Completeness of derived interleaving distances and sheaf quantization of non-smooth objects

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Abstract

We develop sheaf-theoretic methods to deal with non-smooth objects in symplectic geometry. We show the completeness of a derived category of sheaves with respect to the interleaving distance and construct a sheaf quantization of a Hamiltonian homeomorphism. We also develop Lusternik–Schnirelmann theory in the microlocal theory of sheaves. With these new sheaf-theoretic methods, we prove an Arnold-type theorem for the image of a compact exact Lagrangian submanifold under a Hamiltonian homeomorphism.

1 Introduction

The microlocal theory of sheaves due to Kashiwara and Schapira [KS90] has been effectively applied to symplectic and contact geometry after the pioneering work by Nadler–Zaslow [NZ09; Nad09] and Tamarkin [Tam18]. In this paper, we apply the theory to symplectic geometry for non-smooth objects, in particular, limits of smooth objects.

1.1 Our results

In this work, we use the Tamarkin category, which was introduced by Tamarkin [Tam18] to prove his non-displaceability theorem. Let $X$ be a connected $C^\infty$-manifold without boundary and let $\pi: T^*X \to X$ denote its cotangent bundle equipped with the canonical symplectic structure. We also let $(t; \tau)$ denote the homogeneous coordinate system on $T^*\mathbb{R}_t$ and fix a field $k$. The Tamarkin category $\mathcal{D}(X)$ is defined to be the left orthogonal of the triangulated subcategory consisting of objects microsupported in $\{\tau \leq 0\} \subset T^*(X \times \mathbb{R}_t)$ in $\mathcal{D}(k_{X \times \mathbb{R}_t})$, the derived category of sheaves on $X \times \mathbb{R}_t$. To generalize Tamarkin’s non-displaceability theorem to a quantitative version, the authors introduced the interleaving-like pseudo-distance on $\mathcal{D}(X)$ in [AI20], motivated by the convolution distance on $\mathcal{D}(k_{\mathbb{R}^n})$ due to Kashiwara–Schapira [KS18]. In this paper, we introduce a modified pseudo-distance $d_{\mathcal{D}(X)}$ on $\mathcal{D}(X)$ that is possibly larger than the previous one. See Subsection 3.2 for the details of the Tamarkin category and the pseudo-distance $d_{\mathcal{D}(X)}$.

In this paper, we first establish the completeness of the Tamarkin category $\mathcal{D}(X)$ with respect to $d_{\mathcal{D}(X)}$ in Section 4.

**Theorem 1.1** (see Corollary 4.5). The Tamarkin category $\mathcal{D}(X)$ is complete with respect to the pseudo-distance $d_{\mathcal{D}(X)}$.

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In fact, we prove the completeness result for a wide class of distances associated with thickening kernels, which was introduced by Petit–Schapira [PS23]. See the body of the paper for a precise statement.

In Section 5, we revisit sheaf quantization of Hamiltonian isotopies [GKS12] and Hamiltonian stability [AI20]. Let $M$ be a $C^\infty$-manifold and $I$ be an open interval containing the closed interval $[0, 1]$. With the Hamiltonian isotopy $\phi^H : T^*M \times I \to T^*M$ associated with a timewise compactly supported function $H : T^*M \times I \to \mathbb{R}$, one can associate a canonical object $K^I - \phi(1) \in \mathcal{D}(k_{M^2 \times I})$, called the sheaf quantization or the Guillermou–Kashiwara–Schapira (GKS) kernel. We prove that the restriction $K^I - \phi(1)\big|_{M^2 \times I \times \{1\}}$ depends only on the time-1 map of $\phi^H$. This allows us to define the sheaf quantization $K^\varphi - \phi \in \mathcal{D}(k_{M^2 \times I})$ of a compactly supported Hamiltonian diffeomorphism $\varphi \in \text{Ham}_c(T^*M)$. The sheaf quantization $K^\varphi$ defines an object $\mathcal{K}^\varphi$ of the Tamarkin category $\mathcal{D}(M^2)$. For these objects, we prove the following.

**Theorem 1.2** (see Theorem 5.11). For compactly supported Hamiltonian diffeomorphisms $\varphi, \varphi' \in \text{Ham}_c(T^*M)$, one has

$$d_{\mathcal{D}(M^2)}(K^\varphi, K^{\varphi'}) \leq d_H(\varphi, \varphi'),$$

where the right-hand side denotes the Hofer distance between $\varphi$ and $\varphi'$.

Theorem 1.2 is stronger than the previous Hamiltonian stability result in [AI20] in the following two points: (1) it is about the distance between GKS kernels and (2) the distance $d_{\mathcal{D}(X)}$ is possibly larger than that in [AI20].

By the completeness result and the stability result, we can associate a sheaf with an element of the completion of $\text{Ham}_c(T^*M)$ with respect to the Hofer metric. Indeed, a Cauchy sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \text{Ham}_c(T^*M)$ with respect to the Hofer metric $d_H$ defines a Cauchy sequence $(K^{\varphi_n})_{n \in \mathbb{N}} \subset \mathcal{D}(M^2)$ with respect to $d_{\mathcal{D}(M^2)}$ by Theorem 1.2. Hence, by applying Theorem 1.1 we obtain a limit object $\mathcal{K}^{\{(\varphi_n)_n\}} \in \mathcal{D}(M^2)$. In particular, we can define the sheaf quantization $\mathcal{K}^{\varphi_\infty}$ of a Hamiltonian homeomorphism $\varphi_\infty$ (see Definition 5.18 for the definition). One of the advantages of this approach is that we may directly use the limit object without explicitly dealing with a limit of some sequence to study $\mathcal{C}^0$-symplectic geometry.

Next, in Section 6 we develop Lusternik–Schnirelmann theory in the microlocal theory of sheaves. More precisely, for an object $F \in \mathcal{D}(M)$, we define the set of spectral invariants $\text{Spec}(F) \subset \mathbb{R}$ and prove the following. Here, $\text{cl}(M)$ denotes the cup-length of $M$ over $k$.

**Theorem 1.3** (see Theorem 6.3 for a more precise statement). Let $t : M \times \mathbb{R}_t \to \mathbb{R}_t$ and $\pi_M : T^*(M \times \mathbb{R}_t) \to M$ be the projections. Let $F \in \mathcal{D}(M)$ and assume that $M$ is compact and $R\Gamma_M(-c, \infty)(M \times \mathbb{R}_t; F) \simeq R\Gamma(M; k_M)$ and $F|_{M \times (c, \infty)}$ is locally constant for $c \gg 0$. If $\# \text{Spec}(F) \leq \text{cl}(M)$, then there exists $c \in \text{Spec}(F)$ such that $\pi_M(\text{SS}(F) \cap \Gamma_{dt} \cap \pi^{-1} t^{-1}(-c))$ is cohomologically non-trivial in $M$, where $\Gamma_{dt} \subset T^*(M \times \mathbb{R}_t)$ is the graph of the 1-form $dt$. That is, for any open neighborhood $U$ of $\pi_M(\text{SS}(F) \cap \Gamma_{dt} \cap \pi^{-1} t^{-1}(-c))$, the restriction map $\bigoplus_{n \geq 1} H^n(M; k) \to \bigoplus_{n \geq 1} H^n(U; k)$ is non-zero.

The theorem above was announced to appear in Humilière–Vichery [HV], which motivated our work.

Finally, in Section 7 by combining the machinery we have developed, we give a purely sheaf-theoretic proof of the following Arnold-type theorem for non-smooth objects (cf. Buhovsky–Humilière–Seyfaddini [BHS22]). We let $0_M$ denote the zero-section of $T^*M$.
Theorem 1.4 (see Theorem 1.1). Assume that $M$ is compact and let $L$ be a compact exact Lagrangian submanifold of $T^*M$. Let $\varphi_\infty$ be a Hamiltonian homeomorphism of $T^*M$. If the number of spectral invariants of $\varphi_\infty(L)$ is smaller than $\text{cl}(M) + 1$, then $0_M \cap \varphi_\infty(L)$ is cohomologically non-trivial in $M$, hence it is infinite.

This theorem is proved with the sheaf quantization of $L$ due to Guillermou [Gui12; Gui22] and Viterbo [Vit19], the sheaf quantization of $\varphi_\infty$, and Theorem 1.3. By combining our machinery with the $C^0$-continuity of the spectral norm, which is obtained in the field of symplectic geometry, we can also construct a sheaf quantization of the image of the zero-section $0_M$ under a $C^0$-limit of Hamiltonian diffeomorphisms. Note that $C^0$-limits of Hamiltonian diffeomorphisms is a more general notion than Hamiltonian homeomorphisms. With the sheaf quantization, we can recover an Arnold-type theorem for a $C^0$-limit of Hamiltonian diffeomorphisms, which is exactly a result by Buhovsky–Humilière–Seyfaddini [BHS22] (see Proposition 7.2).

We also give a sheaf-theoretic proof of a Legendrian analogue of Theorem 1.4. More precisely, we give an Arnold-type theorem for Hausdorff limits of Legendrian submanifolds in a 1-jet bundle (cf. [BHS22, Thm. 1.5]).

In Appendix A we give a Hamiltonian stability result with support conditions, which may be of independent interest.

1.2 Related work

The completeness of a persistence category was studied by Cruz [Cru19] and Scoccola [Sco20]. They showed that if the category admits any sequential (co)limit, then there is a limit object for any Cauchy sequence with respect to the interleaving distance. In this paper, we work with derived categories and need a different argument. Our completeness result (Theorem 1.1) also holds in a triangulated category endowed with a persistence structure. See also Biran–Cornea–Zhang [BCZ21] for persistence structures for triangulated categories.

Recently Fukaya [Fuk21] introduced the Gromov–Hausdorff distance between filtered $A_\infty$ categories, whose idea is based on the interleaving distance, and proved a completeness result. During the preparation of this paper, the authors learned from Stéphane Guillermou that he and Claude Viterbo had independently obtained a similar completeness result in a derived category of sheaves [GV22].

The persistence method has been widely applied to symplectic and contact geometry. After the pioneering work by Polterovich–Shelukhin [PS16], persistence modules have been used to study barcodes of Floer cohomology complexes in Usher–Zhang [UZ16] and the study of spectral norms in Kislev–Shelukhin [KS22], to name a few. In this paper, we also investigate the relation between the interleaving-like distance $d_{D(M)}$ and the spectral norm (see Proposition 6.9). Similar results are independently discovered by Guillermou and Viterbo [GV22].

Kashiwara–Schapira [KS18] studied persistence modules from the point of view of the microlocal theory of sheaves. Motivated by the work, Asano–Ike [AI20] introduced the interleaving-like distance $d_{D(M)}$ and showed that the distance between an object and its Hamiltonian deformation is upper bounded by the Hofer norm. The result was effectively used in Chiu [Chi23] and Li [Li21]. See also Zhang [Zha20] for the interleaving-like distance in the Tamarkin category.

Buhovsky–Humilière–Seyfaddini [BHS18] constructed a counterexample of Arnold’s conjecture for a Hamiltonian homeomorphism of a closed symplectic manifold $M$ with $\dim M \geq 4$. However, one still obtains an Arnold-type theorem for a Hamiltonian homeomorphism if one reformulates the conjecture with the notion of spectral invariants, as
in [BHS21, Kaw22, BHS22]. In these studies, the $C^0$-continuity of persistence modules associated with Hamiltonian diffeomorphisms was effectively used (see also [LSV21]). Guillermou [Gui13] (see also [Gui23, Part VII]) applied the microlocal theory of sheaves to $C^0$-symplectic geometry. He gave a purely sheaf-theoretic proof of the Gromov–Eliashberg theorem, using the involutivity of microsupports.

### 1.3 Organization

This paper is structured as follows. In Section 2, we recall some basics of the microlocal theory of sheaves. In Section 3, we first recall the interleaving distance for sheaves associated with a thickening kernel. We then briefly review the Tamarkin category and sheaf quantization of Hamiltonian isotopies. In Section 4, we prove the completeness of the derived category of sheaves with respect to the distance associated with a thickening kernel. This completeness in particular implies Theorem 1.1. In Section 5, we first prove a Hamiltonian stability theorem in terms of GKS kernels and the modified distance. We then show that the restriction of the sheaf quantization of a Hamiltonian isotopy to time 1 depends only on the time-1 map, which allows us to define the sheaf quantization of a Hamiltonian diffeomorphism. We also prove Theorem 1.2 and construct a sheaf quantization of a Hamiltonian homeomorphism. In Section 6, we develop Lusternik–Schnirelmann theory for the Tamarkin category and prove Theorem 1.3. We prove Theorem 1.4 in Section 7 and its Legendrian analogue in Section 8. In Appendix A, we prove a Hamiltonian stability result with support conditions, by using sheaf quantization of 2-parameter Hamiltonian isotopies.

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### 2 Microlocal theory of sheaves

Throughout this paper, let $k$ be a field. We mainly follow the notation of [KS90]. In this section, let $X$ be a $C^\infty$-manifold without boundary.

#### 2.1 Geometric notions

For a locally closed subset $Z$ of $X$, we denote by $\overline{Z}$ its closure and by $\text{Int}(Z)$ its interior. We also denote by $\delta_X : X \to X \times X, x \mapsto (x, x)$ the diagonal map and by $\Delta_X := \delta_X(X)$ the diagonal of $X \times X$. We often write $\Delta$ for $\Delta_X$ for simplicity. We denote by $TX$ the tangent bundle and by $T^*X$ the cotangent bundle of $X$. We write $\pi : T^*X \to X$ for the projection. For a closed submanifold $M$ of $X$, we denote by $T^*_M X$ the conormal bundle
to $M$ in $X$. In particular, $T_X^*X$ denotes the zero-section of $T^*X$, which we often simply write $0_X$. We set $T^*X := T^*X \setminus 0_X$.

With a morphism of manifolds $f : X \to Y$, we associate the following commutative diagram of morphisms of manifolds:

$$
\begin{array}{ccc}
T^*X & \xrightarrow{f_d} & X \times_Y T^*Y \\
\pi & \downarrow & \downarrow \pi \\
X & \xrightarrow{f} & Y,
\end{array}
$$

(2.1)

where $f_\pi$ is the projection and $f_d$ is induced by the transpose of the tangent map $f' : TX \to X \times_Y TY$.

We denote by $(x; \xi)$ a local homogeneous coordinate system on $T^*X$. The cotangent bundle $T^*X$ is an exact symplectic manifold with the Liouville 1-form $\alpha_{T^*X} = \langle \xi, dx \rangle$. Thus the symplectic form on $T^*X$ is defined to be $\omega = d\alpha_{T^*X}$. We denote by $a : T^*X \to T^*X, (x; \xi) \mapsto (x; -\xi)$ the antipodal map. For a subset $A$ of $T^*X$, $A^a$ denotes its image under the antipodal map $a$. A subset $A$ of $T^*M$ is said to be conic if it is invariant under the action of $\mathbb{R}_{>0}$ on $T^*M$, that is, the scaling of the fibers.

### 2.2 Microsupports of sheaves

We write $k_X$ for the constant sheaf with stalk $k$ and $\text{Mod}(k_X)$ for the abelian category of sheaves of $k$-vector spaces on $X$. Moreover, we denote by $D(k_X)$ the unbounded derived category of $k$-vector spaces on $X$. Although all the results are stated for bounded derived categories in [KS90], we can apply most of them for unbounded categories, which we shall state in this subsection. We refer to [Spa88; KS06; RS18]. One can define Grothendieck’s six operations $R\text{Hom}, \otimes, Rf_*, f^{-1}, Rf_!, f^!$ for a continuous map $f : X \to Y$ with suitable conditions. For a locally closed subset $Z$ of $X$, we denote by $k_Z$ the zero extension of the constant sheaf with stalk $k$ on $Z$ to $X$, whose stalk is 0 on $X \setminus Z$. Moreover, for a locally closed subset $Z$ of $X$ and $F \in D(k_X)$, we define $F_Z, R\Gamma_Z(F) \in D(k_X)$ by

$$
F_Z := F \otimes k_Z, \quad R\Gamma_Z(F) := R\text{Hom}(k_Z, F).
$$

(2.2)

Let us recall the definition of the microsupport $\text{SS}(F)$ of an object $F \in D(k_X)$.

**Definition 2.1 ([KS90], Def. 5.1.2).** Let $F \in D(k_X)$ and $p \in T^*X$. One says that $p \notin \text{SS}(F)$ if there is a neighborhood $U$ of $p$ in $T^*X$ such that for any $x_0 \in X$ and any $C^\infty$-function $\varphi$ on $X$ (defined on a neighborhood of $x_0$) satisfying $d\varphi(x_0) \in U$, one has $R\Gamma_{\{x \in X | \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} \simeq 0$. One also sets $\text{SS}(F) := \text{SS}(F) \cap T^*X$.

For a closed subset $A$ of $T^*X$, we denote by $D_A(k_X)$ the full triangulated subcategory of $D(k_X)$ consisting of objects whose microsupports are contained in $A$.

The following is called the microlocal Morse lemma.

**Proposition 2.2 ([KS90], Prop. 5.4.17 and [RS18], Thm. 4.1]).** Let $F \in D(k_X)$ and $\varphi : X \to \mathbb{R}$ be a $C^\infty$-function. Let moreover $a, b \in \mathbb{R}$ with $a < b$. Assume

1. $\varphi$ is proper on $\text{Supp}(F)$,
2. $d\varphi(x) \notin \text{SS}(F)$ for any $x \in \varphi^{-1}([a, b])$. 

5
Then the canonical morphism
\[ \text{R} \Gamma (\varphi^{-1}((-\infty, b)); F) \to \text{R} \Gamma (\varphi^{-1}((-\infty, a)); F) \]  
(2.3)
is an isomorphism.

We consider the behavior of the microsupports with respect to functorial operations.

**Proposition 2.3** ([KS90, Prop. 5.4.4, Prop. 5.4.13, and Prop. 5.4.5]). Let \( f : X \to Y \) be a morphism of manifolds, \( F \in \mathcal{D}(k_X) \), and \( G \in \mathcal{D}(k_Y) \).

1. Assume that \( f \) is proper on \( \text{Supp}(F) \). Then \( \text{SS}(\text{R} f_* F) \subset f_* f^{-1}_q (\text{SS}(F)) \). Moreover, if \( f \) is a closed embedding, the inclusion is an equality.

2. Assume that \( f \) is non-characteristic for \( \text{SS}(G) \) (see [KS90, Def. 5.4.12] for the definition). Then \( \text{SS}(f^{-1} G) \cap \text{SS}(f^! G) \subset f_* f^{-1}(\text{SS}(G)) \).

For closed conic subsets \( A \) and \( B \) of \( T^* X \), let us denote by \( A + B \) the fiberwise sum of \( A \) and \( B \), that is,
\[ A + B := \left\{ (x; a + b) \mid x \in \pi(A) \cap \pi(B), a \in A \cap \pi^{-1}(x), b \in B \cap \pi^{-1}(x) \right\} \subset T^* X. \]  
(2.4)

**Proposition 2.4** ([KS90, Prop. 5.4.14]). Let \( F, G \in \mathcal{D}(k_X) \).

1. If \( \text{SS}(F) \cap \text{SS}(G)^a \subset 0_X \), then \( \text{SS}(F \otimes G) \subset \text{SS}(F) + \text{SS}(G) \).

2. If \( \text{SS}(F) \cap \text{SS}(G) \subset 0_X \), then \( \text{SS}( \text{R} \text{Hom}(F, G)) \subset \text{SS}(F)^a + \text{SS}(G) \).

We need an estimate for the microsupport of a kind of limit object in \( \mathcal{D}(k_X) \). For that purpose, we use the following estimates.

**Lemma 2.5** (cf. [KS90, Exe. V.7]). Let \( I \) be an index set and \( F_i \in \mathcal{D}(k_X) \) for \( i \in I \). Then, one has
\[ \text{SS} \left( \coprod_{i \in I} F_i \right), \text{SS} \left( \bigoplus_{i \in I} F_i \right) \subset \bigcup_{i \in I} \text{SS}(F_i). \]  
(2.5)

### 2.3 Composition and convolution

We recall the operation called the composition of sheaves.

For \( i = 1, 2, 3 \), let \( X_i \) be a manifold. We write \( X_{ij} := X_i \times X_j \) and \( X_{123} := X_1 \times X_2 \times X_3 \) for short. We denote by \( q_{ij} \) the projection \( X_{123} \to X_{ij} \). Similarly, we denote by \( p_{ij} \) the projection \( T^* X_{123} \to T^* X_{ij} \). We also denote by \( p_{12^3} \) the composite of \( p_{12} \) and the antipodal map on \( T^* X_2 \).

Let \( A \subset T^* X_{12} \) and \( B \subset T^* X_{23} \). We set
\[ A \circ B := p_{13}(p_{12}^{-1}A \cap p_{23}^{-1}B) \subset T^* X_{13}. \]  
(2.6)

We define a composition operation of sheaves by
\[ \circ : \mathcal{D}(k_{X_{12}}) \times \mathcal{D}(k_{X_{23}}) \to \mathcal{D}(k_{X_{13}}), \]
\[ (K_{12}, K_{23}) \mapsto K_{12} \circ_{X_2} K_{23} := \text{R} q_{13}! (q_{12}^{-1} K_{12} \otimes q_{23}^{-1} K_{23}). \]  
(2.7)

If there is no risk of confusion, we simply write \( \circ \) instead of \( \circ_{X_2} \). By Propositions 2.3 and 2.4, we have the following.
Proposition 2.6. Let $K_{ij} \in D(k_{X_{ij}})$ and set $\Lambda_{ij} := SS(K_{ij}) \subset T^X_{X_{ij}}$ ($ij = 12,23$). Assume

1. $q_{ij}$ is proper on $q_{12}^{-1} \text{Supp}(K_{12}) \cap q_{23}^{-1} \text{Supp}(K_{23})$,
2. $p_{12}^{-1} \Lambda_{12} \cap p_{23}^{-1} \Lambda_{23} \cap (T^*_X X_1 \times T^*_X X_2 \times T^*_X X_3) \subset 0_{X_{123}}$.

