On 3-Inflatable Permutations

Tanya Khovanova       Eric Zhang

Abstract
Call a permutation \( k \)-inflatable if it can be “blown up” into a convergent sequence of permutations by a uniform inflation construction, such that this sequence is symmetric with respect to densities of induced subpermutations of length \( k \). We study properties of 3-inflatable permutations, finding a general formula for limit densities of pattern permutations in the uniform inflation of a given permutation. We also characterize and find examples of 3-inflatable permutations of various lengths, including the shortest examples with length 17.

1 Introduction
In permutation sequences, randomness is an interesting and important property. Many different formulations of randomness are equivalent to the notion of a \textit{quasirandom} permutation sequence. One of the definitions of quasirandomness states that in the limit as lengths tend to infinity, the densities of all permutations of length \( k \) should equal \( 1/k! \).

In 2012, Kral and Pikhurko \cite{2} proved that if a permutation has uniform densities for all permutations of length 4, then it is quasirandom. With this result in mind, it is natural to consider permutation sequences that have some set of uniform densities but are not quasirandom.

Inflatable permutations are one example of a general construction for permutation sequences. We define the inflation of two permutations \( \pi \) and \( \tau \) with lengths \( m \) and \( n \) to be a new permutation with length equal to the product \( mn \). For each index \( i \in [m] \) in the first permutation \( \pi \), we replace it with \( n\pi_i + \tau_j \) for \( j \in [n] \). In a sense, we are “inflating” each number in the first permutation by substituting for it \( m \) numbers in the order of the second permutation.

We can use this definition to build convergent permutation sequences. The main idea is to start from a base permutation and take its inflation with a sequence of random permutations with lengths tending to infinity. The \( n \)-th term of this sequence is the inflation of the base permutation with a permutation chosen uniformly at random from all elements of \( S_n \).

A permutation is called \( k \)-inflatable when this construction results in a permutation sequence that has uniform densities of all length-\( k \) subpermutations. Cooper and Petrarca \cite{1} previously studied this topic in 2008, finding examples of 3-inflatable permutations. The result from Kral and Pikhurko \cite{2} implies...
that there do not exist any nontrivial 4-inflatable permutations. In general, k-inflatable permutations of length greater than 1 do not exist for k \geq 4.

We decided to write this paper when we discovered via random sampling that the example of a 3-inflatable permutation of length 9 in [1] is wrong. We found an explicit formula that calculates densities in permutation inflations and used this to find constraints on admissible lengths of such permutations. This proved that the shortest length of a 3-inflatable permutation is 17, and we found many examples of this length via computer search. We also found a way to multiply 3-inflatable permutations that provides an infinite number of examples.

We start with basic definitions in Section 2. This includes formal definitions for permutation density, convergence of permutation sequences, and quasirandom permutation sequences.

In Section 3 we prove a formula for the density of a permutation \( \pi \) of length \( k \) in the inflation of a permutation \( \tau \) with another permutation \( \gamma \), given the densities of permutations of lengths up to \( k \) in \( \tau \) and \( \gamma \). This allows the densities of induced patterns in inflations of multiple permutations to be computed efficiently.

In Section 4 we examine the properties of a given permutation when inflated by uniformly chosen random permutations of increasing length.

In Section 5 we apply the formula to densities of permutations of length 3. We calculate necessary and sufficient conditions for a permutation to be 3-inflatable. From this, we deduce that the lengths of 3-inflatable permutations must have remainders 0, 1, 17, 64, 80, or 81 modulo 144. By computational search, we have found examples of 3-inflatable permutations for each of these lengths, including the shortest possible examples, with length 17.

In Section 6 we show that if two permutations are both 3-inflatable, then the inflation of one with the other is also 3-inflatable. We then discuss how a rotational symmetry helps us find examples of 3-inflatable permutations by reducing the number of needed equivalences.

