ON QUASI-PARA-SASAKIAN MANIFOLDS

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Abstract. In this paper we study quasi-para-Sasakian manifolds. We characterize these manifolds by tensor equations and study their properties. We are devoted to the study of $\eta$–Einstein manifolds. We show that a conformally flat quasi-para-Sasakian manifold is a space of constant negative curvature $-1$ and we prove that if a quasi-para-Sasakian manifold is a space of constant $\varphi$–para-holomorphic sectional curvature $H = -1$, then it is a space of constant curvature. Finally, the object of the present paper is to study a 3-dimensional quasi-para-Sasakian manifold, satisfying certain curvature conditions. Among other, it is proved that any 3-dimensional quasi-para-Sasakian manifold with $\eta$–parallel Ricci tensor is of constant scalar curvature.

1. Introduction

In this paper we study a class of paracontact pseudo-Riemannian manifolds satisfying some special conditions. These manifolds are analogues to the quasi-Sasakian manifolds and they belong of the class $G_5$ of the classification given in [8]. We characterize these manifolds by tensor equations and study their properties. From the definition by means of the tensor equations, it is easily verified that the structure is normal, but not para-Sasakian. We are devoted to the study of $\eta$–Einstein manifolds. We show that a conformally flat quasi-para-Sasakian manifold is a space of constant negative curvature $-1$ and we prove that if a quasi-para-Sasakian manifold is a space of constant $\varphi$–para-holomorphic sectional curvature $H = -1$, then it is a space of constant curvature. In the last section, we study the 3-dimensional quasi-para-Sasakian manifolds. We prove that any 3-dimensional quasi-para-Sasakian manifold satisfying the condition $R(X, Y).Ric = 0$ is a manifold of constant negative curvature, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of manifold ($X, Y$ are tangent vectors). We study locally $\varphi$–symmetric quasi-para-Sasakian manifolds and obtain a necessary and sufficient condition a 3-dimensional quasi-para-Sasakian manifold to be locally $\varphi$–symmetric. We obtain some interesting results about a 3-dimensional quasi-para-Sasakian manifolds with $\eta$–parallel Ricci tensor. We give an example for a 3-dimensional quasi-para-Sasakian manifold with a scalar curvature equal to $-6$.

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2. Preliminaries

A \((2n+1)\)-dimensional smooth manifold \(M^{(2n+1)}\) has an **almost paracontact structure** \((\varphi, \xi, \eta)\) if it admits a tensor field \(\varphi\) of type \((1, 1)\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying the following compatibility conditions

\[
\begin{align*}
(i) & \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\
(ii) & \quad \eta(\xi) = 1, \quad \varphi^2 = id - \eta \otimes \xi, \\
(iii) & \quad \text{distribution } \mathcal{D} : p \in M \rightarrow \mathcal{D}_p \subset T_p M : \\
& \quad \mathcal{D}_p = \text{Ker}\eta = \{X \in T_p M : \eta(X) = 0\} \text{ is called paracontact distribution generated by } \eta.
\end{align*}
\]

The tensor field \(\varphi\) induces an almost paracomplex structure \([3]\) on each fibre on \(\mathcal{D}\) and \((\mathcal{D}, \varphi, g|_{\mathcal{D}})\) is a \(2n\)-dimensional almost paracomplex manifold. Since \(g\) is non-degenerate metric on \(M\) and \(\xi\) is non-isotropic, the paracontact distribution \(\mathcal{D}\) is non-degenerate.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism \(\varphi\) has rank \(2n\), \(\varphi \xi = 0\) and \(\eta \circ \varphi = 0\), (see [1] [2] for the almost contact case).

If a manifold \(M^{(2n+1)}\) with \((\varphi, \xi, \eta)\)-structure admits a pseudo-Riemannian metric \(g\) such that

\[
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

then we say that \(M^{(2n+1)}\) has an almost paracontact metric structure and \(g\) is called **compatible**. Any compatible metric \(g\) with a given almost paracontact structure is necessarily of signature \((n + 1, n)\).

Note that setting \(Y = \xi\), we have \(\eta(X) = g(X, \xi)\).

