Reparameterization Invariance in Heavy Particle Effective Field Theories

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Abstract

In a low energy effective theory, the fields of heavy particles with mass $m$ and total momentum $p$ depend on both a velocity $v$ and a residual momentum $k$ such that $p = mv + k$. However, there is some arbitrariness in such a description because one may also use a slightly different velocity with a compensating change in the residual momentum. This non-uniqueness is the origin of an invariance under such a reparameterization which imposes non-trivial constraints on the theory. The implications of this invariance for effective theories of heavy spin-0, 1/2, and 1 fields is investigated.

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1 Introduction

In recent years, there has been considerable interest in using effective field theory methods to analyse hadronic processes involving a heavy quark in a model-independent way [1, 2, 3, 4, 5]. The utility of this approach, of course, is not limited to heavy quarks. Indeed, the low-momentum dynamics of heavy particles of other spins interacting with a non-Abelian gauge field may also be conveniently and systematically formulated in terms of an effective field theory. When the typical interaction scale $\Lambda_{SI}$ is much less than the mass $m$ of the heavy particle, $\Lambda_{SI} \ll m$, the heavy particle is essentially a featureless source of colour (or other quantum number) which propagates almost on-shell in a nearly straight world-line except for small deviations due to its interactions with the gauge field and other light degrees of freedom which may be bound to it. Since the interactions of such a heavy particle are independent of its quantum numbers (other than that it may be in a bound state), this is the situation which in the case of QCD is the origin the spin-flavour symmetries [1, 2] which have been so useful in the analysis of heavy quark systems. Moreover, in the infinite mass limit the velocity $v$ (with $v^2 = 1$) of the heavy particle is conserved by such soft interactions, and hence, its momentum $p$ may be split into a large part which contains the kinematic dependence on the heavy mass, $mv$, and a small remaining “residual momentum”, $|k^\mu| \sim \Lambda_{SI}$:

$$p = mv + k.$$  \hspace{1cm} (1)

A heavy particle effective field theory (HPEFT) can then be constructed which incorporates the above picture at leading order in a derivative expansion in powers of $k/m$, and the subleading order operators correct for the finite mass of the heavy particle.

The HPEFT is constructed from fields which depend on the velocity of the particle [3, 6]. However, when recoil effects are taken into account at subleading order in $1/m$, there is no unique assignment of a velocity to the heavy particle which leads to an ambiguity in the partitioning of the total momentum $p$ [7, 8]; instead of assigning the particle a velocity and a residual momentum pair, $(v, k)$, as in eq. (1), one may equally well use the pair

$$(\tilde{v}, \tilde{k}) = (v + \Delta v, k - m \Delta v),$$  \hspace{1cm} (2)

where

$$p = m\tilde{v} + \tilde{k};$$  \hspace{1cm} (3)

with $\tilde{v}^2 = 1$ and $|\tilde{k}| \sim \Lambda_{SI}$. The HPEFT expressed in terms of these two sets of variables are physically equivalent, and this indistinguishability gives rise to a velocity reparameterization invariance (VRI) of the theory [7]. It is only the total momentum $p$ that is unique which indicates that explicit factors of the velocity and residual momentum (or the gauge-covariant
derivative in position space) should occur in the invariant combination

\[ \frac{\hat{p}}{m} = \hat{V} = v + iD_m. \] (4)

While such an observation appears to be trivial, it along with some other relations derived below lead to surprisingly powerful constraints on the kinds of operators which can appear at subleading order in the $1/m$ expansion of the HPEFT.

Note that although states in the full theory depend on $p$ while those in the effective theory carry the two labels $v$ and $k$, extra degrees of freedom have not been introduced in the latter because the velocity is no longer a dynamical observable but rather a conserved quantity [8].

While there has been an analysis of reparameterization invariance for HPEFT before [7], we have been unable to follow that presentation in its entirety, and there is some controversy over the predictions of this invariance [9]. In this paper a different approach is used and is found to lead to some results which appear different from previous ones. In the following sections of this paper, the VRI consequences for effective field theories of heavy particles with spin 0, 1/2, and 1 are investigated.

Such an analysis can, in principle, be applied to heavy particles including composite ones of any spin, and the predictions are valid to all orders in perturbation theory. VRI is also an exact invariance of HPEFT — unlike the spin-flavour symmetries that arise in the heavy quark mass limit of QCD, it is not broken by subleading higher-dimensional operators which are suppressed by powers of the large mass. Rather such operators serve to embody this invariance in the effective field theory.

We shall begin by considering a scalar field theory before going onto cases involving the additional complications of spin and constrained fields.

### 2 Reparameterization Invariance of a Heavy Scalar Effective Field Theory

In a previous paper [6], the field theory of a heavy scalar field $\phi$ of mass $m_s$ with the Lagrangian [6, 7, 10]

\[ \mathcal{L}_s = \left( D_\mu \phi \right)^\dagger D^\mu \phi - m_s^2 \phi^\dagger \phi, \] (5)

where

\[ D^\mu \phi = (\partial^\mu - igA^\mu)\phi \] (6)

was examined. Here, terms in the Lagrangian involving external sources have been dropped. In particular, using a functional integral formalism an effective field theory (HSEFT) was constructed which reproduces the physics of the above full theory Lagrangian, eq. (5), at...
scales below the large mass \( m_s \). In this paper, we re-examine the kind of constraints that are placed on this HSEFT by reparameterization invariance.

As in ref. \([4, 3]\) the effective theory is constructed for scalar particles so that it is expressed in terms of the field \( \phi_v \) which creates and annihilates scalars of a definite velocity and residual momentum but does not act on anti-scalars. Under a reparameterization, as described above, the velocity and residual momentum transform, respectively, as

\[
v \to \tilde{v} = v + \Delta v, \quad (7a)
\]

and

\[
k \to \tilde{k} = k - m_s \Delta v. \quad (7b)
\]

Since the full theory field \( \phi_v \) defined by

\[
\phi_v(x) = \phi_v^+(x) + \phi_v^-(x) = e^{im_s v \cdot x} \phi(x), \quad (8a)
\]

transforms by an overall phase factor, and the effective theory field \( \varphi_v = \phi_v^+ \) is related to \( \phi_v \) by \([6]\)

\[
\varphi_v = \left(1 + \frac{i v \cdot D}{2m_s}\right) \phi_v, \quad (8b)
\]

\( \varphi_v \) transforms as

\[
\varphi_v \to \tilde{\varphi}_v = e^{im_s \Delta v \cdot x} \left\{ \varphi_v + \left[ \frac{i \Delta v \cdot D}{2m_s} + \mathcal{O}((\Delta v)^2) \right] \phi_v \right\} \quad (3a)
\]

\[
= e^{im_s \Delta v \cdot x} \left\{ 1 + \frac{i \Delta v \cdot D}{2m_s} \left[ 1 - \frac{1}{2m_s(2m_s + iv \cdot D + (D_\perp)^2)} (D_\perp)^2 \right] + \mathcal{O}((\Delta v)^2) \right\} \varphi_v
\]

\[
= e^{im_s \Delta v \cdot x} \left\{ 1 + \frac{i \Delta v \cdot D}{2m_s} \left[ 1 - \frac{(D_\perp)^2}{4m_s^2} - \frac{2m_s iv \cdot D + (D_\perp)^2}{16m_s^4} (D_\perp)^2 \right. \\
\quad \left. + \mathcal{O}((\Delta v)^2, \Delta v^2) \right] \right\} \varphi_v. \quad (3b)
\]

The relation in eq. \((3b)\) is exact. However, in the two lines following it, a tree-level result from the path integral approach in ref. \([3]\) has been used to express the antiscalar component \( \phi_v^- \) in terms of \( \varphi_v \). The \( \phi_v^- \) component does not contribute to the transformation of \( \varphi_v \) until order \( 1/m_s^3 \) so eq. \((3b)\) will not receive radiative corrections and hence be valid up to at least this order. Such loop corrections which may be calculated, for instance by matching a three-point function consisting of two scalar and a gauge external line, will generally modify the terms in eq. \((3b)\) beyond order \( 1/m_s^3 \).

Since this effective theory field, even though it represents a spinless particle, transforms non-trivially under a velocity reparameterization, it is now clear that not only must the velocity and covariant derivative appear in the combination in eq. \((4)\), but there are also
additional constraints on the effective Lagrangian coming from the above transformation of the fields, eq. \( \text{(3)} \).

