Measure of irregularity for Dirac, Schwinger, Wu-Yang’s bases in the Abelian monopole theory and affecting of the gauge symmetry principle by allowance of singularity in physics

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Abstract

In the literature concerning the monopole matter, three gauges: Dirac, Schwinger, and Wu-Yang’s, have been contrasted to each other, and the Wu-Yang’s often appears as the most preferable one. The article aims to analyse this view by interpreting the monopole situation in terms of the conventional Fourier series theory; in particular, having relied on the eminent Dirichlet theorem. It is shown that the monopole case can be labelled as a very specific and even rather simple class of problems in the frame of that theory: all the three monopole gauges amount to practically the same one-dimenstional problem for functions given on the interval [0, π], having a single point of discontinuity; these three vary only in its location.

Some general aspects of the Aharonov-Bohm effect are discussed; also the way of how any singular potentials such as monopole’s, being allowed in physics, touch the essence of the physical gauge principle itself is considered.
1. Introduction

The study of monopoles has now reached a point where further progress depends on a clearer understanding of this object that had been available so far. As evidenced even by a cursory examination of some popular surveys [1,2], the whole monopole area covers and touches quite a variety of fundamental problems. In particular, following the original and brilliant pattern given by Dirac [3,4], physicists always were especially concerned with relevant singularity problems. Besides, throughout all the history of this matter, conceiving itself of an idea of monopoles has been always associated with concept of singularity. Leaving aside a major part of various monopole problems, much more comprehensive area in itself, just those singularity aspects, notoriously known and generally accepted as difficult, and what is more, hitherto conclusively unsettled, will be a subject of the present study.

In the work, only the monopole’s singularities are discussed. In this connection, it should be emphasized at once that though much more involved irregular (even not monopole-like ones) configurations are consistently invented and reported in the literature; in the same time, it might be hoped, that just above-mentioned, old and familiar, monopole-based peculiarities came to light, again and repeatedly, in a somewhat disguised form, when considering those generalized systems. So, in the light of that connection, a more particular situation, investigated in the paper, is of reasonable interest for the more large number of problems. In any case, as evidenced by all the history of study of monopoles, even this seemingly plain, at first glance, case has turned out to be far too formidable an undertaking theoretically (and all the more experimentally).

Once the monopole had been brought into scientific usage at the quantum level, its main singular properties had been noted and examined. The background of thinking the whole monopole problem in that time can easily be traced; it was obviously tied up with the most outstanding point of hypothesis about a magnetic charge ($g$): the Dirac’s electric charge quantization condition. Just the latter was the first consideration in any assessment of the problem in a whole. Moreover, this quantization condition had occurred from the Dirac’s attempts to get over some difficulties concerning the basic requirements of continuity in quantum mechanics, i.e. in the process of solving again the same singularity problems.

Also, the Schwinger’s attempts to dispose of magnetic charge’s singularities and modify the quantization condition were of significant implication to subsequent discussing the monopole matter. In particular, his seminar paper [5] brought out a sharp separation of characteristics of integral and half-integral $eg$ cases ($e$ and $g$ denote, respectively, electric and magnetic charge) and based the full discussion on a study of such peculiarities.

In essence, the more recent, and of great popularity currently, approach by Wu and Yang [6] adheres closely to the same Dirac and Schwinger’s regard for the importance of continuity requirements in presence of the monopole and for the importance of establishing some reasonable and intelligible rules for handling all singularities encountered. They (Wu and Yang) renewed the old Dirac’s arguments, essentially updated the relevant mathematical techniques, and finally invented, in a sense, a new mathematical and physical object; the latter is designated now often as the Wu-Yang monopole. The crucial

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1. Though evidently, ultimate answers have not been found by this work as well, it might be hoped that a certain exploration into and clearing up this matter have been achieved.

2. Their approach seemingly enables us to surmount the old problem of monopole singularities; though,
moment in their contemplating the problem of monopole peculiarities had been the same old intention to overcome all the singularities occurred. Starting from the observation that the Schwinger’s potential is not well defined at the $x_3$-axis; as well as the Dirac’s potential is undetermined at a half-axis (it is either $x_3^+$ or $x_3^-$), Wu and Yang had suggested a simple trick: instead of a globally given electromagnetic 4-potential \( \{ A_\beta(t, \vec{x}) \}, \vec{x} \in \mathbb{R}_3 \) (in particular, the Dirac or Schwinger’s monopole potentials had been meant), they had said, one could use a pair of non globally given ones, which consists of two Dirac’s type sub-potentials \( \vec{A}^{(N)} \) and \( \vec{A}^{(S)} \) given respectively in two half-spaces (just in their own regions with no singularities) completed each other up to the whole 3-space as follows \( \vec{A}_{WY} = \{ \vec{A}^{(N)} \cup \vec{A}^{(S)} \} \); so that \( \vec{A}^{(N)} \) has no singularity at the positive half-axis $x_3^+$ as well as \( \vec{A}^{(S)} \) does not have any singularity at the negative one $x_3^-$. Thus, as often asserted, the absence of singularity, at least locally, had been achieved and thereby the clouds over this part of the subject had been dispersed. Therefore, the crisis in the scientific picture of this matter had been set out in a seemingly perfect fashion, thereby obviating any further doubts. Such a local charts-based approach to this and a variety of similar situations has been extensively and in great detail elaborated, so as an absolutely new mathematical language and physical methodology have been worked out to date. And now, it is almost generally accepted outlook to this matter that such a locally achieved continuity provides us with a substantial progress in studying and understanding any systems containing some singularities. This is where the subject stands now — very roughly speaking, of course.

The aim of the present work, in particular, is to demonstrate that there exist some grounds to query whether the monopole singularities have been ruled out indeed; the article suggests that such an outlook hardly would stand close examination.

