On sums of $k$-th powers with almost equal primes

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Abstract For “almost all” sufficiently large $N$, satisfying necessary congruence conditions and $k \geq 2$, we show that there is an asymptotic formula for the number of solutions of the equation

$$N = p_1^k + p_2^k + \cdots + p_s^k,$$

$$|p_i - (N/s)^{1/k}| \leq (N/s)^{\theta/k}, \quad (1 \leq i \leq s)$$

with

$$s \geq \frac{k(k+1)}{2} + 1 \quad \text{and} \quad \theta \geq \frac{2}{3} + \varepsilon.$$

This enlarges the effective range of $s$ for which can be obtained by the method of Mátomáki and Xuancheng Shao [12]. [Discorrelation between primes in short intervals and polynomial phase, Int. Math. Res. Not. IMRN 2021, no. 16, 12330-12355.]

The idea is to avoid using the exponential sums (1.2) and Vinogradov mean value theorems in Lemma 2.4 simultaneously. And the main new ingredient is from Kumchev and Liu [9] (see Lemma 2.2).

Keywords Waring-Goldbach problem; exponential sums over primes in short intervals; circle method.

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1. Introduction

For $k \in \mathbb{Z}^+$ and a prime $p$, let $\tau = \tau(k, p)$ be the largest integer such that $p^\tau | k$. Define

$$\eta(k, p) = \tau + 2$$

when $p = 2$ and $\tau > 0$, and $\eta(k, p) = \tau + 1$ otherwise. Then define

$$R = R(k) = \prod_{(p-1)|k} p^{\eta}.$$

For sufficiently large $N$, $N \in \mathcal{N}_s = \{ N \in \mathbb{N} : N \equiv s \pmod{R} \}$ and $9 \nmid N$, for $k = 3$ and $s = 7$,

$$N = p_1^k + p_2^k + \cdots + p_s^k,$$

$$|p_i - (N/s)^{1/k}| \leq (N/s)^{\theta/k}, \quad (1 \leq i \leq s)$$

was investigated by many experts. For example, see [8, 9, 10, 12, 13, 15]. In [8], this was first studied for $s \geq 7$ and $k = 2$. Recently, this was improved by many authors. For example, see [9, 12, 13, 15].

In [9], Kumchev and Liu proved that for sufficiently large $N$, $N \in \mathcal{N}_s = \{ N \in \mathbb{N} : N \equiv s \pmod{R} \}$, there are solutions of the equation (1.1) with $s \geq k^2 + k + 1$ and $\theta \geq 31/40 + \varepsilon$, which gives an asymptotic lower bound of the correct order for the number of the representations. They also proved that “almost all” sufficiently large $N$, $N \in \mathcal{N}_s =$
$\{N \in \mathbb{N} : N \equiv s \pmod{R}\}$, (1.1) is solvable with $s \geq \frac{k(k+1)}{2} + 1$ and $\theta \geq 31/40 + \varepsilon$. Next we will introduce the definition of “almost all”. For primes $p_i$, define

$$E(N, s, x^{k-1}y) = |\mathcal{E}(N, s, x^{k-1}y)|,$$

where

$$\mathcal{E}(N, s, x^{k-1}y) = \{n : n \in (N, N + x^{k-1}y) \cap \mathcal{N}_s, \varrho(n) - \mathfrak{S}(n) \mathfrak{J}(n) \geq y^{s-1}x^{1-k}(\log x)^{-1}\},$$

where $\varrho(n)$ is the number of representations of $n$ in the form of (1.1) and $\mathfrak{S}(n), \mathfrak{J}(n)$ are defined by (3.1). If

$$\lim_{N \to \infty} \frac{E(N, s, x^{k-1}y)}{x^{k-1}y} = 0 \quad (i.e. \ E(N, s, x^{k-1}y) = o \left( x^{k-1}y \right)),$$

then we say that for “almost all” sufficiently large $N$, $N \in \mathcal{N}_s$, there is an asymptotic formula for the number of solutions of the equation (1.1) with certain $s$ and $\theta$.

