Measuring what Matters: A Hybrid Approach to Dynamic Programming with Treewidth

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Abstract

We develop a framework for applying treewidth-based dynamic programming on graphs with “hybrid structure”, i.e., with parts that may not have small treewidth but instead possess other structural properties. Informally, this is achieved by defining a refinement of treewidth which only considers parts of the graph that do not belong to a pre-specified tractable graph class. Our approach allows us to not only generalize existing fixed-parameter algorithms exploiting treewidth, but also fixed-parameter algorithms which use the size of a modulator as their parameter. As the flagship application of our framework, we obtain a parameter that combines treewidth and rank-width to obtain fixed-parameter algorithms for Chromatic Number, Hamiltonian Cycle, and Max-Cut.

1 Introduction

Over the past decades, the use of structural properties of graphs to obtain efficient algorithms for NP-hard computational problems has become a prominent research direction in computer science. Perhaps the best known example of a structural property that can be exploited in this way is the tree-likeness of the inputs, formalized in terms of the decomposition-based structural parameter treewidth \cite{1}. It is now well known that a vast range of fundamental problems admit so-called fixed-parameter algorithms parameterized by the treewidth of the input graph – that is, can be solved in time $f(k) \cdot n^{O(1)}$ on $n$-vertex graphs of treewidth $k$ (for some computable function $f$). We say that such problems are FPT parameterized by treewidth.

On the other hand, dense graphs are known to have high treewidth and hence require the use of different structural parameters; the classical example of such a parameter tailored to dense graphs is clique-width \cite{11}. Clique-width is asymptotically equivalent to the structural parameter rank-width \cite{51}, which is nowadays often used instead of clique-width due to a number of advantages (rank-width is much easier to
compute \cite{35} and can be used to design more efficient fixed-parameter algorithms than clique-width \cite{28,29}. While rank-width (or, equivalently, clique-width) dominates treewidth and can be used to “lift” fixed-parameter algorithms designed for treewidth to well-structured dense graphs for a number of problems, there are also important problems which are FPT parameterized by treewidth but $W[1]$-hard (and hence probably not FPT) parameterized by rank-width. The most prominent examples of such problems are CHROMATIC NUMBER \cite{24}, HAMILTONIAN CYCLE \cite{24}, and MAX-CUT \cite{25}.

Another generic type of structure used in algorithmic design is based on measuring the size of a \textit{modulator} (i.e., a vertex deletion set) \cite{7} to a certain graph class. Basic examples of parameters based on modulators include the vertex cover number (a modulator to edgeless graphs) \cite{21,26} and the feedback vertex set number (a modulator to forests) \cite{4,38}. For dense graphs, modulators to graphs of rank-width 1 have been studied \cite{17,43}, and it is known that for every constant $c$ one can find a modulator of size at most $k$ to graphs of rank-width $c$ (if such a modulator exists) in time $f(k) \cdot n$ \cite{42}. However, the algorithmic applications of such modulators have remained largely unexplored up to this point.

\textbf{Our Contribution.} We develop a class of \textit{hybrid parameters} which combines the foremost advantages of treewidth and modulators to obtain a “best-of-both-worlds” outcome. In particular, instead of measuring the treewidth of the graph itself or the size of a modulator to a graph class $\mathcal{H}$, we consider the treewidth of a (torso of a) modulator to $\mathcal{H}$\cite{32}. This parameter, which we simply call $\mathcal{H}$-treewidth, allows us to lift previously established tractability results for a vast number of problems from treewidth and modulators to a strictly more general setting. As our first technical contribution, we substantiate this claim with a meta-theorem that formalizes generic conditions under which a treewidth-based algorithm can be generalized to $\mathcal{H}$-treewidth; the main technical tool for the proof is an adaptation of \textit{protrusion replacement} techniques \cite{3, Section 4}.

As the flagship application of $\mathcal{H}$-treewidth, we study the case where $\mathcal{H}$ is the class $\mathcal{R}_c$ of graphs of rank-width at most $c$ (an arbitrary constant). $\mathcal{R}_c$-treewidth hence represents a way of lifting treewidth towards dense graphs that lies “between” treewidth and rank-width. We note that this class of parameters naturally incorporates a certain scaling trade-off: $\mathcal{R}_c$-treewidth dominates $\mathcal{R}_{c-1}$-treewidth for each constant $c$, but the runtime bounds for algorithms using $\mathcal{R}_c$-treewidth are worse than those for $\mathcal{R}_{c-1}$-treewidth.

Our first result for $\mathcal{R}_c$-treewidth is a fixed-parameter algorithm for computing the parameter itself. We then develop fixed-parameter algorithms for CHROMATIC NUMBER, HAMILTONIAN CYCLE and MAX-CUT parameterized by $\mathcal{R}_c$-treewidth; moreover, in 2 out of these 3 cases the parameter dependencies of our algorithms are essentially tight. These algorithms represent generalizations of:

1. classical fixed-parameter algorithms parameterized by treewidth \cite{16},

2. polynomial-time algorithms on graphs of bounded rank-width \cite{29}, and

\footnote{Parameter $\alpha$ dominates parameter $\beta$ if for each graph class with bounded $\beta$, $\alpha$ is also bounded.}

\footnote{Formal definitions are provided in Section 3.}
3. (not previously known) fixed-parameter algorithms parameterized by modulators to graph classes of bounded rank-width.

The main challenge for all of these problems lies in dealing with the fact that some parts of the graph need to be handled using rank-width based techniques, while for others we use treewidth-based dynamic programming. We separate these parts from each other using the notion of nice $\mathcal{H}$-tree decompositions. The algorithm then relies on enhancing the known dynamic programming approach for solving the problem on treewidth with a subroutine that not only solves the problem on the part of the graph outside of the modulator, but also serves as an interface by supplying appropriate records to the treewidth-based dynamic programming part of the algorithm. At its core, each of these subroutines boils down to solving an “extended” version of the original problem parameterized by the size of a modulator to $R_c$; in particular, each subroutine immediately implies a fixed-parameter algorithm for the respective problem when parameterized by a modulator to constant rank-width. To give a specific example for such a subroutine, in the case of CHROMATIC NUMBER one needs to solve the problem parameterized by a modulator to $R_c$ where the modulator is furthermore precolored.

To avoid any doubt, we make it explicitly clear that the runtime of all of our algorithms utilizing $R_c$-treewidth has a polynomial dependency on the input where the degree of this polynomial depends on $c$ (as is necessitated by the $W[1]$-hardness of the studied problems parameterized by rank-width).

Related Work. Previous works have used a combination of treewidth with backdoors, a notion that is closely related to modulators, in order to solve non-graph problems such as CONSTRAINT SATISFACTION [52], BOOLEAN SATISFIABILITY [31] and INTEGER PROGRAMMING [30]. Interestingly, the main technical challenge in all of these papers is the problem of computing the parameter, while using the parameter to solve the problem is straightforward. In the algorithmic results presented in this contribution, the situation is completely reversed: the main technical challenge lies in developing the algorithms (and most notably the subroutines) for solving our targeted problems. Moreover, while the aforementioned three papers focus on solving a single problem, here we aim at identifying and exploiting structural properties that can be used to solve a wide variety of graph problems.

Telle and Saether [54] proposed a generalization of treewidth towards dense graphs that is based on explicitly allowing a specific operation (graph splits). The resulting parameter is not related to modulators, and we show that it is incomparable to $R_c$-treewidth. A subset of the authors previously studied modulator-based parameters that used graph splits and rank-width. The resulting well-structured modulators could either dominate rank-width [19], or (depending on the specific constants used) be used to obtain polynomial kernels for a variety of problems [18]. Since $R_c$-treewidth lies “between” treewidth and rank-width, it is clear that our class of parameters is different from these previously studied ones.

2 Preliminaries

For $i \in \mathbb{N}$, let $[i]$ denote the set $\{1, \ldots, i\}$. All graphs in this paper are simple and undirected. We refer to the standard textbook [15] for basic graph terminology.
For a graph \(G\), let \(V(G)\) and \(E(G)\) denote the vertex set and the edge set of \(G\), respectively. For \(S \subseteq V(G)\), let \(G[S]\) denote the subgraph of \(G\) induced by \(S\); if \(G\) contains no edges with precisely one endpoint in \(S\), then we call \(S\) a connected component of \(G\). For \(v \in V(G)\) and \(S \subseteq V(G)\), let \(G - v\) be the graph obtained from \(G\) by removing \(v\), and let \(G - S\) be the graph obtained by removing all vertices in \(S\). For \(v \in V(G)\), the set of neighbors of \(v\) in \(G\) is denoted by \(N_G(v)\) (or \(N(v)\) when \(G\) is clear from the context). For \(A \subseteq V(G)\), let \(N_G(A)\) denote the set of vertices in \(G - A\) that have a neighbor in \(A\). For \(v \in V(G)\) and a subgraph \(H\) of \(G - v\), we say \(v\) is adjacent to \(H\) if it has a neighbor in \(H\). For a vertex set \(A\) of a graph \(G\), an \(A\)-path is a path whose endpoints are contained in \(A\) and all the internal vertices are contained in \(G - A\).

A set \(M\) of vertices in a graph \(G\) is called a modulator to a graph class \(\mathcal{H}\) if \(G - M \in \mathcal{H}\). The operation of collapsing a vertex set \(X\), denoted \(G \circ X\), deletes \(X\) from the graph and adds an edge between vertices \(u, v \in V(G - X)\) if \(uv \notin E(G)\) and there is an \(u-v\) path with all internal vertices in \(G[X]\).

### 2.1 Parameterized Complexity

A parameterized problem \(\mathcal{P}\) is a subset of \(\Sigma^* \times \mathbb{N}\) for some finite alphabet \(\Sigma\). Let \(L \subseteq \Sigma^*\) be a classical decision problem for a finite alphabet, and let \(p\) be a non-negative integer-valued function defined on \(\Sigma^*\). Then \(L\) parameterized by \(p\) denotes the parameterized problem \(\{(x, p(x)) \mid x \in L\}\) where \(x \in \Sigma^*\). For a problem instance \((x, k) \in \Sigma^* \times \mathbb{N}\) we call \(x\) the main part and \(k\) the parameter. A parameterized problem \(\mathcal{P}\) is fixed-parameter tractable (FPT in short) if a given instance \((x, k)\) can be solved in time \(f(k) \cdot |x|^{O(1)}\) where \(f\) is an arbitrary computable function of \(k\); we call algorithms running in this time fixed-parameter algorithms. Similarly, a parameterized problem \(\mathcal{P}\) is in the class \(\text{XP}\) if a given instance \((x, k)\) can be solved in time \(|x|^{f(k)}\) where \(f\) is an arbitrary computable function of \(k\), and we call algorithms running in this time \(\text{XP}\) algorithms.

Parameterized complexity classes are defined with respect to fpt-reducibility. A parameterized problem \(P\) is fpt-reducible to \(Q\) if in time \(f(k) \cdot |x|^{O(1)}\), one can transform an instance \((x, k)\) of \(\mathcal{P}\) into an instance \((x', k')\) of \(\mathcal{Q}\) such that \((x, k) \in \mathcal{P}\) if and only if \((x', k') \in \mathcal{Q}\), and \(k' \leq g(k)\), where \(f\) and \(g\) are computable functions depending only on \(k\). Central to parameterized complexity is the following hierarchy of complexity classes, defined by the closure of canonical problems under fpt-reductions: \(\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \cdots \subseteq \text{XP}\). All inclusions are believed to be strict. In particular, \(\text{FPT} \neq \text{W}[1]\) under the Exponential Time Hypothesis [36].

The class \(\text{W}[1]\) is the analog of \(\text{NP}\) in parameterized complexity. A major goal in parameterized complexity is to distinguish between parameterized problems which are in \(\text{FPT}\) and those which are \(\text{W}[1]\)-hard, i.e., those to which every problem in \(\text{W}[1]\) is fpt-reducible.

We refer the reader to the respective books [13, 16, 49] for more details on parameterized complexity. Finally, we recall that a parameter \(\alpha\) dominates a parameter \(\beta\) if for each graph class where \(\beta\) is bounded, \(\alpha\) is also bounded. Two parameters are incomparable if neither dominates the other.
2.2 Treewidth

A tree decomposition of a graph $G = (V, E)$ is a pair $(T, \{B_t \mid t \in V(T)\})$ where $B_t \subseteq V$ for every $t \in V(T)$ and $T$ is a tree such that:

1. for each edge $\{u, v\} \in E$, there is a $t \in V(T)$ such that $\{u, v\} \subseteq B_t$, and
2. for each vertex $v \in V$, $T[\{t \in V(T) \mid v \in B_t\}]$ is a non-empty (connected) tree.

