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Analysis of novel fractional COVID-19 model with real-life data application

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1. Background and motivation

The epidemiology of infectious diseases examines how the disease occurs under ordinary conditions and so properties of location, environment, and factor are evaluated. The cycle formed by taking these features into consideration is called the infection chain in which there are relationships between the agent, the mode of transmission, and the host. When one of these relationships is prevented, it is not possible to see that infectious disease in the community. Throughout history, epidemics such as cholera, chickenpox, malaria, HIV, and especially plague have deeply affected states and people. Infectious diseases were more devastating effects such as plague and cholera. While typhoid and cholera can be transmitted from person to person with body wastes and secretions, typhus is transmitted to the person when a lice biting the sick person bites a non-sick person, that is, it needs a vector. It is possible to be protected from these diseases by paying attention to hygiene rules, using wastewater infrastructure, providing clean water, and cleaning lives of the survivors have never been the same after this hard period, and people had to continue their lives by suffering severe trauma and fear. In the 19th century, outbreaks of typhoid, cholera, and typhus began to appear. The infectious disease cholera occurred in India in the late 18th and early 19th centuries and was as effective as the plague. Besides, smallpox, another disease affecting the world, was generally seen in children, causing them to bear the traces of the disease on their faces throughout their lives. To cope with the deaths caused by this disease, the Turks applied the vaccine to children. Although the malaria epidemic caused many deaths like other outbreaks, it did not have devastating effects such as plague and cholera. While typhoid and cholera can be transmitted from person to person with body wastes and secretions, typhus is transmitted to the person when a lice biting the sick person bites a non-sick person, that is, it needs a vector. It is possible to be protected from these diseases by paying attention to hygiene rules, using wastewater infrastructure, providing clean water, and cleaning

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## A R T I C L E   I N F O

**Keywords:**
- Fractional operators
- Caputo derivative
- COVID-19
- Epidemiology
- Numerical scheme

## A B S T R A C T

The current work is of interest to introduce a detailed analysis of the novel fractional COVID-19 model. Non-local fractional operators are one of the most efficient tools in order to understand the dynamics of the disease spread. For this purpose, we intend as an attempt at investigating the fractional COVID-19 model through Caputo operator with order $\chi \in (0,1)$. Employing the fixed point theorem, it is shown that the solutions of the proposed fractional model are determined to satisfy the existence and uniqueness conditions under the Caputo derivative. On the other hand, its iterative solutions are indicated by making use of the Laplace transform of the Caputo fractional operator. Also, we establish the stability criteria for the fractional COVID-19 model via the fixed point theorem. The invariant region in which all solutions of the fractional model under investigation are positive is determined as the non-negative hyperoctant $\mathbb{R}_+^n$. Moreover, we perform the parameter estimation of the COVID-19 model by utilizing the non-linear least squares curve fitting method. The sensitivity analysis of the basic reproduction number $R_0$ is carried out to determine the effects of the proposed fractional model’s parameters on the spread of the disease. Numerical simulations show that all results are in good agreement with real data and all theoretical calculations about the disease.
food and beverages. In the 19th century, when diseases were not yet known to spread with pathogens (i.e. disease-causing microorganisms and viruses), doctors could cause pathogens to spread from one patient to another, causing their patients to become infected and even to die from the infection. The view that illnesses could be transmitted through touch, especially through doctors who were used to heal, was not easily accepted at that time. It should be emphasized that the determination of human beings to survive cannot be underestimated. Although great losses have been inflicted, humanity has managed to overcome the most terrible epidemics that have ever happened and continued its kind.

At the present time, the spread of infectious diseases has become easier with the spread of international trade, tourism, and transportation. Additionally, economic conditions, poverty, scarcity, housing problems, etc. are also factors that increase the effect of epidemic diseases and accelerate its spread. Anatolia, which is located especially on the intercontinental transit route, is a region that has been exposed to epidemics of both European and Asian origin since the early ages. One of the periods, when epidemics captured Anatolia was the 19th century, the time of the Ottoman Empire. In this period; plague, cholera, malaria, and smallpox diseases caused huge loss of life and property in almost every part of Anatolia. The government tried to take various measures to prevent epidemics, such as quarantine, drug supply, doctor, pharmacist, and vaccine assignment, in coordination with the world. While these outbreaks affecting the whole world brought great disaster to the society, on the other hand, they negatively affected trade and caused a decrease in tax revenues. In addition, the people leaving sick people and places played a crucial role in spreading the epidemic to other places. The fact that individuals left the outbreaks caused land not to be cultivated and also agriculture and trade could not be handled. Some of the measures taken by the state against the consequences of infectious diseases, such as raising awareness of the public, holidaying schools, starting quarantine practice, banning food and beverages causing illness, were among the most important (see Figs. 1 and 2).

Since ancient times, many qualified studies have been conducted on contagious diseases occurring in the world and in Europe. Some of these studies have been written on only one epidemic disease, and some have been published in a general epidemic history feature involving a large number of outbreaks. While some outbreaks were the subject of a periodic or regional review, numerous books and articles included the history of one or more outbreaks in a city, region, or country. In order to make inquiries in detail about smallpox, measles, diphtheria, Spanish Flu (the deadliest pandemic), Black Death (plague), gonorrhea, tuberculosis, pneumonia, dengue fever, yellow fever, malaria, and some other infectious diseases, we refer the reader to [1–4]. Many new types of such infectious diseases as Lyme (1975), legionary (1976), toxic shock syndrome (1978), AIDS (1981), hepatitis C (1989), hepatitis E (1990), hantavirus (1993), hepatitis G (1995), Nipah (1998), SARS (2003) have emerged in the last 50 years. For more information on these diseases, we refer the reader to [5].

Coronavirus is another dangerous infectious disease that first appeared in the 1960s and causes respiratory infections in humans. This virus is a large group of viruses that cause disease in animals and humans. Coronavirus takes their name from the protein capsules that form a crown, and because of this feature, viruses are called "corona", which means Latin crown. The main causes of transmission of coronavirus, which was seen as the first case in Wuhan, China in December 2019, are similar to other viruses. These reasons can be listed as sneezing, coughing, contacting infected people, touching items used daily, and so on. Owing to Chinese New Year migration, severe epidemic coronavirus spread rapidly to the other parts of China and to many countries. On February 11, 2020, the World Health Care Organization (WHO) has announced the official name of the disease caused by the new type of coronavirus as COVID-19. As experienced in epidemic diseases like SARS, MERS, h5n1 bird flu virus, h1n1 swine flu virus, and Ebola, new type coronavirus (COVID-19) also spread rapidly in a short time and bring about thousands of deaths. Moreover, this new type of virus is known to cause respiratory infections in humans, from the common cold to more severe diseases such as the Middle East Respiratory Syndrome (MERS) and Severe Acute Respiratory Syndrome (SARS) or pneumonia. The Covid-19, which was first seen in December, infects three million people in the early period of the epidemic by causing the death of 200 thousand people. COVID-19 outbreak dramatically changes

Fig. 1. $R_0$ versus sensitive parameters $\mu_1$ and $\alpha$.

