Families of solution curves for some non-autonomous problems

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Abstract
The paper studies families of positive solution curves for non-autonomous two-point problems

\[ u'' + \lambda f(u) - \mu g(x) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0, \]

depending on two positive parameters \( \lambda \) and \( \mu \). We regard \( \lambda \) as a primary parameter, giving us the solution curves, while the secondary parameter \( \mu \) allows for evolution of these curves. We give conditions under which the solution curves do not intersect, and the maximum value of solutions provides a global parameter. Our primary application is to constant yield harvesting for diffusive logistic equation. We implement numerical computations of the solution curves, using continuation in a global parameter, a technique that we developed in [11].

Key words: Families of solution curves, fishing models, numerical computations.

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1 Introduction

We study positive solutions of non-autonomous two-point problems

\[ u'' + \lambda f(u) - \mu g(x) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0, \]

depending on two positive parameters \( \lambda \) and \( \mu \). We assume that \( f(u) \in C^2(\mathbb{R}_+) \), and \( g(x) \in C^1(-1,1) \cap C[-1,1] \) satisfies

\[ g(-x) = g(x), \quad \text{for } x \in (0,1), \]
In case $g(x)$ is a constant, one can use the time map method, see K.C. Hung and S.H. Wang [7], [8] who have studied similar multiparameter problems, or the book by S.P. Hastings and J.B. McLeod [6]. We show that under the conditions (1.2) and (1.3) one can still get detailed results on the solution curves $u = u(x, \lambda)$, where we regard $\lambda$ as a primary parameter, and on the evolution of these curves when the secondary parameter $\mu$ changes. We say that the solution curves $u = u(x, \lambda)$ are the $\lambda$-curves. We also consider the $\mu$-curves, by regarding $\lambda$ as the secondary parameter.

By B. Gidas, W.-M. Ni and L. Nirenberg [4], any positive solution of (1.1) is an even function, and moreover $u'(x) < 0$ for $x \in (0, 1)$. It follows that $u(0)$ is the maximum value of the solution $u(x)$. Our first result says that $u(0)$ is a global parameter, i.e., its value uniquely determines the solution pair $(\lambda, u(x))$ ($\mu$ is assumed to be fixed). It follows that a planar curve $(\lambda, u(0))$ gives a faithful representation of the solution set of (1.1), so that $(\lambda, u(0))$ describes the global solution curve. Then we show positivity of any non-trivial solution of the linearized problem for (1.1). This allows us to compute the direction of turn for convex and concave $f(u)$.

Turning to the secondary parameter $\mu$, we show that solution curves at different $\mu$'s do not intersect, which allows us to discuss the evolution of solution curves in $\mu$.

We apply our results to a logistic model with fishing. S. Oruganti, J. Shi, and R. Shivaji [16] considered a class of general elliptic equations on an arbitrary domain, which includes

$$u'' + \lambda u(1 - u) - \mu g(x) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0.$$ 

They proved that for $\lambda$ sufficiently close to the principal eigenvalue $\lambda_1$, the $\mu$-curves are as in the Figure 2 below. We show that such curves are rather special, with the solution curves as in the Figure 3 below being more common. Our approach is to study the $\lambda$-curves first, leading to the understanding of the $\mu$-curves. We obtain an exhaustive result in case $g(x)$ is a constant. The parameter $\mu > 0$ quantifies the amount of fishing in the logistic model. We also consider the case $\mu < 0$, corresponding to “stocking” of fish.

Using the fact that $u(0)$ is a global parameter, we implement numerical computations of the solution curves, illustrating our results. We use continuation in a global parameter, a technique that we developed in [11].
2 Families of solution curves

The following result is included in B. Gidas, W.-M. Ni and L. Nirenberg [4], see also P. Korman [9] for an elementary proof.

Lemma 2.1 Under the conditions (1.2) and (1.3), any positive solution of (1.1) is an even function, with \( u'(x) < 0 \) for all \( x \in (0, 1] \), so that \( x = 0 \) is a point of global maximum.

We begin by considering the secondary parameter \( \mu \) to be fixed. To stress that, we call \( h(x) = \mu g(x) \), and consider positive solutions of

\[
2.1 \quad u'' + \lambda f(u) - h(x) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0.
\]

Lemma 2.2 Assume that \( f(u) \in C(\bar{R}_+) \) satisfies \( f(u) > 0 \) for \( u > 0 \), and \( h(x) \) satisfies the conditions (1.2) and (1.3). Then \( u(0) \), the maximum value of any positive solution, uniquely identifies the solution pair \( (\lambda, u(x)) \).

