SINGULARITIES OF PLURI-FUNDAMENTAL DIVISORS ON GORENSTEIN FANO VARIETIES OF COINDEX 4

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Abstract. Let $X$ be a Gorenstein canonical Fano variety of coindex 4 and dimension $n$ with $H$ fundamental divisor. Assume $h^0(X, H) \geq n - 2$. We prove that a general element of the linear system $|mH|$ has at worst canonical singularities for any integer $m \geq 1$. When $X$ has terminal singularities and $n \geq 5$, we show that a general element of $|mH|$ has at worst terminal singularities for any integer $m \geq 1$. When $n = 4$, we give an example of Gorenstein terminal Fano fourfold $X$ such that a general element of $|H|$ does not have terminal singularities.

1. Introduction

Throughout the paper, we work over the field $\mathbb{C}$ of complex numbers. Let $X$ be a Gorenstein Fano variety of dimension $n$ with canonical singularities. The index of $X$ is

$$i_X := \max\{t \in \mathbb{Z} \mid -K_X \sim tH \text{ where } H \text{ is an ample Cartier divisor}\}.$$ 

It is well known that

$$1 \leq i_X \leq n + 1.$$ 

The coindex of $X$ is $n + 1 - i_X$. An ample Cartier divisor $H$ on $X$ with $-K_X \sim i_X H$ is called the fundamental divisor of $X$. Since Pic$(X)$ is torsion-free, $H$ is uniquely determined up to linear equivalence. It is a natural problem to study singularities of general members in pluri-fundamental linear systems $|mH|$ for all integers $m \geq 1$.

By Kobayashi–Ochiai, $X$ is a projective space if $i_X = n + 1$, and $X$ is a hyperquadric if $i_X = n$. A Gorenstein canonical Fano variety $X$ with $i_X = n - 1$ is a del Pezzo variety, and del Pezzo varieties were classified by Fujita [Ft1, Ft3]. If $X$ is a del Pezzo variety, then the base locus $B_s |H|$ is empty or consists of a single point neither in Sing $X$ nor in Sing $Y$, where $Y \in |H|$ is a general member. Thus $Y$ has canonical/terminal singularities if $X$ has canonical/terminal singularities. A Gorenstein canonical Fano variety $X$ with $i_X = n - 2$ is a Mukai variety, and smooth Mukai varieties were classified by Mukai [Mu] under the assumption that $|H|$ contains a smooth divisor. Mella [Me] verified this assumption, and moreover, he also proved that if $X$ is a Gorenstein Mukai variety with canonical/terminal singularities, then a general member in $|H|$ has canonical/terminal singularities except when $X$ is a complete intersection in $\mathbb{P}(1,1,1,1,2,3)$ of a quadric

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defined in the first four linear variables and a sextic. Finally, note that if \( i_X \geq n - 2 \), then \( |mH| \) is base point free for every integer \( m \geq 2 \) (see [L1, Remark 4.5]). We can conclude that singularities of general members in \( |mH| \) with \( m \geq 1 \) are well understood when \( i_X \geq n - 2 \).

In this paper, we consider the case \( i_X = n - 3 \), i.e., \( X \) has coindex 4. Floris [Fl] proved that a general member of the linear system \(|H|\) has canonical singularities if \( X \) is a Gorenstein canonical Fano variety of coindex 4 and \( h^0(X, H) \neq 0 \). However, in contrasts to the smaller coindex cases, there is a smooth Fano fourfold \( X \) of coindex 4 such that every member in \(|H|\) is singular (see [HV, Example 2.12]). Heuberger [H] proved that if \( X \) is a smooth Fano fourfold, then a general member in \(|-K_X|\) has only terminal singularities. This is a natural generalization of a classical result of Shokurov [S] for smooth Fano threefolds. Heuberger’s theorem together with aforementioned results implies that a general member in \(|H|\) has only terminal singularities if \( X \) is smooth.

The main result of this paper is the following.

**Theorem 1.1.** Let \( X \) be a Gorenstein canonical Fano variety of coindex 4 and dimension \( n \geq 4 \), and \( H \) be the fundamental divisor of \( X \).

1. Assume that \( h^0(X, H) \geq n - 2 \). Then a general member of the linear system \(|mH|\) has only canonical singularities for every integer \( m \geq 1 \).

2. Assume that \( X \) has terminal singularities and \( h^0(X, H) \geq n - 2 \). Then a general member of the linear system \(|mH|\) has only terminal singularities for every integer \( m \geq 1 \) unless \((n, m) = (4, 1), (4, 2), (4, 3)\).

3. Assume that \( X \) is smooth. Then \( h^0(X, H) \geq n - 2 \), and a general member of the linear system \(|mH|\) has only terminal singularities for every integer \( m \geq 1 \) unless \((n, m) = (4, 2)\).

