Rossby wave equilibria and zonal jets

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The problem of coherent vortex and zonal jet formation in a system of nonlinear Rossby waves is considered from the point of view of the late time steady state achieved by free decay of a given initial state. Statistical equilibrium equations respecting all conservation laws are constructed, generalizing those derived previously for 2D inviscid Euler flow. Jet-like solutions are ubiquitous, with large coherent vortices existing only when there is uniform background flow with the precise velocity to cancel the beta effect.

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The formation of jets (localized, elongated energetic flows) in rotating 2D flow, such as planetary atmospheres and oceans, is a ubiquitous phenomenon. Although jet-like structures are often dictated by external forcing, or by boundary constraints, zonal (east-west) jets may also form via unforced evolution of essentially isotropic initial conditions in large systems where boundaries are unimportant—the remarkable band structure of Jupiter is a famous example. Although the planetary rotation axis clearly distinguishes the zonal and meridional directions, its effect on the dynamics is rather subtle and there is no simple argument why this should lead to highly elongated structures. In particular, conserved quantities like energy \( E \) and enstrophy \( \Omega_2 \), as well nonlinear advection effects, remain isotropic.

Recently, a new, highly anisotropic adiabatic invariant \( B \), special to systems of interacting Rossby waves \( \beta \), has been argued to provide the required dynamical mechanism \( \beta \). In particular, for the ever larger scale flows generated by the well-known inverse cascade of energy, \( B \) increases strongly with scale if the energy spectrum remains isotropic. Conservation of \( B \) requires that the spectrum concentrate in the meridional direction, leading to zonally concentrated flows. Well defined Rossby waves exist only at mid-latitudes, so this offers only a partial explanation since jet formation is also seen in near-equatorial flows.

In this work a very general, complementary approach is considered. Rather than trying to follow the very complicated turbulent dynamics from some given initial condition, one seeks to understand only the range of possible long-time steady states produced by free decay of such flows. In particular, equilibrium steady states are considered, which are completely specified by the values of all the conserved quantities. In standard 3D thermodynamic systems, often only energy and particle number are conserved. In the 2D flows of interest, there are an infinite number, providing a far greater range of interesting equilibria with nontrivial spatial structure.

In the following, an exact nonlinear PDE that determines the equilibrium flows is derived, and solved in various limits. It will be shown, quite generally, that the latitude dependence of the Coriolis force (the “beta effect”) destabilizes large coherent vortex structures: the only stable equilibrium flows are those that are zonally translation invariant, i.e., jet-like. Only in a background flow that precisely cancels the beta effect are coherent vortices stable. Interestingly, conservation of \( B \), which requires a beta effect, also fails in this limit, and therefore does not hinder the formation of such structures. The equilibrium arguments are independent of all dynamical considerations, and hence do not depend on the existence of \( B \), but the fact that they are so consistent suggests a relation. This will be a subject of future investigation.

For generality, consider a class of equations (examples to follow) which may be written in the form

\[ \partial_t Q + \mathbf{v} \cdot \nabla Q = 0. \]

The incompressible velocity field \( \mathbf{v} = \nabla \times \psi \equiv (\partial_y \psi, -\partial_x \psi) \). The stream function \( \psi \) is assumed related to the convectively conserved “charge field” \( Q \) through an energy functional \( \mathcal{H}[Q] \):

\[ \psi(r) = \delta \mathcal{H}/\delta Q(r). \]

An example is the Charny-Hasagawa-Mima (CHM) equation in which \( Q = \omega + k_R^2 \psi + f \), where \( \omega = \nabla \times \mathbf{v} \equiv \partial_y v_x - \partial_x v_y = -\nabla^2 \psi \) is the vorticity. The Coriolis function \( f = 2\Omega_E \sin[\theta(y)] \), where \( \theta(y) \) is the latitude, and \( \Omega_E = 2\pi/(24 \text{ hr}) \) is the rotation frequency of the Earth, and coordinates are chosen so that the \( x \)-axis points eastward and the \( y \)-axis northward. In this case,

\[ \mathcal{H} = \frac{1}{2} \int d^2 r \int d^2 r' (Q(r) - f(r))g(r,r')(Q(r') - f(r')) \]

in which \( g \) is the Green function of the operator \(-\nabla^2 + k_R^2\) satisfying free slip and/or periodic boundary conditions. In the beta-plane approximation, \( f = f_0 + \beta_f y \), where \( f_0 = 2\Omega_E \sin(\theta_0) \), \( \beta_f = 2(\Omega_E/R_E) \cos(\theta_0) \), where \( R_E \) is Earth’s radius, and \( \theta_0 \) is a reference latitude. The Rossby radius of deformation is \( R_0 = 1/k_R = c/f \), where \( c = 3 \text{ m/s} \) is the speed of internal gravity waves. In principle \( k_R = k_R(y) \), but usually one sets \( k_R = 1/R_0(0) \).

