Temperature dependence of butterfly effect in a classical many-body system

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We study the chaotic dynamics in a classical many-body system of interacting spins on the kagome lattice. We characterise many-body chaos via the butterfly effect as captured by an appropriate out-of-time-ordered correlator. Due to the emergence of a spin liquid phase, the chaotic dynamics extends all the way to zero temperature. We thus determine the full temperature dependence of two complementary aspects of the butterfly effect: the Lyapunov exponent, \( \mu \), and the butterfly speed, \( v_b \), and study their interrelations with usual measures of spin dynamics such as the spin-diffusion constant, \( D \) and spin-autocorrelation time, \( \tau \). We find that they all exhibit power law behaviour at low temperature, consistent with scaling of the form \( D \sim v_b^2/\mu \) and \( \tau^{-1} \sim T \). The vanishing of \( \mu \sim T^{0.48} \) is parametrically slower than that of the corresponding quantum bound, \( \mu \sim T \), raising interesting questions regarding the semi-classical limit of such spin systems.

*Introduction:* Chaos [1–11] underpins much of statistical mechanics, providing the basis for ergodicity, thermalization and transport in many-body systems. Perhaps its most striking feature that has captured public imagination is the butterfly effect [12–15]: an infinitesimal local change of initial condition is amplified exponentially (Lyapunov exponent \( \mu \)) and spreads out ballistically (butterfly speed \( v_b \)) to dramatically affect global outcomes.

Quantitative connections between characteristic time and length scales of the chaotic dynamics of a many-body system and those related to its thermalization and transport are far from settled, receiving renewed attention particularly for quantum many-body systems [16–37]. There, diagnostic tools of chaos akin to out-of-time-ordered commutators (OTOC) [38–40]. Further, in a recent study of classical spin chain at infinite temperature [41], the classical limit of OTOC’s has been shown to characterize these features of the butterfly effect.

Here, we study the evolution of the chaotic dynamics as a function of temperature, \( T \), of the many body system, and its interrelation with thermalization and transport quantities such as relaxation and diffusion. This is interesting as correlations develop due to interactions as \( T \) is lowered, thereby affecting the dynamics of the system. Generally, one expects that at low \( T \), the effect of chaos may be weakened due to emergence of long-lived quasiparticles (with or without spontaneously broken symmetry) that dominate the dynamics. Indeed, recent studies of the quantum Sachdev-Ye-Kitaev (SYK) [42–45] and finite density fermions coupled to gauge fields [46] (both in the large \( N \) limit), show that the absence of quasiparticles due to interactions can lead to chaos as manifested in OTOCs even at the lowest \( T \).

We explore these issues in an interacting many-body classical spin system with local interactions on a two dimensional kagome lattice. We elucidate the \( T \)-dependence of Lyapunov exponent and butterfly speed and find connections between diffusion and chaos over the entire temperature range. At low \( T \), Lyapunov exponent (\( \mu \sim T^{0.48} \)) and butterfly speed (\( v_b \sim T^{0.23} \)), extracted from a classical OTOC [41], show novel algebraic scaling with \( T \). This behaviour is qualitatively distinct from that of the quantum counterparts since the observed sub-linear scaling of the Lyapunov exponent is at odds with the quantum low-\( T \) bound (\( \mu \leq 2\pi k_B T/\hbar \) [16]). This raises questions regarding the presumably singular nature of semi-classical (in \( 1/S \) sense) corrections [47, 48] in this spin system. However, in spite of the seeming “violation” of the quantum bound, our results are consistent with a recently identified connections of these microscopic measures of chaos and the macroscopic phenomenon of transport, where for the SYK-model and “strange” metals the energy diffusion constant was found to scale as \( D \sim v_b^2/\mu \) [49–53].