The width of a tree decomposition is $\max_{t \in V(T)} |B_t| - 1$. The treewidth $\text{tw}(G)$ of $G$ is the minimum width taken over all tree decompositions of $G$ and it is denoted by $\text{tw}(G)$. We call the elements of $V(T)$ nodes and $B_t$ bags.

**Fact 1**. There exists an algorithm which, given an $n$-vertex graph $G$ and an integer $k$, runs in time $k^{O(k^2)} \cdot n$, and either outputs a tree decomposition of $G$ of width at most $k$ or correctly determines that $\text{tw}(G) > k$.

A tree decomposition $(T, \{B_t \mid t \in V(T)\})$ of a graph $G$ is nice if $T$ can be rooted such that the following conditions hold:

1. Every node of $T$ has at most two children.
2. If a node $t$ of $T$ has two children $t_1$ and $t_2$, then $B_t = B_{t_1} = B_{t_2}$; in this case we call $t$ a join node.
3. If a node $t$ of $T$ has exactly one child $t'$, then either of the following holds:
   - (a) $|B_t| = |B_{t'}| + 1$, in which case we call $t$ an introduce node, or
   - (b) $|B_t| = |B_{t'}| - 1$ in which case we call $t$ a forget node
4. If a node $t$ is a leaf, then $|B_t| = 1$; in this case we call $t$ a leaf node.

The advantage of nice tree decompositions is that they allow the design of much more transparent dynamic programming algorithms, since one only needs to deal with four specific types of nodes. It is well known (and easy to see) that given a tree decomposition of a graph $G = (V, E)$ of width at most $k$ and with $O(|V|)$ nodes, one can construct in linear time a nice tree decomposition of $G$ with $O(|V|)$ nodes and width at most $k$ [5]. Given a node $t$ in $T$, we let $Y_t$ be the set of all vertices contained in the bags of the subtree rooted at $t$, i.e., $Y_t = B_t \cup \bigcup_{p \in B_t} \{v \mid v$ is separated from the root by $t\}.$

2.3 Rank-width

For a graph $G$ and $U, W \subseteq V(G)$, let $A_{G[U, W]}$ denote the $U \times W$-submatrix of the adjacency matrix over the two-element field $GF(2)$, i.e., the entry $a_{u,w}$, $u \in U$ and $w \in W$, of $A_{G[U, W]}$ is 1 if and only if $\{u, w\}$ is an edge of $G$. The cut-rank function $\rho_G$ of a graph $G$ is defined as follows: For a bipartition $(U, W)$ of the vertex set $V(G)$, $\rho_G(U) = \rho_G(W)$ equals the rank of $A_{G[U, W]}$.

A rank-decomposition of a graph $G$ is a pair $(T, \mu)$ where $T$ is a tree of maximum degree 3 and $\mu : V(G) \rightarrow \{t \mid t$ is a leaf of $T\}$ is a bijective function (See Figure 1).
For an edge $e$ of $T$, the connected components of $T - e$ induce a bipartition $(X,Y)$ of the set of leaves of $T$. The width of an edge $e$ of a rank-decomposition $(T,\mu)$ is $ho_G(\mu^{-1}(X))$. The width of $(T,\mu)$ is the maximum width over all edges of $T$. The rank-width of $G$, $\text{rw}(G)$ in short, is the minimum width over all rank-decompositions of $G$. We denote by $\mathcal{R}_i$ the class of all graphs of rank-width at most $i$. A rooted rank-decomposition is obtained from a rank-decomposition by subdividing an arbitrarily chosen edge, and the newly created vertex is called the root.

![Figure 1: A rank-decomposition of the graph cycle $C_5$.](image)

Unlike clique-width, rank-width can be computed exactly by a fixed-parameter algorithm (which also outputs a corresponding rank-decomposition).

**Theorem 1 (\cite[Theorem 7.3]{35}).** Let $k \in \mathbb{N}$ be a constant and $n \geq 2$. For an $n$-vertex graph $G$, we can output a rank-decomposition of width at most $k$ or confirm that the rank-width of $G$ is larger than $k$ in time $f(k) \cdot n^3$, where $f$ is a computable function.

### 2.4 Monadic Second-Order Logic

Counting Monadic Second-Order Logic (CMSO$_1$) is a basic tool to express properties of vertex sets in graphs. The syntax of CMSO$_1$ includes logical connectives $\land, \lor, \neg, \Rightarrow$, variables for vertices and vertex sets, quantifiers $\exists, \forall$ over these variables, and the relations $a \in A$ where $a$ is a vertex variable and $A$ is a vertex set variable; $\text{adj}(a,b)$, where $a$ and $b$ are vertex variables and the interpretation is that $a$ and $b$ are adjacent; equality of variables representing vertices and sets of vertices; $\text{Parity}(A)$, where $A$ is a vertex set variable and the interpretation is that $|A|$ is even.

The CMSO$_1$ Optimization problem is defined as follows:

| CMSO$_1$-OPT |
|---------------|
| **Instance:** A graph $G$, a CMSO$_1$ formula $\phi(A)$ with a free set variable $A$, and $\text{opt} \in \{\text{min}, \text{max}\}$. |
| **Task:** Find an interpretation of the set $A$ in $G$ such that $G$ models $\phi(A)$ and $A$ is of minimum/maximum (depending on $\text{opt}$) cardinality. |

From the fixed-parameter tractability of computing rank-width \cite{35}, the equivalence of rank-width and clique-width \cite{51} and Courcelle’s Theorem for graphs of bounded clique-width \cite{10} (see also later work that establishes the result directly for rank-width \cite{28}), it follows that:

**Fact 2.** CMSO$_1$-OPT is FPT parameterized by $\text{rw}(G) + |\phi|$, where $G$ is the input graph and $\phi$ is the CMSO$_1$ formula.
3 \( \mathcal{H} \)-Treewidth

The aim of \( \mathcal{H} \)-treewidth is to capture the treewidth of a modulator to the graph class \( \mathcal{H} \). However, one cannot expect to obtain a parameter with reasonable algorithmic applications by simply measuring the treewidth of the graph induced by a modulator to \( \mathcal{H} \) – instead, one needs to measure the treewidth of a so-called torso, which adds edges to track how the vertices in the modulator interact through \( \mathcal{H} \). To substantiate this, we observe that \textsc{Hamiltonian Cycle} would become \textsc{NP}-hard even on graphs with a modulator that (1) induces an edgeless graph, and (2) is a modulator to an edgeless graph, and where (3) each connected component outside the modulator has boundedly many neighbors in the modulator.

\textbf{Observation 1.} \textsc{Hamiltonian Cycle} remains \textsc{NP}-hard if we allow the input to consist, not only of a graph \( G \), but also a modulator \( X \subseteq V(G) \) to an edgeless graph and for each connected component \( C \) of \( G[X] \), \( |N(C)| \leq 3 \).

\textit{Proof.} \textsc{Hamiltonian Cycle} is known to be \textsc{NP}-hard on cubic bipartite graphs [1]. Setting \( X \) to be one side of the bipartition of any graph in this graph class is easily seen to satisfy the given conditions.

The notion of a torso has previously been algorithmically exploited in other settings [30, 32, 48], and its adaptation is our first step towards the definition of \( \mathcal{H} \)-treewidth (see also Figure 2).

\textbf{Definition 2 (\( \mathcal{H} \)-Torso).} Let \( G \) be a graph and \( X \subseteq V(G) \). For a graph class \( \mathcal{H} \), \( G \circ X \) is an \( \mathcal{H} \)-torso of \( G \) if each connected component \( C \) of \( G[X] \) satisfies \( C \in \mathcal{H} \).

\textbf{Definition 3 (\( \mathcal{H} \)-Treewidth).} The \( \mathcal{H} \)-treewidth of a graph \( G \) is the minimum treewidth of an \( \mathcal{H} \)-torso of \( G \). We denote the \( \mathcal{H} \)-treewidth of \( G \) by \( \text{tw}_{\mathcal{H}}(G) \).
Typically, we will want to consider a graph class $\mathcal{H}$ for which certain problems are polynomial-time tractable. Hence, we will assume w.l.o.g. that $(\emptyset, \emptyset) \in \mathcal{H}$. From the definition it is obvious that:

**Observation 2.** For any graph $G$, $\text{tw}_{\mathcal{H}}(G) \leq \text{tw}(G)$.

### 3.1 Nice $\mathcal{H}$-Tree-Decompositions

Just like for tree decompositions, we can also define a canonical form of decompositions which has properties that are convenient when formulating dynamic programs using $\mathcal{H}$-treewidth. Intuitively, a nice $\mathcal{H}$-tree decomposition behaves like a nice tree decomposition on the torso graph (see points 1-3), with the exception that the neighborhoods of the collapsed parts must occur as special boundary leaves (see points 4-5).

**Definition 4 (Nice $\mathcal{H}$-Tree-Decomposition).** A nice $\mathcal{H}$-tree decomposition of a graph $G$ is a triple $(X, T, \{B_t | t \in V(T)\})$ where $X \subseteq V(G)$ such that $G \circ X$ is an $\mathcal{H}$-torso, $(T, \{B_t | t \in V(T)\})$ is a rooted tree decomposition of $G \circ X$, and:

1. Every node in $T$ has at most two children.
2. If a node $t$ of $T$ has two children $t_1$ and $t_2$, then $B_t = B_{t_1} = B_{t_2}$; in this case we call $t$ a join node.
3. If a node $t$ of $T$ has exactly one child $t'$, then either of the following holds:
   
   (a) $|B_t| = |B_{t'}| + 1$, in which case we call $t$ an introduce node, or
   
   (b) $|B_t| = |B_{t'}| - 1$ in which case we call $t$ a forget node.
4. If a node $t$ is a leaf, then one of the following holds:
   
   (a) $|B_t| = 1$: in which case we call $t$ a simple leaf node, or
   
   (b) $B_t = N(C)$ for some connected component $C$ of $G[X]$, in which case we call $t$ a boundary leaf node.
5. For each connected component $C$ of $G[X]$ there is a unique leaf $t$ with $B_t = N(C)$.

An illustration of a nice $\mathcal{H}$-tree decomposition showcasing how it differs from a nice tree decomposition is provided in Figure 3; in line with standard terminology for treewidth, we call the sets $B_t$ bags. The width of a nice $\mathcal{H}$-tree decomposition is simply the width of $(T, \{B_t | t \in V(T)\})$. Below we show that computing a nice $\mathcal{H}$-tree decomposition of bounded width can be reduced to finding an appropriate $\mathcal{H}$-torso.

**Lemma 5.** Given an $n$-vertex graph $G$ and an $\mathcal{H}$-torso $U$ of $G$ with $\text{tw}(U) \leq k$, we can find a nice $\mathcal{H}$-tree decomposition of $G$ with width at most $k$ in time $k^{O(k^3)} \cdot n$.

**Proof.** Compute a nice tree decomposition $(T, \{B_t \mid t \in V(T)\})$ of $U$ with width at most $k$ in time at most $k^{O(k^3)} \cdot n$ using Fact 1. Let $X := V(G) \setminus V(U)$. By the definition of $\circ$, $U = G \circ X$ and the neighbors of each connected component $C$ of $G[X]$ form a clique in $U$. Thus, for each $C$ there is some node, say $t_C$, of $T$ such that $N(C) \subseteq B_{t_C}$. 
Obtain \( T' \) from \( T \) by successively performing the following, for each connected component \( C \) of \( G[X] \): Replace \( t_C \) by a new node \( t'_C \) with two children \( t_1 \) and \( t_2 \), and set \( B_{t'_C} = B_{t_1} = B_{t_2} = B_{t_C} \). Below \( t_1 \), attach the descendants of \( t_C \) (i.e., a copy of the subtree rooted at \( t_C \) taken from the tree that was obtained after the adaptations for the previously considered connected components of \( G[X] \)), and below \( t_2 \) attach a path \( s_1, \ldots, s_{|B_t \setminus N(C)|} \). Set \( B_{s_1} = B_{t_2} \setminus \{v_1\} \) and \( B_{s_i+1} = B_{s_i} \setminus \{v_{i+1}\} \) where \( \{v_1, \ldots, v_{|B_t \setminus N(C)|}\} \) is some enumeration of \( B_t \setminus N(C) \).

It is easy to see that these manipulations result in a nice \( H \)-tree decomposition without introducing bags larger than the ones already occurring in \( (T, \{B_t \mid t \in V(T)\}) \).

Hence, we can state the problem of computing a decomposition as follows:

| \( H \)-Treewidth | Parameter: \( k \) |
|-------------------|------------------|
| **Instance:**     | A graph \( G \), an integer \( k \). |
| **Task:**         | Find an \( H \)-torso \( U \) of \( G \) such that \( tw(U) \leq k \), or correctly determine that no such \( H \)-torso exists. |

### 3.2 An Algorithmic Meta-Theorem

Before proceeding to the flagship application of \( H \)-treewidth where \( H \) is the class of graphs of bounded rank-width, here we give a generic set of conditions that allow fixed-parameter algorithms for problems parameterized by \( H \)-treewidth. Specifically,
we consider graph problems that are finite-state [8] or have finite integer index [3, 6, 27]. Informally speaking, such problems only transfer a limited amount of information across a small separator in the input graph and hence can be solved “independently” on both sides of such a separator. Since these notions are only used in this section, we provide concise definitions below.

