Efficient PAC Reinforcement Learning in Regular Decision Processes

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Abstract

Recently regular decision processes have been proposed as a well-behaved form of non-Markov decision process. Regular decision processes are characterised by a transition function and a reward function that depend on the whole history, though regularly (as in regular languages). In practice both the transition and the reward functions can be seen as finite transducers. We study reinforcement learning in regular decision processes. Our main contribution is to show that a near-optimal policy can be PAC-learned in polynomial time in a set of parameters that describe the underlying decision process. We argue that the identified set of parameters is minimal and it reasonably captures the difficulty of a regular decision process.

1 Introduction

The standard RL setting [Sutton and Barto, 2018] has long been characterised by the Markov assumption, which is arguably limiting. Regular decision processes (RDPs) have been proposed as a formalism with the promise to dispense with the Markov assumption while retaining good computational properties [Brafman and De Giacomo, 2019]. Complete knowledge of an RDP allows one to compute an equivalent Markov decision process (MDP), in a precise formal sense, and then to apply standard solution techniques for MDPs. This correspondence is not immediately applicable in the standard reinforcement learning (RL) setting, where an agent has no prior knowledge of the decision process. Instead, one needs to learn simultaneously a model of the dependencies on the past, and how observations and rewards evolve [Brafman and De Giacomo, 2019]. The possibility for an RL agent to learn such a model in the form of an automaton has been exploited in [Abadi and Brafman, 2020]. The work provides empirical evidence that algorithms based on automata learning can find near-optimal policies for certain RDPs. However, the proposed technique requires exponential time to converge unless only a small number of traces have high probability to be observed: it relies on statistics for single traces that may have exponentially-low probability, and hence require exponential time to be observed a number of times that is sufficient to compute accurate statistics.

In this paper, for the first time to the best of our knowledge, we introduce a technique for RL in RDPs which has formal guarantees analogous to those studied in literature for RL in MDPs [Fiechter, 1994; Brafman and Tennenholtz, 2002; Kearns and Singh, 2002; Auer et al., 2002; Kakade, 2003; Strehl and Littman, 2005; Auer and Ortner, 2006; Strehl et al., 2009; Audibert et al., 2009; Jaksch et al., 2010; Szita and Szepesvári, 2010; Lattimore and Hutter, 2014; Dann and Brunskill, 2015; Dann et al., 2017; Azar et al., 2017; Bai et al., 2020; Wang et al., 2020].

We show that RL in RDPs is feasible in polynomial time with respect to the probably approximately correct (PAC) learning criterion [Valiant, 1984; Kearns and Vazirani, 1994]. We present an algorithm that computes a near-optimal policy with high confidence, in a number of steps that is polynomial in the required accuracy and confidence, and in a set of parameters that describe the underlying RDP. The algorithm first learns a probabilistic deterministic finite automaton (PDFA) that represents the underlying RDP, then it maps it to an MDP, which is solved to compute an intermediate stationary policy, which is then turned into the final policy by composing it with the transition function of the learned PDFA. This process is repeated many times, yielding a better policy as more data becomes available, and converging in polynomial time.

Our results show that the Markov assumption can be largely removed while maintaining polynomial-time guarantees. This is a fundamental positive result, that should boost the confidence in finding effective RL techniques that dispense with the Markov assumption. Furthermore, our results should draw attention on RDPs, which have the potential of becoming the standard formalism for RL in the non-Markov setting, similarly to MDPs in the Markov setting.

Our work shows a strong connection between RDPs and PDFA, which allows one to take advantage of the many fundamental results and learning algorithms available for PDFA [Kearns et al., 1994; Ron et al., 1998; Clark and Tholhammer, 2004; Palmer and Goldberg, 2007; Balle et al., 2013; Balle et al., 2014]. On one hand, the negative results from [Kearns et al., 1994] about the impossibility of polynomial-time learning when state distinguishability is low transfer to RDPs. On the other hand, when state distinguishability is sufficiently high, PDFA techniques allow for polynomial-time learning (cf. [Ron et al., 1998; Balle et al., 2013]), and we
show that such techniques can be leveraged to learn RDPs. In particular, we claim that PDFA techniques are the only known techniques so far to effectively learn states when they determine probability distributions, as it happens in RDPs.

Complete proofs of all our technical results are deferred to the appendix.

2 Preliminaries

Transducers. We follow [Moore, 1956]. A transducer is a tuple \((Q, q_0, \Sigma, \tau, \Gamma, \theta)\) where: \(Q\) is a finite set of states; \(q_0 \in Q\) is the initial state; \(\Sigma\) is the finite input alphabet; \(\tau : Q \times \Sigma \to Q\) is the deterministic transition function; \(\Gamma\) is the finite output alphabet; \(\theta : Q \to \Gamma\) is the output function.

We extend the use of \(\tau\) to strings of length greater than one as \(\tau(q, \sigma_1, \sigma_2, \ldots, \sigma_n) = \tau(\tau(q, \sigma_1), \sigma_2, \ldots, \sigma_n)\), and to the empty string as \(\tau(q, \varepsilon) = q\). We also extend the use of \(\theta\) to arbitrary strings as \(\theta(q, \sigma_1, \ldots, \sigma_n) = \theta(q) \theta(\tau(q, \sigma_1), \sigma_2, \ldots, \sigma_n)\) where the base case is \(\theta(q, \varepsilon) = \theta(q)\). Furthermore, we write
\[
\tau(\sigma_1, \ldots, \sigma_n) \text{ for } \tau(q_0, \sigma_1, \ldots, \sigma_n),
\]
and we write \(\theta(\sigma_1, \ldots, \sigma_n)\) for \(\theta(q_0, \sigma_1, \ldots, \sigma_n)\). Finally, we call \(\theta(\sigma_1, \ldots, \sigma_n)\) the output of the transducer on \(\sigma_1, \ldots, \sigma_n\).

Probabilistic Automata. We follow [Balle et al., 2013]. A Probabilistic Deterministic Automaton (P DFA) is a tuple \(A = (Q, \Sigma, \tau, \lambda, \zeta, q_0)\) where: \(Q\) is a finite set of states; \(\Sigma\) is an arbitrary finite alphabet; \(\tau : Q \times \Sigma \to Q\) is the transition function; \(\lambda : Q \times (\Sigma \cup \{\varepsilon\}) \to [0, 1]\) defines the probability of emitting each symbol from each state \((\lambda(q, \sigma) = 0)\) when \(\sigma \in \Sigma\) and \(\tau(q, \sigma)\) is not defined; \(\zeta\) is a special symbol not in \(\Sigma\) reserved to mark the end of a string; \(q_0 \in Q\) is the initial state. It is required that: (i) \(\lambda(q, \varepsilon) = 0\) when \(\varepsilon \in \Sigma\) and \(\tau(q, \varepsilon)\) is not defined; (ii) for every state \(q \in Q\), \(\sum_{\sigma \in (\Sigma \cup \{\varepsilon\})} \lambda(q, \sigma) = 1\); (iii) for each state \(q \in Q\) that can be reached from the initial state \(q_0\) with non-zero probability, there is a state \(q' \in Q\) with \(\lambda(q', \zeta) > 0\) that can be reached from \(q\). The transition function \(\tau\) is extended to strings as for transducers, and the probability function is extended as \(\lambda(q, \varepsilon) = 1\) and \(\lambda(q, \sigma_1, \sigma_2, \ldots, \sigma_n) = \lambda(q, \sigma_1) \lambda(q, \sigma_2, \ldots, \sigma_n)\). Then, the probability that \(x \in \Sigma^*\) is a string generated by the automaton starting from state \(q\) is \(\lambda(q, x)\), and the probability that it is a prefix \(\lambda(q, x)\). For \(\mu > 0\), we say that \(A\) is \(\mu\)-distinguishable if \(\max_x |\lambda(q_1, x) - \lambda(q_2, x)| \geq \mu\) for every two distinct states \(q_1\) and \(q_2\).

Non-Markov Decision Processes. A Non-Markov Decision Process (NMDP) (cf. [Brafman and De Giacomo, 2019]) is a tuple \(D = (A, S, R, T, \Gamma, \gamma)\) where: \(A\) is a finite set of actions; \(S\) is a finite set of states (or observation states to distinguish them from automata and transducer states); \(R \subseteq \mathbb{R}_{\geq 0}\) is a finite set of non-negative reward values; \(T : S^* \times A \times S \to [0, 1]\) is the transition function which defines a probability distribution \(T(\cdot, a, \cdot)\) over \(S\) for every \(h \in S^*\) and every \(a \in A\); \(R : S^* \times A \times S \to \mathbb{R}\) is the reward function; \(\gamma \in (0, 1)\) is the discount factor. The transition and reward functions can be combined into the dynamics function \(D : S^* \times A \times S \to [0, 1]\) which defines a probability distribution \(D(\cdot, a, \cdot)\) over \(S\) for every \(h \in S^*\) and every \(a \in A\). Namely, \(D(s, r, a, s') = \delta(s', a) \mu, 0)\), and zero otherwise. Every element of \(S^*\) is called a history. A policy is a function \(\pi : S^* \times A \to [0, 1]\) that, for every history \(h\), defines a probability distribution \(\pi(h)\) over the actions \(A\).

A policy \(\pi\) is deterministic on a history \(h\) if \(\pi(a|h) = 1\) for some action \(a\), in which case we write \(\pi(h) = a\); and it is deterministic if it is deterministic on every history. A policy \(\pi\) is uniform on a history \(h\) if \(\pi(a|h) = 1/|A|\) for every action \(a \in A\). We call \(\pi_{un}\) the policy that is uniform on every history.

Every element of \((ASR)^*\) is called a trace. The dynamics of \(P\) under a policy \(\pi\) describe the probability of an upcoming trace, given the history so far, when actions are chosen according to a policy \(\pi\); it can be recursively computed as \(D_{\pi}(\text{asr} h) = \pi(a|h) \cdot D(s, r, a) \cdot D_t(h, s)\), with base case \(D_{\pi}(\varepsilon|h) = 1\) for \(\varepsilon\) the empty trace. The value of a policy \(\pi\) on a history \(h\), written \(v_{\pi}(h)\), is the expected discounted sum of future rewards when actions are chosen according to \(\pi\) given that the history so far is \(h\); it can be recursively computed as \(v_{\pi}(h) = \sum_{asr} \pi(a|h) \cdot D(s, r, a) \cdot (r + \gamma \cdot v_{\pi}(h))\). The optimal value of an action \(a\) on a history \(h\) is \(q_{\pi}(h, a) = \max_a \sum_{asr} D(s, r, a) \cdot (r + \gamma \cdot v_{\pi}(h))\). A policy \(\pi\) is optimal on a history \(h\) if \(v_{\pi}(h) = v_{\pi}(h)\) for every \(\varepsilon > 0\), a policy \(\varepsilon\)-optimal on \(h\) if \(v_{\pi}(h) \geq v_{\pi}(h) - \varepsilon\). A policy is optimal (resp., \(\varepsilon\)-optimal) if it is so on every history. We also say near-optimal to say \(\varepsilon\)-optimal for some \(\varepsilon\).

Regular Decision Processes. A Regular Decision Process (RDP) [Brafman and De Giacomo, 2019] is an NMDP \(P = (A, S, R, T, \Gamma, \gamma)\) whose transition and reward functions can be represented by finite transducers. Specifically, there is a finite transducer that, on every history \(h\), outputs the function \(T_h : A \times S \to [0, 1]\) induced by \(T\) when its first argument is \(h\); and there is a finite transducer that, on every history \(h\), outputs the function \(R_h : A \times S \times R \to [0, 1]\) induced by \(R\) when its first argument is \(h\). Note that the cross-product of such transducers yields a finite transducer for the dynamics function \(D\) of \(P\).

