INFINITESIMAL DEFORMATIONS OF RESTRICTED SIMPLE LIE ALGEBRAS II

FILIPPO VIVIANI

Abstract. We compute the infinitesimal deformations of two families of restricted simple modular Lie algebras of Cartan-type: the Contact and the Hamiltonian Lie algebras.

1. Introduction

Simple Lie algebras over an algebraically closed field of positive characteristic different from 2 and 3 were classified by Wilson-Block (see [BW88]) in the restricted case and by Strade (see [STR98]) and Premet-Strade (see [PS01]) in the general case. The classification remains still open in characteristic 2 and 3 (see [STR04, page 209]).

According to this classification, simple modular (that is over a field of positive characteristic) Lie algebras are divided into two big families, called classical-type and Cartan-type algebras. The algebras of classical-type are obtained by the simple Lie algebras in characteristic zero by first taking a model over the integers (via Chevalley bases) and then reducing modulo $p$ (see [SEL67]). The algebras of Cartan-type were constructed by Kostrikin-Shafarevich in 1966 (see [KS66]) as finite-dimensional analogues of the infinite-dimensional complex simple Lie algebras, which occurred in Cartan’s classification of Lie pseudogroups, and are divided into four families, called Witt-Jacobson, Special, Hamiltonian and Contact algebras. The Witt-Jacobson Lie algebras are derivation algebras of truncated divided power algebras and the remaining three families are the subalgebras of derivations fixing a volume form, a Hamiltonian form and a contact form, respectively. Moreover in characteristic 5 there is one exceptional simple modular Lie algebra called the Melikian algebra (introduced in [MEL80]).

A particular important class of simple modular Lie algebras are the ones which are restricted. These can be characterized as those modular Lie algebras such that the $p$-power of an inner derivation (which in characteristic $p$ is a derivation) is still inner (see [FSSS] or [STR04]). Important examples of restricted Lie algebras are the ones coming from groups schemes. Indeed, there is a bijection between restricted Lie algebras over $k$ and infinitesimal $k$-group schemes of height one (see [DG70 Chap. 2]).

This paper is devoted to the study of the infinitesimal deformations of the restricted simple Lie algebras. The simple Lie algebras of classical type are known to be rigid over a field of characteristic different from 2 and 3 (see [RUD71]), in analogy of what happens in characteristic zero. In the papers [VIV1] and [VIV2], the author computed the infinitesimal deformations of the Witt-Jacobson, Special
and Melikian restricted simple Lie algebras. In this paper, we compute the infinitesimal deformations of the Contact algebras $K(n)$ and the Hamiltonian algebras $H(n)$ over a field $F$ of characteristic different from 2 and 3.

By standard facts of deformation theory, the infinitesimal deformations of a Lie algebra are parametrized by the second cohomology of the Lie algebra with values in the adjoint representation (see for example [GER64]).

Before stating the main results of this paper, we recall that there is a canonical way to produce 2-cocycles in $Z^2(\mathfrak{g}, \mathfrak{g})$ for a modular Lie algebra $\mathfrak{g}$ over a field of characteristic $p > 0$, namely the squaring operation (see [GER64]). Given an element $\gamma \in \mathfrak{g}$, one defines the squaring of $\gamma$ to be

\begin{equation}
\text{Sq}(\gamma)(x, y) = \sum_{i=1}^{p-1} \frac{[\text{ad}(\gamma)]^i(x), [\text{ad}(\gamma)]^{p-i}(y)}{i!(p-i)!} \in Z^2(\mathfrak{g}, \mathfrak{g})
\end{equation}

where $\text{ad}(\gamma)^i$ is the $i$-iteration of the inner derivation $\text{ad}(\gamma)$.

Assuming the (standard) notations from sections 2.1 and 3.1 about the Contact algebras $K$ and Melikian restricted simple Lie algebras. In this paper, we compute the infinitesimal deformations of the Contact algebras $K(n)$ and the Hamiltonian algebras $H(n)$, we can state the main results of this paper.

**Theorem 1.1.** Let $n = 2m + 1 \geq 3$. Then

$$H^2(K(n), K(n)) = \bigoplus_{i=1}^{2m} (\text{Sq}(x_i))_F \oplus (\text{Sq}(1))_F.$$ 

**Theorem 1.2.** Let $n = 2m \geq 2$. Then if $n \geq 4$ we have that

$$H^2(H(n), H(n)) = \bigoplus_{i=1}^{n} (\text{Sq}(x_i))_F \bigoplus_{i<j}^{m} (\Pi_{ij})_F \bigoplus_{i=1}^{m} (\Pi_i)_F \bigoplus (\Phi)_F,$$

where the above cocycles are defined (and vanish outside) by

$$\Pi_{ij}(x^a, x^b) = x^a x_j^{p-1} [D_i(x^a)D_j(x^b) - D_i(x^b)D_j(x^a)] \quad \text{for } j \neq i, i',$$

$$\Pi_i(x^a, x^b) = x^a x^b - (p-1)_{i-1} + (p-1)_{i+1} x_i' x_i$$

if $a + b < \sigma_i$,

$$\Phi(x^a, x^b) = \sum_{\alpha < \delta < b} \left( \begin{array}{c} a \\ \delta \end{array} \right) \left( \begin{array}{c} b \\ \delta \end{array} \right) \sigma(\delta) \delta! x^{a+b-\delta}.$$ 

If $n = 2$ then

$$H^2(H(2), H(2)) = \bigoplus_{i=1}^{2} (\text{Sq}(x_i))_F \bigoplus (\Phi)_F.$$ 

In a forthcoming paper [VIV3], we use the above computations to determine the infinitesimal deformations of the simple finite group schemes in positive characteristic associated to the restricted simple Lie algebras.

The result presented here constitute part of my doctoral thesis. I thank my advisor prof. Schoof for useful advices and constant encouragement. I thank the referee for the suggestion of using the results of [FAR86] and [FS91] in order to simply some of the proofs of this paper.

2. **Contact algebra**

2.1. **Definition and Basic properties.** We first introduce some notations about the set $\mathbb{N}^n$ of $n$-tuple of natural numbers. We consider the order relation defined by $a = (a_1, \ldots, a_n) < b = (b_1, \ldots, b_n)$ if $a_i < b_i$ for every $i = 1, \ldots, n$. We define
the degree of \( a \in \mathbb{N}^n \) as \( |a| = \sum_{i=1}^{n} a_i \) and the factorial as \( a! = \prod_{i=1}^{n} a_i! \). For two multindex \( a, b \in \mathbb{N}^n \) such that \( b \leq a \), we set \( (a)_b := \prod_{i=1}^{n} (a_i)!_{(a-b)_i} \). For every integer \( j \in \{1, \ldots, n\} \) we call \( e_j \) the \( n \)-tuple having \( 1 \) at the \( j \)-th entry and \( 0 \) outside.

Throughout this section we fix a field \( F \) of characteristic \( p \neq 2,3 \) and an odd integer \( n = 2m + 1 \geq 3 \). For any \( j \in \{1, \ldots, 2m\} \), we define the sign \( \sigma(j) \) and the conjugate \( j' \) of \( j \) as follows:

\[
\sigma(j) = \begin{cases} 
1 & \text{if } 1 \leq j \leq m, \\
-1 & \text{if } m < j \leq 2m,
\end{cases}
\quad j' = \begin{cases} 
 j + m & \text{if } 1 \leq j \leq m, \\
 j - m & \text{if } m < j \leq 2m.
\end{cases}
\]

Given a multindex \( \hat{a} = (a_1, \ldots, a_{2m}) \in \mathbb{N}^{2m} \), we define the sign of \( \hat{a} \) as \( \sigma(\hat{a}) = \prod \sigma(a_i) \) and the conjugate of \( \hat{a} \) as the multindex \( \hat{a}' \) such that \( \hat{a}_i = a_{\sigma(i)} \) for every \( 1 \leq i \leq 2m \). We are going to use often the following special \( n \)-tuples: \( \mathbf{2} := (0, \ldots, 0) \), \( \tau := (p-1, \ldots, p-1) \) and \( \sigma := (p-1, \ldots, p-1) \).

Let \( A(n) = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \) the ring of \( p \)-truncated polynomials in \( n \)-variables. Note that \( A(n) \) is a finite \( F \)-algebra of dimension \( p^n \) with a basis given by the monomials \( \{x^a = x_1^{a_1} \cdots x_n^{a_n} \mid a \in \mathbb{N}^n, a \leq \tau\} \).

Consider the operator \( D_H : (\gamma) : W(n) \to F[n] : D_H(A(n)) \) defined as

\[
D_H(f) = \sum_{j=1}^{2m} \sigma(j) D_j(f) D_{j'} = \sum_{i=1}^{m} [D_i(f) D_{i+m} - D_{i+m}(f) D_i],
\]

where, as usual, \( D_i := \frac{\partial}{\partial x_i} \in W(n) \).

We denote with \( K'(n) \) the graded Lie algebra over \( F \) whose underlying \( F \)-vector space is \( A(n) = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \) endowed with the grading defined by \( \deg(x^a) = |a| + a_n - 2 \) and with the Lie bracket defined by

\[
[x^a, x^b]_H = D_H(x^a)(x^b) + [a_n \deg(x^b) - b_n \deg(x^a)] x^{a+b-c_n}.
\]

**Definition 2.1.** The Contact algebra is the derived subalgebra of \( K'(n) \):

\[
K(n) := K'(n)\langle 1 \rangle = [K'(n), K'(n)].
\]

We need the following characterization of \( K(n) \) (see [FSSS, Chap. 4, Theo. 5.5]).

**Proposition 2.2.** Denote with \( K'(n)_{<\tau} \) the sub-vector space of \( K'(n) \) generated over \( F \) by the monomials \( x^a \) such that \( a < \tau \). Then

\[
K(n) = \begin{cases} 
K'(n) & \text{if } p \not| (m+2), \\
K'(n)_{<\tau} & \text{if } p \mid (m+2).
\end{cases}
\]

We can describe explicitly the low degree terms of \( K(n) \) together with their adjoint action. The negative graded pieces of \( K(n) \) are \( K(n)_{-2} = \langle 1 \rangle_F \) whose adjoint action is like the action of \( 2D_n \) on \( A(n) \) and \( K(n)_{-1} = \oplus_{i=1}^{2m} \langle x_i \rangle_F \) where the adjoint action of \( x_i \) is like \( \sigma(i) D_{x_i} + x_i D_n \). The piece \( K(n)_0 \) of degree 0 is generated by the central element \( x_n \) whose adjoint action is given by \( [x_n, x^a] = \deg(x^a)x^n \) and by \( x_i x_j \) (with \( 1 \leq i, j \leq 2m \)) whose adjoint action is like \( \sigma(i)x_j D_{x_i} + \sigma(j)x_i D_{x_j} \). Hence \( K(n)_0 \cong \mathfrak{sp}(2m, F) \oplus \langle x_n \rangle_F \).

The algebra \( K(n) \) admits a root space decomposition with respect to a canonical Cartan subalgebra.

**Proposition 2.3.**

(a) \( T_K := \oplus_{i=1}^{2m} \langle x_i x_{i'} \rangle_F \oplus \langle x_n \rangle_F \) is a maximal torus of \( K(n) \) (called the canonical maximal torus).

(b) The centralizer of \( T_K \) inside \( K(n) \) is the subalgebra \( C_K = \{ x^a \mid a_i = a_{\sigma(i)} \text{ and } \deg(x^a) \equiv 0 \mod p \} \), which is hence a Cartan subalgebra (called the canonical Cartan subalgebra). The dimension of \( C_K \) is \( p^n \) if \( p \not| (m+2) \) and \( p^n - 1 \) otherwise.
(c) Let $\Phi_K := \text{Hom}_F(\bigoplus_{i=1}^n \langle x_i, x_i' \rangle \mathbb{F}_p \oplus \langle x_n \rangle \mathbb{F}_p, \mathbb{F}_p)$, where $\mathbb{F}_p$ is the prime field of $F$. We have a Cartan decomposition $K(n) = C_K \oplus \Phi_K \cdot 0$, where $K(n) \cdot 0 = \{ x^n \mid a_i m - a_i \equiv \phi(x_i, x_i') \forall i = 1, \cdots, m \text{ and } \deg(x^n) \equiv \phi(x_i) \}$. The dimension of every $K(n) \cdot 0$, with $\phi \in \Phi_K \cdot 0$, is $p^m$.

Proof. See [FS88, Chap. 4, Theo. 5.6 and 5.7].

2.2. Proof of the Main Theorem [1.1] In this section, assuming the results of the next section, we give a proof of the Main Theorem [1.1].

Proof of the Main Theorem [1.1] It is easy to see that the cochains appearing in Theorem [1.1] are cocycles and that they are independent in $H^2(K(n), K(n))$. Therefore, we are left with showing that $\dim_F H^2(K(n), K(n)) = n$. We divide the proof in three steps.

STEP I: It is enough to show that $\dim_F H^2(K(n), K'(n)) = n$ since there is an inclusion $H^2(K(n), K(n)) \hookrightarrow H^2(K(n), K'(n))$.

Indeed, if $p$ does not divide $m + 2$ then $K'(n) = K(n)$ and we get the equality. Otherwise there is an exact sequence of $K(n)$-modules

$$(2.1) \quad 0 \to K(n) \to K'(n) \to \langle x^n \rangle_F \to 0$$

where $\langle x^n \rangle_F \cong F$ is the trivial $K(n)$-module. We get the desired inclusion since $H^1(K(n), F) = 0$, which follows from the fact that $[K(n), K(n)] = K(n)$.

STEP II: We have that $H^2(K(n), K'(n)) = H^2(K(n), F_{\lambda - \sigma})$, where $F_{\lambda - \sigma}$ is the one-dimensional representation of $K(n)_{\geq 0}$ on which $x_n$ acts as $-2$ and all the others elements act trivially.

This follows from the general results of [FS91]. Indeed, it is easily seen that $K'(n)$ is the restricted $K(n)$-module induced from the restricted $K(n)_{\geq 0}$-submodule $\langle x^n \rangle_F \subset K'(n)$. In the notation of [FS91], the $K(n)_{\geq 0}$-module $\langle x^n \rangle_F$ is isomorphic to $F_{\lambda}$, where $F_{\lambda}$ is the one dimensional $K(n)_{\geq 0}$-module corresponding to the Lie algebra homomorphism $\lambda : K(n)_{\geq 0} \to F$ whose only non-zero value is $\lambda(x_n) = -2m - 4 = \deg(x^n) = (2m + 2)(p - 1) - 2 \mod p$.

Moreover, consider the Lie algebra homomorphism $\sigma : K(n)_{\geq 0} \to F$ given by $\sigma(x) := \text{tr}(ad_{K(n)/K(n)_{\geq 0}}) x$ for $x \in K(n)_{\geq 0}$ (see [FS91] Pag. 155). It is easily seen that the only non-zero value of $\sigma$ is given by $\sigma(x_n) = -2m - 2$.