Fig. 2. $R_0$ versus sensitive parameters $\mu_1$ and $\mu_A$.

Fig. 3. $R_0$ versus sensitive parameters $\mu_1$ and $\delta$. 
the way millions of people live around the world, and some of these changes will be permanent. It is possible to compare this situation to the Spanish flu, the 1929 economic crisis that opened the door to the second world war, or to the wave of fascism/communism that paralyzed liberal institutions, politics, economies, and social values. The new type of coronavirus, which led to COVID-19 disease, is the seventh variant of human coronaviruses. This new type of coronavirus, which can be transmitted from animal to person and from person to person, was not known before the outbreak began in Wuhan, China in 2019. With the PNAS article published very soon, British researchers are examining how the virus mutates by comparing the genomes of viruses in 160 different samples worldwide using their own phylogenetic techniques. Accordingly, there are three main types of COVID-19: A, B, and C. Type A is the most similar variant to the natural virus from the bat. Interestingly, type A virus is not the most common type seen in Wuhan and Chinese patients. This type mostly caused illness in America and Australia. Type B is the main type that infects Chinese and East Asian people. Type C is the subtype spreading in Italy, France, England, Sweden, in Europe. When and how did A transform into B and C with a mutation, why it caused more infections in Americans rather than where it was born, why certain types prefer certain populations, perhaps these are explained by the adaptive mutation feature of the virus? Especially those with advanced age and chronic disease are known to experience COVID-19 more severely. Many people (80 percent) survive the disease with mild symptoms and at home. According to WHO, the "incubation period" of the new virus, that is, from the moment the infection is infected until the symptoms appear, is about 2–14 days. Many scientists warn that some people spread the infection even without symptoms. In the SARS epidemic, which appeared in 2003 and has similarities with COVID-19, 1 in 10 patients died, so the mortality rate was around 10 percent. MERS was an even more fatal disease. The death rate among those caught in

Fig. 4. \( R_0 \) versus sensitive parameters \( \rho_I \) and \( \mu_I \).

Fig. 5. \( R_0 \) versus sensitive parameters \( \mu_A \) and \( \delta \).

Fig. 6. \( R_0 \) versus sensitive parameters \( \rho_I \) and \( \mu_A \).

Fig. 7. \( R_0 \) versus sensitive parameters \( a \) and \( \delta \).

Fig. 8. \( R_0 \) versus sensitive parameters \( \mu_A \) and \( \alpha \).
COVID-19 varies greatly from country to country. This crisis, besides being a disaster that is not in the account of almost any country, society, family, at a time when science and technology were so advanced, the first reactions to the attack of a virus, the source of which was still not fully understood, were denial, shock, and surprise.

The fractional analysis is an important field of mathematical analysis that allows the order of the derivative or integral to be real or complex. Recently, fractional derivatives and integrals have been used frequently to critically enunciate the main characteristics of the problems in nature. Multifarious authors have presented that the fractional operators can more accurately express the natural phenomena than traditional derivation.

As we see in the model (4), the population consists of seven compartments, where:

- $S(t)$ = susceptible individuals at time $t$,
- $E(t)$ = exposed individuals who carry the coronavirus at low-level (there is no infectiousness),
- $I(t)$ = infected individuals with certain symptoms of COVID-19,
- $R(t)$ = recovered individuals at time $t$,
- $Q(t)$ = individuals who are in the quarantine process at time $t$ (they do not contact with the infected people),
- $A(t)$ = asymptomatic infected individuals at time $t$,
- $D(t)$ = the number of medically verified cases in which patients are in quarantine treatment at time $t$,
- $\alpha$ = contact rate to individuals,
- $\delta$ = the rate of spread among infectious people with symptoms of COVID-19 and asymptomatic infected class is different. Hence, the parameter $\delta \in (0,1)$ represents this main difference,
- $k$ = the quarantine rate of individuals,
- $\gamma$ = the release rate of quarantined class $Q$,
- $\eta$ = the transmission rate of exposed people to infected class,
- $\eta$ = after infected, the proportion of getting symptomatic is denoted by $\eta$, and it is $1 - \eta$ when getting asymptomatic,
- $\mu_A$ = the diagnostic rate of asymptomatic infectious individuals,
- $\mu_I$ = the diagnostic rate of symptomatic infectious individuals,
- $1/\psi_A$ = the mean recovery period of compartment $A$,
- $1/\psi_I$ = the mean recovery period of compartment $I$,
- $1/\psi_D$ = the mean recovery period of compartment $D$,
- $\rho_I$ = the disease-induced death rate of compartment $I$,
- $\rho_D$ = the disease-induced death rate of compartment $D$.

After introducing above essential information on the model (4), we wish to propose the corresponding fractional type model employing Caputo derivative as below:

2. Formulation of the fractional COVID-19 model

The extended version of classical SEIR model is provided a description for the spread of COVID-19 under the quarantine strategy applied by the government [30]. Some theoretically necessary assumptions required for simplification of the current problem can be listed as follows:

- In this model, once a patient has been treated well, the probability of getting infected a second time is not considered.
- Since it is a short-term model, the natural birth and death rate is ignored.
- All coefficients of the model are positive constants.

The fractional analysis is an important field of mathematical analysis which has a very important place in fractional calculus, it has some shortcomings in the application due to initial conditions. To overcome these shortcomings and gain an advantage in the application, the Caputo derivative has been defined by making a modification to the Riemann–Liouville operator. This favorable fractional derivative, which is often preferred to solve real-world problems, is defined as follows:

\[
(\text{RL} D^n)^\alpha \phi(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{\phi(\tau)}{(t-\tau)^{\alpha+n-1}} d\tau, \quad \Re(\alpha) > 0, \quad n = \lfloor \Re(\alpha) \rfloor + 1, \quad \Re(\alpha) \geq 0.
\]

Although the definition of Riemann–Liouville has a very important place in fractional calculus, it has some shortcomings in the application due to initial conditions. To overcome these shortcomings and gain an advantage in the application, the Caputo derivative has been defined by making a modification to the Riemann–Liouville operator. This favorable fractional derivative, which is often preferred to solve real-world problems, is defined as follows:

\[
(\text{C} D^n)^\alpha \phi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\phi^{(n)}(\tau)}{(t-\tau)^{\alpha+n-1}} d\tau,
\]

where $\Re(\alpha) > 0$ and $n = \lfloor \Re(\alpha) \rfloor + 1$ [6]. In addition to these definitions we have mentioned, many more fractional operator definitions and various applications have been presented in the literature as can be seen in [7–29]. Some of these derivatives are obtained by iteration of local derivatives, while others are generalized versions of existing fractional derivatives. On the other hand, some fractional operators have also been obtained using singular or non-singular kernels. One of the most important features of all these fractional derivatives and integrals is that they have a memory effect. Hence, we feel inspired to analyze the fractional version of a disease model called COVID-19 in [30] with the efficient fractional operator of Caputo. In order to capture certain properties of the dynamical model of COVID-19 in detail, we divide the population into seven classes to forecast the impact of the home quarantine which is one of the most crucial measures for preventing the transmission of COVID-19.

