LIE SUPERALGEBRAS OF STRING THEORIES

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Abstract. We describe simple complex Lie superalgebras of vector fields on “supercircles” — stringy superalgebras. There are four series of such algebras (one depends on a complex parameter as well as on an integer one) and four exceptional stringy superalgebras. Two of the exceptional algebras are new in the literature.

The 13 of the simple stringy Lie superalgebras are distinguished: only they have nontrivial central extensions and since two of the distinguished algebras have 3 nontrivial central extensions each, there are exactly 16 superizations of the Liouville action, Schrödinger equation, KdV hierarchy, etc. We also present the three nontrivial cocycles on the $N = 4$ extended Neveu–Schwarz and Ramond superalgebras in terms of primary fields.

In the literature the stringy superalgebras are often referred to as superconformal ones. We discuss how superconformal stringy superalgebras really are.

Introduction

I.1. The discovery of stringy Lie superalgebras. The discovery of stringy superalgebras was not a one-time job and their further study was not smooth either. Even the original name of these Lie superalgebras is unfortunate. (Physicists dubbed them “superconformal” in analogy with the stringy Lie algebra $witt$ of conformal transformations, but, as we will show, not all stringy super algebras are superconformal. Besides, the diversity of them requires often to refer to them using their “given names” that describe them more precisely, rather than the common term. If the common name is needed, nevertheless, then stringy is, at least, not selfcontradictory and suggestive.) We give here an intrinsic description borrowed from [Ma].

The physicists who discovered stringy superalgebras ([NS], [R], [Ad]) were primarily interested in unitary representations, so they started with real algebras which are more difficult to classify than complex algebras. So they gave a number of examples, not a classification.

Observe also that physicists who studied superstrings were mainly interested in nontrivial central extensions of “superconformal” superalgebras. Only several first terms of the four series of stringy superalgebras — the 13 distinguished superalgebras — have such extensions, the other algebras were snubbed at. For a review how distinguished stringy superalgebras are used in string theory see [GSW]. For some other applications see [LX] and [LSX] (where some of the results are quite unexpected). Mathematically, nondistinguished simple stringy superalgebras are also of interest, see [CLL], [GL2] and [LS].

Ordered historically, the steps of classification are: [NS] and [R] followed by [Ad], where 4 series of the stringy superalgebras (without a continuous parameter) and most of the central extensions of the distinguished superalgebras were found for one real form of each algebra; [FL], where the complexifications of the algebras from [Ad] were interpreted geometrically and where the classifications of simple stringy superalgebras and their central extensions expressed in terms of superfields were announced. Regrettably, each classification had a gap. During the past years these gaps were partly filled in by several authors, in this paper the repair is completed.

Poletaeva [P] in 1983 and, independently, Schoutens [Sc] in 1986, found 3 nontrivial central extensions of $L^0(1|4)$ and $M^1(1|4)$, i.e., of the 4-extended Neveu-Schwarz and Ramond superalgebras (the importance of [P], whose results were expressed in unconventional terms, was not recognized in time and the paper was never properly published).

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Schwimmer and Seiberg [SS] found a deformation of the divergence-free series.

In [KvL] the completeness of the list of examples from [FL] amended with the deformation from [SS] was conjectured and the statements from [FL] and [Sc] on the nontrivial central extensions reproved; [KvL] contains the first published proof of the classification of the nontrivial central extensions of the simple superalgebras considered.

Other important steps of classification: [K], [L1] and [Sch], where the vectorial Lie superalgebras with polynomial coefficients are considered, and [Ma], where a characterisation of stringy algebras is introduced.

Observe, that after [Sc] there appeared several papers in which only two of the three central extensions of \( \mathfrak{f}^{(0)}(1|4) \) and \( \mathfrak{f}^{M}(1|4) \) were recognized; the controversy is occasioned, presumably, by the insufficiently lucid description of the superalgebras involved and ensuing confusion between the exceptional simple superalgebra \( \mathfrak{f}^{(0)}(1|4) \) and \( \mathfrak{f}^{(1)}(1|4) \), see below. Besides, the cocycles that the physicists need should be expressed in terms of the primary fields; so far, this was not done.

I.2. Our results. Here we define classical stringy Lie superalgebras, formally superconformal ones, and announce the conjectural list of all simple stringy superalgebras. We thus repair the classification result of [FL] with the help of later discovery [Sch]. Two series of the simple stringy superalgebras and several exceptional ones, as well as two nontrivial cocycles, seem to be new. Proof of the completeness of the classification will be given elsewhere.

We also answer a question of S. Krivonos: we replace the three cocycles found in [P] and [Sh] with the cohomologic ones but expressed in terms of primary fields.

I.3. Related latest results. This paper is accompanied with several related papers.

1) Real forms. A 1986 result of Serganova [S], completed with the real forms of the distinguished and simple stringy superalgebras unknown to her at that time. Crucial there is the discovery of 3, not 2, types of real forms of stringy and Kac–Moody superalgebras, cf. [S] with [K1], where only two types of real forms of Kac–Moody algebras are recognized. Another important result of Serganova pertaining here: the discovery of three basic types of unitarity, one of them with an odd form.

2) Semi-infinite cohomology and the critical dimension. See [LSX].

3) Integrable systems. See [LX].

Remark. The results of this paper were obtained in Stockholm in June 1996 (except 4.3, explicit formula 1.1.8 and the answer to Krivonos’ question) and delivered at the seminar of E. Ivanov, JINR, Dubna (July, 1996) and Voronezh winter school Jan. 12–18, 1997. Kac’s questions concerning exceptional Lie superalgebras in his numerous letters to I. Shchepochkina in October-November 1996 encouraged us to struggle with the crashing computer systems and \( \TeX \). While the \( \TeX \)-file was being processed, we got a recent preprint [CK] by Cheng Shun-Jen and V. Kac, where our example \( \mathfrak{t} \) is described in different terms. We thank Kac for the kind letter (sent to Leites) that acknowledges his receiving of a preprint of [Sch] and interesting preprints [CK] and [K2] (where several results from [L2] get a new perspective).

§0. Background

0.1. Linear algebra in superspaces. Generalities. Superization has certain subtleties, often disregarded or expressed as in [L], [L3] or [M]: too briefly. We will dwell on them a bit.

A superspace is a \( \mathbb{Z}/2 \)-graded space; for a superspace \( V = V_{\bar{0}} \oplus V_{\bar{1}} \) denote by \( \Pi(V) \) another copy of the same superspace: with the shifted parity, i.e., \( (\Pi(V))_{\bar{i}} = V_{\bar{i} + 1} \). The superdimension of \( V \) is \( \dim V = p + q\varepsilon \), where \( \varepsilon^2 = 1 \) and \( p = \dim V_{\bar{0}}, q = \dim V_{\bar{1}} \). (Usually, \( \dim V \) is expressed as a pair \( (p, q) \) or \( p|q \); this obscures the fact that \( \dim V \otimes W = \dim V \cdot \dim W \), clear with the help of \( \varepsilon \).

A superspace structure in \( V \) induces the superspace structure in the space \( \text{End}(V) \). A basis of a superspace is always a basis consisting of homogeneous vectors; let \( \text{Par} = \{p_1, \ldots, p_{\dim V}\} \) be an ordered collection of their parities. We call \( \text{Par} \) the format of (the basis of) \( V \). A square supermatrix of format (size) \( \text{Par} \) is a \( \dim V \times \dim V \) matrix whose \( i \)th row and \( i \)th column are of the same parity \( p_i \). The matrix unit \( E_{ij} \) is supposed to be of parity \( p_i + p_j \) and the bracket of supermatrices (of the same format) is defined via Sign Rule: if something of parity \( p \) moves past something of parity \( q \) the sign \( (-1)^{pq} \) accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity. For example, setting \( [X, Y] = XY - (-1)^{\text{Par}(X)\text{Par}(Y)}YX \) we get the notion of the supercommutator and the ensuing notion of the Lie superalgebra (that satisfies the superskew-commutativity and super Jacobi identity).

We do not usually use the sign \( \wedge \) for differential forms on supermanifolds: in what follows we assume that the exterior differential is odd and the differential forms constitute a supercommutative superalgebra; still, we keep using it on manifolds, sometimes, not to divagate too far from conventional notations.
Usually, \( \text{Par} \) is of the form \((0, \ldots, 0, \bar{1}, \ldots, \bar{1})\). Such a format is called \textit{standard}. In this paper we can do without nonstandard formats. But they are vital in the study of systems of simple roots that the reader might be interested in connection with applications to \( q \)-quantization or integrable systems.

The \textit{general linear} Lie superalgebra of all supermatrices of size \( \text{Par} \) is denoted by \( \mathfrak{gl}(\text{Par}) \); usually, \( \mathfrak{gl}(0, \ldots, 0, 1, \ldots, 1) \) is abbreviated to \( \mathfrak{gl}(\dim V_0, \dim V_1) \). Any matrix from \( \mathfrak{gl}(\text{Par}) \) can be expressed as the sum of its even and odd parts; in the standard format this is the following block expression:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} =
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix} + \begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}, \quad \text{p}\left( \begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix} \right) = 0,
\text{p}\left( \begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix} \right) = \bar{1}.
\]

The \textit{supertrace} is the map \( \mathfrak{gl}(\text{Par}) \to \mathbb{C} \), \((A_{ij}) \to \sum_{(-1)^{|i|}p} A_{ji}\). Since \( \text{str}[x, y] = 0 \), the subsuperspace of supertraceless matrices constitutes the \textit{special linear} Lie subsuperalgebra \( \mathfrak{sl}(\text{Par}) \).

**Superalgebras that preserve bilinear forms: two types.** To the linear map \( F \) of superspaces there corresponds the dual map \( F^* \) between the dual superspaces; if \( A \) is the supermatrix corresponding to \( F \) in a basis of format \( \text{Par} \), then to \( F^* \) the \textit{supertransposed} matrix \( A^* \) corresponds:

\[
(A^*)_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.
\]

The supermatrices \( X \in \mathfrak{gl}(\text{Par}) \) such that

\[
X^* B + (-1)^{|X||B|} B X = 0
\]

constitute the Lie superalgebra \( \mathfrak{aut}(B) \) that preserves the bilinear form on \( V \) with matrix \( B \). Most popular is the nondegenerate supersymmetric form whose matrix in the standard format is the canonical form \( B_{ev} \) or \( B'_{ev} \):

\[
B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \text{or } B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \ldots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.
\]

The usual notation for \( \mathfrak{aut}(B_{ev}(m|2n)) \) is \( \mathfrak{osp}(m|2n) \) or \( \mathfrak{osp}^{sy}(m|2n) \). Observe that the passage from \( V \) to \( \Pi(V) \) sends the supersymmetric forms to superskew-symmetric ones, preserved by the “symplectic-orthogonal” Lie superalgebra, \( \mathfrak{sp}'(2n|m) \) or better \( \mathfrak{osp}^{sk}(m|2n) \), which is isomorphic to \( \mathfrak{osp}^{sy}(m|2n) \) but has a different matrix realization. We never use notation \( \mathfrak{sp}'(2n|m) \) in order not to confuse with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:

\[
\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & -X^t \\ X & A & B \\ Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y^t & -X^t & E \end{pmatrix} \right\},
\]

where \( \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n) \), \( E \in \mathfrak{o}(m) \) and \( ^t \) is the usual transposition.

A nondegenerate supersymmetric odd bilinear form \( B_{odd}(n|n) \) can be reduced to a canonical form whose matrix in the standard format is \( J_{2n} \). A canonical form of the superskew odd nondegenerate form in the standard format is \( \Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \). The usual notation for \( \mathfrak{aut}(B_{odd}(\text{Par})) \) is \( \mathfrak{pe}(\text{Par}) \). The passage from \( V \) to \( \Pi(V) \) sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism \( \mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^{sk}(\text{Par}) \). This Lie superalgebra is called, as A. Weil suggested, \textit{periplectic}, i.e., odd-plectic.

The matrix realizations in the standard format of these superalgebras is shorthanded to:

\[
\mathfrak{pe}^{sy}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}, \text{where } B = -B^t, C = C^t;
\]

\[
\mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}, \text{where } B = B^t, C = -C^t.
\]

The \textit{special periplectic} superalgebra is \( \mathfrak{spe}(n) = \{ X \in \mathfrak{pe}(n) : \text{str}X = 0 \} \).

### 0.2. Vectorial Lie superalgebras. The standard realization

The elements of the Lie algebra \( \mathcal{L} = \mathfrak{d} \mathfrak{e} \mathfrak{r} \mathfrak{t} \mathfrak{c} \mathfrak{h} \mathfrak{e} \mathfrak{n} \mathfrak{e} \mathfrak{r} \mathfrak{i} \mathfrak{o} \mathfrak{n} \mathfrak{e} \mathfrak{r} \mathfrak{i} \mathfrak{s} \mathfrak{i} \mathfrak{a} \mathfrak{s} \mathfrak{f} \mathfrak{i} \mathfrak{o} \mathfrak{d} \mathfrak{i} \mathfrak{n} \mathfrak{o} \mathfrak{t} \mathfrak{i} \mathfrak{o} \mathfrak{n} \mathfrak{e} \mathfrak{r} \} \) are considered as vector fields. The Lie algebra \( \mathcal{L} \) has only one maximal subalgebra \( \mathcal{L}_0 \) of finite codimension (consisting of the fields that vanish at the origin). The subalgebra \( \mathcal{L}_0 \) determines a filtration of \( \mathcal{L} \): set

\[
\mathcal{L}_{-1} = \mathcal{L}; \quad \mathcal{L}_i = \{ D \in \mathcal{L}_{i-1} : [D, \mathcal{L}] \subset \mathcal{L}_{i-1} \} \text{ for } i \geq 1.
\]

The associated graded Lie algebra \( \mathcal{L} = \bigoplus_{i \geq -1} \mathcal{L}_i \), where \( \mathcal{L}_i = \mathcal{L}_i/\mathcal{L}_{i+1} \), consists of the vector fields with \textit{polynomial} coefficients.
Superization and the passage to a subalgebras of \( \mathfrak{der} \mathbb{C}[[u]] \) brings new phenomena. Suppose \( L_0 \subset L \) is a maximal subalgebra of finite codimension and containing no ideals of \( L \). For the Lie superalgebra \( L = \mathfrak{der} \mathbb{C}[u, \xi] \) the minimal \( L_0 \)-invariant subspace of \( L \) strictly containing \( L_0 \) coincides with \( L \). Not all the subalgebras \( L \) of \( \mathfrak{der} \mathbb{C}[u, \xi] \) have this property. Let \( L_{i-1} \) be a minimal subspace of \( L \) containing \( L_0 \), different from \( L_0 \) and \( L_0 \)-invariant. A Weisfeiler filtration of \( L \) is determined by the formula

\[
L_{-i-1} = [L_{-i}, L_{-i}] + L_{-i} \quad L_i = \{ D \in L_{i-1} : [D, L_{i-1}] \subset L_{i-1} \} \text{ for } i > 0.
\]

Since the codimension of \( L_0 \) is finite, the filtration takes the form

\[
L = L_{-d} \supset L_0 \supset \ldots
\]

for some \( d \). This \( d \) is the depth of \( L \) or of the associated graded Lie superalgebra \( L \). We call all filtered or graded Lie superalgebras of finite depth \( \text{vectoral} \), i.e., realizable with vector fields on a finite dimensional supermanifold. Considering the subspaces \( (0) \) as the basis of a topology, we can complete the graded or filtered Lie superalgebras \( L \) or \( L \); the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Unlike Lie algebras, simple vectoral superalgebras possess several maximal subalgebras of finite codimension. We describe them, together with the corresponding gradings, in sec. 0.4.

