Distinguishing Quantum and Classical Many-Body Systems

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Controllable systems relying on quantum behavior to simulate distinctly quantum models so far rely on increasingly challenging classical computing to verify their results. We develop a general protocol for confirming that an arbitrary many-body system, such as a quantum simulator, can entangle distant objects. The protocol verifies that distant qubits interacting separately with the system can become mutually entangled, and therefore serves as a local test that excitations of the system can create non-local quantum correlations. We derive an inequality analogous to Bell’s inequality[1,2] which can only be violated through entanglement between distant sites of the many-body system. Although our protocol is applicable to general many-body systems, it requires finding system-dependent local operations to violate the inequality. A specific example in quantum magnetism is presented.

Quantum simulators can efficiently model quantum systems [3,4]. However, characterizing and validating such devices is in general difficult. Indeed, quantum state tomography[5] for even eight qubits has required weeks of classical computational processing time[6] (in addition to exponentially growing measurement requirements). This issue is also present in process tomography, the analogue for quantum channels[7,8]. Notwithstanding the number of measurements growing exponentially in system size, for systems with 10+ constituents reliable tomography is expected to break down from systematic errors in preparation and measurement[9]. Although these resource scaling problems are partially alleviated with methods based on compressed sensing[10–14], the cost still scales at least linearly with the Hilbert space dimension, and thus exponentially in the number of constituents. Added difficulties arise when data is limited to experimentally accessible local observables[15].

Despite the costs of completely characterizing large quantum systems, there do exist scalable tests giving incomplete – though useful – descriptions of system behavior. For example, techniques such as randomized benchmarking[16–19] and fidelity estimation[20,21] require a number of measurements polynomial in system size, while still quantifying useful information such as error rates or average gate fidelities. In the case of locally correlated errors, this is sufficient to guarantee the operation of error-corrected quantum computer[22–25]. Recent linear optics experiments have likewise partially verified implementations of boson sampling, a task which currently cannot be carried out classically[26]. Although it is impossible to efficiently verify this sampling, by using the results of Ref. [27] the authors of Ref. [28] were able to distinguish the sampled distribution from a uniform one. The experiment of Ref. [29] produced similar results, while also checking whether the photon statistics corresponded to indistinguishable particles.

This paper presents another incomplete test, in the context of quantum simulators of many-body systems (MBS)[30,31]. It derives from a constructive procedure to entangle two distant ancilla qubits through local interactions with the many-body system. It starts by preparing the system in an initial, known state, composed of many spatially-fixed sites (see Fig. 1). We apply a local perturbation to a single site, conditioned on the state of an ancilla qubit in superposition. The simulator then propagates the system forward in time, correlating the ancilla with other sites of the MBS. At one of these distant sites, we apply a second perturbation controlled by a second ancilla qubit. This interaction is chosen to increase the probability amplitude between the current excited MBS state and its initial state, but only does so if both qubits are in the same control state. Qualitatively, excitations induced by the first qubit are conditionally removed by the second. Although both ancillas are now correlated, their correlation with the MBS prevent them from being entangled. We therefore measure the MBS and post-select for it being in its original state. This both disentangles the MBS from the ancillas and increases the probability that they are in the same eigenstate. Finally, we can directly verify that the qubits are entangled (e.g., through state tomography).

As a toy model, we consider a one dimensional transverse field Ising Hamiltonian,

$$\hat{H} = B \sum_i \sigma_x^{(i)} - J \sum_i \sigma_z^{(i)} \otimes \sigma_z^{(i+1)} . \quad (1)$$

We write the evolution for a time $\tau$ as $V_\tau = \exp(-i\tau \hat{H})$. In the $B/J \ll 1$ limit, the eigenstates of $\hat{H}$ are well approximated by eigenstates of the transverse field term, $H_J = -J \sum_i \sigma_z^{(i)} \otimes \sigma_z^{(i+1)}$. We therefore assume an initial state of the form

$$|g\rangle = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle , \quad (2)$$

where $\sigma_z |0\rangle = |0\rangle$, which minimizes the energy $\langle \hat{H} \rangle$ up to corrections of order $B \cdot (B/J)^N$. The lowest lying excitations of $H_J$ are described by domain walls. For
our protocol to produce entanglement, we use excitations that are indistinguishable from \(|g|\) outside a finite region. We therefore consider the evolution of the next lowest excitations of \(\hat{H}_J\), involving pairs of domain walls of the form,

\[ |e_{i,j}⟩ = \prod_{i\leq k \leq j} σ_x^{(k)} |g⟩ , \tag{3} \]

where \(1 < i \leq j < n\). Evolution under \(\hat{H} = \hat{H}_J + \hat{H}_B\) disperses these states across the chain\[32\].

