary loop. Semiclassically, the equations of motion are the equations of the minimal surface in the constant negative curvature space with prescribed boundary; and the action is the area of such minimal surface (we sum over minimal surfaces when the solution is not unique). In the paper, we present the equations of motion for such string theory in a geometrical format, which reveals, that those equations are integrable, since they can be written via a Lax pair with a spectral parameter, and therefore, the methods of boundary integrable models might be applicable, either modern ones like boundary inverse scattering [9], [10]; boundary r-matrices, the bootstrap, etc [5]; or rather more geometrical ones, as they used more then 100 years ago to solve nonlinear partial differential equations such as Liouville, Monge Ampere, and sine Gordon. We also show that the issue of the loop equation being satisfied rests solely on the properly posing the boundary condition, since the terms in the second variation containing \( \delta \) function is given by a line integral over the boundary.

Some other geometric and integrability aspects of the methods used in the paper will be published elsewhere, [8], [10].

2 Moving Frames; surfaces in \( H_{(n)} \); minimal surfaces.

2.1 Frames in \( M^{n+1} \)

Let \( M^{n+1} \) be Minksowsky space: a real vector space with the metric \( \eta = (-1, 1, 1, \ldots, 1) \). A frame in \( M^{n+1} \) is an \( (n+1) \)-tuple of vectors

\[
F = (f_{(0)}, f_{(1)}, f_{(2)}, \ldots, f_{(n)}), \quad \text{such that}
\]

\[
<f_{(0)}, f_{(0)}> = -1, \quad <f_{(\alpha)}, f_{(\alpha)}> = 1, \quad \alpha = 1, 2, \ldots, n.
\]

(1)

A standard frame is the following \( (n+1) \) tuple of vectors:

\[
e_{(0)} = (1, 0, 0, \ldots, 0)
\]

\[
e_{(1)} = (0, 1, 0, \ldots, 0)
\]

\[...
\]

\[
e_{(n)} = (0, 0, 0, \ldots, 1)
\]
Any frame is obtained from the standard frame by an action of the group \( SO(1, n) \), 
\( f(i) = g_{ij} e(j) \), or in components,

\[
\begin{pmatrix}
  f_0(0) & f_1(0) & f_2(0) & \ldots & f_n(0) \\
  f_0(1) & f_1(1) & f_2(1) & \ldots & f_n(1) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  f_0(n) & f_1(n) & f_2(n) & \ldots & f_n(n)
\end{pmatrix}
= G
\begin{pmatrix}
  1 & 0 & 0 & \ldots & 0 \\
  0 & 1 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 1
\end{pmatrix},
\tag{2}
\]

since (2) defines a matrix \( G \) with \( G \eta G^T = \eta \), \( \eta = Diag (-1,1,1,\ldots,1) \), and therefore, \( G \in SO(1, n) \).

### 2.2 The hyperboloid \( H(n) \)

Since \( <f(0), f(0)> \equiv -f_0(0)^2 + f_1(0)^2 + \ldots + f_n(0)^2 = -1 \), the vector \( f(0) \) can be identified with a point on a hyperboloid \( H(n) \) in \( M^{n+1} \),

\[
-(x_0)^2 + (x_1)^2 + (x_2)^2 + \ldots + (x_n)^2 = -1
\tag{3}
\]

and for a fixed \( f(0) \), the tangent space to the hyperboloid at the point \( f(0) \) is spanned by \{ \( f(1), f(2), \ldots, f(n) \) \}; indeed, for any \( \alpha = 1, 2, \ldots, n \),

\[
<f(0) + \epsilon f(\alpha), f(0) + \epsilon f(\alpha)> = <f(0), f(0)> + 2\epsilon <f(0), f(\alpha)> + \epsilon^2 <f(\alpha), f(\alpha)> = -1 + \epsilon^2,
\]
as \( <f(0), f(\alpha)> = 0 \).

