On the existence of self-similar spherically symmetric wave maps coupled to gravity

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Abstract
We present a detailed analytical study of spherically symmetric self-similar solutions in the SU(2) sigma model coupled to gravity. Using a shooting argument, we prove that there is a countable family of solutions which are analytic inside the past self-similarity horizon. In addition, we show that for sufficiently small values of the coupling constant these solutions possess a regular future self-similarity horizon and thus are examples of naked singularities. One of the solutions constructed here has been recently found as the critical solution at the threshold of black-hole formation.

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1. Introduction
In this paper, we continue our investigations, started in [1] (referred to as I), of wave maps coupled to gravity, that is, solutions of Einstein’s equations with an SU(2) sigma field as matter. We found numerically in I that for \( \alpha < 1/2 \) (\( \alpha \) is the dimensionless coupling constant) the model admits a countable family of continuously self-similar (CSS) solutions, labelled by an integer nodal index \( n = 0, 1, \ldots \), that are analytic inside the past light cone of the singularity. We also provided evidence that the \( n \)th CSS solution can be extended to the future light cone of the singularity if \( \alpha < \alpha_n \), where \( \{ \alpha_n \} \) is an increasing sequence bounded above by \( 1/2 \). The purpose of this paper is to make the results of I (except for the ordering of \( \alpha_n \)) into theorem–proof rigorous mathematics. This is accomplished by applying a shooting argument to the resulting dynamical system. We note that the case \( \alpha = 0 \) was previously analysed in [2].

The physical importance of the CSS solutions considered here was discussed in I. In particular, we conjectured that in a certain parameter range \(( \alpha_0 < \alpha < \alpha_1 \) the \( n = 1 \) solution is a critical solution at the threshold of black-hole formation. This conjecture has been recently confirmed in numerical studies of the critical behaviour [3] and in the linear stability analysis [4]. As far as we know, this is the only case where the existence of a self-similar solution,
which was numerically found as the critical solution in gravitational collapse, has been established rigorously.

2. Setup

For the reader’s convenience, we repeat from I the basic setting for the problem. Let $X : M \rightarrow N$ be a map from a spacetime $(M, g_{ab})$ into a Riemannian manifold $(N, G_{AB})$. Wave maps coupled to gravity are defined as extrema of the action

$$S = \int_M \left( \frac{R}{16\pi G} + L_{WM} \right) d\nu_g$$

with the Lagrangian density

$$L_{WM} = -\frac{f^2}{2} \pi^2 g^{ab} \partial_a X^A \partial_b X^B G_{AB}.$$  

Here $G$ is Newton’s constant and $f^2$ is the wave map coupling constant. The product $\alpha = 4\pi G f^2$ is dimensionless. The field equations derived from (1) are the wave map equation

$$\Box g X^A + \nabla^A \nabla^B (X) \partial_a X^B \partial_b X^C g^{ab} = 0,$$

where $\Gamma^A_{BC}(X)$ are the Christoffel symbols of the target metric $G_{AB}$ and $\Box$ is the d’Alembertian associated with the metric $g_{ab}$, and the Einstein equations $R_{ab} - \frac{1}{2} R g_{ab} R = 8\pi G T_{ab}$ with the stress–energy tensor

$$T_{ab} = f^2 \pi (\partial_a X^A \partial_b X^B - \frac{1}{2} g_{ab} (g^{cd} \partial_c X^A \partial_d X^B)) G_{AB}.$$  

As a target manifold, we take the 3-sphere $S^3$ with the standard metric in polar coordinates $X^A = (F, \Theta_1, \Phi_1)$,

$$G_{AB} dX^A dX^B = dF^2 + \sin^2 F (d\Theta^2 + \sin^2 \Theta d\Phi^2).$$

For the domain manifold, we assume spherical symmetry and use Schwarzschild coordinates

$$g_{ab} dx^a dx^b = -e^{-2A} dr^2 + A^{-1} dr^2 + r^2 (d\Theta^2 + \sin^2 \Theta d\Phi^2),$$

where $\delta$ and $A$ are functions of $(t, r)$. Next, we assume that the wave maps are corotational, that is,

$$F = F(t, r), \quad \Theta = \theta, \quad \Phi = \phi.$$  

Equation (3) reduces then to the single semilinear wave equation

$$\Box g F = \frac{\sin(2F)}{r^2} = 0,$$

where

$$\Box g = -e^\delta \partial_t \left( e^A A^{-1} \partial_t \right) + \frac{\delta}{r} r^2 e^{-\delta} (r^2 e^{2A} \partial_r),$$

and the Einstein equations become

$$\partial_t A = -2\alpha r A (\partial_t F)(\partial_t F),$$

$$\partial_t \delta = -\alpha r \left( (\partial_t F)^2 + A^{-2} e^{2A} (\partial_t F)^2 \right),$$

$$\partial_r A = \frac{1 - A}{r} - \alpha r \left( A (\partial_r F)^2 + A^{-1} e^{2A} (\partial_r F)^2 + 2 \frac{\sin^2 F}{r^2} \right).$$

