Abstract. We give a geometrical set up for the semiclassical approximation to euclidean field theories having families of minima (instantons) parametrized by suitable moduli spaces $\mathcal{M}$. The standard examples are of course Yang-Mills theory and non-linear $\sigma$-models. The relevant space here is a family of measure spaces $\tilde{\mathcal{N}} \to \mathcal{M}$, with standard fibre a distribution space, given by a suitable extension of the normal bundle to $\mathcal{M}$ in the space of smooth fields. Over $\tilde{\mathcal{N}}$ there is a probability measure $d\mu$ given by the twisted product of the (normalized) volume element on $\mathcal{M}$ and the family of gaussian measures with covariance given by the tree propagator $C_\phi$ in the background of an instanton $\phi \in \mathcal{M}$. The space of “observables”, i.e. measurable functions on $(\tilde{\mathcal{N}}, d\mu)$, is studied and it is shown to contain a topological sector, corresponding to the intersection theory on $\mathcal{M}$. The expectation value of these topological “observables” does not depend on the covariance; it is therefore exact at all orders in perturbation theory and can moreover be computed in the topological regime by setting the covariance to zero.
1. Introduction.

The basic idea of quantizing a field theory in presence of instantons goes back to 't Hooft [‘tH] in his pioneering papers on Yang Mills theory. It includes from the very beginning some non linearity in the space of fields, represented by the instanton moduli space $\mathcal{M}$. More recently topological field theories [W] were studied as an attempt to give a field theoretical meaning to the intersection theory on certain moduli spaces.

In this paper we will show that in fact topological "observables" arise at the semiclassical level also in non topological field theories, provided the minima of the classical action occur in families. Our set up is a generalization of the result of Labastida [L] and at the same time it gives a full mathematical status to the path integrals entering the game. At the semiclassical level this can be done in a quite standard way in terms of gaussian measures, for which we refer e.g. to Gel'fand and Vilenkin’s book [GV]. The non linearity entailed by the instanton moduli space $\mathcal{M}$ gives rise to a non gaussian measure $d\mu$ on the dual $\tilde{\mathcal{N}}$ of the normal bundle to $\mathcal{M}$ in the space of smooth fields. Roughly speaking $d\mu$ is the “product” of a family $d\mu_\phi$ of gaussian measures on the fibres $\tilde{N}_\phi$ of $\tilde{\mathcal{N}}$ times a volume element $dv$ on $\mathcal{M}$ which represents the non linear contribution to the measure. Section 2 is devoted to the geometric construction of the measure $d\mu$. Concrete examples will be discussed in sect 3.

Although $\tilde{\mathcal{N}}$ has to be considered as a family $\bigcup_{\phi \in \mathcal{M}} \tilde{N}_\phi$ of a measure spaces, we will think of it as if it was a vector bundle over $\mathcal{M}$. With this analogy in mind, it turns out that the family $d\mu_\phi$ of gaussian measures plays the rôle of the Thom class of $\tilde{\mathcal{N}}$. To make this concrete, we will introduce in section 4 a cohomology explicitly designed for the job. From the physical point of view, the coboundary operator $\tilde{d}$ of this cohomology is the functional analogue of the BRS operator. Moreover, an analogue of Thom isomorphism goes through, identifying the BRS cohomology of $\tilde{d}$ with the cohomology of the moduli space $\mathcal{M}$. In particular, to any cycle on $\mathcal{M}$ one can associate an observable and to the expectation value of products of such observable equals the intersection number of the corresponding cycles, whenever defined.

2. Geometric set up

The basic ingredients to set up a “euclidean” classical field theory are

i) a space of classical field $\Phi = \{\phi : X \rightarrow Y\}$ defined on a compact manifold $X$ without boundary with values in a manifold $Y$. $X$ and $Y$ are assumed to have further structures, as Riemannian metrics etc., such that one can give to $\Phi$ the structure of a Hilbert manifold. In particular the tangent space $T_\phi \Phi$ will be assumed to be a Hilbert space. In all the physical examples it can be identified with the Sobolev space of section of a suitable vector bundle, $E_\phi$ say, over $X$ which are square integrable together with their derivatives up to order $s$. We will denote by $\langle , \rangle_s$ the inner product of Sobolev index $s$. Since $X$ is compact without boundary, the family of inner products $\langle , \rangle_s$ for $s \geq 0$ gives $T_\phi \Phi$ the structure of a nuclear space. This will be crucial in order to construct the gaussian measures we will
ii) an action functional $S : Φ → R$. When this is $C^2$ on $Φ$, one can write the first as well as the second variation of $S$ as

$$dS(δφ)|_φ = < EL_φ, δφ >_0$$

$$QS(δφ, δφ)|_φ = < δφ, H_φδφ >_0,$$

where $EL_φ, H_φ$ are the Euler–Lagrange expression and the Hessian operator of $S$ at $φ ∈ Φ$.