So, our further work is laid out as follows. Sec.2 treats, in a fairly unusual way, an indetermined character of the above potentials. It is convenient first to discuss in detail one gauge — for definiteness we start with the Schwinger’s; the considering of two others is deferred to Sec.3. In so doing, a special notice is given to comparison of the representations of the Schwinger’s monopole in Cartesian and spherical coordinates; at this we trace a delicate cancellation between different terms in the process of this coordinate change. The spherical picture is treated as preferable to Cartesian one; the reasons to this are that in spherical basis a major part (though not the most essential one) of singular manifestations of the monopole’s 4-potential is hidden (effectively) by known $\theta, \phi$-coordinates’ singularity. As shown, through the use of the conventional generally-covariant tensor formalism, the monopole singularity problem is reduced to a single function \( f(\theta) \) \( (A_\phi = g \cos \theta) \) given on the interval \( [0, \pi] \). In this connection, one ought to keep in mind the known (and apparently hidden) indeterminacy at the axis $x_3$ for the spherical vector $\vec{e}_\phi$. For the case as may be noticed in more close investigating (see below), it does not explain away all its concomitant doubts and obscurities. It would carry us very far afield to discuss at any length such purely mathematical considerations; instead, the working language of the paper is going to be much more conventional, intuitive, and physically felt.

So, the tranquillity dominating among majority of physicists on this problem is not justified anyhow. To avoid any misunderstanding, it must be emphasized at once that this work is in no way a strenuous objection against the Wu-Yang formalism and its concomitant methodology. Also, author does not claim that the method by Wu and Yang is mistaken or misleading anyhow; instead, the article just points certain inherent features which delimit its powers to some natural bounds, and puts forward a possible development complementary to it.
under consideration, this circumstance implies that to any non-singular physical situation there must correspond a function $A_{\phi}^{\text{regul.}} = f(\theta)^{\text{regul.}}$ with zero-boundary conditions at these points $\theta = 0, \pi$.

In Sec.3, the other two gauges (Dirac and Wu-Yang’s) are looked at, of course on the line used in Sec. 2. Then, the prime question is that concerns the hierarchy (if any) among three of them; i.e. — whether or not these three gauges are unequally singular ones.

To produce any constructive and justified criterion for counting a quantity of singularity, we contrast the above three functions ($A_S^{\phi}, A_D^{\phi}, A_{WY}^{\phi}$) and accompanying boundary conditions (for definiteness, here the Schwinger’s case is taken)

$$f^S(\theta) = A_S^{\phi} = g \cos \theta : \quad \theta \in [0, \pi], \quad f^S(0) = +1, \quad f^S(\pi) = -1$$

with their counterpart in absence of any singularity at the axis $x_3$; namely

$$f^0(\theta) = A_0^{\phi} : \quad \theta \in [0, \pi], \quad f^0(0) = 0, \quad f^0(\pi) = 0.$$  \hspace{1cm} (1.1)

This may be expressed as follows: while a non-singular problem being associated with a definite case in the frame of the Fourier series analysis, for which the boundary conditions are specified as null ones, the monopole problem should be referred to its own type of Fourier problem. Definitely, all the differences concern and come from variations in boundary conditions and continuity properties, which either remain the same or get violated.

To formalize mathematically this observation (see (1.1) and (1.2)), we have determined a quantity (designated by $\mu_{\text{inv.}}(\vec{A})$ which might be treated as a measure of singularity for electromagnetic potential $\vec{A}$). Besides, and what is more, we show that this $\mu_{\text{inv.}}$ has the same one value for all the three monopole potentials:

$$\mu_{\text{inv.}}(\vec{A}_S) = \mu_{\text{inv.}}(\vec{A}_D) = \mu_{\text{inv.}}(\vec{A}_{WY}).$$

Therefore, in that sense, all three gauges amount to each other and there are no reasons to prefer any one of them. Extending this observation, it is reasonable to conjecture that the $\mu(\vec{A})$ is gauge-invariant quantity, i.e. it will not change when we perform an arbitrary $U(1)$ gauge transformation with any type of singularity involved.

In addition, else one type of measure of singularity of electromagnetic potential (it called an ‘additive’ measure $\mu_{\text{addit.}}(\vec{A})$) has been introduced. In contrast to $\mu_{\text{inv.}}$ the latter must substantially vary when any piecewise continuous (in the sense of functions of spatial coordinates) gauge transformations are used; for more detail see below in Sec.4).

In sec.5, in terms of those measures $\mu_{\text{inv.}}$ and $\mu_{\text{addit.}}(\vec{A})$, we consider several aspects of the Aharonov-Bohm effect and discuss some inherent requirements implied by the conventional gauge principle. Particularly, we take special notice of the fact that any singular potentials such as monopole’s, being allowed in physics, significantly touch the essence of the physical gauge principle itself.

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5So, merely the reformulation of the monopole problem in other terms unables us to make good use for the conventional mathematical theory of Fourier series in studying the monopole problem. In particular, as a general basic point representing this theory and obviously touching the problem under consideration, the well-known Dirichlet theorem has been taken.

6This contrasts somewhat with a common viewpoint in the literature, when the Wu-Yang approach to monopole problem has been regarded often as having some advantage.
2. Schwinger’s potential in Cartesian and spherical coordinates

First let us consider more closely some facts on Schwinger’s gauge which are to be counted in the following. As well known, the Schwinger’s potential [5] is given by

\[ \vec{A}_S(x) = -g \frac{[\vec{r} \times \vec{n}] \cdot (\vec{r} \vec{n})}{r (r^2 - (\vec{r} \vec{n})^2)} \]  

(2.1)

where \( \vec{n} \) stands for an arbitrary 3-vector and thereby it represents an additional parametre fixing a certain geometrical orientation of the monopole in the 3-space. At once, it should be noted that this potential \( \vec{A}_S(x) \) is not a well defined quantity at the whole \( x_3 \)-axis; it is only a \((0/0)\)-expression when \( \vec{r} = \pm \vec{r} \vec{n} \).

Setting \( \vec{n} = (0, 0, +1) \) and translating this \( \vec{A}_S \) to the usual spheric coordinates, one gets

\[ A^S_{\phi} = g \cos \theta \]  

(2.2)

The \( A^S_{\phi} \) has non-vanishing values at \( \theta = 0 \) and \( \theta = \pi \):

\[ A^S_{\phi} = +g \text{ if } \theta = 0 \text{ and } -g \text{ if } \theta = \pi . \]

It is the point to remember that as \( \theta = 0 \) or \( \theta = \pi \), then the basis spherical vector \( \vec{e}_{\phi} \) has no single sense: there exists a set of possibilities for \( \vec{e}_{\phi} \) rather than only one that. This circumstance obviously comes from an original indeterminacy of the spherical coordinate \( \phi \) at the \( x_3 \)-axis. For this reason, a genuine sense of \( A_{\phi} \) at the axis \( x_3 \) should constitute just a characterization of \( A_{\phi} \) in a neighborhood of this axis rather than any specific values for it at the points lying in the axis \( x_3 \). In other words, the potential from (2.2) provides us with a non-single-valued function of spatial points just at the axis \( x_3 \).