1) General algebras. Let \( x = (u_1, \ldots, u_n, \theta_1, \ldots, \theta_m) \), where the \( u_i \) are even indeterminates and the \( \theta_j \) are odd ones. Set \( \text{vect}(n|m) = \mathfrak{der} \mathbb{C}[x] \); it is called the general vectoral Lie superalgebra.

2) Special algebras. The divergence of the field \( D = \sum f_i \frac{\partial}{\partial u_i} + \sum g_j \frac{\partial}{\partial \theta_j} \) is the function (in our case: a polynomial, or a series)

\[
\text{div}D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_j}{\partial \theta_j}
\]

- The Lie superalgebra \( \text{svect}(n|m) = \{ D \in \text{vect}(n|m) : \text{div}D = 0 \} \) is called the special or divergence-free vectoral superalgebra.

It is clear that it is also possible to describe \( \text{svect}(n|m) \) as \( \{ D \in \text{vect}(n|m) : L_D \text{vol}_{x} = 0 \} \), where \( \text{vol}_{x} \) is the volume form with constant coefficients in coordinates \( x \) and \( L_D \) the Lie derivative with respect to \( D \).

- The Lie superalgebra \( \text{svect}_{\lambda}(0|m) = \{ D \in \text{vect}(0|m) : \text{div}(1 + \lambda \theta_1 \cdots \theta_m)D = 0 \} \) — the deform of \( \text{svect}(0|m) \) — is called the special or divergence-free vectoral superalgebra. Clearly, \( \text{svect}_{\lambda}(0|m) \cong \text{svect}_{\mu}(0|m) \) for \( \lambda \mu \neq 0 \). Observe that \( p(\lambda) \equiv m \mod(2) \), i.e., for odd \( m \) the parameter of deformation \( \lambda \) is odd.

**Remark.** Sometimes we write \( \text{vect}(x) \) or even \( \text{vect}(V) \) if \( V = \text{Span}(x) \) and use similar notations for the subalgebras of \( \text{vect} \) introduced below. Algebraists sometimes abbreviate \( \text{vect}(n) \) and \( \text{vect}(n) \) to \( W_n \) (in honor of Witt) and \( S_n \), respectively.

3) The algebras that preserve Pfaff equations and differential 2-forms.
- Set \( u = (t, p_1, \ldots, p_n, q_1, \ldots, q_n) \); let

\[
\hat{\alpha}_1 = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq m} \theta_j d\theta_j \quad \text{and} \quad \omega_0 = d\alpha_1.
\]

The form \( \hat{\alpha}_1 \) is called contact, the form \( \omega_0 \) is called symplectic. Sometimes it is more convenient to redenote the \( \theta \)'s and set

\[
\xi_j = \frac{1}{\sqrt{2}} (\theta_j - i \theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}} (\theta_j + i \theta_{r+j}) \quad \text{for } j \leq r = [m/2] \quad \text{(here } i^2 = -1) \quad \text{and} \quad \theta = \theta_{2r+1}
\]

and in place of \( \omega_0 \) or \( \hat{\alpha}_1 \) take \( \alpha_1 \) and \( \omega_0 = d\alpha_1 \), respectively, where

\[
\alpha_1 = \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) \quad \text{if } m = 2r
\]

\[
\alpha_1 = \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + d\theta \quad \text{if } m = 2r + 1.
\]

The Lie superalgebra that preserves the Pfaff equation \( \alpha_1 = 0 \), i.e., the superalgebra

\[
\mathfrak{t}(2n+1|m) = \{ D \in \text{vect}(2n+1|m) : L_D \alpha_1 = f_D \alpha_1 \},
\]

(here \( f_D \in \mathbb{C}[t, p, q, \xi] \) is a polynomial determined by \( D \)) is called the contact superalgebra. The Lie superalgebra

\[
\mathfrak{po}(2n|m) = \{ D \in \mathfrak{t}(2n+1|m) : L_D \alpha_1 = 0 \}
\]
is called the Poisson superalgebra. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form $\alpha$ in the line bundle over a symplectic supermanifold with the symplectic form $d\alpha$.)

- Similarly, set $u = q = (q_1, \ldots, q_n)$, let $\theta = (\xi_1, \ldots, \xi_n; \tau)$ be odd. Set
  $$\alpha_0 = d\tau + \sum_i (\xi_i dq_i + q_i d\xi_i), \quad \omega_1 = d\alpha_0$$

and call these forms the odd contact and periplectic, respectively.

The Lie superalgebra that preserves the Pfaff equation $\alpha_0 = 0$, i.e., the superalgebra
  $$\mathfrak{m}(n) = \{ D \in \text{vect}(n|n+1) : L_D \alpha_0 = f_D \cdot \alpha_0 \}, \quad \text{where } f_D \in \mathbb{C}[q, \xi, \tau],$$
is called the odd contact superalgebra.

The Lie superalgebra
  $$\mathfrak{b}(n) = \{ D \in \mathfrak{m}(n) : L_D \alpha_0 = 0 \}$$
is called the Buttin superalgebra ([L3]). (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the Pfaff equation $\alpha$, and call these forms the supermanifold, i.e., a supermanifold with the periplectic form $d\alpha_0$.)

The Lie superalgebras
  $$\mathfrak{sm}(n) = \{ D \in \mathfrak{m}(n) : \text{div } D = 0 \}, \quad \mathfrak{sb}(n) = \{ D \in \mathfrak{b}(n) : \text{div } D = 0 \}$$

are called the divergence-free (or special) odd contact and special Buttin superalgebras, respectively.

Remark. A relation with finite dimensional geometry is as follows. Clearly, $\ker \alpha_1 = \ker \alpha_1$. The restriction of $\omega_0$ to $\ker \alpha_1$ is the orthosymplectic form $B_{ev}(m|2n)$; the restriction of $\omega_0$ to $\ker \alpha_1$ is $B_{ev}(m|2n)$. Similarly, the restriction of $\omega_1$ to $\ker \alpha_0$ is $B_{odd}(n|n)$.

### 0.3. Generating functions.

A laconic way to describe $\mathfrak{t}$, $\mathfrak{m}$ and their subalgebras is via generating functions.

- Odd form $\alpha_1$. For $f \in \mathbb{C}[t, p, q, \xi]$ set:
  $$K_f = \Delta(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,$$

where $E = \sum_i y_i \frac{\partial}{\partial y_i}$ (here the $y$ are all the coordinates except $t$) is the Euler operator (which counts the degree with respect to the $y$), $\Delta(f) = 2f - E(f)$, and $H_f$ is the hamiltonian field with Hamiltonian $f$ that preserves $d\alpha_1$:

$$H_f = \sum_{i \leq n} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - (-1)^p(f) \left( \sum_{j \leq m} \frac{\partial f}{\partial \theta^j} \frac{\partial}{\partial \theta^j} \right), \quad f \in \mathbb{C}[p, q, \theta].$$

The choice of the form $\alpha_1$ instead of $\tilde{\alpha}_1$ only affects the form of $H_f$ that we give for $m = 2k + 1$:

$$H_f = \sum_{i \leq n} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - (-1)^p(f) \sum_{j \leq k} \left( \frac{\partial f}{\partial \theta^j} \frac{\partial}{\partial \theta^j} + \frac{\partial f}{\partial \eta^j} \frac{\partial}{\partial \eta^j} + \frac{\partial f}{\partial \eta^j} \frac{\partial}{\partial \theta^j} \right), \quad f \in \mathbb{C}[p, q, \xi, \eta, \theta].$$

- Even form $\alpha_0$. For $f \in \mathbb{C}[q, \xi, \tau]$ set:
  $$M_f = \Delta(f) \frac{\partial}{\partial \tau} - Le_f - (-1)^p(f) \frac{\partial f}{\partial \tau} E,$$

where $E = \sum_i y_i \frac{\partial}{\partial y_i}$ (here the $y$ are all the coordinates except $\tau$) is the Euler operator, $\Delta(f) = 2f - E(f)$, and

$$Le_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial \xi_i} + (-1)^p(f) \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial q_i} \right), \quad f \in \mathbb{C}[q, \xi].$$

Since

$$L_{K_f}(\alpha_1) = 2 \frac{\partial f}{\partial t} \alpha_1, \quad L_{M_f}(\alpha_0) = -(-1)^p(f) 2 \frac{\partial f}{\partial \tau} \alpha_0,$$

it follows that $K_f \in \mathfrak{t}(2n + 1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$p(Le_f) = p(M_f) = p(f) + 1.$$
To the (super)commutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond contact brackets of the generating functions:

$$[K_f, K_g] = K_{\{f,g\}_{k.b.}}; \quad [M_f, M_g] = M_{\{f,g\}_{m.b.}}.$$  

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on $t$ (resp. $\tau$).

The Poisson bracket $\{\cdot, \cdot\}_{P.b.}$ (in the realization with the form $\omega_0$) is given by the formula

$$\{f, g\}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j},$$

and in the realization with the form $\omega_0$ for $m = 2k + 1$ it is given by the formula

$$\{f, g\}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial q_i} \right).$$

The Buttin bracket $\{\cdot, \cdot\}_{B.b.}$ is given by the formula

$$\{f, g\}_{B.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial q_i} + (-1)^{p(f)} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \xi_i} \right).$$

Remark. The what we call here Buttin bracket was discovered in pre-super era by Schouten; Buttin proved that this bracket establishes a Lie superalgebra structure and the interpretation of this superalgebra similar to that of Poisson algebra was given in [L1]. The Schouten bracket was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra $\mathcal{F}$ of functions) of the tangent bundle, $\Gamma(\Lambda^*(T(M))) \cong \Lambda_{\mathbb{Z}}(\text{Vect}(M))$. The explicit formula of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l. \quad (*)$$

With the help of Sign Rule we easily superize formula $(*)$ for the case when manifold $M$ is replaced with supermanifold $M$. Let $x$ and $\xi$ be the even and odd coordinates on $M$. Setting $\theta_i = \Pi(\partial x_i) = \delta_i$, $\eta_i = \Pi(\partial \xi_i) = \xi_i$ we get an identification of the Schouten bracket of polyvector fields on $M$ with the Buttin bracket of functions on the supermanifold $M$ whose coordinates are $x, \xi$ and $\xi$ and $\delta = \Pi(\partial x)$, $\xi = \Pi(\partial \xi)$; the transformation of $x, \xi$ induces from that of the checked coordinates.

In terms of the Poisson and Buttin brackets, respectively, the contact brackets take the form

$$\{f, g\}_{k.b.} = \Delta(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \Delta(g) - \{f, g\}_{P.b.},$$

and, respectively,

$$\{f, g\}_{m.b.} = \Delta(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} \Delta(g) - \{f, g\}_{B.b.}.$$

The Lie superalgebras of Hamiltonian fields (or Hamiltonian superalgebra) and its special subalgebra (defined only if $n = 0$) are

$$\mathfrak{h}(2n|m) = \{D \in \text{Vect}(2n|m) : L_D \omega_0 = 0\} \quad \text{and} \quad \mathfrak{sh}(m) = \{H_f \in \mathfrak{h}(0|m) : \int f \text{vol} = 0\}.$$  

Its odd analogues are the Lie superalgebra of Leitesian fields introduced in [L1] and its special subalgebra:

$$\mathfrak{le}(n) = \{D \in \text{Vect}(n|n) : L_D \omega_1 = 0\} \quad \text{and} \quad \mathfrak{se}(n) = \{D \in \mathfrak{le}(n) : \text{div} D = 0\}.$$  

It is not difficult to prove the following isomorphisms (as superspaces):

$$\mathfrak{t}(2n + 1|m) \cong \text{Span}(K_f : f \in \mathbb{C}[t, p, q, \xi]); \quad \mathfrak{le}(n) \cong \text{Span}(L_{ef} : f \in \mathbb{C}[q, \xi]);$$

$$\mathfrak{m}(n) \cong \text{Span}(M_f : f \in \mathbb{C}[r, q, \xi]); \quad \mathfrak{h}(2n|m) \cong \text{Span}(H_f : f \in \mathbb{C}[p, q, \xi]).$$

Remark. 1) It is obvious that the Lie superalgebras of the series vect, svect, $\mathfrak{h}$ and $\mathfrak{p}$ for $n = 0$ are finite dimensional.

2) A Lie superalgebra of the series $\mathfrak{h}$ is the quotient of the Lie superalgebra $\mathfrak{p}$ modulo the one-dimensional center $\mathfrak{z}$ generated by constant functions. Similarly, $\mathfrak{le}$ and $\mathfrak{se}$ are the quotients of $\mathfrak{h}$ and $\mathfrak{sh}$, respectively, modulo the one-dimensional (odd) center $\mathfrak{z}$ generated by constant functions.
Set \(\mathfrak{sp}(m) = \{K_f \in \mathfrak{po}(0|m) : \int f \nu = 0\}\); clearly, \(\mathfrak{sh}(m) = \mathfrak{sp}(m)/\mathfrak{z}\).

Since

\[
\text{div} M_f = (-1)^{p(f)} 2 \left( (1 - E) \frac{\partial f}{\partial t} - \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right),
\]

it follows that

\[
\mathfrak{sm}(n) = \text{Span} \left( M_f \in \mathfrak{m}(n) : (1 - E) \frac{\partial f}{\partial t} = \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right).
\]

In particular,

\[
\text{div} L e_f = (-1)^{p(f)} 2 \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i}.
\]

The odd analog of the Laplacian, namely, the operator

\[
\Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}
\]

on a periplectic supermanifold appeared in physics under the name of **BRST operator**, cf. [GPS]. The divergence-free vector fields from \(\mathfrak{sl}(n)\) are generated by **harmonic** functions, i.e., such that \(\Delta(f) = 0\).

Lie superalgebras \(\mathfrak{sl}(n), \mathfrak{sb}(n)\) and \(\mathfrak{svect}(1|n)\) have ideals \(\mathfrak{sl}^c(n), \mathfrak{sb}^c(n)\) and \(\mathfrak{svect}^c(n)\) of codimension 1 defined from the exact sequences

\[
\begin{align*}
0 &\rightarrow \mathfrak{sl}^c(n) \rightarrow \mathfrak{sl}(n) \rightarrow \mathbb{C} : \mathfrak{L} e_{\xi_1 \ldots \xi_n} \rightarrow 0, \\
0 &\rightarrow \mathfrak{sb}^c(n) \rightarrow \mathfrak{sb}(n) \rightarrow \mathbb{C} : \mathfrak{M}_{\xi_1 \ldots \xi_n} \rightarrow 0, \\
0 &\rightarrow \mathfrak{svect}^c(n) \rightarrow \mathfrak{svect}(1|n) \rightarrow \mathbb{C} : \mathfrak{K}_{\xi_1 \ldots \xi_n} \frac{\partial}{\partial t} \rightarrow 0.
\end{align*}
\]

**0.4. Nonstandard realizations.** In [LSh] we proved that the following are all the nonstandard gradings of the Lie superalgebras indicated. Moreover, the gradings in the series \(\mathfrak{svect}\) induce the gradings in the series \(\mathfrak{svect}^c\); the gradings in \(\mathfrak{m}\) induce the gradings in \(\mathfrak{sm}, \mathfrak{le}, \mathfrak{sl}^c, \mathfrak{sb}^c\); the gradings in \(\mathfrak{t}\) induce the gradings in \(\mathfrak{po}, \mathfrak{h}\). In what follows we consider \(\mathfrak{t}(2n + 1|m)\) as preserving Pfaff eq. \(\alpha_1 = 0\).

