Classical capacity of a noiseless quantum channel assisted by noisy entanglement

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We derive the general formula for the capacity of a noiseless quantum channel assisted by an arbitrary amount of noisy entanglement. In this capacity formula, the ratio of the quantum mutual information and the von Neumann entropy of the sender’s share of the noisy entanglement plays the role of mutual information in the completely classical case. A consequence of our results is that bound entangled states cannot increase the capacity of a noiseless quantum channel.

Keywords: Capacity of a quantum channel, quantum entanglement, superdense coding

1. Introduction

One of manifestations of the power of quantum entanglement is superdense coding. In this communication problem, the sender Alice and the receiver Bob share a pair of two-level quantum systems (qubits) in a maximally entangled state, such as the singlet,

\[ \psi = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \]

Alice can transmit two classical bits to Bob by sending only one qubit: to encode one of four messages, Alice applies one of four unitary operations, \( \sigma_x, \sigma_y, \sigma_z, I \) to her half of the singlet and produces one of four mutually orthogonal states. Then, if Alice sends her half to Bob, he can distinguish between the four states and determine which operation was applied.

Both classical and quantum communication can suffer from imperfections of the channel. With the use of coding, information can still be transmitted almost perfectly through a noisy quantum or classical channel, even though the information transmission rate per signal is smaller than that of a perfect channel. The highest transmission rate which is
attainable with a given channel is called the capacity of the channel. Determining the capacities of quantum transmission schemes is one of the central issues in the domain of quantum information theory.

With the above in mind, it is interesting to study imperfect superdense coding schemes. First, the quantum channel (that transmits the qubit) can be noisy, see Ref. [2]. Second, the shared quantum state can suffer from the phenomenon of decoherence that turns pure entangled quantum states into probabilistic mixtures of states, see Refs. [4,5,6].

The generalization of superdense coding for noisy channel was considered in Ref. [2]. The formula for the capacity of an arbitrary noisy quantum channel assisted by unlimited pure entanglement was determined: it is given by a certain maximization of the quantum mutual information of the quantum channel, cf. Ref. [3].

Along the other direction of generalization, Alice and Bob share a noiseless quantum channel, but a mixed (noisy) quantum state. The notion of capacity still applies in this situation. Some partial results were obtained in Refs. [4,5,6]. This is the direction we focus on in this paper. Central to our discussion is a quantity called the coherent information \( I_{sd} \), defined as

\[
I_{sd}(\rho) := \max \{ S(\rho_B) - S(\rho), 0 \},
\]

\( \rho \) and \( \rho_B \) are the reduced density matrices for Alice and Bob respectively.

Let us first summarize the results in Ref. [6]. The achievable classical capacity of a noiseless quantum channel of dimension \( d \) was determined in the following restricted setting. In addition to \( n \) uses of the channel, Alice and Bob share \( n \) copies of the state \( \rho \) acting on \( \mathcal{H}_A \otimes \mathcal{H}_B \), a tensor product of two \( d \)-dimensional Hilbert spaces. The following assumptions were made about the encoding of the classical data: 

1. Alice uses at most one copy of \( \rho \) per use of the \( d \)-dimensional channel.
2. If \( I_{sd}(\rho) = 0 \), Alice does not use the state \( \rho \) in the encoding process and if \( I_{sd}(\rho) > 0 \), the classical data is encoded via a unitary transformation on Alice’s share of \( \rho \) which is sent through the noiseless channel.

Note that each copy of \( \rho \) is encoded independently. In this setting the following capacity formula was derived:

\[
C = \log d + I_{sd}(\rho).
\]

(2)

In this paper we generalize the setting in Ref. [6], namely we remove any restrictions on the encoding and decoding procedure. In our generalized setting, Alice and Bob are still connected by a noiseless channel and they possess an unlimited amount of noisy quantum entanglement \( \rho^\otimes \infty \). We derive the following expression for the capacity of the noisy entanglement assisted channel, given as the rate of information transmission per qubit transmission

\[
C_{sd}(\rho) = \sup_{n} \sup_{\Lambda_A} \left( I_{sd}( (\Lambda_A \otimes I_B)(\rho^\otimes n)) \right),
\]