2 Preliminaries

A permutation is an ordering of the elements of a set. We represent a permutation of length \( n \) as an \( n \)-tuple of distinct positive integers, representing the image of \([n]\) under application of the permutation. For example, \((1, 2, 3)\) is the identity permutation of length 3, while \(\tau = (3, 2, 1)\) is the permutation that swaps 1 and 3. For convenience, we can leave out the parentheses and write \(\tau = 321\). We denote the length of the permutation \(\tau\) as \(|\tau|\).

We are interested in patterns that are formed by subsets in a given permutation.

Suppose we have an ordered \( k \)-tuple \((a_1, \ldots, a_k)\) of distinct positive integers. We say that this tuple is order isomorphic to a permutation \(\pi\) of size \( k \), if and only if \(a_i < a_j \iff \pi_i < \pi_j\) for all \( i, j \in [k] \).

The density of a pattern permutation \(\pi\) of length \( k \) in a permutation \(\tau\) is defined as the probability that a randomly-selected \( k \)-point subset of \(\tau\) is order-isomorphic to \(\pi\).
isomorphic to $\pi$. We denote this density by $t(\pi, \tau)$. For example, $t(12, 132) = 2/3$.

A sequence of permutations $\{\tau_j\}$ is convergent if and only if as $j \to \infty$, $|\tau_j| \to \infty$, and for every permutation $\pi$, the density $t(\pi, \tau_j)$ converges. The number $t(\pi, \tau_j)$ is called the limit density of $\pi$ in the sequence $\{\tau_j\}$.

The notion of density can be used to describe property of “randomness” in a permutation.

A convergent permutation sequence $\{\tau_j\}$ is called quasirandom if for every permutation $\pi$, the limit density of $\pi$ in the sequence is $\frac{1}{|\pi|!}$.

3 Inflation

We define an inflation of one permutation with respect to another permutation. Cooper and Petrarka call this the tensor product of two permutations [1].

**Definition 1.** The inflation of $\tau \in S_n$ with respect to a permutation $\gamma \in S_m$ is defined as a permutation $\text{inflate}(\tau, \gamma)$ of length $mn$ that consists of $n$ blocks of length $m$, which are each order-isomorphic to $\gamma$, and any restriction of the permutation to one element in each block is order-isomorphic to $\tau$.

For example, if $\tau = 12$ and $\gamma = 312$, then $\text{inflate}(\tau, \gamma) = 312645$. The resulting permutation consists of two blocks 312 and 645, where each is order-isomorphic to $\gamma$. Also, each number in the second block is greater than every number in the first block as dictated by permutation $\tau$. As another example $\text{inflate}(\gamma, \tau) = 561234$.

Inflations of permutations are interesting objects of study, as it is possible to calculate densities in $\text{inflate}(\tau, \gamma)$ through densities in $\tau$ and $\gamma$. This later allows for the construction of interesting examples of permutation sequences with some set densities.

There is a more general definition of inflation, which turns out to be useful for computing densities. For some permutation $\tau \in S_n$, rather than inflating $\tau$ with respect to a single permutation $\pi$, we can define an operation on $\tau$ with respect to a sequence of $n$ permutations $\gamma_1, \ldots, \gamma_n$ as follows.

**Definition 2.** The generalized inflation of $\tau \in S_n$ with respect to a sequence of $n$ permutations $\gamma_1, \ldots, \gamma_n$ is defined as a permutation of length $|\gamma_1| + \ldots + |\gamma_n|$ that consists of $n$ blocks, such that the $i$-th block is order-isomorphic to $\gamma_i$, and any restriction of the permutation to one element in each block is order-isomorphic to $\tau$.

In particular, inflation is a special case of the generalized inflation, where all the $\gamma_i$ for $i \in [n]$ are equal to some single permutation $\gamma$, hence the name.