These equations are invariant under dilations $(t, r) \rightarrow (\lambda t, \lambda r)$, so it is natural to look for continuously self-similar (CSS) solutions, that is solutions which are left invariant by the
action of the homothetic Killing vector $K = t \partial_t + r \partial_r$. To study such solutions, it is convenient to use similarity variables $\rho = r/(−t)$ and $\tau = −\ln(−t)$. Then $K = −\partial_\tau$, so CSS solutions do not depend on $\tau$. Assuming this and using an auxiliary function $Z = e^\rho /A$, we reduce equations (8)–(12) to the system of ordinary differential equations (where prime is $d/d\rho$):

$$F'' + \frac{2}{\rho}F' - \alpha (1 + Z^2) \rho F' + \frac{\sin(2F)}{A\rho^2 (1 - Z^2)} = 0,$$

$$A' = -2\alpha \rho A F'^2,$$

$$\rho Z' = Z(1 + \alpha (1 - Z^2) \rho^2 F'^2),$$

$$\rho A' = 1 - A - \alpha (\rho^2 A (1 + Z^2) F'^2 + 2 \sin^2 F).$$

The combination of (14) and (16) yields the constraint

$$1 - A - 2\alpha \sin^2 F + \alpha A \rho^2 F'^2 (1 - Z^2) = 0.$$ (17)

This system of equations has a fixed singularity at the centre $\rho = 0$ and moving singularities at points where $Z(\rho) = ± 1$ and/or $A(\rho) = 0$. In terms of the similarity coordinate $\rho$, the metric (6) takes the form

$$ds^2 = A^{-3}(1 - Z^2)^2 \rho^2 \text{d}r^2 + 2 A^{-1} \rho \text{d}\rho \text{d}t + A^{-1} \rho^2 \text{d}\rho^2 + \rho^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),$$

hence the hypersurfaces $Z = ± 1$ are null (provided that $A > 0$). The first $\rho_1$ where $Z(\rho_1) = 1$ is the locus of the past light cone of the singularity at the origin $(t = 0, r = 0)$ (in what follows, we shall refer to the past and future light cones of the singularity as the past and future self-similarity horizons (SSH)). By rescaling, $\rho \to \rho/\rho_1$, one can always locate the past self-similarity horizon at $\rho_1 = 1$, that is $Z(1) = 1$. To ensure regularity of solutions in the interval $0 \leq \rho \leq 1$, equations (13)–(17) must be supplemented by the boundary conditions at both endpoints,

$$F(0) = 0, \quad F'(0) = c, \quad Z(0) = 0, \quad A(0) = 1,$$

$$F(1) = \frac{\pi}{2}, \quad F'(1) = b, \quad Z(1) = 1, \quad A(1) = 1 - 2\alpha,$$ (19–20)

where $c$ and $b$ are free parameters. At this point, it might not be obvious why the boundary condition $F(1) = \pi/2$ in (20) needs to be chosen, as one could naively think of any solution of $\sin(2F(1)) = 0$. We shall show below that $F(1) = \pi/2$ is the only possibility.

Our main result is the following theorem:

**Theorem 1.** For any $0 \leq \alpha < 1/2$ and any non-negative integer $n$, equations (13)–(17) have an analytic solution $(F_n, A_n, Z_n)$ which satisfies the boundary conditions (19)–(20) and has precisely $n$ oscillations of $F_n(\rho)$ around $\pi/2$.

In the next section, we shall prove this theorem using a shooting technique. The numerical evidence for theorem 1 was given in I. The case $\alpha = 0$ was proved previously in [2], so hereafter we assume that $0 < \alpha < 1/2$.

3. **Proof of theorem 1**

For convenience, we rewrite equations (13)–(15) in terms of $H = F - \pi/2$:

$$H'' + \frac{2}{\rho}H' - \alpha (1 + Z^2) \rho H' + \frac{\sin(2H)}{A\rho^2 (1 - Z^2)} = 0.$$ (21)
\[ A' = -2\alpha \rho A H'^2, \]
\[ \rho Z' = Z(1 + \alpha(1 - \rho^2)\rho^2 H'^2). \]

The constraint becomes
\[ 1 - 2\alpha - A + 2\alpha \sin^2 H + \alpha A \rho^2 H'^2(1 - \rho^2) = 0. \]

The initial conditions at \( \rho = 0 \) are
\[ H(0) = -\frac{\pi}{2}, \quad H'(0) = c, \quad A(0) = 1, \quad Z(0) = 0, \quad Z'(0) = 1. \]

Note that the above equations have a residual scaling symmetry \( \rho \to \lambda \rho \). The initial condition \( Z'(0) = 1 \) is imposed temporarily in order to fix the scale. We shall refer to solutions of equations (21)–(24) satisfying the initial conditions (25) as \( c \)-orbits. In the appendix, we show that \( c \)-orbits exist locally and are analytic in \( \rho \) and \( c \). Now we shall show that \( c \)-orbits can be extended up to a point \( \rho_1 \) at which \( Z(\rho_1) = 1 \).

**Proposition 2.** For any \( 0 < \alpha < 1/2 \) and \( c > 0 \) there is a \( \rho_1(c) \in (\sqrt{1 - 2\alpha}, 1) \), such that the \( c \)-orbit is defined for all \( \rho < \rho_1 \) and \( \lim_{\rho \to \rho_1} Z(\rho) = 1 \). Furthermore, the following limits exist:
\[ -\frac{\pi}{2} < \tilde{H} \overset{\text{def}}{=} \lim_{\rho \to \rho_1} H(\rho) < \frac{\pi}{2}, \quad \tilde{A} \overset{\text{def}}{=} \lim_{\rho \to \rho_1} A(\rho) = 1 - 2\alpha \cos^2 \tilde{H}, \]
\[ \lim_{\rho \to \rho_1} (1 - \rho^2) H'^2 = 0. \]