Then

$$SS(K_{12} \circ_{X_2} K_{23}) \subset \Lambda_{12} \circ \Lambda_{23}. \quad (2.8)$$

In this work, we often use sheaves on $X \times \mathbb{R}$. We introduce the operation of convolution for objects of $D(k_{X \times \mathbb{R}})$. Define the maps

$$\tilde{q}_1, \tilde{q}_2, m: X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}, \quad \tilde{q}_1(x,t_1,t_2) = (x,t_1), \quad \tilde{q}_2(x,t_1,t_2) = (x,t_2), \quad m(x,t_1,t_2) = (x,t_1 + t_2). \quad (2.9)$$

We define a convolution operation by

$$*: D(k_{X \times \mathbb{R}}) \times D(k_{X \times \mathbb{R}}) \to D(k_{X \times \mathbb{R}}), \quad (F,G) \mapsto F * G := \text{Rm}_t(\tilde{q}_1^{-1}F \otimes \tilde{q}_2^{-1}G). \quad (2.10)$$

We also introduce a right adjoint to the convolution functor. Set $i: X \times \mathbb{R} \to X \times \mathbb{R}, (x,t) \mapsto (x,-t)$.

Definition 2.7. For $F,G \in D(k_{X \times \mathbb{R}})$, one sets

$$\text{Hom}^*(F,G) := \text{R} \tilde{q}_1^* \text{RHom}(\tilde{q}_2^{-1}F, m^1 G) \quad (2.11)$$

$$\simeq \text{Rm}_t \text{RHom}(\tilde{q}_2^{-1}i^{-1}F, \tilde{q}_1^1 G). \quad (2.12)$$

Lemma 2.8. Let $F,G \in D(k_{X \times \mathbb{R}})$ and assume that there exist two closed cones $A, B \subset \mathbb{R}$ such that $SS(F) \subset T^X X \times \mathbb{R} \times A$ and $SS(G) \subset T^X X \times \mathbb{R} \times B$. Then, one has

$$SS(F * G) \subset T^X X \times \mathbb{R} \times (A \cap B), \quad SS(\text{Hom}^*(F,G)) \subset T^X X \times \mathbb{R} \times (A \cap B). \quad (2.13)$$

It is useful to define a more general operation, which combines composition and convolution. Set

$$\tilde{q}_{12}: X_{123} \times \mathbb{R}^2 \to X_{12} \times \mathbb{R}, (x_1,x_2,x_3,t_1,t_2) \mapsto (x_1,x_2,t_1), \quad (2.14)$$

$$\tilde{q}_{23}: X_{123} \times \mathbb{R}^2 \to X_{23} \times \mathbb{R}, (x_1,x_2,x_3,t_1,t_2) \mapsto (x_2,x_3,t_2), \quad (2.15)$$

$$m_{13}: X_{123} \times \mathbb{R}^2 \to X_{13} \times \mathbb{R}, (x_1,x_2,x_3,t_1,t_2) \mapsto (x_1,x_3,t_1 + t_2) \quad (2.16)$$

and define

$$\bullet_{X_2} : D(k_{X_{12} \times \mathbb{R}}) \times D(k_{X_{23} \times \mathbb{R}}) \to D(k_{X_{13} \times \mathbb{R}}), \quad (K_{12}, K_{23}) \mapsto K_{12} \bullet_{X_2} K_{23} := \text{Rm}_{13t} (\tilde{q}_{12}^{-1} K_{12} \otimes \tilde{q}_{23}^{-1} K_{23}). \quad (2.17)$$

If there is no risk of confusion, we simply write $\bullet$ instead of $\bullet_{X_2}$.\]
2.4 Homotopy colimits

Here, we recall the definition of homotopy colimits (cf. [BN93] and [KS06]) and estimate their microsupports.

Let \((F_n)_{n \in \mathbb{N}}\) be a sequence of objects of \(D(k_X)\) together with morphisms \(f_n : F_n \to F_{n+1}\) \((n \in \mathbb{N} = \mathbb{Z}_{\geq 0})\), that is, \((F_n, f_n)_{n \in \mathbb{N}}\) is an inductive system in \(D(k_X)\). Define a morphism \(s : \bigoplus_{n \in \mathbb{N}} F_n \to \bigoplus_{n \in \mathbb{N}} F_n\) as the composite

\[
\bigoplus_{n \in \mathbb{N}} F_n \xrightarrow{\bigoplus_{n \in \mathbb{N}} f_n} \bigoplus_{n \in \mathbb{N}} F_{n+1} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 1}} F_n \to \bigoplus_{n \in \mathbb{N}} F_n. \tag{2.18}
\]

Then one can define the homotopy colimit of the inductive system \((F_n, f_n)_{n \in \mathbb{N}}\) as the cone of the morphism

\[
\text{id} - s : \bigoplus_{n \in \mathbb{N}} F_n \to \bigoplus_{n \in \mathbb{N}} F_n, \tag{2.19}
\]

which we write \(\text{hocolim}(F_n) \in D(k_X)\). We have a canonical morphism \(\rho_n : F_n \to \text{hocolim}(F_n)\).

Given a sequence of morphisms \(g_n : F_n \to G\) \((n \in \mathbb{N})\) such that \(g_{n+1} \circ f_n = g_n\) for any \(n\), we get a morphism \(g : \text{hocolim}(F_n) \to G\) satisfying \(g_n = g \circ \rho_n\).

**Lemma 2.9.** Let \((F_n, f_n)_{n \in \mathbb{N}}\) be an inductive system in \(D(k_X)\). Then

\[
\text{SS}(\text{hocolim}(F_n)) \subset \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \text{SS}(F_n). \tag{2.20}
\]

**Proof.** By Lemma 2.9, \(\text{SS}(\bigoplus_{n \in \mathbb{N}} F_n) \subset \bigcup_n \text{SS}(F_n)\). Note that we have \(\text{hocolim}_n(F_n) \simeq \text{hocolim}_n(G_n)\) with \(G_n = F_{n+N}\) for any \(N \in \mathbb{N}\). Hence, the result follows from the triangle inequality for microsupports and the definition of homotopy colimits.

3 Interleaving distance for sheaves and sheaf quantization

In this section, we review the interleaving distance for sheaves following Petit–Schapira [PS23]. We also briefly review the Tamarkin category [Tam18] and sheaf quantization of Hamiltonian isotopies [GKS12]. See also Guillermou–Schapira [GS14] and Zhang [Zha20] for details of the Tamarkin category.

3.1 Interleaving distance for sheaves

We recall some notions from Petit–Schapira [PS23]. A topological space is said to be good if it is Hausdorff, locally compact, countable at infinity, and finite flabby dimension.

**Definition 3.1.** Let \(X\) be a good topological space. A thickening kernel on \(X\) is a monoidal presheaf \(\mathfrak{R}\) on \((\mathbb{R}_{>0}, +)\) with values in the monoidal category \(D(k_X \times X)\). In other words, it is a family of kernels \(\mathfrak{R}_a \in D(k_X \times X)\) with a morphism \(\rho_{b,a} : \mathfrak{R}_b \to \mathfrak{R}_a\) for \(a \leq b\) together with isomorphisms

\[
\mathfrak{R}_a \circ \mathfrak{R}_b \simeq \mathfrak{R}_{a+b}, \quad \mathfrak{R}_0 \simeq k_{\Delta_X} \tag{3.1}
\]

satisfying the compatibility conditions (see [PS23, Def. 1.2.2] for details).

For a thickening kernel \(\mathfrak{R}\) on \(X\) and \(F \in D(k_X)\), we write \(\rho_{b,a}(F)\) for the morphism \(\rho_{b,a} \circ \text{id}_F : \mathfrak{R}_b \circ F \to \mathfrak{R}_a \circ F\) \((a \leq b)\).
**Definition 3.2.** Let \( \mathcal{K} \) be a thickening kernel on \( X, F, G \in D(k_X) \), and \( a, b \in \mathbb{R}_{\geq 0} \).

(i) The pair \((F, G)\) is said to be \((a, b)\)-isomorphic if there exist morphisms \( \alpha : \mathcal{K}_a \circ F \to G \) and \( \beta : \mathcal{K}_b \circ G \to F \) such that

1. the composite \( \mathcal{K}_a \circ \beta \circ \mathcal{K}_a \circ F \to \mathcal{K}_b \circ G \) is equal to \( \rho_{a+b,0}(F) \),
2. the composite \( \mathcal{K}_b \circ \alpha \circ \mathcal{K}_b \circ G \to \mathcal{K}_a \circ F \) is equal to \( \rho_{a+b,0}(G) \).

In this case, the pair of morphisms \((\alpha, \beta)\) is called an \((a, b)\)-isomorphism.

If \((F, G)\) is \((a, a)\)-isomorphic then \( F \) and \( G \) are called \( a \)-isomorphic.

(ii) One sets

\[
d_{\mathcal{R}}(F, G) := \inf \left\{ a + b \in \mathbb{R}_{\geq 0} \mid a, b \in \mathbb{R}_{\geq 0}, \text{ \((F, G)\) is \((a, b)\)-isomorphic} \right\},
\]

which defines a pseudo-distance on the category \( D(k_X) \).

(iii) The pair \((F, G)\) is said to be weakly \((a, b)\)-isomorphic if there exist morphisms \( \alpha, \delta : \mathcal{K}_a \circ F \to G \) and \( \beta, \gamma : \mathcal{K}_b \circ G \to F \) such that

1. the composite \( \mathcal{K}_a \circ \delta \circ \mathcal{K}_a \circ F \to \mathcal{K}_b \circ G \) is equal to \( \rho_{a+b,0}(F) \),
2. the composite \( \mathcal{K}_b \circ \gamma \circ \mathcal{K}_b \circ G \to \mathcal{K}_a \circ F \) is equal to \( \rho_{a+b,0}(G) \),
3. \( \alpha \circ \rho_{2a,a}(F) = \delta \circ \rho_{2a,a}(F) \) and \( \beta \circ \rho_{2b,b}(G) = \gamma \circ \rho_{2b,b}(G) \).

(iv) One says that \( F \) is \( a \)-torsion or \( a \)-trivial if \( \rho_{a,0}(F) : \mathcal{K}_a \circ F \to F \) is the zero morphism.

**Remark 3.3.** (i) One can see that

\[
(a, b)\text{-isomorphic} \Rightarrow \text{weakly } (a, b)\text{-isomorphic}
\]

and

\[
\text{weakly } (a, b)\text{-isomorphic} \Rightarrow (2a, 2b)\text{-isomorphic} \Rightarrow 2\max(a, b)\text{-isomorphic}.
\]

(ii) In [PS23], the authors define

\[
dist(\mathcal{R})(F, G) := \inf \{ a \in \mathbb{R}_{\geq 0} \mid F \text{ and } G \text{ are } a\text{-isomorphic} \}
\]

and call it the *interleaving distance* associated with \( \mathcal{R} \). By (i), the pseudo-distances \( d_{\mathcal{R}} \) and \( dist_{\mathcal{R}} \) are equivalent. Indeed,

\[
d_{\mathcal{R}}(F, G) \leq 2 dist_{\mathcal{R}}(F, G) \leq 2d_{\mathcal{R}}(F, G).
\]

(iii) The interleaving distance above is a generalization of the convolution distance \( d_C \) on \( D(k_X) \) introduced by Kashiwara–Schapira [KS18] and later investigated by [KS21; BG21; BP21; BGO19] and others. Indeed, when \( X = \mathbb{R} \) and \( \mathcal{R}_a = k_{\Delta_a} \), where \( \Delta_a := \{(x, y) \in \mathbb{R}^2 \mid \|x - y\| \leq a\} \subset \mathbb{R}^2 \) is the thickened diagonal, we find that \( d_C = dist_{\mathcal{R}} \).
In what follows, let $\mathcal{R}$ be a thickening kernel on $X$. The statement of the following lemma is slightly stronger than [AI20, Lem. 4.14], but the proof itself is almost the same. We need the stronger result in this paper, so we reproduce the proof for the convenience of the reader.

**Lemma 3.4** (cf. [AI20, Lem. 4.14]). Let $\xymatrix{F \ar[r]^u & G \ar[r]^v & H \ar[r]^w & F[1]}$ be an exact triangle in $\mathcal{D}(k_X)$ and assume that $F$ is $e$-torsion. Then $(G, H)$ is weakly $(0, c)$-isomorphic.

**Proof.** By assumption, we have $w \circ \rho_{c,0}(H) = \rho_{c,0}(F[1]) \circ (\mathcal{R}_c \circ_X w) = 0$. Hence, we get a morphism $\gamma : \mathcal{R}_c \circ H \to G$ satisfying $\rho_{c,0}(H) = v \circ \gamma$:

$$
\begin{array}{c}
\xymatrix{ & \mathcal{R}_c \circ F \ar[r]^{\mathcal{R}_{c,0}u} & \mathcal{R}_c \circ G \ar[r]^{\mathcal{R}_{c,0}v} & \mathcal{R}_c \circ H \ar[r]^{\mathcal{R}_{c,0}w} & \mathcal{R}_c \circ F[1] \\
\mathcal{R}_c \circ F \ar[r] & \mathcal{R}_c \circ G \ar[r] & \mathcal{R}_c \circ H \ar[r] & \mathcal{R}_c \circ F[1] \\
F \ar[r]^u & G \ar[r]^v & H \ar[r]^w & F[1].}
\end{array}
$$

(3.7)

On the other hand, since $\rho_{c,0}(G) \circ (\mathcal{R}_c \circ_X u) = u \circ \rho_{c,0}(F) = 0$, there exists a morphism $\beta : \mathcal{R}_c \circ H \to G$ satisfying $\rho_{c,0}(G) = \beta \circ (\mathcal{R}_c \circ_X v)$:

$$
\begin{array}{c}
\xymatrix{ & \mathcal{R}_c \circ F \ar[r]^{\mathcal{R}_{c,0}u} & \mathcal{R}_c \circ G \ar[r]^{\mathcal{R}_{c,0}v} & \mathcal{R}_c \circ H \ar[r]^{\mathcal{R}_{c,0}w} & \mathcal{R}_c \circ F[1] \\
\mathcal{R}_c \circ F \ar[r] & \mathcal{R}_c \circ G \ar[r] & \mathcal{R}_c \circ H \ar[r] & \mathcal{R}_c \circ F[1] \\
F \ar[r]^u & G \ar[r]^v & H \ar[r]^w & F[1].}
\end{array}
$$

(3.8)

Moreover, we obtain

$$
\begin{align*}
\beta \circ \rho_{2c,c}(H) &= \beta \circ (\mathcal{R}_c \circ_X \rho_{c,0}(H)) \\
&= \beta \circ (\mathcal{R}_c \circ_X v) \circ (\mathcal{R}_c \circ_X \gamma) \\
&= \rho_{c,0}(G) \circ (\mathcal{R}_c \circ_X \gamma) \\
&= \gamma \circ (\mathcal{R}_c \circ_X \rho_{c,0}(H)) = \gamma \circ \rho_{2c,c}(H),
\end{align*}
$$

(3.9)

which proves the lemma. \hfill \Box

In particular, if $F \to G \to H \to [1]$ is an exact triangle, $F$ is $a$-torsion, and $G$ is $b$-torsion, then $H$ is $(a + b)$-torsion.

In our later applications, we mainly focus on the interleaving distance on the derived category $\mathcal{D}(k_{X \times \mathbb{R}_t})$. For $c \in \mathbb{R} \geq 0$, define

$$
\mathcal{R}_c := k_{\Delta_X \times \Delta_c} \in \mathcal{D}(k_{(X \times \mathbb{R})^2}),
$$

(3.10)

where $\Delta_c := \{(t_1, t_2) \mid \|t_1 - t_2\| \leq c\} \subset \mathbb{R}^2$ is the thickened diagonal of $\Delta_{\mathbb{R}} \subset \mathbb{R}^2$. Then the assignment $c \mapsto \mathcal{R}_c$ defines a thickening kernel on $X \times \mathbb{R}_t$. Hence, we can consider the pseudo-distance $d_{\mathcal{R}}$ on $\mathcal{D}(k_{X \times \mathbb{R}_t})$ associated with $\mathcal{R}$, which we denote by $d_{X \times \mathbb{R}_t}$. This is a slight modification of the relative distance $\text{dist}_{X \times \mathbb{R}_t/X}$ studied in [PS23]. Note also that $\mathcal{R}_c \circ F \simeq k_{X \times [-c, c]} \ast F$. With the notation in Subsection 22.3 for $F, F' \in \mathcal{D}(k_{X \times \mathbb{R}_t})$ and $G, G' \in \mathcal{D}(k_{X \times \mathbb{R}_t})$,

$$
d_{X_{13} \times \mathbb{R}_t}(F \bullet G, F' \bullet G') \leq d_{X_{12} \times \mathbb{R}_t}(F, F') + d_{X_{23} \times \mathbb{R}_t}(G, G').
$$

(3.11)
3.2 Tamarkin category

Let $X$ be a manifold without boundary. We let $(t; \tau)$ denote the homogeneous coordinate system on $T^*\mathbb{R}_t$. It is proved by Tamarkin [Tam18] that the functor $P_l := k_{X \times [0, \infty]} \ast (\ast) : D(k_{X \times \mathbb{R}_t}) \to D(k_{X \times \mathbb{R}_t})$ defines a projector onto $^{\perp}D_{\tau \leq 0}(k_{X \times \mathbb{R}_t})$, where $\{ \tau \leq 0 \} = \{(x, t; \xi, \tau) \mid \tau \leq 0 \} \subset T^*(X \times \mathbb{R}_t)$ and $^\perp(\ast)$ denotes the left orthogonal.

**Definition 3.5.** One defines

$$D(X) := ^\perp D_{\{\tau \leq 0\}}(k_{X \times \mathbb{R}_t}),$$

(3.12)

and call it the Tamarkin category of $X$.

For an object $F \in D(k_{X \times \mathbb{R}_t})$, $F \in D(X)$ if and only if $P_l(F) \simeq F$. Note also that $D(X) \subset D_{\{\tau \geq 0\}}(k_{X \times \mathbb{R}_t})$ by Lemma 2.3.

**Definition 3.6.** One defines $d_{D(X)}$ as the restriction of the pseudo-distance $d_{X \times \mathbb{R}_t}$ on $D(k_{X \times \mathbb{R}_t})$ to $D(X)$.

We will describe the pseudo-distance using the translation to the $\mathbb{R}_t$-direction. For $c \in \mathbb{R}$, let $T_c : X \times \mathbb{R}_t \to X \times \mathbb{R}_t, (x, t) \mapsto (x, t + c)$ be the translation map by $c$ to the $\mathbb{R}$-direction. In what follows, we write $T_c$ instead of $T_{c \ast}$ for simplicity. Recall that we have set $\mathcal{R}_c := k_{\Delta X \times \Delta c} \in D(k_{X \times \mathbb{R}_t})$.

**Lemma 3.7.** Let $F \in D_{\{\tau \geq 0\}}(k_{X \times \mathbb{R}_t})$. Then $\mathcal{R}_c \circ F \simeq T_{-c}F$.

**Proof.** First we recall that $\mathcal{R}_c \circ F \simeq k_{X \times [-c, c]} \ast F$. Since there exist an exact triangle $k_{X \times [-c, c]} \to k_{X \times [-c, c]} \to k_{X \times \{-c\}} \to F$ and an isomorphism $T_{-c}F \simeq k_{X \times \{-c\}} \ast F$, it suffices to show that $k_{X \times \{-c\}} \ast F \simeq 0$. By Lemma 5.3, there exists $H \in D(k_X)$ such that $k_{X \times \{-c\}} \ast F \simeq H \otimes k_{\mathbb{R}_t}$. Thus, we obtain

$$k_{X \times \{-c\}} \ast F \simeq k_{X \times [-c, c]} \ast k_{X \times \{-c\}} \ast F$$

$$\simeq k_{X \times \{-c\}} \ast (H \otimes k_{\mathbb{R}_t})$$

$$\simeq H \otimes (k_{(-c, c)} \ast k_{\mathbb{R}_t}) \simeq 0,$$

(3.13)

which completes the proof. \hfill \Box

Let $F \in D_{\{\tau \geq 0\}}(k_{X \times \mathbb{R}_t})$. Then, we have an isomorphism

$$R_m(\bar{q}_1^{-1}k_{X \times [0, \infty]} \otimes \bar{q}_2^{-1}F) \simeq R_m(\bar{q}_1^{-1}k_{X \times \{0\}} \otimes \bar{q}_2^{-1}F) \simeq F.$$

(3.14)

Hence, for $c, d \in \mathbb{R}_{\geq 0}$ with $c \leq d$, the canonical morphism $k_{X \times [c, \infty)} \to k_{X \times [d, \infty)}$ induces a canonical morphism

$$\tau_{c, d}(F) : T_c F \simeq R_m(\bar{q}_1^{-1}k_{X \times [c, \infty)} \otimes \bar{q}_2^{-1}F)$$

$$\to R_m(\bar{q}_1^{-1}k_{X \times [d, \infty)} \otimes \bar{q}_2^{-1}F) \simeq T_d F.$$

(3.15)

By Lemma 3.7, the morphism is identified with

$$T_{c+d}(\rho_{d, c}(F)) : T_{c+d}(\mathcal{R}_d \circ F) \to T_{c+d}(\mathcal{R}_c \circ F).$$

(3.16)

Hence, a pair $(F, G)$ of objects of $D(X)$ is $(a, b)$-isomorphic if and only if there exist morphisms $\alpha, \delta : F \to T_a G$ and $\beta, \gamma : G \to T_b F$ such that
(1) \( F \xrightarrow{\alpha} T_aG \xrightarrow{T_{ab}} T_{a+b}F \) is equal to \( \tau_{0,a+b}(F) : F \to T_{a+b}F \) and

(2) \( G \xrightarrow{\beta} T_bF \xrightarrow{T_{ab}} T_{a+b}G \) is equal to \( \tau_{0,a+b}(G) : G \to T_{a+b}G \).