We now consider how to use the MBS to entangle two distant qubits. First, an ancilla qubit (labeled \(A\)) in state \(|+⟩ = (|0⟩ + |1⟩)/\sqrt{2}\) applies controlled unitary \(\hat{U}_1\) to site 1 of the initial MBS state \(|ψ⟩_{12}\), producing a conditional excited state \((|0⟩_A ⊗ |ψ⟩_{12} + |1⟩_A ⊗ U_1 |ψ⟩_{12})/\sqrt{2}\).

Time evolution \(\hat{V}_r\) propagates the MBS, spreading the excitation across the system. Ancilla qubit \(B\) (also in superposition) then applies controlled unitary \(\hat{U}_2\) to site 2, removing the excitation (conditional on its initial state). Finally, a post-selection conditioned on projector \(\hat{P}\) is carried out, confirming the MBS has returned to its original state. This projects the ancillas into a (possibly entangled) correlated state. c) An analogous protocol using only one ancilla qubit. After post-selection, measurements of the single ancilla are sufficient to determine whether the previous protocol would have entangled the two ancillas.

At this point, the ancilla state is correlated with the chain, conditioned on the state of the ancilla qubit, \(\hat{U}_{A,2} |+⟩_A \otimes |g⟩\) dispersion is equivalent to that of a continuous-time quantum walk in two dimensions\[34\]. The distinct behaviors of \(|g⟩\) and \(|e_{2,2}⟩\) under \(\hat{V}_r\) will allow us to distinguish the states of the ancilla through a local operation at spin \(N - 1\). After evolving for a time \(τ \sim N/B\), the excitation \(\hat{V}_r |e_{2,2}⟩\) has a probability \(\sim 1/N^2\) of being localized in state \(|e_{1,N-1,1}⟩\). We verify this numerically by using the quantum walk analogy above. As seen in Fig. 2a, the peak transition probability \(|⟨e_{1,N-1,1} | \hat{V}_r |e_{2,2}⟩|^2\) scales as \(\sim 1/N^2\). Key to the success of our protocol, we note that the unitary \(σ_x\) at site \(N - 1\) maps the excitation \(e_{1,N-1,1}\) back to the ground state \(|g⟩\). Hence we can applying \(σ_x^{(N-1)}\) to \(\hat{V}_r |e_{2,2}⟩\) to give it a non-zero overlap with the ground state.

\[ r = ⟨g|σ_x^{(N-1)} \hat{V}_r |e_{2,2}⟩ = ⟨e_{N-1,N-1} | \hat{V}_r |e_{2,2}⟩ ≠ 0 , \tag{7} \]

where \(|r|^2 \sim 1/N^2\). The time \(τ\) required for the overlap \(|r|^2\) to reach its peak scales linearly with \(N\) (Fig. 2b), as opposed to a time scale \(\sim N^2/B\) observed in diffusive propagation\[35\].

We can use the fact that \(|e_{2,2}⟩\) can transition to \(|e_{1,N-1,1}⟩\) via \(σ_x^{(N-1)}\) to entangle ancilla \(A\) with a second ancilla, \(B\). After time evolution \(\hat{V}_r\), we apply a second controlled-NOT gate between \(B\) and spin \(N - 1\),

\[ \hat{U}_{N-1,B} |0⟩_B ⊗ \hat{1} + |1⟩_B ⊗ σ_x^{(N-1)} , \tag{8} \]

The resulting state displays correlations between both ancillas and the MBS,

\[ \hat{U}_{N-1,B} \hat{V}_r \hat{U}_{A,2} |+⟩_A ⊗ |g⟩ = \frac{1}{2} \left( |0⟩_A ⊗ e^{iθ} |g⟩ + |1⟩_A ⊗ \hat{V}_r |e_{2,2}⟩ + |0⟩_A ⊗ e^{iθ} σ_x^{(N-1)} |g⟩ + |1⟩_A ⊗ σ_x^{(N-1)} \hat{V}_r |e_{2,2}⟩ \right) . \]