First of all, we will need the notion of boundaried graphs and gluing. A graph \( \bar{G} \) is called \( t \)-boundaried if it contains \( t \) distinguished vertices identified as \( b_{\bar{G}}^1, \ldots, b_{\bar{G}}^t \). The gluing operation \( \oplus \) takes two \( t \)-boundaried graphs \( \bar{G} \) and \( \bar{H} \), creates their disjoint union, and then alters this disjoint union by identifying the boundaries of the two graphs (i.e. by setting \( b_{\bar{G}}^i = b_{\bar{H}}^i \) for each \( i \in [t] \)).

Consider a decision problem \( \mathcal{P} \) whose input is a graph. We say that two \( t \)-boundaried graphs \( \bar{C} \) and \( \bar{D} \) are equivalent, denoted by \( \bar{C} \sim_{\mathcal{P}, t} \bar{D} \), if for each \( t \)-boundaried graph \( \bar{H} \) it holds that \( \bar{C} \oplus \bar{H} \in \mathcal{P} \) if and only if \( \bar{D} \oplus \bar{H} \in \mathcal{P} \).

We say that \( \mathcal{P} \) is finite-state (or FS, in brief) if, for each \( t \in \mathbb{N} \), \( \sim_{\mathcal{P}, t} \) has a finite number of equivalence classes.

Next, consider a decision problem \( \mathcal{Q} \) whose input is a graph and an integer. In this case we say that two \( t \)-boundaried graphs \( \bar{C} \) and \( \bar{D} \) are equivalent (denoted by \( \bar{C} \sim_{\mathcal{Q}, t} \bar{D} \)) if there exists an offset \( \delta(\bar{C}, \bar{D}) \in \mathbb{Z} \) such that for each \( t \)-boundaried graph \( \bar{H} \) and each \( q \in \mathbb{Z} \):

\[(\bar{C} \oplus \bar{H}, q) \in \mathcal{Q} \text{ if and only if } (\bar{D} \oplus \bar{H}, q + \delta(\bar{C}, \bar{D})) \in \mathcal{Q}.
\]

We say that \( \mathcal{Q} \) has finite integer index (or is FII, in brief) if, for each \( t \in \mathbb{N} \), \( \sim_{\mathcal{Q}, t} \) has a bounded number of equivalence classes.

We note that a great number of natural graph problems are known to be FS or FII. For instance, all problems definable in Monadic Second Order logic are FS [3, Lemma 3.2], while examples of FII problems include VERTEX COVER, INDEPENDENT SET, FEEDBACK VERTEX SET, DOMINATING SET, CONNECTED DOMINATING SET, and EDGE DOMINATING SET [27].

We say that a FS or FII problem \( \mathcal{P} \) is efficiently extendable on a graph class \( \mathcal{H} \) if there is a fixed-parameter algorithm (parameterized by \( t \)) that takes as input a \( t \)-boundaried graph \( \bar{G} \) such that the boundary is a modulator to \( \mathcal{H} \) and outputs the equivalence class of \( \bar{G} \) w.r.t. \( \sim_{\mathcal{P}, t} \).

**Theorem 6.** Let \( \mathcal{P} \) be a FS or FII graph problem and \( \mathcal{H} \) be a graph class. If

1. \( \mathcal{P} \) is efficiently extendable on \( \mathcal{H} \),
2. \( \mathcal{P} \) is FPT parameterized by treewidth, and
3. \( \mathcal{H} \)-treewidth is FPT,

then \( \mathcal{P} \) is FPT parameterized by \( \mathcal{H} \)-treewidth.

**Proof.** \( \mathcal{P} \) can be solved using the following algorithm. First of all, we use Point 3. to compute an \( \mathcal{H} \)-torso \( G \circ X \) of treewidth \( k \), where \( k \) is the \( \mathcal{H} \)-treewidth of the input graph \( G \). Next, for each connected component \( C \) of \( G[X] \), we use the fact that \( C \in \mathcal{H} \) and
Point 1. to compute the equivalence class of the boundaried graph \( \overline{H} = G[C \cup N(C)] \)
where the boundary is \( N(C) \). Note that since \( \text{tw}(G \circ X) \leq k \) and \( N(C) \) forms a clique
in \( G \circ X \), \(|N(C)| \leq k\) and hence this step takes only fixed-parameter time.

Our next task is to compute a minimum-size representative of the equivalence class \( \alpha \) of \( \overline{H} \); we remark that the procedure used to do so in the general setting of this
meta-theorem does not allow us to give any explicit upper bound on the parameter
dependency. We describe the procedure for the case of finite-state problems; FII prob-
lems are handled analogously, with the sole distinction being that we also keep track
of the offset \( \delta \). In particular, we use a brute-force enumeration argument that has pre-
viously been applied in the kernelization setting [33]: we enumerate all boundaried
graphs (by brute force and in any order with a non-decreasing number of vertices), and
for each graph we check whether its equivalence class w.r.t. \( \sim_{P,k} \) is \( \alpha \). Observe that
since the size of a minimum-cardinality representative of \( \alpha \) depends only on
\( P \) and \( k \), the number of graphs that need to be checked in this way as well as the size of each
such graph depends only on \( P \) and \( k \), and hence there exists some function of \( P \) and \( k \) which upper-bounds the runtime of this brute-force procedure. Let \( \overline{G}_\alpha \) be the computed
minimum-cardinality representative of \( \alpha \).

In the final step, for each \( C \) the algorithm glues \( G \circ X \) (with boundary \( N(C) \)) with
the computed representative \( \overline{G}_\alpha \). Let \( G' \) be the graph obtained after processing each
connected component \( C \) in the above manner. Note that since \( \text{tw}(G \circ X) \leq k \) and
each \( \overline{G}_\alpha \) has size bounded by a function of \( k \), the graph \( G' \) has treewidth bounded by
a function of \( k \). Indeed, one can extend a tree decomposition of \( G \circ X \) by adding each
\( \overline{G}_\alpha \) into a new bag (a leaf in the decomposition) adjacent to the leaf of the nice \( \mathcal{H} \)-tree
decomposition we obtain containing \( N(C) \). At this stage, the algorithm invokes Point
2. and outputs \text{YES} iff \( G' \in P \). In the case of FII problems where the input was \((G, \ell)\),
the algorithm instead outputs \text{YES} iff \((G', \ell') \in P \), where \( \ell' \) is obtained from \( \ell \) by
subtracting all the individual offsets \( \delta(G[C \cup N(C)], \overline{G}_\alpha) \). Correctness follows by the
fact that \( G_\alpha \sim_{P,k} G[C \cup N(C)] \).

\[ \blacksquare \]

4 \( \mathcal{R}_c \)-Treewidth

This section focuses on the properties of \( \mathcal{R}_c \)-treewidth, a hierarchy of graph parameters
that represent our flagship application of the generic notion of \( \mathcal{H} \)-treewidth.

4.1 Comparison to Known Parameters

It follows from Observation[2] that \( \mathcal{R}_c \)-treewidth dominates treewidth (for every \( c \in \mathbb{N} \)).
Similarly, it is obvious that \( \mathcal{R}_c \)-treewidth dominates the size of a modulator to \( \mathcal{R}_c \) (also
for every \( c \in \mathbb{N} \)). The following lemma shows that, for every fixed \( c \), \( \mathcal{R}_c \)-treewidth is
dominated by rankwidth.

\textbf{Lemma 7.} Let \( c \in \mathbb{N} \). If \( \text{tw}_{\mathcal{R}_c}(G) = k \) then \( \text{rw}(G) \leq c + k + 1 \).

\textbf{Proof.} Let the \( \mathcal{R}_c \)-treewidth of \( G \) be witnessed by some nice \( \mathcal{R}_c \)-tree-decomposition
\((X, T, \{B_t \mid t \in V(T)\})\) of width \( k \).
We can obtain a rank-decomposition \((T', \mu)\) of \(G\) from \((X, T, \{B_t \mid t \in V(T)\})\) as follows:

For vertices \(v\) of \(G \circ X\) such that there is no leaf node \(t \in V(T)\) with \(B_t = \{v\}\), let \(t \in V(T)\) be a forget node with child \(t'\) such that \(B_t \cup \{v\} = B_{t'}\). Turn \(t'\) into a join node by introducing \(t_1, t_2\) with \(B_{t_1} = B_{t_2} = B_v\) as children of \(t'\), attaching the former child of \(t'\) to \(t_1\) and a new leaf node \(v\) with \(B_{t_v} = \{v\}\) below \(t_2\). Note that this preserves the fact that for any \(v \in V(G) \setminus X, T[\{u \in V(T) \mid v \in B_u\}]\) is a tree. Now we can choose for each \(v \in V(G \circ X)\) some \(\mu(v) \in V(T)\) such that \(B_\mu = \{v\}\). This defines an injection from \(V(G \circ X)\) to the leaves of \(T\). However not every leaf of \(T\) is mapped to by \(\mu\). On one hand there are the boundary leaf nodes, below which we will attach subtrees to obtain a rank-decomposition of \(G\). On the other hand there may be \(v \in V(G \circ X)\) for which the choice of \(\mu(v)\) was not unique, i.e. there is \(t \neq \mu(v)\) with \(B_t = \{v\}\). For all such \(v\) and \(t\) we delete all nodes on the root-\(t\)-path in \(T\) that do not lie on a path from the root to a vertex in \(\{\mu(w) \mid w \in V(G \circ X)\} \cup \{t' \mid t'\text{ boundary leaf node}\}\). This turns \(\mu\) into an injection from \(V(G \circ X)\) to the leaves of \(T\), that is surjective on the non-boundary leaf nodes.

Next, we extend \((T, \mu)\) to a rank-decomposition of \(G\) by proceeding in the following way for each connected component \(C\) of \(G[X]\):

Let \(t_C \in V(T)\) be the boundary leaf node with \(B_t = N(C)\). Since \(rw(C) \leq c\), we find a rank-decomposition \((T_C, \mu_C)\) of \(C\) with width at most \(c\). Attach \(T_C\) below \(t_C\). Let \(T'\) be the tree obtained by performing these modifications for all connected components \(C\). Consider the rank-decomposition of \(G\) given by

\[
(T', v) \mapsto \begin{cases} 
\mu(v) & \text{if } v \in V(G \circ X) \\
\mu_C(v) & \text{if } v \in C \text{ for some } C \text{ as above}
\end{cases}
\]

We show that its width is at most \(c + k + 1\). Any edge \(e\) of \(T'\) is of one of the following types:

- \(e\) corresponds to an edge already contained in \(T\): Then \(e\) induces a bipartition \((X_G, Y_G)\) of \(V(G)\). Fix \(t \in V(T)\) to be the vertex in which \(e\) starts. Let \(x \in X_G\) and \(y \in Y_G\) be such that \(xy \in E(G)\). Observe that \(e\) does not separate neighbors in \(X\) as these lie within the same connected component of \(G[X]\) whose rank-decomposition is, by construction, attached completely within one of the two subtrees of \(T'\) separated by \(e\). So, we consider \(x \in V(G \circ X)\). If \(y \in V(G \circ X)\) then \(x\) and \(y\) occur together in some bag of the original tree of the tree decomposition, and as remarked earlier this is still the case modified tree. This implies that at least one of \(\{x, y\}\) must be present in \(B_t\). If \(y \notin V(G \circ X)\), by the construction of \(T'\), \(y\) corresponds to a leaf \(t_y \in V(T')\) in a subtree attached to \(T\) rooted at \(t_C \in V(T)\) with \(B_{t_C} = N(C) \ni x\) and \(t_C\) and \(t_y\) are not separated by the removal of \(e\). Also, \(x\) corresponds to a leaf \(t_x \in V(T)\) which is in the subtree of \(T' - e\) not containing \(t_y\), i.e. the subtree not containing \(t_C\). This means any subtree containing \(t_C\) and \(t_x\) also contains \(t\), and since \((T, \{B_t \mid t \in V(T)\})\) is a tree-decomposition and \(x \in B_{t} \cap B_{t_C}\) this means \(x \in B_t\).

In both cases at least one of \(\{x, y\}\) is in \(B_t\). Since this argument applies for every edge crossing the bipartition \((X_G, Y_G)\), it follows that \(A_G[X_G, Y_G]\) may
only contain “1” entries in rows and columns that correspond to the vertices in \(B_t\). Since \(|B_t| \leq k + 1\), it holds that \(A_G[X_G, Y_G] \) can be converted into a zero matrix by deleting at most \(k + 1\) rows plus columns, which is a sufficient condition for \(A_G[X_G, Y_G] \) having rank at most \(k + 1\).

