Vacuum Polarization Effects in the Global Monopole Spacetime in the Presence of Wu-Yang Magnetic Monopole

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Abstract

In this paper we consider the presence of the Wu-Yang magnetic monopole in the global monopole spacetime and their influence on the vacuum polarization effects around these two monopoles placed together. According to Wu-Yang [Nucl. Phys. B107, 365 (1976)] the solution of the Klein-Gordon equation in such an external field will not be an ordinary function but, instead, section. Because of the peculiar radial symmetry of the global monopole spacetime, it is possible to cover its space section by two overlapping regions, needed to define the singularity free vector potential, and to study the quantum effects due to a charged scalar field in this system. In order to develop this analysis we construct the explicit Euclidean scalar Green section associated with a charged massless field in a global monopole spacetime in the presence of the Abelian Wu-Yang magnetic monopole. Having this Green section it is possible to study the vacuum polarization

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effects. We explicitly calculate the renormalized vacuum expectation value $\langle \Phi(x)^\dagger \Phi(x) \rangle_{\text{Ren.}}$, associated with a charged scalar field operator and the respective energy-momentum tensor, $\langle T_{\mu\nu}(x) \rangle_{\text{Ren.}}$, which are expressed in terms of the parameter which codify the presence of the global and magnetic monopoles.

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1. Introduction

In this paper we calculate the Euclidean scalar Green section associated with a charged massless field in the global monopole spacetime in presence of an Abelian Wu-Yang magnetic monopole.

As it is well known the Wu-Yang magnetic monopole \([1]\) does not possess strings of singularities around it. The vector potential \(A_\mu\) is defined in two overlapping region \(R_\alpha\) and \(R_\beta\) which cover the whole space. Using spherical coordinates system \(r, \theta, \phi\) with the monopole at origin we choose:

\[
R_\alpha : 0 \leq \theta < \frac{\pi}{2} + \delta, \ r > 0, \ 0 \leq \phi < 2\pi , \quad (1)
\]
\[
R_\beta : \frac{\pi}{2} - \delta < \theta \leq \pi, \ r > 0, \ 0 \leq \phi < 2\pi , \quad (2)
\]
\[
R_{ab} : \frac{\pi}{2} - \delta < \theta < \frac{\pi}{2} + \delta, \ r > 0, \ 0 \leq \phi < 2\pi , \quad (3)
\]

with \(0 < \delta \leq \frac{\pi}{2}\). \(R_{ab}\) is the overlapping region.

The non vanishing components of vector potentials are

\[
(A_\phi)_a = g(1 - \cos \theta), \\
(A_\phi)_b = -g(1 + \cos \theta), \quad (4)
\]

being \(g\) the monopole strength. In the overlapping region the non vanishing components are related by a gauge transformation

\[
(A_\phi)_a = (A_\phi)_b + i e S_{ab} \partial_\phi S^{-1}_{ab} , \quad (5)
\]

where \(S = S_{ab} = e^{2i q \phi}\) being \(q = eg = n/2\) in units \(\hbar = c = 1\).

The global monopole is a heavy object formed in the phase transition of a system composed of a self-coupling scalar field triplet \(\phi^a\) whose original global \(O(3)\) symmetry is spontaneously broken to \(U(1)\) \([2]\). The Euclidean version of the metric tensor associated with a pointlike global monopole can be given by the line element below:

\[
ds^2 = d\tau^2 + \frac{dr^2}{\alpha^2} + r^2 d\Omega_2^2 = g_{\mu\nu}(x)dx^\mu dx^\nu . \quad (6)
\]

The parameter \(\alpha\) which codify the presence of the global monopole is smaller than unity. This spacetime it is not flat: the scalar curvature \(R = 2(1 - \)
\( \alpha^2/r^2 \), and the solid angle associated with a sphere with unity radius is \( \Omega = 4\pi\alpha^2 \), so smaller than ordinary one. The energy-momentum tensor associated with this object has a diagonal form and its non-vanishing components read \( T_0^0 = T_1^1 = (\alpha^2 - 1)/r^2 \).