To motivate the final step, observe that if the ancillas are in either state \(|0⟩_A\) or \(|1⟩_A\), the MBS is necessarily in an excited state. Thus if we measure and post-select the MBS to be in the ground state \(|g⟩\), we project the ancilla qubits into an entangled superposition of states

\[ \frac{1}{\sqrt{2}} \left( |0⟩_A ⊗ e^{iθ} |g⟩ + |1⟩_A ⊗ \hat{V}_r |e_{2,2}⟩ \right) . \]
excited state, though the combined evolution \( \hat{\rho}_C \) produces a non-zero overlap with the ground state. After presenting the rest of the system. As in the experimental example, this post-selection imparts a known correlation to the ancilla qubits.

The actual test of the MBS derives from verifying that the ancilla qubits are entangled. Writing out the qubit density matrix in the \( \sigma_z \) basis, we have

\[
\rho_{ij,i'j'} = \frac{1}{\mathcal{P}} \left\langle \left( \hat{U}_1 \right)^{i'} \hat{V}_T^\dagger \hat{U}_2 \hat{V}_T \right\rangle, \tag{11}
\]

where \( \langle \hat{O} \rangle = \langle \psi_{12E} | \hat{O} | \psi_{12E} \rangle \) refers to an average over the MBS state alone, and \( \mathcal{P} \) is the probability of making the projective measurement \( \hat{P} \) for the state of Equation (10). The product \( (\hat{U}_2)_{ij} \hat{V}_T^\dagger \hat{U}_1^\dagger \) (with \( i, j \) either 0 or 1) represents the unitary evolution of the MBS conditioned on the ancillas being in initial state \( |ij\rangle \). Hence to generically determine whether the qubits are entangled, one may carry out full state tomography of the qubits’ density matrix which can be done through concurrent local measurements on the individual qubits.

The protocol outlined above is the central result of our paper. Given the ability to prepare and measure an initial state, it provides a local test verifying that the MBS propagator can generate non-local entanglement. Such a result precludes a description in which subsystems are locally quantum but all correlations between subsystems are essentially classical. In this light, the protocol is akin to a Bell’s inequality applied to the system and its dynamics as a whole. Like Bell’s inequality, it uses only local operations. It also has a ‘loophole’: we require that the sites of the MBS are spatially stationary. Otherwise a single site could migrate through the MBS and interact with both ancilla to entangle them, bypassing the need for quantum information to be passed between different sites of the MBS.

Although the procedure we have presented generically requires two ancilla qubits to be carried out, under certain assumptions this requirement may be loosened. Indeed, although full state tomography on the ancillas is required to generically verify entanglement, we note that by equation (11) the qubit density matrix is completely determined by averages over the MBS alone. This means that in certain cases, even in the absence of one of the ancilla qubits, it is possible to test whether entanglement would have occurred. Such a test derives from the Peres-Horodecki criterion[36, 37], which states that the ancillas are entangled if and only if the partial transpose matrix \( \rho^T \) has a negative eigenvalue. This property is characterized by Sylvester’s criterion, which states that a square matrix \( A \) has no negative eigenvalues if and only if its 

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
principal minors are all non-negative[38]. Hence to confirm that the ancilla qubits are entangled, it is sufficient to check that a single principal minor of $\rho^F$ is negative: $\rho_{01,01}^F \rho_{10,10}^F - \rho_{01,10}^F \rho_{10,01}^F < 0$. We map this statement to an expression on the MBS by using $\rho_{ij,i'j'}^F = \rho_{i'j'}^E \rho_{ij}^F$ and equation (11):

$$\langle \hat{V} \hat{U} \hat{P} \hat{V} \hat{U} \rangle < \left| \langle \hat{V} \hat{P} \hat{U} \hat{V} \hat{U} \rangle \right|^2.$$  

(12)

When this inequality holds, the ancilla qubits become entangled under our protocol. It can only be satisfied when quantum correlations are propagated between spatially distant sites. Importantly, since $\langle O \rangle = \langle \psi_{12E} \mid O \mid \psi_{12E} \rangle$ represents averages over only the state $\mid \psi_{12E} \rangle$, so it characterizes the many-body system and its evolution alone.

