Abstract. In this paper, we study the structure of the set of local Arthur parameters such that the corresponding local Arthur packets all contain a given representation of the split classical groups $G_n = \text{Sp}_{2n}, \text{SO}_{2n+1}$ over a non-Archimedean local field of characteristic zero. We show that there exist a pair of elements in this set which are the unique maximal and minimal elements under various orderings. As an application, we prove the closure ordering conjecture on the local $L$-parameters of representations in local Arthur packets of $G_n$, namely, given any representation $\pi$ in a local Arthur packet $\Pi_\psi$, the closure of the local $L$-parameter of $\pi$ in the Vogan variety must contain the local $L$-parameter corresponding to $\psi$. This conjecture is inspired by the work of Adams, Barbasch, and Vogan, and Cunningham et al. on ABV-packets. Its ABV-analogue has been proved by Cunningham et al. Our result provides evidence for the coincidence of the local Arthur packets with the ABV-packets as conjectured by Cunningham et al. in [CFMMX22, Section 8.3, Conjecture 1]. Our key ingredients to overcome the difficulty which lies in the nontrivial intersections of local Arthur packets are the operators developed in [HLL22] and a reduction to the unique maximal element above. The closure ordering conjecture reveals a very interesting geometric nature of local Arthur packets and is expected to have many important implications. We show that it implies the Enhanced Shahidi’s Conjecture for general quasi-split connected reductive groups, under certain reasonable assumptions. We verify these assumptions for $G_n$, giving a new proof of the Enhanced Shahidi’s Conjecture. As a byproduct of the proof of the closure ordering conjecture, we give new algorithms to determine whether a given representation is of Arthur type, which have their own interests and theoretical importance.

1. Introduction

Let $F$ be a non-Archimedean local field of characteristic zero. Let $q = q_F$ be the cardinality of the residue field of $F$ and $W_F$ be the Weil group of $F$. Let $G$ be a connected reductive group defined over $F$ and let $G = G(F)$. Denote $\hat{G}(\mathbb{C})$ the complex dual group of $G$, and $L^G$ the Langlands $L$-group of $G$. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of $G$.

An admissible homomorphism $\phi$ of $G$ is a continuous homomorphism from $W_F \times \text{SL}_2(\mathbb{C})$ to $L^G$ such that the followings hold:

1. $\phi$ commutes with the projections $W_F \times \text{SL}_2(\mathbb{C}) \to W_F$ and $L^G \to W_F$;
2. the restriction of $\phi$ to $W_F$ consists of semi-simple elements;
3. the restriction of $\phi$ to $\text{SL}_2(\mathbb{C})$ is a morphism of complex algebraic groups;
4. if the image of $\phi$ is contained in the Levi subgroup of some parabolic subgroup $P$ of $L^G$, then $P$ is relevant for $G$ (see [Bor79, 8.2(ii)] for notation).

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The $\hat{G}(\mathbb{C})$-conjugacy class of $\phi$ is called a local $L$-parameter, or Langlands parameter, for $G$ and we denote the set of all local $L$-parameters of $G$ by $\Phi(G)$. The local Langlands conjecture asserts that there is a map $\phi \mapsto \Pi_\phi$, called the Local Langlands Correspondence, such that $\Pi_\phi$ is a finite subset of $\Pi(G)$ and the set $\{\Pi_\phi \mid \phi \in \Phi(G)\}$ forms a partition of $\Pi(G)$. $\Pi_\phi$ is called the local $L$-packet attached to $\phi$. Note that for $\pi \in \Pi(G)$, there exists a unique $\phi_\pi \in \Phi(G)$ such that $\pi \in \Pi_{\phi_\pi}$. We call $\phi_\pi$ the local $L$-parameter for $\pi$. For quasi-split symplectic and special orthogonal groups, Arthur established the local Langlands correspondence using functoriality ([Art13]). We detail some of his work below. For simplicity, let $G_n = \text{Sp}_{2n}$, $SO_{2n+1}$ be a symplectic or split odd special orthogonal group and let $G_n = G_n(F)$. In this case, $L G_n = \hat{G}_n(\mathbb{C}) \times W_F$ and we will just use $\hat{G}_n(\mathbb{C})$ in places of $L G_n$.

A local Arthur parameter $\psi : W_F \times \text{SL}_2^D(\mathbb{C}) \times \text{SL}_2^A(\mathbb{C}) \to \hat{G}_n(\mathbb{C})$ is a direct sum of irreducible representations

$$\psi = \bigoplus_{i=1}^r \phi_i \otimes S_{a_i} \otimes S_{b_i},$$

satisfying the following conditions:

1. $\phi_i(W_F)$ is bounded and consists of semi-simple elements;
2. $\text{SL}_2^D(\mathbb{C})$ and $\text{SL}_2^A(\mathbb{C})$ are two copies of $\text{SL}_2(\mathbb{C})$, the restriction of $\psi$ to either copy is analytic, $S_k$ is the unique $k$-dimensional irreducible representation of $\text{SL}_2(\mathbb{C})$. Here, we call $\text{SL}_2^D(\mathbb{C})$ the Deligne-$\text{SL}_2(\mathbb{C})$ and $\text{SL}_2^A(\mathbb{C})$ the Arthur-$\text{SL}_2(\mathbb{C})$.

We denote the set of all local Arthur parameters for $G_n$ by $\Psi(G_n)$. By the Local Langlands Correspondence for general linear groups, the bounded representation $\phi_i$ of $W_F$ can be identified with an irreducible unitary supercuspidal representation $\rho_i$ of $\text{GL}_d(F)$ for some $d \in \mathbb{Z}_{\geq 1}$ ([Hen00, HT01, Sch13]). Consequently, we often write

$$\psi = \bigoplus_{\rho} \left( \bigoplus_{i \in I_{\rho}} \rho \otimes S_{a_i} \otimes S_{b_i} \right),$$

where the first sum runs over certain irreducible unitary supercuspidal representations $\rho$ of $\text{GL}_n(F)$, $n \in \mathbb{Z}_{\geq 1}$, and $I_{\rho}$ denotes an indexing set. Occasionally, when we are working with multiple local Arthur parameters, we denote the indexing set by $I_{\rho}(\psi)$.

We also consider $\hat{\psi}$, the dual of $\psi$, defined by

$$\hat{\psi}(w, x, y) := \psi(w, y, x).$$

Equivalently, $\hat{\psi}$ can be obtained from $\psi$ by switching the $a_i$’s and $b_i$’s in the decomposition (1.1).

For a local Arthur parameter $\psi$ of quasi-split classical groups, Arthur attached a local Arthur packet $\Pi_\psi$. This is a finite multi-set of smooth irreducible representations, which satisfy certain twisted endoscopic character identities ([Art13, §2]). Assuming the Ramanujan Conjecture, Arthur showed that these local Arthur packets characterize the local components of square-integrable automorphic representations. Mœglin explicitly constructed each local Arthur packet $\Pi_\psi$ and showed that it is multiplicity free ([Moe06a, Moe06b, Moe09a, Moe10, Moe11a]). However, both Arthur’s and Mœglin’s constructions have difficulties when trying to compute the representations in the local Arthur packet using the Langlands classification. To remedy this, for symplectic or split odd special orthogonal groups, Atobe gave a reformulation of Mœglin’s construction ([Ato20]) based on the derivatives introduced in [AM20] and gave an algorithm to explicitly compute the Langlands classification for the representations in a local Arthur packet.
To each local Arthur parameter \( \psi \), one can associate a local \( L \)-parameter \( \phi_\psi \) defined as follows:

\[
\phi_\psi(w, x) := \psi \left( w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \right) .
\]

Arthur showed that the map \( \psi \mapsto \phi_\psi \) is injective. Furthermore, Arthur defined the local \( L \)-packet \( \Pi_\phi_\psi \) ([Art13, Theorem 1.5.1]) and showed that \( \Pi_\phi_\psi \) is contained entirely inside of the local Arthur packet \( \Pi_\psi \). We say that a local \( L \)-parameter \( \phi \) is of Arthur type if \( \phi = \phi_\psi \) for some local Arthur parameter \( \psi \). We also say that a representation \( \pi \) is of Arthur type if \( \pi \in \Pi_\psi \) for some local Arthur parameter \( \psi \) and let \( \Pi_\psi(G_n) \) denote the set of representations of \( G_n \) of Arthur type.

The study of local Arthur packets has many mysteries and difficulties. While local \( L \)-packets are always disjoint, local Arthur packets often have nontrivial intersections. Therefore, for \( \pi \in \Pi_\psi(G_n) \), it is an interesting question to understand the structure of the set

\[
\Psi(\pi) := \{ \psi \in \Psi(G_n) \mid \pi \in \Pi_\psi \}.
\]

When \( G \) is symplectic or a split odd special orthogonal group, different algorithms to compute \( \Psi(\pi) \) were developed independently in [Ato22] and [HLL22]. In this paper, our first goal is to study the extremal elements in \( \Psi(\pi) \) under certain orderings, which is a continuation of the study of the first three named authors in [HLL22] on the intersection of local Arthur packets. We outline some of the related results in [HLL22] below.

Let \( \pi \in \Pi_\psi(G_n) \) and \( \psi \in \Psi(\pi) \) as in (1.1). Following [HLL22], we consider three types of operators

(i) \( ui_{i,j} \),

(ii) \( \text{dual} \circ ui_{j,i} \circ \text{dual} \),

(iii) \( \text{dual}_k^{-} \),

\((i, j, k \text{ are indices in } I_\rho \text{ for some } \rho)\) and their inverses on the set of local Arthur parameters (see §2.4 for details). These operators have been applied in loc. cit. to determine whether two local Arthur packets intersect.

**Theorem 1.1** ([HLL22, Theorem 1.4]). If \( \pi \in \Pi_\psi(G_n) \) and \( \psi_1, \psi_2 \in \Psi(\pi) \), then there exists a sequence of operators \( \{T_i\}_{i=1}^m \) such that

\[
\psi_1 = T_1 \circ \cdots \circ T_m(\psi_2),
\]

where each \( T_i \) is one of the operators \( ui_{i,j} , \text{dual} \circ ui_{j,i} \circ \text{dual} , \text{dual}_k^{-} , \) or their inverses.

We remark that given \( \psi \in \Psi(\pi) \) and an operator \( T, \Pi_\psi \) and \( \Pi_{T(\psi)} \) may not have nontrivial intersection, see Remark 2.11 (3).

We observe that among these six types of operators (including the inverses), \( ui_{i,j}^{-1} \), \( \text{dual} \circ ui_{j,i} \circ \text{dual} \) and \( \text{dual}_k^{-} \) raise the “temperedness” of local Arthur parameters under a certain measurement of temperedness (see Theorem 1.7 (1) below). This idea leads us to the following definition.

**Definition 1.2.** We say an operator \( T \) is a raising operator if it is of the form \( ui_{i,j}^{-1} \), \( \text{dual} \circ ui_{j,i} \circ \text{dual} \), or \( \text{dual}_k^{-} \).

For \( \pi \in \Pi_\psi(G_n) \), the raising operators induce a partial order on \( \Psi(\pi) \).

**Definition 1.3.** For \( \pi \in \Pi_\psi(G_n) \), we define a partial order \( \geq_O \) on \( \Psi(\pi) \) by \( \psi_1 \geq_O \psi_2 \) if \( \psi_1 = \psi_2 \) or there exists a sequence of raising operators \( \{T_i\}_{i=1}^m \) such that

\[
\psi_1 = T_1 \circ \cdots \circ T_m(\psi_2).
\]
Another main result of [HLL22] can be rephrased as follows.

**Theorem 1.4 ([HLL22, §11]).** There exists a unique maximal (resp. minimal) element in $\Psi(\pi)$ under the partial order $\geq_O$, which we denote by $\psi^{\max}(\pi)$ (resp. $\psi^{\min}(\pi)$).

Thus intuitively, $\psi^{\max}(\pi)$ (resp. $\psi^{\min}(\pi)$) is the most (resp. least) tempered local Arthur parameter in the set $\Psi(\pi)$. The following properties of $\psi^{\max}(\pi)$ and $\psi^{\min}(\pi)$ are proved in [HLL22].

**Proposition 1.5 ([HLL22, §11]).** Let $\pi \in \Pi_A(G_n)$.

1. $\phi_\pi$ is of Arthur type if and only if $\pi \in \Pi_{\phi^{\max}(\pi)}$.
2. $\psi^{\min}(\pi) = \psi^{\max}(\hat{\pi})$, where $\hat{\pi}$ is the Aubert-Zelevinsky dual of $\pi$.
3. $\psi^{\max}(\pi) = \psi^{\min}(\pi)$ if and only if $\Psi(\pi)$ is a singleton.

Since local Arthur packets can have nontrivial intersections, given an irreducible representation $\pi$ of Arthur type lying in several local Arthur packets, it is a very mysterious question to determine which of these local Arthur parameters could be called “the” local Arthur parameter for $\pi$. Proposition 1.5 (1) indicates that we can call $\psi^{\max}(\pi)$ “the” local Arthur parameter of $\pi$ if $\pi$ is of Arthur type. We expect $\psi^{\max}(\pi)$ to play an important role in the theories of local Arthur packets and automorphic forms. One of the main goals of this paper is to provide more representation theoretic characterizations of $\psi^{\max}(\pi)$. Proposition 1.5 (3) gives a criterion on $\Psi(\pi)$ being a singleton, namely, $\pi$ lying in a unique local Arthur packet.

In [HLL22, Section 12], we gave two characterizations of $\psi^{\max}(\pi)$ and $\psi^{\min}(\pi)$. The first one used the partitions considered in [Jia14, Conjecture 4.2] and [LS22, Conjecture 1.6], and the second one used the order of zeros of the Arthur normalized intertwining operators considered in [Moe08, Moe10, Moe11b, Moe12, Art13]. We recall the first one now, which serves as a measurement of temperedness mentioned above.

For each local Arthur parameter of $G_n$,

$$\psi = \bigoplus_{i=1}^r \rho_i \otimes S_{a_i} \otimes S_{b_i},$$

let $d_i = \dim \rho_i$. Then we consider the partition

$$P^A(\psi) := [b_1^{d_1 a_1}, \ldots, b_r^{d_r a_r}],$$

which is exactly the partition corresponding to the nilpotent orbit of $\tilde{G}_n(\mathbb{C})$ of the element

$$d(\psi|_{\SL_2(\mathbb{C})}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  

We remark that $P^A(\psi)$ is a key ingredient considered in Jiang’s Conjecture ([Jia14, Conjecture 4.2] and [LS22, Conjecture 1.6]). Also, $\psi$ is tempered if and only if $P^A(\psi) = [1^N]$ ($N = 2n$ or $2n + 1$), which is the smallest partition under the dominance order (see Definition 3.5). Now we can equip the set $\Psi(\pi)$ with a second ordering.

**Definition 1.6.** We define a preorder $\geq_A$ on $\Psi(G_n)$ by $\psi_1 \geq_A \psi_2$ if $P^A(\psi_1) \leq P^A(\psi_2)$ under the dominance order.

It is possible that $P^A(\psi_1) = P^A(\psi_2)$ but $\psi_1 \neq \psi_2$, and hence $\geq_A$ is only a preorder but not a partial order. We have the following theorem.

**Theorem 1.7 ([HLL22, §12]).**
(1) If $T$ is a raising operator, then

$$T(\psi) \geq_A \psi,$$

for any $\psi \in \Psi(G_n)$. In other words, if $\psi \geq_O \psi'$, then $\psi \geq_A \psi'$.

(2) Let $\pi \in \Pi_A(G_n)$. Then $\psi^{\max}(\pi)$ and $\psi^{\min}(\pi)$ are the unique elements in $\Psi(\pi)$ satisfying the following inequality

$$\psi^{\max}(\pi) \geq_A \psi \geq_A \psi^{\min}(\pi),$$

for any $\psi \in \Psi(\pi)$.

From (1) above, we see that $\geq_A$ is finer than $\geq_O$. (2) is a direct consequence of (1) (together with the computation that $T(\psi) \geq_A \psi$ if $T(\psi) \neq \psi$), and it gives a representation theoretic characterization of $\psi^{\max}(\pi)$ and $\psi^{\min}(\pi)$.

It is natural to ask whether there is similar phenomenon if we restrict to the Deligne-$\operatorname{SL}_2(\mathbb{C})$ instead of the Arthur-$\operatorname{SL}_2(\mathbb{C})$. Let

$$\underline{p}^D(\psi) = [a_1^{d_1 b_1}, \ldots, a_r^{d_r b_r}],$$

which is exactly the partition corresponding to the nilpotent orbit of $\hat{G}_n(\mathbb{C})$ of the element

$$d(\psi|_{\operatorname{SL}_2^D(\mathbb{C})}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
which is an *infinitesimal parameter* of $G$, i.e., it is continuous and its image consists of semi-simple elements (see [CFMMX22, Section 4.1]). Conversely, for each infinitesimal parameter $\lambda$, we denote

$$\Phi(G)_\lambda := \{ \phi \in \Phi(G) \mid \lambda_\phi = \lambda \}.$$ 

Following [Vog93], we consider the *Vogan variety* $V_\lambda$ for each infinitesimal parameter $\lambda$

$$V_\lambda := \{ x \in \hat{g} \mid \text{Ad}(\lambda(w))x = |w|x \},$$

where $\hat{g}$ is the Lie algebra of $\hat{G}(\mathbb{C})$. $V_\lambda$ admits an action of the group

$$H_\lambda := \{ g \in \hat{G}(\mathbb{C}) \mid \lambda(w)g = g\lambda(w), \forall w \in W_F \}$$

with finitely many orbits (see [CFMMX22, Proposition 5.6]), and for each $\phi \in \Phi(G)_\lambda$, the element

$$X_\phi := d(\phi|_{\text{SL}_2(\mathbb{C})}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is in $V_\lambda$. Let $C_\phi$ denote the $H_\lambda$-orbit of $X_\phi$. Then we obtain a map

$$\Phi(G)_\lambda \rightarrow V_\lambda/H_\lambda,$$

$$\phi \mapsto C_\phi,$$

which is in fact a bijection (see [CFMMX22, Proposition 4.2]). The geometric structure of $V_\lambda$ now enables us to equip the set $\Phi(G)_\lambda$ with a partial order $\geq_C$ by the closure ordering as follows.

**Definition 1.10.** For each infinitesimal parameter $\lambda$ of $G$, we define a partial order $\geq_C$ on $\Phi(G)_\lambda$ by $\phi_1 \geq_C \phi_2$ if $\overline{C_{\phi_1}} \supseteq C_{\phi_2}$.

