Dark Energy Models and Laws of Thermodynamics in Bianchi I Model

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Abstract

This paper is devoted to check validity of the laws of thermodynamics for LRS Bianchi type I universe model which is filled with combination of dark matter and dark energy. We take two types of dark energy models, i.e., generalized holographic dark energy and generalized Ricci dark energy. It is proved that the first and generalized second law of thermodynamics are valid on the apparent horizon for both the models. Further, we take fixed radius $L$ of the apparent horizon with original holographic or Ricci dark energy. We conclude that the first and generalized second laws of thermodynamics do not hold on the horizon of fixed radius $L$ for both the models.

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1 Introduction

Our universe has a phase transition from decelerating to accelerating. Type Ia supernova [1]-[5] indicates that the universe has accelerated expansion, i.e., the universe is expanding with accelerating velocity. The main reason of this expansion is said to be a mysterious energy with large negative pressure

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known as dark energy (DE). Recent data [6]-[8] shows that DE occupies 76% of the universe and 24% is occupied by some other matter. The cosmological constant is the best identification of this mysterious energy, but it raises some other issues like fine-tuning and cosmic-coincidence puzzle. The equation of state parameter (EoS), \( \omega = -1 \) is the most acceptable candidate for DE. Some dynamical models also help to understand its nature like quintessence [9, 10], K-essence [11], tachyon field [12, 13], Chaplygin gas [14, 15], phantom model [16]-[18] etc.

Recent literature indicate keen interest to check the validity of first and second law of thermodynamics. In different scenarios, Wang et al. [19] investigated that these laws are valid on the apparent horizon, when different from the event horizon, but do not hold for event horizon. Mazumder and Chakraborty [20, 21] explored some conditions for the validity of generalized second law of thermodynamics (GSLT) by using the first law of thermodynamics. Debnath [22] investigated validity of GSLT by using holographic DE (HDE) interacting with two fluids for FRW model. Mubasher et al. [23] proved that GSLT holds for all time and does not depend upon geometry and EoS parameter. The validity of GSLT has also been checked on Kaluza-Klein cosmology with modified HDE (MHDE) [24]. In a recent paper [25], the validity of GSLT is investigated when anisotropic DE is interacting with anisotropic radiations and DM in BI universe model.

Holographic DE principle is also used to study the nature of DE. This principle states that in a bounded system the number of degrees of freedom should not be infinite and system is scaled by its boundary area but not by its volume [26]. Cohen and his collaborators [27] explained the relationship between short distance cutoff \( \Lambda \) and long distance cutoff \( L \) by considering the quantum field theory. They proposed a limiting energy bound, a system with size \( L \) cannot form black hole if the vacuum energy of the system exceeds than its mass of the same size \( L \). This can be written as \( L^3 \rho_\Lambda \leq L M_p^2 \), where \( \rho_\Lambda \) is the quantum zero point energy density, \( L \) is infrared cutoff and \( M_p \) is the reduced Planck mass expressed as \( M_p = \frac{(8\pi G)^{-1/2}}{} \). This inequality is possible only for large \( L \), so the HDE density can be expressed as \( \rho_\Lambda = 3c^2 M_p^2 L^{-2} \), where \( 3c^2 \) is a dimensionless constant.

Ricci DE (RDE) [28] is a type of DE obtained by taking square root of the inverse Ricci scalar as its infrared cutoff. Gao et al. [29] explored that the DE is proportional to the Ricci scalar. Some recent work [30]-[33] shows that the RDE model fits well with observational data. Xu et al. [34] defined
two types of DE models, i.e., generalized HDE (GHDE) and generalized RDE (GRDE) models.

In this paper, we use LRS BI universe model composed of DM and DE with GHDE and GRDE models. The paper is organized as follows: In section 2, the density and pressure for GHDE and GRDE models are found. Section 3 is devoted to check the validity of the first and GSLT on the apparent horizon and also by taking GHDE or GRDE as the original HDE or RDE. In the last section, we conclude the results.

2 Density and Pressure for GHDE and GRDE models

In this section, we formulate the field equations for LRS BI universe model. We then evaluate density and pressure for GHDE as well as GRDE models. The line element of LRS BI model is given as follows

\[ ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2), \]

(1)

where \( A(t) \) and \( B(t) \) are scale factors. We use the well-known condition \( A = B^m \) \([35]-[38]\), where \( m \) is a positive constant. Consequently, the above metric reduces to

\[ ds^2 = -dt^2 + B^{2m}(t)dx^2 + B^2(t)(dy^2 + dz^2). \]

(2)

The first field equation corresponding to this metric gives us

\[ (2m + 1) \frac{\dot{B}^2}{B^2} = 8\pi \rho, \]