Evidently, the above peculiarities of the monopole potential (2.2) do not originate in the irregularity properties of the spherical coordinates \((\theta, \phi)\). Indeed, some discontinuity occurs likewise in the Cartesian coordinates, when the monopole potential is described by (2.1); there it exhibits its own indeterminacy of the \((0/0)\)-kind at the axis \( x_3 \). Let us look at this more closely.

Because of the potential \( \vec{A}_S \) from (2.1) is formally meaningless at the axis \( x_3 \), we should look at its values in the adjoining neighborhood defined by

\[ \vec{r} = (0, 0, z) + \epsilon (m_1, m_2, m_3) = \vec{z} + \epsilon \vec{m} , \quad \left( \vec{m}^2 = 1 , m_3 \neq \pm 1 , \epsilon \to 0 \right) . \]

So, as a representative of the monopole potential at \( x_3 \), one has the quantity depending additionally on the vector \( \vec{m} \):

\[ \vec{A}^S(z, \vec{m}) \equiv \lim_{\epsilon \to 0} \vec{A}^S(z + \epsilon \vec{m}) \]

that is

\[ \vec{A}^S(z, \vec{m}) = \lim_{\epsilon \to 0} \left[ -g \frac{\epsilon (m_2 \vec{e}_1 - m_1 \vec{e}_2) \cdot (z + \epsilon m_3)}{\sqrt{\epsilon^2 m_1^2 + \epsilon^2 m_2 + (z + \epsilon m_3)^2} \epsilon^2 (m_1^2 + m_2^2)} \right] \]

where \( \vec{e}_i \) denotes the usual Cartesian orthonormal vectors. Further we have to draw distinction between \( z = 0 \) and \( z \neq 0 \). First,

\[ z \neq 0 : \quad \vec{A}^S(z, \vec{m}) = -\infty g \text{ Sgn } z \left( \frac{m_2 \vec{e}_1 - m_1 \vec{e}_2}{m_1^2 + m_2^2} \right) \text{ where } \infty = \lim_{\epsilon \to 0} \frac{1}{\epsilon} . \]  

(2.3a)
The unit vector $\vec{m}$ can be characterized by
\[ m_1 = \sin \Theta \cos \phi, \quad m_2 = \sin \Theta \sin \phi, \quad m_3 = \cos \Theta \]
where the quantity $\Theta$ does not coincide with the spatial coordinate variable $\theta$, whereas the $\phi$ is the usual spherical coordinate. In a sequence, the above vector $\vec{A}^S(\vec{z}, \vec{m})$ can be reexpressed as
\[ \vec{A}^S(\vec{z}, \vec{m}) = g \text{ sgn } z \frac{\infty}{\sin \Theta} \vec{e}_\phi \]  \hspace{1cm} (2.3b)
where $\vec{e}_\phi$ designates a combination ($\vec{e}_\phi = \sin \phi \vec{e}_1 - \cos \phi \vec{e}_2$). It should be noted that the factor $\sin^{-1} \Theta$ (in the (2.3b)) is not essential one in the sense that the symbol $\infty$ (having remembered that $\Theta \neq 0, \pi$) is presented there as well. So, instead of (2.3b) one may write down
\[ \vec{A}^S(\vec{z}, \vec{m}) = g \text{ sgn } z \infty \vec{e}_\phi . \]  \hspace{1cm} (2.3c)

In turn, for the $z = 0$ case one produces
\[ \vec{A}^S(\vec{r} = 0, \vec{m}) = -g \frac{\infty}{\sqrt{m_1^2 + m_2^2}} \frac{m_2 \vec{e}_1 - m_1 \vec{e}_2}{\sqrt{m_1^2 + m_2^2}} \]  \hspace{1cm} (2.4a)
or further
\[ \vec{A}^S(\vec{r} = 0, \vec{m}) = + g \frac{\infty}{\sin \theta} \cos \theta \vec{e}_\phi \sim + g \infty \cos \theta \vec{e}_\phi \]  \hspace{1cm} (2.4b)

Evidently, contrasting that Cartesian representation (see (2.3) and (2.4)) with an alternative spherical one
\[ A^S_\phi(\vec{r} = \vec{z}) = g \text{ sgn } z, \quad A^S_\phi(\vec{r} = 0) = g \cos \theta \]  \hspace{1cm} (2.5)
we conclude that the spherical description seems formally a bit less singular than Cartesian’s: the $\infty$ is absent in spherical picture. In other words, the singularity properties of the monopole at the axis $x_3$ fall naturally into two groups, one of which (the $\infty$ is meant) is subject to an incidental coordinate choice and another one (the factor $g \cos \theta$) reflects a properly monopole’s essence. So, it seems that just the factor $g \cos \theta$ carries a monopole quality after leaving out the all complications originating in the Cartesian coordinate system\footnote{It is somewhat surprising that so simple function as $g \cos \theta$ tells us a lot about the monopole and contains potentially a great deal of information concerning the magnetic charge.}.

It is of primary significance to the further exposition, that the Schwinger gauge exhibits a singularity both in the positive and negative half-axes $x_3$, as well as in the zero-point (0,0,0). One should repeat again: these singularities consist solely in the fact that the values of the monopole potential at the axis $x_3$ are a function of spatial directions that characterize possible ways of approaching these points $(0, 0, z)$. The same may be expressed as assertion that the monopole potential provides us with an example of quantity which is not a single-valued function of spatial points at the axis $x_3$.

All points of the positive half-axis $x_3$ are exactly alike with respect to their discontinuity properties. Therefore, as a possible method to describe this, one may try the following formulation: the half-axis $x_3^+$ provides $2\pi$ directions of discontinuity. A completely analogous statement concerns the negative half-axes $x_3^-$. Finally, the null point gives us...
irregularity directions. Thus, the Schwinger monopole potential, in a whole, can be schematically sketched by

\[
A^S_\phi \rightarrow \begin{cases} 
  x_3^+ & \sim (+2\pi) g \\
  0 & \sim (+2\pi \otimes \pi) g \\
  x_3^- & \sim (-2\pi) g 
\end{cases}
\] (2.6)

where the signs “+” and “−” serve to remind us of the \(\text{sgn}(z)\) in the \(A_\phi(z) = g \text{sgn}(z)\); just the function \(\text{sgn}(z)\) leads us to distinguish the \(x_3^+\) and \(x_3^-\) half-axes when characterizing the monopole singularities.