Recently, these results were improved by Matomäki-Shao [12] and Salmensuu [13] respectively. Matomäki-Shao [12] use the exponential sums. In order to use the exponential sums

$$\sum_{x<n \leq x^{2/3+\varepsilon}} \Lambda(n)e(\alpha n^k), \quad (1.2)$$

they need more variables to apply a better type Vinogradov mean value theorem related to [3] instead of Lemma 2.4. By using the Harman sieve, Salmensuu got some even better results. Unfortunately, both their results do not contain the situation of “almost all” for $s \leq k^2 + k$. However, the method of Matomäki-Shao [12] implies that “almost all” sufficiently large $N$, $N \in \mathcal{N}_s = \{N \in \mathbb{N} : N \equiv s \pmod{R}\}$, there is an asymptotic formula for the number of solutions of the equation (1.1) with $s \geq (k^2 + k)/2 + 2$ and $\theta \geq 2/3 + \varepsilon$. This improves the previous results for the aspect of $\theta$ but with worse $s$ (as cited in [12]). The aim of this paper is to reduce the $\frac{k(k+1)}{2} + 2$ to $\frac{k(k+1)}{2} + 1$. Then the range of $s$ is the same as previous results in [15, 9]. The idea is to avoid using the exponential sums (1.2) and the Vinogradov mean value theorems in Lemma 2.4 simultaneously.

**Theorem 1.1.** For “almost all” sufficiently large $N$, $N \in \mathcal{N}_s = \{N \in \mathbb{N} : N \equiv s \pmod{R}\}$, there is an asymptotic formula for the number of solutions of the equation

$$N = p_1^k + p_2^k + \cdots + p_s^k,$$

$$|p_i - (N/s)^{1/k}| \leq (N/s)^{\theta/k}, \quad (1 \leq i \leq s)$$

with

$$s \geq \frac{k(k+1)}{2} + 1 \quad \text{and} \quad \theta \geq 2/3 + \varepsilon.$$

In fact, such an idea can also be used to enlarge the effective range of $s$ of [12]. Though there are some other results in [13] with much better $\theta$. However, such type results cannot be used to give something related to asymptotic formulas for the number of solutions of the equation (1.1). By similar arguments, one can get the following corollary.
Corollary 1.2. For all sufficiently large $N$, $N \in \mathcal{N}_s = \{N \in \mathbb{N} : N \equiv s (\text{mod } R)\}$, there is an asymptotic formula for the number of solutions of the equation

$$N = p_1^k + p_2^k + \cdots + p_s^k,$$

with

$$|p_i - (N/s)^{1/k}| \leq (N/s)^{\theta/k}, \quad (1 \leq i \leq s)$$

with

$$s \geq k(k+1) + 1 \quad \text{and} \quad \theta \geq 2/3 + \varepsilon.$$

2. Preliminaries

For $v \in \mathbb{Z}^+$, let $w_k(q)$ be defined as

$$w_k(p^{ku+v}) = \begin{cases} kp^{-u-1/2} & \text{for } u \geq 0 \text{ and } v = 1, \\
p^{-u-1} & \text{for } u \geq 0 \text{ and } 2 \leq v \leq k. \end{cases} \quad (2.1)$$

Then we will introduce two lemmas. These lemmas are from [16] (in Chinese) and [9] respectively.

**Lemma 2.1.** Let $\log D \ll \log x$ and $1 \leq \max\{x^{1/2}, M\} \leq y \leq x$. Then there exist a constant $c_0 > 0$ uniformly for $\gamma \in \mathbb{R}$ such that

$$\sum_{q \leq M} \sum_{1 \leq a \leq q} \int_{|\alpha - a/q| \leq 1} \frac{w_k(q)^2 \sum_{x < p \leq x+y} (\log p) e(p^k(\alpha + \gamma))|^2}{1 + D|\alpha - a/q|} \, d\alpha \ll y^2 D^{-1} \log^{c_0} x.$$

**Proof.** One can refer to the case $k = 3$ and $k = 4$ in [17] and [18], respectively. And this is Lemma 2.1 in [16] (in Chinese). So we only sketch the proof. For $\alpha = a/q + \gamma_1$, for the properties of geometric series, we have

$$\sum_{1 \leq a \leq q} \left| \sum_{x < p \leq x+y} (\log p) e(p^k(\alpha + \gamma)) \right|^2 \leq q \sum_{x < p_1, p_2 \leq x+y, p_1^k \equiv p_2^k (\text{mod } q)} (\log p_1)(\log p_2).$$

