Differential geometry of the quantum Lie superalgebra of the extended quantum superplane and its $\mathbb{Z}_2$-graded Hopf algebra structure is obtained. Its $\mathbb{Z}_2$-graded dual Hopf algebra is also given.

1. Introduction

Noncommutative differential geometry has attracted considerable interest both mathematically and also from theoretical physics side over the past decade. Especially, there is much activity in differential geometry on quantum groups. For references to the literature for quantum groups we refer to the recent book by Majid [1]. The basic structure giving a direction to the noncommutative geometry is a differential calculus on an associative algebra. A noncommutative differential calculus on quantum groups has been introduced by Woronowicz [2]. Wess and Zumino [3] has been reformulated to fit this general theory, in less abstract way. Some other methods to define a differential geometric structure (or a De Rham complex) on a given noncommutative associative algebra or to construct a noncommutative geometry on a quantum group have been proposed and investigated by several authors [4-9].

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It is known that, in order to construct a noncommutative differential calculus on quantum groups and Hopf algebras, one takes into consideration the associative algebra of functions on the group. The starting point of this work is its Lie algebra. We present here a differential calculus on the Lie superalgebra of the associative algebra of functions on the extended $q$-superplane.

The paper is organized as follows. In the second section, we state briefly the properties of $\mathbb{Z}_2$-graded quantum superplane which are described in Ref. 10. In the third section, using the fact that an element of a Lie group can be represented by exponential of an element of its Lie algebra, we shall write the generators of the extended $q$-superplane as exponential of some elements. The obtained new elements which are to be generators of the Lie superalgebra [11]. We give a differential calculus on the Lie superalgebra and its Hopf algebra structure. We also obtain its $\mathbb{Z}_2$-graded dual Hopf algebra.

2. Review of hopf algebra $\mathcal{A}$

The quantum superplane [12] is defined as an associative algebra whose the even coordinate $x$ and the odd coordinate $\theta$ satisfying

$$x\theta - q\theta x = 0 \quad \theta^2 = 0 \quad (1)$$

where $q$ is a nonzero complex deformation parameter. This algebra is known as the algebra of polynomials over the quantum superplane and we shall denote by $\mathcal{A} = \text{Fun}_q(R(1|1))$.

We know that the algebra $\mathcal{A}$ is a $\mathbb{Z}_2$-graded Hopf algebra with the following costructures [10]: the coproduct $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is defined by

$$\Delta(x) = x \otimes x, \quad \Delta(\theta) = \theta \otimes x + x \otimes \theta. \quad (2a)$$

The counit $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$ is given by

$$\epsilon(x) = 1, \quad \epsilon(\theta) = 0. \quad (2b)$$

We extend the algebra $\mathcal{A}$ by including the inverse of $x$ which obeys

$$xx^{-1} = 1 = x^{-1}x.$$
If we extend the algebra $\mathcal{A}$ by adding the inverse of $x$ then the algebra $\mathcal{A}$ admits a coinverse $\kappa : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$
\kappa(x) = x^{-1}, \quad \kappa(\theta) = -x^{-1}\theta x^{-1}.
$$

(2c)

It is not difficult to verify the following properties of costructures:

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,$$

$$m_{\mathcal{A}} \circ (\epsilon \otimes \text{id}) \circ \Delta = m_{\mathcal{A}} \circ (\text{id} \otimes \epsilon) \circ \Delta,$$

$$m_{\mathcal{A}} \circ (\kappa \otimes \text{id}) \circ \Delta = \epsilon = m_{\mathcal{A}} \circ (\text{id} \otimes \kappa) \circ \Delta$$

where $\text{id}$ denotes the identity mapping and $m_{\mathcal{A}}$ is the multiplication map

$$m_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad m_{\mathcal{A}}(a \otimes b) = ab.$$

The multiplication in $\mathcal{A} \otimes \mathcal{A}$ follows the rule

$$(A \otimes B)(C \otimes D) = (-1)^{\hat{f}_{\mathcal{C}}} AC \otimes BD,$$

(4)

where $\hat{f}$ denotes the $\mathbb{Z}_2$-grading of $f$.