Therefore we have that $\gamma - \sigma : K(n)_{\geq 0} \to F$ is the Lie algebra homomorphism sending $x_n$ to $-2$ and vanishing on the other elements. Moreover, using Lemma 2.5 it is straightforward to check that, in the notation of [FS91], we have the equality $\{ [x, y] - (\lambda - \sigma)(x)y + (\lambda - \sigma)(y)x, x, y \in K(n)_{\geq 0} \} := (K(n)_{\geq 0})_{\lambda - \sigma}^{(1)} = I := \ker(\lambda - \sigma)$.

We conclude using [FS91] Thm. 3.6(1).

STEP III: In section 2.3, we prove that

$$\dim_F H^2(K(n)_{\geq 0}, F_{\lambda - \sigma}) = \dim_F H^2(K(n)_{\geq 1}, F_{\lambda - \sigma}) K(n)_{\geq 0} = n.$$

2.3. Computation of $H^2(K(n)_{\geq 0}, F_{\lambda - \sigma})$. This section is devoted to complete the third step of the proof of the Main Theorem as outlined in section 2.2, that is the computation of $H^2(K(n)_{\geq 0}, F_{\lambda - \sigma})$. This is done in the Propositions 2.3 and 2.6 below.

Recall that $F_{\lambda - \sigma}$ is the one-dimensional representation of $K(n)_{\geq 0}$ on which $x_n$ acts as $-2$ and all the others elements act trivially. This action becomes homogeneous with respect to the weight decomposition of $K(n)_{\geq 0}$ if we give the weight
-2\epsilon_n to the generator of $F_{\lambda - \sigma}$. As remarked in [V191 Sec. 2.1], in this situation we have that

\[ H^2(K(n)_{g0}, F_{\lambda - \sigma}) = H^2(K(n)_{g1}, F_{\lambda - \sigma}) \]

where the subscript $0$ means that we consider only homogeneous cochains with respect to the natural action of the maximal torus $T_K$ (see Prop. 2.3).

**Proposition 2.4.** We have that

\[ H^2(K(n)_{g0}, F_{\lambda - \sigma}) = H^2(K(n)_{g1}, F_{\lambda - \sigma}) K(n)_g. \]

**Proof.** Consider the Hochschild-Serre spectral sequence (see [HS53]) associated to the ideal $K(n)_{g1} \triangleleft K(n)_{g0}$:

\[ E_2^{r,s} = H^r(K(n)_0, H^s(K(n)_{g1}, F_{\lambda - \sigma}))) \Rightarrow H^{r+s}(K(n)_{g0}, F_{\lambda - \sigma}). \]

We are going to prove that the first two lines of the above spectral sequence vanish, which clearly imply the Proposition.

The first line $E_2^{r,s} = H^r(K(n)_0, H^s(K(n)_{g1}, F_{\lambda - \sigma})))$ vanish for homogeneity reasons. Indeed, the weight of $F_{\lambda - \sigma}$ is $-2\epsilon_n \neq 0$, while the weights occurring on $K(n)_0$ are $\{ \pm \epsilon_i \pm \epsilon_j, 1 \leq i, j \leq 2m \}$ and hence the weights that occur on $K(n)_{0}^{g_k}$ cannot contain $\epsilon_n$ with a non-trivial coefficient.

On the other hand, since $F_{\lambda - \sigma}$ is a trivial $K(n)_{g1}$-module and $K(n)_{g1}, K(n)_{g2}$ = $K(n)_{g2}$ by the Lemma 2.5 below, we have that

\[ H^1(K(n)_{g1}, F_{\lambda - \sigma}) = C^1(K(n)_{g1}, F_{\lambda - \sigma}). \]

From this equality, we deduce that the second line $E_2^{r,s} = H^r(K(n)_0, H^s(W(n)_{g1}, F_{\lambda - \sigma})))$ vanish again for homogeneity reasons. Indeed the $n$-component of the weights appearing in $H^1(K(n)_{g1}, F_{\lambda - \sigma}) = C^1(K(n)_{g1}, F_{\lambda - \sigma})$ is $-3\epsilon_n \neq 0$ (because $p \geq 5$) while the weights appearing in $K(n)_{0}^{g_k}$ have trivial $n$-component. \(\square\)

**Lemma 2.5.** Let $d$ be an integer greater or equal to $-2$. Then

\[ [K(n)_1, K(n)_d] = K(n)_{d+1}. \]

**Proof.** The inclusion $[K(n)_1, K(n)_d] \subset K(n)_{d+1}$ is clear. In order to prove the other inclusion, we consider an element $x^a \in K(n)_{d+1}$ and we have to show that it belongs to the commutators $[K(n)_1, K(n)_d]$.

The elements of $K(n)_1$ are of the form $x_i x_j x_k$ or $x_i x_n$ (for some $1 \leq i, j, k \leq 2m$). The first ones act, via adjoint action, as $D_H(x_i, x_j) x_k - x_i x_j x_k$ while the latter ones act as $\sigma(i) x_n D_{\nu} + x_i \text{deg} - x_i x_n D_{\nu}$. Consider the decomposition $K(n) = \otimes_{n=0}^{p-1} A(2m) x_n^a$. The proof is by induction on the coefficient $a_n$, which in what follows it is called the $x_n$-degree of $x^a$.

First of all consider the case of $x_n$-degree equal to 0, that is the case $x^a \in A(2m)$. If there exists an index $1 \leq i \leq 2m$ such that $a_i \geq 2$ and $a_{i'} < p - 1$, then we conclude by mean of the following formula

\[ [x_i^a, x^{a-2\epsilon_i + \epsilon_{i'}}] = 3\sigma(i)(a_{i'} + 1)x^a. \]

Therefore it remains to consider the elements $x^a$ for which $a_i = a_{i'} = p - 1$ or $0 \leq a_i, a_{i'} \leq 1$ for every $1 \leq i \leq 2m$. If there exists a couple $(a_i, a_{i'}) = (1, 1)$, we are done by the formula

\[ [x_i^2 x_{i'}, x^{a-\epsilon_i + \epsilon_{i'}}] = \sigma(i)(2a_{i'} - a_i + 1)x^a = 2\sigma(i)x^a. \]

If there exists a couple $(a_i, a_{i'}) = (1, 0)$, then there are two possibilities: either $x^a = x_i$ or there exists an index $j \neq i, i'$ such that $a_j \geq 1$. In the first case we use $[x_i x_n, 1] = -2\epsilon_i$ while in the second we conclude by mean of the following formula

\[ [x_i^2 x_j, x^{a-\epsilon_i + \epsilon_{i'} - \epsilon_j}] = 2\sigma(i)x^a + \sigma(j)x^{a+\epsilon_i + \epsilon_{i'} - \epsilon_j - \epsilon_i}. \]
together with the fact that the second element in the right hand side belongs to \([K(n)_{1}, K(n)_{d}]\) by what proved above. Hence we are left with considering the elements \(x^a\) for which every couple of conjugated coefficients \((a_i, a_{i'})\) is equal to \((0, 0)\) or \((p - 1, p - 1)\). If there are two indexes \(1 \leq i \neq j \leq m\) such that \((a_i, a_{i'}) = (0, 0)\) and \((a_j, a_{j'}) = (p - 1, p - 1)\) we use the formula

\[
[x^j_2 x^i, x^{a+\varepsilon_i, -2\varepsilon_i}] = 2(p - 1)x^{a+\varepsilon_i, \varepsilon_i - \varepsilon_i' + a^d},
\]

together with the fact that the first term on the right hand side belongs to \([K(n)_{1}, K(n)_{d}]\) by what proved above. Since the case \(x^a = 1\) is excluded by the hypothesis \(d + 1 \geq -1\), it remains to consider the element \(x^a = x^\sigma\) for which we can take an appropriate linear combination of the two equations (with \(k = 0\)):

\[
(x^\sigma^{a - 2\varepsilon_i} x^{k - 1}) = -3\sigma(i)x^{a - \varepsilon_i, \varepsilon_i} x^{k + 1} - (k + 1)x^\sigma x^k,
\]

(2.2)

\[
(x^\sigma^{a - 2\varepsilon_i} x^{k - 1}) = -3\sigma(i)x^{a - \varepsilon_i, \varepsilon_i} x^{k + 1} - (k + 1)x^\sigma x^k.
\]

(2.3)

For the inductive step, suppose that \(a_n = k \geq 1\) and that we have already proved the desired inclusion for the elements of \(x_n\)-degree less than or equal to \(k - 1\). If there exists an index \(i\) such that \(a_i < p - 1\), then the formula

\[
[x^\sigma x^i, x^{\sigma - \varepsilon_i \varepsilon_i} x^{a - 2\varepsilon_i} x^{k - 1}] = \sigma(i)(a_i + 1)x^a + (d - a_n + 1)x^{a + \varepsilon_i, \varepsilon_i - \varepsilon_i},
\]

together with the induction hypothesis, gives the conclusion. Otherwise our element is equal to \(x^a x^{p - 1}\). If \(k < p - 1\), then one concludes by taking an appropriate linear combination of the above formulas (2.2) and (2.3). Finally for the element \(x^a x^{p - 1} = x^\tau\) (which can occur only if \(p \not| m + 2\)), the conclusion follows from the formula (for an arbitrary chosen \(1 \leq i \leq 2m\))

\[
[x^\tau x^i, x^{\sigma - \varepsilon_i} x^{p - 1}] = -2(m + 2)x^\tau.
\]

\[\square\]

For the remaining part of this subsection, we identify the \(K(n)_{0}\)-module \(F_{\lambda, -\sigma}\) with the \(K(n)_{0}\)-module \(F \cong (1)_F = K(n)_{-2}\). On both these modules, \(K(n)_{\geq 1}\) acts trivially.

**Proposition 2.6.** We have that

\[
H^2(K(n)_{\geq 1}, F)^{K(n)_{0}} = \bigoplus_{i=1}^{2m} \langle \text{Sq}(x^i) \rangle_F \oplus \langle \text{Sq}(1) \rangle_F,
\]

where \(\text{Sq}(x^i)\) is the projection of \(\text{Sq}(x)\) onto \((1)_F \cong F\) (analogously for \(\text{Sq}(1)\)).

**Proof.** It is easy to check that the above cocycles are independent modulo coboundaries, so we have to prove that they generate the whole cohomology group.

The strategy of the proof is exactly the same as that of proposition [VIV1 Prop. 3.10], that is to compute, step by step as \(d\) increases, the truncated invariant cohomology groups

\[
H^2 \left( \frac{K(n)_{\geq 1}}{K(n)_{\geq d + 1}}, F \right)^{K(n)_{0}}.
\]

Observe that if \(d\) is big enough (at least \(2(m + 1)(p - 1) - 1\)) then \(K(n)_{\geq d + 1} = 0\) and hence we get the cohomology we are interested in. On the other hand, by homogeneity, we get that

\[
H^2 \left( \frac{K(n)_{\geq 1}}{K(n)_{\geq 2}}, F \right)^{K(n)_{0}} = C^2(K(n)_{1}, F)^{K(n)_{0}} = 0.
\]
By taking the Hochschild-Serre spectral sequence (see [HS53]) associated to the ideal \( K(n)_d = \frac{K(n)_{>4}}{K(n)_{>2d+3}} \leq \frac{K(n)_{>1}}{K(n)_{>d+1}} \) (for \( d \geq 2 \)):

\[
E_2^{r,s} = H^r\left( \frac{K(n)_{>1}}{K(n)}_{>d} H^s(K(n)_d, F) \right) \Rightarrow H^{r+s}\left( \frac{K(n)_{>1}}{K(n)}_{>d+1}, F \right),
\]

we get the same diagram as in [VIVI] Prop. 3.10 (the vanishing of \( E_2^{0,2} \) and the injectivity of the map \( \alpha \) are proved in exactly the same way).

By taking the cohomology with respect to \( K(n)_0 \) and using the Lemmas 2.8, 2.9 below, we see that the only cocycles that contribute to the required cohomology in degree \( 2m(p-1) + 2 \mu - 3 \) is annihilated by \( \epsilon \in C^1(K(n)_{2m(p-1)+2\mu-2}, F)K(n)_0 \).

The remaining part of this section is devoted to the proof of the Lemmas that were used in the proof of Proposition 2.9.

In the next two Lemmas, we are going to compute the \( K(n)_0 \)-invariant terms \( (E_2^{1,1})^{K(n)_0} \) and \( (E_1^{1,1})^{K(n)_0} \). Observe that, since \( K(n)_d \) is in the center of \( \frac{K(n)_{>1}}{K(n)_{>d+1}} \), we have that \( H^*(K(n)_d, F) = C^*(K(n)_d, F) \) and \( \frac{K(n)_{>1}}{K(n)_{>d+1}} \) acts trivially on it. Therefore, using Lemma 2.8 we deduce that \( E_2^{1,1} = C^1(K(n) \times K(n)_d, F) \).

**Lemma 2.7.** Let \( 0 \leq \mu, \nu \leq p-1 \) such that \( \mu \equiv m \mod p \) and \( \nu \equiv (m + 1) \mod p \). Then we have that

\[
\bigoplus_{d \geq 2} C^1(K(n)_1 \times K(n)_d, F)^{K(n)_0} = (\Phi_1)_F \oplus (\Phi_2)_F \oplus (\Psi_1)_F \oplus (\Psi_2)_F,
\]

where the above cochains are defined by

\[
\begin{align*}
\Phi_1(x_1x_n, x_2x_n^{-1}) &= \sigma(i), \\
\Phi_2(x^a, x^a_n x_n^{-2}) &= \sigma(a) a! \quad \text{if } a \in \mathbb{N}^{2m} \text{ and } |a| = 3, \\
\Psi_1(x_1x_n, x^{a-c}x_n^a) &= 1, \\
\Psi_2(x^a, x^{a-c}x_n^a) &= 1 \quad \text{if } a \in \mathbb{N}^{2m} \text{ and } |a| = 3.
\end{align*}
\]

**Proof.** An easy verification shows that the four cochains of above are \( K(n)_0 \)-invariants and linearly independent. We will conclude by showing that the dimension over the base field \( F \) of the space \( \bigoplus_{d \geq 2} C^1(K(n)_1 \times K(n)_d, F)^{K(n)_0} \) of all invariant homogeneous cochains is less than or equal to 4.

The space \( K(n)_1 \) admits the decomposition \( K(n)_1 = A(2m)_{-1} \cdot x_n \oplus A(2m)_{1} \) which is invariant under the adjoint action of \( K(n)_0 = A(2m)_{0} \oplus \langle x_n \rangle_F \). Moreover the action of \( K(n)_0 \) is transitive in both the summands \( A(2m)_{-1} \cdot x_n \) and \( A(2m)_{1} \).

Therefore a \( K(n)_0 \)-invariant homogeneous cochain \( g \in C^1(K(n)_1 \times K(n)_d, F)^{K(n)_0} \) is determined by the values on any two elements of \( A(2m)_{-1} \cdot x_n \) and \( A(2m)_{1} \), let’s say \( x_1x_n \) and \( x_1^3 \).