In this form, we can see that \( d_{\mathcal{D}(X)} \) is similar to the pseudo-distance introduced in \[AI20\] (see Remark 3.8 below).

**Remark 3.8.** The terminology has been changed from that in \[AI20\]. In that paper, “weakly \((a,b)\)-isomorphic” in this paper was called “\((a,b)\)-isomorphic”. Moreover, we defined the notion of “\((a,b)\)-interleaved” as follows: a pair \((F,G)\) of objects of \(\mathcal{D}(X)\) is said to be \((a,b)\)-interleaved if there exist morphisms \(\alpha, \delta : F \to T_aG\) and \(\beta, \gamma : G \to T_bF\) satisfying

(1) \( F \xrightarrow{\alpha} T_aG \xrightarrow{T_{ab}} T_{a+b}F \) is equal to \( \tau_{0,a+b}(F) : F \to T_{a+b}F \) and

(2) \( G \xrightarrow{\beta} T_bF \xrightarrow{T_{ab}} T_{a+b}G \) is equal to \( \tau_{0,a+b}(G) : G \to T_{a+b}G \).

One can see that

\((a,b)\)-isomorphic \(\Rightarrow\) weakly \((a,b)\)-isomorphic \(\Rightarrow\) \((a,b)\)-interleaved.

We also remark that the distance \(d_{\mathcal{D}(X)}\) in \[AI20\] is defined by the relation “\((a,b)\)-interleaved” instead of “\((a,b)\)-isomorphic”, and hence it is different from that in Definition 3.6. Later we will prove the main result in \[AI20\] also holds for the modified \(d_{\mathcal{D}(X)}\) (see Theorem 5.1).

The following proposition is slightly stronger than the similar results in the published version of \[AI20\].

**Proposition 3.9 (cf. \[AI20\], Prop. 4.15).** Let \( I \) be an open interval containing the closed interval \([0,1]\) and \( \mathcal{H} \in \mathcal{D}_{\{\tau \geq 0\}}(k \times \mathbb{R}_x \times I) \). Assume that there exist continuous functions \( f, g : I \to \mathbb{R}_{\geq 0} \) satisfying

\[
SS(\mathcal{H}) \subset T^*X \times \{(t,s;\tau,\sigma) \mid -f(s) \cdot \tau \leq \sigma \leq g(s) \cdot \tau\}. \tag{3.17}
\]

Then \( (\mathcal{H}|_{\mathbb{R}_x \times \{0\}}, \mathcal{H}|_{\mathbb{R}_x \times \{1\}}) \) is weakly \( (\int_0^1 g(s)ds + \varepsilon, \int_0^1 f(s)ds + \varepsilon) \)-isomorphic for any \( \varepsilon \in \mathbb{R}_{>0} \).

**Proof.** The proof is similar to that of \[AI20\], Prop. 4.15. We only need to replace \[AI20\], Lem. 4.14 with Lemma 5.4. \[\square\]

### 3.3 Sheaf quantization of Hamiltonian isotopies

In this subsection, we first recall the existence and uniqueness result of a sheaf quantization of a Hamiltonian isotopy due to Guillermou–Kashiwara–Schapira \[GKS12\].

Let \( M \) be a connected manifold without boundary and \( I \) an open interval of \( \mathbb{R} \) containing the closed interval \([0,1]\). We say that a \( C^\infty \)-function \( H = (H_s)_{s \in I} : T^*M \times I \to \mathbb{R} \) is **timewise compactly supported** if \( \text{supp}(H_s) \) is compact for any \( s \in I \). A compactly supported **Hamiltonian isotopy** is a flow of the Hamiltonian vector field of a timewise compactly supported \( C^\infty \)-function \( H \). In this paper, the isotopy associated with \( H \) is denoted by \( \phi^H = (\phi^H_s)_{s \in I} : T^*M \times I \to T^*M \). Note that \( (\phi^H_s)^{-1} = \phi^{-H}_s \) with \( \overline{H}_s(p) := -H_s(\phi^H_s(p)) \). Moreover, for two timewise compactly supported functions \( H, H' : T^*M \times I \to \mathbb{R} \), we have

\[
\phi^H_s \circ \phi^H_{s'} = \phi^{HsH'}_{s'}, \tag{3.18}
\]
where \((H_s^* H')_s(p) := H_s(p) + H_s'((\phi_s^H)^{-1}(p))\). In particular, for two timewise compactly supported functions \(H, H' : T^* M \times I \to \mathbb{R}\),

\[
(\phi_s^H)^{-1} \circ \phi_s^{H'} = \phi_s^{H \circ H'},
\]

where \((\overline{H}_s^* H')_s(p) = (H' - H)_s(\phi_s^H(p))\).

**Definition 3.10.** Let \(\phi^H = (\phi_s^H)_{s \in I} : T^* M \times I \to T^* M\) be the compactly supported Hamiltonian isotopy associated with a timewise compactly supported function \(H : T^* M \times I \to \mathbb{R}\).

(i) One defines \(\hat{H} : \hat{T}^*(M \times \mathbb{R}_t) \times I \to \mathbb{R}\) by

\[
\hat{H}((x, t; \xi, \tau), s) := \begin{cases} 
\tau H((x; \xi/\tau), s) & (\tau \neq 0) \\
0 & (\tau = 0)
\end{cases}
\]

and \(\hat{\phi} = (\hat{\phi}_s)_{s \in I} : \hat{T}^*(M \times \mathbb{R}_t) \times I \to \hat{T}^*(M \times \mathbb{R}_t)\) to be the homogeneous Hamiltonian flow of \(\hat{H}\).

(ii) One defines a conic Lagrangian submanifold \(\Lambda_{\hat{\phi}}\) of \(\hat{T}^*(M \times \mathbb{R})^2 \times T^* I\) by

\[
\Lambda_{\hat{\phi}} := \left\{ \left( \hat{\phi}((x, t; \xi, \tau), s), (x, t; -\xi, -\tau), (s, -\hat{H}(\hat{\phi}((x, t; \xi, \tau), s), s)) \right) \right\},
\]

For a timewise compactly supported function \(H : T^* M \times I \to \mathbb{R}\), we also define \(u = (u_s)_{s \in I} : T^* M \times I \to \mathbb{R}\) by \(u_s(p) = \int_0^s (H_\tau - \alpha(X_p))((\phi_s^H(p))ds', \) where \((X_s)_{s \in I}\) is the Hamiltonian vector field for \(H\). Then \(\hat{\phi}\) can be written as

\[
\hat{\phi}_s(x, t; \xi, \tau) = (x', t + u_s(x; \xi/\tau); \xi', \tau),
\]

where \((x', \xi'/\tau) = \hat{\phi}_s(x; \xi/\tau)\) for \(\tau \neq 0\), and \(\hat{\phi}_s(x, t; \xi, 0) = (x, t; \xi, 0)\). Hereafter, we use the convention that \(\tau H_\tau((\phi_s^H(x; \xi/\tau)) = 0\) and \(u_s(x; \xi/\tau) = 0\) when \(\tau = 0\). Moreover, we write \((x', \xi'/\tau) = \phi_s(x; \xi/\tau)\) also for \(\tau = 0\), in which case it is understood that \((x', \xi') = (x; \xi)\).

We have \(du_s = \alpha - (\phi_s)^* \alpha\), which gives the following properties.

**Lemma 3.11.** If \(\phi_1 = \text{id}_{T^* M}\), then \(u_1 \equiv 0\).

The main theorem of [GKS12] is the following.

**Theorem 3.12** ([GKS12, Thm. 3.7]). Let \(\phi : T^* M \times I \to T^* M\) be a compactly supported Hamiltonian isotopy and \(H : T^* M \times I \to \mathbb{R}\) be a \(C^\infty\)-function with Hamiltonian flow \(\phi\). Then there exists a unique simple object \(\tilde{K}^H \in \mathcal{D}(k_{(M \times \mathbb{R})^2 \times I})\) such that \(\text{SS}(\tilde{K}^H) = \Lambda_{\hat{\phi}}\) and \(\hat{K}^H\mid_{(M \times \mathbb{R})^2 \times \{0\}} \simeq k_{\Lambda_{\hat{\phi}} \times \mathbb{R}}\).

Set \(\tilde{K}^H_s := \tilde{K}^H\mid_{(M \times \mathbb{R})^2 \times \{s\}} \in \mathcal{D}(k_{(M \times \mathbb{R})^2})\). Note that \(\text{SS}(\tilde{K}^H_s) \subset \Lambda_{\hat{\phi}_s} \circ T^* I\). We also have

\[
\tilde{K}^H_s \circ \tilde{K}^H_{s'} \simeq \tilde{K}^H_{s + s'}.
\]

It is also proved by Guillermou–Schapira [GS14, Prop. 4.29] that the composition with \(\tilde{K}^H_s\) defines a functor

\[
\tilde{K}^H_s \circ (*): \mathcal{D}(M) \to \mathcal{D}(M).
\]
Define \( q: (M \times \mathbb{R})^2 \times I \to M^2 \times \mathbb{R} \times I, (x_1, t_1, x_2, t_2, s) \mapsto (x_1, x_2, t_1 - t_2, s) \) and set
\[
\Lambda_H := q_{1,0}(\Lambda_{\hat{q}}) = \{(x', \xi'), (x, -\xi), (u_s(x; \xi/\tau); \tau), (s; -\tau H_s(\phi_{s}(x; \xi/\tau))) \mid (x; \xi) \in T^n M, s \in I, \tau \in \mathbb{R}, (x'; \xi'/\tau) = \phi_{s}(x; \xi/\tau) \}
\]
\[
\subset \hat{\tau}^*(M^2 \times \mathbb{R} \times I).
\]

Then the inverse image functor \( q^{-1} \) gives an equivalence (see \cite[Cor. 2.3.2]{Gru23})
\[
\{ K \in D(k_{M^2 \times \mathbb{R} \times I}) \mid SS(K) = \Lambda_H \} \xrightarrow{\sim} \{ \tilde{K} \in D(k_{(M \times \mathbb{R})^2 \times I}) \mid SS(\tilde{K}) = \Lambda_{\hat{q}} \}. \tag{3.26}
\]

Recall that we have a projector \( P_l: D(k_{M^2 \times \mathbb{R}^l}) \to \mathcal{D}_{I^\tau \leq 0}(k_{M^2 \times \mathbb{R}^l}) = D(M^2) \) and similarly for \( D(M^2 \times I) \).

**Definition 3.13.** Let \( H: T^n M \times I \to \mathbb{R} \) be a timewise compactly supported function.

(i) One defines \( K^H \in D(k_{M^2 \times \mathbb{R} \times I}) \) to be the object such that \( SS(K^H) = \Lambda_H \) and \( K^H|_{M^2 \times \mathbb{R} \times \{0\}} \simeq k_{\Delta M \times \{0\}} \), that is, the object \( K^H \) satisfying \( q^{-1}K^H \simeq \tilde{K}^H \). One also sets \( \hat{K}^H_s := K^H|_{M^2 \times \mathbb{R} \times \{s\}} \) for simplicity.

(ii) One defines \( \mathcal{K}^H_s := P_l(K^H_s) \in D(M^2) \) and \( \mathcal{K}^H := P_l(K^H) \in D(M^2 \times I) \).

Note that \( \hat{K}^H_s \circ F \simeq K^H_s \bullet F \) for any \( F \in D(M) \) and \( s \in I \). By the associativity, for any \( F \in D(M) \) and \( s \in I \), we have
\[
K^H_s \bullet F \simeq (K^H_s \bullet (k_{M \times [0, \infty)} \bullet F) \simeq (K^H_s \bullet k_{M \times [0, \infty)}) \bullet F \simeq \mathcal{K}^H_s \bullet F. \tag{3.27}
\]

## 4 Completeness of derived category of sheaves

In this section, we prove the completeness of the derived category \( D(k_X) \) with respect to the pseudo-distance \( d_\xi \) associated with a thickening kernel \( \xi \). If a category with a persistence structure admits any sequential colimit, then the category is complete with respect to the interleaving distance (Cruz \cite{Cru19} and Scoccola \cite{Sco20}). However, the derived category does not admit sequential colimits. Hence, we construct a limit object by using a homotopy colimit instead. Let \( X \) be a manifold throughout this section.

In Lemma \cite{L4}, we saw that for an exact triangle \( F \to G \to H \xrightarrow{+1} \), if \( H \) is \( c \)-torsion, then \( (F, G) \) is weakly \((0, c)\)-isomorphic. Conversely, we obtain Proposition \cite{L2} below, which is a key to our construction of limit objects.

**Lemma 4.1.** Let \( F \in D(k_X) \) and \( \alpha \in \mathbb{R}_{\geq 0} \) and consider the exact triangle
\[
\xi_\alpha \circ F \xrightarrow{\rho_{a,0}(F)} F \to \text{Cone}(\rho_{a,0}(F)) \xrightarrow{+1}. \tag{4.1}
\]
Then \( \text{Cone}(\rho_{a,0}(F)) \) is \( 2a \)-torsion.
Proof. Set $C := \text{Cone}(\rho_{a,0})$ and consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
\mathcal{R}_{3a} \circ F & \longrightarrow & \mathcal{R}_{2a} \circ F & \longrightarrow & \mathcal{R}_{2a} \circ C & \longrightarrow & \mathcal{R}_{3a} \circ F[1] & \longrightarrow & \mathcal{R}_{2a} \circ F[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_a \circ F & \longrightarrow & \mathcal{R}_a \circ C & \longrightarrow & \mathcal{R}_{2a} \circ F[1] & \longrightarrow & \mathcal{R}_a \circ F[1]. \\
\end{array}
\] (4.2)

The composite morphism $\mathcal{R}_{2a} \circ C \to \mathcal{R}_{3a} \circ F[1] \to \mathcal{R}_{2a} \circ F[1]$ is zero since $\mathcal{R}_{3a} \circ F \to \mathcal{R}_{2a} \circ F \to \mathcal{R}_{2a} \circ C \to \mathcal{R}_{2a} \circ F[1]$ is also zero. By the commutativity, the composite $\mathcal{R}_{2a} \circ C \to \mathcal{R}_a \circ C \to \mathcal{R}_{2a} \circ F[1]$ is also zero. Hence, there exists a morphism $\mathcal{R}_{2a} \circ C \to \mathcal{R}_a \circ F$ that makes the lower triangle commutative. Therefore, the morphism $\mathcal{R}_{2a} \circ C \to C$ factors the composite $\mathcal{R}_a \circ F \to F \to C$ and hence it is zero. \hfill \Box

**Proposition 4.2.** Let $F, G \in \mathcal{D}(k_X)$ and assume that the pair $(F, G)$ is $(a, b)$-isomorphic. Let $(\alpha : \mathcal{R}_a \circ F \to G, \beta : \mathcal{R}_b \circ G \to F)$ be an $(a, b)$-isomorphism for $(F, G)$ and consider the exact triangle

\[
\mathcal{R}_a \circ F \to G \to \text{Cone}(\alpha) \xrightarrow{+1}.
\] (4.3)

Then $\text{Cone}(\alpha)$ is $3(a + b)$-torsion.

*Proof.* Consider the three exact triangles:

\[
\begin{array}{ccccccccc}
\mathcal{R}_{a+b} \circ F & \overset{\mathcal{R}_{a+b}(F)}{\longrightarrow} & \mathcal{R}_b \circ G & \longrightarrow & \text{Cone}(\alpha) & \overset{u}{\longrightarrow} & \mathcal{R}_{a+b} \circ F[1], \\
\mathcal{R}_{a+b} \circ F & \overset{\mathcal{R}_{a+b}(F)}{\longrightarrow} & F & \longrightarrow & \text{Cone}(\rho_{a+b,0}(F)) & \longrightarrow & \mathcal{R}_{a+b} \circ F[1], \\
\mathcal{R}_b \circ G & \overset{\beta}{\longrightarrow} & F & \longrightarrow & \text{Cone}(\beta) & \longrightarrow & \mathcal{R}_b \circ G[1].
\end{array}
\] (4.4)

Note that $\beta \circ (\mathcal{R}_b \circ \alpha) = \rho_{a+b,0}(F)$. By the octahedral axiom, we have the following commutative diagram, where the bottom row is also an exact triangle:

\[
\begin{array}{ccccccccc}
\mathcal{R}_{a+b} \circ F & \overset{\mathcal{R}_{a+b}(F)}{\longrightarrow} & \mathcal{R}_b \circ G & \longrightarrow & \text{Cone}(\alpha) & \overset{u}{\longrightarrow} & \mathcal{R}_{a+b} \circ F[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_{a+b} \circ F & \overset{\mathcal{R}_{a+b}(F)}{\longrightarrow} & F & \longrightarrow & \text{Cone}(\rho_{a+b,0}(F)) & \longrightarrow & \mathcal{R}_{a+b} \circ F[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_b \circ G & \overset{\beta}{\longrightarrow} & F & \longrightarrow & \text{Cone}(\beta) & \longrightarrow & \mathcal{R}_b \circ G[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Cone}(\alpha) & \longrightarrow & \text{Cone}(\rho_{a+b,0}(F)) & \longrightarrow & \text{Cone}(\beta) & \longrightarrow & \text{Cone}(\alpha)[1].
\end{array}
\] (4.5)

In particular, the morphism $u : \mathcal{R}_b \circ \text{Cone}(\alpha) \to \mathcal{R}_{a+b} \circ F[1]$ factors through $\text{Cone}(\rho_{a+b,0}(F))$. Then the commutative diagram

\[
\begin{array}{ccccccccc}
\mathcal{R}_{2a+3b} \circ \text{Cone}(\alpha) & \overset{\mathcal{R}_{2a+3b}(\alpha)}{\longrightarrow} & \mathcal{R}_b \circ \text{Cone}(\alpha) & \overset{u}{\longrightarrow} & \mathcal{R}_{a+b} \circ F[1] \\
\mathcal{R}_{2a+3b} \circ \text{Cone}(\alpha) & \overset{\mathcal{R}_{2a+3b}(\alpha)}{\longrightarrow} & \mathcal{R}_b \circ \text{Cone}(\alpha) & \overset{u}{\longrightarrow} & \mathcal{R}_{a+b} \circ F[1] \\
\mathcal{R}_{2a+2b} \circ \text{Cone}(\rho_{a+b,0}(F)) & \longrightarrow & \text{Cone}(\rho_{a+b,0}(F))
\end{array}
\] (4.6)
proves that the composite morphism in the first row is zero by Lemma 4.1. Thus, we obtain a morphism $\mathcal{R}_{3a+3b} \circ \text{Cone}(\alpha) \to \mathcal{R}_{3a+b} \circ G$ that makes the following diagram commute, where the vertical arrows are the corresponding $\rho$'s:

$$
\begin{array}{cccc}
\mathcal{R}_{3a+3b} \circ \text{Cone}(\alpha) & \xrightarrow{\beta} & \mathcal{R}_{3a+b} \circ \text{Cone}(\alpha) & \xrightarrow{\mathcal{R}_{k \alpha} \circ \beta} & \mathcal{R}_{2a+b} \circ \text{Cone}(\alpha) \\
\mathcal{R}_{a+b} \circ G & \xrightarrow{\mathcal{R}_{a} \circ \beta} & \mathcal{R}_{a+b} \circ \text{Cone}(\alpha) & \xrightarrow{\mathcal{R}_{k \alpha} \circ \beta} & \mathcal{R}_{a+b} \circ \text{Cone} \circ \text{Cone}(\alpha) \\
\mathcal{R}_{a} \circ F & \xrightarrow{\alpha} & G & \xrightarrow{\text{Cone}(\alpha)} & \mathcal{R}_{a} \circ \text{Cone} \circ \text{Cone}(\alpha).
\end{array}
$$

(4.7)

Hence, the morphism $\mathcal{R}_{3a+3b,0}(\text{Cone}(\alpha))$ factors the composite morphism $\mathcal{R}_{a} \circ F \to G \to \text{Cone}(\alpha)$ and thus it is zero. □

**Theorem 4.3.** Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of objects in $\mathcal{D}(k_X)$ and assume that $F_n$ and $F_{n+1}$ are $\alpha_n$-isomorphic with $\sum_n \alpha_n < \infty$. Set $a_{\geq n} := \sum_{k \geq n} \alpha_k$. Then there exists an object $F_\infty \in \mathcal{D}(k_X)$ such that $(F_n, F_\infty)$ is $(2a_{\geq n}, 24a_{\geq n})$-isomorphic for any $n \in \mathbb{N}$. In particular, $d_{\mathcal{A}}(F_n, F_\infty) \to 0$ ($n \to \infty$).

