Local Search is a PTAS for Feedback Vertex Set in Minor-free Graphs

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Abstract

Given an undirected graph, the Feedback Vertex Set (FVS) problem asks for a minimum set of vertices that hits all the cycles of the graph. Fomin, Lokshtanov, Rauman and Saurabh gave a PTAS for the FVS problem in $H$-minor-free graphs. Their algorithm relies on many complicated algorithmic tools. In this work, we show that a simple local search algorithm, that tries to improve the solution by exchanging only constant number of vertices, is a PTAS for the FVS problem in $H$-minor-free graphs.

1 Introduction

Given an undirected graph, the Feedback Vertex Set (FVS) problem asks for a minimum set of vertices such that after removing this set, the resulting graph has no cycle. This problem arises in a variety of applications, including deadlock resolution, circuit testing, artificial intelligence, and analysis of manufacturing processes. Because of its importance, the FVS problem has been studied for a long time in the algorithm area. It is one of Karp’s 21 original NP-Complete problems and is proved to be NP-hard even in planar graphs. The current best approximation ratio for the FVS problem in general graphs is 2 due to Becker and Geiger and Bafna, Berman, and Fujito.

For some special classes of graphs, we can obtain better approximation algorithms. A polynomial-time approximation scheme (PTAS) is an algorithm that for any fixed $\epsilon > 0$, finds an $(1 + \epsilon)$-approximation of the optimal solution in polynomial time. Kleinberg and Kumar gave the first PTAS for the FVS problem in planar graphs, and Cohen-Addad, Colin de Verdière, Klein, Mathieu, and Meierfrankenfeld gave a PTAS for the weighted version of this problem in bounded-genus graphs. Demaine and Hajiaghayi gave an efficient PTAS (EPTAS) for single-crossing-minor-free graphs. Fomin, Lokshtanov, Rauman and Saurabh gave an EPTAS for $H$-minor-free graphs. However, their algorithm for $H$-minor-free graphs relies on a tree decomposition construction and other algorithmic tools based on bounded treewidth, such as Courcelle’s theorem or the similar result of Borie, Parker and Tovey, which makes the algorithm hard to apply in

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1 A PTAS is efficient if it running time is bounded by a polynomial whose degree is independent of $\epsilon$. 
practice. On the other hand, local search is a simple heuristic and there have been several experimental works on applying local search to FVS problem in general graphs, which show that local search gave good approximate solutions. However, no theoretical analysis of local search for the FVS problem was known before.

In this work, we show that a simple local search algorithm is a PTAS for the FVS problem in $H$-minor-free graphs. The algorithm is depicted in Algorithm 1. Intuitively, the local search algorithm starts with an arbitrary solution for the problem and tries to change a constant number (depending on $\epsilon$) of vertices in the current solution to obtain a better solution. The algorithm outputs the current solution when it cannot obtain a better solution in this way.

Algorithm 1 \textsc{LocalSearch}(G, \epsilon)

1: $S \leftarrow$ an arbitrary solution of $G$
2: $c \leftarrow$ a constant depending on $\epsilon$
3: \textbf{while} there is a solution $S'$ such that $|S \setminus S'| \leq c$, $|S' \setminus S| \leq c$ and $|S'| < |S|$ \textbf{do}
4: \hspace{1em} $S \leftarrow S'$
5: \hspace{1em} output $S$

\textbf{Theorem 1.1.} For any fixed $\epsilon > 0$, there is a local search algorithm that finds a $(1+\epsilon)$-approximate solution for the FVS problem in $H$-minor-free graphs with running time $O(n^c)$ where $c = \frac{\text{poly}(|V(H)|)}{\epsilon^2}$.