(3)

\[ H_2^2 = \frac{8\pi}{1 + 2m}\rho, \]

(4)

where \( H_2 = \frac{\dot{B}}{B} \) is the directional Hubble parameter. The conservation equation yields

\[ \dot{\rho} + (m + 2)H_2(\rho + P) = 0. \]

(5)

where \( \rho \) and \( p \) are the energy density and pressure of the fluid, respectively. Taking derivative of Eq.(4) and using Eq.(5), it follows that

\[ \dot{H}_2 = -\frac{4\pi(m + 2)}{1 + 2m}(\rho + P). \]

(6)
We consider that the fluid is a combination of DM and DE, i.e., \( \rho = \rho_m + \rho_{DE} \) and \( P = P_m + P_{DE} \) with \( P_m = 0 \). Assuming that there is no interaction between DM and DE so that these are separately conserved. Thus the conservation equation (6) gives

\[
\dot{\rho}_m + 2(m + 2)\rho_m H^2 = 0, \tag{7}
\]
\[
\dot{\rho}_{DE} + (m + 2)(\rho_{DE} + P_{DE}) H^2 = 0. \tag{8}
\]

Equation (7) gives the energy density of matter as

\[
\rho_m = \rho_{m0}(1 + z)^3, \tag{9}
\]

where \( \rho_{m0} \) is an integration constant which gives the present value of DE density and \( z \) is the red shift given as

\[
z = \frac{1}{B^{\frac{m+2}{3}}} - 1.
\]

In the following, we evaluate energy density and pressure for GHDE and GRDE models.

### 2.1 Generalized Holographic Dark Energy Model

The energy density of GHDE model is given by [33]

\[
\rho_h = \rho_{DE} = \frac{3c^2}{8\pi} H^2 f\left(\frac{R}{H^2}\right), \tag{10}
\]

where \( H = \frac{1}{3}(m + 2)\frac{\dot{B}}{B} \), \( c(\neq 0) \) is an arbitrary constant and \( f(x) > 0 \) such that \( f(x) = \alpha x + (1 - \alpha), \ 0 \leq \alpha \leq 1 \). The Ricci scalar is given by

\[
R = -2 \left[ (m^2 + 2m + 3)H_2^2 + (m + 2)\dot{H}_2 \right]. \tag{11}
\]

When \( \alpha = 0 \), the energy density of the original HDE is recovered, while for \( \alpha = 1 \), we obtain energy density of the original RDE. Using Eq. (11) in (10), it follows that

\[
\rho_h = \frac{c^2}{24\pi} \left[-18(m + 2)\alpha \dot{H}_2 + [(m + 2)^2 - (19m^2 + 40m + 58)\alpha] H_2^2 \right]. \tag{12}
\]
Inserting Eqs. (9) and (12) in (4), we obtain first order linear differential equation whose solution is

\[ H_2^2 = \frac{2}{24\pi \rho_{mo}} \gamma (1 + z)^3 + H_0^2(1 + z)^\beta, \]  

(13)

where

\[ \beta = \frac{1 + 2m}{2(m + 2)^2 \alpha c^2} \left[ 1 - \frac{c^2}{3(1 + 2m)}[(m + 2)^2 - (19m^2 + 40m + 58)\alpha] \right], \]

\[ \gamma = 3(1 + 2m) - [(m + 2)^2 - (m^2 - 32m - 14)\alpha]c^2, \]

\( H_0 \) is an integration constant. Differentiating Eq. (13) with respect to \( t \), we get

\[ \dot{H}_2 = -\frac{12\pi \rho_{mo}(m + 2)}{\gamma}(1 + z)^3 - H_0^2 \frac{\beta}{6(m + 2)}(1 + z)^\beta. \]  

(14)

Substituting \( H_2^2 \) and \( \dot{H}_2 \) in Eqs. (11), (12) and (8), we obtain the Ricci scalar, density \( \rho_h \) and pressure \( P_h \), respectively

\[ R = \frac{-24\pi (m^2 + 2) \rho_{mo}}{\gamma}(1 + z)^3 - \frac{H_0^2}{18\alpha c^2} \]

\[ \times [(m + 2)^2 + (17m^2 + 32m + 50)\alpha]c^2 - 3(1 + 2m)](1 + z)^\beta, \]  

(15)

\[ \rho_h = \frac{[(m + 2)^2 - (10m^2 + 4m + 22)\alpha]\rho_{mo}c^2}{\gamma}(1 + z)^3 \]

\[ + \frac{H_0^2(1 + z)^\beta}{48\pi} \left[ 3(1 + 2m) + [(m + 2)^2 - (19m^2 + 40m + 58)\alpha]c^2 \right]. \]  

(16)

\[ P_h = \frac{H_0^2 \gamma(1 + z)^\beta}{864\pi (m + 2)^2 \alpha c^2} \left[ 3(1 + 2m) - [(m + 2)^2 - (m^2 - 32m - 14)\alpha]c^2 \right]. \]  

(17)

Equation (16) and (17) represent the density and pressure in the form of red shift \( z \).