(3)

where the supremum is taken over all trace-preserving completely positive maps \( \Lambda_A \) (with arbitrary output dimension) which are applied to Alice’s half of the state \( \rho^\otimes n \). In Eq. (3), the role of mutual information is played the quantity \( I_{sd} \) defined for any bipartite state \( \eta \):

\[
I_{sd}(\eta) := \frac{S(\eta_A) + S(\eta_B) - S(\eta)}{S(\eta_A)},
\]

(4)

original definition of the coherent information which appeared in the context of quantum channels was extended to bipartite states in Ref. [7].
where η_{A,B} again denote the reduced density matrices. The mutual information I_{sd} in Eq. (3) has an unusual structure: it represents the interplay between the transmission capability of a state (the numerator) and the number of states that can be sent per transmitted qubit (the denominator). Another feature in Eq. (3) is that the usual maximization over input sources in the classical capacity of a quantum or classical channel is generalized to a maximization over local operations Λ_A.

An important conclusion one can draw from the capacity expression, Eq. (3), is that bound entangled states, i.e. entangled quantum states which are not distillable, are not useful for superdense coding. In other words, they do not provide a capacity greater than 1 which is attainable without the use of entanglement. The expression for the channel capacity, Eq. (3), can be rewritten as

\[ C_{sd}(g) = \sup_n \left[ 1 + \sup_{\Lambda_A} \frac{I^B((\Lambda_A \otimes I_B)(g^{\otimes n}))}{S(g_A)} \right] \]  

(5)

If g is bound entangled, the state (Λ_A \otimes I_B)(g^{\otimes n}) is bound entangled as well. As was shown in Ref. 13, bound entangled states satisfy the reduction criterion 13,14. This implies that they have zero coherent information 14 and thus C_{sd}(g) = 1. In Ref. 12 it was found that bound entangled states are useless as an entanglement resource for quantum teleportation. The results of this paper thus form another demonstration of the qualitative difference between bound and free entanglement in quantum information theory.

2. The channel capacity

Let us start by defining the channel capacity of a noiseless channel assisted by (unlimited) noisy entanglement. Recall that the capacity is defined as the highest rate of faithful transmission per signal sent. More formally, let x \in \{0,1\}^m be an m-bit string to be communicated. Let A^{(d)}_{g^{\otimes n}} be an encoding scheme for Alice which uses her share of g^{\otimes n} and outputs a quantum state in \mathcal{H}_d that is sent through a noiseless d-dimensional quantum channel. Let B^{(d)}_{g^{\otimes n}} be a corresponding decoding scheme for Bob which uses his share of g^{\otimes n} and the received state. Bob should decode the message x with high probability. The capacity, C_{sd}(g), expresses the optimal rate

\[ C_{sd}(g) := \sup_{d,n} \frac{1}{\log d} \left[ \lim_{k \to \infty} \limsup_{m} \{ \exists A^{(d)}_{g^{\otimes nk}} \exists B^{(d)}_{g^{\otimes nk}} \forall x \in \{0,1\}^m F(x, A^{(d)}_{g^{\otimes nk}}, B^{(d)}_{g^{\otimes nk}}) \geq 1 - \epsilon \} \right] \]  

(6)

where F(x, A^{(d)}_{g^{\otimes nk}}, B^{(d)}_{g^{\otimes nk}}) = \langle x | B^{(d)}_{g^{\otimes nk}} A^{(d)}_{g^{\otimes nk}} (|x\rangle \langle x|) |x\rangle is the probability for Bob to receive the correct message. The defining formula can be understood in the following manner: the channel used by Alice and Bob is d-dimensional, and the number of entangled mixed states g used per single channel is n. Given d and n, we consider a large number k of uses of the d-dimensional channel and some corresponding block encoding that uses nk copies of g. Note that the possibility of entangling inputs at the encoding stage, which in the case of the classical capacity of quantum channels may give rise to a nonadditive capacity, see e.g. Ref. 15, is included in the final two suprema above.
The problem of determining the channel capacity \( C_{sd}(\rho) \) thus decomposes into two parts. Let \( C_d(\rho^\otimes n) \) be the expression inside the square brackets on the right-hand-side of Eq. (6) so that

\[
C_{sd}(\rho) = \sup_{d,n} \frac{C_d(\rho^\otimes n)}{\log d}. \tag{7}
\]

First, for fixed \( d \) and \( n \), we will determine the capacity \( C_d(\rho^\otimes n) \). Second, we will consider what happens when we take the supremum over \( d \) and \( n \).