The constraints imposed by such a reparameterization symmetry in the effective theory may be determined by examining its effects on the most general set of operators. The most general effective Lagrangian subject to the general requirements of gauge, space inversion, and time reversal invariance, locality, and hermiticity, which corresponds to the full theory Lagrangian above is

\[
\mathcal{L}_{\text{HSEFT}}^{\text{gen}} = \sum_v \mathcal{L}_{\text{HSEFT},v}^{\text{gen}},
\]

where

\[
\mathcal{L}_{\text{HSEFT},v}^{\text{gen}} = \sum_{j=0}^{\infty} \frac{\mathcal{L}_{\text{HSEFT},v}^{\text{gen}(j)}}{(m_s)^{j-1}},
\]

and the first several terms are

\[
\begin{align*}
\mathcal{L}_{\text{HSEFT},v}^{\text{gen}(0)} & = 2\varphi_v^{\dagger}iv \cdot D\varphi_v, \\
\mathcal{L}_{\text{HSEFT},v}^{\text{gen}(1)} & = \varphi_v^{\dagger} \left( a_{1a} D^2 + a_{1b} (v \cdot D)^2 \right) \varphi_v, \\
\mathcal{L}_{\text{HSEFT},v}^{\text{gen}(2)} & = i\varphi_v^{\dagger} \left[ a_{2a} D^2 v \cdot D D^\mu + a_{2b} \left( D^2 (v \cdot D) + v \cdot DD^2 \right) + a_{2c} (v \cdot D)^3 \right] \varphi_v, \\
\mathcal{L}_{\text{HSEFT},v}^{\text{gen}(3)} & = \varphi_v^{\dagger} \left\{ a_{3a} D^4 + a_{3b} \left[ D^2 (v \cdot D)^2 + (v \cdot D)^2 D^2 \right] + a_{3c} (v \cdot D)^4 \\
& \quad + a_{3d} (D^2 v \cdot DD^\mu v \cdot D + v \cdot DD^\mu v \cdot DD^\mu) + a_{3e} D^\mu (v \cdot D)^2 D^\mu \\
& \quad + a_{3f} D^\mu D^2 D^\mu + a_{3g} D^\mu D^\mu D^\nu D^\nu \right\} \varphi_v, \\
\mathcal{L}_{\text{HSEFT},v}^{\text{gen}(4)} & = i\varphi_v^{\dagger} \left\{ a_{4a} (D^4 v \cdot D + v \cdot D D^4) + a_{4b} D^2 v \cdot D D^2 \\
& \quad + a_{4c} \left( D^2 (D^2 D^\mu v \cdot DD^\mu + D^\mu v \cdot DD^\mu D^2) \right) + a_{4d} \left( D^\mu D^2 v \cdot DD^\mu + D^\mu v \cdot DD^2 D^\mu \right) \\
& \quad + a_{4e} D^\mu D^\nu D^\mu v \cdot DD^\nu D^\mu + a_{4f} D^\mu D^\nu v \cdot DD^\mu D^\nu D^\mu \\
& \quad + a_{4g} D^\mu D^\nu D^\mu v \cdot DD^\nu + D^\nu v \cdot DD^\mu D^\mu D^\mu \right\} + a_{4h} \left[ D^2 (v \cdot D)^3 + (v \cdot D)^3 D^2 \right] \\
& \quad + a_{4i} D^\mu (v \cdot D)^3 D^\mu + a_{4j} v \cdot DD^\mu v \cdot DD^\mu v \cdot DD^\nu D^\mu + a_{4k} \left[ v \cdot DD^2 (v \cdot D)^2 + (v \cdot D)^2 D^2 v \cdot D \right] + a_{4l} (v \cdot D)^5 \right\} \varphi_v.
\end{align*}
\]

The coefficients denoted by “a” have been introduced to take into account short-distance contributions, and the sum over the different velocities is needed to recover Lorentz covariance.

The field may be redefined to eliminate the subleading terms in \( \mathcal{L}_{\text{HSEFT},v}^{\text{gen}(j)} \) which vanish by the leading order equation of motion, namely \( iv \cdot D\varphi_v = 0 \), and would lead to an operator basis considerably simpler than the one above. However, the redefined field will then transform under a reparameterization not as in eq. \( \text{(3b)} \) but in a more complicated way with extra terms containing \( v \cdot D\varphi_v \) so that the analysis is perhaps even more labourious. Moreover the field redefinition must be done at each successive order in \( 1/m_s \) which is not very convenient.
Hence in this and the subsequent analyses, operators which vanish by the leading order equation of motion are retained.

This Lagrangian must remain invariant under a velocity reparameterization as given by eq. (7a-7b, 3a-3b):

\[
\Delta L_{\text{HSEFT}, v}^{\text{gen}} = L_{\text{HSEFT}, \tilde{v}}^{\text{gen}} - L_{\text{HSEFT}, v}^{\text{gen}} = 0. \tag{7}
\]

Imposing this condition on \(L_{\text{HSEFT}, v}^{\text{gen}}\) at each order in \(1/m_s\) up to operators of order \(1/m_s^2\), which is certainly within the range of validity of eq. (3b), yields the following constraints for the coefficients.

\[
\begin{align*}
a_{1a} &= -1, \\
a_{1b} - a_{2a} - 2a_{2b} &= 1, \\
a_{2b} + 2a_{3a} + a_{3f} + a_{3g} &= \frac{1}{2}, \\
\frac{a_{1b}}{2} + a_{2c} + 2a_{3b} + a_{3d} + a_{3e} &= 0, \\
a_{2a} + 2a_{3f} + 2a_{3g} &= 0, \\
a_{2c} + 2a_{3d} &= 0, \\
a_{3b} - 2a_{4a} - a_{4c} &= -\frac{1}{4}, \\
a_{3c} - a_{4j} - 2a_{4k} &= \frac{1}{4}, \\
-\frac{a_{2a}}{2} + a_{3d} - 2a_{4c} - a_{4e} - a_{4f} - a_{4g} &= 0, \\
-\frac{a_{2b}}{2} + a_{3b} - 2a_{4b} - a_{4c} - a_{4d} - a_{4f} &= 0, \\
a_{2b} + 2a_{4a} + a_{4d} &= 0, \\
-\frac{a_{2c}}{2} + a_{3c} - 2a_{4b} - a_{4i} &= 0, \\
a_{3d} &= a_{4g}, \\
a_{3e} - 2a_{4d} - a_{4e} - a_{4f} - a_{4g} &= 0. \tag{8}
\end{align*}
\]

The VRI constraints for this scalar field theory derived here appear to differ from those obtained by Luke and Manohar in ref. [7]. There they found that the effective theory field transforms by a phase under a velocity reparameterization and hence concluded that for the scalar field theory Lagrangian to be reparameterization invariant, it was necessary and sufficient for the factors of \(v\) and \(D\) to always occur in the combination in eq. (4). Here we find that the field in the effective theory transforms under a reparameterization not by just a phase but in accordance to eq. (3a-3b), so that their condition seems to be necessary but not sufficient. Since their transformation does not have any contributions which are suppressed by powers of \(m_s\), it now easy to see that while their leading order prediction, namely, \(a_{1a} = -1\), is in agreement with what is found here, the other constraints would be
expected to differ. The VRI predictions for the operators in \( \mathcal{L}^{\text{gen}(j)}_{\text{HSEFT},v} \) for \( j \geq 2 \) were not calculated in ref. [7], and since the interpretation of eq. (2.9) in that paper is not completely clear to us (i.e., whether the most general reparameterization invariant effective Lagrangian can be constructed by writing down all possible operators assembled out of their \( \phi_v \) and \( \mathcal{V} \) or otherwise), we are unable to make a direct comparison.

In ref. [6], the tree-level matching between the full and effective theory Lagrangians was performed. It is important to compare the results from that calculation with the ones here required by VRI. The tree-level coefficients were found to be

\[
\begin{align*}
-a_1 a &= a_1 b = 1 \\
-a_2 a &= a_2 b = a_2 c = 0 \\
-a_3 a &= a_3 b = a_3 c = \frac{1}{4} \\
-a_3 d &= a_3 e = a_3 f = a_3 g = 0 \\
-a_4 b &= a_4 b = a_4 c = 1 \\
-a_4 f &= a_4 g = a_4 j = a_4 k = 0
\end{align*}
\]

which are consistent with the VRI constraints obtained above.

Next we shall apply these considerations to particles with spin.

### 3 Reparameterization Invariance of a Heavy Spin-\( \frac{1}{2} \) Fermion Effective Field Theory

In this section we formulate a framework for applying a reparameterization invariance analysis to heavy spin-\( \frac{1}{2} \) particles. While this method can be used for any such spin-\( \frac{1}{2} \) particle, for concreteness we shall focus specifically on the physical system where it has been most useful — namely to the low-energy strong interactions of heavy quarks.