Proof: Observe from (2.1) that \( f(u(0)) > 0 \) for any positive solution \( u(x) \). Let \( (\lambda_1, v(x)) \) be another solution of (2.1), with \( v(0) = u(0), v'(0) = u'(0) = 0 \), and \( \lambda_1 > \lambda \). From the equation (2.1), \( v''(0) < u''(0) \), and hence \( v(x) < u(x) \) for small \( x > 0 \). Let \( \xi \leq 1 \) be their first point of intersection, i.e., \( u(\xi) = v(\xi) \).

Clearly

\[
2.2 \quad |v'(\xi)| \leq |u'(\xi)|.
\]

Multiplying the equation (2.1) by \( u' \), and integrating over \((0, \xi)\), we get

\[
\frac{1}{2} u^2(\xi) + \lambda [F(u(\xi)) - F(u(0))] - \int_0^\xi h(x)u'(x)dx = 0,
\]

where \( F(u) = \int_0^u f(t)dt \). Integrating by parts, we conclude

\[
\frac{1}{2} u^2(\xi) = \lambda [F(u(0)) - F(u(\xi))] + h(\xi)u(\xi) - h(0)u(0) - \int_0^\xi h'(x)u(x)dx.
\]

Similarly,

\[
\frac{1}{2} v^2(\xi) = \lambda_1 [F(u(0)) - F(u(\xi))] + h(\xi)v(\xi) - h(0)v(0) - \int_0^\xi h'(x)v(x)dx,
\]

and then, subtracting,

\[
\frac{1}{2} [u^2(\xi) - v^2(\xi)] = (\lambda - \lambda_1) [F(u(0)) - F(u(\xi))] + \int_0^\xi h'(x)(v(x) - u(x))dx.
\]
Since \( v(x) < u(x) \) on \((0, \xi)\), the second term on the right is non-positive, while the first term on the right is negative, since \( F(u) \) is an increasing function. It follows that \(|u'((\xi))| < |v'((\xi))|\), which contradicts (2.2). 

Lemma 2.3 Assume that \( f(u) \in C(\bar{\mathbb{R}}_+) \), and \( h(x) \) satisfies the conditions (1.2) and (1.3). Then the curves of positive solutions of (1.1) in \((\lambda, u(0))\) plane, computed at different \( \mu \)'s, do not intersect.

Proof: Assume, on the contrary, that \( v(x) \) is a solution of
\[
(2.3) \quad v'' + \lambda f(v) - \mu_1 g(x) = 0, \quad -1 < x < 1, \quad v(-1) = v(1) = 0,
\]
with \( \mu_1 > \mu \), but \( u(0) = v(0) \), where \( u(x) \) is a solution of (1.1). Then \( u''(0) < v''(0) \), and hence \( u(x) < v(x) \) for small \( x > 0 \). Let \( \xi \leq 1 \) be their first point of intersection, i.e., \( u(\xi) = v(\xi) \). Clearly
\[
(2.4) \quad |u'(\xi)| \leq |v'(\xi)|.
\]

Multiplying the equation (1.1) by \( u' \), and integrating over \((0, \xi)\), we get
\[
\frac{1}{2} u'^2(\xi) + \lambda [F(u(\xi)) - F(u(0))] = \mu \int_0^\xi g(x) u'(x) \, dx.
\]
Similarly, using (2.3), we get
\[
\frac{1}{2} v'^2(\xi) + \lambda [F(u(\xi)) - F(u(0))] = \mu_1 \int_0^\xi g(x) v'(x) \, dx.
\]
Subtracting, we obtain
\[
\frac{1}{2} \left[ u'^2(\xi) - v'^2(\xi) \right] = \mu \int_0^\xi g(x) u'(x) \, dx - \mu_1 \int_0^\xi g(x) v'(x) \, dx > \mu_1 \left[ \int_0^\xi g(x) u'(x) \, dx - \int_0^\xi g(x) v'(x) \, dx \right] = \mu_1 \int_0^\xi g'(x) (v(x) - u(x)) \, dx > 0.
\]
Hence, \(|u'((\xi))| > |v'((\xi))|\), which contradicts (2.4). 

Corollary 1 Assume that \( \lambda \) is fixed in (1.1), and \( \mu \) is the primary parameter. Assume that \( f(u) \in C(\bar{\mathbb{R}}_+) \) satisfies \( f(u) > 0 \) for \( u > 0 \), and \( g(x) \) satisfies the conditions (1.2) and (1.3). Then the maximum value of solution \( u(0) \) is a global parameter, i.e., it uniquely identifies the solution pair \((\mu, u(x))\).
Proof: If at some $\lambda_0$ we had another solution pair $(\mu_1, u_1(x))$ with $u(0) = u_1(0)$, then the $\lambda$-curves at $\mu$ and $\mu_1$ would intersect at $(\lambda_0, u(0))$, contradicting Lemma 2.2.