Consider an element \( x^a \in K(n)_{>2} \) such that \( g(x_1x_n, x^a) \neq 0 \). By homogeneity the element \( x^a \) must satisfy \( a_1 \equiv a_1 + 1 \mod p, a_j = a_j \) for every \( j \notin \{1, 1' \} \) and \( \text{deg}(x^a) \equiv -3 \mod p \). If the couple \( (a_1, a_1') \) would be different from \((0, 1)\) or \((p-2, p-1)\) then the following invariance condition

\[
0 = (x_1^3 \circ g)(x_1x_n, x^{a-c_1+c_1'}) = -2(a_1 + 1)g(x_1x_n, x^a)
\]

would contradict the hypothesis of non-vanishing. Therefore we can assume that \((a_1, a_1') = (0, 1)\) or \((p-2, p-1)\). If the first case holds, then necessarily \((a_j, a_j') = \)
(0, 0) for every \( j \not\in \{1, 1'\} \). Indeed if this is not the case, then we get a contradiction with the non-vanishing hypothesis by means of the following invariance condition
\[
0 = (x_1 x_j \circ g)(x_1 x_n, x^{a + c_i' - c_j}) = -2g(x_1 x_n, x^a).
\]
Analogously, if \( (a_1, a_1') = (p - 2, p - 1) \) then \( (a_j, a_j') \) for every \( j \not\in \{1, 1'\} \) because of the following invariance condition
\[
0 = (x_1 x_j \circ g)(x_1 x_n, x^{a - c_i + c_j'}) = -\sigma(j)(a_j + 1)g(x_1 x_n, x^a).
\]
Taking into account the homogeneity condition \( \deg(x^a) = -3 \mod p \), we get that the only non-zero values of \( g(x_1 x_n, -) \) can be \( g(x_1 x_n, x_{2 - \nu}) \) and \( g(x_1 x_n, x_{\sigma - \epsilon_1} x_{2 - \nu}') \).

In exactly the same way, one proves that the only non-zero values of \( g(x_1', -) \) can be \( g(x_1', x_1^{\nu}, x_{2 - \nu}') \) and \( g(x_1', x_{\sigma - \epsilon_1} x_{2 - \nu}') \) and therefore we get
\[
\dim F C^1(K(n)_1 \times K(n)_{2}, F)^{K(n)_{0}} \leq 4.
\]
\[\square\]

**Lemma 2.8.** In the above spectral sequence \( [2, 4] \), we have that
\[
(E^{1,1}_\infty)^{K(n)_{0}} = \begin{cases} (\Sigma(1))_F & \text{if } d = 2p - 3, \\ \langle \text{inv } [-, -] \rangle_F & \text{if } d = 2m(p - 1) + 2\mu - 3, \\ 0 & \text{otherwise,} \end{cases}
\]
where \( \text{inv} \in C^1(K(n)_2m(p - 1) + 2\mu - 2, F)^{K(n)_{0}} \) is defined in the Lemma 2.7 below and \( \Sigma(1) \) is the restriction of \( \Sigma(1) \) to \( K(n)_1 \times K(n)_{2p - 3} \).

**Proof.** \( (E^{1,1}_\infty)^{K(n)_{0}} \) is the subspace of \( (E^{1,1}_2)^{K(n)_{0}} = C^1(K(n)_1 \times K(n)_{d}, F)^{K(n)_{0}} \) consisting of cocycles that can be lifted to \( Z^2(K(n)_{2m(p - 1) + 2\mu - 2}, F) \). A direct computation shows that, with the notations of Lemma 2.7 \( \Sigma(1) = 2\Phi_1 \) and \( \text{inv} [-, -] = -(\nu + 2)\Psi_1 - \nu\Psi_2 \) and clearly these two cocycles can be lifted. We want two show that any other liftable cocycle is a linear combination of them.

First of all we show that the cocycle \( \Phi_2 \) is not liftable. By absurd, suppose that a lifting exists and call it again \( \Phi_2 \). We get a contradiction by mean of the following cocycle conditions
\[
\begin{align*}
0 = d\Phi_2(x_1^3, x_1^3, x_{2 - \nu}) &= -9\sigma(i)\Phi_2(x_1^3, x_1^3, x_{2 - \nu}) + \Phi_2(x_1^3, x_1^{2 - \nu}, x_1^3) + \\
-\Phi_2(x_1^3, x_1^{2 - \nu}, x_1^3) &= -9\sigma(i)\Phi_2(x_1^3, x_1^{3}, x_{2 - \nu}) + 12\sigma(i),
\end{align*}
\]
\[
\begin{align*}
0 = d\Phi_2(x_1^3, x_1^3, x_{2 - \nu}) &= -3\sigma(i)\Phi_2(x_1^3, x_1^{3}, x_{2 - \nu}) + \Phi_2(x_1^3, x_1^{2 - \nu}, x_1^3) + \\
-\Phi_2(x_1^3, x_1^{2 - \nu}, x_1^3) &= -3\sigma(i)\phi_2(x_1^3, x_1^{3}, x_{2 - \nu}) - 4\sigma(i).
\end{align*}
\]
Finally, consider the cocycle \( \Psi \in \langle \Psi_1, \Psi_2 \rangle_F \) defined in degree \( 2m(p - 1) + 2\nu - 5 \) by
\[
\Psi = \begin{cases} a\Psi_1 + b\Psi_2 & \text{if } \nu \neq 0, \\ b\Psi_2 & \text{if } \nu = 0,
\end{cases}
\]
for certain \( a, b \in F \). We will show that \( \Psi \) can be lifted to \( Z^2(K(n)_{2m(p - 1) + 2\nu - 4}, F) \) if and only if \( b(\nu + 2) \equiv \nu \mod p \) and this will conclude our proof. Indeed this imply that if \( \nu \neq 0 \) then \( \Psi \) is liftable if and only if it is a multiple of \( \text{inv} [-, -] \), while if \( \nu = 0 \) it implies that \( \Psi_2 \) is not liftable and hence again that \( \text{inv} [-, -] = -2\Psi_1 \) is the only liftable cocycle in the span of \( \Psi_1 \) and \( \Psi_2 \).

So suppose that a lift exists and call it again \( \Psi \). From the following cocycle condition
\[
0 = d\Psi(x_1^3, x_1^3, x_{\sigma - 2\nu}, x_{\sigma - 2\nu}) = -9\sigma(i)\Psi(x_1^3, x_1^3, x_{\sigma - 2\nu}, x_{\sigma - 2\nu}) + \\
-9\sigma(i)\Psi(x_{\sigma - 3\nu} x_{\sigma - 3\nu}, x_1^3) + 9\sigma(i)\Psi(x_{\sigma - 3\nu} x_{\sigma - 3\nu}, x_1^3),
\]
we deduce that $\Psi(x_1^2 x_1^n, x^\sigma_{-2i'}-2\epsilon, x_1^{\nu'}) = 2b$. Using this, we get the following

$$0 = d\Psi(x_i x_j, x_1 x_1^n, x^\sigma_{-2i'}-2\epsilon, x_1^{\nu'}) = -2\sigma(1')\Psi(x_i x_j x_1^n, x_1 x_1^n, x^\sigma_{-2i'}-2\epsilon, x_1^{\nu'}) +$$

$$+ \left(-4 + \nu\right)\Psi(x^\sigma_{-\epsilon', -2\epsilon', x_1^\mu}, x_1 x_1^n) + \nu\Psi(x^\sigma_{-\epsilon', x_1^\mu}, x_1 x_1^n) =$$

$$= -2\sigma(1')\Psi(x_i x_j x_1^n, x^\sigma_{-2i'}-2\epsilon, x_1^{\nu'}) + (\nu + 2)b - va.$$ Exchanging $i$ with $i'$ and summing the two expressions, we obtain the required congruence $(\nu + 2)b \equiv va \mod p$. □

In the next Lemma, we compute the $K(n)_0$-invariants of the term $E_2^{0,1} = C^1(K(n)_d, F)$ of the above spectral sequence (2.4).

**Lemma 2.9.** Let $\mu$ the integer defined in Lemma 2.7. We have that

$$C^1(K(n)_d, F)^{K(n)_0} = \begin{cases} \langle \text{inv}\rangle_F & \text{if } d = 2m(p-1) + 2\mu - 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{inv} \in C^1(K(n)_{2m(p-1)+2\mu-2}, F)$ sends $x^\sigma x_1^{\mu}$ into 1 and vanish on the other elements.

**Proof.** First of all observe that if $p$ divide $(m + 2)$, then $\mu = p - 2$ and hence $x^\sigma x_1^{\mu} \in K(n)$. The (well-defined) cochain $\text{inv}$ is $K(n)_0$-invariant. Indeed it is homogeneous and the invariance with respect to an element $x_i x_j \in K(n)_0 \setminus T_K$ (hence with $j \neq i'$) follows from the fact that $[x_i x_j, x^\sigma x_1^{\mu}] = [x_i x_j, 1] = 0$ together with the fact that $x^\sigma x_1^{\mu} \notin [x_i x_j, K(n)]$.

Consider next an invariant cochain $f \in C^1(K(n)_d, F)^{K(n)_0}$ and let $x^\sigma \in K(n)_d$ be an element such that $f(x^\sigma) \neq 0$. Then by homogeneity it must hold that $a_i = a_{i'}$, for every $1 \leq i \leq m$ and deg$(x^\sigma) \equiv -2 \mod p$. Using the invariance with respect to $x_1^2$ or $x_1^n$, we obtain that $a_i \equiv a_{i'} = 0$ or $p - 1$. Otherwise, assuming, up to interchanging $i$ with $i'$, that $a_i > 0$ and $a_{i'} < p - 1$, one gets the vanishing as follows

$$0 = (x_1^2 \circ f)(x^\sigma_{-\epsilon', +\epsilon'}) = -2\sigma(1')(a_{i'} + 1)f(x^\sigma).$$

Moreover, if there are two pairs verifying $(a_i, a_{i'}) = (0, 0)$ and $(a_j, a_{j'}) = (p-1, p-1)$ (for $j \neq i, i'$), then we obtain the vanishing by means of the following

$$0 = (x_i x_j \circ f)(x^\sigma_{+\epsilon', -\epsilon'}) = -\sigma(1)f(x^\sigma).$$

Finally, by imposing deg$(x^\sigma) \equiv -2 \mod p$, we deduce that $x^\sigma = 1$ (which we can exclude since deg$(x^\sigma) = d \geq 2$) or $x^\sigma = x^\sigma x_1^{\mu}$. □

In the next Lemma, we compute the first cohomology group with respect to $K(n)_0$ of the term $E_2^{0,1} = C^1(K(n)_d, F)$ of the above spectral sequence (2.4).

**Lemma 2.10.** Let $\mu$ be the integer defined in Lemma 2.7. We have that

$$H^1(K(n)_0, C^1(K(n)_d, F)) = \begin{cases} \bigoplus_{i=1}^{2m} \langle \text{Sq}(x_i) \rangle_F & \text{if } d = p - 2, \\ \bigoplus_{j=1}^{2m} \langle \omega_j \rangle_F & \text{if } d = 2m(p-1) + 2\mu - 2, \\ \langle x_0 \mapsto \text{inv} \rangle_F & \text{if } d = 2m(p-1) + 2\mu - 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{Sq}(x_i)$ denotes the restriction of $\text{Sq}(x_i)$ to $K(n)_0 \times K(n)_{p-2}$ and the cocycle $\omega_j$ is defined by $(with \ j \neq i, i')$

$$\begin{cases} \omega_1(x_i^2, x^\sigma_{-\epsilon', -\epsilon', x_1^{\mu}}) = 2, \\ \omega_1(x_i x_j, x^\sigma_{-\epsilon', -\epsilon', x_1^{\mu}}) = 1. \end{cases}$$
Proof. By homogeneity we can restrict to the case $d \equiv -2 \mod p$. First of all I claim that $f_{x_i,x_j} = 0$ for every $1 \leq i \leq n$ and $f_{x_i}$ takes a non zero-value only on the element $x_i x_n^{n-i}$. Indeed, by the homogeneity assumption, we get for an element $\gamma \in F_k$ that $0 = d f_{\gamma} = \gamma \circ f - d(\gamma)$, that is $f_{\gamma} \in C^1(K(n), F) K(n)_0 = (\text{inv}) F$ (see Lemma 2.9). Moreover the cocycle condition

$$0 = df_{(x_i^j,x_j^k)}(x^\sigma x_n^\mu) = -4\sigma(i) f_{x_i,x_j}(x^\sigma x_n^\mu)$$

gives $f_{x_i,x_j} = 0$ for every $1 \leq i \leq n$, while the fact that $x_n \notin [K(n)_0, K(n)_0]$ implies that $f_{x_n}(x^\sigma x_n^\mu)$ can be different from zero.

Now we split the proof into two parts according to the cases $d = (2r + 1)p - 2$ or $d = 2rp - 2$ for some integer $r$.

I CASE : $d = (2r + 1)p - 2$.

Note that in this case, there are not coboundary elements since, by reasons of parity, $C^1(K(n)_0, F) = 0$. Moreover, for a homogeneous cocycle $f \in C^1(K(n)_0)$, $C^1(K(n)_0, F) = 0$, the value $f_{x_i}(x^a)$ can be different from 0 only if one of the following possibilities occur

(A) $(a_i, a_{i'}) = (p - 1, 1)$ and $a_j = a_{j'}$ for every $j \neq i, i'$,

(B) $(a_i, a_{i'}) = (p - 2, 0)$ and $a_j = a_{j'}$ for every $j \neq i, i'$.

Analogously, if $j \neq i, i'$, then $f_{x_i,x_j}(x^a)$ can be different from 0 only if (up to interchanging $i$ and $j$)

(C) $(a_i, a_{i'}) = (p - 1, 0)$, $1 \leq a_{j'} = a_j + 1 \leq p - 1$ and $a_k = a_{k'}$ for $k \neq i, i', j, j'$.

The values of types (C) are determined by the values of types (A) and (B) by mean of the following cocycle condition (where $a$ is a multindex as in (C))

$$0 = df_{(x_i,x_j,x_k)}(x^{a_{i-j}+\epsilon_{i'}}) = -\sigma(j) a_j f_{x_i,x_j}(x^{a_{i-j}+\epsilon_{j'}}) - \sigma(i) f_{x_i,x_j}(x^{a_{i-j}+\epsilon_{i}}) + 2\sigma(i) f_{x_i,x_j}(x^a)$$

where in the last equation the first term is of type (A) (or vanish) and the second is of type (B) (or vanish).

The values of type (A) vanish if there exists an index $j \neq i, i'$ such that $a_j = a_{j'} \neq 0$, because of the following condition (where $a$ satisfies the conditions in (A))

$$0 = df_{(x_i,x_j,x_k)}(x^{a_{i-j}+\epsilon_{i'}}) = -2\sigma(i) f_{x_i,x_j}(x^a).$$

On the other hand, the values of type (B) vanish if there exists a $j \neq i, i'$ such that $a_j = a_{j'} \neq p - 1$ because the following cocycle condition (where $a$ satisfies the conditions of (B))

$$0 = df_{(x_i,x_j,x_k)}(x^{a_{i-j}+\epsilon_{i'}}) = -\sigma(j)(a_{j'} + 1) f_{x_i,x_j}(x^a).$$

Therefore a cocycle $f$ is completely determined by the values $f_{x_i}(x^{p-1} x_i)$ and $f_{x_i}(x^{s_{i-j}-(p-1)i-j} x_n^\mu)$, whose values determine also the cocycles $\text{Sq}(x_i)$ and $\omega_i$ (respectively), and hence $f$ is a linear combination of $\text{Sq}(x_i)$ or $\omega_i$.

