On the Assouad dimension of self-similar sets with overlaps

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Abstract

It is known that, unlike the Hausdorff dimension, the Assouad dimension of a self-similar set can exceed the similarity dimension if there are overlaps in the construction. Our main result is the following precise dichotomy for self-similar sets in the line: either the weak separation property is satisfied, in which case the Hausdorff and Assouad dimensions coincide; or the weak separation property is not satisfied, in which case the Assouad dimension is maximal (equal to one). In the first case we prove that the self-similar set is Ahlfors regular, and in the second case we use the fact that if the weak separation property is not satisfied, one can approximate the identity arbitrarily well in the group generated by the similarity mappings, and this allows us to build a weak tangent that contains an interval. We also obtain results in higher dimensions and provide illustrative examples showing that the ‘equality/maximal’ dichotomy does not extend to this setting.

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1 Introduction

Self-similar sets are arguably the most important examples of fractal sets. This is due to their simple description and the fact that they exhibit many of the properties one expects from fractals. Although they have been studied extensively in the literature over the past half century, there are still many fascinating and challenging open problems available, particularly relating to the overlapping case, that is, self-similar sets \(F\)
such that every iterated function system with attractor $F$ fails to satisfy the open set condition. Our paper focuses on the Assouad dimension of such sets.

Given a bounded set $E \subseteq \mathbb{R}^d$ where $d \in \mathbb{N}$ and $\rho > 0$, let $\mathcal{N}_E(\rho)$ be the smallest number of open balls of radius $\rho$ required to cover $E$. Given a (potentially unbounded) set $X \subseteq \mathbb{R}^d$, let

$$\mathcal{N}_X(r, \rho) = \max \{ \mathcal{N}_{X \cap B_r(x)}(\rho) : x \in X \},$$

denotes the open ball of radius $r$ centered at $x$. The Assouad dimension $\dim_A X$ is then defined to be the infimum over all $s$ for which there exists $K_s$ such that $\mathcal{N}_X(r, \rho) \leq K_s (r/\rho)^s$ for all $0 < \rho < r \leq 1$. We note that some authors allow $r$ (and $\rho < r$) to be arbitrarily large, but for bounded sets $X$ this does not change the dimension.

This dimension was introduced by Bouligand [5] to study bi-Lipschitz embeddings and was further studied by Assouad in [1, 2]. Although an interesting notion in its own right, it has found most prominence in the literature due to its relationship with embeddability problems and quasi-conformal mappings. For example, Olson [25] proved that for $X \subseteq \mathbb{R}^d$, $\dim_A (X - X) < s$ implies almost every orthogonal projection of rank $s$ is injective on $X$ and results in an embedding that is bi-Lipschitz except for a logarithmic correction. This result was extended to subsets of infinite-dimensional Hilbert spaces by Olson and Robinson in [26]. Further work relating to Assouad dimension and embeddings may be found in Heinonen [12], Luukkainen [17], Mackay and Tyson [19] and Robinson [28]. Concerning quasi-conformal mappings, there is widespread interest in computing the conformal (Hausdorff or Assouad) dimension in various settings. Here the conformal dimension of $X$ is the infimum of the dimension of quasi-conformal images of $X$. Keith and Laakso [15] study Ahlfors regular metric spaces whose Assouad dimension cannot be lowered by quasi-conformal maps and Bonk and Kleiner [4] apply this work to resolve Cannon’s conjecture in certain settings. Tyson [30] proved that if the conformal Assouad dimension of a bounded set in Euclidean space is strictly less than 1, then it is 0. Recently, the Assouad dimension has also been gaining substantial attention in the literature on fractal geometry.

Let $I = \{1, \ldots, |I|\}$ be a finite index set and $\{S_i\}_{i \in I}$ be a set of contracting self-maps on some compact subset of $\mathbb{R}^d$. Such a collection of maps is called an iterated function system (IFS). It is well known, see for example Edgar [6] or Falconer [7], that there exists a unique non-empty compact set $F \subseteq \mathbb{R}^d$ satisfying

$$F = \bigcup_{i \in I} S_i(F) \quad (1.1)$$

which is called the attractor of the iterated function system. In this paper we shall work exclusively in the setting where the maps are similarities. A map $S : \mathbb{R}^d \to \mathbb{R}^d$ is called a similarity if there exists a ratio $c > 0$ such that

$$\|S(x) - S(y)\| = c\|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$ 

If $c \in (0, 1)$ then $S$ is called a contracting similarity. A self-similar set is any set $F \subseteq \mathbb{R}^d$ for which there exists an iterated function system of contracting similarities $\{S_i\}_{i \in I}$ that has $F$ as its attractor. Without loss of generality we will assume from now on that $F \subseteq [0,1]^d$. We say $F$ is non-trivial if it is not a singleton. The trivial
situations can only occur if all the defining maps have a common fixed point, in which case the attractor is just this point.

Given an iterated function system of contracting similarities, the similarity dimension \( \text{dms}(\{S_i\}_{i \in I}) \) is defined to be the maximum of the spatial dimension \( d \) and the unique solution \( s \) of the Hutchinson–Moran formula

\[
\sum_{i \in I} c_i^s = 1.
\]

Note that the similarity dimension is always an upper bound for the Hausdorff dimension of \( F \) which we shall denote by \( \text{dim}_H F \). The Assouad dimension is also an upper bound on the Hausdorff dimension (see [28], for example).

Most of the commonly used notions of dimension coincide for arbitrary self-similar sets, regardless of separation conditions. Indeed, equality of the box, packing and Hausdorff dimensions has been long known and follows, for example, from the implicit theorems of Falconer [8] and McLaughlin [20]. The additional equality with the lower dimension, which is the natural dual to the Assouad dimension and is \( a \, \text{priori} \) just a lower bound for the Hausdorff dimension, was proved by Fraser [10, Theorem 2.11]. For this reason we will omit discussion of all notions of dimension apart from the Hausdorff and Assouad dimensions. For a review of the other notions see Falconer [9, Chapters 2-3] or Robinson [28].

In [21] Moran introduced the open set condition (OSC) which, if satisfied, guarantees equality of the Hausdorff dimension and the similarity dimension, see for example [9, Section 9.3]. It can occur that the Hausdorff dimension is strictly less than the similarity dimension if different iterates of maps in the iterated function system overlap exactly. An example of this dimension drop is given by the modified Sierpiński triangle of Ngai and Wang [22, Figure 2, page 670]. It is an important open problem to decide if exact overlaps are the only way such a dimension drop can occur, see for example Peres and Solomyak [27, Question 2.6]. Recently an important step towards solving this problem has been made by Hochman [14], where it is shown in \( \mathbb{R} \), assuming the defining parameters for the iterated function system are algebraic, that the only cause for a dimension drop is exact overlaps. Hochman’s arguments rely on a careful application of ergodic theory and in establishing inverse theorems for the growth of certain entropy functions.

**Definition 1.1.** An IFS \( \{S_i\}_{i \in I} \) satisfies the open set condition if there exists a non-empty open set \( U \) such that

\[
\bigcup_{i \in I} S_i(U) \subseteq U
\]

with the union disjoint.

Note that the open set \( U \) need not be unique. Falconer [7, page 122] shows that if the OSC is satisfied for some \( U \), then the attractor \( F \subseteq U \) and a result of Schief [29] is that we may assume that \( F \cap U \neq \emptyset \). Notably, if the OSC is satisfied, then the Assouad dimension also coincides with the Hausdorff, and thus similarity, dimension. This result, which has been known in the folklore and essentially dates back to Hutchinson, is due to the fact that self-similar sets are Ahlfors regular, which is a sufficient condition for equality of Hausdorff and Assouad dimension. With \( H^s \) denoting the \( s \)-dimensional
Hausdorff measure, recall that a set $F \subseteq \mathbb{R}^d$ is called Ahlfors regular if there exist constants $a, b > 0$ such that
\[ a r^s \leq \mathcal{H}^s(B_r(x) \cap F) \leq b r^s \]
for $s = \dim_H F$, all $x \in F$ and all $0 < r < 1$, see Heinonen [12, Chapter 8]. We note that it is sufficient to show that there exists some $r_0 > 0$ such that these bounds hold for all $r \in (0, r_0)$. Without relying on the sophisticated notion of Ahlfors regularity, simple direct proofs are also possible, see for example Olson [25], Olsen [24], Henderson [13], and Fraser [10]. Olsen [24, Question 1.3] asked the natural question of whether $\dim_H F = \dim_A F$ holds for any self-similar set $F$, regardless of separation properties. This was answered in the negative by Fraser [10, Section 3.1]. Similar techniques were used by Henderson [13] to find examples of self-similar sets such that $\dim_A F$ is arbitrarily small but where $\dim_A(F - F) = 1$. The Assouad dimension of more general attractors has also been considered by, for example, Mackay [18] and Fraser [10] who studied certain planar self-affine constructions.