The standard realizations are marked by \((\ast)\) and in this case indication to \(r = 0\) is omitted; note that (bar several exceptions for small \(m, n\)) it corresponds to the case of the minimal codimension of \(\mathcal{L}_0\).

| Lie superalgebra | its Z-grading |
|-----------------|---------------|
| \(\mathfrak{vect}(n|m; r)\), \(0 \leq r \leq m\) | \(\deg u_i = \deg \xi_j = 1\) for any \(i, j\) \((\ast)\) |
| \(\mathfrak{m}(n;r), 0 \leq r \leq n\) | \(\deg \xi_j = 0\) for \(1 \leq j \leq r\); \(\deg u_i = \deg \xi_{n+1} = 1\) for any \(i, s\) |
| \(\mathfrak{t}(2n + 1|m; r), 0 \leq r \leq \left[\frac{m}{2}\right]\) | \(\deg t = 2\), \(\deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1\) for any \(i, j, k\) \((\ast)\) |
| \(\mathfrak{t}(1|2m; m)\) | \(\deg t = \deg \xi_i = 1\), \(\deg \eta_i = 0\) for \(1 \leq i \leq m\) |

Observe that \(\mathfrak{t}(1|2; 2) \cong \mathfrak{vect}(1|1)\) and \(\mathfrak{m}(1; 1) \cong \mathfrak{vect}(1|1)\).

**The exceptional nonstandard gradings.** Denote the indeterminates and their respective exceptional degrees as follows (here \(\mathfrak{t}(1|2)\) is considered in the realization that preserves the Pfaff eq. \(\alpha_1 = 0\)):

| \(\mathfrak{vect}(1|1)\) | \(t, \xi\) |
|-----------------|---------------|
| \(\mathfrak{t}(1|2)\) | \(t, \xi, \eta\) |
| \(\mathfrak{m}(1)\) | \(\tau, q, \xi\) |

| \(\mathfrak{vect}(1|1)\) | \(t, \xi\) |
|----------------------------|----------------|
| \(\mathfrak{t}(1|2)\) | \(t, \xi, \eta\) |
| \(\mathfrak{m}(1)\) | \(\tau, q, \xi\) |
Denote the nonstandard exceptional realizations by indicating the above degrees after a semicolon. We get the following isomorphisms:

\[
\begin{align*}
\text{vect}(1|1; 2, 1) &\cong \mathfrak{t}(1|2); \\
\mathfrak{t}(1|2; 1, 2, -1) &\cong \mathfrak{m}(1); \\
\text{vect}(1|1; -1) &\cong \mathfrak{m}(1); \\
\mathfrak{m}(1; 1, 2, -1) &\cong \mathfrak{t}(1|2).
\end{align*}
\]

Observe that the Lie superalgebras corresponding to different values of \(r\) are isomorphic as abstract Lie superalgebras, but as filtered ones they are distinct.

### 0.5. The Cartan prolongs

We will repeatedly use the Cartan prolong. Let us recall the definition and generalize it somewhat. Let \(\mathfrak{g}\) be a Lie algebra, \(V\) a \(\mathfrak{g}\)-module, \(S^i\) the operator of the \(i\)-th symmetric power. Set \(\mathfrak{g}_{-1} = V, \mathfrak{g}_0 = \mathfrak{g}\) and define the \(i\)-th Cartan prolongation for \(i > 0\) as

\[
\mathfrak{g}_i = \{X \in \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) : X(v)(w, ...) = X(w)(v, ...) \text{ for any } v, w \in \mathfrak{g}_{-1} \}
= (S^i(\mathfrak{g}_{-1}^*)^* \otimes \mathfrak{g}_0) \cap (S^{i+1}(\mathfrak{g}_{-1}^*)^* \otimes \mathfrak{g}_{-1}).
\]

The Cartan prolong (the result of Cartan’s prolongation) of the pair \((V, \mathfrak{g})\) is \((\mathfrak{g}_{-1}, \mathfrak{g}_0)^* = \bigoplus_{i \geq -1} \mathfrak{g}_i\). (In what follows \(\cdot\) in superscript denotes, as is now customary, the collection of all degrees, while \(\ast\) is reserved for dualization; in the subscripts we retain the oldfashioned \(\ast\) instead of \(\cdot\) to avoid too close a contact with the punctuation marks.)

Suppose that the \(\mathfrak{g}_0\)-module \(\mathfrak{g}_{-1}\) is faithful. Then, clearly,

\[(\mathfrak{g}_{-1}, \mathfrak{g}_0)^* \subset \text{vect}(n) = \text{der} C[x_1, ..., x_n], \text{ where } n = \text{dim } \mathfrak{g}_{-1} \text{ and } \mathfrak{g}_i = \{D \in \text{vect}(n) : \deg D = i, [D, X] \in \mathfrak{g}_{-1} \text{ for any } X \in \mathfrak{g}_{-1}\}.
\]

It is subject to an easy verification that the Lie algebra structure on \(\text{vect}(n)\) induces same on \((\mathfrak{g}_{-1}, \mathfrak{g}_0)^*\).

Of the four simple vectorial Lie algebras, three are Cartan prolongs: \(\text{vect}(n) = (\text{id}, gl(n))^*, \text{svect}(n) = (\text{id}, \mathfrak{sl}(n))^*, \text{svect}(2n) = (\text{id}, \mathfrak{sp}(n))^*\). The fourth one — \(\mathfrak{t}(2n + 1)\) — is also the prolong under a trifle more general construction described as follows.

**A generalization of the Cartan prolong.** Let \(\mathfrak{g}_{-1} = \bigoplus_{-d \leq \ell \leq -1} \mathfrak{g}_\ell\) be a nilpotent \(\mathbb{Z}\)-graded Lie algebra and \(\mathfrak{g}_0 \subset \text{der}_\mathfrak{g} \mathfrak{g}\) a Lie subalgebra of the \(\mathbb{Z}\)-grading-preserving derivations. For \(i > 0\) define the \(i\)-th prolong of the pair \((\mathfrak{g}, \mathfrak{g}_0)\) to be:

\[
\mathfrak{g}_i = ((S^\ell(\mathfrak{g}_{-1}^*)^* \otimes \mathfrak{g}_0) \cap (S^{\ell+1}(\mathfrak{g}_{-1}^*)^* \otimes \mathfrak{g}_{-1}))_{i},
\]

where the subscript \(i\) in the rhs singles out the component of degree \(i\).

Define \(\mathfrak{g}_0\), or rather, \((\mathfrak{g}_{-1}, \mathfrak{g}_0)^*\), to be \(\bigoplus_{i \geq -d} \mathfrak{g}_i\); then, as is easy to verify, \((\mathfrak{g}_{-1}, \mathfrak{g}_0)^*\) is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by \(\mathfrak{hei}(2n)\) the Heisenberg Lie algebra: its space is \(W \oplus \mathbb{C} \cdot z\), where \(W\) is a \(2n\)-dimensional space endowed with a nondegenerate skew-symmetric bilinear form \(B\) and the bracket in \(\mathfrak{hei}(2n)\) is given by the following conditions: \(z\) is in the center and \([v, w] = B(v, w) \cdot z\) for any \(v, w \in W\).

Clearly, \(\mathfrak{t}(2n + 1) = (\mathfrak{hei}(2n), \mathfrak{osp}(2n))^*\), where for any \(\mathfrak{g}\) we write \(\mathfrak{cg} = \mathfrak{g} \oplus \mathbb{C} \cdot z\) or \(\mathfrak{c}(\mathfrak{g})\) to denote the trivial central extension with the 1-dimensional even center generated by \(z\).

### 0.6. Lie superalgebras of vector fields as Cartan’s prolongs

The superization of the constructions from sec. 0.5 are straightforward: via Sign Rule. We thus get

\[
\begin{align*}
\text{vect}(m|n) &\cong (\text{id}, gl(m|n))^*, \text{svect}(m|n) = (\text{id}, \mathfrak{sl}(m|n))^*; \\
\mathfrak{le}(n) &\cong (\text{id}, \mathfrak{sp}(n))^*; \\
\mathfrak{fe}(n) &\cong (\text{id}, \mathfrak{spe}(n))^*.
\end{align*}
\]

Remark. Observe that the Cartan prolongs \((\text{id}, \mathfrak{osp}(m|2n))^*\), and \((\text{id}, \mathfrak{spe}(n))^*\) are finite dimensional.

The generalization of Cartan’s prolongations described in 0.5 has, after superization, two analogs associated with the contact series \(\mathfrak{t}\) and \(\mathfrak{m}\), respectively.

- First we define \(\mathfrak{hei}(2n|m)\) on the direct sum of a \((2n, m)\)-dimensional superspace \(W\) endowed with a nondegenerate skew-symmetric bilinear form and a \(1, 0\)-dimensional space spanned by \(z\).

  Clearly, we have \(\mathfrak{t}(2n + 1|m) = (\mathfrak{hei}(2n|m), \mathfrak{osp}(m|2n))\), and, given \(\mathfrak{hei}(2n|m)\) and a subalgebra \(\mathfrak{g}\) of \(\mathfrak{osp}(m|2n)\), we call \((\mathfrak{hei}(2n|m), \mathfrak{g})\) the \(k\)-prolong of \((W, \mathfrak{g})\), where \(W\) is the identity \(\mathfrak{osp}(m|2n)\)-module.

- The odd analog of \(\mathfrak{t}\) is associated with the following odd analog of \(\mathfrak{hei}(2n|m)\). Denote by \(ab(n)\) the antibracket Lie superalgebra: its space is \(W \oplus \mathbb{C} \cdot z\), where \(W\) is an \(n|n\)-dimensional superspace endowed
with a nondegenerate skew-symmetric odd bilinear form $B$; the bracket in $\mathfrak{ab}(n)$ is given by the following formulas: $z$ is odd and lies in the center; $[v, w] = B(v, w) \cdot z$ for $v, w \in W$.

Set $\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{pe}^{sk}(n))$, and, given $\mathfrak{ab}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{pe}^{sk}(n)$, we call $(\mathfrak{ab}(n), \mathfrak{g})_*$ the $m$-prolong of $(W, \mathfrak{g})$, where $W$ is the identity $\mathfrak{pe}^{sk}(n)$-module.

Generally, given a nondegenerate form $B$ on a superspace $W$ and a superalgebra $\mathfrak{g}$ that preserves $B$, we refer to the above generalized prolongations as to $mk$-prolongations of the pair $(W, \mathfrak{g})$.

A partial Cartan prolong or the prolong of a positive part. Take a $\mathfrak{g}_0$-submodule $\mathfrak{h}_1$ in $\mathfrak{g}_1$. Suppose that $[\mathfrak{g}_{-1}, \mathfrak{h}_1] = \mathfrak{g}_0$, not a subalgebra of $\mathfrak{g}_0$. Define the 2nd prolongation of $(\oplus_{i \leq 0} \mathfrak{h}_i)$ to be $\mathfrak{h}_2 = \{ D \in \mathfrak{g}_2 : [D, \mathfrak{g}_{-1}] \in \mathfrak{h}_1 \}$. The terms $\mathfrak{h}_i$ are similarly defined. Set $\mathfrak{h}_1 = \mathfrak{g}_i$ for $i < 0$ and $\mathfrak{h}_* = \sum \mathfrak{h}_i$.

Examples: $\text{vect}(1|n)$ is a subalgebra of $\mathfrak{gl}(1|2n)$, the latter being a deformation of $\mathfrak{gl}(1|2n)$ by $\mathfrak{h}_1$. The former is obtained as Cartan’s prolong of the same nonpositive part as $\mathfrak{gl}(1|2n)$ and a submodule of $\mathfrak{gl}(1|2n)_1$, cf. Table 0.9. The simple exceptional superalgebra $\mathfrak{f}as$ introduced in 0.9 is another example.

0.7. The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over $\mathfrak{vect}(m|n)$ and its subalgebras, see [BL]. Let $\mathfrak{g} = \mathfrak{vect}(m|n)$ (for any other $\mathbb{Z}$-graded vectorial Lie superalgebra the construction is identical) and $\mathfrak{g}_+ = \oplus_{i > 0} \mathfrak{g}_i$. Clearly, $\mathfrak{vect}_0(m|n) \cong \mathfrak{gl}(m|n)$. Let $V$ be the $\mathfrak{gl}(m|n)$-module with the lowest weight $\lambda = \text{hw}(V)$. Make $V$ into a $\mathfrak{g}_2$-module setting $\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \oplus_{i > 0} \mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathbb{C}^{m|n}$ with coordinates $x = (u, \xi)$. The superspace $T(V) = \text{Hom}_{\mathfrak{gl}(\mathbb{C}^{m|n})}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[x] \otimes V$. Its elements have a natural interpretation as formal tensor fields of type $V$. When $\lambda = (a, \ldots, a)$ we will simply write $T(\bar{a})$ instead of $T(\lambda)$. We usually consider irreducible $\mathfrak{g}_0$-modules.

Examples: $T(\bar{0})$ is the superspace of functions; $\text{Vol}(m|n) = T(1, \ldots, 1; -1, \ldots, -1)$ (the semicolon separates the first $m$ coordinates of the weight with respect to the matrix units $E_{ii}$ of $\mathfrak{gl}(m|n)$) is the superspace of densities or volume forms. We denote the generator of $\text{Vol}(m|n)$ corresponding to the ordered set of coordinates $x$ by $\text{vol}(x)$. The space of $\lambda$-densities is $\text{Vol}^\lambda(m|n) = T(\lambda, \ldots, \lambda; -\lambda, \ldots, -\lambda)$. In particular, $\text{Vol}^\lambda(m|0) = T(\bar{\lambda})$ but $\text{Vol}^\lambda(m|n) = T(\bar{-\lambda})$.

Remark. To view the volume element as “$d^m u d^m \xi$” is totally wrong. Though under linear transformations that do not intermix the even $u$ with the odd $\xi$ the volume element $\text{vol}(x)$ can be viewed as the fraction $\frac{du_1 \cdots du_m}{d\xi_1 \cdots d\xi_n}$, here all the differentials anticommute. How could this happen? If we consider the usual, exterior, differential forms, then the $d\xi_i$ commute, if we consider the symmetric product of the differentials, as in the metrics, then the $dx_i$ commute. However, the $\frac{\partial}{\partial u_i}$ anticommute and, from transformation point of view, $\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i}$. The notation, $\frac{du_1 \cdots du_m}{d\xi_1 \cdots d\xi_n}$, is, nevertheless, still wrong: almost any transformation $A : (u, \xi) \mapsto (u, \eta)$ sends $\frac{du_1 \cdots du_m}{d\xi_1 \cdots d\xi_n}$ to the correct element, $\text{ber}(A) \frac{du_1 \cdots du_m}{d\eta_1 \cdots d\eta_n}$, plus extra terms. Indeed, the fraction $\frac{du_1 \cdots du_m}{d\xi_1 \cdots d\xi_n}$ is the highest weight vector of an indecomposable $\mathfrak{gl}(m|n)$-module and $\text{vol}(x)$ is the image of this vector in the 1-dimensional quotient module modulo the invariant submodule that consists precisely of the extra terms.

0.8. Deformations of the Buttin superalgebra. Here we reproduce a result of Kotchetkov [Ko1] with corrections from [Ko2], [L3], [LSb]. As is clear from the definition of the Buttin bracket, there is a regrading (namely, $\mathfrak{b}(n)$ given by $\deg \xi_i = 0, \deg g_i = 1$ for all $i$) under which $\mathfrak{b}(n)$, initially of depth 2, takes the form $\mathfrak{g} = \oplus_{i \geq 1} \mathfrak{g}_i$ with $\mathfrak{g}_0 = \mathfrak{vect}(0|n)$ and $\mathfrak{g}_{-1} \cong \Pi(\mathbb{C}[\xi])$.