If CASE : $d = 2rp - 2$.

In this case we will prove that $f$ vanish (up to adding a coboundary $dg$) except for the value $f_{x_n}(x^\sigma x_n^\mu)$ (which can be non-zero as seen before). We have already seen that $f_{x_i,x_j}$ vanish for every $1 \leq i \leq n$.

We first prove that, by adding coboundaries, we can modify the cocycle $f$ (without changing its cohomological class) in such a way that it satisfies $f_{x_i} = 0$ for every $1 \leq i \leq m$. The proof is by induction on $i$. So suppose that for a certain $k$, we have that $f_{x_i} = 0$ for every $i < k$. We want to prove that, by adding coboundaries, we can modify $f$ in such a way that it verifies $f_{x_k} = 0$. 


Moreover we say that a jump in position a if (1 ≤ a_k' = a_k + 2 ≤ p - 1 and a_k' = a_k for h ≠ k, k').

Moreover if there exists an index 1 ≤ h < k ≤ m such that a_h = a_h' ≠ 0, (p - 1), then f_{x_k}(x^a) = 0 because of the following cocycle condition

\[ 0 = df_{x_k}(x^a)(x^{a_0} + \epsilon + \epsilon') = -2(a_{h'} + 1)f_{x_k}(x^a) \]

where we used that f_{x_{k'}} = 0 by induction. Therefore we can suppose that for 1 ≤ h < k ≤ m, a_h = a_{h'} = 0 or (p - 1). Fix one of these elements x^a. Define an element g ∈ C^1(K(n)_d, F)_0 as follows:

\[
\begin{align*}
    g(x^{a_0} + \epsilon + \epsilon') &= f_{x_k}(x^a) \\
g(x^{b_0}) &= 0
\end{align*}
\]

By construction, if 1 ≤ h < k ≤ m then (x_h^a ∘ g) = 0 while (x_{h'}^a ∘ g)(x^a) = f_{x_k}(x^a).

Therefore the new cocycle f := f + dg satisfies the same inductive hypothesis as before and moreover it verifies f_{x_k}(x^a) = 0. Repeating these modifications for all the elements x^a as before, eventually we obtain a new cocycle homologous to the old one (which, by an abuse of notation, we continue to call f) and which satisfies f_{x_k} = 0 for every 1 ≤ i ≤ k, as required.

Using the above conditions, we want to show that the cocycle f must satisfy also f_{x_i'} = 0 for every 1 ≤ i ≤ m (and hence that f_{x_i} = 0 for every j). Indeed, as before, we have that f_{x_i'} ≠ 0 can be different from 0 only if 2 ≤ a_i = a_i' + 2 ≤ p - 1 and a_j = a_j' for every j ≠ i, i'. Hence the required vanishing follows from the following cocycle condition

\[ 0 = df_{x_i'}(x^{a_0} + \epsilon + \epsilon') = -2(a_{i'} + 1)f_{x_i'}(x^a). \]

Finally we have to show that we can modify once more (by adding coboundaries) the cocycle f in such a way that the previous vanishings f_{x_k} = 0 are still satisfied and moreover also f_{x_i, x_j} vanish for every j ≠ i, i'.

First of all, note that using cocycle conditions of type 0 = df_{x_i, x_j}(x^a) with h ≠ i', j' and the fact that f_{x_k} = 0, we obtain the vanishing of f_{x_i, x_j}(x^a) for all the elements x^a ∈ [x_{i'}^a, K(n)_d] ∩ K(n)_{-\epsilon_i - \epsilon_j} (for h ≠ i', j'), that is for all the elements of x^a ∈ K(n)_d with the exception of the ones that verify

\[
(a_k, a_{k'}) = \begin{cases} (0, 0) \text{ or } (p - 2, p - 1) & \text{if } k = i \text{ or } j, \\ (0, 0) \text{ or } (p - 1, p - 1) & \text{otherwise}. \end{cases}
\]

Therefore, we can assume that our x^a verifies these conditions. For the rest of the proof, we introduce the following definitions. We say that a couple (a_k, a_{k'}) is small if it is equal to (0, 0) or (0, 1) or (1, 0) according to the conditions above, while we say that it is big if it is equal to (p - 1, p - 1) or (p - 2, p - 1) or (p - 1, p - 2).

Moreover we say that x^a has an ascending jump in position k (with 1 ≤ k ≤ m - 1) if (a_k, a_{k'}) is small and (a_{k+1}, a_{(k+1)'}{y}) is big, while we say that it has a descending jump in position k if (a_k, a_{k'}) is big and (a_{k+1}, a_{(k+1)'}{y}) is small.

We want to modify our cocycle f, by adding coboundaries, in such a way that f_{x_i, x_j}(x^a) vanish if x^a has a jump.

We prove this for the elements f_{x_i, x_{i+1}} with 1 ≤ i ≤ m - 1. It is enough to prove that f_{x_i, x_j}(x^a) = 0 if there is a jump in a position less than or equal to i. Indeed if the jump on x^a occurs for h > i, then one obtains the vanishing using the cocycle condition 0 = df_{x_{i+1}, x_{i+1}}(x^a). Hence, by induction on i, suppose that we have already proved this for the elements i ≤ k - 1 and we want to prove it for f_{x_k, x_{k+1}}.

If there is a jump in the element x^a occurring in a position h < k then the vanishing
follows from a cocycle condition of type $0 = \text{d} f_{(x_1 x_{k+1})}$ plus the induction hypothesis. If the first jump occurring in $x^a$ is in the $k$-th position, then we define an element $g \in C^1(K(n,d), F_2)$ as follows:

$$
\begin{align*}
&D_f(x_{(0),\alpha + r + k + 1}) = f_{x_k x_{k+1}}(x^a), \quad \text{if the jump is ascending}, \\
&D_f(x_{(0),\alpha - r + k + 1}) = f_{x_k x_{k+1}}(x^a), \quad \text{if the jump is descending}, \\
&D_f(x^a) = 0, \quad \text{otherwise}.
\end{align*}
$$

By construction (and the hypothesis on $x^a$), for every $1 \leq j \leq 2m$ we have that $(x^j \circ g) = 0$ and if $1 \leq h < k \leq m$ then $(x_h x_{h+1} \circ g) = 0$ while $(x_h x_{h+1} \circ g)(x^a) = -f_{x_k x_{k+1}}(x^a)$. Therefore the new cocycle $\tilde{f} = f + dg$ satisfies the same vanishing conditions of $f$ (namely $\tilde{f}_g = 0$ for every $j$ and $\tilde{f}_{x_k x_{k+1}} = 0$ for $1 \leq h < k$) plus the new one $\tilde{f}_{x_k x_{k+1}}(x^a) = 0$. Repeating these modifications for all the elements $x^a$ as above, we find a new cocycle (which, by an abuse of notation, we will still call $\tilde{f}$) that satisfies $f_{x_k x_{k+1}} = 0$, concluding thus the inductive step.

From the previous special cases, it follows also the vanishing of $f_{x_k x_{k+1}}$ (always under the presence of a jump) if $1 \leq i, j \leq m$. Indeed, if an element $x^a$ as before has a jump in position $k$ then the coboundary condition

$$
0 = \text{d} f_{(x_i x_j, x_{k+1})}(x^a) = -\sigma(k)(a_k + 1)f_{x_i x_j}(x^a),
$$

in the case of an ascending jump, and

$$
0 = \text{d} f_{(x_i x_j, x_{k+1})}(x^a) = -\sigma(k + 1)(a_{k+1} + 1)f_{x_i x_j}(x^a),
$$

in the case of a descending jump, gives the required vanishing.

Finally, the general case (in which $i$ and $j$ can vary from $1$ to $2m$) follows from cocycle conditions of type $0 = \text{d} f_{(x_i x_j, x_{h})} = -(x^j \circ f_{x_i x_j}) - 2\sigma(i)f_{x_i x_j}$.

So it remains to consider only the elements $x^a$ without jumps or, in other words, it remains to prove the vanishing of the following values of $f$: $f_{x_i x_j}(x^a x_j x_i x_{k}) = \alpha_{ij} \cdot 1$ and $f_{x_i x_j}(x^a_{(0),\alpha} x_i x_j x_{k}) = \beta_{ij} \cdot 1$, where $\nu \equiv m + 1 \mod p$ and $0 \leq \nu \leq p - 1$. The first ones vanish because of the following two cocycle conditions

$$
\begin{align*}
&D_f(x_{(0),\nu}) = -2\sigma(i')\alpha_{ij} - 2\sigma(i')\alpha_{ij'}, \\
&D_f(x_{(0),\nu}) = -2\sigma(j')\alpha_{ij'} + 2\sigma(j')\alpha_{ij}.
\end{align*}
$$

The second ones vanish because of the following two cocycle conditions

$$
\begin{align*}
&D_f(x_{(0),\nu}) = -2\sigma(i')(p - 1)\beta_{ij} - 2\sigma(i')\beta_{ij'}, \\
&D_f(x_{(0),\nu}) = -\sigma(i)(p - 1)\beta_{ij} + \sigma(i)(p - 1)\beta_{ij'}.
\end{align*}
$$

In the next (and last) Lemma, we consider the differential map

$$
\text{d} : E_2^{0,1} = C^1(K(n,d), F) \rightarrow E_2^{2,0} = H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, F\right),
$$

induced by the above spectral sequence \((2.3)\). We compute the kernel of the induced map on the first cohomology group with respect to $K(n,d)$.

**Lemma 2.11.** Consider the map

$$
\text{d}^{(1)} : H^1(K(n)_0, C^1(K(n,d), F)) \rightarrow H^1\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, F\right)
$$

induced by the differential map \((2.5)\). The kernel of $\text{d}^{(1)}$ is given by

$$
\text{Ker}(\text{d}) = \left\{2m \sum_{i=1}^{m} \text{Sq}(x_i) F \quad \text{if} \quad d = p - 2, \\
0 \quad \text{otherwise}
\right\}
$$
where $\text{Sq}(x_i)$ denotes the restriction of $\text{Sq}(x_i)$ to $K(n)_0 \times K(n)_{p-2}$.

**Proof.** Clearly the cocycles $\text{Sq}(x_i)$, being the restriction of global cocycles, belong to the kernel of $d$. We want to show that the other generators of $H^1(K(n)_0, C^1(K(n)_0, F))$ (see Lemma 2.10) do not belong to Ker($d(1)$). First of all we have that

$$d(1)\langle x_n \mapsto \text{inv} \rangle_F = \langle x_n \mapsto \text{inv} \circ [-], - \rangle,$$

and this last cocycle is not a coboundary since inv $\circ [-, -] \in H^2\left(\frac{K(n)_0}{K(n)_0^p}, F\right)$ and $x_n$ acts trivially on this space.

Consider the cocycles $\omega_i$ for $1 \leq i \leq 2m$. At least one of the following values is non-zero (depending on $\mu$):

$$\begin{cases} 
(d\omega_i)\omega_i^2 (x_i x_n, x^\sigma - 2\epsilon_i - (p-1)\epsilon_i, x_n^\mu) = (\omega_i)\omega_i^2 ([x_i x_n, x^\sigma - 2\epsilon_i - (p-1)\epsilon_i, x_n^\mu]) = \\
\quad = (-3 - \mu)(\omega_i)\omega_i^2 (x^\sigma - \epsilon_i - (p-1)\epsilon_i, x_n^\mu) = 2(-3 - \mu), \\
(d\omega_i)\omega_i^2 (x_i^2, x^\sigma - 4\epsilon_i - (p-1)\epsilon_i, x_n^\mu) = (\omega_i)\omega_i^2 ([x_i^2, x^\sigma - 4\epsilon_i - (p-1)\epsilon_i, x_n^\mu]) = \\
\quad = -\mu(\omega_i)\omega_i^2 (x^\sigma - \epsilon_i - (p-1)\epsilon_i, x_n^\mu) = -2\mu.
\end{cases}$$

On the other hand, for every $g \in H^2\left(\frac{K(n)_0}{K(n)_0^p}, F\right)$, it holds that

$$\begin{cases} 
(x_i^2 \circ g)(x_i x_n, x^\sigma - 2\epsilon_i - (p-1)\epsilon_i, x_n^\mu) = 0, \\
(x_i^2 \circ g)(x_i^2, x^\sigma - 4\epsilon_i - (p-1)\epsilon_i, x_n^\mu) = 0.
\end{cases}$$

since $[x_i^2, x_i x_n] = [x_i^2, x^\sigma - 2\epsilon_i - (p-1)\epsilon_i, x_n^\mu] = [x_i^2, x_i^3] = [x_i^2, x^\sigma - 4\epsilon_i - (p-1)\epsilon_i, x_n^\mu] = 0$. □

3. **Hamiltonian algebra**

3.1. **Definition and Basic properties.** Throughout this section we fix a field $F$ of characteristic $p \neq 2, 3$ and an even integer $n = 2m \geq 2$.

We are going to use all the notations about multiindices introduced at the beginning of section 2.1. We are going to use often the following special $n$-tuples: $\underline{i} : = (0, \cdots, 0)$, $\sigma : = (p-1, \cdots, p-1)$ and $\sigma^i : = \sigma - (p-1)\epsilon_i - (p-1)\epsilon_i'$.

The vector space $A(n) = F[x_1, \cdots, x_n]/(x_1^p, \cdots, x_n^p)$, endowed with the grading defined by $\deg(x^a) = |a| - 2$, becomes a graded Lie algebra by mean of $[x^a, x^b] = D_H(x^a)(x^b)$, where $D_H : A(n) \to W(n) = \text{Der}_F A(n)$ is defined by

$$D_H(f) = \sum_{j=1}^{2m} \sigma(j) D_j(f) D_j' = \sum_{i=1}^m [D_i(f) D_{i+m} - D_{i+m}(f) D_i].$$

We denote with $H'(n)$ the quotient of $A(n)$ by the central element $1 = z \underline{0}$ so that there is an exact sequence of $H'(n)$-modules

$$0 \to \langle 1 \rangle_F \to A(n) \to H'(n) \to 0,$$

where $\langle 1 \rangle_F \cong F$ is the trivial $H'(n)$-module.