**Proof.** Set $G_n := \mathcal{R}_{a_{\geq n}} \circ F_n$. Then we have a morphism $\alpha_{n,m} : G_n \to G_m$ for $n \leq m$ and get an inductive system $(G_n)_{n \in \mathbb{N}}$. Let $F_\infty := \text{hocolim}_n G_n \in \mathcal{D}(k_X)$.

We fix $n$ and consider the cone $C_{n,m}$ of the morphism $\alpha_{n,m} : G_n \to G_m$. By composing $\beta_m$'s, we obtain $\beta_{m,n} : G_m \to G_n$. Since $(\alpha_{n,m}, \beta_{m,n})$ gives an $a_{\geq n}$-isomorphism between $G_n$ and $G_m$, the cone $C_{n,m}$ is $6a_{\geq n}$-torsion by Proposition 4.2. Consider the following commutative diagram with solid arrows:

$$
\begin{array}{cccc}
G_n^{\oplus \mathbb{N}} & \xrightarrow{} & \bigoplus_{m \geq n} G_m & \xrightarrow{} & \bigoplus_{m \geq n} C_{n,m} & \xrightarrow{} & G_n^{\oplus \mathbb{N}}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_n^{\oplus \mathbb{N}} & \xrightarrow{} & \bigoplus_{m \geq n} G_m & \xrightarrow{} & \bigoplus_{m \geq n} C_{n,m} & \xrightarrow{} & G_n^{\oplus \mathbb{N}}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{hocolim}_m G_n & \xrightarrow{} & \text{hocolim}_{m \geq n} G_m & \xrightarrow{} & H & \xrightarrow{} & \text{hocolim}_m G_n[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_n^{\oplus \mathbb{N}}[1] & \xrightarrow{} & \bigoplus_{m \geq n} G_m[1] & \xrightarrow{} & \bigoplus_{m \geq n} C_{n,m}[1] & \xrightarrow{} & G_n^{\oplus \mathbb{N}}[2].
\end{array}
$$

(4.8)

Then by [KS06, Exercise 10.6], the dotted arrows can be completed so that the right bottom square is anti-commutative, all the other squares are commutative, and all the rows and all the columns are exact triangles. Since $\bigoplus_{m \geq n} C_{n,m}$ is $6a_{\geq n}$-torsion, $H$ is $12a_{\geq n}$-torsion. Noticing that $\text{hocolim}_m G_n \simeq G_n$ and $\text{hocolim}_{m \geq n} G_m \simeq F_\infty$, by Lemma 3.4 $(G_n, F_\infty)$ is weakly $(0, 12a_{\geq n})$-isomorphic. Since we set $G_n = \mathcal{R}_{a_{\geq n}} \circ F_n$, we find that $(F_n, F_\infty)$ is weakly $(a_{\geq n}, 12a_{\geq n})$-isomorphic, which implies that $(F_n, F_\infty)$ is $(2a_{\geq n}, 24a_{\geq n})$-isomorphic (see Remark 3.3). □

**Remark 4.4.** One can prove a similar completeness result in a more general setting, i.e., for a triangulated category with a persistence structure (cf. persistence triangulated category by Biran–Cornea–Zhang [BCZ2]). Here we do not go into details.
Corollary 4.5. The derived category of sheaves $\mathcal{D}(k_X)$ is complete with respect to the pseudo-distance $d_R$. In particular, the Tamarkin category $\mathcal{D}(X)$ is complete with respect to the pseudo-distance $d_{\mathcal{D}(X)}$.

Proof. For the latter claim, it suffices to show that for a Cauchy sequence $(F_n)_{n\in\mathbb{N}}$ in the Tamarkin category $\mathcal{D}(X)$, the limit object $F_\infty$ constructed in Theorem 4.3 is also in $\mathcal{D}(X)$. By construction and Lemma 3.7 after taking a subsequence, we have $F_\infty = \text{holim}_m T_{a\geq n} F_n$, where $(a\geq n)_{n\in\mathbb{N}}$ is as in Theorem 4.3. Set $G_n := T_{a\geq n} F_n$. Since the functor $k_{\times [0,\infty]} \ast (\ast)$ is defined as the composite of left adjoint functors, it commutes with direct sums. Since each $G_n$ is an object of the left orthogonal $\perp_{\mathcal{D}(X \times \mathbb{R}_1)} (k_{\times \mathbb{R}_1}) = \mathcal{D}(X)$, in the following commutative diagram the first and the second vertical arrows are isomorphisms:

$$
\begin{array}{ccc}
\mathcal{D}(X \times [0,\infty]) \ast \bigoplus_n G_n & \longrightarrow & \mathcal{D}(X \times [0,\infty]) \ast \bigoplus_n G_n \\
& & \downarrow \\
\bigoplus_n G_n & \longrightarrow & \bigoplus_n G_n
\end{array}
\xrightarrow{1} 
\begin{array}{ccc}
\mathcal{D}(X \times [0,\infty]) \ast \bigoplus_n G_n & \longrightarrow & \mathcal{D}(X \times [0,\infty]) \ast F_\infty \\
\downarrow & & \downarrow \\
\bigoplus_n G_n & \longrightarrow & F_\infty
\end{array}
\xrightarrow{1}.
$$

(4.9)

Thus, we have an isomorphism $k_{\times [0,\infty]} \ast F_\infty \xrightarrow{\sim} F_\infty$ and find that $F_\infty \in \mathcal{D}(X)$. □

Remark 4.6. As mentioned in Remark 3.3, $d_R$ and $\text{dist}_R$ are equivalent. Hence $\mathcal{D}(k_X)$ is also complete with respect to $\text{dist}_R$.

5 Sheaf quantization of Hamiltonian diffeomorphisms and homeomorphisms

In this section, we give a refined Hamiltonian stability result, which state the distance between sheaf quantizations of Hamiltonian diffeomorphisms is at most the Hofer distance. By using the stability result, we also construct a sheaf quantization of a Hamiltonian homeomorphism.

5.1 Hamiltonian stability theorem

In this subsection, we give a generalization of our previous result [AI20, Thm. 4.16].

For a timewise compactly supported function $H: T^* M \times I \rightarrow \mathbb{R}$, we define

$$
E_+(H) := \int_0^1 \max_{p \in T^* M} H_s(p) ds,
\quad
E_-(H) := -\int_0^1 \min_{p \in T^* M} H_s(p) ds,
\quad
\|H\|_{\text{osc}} := E_+(H) + E_-(H) = \int_0^1 \left( \max_{p \in T^* M} H_s(p) - \min_{p \in T^* M} H_s(p) \right) ds.
$$

(5.1)

When $M = pt$, we need to modify $E_+(H)$ and $E_-(H)$ so that they are non-negative.

**Theorem 5.1.** Let $H: T^* M \times I \rightarrow \mathbb{R}$ be a timewise compactly supported function. Then $(K_0^H, K_1^H)$ is $(E_-(H) + \epsilon, E_+(H) + \epsilon)$-isomorphic for any $\epsilon \in \mathbb{R}_{>0}$, where $0$ denotes the zero function on $T^* M \times I$. In particular, $d_{\mathcal{D}(M^2)}(K_0^H, K_1^H) \leq \|H\|_{\text{osc}}$.

**Proof.** Let $\epsilon > 0$ be an arbitrary positive number. We apply Proposition 4.4 to $K^H$. Then by (3.22), we find that $(K_0^H, K_1^H) = (K_0^H, K_1^H)$ is weakly $(E_-(H) + \epsilon, E_+(H) + \epsilon)$-isomorphic. It is enough to show that $(K_0^H, K_1^H)$ is $(E_-(H) + \epsilon, E_+(H) + \epsilon)$-isomorphic.

We set $a := E_-(H) + \epsilon$ and $b := E_+(H) + \epsilon$. Then, by definition, there exist morphisms $\alpha, \delta: K_0^H \rightarrow T_a K_1^H$ and $\beta, \gamma: K_1^H \rightarrow T_b K_1^H$ such that
(1) \( \mathcal{K}_1^0 \xrightarrow{\alpha} T_d \mathcal{K}_1^H \xrightarrow{T_a \beta} T_{a+b} \mathcal{K}_1^0 \) is equal to \( \tau_{0,a+b}(\mathcal{K}_1^0) : \mathcal{K}_1^0 \rightarrow T_{a+b} \mathcal{K}_1^0 \),

(2) \( \mathcal{K}_1^H \xrightarrow{\tau} T_b \mathcal{K}_1^0 \xrightarrow{T_b \delta} T_{a+b} \mathcal{K}_1^H \) is equal to \( \tau_{0,a+b}(\mathcal{K}_1^H) : \mathcal{K}_1^H \rightarrow T_{a+b} \mathcal{K}_1^H \), and

(3) \( \tau_{a,2a}(\mathcal{K}_1^H) \circ \alpha = \tau_{a,2a}(\mathcal{K}_1^H) \circ \delta \) and \( \tau_{b,2b}(\mathcal{K}_1^0) \circ \beta = \tau_{b,2b}(\mathcal{K}_1^0) \circ \gamma \).

Now we let Tor be the full triangulated subcategory of \( \mathcal{D}(M^2) \) consisting of torsion objects \( \{ F \mid d_{\mathcal{D}(M^2)}(F, 0) < \infty \} \). Then, by [GS14, Prop. 6.7], the Hom set of the localized category \( \mathcal{D}(M^2)/\operatorname{Tor} \) is computed as

\[
\operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(F, G) \simeq \lim_{c \to \infty} \operatorname{Hom}_{\mathcal{D}(M^2)}(F, T_c G).
\]

For the objects \( \mathcal{K}_1^0, \mathcal{K}_1^H \) and \( d \in \mathbb{R} \), we have

\[
\operatorname{Hom}_{\mathcal{D}(M^2)}(\mathcal{K}_1^H, T_d \mathcal{K}_1^0) \simeq \begin{cases} k & (d \geq 0) \\ 0 & (d < 0). \end{cases}
\]

Hence, \( \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^H, \mathcal{K}_1^0) \simeq \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^H, \mathcal{K}_1^H) \simeq k \) and the canonical morphism

\[
\operatorname{Hom}_{\mathcal{D}(M^2)}(\mathcal{K}_1^H, T_d \mathcal{K}_1^0) \rightarrow \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^H, \mathcal{K}_1^H), \alpha' \mapsto \alpha'
\]

is injective for \( d \geq 0 \).

By the condition (3), we have

\[
\overline{\alpha} = \delta \in \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^0, \mathcal{K}_1^H)
\]

(5.5)

\[
\overline{\beta} = \gamma \in \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^H, \mathcal{K}_1^0).
\]

(5.6)

Hence, through the isomorphism \( \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^H, \mathcal{K}_1^0) \simeq k \), we get

\[
1 = \delta \circ \gamma = \delta \circ \overline{\beta} \in \operatorname{Hom}_{\mathcal{D}(M^2)/\operatorname{Tor}}(\mathcal{K}_1^H, \mathcal{K}_1^H),
\]

(5.7)

by the condition (2). Since \( T_b \alpha \circ \beta \in \operatorname{Hom}_{\mathcal{D}(M^2)}(\mathcal{K}_1^H, T_{a+b} \mathcal{K}_1^H) \) is sent to \( \overline{\alpha} \circ \overline{\beta} \) and \( \tau_{0,a+b}(\mathcal{K}_1^H) \) is sent to 1, by the injectivity we obtain \( T_b \alpha \circ \beta = \tau_{0,a+b}(\mathcal{K}_1^H) \). Thus, combining this with the condition (1), we find that the pair \( (\alpha, \beta) \) gives an \((a, b)\)-isomorphism for the pair \( (\mathcal{K}_1^0, \mathcal{K}_1^H) \), which completes the proof.

For two timewise compactly supported functions \( H, H' : T^* M \times I \rightarrow \mathbb{R} \), by (3.11) and (3.23), we have

\[
d_{\mathcal{D}(M^2)}(\mathcal{K}_1^H, \mathcal{K}_1^{H'}) = d_{\mathcal{D}(M^2)}(\mathcal{K}_1^0, \mathcal{K}_1^{H'}) \leq \| H - H' \|_{\text{osc}}.
\]

(5.8)

### 5.2 Sheaf quantization of Hamiltonian diffeomorphisms

In this subsection, we investigate sheaf quantization of Hamiltonian diffeomorphisms. We keep the symbols \( M \) for a connected manifold without boundary and \( I \) for an open interval containing \([0, 1]\). First, we prove the following, an analogue of Theorem 5.1 in the derived category \( \mathcal{D}(k M^2 \times \mathbb{R}_u) \) not in the Tamarkin category \( \mathcal{D}(M^2) \).

**Proposition 5.2.** Let \( H : T^* M \times I \rightarrow \mathbb{R} \) be a timewise compactly supported function. Then, one has an inequality

\[
d_{\mathcal{M}^2 \times \mathbb{R}_u}(\mathcal{K}_1^0, \mathcal{K}_1^H) \leq 4 \int_0^1 \| H_s \|_{\text{osc}} ds \leq 4 \| H \|_{\text{osc}}.
\]

(5.9)
Similarly to (5.8), for two timewise compactly supported functions $H, H': T^*M \times I \to \mathbb{R}$, we have

$$d_{M^2 \times \mathbb{R}_+}(K^H, K^{H'}) \leq 4\|H - H'\|_{\text{osc}}.$$  \hfill (5.10)

To prove the proposition, we prepare some lemmas. Let $X$ be a manifold and let $q: X \times \mathbb{R}_+ \to \mathbb{R}_+$ denote the projection.

**Lemma 5.3.** Let $F \in D_{\{\tau \leq 0\}}(k_X \times \mathbb{R}_+)$ and $G \in D_{\{\tau \geq 0\}}(k_X \times \mathbb{R}_+)$. Then there exist $H, H' \in D(k_X \times \mathbb{R}_+)$ such that $F \star G \simeq H \boxtimes k_{\mathbb{R}_+}$ and $\mathcal{H}om^*(F, G) \simeq H' \boxtimes k_{\mathbb{R}_+}$.

**Proof.** By Lemma 2.8, we find that $SS(F \star G) \subset \{\tau = 0\}$ and $SS(\mathcal{H}om^*(F, G)) \subset \{\tau = 0\}$, which imply the result. \hfill □

For $a, b \in \mathbb{R}_+$, we set

$$D(a, b) := \bigcup_{-a \leq c \leq b} \{ (\tau, \sigma) \in \mathbb{R}^2 \mid \sigma = c \cdot \tau \}. \hfill (5.11)$$

Note that $D(a, b)$ is a closed cone in $\mathbb{R}^2$, which is not necessarily convex. See Figure 5.1. We set $D^+(a, b) := D(a, b) \cap \{ (\tau, \sigma) \mid \tau \geq 0 \}$ and $D^-(a, b) := D(a, b) \cap \{ (\tau, \sigma) \mid \tau \leq 0 \}$.

![Figure 5.1: $D(a, b)$](image-url)

**Lemma 5.4.** Let $F \to G \to H \xrightarrow{\cong} H'$ be an exact triangle in $D(k_X \times \mathbb{R}_+)$ and $a, b \in \mathbb{R}_+$. Assume

1. $F \in D_{\{\tau \leq 0\}}(k_X \times \mathbb{R}_+)$ and $F$ is $a$-torsion,
2. $G \in D_{\{\tau \geq 0\}}(k_X \times \mathbb{R}_+)$ and $G$ is $b$-torsion.

Then, $\mathcal{R}Hom(F, G) \simeq 0$. In particular, $H \simeq G \oplus F[1]$ is max($a$, $b$)-torsion.

**Proof.** By Lemma 5.3, there exists $H' \in D(k_X)$ such that $\mathcal{H}om^*(F, G) \simeq H' \boxtimes k_{\mathbb{R}_+}$. Moreover by the isomorphism $T_b \mathcal{H}om^*(F, G) \simeq \mathcal{H}om^*(F, T_b G)$, we find that $\mathcal{H}om^*(F, G)$ is $b$-torsion. Hence we have $\mathcal{H}om^*(F, G) \simeq 0$ and

$$\mathcal{R}Hom(F, G) \simeq \mathcal{R}Hom(k_X \times \{0\} \star F, G) \simeq \mathcal{R}Hom(k_X \times \{0\}, \mathcal{H}om^*(F, G)) \simeq 0,$$ \hfill (5.12)

which proves the result. \hfill □

Next, we give a microlocal cut-off result, which we use to reduce the problem to Proposition 3.9.
Lemma 5.5. Let $\mathcal{H} \in \text{D}(k_{X \times \mathbb{R} \times \mathbb{R}})$ and assume that there exist $a, b \in \mathbb{R}_{\geq 0}$ such that

$$\text{SS}(\mathcal{H}) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D(a, b).$$

(5.13)

Then there exists an exact triangle $\mathcal{H}^- \to \mathcal{H}^+ \to \mathcal{H} \xrightarrow{+1} \in \text{D}(k_{X \times \mathbb{R} \times \mathbb{R}})$ such that $\text{SS}(\mathcal{H}^-) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D^-(a, b)$ and $\text{SS}(\mathcal{H}^+) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D^+(a, b)$.

Proof. Let $\lambda := \{(t, s) \mid t \geq 0, s = 0\}$. Then we get an exact triangle

$$k_{X \times (\lambda \setminus \{0\})} \ast \mathcal{H} \longrightarrow k_{X \times \lambda} \ast \mathcal{H} \longrightarrow \mathcal{H} \longrightarrow +1$$

(5.14)

with $\text{SS}(k_{X \times \lambda} \ast \mathcal{H}) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D(a, b) \cap \{(\tau, \sigma) \mid \tau \geq 0\}$. Moreover $k_{X \times \lambda} \ast F \to F$ is an isomorphism on $T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times \text{Int}(\lambda)$, where $\lambda$ denotes the polar cone of $\lambda$. Thus, we conclude that $\text{SS}(k_{\lambda \setminus \{0\}} \ast \mathcal{H}) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times (D(a, b) \setminus \text{Int}(\lambda^0)) = T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D^-(a, b)$.

The following is a variant of [Al20, Prop. 4.3].

Proposition 5.6. Let $\mathcal{H} \in \text{D}(k_{X \times \mathbb{R} \times \mathbb{R}^1})$ and $s_1 < s_2 \in I$. Assume that there exist $a, b \in \mathbb{R}_{\geq 0}$ and $r \in \mathbb{R}_{>0}$ such that

$$\text{SS}(\mathcal{H}) \cap \pi^{-1}(X \times \mathbb{R}_t \times (s_1 - r, s_2 + r)) \subset T^* X \times (\mathbb{R}_t \times I) \times D(a, b).$$

(5.15)

(i) Let $q: X \times \mathbb{R}_t \times I \to X \times \mathbb{R}_t$ be the projection. Then $Rq_* \mathcal{H}_{X \times \mathbb{R} \times [s_1, s_2]}$ and $Rq_* \mathcal{H}_{X \times \mathbb{R} \times [s_1, s_2]}$ are $(\text{max}(a, b)(s_2 - s_1) + \varepsilon)$-torsion for any $\varepsilon \in \mathbb{R}_{>0}$.

(ii) One has $d_{X \times \mathbb{R}_t}(\mathcal{H}|_{X \times \mathbb{R} \times \{s_1\}}, \mathcal{H}|_{X \times \mathbb{R} \times \{s_2\}}) \leq 4 \text{max}(a, b)(s_2 - s_1)$.

Proof. (i) Choose a diffeomorphism $\varphi: (s_1 - r, s_2 + r) \xrightarrow{\sim} \mathbb{R}$ satisfying $\varphi|_{[s_1, s_2]} = \text{id}_{[s_1, s_2]}$ and $d\varphi(s) \geq 1$ for any $s \in (s_1 - r, s_2 + r)$. Set $\Phi := \text{id}_{X \times \mathbb{R}_t} \times \varphi: X \times \mathbb{R}_t \times (s_1 - r, s_2 + r) \xrightarrow{\sim} X \times \mathbb{R}_t \times \mathbb{R}$ and $\mathcal{H}' := \Phi_* \mathcal{H} \in \text{D}(k_{X \times \mathbb{R} \times \mathbb{R}})$. Then by the assumption on $\varphi$, we have

$$\text{SS}(\mathcal{H}') \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D(a, b)$$

and $\mathcal{H}'|_{X \times \mathbb{R} \times [s_1, s_2]} \simeq \mathcal{H}|_{X \times \mathbb{R} \times [s_1, s_2]}$. Hence, we may assume $I = \mathbb{R}$ from the beginning.

Applying Lemma 5.5, we have an exact triangle $\mathcal{H}^- \to \mathcal{H}^+ \to \mathcal{H} \xrightarrow{+1} \in \text{D}(k_{X \times \mathbb{R} \times \mathbb{R}})$ with $\text{SS}(\mathcal{H}^-) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D^-((a, b))$ and $\text{SS}(\mathcal{H}^+) \subset T^* X \times (\mathbb{R}_t \times \mathbb{R}) \times D^+(a, b)$.