- \(e\) corresponds to an edge in a rank-decomposition of some connected component \(C\) of \(G[X]\): Then \(e\) induces a bipartition \((X_G, Y_G)\) of \(V(G)\) and a bipartition \((X_C, Y_C)\), where \(X_C = X_G \cap C\) and \(Y_C = Y_G \cap C\), of \(C\). Since vertices of \(C\) are only connected to vertices in \(N(C)\) outside of \(C\) and \(N(C) \subseteq B_t\) for some \(t \in V(T)\), we have \(\rho_G(X_G) \leq \rho_C(X_C) + |N(C)| \leq c + k + 1\).

- \(e\) corresponds to an edge connecting the rank-decomposition of some connected component \(C\) of \(G[X]\) to \(T\): Then the bipartition induced is \((C, V(G) \setminus C)\) and as \(N(C) \subseteq B_t\) for some \(t \in V(T)\) \(\rho_C(C) \leq |N(C)| \leq c + k + 1\).

Next, we compare \(R_c\)-treewidth to Telle and Saether’s notion of \(sm\)-width [54], which is another parameter that lies between treewidth and rank-width. The rough idea behind the definition of this parameter is to generalize treewidth towards denser graph classes by allowing the use of graph splits in an explicit way without increasing the parameter value. For completeness, we provide a formal definition for \(sm\)-width below using the general notions of branch-width and graph splits.

**Definition 8 (Branch-Width [53])**. A branch decomposition of the vertex set of a graph \(G\) is a binary tree \(T\) with \(V(G)\) as leaf set. Each edge \(e\) in \(T\) partitions the vertices of \(V(G)\) into two sets (say \(X_e\) and \(V(G) \setminus X_e\)), corresponding to the leaves present in the two connected components of \(T - e\).

A cut function is a function \(c : P(V(G)) \rightarrow \mathbb{N}\) such that

\[
\forall A \subseteq V(G) \quad c(A) = c(V(G) \setminus A).
\]

The width of a branch decomposition with respect to the cut function \(c\) is given by \(\max_{e \in E(T)} c(X_e)\). The branch width of a graph with respect to a cut function is the minimum of the branch width with respect to the cut function over all branch decompositions of the vertex set of this graph.

**Definition 9 (Split)**. A \(A \subseteq V(G)\) is a split of a graph \(G\) if

1. \(|A| \geq 2, |V(G) \setminus A| \geq 2\) and
2. \(\forall v, w \in A \quad N(v) \cap (V(G) \setminus A) \neq \emptyset \land N(w) \cap (V(G) \setminus A) \neq \emptyset \implies N(v) \cap (V(G) \setminus A) = N(w) \cap (V(G) \setminus A)\).

**Definition 10 (Split-Matching-Width [54])**. The split-matching-width (or \(sm\)-width) of a graph \(G\), denoted by \(smw(G)\), is the branch width of \(G\) with respect to the cut function given by

\[
c(A) = \begin{cases} 
1 & \text{if } A \text{ is a split} \\
\max\{|M| \mid M \text{ is a matching of } G[A, V(G) \setminus A]\} & \text{otherwise}
\end{cases}
\]
For clarity, we remark that omitting the first case for \(c(A)\) in the definition of sm-width results in a parameter called \(mm\)-width which is asymptotically equivalent to treewidth \([41]\).

**Lemma 11.** \(\mathcal{R}_c\)-treewidth and smw are incomparable. More formally, the following hold:

1. For all \(c \in \mathbb{N}\) there is a sequence of graphs \((G_i)_{i \in \mathbb{N}}\) such that for all \(i \in \mathbb{N}\)
   \[\text{tw}_{\mathcal{R}_c}(G_i) \geq i, \text{ but smw}(G_i) \leq c + 3.\]

2. For all \(c \geq 3\) there is a sequence of graphs \((G_i)_{i \in \mathbb{N}}\) such that for all \(i \in \mathbb{N}\)
   \[\text{smw}(G_i) \geq i, \text{ but tw}_{\mathcal{R}_c}(G_i) = 0.\]

**Proof.** Consider the following construction: For arbitrary \(c\) there is a graph \(H_c\) such that \(\text{tw}(H_c) = c + 2\) and \(\text{rw}(H_c) = c + 1\); notably, this can be achieved by setting \(H_c\) to be the \((c+2) \times (c+2)\) square grid \([40]\). Now let \(G_i\) be given by taking \(2 \cdot (i+1)\) copies, \(H_c^{(1)}, \ldots, H_c^{(2 \cdot (i+1))}\) of \(H_c\) and completely connecting all \(H_c^{(j)}\) with \(j\) even to all \(H_c^{(j)}\) with \(j\) odd. The split-matching-width of \(G_i\) is at most \(c + 3\) since each copy of \(H_c\) iteratively defines a split and thus can be treated independently and it is known that the maximum matching width of a graph is at most its treewidth + 1 \([55]\).

On the other hand, we claim that \(\text{tw}_{\mathcal{R}_c}(G_i) \geq i:\)

- If \(G_i\) has no non-trivial \(\mathcal{R}_c\)-torso, then since \(G_i\) is at least \(((i + 1) \cdot |V(H_c)|)\)-degenerate because of the complete connections between odd and even copies of \(H_c\), we know that
  \[\text{tw}_{\mathcal{R}_c}(G_i) = \text{tw}(G_i) \geq (i + 1) \cdot |V(H_c)| \geq i.\]

- If the \(\mathcal{R}_c\)-treewidth of \(G_i\) is given by \(\text{tw}(G_i \circ X)\) and it holds that \(X\) only intersects odd (symmetrically only even) copies of \(H_c\) then, by definition of \(G_i \circ X\), the even (respectively odd) copies of \(H_c\) form a clique in \(G_i \circ X\) and thus
  \[\text{tw}_{\mathcal{R}_c}(G_i) = \text{tw}(G_i \circ X) \geq (i + 1) \cdot |V(H_c)| - 1 \geq i.\]

- If the \(\mathcal{R}_c\)-treewidth of \(G_i\) is given by \(\text{tw}(G_i \circ X)\) and it holds that \(X\) intersects even and odd copies of \(H_c\) and \(|G - X| \geq i + 1\), then \(G_i \circ X\) is a clique of size at least \(i + 1\) and thus
  \[\text{tw}_{\mathcal{R}_c}(G_i) = \text{tw}(G_i \circ X) = |V(G_i) \setminus X| - 1 = i.\]

- Otherwise the \(\mathcal{R}_c\)-treewidth of \(G_i\) is given by \(\text{tw}(G_i \circ X)\) and it holds that \(|G - X| \leq i\). Then by the pigeon hole principle we find a \(1 \leq j \leq i\) such that
  \[|V(H_c^{(j)} \cap X)| \geq \frac{|V(G_i)| - i}{2 \cdot (i + 1)} = \frac{2 \cdot (i + 1) \cdot |V(H_c)| - i}{2 \cdot (i + 1)} > |V(H_c)| - 1.\]
  Since \(\text{rw}(H_c^{(j)}) = c + 1\), this is a contradiction to \(\text{rw}(G[X]) \leq c\).
For 2, let \( c \geq 3 \) and consider the following construction: Fix a cycle \( C_i \) on \( 3 \cdot i + 5 \) vertices. Now let \( G_i \) to be the complement graph of \( C_i \). Since \( C_i \) has rank-width 2 and the complement graph’s rank-width is at most one larger \( [34] \), we know \( \text{rw}(G_i) \leq 3 \) and we can use \( V(G_i) \) to build an \( \mathcal{R}_c \)-torso of treewidth 0. On the other hand, there is no split of \( G_i \) and the maximum matching width of a graph \( G \) is known to be at least \( \frac{1}{3} \cdot (\text{tw}(G) + 1) \) \( [55] \). So \( \text{smw}(G_i) \geq \frac{1}{3} \cdot (\text{tw}(G_i) + 1) \geq i \). ■

4.2 Computing \( \mathcal{R}_c \)-Treewidth

Our aim here is to determine the complexity of computing our parameters, i.e., finding a torso of small treewidth. Obtaining such a torso is a base prerequisite for our algorithms. We formalize the problem below.

*\( \mathcal{R}_c \)-Treewidth

| Parameter: \( k \) |
|-------------------|
| **Instance:** A graph \( G \), an integer \( k \). |
| **Task:** Find a \( \mathcal{R}_c \)-torso \( U \) of \( G \) such that \( \text{tw}(U) \leq k \), or correctly determine that no such \( \mathcal{R}_c \)-torso exists. |

**Lemma 12.** \( \mathcal{R}_c \)-Treewidth is FPT.

*Proof.* We begin by noting that there exists an CMSO$_1$ formula \( \phi_c(G) \) that is true if and only if a graph \( G \) has rank-width at most \( c \) (and, equivalently, if each connected component of \( G \) has rank-width at most \( c \)). This was observed already by Kanté, Kim, Kwon and Paul \([42]\) and follows from the fact that the property of having rank-width at most \( c \) can be characterized by a finite set of vertex minors \([50]\), a property which can be expressed in CMSO$_1$ \([12]\). For clarity, we explicitly remark that \( |\phi_c(G)| \) depends only on \( c \).

Next, we also note that there exists an CMSO$_1$ formula \( \psi_k(Z) \) (of size depending only on \( k \)) that is true if and only if a graph \( G \) has treewidth at most \( k \) \([46]\). Consider now the formula \( \psi_k(Z) \) for some fixed \( Z \subseteq V(G) \) which operates precisely like \( \psi_k \), but only on the graph \( G \circ Z \). In particular, \( \psi_k(Z) \) can be obtained from \( \psi_k \) by:

- For each vertex variable \( a \) occurring in \( \psi_k \), adding a condition forcing \( v \notin Z \);
- For each set variable \( A \) occurring in \( \psi_k \), adding a condition forcing \( A \cap Z = \emptyset \);
- Replacing each occurrence of \( \text{adj}(a, b) \) with an expression that is true if \( \text{adj}(a, b) \) or if there exists an \( a-b \) path with all internal vertices in \( G[Z] \).

Let \( \phi_{c,k}(Z) = \phi_c(G[Z]) \land \psi_k(Z) \). Consider now a vertex set \( Z \) (or, more precisely, an interpretation of \( Z \)) which satisfies \( \phi_{c,k} \). Since \( G[Z] \) satisfies \( \phi_{c,k} \), it holds that each connected component of \( G[Z] \) has rank-width at most \( c \), and hence \( G \circ Z \) is an \( \mathcal{R}_c \)-torso of \( G \). Moreover, since \( \psi_k(Z) \) is true, \( G \circ Z \) has treewidth at most \( k \).

To conclude the proof, note that Fact \([2]\) alongside Lemma \([7]\) provide an FPT algorithm (with parameter \( k \)) to find an interpretation of \( Z \) satisfying \( \phi_{c,k}(Z) \), or correctly identify that no such interpretation exists. ■

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5 Algorithms Exploiting Modulators to $\mathcal{R}_c$

For a problem $\mathcal{P}$ to be FPT parameterized by $\mathcal{R}_c$-treewidth, $\mathcal{P}$ must necessarily be FPT parameterized by treewidth and also FPT parameterized by the size of a modulator to $\mathcal{R}_c$. However, it is important to note that the latter condition is not sufficient; indeed, one can easily invent artificial problems that are defined in a way which make them trivial in both of the cases outlined above, but become intractable (or even undecidable) once parameterized by $\mathcal{R}_c$-treewidth. That is, after all, why we need the notion of efficient extendability in Theorem 6.

Hence, in order to develop fixed-parameter algorithms for CHROMATIC NUMBER, HAMILTONIAN CYCLE and MAX-CUT parameterized by $\mathcal{R}_c$-treewidth, we first need to show that they are not only FPT parameterized by the size of a modulator to $\mathcal{R}_c$, but they are also efficiently extendable. Such a result would be sufficient to employ Theorem 6 together with Lemma 12 in order to establish the desired fixed-parameter tractability results. That is also our general aim in this section, with one caveat: in order to give explicit and tight upper bounds on the parameter dependency of our algorithms, we provide algorithms that solve generalizations of CHROMATIC NUMBER, HAMILTONIAN CYCLE and MAX-CUT parameterized by the size of a modulator to $\mathcal{R}_c$, whereas it will become apparent in the next section that these generalizations precisely correspond to the records required by the treewidth-based dynamic program that will be used in the torso. In other words, the efficient extendability of our problems on $\mathcal{R}_c$ is not proved directly but rather follows as an immediate consequence of our proofs in this section and the correctness of known treewidth-based algorithms.