Only \( \beta_f \) then produces anisotropy. The free space Green function is the modified Bessel function, \( g(r,r') = \)
$K_0(k_R |r - r'|)/2\pi$, and linearizing (1) produces the usual Rossby wave dispersion relation $\omega = -\beta_j k_z/(k_B^2 + k^2)$. The CHM equation (reducing to the Euler equation for $k_R = 0$) is an approximation to the shallow water equations in which propagating gravity waves are neglected. The surface height is then proportional to $\psi$, and adiabatically follows the flow. This equation also describes drift wave plasmas, in the 2D plane perpendicular to an applied magnetic field, where $\psi$ is the electrostatic potential and $\omega$ the charge density.

For later purposes, it is useful to define the Legendre transform of $\mathcal{H}$,

$$L[\psi] = \mathcal{H}[Q] - \int d^2r\psi(r)Q(r)$$

in which (2) is used to substitute $\psi$ for $Q$. The relation may be inverted via the implied identity

$$Q(r) = -\delta L/\delta \psi(r).$$

From (3) one obtains the simple result

$$L = -\int d^2r \left[ \frac{1}{2}\nabla \psi^2 + \frac{1}{2}k_B^2 \psi^2 + f\psi \right],$$

the first two terms of which may be recognized as the kinetic and potential energies. Since $\mathbf{v} \cdot \nabla \psi \equiv 0$, it follows that $\mathcal{H}$ is conserved. With periodic or free slip boundary conditions, it follows also from (1) that $\Omega_h = \int d^2r h(Q(r))$ is conserved for any 1D function $h(\sigma)$, conveniently summarized by

$$g(\sigma) = \int d^2r \delta[\sigma - Q(r)],$$

conserved for any $\sigma$. One recovers $\Omega_h = \int d\sigma h(\sigma)g(\sigma)$. Certain “momentum functionals”

$$P = \int d^2r h(r)Q(r)$$

are also conserved if $\mathcal{H}$ has appropriate translation symmetries. For rotational symmetry, the conserved (vertical component of) angular momentum corresponds to $\lambda = \hat{\mathbf{l}} \times \mathbf{r}$. For translation symmetry along direction $\hat{\mathbf{l}}$, the conserved linear momentum corresponds to $\lambda = \hat{\mathbf{l}} \times \mathbf{r}$.

For the CHM equation in the open beta-plane there is an additional adiabatically conserved quantity, which to quadratic order in $\psi$ takes the Fourier space form

$$B = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} b(k) \hat{Q}(-k) \hat{Q}(k),$$

in which $\hat{Q}(k) = (k^2 + k_R^2)\hat{\psi}(k)$ is the Fourier transform of $Q - f = (-\nabla^2 + k_B^2)\psi$. The key properties are:

1. $\hat{b}(k) = \mathcal{O}([k_R/k]^0$) decays rapidly for $k/k_R \gg 1$, and
2. for $k/k_R \ll 1$, $\hat{b}(k) = \mathcal{O}(1)$ for $k$-space directions

$$\pi/3 < \theta < 2\pi/3$$

(but vanishing for $|\theta| \to \pi/2$) while $\hat{b}(k) = \mathcal{O}(k_R/k)$ otherwise. Property (1) implies that $B$,

even more so than $\mathcal{H}$, is a large scale quantity, insensitive to small scale variations in $Q$. Property (2) then implies that the inverse cascade must focus the small $k$ part of the spectrum close to $|\theta| = \pi/2$, leading (through the curl relation with $\mathbf{v}$) to large scale zonal flows.

When $k_R \to 0$ or $\beta_j \to 0$, the adiabatic conditions fail, and $B$ is no longer conserved, so this form should be used only at mid-latitudes. Since $B$ is not exactly conserved, it will not be included explicitly in the equilibrium calculation, but its consistency with various proposed equilibrium states will be confirmed at the end.

