Riemann-Roch and index formulae in twisted $K$-theory

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Abstract. In this paper, we establish the Riemann-Roch theorem in twisted $K$-theory extending our earlier results. We also give a careful summary of twisted geometric cycles explaining in detail some subtle points in the theory. As an application, we prove a twisted index formula and show that D-brane charges in Type I and Type II string theory are classified by twisted KO-theory and twisted $K$-theory respectively in the presence of $B$-fields as proposed by Witten.

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1. Introduction

1.1. String geometry. We begin by giving a short discussion of the physical background. Readers uninterested in this motivation may move to the next subsection. In string theory D-branes were proposed as a mechanism for providing boundary conditions for the dynamics of open strings moving in space-time. Initially they were thought of as submanifolds. As D-branes themselves can evolve over time one needs to study equivalence relations on the set of D-branes. An invariant of the equivalence class is the topological charge of the D-brane which should be thought of as an analogue of the Dirac monopole charge as these D-brane charges are associated with gauge fields (connections) on vector bundles over the D-brane. These vector bundles are known as Chan-Paton bundles.

In [MM] Minasian and Moore made the proposal that D-brane charges should take values in $K$-groups and not in the cohomology of the space-time or the D-brane. However, they proposed a cohomological formula for these charges which might be thought of as a kind of index theorem in the sense that, in general, index theory associates to a $K$-theory class a number which is given by an integral of a closed differential form. In string theory there is an additional field on space-time known as the $H$-flux which may be thought of as a global closed three form. Locally it is given by a family of ‘two-form potentials’ known as the $B$-field. Mathematically we think of these $B$-fields as defining a degree three integral Čech class on the space-time, called a ‘twist’. Witten [Wit], extending [MM], gave a physical argument for the idea that D-brane charges should be elements of $K$-groups and, in addition, proposed that the D-brane charges in the presence of a twist should take values in twisted $K$-theory (at least in the case where the twist is torsion). The mathematical ideas he relied on were due to Donovan and Karoubi [DK]. Subsequently Bouwknegt and Mathai [BouMat] extended Witten’s proposal to the non-torsion case using ideas from [Ros]. A geometric model (that is, a ‘string geometry’ picture) for some of these string theory constructions and for twisted $K$-theory was proposed in [BCMMS] using the notion of bundle gerbes and bundle gerbe modules. Various refinements of twisted $K$-theory that are suggested by these applications are also described in the article of Atiyah and Segal [AS1] and we will need to use their results here.

1.2. Mathematical results. From a mathematical perspective some immediate questions arise from the physical input summarised above. When there is no twist it is well known that $K$-theory provides the main topological tool for the index theory of elliptic operators. One version of the Atiyah-Singer index theorem due to Baum-Higson-Schick [BHS] establishes a relationship between the analytic viewpoint provided by elliptic differential operators and the geometric viewpoint provided by the notion of geometric cycle introduced in the fundamental paper of Baum and Douglas [BD2]. The viewpoint that geometric cycles in the sense of [BD2] are a model for D-branes in the untwisted case is expounded in [RS, RSV, Sz]. Note that in this viewpoint D-branes are no longer submanifolds but the images of manifolds under a smooth map.

It is thus tempting to conjecture that there is an analogous picture of D-branes as a type of geometric cycle in the twisted case as well. More precisely we ask the question of whether there is a way to formulate the notion of ‘twisted geometric cycle’ (cf [BD1] and [BD2]) and to prove an index theorem in the spirit of [BHS] for twisted $K$-theory. This precise question was answered in the positive in [Wa]. It is important to emphasise that string geometry ideas from [FreWit] played a key role in finding the correct way to generalise [BD1].
Our purpose here in the present paper is threefold. First, we explain the results in \cite{Wa} (see Section 5) in a fashion that is more aligned to the string geometry viewpoint. Second, we prove an analogue of the Atiyah-Hirzebruch Riemann-Roch formula in twisted $K$-theory by extending the results and approach of \cite{CMW}. An interesting by-product of our approach in Section 5 is a discussion of the Thom class in twisted $K$ theory. Third, in Section 6 we prove an index formula using our twisted Riemann-Roch theorem. It will be clear from our approach to this twisted index theory that our twisted geometric cycles provide a geometric model for D-branes and we give details in Section 7.

Our main new results are stated as two theorems, Theorem 5.3 (twisted Riemann-Roch) and Theorem 6.1 (the index pairing). We remark that the Minasian-Moore formula \cite{MM} arises from the fact that the index pairing they discuss may be regarded as a quadratic form on $K$-theory. In the twisted index formula that we establish, the pairing is asymmetric and may be thought of as a bilinear form, from which there is no obvious way to extract a twisted analogue of the Minasian-Moore formula. Nevertheless we interpret our results in terms of the physics language in Section 7 explaining the link to Witten’s original ideas on D-brane charges.

2. Twisted $K$-theory: preliminary review

2.1. Twisted $K$-theory: topological and analytic definitions. We begin by reviewing the notion of a ‘twisting’. Let $\mathcal{H}$ be an infinite dimensional, complex and separable Hilbert space. We shall consider locally trivial principal $PU(\mathcal{H})$-bundles over a paracompact Hausdorff topological space $X$, the structure group $PU(\mathcal{H})$ is equipped with the norm topology. The projective unitary group $PU(\mathcal{H})$ with the topology induced by the norm topology on $U(\mathcal{H})$ (Cf. \cite{Kui}) has the homotopy type of an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. The classifying space of $PU(\mathcal{H})$, denoted $BPU(\mathcal{H})$, is a $K(\mathbb{Z}, 3)$. The set of isomorphism classes of principal $PU(\mathcal{H})$-bundles over $X$ is given by (Proposition 2.1 in \cite{AS1}) homotopy classes of maps from $X$ to any $K(\mathbb{Z}, 3)$ and there is a canonical identification

$$[X, BP(\mathcal{H})] \cong H^3(X, \mathbb{Z}).$$

A twisting of complex $K$-theory on $X$ is given by a continuous map $\alpha : X \to K(\mathbb{Z}, 3)$. For such a twisting, we can associate a canonical principal $PU(\mathcal{H})$-bundle $P_\alpha$ through the usual pull-back construction from the universal $PU(\mathcal{H})$ bundle denoted by $EK(\mathbb{Z}, 2)$, as summarised by the diagram:

$$\begin{array}{ccc}
P_\alpha & \longrightarrow & EK(\mathbb{Z}, 2) \\
\downarrow & & \downarrow \\
X & \underset{\alpha}{\longrightarrow} & K(\mathbb{Z}, 3).
\end{array}$$

We will use $PU(\mathcal{H})$ as a group model for a $K(\mathbb{Z}, 2)$. We write $\text{Fred}(\mathcal{H})$ for the connected component of the identity of the space of Fredholm operators on $\mathcal{H}$ equipped with the norm topology. There is a base-point preserving action of $PU(\mathcal{H})$ given by the conjugation action of $U(\mathcal{H})$ on $\text{Fred}(\mathcal{H})$:

$$PU(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \longrightarrow \text{Fred}(\mathcal{H}).$$

The action \cite{2} defines an associated bundle over $X$ which we denote by

$$P_\alpha(\text{Fred}) = P_\alpha \times_{PU(\mathcal{H})} \text{Fred}(\mathcal{H})$$

We write $\{\Omega^n X P_\alpha(\text{Fred}) = P_\alpha \times_{PU(\mathcal{H})} \Omega^n \text{Fred}\}$ for the fiber-wise iterated loop spaces.
DEFINITION 2.1. The (topological) twisted K-groups of $(X, \alpha)$ are defined to be
\[ K^{-n}(X, \alpha) := \pi_0(C_c(X, \Omega_X^\infty P_\alpha(Fred))) \]
the set of homotopy classes of compactly supported sections (meaning they are the identity operator in Fred off a compact set) of the bundle of $P_\alpha(Fred)$.

Due to Bott periodicity, we only have two different twisted K-groups $K^0(X, \alpha)$ and $K^1(X, \alpha)$. Given a closed subspace $A$ of $X$, then $(X, A)$ is a pair of topological spaces, and we define relative twisted K-groups to be
\[ K^{ev/odd}(X, A; \alpha) := K^{ev/odd}(X - A, \alpha). \]

Take a pair of twistings $\alpha_0, \alpha_1 : X \to K(\mathbb{Z}, 3)$, and a map $\eta : X \times [1, 0] \to K(\mathbb{Z}, 3)$ which is a homotopy between $\alpha_0$ and $\alpha_1$, represented diagrammatically by

\[ \begin{array}{ccc}
X & \xrightarrow{\alpha_0} & K(\mathbb{Z}, 3) \\
\downarrow_{\eta} & & \downarrow_{\sim} \\
\downarrow_{\psi} & & \\
K(\mathbb{Z}, 3) & \xrightarrow{\sim} & K^{ev/odd}(X, \alpha_1),
\end{array} \]

Then there is a canonical isomorphism $P_{\alpha_0} \cong P_{\alpha_1}$ induced by $\eta$. This canonical isomorphism determines a canonical isomorphism on twisted K-groups
\[ (2.3) \quad \eta_* : K^{ev/odd}(X, \alpha_0) \cong K^{ev/odd}(X, \alpha_1), \]

This isomorphism $\eta_*$ depends only on the homotopy class of $\eta$. The set of homotopy classes of maps between $\alpha_0$ and $\alpha_1$ is labelled by $[X, K(\mathbb{Z}, 2)]$. Recall the first Chern class isomorphism
\[ \operatorname{Vect}_1(X) \cong [X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z}) \]
where $\operatorname{Vect}_1(X)$ is the set of equivalence classes of complex line bundles on $X$. We remark that the isomorphisms induced by two different homotopies between $\alpha_0$ and $\alpha_1$ are related through an action of complex line bundles.

Let $\mathcal{K}$ be the $C^*$-algebra of compact operators on $\mathcal{H}$. The isomorphism $PU(\mathcal{H}) \cong Aut(\mathcal{K})$ via the conjugation action of the unitary group $U(\mathcal{H})$ provides an action of a $K(\mathbb{Z}, 2)$ on the $C^*$-algebra $\mathcal{K}$. Hence, any $K(\mathbb{Z}, 2)$-principal bundle $P_\alpha$ defines a locally trivial bundle of compact operators, denoted by $P_\alpha(K) = P_\alpha \times_{PU(\mathcal{H})} \mathcal{K}$.

Let $C_0(X, P_\alpha(K))$ be the $C^*$-algebra of sections of $P_\alpha(K)$ vanishing at infinity. Then $C_0(X, P_\alpha(K))$ is the (unique up to isomorphism) stable separable complex continuous-trace $C^*$-algebra over $X$ with Dixmier-Douday class $[\alpha] \in H^2(X, \mathbb{Z})$ (here we identify the Čech cohomology of $X$ with its singular cohomology, cf [Ros] and [AS1]).

THEOREM 2.2. ([AS1] and [Ros]) The topological twisted K-groups $K^{ev/odd}(X, \alpha)$ are canonically isomorphic to analytic K-theory of the $C^*$-algebra $C_0(X, P_\alpha(K))$
\[ K^{ev/odd}(X, \alpha) \cong K^{ev/odd}(C_0(X, P_\alpha(K))) \]
where the latter group is the algebraic K-theory of $C_0(X, P_\alpha(K))$, defined to be
\[ \lim_{k \to -\infty} \pi_1(GL_k(C_0(X, P_\alpha(K)))) \]

Note that the algebraic K-theory of $C_0(X, P_\alpha(K))$ is isomorphic to Kasparov’s KK-theory ([Kas1] and [Kas2])
\[ KK^{ev/odd}(C, C_0(X, P_\alpha(K))). \]
It is important to recognise that these groups are only defined up to isomorphism by the Dixmier-Douady class \([\alpha] \in H^3(X, \mathbb{Z})\). To distinguish these two equivalent definitions of twisted \(K\)-theory if needed, we will write
\[
K_{\text{top/odd}}(X, \alpha) \quad \text{and} \quad K_{\text{an/odd}}(X, \alpha)
\]
for the topological and analytic twisted \(K\)-theories of \((X, \alpha)\) respectively. Twisted \(K\)-theory is a 2-periodic generalized cohomology theory: a contravariant functor on the category consisting of pairs \((X, \alpha)\), with the twisting \(\alpha : X \to K(\mathbb{Z}, 3)\), to the category of \(\mathbb{Z}_2\)-graded abelian groups. Note that a morphism between two pairs \((X, \alpha)\) and \((Y, \beta)\) is a continuous map \(f : X \to Y\) such that \(\beta \circ f = \alpha\).

### 2.2. Twisted \(K\)-theory for torsion twistings.

There are some subtle issues in twisted \(K\)-theory and to handle these we have chosen to use the language of bundle gerbes, connections and curvings as explained in [Mur]. We explain first the so-called ‘lifting bundle gerbe’ \(G_{\alpha}\) [Mur] associated to the principal \(PU(H)\)-bundle \(\pi : \mathcal{P}_{\alpha} \to X\) and the central extension
\[
(2.4) \quad 1 \to U(1) \to U(H) \to PU(H) \to 1.
\]
This is constructed by starting with \(\pi : \mathcal{P}_{\alpha} \to X\), forming the fibre product \(\mathcal{P}_{\alpha}^{[2]}\) which is a groupoid
\[
\mathcal{P}_{\alpha}^{[2]} = \mathcal{P}_{\alpha} \times_X \mathcal{P}_{\alpha} \xrightarrow{\pi_1} \mathcal{P}_{\alpha}
\]
with source and range maps \(\pi_1 : (y_1, y_2) \mapsto y_1\) and \(\pi_2 : (y_1, y_2) \mapsto y_2\). There is an obvious map from each fiber of \(\mathcal{P}_{\alpha}^{[2]}\) to \(PU(H)\) and so we can define the fiber of \(G_{\alpha}\) over a point in \(\mathcal{P}_{\alpha}^{[2]}\) by pulling back the fibration \((2.4)\) using this map. This endows \(G_{\alpha}\) with a groupoid structure (from the multiplication in \(U(H)\)) and in fact it is a \(U(1)\)-groupoid extension of \(\mathcal{P}_{\alpha}^{[2]}\).

A torsion twisting \(\alpha\) is a map \(\alpha : X \to K(\mathbb{Z}, 3)\) representing a torsion class in \(H^3(X, \mathbb{Z})\). Every torsion twisting arises from a principal \(PU(n)\)-bundle \(\mathcal{P}_{\alpha}(n)\) with its classifying map
\[
X \to BU(n),
\]
or a principal \(PU(H)\)-bundle with a reduction to \(PU(n) \subset PU(H)\). For a torsion twisting \(\alpha : X \to BU(n) \to BU(n)\), the corresponding lifting bundle gerbe \(G_{\alpha}\)
\[
\begin{array}{ccc}
\mathcal{P}_{\alpha}(n)^{[2]} & \xrightarrow{\pi_1} & \mathcal{P}_{\alpha}(n) \\
\mathcal{P}_{\alpha}(n) & \xrightarrow{\pi_2} & M \\
\downarrow & & \downarrow \pi \\
& & X
\end{array}
\]

is defined by \(\mathcal{P}_{\alpha}(n)^{[2]} \cong \mathcal{P}_{\alpha}(n) \times PU(n) \cong \mathcal{P}_{\alpha}(n)\) (as a groupoid) and the central extension
\[
1 \to U(1) \to U(n) \to PU(n) \to 1.
\]

There is an Azumaya bundle associated to \(\mathcal{P}_{\alpha}(n)\) arising naturally from the \(PU(n)\) action on the \(n \times n\) matrices. We denote this associated Azumaya bundle by \(A_{\alpha}\). An
$\mathcal{A}_\alpha$-module is a complex vector bundle $\mathcal{E}$ over $M$ with a fiberwise $\mathcal{A}_\alpha$ action
\[ \mathcal{A}_\sigma \times_M \mathcal{E} \longrightarrow \mathcal{E}. \]

The $C^*$-algebra of continuous sections of $\mathcal{A}_\alpha$, vanishing at infinity if $X$ is non-compact, is Morita equivalent to a continuous trace $C^*$-algebra $C_0(X, \mathcal{P}_\alpha(K))$. Hence there is an isomorphism between $K^0(X, \alpha)$ and the $K$-theory of the bundle modules of $\mathcal{A}_\alpha$.

