Degenerate spectrum in the neutrino mass anarchy with Wishart matrices and implications for $0\nu\beta\beta$ and $\delta_{\text{CP}}$

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Abstract

We show that a degenerate neutrino mass spectrum can be realized in the neutrino mass anarchy hypothesis, if the neutrino Yukawa and right-handed neutrino mass matrices are given by the Wishart matrix, i.e. products of $N \times 3$ rectangular random matrices, whose eigenvalue distribution approaches the Marchenko-Pastur distribution for large $N$. The mixing angle and CP phase distributions are determined by either the Haar measure of $U(3)$ or that of $SO(3)$. We study how large $N$ can be without tension with the observed neutrino mass squared differences, and find that the predicted value of $m_{ee}$ is likely within the reach of future $0\nu\beta\beta$ experiments especially for $N$ on the high side of the allowed range.
I. INTRODUCTION

The standard model (SM) of particle physics has been overwhelmingly successful for decades, and the long-sought Higgs boson, the last missing piece of the SM, was finally discovered at the LHC [1, 2]. Despite the great success of the SM, there are many puzzles left unanswered; one of them is the origin of the flavor structure.

While neutrinos are massless in the SM, atmospheric and solar neutrino oscillation experiments revealed that neutrinos have tiny but non-zero masses (see e.g. Refs. [3, 4] for the latest results). In particular, a mild mass hierarchy and large mixing angles for the neutrino sector are in sharp contrast with quarks and charged leptons. If we are to understand the neutrino flavor structure based on symmetry principles, it seems to require rather contrived flavor models. The observed large mixing angles rather suggest structureless mass matrix for neutrinos, implying that all the neutrino species have the same quantum number.

The squared mass differences and mixing angles are measured by various neutrino oscillation experiments [6–11] and the recent best-fit values for normal (inverted) hierarchy are obtained as [3]

\[ \Delta m_{21}^2 = 7.60 \times 10^{-5} \text{ eV}^2, \quad |\Delta m_{31}^2| = 2.48 (2.38) \times 10^{-3} \text{ eV}^2 \]
\[ \sin^2 \theta_{12} = 0.323, \quad \sin^2 \theta_{23} = 0.567 (0.573), \quad \sin^2 \theta_{13} = 0.0234 (0.0240), \]

and the favored value of the Dirac CP phase is around $3\pi/2$. Besides the neutrino oscillation experiments, further information can be obtained from the cosmic microwave background (CMB) observations and the neutrinoless double beta decay ($0\nu\beta\beta$) experiments. In particular, the CMB observations by Planck, WMAP and other ground-based experiments set the upper limit on the sum of the neutrino masses as $\sum m_i < 0.66 \text{ eV} \ (95\% \text{ CL})$ [12].

1 While it is possible to understand the hierarchical mass pattern of quarks and charged leptons based on symmetry principles, a variety of flavor symmetries and charge assignments are allowed. For an alternative approach without flavor symmetry, see e.g. Ref. [5].
One of the attractive explanations for the observed large neutrino mixing is the neutrino mass anarchy [13–16], which gained momentum especially after the discovery of a non-zero value of $\theta_{13}$ by the Daya-Bay experiment [6]. The basic idea of the neutrino mass anarchy is simple. Suppose that all the Yukawa couplings and/or right-handed neutrino masses are determined by a UV theory, which has a sufficiently large landscape of vacua. If each coupling is allowed to take values of order unity in the landscape, the Yukawa couplings and/or right-handed neutrino masses may be modeled by some functions of random matrices. The simplest possibility is the linear measure:

$$h, M \sim X,$$

where the neutrino Yukawa matrix $h$ as well as the right-handed neutrino mass matrix $M$ are proportional to $3 \times 3$ random matrices represented by $X$. Phenomenological and cosmological aspects of the neutrino mass anarchy have been studied; e.g. two of the present authors (KSJ and FT) studied the implications of neutrino mass anarchy for leptogenesis in Ref. [17], and it was also recently revisited in Ref. [18]. See also Refs. [19, 20] for phenomenological study of the neutrino mass anarchy.