**Proof.** Let the maximum domain of definition of the \( c \)-orbit be \( 0 \leq \rho < \rho_1 \) and assume that \( Z(\rho) < 1 \) in this interval. Then, from constraint (24) we have \( A \geq 1 - 2\alpha > 0 \) and hence \( \tilde{A} = \lim_{\rho \to \rho_1} A(\rho) > 0 \) (\( \tilde{A} \) exists since \( A(\rho) \) is monotone decreasing). By (23) \( Z' \geq 0 \), hence \( \tilde{Z} = \lim_{\rho \to \rho_1} Z(\rho) \) exists. If \( \tilde{Z} < 1 \), then from constraint (24) \( H'^2 \) is bounded so \( \tilde{H} = \lim_{\rho \to \rho_1} H(\rho) \) exists, which in turn implies, again by (24), that \( \lim_{\rho \to \rho_1} H' \) exists. Thus, \( \tilde{H}, \tilde{H}', \tilde{A} \) and \( \tilde{Z} \) all have finite limits at \( \rho_1 \) and therefore the \( c \)-orbit may be continued beyond \( \rho_1 \) contradicting the maximality of \( \rho_1 \). We conclude that \( \tilde{Z} = 1 \).

Now, we must show that \( \tilde{H} \in (-\pi/2, \pi/2) \) exists. Since \( \tilde{Z} = 1 \), we may no longer conclude that \( H'^2 \) is bounded but from equation (22) we get (in \( A' = -2\alpha \rho H'^2 \), so \( H'^2 \) is integrable near \( \rho_1 \) which implies that \( H' \) is absolutely integrable (\( |H'| < 1 + H'^2 \)) and thus \( \tilde{H} \) exists. From constraint (24), \( H(\rho) = \pm \pi/2 \) for some \( 0 < \rho < \rho_1 \) is not possible since \( 1 - A > 0 \). Thus, \( -\pi/2 < \tilde{H}(\rho) < \pi/2 \) and so \( -\pi/2 < \tilde{H} \leq \pi/2 \). In fact, for \( \rho > \rho_1/2 \) we have \( 1 - A > \sigma > 0 \), so \( 2\alpha \cos^2 \tilde{H} > \sigma > 0 \) (remember that we assume \( \alpha > 0 \)), hence \( H \) is uniformly bounded away from \( \pm \pi/2 \), and thus \( -\pi/2 < \tilde{H} < \pi/2 \).

To prove \( \tilde{A} = 1 - 2\alpha \cos^2 \tilde{H} \), note that by (24) \( d = \lim_{\rho \to \rho_1} H'^2(1 - Z^2) \) exists and is finite. Hence, by (23) \( \lim_{\rho \to \rho_1} Z' \) exists and is finite, so \( 1 - Z^2 = O(\rho - \rho_1) \) near \( \rho_1 \). If \( d \neq 0 \), then \( H'^2(\rho) \sim d/(\rho_1 - \rho) \) would not be integrable near \( \rho_1 \), thus \( d \) must be zero. Inserting this into (24) we get \( \tilde{A} = 1 - 2\alpha \cos^2 \tilde{H} \).

Next, \( (Z/\rho)' > 0 \) by (23) and \( \lim_{\rho \to 0} (Z/\rho) = 1 \) by L’Hôpital’s rule, hence \( Z > \rho \) for all \( \rho > 0 \), and thus \( \rho_1 < 1 \). Finally, from (22) and (23)
\[ \frac{(AZ^2/\rho^2)'}{\rho^2} = -2Z^4 \frac{A H'^2}{\rho} < 0, \]
and since \( \lim_{\rho \to 0} (AZ^2/\rho^2) = 1 \), we have \( (AZ^2/\rho^2) \leq 1 \) and hence \( \rho_1 > \sqrt{\tilde{A}} \geq \sqrt{1 - 2\alpha} \).

If \( \rho(\rho_2) = 1 \) for some \( \rho_2 < \rho_1 \), we replace \( \rho_1 \) by \( \rho_2 \) in the above arguments. \( \square \)

**Corollary 3.** The function \( \rho_1(c) \) is continuous.
Lemma 4. $H'(\rho)$ is bounded near $\rho_1$ if and only if $\bar{H} = 0$.

Proof. Suppose that $\bar{H} \neq 0$ and $H'(\rho)$ is bounded. Then, in (21) we have

$$H'' = \text{bounded terms} - \frac{\sin 2H}{A\rho^2(1 - Z^2)} \sim \frac{b}{\rho_1 - \rho},$$

where $b \neq 0$. This contradicts that $H'(\rho)$ is bounded near $\rho_1$ and concludes the ‘only if’ part of lemma 4.

Suppose now that $H(\rho_1) = 0$ and $H'(\rho)$ is unbounded. Without the loss of generality, we consider the case that $H(\rho) < 0$ and $H'(\rho) > 0$ near $\rho_1$. Dividing equation (21) by $H'$ and integrating from $\rho$ to $\rho_1$, we obtain

$$\int_\rho^{\rho_1} \left( \frac{H''}{H'} + \frac{2}{\rho} - \alpha(1 + Z^2) \rho H'' + \frac{\sin(2H)}{HH'A\rho^2(1 - Z^2)} \right) d\rho = 0.$$  \hspace{1cm} (28)

The first integral is divergent because $\lim_{\rho \to \rho_1} \ln H' = \infty$. The second and third terms are integrable (remember that $H''$ is integrable). Thus, to get a contradiction it suffices to show that the last term is integrable. We write this term as

$$\frac{\sin(2H)}{HH'A\rho^2(1 - Z^2)} = \frac{\sin(2H)}{H'A\rho^2(1 - Z^2)^2} \frac{H'}{H'}.$$

The first factor is continuous and we now show that the second factor is also continuous. Applying L'Hôpital's rule, we get

$$\lim_{\rho \to \rho_1} \frac{H}{(1 - Z^2)H'} = \lim_{\rho \to \rho_1} \frac{H'}{-2ZZ'H' + (1 - Z^2)H''} = \lim_{\rho \to \rho_1} \frac{1}{-2ZZ' + (1 - Z^2)H''/H'}.$$  \hspace{1cm} (30)

