Brain Dynamics, Chaos and Bessel Functions

W J Freeman¹, A Capolupo², R Kozma³, A Olivares del Campo⁴ and G Vitiello²

¹ Department of Molecular and Cell Biology, University of California Berkeley, CA 94720-3206 USA
² Dipartimento di Fisica “E.R.Caianiello” Università di Salerno and INFN Gruppo collegato di Salerno, Fisciano (SA) - 84084, Italy
³ Department of Mathematics, University of Memphis, Memphis TN 38152, USA
⁴ The Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BZ, UK

E-mail: vitiello@sa.infn.it

Abstract. By resorting to Freeman’s observations showing that the distribution functions of impulse responses of cortex to sensory stimuli resemble Bessel functions, we study brain dynamics by considering the equivalence of spherical Bessel equation, in a given parametrization, to two oscillator equations, one damped and one amplified oscillator. The study of such a couple of equations, which are at the basis of the formulation of the dissipative many-body model, reveals the structure of the root loci of poles and zeros of solutions of Bessel equations, which are consistent with results obtained using ordinary differential equation techniques. We analyze stable and unstable limit cycles and consider thermodynamic features of brain functioning, which in this way may be described in terms of transitions between chaotic gas-like and ordered liquid-like behaviors. Nonlinearity dominates the dynamical critical transition regimes. Linear behavior, on the other hand, characterizes superpositions within self-organized neuronal domains in each dynamical phase. The formalism is consistent with the observed coexistence in circular causality of pulse density fields and wave density fields.

1. Introduction

The distribution functions of impulse responses of cortical evoked potentials has been analyzed in terms of ordinary differential equations (ODE) [1]. The present study starts with the observation that impulse responses resemble Bessel functions (see e.g. Figure 2.24a, Section 2, page 111 of ref. [1]). In addition, it is commonly assumed that pulse and wave modes coexist simultaneously in circular causality. The resemblance of functions representing neurological processes with Bessel functions is a motivation for us to investigate the equivalence of the spherical Bessel equation and a system of two equations, one for a damped harmonic oscillator, the other one for an amplified harmonic oscillator, provided a specific parametrization is used [2]. The analysis of the system of such equations, which are at the basis of the formulation of the dissipative many-body model [3]-[8], reveals the structure of the root loci of poles and zeros of solutions of Bessel equations. The results we find are consistent with the results obtained using ordinary differential equation techniques. Moreover, we find that consideration of wave modes cannot be omitted in favor of pulse mode, as often done in the study of the individual neurons. Although necessary, the knowledge of single neuron properties is not sufficient for the comprehension of global functional activity of the brain.
We study stable and unstable limit cycles and consider thermodynamic features of brain functions, which may be described in terms of transitions between chaotic gas-like and ordered liquid-like behaviors [7, 8, 9]. Nonlinearity dominates the dynamical critical transition regimes. Linear behavior instead characterizes superpositions within self-organized neuronal domains in each dynamical phase. The dissipative character of the brain system dynamics leads us, following refs. [3, 4, 5, 10], to consider collective properties in the many-body physics framework, focusing on the quantum degrees of freedom associated with the electrical dipole vibrational modes of water molecules and other molecules of biological interest. We stress, however, that neurons and glia cells are considered to be classical objects in the dissipative many-body model. Remarkably, Bessel equations are classical equations, which we thus understand as the macroscopic manifestation of the underlying quantum field theory dynamics [11].

2. Brain dynamics and Bessel equations

As already mentioned, laboratory observations in brief time windows of brain activity show [1] that pulses and wave modes coexist simultaneously in circular causality determining the dynamics of neuronal populations. Much attention is given in current neuroscience to the study of pulse modes. In this paper our attempt is to combine in one unified dynamical frame the continuum of the wave mode and the action potential point processes of pulse modes. Our study of pulse modes. In this paper our attempt is to combine in one unified dynamical frame the properties intrinsic to the system, such as biological, physiological or behavioral properties. The time-dependent frequencies appears to be \( \omega_n(t) = \epsilon_n e^{-\frac{t}{\tau_n}} = \epsilon_0 e^{-\frac{t}{\tau_n} c} \), with \( \omega_0 \equiv \epsilon_n / \alpha_n = k e \equiv \omega_{0,k} \); \( k \) is the wave-number \( k = 2 \pi / \lambda \). 