There is an equivalent definition of twisted $K$-theory using bundle gerbe modules (Cf. [BCMMS] and [CWI]). A bundle gerbe module $E$ of $\mathcal{G}_\alpha$ is a complex vector bundle $E$ over $\mathcal{P}_\alpha(n)$ with a groupoid action of $\mathcal{G}_\alpha$, i.e., an isomorphism
\[ \phi : \mathcal{G}_\alpha \times (\pi_2, p) E \longrightarrow E \]
where $\mathcal{G}_\alpha \times (\pi_2, p) E$ is the fiber product of the source $\pi_2 : \mathcal{G}_\alpha \to \mathcal{P}_\alpha(n)$ and $p : E \to \mathcal{P}_\alpha(n)$ such that

1. $p \circ \phi(g, v) = \pi_1(g)$ for $(g, v) \in \mathcal{G}_\alpha \times (\pi_2, p) E$, and $\pi_1$ is the target map of $\mathcal{G}_\alpha$.
2. $\phi$ is compatible with the bundle gerbe multiplication $m : \mathcal{G}_\alpha \times (\pi_2, \pi_1) \mathcal{G}_\alpha \to \mathcal{G}_\alpha$, which means
\[ \phi \circ (id \times \phi) = \phi \circ (m \times id). \]

Note that the natural representation of $U(n)$ on $\mathbb{C}^n$ induces a $\mathcal{G}_\alpha$ bundle gerbe module
\[ S_n = \mathcal{P}_\alpha(n) \times \mathbb{C}^n. \]

Here we use the fact that $\mathcal{G}_\alpha = \mathcal{P}_\alpha(n) \times U(n) \cong \mathcal{P}_\alpha(n)$ (as a groupoid). Similarly, the dual representation of $U(n)$ on $\mathbb{C}^n$ induces a $\mathcal{G}_\alpha$ bundle gerbe module $S_n^* = \mathcal{P}_\alpha(n) \times \mathbb{C}^n$.

Note that $S_n^* \otimes S_n \cong \pi^* \mathcal{A}_\alpha$ descends to the Azumaya bundle $\mathcal{A}_\alpha$. Given a $\mathcal{G}_\alpha$ bundle gerbe module $E$ of rank $k$, then as a $PU(n)$-equivariant vector bundle, $S_n^* \otimes E$ descends to an $\mathcal{A}_\alpha$-bundle over $M$. Conversely, given an $\mathcal{A}_\alpha$-bundle $E$ over $M$, $S_n \otimes \pi^* \mathcal{A}_\alpha \pi^* E$ defines a $\mathcal{G}_\alpha$ bundle gerbe module. These two constructions are inverse to each other due to the fact that
\[ S_n^* \otimes (S_n \otimes \pi^* \mathcal{A}_\alpha \pi^* E) \cong (S_n^* \otimes S_n) \otimes \pi^* \mathcal{A}_\alpha \cong \pi^* \mathcal{A}_\alpha \otimes \pi^* \mathcal{A}_\alpha \cong \pi^* E. \]

Therefore, there is a natural equivalence between the category of $\mathcal{G}_\alpha$ bundle gerbe modules and the category of $\mathcal{A}_\alpha$ bundle modules, as discussed in [CWI]. In summary, we have the following proposition.

**Proposition 2.3.** ([BCMMS], [CWI]) For a torsion twisting $\alpha : X \to BPU(n) \to BPU(H)$, twisted $K$-theory $K^0(X, \alpha)$ has another two equivalent descriptions:

1. The Grothendieck group of the category of $\mathcal{G}_\alpha$ bundle gerbe modules.
2. The Grothendieck group of the category of $\mathcal{A}_\alpha$ bundle modules.

One important example of torsion twistings comes from real oriented vector bundles. Consider an oriented real vector bundle $E$ of even rank over $X$ with a fixed fiberwise inner product. Denote by
\[ \nu_E : X \to BSO(2k) \]
the classifying map of $E$. The following twisting
\[ o(E) := W_3 \circ \nu_E : X \longrightarrow BSO(2k) \longrightarrow K(\mathbb{Z}, 3), \]
will be called the orientation twisting associated to $E$. Here $W_3$ is the classifying map of the principal $BU(1)$-bundle $BSpin^c(2k) \to BSO(2k)$. Note that the orientation twisting $o(E)$ is null-homotopic if and only if $E$ is $K$-oriented.
**Proposition 2.4.** Given an oriented real vector bundle $E$ of even rank over $X$ with an orientation twisting $o(E)$, then there is a canonical isomorphism

$$K^0(X, o(E)) \cong K^0(X, W_3(E))$$

where $K^0(X, W_3(E))$ is the $K$-theory of the Clifford modules associated to the bundle $\text{Cliff}(E)$ of Clifford algebras.

**Proof.** Denote by $\mathcal{F}r$ the frame bundle of $V$, the principal $SO(2k)$-bundle of positively oriented orthonormal frames, i.e.,

$$E = \mathcal{F}r \times_{\rho_n} \mathbb{R}^{2k},$$

where $\rho_n$ is the standard representation of $SO(2k)$ on $\mathbb{R}^n$. The lifting bundle gerbe associated to the frame bundle and the central extension

$$1 \to U(1) \to Spin^c(2k) \to SO(2k) \to 1$$

is called the $Spin^c$ bundle gerbe $\mathcal{G}_{W_3(E)}$ of $E$, whose Dixmier-Douady invariant is given by the integral third Stiefel-Whitney class $W_3(E) \in H^3(X, \mathbb{Z})$. The canonical representation of $Spin^c(2k)$ gives a natural inclusion

$$Spin^c(2k) \subset U(2^k)$$

which induces a commutative diagram

$$
\begin{array}{ccc}
U(1) & \longrightarrow & Spin^c(2k) \\
\downarrow & & \downarrow \\
U(1) & \longrightarrow & SO(2k) \\
\downarrow & & \downarrow \\
U(1) & \longrightarrow & U(H) \\
\downarrow & & \downarrow \\
U(1) & \longrightarrow & PU(H).
\end{array}
$$

This provides a reduction of the principal $PU(H)$-bundle $\mathcal{P}_{o(E)}$. The associated bundle of Azumaya algebras is in fact the bundle of Clifford algebras, whose bundle modules are called Clifford modules ($\text{Cliff}(E)$). Hence, there exists a canonical isomorphism between $K^0(X, o(E))$ and the $K$-theory of the Clifford modules associated to the bundle $\text{Cliff}(E)$.

**2.3. Twisted $K$-theory: general properties.** Twisted $K$-theory satisfies the following properties whose proofs are rather standard for a 2-periodic generalized cohomology theory ($\text{AS1}$ $\text{[CW1]}$ $\text{[Kar]}$ $\text{[Wa]}$). (Note that when we write $(X, A)$ for a pair of spaces we assume $A \subset X$.)

1. **(The homotopy axiom)** If two morphisms $f, g : (Y, B) \to (X, A)$ are homotopic through a map $\eta : (Y \times [0, 1], B \times [0, 1]) \to (X, A)$, written in terms of the following homotopy commutative diagram

$$
\begin{array}{ccc}
(Y, B) & \overset{f}{\longrightarrow} & (X, A) \\
\downarrow & \overset{\eta}{\cong} & \downarrow \\
(X, A) & \overset{\alpha}{\longrightarrow} & K(\mathbb{Z}, 3),
\end{array}
$$

$$\text{where } K^0(X, o(E)) \cong K^0(X, W_3(E))$$

is the $K$-theory of the Clifford modules associated to the bundle $\text{Cliff}(E)$ of Clifford algebras.
then we have the following commutative diagram

\[
\begin{array}{ccc}
K^{ev/odd}(X, A; \alpha) & \xrightarrow{f^*} & K^{ev/odd}(Y, B; \alpha \circ f) \\
\downarrow \eta^* & & \downarrow \eta^*
\end{array}
\]

where

\[
\eta^* = \text{the canonical isomorphism induced by the homotopy } \eta.
\]

(II) (The exact axiom) For any pair \((X, A)\) with a twisting \(\alpha : X \to K(\mathbb{Z}, 3)\), there exists the following six-term exact sequence

\[
\begin{array}{cccccc}
K^0(X, A; \alpha) & \longrightarrow & K^0(X, \alpha) & \longrightarrow & K^0(A, \alpha|_A) & \\
\uparrow & & \uparrow & & \uparrow & \\
K^1(A, \alpha|_A) & \longrightarrow & K^1(X, \alpha) & \longrightarrow & K^1(X, A; \alpha)
\end{array}
\]

here \(\alpha|_A\) is the composition of the inclusion and \(\alpha\).

(III) (The excision axiom) Let \((X, A)\) be a pair of spaces and let \(U \subset A\) be a subspace such that the closure \(\overline{U}\) is contained in the interior of \(A\). Then the inclusion \(\iota : (X - U, A - U) \to (X, A)\) induces, for all \(\alpha : X \to K(\mathbb{Z}, 3)\), an isomorphism

\[
K^{ev/odd}(X, A; \alpha) \longrightarrow K^{ev/odd}(X - U, A - U; \alpha \circ \iota).
\]

(IV) (Multiplicative property) Let \(\alpha, \beta : X \to K(\mathbb{Z}, 3)\) be a pair of twistings on \(X\). Denote by \(\alpha + \beta\) the new twisting defined by the following map\(^1\)

\[
\alpha + \beta : X \xrightarrow{(\alpha, \beta)} K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3) \xrightarrow{m} K(\mathbb{Z}, 3),
\]

where \(m\) is defined as follows

\[
BP\mathcal{U}(\mathcal{H}) \times BP\mathcal{U}(\mathcal{H}) \cong B(PU(\mathcal{H}) \times PU(\mathcal{H})) \longrightarrow BP\mathcal{U}(\mathcal{H}),
\]

for a fixed isomorphism \(\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}\). Then there is a canonical multiplication

\[
K^{ev/odd}(X, \alpha) \times K^{ev/odd}(X, \beta) \longrightarrow K^{ev/odd}(X, \alpha + \beta),
\]

which defines a \(K^0(X)\)-module structure on twisted K-groups \(K^{ev/odd}(X, \alpha)\).

(V) (Thom isomorphism) Let \(\pi : E \to X\) be an oriented real vector bundle of rank \(k\) over \(X\), then there is a canonical isomorphism, for any twisting \(\alpha : X \to K(\mathbb{Z}, 3)\),

\[
K^{ev/odd}(X, \alpha + o_E) \cong K^{ev/odd}(E, \alpha \circ \pi),
\]

with the grading shifted by \(k(\text{mod} \ 2)\).

(VI) (The push-forward map) For any differentiable map \(f : X \to Y\) between two smooth manifolds \(X\) and \(Y\), let \(\alpha : Y \to K(\mathbb{Z}, 3)\) be a twisting. Then there is a canonical push-forward homomorphism

\[
f^K_f : K^{ev/odd}(X, (\alpha \circ f) + o_f) \longrightarrow K^{ev/odd}(Y, \alpha),
\]

with the grading shifted by \(n \text{ mod}(2)\) for \(n = \dim(X) + \dim(Y)\). Here \(o_f\) is the orientation twisting corresponding to the bundle \(TX \oplus f^*TY\) over \(X\).

---

\(^1\)In terms of bundles of projective Hilbert space, this operation corresponds to the Hilbert space tensor product, see [ASI].
(VII) **(Mayer-Vietoris sequence)** If $X$ is covered by two open subsets $U_1$ and $U_2$ with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$, then there is a Mayer-Vietoris exact sequence

$$
\begin{array}{ccc}
K^0(X, \alpha) & \longrightarrow & K^1(U_1 \cap U_2, \alpha_{12}) \longrightarrow K^1(U_1, \alpha_1) \oplus K^1(U_2, \alpha_2) \\
\downarrow & & \downarrow \\
K^0(U_1, \alpha_1) \oplus K^0(U_2, \alpha_2) & \longleftarrow & K^0(U_1 \cap U_2, \alpha_{12}) \longleftarrow K^1(X, \alpha)
\end{array}
$$

where $\alpha_1$, $\alpha_2$ and $\alpha_{12}$ are the restrictions of $\alpha$ to $U_1$, $U_2$ and $U_1 \cap U_2$ respectively.

3. **Twisted $K$-homology**

Complex $K$-theory, as a generalized cohomology theory on a CW complex, is developed by Atiyah-Hirzebruch using complex vector bundles. It is representable in the sense that there exists a classifying space $\mathbb{Z} \times BU(\infty)$, where $BU(\infty) = \lim_k BU(k)$, such that

$$K^0(X) = [X, \mathbb{Z} \times BU(\infty)]$$

for any finite CW complex $X$. The classifying space for complex $K$-theory is referred to as the $BU(\infty)$-spectrum with even term $\mathbb{Z} \times BU(\infty)$ and odd term $U(\infty)$. They are also called the ‘complex $K$-spectra’ in the literature. The advantage of using spectra is that there is a natural definition of a homology theory associated to a classifying space of each generalized cohomology theory. Hence, the topological $K$-homology of a CW complex $X$, dual to complex $K$-theory, is defined by the following stable homotopy groups

$$K^0_{\text{ev}}(X) = \lim_{k \to \infty} \pi_{2k}(BU(\infty) \wedge X^+)$$

and

$$K^0_{\text{odd}}(X, \alpha) = \lim_{k \to \infty} \pi_{2k+1}(BU(\infty) \wedge X^+).$$

Here $X^+$ is the space $X$ with one point added as a based point, and the wedge product of two based CW complexes $(X, x_0)$ and $(Y, y_0)$ is defined to be

$$X \wedge Y = \frac{X \times Y}{(X \times \{y_0\} \cup \{x_0\} \times Y)}.$$

All the properties of $K$-homology, as a generalized homology theory, can be obtained in a natural way see for example in [Swi]. There are two other equivalent definitions of $K$-homology, called analytic $K$-homology developed by Kasparov, and geometric $K$-homology by Baum and Douglas. We now give a brief review of these two definitions.

Kasparov’s analytic $K$-homology $KK_{\text{ev/odd}}(C(X), \mathbb{C})$ is generated by unitary equivalence classes of (graded) Fredholm modules over $C(X)$ modulo an operator homotopy relation ([Kas] and [HigRoe]). For brevity we will use the notation $K^0_{\text{an}}(X)$ for this $K$-homology. A cycle for $K^0_{\text{an}}(X)$, also called a $\mathbb{Z}_2$-graded Fredholm module, consists of a triple $(\phi_0 \oplus, \phi_1, \mathcal{H}_0 \oplus \mathcal{H}_1, F)$, where

- $\phi_i : C(X) \to B(\mathcal{H}_i)$ is a representation of $C(X)$ on a separable Hilbert space $\mathcal{H}_i$;
- $F : \mathcal{H}_0 \to \mathcal{H}_1$ is a bounded operator such that
  $$\phi_1(a)F - F\phi_0(a), \quad \phi_0(a)(F^*F - Id) \quad \phi_1(a)(FF^* - Id)$$
  are compact operators for all $a \in C(X)$.

A cycle for $K^0_{\text{an}}(X)$, also called a trivially graded or odd Fredholm module, consists of a pair $(\phi, F)$, where
• $\phi : C(X) \to B(\mathcal{H})$ is a representation of $C(X)$ on a separable Hilbert space $\mathcal{H}$;
• $F$ is a bounded self-adjoint operator on $\mathcal{H}$ such that
  
  \[ \phi(a)F - F\phi(a), \quad \phi(a)(F^2 - I) \]

  are compact operators for all $a \in C(X)$.