Here in this paper we shall consider the quantum analysis of a massless charged scalar field in the spacetime of a global monopole and in the presence of a magnetic monopole. Specifically we are interested in calculation of the vacuum polarization effect on this field. In order to develop this analysis we shall consider that both monopoles, the global and the magnetic ones, are at the same position which we take as the origin of our reference system. Doing this we try to reproduce the quantum analysis of a charged scalar field in the presence of a more general object which presents the main aspects of both monopoles: a solid angle metric spacetime and a radial magnetic field. The complete treatment about this system must take into account the contribution on the geometry of the spacetime due to the energy density associated with the magnetic field. However at this first moment, seems appropriate to us to start this analysis in a simpler way, i.e., considering only the influence of the global monopole on the geometry of the spacetime.

As in the flat case, it is possible to cover the space section of the global monopole spacetime into two overlapping regions \( R_a \) and \( R_b \) as in \( (1) \) - \( (3) \), with the metric tensor given by the same expressions as before in both regions, and define a vector potential as in \( (4) \). In this way the magnetic field is expressed by the same expressions in both regions: \( \vec{B} = g\hat{r}/r^2 \).

The Klein-Gordon equation associated with a massless charged field in curved spacetime and in presence of an external four-vector potential \( A_\mu \), can be obtained from the covariant form of the Klein-Gordon equation replacing the partial derivative \( \partial_\mu \) by extended derivative \( D_\mu = \partial_\mu - ieA_\mu \):

\[
\left[ \Box - \frac{ieA^\mu}{\sqrt{g}} (\partial_\mu \sqrt{g}) - ie(\partial_\mu A^\mu) - 2ieA^\mu \partial_\mu - e^2 A^\mu A_\mu + \xi R \right] \Psi(x) = 0 , \quad (7)
\]

\(^1\)The metric tensor associated with a pointlike compost topological object which takes into account the presence of a magnetic charge in a solid angle deficit geometry, has been obtained recently \( \footnote{3} \). The structure of the respective manifold corresponds to a Reissner-Nordström spacetime with a solid angle deficit factor. For large distance, the contribution due to the magnetic field can be neglected.
with
\[ \Box \Psi = \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} g^{\mu \nu} \partial_\nu \Psi \right). \]

In the above equation we have introduced a non-minimal coupling between the field with the geometry, \( \xi R \), where \( \xi \) is an arbitrary numerical factor and \( R \) is the Ricci scalar curvature. Being the four-vector representing the Wu-Yang magnetic monopole, this equation must be analysed in both regions, \( R_a \) and \( R_b \), separately. The solution \( \Psi \) is not an ordinary function but, instead, a section assuming values \( \Psi_a \) and \( \Psi_b \) in \( R_a \) and \( R_b \), respectively, and satisfying the gauge transformation
\[ \Psi_a = S_{ab} \Psi_b \]
in the overlapping region \( R_{ab} \).

In order to analyse this Klein-Gordon equation we shall adopt the usual approach: We write the solution in the form \( \Psi(x) = e^{-iEt} R(r) \Theta(\theta) \Phi(\phi) \), where the angular part can be expressed in terms of the monopole harmonics, \( Y_{lm}^q(\theta, \phi) \) [4]. The monopole harmonics are eigensections satisfying the eigenvalues equations below:
\[ \bar{L}_q^2 Y_{lm}^q = l(l+1)Y_{lm}^q, \]
and
\[ L_z Y_{lm}^q = mY_{lm}^q \]
with \( l = |q|, |q| + 1, |q| + 2, \ldots \) and \( m = -l, -l + 1, \ldots, l \). Being
\[ \bar{L}_q = \bar{r} \times (\bar{p} - e \bar{A}) - q \bar{r}. \]

The temporal and radial functions, solutions of (7), have the same expressions in whole space.

In section 2 we derive the Euclidean Green section associated with this system. In fact this Green section is given in terms of an infinity sum of continuous section. In section 3 we compute explicitly the renormalized vacuum expectation value \( \langle \Phi^*(x) \Phi(x) \rangle \) of a charged massless scalar field operator for the case where \( \eta^2 = 1 - \alpha^2 \ll 1 \) and also present a formal expression to \( \langle T^\nu_\mu(x) \rangle_{\text{Ren.}} \). The scale dependent term in \( \langle T^\nu_\mu \rangle \) is explicitly calculated up to the first order in \( \eta^2 \) and \( q^2 \). In section 4 we present our conclusions and the most important remarks about this system.
2. Green Section