In the neutrino mass anarchy hypothesis, the mixing angle and CP phase distributions are determined by the invariant Haar measure of the underlying symmetry group such as U(3) or SO(3) [14], and so, they are rather robust predictions. Interestingly, the observed large mixing angles can be nicely explained in the neutrino mass anarchy [16].² On the other hand, the neutrino mass spectrum depends sensitively on the weighting functions. In the case of the linear measure, normal mass hierarchy is highly favored over the inverted or quasi-degenerate one. In addition, the observed mild hierarchy of the mass squared differences can be nicely explained by the neutrino mass anarchy together with the seesaw mechanism [13, 14]. The estimated $m_{ee}$ turned out to be too small to be detected by future $0\nu\beta\beta$ experiments [17], but this result can be modified for more general measure functions [22].³

² See, however, Ref. [21].
³ Our analysis is different from Ref. [22] in which the adopted measure is not applicable to the case of the seesaw mechanism with neutrino mass anarchy.
In this letter we study the next simplest possibility: the neutrino Yukawa couplings and the right-handed neutrino masses are given by the random matrix squared, or more precisely, the Wishart matrices:

$$h, M \sim X^\dagger X \text{ or } X^T X$$

(3)

where $X$ represents $N \times 3$ complex or real random matrices. In general, $N$ does not have to be equal to 3. For $N > 3$, the neutrino Yukawa and right-handed neutrino mass matrices are given by products of rectangular matrices. We shall see that the observed neutrino mass squared differences can be explained for $N$ as large as 35. Interestingly, the eigenvalue distribution of the Wishart matrix is known to approach the Marchenko-Pastur distribution [23], where the eigenvalues tend to be degenerate. Therefore, quasi-degenerate neutrino mass spectrum can be realized in the neutrino mass anarchy with the Wishart matrix if $N \gg 3$, which should be contrasted to the case of the linear measure (2). We will discuss its implications for the $0\nu\beta\beta$ experiments. We will also show that the mixing angle and CP phase distributions of our scenario are determined by either the Haar measure of $U(3)$ or that of $SO(3)$.

The rest of this letter is organized as follows. In Sec. II we first explain our set-up and see how the neutrino mass spectrum changes as the size of the rectangular matrices $N$ increases. Then we study the implication for the Dirac CP phase and the $0\nu\beta\beta$ experiments. The last section is devoted for discussion and conclusions.

II. NEUTRINO MASS ANARCHY

In this section, we consider the neutrino mass anarchy based on the Wishart matrices as a simple extension of the linear measure. We focus on the case of the Majorana neutrino mass with the seesaw mechanism [24–27].

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4 Our set-up can be straightforwardly applied to the case of the Dirac neutrino mass, and most of our results (except for the $0\nu\beta\beta$) will remain qualitatively valid. In particular, the quasi-degenerate spectrum can be realized.
A. Preliminaries

The seesaw Lagrangian is given by

\[ \mathcal{L} = f_{ij} \bar{e}_R \ell_j \tilde{H} + h_{ij} \bar{N}_i \ell_j H + \frac{1}{2} M_{ij} \bar{N}_i \bar{N}_j + \text{h.c.}, \]  

(4)

where \( \ell, H(\tilde{H}), e_R \) and \( N \) are respectively the left-handed lepton doublet, the Higgs doublet (its SU(2) conjugate), the right-handed charged lepton and the right-handed neutrino, \( f_{ij}, h_{ij} \) are Yukawa matrices for charged leptons and neutrinos respectively and \( M_{ij} \) represents the Majorana mass matrix for right-handed neutrinos. The subscripts represent the generation, \( i, j = 1, 2, 3 \).

