Fixed-Time Synchronization of Delayed Fractional-Order Memristor-Based Fuzzy Cellular Neural Networks

YE GUO SUN AND YIHONG LIU

1School of Finance and Mathematics, Huainan Normal University, Huainan 232038, China
2School of Computer Science, Huainan Normal University, Huainan 232038, China

Corresponding author: Yeguo Sun (yeguosun@126.com)

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Abstract In this article, we discussed the fixed-time synchronization of fractional-order memristor-based fuzzy cellular neural networks (FMFCNNs) with time-varying delays. Based on the differential inclusion theory and discontinuous state feedback control technique, some useful novel criteria of the fixed-time synchronization for FMFCNNs are derived by constructing appropriate Lyapunov function. The results in our paper improve and generalize current literature research results significantly. Finally, the effectiveness of our control schemes for the fixed-time synchronization is demonstrated through numerical simulations.

Index Terms Fractional-order memristor-based fuzzy cellular neural networks, fixed-time synchronization, state feedback control technique, Lyapunov direct method.

I. INTRODUCTION

As an extension of integer derivative and integral to arbitrary order, fractional-order calculus can date from 300 years ago [1]. There are many kinds of definitions for fractional integration and fractional differentiation such as Caputo derivative, Riemann-Liouville derivative and etc. Fractional differential equation has been deemed to be a powerful tool for the modeling of practical problems in biology, chemistry, physics, medicine, economics and other sciences [2]. Contrasting with classical integer-order systems, the reality can be better described by fractional-order systems for the reason that fractional-order differentiation takes into account the present state and all the history of its previous states [3], [4]. In other words, fractional-order systems have memory and heredity. Therefore, many scholars have applied fractional operators to neural networks to build fractional models [5].

On the other hand, since Chua and Yang first proposed cellular neural networks in 1988 [6], [7], many scholars have conducted extensive research on cellular neural networks owing to the widespread application of cellular neural networks in many fields such as quadratic optimization, image processing, pattern recognition, associative memories and etc.. In 1996, based on the traditional cellular neural network, T. Yang and L. B. Yang first proposed the fuzzy cell neural network. Compared with the traditional cellular neural network, the fuzzy cellular neural network adds fuzzy logic (fuzzy AND and fuzzy OR) to its structure, and maintains the local connection between cells. The research results show that fuzzy cell neural networks have better applications in pattern recognition and image processing [8]–[11].

Among the neural network cluster behaviors, synchronization is the most important one and its manifestations are various, such as asymptotic synchronization [12], [13], exponential synchronization [14], [15], robust synchronization [16], [17], finite time synchronization [18]–[20], and etc. Among the existing references related to the synchronization of fuzzy cellular neural networks, [21] studied the asymptotic synchronization of non-identical chaotic fuzzy cellular neural networks with time-varying delay based on sliding mode control. Based on Lyapunov functional theory and inequality techniques, [22] discussed the exponential lag synchronization of delayed fuzzy cellular neural networks by periodic intermittent control methods. Reference [23] discussed the exponential lag synchronization of neural networks with time-varying delays. Based on the linear matrix
To the best of our knowledge, fixed-time synchronization control of FMFCNNs with time-varying delay is not yet completely studied, which motivated our research. This article aims to establish some novel fixed-time synchronization criteria for FMFCNNs with time-varying delay. The main contributions of this article are summarized as follows.

- For the first time, the fixed-time synchronization problem for FMFCNNs with time-varying is studied. In practical applications, fixed-time synchronization is more general and practical than finite-time synchronization and asymptotic synchronization.
- By constructing a nonlinear feedback controller and choosing a simple Lyapunov function, some sufficient conditions which are easy to verify are obtained to ensure the fixed-time stability of FMFCNNs and the fixed-time synchronization of the drive-response FMFCNNs systems.
- The theoretical results obtained are more general and can improve or supplement previous results effectively. Moreover, the existing FMFCNNs model with no fuzzy logic, no time-varying delay, or no memristor can all be regarded as the special case of our model.
- The settling time in this article is easy to estimate. In addition, compared with the classical results, the estimation bound of the settling time given in our paper is more accurate and effective. Numerical examples are given to demonstrate the effectiveness of the proposed approaches.

The remainder of the paper is as follows. The response-drive system introduced in Section 2. Furthermore, few definitions and assumptions are presented, and some useful lemmas needed are also introduced. In Section 3, some fixed-time synchronization criteria for delayed FMFCNNs are proposed on the basis of the fixed-time stability theory. Section 4 presents two simulation examples. Finally, some conclusions of this article are drawn, and some future research works are proposed in the last section.

II. PROBLEM FORMULATION AND PRELIMINARIES

The fractional-order integral-differentiation can be considered as an extension of integer integral-differentiation. The most common definitions of fractional calculus are Caputo, Riemann-Liouville, and Grünwald-Letnikov. In practical engineering applications, we generally use the Caputo derivative instead of the Riemann-Liouville derivative. The reason is that Laplace transform of Caputo’s derivative only requires integer-order derivatives, while the Laplace transform of Riemann-Liouville involves fractional-order derivatives which are difficult to be physically interpreted. In this article, the Caputo’s derivative is considered. Let the lower limit of the fractional calculus be 0. Then the fractional-order integral with fractional order $\alpha$ can be defined as

$$0I_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$
where $\Gamma(\cdot)$ is the Euler’s Gamma function. The corresponding Caputo fractional derivative can be expressed by

$$0D^\alpha_t f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t f^{(n)}(\tau)(t - \tau)^{\alpha - 1 - n} d\tau,$$

where $\alpha$ is the fractional order, and $n$ is an integer satisfying $n - 1 \leq \alpha < n$.