Markov Decision Processes. A Markov Decision Process (MDP) [Bellman, 1957; Puterman, 1994] is an NMDP \(\langle A, S, R, T, \Gamma, \gamma \rangle\) where both the transition function and the reward function (and hence the dynamics function) depend only on the last state in the history, which is never empty since every history starts with a state that is chosen arbitrarily. Specifically, for every pair of non-empty histories \(h_1, s\) and \(h_2, s\), it holds that \(T(h_1, s) = T(h_2, s)\) and \(R(h_1, s) = R(h_2, s)\). Similarly, we say that a policy \(\pi\) is stationary if \(\pi(h_1, s) = \pi(h_2, s)\) for every pair of non-empty histories \(h_1, s\) and \(h_2, s\). In the case of MDPs, we can see all history-
dependent functions—e.g., transition and reward functions, value functions, policies—as taking a single state in place of a history.

3 PAC-RL in RDPs

We consider reinforcement learning (RL) as the problem of an agent that has to learn an optimal policy for an unknown RDP $\mathcal{P}$, by acting and receiving observation states and rewards according to the transition and reward functions of $\mathcal{P}$. The agent performs a sequence of actions $a_1 \ldots a_n$, receiving an observation state $s_i$ and a reward $r_i$ after each action $a_i$. The agent has the opportunity to stop and possibly start over. This process generates strings of the form $a_1s_1r_1 \ldots a_n s_n r_n$, which we call episodes. They form the experience of the agent, which is the basis to learn an optimal policy.

**Example 1.** As a running example, we consider an agent that has to cross a $2 \times m$ grid while avoiding enemies. Every enemy guards a two-cell column: enemy $i$ guards cells $(0, i)$ and $(1, i)$, and it is found in cell $(0, i)$ with probability $p_{i0}$ or $p_{i1}$. Initially the probability is $p_{i0}$, and the two probabilities are swapped every time the agent hits an enemy. At every step, say $i$, the agent has to decide whether to go through cell $(0, i)$ or $(1, i)$, taking action $a_0$ or $a_1$, respectively. Specifically, when in cell $(0, i)$ or $(1, i)$, action $a_0$ leads to $(0, i + 1)$ and action $a_1$ to $(1, i + 1)$. From the last column, the agent is brought back to the first one. The agent receives reward one every time it avoids an enemy. Each observation state $s$ is a triple $(i, j, e)$ where $i \in \{0, 1\}$ and $j \in \{0, m-1\}$ are the coordinates of the agent and $e \in \{\text{enemy, clear}\}$ denotes whether an enemy is in the agent’s current cell. To maximise rewards, the agent has to learn to predict the enemies’ positions, which depend on the history of observation states in a regular manner. Such a dependency is described by a transducer with $2m$ states of the form $(q_1, q_2)$ where $q_1 \in \{0, m-1\}$ stores the agent’s current column and $q_2$ is a bit that keeps track of whether probabilities $p_{i0}$ or $p_{i1}$ are being used.

Our goal is to understand how fast an agent can learn a near-optimal policy. In particular, we want the agent to find near-optimal policies in the shortest possible time, and we are not interested in the agent maximising the collected rewards during learning. Thus, we require the agent to return policies $\pi_1, \pi_2, \ldots$ that improve over time. The idea is that these policies can be passed to an executing agent, living in such an environment, whose behaviour will improve as it receives better policies. Then, the central question is how fast an agent can reach a point after which it outputs only good policies.

Now we make our discussion more precise. To measure the learning performance of an agent, we consider the number of steps it performs.

**Definition 1.** An action step consists in performing an action and collecting the resulting observation-state and reward. A step is an action step or an elementary computation step.

Then, to have a reasonable notion of learning, we borrow from the Probabilistically Approximately Correct (PAC) framework [Valiant, 1984; Kearns and Vazirani, 1994], similarly to what is done in previous work on RL in MDPs (cf. [Fiechter, 1994; Kearns and Singh, 2002; Strehl et al., 2009]). The PAC framework is based on the observation that exact learning is infeasible. Thus, it introduces two parameters $\epsilon > 0$ and $\delta \in (0, 1)$ that describe the required accuracy and the confidence of success, respectively. In the RL setting it translates into looking for policies that are $\epsilon$-optimal with probability at least $1 - \delta$. Then, RL is considered feasible if these policies can be found in a number of steps that is polynomial in $1/\epsilon$ and $\ln(1/\delta)$, and in other parameters that describe the underlying RDP.

**Definition 2.** An RL agent (or algorithm) is said to reach accuracy $\epsilon$ and confidence $\delta$ in the moment it returns the first policy $\pi_*$ such that $\pi_*$ is $\epsilon$-optimal with probability at least $1 - \delta$ and the same holds for every policy returned after $\pi_*$.\n
**Definition 3.** Let $d_\mathcal{P}$ be a list of parameters describing $\mathcal{P}$, let $A$ be its actions, and let $\gamma$ be its discount factor. An RL agent (or algorithm) is PAC-RL with respect to $d_\mathcal{P}$ if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, given $(A, \gamma, \epsilon, \delta)$ as input, it reaches accuracy $\epsilon$ and confidence $\delta$ in poly$(1/\epsilon, \ln(1/\delta), d_\mathcal{P})$ steps.

We propose to describe an RDP $\mathcal{P} = (A, S, R, T, R, \gamma)$ using the following parameters.

$$d_\mathcal{P} = \left( |A|, \frac{1}{1 - \gamma}, R_{\text{max}}, n, \frac{1}{\rho}, \frac{1}{\mu}, \frac{1}{\eta} \right)$$

The first three parameters take into account the number of actions $|A|$, the discount factor $\gamma$, the maximum reward value $R_{\text{max}}$, and the number of states $n$ of the minimum transducer for the dynamics of $\mathcal{P}$. Then, the reachability $\rho$ of an RDP measures how easy it is to reach states of the dynamics transducer $T$ when actions are taken uniformly at random.

**Definition 4.** The reachability of $\mathcal{P}$ is the minimum non-zero probability $\rho$ that a given state of the dynamics transducer $T$ is reached from the initial state within $n$ steps (with $n$ the number of states) when actions are chosen uniformly at random.

The distinguishability $\mu$ of an RDP measures how easy it is to distinguish states of the dynamics transducer $T$ when actions are taken uniformly at random. It is a parameter inherited from the PDFA literature [Ron et al., 1998; Balle et al., 2013].

**Definition 5.** The distinguishability of $\mathcal{P}$ is the minimum $\mu \in (0, 1)$ such that one of the following conditions holds for every two histories $h_1, h_2$, where $\pi_*$ is the uniform policy:

- $D_{\pi_*}(t|h_1) = D_{\pi_*}(t|h_2)$ for every trace $t$;
- $|D_{\pi_*}(t|h_1) - D_{\pi_*}(t|h_2)| > \mu$ for some trace $t$.

The degree of determinism $\eta$ measures how easy it is to discover transitions. If $\eta$ is small, then there is some transition that is possible, but unlikely to be observed. Note that this parameter takes value one when the RDP is deterministic.

**Definition 6.** The degree of determinism $\eta$ of $\mathcal{P}$ is the minimum non-zero probability that $T(\cdot|h, a)$ assigns to an observation state.

**Example 2.** Consider the RDP of Example 1. Its reachability $\rho$ is 1/2, since, by an inductive argument, every state reachable in $i$ steps has probability at least 1/2 to be visited at step $i$ and, from there, each of the two next states is visited...
with probability 1/2. The degree of determinism $\eta$ is the minimum, for any $i$, among the probabilities $p_i^0$ and $p_i^1$, and their complements. The distinguishability $\mu$ is given by the uniform probability of an action 1/2 times the minimum among $\eta$ and the values $|p_i^0 - p_i^1|$. 

One can show that all the above parameters are necessary. For the number of actions, discount, and maximum reward parameters (and also for the accuracy and confidence parameters) the result follows from a known lower bound for the MAB problem [Mannor and Tsitsiklis, 2004]. Then, the number of transducer states, reachability, and degree of determinism can be shown to increase the number of action steps required to visit the relevant parts of the dynamics transducer a sufficient number of times. Finally, for the distinguishability parameter, it is straightforward to adapt the hardness proof for PDFA given in [Kearns et al., 1994], which assumes hardness of learning noisy parity function, a standard cryptographic assumption.

**Theorem 1.** There is no algorithm that is PAC-RL with respect to a strict subset of the parameters $d_P$ given in (1), if it is hard to learn noisy parity functions.

4 PAC-RL via Probabilistic Automata

We present an RL algorithm for RDPs that relies on probabilistic automata. We consider an RDP $\mathcal{P} = \langle A, S, R, T, \mathbf{r}, \eta \rangle$, its dynamics function $\mathbf{D}$, and the minimum transducer $T = \langle \hat{Q}, q_0, S, \tau, T, \mathbf{D}(s, r|\theta) \rangle$ that represents $\mathbf{D}$ in the sense that $\theta(h)(a, s, r) = \mathbf{D}(s, r|h, a)$.

**Representing RDPs as PDFA.** When learning, the agent has the possibility to experience multiple episodes. In particular, the agent has an extra action, called stop action, that ends the current episode and starts a new one. To take advantage of that, the agent generates episodes following a stationary policy that chooses to stop with a non-zero probability $p$, and it chooses an action from $A$ uniformly if it does not stop.

**Definition 7.** The stop action is a special action $\zeta$ that allows the agent to terminate the current episode and start a new one. The exploration policy with stop probability $p > 0$, written $\pi_p$, is the policy that selects the stop action $\zeta$ with probability $p$, and each action from $A$ with probability $(1 - p)/|A|$.

Under an exploration policy, an RDP determines a probability distribution on traces, and hence can be seen as a PDFA. Specifically, the dynamics of $\mathcal{P}$ under $\pi_p$ are captured by the PDFA $\mathcal{A} = \langle \hat{Q}, \hat{\Sigma}, \hat{\mathbf{r}},\hat{\lambda},\hat{\zeta},\hat{q}_0 \rangle$ where:

- alphabet $\hat{\Sigma} = \{asr \in ASR \mid \exists h. \mathbf{D}(s, r|h, a) > 0\}$,
- transitions $\hat{\mathbf{r}}(q, asr) = \mathbf{r}(q, s)$,
- probability function:
  - $\hat{\lambda}(q, asr) = ((1 - p)/|A|) \cdot \theta(q)(a, s, r)$,
  - $\hat{\lambda}(q, \zeta) = p$.

The dynamics of $\mathcal{P}$ are captured in the sense that the following holds for every history $h$, trace $t$, and string $h'$ such that the projection of $h'$ on observation states coincides with $h$:

$\mathbf{D}_{\pi_p}(t|h) = \lambda(\mathbf{r}(q_0, h'), t)$.