### 2.2 Generalized Ricci Dark Energy Model

The energy density of GRDE model is given by

\[ \rho_r = \frac{3c^2}{8\pi} R g \left( \frac{\frac{1}{2}(m + 2)^2 H_2^2}{R} \right), \]  

(18)
where \( g(y) = \beta y + (1 - \beta) > 0, \ 0 \leq \beta \leq 1 \). For \( \beta = 0 \), we recover energy density of the original RDE whereas \( \beta = 1 \) leads to energy density of the original HDE. Comparing Eqs. (11) and (18), we see that the GRDE reduces to the GHDE and vice versa for \( \beta = 1 - \alpha \). If we replace \( \alpha \) by \( (1 - \beta) \) in Eqs. (13)-(17), we obtain similar solutions for GRDE model. This implies that these equations are also solutions of the GRDE model with \( \alpha = 1 - \beta \).

3 First and Generalized Second Law of Thermodynamics

First we discuss validity of the first and GSLT in BI universe bounded by apparent horizon. For this purpose, we use entropy given by Gibb's law

\[
T_A dS_I = pdV + d(E_A),
\]

where \( S_I, V, p, E_A \) and \( T_A \) are internal entropy, volume, pressure, internal energy within the apparent horizon and temperature of the apparent horizon, respectively. The internal energy is \( E_A = \rho V \) and \( V = B^{m+2} \) while the radius of the apparent horizon in the case of flat geometry coincides with Hubble horizon is given by

\[
R_A = L = \frac{1}{H} = \frac{3}{(m+2)H_2}.
\]

We assumed that the system is in equilibrium so that the fluid and the horizon has the same temperature. The temperature and entropy of the apparent horizon are defined as

\[
T_A = \frac{1}{2\pi L}, \quad S_A = S_h = \kappa \pi L^2.
\]

The entropy of the horizon is \( S_h = \frac{\kappa A^*}{4} \), \( \kappa \) can be taken 1 in energy units, \( A^* = 4\pi L^2 \) is the area of black hole.

Now we check validity of the first law of thermodynamics on the apparent horizon given by

\[
-dE_A = T_A dS_A.
\]

The energy crossing on the apparent horizon for BI universe model can be
found as follows [12]

\[-dE_A = \frac{4\pi}{3}(m + 2)L^3H_2\dot{T}_{\mu\nu}K^\mu K^\nu dt = \frac{8\pi}{3}(m + 2)L^3H_2\rho dt\]

\[= -\frac{(1 + 2m)}{3}H_2\dot{H}_2L^3 dt. \quad (23)\]

Inserting the value of $L$ from Eq. (20) in this equation, we obtain

\[-dE_A = -\frac{9(1 + 2m)}{(m + 2)^3}\left(\frac{\dot{H}_2}{H_2^2}\right) dt, \quad (24)\]

Also, we have

\[T_A dS_A = \dot{\mathcal{L}} dt = -\frac{3}{(m + 2)}\left(\frac{\dot{H}_2}{H_2^2}\right) dt. \quad (25)\]

These two equations lead to

\[-dE_A = kT_A dS_A, \quad (26)\]

where $k = \frac{3(1+2m)}{(m+2)^2}$. This shows that the first law of thermodynamics always holds on the apparent horizon for all kinds of energies as it is independent of DE.

For the validity of GSLT on the apparent horizon, we evaluate the rate of change of internal entropy from Eq. (19) as follows

\[\dot{S}_I = \frac{(\rho + P)\dot{V} + V\dot{\rho}}{T_A}. \quad (27)\]

Substituting the values of $\dot{V}$, $T_A$ and $\dot{\rho}$, it follows that $\dot{S}_I = 0$. According to SLT, entropy of the thermodynamical system can never be decreased. This is generalized in such a way that the derivative of any entropy is always increasing, i.e., $\dot{S}_I + \dot{S}_A \geq 0$. Thus we have

\[\dot{S}_I + \dot{S}_A = -\frac{18\pi}{(m + 2)^2}\left(\frac{\dot{H}_2}{H_2^2}\right) dt \geq 0. \quad (28)\]