Before we proceed to the three algorithms, we introduce some common notation that will be useful to describe the operation of dynamic programming algorithms using rank-width. For a node $t$ of a rooted rank decomposition $(T, \mu)$ of a graph $G$, let $S_t \subseteq V(G)$ be the set of vertices mapped by $\mu$ to leaves that $t$ separates from the root (i.e., $S_t$ contains the “vertices below $t$”). For disjoint vertex sets $S$ and $Q$ of $G$, two vertices $v$ and $w$ in $S$ are twins with respect to $Q$ if $N(v) \cap Q = N(w) \cap Q$. A twin class of $S$ with respect to $Q$ is a maximal subset of $S$ that consists of pairwise twins w.r.t. $Q$. Twin classes are a useful concept for designing dynamic programming algorithms that run on rank-decompositions, since there are at most $2^{ rw(G) }$ twin classes of the set $S_t$ of vertices assigned in a subtree rooted at an arbitrary node $t \in V(T)$ with respect to the remaining vertices in the graph. For each node $t$ of $T$ we enumerate these twin classes as $R_t^1, R_t^2, \ldots, R_t^z$, where $z = 2^{ rw(G) }$ is a bound on the number of twin classes of $S_t$ w.r.t. $V(G) \setminus (X \cup S_t)$; in the case where there are only $i < z$ twin classes, we let $R_t^i = \emptyset$ for all $i < j < z$.

For two children $t_1$ and $t_2$ of some node, we define $L_{t_1,t_2}$ as a matrix whose rows are indexed by the indices of twin classes of $S_{t_1}$ and columns are indexed by the indices of twin classes of $S_{t_2}$ such that for $L_{t_1,t_2}[j_1, j_2] = 1$ if $R_{t_1}^{j_1}$ is complete to $R_{t_2}^{j_2}$, and $L_{t_1,t_2}[j_1, j_2] = 0$ otherwise. For a child $t'$ of a node $t$, we define a function $U_{t', t}$ from the indices of twin classes of $S_{t'}$ to the indices of twin classes of $S_t$ such that for $U_{t', t}(j_1) = j_2$ if $R_{t'}^{j_1}$ is fully contained in $R_t^{j_2}$. This function is well defined because every twin class at $S_{t'}$ is fully contained in one of twin classes at $S_t$. Informally, $L_{t_1,t_2}$ specifies which edges are created between twin classes at $t$, while $U_{t_1}$ and $U_{t_2}$ specifies how twin-classes are “reshuffled” at $t$; note that all of these matrices can be
pre-computed in polynomial time.

5.1 Chromatic Number

In Chromatic Number, we are given a graph $G$ and asked for the smallest number $\chi(G)$ such that the vertex set of $G$ can be properly colored using $\chi(G)$ colors, i.e., the smallest number $\chi(G)$ such that $V(G)$ can be partitioned into $\chi(G)$ independent sets. Our aim in this section is to solve a variant of Chromatic Number on graphs with a $k$-vertex modulator $X$ to $R_c$ where $X$ is precolored:

| $R_c$-Precoloring Extension | Parameter: $k$
|-------------------------------|------------------|
| **Instance:** A graph $G$, a $k$-vertex modulator $X \subseteq V(G)$ to $R_c$ and a coloring of $X$. |  |
| **Task:** Compute the smallest number of colors required to extend the coloring of $X$ to a proper coloring of $G$. |  |

Theorem 13. $R_c$-Precoloring Extension can be solved in time $2^{O(k)} n^{O(1)}$.

**Proof.** Let $G$ be a graph together with a $k$-vertex modulator $X$ to $R_c$ and a proper coloring of $X$ by colors $\{1, 2, \ldots, k\} = [k]$; let $col_X$ be the set of colors assigned to at least one vertex in $X$. Our starting point is a rooted rank-decomposition $(T, \mu)$ of $G - X$ of width at most $c$, which may be computed in time $O(n^3)$, using Theorem 1.

On a high level, our algorithm will proceed by dynamic programming along $(T, \mu)$ and group colors together based on which twin classes they occur in (analogously as in the XP algorithm for Chromatic Number parameterized by clique-width, due to Kobler and Rotics [45]), but it keeps different (more detailed) records about the at most $k$ colors used in $X$.

Recall that, for any $t \in V(T)$, $\rho_{G - X}(S_t) \leq c$ and that there are at most $z = 2^c$ twin classes of $S_t$ w.r.t. $V(G) \setminus (X \cup S_t)$.

**The Table.** We are now ready to formally define the dynamic programming table $M_t$ that stores the information we require at a node $t$ of $T$. For $b_1, b_2, \ldots, b_k \subseteq [z]$ and $\{ d_Z \in [n] \mid Z \subseteq [z] \}$, we let

$$M_t(b_1, b_2, \ldots, b_k, \{ d_Z \mid Z \subseteq [z] \}) = 1$$

if there is a proper coloring of $G_t$ such that

- for every $i \in [k]$, the color $i$ appears in twin classes in $\{ R^t_j : j \in b_i \}$ and does not appear in other twin classes, and

- for every $Z \subseteq [z]$, $d_Z$ is the number of colors from $\{k + 1, k + 2, \ldots, n\}$ that appear in twin classes of $\{ R^t_j : j \in Z \}$ and do not appear in other twin classes;

on the other hand, if no such proper coloring exists, we let $M_t(b_1, b_2, \ldots, b_k, \{ d_Z \mid Z \subseteq [z] \}) = 0$.

The table $M_t$ will be filled in a leaf-to-root fashion. Observe that by definition of $d_Z$’s, for distinct subsets $Z_1, Z_2$ of $[z]$, $d_{Z_1}$ and $d_{Z_2}$ count disjoint sets of colors. This
provides an easy way to count the total number of colors used. Since all vertices in 
\( G - X \) appear below the root node \( r \), the minimum number of colors required for a 
proper coloring of \( G \) will be the minimum value of 
\[
|\text{col}_X \cup \{ i \in [k] \mid b_i \neq \emptyset \}| + \sum_{Z \subseteq [z]} d_Z,
\]
over all tuples \((b_1, b_2, \ldots, b_k, \{ d_Z \mid Z \subseteq [z] \})\) whose \( M_r \) value is 1. Thus, in order 
to complete the proof, it suffices to show how to fill in the table \( M_t \) for each node 
\( t \in V(T) \).

**Leaf Nodes.** Consider a leaf node \( t \), and assume that a vertex \( v \) is assigned to \( t \). To 
be a proper coloring, \( v \) has to have a color that differs from the colors in \( N_G(v) \cap X \). 
So, either we choose a color \( i \in [k] \) that does not appear in \( N_G(v) \cap X \), in which case 
we let \( M_t \) assign the tuple \((b_1 = 1 \text{ and all other values are } 0) \) to \( t \), or we choose a 
color outside of \([k]\), in which case we let \( M_t \) assign the tuple \((\text{all of } b_i \text{'s and } d_Z \text{'s are } 0, \text{ except } d_{\{j\}} = 1 \) \( j \) is the label of the twin class containing \( v \)) \) to \( t \). All other 
tuples are assigned a value of 0 by \( M_t \).

**Internal Nodes.** Let us now consider an internal node \( t \) with children \( t_1, t_2 \). To check 
whether \( M_t(b_1, b_2, \ldots, b_k, \{ d_Z \mid Z \subseteq [z] \}) = 1 \) for some tuple \( T = (b_1, b_2, \ldots, b_k, \{ d_Z \mid Z \subseteq [z] \}) \) at \( t \), we proceed as below, for all pairs of tuples \( \mathcal{T}_i = (b_{i1}, \ldots, b_{ik}, \{ d_{Zi} \mid Z \subseteq [z] \}) \) stored at \( t_i, i \in [2] \).

First, for the colors in \([k]\), we have to check whether they are used in the twin 
classes in \( R_{j_1}^{t_i} \) and \( R_{j_2}^{t_i} \) that are complete to each other (meaning that an edge will be 
created connecting vertices of the same color), and if yes then we discard this tuple and 
proceed to the next. More precisely, we check the following:

- (Compatibility 1) For every \( i \in [k] \) with \( b_{i1}^{t_1} \neq \emptyset \text{ and } b_{i2}^{t_2} \neq \emptyset \), there are no 
  \( j_1 \in b_{i1}^{t_1} \) and \( j_2 \in b_{i2}^{t_2} \) such that \( L_{t_1,t_2}[j_1,j_2] = 1 \).

Second, we need to deal with the \( d_{Zi}^{t_i} \) components in our tables. Here, we branch 
on (“guess”) how many colors that appear in twin classes with indices in \( d_{Zi}^{t_i} \) are the 
same as the colors that appear in in twin classes with indices in \( d_{Zj}^{t_j} \) for all pairs \( Z_1 \) and 
\( Z_2 \). In particular, for all \( Z_1, Z_2 \subseteq [z] \), let \( \tau(Z_1, Z_2) \) be a non-negative integer; this will 
correspond to the number of common colors in \( d_{Z1}^{t_1} \) and \( d_{Z2}^{t_2} \). Similar to (Compatibility 
1) we have to check that for two complete twin classes, the same color does not appear 
at the same time. So, for each such mapping \( \tau \), we check the following:

- (Compatibility 2) For every \( Z_1, Z_2 \subseteq [z] \) with \( \tau(Z_1, Z_2) > 0 \), there are no 
  \( j_1 \in Z_1 \) and \( j_2 \in Z_2 \) such that \( L_{t_1,t_2}[j_1,j_2] = 1 \).

Observe that if Compatibility 2 would be violated, then \( R_{j_1}^{t_1} \) and \( R_{j_2}^{t_2} \) contain the same 
color while being complete, meaning that the result would not be a proper coloring. 
Hence if Compatibility 2 is violated, we discard and proceed to the next available 
branch (i.e., choice of \( \tau \)).
Third, we check whether \( T_1 \) and \( T_2 \) will be combined into \( T \) with the function \( \tau \). Recall that the function \( U_{t_1,t} \) provides us with information about which set \( R^i_j \) will contain \( R^i_{j,t} \). We will use this function to rename our twin classes accordingly. When we perform this renaming, we also update \( \tau, d^1_Z, d^2_Z \) and obtain new functions \( \tau', dm^1_Z, dm^2_Z \), respectively. For example, if index 2 were to be renamed to index 1 at node \( t_1 \) and \( \tau(\{1, 2\}, \{3, 4\}) = 3 \), then \( \tau'(\{1, 2\}, \{3, 4\}) \) becomes 0 (since twin class 2 is empty after renaming) and we apply \( \tau'(\{1\}, \{3, 4\}) \leftarrow \tau(\{1\}, \{3, 4\}) + \tau(\{1, 2\}, \{3, 4\}) \).

- (Renaming) For each \( i \in \{1, 2\} \) and \( j \in [z] \), we rename twin class \( R^i_j \) to \( Rm^i_{U_{t_1,t}(j)} \), and update \( \tau, d^1_Z, d^2_Z \) to \( \tau', dm^1_Z, dm^2_Z \) accordingly.

Finally, for each \( Z \subseteq [z] \) we check whether the number of colors in \( dm^1_Z \) and \( dm^2_Z \) correctly add up to \( d^2_Z \) when taking \( \tau \) into account. This is carried out by checking the following equality holds.

\[
d^2_Z = dm^1_Z - \sum_{Z' \subseteq [z]} \tau'(Z, Z') \\
+ dm^2_Z - \sum_{Z' \subseteq [z]} \tau'(Z', Z) \\
+ \sum_{Z_1 \subseteq [z]} \sum_{Z_2 \subseteq [z]} \tau'(Z_1, Z_2).
\]

Informally, the first row counts the number of colors that are unique to \( dm^1_Z \), i.e., do not appear on any twin classes in \( t_2 \). The second row then does the same for \( dm^2_Z \). To correctly determine the actual number of colors in \( d^2_Z \) that we obtain by following \( \tau \), we now need to add colors which occur in both \( t_1 \) and \( t_2 \) and which end up precisely in \( Z \) — that is what the third row counts. Summarizing, we assign \( M_i(b_1, b_2, \ldots, b_k, \{ d^2_Z \ | \ Z \subseteq [z] \}) = 1 \) if all the above conditions are satisfied in at least one branch, and use 0 otherwise.

We conclude the proof by analyzing the running time of the described algorithm. For a leaf node \( t \), we can fill in the table in time \( O(k) \). For an internal node \( t \), there are at most \( z^k \cdot n^2z \) different tuples \( \{ b_1, b_2, \ldots, b_k, \{ d^2_Z \ | \ Z \subseteq [z] \} \} \). For each, (Compatibility 1) can be checked in time \( O(k) \). The number of possible functions \( \tau \) is the same as the number of possible assignments of a number from \([n]\) to each pair \( Z_1, Z_2 \subseteq [z] \). Thus, it is bounded by \( n^{2z^2} \). Finally, updating \( \tau, d^1_Z, d^2_Z \) to \( \tau', dm^1_Z, dm^2_Z \) can be done in time \( O(k) \). So, for a given coloring on \( X \), we can find a minimum number of necessary colors in time \( 2^{zk}n^{O(1)} \).

