Upper bound on cubicity in terms of boxicity for graphs of low chromatic number

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April 30, 2014

Abstract

The boxicity (respectively cubicity) of a graph $G$ is the minimum non-negative integer $k$, such that $G$ can be represented as an intersection graph of axis-parallel $k$-dimensional boxes (respectively $k$-dimensional unit cubes) and is denoted by box($G$) (respectively cub($G$)). It was shown by Adiga and Chandran (Journal of Graph Theory, 65(4), 2010) that for any graph $G$, cub($G$) $\leq$ box($G$) $\lceil \log_2 \alpha \rceil$, where $\alpha = \alpha(G)$ is the cardinality of the maximum independent set in $G$. In this note we show that cub($G$) $\leq$ $2 \lceil \log_2 \chi(G) \rceil$ box($G$) + $\chi(G) \lceil \log_2 \alpha(G) \rceil$. In general, this result can provide a much better upper bound than that of Adiga and Chandran for graph classes with bounded chromatic number. For example, for bipartite graphs we get, cub($G$) $\leq$ $2(\text{box}(G) + \lceil \log_2 \alpha(G) \rceil)$.

Moreover we show that for every positive integer $k$, there exist graphs with chromatic number $k$, such that for every $\epsilon > 0$, the value given by our upper bound is at most $(1 + \epsilon)$ times their cubicity. Thus, our upper bound is almost tight.

Keywords: Boxicity, Cubicity, Chromatic Number.

1 Introduction

An axis-parallel $k$-dimensional box, or $k$-box in short, is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where each $R_i$ is an interval of the form $[a_i, b_i]$ on the real line. A $k$-unit-cube (or $k$-cube in short) is a $k$-box where each $R_i$ is an interval of the form $[a_i, a_i + 1]$. A graph $G(V, E)$ is said to be an intersection graph of $k$-boxes (respectively $k$-cubes) if there is a mapping $f$ that maps the vertices of $G$ to $k$-boxes (respectively $k$-cubes) such that for any two vertices $u,v \in V$, $(u,v) \in E(G) \iff f(u) \cap f(v) \neq \emptyset$. Then $f$ is called a $k$-box representation of $G$ (respectively $k$-cube representation). The boxicity (respectively cubicity) of a graph $G$, denoted by box($G$) (respectively cub($G$)) is the minimum non-negative integer $k$ such that $G$ has a $k$-box representation (respectively $k$-cube representation). Only complete graphs have boxicity (cubicity) 0. The class of graphs with boxicity at most 1 is the class of interval graphs, and the class of graphs with cubicity at most 1 is the class of unit interval graphs.

\textsuperscript{*}Supported by VATAT Post-doctoral Fellowship, Council of Higher Education, Israel.
In the rest of the paper we will always use $n$ to denote the number of vertices of the graph being discussed. Logarithms will be to the base 2, unless otherwise specified.

Let $H_1$ and $H_2$ be two graphs such that $V(H_1) = V(H_2) = V(G)$ and $E(G) = E(H_1) \cap E(H_2)$. Then we write $G = H_1 \cap H_2$. The following observation was made by F. S. Roberts [23].

Lemma 1 ([23]). Boxicity of a non-complete graph $G$ is the minimum positive integer $k$ such that there exists interval graphs $I_1, \ldots, I_k$ such that $G = I_1 \cap \cdots \cap I_k$. Cubicity of a non-complete graph $G$ is the minimum positive integer $k$ such that there exists $k$ unit interval graphs $U_1, \ldots, U_k$ such that $G = U_1 \cap \ldots \cap U_k$.

### 1.1 Brief literature survey

The concepts of boxicity and cubicity were introduced by F. S. Roberts [23] in 1968 for studying some problems in ecology. The computational complexity of finding the boxicity of a graph was studied by [26] [16]. It is known that it is NP-hard to decide whether the boxicity of a graph is at most 2 [22]. Recently it was shown that it is hard to even approximate within $n^{1-\epsilon}$, for any $\epsilon > 0$ unless NP=ZPP [8]. The best known approximation factor for boxicity is $O\left(\frac{n \sqrt{\log \log n}}{\sqrt{\log n}}\right)$ [2], and that for cubicity is $O\left(\frac{n (\log \log n)^{3/2}}{\sqrt{\log n}}\right)$ [2].