Let us replace the $\mathfrak{vect}(0|n)$-module $\mathfrak{g}_{-1}$ of functions (with reversed parity) with the module of $\lambda$-densities, i.e., set $\mathfrak{g}_{-1} \cong \mathbb{C}[\xi] \langle \text{vol} \xi \rangle^\lambda$, where

$$L_D(\text{vol} \xi)^\lambda = \lambda \text{div} D \cdot \text{vol} \xi^\lambda$$

and $p(\text{vol} \xi)^\lambda = \bar{1}$.

Then the Cartan’s prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is a deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$. The collection of these deformations for various $\lambda \in \mathbb{C}$ constitutes a deformation of $\mathfrak{b}(n)$; we called it the main deformation. (Though main, it is not the quantization of the Buttin bracket, cf. [L3].) The deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$ is the regrading of $\mathfrak{b}_\lambda(n)$ inverse to the regrading of $\mathfrak{b}(n)$ into $\mathfrak{b}(n)$.

Another description of the main deformation is as follows. Set

$$\mathfrak{b}_{a, b}(n) = \{ M_f \in \mathfrak{m}(n) : a \text{ div } M_f = (-1)^{b(f)} 2(a - bn) \frac{\partial f}{\partial r} \}.$$
It is subject to a direct check that \( b_{a,n}(b) \cong b_{\lambda}(n) \) for \( \lambda = \frac{2a}{m(a - b)} \). This isomorphism shows that \( \lambda \) actually runs over \( \mathbb{C} \mathbb{P}^1 \), not \( \mathbb{C} \). Observe that for \( a = nb \), i.e., for \( \lambda = \frac{n}{2} \), we have \( b_{nb,n}(b) \cong \text{sm}(n) \).

As follows from the description of \( \text{vect}(m|n) \)-modules ([BL]), the Lie superalgebras \( b_\lambda(n) \) are simple for \( n > 1 \) and \( \lambda \neq 0, -1, \infty \). It is also clear that \( b_\lambda(n) \) are nonisomorphic for distinct \( \lambda \)'s. (Notice, that at some values of \( \lambda \) the Lie superalgebras \( b_\lambda(n) \) have additional deformations distinct from the above. These deformations are partly described in [L3].)

**0.9. The exceptional Lie subsuperalgebra \( \mathfrak{tas} \) of \( \mathfrak{f}(1|6) \).** The Lie superalgebra \( \mathfrak{g} = \mathfrak{f}(1|2n) \) is generated by the functions from \( \mathbb{C}[t, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n] \). The standard \( \mathbb{Z} \)-grading of \( \mathfrak{g} \) is induced by the \( \mathbb{Z} \)-grading of \( \mathbb{C}[t, \xi, \eta] \) given by \( \deg t = 2, \deg \xi_i = \deg \eta_i = 1 \); namely, \( \deg K_f = \deg f - 2 \). Clearly, in this grading \( \mathfrak{g} \) is of depth 2. Let us consider the functions that generate several first homogeneous components of \( \mathfrak{g} = \bigoplus_{i \geq 2} \mathfrak{g}_i \):}

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{component} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\
\hline
\text{its generators} & 1 & \Lambda^2(\xi, \eta) \oplus \mathbb{C} \cdot t & \Lambda^3(\xi, \eta) & \mathfrak{g}_1 \\
\hline
\end{array}
\]

As one can prove directly, the component \( \mathfrak{g}_1 \) generates the whole subalgebra \( \mathfrak{g}_+ \) of elements of positive degree. The component \( \mathfrak{g}_1 \) splits into two \( \mathfrak{g}_0 \)-modules \( \mathfrak{g}_{11} = \Lambda^3 \) and \( \mathfrak{g}_{12} = t\Lambda^1 \). It is obvious that \( \mathfrak{g}_{12} \) is always irreducible and the component \( \mathfrak{g}_{11} \) is trivial for \( n = 1 \).

Recall that if the operator \( d \) that determines a \( \mathbb{Z} \)-grading of the Lie superalgebra \( \mathfrak{g} \) does not belong to \( \mathfrak{g} \), we denote the Lie superalgebra \( \mathfrak{g} \oplus \mathbb{C} \cdot d \) by \( \mathfrak{d} \mathfrak{g} \). Recall also that \( \mathfrak{c}(\mathfrak{g}) \) or just \( \mathfrak{c} \mathfrak{g} \) denotes the trivial 1-dimensional central extension of \( \mathfrak{g} \) with the even center.

The Cartan prolongations from these components are well-known:

\[
\begin{align*}
(\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{11} \oplus \mathfrak{g}_{12})_{m,k}^\text{\tiny \mathfrak{tas}} & \cong \mathfrak{po}(0|2n) \oplus \mathbb{C} \cdot K_f \cong \mathfrak{d}(\mathfrak{po}(0|2n)) \\
(\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{12})_{m,k}^\text{\tiny \mathfrak{tas}} & \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{12} \oplus \mathbb{C} \cdot K_{f^2} \cong \mathfrak{osp}(2n|2). 
\end{align*}
\]

Observe a remarkable property of \( \mathfrak{f}(1|6) \). For \( n > 1 \) and \( n \neq 3 \) the component \( \mathfrak{g}_{11} \) is irreducible. For \( n = 3 \) it splits into 2 irreducible conjugate modules that we will denote \( \mathfrak{g}_{11}^1 \) and \( \mathfrak{g}_{11}^2 \). Observe further, that \( \mathfrak{g}_0 = \mathfrak{co}(6) \cong \mathfrak{gl}(4) \). As \( \mathfrak{gl}(4) \)-modules, \( \mathfrak{g}_{11}^1 \) and \( \mathfrak{g}_{11}^2 \) are the symmetric squares \( S^2(\text{id}) \) and \( S^2(\text{id}^*) \) of the identity 4-dimensional representation and its dual, respectively.

**Theorem.** The Cartan prolong \( (\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{11} \oplus \mathfrak{g}_{12})_{m,k}^\text{\tiny \mathfrak{tas}} \) is infinite dimensional and simple. It is isomorphic to \( (\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{11}^1 \oplus \mathfrak{g}_{12}^1)_{m,k}^\text{\tiny \mathfrak{tas}} \).

We will denote the simple exceptional vectorial Lie superalgebra described in Theorem 0.9 by \( \mathfrak{tas} \).

**0.10. Several first terms that determine the Cartan and \( m \cdot k \)-prolongations.** To facilitate the comparison of various vectorial superalgebras, consider the following Table. The central element \( z \in \mathfrak{g}_0 \) is supposed to be chosen so that it acts on \( \mathfrak{g}_k \) as \( k \cdot \text{id} \). The sign \( \oplus \) (resp. \( \boxplus \)) denotes the semidirect sum with the subspace or ideal on the left (right) of it; \( \Lambda(r) = \mathbb{C}[\xi_1, \ldots, \xi_r] \) is the Grassmann superalgebra of the elements of degree 0.

\[
\begin{array}{|c|c|c|c|}
\hline
\mathfrak{g} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \\
\hline
\text{vect}(n|m; r) & - & \text{id} \otimes \Lambda(r) & \mathfrak{gl}(n|m - r) \otimes \Lambda(r) \boxplus \text{vect}(0|r) \\
\text{vect}(n|m; r) & - & \text{id} \otimes \Lambda(r) & \mathfrak{sl}(n|m - r) \otimes \Lambda(r) \boxplus \text{vect}(0|r) \\
\text{vect}(1|m; m) & - & \Lambda(m) & \Lambda(m) \boxplus \text{vect}(0|m) \\
\text{vect}^\circ(1|m; m) & - & v \in \text{Vol}(0|m) & \text{vect}(0|m) \\
\text{vect}^\circ(1|2) & - & T^0(0) \cong \Lambda(2)/\mathbb{C} \cdot 1 & \text{vect}(0|2) \cong \mathfrak{sl}(1|2) \\
\text{vect}(2|1) & - & \Pi(T^0(0)) & \text{vect}(0|2) \cong \mathfrak{sl}(2|1) \\
\mathfrak{f}(2n + 1|m; r) & \Lambda(r) & \text{id} \otimes \Lambda(r) & \mathfrak{osp}(m - 2r|2n) \otimes \Lambda(r) \boxplus \text{vect}(0|r) \\
\mathfrak{h}(2n|m; r) & \Lambda(r)/\mathbb{C} \cdot 1 & \text{id} \otimes \Lambda(r) & \mathfrak{osp}(m - 2r|2n) \otimes \Lambda(r) \boxplus \text{vect}(0|r) \\
\mathfrak{f}(1|2m; m) & - & \Lambda(m) & \Lambda(m) \boxplus \text{vect}(0|m) \\
\mathfrak{f}(1|2m + 1; m) & \Lambda(m) & \Pi(\Lambda(m)) & \Lambda(m) \boxplus \text{vect}(0|m) \\
\hline
\end{array}
\]
Recall that \( b_{a,b}(n) \cong b_\lambda(n) \) for \( \lambda = \frac{2n}{a-b} \). Recall that \( d \) denotes the operator that determines a Z-grading of the Lie superalgebra \( \mathfrak{g} \); \( c(\mathfrak{g}) \) or just \( \mathfrak{c g} \) denotes the trivial 1-dimensional central extension of \( \mathfrak{g} \) with the even center.

| \( \mathfrak{g} \)  | \( \mathfrak{g}_{-2} \)  | \( \mathfrak{g}_{-1} \) | \( \mathfrak{g}_0 \) |
|-----------------|-----------------|-----------------|-----------------|
| \( b_\lambda(n; r) \) | \( \Pi(\lambda(r)) \) | \( \text{id} \otimes \Lambda(r) \) | \( (\text{spe}(n-r) \oplus \mathbb{C}(az+bd)) \otimes \Lambda(r) \oplus \text{vect}(0|r) \) |
| \( b_\lambda(n; n) \) | \( - \) | \( \Pi(\text{Vol}^\lambda(0|n)) \) | \( \text{vect}(0|n) \) |
| \( \mathfrak{m}(n; r) \) | \( \Pi(\lambda(r)) \) | \( \text{id} \otimes \Lambda(r) \) | \( \text{cpe}(n-r) \otimes \Lambda(r) \oplus \text{vect}(0|r) \) |
| \( \mathfrak{m}(n; n) \) | \( - \) | \( \Pi(\lambda(n)) \) | \( \Lambda(n) \oplus \text{vect}(0|n) \) |
| \( \mathfrak{s b}^\la(n; n) \) | \( - \) | \( \Pi(\text{Vol}(0|n)) \) | \( \frac{1}{(1+\lambda_1 \cdots \lambda_n)\text{vol}(\xi)} \otimes \text{vect}(0|n) \) |
| \( \mathfrak{l e}(n; r) \) | \( \Pi(\lambda(r))/\mathbb{C} \cdot 1 \) | \( \text{id} \otimes \Lambda(r) \) | \( \text{pe}(n-r) \otimes \Lambda(r) \oplus \text{vect}(0|r) \) |
| \( \mathfrak{l e}(n; n) \) | \( - \) | \( \Pi(\lambda(n))/\mathbb{C} \cdot 1 \) | \( \text{vect}(0|n) \) |
| \( \mathfrak{s l e}^\la(n; r) \) | \( \Pi(\lambda(r))/\mathbb{C} \cdot 1 \) | \( \text{id} \otimes \Lambda(r) \) | \( (\text{spe}(n-r) \otimes \Lambda(r)) \oplus \text{vect}(0|r) \) |
| \( \mathfrak{s l e}^\la(n; n) \) | \( - \) | \( \Pi(T^0(0)) \) | \( \text{vect}(0|n) \) |

### §1. Stringy superalgebras

These superalgebras are particular cases of the Lie superalgebras of vector fields, namely, the ones that preserve a structure on a what physicists call superstring, i.e., the supermanifold associated with a vector bundle on the circle. These superalgebras themselves are “stringy” indeed: as modules over the Witt algebra \( \mathfrak{witt} = \text{der} \mathbb{C}[t^{-1}, t] \) they are direct sums of several “strings”, the modules \( \mathcal{F}^\lambda \) described in sec. 2.3.

This description, sometimes taken for definition of the stringy superalgebra \( \mathfrak{g} \), depends on the embedding \( \mathfrak{witt} \rightarrow \mathfrak{g} \) and the spectrum of \( \mathfrak{witt} \)-modules constituting \( \mathfrak{g} \) might vary hampering recognition. Rigorous is a deep definition of a deep superalgebra due to Mathieu. He separates the deep algebras of which stringy is a particular case Lie algebras from affine Kac–Moody ones. Both are of infinite depth (see 0.2) but while for the loop algebras all root vectors act locally nilpotently, whereas \( \mathfrak{g} \) is stringy if

\[
\mathfrak{g} \text{ is of polynomial growth and there is a root vector which does not act locally nilpotently.} \tag{1.0}
\]

Similarly, we say that a Lie superalgebra \( \mathfrak{g} \) of infinite depth is of the loop type if it satisfies (1.0) and stringy otherwise. Observe, that a stringy superalgebra of polynomial growth can be a Kac–Moody superalgebra, i.e., have a Cartan matrix, but not be a (twisted) loop superalgebra. (Roughly speaking, the stringy superalgebras have the root vector \( \frac{d}{dt} \).)

1.1. Let \( \varphi \) be the angle parameter on the circle, \( t = \exp(i \varphi) \). The only stringy Lie algebra is \( \mathfrak{witt} \).

Examples of stringy Lie superalgebras are certain subalgebras of the Lie superalgebra of superderivations of either of the two supercommutative superalgebras

\[
R^L(1|n) = \mathbb{C}[t^{-1}, t, \theta_1, \ldots, \theta_n] \quad \text{or} \quad R^M(1|n) = \mathbb{C}[t^{-1}, t, \theta_1, \ldots, \theta_{n-1}, \sqrt{t} \xi].
\]

\( R^L(1|n) \) can be considered as the superalgebra of complex-valued functions expandable into finite Fourier series or, as superscript indicates, Laurent series. These functions are considered on the real supermanifold \( S^{1|n} \) with the even center over the trivial bundle over the circle. We can forget about \( \varphi \) and think in terms of \( t \) considered as the even coordinate on \( \mathbb{C}^* \).

\( R^M(1|n) \) can be considered as the superalgebra of complex-valued functions (expandable into finite Fourier series) on the supermanifold \( S^{1|n-1, M} \) associated with the Whitney sum of the Möbius bundle and the trivial bundle of rank \( n-1 \). Since the Whitney sum of two Möbius bundles is isomorphic to the trivial bundle of rank 2, it suffices to consider one Möbius bundle.

Let us introduce first stringy Lie superalgebras. These are analogues of \( \text{vect}, \text{s vect} \) and \( \mathfrak{k} \) obtained by replacing \( R(1|n) = \mathbb{C}[t, \theta_1, \ldots, \theta_n] \) with \( R^L(1|n) \):

\[
\text{vect}^L(1|n) = \text{der} R^L(1|n); \quad \text{s vect}^L(1|n) = \{ D \in \text{vect}^L(1|n) : \text{div}(t^L D) = 0 \} = \{ D \in \text{vect}^L(1|n) : L_D(t^L \text{vol}(t, \theta)) = 0 \};
\]

\[\text{e}^L(1|n) = \{ D \in \text{vect}^L(1|n) : D(\alpha) = f_D \alpha \text{ for } \alpha = dt + \sum \theta_i d \theta_i \text{ and } f_D \in R^L(1|n) \}.\]

We abbreviate \( \text{s vect}_0^L(1|n) \) to \( \text{s vect}^L(1|n) \).
The routine arguments prove that the elements that the functions $f \in R^k(n)$ generate $t^k(n)$, and the formulas for $K_f$ and the contact bracket are the same as for $\mathfrak{k}(1|n)$ and the polynomial $f$.