The weak separation property of Lau and Ngai [16] and Zerner [31] will be our primary tool for studying self-similar sets with overlaps and, indeed, provides a pleasing dichotomy concerning Assouad dimension. Before describing the weak separation property we introduce some notation.

Let $I^* = \bigcup_{k \geq 1} I^k$ be the set of all finite sequences with entries in $I$. For
\[ \alpha = (i_1, i_2, \ldots, i_k) \in I^* \]
write
\[ S_\alpha = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k} \]
and
\[ c_\alpha = c_{i_1} c_{i_2} \cdots c_{i_k}. \]
Note that $S_\alpha$ is a contracting similarity with ratio $c_\alpha$. Let
\[ \overline{\alpha} = (i_1, i_2, \ldots, i_{k-1}) \in I^* \cup \{\omega\}, \]
where $\omega$ is the empty sequence of length zero. For notational convenience define $S_\omega = I$ to be the identity map and $c_\omega = 1$. Let
\[ \mathcal{E} = \{ S_{\alpha}^{-1} \circ S_{\beta} : \alpha, \beta \in I^* \text{ with } \alpha \neq \beta \} \]
and equip the group of all similarities on $\mathbb{R}^d$ with the topology induced by pointwise convergence. We now define the weak separation property.

**Definition 1.2** (Lau and Ngai [16], Zerner [31]). An IFS $\{S_i\}_{i \in I}$ satisfies the weak separation property if
\[ I \notin \overline{\mathcal{E}} \setminus \{I\}. \]

The weak separation property says that we cannot find a sequence of pairs of maps $(S_\alpha, S_\beta)$ such that $S_\alpha^{-1} \circ S_\beta$ gets arbitrarily close but not equal to the identity. Intuitively, this means that the images $S_\alpha(F)$ of the attractor $F$ may only overlap in a limited number of ways.

It was shown by Zerner [31] that the open set condition is strictly stronger than the weak separation property. In particular using results from Schief [29] it was shown
that the open set condition is equivalent to \( I \notin \mathcal{E} \). Since all maps in \( \mathcal{E} \) are similarities, taking the pointwise closure of \( \mathcal{E} \) is equivalent to taking the closure in the space of similarities \( S: [0,1]^d \to \mathbb{R}^d \) equipped with the uniform norm \( \|\cdot\|_{L^\infty([0,1]^d)} \) which we denote by \( \|\cdot\|_\infty \) throughout, and where the value of \( d \) will be clear from the context.

Our main results may be stated as follows. In the case of self-similar sets in the real line we obtain a precise dichotomy.

**Theorem 1.3.** Let \( F \) be a non-trivial self-similar set in \( \mathbb{R} \). If the defining IFS for \( F \) satisfies the weak separation property, then \( \dim_A F = \dim_H F \); otherwise, \( \dim_A F = 1 \).

In the case of self-similar sets in \( \mathbb{R}^d \) we obtain a less precise dichotomy.

**Theorem 1.4.** Let \( F \) be a self-similar set in \( \mathbb{R}^d \) that is not contained in any \((d-1)\)-dimensional hyperplane. If the defining IFS for \( F \) satisfies the weak separation property, then \( \dim_A F = \dim_H F \); otherwise, \( \dim_A F \geq 1 \).

It is straightforward to see that our assumption that \( F \) is not contained in a hyperplane is required. For example, let \( F \) be the middle third Cantor set, embedded in the natural way in \( \mathbb{R}^3 \), and view it as the attractor of an IFS consisting of three mappings, the first two being the expected maps (the natural extensions of the maps in the one dimensional IFS to \( \mathbb{R}^3 \)) and the third one being one of the first two maps composed with an irrational rotation around the one dimensional subspace containing \( F \). Indeed, the WSP is not satisfied for this system, but \( \dim_A F < 1 \). The problem is that the WSP is satisfied when one restricts the defining maps to the one dimensional subspace containing \( F \). It is also easy to see that this condition is not restrictive at all because one can always restrict to a linear space with the appropriate dimension, i.e. where one cannot further restrict to any hyperplane, and then apply Theorem 1.4.

Theorems 1.3 and 1.4 may be broken into two parts. The first case covers what happens when the weak separation property holds and the second case covers when the weak separation property fails. Section 2 treats the first case; Section 3 treats second case. After proving our main results we explore some examples which illustrate our theorems and in particular show that the ‘equality/maximal’ dichotomy seen in Theorem 1.3 does not extend to \( \mathbb{R}^d \), i.e., there exist self-similar sets in \( \mathbb{R}^d \) (not contained in any hyperplane) with Assouad dimension strictly between the Hausdorff dimension and the ambient spatial dimension. We close with some remarks concerning how we might improve our understanding of the Assouad dimension of a self-similar set in \( \mathbb{R}^d \) when the weak separation property is not satisfied.

## 2 Systems with the Weak Separation Property

Our first result shows that the weak separation property implies that the Hausdorff and Assouad dimensions coincide.

**Theorem 2.1.** Let \( F \) be a self-similar set in \( \mathbb{R}^d \) not contained in any \((d-1)\)-dimensional hyperplane. If the defining IFS for \( F \) satisfies the weak separation property, then \( F \) is Ahlfors regular and, in particular, \( \dim_H F = \dim_A F \).

Before beginning our proof we introduce the notations

\[
I_r = \{ \alpha \in I^*: c_\alpha < r < c_\alpha \}
\] (2.1)
and

\[ I_r(x) = \{ \alpha \in \mathcal{I}_r : B_r(x) \cap S_\alpha(F) \neq \emptyset \}. \]

Note that for \( r \in (0,1) \) we have

\[ F = \bigcup_{\alpha \in \mathcal{I}_r} S_\alpha(F) \quad \text{and} \quad B_r(x) \cap F \subseteq \bigcup_{\alpha \in \mathcal{I}_r(x)} S_\alpha(F). \]

We remark that the common value for the Hausdorff and Assouad dimensions given by Theorem 2.1 may be strictly less than the similarity dimension. This happens when exact overlaps allow certain compositions of maps to be deleted. In fact, if one takes the infimum over all such ‘reduced IFSs’, then Zerner [31, Theorem 2] shows that this gives the common value. In particular, if the WSP is satisfied and \( F \) is not contained in a hyperplane, then

\[ \dim_H F = \dim_A F = \lim_{r \to 0} \dim_S(\{ S_\alpha : \alpha \in \mathcal{I}_r \}) = \inf_{r \in (0,1)} \dim_S(\{ S_\alpha : \alpha \in \mathcal{I}_r \}). \]

To see why this number drops from the original similarity dimension if there are exact overlaps, observe that \( \{ S_\alpha : \alpha \in \mathcal{I}_r \} \) is a set of maps rather than a multiset, i.e. repeated maps are included only once. Zerner [31, Proposition 2] also shows that the number on the right of the above equation is equal to the original similarity dimension if and only if the OSC is satisfied. The problem is that this value may be very difficult to compute. Ngai and Wang [22] introduced the finite type condition which is strictly stronger than the WSP (see Nguyen [23]), but allows the above value to be computed explicitly.

**Proof of Theorem 2.1.** Let \( F \) be a self-similar set in \( \mathbb{R}^d \) not contained in any \((d-1)\)-dimensional hyperplane and assume that the weak separation property is satisfied. Also, without loss of generality, assume that \( 0 \in F \). Let \( c_{\min} = \min\{ c_i : i \in \mathcal{I} \} \) and write \( |F| \) for the diameter of \( F \). To prove the result, it suffices to show that \( F \) is Ahlfors regular. Let \( s = \dim_H F \) denote the Hausdorff dimension of \( F \). First observe that the weak separation property implies that \( F \) is an \( s \)-set, i.e.

\[ 0 < \mathcal{H}^s(F) < \infty, \]

which is a necessary condition for Ahlfors regularity. Indeed, any self-similar set has finite Hausdorff measure in the critical dimension and Zerner shows that, together with \( F \) not being contained in a hyperplane, the weak separation property implies that this measure is also positive [31, Corollary to Proposition 2]. Let \( x \in F \) and \( r \in (0, \min\{1, |F|\}) \) be arbitrary. The lower bound in the definition of Ahlfors regularity is easy to establish. Choose \( \alpha \in \mathcal{I}_r^* \) such that

\[ x \in S_\alpha(F) \quad \text{and} \quad c_\alpha \leq r/|F| < c_\alpha. \]