To complement our positive result, we provide several negative results. First, we show that the FVS problem is APX-hard in 1-planar graphs. That implies the FVS problem is unlikely to have a PTAS beyond $H$-minor-free graphs. Our negative result contrasts with the positive results of Har-Peled and Quanrud, who show that local search provides PTASes for many problems including vertex cover, independent set, dominating set and connected dominating set, in graphs with polynomial expansion, which generalize 1-planar graphs and $H$-minor-free graphs. Second, we show that two closely related variants of the FVS problem, namely: odd cycle transversal and subset feedback vertex set, do not have such simple local search PTASes even in planar graphs. The odd cycle transversal (also called bipartization) problem asks for a minimum set of vertices in an undirected graph whose removal results in a bipartite graph. Given an undirected graph and a subset $U$ of vertices, the subset feedback vertex set problem asks for a minimum set $S$ of vertices such that after removing $S$ the resulting graph contains no cycle that passes through any vertex of $U$. We show by examples, that the simple local search with constant exchanges cannot give a constant approximation in planar graphs.

1.1 Local Search Algorithms

Local search has been used before to obtain PTAS for other problems in $H$-minor-free graphs. Cabello and Gajser gave local search PTAS for maximum independent set problem, minimum vertex cover problem and minimum dominating set problem in $H$-minor-free graphs. Cohen-Addad, Klein and Mathieu showed that local search yields PTAS for $k$-means, $k$-median and uniform uncapacitated facility location in $H$-minor-free graphs. All their analysis relies on the exchange graph, a graph constructed from the optimal solution $O$ and the local search solution $L$. For $k$-means and $k$-median, the exchange graph is constructed from $L$ and a nearly optimal solution $O'$, which is obtained by removing some vertices of $O$. For
independent set and vertex cover, the exchange graph is the subgraph induced by \( O \cup L \), and for the other problems, the exchange graph is built by contracting other vertices to nearest vertices in \( O \cup L \). Then the local properties of these problems naturally appear in the exchange graphs: if we consider a small neighborhood \( R \) in the exchange graph and replace the vertices of \( L \) in \( R \) with the vertices of \( O \) in \( R \) and the boundary of \( R \), the resulting vertex set is still a feasible solution for the original graph. Then by decomposing the exchange graph into small neighborhoods, we can bound the size of \( L \) by the size of \( O \) and all the boundaries of those neighborhoods.

However, the FVS problem does not have such local property if we construct exchange graph only by deletion or only by contraction. This is because for a cycle \( C \) in the original graph, the vertex of \( L \) that covers \( C \) may be inside of some neighborhood but the vertex of \( O \) that covers \( C \) may be outside of that neighborhood. One may try to argue the boundary of the neighborhood could cover \( C \). But unfortunately, the boundary may not be helpful since the crossing vertices of \( C \) and the boundary may not be in both solutions and then they may be deleted or contracted to other vertices.

To solve this problem, we will construct an exchange graph with the following property: for any cycle \( C \) of the original graph, in our exchange graph there is either a vertex in \( O \cap L \cap C \), or an edge between a vertex in \( O \cap C \) and a vertex in \( L \cap C \), or another cycle \( C' \) such that vertices in \( C' \) is a subset of vertices in \( C \) and \( C' \cap (O \cup L) = C \cap (O \cup L) \). To achieve this goal, we will apply both deletion and contraction to construct our exchange graph. Furthermore, we need to introduce vertices that are not in both solutions into the exchange graph, which is different from all previous exchange graph constructions. Meanwhile, we need to guarantee that the number of such vertices is linear to the size of \( O \cup L \). The linear size bound is essential to the correctness of our algorithm and we prove this size bound by a structural lemma which may be of independent interest.

2 Preliminaries

For a graph \( G \), we denote the vertex set and the edge set of \( G \) by \( V(G) \) and \( E(G) \), respectively. For a subgraph \( H \) of \( G \), the boundary of \( H \) is the set of vertices that are in \( H \) but have at least one incident edge that is not in \( H \). We denote by \( int(H) \) the set of vertices of \( H \) that are not in the boundary of \( H \). The degree of a vertex is the number of its incident edges.

