From emission to inertial coordinates: an analytical approach

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Abstract. In a previous work [Class. Quantum Grav. 27 (2010) 065013] relativistic positioning systems in Minkowski space-time have been studied, and the transformation from emission to inertial coordinates have been obtained for an arbitrary configuration of the emitters. The formula giving this transformation applies in all the emission coordinate region and involves the orientation of the positioning system (the Jacobian sign of the map which gives the emission coordinates of an event). Nevertheless, there exists an inherent limitation on the applicability of this formula: only the users in the central region of a positioning system can obtain the orientation from the sole emission data. Here an observational method to determine the orientation of a relativistic positioning system is presented. In this procedure, a certain additional information allows any user to obtain its inertial coordinates, irrespectively of its location in the emission region of the positioning system.

1. The coordinate transformation from emission to inertial coordinates
A relativistic positioning system consists of four emitters \( \gamma_A(\tau^A) \) \( (A = 1, 2, 3, 4) \) broadcasting their respective proper times \( \tau^A \) by means of electromagnetic signals. Every event reached by the signals is labelled by the four times \( \{\tau^A\} \): the emission coordinates of this event. With an eye on the development of a relativistic theory for Global Navigation Satellite Systems (GNSS), we consider here relativistic positioning systems in Minkowski space-time. The space-time region covered by the signals is called the emission region, \( \mathcal{R} \), of the relativistic positioning system. Then, if \( x \in \mathcal{R} \), a user at \( x \) receives the four broadcast times \( \{\tau^A\} \), and then
\[
m_A \equiv x - \gamma_A(\tau^A), \quad (A = 1, \ldots, 4),
\]
denote the future oriented light-like vectors that represent the trajectories followed by the electromagnetic signals from the emitters \( \gamma_A(\tau^A) \) to the reception event \( x \in \mathcal{R} \).

The emission data \( \{\tau^A\} \) received at \( x \) are the emission coordinates of the event \( x \in \mathcal{R} \) and were broadcast when the emitters were at the events \( \{\gamma_A(\tau^A)\} \): the configuration of the emitters for the event \( x \). Generically, these four events determine an hyperplane (the configuration hyperplane) whose orthogonal direction is that of the vector \( \chi \).\(^1\)

\(^1\) Here, we always consider that the four events \( \{\gamma_A(\tau^A)\} \) are neither coplanar nor collinear (i. e, that the emitter configuration is regular, \( \chi \neq 0 \)). The study of non-regular (or degenerate) configurations is considered elsewhere.
The space-time region where the gradients $d\tau^A$ are well defined and linearly independent is called the emission coordinate region, $C$, of the positioning system. Indeed, the emitter worldlines are excluded from $C$ because $d\tau^A$ are not defined at the emission events (cone emission vertices). For all $x \in C$, the transformation from emission to inertial coordinates is locally well defined.

From now on, let us suppose that the world-lines $\gamma_A(\tau^A)$ of the emitters in an inertial system $\{x^\alpha\}$ are known. In [1], we have obtained the coordinate transformation $x = \kappa(\tau^A)$ from an emission coordinate system $\{\tau^A\}$ to an inertial one $\{x^\alpha\}$, which is covariantly expressed by the following formula:

$$x = \gamma_4 + y_* - \frac{y_*^2 \chi}{y_* \cdot \chi + \hat{\chi} \sqrt{(y_* \cdot \chi)^2 - y_*^2 \chi^2}} \quad (2)$$

where $\gamma_4(\tau^4)$ has been chosen as the reference emitter.

Quantities $y_*$ and $\chi$ (the configuration vector) are both computable from the emission data $\{\tau^A\}$ by applying the following expressions:

$$y_* = \frac{1}{\xi \cdot \chi} i(\xi) H, \quad H = * (\Omega_1 e_2 \wedge e_3 + \Omega_2 e_3 \wedge e_1 + \Omega_3 e_1 \wedge e_2) \quad (3)$$

$$\Omega_a = \frac{1}{2} (e_a)^2, \quad e_a = \gamma_a(\tau^a) - \gamma_4(\tau^4), \quad \chi = *(e_1 \wedge e_2 \wedge e_3) \quad (4)$$

with $a = 1, 2, 3$, and where $\xi$ is any vector transversal to the configuration, $\xi \cdot \chi \neq 0$; $i()$ and $\wedge$ are, respectively, the interior and the wedge products, and $\ast$ stands for the Hodge dual operator.\(^2\)

Quantity $\hat{\chi}$ (the orientation of the positioning system with respect to $x$) is defined by

$$\hat{\chi} \equiv \text{sgn} * (m_1 \wedge m_2 \wedge m_3 \wedge m_4). \quad (5)$$

Note that to obtain $x$ from (2) one needs to determine the orientation $\hat{\chi}$, which involves, by substituting (1) in (5), the unknown $x$. Therefore, in order to show that Eq. (2) does not chase its own tail, one must be able to determine the orientation $\hat{\chi}$ at $x$ by using a procedure that does not involve the previous knowledge of $x$. Next, we study the region where $\hat{\chi}$ is computable by the positioning data (the central region of the positioning system) which does not cover the whole emission coordinate region $C$.