By [Al20, Prop. 4.3], $Rq_* \mathcal{H}^+_{X \times \mathbb{R}_t \times [s_1, s_2]}$ is $(b(s_2 - s_1) + \varepsilon)$-torsion. Similarly we find that $Rq_* \mathcal{H}^-_{X \times \mathbb{R}_t \times (s_1, s_2)}$ is $(a(s_2 - s_1) + \varepsilon)$-torsion. Here we have an exact triangle

$$Rq_* \mathcal{H}^-_{X \times \mathbb{R}_t \times (s_1, s_2)} \longrightarrow Rq_* \mathcal{H}^+_{X \times \mathbb{R}_t \times (s_1, s_2)} \longrightarrow Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times [s_1, s_2]} \longrightarrow +1$$

(5.17)

with $Rq_* \mathcal{H}^+_{X \times \mathbb{R}_t \times (s_1, s_2)} \in D_{\{\tau \geq 0\}}(k_{X \times \mathbb{R}_t})$ and $Rq_* \mathcal{H}^-_{X \times \mathbb{R}_t \times (s_1, s_2)} \in D_{\{\tau \leq 0\}}(k_{X \times \mathbb{R}_t})$. Hence, by Lemma 5.4 we find that $Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times (s_1, s_2)}$ is $(\text{max}(a, b)(s_2 - s_1) + \varepsilon)$-torsion. The proof for the other case is similar.

(ii) We have the following two exact triangles

$$Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times [s_1, s_2]} \longrightarrow Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times (s_1, s_2)} \longrightarrow \mathcal{H}|_{X \times \mathbb{R} \times \{s_1\}} \xrightarrow{+1},$$

(5.18)

$$Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times [s_1, s_2]} \longrightarrow Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times (s_1, s_2)} \longrightarrow \mathcal{H}|_{X \times \mathbb{R} \times \{s_2\}} \xrightarrow{+1}.$$ (5.19)

Hence, by the result of (i) and Lemma 5.3, the two pairs $(Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times [s_1, s_2]}, \mathcal{H}|_{X \times \mathbb{R} \times \{s_1\}})$ and $(Rq_* \mathcal{H}_{X \times \mathbb{R}_t \times [s_1, s_2]}, \mathcal{H}|_{X \times \mathbb{R} \times \{s_2\}})$ are weakly $(0, (\text{max}(a, b)(s_2 - s_1) + \varepsilon))$-isomorphic. Hence, the result follows from the triangle inequality. □
The following proposition is a variant of Proposition 5.9.

**Proposition 5.7.** Let $I$ be an open interval containing the closed interval $[0, 1]$ and $\mathcal{H} \in \mathcal{D}(k_{X \times \mathbb{R}_t \times I})$. Assume that there exist continuous functions $f, g: I \to \mathbb{R}_{\geq 0}$ satisfying

$$\text{SS}(\mathcal{H}) \subset T^*X \times \{(t, s; \tau, \sigma) \mid (\tau, \sigma) \in D(f(s), g(s))\}. \quad (5.20)$$

Then $d_{X \times \mathbb{R}_t}(\mathcal{H}|_{X \times \mathbb{R}_t \times \{0\}}, \mathcal{H}|_{X \times \mathbb{R}_t \times \{1\}}) \leq 4 \int_0^1 \max(f, g)(s)ds$.

**Proof.** We can apply an argument similar to [Al20, Prop. 4.15]. We only need to replace [Al20, Prop. 4.3] with Proposition 5.6. 

**Proof of Proposition 5.7.** The result follows from (3.25) and Proposition 5.7.

**Remark 5.8.** One could prove an inequality

$$d_{M^2 \times \mathbb{R}_t}(K_0^1, K_1^H) \leq 2 \int_0^1 \|H_s\|_{\infty}ds, \quad (5.21)$$

which is stronger than Proposition 5.2. Indeed, by the proof of Proposition 5.6 we have proved that under the assumption of Proposition 5.7 ($\mathcal{H}|_{X \times \mathbb{R}_t \times \{0\}}, \mathcal{H}|_{X \times \mathbb{R}_t \times \{1\}}$) is weakly ($\int_0^1 \max(f, g)(s)ds + \varepsilon, \int_0^1 \max(f, g)(s)ds + \varepsilon$)-isomorphic for any $\varepsilon \in \mathbb{R}_{>0}$. Hence, we find that $(K_0^1, K_1^H)$ is weakly ($\int_0^1 \|H_s\|_{\infty}ds + \varepsilon, \int_0^1 \|H_s\|_{\infty}ds + \varepsilon$)-isomorphic for any $\varepsilon \in \mathbb{R}_{>0}$. It remains to apply an argument similar to the proof of Theorem 5.1. However, we do not need this stronger inequality in this paper.

It is proved in [Zha20, Prop. 4.3] that the restriction of the sheaf quantization $K^H$ to $s = 1$ depends only on the relative homotopy class of the path $[s \mapsto \phi^H_s]$. Now we prove the following stronger result, which claims that the restriction depends only on the time-1 map.

**Proposition 5.9.** Let $H: T^*M \times I \to \mathbb{R}$ be a timewise compactly supported function. Then the objects $K^H_1, K^H, K^H$ are determined by the time-1 map $\phi^H_1$.

**Proof.** We shall prove $K^H_1 \equiv K^H_1$ assuming that $\phi^H_1 = \phi^H_1$. By (3.23), it suffices to show that $K^H_1 \simeq k_{\Delta^2 \times \{0\}}$. Hence, we may assume that $H \equiv 0$ and $\phi^H_1 = id_{T^*M}$.

Since $u_s \equiv 0$ by Lemma 3.11 we find that $\text{SS}(K^H_1) = T^*_{\Delta^2 \times \{0\}}(M^2 \times \mathbb{R}_t)$ and $K^H_1$ is simple along the subset. Moreover, since $K^H_0 \simeq k_{\Delta^2 \times \{0\}}$ and $u$ is compactly supported and hence bounded, $K^H|_{M^2 \times \{R\} \times I}$ and $K^H|_{M^2 \times \{-R\} \times I}$ are 0 for sufficiently large $R$. Hence, there exists a rank one local system $\mathcal{L}$ on $M$ such that $K^H \simeq \delta_{M \times \mathcal{L} \boxtimes k_{\{0\}}[m]}$, where $m$ is some integer. By Proposition 5.6 we have

$$d_{M^2 \times \mathbb{R}_t}(k_{\Delta^2 \times \{0\}}, \delta_{M \times \mathcal{L} \boxtimes k_{\{0\}}[m]}) = d_{M^2 \times \mathbb{R}_t}(K^0_0, K^H_1) \leq 4\|H\|_{\infty} < \infty. \quad (5.22)$$

By restricting to $\{(x, x)\} \times \mathbb{R} \subset M^2 \times \mathbb{R}_t$ for some $x \in M$, we obtain $d_{\mathbb{R}_t}(k_{\{0\}}, k_{\{0\}}[m]) < \infty$ and find that $m = 0$. Then, we have

$$\text{RG}(M; \mathcal{L}) \simeq \text{RG}(M^2 \times \mathbb{R}_t; K^H_1) \simeq \text{RG}(M^2 \times \mathbb{R}_t; K^0_0) \simeq \text{RG}(M; k_M). \quad (5.23)$$

In particular, $H^0(M; \mathcal{L}) \simeq H^0(M; k_M)$, which implies that $\mathcal{L}$ is trivial. 

The proposition above shows the well-definedness in the following definition.
Definition 5.10. (i) A diffeomorphism \( \varphi: T^*M \to T^*M \) is said to be a compactly supported Hamiltonian diffeomorphism if it is the time-1 map of some compactly supported Hamiltonian isotopy \( \phi^H \), that is, \( \varphi = \phi^H_1 \). The set of compactly supported Hamiltonian diffeomorphisms is denoted by \( \text{Ham}_c(T^*M, \omega) \).

(ii) The Hofer metric between Hamiltonian diffeomorphisms is defined by

\[
 d_H(\varphi, \varphi') := \inf \{ \|H\|_{\text{osc}} \mid \phi^H_1 = \varphi^{-1}\varphi' \} \tag{5.24}
\]

for \( \varphi, \varphi' \in \text{Ham}_c(T^*M, \omega) \).

(iii) For \( \varphi \in \text{Ham}_c(T^*M, \omega) \), one sets \( \overline{K}^\varphi := \overline{K}^H_1, K^\varphi := K^H_1 \), and \( \mathcal{K}^\varphi := K^H_1 \), where \( H \) is any timewise compactly supported function with \( \varphi = \phi^H_1 \).

By (5.10), we have

\[
 K_{n+1} \to K_n \text{ as } n \to \infty \text{ and the endofunctor } K_{\infty} \ast (\ast) \text{ on } D(k^M_{\infty}) \text{ gives an equivalence of categories.}
\]

5.3 Sheaf quantization of Hamiltonian homeomorphisms

In this subsection, we construct a sheaf quantization of a limit of Hamiltonian diffeomorphisms with respect to the Hofer metric.

Proposition 5.12. Let \( (\varphi_n)_{n \in \mathbb{N}} \subseteq \text{Ham}_c(T^*M, \omega) \) be a sequence of compactly supported Hamiltonian diffeomorphisms and \( K_n := \varphi_n \in D(k^{M^2}_{\infty}) \) the sheaf quantization of \( \varphi_n \). Assume that it is a Cauchy sequence with respect to the Hofer metric \( d_H \). Then there exists an object \( K_{\infty} \in D(k^{M^2}_{\infty}) \) such that \( d_{M^2 \times \mathbb{R}_1}(K_n, K_{\infty}) \to 0 \) \( (n \to \infty) \). Moreover, such an object is unique up to isomorphism, and the endofunctor \( K_{\infty} \ast (\ast) \) on \( D(k^M_{\infty}) \) gives an equivalence of categories.

Proof. By (5.10), we have

\[
 d_{M^2 \times \mathbb{R}_1}(K_n, K_m) \leq 4d_H(\varphi_n, \varphi_m), \tag{5.26}
\]

which proves that \( (K_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( D(k^{M^2}_{\infty}) \) with respect to \( d_{M^2 \times \mathbb{R}_1} \). Hence, Corollary 4.5 shows the existence of a limit object.

Let \( K'_{\infty} \) be another limit object that satisfies \( d_{M^2 \times \mathbb{R}_1}(K_{\infty}, K'_{\infty}) = 0 \). There exists \( \overline{K}_n \in D(k^{M^2}_{\infty}) \) such that \( \overline{K}_n \ast K_n \simeq K_n \ast \overline{K}_n \simeq k_{\Delta M \times \{0\}} \) for each \( n \). Then we find that the sequence \( (\overline{K}_n)_{n \in \mathbb{N}} \) is also Cauchy, and we can take a limit object \( \overline{K}_{\infty} \). The Cauchy sequence \( \overline{K}_n \ast K_n \) converges to both \( \overline{K}_{\infty} \ast K_{\infty} \) and \( k_{\Delta M \times \{0\}} \). Hence we have \( d_{M^2 \times \mathbb{R}_1}(\overline{K}_{\infty} \ast K_{\infty}, k_{\Delta M \times \{0\}}) = 0 \). Similarly, we have \( d_{M^2 \times \mathbb{R}_1}(K'_{\infty} \ast \overline{K}_{\infty}, k_{\Delta M \times \{0\}}) = 0 \). By Lemma 5.10, we have \( \overline{K}_{\infty} \ast K_{\infty} \simeq k_{\Delta M \times \{0\}} \simeq K'_{\infty} \ast \overline{K}_{\infty} \). Hence, we conclude that \( K'_{\infty} \simeq K_{\infty} \ast \overline{K}_{\infty} \simeq K_{\infty} \ast k_{\Delta M \times \{0\}} \ast \overline{K}_{\infty} \ast (\ast) \) gives the inverse. \( \square \)

Note that we have an inequality similar to (3.11) for \( d_{X \times \mathbb{R}_1} \).

Lemma 5.13. Let \( F, G \in D_{\{r \geq 0\}}(k_{\mathbb{R}_1}) \). If \( d_{\mathbb{R}_1}(F, G) = 0 \), then \( \text{SS}(F) = \text{SS}(G) \).
Proof. We prove $SS(F) \subset SS(G)$ by contradiction. Choose $(t_0; 1) \in SS(F) \setminus SS(G)$. There exists a neighborhood $(a, b)$ of $t_0$ in $\mathbb{R}_t$ such that $(a, b) \cap \pi(SS(G)) = \emptyset$. There also exist $t_1, t_2 \in \mathbb{R}$ such that $a < t_1 < t_2 < b$ and the restriction map $\Gamma((−\infty, t_2); F) \to \Gamma((−\infty, t_1); F)$ is not an isomorphism. We may choose $t_1$ and $t_2$ arbitrarily close to $t_0$. On the other hand, $\Gamma((−\infty, t')\cap G) \to \Gamma((−\infty, t'); G)$ is an isomorphism for any $a < t' < t'' < b$ by the microlocal Morse lemma. Thus, an interleaving between $(\Gamma((−\infty, t); F))_{t \in \mathbb{R}}$ and $(\Gamma((−\infty, t); G))_{t \in \mathbb{R}}$ leads to a contradiction. □

Lemma 5.14. Let $F, G \in D(k_{\mathbb{R}_t})$ and assume that $d_{\mathbb{R}_t}(F, G) = 0$. Then $SS(F) = SS(G)$ and hence $\text{Supp}(F) = \text{Supp}(G)$.

Proof. Since $d_{\mathbb{R}_t}(F \ast k_{[0, \infty)}, G \ast k_{[0, \infty)}) = 0$, we have

$$SS(F) \cap \{\tau > 0\} = SS(F \ast k_{[0, \infty)}) = SS(G \ast k_{[0, \infty)}) = SS(G) \cap \{\tau > 0\} \quad (5.27)$$

by Lemma 5.13. Similarly, we obtain

$$SS(F) \cap \{\tau < 0\} = SS(F \ast k_{[0, \infty)}) = SS(G \ast k_{[0, \infty)}) = SS(G) \cap \{\tau < 0\} \quad (5.28)$$

and hence $\text{Supp}(F) = \text{Supp}(G)$. Note that $U := \mathbb{R}_t \setminus \pi(SS(F)) = \mathbb{R}_t \setminus \pi(SS(G))$ is an open subset of $\mathbb{R}_t$, and both $F|_{U}$ and $G|_{U}$ are locally constant. Thus $d_{X \times \mathbb{R}_t}(F, G) = 0$ implies $F|_{U} \simeq G|_{U}$. □

Lemma 5.15. For $F, G \in D(k_{X \times \mathbb{R}_t})$, $d_{X \times \mathbb{R}_t}(F, G) = 0$ implies $\text{Supp}(F) = \text{Supp}(G)$.

Proof. Let $(x, t) \in X \times \mathbb{R}_t \setminus \text{Supp}(G)$. Take an open neighborhood $x \in U$ in $X$ and $\varepsilon > 0$ so that $U \times (t - \varepsilon, t + \varepsilon) \subset X \times \mathbb{R}_t \setminus \text{Supp}(G)$. Note that for any $y \in X$, $0 \leq d_{\mathbb{R}_t}(F|_{(y) \times \mathbb{R}_t}, G|_{(y) \times \mathbb{R}_t}) \leq d_{X \times \mathbb{R}_t}(F, G) = 0$ and $\text{Supp}(G|_{(y) \times \mathbb{R}_t}) \subset \text{Supp}(G) \cap ((y) \times \mathbb{R}_t)$. By Lemma 5.14, $F|_{(y) \times (t-\varepsilon, t+\varepsilon)} \simeq 0$ for any $y \in U$. Hence, we obtain $F|_{U \times (t-\varepsilon, t+\varepsilon)} \simeq 0$ and $\text{Supp}(F) \subset \text{Supp}(G)$. We obtain the converse inclusion $\text{Supp}(F) \supset \text{Supp}(G)$ similarly. □

Lemma 5.16. Let $F, G \in D(k_{X \times \mathbb{R}_t})$ and assume that $d_{X \times \mathbb{R}_t}(F, G) = 0$ and $\text{Supp}(G) \subset X \times \{0\}$. Then $F \simeq G$.

Proof. By Lemma 5.15, we have $\text{Supp}(F) = \text{Supp}(G) \subset X \times \{0\}$. We may write $F \simeq F' \boxtimes k_{[0]}$ and $G \simeq G' \boxtimes k_{[0]}$ with some $F', G' \in D(k_X)$. Take $\varepsilon > 0$ arbitrarily. Note that $\mathcal{R}_x \circ F \simeq F' \boxtimes k_{[-\varepsilon, \varepsilon]}$ and $\mathcal{R}_x \circ G \simeq G' \boxtimes k_{[-\varepsilon, \varepsilon]}$. Hence, there exist morphisms $\alpha: F' \boxtimes k_{[-\varepsilon, \varepsilon]} \to G' \boxtimes k_{[0]}$ and $\beta: G' \boxtimes k_{[-\varepsilon, \varepsilon]} \to F' \boxtimes k_{[0]}$ such that $\alpha(\varepsilon) \simeq G' \boxtimes k_{[0]}$ are $\varepsilon$-isomorphism of $(F, G)$. Restricting $\alpha$ and $\beta$ on $X \times \{0\}$, we obtain isomorphisms between $F'$ and $G'$.

Note that Petit–Schapira–Wasa [PSW21] proved that $\text{dist}(F, G) = 0$ if and only if $F \simeq G$ when $F$ and $G$ are constructible sheaves up to infinity on a real analytic manifold. This result, as well as its proof, is different from ours and is not related to the Tamarkin category.

Proposition 5.12 shows that we can associate a sheaf with an element of the metric completion of $\text{Ham}_c(T^*M, \omega)$ with respect to the Hofer metric $d_H$ as follows. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of Hamiltonian diffeomorphisms and $K_n := K^{\varphi_n}$ for $n \in \mathbb{N}$. By Proposition 5.12, we obtain a limit object $K_\infty$ of the sequence $(K_n)_{n \in \mathbb{N}}$. The argument in the proof of the proposition also shows that another Cauchy sequence equivalent to $(\varphi_n)_{n \in \mathbb{N}}$ gives the same limit object $K_\infty$ up to isomorphism.
Definition 5.17. Let \( [(\varphi_n)_{n \in \mathbb{N}}] \) be an element of the completion of \( \text{Ham}_c(T^*M, \omega) \) with respect to \( d_H \). The limit sheaf \( K_\infty \) defined as above is denoted by \( K_{[(\varphi_n)_{n \in \mathbb{N}}]} \) and called the sheaf quantization of \( [(\varphi_n)_{n \in \mathbb{N}}] \).

We use the above construction to obtain a sheaf quantization of a Hamiltonian homeomorphism of \( T^*M \).

Definition 5.18 (Oh–Müller [OM07]). Let \( \phi = (\phi_n)_n : T^*M \times I \to T^*M \) be an isotopy of homeomorphisms of \( T^*M \). The isotopy \( \phi \) is said to be a continuous Hamiltonian isotopy if there exist a compact subset \( C \subset T^*M \) and a sequence of smooth functions \( H_n : T^*M \times I \to \mathbb{R} \) timewisely supported in \( C \) satisfying the following two conditions.

1. The sequence of flows \( (\phi^H_n)_{n \in \mathbb{N}} \) \( C^0 \)-converges to \( \phi \), uniformly in \( s \in I \).
2. The sequence \( (H_n)_{n \in \mathbb{N}} \) converges uniformly to a continuous function \( H : T^*M \times I \to \mathbb{R} \). That is, \( \|H_n - H\|_\infty \to 0 \).

In this case, \( H \) is said to generate \( \phi \). A homeomorphism of \( T^*M \) is called a Hamiltonian homeomorphism if it is the time-1 map of a continuous Hamiltonian isotopy.

The following uniqueness theorems hold.

(i) A continuous Hamiltonian function generates a unique continuous Hamiltonian isotopy (Oh–Müller [OM07]).

(ii) A continuous Hamiltonian isotopy can be generated by a unique continuous Hamiltonian function up to addition of a function of time (Viterbo [Vit06]).

A continuous Hamiltonian isotopy \( \phi \) defines an element of the metric completion of \( \text{Ham}_c(T^*M, \omega) \) with respect to \( d_H \). Indeed, for a sequence \( (H_n)_{n \in \mathbb{N}} \) satisfying the condition (2) in Definition 5.18, \( (\phi^H_n)_{n \in \mathbb{N}} \) forms a Cauchy sequence with respect to the Hofer metric \( d_H \). Moreover, the element in the metric completion is independent of the choice of a sequence \( (H_n)_{n \in \mathbb{N}} \). Thus we obtain a sheaf \( K_\infty \) as in Definition 5.17. We give a bound of the microsupport of \( K_\infty \). Set \( \varphi_n = \phi^H_n \) and \( K_n := K_{\varphi_n}^\dag \). By construction, after taking a subsequence of \( (K_n)_{n \in \mathbb{N}} \) if necessary, \( K_\infty \simeq \text{hocolim}_n K_{[-a_n, a_n]}(\varphi_n K_n) \), where \( (a_n)_{n \in \mathbb{N}} \) is as in Theorem 4.3. Hence, we have

\[
\text{SS}(K_\infty) \subset \left\{ ((x'; \xi'), (x; -\xi), (t; \tau)) \middle| \begin{aligned} (x; \xi) \in T^*M, \\ (x'; \xi'/\tau) = \varphi_\infty(x; \xi/\tau) \end{aligned} \right\} 
\]

by the condition (1) and Lemma 4.3.

Remark 5.19. For the microsupport estimate above, we do not need the full convergence of flows, but only the convergence of the time-1 maps, that is, \( \varphi_n \to \varphi_\infty \) \( (n \to \infty) \).

Definition 5.20. Let \( \phi \) be a continuous Hamiltonian isotopy associated with a continuous function \( H : T^*M \times I \to \mathbb{R} \) and \( \varphi_\infty := \phi_1 \) a Hamiltonian homeomorphism. The limit sheaf \( K_\infty \) defined as above is denoted by \( K_1^H = K_{\varphi_\infty} \) and called the sheaf quantization of the Hamiltonian homeomorphism \( \varphi_\infty \). One also sets \( K_{\varphi_\infty} := P_1(K_{\varphi_\infty}) \in D(M^2) \).

