Final state interactions & the Sivers function

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Abstract. The non-vanishing of naïve T-odd parton distributions function can be explained by the existence of the gauge link which emerges from the factorized description of the deep inelastic scattering cross section into perturbatively calculable and non-perturbative factors. This path ordered exponential describes initial / final-state interactions of the active parton due to soft gluon exchanges with the target remnants. Although these interactions are non-perturbative, studies of final state interactions have been approximated by perturbative one-gluon exchange in Abelian models. We include higher-order gluonic contributions from the gauge link by applying non-perturbative eikonal methods, incorporating color degrees of freedom in a calculation of the Sivers function. In this context we study the effects of color by considering the FSIs with Abelian and non-Abelian gluon interactions. We confirm the large Ng QCD scaling behavior of Sivers functions and further uncover the deviations for finite Ng. Within this framework of FSIs we perform a quantitative check of approximate relations between T-odd TMDs and GPD which goes beyond the discussion of overall signs.

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Over the past two decades the transverse partonic structure of hadrons has been the subject of a great deal of theoretical and experimental study. Central to these investigations are the observations of large transverse single spin asymmetries (TSSAs) and azimuthal asymmetries in hadronic reactions—from inclusive hadron production [1, 2, 3] to Drell-Yan Scattering [4, 5], and in semi-inclusive deep inelastic lepton-nucleon scattering experiments [6, 7]. Two explanations to account for TSSAs in QCD have emerged which are based on the twist-three [8, 9, 10] and twist-two [11, 12, 13] approaches. We focus on the twist–two approach in the factorized picture of semi-inclusive deep inelastic lepton-hadron scattering (SIDIS) [13, 14] at small transverse momenta of the produced hadron, \( P_T \sim k_T << \sqrt{Q^2} \), where \( \sqrt{Q^2} \) is the hard scale. In this kinematic regime the Sivers effect describes a twist-two transverse target spin asymmetry through the “naïve” time reversal odd (T-odd) structure, \( \Delta f(x, k_T) \sim S_T \cdot (P \times \bar{k}_T) f_{1T}^+(x, k_T^2) \) [11]. \( k_T \) is the quark intrinsic transverse momentum and \( P \) is the momentum of the target. The Sivers asymmetry has been the focus of much theoretical work on QCD factorization theorems. Among the most interesting results is that the Sivers function is not universal. It is predicted that there is a relative sign between the Sivers function from inclusive hadron production and from the time-reversal behavior of the FSI/ISI implemented by the gauge link operator in (1) and is expressed in terms of the gluonic pole matrix element which we express

\[
\langle k_T \rangle(x) = \int \frac{dz^-}{2\pi} e^{iP_T z^-} \langle P, S_T | \bar{q}(z^- n/2) \gamma^+ [z^- n/2; z^- n/2] \bar{q}(z^- n/2) | P, S_T \rangle,
\]

where \( \langle k_T \rangle(x) \equiv 2M e^{iS_T} f_{1T}^{+(1)}(x) \). The light-like vector \( n = (n^-, n^+, n_\perp = 0) \) represents a specific direction on the light-cone, and \( [x, y] \) denotes a gauge link operator connecting the two locations \( x \) and \( y \). The operator \( I \) originates from the time-reversal behavior of the FSI/ISI implemented by the gauge link operator in (1) and is expressed in terms of the gluonic field strength tensor

\[
2\hat{F}(z^- n/2) = \int dy^- [z^- n/2, y^- n] g F^{+i}(y^- n) [y^- n, z^- n/2].
\]
The impact parameter of the quark-quark correlation function in terms of sum of spectators states is most clearly displayed by transforming Eq. (1) into a mixed coordinate-momentum space [21]. This is captured by the distortion and the color dependence is determined by the chromodynamic GPDs. While one cannot achieve a general factorization between TMDs and impact parameter GPDs; that is for the element as the factorization of FSIs convoluted with transverse impact parameter distortion (via impact parameter space [19]) by inserting a complete set of momentum states and demand that the operator is expressed in terms of the matrix element which is calculated in a generalized ladder approximation [20]. Right: The quark-quark correlator in momentum eigenstates, that is

\[ \langle k^z_\gamma(x) \rangle = \int d^2b_T \int \frac{dz^2}{2\pi} e^{iz^\mu x} \langle P^+, \vec{0}T; S_T | \bar{q}(z_1) \gamma^\gamma \bar{\gamma}(z_1; z_2) | P^+, \vec{0}T; S_T \rangle. \]  

