On the frequency domain detection of high dimensional time series
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M–dimensional complex time series \((y_n)_{n \in \mathbb{Z}}\) modeled as

\[
y_n = \sum_{k=0}^{+\infty} H_k \epsilon_{n-k} + v_n \in \mathbb{C}^M
\]

where \(\epsilon_n \sim \mathcal{N}_K(0, I_K)\) is white noise, \(v_n\) are mutually independent additive noise stationary complex Gaussian time series components, \((u_n)_{n \in \mathbb{Z}}\) is the useful signal formed by the output of a causal and stable MIMO filter driven by \(\epsilon_n\).
Introduction - Setting

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- components time series \((v_{1,n})_{n \in \mathbb{Z}}, \ldots, (v_{M,n})_{n \in \mathbb{Z}}\) are mutually independent.
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**Frequency domain detection hypothesis test - \(S_y\)**

- \(H_0 : S_y(\nu) = \text{diag}(S_y(\nu)) = S_v(\nu)\) (noise only) vs
- \(H_1 : S_y(\nu) = H(\nu)H(\nu)^* + S_v(\nu) \neq \text{diag}(S_y(\nu))\) (signal+noise)
- \((H(\nu)\) is the Fourier transform of \((H_k)_{k}\)
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With \(C_y(\nu) := \text{diag}(S_y(\nu))^{-\frac{1}{2}} S_y(\nu) \text{diag}(S_y(\nu))^{-\frac{1}{2}}\)

**Frequency domain detection hypothesis test - \(C_y\)**

\(\mathcal{H}_0 : C_y = I_M\) (pure noise) vs \(\mathcal{H}_1 : C_y \neq I_M\) (signal + noise). Use frequency domain estimators of \(C_y\) to test if \(u_n = 0\).
Introduction - Signal detection context

High dimensional regime: \( K \) fixed \( \ll M, N \to +\infty \)
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Relevant large dimensional regime in econometrics
- late 90’s: Generalized dynamic linear factor models
- other underlying assumptions are not relevant in our context
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Relevant large dimensional regime in array processing
- finite $K \times \mathcal{O}(1)$ signal eigenvalues vs $M \times \mathcal{O}(1)$ noise eigenvalues
- $\text{SNR} \ \rho = \frac{\mathbb{E} \|u_n\|^2}{\mathbb{E} \|v_n\|^2} = \mathcal{O}\left(\frac{1}{M}\right)$ is of special interest.
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(additive noise is temporally and spatially white and signal is $u_n = H_0 \epsilon_n$)
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- Considerable work still needed for dynamic / wideband models
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- Considerable work still needed for dynamic / wideband models
- Temporal approaches also possible, but frequency ones turn out to be simpler.
Fourier frequencies set: \( \mathcal{V}_N = \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\} \)
Introduction - Notations & Smoothed periodogram estimator

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Finite Fourier transform:

$$\xi_y(\nu) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} y_n e^{-i2\pi \nu (n-1)}$$
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**Smoothed periodogram estimator of the spectral density matrix:**

\[
\hat{S}_y(\nu) = \frac{1}{B+1} \sum_{b=-B/2}^{B/2} \xi_y \left( \nu + \frac{b}{N} \right) \xi_y \left( \nu + \frac{b}{N} \right)^* \quad (B: \text{smoothing span})
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Estimator of the spectral coherency matrix:
\[
\hat{C}_y(\nu) = \text{diag}(\hat{S}_y(\nu))^{-\frac{1}{2}} \hat{S}_y(\nu) \text{diag}(\hat{S}_y(\nu))^{-\frac{1}{2}}
\]
Main result on $\hat{C}_y$

**High dimensional regime**: consider $B := B(N)$, $M := M(N)$ such that

$$M, B, N \xrightarrow{N \to \infty} +\infty, \quad \frac{B}{N} \xrightarrow{N \to \infty} 0, \quad \frac{M}{B} \xrightarrow{N \to \infty} c \in (0, 1)$$
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**Theorem - Wishart approximation of $\hat{C}_y$**

Under proper technical assumptions on the signal and noise, there exists a $M \times (B + 1)$ random matrix $X(\nu)$ with i.i.d. $\mathcal{N}(0, 1)$ entries such that

$$\max_{\nu \in \mathcal{V}_N} \left\| \hat{C}_y(\nu) - \Xi(\nu)^{\frac{1}{2}} \frac{X(\nu)X(\nu)^*}{B + 1} \Xi(\nu)^{\frac{1}{2}} \right\| \xrightarrow{a.s. \ N \to \infty} 0 \quad (1)$$

where $\Xi(\nu) = S_v(\nu)^{-\frac{1}{2}} H(\nu) H(\nu)^* S_v(\nu)^{-\frac{1}{2}} + I_M$ and $H(\nu) := \sum_{k=0}^{+\infty} H_k e^{-i2\pi \nu k}$
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$$N \to \infty$$

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**Key idea**:  
- $\Xi(\nu)$ fixed rank $K$ perturbation of the identity matrix. This is not the case with temporal approaches.
- first order behaviour of $\Xi(\nu)^{\frac{1}{2}} \frac{X(\nu)X(\nu)^*}{B+1} \Xi(\nu)^{\frac{1}{2}}$ known.
Simulation in the pure noise case ($K = 0$)

- $K = 0$ ($y_n = v_n$ as MA(1)), $M = 100$, $B = 200$, $N = 4000$
- asymptotically, eigenvalues of $\hat{C}_y(\nu) \in [(1 - \sqrt{c})^2, (1 + \sqrt{c})^2]$ (Marchenko & Pastur, 1967)
- good fit even for small dimensions (20 realisations)
Application - Spectral behaviour of $\hat{C}_y$