Suppose at each point of a hyperboloid a frame \( F \) is choosen, with \( f(0) \) corresponding to the point itself, and other \( f(\alpha) \) choosen arbitrary in the tangent space to the hyperboloid; obviously it is always possible to do. Since \( f = G e \), ( which is components is \( f(i) = \sum_j g_{ij} e(j) \)),

\[
df = dG e = \left( dG G^{-1} \right) f \equiv \omega f,
\]
or \( df(i) = \omega_{ij} f(j) \), where \( \omega_{ij} = (dG G^{-1})_{ij} \); therefore,

\[
d\omega = d(dG G^{-1}) = dG G^{-1} \wedge dG G^{-1} = \omega \wedge \omega,
\]
and we arrived at the Maurer-Cartan equations:

\[
df = \omega f \tag{4}
\]
\[
d\omega = \omega \wedge \omega \tag{5}
\]
In fact, if (5) is satisfied, then equations (4) are compatible. It’s easy to see that (4) is necessary, as if \( df = \omega f \), it follows that \( 0 = d(\omega f) = (d\omega \wedge \omega) f \). Therefore,
the crucial part in what follows will be to construct one forms satisfying (4), as then the frame can be found by integrating the compatible first order equations (4).

From (1), it follows that

\[ \begin{align*}
\omega_{0,0} &= 0 \\
\omega_{0,\alpha} &= \omega_{\alpha,0} \\
\omega_{\alpha,\beta} &= -\omega_{\beta,\alpha}, \quad \alpha, \beta = 1, 2, \ldots n.
\end{align*} \tag{6} \]

The induced metric on the hyperboloid is just

\[ h := \langle df(0), \otimes df(0) \rangle = \sum \omega_{0,i} \otimes \omega_{0,j} \langle f(i), f(j) \rangle = \sum \omega_{0,i} \otimes^{2} \tag{7} \]

Since the curvature 2-form on the hyperboloid is

\[ -\Omega_{\alpha,\beta} = d\omega_{\alpha,\beta} - \omega_{\alpha,\gamma} \wedge \omega_{\gamma,\beta}, \quad \alpha, \beta, \gamma = 1, 2, \ldots n, \]

and from Maurer Cartan (3) it follows that

\[ \Omega_{\alpha,\beta} = -\omega_{0,\alpha} \wedge \omega_{0,\beta} \]

It is easy to see that the hyperboloid has constant negative curvature in this language: choose the basis of tangent vectors \( X_{i}, i = 1, 2, \ldots \), such that \( \omega_{0,i}(X_{j}) = \delta_{ij} \). Then such vectors are orthonormal in the induced metric, \( h(X_{i}, X_{j}) = \delta_{ij} \). The Riemann tensor in this basis is \( R_{ijkl} = \Omega_{ij}(X_{k}, X_{l}) \). Contracting two indices, \( R_{ikij} = \Omega_{ki}(X_{k}, X_{j}) = -\omega_{0,k} \wedge \omega_{0,i}(X_{k}, X_{j}) = -(n-1)\delta_{ij} = -(n-1)h(X_{i}, X_{j}). \)

### 2.3 A surface in \( H(n) \)

Since \( f(0) \) is identified with a point on a hyperboloid, a vector-valued function of 2 real variables \( f(0) \) \((u, v)\) defines a surface on the hyperboloid. At each point on the surface, we choose the frame \( F(u, v) \), (4), in such a way, that \( f(1) \) and \( f(2) \) will span the tangent space of the surface (of course, there are many ways to do it); as before, all \( \{f(i)(u, v)\} \), \( i = 1, 2, \ldots n \) span the tangent space at \( f(0), T_{f(0)} \) in the hyperboloid \( H(n) \). With this choice,

\[ df(0)(u, v) = \left(\omega_{01}f(1) + \omega_{02}f(2)\right)(u, v) \tag{8} \]

and

\[ \omega_{0,\mu} = 0, \mu = 3, 4, \ldots n; \tag{9} \]
From Maurer-Cartan also $d\omega_{0\mu} = \omega_{01} \wedge \omega_{1\mu} + \omega_{02} \wedge \omega_{2\mu} = 0$ From this follows that

$$\begin{align*}
\omega_{1\mu} &= b_{(\mu),1}\omega_{01} + c_{(\mu)}\omega_{02} \\
\omega_{2\mu} &= c_{(\mu)}\omega_{01} + b_{(\mu),2}\omega_{02},
\end{align*}$$

(10)

where $b_{(\mu),\alpha}$ and $c_{(\mu)}$ are some functions on the surface.