In [BD1] and [BD2], Baum and Douglas gave a geometric definition of $K$-homology using what are now called geometric cycles. The basic cycles for $K_{\text{geo}}^{\text{ev}}(X)$ (respectively $K_{\text{odd}}^{\text{geo}}(X)$) are triples

\[ (M, \iota, E) \]

consisting of even-dimensional (resp. odd-dimensional) closed smooth manifolds $M$ with a given $\text{Spin}^c$ structure on the tangent bundle of $M$ together with a continuous map $\iota : M \to X$ and a complex vector bundle $E$ over $M$. The equivalence relation on the set of all cycles is generated by the following three steps (see [BD1] for details):

(i) Bordism.
(ii) Direct sum and disjoint union.
(iii) Vector bundle modification.

Addition in $K_{\text{geo}}^{\text{ev/odd}}(X)$ is given by the disjoint union operation of geometric cycles.

Baum-Douglas in [BD2] showed that the Atiyah-Singer index theorem is encoded in the following commutative diagram

\[
\begin{array}{ccc}
K_{\text{top}}^{\text{ev/odd}}(X) & \cong & K_{\text{an}}^{\text{ev/odd}}(X) \\
\cong & & \\
K_{\text{ev/odd}}^{\text{geo}}(X) & \mu & K_{\text{ev/odd}}^{\text{an}}(X)
\end{array}
\]

where $\mu$ is the assembly map assigning an abstract Dirac operator $\iota_*(\left[B_{\text{M}}^E\right]) \in K_{\text{ev/odd}}^{\text{an}}(X)$ to a geometric cycle $(M, \iota, E)$.

For a paracompact Hausdorff space $X$ with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$, all these three versions of twisted $K$-homology were studied in [Wa]. They are called there the twisted topological, analytic and geometric $K$-homologies, and denoted respectively by $K_{\text{top}}^{\text{ev/odd}}(X, \alpha)$, $K_{\text{an}}^{\text{ev/odd}}(X, \alpha)$ and $K_{\text{ev/odd}}^{\text{geo}}(X, \alpha)$. Our first task in this Section is to review these three definitions, see [Wa] for greater detail.

### 3.1. Topological and analytic definitions of twisted $K$-homology

Let $X$ be a CW complex (or paracompact Hausdorff space) with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$. Let $\mathcal{P}_\alpha$ be the corresponding principal $K(\mathbb{Z}, 2)$-bundle. Any base-point preserving action of a $K(\mathbb{Z}, 2)$ on a space defines an associated bundle by the standard construction. In particular, as a classifying space of complex line bundles, a $K(\mathbb{Z}, 2)$ acts on the complex $K$-theory spectrum $\mathbb{K}$ representing the tensor product by complex line bundles, where

\[ \mathbb{K}_{\text{ev}} = \mathbb{Z} \times BU(\infty), \quad \mathbb{K}_{\text{odd}} = U(\infty). \]

Denote by $\mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \mathbb{K}$ the bundle of based $K$-theory spectra over $X$. There is a section of $\mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \mathbb{K}$ defined by taking the base points of each fiber. The image of this section can be identified with $X$ and we denote by $\mathcal{P}_\alpha(\mathbb{K})/X$ the quotient space of $\mathcal{P}_\alpha(\mathbb{K})$ obtained by collapsing the image of this section.
The stable homotopy groups of $\mathcal{P}_\alpha(\mathbb{K})/X$ by definition give the topological twisted $K$-homology groups $K^{\text{top}}_{\text{ev/odd}}(X, \alpha)$. (There are only two due to Bott periodicity of $\mathbb{K}$.) Thus we have
\[
K^{\text{top}}_{\text{ev}}(X, \alpha) = \lim_{k \to \infty} \pi_{2k}(\mathcal{P}_\alpha(BU(\infty))/X)
\]
and
\[
K^{\text{top}}_{\text{odd}}(X, \alpha) = \lim_{k \to \infty} \pi_{2k+1}(\mathcal{P}_\alpha(BU(\infty))/X).
\]
Here the direct limits are taken by the double suspension
\[
\pi_{n+2k}(\mathcal{P}_\alpha(BU(\infty))/X) \to \pi_{n+2k+2}(\mathcal{P}_\alpha(S^2 \wedge BU(\infty))/X)
\]
and then followed by the standard map
\[
\pi_{n+2k+2}(\mathcal{P}_\alpha(S^2 \wedge BU(\infty))/X) \xrightarrow{b\wedge 1} \pi_{n+2k+2}(\mathcal{P}_\alpha(BU(\infty) \wedge BU(\infty))/X)
\]
\[
\xrightarrow{m} \pi_{n+2k+2}(\mathcal{P}_\alpha(BU(\infty))/X)
\]
where $b : \mathbb{R}^2 \to BU(\infty)$ represents the Bott generator in $K^0(\mathbb{R}^2) \cong \mathbb{Z}$, $m$ is the base point preserving map inducing the ring structure on $K$-theory.

For a relative CW-complex $(X, A)$ with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$, the relative version of topological twisted $K$-homology, denoted $K^{\text{top}}_{\text{ev/odd}}(X, A, \alpha)$, is defined to be $K^{\text{top}}_{\text{ev/odd}}(X/A, \alpha)$ where $X/A$ is the quotient space of $X$ obtained by collapsing $A$ to a point. Then we have the following exact sequence
\[
K^{\text{top}}_{\text{odd}}(X, A; \alpha) \xrightarrow{} K^{\text{top}}_{\text{ev}}(A, \alpha|A) \xrightarrow{} K^{\text{top}}_{\text{ev}}(X, \alpha)
\]
\[
\xleftarrow{} K^{\text{top}}_{\text{odd}}(X, \alpha) \xleftarrow{} K^{\text{top}}_{\text{odd}}(A, \alpha|A) \xleftarrow{} K^{\text{top}}_{\text{ev}}(X, A; \alpha)
\]
and the excision properties
\[
K^{\text{top}}_{\text{ev/odd}}(X, B; \alpha) \cong K^{\text{top}}_{\text{ev/odd}}(A, A - B; \alpha|A)
\]
for any CW-triad $(X; A, B)$ with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$. A triple $(X; A, B)$ is a CW-triad if $X$ is a CW-complex, and $A, B$ are two subcomplexes of $X$ such that $A \cup B = X$.

For the analytic twisted $K$-homology, recall that $\mathcal{P}_\alpha(K)$ is the associated bundle of compact operators on $X$. Analytic twisted $K$-homology, denoted by $K^{\text{an}}_{\text{ev/odd}}(X, \alpha)$, is defined to be
\[
K^{\text{an}}_{\text{ev/odd}}(X, \alpha) := KK^{\text{ev/odd}}(\mathcal{C}_0(X, \mathcal{P}_\alpha(K)), \mathbb{C}),
\]
Kasparov’s $\mathbb{Z}_2$-graded $K$-homology of the $C^*$-algebra $\mathcal{C}_0(X, \mathcal{P}_\alpha(K))$.

For a relative CW-complex $(X, A)$ with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$, the relative version of analytic twisted $K$-homology $K^{\text{an}}_{\text{ev/odd}}(X, A, \alpha)$ is defined to be $K^{\text{an}}_{\text{ev/odd}}(X - A, \alpha)$. Then we have the following exact sequence
\[
K^{\text{an}}_{\text{odd}}(X, A; \alpha) \xrightarrow{} K^{\text{an}}_{\text{ev}}(A, \alpha|A) \xrightarrow{} K^{\text{an}}_{\text{ev}}(X, \alpha)
\]
\[
\xleftarrow{} K^{\text{an}}_{\text{odd}}(X, \alpha) \xleftarrow{} K^{\text{an}}_{\text{odd}}(A, \alpha|A) \xleftarrow{} K^{\text{an}}_{\text{ev}}(X, A; \alpha)
\]
and the excision properties
\[ K_{ev/odd}^{an}(X, B; \alpha) \cong K_{ev/odd}^{an}(A, A - B; \alpha|_A) \]
for any CW-triad \((X; A, B)\) with a twisting \(\alpha : X \to K(\mathbb{Z}, 3)\).

**Theorem 3.1.** (Theorem 5.1 in [Wa]) There is a natural isomorphism
\[ \Phi : K_{ev/odd}^{top}(X, \alpha) \longrightarrow K_{ev/odd}^{an}(X, \alpha) \]
for any smooth manifold \(X\) with a twisting \(\alpha : X \to K(\mathbb{Z}, 3)\).

The proof of this theorem requires Poincaré duality between twisted \(K\)-theory and twisted \(K\)-homology (we describe this duality in the next theorem), and the isomorphism (Theorem 2.2) between topological twisted \(K\)-theory and analytic twisted \(K\)-theory.

Fix an isomorphism \(H \otimes H \cong H\) which induces a group homomorphism \(U(H) \times U(H) \longrightarrow U(H)\) whose restriction to the center is the group multiplication on \(U(1)\). So we have a group homomorphism
\[ PU(H) \times PU(H) \longrightarrow PU(H) \]
which defines a continuous map, denoted \(m_*\), of CW-complexes
\[ BPU(H) \times BPU(H) \longrightarrow BPU(H). \]
As \(BPU(H)\) is identified as \(K(\mathbb{Z}, 3)\), we may think of this as a continuous map taking \(K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3)\) to \(K(\mathbb{Z}, 3)\), which can be used to define \(\alpha + o_X\).

There are natural isomorphisms from twisted \(K\)-homology (topological resp. analytic) to twisted \(K\)-theory (topological resp. analytic) of a smooth manifold \(X\) where the twisting is shifted by
\[ \alpha \mapsto \alpha + o_X \]
where \(\tau : X \to BSO\) is the classifying map of the stable tangent space and \(\alpha + o_X\) denotes the map \(X \to K(\mathbb{Z}, 3)\), representing the class \([\alpha] + W_3(X)\) in \(H^3(X, \mathbb{Z})\).

**Theorem 3.2.** Let \(X\) be a smooth manifold with a twisting \(\alpha : X \to K(\mathbb{Z}, 3)\). There exist isomorphisms
\[ K^{top}_{ev/odd}(X, \alpha) \cong K^{an}_{ev/odd}(X, \alpha + o_X) \]
and
\[ K^{ev/odd}_{ev/odd}(X, \alpha) \cong K^{ev/odd}_{ev/odd}(X, \alpha + o_X) \]
with the degree shifted by \(\dim X \pmod 2\).

Analytic Poincaré duality was established in [EEK] and [Tu], and topological Poincaré duality was established in [Wa]. Theorem 3.1 and the exact sequences for a pair \((X, A)\) imply the following corollary.

**Corollary 3.3.** There is a natural isomorphism
\[ \Phi : K^{top}_{ev/odd}(X, A, \alpha) \longrightarrow K^{an}_{ev/odd}(X, A, \alpha) \]
for any smooth manifold \(X\) with a twisting \(\alpha : X \to K(\mathbb{Z}, 3)\) and a closed submanifold \(A \subset X\).

**Remark 3.4.** In fact, Poincaré duality as in Theorem 3.2 holds for any compact Riemannian manifold \(W\) with boundary \(\partial W\) and a twisting \(\alpha : W \to K(\mathbb{Z}, 3)\). This duality takes the following form
\[ K^{top}_{ev/odd}(W, \alpha) \cong K^{ev/odd}_{top}(W, \partial W, \alpha + o_W) \]
and

$$K_{ev/odd}^{an}(W, \alpha) \cong K_{ev/odd}^{an}(X, \partial X, \alpha + \partial W)$$

with the degree shifted by \(\dim W (\text{mod } 2)\). From this, we have a natural isomorphism ([BW])

$$\Phi : K_{ev/odd}^{top}(X, A, \alpha) \to K_{ev/odd}^{an}(X, A, \alpha)$$

for any CW pair \((X, A)\) with a twisting \(\alpha : X \to K(\mathbb{Z}, 3)\) using the Five Lemma.

### 3.2. Geometric cycles and geometric twisted \(K\)-homology.

Let \(X\) be a paracompact Hausdorff space and let \(\alpha : X \to K(\mathbb{Z}, 3)\) be a twisting over \(X\).

**Definition 3.5.** Given a smooth oriented manifold \(M\) with a classifying map \(\nu\) of its stable normal bundle then we say that \(M\) is an \(\alpha\)-twisted \(Spin^c\) manifold over \(X\) if \(M\) is equipped with an \(\alpha\)-twisted \(Spin^c\) structure, that means, a continuous map \(\iota : M \to X\) such that the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BSO} \\
\downarrow \iota & & \downarrow W_3 \\
X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3), \\
\end{array}
\]

commutes up to a fixed homotopy \(\eta\) from \(W_3 \circ \nu\) and \(\alpha \circ \iota\). Such an \(\alpha\)-twisted \(Spin^c\) manifold over \(X\) will be denoted by \((M, \nu, \iota, \eta)\).

**Proposition 3.6.** \(M\) admits an \(\alpha\)-twisted \(Spin^c\) structure if and only if there is a continuous map \(\iota : M \to X\) such that

$$\iota^*([\alpha]) + W_3(M) = 0.$$

If \(\iota\) is an embedding, this is the anomaly cancellation condition obtained by Freed and Witten in [FreWit].

**Proof.** This is clear. \(\square\)

A morphism between \(\alpha\)-twisted \(Spin^c\) manifolds \((M_1, \nu_1, \iota_1, \eta_1)\) and \((M_2, \nu_2, \iota_2, \eta_2)\) is a continuous map \(f : M_1 \to M_2\) where the following diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3), \\
\end{array}
\]

is a homotopy commutative diagram such that

1. \(\nu_1\) is homotopic to \(\nu_2 \circ f\) through a continuous map \(\nu : M_1 \times [0, 1] \to \text{BSO}\);
2. \(\iota_2 \circ f\) is homotopic to \(\iota_1\) through continuous map \(\iota : M_1 \times [0, 1] \to X\);
3. the composition of homotopies \((\alpha \circ \iota) \circ (\eta_2 (f \times \text{Id})) \circ (W_3 \circ \nu)\) is homotopic to \(\eta_1\).

Two \(\alpha\)-twisted \(Spin^c\) manifolds \((M_1, \nu_1, \iota_1, \eta_1)\) and \((M_2, \nu_2, \iota_2, \eta_2)\) are called isomorphic if there exists a diffeomorphism \(f : M_1 \to M_2\) such that the above holds. If the identity map on \(M\) induces an isomorphism between \((M, \nu_1, \iota_1, \eta_1)\) and \((M, \nu_2, \iota_2, \eta_2)\), then these two \(\alpha\)-twisted \(Spin^c\) structures are called equivalent.
Orientation reversal in the Grassmannian model defines an involution
\[ r : \text{BSO} \rightarrow \text{BSO}. \]
Choose a good cover \( \{ V_i \} \) of \( M \) and hence a trivialisation of the universal bundle over \( \text{BSO}(n) \) with transition functions
\[ g_{ij} : V_i \cap V_j \rightarrow SO(n). \]
Let \( \tilde{g}_{ij} : V_i \cap V_j \rightarrow Spin^c(n) \) be a lifting of \( g_{ij} \). Then \( \{ c_{ijk} \} \), obtained from
\[ \tilde{g}_{ij} \tilde{g}_{jk} = c_{ijk} \tilde{g}_{ik}, \]
defines \( [W_3] \in H^3(\text{BSO}, \mathbb{Z}) \). Let \( h \) be the diagonal matrix with the first \((n - 1)\) diagonal entries 1 and the last entry \(-1\). Then \( \{ h g_{ij} h^{-1} \} \) are the transition functions for the universal bundle over \( \text{BSO}(n) \) with the opposite orientation. Note that \( \{ h g_{ij} h^{-1} \} \) is a lifting of \( \{ g_{ij} h^{-1} \} \), which leaves \( \{ c_{ijk} \} \) unchanged. We have \( [W_3] = [W_3 \circ r] \in H^3(\text{BSO}, \mathbb{Z}) \).
Hence there is a homotopy connecting \( W_3 \) and \( W_3 \circ r \). (It is unique up to homotopy as \( H^2(\text{BSO}, \mathbb{Z}) = 0 \).) Given an \( \alpha \)-twisted \( Spin^c \) manifold \( (M, \nu, \iota, \eta) \), let \( -M \) be the same manifold with the orientation reversed. Then the homotopy commutative diagram
\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BSO} \\
\downarrow{\iota} & \quad & \downarrow{r} \\
X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3)
\end{array}
\]
determines a unique equivalence class of \( \alpha \)-twisted \( Spin^c \) structure on \( -M \), called the opposite \( \alpha \)-twisted \( Spin^c \) structure, simply denoted by \( -(M, \nu, \iota, \eta) \).