**Exercise.** The algebras $\mathfrak{vect}(1|n)$ and $\hat{\mathfrak{vect}}(1|n)$ obtained by replacing $R(n)$ with $R^M(1|n)$ are isomorphic to $\mathfrak{vect}^L(1|n)$ and $\hat{\mathfrak{vect}}^L(1|n)$, respectively. Moreover, $\mathfrak{vect}^L(1|n) \cong \mathfrak{vect}^R(1|n)$ if and only if $\lambda - \mu \in \mathbb{Z}$.

The following formula is convenient:

$$D = f\partial_t + \sum f_i \partial_i \in \mathfrak{vect}^L(1|n) \quad \text{if and only if} \quad \lambda f = -t \text{div} D. \quad (1.1.1)$$

If $\lambda \in \mathbb{Z}$, the Lie superalgebra $\mathfrak{vect}^L(1|n)$ has the simple ideal $\mathfrak{vect}^L(1|n)$ of codimension $\varepsilon^n$:

$$0 \longrightarrow \mathfrak{vect}^L(1|n) \longrightarrow \mathfrak{vect}^L(1|n) \longrightarrow \theta_1 \cdots \theta_n \partial_i \longrightarrow 0.$$

• The lift of the contact structure from $S^L(1|n)$ to its two-sheeted covering, $S^L(1|n.M)$, brings a new structure. Indeed, this lift means replacing $\theta_n$ with $\sqrt{n}\theta$; this replacement sends the form $\hat{\alpha}_1$ into the Möbius form

$$\hat{\alpha} = dt + \sum_{i=1}^{n-1} \theta_i dt_i + t \theta d\theta. \quad (1.1.2)$$

It is often convenient to pass to another canonical expression of the Möbius form:

$$\hat{\alpha} = \begin{cases} dt + \sum_{i<k} (\xi_i d\eta_i + \eta_i d\xi_i) + t \theta d\theta & \text{if } n = 2k + 1 \\ dt + \sum_{i<k} (\xi_i d\eta_i + \eta_i d\xi_i + \zeta d\zeta) + t \theta d\theta & \text{if } n = 2k + 2 \end{cases} \quad (1.1.2)$$

Now, we have two ways for describing the vector fields that preserve $\hat{\alpha}$ or $\hat{\alpha}$:

1) We can set:

$$\mathfrak{t}^M(1|n) = \{ D \in \mathfrak{vect}^M(1|n) : L_D(\alpha_1) = f_D \cdot \alpha_1, \quad \text{where} \quad f_D \in R^M(1|n) \}. \quad (\text{aut}_{R^M}(\alpha_1))$$

In this case the fields $K_f$ are given by the same formulas as for $\mathfrak{k}(1|n)$ but the generating functions belong to $R^M(n)$. The contact bracket between the generating functions from $R^M(n)$ is also given by the same formulas as for the generating functions of $\mathfrak{k}(1|n)$.

2) We can set:

$$\mathfrak{t}^M(1|n) = \{ D \in \mathfrak{vect}^L(1|n) : L_D(\hat{\alpha}) = f_D \cdot \hat{\alpha}, \quad \text{where} \quad f_D \in R^L(1|n) \}. \quad (\text{aut}_{R^L}(\hat{\alpha}))$$

It is not difficult to verify that $\mathfrak{t}^M(1|n) = \text{Span}(\hat{K}_f : f \in R^L(1|n))$, where the Möbius contact field is given by the formula

$$\hat{K}_f = \Delta(f) D + D(f) E + \hat{H}_f, \quad (1.1.3)$$

in which, as in the case of a cylinder $S^1(1|n)$, we set $\Delta = 2 - E$ and $E = \sum_{i \leq n-1} \theta_i \frac{\partial}{\partial \theta_i} + \theta \frac{\partial}{\partial \theta}$, but where

$$D = \frac{\partial}{\partial t} - \frac{\theta}{2t} \frac{\partial}{\partial \theta} = \frac{1}{2} \hat{K}_1$$

and where

$$\hat{H}_f = (-1)^{\rho(f)} \left( \sum \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i} + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right) \quad \text{in the realization with form } \hat{\alpha};$$

$$\hat{H}_f = (-1)^{\rho(f)} \left( \sum \left( \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial}{\partial \xi_i} \right) + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right) \quad \text{in the realization with form } \hat{\alpha} \text{ for } n = 2k,$$

$$\hat{H}_f = (-1)^{\rho(f)} \left( \sum \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial}{\partial \xi_i} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right) \quad \text{in the realization with form } \hat{\alpha} \text{ for } n = 2k + 1.$$

The corresponding bracket of generating functions will be called the Ramond bracket; its explicit form is

$$(f,g)_{R.b.} = \Delta(f) D(g) - D(f) \Delta(g) - \{ f,g \}_{MP,b}, \quad (1.1.4)$$

where the Möbius-Poisson bracket $\{ \cdot, \cdot \}_{MP,b}$ is

$$\{ f,g \}_{MP,b} = (-1)^{\rho(f)} \left( \sum \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right) \quad \text{in the realization with form } \hat{\alpha}. \quad (1.1.5)$$

Observe that

$$L_{K_f}(\alpha_1) = K_f(\cdot) \cdot \alpha_1, \quad L_{\hat{K}_f}(\hat{\alpha}) = \hat{K}_f(\cdot) \cdot \hat{\alpha}. \quad (1.1.6)$$
Remark. Let us give a relation of the brackets with the dot product on the space of functions. (More exactly, for the Buttin superalgebra the Lie superalgebra structure is determined on \( \Pi(\mathcal{F}) \); for \( t \) it is defined on \( \mathcal{F}_1 \), see sec. 1.3, etc.) This relation is often listed as part of the definition of the Poisson algebra which is, certainly, pure noncence.

\[
\begin{align*}
\{ f, gh \} & = \{ f, g \} h + (1-p(f)p(g)) \{ f, g \} h + K_1(f)gh; \\
\{ f, gh \} & = \{ f, g \} h + (1-p(f)p(g)) \{ f, g \} h + K_1(f)gh; \\
\{ f, gh \} & = \{ f, g \} h + (1-p(f)p(g)) \{ f, g \} h + M_1(f)gh.
\end{align*}
\]

The corresponding formulas for the Poisson and Buttin superalgebras are without the third term.

Explicitly, the embedding \( i : \text{vect}^L(1|n) \rightarrow \mathfrak{t}^L(1|2n) \) is given by the following formula in which \( \Phi = \sum_{i \leq n} \xi_i \eta_i \):

\[
\begin{array}{c|c}
D \in \text{vect}^L(1|n) & \text{the generator of } i(D) \\
\hline
f(\xi)t^m \partial_t & (1-p(f)) f(\xi)(t - \Phi)^m, \\
f(\xi)t^m \partial_{\xi} & (1-p(f)) f(\xi)\eta_t(t - \Phi)^m.
\end{array}
\]

Clearly, \( \text{svect}^L(1|n) \) is the subsuperspace of \( \text{vect}^L(1|n) \) spanned by the expressions

\[
f(\xi)(t - \Phi)^m + \sum f_i(\xi)\eta_i(t - \Phi)^{m-1} \text{ such that } (\lambda + n)f(\xi) = - \sum(1-p(f))\partial f_i \partial_{\xi_i}.
\]

The four series of classical stringy superalgebras are: \( \text{vect}^L(1|n) \), \( \text{svect}^L(1|n) \), \( \mathfrak{t}^L(1|n) \) and \( \mathfrak{t}^M(1|n) \).

1.2. Nonstandard gradings of stringy superalgebras. The Weisfeiler filtrations of vectoral superalgebras with polynomial or formal coefficients are determined by the maximal subalgebra of finite codimension not containing ideals of the whole algebra. The corresponding gradings are natural. We believe that the filtrations and \( \mathbb{Z} \)-gradings of stringy superalgebras induced by Weisfeiler filtrations of the corresponding vectoral superalgebras with polynomial coefficients are distinguished but we do not know how to characterize such \( \mathbb{Z} \)-gradings intrinsically.

The gradings described in sec. 0.4 for \( \text{vect}(1|n) \) and \( \mathfrak{t}(1|n) \) induce gradings of \( \text{vect}^L(1|n) \) and \( \mathfrak{t}^L(1|n) \) and even \( \text{svect}^L(1|n) \) since the elements of latter have as coefficients the usual Laurent polynomials. For \( \mathfrak{t}^M(1|n) \) the preserved form \( \hat{\alpha} \) should be homogeneous, so \( \text{deg } \theta \) is always equal to 0. In realization \( \mathfrak{t}^M(1|n) = \text{aut}_{\mathfrak{t}^L}(\hat{\alpha}) \) these \( \mathbb{Z} \)-gradings are as follows (here \( * \) marks the “standard” grading):

\[
\begin{array}{c|c}
\mathfrak{t}^M(1|n) & \text{deg } t = 2, \text{ deg } \theta = 0, \text{ deg } \xi_i = 1 \text{ for all } i (*) \\
\mathfrak{t}^M(1|2n; r) & \begin{array}{c}
\text{deg } t = \text{deg } \eta_1 = \ldots = \text{deg } \eta_r = 2 \text{ deg } \xi_1 = \ldots = \text{deg } \xi_r = 0 \\
\text{deg } \xi_{r+i} = \text{deg } \eta_{r+i} = \text{deg } \zeta = 1 \text{ for all } i
\end{array} \\
1 \leq r < n & \\
\mathfrak{t}^M(1|2n + 1; n) & \begin{array}{c}
\text{deg } t = \text{deg } \eta_1 = \ldots = \text{deg } \eta_n = 1 \\
\text{deg } \theta = \text{deg } \xi_1 = \ldots = \text{deg } \xi_n = 0
\end{array}
\end{array}
\]

1.3. Modules of tensor fields over stringy superalgebras. Denote by \( T^L(V) = \mathbb{C}[t^{-1}, t] \otimes V \) the \( \text{vect}(1|n) \)-module that differs from \( T(V) \) by allowing the Laurent polynomials as coefficients of its elements instead of just polynomials. Clearly, \( T^L(V) \) is a \( \text{vect}^L(1|n) \)-module. Define the twisted with weight \( \mu \) version of \( T^L(V) \) by setting:

\[
T^L_\mu(V) = \mathbb{C}[t^{-1}, t] t^\mu \otimes V.
\]

- The “simplest” modules — the analogues of the standard or identity representation of the matrix algebras. The simplest modules over the Lie superalgebras of series \( \text{vect} \) are, clearly, the modules of \( \lambda \)-densities, cf. sec. 0.7. These modules are characterized by the fact that they are of rank 1 over \( \mathcal{F} \), the algebra of functions. Over stringy superalgebras, we can also twist these modules and consider \( \text{Vol}^\lambda \).

Observe that for \( \mu \notin \mathbb{Z} \) this module has only one submodule, the image of the exterior differential \( d \), see [BL], whereas for \( \mu \in \mathbb{Z} \) there is, additionally, the kernel of the residue:

\[
\text{Res } : \text{Vol} \rightarrow \mathbb{C}, \ f \text{vol}_{t, \xi} \mapsto \text{ the coefficient of } \frac{\xi_1 \ldots \xi_n}{t} \text{ in the power series expansion of } f.
\]

- Over \( \text{svect}^L(1|n) \) all the spaces \( \text{Vol}^\lambda \) are, clearly, isomorphic, since their generator, \( \text{vol}(t, \theta) \), is preserved. So all rank 1 modules over the module of functions are isomorphic to the module of twisted functions \( \mathcal{F}_\mu \).
Over \textit{svect}\(_L\)(1|n), the simplest modules are generated by \(t^\lambda vol(t,\theta)\). The submodules of the simplest modules over \textit{svect}\(_L\)(1|n) and \textit{svect}\(_L\)(1|n) are the same as those over \textit{svect}\(_L\)(1|n) but if \(\mu \in \mathbb{Z}\) there is, additionally, the trivial submodule generated by (the \(\Lambda\)-th power of) \(vol(t,\theta)\) or \(t^\lambda vol(t,\theta)\), respectively.

- Over contact superalgebras \(\mathfrak{t}(2n+1|m)\), it is more natural to express the simplest modules not in terms of \(\lambda\)-densities but via powers of the form \(\alpha = \alpha_1\):

\[
\mathcal{F}_\lambda = \begin{cases} 
\mathcal{F}_{\alpha^2} & \text{for } n = m = 0 \\
\mathcal{F}_{\alpha^2/2} & \text{otherwise} .
\end{cases}
\]

Observe that \(\text{Vol}^\lambda \cong \mathcal{F}_{\lambda(2n+2-m)}\), as \(\mathfrak{t}(2n+1|m)\)-modules. In particular, the Lie superalgebra of series \(\mathfrak{t}\) does not distinguish between \(\frac{\partial}{\partial \alpha}\) and \(\alpha^{-1}\): their transformation rules are identical. Hence, \(\mathfrak{t}(2n+1|m) \cong \mathcal{F}_{\lambda(2n+2-m)}\).

- For \(n = 0, m = 2\) (we take \(\alpha = dt - \xi d\eta - \eta d\xi\)) there are other rank 1 modules over \(\mathcal{F}\), the algebra of functions, namely:

\[
T(\lambda,\nu)_\mu = \mathcal{F}_{\lambda,\mu} \cdot \left(\frac{d\xi}{d\eta}\right)^{\nu/2} .
\]

- Over \(\mathfrak{f}^M\), we should replace the form \(\alpha\) with \(\tilde{\alpha}\) and the definition of the \(\mathfrak{f}^L(1|m)\)-modules \(\mathcal{F}_{\lambda,\mu}\) should be replaced with

\[
\mathcal{F}_{\lambda,\mu}^M = \begin{cases} 
\mathcal{F}_{\lambda,\mu}(\tilde{\alpha})^\lambda & \text{for } m = 1 \\
\mathcal{F}_{\lambda,\mu}(\tilde{\alpha})^{\lambda/2} & \text{for } m > 1.
\end{cases}
\]

- For \(m = 3\) and \(\alpha = dt - \xi d\eta - \eta d\xi - \theta d\theta\) there are other rank 1 modules over the algebra of functions \(\mathcal{F}\), namely:

\[
T^M(\lambda,\nu)_\mu = \mathcal{F}_{\lambda,\mu}^M \cdot \left(\frac{d\xi}{d\eta}\right)^{\nu/2} .
\]

\textit{Examples}. 1) The \(\mathfrak{f}(2n+1|m)\)-module of volume forms is \(\mathcal{F}_{2n+2-m}\). In particular, \(\mathfrak{f}(2n+1|2n+2) \subset \text{svect}(2n+1|2n+2)\).

2) As \(\mathfrak{f}^L(1|m)\)-module, \(\mathfrak{f}^L(1|m)\) is isomorphic to \(\mathcal{F}_{-1}\) for \(m = 0\) and \(\mathcal{F}_{-2}\) otherwise. As \(\mathfrak{f}^M(1|m)\)-module, \(\mathfrak{f}^M(1|m)\) is isomorphic to \(\mathcal{F}_{-1}\) for \(m = 1\) and \(\mathcal{F}_{-2}\) otherwise. In particular, \(\mathfrak{f}^L(1|4) \cong \text{Vol}\) and \(\mathfrak{f}^M(1|5) \cong \Pi(\text{Vol})\).

1.4. The four exceptional stringy superalgebras. The “status” of these exceptions is different: the first algebra is a true exception, the second one is an exceptional regrading; the last two are “drop outs” from the series.