**Definition 3.1.** The Hamiltonian algebra is the derived subalgebra of $H'(n)$:

$$H(n) := H'(n)^{(1)} = [H'(n), H'(n)].$$

There is an exact sequence of $H(n)$-modules (see [ESSS, Chap. 4, Prop. 4.4]):

$$0 \to H(n) \to H'(n) \to \langle x^\sigma \rangle_F \to 0,$$

where $\langle x^\sigma \rangle_F \cong F$ is the trivial $H(n)$-module.
Note that the unique term of negative degree is $H(n)_{-1} = \oplus_{i=1}^{n}(x_i)_F$ where $x_i$ acts, via the adjoint action, as $D_H(x_i) = \sigma(i)D_x$. The term of degree 0 is $H(n)_0 = \oplus_{1 \leq i \leq n}(x_ix_j)_F$ and its adjoint action on $H(n)_{-1}$ induces an isomorphism $H(2m)_0 \cong sp(2m,F)$.

The algebra $H(n)$ admits a root space decomposition with respect to a canonical Cartan subalgebra.

**Proposition 3.2.** (a) $T_H := \oplus_{i=1}^{n}(x_i, x_i)_F$ is a maximal torus of $H(n)$ (called the canonical maximal torus).

(b) The centralizer of $T_H$ inside $H(n)$ is the subalgebra $C_H = \{x^a | a_i = a_i\}$, which is hence a Cartan subalgebra (called the canonical Cartan subalgebra).

The dimension of $C_H$ is $p^n - 2$.

(c) Let $\Phi_H := \text{Hom}_F(\oplus_{i=1}^{n}(x_i, x_i)_F, F_p)$, where $F_p$ is the prime field of $F$. We have a Cartan decomposition $H(n) = C_H \oplus \Phi_H - 2$, where $H(n)_\phi = \{x^a | a_i + m - a_i \equiv \phi(x_i, x_i) \mod p\}$. The dimension of every $H(n)_\phi$, with $\phi \in \Phi_H - 2$, is $p^n$.

Proof. See [FS88, Chap. 4, Theo. 4.5 and 4.6].

**3.2. Proof of the Main Theorem 1.2.** In this section, assuming the results of the next two sections, we give a proof of the Main Theorem 1.2. As a first step towards the proof, we compute the cohomology group of the second cohomology group of the $H(n)$-module $H'(n)$.

**Proposition 3.3.** The second cohomology group of $H'(n)$ is given by

$$H^2(H(n), H'(n)) = \bigoplus_{i=1}^{n}(\text{Sq}(x_i))_F \bigoplus_{i<j}(\Pi_{ij})_F \bigoplus(\Phi)_F,$$

where $\Pi_{ij}$ and $\Phi$ are the cocycles appearing in Theorem 1.2

Proof. From the exact sequence (3.1) and using Propositions 3.4 and 3.10 we get the exact sequence

$$0 \to \bigoplus_{i=1}^{n}(\text{Sq}(x_i))_F \bigoplus_{i<j}(\Pi_{ij})_F \bigoplus(\Phi)_F \to H^2(H(n), H'(n)) \to H^3(H(n), H'(n))$$

We have to verify that the coboundary map $\partial$ is equal to zero, or in other words that the cocycles which generate $H^3(H(n), \{1\}_F)$ (see Proposition 3.5) do not become zero in the group $H^3(H(n), H'(n))$.

The cocycle $\Gamma_{ij}$ (for certain $i < j, j \neq i'$) cannot be the coboundary of an element $h \in C^2(H(n), A(n))$. Indeed we have that $\Gamma_{ij}(x_i, x_j, x_i, x_j, x_i, x_j, x_i, x_j) = 4$ while the element $dh(x_i, x_j, x_i, x_j, x_i, x_j)$ cannot contain the monomial 1 since the bracket of any two of the above elements vanish and all the three elements have degree greater or equal to 0.

Assume now that $n \equiv -4 \mod p$ and suppose, by absurd, that the cocycle $\Xi$ is the coboundary of a cochain $f \in C^2(H(n), A(n))$. For a multindex $0 \leq a \leq \sigma$, call $\phi(x^a) = -\phi(x^{\sigma-a})$ the coefficient of 1 in the element $f(x^a, x^{\sigma-a})$. Consider a triple of elements $(x_i, x_i, x_i) \in H(n) \times H(n) \times H(n)$ such that $a + b + c = \sigma_0 + \epsilon_0 + \epsilon_0$ (for a certain $k$) and $\deg(x^a) = \deg(x^b) = \deg(x^c) \geq 0$. By taking the coefficient of 1 in the equality $\Xi(x_i, x_i, x_i) = df(x_i, x_i, x_i)$ and using the relations $c_k = -a_k - b_k \mod p$ and $c_k = -a_k - b_k \mod p$, we get that

$$\phi(x^a) + \phi(x^b) = \phi(x^{a+b+c}) + 1 \quad \text{if} \quad a_kb_k - a_kb_k \not\equiv 0 \mod p.$$

By considering triples as above with $\deg(x^a) = \deg(x^b) = 0$, we get the relations $2\phi(x_i, x_j) = \phi(x_i^2) + \phi(x_j^2)$ and $2 = \phi(x_i^2) + \phi(x_j^2)$, from which we deduce that
the restriction of \( \phi \) to \( H(n)_0 \) is determined by the values \( \phi(x_i^2) \) for \( 1 \leq i \leq m \).

Analogously, by taking \( \deg(x^a) = 0 \) and \( \deg(x^b) = 1 \), one gets that the restriction of \( \phi \) to \( H(n)_1 \) is determined by the value \( \phi(x_1^1) \) together with the restriction of \( \phi \) to \( H(n)_0 \). Finally, by taking \( \deg(x^a) = 1 \) and \( \deg(x^b) = d \leq n(p - 1) - 5 \), one gets that the values of \( \phi \) on \( H(n)_{d+1} \) are determined by the values of \( \phi \) on \( H(n)_1 \) and on \( H(n)_d \). Therefore the values of \( \phi \) on the elements having degree \( 0 \leq d \leq n(p - 1) - 4 \) is determined by the values \( \phi(x_i^2) \) for \( 1 \leq i \leq m \) and \( \phi(x_1^1) \).

Explicitly, for an element \( x^n \in H(n) \) such that \( 0 \leq \deg(x^n) \leq n(p - 1) - 4 \), one gets the following formula

\[
\phi(x^n) = \left( \sum_{i=1}^{n} a_i - 2 \right) \phi(x_1^1) + \left( -a_1 - 2a_{1'} - \sum_{j \neq 1, 1'} 3a_j + 3 \right) \phi(x_1^2) + \\
+ \sum_{k=2}^{m} \frac{a_k - a_{k'}}{2} \phi(x_k^2) + \sum_{h=m+1}^{n} a_h.
\]

Imposing the antisymmetric relation \( \phi(x^{σ−a}) = −\phi(x^a) \), we get the relation

\[
-(n + 4)\phi(x_1^1) + \frac{3(n + 4)}{2} \phi(x_1^2) - \frac{n}{2} = 0,
\]

which is impossible by the hypothesis \( n \equiv -4 \pmod{p} \) and \( p \neq 2 \).

Finally, the cocycles belonging to \( H^3(H(n), H(n)_{-1}; (1)_F) \) are not in the image of the coboundary map \( \partial \) of above. Indeed, consider a cohomology class of \( H^3(H(n), (1)_F) \) coming from \( H^2(H(n), H'(n)) \) and choose a representative \( f \in Z^2(H(n), (1)_F) \) such that \( f = \partial g \) where \( g \in Z^2(H(n), H'(n)) \). Since \( g \) takes values in \( H'(n) = A(n)_{\geq 0} \), then the cocycle \( f \) vanish on the 3-tuples of elements having non-negative degree. On the other hand, if \( f \) belongs to \( Z^2(H(n), H(n)_{-1}; (1)_F) \), then by definition it must vanish on the 3-tuples of elements such that at least one has negative degree. Putting together these two vanishings, we deduce that \( f = 0 \).

\[
\square
\]

Now, using the above Proposition, we can prove the Main Theorem \[12\].

**Proof of Theorem \[12\]** From the exact sequence \[52\] and using that \( H^1(H(n), \langle x^a \rangle_\bar{F}) = 0 \), we get the exact sequence

\[
0 \to H^2(H(n), H(n)) \to H^2(H(n), H'(n)) \to H^2(H(n), \langle x^a \rangle_\bar{F}),
\]

so that we have to check which of the cocycles of the above Proposition \[53\] go to 0 under the projection onto \( H^2(H(n), \langle x^a \rangle_\bar{F}) \). Clearly the cocycles \( \text{Sq}(x_i) \) take values in \( H(n) = [H'(n), H'(n)] \) by definition.

Consider the cocycles \( \Pi_{ij} \in H^2(H(n), H'(n)) \). If \( j \neq i, i' \) then the projection of \( \Pi_{ij} \) onto \( H^2(H(n), \langle x^a \rangle_\bar{F}) \) is 0. In fact \( \Pi_{ij}(x^a, x^{b}) \subset \langle x^a \rangle_\bar{F} \) if and only if \( a + b = \sigma - (p - 1)\epsilon_i - (p - 1)\epsilon_{i'} + \epsilon_j + \epsilon_{j'} \) but, for these pairs of elements, it is easily checked that \( D_i(x^a)D_j(x^b) - D_j(x^a)D_i(x^b) = (a_i b_j - a_j b_i) x^{a+b-\epsilon_i-\epsilon_j} = 0 \). On the other hand, if \( i = i' \), then the only non-zero values of \( \Pi_{ii} \) are given by

\[
\Pi_{ii}(x^a, x^{b}) = x^{a+b+(p-1)\epsilon_i+(p-1)\epsilon_{i'}} \quad \text{for} \quad a + b \leq \sigma_i.
\]

Therefore, if \( n = 2 \), the cocycle \( \Pi_{12} \) satisfy \( \Pi_{12}(x_1, x_2) = x^a \) and hence it cannot be lifted to \( H^2(H(n), H(n)) \). On the other hand, for \( n \geq 4 \), if we define \( g_i \in C^1(H(n), H'(n)) \) by \( g_i(x^a) = x^a \), then the only non-zero values of the coboundary \( d g_i \) (for \( 1 \leq i \leq m \)) can be

\[
\begin{align*}
d g_i(x^a, x^{b}) &= -g_i([x^a, x^{b}]) = -x^a \quad \text{if} \quad a + b = \sigma_i, \\
d g_i(x, x^a) &= [x, g_i(x^a)] = -\sigma (k)x^{a-\epsilon_i} \quad \text{for} \quad 1 \leq k \leq n.
\end{align*}
\]
Therefore $\Pi_i = \Pi_{i\sigma'} + dq_i$ and clearly $\Pi_i \in H^2(H(n), H(n))$ since it vanish on the pairs $(x, x^a, x^{a'} x^b)$ such that $a + b = \sigma'$.

Consider now the coycle $\Phi$. We want to prove that its projection onto $H^2(H(n), \langle x^\sigma \rangle_F)$ vanish. From the explicit description of $\Phi$, it follows that its projection onto $\langle x^\sigma \rangle_F$ is given by

$$\Phi(x^a, x^b) = \sum_{|\delta| \geq 3, \hat{\delta} < a, \hat{\delta} + \delta = a + b} \binom{a}{\delta} \binom{b}{\hat{\delta}} \sigma(\delta) \delta! x^\sigma,$$

where the above sum is set equal to 0 if there are no elements $\delta$ verifying the hypothesis. Each element $\delta$ verifying the above hypothesis contributes to the summation with the coefficient

$$\sigma(\delta) \delta! \binom{a}{\delta} \binom{b}{\hat{\delta}} = -\sigma(\delta) \delta! \binom{a}{\delta} \binom{a - \delta}{\delta} = -\frac{\sigma(\delta)}{\delta!} \frac{a!}{(a - \delta - \hat{\delta})!},$$

where in the first equality we substitute $b = \sigma - a + \delta + \hat{\delta}$ and we use the relation $(\sigma - \hat{\delta}) = (-1)^{|\delta| + |\hat{\delta}|}$ which follows from the congruence $k!(p - 1 - k)! \equiv (-1)^{k+1} \mod p$ (for $0 \leq k \leq p - 1$). Now note that if a certain $\delta$ appears in the above summation, then it also appears its conjugate $\hat{\delta}$ and we have that $\delta \neq \hat{\delta}$ because of the oddness of the degree $|\delta|$. Using the easy relations $\delta! = \hat{\delta}!$ and $\sigma(\delta) = (-1)^{|\delta|} \sigma(\hat{\delta}) = -\sigma(\hat{\delta})$, it follows that the contributions of $\delta$ and $\hat{\delta}$ are opposite and therefore the sum vanish.

$\square$

3.3. Cohomology of the trivial module. In this section we compute the second and third cohomology group of $H(n)$ with coefficients in the trivial module $F$.

**Proposition 3.4.** The second cohomology group of the trivial module is equal to

$$H^2(H(n), F) = \begin{cases} \bigoplus_{i=1}^n \langle \Omega_i \rangle_F \bigoplus \langle \Sigma \rangle_F & \text{if } n \not\equiv -4 \mod p, \\ \bigoplus_{i=1}^n \langle \Omega_i \rangle_F \bigoplus \langle \Sigma \rangle_F \bigoplus \langle \Delta \rangle_F & \text{otherwise}, \end{cases}$$

where the only non-zero values of the above cocycles are

$$\begin{cases} \Omega_i(x^a, x^b) = a_i & \text{if } a + b = \sigma + \epsilon_i - (p - 1)\epsilon_i', \\ \Sigma(x_k, x_{k'}) = \sigma(k), \\ \Delta(x^a, x^b) = \deg(x^a) & \text{if } a + b = \sigma. \end{cases}$$

**Proof.** Note that the cochain $\Delta$ is antisymmetric if and only if $n \equiv -4 \mod p$, because if $a + b = \sigma$ then $\deg(x^a) + \deg(x^b) = n(p - 1) - 4 \equiv -n - 4 \mod p$.

The verification that the above cocycles are cocycles and are independent in $H^2(H(n), F)$ is straightforward and is left to the reader. We conclude by [FARS64] Thm 2.4, which gives that

$$\dim_F H^2(H(n), F) = \begin{cases} n + 1 & \text{if } n \not\equiv -4 \mod p, \\ n + 2 & \text{otherwise}. \end{cases}$$

$\square$

In order to compute $H^3(H(n), F)$, we use the Hochschild-Serre spectral sequence (see [HS53]) relative to the subalgebra $H(n)_{-1} < H(n)$:

$$E_2^{s,t} = H^s(H(n)_{-1}; C^t(H(n)/H(n)_{-1}, F)) \Rightarrow H^{s+t}(H(n), F).$$

For the first line of the second page of the above spectral sequence, we have the equality

$$E_2^{0,0} = H^n(H(n), H(n)_{-1}; F),$$

and clearly $H^1(H(n)_{-1}; C^0(H(n)/H(n)_{-1}, F)) = 0$ since $H^1(H(n)_{-1}; F) = 0$. Therefore, by [HS53] Thm 3.1,$ E_2^{0,0} = H^0(H(n), H(n)_{-1}) = 0$. Hence

$$E_2^{0,0} = H^n(H(n), H(n)_{-1}; F) = 0.$$
where $H^*(H(n), H(n)_{-1}; F)$ are the relative cohomology groups of $H(n)$ with respect to the subalgebra $H(n)_{-1}$ with coefficients in the trivial module $F$ (as defined in \cite{C-FAS}).