### A. THE DRIVE-RESPONSE FMFCNNs MODELS

The fractional-order memristor-based fuzzy cellular neural networks with time-varying delay in this article is as follows:

$$0D^\alpha_t x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} x_j(t - \tau_{ij}(t)) + \sum_{j=1}^n \tilde{b}_{ij} x_j(t - \tau_{ij}(t))$$

$$+ \sum_{j=1}^n d_{ij} v_j + \sum_{j=1}^n \alpha_{ij} g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n S_{ij} v_j + \sum_{j=1}^n \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i$$

$$x_i(t) = \phi_i(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \ldots, n$$

where $x_i(t)$ is the state of the $i$th neuron at time $t$, $0 < \alpha \leq 1$ is the fractional order, $c_i$ represents the passive decay rates of the $i$th neuron, $I_i$ and $v_j$ represent the bias value and input of the $i$th neuron, respectively; $\tau_{ij}$ and $\sigma_{ij}$ denote the time delay of fuzzy feed-forward MIN and MAX template, respectively; $\alpha_{ij}$ and $\beta_{ij}$ are the element of fuzzy feedback MIN and MAX template, respectively; $\vee$ and $\wedge$ represent the fuzzy AND operation and fuzzy OR operation, respectively; $f_j$ and $g_l$ are activation functions; $\phi_i(t) \in C([-\tau, 0], R)$ is the initial value of the model (1), where $\tau = \max_{i \neq j} (\tau_{ij})$, $C([-\tau, 0], R)$ represents a continuous function set from the interval $[-\tau, 0]$ to $R$, $\tilde{a}_{ij}(x(t))$ and $\tilde{b}_{ij}(x(t - \tau_{ij}(t)))$ denote the non-delayed and delayed memristor-based synaptic connection weights, respectively, and given by

$$\tilde{a}_{ij}(x(t)) = \begin{cases} \tilde{a}_{ij}^1, & |x(t)| \leq T_j \\ \tilde{a}_{ij}^2, & |x(t)| > T_j \end{cases}$$

$$\tilde{b}_{ij}(x(t - \tau_{ij}(t))) = \begin{cases} \tilde{b}_{ij}^1, & |x(t - \tau_{ij}(t))| \leq T_j \\ \tilde{b}_{ij}^2, & |x(t - \tau_{ij}(t))| > T_j \end{cases}$$

where $T_j$ is the switching jump, $\tilde{a}_{ij}^1, \tilde{a}_{ij}^2, \tilde{b}_{ij}^1, \tilde{b}_{ij}^2$, $i, j = 1, 2, \ldots, n$ are all constant numbers.

**Remark 1:** The weight of FMFCNNs includes a memristor, and the resistance of the memristor can be changed according to the state of the system. Therefore, FMFCNNs can be regarded as a special switching neural networks.

From the definitions of $\tilde{a}_{ij}(x(t))$ and $\tilde{b}_{ij}(x(t - \tau_{ij}(t)))$, the model (1) is a switching differential equation with discontinuous right-hand sides. For this kind of equation, Filippov gave an effective treatment [36]. In order to simplify the processing of such equations, we define the following set-valued mappings

$$K[I\tilde{a}_{ij}(x(t))] = \begin{cases} I\tilde{a}_{ij}^1, & |x(t)| < T_j \\ I\tilde{a}_{ij}^2, & |x(t)| = T_j \\ I\tilde{a}_{ij}^2, & |x(t)| > T_j \end{cases}$$

and

$$K[I\tilde{b}_{ij}(x(t)) = \begin{cases} I\tilde{b}_{ij}^1, & |x(t - \tau_{ij}(t))| < T_j \\ I\tilde{b}_{ij}^2, & |x(t - \tau_{ij}(t))| = T_j \\ I\tilde{b}_{ij}^2, & |x(t - \tau_{ij}(t))| > T_j \end{cases}$$

where $a_{ij} = \min(\tilde{a}_{ij}^1, \tilde{a}_{ij}^2)$, $\bar{a}_{ij} = \max(\tilde{a}_{ij}^1, \tilde{a}_{ij}^2)$, $b_{ij} = \min(\tilde{b}_{ij}^1, \tilde{b}_{ij}^2)$, $\bar{b}_{ij} = \max(\tilde{b}_{ij}^1, \tilde{b}_{ij}^2)$, $\sigma_{ij}$ denotes the convex closure of a set, and $K[I\tilde{a}_{ij}(x(t))], K[I\tilde{b}_{ij}(x(t - \tau_{ij}(t)))]$ are all closed, convex and compact about $x(t), x(t - \tau_{ij}(t)).$

Based on the differential inclusions theory, the model (1) can be rewritten

$$0D^\alpha_t x_i(t) \in -c_i x_i(t) + \sum_{j=1}^n I\tilde{a}_{ij}(x_j(t)) f_j(x_j(t))$$

$$+ \sum_{j=1}^n \tilde{b}_{ij}(x_j(t - \tau_{ij}(t))) g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n d_{ij} v_j + \sum_{j=1}^n \alpha_{ij} g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n S_{ij} v_j + \sum_{j=1}^n \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i$$