Persistence of chaotic dynamics, usually characteristic to high \( T \), in our spin system all the way down to \( T = 0 \) owes its origin to competing (frustrated) local interactions that completely suppresses magnetic ordering. **Model:** We study classical \( O(3) \) Heisenberg spins of unit length, \( \mathbf{S}_x \), on the sites \( x \) of the kagome lattice,

\[
H = J \sum_{x, x' \in \mathcal{O}} \mathbf{S}_x \cdot \mathbf{S}_{x'} = \frac{J}{2} \sum_{\alpha} (L_{\alpha})^2 + \text{const},
\]

**FIG. 1.** Left: The classical Heisenberg model hosting the \( Z_2 \) spin liquid is defined on the Kagome lattice with couplings fully connecting all hexagons (only shown for central one). Indicated are the basis vectors \( a_1, a_2 \) and a possible unit cell (light gray vertices). Right: Snapshot of the dynamics of the de-correlator \( D(x, t) \) showing ballistic isotropic spreading of a perturbation initially localised in the centre of the system.
where each spin interacts equally with all the spins with which it shares a hexagon, \( \circ \) [54], whose total spin is denoted by \( \mathbf{L}_\alpha = \sum_{\mathbf{x} \in \mathcal{O}} \mathbf{S}_\mathbf{x} \), schematically illustrated in Fig. 1.

For antiferromagnetic interactions \( (J > 0) \), ground states satisfy the local constraints \( \mathbf{L}_\alpha = 0 \) for each hexagon, which leads to a macroscopically degenerate ground state manifold. The system remains in a paramagnetic state all the way down to \( T = 0 \) which has a finite spin correlation length and exhibits fractionalization [55]. Interestingly, the system does not freeze or fall out of equilibrium in the entire temperature range. Such a phase has been dubbed a classical ‘\( \mathbb{Z}_2 \)’ spin-liquid.

The dynamics is that of spins precessing around their local exchange fields, which conserves total energy \( E \), magnetization \( M \), as well as the spin norm:

\[
\frac{d\mathbf{S}_\mathbf{x}(t)}{dt} = -\mathbf{S}_\mathbf{x}(t) \times \sum_j J_{\mathbf{x}\mathbf{x}'} \mathbf{S}_{\mathbf{x}'}(t). \tag{2}
\]

**Numerical simulations:** These were performed over a range of temperature \( T = 10^{-3} \) to \( T = 100 \), and linear system size \( L = 25 \) to \( L = 201 \) with \( N_s = 3L^2 \) spins and periodic boundary conditions. Results shown are for \( L = 101 \) unless indicated otherwise. The spin dynamics is integrated using an eighth-order Runge-Kutta solver with a time-step chosen such that energy/site and magnetisation/site are conserved to better than \( \sim 10^{-8} \). Results are averaged over \( 10^4 \) initial states sampled from the Boltzmann distribution via Monte-Carlo. Details on the fitting procedure and exemplary raw data fits can be found in the suppl.mat. [56]. We measure energy in units of \( J = 1 \), and distances in units of the lattice spacing \( a = 1 \).

**Temperature dependence of dynamics:** We begin by discussing the two point spin correlator

\[
C(\mathbf{x}, t) = \langle \mathbf{S}_\mathbf{x}(t) \cdot \mathbf{S}_\mathbf{0}(0) \rangle, \tag{3}
\]

its Fourier-transform, the dynamical structure factor \( \mathcal{S}(q, \omega) \), and the auto-correlator \( A(t) = \sum_\mathbf{x} \langle \mathbf{S}_\mathbf{x}(t) \cdot \mathbf{S}_\mathbf{x}(0) \rangle \).

At all \( T \), the autocorrelator exhibits an initial exponential decay \( A(t) \sim e^{-\kappa t} \), with diffusion at long wavelengths seen in the tail at long times, \( A(t) \sim 1/t \), as well as in the decay of the dynamical structure factor close to the \( \Gamma \)-point, \( \mathcal{S}(q, \omega) \sim 1/[(Dq)^2 + \omega^2] \) [56].

The \( T \) dependence of relaxation rate \( \kappa \) and diffusion constant \( D \) is shown Fig. 2. We observe a linear scaling \( \kappa \sim T \), and saturation of the diffusion constant to a constant value, in the low \( T \) spin-liquid regime in conformity with the large-\( N \) results [56].

Having established and characterised the diffusive behaviour of the spin correlators, we now turn to the main subject of this work, the many-body chaos in this many-body system, in the form of the butterfly effect.
the initially perturbed site from a fully correlated unper-
turbed region with $D \sim 0$. The speed of the ballistically
propagating light-cone of the perturbation allows us to
define the butterfly speed, $v_b$ [63–66]. The wave-front af-
after an initial transient remains circular over the course of
the dynamics and the full temperature range we con-
sider even though the underlying lattice only has a six-
fold rotational symmetry. This constitutes a non-trivial
model-dependent feature of OTOC’s, which generically
only need to respect, even at late times, the discrete lat-
tice symmetries [31]. Thus, it is sufficient to restrict to
1D cuts in the following discussions.