A graph \( H \) is a minor of \( G \) if \( H \) can be obtained from \( G \) by a sequence of vertex deletions, edge deletions and edge contractions. \( G \) is \( H \)-minor-free if \( G \) does not contain a fixed graph \( H \) as a minor. The following theorem of Mader shows that \( H \)-minor-free graph is sparse.

**Theorem 2.1** (Mader \[27\]). An \( H \)-minor-free graph of \( n \) vertices has \( O(\sigma_H n) \) edges where \( \sigma_H = |V(H)| \sqrt{\log |V(H)|} \).

A balanced separator of a graph is a set of vertices whose removal partitions the graph roughly in half. A separator theorem typically provides bounds for the size of each part and the size of the balanced separator. Usually, the size of the balanced separator is sublinear w.r.t. the size of the graph. Separator theorems have been found for planar graphs \[12, 14, 18, 26\], bounded-genus graphs \[13, 19, 24\], and minor-free graphs \[14, 23, 29, 30\]. An \( r \)-division is a decomposition of a graph, which was first introduced by Frederickson \[17\] for planar graphs to speed up planar shortest path algorithm.

**Definition 2.2.** For an integer \( r \), an \( r \)-division of a graph \( G \) is a collection of edge-disjoint subgraphs of \( G \), called regions, with the following properties.
1. Each region contains at most \( r \) vertices.
2. The number of regions is at most \( c_d \frac{n}{r} \).
3. The number of boundary vertices, summed over all regions, is at most \( c_d \frac{n}{\sqrt{r}} \).

where \( c_d \) is a constant.

We say a graph is \( r \)-divisible if it has an \( r \)-division. Frederickson [17] gave a construction for the \( r \)-division of a planar graph which only relies on the separator theorem in planar graphs [26]. It is straightforward to extend the construction to any family of graphs with balanced separator of sublinear size. Since \( H \)-minor-free graphs are known to have balanced separators (Alon, Seymour, and Thomas [1]), we have:

**Theorem 2.3** (Alon, Seymour, and Thomas [1] + Frederickson [17]). *Minor-free graphs are \( r \)-divisible with \( c_d = \text{poly}(|V(H)|) \).*

### 3 Exchange Graph Implies PTAS by Local Search

In this section, we show that if for an \( H \)-minor-free graph \( G \), we can construct another graph, called exchange graph, such that it is \( r \)-divisible, then Algorithm \[1\] is a PTAS for the FVS problem. Let \( O \) be an optimal solution of the FVS problem and \( L \) be the output of the local search algorithm. We say a vertex \( u \) a solution vertex if \( u \in O \cup L \) and a Steiner vertex otherwise. Unlike prior works [6,20], we allow Steiner vertices in our exchange graphs.

**Definition 3.1.** A graph \( \text{EX} \) is an exchange graph for the optimal solution \( O \) and the local solution \( L \) of the FVS problem in a graph \( G \) if it satisfies the following properties:

1. \( L \cup O \subseteq V(\text{EX}) \subseteq V(G) \).
2. \( |V(\text{EX})| \leq c_e(|L| + |O|) \) for some constant \( c_e \).
3. For every cycle \( C \) of \( G \), there is (3a) a vertex of \( C \) in \( O \cap L \) or (3b) an edge \( uv \in E(\text{EX}) \) between a vertex \( u \in L \) and a vertex \( v \in O \) in \( C \) or (3c) a cycle \( C' \) of \( \text{EX} \) such that \( V(C') \subseteq V(C) \) and \( C \cap (O \cup L) = C' \cap (O \cup L) \).

**Theorem 3.2.** If graph \( G \) has an \( r \)-divisible exchange graph for an optimal solution \( O \) and a local solution \( L \), then Algorithm \[1\] is a polynomial-time approximation scheme for the FVS problem in \( G \), whose running time is \( n^O(1/\epsilon^2) \).

