UNIFORM SYMBOLIC TOPOLOGIES IN NORMAL TORIC RINGS

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Abstract. Given a normal toric algebra \( R \), we compute a uniform integer \( D = D(R) > 0 \) such that the symbolic power \( P^{(DN)} \subseteq P^N \) for all \( N > 0 \) and all monomial primes \( P \). We compute the multiplier \( D \) explicitly in terms of the polyhedral cone data defining \( R \). In this toric setting, we draw a connection with the F-signature of \( R \) in positive characteristic.

1. Introduction and Conventions for the Paper

Given a Noetherian commutative ring \( R \), when is there an integer \( D \), depending only on \( R \), such that the symbolic power \( P^{(Dr)} \subseteq P^r \) for all prime ideals \( P \subseteq R \) and all positive integers \( r \)?

In short, when does \( R \) have uniform symbolic topologies on primes \([18]\)? Moreover, can we effectively compute the multiplier \( D \) in terms of simple data about \( R \)? In this paper, we answer this last question in the setting of torus-invariant primes in a normal toric (or semigroup) algebra.

The Ein-Lazarsfeld-Smith Theorem \([10]\), as extended by Hochster and Huneke \([16]\), says that if \( R \) is a \( d \)-dimensional regular ring containing a field, and \( D = \max \{ 1, d - 1 \} \), then \( Q^{(Dr)} \subseteq Q^r \) for all radical ideals \( Q \subseteq R \) and all \( r > 0 \). To what extent does this theme ring true for non-regular rings? Under mild stipulations, a local ring \( R \) regular on the punctured spectrum has uniform symbolic topologies on primes \([17, Cor. 3.10]\), although explicit values for \( D \) remain elusive. Far less is known for rings with non-isolated singularities.

This manuscript approaches the above questions for the coordinate rings of normal affine toric varieties. Such algebras are combinatorially-defined, finitely generated, Cohen-Macaulay, normal, and have rational singularities. We deduce an explicit multiplier \( D \) such that for any torus-invariant prime \( P \) in \( R \), \( P^{(Dr)} \subseteq P^r \) for all \( r > 0 \); see Theorems \([1, 3.1] \) and \([3, 3.1] \) for precise statements. Prior work towards explicit \( D \) includes the papers \([26, Table 3.3] \) and \([28, Thm. 1.2] \) for ADE rational surface singularities and for select domains with non-isolated singularities, respectively; see also \([8, Thm. 3.29, Cor. 3.30] \) for module-finite direct summands of affine polynomial rings.

To state our main results in a special case, fix a ground field \( F \). We fix a full-dimensional pointed convex polyhedral cone \( C \subseteq \mathbb{R}^n \) generated by a finite set \( G \subseteq \mathbb{Z}^n \), and its dual \( C^\vee \subseteq \mathbb{R}^n \). Let \( R_F \) be the semigroup algebra of the semigroup \( C^\vee \cap \mathbb{Z}^n \). This toric \( F \)-algebra associated to \( C \) is a normal domain of finite type over \( F \) \([7, Thm. 1.3.5] \) with an \( F \)-basis of monomials \( \{ \chi^\ell : \ell \in C^\vee \cap \mathbb{Z}^n \} \).

An ideal of \( R_F \) is monomial (or torus-invariant) if it is generated by a subset of these monomials. In what follows, we use \( \bullet \) to denote dot products in \( \mathbb{R}^n \).

\footnotesize
\begin{itemize}
  \item \textbf{2010 Mathematics Subject Classification:} 13H10, 14C20, 14M25.
  \item \textbf{Keywords:} symbolic powers, rational singularity, toric ring, monomial primes, F-signature, Segre-Veronese.
  \item For a definition of symbolic powers of prime and radical ideals, see \([8]\).
  \item This result has been extended to all excellent regular rings, even in mixed characteristic, by Ma-Schwede \([20]\).
\end{itemize}
Theorem 1.1 (Cf., Theorem 3.1). Suppose $C \subseteq \mathbb{R}^n$ is a full-dimensional pointed polyhedral cone as above. Set $D := \max_{w \in B} (w \cdot v_C) \in \mathbb{Z}_{>0}$, where $B$ is a generating set for the semigroup $C^\vee \cap \mathbb{Z}^n$ and $v_C \in \mathbb{Z}^n$ is the sum of any (finite) set $G$ of vectors in $\mathbb{Z}^n$ generating $C$. Then
\[ P(Dr) \subseteq P(D(r-1)+1) \subseteq P^r, \]
for all $r > 0$ and all monomial prime ideals $P$ in the toric ring $R_F = \mathbb{F}[C^\vee \cap \mathbb{Z}^n]$.

We get the best result in Theorem 1.1 by taking $G$ to consist of the unique set of primitive generators for $C$, in which case we write $v_C$ in place of $v_G$; see Section 2. For example, if $R_F$ is the $E$-th Veronese subring of a polynomial ring of finite type over $\mathbb{F}$, then our $D = E$ in Theorem 1.1. As another example, for the Segre product of Veronese rings of respective degrees $E_1, \ldots, E_k$, we can take $D = \sum_{i=1}^k E_i$ in Theorem 1.1; see Theorem 4.8.

The next result covers select non-monomial primes. We note that a toric algebra satisfying the additional hypothesis below is called simplicial; see the discussion around Theorem 2.5.