The remaining parts of this study are designed as follows. In Section “Formulation of the fractional COVID-19 model”, the formulation of the COVID-19 model is presented by means of Caputo fractional operator. The real incidence data is given for Turkey in Section “Real-life data”. We perform such detailed analysis as the existence and uniqueness of the solutions, stability analysis, iterative solutions, the positiveness of the solutions, basic reproduction number, sensitivity analysis of the fractional model under investigation in Section “In-depth analysis of COVID-19 model”. In Section “Parameter estimation based on real-life data”, the parameters’ estimation of the COVID-19 model is presented by using real data of Turkey. Finally, in Section “…” , we introduce the numerical simulation and discussions to grasp all results of the current work.
In order to make use of the advantages of the fractional calculus, we investigate some crucial theoretical and numerical properties of the dynamical system (5) in the following sections. Therefore, we obtain more sensitive results for the spread of COVID-19 and the effect of quarantine on the transmission of disease thanks to the non-integer order operator called Caputo derivative.

3. Real-life data

The first case of COVID-19 identified by the Ministry of Health in Turkey announced on March 19, 2020, and the first virus-related death in the country occurred on March 15, 2020. Also, he announced that cases of COVID-19 spread all over Turkey On April 1, 2020. As of June 21, 2020, in Turkey, it was declared that 4,950 people lost their lives owing to COVID-19 and the total number of cases reached 187,685. Moreover, the head of the Turkish Ministry of Health expressed that the spread of COVID-19 in Turkey reached its peak in the fourth week on April 14, 2020. According to the published data on the official website, we have collected these data from April 1 to May 1, 2020, in order to perform parameter estimation. The period we determine is 31 days Turkey faced with a large number of COVID-19 cases. Table 1.2 shows these reported cases of COVID-19 across Turkey. In the light of these data, it is aimed to carry out the most appropriate estimates for the parameters in the model to obtain much better numerical results.

4. In-depth analysis of COVID-19 model

4.1. Existence and uniqueness of solutions through Caputo fractional derivative

In this segment, we present the existence and uniqueness of the solution for the non-linear fractional system (5) via Caputo derivative by utilizing the theory of fixed-point. Let us suppose that $B(J)$ is a Banach space for the continuous real-valued functions defined on $J = [0, a]$ with sub norm and $Q = B(J) 	imes B(J) 	imes B(J) 	imes B(J) 	imes B(J) 	imes B(J)$ with the norm $||S, Q, E, A, D, R|| = |S| + |Q| + |E| + |A| + |D| + |R|, |S| = \sup_{t \in J} |S|, |Q| = \sup_{t \in J} |Q| = \sup_{t \in J} |E|, |A| = \sup_{t \in J} |A|, |D| = \sup_{t \in J} |D|, |R| = \sup_{t \in J} |R|$. Applying Caputo fractional integral to the COVID-19 model (5), we get

$$S(t) - S(0) = c \int_0^t \left[ \alpha S(t)(t) + \beta A(t) - \mu S(t) + \gamma \phi Q(t) \right] \, dt,$$

$$Q(t) - Q(0) = \int_0^t \left[ \phi \beta A(t) + \gamma \phi D(t) \right] \, dt,$$

$$I(t) - I(0) = \int_0^t \left[ \phi \beta A(t) + \gamma \phi D(t) \right] \, dt,$$

$$R(t) - R(0) = \int_0^t \left[ \phi \beta A(t) + \gamma \phi D(t) \right] \, dt.$$

It should be stressed that $X_1(S, \Theta), X_2(Q, \Theta), X_3(E, \Theta), X_4(A, \Theta), X_5(I, \Theta), X_6(D, \Theta)$ and $X_7(R, \Theta)$ satisfy the Lipschitz condition if and only if $S(t), Q(t), E(t), A(t), I(t), D(t)$ and $R(t)$ have an upper bound. Let $S(t)$ and $S'(t)$ be couple functions, then we get as follows

$$\| X_1(g, t, S(t)) - X_1(g, t, S'(t)) \| = \| -\alpha S(t) - \alpha S'(t) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

$$\| X_2(g, t, Q(t)) - X_2(g, t, Q'(t)) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

$$\| X_3(g, t, E(t)) - X_3(g, t, E'(t)) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

$$\| X_4(g, t, A(t)) - X_4(g, t, A'(t)) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

$$\| X_5(g, t, I(t)) - X_5(g, t, I'(t)) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

$$\| X_6(g, t, D(t)) - X_6(g, t, D'(t)) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

$$\| X_7(g, t, R(t)) - X_7(g, t, R'(t)) \| = \| \phi \beta A(t) - \phi \beta A'(t) \| = \| -\phi \beta A(t) - \phi \beta A'(t) \|,$$

Table 2

| Parameter | S. Index | Value | Parameter | S. Index | Value |
|-----------|----------|-------|-----------|----------|-------|
| $\alpha$  | $S_\alpha$ | 1.00000 | $\delta$  | $S_\delta$ | 0.03729 |
| $\eta$    | $S_\eta$  | 0.68924 | $\mu_\alpha$ | $S_{\mu_\alpha}$ | -0.21690 |
| $\mu_1$   | $S_{\mu_1}$ | -0.01491 | $\rho_1$  | $S_{\rho_1}$ | -0.02277 |
| $\nu_1$   | $S_{\nu_1}$ | -0.02237 | $\phi_1$  | $S_{\phi_1}$ | -0.21690 |

Table 1

| COVID-19 incidence data, Turkey, 2020. |
|-----------------|-----------------|
| Month | Date | Total cases | Month | Date | Total cases |
| April | 1 | 2148 | April | 17 | 4353 |
| April | 2 | 2456 | April | 18 | 3798 |
| April | 3 | 2786 | April | 19 | 3977 |
| April | 4 | 3103 | April | 20 | 4674 |
| April | 5 | 3135 | April | 21 | 4611 |
| April | 6 | 3148 | April | 22 | 3083 |
| April | 7 | 3892 | April | 23 | 3166 |
| April | 8 | 4117 | April | 24 | 3122 |
| April | 9 | 4056 | April | 25 | 2661 |
| April | 10 | 4747 | April | 26 | 2357 |
| April | 11 | 5138 | April | 27 | 2131 |
| April | 12 | 4789 | April | 28 | 2392 |
| April | 13 | 4939 | April | 29 | 2936 |
| April | 14 | 4922 | April | 30 | 2615 |
| April | 15 | 4281 | May | 1 | 2188 |
| April | 16 | 4801 | | | |
Thus, it can be concluded that the Lipschitz condition is satisfied for $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_6$ and $\mathcal{R}_7$.