**Proof.** We set the constant \( c \) in Algorithm \[1\] as \( 1/\delta^2 \) where \( \delta = \frac{\epsilon}{2c_d c_e (2 + \epsilon)} = O(\frac{\epsilon}{c_d c_e}) \). Note that \( c_d \) and \( c_e \) are constants in Definition 2.2 and Definition 3.1, respectively. Since in each iteration, the size of the solution is reduced by at least one, there are at most \( n \) iterations. Since each iteration can be implemented in \( n^{O(c)} \) time by enumerating all possibilities, the total running time is \( n^{O(c)} = n^{O(1/\epsilon^2)} \). We now show that the output \( L \) has size at most \( (1 + \epsilon)|O| \).

Let \( \text{EX} \) be an \( r \)-divisible exchange graph for \( O \) and \( L \). Since \( \text{EX} \) is \( r \)-divisible, we can find an \( r \)-division of \( \text{EX} \) for \( r = c = \lceil 1/\delta^2 \rceil \). Let \( B \) be the multi-set containing all the boundary vertices in the \( r \)-division. By the third property of \( r \)-division, \(|B|\) is at most \( c_d \frac{|V(\text{EX})|}{\sqrt{r}} \). By the second property of exchange graph, \(|V(\text{EX})| \leq c_e(|O| + |L|) \). Thus, we have:

\[
|B| \leq c_d c_e \delta (|O| + |L|)
\]
In the following, we will show that:
\[ |L| \leq |O| + 2|B| \]  \tag{2}

Then by Equation \(1\) we have:
\[ |L| \leq |O| + 2c_d c_e \delta(|O| + |L|) = |O| + \frac{\epsilon}{2 + \epsilon} (|O| + |L|) \]

that implies \[ |L| \leq (1 + \epsilon)|O| \].

To prove Equation (2), we need to study some properties of Ex. For any region \(R_i\) of the \(r\)-division, let \(B_i\) be the boundary of \(R_i\) and \(M_i\) be the union of \(L \setminus R_i\), \(O \cap R_i\) and \(B_i\).

**Claim 3.3.** \(M_i\) is a feedback vertex set of \(G\).

**Proof.** For a contradiction, assume that there is a cycle \(C\) of \(G\) that is not covered by \(M_i\). Then \(C\) does not contain any vertex of \(L \setminus R_i\), \(O \cap R_i\) and \(B_i\). So \(C\) can only be covered by some vertices of \((L \setminus O) \cap int(R_i)\) and some vertices of \(O \setminus (L \cup R_i)\). This implies that \(C\) does not contain any vertex of \(O \cap L\) and there is no edge in Ex between \(C \cap O\) and \(C \cap L\). By the third property of exchange graph, there must be a cycle \(C'\) in Ex such that \(V(C') \subseteq V(C)\) and \(C \cap (O \cup L) = C' \cap (O \cup L)\). Let \(u\) be the vertex of \((L \setminus O) \cap int(R_i)\) in \(C\) and \(v\) be the vertex of \(O \setminus (L \cup R_i)\) in \(C\). Then cycle \(C'\) contains both \(u\) and \(v\), which implies \(C'\) crosses the boundary of \(R_i\), that is \(C' \cap B_i \neq \emptyset\). Let \(w\) be a vertex in \(C' \cap B_i\), then \(w\) also belongs to \(C\) in \(G\). This implies \(M_i\) contains a vertex of \(C\), a contradiction. \(\square\)

By the construction of \(M_i\), we know the difference between \(L\) and \(M_i\) is bounded by the size of the region \(R_i\), that is \(r\). Recall that \(c = r = 1/\delta^2\). Since \(L\) is the output of Algorithm \(1\) we know \(L\) cannot be improved by changing at most \(r\) vertices. So we have \(|L| \leq |M_i|\). By the construction of \(M_i\), this implies
\[ |L \cap R_i| \leq |M_i \cap R_i| \leq |O \cap int(R_i)| + |B_i|. \]