It is clear that such an \( \alpha \in \mathcal{I}_r^* \) exists, that it satisfies \( c_\alpha > c_{\min}r/|F| \) and that \( S_\alpha(F) \subseteq B_r(x) \). As such, using the scaling property for Hausdorff measure,

\[ \mathcal{H}^s(B_r(x) \cap F) \geq \mathcal{H}^s(S_\alpha(F)) \geq c_\alpha^s \mathcal{H}^s(F) \geq \left( c_{\min}^s \mathcal{H}^s(F) / |F|^s \right) r^s. \]
The upper bound is more awkward to establish and relies on the weak separation property. Observe that
\[
\mathcal{H}^s(B_r(x) \cap F) \leq \mathcal{H}^s \left( \bigcup_{\alpha \in \mathcal{I}_r(x)} S_\alpha(F) \right) \\
\leq \sum_{\alpha \in \mathcal{I}_r(x)} c_\alpha \mathcal{H}^s(F) \\
\leq \mathcal{H}^s(F) |\mathcal{I}_r(x)| r^s.
\]
Thus in order to finish the proof, it suffices to bound $|\mathcal{I}_r(x)|$ independently of $x$ and $r$. Of course, if the defining system has complicated overlaps, this is impossible, but we can prove it in our case by applying one of the equivalent formulations of the weak separation property given by Zerner. In particular, [31, Theorem 1 (3a), (5a)] shows the definition we use in this paper is equivalent to the following (stated in our notation): for all $y \in \mathbb{R}^d$, there exists $l(y) \in \mathbb{N}$ such that for any $\beta \in \mathcal{I}^*$ and $r > 0$, every ball with radius $r$ contains at most $l(y)$ elements of the set
\[
\{S_\alpha(S_\beta(y)) : \alpha \in \mathcal{I}_r\}.
\]
Set $y = 0$ and choose $\beta \in \mathcal{I}$ arbitrarily and observe that if $\alpha \in \mathcal{I}_r(x)$, then
\[
S_\alpha(S_\beta(0)) \in S_\alpha(F) \subseteq B(x, r + r|F|).
\]
Since $\mathbb{R}^d$ is a doubling metric space we can find a cover of $B(x, r + r|F|)$ by fewer than $L$ balls of radius $r$, where $L$ is a uniform constant independent of $x$ and $r$. Each of these $r$-balls can contain no more than $l(0)$ of the points $\{S_\alpha(S_\beta(0)) : \alpha \in \mathcal{I}_r\}$ and so we deduce that $|\mathcal{I}_r(x)| \leq l(0) L$, completing the proof.

3 Systems without the Weak Separation Property

Our second result shows that if the weak separation property does not hold, then the Assouad dimension is bounded below by one. On the real line this gives a set of maximal dimension.

**Theorem 3.1.** Let $F$ be a non-trivial self-similar set in $\mathbb{R}$. If the defining IFS for $F$ does not satisfy the weak separation property, then $\dim_A F = 1$.

In $\mathbb{R}^d$ we can only show that the dimension is at least one.

**Theorem 3.2.** Let $F$ be a self-similar set in $\mathbb{R}^d$ not contained in any $(d-1)$-dimensional hyperplane. If the defining IFS for $F$ does not satisfy the weak separation property, then $\dim_A F \geq 1$.

Theorem 3.1 is the special case of Theorem 3.2 when $d = 1$, however, for clarity of exposition we give a separate proof of Theorem 3.1 first. Another reason for doing this is that Theorem 1.3 is the main result of the paper as it provides a precise result in $\mathbb{R}$ and so it is expedient to give a clear proof of Theorem 3.1 without the extra technical details required in $\mathbb{R}^d$. Note that combining Theorem 2.1 with Theorem 3.1 yields Theorem 1.3 which provides a precise dichotomy for self-similar sets in the line. Combining Theorem 2.1 with Theorem 3.2 yields Theorem 1.4.
One of the most powerful techniques for proving lower bounds for Assouad dimension is to construct weak tangents to the set. This approach has been pioneered by Mackay and Tyson [19], but also harks back to Furstenberg’s notion of microsets [11]. We first recall the definition of the Hausdorff distance and then give the definition of weak tangent from [19], see also [18].

**Definition 3.3.** Given compact subsets $X$ and $Y$ of $\mathbb{R}^d$, the Hausdorff distance between $X$ and $Y$ is defined as

$$d_H(X,Y) = \max \{ p_H(X,Y), p_H(Y,X) \}$$

where

$$p_H(X,Y) = \sup_{x \in X} \inf_{y \in Y} \| x - y \|.$$

**Definition 3.4.** Let $X$ be a compact subset of $\mathbb{R}^d$ and let $F$ and $\hat{F}$ be compact subsets of $X$. We say that $\hat{F}$ is a weak tangent of $F$ if there exists a sequence of similarity maps $T_k: \mathbb{R}^d \to \mathbb{R}^d$ such that $d_H(T_k(F) \cap X, \hat{F}) \to 0$ as $k \to \infty$.

**Proposition 3.5.** If $\hat{F}$ is a weak tangent of $F$ then $\dim_A \hat{F} \leq \dim_A F$.

The proof of Proposition 3.5 can be found in [19 Proposition 6.1.5] and [18 Proposition 2.1]. In practise, the similarity maps $T_k$ in Definition 3.4 are usually taken to be expanding with expansion ratios tending to infinity with $k$. The result is still true if the similarity maps are contracting with contraction ratios tending to zero, but then the limit set $\hat{F}$ is often referred to as an asymptotic cone, rather than a tangent. In our work we shall employ the following slightly weaker notion of a tangent than the weak tangent of Definition 3.4.

**Definition 3.6.** Let $F$ and $\hat{F}$ be compact subsets of $\mathbb{R}^d$. We say $\hat{F}$ is a weak pseudo-tangent of $F$ if there exists a sequence of similarity maps $T_k: \mathbb{R}^d \to \mathbb{R}^d$ such that $p_H(\hat{F}, T_k(F)) \to 0$ as $k \to \infty$.

We choose the name weak pseudo-tangent because the main difference between Definition 3.6 and Definition 3.4 is the replacement of the Hausdorff metric $d_H$ with the one-sided Hausdorff pseudo-metric $p_H$. Note that we also do not need to take the intersection with some explicitly chosen set $X$ in the definition of the weak pseudo-tangent. Fortunately for our analysis the Assouad dimension of a pseudo-tangent still forms a lower bound on the Assouad dimension of the original set.

**Proposition 3.7.** If $\hat{F}$ is a weak pseudo-tangent of $F$ then $\dim_A \hat{F} \leq \dim_A F$.

The proof of Proposition 3.7 follows the proofs in the weak-tangent setting given by Mackay and Tyson, but we include the details for completeness. In fact our proof is slightly simpler since we do not have to take intersections with some previously chosen $X$ and we need only control ‘one side’ of the convergence. Moreover, Proposition 3.5 follows from Proposition 3.7 by choosing an appropriate reference set $X$.

**Proof.** Let $s > \dim_A F$. By definition there exists $K_s$ such that

$$\mathcal{N}_F(r, \rho) \leq K_s(r/\rho)^s \quad \text{for all} \quad 0 < \rho < r \leq 1.$$
Since \( \hat{F} \) is a weak pseudo-tangent there exist similarities \( T_k \) with corresponding contraction ratios \( u_k \) such that
\[
p_H(\hat{F}, T_k(F)) \to 0 \quad \text{as} \quad k \to \infty.
\]
Since any cover of an \( r \)-ball in \( T_k(F) \) by \( \rho \)-balls gives rise to a cover of a corresponding \( r/u_k \)-ball in \( F \) by the same number of \( \rho/u_k \)-balls, and vice-versa, it follows that
\[
N_{T_k(F)}(r, \rho) = N_F(r/u_k, \rho/u_k) \leq K_s(r/\rho)^s \quad \text{for all} \quad 0 < \rho < r \leq 1.
\]
Note that \( K_s \) is independent of \( k \). Given \( r \in (0, 1/2) \) and \( \rho \in (0, r) \) choose \( k \) so large that
\[
p_H(\hat{F}, T_k(F)) \leq \rho/4.
\]
Thus, every point of \( \hat{F} \) is within \( \rho/2 \) distance from some point of \( T_k(F) \) and in particular
\[
\hat{F} \subseteq \bigcup_{\eta \in T_k(F)} B_{\rho/2}(\eta).
\]

Given \( x \in \hat{F} \) choose \( y \in T_k(F) \) such that \( \|x - y\| < \rho/2 \). Then
\[
B_r(x) \subseteq B_{r+\rho/2}(y) \subseteq B_{2r}(y).
\]
Let \( \{ B_{\rho/2}(y_i) : i = 1, \ldots, N \} \) be a cover of \( T_k(F) \cap B_{2r}(y) \) with
\[
N \leq K_s \left( \frac{2r}{\rho/2} \right)^s.
\]
It follows that
\[
\hat{F} \cap B_r(x) \subseteq \bigcup_{\eta \in T_k(F) \cap B_{2r}(y)} B_{\rho/2}(\eta) \subseteq \bigcup_{i=1}^N \bigcup_{\eta \in B_{\rho/2}(y_i)} B_{\rho/2}(\eta) = \bigcup_{i=1}^N B_{\rho}(y_i).
\]
Therefore
\[
N_{\hat{F} \cap B_r(x)}(\rho) \leq N \leq K_s A^s(r/\rho)^s
\]
for all \( x \in \hat{F} \) and \( 0 < \rho < r < 1/2 \) which is sufficient to prove \( \dim_A \hat{F} \leq s \). It follows that \( \dim_A \hat{F} \leq \dim_A F \). \( \square \)

3.1 Proof of Theorem 3.1

Similarities \( S_i \) on the real line with ratio \( c_i > 0 \) come in two types: with reflection and without. Namely,
\[
S_i(x) = -c_i x + b_i \quad \text{and} \quad S_i(x) = c_i x + b_i.
\]
In the first case the derivative \( S_i' = -c_i < 0 \). In order to treat iterated function systems that contain both types of similarities we first prove the following result.
Lemma 3.8. Suppose that there exist $\alpha_k, \beta_k \in \mathcal{I}^*$ such that

$$\|S_{\alpha_k}^{-1} \circ S_{\beta_k} - I\|_\infty \to 0 \quad \text{as} \quad k \to \infty.$$  

Then there exist $\alpha'_k, \beta'_k \in \mathcal{I}^*$ such that

$$\|S_{\alpha'_k}^{-1} \circ S_{\beta'_k} - I\|_\infty \to 0 \quad \text{as} \quad k \to \infty$$

and $S_{\alpha'_k} = c_{\alpha'_k} > 0$ for all $k$.