All statistical equilibrium states are contained in the free energy

$$F = -\frac{1}{\beta} \ln \left\{ \int DQ \Delta_\beta [Q]e^{-\beta[\mathcal{H}[Q] - \alpha_0 P[Q]} \right\},$$

in which $\beta = 1/T$ is the inverse temperature, $\alpha_0$ is a Lagrange multiplier for $P$, $\int DQ$ is a functional integral over all possible configurations of $Q$, and $\Delta_\beta = \prod_\alpha \delta(g(\sigma) - g_{\beta}(\alpha))$ represents the infinite product of delta-functions required to impose the chosen values $g(\sigma)$ of all the conserved integrals $g_{\beta}(\alpha)$ represented by the right hand side of (4). If $Q(r) \to Q_0$ is discretized on a grid with microscopic spacing $a$, then Liouville’s theorem specifies the measure $\int DQ = \lim_{a \to 0} \prod_\alpha \int_{-\infty}^{\infty} dQ_\alpha$. It transpires that $F$ may be computed exactly under the very general assumption that $\mathcal{H}$ and $P$ are insensitive to very small scale fluctuations in $Q$. Due to fine-scale mixing, the equilibrium $Q$ will fluctuate wildly from grid point to grid point, but its average $Q_0(r)$ over a small area $l^2 \gg a^2$ (with $l \to 0$ also at the end), will vary smoothly. If $Q_0(r, \sigma)$ is the equilibrium probability density for finding a parcel of fluid with charge $\sigma$ in the area $l^2$ about $r$. Then $Q_0(r, \sigma) = \int d\sigma_0 Q_0(r, \sigma)$. It is assumed that $\mathcal{H}$ is smooth on scale $l$, and hence $\mathcal{H}[Q] = \mathcal{H}[Q_0]$.

In this is provided by the smoothness of $g$: although it diverges at the origin, the logarithmic singularity is sufficiently weak that this assumption remains valid. By integrating out the small-scale fluctuations, which may be treated as independent from grid point to grid point, one obtains an entropy contribution

$$S = \frac{1}{a^2} \int d^2r d\sigma_0(n_0(r, \sigma) \ln[n_0(r, \sigma)],$$

in terms of which the free energy is $F[n_0] = \mathcal{H}[Q] - TS[n_0]$. One now observes that if $TS$ is to remain finite as $a \to 0$ (so that nontrivial equilibria are obtained in which entropy and energy compete), one must adopt the scaling $T = Ta^2$, $\beta = \beta/2$, with fixed $\beta = 1/T$.

To self consistently determine $n_0(r, \sigma)$, Lagrange multipliers are introduced via the Gibbs free energy

$$G = F - \int d^2r \mu(\sigma) n_0(r, \sigma),$$

in which $\mu(\sigma)$ is used to tune $g(\sigma)$. One may now freely extremalize $G$ over all $n_0$, constrained only by the normalization $\int d\sigma n_0(r, \sigma) = 1$, to obtain

$$n_0(r, \sigma) = e^{-\beta W(\Psi_0 - \alpha)}e^{\beta[\mu(\sigma) - \sigma(\Psi_0 - \alpha)]},$$

where $\beta W(\Psi_0 - \alpha)$ is the Gibbs free energy.
where $\Psi_0(\mathbf{r}) = \delta\mathcal{H}/\delta Q_0(\mathbf{r})$ is the equilibrium stream function, $a(\mathbf{r}) = a_0(\mathbf{r})$, and

$$ W(\tau) = \frac{1}{\beta} \ln \left\{ \int d\sigma e^{\beta [\mu(\sigma) - \sigma\tau]} \right\}. \quad (14) $$

Substituting (13) back into (11) and using (4), finally determines $G$ as a functional of $\Psi_0$:

$$ G[\Psi_0] = \mathcal{L}[\Psi_0] - \int d^2 r W(\Psi_0 - \alpha). \quad (15) $$

The latter is finally determined by the (minimum for $\tilde{T} > 0$, maximum for $\tilde{T} < 0$) condition

$$ -\delta\mathcal{L}/\delta \Psi_0(\mathbf{r}) \equiv Q_0(\mathbf{r}) = F[\Psi_0(\mathbf{r}) - a(\mathbf{r})], \quad (16) $$

where $F(\tau) = -\partial_\tau W(\tau)$. Since $W(\tau)$ is convex, $F(\tau)$ is monotonic. Using (13) and (14), the left hand side is $Q_0 = \left( -\nabla^2 + k_f^2 \right) \Psi_0 + f$, and (16) represents a nonlinear PDE for $\Psi_0$. The conserved quantities are set by the derivatives $P = -\partial G/\partial Q_0 = \int d^2 r F(\Psi_0 - \alpha)$, $g(\sigma) = -\partial G/\partial \mu(\sigma) = \int d^2 r \sigma(\mathbf{r}, \sigma)$.