That implies
\[ |L \cap int(R_i)| \leq |L \cap R_i| \leq |O \cap int(R_i)| + |B_i|. \]

Since \(int(R_i)\) and \(int(R_j)\) are vertex-disjoint for any two distinct \(i\) and \(j\), by summing over all regions in the \(r\)-division, we can have
\[ |L| - |B| \leq \sum_i |L \cap int(R_i)| \leq \sum_i (|O \cap int(R_i)| + |B_i|) \leq |O| + |B|. \]

This proves Equation (2). \(\square\)

## 4 Exchange Graph Construction

Recall that \(\sigma_H = |V(H)| \sqrt{\log |V(H)|}\) is the sparsity of \(H\)-minor-free graphs. In this section, we will show that:

**Theorem 4.1.** \(H\)-minor-free graphs have exchange graphs for the FVS problem with \(c_e = O(\sigma_H)\).

Observe that Theorem \(1.1\) immediately follows from Theorem \(4.1\) and Theorem \(3.2\). The running time is \(n^{O(c)}\) where \(c = \frac{\text{poly}(\log |V(H)|)}{c^2}\) since both \(c_d\) and \(c_e\) are polynomial in \(|V(H)|\).

We construct the exchange graph in three steps:
Step 1 We delete all edges in $G$ that are incident to vertices of $O \cap L$. We then remove all components that do not contain any solution vertex. Note that the removed components are acyclic.

Step 2 We contract edges that have an endpoint that is not a solution vertex and has degree at most two until there is no such an edge left. Since $L$ and $O$ are both feedback vertex set of $G$, every cycle after the contraction must contain a vertex from $L$ and a vertex from $O$. Since edges incident to vertices of $O \cap L$ are removed, there is no self-loop after this step.

Step 3 We keep the graph simple by removing all but one edge in each maximal set of parallel edges.

Let $K$ be the resulting graph. We now show that $K$ satisfies three properties in Definition 3.1. Property (1) is obvious because we never delete a vertex in $L \cup O$ from $K$. To show property (3), let $C$ be a cycle of $G$. If any edge of $C$ is removed in Step 1, $C$ must contain a vertex in $O \cap L$; implying (3a). Thus, we can assume that no edge of $C$ is deleted after Step 1. Since contraction does not destroy cycles, after the contraction in Step 2, there is a cycle $C'$ such that $V(C') \subseteq V(C)$. If $|V(C')| = 2$ ($C'$ is a cycle of two parallel edges), then (3b) holds. Thus, we can assume that every edge of $C'$ remains intact after removing parallel edges. But that implies (3c) since we never remove solution vertices from $G$. Thus, $K$ satisfies property (3).

It remains to show $K$ satisfies property (2) in Definition 3.1, that is, $|V(K)| \leq O(\sigma_H)(|L| + |O|)$. By Step 2 we can have the following observation.

**Observation 4.2.** Every Steiner vertex of $K$ has degree at least 3.

Since $O \cup L$ is a feedback vertex set of $K$, $K \setminus (O \cup L)$ is a forest $F$ containing only Steiner vertices. For each tree $T$ in $F$, we define the degree of $T$, denoted by $\deg_K(T)$, as the number of edges in $K$ between $T$ and $O \cup L$. Let $\ell(T)$ be the number of leaves of $T$. By Observation 4.2, every internal vertex of $T$ has degree at least 3. Thus, $|V(T)| \leq 2\ell(T)$. That implies:

$$|V(T)| \leq 2\deg_K(T).$$ (3)

We contract each tree $T$ of $F$ into a single Steiner vertex $s_T$. Let $K'$ be the resulting graph. Then we have the following observation.

**Observation 4.3.** Graph $K'$ is simple.

**Proof.** Since every cycle of $K$ must contains a vertex from $L$ and a vertex from $O$, there cannot be any solution vertex in $K$ that is incident to more than one vertex of a tree $T$ of $F$. So there cannot be parallel edges in $K'$.