The first fundamental form on the surface is

$$I = \langle df_{(0)}, \otimes df_{(0)} \rangle = \omega_{01}^{\otimes 2} + \omega_{02}^{\otimes 2}$$

(11)

To each normal (in $M^{n+1}$) direction $\mu = 3, 4, \ldots$ there correspond a second fundamental form,

$$II_{\mu} = \omega_{1\mu} \otimes \omega_{01} + \omega_{2\mu} \otimes \omega_{02}$$

(12)

It’s convenient to introduce notations

$$\omega_{ij} = \alpha_{ij} du + \beta_{ij} dv$$

(13)

We choose the conformal coordinates on the surfaces, such that

$$I = e^{2\phi} (du^2 + dv^2)$$

(14)

In this coordinates, $\langle \frac{\partial}{\partial u} f_{(0)}, \frac{\partial}{\partial v} f_{(0)} \rangle = 0$, and therefore we can choose $f_{(1)}$ and $f_{(2)}$ in such a way that

$$df_{(0)}(u, v) = \left( \omega_{01} f_{(1)} + \omega_{02} f_{(2)} \right)(u, v),$$

with $\alpha_{02} = 0$, $\beta_{01} = 0$ in notations (13). In conformal coordinates (14), we also have $(\alpha_{01})^2 = e^{2\phi}$ and $(\beta_{02})^2 = e^{2\phi}$, thus we can make a choice

$$\omega_{01} = e^{\phi} du; \quad \omega_{02} = e^{\phi} dv$$

(15)

From (5), (8) we have

$$\begin{align*}
d\omega_{01} &= -\omega_{02} \wedge \omega_{12} \\
d\omega_{02} &= \omega_{01} \wedge \omega_{12},
\end{align*}$$

and from (13), (14) it follows

$$\begin{align*}
\alpha_{01} &= e^{\phi}; \quad \beta_{01} = 0 \\
\alpha_{02} &= 0; \quad \beta_{02} = e^{\phi} \\
\alpha_{12} &= -\frac{\partial \phi}{\partial u}; \quad \beta_{12} = \frac{\partial \phi}{\partial u} \\
\alpha_{1\mu} &= b_{(\mu),1} e^{\phi}; \quad \beta_{1\mu} = c_{(\mu)} e^{\phi} \\
\alpha_{2\mu} &= c_{(\mu)} e^{\phi}; \quad \beta_{2\mu} = b_{(\mu),2} e^{\phi}
\end{align*}$$

(16)

From (5), (8)

$$d\omega_{12} = \omega_{01} \wedge \omega_{02} - \sum_{\mu=3,4,\ldots} \omega_{1\mu} \wedge \omega_{2\mu},$$
and therefore, using \((10)\),

\[
\left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \phi = e^{2\phi} \left( 1 + \sum_{\mu=3,4,\ldots} \text{det} \left( \begin{array}{cc}
(b_{(\mu),1} & c_{(\mu)} \\
c_{(\mu)} & b_{(\mu),2} \end{array} \right) \right).
\]

\(d\omega_{1\mu} = \omega_{12} \wedge \omega_{2\mu} + \sum_{\nu=3,4,\ldots} \omega_{1\nu} \wedge \omega_{\nu\mu}\)

\(d\omega_{2\mu} = -\omega_{12} \wedge \omega_{1\mu} + \sum_{\nu=3,4,\ldots} \omega_{2\nu} \wedge \omega_{\nu\mu}\)

\(2c_{(\mu)} \frac{\partial \phi}{\partial u} - \left( b_{(\mu),1} - b_{(\mu),2} \right) \frac{\partial \phi}{\partial v} + \frac{\partial c_{(\mu)}}{\partial u} - \frac{\partial b_{(\mu),1}}{\partial v} = \sum_{\nu=3,4,\ldots} \left( b_{(\nu),1} \beta_{\nu\mu} - c_{(\nu)} \alpha_{\nu\mu} \right) \)

\(\left( b_{(\mu),2} - b_{(\mu),1} \right) \frac{\partial \phi}{\partial u} + \frac{\partial b_{(\mu),1}}{\partial v} = \sum_{\nu=3,4,\ldots} \left( c_{(\nu)} \beta_{\nu\mu} - b_{(\nu),2} \alpha_{\nu\mu} \right) \)

\(\mu = 3, 4, \ldots\)

\(d\omega_{\mu\nu} = -\omega_{1\mu} \wedge \omega_{1\nu} - \omega_{2\mu} \wedge \omega_{2\nu} + \sum_{\eta=3,4,\ldots} \omega_{\mu\eta} \wedge \omega_{\eta\nu}\)