A natural way to describe the low-momentum QCD interactions of heavy quarks is in terms of an effective field theory (HQEFT) which is formulated in terms of an expansion in inverse powers of the large mass with operators containing velocity-dependent heavy quark fields [3]. Using the notation in ref. [6], one starts with the QCD Lagrangian for the heavy quark:

\[
\mathcal{L}_{H,QCD} = \bar{\psi} (i \not{D} - m_Q) \psi.
\]

The heavy quark effective field \( h_v^+ \) of HQEFT defined through

\[
\begin{align*}
\psi_v' (x) &= e^{imQv \cdot x} \psi_v (x) \\
h_v^+ (x) &= \frac{1 \pm \gamma_5}{2} \psi_v' (x)
\end{align*}
\]

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acts on quarks of velocity $v$ but not on antiquarks. The transformation of this field under
the above velocity reparameterization eq. (7a-7b) follows from the observation that the field
$\psi'_{v}$ transforms by a global phase factor so that the definitions in eq. (11b) and ref. [6] give

$$h_{v}^+ \rightarrow \tilde{h}_{v}^+ = e^{imQ_{v}\Delta_{v}x} \left[ h_{v}^+ + \frac{\gamma_{\mu}(\Delta_{v})_{\mu}}{2} h_{v} \right]$$ (12a)

$$= e^{imQ_{v}\Delta_{v}x} \left[ 1 + \frac{\gamma_{\mu}(\Delta_{v})_{\mu}}{2} \left( 1 + \frac{1}{2m_{Q}} + \frac{i\not{v}}{D} \right) \right] h_{v}^+$$ (12b)

$$= e^{imQ_{v}\Delta_{v}x} \left[ 1 + \frac{\gamma_{\mu}(\Delta_{v})_{\mu}}{2} \left( 1 + \frac{i\not{D}Q_{v}}{2m_{Q}} - \frac{i\not{v}}{2m_{Q}} \frac{i\not{D}Q_{v}}{2m_{Q}} \right) \right]$$

$$+ O \left( \frac{(\Delta_{v})^{2}}{m_{Q}^{3}} \right) \right] h_{v}^+.$$ (12c)

Eq. (12a) is an exact expression but in the last two equalities above, $h_{v}^-$ has been expressed
in terms of $h_{v}^+$ using a relation in ref. [6] that holds to order $1/m_{Q}$. Once again, radiative
contributions will likely introduce corrections beyond this order.

To simplify the notation, we shall set

$$Q_{v} = h_{v}^+.$$ (13)

Then to investigate the implications of reparameterization invariance on the effective La-
grangian, we first determine the most general form of this Lagrangian consistent with the
QCD symmetries of the theory:

$$\mathcal{L}_{\text{HQEFT}} = \sum_{v} \mathcal{L}_{\text{HQEFT}, v}^{\text{gen}},$$ (14)

with

$$\mathcal{L}_{\text{HQEFT}, v}^{\text{gen}} = \tilde{Q}_{v} i\not{v} \cdot D Q_{v}$$

$$+ \frac{1}{2m_{Q}} \tilde{Q}_{v} \left[ b_{1a}(iD)^{2} - b_{1b}(i\not{v} \cdot D)^{2} + \frac{b_{1c}}{2} g_{s} \sigma^{\mu\nu} G_{\mu\nu} \right] Q_{v}$$

$$+ \frac{1}{4m_{Q}^{2}} \tilde{Q}_{v} \left[ b_{2a} \sfrac{g_{s} v^{\mu} [D_{\nu}, G_{\mu\nu}]}{2} + \frac{ib_{2b}}{2} g_{s} \sigma^{\alpha\mu\nu} v^{\nu} \{ D_{\alpha}, G_{\mu\nu} \} + \frac{ib_{2c}}{2} \{ D^{2}, \not{v} \cdot D \} \right]$$

$$+ i b_{2d} (v \cdot D)^{3} + \frac{ib_{2e}}{2} g_{s} \{ \sigma^{\mu\nu} G_{\mu\nu}, v \cdot D \} \right] Q_{v}.$$ (15)

In anticipation of their future use, we define the following operators.

$$\hat{O}_{1a} = \frac{1}{2m_{Q}} Q_{v} (iD)^{2} Q_{v},$$

$$\hat{O}_{1c} = \frac{g_{s}}{4m_{Q}} \tilde{Q}_{v} \sigma^{\mu\nu} G_{\mu\nu} Q_{v},$$

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\[ \hat{O}_{2a} = \frac{g_s}{8m_Q^2} \bar{Q}_v v^\mu [D^\nu, G_{\mu\nu}] Q_v, \]
\[ \hat{O}_{2b} = \frac{ig_s}{8m_Q^2} \bar{Q}_v \sigma^{\alpha\mu} v^\nu \{D_\alpha, G_{\mu\nu}\} Q_v. \]

Now consider a small reparameterization of the velocity and residual momentum as given by eq. (16), then for physical results to remain invariant, it is necessary to require that
\[ \Delta L_{\text{gen}, \overline{\text{HQEFT}}, v} = L_{\text{gen}, \overline{\text{HQEFT}}, \overline{v}} - L_{\text{gen}, \overline{\text{HQEFT}}, v} = 0. \]

For a small but arbitrary shift of \( v \) by \( \Delta v \) and \( k \) by \( \Delta k = -m_Q \Delta v \), this relation requires that up to order \( 1/m_Q \)
\[ b_{1a} = 1, \]
\[ b_{2b} = 2b_{1c} - 1, \]
\[ b_{2c} = 2b_{1b} - 1. \]

Since these relations should hold to arbitrary order in the coupling, it is essential to verify that they are satisfied by perturbative calculations. The coefficients of the \( \mathcal{O}(1/m_Q) \) operators have been calculated to one-loop order in \([4, 5, 11]\) and are found to be
\[ b_{1a}(\mu) = 1 + \mathcal{O}\left(\alpha_s(\mu)^2\right), \]
\[ b_{1b}(\mu) = 3\left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-\frac{8}{33-2n_f}} - 2, \]
\[ b_{1c}(\mu) = \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-\frac{9}{33-2n_f}}, \]
where \( n_f \) is the number of effective light quark flavors in the momentum interval between \( \mu \) and \( m_Q \). Coefficients of some of the \( \mathcal{O}\left(1/m_Q^2\right) \) operators have also been calculated to the same order; in particular \([11]\)
\[ b_{2b}(\mu) = 2\left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-\frac{9}{33-2n_f}} - 1, \]
so these results are consistent with the VRI predictions above, eq. (18), as required.

The transformation of \( Q_v \) under a reparameterization and some of the constraints required by such a change seem to differ again from those in ref. \([7]\). However, in this case, it was found in ref. \([8]\) that to order \( 1/m_Q^2 \), the difference is in a field redefinition so that to this order these two apparently disparate formulations are physically equivalent \([12]\).

The above analysis is not limited to the effective Lagrangian — it can equally well be applied to other reparameterization invariant combinations of operators such as currents. So next we turn to a similar analysis of the VRI predictions for the weak currents involving...
a heavy quark to order $1/m_Q^2$ which extends some previous results at order $1/m_Q$ \[7, 13, 14\]. We shall begin with an investigation of the heavy-light currents which in QCD are of the form $\bar{q} \Gamma^\alpha \psi_Q$ \[13\]. They are constructed from a heavy quark and a light quark field denoted generically by, $\psi_Q$ and $q$, respectively. These are the vector and axial-vector currents, respectively,

\[
V^\alpha(x) = \bar{q}(x) \gamma^\alpha \psi_Q(x), \quad (\Gamma^\alpha = \gamma^\alpha), \quad (21a)
\]

\[
A^\alpha(x) = \bar{q}(x) \gamma^\alpha \gamma_5 \psi_Q(x), \quad (\Gamma^\alpha = \gamma^\alpha \gamma_5), \quad (21b)
\]

which are clearly reparameterization invariant. As in the preceding analysis, the corresponding heavy quark fields in HQEFT will be represented by $Q_v$.