To bound the size of $K'$, we need the following structural lemma. We remark that this lemma holds for general graphs.

**Lemma 4.4.** For a graph $G$ and its two disjoint nonempty vertex subsets $A$ and $B$, let $D = V(G) \setminus (A \cup B)$. If (i) $D$ is an independent set, (ii) every vertex in $V(G) \setminus (A \cup B)$ has degree at least 3 in $G$ and (iii) for every cycle $C$ in $G$, we have $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$, then we have $|V(G)| \leq 2(|A| + |B|)$. 


Proof. We remove every edge that only has endpoints in \( A \cup B \) and let the resulting graph be \( G' \). Then \( G' \) is a bipartite graph with \( A \cup B \) in one side and \( D \) in the other side since \( D \) is an independent set. Let \( D_A \) (\( D_B \)) be the subset of \( D \) only containing the vertices that have at least two neighbors in \( A \) (\( B \)). Since every vertex of \( D \) has degree at least 3, we have \( D_A \cup D_B = D \).

Let \( H_A \) be the subgraph of \( G' \) induced by \( A \cup D_A \). Then \( H_A \) is acyclic since otherwise every cycle of \( H_A \) would correspond to a cycle in \( G \) that does not contain any vertex in \( B \). We now construct a graph \( H_A^* \) on vertex set \( A \). For each vertex \( v \in D_A \), we arbitrarily choose its two neighbors \( x \) and \( y \) in \( A \) and add an edge between \( x \) and \( y \) in \( H_A^* \). By construction, there is a one-to-one mapping between edges of \( H_A^* \) and vertices of \( D_A \).

Since \( H_A \) is acyclic, \( H_A^* \) is also acyclic. Thus, \(|E(H_A^*)| \leq |V(H_A^*)| = |A| \). That implies \(|D_A| \leq |A| \). By a similar argument, we can show that \(|D_B| \leq |B| \). Thus, \(|D| = |D_A \cup D_B| \leq |A| + |B| \), and the lemma follows.

Let \( Z \) be an arbitrary component of \( K' \) that contains at least one Steiner vertex. Then the two sets \( V(Z) \cap O \) and \( V(Z) \cap L \) must be disjoint since any vertex in \( O \cap L \) is isolated in \( K' \). And each of the two sets cannot be empty since there must be a cycle in \( Z \) through the Steiner vertex which also contains a vertex of \( O \) and a vertex of \( L \) respectively. Let \( X \) be the set of Steiner vertices in \( Z \). By the construction of \( K' \), vertex set \( X \) is an independent set of \( Z \). By Observation 4.2 every vertex of \( X \) has degree at least 3. So we can apply Lemma 4.4 for \( Z \), \( V(Z) \cap O \) and \( V(Z) \cap L \), and obtain \(|V(Z)| \leq 2(|V(Z) \cap O| + |V(Z) \cap L|) = 2(|V(Z) \cap O| + |V(Z) \cap (L \setminus O)|) \). Note that this bound holds trivially if \( Z \) does not contain any Steiner vertex. Thus, summing over all components of \( K' \), we have \(|V(K')| \leq 2(|V(K') \cap O| + |V(K') \cap (L \setminus O)|) \leq 2(|O| + |L|) \). Since \( K' \) is a minor of \( G \), it is also \( H \)-minor-free. By Theorem 2.1 we have

\[
|E(K')| = O(\sigma_H |V(K')|) = O(\sigma_H (|O| + |L|)) \tag{4}
\]

We now ready to bound the size of \( V(K) \). We have:

\[
|V(K \setminus (O \cup L)| \leq \sum_{T \in F} |V(T)| \leq 2 \sum_{T \in F} \deg_K(T) \leq 2 \sum_{T \in F} \deg_{K'}(st) \leq 2 |E(K')| = O(\sigma_H (|O| + |L|)) \tag{4}
\]