We are interested in studying the cases in which $H_φ$ has kernel. In all physical examples $H_φ$ turns out to be selfadjoint elliptic operator ( may be modulo the action of a group $G$ as for gauge theories, in which cases $Φ$ will be the space of orbits of the group $G$, see sect. 3.1 for more details). We have then an obvious exact sequence

$$0 → ker H_φ → T_φ Φ → N_φ → 0,$$

which gives the space $N_φ$ of the classical variations of $φ$ along which the Hessian is invertible. We say that $φ$ is a minimum (an instanton) for $S$ if $EL_φ = 0$ and $H_φ$ is positive definite on $N_φ$. Any element $ξ ∈ ker H_φ$ is a Jacobi field along $φ$, i.e. it is such that $φ + εξ$ is still a minimum (at the first order in $ε$). We will denote by $M ⊂ Φ$ a connected component of the moduli space of instantons. This will be assumed to be a manifold, i.e. we work on the subset of instantons which can be given a manifold structure. Then $ker H_φ$ can be identified with the tangent space $T_φ M$ and $N_φ$ is the fibre at $φ$ of the normal bundle $N$ w.r.t. the embedding $M → Φ$ of the moduli space in the space of all classical fields.

To construct the semiclassical approximation to the quantized version of the theory, we need first to give meaning to formal path integrals as

"$∫ Dη e^{−η. H_φη}.$"

This can be done by introducing well defined gaussian measures; i.e. by defining the covariance operator (i.e. the “euclidean propagator”) on the space $N_φ^*$ dual to $N_φ$. $N_φ^*$ will play the rôle of the space of the currents (i.e. test function) which we want to keep as smooth as possible. Let $E_φ^*$ be the bundle on $X$ dual to $E_φ$; any $η ∈ T_φ Φ = C^∞(X, E_φ)$ defines a functional on $C^∞(X, E_φ^*)$ by setting $F_η(j) = ∫ j · η * 1; * 1$ being a fixed volume element on $X$. Accordingly we will identify $C^∞(X, E_φ^*)$ with the cotangent space $T_φ^* Φ$. The Hessian gives us a transposed operator $H′_φ$ such that $F_{H_φη}(j) = F_η(H′_φj)$. Again we have an exact sequence

$$0 → ker H′_φ → T_φ^* Φ → N_φ^* → 0,$$

which identifies $ker H′_φ$ with the cotangent space $T_φ^* M$ and $N_φ^*$ with the fibre at $φ$ of the conormal bundle to $M$ in $Φ$. As $E_φ$ is assumed to have smooth fibre metric $h_φ$ we have an isomorphism $C^∞(X, E_φ^*) ≃ C^∞(X, E_φ)$ given by $j → h_φ(j, ·)$. Under such an isomorphism $H′_φ$ transforms into the adjoint $H^*_φ = H_φ$ so $ker H^*_φ ≃ ker H′_φ$ is actually the isomorphism $T_φ^* M ≃ T_φ M$. 

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The $L^2$ inner product $<\cdot,\cdot>_0$ on $C^\infty(X,E_\phi)$ given by
\[
<\eta_1,\eta_2>_0 = \int_X h^{-1}_\phi(\eta_1,\eta_2) * 1
\]
induces a metric $g$ on $\mathcal{M}$, called the Weil–Peterson metric, which we assume to be smooth. Thus we have a smooth volume element $*_g 1$ on $\mathcal{M}$. We will also assume, as it is the case in all concrete examples, that $\mathcal{M}$ has a natural compactification $\overline{\mathcal{M}}$ and that $\int_{\overline{\mathcal{M}}} *_g 1 = \text{vol}(\overline{\mathcal{M}})$ is finite. Thus, with the normalized measure $dv = *_g 1/\text{vol}(\overline{\mathcal{M}})$, $\overline{\mathcal{M}}$ itself is assumed to be a probability space.