Since the ensuing reparameterization analysis is virtually the same whether it is the vector or axial-vector current, we shall choose to treat the vector current in detail — the corresponding results for the axial current are simply obtained by replacing $\bar{q}$ with $-\bar{q} \gamma_5$ \[15\]. While the vector current in QCD is conserved, radiative corrections in HQEFT not only effect current renormalization but can also induce additional operators not present at tree level. Hence the matrix element of the vector current in the full theory must be matched to a complete basis of current operators in the effective field theory with the same symmetries:

\[
\bar{q} \gamma^\alpha \psi_Q \simeq \sum c_i^{(0)}(\mu) j_i^{(0)\alpha} + \frac{1}{2m_Q} \left[ \sum_j c_{1,j}^{(1)}(\mu) j_{1,j}^{(1)\alpha} + \sum_k c_{2,k}^{(1)}(\mu) j_{2,k}^{(1)\alpha} \right] + \frac{1}{(2m_Q)^2} \left[ \sum_l c_{1,l}^{(2)}(\mu) j_{1,l}^{(2)\alpha} + \sum_n c_{2,n}^{(2)}(\mu) j_{2,n}^{(2)\alpha} \right] + \mathcal{O} \left( \frac{1}{m_Q^3} \right), \quad (22)
\]

where “$\simeq$” indicates an equality that holds only at the matrix element level, and $\mu$ is the renormalization scale. In eq. (22), a convenient operator basis is

\[
\begin{align*}
\tilde{j}_1^{(0)\alpha} &= \bar{q} \gamma^\alpha Q_v, \\
\tilde{j}_2^{(0)\alpha} &= \bar{q} i \sigma^\alpha Q_v,
\end{align*}
\]

\[
\begin{align*}
J_{1,1}^{(1)\alpha} &= \bar{q} \gamma^\alpha i \slashed{D} Q_v, \\
J_{1,4}^{(1)\alpha} &= -\bar{q} i \slashed{D} \gamma^\alpha Q_v, \\
J_{1,6}^{(1)\alpha} &= \bar{q} \gamma^\alpha i v \cdot \slashed{D} Q_v, \\
J_{1,5}^{(1)\alpha} &= -\bar{q} i \slashed{D} v^\alpha Q_v, \\
J_{1,2}^{(1)\alpha} &= \bar{q} i \sigma^\alpha \slashed{D} Q_v, \\
J_{1,6}^{(1)\alpha} &= -\bar{q} i v \cdot \slashed{D} \gamma^\alpha Q_v, \\
J_{1,2b}^{(1)\alpha} &= \bar{q} v^\alpha i v \cdot \slashed{D} Q_v, \\
J_{1,7}^{(1)\alpha} &= -\bar{q} i v \cdot \slashed{D} v^\alpha Q_v, \\
J_{1,3}^{(1)\alpha} &= \bar{q} i D^\alpha Q_v, \\
J_{1,8}^{(1)\alpha} &= -\bar{q} i \slashed{D}^\alpha Q_v,
\end{align*}
\]

\[
\begin{align*}
\frac{j_{2,k}^{(1)\alpha}(x)}{2m_Q} &= i \int T \left[ j_{l}^{(0)\alpha}(x) \hat{O}_n(y) \right] d^4y, \\
&\quad \text{for } l = 1, 2, n = 1a, 1c, \text{ and } k = 1, \ldots, 4, \quad (23c)
\end{align*}
\]
\[ j_{1,1}^{(2)\alpha} = \bar{q} \gamma^\alpha (iD)^2 Q_v, \quad \quad j_{1,10}^{(2)\alpha} = -\bar{q} i \not \! D \gamma^\alpha \not \! Q Q_v, \]
\[ j_{1,2}^{(2)\alpha} = \bar{q} v^\alpha (iD)^2 Q_v, \quad \quad j_{1,11}^{(2)\alpha} = -\bar{q} i \not \! D v^\alpha \not \! Q Q_v, \]
\[ j_{1,3}^{(2)\alpha} = \bar{q} \gamma^\alpha (i\not \! D)^2 Q_v, \quad \quad j_{1,12}^{(2)\alpha} = -\bar{q} i v \cdot \not \! D \gamma^\alpha \not \! Q Q_v, \]
\[ j_{1,4}^{(2)\alpha} = \bar{q} v^\alpha (i\not \! D)^2 Q_v, \quad \quad j_{1,13}^{(2)\alpha} = -\bar{q} i v \cdot \not \! D v^\alpha \not \! Q Q_v, \]
\[ j_{1,5}^{(2)\alpha} = \bar{q} \gamma^\alpha iv \cdot D \not \! Q Q_v, \quad \quad j_{1,14}^{(2)\alpha} = -\bar{q} i \not \! D iv \cdot D \not \! Q Q_v, \]
\[ j_{1,6}^{(2)\alpha} = \bar{q} v^\alpha iv \cdot D \not \! Q Q_v, \quad \quad j_{1,15}^{(2)\alpha} = -\bar{q} i \not \! D v^\alpha \not \! Q Q_v, \]
\[ j_{1,6b}^{(2)\alpha} = \bar{q} v^\alpha \not \! D iv \cdot D \not \! Q Q_v, \quad \quad j_{1,16}^{(2)\alpha} = -\bar{q} i \not \! D v^\alpha \not \! D \not \! Q Q_v, \]
\[ \cdots , \]
\[ \frac{j_{2,k}^{(2)\alpha}(x)}{4m_Q^2} = \begin{cases} 
  i \int T \left[ j_{l}^{(0)\alpha}(x) \hat{O}_n(y) \right] d^4 y, & \text{for } l = 1, 2, n = 2a, 2b, \text{ and } k = 1, \ldots, 4, \\
  \left( \frac{\delta_{a,b} - 1} {2} \right) i \int T \left[ j_{l}^{(0)\alpha}(x) \hat{O}_m(y), \hat{O}_n(z) \right] d^4 y d^4 z, & \text{for } l = 1, 2, (m, n) = (1a, 1a), (1a, 1c), (1c, 1c), \text{ and } k = 5, \ldots, 10, \\
  i \int T \left[ j_{l}^{(1)\alpha}(x) \hat{O}_n(y) \right] d^4 y, & \text{for } l = 1, \ldots, 8, n = 1a, 1c, \text{ and } k = 11, \ldots, 26, 
\end{cases} \]

where the ellipsis in eq. (23d) denotes dimension-5 operators which are not related to the lower-dimensional ones through reparameterization invariance constraints. In eq. (23e), operators which vanish by the equation of motion, \( iv \cdot D Q_v = 0 \), have not been included because some of the time-ordered products involving such operators can be expressed in terms of other operators by contraction of internal heavy quark fields \[Q\]; they have also been excluded from eq. (23c). Moreover, as explained next, reparameterization invariance does not further constrain time-ordered products of operators.

In eq. (22), the short-distance coefficients of those operators which are time-ordered products are easily obtained; they are just the product of the individual component operators which form it. So, for example, the coefficient of the dimension-4 operator \( j_{2,1}^{(1)\alpha} \) with \( k = 1, l = 1, n = 1a, \) in eq. (23c) above is
\[ c_{2,1}^{(1)} = c_{1}^{(0)} b_{1a}, \]
while for the dimension-5 operator \( j_{2,4}^{(2)\alpha} \) with \( k = 4, l = 2, n = 2b \) in eq. (23c) it is
\[ c_{2,4}^{(2)} = c_{2}^{(0)} b_{2b}. \]
As for the effective Lagrangian above, the variation of the operators in this current expansion under a reparameterization transformation is required to vanish. When the variation is evaluated at the space-time origin for simplicity, this condition then yields the following constraints.

\[
\begin{align*}
2c_2^{(0)} &= 2c_{1,2}^{(1)} = c_{1,3}^{(1)} \\
n - c_{1,1b}^{(1)} + c_{1,3}^{(1)} + c_{1,3b}^{(2)} &= 0 \\
2c_2^{(0)} - c_{1,2b}^{(1)} + c_{1,15}^{(2)} + c_{1,5b}^{(2)} &= 0 \\
2c_1^{(1)} - c_{1,1b}^{(2)} - c_{1,1}^{(2)} - c_{1,3}^{(2)} &= 0 \\
2c_1^{(1)} - c_{1,2}^{(2)} - c_{1,8}^{(2)} &= 0 \\
c_{1,2}^{(1)} + c_{1,2b}^{(1)} - c_{1,2}^{(2)} - c_{1,4}^{(2)} &= 0 \\
2c_1^{(1)} - c_{1,7b}^{(2)} - c_{1,9}^{(2)} &= 0 \\
c_{1,3}^{(1)} - c_{1,7}^{(2)} - c_{1,8}^{(2)} &= 0 \\
c_{1,4}^{(1)} &= c_{1,10}^{(2)} \\
2c_1^{(1)} &= 2c_{1,11}^{(2)} = c_{1,14}^{(2)} \\
c_{1,6}^{(1)} &= c_{1,12}^{(2)} \\
2c_1^{(1)} &= 2c_{1,13}^{(2)} = c_{1,16}^{(2)} \\
c_{1,8}^{(1)} &= c_{1,15}^{(2)}
\end{align*}
\]

These relations are consistent with those obtained for the coefficients of the dimension-three and four operators in ref. [13].