**Definition 3.7.** A geometric cycle for \((X, \alpha)\) is a quintuple \((M, \iota, \nu, \eta, [E])\) where \([E]\) is a \( K \)-class in \( K^0(M) \) and \( M \) is a smooth closed manifold equipped with an \( \alpha \)-twisted \( Spin^c \) structure \((M, \iota, \nu, \eta)\).

Two geometric cycles \((M_1, \iota_1, \nu_1, \eta_1, [E_1])\) and \((M_2, \iota_2, \nu_2, \eta_2, [E_2])\) are isomorphic if there is an isomorphism \( f : (M_1, \iota_1, \nu_1, \eta_1) \rightarrow (M_2, \iota_2, \nu_2, \eta_2) \), as \( \alpha \)-twisted \( Spin^c \) manifolds over \( X \), such that \( f([E_1]) = [E_2] \).

Let \( \Gamma(X, \alpha) \) be the collection of all geometric cycles for \((X, \alpha)\). We now impose an equivalence relation \( \sim \) on \( \Gamma(X, \alpha) \), generated by the following three elementary relations:

1. **Direct sum - disjoint union**
   If \((M, \iota, \nu, \eta, [E_1])\) and \((M, \iota, \nu, \eta, [E_2])\) are two geometric cycles with the same \( \alpha \)-twisted \( Spin^c \) structure, then
   \[(M, \iota, \nu, \eta, [E_1]) \cup (M, \iota, \nu, \eta, [E_2]) \sim (M, \iota, \nu, \eta, [E_1] + [E_2]).\]

2. **Bordism**
   Given two geometric cycles \((M_1, \iota_1, \nu_1, \eta_1, [E_1])\) and \((M_2, \iota_2, \nu_2, \eta_2, [E_2])\), if there exists a \( \alpha \)-twisted \( Spin^c \) manifold \((W, \iota, \nu, \eta)\) and \([E] \in K^0(W)\) such that
   \[\partial(W, \iota, \nu, \eta) = -(M_1, \iota_1, \nu_1, \eta_1) \cup (M_2, \iota_2, \nu_2, \eta_2)\]
   and \( \partial([E]) = [E_1] \cup [E_2] \). Here \(-(M_1, \iota_1, \nu_1, \eta_1)\) denotes the manifold \( M_1 \) with the opposite \( \alpha \)-twisted \( Spin^c \) structure.
(3) 
**Spin\(^c\)** vector bundle modification

Suppose we are given a geometric cycle \((M, \iota, \nu, \eta, [E])\) and a \(\text{Spin}\(^c\)) vector bundle \(V\) over \(M\) with even dimensional fibers. Denote by \(\mathbb{R}\) the trivial rank one real vector bundle. Choose a Riemannian metric on \(V \oplus \mathbb{R}\), let
\[
\hat{M} = S(V \oplus \mathbb{R})
\]
be the sphere bundle of \(V \oplus \mathbb{R}\). Then the vertical tangent bundle \(T^v(\hat{M})\) of \(\hat{M}\) admits a natural \(\text{Spin}\(^c\)) structure with an associated \(\mathbb{Z}_2\)-graded spinor bundle \(S^+_V \oplus S^-_V\). Denote by \(\rho : \hat{M} \to M\) the projection which is K-oriented. Then
\[
(M, \iota, \nu, \eta, [E]) \sim (\hat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^*E \otimes S^+_V]).
\]

**Definition 3.8.** Denote by \(K^\text{geo}\(\alpha\)(X, \alpha) = \Gamma(X, \alpha)/\sim\) the geometric twisted K-homology. Addition is given by disjoint union - direct sum relation. Note that the equivalence relation \(\sim\) preserves the parity of the dimension of the underlying \(\alpha\)-twisted \(\text{Spin}\(^c\)) manifold. Let \(K^\text{geo}\(\alpha\)(X, \alpha)\) (resp. \(K^\text{ev/odd}\(\alpha\)(X, \alpha)\)) the subgroup of \(K^\text{geo}\(\alpha\)(X, \alpha)\) determined by all geometric cycles with even (resp. odd) dimensional \(\alpha\)-twisted \(\text{Spin}\(^c\)) manifolds.

**Remark 3.9.**

1. If \(M\), in a geometric cycle \((M, \iota, \nu, \eta, [E])\) for \((X, \alpha)\), is a compact manifold with boundary, then \([E]\) has to be a class in \(K^0(M, \partial M)\).
2. If \(f : X \to Y\) is a continuous map and \(\alpha : Y \to K(\mathbb{Z}, 3)\) is a twisting, then there is a natural homomorphism of abelian groups
\[
f_* : K^\text{geo/odd}(X, \alpha \circ f) \to K^\text{geo/odd}(Y, \alpha)
\]
sending \([M, \iota, \nu, \eta, E]\) to \([M, f \circ \iota, \nu, \eta, E]\).
3. Let \(A\) be a closed subspace of \(X\), and \(\alpha\) be a twisting on \(X\). A relative geometric cycle for \((X, A; \alpha)\) is a quintuple \((M, \iota, \nu, \eta, [E])\) such that
   a. \(M\) is a smooth manifold (possibly with boundary), equipped with an \(\alpha\)-twisted \(\text{Spin}\(^c\)) structure \((M, \iota, \nu, \eta)\);
   b. \(f\) has a non-empty boundary, then \(\iota(\partial M) \subset A\);
   c. \([E]\) is a K-class in \(K^0(M)\) represented by a \(\mathbb{Z}_2\)-graded vector bundle \(E\) over \(M\), or a continuous map \(M \to BU(\infty)\).

The relation \(\sim\) generated by disjoint union - direct sum, bordism and \(\text{Spin}\(^c\)) vector bundle modification is an equivalence relation. The collection of relative geometric cycles, modulo the equivalence relation is denoted by
\[
K^\text{geo/odd}(X, A; \alpha).
\]

There exists a natural homomorphism, called the assembly map
\[
\mu : K^\text{geo/odd}(X, \alpha) \to K^\text{an}(X, \alpha)
\]
whose definition (which we will now explain) requires a careful study of geometric cycles.

Given a geometric cycle \((M, \iota, \nu, \eta, [E])\), equip \(M\) with a Riemannian metric. Denote by \(\text{Cliff}(TM)\) the bundle of complex Clifford algebras of \(TM\) over \(M\). The algebra of sections, \(C(M, \text{Cliff}(TM))\), is Morita equivalent to \(C(M, \tau^*\text{BSpin}\(^c\)(K))\). Hence, we have a canonical isomorphism
\[
K^\text{an}(X, \alpha) \simeq K^\text{geo/odd}(C(M, \text{Cliff}(M)), \mathbb{C})
\]
with the degree shift by \(\dim M \mod 2\). Applying Kasparov’s Poincaré duality (Cf. [Kas2])
\[
K^\text{geo/odd}(C, C(M)) \cong K^\text{geo/odd}(C(M, \text{Cliff}(M)), \mathbb{C}),
\]
we obtain a canonical isomorphism

\[ PD : K^0(M) \cong K_{ev/odd}^\an(M, o_M), \]

with the degree shift by \( \dim M \pmod 2 \). The fundamental class \([M] \in K_{ev/odd}^\an(M, o_M)\) is the Poincaré dual of the unit element in \( K^0(M) \). Note that \([M] \in K_{ev}^\an(M, o_M)\) if \( M \) is even dimensional and \([M] \in K_{odd}^\an(M, o_M)\) if \( M \) is odd dimensional. The cap product \( \cap : K_{ev/odd}^\an(M, o_M) \otimes K^0(M) \to K_{ev/odd}^\an(M, o_M) \)

is defined by the Kasparov product. We remark that Poincaré duality is given by the cap product of the fundamental \( K \)-homology class \([M]\)

\[ [M] \cap : K^0(M) \cong K_{ev/odd}^\an(M, o_M). \]

Choose an embedding \( i_k : M \to \mathbb{R}^{n+k} \) and take the resulting normal bundle \( \nu_M \). The natural isomorphism

\[ TM \oplus \nu_M \cong \mathbb{R}^{n+k} \oplus \nu_M \]

and the canonical \( Spin^c \) structure on \( \nu_M \) define a canonical homotopy between the orientation twisting \( o_M \) of \( TM \) and the orientation twisting \( o_{\nu_M} \) of \( \nu_M \). This canonical homotopy defines an isomorphism

\[ I_\ast : K_{ev/odd}^\an(M, o_M) \cong K_{ev/odd}^\an(M, o_{\nu_M}). \]

Given an \( \alpha \)-twisted \( Spin^c \) manifold \((M, \nu, \iota, \eta)\) over \( X \), the homotopy \( \eta \) induces an isomorphism \( \nu^* BSpin^c \cong \iota^* \mathcal{P}_\alpha \) as principal \( K(\mathbb{Z}, 2) \)-bundles on \( M \). Hence there is an isomorphism

\[ \nu^* BSpin^c(M) \xrightarrow{\eta_*} \iota^* \mathcal{P}_\alpha(M) \]

as bundles of \( C^* \)-algebras on \( M \). This isomorphism determines a canonical isomorphism between the corresponding continuous trace \( C^* \)-algebras

\[ C(M, \nu^* BSpin^c(M)) \cong C(M, \iota^* \mathcal{P}_\alpha(M)). \]

Hence, we have a canonical isomorphism

\[ \eta_* : K_{ev/odd}^\an(M, o_{\nu_M}) \cong K_{ev/odd}^\an(M, \alpha \circ \iota). \]

Now we can define the assembly map as

\[ \mu(M, \iota, \nu, \eta, [E]) = \iota_* \circ \eta_* \circ I_\ast([M] \cap [E]) \]

in \( K_{ev/odd}^\an(X, \alpha) \). Here \( \iota_* \) is the natural push-forward map in analytic twisted \( K \)-homology.

**Theorem 3.10.** (Theorem 6.4 in [Wa]) The assembly map \( \mu : K^\text{geo}^\top(X, \alpha) \to K^\text{geo}^\an(X, \alpha) \) is an isomorphism for any smooth manifold \( X \) with a twisting \( \alpha : X \to K(\mathbb{Z}, 3) \).

The proof follows by establishing the existence of a natural map \( \Psi : K^\top_{ev/odd}(X, \alpha) \to K^\text{geo}^\top_0(X, \alpha) \) such that the following diagram

\[ \begin{array}{ccc}
K^\top_{ev/odd}(X, \alpha) & \xrightarrow{\Psi} & K^\text{geo}^\top_0(X, \alpha) \\
\Phi \downarrow \quad \quad & \quad \quad & \quad \quad \uparrow \Theta \\
K^\text{geo}^\an(X, \alpha) & \xrightarrow{\mu} & K^\text{geo}^\an_{ev/odd}(X, \alpha)
\end{array} \]

commutes. All the maps in the diagram are isomorphisms.
**Remark 3.11.** This theorem is generalised in [BW] to the case of any CW pair $(X, A)$. That is, it is shown that the equivalence between the geometric twisted $K$-homology and the analytic twisted $K$-theory holds in this more general situation.

**Corollary 3.12.** $K_{ev/odd}^\text{an}(X, \alpha) \cong K_{ev/odd}^\text{an}(X, -\alpha)$.

**Proof.** By the Brown representation theorem ([Swi]), there is a continuous map $\hat{i} : K(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$ (unique up to homotopy as $H^2(K(\mathbb{Z}, 3), \mathbb{Z}) = 0$) such that

$$[\hat{i} \circ \alpha] = -[\alpha] \in H^3(X, \mathbb{Z})$$

for any map $\alpha : X \to K(\mathbb{Z}, 3)$. Then we have

$$[\hat{i} \circ W_3] = -[W_3] \in H^3(BSO, \mathbb{Z}).$$

As $[W_3]$ is 2-torsion, we know that $[\hat{i} \circ W_3] = -[W_3] = [W_3]$. Therefore, there is a homotopy $\eta_0$ connecting $i \circ W_3$ and $W_3$, that is, the following diagram is homotopy commutative

\[
\begin{array}{ccc}
BSO & \xrightarrow{W_3} & K(\mathbb{Z}, 3) \\
\downarrow \eta_0 & & \downarrow \hat{i} \\
K(\mathbb{Z}, 3) & \xrightarrow{i} & K(\mathbb{Z}, 3)
\end{array}
\]

Note that the homotopy class of $\eta_0$ as a homotopy connecting $W_3$ and $\hat{i} \circ W_3$ is unique due to the fact that $H^2(BSO, \mathbb{Z}) = 0$.

Given an $\alpha$-twisted $Spin^c$ manifold $(M, \iota, \nu, \eta)$, then the following homotopy commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & BSO \\
\downarrow \iota & & \downarrow W_3 \\
X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3)
\end{array}
\]

defines a unique (due to $H^2(BSO, \mathbb{Z}) = 0$) equivalence class of $(-\alpha)$-twisted $Spin^c$ structures. Here $-\alpha = \hat{i} \circ \alpha$. We denote by $\hat{i}(M, \iota, \nu, \eta)$ this $(-\alpha)$-twisted $Spin^c$ manifold. Obviously,

$$\hat{i}(\hat{i}(M, \iota, \nu, \eta)) = (M, \iota, \nu, \eta).$$

The isomorphism $K_{ev/odd}^\text{an}(X, \alpha) \cong K_{ev/odd}^\text{an}(X, -\alpha)$ is induced by the involution $\hat{i}$ on geometric cycles. $\square$

**4. The Chern character in twisted $K$-theory**

In this Section, we will review the Chern character map in twisted $K$-theory on smooth manifolds developed in [CMW] using gerbe connections and curvings. For the topological and analytic definitions, see [AS2] and [MatSte] respectively. Recently, Gomi and Terashima in [GoTe] gave another construction of a Chern character for twisted $K$-theory using a notion of connection on a finite-dimensional approximation of a twisted family of Fredholm operators developed by Gomi ([Gomi]).
4.1. Twisted Chern character. For a fibration \( \pi^* : Y \to X \), let \( Y^{[p]} \) denote the \( p \)th fibered product. There are projection maps \( \pi_i : Y^{[p]} \to Y^{[p-1]} \) which omit the \( i \)th element for each \( i = 1 \ldots p \). These define a map

\[
\delta : \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]})
\]

by

\[
\delta(\omega) = \sum_{i=1}^{p} (-1)^i \pi_i^*(\omega).
\]

Clearly \( \delta^2 = 0 \). In fact, the \( \delta \)-cohomology of this complex vanishes identically, hence, the sequence

\[
0 \to \Omega^q(X) \xrightarrow{\pi^*} \Omega^q(Y) \cdots \xrightarrow{\delta} \Omega^q(Y^{[p-1]}) \xrightarrow{\delta} \Omega^q(Y^{[p]}) \to \cdots
\]

is exact.