The Euclidean Green section associated with a charged massless scalar field in the global monopole spacetime in presence of the Wu-Yang magnetic monopole, can be obtained by the Schwinger-DeWitt formalism as follows:

\[ G_E(x, x') = \int_0^\infty dsK(x, x'; s) , \]  

(13)

where the heat kernel, \( K(x, x'; s) \), can be expressed in terms of the eigen-sections of the Klein-Gordon operator defined in (7) as shown below:

\[ K(x, x'; s) = \sum_\sigma \Psi_\sigma(x)\Psi_\sigma^*(x')e^{-s\sigma^2} , \]  

(14)

\( \sigma^2 \) being the corresponding positively defined eigenvalue. Because of the completeness of the monopole harmonics [4], the respective normalized eigen-section is:

\[ \Psi^q_\sigma(x) = \sqrt{\frac{\alpha p}{2\pi}} e^{-i\omega r} J_{\lambda_l}(pr) Y_{lm}^q(\theta, \phi) , \]  

(15)

with

\[ \sigma^2 = \omega^2 + \alpha^2 p^2 . \]  

(16)

\( J_{\lambda_l} \) is the Bessel function of order

\[ \lambda_l = \alpha^{-1}\sqrt{(l + 1/2)^2 - q^2 + 2(\alpha^2 - 1)(\xi + 1/8)} . \]  

(17)

So according to (13) the heat kernel is given by

\[ K(x, x'; s) = \int_{-\infty}^{\infty} d\omega \int_0^\infty dp \sum_{l,m} \Psi_\sigma(x)\Psi_\sigma^*(x')e^{-s\sigma^2} \]

\[ = \frac{1}{4\alpha s^{3/2}} \frac{1}{\sqrt{\pi r r'}} e^{-\Delta r^2 s^{2} + \Delta r r' s^2} \times \]

\[ \sum_{l=-|q|}^{\infty} I_{\lambda_l} \left( \frac{rr'}{2\alpha^2 s} \right) \sum_{m=-l}^{l} Y_{lm}^q(\theta, \phi) (Y_{lm}(\theta', \phi'))^* , \]  

(18)

\( I_{\lambda_l} \) being the modified Bessel function. Tai Tsun Wu and Chen Ning Yang in [5] have derived some properties of monopole harmonics, including the generalization of spherical harmonics addition theorem. However, because
we are interested to calculate the renormalized value of the Green function in the coincidence limit, a simpler expression is obtained taking \( \theta = \theta' \) and \( \phi = \phi' \) in (18). Moreover, substituting this simplified expression into (13) we obtain, with the help of [6], the following Green function:

\[
G^q(r, \tau; r', \tau') = \frac{1}{8\pi^2 rr'} \sum_{l=|q|}^{\infty} (2l + 1)Q_{\lambda, l-1/2} \left( \frac{\alpha^2 \Delta \tau^2 + r^2 + r'^2}{2rr'} \right),
\]

\( Q_\lambda \) being the Legendre function. Unfortunately because of the non-trivial dependence of the order of the Legendre function with the parameter \( \alpha \) and \( q \), it is not possible to obtain a closed expression to the above Green section, even for the case \( \xi = -1/8 \). The best that we can do is to develop an approximated expression to it. This development will be presented in the next subsection.

Having now this Green section it is possible to obtain the vacuum polarization effect in this gravitational background in the presence of the Wu-Yang magnetic monopole.

### 2.1 Computation of \( \langle \Phi^*(x) \Phi(x') \rangle_{\text{Ren.}} \)

Here in this subsection we shall develop the calculation of the vacuum expectation value of the square of the modulus of the field operator. This quantity can be obtained calculating the Green section in the coincidence limit as shown below:

\[
\langle \Phi^*(x) \Phi(x) \rangle = \lim_{x' \to x} G^q(x, x').
\]

However this procedure provides a divergent result. In order to obtain a finite and well defined result we must apply in this calculation some renormalization procedure. Here in this paper we shall adopt the point-splitting renormalization one. This procedure is based upon a divergence subtraction scheme in the coincidence limit of the Green function. Adler et al. \[7\] have pointed out that the singular behavior of the Green function has the same structure as the Hadamard function. So in order to obtain a finite result to the vacuum expectation value above, we subtract from the Green section the