Theorem 1.2 (Cf., Theorem 3.2). With notation as in Theorem 1.1, assume moreover that the divisor class group $\text{Cl}(R_F)$ is finite. Set $U := \text{lcm}\{\max_{w \in B} (w \cdot v_C), \# \text{Cl}(R_F)\} \in \mathbb{Z}_{>0}$, where $B$ and $v_C$ are as in Theorem 1.1. Then
\[ P(U(r-1)+1) \subseteq P^r \]
for all $r > 0$, all monomial primes in $R_F$, and all height one primes in $R_F$.

Tighter containments of the type $I^{(E(r-1)+1)} \subseteq I^r$, as in Theorems 1.1 and 1.2 and first promoted by Harbourne, hold for all monomial ideals in an affine polynomial ring over any field [5, Ex. 8.4.5]; see also recent work of Grifo-Huneke [15].

At the end of Section 3, we draw connections between the multiplier $U$ and the so-called F-signature of $R_F$. In Section 4, we discuss the extent to which the multipliers $D$ and $U$ in Theorems 1.1 and 1.2 are sharp.

Conventions: Throughout, $\mathbb{F}$ denotes an arbitrary ground field of arbitrary characteristic. All rings are commutative with identity—indeed, they are normal domains of finite type over $\mathbb{F}$.

Acknowledgements: This paper is part of my Ph.D. thesis at the University of Michigan-Ann Arbor. My thesis adviser Karen E. Smith, along with Daniel Hernández and Jack Jeffries, suggested studying precursors for the ideals $I_4(E)$ in the proof of Theorem 3.1. I am grateful for this idea. I acknowledge support from a NSF GRF (Grant No. PGF-031543), NSF RTG grant DMS-0943832, and a 2017 Ford Foundation Dissertation Fellowship. Several computations were performed using the Polyhedra package in Macaulay2 [14] to gain an incisive handle on the polyhedral geometry.

2. Toric Algebra Preliminaries

We review notation and relevant facts from toric algebra, citing Cox-Little-Schenck [7, Ch.1,3,4] and Fulton [12, Ch.1,3]. A lattice is a free abelian group of finite rank. We fix a perfect bilinear pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ between two lattices $M$ and $N$; this identifies $M$ with $\text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and $N$ with $\text{Hom}_\mathbb{Z}(M, \mathbb{Z})$. Our pairing extends to a perfect pairing of finite-dimensional vector spaces $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$, where $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.
Fix an \( N \)-rational polyhedral cone and its \( M \)-rational dual: respectively, for some finite subset \( G \subseteq N - \{0\} \) these are closed, convex sets of the form

\[
C = \text{Cone}(G) := \left\{ \sum_{v \in G} a_v \cdot v : \text{ each } a_v \in \mathbb{R}_{\geq 0} \right\} \subseteq N_\mathbb{R}, \quad \text{and }
\]

\[
C^\vee := \{ w \in M_\mathbb{R} : \langle w, v \rangle \geq 0 \text{ for all } v \in C \} = \{ w \in M_\mathbb{R} : \langle w, v \rangle \geq 0 \text{ for all } v \in G \}.
\]

By definition, the dimension of a cone in \( M_\mathbb{R} \) or \( N_\mathbb{R} \) is the dimension of the real vector subspace it spans; a cone is full-(dimensional) if it spans the whole ambient space. A cone in \( M_\mathbb{R} \) or \( N_\mathbb{R} \) is pointed (or strongly convex) if it contains no line through the origin. A face of \( C \) is a convex polyhedral cone \( F \) in \( N_\mathbb{R} \) obtained by intersecting \( C \) with a hyperplane which is the kernel of a linear functional \( m \in C^\vee \); \( F \) is proper if \( F \neq C \). When \( C \) is both \( N \)-rational and pointed, so is every face \( F \). Each such face \( F \neq \{0\} \) has a uniquely-determined set \( G_F \) of primitive generators. By definition, \( v \in N \) is primitive if \( \frac{1}{k} \cdot v \notin N \) for all \( k \in \mathbb{Z}_{>1} \).

There is a bijective inclusion-reversing correspondence between faces \( F \) of \( C \) and faces \( F^* \) of \( C^\vee \), where \( F^* = \{ w \in C^\vee : \langle w, v \rangle = 0 \text{ for all } v \in F \} \) is the face of \( C^\vee \) dual to \( F \) \([12, \text{Sec. 1.2}]\). Under this correspondence, either cone is pointed if and only if the other is full, and

\[
\dim(F) + \dim(F^*) = \dim(N_\mathbb{R}) = \dim(M_\mathbb{R}). \quad (2.0.1)
\]

Fix an arbitrary ground field \( \mathbb{F} \) and a cone \( C \) as above in \( N_\mathbb{R} \). The semigroup ring \( R_F = \mathbb{F}[C^\vee \cap M] \) is the toric \( \mathbb{F} \)-algebra associated to \( C \). This ring \( R_F \) is a normal domain of finite type over \( \mathbb{F} \) \([7, \text{Thm. 1.3.5}]\). Note that \( R_F \) has an \( \mathbb{F} \)-basis \( \{ \chi^m : m \in C^\vee \cap M \} \) of monomials, giving \( R_F \) an \( M \)-grading, where \( \deg(\chi^m) := m \). A monomial ideal (also called an \( M \)-homogeneous or torus-invariant ideal) in \( R_F \) is an ideal generated by a subset of these monomials. When \( C^\vee \) is pointed, \( R_F \) also has a non-canonical \( \mathbb{N} \)-grading obtained by fixing any group homomorphism \( M \to \mathbb{Z} \) taking positive values \( C^\vee \cap M - \{0\} \). The set \( \{ \chi^m : m \in C^\vee \cap M - \{0\} \} \) generates the unique homogeneous maximal ideal \( \mathfrak{m} \) under this \( \mathbb{N} \)-grading.