The expression \( C_d(\rho^\otimes n) \) can be determined using the general framework of transmitting classical information using quantum resources. Classical messages \( i \), occurring with probability \( p_i \), are encoded into quantum states \( \psi_i \), and are sent to Bob, who receives the states \( \rho_i \). In general, \( \rho_i \) can be different from \( \psi_i \) and can be mixed if the transmission is noisy. Bob applies an optimal measurement, possibly a joint measurement on blocks of states, to recover the encoded classical information. The classical capacity of the quantum channel is given by a maximization of the Holevo information:

\[
I_H(\{p_i, \rho_i\}) = S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i). \tag{8}
\]

We can apply Eq. (8) to our problem of superdense coding with a noiseless channel and noisy entanglement, when identifying \( \rho_i \) with the final state possessed by Bob. We have depicted the most general communication protocol for \( C_d(\rho^\otimes n) \) in Fig. 1. The most general encoding that Alice can do is to apply to her half of the state \( \rho^\otimes n \) a trace-preserving quantum operation \( \Lambda_i \) with probability \( p_i \), corresponding to the classical data \( i \) that she would like to transmit. We require that the output of \( \Lambda_i \) for every \( i \) acts on a \( d \)-dimensional space, so that it can be sent through the channel, see Fig. 1. After the transmission, Bob possesses the state \( (\Lambda_i \otimes I)(\rho^\otimes n) \) when the message is \( i \), with apriori probability \( p_i \). We maximize the Holevo information of this ensemble, indicated with the dashed line in the figure, under all possible encoding schemes (i.e. local operations and probability distributions \( \{p_i, \Lambda_i\} \))

\[
C_d(\rho^\otimes n) = \sup_{\{p_i, \Lambda_i\}} S\left(\sum_i p_i \rho_i^{(n)}\right) - \sum_i p_i S(\rho_i^{(n)}), \tag{9}
\]

where \( \rho_i^{(n)} := (\Lambda_i \otimes I)(\rho^\otimes n) \). This expression can be considerably simplified and we will show that it is in fact equal to

\[
C_d(\rho^\otimes n) = \log d + \sup_{\Lambda^{(d)}} I^B(\Lambda^{(d)} \otimes I)(\rho^\otimes n)), \tag{10}
\]

where the maximum is taken over all trace-preserving completely positive maps \( \Lambda^{(d)} \) with an output acting on a \( d \)-dimensional Hilbert space.

To prove Eq. (10), we will first estimate it from above, by using the encoding scheme \( \{p_i, \Lambda_i\} \) which maximizes the formula. Using the subadditivity of entropy and the fact that Alice’s actions do not affect Bob’s part of the states, we obtain

\[
C_d(\rho^\otimes n) \leq S(\sum_i p_i \rho_i^{(n)}_{A}) + S(\rho_B^{\otimes n}) - \sum_i p_i S(\rho_i^{(n)}). \tag{11}
\]
Here $\rho_A^{(n)}$ and $\rho_B^{\otimes n}$ are the reduced density matrices of $\rho_i^{(n)}$ for Alice and Bob. The entropy of $\sum_i p_i \rho_i^{(n)}$ cannot exceed $\log d$ since it acts on a $d$-dimensional Hilbert space. Furthermore, we can estimate
\[ \sum_i p_i S(\rho_i^{(n)}) \geq \min_{\Lambda^{(d)}} S((\Lambda^{(d)} \otimes I)(\rho^{\otimes n})). \] (12)
These two bounds together give
\[ C_d(\rho^{\otimes n}) \leq \log d + \sup_{\Lambda^{(d)}} \left[ S(\rho_B^{\otimes n}) - S((\Lambda^{(d)} \otimes I)(\rho^{\otimes n})) \right]. \] (13)

Note that one particular choice of $\Lambda^{(d)}$ is for Alice to trace over $\rho^{\otimes n}$ locally and to transmit classical signals of length $\log d$, which corresponds to $S(\rho_B^{\otimes n}) - S((\Lambda^{(d)} \otimes I)(\rho^{\otimes n})) = 0$. The last term in Eq. (13) which is a supremum over $\Lambda^{(d)}$ is therefore nonnegative, so that we can replace it with $\sup_{\Lambda^{(d)}} I^B((\Lambda^{(d)} \otimes I)(\rho^{\otimes n}))$.