According to the measurable selection theorem [37], there exist measurable functions $\tilde{a}_{ij} \in K[I\tilde{a}_{ij}(x(t))], \tilde{b}_{ij} \in K[I\tilde{b}_{ij}(x(t - \tau_{ij}(t)))]$ for all $t \geq 0$ such that

$$0D^\alpha_t x_i(t) = -c_i x_i(t) + \sum_{j=1}^n I\tilde{a}_{ij}(x_j(t)) f_j(x_j(t))$$

$$+ \sum_{j=1}^n \tilde{b}_{ij}(x_j(t - \tau_{ij}(t))) g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n d_{ij} v_j + \sum_{j=1}^n \alpha_{ij} g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n S_{ij} v_j + \sum_{j=1}^n \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i$$

Let the FMFCNNs (1) be the drive system. The corresponding response system is selected as follows:

$$0D^\alpha_t y_i(t) = -c_i y_i(t) + \sum_{j=1}^n I\tilde{a}_{ij}(y_j(t)) f_j(x_j(t))$$

$$+ \sum_{j=1}^n \tilde{b}_{ij}(y_j(t - \tau_{ij}(t))) g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n d_{ij} v_j + \sum_{j=1}^n \alpha_{ij} g_j(x_j(t - \tau_{ij}(t)))$$

$$+ \sum_{j=1}^n S_{ij} v_j + \sum_{j=1}^n \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i + u_i(t)$$

$$y_i(t) = \psi_i(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \ldots, n$$

where $u_i(t)$ is the fixed-time synchronization controller to be designed, $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ are as follows.

$$\tilde{a}_{ij}(y(t)) = \begin{cases} \tilde{a}_{ij}^1, & |y(t)| \leq T_j \\ \tilde{a}_{ij}^2, & |y(t)| > T_j \end{cases}$$

$$\tilde{b}_{ij}(y(t - \tau_{ij}(t))) = \begin{cases} \tilde{b}_{ij}^1, & |y(t - \tau_{ij}(t))| \leq T_j \\ \tilde{b}_{ij}^2, & |y(t - \tau_{ij}(t))| > T_j \end{cases}$$
Similarly, we have

\[ K[\tilde{a}_{ij}(y_j(t))] = \begin{cases} a_{ij}^1 \text{ if } |y_j(t)| < T_j \\ \infty \text{ if } |y_j(t)| \geq T_j \end{cases} \quad \text{and} \quad K[\overline{a}_{ij}(y_j(t))] = \begin{cases} a_{ij}^2 \text{ if } |y_j(t)| > T_j \\ \infty \end{cases} \]

Similarly, there exist measurable functions \( \tilde{b}_{ij} \in K[\tilde{a}_{ij}(y_j(t))] \), \( \overline{b}_{ij} \in K[\overline{a}_{ij}(y_j(t))] \) for a.e. \( t \geq 0 \) such as

\[
0D^\mu_t y_i(t) = -c_i y_i(t) + \sum_{j=1}^n \tilde{a}_{ij}(y_j(t)) f_j(y_j(t)) + \sum_{j=1}^n \overline{b}_{ij}(y_j(t)) g_j(y_j(t) - y_i(t)) + \sum_{j=1}^n d_{ij} y_j(t) + T y_j(t) + \sum_{j=1}^n \alpha_{ij} g_j(y_j(t) - y_i(t)) + I_i + u_i(t)
\]

(5)

For the drive-response systems (1) and (4), we have the following assumptions.

**Assumption 1:** The activation functions \( f_i(x) \) and \( g_i(x) \) are Lipschitz continuous on \( R \), i.e., there exist positive constants \( l_i, m_i \) that make the following inequality hold

\[
|f_i(y) - f_i(x)| \leq l_i |y - x| \\
g_i(y) - g_i(x) \leq m_i |y - x|, \quad i = 1, 2, \ldots, n.
\]

**Assumption 2:** For any switching jump \( T_i, i = 1, 2, \ldots, n \), the activations \( f_i(x) \) and \( g_i(x) \) satisfy the following conditions

\[
f_i(\pm T_i) = g_i(\pm T_i) = 0, \quad i = 1, 2, \ldots, n.
\]

**Remark 2:** Generally speaking, in order to study the stability or synchronization control of the memristor-based neural networks, we must first put forward relevant assumptions about the activation function of the memristor-based neural networks. Assumption 1 and 2 are rational, and one can see the references [33], [38], [41].

**B. THE DEFINITION OF FIXED-TIME SYNCHRONIZATION**

In this subsection, the definition of fixed-time synchronization and some useful lemmas are given. According to the drive system (3) and response system (5), we give the definition of the error system as follows

\[
\begin{align*}
0D^\mu_t e_i(t) &= 0D^\mu_t y_i(t) - 0D^\mu_t x_i(t), \quad t \in (0, T) \\
0D^\mu_T e_i(t) &= 0D^\mu_T y_i(t) - 0D^\mu_T x_i(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \ldots, n.
\end{align*}
\]

(6)

**Definition 1:** (See [39].) The origin of error system (6) is said to be globally fixed-time stable if it is globally uniformly finite-time stable and the settling time \( T \) is globally bounded, i.e., \( \exists T_{max} \in R^+ \) such that \( T(e_0(t)) \leq T_{max}, \forall e_0(t) \in R^n \).

**Remark 3:** From the above definition, we can see that fixed-time stability is actually a special finite-time stability, but the corresponding the settling time \( T \) has a certain upper bound, which does not depend on the initial value of the system, and it is only related to system parameters and controller parameters.