**Remark 2.1.** In forming the toric algebra \( \mathbb{F}[C^\vee \cap M] \), there is no loss of generality in assuming \( C \) is pointed in \( N_\mathbb{R} \). Indeed, because \( C^\vee \cap M = C^\vee \cap M' \) where \( M' = M \cap \{ \mathbb{R} \text{-span of } C^\vee \in M_\mathbb{R} \} \), we may replace \( M \) by \( M' \) to assume \( C^\vee \) is full in \( (M')_\mathbb{R} \). Now, replacing \( N \) and \( C \) by the duals of \( M' \) and \( C^\vee \), we may assume that \( C \) is pointed in \( N' = \text{Hom}_\mathbb{Z}(M', \mathbb{Z}) \). See \([7, \text{Thm. 1.3.5}]\) for details.

Fix a face \( F \) of a pointed cone \( C \): \([12, \text{p.53}]\) records a surjective \( M \)-graded ring map

\[
\phi_F : R_F = \mathbb{F}[C^\vee \cap M] \twoheadrightarrow \mathbb{F}[F^* \cap M], \quad \phi_F(\chi^m) = \begin{cases} \chi^m & \text{if } \langle m, v \rangle = 0 \text{ for all } v \in F \\ 0 & \text{if } \langle m, v \rangle > 0 \text{ for some } v \in F. \end{cases}
\]

Both rings are domains. The monomial prime ideal of \( F \), \( P_F := \ker(\phi_F) \), has height equal to \( \dim(F) \). Conversely, any monomial prime of \( R_F \) corresponds bijectively to a face of \( C \).

**Lemma 2.2.** Fix a face \( F \) of a pointed cone \( C \), and the monomial prime \( P_F \subseteq R_F \) above. Let \( G_F \) be the set of primitive generators of \( F \), and set \( v_F := \sum_{v \in G_F} v \in F \cap N \). Then

\[
P_F = (\{ \chi^m : m \in C^\vee \cap M \text{ and the integer } \langle m, v_F \rangle > 0 \}) R_F. \quad (2.0.2)
\]

**Proof.** First, in defining \( \phi_F(\chi^m) \) above, notice we can work with \( v \in G_F \) without loss of generality. Now, fix \( m \in C^\vee \cap M \). Then \( \langle m, v \rangle \in \mathbb{Z}_{\geq 0} \) for all \( v \in C \cap N \). As \( \langle \cdot, \cdot \rangle \) is bilinear, \((2.0.2)\) follows since a sum of nonnegative integers is positive if and only if one of the summands is positive. \( \square \)
2.0.1. **Hilbert Bases.** First, suppose the pointed cone $C$ from Remark 2.1 is full. Then there is a uniquely-determined minimal generating set $B$ for $C^\vee \cap M$, in the sense that any other generating set contains $B$. The set $B$ is called the **Hilbert basis** of the semigroup, and consists of the irreducible vectors $m \in C^\vee \cap M - \{0\}$; a vector $v \in C^\vee \cap M$ is irreducible if it cannot be expressed as a sum of two vectors $m \in C^\vee \cap M - \{0\}$. See [7] Prop. 1.2.17 and [7] Prop. 1.2.23 for details.

In general, the pointed cone $C$ need not be full. Thus the next proposition is handy.

**Proposition 2.3.** Let $N'_R$ by the $\mathbb{R}$-span of a pointed cone $C \subseteq N_R$. Set $N' = N'_R \cap N$, and consider $C$ as a full-dimensional cone in $N'_R$ (relabeled as $C'$). Let $M' = \text{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$ be the dual lattice. The toric ring $R_{\mathbb{F}} := \mathbb{F}[C^\vee \cap M]$ is isomorphic to $R_{\mathbb{F}}' \otimes_{\mathbb{F}} L$ where the toric ring $R_{\mathbb{F}}' := \mathbb{F}[C']^\vee \cap M'$ and $L$ is a Laurent polynomial ring over $\mathbb{F}$. In particular, there is a bijective correspondence between the monomial primes of $R_{\mathbb{F}}'$ and $R_{\mathbb{F}}$ given by expansion and contraction of ideals along the faithfully flat ring map $\varphi : R_{\mathbb{F}}' \hookrightarrow R_{\mathbb{F}} \otimes L = R_{\mathbb{F}}$.

**Proof.** Combine [7] Proof of Prop. 3.3.9] with [27] Discussion/Proof preceding Lem. 3.1]. □

2.0.2. **Toric Divisor Theory.** Given a Noetherian normal domain $R$, the **divisor class group** $\text{Cl}(R) = \text{Cl}(\text{Spec}(R))$ is the free abelian group on the set of height one prime ideals of $R$ modulo relations $\sum_{i=1}^r a_i P_i = 0$ when the ideal $\bigcap_{i=1}^r P_i^{(a_i)}$ is principal. Note $\text{Cl}(R)$ is trivial if and only if $R$ is a UFD, i.e., all height one primes in $R$ are principal. We recall the following

**Lemma 2.4 (Cf., Lem. 1.1 of [27]).** When every element of $\text{Cl}(R) := \text{Cl}(\text{Spec}(R))$ is annihilated by an integer $D > 0$, written as $D : \Cl(R) = 0$, the symbolic power $P^{(D(r-1)+1)} \subseteq P^r$ for all $r > 0$ and all prime ideals $P \subseteq R$ of height one.