\textbf{A) } \textit{fas}^L. Certain polynomial functions described out in \S 2 generate \(\textit{fas} \subset \mathfrak{f}(1|6)\). Inserting Laurent polynomials in the formulas for the generators of \(\textit{fas}\) we get the exceptional stringy superalgebra \(\textit{fas}^L \subset \mathfrak{f}^L(1|6)\).

\textbf{B) } \textit{m}^L(1) On the complexification of \(S^{1,2}\), let \(q\) be the even coordinate, \(\tau\) and \(\xi\) the odd ones. Set

\[
\textit{m}^L(1) = \{ D \in \textit{vect}^L(1|2) : D\alpha_0 = f D\alpha_0, \text{ where } f_D \in \textit{R}^L(1|1), \alpha_0 = dt + q d\xi + \xi dq\}.
\]

\textbf{C, D) } \textit{t}^L(0,4) and \(\textit{m}^M(0,1|5)\) It follows from Example 2) in sec. 2.3 that the functions with zero residue on \(S^{1,4}|M\) generate an ideal in \(\mathfrak{f}^L(4)\) (resp. \(\mathfrak{f}^M(5)\)). These ideals are, clearly, simple Lie superalgebras denoted in what follows by \(\textit{t}^L(0,1|4)\) and \(\textit{m}^M(0,1|5)\), respectively.

1.5. Deformations. The superalgebras \(\textit{svect}(1|n)\) and \(\textit{svect}^0(1|n)\) do not have deformations that preserve the \(\mathbb{Z}\)-gradings (other deformations may happen; one has to calculate, this is a research problem). The stringy superalgebras \(\textit{svect}^L(1|n)\) do have \(\mathbb{Z}\)-grading preserving deformations discovered by Schwimmer and Seiberg [SS]. More deformations (none of which preserves \(\mathbb{Z}\)-grading) are described in [KV]; the complete description of deformations of \(\textit{svect}^L(1|n)\) is, nevertheless, unknown.

\textit{Conjecture}. \(\textit{svect}^L(1|n), \mathfrak{f}^L(1|n)\) and the four exceptional stringy superalgebras are rigid.
1.6. Distinguished stringy superalgebras.

**Theorem.** The only nontrivial central extensions of the simple stringy Lie superalgebras are those given in the following table.

Let in this subsection and in sec. 2.1 $K_f$ be the common term for both $K_f$ and $\tilde{K}_f$. Let further $\mathcal{K} = (2\theta \partial_{\theta g} - 1) \frac{\partial}{\partial x}$.

| algebra | the cocycle $c$ | The name of the extended algebra |
|---------|----------------|----------------------------------|
| $\mathfrak{t}^L(1|0)$ | $K_f, K_g \mapsto \text{Res} f K_1^3(g)$ | Virasoro or $\mathfrak{vir}$ |
| $\mathfrak{t}^L(1|1)$, $\mathfrak{t}^M(1|1)$ | $K_f, K_g \mapsto \text{Res} f K_\theta K_\theta K_1(g)$ | Neveu-Schwarz or $\mathfrak{ns}$ Ramond or $\mathfrak{r}$ |
| $\mathfrak{vect}^L(1|1)$ | $D_1 = f \frac{\partial}{\partial \theta} + g_1 \frac{\partial}{\partial x}$, $D_2 = f \frac{\partial}{\partial \theta} + \tilde{g}_1 \frac{\partial}{\partial x}$ \mapsto Res$(f \mathcal{K}(\tilde{g})) + (-1)p(D_1)p(D_2)g\mathcal{K}(\tilde{f}) + 2(-1)p(D_1)p(D_2) + p(D_2)g\frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta}(\tilde{g})$ | $\mathfrak{vect}^L(1|1)$ |
| $\mathfrak{vect}^L(1|2)$, $\mathfrak{vect}^M(1|2)$ | $K_f, K_g \mapsto \text{Res} f K_\theta K_\theta K_1(g)$ | 2-Neveu-Schwarz or $\mathfrak{ns}(2)$ 2-Ramond or $\mathfrak{r}(2)$ |
| $\mathfrak{m}^L(1)$ | $M_f, M_g \mapsto \text{Res} f (M_g)^3(g)$ | $\mathfrak{m}^L(1)$ |
| $\mathfrak{vect}^L(1|3)$, $\mathfrak{vect}^M(1|3)$ | $K_f, K_g \mapsto \text{Res} f K_\theta K_\theta K_1(g)$ | 3-Neveu-Schwarz or $\mathfrak{ns}(3)$ 3-Ramond or $\mathfrak{r}(3)$ |
| $\mathfrak{vect}^L(1|4)$, $\mathfrak{vect}^M(1|4)$ | $K_f, K_g \mapsto \text{Res} f K_\theta K_\theta K_\theta K_\theta K_1^{-1}(g)$ | 4-Neveu-Schwarz or $\mathfrak{ns}(4)$ 4-Ramond or $\mathfrak{r}(4)$ |

Observe that the restriction of the only cocycle on $\mathfrak{vect}^L(1|2)$ to its subalgebra $Span(f(t) \frac{\partial}{\partial x}) \cong \mathfrak{witt}$ is trivial while the the restriction of the only cocycle on $\mathfrak{vect}_\Lambda^L(1|2)$ to its unique subalgebra $\mathfrak{witt}$ is nontrivial. The riddle is solved by a closer study of the embedding $\mathfrak{vect}(1|m) \longrightarrow \mathfrak{t}(1|2m)$: it involves differentiations, see formulas (1.1.8).

The nonzero values of the cocycle $c$ on $\mathfrak{vect}^L(1|2)$ in monomial basis are:

$$
c(t^k \theta_1 \frac{\partial}{\partial \theta_1}, t^l \theta_2 \frac{\partial}{\partial \theta_2}) = k \delta_{k,-l}, \quad c(t^k \theta_1 \frac{\partial}{\partial \theta_1}, t^l \theta_2 \frac{\partial}{\partial \theta_2}) = -k \delta_{k,-l},
$$

$$
c(t^k \theta_1 \theta_2 \frac{\partial}{\partial \theta_1}, t^l \frac{\partial}{\partial \theta_2}) = k \delta_{k,-l}, \quad c(t^k \theta_1 \theta_2 \frac{\partial}{\partial \theta_1}, t^l \theta_2 \frac{\partial}{\partial \theta_2}) = k \delta_{k,-l}.
$$

In $\mathfrak{vect}_\Lambda^L(1|2)$, set:

$$
L_m = t^m \left( \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \right), \quad S_m^j = t^m \theta_j \left( \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \right).
$$

The nonzero values of the cocycles on $\mathfrak{vect}_\Lambda^L(1|2)$ are:

$$
c(L_m, L_n) = \frac{1}{2} m(m^2 - (\lambda + 1)^2) \delta_{m,-n}, \quad c(t^k \frac{\partial}{\partial \theta_1}, S_m^j) = -m(m - (\lambda + 1)) \delta_{m,-n} \delta_{i,j},
$$

$$
c(t^m (\theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}), t^n (\theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2})) = m \delta_{m,-n}, \quad c(t^m \theta_1 \frac{\partial}{\partial \theta_1}, t^n \theta_2 \frac{\partial}{\partial \theta_1}) = m \delta_{m,-n}.
$$
When are nontrivial central extensions of the stringy superalgebras possible? We find the following quantitative discussion instructive, though it neither replaces the detailed proof (that can be found in [KvL] for all cases except \( \mathfrak{fas}^2 \); the arguments in the latter case are similar) nor explains the number of nontrivial cocycles on \((1/4)\)-dimensional supercircle with a contact structure.

When we pass from simple finite dimensional Lie algebras to loop algebras, we enlarge the maximal toral subalgebra of the latter to make the number of generators of weight 0 equal to that of positive or negative generators corresponding to simple roots. In this way we get the nontrivial central extensions of the loop algebras called Kac–Moody algebras. (Actually, the latter have one more operator of weight 0: the exterior derivation.)

Similarly, for the Witt algebra \( \mathfrak{witt} \) we get:

\[
\begin{array}{cccccccc}
\deg K_f & \vdots & -2 & -1 & 0 & 1 & 2 & \cdots \\
\hline
f & \vdots & t & \frac{1}{t} & t^2 & t^3 & \cdots \\
\end{array}
\]

The depicted elements generate \( \mathfrak{witt} \) more explicitly,

(a) the elements of degrees \(-1, 0, 1\) generate \( \mathfrak{sl}(2) \);

(b) \( \mathfrak{witt} \), as \( \mathfrak{sl}(2) \)-module, is glued from the three modules: the adjoint module and the Verma modules \( M^{-2} \) and \( M_2 \) with highest and lowest weights as indicated: \(-2\) and \(2\), respectively;

it is natural to expect a central element to be obtained by pairing of the dual \( \mathfrak{sl}(2) \)-modules \( M^{-2} \) and \( M_2 \).

This actually happens; one of the methods to find the cocycle is to compute \( H^1(\mathfrak{g}; \mathfrak{g}^*) \) and identify it with \( H^2(\mathfrak{g}) \). (The former and the latter cohomology is what is calculated in [P] and [Sc], respectively.)

- Further on, consider the subalgebra \( \mathfrak{osp}(n|2) \) in \( \mathfrak{t}^L(1|n) \) and decompose \( \mathfrak{t}^L(1|n) \), as \( \mathfrak{osp}(n|2) \)-module, into irreducible modules. Denote by \((\chi_0; \chi_1, \ldots, \chi_r)\) the highest (lowest) weight of the irreducible \( \mathfrak{osp}(n|2) \)-module with respect to the standard basis of Cartan subalgebra of \( \mathfrak{sl}(2) \oplus \mathfrak{o}(n) \); here \( r = \lfloor n/2 \rfloor \).

These modules and their generators are as follows. Set

\[
\alpha = \begin{cases} 
2t - \sum (\xi_i d\eta_i + \eta_i d\xi_i) & \text{if } n \text{ is even} \\
2t - \sum (\xi_i d\eta_i + \eta_i d\xi_i) - \theta d\theta & \text{if } n \text{ is odd} \\
\end{cases} \quad \text{and} \quad \zeta = \begin{cases} 
(\xi, \eta) & \text{if } n \text{ is even} \\
(\xi, \eta, \theta) & \text{if } n \text{ is odd}.
\end{cases}
\]

Let \( \langle f \rangle \) be a shorthand for the generator \( K_f \) of the Verma module \( M \) with the highest (lowest) weight as indicated by the sub- or superscript, respectively: we denote the quotient of \( M \) modulo the maximal submodule by \( L \) with the same indices. Then the Jordan–Hölder factors of the \( \mathfrak{osp}(n|2) \)-module \( \mathfrak{t}^L(1|n) \) are

\[
\begin{array}{cccc}
n & \text{irreducible factors} & \text{of } \mathfrak{t}^L(1|n) & \text{as } \mathfrak{osp}(n|2) \text{-module} \\
0 & \langle t^{-1} \rangle = M^{-2} & \mathfrak{sl}(2) & M_2 = (t^3) \\
1 & \langle t^{-1} \theta \rangle = M^{-3} & \mathfrak{osp}(1|2) & M_3 = (t^2 \theta) \\
2 & \langle t^{-1} \xi \eta \rangle = M^{-2;0} & \mathfrak{osp}(2|2) & M_{2;0} = (t \xi \eta) \\
3 & \langle t^{-1} \xi \eta \theta \rangle = M^{-1;0} & \mathfrak{osp}(3|2) & M_{1;0} = (\xi \eta \theta) \\
4 & \langle t^{-1} \xi_2 \eta_2 \rangle = M^{-1;e_1} & \mathfrak{osp}(4|2) & M_{1;e_2} = (\xi \eta \eta_2) \\
5 & \langle t^{-1} \xi_1 \ldots \xi_n \rangle = M^{1;0} & \mathfrak{osp}(5|2) & M_{1;e_2} = (\eta \eta_2) \\
6 & \langle t^{-1} \xi_1 \ldots \xi_n \rangle = M^{n;4;0} & \mathfrak{osp}(n|2) & M_{1;e_2} = (\eta \eta_2) \\
\end{array}
\]

For \( n > 5 \) the module \( M_{-1,-e_1-e_2-e_3} \) is always irreducible whereas \( M^{n-4;0} \) is always reducible:

\[
[M^{n-4;0}] = [M^{n-7;e_1+e_2+e_3}] \in [L^{n-4;0}].
\]

Exceptional cases:

\( n = 4 \). In this case the Verma module \( M^{0;0} \) induced from the Borel subalgebra \( M^{-1;e_1} \) dual to \( M_{1;e_1} \): the subspace of \( \mathfrak{t}^L(1|4) \) spanned by all functions except \( t^{-1} \xi_2 \eta_2 \) is an ideal. An explanation of this phenomenon is given in 1.3.

\( n = 6 \). The following miracle happens: \( M^{6-7;e_1+e_2+e_3} = \langle M_{1;-(e_1+e_2+e_3)} \rangle^* \) and \( L^{6-4;0} \cong \mathfrak{osp}(6|2) \). But the bilinear form obtained is supersymmetric, see §2.

For \( n > 6 \) there is no chance to have a nondegenerate bilinear form on \( \mathfrak{t}^L(1|n) \). The above arguments, however, do not exclude a degenerate form on \( \mathfrak{t}^L(1|n) \) such as a cocycle. For a proof see [KvL].

1.7. Root systems and simple roots for \( \mathfrak{svect}^L_\lambda(1|2) \). Set \( \partial = \frac{\partial}{\partial \xi} \), \( \delta_1 = \frac{\partial}{\partial \xi_1} \), \( \delta_2 = \frac{\partial}{\partial \xi_2} \).
The generators of the *distinguished* system of simple roots are:

\[
\begin{align*}
X_1^+ &= \xi_1 \delta_2 & X_2^+ &= t \delta_1 & X_3^+ &= \xi_2 t \partial - (\lambda + 1) \xi_1 \delta_1 \\
X_1^- &= \xi_2 \delta_1 & X_2^- &= \lambda \xi_1 \delta_2 + \xi_1 \partial & X_3^- &= \delta_2
\end{align*}
\] (G1)

The reflection in the 2nd root sends \((G1)\) into the following system that, to simplify the expressions, we consider them up to factors in square brackets \([\cdot]\).

\[
\begin{align*}
X_1^+ &= t \delta_2 & X_2^+ &= \lambda \xi_1 \delta_2 + \xi_1 \partial & X_3^+ &= [-\lambda] t \xi_2 \delta_1 \\
X_1^- &= \lambda \xi_1 \delta_2 - \xi_2 \partial & X_2^- &= t \delta_1 & X_3^- &= [-\lambda] \lambda \xi_2 \delta_1
\end{align*}
\] (G2)

The reflection in the 3rd root sends \((G1)\) into the following system. To simplify the expressions we consider them up to factors in square brackets \([\cdot]\).

\[
\begin{align*}
X_1^+ &= 2^{- \lambda - 1} & X_2^+ &= [\lambda + 2] t \xi_2 \delta_1 & X_3^+ &= (\lambda + 1) \xi_2 \xi_1 \delta_1 - \xi_2 t \partial \\
X_1^- &= 2^{- \lambda - 1} t \xi_2 \partial - (\lambda + 1) \xi_1 \xi_2 \delta_1 & X_2^- &= [-\lambda] \lambda \xi_1 \delta_2 & X_3^- &= \delta_1
\end{align*}
\] (G3)

The corresponding Cartan matrices are:

\[
\begin{pmatrix}
2 & -1 & -1 \\
1 - \lambda & 0 & \lambda \\
1 + \lambda & -\lambda & 0
\end{pmatrix},
\begin{pmatrix}
0 & -\lambda + 1 & -2 + \lambda \\
1 - \lambda & 0 & \lambda \\
-1 & -1 & 2
\end{pmatrix},
\begin{pmatrix}
0 & -\lambda & \lambda + 1 \\
-1 & 2 & -1 \\
1 + \lambda & -\lambda - 2 & 0
\end{pmatrix}.
\]

To compare these matrices, let us reduce them to the following canonical forms \((C1) - (C3)\), respectively, by renumbering generators and rescaling. (Observe that by definition, \(\lambda \neq 0, \pm 1\), so the fractions are well-defined.) We get

\[
\begin{pmatrix}
2 & -1 & -1 \\
1 - \lambda & 0 & \lambda \\
1 + \lambda & -\lambda & 0
\end{pmatrix},
\begin{pmatrix}
2 & -1 & -1 \\
1 - \lambda & 0 & \lambda \\
1 + \lambda & -\lambda & 0
\end{pmatrix},
\begin{pmatrix}
2 & -1 & -1 \\
1 - \lambda & 0 & \lambda \\
1 + \lambda & -\lambda & 0
\end{pmatrix}.
\]

We see that the transformations \(\lambda \rightarrow \lambda + 1\) and \(\lambda \rightarrow 1 - \lambda\) establish isomorphisms. We may, therefore, assume that \(\text{Re} \lambda \in [0, \frac{1}{2}]\).

