The Double Pendulum of Variable Mass: Numerical Study for different cases.

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The purpose of this project is to theoretically and numerically analyze the double pendulum with variable mass system, where different cases are presented: The first one corresponds to the classical analysis of the double pendulum, the second case corresponds to the double pendulum with variable mass, considering the upper mass as the variable one. The third case concerns the double pendulum with variable mass, where the lower mass varies. Finally, the fourth case corresponds to the double pendulum with variable mass, considering both of the masses as variable. For every case it is verified that $m_s >> m_i$ which guarantees the general double pendulum behavior. We obtain the coupled equations of motion for every case by using the Lagrange formulation. The system equations are developed by means of simulations in order to solve the equations of motion, applying the fourth order Runge Kutta (RK4) numerical method. The outcomes for the numerical solution and the observed effects due to the variable mass are presented for every case.

1. Introduction

The experimental study of variable mass systems in applied sciences has not been receiving much of attention, maybe because of its practical complexity and the difficulty for directly obtaining the implied variables in them. These systems encompass a wide study field because of its great applicability and the vast physics phenomena they are indirectly involved with. A great example are the spacecrafts used to put satellites into orbit or often used for space exploration. The articles or experiments found in literature related to the analysis of variable mass systems are very rare, where these can be found among others: rocket modeling, Atwood machines with variable mass deposits, oscillators and torsion pendulums with variable mass. Such systems are complicated to study in an experimental manner. In this paper we analyze various variable mass systems as a consequence of studying the double pendulum system.

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Interesting applications of the PDMV movement, as well as its physical and mathematical background can be found in the corn or granular material silo unloading. This movement can be used for a non-uniform discharge manner. Another application of the PDMV can be seen in earthquakes simulations; the first seismographs were built in base of pendular motion [1]. PDMV offers deadening factors that can be used in buildings, for a better structure and balance. Determinist chaos systems are another application of the PDMV, the forced pendulum and the damped pendulum, for example, both are used to encrypt information [2]. The PDMV is another determinist chaos system that can be used for encrypting information as well, and thanks to the variable mass factors, this encryption can be improved.

Our principal interest lies upon exploring the behavior of the variable mass system by applying Classical Lagrange Mechanics and through using the Lagrangian formulations, particularly for each case where the equations of motion are solved by simulating the numerical method.

2. Theory
The system to analyze consists of the double pendulum for four particular cases, first of which is formed by two simple pendulums with lengths l1 and l2 (l1 ≠ l2) from where they hang two spherical container masses: m1 and m2, being m1 the upper mass and m2 the lower mass. At a certain time, the strings that are inextensible, form angles θ1 and θ2 with respect to the vertical axis [3, 4, 5, 6].

For the variable mass cases with respect to time, we define a new linear function that describes the behavior of the varying mass (1),

\[ m(t) = m_c + m_g(1 + \lambda t), \]

where \( m(t) \) is the variable mass, \( m_c \) is the container mass, \( m_g \) is the granular mass, \( \lambda \) is the velocity in which the granular mass exits the container.

Later on, we obtain the kinetic energy (T) and the gravitational potential energy (V) of the system, considering the suspension point of the upper pendulum (\( m_1 \)) as the origin reference of the coordinate system. Then the Lagrangian is obtained \( (L = T - V) \) and by employing the Euler-Lagrange equation \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \) the equations of motion are calculated [7, 8, 9, 10].

2.1. Case 1. Double Pendulum without variable mass.
The equations of motion are described as follows:

\[ \ddot{\theta}_1 + \frac{g}{l_1} \sin \theta_1 = -\frac{l_2}{l_1} \frac{m_2}{m_1 + m_2} \left( \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \right), \]

Figure 1. Double Pendulum system.
\[
\ddot{\theta}_2 + \frac{g}{l_2} \sin \theta_2 = \frac{l_1}{l_2} \dot{\theta}_1 \cos(\theta_2 - \theta_1) - \dot{\theta}_2 \frac{l_1^2}{l_2^2} \sin(\theta_2 - \theta_1),
\]

(3)
equations that represent a coupled motion between \(\theta_1\) and \(\theta_2\).