**Algorithm 1** Reinforcement Learning RL($A, \gamma, \epsilon, \delta$)

**Input:** Actions $A$, discount factor $\gamma$, required precision $\epsilon$, confidence parameter $\delta$.

**Output:** Policies.

1: for $\ell = 1, 2, \ldots$ do
2: $p \leftarrow 1/(10\ell + 1)$; $k \leftarrow (2/p) \cdot \ell^2 \cdot (\ell + 5 \ln \ell)$
3: $X \leftarrow \emptyset$; $i \leftarrow 0$; hardStop $\leftarrow$ false
4: while $\neg$hardStop do
5: $x$, hardStop $\leftarrow$ generate an episode under policy $\pi_p$ with a hard stop after $k - i$ actions
6: $i \leftarrow$ increase $i$ by the number of actions in $x$
7: if $\neg$hardStop then
8: $X \leftarrow X \cup \{x\}$
9: end if
10: end while
11: $\Sigma \leftarrow$ symbols in $X$; $\hat{R}_{\text{max}} \leftarrow$ max reward in $X$
12: $\hat{A}$ $\leftarrow$ learn PDFA by calling AdaCT($\ell$, |$\Sigma$|, $\delta/2$, $X$)
13: $\hat{M}$ $\leftarrow$ compute the MDP induced by $\hat{A}$ and $\gamma$
14: $m \leftarrow \frac{1}{1 - \gamma} \cdot \ln \left(\frac{2 \hat{R}_{\text{max}}}{\epsilon (1 - \gamma)^2}\right)$
15: $\pi$ $\leftarrow$ solve $\hat{M}$ by calling ValueIteration($\hat{M}, m$)
16: return transducer for the composition of $\pi$ with the projection-on-states of the transition function of $\hat{A}$
17: end for

**PAC-RL Algorithm.** Algorithm 1 provides the pseudocode of our RL algorithm. The algorithm repeats the following operations for increasing values of an integer variable $\ell$, starting from 1. (Line 2) It computes the stop probability $p$, and the maximum number $k$ of actions to perform during the current iteration. (Line 3) It initialises the set of episodes $X$ to the empty set, a counter $i$ for the actions to zero, and a flag hardStop to know when a hard stop has occurred. (Lines 4–10) It generates episodes following the exploration policy with stop probability $p$, and with a hard stop if the number of performed actions reaches $k$. Episodes are stored in variable $X$. (Line 11) It reads the set $\Sigma$ of action-state-reward symbols from $X$, and the maximum reward $\hat{R}_{\text{max}}$ occurring in $X$. (Line 12) It learns a PDFA $\hat{A} = \langle \hat{Q}, \hat{\Sigma}, \hat{\mathbf{r}},\hat{\lambda},\hat{\zeta},\hat{q}_0 \rangle$ via the AdaCT algorithm [Balle et al., 2013] instantiated with $\ell$ as an upper bound on the number of automaton states, $|\Sigma|$ as an estimate of the size of the alphabet, confidence parameter $\delta/2$, and set of strings $X$. (Line 13) Starting from $\hat{A}$, the algorithm computes the MDP $\hat{M} = \langle A, \hat{Q}, \hat{R}, \hat{D},\hat{\lambda},\hat{q}_0 \rangle$ where $\hat{R}$ consists of each reward value occurring as the third component of an element of $\hat{\Sigma}$, and the dynamics function is as follows:

$\hat{D}(q_2, r|q_1, a) = (|A|/(1 - p)) \sum_{q':\mathbf{r}(q_1, asr)} \hat{\lambda}(q_1, asr)$.

(Lines 14–15) The algorithm then solves $\hat{M}$ via $m$ iterations of the classic value iteration algorithm with action-value estimates initialised to zero. Specifically, value iteration computes an approximation $\hat{q}_s^M$ of the optimal action-value function $q_s^M$ of $\hat{M}$, and then computes the greedy policy $\pi$ with respect $\hat{q}_s^M$, which is a stationary policy on state space $\hat{Q}$.
Definition

\[ e' = (1 - \gamma) \rho \epsilon M \delta_0 \]

\[ e'' = (1 - \gamma)^2 \rho \epsilon M \delta_0 \]

\[ \delta_0 = \frac{\pi \delta}{|A| - 2|\Sigma|} \]

Table 1: Quantities used in the PAC analysis.

| Definition | Value |
|------------|-------|
| \[ e' \] | \((1 - \gamma) \rho \epsilon M \delta_0 \) |
| \[ e'' \] | \((1 - \gamma)^2 \rho \epsilon M \delta_0 \) |
| \[ \delta_0 \] | \(\frac{\pi \delta}{|A| - 2|\Sigma|}\) |

(16) The algorithm returns a transducer that represents the policy function \( \pi(\tau(\cdot)) \) obtained by composing the stationary policy \( \pi \) and the ‘projection’ transition function \( \tau \) defined as \( \tau(q, s) = \hat{\tau}(q, asr) \) for an arbitrary choice of \( a \) and \( r \) such that \( \hat{\lambda}(q, asr) \geq 0 \); note that every choice yields the same result and a choice always exists. Specifically, the returned transducer is \( (\hat{Q}, \hat{q}_0, \hat{S}, \hat{\tau}, A, \theta') \) where \( \theta'(\hat{q}) = \pi(\hat{q}) \) and \( \hat{S} \) consists of each observation-state occurring as the second component of an element of \( \Sigma \).

5 PAC Analysis

Algorithm 1 shows that RL in RDPs is feasible in polynomial time.

Theorem 2. Algorithm 1 is PAC-RL with respect to the parameters \( d_P \) given in (1).

We analyse Algorithm 1 to show that the theorem holds. We first show that we can compute a near-optimal policy for \( P \) starting from a near-optimal policy for the MDP induced by the dynamics transducer \( T \), that is defined as \( M = \langle A, Q, \hat{R}, D^M, \gamma \rangle \) where the dynamics function is as follows:

\[ D^M(q_2, r|q_1, a) = \sum_{s: s' = q_2} \gamma(s_1, s)|q_1\rangle(a, s, r). \]

We call it the ideal MDP, since it is the MDP that the algorithm would build if it learned the automaton \( A \) perfectly. Its key property is that an \( \epsilon \)-optimal policy for \( P \) can be obtained from an \( \epsilon \)-optimal policy for \( M \) by composing it with the transition function of \( T \).

Lemma 1. If \( \pi \) is an \( \epsilon \)-optimal policy for \( M \), then \( \pi(\tau(\cdot)) \) is an \( \epsilon \)-optimal policy for \( P \).

Next we derive the number of episodes and the stop probability (which determines the length of episodes) under which the AdaCT algorithm guarantees to learn \( e'' \)-approximation of \( A \). The accuracy of the algorithm depends on the distinguishability of the automaton to learn, on the minimum probability that a state is visited during a run, and on the minimum transition probability. Thus, we need establish lower bounds for the former three quantities. First, the distinguishability \( A \) is close to the distinguishability \( \mu \) of \( P \) if \( \mu \) is sufficiently small, since states can be distinguished by looking at strings of length at most the number of states \( n \). In fact, for any pair of longer strings witnessing the distinguishability, we can take their substrings of length at most \( n \) obtained by removing ‘cycles’, similarly to what is done in the pumping lemma for regular languages. The bound then follows by taking into account the probability \( (1 - \rho)^n \) of generating a string of length at least \( n \). Second, the probability of visiting a state of \( A \) is at least the probability of generating a string of length at least \( n \) times the probability of visiting the corresponding state in \( T \) while generating the string, which is at least the reachability \( \rho \) of \( P \). Third, the minimum transition probability in \( A \) is at least the degree of determinism \( \eta \) of \( P \) times the probability of picking an action uniformly when not stopping.

Lemma 4. The distinguishability of \( A \) is at least \( \mu \cdot (1 - \rho)^n \). The minimum non-zero probability that a state of \( A \) is visited during a run is at least \( \rho \cdot (1 - \rho)^n \). The minimum non-zero probability of a non-\( \xi \) transition of \( A \) is at least \( \eta \cdot (1 - \rho)^{|A|} \).

Given the bounds above, the next lemma follows from the guarantees for the AdaCT algorithm [Balle et al., 2013]. In
particular, the guarantees require a number of episodes that depends on the specified upper bound on the states, on the specified alphabet size, on the required accuracy and confidence, and on the three quantities of the previous lemma, which applies considering that \((1 - p)^n \geq 0.9\) because of the condition on \(p\).

**Lemma 5.** If \(p \leq 1/(10n + 1)\), then \(\hat{n}\) is an upper bound on the number \(n\) of states of an A, and X are strings generated by A with \(|X| \geq \max(N', N'')\), then \(\text{AdaCT}(\hat{n}, \delta, \delta/2, X)\) returns an \(\epsilon''\)-approximation of A with probability \(1 - \delta/2\).

The required sample size and stop probability are achieved in a polynomial number \(\ell\) of iterations of the algorithm. The sample size also guarantees \(\Sigma = \Sigma\) and \(R_{\text{max}} = R_{\text{max}}\). Furthermore, introducing a logarithmic dependency on \(\delta\) ensures that a hard stop occurs with probability at most \(\delta/2\), which combined with the probability of failure of \(\text{AdaCT}\) is still less than \(\delta\). The \(\ell\)-th iteration of the algorithm performs the number of actions \(k\) specified in Line 2, which is \(O(\ell^2)\), and hence the algorithm performs \(O(\ell^3)\) action steps in \(\ell\) iterations, which is:

\[
\tilde{O} \left( n + \frac{|\Sigma|^5 \cdot |A|^5}{\rho^2 \cdot n^9} \right)
\]

The bound mentions \(|\Sigma|\) which is not in \(d_P\), but satisfies \(|\Sigma| \leq n \cdot |A| \cdot |1/\eta|\). Then, a polynomial number of action steps immediately implies a polynomial number of steps overall. In particular, \(\text{AdaCT}\) runs in time polynomial in the size of the input sample and in the specified values for the number of states and alphabet size, and also value iteration \(\text{ValueIteration}\) runs in time polynomial in the size of the input MDP and in the number of iterations. We conclude that Algorithm 1 reaches accuracy \(\epsilon\) and confidence \(\delta\) in a polynomial number of steps. Therefore, Algorithm 1 is PAC-RL.

6 Exploiting Prior Knowledge

We observe that if we know (or we can estimate) the number of states in the dynamics transducer, we can devise a simpler algorithm, Algorithm 2, with better performance bounds. Algorithm 2 is simpler than Algorithm 1, since it does need to search for the number of transducer states. The stop probability is constant throughout a run, and computed based on the upper bound on the states. Since the stop probability is uniform across iterations, all episodes can be seen as generated by the same automaton, and hence we can accumulate episodes instead of deleting previous ones. Furthermore, we can now call \(\text{AdaCT}\) directly with the given parameter \(\hat{n}\). A simplified analysis, again based on Lemmas 1–5, yields the following polynomial bound on the expected number of action steps, which immediately implies a polynomial bound on the expected number of steps overall.

**Theorem 3.** Algorithm 2 on input \((A, \gamma, \epsilon, \delta, \hat{n})\) reaches accuracy \(\epsilon\) and confidence \(\delta\) within an expected number of action steps that is:

\[
\tilde{O} \left( \frac{|A| \cdot \hat{n} \cdot |\Sigma|}{\rho \cdot \eta} \cdot \left( 1 + \frac{n^2 \cdot |\Sigma|^2 \cdot |A|^2 \cdot R_{\text{max}}^2}{1 - \gamma^6 \cdot \epsilon^2} \right) \right).
\]

**Algorithm 2 Reinforcement Learning**

**Input:** Actions \(A\), discount factor \(\gamma\), required precision \(\epsilon\), confidence parameter \(\delta\), upper bound \(\hat{n}\) on transducer states.

**Output:** Policies.

1. \(p \leftarrow 1/(10 \cdot \hat{n} + 1)\)
2. \(X \leftarrow \emptyset\)
3. loop
4. \(x \leftarrow \text{generate an episode under exploration policy } \pi_p\)
5. \(X \leftarrow X \cup \{x\}\)
6. \(\Sigma \leftarrow \text{symbols in } X\)
7. \(\hat{R}_{\text{max}} \leftarrow \max \text{ reward in } X\)
8. \(\hat{M} \leftarrow \text{compute the MDP induced by } A \text{ and } \gamma\)
9. \(m \leftarrow \left[ \frac{1}{1 - \gamma} \cdot \ln \left( \frac{2 \hat{R}_{\text{max}}}{\gamma} \right) \right] \)
10. \(\pi \leftarrow \text{solve } \hat{M} \text{ by calling } \text{ValueIteration}(\hat{M}, m)\)
11. return transducer for the composition of \(\pi\) with the projection-on-states of the transition function of \(A\)
12. end loop

7 Discussion

We have presented RL algorithms that can learn near-optimal policies for RDPs in polynomially-many steps, in the parameters that describe the underlying RDP.