1.8. Simplicity and occasional isomorphisms.

**Statement.** 1) The Lie superalgebras \(\text{vect}^L(1|n)\) for any \(n\), \(\text{svect}^I(1|n)\) for \(n \not\in \mathbb{Z}\) and \(n > 1\), \(\text{svect}^{L0}(1|n)\) for \(n > 1\), \(\text{vect}^{L}(1|n)\) for \(n \neq 5\) and \(\text{vect}^{L}(1|n)\) for \(n \neq 4\); and the five exceptional stringy superalgebras are simple.

2) The Lie superalgebras \(\text{vect}^L(1|1), t^L(1|2)\) and \(m^L(1|1)\) are isomorphic.

3) The Lie superalgebras \(\text{vect}^I(1|2) \cong \text{vect}^L(1|2)\) if \(\mu\) can be obtained from \(\lambda\) with the help of transformations \(\lambda \rightarrow \lambda + 1\) and \(\lambda \rightarrow 1 - \lambda\). The Lie superalgebras \(\text{svect}^I(1|2)\) from the strip \(\text{Re} \lambda \in [0, \frac{1}{2}]\) are nonisomorphic.

The statement on simplicity follows from a criterion similar to the one Kac applied for Lie (super)algebras with polynomial coefficients ([K]). The isomorphism is determined by the gradings listed in sec. 0.4 and arguments of 1.7.

1.9. Remarks. 1) A relation with Kac–Moody superalgebras. An unpublished theorem of Serganova states that the *only* Kac–Moody superalgebras \(\mathfrak{g}(A)\) of polynomial growth with nonsymmetrizable Cartan matrix \(A\) are: \(\mathfrak{psq}(n)^{(2)}\) (the corresponding Dynkin–Kac diagram is the same as that of \(\mathfrak{sl}(n)^{(1)}\)) but with any odd number of nodes replaced with "grey" nodes corresponding to the odd simple roots, see [FLS]) and an exceptional parametric family (found by J. van de Leur around 1986) with the matrix

\[
A = \begin{pmatrix}
2 & -1 & -1 \\
1 - \alpha & 0 & \alpha \\
1 + \alpha & -\alpha & 0
\end{pmatrix}.
\]
The Lie superalgebra $\mathfrak{g}(A)$ can be realized as the distinguished stringy superalgebra $\mathfrak{svect}_L^{(1|2)}$. For the description of the relations between its generators see [GL1].

Observe, that unlike the Kac–Moody superalgebras of polynomial growth with symmetrizable Cartan matrix, $\hat{\mathfrak{g}} = \mathfrak{svect}_L^{(1|2)}$ cannot be interpreted as a central extension of any twisted loop algebra. Indeed, the root vectors of the latter are locally nilpotent, whereas the former contains the operator $\partial_t$ with nonzero image of every $\hat{\mathfrak{g}}_i$.

2) How conformal are stringy superalgebras. Recall that a Lie algebra is called conformal if it preserves a metric (or, more generally a bilinear form) $B$ up to a factor. It is known that given a metric $B$ on the real space of dimension $\neq 2$, the algebras conformal wrt $B$ are isomorphic to $\mathfrak{so}(V, B)$. If $\dim V = 2$ we can consider $V$ as the complex line $\mathbb{C}^1$ with coordinate $t$ and identify $B$ with $dt \cdot d\bar{t}$ (the symmetric product of the differentials). The element $f \frac{d}{dt}$ from $\mathfrak{wit}$ multiplies $dt$ by $f'$ and, therefore, it multiplies $\partial_t \cdot dt$ by $f' \bar{f}'$, so $\mathfrak{wit}$ is conformal.

On superspaces $V$, metrics $B$ can be even and odd, the Lie superalgebras $\mathfrak{aut}(V, B)$ that preserve them are $\mathfrak{osp}^w(\text{Par})$ and $\mathfrak{pe}^w(\text{Par})$ and the corresponding conformal superalgebras are just trivial central extensions of $\mathfrak{aut}(V, B)$ for any dimension.

Suppose now that we consider a real form of each of the stringy superalgebras considered above and an extension of the complex conjugation (for possibilities see [M]). Let the contact superalgebras $\mathfrak{t}^L$, $\mathfrak{t}^M$ and $\mathfrak{m}^L$ preserve the Pfaff equation with form $\alpha$. From formulas (0.3) and (1.1.6) we deduce that the elements of these superalgebras multiply the symmetric product of forms $\alpha \cdot \bar{\alpha}$ by a factor of the form $F \bar{F}$, where $F$ is the function determined in (0.3) and (1.1.6). Every element $D$ of the general and divergence-free superalgebra $\mathfrak{svect}_L^{(1|n)}$ multiplies the symmetric product of volume forms $\det(t, \theta) \cdot \det(t, \bar{\theta})$ by $\det D \cdot \det \bar{D}$.

None of the tensors considered can be viewed as a metric. The series $\mathfrak{svect}^{(1|n)}$ ($\lambda \in \mathbb{Z}$) can not be considered as a superconformal even in the above sense.

There is, however, a possibility to consider $\mathfrak{t}^L(1|1)$ and $\mathfrak{t}^M(1|1)$ as conformal superalgebras, since the volume form “$dt \frac{d}{dt}$” can be considered as, more or less, $d\bar{t}$: consider the quotient $\Omega^1 / \mathfrak{F} \alpha$ of the superspace of differential 1-forms modulo the subspace spanned over functions by the contact form. Therefore, the tensor $dt \frac{\partial}{\partial \theta} \cdot d\bar{t} \frac{\partial}{\partial \bar{\theta}}$ can be viewed as the bilinear form $d\theta \cdot d\bar{\theta}$.

§2. INVARIANT BILINEAR FORMS ON STRINGY LIE SUPERALGEBRAS

Statement. An invariant (with respect to the adjoint action) nondegenerate supersymmetric bilinear form on a simple Lie superalgebra $\mathfrak{g}$, if exists, is unique up to proportionality.

Observe that the form can be odd.

The invariant nondegenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ exists if and only if $\mathfrak{g} \cong \begin{cases} \mathfrak{g}^* & \text{if} \ (\cdot, \cdot) \text{ is even} \\ \Pi(\mathfrak{g}) & \text{if} \ (\cdot, \cdot) \text{ is odd} \end{cases}$. Let us compare $\mathfrak{g}$ with $\mathfrak{g}^*$. Recall the definition of the modules $\mathfrak{F}_\lambda$ from sec. 1.3.

| $\mathfrak{g}$ | $\mathfrak{F}_{-1}$ | $\mathfrak{F}_{-2}$ | $\mathfrak{F}_{-3}$ | $\mathfrak{F}_{-4}$ | $\mathfrak{F}_{-5}$ | $\mathfrak{F}_{-6}$ | $\mathfrak{F}_{-7}$ | $n$ |
|---|---|---|---|---|---|---|---|---|
| $\mathfrak{F}_{1}$ | $\Pi(\mathfrak{F}_{0})$ | $\mathfrak{F}_{-1}$ | $\Pi(\mathfrak{F}_{-2})$ | $\Pi(\mathfrak{F}_{-3})$ | $\Pi(\mathfrak{F}_{-4})$ | $\Pi(\mathfrak{F}_{-5})$ | $\Pi(\mathfrak{F}_{-6})$ | $\Pi(\mathfrak{F}_{-7})$ |
| $\mathfrak{g}^*$ | $\mathfrak{F}_{2}$ | $\Pi(\mathfrak{F}_{3})$ | $\bar{\mathfrak{F}}_{2}$ | $\Pi(\mathfrak{F}_{1})$ | $\mathfrak{F}_{0}$ | $\Pi(\mathfrak{F}_{-1})$ | $\Pi(\mathfrak{F}_{-2})$ | $\Pi(\mathfrak{F}_{-3})$ | $\Pi(\mathfrak{F}_{-4})$ | $\Pi(\mathfrak{F}_{-5})$ |

$(\mathfrak{g} = \mathfrak{t}^L(1|n))$

| $\mathfrak{g}$ | $\mathfrak{F}_{-1}$ | $\mathfrak{F}_{-2}$ | $\mathfrak{F}_{-3}$ | $\mathfrak{F}_{-4}$ | $\mathfrak{F}_{-5}$ | $\mathfrak{F}_{-6}$ | $\mathfrak{F}_{-7}$ | $n$ |
|---|---|---|---|---|---|---|---|---|
| $\Pi(\mathfrak{F}_{1})$ | $\mathfrak{F}_{2}$ | $\Pi(\mathfrak{F}_{0})$ | $\mathfrak{F}_{-1}$ | $\Pi(\mathfrak{F}_{-2})$ | $\Pi(\mathfrak{F}_{-3})$ | $\Pi(\mathfrak{F}_{-4})$ | $\Pi(\mathfrak{F}_{-5})$ | $\Pi(\mathfrak{F}_{-6})$ | $\Pi(\mathfrak{F}_{-7})$ |
| $\mathfrak{g}^*$ | $\Pi(\mathfrak{F}_{2})$ | $\mathfrak{F}_{3}$ | $\Pi(\mathfrak{F}_{2})$ | $\mathfrak{F}_{1}$ | $\Pi(\mathfrak{F}_{0})$ | $\mathfrak{F}_{-1}$ | $\Pi(\mathfrak{F}_{-2})$ | $\Pi(\mathfrak{F}_{-3})$ | $\Pi(\mathfrak{F}_{-4})$ |

$(\mathfrak{g} = \mathfrak{t}^M(1|n))$

A comparison of $\mathfrak{g}$ with $\mathfrak{g}^*$ shows that there is a nondegenerate bilinear form on $\mathfrak{g} = \mathfrak{t}^L(1|6)$ and $\mathfrak{g} = \mathfrak{t}^M(1|7)$, even and odd, respectively. These forms are symmetric and given by the formula

$$(K_f, K_g) = \text{Res} f g.$$ 

The restriction of the bilinear form to $\mathfrak{as}_L$ is identically zero.
§3. THE THREE COCYCLES ON $\mathfrak{t}^{L/0}(1|4)$ AND PRIMARY FIELDS

Set

| $L_n = K_{1^{n+1}}$; $T_{ij}^n = K_{\theta_i \theta_j}$; $S_n = K_{\theta_1^{2n} \theta_3 \theta_4}$ | their degree | their parity |
|-----------------------------------------------|--------------|-------------|
| $E_n^i = K_{1^{n+1} \theta_i}$; $F_n^i = K_{\theta_i^{2n+1} \theta_3 \theta_4}$ | $2n$          | $0$;        |
|                                              | $2n + 1$     | $1$.        |

In the above “natural” basis the nonzero values of the cocycles are (see [KvL]; here $A_n$ is the group of even permutations):

| $c(L_m, L_n) = \alpha \cdot m(m^2 - 1)\delta_{m+n, 0}$ | 
|-------------------------------------------------|
| $c(E_m^i, E_n^i) = \alpha \cdot m(m + 1)\delta_{m+n+1, 0}$ | 
| $c(T_{ij}^m, T_{ij}^n) = \alpha \cdot m\delta_{m+n, 0}$ | 
| $c(F_m^i, F_n^i) = \alpha \cdot \delta_{m+n+1, 0}$ | 
| $c(S_m, S_n) = \alpha \cdot \frac{1}{m}\delta_{m+n, 0}$ | 

$c(L_m, S_n) = (\gamma + \beta \cdot m)\delta_{m+n, 0}$

$c(E_m^i, F_n^i) = (\frac{1}{2}\gamma + \beta \cdot (m + \frac{1}{2}))\delta_{m+n+1, 0}$

c($T_{ij}^m, T_{kl}^n$) $= -\beta \cdot m\delta_{m+n, 0}$, where $(i, j, k, l) \in A_4$.

To express the cocycle in terms of primary fields (i.e., the elements of Witt-modules such that the cocycle does not vanish on the pair of elements from one module only), let us embed Witt differently and, simultaneously, suitably internex the odd generators:

$\hat{L}_m = L_m + a_m S_m$ for $a_m = -\frac{2}{\alpha} \cdot m^2 - \frac{2}{\alpha} \cdot m$;

$\hat{E}_m^i = E_m^i + b_m F_m^i$ for $b_m = \frac{\beta}{2\alpha} \cdot (2m + 1) + \frac{\gamma}{2\alpha}$.

In the new basis the cocycle is of the form:

$c(\hat{L}_m, \hat{L}_n) = \left(\frac{\alpha^2 - 4\alpha}{\alpha} \cdot m^3 - \frac{\alpha^2 - 4\alpha}{\alpha} \cdot m\right)\delta_{m+n, 0}$

$c(\hat{E}_m^i, \hat{E}_n^i) = \left(\frac{\alpha^2 - 3\alpha^2}{\alpha} \cdot (m + \frac{1}{2})^2 - \frac{\alpha^2 - 3\alpha^2}{4\alpha}\right)\delta_{m+n+1, 0}$

$c(\hat{T}_{ij}^m, \hat{T}_{ij}^n) = \alpha \cdot m\delta_{m+n, 0}$

$c(\hat{F}_m^i, \hat{F}_n^i) = \alpha \cdot \delta_{m+n+1, 0}$

$c(S_m, S_n) = \alpha \cdot \frac{1}{m}\delta_{m+n, 0}$

$c(\hat{E}_m^i, \hat{F}_n^i) = (\gamma + \beta \cdot (2m + 1))\delta_{m+n+1, 0}$

c($\hat{T}_{ij}^m, \hat{T}_{kl}^n$) $= -\beta \cdot m\delta_{m+n, 0}$, where $(i, j, k, l) \in A_4$.

It depends on the three parameters and is expressed in terms of primary fields. Observe that the 3-dimensional space of parameters is not $\mathbb{C}^3 = \{(\alpha, \beta, \gamma)\}$ but $\mathbb{C}^3$ without a line, since $\alpha$ can never vanish.

