Lattice QCD calculation of the pion charge radius using a model-independent method

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August 05, 2020

Asia-Pacific Symposium for Lattice Field Theory

Reference: Phys. Rev. D 101, 051502 (2020)
Outline

1. **Introduction and motivation**
   - What is charge radius?
   - Traditional lattice approach: fitting the form factor

2. **Model-independent method**
   - Calculating the slope directly on lattice
   - Results: smaller statistical errors

3. **Conclusions and outlook**
Proton radius puzzle

Over $5\sigma$ discrepancy between muon and electron based measurements;

The debate over this puzzle mainly comes from experiments;

Lattice simulations have not been able to give a comparable result.

$\Rightarrow$ Still affected by systematic uncertainties.

JP Karr, Nature, 2019, 575, 61-62
Why pion charge radius?

Pion structure is simple, no signal-to-noise problem, ...
⇒ Well suited for a high precision benchmark of new methods.

What is charge radius?

Defined as the derivative of the form factor $F(q^2)$ at zero momentum transfer

$$\langle r^2 \rangle = 6 \left\{ \frac{dF(q^2)}{dq^2} \right\}_{q^2=0}$$

Pion form factor ↔ matrix element of the vector current

$$\langle \pi^+(p')|J_{\mu}|\pi^+(p)\rangle = F_\pi(q^2)(p + p')_{\mu}$$
Traditional approach: form factor from lattice

- Form factor under a certain momentum transfer

Construct ratio from two- and three-point functions

\[
\frac{C^{(3)}(t, t', \vec{p}, -\vec{p})}{C^{(2)}(t', \vec{p})} \xrightarrow{(t' - t) \to \infty} \frac{\langle \pi^+(p') | J_4 | \pi^+(p) \rangle}{2E_\pi(\vec{p})} = F_\pi(q^2)
\]

- Fit \( F(q^2) \) to get charge radius \( \rightarrow \) model dependence.
- Widely used for over a decade!

C. Alexandrou, Phys. Rev. D 97, 014508 (2018)
Basic idea

- We start with this hadronic function
  \[ H(x) = \langle 0 | O_\pi(t, \vec{x}) J_\mu(0) | \pi(0) \rangle \]
  In principle, this function contains all information about EM interactions.

- How to extract the part of interest?

By using an appropriate weight function \( \omega(x) \)

\[ \langle A \rangle = \int d^3\vec{x} \, \omega(x) H(x) \]

- E.g, \( \omega(x) = e^{i\vec{p} \cdot \vec{x}} \) will extract the part under a certain momentum

\[ \tilde{H}(t, \vec{p}) = \int d^3\vec{x} e^{i\vec{p} \cdot \vec{x}} H(x) \sim F_\pi(q^2) \]

- Charge radius \( \rightarrow \) derivative of the form factor \( \rightarrow \frac{d}{d|\vec{p}|^2} \int d^3\vec{x} e^{i\vec{p} \cdot \vec{x}} H(x) \)

\[ \Rightarrow \omega(x) \sim |\vec{x}|^2 \]
In the infinite volume, following the above idea, one can construct

\[ D(t) \equiv -\frac{m^2_\pi}{3!} \int d^3 \vec{x} |\vec{x}|^2 H(x), \quad \tilde{H}(t, \vec{0}) \equiv \int d^3 \vec{x} H(x) \]

At large time \( t \), the ratio of this two functions gives

\[ R(t) \equiv \frac{D(t)}{\tilde{H}(t, \vec{0})} \to \frac{1}{4} - \frac{m_\pi t}{2} - c_1 \]

where \( c_n \) is the expansion coefficients of the form factor

\[ F_\pi(q^2) = \sum_n c_n \left( \frac{q^2}{m^2_\pi} \right)^n \]

with \( c_0 = 1 \) and \( c_1 = \frac{m^2_\pi}{6} \langle r^2_\pi \rangle \).

One can determine \( c_1 \) directly using \( H(x) \) as input.
Figure: determine $c_1$ from the formula in continuum theory

- No plateau at all.
- Typical lattice size $m_\pi L \approx 4$, the integrand in $D(t)$ at the edge scales as
  \[ m_\pi^2 |\vec{x}|^2 \exp\left(-m_\pi \sqrt{|\vec{x}|^2 + t^2}\right) \approx 0.5 \ll 1. \]

$\Rightarrow$ Significant finite-volume effects!
On the lattice, one can still construct

\[ D^{(L)}(t) \equiv -\frac{m^2}{3!} \sum_{\mathbf{x} \in \mathbb{L}} |\mathbf{x}|^2 H(x), \quad \tilde{H}^{(L)}(t, \mathbf{0}) \equiv \sum_{\mathbf{x} \in \mathbb{L}} H(x) \]

In continuum theory, the ratio only contains contributions to the first order

\[ R^{(\infty)}(t) \equiv \frac{D^{(\infty)}(t)}{\tilde{H}^{(\infty)}(t, \mathbf{0})} \to \beta_0^{(\infty)}(t) + \beta_1^{(\infty)}(t) c_1 \]