**Example 3.** Algorithm 1 and Algorithm 2 (with a bound on the number of states that is polynomial in the grid length) compute a near-optimal policy in each of the RDPs introduced in Example 1 in a number of steps that is polynomial in the following quantities: (i) the grid length \(m\), (ii) the inverse of the minimum among \(p_0, p_1, 1 - p_0, 1 - p_1\), (iii) the inverse of the minimum value \(|p_0 - p_1|\).

Adopting PDFA techniques takes us into a different direction from existing approaches based on a direct clustering of histories such as [Abadi and Brafman, 2020]. There, histories are clustered according to the probability of the following observation state. Since the algorithm compares single histories, the accuracy of the algorithm depends on the probability of single histories, which can be exponentially-low in their length. In turn, the required length of histories can grow with the number of transducer states, and hence the approach can require exponentially-many episodes in order to achieve high accuracy. For instance, in our running example, a history of length \(m\) has probability at most \(g^m\) under any policy, with \(g\) the maximum probability among \(p_0, p_1, 1 - p_0, 1 - p_1\). Histories of length \(m - 1\) are necessary to determine the best action at the \(m\)-th step. To address the issue, PDFA algorithms build states incrementally while relying on their distinguishability. This way, each state gathers the probability of all the histories it represents. In light of our results, we believe that these PDFA techniques will be instrumental in developing the next generation of tools for RL in RDPs.

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[Wang et al., 2020] Ruosong Wang, Simon S. Du, Lin F. Yang, and Sham M. Kakade. Is long horizon RL more difficult than short horizon RL? In *NeurIPS*, 2020.
Lemma 6. Under the assumption that noisy parity functions cannot be PAC-learned in polynomial time, for every algorithm there is an RDP $P$ such that the algorithm does not reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps $\text{poly}(1/\epsilon, \ln(1/\delta), d_P)$ if $d'_P$ is $d_P$ after removing the distinguishability parameter.

Proof. We show a class of RDPs such that the existence of an algorithm that reaches accuracy $\epsilon$ and confidence $\delta$ in the claimed number of actions steps in all such RDPs contradicts the conjecture about noisy parity functions. These RDPs are inspired by the construction given in Theorem 16 of [Kearns et al., 1994]. Consider a parity function $f_S : \{0, 1\}^m \rightarrow \{0, 1\}$ where $S \subseteq \{x_1, \ldots, x_m\}$ and $f_S(x_1, \ldots, x_m) = 1$ iff the parity of $x = x_1, \ldots, x_m$ on the set $S$ is 1. We then build an RDP $P_S$ where learning a 1-optimal policy amounts to learning $f_S$ from noisy samples of $f_S$ for a given noise rate $\alpha \in (0, 1/2)$, which contradicts the noisy parity conjecture. The RDP $P_S$ has observation states $S = \{0, 1\}$ and actions $A = \{a_0, a_1\}$. The initial observation state is 0. When the history of observation states has length at most $m + 1$, the next observation state is chosen uniformly at random regardless of the action, and no reward is issued. When the history $h$ of observation states has length $m + 2$, the next observation state is $f_S(h)$ with probability $1 - \eta$, and the reward is 1 if the chosen action is $a_0$ with $b = f_S(h)$—i.e., the agent has guessed the value of the function correctly. From this point onwards, the observation state is always 0 and no reward is issued. Note that the transducer of the dynamics described above has states $Q = \{q_0, q_1^0, q_1^1, \ldots, q_m^0, q_m^1, q_{m+1}\}$, with the following specification: $q_0$ specifies probability 1 for observation 0, and reward zero. The output in state $q_{m+1}$ specifies probability 1 for observation 1, and reward zero if the action is $a_0$ regardless of the observation. The output in state $q_{m+1}$ specifies probability 1 for observation 0 and reward zero.

If an algorithm could reach accuracy 1 and confidence $\delta$ in $N = \text{poly}(1/\epsilon, \ln(1/\delta), d_P)$ action steps, then the learned policy would encode $f_S$ with confidence $\delta$, which contradicts the noisy parity conjecture because $N$ is polynomial—the only parameter in $d_P$ to grow is the number of states, which grows linearly with $m$. \hfill \Box

The following lemma shows that the required number of action steps increases with the inverse of the degree of determinism. In other words, the less likely transitions are, the more the agent has to explore. This is due to the fact that an agent that has not experienced some transition of the dynamics transducer will not be able to determine the resulting transducer state when the transition occurs.

Lemma 7. For every algorithm there is an RDP $P$ such that the algorithm does not necessarily reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps $\text{poly}(1/\epsilon, \ln(1/\delta), d_P)$ if $d'_P$ is $d_P$ after removing the parameter for the degree of determinism.

Proof. Consider $i \in \{1, 2\}$, $\eta \in (0, 1)$, and an RDP $P_i^n$ with two observation states $S = \{s_1, s_2, s_3\}$, two actions $A = \{a_1, a_2\}$, and dynamics as described next. The initial observation state is $s_1$. When the last observation state is $s_1$, both actions yield observation state $s_2$ with probability $1 - \eta$ and observation state $s_3$ with probability $\eta$. When the last observation state is $s_2$, both actions yield observation state $s_1$ with probability $1 - \eta$ and observation state $s_3$ with probability $\eta$. When the last observation state is $s_3$, both actions yield observation state $s_1$ with probability $1 - \eta$ and observation state $s_3$ with probability $\eta$. Note that the dynamics of both RDPs are represented by a transducer with two states. The dynamics transducer of $P_i^n$ has two states because when the last observation state is $s_3$ it is the same as when the last observation state is $s_2$. Thus, in $P_i^n$, action $a_2$ is optimal when the last observation state is $s_3$; it is instead $a_1$ in $P_2^n$. This assessment is required in order to compute an $\epsilon$-optimal
policy with $\epsilon < 1$ when the underlying RDP can be $P_1^0$ or $P_2^0$. Therefore, it is necessary to observe at least one transition when the last observation state is $s_N$. However, the probability to never see $s_N$ in $N$ steps is $(1 - \eta)^N$. If $N = \text{poly}(1/\epsilon, \ln(1/\delta), d_P^{\epsilon})$, then $(1 - \eta)^N$ can be made arbitrarily high by choosing an arbitrarily small value for $\eta$, since none of the parameters in $d_P^{\epsilon}$ increases when $\eta$ decreases.

The following lemma shows that the length of episodes to consider can grow with the number of transducer states $n$.

**Lemma 8.** For every algorithm there is an RDP $P$ such that the algorithm does not necessarily reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps $\text{poly}(1/\epsilon, \ln(1/\delta), d_P^{\epsilon})$ if $d_P^{\epsilon}$ is $d_P$ after removing number of transducer states.

**Proof.** Consider $i \in \{1, 2\}$, $n \geq 1$, states $S = \{s_1, \ldots, s_n\}$, and actions $A = \{a_1, a_2\}$. We describe an RDP $P_i^n$. The initial observation state is $s_1$. When the last observation state is $s_j$ with $j \in [1, n)$, all actions yield observation state $s_{j+1}$ with probability 1. The reward in all the above transitions is zero. When the last observation state is $s_n$, action $a_1$ yields reward 1 and observation state $s_n$ with probability 1, and the other actions yields reward zero and observation state $s_n$ with probability 1. The minimum dynamics transducer of $P_i^n$ has states $Q = \{q_0, q_1, \ldots, q_n\}$, where $q_0$ is the initial state, and $q_j$ for $j \in [1, n]$ is the state when the last observation is $s_j$.

To determine the best action when the last observation state is $s_n$ and hence compute an $\epsilon$-optimal policy with $\epsilon < 1$, it is required to see at least one transition when the last observation state is $s_n$, to assess whether the underlying RDP is $P_i^0$ or $P_i^2$. This requires an episode of length $n$. The value of $n$ can be chosen such that $n = \text{poly}(1/\epsilon, \ln(1/\delta), d_P^{\epsilon})$, since none of the parameters in $d_P^{\epsilon}$ increases with $n$.

The following lemma shows that the required number of action steps can grow with the reachability parameter $1/\rho$.

**Lemma 9.** For every algorithm there is an RDP $P$ such that the algorithm does not necessarily reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps $\text{poly}(1/\epsilon, \ln(1/\delta), d_P^{\epsilon})$ if $d_P^{\epsilon}$ is $d_P$ after removing the reachability parameter.

**Proof.** Consider $i \in \{1, 2\}$, $n \geq 1$, states $S = \{s_1, \ldots, s_n, \text{ended}\}$, and actions $A = \{a_1, a_2\}$. We describe an RDP $P_i^n$. The initial observation state is $s_1$. When the last observation state is $s_j$ with $j \in [1, n)$, all actions yield observation state $s_{j+1}$ with probability 1/2, and observation state $\text{ended}$ with probability 1/2. When the last observation state is $\text{ended}$, all actions yield observation state $\text{ended}$ with probability 1. The reward in all the above transitions is zero. When the last observation state is $s_n$, action $a_1$ yields reward 1 and observation state $s_n$ with probability 1, and the other actions yields reward zero and observation state $s_n$ with probability 1. The minimum dynamics transducer of $P_i^n$ has states $Q = \{q_0, q_1, \ldots, q_n, \text{sink}\}$, where $q_0$ is the initial state, and $q_j$ for $j \in [1, n]$ is the state when the last observation is $s_j$.

To determine the best action when the last observation state is $s_n$ and hence compute an $\epsilon$-optimal policy with $\epsilon < 1$, it is required to see at least one transition when the last observation state is $s_n$, to assess whether the underlying RDP is $P_i^0$ or $P_i^2$. The probability of seeing observation state $s_n$ in an episode is $(1/2)^n$. The probability of seeing observation state $s_n$ in $N$ episodes is at most $N \cdot (1/2)^n$, by a union bound. If $N = \text{poly}(1/\epsilon, \ln(1/\delta), d_P^{\epsilon})$, we can choose $n$ such that $N \cdot (1/2)^n$ is arbitrarily small.

The remaining lemmas are based on a core lower bound for the multi-armed bandit problem given in [Mannor and Tsitsiklis, 2004].

**Proposition 1** (Mannor and Tsitsiklis, 2004; Theorem 1). There exist positive constants $c_1$, $c_2$, $c_0$, and $\delta_0$ such that for every $K \geq 2$, $\epsilon \in (0, c_0)$, $\delta \in (0, \delta_0)$, and for every algorithm, there is a $K$-armed bandit problem for which the number of action steps for the algorithm to reach accuracy $\epsilon$ and confidence $\delta$ in the worst case is at least

$$c_1 \cdot \frac{K}{c_2} \log \frac{c_2}{\delta}.$$  

In particular, $c_0$ and $\delta_0$ can be taken equal to $1/8$ and $e^{-4}/4$, respectively. Furthermore, the mentioned $K$-armed bandit problem has rewards in $[0, 1]$, and its maximum reward probability is $1/2 + \epsilon$.

Note that the bound was originally stated for the expected number of action steps, which is a lower bound for the actual number of action steps in the worst-case—this was also pointed out in [Strehl et al., 2009]. Note also that the original theorem was for algorithms that, as a last step, output a policy and then stop. However, the proof of the theorem shows that, when the number of action steps is lower than the claimed bound, the probability of selecting the wrong arm is bigger than $\delta$, regardless of whether the algorithm stops or not.