\( L \) can be expressed as \( L_k = \frac{(2n+1)}{\epsilon_n} k e \). This is done in such a way that, by fixing \( \epsilon_n \), \( L_k \) and \( \omega_{0,k} \) are independent of \( n \) (and of time). The transformation \( n \to -(n+1) \) leads to solutions (corresponding to \( M_{-(n+1)} \)) which can be obtained by time-reversal \( t \to -t \) in \( \omega_n(t) \). The \( k \) index will be omitted for simplicity whenever no misunderstanding arises.

We remark that Eqs. (2) and (3) are nothing else than the couple of equations at the basis of the dissipative model of brain [3] with time-dependent frequency \( \omega(t) \). One can reduce them to the single parametric oscillator equation [14]

\[
\ddot{r}_n + \Omega_n^2(t)r_n = 0 ,
\]
where \( \xi \equiv q_{n,-(n+1)} = \frac{1}{\sqrt{2}} r_n e^{-Lt} \), \( \zeta \equiv q_{n,n} = \frac{1}{\sqrt{2}} r_n e^{-Lt} \) and

\[
\Omega_n(t) = \left[ \omega_n^2(t) - \frac{L^2}{4} \right]^{\frac{1}{2}} \geq 0 ,
\]

is the common frequency of oscillation. Proof of the mechanism is the demonstration that the signals are in quadrature; the frequency and decay rate are always identical, and the phase of the inhibitory signal lags a quarter cycle behind the excitatory signal. Note that \( \xi = q_{n,-(n+1)} \) and \( \zeta = q_{n,n} \) are “harmonically conjugated”. \( \Omega(t) \), assumed to be real in order to avoid the overdamping regime, becomes time-independent in the limit \( t \to \infty \) and it is fully specified by characteristic parameters of the system. We remark that temporal frequency modulation (smooth decrease of 10% of the mean, but rarely an increase) is observed in gamma bursts of oscillation in the ECoG lasting long enough (> 3 to 5 cycles). The impulse responses (averaged evoked potentials) has fixed time-invariant frequencies. This is equivalent to time \( t \) going to infinity.

Quantization of Eqs. (2) and (3) leads to the introduction of modes \( a_k \) and \( \tilde{a}_k \), respectively. In the dissipative quantum model of brain [3], they represent molecular electric dipole vibrational modes called dipole wave quanta (DWQ). They are identified with the Nambu-Godstone (NG) boson quanta generated in the dynamical mechanism of the breakdown of the rotational symmetry of the electrical dipoles of the molecules of water and other biomolecules [3, 4, 6]. Through sensorial perception, the brain is permanently coupled to its environment (called its Double [3, 4, 6]). External stimuli induce the symmetry breakdown with the consequent dynamical process of condensation of NG boson modes in the system ground state. Since these are (quasi-)massless modes, they propagate through the whole system volume and are thus responsible for long range correlations among its molecular components (formation of ordered coherent patterns) [3, 4, 10, 11]. These correlations trigger and promote the ephaptic interaction among neurons implying the formation of the observed coherently oscillating neuronal amplitude modulated (AM) assemblies [1, 5, 6, 15].

In the dissipative model, the constrain \( N_{a_k} - N_{\tilde{a}_k} = 0 \), with \( N_{a_k} \) and \( N_{\tilde{a}_k} \) denoting the number of \( a_k \) and \( \tilde{a}_k \) modes, respectively, is imposed on the NG boson condensation. It guaranties that the ground states has the lowest energy. However, the number of \( k \)-modes, for each \( k \), in the ground state is not fixed. Infinitely many lowest energy states (called the memory states in the dissipative model) with distinct \( N \equiv \{ N_{a_k} = N_{\tilde{a}_k}, \forall k \} \) are then possible. \( N \) is then referred to as the “order parameter”.