On the fibres $N'_\phi$ the transposed Hessian $H'_\phi$ can be inverted, giving a positive definite operator $C_\phi = H'^{-1}_\phi$ and a positive quadratic form $<j,C_\phi j>$. Now $\forall \phi \in \mathcal{M}$, $N'_\phi$ is a nuclear space and the functional $W_\phi(j) = e^{-<j,C_\phi j>}$ is continuous, positive and $W_\phi(0) = 1$. Accordingly, it is the Fourier transform of a positive countably additive measure on the topological dual $\tilde{N}_\phi$ of $N_\phi$, i.e.
\[
W_\phi(j) = \int d\mu_\phi e^{i\eta(j)}.
\]
The measure $\mu_\phi$ is called the gaussian measure with covariance $C_\phi$. Recall that generically $\tilde{N}_\phi$ will be a distribution space given by the union of the completions $N_{\phi,s}$ of $N_\phi$ w.r.t. the Sobolev norm $<\cdot,\cdot>_s$ with $s \in \mathbb{Z}$. Summing up we have the following:

**Proposition.** Let $S : \Phi \to \mathbb{R}$ be a classical action, with a selfadjoint elliptic non negative Hessian operator $H_\phi$ along a family of instantons parametrized by a smooth manifold $\mathcal{M}$. Then there is a family of gaussian measures $d\mu_\phi$ on the topological dual $\tilde{N}_\phi$ of the fibres $N_\phi$ of the nuclear completion of the conormal bundle to $\mathcal{M}$ in $\Phi$.

As the measure $d\mu_\phi$ is automatically normalized, we have a family of probability spaces, parametrized by $\phi \in \mathcal{M}$. We recall that each of them is a triple $(\tilde{N}_\phi, \Sigma_\phi, \mu_\phi)$, where $\Sigma_\phi$ is the smallest $\sigma$-algebra generated by the cylinder sets $[Y]$. The normal “bundle” $\tilde{N} = \cup_{\phi \in \mathcal{M}} \tilde{N}_\phi$ has a natural probability measure $d\mu$, given by the family $d\mu_\phi$ and the (normalized) volume element $dv$ on $\mathcal{M}$. This is constructed as follows. A subset $U \subset \tilde{N}$ is measurable if $\pi(U)$ is a Borel set in $\mathcal{M}$ and if $U_\phi = U \cap \tilde{N}_\phi$ is measurable with measure $f(\phi) =: \mu_\phi(U_\phi) \in \mathcal{L}^1(\mathcal{M},dv)$. We will denote by $\Sigma$ the smallest $\sigma$-algebra generated by such sets. Then by definition the measure $d\mu$ is such that
\[
\int_U d\mu = \int_{\pi(U)} dv f(\phi).
\]
The triple $(\tilde{N}, \Sigma, \mu)$ is a probability space.

Notice that formally
\[
\int_{\tilde{N}} ... d\mu = \int_{\mathcal{M}} dv \int_{\tilde{N}_\phi} D\eta ... e^{-<\eta,H(\phi)\eta>}.
\]
As is well known, neither the "Lebesgue measure" $D\eta$ nor the quadratic form $<\eta,H_\phi\eta>$ exist separately on $\tilde{N}_\phi$, nevertheless $d\mu$ is well defined. The physical consequence of such a construction is that, whenever all the assumptions made above hold true, the semiclassical measure $d\mu$ is well defined and gives a mathematical meaning to our path integrals.
3. Examples

In this section we briefly discuss some examples of the geometric set up given above.

3.1 – As mentioned in the introduction, the standard example is euclidean gauge theory on an SU(2)–principal bundle $P$ over a compact Riemannian 4–manifold $X$. We will take as space of classical fields the space of gauge orbits $\Phi = \bar{C}/\bar{G}$, i.e. the space $\bar{C}$ of irreducible connections on $P$ modulo the action of the quotient $\bar{G} = G/Z$ of the group $G$ of gauge transformations by its centre $Z$. We refer to Singer [S], Donaldson and Kronheimer [DK] or to Grossier and Parker [GP] for details. The tangent bundle $T\Phi = \bar{C} \times_G Ker d_A^*$ has the typical fibre at the orbit through $\phi = [A]$ isomorphic to $Ker d_A^* \subset \Omega^1(adP)$. Here $\Omega^p(adP)$ denotes the space of smooth $p$–forms on $X$ with values in the bundle $adP$ associated to $P$ via the adjoint action of its structure group on its Lie algebra. The Yang–Mills action $S : \Phi \to \mathbb{R}$, given by $S(\phi) = \|F_A\|^2$, has first and second variations $dS(\delta \phi) =< d_A^* F, \delta \phi >, QS(\delta \phi, \delta \phi) =< \delta \phi, L_A \delta \phi >$ with Hessian