3. The Dirac and Wu-Yang’s representations

Now, from the same point of view, we are to analyze Cartesian and spherical pictures for the Dirac gauge. The Dirac potential is as follows (in the following, let \(\vec{n}\) be equal \((0, 0, +1)\))

\[
\vec{A}^{D(+)} = g \frac{[\vec{n} \times \vec{r}]}{r (r + \vec{r} \cdot \vec{n})} = g \frac{-x_1 \vec{e}_2 + x_2 \vec{e}_1}{r (r + x_3)}.
\] (3.1)

In contrast to the Schwinger’s case, here an indeterminacy \(0/0\) is located only at the negative half-axis \(x_3^-\) as well as at the zero-point, while the \(\vec{A}^{D(+)}\) has no discontinuity at the positive one: \(\vec{A}^{D(+)}(x_3^+) \equiv 0\).

Applying the limiting procedure above to the \(\vec{A}^{D(+)}\), one easily produces

\[
\vec{A}^{D(+)}(x_3^-, \vec{m}) = \infty (-2g) (m_2 \vec{e}_1 - m_1 \vec{e}_2) \sim \infty (-2g) \vec{e}_\phi,
\]

\[
\vec{A}^{D(+)}(\vec{r} = 0, \vec{m}) = \infty g (m_3 - 1) \vec{e}_\phi.
\] (3.2a)

In the spherical picture, the Dirac potential is

\[
A^{D(+)}_\phi = g (\cos \theta - 1)
\]

correspondingly, its singularities are characterized by

\[
A^{D(+)}_\phi(\vec{r} = 0) = g (\cos \theta - 1), \quad A^{D(+)}_\phi(x_3^-, \vec{m}) = -2g.
\] (3.2b)

It is the absence of discontinuity at the positive half-axis \(x_3^+\) (when \(\vec{n} = (0, 0, +1)\)) that singles out the Dirac gauge \(\vec{A}^{D(+)}\). In comparison with the Schwinger’s that is singular both in the \(x_3^+\) and \(x_3^-\) half-axis, the Dirac gauge seems less singular.

So, at first glance, the \(D\)-gauge looks preferable to the \(S\)-gauge; but on closer examination we will see that it is hardly so. In particularly, this (\emph{good} at the \(x_3^+\)-axis) gauge can be sketched by (compare it with (2.6))

\[
A^{D(+)}_\phi \rightarrow \begin{cases} 
  x_3^+ & \sim 0 \\
  0 & \sim (+2\pi \otimes \pi) g \\
  x_3^- & \sim (-2\pi) 2g
\end{cases}
\] (3.2c)

In the same time it should be noted that, by some intuitive considerations, the Dirac gauge appears to be equivalent to Schwinger’s because, in a sense, the Dirac discontinuity at \(x_3^-\) looks more intense than Schwinger’s: to realize this it suffices to take notice of
the factor $2g$ at (3.2c) in contrast to the factors: $+g$ and $-g$ in (2.2). This would mean that through the transformation $S$-gauge into $D$-gauge one has managed to reduce the discontinuity set from \{ $x^+_3 \oplus (0, 0, 0) \oplus x^-_3$ \} into \{ $(0, 0, 0) \oplus x^-_3$ \}, but in the same time one has augmented a power (or intensity) of the remaining discontinuity set.

In addition to the above, it should be reminded that the Dirac potential (with $\vec{\mathbf{n}}$ specified as $(0, 0, -1)$) is given by

$$A_D^{(-)} = g \frac{[ -\vec{n} \times \vec{r} ]}{r (r - \vec{n} \cdot \vec{r})} = -g \frac{-x_1 \vec{e}_2 + x_2 \vec{e}_1}{r (r - x^+_3)}$$

and, in turn, it has a $0/0$ indeterminacy at the positive half-axis $x^+_3$, which leads to

$$A_D^{(-)}(x^+_3, \vec{m}) = \infty ( + 2 \cos \theta ) \left( m_2 \vec{e}_1 - m_1 \vec{e}_2 \right) \sim \infty ( + 2g ) \vec{e}_\phi ,$$

$$A_D^{(-)}(\vec{r} = 0, \vec{m}) = \infty g \left( m_3 + 1 \right) \vec{e}_\phi .$$

Instead of (3.2b) now we have

$$A^{D(-)} = g( \cos \theta - 1 ) \rightarrow \begin{cases} A^{D(-)}(x^+_3, \vec{m}) = +2g \\ A^{D(-)}(0, \vec{m}) = g(\cos \theta + 1). \end{cases}$$

Now, it is the point to introduce the Wu-Yang potential [6]. It is determined by the following constituent form

$$A_{WY} = \begin{cases} A^{(N)} = A^{D(+)} & \text{if } 0 \leq \theta < \pi/2 , \\ A^{(S)} = A^{D(-)} & \text{if } \pi/2 < \theta \leq \pi . \end{cases}$$

As evidenced by its definition, this potential $A_{WY}$ has no discontinuity both in the $x^+_3$ and $x^-_3$ half-axes. But, in author’s opinion, it would be untenable to justify preferable utilizing the latter gauge only. The reason is that one should give special attention to the following: in $(WY)$-gauge, some discontinuity occurs at the $(x_1 - x_2)$-plane and this must be taken into account. One should remember that the term ‘discontinuity’ itself implies that there is, at certain points, any dependence on possible directions of approaching them; and just so the Wu-Yang potential looks at the $(x_1 - x_2)$-plane:

$$A^{WY}_\phi (\theta = \pi/2 + 0) = g (-1) , \quad A^{WY}_\phi (\theta = \pi/2 - 0) = g (+1) .$$

4. Hierarchi among the Dirac, Schwinger, Wu-Yang’s gauges and measure of the monopole irregularity

The whole situation with the monopole gauges (described in Sections 2 and 3) can be reformulated and summarized as follows. Original $S$- and $D$-gauges provide us with strong singularities concentrated along the $x_3$-axis. In going from the $S$- and $D$-gauges to the Wu-Yang’s we scatter the points of discontinuity over the $(x_1 - x_2)$-plane, so that this plane turns out to be filled up with irregularity points. These latter are less singular that former ones because these new irregularities are only two-valued ones; however, as a way of compensation, the number of such irregular points becomes much more greater. So, each of these three gauges is equally singular, with its own character of discontinuity, varying only in location. The case may be illustrated by the following picture
Certainly, such considerations are hinting and intuitive rather than exact and conclusively formulated arguments. Evidently, that the whole situation would be more satisfactory if we could determine a mathematically more strict characterization for measuring certain amount of singularity carrying by those monopole potentials. It is understandable that one should expect an $U(1)$-invariant character of that desirable measure of singularity.