3. Differential calculus on the Lie superalgebra

It is known that an element of a Lie group can be represented by exponential of an element of its Lie algebra. Using this fact, one can define the generators of $\mathcal{A}$ as

$$x = e^u, \quad \theta = e^u\eta.$$  

(5)

Let

$$q = e^h.$$  

(6)

Then we obtain the relations

$$[u, \eta] = h\eta, \quad \eta^2 = 0,$$

(7)

where

$$[a, b]_\pm = ab \pm ba.$$
These are the relations of a Lie superalgebra and we shall denote it by $\mathcal{L}(A)$. The $\mathbb{Z}_2$-graded Hopf algebra structure of $\mathcal{L}(A)$ can be read off from (2),

$$
\Delta(u) = u \otimes 1 + 1 \otimes u, \quad \Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta,
$$

$$
\epsilon(u) = 0, \quad \epsilon(\eta) = 0,
$$

$$
\kappa(u) = -u, \quad \kappa(\eta) = -\eta.
$$

(8)

An interesting case is that these costructures are the same with the Hopf algebra structure of one-forms on $A$ which is given in Ref. 10.

We want build up a noncommutative differential calculus on the Lie superalgebra $\mathcal{L}(A)$. This may be involve functions on the Lie superalgebra $\mathcal{L}(A)$, differentials and differential forms. So we have to define a linear operator $d$ which acts on the functions of the elements of $\mathcal{L}(A)$.

In order to establish a noncommutative differential calculus on the algebra $\mathcal{L}(A)$, we assume that the commutation relations between the elements of $\mathcal{L}(A)$ and their differentials are of the following form:

$$
u d\eta = A_{11} d u + B_1 d u + B_2 d \eta,$$

$$
u d\eta = A_{12} d \eta + B_3 d u + B_4 d \eta,$$

$$\eta d\nu = A_{21} d u + B_5 d u + B_6 d \eta,$$

$$\eta d\eta = A_{22} d \eta + B_7 d u + B_8 d \eta.$$

(9)

The coefficients $A_{ij}$ and $B_i$ will be determined in terms of the ”new” deformation parameter $h$. To find them we shall use the consistency of calculus.

We first note that the following properties of the exterior differential $d$: the nilpotency

$$
d^2 = 0,
$$

(10a)

and the $\mathbb{Z}_2$-graded Leibniz rule

$$
d(fg) = (df)g + (-1)^{\tilde{d}} f(dg).
$$

(10b)
From the consistency conditions
\[ d(u\eta - \eta u - h\eta) = 0, \quad d(\eta^2) = 0 \]
we find
\[
A_{12} = 1, \quad B_3 + B_5 = 0, \quad A_{22} = 1, \quad (11a)
\]
\[ A_{21} = -1, \quad B_4 + B_6 = h, \quad B_7 = 0 = B_8. \]