On the other hand, we have the estimate

$$q \sum_{x < p_1, p_2 \leq x+y, p_1^k \equiv p_2^k (\text{mod } q)} 1 \ll y^2 d(q)^{c_{k,1}}.$$

As (see Lemma 2.1 in [19])

$$\sum_{q \leq M} w_k(q)^2 d(q)^{c_{k,2}} \ll (\log M)^{c_{k,3}}.$$

Then the desired conclusion follows. For similar proofs, one can refer to [2, 1, 17, 18, 19]. \hfill \Box

Next Lemma is from [9].

**Lemma 2.2.** Suppose that $y = x^\theta$, $0 < \rho \leq t_k^{-1}$, $\frac{1}{2 - t_k \rho} \leq \theta \leq 1$ and $A := (x, x+y)$, where

$$t_k = \begin{cases} 2, & \text{if } k = 2, \\
k^2 - k + 1, & \text{if } k \geq 3. \end{cases} \quad (2.2)$$
Then either
\[ \sum_{n \in A} e(\alpha n^k) \ll y^{1-\rho+\varepsilon}, \]  
(2.3)

or there exist integers \(a\) and \(q\) such that
\[ 1 \leq q \leq y^{k\rho}, \quad (a, q) = 1, \quad |q\alpha - a| \leq x^{1-k} y^{k\rho-1}, \]  
(2.4)

and
\[ \sum_{n \in A} e(\alpha n^k) \ll \frac{w_k(q)y}{1 + y x^{k-1} |\alpha - a/q|} + y^{1-\rho+\varepsilon}. \]  
(2.5)

Proof. Similar as the proof in [9, 4, 3], but estimate
\[ I(\beta) = \int_{x-y}^{x+y} e(\beta \gamma^k) d\gamma \]
in another way (as the way in [7]), where \(\beta = \alpha - a/q\). By Lemma 6.2 in [14], we have
\[ I(\beta) \ll w_k(q)y(1 + |\beta| y x^{k-1})^{-1} \]
instead of the estimates in [9] and [3], i.e.
\[ I(\beta) \ll y(q + |\beta| y x^{k-1})^{-1/k}. \]
Combining the analysis in [9, 4], we finally complete the proof. \(\square\)

Lemma 2.3 (see [12], see also in [11] for the case of \(k = 2\)). Let \(y = x^\theta\) for some fixed \(\theta \geq 2/3 + \varepsilon\). Let \(\alpha \in \mathbb{R}\) and let \(k\) be a positive integer. Suppose that
\[ \left| \sum_{x < n \leq x+y} \Lambda(n) e(\alpha n^k) \right| \geq \frac{y}{(\log x)^A} \]
for some \(A \geq 2\). Then there exists a positive integer \(q \leq (\log x)^{O_k(A)}\) such that
\[ |q\alpha - a| x^{k-1} y \leq (\log x)^{O_k(A)}. \]

Lemma 2.4 (see [9]). For \(y \geq x^{1/2+\varepsilon}, s \geq k^2 + k\) and \(k \geq 2\), we have
\[ \int_{[0,1]} \left| \sum_{x < n \leq x+y} (\log p) e(\alpha p) \right|^s d\alpha \ll y^{s-1} x^{1-k+\varepsilon}. \]

Next we will prove another lemma to achieve our goals. The proof involves some ideas in [6, 17, 19].