\(\frac{\partial \beta_{\mu\nu}}{\partial u} - \frac{\partial \alpha_{\mu\nu}}{\partial v} = -\frac{1}{2}D_{\nu}(\phi) = -e^{2\phi} \left( \left( b_{(\mu),1} - b_{(\mu),2} \right) c_{(\nu)} - \left( b_{(\nu),1} - b_{(\nu),2} \right) c_{(\mu)} \right) + \sum_{\eta=3,4,\ldots} \left( \alpha_{\mu\eta} \beta_{\eta\nu} - \beta_{\mu\eta} \alpha_{\eta\nu} \right)\)

### 2.4 Lagrangians, Variational derivative, minimal Surface

The equations of motion are just the equations for the minimal surface. They can be obtained from the condition that the variation of the Lagrangian

\[
L = \int \sqrt{\text{Det}_{\alpha\beta} \left( <\frac{\partial}{\partial u_{\alpha}} f_{(0)}, \frac{\partial}{\partial u_{\beta}} f_{(0)}> \right)} du_{1} \wedge du_{2};
\]

is zero, \(\frac{\delta L}{\delta f_{(0)}} = 0\), subject to \(<\delta f_{(0)}, f_{(0)}> = 0\); the last ensures that we stay on the hyperboloid. In the conformal coordinates \((14)\), those equations are

\(0 = <\delta f_{(0)}, \frac{\partial}{\partial u} f_{(0)} + \frac{\partial}{\partial v} f_{(0)}> =
\)

\(<\delta f_{(0)}, \frac{\partial}{\partial u} \left( e^{\phi} f_{(1)} \right) + \frac{\partial}{\partial v} \left( e^{\phi} f_{(2)} \right)> = <\delta f_{(0)}, e^{2\phi} \left( \# f_{(0)} + \sum_{\mu=3,4,\ldots} \left( b_{(\mu),1} + b_{(\mu),2} \right) f_{(\mu)} \right)>\)

we made the computation in the conformal basis, \((13)\), and used the Maurer Cartan equations, \((16)\). Since \(<\delta f_{(0)}, f_{(0)}> = 0\), and otherwise arbitrary, it follows that

\((b_{(\mu),1} + b_{(\mu),2}) = 0, \mu = 3, 4, \ldots\)
2.5 Minimal surface in $H(3)$ as an integrable system

The Maurer-Cartan equations for the minimal surface (23) simplify, and for the surface in $H(3)$ they are

\[
\frac{\partial^2}{\partial u^2} \phi + \frac{\partial^2}{\partial v^2} \phi = e^{2\phi} (1 + b^2 + c^2),
\]

(24)

\[
2c\phi_u - 2b\phi_v + c_u - b_v = 0,
\]

\[
2b\phi_u + 2c\phi_v + b_u + c_v = 0,
\]

(25)

where $b \equiv b_{(3)1} = -b_{(3)2}$, $c \equiv c_{(3)}$; and $\alpha_{12}, \beta_{12}$ are determined by $\phi$, $\alpha_{12} = -\phi_v, \beta_{12} = \phi_u$. The system (25) is integrable; it has a Lax pair, with a spectral parameter $\lambda \in \mathbb{C}$, for example this one (there is in fact a much better one for purposes of inverse scattering; but the one below is more geometric):

\[
\frac{\partial}{\partial u} \Phi = \begin{bmatrix}
0 & \lambda^2 + 1 e^{\phi} & -i\lambda^2 + i e^{\phi} & 0 \\
-\lambda^2 (b - i c) + (b + ic) e^{\phi} & 0 & -\phi_v & 0 \\
0 & -\lambda^2 (c + ib) + (c - ib) e^{\phi} & 0 & 0 \\
\lambda^2 (b - ic) + (b + ic) e^{\phi} & \phi_v & 0 & 0 \\
\end{bmatrix} \Phi
\]

(26)

\[
\frac{\partial}{\partial v} \Phi = \begin{bmatrix}
0 & -\lambda^2 - i e^{\phi} & \lambda^2 + 1 e^{\phi} & 0 \\
-\lambda^2 (c + ib) + (c - ib) e^{\phi} & 0 & -\phi_u & 0 \\
0 & -\lambda^2 (b - ic) + (b - ic) e^{\phi} & 0 & 0 \\
\lambda^2 (b + ic) + (-b - ic) e^{\phi} & \phi_u & 0 & 0 \\
\end{bmatrix} \Phi
\]

In fact, the integrable system here is something quite familiar. It follows from (25), that

\[
\phi_u = -bb_u + cc_u + bc_v - cb_v \\
2(b^2 + c^2)
\]