$$L_A = d_A^* d_A + \hat{F}$$

(see e.g. Atiyah–Bott [AB]). As is well known, $L_A$ is elliptic modulo the action of the gauge group, i.e. it becomes elliptic on the gauge fixing plane $ker d_A^* \subset T_{A}\bar{C}$. Indeed on this plane $L_A = H_A := d_A^* d_A + d_A d_A^* + \hat{F}$, which is obviously elliptic. Hence is kernel is finite dimensional and there is an isomorphism $ker H_A \simeq T_{[A]} \bar{M}$.

Let $\bar{M}$ be a component of the space of all (irreducible) instantons in $\bar{C}$. As is well known, it is a principal $\bar{G}$–bundle over the moduli space $\bar{M}$. At any $A \in \bar{M}$ we have a $\bar{G}$–equivariant decomposition $\Omega^1(adP) = ker H_A \oplus N_A \oplus Imd_A$ into orthogonal closed subspaces. Here $Imd_A$ is the tangent space to the gauge orbit through $A$, and $N_A$ is isomorphic to the fibre of the normal bundle $N_{[A]}$ at $[A]$ with respect to the embedding of the moduli space $\bar{M}$ in the space of gauge orbits $\Phi$. Accordingly, the vector bundle $V = \bar{M} \times_G \Omega^1(adP)$ on $\bar{M}$ decomposes as $V \simeq T\bar{M} \oplus N \oplus I$, where $I = \bar{M} \times_G \Omega^0(adP)$.

Now the gauge fixed Hessian $H_A$ is invertible on the much larger space $T_A\bar{C}/ker H_A \simeq N_A \oplus Imd_A$, and

$$H_A = \begin{cases} L_A & \text{on } N_A \\ d_A d_A^* & \text{on } Imd_A. \end{cases}$$

Decomposing $\eta \in N_A \oplus Imd_A$ as $\eta = \eta_0 + \eta_1$, the quadratic form associated to $H_A$ reads

$$< \eta, H_A \eta > = < \eta_0, H_A \eta_0 > + < d_A^* \eta_1, d_A^* \eta_1 >$$

and is obviously positive on this domain. Being $A$ irreducible, $Imd_A$ is isomorphic to the Lie algebra $\Omega^0(adP)$ of the group of gauge transformations, and ”changing variables” from $Imd_A$ to $\Omega^0(adP)$ is the basis of the Faddeev-Popov procedure. Indeed, if we set $\eta_1 = d_A \xi$, $\xi \in \Omega^0(adP)$, the quadratic form above becomes

$$< \eta, H_A \eta > = < \eta_0, H_A \eta_0 > + < \xi, \Delta_A^2 \xi >$$

on $N_A \oplus \Omega^0(adP)$, the operator $\Delta_A = d_A^* d_A$ being the laplacian on $\Omega^0(adP)$.
Now the insertion of the Faddeev-Popov determinant in the formal path integral for gauge theories has the effect of "normalizing" the formal gaussian measure on the \( \xi \)-variables; indeed

\[
\text{"} \det \Delta_A \int D\xi e^{-\langle \xi, \Delta_A^2 \xi \rangle} = 1. \text{"} 
\]

The analogue of this procedure in our set up is then to consider the complete gaussian measure associated to the operator \( H_A \oplus \Delta_A^2 \) on \( N_A \oplus \Omega^0(adP) \). The transposed of this operator gives rise to a covariance which is the direct sum of the covariance induced by \( H'_A \) on \( N'_A \) plus that induced by \( \Delta_A^2 \) on \( \Omega^0(adP)' \). The associated generating functionals \( W_A(j) \) induce gaussian measures on the topological duals \( N_A \) and \( \Omega^0(adP) \) and the product measure on their direct sum. Since \( W_A(j) \) is invariant under the action of the group of gauge transformations, the induced measures are invariant and descend to a family of measures on \( \tilde{N} \oplus \tilde{\mathcal{I}} \).