Proof. If there are an infinite number of $k$ such that $S_{\alpha'_k} > 0$ then let $\alpha'_k$ be this subsequence and we are done. Otherwise, there must be an infinite number of $k$ such that $S_{\alpha'_k} < 0$. By taking a subsequence we may assume, in fact, that $S_{\alpha'_k} < 0$ for all $k$.

Now, since $S_{\alpha_k}$ contains a reflection for every $k$, there must be at least one similarity in the iterated function system that contains a reflection. Without loss of generality, assume that $S_1' = -c_1 < 0$. Define $\alpha'_k$ and $\beta'_k$ so that

$$S_{\alpha'_k}^{-1} = S_1^{-1} \circ S_{\alpha_k}^{-1} \quad \text{and} \quad S_{\beta'_k} = S_{\beta_k} \circ S_1.$$  

It follows that $S_{\alpha'_k} > 0$ for all $k$ and furthermore that

$$|(S_{\alpha'_k}^{-1} \circ S_{\beta'_k} - I)(x)| = |S_1^{-1}(S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ S_1(x)) - S_1^{-1}(S_1(x))|$$  

$$= c_1^{-1}|S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ S_1(x) - S_1(x)|$$  

$$= c_1^{-1}|(S_{\alpha_k}^{-1} \circ S_{\beta_k} - I)(S_1(x))|.$$  

Since $F \subseteq [0, 1]$ it follows that $S_1(x) \in [0, 1]$ for $x \in [0, 1]$. Therefore

$$\|S_{\alpha'_k}^{-1} \circ S_{\beta'_k} - I\|_\infty \leq c_1^{-1}\|S_{\alpha_k}^{-1} \circ S_{\beta_k} - I\|_\infty \to 0,$$

which finishes the proof of the lemma.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Since $F$ is non-trivial there must be two distinct points $a, b \in F$ and similarities $S_1$ and $S_2$ such that

$$S_1(a) = a \quad \text{and} \quad S_2(b) = b. \quad (3.2)$$  

Choose $\rho > 0$ small enough that $B_{2\rho}(a) \cap B_{2\rho}(b) = \emptyset$.

Since the Weak Separation Property does not hold, $I \in \mathcal{E}\setminus\{I\}$ and we can choose $\alpha_k, \beta_k \in \mathcal{I}^*$ such that

$$0 < \|S_{\alpha_k}^{-1} \circ S_{\beta_k} - I\|_\infty \to 0 \quad \text{as} \quad k \to \infty.$$  

By Lemma 3.8 we may assume that

$$S_{\alpha_k}' = c_{\alpha_k} > 0 \quad \text{for all} \quad k.$$  

Define $\varphi_k = S_{\alpha_k}^{-1} \circ S_{\beta_k} - I$. Note that $\varphi_k \neq 0$ for all $k$. Therefore either $\varphi_k$ is a similarity or a non-zero constant function. Since similarities are injective, images of the disjoint $2\rho$ balls $B_{2\rho}(a)$ and $B_{2\rho}(b)$ are disjoint. In particular, the origin can lie in at most one
of these images. On the other-hand, if \( \varphi_k \) is a non-zero constant function, then the image of both of these balls consists of the same non-zero point and in particular does not contain the origin. It follows that each \( \varphi_k \colon \mathbb{R} \to \mathbb{R} \) must satisfy at least one of the following four conditions:

\[
\varphi_k(B_{2\rho}(a)) \subseteq (0, \infty), \quad \varphi_k(B_{2\rho}(b)) \subseteq (0, \infty),
\]

\[
\varphi_k(B_{2\rho}(a)) \subseteq (-\infty, 0) \quad \text{or} \quad \varphi_k(B_{2\rho}(b)) \subseteq (-\infty, 0).
\]

Moreover, by taking a subsequence we may assume that all \( \varphi_k \) satisfy the same inclusion. For definiteness assume \( \varphi_k(B_{2\rho}(a)) \subseteq (0, \infty) \) for all \( k \). The other cases may be obtained from the first by relabeling \( a \) and \( b \) and making a change of coordinates to reverse the direction of the real line if necessary.

Let \( f = S_1 \circ S_1 \). Then \( f' = c = c_1^2 > 0 \). This avoids the changes in sign that might occur when \( S_1 \) contains a reflection. Fix \( M \) large enough that \( f^M(F) \subseteq B_{\rho}(a) \) (see (3.2)). Define

\[
\zeta_k = \inf \{ \varphi_k(x) : x \in B_{2\rho}(a) \},
\]

\[
\delta_k = \inf \{ \varphi_k(x) : x \in B_{\rho}(a) \},
\]

and \( \Delta_k = \sup \{ \varphi_k(x) : x \in B_{\rho}(a) \} \).

Then

\[
\delta_k - \zeta_k = \rho |\varphi_k'| \quad \text{and} \quad \Delta_k - \delta_k = 2\rho |\varphi_k'|.
\]

Therefore (since \( \zeta_k \geq 0 \))

\[
\rho |\varphi_k'| \leq \delta_k \quad \text{and} \quad \Delta_k \leq 3\delta_k.
\]

While the inequality \( \Delta_k \leq 3\delta_k \) holds whether \( \varphi_k \) is a similarity or a non-zero constant function, note that in the latter case \( \varphi_k' = 0 \) and then \( \Delta_k = \delta_k \). In either case, since we know \( \|\varphi_k\|_\infty \to 0 \) then \( \delta_k \to 0 \) as \( k \to \infty \).

We now derive an estimate that will be used as the basis for our inductive construction in the next paragraph. First note that

\[
(f^{-m} \circ S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m - I)(x)
\]

\[
= f^{-m}(S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m(x)) - f^{-m}(f^m(x))
\]

\[
= c^{-m}(S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m(x) - f^m(x))
\]

\[
= c^{-m}\varphi_k(f^m(x)).
\]

Since \( f^m(x) \in B_{\rho}(a) \) for every \( x \in F \) and \( m \geq M \) we obtain

\[
c^{-m}\delta_k \leq (f^{-m} \circ S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m - I)(x) \leq 3c^{-m}\delta_k
\]

(3.3) for every \( x \in F \) and \( m \geq M \).

Given \( \epsilon > 0 \) we now define natural numbers \( k_j, m_j \), and maps \( g_j \) and \( h_j \) by induction on \( j \). First, choose \( k_1 \) and \( m_1 \geq M \) so large that

\[
c^{-m_1}\delta_{k_1} < \epsilon \leq c^{-m_1-1}\delta_{k_1}
\]

(this can be done by first choosing \( k_1 \) large enough that \( \delta_{k_1} < \epsilon c^M \) and then choosing \( m_1 \) appropriately) and define

\[
g_1 = f^{-m_1} \circ S_{\alpha_k}^{-1} \quad \text{and} \quad h_1 = S_{\beta_k} \circ f^{m_1}.
\]
From the estimate (3.3) and our choice of $k_1$ and $m_1$ it follows that

$$c\epsilon \leq (g_1 \circ h_1 - I)(x) \leq 3\epsilon$$

holds for every $x \in F$. Let $d_j = g'_j$ be the derivative of $g_j$. Since both $f$ and $S_{\alpha k_1}$ have positive derivatives, we know that $d_1 > 0$. Thus $d_1$ is the ratio corresponding to the similarity $g_1$ such that $|g_1(x) - g_1(y)| = d_1|x - y|$ for every $x, y \in \mathbb{R}$. By the construction below, it will also follow that $d_j > 0$ for every $j \in \mathbb{N}$.