If \( X \) is a smooth Fano variety of coindex 4 and dimension \( n \), then Floris [Fl, Theorem 1.2] and Liu [L1, Theorem 1.2] showed that \( h^0(X, H) \geq n - 2 \). If \( X \) is singular, then we do not know whether \( H^0(X, H) \neq 0 \). This nonvanishing follows from the following:

**Conjecture 1.2** (Ambro–Kawamata effective nonvanishing conjecture [A], [K2]). Let \((X, \Delta)\) be a klt pair, and \( D \) be a Cartier divisor on \( X \). If \( D \) is nef and \( D - (K_X + \Delta) \) is nef and big, then \( H^0(X, D) \neq 0 \).

This conjecture has been verified for low dimensional varieties [K2] and Fano weighted complete intersections [PST]. Especially, [K2, Proposition 4.1 and Theorem 5.2] say that if \( X \) is a Gorenstein Fano fourfold with canonical singularities, then \( h^0(X, H) \geq 2 \). Although the methods of the present paper do not yield the results for higher coindex cases directly, we may still expect that Theorem 1.1 for higher coindex would follow from the effective nonvanishing conjecture (cf. [HS]).

In Theorem 1.1 (2), when \( n = 4 \), one cannot expect that a general member in \(|H|\) has terminal singularities. We give an example of Gorenstein terminal Fano fourfold \( X \) such that a general member of the linear system \(|H|\) does not have terminal singularities.
In Theorem 1.1 (3), we do not know whether there is an example of a smooth Fano fourfold such that a general element in \(|2H|\) does not have terminal singularities. See Remark 3.5 for some partial result.

By [L2, Corollary 3], if \(X\) is a Gorenstein Fano variety of coindex 4 and dimension \(n\) with canonical singularities and \(h^0(X, H) \geq n - 2\), then \(|mH|\) is base point free for any integer \(m \geq 4\) (see Remark 2.2). In particular, if \(X\) is a smooth Fano variety of coindex 4, then a general member in \(|mH|\) is smooth for any integer \(m \geq 4\).

One may expect that if a general member in \(|H|\) has only mild singularities, then so does a general member in \(|mH|\) for any \(m \geq 2\). More generally, we may ask the following:

**Question 1.3.** Let \(X\) be a smooth projective variety, and \(L, M\) be divisors on \(X\). Suppose that general members of \(|L|\) and \(|M|\) have only canonical/terminal singularities. Then does a general member of \(|L + M|\) have also canonical/terminal singularities?

The answer is “NO” of course. Some counterexamples are given in Example 3.1.

**Organization.** The paper is organized as follows. Section 2 is devoted to proving Theorem 1.1 (1) and (2). We also give some examples of terminal Fano fourfolds in which a general member of the fundamental linear system does not have terminal singularities (see Example 2.4). In Section 3, we negatively answer Question 1.3 in Example 3.1, and we prove Theorem 1.1 (3).

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## 2. Pluri-fundamental divisors on singular Fano varieties

In this section, we prove Theorem 1.1 (1) and (2). For the definitions and basic properties of singularities of pairs, we refer to [KM]. We begin with fixing some notations. Let \(X\) be a Gorenstein Fano variety of coindex 4 and dimension \(n \geq 4\) with canonical singularities, and \(H\) be the fundamental divisor on \(X\). We have \(-K_X = (n - 3)H\). Assume that \(h^0(X, H) \geq n - 2\). Then \(|mH| \neq \emptyset\) for each integer \(m \geq 1\). Take a log resolution

\[ f_m : X_m \longrightarrow X \]

of the ideal of the base locus \(Bs \, |mH|\). We may assume that \(f_m\) is obtained by a sequence of blow-ups along smooth centers. We write

\[ K_{X_m} = f_m^*K_X + \sum_i a_{m,i}E_{m,i} \quad \text{and} \quad |f_m^*mH| = |M_m| + \sum_i r_{m,i}E_{m,i}, \]

where all \(E_{m,i}\) are prime divisors, \(|M_m|\) is the free part of \(|f_m^*mH|\), and \(\sum_i r_{m,i}E_{m,i}\) is the fixed part of \(|f_m^*mH|\). By [F, Theorem 1.1], a general member of \(|H|\) is irreducible. Since \(h^0(X, H) \geq 2\), it follows that \(\dim Bs \, |H| \leq n - 2\); thus \(\dim Bs \, |mH| \leq n - 2\) for each integer \(m \geq 1\). Hence every \(E_{m,i}\) is an \(f_m\)-exceptional divisor. Since \(H\) is a Cartier divisor, all \(a_{m,i}\) and \(r_{m,i}\) are nonnegative integers.
Lemma 2.1. \( \dim \text{Bs} |mH| \leq 2 \) for any integer \( m \geq 1 \).