Next, using (21) we get

$$\frac{1 - Z^2}{H'} H'' = -\frac{2(1 - Z^2)}{\rho} + \alpha \rho(1 + Z^2)(1 - Z^2)H' = \frac{\sin(2H)}{A\rho^2H'}. $$

In the limit $\rho \to \rho_1$, the first term on the rhs of (31) obviously goes to zero, the second does by proposition 2 and the third does by the assumption that $H' \to \infty$. Thus, limit (30) is finite and consequently so is (29). This contradicts (28) and thus concludes the proof of the ‘if’ part of lemma 4. \hfill \Box

Corollary 5. A c-orbit which has $\bar{H}(c) = 0$ is analytic on the whole interval $0 \leq \rho \leq \rho_1$.

Proof. The boundedness of $H'(\rho)$ implies by (21) that $H'' > -2H'/\rho$ is bounded below (remember that $H(\rho) < 0$ and $H'(\rho) > 0$ near $\rho_1$), hence $\lim_{\rho \to \rho_1} H'(\rho)$ exists. Having that, it follows that $\lim_{\rho \to \rho_1} \frac{\sin(2H)}{(1 - Z^2)}$ has a finite limit (since $\lim Z' = 1/\rho_1 \neq 0$), and therefore the solution $(H, A, Z)$ is $C^2$ near $\rho_1$. By a routine contraction mapping argument, one can show that $C^2$ solutions are unique, hence a c-orbit must belong to the one-parameter family of analytic solutions from proposition 14 (see the appendix). \hfill \Box
Next, we describe the behaviour of $c$-orbits for small and large values of the shooting parameter $c$. We define a nodal number of a $c$-orbit $N(c)$ as the number of zeros of the function $H(\rho)$ on the interval $0 \leq \rho < \rho_1$. We first show that $c$-orbits with small $c$ have no nodes.

**Proposition 6.** If $c$ is sufficiently small then $N(c) = 0$.

**Proof.** For $c = 0$ we have $H(\rho) \equiv -\pi/2$ and $Z(\rho) = \rho$, so $\rho_1(c = 0) = 1$. By continuity, for any $\epsilon > 0$ and sufficiently small $c$ we can find $\rho_0$ such that $1 - \epsilon < \rho_0 < \rho_1(c) < 1$ and $H(\rho_0) < -\pi/2 + \epsilon$. We know from the proof of proposition 2 that $\lim_{\rho \rightarrow \rho_1} \sqrt{\rho_1 - \rho} H' = 0$, hence

$$H(\rho_1) - H(\rho_0) = \int_{\rho_0}^{\rho_1} H'(\rho) \, d\rho < \text{const} \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\rho_1 - \rho}} < \text{const} \sqrt{\epsilon}. \quad (32)$$

Thus, $H(\rho)$ stays arbitrarily close to $-\pi/2$ all the way up to $\rho_1$ if $c$ is sufficiently small and therefore $N(c) = 0$. We remark that using a scaling argument one can derive the precise asymptotic behaviour of $c$-orbits for small $c$. We omit this argument since it is not needed for the proof. \(\square\)

We show next that $c$-orbits with large $c$ have arbitrarily many nodes.

**Proposition 7.** $N(c) \rightarrow \infty$ for $c \rightarrow \infty$.

**Proof.** We rescale the variables, setting

$$x = c \rho, \quad \tilde{H}(x) = H(\rho), \quad \tilde{A}(x) = A(\rho), \quad \tilde{Z}(x) = cZ(\rho). \quad (33)$$

Then, equations (21)–(24) become

$$\tilde{H}'' + \frac{2}{x} \tilde{H}' - \alpha \left(1 + \frac{\tilde{Z}^2}{c^2}\right) x H^3 + \frac{\sin(2\tilde{H})}{Ax^2 \left(1 - \frac{\tilde{Z}^2}{c^2}\right)} = 0, \quad (34)$$

$$\tilde{A}' = -2ax \tilde{H}', \quad (35)$$

$$x \tilde{Z}' = \tilde{Z} \left(1 + \alpha \left(1 - \frac{\tilde{Z}^2}{c^2}\right) x^2 H^2\right), \quad (36)$$

with the constraint

$$1 - 2\alpha - \tilde{A} + 2\alpha \sin^2 \tilde{H} + \alpha \tilde{A} \tilde{H} \left(1 - \frac{\tilde{Z}^2}{c^2}\right) = 0, \quad (37)$$

and the initial conditions at $x = 0$

$$\tilde{H}(0) = -\frac{\pi}{2}, \quad \tilde{H}'(0) = 1, \quad \tilde{A}(0) = 1, \quad \tilde{Z}(0) = 0, \quad \tilde{Z}'(0) = 1. \quad (38)$$

As $c \rightarrow \infty$, the solutions of equations (34)–(38) tend uniformly on compact intervals to solutions of the limiting equations

$$h'' + \frac{2}{x} h' - \alpha x h^3 + \frac{\sin(2h)}{ax^2} = 0, \quad (39)$$

$$a' = -2axah^2, \quad (40)$$

$$xz' = z(1 + axh^2), \quad (41)$$
with the constraint
\[ 1 - 2a - a + 2a \sin^2 h + ax^2 h^2 = 0, \quad (42) \]
and the same initial conditions at \( x = 0 \),
\[
h(0) = -\frac{\pi}{2}, \quad h'(0) = 1, \quad a(0) = 1, \quad z(0) = 0, \quad z'(0) = 1. \quad (43)
\]
We observe first that the function \( a(x) \) is monotone decreasing by (40) and bounded below, \( a > 1 - 2a \), by (42). Thus, no singularity can develop due to \( a \) going to zero. Also, by (42) no singularity can develop due to \( h' \) becoming unbounded. Thus, solutions exist for all \( x > 0 \) (assuming the existence of a solution for small \( x \)). In order to complete the proof it is sufficient to show that the function \( h(x) \) has an infinite number of zeros for \( x > 0 \). Since \( a < 1 \), it follows from (42) that \( -\pi/2 < h(x) < \pi/2 \) for all \( x > 0 \). To show that \( h(x) \) oscillates around zero we consider three cases:

(i) Assume that \( \lim_{x \to \infty} h(x) \) does not exist. Then, there must be a sequence \( y_k < x_k < y_{k+1} < \cdots \) such that \( h \) has a local minimum at \( x_k \) and a local maximum at \( y_k \). By (39), \( h'(x_k) = 0, h''(x_k) \geq 0 \) imply that \( \sin(2h(x_k)) \leq 0 \), hence \( h(x_k) \leq 0 \). By a similar argument, \( h(y_k) \geq 0 \). Thus, \( h(x) \) has a zero in each interval \( x_k < x < y_k \).

(ii) Assume that a nonzero \( \lim_{x \to \infty} h(x) \) exists. Then, from (42) \( \lim_{x \to \infty} x^2 h^2 \) exists and, in fact, equals zero because \( \lim_{x \to \infty} h(x) \) exists. This implies by (39) that \( \lim_{x \to \infty} x^2 h'(x) = -\sin(2h(\infty))/A(\infty) \neq 0 \), hence \( \lim_{x \to \infty} x^2 h'(x) \neq 0 \). Thus case (ii) does not arise.

(iii) Assume that \( \lim_{x \to \infty} h(x) = 0 \). We define the rotation function \( \theta(x) \) by

\[
\tan \theta(x) = \frac{x h'(x)}{h(x)}, \quad \theta(0) = 0. \quad (44)
\]

**Remark 1.** The rotation function \( \theta(x) \) is well defined because the case \( h(x) = h'(x) = 0 \) is impossible for solutions satisfying the initial conditions (43). To see this, assume that \( h(x_0) = h'(x_0) = 0 \) for some \( x_0 > 0 \). Then, by (42) \( a(x_0) = 1 - 2a \) and the unique solution with these initial conditions at \( x_0 \) is \( h(x) = 0, a(x) = 1 - 2a \) for all \( x \), contradicting the initial conditions (43).

We want to show that \( \lim_{x \to \infty} \theta(x) = -\infty \). Using (39) we obtain

\[
x \theta'(x) = -\sin^2 \theta - \frac{\sin 2h}{2h} \frac{2 \cos^2 \theta}{a} - \frac{(1 - 2a \cos^2 h) \sin \theta \cos \theta}{a} \quad (45)
\]

Under the assumption \( \lim_{x \to \infty} h(x) = 0 \), it follows from (42) that \( \lim_{x \to \infty} a(x) = 1 - 2a \), hence for sufficiently large \( x \)

\[
\theta'(x) \approx -\frac{1}{x} \left( \sin^2 \theta + \sin \theta \cos \theta + \frac{2 \cos^2 \theta}{1 - 2a} \right) < -\frac{3}{4x}. \quad (46)
\]

so \( \lim_{x \to \infty} \theta(x) = -\infty \). Thus, given any integer \( k \) there exists an \( x_k \) such that \( h(x) \) has at least \( k \) zeros for \( x < x_k \). By continuous dependence on initial conditions, we may choose \( c > x_k / \sqrt{1 - 2a} \) so that the \( c \)-solution has \( k \) zeros also for \( x < x_k \). In terms of the variable \( \rho = x/c \) the \( c \)-solution has \( k \) zeros for \( \rho < \sqrt{1 - 2a} < \rho_1(c) \). This completes the proof of proposition 7.

Next, we need two lemmas which tell us how the number of nodes \( N(c) \) changes under small variations of \( c \).

**Lemma 8.** If \( \hat{H}(\hat{c}) = 0 \), then \( N(c) = N(\hat{c}) \) or \( N(c) = N(\hat{c}) + 1 \) for \( c \) sufficiently close to \( \hat{c} \).
Proof. First note that if $H(\rho, \tilde{c})$ has a zero at some $\rho_0 < \rho_1(\tilde{c})$, then $H'(\rho_0, \tilde{c}) \neq 0$ (see remark 1), so by continuity of $H(\rho, c)$ with respect to $c$, $H(\rho, c)$ also has a zero if $c$ is sufficiently close to $\tilde{c}$. Thus $N(c) \geq N(\tilde{c})$ and it suffices to show that $N(c) \leq N(\tilde{c}) + 1$.

Let $\tilde{\alpha} < \rho_1(\tilde{c})$ be the last node of the $\tilde{c}$-orbit, that is $H(\tilde{\alpha}, \tilde{c}) = 0$ and, for concreteness, $H(\rho, \tilde{c}) < 0$ for $\tilde{\alpha} < \rho < \rho_1$. By continuity with respect to $c$, $H(\rho, c)$ will also have a zero at $\tilde{\alpha}$ near $\tilde{c}$ if $c$ is near $\tilde{c}$. In order to prove that $H(\rho, c)$ cannot have more than one zero in the interval $a < \rho < \rho_1(c)$, we now show that if $H(\rho, c)$ becomes positive for some $\rho > a$, then it would not have time to change the sign again before going singular. Assume for contradiction that there is a segment $a < \rho_N < \rho \leq \rho_D$ of the $\tilde{c}$-orbit in which the function $H(\rho)$ is monotone decreasing from a local maximum $H(\rho_N) > 0$ to $H(\rho_D) = 0$.