### 3. Linearity and nonlinearity in brain activity

Laboratory observations (see in particular Section 2.5.3 in ref. [1]) have shown that linearity plays a crucial role in brain activity, so that linearity cannot be neglected in favor of nonlinearity, which also plays a crucial role, as we will see. Brain functional activity presents an intriguing interplay between linear and nonlinear dynamical regimes. Such an interplay depicts a scenario neither new nor surprising if compared to what normally observed in condensed matter physics. Unfortunately, in recent years neuroscience studies have mostly focused on fashionable nonlinear dynamical models, which, we repeat play a most important role in brain activity, but, nevertheless, are not sufficient for the comprehension of the brain functional activity. One aspect of the relevance of linearity has to do with the fact that the fitting functions describing oscillatory processes of assemblies of neurons are linear superpositions of the form [1])

\[
v(t) = \sum_{j}^{n} A_j e^{-p_j t} .
\]
$A_j, p_j \in \mathbb{C}$. They have positive coefficients, i.e. $Re(p_j) \geq 0$ and $Im(p_j) \geq 0$. When $p_j$ is real, $A_j$ is also real. Complex $p_j$ and $A_j$ always come in conjugate pairs, yielding damping $sin$ or $cos$ terms, such as $\sin(\Gamma_j t + \varphi_j) e^{-Re(p_j) t}$, with $\varphi_j$ a constant phase term in $A_j$. For example, $\sin(\gamma_j t + \varphi_j) e^{-\gamma_j t}$ can be rewritten as $A_j e^{-\gamma_j t} + A_j' e^{-\gamma_j t}$, where $A_j = e^{-\mu j^2/2}$, $p_j = \gamma_j + i\Gamma_j$ and similarly for a damping $cos$ term. Note that $A_j \equiv A_k$, and $p_j \equiv p_k$, i.e. $A_j$ and $p_j$ depend on the wave-number $k_j$, for each $j$, which in our field theory frame varies over a continuum set of values. Whenever no misunderstanding arises, the simplified notation $A_j$ and $p_j$ will be used. Eq. (6) has the Laplace transform

$$V(s) = \prod_{j=1}^{m} \frac{s + z_j}{s + p_j},$$

(7)

The zeros are at $z_i$ and the poles are lying in the left-hand side of the complex $s$-plane.

Each term in Eq. (6) satisfies Eq. (2) and corresponds to a DWQ $a_{k_j}$ mode. The modeled output potential is thus described by a linear superposition of a continuum set of values. Whenever no misunderstanding arises, the simplified notation $A_j$ and $p_j$ will be used. Eq. (6) has the Laplace transform

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Each term in Eq. (6) satisfies Eq. (2) and corresponds to a DWQ $a_{k_j}$ mode. The modeled output potential is thus described by a linear superposition of $a_{k_j}$ modes, representing solutions of the $n$-order Bessel equation. The meaning of this is that a particular stimulus will excite an $N$-set, to which different $N_{ak}$ components belong [3].

We remark that, due to mathematical completeness, we need to “double” the degrees of freedom of the system by introducing the basis functions $e^{+p_j t}$, so to complete the basis $\{e^{-p_j t}, e^{-p_j t}\}$. Thus, one cannot avoid to consider also the basis element $e^{+p_j t}$.

Use of $A_j(t) = A_j e^{-p_j t}$, $A_j \neq 0$, in Eq. (2) eventually gives:

$$\gamma_{k_j} = \frac{L_{k_j}}{2}, \quad \Gamma_{k_j} = \left[\omega_{n,k_j}^2(t) - \gamma_{k_j}^2\right]^{1/2} \equiv \Omega_{k_j,n}(t),$$

(8)

where the $k$ subscript notation has been restored. The requirement of doubling the degrees of freedom, i.e. introducing terms with $e^{+p_j t}$ basis as solutions of Eq. (3), the same equations as in (8) are obtained. The same common frequency $\Omega_{k_j,n}(t)$ thus establishes for each $k$-mode and a particular $n$ the resonating link between the damped/amplified systems.