The linearized problem for (1.1) is

\begin{equation}
(2.5) \quad u'' + \lambda f'(u)u = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0.
\end{equation}

We call the solution of (1.1) singular if (2.5) has non-trivial solutions. Since the solution set of (2.5) is one-dimensional (parameterized by $w'(-1)$), it follows that $w(-x) = w(x)$, and $w'(0) = 0$.

Lemma 2.4 Assume that $f(u) \in C^1(\mathbb{R}^+)$, and $g(x)$ satisfies the conditions (1.2) and (1.3), and let $u(x)$ be a positive solution of (1.1). Then any non-trivial solution of (2.5) is of one sign, i.e., we may assume that $w(x) > 0$ for all $x \in (-1, 1)$.

Proof: Assuming the contrary, we can find a point $\xi \in (0, 1)$ such that $w(\xi) = w(1) = 0$, and $w(x) > 0$ on $(\xi, 1)$. Differentiate the equation (1.1)

\begin{equation}
(2.6) \quad u''' + \lambda f'(u)u' - \mu g'(x) = 0.
\end{equation}

Combining this with (2.5),

\begin{equation}
(u'w' - u''w)' + \mu g'(x) = 0.
\end{equation}

Integrating over $(\xi, 1)$,

\begin{equation}
u'(1)w'(1) - u'(\xi)w'(\xi) + \mu \int_{\xi}^{1} g'(x) \, dx = 0.
\end{equation}

All three terms on the left are non-negative, and the second one is positive, which results in a contradiction.

Lemma 2.5 Assume that $f(u) \in C(\mathbb{R}^+)$, and $g(x)$ satisfies the conditions (1.2) and (1.3). Let $u(x)$ be a positive and singular solution of (1.1), with $u'(1) < 0$, and $w(x) > 0$ a solution of (2.5). Then

\begin{equation}
\int_{0}^{1} f(u) w \, dx > 0.
\end{equation}
Proof: By (2.6), the function \( u'w' - u''w \) is non-increasing on \((0,1)\), and hence

\[
u'w' - u''w \geq u'(1)w'(1)\,.
\]

Integrating over \((0,1)\), and expressing \( u'' \) from (1.1), gives

\[
2\int_0^1 w (\lambda f(u) - \mu g(x)) \, dx \geq u'(1)w'(1) > 0,
\]

which implies the lemma. \(\Box\)

Theorem 2.1 Assume that \( f(u) \in C^1(\mathbb{R}_+) \), and \( g(x) \) satisfies the conditions (1.2) and (1.3). Then positive solutions of the problem (1.1) can be continued globally either in \( \lambda \) or in \( \mu \), on smooth solution curves, so long as \( u'(1) < 0 \).

Proof: At any non-singular solution of (1.1), the implicit function theorem applies (see e.g., L. Nirenberg [13], or P. Korman [10] for more details), while at the singular solutions the Crandall-Rabinowitz [2] bifurcation theorem applies, with Lemma 2.5 verifying its crucial “transversality condition”, see e.g., P. Korman [10] (or [12], [15]) for more details. In either case we can always continue the solution curves. \(\Box\)

Theorem 2.2 (i) Assume that \( f(u) \in C^2[0,\infty) \) is concave. Then only turns to the right are possible in the \((\lambda, u(0))\) plane, when solutions are continued in \( \lambda \), and only turns to the left are possible in the \((\mu, u(0))\) plane, when solutions are continued in \( \mu \).

(ii) Assume that \( f(u) \in C^2[0,\infty) \) is convex. Then only turns to the left are possible in the \((\lambda, u(0))\) plane, when solutions are continued in \( \lambda \), and only turns to the right are possible in the \((\mu, u(0))\) plane, when solutions are continued in \( \mu \).

Proof: Assume that when continuing in \( \lambda \), we encounter a critical point \((\lambda_0, u_0)\), i.e., the problem (2.5) has a non-trivial solution \( w(x) > 0 \). By Lemma 2.5 the Crandall-Rabinowitz [2] bifurcation theorem applies. This theorem implies that the solution set near \((\lambda_0, u_0)\) is given by a curve \((\lambda(s), u(s))\) for \( s \in (-\delta, \delta) \), with \( \lambda(s) = \lambda_0 + \frac{1}{2} \lambda''(0)s^2 + o(s^2) \), and

\[
\lambda''(0) = -\lambda_0\frac{\int_{-1}^1 f''(u)w^3 \, dx}{\int_{-1}^1 f(u)w \, dx},
\]

6
see e.g., [10]. When \( f(u) \) is concave (convex), \( \lambda''(0) \) is positive (negative), and a turn to the right (left) occurs on the solution curve.