We define

$$W = \frac{1}{2} \rho^2 A H''(1 - Z^2) + \sin^2 H.$$  \hfill(47)

From (24) $W = (A - 1 + 2a)/(2a)$, hence by (22) $W' < 0$. We have

$$\frac{H'}{W - \sin^2 H} = \frac{2}{\rho^2 A (1 - Z^2)}.$$  \hfill(48)

Integrating the left-hand side from $\rho_D$ to $\rho_N$, we get (using $H_N = H(\rho_N)$)

$$\int_{\rho_N}^{\rho_D} \frac{-H'}{W - \sin^2 H} \, d\rho = \int_0^{H_N} \frac{dH}{\sqrt{W - \sin^2 H}} \geq \int_0^{\sqrt{2}} \frac{dH}{\sqrt{\sin^2 H_N - \sin^2 H}} > \frac{\pi}{2},$$  \hfill(49)

where the first inequality follows from $W(\rho) < W(\rho_N) = \sin^2 H_N$ (since $W'$ decreases) and the second inequality is a simple calculation using a substitution $\sin H = u$ sin $H_N$ (remember that $H_N < \pi/2$).

Next, we derive an upper bound for the integral of the right-hand side of (48). We have

$$\int_{\rho_N}^{\rho_D} \frac{d\rho}{\rho \sqrt{A (1 - Z^2)}} \leq \frac{1}{\rho_N \sqrt{1 - 2a}} \int_{\rho_N}^{\rho_D} \frac{d\rho}{\sqrt{1 - Z^2}} \leq \frac{1}{\rho_N \sqrt{1 - 2a}} \int_{\rho_N}^{\rho_D} \frac{d\rho}{\sqrt{1 - Z}}.$$  \hfill(50)

We showed above that $Z' > 1$, hence $1 - Z \geq 1 - \rho$. Therefore

$$\int_{\rho_N}^{\rho_D} \frac{d\rho}{\sqrt{1 - Z}} \leq \int_{\rho_N}^{\rho_D} \frac{d\rho}{\sqrt{1 - 1 - \rho}} = 2(\rho_N - \rho_D - \rho_N - \rho_D) < 2(\rho_N - \rho_D).$$  \hfill(51)

By continuity of solutions with respect to $c$ and by corollary 3, $\rho_N$ is arbitrarily close to $\rho_1(c)$ if $c$ is sufficiently close to $\tilde{c}$, hence it follows from (51) that the integral of the right-hand side of (48) is arbitrarily small. This is in contradiction with (49), hence $H(\rho, c)$ cannot have a second additional zero, which completes the proof of lemma 8.

Lemma 9. If $\tilde{H}(\tilde{c}) \neq 0$, then $N(c) = N(\tilde{c})$ for $c$ sufficiently close to $\tilde{c}$.

Proof. Without the loss of generality, we assume that $\tilde{H}(\tilde{c}) < 0$. As above, let $\tilde{\alpha} < \rho_1(\tilde{c})$ be the last node of the $\tilde{c}$-orbit, that is $H(\tilde{\alpha}, \tilde{c}) = 0$ and $H(\rho, \tilde{c}) < 0$ for $\tilde{\alpha} < \rho < \rho_1$. Let $a$ be the corresponding zero of $H(\rho, c)$ for $c$ near $\tilde{c}$. We want to show that $H(\rho, c)$ cannot have an extra zero for $\rho > a$. Suppose for contradiction that $H(b, c) = 0$ for some $b > a$.

For $\tilde{H}(\tilde{c}) < 0$ we have $H'(\rho, \tilde{c}) > 0$ near $\rho_1(\tilde{c})$, so for solutions with $c$ sufficiently close to $\tilde{c}$ there must be a $\delta < b$ such that $H(\delta, c) = \tilde{H}(\tilde{c})$. Let us integrate the identity

$$\frac{H'}{\sqrt{W - \sin^2 H}} = \frac{\sqrt{2}}{\rho \sqrt{A (1 - Z^2)}}$$  \hfill(52)

from $\delta$ to $b$. For the left-hand side, we get

$$\int_{\delta}^{b} \frac{H'}{\sqrt{W - \sin^2 H}} \, d\rho = \int_{\delta}^{\tilde{H}} \frac{dH}{\sqrt{W - \sin^2 H}}$$  \hfill(53)
From proposition 2 we know that \( \lim_{\rho \to \rho_1} (1 - Z^2)H^2 = 0 \), so \( W(\rho, c) < (1 + \epsilon/2)\sin^2 \tilde{H} \) for \( \rho \) near \( \rho_1 \) and hence \( W(\rho, c) < (1 + \epsilon)\sin^2 \tilde{H} \) for \( c \) near \( \tilde{c} \). Since \( W \) is decreasing, \( W(\delta, c) < W(\rho, c) < (1 + \epsilon)\sin^2 \tilde{H} \) for \( (\delta, c) < (\rho, c) \). Hence the integral of the left-hand side of equation (52) is arbitrarily small. This contradicts (54) and completes the proof of lemma 9.