The following lemma is implied by the previous proposition since the number of actions corresponds to the number of arms.

**Lemma 10.** For every algorithm there is an RDP $P$ such that the algorithm does not necessarily reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps $\text{poly}(1/\epsilon, \ln(1/\delta), d_P^{\epsilon})$ if $d_P^{\epsilon}$ is $d_P$ after removing the number of actions.
Proof. We show RDPs that encode MAB problems and argue that if this lemma were false then Proposition 1 would be false. Consider a $K$-armed bandit problem with arm probabilities $p_1, \ldots, p_K$, rewards in $\{0, 1\}$, and maximum reward probability $1/2 + \epsilon$. The corresponding RDP has observation states $S = \{s_0, s^+, s^-\}$ and actions $A = \{a_1, \ldots, a_K\}$. The initial observation state is $s_0$. Regardless of the last observation state, when action $a_i$ is performed, the agent observes $s^+$ with probability $p_i$ receiving reward 1 and, it observes $s^-$ with probability $1 - p_i$ and receiving reward zero. Note that there is one set of parameters $d^*_p$ that describes all such RDPs. In particular, the discount factor is irrelevant and can be taken equal to zero, the maximum reward is one, the reachability is one and the distinguishability is one because the dynamics function is state-less, and the degree of determinism is $1/2 - \epsilon$ because it is the minimum probability that an arm issues a reward.

Assume by contradiction that, in each of the above RDPs, an agent can reach accuracy $\epsilon \cdot r$ and confidence $\delta$ in poly$(1/\epsilon, \ln(1/\delta), d^*_p)$ action steps. It amounts to solving the corresponding MAB problem with accuracy $\epsilon$ and confidence $\delta$ in poly$(1/\epsilon, \ln(1/\delta), d^*_p)$ action steps. For any given $\epsilon$ and $\delta$, the former quantity is constant, and hence will be smaller than the bound of Proposition 1 for a sufficiently big number of arms $K$. This contradicts Proposition 1.

The following lemma is based on the observation that, if the number of action steps did not depend on the maximum reward value $R_{\max}$, then we could artificially increase all the rewards given to the agent by an arbitrary factor in order to obtain a higher accuracy in the same number of steps.

Lemma 11. For every algorithm there is an RDP $\mathcal{P}$ such that the algorithm does not necessarily reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps poly$(1/\epsilon, \ln(1/\delta), d^*_p)$ if $d'$ is $d_\mathcal{P}$ after removing the maximum reward.

Proof. Consider the encoding of the MAB problem introduced in the proof of Lemma 10, with the difference that, each arms yields reward $R_{\max}$ instead of one. Note that the number of actions $|A|$ is the only parameter that can change in $d^*_p$ for different RDPs.

Assume by contradiction that, in each of the above RDPs, an agent can reach accuracy $\epsilon \cdot R_{\max}$ and confidence $\delta$ in poly$(1/(\epsilon \cdot R_{\max}), \ln(1/\delta), d^*_p)$ action steps. We can use this agent to solve the original MAB problem—the one with rewards in $\{0, 1\}$—with accuracy $\epsilon$ and confidence $\delta$ in $N = \text{poly}(1/(\epsilon \cdot R_{\max}), \ln(1/\delta), d^*_p)$ action steps. It suffices to give the agent reward $r \cdot R_{\max}$ whenever the original MAB issues reward $r$—i.e., to give $R_{\max}$ when reward one is issued by the original MAB, and zero otherwise. Then, if the computed $(\epsilon \cdot R_{\max})$-optimal policy selects action $a_i$, the $i$-th arm in the original MAB is $\epsilon$-optimal, which solves the original MAB. This contradicts Proposition 1 because the number $N$ of actions steps can be made arbitrarily small by increasing the value of $R_{\max}$.

The proof of the following lemma is a variation of the previous one. It builds on the observation that, in the setting with discounted rewards, even if $R_{\max} = 1$, there can be a state whose value is arbitrarily large, for decreasing values of the discount factor.

Lemma 12. For every algorithm there is an MDP $\mathcal{P}$ such that the algorithm does not necessarily reach accuracy $\epsilon$ and confidence $\delta$ in a number of steps poly$(1/\epsilon, \ln(1/\delta), d^*_p)$ if $d'_p$ is $d_\mathcal{P}$ after removing the discount parameter.

Proof. The proof is based on Proposition 1 similarly to the proof of Lemma 11. However, it considers a different encoding of the MAB problem, to ensure that there is an action whose value increases with the discount parameter—instead of the maximum reward parameter.

Consider a $K$-armed bandit problem with arm probabilities $p_1, \ldots, p_K$, rewards in $\{0, 1\}$, and maximum reward probability $1/2 + \epsilon$. The corresponding RDP has observation states $S = \{s_0, s^+, s^-\}$ and actions $A = \{a_1, \ldots, a_K\}$. The initial observation state is $s_0$. When the last observation state is $s_0$, action $a_i$ yields observation state $s^+$ with probability $p_i$ and observation state $s^-$ with probability $1 - p_i$, and the reward is zero in both cases. When the last observation state is $s^+$, all actions yield $s^+$ with reward one. When the last observation state is $s^-$, all actions yield $s^-$ with reward zero. Clearly, during exploration, the agent can always go back to state $s_0$ by performing a stop action—as assumed throughout the paper.

The key observation is that the value of every action in state $s_0$ is given by its probability of leading to state $s^+$ times the value $g = \sum_{i=2}^{\infty} \gamma^i$, which is finite and it amounts to $\gamma^2/(1 - \gamma)$. In particular, $g$ can be made arbitrarily big by making $\gamma$ arbitrarily close to 1. Thus, $g$ plays the same role as $R_{\max}$ in Lemma 11, and we omit the rest of the proof since it proceeds similarly to the one of Lemma 11.

A.2 Proof of Lemma 1

The goal of this section is to prove Lemma 1. Let the target RDP $\mathcal{P}$ and the ideal MDP $M$ be defined as in Section 4. First, the optimal value functions $v_*$ and $q_*$ of $\mathcal{P}$ can be expressed in terms of the optimal value functions $v_*^M$ and $q_*^M$ of $M$.

Lemma 13. The value functions $v_*$ and $q_*$ of $\mathcal{P}$ can be expressed as $v_*(h) = v_*^M(\tau(h))$ and $q_*(h, a) = q_*^M(\tau(h), a)$.
Lemma 14. Let $\pi$ be an $\epsilon$-optimal policy for $M$. Then $v_{\pi}(h) = v^*_\pi(h)$ for all $h \in M$. Assume $\max_h |v^*_\pi(h) - v^M(h)| \leq \epsilon$. The former is equivalent to $\max_h |v^M(\tau(h)) - v^*_\pi(\tau(h))| \leq \epsilon$. The former can be rewritten as $\max_h |v^\pi(h) - v^M(\tau(h))| \leq \epsilon$ by Theorem 14. The former can be rewritten as $\max_h |v^\pi(h) - v^*_\pi(h)| \leq \epsilon$ by Theorem 13.

Proof. The proof technique is the one from Lemma 13.

$\pi(h) = \max_a \sum_{sr} D(s,r|h,a) \cdot (r + \gamma \cdot v^\pi(h))$

$\pi(h) = \max_a \sum_{sr} \theta(\tau(h))(a,s,r) \cdot (r + \gamma \cdot v^\pi(h))$

The equation above can be rewritten in terms of a new function $v^M$. Then $v^\pi(h) = v^M(h)$ as claimed.

Proof. The proof technique is the one from Lemma 13.

$\pi(h) = \max_a \sum_{sr} D(s,r|h,a) \cdot (r + \gamma \cdot v^\pi(h))$

$\pi(h) = \max_a \sum_{sr} \theta(\tau(h))(a,s,r) \cdot (r + \gamma \cdot v^\pi(h))$

The equation above can be rewritten in terms of a new function $v^M$. Then $v^\pi(h) = v^M(h)$ as claimed.
A.3 Proof of Lemma 2

The goal of this section is to prove Lemma 2, by applying known results for MDPs [Singh and Yee, 1994; Strehl and Littman, 2005; Strehl et al., 2009] similar to what is done in [Strehl et al., 2009]. First, similar MDPs have similar action-value functions. Note that $|| \cdot ||_1$ is the $L_1$-norm.

**Proposition 2** (Strehl and Littman, 2005; Lemma 4). Let $M_1 = \langle A, S, R, T_1, R_1, \gamma \rangle$ and $M_2 = \langle A, S, R, T_2, R_2, \gamma \rangle$ be two MDPs. Let $R_{\text{max}}$ be the maximum value in $R$, let $r_1(s, a)$ be the expected reward after doing action $a$ in state $s$ in the MDP $M_i$, and let $q_1^i$ and $q_2^i$ be the action-value functions of $M_1$ and $M_2$ for a policy $\pi$. If $|r_1(s, a) - r_2(s, a)| \leq \alpha$ and $|T_1(\cdot | s, a) - T_2(\cdot | s, a)|_1 \leq 2 \beta$ for every state $s$ and action $a$:

$$|q_1^i(s, a) - q_2^i(s, a)| \leq \frac{(1 - \gamma) \cdot \alpha + \gamma \cdot \beta \cdot R_{\text{max}}}{(1 - \gamma) \cdot (1 - \gamma + \beta \cdot \gamma)}.$$

In our case we have a bound on the accuracy of the dynamics function, which transfers as follows.

**Proposition 3.** Let $M_1, M_2, r_1, r_2, \text{ and } R_{\text{max}}$ be as in Proposition 2, and let $D_i$ be the dynamics function of $M_i$. The following holds, for every state $s$ and action $a$:

$$|r_1(s, a) - r_2(s, a)| \leq R_{\text{max}} \cdot ||D_1(\cdot | s, a) - D_2(\cdot | s, a)||_1.$$

**Proof.** We have $r_i(s_1, a) = \sum_r r \cdot \sum_{s_2} D_i(s_2, r | s_1, a)$. Then,

$$|r_1(s, a) - r_2(s, a)| =$$

$$\left| \sum_r r \cdot \sum_{s_2} D_1(s_2, r | s_1, a) - D_2(s_2, r | s_1, a) \right| \leq$$

$$\sum_r r \cdot \sum_{s_2} |D_1(s_2, r | s_1, a) - D_2(s_2, r | s_1, a)| \leq$$

$$R_{\text{max}} \cdot \sum_r \sum_{s_2} |D_1(s_2, r | s_1, a) - D_2(s_2, r | s_1, a)| =$$

$$R_{\text{max}} \cdot ||D_1(\cdot | s, a) - D_2(\cdot | s, a)||_1.$$

This concludes the proof.

**Proposition 4.** Let $M_1$ and $M_2$ be as in Proposition 2, and let $D_i$ be the dynamics function of $M_i$. The following holds, for every state $s$ and action $a$:

$$||T_1(\cdot | s, a) - T_2(\cdot | s, a)||_1 \leq ||D_1(\cdot | s, a) - D_2(\cdot | s, a)||_1.$$

**Proof.** We have $D_i(s_2 | s_1, a) = \sum_r D_i(s_2, r | s_1, a)$. Then, for every state $s_1$ and action $a$,

$$\left| T_1(\cdot | s_1, a) - T_2(\cdot | s_1, a) \right| =$$

$$\sum_{s_2} \left| T_1(s_2 | s_1, a) - T_2(s_2 | s_1, a) \right| =$$

$$\sum_{s_2} \sum_r \left| D_1(s_2, r | s_1, a) - D_2(s_2, r | s_1, a) \right| \leq$$

$$\sum_{s_2} \sum_r \left| D_1(s_2, r | s_1, a) - D_2(s_2, r | s_1, a) \right| =$$

$$||D_1(s_2, r | s_1, a) - D_2(s_2, r | s_1, a)||_1.$$

This concludes the proof.