**Temperature dependence of $v_b$ and $\mu$:** The next central
result of this study, the full $T$ dependence of Lyapunov
exponent $\mu$ and the butterfly velocity $v_b$, is shown in
Fig. 2. The exponential growth of the decorrelation
throughout the entire temperature range confirms the
consistency of the chaotic dynamics down to the lowest $T$,
in keeping with the persistence of the spin liquid phase.

We employ two methods to extract $\mu$ and $v_b$, a fit to the
scaling form of the wavefronts, Eq. 5, and independent
fits discussed in detail below. Generically, the Lyapunov
exponent and butterfly speed extracted from the fit to
the full scaling form are slightly lower than those deter-
mined independently, which we attribute to subleading
prefactors not contained in the scaling form (Eq. 5). Ad-
ditionally, the independent fits can be extended to lower
temperatures than fitting the full wavefront.

For both methods we observe algebraic behaviour at
low temperatures: the butterfly speed scales as $\mu \sim
T^{0.48 \pm 0.006}$ ($T^{0.47 \pm 0.005}$) for the independent fit (scaling
form) and the Lyapunov exponent as $v_b \sim T^{0.23 \pm 0.01}$.

The observed scaling of the Lyapunov exponent, $\mu \sim
T^{0.48}$, is parametrically larger at low temperatures than
the bound on quantum chaos $\mu \leq k_B T / \hbar$ [16]. While this
bound is not directly applicable to our classical model, it
implies the semi-classical (say in the form of $1/S$) correc-
tions are singular in the low $T$ limit [47, 48]. At the same
time, we note that the observed scaling is consistent with
a recently suggested $\sqrt{E}$ behaviour [47].

Interestingly, however, $v_b^2 / \mu$ is approximately constant
in the low $T$ regime. This is consistent with the con-
jected relation between the diffusion constant $D$, the
Lyapunov exponent and butterfly velocity $v_b$ as $D \sim v_b^2 / \mu
[49–53]$, and the fact that we obtain a $T$-independent dif-
fusion constant in the spin-liquid regime (Fig. 2).

We now turn to a detailed discussion of the wavefronts,
their ballistic propagation, and scaling form. This also
illustrates how the discussed quantities are obtained from
the OTOC.

**Scaling form:** Fig. 3 (top panel) shows the scaling form
of the de-correlator, according to Eq. 5, and the build-
up and propagation of the wavefronts (inset). Close to
the wavefront we observe approximate scaling collapse of
the de-correlator. In contrast, inside the light-cone we
observe deviations from the scaling form due to the sat-
uration of the bounded de-correlators. This is avoided in
the linearised version of the dynamics [41], which al-
 lows us to directly access the limit of vanishingly small
perturbation strength $\epsilon$.

**Individual fits:** Complementary to fitting the full spa-
tiotemporal profile of $D(x, t)$, which allows access to the
velocity dependent Lyapunov exponent $\lambda(v)$, we extract
$\mu = \lambda(v = 0)$ from an exponential fit to $D(x = 0, t)$ and
the butterfly velocity $v_b$ from the arrival times $t_{D_0}(x)$ of
the wave-front via $v_b = x / t_{D_0}$, where $D(x, t_{D_0}) > D_0
[41]$.

We demonstrate the expected exponential growth
$D(x = 0, t) \sim e^{2\mu t}$ in the lower panel of Fig. 3 compar-
ing the linearised dynamics to the non-linear dynamics
with $\epsilon = 10^{-4}$. In particular, the linearised dynamics
shows exponential growth for all times, whereas the lin-
erised dynamics only shows exponential growth over a
finite time increasing with smaller perturbation strength
as $\log(\epsilon)$ before the decorrelator saturates.

The results for $v_b$ and $\mu$ from the non-linear dynamics
converge with decreasing perturbation strength $\epsilon$ to the

![Graph showing the scaling form and ballistic propagation of the wavefront](image-url)
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Many follow-on questions naturally pose themselves, e.g., concerning the role of phase transitions and order, the nature of the semiclassical limit, and the ‘transition’ into an integrable regime with an increasing number of conserved quantities. Even at the classical level, small perturbations may lead to ordering and a reduction of the dimensionality of the ground state manifold. In such cases the observed effects are expected to survive above the concomitant ordering temperatures, which are often much smaller than the leading interaction scale, opening up a robust spin liquid regime.