That implies \( V(K) \leq O(\sigma_H (|O| + |L|) \). Thus \( K \) satisfies property (2) in Definition 3.1 with \( c_e = O(\sigma_H) \), thereby, implying Theorem 4.1

5 Negative Results

In this section, we show some negative results for the FVS problem and its variants. A graph is \emph{1-planar} if it can be drawn in the Euclidean plane such that every edge has at most one crossing, where it crosses a single additional edge. We first show that FVS is APX-hard in 1-planar graphs. Then for the two variants, odd cycle transversal and subset feedback vertex set, we construct examples where local search with constant exchanges cannot give a constant approximation in planar graphs.
**Theorem 5.1.** Given a graph $G$, we can construct a 1-planar graph $H$ in polynomial time, such that $G$ has a feedback vertex set of size at most $k$ if and only if $H$ has a feedback vertex set of size at most $k$.

**Proof.** Consider a drawing of $G$ on the plane where each pair of edges can cross at most once. For each crossed edge $e$ in $G$, we subdivide $e$ into edges so that there is exactly one crossing per new edge. Let $H$ be the resulting graph. By construction, graph $H$ is 1-planar.

Let $n$ be the size of $G$. Since there are at most $O(n^2)$ crossings per edge in the drawing, the size of $H$ is at most $O(n^4)$. Since we only subdivide edges, there is a one-to-one mapping between cycles of $G$ and cycles of $H$. It is straightforward to see that any feedback vertex set of $G$ is also a feedback vertex set of $H$.

Let $S$ be a feedback vertex set of $H$. If $S \subseteq V(H) \cap V(G)$, then it is also a feedback vertex set for $G$. Otherwise, let $v \in V(H) \setminus V(G)$ be a vertex in $S$. Then $v$ must be added to subdivide an edge, say $e$, in $G$. We remove $v$ from $S$ and add an arbitrary endpoint of $e$ in $G$ to $S$. Then $S$ is still a feedback vertex set for $H$. We repeat this process until $S$ is a subset of $V(H) \cap V(G)$. Observe that $S$ is a feedback vertex set of size at most $k$ for $G$. Thus, the lemma holds.

Since the FVS problem is APX-hard in general graphs (by an approximation preserving reduction [22] from vertex cover problem, which is APX-hard [11]), Theorem 5.1 implies that FVS is APX-hard in 1-planar graphs.

To show that simple local search cannot give a constant approximation for the odd cycle transversal problem and the subset feedback vertex set problem, we construct a counter-example from a $k \times k$ grid as shown in Figure 1.
Figure 1: Counterexamples for local search on odd cycle transversal and subset feedback vertex set. Circle vertices represent vertices of the optimal solution, and triangle vertices represent vertices of the local search solution. The grid could be arbitrarily large. We add one edge in some diagonal cells of the grid. Left: counterexample for odd cycle transversal. Since any grid is bipartite and does not contain any odd cycle, any odd cycle in the example must contain an edge in the diagonal cell. All the vertices in the diagonal, represented by triangles, give a solution that is locally optimal, that is, we cannot improve this solution by changing a small number of vertices. This is because each triangle vertex and each new edge, together with some other edges, can form at least one odd cycle in the graph. For a constant $c$ that is smaller than the size of optimal solution, if we remove $c$ triangle vertices, say $V'$, in the locally optimal solution, there will be $c$ vertex-disjoint odd cycles in the resulting graph, each of which contains one removed triangle. Thus, there is no subset of size less than $c$ that can replace $V'$. Then the ratio between the two solutions could be arbitrarily big if the grid is arbitrarily big and the number of added diagonal edges is super-constant and sublinear to the size of the diagonal. Right: counterexample for subset feedback vertex set. The diamonds represent the vertices in the given set $U$. Similarly, any cycle through a given vertex must contain the two edges in the diagonal cell. By the same reason, the local search solution cannot be improved.
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