According to the definition of the fixed-time stability, the following definition is given.

**Definition 2:** The drive-response systems (1) and (4) are said to achieve fixed-time synchronization if there exists \( T(e_0(t)) \) in some finite time such that

\[
\lim_{t \to T(e_0(t))} \| e(t) \| = 0 \quad \forall t \geq T(e_0(t)) \quad (7)
\]

where \( T_{max} \) is the settling time, \( \| \cdot \| \) represents the Euclidean norm.

**Remark 4:** In order to study the fixed-time synchronization problem between the driving system (1) and the response system (4), we convert the fixed-time synchronization problem into the fixed-time stability problem of the error system (6).

**Lemma 1** (See [31]): If there exists a regular, positive definite and radially unbounded function \( V(x) : R^n \to R \) and constants \( a, b, \delta, k > 0 \) and \( \delta k > 1 \) meet

\[
\dot{V}(e(t)) \leq - \left( aV^\delta(e(t)) + b \right)^k, \quad e(t) \in R^n \setminus 0
\]

(8)

then the origin is fixed-time stable, and the settling time \( T_{max} \) is estimated by

\[
T(e_0(t)) \leq T_{max} = \frac{1}{b^k} \left( \frac{b}{a} \right)^\frac{1}{k} \left( 1 + \frac{1}{\delta k - 1} \right)
\]

(9)

**Lemma 2** (See [6]): Suppose \( y_j(t) \) and \( y_j(t) \) are two states of system(6), then we have

\[
\begin{align*}
\left| \sum_{j=1}^n \alpha_{ij} f_j(x_j) - \sum_{j=1}^n \alpha_{ij} f_j(y_j) \right| &\leq \sum_{j=1}^n \left| \alpha_{ij} \right| \left| f_j(x_j) - f_j(y_j) \right| \\
\left| \sum_{j=1}^n \beta_{ij} f_j(x_j) - \sum_{j=1}^n \beta_{ij} f_j(y_j) \right| &\leq \sum_{j=1}^n \left| \beta_{ij} \right| \left| f_j(x_j) - f_j(y_j) \right|
\end{align*}
\]

**Lemma 3** (See [40]): Let \( b_1, b_2, \ldots, b_N \geq 0, q \geq 1 \), then the following inequality holds

\[
\sum_{i=1}^N b_i^q \geq N^{1-q} \left( \sum_{i=1}^N b_i \right)^q
\]
Lemma 4 (See [41]): If Assumptions 1, 2 hold, then we have
\[
\begin{align*}
K[a_{ij}(y_j(t))]f_j(y_j) - K[a_{ij}(x_j(t))]f_j(x_j) & \leq a_{ij}^m \|y_j(t) - x_j(t)\| \\
K[b_{ij}(y_j(t))]g_j(y_j) - K[b_{ij}(x_j(t))]g_j(x_j) & \leq b_{ij}^m m_j \|y_j(t) - x_j(t)\|
\end{align*}
\]
i.e. \(v_j(x_j(t)) \in K[a_{ij}(x_j(t))], \omega_j(y_j(t)) \in K[a_{ij}(y_j(t))], i = 1, 2, \ldots, n,\) we also have
\[
\begin{align*}
|\omega(y_j(t))f_j(y_j) - v(x_j(t))f_j(x_j)| & \leq a_{ij}^m \|y_j(t) - x_j(t)\| \\
|\omega(y_j(t))g_j(y_j) - v(x_j(t))g_j(x_j)| & \leq b_{ij}^m m_j \|y_j(t) - x_j(t)\|
\end{align*}
\]
where \(a_{ij}^m = \max(|\tilde{a}_{ij}^1|, |\tilde{a}_{ij}^2|)\) and \(b_{ij}^m = \max(|\tilde{b}_{ij}^1|, |\tilde{b}_{ij}^2|)\).

Lemma 5 (See [42]): Let \(x(t) \in C^1[0, T]\), where \(T\) is a positive constant, then
\[
oD_t^\alpha 0D_t^{\alpha 2} x(t) = -c_i x_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} \tilde{b}_{ij}(t) g_j(y_j(t) - \tau_j(t)) + \sum_{j=1}^{n} d_{ij} v_j
\]
and
\[
oD_t^\alpha 0D_t^{\alpha 2} x(t) = x(t).
\]

Remark 5: In [31]–[35], the fixed-time synchronization of FCNNs are studied. Our results differ from others in two aspects at least. First, the synchronization controller used in the paper is more complex, and the derivation process is more sophisticated. Second, our research object is the fractional-order neural network, which is superior in describing the complicated input-output signal relations and the dynamics of neural networks.

III. THE FIXED-TIME SYNCHRONIZATION OF THE DRIVE-RESPONSE FMFCNNs

In this section, the sufficient conditions of fixed-time synchronization of the drive-response FMFCNNs will be given. In order to obtain these synchronization criteria, the fixed-time synchronization controller \(u_i(t)\) is designed as follow
\[
u_i(t) = -k_i e_i(t) - \text{sign}(e_i(t))(\gamma_i + \mu_i |e_i(t) - \tau_i(t)|) + \rho_i |\dot{e}_i(t)| |e_i(t)|^\frac{\mu_i}{\gamma_i + \mu_i}, i = 1, 2, \ldots, n.
\]
where \(k_i \geq 0, \gamma_i > 0, \mu_i > 0, \rho_i > 0, \) and \(l > 1.\) \text{sign}(\cdot) is the symbolic function.