Recursively, (8) can be written as follows

\begin{align}
S_n(t) &= \mathcal{M}(\chi) \int_0^t \mathcal{R}_3(\chi, \Theta, \chi, \Theta, 0, \Theta) d\Theta,
Q_n(t) &= \mathcal{M}(\chi) \int_0^t \mathcal{R}_4(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
E_n(t) &= \mathcal{M}(\chi) \int_0^t \mathcal{R}_5(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
A_n(t) &= \mathcal{M}(\chi) \int_0^t \mathcal{R}_6(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
I_n(t) &= \mathcal{M}(\chi) \int_0^t \mathcal{R}_7(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
\end{align}

associated with the initial conditions $S_0(t) = S(0), Q_0(t) = Q(0), E_0(t) = E(0), A_0(t) = A(0), I_0(t) = I(0), D_0(t) = D(0), R_0(t) = R(0).$ By subtracting the successive terms, we attain

\begin{align}
\Psi_{S_n}(t) &= S_n(t) - S_{n-1}(t) = \mathcal{M}(\chi) \int_0^t \mathcal{R}_3(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
\Psi_{Q_n}(t) &= Q_n(t) - Q_{n-1}(t) = \mathcal{M}(\chi) \int_0^t \mathcal{R}_4(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
\Psi_{E_n}(t) &= E_n(t) - E_{n-1}(t) = \mathcal{M}(\chi) \int_0^t \mathcal{R}_5(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
\Psi_{A_n}(t) &= A_n(t) - A_{n-1}(t) = \mathcal{M}(\chi) \int_0^t \mathcal{R}_6(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
\Psi_{I_n}(t) &= I_n(t) - I_{n-1}(t) = \mathcal{M}(\chi) \int_0^t \mathcal{R}_7(\chi, \Theta, 0, \Theta, 0, \Theta) d\Theta,
\end{align}

and utilizing the Eqs. (10), (11) considering $\Psi_{S_{N-1}}(t) = S_{N-1}(t) - S_{N-2}(t), \Psi_{Q_{N-1}}(t) = Q_{N-1}(t) - Q_{N-2}(t), \Psi_{E_{N-1}}(t) = E_{N-1}(t) - E_{N-2}(t), \Psi_{A_{N-1}}(t) = A_{N-1}(t) - A_{N-2}(t), \Psi_{I_{N-1}}(t) = I_{N-1}(t) - I_{N-2}(t), \Psi_{D_{N-1}}(t) = D_{N-1}(t) - D_{N-2}(t), \Psi_{R_{N-1}}(t) = R_{N-1}(t) - R_{N-2}(t)$, it can be obtained the following relations

\begin{align}
\|\Psi_{S_n}(t)\| &= \mathcal{M}(\chi) u_1 \int_0^t \|\Psi_{S_{n-1}}(\Theta)\| d\Theta,
\|\Psi_{Q_n}(t)\| &= \mathcal{M}(\chi) u_2 \int_0^t \|\Psi_{Q_{n-1}}(\Theta)\| d\Theta,
\|\Psi_{E_n}(t)\| &= \mathcal{M}(\chi) u_3 \int_0^t \|\Psi_{E_{n-1}}(\Theta)\| d\Theta,
\|\Psi_{A_n}(t)\| &= \mathcal{M}(\chi) u_4 \int_0^t \|\Psi_{A_{n-1}}(\Theta)\| d\Theta,
\|\Psi_{I_n}(t)\| &= \mathcal{M}(\chi) u_5 \int_0^t \|\Psi_{I_{n-1}}(\Theta)\| d\Theta,
\|\Psi_{D_n}(t)\| &= \mathcal{M}(\chi) u_6 \int_0^t \|\Psi_{D_{n-1}}(\Theta)\| d\Theta,
\|\Psi_{R_n}(t)\| &= \mathcal{M}(\chi) u_7 \int_0^t \|\Psi_{R_{n-1}}(\Theta)\| d\Theta.
\end{align}

Consequently, we can prove the theorem below:

**Theorem 1.** The fractional COVID-19 model (5) has a unique solution under the condition that

\begin{equation}
\mathcal{M}(\chi) \leq M, \quad i = 1, 2, \ldots, 7
\end{equation}

when $t \in [0, t]$. 

**Proof.** As we showed above, the functions $S(t), Q(t), E(t), A(t), I(t), D(t)$ and $R(t)$ are bounded and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_6, \mathcal{R}_7$ satisfy the
Lipschitz condition. So, with the help of the recursive principle and (15), we reach

\[ ||\Psi_{s,n}(t)|| = \left( \frac{d(s)}{X} \right)^n ||k^1 \right)^n, \]

\[ ||\Psi_{d,n}(t)|| = \left( \frac{d(s)}{X} \right)^n ||k^2 \right)^n, \]

\[ ||\Psi_{e,n}(t)|| = \left( \frac{d(s)}{X} \right)^n ||k^3 \right)^n, \]

\[ ||\Psi_{a,n}(t)|| = \left( \frac{d(s)}{X} \right)^n ||k^4 \right)^n, \]

\[ ||\Psi(t)|| = \left( \frac{d(s)}{X} \right)^n ||k^5 \right)^n. \]  

(17)

Thereby, it can be considered that \( ||\Psi_{s,n}(t)|| \rightarrow 0, ||\Psi_{q,n}(t)|| \rightarrow 0, ||\Psi_{e,n}(t)|| \rightarrow 0, ||\Psi_{a,n}(t)|| \rightarrow 0 \) and \( ||\Psi_{n,n}(t)|| \rightarrow 0 \) for \( n \rightarrow \infty \). On the other hand, employing the triangle inequality and the system (17) for any \( p \), one can get

\[ ||S_{n+1}(t) - S_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]

\[ ||Q_{n+1}(t) - Q_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]

\[ ||E_{n+1}(t) - E_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]

\[ ||A_{n+1}(t) - A_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]

\[ ||I_{n+1}(t) - I_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]

\[ ||D_{n+1}(t) - D_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]

\[ ||R_{n+1}(t) - R_n(t)|| \leq \sum_{j=1}^{n-p} k_j^p = \frac{k^{p+1} - k^{p+1}}{1-k^p}, \]  

such that \( k_j = \frac{d(s)}{X} \). Thus, \( S_n, Q_n, E_n, A_n, I_n, D_n, R_n \) are Cauchy sequences in \( B(f) \). For this reason, it can be said that they are uniformly convergent. Through the limit theorem, the limit of the sequences (12) is the unique solution of the fractional system (5).