Adiga et al. [3] showed that boxicity is closely related to the well-studied concept of partial order dimension. Chandran and Sivadasan [13] proved that for any graph $G$, $\text{box}(G) \leq \text{treewidth}(G) + 2$. It is also known to be related to graph minors: Esperet and Joret [18] showed that for any graph $G$, $\text{box}(G) \in O(\eta^2 \log^2 \eta)$, where $\eta$ is the number of vertices in the largest clique minor of $G$. Chatterjee and Ghosh [15] related boxicity with Ferrer’s dimension. The upper bound for boxicity of a non-complete graph in terms of maximum degree of $G$ (denoted by $\Delta(G)$) was studied in [9] [17]. The current best known upper bound for boxicity of a graph in terms of $\Delta(G)$ is $O(\Delta \log^2 \Delta)$ [3], which follows from a corresponding upper bound for partial order dimension in terms of the maximum degree of the comparability graph of the partial order [20].

Boxicity of outerplanar graphs is known to be at most 2 [24]. Thomassen [25] showed that the boxicity of planar graphs is at most 3. Recently Felsner and Francis [10] gave a different proof for the above theorem. More proofs of this theorem are given in [4]. Hartman et al. showed that the boxicity of bipartite planar graphs is at most 2 [21]. Lower bounds for boxicity were studied in [6].

Chandran et al. [10] showed that for any graph $G$, $\text{cub}(G) \leq b + 1$, where $b$ denotes the bandwidth of $G$. They also showed that $\text{cub}(G) = O(\Delta \log b)$. Chandran and Mathew [5] showed that for any graph $G$, $\text{cub}(G) \leq (k + 2) \lceil 2e \log n \rceil$, where $k$ is the degeneracy of $G$. Chandran and Sivadasan [14] showed that for $d$-dimensional hypercubes $H_d$, $\text{cub}(H_d) = \Theta\left(\frac{d}{\log d}\right)$. Adiga and Chandran [4] showed that for an interval graph $G$, $\lceil \log \psi \rceil \leq \text{cub}(G) \leq \lceil \log \psi \rceil + 2$, where $\psi$ denotes the number of leaves in the largest induced star in $G$.

### 1.2 Cubicity vs. boxicity

Clearly for any graph $G$, $\text{box}(G) \leq \text{cub}(G)$. Then the following question becomes relevant: Does there exist a function $g$ such that $\text{cub}(G) = g(\text{box}(G))$? It is easy to see that the answer is negative: Consider a star graph on $n + 1$ vertices. Its cubicity is $\lceil \log n \rceil$ [23] whereas its boxicity is 1, since a star graph is an interval graph. Chandran and Mathew [12] showed that for any graph $G$, $\text{cub}(G) \leq \text{box}(G) \lceil \log n \rceil$, where $n$ is the number of vertices. Adiga and Chandran [4] improved this result by showing that we can use the cardinality of the maximum independent set in $G$, denoted by $\alpha(G)$, in the place of $n$. 

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Lemma 2. For any graph $G$, $\text{cub}(G) \leq \text{box}(G) \lceil \log \alpha(G) \rceil$.

Remark. We demonstrate next that the bound in the above lemma is tight, i.e., given any two positive integers $b$ and $a$, we show that there exists a graph $G$ with $\text{box}(G) = b$, $\alpha(G) = a$ and $\text{cub}(G) = b\lceil \log a \rceil$. It is known that a complete $p$-partite graph, $p \geq 2$, with $n_i, i \in [p]$, vertices in each part has boxicity $p$ and cubicity $\sum_{i=1}^p \lceil \log n_i \rceil$ [23]. Hence, given positive integers $b$ and $a$, if $b \geq 2$, the complete $b$-partite graph with $a$ vertices in each part will serve our purpose. If $b = 1$, then a star graph on $a + 1$ vertices gives the required graph.