In order to facilitate the main results of our paper, we first simplified the error system (6). Combined with Eqs. (3) and (5), we have
\[
oD_t^\alpha e_i(t) = \dot{oD_t^\alpha y_i(t)} - \dot{oD_t^\alpha x_i(t)} = -c_i y_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{n} \tilde{b}_{ij}(t) g_j(y_j(t) - \tau_j(t)) + \sum_{j=1}^{n} d_{ij} v_j
\]
and
\[
\begin{align*}
&+ \sum_{j=1}^{n} \tilde{a}_{ij}(t) f_j(y_j(t) - \tau_j(t)) + \sum_{j=1}^{n} \tilde{b}_{ij}(t) g_j(y_j(t) - \tau_j(t)) + \sum_{j=1}^{n} d_{ij} v_j
\end{align*}
\]
According to Assumption 1, Lemmas 2 and 4, we obtain
\[
oD_t^\alpha e_i(t) \leq \sum_{j=1}^{n} k_i e_i(t) - \sum_{j=1}^{n} \mu_i |e_i(t)| |e_i(t)|^\frac{\mu_i}{\gamma_i + \mu_i} + \rho_i |\dot{e}_i(t)| |e_i(t)|^\frac{\mu_i}{\gamma_i + \mu_i}
\]
and
\[
\begin{align*}
&+ \sum_{j=1}^{n} \tilde{a}_{ij}(t) f_j(y_j(t) - \tau_j(t)) + \sum_{j=1}^{n} \tilde{b}_{ij}(t) g_j(y_j(t) - \tau_j(t)) + \sum_{j=1}^{n} d_{ij} v_j
\end{align*}
\]
Obviously, where \( V_0 \).

**Theorem 1:** Suppose that Assumptions 1 and 2 hold, then the error system (6) is fixed-time stable based on the controller (10) if the design parameters are selected as follows

\[
\begin{align*}
\sum_{i=1}^{n} d_{ij}^m l_i - c_i - k_i &< 0 \\
\sum_{i=1}^{n} b_i^m + |\alpha_i| + |\beta_i| m_i - \mu_i &< 0
\end{align*}
\]

(14)

Furthermore, the settling time can be calculated by the approximate formula

\[
T_s \leq T_{\max} = \frac{1}{\theta(l - 1)} \left( \frac{\theta}{\lambda n^{1-l}} \right)^{\frac{1}{l}}
\]

(15)

where \( \lambda = \min_{1 \leq i \leq n} \rho_i \), \( \theta = \sum_{i=1}^{n} \gamma_i \).

**Proof:** Consider a Lyapunov function defined by

\[
V(e(t)) = \sum_{i=1}^{n} aD_t^{\alpha_i} |e_i(t)|
\]

(16)

Obviously, \( V(e(t)) \geq 0 \) and \( V(e(t)) = 0 \) if and only if \( e(t) = 0 \). \( V(e(t)) \) is regular, positive definite and radially unbounded. According to Lemmas 5 and 6, we obtain

\[
\dot{V}(e(t)) = aD_t^{\alpha_i} \left( D_t^{1-\alpha_i} V(e(t)) \right)
\]

\[
= aD_t^{\alpha_i} \left( D_t^{1-\alpha_i} \sum_{i=1}^{n} aD_t^{\alpha_i-1} |e_i(t)| \right)
\]

\[
= aD_t^{\alpha_i} \left( D_t^{1-\alpha_i} \left( D_t^{\alpha_i-1} \sum_{i=1}^{n} |e_i(t)| \right) \right)
\]

\[
= \sum_{i=1}^{n} aD_t^{\alpha_i} |e_i(t)| \leq \sum_{i=1}^{n} \text{sign}(e_i(t)) aD_t^{\alpha_i} e_i(t)
\]

Calculating the upper right-hand derivative of Eq.(16) along the error system (6) and replacing the \( aD_t^{\alpha_i} e_i(t) \) with the inequality (13), we have

\[
\dot{V}(e(t)) \leq \sum_{i=1}^{n} \text{sign}(e_i(t)) aD_t^{\alpha_i} e_i(t)
\]

\[
\leq \sum_{i=1}^{n} \left( \langle c_i - k_i \rangle e_i(t) \right) + \sum_{i=1}^{n} d_{ij}^m l_i |e_i(t)|
\]

\[
\leq \sum_{i=1}^{n} \left( \langle c_i - k_i \rangle e_i(t) \right) + \sum_{i=1}^{n} d_{ij}^m l_i |e_i(t)|
\]

When choosing the appropriate controller parameters to meet the condition (14), we have the following inequality

\[
\dot{V}(e(t)) \leq -\sum_{i=1}^{n} \gamma_i + \sum_{i=1}^{n} \rho_i \left( D_t^{\alpha_i-1} |e_i(t)| \right)
\]

(14)

Let \( \lambda = \min_{1 \leq i \leq n} \rho_i \), \( \theta = \sum_{i=1}^{n} \gamma_i \), and by Lemma 3, we obtain

\[
\dot{V}(e(t)) \leq -\theta - \lambda n^{1-l} (V(e(t)))
\]

(14)

Let \( k = 1 \) in Lemma 1, we know the origin is fixed-time stable, and the settling time \( T_{\max} \)

\[
T_s \leq T_{\max} = \frac{1}{\theta(l - 1)} \left( \frac{\theta}{\lambda n^{1-l}} \right)^{\frac{1}{l}}
\]

(15)

**Remark 6:** In [25]-[30], the finite-time synchronization problems of fuzzy cellular neural networks were investigated. Compared with the finite-time synchronization, the settling time of fixed-time synchronization problem is independent of the initial synchronization errors, and the upper bound of the settling time can be given in advance. From this point of view, the fixed time synchronization is more favorable and applicable.