§4. THE EXPLICIT RELATIONS BETWEEN THE CHEVALLERY GENERATORS OF $\mathfrak{t}^{L/6}$

Let $\Lambda^k$ be the subsuperspace of $\mathfrak{t}^{L/6}(1|6)$ generated by the $k$-th degree monomials in the odd indeterminates $\theta_i$. Then the basis elements of $\mathfrak{t}^{L/6}(1|6)$ with their degrees wrt to $K_t$ is given by the following table:

| $\ldots$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $\ldots$ |
|-----------|------|------|-----|-----|-----|---------|
| $\ldots$ | $\Lambda$ | $t$ | $t\Lambda$ | $t^2$ | $\ldots$ |
| $\ldots$ | $\Lambda^2$ | $\Lambda^2$ | $t\Lambda^2$ | $\ldots$ |
| $\ldots$ | $\Lambda^3$ | $\Lambda^3$ | $\ldots$ |
| $\ldots$ | $\Lambda^4$ | $\Lambda^4$ | $\ldots$ |
| $\ldots$ | $\Lambda^5$ | $\Lambda^5$ | $\ldots$ |
| $\ldots$ | $\Lambda^6$ | $\Lambda^6$ | $\ldots$ |
4.1. The Chevalley generators in \(\mathfrak{tas}^L\) in terms of \(\mathfrak{o}(6)\). Explicitly, in terms of the generating functions, the basis elements of \(\mathfrak{tas}^L\) are given by the following formulas, where \(\Theta = \xi_1 \xi_2 \xi_3 \eta_3 \eta_2 \eta_1\), \(\eta_i = \xi_i\), and \(\xi_i = \eta_i\). Let \(T^{ij}(i = 1, \ldots, 6)\) be the matrix skew-symmetric with respect to the side diagonal with only \((i, j)\)-th and \((j, i)\)-th nonzero entries equal to \(\pm 1\); let \(G^i = \theta_i\), where \(\theta = (\xi_1, \xi_2, \xi_3, \eta_3, \eta_2, \eta_1)\). Let the \(\hat{S}\) denote the generators of one of the two irreducible components in the \(\mathfrak{o}(6)\)-module \(\Lambda^3(\text{id})\). We will later identify \(G\) with the space of skew-symmetric \(4 \times 4\) matrices and \(\hat{S}\) with the \(\mathfrak{sl}(4)\)-module \(S^2(\text{id})\) of symmetric \(4 \times 4\) matrices, namely, \(\hat{S}^{\pm \varepsilon_i}\) for \(i = 1, 2, 3\) will be the symmetric off-diagonal matrices; \(S^{2,0,0}, S^{-2,2,0}, S^{0,-2,2}\) and \(S^{0,0,-2}\) the diagonal matrix units (the superscripts of \(\hat{S}\) are the weights of the matrix elements of the symmetric bilinear form wrt \(\mathfrak{sl}(4)\), see sec. 4.2). Set

| element | its generator |
|---------|---------------|
| \(L(2n - 2)\) | \(t^n - n(n - 1)(n - 2)t^{n-3}\Theta\), |
| \(G^i(2n - 1)\) | \(t^n\theta_i - n(n - 1)t^{n-2}\frac{\partial}{\partial \theta_i}\), |
| \(T^{ij}(2n)\) | \(t^n\theta_i \theta_j - n t^{n-1}\frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j}\) |
| \(\hat{S}^{\varepsilon_i}(2n + 1)\) | \(t^n \xi_i (\xi_j \eta_j + \xi_k \eta_k)\) |
| \(\hat{S}^{-\varepsilon_i}(2n + 1)\) | \(t^n \eta_i (\xi_j \eta_j - \xi_k \eta_k)\) |
| \(\hat{S}^{2,0,0}(2n + 1)\) | \(t^n \xi_1 \xi_2 \xi_3\) |
| \(\hat{S}^{-2,2,0}(2n + 1)\) | \(t^n \xi_1 \eta_2 \eta_3\) |
| \(\hat{S}^{0,-2,2}(2n + 1)\) | \(t^n \xi_2 \eta_1 \eta_3\) |
| \(\hat{S}^{0,0,-2}(2n + 1)\) | \(t^n \xi_3 \eta_1 \eta_2\) |

where \(\hat{G}\) and \(\hat{S}\) are the following skew-symmetric and symmetric matrices, respectively; we set \(\hat{G}^i = \hat{G}^{\varepsilon_i}\), where \(\varepsilon_i\) and \(-\varepsilon_i\) is the weight of \(\xi_i\) and \(\eta_i\), respectively, wrt \((H^1, H^2, H^3)\) \(\in \mathfrak{sl}(4)\).

\[
\hat{G} = \begin{pmatrix}
0 & -\xi_1 & -\xi_2 & \eta_1 \\
\xi_1 & 0 & \xi_2 & \eta_2 \\
0 & -\xi_2 & 0 & \eta_3 \\
\eta_1 & \eta_2 & \eta_3 & 0
\end{pmatrix}, \quad \hat{S} = \begin{pmatrix}
\xi_1 \xi_2 \xi_3 & \xi_1 (\xi_2 \eta_2 + \xi_3 \eta_3) & \xi_2 (\xi_1 \eta_1 + \xi_3 \eta_3) & \xi_3 (\xi_1 \eta_1 + \xi_2 \eta_2) & \eta_3 (\xi_1 \eta_1 - \xi_2 \eta_2) \\
\eta_1 \eta_2 & \xi_1 \eta_3 & \xi_2 \eta_3 & \eta_3 \eta_1 \\
-\xi_3 \eta_1 & \xi_1 - \eta_2 & -\xi_1 \xi_3 & -\xi_2 \eta_3 \\
\eta_1 \eta_2 & \eta_1 \eta_3 & \eta_2 \eta_3 & 0
\end{pmatrix}.
\]

These generators, expressed via monomial generators of \(\mathfrak{f}^L(1|6)\), are rather complicated. Let us pass to simpler ones using the isomorphism \(\mathfrak{sl}(4) \cong \mathfrak{o}(6)\). Explicitly, this isomorphism is defined as follows:

\[
\begin{pmatrix}
\xi_2 \eta_3 & \xi_1 \eta_3 & \xi_1 \xi_2 & \eta_3 \eta_1 - \xi_2 \eta_2 \\
\xi_3 \eta_2 & -\eta_1 & -\xi_1 \xi_3 & -\xi_2 \eta_3 \\
-\xi_3 \eta_1 & -\xi_2 \eta_1 & \eta_3 \eta_2 & \eta_2 \eta_3 \\
\eta_1 \eta_2 & \eta_1 \eta_3 & \eta_2 \eta_3 & 0
\end{pmatrix} \cdot \begin{pmatrix}
H^1 = - (\xi_2 \eta_2 - \xi_3 \eta_3), \\
H^2 = -(\xi_1 \eta_1 - \xi_2 \eta_2), \\
H^3 = (\xi_2 \eta_2 + \xi_3 \eta_3).
\end{pmatrix}
\]

4.2. The multiplication table in \(\mathfrak{tas}^L\). In terms of \(\mathfrak{sl}(4)\)-modules we get a more compact expression of the elements of \(\mathfrak{tas}^L\). Let \(\mathfrak{ad}\) be the adjoint module, \(\mathfrak{S}\) the symmetric square of the identity 4-dimensional module id and \(G = \Lambda^2(\text{id})\); let \(\mathbb{C}\cdot 1\) denote the trivial module. Then the basis elements of \(\mathfrak{tas}^L(1|6)\) with their degrees wrt to \(K_i\) is given by the following table in which the degrees of \(t\) indicate the grading:

| degree | \(-2\) | \(-1\) | \(0\) | \(1\) | \(2\) | \(\ldots\) |
|--------|--------|--------|--------|--------|--------|--------|
| space  | \(\mathbb{C} \cdot t^{-1}, \text{ad} \cdot t^{-1}\) | \(\mathbb{C} \cdot t^{-1}, G \cdot t^{-1}\) | \(\mathbb{C} \cdot 1, \text{ad}\) | \(\mathbb{C} \cdot t, \text{ad} \cdot t\) | \(\mathbb{C} \cdot t\) |

Though it is impossible to embed \(\mathfrak{witt} \cong \mathfrak{gl}(4)\) into \(\mathfrak{tas}^L\), it is convenient to express the brackets in \(\mathfrak{tas}^L\) in terms of the matrix units of \(\mathfrak{gl}(4)\) that we will denote by \(T^i_j(a)\); we further set \(H_1(a) = T^1_1(a) - T^2_2(a)\), \(H_2(a) = T^2_2(a) - T^3_3(a)\) and \(H_3(a) = T^3_3(a) - T^1_1(a)\). Clearly, the rhs in the last line of the following multiplication table can be expressed via the \(H_i(a)\). We denote the basis elements of the trivial \(\mathfrak{sl}(4)\)-module of degree \(a\) by \(L(a)\) and norm them so that they commute as the usual basis elements of \(\mathfrak{witt}\).
The relations between Chevalley generators in \( \mathfrak{tas}^c \) is given by the following table:

\[
\begin{align*}
[L(a), L(b)] &= (b - a)L(a + b), \\
[L(a), T^i_j] &= bT^i_j(a + b), \\
[L(a), S^i_j] &= (b + \frac{1}{2}a)S^i_j(a + b), \\
[L(a), G_{ij}(b)] &= (b - \frac{1}{2}a)G_{ij}(a + b), \\
[T^i_j(a), T^k_l] &= \delta^i_kT^j_l(a + b) - \delta^j_lT^i_k(a + b), \\
[T^i_j(a), S^k_l] &= \delta^i_kS^j_l(a + b) + \delta^j_lS^i_k(a + b), \\
[T^i_j(a), G_{kl}] &= \delta^i_kG_{ij}(a + b) + \delta^j_lG_{jk}(a + b) + a\sigma(j, k, l, m)S^{lm}(a + b), \\
[S^i_j(a), S^k_l] &= 0, \\
[S^i_j(a), G_{kl}] &= (2\delta^i_kT^j_l - 2\delta^j_lT^i_k + 2\delta^i_kT^j_l - 2\delta^j_lT^i_k)(a + b), \\
\end{align*}
\]

\[
[G_{ij}(a), G_{kl}(b)] = 2(b - a)(\delta_{i,j,k}\sigma(i, j, l, m)T^m_j(a + b) + \delta_{i,j,k}\sigma(i, j, l, m)T^m_l(a + b) + \\
\delta_{i,j,k}\sigma(i, k, l, m)T^m_j(a + b) + \delta_{i,j,k}\sigma(i, j, k, m)T^m_l(a + b) + \\
\sigma(i, j, k, l)\left(-4L(a + b) + (b - a)(T^i_j(a + b) + T^j_i(a + b) - T^k_k(a + b) - T^k_k(a + b))\right)
\]

4.3. The relations between Chevalley generators in \( \mathfrak{tas}^c \) in terms of \( \mathfrak{sl}(4) \). Denote: \( T^a_{ij} = T^i_j(a) \).

For the positive Chevalley generators we take same of \( \mathfrak{sl}(4) = \text{Span}(T_{ij}) \) and the lowest weight vectors \( S^1_{11} \) and \( G^1_{12} \) of \( S^1 \) and \( G^1 \), respectively. For the negative Chevalley generators we take same of \( \mathfrak{sl}(4) \) and the highest weight vectors \( S^1_{11} \) and \( G^1_{12} \) of \( S^1 \) and \( G^1 \), respectively. Then the defining relations, stratified by weight, are the following ones united with the usual Serre relations in \( \mathfrak{sl}(4) \) (we skip them) and the relations that describe the highest (lowest) weight vectors:

\[
\begin{align*}
[T^{0}_{23}, [T^{0}_{23}, G^1_{12}]] &= 0 & [T^{0}_{34}, [T^{0}_{34}, [T^{0}_{34}, S^1_{44}]]] &= 0 \\
[[G^1_{12}, [T^{0}_{23}, G^1_{12}]], [T^{0}_{34}, S^1_{44}]] &= 0 & [[T^{0}_{23}, G^1_{12}], [T^{0}_{34}, S^1_{44}]] &= 0 \\
[[G^1_{12}, [T^{0}_{23}, T^{0}_{34}]], [T^{0}_{23}, G^1_{12}], [T^{0}_{34}, S^1_{44}]] &= 0 & [[[T^{0}_{23}, G^1_{12}], [T^{0}_{34}, G^1_{12}]], [T^{0}_{34}, S^1_{44}]] &= 0 \\
[L^0, S^1_{11}] &= -S^1_{11} & [L^0, S^1_{44}] &= S^1_{44} \\
[L^0, G^1_{12}] &= -G^1_{12} & [L^0, G^1_{12}] &= -G^1_{12} \\
[H^1_0, S^1_{11}] &= 2S^1_{11} & [H^1_0, S^1_{44}] &= 0 \\
[H^1_0, G^1_{12}] &= 0 & [H^1_0, G^1_{12}] &= 0 \\
[H^2_0, S^1_{11}] &= 0 & [H^2_0, S^1_{44}] &= 0 \\
[H^2_0, G^1_{12}] &= -G^1_{12} & [H^2_0, G^1_{12}] &= -G^1_{12} \\
[H^3_0, S^1_{11}] &= 0 & [H^3_0, S^1_{44}] &= -2S^1_{44} \\
[H^3_0, G^1_{12}] &= 0 & [H^3_0, G^1_{12}] &= 0 \\
[T^{0}_{12}, S^1_{11}] &= 0 & [S^1_{44}, T^{0}_{21}] &= 0 \\
[T^{0}_{12}, G^1_{12}] &= 0 & [S^1_{44}, T^{0}_{32}] &= 0 \\
[T^{0}_{23}, S^1_{11}] &= 0 & [S^1_{44}, T^{0}_{43}] &= 0 \\
[T^{0}_{23}, G^1_{12}] &= 0 & [G^1_{12}, T^{0}_{21}] &= 0 \\
[T^{0}_{34}, S^1_{11}] &= 0 & [G^1_{12}, T^{0}_{32}] &= 0 \\
[T^{0}_{34}, G^1_{12}] &= 0 & [G^1_{12}, T^{0}_{43}] &= 0 \\
[T^{0}_{12}, S^1_{44}] &= 0 & [T^{0}_{21}, G^1_{12}] &= 0 \\
[T^{0}_{12}, G^1_{12}] &= 0 & [T^{0}_{32}, S^1_{11}] &= 0 \\
[T^{0}_{23}, S^1_{44}] &= 0 & [T^{0}_{34}, S^1_{11}] &= 0 \\
[T^{0}_{34}, G^1_{12}] &= 0 & [T^{0}_{34}, G^1_{12}] &= 0 \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{44}, S^1_{11}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [S^1_{44}, G^1_{44}] &= -4T^{0}_{43} \\
[G^1_{12}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= 4T^{0}_{12} \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{11}, S^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= -4L^0 - 2H^0 - 2H^0 - 2H^0 \\
[G^1_{34}, G^1_{34}] &= 0 & [G^1_{34}, G^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [S^1_{44}, G^1_{44}] &= -4T^{0}_{33} \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{11}, S^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= 4T^{0}_{12} \\
[G^1_{34}, G^1_{34}] &= 0 & [G^1_{34}, G^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [S^1_{44}, G^1_{44}] &= 0 \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{11}, S^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= 4T^{0}_{12} \\
[G^1_{34}, G^1_{34}] &= 0 & [G^1_{34}, G^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [S^1_{44}, G^1_{44}] &= 0 \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{11}, S^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= 4T^{0}_{12} \\
[G^1_{34}, G^1_{34}] &= 0 & [G^1_{34}, G^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [S^1_{44}, G^1_{44}] &= 0 \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{11}, S^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= 4T^{0}_{12} \\
[G^1_{34}, G^1_{34}] &= 0 & [G^1_{34}, G^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [S^1_{44}, G^1_{44}] &= 0 \\
[S^1_{11}, S^1_{11}] &= 0 & [S^1_{11}, S^1_{44}] &= 0 \\
[S^1_{11}, G^1_{12}] &= 0 & [G^1_{12}, G^1_{44}] &= 4T^{0}_{12} \\
[G^1_{34}, G^1_{34}] &= 0 & [G^1_{34}, G^1_{44}] &= 0 \\
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