So, the immediate task to solve is invention of a certain mathematical procedure clarifying and rationalizing this matter through some special heuristic construction. To begin with some summarizing steps — one may list all three gauges through the following schematic graphs (of which exact form does not matter to us, rather location of points of discontinuity is essential only)
Continuing this series of graphs, else one type of picture may be naturally suggested (it might be called anti-(Wu-Yang) gauge)

Obviously, $U(1)$-gauge transformations act effectively just within a fixed value of the parametre $g$, and all the more — the separation of $g$'s into positive and negative ones is very substantial (in the above figures, the positive values $g$'s are meant).

So, the question of special interest is — how would one substantiate correctness of the above claim that all three potentials are equally singular ones? What could serve as a measure for proper quantitative evaluation of their singular properties?

A possible answer is almost evident at once. Indeed, after all the above steps and transformations, the problem has been effectivelly reduced to a neat if not trivial task: namely, for exploration into the singularities one should compare all the monopole functions $A_\phi$ (for definiteness we will discuss the case $g > 0$ ) with a non-singular potential.

It should be remembered that all those functions $f_g(\theta)$ ought to be contrasted with the situation free of singularity, i. e. when a function $f^0(\theta)$ has the regular boundary conditions:

$$f^0(\theta = 0) = f^0(\theta = \pi) = 0.$$  \hfill (4.1)

The latter indicates that dealing with the mathematical problem of singularity, we should rather regard those two values $\theta = 0$ and $\theta = \pi$ of the interval $[0, \pi]$ as identified ones.\footnote{They certainly represent different regions in the geometric 3-space $x_1$, $x_2$, $x_3$; but instead we mean something very different: a space of functions with specific (null) boudnery properties and given at the interval $[0, \pi]$, which admit identification of its bounding points.}
Thus, all this may be reformulated mathematically as follows: any regular problem may be associated with a space of continuous functions on that interval. That is, every Abelian situation, not having at all any discontinuity at the $x_3$ axis, can be associated with a function $f^0(\theta)$ of null boundary conditions at this axis

$$\lim_{\text{at } x_3-\text{axis}} A_{\text{Reg.}} = 0 : A_{\text{Reg.}}^\theta(0) \rightarrow f(0) = 0 , \ f(\pi) = 0 . \quad (4.2)$$

Evidently, the above statement reflects only the following requirement: (of course, in its the most simple and particular form): in any regular case, the electromagnetic potential $A^0_\phi$ must approach zero as we approach the $x_3$-axis (along any direction).

It is natural (if not obvious) the further assertion: in any irregular case, the electromagnetic potential $A_\phi$ may not approach zero as we approach the $x_3$-axis:

$$\lim_{\text{at } x_3-\text{axis}} A_{\text{Irreg.}} \neq 0 : A_{\text{Irreg.}}^\theta(0) \rightarrow f(0) \neq 0 , \ f(\pi) \neq 0 . \quad (4.3)$$

One remark of principle must be given. Indeed, if the original electromagnetic potential $A_{\text{Reg.}}^\phi$ has been previously submitted to a gauge transformation in accordance with

$$A_{\text{Reg.}}^\phi \rightarrow A'_{\text{Reg.}}^\phi = A_{\text{Reg.}}^\phi - i \frac{e\hbar}{c} S \partial_\phi S^{-1} \quad (4.4a)$$

then the question as to whether a given potential $A_{\phi}$ is regular or irregular — is to be solved in a different way: namely, in any regular case, the electromagnetic potential $A_{\phi}^{0}$ must behave so that the relation of the form

$$\lim_{\text{at } x_3-\text{axis}} \left[ A_{\text{Reg.}}^{\prime \ Reg.} + i \frac{e\hbar}{c} S \partial_\phi S^{-1} \right] = 0 \quad (4.4b)$$

holds. In turn for all irregular cases this relation must be violated (by definition):

$$\lim_{\text{at } x_3-\text{axis}} \left[ A_{\text{Irreg.}}^{\prime \ Irreg.} + i \frac{e\hbar}{c} S \partial_\phi S^{-1} \right] \neq 0 . \quad (4.4c)$$

Now, turning again to the monopole case, one can easily realize that all the monopole potentials, being listed above, have the same one feature: each of them may be related to space of functions given on the interval $[0, \pi]$ (when the point $\theta = 0$ is identified with $\theta = \pi$ ) and having only one point of discontinuity. Moreover, it is evident at a glance that the intensity of the relevant discontinuity remains the same as we go over from one potential to another, just varying in their location. An additional remark should be given: significant as the locally achieved continuity according to Wu-Yang approach might seen, it is not as important as the plain mathematical fact that to each of gauges used there corresponds almost just the same Fourier $[0, \pi]$ problem with only single point of discontinuity.

This observation should be formalized in a suitable notation. To this end, one may take the following definition for a measure of singularity (as will be seen, it is invariant under any gauge transformations):

$$\mu_{\text{inv.}}(A_\phi) = \mu_{\text{inv.}}[f(\theta)] \overset{\text{def}}{=} \frac{1}{2} \left[ f(x_0 + 0) - f(x_0 - 0) \right] \quad (4.5)$$
where \( x_0 \) denotes a point of discontinuity, and \( \mu_{\text{inv.}}(A_\phi) \) designates a measure of singularity of the potential \( A_\phi \) at such a point. Then, for the monopole potentials described above, we will have

\[
\begin{align*}
S - \text{gauge} : & \quad f(\pi - 0) = -g , \quad f(0 + 0) = +g , \quad \mu_{\text{inv.}}(A_\phi^S) = +g ; \\
D - \text{gauge} : & \quad f(\pi - 0) = -2g , \quad f(0 + 0) = 0 , \quad \mu_{\text{inv.}}(A_\phi^D) = +g ; \\
D' - \text{gauge} : & \quad f(\pi - 0) = 0 , \quad f(0 + 0) = +2g , \quad \mu_{\text{inv.}}(A_\phi^{D'}) = +g ; \\
(WY) - \text{gauge} : & \quad f(\pi/2 - 0) = -g , \quad f(\pi/2 + 0) = +g , \quad \mu_{\text{inv.}}(A_\phi^{WY}) = +g .
\end{align*}
\]

All the above types of discontinuous functions in the interval \([ 0, \pi ]\) come under the eminent Dirichlet theorem’s conditions: — let us write out it in full.