Similarly, from
\[
(u\eta - \eta u - h\eta)du = 0, \quad (u\eta - \eta u - h\eta)d\eta = 0
\]
one has
\[
B_2 = 0 = B_3, \quad A_{11}B_5 = 0, \quad (B_1 - A_{11})B_5 = 0 = (1 - A_{11})B_6, \quad (11b)
\]
\[
(B_1 - B_4 + h)B_6 = 0, \quad (B_1 + h)B_5 = B_3B_6.
\]
The system (11) has many solutions and we shall only discuss one of them below. Most of the coefficients in the relations (9) are already determined. The remaining coefficients can be determined from the following equations
\[
B_4 + B_6 = h, \quad (1 - A_{11})B_6 = 0, \quad (B_1 - B_4 + h)B_6 = 0. \quad (11c)
\]
The system (11c) admits many solutions. We here consider only the solution
\[
B_1 = 2h, \quad B_4 = h, \quad B_6 = 0 \quad \text{and} \quad A_{11} = 1. \quad (11d)
\]
In this case, the relations (9) are of the following form
\[
[u, du] = 2hd\eta, \quad \quad [\eta, du]_+ = 0,
\]
\[
[u, d\eta] = h d\eta, \quad \quad [\eta, d\eta] = 0. \quad (12)
\]
Applying the exterior differential \(d\) to the first and second (or third) of the relations (12) one gets
\[
[du, d\eta] = 0, \quad (du)^2 = 0. \quad (13)
\]
Note. The two superalgebras (1) and (7) are closely related. Therefore the
differential calculi on these two superalgebras are also closely related. Indeed,
the differentials of \( u \) and \( \eta \) in terms of \( x \) and \( \theta \) are
\[
\begin{align*}
\text{du} &= \frac{2h}{e^{2h} - 1} \text{dx} \ x^{-1}, \\
\text{d}\eta &= x^{-1}(\text{d}\theta - \text{d}x\ x^{-1}\theta)
\end{align*}
\]
so that replacing these into (12) one obtains
\[
\begin{align*}
x \ \text{dx} &= q^2 \text{dx} \ x, \\
\theta \ \text{dx} &= -q \text{d}x \ \theta, \\
x \ \text{d}\theta &= q \text{d}\theta \ x + (q^2 - 1) \text{dx} \ \theta, \\
\theta \ \text{d}\theta &= \text{d}\theta \ \theta.
\end{align*}
\]
This differential structure is invariant under action of \( GL_q(1|1) \) (see, e.g. [13]).
Thus the differential structure (12) must be invariant under action of \( gl_h(1|1) \)
with \( q = e^h \) (see, Ref. 11).

A differential algebra of the associative algebra \( B \) is a \( \mathbb{Z}_2 \)-graded asso-
ciative algebra \( \Gamma \) equipped with an operator \( \text{d} \) that has the properties (10).
Furthermore, the algebra \( \Gamma \) has to be generated by \( \Gamma^0 \cup \Gamma^1 \cup \Gamma^2 \), where \( \Gamma^0 \)
is isomorphic to \( B \). For \( B \) we write \( \mathcal{L}(A) \), the Lie superalgebra of \( A \). Let us
denote the algebra generated by \( \text{du} \) and \( \text{d}\eta \) with the relations (12) by \( \Gamma^1 \), where
\( \Gamma^1 \) is isomorphic to \( \mathcal{dL}(A) \), and the algebra (13) by \( \Gamma^2 \). Let \( \Gamma \) be the quoitent
algebra of the free associative algebra on the set \{u, \eta, \text{du}, \text{d}\eta\} modulo the ideal
\( J \) that is generated by the relations (7), (12) and (13). Then the differential
algebra \( \Gamma \) is a \( \mathbb{Z}_2 \)-graded Hopf algebra with the following costructures:
\[
\begin{align*}
\Delta(\text{du}) &= \text{du} \otimes 1 + 1 \otimes \text{du}, \\
\Delta(\text{d}\eta) &= \text{d}\eta \otimes 1 + 1 \otimes \text{d}\eta, \\
\epsilon(\text{du}) &= 0, \\
\epsilon(\text{d}\eta) &= 0, \\
\kappa(\text{du}) &= -\text{du}, \\
\kappa(\text{d}\eta) &= -\text{d}\eta.
\end{align*}
\]