**Remark 7:** The settling time in this article is based on Lemma 1, and the algorithm of the settling time in Lemma 1 itself is an optimized result. In fact, in the proof of Lemma 1, an estimation of the settling time is given by

\[
T_s \leq \int_{0}^{+\infty} \frac{1}{(a0^\beta + b)^k} d\omega = \int_{0}^{r} \frac{1}{(a0^\beta + b)^k} d\omega
\]

\[
+ \int_{r}^{+\infty} \frac{1}{(a0^\beta + b)^k} d\omega
\]
\[
\varphi(r) = \frac{r}{b^k} + \frac{1}{a^\delta(k - 1)} r^{1-\delta k}
\]

where \(r\) is an arbitrary positive number. To give a less conservative and more accurate estimation of \(T_s\), let

\[
\varphi(r) = \frac{1}{b^k} + \frac{1}{a^\delta(k - 1)} r^{-\delta k}
\]

which shows that \(\varphi(r)\) reaches to its minimum value \(\hat{\varphi}\) given by

\[
\hat{\varphi} = \frac{1}{b^k} \left( \frac{b}{a} \right) \frac{1}{\left( 1 + \frac{1}{\delta k - 1} \right)}
\]

On the other hand, we can optimize the estimation of the settling time by choosing the suitable parameters \(a, b, k, \delta\) in applications. In fact, if \(a > b\) and \(\delta = \frac{\ln \frac{a}{b}}{\ln \frac{a}{b} + 1}\), a less conservative estimation of the settling time function \(T_s\) can be obtained by

\[
T_s \leq T_{\max} = \frac{\ln \frac{b}{a} \left( \frac{b}{a} \right) \left( 1 + \frac{1}{\ln \frac{a}{b}} \right)}{\ln \frac{a}{b}}
\]

**Corollary 1:** It should be noted that, when the conditions (14) hold, the fixed-time synchronization between system (1) and (4) can be achieved, and the settling time can be calculated by Eq. (15).

If there are not fuzzy feedback MAX template, fuzzy feedback MIN template and delayed feedback template in the system (1), i.e. \(\beta_{ij} = \alpha_{ij} = \tilde{\beta}_{ij} = 0\), then the corresponding drive-response system can be rewritten as follow

\[
\begin{align*}
\dot{D}_{ij}\dot{x}_i(t) &= -c_i x_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij}(x_j(t)) f_j(x_j(t)) \\
+ \sum_{j=1}^{n} d_{ij}v_j + \sum_{j=1}^{n} T_{ij}v_j + \sum_{j=1}^{n} S_{ij}v_j + I_i \\
\dot{D}_{ij}\dot{y}_j(t) &= -c_j y_j(t) + \sum_{i=1}^{n} \tilde{a}_{ij}(y_i(t)) f_j(y_i(t)) \\
+ \sum_{i=1}^{n} d_{ij}v_i + \sum_{i=1}^{n} T_{ij}v_j + \sum_{i=1}^{n} S_{ij}v_j + I_i
\end{align*}
\]

Here, the fixed-time synchronization controller (12) can also be reduced to the following form

\[
u_i(t) = -k_i e_i(t) - \text{sign}(e_i(t))\beta_i + \rho_0(\partial D_{ij}^{\alpha - 1}|e_i(t)|)^{\delta_i}, \quad i = 1, 2, \ldots, n.
\]

**Corollary 2:** Under Assumptions 1, 2 and the controller (18), the drive-response system (17) can achieve fixed-time synchronization if

\[
\sum_{j=1}^{n} a_{ij}^m l_j - c_i - k_i < 0, \quad i = 1, 2, \ldots, n.
\]

Moreover, the settling time \(T_{\max}\) can be obtained from Eq. (15)

**IV. NUMERICAL SIMULATIONS**

In this section, two numerical examples are given in order to verify the effectiveness of the theoretical results of Theorem 1 and Corollary 2.

**Example 1:** Consider the following three-dimensional drive-response memristor-based fractional-order fuzzy cellular neural network:

\[
\begin{align*}
\dot{D}_{11}\dot{x}_1(t) &= -c_1 x_1(t) + \sum_{j=1}^{3} \tilde{a}_{ij}(x_j(t)) f_j(x_j(t)) \\
+ \sum_{j=1}^{3} \tilde{b}_{ij}(x_j(t - \tau_j(t))) g_j(x_j(t - \tau_j(t))) \\
+ \sum_{j=1}^{3} \tilde{d}_{ij}v_j + \sum_{j=1}^{3} T_{ij}v_j \\
+ \sum_{j=1}^{3} \alpha_{ij}g_j(x_j(t - \tau_j(t))) + S_{ij}v_j \\
+ \sum_{j=1}^{3} \beta_{ij}g_j(x_j(t - \tau_j(t))) + I_i \\
\dot{D}_{12}\dot{x}_2(t) &= -c_2 x_2(t) + \sum_{j=1}^{3} \tilde{a}_{ij}(x_j(t)) f_j(x_j(t)) \\
+ \sum_{j=1}^{3} \tilde{b}_{ij}(x_j(t - \tau_j(t))) g_j(x_j(t - \tau_j(t))) \\
+ \sum_{j=1}^{3} \tilde{d}_{ij}v_j + \sum_{j=1}^{3} T_{ij}v_j \\
+ \sum_{j=1}^{3} \alpha_{ij}g_j(x_j(t - \tau_j(t))) + \sum_{j=1}^{3} S_{ij}v_j \\
+ \sum_{j=1}^{3} \beta_{ij}g_j(x_j(t - \tau_j(t))) + I_i + u_i(t), \quad i = 1, 2, 3.
\end{align*}
\]

where the non-delayed and delayed memristor-based synaptic connection weights are listed as follows.