Although inequality (12) implies the pair of qubits in the protocol become entangled, it can actually be measured using a single ancilla qubit. First, we note that the product of terms on the left hand side require no ancilla qubit and site 1 of the MBS. As before, the right hand side requires an ancilla qubit to measure. Preparing the ancilla in state $| \psi \rangle$, we follow the same unitary protocol as in equation (10), except in this case we use the single ancilla as the control for both unitaries $\hat{U}_1$ and $\hat{U}_2$. Written out, this produces the state

$$| \phi \rangle = \frac{1}{\sqrt{2}} \left( | 0 \rangle_A \otimes | \psi \rangle + | 1 \rangle_A \otimes \hat{U}_2 \hat{V} \hat{U}_1 | \psi \rangle \right).$$  

(13)

The real and imaginary parts of $\langle \hat{V} \hat{P} \hat{U}_2 \hat{V} \hat{U}_1 \rangle$ for the many-body state $| \psi \rangle$ are then the means of $\sigma_x \otimes \hat{P}$ and $\sigma_y \otimes \hat{P}$ for the compound state $| \phi \rangle$,

$$\langle \hat{V} \hat{P} \hat{U}_2 \hat{V} \hat{U}_1 \rangle = \langle \phi \rangle | \sigma_x \otimes \hat{P} \rangle + i \langle \phi \rangle | \sigma_y \otimes \hat{P} \rangle.$$

(14)

Preparing the state $| \phi \rangle$ of Equation (13) requires the ancilla qubit to interact with both sites of the many-body system, but certain cases require only a single site interaction to measure $\langle \hat{V} \hat{P} \hat{U}_2 \hat{V} \hat{U}_1 \rangle$. This occurs when the post-selection projector takes a tensor product form,

$$\hat{P} = \hat{P}_2 \otimes \hat{P}_{1E} \otimes \hat{1}_{E'}.$$  

(15)

The product $\hat{P} \hat{U}_2$ can also be written in this way,

$$\hat{P} \hat{U}_2 = \left( \hat{B}_+ + i \hat{B}_- \right) \otimes \hat{P}_{1E} \otimes \hat{1}_{E'}.$$  

(16)

where we have written $\hat{P}_2 \hat{U}_2$ in terms of its Hermitian and anti-Hermitian parts, and as before we let $E'$ denote the (inaccessible) environmental degrees of freedom. Using this decomposition, it suffices to prepare the state

$$| \phi' \rangle = \frac{1}{\sqrt{2}} \left( | 0 \rangle_A \otimes \hat{V} | \psi \rangle + | 1 \rangle_A \otimes \hat{V} \hat{U}_1 | \psi \rangle \right),$$

(17)

which requires only a controlled unitary between the ancilla qubit and site 1 of the MBS. As before, the right hand side of (12) can then be written as a sum of observables,

$$\langle \hat{V} \hat{P} \hat{U}_2 \hat{V} \hat{U}_1 \rangle = \langle \phi' \rangle \left( \sigma_x \otimes \hat{B}_+ - \sigma_y \otimes \hat{B}_- \right) \langle \phi' \rangle + i \langle \phi' \rangle \left( \sigma_x \otimes \hat{B}_+ + \sigma_y \otimes \hat{B}_- \right) \langle \phi' \rangle.$$  

(18)

We note that with identity (11), both of these procedures can be generalized to do complete state tomography.

The ideas we have presented have potential applications in a variety of existing experimental setups. For example, current ion trap experiments have the potential to simulate spin models displaying long-range propagation of correlations[39–44], making them amenable to the single ancilla protocol described above. The models studied have ground states that can be both prepared and measured through direct fluorescence spectroscopy following (if necessary) adiabatic passage. The protocol’s ancilla qubit can take the form of either one of the ions present in the system or one of the global motional modes associated with the ion trap. A key requirement in these setups is the ability to individually address single ions in the experiment[45–48]. Alternatively, in optical lattice many-body simulators[49–53] it is possible to use polarization of light as the ancilla qubit. Based on selection rules arising out of angular momentum conservation, the controlled unitary operation of the ancilla would correspond to a polarization-dependent interaction with a localized subsystem. Between interactions the light must be sent through a delay line (e.g., a Fabry-Perot cavity), so that correlations between spatially distant MBS sites are given enough time develop[54, 55].

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