Another interesting aspect is the effect of the quantum fluctuations on this classical model. Such quantum fluctuations would, again, generally quench the ground state entropy. However, this may not necessarily lead to an ordered state, but a long range quantum entangled spin liquid state expected for this system in $S = 1/2$ limit [54]. The crossover to such a quantum coherent regime would then be accompanied by sharp signature in the indicators of many-body chaos studied above.

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Supplemental Material:

DYNAMICS OF THE $\mathbb{Z}_2$ SPIN LIQUID

Large-N analytics

The large-N limit relaxes the condition of unit length $O(N)$ spins in the limit of $N \to \infty$ [67]. It reproduces well the static properties of the highly frustrated kagome and pyrochlore Heisenberg models [68, 69]. Its extension via a stochastic Langevin dynamics allows accurate predictions also for the dynamics of the classical pyrochlore and kagome models [69, 70].

Eventhough the quantitative agreement has been shown to be slightly worse for the $\mathbb{Z}_2$ model under consideration here [55], it provides an analytically tractable starting point from which to approach the full model dynamics.

Large-N calculation: In the large-N calculation the soft spins follow the (unnormalised) probability distribution $e^{-\beta E}$ with the energy

$$\beta E = \frac{1}{2} \sum_i \lambda s_i^2 + \frac{1}{2} \beta J \sum_{\alpha} l_i^\alpha$$  \hspace{1cm} (S1)

where $l_i = \sum_{\alpha \in \alpha} s_i$ is the sum of the “soft” spins $s_i$ over the hexagon $\alpha$ and $\lambda$ is a lagrange multiplier ensuring the length constraint $\langle s_i^2 \rangle = 1/3$ (Heisenberg spins).

We may rewrite the interaction term as $\sum_{i,j} s_i (A_{ij} + 2\delta_{ij}) s_j$ where $A_{ij}$ is the connectivity matrix of the model. We call $M = (A_{ij} + 2\delta_{ij})$ the interaction matrix.

Since the model is translationally invariant, the eigenbasis is labelled by a momentum $\mathbf{q}$ and a sublattice index $\nu \in \{1, 2, 3\}$. We obtain two flat bands $\nu_1, \nu_2(\mathbf{q}) = 0$ and one dispersive gapped band $\nu_3(\mathbf{q})$.

Dynamics is introduced via the Langevin equation

$$\frac{ds_i(t)}{dt} = \Gamma \sum_j (A_{ij} - z\delta_{ij}) \frac{\partial E}{\partial s_i} + \zeta_i(t)$$  \hspace{1cm} (S2)

with the coordination number $z$ ($z = 10$ in this model), a noise term $\zeta_i(t)$ and a free constant $\Gamma$ determining the overall timescale of dynamical processes.

Solving this equation in the eigenbasis $\tilde{s}(\mathbf{q})$ of the interaction matrix $M$ we obtain

$$\langle \tilde{s}_\mu^*(t) \tilde{s}_\nu^0(0) \rangle = \frac{\delta_{\mu,\nu} T}{M_{\mu,\nu} + \lambda T} e^{-\Gamma(12 - \nu_\mu)(J_{\nu_\mu} + \lambda T) t}$$  \hspace{1cm} (S3)

with a characteristic decay rate $\kappa_\mu = \Gamma (12 - \nu_\mu) (J_{\nu_\mu} + \lambda T)$.

The dynamical structure factor is given by

$$S(\mathbf{q}, t) = \sum_{\mu,\nu} \langle \tilde{s}_\mu^*(t) \tilde{s}_\nu^0(0) \rangle$$  \hspace{1cm} (S4)

$$= \sum_{\alpha} \left( \sum_{\mu,\nu} U_{\alpha q}^{\mu,\nu} \right) \langle \tilde{s}_\alpha^0(t) \tilde{s}_\alpha^0(0) \rangle$$  \hspace{1cm} (S5)

$$= \sum_{\alpha} g_{\alpha q} \langle \tilde{s}_\alpha^0(t) \tilde{s}_\alpha^0(0) \rangle$$  \hspace{1cm} (S6)

with the factors $g_{\alpha q}^2$ defined via the matrix of eigenvectors $U_{\alpha q}^{\mu,\nu}$. These factors satisfy the sum rule $\sum_{\alpha} g_{\alpha q}^2 = 3$.