Working over an algebraically closed field $\mathbb{F}$, fix a pointed cone $C$ as in Remark 2.1 and the pair of rings $R_{\mathbb{F}}$ and $R_{\mathbb{F}}'$ as in Proposition 2.3. When $C \neq \{0\}$, each $\rho \in \Sigma(1)$, the collection of rational rays (one-dimensional faces) of $C$, yields a unique primitive generator $u_{\rho} \in \rho \cap N$ for $C$ and a torus-invariant height one prime ideal $P_{\rho}$ in $R_{\mathbb{F}}'$; cf., [7] Thm. 3.2.6]. The torus-invariant height one primes generate a free abelian group $\bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} P_{\rho}$ which maps surjectively onto the divisor class group of $R_{\mathbb{F}}'$. More precisely, we record the following well-known theorem; see [7, Ch. 4].

**Theorem 2.5.** With notation as in Proposition 2.3 let $C \subseteq N_R$ be a pointed cone with primitive generators $\Sigma(1)$ as described above. Then there is a short exact sequence

$$0 \to M' \xrightarrow{\phi} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} P_{\rho} \to \text{Cl}(R_{\mathbb{F}}') \to 0,$$

where $\phi(m) = \text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} (m, u_{\rho}) P_{\rho}$. Furthermore, $\text{Cl}(R_{\mathbb{F}})$ and $\text{Cl}(R_{\mathbb{F}}')$ are isomorphic, $\text{Cl}(R_{\mathbb{F}})$ is finite abelian if and only if $C$ is simplicial, and trivial if and only if $C$ is smooth.

By definition, the cone $C \subseteq N_R$ is simplicial (respectively, smooth) if $C = \{0\}$ or the primitive ray generators form part of an $\mathbb{R}$-basis for $N_R$ (resp., a $\mathbb{Z}$-basis for $N$). We also apply the adjectives simplicial and smooth to the corresponding toric algebra $R_{\mathbb{F}}$ and the toric $\mathbb{F}$-variety $\text{Spec}(R_{\mathbb{F}})$.

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3This result follows from [7] Prop. 3.3.9, Prop. 4.1.1-4.1.2, Thm. 4.1.3, Exer. 4.1.1-4.1.2, Prop. 4.2.2, Prop. 4.2.6, and Prop. 4.2.7], essentially consolidating what facts we need to bear in mind going forward in the manuscript.
3. Main Results

Maintaining all notation conventions from the last section, we now state our main results.

**Theorem 3.1.** Let $C \subseteq \mathbb{N}_\mathbb{Z}$ be a full pointed rational polyhedral cone. Let $R_F = \mathbb{F}[C^\vee] \cap M$ be the associated toric algebra over a field $\mathbb{F}$. Set $D := \max_{m \in B} \langle m, v_C \rangle$, where $B$ is the minimal generating set for $C^\vee \cap M$ and $v_C \in N$ is the sum of the primitive generators for $C$. Then

$$p^{(D(r-1)+1)} \leq p^r$$

for all $r > 0$, and all monomial primes $P$ in $R_F$.

**Theorem 3.2.** With notation as in Theorem 3.1, we assume further that $C$ is simplicial. Define $T := \max \{ \max_{m \in B} \langle m, v_C \rangle, D \}$. Then

$$p^{(T(r-1)+1)} \leq p^r$$

for all $r > 0$, all monomial primes, and all height one primes in $R_F$. In particular, we can take $D = \# Cl(R_F)$.

**Remark 3.3.** If the cone $C$ in Theorem 3.2 is smooth, then $T = 1$ and $p^{(r)} = p^r$ for all $r > 0$, all monomial primes, and all height one primes in $R_F$. As $C$ is smooth, $C$ and $C^\vee$ are generated by a $\mathbb{Z}$-basis for $N$ and the dual basis for $M$, respectively. Also, $\# Cl(R_F) = \# Cl(R_F^\mathbb{Z}) = 1$. Note that in general, this means our multiplier $T$ will not confirm uniform symbolic topologies for all primes $P$ in a toric algebra. For example, even in a polynomial ring of dimension three, there are height two primes for which $P^{(r)} \neq p^r$ for some $r \geq 2$; [8, p.2 of Intro] gives an example.

**Remark 3.4.** Two-dimensional toric algebras are always simplicial with cyclic class group. In this case, the conclusion of Theorem 3.2 holds using the multiplier $\# Cl(R_F)$. This multiplier is sharp by Proposition 4.1.

**Remark 3.5.** Theorems 3.1-3.2 can be adapted to the non-full case by replacing $R_F$ with $R_F^\mathbb{Z}$ as in Proposition 2.3 and applying [28, Prop. 2.6] to the faithfully flat map $\varphi$ from Proposition 2.3.

**Proof of Theorem 3.1.** We may fix a face $F \neq \{0\}$ of $C$, and $P = P_F$ the corresponding monomial prime in $R = R_F$. Per Lemma 2.2 (2.0.2), a monomial $\chi^m \in P = P_F$ if and only if $\langle m, v_F \rangle \in \mathbb{Z}_{\geq 0}$, where $v_F \in F \cap N$ is the sum of the primitive generators for $F$.

**Lemma 3.6.** For each integer $E \geq 1$, $P_F^{(E)} \subseteq I_F(E)$, where $I_F(E) := (\chi^m : \langle m, v_F \rangle \geq E) R$.