Proof. By [Fl, Proposition 4.1], \((X, X_{n-1})\) is a plt (=purely log terminal) pair, where \( X_{n-1} \in |H| \) is a general member. As \( X_{n-1} \) is connected, [KM, Proposition 5.51] shows that \( X_{n-1} \) is irreducible and normal. By [KM, Theorem 5.50], \( X_{n-1} \) has Gorenstein canonical singularities. Note that \(-K_{X_{n-1}} = ((n-1) - 3)H_{n-1} \), where \( H_{n-1} := H|_{X_{n-1}} \).

If \( n \geq 5 \), then \( X_{n-1} \) is an \((n-1)\)-dimensional Gorenstein canonical Fano variety of index \( i_{X_{n-1}} \geq (n-1) - 3 \) with \( h^0(X_{n-1}, H_{n-1}) \geq (n-1) - 2 \). If \( i_{X_{n-1}} > (n-1) - 3 \), then \( |H_{n-1}| \) is base point free (cf. [L1, Remark 4.5]) so that \((X_{n-1}, X_{n-2})\) is a plt pair, where \( X_{n-2} \in |H_{n-1}| \) is a general member. If \( i_{X_{n-1}} = (n-1) - 3 \), then by [Fl, Proposition 4.1], \((X_{n-1}, X_{n-2})\) is also a plt pair. Continuing this process, we finally obtain a Calabi–Yau threefold \( X_3 \) with canonical singularities and \( h^0(X_3, H_4|_{X_3}) \geq 1 \).

Notice that
\[
\text{Bs} |H| = \text{Bs} |H_{n-1}| = \cdots = \text{Bs} |H_4| = \text{Bs} |H_4|_{X_3}.
\]
This shows \( \dim \text{Bs} |H| \leq 2 \). Note that \( \text{Bs} |mH| \subseteq \text{Bs} |H| \) for any \( m \geq 2 \). Then the lemma follows. \( \square \)

Remark 2.2. (1) If the Ambro–Kawamata effective nonvanishing conjecture is true for Gorenstein Fano variety of coindex 4 with canonical singularities, then [Fl, Proposition 4.1] and the “ladder” argument as in the proof of Lemma 2.1 show that \( h^0(X, H) \geq n-2 \).

(2) If \( X \) is a Gorenstein Fano variety of coindex 4 with canonical singularities and \( h^0(X, H) \geq n-2 \), then the “ladder” argument and [L2, Theorem 2] show that \( |mH| \) is base point free for every integer \( m \geq 4 \).

The following proposition, inspired by [H, Proposition 9], is the key ingredient of the proof of Theorem 1.1.

Proposition 2.3. For an integer \( m \geq 1 \), we have
\[
a_{m, i} \geq \frac{m}{m+n-3} r_{m,i} - 1 \quad \text{for all } i.
\]

Proof. Suppose that \( a_{m,i} - \frac{m+n-3}{m} r_{m,i} < -1 \) for some \( i \). Let
\[
c_0 := \inf \{ c \mid a_{m,i} - cr_{m,i} \leq -1 \text{ for some } i \}.
\]
Then \( 0 < c_0 < \frac{m+n-3}{m} \). For an integer \( k > c_0 \), choose \( k \) general members \( D_1, \ldots, D_k \in |mH| \), and let \( \Delta := c_0 \cdot \frac{D_1 + \cdots + D_k}{k} \). Then the pair \((X, \Delta)\) is lc (=log canonical) but not klt (=Kawamata log terminal). Let \( W \) be a minimal lc center of the lc pair \((X, \Delta)\). Since \( D_1, \ldots, D_k \in |mH| \) are general, \((X, \Delta)\) is klt outside the base locus \( \text{Bs} |mH| \) (cf. [A, Lemma 5.1]). Thus \( W \) is contained in \( \text{Bs} |mH| \), so \( \dim W \leq 2 \) by Lemma 2.1.

By the generalization of Kawamata’s subadjunction formula [FG, Theorem 1.2], there exists an effective divisor \( \Gamma \) on \( W \) such that
\[
(K_X + \Delta)|_W \sim_\mathbb{Q} K_W + \Gamma
\]
and the pair \((W, \Gamma)\) is klt. Note that
\[
mH - (K_X + \Delta) \sim_\mathbb{Q} (m+n-3 - c_0 m)H.
\]
Since \( c_0 m < m + n - 3 \), it follows that \( m H - (K_X + \Delta) \) is ample. Then \( m H|_W - (K_W + \Gamma) \) is ample. Recall that \( \dim W \leq 2 \). By [K2, Theorem 3.1], \( H^0(W, m H|_W) \neq 0 \). Now, since \( m H - (K_X + \Delta) \) is ample and \( W \) is an lc center of the lc pair \((X, \Delta)\), we can apply [F, Theorem 2.2] to see that the restriction map

\[
H^0(X, m H) \longrightarrow H^0(W, m H|_W)
\]

is surjective. However, \( W \subseteq \text{Bs}|m H| \), so this restriction map is the zero map. We obtain \( H^0(W, m H|_W) = 0 \), which is a contradiction. Thus the proposition holds. \( \square \)

We are ready to prove Theorem 1.1 (1) and (2).