For an infinitesimal parameter $\lambda$ of $G$, we define the set

$$\Psi(G)_\lambda := \{ \psi \in \Psi(G) \mid \lambda_{\phi_\psi} = \lambda \}.$$ 

Since the map from $\psi$ to $\phi_\psi$ is injective, $\Psi(G)_\lambda$ can be viewed as a subset of $\Phi(G)_\lambda$. Mœglin proved that all representations in a local Arthur packet $\Pi_\psi$ share the same extended cuspidal support (see [Moe99b, Proposition 4.1]). Hence, for any $\pi \in \Pi_\psi$, we have $\lambda_{\phi_\pi} = \lambda_{\phi_\psi}$. Thus, for a fixed $\pi \in \Pi_A(G_n)$, we have an inclusion $\Psi(\pi) \subseteq \Psi(G)_{\lambda_{\phi_\pi}}$. Then we may restrict the partial order $\geq_C$ from $\Phi(G)_\lambda$ to $\Psi(G)_\lambda$ or $\Psi(\pi)$.

**Definition 1.11.** For each infinitesimal parameter $\lambda$ of $G$, we define a partial order $\geq_C$ on the set $\{ \psi \in \Psi(G) \mid \lambda_{\phi_\psi} = \lambda \}$ by $\psi_1 \geq_C \psi_2$ if $\phi_{\psi_1} \geq_C \phi_{\psi_2}$.

Now we state the first main result of this paper which is proved in §3.3.

**Theorem 1.12.** Let $G_n$ be $\text{Sp}_{2n}$ or split $\text{SO}_{2n+1}$.

1. If $T$ is a raising operator, then

$$T(\psi) \geq_C \psi,$$

for any $\psi \in \Psi(G_n)$. In other words, if $\psi \succeq_0 \psi'$, then $\psi \succeq_C \psi'$.

2. Let $\pi \in \Pi_A(G_n)$. $\psi^{\text{max}}(\pi)$ and $\psi^{\text{min}}(\pi)$ are the unique elements in $\Psi(\pi)$ satisfying the following inequality

$$(1.3) \quad \psi^{\text{max}}(\pi) \geq_C \psi \geq C \psi^{\text{min}}(\pi),$$

for any $\psi \in \Psi(\pi)$. 

We remark that Part (2) of the above theorem follows from Part (1) and Theorem 1.4. From the definition of raising operators, Part (1) reduces to explicit computation.

In summary, we have the following relations among the four orderings. The operator ordering \( \geq_O \) implies \( \geq_A, \geq_D, \) and \( \geq_C, \) but the converse is not true in general (see Example 3.11). The closure ordering \( \geq_C \) implies \( \geq_D, \) but the converse is also not true in general (again, see Example 3.11). The orderings \( \geq_A \) and \( \geq_D \) are related by taking the dual, i.e., \( \psi \geq_A \psi' \) if and only if \( \hat{\psi} \geq_D \hat{\psi} \). It is certainly an interesting question to study how the non-extremal elements in \( \Psi(\pi) \) behave under these four orderings, particularly the partial order \( \geq_C. \) This is a work in progress of the authors.

Assume that \( \lambda = \lambda_{\phi_0} \) for some local Arthur parameter \( \psi. \) Then the unique tempered parameter \( \phi^0 \) in \( \Phi(G)_\lambda \) corresponds to the unique open orbit in \( V^\lambda \) (see [CFMMX22, Proposition 6.1] and [CFMZ22], also see Proposition 6.4 and Lemma 6.5). Consequently, for any \( \phi \in \Phi(G)_\lambda \) we have

\[
\phi^0 \geq_C \phi.
\]

On the other hand, there exists a unique L-parameter \( \phi_0 \) in \( \Phi(G)_\lambda \) corresponding to the zero orbit in \( V^\lambda \) (see Lemma 6.5). Moreover, \( \phi_0 \) is of Arthur type and its Arthur parameter is anti-tempered (meaning the Aubert-Zelevinsky dual of a tempered local Arthur parameter) and satisfies that for any \( \phi \in \Phi(G)_\lambda \)

\[
\phi \geq_C \phi_0.
\]

Comparing the settings of \( \Phi(G)_\lambda \) and \( \Psi(\pi) \), we can see from (1.3) that \( \phi_{\psi_{\max}(\pi)} \) and \( \phi_{\psi_{\min}(\pi)} \) are the closest L-parameters (coming from \( \Psi(\pi) \)) to \( \phi^0 \) and \( \phi_0 \), respectively. Hence, we view \( \psi_{\max}(\pi) \) (resp. \( \psi_{\min}(\pi) \)) as the most (resp. least) tempered member in the set \( \Psi(\pi) \).

Theorem 1.12 gives a new characterization for \( \psi_{\max}(\pi) \), namely, \( \psi_{\max}(\pi) \) is the unique maximal local Arthur parameter in \( \Psi(\pi) \) under the closure ordering in (1.3). One significance of Theorem 1.12 is that its Part (2) is expected to be generalizable to general groups, namely, we expect that there exist unique elements \( \psi_{\max}(\pi) \) and \( \psi_{\min}(\pi) \) in \( \Psi(\pi) \) satisfying (1.3) for any irreducible representation \( \pi \) of Arthur type of any connected reductive groups. More precisely, we have a conjecture below, where we assume that there is a theory of local Arthur packets in the sense that the conjectures in [Art89] are known. Note that for our purposes, it is enough to only have the conjectures needed to construct the local Arthur packets. That is, for a local Arthur parameter \( \psi \), we can define a finite set \( \Pi_\psi \) consisting of irreducible smooth representations of \( G \) satisfying certain twisted endoscopic character identities. Then, we can define \( \Pi_A(G), \phi_\psi, \) and \( \Psi(\pi) \) for \( \pi \in \Pi_A(G) \) similarly as above. Inspired by [ABV92, Vog93, CFMMX22], it is also natural to conjecture (see [Xu21, Conjecture 2.1] and Conjecture 1.14 below) that all representations in \( \Pi_\psi \) share the same infinitesimal parameter, which is known for classical groups by [Moe09b, Proposition 4.1] as mentioned above.

**Conjecture 1.13.** Let \( G \) be a connected reductive group defined over a non-Archimedean local field. Assume that there is a theory of local Arthur packets for \( G \). Let \( \pi \in \Pi_A(G). \)

Then, for any \( \psi_1, \psi_2 \in \Psi(\pi) \), we have \( \lambda_{\psi_1} = \lambda_{\psi_2} \). Furthermore, there are unique elements \( \psi_{\max}(\pi) \) and \( \psi_{\min}(\pi) \) in \( \Psi(\pi) \) such that

\[
\psi_{\max}(\pi) \geq_C \psi \geq_C \psi_{\min}(\pi),
\]

for any local Arthur parameter \( \psi \in \Psi(\pi) \).

We expect that there is an analogous conjecture for the Archimedean case and it is a very interesting question to define “the” local Arthur parameters for irreducible representations of Arthur type for any real or complex connected reductive group.
1.1. **The closure ordering conjecture of local $L$-parameters in local Arthur packets.** As an application of Theorem 1.12, we prove a conjecture below on the closure ordering of local $L$-parameters for representations in local Arthur packets of $G_n$, which provides evidence for the coincidence of the local Arthur packets with the ABV-packets as conjectured in [CFMMX22, Section 8.3, Conjecture 1].

Let $\psi$ be a local Arthur parameter of a classical group $G$. It is known that the local $L$-packet $\Pi_{\phi_\psi}$ is usually a proper subset of the whole local Arthur packet $\Pi_\psi$. Hence, it is an interesting question to figure out the local $L$-parameters for all representations of $\psi$ and understand their relations with $\psi$ and $\phi_\psi$. Towards this question, suggested by the work in [ABV92, Vog93, CFMMX22], there is a general conjecture as follows, which we call the closure ordering conjecture.

**Conjecture 1.14** ([Xu21, Conjecture 2.1]). Let $G$ be a connected reductive group over $F$. Assume that there is a local Arthur packets theory for $G$ as conjectured in [Art89].

Let $\psi$ be a local Arthur parameter of $G$. Then for any $\pi \in \Pi_\psi$, we have

$$\phi_\pi \geq_C \phi_\psi.$$  

Conjecture 1.14 reveals a very interesting geometric nature of local Arthur packets and is expected to have many important implications. Under certain assumptions (Working Hypotheses 6.6), Conjecture 1.14 implies that a local Arthur packet of a quasi-split connected reductive group $G$ is tempered if it has a generic member, which is an essential part of the Enhanced Shahidi’s Conjecture (see §1.3 below for more details). We expect that non-extremal elements in $\Psi(\pi)$ can be better understood under the partial order $\geq_C$.

As an application of Theorem 1.12, we prove Conjecture 1.14 for the groups $G_n = \text{Sp}_{2n}$ or split $\text{SO}_{2n+1}$, which is the second main result of this paper.

**Theorem 1.15.** Conjecture 1.14 is true for $G_n = \text{Sp}_{2n}$ or split $\text{SO}_{2n+1}$.

Theorem 1.15 follows directly from Theorem 1.12 and the following Theorem 1.16, which is a key reduction to $\psi^{\text{max}}(\pi)$ and is proved in §4. This shows that a feasible way to overcome the difficulty of a problem related to the nontrivial intersections of local Arthur packets is to reduce it to $\psi^{\text{max}}(\pi)$.

**Theorem 1.16.** Let $G_n$ be $\text{Sp}_{2n}$ or split $\text{SO}_{2n+1}$. If $\pi \in \Pi_A(G_n)$, then we have

$$\phi_\pi \geq_C \phi_{\psi^{\text{max}}(\pi)}.$$  

Theorem 1.16 is proved by induction on the rank of $G_n$ and one ingredient is an important property of $\psi^{\text{max}}(\pi)$ (Proposition 4.3) which implies that the local $L$-parameter of $\pi$ and $\phi_{\psi^{\text{max}}(\pi)}$ share certain common direct summands.

Although playing an important role in the study of automorphic representations and number theory, local Arthur packets [Art13] are very mysterious. Thus it is natural to attempt to give different descriptions of local Arthur packets. The geometry of the Vogan variety $V_\lambda$ together with the action by $H_\lambda$ is a central tool in such existing attempts. In [ABV92], Adams, Barbasch and Vogan proposed a purely local geometric construction of local Arthur packets (called ABV-packets in follow-up literature) for connected real reductive groups by studying the geometry of Vogan varieties. Recently, the ABV-packets have been shown to be the same as the local Arthur packets constructed by Arthur in [Art13] for real classical groups by Adams, Arancibia, and Mezo ([AAM21]), and for real quasi-split unitary groups by Arancibia and Mezo ([AM22]).

In [CFMMX22], Cunningham *et al.* extended the work of [ABV92] to $p$-adic reductive groups and defined a packet $\Pi^{\text{ABV}}_\phi$ using micro-local vanishing cycle functors, for
any $L$-parameter $\phi$ of any $p$-adic reductive group $G$. As in the real reductive groups cases, it is expected that $\Pi^\text{ABV}_\phi = \Pi_\psi$ if $\phi = \phi_\psi$, see [CFMMX22, Section 8.3, Conjecture 1]. For $GL_n$, some special case of this conjecture was recently proved in [CR22]. By [CFMMX22, Theorem 7.22 (b)], for $\pi \in \Pi^\text{ABV}_\phi$, one must have $\phi_\pi \geq C \phi$. Conjecture 1.14 is the local Arthur packets analogue of this. Therefore, Theorem 1.15 provides evidence for [CFMMX22, Section 8.3, Conjecture 1]. We remark that comparing to the analogue for ABV-packets in [CFMMX22, Theorem 7.22 (b)], there are no geometric tools developed for Theorem 1.15, which increases the difficulty of study. Instead, our novelty is to apply the operators developed in [HLL22] to overcome the difficulty which lies in the nontrivial intersections of local Arthur packets and reduce to the study of $\psi^{\max}(\pi)$.

1.2. New algorithms to determine a representation is of Arthur type. It is known that the set of representations in local Arthur packets is just a proper subset of the whole unitary dual. Given an irreducible unitary representation $\pi$ of a classical group $G$, it is desirable to have algorithms to determine whether $\pi$ is of Arthur type. Recently, there are two different algorithms given independently. One algorithm is given by Atobe in [Ato22, Algorithm 3.3], which constructed several representations $\pi_i$ of smaller rank, and used the information from the derivatives (see §2.2) of $\pi$ and $\pi_i$ together with the construction of $\Psi(\pi_i)$. The other algorithm is given by the first three named authors in [HLL22, Algorithm 7.9], which used the derivatives information of $\pi$ only to construct a local Arthur parameter $\psi$ such that $\pi$ is of Arthur type if and only if $\pi \in \Pi_\psi$. We remark that both algorithms above end up with constructing a possibly non-tempered packet $\Pi_\psi$, and checking whether $\pi$ or $\pi_i$ is in this packet or not. As an application of our main results, we generalize the properties of $\psi^{\max}(\pi)$ in Proposition 4.3 and Corollary 4.4 (see Theorem 5.4, Lemmas 5.5 and 5.6) and give new algorithms (Algorithms 5.12 and 5.22) to determine whether a representation $\pi$ of $G_n$ is of Arthur type. In contrast, the new algorithms do not require computing highest derivatives of representations, but use the construction of $\Psi(\pi_i)$ only, where $\pi_i$ are several representations of smaller rank constructed from $\pi$ without using highest derivatives. Moreover, there is no construction of a non-tempered local Arthur packet involved. Hence, these new algorithms have their own interests and theoretical importance.

1.3. A new proof of the Enhanced Shahidi’s Conjecture. The famous Shahidi’s Conjecture states that for any quasi-split reductive group $G$, tempered $L$-packets have generic members ([Sha90, Conjecture 9.4]). This conjecture has been generalized by Jiang ([Jia14], see also [LS22, Conjecture 1.6]) to non-tempered local Arthur packets. Recently, Shahidi made an enhanced conjecture (see [LS22, Conjecture 1.5] and Conjecture 6.1) as follows.

**Conjecture 1.17** ([LS22, Conjecture 1.5], Enhanced Shahidi’s Conjecture). For any quasi-split reductive group $G$, a local Arthur packet is tempered if and only if it has a generic member.

In [LS22], the second named author and Shahidi proved Conjecture 1.17 for quasi-split classical groups, with certain assumption.

**Theorem 1.18** ([LS22, Theorem 1.7]). Under an assumption on wavefront sets of certain bitorsor representations ([LS22, Conjecture 1.3]), Conjecture 1.17 is valid for symplectic and quasi-split special orthogonal groups.

In [HLL22], the first three named authors proved Conjecture 1.17 for symplectic and split odd special orthogonal groups, without any assumption.
Theorem 1.19 ([HLL22, Theorem 1.8]). Conjecture 1.17 is valid for symplectic and split odd special orthogonal groups.

As an application of Theorem 1.15, we give a new proof of the Enhanced Shahidi’s Conjecture 1.17 for symplectic and split odd special orthogonal groups (see Theorem 6.13). More precisely, for any quasi-split connected reductive group $G$, suppose that there is a theory of local Arthur packets for $G$ as conjectured in [Art89], we show that Conjecture 1.14, along with certain assumptions (Working Hypotheses 6.6), implies that a local Arthur packet of $G$ is tempered if it has a generic member, which is an essential part of the Enhanced Shahidi’s Conjecture (see Theorem 6.7). Then, we verify these assumptions for symplectic and split odd special orthogonal groups (see Theorem 6.8 and Lemma 6.12). The merit of this new proof is that it provides a framework of proving the Enhanced Shahidi’s Conjecture for general quasi-split connected reductive groups.

We remark that the methods in this paper are expected to extend to other classical groups.

Following is the structure of this paper. In §2, we recall necessary notation and preliminaries and prove Theorem 1.9. In §3, we prove Theorem 1.12. In §4, we prove Theorem 1.16, hence prove Theorem 1.15. In §5, we give new algorithms (Algorithms 5.12 and 5.22) to determine whether a representation $\pi$ of $G_n$ is of Arthur type. In §6, we give our new proof of Conjecture 1.17 for symplectic and split odd special orthogonal groups.

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2. Notation and preliminaries

Let $F$ be a non-Archimedean local field of characteristic zero, and let $G_n$ denote one of the split groups $\text{Sp}_{2n}(F)$ or $\text{SO}_{2n+1}(F)$. In this section, we introduce necessary notation and preliminaries and prove Theorem 1.9.

2.1. Langlands classification. In this subsection, we recall the Langlands classification for $\text{GL}_n(F)$ and $G_n$. These follow from the Langlands classification for $p$-adic reductive groups. For a more general setup, we refer to [Kon03].

Let $n$ be a positive integer and fix a Borel subgroup of $\text{GL}_n$ to be the subgroup of upper triangular matrices. Let $P$ be a standard parabolic subgroup of $\text{GL}_n(F)$ with Levi subgroup $M \cong \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_r}(F)$. Let $\tau_i \in \Pi(\text{GL}_{n_i}(F))$ for $i = 1, 2, \ldots, r$. We set

$$\tau_1 \times \cdots \times \tau_r := \text{Ind}_{M}^{\text{GL}_n(F)}(\tau_1 \otimes \cdots \otimes \tau_r)$$

to be the normalized parabolic induction. Let $\rho$ be an irreducible unitary supercuspidal representation of $\text{GL}_n(F)$. For $x, y \in \mathbb{R}$ such that $x - y$ is a non-negative integer, we define a segment, denoted $[x, y]_{\rho}$, by

$$[x, y]_{\rho} = \left\{ \rho \cdot |^x \rho \cdot |^{x-1} \rho \cdots |^y \rho \right\}.$$ 

We denote the Steinberg representation attached to the segment $[x, y]_{\rho}$ by $\Delta_{\rho}[x, y]$. This is the unique irreducible subrepresentation of $\rho \cdot |^x \rho \cdots |^y \rho$. It is an essentially discrete series representation of $\text{GL}_{n(x-y+1)}(F)$. When it is clear in context, we refer to
both \([x, y]\rho\) and \(\Delta_{\rho}[x, y]\) as segments. We also set \(Z_{\rho}[y, x]\) to be the unique irreducible quotient of \(\rho \cdot |x| \times \cdots \times \rho \cdot |y|\). In the case \(y = x + 1\), we set \(\Delta_{\rho}[x, x + 1] = Z_{\rho}[x + 1, x]\) to be the trivial representation of \(GL_n(F)\).