For example, the generalized inflation of the permutation $\tau = 231$ with the sequence $\{\gamma_j\} = (12, 231, 1)$ is the length-6 permutation $23|546|1$. Note how each element in the original permutation $\tau$ corresponds to a “block” of elements in the new permutation.
Given a permutation $\pi$, it is useful to see how $\pi$ can be represented as a generalized inflation. To this end, consider pairs $(b, \sigma)$, where $\sigma$ is a permutation and $b$ is a sequence of permutations $(\sigma_1, \ldots, \sigma_k)$ so that $\pi$ is the generalized inflation of $\sigma$ with respect to $\sigma_1, \ldots, \sigma_k$. We call the pair $(b, \sigma)$ a block-partition of $\pi$, and define the function $B(\pi)$ to be the set of all block-partitions of $\pi$. Each block-partition of a permutation is a way of representing it as a generalized inflation.

For example, if $\pi = 132$, then we have three block-partitions: first, $1|3|2$ with $b = (1, 1, 1)$ and $\sigma = 132$; second, $1|32$ with $b = (1, 21)$ and $\sigma = 12$; and third, $132$ with $b = (132)$ and $\sigma = 1$.

This definition allows us to derive an explicit formula for the density of pattern permutations in the inflation of two permutations.

**Theorem 1.** The density of $\pi$ in $\text{infl}(\tau, \gamma)$ is

$$t(\pi, \text{infl}(\tau, \gamma)) = \frac{|\pi|!}{|\tau||\gamma|} \sum_{(b, \sigma) \in B(\pi)} \left( \frac{|\tau|}{|\sigma|} t(\sigma, \tau) \cdot \prod_{\alpha \in b} \frac{t(\alpha, \gamma)}{|\alpha|!} \right).$$

**Proof.** There are $|\tau||\gamma|$ equiprobable ways of selecting (counting distinct orderings) $|\pi|$ blocks independently from the inflation of $\tau$.

For a given pair $(b, \sigma) \in B(\pi)$, let $k = |\sigma|$. There exist $\binom{|\tau|}{k} \cdot t(\sigma, \tau)$ order-respecting assignments of distinct blocks $a_1, a_2, \ldots, a_k$ in the inflation of $\tau$ to the corresponding indices in $\sigma$. Furthermore, since ordering counts, the number of ways to select these $k$ blocks with their respective multiplicities is given by the multinomial coefficient

$$\binom{|\tau|}{|\sigma_1|:|\sigma_2|:\ldots:|\sigma_k|} = |\pi|! \cdot \prod_{\alpha \in b} \frac{1}{|\alpha|!}.$$

The probability that a random $|\pi|$-point subset from the inflation of $\tau$ is order-isomorphic to $\pi$, and also blocked according to $a_1, a_2, \ldots, a_k$, is thus equal to

$$\frac{1}{|\tau||\gamma|} \left( \frac{|\tau|!}{\prod_{\alpha \in b} |\alpha|!} \right) \prod_{\alpha \in b} t(\alpha, \gamma) = \frac{|\pi|!}{|\tau||\gamma|} \prod_{\alpha \in b} \frac{t(\alpha, \gamma)}{|\alpha|!}.$$

Summing these probabilities for all assignments of $\{a_j\}$ mentioned above, as well as for each block-partitioning in $B(\pi)$, we arrive at the final formula.

For example, let $\pi = 12$. Then $B(\pi)$ consists of two elements $((1, 1), 12)$ and $((12), 1)$. Plugging this into the formula above, we get

$$t(12, \text{infl}(\tau, \sigma)) = \frac{2!}{|\tau|^2} \left( \binom{|\tau|}{2} t(12, \tau)/2 + \frac{|\tau|}{4} \cdot t(12, \sigma) \right).$$

In particular, if $t(12, \tau) = t(12, \sigma) = 1/2$, then $t(12, \text{infl}(\tau, \sigma)) = 1/2$. However, it is possible that $t(12, \text{infl}(\tau, \sigma)) = 1/2$, while $t(12, \tau) \neq t(12, \sigma)$. 