First, $I_F(E)$ is $P_F$-primary for all $E \geq 1$, i.e., if $sf \in I_F(E)$ for some $s \in R - P_F$, then $f \in I_F(E)$. As $I_F(E)$ is monomial, we may test this by fixing $\chi^m \in I_F(E) R_P \cap R$ and $\chi^q \in R - P_F$ such that $\chi^m \cdot \chi^q = \chi^{m+q} \in I_F(E)$: $\langle q, v_F \rangle = 0$, while $E \leq \langle m + q, v_F \rangle = \langle m, v_F \rangle + \langle q, v_F \rangle = \langle m, v_F \rangle$, so $\chi^m \in I_F(E)$. Thus all $I_F(E)$ are $P_F$-primary, and certainly $P_F^{(E)} \subseteq I_F(E)$. Thus $P_F^{(E)} \subseteq I_F(E)$, being the smallest $P_F$-primary ideal containing $P_F^{E}$, proving the lemma.

Certainly, $1 \leq \max_{m \in B} \langle m, v_F \rangle \leq \max_{m \in B} \langle m, v_C \rangle$, for $B$ and $D$ as above. The conclusion of the theorem follows by [27, Lem. 3.3], once we verify the left-hand containments in the next

**Lemma 3.7.** For each integer $E \geq 1$, $I_F(E) \subseteq P_F^{(E[D'])} \subseteq P_F^{(E[D])}$ where $D' = \max_{m \in B} \langle m, v_F \rangle$.

Fix any monomial $\chi^\ell \in I_F(E)$, say $\ell = \sum_{m \in B} a_m \cdot m$ with $a_m \in \mathbb{Z}_{\geq 0}$. Let $S \subseteq B$ consist of those $m \in B$ such that the monomials $\chi^m$ form a minimal generating set for $P$. By linearity of $\langle \cdot, v_F \rangle$,

$$E \leq \langle \ell, v_F \rangle = \sum_{m \in B} a_m \langle m, v_F \rangle = \sum_{m \in S} a_m \langle m, v_F \rangle \leq \sum_{m \in S} a_m \cdot D' \implies \sum_{m \in S} a_m \geq \lceil E/D' \rceil.$$
Thus $\chi^m \in P_F^{\sum_{m \in S} a_m} \subseteq P_F^{[E/D]}$. Being a monomial ideal, it follows that $I_F(E) \subseteq P_F^{[E/D]}$. \hfill $\Box$

**Remark 3.8.** In passing, we invite the reader to compare the ideals $I_F(\bullet)$ in Lemma 3.6 with Bruns and Gubeladze’s terminology and description [6, Ch. 4, p. 149] for the symbolic powers of the height one monomials primes in terms of a full pointed cone. Lemma 3.6 works in any height.

**Lemma 3.9.** With notation as in Theorem 3.7, the class groups $\text{Cl}(R_\mathbb{F}) \cong \text{Cl}(R_\mathbb{F}')$ are isomorphic.

**Proof.** Since $C$ is a full pointed cone in Theorem 3.1 $R_\mathbb{F}$ admits an $\mathbb{N}$-grading; see the passage above Remark 2.1. We may then cite Fossum [11, Cor. 10.5 on p.43] to conclude that up to isomorphism, $\text{Cl}(R_\mathbb{F}) \subseteq \text{Cl}(R_\mathbb{F}')$ as a subgroup. This is an equality for toric rings because the divisor classes of height one monomial primes belong to both groups and generate the latter by Theorem 2.5. \hfill $\Box$

**Proof of Theorem 3.2.** Since $C$ is simplicial, $\# \text{Cl}(R_\mathbb{F})$ is finite by Lemma 3.9 and Theorem 2.5. Now we simply combine Theorem 3.1 with Lemma 2.4 and take the maximum of the values. \hfill $\Box$

**Example 3.10.** Fix an arbitrary ground field $\mathbb{F}$ and integers $n \geq 2$ and $E \geq 2$. Let

$$R = \mathbb{F}[x_1, \ldots, x_n, z]/(z^E - x_1 \cdots x_n).$$

Then $P^{(T(r-1)+1)} \subseteq P^r$ for all $r > 0$, all monomials primes, and all height one primes in $R$, where $T = \max\{n, E\}$. Indeed, $R$ is a toric algebra arising from the simplicial full pointed cone $C \subseteq \mathbb{R}^n$ spanned by

$$\{e_n, E \cdot e_i + e_n : i = 1, \ldots, n-1\} \subseteq \mathbb{Z}^n,$$

where $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. We may apply Theorem 3.2 noting that:

- In the notation of Theorem 3.1 $B = \{e_1, \ldots, e_{n-1}, e_n, E \cdot e_n - e_1 - \cdots - e_{n-1}\} \subseteq \mathbb{Z}^n$ and the vector $v_C = n \cdot e_n + E \cdot (e_1 + \cdots + e_{n-1}) \in \mathbb{Z}^n$.
- $\text{Cl}(R) \cong (\mathbb{Z}/E\mathbb{Z})^{n-1}$, so $E \cdot \text{Cl}(R) = 0$; see [27 Sec. 4] for this class group computation.