If a critical point \((\mu_0, u_0)\) is encountered when continuing in \( \mu \), the Crandall-Rabinowitz [2] bifurcation theorem implies that the solution set near \((\mu_0, u_0)\) is given by \((\mu(s), u(s))\) for \( s \in (-\delta, \delta) \), with \( \mu(s) = \mu_0 + \frac{1}{2} \mu''(0) s^2 + o(s^2) \), and

\[
\mu''(0) = \lambda \frac{\int_{-1}^{1} f''(u) w^3 \, dx}{\int_{-1}^{1} g(x) w \, dx},
\]

see e.g., [14]. When \( f(u) \) is concave (convex), \( \mu''(0) \) is negative (positive), and a turn to the left (right) occurs on the solution curve.

3 Numerical computation of the solution curves

In this section we present computations of the global curves of positive solutions for the problem (1.1), which are based on our paper [11]. We assume that the conditions of Lemma 2.2 hold, so that \( \alpha \equiv u(0) \) is a global parameter. We think of the parameter \( \mu \) as secondary, and we begin with the problem (2.1) (i.e., we set \( \mu g(x) = h(x) \)). Since any positive solution \( u(x) \) is an even function, we shall compute it on the half-interval \((0, 1)\), by solving

\[
(3.1) \quad u'' + \lambda f(u) - h(x) = 0 \quad \text{for } 0 < x < 1, \quad u'(0) = u(1) = 0.
\]

A standard approach to numerical computations involves continuation in \( \lambda \) by using the predictor-corrector methods, see e.g., E.L. Allgower and K. Georg [1]. These methods are well developed, but not easy to implement, because the solution curve \( u = u(x, \lambda) \) may consist of several parts, each having multiple turns. Here \( \lambda \) is a local parameter, but not a global one.

Since \( \alpha = u(0) \) is a global parameter, we shall compute the solution curve \((\lambda, u(0))\) of (3.1) in the form \( \lambda = \lambda(\alpha) \), with \( \alpha = u(0) \). If we solve the initial value problem

\[
(3.2) \quad u'' + \lambda f(u) - h(x) = 0, \quad u(0) = \alpha, \quad u'(0) = 0,
\]

then we need to find \( \lambda \), so that \( u(1) = 0 \), in order to obtain the solution of (3.1), with \( u(0) = \alpha \). Rewrite the equation (3.2) in the integral form

\[
u(x) = \alpha - \lambda \int_{0}^{x} (x - t) f(u(t)) \, dt + \int_{0}^{x} (x - t) h(t) \, dt,
\]

7
and then the equation for $\lambda$ is

\[(3.3) \quad F(\lambda) \equiv u(1) = \alpha - \lambda \int_0^1 (1-t)f(u(t)) \, dt + \int_0^1 (1-t)h(t) \, dt = 0.\]

We solve this equation by using Newton’s method

\[
\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)}.
\]

We have

\[
F(\lambda_n) = \alpha - \lambda_n \int_0^1 (1-t)f(u(t, \lambda_n)) \, dt + \int_0^1 (1-t)h(t) \, dt,
\]

\[
F'(\lambda_n) = -\int_0^1 (1-t)f(u(t, \lambda_n)) \, dt - \lambda_n \int_0^1 (1-t)f'(u(t, \lambda_n))u \, dt,
\]

where $u(x, \lambda_n)$ and $u_\lambda$ are respectively the solutions of

\[(3.4) \quad u''(x) + \lambda_n f(u) - h(x) = 0, \quad u(0) = \alpha, \quad u'(0) = 0,\]

\[(3.5) \quad u''(x) + \lambda_n f'(u(x, \lambda_n))u + f(u(x, \lambda_n)) = 0, \quad u(0) = 0, \quad u'_0(0) = 0.\]

(As we vary $\lambda$, we keep $u(0) = \alpha$ fixed, that is why $u_\lambda(0) = 0$.) This method is very easy to implement. It requires repeated solutions of the initial value problems (3.4) and (3.5) (using the NDSolve command in Mathematica).

**Example** Using Mathematica software, we have computed the solution curves in $(\lambda, u(0))$ plane for the problem

\[(3.6) \quad u'' + \lambda(u(10 - 2u) - \mu(1 + 0.2x^2)) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,\]

at $\mu = 0.9$, $\mu = 1.5$ and $\mu = 2.2$. Results are presented in the Figure 1. (The curve in the middle corresponds to $\mu = 1.5$.)