We now state the Langlands classification for \(GL_n(F)\). Let \(\tau \in \Pi(GL_n(F))\), then \(\tau\) is a unique irreducible subrepresentation of \(\Delta_{\rho_{\tau}}[x_1, y_1] \times \cdots \times \Delta_{\rho_{\tau}}[x_r, y_r]\), where \(\rho_{\tau}\) is an irreducible unitary supercuspidal representation of \(GL_{n_r}(F)\), \([x_i, y_i]_{\rho_{\tau}}\) is a segment, and \(x_1 + y_1 \leq \cdots \leq x_r + y_r\). In this setting, we write

\[
\tau = L(\Delta_{\rho_{\tau}}[x_1, y_1], \ldots, \Delta_{\rho_{\tau}}[x_r, y_r]).
\]

Next, fix an \(F\)-rational Borel subgroup of \(G_n\) and let \(P\) be a standard parabolic subgroup of \(G_n\) with Levi subgroup \(M \cong GL_{n_1}(F) \times \cdots \times GL_{n_r}(F) \times G_m\). Let \(\tau_i \in \Pi(GL_{n_i}(F))\) for \(i = 1, 2, \ldots, r\) and \(\sigma \in \Pi(G_m)\). We set

\[
\tau_1 \times \cdots \times \tau_r \times \sigma := \text{Ind}_{\prod P}^G(\tau_1 \times \cdots \times \tau_r \times \sigma)
\]

to be the normalized parabolic induction.

The Langlands classification for \(G_n\) states that any \(\pi \in \Pi(G_n)\) is a unique irreducible subrepresentation of \(\Delta_{\rho_{\pi}}[x_1, y_1] \times \cdots \times \Delta_{\rho_{\pi}}[x_r, y_r] \times \pi_0\), where \(\rho_{\pi}\) is an irreducible unitary supercuspidal representation of \(GL_{n_1}(F)\), \(x_1 + y_1 \leq \cdots \leq x_r + y_r < 0\), and \(\pi_0\) is an irreducible tempered representation of \(G_m\). We denote this by

\[
\pi = L(\Delta_{\rho_{\pi}}[x_1, y_1], \ldots, \Delta_{\rho_{\pi}}[x_r, y_r]; \pi_0)
\]

and call \((\Delta_{\rho_{\pi}}[x_1, y_1], \ldots, \Delta_{\rho_{\pi}}[x_r, y_r]; \pi_0)\) the Langlands data, or \(L\)-data, of \(\pi\).

### 2.2. Derivatives

We need the following notation of derivatives considered in [AM20] and [Ato20].

Let \(\pi\) be a smooth representation of \(G_n\) of finite length. We let \(\text{Jac}_P(\pi)\) denote the Jacquet module of \(\pi\) with respect to a parabolic subgroup \(P\) of \(G_n\). We also denote the semisimplification of \(\text{Jac}_P(\pi)\) by \([\text{Jac}_P(\pi)]\).

**Definition 2.1.**

1. Let \(P_d\) be a standard parabolic subgroup of \(G_n\) with Levi subgroup isomorphic to \(GL_{d}(F) \times G_{n-d}\), \(x \in \mathbb{R}\), and \(\rho\) be an irreducible unitary self-dual supercuspidal representation of \(GL_{d}(F)\). We define the \(|x|\)-derivative of \(\pi\), denoted \(D_{\rho,|x|}(\pi)\), to be a semisimple representation satisfying

\[
[\text{Jac}_{P_d}(\pi)] = \rho \cdot |x| \otimes D_{\rho,|x|}(\pi) + \sum_i \tau_i \otimes \pi_i,
\]

where the sum is over all irreducible representations \(\tau_i\) of \(GL_{d}(F)\) such that \(\tau_i \not\cong \rho \cdot |x|\).

2. We define \(D_{\rho,|x|}(\pi)\) recursively by \(D_{\rho,|x|}(\pi) = D_{\rho,|x|}(\pi)\) and

\[
D_{\rho,|x|}(\pi) = \frac{1}{k} D_{\rho,|x|} \circ D_{\rho,|x|}^{(k-1)}(\pi).
\]

3. For a sequence of real number \(\{x_1, \ldots, x_r\}\), we denote the composition of derivatives by

\[
D_{\rho,|x_1, \ldots, x_r|}(\pi) := D_{\rho,|x_r|} \circ \cdots \circ D_{\rho,|x_1|}(\pi).
\]

4. We say that \(D_{\rho,|x|}(\pi)\) is the highest \(\rho \cdot |x|\)-derivative of \(\pi\) if \(D_{\rho,|x|}(\pi) \neq 0\), but \(D_{\rho,|x|}^{(k+1)}(\pi) = 0\).

We recall the following properties of derivatives.
Theorem 2.2 ([Jan14, Lemma 3.1.3], [AM20, Proposition 6.1, Theorem 7.1]). Let $\rho$ be an irreducible unitary self-dual supercuspidal representation of $GL_d(F)$, $\pi \in \Pi(G_n)$, and $x \in \mathbb{R} \setminus \{0\}$. Then the highest $\rho | \cdot^x$-derivative of $\pi$, say $D^{(k)}_{\rho | \cdot^x}(\pi)$, is irreducible. Moreover, the $L$-data of $D^{(k)}_{\rho | \cdot^x}(\pi)$ can be explicitly computed from the $L$-data of $\pi$.

We also need the following lemma.

Lemma 2.3 ([HLL22, Lemma 2.6]). Suppose $\rho$ is a self-dual supercuspidal representation of $GL_d$, and $\pi$ is an irreducible representation of $G_{n+dt}$. Then there exists a representation $\sigma$ of $G_n$ and $x_1, \ldots, x_t \in \mathbb{R}$ such that

$$\pi \leftrightarrow \rho | \cdot^{x_1} \times \ldots \times \rho | \cdot^{x_t} \times \sigma$$

if and only if

$$D_{\rho | \cdot^{x_1} \times \ldots \times \cdot^{x_t}}(\pi) \neq 0.$$ 

In this case, if $\sigma$ has a unique irreducible subrepresentation $\sigma'$, then

$$D_{\rho | \cdot^{x_1} \times \ldots \times \cdot^{x_t}}(\pi) \geq \sigma'$$

in the sense of Grothendieck group.

2.3. Local Arthur packets and extended multi-segments. Recall from (1.1) that we can identify a local Arthur parameter $\psi$ with

$$(2.1) \quad \psi = \bigoplus_{\rho} \left( \bigoplus_{i \in I_{\rho}} \rho \otimes S_{a_i} \otimes S_{b_i} \right),$$

where the first sum runs over the isomorphism class of irreducible unitary supercuspidal representations $\rho$ of $GL_d(F)$ for $d_{\rho} \in \mathbb{Z}_{\geq 1}$, the restrictions of $\psi$ to the two copies of $SL_2(\mathbb{C})$ are analytic, $S_k$ is the $k$-dimensional irreducible representation of $SL_2(\mathbb{C})$, and

$$\sum_{\rho} \sum_{i \in I_{\rho}} d_{\rho} a_i b_i = N := \begin{cases} 2n + 1 & \text{when } G_n = \text{Sp}_{2n}, \\
2n & \text{when } G_n = \text{SO}_{2n+1}. \end{cases}$$

Occasionally, we write $\rho \otimes S_a = \rho \otimes S_{a_1} \otimes S_{a_2}$.

Let $\psi$ be a local Arthur parameter as in (2.1), we say that $\psi$ is of good parity if every summand $\rho \otimes S_{a_i} \otimes S_{b_i}$ is self-dual and of the same type as $\psi$. That is, $\rho$ is self-dual and

- if $G_n = \text{Sp}_{2n}(F)$ and $\rho$ is orthogonal (resp. symplectic), then $a_i + b_i$ is even (resp. odd);
- if $G_n = \text{SO}_{2n+1}(F)$ and $\rho$ is orthogonal (resp. symplectic), then $a_i + b_i$ is odd, (resp. even).

Consider a local Arthur parameter $\psi$ with decomposition given by Equation (2.1). We set

$$\tau_\psi := \bigotimes_{\rho} \bigotimes_{i \in I_{\rho}} L \left( \Delta_{\rho} \left[ \frac{a_i - b_i}{2}, \frac{a_i + b_i}{2} - 1 \right], \ldots, \Delta_{\rho} \left[ \frac{-a_i - b_i}{2} + 1, \frac{b_i - a_i}{2} \right] \right).$$

Theorem 2.4 ([Mo06a, Theorem 6], [Xu17, Proposition 8.11]). Let $\psi$ be a local Arthur parameter. We have the decomposition

$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$$

where $\psi_1$ is a local Arthur parameter which is not of good parity, $\psi_0$ is a local Arthur parameter of good parity, and $\psi_1^\vee$ denotes the dual of $\psi_1$. Furthermore, for $\pi \in \Pi_{\psi_0}$
the induced representation $\tau_{\psi_1} \rtimes \pi$ is irreducible, independent of choice of $\psi_1$, and we have

$$\Pi_{\psi} = \{ \tau_{\psi_1} \rtimes \pi \mid \pi \in \Pi_{\psi_0} \}.$$ 

The above theorem allows us to construct all local Arthur packets provided that we can construct local Arthur packets of good parity. Consequently, we are often interested in representations which lie in a local Arthur packet of good parity.

**Definition 2.5.** We say an irreducible representation

$$\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi_0)$$

of $G_n$ is of good parity if

- the $L$-parameter of $\pi_0$, say $\phi$, is of good parity. More precisely, the local Arthur parameter defined by $\psi := \phi \otimes S_1$ is of good parity. Note that $\phi_\psi = \phi$.
- For $1 \leq i \leq r$, $x_i, y_i \in \frac{1}{2} \mathbb{Z}$ and $\rho_i \otimes S_{x_i - y_i + 1} \otimes S_1$ is self-dual of the same type as $\phi$.

We define $\Pi_A^\rho(G_n)$ to be the subset of $\Pi_A(G_n)$ which consists of representations of good parity.

Assuming that $\psi$ is of good parity, we can define the enhanced component group of $\psi$ to be

$$A_\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} (\mathbb{Z}/2\mathbb{Z}) \alpha_{\rho,i}.$$ 

That is, $A_\psi$ is the finite vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\alpha_{\rho,i}$ corresponding to the summands $\rho \otimes S_{a_i} \otimes S_{b_i}$ of Equation (2.1). While it is possible that for some $i, j \in I_\rho$, we have $\rho \otimes S_{a_i} \otimes S_{b_i} = \rho \otimes S_{a_j} \otimes S_{b_j}$, we do distinguish these summands in $A_\psi$. That is, $\alpha_{\rho,i} \neq \alpha_{\rho,j}$ in $A_\psi$. The central element of $A_\psi$ is $z_\psi := \sum_{\rho} \sum_{i \in I_\rho} \alpha_{\rho,i}.$

We define the component group $S_\psi$ of $\psi$ to be the quotient of $A_\psi$ by the subgroup generated by the central element and the elements $\alpha_{\rho,i} + \alpha_{\rho,j}$ such that $i, j \in I_\rho$ with $\rho \otimes S_{a_i} \otimes S_{b_i} = \rho \otimes S_{a_j} \otimes S_{b_j}$. We let $\hat{S}_\psi$ and $\hat{A}_\psi$ be the Pontryagin duals of $S_\psi$ and $A_\psi$, respectively. For $\varepsilon \in \hat{A}_\psi$, we write $\varepsilon(\rho \otimes S_{a_i} \otimes S_{b_i}) = \varepsilon(\alpha_{\rho,i}).$

The following theorem gives Arthur’s classification of tempered representations.

**Theorem 2.6 ([Art13, Theorem 1.5.1]).** Any irreducible tempered representation of $G_n$ lies in $\Pi_{\psi}$ for some tempered local Arthur parameter $\psi$. Moreover, if $\psi_1$ and $\psi_2$ are two non-isomorphic tempered local Arthur parameters, then $\Pi_{\psi_1} \cap \Pi_{\psi_2} = \emptyset$.

Finally, if one fixes a choice of Whittaker datum for $G_n$ and $\psi$ is tempered, then there is a bijective map between the tempered local Arthur packet $\Pi_{\psi}$ and $\hat{S}_\psi$.

Hereinafter, we implicitly fix a choice of Whittaker datum for $G_n$. When $\psi$ is tempered and of good parity, we write $\pi(\psi, \varepsilon)$ or $\pi(\phi_\psi, \varepsilon)$ for the element of $\Pi_{\psi}$ corresponding to $\varepsilon \in \hat{S}_\psi$ via the bijection in Theorem 2.6. In examples, when $\rho$ is fixed, we set $\pi(x_1^1, \ldots, x_r^r) := \pi(\phi, \varepsilon)$ where $\phi = \rho \otimes S_{2x_1+1} + \cdots + \rho \otimes S_{2x_r+1}$ and $\varepsilon(\rho \otimes S_{2x_i+1}) = \varepsilon_i$ for $i = 1, \ldots, r$.

For $\pi \in \Pi_A^\rho(G_n)$, we describe the possible changes of the non-tempered part of the $L$-data of $\pi$ under the highest derivative $D^{(k)}_{\rho|\alpha}$ when $k = 1$ in the following lemma.

**Lemma 2.7.** Suppose

$$\pi = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_r}[x_r, -y_r]; \pi(\phi, \varepsilon))$$
is a representation of good parity of $G_n$ and $\rho \otimes S_{2\alpha+1}$ is of the same type as $G_n$. We further assume that $D_{\rho|\alpha}^{(1)}(\pi)$ is a highest derivative. Then the derivative $D_{\rho|\alpha}$ either leaves the multi-set

$$\{\Delta_{\rho_1}[x_1,-y_1], \ldots, \Delta_{\rho_f}[x_f,-y_f]\}$$

of $\pi$ unchanged, or changes it in one of the following ways.

(i) Replace $\Delta_{\rho_i}[x_i,-y_i]$ by $\Delta_{\rho_i}[x_i-1,-y_i]$ for some $i$ such that $\rho_i \cong \rho$ and $x_i = \alpha$. 
(ii) Replace $\Delta_{\rho_i}[x_i,-y_i]$ by $\Delta_{\rho_i}[x_i,-y_i+1]$ for some $i$ such that $\rho_i \cong \rho$ and $y_i = \alpha$. 
(iii) Insert $\Delta_{\rho}[-1,-\alpha]$.

Proof. This follows from the explicit formula in [AM20, Theorem 7.1].

Next, we recall Atobe’s construction of local Arthur packets of good parity from [Ato20].

**Definition 2.8** ([Ato20, Definition 3.1]). *(Extended multi-segments)*

1. An extended segment is a triple $([A,B]_\rho, l, \eta)$, where
   - $[A,B]_\rho = \{\rho, |A, \rho|, |A^{-1}, \ldots, \rho|, |B\}$ is a segment for an irreducible unitary supercuspidal representation $\rho$ of some $\text{GL}_d(F)$;
   - $l \in \mathbb{Z}$ with $0 \leq l \leq \frac{b}{2}$, where $b = \#(A,B)_\rho = A - B + 1$;
   - $\eta \in \{\pm 1\}$.

2. Consider a multi-set of extended segments of the form $\{(A_i, B_i)_\rho, l_i, \eta_i\}_{i \in I_\rho}$. We say a total order $>_{\text{ad}}$ on $I_\rho$ is admissible if
   $$A_i < A_j, B_i < B_j \implies i < j.$$ 
   We say an admissible order $>_{\text{ad}}$ satisfies (P) if
   $$B_i < B_j \implies i < j.$$ 

3. An extended multi-segment for $G_n$ is an equivalence class (via the equivalence defined below) of multi-sets of extended segments
   $$\mathcal{E} = \bigcup_{\rho} \{(A_i, B_i)_\rho, l_i, \eta_i\}_{i \in (I_\rho, >_{\text{ad}})}$$
   such that
   - $I_\rho$ is a totally ordered finite set with a fixed total order $>_{\text{ad}}$ satisfies (P);
   - $A_i + B_i \geq 0$ for all $\rho$ and $i \in I_\rho$;
   - as a representation of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$,
   $$\psi_\mathcal{E} = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$$
   where $(a_i, b_i) = (A_i + B_i + 1, A_i - B_i + 1)$, is a local Arthur parameter for $G_n$ of good parity. We shall denote $\psi_\mathcal{E}$ the local Arthur parameter associated with $\mathcal{E}$.
   - The sign condition
   $$\prod_{\rho} \prod_{i \in I_\rho} (-1)^{|a_i|+l_i} \eta_i^{-b_i} = 1$$
   holds.

4. Two extended segments $([A,B]_\rho, l, \eta)$ and $([A',B']_\rho, l', \eta')$ are weakly equivalent if
   - $[A,B]_\rho = [A',B']_\rho$;
   - $l = l'$; and
   - $\eta = \eta'$ whenever $l = l' < \frac{b}{2}$. 

We say that two extended multi-segments $E = \bigcup_{\rho} \{([A_i, B_i], l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ and $E' = \bigcup_{\rho} \{([A'_i, B'_i], l'_i, \eta'_i)\}_{i \in (I_{\rho}, >)}$ are weakly equivalent if for any $\rho$ and $i \in I_{\rho}$, the extended segments $([A_i, B_i], l_i, \eta_i)$ and $([A'_i, B'_i], l'_i, \eta'_i)$ are weakly equivalent.

(5) For each extended multi-segment $E$, we let $\pi(E)$ denote the representation associated with $E$ as in [Ato20, §3.2] (or see [HLL22, Definition 3.4]). $\pi(E)$ is either irreducible or zero. We denote $\text{Rep}$ the set of extended multi-segments that give nonzero representations, and $\text{Rep}^{(P')}$ the subset of $\text{Rep}$ consists of extended multi-segments whose total order on any $I_\rho$ satisfies $(P')$.

The following theorem of Atobe shows how to use extended multi-segments to construct local Arthur packets of good parity.

**Theorem 2.9** ([Ato20, Theorem 3.4]). Suppose $\psi = \bigoplus_{\rho \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$ is a local Arthur parameter of $G_n$, of good parity. Fix an admissible order $>$ on $I_\rho$ for each $\rho$ that satisfies $(P')$ if $\frac{a_i - b_i}{2} < 0$ for some $i \in I_\rho$. Then

$$\bigoplus_{\pi \in \Pi_{\psi}} \pi = \bigoplus_{\pi \in \Pi_{\psi}} \pi(E),$$

where $E$ runs over all extended multi-segments with $\psi_E = \psi$ and $\pi(E) \neq 0$, and the total orders are the ones fixed above.

2.4. **Operators on extended multi-segments and local Arthur parameters.** In this paper, we use the following operators on extended multi-segments, whose precise definitions can be found in [Ato20] and [HLL22].

- **Row exchange:** $R_k$ ([Ato20, Section 4.2] or [HLL22, Definition 3.14]).
- **Union-intersection:** $u_{i,j}$ ([Ato20, Section 5.2] or [HLL22, Definitions 3.22, 5.1]).
- **Aubert-Zelevinsky dual:** dual ([Ato20, Definition 6.1] or [HLL22, Definition 3.27]).
- **Partial dual:** $dual_{k}^{-}$, $dual_{k}^{+}$ ([HLL22, Definition 6.5]).

By abuse of notation, we also regard these operators as operators on local Arthur parameters of good parity by $T(\psi_E) = \psi_{T(E)}$, and extend the definition to general local Arthur parameters. We recall [HLL22, Definition 12.1] as follows.