3.2 - Next we briefly mention another example, namely semiclassical gravity around Einstein metrics. The space of classical fields is the space of metrics \( g \) with unit volume and with no conformal automorphisms on a compact closed \( n \)-dimensional manifold \( (n \geq 3) \). On this space, Einstein metrics are critical points of the Hilbert-Einstein action

\[
S(g) = \int R_g \ast_g 1,
\]

where \( R_g \) is the scalar curvature and \( \ast_g 1 \) is the volume element of \( g \). As is well known [Be], the second variation of this functional is the quadratic form associated to an Hessian operator \( H_g \) which is elliptic modulo the action of the group of diffeomorphisms. The standard gauge fixing procedure (which amounts to restricting the space of variations of the metrics to the orthogonal complement of the image of the traceless part of their Lie derivatives, see e.g. [E]) yields an elliptic operator whose kernel is isomorphic to the tangent space to the moduli space of Einstein metrics. Although there are scalings of the metric along which the action \( S(g) \) decreases, restricting on the space of metrics with unit volumes gets rid of such modes, makes the Einstein metrics minima for the action and hence yields a positive semidefinite Hessian. So, there is a strict analogy with the previous example. As a word of caution, notice that in this case the space of all fields is not an affine space and that the gauge fixing is more complicated due to the constant volume condition; these facts may imply subtelties which deserve a closer study.

3.3 – We will spend some more words for the case of \( \sigma \)-models, i.e. harmonic theory for maps \( \phi : C \to Y \) of an algebraic curve \( C \) with values in a compact Kähler manifold \( Y \). To be concrete we will stick to the case of \( \phi : \mathbb{P}^1 \to \mathbb{P}^n, \mathbb{P}^n \) being the n–dimensional complex projective space. We refer to Palais [P] for the construction of the manifold \( \Phi \) of such maps of a given degree \( d \), and we limit ourselves to notice that the local model for \( \Phi \) around \( \phi \), i.e. the tangent space \( T_{\phi} \Phi \) is actually isomorphic to \( C^\infty(\phi^*T\mathbb{P}^n), T\mathbb{P}^n \) being the real tangent bundle to \( \mathbb{P}^n \). The Dirichelet functional \( S : \Phi \to \mathbb{R} \) reads \( S(\phi) = \int_{\mathbb{P}^1} \text{tr} \alpha_{\phi} \), where \( \alpha_{\phi} \) is the first fundamental form of \( \phi \) w.r.t. the Fubini–Study metric on \( \mathbb{P}^n \), and \( \text{tr} \) denotes the trace w.r.t. the Fubini–Study metric on \( \mathbb{P}^1 \) (or any other metric conformally equivalent
to it). The explicit form of the first and the second variations of \( S \) are given by e.g. Siu and Yau \([SY]\) Eells and Wood \([EW]\).

Thanks to the complex structures of both the source and the target space, one easily sees that \( S(\phi) \geq \deg \phi = \int_{\mathbb{P}^1} \phi^* \omega \), \( \omega \) being the Fubini–Study Kähler form, and that the equality holds precisely if \( \phi \) is holomorphic (or antiholomorphic). The holomorphic maps are called instantons, and let \( \mathcal{M} \) denote the set of such maps of a fixed degree \( d \). Since \( S(\phi) = d \) for any \( \phi \in \mathcal{M} \) and \( S(\phi) > d \) for \( \phi \in \Phi - \mathcal{M} \), we see that the Hessian has as kernel at \( \phi \) the space of holomorphic deformations of \( \phi \), i.e. \( \ker H_\phi \cong H^0(\mathbb{P}^1, \phi^* T'\mathbb{P}^n) \) (where \( T'\mathbb{P}^n \) is the holomorphic tangent bundle) and we have an exact sequence

\[
0 \to H^0(\mathbb{P}^1, \phi^* T'\mathbb{P}^n) \to C^\infty(\mathbb{P}^1, \phi^* T'\mathbb{P}^n) \to N_\phi \to 0
\]

where \( H^0(\mathbb{P}^1, \phi^* T'\mathbb{P}^n) \) and \( N_\phi \) play the rôle of the tangent space \( T_\phi \mathcal{M} \) and the normal space to \( \mathcal{M} \) in \( \Phi \), assuming of course that \( \mathcal{M} \) is a manifold. This is indeed the case, as shown for instance by \([St]\) by applying the techniques of Quot schemes of Groethendieck.