Further, any almost paracontact structure admits a compatible metric.

**Definition 2.1.** If \(g(\varphi X, \varphi Y) = d\eta(X, Y)\) (where \(d\eta(X, Y) = \frac{1}{2}(X \eta(Y) - Y \eta(X) - \eta([X, Y]))\)) then \(\eta\) is a paracontact form and the almost paracontact metric manifold \((M, \varphi, \eta, \xi, g)\) is said to be a **paracontact metric manifold**.

A paracontact metric manifold for which \(\xi\) is Killing is called a **K-paracontact manifold**. A paracontact structure on \(M^{(2n+1)}\) naturally gives rise to an almost paracomplex structure on the product \(M^{(2n+1)} \times \mathbb{R}\). If this almost paracomplex structure is integrable, then the given paracontact metric manifold is said to be a **para-Sasakian manifold**. Equivalently, (see [7]) a paracontact metric manifold is a para-Sasakian manifold if and only if

\[
(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X,
\]

for all vector fields \(X\) and \(Y\) (where \(\nabla\) is the Livi-Civita connection of \(g\)).

**Definition 2.2.** If \((\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X\), then the manifold \((M, \varphi, \eta, \xi, g)\) is said to be a **quasi-para-Sasakian manifold**.

From **Definition 2.2** (see [5]) we have

\[
\nabla_X \xi = \varphi X.
\]
In [8], it is proved that \((M, \varphi, \eta, \xi, g)\) is normal, since \(\xi\) is a Killing vector field and the manifold is not para-Sasakian. Thus we have

**Proposition 2.3.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold. Then \((M, \varphi, \eta, \xi, g)\) is normal but not para-Sasakian.

Denoting by \(\mathcal{L}\) the Lie differentiation of \(g\), we see

**Proposition 2.4.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold. Then we have

\[(\nabla_X \eta)Y = -g(X, \varphi Y),\]
\[(\mathcal{L}_\xi g) = 0,\]
\[(\mathcal{L}_\xi \varphi) = 0,\]
\[(\mathcal{L}_\xi \eta) = 0,\]
\[(d\eta(X, Y) = -g(X, \varphi Y),\]

where \(X, Y \in T_pM\).

Since the proof of **Proposition 2.4** follows by routine calculation, we shall omit it.

Denoting by \(R\) the curvature tensor of \(\nabla\), we have the following

**Definition 2.5.** An almost paracontact structure \((\varphi, \xi, \eta, g)\) is said to be **locally symmetric** if \((\nabla_W R)(X, Y, Z) = 0\), for all vector fields \(W, X, Y, Z \in T_pM\).

**Definition 2.6.** An almost paracontact structure \((\varphi, \xi, \eta, g)\) is said to be **locally \(\varphi\)-symmetric** if \(\varphi^2(\nabla_W R)(X, Y, Z) = 0\), for all vector fields \(W, X, Y, Z\) orthogonal to \(\xi\).

Finally, the sectional curvature \(K(\xi, X) = \epsilon_X R(X, \xi, \xi, X)\), where \(|X| = \epsilon_X = \pm 1\), of a plane section spanned by \(\xi\) and the vector \(X\) orthogonal to \(\xi\) is called \(\xi\)-**sectional curvature**, whereas the sectional curvature \(K(X, \varphi X) = -R(X, \varphi X, \varphi X, X)\), where \(|X| = -|\varphi X| = \pm 1\), of a plane section spanned by vectors \(X\) and \(\varphi X\) orthogonal to \(\xi\) is called a \(\varphi\)-**para-holomorphic sectional curvature**.

### 3. Some properties of quasi-para-Sasakian manifolds

The following result is well-known from the theory of para-Sasakian manifolds: \(K(X, \xi) = -1\) and if a para-Sasakian manifold is locally symmetric, then it is of constant negative curvature \(-1([5])\). For quasi-para-Sasakian manifolds we get

**Proposition 3.1.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold. Then we have

\[(3.10) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,\]
\[(3.11) \quad R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,\]
\[(3.12) \quad Ric(X, \xi) = -2n\eta(X),\]
\[(3.13)\quad K(X, \xi) = -1,\]

\[(3.14)\quad (\nabla_Z R)(X, Y, \xi) = -R(X, Y)\varphi Z + g(X, \varphi Z)Y - g(Y, \varphi Z)X,\]

where Ric is the Ricci tensor and \(X, Y, Z \in T_p M\).