**Definition 2.10.** Suppose $\psi$ is a local Arthur parameter of $G_n$. Decompose $\psi = \psi_0 \oplus \psi_1 \oplus \psi_1^\dual$ as in Theorem 2.4 and write

$$\psi_0 = \bigoplus_{\rho \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}.$$

Then for $i, j, k \in I_\rho$, we define the operators dual, $u_{i,j}$ and $u_{i,j}^-$ as follows.

1. $\text{dual}(\psi) := \hat{\psi}$. We identify the index set $I_\rho(\psi_0)$ with $I_\rho(\hat{\psi_0})$ in the obvious way.
2. For $\rho \in I_\rho$, let $A_r = \frac{a_i + b_i}{2} - 1$ and $B_r = \frac{a_r - b_r}{2}$. Then we may rewrite the decomposition of $\psi_0$ as

$$\psi_0 = \bigoplus_{\rho \in I_\rho} \rho \otimes S_{A_i + B_i + 1} \otimes S_{A_i - B_i + 1}.$$

The operator $u_{i,j}$ is applicable on $\psi$ if the following conditions hold.

- $A_j \geq A_i + 1 \geq B_j > B_i$.
- For any $\rho \in I_\rho$, if $B_i < B_r < B_j$, then $A_r \leq A_i$ or $A_r \geq A_j$.
In this case, we define $u_{i,j}(\psi_0)$ by replacing the summands
\[
\rho \otimes S_{A_i+B_i+1} \otimes S_{A_j-B_j+1} + \rho \otimes S_{A_j+B_j+1} \otimes S_{A_i-B_i+1}
\]
of $\psi_0$ with
\[
\rho \otimes S_{A_i+B_i+1} \otimes S_{A_j-B_j+1} + \rho \otimes S_{A_i+B_i+1} \otimes S_{A_i-B_i+1}.
\]
If $A_i+1-B_j=0$, then we omit the last summand, and say this $u_{i,j}$ is of type $3'$. Finally, we define $u_{i,j}(\psi) = \psi_1 \otimes u_{i,j}(\psi_0) \otimes \psi_1'$. (3) The operator $\operatorname{dual}_{k}^-(\psi)$ is applicable on $\psi$ if $b_k = a_k + 1$. In this case, we define $\operatorname{dual}_{k}^-(\psi_0)$ by replacing the summand
\[
\rho \otimes S_{A_k} \otimes S_{A_k+1}
\]
of $\psi_0$ with
\[
\rho \otimes S_{A_k+1} \otimes S_{A_k},
\]
and we define $\operatorname{dual}_{k}^-(\psi) = \psi_1 \otimes \operatorname{dual}_{k}^-(\psi_0) \otimes \psi_1'$.

(4) Let $T$ be any of the operators above or their inverses. If $T$ is not applicable on $\psi$, then we define $T(\psi) = \psi$.

**Remark 2.11.** (1) We say $u_{i,j}^{-1}$ is applicable on $\psi$ if there exists $\psi'$ such that $u_{i,j}(\psi') = \psi$, and we define $u_{i,j}^{-1}(\psi) = \psi'$ in this case. Note that $i,j$ are indices of $I_{\rho}(\psi')$ but not of $I_{\rho}(\psi)$. However, there is an obvious way to identify $I_{\rho}(\psi')$ with $I_{\rho}(\psi) \sqcup \{j\}$ if $u_{i,j}$ is of type $3'$.

(2) The composition $\operatorname{dual} \circ u_{i,j} \circ \operatorname{dual} =: T$ can be described as follows. Keep the notation in above definition. $T$ is applicable on $\psi$ only if
\[
A_i \geq A_j + 1 \geq -B_i > -B_j.
\]
When it is applicable, it replaces the summands
\[
\rho \otimes S_{A_i+B_i+1} \otimes S_{A_j-B_j+1} + \rho \otimes S_{A_j+B_j+1} \otimes S_{A_i-B_i+1}
\]
of $\psi$ with
\[
\rho \otimes S_{A_i+B_i+1} \otimes S_{A_j-B_j+1} + \rho \otimes S_{A_i+B_i+1} \otimes S_{A_i-B_i+1}.
\]
If $A_j+1 = -B_i$, then we omit the last summand.

(3) Note that it is possible that $\Pi_{\psi_1} \cap \Pi_{T(\psi)} = \emptyset$. For example, let $\rho$ be a symplectic representation, $\psi_1 = \rho \otimes S_1 \otimes S_1 + \rho \otimes S_3 \otimes S_1$, and $\psi_2 = \rho \otimes S_2 \otimes S_2$. Then $u_{1,2}(\psi_1) = \psi_2$. However, $\Pi_{\psi_1} \cap \Pi_{\psi_2} = \emptyset$, since $\Pi_{\psi_1}$ consists of 2 tempered representations while $\Pi_{\psi_2}$ consists of a single non-tempered representation. One way to see this is that $u_{1,2}$ is not applicable on any extended multi-segment $E$ such that $\gamma_{E} = \psi_1$.

Now we recall the extended multi-segment version of Theorem 1.1, 1.4. We say an extended multi-segment $E \in \operatorname{Rep}$ is absolutely maximal if there is no raising operators applicable on $E$, and we say $E$ is absolutely minimal if there is no inverse of raising operators applicable on $E$. Note that $E$ is absolutely minimal if and only if $u_{i,j}$ is applicable on $E$. With the notation introduced so far, we summarize the main results of [HLL22] we need in the following theorem.

**Theorem 2.12** ([HLL22, §6 and §11]). Suppose $\pi \in \Pi_{A}(G_{n})$ is of good parity, and take $E \in \operatorname{Rep}(\rho)$ such that $\pi = \pi(E)$.

(1) $\pi(E) \cong \pi(E')$ if and only if $E'$ can be obtained from $E$ by applying a sequence of operators involving $R_{k}, u_{i,j}, \operatorname{dual} \circ u_{i,j} \circ \operatorname{dual}$ and $\operatorname{dual}_{k}^-$ and their inverses.
Lemma 2.15.

(2) There exists a unique absolutely maximal (minimal) member in the set
\[ \Psi(\mathcal{E}) := \{ \mathcal{E}' \mid \pi(\mathcal{E}') = \pi(\mathcal{E}) \} / \text{(row exchanges)}, \]
which we denote by \( \mathcal{E}^{[\max]} \) (resp. \( \mathcal{E}^{[\min]} \)).

(3) After row exchanges, there exists a sequence of raising operators \( \{T_i\}_{i=1}^m, 1 \leq r \leq m \) such that
\[ \mathcal{E}^{[\max]} = T_1 \circ \cdots \circ T_r(\mathcal{E}), \]
\[ \mathcal{E}^{[\min]} = T_m^{-1} \circ \cdots \circ T_{r+1}^{-1}(\mathcal{E}). \]

From Part (2) of above theorem, we give the following definition.

Definition 2.13. Let \( \pi \in \Pi_A(G_n) \). Write \( \pi = \tau_{\psi_1} \times \pi_0 \) as in Theorem 2.4, where \( \pi_0 \) is of good parity. Take any \( \mathcal{E} \) such that \( \pi(\mathcal{E}) = \pi_0 \). Then we define
\[ \psi^{[\max]}(\pi) := \psi_1 + \psi_1' + \psi_{\mathcal{E}^{[\max]}}, \]
\[ \psi^{[\min]}(\pi) := \psi_1 + \psi_1' + \psi_{\mathcal{E}^{[\min]}}. \]

We recall the definition of the operators \( sh^d_j, add^d_j \) on extended multi-segments.

Definition 2.14. \((\text{shift, add})\)

Let \( \mathcal{E} = \cup_\rho \{ ([A_i, B_i]_\rho, l_i, \eta_i) \}_{i \in (I_\rho, >)} \) be an extended multi-segment. For \( j \in I_\rho' \) and \( d \in \mathbb{Z} \), we define the following operators. It is immediate that the operators commute with each other and so we denote the composition by summation.

1. \( sh^d_j(\mathcal{E}) = \cup_\rho \{ ([A_i', B_i']_\rho, l_i', \eta_i) \}_{i \in (I_\rho, >)} \) with
\[ [A_i', B_i']_\rho = \begin{cases} [A_i + d, B_i + d]_\rho & \text{if } \rho = \rho' \text{ and } i = j, \\ [A_i, B_i]_\rho & \text{otherwise}, \end{cases} \]
and \( sh^d_\rho = \sum_{j \in I_\rho} sh^d_j \).

2. \( add^d_j(\mathcal{E}) = \cup_\rho \{ ([A_i', B_i']_\rho, l_i', \eta_i) \}_{i \in (I_\rho, >)} \) with
\[ ([A_i', B_i']_\rho, l_i') = \begin{cases} ([A_i + d, B_i - d]_\rho, l_i + d) & \text{if } \rho = \rho' \text{ and } i = j, \\ ([A_i, B_i]_\rho, l_i) & \text{otherwise}, \end{cases} \]
and \( add^d_\rho = \sum_{j \in I_\rho} add^d_j \). We remove the extended segments in \( add^d_j(\mathcal{E}) \) of the form \( ([A, A - 1]_\rho, 0, *) \).

We only use these notations in the case that the resulting object is still an extended multi-segment.

Here are some identities between these operators.

Lemma 2.15 ([HLL22, Lemma 3.30(ii), and Corollary 5.5]). Let
\[ \mathcal{E} = \cup_\rho \{ ([A_i, B_i]_\rho, l_i, \eta_i) \}_{i \in (I_\rho, >)} \]
be an extended multi-segment.

(1) For any \( k \in I_\rho \) and \( t \in \mathbb{Z} \), we have
\[ \text{dual} \circ sh^t_k(\mathcal{E}) = add^t_k \circ \text{dual}(\mathcal{E}) \]
if any side of the equation is still an extended multi-segment.

(2) Suppose \( \mathcal{E} \in \text{Rep}^{(P)} \) and \( u_{i,j} \) is applicable on \( \mathcal{E} \) not of type 3’. Then we have
\[ \text{dual} \circ u_{i,j} \circ \text{dual} \circ u_{i,j}(\mathcal{E}) = \mathcal{E}. \]
2.5. **Proof of Theorem 1.9.** For \( \psi \in \Psi(G_n) \), it is shown in [Xu17, §A] that
\[
\Pi_\psi = \{ \hat{\pi} \mid \pi \in \Pi_\psi \},
\]
and hence for any \( \pi \in \Pi_A(G_n) \), we have a map
\[
(2.2) \quad \Psi(\pi) \to \Psi(\hat{\pi}),
\]
\( \psi \mapsto \hat{\psi} \).

We first show the following lemma.

**Lemma 2.16.** If \( T \) is a raising operator applicable on \( \psi \in \Psi(G_n) \), then there exists another raising operator \( T' \) applicable on \( \hat{T}(\hat{\psi}) \) such that
\[
\hat{\psi} = T'(T(\psi)).
\]

In particular, for any \( \pi \in \Pi_A(G_n) \), the map \((2.2)\) is order reversing with respect to \( \geq_O \).

**Proof.** For the first part, it follows from definition that if \( T = u_i^{-1} \) (resp, \( u_i \), \( \text{dual} \), \( \text{dual}_k^{-1} \)), then \( T' = \text{dual} \circ \text{dual}_i \circ \text{dual} \) (resp. \( \text{dual}_i^{-1} \), \( \text{dual}_k^{-1} \)).

For the second part, suppose \( \psi_1 \geq_O \psi_2 \), then we may write
\[
\psi_1 = T_1 \circ \cdots \circ T_m(\psi_2),
\]
where \( T_i \)s are raising operators. We apply induction on \( m \) to show that \( \hat{\psi}_2 \geq_O \hat{\psi}_1 \). The case \( m = 0 \) is clear. For \( m \geq 1 \), let \( \psi_3 := T_2 \circ \cdots \circ T_m \) so that \( \psi_1 = T_1(\psi_3) \).

Part (1) gives a raising operator \( T' \) such that
\[
\hat{\psi}_3 = T'(\hat{\psi}_1),
\]
and hence \( \hat{\psi}_3 \geq_O \hat{\psi}_1 \). The induction hypothesis then gives \( \hat{\psi}_2 \geq_O \hat{\psi}_3 \geq_O \hat{\psi}_1 \). This completes the proof of the lemma. \( \square \)

Now we prove Theorem 1.9.

**Proof of Theorem 1.9.** Recall that \( \psi_1 \geq_D \psi_2 \) if and only if \( \hat{\psi}_1 \leq_A \hat{\psi}_2 \). Thus Part (1) follows from Theorem 1.7 and Lemma 2.16, and Part (2) follows from Proposition 1.5 Parts (2), (3). \( \square \)

3. **Proof of Theorem 1.12**

Let \( T \) be a raising operator applicable on a local Arthur parameter \( \psi \). The relation between \( \phi_1 := \phi_T(\psi) \) and \( \phi_2 := \phi_\psi \) is stated explicitly in Definition 2.10 and Remark 2.11. In this section, first, we develop two reduction lemmas that construct unramified \( L \)-parameters (i.e., trivial on the inertia group) \( \phi_{1,ur} \) and \( \phi_{2,ur} \) such that \( \phi_1 \geq_C \phi_2 \) if \( \phi_{1,ur} \geq_C \phi_{2,ur} \) (see §3.1, Lemmas 3.2 and 3.4 below). Then we recall the notion of rank triangles from [CFMMX22, §10.2.1] that determines the closure ordering for unramified \( L \)-parameters (see §3.2). Finally, we show that \( \phi_{1,ur} \geq_C \phi_{2,ur} \) by explicit computation and prove Theorem 1.12 in §3.3.

Theorem 1.12 (2) follows directly from its Part (1) and Theorem 1.4. Let us demonstrate how to show Theorem 1.12 (1) using the reduction lemmas below in the following example.

**Example 3.1.** Let \( \rho_1 \) (resp. \( \rho_2 \)) be an irreducible symplectic (resp. orthogonal) representation of dimension \( 2m_1 \) (resp. \( 2m_2 + 1 \)) of \( W_F \). Then consider the following two
local Arthur parameters of $\text{Sp}_{4m_1+2m_1}(F)$ of good parity
\[
\psi_1 = \rho_1 \otimes S_1 \otimes S_2 + \rho_2 \otimes S_1 \otimes S_1,
\psi_2 = \rho_1 \otimes S_2 \otimes S_1 + \rho_2 \otimes S_1 \otimes S_1.
\]

Note that $\psi_2 = \text{dual}_1(\psi_1)$, where the index 1 is the unique element in $I_{\rho_1}$. We have
\[
\phi_{\psi_1} = \rho_1 | \cdot |^2 S_1 + \rho_1 | \cdot |^2 S_1 + \rho_2 \otimes S_1,
\phi_{\psi_2} = \rho_1 \otimes S_2 + \rho_2 \otimes S_1.
\]

First, Lemma 3.2 below allows us to cancel the common part $\rho_2 \otimes S_1$. To be explicit, consider the following local $L$-parameters of $\text{SO}_{4m_1}(F)$
\[
\phi_1' = \rho_1 | \cdot |^2 S_1 + \rho_1 | \cdot |^2 S_1,
\phi_2' = \rho_1 \otimes S_2.
\]
Then $\phi_{\psi_2} \geq C \phi_{\psi_1}$ if $\phi_2' \geq C \phi_1'$. Next, Lemma 3.4 below allows us to reduce to the unramified case. To be explicit, consider the following local $L$-parameters of $\text{SO}_3(F)$
\[
\phi_{1,ur}' = | \cdot |^2 S_1 + | \cdot |^2 S_1,
\phi_{2,ur}' = | \cdot |^0 S_2.
\]
Then $\phi_2' \geq C \phi_1'$ if and only if $\phi_{2,ur}' \geq C \phi_{1,ur}'$. Finally, we compute their rank triangles to conclude that $\phi_{2,ur}' \geq C \phi_{1,ur}'$. In conclusion, we have
\[
\text{dual}_1(\psi_1) = \psi_2 \geq C \psi_1.
\]

3.1. Two reduction lemmas. In this subsection, we let $G_n$ denote one of the split groups $\text{Sp}_{2n}, \text{SO}_{2n+1}$ or $\text{SO}_{2n}$, and let $\phi_1, \phi_2$ be two local $L$-parameters of $G_n$ that share the same infinitesimal character $\lambda$. We introduce two reduction lemmas for comparing the closure ordering for $\phi_1$ and $\phi_2$.

The first lemma allows us to cancel the common part of $\phi_1$ and $\phi_2$ that is self-dual.

**Lemma 3.2.** If $\phi_1 = \phi + \phi_1'$ and $\phi_2 = \phi + \phi_2'$ are local $L$-parameters of $G_n$ such that $\lambda_{\phi_1} = \lambda_{\phi_2}$ and $\phi, \phi_1', \phi_2'$ are all self-dual, then $\phi_1 \geq C \phi_2$ if $\phi_1' \geq C \phi_2'$.

**Proof.** Denote $\lambda' = \lambda_{\phi_1}'. Then we may write $\lambda := \lambda_{\phi_1} = \lambda_{\phi} \oplus \lambda'$. Let $E$ (resp. $E_{\phi}, E'$) be the space that the image of $\lambda$ (resp. $\lambda_{\phi}, \lambda'$) acts on. Then we have $E = E_{\phi} \oplus E'$, and both $\phi_1$ and $\lambda$ have image in $\text{Aut}(E_{\phi}) \times \text{Aut}(E') \subseteq \text{Aut}(E)$. In other words, we have the following after choosing a basis
\[
\phi_1 = \left(\begin{array}{c}
\phi \\
\phi_1'
\end{array}\right), \quad \lambda_{\phi_1} = \left(\begin{array}{c}
\lambda_{\phi} \\
\lambda'
\end{array}\right).
\]
Let $\iota$ be the injection from $\text{End}(E_{\phi}) \times \text{End}(E')$ to $\text{End}(E)$, and denote
\[
X := d(\phi|_{\text{SL}_2(C)}) \left(\begin{array}{c}
0 \\
1 \\
0
\end{array}\right).
\]
Then we consider the following sets
\[
H' := \{\iota(I, h) \in \text{Aut}(E) \mid h \in H_{\lambda'}\}, \quad V' := \{\iota(X, v) \in \text{End}(E) \mid v \in V_{\lambda'}\}.
\]
That is
\[
H' = \left(\begin{array}{c}
I \\
H_{\lambda'}
\end{array}\right), \quad V' = \left(\begin{array}{c}
X \\
V_{\lambda'}
\end{array}\right).
\]
One can check that $H'$ is a subgroup of $H_\lambda$, and $V'$ is a closed subvariety of $V_\lambda$. Clearly, we have $H' \cong H_\lambda$, $V' \cong V_\lambda$, and the conjugation action of $H'$ on $V'$ can be identified with the adjoint action of $H_\lambda$ on $V_\lambda$. Let

$$X_i := d(\phi_i|_{\text{SL}_2(\mathbb{C})}) \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right), \quad X'_i := d(\phi'_i|_{\text{SL}_2(\mathbb{C})}) \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

Then $X = \phi(X, X'_i)$. That is

$$X_i = \begin{pmatrix} X \\ X'_i \end{pmatrix}.$$

The assumption $\phi'_i \geq_C \phi''_2$ implies that $X'_i$ is in the closure of the $H'$ orbit of $X_i$. Therefore, $X_2$ is in the closure of the $H_\lambda$ orbit of $X_1$, and hence $\phi_1 \geq_C \phi_2$. This completes the proof of the corollary. \hfill \Box

As a corollary, we show that it suffices to consider representations of good parity for Theorems 1.12, 1.16.