For $j \geq 2$ choose $k_j$ and $m_j \geq M$ so large that

$$d_{j-1}e^{-m_j}d_{k_j} < \epsilon \leq d_{j-1}e^{-m_j}d_{k_j}$$

(this can be done in a similar way to the choice of $k_1$ and $m_1$, above) and define

$$g_j = g_{j-1} \circ f^{-m_j} S_{\alpha k_j}^{-1} \quad \text{and} \quad h_j = S_{\beta k_j} \circ f^{m_j} \circ h_{j-1}.$$  

Since $d_j = d_{j-1}e^{-m_j}c_{\alpha k_j}$ and $d_1 > 0$, it follows by induction that $d_j > 0$. Since $h_{j-1}(x) \in F$ we have $f^{m_j}(h_{j-1}(x)) \in B_p(a)$ and so (recalling that $\varphi_k = S_{\alpha k}^{-1} \circ S_{\beta k} - I$)

$$(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) = g_{j-1} \circ f^{-m_j}(S_{\alpha k_j}^{-1} \circ S_{\beta k_j} \circ f^{m_j} \circ h_{j-1}(x))$$

$$= g_{j-1} \circ f^{-m_j}(f^{m_j} \circ h_{j-1}(x))$$

$$= d_{j-1}e^{-m_j} \varphi_{k_j}(f^{m_j} \circ h_{j-1}(x))$$

implies for every $x \in F$ that

$$c\epsilon \leq (g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) \leq 3\epsilon.$$  

We next claim for all $n \in \mathbb{N}$ that

$$\{a\} \cup \{ g_j \circ h_j(a) : j = 1, \ldots, n \} \subseteq \{ g_n \circ S_\beta(a) : \beta \in \mathcal{I}^* \}.$$  

(3.4)

This follows from induction on $n$. For $n = 1$ we readily see this is true by taking $\beta \in \mathcal{I}^*$ so that $S_\beta = h_1$ and then taking $\beta$ so that $S_\beta = g_1^{-1}$. Suppose the claim holds true for $n - 1$, then

$$\{a\} \cup \{ g_j \circ h_j(a) : j = 1, \ldots, n - 1 \} \subseteq \{ g_{n-1} \circ S_\beta(a) : \beta \in \mathcal{I}^* \}.$$  

Choose $\gamma \in \mathcal{I}^*$ so that $S_\gamma = S_{\alpha n} \circ f^{m_n}$. Since $g_n = g_{n-1} \circ f^{-m_n} \circ S_{\alpha n}^{-1}$, it follows that

$$\{ g_n \circ S_\beta(a) : \beta \in \mathcal{I}^* \} \supseteq \{ g_n \circ S_\gamma \circ S_\beta(a) : \beta \in \mathcal{I}^* \}$$

$$\quad = \{ g_{n-1} \circ S_\beta(a) : \beta \in \mathcal{I}^* \}$$

$$\supseteq \{a\} \cup \{ g_j \circ h_j(a) : j = 1, \ldots, n - 1 \}.$$  

Additionally taking $\beta \in \mathcal{I}^*$ so that $S_\beta = h_n$ completes the induction.

We finish the proof by constructing a weak pseudo-tangent to $F$ that has Assouad dimension 1. Let $g_{j,n}$ be the functions $g_j$ defined above when $\epsilon = (cn)^{-1}$ and define $T_n = g_{n,n}$. It follows from (3.4) that

$$T_n F \supseteq \{ g_{n,n} \circ S_\beta(a) : \beta \in \mathcal{I}^* \}$$

$$\quad \supseteq \{a\} \cup \{ g_{j,n} \circ h_{j,n}(a) : j = 1, \ldots, n \}.$$  

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Since \( n^{-1} \leq g_{j,n} \circ h_{j,n}(a) - g_{j-1,n} \circ h_{j-1,n}(a) \leq 3(cn)^{-1} \) we obtain

\[
p_H([a, a + 1], T_n(F)) \leq 3(cn)^{-1} \to 0.
\]

Therefore \([a, a + 1]\) is a weak pseudo-tangent of \( F \) and

\[
\dim_A F \geq \dim_A [a, a + 1] = 1.
\]

This finishes the proof of our lower bound for self-similar sets on the real line that do not satisfy the weak separation condition. \(\square\)

### 3.2 Proof of Theorem 3.2

In this section we prove the more complicated case concerning self-similar sets \( F \) in \( \mathbb{R}^d \). Thus, \( F \) is the attractor of an iterated function system \( \{S_i\}_{i \in I} \) given by

\[
S_i(x) = c_iO_ix + b_i
\]

where \( O_i \in O(d) \) is a \( d \times d \) orthogonal matrix, \( c_i \in (0,1) \) and \( b_i \in \mathbb{R}^d \) for each \( i \in I \). In the case of the real line \( O_i = \pm 1 \), and so \( O_i^2 \) always yielded the identity. However, in higher dimensions there are orthogonal matrices such that \( O^m \neq I \) for all \( m \geq 0 \). To deal with this we make use of Lemma 3.9 below. Given any \( \gamma \in I^* \), define \( O_\gamma \in O(d) \) and \( b_\gamma \in \mathbb{R}^d \) such that

\[
S_\gamma(x) = c_\gamma O_\gamma x + b_\gamma.
\]

Note that \( S_\gamma \) is a similarity with ratio \( c_\gamma \) such that

\[
\|S_\gamma(x) - S_\gamma(y)\| = c_\gamma \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^d.
\]

We now define

\[
G = \{ O_\alpha : \alpha \in I^* \} \quad \text{and} \quad \mathcal{G} = \overline{G}.
\]

and show that for any \( \epsilon > 0 \) there exists a finite collection of elements of the form \( O_\alpha \) with \( \alpha \in I^* \) which can be used to approximate every element of \( G \) to within \( \epsilon \).

**Lemma 3.9.** The set \( G \) is a compact group. Consequently, given \( \epsilon > 0 \) there exists a finite set \( \mathcal{J} \subseteq I^* \) such that for every \( U \in G \) there is \( \alpha \in \mathcal{J} \) such that \( \|U - O_\alpha\| < \epsilon \).

**Proof.** Let \( O(d) \) be the group of all \( d \times d \) orthogonal matrices. We claim that \( G \) is a subgroup of \( O(d) \). Indeed, since \( G \subseteq O(d) \) and \( \mathcal{G} \) is a semigroup, then \( G \) is at least a semigroup. Let \( U \in G \) and consider the sequence of iterates \( U^n \) where \( n \in \mathbb{N} \). Since \( O(d) \) is compact and \( G \) is closed, there exists a subsequence \( U^{n_k} \to V \) where \( V \in G \). It follows that \( U^{n_{k+1} - n_k} \to I \) and consequently \( I \in G \). We may further assume that \( n_{k+1} - n_k \geq 2 \) for every \( k \), from which it follows that \( U^{n_{k+1} - n_k - 1} \to U^{-1} \) and consequently \( U^{-1} \in G \). This proves the claim.

Now, since \( G \) is compact,

\[
G \subseteq \bigcup_{\alpha \in I^*} B_\epsilon(O_\alpha) \quad \text{implies that} \quad G \subseteq \bigcup_{\alpha \in \mathcal{J}} B_\epsilon(O_\alpha)
\]

for some finite subset \( \mathcal{J} \subseteq I^* \). Let \( U \in G \). Then \( U \in B_\epsilon(O_\alpha) \) for some \( \alpha \in \mathcal{J} \) and consequently \( \|U - O_\alpha\| < \epsilon \). \(\square\)
For self-similar sets in $\mathbb{R}^d$ we insist that $F$ not lie in any $(d-1)$-dimensional hyperplane of $\mathbb{R}^d$. To make use of this hypothesis we first show that we can find a collection of fixed points of the similarities which span $\mathbb{R}^d$, after a suitable translation.

**Lemma 3.10.** Suppose that $F$ is a self-similar set in $\mathbb{R}^d$ that is not contained in any hyperplane. Then there exist $\gamma_n \in \mathcal{I}^*$, where $n = 1, \ldots, d + 1$, such that the fixed points $a_n$ of the maps $S_{\gamma_n}$ satisfy $\text{span}\{a_n - a_{d+1} : n = 1, \ldots, d\} = \mathbb{R}^d$.

*Proof.* Given $a_n \in \mathbb{R}^d$ let $A$ be the $d \times d$ matrix given by the vectors $a_n - a_{d+1}$ as

$$A = \begin{bmatrix} a_1 - a_{d+1} & a_2 - a_{d+1} & \cdots & a_d - a_{d+1} \end{bmatrix}.$$ 

We are looking for $a_n$ such that rank $A = d$ or equivalently such that $\det A \neq 0$. Since $\det A$ depends continuously on the $a_n$ then

$$\mathcal{R} = \{ (a_1, \ldots, a_{d+1}) : \det A \neq 0 \} \subseteq [\mathbb{R}^d]^{d+1}$$

is open. Since $F$ is not contained in any hyperplane, there exist $b_n \in F$, $n = 1, \ldots, d+1$, such that $(b_1, \ldots, b_{d+1}) \in \mathcal{R}$. Moreover, since $\mathcal{R}$ is open, there is $\epsilon > 0$ such that $|a_n - b_n| < \epsilon$ for $n = 1, \ldots, d+1$ implies that $(a_1, \ldots, a_{d+1}) \in \mathcal{R}$.