(27)

\[
\phi_v = -bb_v + cc_v - bc_u + cb_u \\
2(b^2 + c^2)
\]

here subscripts $u$ and $v$ denote derivatives with respect to $u, v$. Let’s introduce $\rho$ and $\Theta$, such that $b = \rho \cos \Theta, c = \rho \sin \Theta$. Then it follows from (27) that

\[
\left(\phi + \frac{1}{2} \log \rho\right)_u = -\Theta_v, \\
\left(\phi + \frac{1}{2} \log \rho\right)_v = \Theta_u.
\]

Therefore, $\Theta$ must be a harmonic function of $(u, v)$, ( as well as $(\phi + \frac{1}{2} \log \rho)$), so whenever the only harmonic functions are constants; say if a surface is an imbedding
of a sphere; then \((\phi + \frac{1}{2}\log \rho)\) is some constant \(\kappa\) as well; and so the equation (24) is in fact a \(\cosh\)-Gordon,
\[
\left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \phi = e^{2\phi} + \kappa e^{-2\phi}.
\] (28)

2.6  A remark on minimal surfaces in \(\mathbb{R}^3\) and the Liouville equation.

It is well known that a minimal surface in \(\mathbb{R}^3\) is a surface with the mean curvature equal to zero. In the Maurer Cartan format, the equations for a surface in \(\mathbb{R}^3\) are
\[
\begin{align*}
\frac{dx}{\omega_i f_{(i)}} &= \omega_{ij} f_{(j)} \\
\frac{df_{(i)}}{\omega_{ij}} &= \omega_i j \wedge \omega_{ji} \\
\frac{d\omega_i}{\omega_{i j} \wedge \omega_{k j}} &= \omega_{i k} \wedge \omega_{k j},
\end{align*}
\] (29)

where \(x \in \mathbb{R}^3\), and \(\{f_{(\mu)}\}\) is an orthonormal frame; the group is the group of Euclidean motions of \(\mathbb{R}^3\), instead of Lorentz group which we work with. For a minimal surface, choosing the conformal coordinates (14), writing the Maurer-Cartan equations, and taking into account that the mean curvature is zero, similar to what we did in constant negative curvature space above, we would arrive at the Liouville equation,
\[
\left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \phi = e^{2\phi},
\]
for which a solution can be written explicitly, as it is well known for a very long time; but the corresponding quantum field theory is regarded to be notoriously difficult [5].

I find quite amusing the following set of facts: a) Maurer Cartan plays a major role in the geometry of frames on surfaces, and in particular it is responsible for the Liouville equation; b) some Maurer Cartan shows up in the celebrated deformation quantization construction of associative algebras, c) the way they approach quantum Liouville in [3] is via associativity of the operator product algebra, and d) they seem to be using the same software to draw their pictures in their texts in b and c, and if you look at those pictures from far away, they look alike; but I do not know what exactly to make of those observations.

For a surface of constant mean curvature \(h\), we would obtain the \(\sinh\) Gordon,
\[
\left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \phi = -h^2 e^{2\phi} + e^{-2\phi}.
\]
This and other integrable surfaces in \(\mathbb{R}^3\) were studied in [7].
2.7 Minimal surface in $H_{(5)}$ as an integrable system

The Maurer-Cartan equations for the minimal surface (23) in $H_{(5)}$ are

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \phi = e^{2\phi} \left(1 + b_3^2 + c_3^2 + b_4^2 + c_4^2 + b_5^2 + c_5^2\right),$$

$$L_{1\mu} \overset{(\text{def.})}{=} 2c_{(\mu)} \frac{\partial \phi}{\partial u} - 2b_{(\mu)} \frac{\partial \phi}{\partial v} + \frac{\partial c_{(\mu)}}{\partial u} \phi - \frac{\partial b_{(\mu)}}{\partial v} \phi = \sum_{\nu=3,4,5, \nu \neq \mu} \left(b_{(\nu)} \beta_{\nu \mu} - c_{(\nu)} \alpha_{\nu \mu}\right)$$

$$L_{2\mu} \overset{(\text{def.})}{=} -2b_{(\mu)} \frac{\partial \phi}{\partial u} - 2c_{(\mu)} \frac{\partial \phi}{\partial v} - \frac{\partial b_{(\mu)}}{\partial u} \phi - \frac{\partial c_{(\mu)}}{\partial v} \phi = \sum_{\nu=3,4,5, \nu \neq \mu} \left(c_{(\nu)} \beta_{\nu \mu} + b_{(\nu)} \alpha_{\nu \mu}\right), \mu = 3, 4, 5;$$