### 5.2 Hamiltonian Cycle

In Hamiltonian Cycle, we are given an \( n \)-vertex graph \( G \) and asked whether \( G \) contains a cycle of length \( n \) as a subgraph. Note that if we restrict \( G \) to some subset of vertices \( Y \subseteq V(G) \), then what remains from a Hamiltonian Cycle in \( G \) is a set of paths that start and end in the neighborhood of \( V(G) \setminus Y \). Hence, the aim of this section is to solve the following generalization of Hamiltonian Cycle:
**Theorem 14.** \( \mathcal{R}_c \)-Disjoint Paths Cover can be solved in time \( 2^{O(k)} n^{O(1)} \).

**Proof.** Let \( G \) be a graph together with a \( k \)-vertex modulator \( X \subseteq V(G) \) to \( \mathcal{R}_c \), and \( m \leq k \) pairs \((s_1,t_1), \ldots, (s_m,t_m)\) of vertices from \( X \) with \( s_i \neq t_i \) for all \( i \in [m] \).

**Task:** Decide whether there are internally vertex-disjoint paths \( P_1, P_2, \ldots, P_m \) in \( G \) such that \( P_i \) is a path from \( s_i \) to \( t_i \) and every vertex in \( G - X \) belongs to precisely one path in \( P_1, P_2, \ldots, P_m \).

We define a DP table \( M \) in \( (\mathcal{R}_c \text{-Disjoint Paths Cover}) \).

**Instance:** A graph \( G \), a \( k \)-vertex modulator \( X \subseteq V(G) \) to \( \mathcal{R}_c \), and \( m \leq k \) pairs \((s_1,t_1), \ldots, (s_m,t_m)\) of vertices from \( X \) with \( s_i \neq t_i \) for all \( i \in [m] \).

**Parameter:** \( k \)

**Task:** Decide whether there are internally vertex-disjoint paths \( P_1, P_2, \ldots, P_m \) in \( G \) such that \( P_i \) is a path from \( s_i \) to \( t_i \) and every vertex in \( G - X \) belongs to precisely one path in \( P_1, P_2, \ldots, P_m \).

We define a DP table \( M_t \) such that \( M_t(D, SP) = 1 \) if there is an \( X \)-lenient path-partition \( \mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2 \) of \( G_t \) such that

- \( \mathcal{P}_1 \) is the subset of \( \mathcal{P} \) that consists of all paths fully contained in \( G_t - X \),

- \( \mathcal{P}_2 \) is the subset of \( \mathcal{P} \) that consists of all paths that start in \( s_t \) and end in \( t_t \) in \( R_x^t \). Similarly, \( \mathcal{P}_1 \) will indicate the existence of a path starting in \( s_t \) and ending in \( t_t \) in \( R_x^t \). The tuple \((i,i)\) will indicate the existence of a path starting in \( s_t \) and ending in \( t_t \). Note that there are at most \((n + 1)z^2 \) possibilities for \( D \). For an element of \( SP \), there are \( 2mz + 1 \) possible elements in \( SP \), and thus there are at most \( 2^{2kz+1} \) possibilities for \( SP \). It implies that the number of possible tuples \((D, SP)\) is bounded by \((n + 1)z^2 \) possibilities for \( SP \).
for every $d_{b_1,b_2} \in D$, there are exactly $d_{b_1,b_2}$ distinct paths in $\mathcal{P}_1$ whose endpoints are contained in $R_{b_1}$ and $R_{b_2}$,

- for every $(\ell, r, p)$ or $(\ell, r) \in SP$, there is a unique path $P \in \mathcal{P}_2$ such that
  - if $\ell = i > 0$, then $\ell(P) = s_i$, and if $r = i > 0$, then $r(P) = t_i$,
  - if $\ell = 0$, then $\ell(P) \in R^\ell_{p}$, and if $r = 0$, then $r(P) \in R^r_{p}$.

In this case, we say that the $X$-lenient path-partition $\mathcal{P}$ is a partial solution with respect to $(D, SP)$, and also $(D, SP)$ is a characteristic of $\mathcal{P}$. We say that paths in $\mathcal{P}_1$ are normal paths, and paths in $\mathcal{P}_2$ are special paths. The value $M_t(D, SP) = 0$ if there is no such $X$-lenient path-partition of $G_t$. We define $Q_t$ as the set of all tuples $(D, SP)$ where $M_t(D, SP) = 1$.

The table $M_t$ will be filled in a leaf-to-root fashion. Since all vertices in $G - X$ appear below the root node $m$, to decide whether there is a desired $X$-lenient path-partition, it suffices to confirm that there are $D$ and $SP$ such that

- $M_{ro}(D, SP) = 1$,
- for every $d_{b_1,b_2} \in D$, $d_{b_1,b_2} = 0$,
- for every $i \in \{1, 2, \ldots, m\}$, $(i, i) \in SP$.

Thus, in order to complete the proof, it suffices to show how to fill in the table $M_t$ for each node $t \in V(T)$.

**Leaf Nodes.** First consider a leaf node $t$. Let $J = \{i \in [m] \mid \{s_i, t_i\} \in E(G)\}$. For every pair $i \in J$, we can branch on whether a $s_i$-$t_i$ path will be only the edge $\{s_i, t_i\}$, or it will contain also some vertices from $G - X$. For $I \subseteq J$, let us denote by $SP_I$ the set $\{(i, i) \mid i \in I\}$. Moreover, let a vertex $v$ be assigned to $t$, and $x$ be the index of the twin class $\{v\}$. If it is adjacent to $s_i$ or $t_i$ for some $i$, then we may select it as a starting point or an ending point of a desired path or both. So, for every tuple $(D, SP)$ such that $d_{b_i,b_j} = 0$ for all $b_i, b_j \in [z]$, and $SP = SP_I \cup SP_i$, where $I \subseteq J \setminus \{i\}$ and

$$SP_i = \begin{cases} 
\{(i, 0, x)\} & \text{if } v \in N_G(s_i) \\
\{(0, i, x)\} & \text{if } v \in N_G(t_i) \\
\{(i, i)\} & \text{if both holds,}
\end{cases}$$

we give $M_t(D, SP) = 1$. Note, that if $v \in N_G(s_i) \cap N_G(t_i)$, then there is a tuple for each of the three possibilities for $SP_i$. Also, we could just keep it as a path whose endpoints are not linked to $X$ in the final solution. In this case, for a tuple $(D, SP)$ where

- $d_{x,x} = 1$ and all other $d_{b,y}$’s have values 0,
- $SP = SP^I$ for some $I \subseteq J$,

we give $M_t(D, SP) = 1$. Other tuples get the value 0.
Internal Nodes. Let $t$ be an internal node, and let $t_1, t_2$ be the children of $t$. We assume that $Q_{t_1}$ and $Q_{t_2}$ were already computed, and we want to generate $Q_t$ from these sets.

For all pairs of tuples $(D^1, SP^1) \in Q_{t_1}$ and $(D^2, SP^2) \in Q_{t_2}$, we will generate all possible tuples $(D, SP)$ such that for an $X$-lenient path-partition $P$ with respect to $(D, S)$, its restriction to $G_t$ has $(D^i, SP^i)$ as a characteristic.

We first test basic conditions that the two partial solutions can be a partial solution $P$ with respect to those tuples. More precisely, we take that

- (Special terminal) Each integer in $\{0, \ldots, m\}$ appears at most once as an $\ell$ among all tuples $(\ell, 0, p)$ or $(\ell, \ell)$ in $SP^1 \cup SP^2$, and similarly, each integer in $\{0, \ldots, m\}$ appears at most once as an $r$ among all tuples $(0, r, p)$ or $(r, r)$ in $SP^1 \cup SP^2$.

If (Special terminal) condition is not satisfied, then we skip this pair of tuples. In the rest, we assume that (Special terminal) condition is satisfied.

Secondly, we rename twin classes to consider both partial solutions together.

- (Renaming) For each $i \in \{1, 2\}$ and $j \in \{0\}$, we rename twin class $R_{i,j}$ to $R_{i,j}(\ell)$, and update $D^1, D^2, SP^1, SP^2$ to $Dm^1, Dm^2, SPm^1, SPm^2$ accordingly.

For example, if index 2 were to be renamed to index 1 at node $t_1$ and $d^2_{1,3} = 3$, then $dm_{2,3}^1$ in $Dm^1$ becomes 0 and apply $dm_{1,3}^1$ to $d_{1,3}^1 + d_{2,3}^1$. Similarly, if there was a tuple $(i, 0, 2)$ in $SP^1$, then we add a tuple $(i, 0, 1)$ to $SPm^1$. Because this simply change the indices of twin classes, (Special terminal) condition is still valid. After then we take the sum of $(Dm^1, SPm^1)$ and $(Dm^2, SPm^2)$ which correspond to the disjoint union of partial solutions with respect to those tuples. More precisely, we take that

- $D_{begin} = \{a_{begin}^{\begin{bmatrix} x & y \end{bmatrix}} \mid x, y \in \{0\}\}$ where $a_{begin}^{\begin{bmatrix} x & y \end{bmatrix}} = dm_{x,y}^1 + dm_{x,y}^2$ for all $x, y \in \{0\}$
  and $dm_{x,y}^1 \in Dm^1$ and $dm_{x,y}^2 \in Dm^2$, and

- $SP_{begin} = SPm^1 \cup SPm^2$.

Because of the condition (Special terminal), no integer in $\{0, \ldots, m\}$ appears twice in $SP_{begin}$. Therefore, $(D_{begin}, SP_{begin})$ is a possible tuple for $G_t$, which corresponds to exactly the disjoint union of partial solutions with respect to $(D^1, SP^1)$ and $(D^2, SP^2)$.

We put it to $Q_t$.

Next, we consider to merge two paths in an $X$-lenient path-partition, using edges between $V(G_{t_1}) \setminus X$ and $V(G_{t_2}) \setminus X$. There are three types of merging operations: merging two normal paths, merging a normal path and a special path, and merging two special paths.

- (Merging two normal paths) Suppose there are $x_1, y_1, x_2, y_2 \in \{0\}$ with $x_1' = U_{t_1,t}(x_1), y_1' = U_{t_1,t}(y_1), x_2' = U_{t_2,t}(x_2), y_2' = U_{t_2,t}(y_2)$ such that $R^i_{x_i'}$ is complete to $R^i_{x'_i, y'_i} > 0$ and $d^i_{x'_i, y'_i} > 0$. Then we subtract 1 from each of $d^b_{x_i', y'_i}$ and $d^b_{x'_i, y'_i}$ and add 1 to $d^b_{y_i', y'_i}$.
• (Merging a normal path and a special path) Suppose there are \( x, x_2, y_2 \in [z] \) with 
\[
x' = U_{t_1,t}(x), x'_2 = U_{t_2,t}(x_2), y'_2 = U_{t_2,t}(y_2)
\]
such that \( R^1_{t_1} \) is complete to \( R^2_{t_2} \) and \((i, 0, x) \in SP^1 \) and \( d^2_{x_2,y_2} > 0 \). Then we subtract 1 from \( d^\text{begin} \) and replace 
\((i, 0, x) \) with \((i, 0, y'_2) \) in \( SP^\text{begin} \). The symmetric operation is also applied.

• (Merging two special paths) Suppose there are \( x, y \in [z] \) with 
\[
x' = U_{t_1,t}(x), y' = U_{t_2,t}(y)\]
such that \( R^2_{t_2} \) is complete to \( R^2_{t_2} \) and \((i, 0, x) \in SP^1 \) and \((0, i, y) \in SP^2 \). Then we remove \((i, 0, x') \) and \((0, i, y') \) from \( SP^\text{begin} \) and add \((i, i) \) to
\( SP^\text{begin} \).

To keep track of the information on how many edges are used, we also have to change
the information from \((D^1, SP^1)\) and \((D^2, SP^2)\). We recursively apply this process of
merging two paths until we can no longer get a tuple \((D', SP')\) that is not yet in \( Q_t \).
Because the number of possible tuples \((D, SP)\) is bounded by \((n + 1)^{2^2}2^{2kz+1}\), this
process can be done in time \(O((n + 1)^{2^2}2^{2kz+1})\) for a fixed pair of tuples \((D^1, SP^1)\)
and \((D^2, SP^2)\). Considering all pairs of tuples in \( Q_{t_1} \) and \( Q_{t_2} \), we can update \( Q_t \)
in time \(O((n + 1)^{3^2}2^{6kz+3})\). Over all, we can update \( Q_t \) for all \( t \) in time \(O((n + 1)^{3^2}2^{6kz+3})\). As \( z \) is constant, this problem can be solved in time \(2^{O(k)}n^{O(1)}\).