**Corollary 3.3.** Theorem 1.12 (resp. Theorem 1.16) holds for any representation $\pi \in \Pi_A(G_n)$ if and only if it holds for any representation $\pi_0 \in \Pi_{A}'(G_n)$.

**Proof.** For an arbitrary representation $\pi \in \Pi_\psi$, we want to show that

$$\phi_\pi \geq_C \phi_{\psi^{\max}(\pi)} \geq_C \phi_{\psi^{\min}(\pi)},$$

By Theorem 2.4, there exists a representation $\psi_1$ of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ such that $\pi = \tau_{\psi_1} \times \pi_0$, where $\pi_0 \in \Pi_{A}'(G_n)$. Also, we have

\[
\begin{align*}
\psi &= (\psi_1 \oplus \psi_1') \oplus \psi_0, \\
\psi^{\max}(\pi) &= (\psi_1 \oplus \psi_1') \oplus \psi^{\max}(\pi_0), \\
\psi^{\min}(\pi) &= (\psi_1 \oplus \psi_1') \oplus \psi^{\min}(\pi_0),
\end{align*}
\]

where $\pi_0 \in \Pi_\psi$. Therefore, Lemma 3.2 implies that (3.1) holds if

$$\phi_{\pi_0} \geq_C \phi_{\psi^{\max}(\pi_0)} \geq_C \phi_{\psi_0} \geq_C \phi_{\psi^{\min}(\pi_0)},$$

which follows from the assumption. This completes the proof of the corollary. \hfill \Box

Now assume the restriction of $\phi_1, \phi_2$ on $W_F$ is a direct sum of twists of the same irreducible self-dual representation $\rho$ of $W_F$. To show $\phi_1 \geq_C \phi_2$, the following lemma allows us to reduce to show $\phi_{1,\text{ur}} \geq_C \phi_{2,\text{ur}}$ where both $\phi_{1,\text{ur}}, \phi_{2,\text{ur}}$ are unramified.

**Lemma 3.4.** Suppose $\phi_1, \phi_2$ are local $L$-parameters of $G_n$ that share the same infinitesimal character. We assume they are of the form

$$\phi_i = \bigoplus_{j \in I_i} \rho_j \cdot |x_j| \otimes S_{a_j},$$

where $\rho$ is a self-dual irreducible representation of $W_F$ and $x_j \in \mathbb{R}$. Then we define

$$\phi_{i,\text{ur}} := \bigoplus_{j \in I_i} |x_j| \otimes S_{a_j},$$

which are local $L$-parameters of split $\text{Sp}_{2m}(F), \text{SO}_{2m+1}(F)$ or $\text{SO}_{2m}(F)$. Then $\phi_{1,\text{ur}}$ and $\phi_{2,\text{ur}}$ also share the same infinitesimal character, and $\phi_1 \geq_C \phi_2$ if and only if $\phi_{1,\text{ur}} \geq_C \phi_{2,\text{ur}}$.\hfill \Box
Proof. First, note that $\rho \otimes \lambda_{\phi_{i,ur}} = \lambda_{\phi_{i}}$, so $\lambda_{\phi_{i,ur}} = \lambda_{\phi_{2,ur}}$. We denote $\lambda_{ur} := \lambda_{\phi_{1,ur}}$.

Let $E$ (resp. $E_{\rho}, E_{ur}$) be the space that the image of $\lambda$ (resp. $\rho, \lambda_{ur}$) acts on. That is, we view

\[
\phi_{i} : W_{F} \times \text{SL}_2(\mathbb{C}) \to \text{GL}(E),
\]

\[
\rho \otimes S_{1} : W_{F} \times \text{SL}_2(\mathbb{C}) \to \text{GL}(E_{\rho}),
\]

\[
\phi_{i,ur} : W_{F} \times \text{SL}_2(\mathbb{C}) \to \text{GL}(E_{ur}).
\]

Then we may identify $\phi_{i} = (\rho \otimes S_{1}) \otimes \phi_{i,ur}$ and $E = E_{\rho} \otimes E_{ur}$. Consider

\[
H' := \{ I \otimes h \in \text{Aut}(E) \mid h \in H_{\lambda_{ur}} \}, \quad V' := \{ I \otimes v \in \text{End}(E) \mid v \in V_{\lambda_{ur}} \},
\]

where we view $V_{\lambda_{ur}}$ as a closed subset of $\text{End}(E_{ur})$, and $H_{\lambda_{ur}}$ as a subgroup of $\text{Aut}(E_{ur})$. We have $H'$ is isomorphic to $H_{\lambda_{ur}}$ and $V'$ is isomorphic to $V_{\lambda_{ur}}$. The conjugation action of $H'$ on $V'$ can be identified with the action of $H_{\lambda_{ur}}$ on $V_{\lambda_{ur}}$. Note that

\[
\text{End}(E_{ur}) \leftrightarrow \text{End}(E),
\]

\[
h \mapsto I \otimes h
\]

is a closed immersion. It follows that $V'$ is closed in $\text{End}(E)$.

The conclusion follows from the following claims:

1. $H' = H_{\lambda}$, and
2. $V' = V_{\lambda}$.

Now we prove the claims. For (1), we first show that $H' \subseteq H_{\lambda}$, i.e., for any $h \in H_{\lambda_{ur}}$, $I \otimes h$ is also in $\hat{G}_{n}(\mathbb{C})$ and commutes with the image of $\lambda$.

Since the identity preserves the bilinear form on $E_{\rho}$ and $h$ preserves the bilinear form on $E_{ur}$, we see that $I \otimes h$ preserves the symplectic or orthogonal form on $E = E_{\rho} \otimes E_{ur}$. This shows that $I \otimes h \in \hat{G}_{n}(\mathbb{C})$.

To check $I \otimes h$ commutes with the image of $\lambda$, it suffices to check that

\[
\lambda(w) \circ (I \otimes h)(e_{1} \otimes e_{2}) = (I \otimes h) \circ \lambda(w)(e_{1} \otimes e_{2}),
\]

for any $w \in W_{F}, e_{1} \in E_{\rho}$ and $e_{2} \in E_{ur}$. Recall that we have $\lambda = \rho \otimes \lambda_{ur}$. Thus

\[
\lambda(w) \circ (I \otimes h)(e_{1} \otimes e_{2}) = \lambda(w)(e_{1} \otimes h(e_{2})),
\]

\[
= \rho(w)(e_{1}) \otimes (\lambda_{ur}(w) \circ h(e_{2}))
\]

\[
= \rho(w)(e_{1}) \otimes (h \circ \lambda_{ur}(w)(e_{2}))
\]

\[
= (I \otimes h)(\rho(w)(e_{1}) \otimes \lambda_{ur}(w)(e_{2}))
\]

\[
= (I \otimes h) \circ \lambda(w)(e_{1} \otimes e_{2}).
\]

This completes the verification that $H' \subseteq H_{\lambda}$.

Now we show that $H_{\lambda} \subseteq H'$. Write $\lambda_{ur} = \oplus_{i=1}^{r} |y_{i}|$ and choose a basis $\{e_{1}, \ldots, e_{r}\}$ of $E_{ur}$ such that

\[
\lambda(w)(e_{\rho} \otimes e_{i}) = |w|^{y_{i}} \rho(w)(e_{\rho}) \otimes e_{i},
\]

for any $e_{\rho} \in E_{\rho}$ and $w \in W_{F}$. Denote $E_{i} := E_{\rho} \otimes \mathbb{C}e_{i}$, which is the underlying space of the irreducible representation $\rho \otimes |y_{i}|$.

Take an $T \in \text{Aut}(E_{\rho} \otimes E_{ur})$ that commutes with the image of $\lambda$. From the assumptions, $T(E_{i})$ is invariant under the action of $W_{F}$, and hence $T(E_{i}) = E_{j}$ for some $j$ such that $y_{i} = y_{j}$. Then we may write

\[
T|_{E_{i}}(e_{\rho} \otimes e_{i}) = T_{i}(e_{\rho}) \otimes e_{j},
\]

for some $T_{i} \in \text{Aut}(E_{\rho})$. Since $T$ commutes with the image of $\lambda$, $T_{i}$ also commutes with the image of $\rho$. By Schur’s lemma, the irreducibility of $\rho$ implies that $T_{i}$ is a scalar
multiplication. Therefore, we may write $T = I \otimes h$ for some $h \in \text{Aut}(E_{ur})$. Moreover, since $T$ sends $E_i$ to $E_j$ such that $y_i = y_j$, $h$ also commutes with the image of $\lambda_{ur}$.

Finally, we further assume $T$ is in $G_n(\mathbb{C})$, i.e., $T = I \otimes h$ preserves the symplectic or orthogonal form on $E_{\rho} \otimes E_{ur}$. Then $h$ must also preserve the symplectic or orthogonal form on $E_{ur}$. This completes the proof that $H_\lambda \subseteq H'$.

For (2), we first check that $V' \subseteq V_\lambda$, i.e., $V' \subseteq \widehat{g}_n$, and any element $I \otimes v$ in $V'$ satisfies the following equation

$$\text{Ad}(\lambda(w))(I \otimes v) = |w|I \otimes v,$$

for any $w \in W_F$.

Let $\widehat{G}_{\lambda_{ur}}(\mathbb{C})$ be the subgroup of $\text{Aut}(E_{ur})$ that preserves the symplectic or orthogonal form on $E_{ur}$. From the proof of (1) above, it follows that

$$\widehat{G}_{\lambda_{ur}}(\mathbb{C}) \hookrightarrow \widehat{G}_n(\mathbb{C}),$$

$$g \mapsto I \otimes g$$

is a Lie group homomorphism. Thus $V'$, a subset of the image of differential of above map, is contained in the Lie algebra of $\widehat{G}_n(\mathbb{C})$.

Next, we check that

$$\lambda(w) \circ (I \otimes v) \circ \lambda(w)^{-1}(e_1 \otimes e_2) = (|w|I \otimes v)(e_1 \otimes e_2),$$

for any $w \in W_F$, $e_1 \in E_{\rho}$ and $e_2 \in E_{ur}$. Using the relation $\lambda = \rho \otimes \lambda_{ur}$, we have

$$\lambda(w) \circ (I \otimes v) \circ \lambda(w)^{-1}(e_1 \otimes e_2) = (\rho(w) \circ (I \otimes v \circ \rho(w^{-1}))(e_1) \otimes (\lambda_{ur}(w) \circ v \circ \lambda_{ur}(w)^{-1})(e_2))$$

$$= e_1 \otimes |w|v(e_2)$$

$$= (|w|I \otimes v)(e_1 \otimes e_2).$$

This completes the verification that $V' \subseteq V_\lambda$.

Finally, we show that $V_\lambda \subseteq V'$. Continuing the notation in the proof of (1), a similar argument shows that if $T \in \text{End}(E_{\rho} \otimes E')$ satisfies the equation

$$\text{Ad}(\lambda(w))T = |w|T,$$

for any $w \in W_F$, then $T$ sends $E_i$ to $E_j$ such that $y_j = y_i + 1$, and $T = I \otimes v$ for some $v \in \text{End}(E_{ur})$. Moreover, if $T$ is in $\widehat{g}_n$, then $v$ is in $\widehat{g}_{\lambda_{ur}}$, the Lie algebra of $\widehat{G}_{\lambda_{ur}}(\mathbb{C})$. This completes the verification that $V_\lambda \subseteq V'$.

Now the proof of the claims, and hence of the lemma, is complete. $\square$

3.2. Closure Ordering for unramified $L$-parameters. In this subsection, we let $G_n$ denote one of the split group $Sp_{2n}$, $SO_{2n+1}$ or $SO_{2n}$, and let $\phi_1, \phi_2$ be two unramified local $L$-parameters of $G_n$ that share the same infinitesimal character $\lambda$. We first recall that the closure ordering of $\phi_1, \phi_2$ is determined by rank triangles (see Definition 3.8 below) from [CFMMX22, §10.2.1], and how to compute the rank triangle of each $\phi_i$ from [CFK21, §1.5] and [CR22, §3.3]. Finally, we compare the orderings $\geq_C$ and $\geq_D$.

Let $\mathfrak{g}$ be a classical Lie algebra. Recall that there is a correspondence between the set of nilpotent orbits of $\mathfrak{g}$ (over $\mathbb{C}$) and certain set of partitions (see [CM93, §5.1]). Denote $p(O)$ the partition corresponds to the nilpotent orbit $O$. We recall the dominance order, which is a partial order on the set of partitions, in the definition below.

**Definition 3.5.** Suppose $\underline{p} = [p_1, \ldots, p_r]$ and $\underline{q} = [q_1, \ldots, q_s]$ are two partitions of $n$, i.e.,

- $\sum_{i=1}^r p_i = n = \sum_{j=1}^s q_j$, and
- $\{p_i\}_{i=1}^r$ and $\{q_j\}_{j=1}^s$ are both non-raising sequence of positive integers.
If for any $1 \leq k \leq r$, we have

$$\sum_{i=1}^{k} p_i \geq \sum_{j=1}^{k} q_j,$$

then we say $\pi$ precedes $\eta$ in the dominance order and write $\pi \geq \eta$.

A classical theorem due to Gerstenhaber and Hesselink (see \[CM93, \text{Theorem 6.2.5}\]) states that the closure ordering for nilpotent orbits of $\mathfrak{g}$ is equivalent to the dominance order of the corresponding partitions.

**Theorem 3.6** (Gerstenhaber, Hesselink). Let $O_1, O_2$ be two nilpotent orbits of a classical Lie algebra $\mathfrak{g}(\mathbb{C})$. Then $O_1 \supseteq O_2$ if and only if $\underline{p}(O_1) \geq \underline{p}(O_2)$.

One direction of the above theorem follows from the fact that for $X \in O$, the rank sequence $(\text{rank}(X^k))_{k \in \mathbb{Z}_{\geq 0}}$ can be computed from the partition $\underline{p}(O)$ (see \[CM93, \text{6.2.3}\]).

Let $\phi_i = \oplus_{j \in I_i} \cdot [\chi_j] \otimes S_{\alpha_j}$, for $i = 1, 2$, be unramified local $L$-parameters of a split classical group $G$ that share the same infinitesimal character $\lambda$. We may associate $\phi_i$ a partition $\underline{p}(\phi_i)$ by

$$\underline{p}(\phi_i) := [a_j]_{j \in I_i},$$

which is exactly the corresponding partition of the $\hat{G}(\mathbb{C})$-orbit of $d(\phi_i)_{\text{SL}_2(\mathbb{C})} \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$. Note that if $\phi = \phi_\psi$ for some local Arthur parameter $\psi$, then $\underline{p}(\phi) = \underline{p}^D(\psi)$.

By Theorem 3.6, if $\phi_1 \succeq_C \phi_2$, then $\underline{p}(\phi_1) \geq \underline{p}(\phi_2)$ since $H_\lambda$ is a subgroup of $\hat{G}(\mathbb{C})$. However, the converse does not hold in general. See Example 3.11 (2) below for a counter example.

To get a complete description of the closure ordering for $H_\lambda$-orbit on $V_\lambda$, we recall the concept of rank triangle considered in \[CFMMX22\] now. The rank triangle of $\phi$ not only keeps the partition information for $\phi_{\text{SL}_2(\mathbb{C})}$, but also contains the twist information for $\phi_{|W_F}$.

Let $\hat{G}_n(\mathbb{C}) \rightarrow \text{GL}(E)$ be the standard representation of $\hat{G}_n(\mathbb{C})$. Therefore, $E$ is $(2n + 1)$-dimensional if $G_n = \text{Sp}_{2n}$, and $2n$-dimensional otherwise. Let $\{q_{\lambda_1}^1, \ldots, q_{\lambda_n}^1\}$ be the set of distinct eigenvalues of $\lambda(\text{Fr}) : W_F \rightarrow \text{GL}(E)$, and denote $E_\alpha$ the $q_{\lambda_\alpha}$-eigenspace of $\lambda(\text{Fr})$ in $E$.

For simplicity, we keep the following assumptions.

**Assumption 3.7.** Following the notation above, we assume the following:

- $\lambda_\alpha \in \frac{1}{2} \mathbb{Z}$,
- $\lambda_\alpha - \lambda_\beta \in \mathbb{Z}_{>0}$ for $\alpha < \beta$, and
- $\lambda_\alpha = -\lambda_\beta - \alpha$.

These assumptions are natural in the case we consider. To be explicit, let $\pi \in \Pi_{\psi}$, where $\psi$ is a local Arthur parameter of good parity of $\text{Sp}_{2n}$, or split $\text{SO}_{2n+1}$. Then for any subrepresentation $\phi'$ of $\phi_\pi$ of the form $\phi' = \oplus_{j \in I_\pi} [\chi_j] \otimes S_{\alpha_j}$, the parameter

$$\phi'_{|W_F} := \oplus_{j \in I_\pi} [\chi_j] \otimes S_{\alpha_j},$$

coming from the reduction in Lemma 3.4, satisfies the three assumptions. Note that the third assumption follows from the first two assumptions and that $\lambda_\alpha$ has image in $\hat{G}_n(\mathbb{C})$.

In the general case, we only have $\lambda_\alpha \in \mathbb{C}$. One can modify the argument below by splitting $\{\lambda_0, \ldots, \lambda_r\}$ into residue classes mod $\mathbb{Z}$.
Recall that when \( \lambda \) is unramified, 
\[
V_\lambda := \{ x \in \widehat{\mathfrak{g}}_n \mid \text{Ad}(\lambda(\Fr)) x = qx \}.
\]

Therefore, we may identify \( V_\lambda \) as a subset of \( \text{Hom}(E_0, E_1) \times \cdots \times \text{Hom}(E_{r-1}, E_r) \) and write each element \( v \in V_\lambda \) by \( v = (v_\alpha)_{\alpha=1}^r \) where \( v_\alpha \in \text{Hom}(E_{\alpha-1}, E_\alpha) \). Clearly, \( v_\alpha = 0 \) if \( \lambda_{\alpha} - \lambda_{\alpha-1} > 1 \). We consider the following definition of rank triangles, where we put the triangle into an upper triangular matrix.