We apply the previous proposition to bound the action-value function of an MDP $\hat{M}$ that is an approximation of $M$. In the following, to simplify the notation, we identify the states of $\hat{M}$ with the ones of $M$; this is w.l.o.g. since we could rename each state $q$ of $\hat{M}$ with $\phi(q)$ where $\phi$ is the bijection mentioned in Definition 8.

**Lemma 15.** If $\hat{M}$ is an $\alpha$-approximation of $M$, then the following holds for every state $q$, action $a$, and stationary policy $\pi$:

$$|q_\pi^{\hat{M}}(q, a) - q_\pi^M(q, a)| \leq \frac{\alpha \cdot R_{\text{max}}}{(1 - \gamma)^2}.$$
Lemma 2. If $\hat{M}$ is an $\epsilon'$-approximation of $M$ (with $\epsilon'$ as in Table 1), then the greedy policy obtained via

$$\left[\frac{1}{1-\gamma} \cdot \ln \left( \frac{\gamma}{\epsilon \cdot (1-\gamma)^2} \right) \right]$$

iterations of the value iteration algorithm, with action-value estimates initialised to zero, is an $\epsilon$-optimal policy for $M$. 

Proof. Let $R_{\text{max}}$ be the maximum value in $R$, and let $r(s,a)$ and $\tilde{r}(s,a)$ be the expected reward after doing action $a$ in state $s$ in the MDP $M$ and $\hat{M}$, respectively. Let $\alpha'$ and $\beta$ be such that $|r(s,a) - \tilde{r}(s,a)| \leq \alpha'$ and $\|T(s,a) - \tilde{T}(s,a)\|_1 \leq 2\beta$ for every state $s$ and action $a$. Then,

$$|q^*_\hat{M}(q,a) - q^*_M(q,a)| \leq \frac{(1-\gamma) \cdot \alpha + \gamma \cdot \beta \cdot R_{\text{max}}}{(1-\gamma) \cdot (1-\gamma + \beta \cdot \gamma)} \leq \frac{(1-\gamma) \cdot R_{\text{max}} \cdot \epsilon + \gamma \cdot \epsilon \cdot R_{\text{max}}}{(1-\gamma) \cdot (1-\gamma + \epsilon \cdot \gamma)} = \frac{\alpha \cdot R_{\text{max}}}{(1-\gamma)^2}.$$ 

The first inequality holds by Proposition 2, the second one by Propositions 3 and 4, and the last one because it is obtained by removing a positive additive term at the denominator. 

A near-optimal policy for $M$ can be computed via the value iteration algorithm. Specifically, value iteration allows us to compute a close approximation of the action-value function of $M$, and then the corresponding greedy policy—that picks actions with maximum value—is near-optimal. The proposition is a straightforward generalisation of Proposition 4 from [Strehl et al., 2009] to the case where $R_{\text{max}}$ is not necessarily 1. We rewrite their proof to include $R_{\text{max}}$.

Proposition 5 (Generalisation of Proposition 4 of [Strehl et al., 2009]). Consider an MDP $M'$, let $\gamma$ be its discount factor, let $R_{\text{max}}$ be its maximum reward value, and let $q^*_M$ be its optimal action-value function. Let $\alpha > 0$ be any real number satisfying $\alpha < R_{\text{max}} / (1-\gamma)$. Suppose that value iteration is run on $M'$ for $\left\lfloor \frac{1}{1-\gamma} \cdot \ln \left( \frac{R_{\text{max}}}{\alpha (1-\gamma)} \right) \right\rfloor$ iterations where the action-value estimates are initialized to some value between 0 and $R_{\text{max}} / (1-\gamma)$. Then, the resulting action value estimates $q^*_M$ satisfy $|q^*_M(s,a) - q^*_M(s,a)| \leq \alpha$ for every state $s$ and action $a$. 

Proof. Let $q^*_M$ denote the initial action-value estimates, and let $q^*_M$ denote the action-value estimates after the $i$-th iteration of value iteration. Then, let $\Delta_i = \max_{s,a} |q^*_M(s,a) - q^*_M(s,a)|$. We have that

$$\Delta_i = \max_{s,a} \left| \sum_{s'} D^M(s',r|s,a) \cdot (r + \gamma \cdot v^M(s')) - \sum_{s'} D^M(s',r|s,a) \cdot (r + \gamma \cdot \tilde{v}^M_i(s')) \right|$$

$$= \gamma \cdot \max_{s,a} \left| \sum_{s'} D^M(s',r|s,a) \cdot (v^M(s') - \tilde{v}^M_i(s')) \right|$$

$$\leq \gamma \cdot \Delta_{i-1}.$$ 

Using this bound along with the fact that $\Delta_0 \leq R_{\text{max}} / (1-\gamma)$ shows that $\Delta_i \leq (R_{\text{max}} \cdot \gamma^i) / (1-\gamma)$. Thus, in order to have $\Delta_i \leq \alpha$, it suffices that $i \geq \ln((\alpha \cdot (1-\gamma)) / R_{\text{max}}) / \ln \gamma$. Then, the theorem follows since

$$\frac{\ln \left( \frac{\alpha \cdot (1-\gamma)}{R_{\text{max}}} \right)}{\ln \gamma} = \frac{\ln \left( \frac{R_{\text{max}}}{\alpha (1-\gamma)} \right)}{-\ln \gamma} \leq 1 - \gamma \cdot \ln \left( \frac{R_{\text{max}}}{\alpha \cdot (1-\gamma)} \right).$$

To see that the former inequality holds, it suffices to note that the inequality $-\ln \gamma \geq 1 - \gamma$ follows from the inequality $e^x \geq 1 + x$. 

Proposition 6 (Singh and Yee, 1994; Corollary 2). Consider an MDP $M'$, let $\gamma$ be its discount factor, and let $q^*_M$ be its optimal action-value function. Furthermore, let $q_*$ be an estimate of $q^*_M$, and let let $\pi$ be the greedy policy with respect to $q_*$, i.e., $\pi(s) = \arg \max_a q_*(s,a)$. For any $\alpha > 0$, if $|q_*(s,a) - q^*_M(s,a)| \leq \alpha$ for every state $s$ and action $a$, then $|v^*_M(s) - v^*_M(s)| \leq 2\alpha / (1-\gamma)$ for every state $s$. 

To prove the main result of this section, we apply Lemma 15 and the previous propositions above. Note that $\epsilon' = \frac{(1-\gamma)^3 \cdot x}{3 \cdot R_{\text{max}}}$ as specified in Table 1.
Proposition 5, it yields $\hat{q}_{\pi}(s, a) = q^*_{\pi}(s, a)$.

To simplify the notation, we identify the states of $A$ with those of $\hat{A}$, where $\phi(q)$ is an isomorphism mentioned in Definition 9. Note that $\hat{q}_{\pi}$ is an $\epsilon$-approximation of $q^*_{\pi}$.

Proof. Assume that $\hat{M}$ is an $\epsilon'$-approximation of $M$. By Lemma 15,

$$|q^*_\pi(q, a) - q^*_\pi(q, a)| \leq \frac{\epsilon'}{\gamma}.$$  

Consider to run value iteration on $\hat{M}$ for $\left[ \frac{1}{1-\gamma} \ln \left( \frac{2R_{\max}(1-\gamma)}{\epsilon} \right) \right]$ iterations, initialising all action-value estimates to zero. By Proposition 5, it yields $\hat{q}_{\pi}$ such that $|q^*_\pi(s, a) - q^*_\pi(s, a)| \leq \frac{1}{2} \cdot \epsilon'$. Note that there are two levels of approximation; namely, $q^*_\pi$ is an approximation of the action-value function $q^*_\pi$ of an approximation $\hat{M}$ of the MDP $M$. The former bound together with (2) yields the following:

$$|q^*_\pi(s, a) - q^*_\pi(s, a)| \leq \frac{1}{2} \cdot \epsilon' \cdot \frac{R_{\max}}{(1-\gamma)^2} + \frac{1}{2} \cdot \epsilon' \cdot \frac{R_{\max}}{(1-\gamma)^2} = \frac{3}{2} \cdot \epsilon' \cdot \frac{R_{\max}}{(1-\gamma)^2} = \frac{(1-\gamma) \cdot \epsilon}{2}. $$

Let $\pi$ be the greedy policy $\pi(q) = \arg \max_a \hat{q}_\pi(q, a)$, hence $\pi$ is $\epsilon$-optimal for $M$.

A.4 Proof of Lemma 3

To simplify the notation, we identify the states of $\hat{A}$ with the ones of $A$; this is w.l.o.g. since we could rename each state $q$ of $\hat{A}$ with $\phi(q)$ where $\phi$ is the isomorphism mentioned in Definition 9. Note that $\epsilon'' = \frac{(1-p)\epsilon'}{|A| \cdot n |\Sigma|}$ as defined in Table 1.

Lemma 3. If $\hat{A}$ is an $\epsilon''$-approximation of $A$, then $\hat{M}$ is an $\epsilon'$-approximation of $M$ (with $\epsilon''$ as in Table 1).

Proof. For any state $q$ and action $a$,

$$\|D^M(\cdot|q, a) - D^M(\cdot|q, a)\|_1 = \sum_{q'} \sum_r |D^M(q', r|q, a) - D^M(q', r|q, a)| = \sum_{q'} \sum_{r: \pi(q,s) = q'} |\theta(q)(a, s, r) - \frac{|A|}{1-p} \cdot \theta(q, asr)| = \frac{|A|}{1-p} \cdot \sum_{q'} \sum_{r: \pi(q,s) = q'} |\lambda(q, asr) - \lambda(q, asr)| \leq \frac{|A|}{1-p} \cdot \sum_{q'} \sum_{r: \pi(q,s) = q'} |\lambda(q, asr) - \lambda(q, asr)| \leq \epsilon'' \leq \epsilon'.$$

The first equality holds by the definition of $L_1$-norm. The second equality holds by the definition of $D^M$ and $D^\hat{M}$. The third equality holds by the definition of $\lambda$. The fourth equality is simply a partitioning of the elements $q' sr$. The first inequality holds because $\hat{A}$ is an $\epsilon''$-approximation of $A$; in particular, for all elements $q' sr$ such that $asr \notin \Sigma$, $\lambda(q, asr) = 0$ and hence $\hat{\lambda}(q, asr) = 0$. The second inequality holds because there are at most $n$ elements $q'$ and at most $|\Sigma|$ elements sr.

A.5 Proof of Lemma 4

We split the lemma in three separate ones. Note that $\mu$ is the distinguishability of $P$, $n$ is the number of states of the dynamics of transducer $T$, and $p$ is the stop probability.

Lemma 4.1. The distinguishability of $A$ is at least $\mu \cdot (1-p)^n$.

Proof. Consider two distinct states $q_1$ and $q_2$ of $A$. It suffices to show a string $x \in \Sigma^*$ such that $|\lambda(q_1, x) - \lambda(q_2, x)| \geq \mu \cdot (1-p)^n$. Since $q_1$ and $q_2$ are distinct and $T$ is minimum, there are strings $y_1, y_2, z \in \Sigma^*$ such that:

(i) $y_1z$ or $y_2z$ have non-zero probability of being prefixes of strings generated by $A$, i.e., $\lambda(q_0, y_1z) > 0$ or $\lambda(q_0, y_2z) > 0$;
(ii) $y_1$ and $y_2$ lead to $q_1$ and $q_2$, respectively, starting from the initial state $q_0$, i.e., $\tau'(q_0, y_1) = q_1$ and $\tau'(q_0, y_1) = q_1$;

where $\tau'$ is the action taken by the dynamics of transducer $T$.
(iii) \( z \) has non-zero probability of being a prefix of a string generated by \( \mathcal{A} \) from state \( q_1 \) or \( q_2 \), i.e., \( |\lambda(q_1, z) - \lambda(q_2, z)| > 0 \).