Autocorrelation: The autocorrelation function shows exponential decay $A(t) = e^{-\kappa t}$ in the low-temperature limit with $\kappa = 12\Gamma T$, i.e. a linear dependence on temperature.

Structure factor: In the low-temperature limit the dynamical structure factor at generic wave-vector is exponentially decaying with a $\mathbf{q}$-independent decay rate scaling with $T$, $\kappa(\mathbf{q}) = 12\Gamma T$.

Around $\mathbf{q} = 0$ the full weight is in the dispersive band, and we can extract a diffusion constant from $\kappa(\mathbf{q}) = Dq^2$ with $D = 9\Gamma J$, i.e. a temperature independent diffusion constant.

NUMERICS

Structure factor

We consider the dynamical structure factor $S(\mathbf{q}, t) = \sum_{ij} e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \langle \mathbf{S}_i(t) \cdot \mathbf{S}_j(0) \rangle$ and its fourier transform $S(\mathbf{q}, \omega)$ which provides spatially and frequency resolved information on the dynamics.

Static structure factor: The static structure factor $S(\mathbf{q}, t = 0)$ in Fig. S1 for temperatures $T = 0.01, 0.1, 1, 10$ shows the transition from the high-temperature paramagnet to the spin liquid regime at low temperatures.

The static structure factor remains essentially unchanged below $T \sim 0.1 - 1$. In particular, it shows no
pinch points or lines and no sign of ordering. The results in the spin-liquid regime are in good agreement with the predictions of the large-N calculations (not shown).

**Dynamical structure factor:** Since the dynamics, Eq. 2, conserves the total magnetisation, we expect diffusion to occur at small wavevectors.

To test this expectation we perform a scaling collapse of the dynamical structure factor via

\[
\beta q^2 S(q, \omega) \sim \frac{D}{\omega q^2} + D^2
\]  

(S7)
appropriate diffusion in 2D. We find an approximate scaling collapse of this form in Fig. S2.

To extract the diffusion constant we fit the dynamic structure factor via \( S(q, \omega) \sim 1/(\omega^2 + \kappa(q)^2) \), corresponding to an exponentially decaying dynamical structure factor with decay rate \( \kappa(q) \), i.e. \( S(q, t) \sim e^{-\kappa(q) t} \).

For diffusive behaviour we expect \( \kappa(q) = Dq^2 \) for momenta \( q \) close to the \( \Gamma \)-point.

Fig. S3 presents the results for \( \kappa(q) \) and the quadratic fits to extract the diffusion constant \( D \). Already on the level of the raw data we observe a clear separation into the high-temperature paramagnetic phase \( T > 1 \) and the low-temperature spin-liquid regime \( T < 0.1 \).

We also note that with decreasing temperature the range of validity of the quadratic fit shrinks which limits the extraction of the diffusion constant to temperatures \( T \geq 0.002 \) on the available system sizes, and results in increasing uncertainties at lower temperatures.

**BUTTERFLY EFFECT IN THE \( \mathbb{Z}_2 \) SPIN LIQUID**

**Fitting the full propagating wavefront**

As stated in the main text the de-correlator \( D(x, t) \) is fit well by the scaling form

\[
D(x, t) \sim \exp[2\mu(1 - (v/v_b)^\nu)t]
\]  

(S8)
with the Lyapunov exponent \( \mu \), the butterfly velocity \( v_b \) and an exponent \( \nu \), which will all generically be temperature dependent.

In Fig. S4 we show typical fits of the scaling form to the results of the de-correlator \( D(x, t) \) for the non-linear (left row) and linearised dynamics (right row) for temperatures \( T = 100 \) (top) and \( T = 0.1 \) (bottom).

The non-linear data shows saturation effects inside the light-cone for \( v < v_b \) when the decorrelation reaches \( D \sim 1 \). Thus, it only fits the scaling form around \( v \sim v_b \) and the estimate for \( \mu \) is biased to smaller values. The linearised dynamics shows no saturation effects, and therefore provides considerably less spread around the scaling form. Moreover, the remaining spread is further reduced when performing the simulations on larger system sizes. Smaller temperatures inherently require larger systems.
for a wave front to build up since the perturbation propagates relatively faster than it grows, thus, reaching the boundaries of the system before it saturates at the initial site. This limits the temperature range we can reliably perform fits to the full wavefronts to $T > 0.02$ on systems up to $L = 101$.