### 2. Emission configuration regions and orientation of a positioning system

Let us consider the map $\Theta$ that to every event $x \in R$ associates its emission coordinates $\{\tau^A\}$, that is $(\tau^A) = \Theta(x)$, and let us denote by $j_\Theta(x)$ its Jacobian determinant,

$$j_\Theta(x) = *(d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4). \quad (6)$$

For each value of $A$, $d\tau^A$ and $m^A$ define the same future oriented null direction and then, from (5) and (6), the orientation $\hat{\chi}$ of the positioning system is nothing but the sign of this Jacobian determinant, $\hat{\chi} = \text{sgn} \ j_\Theta(x)$. The hypersurface $\mathcal{J}$ where this Jacobian vanishes, $\mathcal{J} = \{x \mid j_\Theta(x) = 0\}$, has been studied by Coll and Pozo [2, 3] who stated that $\mathcal{J}$ is constituted by those events for which any user in them can see the four emitters on a circle in his celestial sphere.

The vector $\chi$ provides the causal character of the configuration hyperplane. Taking the metric signature equal to $2\epsilon$, $\epsilon = \pm 1$, the configuration hyperplane is space-like, light-like or time-like.

\(^2\) Using index notation, one write, for instance, $\Lbrack i(\xi)H\Rbrack_{\nu} = \xi^\nu H_{\mu\nu}$, $(e_1 \wedge e_2)^{\mu\nu} = e_1^\nu e_2^\mu - e_2^\nu e_1^\mu$, $[* (e_1 \wedge e_2)]_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} (e_1 \wedge e_2)^{\mu\nu}$, $\eta_{\alpha\beta\mu\nu} e_1^\mu e_2^\nu = \eta_{\alpha\beta\mu\nu} e_1^\nu e_2^\mu$, $\chi_{\alpha} = \eta_{\alpha\mu\nu} e_1^\nu e_2^\mu$, where $\eta$ is the metric volume element.
if $\epsilon \chi^2$ is, respectively, negative, zero or positive. Thus, $\mathcal{C}$ is the union of three disjoint regions ($\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_{t} \cup \mathcal{C}_\ell$): the space-like $\mathcal{C}_s$, the null $\mathcal{C}_t$ and the time-like $\mathcal{C}_\ell$ configuration region, which are respectively defined by the condition $\epsilon \chi^2 < 0$, $\chi^2 = 0$ and $\epsilon \chi^2 > 0$. At every event $x$ in $\mathcal{C}_s$ (respectively, in $\mathcal{C}_t$ or in $\mathcal{C}_\ell$) a user receives the signals from four emission events that generate a space-like (respectively, null or time-like) hyperplane.

From the emission data $\{\tau^A\}$, the user is able to determine the sign of $\epsilon \chi^2$ and then, from the above definitions, he knows in what of the configuration region, $\mathcal{C}_s$, $\mathcal{C}_t$ or $\mathcal{C}_\ell$ the positioning system he is. Clearly, one has that for a future pointing time-like vector $u$, $u \cdot \chi \neq 0$ at any event $x \in \mathcal{C}_s \cup \mathcal{C}_\ell$ because $\chi$ is not space-like in this region. On the other hand, from (1) and (4) one obtains:

$$\ast (m_1 \wedge m_2 \wedge m_3 \wedge m_4) = - \ast (e_1 \wedge e_2 \wedge e_3 \wedge m_4) = i(m_4)\chi,$$

and then, the sign of $u \cdot \chi$ is the same for any $u$ and for all $x \in \mathcal{C}_s \cup \mathcal{C}_t$, and this sign is precisely the orientation of the positioning system in both, the space-like and the null configuration regions.

This region $\mathcal{C}^C \equiv \mathcal{C}_s \cup \mathcal{C}_\ell$, where the orientation is constant and obtainable from the sole emission data, is the central region of the positioning system. Then, one has:

$$\forall x \in \mathcal{C}^C, \quad \hat{\epsilon} = \text{sgn} (u \cdot \chi)$$

(8)

for any future pointing time-like vector $u$.

Therefore, from (8) any user in the central region is able to compute the orientation of the positioning system in this region, and then, from (2), (3) and (4), he can obtain his position $x$ in the inertial system from the sole emission data $\{\tau^A\}$, by substituting $\hat{\epsilon} = \text{sgn} (u \cdot \chi)$ in (2).