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\(^2\)In Appendix A we show the simplified result to the sum of product of monopole harmonics in [19] when we take the coincidence limit in the angular variables.
Hadamard function, before applying the coincidence limit:

\[ \langle \Phi^*(x)\Phi(x) \rangle_{\text{Ren.}} = \lim_{x' \to x} \left[ G^q(x, x') - G_H(x, x') \right] . \]  

(21)

In order to develop this calculation, let us take first in (19) \( \Delta \tau = 0 \) and use for the Legendre function its integral representation [6]:

\[ Q_{\lambda-1/2}(\cosh \rho) = \frac{1}{\sqrt{2}} \int_\rho^\infty dt \frac{e^{-\lambda t}}{\cosh t - \cosh \rho} , \]  

(22)

with \( \cosh \rho = \frac{r^2 + r'^2}{2rr'} \). So, substituting this expression into (19) we get:

\[ G^q(r, r') = \frac{1}{8\pi^2 rr'} \frac{1}{\sqrt{2}} \int_\rho^\infty dt \frac{1}{\cosh t - \cosh \rho} \sum_{l=|q|}^{\infty} (2l + 1)e^{-\lambda t} . \]  

(23)

Unfortunately because of the dependence of \( \lambda_l \) with \( \alpha^2 \) and \( q^2 \), it is not possible to develop the summation above and to express (23) in a closed form. So, in order to obtain a more workable expression, it is necessary to make some approximations: The first one is to consider the parameter \( \eta^2 = 1 - \alpha^2 \ll 1 \) and to develop a series expansion in powers of this parameter. The second procedure to be adopted, is to develop again another expansion in the resulting expressions in powers of \( q^2 (l+1/2)^2 \). In this way for each order in \( \eta^2 \) in the summation, we procedure another expansion. After doing this it is possible to develop the summation. Our approximated expression was developed only up to the first order in \( \eta^2 \) and the next expansion, which involves \( \sqrt{1 - q^2/(l+1/2)^2} \), was developed up to the first order too. The final result presents also an exponential dependence on \( q \), \( e^{\eta t} \), due the fact that all terms in the summations start from \( l = q \).

The summation above can be written as having two mains contributions as shown below:

\[ S = \sum_{l=q}^{\infty} (2l + 1)e^{-\lambda_l t} \simeq S_1 - \frac{\eta^2 t}{2} S_2 \]  

(24)

---

3In the calculation of renormalized vacuum expectation value of the energy-momentum operator, \( \langle T_{\mu\nu} \rangle \), Wald [8] has detected an error in [7] and modified the prescription adding an extra term proportional to \( a_2(x) \) in order to provide the correct result for the trace anomaly.

4In fact for a typical grand unified theory where this topological defect appears as a consequence of spontaneously break of the global symmetry, \( \eta^2 \) is of order \( 10^{-6} \).

5From now on in this paper we shall consider \( q \geq 0 \). The extension of our results to negative values of \( q \) are straightforward.
with
\[ S_1 = \frac{e^{-qt}}{2\sinh(t/2)} \left[ \coth(t/2) + 2q \right] + \frac{q^2e^{-qt}t}{2\sinh(t/2)}, \quad (25) \]
and after some calculations,
\[
S_2 = \frac{e^{-qt}}{8\sinh^2(t/2)} \left[ 8q\cosh(t/2) + 8q^2\sinh(t/2) + \frac{3 + \cosh(t)}{\sinh(t/2)} \right] + \frac{q^2e^{-qt}}{2\sinh(t/2)} \left[ \frac{t}{2} \coth(t/2) - 1 \right] - 2(\xi + 1/8)e^{-qt} \frac{1}{\sinh(t/2)} \times \left[ 1 + q^2te^{-t/2}\sinh(t/2)\Phi(e^{-t}, 1, q + 1/2) + \frac{4q^2}{(2q + 1)^2}e^{-t/2} \times \sinh(t/2)_{3F2}(1, 1/2 + q, 1/2 + q; 3/2 + q, 3/2 + q; e^{-t}) \right], \quad (26)
\]
where \(3F2\) is an hypergeometric function and \(\Phi(z, s, v)\) the Phi-function [3].

The Hadamard function in a curved four dimensional spacetime contains a logarithmic term which depends on the arbitrary cutoff scale \(\mu\) and is expressed in terms of the square of the geodesic distance \(2\sigma(x, x')\) as shown below:
\[
G_H(x, x') = \frac{1}{16\pi^2} \left[ \frac{2}{\sigma(x, x')} - (\xi + 1/6)R \ln \left[ \frac{\mu^2}{2\sigma(x, x')} \right] \right]. \quad (27)
\]
For the radial point splitting we have \(\sigma(x, x') = \frac{(r - r')^2}{2\alpha^2}\).