If we think of \( \mathbb{P}^n \) as the dual projective space, \( \mathcal{M} \) is actually isomorphic to a projective space itself, i.e. \( \mathcal{M} \cong \mathbb{P}^{(n+1)d+n} \).

These results actually hold true for more general Grassmannian–valued \( \sigma \)-models (see again \([St]\) for more details). The cohomology ring of such \( \mathcal{M} \)'s is also known, being a quotient of the cohomology ring of a product of two Grassmannians. In view of the results of Gepner’s \([G]\), it is tempting to imagine that this ring may be identified with some chiral ring in superconformal field theory.

3.4 - A somewhat different class of examples which fits in a similar scheme has to do with “free” fields on an external background. To quote a couple of these, one may consider some bosonic fields \( f \), e.g. \( f \in C^\infty(X,F) \), \( F \) being an hermitean vector bundle associated to \( P \) via some unitary representation \( \rho \) and taking an action \( S_A(f) = \|d_A f\|^2 \) in the background of an instanton \( A \). Or else on a Riemann surface \( C \) of genus \( g \geq 2 \), pick up a metric \( m \) of constant curvature \( R = -1 \) modulo diffeomorphisms. If \( t \) is a smooth tensor field on \( C \) one can study the action \( S_g(t) = \|d_g t\|^2 \), where, as \( g \) varies in the Teichmüller space, \( d_g \) is the Levi–Civita connection. In all such cases one has a family of distribution spaces parametrized by suitable moduli and a gaussian measure on each of them. Of course there will be no more normal bundles, but the ideas of this paper obviously apply.

4. Gaussian cohomology and BRS operator

Having constructed a measure on \( \tilde{\mathcal{N}} \), we next turn to study the “observables”, which we assume to be all the measurable functions \( O : \tilde{\mathcal{N}} \to \mathbb{R} \), i.e. \( O \in \mathcal{L}^1(\tilde{\mathcal{N}}, d\mu) \). Here we use quotation marks because, as we will shortly see, such measurable functions include objects which are by no means local in the usual sense. Although there is no perturbative clue as for their physical meaning, they will contain the “topological sector” of the theory.

To see how this works, let us first notice that the family of measures \( d\mu_\phi \) plays the rôle of the Thom class of \( \tilde{\mathcal{N}} \). To avoid confusion, in our case \( \tilde{\mathcal{N}} \) is simply a family of
measure spaces, with a projection \( \pi : \tilde{N} \rightarrow M \). We will not need to give \( \tilde{N} \) the structure of a vector bundle, even if sometimes we will speak as if it was. Let us introduce the space

\[
\mathcal{L}^1(\tilde{N}, d\mu_\phi) = \left\{ W : \tilde{N} \rightarrow \mathbb{R} \mid \omega(\phi) = \int_{\tilde{N}_\phi} W d\mu_\phi \in C^\infty(M) \right\}.
\]

Any object of the form \( W d\mu_\phi \), with \( W \in \mathcal{L}^1(\tilde{N}, d\mu_\phi) \) will be called a vertical ”form” on \( \tilde{N} \), with gaussian behavior on the fibers and vertical ”codimension” 0. This last statement means that its restriction to each fibre \( \tilde{N}_\phi \) is actually a measure on it. We will denote by \( \Omega^{0,0}(\tilde{N}) \) a \( C^\infty(M) \) module, since \( \forall f \in C^\infty(M), \int f W d\mu_\phi = f \int W d\mu_\phi \). Accordingly we can construct the tensor products

\[
\Omega^p,0(\tilde{N}) = \Omega^p(M) \otimes_{C^\infty(M)} \Omega^{0,0}(\tilde{N}),
\]

\( \Omega^p(M) \) being the \( C^\infty(M) \)–module of smooth \( p \)–forms on \( M \). The first step in making contact with the Thom class is to notice that there is a map

\[
T : \Omega^p(M) \rightarrow \Omega^{p,0}(\tilde{N})
\]

\[
\omega \mapsto \omega \otimes d\mu_\phi,
\]

such that \( \int_{\tilde{N}_\phi} \cdot T = \text{id} \) on \( \Omega^p(M) \).