Now we are ready to make a shooting argument. We define a set
\[
C_0 = \{ c \mid N(c) = 0 \}
\]
and let \( c_0 = \sup C_0 \). The set \( C_0 \) is nonempty (by proposition 6) and bounded above (by proposition 7) so \( c_0 \) exists. We claim that the \( c_0 \)-orbit has no nodes and satisfies the boundary condition \( \hat{H}(c_0) = 0 \). To see this, note that the \( c_0 \)-orbit cannot have a node because then by lemmas 8 and 9 all nearby \( c \)-orbits would have a node, so there would be an interval around \( c_0 \) with no elements of \( C_0 \) in it, contradicting the assumption that \( c_0 \) is the least upper bound. Thus, \( N(c_0) = 0 \). Now, if \( \hat{H}(c_0) < 0 \), then by lemma 9 all nearby \( c \)-orbits would have no nodes, so there would be an interval around \( c_0 \) consisting of elements of \( C_0 \), contradicting the assumption that \( c_0 \) is an upper bound of \( C_0 \). Thus \( \hat{H}(c_0) = 0 \).

Next, we define \( C_1 = \{ c > c_0 \mid N(c) = 1 \} \). This set is nonempty by the previous step and lemma 8 and bounded above by proposition 7, hence \( c_1 = \sup C_1 \) exists. By the same argument as above, the \( c_1 \)-orbit has exactly one node and satisfies \( \hat{H}(c_1) = 0 \). The construction of subsequent \( c_n \)-orbits proceeds by induction.

3.1. Conclusion of the proof of theorem 1

Returning to the original variable \( F(\rho) \) and rescaling \( \rho \to \rho_1/c_n \) we get the solution of equations (13)–(17) which satisfies the boundary conditions (19) and (20) and has exactly \( n \) intersections with the line \( F = \pi/2 \). By corollary 5 this solution is analytic in the whole interval \( 0 \leq \rho \leq 1 \).

4. Beyond the past self-similarity horizon

In this section, we consider the behaviour of the CSS solutions of theorem 1 outside the past SSH; in particular, we ask the question: do these solutions possess a regular future self-similarity horizon? Note that \( \rho = \infty \) corresponds to the hypersurface \( (t = 0, r > 0) \) so in order to analyse the global behaviour of solutions (for \( t > 0 \)) we need to go 'beyond \( \rho = \infty \'). To this end, we define, after I, a new coordinate \( x \) by
\[
\frac{d}{dx} = \rho \frac{d}{d\rho}, \quad \rho(x = 1) = 0.
\]
We also define an auxiliary function \( w(x) = 1/Z(\rho) \). In these new variables, the past SSH where \( w = 1 \) is at \( x = 0 \), while the future SSH (if it exists) is at some \( x_A > 0 \) where \( w(x_A) = -1 \).

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Proof. It was shown in [2] (see lemma 4 in that reference) that for equations (57)–(61) for sufficiently small \(\alpha\) have the solutions to equations (57)–(60) has \(w(x_0) < 0\) and \(A(x_0) > 1/2\) for some \(x_0\), then there is \(x_A > x_0\) such that \(\lim_{x \to x_A} w(x) = -1\), i.e., the solution is of type A.

Proof. By (58) \(A\) is increasing for \(w < 0\). Thus, using equation (59) and the constraint (60) we get for \(x > x_0\)

\[
w' = -1 + \alpha(1 - w^3)H^2 = -1 + \frac{1 - A - 2\alpha \cos^2 H}{A} < -2 + \frac{1}{A(x_0)} < 0,
\]

hence \(w\) must hit \(-1\) for some finite \(x_A > x_0\).

Proposition 12. The \(c_n(\alpha)\)-orbits are of type A if \(\alpha\) is sufficiently small.

Proof. For \(\alpha = 0\) and any \(b\) we have \(w(x) = 1 - x\) and \(A(x) \equiv 1\); in particular, \(A(3/2) = 1 > 1/2\) and \(w(3/2) = -1/2 < 0\). By continuous dependence on initial conditions, there exists a \(\delta(b)\) such that such that the solutions with \(\alpha < \delta(b)\) and \(|b - b'| < \delta(b)\), then \(A(3/2, b') > 1/2\) and \(w(3/2, b') < 0\). This implies by lemma 11 that the solutions corresponding to such values of \(\alpha\) and \(b'\) are of type A. By a standard theorem of advanced calculus, there is a \(\delta' > 0\) (independent of \(b\)) such that the solutions with \(\alpha < \delta'\) and \(|b| \leq B\) are of type A. By lemma 10 any \(c_n\)-orbit has \(|b| \leq B\), so for \(\alpha < \delta'\) the \(c_n\)-orbits are of type A.
By a similar argument as in the proof of proposition 2, one can easily show that the type A solutions are generically only \( C^0 \) at the future SSH (for isolated values of \( \alpha \) there are solutions that go smoothly through the future SSH). In I we showed that the leading-order asymptotic behaviour at the future SSH is (using \( y = x_A - x \))

\[
\begin{align*}
w & \sim -1 + y, & A & \sim A_0 - 2\alpha A_0 C^2 y \ln^2(y), & H & \sim H_0 - C y \ln(y),
\end{align*}
\]

where \( A_0 = 1 - 2\alpha \cos^2 H_0, \ C = \sin(2H_0)/2A_0 \) and \( H_0 \) is a free parameter. Using this expansion, one can check that the curvature is finite as \( y \to 0 \) which means that the type A solutions are examples of naked singularities.

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Appendix (local existence theorems)

In [5] (proposition 1) Breitelohner, Forgács and Maison derived the following result concerning the behaviour of solutions of a system of ordinary differential equations near a singular point (see also [6] for a similar result).