**Proof.** The equation (3.10) follows directly from (2.4), (2.5) and the definition of the curvature \(R\). The equations (3.11), (3.12) and (3.13) are a consequence of (3.10). By virtue of (2.4) (2.5) and (3.10) we get (3.14):

\[
(\nabla_Z R)(X, Y, \xi) = Z(R(X, Y)\xi) - R(\nabla_Z X, Y)\xi - R(X, \nabla_Z Y)\xi - R(X, Y)\nabla_Z \xi = -R(X, Y)\varphi Z + g(X, \varphi Z)Y - g(Y, \varphi Z)X.
\]

\[\square\]

**Corollary 3.2.** If \((M, \varphi, \eta, \xi, g)\) is locally symmetric, then it is of constant negative curvature \(-1\).

**Proof.** Corollary 3.2 follows from (3.14).

\[\square\]

### 4. \(\eta\)-Einstein manifolds

An almost paracontact pseudo-Riemannian manifold is called \(\eta\) – *Einstein*, if the Ricci tensor \(Ric\) satisfies \(Ric = a.\eta + b.\eta \otimes \eta\), where \(a, b\) are smooth scalar functions on \(M\). If a para-Sasakian manifold is \(\eta\)–Einstein and \(n > 1\), then \(a\) and \(b\) are constant (see [7]).

**Proposition 4.1.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold. If \(M\) is an \(\eta\)–Einstein manifold, we have

\[(4.15)\quad a + b = -2n,\]

\[(4.16)\quad a = \text{constant}\quad \text{and}\quad b = \text{constant},\quad n > 1.\]

**Proof.** The equation (4.15) follows from \(Ric(X, \xi) = -2n\eta(X)\) which is derived from (3.10). As \(M\) is an \(\eta\)–Einstein manifold, the scalar curvature \(scal\) is equal to \(2n(a - 1)\). We define the Ricci operator \(Q\) as follows: \(g(QX, Y) = Ric(X, Y)\). By the identity \(Y(scal) = 2nY(a)\) and the trace of the map \([X \rightarrow (\nabla_X Q)Y]\), we have \(\xi(scal) = 0\), \(\xi(a) = \xi(b) = 0\). From the identity

\[
(\nabla_Z Ric)(X, Y) = Z(a)g(X, Y) + Z(b)\eta(X)\eta(Y) - b\eta(Y)g(Z, \varphi X) - b\eta(X)g(Z, \varphi Y)
\]

we have \(Y(scal) = 2Y(a)\), but \(Y(scal) = 2nY(a)\) and therefore \(2(n - 1)Y(a) = 0\). For \(n > 1\) we obtain \(a = \text{constant}\) and \(b = \text{constant}\).

\[\square\]
5. Curvature tensor

Now we prove the following

**Proposition 5.1.** Let $(M, \varphi, \eta, \xi, g)$ be a quasi-para-Sasakian manifold. Then we have the following identities

\[(5.17) \quad R(X,Y)\varphi Z - \varphi R(X,Y)Z = g(Y,Z)\varphi X - g(X,Z)\varphi Y - g(Y,\varphi Z)X + + g(X,\varphi Z)Y,\]

\[(5.18) \quad R(\varphi X,\varphi Y)Z = -R(X,Y)Z - g(Y,Z)X + g(X,Z)Y + + g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y,\]

where $X,Y,Z \in T_pM$.