The above expression can be written in a more convenient way expressing \(1/(r - r')^2\) in terms of an integral similar to that one which appears in (23), and the logarithm in terms of \(Q_0(\cosh \rho)\). So, the final expression is
\[
G_H(r, r') = \frac{1 - \eta^2}{16\pi^2} \frac{1}{rr'\sqrt{2}} \int_{\rho}^{\infty} \frac{dt}{\sqrt{\cosh t - \cosh \rho}} \frac{1}{\sinh^2(t/2)} \cosh(t/2) + \frac{1}{4\pi^2\sqrt{2}}(\xi + 1/6)\frac{\eta^2}{r^2} \int_{\rho}^{\infty} \frac{dt}{\sqrt{\cosh t - \cosh \rho}} e^{-t/2} - \frac{1}{8\pi^2}(\xi + 1/6)\frac{\eta^2}{r^2} \ln \left[ \frac{\mu^2(r + r')^2}{4\alpha^2} \right]. \quad (28)
\]
Finally below we present the renormalized vacuum expectation value associated with the square of the modulus of the scalar field. Because this
expression is a long one, we shall present it written in terms of four different contributions: The purely magnetic part, followed by two other proportional to the parameter $\eta^2$ which also depend on $q$. The last contribution is the cutoff dependent term which disappears when we take $\xi = -1/6$:

$$\langle \Phi^*(x) \Phi(x) \rangle_{\text{Ren.}} = \frac{1}{32\pi^2 r^2} I_1(q) - \frac{\eta^2}{32\pi^2 r^2} I_2(q) + \frac{\eta^2}{32\pi^2 r^2} (\xi + 1/8) I_3(q) + \frac{\eta^2}{4\pi^2 r^2} (\xi + 1/6) \ln(\mu r) ,$$

(29)

where

$$I_1(q) = \int_0^\infty dt \frac{1}{\sinh(t/2)} \left[ e^{-qt} \sinh(t/2) \left[ \coth(t/2) + 2q \right] + \frac{q^2 e^{-qt}}{\sinh(t/2)} \right] ,$$

(30)

$$I_2(q) = \int_0^\infty dt \frac{1}{\sinh(t/2)} \left[ \frac{te^{-qt}}{8 \sinh^2(t/2)} \left[ 8q \cosh(t/2) + 8q^2 \sinh(t/2) \right] + \frac{3 + \cosh(t)}{\sinh(t/2)} \right] + \frac{q^2 e^{-qt}}{2 \sinh(t/2)} \left[ t/2 \coth(t/2) - 1 - \frac{\coth(t/2)}{\sinh(t/2)} \right] + \frac{e^{-t/2}}{6} ,$$

(31)

and

$$I_3(q) = \int_0^\infty dt \frac{1}{\sinh(t/2)} \left[ \frac{2te^{-qt}}{\sinh(t/2)} \left[ 1 + q^2 e^{-t/2} \sinh(t/2) \right] \times \Phi(e^{-t}, 1, q + 1/2) + \frac{4q^2}{(2q + 1)^2} e^{-t/2} \sinh(t/2) \times 3F_2(1, 1/2 + q, 1/2 + q; 3/2 + q, 3/2 + q; e^{-t}) - 4e^{-t/2} \right] .$$

(32)

From the above expressions it is possible to observe that all the integrand are regular at $t \to 0$ and vanish for $t \to \infty$. This is a consequence of all the divergences of $G^q(x, x')$ in the coincidence limit, do not depend on the parameter $q$. On the other hand in $(1 + 3)$–dimensional spacetime the
The Hadamard function does not depend on the gauge field, consequently if we had developed an expansion in higher powers of $\frac{q^2}{(l+1/2)^2}$ in the summation of (23), all these corrections, besides to be subdominant, would provide finite results to the renormalized vacuum expectation value (21).

The calculation of the renormalized vacuum expectation value associated with a massless scalar field in the global monopole spacetime has been calculated a few years ago by Mazzitelli and Lousto [10]. From our results it is possible to see that taking $q = 0$ into (24), the integral $I_1(q)$ vanishes and we obtain the same expression as found in equation (2.17) of [10].