We observe that, in a loose sense, the two terms in the upper bound on cub$(G)$ given in Lemma 2, namely $\lceil \log \alpha(G) \rceil$ and box$(G)$, by themselves contribute in keeping the cubicity of a graph high. Clearly box$(G)$ is a lower bound for cub$(G)$ since cubes are specialised boxes. The other term $\lceil \log \alpha(G) \rceil$ can make cub$(G)$ high due to a geometric reason, captured in the so-called ‘volume argument’, which we reproduce here (also see [11]): Let cub$(G) = k$. Then the vertices in the biggest independent set should correspond to pairwise non-intersecting $k$-dimensional unit cubes in the cube representation of $G$. Thus if we consider the minimal bounding box for the cube representation, that bounding box should have a volume of at least $\alpha(G)$ units. On the other hand, the width of this bounding box on any of the dimensions (i.e. the distance between the extreme points of the projection of the cube representation on the corresponding axis) can be at most $d + 1$ where $d$ is the diameter of $G$. (To see this note that each vertex correspond to a unit length interval in the projection and the pair of vertices containing the two extreme points in their respective intervals, are at a distance of at most $d$ in the graph, and thus it follows that the geometric distance between the two extreme points is at most $d + 1$.) From this it is clear that the volume of the minimal bounding box is at most $(d + 1)^k$. It follows that $(d + 1)^k \geq \alpha(G)$. From this we get cub$(G) = k \geq \lceil \log_{d+1} \alpha(G) \rceil$. Thus we have $M = \max(\text{box}(G), \lceil \log_{d+1} \alpha(G) \rceil) \leq \text{cub}(G)$. It is natural to investigate whether there exists a function $g$ such that cub$(G) \leq g(M)$. But the answer is negative since we can increase the diameter of a graph unboundedly without affecting its cubicity. For example, if $G$ is the graph obtained by identifying one end point of a path on $2n + 1$ vertices with the leaf of a star graph on $n + 1$ vertices, it is easy to check that box$(G) = 1$, $\alpha(G) = 2n$, diameter of $G$ $d = 2n + 2$ and hence $M = \max\{\text{box}(G), \lceil \log \alpha(G) / \log(d+1) \rceil \} = 1$, whereas cub$(G) = \lceil \log n \rceil$ which is far higher.

In this paper we ask a simpler question: Let $\bar{M} = \max(\text{box}(G), \lceil \log \alpha(G) \rceil)$. Lemma 2 tells us that cub$(G) \in O(M^3)$ and the remark after the lemma indicates that we cannot have anything better in general (choosing $\alpha = 2^k$ there illustrates the point). But can we show that cub$(G) \in O(\bar{M})$ for some restricted graph classes? In this paper we show that if we restrict ourselves to classes of graphs whose chromatic number is bounded above by a constant, such a result can indeed be proved. In fact our main theorem is a general upper bound for cubicity in terms of boxicity, the independence number and chromatic number:

Theorem 3. Let $G$ be a graph with chromatic number $\chi(G)$ and the cardinality of the maximum independent set $\alpha(G)$. Then cub$(G) \leq 2 \lceil \log \chi(G) \rceil \times \text{box}(G) + \chi(G) \lceil \log \alpha(G) \rceil$.

For graphs of low chromatic number, this result can be in general far better than that of Adiga et al. [4]. The most interesting case is that of bipartite graphs:

Corollary 4. For a bipartite graph $G$, cub$(G) \leq 2(\text{box}(G) + \lceil \log \alpha(G) \rceil)$.

Remark. The reader may naturally wonder whether chromatic number is an upper bound for the boxicity of a graph or not, in which case, Theorem 3 cannot be an improvement over Lemma 2. On the contrary there are several graphs with boxicity greater than chromatic number. In fact it looks most graphs are like that: In [6] it is shown that almost all balanced bipartite graphs (on $2n$ vertices) have boxicity $\Omega(n)$. The proof can be modified to show that almost
all bipartite graphs with \( n \) vertices on one side and \( m \) vertices on the other, have boxicity \( \Omega(\min(n, m)) \).

### 1.3 Preliminaries

A graph \( G \) is a co-bipartite graph if the complement of it, \( \overline{G} \) is a bipartite graph. Thus \( G \) is a co-bipartite graph if and only if the vertex set \( V(G) \) can be partitioned into two sets \( A \) and \( B \) such that \( A \) and \( B \) both induce cliques in \( G \). It is clear that for a co-bipartite graph \( G \), \( \alpha(G) \leq 2 \). Applying Lemma 2 we can infer the following:

**Lemma 5.** For a co-bipartite graph \( G \), \( \text{cub}(G) = \text{box}(G) \).

Let \( G \) be a graph and let its vertex set \( V(G) \) be partitioned into \( A \) and \( B \). Now construct a graph \( H_i \), with \( V(H_i) = V(G) \) and \( E(H_i) = E(G) \cup \{(u, v) : u, v \in A\} \cup \{(u, v) : u, v \in B\} \). Note that \( H_i \) is obtained from \( G \) by adding more edges so that \( A \) as well as \( B \) induce cliques in \( H \) and the edges across \( A \) and \( B \) are as in \( G \). The following observation is from [1].