**Proof.** The equation (5.17) follows from the Ricci's identity:

\[\nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi = R(X,Y)\varphi Z - \varphi R(X,Y)Z.\]

We verify (5.18): By (5.17), we have

\[R(X,Y)\varphi Z - \varphi R(X,Y)Z = g(Y,Z)\varphi X - g(X,Z)\varphi Y - g(Y,\varphi Z)X + + g(X,\varphi Z)Y,\]

\[R(\varphi X,\varphi Y)Z = -R(X,Y)Z - g(Y,Z)X + g(X,Z)Y + + g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y,\]

Using $\eta(R(X,Y)Z) = -\eta(X)(g(Y,Z) + \eta(Y)g(X,Z)$, the above formula takes the form

\[R(\varphi Z,\varphi W,X,Y) = -R(Z,W,X,Y) - g(Y,Z)g(X,W) + g(X,Z)g(Y,W) - -g(Y,\varphi Z)g(X,\varphi W) + g(X,\varphi Z)g(Y,\varphi W).\]

\[\square\]

As an application of Proposition 5.1, we shall prove the following proposition.

**Proposition 5.2.** Let $(M, \varphi, \eta, \xi, g)$ be a quasi-para-Sasakian manifold of dimension greater than 3. If $M$ is conformally flat, then $M$ is a space of constant negative curvature $-1$.

**Proof.** Since $M$ is conformally flat, the curvature tensor of $M$ is written as

\[(5.19) \quad R(X,Y)Z = \frac{1}{2n-1}(Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY) - + \frac{scal}{2n(2n-1)}(g(X,Z)Y - g(Y,Z)X).\]

We calculate $R(\xi,Y)\xi$ using the previous formula. Using (3.10) and $Ric(X,\xi) = -2n\eta(X)$,

we get

\[(5.20) \quad 2nRic(Y,Z) = (scal + 2n)g(Y,Z) - (scal + 4n^2 + 2n)\eta(Y)\eta(Z).\]

By virtue of (5.17), (5.19) and (5.20), we have

\[(5.21) \quad (scal + 4n^2 + 2n)(g(Y,\varphi Z)X - g(X,\varphi Z)Y + g(X,Z)\varphi Y - g(Y,Z)\varphi X + + ...\]
Replacing $R(\varphi Z)\eta(Y)\xi - g(Y, \varphi Z)\eta(X)\xi + \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y = 0$.

Let $(e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \xi)$ be an orthonormal basis of $T_pM$. Setting $X = e_1, Y = e_2$ and $Z = \varphi e_2$ in (5.21), we see scal $= -2n(2n + 1)$. Thus we have $\text{Ric} = -2ng$.

**Proposition 5.3.** Let $(M, \varphi, \eta, \xi, g)$ be a quasi-para-Sasakian manifold. The necessary and sufficient condition for a para-Sasakian manifold to have constant $\varphi$–para-holomorphic sectional curvature $H$ is

\begin{equation}
4R(X, Y)Z = (H - 3)(g(Y, Z)X - g(X, Z)Y) + (H + 1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \\
+ \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(Y, \varphi Z)\varphi X - g(X, \varphi Z)\varphi Y + 2g(\varphi X, Y)\varphi Z),
\end{equation}

where $X, Y, Z \in T_pM$.

**Proof.** For any vector fields $X, Y \in \mathcal{D}$, we have

\begin{equation}
R(X, \varphi X, X, \varphi X) = Hg^2(X, X)
\end{equation}

By identity (5.17) we get

\begin{equation}
R(X, Y, X, \varphi Y) = R(X, \varphi Y, Y, \varphi X) - g^2(X, \varphi Y) + g^2(X, Y) - g(X, X)g(Y, Y),
\end{equation}

\begin{equation}
R(X, \varphi X, Y, X, \varphi Y) = R(X, \varphi Y, Y, \varphi X).
\end{equation}

Substituting $X + Y$ in (5.17) and using the Bianchi identity, we obtain

\begin{equation}
2R(X, \varphi X, X, \varphi Y) + 2R(Y, \varphi Y, Y, \varphi X) + 3R(X, \varphi Y, Y, \varphi X) - R(X, Y, X, Y) = \\
= H(2g^2(X, Y) + g(X, X)g(Y, Y) + g(Y, Y)g(X, X) + 2g(X, Y)g(Y, Y)).
\end{equation}

Replacing $Y$ by $-Y$ in (5.26) and summing it to (5.26) we have

\begin{equation}
3R(X, \varphi Y, Y, \varphi X) - R(X, Y, X, Y) = H(2g^2(X, Y) + g(X, X)g(Y, Y)).
\end{equation}