Let us first diagonalize the charged lepton Yukawa matrix as

\[ f = U_{fR}^\dagger D_e U_{fL} \]  

(5)

with

\[ D_e \equiv \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \]  

(6)

where \( U_{fR} \) and \( U_{fL} \) are unitary matrices, and \( y_{e,\mu,\tau}(>0) \) denotes the charged lepton Yukawa couplings.\(^5\) In the basis where the charged lepton Yukawa matrix is diagonalized, the Lagrangian becomes

\[ \mathcal{L} = (y_\alpha \delta_{\alpha\beta}) \bar{e}_{R\alpha} \ell_\beta \tilde{H} + h_{i\alpha} \bar{N}_i \ell_\alpha H + \frac{1}{2} M_{ij} \bar{N}_i \bar{N}_j + \text{h.c.}, \]  

(7)

where \( \alpha, \beta \) run over the lepton flavor indices (\( e, \mu, \tau \)), and we have defined

\[ h_{i\alpha} \equiv h_{ij} \left( U_{fL}^\dagger \right)_{j\alpha}, \]  

(8)

\[ \ell_\alpha \equiv (U_{fL})_{\alpha i} \ell_i, \]  

(9)

\[ e_{R\alpha} \equiv (U_{fR})_{\alpha i} e_{Ri}. \]  

(10)

\(^5\) Throughout this letter we do not try to interpret the charged lepton mass hierarchy in our scheme because there could be additional selection (anthropic) effects.
After the Higgs field acquires the vacuum expectation value (VEV), one obtains the effective Lagrangian for active neutrinos by integrating out the heavy right-handed neutrinos,

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{2} (m_{\nu})_{\alpha\beta} \nu_\alpha \nu_\beta + \text{h.c.}, \]

(11)

where \( \nu_\alpha \) is the light left-handed neutrino, the neutrino mass matrix is given by

\[ (m_{\nu})_{\alpha\beta} = v^2 (h^T)_{\alpha i} (M^{-1})_{ij} h_{j\beta} \]

and \( v \simeq 174 \text{ GeV} \) is the VEV of the Higgs field. The neutrino mass matrix \( m_{\nu} \) is generically a complex-valued symmetric matrix, and it can be diagonalized by a unitary matrix \( U_{\text{MNS}} \) as

\[ m_{\nu} = U_{\text{MNS}}^* \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U_{\text{MNS}}^\dagger. \]

(12)

Here \( m_1, m_2 \) and \( m_3 \) are real and positive values with \( m_1 < m_2 < m_3 \). This numbering is for the normal hierarchy, whereas in the inverted hierarchy case, one should relabel them as \( m_3 \rightarrow m_2, m_2 \rightarrow m_1 \) and \( m_1 \rightarrow m_3 \) in order to compare our results with the observations (1). In fact, however, mostly either normal or quasi-degenerate (normal-ordering) mass hierarchy is realized in our scheme, and so, the inverted hierarchy case is practically negligible.

The neutrino oscillation experiments provide us with only the squared mass differences, \( \Delta m^2_{ij} = m_i^2 - m_j^2 \). In order to compare our results with observations, we use the dimensionless parameter \( R \) defined by the ratio of the squared mass difference between the heaviest and the second heaviest neutrinos to that between the second heaviest and the lightest ones:

\[ R = \frac{\Delta m^2_{21}}{\Delta m^2_{32}} \text{ (normal)} \quad \text{or} \quad \frac{\Delta m^2_{13}}{\Delta m^2_{21}} \text{ (inverted)}. \]

(13)

The observed value of \( R \) is given by \( R \sim 1/30 \) for normal-ordering hierarchy and \( R \sim 30 \) for inverted hierarchy.

The neutrino mixing matrix \( U_{\text{MNS}} \) can be expressed in terms of the mixing angles, \( \theta_{ij} \), with \( (i, j) = (1, 2), (2, 3) \) and \( (3, 1) \), and the Dirac and Majorana CP phases, \( \delta, \alpha_{21} \) and \( \alpha_{31} \).
after absorbing the unphysical phases by redefinition of the fields, and it is conventionally written as

$$U_{\text{MNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \text{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$$

(14)

where we abbreviate \(\sin \theta_{ij}\) and \(\cos \theta_{ij}\) as \(s_{ij}\) and \(c_{ij}\), respectively, and the mixing angles and the CP phases satisfy \(\theta_{ij} \in [0, \pi/2)\) and \(\delta, \alpha_{21}, \alpha_{31}, \in [0, 2\pi)\).