4
4 Uniform inflation

A special case of inflation is of particular interest to us, when we inflate a fixed permutation with a random second permutation. In particular, given a permutation $\pi$ and a uniform random permutation $\gamma$ of length not less than $|\pi|$, the expected value of the density $\mathbb{E}t(\pi, \gamma) = \frac{1}{|\pi|!}$ for symmetry reasons.

Using this idea, we can define the uniform inflation of a single permutation as the convergent sequence of its inflations with random permutations of a set length tending to infinity.

**Definition 3.** The uniform inflation of $\tau$, denoted by inflate($\tau$), is a sequence of random permutations $\{\text{inflate}(\tau, \lambda_j)\}$, where $\lambda_j$ is selected uniformly at random from $S_j$. In other words, the $j$-th term of the sequence is the inflation of $\tau$ with a uniform random permutation of length $j$.

It is not difficult to show that the densities of pattern permutations in the uniform inflation of $\tau$ converge for any permutation $\tau$, so the sequence is convergent. This convergence fact allows us to extend Theorem 1 to the case of uniform inflation by plugging in $t(\alpha, \gamma) = \frac{1}{|\alpha|!}$, which yields the following result.

**Corollary 2.** If $\{\lambda_j\}$ is a sequence of random permutations with lengths tending to infinity, then the limit density of $\pi$ in the sequence of permutations $\{\text{inflate}(\tau, \lambda_j)\}$ is

$$t(\pi, \text{inflate}(\tau)) = \frac{|\pi|!}{|\tau||\pi|} \sum_{(b, \sigma) \in B(\pi)} \left(\frac{|\tau|}{|\sigma|} \cdot \prod_{\alpha \in b} \frac{1}{|\alpha|!} \cdot t(\sigma, \tau) \cdot \prod_{\alpha \in b} \frac{1}{|\alpha|!} \cdot \prod_{\alpha \in b} \frac{1}{|\alpha|!} \right).$$

As we are primarily interested in the densities of length-3 pattern permutations, for our next example we choose $\pi = 132$. Consider some general permutation $\tau$ of length $n$.

The permutation 132 admits 3 block-partitions: 132, 1|32, and 1|3|2. Our formula yields for the limit density of 132 in the inflation of $\tau$ with a random sequence:

$$\frac{3!}{n^3} \left( \binom{n}{1} t(1, \tau) \cdot \frac{1}{3!^2} + \binom{n}{2} t(12, \tau) \cdot \frac{1}{12!2!} + \binom{n}{3} t(132, \tau) \cdot \frac{1}{12!12!12!} \right) = \frac{3!}{n^3} \left( \binom{n}{1} \left( \frac{n}{3} \right) t(12, \tau) + \binom{n}{2} t(12, \tau) + \binom{n}{3} t(132, \tau) \right).$$

Notice from this expression that there exists additional structure if we focus our attention on length-3 pattern permutations. Let us denote first $\frac{3!}{n^3} \binom{n}{1}$ by $A(n)$, second $\frac{2!}{3!^2} \binom{n}{2}$ by $B(n)$, and third $\frac{2!}{12!2!} \binom{n}{3}$ by $C(n)$. This gives us

$$t(132, \text{inflate}(\tau)) = A(n)t(132, \tau) + B(n)t(12, \tau) + C(n).$$

Then the limit densities of permutations 213, 312 and 231 in the inflation of $\tau$ are given by analogous formulas:

$$t(213, \text{inflate}(\tau)) = A(n)t(213, \tau) + B(n)t(12, \tau) + C(n).$$
\[ t(312, \text{inflate}(\tau)) = A(n)t(312, \tau) + B(n)t(21, \tau) + C(n), \]
and
\[ t(231, \text{inflate}(\tau)) = A(n)t(231, \tau) + B(n)t(21, \tau) + C(n). \]

The reason for this similarity is that the block-partitions of 213, 312, and 132 all follow the same pattern. Each has a partition into one block of size 3, two blocks of sizes 1 and 2, and three blocks of size 1.