2.2. Case 2. Double Pendulum where the upper pendulum varies its mass.

Here is suggested that the upper pendulum ejects mass, knowing that \((m_1 >> m_2)\), which guarantees the double pendulum behaviour, the equations of motion are [6]:

\[
\ddot{\theta}_1 + \frac{m_1(t)}{m_2 + m_1(t)} \dot{\theta}_1 + \frac{g}{l_1} \sin \theta_1 = -\frac{l_2}{l_1} \frac{m_2}{m_1 + m_2(t)} (\dot{\theta}_2 \cos(\theta_2 - \theta_1) + \dot{\theta}_2^2 \sin(\theta_2 - \theta_1)),
\]

(4)
\[
\ddot{\theta}_2 + \frac{g}{l_2} \sin \theta_2 = \frac{l_1}{l_2} \dot{\theta}_1 \cos(\theta_2 - \theta_1) - \frac{l_1}{l_2} \dot{\theta}_1^2 \sin(\theta_2 - \theta_1),
\]

(5)
As we compare equations (2) and (4) it is observed that a new term is provided by the mass variation, complementing the angular velocity variable \(\dot{\theta}_1\). This term represents a damping term when comparing to the mathematical pendulum [4, 5, 7], which is given by \(b_{i,j,k}\), where the sub-index \((i, j, k)\) corresponds to: case number, angular velocity of associated pendulum, equation of motion of the given case:

\[
b_{2,1,1} = -\frac{m_1(t)}{m_2 + m_1(t)}
\]

(6)

2.3. Case 3. Double Pendulum where the lower pendulum varies its mass.

Now the lower pendulum varies its mass, and the equations of motion are given by:

\[
\ddot{\theta}_1 + \frac{m_1(t)}{m_1 + m_2(t)} \dot{\theta}_1 + \frac{g}{l_1} \sin \theta_1 = -\frac{l_2}{l_1} \frac{m_2(t)}{m_1 + m_2(t)} \left[ \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \right] + \frac{m_2(t)}{m_1 + m_2(t)} \dot{\theta}_1 \cos(\theta_2 - \theta_1),
\]

(7)
\[
\ddot{\theta}_2 + \frac{m_1(t)}{m_2(t)} \dot{\theta}_2 + \frac{g}{l_2} \sin \theta_2 = -\frac{l_1}{l_2} \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{l_1}{l_2} \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - \frac{l_1 m_2(t)}{l_2 m_2(t)} \dot{\theta}_1 \cos(\theta_2 - \theta_1),
\]

(8)
If we compare equations (2) and (7) we can identify two additional terms; the upper pendulum suffers a damping due to the mass variation of the lower pendulum and the lower pendulum also suffers a damping due to the variation of its own mass, where the damping factors are given as follows:

\[
b_{3,1,1} = \frac{m_2(t)}{m_1 + m_2(t)}
\]

(9)
\[
b_{3,2,1} = \frac{l_2}{l_1} \frac{m_1(t)}{m_1(t) + m_2(t)} \cos(\theta_2 - \theta_1)
\]

If we also compare equations (3) and (8) we can observe two new terms that seem to represent a damping motion given by:

\[
b_{3,2,2} = \frac{m_2(t)}{m_2(t)}
\]

(10)
\[
b_{3,1,2} = \frac{l_1 m_2(t)}{l_2 m_2(t)} \cos(\theta_2 - \theta_1)
\]

2.4. Case 4. Double Pendulum with variable mass for both individual pendulums.

Here both of the pendulums are varying their mass and the equations of motion are given as follows:

\[
\ddot{\theta}_1 + \frac{m_1(t) + m_2(t)}{m_1(t) + m_2(t)} \dot{\theta}_1 + \frac{g}{l_1} \sin \theta_1 = -\frac{l_2}{l_1} \frac{m_2(t)}{m_1(t) + m_2(t)} \left[ \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \right] + \frac{m_1(t)}{m_1(t) + m_2(t)} \dot{\theta}_1 \cos(\theta_2 - \theta_1),
\]

(11)
\[
\ddot{\theta}_2 + \frac{m_1(t)}{m_2(t)} \dot{\theta}_2 + \frac{g}{l_2} \sin \theta_2 = -\frac{l_1}{l_2} \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{l_1}{l_2} \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - \frac{l_1 m_2(t)}{l_2 m_2(t)} \dot{\theta}_1 \cos(\theta_2 - \theta_1),
\]