Moreover, as remarked in \cite{V-W1} Sec. 2.1, we can restrict ourselves to consider homogeneous cohomology with respect to the maximal torus $T_H \subset H(n)$ (see Prop. 3.2).

**Proposition 3.5.** The third cohomology group of the trivial module is equal to

\[
H^3(H(n), F) = \begin{cases} 
H^3(H(n), H(n)_{-1}; F) \bigoplus_{i \leq j, i \neq j'} (\Gamma_{ij})_F & \text{if } n \not\equiv -4 \mod p, \\
H^3(H(n), H(n)_{-1}; F) \bigoplus_{i \leq j, i \neq j'} (\Gamma_{ij})_F \bigoplus (\Xi)_F & \text{otherwise}.
\end{cases}
\]

where, by definition, the only non-zero values of the above cocycles are (for $j \neq i, i'$)

\[
\Gamma_{ij}(x^a, x^b, x^c) = a_ib_j - a_jb_i \quad \text{if } a + b + c = \sigma - (p-1)\epsilon_i - (p-1)\epsilon_{i'} + \epsilon_i + \epsilon_j,
\]

\[\Xi(x^a, x^b, x^c) = \sigma(k)[a_kb_j - a_jb_k] \quad \text{if } a + b + c = \sigma + \epsilon_k + \epsilon_{k'} \quad \text{for some } k.
\]

**Proof.** The verification that the above cochains are cocycles is straightforward and left to the reader. In order to show that they freely generate the third cohomology group, we divide the proof into four steps according to the spectral sequence (3.3).

**STEP I** : \((E_1^{0,3})_0 = H^0(H(n), F)_0 \equiv 0\) by homogeneity.

**STEP II** : \((E_1^{0,3})_2 = \bigoplus_{i < j, i \neq j'} (\Gamma_{ij})_F\).

With the notations of Lemma 3.7 below, consider a cochain $\zeta = \sum_{k=1}^n d_k\zeta_k \in (E_1^{0,3})_2$ and suppose that it can be lifted to a global cocycle in $Z^3(H(n), F)_2$ (which we continue to call $\zeta$). Consider the following cocycle condition

\[
d\zeta(x_i, x_{i'}, x_i^2, x_{i'}^2) = -4\sigma(i)\zeta(x_i, x_{i'}, x_i, x_{i'}) = -4\sigma(i)[\sigma(i')d_i - \sigma(i)d_i] = 4[d_i + d_{i'}],
\]

from which we deduce that $\zeta$ is the coboundary of the cocycle $f \in (E_1^{0,3})_2 = H^2(H(n), F)_2$ defined by $f(x_i, x_{i'}) = -d_i - d_{i'}$, since we have (for $j \neq i, i'$)

\[
\zeta(x_i x_{i'}, x_i, x_{i'}) = \sigma(i')d_{i'} - \sigma(i)d_i = d(f)(x_i, x_{i'}, x_i, x_{i'}).
\]

Suppose now that $n \geq 4$. The cocycles $\Gamma_{ij}$ with $j \neq i, i'$ appearing in Lemma 3.7 are clearly lifted by the cocycles $\Gamma_{i'i}$. On the other hand, the cocycles $\Gamma_{ii'}$ cannot be lifted to $Z^3(H(n), F)_2$. Indeed, by absurd, suppose that we can find such a lift and call it $\Gamma_{ii'} \in Z^3(H(n), F)_2$. We can suppose that $\Gamma_{ii'}$ takes its non-zero values on the triples $(x^a, x^b, x^c)$ such that $a + b + c = \sigma^i + \epsilon_i + \epsilon_{i'}$, where $\sigma^i := \sigma - (p-1)\epsilon_i - (p-1)\epsilon_{i'}$. Consider the following cocycle condition (where $a, b, c$ are multindices verifying $a + b + c = \sigma^i$):

\[
0 = \sigma(i)\Gamma_{ii'}(x_i^2 a_i, x_i, x_i^2 b_i, x_i^2) = -2\Gamma_{ii'}(x_i x^a, x_i x^b, x^c) = 2\Gamma_{ii'}(x_i x^{a+b}, x_i x^{b}, x^c).
\]

We deduce that the value of $\Gamma_{ii'}(x_i x^a, x_i x^b, x^c)$ depends only on the multindex $c$ and therefore, for every $0 \leq c \leq \sigma^i$, we can define $\omega(c) := \Gamma_{ii'}(x_i x^a, x_i x^b, x^c)$ for every pair of indices $a, b$ such that $a + b + c = \sigma^i$. By the fact that $\Gamma_{ij}|_{H(n)_{-1} \times H(n)_{-1}} = \Gamma_{ij}$, we get

\[
\omega(\sigma^i) = \Gamma_{ij}(x_i, x_{i'}, x_i^\sigma^i) = \Gamma_{ij}(x_i, x_{i'}, x_i^\sigma^i) = 1,
\]

\[
\omega(\epsilon) = \Gamma_{ij}(x_i x^{\sigma^i - \epsilon}, x_i, x_i) = \Gamma_{ij}(x_i x^{\sigma^i - \epsilon}, x_i, x_i) = 0.
\]

Finally consider the following cocycle condition where $j \neq i, i'$ and $0 < d \leq \sigma^i$ is a multindex such that $d_{i'} > 0$:

\[
0 = \sigma(j)\Gamma_{ii'}(x_i, x_i^{\sigma^i - d + \epsilon_i}, x_i, x_i, x_i^d) = d_j[\omega(d) - \omega(d - \epsilon_{i'})],
\]
where we used that \( \omega(\epsilon_j) = 0 \). We deduce that the value \( \omega(d) \) does not depend on the coefficient \( d_j \), and, by repeating for every index \( j \neq i', \) we conclude that \( \omega \) must be constant. But this contradicts with \( \omega(\epsilon_j) = 0 \) and \( \omega(\sigma') = 1 \).

**STEP III**: \((E^{2,1}_\infty) = \begin{cases} \Xi_F & \text{if } n \equiv -4 \mod p, \\ 0 & \text{otherwise.} \end{cases}\)

With the notations of Lemma 3.3 below, consider the cochain \( \xi = \sum_{k=1}^n e_k \xi_k \) and suppose that it can be lifted to a global cocycle of \( Z^3(H(n), F)_\infty \) (which, as usual, we continue to call \( \xi \)). From the following cocycle condition

\[
0 = d\xi(x_k', x^2_k, x^2_k, x^{\sigma - \epsilon_k'}') = \sigma(k')\xi(x^2_k, x^2_k, x^{\sigma - \epsilon_k - \epsilon_k'}) + 4\epsilon_k
\]
together with the analogues one obtained interchanging \( k \) with \( k' \), we get that \( e_k = e_{k'} \).

If \( n \equiv -4 \mod p \), then the cochain \( \sum_{i=1}^n \xi_i \) is lifted by the global cocycle \( \Xi \in (E^{2,1}_\infty) \). We will show that under the assumption that either \( n \not\equiv -4 \mod p \) or \( n \equiv -4 \mod p \) and \( \sum_{i=1}^n e_i = 0 \), then \( \xi \) belongs to the image of the differential map

\[
d : (E^{1,1}_1) \to (E^{2,1}_1)_,
\]
coming from the spectral sequence (3.3).

Consider the cochains \( \eta_k \in (E^{1,1}_1)_\infty = H^1(H(n)-1, C^1(H(n)/H(n)-1, F))_\infty \) (for \( 1 \leq k \leq m \)), whose only non-zero values are given by

\[
\eta_k(x_k, x^{\sigma - \epsilon_k'}) = \eta_k(x_k', x^{\sigma - \epsilon_k'}) = -1.
\]

Form the cochain \( \eta := \sum_{i=1}^m (\frac{\nu}{2} + \beta) \eta_i \), where \( \beta \in F \) is defined as

\[
\beta = \begin{cases} \\
-\sum_{i=1}^m \frac{e_i}{n+4} & \text{if } n \not\equiv -4 \mod p, \\
0 & \text{if } n \equiv -4 \mod p \text{ and } \sum_{i=1}^m e_i = 0.
\end{cases}
\]

It is straightforward to check that the cocycle \( \xi - d\eta \in C^1(H(n)-1, C^2(H(n)/H(n)-1, F))_\infty \) is the coboundary of the cocycle \( g(x^a, x^{\sigma - a}) = \sum_{i=1}^m (a_i + a_j) \frac{e_i}{2} + deg_H(x^a) \beta \) if \( |a|, |a - \sigma| \geq 2 \).

This shows that \( \lbrack \xi \rbrack = \lbrack d\eta \rbrack \in (E^{2,1}_\infty) \).

Suppose next that the cocycle \( \rho_{ij} \) (for certain \( i < j \)) can be lifted to a global cocycle of \( Z^3(H(n), F) \), which we continue to call \( \rho_{ij} \). For \( l = 2, \cdots, p - 1 \), we define \( f_l := \rho_{ij}(x_i x_j', x^{\sigma - l \epsilon_j'}, x^{\sigma - (p - 1) \epsilon_i - (p + 1 - l) \epsilon_j'}) \). Consider the following cocycle condition for \( 1 \leq l \leq p - 1 \):

\[
0 = d\rho_{ij}(x_i x_j', x'_j, x^{\sigma - l \epsilon_j'}, x^{\sigma - (p - 1) \epsilon_i - (p + 1 - l) \epsilon_j'}) = \frac{1 + \delta_{j'}}{l} + \delta_{1l}(1 + \delta_{j'}) - \sigma(j)(l + 1)f_{l+1} + \sigma(j)(l - 1)f_l.
\]

The above equation (*) with \( l = 1, \cdots, p - 2 \) gives that

\[
f_l = \frac{-(1 + \delta_{j'})(l - 2)\sigma(j)}{(l - 1)l} \quad \text{for } l = 2, \cdots, p - 1.
\]

Substituting in the above equation (*) with \( l = p - 1 \), we get

\[
0 = \frac{-(1 + \delta_{j'})(p - 2)\sigma(j)}{p - 1} + \sigma(j)(p - 2)\frac{-(1 + \delta_{j'})(p - 3)\sigma(j)}{(p - 2)(p - 1)} = (1 + \delta_{j'}) - (1 + \delta_{j'}),
\]

which is impossible since \( p \neq 2 \). Therefore the cocycles \( \rho_{ij} \) do not belong to \((E^{2,1}_\infty) \).
From Proposition 3.4 one can easily deduce that
\[
\begin{align*}
(E_{0}^{0,2})_{\mathfrak{h}} &= (E_{2}^{0,2})_{\mathfrak{h}} = (\Sigma)_{F}, \\
(E_{1}^{1,1})_{\mathfrak{h}} &= (E_{1}^{1,1})_{\mathfrak{h}} = \left\{ \begin{array}{ll}
\bigoplus_{n=1}^{n} (\Omega_{n})_{F} \oplus (\Delta)_{F} & \text{if } n \equiv -4 \mod p, \\
\bigoplus_{n=1}^{n} (\Omega_{n})_{F} & \text{otherwise}.
\end{array} \right.
\end{align*}
\]
This implies that \((E_{3}^{3,0})_{\mathfrak{h}} = (E_{3}^{3,0})_{\mathfrak{h}}\) and the result follows from equality (3.3).

\[\square\]

Remark 3.6. It can be proved that \(H^{3}(H(n), H(n)_{-1}; F) = \oplus_{i=1}^{n} (\Upsilon_{i})_{F}\) where the cocycles \(\Upsilon_{i}\) are defined by
\[
\Upsilon_{i}(x^{a}, x^{b}, x^{c}) = \sigma(k)[a_{k}b_{k} - a_{k}b_{k}]\text{ if } a + b + c = \sigma + p\varepsilon_{k} + \varepsilon_{k} + \varepsilon_{k}\text{ for some } k.
\]
We omit the proof, since we do not need this result to prove the Main Theorem 7.2.

Lemma 3.7. In the above spectral sequence (3.3), we have that
\[
(E_{1}^{1,2})_{\mathfrak{h}} = \left\{ \begin{array}{ll}
\bigoplus_{k=1}^{n} (\eta_{k})_{F} \oplus (\Gamma_{ij})_{F} & \text{if } n \geq 4, \\
\bigoplus_{k=1}^{n} (\eta_{k})_{F} & \text{if } n = 2,
\end{array} \right.
\]
where the only non-zero values of the above cocycles are
\[
\begin{align*}
\eta_{k}(x_{i}, x_{j}, x^{\sigma-(p-1)c_{i} - (p-1)c_{j}}) &= 1, \\
\eta_{k}(x_{h}, x_{h}, x_{h}) &= \sigma(h) \text{ for } h = 1, \ldots, n.
\end{align*}
\]
Proof. Consider the exact sequence
\[(3.5)\quad 0 \rightarrow C^{1}(x^{a}, F) \rightarrow C^{1}(H'(n)/H'(n)_{-1}, F) \rightarrow C^{1}(H(n)/H(n)_{-1}, F) \rightarrow 0,
\]
where \(C^{1}(x^{a}, F)\) is a trivial \(H(n)_{-1}\)-module. The coboundary map
\[
H^{1}(H(n)_{-1}, C^{1}(H'(n)/H'(n)_{-1}, F))_{\mathfrak{h}} \stackrel{\partial(2)}{\rightarrow} H^{2}(H(n)_{-1}, C^{1}(x^{a}, F))_{\mathfrak{h}}
\]
is surjective. Indeed consider the cocycles \(\eta_{k} \in H^{1}(H(n)_{-1}, C^{1}(H(n)/H(n)_{-1}, F))_{\mathfrak{h}}\) (with \(1 \leq k \leq n\), defined as
\[
\eta_{k}(x_{k}, x^{\sigma+\varepsilon_{i}}) = \eta_{k}(x_{k}, x^{\sigma+\varepsilon_{i}}) = -1.
\]
It is easy to check that \(\partial(2)\) sends \(\eta_{k}\) into the cocycles \(\{(x_{k}, x_{k'}, x^{a}) \mapsto -2\}\) which generate the last group \(H^{2}(H(n)_{-1}, C^{1}(x^{a}, F))_{\mathfrak{h}}\).

Using the above surjectivity, together with the vanishing \(H^{3}(H(n)_{-1}, C^{1}(x^{a}, F))_{\mathfrak{h}} = 0\) which follows directly by homogeneity considerations, we get that
\[(E_{1}^{1,2})_{\mathfrak{h}} = H^{2}(H(n)_{-1}, C^{1}(H'(n)/H'(n)_{-1}, F))_{\mathfrak{h}}.
\]
Consider now the following exact sequence of \(H(n)_{-1}\)-modules
\[(3.6)\quad 0 \rightarrow C^{1}(H'(n)/H'(n)_{-1}, F) \rightarrow C^{1}(A(n), F) \rightarrow C^{1}(A(n)_{<0}, F) \rightarrow 0,
\]
obtained from the fact that \(H'(n)/H'(n)_{-1} = A(n)/A(n)_{<0}\) (see (3.1)). Using the following isomorphism of \(H(n)_{-1}\)-modules
\[
\chi : A(n) \xrightarrow{\cong} C^{1}(A(n), F)
\]
\[(3.7)\quad x^{a} \mapsto \chi x^{a}(x^{b}) = \begin{cases} 1 & \text{if } b = \sigma - a, \\
0 & \text{otherwise},
\end{cases}
\]
together with [VIIV Prop. 3.4], we get that
\[
H^{1}(H(n)_{-1}, C^{1}(H'(n)/H'(n)_{-1}, F))_{\mathfrak{h}} = H^{1}(H(n)_{-1}, C^{1}(A(n), F))_{\mathfrak{h}} = \bigoplus_{i=1}^{n} (\Omega_{i})_{F},
\]
where the cocycles $\delta k_i$ are defined by (and vanish outside) $\delta k_i(x_i, x_{\sigma^8}) = 1$.