Replacing $Y$ by $\varphi Y$ in (5.27) and from identities (5.28), (5.24) and (5.21), we get

\begin{equation}
4R(X, Y, X, Y) = (H - 3)(g^2(X, Y) - g(X, X)g(Y, Y)) + (H + 1)g^2(X, \varphi Y).
\end{equation}

Let $X, Y, Z, W \in \mathcal{D}$. Calculating $R(X + Z, Y + W, X + Z, Y + W)$ and using (5.28) we see

\begin{equation}
4R(X, Y, Z, W) + 4R(X, W, Z, Y) = (H - 3)(g(X, Y)g(Z, W) + g(X, W)g(Y, Z) - \\
- 2g(X, Z)g(Y, W)) + 3(H + 1)(g(X, \varphi Y)g(Z, \varphi W) + g(X, \varphi W)g(Z, \varphi Y))
\end{equation}
and we have
\[(5.30)\]
\[-4R(X, Z, Y, W) - 4R(X, W, Y, Z) = -(H - 3)(g(X, Z)g(Y, W) + g(X, W)g(Y, Z) -
-2g(X, Y)g(Z, W)) - 3(H + 1)(g(X, \varphi Z)g(Y, \varphi W) + g(X, \varphi W)g(Y, \varphi Z)).\]
Adding (5.29) to (5.30) and using Bianchi identity, we get
\[(5.31)\]
\[4R(X, Y, Z, W) = (H - 3)(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)) +
+(H + 1)(g(X, \varphi W)g(\varphi Y, Z) - g(X, \varphi Z)g(\varphi Y, W) + 2g(X, \varphi Y)g(Z, \varphi W)).\]
For any vector fields \(X, Y, Z, W \in T_pM\) we have \(\varphi X, \varphi Y, \varphi Z, \varphi W \in \mathbb{D}\), and using (5.31), (3.10), (5.17) and (5.18), we get (5.22).

Recall that, if a para-Sasakian manifold has a constant \(\varphi\)--para-holomorphic sectional curvature, then it is \(\eta\)-Einstein. Similarly, we have the following theorem in our case:

**Theorem 5.4.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold. If \(M\) is a space of constant \(\varphi\)--para-holomorphic sectional curvature \(H\), then \(M\) is \(\eta\)-Einstein.

**Proof.** By virtue of Proposition 5.3, \(M\) is an \(\eta\)--Einstein manifold and
\[\text{Ric} = \frac{1}{2}(n(H - 3) + H + 1)g - \frac{1}{2}(n + 1)(H + 1)\eta \otimes \eta.\]

From the Theorem 5.4 we have the following

**Corollary 5.5.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold of a constant \(\varphi\)--para-holomorphic sectional curvature \(H = -1\). Then \(M\) is a space of constant curvature.

We can generalize Corollary 3.2 slightly as follows:

**Proposition 5.6.** Let \((M, \varphi, \eta, \xi, g)\) be a quasi-para-Sasakian manifold. If \(M\) satisfies the Nomizu’s condition, i.e., \(R(X, Y).R = 0\), for any \(X, Y \in T_pM\), then it is of constant negative curvature \(-1\).

**Proof.** Let \(X, Y \in \mathbb{D}\) and \(g(X, Y) = 0\). Then, using (3.10) and (3.11) above, we obtain
\[\text{R}(X, \xi) = g(X, Y)\]
\[\text{R}(X, Y) = R(X,\xi)R(X, Y)Y - R(R(X, \xi)X, Y)Y - R(X, R(X, \xi)Y)Y -
-R(X, Y)R(X, \xi)Y = R(X, Y, X, \xi) - R(X, Y, Y, \xi)X - g(X, X)R(\xi, Y)Y =
= (R(X, Y, X) + g(X, X)g(Y, Y))\xi.\]
From the identity \(R(X, Y).R = 0\), we get \(R(X, Y, X) = -g(X, X)g(Y, Y)\), which implies that \((M, \varphi, \eta, \xi, g)\) is of constant \(\varphi\)--para-holomorphic sectional curvature \(-1\), and hence it is of constant curvature \(-1\).
The PC-Bochner curvature tensor on $M$ is defined by [6]