4.2. Stability analysis and iterative solutions through Caputo fractional derivative

Here, iterative solutions are introduced by utilizing the Laplace transform of Caputo fractional derivative. On the other hand, stability criteria for the fractional COVID-19 model is presented with the help of fixed point theorem. Let \((B, ||||)||\) be a Banach space and \( Q^* \) be a self-map of \( B \). Moreover, let us take into account the recursive procedure in the form of the \( y_{n+1} = h(Q^* y_n) \) and \( G(Q^*) \) be a fixed point set of non-empty \( Q^* \). It should be expressed the sequence \( y_n \) converges to the point of \( y^* \in G(Q^*) \). We define \( ||y_n - h(Q^* y_n)|| \) where \( y_n \subseteq B \). The iterative approach, \( y_{n+1} = h(Q^* y_n) \), is \( Q^* \)-stable if \( \lim_{n \to \infty} C_n = 0 \), that is, \( \lim_{n \to \infty} C_n = p^* \). For the sequence \( y_n \) to converge, it must have an upper limit. If all the conditions mentioned above are satisfied for \( y_{n+1} = Q^* y_n \), then the iteration is \( Q^* \)-stable. Hence, we can express the theorem below:

Theorem 2. Let \((B, ||||)||\) be a Banach space and \( Q^* \) be a self-map on \( B \), then for all \( x, y \in B \), the following inequality is satisfied

\[ ||Q^* x - Q^* y|| \leq K ||x - Q^* y|| + k ||x - y||, \]  

where \( K > 0, k \in [0, 1) \). If we assume that \( Q^* \) is Picard \( Q^* \)-stable, then recursive formula can be given as follows

\[ S_{n+1}(t) = S_n(t) + \frac{1}{\lambda} \left( \frac{-\alpha S_n(t) I_n(t) + \alpha A_n(t)}{\rho} + \rho^* Q_n(t) \right), \]

\[ Q_{n+1}(t) = Q_n(t) + \frac{1}{\lambda} \left( \frac{-\beta S_n(t) I_n(t) + \beta A_n(t)}{\rho} \right), \]

\[ E_{n+1}(t) = E_n(t) + \frac{1}{\lambda} \left( \frac{-\gamma E_n(t) I_n(t) - \gamma E_n(t) - \gamma E_n(t)}{\rho} \right), \]

\[ A_{n+1}(t) = A_n(t) + \frac{1}{\lambda} \left( \frac{-\delta A_n(t) I_n(t) - \delta A_n(t) + \delta A_n(t)}{\rho} \right), \]

\[ I_{n+1}(t) = I_n(t) + \frac{1}{\lambda} \left( \frac{-\delta I_n(t) - \delta I_n(t) - \delta I_n(t)}{\rho} \right), \]

\[ D_{n+1}(t) = D_n(t) + \frac{1}{\lambda} \left( \frac{-\delta D_n(t) - \delta D_n(t) + \delta D_n(t)}{\rho} \right), \]

\[ R_{n+1}(t) = R_n(t) + \frac{1}{\lambda} \left( \frac{-\delta R_n(t) + \delta R_n(t)}{\rho} \right). \]  

(20)

Theorem 3. Let \( \mathcal{F} \) be a self-map, then it is defined as follows

\[ ||\mathcal{F} x - \mathcal{F} y|| \leq K ||x - \mathcal{F} y|| + k ||x - y||. \]
that is \(\mathcal{F}\)-stable in the space of \(L^1(a, b)\), if the following conditions are satisfied

\[
\begin{bmatrix}
1 - \alpha'(P + Q)\eta(p) - \alpha'\delta'(P + R)\xi(p) + (\gamma' - \kappa')k_3(p) & 1, \\
1 + (\alpha' - \gamma')k_5(p) & 1, \\
1 + \alpha'(P + Q)\eta(p) + \alpha'\delta'(P + R)\xi(p) - \gamma'k_5(p) & 1, \\
1 + \gamma'(I_n(t) - \eta') - \mu'_3 - \phi'_5k_4(p) & 1, \\
1 + \gamma'\eta' - \gamma'\delta' - \mu'_3 - \phi'_5k_4(p) & 1, \\
1 + \mu'_3 + \mu'_3 - \phi'_5k_4(p) & 1, \\
1 + \phi'_3 + \phi'_3 + \phi'_5k_4(p) & 1, \\
1 - \kappa'k_3(p) & 1
\end{bmatrix}
\]

(22)

Proof. It is clear that \(\mathcal{F}\) is a fixed point. Thus, we can write the following iterations for all \((m, n) \in \mathbb{N} \times \mathbb{N}\),

\[
\mathcal{F}\left(S_n(t) - S_m(t)\right) = \mathcal{F}\left(S_n(t)\right) - \mathcal{F}\left(S_m(t)\right) + \mathcal{F}\left(S_{m+1}(t)\right) + \mathcal{F}\left(S_{m+2}(t)\right) + \cdots + \mathcal{F}\left(S_{n-1}(t)\right),
\]

\[
\mathcal{F}\left(Q_n(t) - Q_m(t)\right) = \mathcal{F}\left(Q_n(t)\right) - \mathcal{F}\left(Q_m(t)\right) + \mathcal{F}\left(Q_{m+1}(t)\right) + \mathcal{F}\left(Q_{m+2}(t)\right) + \cdots + \mathcal{F}\left(Q_{n-1}(t)\right),
\]

\[
\mathcal{F}\left(E_n(t) - E_m(t)\right) = \mathcal{F}\left(E_n(t)\right) - \mathcal{F}\left(E_m(t)\right) + \mathcal{F}\left(E_{m+1}(t)\right) + \mathcal{F}\left(E_{m+2}(t)\right) + \cdots + \mathcal{F}\left(E_{n-1}(t)\right),
\]

\[
\mathcal{F}\left(A_n(t) - A_m(t)\right) = \mathcal{F}\left(A_n(t)\right) - \mathcal{F}\left(A_m(t)\right) + \mathcal{F}\left(A_{m+1}(t)\right) + \mathcal{F}\left(A_{m+2}(t)\right) + \cdots + \mathcal{F}\left(A_{n-1}(t)\right),
\]

\[
\mathcal{F}\left(I_n(t) - I_m(t)\right) = \mathcal{F}\left(I_n(t)\right) - \mathcal{F}\left(I_m(t)\right) + \mathcal{F}\left(I_{m+1}(t)\right) + \mathcal{F}\left(I_{m+2}(t)\right) + \cdots + \mathcal{F}\left(I_{n-1}(t)\right),
\]

\[
\mathcal{F}\left(D_n(t) - D_m(t)\right) = \mathcal{F}\left(D_n(t)\right) - \mathcal{F}\left(D_m(t)\right) + \mathcal{F}\left(D_{m+1}(t)\right) + \mathcal{F}\left(D_{m+2}(t)\right) + \cdots + \mathcal{F}\left(D_{n-1}(t)\right). 
\]