What is important is that the bound in Eq. (13) can be achieved. In other words, there are $\{p_i, \Lambda_i\}$ that make Eq. (9) equal to Eq. (13). These can be found as follows. Let $U_i$ for $i = 1, \ldots, d^2$ be a set of unitary operations on $\mathcal{H}_d$ such that $\sum_i U_i M U_i^\dagger = 0$ for all traceless matrices $M$, see Refs. [13, 20]. Then $p_i = d^{-2}$ and $\Lambda_i(\eta) = U_i \Lambda^{(d)}(\eta) U_i^\dagger$, where $\Lambda^{(d)}$ is the optimal map defined by Eq. (13). To this end, Alice first applies the optimal map $\Lambda^{(d)}$ defined by Eq. (13), and then subjects the resulting states to the “unitary encoding scheme”, which applies $U_i$ with uniform probability. It is immediate that the first term in Eq. (9) becomes $\log d + S(\rho_B^{\otimes n})$, and the second term to be subtracted becomes $S((\Lambda^{(d)} \otimes I)(\rho^{\otimes n}))$, so that Eq. (9) indeed equals Eq. (13).

The expression that we have found for $C_d(\rho^{\otimes n})$ reduces an optimization over an ensemble of encoding to a single one for Alice that maximizes the coherent information of
the output has to fit into the
tion of the resulting joint state under a local action which is constra infined in the sense that 

I coherent information space $H$

yes, as shown by the following example due to Bennett. Let Alice’s part of the Hilbert space $\mathcal{H}_A$ be of the form $\mathcal{H}_A = \mathcal{H}_{A'} \otimes \mathcal{H}_{A''}$, and the state $\varrho$ be of the form $\varrho_{A'} \otimes \varrho_{A''B}$ with $S(\varrho_{A'}) > 0$ and $I^B(\varrho_{A''B}) > 0$. Then the total coherent information $I^B(\varrho)$ can be increased by discarding the state $\varrho_{A'}$. It is an open question whether this example is generic, i.e. whether an increase of $I^B$ necessarily involves discarding a part of Alice’s system. If this is the case, then it should be impossible to increase $I^B$ for a two-qubit state, where Alice cannot discard part of the system (the latter is already the smallest possible one).

We have performed some numerical work to explore this question for two-qubit states $\varrho$. Because of the convexity of $I^B$ in $\varrho$, we can restrict ourselves to extremal maps $\Lambda$. The results of random sampling over states $\varrho$ and local extremal maps $\Lambda: B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_2)$ suggest that there are no examples for which $I^B((\Lambda \otimes 1)(\varrho)) > I^B(\varrho) \geq 0$.

Let us now consider the fully general case when Alice and Bob share the states $\varrho^\otimes_n$ for free and we consider the information transmission rate through a noiseless quantum channel. In other words, we consider the suprema over $n$ and $d$. Before we proceed with a mathematical derivation of Eq. (3), we argue how Alice can improve her encoding of the classical data. Suppose that after maximizing the coherent information as in Eq. (1), the resulting state $\varrho_A^{(n)}$ has some local entropy $S(\varrho_A^{(n)})$, and some positive coherent information $I^B$. Then, after the final unitary encoding, Alice sends her half of this state through the $d$-dimensional channel then the rate of information transmission will be $\log d + I^B$ per use of the $d$-dimensional channel, or $1 + I^B/\log d$ per qubit sent. However, a quantum state (which can be part of a larger quantum system) such as $\varrho_A^{(n)}$ can be transmitted in fewer qubits by using Schumacher compression [2]. Suppose Alice performs her local operation on, say, $k$ blocks of states, see Fig. 2 and then applies a compression step which has $kS(\varrho_A^{(n)})$ qubits of output. She will perform the unitary encoding after this compression step. The input dimension of the channel is now equal to $2^{kS(\varrho_A^{(n)})}$ and therefore the corresponding transmission rate is $kS + kI^B$, which gives a value of $1 + I^B/S$ per transmitted qubit. Thus if $S(\varrho_A^{(n)})$ is smaller than $\log d$ then the described strategy enhances the transmission rate.