**Lemma 6.** \( \text{box}(H_i) \leq 2\text{box}(G) \)

### 2 Proof of Theorem 3

Let \( G \) be a graph with \( \text{box}(G) = b \). Consider a proper coloring of \( G \) using \( \chi(G) = \chi \) colors: Let \( C_0, \ldots, C_{\chi-1} \) be the color classes with respect to this coloring. First we define \( \lfloor \log \chi \rfloor \) bipartitions of \( V(G) \) by the following rule: The bipartition \( (A_i, B_i) \) for \( 1 \leq i \leq \lfloor \log \chi \rfloor \) is obtained by setting \( A_i \) as the union of all the color classes \( C_j \) such that the \( i \)-th bit in the binary representation of \( j \) is 1. (Here we consider binary representation of numbers in \( \{0,1,\ldots,\chi-1\} \) using \( \lfloor \log \chi \rfloor \) bits). \( B_i \) is defined as the union of the remaining color classes. Define \( H_i \) to be the co-bipartite graph obtained by defining \( V(H_i) = V(G) \) and the edge set \( E(H_i) = E(G) \cup \{(u, v) : u, v \in A_i\} \cup \{(u, v) : u, v \in B_i\} \). That is, \( H_i \) is obtained by adding edges to \( G \) such that \( A_i \) and \( B_i \) induce cliques in the resulting graph, and the edges across \( A_i \) and \( B_i \) are as in \( G \). Since \( H_i \) is a co-bipartite graph, by Lemma 5 \( \text{cub}(H_i) = \text{box}(H_i) \). By Lemma 6 the \( \text{box}(H_i) \leq 2\text{box}(G) = 2b \). Thus \( \text{cub}(H_i) \leq 2b \). Therefore by Lemma 3 there exist \( 2b \) unit interval graphs, say \( U_i^1, \ldots, U_i^{2b} \) such that

\[
U_i^1 \cap \cdots \cap U_i^{2b} = H_i
\]

**Observation 7.** For \( 1 \leq i \leq \lfloor \log \chi(G) \rfloor \) and \( 1 \leq j \leq 2b \), \( U_i^j \) is a super graph of \( G \).

**Proof.** From equation 1 it is clear that \( U_i^j \) is a super graph of \( H_i \) which in turn is a super graph of \( G \). \( \square \)

Also define for each color class \( C_i \), \( 0 \leq i \leq \chi - 1 \), \( t_i = \lfloor \log |C_i| \rfloor \) unit interval graphs, \( W_i^1, \ldots, W_i^{t_i} \) in the following way: First number the vertices of \( C_i \) from 0 to \( |C_i| - 1 \). Let \( n_i(u) \) be the number given to a vertex \( u \in C_i \) as per this numbering. Now to define the unit interval graph \( W_i^j \), for \( 0 \leq i \leq \chi - 1 \) and \( 1 \leq j \leq t_i \), associate with each vertex \( v \in V(G) \) an interval \( f_i^j(v) \) as follows:

For each vertex \( u \in V - C_i \), \( f_i^j(v) = [1, 2] \)

For each \( u \in C_i \), \( f_i^j(u) = [0, 1] \) if the \( j \)-th bit in the binary representation of \( n_i(u) \) is 1, else \( f_i^j(u) = [2, 3] \). Define \( W_i^j \) to be the corresponding unit interval graph.