Therefore we get the exact sequence

$$0 \to H^1(H(n-1), C^1(A(n), F)) \to H^2(H(n-1), C^1(A(n), F)) \to \cdots$$

The first group on the left is generated over $F$ by the cocycles $\delta k_i (k = 1, \ldots, n)$ defined by $\delta k_i(x_k, x_k) = 1$ and subject to the relation $\sum_{k=1}^n \sigma(k') \delta k_i = 0$ coming from the element $(1 \mapsto 1)F \in C^1(A(n), F).$ It is easily checked that $\partial(\delta k_i) = \zeta.$

Moreover, using the isomorphism (3.7) of $H(n-1)$-modules $A(n) \cong C^1(A(n), F)$ and [VIV1 Prop. 3.4], we get that $H^2(H(n-1), C^1(A(n), F))$ can be lifted to $H^2(H(n-1), C^1(H(n)/H(n-1), F)) \to 0$ if and only if $n \geq 4.$

**Lemma 3.8.** In the above spectral sequence (3.3), we have that

$$(E^{2,1}_1) = \begin{cases} \sum_{k=1}^n (\zeta_k)_F \oplus \rho_{i<j}(\rho_{i,j})_F & \text{if } n \geq 4, \\ \sum_{k=1}^n (\zeta_k)_F & \text{if } n = 2, \end{cases}$$

where the only non-zero values of the above cocycles are

$$\begin{cases} \zeta(x_i, x_{\sigma^8}) = \sigma(h)(1 + \delta h, k) & \text{for every } h = 1, \ldots, n, \\ \rho_{i,j}(x_i, x_{\sigma^8}) = \frac{\sigma(j)}{l} & \text{for every } l = 1, \ldots, p - 1, \\ \rho_{i,j}(x_i, x_{\sigma^8}) = \frac{\sigma(j)}{l} & \text{for every } l = 1, \ldots, p - 1. \end{cases}$$

**Proof.** Consider the following exact sequence of $H(n-1)$-modules

$$(3.8) 0 \to C^2(H(n)/H(n-1), F) \to C^2(A(n), F) \to C^1(A(n), F) \to 0,$$

It is easy to see that $C^1(A(n), F) \cong A(n)$ is generated by the cocycle $\zeta$ defined by $\zeta(1, x) = \zeta(x_i, x_{\sigma^8}) = 1$ (for every $i = 1, \ldots, n$) and that the image of $\zeta$ under the first coboundary map is non-zero and equal to $-\sum_{k=1}^n \zeta_k.$ Therefore, using the Lemma 3.3 below, we get that

$$(3.9) H^1(H(n-1), C^2(H(n)/H(n-1), F)) = \sum_{k=1}^n (\zeta_k)_F.$$

Consider finally the following exact sequence

$$(3.10) C^2(H(n)/H(n-1), F) \delta \to C^2(H(n)/H(n-1), \to C^2(H(n)/H(n-1), F),$$

where the map $\theta$ sends the cocycle $g$ into the cocycle $\theta(g)$ defined by $\theta(g)(x_i, x_{\sigma^8}) = g(x^8).$ By taking cohomology, we get the exact sequence

$$H^1(H(n-1), C^2(H(n)/H(n-1), F)) \to (E^{2,1}_1) = \sum_{k=1}^n (\zeta_k)_F.$$
Take a homogeneous cochain $f \in Z^1(H(n)_{-1}, C^2(A(n), F))$. The cocycle conditions for $f$ are $D_i \circ f_{D_j} = D_j \circ f_{D_i}$ for every $1 \leq i, j \leq n$.

**STEP I:** The cocycle $f$ verifies the following condition

\[0 = \sum_{k=-\alpha_i}^{\beta_i} (-1)^k \tilde{f}_{D_i}(x^{\alpha+k\epsilon_i}, x^{\beta-k\epsilon_i})\]

for every $(x^{\alpha}, x^{\beta})$ such that $\alpha_i + \beta_i \leq p - 1$.

For every pair $(x^{\alpha}, x^{\beta})$ as above (such that $\alpha_i + \beta_i \leq p - 1$), define

\[\phi_i(x^{\alpha}, x^{\beta}) := \sum_{k=-\alpha_i}^{\beta_i} (-1)^k \tilde{f}_{D_i}(x^{\alpha+k\epsilon_i}, x^{\beta-k\epsilon_i}).\]

We have to prove that $\phi_i(x^{\alpha}, x^{\beta}) = 0$. Using the cocycle conditions $D_j \circ \tilde{f}_{D_i} = \tilde{f}_{D_j} \circ D_i$ and a telescopic sum, it is easy to see that $(D_j \circ \phi_i)(x^{\alpha}, x^{\beta}) = 0$ for every $j \neq i$. From these conditions, we get that (for every index $j \neq i$)

\[\phi_i(x^{\alpha}, x^{\beta}) = \begin{cases} 0 & \text{if } \alpha_j + \beta_j < p - 1 \text{ for some } j \neq i, \\ (-1)^{|\beta| - \beta_i} \phi_i(x^{\alpha+\beta-\sigma+(p-1-\beta_i)\epsilon_i}, x^{\sigma-(p-1-\beta_i)\epsilon_i}) & \text{otherwise}. \end{cases}\]

So assume we are in the second case, that is $\alpha_j + \beta_j \geq p - 1$ for every $j \neq i$. Consider the same formula of above for the couple $(x^\alpha, x^\beta)$. By using the antisymmetry of $\phi_i$ and the property $\phi_i(x^{\alpha+\alpha_i, x^{\beta+\beta_i}}) = (-1)^{|\beta|} \phi_i(x^\alpha, x^\beta)$ for $-\alpha_i \leq d \leq p - 1 - \alpha_i$ and $-\beta_i \leq \beta \leq p - 1 - \beta_i$, we obtain

\[\left[(-1)^{|\beta|} + (-1)^{|\alpha|}\right] \phi_i(x^{\alpha+\beta-\sigma+(p-1-\beta_i)\epsilon_i}, x^{\sigma-(p-1-\beta_i)\epsilon_i}) = 0.\]

Now recall that $f$ is homogeneous and therefore we have to consider only the pairs $(x^{\alpha}, x^{\beta})$ such that the sum of the weights of $x^{\alpha}$, $x^{\beta}$ and $D_i$ is $\emptyset$. Using the conditions $\alpha_i + \beta_i \leq p - 1$ and $\alpha_j + \beta_j \geq p - 1$ for every $j \neq i$, we find the equalities (and not merely the congruences modulo $p$):

\[\alpha_i + \beta_i = \alpha_{i'} + \beta_{i'} + (p - 1)\quad \text{and} \quad \alpha_j + \beta_j = \alpha_{i'} + \beta_{i'} \quad \text{for every } j \neq i, i'.\]

We deduce that $|\alpha| + |\beta|$ is even and, substituting in the expression above, we get the required vanishing.

**STEP II:** The cocycle $f$ is a coboundary.

We have to find an element $g \in C(A(n), F)$ such that $f_{D_i} = D_i \circ g$. For a homogeneous pair $(x^a, x^b)$ (that is a pair such the sum of the weights of $x^a$ and $x^b$ is $\emptyset$), we define

\[\tilde{g}(x^a, x^b) = \begin{cases} \sum_{k=0}^{\gamma} (-1)^k \tilde{f}_{D_i}(x^{a+k\epsilon_i}, x^{b+(k+1)\epsilon_i}) & \text{if } a_i + b_i < p - 1, \\ \sum_{\emptyset \neq \mathcal{I} \subseteq \{1, \ldots, n\}} \left(-1\right)^{|\mathcal{I}| - 1} \text{sign}(c, d) \tilde{f}_{D_i}(x^a, x^b) & \text{if } a + b \geq \sigma, \end{cases}\]

where for a non-empty subset $I$ of $\{1, \ldots, n\}$ (of cardinality $|I|$), we define $S_I(a, b)$ to be the set of pairs $(c, d)$ of multindices verifying: $c_i + d_i = a_i + b_i + 1$ and $\min(a_i, b_i) + 1 \leq c_i, d_i \leq \max(a_i, b_i)$ if $i \in I$ and $c_j = a_j, d_j = b_j$ if $j \notin I$ (in particular $S_\emptyset(a, b) = \emptyset$ if and only if $a_i \neq b_i$ for $i \in I$). Moreover, if $(c, d) \in S_I(a, b)$, we put $\text{sign}(c, d) = \prod_{i \in I} \text{sign}_i(c, d)$ and $\text{sign}_i(c, d) = d_i - b_i$ or $c_i - a_i$ according, respectively, to the cases $b_i < a_i$ and $a_i < b_i$. Finally, if $I = \{i_1, \ldots, i_r\}$, we put $f_{D_I} := D_{i_1} \circ \cdots \circ D_{i_r} \circ f_{D_i}$, which does not depend upon the order of the elements of $I$ by the cocycle conditions verified by $f$. 
The cochain $g$ is well-defined because if $i$ and $j$ are two indices such that $a_i + b_i < p - 1$ and $a_j + b_j < p - 1$, then the following expression

$$\tilde{g}(x^a, x^b) = \sum_{k=0}^{a_i} \sum_{h=0}^{a_j} (-1)^{k+h} (D_i \circ f_{D_j})(x^{a-k_{e_i}-h_{e_j}}, x^{b+(k+1)e_i+(h+1)e_j})$$

is symmetric in $i$ and $j$ because of $D_i \circ f_{D_j} = D_j \circ f_{D_i}$ and reduces, via a telescopic sum, to the first expression occurring in the definition of $\tilde{g}$.

Moreover it is clear from the definition that $\tilde{g}$ is antisymmetric in the case $a + b \geq \sigma$, while in the case $a_i + b_i < p - 1$ (for a certain $i$) the antisymmetry follows from the condition $(\ast)$ of above.

Finally we have to check that $(D_i \circ \tilde{g})(x^a, x^b) = \int_{D_i}(x^a, x^b)$ for every index $i$ and every pair $(x^a, x^b)$ such that the sum of the weights of $x^a$, $x^b$ and of $D_i$ is $\tilde{g}$.

If $\alpha_i = \beta_i = 0$ then $(D_i \circ \tilde{g})(x^a, x^b) = 0$ and $\int_{D_i}(x^a, x^b) = 0$ by the condition $(\ast)$ of above. If $\alpha_i = 0$ and $\beta_i < 0$ then we get that $(D_i \circ \tilde{g})(x^a, x^b) = -\tilde{g}(x^a, x^{b-c_i})$ is equal to $\int_{D_i}(x^a, x^b)$ by the first case of the definition of $\tilde{g}$. The case $\alpha_i > 0$ and $\beta_i = 0$ follows from the preceding one by the antisymmetry of $g$. Therefore we are left with the case $\alpha_i, \beta_i > 0$.

Suppose first that $\alpha + \beta - c_i \neq \sigma$. Take an index $j$ (may be equal to $i$) such that $(\alpha + \beta - c_i)_j < p - 1$. Using the first case of the definition of $g$, we have

$$(D_i \circ \tilde{g})(x^a, x^b) = \sum_{k=0}^{\alpha_i} (-1)^{k+1}(D_i \circ \int_{D_j})(x^{a-k_{e_i}}, x^{b+(k+1)e_i}) =$$

$$= \sum_{k=0}^{\alpha_i} (-1)^{k+1}(D_i \circ \int_{D_j})(x^{a-k_{e_i}}, x^{b+(k+1)e_i}) = \int_{D_i}(x^a, x^b),$$

where in the last equality we used a telescopic summation.

On the other hand, suppose that $\alpha + \beta - c_i \geq \sigma$. We need two auxiliary facts before proving the required equality in this case. First of all, observe that the hypothesis $\alpha + \beta - c_i \geq \sigma$ forces the equalities (and not merely the congruences modulo $p$) $\alpha + \beta - 1 = \alpha_i + \beta_i$ and $\alpha_j + \beta_j = \alpha_i + \beta_i$ for every $j \neq i, i'$. Therefore the sum of the degrees of the multindices $|\alpha| + |\beta|$ must be odd. Moreover, we can re-write the second expression occurring in the definition of $g$ in a way that will be more suitable for our purpose. Fix an index $i$, a homogeneous pair $(x^a, x^b)$ satisfying $a + b \geq \sigma$ and suppose that $a_i < b_i$. By splitting the summation occurring in the definition of $\tilde{g}(x^a, x^b)$ according to the cases $I = \{i\}$, $I = \{i\} \cup J$ and $I = J$ with $i \notin J \neq \emptyset$, and using a telescopic summation, we get

$$(**): 2\tilde{g}(x^a, x^b) = \sum_{k=1}^{b_i-a_i} (-1)^k \int_{D_i}(x^{a+k_{e_i}}, x^{b+(1-k)e_i}) +$$

$$+ \sum_{i \notin J \neq \emptyset} \sum_{(c,d) \in S_i(a,b)} (-1)^{|J|+b_i-a_i+1}\text{sign}(c,d) \int_{D_j}(x^{c+(b_i-a_i)e_i}, x^{d-(b_i-a_i)e_i}).$$

If $a_i = b_i$ then the above expression is trivially true while if $a_i > b_i$ then we get an analogous expression using the antisymmetry of $g$.