This shows that the GSLT is always satisfied on the apparent horizon. Notice that these laws always hold independent of choice of DE on the apparent horizon.
Now we take GHDE or GRDE model as an original density of HDE and RDE to check validity of the first and GSLT on the horizon of radius $L$. The original HDE or RDE density is given by
\[ \rho_\Lambda = \frac{3c^2}{8\pi} L^{-2}. \]  \hfill (29)

Comparing this value with the energy density of HDE (11), it follows that
\[ L^2 = \frac{1}{\alpha R + \frac{1}{9}(1 - \alpha)(m + 2)^2 H_2^2}. \]  \hfill (30)

Substituting the values of $R$ and $H_2^2$ from Eqs. (12) and (14), we can write this in the form of red shift
\[ L^2 = \frac{18c^2\gamma}{48\pi \rho_{\text{ms}} c^2 \delta (1 + z)^3 + H_0^2 \gamma \mu (1 + z)^3}, \]  \hfill (31)

where
\[
\begin{align*}
\delta &= [(m + 2)^2 - 2(5m^2 + 2m + 11)\alpha], \\
\mu &= [(m + 2)^2 - (19m^2 + 40m + 58)\alpha]c^2 + 3(1 + 2m).
\end{align*}
\]

Here the temperature and entropy on the horizon are similar to Eq. (21), i.e., $T_L = \frac{1}{2\pi L}$, $S_L = \pi L^2$. The amount of energy crossing on the horizon is also similar to Eq. (23) with the difference that $dE_L$ is written instead of $dE_A$. We can write
\[ T_L dS_L = \dot{L} dt. \]  \hfill (32)

For the first law, we must have $-dE_L = T_L dS_L$. Equations (23) and (32) imply that
\[ -dE_L = T_L dS_L - \left[ \frac{(1 + 2m)}{3} H_2 \dot{H}_2 L^3 + \dot{L} \right] dt. \]  \hfill (33)

Since the second term on right hand side is time dependent, so during evolution of the universe, it can never be zero, hence
\[ -dE_L \neq T_L dS_L. \]

This indicates that the first law of thermodynamics does not hold on the horizon of radius $L$ in Einstein’s gravity.
Figure 1: The graph shows the variation of \((\dot{S}_I + \dot{S}_L)\) against red shift \(z\) for \(c = 0.5, \ \rho_{m_0} = 1, \ \alpha = 0.7, \ \beta = 0.7, \ H_0 = 70\). The green colour represents GHDE and red represents GRDE.

For the validity of GSLT on the horizon of radius \(L\), the rate of change of total entropy is

\[
\dot{S}_I + \dot{S}_L = 2\pi L \dot{L}.
\]  

(34)

According to the GSLT, the total entropy of the thermodynamical system always increases, i.e., \(L \dot{L} \geq 0\) indicating its dependence only on \(L\) in the DE model. In GHDE model, the rate of change of total entropy on the horizon is

\[
\dot{S}_I + \dot{S}_L = -\frac{(m + 2)(1 + z) dL^2}{3} \left[ \frac{24\pi \rho_{m_0}}{\gamma} (1 + z)^3 + H_0^2 (1 + z)^2 \right]^\frac{1}{2},
\]  

(35)

where \(L^2\) is given in Eq.(31). This is very complicated expression in \(z\) which does not provide any indication whether it increases or decreases. To get insight, we draw the total entropy \((\dot{S}_I + \dot{S}_L)\) versus red shift \(z\) as shown in Figure 1. The graph indicates that \((\dot{S}_I + \dot{S}_L) < 0\) and hence the GSLT does not hold on the horizon of radius \(L\). The green and red lines represent the GHDE and GRDE models respectively. Consequently, the GSLT does not hold for both kinds of the energy models.

4 Concluding Remarks

In this paper, we have considered LRS BI universe model by assuming that our universe is filled with DM and DE. We have taken two types of DE models, i.e., GHDE and GRDE models. Notice that the GRDE model can be converted to GHDE model if we replace \(\beta\) by \(1 - \alpha\). Also, the original density
of HDE and RDE models is obtained for $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$ respectively. The density and pressure for GHDE and GRDE models in terms of red shift $z$ are evaluated.

The main purpose of this paper is to investigate validity of the first and GSLT on the apparent horizon in this scenario. It turns out that these laws are independent of the choice of DE models and hence hold for any kind of DE models on the apparent horizon. Further, we have considered the GHDE and GRDE as the original HDE and RDE and found $L$ to check validity of these laws on this horizon of radius $L$. It is found that the first and GSLT do not hold on this horizon for both DE models. We would like to mention here that in a recent work [33], similar type of investigation has been done in FRW universe model. We have extended this work to LRS BI universe model with same scenario. Here we can check validity of the laws only on apparent horizon due to the flat geometry rather than particle and event horizons as in the case of FRW universe.

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