**Definition 3.8.** Suppose \( \lambda \) satisfies Assumption 3.7. Under the notation in previous paragraphs, for each \( v = (v_1, \ldots, v_r) \in V_\lambda \), we define \( v_{\alpha\beta} \in \text{Hom}(E_{\alpha-1}, E_\beta) \) by
\[
v_{\alpha\beta} := \begin{cases} v_\beta \circ \cdots \circ v_\alpha, & \text{if } \alpha \leq \beta, \\ 0, & \text{otherwise.} \end{cases}
\]

Then we define an upper triangular \( r \times r \) matrix \( r(v) = (r_{\alpha\beta}(v))_{\alpha\beta} \) by
\[
r_{\alpha\beta}(v) := \text{rank}(v_{\alpha\beta}).
\]

Suppose \( \phi \) is an unramified local \( L \)-parameter where \( \lambda_\phi \) satisfies Assumption 3.7. Take any element \( v \in C_\phi \subseteq V_{\lambda_\phi} \) and define the rank triangle \( r(\phi) = (r_{\alpha\beta}(\phi))_{\alpha\beta} \) of \( \phi \) by
\[
r_{\alpha\beta}(\phi) := r_{\alpha\beta}(v).
\]

The following lemma is contained in [CFMMX22, §10.2.1].

**Lemma 3.9.** Suppose \( \phi_1, \phi_2 \) are unramified local \( L \)-parameters of the group \( G_n \) that share the same infinitesimal character \( \lambda \) satisfying Assumption 3.7. Then \( \phi_1 \geq_C \phi_2 \) if and only if \( r_{\alpha\beta}(\phi_1) \geq r_{\alpha\beta}(\phi_2) \) for all \( \alpha, \beta \in \{1, \ldots, r\} \).

Now we rephrase how to compute the rank triangle \((r_{\alpha\beta}(\phi))_{\alpha\beta}\) from the decomposition of \( \phi \) given in [CFK21, §1.5] and [CR22, §3.3], and give a proof of it for completeness. Assume \( \lambda_\phi \) satisfies Assumption 3.7 and continue the notation from above. Write
\[
\phi = \bigoplus_{i \in I} | \cdot | x_i \otimes S_{a_i}.
\]

To the irreducible representation \( S_{a_i} \), we associate set \( Q_{S_{a_i}} := \{ q^{-1} \frac{a}{2}, q^{-3} \frac{a}{2}, \ldots, q^{-a} \} \). When we have a twist \( | \cdot | x \otimes S_{a_i} \), we set \( Q_{|x| \otimes S_{a_i}} := \{ q^{-1} x \frac{a}{2} + x, q^{-3} x \frac{a}{2} + x, \ldots, q^{-a} x \} \). To each direct summand \( | \cdot | x \otimes S_{a_i} \), we define an \( r \times r \) matrix \( M_{|x| \otimes S_{a_i}} := (m_{\alpha\beta})_{\alpha\beta} \) as follows. If \( \alpha > \beta \), then \( m_{\alpha\beta} := 0 \). That is, \( M_{|x| \otimes S_{a_i}} \) is upper triangular. If \( \alpha \leq \beta \), then we define \( m_{\alpha\beta} := 1 \) if \( \{ q^{\lambda_{\alpha-1}}, q^{\lambda_{\alpha}}, \ldots, q^{\lambda_{\beta}} \} \subseteq Q_{|x| \otimes S_{a_i}} \). Otherwise, we set \( m_{\alpha\beta} := 0 \).

By convention, if \( a = 1 \), then we set \( M_{|x|} := 0 \).

**Lemma 3.10.** If \( \phi = \bigoplus_{i \in I} | \cdot | x_i \otimes S_{a_i} \) and \( \lambda_\phi \) satisfies Assumption 3.7, then \( r(\phi) = \sum_{i \in I} M_{|x|^i \otimes S_{a_i}} \).

**Proof.** For \( i \in I \), let \( E_i \subseteq E \) be the subspace that \( | \cdot | x_i \otimes S_{a_i} \) acts on. Then for each eigenspace \( E_\alpha \) of \( \lambda(\Fr) \), we have a decomposition
\[
E_\alpha = \bigoplus_{i \in I} E^i_\alpha,
\]
where \( E^i_\alpha = E_\alpha \cap E^i \).

Let \( X := d(\phi|_{\text{SL}_2(\mathbb{C})}) \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = (v_1, \ldots, v_r) \). We compute the rank of
\[
v_{\alpha\beta} := v_\beta \circ \cdots \circ v_\alpha \in \text{Hom}(E_{\alpha-1}, E_\beta) = \bigotimes_{i,j \in I} \text{Hom}(E^i_\alpha, E^j_\beta)
\]
for any $\beta \geq \alpha$. Since each $\cdot | x_i \otimes S_a_i$ is a subrepresentation, we see that $X$ leaves the subspaces $E_i$ invariant, and hence $v_{\alpha\beta}$ is indeed in a smaller set

$$v_{\alpha\beta} \in \bigotimes_{i \in I} \text{Hom}(E_{\alpha-1}^i, E_{\beta}^i).$$

Denote $v_{i,\alpha\beta}$ the projection of $v_{\alpha\beta}$ to $\text{Hom}(E_{\alpha-1}^i, E_{\beta}^i)$ . Then we have

$$\text{rank}(v_{\alpha\beta}) = \sum_{i \in I} \text{rank}(v_{i,\alpha\beta}).$$

By definition, we have

$$r_{\alpha\beta}(\phi) = \text{rank}(v_{\alpha\beta}) = \sum_{i \in I} \text{rank}(v_{i,\alpha\beta}),$$

so it remains to check that $\text{rank}(v_{i,\alpha\beta}) = (M_{| \cdot | x_i \otimes S_a_i})_{\alpha\beta}$.

Suppose $(M_{| \cdot | x_i \otimes S_a_i})_{\alpha\beta} = 1$. Then the subspaces $E_{\alpha-1+t}^i$ are all 1-dimensional for $t = 0, \ldots, \beta - \alpha + 1$. In this case, $v_{i,\alpha-1+t}^i \in \text{Hom}(E_{\alpha-1+t}^i, E_{\alpha+t}^i)$ are all isomorphisms for $t = 0, \ldots, \beta - \alpha$, so $\text{rank}(v_{i,\alpha\beta}) = 1$.

Suppose $(M_{| \cdot | x_i \otimes S_a_i})_{\alpha\beta} = 0$. Then at least one of $E_{\alpha-1}^i$ or $E_{\beta}^i$ is zero, and hence $v_{i,\alpha\beta} \in \text{Hom}(E_{\alpha-1}^i, E_{\beta}^i)$ automatically has rank zero.

This completes the verification that $\text{rank}(v_{i,\alpha\beta}) = (M_{| \cdot | x_i \otimes S_a_i})_{\alpha\beta}$ and the proof of the lemma. □

Now we give several examples for the computation of rank triangles and the comparison between four orderings $\succeq_O$ (Definition 1.3), $\succeq_C$ (Definition 1.11), $\succeq_A$ (Definition 1.6), and $\succeq_D$ (Definition 1.8).

**Example 3.11.** Let $\rho$ be the trivial representation of $W_F$.

(1) Consider the tempered representation $\pi_1$ of $\text{SO}_9(F)$ given by

$$\pi_1 = \pi \left( \begin{array}{ccc} 1 & -1 & 3^+ \\ 2 & 2 & 2 \end{array} \right).$$

Then applying [HLL22, Algorithm 7.9, Theorem 7.4], we see $\Psi(\pi_1)$ consists of four local Arthur parameters given as follows.

$$\psi_1 = \rho \otimes S_2 \otimes S_1 + \rho \otimes S_2 \otimes S_1 + \rho \otimes S_4 \otimes S_1,$$
$$\psi_2 = \rho \otimes S_1 \otimes S_2 + \rho \otimes S_2 \otimes S_1 + \rho \otimes S_4 \otimes S_1,$$
$$\psi_3 = \rho \otimes S_2 \otimes S_1 + \rho \otimes S_3 \otimes S_2,$$
$$\psi_4 = \rho \otimes S_1 \otimes S_2 + \rho \otimes S_3 \otimes S_2.$$

Here is the diagram for $\succeq_O$ on $\Psi(\pi_1)$.

$$\begin{array}{c}
\psi_1 \\
\psi_2 \\
\psi_4 \\
\psi_3
\end{array}$$

Now we compute the rank triangles for

$$\phi_{\psi_3} = \rho \otimes S_2 + \rho | \cdot | \frac{3}{2} + \rho | \cdot | \frac{1}{2} \otimes S_3.$$
The multi-set of eigenvalue of \( \lambda_{\phi_{\psi_1}} \) is \( \{q_1^1, q_1^2, q_1^2, q_1^1, q_2^{-1}, q_2^{-1}, q_2^{-1}, q_2^{-3}\} \), and hence we set \( (\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \). Then it follows from the definition that

\[
M_{S_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{|\mathbb{Z}|_{2}\otimes S_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{|\mathbb{Z}|_{2}\otimes S_3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

and hence

\[
r(\phi_{\psi_3}) = M_{S_2} + M_{|\mathbb{Z}|_{2}\otimes S_3} + M_{|\mathbb{Z}|_{2}\otimes S_3} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]

Similarly, one can compute that

\[
r(\phi_{\psi_1}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad r(\phi_{\psi_2}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad r(\phi_{\psi_4}) = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.
\]

Therefore, \( \geq_C \) is identical with \( \geq_O \) on \( \Psi(\pi_1) \) in this case.

Next, it follows from definition that

\[
p^A(\psi_1) = [1^8], \quad p^A(\psi_2) = [2^1, 6^1], \quad p^A(\psi_3) = [2^2, 6^2], \quad p^A(\psi_4) = [2^4],
\]

\[
p^D(\psi_1) = [4, 2^2], \quad p^D(\psi_2) = [4, 2, 1^2], \quad p^D(\psi_3) = [3^2, 2], \quad p^D(\psi_4) = [3^2, 1^2],
\]

and hence \( \geq_D \) is also identical with \( \geq_O \) on \( \Psi(\pi_1) \), while \( \geq_A \) is a total order given by

\[
\psi_1 \geq_A \psi_2 \geq_A \psi_3 \geq_A \psi_4.
\]

Under any of the orderings on \( \Psi(\pi_1) \), \( \psi_{\max}(\pi) = \psi_1 \) (resp. \( \psi_{\min}(\pi) = \psi_4 \)) is the unique maximal (resp. minimal) element.

(2) We take \( \pi_2 \) to be the Aubert-Zelevinsky dual of \( \pi_1 \) above. Then \( \Psi(\pi_2) = \{\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4\} \). Here is the diagram for \( \geq_O \) on \( \Psi(\pi_2) \).

In this example, \( \geq_A \) is identical with \( \geq_O \) on \( \Psi(\pi_2) \), while \( \geq_C \) and \( \geq_D \) are identical total orders given by

\[
\hat{\psi}_4 \geq_D \hat{\psi}_3 \geq_D \hat{\psi}_2 \geq_D \hat{\psi}_1.
\]

We remark that \( \hat{\psi}_3 \geq_C \hat{\psi}_2 \) but \( \psi_2 \not\geq_C \psi_3 \), i.e., the map \( \psi \mapsto \hat{\psi} \) is not order reversing with respect to \( \geq_C \). On the other hand, it is order reversing with respect to \( \geq_O \) by Lemma 2.16.

(3) The following example shows that \( \geq_C \) is not always identical with \( \geq_D \). Let \( \pi \) be the unique supercuspidal representation in the tempered local \( L \)-packet of

\[
\phi = \rho \otimes S_2 + \rho \otimes S_4 + \rho \otimes S_6
\]

of \( SO_{13}(F) \). We have \( |\Psi(\pi)| = 18 \). Consider \( \psi_1, \psi_2 \in \Psi(\pi) \) given by

\[
\psi_1 = \rho \otimes S_1 \otimes S_2 + \rho \otimes S_1 \otimes S_4 + \rho \otimes S_6 \otimes S_1,
\]

\[
\psi_2 = \rho \otimes S_1 \otimes S_6 + \rho \otimes S_3 \otimes S_2.
\]
One can compute that $\psi_1 \geq_D \psi_2$, but $\psi_1$ and $\psi_2$ are not comparable under $\geq_C$.

In the corollary below, we show how to recover the partition $p(\phi)$ from the rank triangle $r(\phi)$. Also, we prove that $\phi \geq_C \phi'$ only if $p(\phi) \geq p(\phi')$ purely combinatorially.

**Corollary 3.12.** Suppose $\phi \in \Phi(G)_{\lambda}$.

1. Let
   
   $$d_s := \sum_{t=1}^{r+1-s} r_t(t+s-1)(\phi),$$
   
   which is the sum of the $(s - 1)$-th off-diagonal entries of $r(\phi)$, and write
   
   $$p(\phi) = [(r+1)^{m_{r+1}}, \ldots, 2^{m_2}, 1^{m_1}],$$
   
   where $m_s$ is the multiplicity of $s$. Then $m_s$ is given recursively by
   
   $$(3.2) \quad \begin{cases} m_{r+1} = d_r, \\ m_s = d_{s-1} - \sum_{t=s+1}^{r+1} (t-s+1)m_t, \end{cases}$$
   
   for $s > 1$, and $m_1$ can be computed from $m_2, \ldots, m_{r+1}$.

2. $\phi \geq_C \phi'$ implies that $p(\phi) \geq p(\phi')$.

3. In particular, for local Arthur parameters $\psi, \psi'$ of $G_n$, $\psi \geq_C \psi'$ implies that $\psi \geq_D \psi'$.

**Proof.** Part (1) follows directly from the fact that each summand $M_{[t]} \otimes S_a$ contributes $\max \{0, a - s \}$ to $d_s$.

For Part (2), we remark that for any element $X$ in $C_{\phi}$, we have

$$\text{rank}(X^k) = d_k,$$

and one can compare the proof below with [CM93, Lemma 6.2.2].

Equivalently, we show that $p(\phi) \not\geq p(\phi')$ implies that $\phi \not\geq_C \phi'$. Write

$$p(\phi) = [p_1, \ldots, p_x] = [(r+1)^{m_{r+1}}, \ldots, 1^{m_1}],$$

$$p(\phi') = [q_1, \ldots, q_y] = [(r+1)^{m_{r+1}'}, \ldots, 1^{m_1'}],$$

and let

$$d_s := \sum_{t=1}^{r-s+1} r_t(t+s-1)(\phi), \quad d_s' := \sum_{t=1}^{r-s+1} r_t(t+s-1)(\phi').$$

Take the minimal $j$ such that

$$\sum_{i=1}^{j} p_i < \sum_{i=1}^{j} q_i.$$

Then the minimality of $j$ gives that $q_j > p_j$, and

$$\sum_{t=p_j}^{r+1} (t-p_j) m_t = \sum_{i=1}^{j} (p_i - p_j) < \sum_{i=1}^{j} (q_i - p_j) < \sum_{t=q_j}^{r+1} (t-p_j) m'_t < \sum_{t=p_j}^{r+1} (t-p_j) m'_t.$$

From (3.2), the left hand side and the right hand side of above inequality are exactly $d_{p_j}$ and $d'_{p_j}$ respectively. Therefore, we obtain

$$\sum_{t=1}^{r+1-p_j} r_t(t+p_j-1)(\phi) = d_{p_j} < d'_{p_j} = \sum_{t=1}^{r+1-p_j} r_t(t+p_j-1)(\phi').$$

It follows that there exists some $1 \leq t \leq r + 1 - p_j$ such that

$$r_t(t+p_j-1)(\phi) < r_t(t+p_j-1)(\phi'),$$
and we conclude that \( \phi \not\succeq_{C} \phi' \) by Lemma 3.10.

Part (3) follows from Part (2) since \( p_{\phi_{\psi}}(\phi) = p_{D}(\psi) \). The proof of the corollary is now complete. \( \square \)

3.3. Proof of Theorem 1.12. Theorem 1.12 (2) follows directly from its Part (1) and Theorem 1.4. We prove Theorem 1.12 (1) in the following, case by case.

First we fix some notations. In each case, we construct an unramified infinitesimal character \( \lambda \) of split \( \text{Sp}_{2n}, \text{SO}_{2n+1} \) or \( \text{SO}_{2n} \) that satisfies Assumption 3.7. Denote \( \{q^{\lambda_1}, \ldots, q^{\lambda_r}\} \) the set of distinct eigenvalues of \( \lambda(F) \), and use \( \alpha, \beta \) to denote two numbers in \( \{0, \ldots, r\} \) with \( \lambda_{\alpha-1} = y, \lambda_{\beta} = x \) such that \( x \geq y \).

The key of the proof is the following computation. Consider a representation \( \psi' = | \cdot |^{0} \otimes S_{A+B+1} \otimes S_{A-B+1} \) of the group \( W_{F} \times \text{SL}_{2} \times \text{SL}_{2} \), which is a subrepresentation of some local Arthur parameter \( \psi \). We consider

\[
\phi_{\psi'} := \bigoplus_{t=0}^{A-B} | \cdot |^{A-B-t} \otimes S_{A+B+1},
\]

a subrepresentation of the local \( L \)-parameter \( \phi_{\psi} \). Then by Lemma 3.10, we have

\[
(3.3) \quad r_{\alpha \beta}(\phi_{\psi'}) = \# \{ t = 0, \ldots, A-B | A-t \geq x, y \geq -B-t \} = \max \{ \min \{ A-x, A-B \} - \max \{ 0, -y-B \} + 1, 0 \} \]

\[
(3.4) \quad r_{\alpha \beta}(\phi_{\psi}) = \max \{ A+B-\max \{ x, B \} - \max \{ -y, B \} + 1, 0 \}.
\]

Now we start the discussion of each case.