\[ B(X, Y, Z, W) = R(X, Y, Z, W) + \frac{1}{2n+4} (Ric(X, Z)g(Y, W) - Ric(Y, Z)g(X, W) + \\
+ Ric(Y, W)g(X, Z) - Ric(X, W)g(Y, Z) + Ric(\varphi X, Z)g(Y, \varphi W) - \\
- Ric(\varphi Y, Z)g(X, \varphi W) + Ric(\varphi Y, W)g(X, \varphi Z) - Ric(\varphi X, W)g(Y, \varphi Z) + \\
+ 2Ric(\varphi X, Y)g(Z, \varphi Z) + 2Ric(\varphi Z, W)g(X, \varphi Y) - Ric(X, Z)\eta(Y)\eta(W) + \\
+ Ric(Y, Z)\eta(X)\eta(W) - Ric(Y, W)\eta(X)\eta(Z) + Ric(X, W)\eta(Y)\eta(Z) + \\
+ \frac{k-4}{2n+4} (g(X, Z)g(Y, W) - g(Y, Z)g(X, W)) - \frac{k+2n}{2n+4} (g(Y, \varphi W)g(X, \varphi Z) - \\
- g(X, \varphi W)g(Y, \varphi Z) + 2g(X, \varphi Y)g(Z, \varphi W) - \frac{k}{2n+4} (g(Y, Z)\eta(X)\eta(W) - \\
- g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)), \\
\]

where $k = -\frac{\text{scal} - 2n}{2n+2}$.

Using the PC-Bochner curvature tensor we have

**Theorem 5.7.** Let $(M, \varphi, \eta, \xi, g)$ be a quasi-para-Sasakian manifold. Then $M$ is with constant $\varphi-$para-holomorphic sectional curvature if and only if it is $\eta$-Einstein and with vanishing PC-Bochner curvature tensor.

Since the proof of this Theorem is done using the same method as in the proof of the Theorem 5.5 in [6], we shall omit it.

6. 3-dimensional quasi-para-Sasakian manifolds

In a 3-dimensional pseudo-Riemannian manifold, we have

\[ R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \\
- \frac{\text{scal}}{2} (g(Y, Z)X - g(X, Z)Y). \]

Setting $Z = \xi$ in (5.31) and using (3.10) and (3.12), we have

\[ \eta(Y)QX - \eta(X)QY = (\frac{\text{scal}}{2} + 1)(\eta(Y)X - \eta(X)Y). \]

Setting $Y = \xi$ in (6.33) and then using (3.12) (for $n=1$), we get

\[ QX = \frac{1}{2} [(\text{scal} + 2)X - (\text{scal} + 6)\eta(X)\xi] \]

i.e.,

\[ Ric(Y, Z) = \frac{(\text{scal} + 2)}{2} g(Y, Z) - \frac{(\text{scal} + 6)}{2} \eta(Y)\eta(Z). \]

**Lemma 6.1.** A 3-dimensional quasi-para-Sasakian manifold is a manifold of constant negative curvature if and only if the scalar curvature $\text{scal} = -6$. 

Proof. Using (6.34) in (6.32), we get

(6.35) \[ R(X, Y)Z = \frac{(\text{scal} + 4)}{2}(g(Y, Z)X - g(X, Z)Y) - \frac{(\text{scal} + 6)}{2}(g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \]

and now the Lemma is obvious.

Let us consider a 3-dimensional quasi-para-Sasakian manifold which satisfies the condition

(6.36) \[ R(X, Y).\text{Ric} = 0, \]

for any vector fields \( X, Y \in T_pM \).

**Theorem 6.2.** A 3-dimensional quasi-para-Sasakian manifold \((M, \varphi, \eta, \xi, g)\) satisfying the condition \( R(X, Y).\text{Ric} = 0 \) is a manifold of constant negative curvature \(-1\).