The complete form for $I_1(q)$ and $I_2(q)$ can be obtained analytically in terms of $q$ using the computer program MAPLE; however they are very long expressions. Unfortunately as to $I_3(q)$, we could not find the respective integral. Because this result is one of the most important one of this paper we decided to evaluate all the integral for specific values of the parameter $q$: $q = 1/2$ and $q = 1$. For these two values the two first integrals acquire a closed and short results, however $I_3$ can only be evaluated numerically. In the Appendix B we present more workable expressions to $\Phi(e^{-t}, 1, q + 1/2)$ for the case of interest when $q$ is an integer or half-integer. Our numerical results are given below:

For $I_1(q)$:

\[ q = 1/2, \quad I_1 = -\frac{\pi^2}{4} + \frac{3}{2}, \]  
\[ q = 1, \quad I_1 = -\frac{2\pi^2}{3} + 4. \]  

For $I_2(q)$:

\[ q = 1/2, \quad I_2 = \frac{1}{144}(44 + 126\zeta(3) - 96\ln(2) - 18\pi^2), \]  
\[ q = 1, \quad I_2 = \frac{1}{18}(16 + 36\zeta(3) - 9\pi^2). \]  

For $I_3(q)$:

\[ q = 1/2, \quad I_3 \approx 1.46785 \]  

\[ For \ higher \ even \ dimensional \ spacetime \ as \ (1+5) \ case, \ the \ Hadamard \ function \ depends \ on \ the \ coefficient \ a_2(x), \ which \ by \ its \ turn \ depend \ on \ the \ gauge \ field \ Lagrangian \ density. \]
\[ q = 1, \ I_3 \approx -2.12959. \] (38)

### 2.2 Dimensional Arguments of \( \langle T_{\mu\nu} \rangle \)

In this paper we are analysing under a quantum point of view, the behavior of a massless charged scalar field in the pointlike global monopole spacetime, defined by the metric tensor given in (3), in the presence of a magnetic field defined by (4). In previous subsection we have obtained the renormalized vacuum expectation value of the square of the modulus of the scalar quantum field. Our result shows an explicit dependence of this term with the two fundamental dimensionless parameters, \( \alpha \) and \( q \), and also with radial coordinate \( r \) and the renormalization mass scale \( \mu \).

Although we are not going to develop the explicit calculation of the renormalized vacuum expectation value of the operator energy-momentum tensor associated with this system, \( \langle T_{\mu\nu}(x) \rangle_{\text{Ren.}} \), we expect by dimensional arguments only that this quantity be proportional to \( 1/r^4 \). The factor of proportionality should be given in terms of \( \alpha \), \( q \), \( \xi \) and also depends on the renormalization mass scale. So briefly speaking we can say that

\[
\langle T_{\mu\nu}(x) \rangle_{\text{Ren.}} = G_{\nu\mu}(q, \alpha, \xi, \mu r). \tag{39}
\]

On the other hand, by symmetry of this model, the above vacuum expectation value should be diagonal. Moreover this quantity must be conserved,

\[
\nabla_\nu \langle T_{\mu\nu}(x) \rangle_{\text{Ren.}} = 0, \tag{40}
\]

and satisfies the trace anomaly

\[
\langle T^\mu_{\mu}(x) \rangle_{\text{Ren.}} = \frac{1}{16\pi^2} a_2(x). \tag{41}
\]

In order to explore the above conditions in a more appropriate way, let us write the vacuum expectation value by

\[
\langle T_{\mu\nu}(x) \rangle_{\text{Ren.}} = \frac{1}{16\pi^2 r^4} \left[ A_{\mu\nu}(q, \alpha, \xi) + B_{\mu\nu}(q, \alpha, \xi) \ln(\mu r) \right]. \tag{42}
\]

\[7\] This dependence is a consequence of the fact that there is no other dimensional parameter, since we are working with natural units \( \hbar = c = 1 \).
where $A_\mu^\nu$ and $B_\mu^\nu$ in principal are arbitrary numbers. However because the cutoff factor $\mu$ is completely arbitrary, there is an ambiguity in the definition of this renormalized vacuum expectation value. Moreover the change in this quantity under a change of the renormalized scale is given in terms of the tensor $B_\mu^\nu$ as shown below:

$$\langle T_\mu^\nu(x) \rangle_{\text{Ren.}(\mu)} - \langle T_\mu^\nu(x) \rangle_{\text{Ren.}(\mu')} = \frac{1}{16\pi^2 r^4} B_\mu^\nu(q, \alpha, \xi) \ln(\mu/\mu') .$$