(3)

The impact parameter \( b_T \) sits in the arguments of the quark fields, \( z_{1/2} = (\pm \frac{z^+}{2}, \pm \frac{z^-}{2}, \pm \frac{z^\perp}{2}) \) and the lensing operator is

\[ 2 \bar{P}(z_2) = \int dy^- [z_2; y] gF^{+\gamma}(y) [y; z_2] \]  

where, \( y^\mu = y^- n^\mu + b_T^\perp \). For each Fock space state we have the phenomenological relation,

\[ \langle k^z_\gamma(x) \rangle \approx 2 \int d^2b_T \bar{F}(x, \vec{b}_T) \frac{\partial}{\partial \vec{b}_T} \delta(x, \vec{b}_T). \]  

(4)

We utilize this picture to calculate the gluonic pole matrix element using a soft approximation for the lensing function [20, 22] which we then convolute with the parameterizations of impact parameter GPDs [23, 24].

First we model the target remnant in terms of the sum of spectators the quark correlation function

\[ \Phi_{ij}(x, \vec{k}_T) = \frac{1}{2(2\pi)^3(1-x)P^n} \sum_{\sigma, \delta} \bar{W}_{ij}^\beta (P, k; \sigma) W_i^\alpha (P, k; \sigma), \]  

(5)

which is expressed in terms of the matrix element \( W_i^{\alpha, \delta}(P, k; \sigma) = \langle P - k, \sigma, \delta | \phi_{in}^\alpha | 0 \rangle g_{\alpha \beta} W_i^\beta (P, k; \sigma) \) where \( \sigma \) and \( \delta \) represent the helicity and color of the intermediate states (see Fig. 1). The FSIs – generated by the gauge link -- are described by a non-perturbative amputated scattering amplitude \( (M)^{\alpha \beta}_{g\delta} \) with \( \beta, \alpha (\gamma, \delta) \) color indices of incoming eikonal quark and out going spectator remnant. In momentum space \( W \) is given in terms of \( M \)

\[ \Delta W_i^{\alpha \beta}(P, k, S) = \int \frac{d^4q}{(2\pi)^4} i \text{tr} G \left( (P - q)^2 \right) \frac{[(P - q + m_q)u(P, S)] \langle M \rangle^{\alpha \delta}_{g\beta} (q, P - k)}{[n \cdot (P - k - q) + i0] [(P - q)^2 - m_q^2 + i0] [q^2 - m_q^2 + i0]}, \]  

(6)

where \( i(n \cdot (P - k - q) + i0)^{-1} \) represents the eikonal propagator and \( \Delta W = W - W^0 \), where \( W^0 \) denotes the contribution without final-state interactions. Tracing Eq. 5 with \( \gamma^+ \) and weighting, and integrating with respect to \( k_T \) yields the first

FIGURE 1. Left: The matrix element \( W = \langle P - k | [\infty n; 0] | q(0) | P \rangle \) dressed with the FSIs. The FSIs are described by a non-perturbative scattering amplitude \( M \) that is calculated in a generalized ladder approximation [20]. Right: The quark-quark correlator with FSIs.
energetic particles on the light-cone in the calculation of
Eq. (8) the moment of the Sivers function [22]
\[ e^{ij}_{\tau}s_{\tau}f^{(1)\infty}_{\tau}(x) = -\frac{1}{2(1-x)\Lambda^{2}} \int \frac{d^{2}k}{(2\pi)^{2}} e^{ij}_{\bar{k}n}s_l f^{l}(x,\bar{k})E(x,0,-\frac{\bar{k}}{1-x}), \] (7)
where the lensing function \( f^{l} \) can be expressed in terms of the real and imaginary part of the scattering amplitude \( M \) [20].