Next we need to introduce the space \( \Omega^{p,1}(\tilde{N}) \) of gaussian ”forms” on \( \tilde{N} \) of vertical ”codimension” 1. For this we will assume that \( N' \) has trivial subline bundles \( L_i \), or in other words that there are global non vanishing sections \( j_i \) of \( N' \). Although one might avoid such an assumption, we notice that this is the case for the physical examples (see appendix A), and that it helps in making the following arguments quite simple. Let \( A_i(\phi) = \{ \eta \in \tilde{N}_\phi \mid \eta(j_i)|_\phi = 0 \} \) be the annihilator of \( j_i \) at \( \phi \). Projecting the covariance to \( N'_\phi / \mathbb{R}\{j_i|_\phi\} \) we get by Fourier transform a gaussian measure \( d\mu_i \) on \( A_i(\phi) \). Now \( \tilde{N}_\phi \) is a cylinder over the real line \( \tilde{N}_\phi / A_i(\phi) \) and we can consider the family of the (unnormalized) gaussian ”measures” \( e^{-q_i^2/2}d\mu_i \) on \( \tilde{N}_\phi \), parametrized by \( q_i = \eta(j_i), \eta \in \tilde{N}_\phi / A_i(\phi) \). Clearly enough \( \int_{\tilde{N}_\phi} e^{-q_i^2/2}d\mu_i = 0 \). We will consider objects of the form

\[
\alpha = \sum_{i=1}^n W_i e^{-q_i^2/2}d\mu_i,
\]

with \( n < \infty, W_i \in \mathcal{L}^1(\tilde{N}, d\mu_\phi) \). Notice that, as a function of \( q_i, W_i \) is a linear combination of squared Hermite polynomials, and is therefore differentiable w.r.t. \( q_i \). We will denote by \( \Omega^{0,1}(\tilde{N}) \) the space of such ”forms” of vertical ”codimension” 1. Again we set \( \Omega^{p,1}(\tilde{N}) = \Omega^p(M) \otimes \Omega^{0,1}(\tilde{N}) \). We will not need to introduce ”forms” of higher codimension.

Let us introduce the operator (for \( q=0,1 \))

\[
\tilde{d} : \Omega^{p,q} \rightarrow \Omega^{p+1,q} \otimes \Omega^{p,q+1}
\]
by linearly extending
\[\tilde{d}(\omega \otimes Wd\mu_\phi) = d\omega \otimes Wd\mu_\phi + (-)^p \omega \wedge dWd\mu_\phi,\]
\[\tilde{d}(\omega \otimes W_i e^{-q_i^2/2}d\mu_i) = d\omega \otimes W_i e^{-q_i^2/2}d\mu_i + (-)^p \omega (W'_i - q_i W_i) d\mu_i.\]

Notice \(W' = \frac{\partial W}{\partial q_i}\) is an odd function of \(q_i\). We also set \(\tilde{d}d\mu_\phi = 0\).

The rationale for this is the following. Let \(\pi_*\) denote the integration along the fibres of \(\tilde{N}\). We have
\[d\pi_*(\omega \otimes W_i e^{-q_i^2/2}d\mu_i) = 0\]
\[\pi_*\tilde{d}(\omega \otimes W_i e^{-q_i^2/2}d\mu_i) = \pi_*(d\omega \otimes W_i e^{-q_i^2/2}d\mu_i + (-)^p \omega (W'_i - q_i W_i)d\mu_i) = 0\]
and \(d\pi_*(\omega \otimes Wd\mu_\phi) = d(f\omega) = f d\omega + df\omega\) with \(f = \int Wd\mu_\phi\). But now
\[\pi_*\tilde{d}(\omega \otimes Wd\mu_\phi) = \pi_*(d\omega \otimes Wd\mu_\phi + dW \wedge \omega d\mu_\phi) =
= d\omega f + df \omega\]
and in all cases \(d\pi_* = \pi_*\tilde{d}\).

The space \(H^{p,0}(\tilde{d}) = \text{Ker} \tilde{d}/\text{Im} \tilde{d}\) is then obviously isomorphic to \(H^p(M)\). Accordingly, we identify \(\tilde{d}\) with the functional counterpart of the BRS operator: up to gaussian integrations it coincides with the exterior differential on the moduli spaces.