Choose $r < \rho_{\min}$ such that $r \text{diam } F < \epsilon$. Then

$$b_n \in F = \bigcup_{\alpha \in \mathcal{I}_r} S_{\alpha}(F) \implies b_n \in S_{\gamma_n}(F)$$

for some $\gamma_n \in \mathcal{I}_r$ (see (2.1) for the definition of $\mathcal{I}_r$). Let $a_n$ be the fixed point of $S_{\gamma_n}$. Then

$$a_n \in F \implies a_n = S_{\gamma_n}(a_n) \in S_{\gamma_n}(F).$$

Therefore

$$|a_n - b_n| \leq \text{diam } S_{\gamma_n}F = c_{\gamma_n} \text{diam } F \leq r \text{diam } F < \epsilon$$

and consequently $(a_1, \ldots, a_{d+1}) \in \mathcal{R}$. \hfill $\square$

As the geometry in $\mathbb{R}^d$ is more complicated than on the real line we will need to prove one more lemma before commencing with the proof of Theorem 3.2 itself.

**Lemma 3.11.** Let $F$ be a self-similar set in $\mathbb{R}^d$ not contained in any $(d-1)$-dimensional hyperplane. Let $\Phi_k$ be a sequence of affine linear maps on $\mathbb{R}^d$ such that $0 < \|\Phi_k\|_\infty \to 0$ as $k \to \infty$. Then there exist $\rho > 0$, $\gamma \in \mathcal{I}^*$ and $a \in \mathbb{R}^d$ such that $S_{\gamma}(a) = a$ and a subsequence $k_j$ such that for every $j$ we have

$$\sup\{ \|\Phi_{k_j}(x)\| : x \in B_\rho(a) \} \leq 3 \inf\{ \|\Phi_{k_j}(x)\| : x \in B_\rho(a) \}$$

and $\rho \|D\Phi_{k_j}\| \leq \|\Phi_{k_j}(a)\|.$

*Proof.* Note that the derivative $D\Phi_k$ is a $d \times d$ matrix. We separate our proof into two cases: the first case is when there are an infinite number of $k$ such that the derivative $D\Phi_k = 0$ and we may assume by taking subsequences that $D\Phi_{k_j} = 0$ for all $k_j$; the second case is when there are only a finite number of $k$ such that $D\Phi_k = 0$ and we may assume $D\Phi_{k_j} \neq 0$ for all $j$.

If $D\Phi_{k_j} = 0$ for all $j$, then $\Phi_{k_j} = B_j$ where $B_j \in \mathbb{R}^d$ and $B_j \neq 0$. In this case any $\rho > 0$ and $\gamma \in \mathcal{I}^*$ yields a similarity $S_{\gamma}$ with fixed point $a$ such that $0 \notin \Phi_{k_j}(B_{2\rho}(a))$.  

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If $D\Phi_{k_j} \neq 0$ for all $j$, write the singular value decomposition of $D\Phi_{k_j}$ as

$$D\Phi_{k_j} = U_j \Sigma_j V_j^{-1}$$

where $U_j, V_j \in O(d)$ are $d \times d$ orthogonal matrices and $\Sigma_j$ is a diagonal matrix with singular values ordered such that $\sigma_{j,1} \geq \cdots \geq \sigma_{j,d} \geq 0$. Since $O(d)$ is compact, we may assume by taking further subsequences, that $V_j \rightarrow V$ for some $V \in O(d)$. Define

$$s_j = \sigma_{j,1} = \|D\Phi_{k_j}\|, \quad u_j = U_j e_1, \quad v_j = V_j e_1 \quad \text{and} \quad v_* = Ve_1,$$

where $e_1 = (1, 0, \ldots, 0)$. Then $0 < \|D\Phi_{k_j}\| \rightarrow 0$ implies that $0 < s_j \rightarrow 0$ as $j \rightarrow \infty$.

Since $F$ is not contained in any hyperplane, Lemma 3.10 implies that there exist $\gamma_n \in I^*$ and $a_n \in F$ such that

$$S_{\gamma_n}(a_n) = a_n \quad \text{and} \quad \text{span}\{a_n - a_{d+1} : n = 1, \ldots, d\} = \mathbb{R}^d.$$  \hspace{1cm} (3.5)

By (3.5) there exists $n$ such that

$$(a_n - a_{d+1}) \cdot v_* \neq 0.$$

Define $\varphi_j(x) = u_j \cdot \Phi_{k_j}(x)$. Note that

$$\varphi_j(x) = u_j \cdot (D\Phi_{k_j})x + z_j = U_j e_1 \cdot U_j \Sigma_j V_j^{-1}x + z_j = s_j v_j \cdot x + z_j$$

for the scalars $z_j = u_j \cdot \Phi_{k_j}(0)$. Therefore

$$\varphi_j(a_n) - \varphi_j(a_{d+1}) = s_j v_j \cdot (a_n - a_{d+1}).$$

Since $v_j \rightarrow v_*$ and $s_j > 0$ it follows for $j$ large enough that $\varphi_j(a_n) - \varphi_j(a_{d+1}) \neq 0$. By choosing $\rho > 0$ small enough and taking another subsequence, we may conclude that

$$\varphi_j(B_{2\rho}(a_n)) \cap \varphi_j(B_{2\rho}(a_{d+1})) = \emptyset \quad \text{for every} \quad j.$$

For each $j$ at least one of the statements $0 \notin \varphi_j(B_{2\rho}(a_n))$ or $0 \notin \varphi_j(B_{2\rho}(a_{d+1}))$ is true. By taking yet one more subsequence we may fix $a$ to be either $a_n$ or $a_{d+1}$ and $\gamma$ to be $\gamma_n$ or $\gamma_{d+1}$, respectively, so that $0 \notin \varphi_j(B_{2\rho}(a))$ for every $j$. It follows that $S_{\gamma}(a) = a$ and $0 \notin \Phi_{k_j}(B_{2\rho}(a))$ for every $j$.

Define

$$\delta_k = \inf \{ \|\Phi_k(x)\| : x \in B_{\rho}(a) \} \quad \text{and} \quad \Delta_k = \sup \{ \|\Phi_k(x)\| : x \in B_{\rho}(a) \}. \hspace{1cm} (3.6)$$

If $D\Phi_k = 0$ then $\delta_k = \Delta_k$. Otherwise $0 \notin \varphi_k(B_{2\rho}(a))$ implies that for $x \in B_{\rho}(a)$

$$\|\Phi_k(x)\| \geq |u_k \cdot \Phi_k(x)| = |s_k v_k \cdot x + z_k| \geq s_k \rho,$$

and therefore $\delta_k \geq s_k \rho$. On the other hand $\|\Phi_k(x) - \Phi_k(y)\| \leq s_k \|x - y\|$ implies that

$$\Delta_k \leq \delta_k + 2s_k \rho \leq 3\delta_k.$$

This completes the proof of the lemma.  \hfill \Box
We are now ready begin the proof of Theorem 3.2. The proof will construct a weak tangent to $F$ that has dimension greater than one; we recover a ‘one-dimensional’ structure by constructing a set all of whose points lie in a particular cone in $\mathbb{R}^n$. For a set $U \in \mathbb{R}^d$ we use

$$
C(U) = \{ \lambda u : \lambda > 0 \text{ and } u \in U \}
$$

for the cone generated by $U$.

**Proof of Theorem 3.2.** Since the weak separation condition is not satisfied, there exist $\alpha_k, \beta_k \in \mathcal{I}^\ast$ such that $0 < \|S_{\alpha_k}^{-1} \circ S_{\beta_k} - I\|_\infty \to 0$. Define $\Phi_k = S_{\alpha_k}^{-1} \circ S_{\beta_k} - I$. After taking subsequences we may assume by Lemma 3.11 that there exist $\gamma \in \mathcal{I}^\ast$ and $a \in \mathbb{R}^d$ such that $S_{\gamma}(a) = a$ and $\rho > 0$ such that the inequalities

$$
\sup\{ \|\Phi_k(x)\| : x \in B_\rho(a) \} \leq 3 \inf\{ \|\Phi_k(x)\| : x \in B_\rho(a) \}
$$

(3.7)

and $\rho \|D\Phi_k\| \leq \|\Phi_k(a)\|$ are satisfied for all $k$. Defining $\Delta_k$ and $\delta_k$ as in (3.6) we may further express (3.7) as $\Delta_k \leq 3\delta_k$.