and

$$\frac{\partial \beta_{\mu \nu}}{\partial u} - \frac{\partial \alpha_{\mu \nu}}{\partial v} = -2e^{2\phi} \left(b_{(\mu)} c_{(\nu)} - b_{(\nu)} c_{(\mu)}\right) + \sum_{\eta=3,4,5, \eta \neq \mu, \nu} \left(\alpha_{\mu \eta} \beta_{\eta \nu} - \beta_{\mu \eta} \alpha_{\eta \nu}\right)$$

$$\mu, \nu = 3, 4, 5.$$ (31)

This system of equations appear integrable, and posess a Lax pair with spectral parameter, as follows. We assume that say $\alpha_{45}, \beta_{45}$ can be represented in the form

$$\alpha_{45} = \psi_u + \chi_v, \quad \beta_{45} = \psi_v - \chi_u$$

with certain functions $\psi(u, v), \chi(u, v)$; which doesnot seem to be terribly restrictive. There is a Lax pair, reproducing the Maurer Cartan equations; it involves a spectral parameter $\lambda \in \text{fontr C}$, and the unknowns: $\psi$, $\chi$, the conformal factor $\phi(u, v)$, as well as $\{c_m(u, v), b_m(u, v)|m = 3, 4, 5\}$, see (10), (12), (where $b_m(u, v)$ (def) $b_{(m)}(u, v) = -b_{(m),2}(u, v)$, as the surface is minimal); that’s all we need to know to be able to find the Maurer-Cartan 1-forms, (9), and then the surface itself is obtained by solving linear compatible first order equations (4). Possibly, there are better, for purposes of boundary inverse scattering, Lax pairs; this is under investigation; but at least, there is some Lax pair.
\[
\frac{\partial}{\partial u} \Phi = \begin{bmatrix}
0 & \frac{\lambda^2 + 1}{2\lambda} e^\phi & \frac{-i\lambda^2 + i}{2\lambda} e^\phi \\
\frac{\lambda^2 + 1}{2\lambda} e^\phi & 0 & -\phi_v \\
\frac{-i\lambda^2 + i}{2\lambda} e^\phi & \phi_v & 0 \\
0 & -\frac{\lambda^2(b_3 - ic_3) + (b_3 + ic_3)}{2\lambda} e^\phi & -\frac{\lambda^2(c_3 + ib_3) + (c_3 - ib_3)}{2\lambda} e^\phi \\
0 & -\frac{\lambda^2(b_4 - ic_4) + (b_4 + ic_4)}{2\lambda} e^\phi & -\frac{\lambda^2(c_4 + ib_4) + (c_4 - ib_4)}{2\lambda} e^\phi \\
0 & -\frac{\lambda^2(b_5 - ic_5) + (b_5 + ic_5)}{2\lambda} e^\phi & -\frac{\lambda^2(c_5 + ib_5) + (c_5 - ib_5)}{2\lambda} e^\phi
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\lambda^2(b_4 - ic_4) + (b_4 + ic_4) e^\phi \\
\lambda^2(c_4 + ib_4) + (c_4 - ib_4) e^\phi \\
\lambda^2(b_5 - ic_5) + (b_5 + ic_5) e^\phi \\
\lambda^2(c_5 + ib_5) + (c_5 - ib_5) e^\phi \\
\lambda^2(b_3 - ic_3) + (b_3 + ic_3) e^\phi \\
\lambda^2(c_3 + ib_3) + (c_3 - ib_3) e^\phi
\end{bmatrix} \frac{\lambda^2(b_3 - ic_3) + (b_3 + ic_3)}{2\lambda} e^\phi \mid_{\mu=3,4,5} \frac{\lambda^2(c_3 + ib_3) + (c_3 - ib_3)}{2\lambda} e^\phi \mid_{\mu=3,4,5} \Phi
\]

\[
A = \begin{bmatrix}
0 & a[3, 4] & a[3, 5] \\
-a[3, 4] & 0 & (\psi_u + \chi_v) \\
a[3, 5] & -(\psi_u + \chi_v) & 0
\end{bmatrix}
\]