**The Dirichlet theorem:**

If a function \( f(x) \) is given on segment \([-\pi, \pi]\), being bounded, piecewise continuous and piecewise monotonic one, then its trigonometric series converges at all the points of the segment. If \( S(x) \) represents a sum of the trigonometric series for the function \( F(x) \), then at all the points of continuity of this function, the equality \( S(x) = F(x) \) holds; whereas at all points of discontinuity (there must exist just a finite number of them) one gets only

\[
S(x) = \frac{1}{2} \left[ F(x - 0) + F(x + 0) \right].
\]

In addition, the identity

\[
S(\pi) = S(-\pi) = \frac{1}{2} \left[ F(\pi - 0) + F(\pi + 0) \right]
\]

holds.

The above limitations on functions assumed in this theorem are often called the Dirichlet conditions. It should be emphasized that they include essentially both piecewise continuity and piecewise monotoney, and none of them cannot be violated or waived. It is obvious that in (reasonable) physical applications, likewise in the situation under consideration, these Dirichlet conditions are likely to be satisfied.

Finally, turning to the anti Wu-Yang potential, we notice two points of discontinuity: those are \( \theta = 0(\pi) \) and \( \theta = \pi/2 \). Taking in mind this example, a more extendent definition for \( \mu(A_\phi) \) might be suggested:

\[
\mu_{\text{inv.}}(A_\phi) = \mu_{\text{inv.}}[f(\theta)] \overset{\text{def}}{=} \sum \frac{1}{2} \left[ f(x_i + 0) - f(x_i - 0) \right] \tag{4.7}
\]

where \( x_i \) denotes all points of discontinuity (here there are two ones). Thus, one has

\[
f(\pi/2 - 0) = +g , \quad f(\pi/2 + 0) = -g , \quad \text{and} \quad f(\pi - 0) = -2g , \quad f(0 + 0) = +2g
\]

and further

\[
\mu_{\text{inv.}}(A_{\phi}^{\text{anti}(WY)}) = \frac{1}{2} g \left[ (-1 - 1) + (2 + 2) \right] = +g ;
\]

i.e., the same value \( \mu_{\text{inv.}}(A_{\phi}^{\text{anti}(WY)}) = +g \) has been found again.

The latter example has been of unexpected interest because it shows some pecularity of the above quantity \( \mu(A_\phi) \). The matter is that from intuitive viewpoint, the case of the potential \( A_{\phi}^{\text{anti}(WY)} \) seems much more discontinuous in comparison with all three
previous ones, as it contains two singular points whereas each of the others exhibit only one point. However, the result has turned out to be exactly the same. What is the matter?

This example points to a reasonable requirement for else one additional characteristic to describe other sides of the singularities encountered above:

\[
\mu_{\text{addit.}}(A_\phi) = \mu_{\text{addit.}}[f(\theta)] \overset{\text{def}}{=} \sum \frac{1}{2} | f(x_i + 0) - f(x_i - 0) | .
\] (4.8)

Thus, one has

\[
\mu_{\text{addit.}}(A_\phi)^{\text{anti-WY}} = +4 \, g .
\] (4.9)

To clarify and spell out all the sense of the two quantities \(\mu_{\text{inv.}}\) and \(\mu_{\text{addit.}}\), let us introduce, for heuristical purposes, certain analogues of the used above monopole gauges for an artificial situation when any electromagnetic field (i.e. \(\vec{E}\) and \(\vec{B}\)) vanish. Those imaginary electromagnetic fields may be represented just by certain unphysical potentials. Those may be sketched by the following figures (supposing that for the Schwinger’s gauge, the electromagnetic potential vanishes: \(A^{(0)S}_\phi \equiv 0\))

Correspondingly, one has

\[
\mu_{\text{inv.}}(A^{(0)WY}_\phi) = g[ ( f(0 + 0) - f(0 - 0) ) + ( f(\pi/2 + 0) - f(\pi/2 + 0) ) ] = g \, [-1 - 1 + 1 + 1] = 0 , \text{ and } \mu_{\text{addit.}}(A^{(0)WY}_\phi) = +4 \, g
\]

those two relations can be interpreted as follows: (A) \(\mu_{\text{inv.}}(A^{(0)WY}_\phi) = 0\) points to the absence of any real singularity at vacuum-like state of electromagnetic field, though \(A^{(0)WY}_\phi\) is not null; (B) \(\mu_{\text{addit.}}(A^{(0)WY}_\phi) \neq 0\) proves \(\mu_{\text{addit.}}\) as a characteristic of singularity properties of gauge transformations involved here.

In case of \(A^{(0)D}_\phi\), two relations

\[
\mu_{\text{inv.}}(A^{(0)D}_\phi) = (g - g) = 0 , \quad \mu_{\text{addit.}}(A^{(0)D}_\phi) = | g - g | = 0 .
\]

may be interpreted in a similar way: \(\mu_{\text{inv.}}(A^{(0)D}_\phi) = 0\) conforms to (A) above; and \(\mu_{\text{addit.}}(A^{(0)D}_\phi) = 0\) corresponds with that \(A^{(0)D}_\phi\) has no singularity in continuity properties at the interval \([0, \pi]\) (thereby, it is in accoradance with (B) above.
5. On singularities, Aharonov-Bohm effects and some inherent requirements implied by the conventional gauge principle

Some immediate consequences of the above constructed two mathematical quantities $\mu_{\text{int.}}$ and $\mu_{\text{addit.}}$ might be of noticeable utility in quite other physical phenomena of much more generalized nature. At that point, we are going to pass away from the monopole matter and to deal with certain aspects of the well known, and extensively learnt in the literature, Aharonov-Bohm effect and will discuss it in somewhat new terms. Simultaneously we shall touch on the conventional gauge principle’s inherent structure.