\[
\begin{align*}
\tilde{a}_{11}(x_1(t)) &= \begin{cases} -0.8, & |x_1(t)| \leq 1 \\
-1, & |x_1(t)| > 1 \end{cases} \\
\tilde{a}_{12}(x_1(t)) &= \begin{cases} 2.2, & |x_1(t)| \leq 1 \\
2, & |x_1(t)| > 1 \end{cases} \\
\tilde{a}_{13}(x_1(t)) &= \begin{cases} 1.2, & |x_1(t)| \leq 1 \\
1.8, & |x_1(t)| > 1 \end{cases} \\
\tilde{a}_{21}(x_2(t)) &= \begin{cases} 1, & |x_2(t)| \leq 1 \\
0.8, & |x_2(t)| > 1 \end{cases} \\
\tilde{a}_{22}(x_2(t)) &= \begin{cases} -1, & |x_2(t)| \leq 1 \\
-0.8, & |x_2(t)| > 1 \end{cases}
\end{align*}
\]
The elements of fuzzy feedback MIN template, fuzzy feedback MAX template, feed-forward template, fuzzy feed-forward MIN template and fuzzy Feed-forward MAX template are listed as follows.

\[
A = (a_{ij})_{3 \times 3} =
\begin{bmatrix}
-0.1 & -0.01 & 0.2 \\
-0.2 & -0.1 & 0.1 \\
-0.04 & -0.2 & 0.4 \\
-0.1 & -0.01 & 0.3 \\
-0.1 & -0.2 & 0.2 \\
-0.1 & -0.2 & 0.3 \\
0.5 & 0.1 & -0.4 \\
0.1 & 0.5 & -0.2 \\
0.2 & 0.4 & 0.5 \\
0.2 & 0.1 & 0.4 \\
0.2 & 0.2 & 0.6 \\
0.5 & 0.3 & 0.2 \\
0.2 & 0.1 & 0.2 \\
0.2 & 0.2 & 0.4 \\
0.8 & 0.1 & 0.2
\end{bmatrix}
\]

\[
B = (b_{ij})_{3 \times 3} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
D = (d_{ij})_{3 \times 3} =
\begin{bmatrix}
0.5 & 0.1 & -0.4 \\
0.1 & 0.5 & -0.2 \\
0.2 & 0.4 & 0.5 \\
0.2 & 0.1 & 0.4 \\
0.2 & 0.2 & 0.6 \\
0.5 & 0.3 & 0.2 \\
0.2 & 0.1 & 0.2 \\
0.2 & 0.2 & 0.4 \\
0.8 & 0.1 & 0.2
\end{bmatrix}
\]

\[
T = (T_{ij})_{3 \times 3} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
S = (S_{ij})_{3 \times 3} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

The fractional order \(\alpha\), the passive decay rates \(c_i\), the input \(v_i\), and the bias value \(I_i\) of the \(i\)th neuron are chosen as \(\alpha = 0.9\), \([c_1; c_2; c_3] = [1; 1.5; 2]\), \([v_1; v_2; v_3] = [1; 2; 1]\), and \([I_1; I_2; I_3] = [0; 0; 0]\), respectively.

The activation functions \(f_i(x)\), \(g_i(x)\), the initial values \(\phi_i(s)\), \(\psi_i(s)\), and the time-varying delay \(\tau_i(t)\) of system (20) are given by

\[
\begin{align*}
  f_i(x) &= g_i(x) = 0.5|x + 1| - |x - 1| - 1 \\
  \phi_1(t) &= 0.2 + 0.1 \sin(t), \quad \phi_2(t) = -0.4 + 0.1 \cos(t), \\
  \phi_3(t) &= 0.6 + 0.1 \sin(t), \quad t \in [-\tau, 0] \\
  \psi_1(t) &= 1.2 - 0.1 \sin(t), \quad \psi_2(t) = -0.6 - 0.1 \cos(t), \\
  \psi_3(t) &= 1.0 - 0.1 \sin(t), \quad t \in [-\tau, 0] \\
  \tau_i(t) &= e^t/(1 + e^t)
\end{align*}
\]

The following controller \(u_i(t)\) is designed as follow

\[
u_i(t) = -k_i e_i(t) - \text{sign}(e_i(t))\gamma_i + \mu_i |e_i(t - \tau_i(t))| + \rho_i(\phi_i D_t^{1-\alpha} |e_i(t)|)^eta, \quad i = 1, 2, 3. \quad (21)
\]
Example 2: Consider the following three-dimensional drive-response memristor-based fractional-order fuzzy cellular neural network without delay items.

\[
0D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^{3} \tilde{a}_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{3} d_{ij} v_j + \sum_{j=1}^{3} T_{ij} v_j + \sum_{j=1}^{3} S_{ij} v_j + I_i, \quad i = 1, 2, 3.
\]

\[
0D_t^\alpha y_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^{3} d_{ij} v_j + \sum_{j=1}^{3} T_{ij} v_j + \sum_{j=1}^{3} S_{ij} v_j + I_i + u_i(t), \quad i = 1, 2, 3.
\]

\(T_{\text{max}} = 0.7211\), which can be also confirmed by error system curves in Fig. 6.
FIGURE 6. The evolutions of synchronization error with controller.