3.0.1. **Von Korff’s Toric F-Signature Formula.** Now fix a perfect field $\mathbb{K}$ of positive characteristic $p$. Given an $\mathbb{F}$-finite $\mathbb{N}$-graded domain $R$ of finite type over $\mathbb{K}$, for each integer $e \geq 0$, we have an $R$-module isomorphism $R^1/p^e R \cong R^{ae} \oplus M$ where $M$ has no free summand, and the integer $ae \leq p^{ed}$ where $d = \dim R$. By definition, the **F-signature** of $R$ is (see [19] and [24])

$$s(R) := \lim\sup_{e \to \infty} \frac{a_e}{p^{ed}} = \lim\inf_{e \to \infty} \frac{a_e}{p^{ed}}, \quad 0 \leq s(R) \leq 1.$$ 

The F-signature has ties to measuring F-singularities: for instance, $s(R)$ is positive if and only if $R$ is strongly F-regular [1], and $s(R) = 1$ if and only if $R$ is regular [24 Thm. 4.16].

Over the perfect field $\mathbb{K}$, any normal toric ring is strongly F-regular and its F-signature is rational [23]. We now state Von Korff’s result [25 Thm. 3.2.3]; see also Watanabe-Yoshida [29 Thm. 5.1] and Yao [30 Rem. 2.3(4)].

**Theorem 3.11 (cf., Von Korff [25 Thm. 3.2.3]).** With notation as in Proposition 2.3, we define a convex polytope, $P_{C'} := \{w \in M_\mathbb{R}' : 0 \leq \langle w, v \rangle < 1, \forall v \in G\} \subseteq \mathbb{R}^\vee$, where $G$ is the set of primitive generators of $C' \neq \{0\}$. Then over any perfect field $\mathbb{K}$ of positive characteristic, the F-signature $s(R_\mathbb{K}) = s(R_\mathbb{K}') = \text{Vol}(P_{C'}) \in \mathbb{Q}_{>0}$, where the volume form Vol on $M_\mathbb{R}'$ is chosen so that a hypercube spanned by primitive generators of $M'$ has volume one.

**Remark 3.12.** When the cone $C'$ in Theorem 3.11 is simplicial, $s(R_\mathbb{K}) = 1/D$ where the integer $D = \# \text{Cl}(R_\mathbb{K}') = \# \text{Cl}(R_\mathbb{K})$; cf., [26 Cor. 3.2]. The latter equality holds by Lemma 3.9.
Given Theorem 3.11 we could opt to replace the invariant $T$ from Theorem 3.2 with the possibly larger invariant $U = \text{lcm}\{\max_{m \in B}\langle m, v_C \rangle, \# \text{Cl}(R_{\mathbb{K}})\} \in \mathbb{Z}_{>0}$, the least common multiple considered in Theorem 1.2. The latter is an integer multiple of the reciprocal $1/s(R_{\mathbb{K}}) = \# \text{Cl}(R_{\mathbb{K}})$ of the F-signature of $R_{\mathbb{K}}$ that controls the asymptotic growth of symbolic powers.

4. Examples of (Non-)Sharp Multipliers; Segre-Veronese algebras

To start, we deduce a result that occasionally provides sharp multipliers in the toric setting.

**Proposition 4.1.** With notation as in Theorem 3.2, we assume $C$ is a simplicial full pointed rational polyhedral cone. We now set $B := \max_{w \in PG} \langle w, v_C \rangle$ where $PG \subseteq B$ consists of the primitive generators of $C^{\vee}$. There exists a monomial prime $P$ in $R = R_{\mathbb{K}}$ of height one such that:

1. $P^{(B(r-1))} \not\subseteq P^r$ for some $r \geq 2$.
2. There is no positive integer $D' < B$ such that $P^{(D'(r-1)+1)} \subseteq P^r$ for all $r > 0$.

**Proof.** Let $v_1, \ldots, v_n \in N$ and $w_1, \ldots, w_n \in M$ denote the primitive generators for $C$ and for $C^{\vee}$, respectively. We index these generators so that the nonnegative integer $\langle w_j, v_i \rangle$ is positive if and only if $i = j$; we may do this citing the notion of facet normals [7] after Prop. 1.2.8. In deference to Lemma 2.2, let $P_j (1 \leq j \leq n)$ be the height one monomial prime in $R_{\mathbb{K}}$ such that a monomial $\chi^m \in P_j$ if and only if $\langle m, v_j \rangle > 0$. In particular, $\chi^w \in P_j$ for each $j$.

**Lemma.** For each $1 \leq j \leq n$, $\langle w_j, v_j \rangle$ is the order of the element in $\text{Cl}(R_{\mathbb{K}})$ corresponding to $P_j$.

We may leverage exact sequence (2.0.3) from Theorem 2.5 since Lemma 3.2 allows us to reduce to the case where $\mathbb{F}$ is algebraically closed. For $1 \leq j \leq n$, we have $0 = [\text{div}(\chi^{w_j})] = \langle w_j, v_j \rangle[D_{\rho_j}]$, where $\rho_j$ is the rational ray of $C$ generated by $v_j$. Thus $P_j^{\langle w_j, v_j \rangle} = \langle \chi^{w_j} \rangle R$. Since the order of $[D_{\rho_j}]$ is the smallest $E_j > 0$ such that $P_j^{(E_j)} = (\chi^{m_j})R$ is principal for some $m_j \in C^{\vee} \cap M - \{0\}$, $E_j$ divides $\langle w_j, v_j \rangle$, and $P_j^{\langle w_j, v_j \rangle} = (P_j^{(E_j)})^L$ where $L = \langle w_j, v_j \rangle/E_j$. Since $C^{\vee}$ is strongly convex, we may conclude that $\chi^{w_j} = \chi^{L-m_j}$, and $L = 1$ since $w_j$ is an irreducible vector in $C^{\vee} \cap M$; see Subsection 2.0.1. This proves the lemma.