Proof of Theorem 1.1 (1) and (2). Recall that \( X \) is a Gorenstein Fano variety of coindex 4 and dimension \( n \) with canonical singularities. We assume that \( h^0(X, H) \geq n - 2 \). Let \( Y_m \in |m H| \) be a general element for an integer \( m \geq 1 \).

(1) We want to prove that \( Y_m \) has canonical singularities. If \((X, Y_m)\) is a plt pair, then [KM, Theorem 5.50 and Proposition 5.51] imply that \( Y_m \) has canonical singularities since \( Y_m \) is Gorenstein. Thus it is enough to show that the pair \((X, Y_m)\) is plt. The birational morphism \( f_m : X_m \rightarrow X \) is a log resolution of \((X, Y_m)\). We have

\[
K_{X_m} + f_m^{-1} Y_m = f_m^* (K_X + Y_m) + \sum_i (a_{m,i} - r_{m,i}) E_{m,i}.
\]

If \( r_{m,i} = 0 \), then \( a_{m,i} - r_{m,i} \geq 0 > -1 \). If \( r_{m,i} \geq 1 \), then Proposition 2.3 implies that

\[
a_{m,i} - r_{m,i} \geq \frac{n - 3}{m} r_{m,i} - 1 > -1.
\]

Thus \((X, Y_m)\) is a plt pair.

(2) Assume that \( X \) has terminal singularities and \( n = \dim X \geq 5 \) or \( m \geq 4 \). We want to show that \( Y_m \) has terminal singularities. If \( m \geq 4 \), then [L2, Theorem 2] (see also Remark 2.2 (2)) implies that \(|m H|\) is base point free; hence \( Y_m \) has terminal singularities. From now on, assume that \( n \geq 5 \) and \( 1 \leq m \leq 3 \). We know that \( Y_m \) is a normal projective variety with canonical singularities. Let \( Y'_m := f_m^{-1} Y_m \) be the strict transform of \( Y_m \) under \( f_m \). Since \( Y'_m \in |M_m| \) is a general element, \( Y'_m \) is smooth. Then

\[
f'_m := f_m|_{Y'_m} : Y'_m \longrightarrow Y_m
\]

is a log resolution of \( Y_m \). We have

\[
K_{Y'_m} = f'_m^* K_{Y_m} + \sum_i (a_{m,i} - r_{m,i}) E_{m,i}|_{Y'_m}.
\]

Since \( X \) has terminal singularities, we have \( a_{m,i} \geq 1 \) for all \( i \).

Consider the case \( m = 1 \). Note that \( \frac{n + n - 3}{m} = n - 2 \geq 3 \). If \( r_{1,i} \geq 1 \), then Proposition 2.3 implies that \( a_{1,i} - r_{1,i} \geq 2 r_{1,i} - 1 > 0 \). If \( r_{1,i} = 0 \), then \( a_{1,i} - r_{1,i} > 0 \). Thus \( Y_1 \) has terminal singularities.

Suppose now that \( Y_m \) does not have terminal singularities for some \( 2 \leq m \leq 3 \). Then there is some \( i_0 \) such that \( a_{m,i_0} = r_{m,i_0} \geq 1 \) and \( E_{m,i_0}|_{Y'_m} \) is an \( f_m|_{Y'_m} \)-exceptional divisor.
Since $Y_m$ has terminal singularities outside $Bs|mH|$, we see that $f_m(E_{m,i_0}) \subseteq Bs|mH|$. Since $n \geq 5$ and $2 \leq m \leq 3$, we have $m+3-n \geq \frac{5}{3}$. If $r_{m,i} \geq 2$, then Proposition 2.3 implies that $a_{m,i} - r_{m,i} \geq \frac{5}{3}r_{m,i} - 1 > 0$. Thus $a_{m,i_0} = r_{m,i_0} = 1$. If $f_m(E_{m,i_0}) \not\subseteq \text{Sing} X$, then $\dim f_m(E_{m,i_0}) = n - 2$ since $f_m$ is a composition of smooth center blow-ups. This means that $f_m(E_{m,i_0})$ is a divisor on $Y_m$ and $E_{m,i_0}|_{Y_m}$ is not an $f_m|_{Y_m}$-exceptional divisor. Thus $f_m(E_{m,i_0}) \subseteq X$, and $\dim f_m(E_{m,i_0}) \leq n - 3$ because $X$ has terminal singularities. By taking further blow-ups, we may assume that $f_1 = f_m$ and $X_1 = X_m$. Then there is an $i_1$ such that $E_{1,i_1} = E_{m,i_0}$. We have $a_{1,i_1} = a_{m,i_0} = 1$. Now, since $f_m(E_{m,i_0}) \subseteq Bs|mH| \subseteq Bs|H|$, it follows that $r_{1,i_1} \geq 1$. Thus $a_{1,i_1} - r_{1,i_1} \leq 0$. Note that $E_{1,i_1}|_{Y_1}$ is an $f_1|_{Y_1}$-exceptional divisor. We get a contradiction to that $Y_1$ has terminal singularities. Hence $Y_m$ has terminal singularities for any $2 \leq m \leq 3$.