Finally, in order to prove the required equality $(D_i \circ \tilde{g})(x^a, x^b) = \int_{D_i}(x^a, x^b)$, we have to distinguish two cases: $\alpha_i < \beta_i - 1$ and $\alpha_i = \beta_i$ (the case $\alpha_i > \beta_i$ follows by antisymmetry). In the first case $\alpha_i < \beta_i$, consider

$$-2(D_i \circ \tilde{g})(x^a, x^b) = 2\tilde{g}(x^{a-c_i}, x^b) + 2\tilde{g}(x^a, x^{b-c_i})$$

and apply formula $(**) \text{to the terms} (x^{a-c_i}, x^b) \text{and} (x^a, x^{b-c_i})$, which verify the required conditions in virtue of our hypothesis. By summing the first terms in the
corresponding expressions (**), we get

\[ \widetilde{f}_D(x^\alpha,x^\beta) + (-1)^{\delta_i} \widetilde{f}_D(x^{\alpha+\delta_i\epsilon_i},x^{\beta-\delta_i\epsilon_i}), \]

where we put \( \delta_i := \beta_i - \alpha_i - 1 \geq 0 \). By summing the last terms in the corresponding expressions (***) and using that if \( i \notin J \) then a pair \((c,d)\) belongs to \( S_J(\alpha,\beta) \) if and only if \((c-\epsilon_i,d) \in S_J(\alpha-\epsilon_i,\beta) \) (and, analogously, if and only if \((c,d-\epsilon_i) \in S_J(\alpha,\beta-\epsilon_i) \)), we obtain

\[ \sum_{i \notin J} \sum_{(c,d) \in S_J(\alpha,\beta)} (-1)^{\delta_i+|J|}\text{sign}(c,d)(D_i \circ \widetilde{f}_D)(x^c+d^\epsilon_i,x^{d-\epsilon_i}). \]

Using that \( D_i \circ \widetilde{f}_D = D_J \circ \widetilde{f}_D \) and iterated telescopic summations, the above expression (***) reduces to

\[ (-1)^{\delta_i} \widetilde{f}_D(x^{\alpha+\delta_i\epsilon_i},x^{\beta-\delta_i\epsilon_i}) - (-1)^{|\beta|-|\alpha|}\widetilde{f}_D(x^{\beta},x^{\alpha}). \]

Summing expressions (***) and (**) and using the fact that \(|\beta| + |\alpha|\) is odd, we end up with \(-\widetilde{f}_D(x^\alpha,x^\beta) + \widetilde{f}_D(x^\beta,x^\alpha) = -2\widetilde{f}_D(x^\alpha,x^\beta),\) q.e.d.

The proof in the other case \( \alpha_i = \beta_i \) is similar apart from the fact that one has to use both the expression (**) and the analogous one with \( a_i > b_i \). We leave the details to the reader. \( \square \)

3.4. Cohomology of \( A(n) \). In this section we compute the second cohomology group of the \( H(n) \)-module \( A(n) \).

**Proposition 3.10.** The second cohomology group of \( A(n) \) is given by

\[ H^2(H(n),A(n)) = \begin{cases} \bigoplus_{i=1}^{n}(\text{Sym}^i(x_1))_F \bigoplus_{i=1}^{n}(\Pi_i)_F \bigoplus_{i<j}(\Pi_{ij})_F \bigoplus(\Phi)_F & \text{if} \ n \neq -4 \mod p, \\ \bigoplus_{i=1}^{n}(\text{Sym}^i(x_1))_F \bigoplus_{i=1}^{n}(\Pi_i)_F \bigoplus(\Pi_{ij})_F \bigoplus(\Phi)_F \bigoplus(\Delta)_F & \text{otherwise}. \end{cases} \]

where \( \Pi_i \) and \( \Delta \) are the cocycles of Proposition 3.4, \( \Phi \) and \( \Pi_{ij} \) (with \( j \neq i' \)) are the cocycles of Theorem 2.2 and the remaining cocycles \( \Pi_{ii'} \) are defined by (and vanish outside):

\[ \Pi_{ij}(x_1,x_2,\ldots,x_n) = x_1^{a+b+(p-1)c_i+(p-1)c_{i'}}, \quad \text{if} \ a+b \leq \sigma_i. \]

**Proof.** It is straightforward to verify that the above cochains are cocycles and that they are independent in \( H^2(H(n),A(n)) \). Therefore it is enough to prove that

\[ \dim_F H^2(H(n),A(n)) = \begin{cases} \binom{n}{2} + 2n + 1 & \text{if} \ n \neq -4 \mod p, \\ \binom{n}{2} + 2n + 2 & \text{otherwise}. \end{cases} \]

It is easily seen that \( A(n) \) is the restricted \( H(n) \)-module induced from the restricted trivial \( H(n) \)-submodule \( F \cong (x_1^a)_F \subset A(n) \). Moreover, it is also easy to see that the Lie algebra homomorphism \( \sigma : H(n)_0 \rightarrow F \) given by \( \sigma(x) := \text{tr}(ad_H(n)/H(n)_0) \) is trivial.

Moreover, using Lemma 3.12 it is straightforward to check that, in the notation of [FS91], we have the equality

\[ H^2(H(n),A(n)) := H^2(H(n)_{\geq 0},F) \oplus \bigwedge^2(H(n)/H(n)_{\geq 0}). \]

Therefore, using [FS91] Thm. 3.6(2), we get that

\[ H^2(H(n),A(n)) = H^2(H(n)_{\geq 0},F) \oplus \bigwedge^2(H(n)/H(n)_{\geq 0}). \]
Since \( \dim_F \Lambda^2(H(n)/H(n)_{>0}) = \binom{n}{2} \), we conclude using Proposition 3.11 below. \qed

**Proposition 3.11.** The second cohomology group of \( H(n)_{>0} \) with coefficients in the trivial module \( F \) is given by

\[
H^2(H(n)_{>0}, F) = \begin{cases} 
\bigoplus_{i=1}^n (\Lambda^i(x_1))_F \bigoplus (\Omega_i)_F \bigoplus (\Phi)_F & \text{if } n \neq -4 \mod p, \\
\bigoplus_{i=1}^n (\Lambda^i(x_1))_F \bigoplus (\Omega_i)_F \bigoplus (\Phi)_F \bigoplus (\Delta)_F & \text{otherwise.}
\end{cases}
\]

where \( \Omega_i \) and \( \Delta \) are the cocycles of Proposition 3.4 \( \Phi \) and \( \Pi_{ij} \) (with \( j \neq i' \)) are the cocycles of Theorem 1.2 and \( \text{Sq}(x_i) \) is the projection of \( \text{Sq}(x_i) \) onto \( (1)_F \cong F \).

**Proof.** We prove first that

\[
H^2(H(n)_{>0}, F) = H^2(H(n)_{>1}, F)^{H(n)_{>0}},
\]

where \( H(n)_{>0} \) acts on \( H(n)_{>1} \) via adjoint action. To this aim, consider the Hochschild-Serre spectral sequence with respect to the ideal \( H(n)_{>1} < H(n)_{>0} \):

(3.11) \[
E_r^{s,t} = H^r(H(n)_{>0}, H^s(H(n)_{>1}, F)) \Rightarrow H^{r+s}(H(n)_{>0}, F).
\]

By the Lemma 3.12 and by homogeneity, it follows that

\[
E_2^{1,1} = H^1(H(n)_{>0}, H^1(H(n)_{>1}, F)) = H^1(H(n)_{>0}, C^1(H(n)_{>1}, F))_{\Phi} = 0.
\]

Therefore we are left with showing that \( H^2(H(n)_{>0}, F) = 0 \). First of all, we prove that \( Z^1(H(n)_{>0}, F)_{\Phi} = 0 \). Indeed, a homogeneous element \( g \in C^1(H(n)_{>0}, F)_{\Phi} \) can only take the following non-zero values \( g(x_i,x_{i'}) = \alpha_i \cdot 1 \), with \( \alpha_i = \alpha_{i'} \in F \). The vanishing of \( g \) follows from the following cocycle condition

(\*\*)

\[
0 = dg(x_i^2,x_{i'}^2) = -4\sigma(i)g(x_i,x_{i'}) = -4\sigma(i)\alpha_i.
\]

Consider now a homogeneous cochain \( f \in C^2(H(n)_{>0}, F)_{\Phi} \). By applying the cocycle condition to the elements of \( T_H \) and using homogeneity, one gets that \( f_{ij}T_{ij} \in Z^1(H(n)_{>0}, 1)_{\Phi} \) which vanishes as proved above. Moreover, by adding to \( f \) a coboundary, we can suppose that \( f(x_i^2,x_{i'}^2) = 0 \) (see equation (\*\*) of above). Therefore the only non-zero values of \( f \) can be \( f(x_i,x_j,x_{i'}x_{j'}) = \alpha_{ij} \cdot 1 \) (for \( j \neq i, i' \)) with the obvious relations \( \alpha_{ij} = \alpha_{ji} \) and \( \alpha_{ij} = -\alpha_{i'j'} \). We conclude by mean of the following cocycle condition

\[
0 = df(x_i^2,x_{i'}x_j,x_{i'}x_{j'}) = -2\sigma(i)\alpha_{ij} + 2\sigma(i)\alpha_{ij'},
\]

which gives \( \alpha_{ij} = \alpha_{i'j'} = -\alpha_{ij} \) and hence \( \alpha_{ij} = 0 \).

In order to compute \( H^2(H(n)_{>1}, F)^{H(n)_{>0}} \), we will use the same strategy of Proposition 2.6 that is to compute, step by step as \( d \) increases, the truncated invariant cohomology groups

\[
H^2 \left( \frac{H(n)_{>1}}{H(n)_{>d+1}}, F \right)^{H(n)_{>0}}.
\]

By using the Hochschild-Serre spectral sequence associated to the ideal

\[
H(n)_d = \frac{H(n)_{>d}}{H(n)_{>d+1}} \subset \frac{H(n)_{>1}}{H(n)_{>d+1}},
\]

we obtain the same diagram as in [VIV1] Prop. 3.10 (the vanishing of \( E_2^{0,2} \) and the injectivity of the map \( \alpha \) are proved in exactly the same way) and then we take the cohomology with respect to \( H(n)_{>0} \). An easy inspection of their proof shows that the Lemmas 2.7, 2.9 and 2.10 of the preceding section (for the algebra \( K(2m+1) \))
can be easily adapted to the present case simply by ignoring the variable \(x_{2m+1}\).

In particular we get that (for \(d \geq 1\))

\[
C^i(H(n)_1 \times H(n)_d, F)^{H(n)_0} = \begin{cases} 
(\Phi_2)_F & \text{if } d = 1, \\
(\Psi_2)_F & \text{if } d = n(p - 1) - 5,
\end{cases}
\]

\[
C^i(H(n)_d, F)^{H(n)_0} = 0,
\]

\[
H^1(H(n)_0, C^i(H(n)_d, F)) = \begin{cases} 
\overline{\langle Sq(x_i) \rangle}_F & \text{if } d = p - 2, \\
\overline{\sum_{i=1}^n \langle \omega_i \rangle}_F & \text{if } d = n(p - 1) - p - 2,
\end{cases}
\]

where \(\Phi_2, \Psi_2\) and \(\omega_i\) are defined as in the case of \(K(n)\) but ignoring the part involving the variable \(x_{2m+1} = x_n\).

By definition \(\text{Sq}(x_i)\) is the restriction of \(\text{Sq}(x_1)\) and it is easy to see that \(\omega_i\) is the restriction of \(\Omega_i\). Moreover if we extend \(\Phi_2\) by 0 outside \(H(n)_1 \times H(n)_1\), then it is clear that \(\Phi_2 \in H^2(H(n)_0, F) \subset H^2(H(n)_1, F)^{H(n)_0}\) and that \(\Phi_2\) is the restriction of the cocycle \(\Phi\) (see also [VII Prop. 3.7]).

Finally, suppose that there is a lifting of \(\Psi_2\) to a global \(H(n)_0\)-invariant cocycle of \(Z^2(H(n)_1, A(n))\), which we will continue to call \(\Psi_2\). Then using the cocycle condition \(0 = d\Psi_2|_{H(n)}\) together with Lemma 3.12 and proceeding by induction on the degree, it is easy to see that \(\Psi_2\) must agree with \(\Delta\) on the couples \((x^a, x^b) \in H(n)_{1} \times H(n)_{1}\) such that \(a + b = \sigma\) and we know from Proposition 3.4 that \(\Delta\) is an antisymmetric cocycle if and only if \(n \equiv -4 \mod p\).

\[\square\]

Lemma 3.12. Let \(d\) be an integer greater or equal to \(-1\). Then

\[ [H(n)_1, H(n)_d] = H(n)_{d+1}. \]

Proof. The proof is the same as the first part of Lemma 2.5 (where we consider elements belonging to \(A(2n) \subset K(2m+1)\) except for the fact that we do not have to consider the elements \(x_i\) because they have degree \(-1\) and the element \(x^n\) which does not belong to \(H(n)\).

\[\square\]

References

[BW88] R.E. Block and R.L. Wilson: Classification of the restricted simple Lie algebras. J. Algebra 114 (1988), 115–259.

[CE48] C. Chevalley and S. Eilenberg: Cohomology theory of Lie groups and Lie algebras. Trans. Amer. Math. Soc. 63, (1948), 85–124.

[DG70] M. Demazure and P. Gabriel: Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs (French). Avec un appendice Corps de classes local par Michiel Hazewinkel. Masson and Cie, Editeur, Paris; North-Holland Publishing Co., Amsterdam, 1970.

[FAR86] R. Farnsteiner: Central extensions and invariant forms of graded Lie algebras. Algebras, Groups, Geom. 3 (1986), 431–455.

[FS88] R. Farnsteiner and H. Strade: Modular Lie algebras and their representation. Monographs and textbooks in pure and applied mathematics, vol. 116 (Dekker, New York, 1988).

[FS91] R. Farnsteiner and H. Strade: Shapiro’s Lemma and its consequences in the cohomology theory of modular Lie algebras. Math. Zeit. 206 (1991), 153–168.

[GER64] M. Gerstenhaber: On the deformation of rings and algebras. Ann. of Math. 79 (1964), 59–103.

[HS53] G. Hochschild and J.-P. Serre: Cohomology of Lie algebras. Ann. of Math. 57 (1953), 591–603.

[KS66] A. I. Kostrikin and I. R. Shafarevich: Cartan’s pseudogroups and the \(p\)-algebras of Lie (Russian). Dokl. Akad. Nauk SSSR 168 (1966), 740–742. English translation: Soviet Math. Dokl. 7 (1966), 715–718.

[MEL80] G. M. Melikian: Simple Lie algebras of characteristic 5 (Russian). Uspekhi Mat. Nauk 35 (1980), no. 1 (211), 203–204.
26 FILIPPO VIVIANI

[PS01] A. Premet and H. Strade: Simple Lie algebras of small characteristic. III. The toral rank 2 case. J. Algebra 242 (2001), 236–337.

[RUD71] A. N. Rudakov: Deformations of simple Lie algebras (Russian). Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1113–1119.

[SEL67] G. B. Seligman: Modular Lie algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40 Springer-Verlag, New York 1967.

[STR98] H. Strade: The classification of the simple modular Lie algebras. VI. Solving the final case. Trans. Amer. Math. Soc. 350 (1998), 2553–2628.

[STR04] H. Strade: Simple Lie algebras over fields of positive characteristic I: Structure theory. De Gruyter Expositions in Mathematics, 38 (Walter de Gruyter, Berlin, 2004).

[VIV1] F. Viviani: Infinitesimal deformations of restricted simple Lie algebras I. To appear on J. Algebra (available at arXiv:math.RA/0612861).

[VIV2] F. Viviani: Deformations of the restricted Melikian algebra. To appear on Comm. Algebra (available at arXiv:math.RA/0702594).

[VIV3] F. Viviani: Deformations of simple finite group schemes, submitted. (available at arXiv:0705.0821).

Institut für Mathematik, Humboldt Universität zu Berlin, 10099 Berlin (Germany). E-mail address: viviani@math.hu-berlin.de