**Proof.** From (6.36), we have

(6.37) \[ \text{Ric}(R(X, Y)U, V) + \text{Ric}(U, R(X, Y)V) = 0. \]

Setting \( X = \xi \) and using (3.11), we obtain

(6.38) \[ \eta(U)\text{Ric}(Y, V) - g(Y, U)\text{Ric}(\xi, V) + \eta(V)\text{Ric}(U, \xi) - g(Y, V)\text{Ric}(\xi, U) = 0. \]

Using (3.12) in (6.38), we have

(6.39) \[ \eta(U)\text{Ric}(Y, V) + 2g(Y, U)\eta(V) + \eta(V)\text{Ric}(Y, U) + 2g(Y, V)\eta(U) = 0. \]

Taking the trace in (6.39), we get

(6.40) \[ \text{Ric}(\xi, V) + 8\eta(V) + \text{scal}\eta(V) = 0. \]

Using (3.12) in (6.40), we obtain

\[ (\text{scal} + 6)\eta(V) = 0. \]

This gives \( \text{scal} = -6 \) (since \( \eta(V) \neq 0 \)), which implies, by Lemma 6.1, that the manifold is of constant negative curvature \(-1\). \( \square \)

**Theorem 6.3.** A 3-dimensional quasi-para-Sasakian manifold \((M, \varphi, \eta, \xi, g)\) is locally \( \varphi \)-symmetric if and only if the scalar curvature \( \text{scal} \) is constant.

**Proof.** Differentiating (6.35) covariantly with respect to \( W \) we get

(6.41) \[ (\nabla_W R)(X, Y, Z) = \frac{W(\text{scal})}{2}(g(Y, Z)X - g(X, Z)Y) - \frac{W(\text{scal})}{2}(g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) - \frac{(\text{scal} + 6)}{2}(g(Y, Z)(\nabla_W \eta)X\xi - g(X, Z)(\nabla_W \eta)Y\xi + g(Y, Z)\eta(X)\nabla_W \xi - g(X, Z)\eta(Y)\nabla_W \eta)X - \eta(Y)(\nabla_W \eta)ZX - - (\nabla_W \eta)X\eta(Z)Y - \eta(X)(\nabla_W \eta)ZY). \]
Taking $X, Y, Z, W$ orthogonal to $\xi$ and using (2.4) and (2.5), we get from the above

\begin{equation}
(\nabla_W R)(X, Y, Z) = \frac{W(\text{scal})}{2}(g(Y, Z)X - g(X, Z)Y) - \frac{\text{(scal) + 6}}{2}(-g(Y, Z)g(X, W)\xi + g(X, Z)g(Y, W)\xi).
\end{equation}

From (6.42) it follows that

\begin{equation}
\varphi^2(\nabla_W R)(X, Y, Z) = \frac{W(\text{scal})}{2}(g(Y, Z)X - g(X, Z)Y).
\end{equation}

Again, if the manifold satisfies the condition $R(X, Y).\text{Ric} = 0$, then we have seen that $\text{scal} = -6$, i.e. $\text{scal} = \text{constant}$ and hence from (6.41) we can state the following

**Theorem 6.4.** A 3-dimensional quasi-para-Sasakian manifold $(M, \varphi, \eta, \xi, g)$ satisfying the condition $R(X, Y).\text{Ric} = 0$ is locally symmetric.

**Definition 6.5.** The Ricci tensor $\text{Ric}$ of a quasi-para-Sasakian manifold $M$ is called $\eta-$parallel if it satisfies $(\nabla_X \text{Ric})(\varphi Y, \varphi Z) = 0$ for all vector fields $X, Y$ and $Z$.

The notation for Ricci-$\eta-$parallelity for Sasakian manifolds was introduce in [4].

**Proposition 6.6.** If a 3-dimensional quasi-para-Sasakian manifold $(M, \varphi, \eta, \xi, g)$ has $\eta-$parallel Ricci tensor, then the scalar curvature $\text{scal}$ is constant.