We now discuss numerical continuation of solutions in the secondary parameter. We consider again

\[(3.7) \quad u'' + \lambda f(u) - \mu g(x) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,\]

and assume $\alpha$ to be fixed, and we continue the solutions in $\mu$. By the Corollary \[\alpha = u(0)\] is a global parameter, and we shall compute the solution curve $(\mu, u(0))$ of (3.1) in the form $\mu = \mu(\alpha)$. As before, to find $\mu = \mu(\alpha)$, we need to solve

\[
F(\mu) \equiv u(1) = \alpha - \lambda \int_0^1 (1-t)f(u(t)) \, dt + \mu \int_0^1 (1-t)g(t) \, dt = 0.
\]
Figure 1: The curve of positive solutions for the problem (3.6) at \( \mu = 0.9 \), \( \mu = 1.5 \) and \( \mu = 2.2 \).

We solve this equation by using Newton’s method

\[
\mu_{n+1} = \mu_n - \frac{F(\mu_n)}{F'(\mu_n)},
\]

with

\[
F(\mu_n) = \alpha - \lambda \int_0^1 (1-t)f(u(t,\mu_n)) \, dt + \mu_n \int_0^1 (1-t)g(t) \, dt,
\]

\[
F'(\mu_n) = -\lambda \int_0^1 (1-t)f'(u(t,\mu_n))u_{\mu} \, dt + \int_0^1 (1-t)g(t) \, dt,
\]

where \( u(x,\mu_n) \) and \( u_{\mu} \) are respectively the solutions of

\[
\begin{align*}
\frac{d^2u}{dx^2} + \lambda f(u) - \mu_n g(x) &= 0, \quad u(0) = \alpha, \quad u'(0) = 0, \\
\frac{d^2u}{dx^2} + \lambda f'(u(x,\mu_n))u_{\mu} - g(x) &= 0, \quad u_{\mu}(0) = 0, \quad u_{\mu}'(0) = 0.
\end{align*}
\]

(As we vary \( \mu \), we keep \( u(0) = \alpha \) fixed, that is why \( u_{\mu}(0) = 0 \).)

**Example** We have continued in \( \mu \) the positive solutions of

\[
\frac{d^2u}{dx^2} + u(4 - u) - \mu(1 + x^2) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0.
\]

The curve of positive solutions is given in Figure 2.
Example We have continued in $\mu$ the positive solutions of

$$u'' + 2.4u(4 - u) - \mu(1 + x^2) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0.$$  

The curve of positive solutions is given in Figure 3. At $\mu \approx 2.28634$, the solutions become sign changing, negative near $x = \pm 1$.

4 Diffusive logistic equation with harvesting

Recall that the eigenvalues of

$$u'' + \lambda u = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0$$

are $\lambda_n = \frac{n^2 \pi^2}{4}$, and in particular $\lambda_1 = \frac{\pi^2}{4}, \quad \lambda_2 = \pi^2$.

We consider positive solutions of

$$u'' + \lambda (1 - u) - \mu = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$
with positive parameters $\lambda$ and $\mu$. It is easy to see that no positive solutions exist if $\lambda \leq \lambda_1 = \frac{\pi^2}{4}$, and by maximum principle any positive solution satisfies $0 < u(x) < 1$. The following result gives a complete description of the set of positive solutions.

**Theorem 4.1** For any fixed $\mu$ the set of positive solutions of \([4.1]\) is a parabola-like curve opening to the right in $(\lambda, u(0))$ plane (the $\lambda$-curves). The upper branch continues for all $\lambda$ after the turn, while the solutions on the lower branch become sign-changing after some $\lambda = \bar{\lambda}$ ($u(x)(\pm 1, \bar{\lambda}) = 0$, see Figure 1). For any fixed $\lambda$ the set of positive solutions of \([4.1]\) is a parabola-like curve opening to the left in $(\mu, u(0))$ plane (the $\mu$-curves). Different $\lambda$-curves (and different $\mu$-curves) do not intersect. The $\lambda$-curves and the $\mu$-curves share the turning points. Namely, if at $\mu = \mu_0$, the $\lambda$-curve turns at the point $(\lambda_0, \alpha)$, then at $\lambda = \lambda_0$, the $\mu$-curve turns at the point $(\mu_0, \alpha)$.

If $\lambda \in (\lambda_1, \lambda_2]$, then the $\mu$-curve joins the point $(0, \mu_1)$, with some $\mu_1 > 0$, to $(0, 0)$, with exactly one turn to the left at some $\mu_0$ (as in Figure 2). If $\lambda > \lambda_2$, then the $\lambda$-curve joins the point $(0, \mu_1)$, with some $\mu_1 > 0$, to some point $(\bar{\mu} > 0, \alpha > 0)$, with exactly one turn to the left at some $\mu_0 > \bar{\mu}$ (as in Figure 3). Solutions on the lower branch become sign-changing for $\mu < \bar{\mu}$.