\[
a[3, 4] = \frac{1}{(b_3^2 + c_3^2)} \left(-c_3(L_{14} - c_5(\psi_u + \chi_v) + b_5(\psi_v - \chi_u)) + b_3(L_{24} + b_5(\psi_u + \chi_v) + c_5(\psi_v - \chi_u))\right)
\]

\[
a[3, 5] = \frac{1}{(b_3^2 + c_3^2)(b_5^2 + c_5^2)} \left(c_5L_{13} - b_5L_{23} + (c_3(b_4b_5 + c_4c_5) + b_3(b_4c_5 - b_5c_4))(L_{14} - c_5(\psi_u + \chi_v) + b_5(\psi_v - \chi_u)) + (c_3(b_4c_5 - b_5c_4) - b_3(c_4c_5 + b_4b_5))(L_{24} + b_5(\psi_u + \chi_v) + c_5(\psi_v - \chi_u))\right)
\]

(32)
\[
\frac{\partial}{\partial \nu} \Phi = \begin{bmatrix}
0 & \frac{i \lambda^2 - 1}{2 \lambda} e^\phi & \frac{\lambda^2 + 1}{2 \lambda} e^\phi & (0 0 0) \\
\frac{i \lambda^2 - 1}{2 \lambda} e^\phi & -\phi_u & \phi_u & \left(\frac{\lambda^2 (c_\mu + ib_\mu) + (c_\mu - ib_\mu)}{2 \lambda}\right) e^\phi \\
\frac{\lambda^2 + 1}{2 \lambda} e^\phi & \phi_u & 0 & \left(\frac{\lambda^2 (-b_\mu + i c_\mu) + (-b_\mu - i c_\mu)}{2 \lambda}\right) e^\phi \\
0 & \left(\frac{\lambda^2 (c_\mu + ib_\mu) + (c_\mu - ib_\mu)}{2 \lambda}\right) e^\phi & \left(\frac{\lambda^2 (-b_\mu + i c_\mu) + (-b_\mu - i c_\mu)}{2 \lambda}\right) e^\phi & B
\end{bmatrix}
\]

\[B = \begin{bmatrix}
0 & b[3, 4] & b[3, 5] \\
1 & 0 & (\psi_v - \chi_u) \\
-1 & 0 & (\psi_v - \chi_u)
\end{bmatrix}\]

\[b[3, 4] = \frac{1}{b_3^2 + c_3^2} \left( b_3 (L_{14} - c_5 (\psi_u + \chi_v)) + b_5 (\psi_v - \chi_u) \right) + c_3 (L_{24} + b_5 (\psi_u + \chi_v) + c_5 (\psi_v - \chi_u)) \]

\[b[3, 5] = \frac{1}{b_3^2 + c_3^2} \left( -b_5 L_{13} - c_5 L_{23} \right) + \frac{1}{(b_3^2 + c_3^2) (b_4^2 + c_4^2)} \left( b_4 (c_3 c_4 - b_3 b_4) + c_5 (b_4 c_3 - b_3 c_4) \right) (L_{14} - c_5 (\psi_u + \chi_v) + b_5 (\psi_v - \chi_u)) + b_4 (b_3 c_4 - b_4 c_3) - c_5 (c_3 c_4 + b_3 b_4) \right) (L_{24} + b_5 (\psi_u + \chi_v) + c_5 (\psi_v - \chi_u)) \]  

where \( L_{ij} \) are defined in (30).

### 3 Some thoughts on the loop equation, in the context of zero mean curvature surfaces.

We currently do not know how to pose an inverse scattering problem for the equations we got; however, an experience with an inverse scattering on an arbitrary domain for integrable equations which have a linear limit, suggests that there exist a \( \overline{\partial} \) problem of a shape

\[ \frac{\partial}{\partial \lambda} \Phi(u, v, \lambda) = S_\gamma(u, v, \lambda) \Phi(u, v, \lambda), \]  