Substituting (14) into (11), one finds $Q(\mathbf{r}, t) = Q_0(\mathbf{r} + v_0 t)$ or $Q_0(\mathbf{r}, \theta + \alpha_0 t)$, depending on the choice of $\lambda$. Fixed momentum solutions are not in general static, but require a background flow with velocity $\alpha_0$ along the symmetry direction.

Equations (13), (13) are the fundamental results of this paper, providing a complete description of the fluid equilibria for a rather general class of problems. Ultimately one is interested in coherent vortex formation, where $Q$ is large in some compact region, and is much smaller outside of it. One sees from (3) that these are high energy configurations, and hence correspond to $\tilde{T} < 0$. For tractable applications “finite level systems” are useful: let $g(\sigma) = \sum_{k=1}^{S} A_k \delta(\sigma - \sigma_k)$, in which $A_k$ is the total area occupied by charge $\sigma_k$. One correspondingly requires only a finite number of Lagrange multipliers $\mu_k$: $e^{\beta g(\sigma)} = \sum_{k=1}^{S} e^{\beta \mu_k \delta(\sigma - \sigma_k)}$, and $W(\tau)$ becomes (the log of) a discrete sum. One obtains $n_k(\mathbf{r}, \sigma) = \sum_{k=1}^{S} \rho_k(\mathbf{r}) \delta(\sigma - \sigma_k)$, $Q_0(\mathbf{r}) = \sum_{k=1}^{S} \sigma_k \rho_k(\mathbf{r})$, where

$$ \rho_k(\mathbf{r}) = \sum_{i=1}^{S} e^{\beta (\mu_k - \sigma_k [\Psi_0(\mathbf{r}) - a(\mathbf{r})])} \int d^2 r \delta(\sigma - \sigma_k), $$

has spatial integral $A_k$, and is therefore the equilibrium number density for charge $\sigma_k$. To illustrate problems of the solutions, consider the two level system, $\sigma_k = 0, \sigma_0$, in the beta-plane where $\alpha = \alpha_0 g/\rho_2 = (\epsilon g[\Psi_0 - a(\mathbf{r})] - \rho_2 + 1)^{-1}$ is the Fermi function, where $\mu = \mu_1 - \mu_0$ is the chemical potential difference. Define the temperature scale $T_0 = -\sigma_0^2/4k_f^2 < 0$, and let $\tilde{T} = T/T_0$. $p_0 = -\sigma_0 [\Psi_0 - a(\mathbf{r})] / 2T_0$, $q_0 = 2Q_0/\sigma_0 - 1$, $\rho = (\rho_2 - \rho_0)/k_f^2$. In terms of these scaled variables, the free energy and equilibrium equation take the form

$$ G = -\frac{\sigma_0^2}{4k_f^2} \int d^2 \rho \left\{ \frac{1}{2} [\nabla \rho p_0]^2 - h_{p_0} + V(\rho_0) + \epsilon_0 \right\} $$

$$ q_0 = -\nabla^2 q_0 - h = \tanh(p_0/\tilde{T}), \quad (18) $$

where $h(\rho) = h_0 + q_0 \rho_g$, with $h_0 = 1 - 2f_0/\sigma_0 + \mu/2T_0$, $g_0 = -2(\alpha_0 k_f^2 + \beta_0)/\sigma_0 k_f R$. The $\rho_0$-independent $\epsilon_0(\rho)$ term is unimportant. The potential $V(\rho_0) = 1/2 \rho_0^2 - \int h_0 \ln(2 \cos\theta(p_0/\tilde{T})$ is an even function with a single minimum at $p_0 = 0$ for $\tilde{T} > 1$, and symmetric double minima at $\pm \rho_{eq}^0(\tilde{T})$ satisfying $p_{eq}^0 = \tanh(p_{eq}^0/\tilde{T}) = 0$ for $\tilde{T} < 1$.

Equation (18) is a standard continuum model of a bi-}

aried fluid in a gravitational field $g_0$, composed of a mixture of heavy ($q_0 = 1, Q_0 = \sigma$) and light ($q_0 = -1, Q_0 = 0$) particles. The $|\nabla p_0|^2$ term is an attractive interaction between like particles which encourages phase separation at low temperatures, $\tilde{T} < 1$. For $\tilde{T} < 1$ there will be a sharp interface centered at $p_0 = -h_0/g_0$ (determined by $\mu$) of width $\Delta_{p_0} \sim \tilde{T}$ (see below) between the segregated phases. The analogy generalizes easily to multiple charge levels, which generate multicomponent fluid mixtures with different combinations of separated and unseparated phases produced as $\tilde{T}, \mu_0$ are varied. However, $g_0$ is unchanged, and in equilibrium the system must again be vertically stratified with phases density ordered along $g_0$. The argument also clearly generalizes to nonconstant $\beta_f(y)$. A varying $g_0(y)$ still produces vertical stratification. This establishes the claim that only equilibria with purely zonal flow are stable in the presence of a beta effect, $\beta_f \neq 0$, and is completely consistent with inverse
cascade arguments based on conservation of $B$.