(23)

After taking the norm of both sides of the first equation in (23), we attain

\[
\left\|\mathcal{F}(S_n(t)) - \mathcal{F}(S_m(t))\right\| = \left\|S_n(t) - S_m(t) + \mathcal{F}\left(-\alpha'\delta S_n(t)(I_n(t) + \delta' A_n(t)) - \kappa' S_n(t) + \gamma' Q_n(t)\right)\right\| - \left\|\mathcal{F}\left(-\alpha'\delta S_n(t)(I_n(t) + \delta' A_n(t)) - \kappa' S_n(t) + \gamma' Q_n(t)\right)\right\|.
\]

and by utilizing the triangular inequality, one can write

\[
\left\|\mathcal{F}(S_n(t)) - \mathcal{F}(S_m(t))\right\| = \left\|S_n(t) - S_m(t) + \mathcal{F}\left(-\alpha'\delta S_n(t)(I_n(t) + \delta' A_n(t)) - \kappa' S_n(t) + \gamma' Q_n(t)\right)\right\| - \left\|\mathcal{F}\left(-\alpha'\delta S_n(t)(I_n(t) + \delta' A_n(t)) - \kappa' S_n(t) + \gamma' Q_n(t)\right)\right\|.
\]

(24)
\[ \| \mathcal{F}(S_i(t)) - \mathcal{F}(S_i(t)) \| \leq \| S_i(t) - S_i(t) \| + \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( - \alpha S_i(t) q(t) \right) + \beta A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( - \alpha S_i(t) q(t) \right) \right) \| = 0. \]

Through some necessary simplifications, (25) takes the form of
\[ \| \mathcal{F}(S_i(t)) - \mathcal{F}(S_i(t)) \| \leq \| S_i(t) - S_i(t) \| + \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) \right) \right) \|. \]

Due to the same behavior of functions inside the fractional system, we suppose that
\[ \| A_i(t) - A_i(t) \| \leq \| S_i(t) - S_i(t) \|. \]

Inserting (27) into the relation (26), one can have
\[ \| \mathcal{F}(S_i(t)) - \mathcal{F}(S_i(t)) \| \leq \| S_i(t) - S_i(t) \| + \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) \right) \right) \|. \]

Owing to the fact that the sequences \( S_i(t), A_i(t) \) and \( A_i(t) \) are convergent and bounded, there are three different constants \( P > 0, Q > 0 \) and \( R > 0 \) for all \( t \). Therefore, we get
\[ \| S_i(t) \| < P, \| A_i(t) \| < Q, \| A_i(t) \| < R. \]

From the relations (28) and (29), we reach
\[ \| \mathcal{F}(S_i(t)) - \mathcal{F}(S_i(t)) \| \leq 1 + \| \alpha S_i(t) q(t) \| - \alpha S_i(t) q(t) \| \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) \right) \right) \|. \]

where \( f_i, g_i \) and \( k_i \) are the functions acquired by the inverse Laplace transform in (28). In a similar way, we also get
\[ \| \mathcal{F}(Q_i(t)) - \mathcal{F}(Q_i(t)) \| \leq 1 + \| \kappa S_i(q(t) - Q_i(t) - Q_i(t)) \|. \]
\[ \| \mathcal{F}(E_i(t)) - \mathcal{F}(E_i(t)) \| \leq 1 + \| \alpha S_i q(t) - \alpha S_i q(t) \| \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) \right) \right) \|. \]
\[ \| \mathcal{F}(A_i(t)) - \mathcal{F}(A_i(t)) \| \leq 1 + \| \alpha S_i(q(t) - q(t)) \| \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) \right) \right) \|. \]
\[ \| \mathcal{F}(L_i(t)) - \mathcal{F}(L_i(t)) \| \leq 1 + \| \alpha S_i q(t) - \alpha S_i q(t) \| \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) \right) \right) \|. \]
\[ \| \mathcal{F}(D_i(t)) - \mathcal{F}(D_i(t)) \| \leq 1 + \| \alpha S_i q(t) - \alpha S_i q(t) \| \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) \right) \right) \|. \]
\[ \| \mathcal{F}(R_i(t)) - \mathcal{F}(R_i(t)) \| \leq 1 + \| \alpha S_i q(t) - \alpha S_i q(t) \| \| \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) + \rho Q_i(t) \right) \right) \| - \mathcal{F}^{-1} \left( \frac{1}{\eta} \mathcal{F} \left( \eta \beta S_i A_i(t) - \kappa S_i(t) \right) \right) \|. \]

where the condition (22) is valid. So, it can be expressed that \( \mathcal{F} \) has a fixed-point. To prove that \( \mathcal{F} \) satisfies the conditions of Theorem 3.3, we presume that (31) and (32) hold and also the following relations are satisfied
\[ k = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0). \]

Hence, we reach the intended result.

4.3. The positivity of solutions for COVID-19 model

Let us determine the invariant region and indicate that all solutions of the fractional COVID-19 system (5) are positive for all \( t \geq 0 \). The main objective is to introduce the convenience of the solutions for the model analyzed by observing whether they enter the invariant region \( Y \). Employing the advantages of the Caputo fractional derivative, we assume that
\[ Y = (S, Q, E, A, I, D, R) \in \mathbb{R}_+^7, \mathbb{R}_+^7 = (n \in \mathbb{R}_+^7 : n \geq 0) \]

be any solution of the model (5) with non-negative initial conditions.

Moreover, we have \( n = (S(t), Q(t), E(t), A(t), I(t), D(t), R(t)) \). We must also demonstrate that the vector field points to \( \mathbb{R}_+^7 \) upon each hyperplane which is bounded by the non-negative hyperoctant. So, one can write
\[ cD^\alpha S(t) = \rho S(t) q(t) \geq 0, \]
\[ cD^\alpha Q(t) = \kappa S(t) q(t) \geq 0, \]
\[ cD^\alpha E(t) = \alpha S(t) q(t) + \beta A(t) q(t) \geq 0, \]
\[ cD^\alpha A(t) = \rho S(t) q(t) \geq 0, \]
\[ cD^\alpha D(t) = \rho S(t) q(t) \geq 0, \]
\[ cD^\alpha R(t) = \rho S(t) q(t) \geq 0. \]

So, we give the convenient region as follows
\[ Y = \{ (S, Q, E, A, I, D, R) : \Delta \geq 0, Q \geq 0, A \geq 0, I \geq 0, D \geq 0, R \geq 0, S + Q + E + A + I + D + R \leq 1 \}. \]

Thereby, the fractional COVID-19 model (5) is biologically appropriate and mathematically well-defined in the region \( Y \). Additionally, this region is positively invariant, that is, solutions of the underlying system (5) are positive for all \( t \).