(34)
where \( S_\gamma(u, v, \lambda) \) is determined from (a yet to be formulated) boundary condition. We do not know how exactly to get this \( \bar{\partial} \) problem here, but since all examples known so far come in this shape, we conjecture it exist here as well. Our Lagrangian is

\[
L = \int_\Omega \left[ \det_{\alpha\beta} \left( \langle \frac{\partial}{\partial u_\alpha} f(0), \frac{\partial}{\partial u_\beta} f(0) \rangle \right) \right] du_1 \wedge du_2 = \int_\Omega \omega_{01} \wedge \omega_{02} \quad (35)
\]

with our choice of frame. We assume we have a family of boundary conditions, for which we can resolve the (34) problem, and therefore we have a family of solutions of (34) depending from the boundary condition \( \Phi_\gamma(u, v, \lambda) \), which give rise to a family of one forms \( \omega_{ij} = \left( (d\Phi_\gamma(u, v, 1))\Phi_\gamma^{-1}(u, v, 1) \right)_{ij} \), depending from the boundary condition. We would like to compute in the second variation of the lagrangian with respect to change of boundary conditions, \( \delta_1 \delta_2 L \equiv \delta f_{\circ(t_1)} \delta f_{\circ(t_2)} \) the term containing a delta function \( \delta(t_1 - t_2) \). We will do it formally, assuming that there exist variation \( \delta \) commuting with the differential. Since

\[
\delta \omega_{01} = \delta((d\Phi)\Phi^{-1})_{01} = (\delta d\Phi)\Phi^{-1}_{01} - ((d\Phi)\Phi^{-1}\delta\Phi\Phi^{-1})_{01} = \left((\delta d\Phi)\Phi^{-1}\right)_{01} - \omega_{02}(\delta\Phi\Phi^{-1})_{21};
\]

here \( \Phi \equiv \Phi(u, v, 1) \in SO(1, n) \), and therefore the symmetry conditions are

\[
(\delta\Phi\Phi^{-1})_{oo} = 0, (\delta\Phi\Phi^{-1})_{oo} = (\delta\Phi\Phi^{-1})_{oo}, (\delta\Phi\Phi^{-1})_{\alpha\beta} = -(\delta\Phi\Phi^{-1})_{\beta\alpha};
\]

we used also our choice of the frame. We remark that \( \delta\Phi\Phi^{-1} \) are zero forms on the tangent space, and \( d\Phi\Phi^{-1} \) are one forms.

Then

\[
\delta(\omega_{01} \wedge \omega_{02}) = ((d\delta\Phi)\Phi^{-1})_{01} \wedge \omega_{02} + \omega_{01} \wedge ((d\delta\Phi)\Phi^{-1})_{02}.
\]

The only terms in the second variation which would contain a \( \delta \) function would come only from terms with the second derivative; as products of the first order derivatives cannot produce a delta function. Therefore (terms with a delta function possible in the second variation) are

\[
((d\Delta\Phi)\Phi^{-1})_{01} \wedge \omega_{02} + \omega_{01} \wedge ((d\Delta\Phi)\Phi^{-1})_{02}
\]

Here \( \Delta = \delta_1 \delta_2 \)

**Proposition** For a mean curvature zero surface \( b_{(\mu),1} + b_{(\mu),2} = 0, \mu = 3, 4, 5 \), it follows from the Maurer Cartan equations, that the terms in the second variation which contain a \( \delta \) function depend only from the boundary condition, and given by

\[
\int_{\partial \Omega} \left( ((\Delta\Phi)\Phi^{-1})_{01} \right) \omega_{02} - \left( ((\Delta\Phi)\Phi^{-1})_{02} \right) \omega_{01}, \quad (36)
\]
since

\[ ((d\Delta\Phi)\Phi^{-1})_{01} \land \omega_{02} + \omega_{01} \land ((d\Delta\Phi)\Phi^{-1})_{02} = \]

\[ = d((\Delta\Phi)\Phi^{-1})_{01} \omega_{02} - ((\Delta\Phi)\Phi^{-1})_{02} \omega_{01}. \]

(37)

Proof: The difference between the right hand side and the left hand side in (37)

is

\[ ((\Delta\Phi)\Phi^{-1})_{01}(\omega_{1\mu} \land \omega_{02} + \omega_{01} \land \omega_{2\mu}) + \]
\[ ((\Delta\Phi)\Phi^{-1})_{01}(d\omega_{02} - \omega_{01} \land \omega_{02}) - \]
\[ -((\Delta\Phi)\Phi^{-1})_{01}(d\omega_{01} - \omega_{02} \land \omega_{21}); \]

the first term is zero since \( (\omega_{1\mu} \land \omega_{02} + \omega_{01} \land \omega_{2\mu}) = (b_{(\mu),1} + (b_{(\mu),1})\omega_{01} \land \omega_{02}, \) and we have a \( (b_{(\mu),1} + (b_{(\mu),1}) = 0 \) surface; the other terms are zero due to the Maurer Cartan, as it looks in our choice of the frame.
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