**Theorem** (BFM). Consider a system of first-order differential equations for \( n+m \) functions \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_m) \),

\[
\begin{align*}
t \frac{du_i}{dt} &= t^{\mu_i} f_i(t, u, v), & t \frac{dv_i}{dt} &= -\lambda_i v_i + t^{\nu_i} g_i(t, u, v),
\end{align*}
\]

where constants \( \lambda_i > 0 \) and integers \( \mu_i, \nu_i \geq 1 \) and let \( C \) be an open subset of \( \mathbb{R}^n \) such that the functions \( f \) and \( g \) are analytic in the neighbourhood of \( t = 0, u = c, v = 0 \) for all \( c \in C \). Then there exists an \( n \)-parameter family of solutions of the system (64) such that

\[
\begin{align*}
u_i(t) &= c_i + O(t^{\mu_i}), & v_i(t) &= O(t^{\nu_i}),
\end{align*}
\]

where \( u_i(t) \) and \( v_i(t) \) are defined for all \( c \in C, |t| < t_0(c) \) and are analytic in \( t \) and \( c \).

We shall use this theorem to prove existence of local solutions of equations (21)–(23) near the singular points \( \rho = 0 \) and \( \rho = 1 \).

**Proposition 13.** Equations (21)–(23) admit a two-parameter family of local solutions near \( \rho = 0 \),

\[
\begin{align*}
H(\rho) &= -\frac{\pi}{2} + c\rho + O(\rho^3),
A(\rho) &= 1 - \alpha c^2 \rho^2 + O(\rho^4),
Z(\rho) &= d\rho + O(\rho^3),
\end{align*}
\]

which are analytic in \( c, d \) and \( \rho \).

**Proof.** Using the variables

\[
\begin{align*}
w_1 &= \frac{H + \pi/2}{\rho}, & w_2 &= H', & w_3 &= \frac{1 - A}{\rho^2}, & w_4 &= \frac{Z}{\rho}
\end{align*}
\]

(69)
we rewrite equations (21)–(23) as the first-order system
\[
\begin{align*}
\rho w'_1 &= -w_1 + w_2, & \rho w'_2 &= 2w_1 - 2w_2 + \rho^2 h_1, \\
\rho w'_3 &= -2w_3 + 2\alpha w_2 + \rho^2 h_2, & \rho w'_4 &= \rho^2 h_3,
\end{align*}
\]
where the functions \(h_i\) are analytic near \(\rho = 0\). Next, substituting
\[
\begin{align*}
w'_1 &= u_1 - v_1, & w'_2 &= u_1 + 2v_1, \\
w'_3 &= v_2 + \alpha(u_1^2 - 2v_1^2 - 8u_1v_1), & w'_4 &= u_2
\end{align*}
\]
we put (70) into the form (64)
\[
\begin{align*}
\rho u'_1 &= \rho^2 f_1, & \rho u'_2 &= \rho^2 f_2, \\
\rho v'_1 &= -3v_1 + \rho^2 g_1, & \rho v'_2 &= -2v_2 + \rho^2 g_2,
\end{align*}
\]
where the functions \(f_i, g_i\) are analytic in an open neighbourhood of \(\rho = 0\), \(u_1 = c\), \(u_2 = d\), \(v_i = 0\) for any \(c\) and \(d\). Thus, according to the BFM theorem, there exists a two-parameter family of solutions such that
\[
\begin{align*}
u_1 &= O(\rho^2), & v_2 &= O(\rho^2),
\end{align*}
\]
which is equivalent to (66)–(68). □

**Proposition 14.** Equations (21)–(23) admit a one-parameter family of local solutions near \(\rho = 1\).
\[
\begin{align*}
H(\rho) &= b(\rho - 1) + O((\rho - 1)^2), \\
A(\rho) &= 1 - 2\alpha - 2\alpha(1 - 2\alpha)b^2(\rho - 1) + O((\rho - 1)^2), \\
Z(\rho) &= \rho + O((\rho - 1)^2)
\end{align*}
\]
which are analytic in \(b\) and \(\rho\).

**Proof.** We define the variables
\[
\begin{align*}
u_1 &= \frac{H'}{\rho - 1} - H', \\
\rho v_2 &= (1 - 2\alpha) - A - 2\alpha(1 - 2\alpha)H^2, & v_3 &= \frac{Z - 1}{\rho - 1} - 1.
\end{align*}
\]
Then, equations (21)–(23) take the form (using \(t = \rho - 1\))
\[
\begin{align*}
tu' &= tf, & tv'_i &= -v_i + tg_i,
\end{align*}
\]
where the functions \(f\) and \(g_i\) are analytic in an open neighbourhood of \(t = 0\), \(u = b\), \(v_i = 0\) for any \(b > 0\). Thus, according to the BFM theorem, there exists a one-parameter family of solutions such that
\[
\begin{align*}
u(t) &= b + O(t), & v_i(t) &= O(t),
\end{align*}
\]
which is equivalent to (75)–(77). □

**References**

[1] Bizoñ P and Wasserman A 2000 Self-similar spherically symmetric wave maps coupled to gravity *Phys. Rev. D* 62 084031

[2] Bizoñ P 2000 Equivariant self-similar wave maps from Minkowski spacetime into 3-sphere *Commun. Math. Phys.* 215 45
On the existence of self-similar spherically symmetric wave maps coupled to gravity

[3] Lechner C, Thomburg J, Husa S and Aichelburg P C 2001 A new transition between discrete and continuous self-similarity in critical gravitational collapse Preprint gr-qc/0112008

[4] Lechner C 2001 PhD Thesis University of Vienna

[5] Breitenlohner P, Forgács P and Maison D 1994 On static spherically symmetric solutions of the Einstein–Yang–Mills equations Commun. Math. Phys. 163 141

[6] Rendall A D and Schmidt B G 1991 Existence and properties of spherically symmetric static fluid bodies with a given equation of state Class. Quantum Grav. 8 985