4.4. Reproduction number

The basic reproduction number denoted by \( R_0 = \rho(FV^{-1}) \) where \( \rho(.) \) is the spectral radius of the matrix \( FV^{-1} \) can be obtained by the next-generation matrix approach. The matrix \( F \) of transmission and matrix \( V \) of transformation for the fractional COVID-19 model (5) are given by
\[ F = \begin{bmatrix} 0 & \alpha S & \alpha S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ V = \begin{bmatrix} 0 & \alpha S & \alpha S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]
\[ V = \begin{bmatrix} \psi \chi & \mu \chi A + \phi \chi A_0 \\ \chi \eta \chi & \psi \chi (1 - \eta \chi) + \phi \chi I + \mu \chi I_0 \\ \eta \chi & \psi \chi \eta \chi + \phi \chi I + \mu \chi I \\ \end{bmatrix}. \] (42)

Thus, we obtain the reproduction ratio for the fractional model of COVID-19 as below

\[ R_0 = \frac{\alpha \delta (1 - \eta \chi) + \alpha \delta \eta \chi}{\mu_A + \mu_I + \mu_I}. \] (43)

4.5. Sensitivity Analysis

The sensitivity analysis of \( R_0 \) has drawn a lot of attention in various scientific areas. As the parameters of a dynamical model are estimated, it is possible to have some uncertainty about their values utilized to draw conclusions about the proposed system. In order to decrease the spread of the infectious disease, it can be carried out the sensitivity analysis by determining the parameters. Sensitivity analysis is a crucial part of the disease model analysis although computation of it can become exhaustive for complex dynamical systems. For this reason, it is very important to evaluate the effects of each parameter on the spread of the disease and thus find the parameters that have the most important effect on the reduction and spread of the outbreak. Here, we perform the sensitivity analysis by means of the sensitivity index for the parameters of the COVID-19 model. This technique helps to measure the most sensitive parameters inside the system for the reproduction number \( R_0 \). The following formula is employed to calculate the sensitivity index of the reproduction number \( R_0 \) of the fractional COVID-19 model presented by the Caputo derivative.

\[ S_\omega = \frac{\omega}{R_0} \times \frac{\partial R_0}{\partial \omega}. \] (44)

It can be used three methods to compute the sensitivity indices: Direct differentiation, Latin hypercube sampling method, and linearizing system, and then it is solved the obtained set of linear algebraic equations. We utilize the direct differentiation method because it provides analytical expressions for the indices. The indices not only gives us the effect of some aspects associated with the spreading of infectious disease but also gives us crucial information about the comparative change between \( R_0 \) and other parameters. So, it helps to reach the control strategies. Table 1 shows that the parameters \( \alpha, \delta \) and \( \eta \) have a positive influence on the reproduction number \( R_0 \), which describe that the growth or decay of these parameters say by 10 percent will increase or decrease the reproduction number by 10 percent, 0.37 percent and 6.89 percent, respectively. However, on the other hand, the index for parameters \( \mu_A, \mu_I, \rho_I, \phi_A \) and \( \phi_I \) show that increasing their values by 10 percent will decrease the values of basic reproduction number \( R_0 \) by

| Parameters | Values | Source |
|------------|--------|--------|
| \( \alpha \) | 0.00000 | Estimated |
| \( \delta \) | 0.09564 | Estimated |
| \( \kappa \) | 0.29140 | Estimated |
| \( \gamma \) | 0.01796 | Estimated |
| \( \psi \) | 0.13100 | Estimated |
| \( \eta \) | 0.76922 | Estimated |
| \( \mu_A \) | 1/5 [30] | |
| \( \rho_I \) | 1/4 [30] | |
| \( \phi_A \) | 0.14960 [30] | |
| \( \phi_I \) | 0.20806 [30] | |
| \( \rho_D \) | 0.29685 | Estimated |

Table 3
Estimated parameter values for the COVID-19 Model (4).

Fig. 9. Model fitting and its residual plots for the classical version of the model.

Fig. 10. Number of COVID-19 cases in Turkey.
2.16 percent, 0.14 percent, 0.227 percent, 0.223 percent and 2.16 percent, respectively.

5. Parameter estimation based on real-life data

Here, we perform the estimation of parameters in the COVID-19 model with the help of the non-linear least squares curve fitting method. All parameters of the underlying model estimated from the reported data in Table 1 will be employed for the numerical simulations in the next section of this paper. The confirmed cases of COVID-19 infected individuals handled in the current study represent those who are tested positive by health care workers. In order to furnish a better fit to the real-life data for the aforementioned system, we obtain the more convenient parameter values by utilizing the least square curve fitting. While the non-linear function fitted contains solving ordinary differential equations employing a numerical integration scheme, the problem is set as a classical non-linear least-squares problem. To carry out the parameters’ estimation, the following steps are applied:

- The model of COVID-19 is solved by means of the ODE45 function and Euler’s method by guessing initial parameters.
- The solution of the system (5) is compared with the real-life data and an optimization algorithm can be utilized to get the estimated parameter values having a much better fit to the real data.
- The model of COVID-19 is solved by using the new parameter values and the results are compared with real-life data.

The initial values of population estimated for parameters’ estimation of the model (4) is $(S_0, Q_0, E_0, A_0, I_0, D_0, R_0) = (921984900, 414225100, 3207, 595, 563, 227, 3)$ and also the initial parameter values is:

\[ p_0 = (a, \delta, \kappa, \psi, \eta, \mu_A, \mu_I, \phi_A, \phi_I, \rho_I, \rho_D) = (5.5010 \times 10^8, 9.0000, 1.90, 1.7, 0.08800, 1.5, 1.496, 0.0998, 0.1496, 0.0046, 0.0031) \].

The estimated values of parameters is given in Table 3.

Here, we simulated each of the $S(t), Q(t), E(t), A(t), I(t), D(t)$ and $R(t)$ state variables. The fitting of the proposed model and the corresponding residual plots are presented in Fig. 9. Also, Fig. 10 represents the number of COVID-19 cases in Turkey for 30 days. (A) and (B) in Fig. 11 are plotted for the susceptible and quarantined individuals, respectively, when $\chi = 1.0, 0.9, 0.85$. Similarly, in Figs. 12–14, the plots...
correspond to the exposed, asymptomatic, infected, diagnosed, and recovered individuals for the same values of $\chi$. These figures show that a shift in the $\chi$ value has an effect on the epidemic dynamics. Examining the depicted figures one can see that the outbreak rate is broader and lower for low $\chi$ values. This outlook is significant considering the epidemiological viewpoint as its definition indicates a longer period over which the health system may be affected by contaminated individuals (see Figs. 15–21).