Given $\eta \in (0, 1)$, cover the unit sphere $\partial B_1(0)$ by a finite number $M$ of balls $B_{\eta/2}(w_m)$ where $\|w_m\| = 1$ and $m = 1, 2, \ldots, M$. It follows that

$$
\mathbb{R}^d = \bigcup_{m=1}^M C(B_{\eta/2}(w_m)).
$$

For each $k$ choose $m_k$ such that $\Phi_k(a)/\|\Phi_k(a)\| \in B_{\eta/2}(w_{m_k})$. Upon taking a subsequence, we obtain a single $w \in \partial B_1(0)$ such that

$$
\Phi_k(a)/\|\Phi_k(a)\| \in B_{\eta/2}(w) \quad \text{for all} \quad k.
$$

Define $f = S_\gamma = c_\gamma O_\gamma x + b_\gamma$ and note that $Df = c_\gamma O_\gamma$. Let $\rho' = \rho \eta/5$ and choose $M$ so large that $m \geq M$ implies that $f^m(x) \in B_{\rho'}(a)$ for every $x \in F$. Let $\epsilon_2 = \eta/4$. By Lemma 3.9 there exists a finite set $\mathcal{J} \subseteq \mathcal{I}^\ast$ such that $U \in G$ implies there is $\alpha \in \mathcal{J}$ such that $\|U - O_\alpha\| < \epsilon_2$.

Given $\epsilon > 0$ we now define natural numbers $k_j, m_j$ and maps $g_j$ and $h_j$ by induction on $j$. Let $g_0 = I$, $h_0 = I$, $d_0 = 1$ and $W_0 = I$. Note that $Dg_0 = d_0W_0$. Choose $k_j$ and $m_j \geq M$ so large that

$$
d_{j-1}^{-1}c_{\gamma_j}^{-1}c_{\gamma_j}^{-m_j}d_{\delta_j} < \epsilon \leq d_{j-1}^{-1}c_{\gamma_j}^{-1}c_{\gamma_j}^{-m_j}d_{\delta_j}
$$

and let $\gamma_j \in \mathcal{J}$ be chosen so $\|O_{\gamma_j}^{-1}O_\gamma^{-m_j}W_{j-1} - I\| = \|W_{j-1}O_{\gamma_j}^{-1}O_\gamma^{-m_j} - I\| \leq \epsilon_2$. Define

$$
g_j = g_{j-1} \circ S_{\gamma_j}^{-1} \circ f^{-m_j} \circ S_{\alpha_j}^{-1} \quad \text{and} \quad h_j = S_{\beta_j} \circ f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}.
$$

Let $d_j$ be the ratio for $g_j$ such that

$$
\|g_j(x) - g_j(y)\| = d_j\|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^d
$$

and let $W_j$ be the $d \times d$ orthogonal matrix such that $Dg_j = d_jW_j$.

We now show that the sequence of functions $g_j$ and $h_j$ defined above satisfy

$$
c_{\gamma_j}c_{\gamma_j} \epsilon \leq \|(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x)\| \leq 3\epsilon
$$

(3.8)
and
\[(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) \in \mathcal{C}(B_\eta(w)) \text{ for all } x \in F\] (3.9)
for every \(j \in \mathbb{N}\). Since
\[f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x) \in B_{\rho'}(a) \subseteq B_\rho(a) \text{ for all } x \in F,
\]
then
\[
(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) = g_{j-1} \circ S_{\gamma_j}^{-1} \circ f^{-m_j}(S_{\alpha_{kj}}^{-1} \circ S_{\beta_{kj}} \circ f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x))
\]
\[
- g_{j-1} \circ S_{\gamma_j}^{-1} \circ f^{-m_j}(f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x))
\]
\[
= d_{j-1} c_{\gamma_j}^{-1} c^{-m_j} W_{j-1} O_{\gamma_j}^{-1} O_{\gamma_j}^{-m_j} \Phi_{kj}(f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x))
\]
which implies
\[
c_{\eta} c_{\gamma} \epsilon \leq d_{j-1} c_{\gamma_j}^{-1} c^{-m_j} \delta_{kj} \leq \|(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x)\| \leq d_{j-1} c_{\gamma_j}^{-1} c^{-m_j} \Delta_{kj} \leq 3 \epsilon
\]
for all \(x \in F\).
Write \(y = f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x)\), \(z = \Phi_{kj}(y)\) and \(\tilde{z} = W_j O_{\gamma_j}^{-1} O_{\gamma_j}^{-m_j} z\). Then
\[
\|z - \Phi_{kj}(a)\| = \|\Phi_{kj}(y) - \Phi_{kj}(a)\| \leq \|D \Phi_{kj}\| y - a \| \leq \rho' \|D \Phi_{kj}\|
\]
which implies that
\[
\|\tilde{z} - z\| \leq \epsilon_2 \|z\| \leq \|z - \Phi_{kj}(a)\| + \|\Phi_{kj}(a)\| \leq \epsilon_2 \rho' \|D \Phi_{kj}\| + \epsilon_2 \|\Phi_{kj}(a)\|.
\]
Since \(\eta < 1, \rho' = \rho \eta / 5, \epsilon_2 = \eta / 4\) and \(\rho \|D \Phi_{kj}\| \leq \|\Phi_{kj}(a)\|\) it follows that
\[
\|\tilde{z} - \Phi_{kj}(a)\| \leq (1 + \eta / 4)(\eta / 5) \rho \|D \Phi_{kj}\| + \eta / 4 \|\Phi_{kj}(a)\|
\]
\[
\leq ((5 / 4)(\eta / 5) + \eta / 4) \|\Phi_{kj}(a)\| = (\eta / 2) \|\Phi_{kj}(a)\|.
\]
Consequently
\[
\frac{W_j O_{\gamma_j}^{-1} O_{\gamma_j}^{-m_j} \Phi_{kj}(f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x))}{\|\Phi_{kj}(a)\|} \in B_{\eta / 2} \left( \frac{\Phi_{kj}(a)}{\|\Phi_{kj}(a)\|} \right) \subseteq B_\eta(w). \] (3.10)
It follows that
\[(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) \in \mathcal{C}(B_\eta(w)) \text{ for all } x \in F.
\]
Figure 1 shows representative locations for \((g_1 \circ h_1 - I)(a)\) and \((g_2 \circ h_2 - g_1 \circ h_1)(a)\) in the cone \(\mathcal{C}(B_\eta(w))\). Note that the norm of the projection of the points \(g_j \circ h_j(a)\) along the \(w\) direction always increases as \(j\) increases. In particular, we have
\[
c_{\eta} c_{\gamma} \xi \epsilon \leq \|w \cdot (g_j \circ h_j - g_{j-1} \circ h_{j-1})(a)\| \leq 3 \epsilon
\]
where \(\xi = \sqrt{1 - \eta^2}\).
We finish by constructing a weak tangent to $F$ that has Assouad dimension 1. The inclusion
$$\{a\} \cup \{ g_j \circ h_j(a) : j = 1, \ldots, n \} \subseteq \{ g_n \circ S_\beta(a) : \beta \in I^* \}$$
holds just as (3.4) held in the proof for the real line. However, unlike our proof for the real line, we will construct a weak tangent rather than a weak pseudo-tangent. This is done in order to make use of the fact that the space of all compact subsets of a compact set in $\mathbb{R}^d$ is a compact metric space with respect to the Hausdorff metric.

Let $g_{j,n}$ be the functions $g_j$ defined above when $\epsilon = (c, c\gamma \xi_n)^{-1}$ and define $T_n = g_{n,n}$. Note that the cone $C(B_\eta(w))$ in the construction does not depend on $\epsilon$. As before, it follows that $T_n F \supseteq \{a\} \cup \{ g_{j,n} \circ h_{j,n}(a) : j = 1, \ldots, n \}$.

Since
$$n^{-1} \leq w \cdot (g_{j,n} \circ h_{j,n} - g_{j-1,n} \circ h_{j-1,n})(a) \leq 3(c, c\gamma \xi_n)^{-1}$$
we obtain
$$p_H([a_w, a_w + 1], w \cdot T_n(F)) \leq 3(c, c\gamma \xi_n)^{-1} \to 0,$$
where $a_w = w \cdot a$. However $w \cdot T_n$ is not a similarity, so this is not a weak pseudo-tangent and we shall make use of one more compactness argument to finish the proof.

Let
$$X = (\{a\} + C(B_\eta(w))) \cap B_{1/\xi}(a)$$
and consider the sets $B_n = T_n(F) \cap X$. Since the space of all compact subsets of $X$ is a compact metric space with respect to the Hausdorff metric, there exists a subsequence $n_k$ and a compact set $\hat{F} \in X$ such that
$$d_H(\hat{F}, T_{n_k}(F) \cap X) \to 0 \quad \text{as} \quad k \to \infty.$$  

Thus, $\hat{F}$ is a weak tangent to $F$. Since (3.11) still holds along any subsequence, we conclude that $\hat{F}$ is not empty and in particular $w \cdot \hat{F} = [a_w, a_w + 1]$. Since
$$N_{\hat{F}}(r, \rho) \geq N_{B_r(a) \cap \hat{F}}(\rho) \geq N_{[a_w, a_w + 1]}(\rho) \geq (2^{-1} \xi)(r/\rho)$$
for all $0 < \rho < r < 1$, it follows immediately that $\dim_A(\hat{F}) \geq 1$. 

\[\square\]
4 Examples

For IFSs satisfying the WSP our result is sharp, and for IFSs not satisfying the WSP our result is sharp if we restrict our attention to $\mathbb{R}$. In this section we consider IFSs in $\mathbb{R}^d$ which do not satisfy the WSP and show that different phenomena are possible. We conclude by mentioning our hope for future work that leads to a refinement of our understanding of the Assouad dimension in $\mathbb{R}^d$ when the weak separation property does not hold.