FIGURE 7. The phase diagram of the drive system and the response system.

The weight parameters of Eq. (22) are listed as follows:

\[
\tilde{a}_{11}(x_1(t)) = \begin{cases} 
2.5, & |x_1(t)| \leq 1 \\
2.6, & |x_1(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{12}(x_2(t)) = \begin{cases} 
-2.0, & |x_2(t)| \leq 1 \\
-2.2, & |x_2(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{13}(x_3(t)) = \begin{cases} 
-3.0, & |x_3(t)| \leq 1 \\
-3.4, & |x_3(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{21}(x_1(t)) = \begin{cases} 
2.2, & |x_1(t)| \leq 1 \\
2.6, & |x_1(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{22}(x_2(t)) = \begin{cases} 
-1.8, & |x_2(t)| \leq 1 \\
-2.2, & |x_2(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{23}(x_3(t)) = \begin{cases} 
2.4, & |x_3(t)| \leq 1 \\
2.2, & |x_3(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{31}(x_1(t)) = \begin{cases} 
2.2, & |x_1(t)| \leq 1 \\
2.8, & |x_1(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{32}(x_2(t)) = \begin{cases} 
-2.4, & |x_2(t)| \leq 1 \\
-2.2, & |x_2(t)| > 1 
\end{cases}
\]

\[
\tilde{a}_{33}(x_3(t)) = \begin{cases} 
3.6, & |x_3(t)| \leq 1 \\
3.2, & |x_3(t)| > 1 
\end{cases}
\]

The fractional order \(\alpha\) is chosen as 0.95, and the parameters \(c_i, d_{ij}, S_{ij}, I_i\) are the same as Example 1. The activation functions \(f_i(x), g_i(x)\) and the initial values \(x_0, y_0\) are given by:

\[
f_i(x) = \begin{cases} 
0.5|x + 1| - |x - 1| - 1
\end{cases}
\]

\[
\begin{align*}
\begin{cases} 
\alpha = 0.2; -0.4; 0.6 \\
y_0 = [-0.2; 0.4; -0.6]
\end{cases}
\end{align*}
\]

Fig. 7 shows the phase diagram of the drive system and response system without controller, respectively.

To guarantee the fixed-time synchronization of drive-response system, we apply the controller scheme in
Corollary 2, and the controller parameter values selected are as follows.

\[
\begin{align*}
  k_1 &= 4.3, \quad k_2 = 5.0, \quad k_3 = 6.8 \\
  \gamma_1 &= 1, \quad \gamma_2 = 1, \quad \gamma_3 = 1 \\
  \rho_1 &= 8, \quad \rho_2 = 8, \quad \rho_3 = 8 \\
  l &= 1.5
\end{align*}
\]

Figs. 8, 9 and 10 present the state trajectories of the drive-response system with controllers (18).

The phase diagrams and the error trajectories of the drive-response system with controller (18) are shown in Fig. 11 and Fig. 12, respectively.

The numerical simulations show that the fixed-time synchronization of the drive-response system are obtained in the settling time \( T_{\text{max}} = 0.5000 \).

V. CONCLUSION

In this article, the problem of fixed-time synchronization control for fractional-order memristor-based fuzzy cellular neural networks (FMFCNNs) with time-varying delay has been studied extensively. By designing an appropriate state feedback controller, based on the fixed time stability theory and the Lyapunov method, we have derived some new and useful fixed-time synchronization criteria for FMFCNNs with time-varying delays. Finally, the simulation results show the effectiveness of the proposed fixed-time synchronization control scheme. It should be pointed out that most of the existing fixed time synchronization results are based on the assumption that the neuron activation functions are Lipschitz continuous. However, this assumption is not necessarily true in the actual environment. Therefore, in future work, based on the research results of this article, we will further study fixed-time synchronization problem of FMFCNNs with discontinuous activation functions.

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**YEGUO SUN** was born in Anhui, China, in 1979. He received the B.S. degree from the Department of Mathematics and Computational Science, Fuyang Normal University, Fuyang, China, in 2001, the M.S. degree from the Department of Mathematics, Shanghai Normal University, Shanghai, China, in 2007, and the Ph.D. degree from the Department of Control Science and Engineering, Beijing University of Aeronautics and Astronautics, Beijing, China, in 2011. He is currently a Professor with the School of Finance and Mathematics, Huainan Normal University. He has published several articles and some of them were indexed by SCI and EI. His current research interests include network control systems, neural networks, finite-time control, and fixed-time control.

**YIHONG LIU** was born in Anhui, China, in 1975. He received the B.S. degree from the Hefei University of Technology, in 2003, and the M.S. degree from East China Normal University, in 2010. He was a Visiting Scholar with the School of Computer Science and Technology, University of Science and Technology of China, from September 2014 to July 2015, under the supervision of Prof. Enhong Chen. He is currently an Associate Professor with the School of Computer Science, Huainan Normal University. His research interests include non-linear dynamics, pattern recognition, and image processing.