Let \( h_1 \) and \( h_2 \) be the histories obtained from \( y_1 \) and \( y_2 \) by removing all action and reward symbols. By the construction of \( \mathcal{A} \), we have \( \tau(q_0, h_1) = q_1 \) and \( \tau(q_0, h_2) = q_2 \), and hence \( |D_{\pi_n}(z|h_1) - D_{\pi_n}(z|h_2)| > 0 \) since the uniform policy \( \pi_n \) does not decrease the probability of a trace \( z \) with respect to the policy \( \pi_p \)—which selects actions uniformly at random like the uniform policy and, additionally, considers to stop at every step. Thus, \( |D_{\pi_n}(z|h_1) - D_{\pi_n}(z|h_2)| > \mu \) because \( \mathcal{P} \) is \( \mu \)-distinguishable. Since \( D \) is represented by a transducer with \( n \) states, and \( \pi_n \) is stateless, also \( D_{\pi_n} \) can be represented by a transducer with \( n \) states. Thus, a pumping lemma argument implies that \( |D_{\pi_n}(z'|h_1) - D_{\pi_n}(z'|h_2)| > \mu \) for some prefix \( z' \) of \( z \) having length at most \( n \). We have that \( |\lambda(q_1, z') - \lambda(q_2, z')| \) is given by \( |D_{\pi_n}(z'|h_1) - D_{\pi_n}(z'|h_2)| \) multiplied by the probability of not stopping for \( |z'| \) times in a row, which is \((1 - p)^n\). Therefore, \( |\lambda(q_1, z') - \lambda(q_2, z')| \geq \mu \cdot (1 - p)^n \), which shows the lemma. \( \square \)

**Lemma 4.2.** The probability that a state of \( \mathcal{A} \) is visited during a run is at least \( \rho \cdot (1 - p)^n \).

**Proof.** A state of \( \mathcal{A} \) is visited if the corresponding state of \( T \) is visited under policy \( \pi_p \). The probability of visiting a state of \( T \) under policy \( \pi_p \) is at least the probability of visiting it in \( n \) steps given that we perform \( n \) steps, times the probability of performing \( n \) steps. The former probability is at least \( \rho \) by definition, and the latter probability is \((1 - p)^n\). \( \square \)

**Lemma 4.3.** The minimum non-zero probability of a non-\( \zeta \) transition of \( \mathcal{A} \) is at least \( \eta \cdot (1 - p)/|A| \).

**Proof.** The minimum non-zero probability of a non-\( \zeta \) transition of \( \mathcal{A} \) can be established from the definition of \( \mathcal{A} \). It is the quantity \((1 - p)/|A|\) multiplied by the minimum non-zero probability of an observation-state in \( \mathcal{P} \) for a given history an action, which is the degree of determinism \( \eta \). \( \square \)

### A.6 Proof of Lemma 5

The following proposition summarises the guarantees of the AdaCT algorithm. In particular, it summarises Definition 3 and Theorem 25 of [Balle et al., 2013]. Differently from the original statements, the following proposition takes \( \epsilon_1, \epsilon_2', \) and \( \epsilon_5 \) as primary quantities.\(^2\) The reason is that Balle et al. were interested in learning likely parts of the automaton, whereas we want to learn all the parts of the automaton, and hence we impose upfront that they are sufficiently likely. Note also that our notion of distinguishability yields a lower bound on their notion of distinguishability for PDFA, and hence can be used in place of theirs. In particular, their distinguishability also takes into account the supremum distance \( L_{\infty} \), which is not relevant to our setting—see Definition 2 of [Balle et al., 2013].

**Proposition 7** ([Balle et al., 2013]). Let \( \epsilon_1, \epsilon_2', \epsilon_5 > 0 \), let \( \delta' \in (0, 1) \), and let \( \mu' > 0 \). Furthermore, let \( \hat{n} \) be a non-negative integer, and let \( X \) be a set of strings generated by \( \mathcal{A} \). Consider the following conditions:

(a) \( \hat{n} \) is an upper bound on the number of states of \( \mathcal{A} \).

(b) \( \mathcal{A} \) is \( \mu' \)-distinguishable.

(c) the minimum probability that a given state of \( \mathcal{A} \) is visited in a run is at least \( \epsilon_2' \).

(d) the minimum probability of a non-\( \zeta \) transition of \( \mathcal{A} \) is at least \( \epsilon_5 \),

(e) the size of \( X \) is greater than \( \max(N_0, N_3) \), where \( N_0 \) and \( N_3 \) are defined in Table 2.

If conditions (a)–(e) hold, then the AdaCT algorithm on input \((\hat{n}, |\Sigma|, \delta', X)\) returns a PDFA \( \hat{A} \) that is an \( \epsilon_1 \)-approximation of \( \mathcal{A} \) with probability \( 1 - \delta' \). Furthermore, the algorithm runs in time \( O(\hat{n}^2 \cdot |\Sigma| \cdot \|X\|) \), even if the above conditions do not hold.\(^3\)

We apply the previous proposition, using the bounds from Lemma 4 to ensure that conditions (b)–(d) hold.

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\(^2\) We have named quantities in a way that highlights the correspondence with [Balle et al., 2013]. Quantities \( \epsilon_1 \) and \( \epsilon_5 \) are as in [Balle et al., 2013]. Quantity \( \epsilon_2' \) corresponds to \( \epsilon_2/(L + 1) \), where \( L \) is the expected length of strings. We have maintained names also for \( N_0, N_3, \) and \( \delta_0 \).

\(^3\) \( \|X\| \) denotes the size of \( X \) defined as the sum (with repetitions) of the length of all its strings in \( X \).
Lemma 5. If $p \leq 1/(10n+1)$, $\tilde{n}$ is an upper bound on the number $n$ of states of $A$, and $X$ are strings generated by $A$ with $|X| \geq \max(N',N'')$, then AdaCT($\tilde{n}, |\Sigma|, \delta/2, X$) returns an $\epsilon'$-approximation of $A$ with probability $1 - \delta/2$.

Proof. It suffices to take $\epsilon_1 = \epsilon''$ and find values for $\mu', \epsilon_2', \epsilon_5, N'$, and $N''$ such that conditions (a)–(e) of Proposition 7 are satisfied. Since $p \leq 1/(10 \cdot n + 1)$ implies $(1-p)^n \geq 0.9$, by Lemma 4.1 we can take $\mu' = 0.9 \cdot \mu$, and by Lemma 4.2 we can take $\epsilon_2' = 0.9 \cdot \rho$. Furthermore, by Lemma 4.3 we can take $\epsilon_5 = \eta \cdot (1-p)/|A|$. Finally, condition (e) is satisfied since $N'$ and $N''$ are $N_0$ and $N_3$, respectively, when we instantiate the parameters specified above. Note also that $L = (1-p)/p$ since the length of episodes has a geometric distribution with parameter $p$. $\square$

A.7 Proof of Theorem 2

Algorithm 1 computes correct estimates for the alphabet and the maximum reward, when the number of collected episodes is sufficiently high.

Lemma 16. Consider the values $\hat{\Sigma}$ and $\hat{R}_{\text{max}}$ computed by Algorithm 1 at Line 11. If $|X| \geq \max(N', N'')$, then $\hat{R}_{\text{max}} = R_{\text{max}}$ and $\hat{\Sigma} = \Sigma$.

Proof. If $|X| \geq \max(N', N'')$, the AdaCT algorithm correctly finds all transitions of $A$, by Proposition 7. For each such transition, the algorithm requires to see at least an example in $X$, and hence $\hat{R}_{\text{max}} = R_{\text{max}}$ and $\hat{\Sigma} = \Sigma$. $\square$

Algorithm 1 computes an accurate automaton in polynomially-many iterations.

Lemma 17. Let $\ell \geq \max(n, \sqrt{8 \ln n} \cdot \ell_1, \ell_2, \ell_3, \ell_4)$ with $\ell_1, \ell_2, \ell_3$, and $\ell_4$ defined as in Table 3. Consider the automaton $\hat{A}$ computed in Line 12 of Algorithm 1 at the $\ell$-th iteration of the main loop. Then, $\hat{A}$ is an $\epsilon''$-approximation of $A$ with probability at least $1 - \delta$.

Proof. Consider the $\ell$-th iteration.

Since $\ell \geq n$, we have that $p \leq 1/(10n+1)$ and AdaCT is instantiated with $\ell \geq n$ as a correct upper bound for the number of states. Thus, the first two conditions of Lemma 5 are satisfied, and it remains to show that $|X| \geq \max(N', N'')$ with sufficient probability; note that probability $1 - \delta^2$ suffices, since when it is multiplied by the probability of success $1 - \delta^2$ of AdaCT, it is still above the required confidence $1 - \delta$. The size of $X$ mentioned above also guarantees that $\hat{\Sigma} = \Sigma$ (Line 11), and hence that AdaCT is called with a correct value for the alphabet size.

We first argue that $\ell \geq \sqrt{8 \ln n} \cdot \ell_1$ ensures that the number of generated episodes $|X|$ is at least $\ell^2 \cdot (\ell + 5 \ln \ell)$ with probability at least $1 - \delta^2$. The algorithm performs $k = 2 \cdot (10\ell + 1) \cdot (\ell + 5 \ln \ell)$ actions. The number of episodes corresponds to the number of stop actions among $k$ actions. Thus, by a Chernoff bound, the probability that the $\ell$-th iteration generates less than $\ell^2 \cdot (\ell + 5 \ln \ell)$ episodes is at most $\exp(-\ell^2 \cdot (\ell + 5 \ln \ell)/4)$, which is at most $\delta^2$ since $\ell \geq \sqrt{8 \ln n} \cdot \ell_1$.

Then, it suffices to show $\ell^2 \cdot (\ell + 5 \ln \ell) \geq \max(N', N'')$. The critical aspect is that also the values of $N'$ and $N''$ depend on $\ell$, because $n$ and $p$ in the definition of $N'$ and $N''$ are two be instantiated as $\tilde{n} = \ell$ and $p = 1/(10 \ell + 1)$. It is easy to verify that $\ell^2 \cdot (\ell + 5 \ln \ell) \geq \max(N', N'')$ when $\ell \geq \max(n, \sqrt{8 \ln n} \cdot \ell_1, \ell_2, \ell_3, \ell_4)$.

The size of alphabet $\Sigma$ is bounded by the parameters in $d_p$. Note that $\Sigma = \{asr \in ASR \mid \exists h. D(s, r|h, a) > 0\}$ as defined in Section 4.

Lemma 18. $|\Sigma| \leq |A| \cdot n \cdot \lceil 1/\eta \rceil$.

Proof. For every history $h$ and action $a$, if $T(\cdot|h, a)$ assigns non-zero probability to an observation state, then it assigns at least probability $\eta$ by definition of $\eta$; thus, $T(\cdot|h, a)$ assigns non-zero probability to at most $|1/\eta|$. We have that $D(s, r|h, a)$ is $T(s|h, a)$ if $r = R(h, a, s)$ and zero otherwise, by definition; hence, there are at most $|1/\eta|$ pairs $s, r$ that are assigned non-zero probability by $D(\cdot|h, a)$. Furthermore, the dynamics function can be expressed in terms of the transducer output, i.e., $D(s, r|h, a) = \theta(\tau(q_0, h))(a, s, r)$, and hence the number of pairs $s, r$ that are assigned non-zero probability when the history
is \( h \) and the action is \( a \) are the pairs that are assigned non-zero probability when the transducer state \( \tau(q_0, h) \) and the action is \( a \). Since there are \( n \) states and \( |A| \) actions, we have \(|\Sigma| \leq |A| \cdot n \cdot [1/\eta]\) as required.