Before closing this section, just as we introduced the derivatives of the gener-
ators of \( A \) in the standard way, let us introduce derivatives of the generators
of \( \mathcal{L}(A) \) and multiply explicit expression of the exterior differential \( \text{d} \) from the
right by \( uf \) and \( \eta f \), respectively. Then, using the \( \mathbb{Z}_2 \)-graded Leibniz rule for
partial derivatives
\[
\partial_i(fg) = (\partial_i f)g + (-1)^i f(\partial_i g)
\]
we get
\[ [\partial_u, u] = \frac{2h}{e^{2h} - 1} + 2h\partial_u, \quad [\partial_u, \eta] = 0, \]
\[ [\partial_\eta, u] = h\partial_\eta, \quad [\partial_\eta, \eta]_+ = 1. \] (17)

The commutation relations between the derivatives can be easily obtained by using \( d^2 = 0 \). So it follows that
\[ 0 = d^2 = du d\eta (\partial_u \partial_\eta - \partial_\eta \partial_u) + (d\eta)^2 \partial_\eta^2 \]
which says that
\[ [\partial_u, \partial_\eta] = 0, \quad \partial_\eta^2 = 0. \] (18)

Finally to find the commutation relations between the differentials and derivatives we shall assume that they have the following form
\[ \partial_u du = C_{11} du \partial_u + C_{12} d\eta \partial_\eta + D_1 du + D_2 d\eta, \]
\[ \partial_u d\eta = C_{21} d\eta \partial_u + C_{22} du \partial_\eta + D_3 du + D_4 d\eta, \]
\[ \partial_\eta du = F_{11} du \partial_\eta + F_{12} d\eta \partial_u + D_5 du + D_6 d\eta, \]
\[ \partial_\eta d\eta = F_{21} d\eta \partial_\eta + F_{22} du \partial_u + D_7 du + D_8 d\eta. \] (19)

After some tedious but straightforward calculations, we find
\[ \partial_u du = e^{-2h} du \partial_u - e^{-2h} du, \quad \partial_u d\eta = e^{-2h} d\eta \partial_u - e^{-2h} d\eta, \]
\[ \partial_\eta du = -du \partial_\eta, \quad \partial_\eta d\eta = d\eta \partial_\eta + \frac{e^h - e^{-h}}{2h} \left\{ (e^h - e^{-h}) du \partial_u + e^{-h} du \right\}. \] (20)

### 4. Hopf algebra structure of forms on \( \mathcal{L}(A) \)

Using the generators of \( \mathcal{L}(A) \) we can define two one-forms as follows:
\[ \phi = \frac{e^{2h} - 1}{2h} du, \quad V = e^h d\eta. \] (21)

We denote the algebra of forms generated by two elements \( \phi \) and \( V \) by \( \Omega \). The generators of the algebra \( \Omega \) with the generators of \( \mathcal{L}(A) \) satisfy the following relations:
\[ [u, \phi] = 2h\phi, \quad [\eta, \phi]_+ = 0, \]
\[ [u, V] = \hbar V, \quad [\eta, V] = 0. \]  
(22)

The commutation rules of the generators of \( \Omega \) are
\[ \phi^2 = 0, \quad [\phi, V] = 0. \]  
(23)

One can make the algebra \( \Omega \) into a \( \mathbb{Z}_2 \)-graded Hopf algebra with the following co-structures: the coproduct \( \Delta : \Omega \rightarrow \Omega \otimes \Omega \) is defined by
\[ \Delta(\phi) = \phi \otimes 1 + 1 \otimes \phi, \quad \Delta(V) = V \otimes 1 + 1 \otimes V. \]  
(24)

The counit \( \epsilon : \Omega \rightarrow \mathbb{C} \) is given by
\[ \epsilon(\phi) = 0, \quad \epsilon(V) = 0 \]  
(25)

and the coinverse \( \kappa : \Omega \rightarrow \Omega \) is defined by
\[ \kappa(\phi) = -\phi, \quad \kappa(V) = -V. \]  
(26)

One can easily check that (22) and (23) are satisfied. Note that the commutation relations (22) and (23) are compatible with \( \Delta, \epsilon \) and \( \kappa \), in the sense that \( \Delta(u\phi) = \Delta(\phi u) + 2\hbar \Delta(\phi) \) and so on.