We remark that in case $\lambda \in (\lambda_1, \lambda_2]$, a more general result was given in J. Shi [16], by a more involved method.

The proof will depend on several lemmas, which we state for a more general problem

\[(4.2) \quad u'' + \lambda f(u) - \mu = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0.\]

**Lemma 4.1** Assume that $f(u) \in C^1(\mathbb{R}_+)$ satisfies $f(0) = 0$, and $f(u(x)) > 0$ for any positive solution of \([4.2]\), for all $x \in (-1, 1)$. Assume that $u(x, \lambda)$ arrives at the point $\lambda_0$ where the positivity of solutions is lost (i.e., $u_x(\pm 1, \lambda_0) = 0$) with the maximum value $u(0, \lambda)$ decreasing along the solution curve. Then the positivity is lost forward in $\lambda$ at $\lambda_0$. (I.e., $u(x, \lambda) > 0$ for all $x \in (-1, 1)$ if $\lambda < \lambda_0$, and $u(x, \lambda)$ is sign-changing for $\lambda > \lambda_0$.)

**Proof:** Since for positive solutions $u_x(x, \lambda_0) < 0$ for $x \in (0, 1)$, the only way for solutions to become sign-changing is to have $u_x(\pm 1, \lambda_0) = 0$. Assume, on the contrary, that positivity is lost backward in $\lambda$. Then by our assumption

\[(4.3) \quad u_\lambda(0, \lambda_0) \geq 0.\]

Differentiating the equation \([4.2]\) in $\lambda$, we have

\[(4.4) \quad u''_\lambda + \lambda f'(u)u_\lambda = -f(u), \quad -1 < x < 1, \quad u_\lambda(-1) = u_\lambda(1) = 0.\]
Differentiating the equation (4.2) in $x$, gives

$$u_x'' + \lambda f'(u)u_x = 0.$$ \hfill (4.5)

Combining the equations (4.4) and (4.5),

$$(u_x' - u_x u_x'')' = -f(u)u_x > 0, \text{ for } x \in (0,1).$$

It follows that the function $q(x) \equiv u_x' u_x' - u_x u_x''$ is increasing, with $q(0) \geq 0$ by (4.3), and $q(1) = 0$, a contradiction. \hfill ♦

**Lemma 4.2** Assume that $f(u) \in C^1(\overline{R}_+)$ satisfies $f(0) = 0$, and $f(u(x)) > 0$ for any positive solution of (4.2), for all $x \in (-1,1)$. Assume that $u(x,\mu)$ arrives at the point $\mu_0$ where the positivity is lost (i.e., $u_x(\pm1,\mu_0) = 0$) with the maximum value $u(0,\mu)$ decreasing along the solution curve. Then the positivity is lost backward in $\mu$ at $\mu_0$. (I.e., $u(x,\mu) > 0$ for all $x \in (-1,1)$ if $\mu > \mu_0$, and $u(x,\mu)$ is sign-changing for $\mu < \mu_0$.)

**Proof:** Assume, on the contrary, that positivity is lost forward in $\mu$. Then by our assumption

$$u_\mu(0,\mu_0) \leq 0.$$ \hfill (4.6)

Differentiating the equation (4.2) in $\mu$, we have

$$u_\mu'' + \lambda f'(u)u_\mu = 1, \quad -1 < x < 1, \quad u_\mu(-1) = u_\mu(1) = 0.$$ \hfill (4.7)

Combining the equations (4.5) and (4.7),

$$(u_\mu' u_\mu'' - u_\mu u_\mu')' = u_x < 0, \text{ for } x \in (0,1).$$

It follows that the function $r(x) \equiv u_\mu' u_\mu' - u_\mu u_\mu''$ is decreasing, with $r(0) \leq 0$ by (4.6), and $r(1) = 0$, a contradiction. \hfill ♦

**Lemma 4.3** Assume that $f(u) \in C^1[0,\infty)$, $f(0) = 0$, and $f'(u)$ is a decreasing function for $u > 0$. Then for any $\lambda > 0$ there is at most one solution pair $(\mu, u(x))$, with $u'(\pm1) = 0$.

**Proof:** Assume, on the contrary, that there are two solution pairs $(\mu_1, u(x))$ and $(\mu_2, v(x))$, with $\mu_2 > \mu_1$, satisfying

$$u'' + \lambda f(u) - \mu_1 = 0, \quad -1 < x < 1, \quad u(\pm1) = u'(\pm1) = 0.$$ \hfill (4.8)
(4.9) \[ v'' + \lambda f(v) - \mu_2 = 0, \quad -1 < x < 1, \quad v(\pm 1) = v'(\pm 1) = 0. \]

Since \( v''(1) = \mu_2 > \mu_1 = u''(1) \), we have \( v(x) > u(x) \) for \( x \) close to 1. Two cases are possible.