Results obtained in that way, though in certain their sides are not without lacking in rigour, seems attractive and quite plausible. In any case, those developments hold promise of appreciable progress in clearing up, even if not explaining away completely, and definitive resolving these long standing paradoxical phenomena on notoriously known and predicted as physically observable manifestations of unphysical and subsidiary field $A_0^\alpha(x)$ related to vanishing field $(\vec{E}, \vec{B})$. Just such a particular aspect of the whole much more comprehensive matter of Aharonov-Bohm effect will be meant in the following. In turn, on that line of arguments, a specific view on various monopole gauges will be worked out.

Let us begin from the very generalities. So, it must be accepted that under all circumstances any entity, if it is considered just as a mathematical construct but not existing in reality, should not be a source of tension and contradiction even at the level of theoretical arguments or mental experiments. If the inverse arises (as it is so now) then, in the first instance, one ought to take notice of a possibly wrong inadequate understanding or interpreting of the situation and hence to look into, in the first, place just those aspects of the problem, rather than to bring out, somewhat routinely, any new confirmations to such an already fixed paradox.

To clarify more exactly what I mean here, let us look at just one side of the matter. That is the following: an arbitrary $U(1)$-gauge transformation — at the level of both the electromagnetic 4-potential $A_\alpha(x)$ and the wave functions for a quantum mechanical particles placed in the field of a magnetic charge — as a matter of fact carries always a certain amount of irregularity (or may be better to say — singularity; the latter term itself might be easily extended so that to cover all the changes produced by those gauge transformations). In other words, any explicitly given picture of a chosen physical system always bears a mark which is in exact correspondance with a respective gauge.

Unfortunately, among physisists, mainly an idea of gauge invariance itself has been fixed in mind — so strongly that, usually, they pass over some its inherent peculiarities and subtleties which accompany this undoubtly grand (mathetically and physicaly) structure. In particular, the substantial affecting of the relevant representing picture of a physical system (i.e. an alteration of regularity properties), though being accepted and recognized as such, seems often less significant. As a sequel, in majority of cases, they incline to detract from the necessity of the accurate following it, so that often this alteration turns out to be not remembered at all. But from this attitude only one step remains to face (unexpectedly just at first sight) the paradox on physical manifestations of not existing fields and further — a variety of Aharonov-Bohm effects.

Evidently, the above question of whether the different gauges (Dirac, Schwinger, Wu-

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9To avoid misunderstanding it should be stressed that the subsequent part of the work bears character of discussion rather than strict conclusive results.
Yang’s) are just different representatives for a unique physical object or not — may be qualified as belonging to the same problems which exist and manifest themselves yet in situations with no magnetic charge. In this connection, the very general view might be claimed that nature hardly produces so extensively really different physical objects, which originate from one that and which can be obtained through such a non-trivial exploiting of the $U(1)$ gauge transformations. Instead I think one should have worked out such a viewpoint that could guarantee any gauge transformation will not be a thing producing physical effects.

So, our immediate concern is the question — how one should reflect and act in order to select a proper representative $A_{\alpha}^{\text{proper}}$ from the whole set of possible candidates

\[
\left\{ A_{\alpha}^{\text{proper}} - i \frac{\hbar c}{e} U(x) \frac{\partial}{\partial x^\alpha} U(x) ; \quad U(x) \in U(1)_{\text{loc}} \right\}.
\]  

But it is the moment to remind that differences between all elements of this set (5.1) are in evidence and they can be immediately seen: those are their boundary condition properties (or in other terms, their singularities). The reasons for passing over, generally, such peculiarities can be quite easily understood. Seemingly those are: first, the authority of a gauge invariance principle itself; second, familiar and imbibed from the very beginning, the quantum-mechanical interpretation of a square modulus $|\Psi(x)|^2$. These both lie equally at the bottom of our understating and even ignoring such minor alterations in boundary properties, especially the second one detracts from the importance of those subtleties. In contrast, further I shall suppose that the true lies within just those boundary condition alterations.

Just in this point, the above introduced quantities $\mu_{\text{inv.}}(A_\alpha)$ and $\mu_{\text{addit.}}(A_\alpha)$ have their practical side. The first measure $\mu_{\text{inv.}}(A_\alpha)$ provides us with a general characteristic for the whole class

\[
\left\{ A_{\alpha}^{\text{proper}} - i \frac{\hbar c}{e} U(x) \frac{\partial}{\partial x^\alpha} U(x) ; \quad \Psi'(x) = U(x)\Psi(x) ; \quad U(x) \in U(1)_{\text{loc}} \right\}
\]  

the $\mu_{\text{inv.}}(A_\alpha)$ remains the same for all those electromagnetic potentials related to each other by means of any gauge transformations (one should remember those may be piecewise continuous as well as monotonic ones). In other words, this measure $\mu_{\text{inv.}}(A_\alpha)$ cannot be changed by the use of any, even with some special purposes constructed, singular gauge transformations; therefore, this quantity can serve the inherent characteristic of physical system itself.

In contrast to this, the second measure $\mu_{\text{addit.}}(A_\alpha)$ generally changes when passing from one gauge to another just through the use of any irregular gauge transformations. Therefore such a measure can serve to trace the use of any singular (piecewise continuous) gauge transformations.

By the way, a quite determined hierarchy between them (measures) might be presupposed in advance: in every separate example of a physical system, after we have calculated

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$^{10}$If the inverse were true, fantasy and ingenuity of nature would exceed any reasonable limits beyond our expectations, to say the least.

$^{11}$It is rather surprising that the technique, based on the use of them, turns out to be geared in such a perfect fashion to handling this matter.
its concomitant quantity $\mu_{\text{inv.}}(A_\alpha)$ and then set all various gauges into correspondence with the values of the measure $\mu_{\text{addit.}}(A_\alpha)$, we may expect that the set

$$\left\{ \mu_{\text{addit.}}(A^U_\alpha) ; \ U \in U(1) \right\}$$

implies a certain minimum value; the lowest bounding evidently exists — this seems more than plausible, and further it should coincide with the value of first (invariant) measure. However, this lowest bounding value of $\mu_{\text{addit.}}(A_\alpha)$ yet does not set just one basis apart from all the others, it provides us only with a set of candidates to preferable one. All these candidates can be referred to each other by means of gauge transformations; and what is more, such transformations are formed by either piecewise continuous or globally continuous functions of 3-space coordinates only; and presupposedly never they are continuous functions of 3-space geometric points.