(12)
where the damping factors are given by:
\[ b_{4,1,1} = \frac{m_3(t) + m_2(t)}{m_1(t) + m_2(t)} \]
\[ b_{4,2,1} = \frac{l_2}{l_1 m_1(t) + m_2(t)} \cos(\theta_2 - \theta_1) \]
and also:
\[ b_{4,2,2} = \frac{m_2(t)}{m_2(t)} \]
\[ b_{4,1,2} = \frac{l_2 m_2(t)}{l_1 m_2(t)} \cos(\theta_2 - \theta_1) \]

The obtained equations for every case are non-linear ordinary differential equations.

3. Simulation

In order to solve the equations of motion two paths can be followed: an analytic solution can be found by integrating the equations of motion, or a simulation based on a numerical method can be implemented. We followed the second path using the fourth order Runge-Kutta numerical method, with the parameters seen at Table 1.

### Table 1. Simulation parameters.

| Case   | \( l_1 \) | \( l_2 \) | \( m_e \) | \( m_g \) | \( m_1 \) | \( m_2 \) |
|--------|-----------|-----------|----------|----------|---------|---------|
| Case 1. | 0.240     | 0.137     | 0.2711   | ---      | 1.5753  | 0.5753  |
| Case 2. | 0.240     | 0.137     | 0.2711   | 0.3047   | \( M_v \) | 0.2711  |
| Case 3. | 0.240     | 0.137     | 0.2711   | 0.3047   | 0.8711  | \( M_l \) |
| Case 4. | 0.240     | 0.137     | 0.2711   | 0.3047   | ---     | ---     |

For case 2 and 3, a new term \( M_l \) exists, corresponding to the value of the variable mass for the simulation, given by equation (1). Where \( r \) represents the radius of the ball, value for \( \lambda \) was constant for the first two cases, and for the last two last cases different values were used.

\( N = 3 \) loops were used for the simulation, with initial conditions for \( \theta_1 \) within the interval \((-\pi/2, \pi/2)\) and for \( \theta_2 \) within the interval \((-\pi/4, \pi/4)\), we integrated in an interval \([0, 2\pi]\), dividing such interval in \( n = 2000 \) simulated data, hence the size of the pace in the simulation is \( \delta t = 0.0047 \).

In order to make the simulation, it’s necessary to obtain \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \), so the equations (1) and (2) are solved as an equation system, obtaining two equations. This process is repeated for each case, equations (15) and (16) correspond to Case 1.

Case 1.

\[ \dot{\theta}_1 = -\frac{g m_e \sin \theta_1 - g m_e \sin(\theta_1 - \theta_2) \cos \theta_2 - \cos(\theta_1 - \theta_2) \sin(\theta_1 - \theta_2) l_1 m_2 \dot{\theta}_1^2 - \sin(\theta_1 - \theta_2) l_1 m_2 \dot{\theta}_2^2}{l_1 (m_1 + m_2 \sin(\theta_1 - \theta_2)^2)} \]  
\[ \dot{\theta}_2 = \frac{\sin(\theta_1 - \theta_2) \left[ l_1 (g \cos \theta_1 + l_1 \dot{\theta}_1^2) + m_2 (g \cos \theta_2 + l_2 \dot{\theta}_2^2) \right]}{l_2 (m_1 + m_2 \sin(\theta_1 - \theta_2)^2)} \]

Finally, a change of variable is made in order to reduce the order of the differential equation, obtaining two first order ordinary differential equations.

4. Simulation Outcomes.

The phase planes for each case are presented:
Case 1:
Figure 2. a) Phase plane of the upper pendulum \( (m_1) \) b) Phase plane of the lower pendulum \( (m_2) \) both with constant mass.

Case 2:

Figure 3. a) Phase plane of the upper pendulum \( (m_1(t)) \) b) Phase plane of the lower pendulum \( (m_2) \).

Case 3:

Figure 4. a) Phase plane of the upper pendulum \( (m_1) \), b) Phase plane of the lower pendulum \( (m_2(t)) \).