4.1 Intermediate Assouad dimension in the plane

In light of Theorems 2.1 and 3.1 one might naively imagine that the Assouad dimension of a self-similar set is either equal to the Hausdorff dimension, or maximal (equal to the ambient spatial dimension). Trivially, this is false by noting that a self-similar set in $\mathbb{R}$ with Assouad dimension equal to 1 and similarity dimension strictly less than 1, is also a self-similar subset of $\mathbb{R}^d$ via the natural embedding and this does not alter the dimensions. However, this is not very enlightening so we give a more convincing counter example, where the self-similar set is not contained in any hyperplane. The key is to make the weak separation property fail ‘only in one direction’. We achieve this by modifying an example of Bandt and Graf [3], which shows that the weak separation property can fail in the line, even if all the contraction ratios are equal. Consider the following four similarity maps on $[0,1]^2$:

$$S_1(x) = x/5, \quad S_2(x) = x/5 + (t/5,0),$$

$$S_3(x) = x/5 + (4/5,0), \quad S_4(x) = x/5 + (0,4/5)$$

where $t \in [0,4]$. Let $F$ denote the attractor of this system and note that it is not contained in any line and has similarity dimension equal to $\log 4/\log 5$, which is an upper bound for the Hausdorff dimension. Let $F_1$ denote the projection of $F$ onto the first coordinate and $F_2$ denote the projection of $F$ onto the second coordinate. $F_1$ is the self-similar attractor of the iterated function system consisting of the three maps

$$S'_1(x) = x/5, \quad S'_2(x) = x/5 + t/5, \quad S'_3(x) = x/5 + 4/5$$

defined on $[0,1]$ and $F_2$ is the self-similar attractor of the iterated function system consisting of the two maps

$$S''_1(x) = x/5, \quad S''_2(x) = x/5 + 4/5$$

defined on $[0,1]$, which satisfies the open set condition and thus has Hausdorff dimension and Assouad dimension equal to $\log 2/\log 5$. Following Bandt and Graf [3, Section 2 (5)], if $t$ is rational, then the iterated function system for $F_1$ satisfies the weak separation property. However, it is possible to construct irrational $t$ for which the weak separation property is not satisfied. Assume for a moment that we have done this for some $t \in [0,4]$. It follows from Theorem 3.1 that $\dim_A F_1 = 1$. Note that $F$ is contained in $F_1 \times F_2$ and contains an isometrically embedded copy of $F_1$. Since the Assouad dimension is subadditive on product sets and monotone, the Assouad dimension of $F$ is bounded above by the sum of the Assouad dimensions of $F_1$ and $F_2$ and below by the Assouad dimension of $F_1$. This gives

$$\dim_H F \leq \log 4/\log 5 < 1 \leq \dim_A F \leq 1 + \log 2/\log 5 < 2$$

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as required. Moreover, it is easy to see that this example could be modified to produce self-similar sets with Assouad dimension arbitrarily close to 1, showing that the lower bound in Theorem 1.4 is sharp.

We will now show how to choose $t$ such that the weak separation property fails for the defining iterated function system for $F_1$. Following Bandt and Graf [3, Section 2 (5)], let

$$t = 4 \sum_{k=0}^{\infty} 5^{-2^k} = 0.9664102400 \ldots$$

and observe that elements $\left((S'_i)^{-1} \circ S'_j\right) \in \mathcal{E}$ where $|i| = |j| = n$ are precisely maps of the form

$$x \mapsto x + \sum_{k=0}^{n-1} 5^k a_k$$

for any sequence $a_k$ over $\{0, \pm 4, \pm t, \pm (4 - c)\}$. Let $n = 2^m + 1$ for some large $m \in \mathbb{N}$ and choose the coefficients $\{a_k\}_{k=0}^{n-1}$ as follows:

$$a_k = \begin{cases} 
-4 & \text{if } k = 2^m - 2^l \text{ for some } l = 0, \ldots, m \\
 t & \text{if } k = n - 1 \\
 0 & \text{otherwise}
\end{cases}$$

This renders the map above equal to

$$x \mapsto x + 5^m t - 4 \sum_{k=0}^{m} 5^{2^m - 2^k} = x + 4 \sum_{k=m+1}^{\infty} 5^{2^m - 2^k}$$

which approximates the identity map arbitrarily closely by making $m$ (and thus $n$) large.

![Figure 2: A self-similar set in the plane with intermediate Assouad dimension where $t = 0.9664102400 \ldots$ is chosen as above to guarantee the failure of the weak separation property for $F_1$. The version on the left includes squares to indicate the images of the four maps in the iterated function system.](image)

### 4.2 Full Assouad dimension in $\mathbb{R}^d$

Here we show how to generalise the example given in Fraser [10, Section 3.1] to arbitrary dimensions. In particular, we show how to construct a self-similar set in $\mathbb{R}^d$ with arbitrarily small Hausdorff dimension, but full Assouad dimension.
Let \(d \in \mathbb{N}\) and \(\Lambda = \{z = (z_1, \ldots, z_d) : z_i \in \{0, 1\} \text{ for } i = 1, \ldots, d\}\) be the set of \(2^d\) corners of the unit hypercube \([0, 1]^d\). Let \(\alpha, \beta, \gamma \in (0, 1)\) be such that \((\log \beta)/(\log \alpha) \not\in \mathbb{Q}\) and define similarity maps \(S_\alpha, S_\beta\) and \(S_\gamma\) (\(z \in \Lambda \setminus \{(0, \ldots, 0)\}\)) mapping \([0, 1]^d\) into itself by
\[
S_\alpha(x) = \alpha x, \quad S_\beta(x) = \beta x \quad \text{and} \quad S_\gamma(x) = \gamma x + (1 - \gamma)z.
\]

Let \(F \subseteq [0, 1]^d\) be the self-similar attractor of the iterated function system consisting of these \(1 + 2^d\) maps, which we will now prove has full Assouad dimension, \(\dim_A F = d\), by showing that \([0, 1]^d\) is a weak tangent to \(F\). Let \(X = [0, 1]^d\) and assume without loss of generality that \(\alpha < \beta\). For each \(k \in \mathbb{N}\) let \(T_k\) on \(\mathbb{R}^d\) be defined by
\[
T_k(x) = \beta^{-k}x
\]
and we will now show that \(T_k(F) \cap [0, 1]^d \to_{d_H} [0, 1]^d\). Let
\[
E_k^d := \bigcup_{z \in \Lambda} \left( \{\alpha^m \beta^n z : m \in \mathbb{N}, n \in \{-k, \ldots, \infty\}\} \cap [0, 1]^d \right)
\]
and observe that \(E_k^d \subseteq T_k(F) \cap [0, 1]^d\) for each \(k \in \mathbb{N}\). However,
\[
E_k^d = \prod_{l=1}^{d} \left( \{\alpha^m \beta^n : m \in \mathbb{N}, n \in \{-k, \ldots, \infty\}\} \cap [0, 1] \right)
\]
where the set \(\{\alpha^m \beta^n : m \in \mathbb{N}, n \in \{-k, \ldots, \infty\}\} \cap [0, 1]\) is precisely the pre-tangent set \(E_k\) which appears in [10, Section 3.1]. There it was shown that \(E_k \to_{d_H} [0, 1]\) from which it follows immediately that \(E_k^d \to_{d_H} [0, 1]^d\) and thus \(T_k(F) \cap [0, 1]^d \to_{d_H} [0, 1]^d\), proving the desired result. Proving that \(E_k \to_{d_H} [0, 1]\) follows by applying Dirchlet’s Theorem on Diophantine approximation and using the fact that \(\alpha\) and \(\beta\) are not log-commensurable. We omit the details but refer the reader to [10, Section 3.1].

The similarity dimension of \(F\), which is an upper bound for the Hausdorff dimension, is the unique solution, \(s\), of
\[
\alpha^s + \beta^s + (2^d - 1)\gamma^s = 1
\]
which, by making \(\alpha\), \(\beta\) and \(\gamma\) small, can be made arbitrarily small (but positive) whilst retaining the non-log-commensurable condition on \(\alpha\) and \(\beta\). Finally, observe that despite the fact we were able to utilise the product substructure of \(F\) and thus the product structure of the tangent, \(F\) itself is not a product, nor could we generalise the example in [10, Section 3.1] by simply taking the \(d\)-fold product of the one dimensional version because the result would not be a self-similar set, but a strictly self-affine set.

4.3 Future work

The lower bound (of 1) in Theorem 3.2 concerning systems which fail the WSP is sharp by virtue of Section 4.1. We suspect, however, that there is a natural additional condition one could add to ‘failure of the weak separation property’ that would guarantee, for example, that \(\dim_A F \geq 2\). Exactly what this additional condition might be is an open question and a topic of further study.
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