(i) \( v(x) > u(x) \) for \( x \in (0, 1) \). Differentiating the equations (4.8) and (4.9), we get

\[
\begin{align*}
    u'' + \lambda f'(u)u_x &= 0, \quad u_x < 0 \text{ on } (0,1), \quad u_x(0) = u_x(1) = 0, \\
v'' + \lambda f'(v)v_x &= 0, \quad v_x < 0 \text{ on } (0,1), \quad v_x(0) = v_x(1) = 0.
\end{align*}
\]

Since \( f'(u) > f'(v) \), we have a contradiction by Sturm’s comparison theorem.

(ii) There is \( \xi \in (0, 1) \) such that \( v(x) > u(x) \) for \( x \in (\xi, 1) \), while \( v(\xi) = u(\xi) \) and \( u'(\xi) = v'(\xi) < 0 \). Multiply the equation (4.8) by \( u' \), and integrate over \((\xi, 1)\) (with \( F(u) = \int_0^u f(t) \, dt \))

\[ -\frac{1}{2} u'^2(\xi) - \lambda F(u(\xi)) + \mu_1 u(\xi) = 0. \]

Similarly, from (4.9)

\[ -\frac{1}{2} v'^2(\xi) - \lambda F(u(\xi)) + \mu_2 u(\xi) = 0. \]

Subtracting

\[ (\mu_2 - \mu_1) u(\xi) = \frac{1}{2} \left( v'^2(\xi) - u'^2(\xi) \right). \]

The quantity on the left is positive, while the one on the right is non-positive, a contradiction.

\[ \blacklozenge \]

**Lemma 4.4** For any \( \alpha \in (0, \frac{3}{4}) \) there exists a unique pair \((\bar{\lambda}, \bar{\mu})\), with \( \bar{\lambda} > \lambda_2 \) and \( \bar{\mu} > 0 \), and a positive solution of (4.1) with \( u(0) = \alpha \) and \( u'(\pm 1) = 0 \). Moreover, if \( \bar{\mu} \to 0 \), then \( \bar{\lambda} \downarrow \lambda_2 = \pi^2 \).

**Proof:** Multiplying the equation (4.1) by \( u' \), we see that the solution with \( u'(\pm 1) = 0 \) satisfies

\[ \frac{1}{2} u'^2 + \bar{\lambda} \left( \frac{1}{2} u'^2 - \frac{1}{3} u^3 \right) - \bar{\mu} u = 0. \]

Evaluating this at \( x = 0 \)

\[ \bar{\lambda} \left( \frac{1}{2} \alpha - \frac{1}{3} \alpha^2 \right) = \bar{\mu}. \]

13
We also express from (4.10)
\[
\frac{du}{dx} = -\sqrt{2\bar{\mu}u - \bar{\lambda} \left( u^2 - \frac{2}{3}u^3 \right)}, \quad \text{for } x \in (0, 1).
\]

We express \(\bar{\mu}\) from (4.11), separate the variables and integrate, getting
\[
\int_0^\alpha \frac{du}{\sqrt{\left( \alpha - \frac{2}{3}\alpha^2 \right) u - (u^2 - \frac{2}{3}u^3)}} = \sqrt{\bar{\lambda}}.
\]

Setting here \(u = \alpha v\), we express \(\bar{\lambda}\)

\[
(4.12) \quad \bar{\lambda} = \left( \int_0^1 \frac{dv}{\sqrt{\left( 1 - \frac{2}{3}\alpha \right) v - (v^2 - \frac{2}{3}\alpha v^3)}} \right)^2.
\]

The formulas (4.12) and (4.11) let us compute \(\bar{\lambda}\), and then \(\bar{\mu}\), for any \(\alpha \in (0, \frac{\lambda}{\lambda_1})\). (For \(\alpha \in (0, \frac{\lambda}{\lambda_1})\), the quantity inside the square root in (4.12), which is \(v(1 - v) \left[ 1 - \frac{2}{3}\alpha(1 + v) \right]\), is positive for all \(v \in (0, 1)\).)

If \(\bar{\mu} \to 0\), then from (4.11), \(\alpha \to 0\) (recall that \(\lambda > \lambda_1\)), and then from (4.12)
\[
\bar{\lambda} \downarrow \left( \int_0^1 \frac{dv}{\sqrt{v - v^2}} \right)^2 = \pi^2,
\]
completing the proof.