Finally, we provide some examples of terminal Fano fourfolds in which a general element in the fundamental linear system does not have terminal singularities.

**Example 2.4.** (1) Let $Z := X_{2,6}$ be a complete intersection in $\mathbb{P}(1,1,1,1,2,3)$ of a general quadric defined in the first four linear variables $x_0, \ldots, x_3$ and a general sextic (cf. [Me, Theorem 1]). Then $Z$ is a Gorenstein terminal Fano threefold of index 1, and $\text{Sing} Z = \{p = (0 : 0 : 0 : 0 : -1 : 1)\}$. A general member of $|H_Z|$ is singular at $p$, where $H_Z = -K_Z$ is the fundamental divisor of $Z$. Let $X := Z \times \mathbb{P}^1$ so that $X$ is a Gorenstein Fano fourfold of index 1 with terminal singularities. Note that $H = -K_X = \pi_1^*(-K_Z) + \pi_2^*(-K_{\mathbb{P}^1})$ is the fundamental divisor of $X$, where $\pi_1 : X \to Z$ and $\pi_2 : X \to \mathbb{P}^1$ are projections. A general element $Y$ in $|H|$ has one dimensional singular locus $\{p\} \times \mathbb{P}^1$. Since $\dim Y = 3$, it follows that $Y$ does not have terminal singularities. Here $Y$ is a Gorenstein Calabi–Yau threefold with canonical singularities.

(2) Let $X := X_9$ be a weighted hypersurface in $\mathbb{P}(1,1,1,1,3,3)$ of degree 9 (cf. quasi-smooth Fano 4-fold hypersurfaces ID 8 in [GRDB] based on [BK]). Then $X$ is a non-Gorenstein $Q$-Fano fourfold with terminal singularities such that $-K_X$ is a hyperplane with $(-K_X)^4 = 1$. Note that $\text{Sing} X$ consists of three terminal singular points of the type $\frac{1}{3}(1,1,1,1)$. A general element in $|-K_X|$ is a weighted hypersurface $S_9$ in $\mathbb{P}(1,1,1,3,3)$ of degree 9, and $S_9$ is a Gorenstein canonical Calabi–Yau threefold. Note that $\text{Sing} S_9$ consists of three (non-terminal) canonical singular points of the type $\frac{1}{3}(1,1,1,1)$.

### 3. Pluri-fundamental divisors on smooth Fano varieties

In this section, we first answer Question 1.3 by constructing smooth projective varieties $X$ and divisors $M$ such that general members in $|M|$ are smooth but all members in $|mM|$ are not normal for some $m \geq 2$, and then prove Theorem 1.1 (3).

**Example 3.1.** (1) If $E$ is an exceptional divisor on a smooth projective variety, then $|mE| = \{mE\}$ for all $m \geq 1$. Now, $E$ is smooth, but $mE$ is non-reduced for any $m \geq 2$.

(2) Let $C$ be a smooth projective curve of genus 2. There are two distinct points $P, Q$ on $C$ such that $2P \sim 2Q \sim K_C$. In particular, $Q - P \in \text{Pic}^0(C)$ is a 2-torsion. We can
also find $\tau \in \Pic^0(C)$ such that $H^0(C, P + \tau) = H^0(C, Q - P + \tau) = H^0(C, 2\tau) = 0$. Let $E := O_C(P) \oplus O_C(\tau)$, and $S := \mathbb{P}(E)$ with the natural projection $\pi: S \to C$ and the tautological divisor $H$, i.e., $O_S(H) = O_{\mathbb{P}(E)}(1)$. Let $A := H$ and $B := H + \pi^*(Q - P)$. Then $A, B$ are sections of $\pi$, so they are smooth irreducible curves isomorphic to $C$. Furthermore, $A, B$ satisfy the following:

- $A^2 = B^2 = A.B = 1$,
- $A \not\sim B$ but $2A \sim 2B$,
- $h^0(S, A) = h^0(S, B) = 1$, and
- $h^0(S, 2A) = h^0(S, 2B) = 2$.

Notice that $A, B$ meet at one point $p$ on $S$ and every member of $|2A| = |2B|$ has multiplicity at least 2 at $p$. Thus every member in $|2A| = |2B|$ is not normal.