We use functional methods to incorporate the color degrees of freedom for soft gauge boson coupling to highly energetic particles on the light-cone in the calculation of \( M \), which is given by the expression
\[
\frac{(M_{\text{color}})^{\alpha\delta}(\chi,\bar{q}_{\tau}+\bar{k}_{\tau})}{2(1-x)\Lambda^{2}} = \int d^{2}z_{\tau}e^{-i\chi\bar{q}_{\tau}+i\bar{k}_{\tau}} \left[ \int d^{N_{c}^{2}-1}\alpha \int d^{N_{c}^{2}-1}u \ e^{-i\alpha u} (\epsilon^{\alpha}_{\beta}(\chi)\epsilon^{\beta}_{\alpha})_{\alpha\delta} \epsilon^{(\tau)\beta}_{\alpha\delta} \right]. \] (8)

In Eq. (8) the \( N_{c}^{2}-1 \) dimensional integrals over the color parameters results from auxiliary fields \( \alpha_{\alpha}(x) \) and \( u^{\alpha}(s) \) that were introduced in the functional formalism of Ref. [25] in order to decouple the gluon fields from the color matrices. The eikonal phase \( \chi(\bar{z}_{\tau}) \) in Eq. (8) represents the amount of soft gluon exchanges that are summed up into an exponential form, and is given in terms of the gluon propagator
\[
\chi(\bar{z}_{\tau}) = g^{2} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta n^{\mu}\bar{n}^{\nu} \partial_{\mu\nu}(z+\alpha n-\beta \bar{n}). \] (9)
\( \partial \) denotes the gluon propagator, and \( g \) the strong coupling. In this form the four-vector \( v \) is related to the complementary light cone vector \( \bar{n}, v = -(1-x)P^{+}/m_{n}\bar{n}, \) with \( n \cdot \bar{n} = 1 \) and \( \bar{n}^{2} = 0 \). We evaluate the color integrals by deriving a power series representation for the expression in brackets in Eq. (8), the color function \( f^{\alpha\beta}(\chi) \)
\[
f^{\alpha\beta}(\chi) = \sum_{n=1}^{\infty} (i\chi)^{n} \sum_{a_{1}=1}^{N_{c}^{2}-1} \sum_{a_{n}=1}^{N_{c}^{2}-1} \sum_{n_{1}=1}^{P_{n}} \cdots \sum_{a_{1}=1}^{N_{c}^{2}-1} \sum_{a_{n}=1}^{N_{c}^{2}-1} \sum_{n_{1}=1}^{P_{n}} (t^{a_{1}} \cdots t^{a_{n}}t^{a_{1}(n)} \cdots t^{a_{n}(n)})_{\alpha\beta}, \] (10)
where \( P_{n} \) represents the sum over all permutations of the set \{1,...,n\}. If we had a direct ladder where gluons were not allowed to cross we would have only factors \( (t^{a_{1}} \cdots t^{a_{n}}t^{a_{1}(n)} \cdots t^{a_{n}(n)})_{\alpha\beta} = C_{F} \delta_{\alpha\beta} \) with \( C_{F} = N_{c}^{2} - 1/2N_{c} \), and we could work in an Abelian theory with an effective replacement \( C_{F} \alpha_{\bar{F}} \) for the fine-structure constant. Since we allow generalized ladders with crossed gluons we have to sum over all permutations in (10), and the simple replacement is not possible. In a large \( N_{c} \) expansion the crossed gluons diagrams would be suppressed such that the direct ladder represents the leading order in \( 1/N_{c} \). In an Abelian theory, the generating matrices \( t \) reduce to identity and since we have \( n! \) permutations of the set \{1,...,n\}, we recover the well-known Abelian result.
For comparison we also plot the perturbative result of Ref. [26] including the eikonalized antiquark spectator with an arbitrary value for the coupling, $\alpha = 0.3$. Right: The first moment of the Sivers function versus $x$ using various models for the GPD $E$.