Lemma 2.5. Suppose that \(y = x^\theta\), \(0 < \rho \leq t_k^{-1}, \frac{1}{2-t_k \rho} \leq \theta \leq 1\). For \(I\), a subinterval of \((x, x + y]\), let \(h(\alpha) = h_I(\alpha) = \sum_{n \in I} (\log n) e(\alpha n^k)\). For \(1 \leq q \leq y^{k\rho}, (a, q) = 1, M_{q,a} = \{\alpha : |q\alpha - a| \leq x^{1-k} y^{k\rho-1}\}\), let \(M\) be the union of the intervals of \(M(q, a)\). Then for any measurable set \(W \subseteq [0,1]\), we have
\[ \int_W |h(\alpha)|^s d\alpha \ll y^2 (\log x)^2 J_0^{1/2} \left( \int_W |h(\alpha)|^{2s-6} \right)^{1/2} + y^{2-\rho+\varepsilon} J(W) \]
where
\[ J_0 = \sup_{\beta \in [0,1]} \int_M \frac{w_k^2(q) |h(\alpha + \beta)|^2}{(1 + x^{k-1} y |\alpha - a/q|)^2} d\alpha \]
and
\[ J(\mathcal{W}) = \int_{\mathcal{W}} |h(\alpha)|^{s-2}d\alpha. \]

**Proof.** Write
\[ \mathcal{M}_\alpha = \mathcal{M}_\alpha(q,a) - \beta = \{ \beta : |q\alpha - q\beta - a| \leq x^{1-k}y^{k-1} \} \]
and
\[ \mathcal{M}_\alpha = \bigcup_{q \leq y^k} \bigcup_{1 \leq \alpha \leq q, \ (a,q) = 1} \mathcal{M}_\alpha(q,a). \]

Following the argument in [17], one can get
\[ \left| \int_{\mathcal{W}} |h(\alpha)|^s d\alpha \right|^2 = \left| \sum_{n \in I} (\log n) \int_{\mathcal{W}} e(\alpha n^k)|h(\alpha)|^{s-2}h(-\alpha)d\alpha \right|^2 \lesssim \left| \log x \sum_{x<n \leq x+y} \left| \int_{\mathcal{W}} e(\alpha n^k)|h(\alpha)|^{s-2}h(-\alpha)d\alpha \right|^2. \]

By Cauchy’s inequality, we have
\[ \left| \int_{\mathcal{W}} |h(\alpha)|^s d\alpha \right|^2 \lesssim y(\log x)^2 \sum_{x<n \leq x+y} \left| \int_{\mathcal{W}} e(\alpha n^k)|h(\alpha)|^{s-2}h(-\alpha)d\alpha \right|^2. \quad (2.6) \]

Expand the square, we can get
\[ \sum_{x<n \leq x+y} \left| \int_{\mathcal{W}} e(\alpha n^k)|h(\alpha)|^{s-2}h(-\alpha)d\alpha \right|^2 \]
\[ = \int_{\mathcal{W}} \int_{\mathcal{W}} \sum_{x<n \leq x+y} e((\alpha - \beta)n^k)|h(\alpha)|^{s-2}h(-\alpha)|h(\beta)|^{s-2}h(\beta) d\alpha d\beta. \]

Write
\[ f(\gamma) = \sum_{x<n \leq x+y} e(\gamma n^k). \]

Note that
\[ |h(-\alpha)h(\beta)| \leq |h(\alpha)|^2 + |h(\beta)|^2. \]

Then
\[ \sum_{x<n \leq x+y} \left| \int_{\mathcal{W}} e(\alpha n^k)|h(\alpha)|^{s-2}h(-\alpha)d\alpha \right|^2 \lesssim \int_{\mathcal{W}} \int_{\mathcal{W}} |f(\alpha - \beta)| \left( |h(\alpha)|^{s-2} |h(\beta)|^s \right) d\alpha d\beta. \]

Thus
\[ \int_{\mathcal{W}} \int_{\mathcal{W}} |f(\alpha - \beta)| |h(\alpha)|^{s-2} |h(\beta)|^s d\alpha d\beta \lesssim \int_{\mathcal{W}} \int_{\mathcal{W} \cap M_\beta} |h(\beta)|^s |h(\alpha)|^{s-2} \frac{w_k(q)y}{1 + x^{k-1}y|\alpha - \beta - a/q|} d\alpha d\beta \]
\[ + y^{1-\rho+\epsilon} \int_{\mathcal{W}} |h(\alpha)|^{s-2} d\alpha \int_{\mathcal{W}} |h(\alpha)|^s d\alpha. \]
Write
\[ T(\beta) = \int_{\mathbb{R} \setminus \mathbb{M}_\alpha} |h(\beta)|^{s-2} \frac{w_k(q)y}{1 + x^{s-1}y|\alpha - \beta - a/q|} d\beta. \]