Returning now to our particular example, a bundle gerbe connection on \( P_\alpha \) is a unitary connection \( \theta \) on the principal \( U(1) \)-bundle \( G_\alpha \) over \( P_\alpha[2] \) which commutes with the bundle gerbe product. A bundle gerbe connection \( \theta \) has curvature \( F_{\theta} \in \Omega^2(P_\alpha[2]) \) satisfying \( \delta(F_{\theta}) = 0 \). There exists a two-form \( \omega \) on \( P_\alpha \) such that

\[
F_{\theta} = \pi_2^*(\omega) - \pi_1^*(\omega).
\]

Such an \( \omega \) is called a curving for the gerbe connection \( \theta \). The choice of a curving is not unique, the ambiguity in the choice is precisely the addition of the pull-back to \( P_\alpha \) of a two-form on \( X \). Given a choice of curving \( \omega \), there is a unique closed three-form on \( \beta \) on \( X \) satisfying \( d\omega = \pi^*\beta \). We denote by

\[
\tilde{\alpha} = (G_\alpha, \theta, \omega)
\]

the lifting bundle gerbe \( G_\alpha \) with the connection \( \theta \) and a curving \( \omega \). Moreover \( H = \beta / 2\pi \sqrt{-1} \) is a de Rham representative for the Dixmier-Douady class \([\alpha]\). We shall call \( \tilde{\alpha} \) the differential twisting, as it is the twisting in differential twisted \( K \)-theory (Cf. [CMW]).

The following theorem is established in [CMW].

**Theorem 4.1.** Let \( X \) be a smooth manifold, \( \pi : P_\alpha \to X \) be a principal \( PU(\mathcal{H}) \) bundle over \( X \) whose classifying map is given by \( \alpha : X \to K(\mathbb{Z},3) \). Let \( \tilde{\alpha} = (G_\alpha, \theta, \omega) \) be a bundle gerbe connection \( \theta \) and a curving \( \omega \) on the lifting bundle gerbe \( G_\alpha \). There is a well-defined twisted Chern character

\[
Ch_{\tilde{\alpha}} : K^* (X, \alpha) \to H^{ev/odd} (X, d-H).
\]

Here the groups \( H^{ev/odd}(X, d-H) \) are the twisted cohomology groups of the complex of differential forms on \( X \) with the coboundary operator given by \( d-H \). The twisted Chern character is functorial under the pull-back. Moreover, given another differential twisting \( \tilde{\alpha} + b = (G_\alpha, \theta, \omega + \pi^* b) \) for a 2-form \( b \) on \( X \),

\[
Ch_{\tilde{\alpha} + b} = Ch_{\tilde{\alpha}} \cdot \exp(\frac{b}{2\pi \sqrt{-1}}).
\]
PROOF. Choose a good open cover \( \{ V_i \} \) of \( X \) such that \( \mathcal{P}_\alpha \rightarrow X \) has trivializing sections \( \phi_i \) over each \( V_i \) with transition functions \( g_{ij} : V_i \cap V_j \rightarrow PU(H) \) satisfying \( \phi_j = \phi_i g_{ij} \). Define \( \{ \sigma_{ijk} \} \) by \( \hat{g}_{ij} \hat{g}_{jk} = \hat{g}_{ik} \sigma_{ijk} \) for a lift of \( g_{ij} \) to \( \hat{g}_{ij} : V_i \cap V_j \rightarrow U(H) \).

Note that the pair \((\phi_i, \phi_j)\) defines a section of \( \mathcal{P}_\alpha^{[2]} \) over \( V_i \cap V_j \). The connection \( \theta \) can be pulled back by \((\phi_i, \phi_j)\) to define a 1-form \( A_{ij} \) on \( V_i \cap V_j \) and the curving \( \omega \) can be pulled-back by the \( \phi_i \) to define two-forms \( B_i \) on \( V_i \). Then the differential twisting defines the triple

\[
(4.3) \quad \{ (\sigma_{ijk}, A_{ij}, B_i) \}
\]

which is a degree two smooth Deligne cocycle. Now we explain in some detail the twisted Chern characters in both the odd and even case following [CMW].

**The even case:** As a model for the \( K^0 \) classifying space, we choose \( \text{Fred} \), the space of bounded self-adjoint Fredholm operators with essential spectrum \( \{ \pm 1 \} \) and otherwise discrete spectra, with a grading operator \( \Gamma \) which anticommutes with the given family of Fredholm operators.

A twisted \( K \)-class in \( K^0(X, \alpha) \) can be represented by \( f : \mathcal{P}_\alpha \rightarrow \text{Fred} \), a \( PU(H) \)-equivariant family of Fredholm operators. We can select an open cover \( \{ V_i \} \) of \( X \) such that on each \( V_i \) there is a local section \( \phi_i : V_i \rightarrow \mathcal{P}_\alpha \) and for each \( i \) the Fredholm operators \( f(\phi_i(x)), x \in V_i \) have a gap in the spectrum at both \( \pm \lambda_i \neq 0 \). Then over \( V_i \) we have a finite rank vector bundle \( E_i \) defined by the spectral projections of the operators \( f(\phi_i(x)) \) corresponding to the interval \([-\lambda_i, \lambda_i]\).

Passing to a finer cover \( \{ U_i \} \) if necessary, we may assume that \( E_i \) is a trivial vector bundle over \( U_i \) of rank \( n_i \). Choosing a trivialization of \( E_i \) gives a \( \mathbb{Z}_2 \) graded parametrix \( q_i \) (an inverse up to finite rank operators) of the family \( f \circ \phi_i \). In the index zero sector the operator \( q_i(x)^{-1} \) is defined as the direct sum of the restriction of \( f(\phi_i(x)) \) to the orthogonal complement of \( E_i \) in \( H \) and an isomorphism between the vector bundles \( E_i^+ \) and \( E_i^- \). Clearly then \( f(\phi_i(x))q_i(x) = 1 \mod \text{rank } n_i \) operators. In the case of nonzero index one defines a parametrix as a graded invertible operator \( q_i \) such that \( f(\phi_i(x))q_i(x) = s_n \mod \text{rank } n_i \), modulo finite rank operators, with \( s_n \) a fixed Fredholm operator of index \( n \) equal to the index of \( f(\phi_i(x)) \).

On the overlap \( U_{ij} \) we have a pair of parametrices \( q_i \) and \( q_j \) of families of \( f \circ \phi_i \) and \( f \circ \phi_j \), respectively. These are related by an invertible operator \( f_{ij} \) which is of the form \( 1 + \) a finite rank operator.

\[
\hat{g}_{ij}q_j(x)\hat{g}_{ij}^{-1} = q_i(x)f_{ij}(x).
\]

The conjugation on the left hand side by \( \hat{g}_{ij} \) comes from the equivariance relation

\[
f(\phi_j(x)) = f(\phi_i(x))q_{ij}(x)) = \hat{g}_{ij}(x)^{-1}f(\phi_i(x))\hat{g}_{ij}(x).
\]

The system \( \{ f_{ij} \} \) does not quite satisfy the Čech cocycle relation needed to define a principal bundle, because of the different local sections \( \phi_i : U_i \rightarrow \mathcal{P}_\alpha \) involved. Instead, we have on \( U_{ijk} \)

\[
\hat{g}_{jk}q_k\hat{g}_{jk}^{-1} = q_jf_{jk} = (\hat{g}_{ij}^{-1}q_if_{ij}\hat{g}_{ij})f_{jk} = \hat{g}_{jk}(\hat{g}_{ik}^{-1}q_i\hat{f}_{ik}\hat{g}_{ik})\hat{g}_{jk}^{-1}.
\]

Using the relation \( \hat{g}_{jk}\hat{g}_{ik}^{-1} = \sigma_{ijk}\hat{g}_{ij}^{-1} \), we get

\[
\hat{g}_{jk}(\hat{g}_{ik}^{-1}q_i\hat{f}_{ik}\hat{g}_{ik})\hat{g}_{jk}^{-1} = \hat{g}_{ij}^{-1}q_i\hat{f}_{ik}\hat{g}_{ij}
\]
multiplying the last equation from right by \( \hat{g}_{ij}^{-1} \) and from the left by \( q_i^{-1}\hat{g}_{ij} \) one gets the twisted cocycle relation

\[
f_{ij}(\hat{g}_{ij}f_{jk}\hat{g}_{ij}^{-1}) = f_{ik},
\]
which is independent of the choice of the lifting $g_{ij}$. For simplicity, we will just write the above twisted cocycle relation as

$$f_{ij}(g_{ij}f_{jk}g_{ij}^{-1}) = f_{ik}. \tag{4.4}$$

This twisted cocycle relation (4.4) actually defines an untwisted cocycle relation for $\{(g_{ij}, f_{ij})\}$ in the twisted product

$$\mathcal{G} = PU(\mathcal{H}) \times GL(\infty),$$

where the group $PU(\mathcal{H})$ acts on the group $GL(\infty)$ of invertible 1+ finite rank operators by conjugation. Thus the product in $\mathcal{G}$ is given by

$$(g, f) \cdot (g', f') = (gg', f(gf'g^{-1})).$$

The cocycle relation for the pairs $\{(g_{ij}, f_{ij})\}$ then encodes both the cocycle relation for the transition functions $\{g_{ij}\}$ of the $PU(\mathcal{H})$ bundle over $X$ and the twisted cocycle relation (4.4). In summary, this cocycle $\{(g_{ij}, f_{ij})\}$ defines a principal $\mathcal{G}$ bundle over $X$.

The classifying space $B\mathcal{G}$ is a fiber bundle over $K(\mathbb{Z}, 3)$. The fiber at each point in $K(\mathbb{Z}, 3)$ is homeomorphic (but not canonically so) to the space $Fred$ of graded Fredholm operators; to set up the isomorphism one needs a choice of element in each fiber. Given a principal $PU(\mathcal{H})$-bundle $P_\alpha$ over $X$ defined by $\alpha : X \to K(\mathbb{Z}, 3)$, the even twisted $K$-theory $K^0(X, P_\alpha)$ is the set of homotopy classes of maps $X \to B\mathcal{G}$ covering the map $\alpha$.

Next we construct the twisted Chern character from a connection $\nabla$ on a principal $\mathcal{G}$ bundle over $X$ associated to the cocycle $\{(g_{ij}, f_{ij})\}$. Locally, on a good open cover $\{U_i\}$ of $X$ we can lift the connection to a connection taking values in the Lie algebra $\mathfrak{g}$ of the central extension $U(\mathcal{H}) \times GL(\infty)$ of $\mathcal{G}$. Denote by $\hat{F}_\nabla$ the curvature of this connection. On the overlaps $\{U_{ij}\}$ the curvature satisfies a twisted relation

$$\hat{F}_{\nabla,j} = Ad(g_{ij}, f_{ij})^{-1} \hat{F}_{\nabla,i} + g_{ij}^* c,$$

where $c$ is the curvature of the canonical connection $\theta$ on the principal $U(1)$-bundle $U(\mathcal{H}) \to PU(\mathcal{H})$.

Since the Lie algebra $\mathfrak{g}(\infty) \oplus \mathbb{C}$ is an ideal in the Lie algebra of $U(\mathcal{H}) \times GL(\infty)$, the projection $F_{\nabla,i}$ of the curvature $\hat{F}_{\nabla,i}$ onto this subalgebra transforms in the same way as $\hat{F}$ under change of local trivialization. It follows that for a $PU(\mathcal{H})$-equivariant map $f : P_\alpha \to Fred$, we can define a twisted Chern character form of $f$ as

$$ch_\alpha(f, \nabla) = e^{B_1} \frac{1}{2\pi i} \frac{1}{e^{\hat{F}_{\nabla,i}}}, \tag{4.5}$$

over $V_i$. Here the trace is well-defined on $\mathfrak{g}(\infty)$ and on the center $\mathbb{C}$ it is defined as the coefficient of the unit operator. Note that $ch_\alpha(f, \nabla)$ is globally defined and $(d-H)$-closed

$$(d - H)ch_\alpha(f, \nabla) = 0,$$

and depends on the differential twisting

$$\tilde{\alpha} = (g_{ij}, \theta, \omega).$$

Let $f_0$ and $f_1$ be homotopic, and $\nabla_0$ and $\nabla_1$ be two connections on the principal $\mathcal{G}$ bundle over $X$, we have a Chern-Simons type form

$$CS((f_0, \nabla_0), (f_1, \nabla_1)),$$

well-defined modulo $(d - H)$-exact forms, such that

$$ch_\alpha(f_1, \nabla_1) - ch_\alpha(f_0, \nabla_0) = (d - H)CS((f_0, \nabla_0), (f_1, \nabla_1)). \tag{4.6}$$
The proof follows directly from the local computation using (4.5). Hence, the \((d - H)\)-
cohomology class of \(ch_\alpha(f, \nabla)\) does not depend choices of a connection \(\nabla\) on the principal
\(\mathbb{G}\) bundle over \(X\), and depends only on the homotopy class of \(f\). We denote the \((d - H)\)-
cohomology class of \(ch_\alpha(f, \nabla)\) by \(Ch_\alpha([f_1])\) which is a natural homomorphism
\[
Ch_\alpha : K^0(X, \alpha) \rightarrow H^{ev}(X, d - H).
\]
From (4.5), we have
\[
Ch_{\tilde{\alpha} + b} = Ch_{\tilde{\alpha}} \cdot \exp\left(\frac{b}{2\pi \sqrt{-1}}\right),
\]
for a differential twisting \(\tilde{\alpha} + b = (\tilde{G}_\alpha, \theta, \omega + \pi^*b)\).

**The odd case:** The odd case is a little easier. First, as a model for the \(K^1\) classifying
space, we choose \(U(\infty) = \lim_{\rightarrow n} U(n)\), the stabilized unitary group. Let \(\Theta\) be the universal
odd character form on \(U(\infty)\) defined by the canonical left invariant \(u(\infty)\)-valued form on
\(U(\infty)\).

Let \(H = H_+ \oplus H_-\) be a polarized Hilbert space and let \(U_{res} = U_{res}(H)\) denotes the
group of unitary operators in \(H\) with Hilbert-Schmidt off-diagonal blocks. The conjugation
action of \(U(H_+) \times U(H_-)\) on \(U_{res}\) defines an action of \(PU_{0}(H) = P(U(H_+) \times U(H_-))\)
on \(U_{res}\). Note that the classifying space of \(U_{res}\) is \(U(\infty)\).

Define
\[
\mathfrak{f}_\Theta = PU_0(H) \rtimes U_{res}.
\]
Then given a principal \(PU_0(H)\)-bundle \(P_\alpha\) over \(X\) defined by \(\alpha : X \rightarrow K(\mathbb{Z}, 3)\), the odd
twisted \(K\)-theory \(K^1(X, \alpha)\) is the set of homotopy classes of maps \(X \rightarrow B\mathfrak{f}_\Theta\) covering
the map \(\alpha\). These are represented by \(PU_0(H)\)-equivariant maps \(f : P_\alpha \rightarrow U(\infty)\). With
respect to trivializing sections \(\phi_i\) over each \(V_i\), Then
\[
\exp(B_i (f \circ \phi_i)^*\Theta)
\]
is a globally defined and \((d - H)\)-closed differential form on \(X\). This defines the odd
version of the twisted Chern character
\[
Ch_\tilde{\alpha} : K^1(X, \alpha) \rightarrow H^{odd}(X, d - H).
\]

\[\square\]

### 4.2. Differential twisted \(K\)-theory.
Recall that the Bockstein exact sequence in complex \(K\)-theory for any finite CW complex:

\[
\begin{array}{cccc}
K^0(X) & \xrightarrow{ch} & H^{ev}(X, \mathbb{R}) & \xrightarrow{\mathbb{R}/\mathbb{Z}} K^0_{\mathbb{R}/\mathbb{Z}}(X) \\
\uparrow & & & \uparrow \\
K^1_{\mathbb{R}/\mathbb{Z}}(X) & \xleftarrow{ch} & H^{odd}(X, \mathbb{R}) & \xleftarrow{ch} K^1(X)
\end{array}
\]

where \(K^0_{\mathbb{R}/\mathbb{Z}}(X)\) is \(K\)-theory with \(\mathbb{R}/\mathbb{Z}\)-coefficients as in [Kar1] and [Ba].