We observe that in each of the damped/amplified oscillators in Eq. (2) and Eq. (3) time-reversal symmetry is broken due to their dissipative/amplified character. The “arrow of time”, in each of the damped and amplified oscillators, thus emerges as a distinctive dynamical feature of the dissipative quantum model. It can be related to the “thermodynamical” and the “psychological” arrows of time. Details on this point may be found in refs. [16, 17].

In closing this Section, we remark that the linearity of Eqs. (2) and (3) allows us to consider the summations in the fitting functions with $\{e^{-p_j t}, e^{+p_j t}\}$ basis. Linearity has originally motivated the analysis in terms of coupled ODEs [1, 19], which results are confirmed by the present analysis in terms of Bessel functions. However, our modeling also implies nonlinearity since the damped/amplified oscillators are in fact nonlinearly coupled in the coherent state they generate at quantum level, where they appear as an entangled two modes system (see [3, 6] for details). On the other hand, the root loci studied in [1, 18] can be also analyzed in the present treatment (cf. next Section). The study of the root loci reveals that the biophysical nonlinearity at the microscopic level allows that phase transition resembling a Hopf bifurcation are energetically possible.

Moreover, we observe that the frequency $\omega_n(t)$ reaches the time independent value $\omega_0$ for higher order $n$ of the Bessel equation, which is the limit of energy conservation, corresponding in turn to the maximal coupling between the damped and the amplified oscillators, forming in that limit a closed system. This is consistent with observations in the limit of stationarity, where most cortical impulse responses (evoked potentials from single shock electrical stimulation) last only 1.5 to 2.5 cycles, those few lasting longer have fixed stationary frequency. Ergodicity has to be also assumed in order to use time ensemble averaging to estimate pulse density in the linear
range, which can be done also in the present case. We thus see that linearity, stationarity and ergodicity are compatible with Bessel functions in the dissipative quantum model.

In conclusion, the complex interplay of linearity and nonlinearity and the coexistence of pulse modes and wave modes are features of brain functional activity which manifest in a very rich phenomenology.

4. Limit cycle

In this Section we consider the limit cycle occurrence as one of the examples discussed in [2], which, together with [1], we will closely follow in our presentation. We reproduce the results obtained in [1] thus obtaining explicit connection with the analysis of the present paper.

From Section 6, Page 380 in ref. [1] one can derive that the dynamics is determined by the rules:

1. If the roots all lie in the left half of the $s$-plane, the nonlinear KII set has a stable equilibrium to which the active states return when input is terminated.
2. If complex conjugate poles lie on root loci in the right of the $s$-plane, the poles move on the root loci in the direction of decreasing $K$ to the $i\omega$ axis and stay there, and the KII set has a stable limit cycle.
3. If two conjugate root loci cross the $i\omega$ axis from the left half into the right half of the $s$-plane with decreasing $K$, the KII set has an unstable limit cycle.

Time evolution carrying through left and right hand side of the locus plots, i.e. damped and amplified solutions, appears to be deeply related with the chaotic dynamics that is observed in the brain [1, 20].

Let $D(s)$ denote the transfer function for a KIII set output from a KII set with negative feedback from a KI set. It is given by

$$D(s) = \frac{C(s)}{1 + KeC(s)P(s)}.$$  \hspace{1cm} (9)

$C(s) = (6.25 \times 10^5)/(s + i250)(s - i250)(s + 100)$ and $P(s) = 100/(s + 100)$ are, respectively, the KII and KI set transfer functions in a stable limit cycle state.