By Cauchy’s inequality, we have
\[ T(\beta) \ll y \left( \int_{\mathbb{R}} |h(\beta)|^{2s-6} d\beta \right)^{1/2} \left( \int_{\mathcal{M}} \frac{w_k^2(q)|h(\alpha - \beta)|^2}{(1 + x^{s-1}y|\beta - a/q|)^2} d\beta \right)^{1/2} \]
\[ \ll y \left( \int_{\mathbb{R}} |h(\beta)|^{2s-6} d\beta \right)^{1/2} J_0^{1/2}. \]

Hence
\[ \sum_{x < n \leq x+y} \left| \int_{\mathbb{R}} e(\alpha n^k)|h(\alpha)|^{s-2} h(-\alpha) d\alpha \right|^2 \ll y \int_{\mathbb{R}} |h(\alpha)|^{s-2} h(-\alpha) d\alpha \left( \int_{\mathbb{R}} |h(\alpha)|^{2s-6} d\alpha \right)^{1/2} J_0^{1/2} \]
\[ + y^{-\rho+\varepsilon} \int_{\mathbb{R}} |h(\alpha)|^{s-2} d\alpha \int_{\mathbb{R}} |h(\alpha)|^s d\alpha. \]

Now the desired result can be deduced from (2.6) and (2.7).

\[ \square \]

3. Proof of Theorem 1.1

Let \( x = (N/s)^{1/k} \) and \( y = x^\theta \), where \( N \) is a sufficiently large natural number. Let \( n \) satisfy \( |n - N| \leq yx^{k-1} \). Let
\[ f(\alpha) = \sum_{x-y < n \leq x+y} (\log p) e(\alpha p^k). \]
denote the summation for primes \( p \). Then
\[ g(n) = \int_{[0,1]} f(\alpha)^s e(-n\alpha) d\alpha. \]

Let
\[ P = (\log x)^A, \quad Q = xy^{k-1} P^{-1}. \]

Write
\[ I(q, a) = \{ \alpha \in [0,1) : |q\alpha - a| \leq Q^{-1} \}. \]

Then define the major arcs \( \mathcal{M} \) and minor arcs \( \mathcal{m} \) as
\[ \mathcal{M} = \bigcup_{q \leq P} \bigcup_{1 \leq a \leq q, (a, q) = 1} I(q, a), \]
\[ \mathcal{m} = \mathcal{m}(P, Q) = [0,1) \setminus \mathcal{M}. \]

Suppose that \( \mathfrak{S} \) is a measurable subset of \([0,1)\). Let
\[ g(n, \mathfrak{S}) = \int_{\mathfrak{S}} f(\alpha)^s e(-n\alpha) d\alpha. \]

Then we can write
\[ g(n) = g(n, \mathcal{M}) + g(n, \mathcal{m}). \]
The analysis of \( \varrho(n, M) \): We follow the analysis in [12] and [15]. The width of our major arc is chosen so that if \( \alpha \in M(q, a) \), then \( f(\alpha) \) can be estimated by counting primes in short intervals in residue classes modulo \( q \). Since \( \theta > 7/12 \), we can use Huxley’s result on primes in short intervals to get

\[
f(a/q + \beta) = \phi(q)^{-1} S(q, a) v(\beta) + O(y/(\log x)^{10})
\]

for

\[
|\beta| \leq P/(yx^{k-1}),
\]

where

\[
S(q, a) = \sum_{\substack{1 \leq b \leq q \\ (b, q) = 1}} e\left( \frac{abk}{q} \right)
\]

and

\[
v(\beta) = k^{-1} \sum_{(x-y)^k \leq m \leq (x+y)^k} m^{-1+1/k} e(\beta m).
\]