**Proof.** From (6.34), we get, by virtue of (2.2) and $\eta \circ \varphi = 0$,

\begin{equation}
\text{Ric}(\varphi X, \varphi Y) = -\frac{(\text{scal} + 2)}{2}(g(X, Y) - \eta(X)\eta(Y)).
\end{equation}

Differentiating (6.44) covariantly along $Z$, we get

\begin{equation}
(\nabla_Z \text{Ric})(\varphi X, \varphi Y) = -\frac{Z(\text{scal})}{2}(g(X, Y) - \eta(X)\eta(Y)) + \frac{Z(\text{scal})}{2}(g(X, Y) - \eta(X)\eta(Y)) + \frac{(\text{scal} + 2)}{2}(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y).
\end{equation}

By using $(\nabla_X \text{Ric})(\varphi Y, \varphi Z) = 0$ and (6.45), we get

\begin{equation}
-Z(\text{scal})(g(X, Y) - \eta(X)\eta(Y)) + (\text{scal} + 2)(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y) = 0.
\end{equation}

Taking the trace in (6.46), we get $Z(\text{scal}) = 0$, for all $Z$. \hfill \Box

By virtue Proposition 6.6 and Theorem 6.3 we have the following

**Theorem 6.7.** If a 3-dimensional quasi-para-Sasakian manifold $(M, \varphi, \eta, \xi, g)$ has $\eta-$parallel Ricci tensor is locally $\varphi-$symmetric.

**Theorem 6.8.** If a 3-dimensional quasi-para-Sasakian manifold $(M, \varphi, \eta, \xi, g)$ has $\eta-$parallel Ricci tensor, then it satisfies the condition

\begin{equation}
(\nabla_X \text{Ric})(Y, Z) + (\nabla_Y \text{Ric})(Z, X) + (\nabla_Z \text{Ric})(X, Y) = 0.
\end{equation}
Proof. Taking the trace in (6.46), we obtain
\[ X(\text{scal}) = 0, \]
for any vector field \( X \). From (6.34), we have
\[ (\nabla_X \text{Ric})(X, Y) = Z(\text{scal}) \frac{g(X, Y) - \eta(X)\eta(Y)}{2} - \frac{(\text{scal} + 6)}{2}(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y). \]

Now using (6.48) and (6.49), we have
\[ (\nabla_Z \text{Ric})(X, Y) = -\frac{(\text{scal} + 6)}{2}(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y). \]

By virtue of (6.50), we get from (2.5) that
\[ (\nabla_X \text{Ric})(Y, Z) + (\nabla_Y \text{Ric})(Z, X) + (\nabla_Z \text{Ric})(X, Y) = 0. \]

Example 6.9. Let \( L \) be a 3-dimensional real connected Lie group and \( \mathfrak{g} \) be its Lie algebra with a basis \( \{E_1, E_2, E_3\} \) of left invariant vector fields (see [8], by the following commutators:
\[ [E_1, E_2] = 2E_3, \quad [E_1, E_3] = 2E_2, \quad [E_2, E_3] = 2E_1. \]

We define an almost paracontact structure \((\varphi, \xi, \eta)\) and a pseudo-Riemannian metric \( g \) in the following way:
\[ \varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0, \]
\[ \xi = E_3, \quad \eta(E_3) = 1, \quad \eta(E_1) = \eta(E_2) = 0, \]
\[ g(E_1, E_1) = g(E_3, E_3) = -g(E_2, E_2) = 1, \]
\[ g(E_i, E_j) = 0, \quad i \neq j \in \{1, 2, 3\}. \]

Then \((L, \varphi, \xi, \eta, g)\) is a 3-dimensional almost paracontact manifold. Since the metric \( g \) is left invariant, the Koszul equality becomes
\[ \nabla_{E_i} E_1 = 0, \quad \nabla_{E_i} E_2 = E_3, \quad \nabla_{E_i} E_3 = E_2, \]
\[ \nabla_{E_3} E_1 = -E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_3 = E_1, \]
\[ \nabla_{E_3} E_1 = -E_3, \quad \nabla_{E_1} E_2 = -E_1, \quad \nabla_{E_3} E_3 = 0. \]

It is not hard to see that the Ricci tensor \( \text{Ric} \) is equal to
\[ \text{Ric}(X, Y) = \frac{\text{scal}}{3}g(X, Y), \]
where \( \text{scal} = -6. \)
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