The poles of Eq. (9) change as $K_e$ changes:

$$D(s) = \frac{6.25 \times 10^5s(s + 100)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)(s + p_5)}.$$  \hspace{1cm} (10)

The $p_n$ poles can be real or complex. Figure (1) shows the root locus plot of $D(s)$. The larger poles cross the imaginary axis from the left to the right as $K_e$ decreases, which signals an unstable limit cycle. The limit cycle will be stable at frequencies where the poles are purely imaginary ($\approx 160$ rad/sec).

As observed in Section 2, decaying solutions and amplifying ones arise in our framework when one removes the degeneracy between the $n$ and $-(n+1)$ solutions of the Bessel equation. The crossing of the imaginary axis thus means the transition between these two sets of Bessel equation solutions as the gain increases (from decaying solutions to amplifying ones).

When the rate of oscillation of the pulses ($\Gamma_{k,n}$) changes with time at a fixed frequency, the stable limit cycle is obtained. This is consistent with our analysis, since identifying the limit cycle frequency with $\Omega(t)$, which, as observed above, links the damped and amplified solutions, the stable equilibrium state (constant pulse frequency) corresponds to fully coupled solutions (obtained for $n \to \infty$, with $\Omega(t)$ becoming indeed time independent in such a limit).
5. Conclusion
We have discussed the laboratory observations showing that impulse responses of cortex to sensory stimuli exhibit distribution functions resembling to Bessel functions and that, in the brief time windows of brain activity accessible to observations, pulses and wave modes coexist simultaneously in circular causality determining the dynamics of neuronal populations. The small-signal linear dynamic range allows the possibility to decompose cortical impulse responses by fitting to them the sums of linear basis functions. These have been found to be solutions of ODE [1, 19] and in the present paper have been shown to be solutions of the spherical Bessel equation of order $n$. This equation turns out to be equivalent to the couple of damped/amplified oscillator equations at the basis of the formulation of the dissipative quantum model of brain [3, 4, 6, 13]. The harmonic conjugated solutions, represented in terms of the quantum modes $a_k$ and their conjugate $\tilde{a}_k$, are expressed in terms of the common frequency of oscillation $\Omega(t)$ and create a mirror image on the imaginary axis in the $s$-space in the Laplace transform representation. The dissipative many-body model offers thus a possibility in the study of pulse-wave duality and in combining the complexity of nonlinear interactions with linear, stationary, ergodic solutions at the observational level. Our results are consistent with the ODE treatment [1, 18, 19] and the connection with such a formalism is made explicit by considering specific applications in [2]. As an example, the occurrence of the limit cycles has been reported in the present paper.

In ref. [2] it has been discussed how the coefficients that can be determined from the plot of the poles of the fitting functions in $s$-space may be related to the lifetime of a specific neuronal

Figure 1. Root locus plot of Equation (9) with poles ($\times$) and zeros ($\bullet$).
(“memory”) process. Here we do not discuss such issues. Rather, we observe that couples of damped/amplified oscillator solutions, \( \{n, k_j\} \) and \( \{n, \tilde{k}_j\} \) (the notation \( \tilde{k}_j \) refers to the \( \tilde{a}_{kj} \)), may be linked through the values of the parameters defining the system \( \{L_{kj}, \omega_{kj}, n\} \). With infinitely many \( k \) modes, one can always pair them up adding modes in pairs in \( s \)-space. If one \( k_j \) mode, for a given \( j \), has a shorter lifetime than the other one in the couple, the linked system becomes unpaired (“widowed”). However, since \( k_j \) and \( \tilde{k}_j \), for each \( j \), belong to continuum sets, new links (new couples) can be created, thus recovering the lost links. Due to the continuum, we may also have couples with different \( i \) and \( j \), for any \( i \) and \( j \), with continuously changeable \( i \) and \( j \). In the observed neuronal assemblies \([5, 6, 15]\), due to the continuous interchange of partners in the couples, no neuron is “wired” to someone else. The neuronal dynamical assembly of correlated neurons is made of constantly changing, unwired neuronal connections, described by coherent states in the dissipative quantum model of brain \([3]-[6]\).

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