(3) [PST, Example 5.9] For an integer $m \geq 1$, let

$$X = X_{(2m+1)(2m+2)} \subseteq \mathbb{P}(\frac{1, \ldots, 1}{1+2m(2m+1)}, 2m + 1, 2m + 2)$$

be a weighted hypersurface of degree $(2m+1)(2m+2)$. Then $X$ is a smooth Fano variety of index 2. If $H$ is the fundamental divisor of $X$, then a general member of $|H|$ is smooth. However, $|−2iH|$ does not contain a smooth member for any $1 \leq i \leq m$. In this case, a general member in $|−2iH|$ has terminal singularities.

We now turn to the proof of Theorem 1.1 (3).

**Proof of Theorem 1.1 (3) except the case $(n, m) = (4, 3)$**. Let $X$ be a smooth Fano variety of coindex 4 and dimension $n$ with fundamental divisor $H$. By Theorem 1.1 (2), we only have to consider the cases $(n, m) = (4, 1), (4, 3)$. If $(n, m) = (4, 1)$, then $H = -K_X$. Now, [H, Theorem 2] says that a general element in $|−K_X|$ has terminal singularities. \(\Box\)

**Remark 3.2**. Let $X$ be a smooth Fano variety of coindex 4, and $H$ be the fundamental divisor of $X$. By [L2, Theorem 4], $|mH|$ is base point free for any integer $m \geq 4$; hence a general element $Y_m \in |mH|$ is smooth in this case. But there is a smooth Fano fourfold $X$ of coindex 4 such that every member in $|H|$ is singular (see [HV, Example 2.12]).

To finish the proof of Theorem 1.1, it only remains to prove that if $H$ is the fundamental divisor of a smooth Fano fourfold $X$ of coindex 4, then a general element $Y \in |3H|$ has terminal singularities. We know that $Y$ has canonical singularities. As in Section 2, take a log resolution $f: X_3 \to X$ of the ideal of the base locus $Bs|3H|$. We may assume that $f$ is isomorphic outside $Bs|3H|$ and it is obtained by a sequence of blow-ups along smooth centers. We write

$$K_{X_3} = f^*K_X + \sum a_iE_i \quad \text{and} \quad |f^*3H| = |M| + \sum r_iE_i,$$

where all $E_i$ are $f$-exceptional prime divisors and $|M|$ is the free part of $|f^*3H|$ and $\sum r_iE_i$ is the fixed part of $|f^*3H|$. We may assume that $f(E_i) \subseteq Bs|3H|$ for all $i$. All $a_i$ and $r_i$ are positive integers.
Lemma 3.3. If a general element $Y$ in $|3H|$ has at worst isolated singularity at $x$ and \( \text{mult}_x Y \leq 2 \), then $Y$ has terminal singularity at $x$.

Proof. We may assume that $f$ factors through the blow-up of $X$ at $x$ with exceptional divisor $E_{i_{\alpha}}$. We have $d_{i_{\alpha}} = 3$ and $r_{i_{\alpha}} \leq 2$, so $a_{i_{\alpha}} - r_{i_{\alpha}} > 0$. For every $f$-exceptional divisor $E_i$ with $f(E_i) = \{x\}$ but $E_i \neq E_{i_{\alpha}}$, we have $a_i \geq 4$ since $f$ is a composition of smooth center blow-ups. It is impossible that $a_i = r_i \geq 4$ because Proposition 2.3 says that $a_i \geq \frac{4}{3}r_i - 1 > r_i$ when $r_i \geq 4$. Thus $a_i - r_i > 0$, and hence, $Y$ has terminal singularity at $x$. \( \square \)

Lemma 3.4. \( \dim \text{Bs} \, |mH| \leq 1 \) for any integer $m \geq 2$. In particular, \( \dim \text{Sing} \, Y \leq 1 \).

Proof. Suppose that $\dim \text{Bs} \, |mH| \geq 2$ for some integer $m \geq 2$. By Lemma 2.1, we have $\dim \text{Bs} \, |mH| = 2$, so there is an irreducible surface $S \subseteq \text{Bs} \, |mH| \subseteq \text{Bs} \, |H|$. Now, take two general elements $D_1, D_2 \in |H|$. By Proposition 2.3, $(X, D_1 + D_2)$ is an lc pair, and $S$ is an lc center of $(X, D_1 + D_2)$. There is a minimal lc center $C$ of $(X, D_1 + D_2)$ contained in $S$. By [FG, Theorem 1.2], there is an effective divisor $\Gamma$ on $C$ such that

\[
(K_X + D_1 + D_2)|_C \sim_K K_C + \Gamma
\]

and $(C, \Gamma)$ is a klt pair. By [K2, Theorem 3.1], $H^0(C, mH|_C) \neq 0$ since $mH - (K_X + D_1 + D_2) \sim (m - 1)H$ is ample. Now, by [Fj, Theorem 2.2], the restriction map

\[
H^0(X, mH) \rightarrow H^0(S, mH|_S)
\]

is surjective. However, $S \subseteq \text{Bs} \, |mH|$, so this restriction map is the zero map. We get a contradiction. Therefore, $\dim \text{Bs} \, |mH| \leq 1$ for any integer $m \geq 2$. Now, since $\text{Sing} \, Y \subseteq \text{Bs} \, |3H|$, it follows that $\dim \text{Sing} \, Y \leq 1$. \( \square \)