As a consequence of (8), the hypersurface $\mathcal{J}$ of vanishing Jacobian is included in the time-like configuration region $\mathcal{C}_\ell$. In [1], we showed that $\mathcal{C}$ is not a coordinate domain but the union of two disjoint coordinate domains ($\mathcal{C} = \mathcal{C}^F \cup \mathcal{C}^B$): the front $\mathcal{C}^F$, and the back $\mathcal{C}^B$ emission coordinate domains. Further, $\mathcal{C}^F$ is a simply-connected domain but $\mathcal{C}^B$ is not. In fact, in the symmetric stationary case where the four emitters define a (regular) tetrahedron, $\mathcal{C}^B$ is the union of four connected components. Then, the common boundary of both domains is the four-leaf hypersurface $\mathcal{J}$. The orientation $\hat{\epsilon}$ of the positioning system only changes across $\mathcal{J}$: it takes different constant value throughout each coordinate domain.

Suppose that a user is in $\mathcal{C}_t$. For a given set of emission data, there are two events (with distinct inertial coordinates) in $\mathcal{C}_t$ which receive the same set of data. The matter is then: from Eq. (2), how can the user determine its truly localization in the inertial system? Can the users in $\mathcal{C}_t$ know the orientation of the positioning system?

3. Observational procedure to obtain the orientation of a positioning system

What we shall see now is that all the users of the coordinate region $\mathcal{C}$, be them in the central region or not, have an observational method to determine the orientation $\hat{\epsilon}$.

Consider an arbitrary user, of unit velocity $u$, at the event $x$ of $\mathcal{C}$; in his relative three-dimensional space $\pi_u$, the apparent directions of propagation of the signals $m_A$ are given by the unit vectors $\overline{m}_A$, such that $m_A = -\epsilon(u \cdot m_A)(u + \overline{m}_A)$, and by direct substitution in the definition (5) of $\hat{\epsilon}$ one has

$$\hat{\epsilon} = \text{sgn} \ast \{ u \wedge [(\overline{m}_1 \wedge \overline{m}_2 + \overline{m}_2 \wedge \overline{m}_3 + \overline{m}_3 \wedge \overline{m}_1) \wedge \overline{m}_4 - \overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3] \}.$$

(9)

And because, for any three-form $F$ in $\pi_u$, one has $\ast (u \wedge F) = i(u) \ast F = \ast_u F$, where $\ast_u$ is the Hodge dual operator on $\pi_u$, it follows

$$\hat{\epsilon} = \text{sgn} \left( i(\overline{m}_4) \ast_u (\overline{m}_1 \wedge \overline{m}_2 + \overline{m}_2 \wedge \overline{m}_3 + \overline{m}_3 \wedge \overline{m}_1) - \ast_u (\overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3) \right).$$

(10)
But, the $\overline{m}_a$ forming a basis of $\pi_u$, their dual $\overline{L}^a$, $\overline{L}^a(\overline{m}_b) = \delta^a_b$, are

$$
\overline{L}^a = \frac{\epsilon^{abc} u_a (\overline{m}_b \wedge \overline{m}_c)}{2 u_b (\overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3)}.
$$

(11)

So that

$$
\dot{e} = \text{sgn}(u_a (\overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3) [i (\overline{m}_4)(\overline{L}^1 + \overline{L}^2 + \overline{L}^3) - 1]).
$$

(12)

That the square bracket expression vanishes when and only when the Jacobian $j_\Theta(x)$ vanishes is known since [3]. It allows to state that, in the emission region $\mathcal{R}$, the events of the emission coordinate region $\mathcal{C}$ are all those for which the four emitters are not on a circle of the celestial sphere of the users at these events. Now, if the satellites 1, 2, 3 are increasingly clock-wise on the circle defined by them on the celestial sphere, the sign of $u_a (\overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3)$ is positive. Then, the bracket in (12) is positive or negative if the satellite 4 is interior or exterior respectively to this circle.

Thus, if in addition to the received data $\{\tau^A\}$ the user can observe the emitters, by applying the above rule he can obtain the orientation $\dot{e}$ and consequently, from (2), he can determine his position in the inertial coordinate system.

Some final considerations are in order. We have remarked that, in general, the four times that a user receives are insufficient to determine his location in an inertial system (for details, see [1]). This lack of uniqueness was pointed out by Schmidt [6] in connection with the location problem in GNSS, and it was studied by Abel and Chaffee [4, 5]. In this communication, we have outlined a method to obtain the orientation of a relativistic positioning system. Applying (2) and this method one can calculate, in any case, the user space-time location in inertial coordinates. A numerical analysis of the quantities appearing in (2) is not hard to implement and it has been recently accomplished [7].

Relativistic positioning concepts have been recently implemented in an algorithm to obtain the Schwarzschild coordinates of a user from his emission coordinates, and viceversa [8]. In building current models, a fundamental assumption consists in picking out an approximate zero order numerical solution. When gravitational fields are not taken into consideration, expression (2) is the exact covariant solution to the location problem. Thus, in a perturbational scheme based on a Minkowskian background, Eq. (2) provides the exact non-perturbed leading solution: it is the better zero order solution to start with, no matter how the accuracy in the approximation methods may be.

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