5.3 Max-Cut

The third problem we consider is Max-Cut, where we are given an integer \( \ell \) together
with an \( n \)-vertex graph \( G \) and asked whether \( V(G) \) can be partitioned into sets \( V_1 \) and
\( V_2 \) such that the number of edges with precisely one endpoint in \( V_1 \) (called the cut size)
is at least \( \ell \).

| \( R_c \)-MAX-CUT EXTENSION | Parameter: k |
|-----------------------------|--------------|
| **Instance:** | A graph \( G \), a \( k \)-vertex modulator \( X \subseteq V(G) \) to \( R_c \), \( s \subseteq X \), and \( \ell \in \mathbb{N} \). |
| **Task:** | Is there a partition of \( V(G) \) into sets \( V_1 \) and \( V_2 \) such that \( X \cap V_1 = s \) and the number of edges between \( V_1 \) and \( V_2 \) is at least \( \ell \). |

**Theorem 15.** \( R_c \)-MAX-CUT EXTENSION can be solved in polynomial time.

**Proof.** Let \( G \) be a graph together with a \( k \)-vertex modulator \( X \) to \( R_c \) and a set \( s \subseteq X \).
Since, we fix the partition of \( X \) beforehand, we can assume that \( X \) is an independent
set and only add the number of edges between \( s \) and \( X \setminus s \) at the end. We can use
**Theorem 1** to obtain a rank-decomposition \((T, \mu)\) of \( G - X \) of width at most \( c \) in time
in \( O(n^3) \).

The general idea is similar to the one for the XP algorithm for Max-Cut [29]: We determine, via a dynamic programming table, how large the largest cut in \( G_t \) can be if
we assume a certain number of vertices of each twin class to be in the same side of the
cut as \( s \).
The Table. The dynamic programming table consists of entries for each node \( t \) of \( T \), each of which in turn describe for any tuple \( (c_1, \ldots, c_z) \in [R^t_1] \times \cdots [R^t_z] \) what the largest possible cut in \( G_t \) is such that \( c_i \) vertices of \( R^t_i \) are in the same side of the cut as \( s \). Formally,

\[
M_t(c_1, \ldots, c_z) = \max_{s \subseteq C \subseteq V(G)} \left\{ \left| \left\{ e \in E(G) \mid e \cap C \neq \emptyset \wedge e \cap (V(G) \setminus C) \neq \emptyset \right\} \right| \right\},
\]

\[\forall 1 \leq i \leq z \left| C \cap R^t_i \right| = c_i \]

Leaf Nodes. Actually, the only adaptation that has to be made for the consideration of \( s \subseteq X \) concerns the leaf nodes. Consider a leaf node \( t \) and let \( v = \mu^{-1}(t) \). There are two possibilities for a cut \( C \supseteq s \) of \( G_t \): Either we include \( v \) into the side of the cut containing \( s \), this corresponds to defining \( M_t(1) = |N_G(v) \cap (X \setminus s)| \), or we exclude \( v \) into the side of the cut not containing \( s \), this corresponds to defining \( M_t(0) = |N_G(v) \cap X| \).

Internal Nodes. Let us now consider an internal node \( t \) with children \( t_1 \) and \( t_2 \). A cut of \( G_t \) then always is the union of a cut in \( G_{t_1} \) and a cut in \( G_{t_2} \). The edges that contribute to the value of these cuts within \( G_{t_1} \) and \( G_{t_2} \) respectively still contribute to the value in \( G_t \). However, edges with an endpoint in \( V(G_{t_1}) \) and an endpoint in \( V(G_{t_2}) \) are not taken into account in \( G_{t_1} \) and \( G_{t_2} \). The number of these edges are not dependant on the specific cutsets within \( G_{t_1} \) and \( G_{t_2} \), but the size of their intersection with each of the twin classes. Thus, each entry of the dynamic programming table at \( t \) is given rise to by a pair of two entries \( M_{t_1}(c_1, \ldots, c_{z_{t_1}}) \) and \( M_{t_2}(d_1, \ldots, d_{z_{t_2}}) \) of the dynamic programming table at \( t_1 \) and \( t_2 \), in the following way:

\[
M\left(t,c_{U_{1,t}(1)} + d_{U_{2,t}(1)}, \ldots, c_{U_{1,t}(z_{t_1})} + d_{U_{2,t}(z_{t_2})}\right)
= M_{t_1}(c_1, \ldots, c_{z_{t_1}}) + M_{t_2}(d_1, \ldots, d_{z_{t_2}})
+ \sum_{i=1}^{z} \left| R_{U_{1,t}(i)} \right| \cdot \left| R_{U_{2,t}(i)} \right| - c_{U_{1,t}(i)} \cdot d_{U_{2,t}(i)}.
\]

for certain \( c_1, \ldots, c_{z_{t_1}}, d_1, \ldots, d_{z_{t_2}} \). To determine the correct \( c_1, \ldots, c_{z_{t_1}}, d_1, \ldots, d_{z_{t_2}} \) we can take the maximum over all possible choices.

We conclude the proof by analyzing the running time of the described algorithm. For a leaf node \( t \), we can fill the table in time in \( \mathcal{O}(k) \). For an internal node \( t \), there are at most \( n^2 \leq n^{2^z} \) different tuples \( (c_1, \ldots, c_z) \) and at most \( n^2 \leq n^{2^z} \) possibilities to write each of the \( c_i \) as a 2-sum.

6 Algorithmic Applications of \( \mathcal{R}_c \)-Treewidth

In this section, we show that Chromatic Number, Hamiltonian Cycle and Max-Cut are \( \mathcal{FPT} \) parameterized by \( \mathcal{R}_c \)-treewidth. As our starting point, recall that each of these problems admits a fixed-parameter algorithm when parameterized by treewidth which is based on leaf-to-root dynamic programming along the nodes of a nice tree decomposition. Notably, the algorithms are based on defining a certain record
\(\delta(P, Q)\) (for vertex sets \(P, Q\)) such that \(\delta(B_t, Y_t)\) captures all the relevant information required to solve the problem on \(G[Y_t]\) and to propagate this information from a node \(t\) to its parent. The algorithms compute these records on the leaves of the tree decomposition by brute force, and then dynamically update these records while traversing a nice tree decomposition towards the root; once the record \(\delta(B_r, Y_r)\) is computed for the root \(r\) of the decomposition, the algorithm outputs the correct answer. We refer to the standard textbooks for a detailed description of dynamic programming along nice tree decompositions \([13, 16]\) and to Subsections \([6, 1, 6, 3]\) for an overview of the definition of the records used for the target problems.

Our general strategy for solving these problems will be to replicate the records employed by the respective dynamic programming algorithm \(A\) used for treewidth, but only for the nice \(R_c\)-tree decomposition of the torso of the input graph \(G\). Recall that aside from the “standard” simple leaf nodes, nice \(R_c\)-tree decompositions also contain boundary leaf nodes, which serve as separators between the torso and a connected component \(C\) with rank-width at most \(c\). For \(A\) to work correctly with the desired runtime, we need to compute the record for each boundary leaf node using a subprocedure that exploits the bounded rank-width of \(C\); in particular, we will see that this amounts to solving the problems defined in Section \([5]\) Before proceeding to the individual problems, we provide a formalization and proof for the general ideas outlined above.

**Lemma 16.** Let \(P\) be a graph problem which can be solved via a fixed-parameter algorithm \(A\) parameterized by treewidth, where \(A\) runs in time \(f(k')\cdot n^\alpha\) and operates by computing a certain record \(\delta\) in a leaves-to-root fashion along a provided nice width-\(k'\) tree decomposition of the \(n\)-vertex input graph.

Let \(Q\) be obtained from \(P\) by receiving the following additional information in the input:

- a nice \(R_c\)-tree decomposition \((X, T, \{B_t \mid t \in V(T)\})\) of width \(k\) for the input \(n\)-vertex graph \(G\), and
- for each boundary leaf node \(t\) corresponding to the neighborhood of a connected component \(C\) of \(G[X]\), the record \(\delta(B_t, B_t \cup C)\).

Then, \(Q\) can be solved in time \(f(k)\cdot n^\alpha\).

**Proof.** Consider the algorithm \(B\) which computes all the records \(\delta\) for simple leaf nodes by calling the respective subroutine used in \(A\), uses the records \(\delta(B_t, B_t \cup C)\) for the boundary leaf nodes, and then proceeds to compute the records in a dynamic leaves-to-root fashion in exactly the same way as \(A\). It is clear that \(B\) terminates in the claimed runtime.

To argue correctness, assume for a contradiction that \(B\) outputs incorrectly. Now consider the tree decomposition \((T', \{B'_t \mid t \in V(T')\})\) of \(G\) obtained by attaching, to each boundary leaf \(t_C\) in \(T\) (where \(B_{t_C} = N(C)\) for some connected component \(C\) in \(G[X]\)), a new leaf whose bag contains \(N(C) \cup C\). While the width of such a decomposition may be linear in the number of vertices, \(A\) must still output the correct solution, and observe that the records computed by \(A\) on \((T', \{B'_t \mid t \in V(T')\})\) must precisely match those computed by \(B\) on \((X, T, \{B_t \mid t \in V(T)\})\). Hence, we would in this case conclude that \(A\) also outputs incorrectly, a contradiction. ■
6.1 Chromatic Number

Chromatic Number is \(W[1]\)-hard parameterized by rank-width \(23\) but can be solved in time \(2^{O(tw(G) \cdot \log tw(G))} \cdot n\) on \(n\)-vertex graphs when a minimum-width tree-decomposition is provided with the input \(39\); moreover, it is known that this runtime is essentially tight, since under ETH the problem cannot be solved in time \(2^{o(tw(G) \cdot \log tw(G))} \cdot n^{O(1)} \) \(47\).

It is well known that the chromatic number is at most \(tw(G) + 1\) (this can be observed, e.g., by greedily coloring a vertex when it is introduced in the nice tree-decomposition). One possible way of defining records in order to achieve a runtime of \(2^{O(tw(G) \cdot \log tw(G))} \cdot n\) is to track, for each proper coloring of vertices in a bag \(B_t\), the minimum number of colors required to extend such a coloring to \(Y_t\). Formally, let \(S_t\) be the set of all colorings of \(B_t\) with colors \([tw(G) + 1]\), and let \(\alpha(B_t, Y_t) : S_t \rightarrow \mathbb{Z}\) be defined as follows:

- \(\alpha(B_t, Y_t)(s) = -1\) if \(s\) is not a proper coloring of \(G[B_t]\).
- \(\alpha(B_t, Y_t)(s) = q\) if \(q\) is the minimum number of colors used by any proper coloring of \(G[Y_t]\) which extends \(s\).

Using Theorem 13, we can compute such \(\alpha(B_t, Y_t)(s)\) for every proper coloring \(s\) of \(B_t\). Hence, combining Lemma 16 and Theorem 13 we obtain:

**Theorem 17.** Chromatic Number can be solved in time \(2^{O(k \cdot \log(k))} \cdot n^{O(1)}\) if a nice \(R_c\)-tree decomposition of width \(k\) is provided on the input.

6.2 Hamiltonian Cycle

Hamiltonian Cycle is \(W[1]\)-hard parameterized by rank-width \(23\) but can be solved in time \(2^{O(tw(G) \cdot \log tw(G))} \cdot n\) on \(n\)-vertex graphs when a minimum-width tree-decomposition is provided with the input via standard dynamic programming (see, e.g., an introduction to advances made for the problem by Ziobro and Pilipczuk \(56\)). This algorithm can be improved to run in time \(2^{O(tw(G))} \cdot n\) by applying the advanced rank-based approach of Cygan, Kratsch and Nederlof \(14\) to prune the number of records.

To simplify our exposition, here we focus on extending the standard dynamic programming algorithm which yields a slightly super-exponential runtime.

One possibility for defining the records for Hamiltonian Cycle is to track all possible ways one can cover \(Y_t\) by paths that start and end in \(B_t\) (intuitively, this corresponds to what remains of a hypothetical solution if we “cut off” everything above \(Y_t\) \(16\)). Formally, let \(B_t^0\) be defined as follows:

- if \(|B_t| > 2\), then \(B_t^0\) is the set of graphs with at most \(|B_t|\) edges and degree at most 2 over vertex set \(B_t\);
- if \(|B_t| = 2\), then \(B_t^0\) contains three (multi)graphs over vertex set \(B_t\): the edgeless graph, the graph with one edge, and the multigraph with two edges and no loops;
- if \(|B_t| = 1\), then \(B_t^0\) contains an edgeless graph and a graph with a single loop, both over the single vertex in \(B_t\);
• if $|B_t| = 0$, then $B_t = \{\text{YES}, \text{NO}\}$.