The main theorem of this section follows from Lemmas 1–5, 17, and 18.

**Theorem 2.** Algorithm 1 is PAC-RL with respect to the parameters \( d, \rho \) given in (1).

**Proof.** By Lemma 1, Algorithm 1 returns a policy that is \( \epsilon \)-optimal with confidence at least \( 1 - \delta \) when the policy \( \pi \) computed in Line 15 is \( \epsilon \)-optimal with confidence at least \( 1 - \delta \). By Lemma 2, this happens when \( \hat{M} \) is an \( \epsilon' \)-approximation of \( M \), assuming that \( \hat{R}_{\max} = R_{\max} \). By Lemma 3, \( \hat{M} \) is an \( \epsilon' \)-approximation of \( M \) if \( \hat{A} \) is an \( \epsilon'' \)-approximation of \( A \). By Lemma 17, the former condition is true in the \( \ell \)-th iteration if \( \ell \geq \max(n, \sqrt{\delta} \cdot \ln \delta, \ell_1, \ell_2, \ell_3, \ell_4) \); this condition also guarantees that \( \hat{R}_{\max} = R_{\max} \). We have that \( \ell \) is polynomial in \( d, \rho \) as given in (1), considering also the bound for \( |\Sigma| \) given in Lemma 18. Since each iteration performs \( O(\ell^4 \ln \ell) \) action steps (see Line 2), and the AdaCT and ValueIteration algorithms run in polynomial time, we have that each iteration performs a polynomial number of steps. Therefore, the end of the \( \ell \)-th iteration is also reached in a a polynomial number of steps, at which point the algorithm returns an \( \epsilon \)-optimal policy with probability at least \( 1 - \delta \). Finally, since the above reasoning applies to the following iterations as well, every policy returned at a later moment is also \( \epsilon \)-optimal with probability at least \( 1 - \delta \). Therefore, Algorithm 1 is PAC-RL. \( \square \)

### A.8 Proof of Theorem 3

**Theorem 3.** Algorithm 2 on input \((A, \gamma, \epsilon, \delta, \hat{n})\) reaches accuracy \( \epsilon \) and confidence \( \delta \) within an expected number of action steps that is:

\[
\hat{O} \left( \frac{|A| \cdot \hat{n} \cdot |\Sigma|}{\rho \cdot \eta} \cdot \left( \frac{1}{\mu^2 + \frac{\hat{n}^2 \cdot |\Sigma|^2 \cdot |A|^2 \cdot R_{\max}^2}{(1 - \gamma)^6 \cdot \epsilon^2} \right) \right).
\]

**Proof.** It suffices to show that the expected number of action steps is at most \( 10\hat{n} \cdot \max(N'_0, N'_3) \), where \( N'_0 \) and \( N'_3 \) are given in Table 4. By the same arguments used for Theorem 2, based on Lemmas 1, 2, 3, and 5, the algorithm outputs an \( \epsilon \)-optimal policy with confidence \( 1 - \delta \) if it has read \( \max(N'_0, N'_3) \) episodes. In particular, the values \( N'_0 \) and \( N'_3 \) can be derived from the values \( N'_0 \) and \( N'_3 \) of Table 1, respectively. The expected length of an episode is \( 10\hat{n} \), since the length of an episode is a geometric random variable with parameter \( 1/(10\hat{n} + 1) \). Therefore, the expected number of action steps is \( 10\hat{n} \cdot \max(N'_0, N'_3) \). \( \square \)

### B Additional Material for The Running Example

**Formal description and transducer.** In Example 1 we have described a family of RDPs \( P_m = \langle A, S, R, D, \gamma \rangle \) where \( A = \{a_1, a_2\}, S = \{0, 1\} \times [0, m - 1] \times \{\text{enemy}, \text{clear}\}, R = \{0, 1\}, \gamma = 0 \) (the discount factor is irrelevant in this case), and the dynamics function \( D \) is represented by the transducer \( \langle Q, q_0, S, \tau, \theta \rangle \) where:

- the states are \( Q = [0, m - 1] \times \{0, 1\} \) where the first component denotes the current agent’s column and the second component is a bit denoting which of the two sets of probabilities are being used by the enemies,
- the initial state is \( q_0 = \langle m - 1, 0 \rangle \),
- the transition function is:
  - \( \tau((i, b), (j, i + 1 \mod m, \text{enemy})) = \langle i + 1 \mod m, b + 1 \mod 2 \rangle \),
  - \( \tau((i, b), (j, i + 1 \mod m, \text{clear})) = \langle i + 1 \mod m, b \rangle \),
- the output function is:
  - \( \theta((i, b))(a_k, (k, j, \text{enemy}), 0) = p_j^k \) where \( j = i + 1 \mod m \),
  - \( \theta((i, b))(a_k, (k, j, \text{clear}), 1) = 1 - p_j^k \) where \( j = i + 1 \mod m \).
Value of the parameters. The reachability \( p \) of the RDP \( \mathcal{P} \) above is 1/2. It suffices to show, by induction, that every state reachable in \( i \) steps has probability at least 1/2 to be visited at step \( i \), under the uniform policy. For the base case we have that the initial state \( \langle m - 1, 0 \rangle \) has probability 1 to be visited at step 0. By the inductive hypothesis, states \( \langle i - 1, 0 \rangle \) and \( \langle i - 1, 1 \rangle \) have probability at least 1/2 to be visited at step \( i \geq 1 \). Thus, the probability of visiting \( \langle i, 0 \rangle \) or \( \langle i, 1 \rangle \) at step \( i + 1 \) is at least

\[
\frac{1}{2} \left( \frac{1}{|A|} \cdot p^0_{i-1} + \frac{1}{|A|} \cdot (1 - p^0_{i-1}) \right) + \frac{1}{2} \left( \frac{1}{|A|} \cdot p^1_{i-1} + \frac{1}{|A|} \cdot (1 - p^1_{i-1}) \right) = \frac{1}{2}.
\]

The degree of determinism \( \eta \) of \( \mathcal{P} \) is the minimum value returned by \( \theta \), which is the minimum, for any \( i \), among \( p^0_i, p^1_i, 1 - p^0_i \), and \( 1 - p^1_i \).

The distinguishability \( \mu \) of \( \mathcal{P} \) is determined by the minimum probability difference of an observation in two different states. For every two states \( \langle i, b \rangle \) and \( \langle j, c \rangle \) with \( i \neq j \), we have that there is an observation \( \langle k, i + 1, e \rangle \) that has probability at least \( \eta \) in the former state and probability zero in the latter state. For every two states \( \langle i, b \rangle \) and \( \langle i, c \rangle \) with \( b \neq c \), we have that the difference in the probability of \( \langle k, i + 1, \text{enemy} \rangle \) is at least \( |p^0_i - p^1_i| \). Taking into account the probability of choosing an action uniformly at random, which is 1/2, we have that the distinguishability is at least 1/2 \( \min(\eta, \min_i(|p^0_i - p^1_i|)) \).

C Comparison with [Abadi and Brafman, 2020]

We compare our approach with the S3M algorithm from [Abadi and Brafman, 2020] on the family of RDPs \( \mathcal{P}_m \) introduced in Example 1 and formally described in Appendix B. Our algorithms output a near-optimal policy within a number of steps that is polynomial in grid length \( m \), whereas S3M requires a number of steps that is exponential in \( m \). The performance of our algorithms follows from Theorems 2 and 3, and from the fact that the parameters describing \( \mathcal{P}_m \) grow polynomially with \( m \)—please refer again to Appendix B for a discussion of the parameters. For the S3M algorithm, we argue that it cannot achieve arbitrary precision and confidence in polynomial time, since the accuracy of its estimates depends on the probability of single histories of length \( m \), which decreases exponentially with \( m \). Our argument to show a lower bound for the S3M algorithm is based on the fact that the probability of generating two identical histories up to the \( m \)-th step in \( N \) episodes can decrease exponentially with \( m \)—proof below.

**Proposition 8.** Let \( g = \max_i(p^0_i, p^1_i, 1 - p^0_i, 1 - p^1_i) \). For every history \( h \in S^m \) and every policy \( \pi \), the probability of observing \( h \) at least twice in \( N \) episodes is at most \( 1/4 \cdot e^2 \cdot (\sqrt{2} \cdot g)^2m \cdot N^2 \).

If \( g < 1/\sqrt{2} \), and \( N \) is polynomial in \( m \), the former bound goes to zero as \( m \) increases, for \( m \) sufficiently large. Thus, for large values of \( m \), there is a high probability that every history of length \( m \) occurs at most once.

We analyse the behaviour of the S3M algorithm when every history of length \( m - 2 \) occurs at most once. These histories are important to determine the action to take in column \( m - 1 \) of the grid. The algorithm compares pairs \( \langle h, a_i \rangle \) of a history \( h \) of length \( m - 2 \) and an action \( a \) based on the empirical distribution on the next observation; the only possible observations are of the form \( \langle m - 1, i, b \rangle \). Thus, the empirical distribution for \( \langle h, a_i \rangle \) is either the distribution \( P_1 \) that assigns probability one to \( \langle m - 1, i, \text{enemy} \rangle \) or the distribution \( P_2 \) that assigns probability one to \( \langle m - 1, i, \text{clear} \rangle \). Let \( \alpha = \min(p^0_{m-1}, p^1_{m-1}, 1 - p^0_{m-1}, 1 - p^1_{m-1}) \). Regardless of whether \( \langle h, a_i \rangle \) is assigned to \( P_1 \) or \( P_2 \), there is probability at least \( \alpha \) that the assigned distribution induces a higher value-estimate for the worse action—e.g., if \( a_i = a_0 \) and enemy \( j \) is in cell \( (0, j) \) with probability \( p^0_{m-1} > 1/2 \), but we do not hit enemy \( j \). In such a case, the error in the estimate is at least \( \alpha \). Therefore, the S3M algorithm introduces an error of at least \( \alpha \) with probability at least \( \alpha \). For instance, if we take every \( p^0_i = 0.7 \) and every \( p^1_i = 0.3 \), we have that \( g = 0.7 < 1/\sqrt{2} \) as required by the proposition, and the error of the S3M algorithm is at least 0.3 with probability at least 0.3.

**Proof of Proposition 8.** First, note that the probability of a given observation \( \langle i, j, b \rangle \) upon performing action \( a_k \) is at most \( g \) if \( j = k \) and zero otherwise. We use a union bound over all histories. There are at most \( 4^m \) histories of length \( m \) having non-zero probability. However, since the probability of actions \( a_0 \) and \( a_1 \) sums to one under every policy, and an observation has probability at most \( g \) if the action index matches its second component (and probability zero otherwise), we can consider only one value for the second component of an observation and bound the probability of the observation by \( g \). Thus, the number of histories to consider in the union bound is \( 2^m \), with each history having probability at most \( g^m \). Overall, the probability that any of the histories occurs in any two episodes, out of \( N \) episodes, is at most:

\[
\sum_h \binom{N}{2} \cdot (\mathbb{P}(h))^2 = \binom{N}{2} \cdot 2^m \cdot g^{2m} \leq \frac{e^2 \cdot N^2}{4} \cdot 2^m \cdot g^{2m} = 1/4 \cdot e^2 \cdot (\sqrt{2} \cdot g)^2m \cdot N^2,
\]

where the binomial coefficient takes into account the possible orders of the episodes in which a history \( h \) can occur. The inequality above makes use of the fact that \( \binom{n}{k} \leq (en/k)^k \).