Case 4:

Figure 5. a) Phase plane of the upper pendulum b) Phase plane of the lower pendulum for the case 4 where both of the pendulums have a variable mass.
The phase diagram in the figures (2 – 5), shown the coupling relation between \( \theta_1 \) and \( \theta_2 \). All cases were generated by the similar initial conditions: \(-\pi/2 < \theta_{10} < \pi/2 \) and \(-\pi/4 < \theta_{20} < \pi/4 \), and the angular velocities are increase by \( \dot{\theta}_{10} = \dot{\theta}_{10} + 0.03 \), \( \dot{\theta}_{10} = \dot{\theta}_{10} + 0.05 \), because the interval where were generated this conditions were the same. The different behaviour shown in the graphics is due to the factors given by the equations (6), (9), (10), (13) and (14).

The simulations were developed in a temporal interval from 0 to \( 2\pi \), for a total \( N = 3 \) different initial conditions, represented by each color in the graphs.

The graphs in figure 2-5, are presented as the phase planes of the behavior for each angular variable, where the angular coupling can be identified, as well as their respective velocities, that exhibit a different trajectory of the ones that can be found in a mathematical pendulum.

Figure 3b-5b show the behavior for the angular variable \( \theta_2 \), which seem to be different that the ones in figure 2a-5a. Apparently, the inferior pendulum tries to stabilize around an equilibrium point, nevertheless, the fact that it is moving in a non-inertial frame makes the equilibrium point to disappear and tries to stabilize in another point, that’s the reason for what seem to be the stability nodes. This can be identified as well in figure 7, were the trajectory is entirely chaotic, that represents the inferior pendulum. The upper pendulum trajectory seems to be an arch of circumference, caused by the offered restriction at his suspension point.

Figure 2a-5a show that the upper pendulum movement describes a set of curves, in the interior of another curve, that’s represented by the phase diagram of a mathematical pendulum.

For case 2 it is observed how the upper pendulum (the one with variable mass) is affected by the damping factor given by (6), although it’s appreciated that the phase diagram has a similar amplitude of case 1, the simulation outputs converge at an amplitude between -0.5, 0.5 rad, and for case 1 the amplitude is of -1.5, 1.5 rad. This damping effect for the upper pendulum allows the inferior pendulum amplitude to expand.

For case 3, we can observe that the upper pendulum trajectories are similar to the ones in case 1, meanwhile the trajectories of the inferior pendulum (with variable mass) are similar to case 2.

![Phase plane](image)

**Figure 6.** a) Phase plane of the lower pendulum for the case 1, b) Phase plane of the lower pendulum for the case 2, c) Phase plane of the lower pendulum for the case 2,d) Phase plane of the lower pendulum for the case 4, all figures were simulated with the same initial conditions.

For case 4, a similar behavior to case 1 can be observed. Finally, in figure 6 we can observe an interesting behavior, regarding the inferior pendulum for all cases since a) constant mass pendulum, to d) variable mass in both pendulums, the simulations are made for the same values, and the output shows that the damping factor of case 1, turns out to be even more chaotic than
case 4, due to the fact that variable mass stops the movement of the pendulums, which can be observed in figures 7a and 7b.

![Figure 7. a) Shown the trajectory of boot pendulums for case 1; red point correspond to the mass $m_1$, orange point is to mass $m_2$, blue curve is the trajectory for $m_2$, the circles have the radii $l_1$ and $l_2$. b) Shown the trajectory of boot pendulums for case 4, the simulation was run since $t = 0$ to $t = 2\pi$.](image)

5. Conclusions
Four different dynamic systems were presented in this paper, based in the double pendulum system, where the equations of motion were obtained for each of the cases by applying the Lagrangian formulation. The factors that differentiate and modify the first classical case with respect of the last ones were identified, that resides in the mass variation. These factors are considered as damping factors, and it seems that they can be seen from to different perspectives: an inertial perspective and a non-inertial perspective. The non-inertial perspective corresponds to the lower pendulum of the system.

The simulation code was written to obtain the numerical solution of the non-linear differential equations, and the outcomes for 10 loops were presented in the phase planes, showing the differences that a variable mass system can make. It is observed that the system is sensible to the change in the mass, and that this causes a very particular behavior.

6. References
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