**B. Neutrino mass anarchy based on the Wishart matrices**

In the neutrino mass anarchy hypothesis with the linear measure, both \(h_{i\alpha}\) and \(M_{ij}\) are taken to be proportional to \(3 \times 3\) complex(or real)-valued random matrices (cf. (2)). The unitary matrix \(U_{fL}\) does not affect the probability distributions of the mixing angles and the CP phases, as they are fixed by the Haar measure of \(U(3)\) (SO(3)). This is the simplest possibility, but it remains unknown how the randomness for these matrices is originated in the landscape. In fact, there are various other basis-independent choices for these matrices. Here we consider the next-to-simplest set-up, in which the neutrino Yukawa matrix and Mayorana mass matrix consist of products of random matrices:\(^6\)

$$h_{ij} = \frac{y_{\nu}}{N} (F^\dagger F)_{ij}, \quad M_{ij} = \frac{M_0}{2N} (G^\dagger G + G^T G^*)_{ij}$$

(15)

where \(F\) and \(G\) are \(N \times 3\) complex (or real) random matrices of order unity, and \(y_{\nu}\) and \(M_0\) represent the typical neutrino Yukawa couplings and the right-handed neutrino mass. For \(y_{\nu} = O(1)\), \(M_0 \sim 10^{15}\) GeV is suggested by the neutrino oscillation experiments and the seesaw mechanism. Note that the above form of the neutrino Yukawa couplings is given in the original basis, and one has to multiply it with the unitary matrix \(U_{fL}\) in the

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\(^6\) If the neutrino Yukawa couplings and the right-handed neutrino mass are given by \(h \sim F^T F\) and \(M \propto G^T G\), where \(F\) and \(G\) are complex-valued \(N \times 3\) random matrices of order unity, there is no degeneracy in the eigenvalues. We do not pursue this case in this letter.
basis where the charged lepton Yukawa matrix is diagonalized (see Eq. (8)). This however does not affect the final mixing and CP phase distributions just as in the previous case.\footnote{In general, any Yukawa matrix can be written as a product of a Hermitian matrix and a unitary matrix by the polar decomposition theorem. Here we consider a case where the Hermitian matrix is of the Wishart-type random matrix.}

The above form of $h_{ij}$ and $M_{ij}$ imply that they are given by the so-called Wishart matrix. Specifically, we will take $F$ and $G$ as a chiral Gaussian Unitary (Orthogonal) Ensemble where each element follows a complex(real)-valued Gaussian distribution with zero mean and a variance of unity. For large $N$, the eigenvalue distribution is given by the well-known Marchenko-Pastur law \cite{23} in random matrix theory, and the eigenvalues of $h$ and $M$ tend to be highly degenerate for $N \gg 3$. As a result, the light neutrino masses are also expected to be degenerate, which is difficult to realize in the case of the linear measure. As we shall see, however, $N$ cannot be arbitrarily large because the predicted value of $R$ tends to be too large compared to the observed value, $R \sim 1/30$.

C. Mass spectrum, mixing angles and CP phases

We have performed numerical calculations of the neutrino mass anarchy based on the Wishart matrices. Specifically, we have generated $10^6 \times 3$ complex and real random matrices, $F$ and $G$, to obtain the distributions of neutrino masses, mixing angles and CP violating phases. The results are shown in Figs. 1 and 2 corresponding to the complex and real Wishart matrices, respectively. We have varied $N$ as $N = 3$ (solid red), $N = 10$ (dashed green), $N = 30$ (dotted blue), and we have set $y_\nu = 1$ and $M_0 = 10^{15}$ GeV. Note that the distribution of $R$ in the right panel is independent of the choice of $y_\nu$ and $M_0$. For comparison, we show the results of the neutrino anarchy with the linear measure as the small-dotted magenta lines in each figure. One can see that the neutrino mass distribution (Figs. 1(a) and 2(a)) tends to be more degenerate as $N$ increases. The probability distribution of $R$ is suppressed at $R > 1$, implying that the inverted hierarchy ($R \sim 30$) is highly disfavored. Thus, the neutrino mass hierarchy is either normal or
FIG. 1: Probability distributions of the neutrino masses (left) and $R$ (right) for complex Wishart matrices are shown. The solid red, dashed green and dotted blue lines correspond to the case with $N = 3, 10$ and $30$ respectively, while the magenta lines represent the anarchy with the linear measure. Here we have taken $y_\nu = 1$ and $M_0 = 10^{15}$ GeV.