Under all these circumstances, the single and only way out may be put forward: namely, that a preferable basis will be discovered if there exist indeed a continuous function of 3-space points (the latter merely could be affected and even destroyed by the use of a singular gauge transformation). Thus, the deciding (and essentially only remaining) step in searching a preferable gauge consist in the following: one ought to find a gauge without singularity\(^\text{12}\).

In other words, looking at the behaviour of relevant potentials or wave functions we would find that those are single-valued functions of 3-space geometric points just in a unique basis. This conjecture (and conclusion) seems quite justified. Moreover, this assumption on existence of a gauge with its concomitant single-valued electromagnetic potentials and likewise single valued wave functions, seems inevitable and even very desirable; otherwise the concept of single-valued physical fields (potentials, wave functions, and so on) even theoretically cannot ve discussed. The problem in issue can be reformulated differently: either we manage to arrive at a determination of a gauge being better than all other or inevitably we have to reconcile ourselves to a variety of physical predictions where just one that would be desirable. No other way out exists.

In this connection, one should take special notice of the fact that such an indeterminacy substantially touches the physical gauge principle itself. Indeed, the situation which we face here may be sketched as follows

$$i f \ g = 0 :$$

$$\{ \text{‘short’ gauge principle } \oplus \text{ a preferable basis} \} = \text{GAUGE PRINCIPLE}$$

$$i f \ g \neq 0 :$$

$$\{ \text{‘short’ gauge principle } \oplus \text{ no preferable basis} \} = \text{What is this?}$$

But not having any preferable basis, in the second case, what is the meaning of the physical identifying of all the pictures as describing the same one situation though in various gauges? The mathematical situation under consideration (specified by allowance of any not removable singularities) does not provide us with a sufficiently justified criterion about a certain physically invariant essence which just can be described in many ways through the use of various gauges. So, in such a new and at the first sight only

\(^{12}\text{Remember that here, for the moment, only a situation free of any magnetic charge or any other singular cases, is discussed.}\)
slightly altered situation, any counterpart of the second essential constituent (see above) in the ordinary gauge-invariance principle cannot be brought out. For these reasons, in my opinion, the case in issue should be associated with a double (two-faced) status for symmetry transformations of the group (here) $U(1)_{\text{loc.}}$ rather than conventional unique one:

$$G = U(1)_{\text{anti-gauge}} \otimes U(1)_{\text{gauge}} \text{ instead } G_{\text{conventional}} = U(1)_{\text{gauge}}.$$  \hfill (5.3)

It is no use blinking at the fact that after any (not removed) singularities had been allowed in physics, then all our subsequent attempts to retain the conventional gauge invariance principle unchanged as such were doomed to failure. In other words, one may say that the ‘innocent’ allowance itself of substantially singular potentials (which cannot be approached by single-valued functions of 3-space geometric points) turns out to be utterly destructive to the essence of old and standard gauge-invariance principle, i.e. translating it into something totally different. Thus, either one should reject the singularities (the simplest representative of which is $\mu_{\text{inv.}}(\vec{A}_\alpha(x)) \neq 0$) considering them as unphysical ones, or one ought to accept physics with an anti-gauge $\otimes$ gauge symmetry as above in (5.3).

In other words, the viewpoint may be advanced (this claim touches certain sides of the Aharonov-Bohm phenomena too) that gauge choice-based paradoxes should be regarded in some extent as a misunderstanding arisen out of not sufficiently exact and elaborated terminology rather than from any actual contradistinction within the physical theory itself. As a matter of fact, just making the terminology used more accurate and precise might explain away paradoxical aspects of all effects of that kind.

### 6. Methodology conclusions and discussion

Thus, the article has brought together such apparently unrelated ideas as to the monopole charge and old conventional Fourier series analysis; in particular, the Dirichlet theorem seems most significant in this connection. In the author opinion, the plain and striking fact is that the monopole situation entirely comes under this purely mathematical theory with many its concomitant subtleties of both mathematics and physics. So, just going back to some classical fundamentals of this theory leads us, as might be hoped, to a constructive reformulation of certain purely physical monopole problems. Such a synthesis is always attractive to theoreticians; the more so as the new insight gained holds promise of further progress, which could be of great importance in, for example, understanding the interrelationship among singularities (in particular, monopole’s), the concept of single-valuedness of wave functions, and gauge principle. Even if this last hope is not fulfilled, or outcomes achieved turn out to be less than expected, one will learn more about the nature of monopoles and how they should be rationalized so as to judge its further evaluation. In any case some interesting lessons can be learnt from the above suggested approach to the monopole matter, which might save us from having unjustified expectations and from dwelling too much on eliminations of the effects of singularities. that cannot be removed in principle.

In addition, one may note that the intrinsic potential power of the approach based on the Fourier analysis is that no assumption regarding the nature of the any underlying equations are necessary, so that various physical systems are automatically included. Definitely, the above treating, while setting a pattern for possible considerations on that line, admits further extening and developing to various situations where some singular potentials could occur. In so doing, certainly, the arrived generalizations can considerably
differ from the present variant in appearance, definitions, terminology, and else in a number of characteristics, but the basic spirit seemingly is going to be the same: namely, that the single-valuedness of potentials and wave functions with respect to 3-space geometric points and requirement to trace accurately where and how this property is modified, both are to be considered as a basis for any assessing extent of singularity of every situation as well as various gauges transformations.

Some more practical sides of the monopole’s presence concerning peculiarities of wave functions of quantum-mechanical particles affected by the monopole potential will be considered in a separate paper. Here I only wish note that the features of the $S$, $D$, $(W - Y)$-gauges find their natural corollary in boundary condition properties of corresponding wave functions. In particularly, the common argument to justify applying the $(W - Y)$-approach to quantum-mechanical particle-monopole problem is that it allows us to get rid of discontinuity of wave function (at $x_3$ axis). In the same time, they pass over the fact that some discontinuity appears at $x_3 = 0$ plane. More exactly, they say that in the region of overlapping, the functions $\Psi^{(N)}(x)$ and $\Psi^{(S)}(x)$ vary in phase factor $\exp(2ieg/h)$, which is not essential to any physically observable quantities. However, it should be stressed that particle wave functions in the $S$- and $D$-gauges have a very special violation, namely, both of them undergo exponential kinds of discontinuity, which are, by the same token, irrelevant to physically observable quantities as if we have utilized the $W - Y$ gauge.

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