From the considerations above we see that in order to achieve a high transmission rate Alice should try (by a local action) to maximize the coherent information of the states that she shares with Bob and at the same time try to minimize the entropy of her part of the states. The coherent information and the local entropy are not independent quantities: Alice can easily make the entropy of her part of $\varrho$ to be zero, for example by throwing away the states, but then she will also make $I^B$ zero.

From these reasonings we expect the formula for the full capacity to contain a trade-off between Alice’s local entropy and the coherent information of the states. An alternative way of illustrating these intuitions is the following. The coherent information of the state describes the information transmission rate offered by the state. On the other hand, the
entropy says how many states can be sent per qubit. As a result we have, roughly speaking,

\[ \text{capacity} = \text{transmission rate per state} \times \text{number of states per qubit}, \quad (14) \]

which is the quantity to be optimized.

Let us now pass to a rigorous mathematical derivation of our capacity formula, Eq. (3). We can upper bound the capacity \( C_{sd}(\varrho) \) in Eq. (3) as follows (using Eq. (11)),

\[
C_{sd}(\varrho) \leq \sup_d \sup_n \left[ 1 + \sup_{\Lambda(d)} \frac{I_B((\Lambda(d) \otimes I)(\varrho^{\otimes n}))}{S(\Lambda(d)(\varrho_A^{\otimes n})))} \right] \leq \sup_n \left[ 1 + \sup_{\Lambda} \frac{I_B((\Lambda \otimes I)(\varrho^{\otimes n}))}{S(\Lambda(\varrho_A^{\otimes n})))} \right]. \quad (15)
\]

The first inequality follows from the fact that after Alice’s action the entropy of Alice’s part cannot exceed \( \log d \). In the right-hand-side of the last inequality the supremum is taken over arbitrary trace-preserving operations by Alice: the constraint on the output dimension is removed. The supremum over \( d \) can then be omitted since the expression no longer depends on the dimension \( d \).

This upper bound can be achieved by using the scheme depicted in Fig. 2. We pick the optimal \( n \) and \( \Lambda \) in the last expression in Eq. (15). Alice subjects \( \varrho^{\otimes n} \) to the optimal map \( \Lambda \) which results in a state with coherent information \( I \) and a local entropy of...
Alice's part which we denote as $S$. As described above applying the unitary encoding on the Schumacher-compressed states leads to a transmission rate of $1 + I/S$ which is the desired result. Instead of writing $1 + I/S$ we can write $I_M/S$ where the quantum mutual information $I_M$ is given by

$$I_M(\rho) = S(\rho_A) + S(\rho_B) - S(\rho). \quad (16)$$

In this way we obtain the capacity expression in Eq. (3). Note that when $\rho$ is an arbitrary entangled pure state $\psi$, we obtain $C_{sd}(|\psi\rangle\langle\psi|) = 2$, as one may expect.

To summarize, we derive $C_{sd}(\rho)$ from first principle in two steps. We first express $C_d(\rho^{\otimes n})$ in terms of Holevo's information, obtain an upper bound, and provide a protocol (in terms of the optimal $\Lambda^{(d)}$ and a unitary encoding) to achieve it. Second, we use the expression for $C_d(\rho^{\otimes n})$ to obtain an upper bound for $C_{sd}(\rho)$, and again provide a protocol (using block encoding with $\Lambda^{(d)}$, Schumacher compression and finally unitary encoding) to achieve it.

It would be interesting to explore how the capacity expression changes when, instead of a noiseless channel, Alice and Bob are using a noisy channel. One expects to find an expression which coincides with the entanglement-assisted capacity of Ref. 2 when the additional entangled states $\rho$ are pure.

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