quasi-degenerate (normal-ordering) in the anarchy based on the Wishart matrices.

Fig. 3 shows the mean value of $R$ as a function of $N$ with 1 and 2 $\sigma$ error bands. It shows that the normal hierarchy ($R \sim 1/30$) is preferred over the inverted hierarchy ($R \sim 30$) and $N$ is bounded from above as $N \lesssim 35$ ($N \lesssim 70$ for real Wishart matrices) in order to be consistent with the observations. This implies that, even if one considers the Wishart matrices, there is an upper bound on the degeneracy of the neutrino masses. We will discuss its implications for the $0\nu\beta\beta$ experiments in the next subsection.

We can also see from Fig. 4 that the mixing angle and CP phase distributions are determined by the Haar measure of $U(3)$. If the random matrices $F$ as well as the charged lepton Yukawa matrix are taken to be real, the resultant distribution is then given by the Haar measure of $SO(3)$. (The right-handed neutrino mass matrix is real by construction.) In this case the Majorana CP phases vanish, and the Dirac CP phase $\delta$ takes a value of either 0 or $\pi$. We note that the currently favored value of $\delta$ is about $3\pi/2$ according to Ref. [4], which corresponds to $\sin^2 2\delta = 0$. Interestingly, the $U(3)$ Haar measure results in the probability distribution of $\delta$ peaked at $\sin^2 2\delta = 0$. 
FIG. 2: Same as Fig. 1 but for real Wishart matrices.

FIG. 3: The mean value of $R$ with $1\sigma$ (green region) and $2\sigma$ (yellow region) error as a function of $N$ corresponding to complex (left) and real (right) random matrices. The blue-dotted horizontal line represents the normal hierarchy taken to be $R = 1/30$. 
FIG. 4: Probability distributions of mixing angles (left) and CP violating phases (right) are shown. $\theta_{ij}$ represents $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ and $\delta_{\text{CP}}$ represents $\delta$, $\alpha_{21}$ and $\alpha_{31}$. Red squares and blue circles correspond to complex and real Wishart matrices respectively and magenta and cyan lines correspond to the U(3) and SO(3) Haar measure respectively. We have taken $N = 30$, but the distributions are unchanged if one varies $N$.

D. Neutrinoless double beta decay

The Majorana nature of the neutrinos can be probed by the $0\nu\beta\beta$ experiments, which is sensitive to $m_{ee}$ defined by

$$m_{ee} \equiv \left| \sum_{i=1}^{3} (U_{\text{MNS}})_{ei}^2 m_i \right|$$

$$= \left| m_1 (c_{12} c_{13})^2 + m_2 (s_{12} c_{13})^2 e^{i \alpha_{21}} + m_3 s_{13}^2 e^{i (\alpha_{31} - 2\delta)} \right|. \quad (16)$$

The current upper bound on $m_{ee}$ by the GERDA experiment using $^{76}\text{Ge}$ reads [28]

$$m_{ee} \lesssim (0.2 - 0.4) \text{ eV} \ (90\% \text{CL}). \quad (17)$$

A similar bound was obtained by EXO-200 using $^{136}\text{Xe}$ [29], and a slightly better bound has been recently obtained by the KamLAND-Zen experiment as [30]

$$m_{ee} \lesssim (0.14 - 0.28) \text{ eV} \ (90\% \text{CL}). \quad (18)$$
The next-generation experiment is expected to reach the level of $m_{ee} \approx 0.01\text{ eV}$ [31].