\begin{align}
\mathcal{F}^{U(1)}(\chi) &= \sum_{n=1}^{\infty} (i\chi)^n / n! = e^{i\chi} - 1. \quad \text{For } N_c = 2, r^a = \sigma^a/2 \text{ and we can calculate the integral analytically. We obtain,}
\mathcal{F}^{SU(2)}(\chi/4) &= \delta_{\alpha\beta} \left( \cos \chi/4 - \chi/4 \sin \chi/4 - 1 \right) + i \delta_{\alpha\beta} \left( 2 \sin \chi/4 + \chi/4 \cos \chi/4 \right). \quad \text{We also calculate numerically the lowest coefficients in the power series (10), and they agree with the coefficients in an expansion in $\chi$ of the analytical result. This serves as a check of both numerical and analytical approaches. For } N_c = 3, \text{ due to difficulty of integrating over the Haar measure we use the power series (10) to obtain the approximative color function which is valid when } a = \chi/4 \text{ is small,}
\mathcal{R}_a^{SU(3)}(\chi) &= \delta_{\alpha\beta}(-c_2 a^2 + c_4 a^4 - c_6 a^6 - c_8 a^8 + \ldots), \quad \mathcal{S}_a^{SU(3)}(\chi) &= \delta_{\alpha\beta}(-c_1 a - c_3 a^3 + c_5 a^5 - c_7 a^7 + \ldots),
\end{align}
with the numerical values $c_1 = 5.333$, $c_2 = 6.222$, $c_3 = 3.951$, $c_4 = 1.934$, $c_5 = 0.680$, $c_6 = 0.198$, $c_7 = 0.047$, $c_8 = 0.00967$. Transforming to coordinate space we can express the lensing function directly in terms of the real and imaginary part of the color function $f$ which is itself a function of the eikonal phase $\chi$ Eq. (9). This results in a lensing function of the form
\begin{align}
\mathcal{I}(x, b_T) &= \frac{(1-x)}{2N_c} \frac{b_T}{|b_T|} \frac{x}{4} C \left[ \frac{\chi}{4} \right],
\end{align}
where we define the function,
\begin{align}
C \left[ \frac{\chi}{4} \right] &= \left[ \text{Tr} \mathcal{S}[f] \right] \left[ \frac{\chi}{4} \right] + \frac{1}{2} \text{Tr} \left[ \left( \mathcal{S}[f] \right) \left( \frac{\chi}{4} \right) \right] - \frac{1}{2} \text{Tr} \left[ \mathcal{S}[f] \right] \left[ \frac{\chi}{4} \right] - \mathcal{S}[f],
\end{align}
and $\chi'$ denotes the first derivative with respect to $|z\tau|$, and $\left( \mathcal{S}[f] \right)'$ and $\mathcal{S}[f]'$ are the first derivatives of the real and imaginary parts of the color function $f$.

In Fig. 3 the function $C \left[ \frac{\chi}{4} \right]$ is plotted versus $\frac{\chi}{4}$ for various approximations. While the convergence of the power series seems to be better for $SU(2)$ than in the $SU(3)$ case where the numerical result calculated with eight coefficients agrees with the analytical result up to $\frac{\chi}{4} \sim 2$, we can trust the numerical result computed with eight coefficients up to $\frac{\chi}{4} \sim 1.5$ for $SU(3)$.

In order to numerically estimate the lensing function and in turn the Sivers function we utilize the infrared behavior of the gluon and the running coupling in the non-perturbative regime where we infer that the soft gluon transverse momentum defines the scale at which the coupling is evaluated. These two quantities have been extensively studied in the infrared limit in the Dyson-Schwinger framework [27] and in lattice QCD [28]. We use calculations of these quantities from Dyson-Schwinger equations [27] where both $\alpha_s$ and $\theta^{-1}$ are defined in the infrared limit (details can be found in a forthcoming publication). This determines the eikonal phase and thus the lensing functions (12) for a $U(1)$, $SU(2)$ and $SU(3)$ color function. We plot the results in Fig. 3 for a color function for $U(1)$, $SU(2)$, $SU(3)$. While we observe that all lensing functions are attractive and fall off at large transverse distances, they are very different in size at small distances.

Using the eikonal model for the lensing function together with different models for the GPD $E$ we present the first moment of the Sivers function from Eq. (4). In Fig. (3) we compare the diquark model result [22], to two
phenomenological models [24, 23]. We find that the lensing function grows with $N_c$ which in turn predicts the growth of the Sivers function. This is consistent with the large $N_c$ QCD scaling behavior of Sivers functions [29] and further we uncover the deviations for finite $N_c$. Further, the sign and size of the Sivers function is consistent with present extractions from data [30]. Within this framework of FSIs we have performed quantitative analysis of approximate relations between T-odd TMDs and GPD which goes beyond the discussion of overall signs.

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