Analogously, in twisted \(K\)-theory, given a smooth manifold \(X\) with a twisting \(\alpha : X \rightarrow K(\mathbb{Z}, 3)\), upon a choice of a differential twisting
\[
\tilde{\alpha} = (\tilde{G}_\alpha, \theta, \omega)
\]

where \(\mathbb{R}/\mathbb{Z}\)-coefficients.
lifting $\alpha$, we have the corresponding Bockstein exact sequence in twisted $K$-theory

\[
\begin{array}{ccccccccc}
K^0(X, \alpha) & \xrightarrow{\text{Ch}_\alpha} & H^{ev}(X, d - H) & \xrightarrow{\text{d}} & K^0_{R/Z}(X, \alpha) \\
\downarrow & & \downarrow & & \downarrow \\
K^1_{R/Z}(X, \alpha) & \xleftarrow{\text{Ch}_\alpha} & H^{odd}(X, d - H) & \xleftarrow{\text{d}} & K^1(X, \alpha)
\end{array}
\]  

(4.8)

Here $K^0_{R/Z}(X, \alpha)$ and $K^1_{R/Z}(X, \alpha)$ are subgroups of differential twisted $K$-theory, respectively $\tilde{K}^0(X, \tilde{\alpha})$ and $\tilde{K}^1(X, \tilde{\alpha})$ (see [CMW] for the detailed construction). Here we give another equivalent construction of differential twisted $K$-theory.

Fix a choice of a connection $\nabla$ on a principal $G$ bundle over $X$. Then $\tilde{K}^0(X, \tilde{\alpha})$ is the abelian group generated by pairs

\[\{(f, \eta)\},\]

modulo an equivalence relation, where $f : \mathcal{P}_\alpha \to \text{Fred}$ is a $PU(H)$-equivariant map and $\eta$ is an odd differential form modulo $(d - H)$-exact forms. Two pairs $(f_0, \eta_0)$ and $(f_1, \eta_1)$ are called equivalent if and only if

\[\eta_1 - \eta_0 = CS((f_1, \nabla), (f_0, \nabla)).\]

The differential Chern character form of $f$ is given by

\[\text{ch}_{\tilde{\alpha}}(f, \nabla) - (d - H)\eta\]

which defines a homomorphism

\[\text{ch}_{\tilde{\alpha}} : \tilde{K}^0(X, \tilde{\alpha}) \to \Omega_{ev}^c(X, d - H),\]

where $\Omega_{ev}^c(X, d - H)$ is the image of $\text{ch}_{\tilde{\alpha}} : \tilde{K}^0(X, \tilde{\alpha}) \to \Omega^c(X)$. The kernel of $\text{ch}_{\tilde{\alpha}}$ is isomorphic to $K^1_{R/Z}(X, \alpha)$.

Similarly, we define the odd differential twisted $K$-theory $\tilde{K}^1(X, \tilde{\alpha})$ with the differential Chern character form homomorphism

\[\text{ch}_{\tilde{\alpha}} : \tilde{K}^1(X, \tilde{\alpha}) \to \Omega_{odd}^c(X, d - H)\]

The kernel of $\text{ch}_{\tilde{\alpha}}$ is isomorphic to $K^0_{R/Z}(X, \alpha)$. The following commutative diagrams were established in [CMW] relating differential twisted $K$-theory with twisted $K$-theory.
and with the diagram (4.8)

\[
\begin{array}{cccccc}
0 & \to & H^{\text{odd}}(X, d-H) & \to & K^1_{\mathbb{R}/\mathbb{Z}}(X, \tilde{\sigma}) & \to & 0 \\
\Omega^{\text{odd}}_0(X, d-H) & \to & \tilde{K}^0(X, \tilde{\sigma}) & \to & K^0(X, \sigma) & \to & 0 \\
\Omega^{\text{odd}}_0(X, d-H) & \to & H^{\text{ev}}(X, d-H) & \to & 0
\end{array}
\]

with exact horizontal and vertical sequences, and exact upper-right and exact lower-left 4-term sequences. We expect that these two commutative diagrams uniquely characterize differential twisted $K$-theory.

4.3. Twisted Chern character for torsion twistings. In this paper, we will only use the twisted Chern character for a torsion twisting and in this case we will give an explicit construction. Let $E$ be a real oriented vector bundle of rank $2k$ over $X$ with its orientation twisting denoted by $o(E) : X \to K(\mathbb{Z}, 3)$.

The associated lifting bundle $G_{o(E)}$ has a canonical reduction to the $\text{Spin}^c$ bundle gerbe $G_{\text{W}_3}(E)$.

Choose a local trivialization of $E$ over a good open cover $\{V_i\}$ of $X$. Then the transition functions

\[ g_{ij} : V_i \cap V_j \to SO(2k). \]
define an element in $H^1(X, SO(2k))$ whose image under the Bockstein exact sequence

$$H^1(X, Spin(2k)) \rightarrow H^1(X, SO(2k)) \rightarrow H^2(X, \mathbb{Z}_2)$$

is the second Stieffel-Whitney class $w_2(E)$ of $E$. Denote the differential twisting by

$$\bar{w}_2(E) = (\mathcal{G}_{W_3(E)}, \theta, 0),$$

the $Spin^c$ bundle gerbe $\mathcal{G}_{W_3(E)}$ with a flat connection $\theta$ and a trivial curving. With respect to a good cover $\{V_i\}$ of $X$ the differential twisting $\bar{w}_2(E)$ defines a Deligne cocycle $\{(\alpha_{ijk}, 0, 0)\}$ with trivial local $B$-fields, here $\alpha_{ijk} = \hat{g}_{ij} \hat{g}_{jk} \hat{g}_{ki}$ where $\hat{g}_{ij} : U_{ij} \rightarrow Spin^c(2n)$ is a lift of $g_{ij}$.

By Proposition 2.4 a twisted K-class in $K^0(X, o(E))$ can be represented by a Clifford bundle, denoted $\mathcal{E}$. Equip $\mathcal{E}$ with a Clifford connection, and $E$ with a $SO(2k)$-connection. Locally, over each $V_i$ we let $\mathcal{E}|_{V_i} \cong S_i \otimes \mathcal{E}_i$ where $S_i$ is the local fundamental spinor bundle associated to $E|_{V_i}$ with the standard Clifford action of $\text{Cliff}(E|_{V_i})$ obtained from the fundamental representation of $Spin(2k)$. Then $\mathcal{E}_i$ is a complex vector bundle over $V_i$ with a connection $\nabla_i$ such that on $V_i \cap V_j$

$$Ch(\mathcal{E}_i, \nabla_i) = Ch(\mathcal{E}_j, \nabla_j).$$

Hence, the twisted Chern character

$$Ch_{\bar{w}_2(E)} : K^0(X, o(E)) \rightarrow H^{ev}(X)$$

is given by $[\mathcal{E}] \mapsto \{[ch(\mathcal{E}_i, \nabla_i)] = ch(\mathcal{E}_i)\}$. The proof of the following proposition is straightforward.

**Proposition 4.2.** The twisted Chern character satisfies the following identities

1. $Ch_{\bar{w}_2(E_1 \oplus E_2)}([E_1 \oplus E_2]) = Ch_{\bar{w}_2(E_1)}([E_1]) + Ch_{\bar{w}_2(E_2)}([E_2]).$
2. $Ch_{\bar{w}_2(E_1 \otimes E_2)}([E_1 \otimes E_2]) = Ch_{\bar{w}_2(E_1)}([E_1]) Ch_{\bar{w}_2(\mathcal{E}_2)}([E_2]).$

In the case that $E$ has a $Spin^c$ structure whose determinant line bundle is $L$, there is a canonical isomorphism

$$K^0(X) \rightarrow K^0(X, o(E)),$$

given by $[V] \mapsto [V \otimes S_E]$ where $S_E$ is the associated spinor bundle of $E$. Then we have

$$Ch_{\bar{w}_2(E)}([V \otimes S_E]) = e^{\frac{i}{2}(1)} ch([V]),$$

where $ch([V])$ is the ordinary Chern character of $[V] \in K^0(X)$.

In particular, when $X$ is an even dimensional Riemannian manifold, and $TX$ is equipped with the Levi-Civita connection, under the identification of $K^0(X, o(E))$ with the Grothendieck group of Clifford modules. Then

$$(4.9) \quad Ch_{\bar{w}_2(X)}([E]) = ch(E/S)$$

where $ch(E/S)$ is the relative Chern character of the Clifford module $E$ constructed in Section 4.1 of [BGV].
5. Thom classes and Riemann-Roch formula in twisted \( K \)-theory

5.1. The Thom class. Given any oriented real vector bundle \( \pi : E \to X \) of rank \( 2k \), \( E \) admits a \( Spin^c \) structure if its classifying map \( \tau : X \to BSO(2k) \) admits a lift \( \tilde{\tau} \)

\[
\begin{array}{ccc}
\text{BSpin}^c & \xrightarrow{\tilde{\tau}} & X \\
\downarrow & & \downarrow \\
BSO(2k) & \xrightarrow{\tau} & \end{array}
\]

As \( \text{BSpin}^c \to BSO(2k) \) is a \( BU(1) \)-principal bundle with the classifying map given by

\[
W_3 : BSO(2k) \to \mathbb{K}(Z, 3),
\]

\( E \) admits a \( Spin^c \) structure if \( W_3 \circ \tau : X \to \mathbb{K}(Z, 3) \) is null homotopic, and a choice of null homotopy determines a \( Spin^c \) structure on \( E \). Associated to a \( Spin^c \) structure \( s \) on \( E \), there is canonical \( K \)-theoretical Thom class

\[
U_E = [\pi^* S^+, \pi^* S^-, cl] \in K^0_c(E)
\]

in the \( K \)-theory of \( E \) with vertical compact supports. Here \( S^+ \) and \( S^- \) are the positive and negative spinor bundle over \( X \) defined by the \( S\text{pin}^c \) structure on \( E \), and \( cl \) is the bundle map \( \pi^* S^+ \to \pi^* S^- \) given by the Clifford action on \( S^\pm \).

Remark 5.1. (1) The restriction of \( U_E^s \) to each fiber is a generator of \( K^0(R^{2k}) \), so a \( Spin^c \) structure on \( E \) is equivalent to a \( K \)-orientation on \( E \). Note that Thom classes and \( K \)-orientation are functorial under pull-backs of \( Spin^c \) vector bundles.

(2) Let \( s \odot L \) be another \( Spin^c \) structure on \( E \) which differs from \( s \) by a complex line bundle \( p : L \to X \), then

\[
U_E^{s'} = U_E^s \cdot p^*[L].
\]

(3) Let \( (E_1, s_1) \) and \( (E_2, s_2) \) be two \( Spin^c \) vector bundles over \( X \), \( p_1 \) and \( p_2 \) be the projections from \( E_1 \oplus E_2 \) to \( E_1 \) and \( E_2 \) respectively, then

\[
U_{E_1 \oplus E_2}^{s_1 \oplus s_2} = p_1^*(U_{E_1}^{s_1}) \cdot p_2^*(U_{E_2}^{s_2}) \in K^0_c(E_1 \oplus E_2).
\]

(4) The Thom isomorphism in \( K \)-theory for a \( Spin^c \) vector bundle \( \pi : E \to X \) of rank \( 2k \) is given by

\[
\Phi^K_E : \quad K^0(X) \to K^0(E) \\
\quad a \mapsto \pi^*(a)U_E^p.
\]

Here for locally compact spaces, we shall consider only \( K \)-theory with compact supports. When \( X \) is compact, \( U_E^p \in K^0(TX) \) and the Thom isomorphism \( \Phi^K_E \) is the inverse of the push-forward map

\[
\pi_1 : K^0(TX) \to K^0(X)
\]

associated to the \( K \)-orientation of \( \pi \) defined the \( S\text{pin}^c \) structure on \( E \).

If an oriented vector bundle \( E \) of even rank over \( X \) does not admit a \( Spin^c \) structure, \( W_3 \circ \tau : X \to \mathbb{K}(Z, 3) \) is not null homotopic. Thus, \( W_3 \circ \tau \) defines a twisting on \( X \) for \( K \)-theory, called the orientation twisting \( o_E \). In this Section we will define a canonical Thom class

\[
U_E \in K^0(E, \pi^*o_E)
\]
such that \( a \mapsto \pi^*(a) \cup U_E \) defines the Thom isomorphism \( K^0(X, o_E) \cong K^0(E) \). In fact, 
\[
K^0(X, \alpha + o_E) \cong K^0(E, \alpha \circ \pi)
\]
for any twisting \( \alpha : X \to K(\mathbb{Z}, 3) \).

Choose a good open cover \( \{ V_i \} \) of \( X \) such that \( E_i = E|_{V_i} \) is trivialized by an isomorphism 
\[
E_i \cong V_i \times \mathbb{R}^{2n}.
\]

This defines a canonical \( Spin^c \) structure \( s_i \) on each \( E_i \). Denote by \( U_{E_i}^s \) the associated Thom class of \( (E_i, s_i) \). Then we have 
\[
U_{E_i}^s = U_{E_i}^s \pi_i^*(\left[ L_{ij} \right]) \in K^0_{cv}(E_{ij})
\]
where \( L_{ij} \) is the difference line bundle over \( V_{ij} = V_i \cap V_j \) defined by \( s_j = s_i \otimes L_{ij} \) on \( E_{ij} = E|_{V_{ij}} \). Recall that these local line bundles \( \{ L_{ij} \} \) define a bundle gerbe [Mur] associated to the twisting \( o_E = W_3 \circ \tau : X \to K(\mathbb{Z}, 3) \) and a locally trivializing cover \( \{ V_i \} \). By the definition of twisted \( K \)-theory, \( \{ U_{E_i}^s \} \) defines a twisted \( K \)-theory class of \( E \) with compact vertical supports and twisting given by 
\[
\pi^*(o_E) = o_E \circ \pi : E \to K(\mathbb{Z}, 3).
\]

We denote this canonical twisted \( K \)-theory class by 
\[
U_E \in K^0_{cv}(E, \pi^*(o_E)).
\]

When \( X \) is compact, then \( U_E \in K^0(E, \pi^*(o_E)) \). One can easily show that the Thom class \( U_E \) does not depend on the choice of the trivializing cover.

Now we can list the properties of the Thom class in twisted \( K \)-theory.

**Proposition 5.2.**

1. If \( E \) is equipped with a \( Spin^c \) structure \( s \), then \( s \) defines a canonical isomorphism 
\[
\phi_s : K^0_{cv}(E, \pi^*(o_E)) \to K^0_{cv}(E)
\]
such that \( \phi_s(U_E) = U_E^s \).

2. Let \( f : X \to Y \) be a continuous map and \( E \) be an oriented vector bundle of even rank over \( Y \), then 
\[
U_{f^*E} = f^*(U_E).
\]

3. Let \( E_1 \) and \( E_2 \) be two oriented vector bundles of even rank over \( X \), \( p_1 \) and \( p_2 \) be the projections from \( E_1 \oplus E_2 \) to \( E_1 \) and \( E_2 \) respectively, that is, we have the diagram 
\[
\begin{array}{ccc}
E_1 \oplus E_2 & \xrightarrow{p_2} & E_2 \\
\downarrow{p_1} & & \downarrow{\pi_2} \\
E_1 & \xrightarrow{\pi_1} & X
\end{array}
\]
then 
\[
U_{E_1 \oplus E_2} = p_1^*(U_{E_1}) \cdot p_2^*(U_{E_2}).
\]

4. Let \( \pi : E \to X \) be an oriented vector bundle of even rank over a compact space \( X \), the Thom isomorphism in twisted \( K \)-theory ([CW1]) 
\[
\Phi_E^K : K^0(X, \alpha + o_E) \cong K^0(E, \pi^*(\alpha))
\]
is given by \( a \mapsto \pi^*(a) \cdot U_E \). Moreover, the push-forward map in twisted \( K \)-theory (\cite{CW1})
\[
\pi_1 : K^0(E, \pi^*(\alpha)) \longrightarrow K^0(X, \alpha + o_E)
\]
satisfies \( \pi_1(\pi^*(a) \cdot U_E) = a \).