We let $\beta(B_t, Y_t) : B_t \to \{0, 1\}$, where for $Q \in B_t$ we set $\beta(B_t, Y_t)(Q) = 1$ if and only if there exists a set $P$ of paths in $G[Y_t]$ and a bijection that maps each $(v_1, \ldots, v_t) \in P$ to an edge $(v_1, v_t) \in E(Q)$ such that each vertex $v \in G[Y_t \setminus B_t]$ is contained in precisely one path in $P$. In the special case where $B_t = \emptyset$, our records explicitly state whether $G[Y_t]$ contains a Hamiltonian cycle or not.

As before, we can now shift our attention to the problem of computing our records in boundary leaf nodes. If $|B_t| \leq 1$, then, because $B_t$ is a separator in $G$, either $Y_t = B_t$, in which case $\beta(B_t, Y_t)$ is trivial to compute, or $\beta(B_t, Y_t)$ can be filled by simply solving HAMILTONIAN CYCLE on $G[Y_t]$ in polynomial time using known algorithms [29]. In all other cases, we loop over all of the at most $k^{2k}$-many graphs $Q \in B_t$ and for each such $Q$ we need to check whether $G[Y_t] - B_t$ can be covered by internally vertex-disjoint paths connecting the pairs of vertices in $B_t$ that form the endpoints of the edges in $Q$. Moreover, note that in this case $Q$ does not contain any loops. Hence, this is precisely the $R_c$-DISJOINT PATHS COVER problem defined in Subsection 5.2 and combining Theorem 14 and Lemma 16, we obtain:

**Theorem 18.** HAMILTONIAN CYCLE can be solved in time $2^{O(k \log(k))} \cdot n^{O(1)}$ if a nice $R_c$-tree decomposition of width $k$ is provided on the input.

### 6.3 Max-Cut

MAX-CUT is another problem that is $\mathsf{W}[1]$-hard parameterized by rank-width [25] but admits a simple fixed-parameter algorithm parameterized by treewidth – notably, it can be solved in time $2^{O(tw(G))} \cdot n$ on $n$-vertex graphs when a minimum-width tree-decomposition is provided with the input via standard dynamic programming [13] [16].

The simplest way of defining the records for MAX-CUT is to keep track of all possible ways the bag $B_t$ can be partitioned into $V_1$ and $V_2$, and for each entry in our table we keep track of the maximum number of crossing edges in $Y_t$ compatible with that entry. Formally, let $\gamma(B_t, Y_t) : 2^{B_t} \to \mathbb{N}_0$, where for each $s \in 2^{B_t}$ it holds that $\gamma(B_t, Y_t)(s)$ is the maximum cut size that can be achieved in $G[Y_t]$ by any partition $(V_1, V_2)$ satisfying $V_1 \cap B_t = s$. As before, from Theorem 15 and Lemma 16 we obtain:

**Theorem 19.** MAX-CUT can be solved in time $2^k \cdot n^{O(1)}$, if a nice $R_c$-tree decomposition of width $k$ is provided on the input.

### 7 Concluding Remarks

While the technical contribution of this paper mainly focused on $R_c$-treewidth, a parameter that allows us to lift fixed-parameter algorithms parameterized by treewidth to well-structured dense graph classes, it is equally viable to consider $\mathcal{H}$-treewidth for other choices of $\mathcal{H}$. Naturally, one should aim at graph classes where problems of interest become tractable, but it is also important to make sure that a (nice) $\mathcal{H}$-tree decomposition can be computed efficiently (i.e., one needs to obtain analogues to our Lemma 12). Examples of graph classes that may be explored in this context include split graphs, interval graphs, and more generally graphs of bounded mim-width [37].
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References

[1] T. Akiyama, T. Nishizeki, and N. Saito. \( \mathcal{NP} \)-completeness of the hamiltonian cycle problem for bipartite graphs. *J. Inform. Process.*, 3:73–76, 01 1980.

[2] H. L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.*, 25(6):1305–1317, 1996.

[3] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. (Meta) kernelization. *J. ACM*, 63(5):44:1–44:69, Nov. 2016.

[4] H. L. Bodlaender, B. M. P. Jansen, and S. Kratsch. Preprocessing for treewidth: A combinatorial analysis through kernelization. *SIAM J. Discrete Math.*, 27(4):2108–2142, 2013.

[5] H. L. Bodlaender and T. Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. *J. Algorithms*, 21(2):358–402, 1996.

[6] H. L. Bodlaender and B. van Antwerpen-de Fluiter. Reduction algorithms for graphs of small treewidth. *Inf. Comput.*, 167(2):86–119, 2001.

[7] L. Cai. Parameterized complexity of vertex colouring. *Discrete Applied Mathematics*, 127(3):415–429, 2003.

[8] B. Courcelle. The monadic second-order logic of graphs. I. recognizable sets of finite graphs. *Inf. Comput.*, 85(1):12–75, 1990.

[9] B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic - A Language-Theoretic Approach*, volume 138 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 2012.

[10] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33(2):125–150, 2000.

[11] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101(1):77 – 114, 2000.

[12] B. Courcelle and S. Oum. Vertex-minors, monadic second-order logic, and a conjecture by seese. *J. Comb. Theory, Ser. B*, 97(1):91–126, 2007.

[13] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Texts in Computer Science. Springer, 2013.
[14] M. Cygan, S. Kratsch, and J. Nederlof. Fast hamiltonicity checking via bases of perfect matchings. *J. ACM*, 65(3):12:1–12:46, 2018.

[15] R. Diestel. *Graph Theory*, volume 173 of Graduate Texts in Mathematics. Springer Verlag, New York, 2nd edition, 2000.

[16] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[17] E. Eiben, R. Ganian, and O. Kwon. A single-exponential fixed-parameter algorithm for distance-hereditary vertex deletion. *J. Comput. Syst. Sci.*, 97:121–146, 2018.

[18] E. Eiben, R. Ganian, and S. Szeider. Meta-kernelization using well-structured modulators. *Discrete Applied Mathematics*, 248:153–167, 2018.

[19] E. Eiben, R. Ganian, and S. Szeider. Solving problems on graphs of high rank-width. *Algorithmica*, 80(2):742–771, 2018.

[20] W. Espelage, F. Gurski, and E. Wanke. Deciding clique-width for graphs of bounded tree-width. *J. Graph Algorithms Appl.*, 7(2):141–180, 2003.

[21] M. R. Fellows, D. Lokshtanov, N. Misra, F. A. Rosamond, and S. Saurabh. Graph layout problems parameterized by vertex cover. In *Proc. ISAAC 2008*, volume 5369 of *Lecture Notes in Computer Science*, pages 294–305. Springer, 2008.

[22] J. Flum and M. Grohe. *Parameterized Complexity Theory*, volume XIV of Texts in Theoretical Computer Science. An EATCS Series. Springer Verlag, Berlin, 2006.

[23] F. V. Fomin, P. A. Golovach, D. Lokshtanov, and S. Saurabh. Clique-width: on the price of generality. In *Proc. SODA 2009*, pages 825–834. SIAM, 2009.

[24] F. V. Fomin, P. A. Golovach, D. Lokshtanov, and S. Saurabh. Intractability of clique-width parameterizations. *SIAM J. Comput.*, 39(5):1941–1956, 2010.

[25] F. V. Fomin, P. A. Golovach, D. Lokshtanov, and S. Saurabh. Almost optimal lower bounds for problems parameterized by clique-width. *SIAM J. Comput.*, 43(5):1541–1563, 2014.

[26] F. V. Fomin, B. M. P. Jansen, and M. Pilipczuk. Preprocessing subgraph and minor problems: When does a small vertex cover help? *J. Comput. Syst. Sci.*, 80(2):468–495, 2014.

[27] J. Gajarský, P. Hlinený, J. Obdržálek, S. Ordyniak, F. Reidl, P. Rossmanith, F. S. Villaamil, and S. Sikdar. Kernelization using structural parameters on sparse graph classes. *J. Comput. Syst. Sci.*, 84:219–242, 2017.

[28] R. Ganian and P. Hlinený. On parse trees and myhill-nerode-type tools for handling graphs of bounded rank-width. *Discrete Applied Mathematics*, 158(7):851–867, 2010.
[29] R. Ganian, P. Hlinený, and J. Obdrzálek. A unified approach to polynomial algorithms on graphs of bounded (bi-)rank-width. *Eur. J. Comb.*, 34(3):680–701, 2013.

[30] R. Ganian, S. Ordyniak, and M. S. Ramanujan. Going beyond primal treewidth for (M)ILP. In *Proc. AAAI 2017*, pages 815–821. AAAI Press, 2017.

[31] R. Ganian, M. S. Ramanujan, and S. Szeider. Backdoor treewidth for SAT. In *Proc. SAT 2017*, volume 10491 of *Lecture Notes in Computer Science*, pages 20–37. Springer, 2017.

[32] R. Ganian, M. S. Ramanujan, and S. Szeider. Combining treewidth and backdoors for CSP. In *Proc. STACS 2017*, volume 66 of *LIPIcs*, pages 36:1–36:17. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

[33] R. Ganian, F. Slivovsky, and S. Szeider. Meta-kernelization with structural parameters. *J. Comput. Syst. Sci.*, 82(2):333–346, 2016.

[34] P. Hlinený, S. Oum, D. Seese, and G. Gottlob. Width parameters beyond treewidth and their applications. *Comput. J.*, 51(3):326–362, 2008.

[35] P. Hliněný and S.-i. Oum. Finding branch-decompositions and rank-decompositions. *SIAM J. Comput.*, 38(3):1012–1032, June 2008.

[36] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*, 63(4):512–530, 2001.

[37] L. Jaffke, O. Kwon, and J. A. Telle. A unified polynomial-time algorithm for feedback vertex set on graphs of bounded mim-width. In *Proc. STACS 2018*, volume 96 of *LIPIcs*, pages 42:1–42:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018.

[38] B. M. P. Jansen and H. L. Bodlaender. Vertex cover kernelization revisited - upper and lower bounds for a refined parameter. *Theory Comput. Syst.*, 53(2):263–299, 2013.

[39] K. Jansen and P. Scheffler. Generalized coloring for tree-like graphs. *Discrete Applied Mathematics*, 75(2):135–155, 1997.

[40] V. Jelínek. The rank-width of the square grid. *Discrete Applied Mathematics*, 158(7):841–850, 2010.

[41] J. Jeong, S. H. Sæther, and J. A. Telle. Maximum matching width: New characterizations and a fast algorithm for dominating set. *Discrete Applied Mathematics*, 248:114–124, 2018.

[42] M. M. Kanté, E. J. Kim, O. Kwon, and C. Paul. An FPT algorithm and a polynomial kernel for linear rankwidth-1 vertex deletion. *Algorithmica*, 79(1):66–95, 2017.
[43] E. J. Kim and O. Kwon. A polynomial kernel for distance-hereditary vertex deletion. In Proc. WADS 2017, volume 10389 of Lecture Notes in Computer Science, pages 509–520. Springer, 2017.

[44] T. Kloks. Treewidth: Computations and Approximations, volume 842 of LNCS. Springer Verlag, Berlin, 1994.

[45] D. Kobler and U. Rotics. Edge dominating set and colorings on graphs with fixed clique-width. Discrete Applied Mathematics, 126(2-3):197–221, 2003.

[46] D. Lapoire. Recognizability equals monadic second-order definability for sets of graphs of bounded tree-width. In Proc. STACS 98, volume 1373 of Lecture Notes in Computer Science, pages 618–628. Springer, 1998.

[47] D. Lokshtanov, D. Marx, and S. Saurabh. Slightly superexponential parameterized problems. SIAM J. Comput., 47(3):675–702, 2018.

[48] D. Marx and P. Wollan. Immersions in highly edge connected graphs. SIAM J. Discrete Math., 28(1):503–520, 2014.

[49] R. Niedermeier. Invitation to Fixed-Parameter Algorithms, volume 31 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2006.

[50] S. Oum. Rank-width and vertex-minors. J. Comb. Theory, Ser. B, 95(1):79–100, 2005.

[51] S. Oum and P. D. Seymour. Approximating clique-width and branch-width. J. Comb. Theory, Ser. B, 96(4):514–528, 2006.

[52] N. Robertson and P. D. Seymour. Graph minors. III. planar tree-width. J. Comb. Theory, Ser. B, 36(1):49–64, 1984.

[53] N. Robertson and P. D. Seymour. Graph minors. X. obstructions to tree-decomposition. J. Comb. Theory, Ser. B, 52(2):153–190, 1991.

[54] S. H. Sæther and J. A. Telle. Between treewidth and clique-width. In Proc. WG 2014, pages 396–407, 2014.

[55] M. Vatshelle. New width parameters of graphs. PhD thesis, University of Bergen, 05 2012.

[56] M. Ziobro and M. Pilipczuk. Finding hamiltonian cycle in graphs of bounded treewidth: Experimental evaluation. In Proc. SEA 2018, volume 103 of LIPIcs, pages 29:1–29:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018.