The linearised dynamics show an extracted exponent $\nu = 2$ at $T = 100$, which then decreases slightly to $\nu \approx 1.9$ at $T = 0.1$. The non-linear dynamics are consistent with a constant exponent $\nu = 2$ over the full temperature range considered, however, can also be fit with the exponent extracted from the linearised dynamics.

**Convergence with $\epsilon$**

Fitting the behaviour of $D(x = 0, t)$ via $D(x = 0, t) \sim e^{2\mu t}$ allows to extract the (leading) Lyapunov exponent $\mu$. The extracted Lyapunov exponent is shown in Fig. S5 versus temperature. We observe convergence of the extracted Lyapunov exponent with decreasing $\epsilon$ towards the results of the linearised equations across the whole temperature range. We note that this is important for two reasons. Firstly, it shows that our results for $\epsilon = 10^{-4}$ are already quite close to the limit of vanishing perturbation strength. Secondly, it implies that the linearised dynamics indeed correctly captures the behaviour of the decorrelator also for finite, but small perturbation strengths.

We may determine the butterfly-speed from the propagation of the wavefront: We define the arrival time $t_{D_0}(x)$ at distance $x$ at which the de-correlator $D(x, t)$ exceeds a given threshold $D_0$. For ballistic propagation we expect a linear relation with $x = v_b t_{D_0}$.

For sufficiently small thresholds $D_0 \sim \epsilon^2$ we observe the expected linear behaviour of the arrival times with distance $t_{D_0} = x/v_b$, at least for sites $x$ sufficiently removed from the initially perturbed site. Moreover, choosing $D_0 = \epsilon^2$ we obtain results for the butterfly speed $v_b$ independent of the chosen perturbation strength, for sufficiently small $\epsilon$, and in agreement with the linearised dynamics as shown in Fig. S6.

**Variation over initial states**

In this section we consider the dependence of the extracted quantities on the initial states keeping information for all $10^4$ simulated states, but restricting to smaller sizes of $L = 51$.

The sample-to-sample variation allows us to determine whether the mean characterises the full state-manifold or whether states at a given temperature might behave differently.

We first consider the variation of the arrival times $t_{D_0}$ in Fig. S7. The relation of $t_{D_0}$ is linear for all samples apart from boundary effects, either at the initial perturbed site or at half the system size when periodic boundary conditions affect the results.

We observe some scatter in the arrival times, increasing considerably at lower temperature. However, the variance of the arrival times actually appears to decrease with increasing distance $x$ from the perturbed site, or equivalently with time. This is confirmed in the inset by the scaling collapse of the data for $t_{D_0}$ with $x^{1/6}$.

However, we emphasise that this might only be true for the accessible times, and this initial decrease could be a transient effect, after which the asymptotic long-time limit could show different behaviour.

Next, we consider the extraction of the Lyapunov exponent from the de-correlator at $x = 0$ via $D(x = 0, t) \sim e^{2\mu t}$. In Fig. S8 we show the variation of $\mu$ at different temperatures $T$ when performing this fit for different initial states individually instead of on the mean of the data. The distribution of $\mu$ over initial states is approximately gaussian at all temperatures and well characterised by
FIG. S7. Arrival times $t_{D_0}$ as a function of $x$ for $D_0 = 10^2$ and $\epsilon = 10^{-4}$ for temperatures $T = 100$ (left) and $T = 0.001$ (right). Gray scatter is the variation over different initial spin-configuration, solid blue line the mean value, and the blue shading the standard deviation of the data. The inset shows the distribution of $t_{D_0}$ at $x = 5, 10, 15, 20, 25$ with an approximate collapse on scaling with $x^{1/6}$. Results obtained on a $L = 51$ system.

FIG. S8. Lyapunov exponent $\mu$ versus temperature $T$, extracted from an exponential fit to the de-correlator at $x = 0$ via $D(x = 0, t) \sim \exp[2\mu t]$ for different initial states. Gray scatter is the variation over initial states, the solid line the mean, and the blue shading marks the standard deviation of the data. Results obtained for the non-linear dynamics with $\epsilon = 10^{-4}$ on a $L = 51$ system.