**Proof.**

1. The \( \text{Spin}^c \) structure \( s \) defines canonical isomorphism
\[
\phi_s : K^0_{cv}(E, \pi^*(a)) \longrightarrow K^0_{cv}(E)
\]
as follows. Given a trivializing cover \( \{ V_i \} \) and the canonical \( \text{Spin}^c \) structure \( s_i \) on \( E_i = E|_{V_i} \), we have
\[
s|_{E_i} = s_i \otimes L_i
\]
for a complex line bundle \( \pi_i : L_i \rightarrow V_i \). This implies \( U^*_{E_i} = U^*_{E_i} \pi^*(|L_i|) \). Note that \( L_{ij} = L_i \otimes L_j^* \).

Any twisted \( K \)-class \( a \) in \( K^0_{cv}(E, \pi^*(a)) \) is given by a local \( K \)-class \( a_i \) with compact vertical support such that \( a_j = a_i \pi^i_j([L_{ij}]) \), then
\[
a_i \pi^i_j([L_{ij}]) = a_j \pi^*[([L_{ij}])]
\]
in \( K^0_{cv}(E|_{V_i}) \). This defines the homomorphism \( \phi_s \), which is obviously an isomorphism sending \( U_E \) to \( U^*_{E} \).

2. Choose a good open cover \( \{ V_i \} \) of \( Y \). By definition, the Thom class \( U_E \) is defined by \( \{ U^*_{E_i} \} \) with
\[
U^*_{E_i} = U^*_{E_i} \pi^i_j([L_{ij}]).
\]
Then \( \{ f^{-1}(V_i) \} \) is an open cover of \( X \), and \( (f^*E)|_{f^{-1}(V_i)} = f^*E_i \) is trivialized with the canonical \( \text{Spin}^c \) structure \( f^*s_i \), thus
\[
U^*_{f^*E_i} = f^*U^*_{E_i}.
\]
This gives \( U_{f^*E} = f^*(U_E) \).

3. The proof is similar to the proof of (2).

4. From (\cite{CW1}), we know that the Thom isomorphism and the push-forward map in twisted \( K \)-theory are both homomorphisms of \( K^0(X, \alpha) \)-modules. There exists an oriented real vector bundle \( F \) of even rank such that
\[
E \oplus F = X \times \mathbb{R}^{2m}
\]
for some \( m \in \mathbb{N} \). Thus, we have
\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & X \times \mathbb{R}^{2m} \\
\downarrow \pi & & \downarrow p \\
X & & \\
\end{array}
\]
From the construction of the push-forward map in (\cite{CW1}), we see that
\[
\pi_1(U_E) = p \circ i_1(U_E) = p(U_E \oplus F) = 1.
\]
As the Thom isomorphism and the push-forward map in twisted \( K \)-theory are both homomorphisms of \( K^0(X, \alpha) \)-modules, we get \( \pi_1(\pi^*(a) \cdot U_E) = a \).

Note that the Thom isomorphism is inverse to the push-forward map \( \pi_1 \), hence, the Thom isomorphism in twisted \( K \)-theory
\[
K^0(X, \alpha + o_E) \cong K^0(E, \pi^*(\alpha))
\]
is given by \( a \mapsto \pi^*(a) \cdot U_E. \)

\[\Box\]

5.2. Twisted Riemann-Roch. By an application of the Thom class and Thom isomorphism in twisted \( K \)-theory, we will now give a direct proof of a special case of the Riemann-Roch theorem for twisted \( K \)-theory. With some notational changes, the argument can be applied to establish the general Riemann-Roch theorem in twisted \( K \)-theory. Denote by \( o_X \) and \( o_Y \) the orientation twistings associated to the tangent bundles \( \pi_X : TX \to X \) and \( \pi_Y : TY \to Y \) respectively.

**Theorem 5.3.** Given a smooth map \( f : X \to Y \) between oriented manifolds, assume that \( \dim Y - \dim X = \) 0 mod 2. Then the Riemann-Roch formula is given by

\[
Ch_{\omega_2(Y)}(f_1)(\hat{A}(Y)) = f_*^H(Ch_{\omega_2(X)}(a)\hat{A}(X)).
\]

for any \( a \in K^0(X, o_X) \). Here \( \hat{A}(X) \) and \( \hat{A}(Y) \) are the A-hat classes of \( X \) and \( Y \) respectively.

**Proof.** For simplicity, assume that both \( X \) and \( Y \) are of even dimension, say \( 2m \) and \( 2n \) respectively, equipped with a Riemannian metric. We will consider Chern character defects in each of the following three squares

\[
\begin{align*}
\begin{array}{c|c|c|c|c}
& Ch_{\omega_2(Y)}(f_1)(\hat{A}(Y)) & f_*^H(Ch_{\omega_2(X)}(a)\hat{A}(X)) & \Phi^K_{TY} & \Phi^K_{TX} \\
\hline
K^0(X, o_X) & K^0(TX) & K^0(TY) & \Phi^K_{TY}^{-1} & \Phi^K_{TX} \\
\hline
H^{ev}(X) & H^{ev}(TX) & H^{ev}(TY) & \Phi^H_{TY}^{-1} & \Phi^H_{TX} \\
\end{array}
\end{align*}
\]

where \( \Phi^K_{TX} \) and \( \Phi^K_{TY} \) are the Thom isomorphisms in twisted \( K \)-theory for \( TX \) and \( TY \), \( \Phi^H_{TX} \) and \( \Phi^H_{TY} \) are the cohomology Thom isomorphisms for \( TX \) and \( TY \). Then we have

1. The push-forward map in twisted \( K \)-theory as established in [CWT]

\[
f^K_1 : K^0(X, o_X) \to K^0(Y, o_Y)
\]

agrees with

\[
(\Phi^K_{TY})^{-1} \circ (df_1)^K \circ \Phi^K_{TX}.
\]

2. The push-forward map in cohomology theory \( f^H_* : H^{ev}(X) \to H^{ev}(Y) \) is given by

\[
f^H_* = (\Phi^H_{TY})^{-1} \circ (df_1)^H \circ \Phi^H_{TX}.
\]

Denote by \( U^H_{TX} \) and \( U^H_{TY} \) the cohomological Thom classes for \( TX \) and \( TY \). Then under the pull-back of the zero section, \( 0^*_X(U^H_{TX}) = e(TX) \) and \( 0^*_Y(U^H_{TY}) = e(TY) \) are the Euler classes for \( TX \) and \( TY \) respectively.

Let the Pontrjagin classes of \( \pi_X : TX \to X \) be symmetric polynomials in \( x_1^2, \cdots, x_m^2 \), then

\[
\hat{A}(X) = \prod_{k=1}^m \frac{x_k/2}{\sinh(x_k/2)}.
\]

The Chern character defect for the left square in (5.1) is given by

\[
Ch(\Phi^K_{TX}(a)) = \Phi^H_{TX}(Ch_{\omega_2(X)}(a)\hat{A}^{-1}(X))
\]

for any \( a \in K^0(X, o_X) \). Here \( \hat{A}(X) \) is the A-hat class of \( TX \).
To prove (5.2), note that

\[(\Phi^H_T X)^{-1} Ch(\Phi^K_{T X}(a))\]

\[= (\Phi^H_T X)^{-1} (Ch(\pi_X^*(a) \cdot U_{T X}))\]

Apply Prop. (4.2)

\[= (\Phi^H_T X)^{-1} (Ch(\pi_X^*(a) \cdot Ch_{\pi_X^*}(U_{T X})))\]

Note that \((\Phi^H_T X)^{-1} = (\pi_X)_*\).

\[= Ch_{w^2(X)}(a)(\Phi^H_T X)^{-1}(Ch_{\pi_X^*}(U_{T X}))\]

By the projection formula.

So the Chern character defect for the square

\[
\begin{array}{ccc}
K^0(X, o_X) & \Phi^K_{T X} & \cong K^0(T X) \\
Ch_{w^2(X)} & \cong & Ch \\
H^{ev}(X) & \Phi^H_{T X} & \cong H^{ev}(T X)
\end{array}
\]

is given by

\[D(X) = (\Phi^H_T X)^{-1} (Ch_{\pi_X^*}(U_{T X})) \in H^{ev}(X).\]

From the cohomology Thom isomorphism, we have

\[0^*_X \circ \Phi^H_{T X}(D(X)) = D(X)e(T X),\]

under the pull-back of the zero section \(0_X\) of the tangent bundle \(T X\).

Therefore, we have

\[D(X) = 0^*_X (Ch_{\pi_X^*}(U_{T X})).\]

By the construction of the Thom class \(U_{T X}\), under the pull-back of the zero section \(0_X\) of \(T X\), \(0^*_X(U_{T X})\) is a twisted K-class in \(K^0(X, o_X)\) and

\[0^*_X Ch_{\pi_X^*}(U_{T X})\]

\[= Ch_{w^2(X)}(0^*_X(U_{T X}))\]

\[= \prod_{k=1}^{m} (e^{x_k/2} - e^{-x_k/2}).\]

Thus, (5.2) follows from

\[D(X) = \prod_{k=1}^{m} \frac{(e^{x_k/2} - e^{-x_k/2})}{x_k} = A^{-1}(X).\]

This implies that the following diagram commutes

\[
\begin{array}{ccc}
K^0(X, o_X) & \Phi^K_{T X} & \cong K^0(T X) \\
Ch_{w^2(X)}(-) \cdot A(X) & \cong & Ch(-) \cdot A^2(X) \\
H^{ev}(X) & \Phi^H_{T X} & \cong H^{ev}(T X)
\end{array}
\]

(5.3)
Similarly, the Chern character defect in
\[
K^0(\mathcal{O}_Y) \xrightarrow{\Phi_{TY}^K} K^0(TY)
\]
is given by
\[
Ch(\Phi_{TY}^K(a)) = \Phi_{TY}^H(Ch_{\tilde{w}_2(Y)}(a)\hat{A}^{-1}(Y))
\]
for any \(a \in K^0(\mathcal{O}_Y)\). This implies that the Chern character defect for the right square in (5.1) is given by
\[
Ch_{\tilde{w}_2(Y)}((\Phi_{TY}^K)^{-1}(c)) \cdot \hat{A}^{-1}(Y) = (\Phi_{TY}^H)^{-1}(Ch(c))
\]
for any \(c \in K^0_c(TY)\). Hence, we have the following commutative diagram
\[
\begin{array}{ccc}
K^0_c(TY) & \xrightarrow{(\Phi_{TY}^K)^{-1}} & K^0(\mathcal{O}_Y) \\
Ch(-) \cdot \pi^*_X \hat{A}^2(Y) & \downarrow & Ch_{\tilde{w}_2(Y)}(-) \cdot \hat{A}(Y) \\
H^c(TY) & \xrightarrow{(\Phi_{TY}^H)^{-1}} & H^c(Y).
\end{array}
\]

The Chern character for the middle square in (5.1) follows from the Riemann-Roch theorem in ordinary \(K\)-theory for the \(K\)-oriented map \(df : TX \to TY\) with the orientation given by canonical \(\text{Spin}^c\) manifolds \(TX\) and \(TY\). Note that the Todd classes of \(\text{Spin}^c\) manifolds of \(TX\) and \(TY\) are given by \(\pi^*_X(\hat{A}^2(X))\) and \(\pi^*_Y(\hat{A}^2(Y))\) respectively. This is due to two facts, that \(T(TX) \cong \pi^*_X(TX \otimes \mathbb{C})\) and that \(Td(TX \otimes \mathbb{C}) = \hat{A}^2(X)\). So we have
\[
Ch((df)_!^X(a)) \cdot \pi^*_X(\hat{A}^2(Y)) = (df)_!^H((Ch(a)\pi^*_X(\hat{A}^2(X))))
\]
for any \(a \in K^0_c(TX)\). Hence, the following diagram commutes
\[
\begin{array}{ccc}
K^0_c(TX) & \xrightarrow{(df)_!^X} & K^0_c(TY) \\
Ch(-) \cdot \pi^*_X \hat{A}^2(X) & \downarrow & Ch(-) \cdot \pi^*_Y \hat{A}^2(Y) \\
H^c_c(TX) & \xrightarrow{(df)_!^H} & H^c_c(TY).
\end{array}
\]

Putting (5.3), (5.4) and (5.7) together, we get the following commutative diagram
\[
\begin{array}{ccc}
K^0(\mathcal{O}_X) & \xrightarrow{f^K} & K^0(\mathcal{O}_Y) \\
Ch_{\tilde{w}_2(X)}(-) \cdot \hat{A}(X) & \downarrow & Ch_{\tilde{w}_2(Y)}(-) \cdot \hat{A}(Y) \\
H^c_c(X) & \xrightarrow{f^H} & H^c_c(Y)
\end{array}
\]
which leads to
\[
Ch_{\tilde{w}_2(Y)}(f^K_!(a))\hat{A}(Y) = f^H_!(Ch_{\tilde{w}_2(X)}(a)\hat{A}(X))
\]
for any \(a \in K^0(\mathcal{O}_X)\). This completes the proof of the Riemann-Roch theorem in twisted \(K\)-theory. \(\square\)
With some notational changes, the above argument can be applied to establish the general Riemann-Roch theorem in twisted $K$-theory. Let $f : X \to Y$ a smooth map between oriented manifolds with $\dim Y - \dim X = 0 \mod 2$. Let $\tilde{\alpha} = (\mathcal{G}_\alpha, \theta, \omega)$ be a differential twisting which lifts $\alpha : Y \to K(\mathbb{Z}, 3)$, $f^*(\tilde{\alpha})$ is the pull-back differential twisting which lifts $\alpha \circ f : X \to K(\mathbb{Z}, 3)$. Then we have the following Riemann-Roch formula

$$\text{Ch}_{\tilde{\alpha}} (f^!K(a)) \hat{A}(Y) = f^!H (\text{Ch}_{f^*\tilde{\alpha} + \hat{w}_2(Y)} + f^*\hat{w}_2(Y))(a) \hat{A}(X) \tag{5.8}$$

for any $a \in K^0(X, \alpha \circ f + o_X + f^*(o_Y))$. In particular, we have the following Riemann-Roch formula for a trivial twisting $\alpha : Y \to K(\mathbb{Z}, 3)$

$$\text{Ch}(f^!K(a)) \hat{A}(Y) = f^!H (\text{Ch}_{\hat{w}_2(Y)} + f^*\hat{w}_2(Y))(a) \hat{A}(X) \tag{5.9}$$

for any $a \in K^0(X, o_X + f^*(o_Y))$.

When $f$ is K-oriented, and equipped with a $\text{Spin}^c$ structure whose determinant bundle is $L$, there is a canonical isomorphism

$$\Psi : K^0(X) \cong K^0(X, o_X + f^*(o_Y))$$

such that $\text{Ch}_{\hat{w}_2(Y)} + f^*\hat{w}_2(Y)(\Psi(a)) = e^{c_1(L)/2} \text{Ch}(a)$ for any $a \in K^0(X)$. Then the Riemann-Roch formula (5.9) agrees with the Riemann-Roch formula for K-oriented maps as established in [AH].

### 6. The twisted index formula

In this Section, we establish the index pairing for a closed smooth manifold with a twisting $\alpha$

$$K^{\text{ev/odd}}(X, \alpha) \times K^{\text{an/odd}}(X, \alpha) \to \mathbb{Z}$$

in terms of the local index formula for twisted geometric cycles.