We show the predicted range of $m_{ee}$ in the $m_{ee}-m_1$ plane in Fig. 5 (complex Wishart) and Fig. 6 (real Wishart), where we have taken $N = 10$ and 30. We have generated $10^7$ Wishart matrices and extracted the subset satisfying the observed $R$ (within $2\sigma$) and $M_0$ is adjusted to realize the best fit value of $\Delta m^2_{21}$. The mixing angles are also adjusted to the best fit values. Thick red (blue) lines are contours of equal probability in which 68% (95%) of the data points are contained. For comparison, we similarly show the prediction of the linear measure case as thin red (blue) lines in the right panel of Fig. 5. The black lines with various line types represent $m_{ee}$ for best-fit values of the neutrino mass differences and mixing angles with vanishing CP-violating phases: $(e^{i\alpha_{21}}, e^{i(\alpha_{31}-\delta)}) = (+1, +1), (+1, -1), (-1, +1)$ and $(-1, -1)$ from top to bottom at $m_1 \gtrsim 10^{-2}\text{eV}$. The horizontal dashed (cyan) line represent the sensitivity of the future experiment, while the shaded (magenta) region is excluded by the current experiments. Since quasi-degenerate mass spectrum is more likely for large values of $N$, relatively large $m_{ee}(\gtrsim 0.01\text{eV})$ is realized with a greater probability compared to the case of the linear measure and a larger fraction of the parameter space will be accessible by the near future experiments. Note however that, since $N$ is bounded from above in order to be consistent with observations, there is an upper bound on the neutrino mass degeneracy. As a result, $m_{ee}$ cannot be arbitrarily large even in the case with the Wishart matrices.

III. DISCUSSION AND CONCLUSIONS

In this letter we have studied in detail the neutrino mass anarchy hypothesis with the Wishart matrices, where the neutrino Yukawa matrices and right-handed neutrino masses are given by products of $N \times 3$ random rectangular matrices. The mixing angle and CP phase distributions are determined by the Haar measure of $U(3)$ or $SO(3)$, depending on whether the Wishart matrices are complex or real. Interestingly, for a large value of $N \gg 3$, the eigenvalues of the Wishart matrix tend to be confined in a narrow range known as the Marchenko-Pastur distribution. As a result, compared to the case
FIG. 5: Contours of probability distribution on $m_{ee}$–$m_1$ plane for $N = 10$ (left) and $N = 30$ (right), where the mixing angles are set to be the best-fit values. The red and blue contours correspond to 68% and 95% CL respectively, and for comparison, the case of the linear measure is shown as the thin red and blue contours in the right panel. The black curves with various line type correspond to the normal hierarchy for best fit values of the neutrino mass differences and mixing angles with vanishing CP phases; $(e^{i\alpha_{21}}, e^{i(\alpha_{31}-\delta)}) = (+1, +1), (+1, -1), (-1, +1)$ and $(-1, -1)$ from top to bottom at $m_1 \gtrsim 10^{-2}$ eV. The horizontal dashed (cyan) line represent the sensitivity of the future experiment, while the shaded (magenta) region is excluded by the current experiments.

Let us briefly discuss if we can understand the structure of the couplings based on

of the neutrino mass anarchy with the linear measure, the neutrino mass spectrum becomes more compressed, in particular, quasi-degenerate (normal-ordering) neutrino mass spectrum can be easily realized without resort to introducing additional constraint (such as successful leptogenesis [17, 18]) or ad hoc choice of the weighting function. We have studied how large $N$ can be to give a reasonable fit to the observed neutrino mass squared differences and found that $N$ can be as large as 35 for complex Wishart matrices and 70 for real Wishart matrices. We have also studied implications of our scenario for the $0\nu\beta\beta$ experiment, and shown that the predicted $m_{ee}$ is likely within the reach of the future experiments especially if $N$ is on the high side of the allowed range.