Since the equations in (25) relate the coefficients of lower-dimensional operators to those of higher dimension, one can then readily obtain the coefficient of the latter by performing the much simpler calculation of the former. For instance, the coefficients of the dimension-4 current operators \(j_{1,i}^{(1)\alpha}\) for \(i = 6, 7, 8\), in eq. (23b) have been constructed to next-to-leading order in renormalization-group-improved perturbation theory [13]; then the last three equations in (25) immediately determine the corresponding dimension-5 operators \(j_{1,j}^{(2)\alpha}\) for \(i = 12, 13, 15, 16\). Furthermore, the coefficient of the dimension-3 operator \(j_{1,1}^{(0)\alpha}\) in eq. (23a), has been calculated to two loops in ref. [7]. Then the first relation in eq. (25) immediately furnishes the coefficient \(c_{1,1}^{(1)}\) of the dimension-4 operator \(j_{1,1}^{(1)\alpha}\) to the same level of accuracy. Alternatively, coefficients calculated by any other method are required to satisfy the VRI constraints, eq. (25).

Another class of weak currents in QCD we shall now consider are the heavy-heavy currents which in QCD are of the form \(\bar{\psi}_Q \Gamma^{\alpha} \psi_Q\) (with \(\Gamma^{\alpha} = \gamma^\alpha\) or \(\gamma^\alpha \gamma_5\)) and may be flavour-conserving \((Q' = Q)\) or flavour-changing \((Q' \neq Q)\) [14]. Such currents are also manifestly
reparameterization invariant in QCD. The heavy quark fields in the effective theory corresponding to \( \psi_Q \) and \( \psi_{Q'} \) will be represented by \( Q_v \) and \( Q'_{v'} \), respectively. The presence of the different velocities \( v \) and \( v' \) introduces a new variable \( w = v \cdot v' \) on which the coefficients of operators in the effective theory can depend on. As for the heavy-light case above, we shall focus on the vector current from which it is easy to obtain the corresponding axial vector results as discussed previously.

Equating matrix elements of the vector current in QCD to a full operator set in HQEFT yields the expansion

\[
\bar{\psi}_{Q'} \gamma^\alpha \psi_Q \simeq \sum_i d_i^{(0)}(\mu, w) J_i^{(0)\alpha} + \sum_j \left[ \frac{d_j^{(1)}(\mu, w)}{2m_Q} + \frac{d_j^{(1)'}(\mu, w)}{2m_{Q'}} \right] J_j^{(1)\alpha} + \sum_k \frac{d_{2,k}^{(1)}(\mu, w) J_{2,k}^{(1)\alpha}}{2M} + \sum_{i,j} \frac{d_{i,j}^{(2)}(\mu, w) J_{i,j}^{(2)\alpha}}{4m_Q m_{Q'}} + \sum_{i,j} \frac{d_{i,j}^{(2)'}(\mu, w) J_{i,j}^{(2)\alpha}}{(2m_{Q'})^2} + O \left( \frac{1}{M^3} \right), \tag{26}
\]

where \( M = m_Q \) or \( m_{Q'} \). A suitable basis for the operators in eq. (26) is

\[
J_{1,\{i,3\}}^{(0)\alpha} = \bar{Q}'_{v'} \Omega^\alpha Q_v,
\]

\[
J_{1,\{4,5,6\}}^{(1)\alpha} = \bar{Q}'_{v'} \Omega^\alpha i\gamma^\mu D Q_v,
\]

\[
J_{1,\{8,9,10\}}^{(1)\alpha} = -\bar{Q}'_{v'} i\gamma^\mu D \Omega^\alpha Q_v,
\]

\[
J_{1,\{11,12,13\}}^{(1)\alpha} = -\bar{Q}'_{v'} i\gamma^\mu \gamma^5 D \Omega^\alpha Q_v,
\]

\[
J_{1,\{16,17,18\}}^{(1)\alpha} = -\bar{Q}'_{v'} i\gamma^\mu \gamma^5 \gamma^a D \Omega^\alpha Q_v,
\]

\[
J_{1,14}^{(1)\alpha} = -\bar{Q}'_{v'} i\gamma^\mu \gamma^5 \gamma^a \gamma^b D \Omega^\alpha Q_v,
\]

\[
\frac{J_{2,k}^{(1)\alpha}(x)}{2M} = \left\{ \begin{array}{ll}
\frac{i}{4} \int T \left[ J_t^{(0)\alpha}(x) \delta_n(y) \right] d^4 y, & \text{for } M = m_Q, \ l = 1, 2, 3, \ n = 1a, 1c, \text{ and } k = 1, \ldots, 6,
\ \frac{i}{4} \int T \left[ J_t^{(0)\alpha}(x) \delta_n'(y) \right] d^4 y, & \text{for } M = m_{Q'}, \ l = 1, 2, 3, \ n = 1a, 1c, \text{ and } k = 7, \ldots, 12,
\end{array} \right. \tag{27c}
\]
\[ J^{(2)\alpha}_{1,\{1,2,3\}} = \bar{Q}' v^\alpha \Omega^\alpha (iD)^2 Q_v, \]
\[ J^{(2)\alpha}_{1,\{4,5,6\}} = \bar{Q}' v^\alpha \Omega^\alpha (i\partial)^2 Q_v, \]
\[ J^{(2)\alpha}_{1,\{7,8,9\}} = \bar{Q}' v^\alpha v^\alpha v^\alpha i v \cdot D i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{10,11,12\}} = \bar{Q}' v^\alpha \Omega^\alpha v^\alpha i v' \cdot D i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{13,14,15\}} = \bar{Q}' v^\alpha \Omega^\alpha i v \cdot D i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{16,17,18\}} = \bar{Q}' v^\alpha \Omega^\alpha (i v' \cdot D)^2 Q_v, \]
\[ J^{(2)\alpha}_{1,\{19\}} = \bar{Q}' v^\alpha i D^\alpha i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{20\}} = \bar{Q}' v^\alpha i D^\alpha i v' \cdot D Q_v, \]
\[ J^{(2)\alpha}_{1,\{21\}} = \bar{Q}' v^\alpha \Omega^\alpha i v \cdot D i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{22\}} = \bar{Q}' v^\alpha v^\alpha v^\alpha i v \cdot D i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{23\}} = \bar{Q}' v^\alpha i v' \cdot D i D^\alpha Q_v, \]
\[ J^{(2)\alpha}_{1,\{47,48,49\}} = -\bar{Q}' v^\alpha i i\partial \Omega^\alpha i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{50,51,52\}} = -\bar{Q}' v^\alpha i i\partial \Omega^\alpha v^\alpha i v' \cdot D Q_v, \]
\[ J^{(2)\alpha}_{1,\{53,54,55\}} = -\bar{Q}' v^\alpha i i\partial \Omega^\alpha v^\alpha i v \cdot D Q_v, \]
\[ J^{(2)\alpha}_{1,\{56\}} = -\bar{Q}' v^\alpha i i\partial D^\alpha Q_v, \]
\[ J^{(2)\alpha}_{1,\{57\}} = -\bar{Q}' v^\alpha \Omega^\alpha i i\partial v^\alpha \cdot D Q_v, \]
\[ J^{(2)\alpha}_{1,\{58,59,60\}} = -\bar{Q}' v^\alpha \Omega^\alpha i i\partial i D Q_v, \]
\[ J^{(2)\alpha}_{1,\{61,62,63\}} = -\bar{Q}' v^\alpha \Omega^\alpha v^\alpha i v' \cdot D i\partial Q_v, \]
\[ J^{(2)\alpha}_{1,\{64\}} = -\bar{Q}' v^\alpha i v' \cdot D i D^\alpha Q_v, \]
\[ J^{(2)\alpha}_{1,\{65\}} = -\bar{Q}' v^\alpha i D^\alpha i v' \cdot D Q_v, \]
\[ J^{(2)\alpha}_{1,\{66\}} = -\bar{Q}' v^\alpha i D^\alpha v^\alpha i v \cdot D Q_v, \]

\[ (27d) \]
Since the short-distance coefficients here also depend on the velocities through the quantity \( v \) vanish by the equation of motion have been omitted from eq. (27c) and (27e).