**Observation 8.** For \( 0 \leq i \leq \chi - 1 \), and \( 1 \leq j \leq t_i \), \( W_i^j \) is a super graph of \( G \)
Proof. If \((u, v) \in E(G)\) then \(u\) and \(v\) do not belong to the same color class. Let \(u \in C_a\) and \(v \in C_b\), where \(a \neq b\). Clearly one of \(a, b \neq i\): Without loss of generality, let \(b \neq i\). Thus \(f_i^a(v) = [1, 2]\). If \(a \neq i\), then \(f_i^a(u) = [1, 2]\) and if \(a = i\), then \(f_i^a(u)\) equals either \([0, 1]\) or \([2, 3]\). In all cases, \(u\) is adjacent to \(v\) in \(W_i^t\). It follows that \(W_i^t\) is a super graph of \(G\).

Claim 9.

\[\bigcap_{1 \leq i \leq \lceil \log \chi \rceil : 1 \leq j \leq 2b} U_i^j \bigcap_{0 \leq i \leq \chi - 1; 1 \leq j \leq t_i} W_i^j = G\].

In view of observations 7 and 8 to prove the above claim it is sufficient to show that if \((u, v) \notin E(G)\) then there exists one unit interval graph \(I \in \{U_i^j : 1 \leq i \leq \lceil \log \chi \rceil ; 1 \leq j \leq 2b\} \cup \{W_i^j : 0 \leq i \leq \chi - 1; 1 \leq j \leq t_i\}\) such that \(u\) is not adjacent to \(v\) in it. We consider two cases:

Case 1 \((u, v) \in C_i, \text{ for some } i \in \{0, \chi - 1\}\).

Recall that we had numbered the vertices of \(i\)th color class from 0 to \(|C_i| - 1\) to define the interval graphs \(W_i^j\), for \(1 \leq j \leq t_i = \lceil \log |C_i| \rceil\). Then there exists an index \(h, 1 \leq h \leq \lceil \log |C_j| \rceil\), such that the bit at the \(h\)-th position differs for \(n_i(u)\) and \(n_i(v)\). Without loss of generality assume that \(h\)-th bit in the binary representation of \(n_i(v)\) is 0, and that of \(n_i(v)\) is 1. Then by construction of \(W_h^i\), \(f_h^i(u) = [2, 3]\) and \(f_h^i(v) = [0, 1]\). It follows that \((u, v) \notin E(W_h^i)\).

Case 2 \((u \in C_i \text{ and } v \in C_j, \text{ where } i \neq j)\).

Let \(h, (1 \leq h \leq \lceil \log \chi \rceil)\), be a position where the binary representations of \(i\) and \(j\) differ. Without loss of generality assume that the \(h\)-th bit in the binary representation of \(i\) is 1 and the \(h\)-th bit in the binary representation of \(j\) is 0. Recalling the construction of the co-bipartite graph \(H_h\), we have \(u \in A_h\) and \(v \in B_h\) and therefore \((u, v) \notin E(H_h)\). Now, from equation 11 it follows that there exists \(j, 1 \leq j \leq 2b\), such that in the unit interval graph \(U_h^j\), \((u, v) \notin E(U_h^j)\).

From Claim 9 and Lemma 11 and noting that \(t_i = \lceil \log \alpha \rceil\) for \(0 \leq i \leq \chi - 1\), it immediately follows that \(\text{cub}(G) \leq 2 \lceil \log \chi(G) \rceil \text{ box}(G) + \chi(G) \lceil \log \alpha(G) \rceil\).

Tightness of Theorem 3: We will show that for every \(\epsilon > 0\), there exists a graph \(G\) such that \(\text{cub}(G) \leq 2 \lceil \log \chi(G) \rceil \text{ box}(G) + \chi(G) \lceil \log \alpha(G) \rceil\) \((1 + \epsilon) G\). Let \(k\) be a positive integer. Let \(T_k\) be the complete \(k\)-partite graph, where each part contains exactly \(\frac{n}{k}\) vertices. (Assume \(n\) to be a multiple of \(k\).) The \(\text{cub}(T_k) = k \lceil \log \frac{n}{k} \rceil\). The upper bound of Theorem 3 for \(T_k\) equals \(2k \lceil \log k \rceil + k \lceil \log \frac{n}{k} \rceil = \text{cub}(T_k)(1 + \frac{2\lceil \log k \rceil}{\lceil \log \frac{n}{k} \rceil}) \leq \text{cub}(T_k)(1 + \epsilon)\), provided we take \(n > k^{2+\epsilon}\). Thus there exist graphs for which the upper bound given by Theorem 3 is arbitrarily close to the true value of their cubicity.

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