(43)

On the other hand, Christensen in his beautiful paper [11] has presented a general expression for this difference in terms of the variation of the effective action which depends on the logarithmic term: it can be calculated by

$$\langle T_\mu^\nu(x) \rangle_{\text{Ren.}(\mu)} - \langle T_\mu^\nu(x) \rangle_{\text{Ren.}(\mu')} = \frac{1}{16\pi^2} \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^\mu\nu} \int d^4x \sqrt{ga_2(x)} \ln(\mu/\mu') ,$$

(44)

where $a_2(x)$, which is fourth order term in derivative of the metric tensor, also presents contribution from the gauge field strength as shown below [9]:

$$a_2(x) = -\frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{6} \left( \frac{1}{5} - \xi \right) \Box R$$

$$- \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 R^2 - \frac{1}{12} \omega_{\alpha\beta} \omega^{\alpha\beta} .$$

(45)

For the global monopole spacetime in the presence of the Wu-Yang magnetic monopole, the above coefficient can be explicitly obtained by using the only non vanishing components of the vector potential (4) and the Riemann tensors $R_{2332} = R_{3223} = (1 - \alpha^2)r^2 \sin^2 \theta$. After some intermediate steps we get:

$$a_2(x) = \frac{1 - \alpha^2}{r^4} \left[ (\alpha^2 - 1) \left( \frac{1}{90} + 2 \left( \frac{1}{6} - \xi \right)^2 \right) + \frac{2}{3} \left( \frac{1}{5} - \xi \right) \alpha^2 \right] + \frac{q^2}{6r^4} .$$

(46)

The geometric part of the tensor $B_\mu^\nu$ associated with the pointlike global monopole, has been obtained in [10]. So the new contribution here is due to the field strength associated with the Wu-Yang magnetic monopole. After some intermediate steps we arrive, up to the first order in the $\eta^2$ parameter, to

$$B_\mu^\nu(q, \eta^2, \xi) = \frac{1}{90} \eta^2 \text{diag}(1,1,-1,-1) + 8\eta^2(\xi + 1/6)^2 \text{diag}(-1/2,1,-1,-1)$$

$$- \frac{q^2}{12} \text{diag}(1,1,-1,-1) .$$

(47)
which has the same expression in whole space.

So for $\xi = -1/6$ the trace anomaly, up to the first order in $\eta^2$, reads:

$$\langle T^\mu_\mu(x) \rangle_{\text{Ren.}} = \frac{\eta^2}{270\pi^2 r^4} + \frac{q^2}{96\pi^2 r^4}.$$

(48)

The conservation conditions on the renormalized vacuum expectation value of the energy-momentum tensor imposes profound restrictions on the tensor $A^\mu_\nu$. After some calculations we arrive at:

$$A^0_0 = A^1_1 - B^1_1 + T + (B^1_1 - B^0_0) \ln(\mu r),$$

(49)

and

$$A^3_3 = A^2_2 = -A^1_1 + \frac{1}{2}B^1_1 - (B^1_1 + B^2_2) \ln(\mu r).$$

(50)

When we take $\xi = -1/6$ we have

$$A^0_0 = A^1_1 + T - B^1_1,$$

(51)

$$A^3_3 = A^2_2 = -A^1_1 + \frac{1}{2}B^1_1,$$

(52)

where

$$T = r^4 a_2(x) = \frac{\eta^2}{45} + \frac{q^2}{6} + O(\eta^4).$$

(53)

So the complete evaluation of $\langle T^\mu_\mu(x) \rangle_{\text{Ren.}}$ requires the knowledge of at least one component of the tensor $A^\mu_\nu$, say $A^1_1$. However this is a very long calculation which we do not attempt to do here.

3. Concluding Remarks

In this paper we have analysed the vacuum polarization effect due to a massless charged scalar field in the idealized global monopole spacetime which presents a Wu-Yang magnetic monopole in its core. In order to develop this analysis we had to obtain the respective Green section, which corresponds to a generalization of the ordinary Green function having as its angular part the monopole harmonics. This formalism was possible to be developed for this case because of the peculiar radial symmetry of the global monopole spacetime, once the Wu-Yang magnetic monopole has been placed at its origin.
In this way the two overlapping regions $R_a$ and $R_b$ as in (1)-(3), needed to define the singularity free vector potential, can be established.