6. Numerical simulations and discussion

Herein, the fractional variant of the model under consideration via Caputo fractional operator is numerically simulated with the help of first order convergent numerical techniques as can be seen in [31–33]. These numerical techniques are accurate, conditionally stable, and convergent for solving arbitrary order linear and non-linear system of ordinary differential equations. To begin the simulation we go as follows: Consider a general Cauchy problem of fractional order with autonomous nature

$$^\alpha D^\chi_{0+}(y(t)) = g(y(t)), \chi \in (0, 1], t \in [0, T], y(0) = y_0,$$

(45)

where $y = (a, b, c, w) \in \mathbb{R}_+^4$ is a real-valued continuous vector function satisfying the Lipchitz condition presented by

$$||g(y_1(t)) - g(y_2(t))|| \leq M ||y_1(t) - y_2(t)||.$$

(46)

where $M$ is a positive real Lipchitz constant.

Employing the fractional integral operators, one can get

$$y(t) = y_0 + \int_{0}^{t} g(y(s)) ds, t \in [0, T].$$

(47)

where $\int_{0}^{t} \cdot$ is the Riemann–Liouville fractional integral. Considering an equi-spaced integration intervals over $[0, T]$ with the fixed step size $h (= 10^{-2}$ for simulation) = $\frac{T}{n} \in \mathbb{N}$. Also, let us suppose that $y_0$ be the approximation of $y(t)$ at $t = t_q$ for $q = 0, 1, \ldots n$. The numerical technique for the governing model under Caputo fractional derivative operator takes the form

$$^\alpha S_p, i = a_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(-\alpha^\chi S(I+\beta^\chi)A - \kappa^\chi S + \xi^\chi Q),$$

$$^\alpha Q_p, i = b_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(S^\chi S - \xi^\chi Q),$$

$$^\alpha E_p, i = d_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(\alpha^\chi S(I+\beta^\chi)A - \psi^\chi E),$$

$$^\alpha A_p, i = e_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(\psi^\chi (1-\eta^\chi)E - \mu^\chi A - \phi^\chi A),$$

$$^\alpha I_p, i = f_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(\eta^\chi (1-\phi^\chi)I - \rho^\chi I - \xi^\chi I),$$

$$^\alpha D_p, i = g_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(\mu^\chi A + \phi^\chi I - \rho^\chi D - \mu^\chi D),$$

$$^\alpha R_p, i = h_i + \int_{0}^{t} \frac{h^\chi}{\Gamma(\chi+1)} \sum_{k=0}^{p} ((p-k+1)\chi - (p-k)\chi)(\phi^\chi A + \varphi^\chi I + \psi^\chi D).$$

(48)

Now we discuss the obtained numerical outcomes of the governing model in respect of the approximate solutions. To this aim, we employed
the effective Euler method under the Caputo fractional operator to do the job. The initial conditions and the estimated parameter values are stated in the immediate section. It is popularly known that the most confusing and standstill affair in the limelight is the subtle characteristics of the COVID-19 pandemic. The deceptive nature of the virus causes scientists, researchers, and medical professionals to constantly analyze the attitudes and attributes to execute the real truth about the virus. Here, by means of some effective numerical scheme, we numerically simulated the model under consideration and physically see how it behaves depending on the scenario.

In Fig. 3, the profile for the behavior of the infectious class with (a) Transition rate of exposure to infected class \( \phi \) and (b) quarantined rate \( \kappa \) under the proposed model with Caputo derivative while considering the increasing and decreasing values of each, have been depicted. While, Fig. 4 shows the profiles for the behavior of the infectious class with (a) the disease-induced death rate \( \rho \) and (b) the difference for the symptoms and asymptomatic infectious class \( \theta \) under the proposed model with Caputo derivative while considering the increasing and decreasing values of each, respectively. In both Figs. 3 and 4, one can see the interacting changes for the infectious class depending on the sensitivity of the parameters. The behavior of each state variable for the Caputo version of the fractional model using the values of the parameters has been depicted in Fig. 5. One can easily see the decreasing-creasing character of each state variable. To clearly see the dynamical characteristics with respect to Caputo fractional derivative, we vary the values of \( \alpha = 1, 0.999, 0.988 \) on each of the state variables. Fig. 6(a) depicts the dynamical behavior for \( S(t) \) (susceptible individuals at time \( t \)) and 6(b) depicts the dynamical behavior of \( Q(t) \) (individuals who are in the quarantine process at time \( t \)). One can see that \( S(t) \) is strongly decreasing while \( Q(t) \) is creasing. In Fig. 7(a), we illustrate the dynamic for \( A(t) \) (asymptomatic infected individuals at time \( t \)) and Fig. 7(b) shows \( E(t) \) (exposed individuals who carry the coronavirus at low-level). In this case, both \( A(t) \) and \( E(t) \) have a monotonic behavior. In Fig. 8(a), we present the dynamic of \( I(t) \) (infected individuals with certain symptoms of COVID-19) and Fig. 8(b) depicts the dynamic of \( D(t) \) (the number of medically verified cases in which patients are in quarantine treatment at time \( t \)). Monotonic behavior can be seen in this case. Fig. 9 represents the outlook of \( R(t) \) (recovered individuals at time \( t \)). It has an increasing behavior.
Fig. 17. Profiles for behavior of each state variable for the Caputo version of the fractional model using the values of the parameters.

Fig. 18. The dynamics of $S(t)$ and $Q(t)$ for different $\chi$. 
7. Concluding remarks

As a summary of the analyzes in this study, the following conclusions can be obtained:

- In this study, we have presented an analysis of the fractional COVID-19 model in detail. In addition to the theoretical calculations for the proposed model, the advantages of the fractional operator have been realized by real data by carrying out numerical simulations.
- In order to analyze the dynamics of the underlying fractional model, we have suggested the fractional-order Caputo operator. Making use of this efficient and advantageous fractional derivative, the existence and uniqueness of the solutions for the fractional COVID-19 model have been explored by means of the fixed point theory and the positiveness of the solutions has been shown.
- We have estimated the parameters of the aforementioned model by using real statistics of the reported cases of the COVID-19 model in Turkey, 2020. We have observed the effect of these estimated parameters on the fractional COVID-19 model by simulating for different values of $\chi$. On the other hand, the model fitting under the estimated parameters has been performed.
To determine the most sensitive parameters of the proposed fractional model, we have carried out the sensitivity analysis of the basic reproduction number. In this way, the impact of the parameters inside the model under investigation has been evaluated.

Moreover, iterative solutions of the fractional COVID-19 model have been presented with the help of the Laplace transform of the Caputo fractional derivative. Also, we have furnished the stability criteria for this fractional model under the fixed point theory.

Finally, the variant of the fractional model under consideration has been simulated by utilizing efficient numerical techniques via Caputo operator in order to observe and grasp the advantages of the novel COVID-19 model with arbitrary order $\chi$. Therefore, the current study indicates that non-local fractional-order models studied in different fields are beneficial because they have the non-integer order derivative that makes the analysis performed stronger especially owing to having the memory-effect.

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Declaration of Competing Interest

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