Similarly, permutations 123 and 321 follow the same formula, except the coefficient of the second term is doubled since there are two ways to partition each of 123 and 321 into two blocks. These limit densities are given by

\[ t(123, \text{inflate}(\tau)) = A(n)t(123, \tau) + 2B(n)t(12, \tau) + C(n), \]
\[ t(321, \text{inflate}(\tau)) = A(n)t(321, \tau) + 2B(n)t(12, \tau) + C(n). \]

To see an example of this symmetry, consider the length-9 permutation \( \tau = 472951836 \). The density of 132 in \( \tau \) is \( \frac{17}{84} \), and the density of 12 in \( \tau \) is \( \frac{1}{2} \). Therefore,
\[ t(132, \text{inflate}(\tau)) = \frac{29}{162}. \]
Similarly, we then have
\[ t(213, \text{inflate}(\tau)) = t(231, \text{inflate}(\tau)) = t(312, \text{inflate}(\tau)) = \frac{29}{162}. \]

Also,
\[ t(123, \text{inflate}(\tau)) = t(321, \text{inflate}(\tau)) = \frac{23}{162}. \]

5 Inflation and \( k \)-symmetry

We now turn our attention to symmetric permutation sequences.

Definition 4. A convergent permutation sequence \( \{\tau_j\} \) is called \( k \)-symmetric if for every permutation \( \pi \) of length \( k \), the limit density of \( \pi \) in \( \tau_j \) is \( \frac{1}{k!} \).

Definition 5. A permutation \( \tau \) is called \( k \)-inflatable if \( \text{inflate}(\tau) \) is \( k \)-symmetric.

As a direct corollary of Theorem 1, permutation \( \tau \) is 2-inflatable if and only if \( t(12, \tau) = \frac{1}{2} \). We are interested in 3-inflatable permutations, that is that the \( \text{inflate}(\tau) \) sequence has 3-point densities all equal to \( \frac{1}{3!} \).

It was claimed in Cooper and Petrarca that the smallest 3-inflatable permutations are of length 9. They provide as examples 472951836 and its inverse. However, by our calculations above, this is not the case. We have also verified this empirically by estimating the limit densities in the inflation using a Monte Carlo method, and the results were consistent with our formula.
5.1 \( k \)-inflatable permutations

Theorem 3 allows us to calculate the densities of the permutation we inflate, so that the result is 3-inflatable.

**Theorem 3.** A permutation \( \tau \) of length \( n \) is 3-inflatable if and only if
\[
t(12, \tau) = \frac{1}{2},
\]
\[
t(123, \tau) = \frac{3}{g_3} \cdot 2 \cdot t(12, \tau) \cdot \frac{1}{2} + \frac{9}{g_3} \cdot \frac{1}{3!}.
\]

**Proof.** First, a 3-inflatable permutation must also be 2-inflatable, so the density of 12 in \( \tau \) must be 1/2.

A specific application of Corollary 2 for three-point densities yields
\[
t(123, \text{inflate}(\tau)) = \frac{9 \cdot 8 \cdot 7}{g_3^3} \cdot t(123, \tau) + \left(\frac{g_3}{2}\right) \cdot \frac{3}{g_3} \cdot 2 \cdot t(12, \tau) \cdot \frac{1}{2} + \frac{9}{g_3} \cdot \frac{1}{3!}.
\]

Using this formula, we can now calculate the density of 123 in a 3-inflatable permutation. We have already showed that the density of inversions should be 1/2. Therefore,
\[
\frac{1}{6} = \frac{n(n-1)(n-2)}{n^3} \cdot t(123, \tau) + \left(\frac{n}{2}\right) \cdot \frac{3}{n^3} \cdot 2 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{n}{n^3} \cdot \frac{1}{3!}.
\]

Multiplying by \( 12n^2 \) and rearranging yields
\[
2n^2 - 9n + 7 = 12(n-1)(n-2) t(123, \tau).
\]

Then, we observe that the density is
\[
t(123, \tau) = \frac{2n^2 - 9n + 7}{12(n-1)(n-2)} = \frac{(2n-7)(n-1)}{12(n-1)(n-2)} = \frac{2n-7}{12(n-2)}.
\]

Analogous applications of this formula for densities of the five other length-3 permutations yields the desired result.