Only for $\beta_f = 0$, hence $\sigma_0 = -\beta_f k_R^2 = O(10 \text{ cm/s})$ does the gravity effect disappear \[11\]. In a background flow moving at speed $\sigma_0$, an effective isotropic $f$-plane is restored. Consider a large vortex region whose boundary is smooth on the scale of its width $\Delta r = 1/\sqrt{2(1 - t)}$.

The free energy increment per unit length $L$ of the interface, $\Delta \mathcal{G}/L = (2|\bar{T}_0|^{3/2}/\sigma_0) \Sigma$ yields the scaled surface tension

$$\Sigma(\bar{t}) = \int_{-\infty}^{\infty} d\xi (\partial_\xi p_0)^2.$$  \hspace{1cm} (19)

In Fig. 1 numerical solutions for $q_0(\xi, \bar{t})$, $\Sigma$ are plotted (and exact asymptotics described), while in Fig. 2 scaled interface profiles $q_0(\xi)$ are plotted for several $\bar{t}$. The true equilibrium solution is finally obtained by minimizing the vortex perimeter $L$ at fixed area $A$ (set by the total charge $\Omega_1$), yielding, not surprisingly, a circular vortex. These arguments again generalize to multiple charge levels, where one could in principle have more than two phases in simultaneous equilibrium, with vortices composed of a central core of one phase, ringed by one or more other phases.

Since $B$ is no longer conserved in the $f$-plane, there is no dynamical barrier to the formation of isotropic structures, again establishing consistency between arguments based on the dynamics of the inverse cascade, and those based on purely equilibrium thermodynamics. It appears likely that conservation of $B$ is a microscopic reflection of the analogy to gravity-induced density stratification. Since vortices drift across $q_0$, rather than accelerating along it, equilibration is necessarily far less direct than in physical binary fluid systems, and $B$ may be one of the mechanisms controlling this. This idea will be investigated more carefully in future work.

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[7] P. Chen and M. C. Cross, Phys. Rev. E 50, 2222 (1994); Phys. Rev. E 50, 2224 (1994).
[8] It follows from \[1\] that any dynamical variable $A(Q)$ obeys $\partial_t A = \{A, H\}$, in which $\{A, B\} = \int dr Q(r) \nabla A(r) / \delta Q(r) \times \nabla B / \delta Q(r)$ is a Poisson bracket.
[9] Formally, one chooses $\alpha$ in such a way that $\{Q, L\} = \partial_\xi Q$, where $\xi$ is the symmetry coordinate, so $L$ is the generator of translations along $\xi$.
[10] The point charge limit is obtained by letting all nonzero $\sigma_k \to \infty$, $A_k \to 0$ with fixed $q_k = \sigma_k A_k$. The temperature scales via $\beta \sigma_k = \beta q_k$ with finite $\beta = 1/T$, and $\sigma_k \rho_k \to q_k \rho_k$, with $\rho_k = \exp(\beta [\mu_k - q_k (\Psi_0 - \alpha)])$, and $\mu_k$ ensuring a unit integral. The nonzero charge region has measure zero. An example is the sinh-Poisson equation, $Q_0 = C \sinh(\beta \Psi_0)$, the symmetric special case of the three level system, $q_k = 0, \pm 1$. F. Spineau and M. Vlad [Phys. Rev. Lett. 94, 235003 (2005)] obtain, through a mysterious sequence of field theoretic mappings, an equilibrium equation with $Q_0 = C_1 \sinh(\beta \Psi_0)(\cosh(\beta \Psi_0) - C_2)$, the symmetric special case of the five level system, $q_k = 0, \pm 1, \pm 2$. All of these are very special cases of the general theory presented here.
[11] Since $\beta_f(y) \equiv f(y)$ decreases with increasing $|y|$, $f(y) - \beta_f(y_0)(y - y_0) \approx f(y_0) - \frac{1}{2} f'(y_0)(y - y_0)^2$ is not perfectly flat, but has an extremum. In the northern (southern) hemisphere it traps lighter (denser) phases, while denser (lighter) phases will be pushed to higher and lower latitudes. Only lower (higher) charged coherent vortices may then be truly stable.