Recall $\Xi(\nu) = S_v(\nu)^{-\frac{1}{2}} H(\nu) H(\nu)^* S_v(\nu)^{-\frac{1}{2}} + I_M \in \mathbb{C}^{M \times M}$, rank $K$. 
Recall $\Xi(\nu) = S_v(\nu) - \frac{1}{2} H(\nu) H(\nu)^* S_v(\nu) - \frac{1}{2} + I_M \in \mathbb{C}^{M \times M}$, rank $K$.
Define $\nu^*_N \in \mathcal{V}_N$ such that:

$$\nu^*_N \in \arg\max_{\nu \in \mathcal{V}_N} \lambda_1 \left( S_v(\nu) - \frac{1}{2} H(\nu) H(\nu)^* S_v(\nu) - \frac{1}{2} \right)$$
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Assumption - Spike

For all $k \in \{1, \ldots, K\}$, there exists $\gamma_k > 0$ such that

$$\lambda_k \left( S_v(\nu^*_N)^{-\frac{1}{2}} H(\nu^*_N) H(\nu^*_N)^* S_v(\nu^*_N)^{-\frac{1}{2}} \right) \xrightarrow{N \to \infty} \gamma_k$$
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Define $\nu_N^* \in V_N$ such that:

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Corollary - Behaviour of the spectrum of $\hat{C}_y(\nu)$

Under proper technical assumptions, for all $k = 1, \ldots, K$ and all $\nu \in V_N$,

$$\lambda_k \left( \hat{C}_y(\nu_N^*) \right) \xrightarrow{\text{a.s.}} \begin{cases} \frac{(\gamma_k+1)(\gamma_k+c)}{\gamma_k} > (1 + \sqrt{c})^2 & \text{if } \gamma_k > \sqrt{c} \\ (1 + \sqrt{c})^2 & \text{if } \gamma_k \leq \sqrt{c} \end{cases}$$

whereas

$$\lambda_{K+1} \left( \hat{C}_y(\nu_N^*) \right) \xrightarrow{\text{a.s.}} (1 + \sqrt{c})^2$$
Application - Spectral behaviour of $\hat{C}_Y$ - Simulation

- rank one signal ($h(\nu)$ vector) + $M$-dimensional noise MA(1) process.
- $K=1$, same $M = 100$, $B = 200$, $N = 4000$, $c = 0.5 \Rightarrow \sqrt{c} \approx 0.7$
- separation starting at $SNR := \gamma_1 = \sqrt{c} \Rightarrow$ detection for low frequencies
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\begin{align*}
\|h(\nu)\|^2 &\quad s_\nu(\nu)/M \\
\lambda_1(\nu) &\quad \sqrt{M/B}
\end{align*}
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![Graphs showing spectral behaviour](image-url)
Application - Simulation - Varying parameters

- $M = 20, B = 40, N = 4000, c = 0.5, \text{ ma parameter } = 0.6, \text{ medium SNR.}$
- as $\frac{B}{N} \to 0$, the finite sample results are closer to the asymptotics

![Graphs showing varying parameters](image-url)

**Figure** – $B/N = 0.5$  **Figure** – $B/N = 0.1$  **Figure** – $B/N = 0.01$
Application - Spectral detection testing

\[ \mathcal{H}_0 : y_n = v_n \quad \text{vs} \quad \mathcal{H}_1 : y_n = u_n + v_n \]
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**New frequency domain detection algorithm**

Consider, for some threshold \( \epsilon > 0 \) the following procedure:

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\begin{aligned}
&\lambda_1(\hat{C}_y(\nu^*_N)) < (1 + \sqrt{c})^2 + \epsilon \quad \text{absence of} \ u \ \text{is decided} \\
&\lambda_1(\hat{C}_y(\nu^*_N)) > (1 + \sqrt{c})^2 + \epsilon \quad \text{presence of} \ u \ \text{is decided}
\end{aligned}
\]

This leads to define the test statistics:

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T_\epsilon = \mathbb{1}
\begin{pmatrix}
(1 + \sqrt{c})^2 + \epsilon, +\infty \\
\max_{\nu \in \mathcal{V}_N} \| \hat{C}_y(\nu) \|
\end{pmatrix}
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\]

Theorem - Spectral detection testing

Under proper assumptions, the previous test is consistent iif \( \gamma_1 > \sqrt{c} \) and \( \epsilon \) small enough.
Conclusion

Contributions:

- In the **high dimensional regime**, $\hat{C}_y$ is approximately a Wishart random matrix with covariance matrix as finite rank perturbation of the identity matrix: **spike model**
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- In the **high dimensional regime**, \( \hat{\mathbf{C}}_y \) is approximately a Wishart random matrix with covariance matrix as finite rank perturbation of the identity matrix: **spike model**
- Well known results provide first order behaviour of its eigenvalues
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- In the **high dimensional regime**, $\hat{C}_y$ is approximately a Wishart random matrix with covariance matrix as finite rank perturbation of the identity matrix: **spike model**

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- Our detection algorithm is based on a phase transition phenomenon of the largest eigenvalues of $\hat{C}_y(\nu)$:
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- In the **high dimensional regime**, $\hat{C}_y$ is approximately a Wishart random matrix with covariance matrix as finite rank perturbation of the identity matrix: **spike model**

- Well known results provide first order behaviour of its eigenvalues

- Our detection algorithm is based on a phase transition phenomenon of the largest eigenvalues of $\hat{C}_y(\nu)$ :
  - weak energy signals $\implies$ eigenvalue absorbed in the noise bulk
  - high energy signals $\implies$ eigenvalue separated from the noise bulk