**Theorem 6.1.** Let $X$ be a smooth closed manifold with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$. The index pairing

$$K^0(X, \alpha) \times K_0(X, \alpha) \to \mathbb{Z}$$

is given by

$$\langle \xi, (M, \iota, \nu, \eta, [E]) \rangle = \int_M \text{Ch}_{\hat{w}_2(M)}(\eta_*(\iota^*\xi \otimes E)) \hat{A}(M)$$

where $\xi \in K^0(X, \alpha)$, and the geometric cycle $(M, \iota, \nu, \eta, [E])$ defines a twisted $K$-homology class on $(X, \alpha)$. Here

$$\eta_* : K^*(M, \iota^*\alpha) \cong K^*(M, o_M)$$

is an isomorphism, and $\text{Ch}_{\hat{w}_2(M)}$ is the Chern character on $K^0(M, o_M)$.

**Proof.** Recall that the index pairing $K^0(X, \alpha) \times K_0(X, \alpha) \to \mathbb{Z}$ can be defined by the internal Kasparov product (Cf. [Kas3] and [ConSka])

$$KK(\mathbb{C}, C(X, \mathcal{P}_\alpha(K))) \times KK(C(X, \mathcal{P}_\alpha(K)), \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z},$$

and is functorial in the sense that if $f : Y \to X$ is a continuous map and $Y$ is equipped with a twisting $\alpha : Y \to \mathbb{Z}$ then

$$\langle f^*b, a \rangle = \langle b, f_*(a) \rangle$$

for any $a \in K_0(Y, f^*\alpha)$ and $b \in K^0(X, \alpha)$. 

Note that under the assembly map, the geometric cycle \((M, \iota, \nu, \eta, [E])\) is mapped to 
\(\iota_\ast \circ \eta_\ast ([M] \cap [E])\), for \(\xi \in K^0(X, \alpha)\). Hence, we have
\[
\langle \xi, (M, \iota, \nu, \eta, [E]) \rangle
= \langle \xi, \iota_\ast \circ \eta_\ast ([M] \cap [E]) \rangle
= \langle \iota^\ast \xi, \eta_\ast ([M] \cap [E]) \rangle
= \langle \eta_\ast (\iota^\ast \xi \otimes E), [M] \rangle.
\]
Here \(\eta_\ast (\iota^\ast \xi \otimes E) \in K^0(M, o_M)\) and \([M]\) is the fundamental class in \(K^\text{an}_\text{ev}(M, o_M)\) which is Poincaré dual to the unit element \(\underline{0}\) in \(K^0(M)\). The index pairing between \(K^0(M, o_M) \times K^\text{an}_\text{ev}(M, o_M)\) can be written as
\[
K^0(M, o_M) \times K^\text{an}_\text{ev}(M, o_M) \to K^0(M, o_M) \times K^0(M) \to K^0(M, o_M) \to \mathbb{Z}
\]
where the first map is given by the Poincaré duality \(K^\text{an}_\text{ev}(M, o_M) \cong K^0(M)\), the middle map is the action of \(K^0(M)\) on \(K^0(M, o_M)\), and the last map is the push-forward map of \(\epsilon : M \to pt\). Therefore, we have
\[
\langle \eta_\ast (\iota^\ast \xi \otimes E), [M] \rangle
= \epsilon^K (\eta_\ast (\iota^\ast \xi \otimes E) \otimes \underline{0})
= \epsilon^K (\eta_\ast (\iota^\ast \xi \otimes E)).
\]
By twisted Riemann-Roch (Theorem 5.3),
\[
\epsilon_1 (\eta_\ast (\iota^\ast \xi \otimes E))
= \epsilon^H (Ch_{\hat{w}_2}(M) (\eta_\ast (\iota^\ast \xi \otimes E)) \hat{A}(M))
= \int_M Ch_{\hat{w}_2}(M) (\eta_\ast (\iota^\ast \xi \otimes E)) \hat{A}(M).
\]
This completes the proof of the twisted index formula. \(\square\)

Note that \(\epsilon : M \to pt\) can be written as \(\iota \circ \epsilon_X : M \to X \to pt\). Applying the Riemann-Roch Theorem 5.3 we can write the above index pairing as
\[
< (M, \iota, \nu, \eta, [E]), \xi >
= \int_M Ch_{\hat{w}_2}(M) (\eta_\ast (\iota^\ast \xi \otimes E)) \hat{A}(M)
= \int_X Ch_{\hat{w}_2}(X) (\iota_! (E) \otimes \xi) \hat{A}(X)
\]
where \(\iota_! : K^0(M) \to K^0(X, -\alpha + o_X)\) is the push-forward map in twisted \(K\)-theory,
\[
K^0(X, \alpha) \times K^0(X, -\alpha + o_X) \longrightarrow K^0(X, o_X)
\]
is the multiplication map (27), and
\[
Ch_{\hat{w}_2}(X) : K^0(X, o_X) \longrightarrow H^\text{ev}(X)
\]
is the twisted Chern character (which agrees with the relative Chern character under the identification \(K^0(X, o_X) \cong K^0(X, W_3(X))\), the \(K\)-theory of Clifford modules on \(X\)).
7. Mathematical definition of D-branes and D-brane charges

Here we give a mathematical interpretation of D-branes in Type II string theory using the twisted geometric cycles and use the index theorem in the previous Section to compute charges of D-branes. In Type II superstring theory on a manifold $X$, a string worldsheet is an oriented Riemann surface $\Sigma$, mapped into $X$ with $\partial \Sigma$ mapped to an oriented submanifold $M$ (called a D-brane world-volume, a source of the Ramond-Ramond flux). The theory also has a Neveu-Schwarz $B$-field classified by a characteristic class $[\alpha] \in H^3(X, \mathbb{Z})$.

In physics, the D-brane world volume $M$ carries a gauge field on a complex vector bundle (called the Chan-Paton bundle), so a D-brane is given by a submanifold $M$ of $X$ with a complex bundle $E$ and a connection $\nabla^E$. This data actually defines a differential $K$-class $[(E, \nabla^E)]$ in differential $K$-theory $\tilde{K}(M)$.

When the $B$-field is topologically trivial, that is $[\alpha] = 0$, D-brane charge takes values in ordinary $K$-theory $K^0(X)$ or $K^1(X)$ for Type IIB or Type IIA string theory (as explained in [MM][Wit]). For a D-brane $M$ to define a class in the $K$-theory of $X$, its normal bundle $\nu_M$ must be endowed with a $\text{Spin}^c$ structure. Equivalently, the embedding $\iota : M \hookrightarrow X$ is $K$-oriented so that the push-forward map in $K$-theory ([AH])

$$\iota^*_K : K^0(M) \rightarrow K^{\text{ev/odd}}(X)$$

is well-defined, (it takes values in even or odd $K$-groups depending on the dimension of $M$). So the D-brane charge of $(\iota : M \rightarrow X, E)$ is

$$\iota^*_K([E]) \in K^{\text{ev/odd}}(X).$$

It was proposed in [MM] that the cohomological Ramond-Ramond charge of the D-brane is given by

$$Q_{RR}(\iota : M \rightarrow X, E) = \text{ch}(f^*_K(E))\sqrt{\hat{A}(X)}$$

when $X$ is a Spin manifold. A natural $K$-theoretic interpretation follows from the fact that the modified Chern character isomorphism

$$K^{\text{ev/odd}}_Q(X) \rightarrow H^{\text{ev/odd}}(X, \mathbb{Q})$$

given by mapping $a \mapsto \text{ch}(a)\sqrt{\hat{A}(X)}$ is an isometry with the natural bilinear parings on $K^*_Q(X) = K^*_R(X) \otimes \mathbb{Q}$ and $H^{\text{ev/odd}}(X, \mathbb{Q})$. Here the pairing on $K(X)$ is given by the index of the Dirac operator

$$(a, b)_K = \text{Index}(\mathcal{D}_{a\otimes b}) = \int_X \text{ch}(a)\text{ch}(b)\hat{A}(X) = (\text{ch}(a)\sqrt{\hat{A}(X)}, (\text{ch}(b)\sqrt{\hat{A}(X)}))_H.$$

When the $B$-field is not topologically trivial, that is $[\alpha] \neq 0$, then $[\alpha]$ defines a complex line bundle over the loop space $LX$, or a stable isomorphism class of bundle gerbes over $X$. Then in order to have a well-defined worldsheet path integral, Freed and Witten in [FreWit] showed that

$$\iota^*[\alpha] + W_3(\nu_M) = 0.$$
When \( \iota^* [\alpha] \neq 0 \), that means \( \iota \) is not \( K \)-oriented, then the push-forward map in \( K \)-theory ([AH])

\[
i^K : K^0(M) \to K^*(X)
\]
is not well-defined. Witten explained in [Wit] that D-brane charges should take values in a twisted form of \( K \)-theory, as supported further by evidence in [BouMat] and [Kap].

In [Wa], the mathematical meaning of (7.1) was discovered using the notion of \( \alpha \)-twisted \( Spin^c \) manifolds for a continuous map

\[
\alpha : X \to K(Z, 3)
\]
representing \([\alpha] \in H^3(X, \mathbb{Z})\. When \( X \) is \( Spin^c \), the datum to describe a D-brane is exactly a geometric cycle for the twisted \( K \)-homology \( K^\text{geo/odd}(X, \alpha) \). By Poincaré duality, we have

\[
K^\text{geo/odd}(X, \alpha) \cong K^0(X, \alpha + o_X)
\]
with the orientation twisting \( o_X : X \to K(Z, 3) \) trivialized by a choice of a \( Spin^c \) structure. Hence,

\[
K^0(X, \alpha + o_X) \cong K^0(X, \alpha).
\]

For a general manifold \( X \), a submanifold \( \iota : M \to X \) with

\[
\iota^* ([\alpha]) + W_3(\nu_M) = 0,
\]
then there is a homotopy commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\nu_M} & BSO \\
\downarrow \iota & & \downarrow W_3 \\
X & \xrightarrow{\alpha} & K(Z, 3)
\end{array}
\]

where \( \nu_M \) also denotes a classifying map of the normal bundle, or a classifying map of the bundle \( TM \oplus \iota^* TX \). This motivates the following definition (see also [CW2]).

**Definition 7.1.** Given a smooth manifold \( X \) with a twisting \( \alpha : X \to K(Z, 3) \), a \( B \)-field of \((X, \alpha)\) is a differential twisting lifting \( \tilde{\alpha} = (G_\alpha, \theta, \omega) \)

which is a (lifting, or local) bundle gerbe \( G_\alpha \) with a connection \( \theta \) and a curving \( \omega \). The field strength of the \( B \)-field \((G_\alpha, \theta, \omega)\) is given by the curvature \( H \) of \( \tilde{\alpha} \).

A Type II (generalized) D-brane in \((X, \alpha)\) is a complex vector bundle \( E \) with a connection \( \nabla^E \) over a twisted \( Spin^c \) manifold \( M \). The twisted \( Spin^c \) structure on \( M \) is given by the following homotopy commutative diagram together with a choice of a homotopy \( \eta \)

\[
\begin{array}{ccc}
M & \xrightarrow{\nu_E} & BSO \\
\downarrow \iota & & \downarrow W_3 \\
X & \xrightarrow{\alpha} & K(Z, 3)
\end{array}
\]

where \( \nu_E \) is the classifying map of \( TM \oplus \iota^* TX \).

**Remark 7.2.** The twisted \( Spin^c \) manifold \( M \) in Definition 7.1 is the D-brane world volume in Type II string theory. The twisted \( Spin^c \) structure given in (7.2) implies that D-brane world volume \( M \subset X \) in Type II string theory satisfies the Freed-Witten anomaly cancellation condition

\[
\iota^* [\alpha] + W_3(\nu_M) = 0.
\]
In particular, if the $B$-field of $(X, \alpha)$ is topologically trivial, then the normal bundle of $M \subset X$ is equipped with a $\text{Spin}^c$ structure given by $T^2$.

Given a Type II D-brane $(M, \iota, \nu, \eta, E, \nabla^E)$, the homotopy $\eta$ induces an isomorphism

$$\eta_* : K^0(M) \to K^0(M, \iota^* \alpha + o).$$

Here $o$ denotes the orientation twisting of the bundle $TM \oplus \iota^* TX$. Note that $\iota^K : K^0(M, \iota^* \alpha + o) \to K^*_{\text{ev/odd}}(X, \alpha)$

is the pushforward map (2.9) in twisted $K$-theory. Hence we have a canonical element in

$$\iota^K(\eta_*([E])),$$

called the D-brane charge of $(M, \iota, \nu, \eta, E)$. We remark that a Type II D-brane

$$(M, \iota, \nu, \eta, E, \nabla^E)$$

defines an element in differential twisted $K$-theory $\hat{K}^*_{\text{ev/odd}}(X, \hat{\alpha})$.

From (7.2), we know that $M$ is an $(\alpha + o_X)$-twisted $\text{Spin}^c$ manifold as we have the following homotopy commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BSO} \\
\downarrow{\iota} & & \downarrow{w_3} \\
X & \xrightarrow{\alpha + o_X} & K(\mathbb{Z}, 3)
\end{array}
\]

where $\nu$ is the classifying map of the stable normal bundle of $M$. Together with the following proposition, we conclude that the Type II D-brane charges, in the present of a $B$-field

$$\hat{\alpha} = (G_\alpha, \theta, \omega)$$

are classified by twisted $K$-theory $K^0(X, \alpha)$.

**Proposition 7.3.** Given a twisting $\alpha : X \to K(\mathbb{Z}, 3)$ on a smooth manifold $X$, every twisted $K$-class in $K^*_{\text{ev/odd}}(X, \alpha)$ is represented by a geometric cycle supported on an $(\alpha + o_X)$-twisted closed $\text{Spin}^c$-manifold $M$ and an ordinary $K$-class $[E] \in K^0(M)$.

For completeness, we also give a definition of Type I D-branes (Cf. [MMS], [RSV] and Section 8 in [Wa]).

**Definition 7.4.** Given a smooth manifold $X$ with a KO-twisting $\alpha : X \to K(\mathbb{Z}_2, 2)$, a Type I (generalized) D-brane in $(X, \alpha)$ is a real vector bundle $E$ with a connection $\nabla^E$ over a twisted $\text{Spin}$ manifold $M$. The twisted $\text{Spin}$ structure on $M$ is given by the following homotopy commutative diagram together with a choice of a homotopy $\eta$

\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BSO} \\
\downarrow{\iota} & & \downarrow{w_2} \\
X & \xrightarrow{\alpha} & K(\mathbb{Z}_2, 2)
\end{array}
\]

where $w_2$ is the classifying map of the principal $K(\mathbb{Z}_2, 1)$-bundle $\text{BSpin} \to \text{BSO}$ associated to the second Stiefel-Whitney class, $\eta$ is a homotopy between $w_2 \circ \nu$ and $\alpha \circ \iota$. Here $\nu$ is the classifying map of $TM \oplus \iota^* TX$. 


Remark 7.5. A Type I D-brane in \((X, \alpha)\) has its support on a manifold \(M\) if and only if there is a differentiable map \(\iota : M \to X\) such that
\[
i^*([\alpha]) + w_2(\nu) = 0.
\]
Here \(\nu\) denotes the bundle \(TM \oplus \iota^*TX\).

Given a Type I D-brane in \((X, \alpha)\), the push-forward map in twisted KO-theory
\[
KO^*(M) \xrightarrow{\eta_*} KO^*(M, \alpha \circ \iota + \nu) \xrightarrow{i^K} KO^*(X, \alpha)
\]
defines a canonical element in \(KO^*(X, \alpha)\). Every class in \(KO^*(X, \alpha)\) can be realized by a Type I (generalized) D-brane in \((X, \alpha)\). Hence, we conclude that the Type I D-brane charges are classified by twisted KO-theory \(KO^{ev/odd}(X, \alpha)\).

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