Proof of Theorem 1.1 (3) for the case $(n, m) = (4, 3)$. We want to prove that a general element $Y \in |3H|$ has terminal singularities. Note that $H = -K_X$ and $\dim \text{Bs} \, |3H| \leq 1$ by Lemma 3.4. We know that $h^0(X, H) \geq 2$.

First, assume that $H^4 \geq 2$. Take a general element $Z \in |H|$, which is a Gorenstein Calabi–Yau threefold with terminal singularities. Suppose that $\dim \text{Bs} \, |3H| = 1$. Then $Z$ is nonsingular at a general point $x$ in $\text{Bs} \, |3H|$. By [K1, Theorem 3.1], $|3H|_{Z}$ is base point free at $x$. But $x \in \text{Bs} \, |3H| = \text{Bs} \, |3H|_{Z}$, so we get a contradiction. Thus $\dim \text{Bs} \, |3H| \leq 0$. Suppose that $Y$ has non-terminal singularity at $x$. By Lemma 3.3, $\text{mult}_x Y \geq 3$. Now, by Proposition 2.3, $(X, Z + Y)$ is an lc pair. Thus $\text{mult}_x Z = 1$, so $Z$ is nonsingular at $x$. By [K1, Theorem 3.1], $|3H|_{Z}$ is base point free at $x$, so we get a contradiction as before. Hence $Y$ has at worst terminal singularities.

Next, assume that $H^4 = 1$. The sectional genus of the polarized pair $(X, H)$ is

\[
g(X, H) = \frac{(K_X + 3H).H}{2} + 1 = 2.
\]

By Fujita’s classification [Ft2, Proposition C], we have $2 \leq h^0(X, H) \leq 4$, and the following hold:

- $h^0(X, H) = 4 \iff X = X_{10} \subseteq \mathbb{P}(1, 1, 1, 1, 2, 5)$ is a hypersurface of degree 10.
- $h^0(X, H) = 3 \iff X = X_{6,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 3)$ is a complete intersection of type (6,6).
If $h^0(X, H) = 4$, then $\mathrm{Bs} \, |H| = \mathrm{Bs} \, |3H| = \{ x \}$ and $|2H|$ is base point free. In this case, mult$_x Y = 1$, so $Y$ is smooth. If $h^0(X, H) = 3$, then $|3H|$ is base point free so that $Y$ is smooth. We now suppose that $h^0(X, H) = 2$. By Riemann–Roch formula, we have
\[
h^0(X, mH) = \frac{m^2(m+1)^2}{24} H^4 + \frac{m(m+1)}{24} H^2 c_2(X) + 1.
\]
Then $H^2 c_2(X) = 10$, and $h^0(X, 2H) = 5$, $h^0(X, 3H) = 12$. Let $Z_1, Z_2 \in |H|$ and $W \in |2H|$ be general members. Then $S := Z_1 \cap Z_2$ is an irreducible Gorenstein surface with $K_S = H|_S$, and $C := Z_1 \cap Z_2 \cap W$ is a Gorenstein curve with $K_C = 3H|_C$. We have $H^i(X, \ell H) = 0$ for $1 \leq i \leq 3$ and $\ell \in \mathbb{Z}$, so we get
\[
\begin{align*}
h^0(Z_1, H|_{Z_1}) &= 1, \quad h^0(Z_1, 2H|_{Z_1}) = 3, \quad h^0(Z_1, 3H|_{Z_1}) = 7, \\
h^0(S, H|_S) &= 0, \quad h^0(S, 2H|_S) = 2, \quad h^0(S, 3H|_S) = 4, \\
h^0(C, H|_C) &= 0, \quad h^0(C, 2H|_C) = 1, \quad h^0(C, 3H|_C) = 4.
\end{align*}
\]
Thus $p_a(C) = h^0(C, 3H|_C) = 4$. As $C.H|_S = 2$, we see that $C$ has at most two irreducible components. If $C$ is non-reduced, then $C = 2H'$ for some $H' \in |H|_S$. However, since $h^0(S, H|_S) = 0$, it follows that $C$ is reduced.