**Case 1:** \( T = w_{i,j}^{-1} \).

In this case, by Definition 2.10 and Remark 2.11, we may write

\[
\begin{cases}
T(\psi) = \psi' + \rho \otimes S_{A+1} \otimes S_{A-B+1} \otimes S_{A+B+1} + \rho \otimes S_{A+1} \otimes S_{A-B+1} \\
\psi = \psi' + \rho \otimes S_{A+1} \otimes S_{A-B+1} \otimes S_{A+B+1} + \rho \otimes S_{A+B+1} \otimes S_{A-B+1}
\end{cases}
\]

where \( A_{j} \geq A_{i} + 1 \geq B_{i} > B_{j} \). If \( A_{i} + 1 = B_{j} \), then we omit the third term in the decomposition of \( \psi \). Applying Lemma 3.2, it suffices to show that \( \phi_{1} \succeq \phi_{2} \), where

\[
\phi_{1} = \bigoplus_{t=0}^{A_{i}-B_{i}} \rho |^{A_{i}-B_{i}-t} \otimes S_{A+1} \bigoplus_{t=0}^{A_{j}-B_{j}} \rho |^{A_{j}-B_{j}-t} \otimes S_{A+B+1},
\]

\[
\phi_{2} = \bigoplus_{t=0}^{A_{i}-B_{i}} \rho |^{A_{i}-B_{i}-t} \otimes S_{A+B+1} \bigoplus_{t=0}^{A_{j}-B_{j}} \rho |^{A_{j}-B_{j}-t} \otimes S_{A+B+1}.
\]

Next, applying Lemma 3.4, we may assume \( \rho \) is the trivial representation. Then \( \lambda_{\phi_{1}} = \lambda_{\phi_{2}} \) satisfies Assumption 3.7, and Lemma 3.9 states that it suffices to show \( r_{\alpha \beta}(\phi_{1}) \geq r_{\alpha \beta}(\phi_{2}) \) for any \( \alpha, \beta \). We shall compare these two numbers explicitly by (3.3) and (3.4).

For \( k, l \in \{ i, j \} \), we denote the sets

\[
T_{kl} := \{ t = 0, \ldots, A_{k} - B_{l} | A_{k} - t \geq x, y \geq -B_{l} - t \}.
\]

Our goal is to show that (by (3.3))

\[
(3.5) \quad r_{\alpha \beta}(\phi_{1}) - r_{\alpha \beta}(\phi_{2}) = \#T_{jj} + \#T_{ii} - \#T_{ji} - \#T_{ij} \geq 0.
\]

It is clear from the definition that \( T_{ij} \) is a subset of \( T_{jj} \). Thus we may assume \( T_{ji} \) is non-empty. We first claim that if \( T_{ij} \) is empty, then \( \#T_{jj} \geq \#T_{ij} \), which implies (3.5) in this situation.
Let \( t_1 \) (resp. \( t_2 \)) be the maximal (resp. minimal) number in \( T_{ij} \). Observe that \( T_{ji} \cap [0, A_j - B_j] \subseteq T_{jj} \), so we may assume \( t_1 > A_j - B_j \). Then we have
\[
x \leq A_j - t_1 < B_j.
\]
On the other hand, since \( A_i \geq B_j + 1 > x \) and 0 is not in \( T_{ij} = \emptyset \), we must have \( y < -B_j \). In summary, we have the following inequalities
\[
\begin{align*}
B_i < A_i & \leq B_j + 1 \leq A_j, \\
A_j - B_j & < t_1 \leq A_j - B_i, \\
x & \leq A_j - t_1, \\
-B_i - t_2 & \leq y < -B_j.
\end{align*}
\]
Then for \(-y - B_j \leq s \leq -y - B_j + t_1 - t_2\), one can check that
- \( 0 < s \leq B_i - B_j + t_1 \leq A_j - B_j \),
- \( x + s \leq x - y - B_j + t_1 - t_2 \leq A_j - y - B_j - t_2 \leq A_j + B_i - B_j < A_j \),
- \( y + s \geq -B_j \).

Therefore, these \( t_1 - t_2 + 1 \) integers are in \( T_{jj} \). This shows that \( \#T_{jj} \geq t_1 - t_2 + 1 = \#T_{ji} \) and completes the proof of the claim.

Next, we deal with the situation that both \( T_{ij}, T_{ji} \) are non-empty. The computation (3.4) shows that
\[
\#T_{kl} = \max\{A_k + B_l + C_l, 0\},
\]
where
\[
C_l := -\max\{x, B_l\} - \max\{-y, B_l\} + 1.
\]
Then since \( T_{ij} \) and \( T_{ji} \) are both non-empty, we have
\[
\begin{align*}
r_{\alpha\beta}(\phi_1) - r_{\alpha\beta}(\phi_2) & = \max\{A_i + B_i + C_i, 0\} + \max\{A_j + B_j + C_j, 0\} \\
& \quad - (A_j + B_i + C_i + A_i + B_j + C_j) \\
& \geq 0.
\end{align*}
\]
This completes the verification of this case.

**Case 2:** \( T = \text{dual} \circ u_{i,j} \circ \text{dual} \).

If the \( u_{i,j} \) is not of type 3', then \( \text{dual} \circ u_{i,j} \circ \text{dual} = u_{i,j}^{-1} \), which is covered in the previous case. Therefore, we may assume that \( u_{i,j} \) is of type 3'. By Definition 2.10 and Remark 2.11, we may write
\[
\begin{align*}
T(\psi) & = \psi' + \rho \otimes S_{A_i + B_i + 1} \otimes S_{A_j - B_j + 1}, \\
\psi & = \psi' + \rho \otimes S_{A_i + B_i + 1} \otimes S_{A_j - B_j + 1} + \rho \otimes S_{A_j + B_j + 1} \otimes S_{A_j - B_j + 1},
\end{align*}
\]
where \( A_i \geq -B_i = A_j + 1 > -B_j \). By the same argument in the previous case, it remains to check that \( r_{\alpha\beta}(\phi_1) \geq r_{\alpha\beta}(\phi_2) \) for any \( \alpha, \beta \), where
\[
\begin{align*}
\phi_1 & = \bigoplus_{t=0}^{A_i - B_j} \left| \cdot \frac{A_i - B_j}{2} - t \right| \otimes S_{A_i + B_i + 1}, \\
\phi_2 & = \bigoplus_{t=0}^{A_j - B_j} \left| \cdot \frac{A_j - B_j}{2} - t \right| \otimes S_{A_j + B_j + 1}.
\end{align*}
\]
We again consider
\[
T_{kl} := \{t = 0, \ldots, A_k - B_l \mid A_k - t \geq x, y \geq -B_l - t\},
\]
for \( k, l \in \{i, j\} \). Our goal is to show the inequality
\[
(3.6) \quad r_{\alpha\beta}(\phi_1) - r_{\alpha\beta}(\phi_2) = \#T_{ij} - (\#T_{ii} + \#T_{jj}) \geq 0.
\]
We first claim that if $T_{jj}$ is empty, then $\#T_{ij} \geq \#T_{ii}$, which implies (3.6) in this situation.

Let $t_1$ (resp. $t_2$) be the maximal (resp. minimal) number in $T_{ii}$. Observe that $T_{ii} \cap [0, A_i - B_j] \subseteq T_{ij}$, so we may assume $A_i - B_j < t_1 \leq A_i - B_i$. Then we have

$$x \leq A_i - t_1 = B_j.$$

On the other hand, since $x \leq B_j \leq A_i$ and $0$ is not in $T_{jj} = \emptyset$, we must have $y < -B_j$.

In summary, we have the following inequalities

\[
\begin{cases}
-B_j \leq A_j = -B_j - 1 < -B_i \leq A_i, \\
A_i - B_j < t_1 \leq A_i - B_i, \\
x \leq A_i - t_1 < B_j, \\
-B_i - t_2 \leq y < -B_j.
\end{cases}
\]

Then for $-y - B_j \leq s \leq -y - B_j + t_1 - t_2$, one can check that

- $0 < s \leq B_i - B_j + t_1 \leq A_i - B_j,$
- $x + s \leq x - y - B_j + t_1 - t_2 \leq A_i - y - B_j - t_2 \leq A_i + B_i - B_j < A_i + 1 \leq A_i,$
- $y + s \geq -B_j.$

Therefore, these $t_1 - t_2 + 1$ integers are in $T_{ij}$. This shows that $\#T_{ij} \geq t_1 - t_2 + 1 = \#T_{jj}$ and completes the proof of the claim.

Next, we deal with the situation that both $T_{ii}$ and $T_{jj}$ are non-empty. The computation (3.4) shows that

$$\#T_{kl} = \max\{A_k + B_l + C_l, 0\},$$

where

$$C_l := -\max\{x, B_l\} - \max\{-y, B_l\} + 1.$$

Since (recall that $x \geq y$)

$$A_j + B_i + C_i \leq (A_j + B_i + 1) + (-x + y) \leq 0,$$

and both $T_{ij}$ and $T_{ji}$ are non-empty, we have

$$r_{\alpha\beta}(\phi_1) - r_{\alpha\beta}(\phi_2) = (\max\{A_i + B_j + C_j, 0\} + 0) - (A_i + B_i + C_i + A_j + B_j + C_j) \geq A_i + B_j + C_j + A_j + B_i + C_i - (A_i + B_i + C_i + A_j + B_j + C_j) = 0.$$

This completes the verification of this case.

**Case 3:** $T = \text{dual}_k$. In this case, by Definition 2.10 (3), we may write

\[
\begin{cases}
T(\psi) = \psi' + \rho \otimes S_{a+1} \otimes S_a, \\
\psi = \psi' + \rho \otimes S_a \otimes S_{a+1}.
\end{cases}
\]

Following the same argument in the first case, it remains to check that $r_{\alpha\beta}(\phi_1) \geq r_{\alpha\beta}(\phi_2)$ for any $\alpha, \beta$ where

$$\phi_1 = \bigoplus_{t=0}^{a-1} | \cdot t^{-1} \otimes S_{a+1},$$

$$\phi_2 = \bigoplus_{t=0}^{a} | \cdot t^{-1} \otimes S_a.$$
Then by (3.3), we have \( r_{\alpha\beta}(\phi_i) = \#T_i \), where
\[
T_1 = \{ t = 0, \ldots, a - 1 \mid a - \frac{1}{2} - t \geq x, y \geq -\frac{1}{2} - t \},
\]
\[
T_2 = \{ t = 0, \ldots, a \mid a - \frac{1}{2} - t \geq x, y \geq \frac{1}{2} - t \}.
\]
Clearly \( T_2 \cap [0, a - 1] \subseteq T_1 \). It remains to check the case that \( a \in T_2 \). In this case, we have
\[
-\frac{1}{2} \geq x, y \geq \frac{1}{2} - a.
\]
Then \( t = -\frac{1}{2} - y \) is in \( T_1 \) but not in \( T_2 \). Therefore, \( r_{\alpha\beta}(\phi_1) = r_{\alpha\beta}(\phi_2) \) in this case. This finishes the verification of this case.

This completes the proof of the theorem. \( \square \)

We remark that Theorem 1.9 (except the uniqueness in Part (2)) also follows directly from Theorem 1.12 and Corollary 3.12 (3).

4. Proof of Theorem 1.16

In this section, we prove Theorem 1.16 by exploring the relations between \( \phi_\pi \) and \( \psi_{\text{max}}(\pi) \). To this end, we prove a key property of \( \psi_{\text{max}}(\pi) \) (Proposition 4.3 below) which implies that the local \( L \)-parameter of \( \pi \) and \( \phi_{\psi_{\text{max}}(\pi)} \) share certain common direct summands.

We first prove a lemma which describes a restriction on the \( L \)-data of \( \pi(\mathcal{E}) \) in terms of an invariant of \( \psi_\mathcal{E} \).

**Lemma 4.1.** Suppose \( \mathcal{E} \in \text{Rep} \). Write \( \psi_\mathcal{E} = \bigoplus_i \rho_i \otimes S_{\alpha_i} \otimes S_{b_i} \),
\[
\pi(\mathcal{E}) = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_f}[x_f, -y_f]; \pi(\phi, \varepsilon)),
\]
and let \( b_\rho = \max\{b_i \mid \rho_i \cong \rho\} \). Then we have
\[
(4.1) \quad -b_\rho + 1 \leq \min (\{x_i - y_i \mid \rho_i \cong \rho \} \cup \{0\}).
\]

In particular, for any positive integer \( A, B \) such that \( B - A \leq -b_\rho + 1 \), the parabolic induction
\[
\sigma := \Delta_{\rho}[B, -A] \rtimes \pi(\mathcal{E})
\]
has a unique irreducible subrepresentation \( \sigma' \), whose \( L \)-data can be obtained from inserting \( \Delta_{\rho}[B, -A] \) in the front of the \( L \)-data of \( \pi(\mathcal{E}) \).

**Proof.** The second assertion follows from the inequality (4.1) and the Langlands classification. Now we show (4.1). By [HLL22, Theorem 3.19 (i)], we may assume \( \mathcal{E} \) is positive and satisfying (P') by replacing \( \mathcal{E} \) by \( \text{sh}^t(\mathcal{E}) \) for a sufficiently large \( t \) and do row exchanges if necessary. If \( b_\rho = 1 \), then the set \( \{x_i - y_i \mid \rho_i \cong \rho \} \) is empty, and hence the equality holds trivially. Therefore, we also assume \( b_\rho > 1 \) in the rest of the proof.

Suppose
\[
\pi = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_f}[x_f, -y_f]; \pi(\phi, \varepsilon)).
\]
We denote the multi-set
\[
\Delta_{\rho}(\pi) := \{\Delta_{\rho_i}[x_i, -y_i] \mid 1 \leq i \leq f, \rho_i \cong \rho\}.
\]
Write \( \mathcal{E} = \bigcup_{\rho} \{(A_i, B_i)_{\rho, i} \} \cup \{(I_{\rho}, >) \} \) and identify \( (I_{\rho}, >) \) with \( \{1, \ldots, n\} \) where \( 1 < \cdots < n \). To prove (4.1), we apply induction on \( n \). The case \( n = 1 \) follows from definition. Assume \( n > 1 \) from now on, and denote \( \mathcal{E}_k = \text{sh}_n^k(\mathcal{E}) \). We separate the proof into two cases: (1) \( b_\rho = A_n - B_n + 1 \); (2) \( b_\rho > A_n - B_n + 1 \).

**Case (1):** \( b_\rho = A_n - B_n + 1 \).
We may assume $l_n \geq 1$ since otherwise, we can split the $n$-th row by $ui^{-1}$ of type 3' (see [HLL22, Corollary 5.7]), then we reduce to the case that $b_\rho > A_n - B_n + 1$ or $b_\rho = 1$.

In this case, we claim that $\Delta_\rho(\pi(\mathcal{E}_k))$ is of the form

$$\Delta_\rho(\pi(\mathcal{E}_k)) = \{\Delta_\rho[B_n + k, -(A_n + k)], \Delta_\rho[x_1, -y_1], \ldots, \Delta_\rho[x_g, -y_g]\},$$

with $-b_\rho + 1 = (B_n + k) - (A_n + k) \leq x_i - y_i$ for all $i = 1, \ldots, g$. This implies (4.1) in this case.

Note that by the definition of $\pi(\mathcal{E}_k)$ and induction hypothesis for $|I_\rho| = n - 1$, we already know that $\Delta_\rho(\pi(\mathcal{E}_k))$ is of the form of (4.2) when $k$ is sufficiently large. Therefore, it suffices to show that if $\pi(\mathcal{E}_k)$ is of the form of (4.2), then so is $\pi(\mathcal{E}_{k-1})$ for $k \geq 1$.

Recall that we have the relation

$$\pi(\mathcal{E}_{k-1}) = D_{\rho|A_n+1} \circ \cdots \circ D_{\rho|B_n+k}(\pi(\mathcal{E}_k)),$$

and each derivative is the highest derivative (see [HLL22, Lemma 4.2]). We first keep track of the change of the segment $\Delta_\rho[B_n + k, -(A_n + k)]$ under the derivatives $D_{\rho|\alpha}$ for $\alpha = B_n + k, \ldots, A_n + k$. The first derivative $D_{\rho|B_n+k}$ has no choice but to replace $\Delta_\rho[B_n + k, -(A_n + k)]$ by $\Delta_\rho[B_n + k - 1, -(A_n + k)]$. Then applying Lemma 2.7 for $\alpha = B_n + k + 1, \ldots, A_n + k - 1$, the derivatives $D_{\rho|\alpha}$ leave the segment $\Delta_\rho[B_n + k - 1, -(A_n + k)]$ unchanged. Finally, for $\alpha = A_n + k$, notice that by our assumption that $\mathcal{E}$ satisfies (P') and $A_n - B_n + 1 = b_\rho$, we know that the multiplicity of $\rho \cdot [A_n+k]$ in $\Omega(\mathcal{E}_{k-1})$ (see [HLL22, Definition 4.1]) is zero. Then in order that $\pi(\mathcal{E}_{k-1})$ satisfies (4.1), we need to show that for any other segment $\Delta_\rho[x, -y]$ in $\Delta_\rho(\pi(\mathcal{E}_{k-1}))$, we have $(B_n + k - 1) - (A_n + k - 1) \leq x - y$. Applying Lemma 2.7 on each derivative in the relation (4.3), we see that $\Delta_\rho[x, -y] \in \mathcal{E}_{k-1}$ is one of the following forms.

(a) $\Delta_\rho[x_i - 1, -y_i]$ where $\Delta_\rho[x_i, -y_i] \in \Delta_\rho(\pi(\mathcal{E}_k))$,
(b) $\Delta_\rho[x_i, -y_i + 1]$ where $\Delta_\rho[x_i, -y_i] \in \Delta_\rho(\pi(\mathcal{E}_k))$,
(c) $\Delta_\rho[x_i - 1, -y_i + 1]$ where $\Delta_\rho[x_i, -y_i] \in \Delta_\rho(\pi(\mathcal{E}_k))$,
(d) $\Delta_\rho[\alpha - 1, -\alpha]$.

Since $\Delta_\rho(\pi(\mathcal{E}_k))$ is of the form (4.2), we have

$$B_n + k - 1 - (A_n + k - 1) = B_n + k - (A_n + k) \leq x_i - y_i,$$

so case (b) and (c) are done. Also, we have assumed $b_\rho > 1$, and hence

$$-b_\rho + 1 = B_n + k - 1 - (A_n + k - 1) \leq -1,$$

so case (d) is also done. It remains to argue that if $\Delta_\rho[x, -y] = \Delta_\rho[x_i - 1, -y_i]$, then

$$B_n + k - 1 - (A_n + k - 1) \leq x_i - y_i.$$

Suppose the contrary. Then we have

$$\begin{cases}
  x_i - y_i = B_n + k - (A_n + k) & \text{by (4.2) for } \pi(\mathcal{E}_k), \\
  B_n + k + 1 \leq x_i \leq A_n + k - 1 & \text{by Lemma 2.7,}
\end{cases}$$

and hence $y_i \geq A_n + k + 1$. Since the multiplicity of $\rho \cdot [A_n+k+1]$ in $\Omega(\mathcal{E}_{k-1})$ is zero, this contradicts to [HLL22, Lemma 4.7]. This completes the proof of the case that $b_\rho = A_n - B_n + 1$.