**Proof of the Theorem 4.1** It is easier to understand the \(\lambda\)-curves, so we assume first that \(\mu\) is fixed. It is well known that for \(\lambda\) large enough the problem (4.1) has a positive stable (“large”) solution, with \(u(0, \lambda)\) increasing in \(\lambda\) (see e.g., [14]). Let us continue this solution for decreasing \(\lambda\). This curve does not continue to \(\lambda\)’s \(\leq \lambda_1\), and it cannot become sign-changing while continued to the left, by Lemma 4.1, hence a turn to the right must occur. After the turn, standard arguments imply that solutions develop zero slope at \(\pm 1\), and become sign-changing for \(\lambda > \bar{\lambda}_\mu\), see e.g., [10]. By Theorem 2.2 exactly one turn occurs on each \(\lambda\)-curve, and by Lemma 4.4 \(\inf_\mu \bar{\lambda}_\mu = \lambda_2 = \pi^2\).

Turning to the \(\mu\)-curves, for any fixed \(\bar{\lambda} > \lambda_1\) we can find a positive solution on the curve \(\mu = 0\) (the curve that bifurcates from the trivial solution at \(\lambda = \lambda_1\)). We now slide down from this point in the \((\lambda, u(0))\) plane, by

\[
\text{the proof, } \diamond
\]
varying $\mu$. As we increase $\mu$ (keeping $\bar{\lambda}$ fixed), we slide to different $\lambda$-curves. At some $\mu$ we reach a $\lambda$-curve which has its turn at $\lambda = \bar{\lambda}$. After that point, $\mu$ begins to decrease on the $\lambda$-curves. If $\bar{\lambda} \in (\lambda_1, \lambda_2]$, we slide all the way to $\mu = 0$, by Lemma 4.3. Hence, the $\mu$-curve at $\bar{\lambda}$ is as in Figure 2. In case $\bar{\lambda} > \lambda_2$, by Lemma 4.4, we do not slide all the way to $\mu = 0$, and hence the $\mu$-curve at $\bar{\lambda}$ is as in Figure 3. By Lemma 4.3 this curve exhausts the set of all positive solutions of (4.1) (any other solution would lie on a solution curve with no place to go, when continued in $\mu$).

\[\text{Remark}\] Our results also imply that the $\mu$-curves described in the Theorem 4.1 continue without turns for all $\mu < 0$. (Observe that Lemma 2.4 holds for autonomous problems, regardless of the sign of $\mu$, see e.g., [10].) Negative $\mu$'s correspond to “stocking” of fish, instead of “fishing”. In Figure 4 we present the solution curve of the problem

\begin{equation}
\tag{4.13}
 u'' + 6u(1 - u) - \mu = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0.
\end{equation}

Observe that $u(0) > 1$ for $\mu < \mu_0$, for some $\mu_0 < 0$.

For the non-autonomous version of the fishing problem

\begin{equation}
\tag{4.14}
 u'' + \lambda u(1 - u) - \mu g(x) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0
\end{equation}

we were not able to extend any of the above lemmas. Still it appears easier to understand the $\lambda$-curves first. Here we cannot rule out the possibility of the $\lambda$-curves losing their positivity backward, and thus never making a turn to the right.

We prove next that a turn to the right does occur for solution curves of (4.14) that are close to the curve bifurcating from zero at $\lambda = \lambda_1$. Let $\bar{\lambda}_\mu$ be the value of $\lambda$, at which positivity is lost for a given $\mu$ (so that $u_\mu(\pm 1, \bar{\lambda}_\mu) = 0$). We claim that $\inf_{\mu > 0} \bar{\lambda}_\mu > \lambda_1$. Indeed, assuming otherwise, we can find a sequence $\{\mu_n\} \to 0$, with $\bar{\lambda}_{\mu_n} \to \lambda_1$. By a standard argument, $\frac{u(x, \bar{\lambda}_{\mu_n})}{u(0, \bar{\lambda}_{\mu_n})} \to w(x) > 0$, where

\begin{equation}
 w'' + \lambda_1 w = 0, \quad w(\pm 1) = w'(\pm 1) = 0,
\end{equation}

which is not possible. It follows that for a fixed $\lambda \in (\lambda_1, \inf_{\mu > 0} \bar{\lambda}_\mu]$, the $\mu$-curve for (4.14) is as in Figure 2. (Notice that this also implies that the $\lambda$-curves for small $\mu$ do turn.) A similar result for general PDE’s was proved in S. Oruganti, J. Shi, and R. Shivaji [14]. In case $\lambda > \inf_{\mu > 0} \bar{\lambda}_\mu$, the $\mu$-curves are different, although we cannot prove in general that they are as in Figure 3. (For example, we cannot rule out the possibility that the $\mu$-curves consist of several pieces.)
Figure 4: The solution curve for the problem (4.13).

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