Considering the global monopole at the same position as the magnetic one, we tried to reproduce the vacuum polarization effect on a massless charged scalar field due to a more general object which contains peculiar aspects of both monopoles: a deficit solid angle spacetime and a radial magnetic field. We are aware that if we had considered the energy density associated with the magnetic field, the geometry of the spacetime would have some contribution due to the latter. So the complete treatment for this problem must take into account both effects on the geometry.

In the calculation of $\langle \Phi^*(x)\Phi(x) \rangle_{\text{Ren.}}$, we had to adopt two different approximations procedure: the first one was to develop an expansion in the parameter $\eta^2 = 1 - \alpha^2 \ll 1$ in the Green section. Doing this expansion it was possible to obtain a more workable expression. However we found another difficulty. The remaining expressions did not allow us to proceed the summations on the angular quantum number $l$. In this way we had to develop the second expansion in $q^2/(l + 1/2)^2$, where then all the summations were possible to be performed. We decided to go in our approximations up to the first order only in both cases. Of course more precise expansions could be developed for both parameters. However these extra terms would be sub-dominant contributions in this specific calculation.

Finally we want to say that in this paper we have considered the magnetic field as an external one, i.e., we did not consider the influence of this gauge field on the geometry of the spacetime. The exact solution of the metric tensor associated with a pointlike non-Abelian magnetic monopole has a Reissner-Nordström form [12].
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4. Appendix A: The Addition Theorem

Here in this appendix we present how the simplified expression to (19) is obtained when we take the coincidence limit in the angular variables in (18). In [5] it is given the generalized monopole harmonics addition theorem which we reproduce below with convenient modifications. For both monopole harmonics defined in $R_a$ we have:

$$
\sum_m Y_{l,m}^q(\theta, \phi)(Y_{l,m}^q(\theta', \phi'))^* = \sqrt{(2l + 1)/4\pi} Y_{l,-q}^q(\gamma, 0) e^{iq(\phi - \phi')} e^{-iq(R + R' - \pi)},
$$

(54)

where

$$
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')
$$

(55)

and $R$ and $R'$ are angles defined in the Fig. 2 of [5]. Taking $\phi = \phi'$, we obtain $R' = 0$ and $R = \pi$. On the other hand

$$
Y_{l,-q}^q(\gamma, 0) = \frac{1}{2\pi} \sqrt{\frac{2l + 1}{4\pi}} (x - 1)^{-q} \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x - 1)^n(x + 1)^{n+2q}],
$$

(56)

with $x = \cos \gamma$ and $n = l - q$. Now taking $\gamma = 0$ into the above equation we get:

$$
Y_{l,-q}^q(0, 0) = \sqrt{\frac{2l + 1}{4\pi}}.
$$

(57)

A similar result would be obtained if we have considered both monopole harmonics in $R_b$. (Of course the situation where both monopole harmonics are in different regions are not pertinent here.)

Another point which deserve to be mentioned in the generalized monopole harmonics addition theorem is that the result is still a section. So the name given to the Green section seems appropriate to us.
5. Appendix B: Some Useful Identity

In the numerical evaluation of $I_3(q)$ in (29) for specific values of $q$ equal to 1/2 and 1, we needed an explicit expression for the $\Phi(e^{-t}, 1, q + 1/2)$:

$$\Phi(e^{-t}, 1, q + 1/2) = \sum_{n=0}^{\infty} \frac{e^{-nt}}{n + q + 1/2}.$$  (58)

Because $q$ assumes only integer or half-integer values this special function can be expressed by elementary functions as:

For integer $q$,

$$\Phi = e^{qt} e^{t/2} \ln(\coth(t/4)) - e^{qt} \sum_{n=0}^{q-1} \frac{e^{-nt}}{n + 1/2}.$$  (59)

and for half-integer $q$,

$$\Phi = -e^{[q]t} e^{t} \ln(1 - e^{-t}) - e^{[q]t} \sum_{n=0}^{[q]-1} \frac{e^{-nt}}{n + 1},$$  (60)

where $[q]$ is the integer part of $q$. 

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