5. The Hopf superalgebra of vector fields on \( \mathcal{L}(\mathcal{A}) \)

In this section, we shall obtain the superalgebra of vector fields on \( \mathcal{L}(\mathcal{A}) \) and their Hopf algebra structure. We first write the Cartan-Maurer forms as
\[ d\phi = \frac{2\hbar}{e^{2\hbar} - 1}\phi, \quad d\eta = e^{-\hbar}V. \]  
(27)

Then, the relations (23) allow us to calculate the superalgebra of the vector fields. Writing the exterior differential \( d \) in the form
\[ d = \phi X + V \nabla \]  
(28)

and considering an arbitrary function \( f \) of the generators of \( \mathcal{A} \) and using the nilpotency of \( d \) one has
\[ \phi dX = Vd\nabla. \]  
(29)
So we find the following commutation relations for the superalgebra of vector fields

\[ [X, \nabla] = 0, \quad \nabla^2 = 0. \quad (30) \]

We also note that the commutation relations (30) of the vector fields should be consistent with monomials of the generators of \( L(A) \). To proceed, we must calculate the actions of the \( \mathbb{Z}_2 \)-graded Leibniz rule by comparing the elements which lie together with each other from the one-forms:

\[
[X, u] = 1 + 2hX, \quad [X, \eta] = 0, \\
[\nabla, u] = h\nabla, \quad [\nabla, \eta]_+ = 1. \quad (31)
\]

Of course, these commutation relations must be consistent.

In order to find the coproduct of this superalgebra, we shall use the fact that the exterior differential operator \( d \) satisfies the Leibniz rule [13]. So using the \( \mathbb{Z}_2 \)-graded Leibniz rule for \( d \) we write

\[ X(fg) = (Xf)g + e^{2hN} f(Xg), \]

\[ \nabla(fg) = (\nabla f)g + e^{hN} f(\nabla g), \]

where \( N \) is a number operator which acting on the monomials of the generators of \( L(A) \). This provides a comultiplication

\[ \Delta(X) = X \otimes I + e^{2hN} \otimes X, \]

\[ \Delta(\nabla) = \nabla \otimes I + e^{hN} \otimes \nabla. \quad (32) \]

Using the following basic axioms of Hopf superalgebra

\[ m(\epsilon \otimes \text{id})\Delta(Y) = Y, \quad m(\text{id} \otimes \kappa)\Delta(Y) = \epsilon(Y) \quad (33) \]

one obtains

\[ \epsilon(X) = 0, \quad \epsilon(\nabla) = 0, \quad (34) \]

\[ \kappa(X) = -e^{-2hN}X, \quad \kappa(\nabla) = -e^{-hN}\nabla. \quad (35) \]
We can now easily obtain the dual Hopf superalgebra as follows: if we introduce the operators $N$ and $\chi$ as
\[
e^{2hN} = I + (e^{2h} - 1)X, \quad \chi = \nabla \left\{ I + (e^{2h} - 1)X \right\}^{-1/2},
\]
then we have
\[
\Delta(N) = N \otimes I + I \otimes N, \\
\Delta(\chi) = \chi \otimes e^{-hN} + I \otimes \chi.
\]
(37a)

and
\[
\epsilon(N) = 0, \quad \epsilon(\chi) = 0, \\
\kappa(N) = -N, \quad \kappa(\chi) = -\chi e^{hN}.
\]
(37b)

(37c)
The operators $N$ and $\chi$ are preserve the commutation relations (30):
\[
[N, \chi] = 0, \quad \chi^2 = 0.
\]
(38)

Note that the relations (38) and the Hopf algebra structure (37) can also be obtained from the approach of Ref. 11 [see, chapter 3].

An interesting problem is the construction of a differential calculus on the Lie superalgebra of the quantum supergroup $GL_q(1|1)$ using the methods of this paper and Ref. 14.

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