Thus the standard theory of the major arc contributions in the Waring-Goldbach problem can be applied to yield the estimate

\[
\varrho(n, M) = \mathcal{S}(n) \mathcal{J}(n) + O\left( \frac{y^{s-1}}{x^{k-1}(\log x)^{10}} \right), \tag{3.1}
\]

where \( \mathcal{S}(n) \) is the singular series

\[
\mathcal{S}(n) = \sum_{q=1}^{\infty} \frac{\phi(q)^{-s}}{q} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} S(q, a)^s e(-an/q),
\]

and \( \mathcal{J}(n) \) is the singular integral

\[
\mathcal{J}(n) = \int_{[0,1]} v(\beta)^s e(-\beta n) d\beta.
\]

On the other hand,

\[
\mathcal{S}(n) \asymp 1, \quad \mathcal{J}(n) \asymp y^{s-1} x^{1-k}.
\]

Then we have

\[
\varrho(n, M) \gg y^{s-1} x^{1-k}. \tag{3.2}
\]

\( E(N, s, x^{k-1} y) \) does not exceed the number of integers \( n \) for which

\[
\varrho(n, M) \ll y^{s-1} x^{1-k}(\log x)^{-2}.
\]

Thus it follows from Bessel’s inequality that

\[
E(N, s, x^{k-1} y) \ll (y^{s-1} x^{1-k})^{-2}(\log x)^4 \int_{m} |f(\alpha)|^{2s} d\alpha,
\]

and theorem follows from the following proposition. Then I would state and prove the proposition, and that will close the paper.

**Proposition 3.1.** Let \( A_0 \geq 1 \) be fixed and

\[
K(t) = \int_{m} |f(\alpha)|^t d\alpha.
\]
For $t \geq k^2 + k + 2$ and $y \geq x^{2/3+\varepsilon}$, we have

$$K(t) \ll y^{-1}x^{1-k}(\log x)^{-A_0},$$

provided that $A$ is sufficiently large in terms of $A_0$.

Proof. Fix $A_0$. Choose $B = A_0 + c_0 + 10$. Choose $A$ sufficiently large in terms of $B$. Then Lemma 2.3 implies that for $y \geq x^{2/3+\varepsilon}$,

$$\sup_{\alpha \in \mathcal{M}} |f(\alpha)| \leq y(\log x)^{-B}. \tag{3.3}$$

Choose $h(\alpha) = f(\alpha)$ and $\mathcal{W} = \mathcal{M}$ in Lemma 2.5. Then we have

$$K(t) \ll y^2(\log x)^2 J_0^{1/2} K(2t - 6)^{1/2} + y^2 - \rho + \varepsilon K(t - 2).$$

(i) For the first term, we have

$$K(t) \ll y^2(\log x)^2 J_0^{1/2} K(2t - 6)^{1/2} \ll y^2(\log x)^2 J_0^{1/2} K(t)^{1/2} \max_{\alpha \in [0,1]} f(\alpha)^{(t-6)/2}.$$  

Hence

$$K(t)^2 \ll y^4(\log x)^4 J_0 K(t) \max_{\alpha \in [0,1]} f(\alpha)^{(t-6)}.$$  

Choosing $D = yx^{k-1}$ in Lemma 2.1, this gives

$$K(t) \ll y^4(\log x)^{4+c_0} (y^2(yx^{k-1})^{-1}) \max_{\alpha \in [0,1]} f(\alpha)^{(t-6)}.$$  

Then by (3.3), we have

$$K(t) \ll y^{-1}x^{1-k}(\log x)^{-A_0}.$$  

(ii) For the second term, by the Vinogradov mean value theorem (see Lemma 2.4)

$$\int_{[0,1]} f^s(\alpha) \, d\alpha \ll y^{s-1}x^{1-k+\varepsilon}, \quad s \geq k^2 + k,$$

we have

$$K(t) \ll y^{-3}x^{1-k+\varepsilon}y^2 - \rho + \varepsilon \ll y^{t-1-\rho+\varepsilon}x^1.$$  

Choosing $\varepsilon > 0$ sufficiently small to ensure that $\rho = 4\varepsilon$ is admissible in Lemma 2.2. Then we can complete the proof.

By the standard argument, we can get

$$E(N, s, x^{k-1}y) \ll (y^{s-1}x^{1-k})^{-2} K(2s).$$

Then Theorem 1.1 can be deduced by the standard argument, (3.2) and Proposition 3.1.

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