However, in contrast to the heavy-light case there is a subtlety which did not arise before. Since the short-distance coefficients here also depend on the velocities through the quantity \( w = v \cdot v' \), they will also transform under a change in the velocity: under the velocity shift \( v \rightarrow v + \Delta v \),

\[
d(v \cdot v') \rightarrow d(v \cdot v') + (\Delta v \cdot v') \frac{\partial d(w)}{\partial w} + \mathcal{O} \left( (\Delta v)^2 \right), \tag{28a}
\]

while for \( v' \rightarrow v' + \Delta v' \),

\[
d(v \cdot v') \rightarrow d(v \cdot v') + (v \cdot \Delta v') \frac{\partial d(w)}{\partial w} + \mathcal{O} \left( (\Delta v')^2 \right), \tag{28b}
\]
The other arguments as well as the subscripts and superscripts of the coefficients have been suppressed here. Hence the right-hand side of the heavy-heavy current expansion, eq. (26), is reparameterization invariant when not only the operators but also the arguments of the coefficients are transformed under a shift in the velocities and the residual momenta. This approach is equivalent to the observation in ref. [14] that both the operators and the coefficients must be written in a form which is invariant under reparameterization. Applying this requirement to eq. (26) and working to linear order in the shifts in the velocity and residual momentum and to order $1/M$ yields the following constraints.

\[ d_{1,j}^{(1)} = 0, \quad \text{for } j = 8, \ldots, 14, \]
\[ d_{1,j}^{(1)'} = 0, \quad \text{for } j = 1, \ldots, 7, \]
\[ d_{1,j}^{(2)} = 0, \quad \text{for } j = 24, \ldots, 65b, \]
\[ d_{1,j}^{(2)'} = 0, \quad \text{for } j = 1, \ldots, 46, \]
\[ d_{1,j}^{(2)''} = 0, \quad \text{for } j = 1, \ldots, 23, \text{ and } 47, \ldots, 65b. \] (29a)

\[ d_{1,1}^{(0)} = d_{1,1}^{(1)} = d_{1,8}^{(1)'} = d_{1,47}^{(2)'} \]
\[ 2d_{1,2}^{(0)} = 2d_{1,2}^{(1)} = d_{1,7}^{(1)'} = 2d_{1,48}^{(1)'} = d_{1,56}^{(2)'} \]
\[ 2d_{1,3}^{(0)} = 2d_{1,3}^{(1)} = 2d_{1,10}^{(1)'} = d_{1,14}^{(1)'} = 2d_{1,49}^{(2)'} = d_{1,57}^{(2)'} \]
\[ 2d_{1,4}^{(1)} = 2d_{1,11}^{(1)'} = 2d_{1,50}^{(2)'} = 2d_{1,53}^{(2)'} = d_{1,58}^{(2)'} \]
\[ 2d_{1,5}^{(1)} = 2d_{1,12}^{(1)'} = 2d_{1,51}^{(2)'} = 2d_{1,54}^{(2)'} = d_{1,59}^{(2)'} = d_{1,64}^{(2)'} \]
\[ 2d_{1,6}^{(1)} = 2d_{1,13}^{(1)'} = 2d_{1,52}^{(2)'} = 2d_{1,55}^{(2)'} = d_{1,60}^{(2)'} = d_{1,65}^{(2)'} \] (29b)

\[ 2 \frac{\partial d_{1,1}^{(0)}}{\partial w} = d_{1,1}^{(1)} = d_{1,11}^{(1)'} = d_{1,53}^{(2)'} \]
\[ 2 \frac{\partial d_{1,2}^{(0)}}{\partial w} = d_{1,1}^{(1)} = d_{1,12}^{(1)'} = d_{1,54}^{(2)'} \]
\[ 2 \frac{\partial d_{1,3}^{(0)}}{\partial w} = d_{1,1}^{(1)} = d_{1,13}^{(1)'} = d_{1,55}^{(2)'} \]
\[
d^{(0)} - d^{(1)}_{1,1b} + d^{(2)}_{1,1b} + d^{(2)}_{1,13b} = 0
\]
\[
d^{(1)}_{1,1} + d^{(1)}_{1,1b} - d^{(2)}_{1,1} - d^{(2)}_{1,14} = 0
\]
\[
d^{(0)} - d^{(1)}_{2,15b} + d^{(2)}_{1,1} + d^{(2)}_{1,14b} = 0
\]
\[
2d^{(1)}_{2,12} - d^{(2)}_{1,19} - d^{(2)}_{1,21} = 0
\]
\[
d^{(1)}_{1,2} + d^{(1)}_{1,5b} - d^{(2)}_{1,12} - d^{(2)}_{1,5} = 0
\]
\[
2d^{(1)}_{3,5} - d^{(2)}_{1,30} - d^{(2)}_{1,22} = 0
\]
\[
d^{(0)} - d^{(1)}_{1,6b} + d^{(2)}_{1,15b} = 0
\]
\[
d^{(1)}_{1,3} + d^{(1)}_{1,6b} - d^{(2)}_{1,3} - d^{(2)}_{1,6} = 0
\]
\[
d^{(1)}_{1,4} - d^{(2)}_{1,10} - d^{(2)}_{1,13} = 0
\]
\[
2d^{(1)}_{1,5} - d^{(2)}_{1,20} - d^{(2)}_{1,23} = 0
\]
\[
d^{(1)}_{1,5} - d^{(2)}_{1,11} - d^{(2)}_{1,14} = 0
\]
\[
d^{(1)}_{1,6} - d^{(2)}_{1,12} - d^{(2)}_{1,15} = 0
\]
\[
d^{(1)}_{1,7} - d^{(2)}_{1,19} - d^{(2)}_{1,21} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,10} - d^{(2)}_{1,13} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,11} - d^{(2)}_{1,14} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,12} - d^{(2)}_{1,15} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,16} - d^{(2)}_{1,16} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,16} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,16} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,17b} - d^{(2)}_{1,17b} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,18} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,18} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,20} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} - d^{(2)}_{1,23} = 0
\]
\[
d^{(1)}_{1,4b} = d^{(2)}_{1,50b}
\]
\[
d^{(1)}_{1,5b} = d^{(2)}_{1,51b}
\]
\[
d^{(1)}_{1,6b} = d^{(2)}_{1,52b}
\]
\[
d^{(1)}_{1,7} = d^{(2)}_{1,56}
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} = d^{(2)}_{1,64}
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} = d^{(2)}_{1,64}
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} = d^{(2)}_{1,51}
\]
\[
2\frac{\partial d^{(1)}_{1,1b}}{\partial w} = d^{(2)}_{1,65}
\]
\[
d^{(0)} - d^{(1)}_{1,11b} + d^{(2)}_{1,30} + d^{(2)}_{1,36b} = 0
\]
\[
d^{(1)}_{1,1b} + d^{(1)}_{1,11b} - d^{(2)}_{1,24} - d^{(2)}_{1,27} = 0
\]
\[
d^{(0)} - d^{(1)}_{1,12b} + d^{(2)}_{1,31} + d^{(2)}_{1,37} = 0
\]
\[
2d^{(1)}_{1,19} - d^{(2)}_{1,42} - d^{(2)}_{1,44} = 0
\]
\[
d^{(1)}_{1,1} + d^{(1)}_{1,12b} - d^{(2)}_{1,25} - d^{(2)}_{1,28} = 0
\]
\[
2d^{(1)}_{1,12b} - d^{(2)}_{1,34} - d^{(2)}_{1,37} = 0
\]
\[
d^{(0)} - d^{(1)}_{1,13b} + d^{(2)}_{1,32} + d^{(2)}_{1,38b} = 0
\]
\[
d^{(1)}_{1,10} + d^{(1)}_{1,13b} - d^{(2)}_{1,26} - d^{(2)}_{1,29} = 0
\]
\[
d^{(1)}_{1,11} - d^{(2)}_{1,33} - d^{(2)}_{1,36} = 0
\]
\[
2d^{(1)}_{1,12} - d^{(2)}_{1,34} - d^{(2)}_{1,46} = 0
\]
\[
2d^{(1)}_{1,12} - d^{(2)}_{1,34} - d^{(2)}_{1,37} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} - d^{(2)}_{1,35} - d^{(2)}_{1,35} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} - d^{(2)}_{1,38} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} - d^{(2)}_{1,39} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} - d^{(2)}_{1,39b} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} - d^{(2)}_{1,40} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} - d^{(2)}_{1,40b} + d^{(2)}_{1,40} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} - d^{(2)}_{1,40} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} - d^{(2)}_{1,41} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} - d^{(2)}_{1,41b} - d^{(2)}_{1,41c} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} - d^{(2)}_{1,43} - d^{(2)}_{1,46} = 0
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} + d^{(2)}_{1,11b} = d^{(2)}_{1,53b}
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} = 2d^{(2)}_{1,54b} = d^{(2)}_{1,64b}
\]
\[
2\frac{\partial d^{(1)}_{1,13b}}{\partial w} = d^{(2)}_{1,55b}
\]
\[
2\frac{\partial d^{(1)}_{1,11b}}{\partial w} = d^{(2)}_{1,57}
\]
\[
2\frac{\partial d^{(1)}_{1,12b}}{\partial w} = d^{(2)}_{1,50}
\]
\[
2\frac{\partial d^{(1)}_{1,13b}}{\partial w} = d^{(2)}_{1,52}
\]
As for the heavy-light current, the coefficients of the operators given in eq. (27c) and (27d) whose definition involve time-ordered products are simply the product of the individual component operators forming them, and hence will not be displayed explicitly.