Suppose that there is an irreducible curve $A$ on $X$ with $A \subseteq \mathrm{Bs} \, |2H| \cap \mathrm{Bs} \, |3H|$. Since $h^0(S, C) = h^0(S, 2H|_S) = 2$, it follows that $C$ has two irreducible components. We write $C = A + B$ on $S$. Since the restriction map $H^0(X, 3H) \to H^0(C, K_C)$ is surjective, we have $A \subseteq \mathrm{Bs} \, |K_C|$. Note that
\[
\deg_A(K_C) = \deg_B(K_C) = 3
\]
since
\[
\deg_A(H|_C) = A.H = 1 \quad \text{and} \quad \deg_B(H|_C) = B.H = 1.
\]
By [FT, Definition 2.1 and Formula (3)], we have
\[
4 = p_a(C) = p_a(A) + p_a(B) + A \cdot B - 1,
\]
where
\[
A \cdot B := \deg_A(K_C) - 2p_a(A) + 2 = \deg_B(K_C) - 2p_a(B) + 2.
\]
If $A \cdot B \geq 2$, then $C$ is numerically 2-connected in the sense of [CFHR, Definition 3.1]. In this case, by [CFHR, Theorem 3.3], $|K_C|$ is base point free, so we get a contradiction to that $A \subseteq \mathrm{Bs} \, |K_C|$. Thus $A \cdot B = 1$, and then, $p_a(A) = p_a(B) = 2$. Consider an exact sequence
\[
0 \to \omega_B \to \omega_C \to \omega_C|_A \to 0,
\]
which induces the following exact sequence
\[
0 \to H^0(B, K_B) \to H^0(C, K_C) \to H^0(A, K_C|_A)
\]
Then $h^0(A, K_C|_A) \geq 2$, which is a contradiction to that $A \subseteq \mathrm{Bs} \, |K_C|$. Thus we obtain
\[
\dim \mathrm{Bs} \, |2H| \cap \mathrm{Bs} \, |3H| \leq 0.
\]

\footnote{It is unknown whether there is a smooth Fano fourfold $X$ with $h^0(X, -K_X) = 2$ (cf. [L2, Question 5]).}
Recall that $Z_1$ is a Gorenstein Calabi–Yau threefold with terminal singularities. Then $\dim \operatorname{Sing} Z_1 \leq 0$. If $Y$ is singular along a curve $D$, then $D \subset \operatorname{Bs} |3H|$ and $D \not\subset \operatorname{Bs} |2H|$. For a general point $x \in D$, we have $\operatorname{mult}_x |H| = 1$ and $\operatorname{mult}_x |2H| = 0$, so $\operatorname{mult}_x Y = 1$ by the upper semicontinuity of the multiplicity. We get a contradiction because $Y$ is singular at $x$. This means that $Y$ cannot be singular along a curve. By Bertini’s theorem, we see that $\operatorname{mult}_x Y \leq 2$ for all $x \in \operatorname{Bs} |2H| \cup \operatorname{Sing} Z_1$. Now, suppose that $Y$ has an isolated non-terminal singular point $x$. By Lemma 3.3, $\operatorname{mult}_x Y \geq 3$, which implies that $x \in \operatorname{Bs} |2H| \cup \operatorname{Sing} Z_1$. Note that every general element $Y' \in |3H|$ has $\operatorname{mult}_x Y' \geq 3$. By Proposition 2.3, $(X, Y + Z_1)$ is an lc pair, so $\operatorname{mult}_x Z_1 = 1$. If $\operatorname{mult}_x W = 1$, then the upper semicontinuity of the multiplicity shows that $\operatorname{mult}_x Y' \leq 2$, which is a contradiction. Thus $\operatorname{mult}_x W \geq 2$. Now, $\dim \operatorname{Bs} |2H| \cap \operatorname{Bs} |3H| \leq 0$ implies that $C \cap Y'$ has dimension zero. Notice that $C \cap Y' = Z_1 \cap Z_2 \cap W \cap Y'$ has length 6, and recall that $\operatorname{mult}_x W \geq 2$ and $\operatorname{mult}_x Y' \geq 3$. Hence $C \cap Y'$ is indeed supported at a single point $x$. Since $H^0(X, 3H) \to H^0(C, 3H|C)$ is surjective, every element in $|3H|_C$ has a single support $x$. But this is impossible since $h^0(C, 3H|C) \geq 2$. We can conclude that $Y$ has at worst terminal singularities. 

\textbf{Remark 3.5.} Let $X$ be a Fano fourfold of coindex 4 with fundamental divisor $H = -K_X$. Suppose that $H^4 \geq 4$, $H^2.S \geq 3$ for every irreducible surface $S$, and $H^3.C \geq 2$ for every irreducible curve $C$. Take a general element $Z \in |H|$. By [K2, Theorem 3.1], $|2H|_Z$ is base point free at every nonsingular point in $Z$. This implies that $\dim \operatorname{Bs} |2H| \leq 0$. In this case, we can easily show that a general member in $|2H|$ has terminal singularities.

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