We can also prove the following, which justifies the notation \( K_{\varphi_\infty} \).

Proposition 5.21. In the situation of Definition 5.20, the object \( K_1^H \) depends only on the time-1 map \( \phi_1 \).
The morphism coincides with the restriction map by the construction, which is a contradiction. Note that we already know that $SS(G \star k_{(0,\infty)}) \subset \{ \tau \geq 0 \}$ by Lemma 2.8. Assume that there exists $(t_0; 1) \in \hat{SS}(G \star k_{(0,\infty)})$ with $t_0 \neq 0$. Then, there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and the restriction map $R\Gamma((t_1, \infty); G \star k_{(0,\infty)}) \to R\Gamma((t_1, \infty); G \star k_{(0,\infty)})$ is not an isomorphism. Moreover, we choose $t_1$ and $t_2$ arbitrary close to $t_0$ and hence may assume $t_2 - t_1 < \min\{t_1, \varepsilon\}$ if $t_0 > 0$ and $t_2 < 0, t_2 - t_1 < \varepsilon$ if $t_0 < 0$. Applying $G \star (\cdot)$ to the exact triangle $k_{\{t_2 - t_1, \infty\}} \to k_{(0,\infty)} \to k_{(t_2 - t_1, \infty)} \to +1$, we have an exact triangle

$$k_{\{t_2 - t_1, \infty\}} \to G \star k_{(0,\infty)} \to T_{t_2 - t_1} G \star k_{(0,\infty)} \to +1.$$  

Since $R\Gamma((\infty, t_2); k_{\{t_2 - t_1, \infty\}}) = 0$, we get an isomorphism

$$R\Gamma((\infty, t_2); G \star k_{(0,\infty)}) \cong R\Gamma((\infty, t_1); G \star k_{(0,\infty)}).$$

The morphism coincides with the restriction map by the construction, which is a contradiction.

Similarly, we obtain $\hat{SS}(G \star k_{(0,\infty)}) \subset \{ \tau \leq 0 \}$.

We get $SS(G) \subset T^*_\mathbb{R} \mathbb{R}$ by applying $G \star (\cdot)$ to the exact triangle $k_{(0,\infty)} \to k_{(0,\infty)} \to k_{(0,\infty)} \to +1$. Hence, we obtain the desired isomorphism from $d_{\mathbb{R}}(G, k_{(0,0)}) \leq 0$.

Now let $\phi$ be a continuous Hamiltonian isotopy generated by a continuous function $H: T^*M \times I \to \mathbb{R}$ and assume that $\phi_1 = id_{T^*M}$. By the microsupport estimate \[15.29\], we get $\pi(\hat{SS}(K^H)) \subset \Delta_M \times \mathbb{R}$. By construction, $d_{M^2 \times \mathbb{R}_\varepsilon}(k_{\Delta_M \times \{0\}}, K^H) < \infty$. Restricting to $(M^2 \setminus \Delta_M) \times \mathbb{R}$, we find that $d_{M^2 \times \mathbb{R}_\varepsilon}(0, K^H |_{(M^2 \setminus \Delta_M) \times \mathbb{R}}) < \infty$, which implies $K^H |_{(M^2 \setminus \Delta_M) \times \mathbb{R}} \cong 0$. Hence, we can write $K^H \cong (\delta_M \times id_{\mathbb{R}_\varepsilon})_* K'$ with $K' \in D(k_{M \times \mathbb{R}})$. Again by the estimate \[15.29\] and Proposition 2.3(i), we find that $SS(K') \subset 0_M \times T^*\mathbb{R}$.

Let us keep the notation of a compact set $C \subset T^*M$ and $K_n = K^{\varepsilon n}$ as above (see also Definition 5.13). For $F \in D(k_{M \times \mathbb{R}})$ such that

$$SS(F) \cap \{(x, t; \xi, \tau) \mid \tau \neq 0, (x; \xi/\tau) \in C\} = \emptyset,$$

we have $K_n \star F \cong F$ for any $n \in \mathbb{N}$, which implies $d_{M \times \mathbb{R}_\varepsilon}(K^H_n \star F, F) = 0$. For each $x_0 \in M$ and $a < b \in \mathbb{R}$ with sufficiently small $b - a$, there exists $F \in D(k_{M \times \mathbb{R}})$ such that (1) its microsupport does not intersect the cone $\{(x, t; \xi, \tau) \mid \tau \neq 0, (x; \xi/\tau) \in C\}$ of $C$ and (2) $F |_{\{x_0\} \times \mathbb{R}_\varepsilon} = k_{(a,b)}$. For example, such $F$ is obtained as the sheaf quantization of an exact Lagrangian immersion of a sphere. Therefore, $d_{\mathbb{R}_\varepsilon}(K^H_n \star k_{(a,b)}, k_{[a,b]}) \leq d_{M \times \mathbb{R}_\varepsilon}(K^H_n \star F, F) = 0$ if $b - a$ is sufficiently small. Applying Lemma 5.14 to $K' |_{\{x_0\} \times \mathbb{R}_\varepsilon} \star k_{[a,b]}$ and $k_{[a,b]}$, we get $K' |_{\{x_0\} \times \mathbb{R}_\varepsilon} \star k_{[a,b]} \cong k_{[a,b]}$. Similarly, we obtain $K' |_{\{x_0\} \times \mathbb{R}_\varepsilon} \star k_{[a,b]} \cong k_{[a,b]}$ for any $x_0 \in M$ and sufficiently small $b - a$.

Hence, for any $x_0 \in M$, $K' |_{\{x_0\} \times \mathbb{R}_\varepsilon} \in D(k_{\mathbb{R}_\varepsilon})$ satisfies the condition of Lemma \[5.22\]. Combining this with the estimate of $SS(K')$, we may write $K' = L \boxtimes k_{\{0\}}$, where $L$ is a rank one local system. Again by the fact that $d_{M^2 \times \mathbb{R}_\varepsilon}(k_{\Delta_M \times \{0\}}, K^H) < \infty$, we conclude that $L = k_M$, which proves $K^H_n \cong k_{\Delta_M \times \{0\}}$. \[25\]
6  Spectral invariants in Tamarkin category

In this section, we define spectral invariants for an object of the Tamarkin category and
develop Lusternik–Schnirelmann theory. Most of the definitions and the results were
announced to appear in Humilière–Vichery [HV].

**Definition 6.1.** Let $F \in \mathcal{D}(M)$.

(i) For $n \in \mathbb{Z}$, one defines

$$Q^n_c(F) := \text{Hom}(k_{M \times [0, \infty)}, T, F[n])$$

$$\simeq H^n \text{RHom}(k_{M \times [-c, \infty)}, F),$$

$$Q^\infty_c(F) := \lim_{c \to \infty} Q^n_c(F).$$

(ii) For $\alpha \in Q^\infty_c(F)$, one defines

$$c(\alpha; F) := \inf\{c \in \mathbb{R} \mid \alpha \in \text{Im } i_c\}$$

and calls it the *spectral invariant* of $F$ for $\alpha$.

(iii) One defines

$$\text{Spec}(F) := \{c(\alpha; F) \in \mathbb{R} \mid \alpha \in Q^\infty_c(F)\} \subset \mathbb{R}.$$  

**Remark 6.2.** In our definition, $c(0; F) = -\infty$. In general, $c(\alpha; F)$ can be $-\infty$ for non-zero $\alpha \in Q^\infty_c(F)$. We give such an example when $M = \text{pt}$. Let $G := \prod_{n \in \mathbb{Z}_{\geq 1}} k_{[-n, n][1]}$ and define $g: k_{[0, \infty)} \to G$ to be the product $\prod_{n \in \mathbb{Z}_{\geq 1}} g_n$, where $g_n: k_{[0, \infty)} \to k_{[-n, n][1]}$ is a non-trivial morphism. Set $F := R_1(G)$, the projection to $\mathcal{D}(\text{pt})$. Then the projector $P_1$ induces a morphism $\tilde{g}: k_{[0, \infty)} \to F$ in $\mathcal{D}(\text{pt})$, which satisfies $[\tilde{g}] \neq 0$ in $Q^0_c(F)$ and $c([\tilde{g}]; F) = -\infty$.

Let $t: M \times \mathbb{R}_t \to \mathbb{R}_d$ be the projection and let $\Gamma_{dt}$ denote the graph of the 1-form $dt$:

$$\Gamma_{dt} := \{(x, t; 0, 1) \mid (x, t) \in M \times \mathbb{R}_d\}.$$  

Note that

$$\text{Spec}(F) \subset \{-c \in \mathbb{R} \mid c \in t\pi(\text{SS}(F) \cap \Gamma_{dt})\}.$$  

In order to state Lusternik–Schnirelmann theory for sheaves, we recall the algebraic
counterpart of cup-length, which is studied in [AI23].

**Definition 6.3.** Let $R$ be an associative (not necessarily commutative) non-unital ring
over $k$. For a right $R$-module $A$, one defines

$$\text{cl}_R(A) := \inf \left\{ k - 1 \left| k \in \mathbb{N}, a_0 \cdot r_1 \cdots r_k = 0 \right. \right.$$  

for any $a_0 \in A$ and $(r_1, \ldots, r_k) \in R^k \left. \right\} \in \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$  

We note that $\text{cl}_R(A) = -1$ if and only if $A = 0$. If there is no risk of confusion, we
simply write $\text{cl}(A)$ for $\text{cl}_R(A)$. By definition, we have the following lemma.
Lemma 6.4. For an exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$, one has
\[ \text{cl}(B) \leq \text{cl}(A) + \text{cl}(C) + 1. \] (6.8)

Let $F \in \mathcal{D}(M)$. Then, we have a right action of $\text{End}(k_{M \times \{0,\infty\}}) \simeq H^*(M; k)$ on $Q^*_\infty(F)$, which induces an action on $Q^*_\infty(F)$. Hereafter we set $R := \bigoplus_{n \geq 1} H^n(M; k)$ and consider the cup-length over $R$. Note that the cup-length over $R$ is always finite.

Theorem 6.5. Let $F \in \mathcal{D}(M)$. Assume that $t$ is proper on $\text{Supp}(F)$ and there exists $c < 0$ satisfying $i_c = 0$. Let $\pi_M : T^*(M \times \mathbb{R}_t) \to M$ denote the projection. If $\# \text{Spec}(F) \leq \text{cl}(Q^*_\infty(F))$, then there exists $c \in \text{Spec}(F)$ such that $\pi_M(\text{SS}(F) \cap \Gamma_{dt} \cap \pi^{-1}t^{-1}(-c))$ is cohomologically non-trivial in $M$. That is, for any open neighborhood $U$ of $\pi_M(\text{SS}(F) \cap \Gamma_{dt} \cap \pi^{-1}t^{-1}(-c))$, the restriction map $\bigoplus_{n \geq 1} H^n(M; k) \to \bigoplus_{n \geq 1} H^n(U; k)$ is non-zero.

For $F \in \mathcal{D}(M)$, if $F|_{M \times (c, \infty)}$ is locally constant for $c \gg 0$, then $i_c = 0$ for $c \ll 0$. If the conclusion holds, then $\pi_M(\text{SS}(F) \cap \Gamma_{dt})$ is also cohomologically non-trivial in $M$.

For the proof of the theorem, we prepare some notation. For $d \in \mathbb{R}$, we define
\[ Q^*_{\infty,d}(F) := \text{Im}(i_d : Q^*_d(F) \to Q^*_\infty(F)) \subset Q^*_\infty(F). \] (6.9)

Then we get the following properties:

(1) If $d < d'$, then $Q^*_{\infty,d}(F) \subset Q^*_{\infty,d'}(F)$.

(2) If $[d,d'] \cap \text{Spec}(F) = \emptyset$, then $Q^*_{\infty,d}(F) \simeq Q^*_{\infty,d'}(F)$.

(3) For $d < d'$, there exists an exact sequence of right $H^*(M; k)$-modules
\[ 0 \to Q^*_{\infty,d}(F) \to Q^*_{\infty,d'}(F) \to Q^*_{\infty,d'}(F)/Q^*_{\infty,d}(F) \to 0. \] (6.10)

Moreover, we have
\[ \text{cl}(H^*_{M \times [-d',-d]}(M \times \mathbb{R}; F)) \geq \text{cl}(Q^*_{\infty,d'}(F)/Q^*_{\infty,d}(F)). \] (6.11)

Proof of Theorem [6.7]. If $Q^*_{\infty}(F) \simeq 0$, then $\text{cl}(Q^*_\infty(F)) = -1$ and there is nothing to prove.

Suppose that $Q^*_\infty(F) \neq 0$. Since $\text{cl}(Q^*_\infty(F))$ is finite and set $\text{Spec}(F) = \{c_1, \ldots, c_N\}$ with $c_1 < c_2 < \cdots < c_N$. Let $d_0, d_1, \ldots, d_N \in \mathbb{R}$ such that $d_0 < c_1 < d_1 < \cdots < d_{N-1} < c_N < d_N$. Note that $Q^*_{\infty,d_0}(F) = 0$ by the assumption and $Q^*_{\infty,d_N}(F) = Q^*_\infty(F)$. Applying Lemma [6.3] to the exact sequence [6.10] with $d = d_{i-1}, d' = d_i$, by induction we get
\[ \text{cl}(Q^*_\infty(F)) \leq N - 1 + \sum_{i=1}^{N} \text{cl}(Q^*_{\infty,d_i}(F)/Q^*_{\infty,d_{i-1}}(F)). \] (6.12)

Hence if $\# \text{Spec}(F) = N \leq \text{cl}(Q^*_\infty(F))$, there exists $i \in \{1, \ldots, N\}$ such that
\[ \text{cl}(Q^*_{\infty,d_i}(F)/Q^*_{\infty,d_{i-1}}(F)) \geq 1. \] (6.13)

For such $i$ above, we claim that $\pi_M(\text{SS}(F) \cap \Gamma_{dt} \cap \pi^{-1}t^{-1}(-c_i))$ is cohomologically non-trivial in $M$. For $c \in \mathbb{R}$ and $I \subset \mathbb{R}$, set
\[ K_c := \pi_M(\text{SS}(F) \cap \Gamma_{dt} \cap \pi^{-1}t^{-1}(-c)), \quad K_I := \bigcup_{c \in I} K_c \subset M. \] (6.14)

Let $U$ be any open neighborhood of $K_{c_i}$ in $M$. We take $K_{c_i} \subset U_0 \subset U_1 \subset U$ and a $C^\infty$-function $\rho : M \to \mathbb{R}$ such that
By the above isomorphism, we get a morphism of exact triangle $s$.

We set $X$.

Morse lemma to $F$.

$x \in U_1 \setminus U_0, t \in [-c_i - \varepsilon', -c_i + \varepsilon']$, and $s \in [0, 1]$. Hence, we can apply the microlocal Morse lemma to $F$ and $(V_s)_{s \in [0, 1]}$, where

$$V_s := \{(x, t) \in M \times \mathbb{R}_t \mid t < -c_i + (s - 1)s' - s\rho'(x)\},$$

and obtain an isomorphism

$$\text{R} \Gamma(M \times (-\infty, -c_i - s'); F) = \text{R} \Gamma(U_0; F) \xrightarrow{\simeq} \text{R} \Gamma(U_1; F).$$

We set $X = M \times \mathbb{R}$ and

$$Z := M \times (-\infty, -c_i + s') \setminus U_1 = \{(x, t) \in M \times \mathbb{R}_t \mid -c_i - \rho'(x) \leq t < -c_i + s'\}.$$ (6.18)

By the above isomorphism, we get a morphism of exact triangles

$$\xymatrix{ \text{R} \Gamma_{M \times [-c_i + s', \infty)}(X; F) \ar[r] & \text{R} \Gamma_{M \times [-c_i + s', \infty)} \cup Z(X; F) \ar[r] & \text{R} \Gamma_Z(X; F) \ar[d] \ar[r] & \text{R} \Gamma_{M \times [-c_i + s', \infty)}(X; F) \ar[d] \\
& \text{R} \Gamma_{M \times [-c_i + s', \infty)}(X; F) \ar[r] & \text{R} \Gamma_{M \times [-c_i - s', -c_i + s']}(X; F) \ar[r] & \text{R} \Gamma_{M \times [-c_i - s', -c_i + s']}(X; F) \ar[r] & ,}$$

(6.19)

where the middle vertical morphism is an isomorphism by the above argument. Hence, by the five lemma, we have an isomorphism

$$\text{R} \Gamma_{M \times [-c_i - s', -c_i + s']}(M \times \mathbb{R}_t; F) \xrightarrow{\sim} \text{R} \Gamma_Z(M \times \mathbb{R}_t; F) \cong \text{R} \text{Hom}(k_Z, F).$$ (6.20)

Since Supp$(k_Z) = \overline{Z} \subset U_1 \times [-c_i - s', -c_i + s']$, the action of $H^*(M; k)$ on $H_Z^*(M \times \mathbb{R}_t; F)$ factors through $H^*(\overline{U}_1)$. Hence, we have

$$\text{cl}(H^*(\overline{U}_1)) \geq \text{cl}(H_Z^*(M \times \mathbb{R}_t; F))$$

$$= \text{cl}(H_M^*(M \times [-c_i - s', -c_i + s'])(M \times \mathbb{R}_t; F))$$

$$\geq \text{cl}(Q_{-\infty, c + s'}(F)/Q_{-\infty, c - s'}(F)) \geq 1.$$ (6.21)

Thus, we conclude that $\overline{U}_1$ is cohomologically non-trivial in $M$, which implies that $U$ is also cohomologically non-trivial in $M$. \hfill \Box

We consider the spectral invariants for the sheaf associated with a Hamiltonian diffeomorphism/homeomorphism and a compact exact Lagrangian submanifold.

Let $L$ be a compact exact Lagrangian submanifold of $T^*M$. Take a function $f : L \rightarrow \mathbb{R}$ satisfying $\alpha_{T^*M}|_L = df$ and define

$$\tilde{L} := \{(x, t; \xi, \tau) \mid \tau > 0, (x; \xi / \tau) \in L, t = -f(x; \xi / \tau)\}.$$ (6.22)
In this setting, Guillermou [Gui12] (see also [Gui23, Vit19]) proved the existence and the uniqueness of an object $F_L \in \mathcal{D}(M)$ that satisfies $SS(F_L) = \hat{L}$ and $F_L|_{M \times (c, \infty)} \simeq k_{M \times (c, \infty)}$ for a sufficiently large $c > 0$. We call $F_L$ the canonical simple sheaf quantization of $L$. When $L = 0_M$ and $f = 0$, we have $F_{0_M} \simeq k_{M \times [0, \infty)}$.

Moreover, let $\phi \in \text{Ham}_c(T^*M, \omega)$ be a compactly supported Hamiltonian diffeomorphism. We define the set of spectral invariants of $\text{Spec}(\phi, L)$ of the Lagrangian submanifold $\phi(L)$ by

$$\text{Spec}(\phi, L) := \text{Spec}(K^\phi \cdot F_L),$$

where $K^\phi \in \mathcal{D}(M^2)$ is the sheaf quantization of $\phi$. This set is well-defined up to shift. By a result of Viterbo [Vit19], $\text{Spec}(\phi, L)$ is equal to the set of the Floer-theoretic spectral invariants associated with $\phi(L)$. If two Hamiltonian diffeomorphisms $\phi, \phi' \in \text{Ham}_c(T^*M, \omega)$ satisfy $\phi(L) = \phi'(L)$, then there exists some constant $C \in \mathbb{R}$ such that

$$\text{Spec}(K^\phi \cdot F_L) = \text{Spec}(K^{\phi'} \cdot F_L) + C. \quad (6.24)$$

Indeed, since both of $K^\phi \cdot F_L$ and $K^{\phi'} \cdot F_L$ are canonical simple sheaf quantizations of $\phi(L)$, by the uniqueness result, we have $K^\phi \cdot F_L \simeq T_c(K^{\phi'} \cdot F_L)$ for some $c \in \mathbb{R}$.

Let $\phi_\infty$ be a Hamiltonian homeomorphism and $(H_n)_{n \in \mathbb{N}}$ a sequence of smooth functions that satisfies the condition in Definition [5.18] For any $n \in \mathbb{N}$, we set $\phi_n := \phi^{H_n}_1$ and consider the sheaf quantization $K^{\phi_n}$ of $\phi_n$. Then, the sequence $(K^{\phi_n})_{n \in \mathbb{N}}$ forms a Cauchy sequence with respect to $d_{\mathcal{D}(M^2)}$ and gives an object $K^{\phi_\infty}$. In this situation, we define the set of spectral invariants $\text{Spec}(\phi_\infty, L)$ by

$$\text{Spec}(\phi_\infty, L) := \text{Spec}(K^{\phi_\infty} \cdot F_L).$$

This is well-defined up to shift.

**Lemma 6.6.** One has

$$\text{Spec}(\phi_\infty, L) = \lim_{n \to \infty} \text{Spec}(\phi_n, L). \quad (6.26)$$

**Proof.** Since $d_{\mathcal{D}(M^2)}(K^{\phi_n}, K^{\phi_\infty}) \to 0 (n \to \infty)$, we have $d_{\mathcal{D}(M)}(K^{\phi_n} \cdot F_L, K^{\phi_\infty} \cdot F_L) \to 0 (n \to \infty)$. Hence, we get the result.  

The spectral norm for Lagrangian submanifolds is defined as follows.