Let us briefly discuss if we can understand the structure of the couplings based on
FIG. 6: Same as Fig. 5 but for real Wishart matrices. We have chosen the case of
$$(e^{i\alpha_{21}}, e^{i(\alpha_{31}-2\delta)}) = (+1, +1).$$

symmetry principles. First let us regard the random matrices $F$ and $G$ as moduli fields
whose VEVs can take various values determined by a UV theory. To be specific we
assume that all the couplings are real, and impose $O(N) \times O(3)$ flavor symmetry, under
which the ordinary leptons and right-handed neutrinos transform as $1 \times 3$ while $F$ and
$G$ transform as $N \times 3$. In this case the unit matrix can be added to the Yukawa and
the right-handed neutrino matrices, satisfying the flavor symmetry. If the contribution
of the unit matrix is negligible compared to that of $F$ and $G$, our results in the text
approximately remain unchanged in this case. On the other hand, if the unit matrix
contribution becomes significant, the mass eigenvalues become more degenerate, whereas
the mixing angle distribution is still determined by the SO(3) Haar measure.\(^8\)

So far we have focused on the neutrino mixing, mass, and CP phase distributions in
the neutrino mass anarchy with the Wishart matrices. It will be interesting to study
cosmological aspects of our scenario, especially in context with leptogenesis, as an ex-
tension of the analysis of Ref. [17]. In particular, in contrast to the case of the linear

\(^8\) This argument suggests another extension of the neutrino mass anarchy with the linear measure: one
may add a unit matrix (with a numerical coefficient) to the neutrino Yukawa and the right-handed
neutrino mass matrices, leading to degenerate mass spectra while the mixing angle and CP phase
distribution are still given by the U(3) or SO(3) Haar measure.
measure, the right-handed neutrinos tend to be degenerate in mass, leading to resonant leptogenesis [32]. The typical mass difference scales as $(M_2 - M_1)/(M_2 + M_1) \sim 1/\sqrt{N}$, and so, we expect that an enhancement of the lepton asymmetry by a factor of 5 or so for $N = 30$. If the value of $N$ is different between the neutrino Yukawa and right-handed neutrino mass matrices, this factor may be even more enhanced. We however expect that it is hard to realize the enhancement by many orders of magnitude in our scenario because the eigenvalues still repel each other even in the limit of large $N$. This difficulty may be eased by allowing a contribution proportional to the unit matrix. We leave the detailed analysis of leptogenesis in this case for future work.

As pointed out in Refs. [13, 14], one can impose a flavor symmetry without modifying the predictions for the light neutrino masses: for instance we can introduce a flavor symmetry on the right-handed neutrinos. Then, while the right-handed neutrinos are hierarchical due to the non-trivial flavor charges, the light neutrinos remain degenerate.

We can consider a possibility that the neutrino Yukawa and the right-handed neutrino mass matrices are given by a more complicated function(s) of random matrices, such as the Wishart matrices squared, and so on. Alternatively one may consider sparse random matrices. It may be interesting to study these possibilities and their implications for the neutrino masses and CP phases.

Acknowledgment

This work was supported by JSPS Grant-in-Aid for Young Scientists (B) (No.24740135 [FT]), Scientific Research (A) (No.26247042 [FT]), Scientific Research (B) (No.26287039 [FT]), the Grant-in-Aid for Scientific Research on Innovative Areas (No.23104008 [NK, FT]), and Inoue Foundation for Science [FT]. This work was also supported by World Premier International Center Initiative (WPI Program), MEXT, Japan [FT]. KSJ was
supported by IBS under the project code, IBS-R018-D1.