For a given integer \( n \), these densities might not be possible for any permutation \( \tau \) due to divisibility. The conditions are described in the following corollary.

**Corollary 4.** If a permutation of length \( n \) is 3-inflatable, then both:

1. \( \binom{n}{2} \) is even.
2. \( \binom{n}{3} \) is divisible by the reduced denominators of \( \frac{2n-7}{12(n-2)} \) and \( \frac{4n-5}{24(n-2)} \).

The above divisibility criteria can be more explicitly described by giving the complete set of admissible residue classes modulo 144.

**Lemma 5.** For any 3-inflatable permutation \( \tau \) with length \( n \),
\[
n \equiv 0, 1, 17, 64, 80, 81 \pmod{144}.
\]
Proof. Let \( k \) be the number of occurrences of 132 in the permutation \( \tau \). By the above criteria, we have that

\[
\frac{k}{\binom{n}{3}} = \frac{4n - 5}{24(n - 2)},
\]

\[
144k = n(n - 1)(4n - 5),
\]

\[
144 \mid n(n - 1)(4n - 5).
\]

Also,

\[
\frac{k}{\binom{n}{3}} = \frac{2n - 7}{12(n - 2)},
\]

\[
72k = n(n - 1)(2n - 7),
\]

\[
72 \mid n(n - 1)(2n - 7).
\]

Together, these two divisibility rules are equivalent to

\[
n \equiv 0, 1, 17, 64, 80, 81 \pmod{144}.
\]

The smallest such number is 17. In this particular case, the binomials \( \binom{17}{2} = 136 \) and \( \binom{17}{3} = 680 \), while \( (2n - 7)/12(n - 2) = 3/20 \) and \( (4n - 5)/24(n - 2) = 7/40 \). Therefore, in a 3-inflatable permutation of length 17, the numbers of occurrences of both 123 and 321 should be 102, while the four other pattern permutations of length 3 should each occur 119 times.

Examples of these minimum-length 3-inflatable permutations were found through an optimized computer search. For example, the permutations

E534BGA9HC2D1687F and B3CE1H76F5A49D2G8

are both 3-inflatable, where capital letters denote numbers greater than nine (A=10, B=11, C=12, etc.).

Although it was intractable to check all 17! of such permutations for the inflation criterion through our available computational resources, we could reduce the search space by checking only those permutations which were centrally symmetric (further discussed in the next section). Of the \( 8 \cdot 2^8 \) centrally symmetric permutations of length 17, a computer search found that 750 of them are 3-inflatable.

Furthermore, additional computer searches found many more examples of centrally symmetric, 3-inflatable permutations of lengths 64, 80, 81, 144, and 145. We have found 3-inflatable permutations of lengths belonging to each admissible residue class modulo 144. There also seems to be a large number of such permutations of each length. These empirical results suggest the following conjecture.

**Conjecture 1.** For any positive integer \( x \) of length belonging to an admissible residue class modulo 144, there exists a 3-inflatable permutation of length \( x \).
6 Structure of 3-inflatable permutations

We start by proving that the inflation of two \( k \)-inflatable permutations is itself \( k \)-inflatable. This gives us an explicit construction that shows there exists an infinite number of 3-inflatable permutations, and also allows us to construct examples of arbitrarily large length.

**Theorem 6.** If \( \tau_1 \) and \( \tau_2 \) are two \( k \)-inflatable permutations, then \( \text{inflate}(\tau_1, \tau_2) \) is also \( k \)-inflatable.