**Definition 6.7.** Let $\phi : T^*M \to T^*M$ be a Hamiltonian diffeomorphism. One defines

$$\gamma(\phi(0_M)) := \max \text{Spec}(K^\phi \cdot k_{M \times [0, \infty)}) + \max \text{Spec}(K^{\phi^{-1}} \cdot k_{M \times [0, \infty)})$$

and calls it the spectral norm of $\phi(0_M)$.

Note that the spectral norm $\gamma(\phi(0_M))$ depends only on the image $\phi(0_M)$. This follows from Proposition [5.9] below and the fact that if $\phi, \phi' \in \text{Ham}_c(T^*M, \omega)$ satisfy $\phi(0_M) = \phi'(0_M)$ then $K^\phi \cdot k_{M \times [0, \infty)} \simeq T_c(K^{\phi'} \cdot k_{M \times [0, \infty)})$ for some constant $c \in \mathbb{R}$. We also remark that $\gamma(\phi(0_M))$ in Definition [6.7] is the same as that in [Vit92; Oh97; Oh99] by the above argument.

We describe this spectral norm in terms of the distance on the Tamarkin category. For that purpose, we introduce an interleaving distance up to shift.

**Definition 6.8.** For $F, G \in \mathcal{D}(M)$, one defines

$$d_{\mathcal{D}(M)}(F, G) := \inf_{c \in \mathbb{R}} d_{\mathcal{D}(M)}(F, T_cG).$$

(6.28)
Proposition 6.9. For a Hamiltonian diffeomorphism \( \varphi : T^* M \to T^* M \), one has

\[
\gamma(\varphi(0)) = d_{D(M)}(k_{M \times [0, \infty)}, K^p \bullet k_{M \times [0, \infty)}).
\] (6.29)

Proof. We will argue similarly to the proof of Theorem 5.1. Again let Tor be the full triangulated subcategory of \( D(M) \) consisting of torsion objects \( \{ F \mid d_{D(M)}(F, 0) < \infty \} \). Then, the \( \text{Hom} \) set of the localized category \( D(M)/\text{Tor} \) is computed as

\[
\text{Hom}_{D(M)/\text{Tor}}(F, G) \simeq \lim_{c \to \infty} \text{Hom}_{D(M)}(F, T_c G).
\] (6.30)

Hence, \( Q^*_\infty(F) \) is isomorphic to \( \bigoplus_n \text{Hom}_{D(M)/\text{Tor}}(k_{M \times [0, \infty)}, F[n]) \).

For an object \( F \in D(M) \) with \( d(k_{M \times [0, \infty)}, F) < \infty, F \) and \( k_{M \times [0, \infty)} \) are isomorphic in \( D(M)/\text{Tor} \). On the other hand, any isomorphism \( \bar{\alpha} \in \text{Hom}_{D(M)/\text{Tor}}(k_{M \times [0, \infty)}, F) \cong Q^0_\infty(F) \) gives an isomorphism \( Q^*_\infty(F) \cong \bigoplus_n \text{Hom}_{D(M)/\text{Tor}}(k_{M \times [0, \infty)}, k_{M \times [0, \infty)}[n]) \cong H^*(M) \) of right \( H^*(M) \)-modules that sends \( \bar{\alpha} \in Q^0_\infty(F) \) to \( 1 \in H^0(M) \cong k \). Note that

\[
Q^0_{\infty, c}(F) \cong \begin{cases} k & (c > c(\bar{\alpha}, F)) \\ 0 & (c < c(\bar{\alpha}, F)) \end{cases}
\] (6.31)

by definition. Since \( Q^*_\infty(F) \subset Q^*_\infty(F) \) is an \( H^*(M) \)-submodule for any \( c, Q^0_{\infty, c}(F) \neq 0 \) if and only if \( Q^0_{\infty, c}(F) = Q^0_{\infty}(F) \). Hence we obtain

\[
\text{max Spec}(F) = c(\bar{\alpha}, F).
\] (6.32)

(i) Let \( a, b \in \mathbb{R} \) and

\[
\begin{align*}
\alpha &: k_{M \times [0, \infty)} \to T_a K^p \bullet k_{M \times [0, \infty)}, \\
\beta &: K^p \bullet k_{M \times [0, \infty)} \to T_b k_{M \times [0, \infty)}
\end{align*}
\] (6.33)

such that \( \alpha \) descends to an isomorphism \( \bar{\alpha} \in \text{Hom}_{D(M)/\text{Tor}}(k_{M \times [0, \infty)}, K^p \bullet k_{M \times [0, \infty)}) \) and \( \beta \) descends to its inverse \( \bar{\beta} \). Since \( \bar{\alpha} \in Q^0_{\infty, c}(K^p \bullet k_{M \times [0, \infty)}) \) is non-zero, \( \alpha \simeq \text{max Spec}(K^p \bullet k_{M \times [0, \infty)}) \). On the other hand, \( \bar{\beta} \) gives a non-zero element of \( Q^0_{\infty, b}(K^p \bullet k_{M \times [0, \infty)}) \) since \( K^p \bullet (*) \) is the inverse functor of \( K^p \bullet (*) \). Hence, we obtain \( b \geq \text{max Spec}(K^p \bullet k_{M \times [0, \infty)}) \).

(ii) For any \( a > \text{max Spec}(K^p \bullet k_{M \times [0, \infty)}) \), there exists \( \alpha \in k_{M \times [0, \infty)} \to T_a K^p \bullet k_{M \times [0, \infty)} \) that descends to an isomorphism \( \bar{\alpha} \in \text{Hom}_{D(M)/\text{Tor}}(k_{M \times [0, \infty)}, K^p \bullet k_{M \times [0, \infty)} \). For any \( b > \text{max Spec}(K^p \bullet k_{M \times [0, \infty)}) \), there also exists \( \beta : K^p \bullet k_{M \times [0, \infty)} \to T_b k_{M \times [0, \infty)} \) that descends to an isomorphism \( \bar{\beta} \). Since \( \text{Hom}_{D(M)/\text{Tor}}(k_{M \times [0, \infty)}, k_{M \times [0, \infty)}) \simeq k \), we may assume that \( \bar{\beta} \) is the inverse of \( \bar{\alpha} \) after multiplying a non-zero element of \( k \) to \( \beta \). The composite \( \bar{\beta} \circ \bar{\alpha} \) can be regarded as a non-zero element of \( Q^0_{\infty, a+b}(k_{M \times [0, \infty)}) \).

Noting that

\[
Q^0_{\infty}(k_{M \times [0, \infty)}) \simeq Q^0_{\infty, c}(k_{M \times [0, \infty)}) \cong \begin{cases} k & (c \geq 0) \\ 0 & (c < 0) \end{cases}
\] (6.35)

we obtain \( a + b \geq 0 \) and the preimage \( T_a \beta \circ \alpha \in Q^0_{a+b}(k_{M \times [0, \infty)}) \) of \( \text{id}_{k_{M \times [0, \infty)}} = \bar{\beta} \circ \bar{\alpha} \) is \( \tau_{a+b}(k_{M \times [0, \infty)}) \). Similarly, we obtain \( T_b \alpha \circ \beta = \tau_{a+b}(K^p \bullet k_{M \times [0, \infty)}) \) using

\[
\text{Hom}_{D(M)}(K^p \bullet k_{M \times [0, \infty)}, T_b K^p \bullet k_{M \times [0, \infty)}) \simeq Q^0_{\infty}(k_{M \times [0, \infty)})
\] (6.36)

and (6.35). This proves \( \gamma(\varphi(0)) \geq d_{D(M)}(k_{M \times [0, \infty)}, K^p \bullet k_{M \times [0, \infty)}). \) ∎
Remark 6.10. One can define Hamiltonian spectral invariants in a sheaf-theoretic way as follows. Let $H: T^*M \times I \to \mathbb{R}$ be a timewise compactly supported function and $K^H \in D(k_{M^2 \times \mathbb{R}^2 \times I})$ the associated sheaf quantization. Then, we define

$$\text{Spec}(H) := \text{Spec}(P(\text{Hom}^*(k_{\Delta M} \times [0, \infty), K^H))).$$

(6.37)

The Hamiltonian spectral norm is also defined by

$$\gamma(H) := \max \text{Spec}(H) + \max \text{Spec}(H)$$

(6.38)

and we obtain

$$\gamma(H) = d_{D(M)}(k_{\Delta M} \times [0, \infty), K^H).$$

(6.39)

We conjecture that $\gamma(H)$ coincides with the Hamiltonian spectral norm of $H$ defined by Frauenfelder–Schlenk [FS07] for a compact manifold $M$.

7 Arnold-type principle for Hamiltonian homeomorphisms

In this section, we use the previous results to prove an Arnold-type theorem for a Hamiltonian homeomorphism of a cotangent bundle in a purely sheaf-theoretic way. Throughout this section, we assume that $M$ is compact.

Let $L$ be a compact exact Lagrangian submanifold of $T^*M$ and $F_L$ be the canonical simple sheaf quantization of $L$. Let $\varphi_\infty$ be a Hamiltonian homeomorphism and $K^{\varphi_\infty}$ be the sheaf quantization of $\varphi_\infty$. We set $F_\infty := K^{\varphi_\infty} \cdot F_L$. Then, by (5.29) we have

$$\text{SS}(F_\infty) \cap \{\tau > 0\} \subset \{(x, t; \xi, \tau) \mid \tau > 0, (x; \xi/\tau) \in \varphi_\infty(L)\}.$$ 

(7.1)

Moreover, by construction and the property of $F_L$, we get $Q^*_{\infty}(F_\infty) \simeq H^*(M; k)$.

Combining the previous results, we obtain the following result by a purely sheaf-theoretic method. In the case $L$ is the zero-section $0_M$, it was proved by [BHS22] in a more general setting (see below). We set $\text{cl}(M) := \text{cl}(H^*(M; k))$, which is called the cup-length of $M$ over $k$.

**Theorem 7.1.** Let $L$ be a compact exact Lagrangian submanifold of $T^*M$ and $\varphi_\infty$ be a Hamiltonian homeomorphism of $T^*M$. If $\# \text{Spec}(\varphi_\infty, L) \leq \text{cl}(M)$, then $0_M \cap \varphi_\infty(L)$ is cohomologically non-trivial in $M$, in particular it is infinite.

**Proof.** Let $t: M \times \mathbb{R} \to \mathbb{R}$ and $\pi_M: T^*(M \times \mathbb{R}) \to M$ be the projections. Let $\Gamma_{dt}$ denote the graph of the 1-form $dt$. Then, by (7.1), we have $\pi_M(\text{SS}(F_\infty) \cap \Gamma_{dt}) \subset 0_M \cap \varphi_\infty(0_M)$. Thus we obtain the result by applying Theorem 6.3 and Lemma 6.6 to $F_\infty$. \qed

By using the spectral norm $\gamma$ and its $C^0$-continuity, we can construct a sheaf quantization the image of the zero-section $0_M$ under a $C^0$-limit of Hamiltonian diffeomorphisms. With the sheaf quantization, we can also recover [BHS22, Thm. 1.1]. Note that the proof is not purely sheaf-theoretic.

**Proposition 7.2.** Let $\varphi_\infty: T^*M \to T^*M$ be a compactly supported homeomorphism. Assume that there exist a compact subset $C \subset T^*M$ and a sequence of Hamiltonian diffeomorphisms $(\varphi_n)_{n \in \mathbb{N}} \subset \text{Ham}_c(T^*M; \omega)$ supported in $C$ that $C^0$-converges to $\varphi_\infty$ for some Riemannian metric.
(i) There exists an object \( F_\infty \in \mathcal{D}(M) \) such that \( d_{\mathcal{D}(M)} (k_{M \times [0, \infty)}, F_\infty) < \infty \) and
\[
\text{SS}(F_\infty) \cap \{ \tau > 0 \} \subset \{(x, t; \xi, t) \mid \tau > 0, (x, \xi/\tau) \in \varphi(0, M)\}.
\] (7.2)

(ii) There exists a sequence of real numbers \((c_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) such that \( \text{Spec}(\varphi, 0_M) := \lim_{n \to \infty} T_{-c_n} \text{Spec}(\varphi_n, 0_M) \) is well-defined up to shift. Moreover, if \( \# \text{Spec}(\varphi, 0_M) \leq \text{cl}(M) \), then \( 0_M \cap \varphi(0, M) \) is cohomologically non-trivial in \( M \), in particular it is infinite.

Proof. (i) Our \( \gamma(\psi(0_M)) \) coincides with \( \gamma(\psi(0_M)) \) in [BHS22] for any compactly supported Hamiltonian diffeomorphism \( \psi \). By [BHS22, Thm. 4.1], for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d_C(\psi, \text{id}_{T^* M}) < \delta \) implies \( \gamma(\psi(0_M)) < \varepsilon \). By the \( C^0 \)-convergence of \( (\varphi_n)_{n \in \mathbb{N}} \), for any \( \delta > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n, m \geq N \), then \( d_{C^0}(\varphi_n^{-1} \varphi_m, \text{id}_{T^* M}) < \delta \). Hence, for any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n, m \geq N \), then \( \gamma(\varphi_n^{-1} \varphi_m(0_M)) < \varepsilon \).

We define \( F_n := K^{\varphi_n} \bullet k_{M \times [0, \infty)} \in \mathcal{D}(M) \). By Proposition 6.9
\[
\gamma(\varphi_n^{-1} \varphi_m(0_M)) = \overline{d_{\mathcal{D}(M)}(k_{M \times [0, \infty)}, K^{\varphi_n^{-1}} \bullet K^{\varphi_m} \bullet k_{M \times [0, \infty)})} = \overline{d_{\mathcal{D}(M)}(F_n, F_m)}.
\] (7.3)
Hence, the sequence \((F_n)_{n \in \mathbb{N}}\) is a Cauchy sequence with respect to \( \overline{d_{\mathcal{D}(M)}} \). This implies that there exists a sequence \((c_n)_{n \in \mathbb{N}}\) of real numbers such that \((T_{c_n} F_n)_{n \in \mathbb{N}}\) is Cauchy with respect to \( d_{\mathcal{D}(M)} \). Thus, there exists a limit object \( F_\infty \) of \((T_{c_n} F_n)_{n \in \mathbb{N}}\) by Corollary [15]. By construction, we obtain \( d_{\mathcal{D}(M)}(k_{M \times [0, \infty)}, F_\infty) < \infty \) and the desired microsupport estimate.

(ii) By construction, we find that
\[
\text{Spec}(F_\infty) = \lim_{n \to \infty} \text{Spec}(T_{c_n} F_n) = \lim_{n \to \infty} T_{-c_n} \text{Spec}(\varphi_n, 0_M).
\] (7.4)
Thus, applying Theorem 6.5 to \( F_\infty \), we obtain the result.

Note that the number of \( \text{Spec}(\varphi, 0_M) \) is the same as that of [BHS22].

8 Arnold-type principle for Hausdorff limits of Legendrians

In this section, we briefly discuss how to prove a Legendrian analogue of Theorem [64] (cf. [BHS22, Thm. 1.5]) by a sheaf-theoretic method. Again, we assume that \( M \) is compact.

Denote by \( J^1 M = T^* M \times \mathbb{R} \) the 1-jet bundle. For a compact Legendrian submanifold \( L \) of \( J^1 M \), we define a conic Lagrangian submanifold \( c(L) \) of \( T^*(M \times \mathbb{R}_t) \) by
\[
c(L) := \{(x, t; \xi, \tau) \mid \tau > 0, (x, \xi/\tau, t) \in L\}.
\] (8.1)
Let \( L \) be a Legendrian submanifold without Reeb chords. Then by the results in [Gui23, Part XII], we can construct \( F_L \in \mathcal{D}(k_{M \times \mathbb{R}_t}) \) such that \( \text{SS}(F_L) = c(L) \).

Consider a sequence \((L_n)_{n \in \mathbb{N}}\) of compact Legendrian submanifolds without Reeb chords of \( J^1 M \). Assume that \((L_n)_{n \in \mathbb{N}}\) converges to a compact subset \( L_\infty \) of \( J^1 M \) with respect to the Hausdorff distance. For each \( L_n \), we can construct \( F_n := F_{L_n} \in \mathcal{D}(k_{M \times \mathbb{R}_t}) \) as above. Following [Gui13] (see also [Gui23, Part VII]), we define \( F_\infty \in \mathcal{D}(k_{M \times \mathbb{R}_t}) \) by the exact triangle
\[
\bigoplus_{n \in \mathbb{N}} F_n \rightarrow \prod_{n \in \mathbb{N}} F_n \rightarrow F_\infty \xrightarrow{\pm 1}.
\] (8.2)
Then we find $F_\infty \in D(M)$ and get $Q^*_\infty (F_\infty) \simeq C \otimes H^*(M; k)$, where $C := \text{Coker}(\bigoplus_{n \in \mathbb{N}} k \to \prod_{n \in \mathbb{N}} k)$. Applying Lemma 2.3 and arguing as in the proof of Lemma 2.9, we have

$$\text{SS}(F_\infty) \subseteq c(L_\infty) = \lim_{n \to \infty} c(L_n) \subseteq T^*(M \times \mathbb{R}_t).$$

Let $t: M \times \mathbb{R}_t \to \mathbb{R}_t$ and $q_\mathbb{R}: J^1M = T^*M \times \mathbb{R}_t \to \mathbb{R}_t$ be the projections. Then we find that

$$\text{SS}(F_\infty) \cap \Gamma_{dt} \subseteq \{(x, t; 0, 1) \mid (x, 0, t) \in L_\infty\} = (L_\infty \cap (0_M \times \mathbb{R}_t)) \times \{1\}$$

and hence

$$- \text{Spec}(F_\infty) \subseteq t\pi(\text{SS}(F_\infty) \cap \Gamma_{dt}) \subseteq q_\mathbb{R}(L_\infty \cap (0_M \times \mathbb{R}_t)).$$

**Proposition 8.1.** In the situation as above, if $\text{Spec}(F_\infty) \subseteq \text{cl}(M)$, then $L_\infty \cap (0_M \times \mathbb{R}_t)$ is cohomologically non-trivial in $M \times \mathbb{R}_t$. In particular, if $\#q_\mathbb{R}(L_\infty \cap (0_M \times \mathbb{R}_t)) \leq \text{cl}(M)$, then $L_\infty \cap (0_M \times \mathbb{R}_t)$ is cohomologically non-trivial in $M \times \mathbb{R}_t$, hence it is infinite.

**Proof.** By applying Theorem 6.5 to the object $F_\infty \in D(M)$, we obtain the result by (8.3) and $Q^*_\infty (F_\infty) \simeq C \otimes H^*(M; k)$.

## A Hamiltonian stability with support conditions

In this appendix, we prove an estimate by the oscillation norm $\|H\|_{osc,A}$ of $H$ restricted to a non-empty closed subset $A$, in the context of the sheaf-theoretic energy estimate.

### A.1 Sheaf quantization of 2-parameter Hamiltonian isotopies

We will use the sheaf quantization of a 2-parameter Hamiltonian isotopy. For that purpose, we first state the main result of [GKS12] in a general form. Let $N$ be a connected non-empty manifold and $W$ a contractible open subset of $\mathbb{R}^n$ with the coordinate system $(w_1, \ldots, w_n)$ containing 0. Let us consider $\psi = (\psi_w)_{w \in W}: T^*N \times W \to T^*N$ be a homogeneous Hamiltonian isotopy, that is, a $C^\infty$-map satisfying (1) $\psi_w$ is homogeneous symplectic isomorphism for each $w \in W$ and (2) $\psi_0 = \text{id}_{T^*N}$. We can define a vector-valued homogeneous function $h: T^*N \times W \to \mathbb{R}^n$ by $h = (h_1, \ldots, h_n)$ with

$$\frac{\partial \psi_w}{\partial w_i} \circ \psi_w^{-1} = X_{h_i(\bullet, w)},$$

where $X_{h_i(\bullet, w)}$ is the Hamiltonian vector field of the function $h_i(\bullet, w): T^*N \to \mathbb{R}$. By using the function $h$, we define a conic Lagrangian submanifold $\Lambda_\psi$ of $T^*(N^2 \times W)$ by

$$\Lambda_\psi := \left\{(\psi_w(y; \eta), (y; -\eta), (w; -h(\psi_w(y; \eta), w))) \mid (y; \eta) \in T^*N, w \in W\right\}$$

A.2

The main theorem of [GKS12] is the following.

**Theorem A.1** ([GKS12, Thm. 3.7 and Rem. 3.9]). Let $\psi: T^*N \times W \to T^*N$ be a homogeneous Hamiltonian isotopy and set $\Lambda_\psi$ as above. Then there exists a unique simple object $\tilde{K} \in D(k_{N^2 \times W})$ such that $SS(\tilde{K}) = \Lambda_\psi$ and $\tilde{K}|_{N^2 \times \{0\}} \simeq k_{\Delta_N}$. 

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For a non-homogeneous compactly supported Hamiltonian isotopy, we can associate a sheaf by homogenizing the isotopy. In the 1-parameter case, it is done as in Definition 3.10. Below we will explain how to homogenize a 2-parameter Hamiltonian isotopy.