The VRI predictions given in ref. [7, 14] for the coefficients of some of the dimension-3 and dimension-4 operators in eq. (27a) and (27b), respectively, are generalized by the relations displayed in eq. (29a) and (29b) above.

Although the complete basis of current operators on the right-hand side of eq. (26) grows dramatically at higher orders in $1/M$, the relations required by VRI provide us with considerable predictive power. As pointed out in ref. [14], reparameterization invariance relates the coefficients of all local dimension-4 operators which do not vanish by the equations of motion in the current expansion, eq. (26), to those of the dimension-3 operators; the precise conditions are given by a subset of eq. (29a) and (29b). While the coefficients of the dimension-5 operators in eq. (26) cannot all be uniquely determined in terms of lower-dimensional ones solely on the basis of VRI, it provides a large number of exact constraints which are displayed above in eq. (29a-29e).

Some dimension-5 operators, however, are completely determined by these equations as we shall now exemplify. The coefficients of the dimension-3 operators, $J_i^{(0)\alpha}$ for $i = 1, 2, 3$, and the dimension-4 operators, $J_i^{(1)\alpha}$ for $i = 1,6,8-13$, were calculated in the leading logarithmic approximation in ref. [15]. The relations in eq. (29b), (29d), and (29e) will then uniquely determine the coefficients of the dimension-5 operators in these equations in terms of them. For instance, the second relation in eq. (29b) gives

$$2\frac{\partial d_2^{(1)}}{\partial w} = 2d_{1,2}^{(1)} = 2d_{1,7}^{(2)} = 2d_{1,19}^{(2)} = d_{1,56}^{(2)} = \frac{\alpha_s(\bar{m})}{\alpha_s(\mu)} a_L(w),$$

while using this result in combination with the eighth relation in eq. (29b) yields

$$2\frac{\partial d_2^{(0)}}{\partial w} = 2d_{1,5}^{(1)} = d_{1,12}^{(1)} = d_{1,54}^{(2)} = 2d_2^{(0)} \frac{\partial a_L(w)}{\partial w} \ln \left[ \frac{\alpha_s(\bar{m})}{\alpha_s(\mu)} \right] a_L(w),$$

where

$$a_L(w) = \frac{8}{33 - 2n_f} [w r(w) - 1],$$

(30)
\[ r(w) = \frac{\ln(w + \sqrt{w^2 - 1})}{\sqrt{w^2 - 1}}. \]

The quantity \( \bar{m} \) is an average mass between \( m_Q \) and \( m_{Q'} \). Furthermore, from the first line of eq. (29e) one obtains
\[ 2 \frac{\partial d_{1,4}^{(1)}}{\partial w} = 2 \frac{\partial d_{1,11}^{(1)'}}{\partial w} = d_{1,61}^{(2)'}, \]
\[ = \frac{8}{33 - 2n_f} \left[ \frac{\ln \alpha_s(\bar{m})}{\alpha_s(\mu)} \right] \frac{\partial}{\partial w} \left[ \frac{r(w) - 1}{w^2 - 1} \left[ \frac{\ln \alpha_s(\bar{m})}{\alpha_s(\mu)} \right] a_L(w) \right]. \quad (33) \]

In these equations, the dependence of the coefficients on \( \mu \) and \( w \) have been omitted for simplicity. With relations such as these, once the coefficients of the dimension-3 operators have been calculated to subleading order, the coefficients of not only some dimension-4 but also some dimension-5 operators are immediately determined to the same accuracy without additional labour.

In the next section, we shall similarly treat systems of higher spin which involve constraints.

## 4 Reparameterization Invariance of a Heavy Vector Effective Field Theory

To illustrate the application of the above methods to a massive spin-1 system, we shall use the model field theory previously considered in ref. [6] with the Lagrangian
\[ \mathcal{L}_V = -\frac{1}{2} (D_\mu A_\nu - D_\nu A_\mu) \left[ (D^\mu A^\nu - D^\nu A^\mu) + (m_V)^2 A^\mu \right]. \quad (34) \]

In \( \mathcal{L}_V \), \( A^\mu \) is the vector field with mass \( m_V \) with interactions prescribed by the gauge-covariant derivative \( D^\mu \). As we saw in ref. [6], the effective field theory for this system can be expressed in terms of the dynamical field \( A_{\mu,v}^\perp \) which acts only on vectors and not antivectors. It is related to the field in the full theory \( A^\mu \) through
\[ A_{\mu,v}^\perp = \left( 1 + \frac{i v \cdot D}{2m_V} \right) e^{im_V\cdot x} A_{\mu,v}^\perp, \quad (35) \]

and its transformation under a velocity reparameterization [6] is determined by
\[ A_{\mu,v}^+ \rightarrow \tilde{A}_{\mu,v}^+ = e^{im_V\Delta v \cdot x} \left[ A_{\mu,v}^+ + \frac{i \Delta v \cdot D}{2m_V} A_{\mu,v}^\perp + \mathcal{O} \left( (\Delta v)^2 \right) \right], \quad (36a) \]
\[ = e^{im_V\Delta v \cdot x} \left\{ A_{\mu,v}^+ + \frac{i \Delta v \cdot D}{2m_V} \left[ A_{\mu,v}^\perp + (B_{2^{-1}})^{\mu,\nu} C_{\nu} \right] + \mathcal{O} \left( (\Delta v)^2 \right) \right\}, \quad (36b) \]
\[ = e^{im_V\Delta v \cdot x} \left\{ A_{\mu,v}^+ + \frac{i \Delta v \cdot D}{2m_V} \left[ A_{\mu,v}^\perp + \left( \frac{1}{4(m_V)^2} - \frac{i v \cdot D}{8(m_V)^3} \right) C_{\mu} \right] + \mathcal{O} \left( (\Delta v)^2, \frac{\Delta v}{(m_V)^2} \right) \right\}. \quad (36c) \]
where

\[(B_2)_{\mu\nu} = [2m_V(2m_V + iv \cdot D) + (D_\perp)^2]g_{\mu\nu} - D_{\perp\nu}D_{\perp\mu} - (v \cdot D - im_V)D_{\perp\mu}[(D_\perp)^2 + (m_V)^2]^{-1}D_{\perp\nu}(v \cdot D - im_V),\]

and

\[C_\nu = (D_\perp)^2A_{\nu,v}^+ - D_\perp^\alpha D_\perp\nu A_{\alpha,v}^+ - (v \cdot D - im_V)D_{\perp\nu}[(D_\perp)^2 + (m_V)^2]^{-1}D_\perp^\alpha (v \cdot D - im_V)A_{\alpha,v}^+.\]

The exact transformation is given by eq. (36a), but in the two equalities following it, a result from the path integral calculation in ref. [3] has once again been used to write \(A_\mu^+\) in terms of \(A_{\mu,v}^+\).

As in the other cases examined above, we shall determine the implications of these transformations on the most general set of operators which respect the symmetries of the theory and hence may appear in the Lagrangian:

\[\mathcal{L}_{\text{HVEFT}}^{\text{gen}} = \sum_v \mathcal{L}_{\text{HVEFT},v}^{\text{gen}},\]

where

\[\mathcal{L}_{\text{HVEFT},v}^{\text{gen}} = \sum_j \frac{\mathcal{L}_{\text{HVEFT},v}^{\text{gen}(j)}}{(m_V)^{j-1}},\]

in which the first five terms in this expansion are

\[\mathcal{L}_{\text{HVEFT},v}^{\text{gen}(0)} = -2g_{\mu\nu}(A_{\mu,v}^+)^\dagger iv \cdot DA_{\nu,v}^+,\]

\[\mathcal{L}_{\text{HVEFT},v}^{\text{gen}(1)} = (A_{\mu,v}^+)^\dagger \left\{ g_{\mu\nu} \left[ d_{1a}D^2 + d_{1b}(v \cdot D)^2 \right] + d_{1c}D^\mu D^\nu + d_{1d}D^\mu D^\nu \right\} A_{\nu,v}^+,\]