Now it contains contributions to all orders

\[ R^{(L)}(t) \equiv \frac{D^{(L)}(t)}{\tilde{H}^{(L)}(t, \mathbf{0})} \to \sum_n \beta_n^{(L)}(t) c_n \]

with \( \beta_n^{(L)}(t) \) known explicitly and can be calculated numerically

\[ \beta_n^{(L)}(t) = -\frac{m^2}{3!} \sum_{\mathbf{x} \in \mathbb{L}^3} |\mathbf{x}|^2 l_n(x) \]

\[ l_n(x) = \frac{1}{L^3} \sum_{\hat{p}} \frac{\hat{E} \hat{m} \hat{E} + \hat{m} \hat{E}}{2\hat{m}} \left( \frac{q^2}{m^2_\pi} \right)^n e^{-(E-m_\pi)t} e^{-i\hat{p} \cdot \mathbf{x}} \]
Naively, one can determine $c_1$ through

$$\frac{R^{(L)}(t) - \beta_0^{(L)}(t)}{\beta_1^{(L)}(t)} \rightarrow c_1 \left( 1 + \sum_{n>1} \frac{\beta_n^{(L)}(t)c_n}{\beta_1^{(L)}(t)c_1} \right)$$

**Figure:** Residual terms as function of $m_\pi L$ with spacing $a = 0$ at fixed $m_\pi t = 1$

- Residual terms can be estimated by VMD model $F_\pi(q^2) = (1 - q^2/m_\rho^2)^{-1}$ with $c_n = (m_\pi/m_\rho)^{2n}$.
- At $m_\pi L = 4$, the lowest order ($n = 2$) of residual term $\sim 5\%$. 
Error reduction: systematic effects

- higher-order terms are not well suppressed at $m_\pi L \approx 4$.

\[ \Rightarrow \text{determine } c_{n \geq 2} \text{ through weight function } \omega(x) = |\vec{x}|^4, |\vec{x}|^6, \cdots \]

- one can generally construct

\[ D_k^{(L)}(t) \sim \sum_{\vec{x} \in L} |\vec{x}|^{2k} H(x), \quad k = 1, 2, \ldots \]

with the ratio truncated to the $m$-th order

\[ R^{(L)}(t) \equiv \frac{\sum_i^m f_i D_i^{(L)}(t)}{\bar{H}^{(L)}(t, 0)} + h \rightarrow c_1 + \mathcal{O}(c_{n \geq m} \text{ terms}) \]

where parameters $f_i$ and $h$ are chosen to remove the $c_0$ and $c_{m \geq n \geq 2}$ terms.

- Our final choice is $m = 2$.
  - Contamination from $c_{n \geq 3}$ are negligibly small. ($\lesssim 0.1\%$)
  - The signal-to-noise ratio decreases as $m$ increases.
Error reduction: statistical uncertainties

- Lattice data near the boundary of the box mainly contribute to the noise rather than signal since \( H(x) \sim \exp(-m_\pi \sqrt{|\vec{x}|^2 + t^2}) \)

\[ \Rightarrow \text{introduce an integral range } \xi_L \text{ to reduce the statistical error.} \]

\[ D_k^{(L,\xi)}(t) \sim \sum_{|\vec{x}| \leq \xi_L} |\vec{x}|^{2k} H(x) \]

- The formula of the ratio is therefore changed to

\[ R_k^{(L,\xi)}(t) \equiv \frac{D_k^{(L,\xi)}(t)}{\tilde{H}^{(L)}(t, 0)} \rightarrow \sum_n \beta_{k,n}^{(L,\xi)}(t) c_n \]

with

\[ \beta_{k,n}^{(L,\xi)}(t) \sim \sum_{|\vec{x}| \leq \xi_L} |\vec{x}|^{2k} I_n(x) \]

- with our final choice \( \xi_L = 1.5\text{fm} \), the statistical uncertainties are reduced by a factor of 1.3 – 1.8.

- We expect the error reduction can be much more significant in the study of nucleon charge radius!
We use 5 DWF ensembles from RBC/UKQCD. Phys. Rev. D 93, 074505 (2016)

- Relatively small statistics $n_{\text{conf}} \approx 30 - 50.$
- Now we can observe clear plateau for each ensemble!
## Results

| Ensemble  | Parameters | New  | Traditional |
|-----------|------------|------|-------------|
|           | $m_\pi$ [MeV] $L$ $a^{-1}$ [GeV] | $\langle r^2 \rangle$ [fm$^2$] | $\langle r^2 \rangle$ [fm$^2$] |
| 24D       | 141.2(4) 24 1.015 | 0.476(18) | 0.466(30) |
| 32D       | 141.3(3) 32 1.015 | 0.480(10) | 0.479(15) |
| 32D-fine  | 143.2(3) 32 1.378 | 0.423(15) | 0.409(28) |
| 48I       | 139.1(3) 48 1.730 | 0.434(20) | 0.395(32) |
| 24D-340   | 340.9(4) 24 1.015 | 0.3485(27) | 0.3495(44) |

- **Traditional**: fit $F_\pi(q^2) = 1 + \frac{1}{6}\langle r^2 \rangle q^2 + cV(q^2)^2$. (To the same order)
- Statistical errors are **1.5 – 1.9 times larger** than the new method.
- **24D and 32D**: finite volume effects are mild.
- **Our final result**

\[
\langle r^2 \rangle = 0.434(20)(13) [\text{fm}^2]
\]

is very consistent with the PDG value $0.434(5)\text{fm}^2$. 
Conclusions and outlook

Model-independent method

- Start with a basic idea: $\langle A \rangle = \sum \omega(x) H(x)$, where the weight function $\omega(x)$ contains all the non-QCD information.
- Overcome the problem of finite-volume effects.
- It also has advantages in statistical uncertainties.

Next step: nucleon charge radius

- Finite-volume effects become insignificant since $m_N \gg m_\pi$.
- Error reduction techniques should be more effective.
- We aim at a result that can be compared with experiments.

Thank you!