Let us now go back to observables. Given cycles \(c_i (i = 1, 2)\) representing homology classes \([c_i] \in H_*(M, \mathbb{Z})\) of complementary dimension, we can compute their intersection \(c_1, c_2 \in \mathbb{Z}\). We can easily construct an observable on \(\tilde{N}\) whose expectation value equals \(c_1, c_2\). Indeed we have
\[c_1.c_2 = \int_M \alpha_1 \wedge \alpha_2,\]
where \([\alpha_i] \in H^*(M, \mathbb{R})\) is the Poincaré dual of \([c_i]\). Setting \(\alpha_1 \wedge \alpha_2 = f dv\), we simply multiply by the "Thom class" \(d\mu_\phi\) getting the measure \(f d\mu\) on \(\tilde{N}\). Accordingly, the expectation value of \(f\) reads
\[< f > = \int_{\tilde{N}} f d\mu = \int_\mathcal{M} f dv = c_1.c_2.\]

Summing up, with the identification made above, we have seen that

**Proposition.** There are observables in the theory whose expectation values is the same as the intersection of cycles in the moduli spaces.
5. Concluding Remarks

The basic result of this paper is that semiclassical field theories with instantons moduli spaces have "observables" corresponding to the intersection theory on $\mathcal{M}$. As it is obvious, such expectation values do not depend on the family $C_\phi$, $\phi \in \mathcal{M}$, of covariances given by the field theory itself and this has the following consequences:

i) as for the expectation values of topological observables, one can safely set $C_\phi = 0$ and kill all quantum degrees of freedom in the theory. This is the same as projecting $\tilde{N}$ onto $\mathcal{M}$ and work out intersection theory there by topological methods. This is the spirit of topological field theories.

ii) the topological sector is by its very definition non perturbative: after all changing the measures and their support (e.g. by renormalization) does not change intersection of cycles. So we see that keeping some non linearity in $\tilde{N}$ (represented by its base space $\mathcal{M}$) has the effect of extracting non perturbative informations.

On the other hand we have seen that quantum field theory and the topology of $\mathcal{M}$ do not mix up at least in our set up. Nor do we believe that this phenomenon arises from the technical assumptions we made in building up our gaussian cohomology. As far as some form of the Thom isomorphism holds in the infinite dimensional case, the conclusion will be invariably the same. Although this makes hard to imagine to compute intersection theory via quantum field theory, it makes feasible to compute non perturbative effects via topological methods.

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Appendix.

In the construction of the "gaussian cohomology" of section 4 we assumed that there are global non vanishing sections of $\mathcal{N}'$. In other words, we need a current on which the covariances $C_\phi$ does not vanish for any $\phi \in \mathcal{M}$. This is the same as finding a section $j$ of $\mathcal{N}$ such that $H_{\phi,j} \neq 0 \forall \phi \in \mathcal{M}$. We will prove here that this is indeed the case for gauge theories and Kahler manifold-valued $\sigma$-models.

A.1 -. As for gauge theories, recall that the measure is carried on the family of dual spaces to the nuclear completion $\mathcal{N}' + \mathcal{I}'$ of the extension of the conormal bundle $\mathcal{N}'_s$ by the bundle $\mathcal{I}'_{s-1}$ on $\mathcal{M}$ with fibre the dual of the Lie algebra of the group of gauge transformations. This plays the rôle of the normal bundle in the present case. Besides gauge fixing problems, this extension is technically usefull for us because any $0 \neq \xi \in \Omega^0(adP)$ gives a section of
the (enlarged) normal bundle on the moduli spaces of irreducible instantons along which the full covariance does not vanish. In fact $\ker \Delta^2_A = \ker \Delta_A = \ker d_A = 0 \in \Omega^0(adP)$ at any irreducible instanton $A$.

A.2 -. For Kahler manifold-valued $\sigma$-models, the kernel of the Hessian operator at an instanton is given by holomorphic vector fields along the instanton itself on the target manifold, i.e. $\ker H_\phi = H^0(C, \phi^*T^1Y)$. The $C^\infty$ structure of the vector bundle $\phi^*T^1Y$ does not depend on the choice of $\phi$ in a given connected component of the moduli instanton space. Although the holomorphic structure of this bundle does depend on $\phi$, any compactly supported section with support contained in a proper open subset of $C$ will never be holomorphic at any $\phi$, and hence, using the fibre metric, will give us a current on which the covariance never vanishes.

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