[1] G. Aad et al. [ATLAS Collaboration], Phys. Lett. B 716, 1 (2012) [arXiv:1207.7214 [hep-ex]].
[2] S. Chatrchyan et al. [CMS Collaboration], Phys. Lett. B 716, 30 (2012) [arXiv:1207.7235 [hep-ex]].
[3] D. V. Forero, M. Tortola and J. W. F. Valle, Phys. Rev. D 90, 093006 (2014) [arXiv:1405.7540 [hep-ph]].
[4] M. C. Gonzalez-Garcia, M. Maltoni and T. Schwetz, JHEP 1411, 052 (2014) [arXiv:1409.5439 [hep-ph]].
[5] L. J. Hall, M. P. Salem and T. Watari, Phys. Rev. D 76, 093001 (2007) [arXiv:0707.3446 [hep-ph]].
[6] F. P. An et al. [DAYA-BAY Collaboration], Phys. Rev. Lett. 108, 171803 (2012) [arXiv:1203.1669 [hep-ex]].
[7] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 107, 041801 (2011) [arXiv:1106.2822 [hep-ex]].
[8] P. Adamson et al. [MINOS Collaboration], Phys. Rev. Lett. 107, 181802 (2011) [arXiv:1108.0015 [hep-ex]].
[9] Y. Abe et al. [DOUBLE-CHOOZ Collaboration], Phys. Rev. Lett. 108, 131801 (2012) [arXiv:1112.6353 [hep-ex]].
[10] Y. Abe et al. [Double Chooz Collaboration], Phys. Rev. D 86, 052008 (2012) [arXiv:1207.6632 [hep-ex]].
[11] J. K. Ahn et al. [RENO Collaboration], arXiv:1204.0626 [hep-ex].
[12] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. (2014) [arXiv:1303.5076 [astro-ph.CO]].
[13] L. J. Hall, H. Murayama and N. Weiner, Phys. Rev. Lett. 84, 2572 (2000) [hep-ph/9911341].
[14] N. Haba and H. Murayama, Phys. Rev. D 63, 053010 (2001) [hep-ph/0009174].
[15] A. de Gouvêa and H. Murayama, Phys. Lett. B 573, 94 (2003) [hep-ph/0301050].
[16] A. de Gouvea and H. Murayama, arXiv:1204.1249 [hep-ph].
[17] K. S. Jeong and F. Takahashi, JHEP 1207, 170 (2012) [arXiv:1204.5453 [hep-ph]].
[18] X. Lu and H. Murayama, JHEP 1408, 101 (2014) [arXiv:1405.0547 [hep-ph]].
[19] B. Feldstein and W. Klem, Phys. Rev. D 85, 053007 (2012) [arXiv:1111.6690 [hep-ph]].
[20] Y. Bai and G. Torroba, JHEP 1212, 026 (2012) [arXiv:1210.2394 [hep-ph]].
[21] G. Altarelli, F. Feruglio, I. Masina and L. Merlo, JHEP 1211, 139 (2012) [arXiv:1207.0587 [hep-ph]].
[22] J. Jenkins, Phys. Rev. D 79, 113003 (2009) [arXiv:0808.1702 [hep-ph]].
[23] V. A. Marcenko and L. A. Pastur, Math. USSR-Sb 1, 457 (1967).
[24] P. Minkowski, Phys. Lett. B 67, 421 (1977).
[25] T. Yanagida, Proceedings: Workshop on the Unified Theories and the Baryon Number in the Universe, Tsukuba, Japan, 13-14 Feb, 95 (1979).
[26] P. Ramond, in a Talk given at Sanibel Symposium, Palm Coast, Fla., 25 Feb.-2 Mar. (1979), hep-ph/9809459.
[27] S. L. Glashow, NATO Sci. Ser. B 59, 687 (1980).
[28] M. Agostini et al. [GERDA Collaboration], Phys. Rev. Lett. 111, no. 12, 122503 (2013) [arXiv:1307.4720 [nucl-ex]].
[29] J. B. Albert et al. [EXO-200 Collaboration], Nature 510, 229-234 (2014) [arXiv:1402.6956 [nucl-ex]].
[30] [The KamLAND-Zen Collaboration], arXiv:1409.0077 [physics.ins-det].
[31] S. Dell’Oro, S. Marcocci and F. Vissani, Phys. Rev. D 90, 033005 (2014) [arXiv:1404.2616 [hep-ph]].
[32] A. Pilaftsis, Nucl. Phys. B 504, 61 (1997) [hep-ph/9702393]; A. Pilaftsis, Phys. Rev. D 56, 5431 (1997) [hep-ph/9707235]; A. Pilaftsis and T. E. J. Underwood, Nucl. Phys. B 692, 303 (2004) [hep-ph/0309342].