To prove (1), notice $B = \langle w_{j_0}, v_{j_0} \rangle$ for some $1 \leq j_0 \leq n$. Then (*): $(\chi^{w_{j_0}})R = P_{j_0}^{(B)} \not\subseteq P_{j_0}^2$. We recall from Section 2 that when the cone $C$ is full-dimensional in $N_{\mathbb{R}}$, the semigroup algebra $R_{\mathbb{K}} = \mathbb{F}[C^{\vee} \cap M]$ can be $\mathbb{N}$-graded. This means that any minimal generator $f$ of a homogeneous ideal $I$ satisfies $f \in I - I^2$ by Nakayama’s lemma. In our situation, $I = P_{j_0}$ and $f = \chi^{w_{j_0}}$. Observation (*) also gives part (2), arguing by contradiction and using [27] Lem. 3.3] accordingly. □

We offer several examples to show that establishing sharpness of our bilinear multipliers is a delicate matter meriting further study.

**Example 4.2.** We fix integers $n \geq 2$ and $E \geq 2$, and an arbitrary field $\mathbb{F}$. Let $V_{E,n}$ be the $E$-th Veronese subalgebra of the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$, that is, the $\mathbb{F}$-algebra generated by all monomials of degree $E$ in $x_1, \ldots, x_n$. Then $P^{(E(r-1)+1)} \subseteq P^r$ for all $r > 0$, all monomial primes, and all primes of height one by Theorem 3.2. See [27] Sec. 4] for details. However, for any $E' < E$, the proof of Proposition 4.1 guarantees that we can find a prime $P \subseteq V_{E,n}$ (monomial, height one) such that $P^{(E'(r-1)+1)} \not\subseteq P^r$ for some $r \geq 2$, namely, for $r = 2$. In fact, this last observation holds for all monomial primes in $V_{E,n}$, aside from the zero ideal and the maximal monomial ideal for which $E' = 1$ will do; the proof of [27] Thm. 4.3] confirms this explicitly.
Despite Example 4.2, Theorem 3.2 does not give sharp multipliers in general. For example,

**Example 4.3.** For any \( n > 2 \), let \( R = \mathbb{F}[Z, X_1, \ldots, X_n]/(Z^2 - X_1 \cdots X_n) \) as in Example 3.10. Citing the proof of [27, Thm. 4.1], when \( P \subseteq R \) is any monomial prime of height at least 2:

- \( P^{(r)} = P^r \) for all \( r > 0 \); however,
- The multiplier \( D' \) corresponding to \( P \) in Lemma 3.7 always satisfies \( D' \geq 2 \).

Theorem 3.1 gives a uniform multiplier \( D \) that works for all monomial primes. Even when this multiplier is sharp across all monomial primes, it need not be best possible for all monomial primes of a given height, contrasting with the situation of Example 4.2. For example,

**Example 4.4.** Let \( R = \mathbb{F}[C^\vee \cap \mathbb{Z}^3] = \mathbb{F}[x, y, z, w]/(xy - zw) \), for the non-simplicial cone \( C \subseteq \mathbb{R}^3 \) with \( e_1, e_2, e_1 + e_3, e_2 + e_3 \) as primitive generators. Theorem 3.1 says \( P^{(2r-1)} \subseteq P^r \) for all \( r > 0 \) and all monomial primes in \( R \), observing that \( C^\vee \cap \mathbb{Z}^3 \) is minimally generated by \( \mathcal{B} = \{e_1, e_2, e_3, (1, 1, -1)\} \). Given any height two monomial prime \( P \) in \( R \), these containments cannot be improved to \( P^{(r)} = P^r \) for all \( r \geq 2 \). For instance, if \( P = (x, y, z)R \), then for any \( s \geq 1 \), \( z^s \in P^{(2s)} - P^{2s} \) and \( z^{s+1} \in P^{(2s+1)} - P^{2s+1} \); indeed, \( w^s \in R - P \) and \( R \) can be standard graded, so the least degree of a homogeneous element of \( P^r \) is \( r \). By contrast, \( P^{(1)} = P \) for all \( r \) and for any height one monomial prime \( P \) in \( R \): the invariant \( D' = 1 \) in Lemma 3.7 via direct computation.

4.0.1. Final Example Computation: Segre-Veronese algebras. In what follows, \( \mathbb{F} \) is a fixed arbitrary field. For more on Segre products, [13] is often cited as a standard reference.

**Definition 4.5.** Fix a family \( A_1, \ldots, A_k \) of \( k \) standard graded algebras of finite type over \( \mathbb{F} \), with \( A_i = \mathbb{F}[a_{i,1}, \ldots, a_{i,b_i}] \) in terms of algebra generators. Their **Segre product** over \( \mathbb{F} \) is the ring \( S = (\#_\mathbb{F})_{i=1}^k A_i \) generated up to isomorphism as an \( \mathbb{F} \)-algebra by all \( k \)-fold products of the \( a_{i,j} \).

**Definition 4.6.** We fix integers \( E \geq 1 \) and \( m \geq 2 \). Suppose \( A = \mathbb{F}[x_1, \ldots, x_m] \) is a standard graded polynomial ring in \( m \) variables over a field \( \mathbb{F} \). Let \( V_{E,m} \subseteq A \) denote the \( E \)-th **Veronese subring** of \( A \), the standard graded \( \mathbb{F} \)-subalgebra generated by all monomials of degree \( E \) in the \( x_i \). There are \( \binom{m-1+E}{E} \) such monomials; this number is the embedding dimension of \( V_{E,m} \).