**Proof.** Note that inflation is associative, so for some random permutation \( \gamma \), \( \text{inflate}(\text{inflate}(\tau_1, \tau_2), \gamma) = \text{inflate}(\tau_1, \text{inflate}(\tau_2, \gamma)) \). Then, by the definition of a \( k \)-inflatable permutation, for any length-\( k \) permutation \( \pi \),

\[
\lim_{|\gamma| \to \infty} t(\pi, \text{inflate}(\tau_2, \gamma)) = \frac{1}{k!}
\]

Since our current discussion only considers densities of pattern permutations with length at most \( k \), we can substitute \( \text{inflate}(\tau_2, \gamma) \) for a random permutation \( \gamma_2 \) of the same length without changing the expected value of the expression. Thus,

\[
\lim_{|\gamma| \to \infty} t(\pi, \text{inflate}(\tau_1, \text{inflate}(\tau_2, \gamma))) = \lim_{|\gamma_2| \to \infty} t(\pi, \text{inflate}(\tau_1, \gamma_2)) = \frac{1}{k!}
\]

So \( \text{inflate}(\tau_1, \tau_2) \) is \( k \)-inflatable.

Given this result, it is natural to ask if the set of admissible lengths of 3-inflatable permutations modulo 144 are closed under multiplication — in other words, if the set of numbers with remainders 0, 1, 17, 64, 80, and 81 modulo 144 is closed under multiplication. This is true, as shown in Figure 1.

For example, we can use the 3-inflatable permutations of size 17 that we found to build 3-inflatable permutations of sizes \( 17^n \).

Now we want to define a symmetry of permutations that is useful for increasing the likelihood of being symmetric. Let us define an operation \( R \) on permutations which we call a rotation: \( R(\pi)_i = n + 1 - \pi_{n+1-i} \). This is equivalent to drawing the permutation as a graph (on a square grid) then rotating it by 180 degrees about its center.

\[
\begin{array}{cccccc}
\times & 0 & 1 & 17 & 64 & 80 & 81 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 17 & 64 & 80 & 81 \\
17 & 0 & 17 & 1 & 80 & 64 & 81 \\
64 & 0 & 64 & 80 & 64 & 80 & 0 \\
80 & 0 & 80 & 64 & 80 & 64 & 0 \\
81 & 0 & 81 & 81 & 0 & 0 & 81 \\
\end{array}
\]

Figure 1: Multiplication table of admissible lengths modulo 144.
We call a permutation $\pi$ of length $n$ centrally symmetric if it is equal to its rotation. In other words, $\pi_i + \pi_j = n + 1$ whenever $i + j = n + 1$. The importance of centrally symmetric permutations is explained by the following lemma.

**Lemma 7.**

$$t(\pi, \gamma) = t(R(\pi), R(\gamma)).$$

**Proof.** Let $k = |\pi|$, and let $S_1$ be the set of $k$-point subsets of $\gamma$ that are order-isomorphic to $\pi$. Similarly, let $S_2$ be the set of $k$-point subsets of $R(\gamma)$ that are order-isomorphic to $R(\pi)$. There is a one-to-one correspondence between $S_1$ and $S_2$ given by the rotation operation.

More formally, for any set of indices $\{i_1, i_2, \ldots, i_k\}$ in $\gamma$, there is a corresponding set of indices, $\{n + 1 - i_1, n + 1 - i_2, \ldots, n + 1 - i_k\}$ in $R(\gamma)$. If the former indices induce some permutation $\pi$ in $\gamma$, then the latter set induces $R(\pi)$ in $R(\gamma)$ from the definition of rotation. Thus, we have $|S_1| = |S_2|$, so the result follows.

**Corollary 8.** If $\pi$ is centrally symmetric, then

$$t(\pi, \gamma) = t(\pi, R(\gamma)).$$

That means centrally symmetric permutations automatically give us some equalities among densities. For example, if $\pi$ is centrally symmetric, then

$$t(132, \pi) = t(231, \pi) \quad \text{and} \quad t(312, \pi) = t(213, \pi).$$

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