\[\mathcal{L}_{\text{HVEFT},v}^{\text{gen}(2)} = i(A_{\mu,v}^+)^\dagger \left\{ g_{\mu\nu} \left[ d_{2a}D\alpha \cdot DD^\alpha + d_{2b}(D^2v \cdot D + v \cdot DD^2) + d_{2c}(v \cdot D)^3 \right] 
+ d_{2d}(D^\mu D^\nu v \cdot D + v \cdot DD^\mu D^\nu) + d_{2e}(D^\nu D^\mu v \cdot D + v \cdot DD^\nu D^\mu) 
+ d_{2f}D^\mu v \cdot DD^\nu + d_{2g}D^\nu v \cdot DD^\mu \right\} A_{\nu,v}^+,\]

\[\mathcal{L}_{\text{HVEFT},v}^{\text{gen}(3)} = (A_{\mu,v}^+)^\dagger \left\{ g_{\mu\nu} \left[ d_{3a}D^4 + d_{3b} \left[ D^2(v \cdot D)^2 + (v \cdot D)^2D^2 \right] + d_{3c}(v \cdot D)^4 \right] + d_{3d}(D_\alpha v \cdot DD^\alpha v \cdot D + v \cdot DD_\alpha v \cdot DD^\alpha) + d_{3e}D_\alpha(v \cdot D)^2D^\alpha + d_{3f}D_\alpha D^\alpha D^\alpha 
+ d_{3g}D_\alpha D_\beta D^\alpha D^\beta + d_{3h}(D^\mu D^\nu D^2 + D^2D^\mu D^\nu) + d_{3i}(D^\nu D^\mu D^2 + D^2D^\nu D^\mu) 
+ d_{3j}D^\mu D^2 D^\nu + d_{3k}D^\nu D^2 D^\mu + d_{3l}D^\mu D_\alpha D^\nu + D^\alpha D_\alpha D^\nu 
+ d_{3m}D_\alpha D^\mu D^\nu D^\alpha + d_{3n}D_\alpha D^\nu D^\mu D^\alpha + d_{3o}(D_\alpha D^\nu D^\mu D^\alpha + D^\nu D^\alpha D^\mu D_\alpha) 
+ d_{3p} \left[ D^\mu D^\nu(v \cdot D)^2 + (v \cdot D)^2D^\mu D^\nu \right] + d_{3q} \left[ D^\nu D^\mu(v \cdot D)^2 + (v \cdot D)^2D^\nu D^\mu \right] 
+ d_{3r}D^\mu(v \cdot D)^2D^\nu + d_{3s}D^\nu(v \cdot D)^2D^\mu + d_{3t}v \cdot DD^\mu v \cdot D \right\} A_{\nu,v}^+,\]

\[\mathcal{L}_{\text{HVEFT},v}^{\text{gen}(4)} = i(A_{\mu,v}^+)^\dagger \left\{ g_{\mu\nu} \left[ d_{4a}(D^4 v \cdot D + v \cdot DD^4) + d_{4b}D^2 v \cdot DD^2 \right] \right\} A_{\nu,v}^+,\]
Constraint-type operators are those where covariant derivatives are contracted with the vectors fields.

Now requiring that the theory be unchanged by the reparameterization in eq. (7a-7b) gives the following relations

\[ d_{1a} = 1, \]
\[ d_{1b} - d_{2a} - 2d_{2b} = -1, \]
\[ d_{2b} + 2d_{3a} + d_{3f} + d_{3g} = -\frac{1}{2}, \]
\[ \frac{d_{1b}}{2} + d_{2c} + 2d_{3b} + d_{3d} + d_{3e} = 0, \]
\[ d_{2a} + 2d_{3f} + 2d_{3g} = 0, \]
\[ d_{2c} + 2d_{3d} = 0, \]
\[ d_{3b} - 2d_{4a} - d_{4c} = \frac{1}{4}, \]
\[ d_{3c} - d_{4j} - 2d_{4k} = -\frac{1}{4}, \] (42a)
\[ -\frac{d_{2a}}{2} + 2d_{4a} - d_{4e} - d_{4f} - d_{4g} = 0, \]
\[ -\frac{d_{2b}}{2} + d_{3b} - 2d_{4b} - d_{4c} - d_{4d} - d_{4g} = 0, \]
\[ \frac{d_{2b}}{2} + 2d_{4a} + d_{4d} = 0, \]
\[ -\frac{d_{2c}}{2} + d_{3c} - 2d_{4h} - d_{4i} = 0, \]
\[ d_{3d} = d_{4g}, \]
\[ d_{3e} - 2d_{4d} - d_{4c} - d_{4f} - d_{4g} = 0. \]

There are also relations between the constraint type terms above; for example,

\[ d_{1c} = -d_{1d}. \] (42b)

For consistency it should verified that the tree-level matching performed in the functional integral formalism of ref. [3] obey these constraints. From that calculation, one finds

\[ d_{1a} = -d_{1b} = d_{1c} = -d_{1d} = d_{2d} = d_{3j} = -d_{3r} = -d_{3t} = 1, \]
\[-d_{3a} = d_{3b} = -d_{3c} = -d_{3h} = d_{3i} = d_{3p} = d_{3q} = \frac{1}{4},
\]
\[d_{4b} = -d_{4h} = d_{4l} = \frac{1}{8},\]

and for \( \rho = 2a-2c, 2e, 2f, 2g, 3d-3i, 3k-3q, 3s, 3u, 4a, 4c-4g, 4i-4k, d_{\rho} = 0, \)

which satisfy the conditions in eq. (42a) and (42b) as they should.

The results presented here seem to be different from those in ref. [7]. In that paper, Luke and Manohar used Lorentz transformation properties to arrive at an effective theory field which simply picks up a phase under reparameterization. Hence they were able to write down the most general reparameterization invariant Lagrangian symbolically in closed form expressed in terms of this field. In the formulation given above, the transformation of the effective theory field \( A_{\mu,v}^{\perp} \) or \( A_{\mu,v}^{+} \) is given by eq. (36a-36c) and there is no field constructed from combinations of \( A_{\mu,v}^{\perp} \) which transforms by only a phase. Thus it is not possible here to achieve what they did except to follow the approach given above.

5 Summary

The velocity reparameterization invariance of heavy particles effective field theories is a consequence of the observation that only the total momentum is unique and not in how it is split into two pieces as in eq. (1). The implications of this invariance has been examined for particles of different spin. It places unusually strong constraints on such theories by relating the short-distance coefficients of operators of different dimension. Although these coefficients can be calculated by other means such as (perturbative) matching with renormalization group running, it is important to note, however, that while the values obtained must be determined order-by-order in the loop expansion, the predictions of VRI are exact results which hold to arbitrary order in perturbation theory.

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References

[1] N. Isgur and M.B. Wise, *Phys. Lett.* **B232**, 113 (1989); **B237**, 527 (1990).

[2] M.B. Voloshin and M.A. Shifman, *Yad. Fiz.* **45**, 463 (1987) [Sov. J. Nucl. Phys. **45**, 292 (1987)].

[3] H. Georgi, *Phys. Lett.* **B240**, 247 (1990).

[4] E. Eichten and B. Hill, *Phys. Lett.* **B243**, 427 (1990).

[5] A.F. Falk, B. Grinstein, and M. Luke, *Nucl. Phys.* **B357**, 194 (1991).

[6] C.L.Y. Lee, UCSD-TH-97-23.

[7] M. Luke and A.V. Manohar, *Phys. Lett.* **B286**, 348 (1992).

[8] M.J. Dugan, M. Golden and B. Grinstein, *Phys. Lett.* **B282**, 142 (1992).

[9] M. Finkemeier, H. Georgi, and M. McIrwin, *Phys. Rev.* **D55**, 6933 (1997).

[10] H. Georgi and M.B. Wise, *Phys. Lett.* **B243**, 279 (1990).

[11] See for example,
    C.L.Y. Lee, CALT-68-1663 (rev), C. Balzereit and T. Ohl, *Phys. Lett.* **B386**, 335 (1996),
    M. Finkemeier and M. McIrwin, *Phys. Rev.* **D55**, 377 (1997).

[12] C. Bauer and A.V.Manohar, UCSD/PTH 97-19, hep-ph/9708300.

[13] M. Neubert, *Phys. Rev.* **D49**, 1542 (1994).

[14] M. Neubert, *Phys. Lett.* **B306**, 357 (1993).

[15] A.F. Falk, M. Neubert, and M. Luke, *Nucl. Phys.* **B388**, 363 (1992).

[16] A.F. Falk, M. Luke, and M.J. Savage, *Phys. Rev.* **D49**, 3367 (1994).

[17] D.J. Broadhurst and A.G. Grozins, *Phys. Rev.* **D52**, 4082 (1995).