**Definition 4.7.** Fix \( k \)-tuples \( \overline{E} = (E_1, \ldots, E_k) \in (\mathbb{Z}_{\geq 1}^k \cap \mathbb{Z}_2^k) \) of integers, with \( k \geq 1 \). Furthermore, we set \( d(j) = (\sum_{i=1}^j m_i) - (j - 1) \) for each \( 1 \leq j \leq k \); \( d(k) \) is the Krull dimension of the Segre product \( SV(\overline{E}, \overline{m}) = (\#_\mathbb{F})_{i=1}^k V_{E_i, m_i} \) of \( k \) Veronese rings in \( m_1, \ldots, m_k \) variables, respectively; this is a **Segre-Veronese algebra with degree sequence** \( \overline{E} \).

Over any perfect field \( \mathbb{K} \), a Segre-Veronese algebra has uniform symbolic topologies on all primes, per [10, Thm. 2.2], [16, Thm. 1.1], and [17, Cor. 3.10]. However, explicit \( D \) are elusive unless \( k = 1 \); see [8, Cor. 3.30]. We provide an effective \( D \) for the monomial primes, which is sharp as suggested by Examples 4.2 and 4.3, the latter being the Segre product of polynomial rings in two variables:

**Theorem 4.8.** Suppose \( A = SV(\overline{E}, \overline{m}) \) is a Segre-Veronese algebra over \( \mathbb{F} \) with degree sequence \( \overline{E} = (E_1, \ldots, E_k) \). Let \( D := \sum_{i=1}^k E_i \). Then \( P^{(D(r-1)+1)} \subseteq P^r \) for all \( r > 0 \) and all monomial primes \( P \) in \( A \).

**Proof.** Given a lattice \( N \cong \mathbb{Z}^d \) we will use \( e_1, \ldots, e_d \in N \) to denote a choice of basis for \( N \) will dual basis \( e'_1, \ldots, e'_d \) for \( M \). In the setup of Theorem 3.1, the cardinality of the minimal generating set \( B \) of \( C^\vee \cap M \) is the embedding dimension of the toric algebra \( R_F = \mathbb{F}[C^\vee \cap M] \); we refer the reader to [7, Sec. 1.0, Proof of Thm. 1.3.10].
Lemma. For $A$ as stated, there exists a lattice $N$ and a full pointed rational polyhedral cone $C \subseteq N_R$ such that $A = F[C^\vee \cap M]$ up to isomorphism.

Proof. Fix $k$-tuples $\underline{e} \in (\mathbb{Z}_{\geq 1})^k$ and $\underline{m} \in (\mathbb{Z}_{\geq 2})^k$. Set $d(j) = \left(\sum_{i=1}^j m_i\right) - (j - 1)$ for $1 \leq j \leq k$, while $d(0) = 0$. Given $SV(\underline{E}, \underline{m}) = (\#F)^k \vert E_{i_{m_i}}\rangle$, we fix a lattice $N \cong \mathbb{Z}^{d(k)}$ and record a cone $C = C(\underline{E}, \underline{m}) \subseteq N_R \cong \mathbb{R}^{d(k)}$ as stipulated with $R_F = F[C^\vee \cap M] \cong SV(\underline{E}, \underline{m})$. Specifically, consider the cone $C \subseteq N_R$ generated by the following irredundant collection of primitive vectors:

$$A = \bigcup_{1 \leq j \leq k} A_j, \text{ where } A_1 = \{e_1, \ldots, e_{m_1-1}, -e_1 - \cdots - e_{m_1-1} + E_1 \cdot e_{m_1}\},$$

and for each $2 \leq j \leq k$, $A_j = \left\{e_h, E_j \cdot e_{m_1} - \sum_{h=d(j-1)+1}^{d(j)} e_h : d(j) - 1 \leq h \leq d(j)\right\}.$

The semigroup $C^\vee \cap M$ is generated by the following set of irreducible vectors:

$$B = \left\{e_{m_1}^* + \sum_{j=1}^k \sum_{\ell=1}^{m_j-1} a_{j,\ell} \cdot e_{d(j-1)+\ell}^* : 0 \leq \sum_{\ell=1}^{m_j-1} a_{j,\ell} E_j \text{ for } 1 \leq j \leq k\right\}.$$

Indeed, $\#B = \prod_{j=1}^k (m_j-1+E_j)$, the embedding dimension of $SV(\underline{E}, \underline{m})$. Finally, one can record a bijection between the monomial generators of $R_F$ and those typically used to present $SV(\underline{E}, \underline{m})$; cf., [27] Proof of Lem. 4.2] for how the bijection would look in the coordinates $a_{j,\ell}$ for each $j$. □

Feeding $v_C = \sum_{u \in A} u = (\sum_{j=1}^k E_j) \cdot e_{m_2}$ and $B$ into Theorem 3.1 yields $D = \sum_{j=1}^k E_j$. We win! □

In passing, we note that the corresponding class of varieties (Segre-Veronese varieties) form a cornerstone of a lot of investigations in classical and applied algebraic geometry, often being varieties for which one has a more incisive handle on some specified computational- or other task. As a sampling of recent investigations in this vein, we include [2, 3, 4, 9, 22] among our references.

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