Let \((G_{s',s})(s',s)\in\mathcal{I})\) be a 2-parameter family of compactly supported smooth functions on \(T^*M\). A 2-parameter family of diffeomorphisms \(\left(\phi_{s',s}(s',s)\right)\in\mathcal{I}\) is determined by \(\phi_{s',0} = \id_{T^*M}\) and \(\frac{\partial \phi_{s',s}}{\partial s} \circ \phi_{s',s}^{-1} = X_{G_{s',s}}\), where \(X_{G_{s',s}}\) is the Hamiltonian vector field corresponding to the function \(G_{s',s}\). We set \(\tilde{G}_{s',s}(x,t;\xi,\tau) := \tau G_{s',s}(x;\xi/\tau)\) and define a 2-parameter homogeneous Hamiltonian isotopy \(\tilde{\phi} = (\tilde{\phi}_{s',s})_{s',s}\) by

\[
\begin{align*}
\tilde{\phi}_{s',0} &= \id_{T^*(M\times\mathbb{R})}, \\
\frac{\partial \tilde{\phi}_{s',s}}{\partial s} \circ \tilde{\phi}_{s',s}^{-1} &= X_{\tilde{G}_{s',s}}.
\end{align*}
\]  

Then, we have

\[
\Lambda_{\tilde{\phi}} = \left\{ \left(\tilde{\phi}_{s',s}(y;\eta), (y;-\eta), (s';-\tilde{F}_{s',s}(\tilde{\phi}_{s',s}(y;\eta))), (s;-\tilde{G}_{s',s}(\tilde{\phi}_{s',s}(y;\eta))) \right) \mid (y;\eta) \in \tilde{T}^*(M\times\mathbb{R}), s', s \in I \right\},
\]

where the 2-parameter family of homogeneous functions \((\tilde{F}_{s',s}(s',s))\) is determined by \(\frac{\partial \tilde{F}_{s',s}}{\partial s'} \circ \tilde{\phi}_{s',s}^{-1} = X_{\tilde{F}_{s',s}}\). By the construction of \(\tilde{\phi}\), there exists a 2-parameter family of timewise compactly supported functions \((F_{s',s}(s',s))\) satisfying \(\tilde{F}_{s',s}(x,t;\xi,\tau) = \tau F_{s',s}(x;\xi/\tau)\) and \(\frac{\partial F_{s',s}}{\partial s'} \circ \phi_{s',s}^{-1} = X_{F_{s',s}}\). A calculation in [Pol12] or [Oh05] (see also [Ban78]) shows that

\[
\frac{\partial F_{s',s}}{\partial s} = \frac{\partial G_{s',s}}{\partial s'} - \{F_{s',s}, G_{s',s}\},
\]

where \(\{-,-\}\) is the Poisson bracket. In this case, we can apply Theorem A.1 to the homogeneous Hamiltonian isotopy \(\tilde{\phi}\) and obtain a simple object \(\tilde{K}\) satisfying \(\SS(\tilde{K}) = \Lambda_{\tilde{\phi}}\).

By using the map \(q: (M\times\mathbb{R})^2 \times I^2 \to M^2 \times \mathbb{R} \times I^2, (x_1,t_1,x_2,t_2,s',s) \mapsto (x_1,x_2,t_1-t_2,s',s)\), we also obtain an equivalence similar to (3.26). Hence, we can define \(K \in \mathcal{D}(k_{M^2\times\mathbb{R}}} \times I^2)\) by the condition \(\SS(K) = q\SS_{\tilde{\phi}}(\Lambda_{\tilde{\phi}})\) and \(K := P_{\tilde{\phi}}(K) \in \mathcal{D}(M^2 \times I^2)\), which we call the *sheaf quantization* of \((\phi_{s',s}(s',s))\) for any \(s'\in I^2\).

**A.2 Statement and proof**

For a closed subset \(A\) of \(T^*M\), we define a full subcategory \(\mathcal{D}_A(M)\) of \(\mathcal{D}(M)\) by

\[
\mathcal{D}_A(M) := \{F \in \mathcal{D}(M) \mid \SS(F) \cap \{\tau > 0\} \subset P_{\tilde{\phi}}^{-1}(A)\}.
\]

where \(P_\tau: T^*M \times T^*M \to T^*M, (x,t;\xi,\tau) \mapsto (x;\xi/\tau)\).

Let \(K^H \in \mathcal{D}(M^2 \times I)\) be the sheaf quantization associated with a timewise compactly supported function \(H: T^*M \times I \to \mathbb{R}\) and \(F \in \mathcal{D}_A(M)\) with \(A\) being a closed subset of \(T^*M\). Then we get \(K^H_s \bullet F \in \mathcal{D}(M \times I)\) and find that

\[
K^H_s \bullet F \simeq (K^H \bullet F)|_{M \times \{s\} \times \mathbb{R}} \in \mathcal{D}_{\phi^H(A)}(M) \quad \text{for any } s \in I.
\]

We shall estimate the distance between \(F \in \mathcal{D}_A(M)\) and \(K^H_1 \bullet F \in \mathcal{D}_{\phi^H(A)}(M)\) up to translation. See also Remark A.3 for a more straightforward but weaker case.
Theorem A.2. Let $A$ be a non-empty closed subset of $T^*M$ and $F \in D(A(M))$. Moreover, let $H: T^*M \times I \to \mathbb{R}$ be a timewise compactly supported function. Then for a continuous function $f: I \to \mathbb{R}$, one has

$$d_{D(M)}(F, T_c K^H_{A}) \leq \int_0^1 \left( \max \left\{ \max_{p \in \phi_s^H(A)} H_s(p), f(s) \right\} - \min \left\{ \min_{p \in \phi_s^H(A)} H_s(p), f(s) \right\} \right) \, ds$$

where $c = \int_0^1 f(s) \, ds$.

Remark A.3. If we take $f \equiv 0$, we obtain

$$d_{D(M)}(F, K^H_{A}) \leq \int_0^1 \left( \max \left\{ \max_{p \in \phi_s^H(A)} H_s(p), 0 \right\} - \min \left\{ \min_{p \in \phi_s^H(A)} H_s(p), 0 \right\} \right) \, ds.$$  

Let $c \in \mathbb{R}$ be a real number satisfying

$$\int_0^1 \min_{p \in \phi_s^H(A)} H_s(p) \, ds \leq c \leq \int_0^1 \max_{p \in \phi_s^H(A)} H_s(p) \, ds.$$  

Then we can take $f$ such that $c = \int_0^1 f(s) \, ds$ and

$$\min_{p \in \phi_s^H(A)} H_s(p) \leq f(s) \leq \max_{p \in \phi_s^H(A)} H_s(p)$$

for any $s \in I$. Hence, by Theorem A.2, we get

$$d_{D(M)}(F, T_c K^H_{A}) \leq \int_0^1 \left( \max_{p \in \phi_s^H(A)} H_s(p) - \min_{p \in \phi_s^H(A)} H_s(p) \right) \, ds.$$  

For simplicity, we introduce a symbol for the right-hand side of (A.8). For a function $H: T^*M \times I \to \mathbb{R}$, a function $f: I \to \mathbb{R}$, and a non-empty closed subset $A$ of $T^*M$, we set

$$B(H, f, A) := \int_0^1 \left( \max_{p \in \phi_s^H(A)} H_s(p), f(s) \right) - \min \left\{ \min_{p \in \phi_s^H(A)} H_s(p), f(s) \right\} \, ds.$$  

Proof of Theorem A.2. Let $\varepsilon > 0$. We can take a smooth family $(\rho_a, b: \mathbb{R} \to \mathbb{R})_{a,b}$ of smooth functions parametrized by $a, b \in \mathbb{R}$ with $a \leq b$ such that

1. $\rho_a(b) = y$ on a neighborhood of $[a, b]$,
2. $a - \varepsilon \leq \inf_y \rho_a(b) < \sup_y \rho_a(b) \leq b + \varepsilon$.

Recall that $I$ denotes an open interval containing the closed interval $[0, 1]$. We take smooth functions $M, m: I \to \mathbb{R}$ satisfying

$$\max_{p \in \phi_s^H(A)} H_s(p) + \frac{\varepsilon}{2} \leq M(s) \leq \max_{p \in \phi_s^H(A)} H_s(p) + \varepsilon$$  

and

$$\min_{p \in \phi_s^H(A)} H_s(p) - \varepsilon \leq m(s) \leq \min_{p \in \phi_s^H(A)} H_s(p) - \frac{\varepsilon}{2}.$$  

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Fix $R > 0$ sufficiently large so that $R > \max_{p,s} H_s^a(p) - \min_{p,s} H_s(p) + 2\varepsilon$. Define $a(s', s) := m(s) - Rs', b(s', s) := M(s) + Rs'$ for $(s', s) \in I^2$. We may assume that $I \subset (-\infty, +\infty)$ by taking $I$ smaller if necessary, and hence that
\[
a(s', s) \leq \min_{p \in \phi_*^p(A)} H_s(p) \leq \max_{p \in \phi_*^p(A)} H_s(p) \leq b(s', s) \tag{A.16}
\]
for all $(s', s) \in I^2$. Take a smooth function $\tilde{f}: I \to \mathbb{R}$ such that $\|\tilde{f} - f\|_{C^0} \leq \varepsilon$. By shrinking $I$, we may assume that $\bigcup_s \text{supp}(H_s)$ is relatively compact. Then we can also take a compactly supported smooth cut-off function $\chi: T^* M \to [0, 1]$ such that $\chi \equiv 1$ on a neighborhood of $\bigcup_s \text{supp}(H_s)$. Using these functions, we define a function $G = (G_{s', s})_{s', s \in I^2}: T^* M \times I^2 \to \mathbb{R}$ by
\[
G_{s', s} := \left(\rho_{a(s', s), b(s', s)} \circ H_s - (1 - s') \tilde{f}(s)\right) \chi.
\tag{A.17}
\]
A 2-parameter family $(\phi_{s', s})_{s', s \in I^2}$ of Hamiltonian diffeomorphisms is determined by $\phi_{s', 0} = \text{id}_{T^* M}$ and $\frac{\partial \phi_{s', s}}{\partial s} \circ \phi_{s', s}^{-1} = X_{G_{s', s}}$, where $X_{G_{s', s}}$ is the Hamiltonian vector field corresponding to the function $G_{s', s}$. Note that $G_{1,s} = H_s$ and $\phi_{s', s}$ is independent of $s'$ on a neighborhood $U$ of $A$. Moreover, we have
\[
\int_0^1 \left( \max_{p \in T^* M} G_{0,s}(p) - \min_{p \in T^* M} G_{0,s}(p) \right) ds
\leq \int_0^1 \left( \max_{p \in T^* M} \left( \rho_{m,s}(M,s) \circ H_s(p) - \tilde{f}(s) \right) \chi(p) - \min_{p \in T^* M} \left( \rho_{m,s}(M,s) \circ H_s(p) - \tilde{f}(s) \right) \chi(p) \right) ds\tag{A.18}
\leq \int_0^1 \left( \max_{p \in \phi_*^p(A)} \left( H_s(p) - \tilde{f}(s) \right), 0 \right) - \min_{p \in \phi_*^p(A)} \left( H_s(p) - \tilde{f}(s) \right), 0 \right) ds + 2\varepsilon
\leq \int_0^1 \left( \max_{p \in \phi_*^p(A)} H_s(p), \tilde{f}(s) \right) - \min_{p \in \phi_*^p(A)} H_s(p), \tilde{f}(s) \right) ds + 2\varepsilon
\leq B(H, f, A) + 4\varepsilon.
\]
For $s' \in I$, we set $G_{s'} := G_{s', s}: T^* M \times I \to \mathbb{R}$. Then, by Theorem 5.1 and the natural inequality for the distance with respect to functorial operations (see (3.11)), we obtain
\[
d_{D(M)}(F, K^G_1 \bullet F) = d_{D(M)}(K^G_1 \bullet F, K^G_1 \bullet F) \leq B(H, f, A) + 4\varepsilon. \tag{A.19}
\]
We set $\tilde{c}(s) := \int_0^s \tilde{f}(t)dt$ and claim that $K^G_1 \bullet F \simeq T_{-\tilde{c}(1)} K^G_1 \bullet F$. By the result recalled in appendix A.1, we can construct the sheaf quantization $K \in D(M^2 \times I^2)$ of the 2-parameter family of diffeomorphisms $(\phi_{s', s}')_{s', s}$. We shall use the same notation as in appendix A.1. Then, $F_{s', 0} = 0$ and $F_{s', s} \mid_{\phi_{s', s}(U) \times I} : \phi_{s', s}(U) \times I \to \mathbb{R}, (p, s') \mapsto F_{s', s}(p)$ is locally constant for each $s$. By (A.18), we find that $\frac{\partial F_{s', s}}{\partial s} = \frac{\partial G_{s', s}}{\partial s} = \tilde{f}(s)$ on $\bigcup_s \phi_{s', s}(U) \times I \times \{s\}$ and that $F_{s', s} = \int_0^s \tilde{f}(t)dt = \tilde{c}(s)$ there. We define $\mathcal{H} := K \bullet F \in D(M \times I^2)$. Then, by the microsupport estimate, we have
\[
\text{SS}(\mathcal{H}) \subset \left\{ (\tilde{\phi}_{s', s}(x, t; \xi, \tau), (s'; -\tau \tilde{c}(s)), (s; -\tau G_{s', s}(\tilde{\phi}_{s', s}(x; \xi/\tau))) \mid (x, t; \xi, \tau) \in \text{SS}(F), s', s \in I \right\} \cup_{0 \times \mathbb{R} \times I^2}.
\tag{A.20}
\]
Hence, $M \times \mathbb{R} \times I \times \{1\}$ is non-characteristic for $\mathcal{H}$ and we get
\[
\text{SS}(\mathcal{H} \mid_{M \times \mathbb{R} \times I \times \{1\}}) \subset \left\{ (\tilde{\phi}_{s', s}(x, t; \xi, \tau), (s'; -\tilde{c}(1)), (s; -\tilde{c}(1))) \mid (x, t; \xi, \tau) \in \text{SS}(F), s' \in I \right\} \cup_{0 \times \mathbb{R} \times I}.
\tag{A.21}
\]
Define a diffeomorphism \( \varphi: M \times \mathbb{R} \times I \xrightarrow{\sim} M \times \mathbb{R} \times I \), \((x, t, s') \mapsto (x, t - \tilde{c}(1)s', s')\). Then we have \(SS(\varphi_\ast \mathcal{H}|_{M \times \mathbb{R} \times I \times \{1\}}) \subset T^* (M \times \mathbb{R}) \times 0_I\), which shows \(\varphi_\ast \mathcal{H}|_{M \times \mathbb{R} \times I \times \{1\}}\) is the pull-back of a sheaf on \(M \times \mathbb{R}\) by [KS90, Prop. 5.4.5]. In particular,

\[
\begin{align*}
K_{G_1} \ast F &= \mathcal{H}|_{M \times \mathbb{R} \times \{0\} \times \{1\}} \\
&\simeq (\varphi_\ast \mathcal{H}|_{M \times \mathbb{R} \times I \times \{1\}})|_{\{s' = 0\}} \\
&\simeq (\varphi_\ast \mathcal{H}|_{M \times \mathbb{R} \times I \times \{1\}})|_{\{s' = 1\}} \\
&\simeq T_{-\tilde{c}(1)} \mathcal{H}|_{M \times \mathbb{R} \times \{1\}} = T_{-\tilde{c}(1)} K_{G_1} \ast F.
\end{align*}
\]  

(A.22)

Since \( |c - \tilde{c}(1)| \leq \varepsilon \), we have \(d_{D(M)}(T_{-c} K_{G_1}^H \ast F, T_{-\tilde{c}(1)} K_{G_1}^H \ast F) \leq \varepsilon \). Combining the result above and noticing \(G_1 = H\), we obtain

\[
d_{D(M)}(F, T_{-c} K_{G_1}^H \ast F) \leq d_{D(M)}(F, T_{-\tilde{c}(1)} K_{G_1}^H \ast F) + \varepsilon
\]

\[
= d_{D(M)}(F, K_{G_1}^G \ast F) + \varepsilon
\]

\[
\leq B(H, f, A) + 5\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this completes the proof. \(\square\)

**Remark A.4.** Under the same assumption as in Theorem A.2, we can prove the weaker result

\[
d_{w-isom}(F, T_{-c} K_{G_1}^H \ast F) \leq B(H, f, A)
\]

(A.24)
morestraightforwardly, without the 2-parameter family, as follows. Here \(d_{w-isom}\) denotes the pseudo-distance on \(D(M)\) defined by

\[
d_{w-isom}(F, G) := \inf \{a + b \mid (F, G) \text{ is weakly } (a, b)\text{-isomorphic}\}.
\]

(A.25)

We set \(\mathcal{H} := K^H \ast F \in D(M \times I)\). Then we have \(\mathcal{H}|_{M \times \mathbb{R} \times \{0\}} \simeq F\) and \(\mathcal{H}|_{M \times \mathbb{R} \times \{1\}} \simeq K_{G_1}^H \ast F\). Moreover, by the microsupport estimate, we find that

\[
SS(\mathcal{H}) \subset T^* M \times \left\{(t, s; \tau, \sigma) \mid -\max_{p \in \phi_\uparrow(A)} H_s(p) \cdot \tau \leq \sigma \leq -\min_{p \in \phi_\uparrow(A)} H_s(p) \cdot \tau\right\}.
\]

(A.26)

Let \( \varepsilon > 0 \) and take a smooth function \( \tilde{f}: I \rightarrow \mathbb{R} \) such that \( \|f - \tilde{f}\|_{C^0} \leq \varepsilon \). We define a function \( \tilde{c}: I \rightarrow \mathbb{R} \) by \( \tilde{c}(s) := \int_0^s \tilde{f}(s')ds'\) and a function \( \varphi: M \times \mathbb{R} \times I \rightarrow M \times \mathbb{R} \times I \) by \( \varphi(x, t, s) := (x, t - \tilde{c}(s), s)\). Then we have \(\varphi_\ast \mathcal{H}|_{M \times \mathbb{R} \times \{0\}} \simeq F\), \(\varphi_\ast \mathcal{H}|_{M \times \mathbb{R} \times \{1\}} \simeq T_{-\tilde{c}(1)} K_{G_1}^H \ast F\), and

\[
SS(\varphi_\ast \mathcal{H}) \subset T^* M \times \left\{(t, s; \tau, \sigma) \mid -\left(\max_{p \in \phi_\uparrow(A)} H_s(p) - \tilde{f}(s)\right) \cdot \tau \leq \sigma \right\}
\]

\[
\leq -\left(\min_{p \in \phi_\uparrow(A)} H_s(p) - \tilde{f}(s)\right) \cdot \tau.
\]

(A.27)

Note that we may have \(\max_{p \in \phi_\uparrow(A)} H_s(p) - \tilde{f}(s) < 0\) and \(\min_{p \in \phi_\uparrow(A)} H_s(p) - \tilde{f}(s) > 0\) in...
general. By applying Proposition A.5 we obtain
\[
\begin{align*}
    d_{\text{w-isom}}(F, T_{-c}(1)\mathcal{K}^H_1 \bullet F) \\
    = d_{\text{w-isom}}(\varphi^1 \mathcal{H}|_{M \times [0, 1]} \circ \varphi^1 \mathcal{H}|_{M \times [0, 1]}^{-1}) \\
    \leq \int_0^1 \left( \max_{p \in \phi_1^H(A)} \left( H_s(p) - \tilde{f}(s) \right) \right) \, ds \\
    = \int_0^1 \left( \max_{p \in \phi_1^H(A)} H_s(p) - \min_{p \in \phi_1^H(A)} H_s(p) \right) \, ds \\
    \leq \int_0^1 \left( \max_{p \in \phi_1^H(A)} H_s(p) - \min_{p \in \phi_1^H(A)} H_s(p) \right) \, ds + 2\varepsilon.
\end{align*}
\]

Hence, we have
\[
\begin{align*}
    d_{\text{w-isom}}(F, T_{-c}\mathcal{K}^H_1 \bullet F) & \leq d_{\text{w-isom}}(F, T_{-c}(1)\mathcal{K}^H_1 \bullet F) + \varepsilon \\
    & \leq \int_0^1 \left( \max_{p \in \phi_1^H(A)} H_s(p) - \min_{p \in \phi_1^H(A)} H_s(p) \right) \, ds + 3\varepsilon,
\end{align*}
\]
which completes the proof.

For a timewise compactly supported function \( H: T^*M \times I \to \mathbb{R} \) and a non-empty closed subset \( A \) of \( T^*M \), we set
\[
\|H\|_{\text{osc, } A} := \int_0^1 \left( \max_{p \in A} H_s(p) - \min_{p \in A} H_s(p) \right) \, ds. \quad (A.30)
\]

**Proposition A.5.** Let \( A \) be a non-empty closed subset of \( T^*M \) and \( F \in \mathcal{D}_A(M) \). Moreover, let \( H: T^*M \times I \to \mathbb{R} \) be a timewise compactly supported function. Then there exists \( c \in \mathbb{R} \) such that
\[
d_{\mathcal{D}(M)}(F, T_{-c}\mathcal{K}^H_1 \bullet F) \leq \|H\|_{\text{osc, } A}. \quad (A.31)
\]

**Proof.** Using the technique in the proof of [Ush15, Theorem 1.3], one can construct a function \( H' \) such that \( \phi_1^H = \phi_1^{H'} \) and
\[
\begin{align*}
    \int_0^1 \left( \max_{p \in A} H_s(p) - \min_{p \in A} H_s(p) \right) \, ds \\
    = \int_0^1 \left( \max_{p \in \phi_1^{H'}(A)} H'_s(p) - \min_{p \in \phi_1^{H'}(A)} H'_s(p) \right) \, ds.
\end{align*}
\]
By Proposition A.5, \( \phi_1^H = \phi_1^{H'} \) implies \( \mathcal{K}^H_1 \simeq \mathcal{K}^{H'}_1 \). Hence, the result follows from Theorem A.2 (see also Remark A.3). \( \square \)

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