Case (2): $b_\rho > A_n - B_n + 1$. 
In this case, we claim that \( \Delta_\rho(\pi(E_k)) \) is of the form
\[
(4.4) \quad \Delta_\rho(\pi(E_k)) = \{ \Delta_\rho[x_1,-y_1], \ldots, \Delta_\rho[x_g,-y_g] \}
\]
where \(-b_\rho + 1 \leq x_i - y_i\) for all \(i = 1, \ldots, g\). Again (4.4) holds for \( \pi(E_k) \) when \( k \) is sufficiently large, and it suffices to show that if \( \Delta_\rho(\pi(E_k)) \) is of the form (4.4), then so is \( \Delta_\rho(\pi(E_{k-1})) \) for \( k \geq 1 \).

Suppose \( \Delta_\rho[x,-y] \in \Delta_\rho(\pi(E_{k-1})) \). Then similarly, it is of one of the following forms.
(a) \( \Delta_\rho[x_i-1,-y_i] \) where \( \Delta_\rho[x_i,-y_i] \) is in \( \Delta_\rho(\pi(E_k)) \).
(b) \( \Delta_\rho[x_i,-y_i+1] \) where \( \Delta_\rho[x_i,-y_i] \) is in \( \Delta_\rho(\pi(E_k)) \).
(c) \( \Delta_\rho[x_i-1,-y_i+1] \) where \( \Delta_\rho[x_i,-y_i] \) is in \( \Delta_\rho(\pi(E_k)) \).
(d) \( \Delta_\rho[a,-\alpha] \).

Again we only need to deal with case (a). In other words, it remains to show that if \( \Delta_\rho[x,-y] = \Delta_\rho[x_i-1,-y_i] \), then
\[-b_\rho + 1 \leq x_i - 1 - y_i.\]

Suppose the contrary. Then we have
\[
\begin{aligned}
 x_i - y_i &= -b_\rho + 1 \quad \text{by (4.4) for } \pi(E_k), \\
 B_n + k &\leq x_i - A_n + k \quad \text{by Lemma 2.7},
\end{aligned}
\]
and hence
\[
y_i \geq B_n + k + b_\rho - 1 > B_n + k + (A_n - B_n + 1) - 1 = A_n + k.
\]
On the other hand, [HLL22, Lemma 4.7] for \( E_{k-1} \) implies the existence of \( j \in I_\rho \) such that \( B_j \leq y_i \leq A_j \). Since we have assumed \( E \) satisfies \( (P') \), we have \( B_j \leq B_n < B_n + k \), and hence
\[
-b_\rho + 1 = x_i - y_i \geq (B_n + k) - A_j > B_j - A_j \geq -b_\rho + 1,
\]
a contradiction.

This completes the proof of the lemma. \( \square \)

The below lemma describes how to relate the \( L \)-data of \( \pi(E) \) with a related representation \( \pi(E^-) \), provided this representation does not vanish. Later, we see that the non-vanishing of \( \pi(E^-) \) is guaranteed when \( E \) is absolutely maximal (see Proposition 4.3).

**Lemma 4.2.** Suppose \( E = \cup_{\rho}\{([A_i,B_i],[l_i,\eta_i]) \}_{i \in (I_\rho, \rho)} \in \text{Rep}(P') \) and fix \( \rho \) such that \( I_\rho \neq \emptyset \). Suppose \( j \in I_\rho \) satisfies the following conditions.

- \( j = \max \{ i \in I_\rho \mid B_i = B_j \} \).
- \( \pi(E^-) \neq 0 \) where \( E^- := add_j^{-1}(E) \).
- Write
  \[ \pi(E^-) = L(\Delta_{\rho_1}[x_1,-y_1], \ldots, \Delta_{\rho_j}[x_j,-y_j]:\pi(\phi,\varepsilon)). \]

We have \( B_j - A_j \leq x_i - y_i \) if \( \rho_i \cong \rho \).

Then the \( L \)-data of \( \pi(E) \) can be obtained from that of \( \pi(E^-) \) by inserting \( \Delta_\rho[B_j,-A_j] \).

**Proof.** Our goal is to show that
\[
\pi(E) = L(\Delta_\rho[B_j,-A_j], \Delta_{\rho_1}[x_1,-y_1], \ldots, \Delta_{\rho_j}[x_j,-y_j]:\pi(\phi,\varepsilon)),
\]
which is equivalent to proving the injection
\[
(4.6) \quad \pi(E) \hookrightarrow \Delta_\rho[B_j,-A_j] \times \pi(E^-).
\]
Identify \((I_\rho, >)\) with \(\{1, \ldots, n\}\) where \(1 < \cdots < n\). Take a sequence of positive integers \(\{t_1, \ldots, t_n\}\) such that
\[
\begin{align*}
B_i + t_i &> 0 \quad \text{for all } i, \\
B_i + t_i &> A_k + t_k \quad \text{for all } i > k,
\end{align*}
\]
and let
\[
\mathcal{E}_k := \left( \sum_{i=k+1}^{n} sh_i^{t_i} \right) (\mathcal{E}), \quad \mathcal{E}_k^- := \left( \sum_{i=k+1}^{n} sh_i^{t_i} \right) (\mathcal{E}^-),
\]
and set \(\mathcal{E}_n = \mathcal{E}, \mathcal{E}_n^- = \mathcal{E}^-\).

To show (4.6), we prove
\[
\pi(\mathcal{E}_k) \hookrightarrow \Delta_\rho[B_j, -A_j] \times \pi(\mathcal{E}_k^-),
\]
for \(j \leq k \leq n\) by applying induction on \(k\).

First, we consider the case of \(k = j\). For \(\alpha = 0, \ldots, t_j\), denote \(\mathcal{E}_{j,\alpha} = sh_j^{-\alpha}(\mathcal{E}_j)\) and \(\mathcal{E}_{j,\alpha}^- = sh_j^{-\alpha}(\mathcal{E}_j^-)\). Then we apply induction on \(\alpha\) to show the injection
\[
\pi(\mathcal{E}_{j,\alpha}) \hookrightarrow \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha)] \times \pi(\mathcal{E}_{j,\alpha}^-).
\]
When \(\alpha = 0\), the injection follows from the definition of \(\pi(\mathcal{E}_j)\) and \(\pi(\mathcal{E}_j^-)\).

The representations \(\pi(\mathcal{E}_{j,\alpha})\) and \(\pi(\mathcal{E}_{j,\alpha-1})\) (resp. \(\pi(\mathcal{E}_{j,\alpha}^-)\) and \(\pi(\mathcal{E}_{j,\alpha-1}^-)\)) are related by the following formulas
\[
\begin{align*}
\pi(\mathcal{E}_{j,\alpha}) &= D_{\rho|} \Delta_{\rho_j + (t_j - \alpha) + 1} \circ D^- \circ D_{\rho_j^+ \rho_j^1 + (t_j - \alpha) + 1} \pi(\mathcal{E}_{j,\alpha-1}), \\
\pi(\mathcal{E}_{j,\alpha}^-) &= D^- \pi(\mathcal{E}_{j,\alpha-1}^-),
\end{align*}
\]
where \(D^- = D_{\rho_i^0 j_i} \rho_j^1 \cdots \rho_j^1 j_i + (t_j - \alpha) + 2\). Therefore, we may apply Lemma 2.3 as follows.

\[
\begin{align*}
\pi(\mathcal{E}_{j,\alpha-1}) &\hookrightarrow \Delta_\rho[B_j + t_j - \alpha + 1, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha-1}^-) \\
&\hookrightarrow \rho_i^1 \rho_j^0 \cdots \rho_j^1 \rho_j^0 \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)] \\
&\quad \times \rho_i^1 \rho_j^0 \cdots \rho_j^1 \rho_j^0 \Delta_\rho[B_j + t_j - \alpha + 1, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha}^-) \\
&= \rho_i^1 \rho_j^0 \cdots \rho_j^1 \rho_j^0 \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha}).
\end{align*}
\]
Then Lemma 4.1 implies that
\[
\sigma := \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha}^-)
\]
has a unique irreducible subrepresentation \(\sigma'\), whose \(L\)-data can be obtained by inserting \(\Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)]\) in the \(L\)-data of \(\pi(\mathcal{E}_{j,\alpha}^-)\). Applying the converse direction of Lemma 2.3, we have that
\[
D_{\rho_j^0 \rho_j^1 j_i} \cdots \rho_j^0 \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha}^-) \geq \sigma'.
\]
Note that each derivative on left hand side is highest (see [HLL22, Lemma 4.2] (iii)), and hence left hand side is irreducible, so the inequality is indeed an equality. Finally, applying the algorithm for positive derivative \(D_{\rho_j^0 \rho_j^1 j_i} \cdots \rho_j^0 \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha})\) on the \(L\)-data of \(\sigma'\), we get
\[
\pi(\mathcal{E}_{j,\alpha}) = D_{\rho_j^0 \rho_j^1 j_i} \cdots \rho_j^0 \Delta_\rho[B_j + t_j - \alpha, -(A_j + t_j - \alpha + 1)] \times \pi(\mathcal{E}_{j,\alpha}^-).
\]
This completes the induction step for (4.8), and the proof of (4.7) for \(k = j\).
Next, we proceed to prove (4.7) for \( k > j \). The representations \( \pi(\mathcal{E}_k) \) and \( \pi(\mathcal{E}_{k-1}) \) (resp. \( \pi(\mathcal{E}_k^-) \) and \( \pi(\mathcal{E}_{k-1}^-) \)) are related by

\[
\pi(\mathcal{E}_k) = c_{t_k}^{1-k} D_{\rho\{B_{k-1}\}}(\pi(\mathcal{E}_{k-1}^-)),
\]

\[
\pi(\mathcal{E}_k^-) = c_{t_k}^{1-k} D_{\rho\{B_{k-1}\}}(\pi(\mathcal{E}_{k-1}^-)).
\]

Therefore, we may apply Lemma 2.3 as follows.

\[
\pi(\mathcal{E}_{k-1}) \hookrightarrow \Delta_\rho[B_j, -A_j] \times \pi(\mathcal{E}_{k-1}^-)
\]

\[
\hookrightarrow \Delta_\rho[B_j, -A_j] \times \left( \rho \cdot |A_{k+1} \times \cdots \times \rho \cdot |B_{k+1} \times \cdots \times \rho \right) \times \pi(\mathcal{E}_{k-1}^-)
\]

\[
= \bigotimes_{s=0}^{t_k-1} \left( \rho \cdot |A_{k+1} \times \cdots \times \rho \cdot |B_{k+1} \times \cdots \times \rho \right) \times \Delta_\rho[B_j, -A_j] \times \pi(\mathcal{E}_{k-1}^-),
\]

where the last equality follows from the fact that any \( \rho \cdot |^x \) in the product commutes with \( \Delta_\rho[B_j, -A_j] \) (see [Tad14, Theorem 1.1] or [LM16, Corollary 6.10]), which is guaranteed by the assumption that \( j = \max\{i \in I_\rho \mid B_i = B_j\} \). Again, Lemma 4.1 implies that

\[
\sigma = \Delta_\rho[B_j, -A_j] \times \pi(\mathcal{E}_k^-)
\]

has a unique irreducible subrepresentation \( \sigma' \). Applying Lemma 2.3, we get

\[
\pi(\mathcal{E}_k) \geq \sigma' \hookrightarrow \Delta_\rho[B_j, -A_j] \times \pi(\mathcal{E}_k^-),
\]

where the inequality is indeed equality since \( \pi(\mathcal{E}_k) \) is irreducible. This completes the induction step for (4.7) and the proof of the lemma.

The following proposition shows that the hypotheses of Lemma 4.2 hold when \( \mathcal{E} \) is absolutely maximal. Note that the condition that \( \mathcal{E} \) is absolutely maximal is crucial for the following proposition. Indeed, if \( \mathcal{E} \) is not absolutely maximal, then Proposition 4.3 may fail (see Example 5.3 in the next section).

**Proposition 4.3.** Suppose \( \mathcal{E} = \cup_\rho\{([A_i, B_i], l_i, \eta_i)\}_{i \in I_\rho} \in \text{Rep}(P') \) is absolutely maximal, i.e., \( \psi_{\text{max}}(\pi) = \psi_{\mathcal{E}} \). Fix \( \rho \) such that \( I_\rho \neq \emptyset \) and \( b_\rho := \max\{A_i - B_i + 1 \mid i \in I_\rho\} > 1 \). Let \( j = \min\{i \in I_\rho \mid A_i - B_i + 1 = b_\rho\} \) and assume \( j = \max\{i \in I_\rho \mid B_i = B_j\} \) by applying row exchanges if necessary, and let \( \mathcal{E}^- := add_{\rho}^1(\mathcal{E}), \) which satisfies (P') by the assumptions. Then, \( \pi(\mathcal{E}^-) \neq 0 \) and the \( L \)-data of \( \pi(\mathcal{E}) \) can be obtained from that of \( \pi(\mathcal{E}^-) \) by inserting \( \Delta_\rho[B_j, -A_j] \).

**Proof.** If \( \pi(\mathcal{E}^-) \neq 0 \), then the first two conditions of Lemma 4.2 hold from assumptions. By applying Lemma 4.1 to \( \mathcal{E}^- \), the third condition also hold. Therefore, it remains to show \( \pi(\mathcal{E}^-) \neq 0 \).

We follow the notation in [HLL22, Section 3.2]. By [HLL22, Theorem 3.29], it is equivalent to show that \( \pi(\text{dual}(\mathcal{E}^-)) \) is nonzero. Now we check conditions (i) and (ii) in [HLL22, Theorem 3.19] for \( \text{dual}(\mathcal{E}^-) \), which is equal to \( sh_j^{-1}((\text{dual}(\mathcal{E}))) \) by Lemma 2.15 (1).

The assumption that \( \mathcal{E} \) is absolutely maximal shows that we can not split the \( j \)-th row of \( \mathcal{E} \) by \( ui \) inverse of type 3’ in the sense of [HLL22, Corollary 5.7], which is equivalent to \( l_j \geq 1 \). Then the dual formula ([HLL22, Definition 3.27]) shows that \( sh_j^{-1}((\text{dual}(\mathcal{E}))) \) satisfies Part (i) of [HLL22, Theorem 3.19].

Now we check Part (ii) of [HLL22, Theorem 3.19], i.e., any adjacent pair \((\alpha, \beta, \gg)\) of \( sh_j^{-1}((\text{dual}(\mathcal{E}))) \) satisfies [HLL22, Proposition 3.12(i)]. We say the adjacent pair \((\alpha, \beta, \gg)\) is good in this case for brevity.
Since
\[ A_j - B_j + 1 = \max\{A_i - B_i + 1 \mid i \in I_\rho\}, \]
we may apply row exchanges to lower the \(j\)-th row of \(\text{dual}(E)\) to the bottom, and hence it suffices to check the case that \(\alpha = j\).

If \((j, \beta, \gg)\) is also an adjacent pair of \(\text{dual}(E)\), then this adjacent pair of \(\text{sh}_{j}^{-1}(\text{dual}(E_\rho))\) is not good only if \(\text{ui}_{\beta,j} \) is applicable on \(\text{dual}(E)\), which implies the applicability of \(\text{dual} \circ \text{ui}_{\beta,j} \circ \text{dual} \) on \(E\) and contradicts to the absolute maximality of \(E\).

If \((j, \beta, \gg)\) is not an adjacent pair of \(\text{dual}(E)\), then we may assume \(j >' \beta + 1 >' \beta\) is adjacent, where \(>'\) is the admissible order of \(\text{dual}(E)\) on \(I_\rho\), and
\[ A_j - 1 = A_{\beta + 1} > A_\beta, \quad -B_j > -B_{\beta + 1} > -B_\beta. \]
The adjacent pair \((j, \beta + 1, >')\) of \(\text{sh}_{j}^{-1}(\text{dual}(E_\rho))\) is good since \(\text{ui}_{\beta + 1,j}\) is not applicable on \(\text{dual}(E)\) by the absolute maximality of \(E\). The adjacent pair \((\beta, \beta + 1, >')\) is good since \(\pi(\text{dual}(E)) \neq 0\). Then [HLL22, Proposition 3.12(ii)] implies that \((j, \beta, \gg)\) is also good. This completes the proof of the proposition.

As a corollary, the inequality in Lemma 4.1 is indeed an equality if \(E\) is absolutely maximal. We rephrase this as follows.

**Corollary 4.4.** Suppose \(\pi\) is of Arthur type with \(L\)-data
\[ \pi = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_f}[x_f, -y_f]; \pi(\phi, \varepsilon)). \]
Denote
\[ b_\rho = -\min\{(x_i - y_i \mid 1 \leq i \leq f, \rho_i \equiv \rho \cup \{0\}) + 1, \]
and write
\[ \psi^{\max}(\pi) = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{\alpha_i} \otimes S_{b_i}. \]
Then we have the equality
\[ b_\rho = \max\{b_i \mid i \in I_\rho\}. \]

We remark that the equality in above corollary does not hold for arbitrary \(\psi \in \Psi(\pi)\). See Example 5.3 in the next section.

Now we prove Theorem 1.16.

**Proof of Theorem 1.16.** By Corollary 3.3, we may assume \(\pi\) is of good parity. Write \(\pi = \pi(E)\) where \(E\) is absolutely maximal. Our goal is to show
\[ (4.9) \quad \phi_\pi \geq c \phi_{\psi}. \]
We apply induction on \(n\), the rank of \(G_n\).
Write \(E = \cup_\rho\{(A_i, B_i, l, \eta_i)\}_{i \in I_\rho, >}\) and consider the quantities
\[ b_\rho(E) := \max_{i \in I_\rho} \{A_i - B_i + 1\}. \]
If \(b_\rho(E) \leq 2\) for every \(\rho\), then \(E\), which is absolutely maximal by assumption, automatically satisfies the conditions in [HLL22, Theorem 10.4]. Therefore, we have \(\phi_\pi = \phi_{\psi}\), so (4.9) trivially holds. Note that when \(n = 1\), we must have \(b_\rho(E) \leq 2\), and hence (4.9) holds.

Suppose \(n > 1\) and \(b_\rho(E) > 2\) for some \(\rho\). We take \(j \in I_\rho\) and construct \(E^- = \text{add}_j^{-1}(E)\) as in Proposition 4.3. Since \(\pi(E^-) \neq 0\) is a representation of \(G_m\) where \(m < n\), the induction hypothesis and Theorem 1.12 imply that
\[ \phi_{\pi(E^-)} \geq c \phi_{\psi^{\max}(\pi(E^-))} \geq c \phi_{\psi}. \]
On the other hand, Proposition 4.3 gives that
\[
\phi_{\pi(\mathcal{E})} = \left( \rho \mid \frac{B_i - A_i}{2} \otimes S_{B_j + A_j + 1} \oplus \rho \mid - \frac{B_j - A_j}{2} \otimes S_{B_j + A_j + 1} \right) \oplus \phi_{\pi(\mathcal{E}^-)},
\]
\[
\phi_{\psi_{\mathcal{E}}} = \left( \rho \mid \frac{B_i - A_i}{2} \otimes S_{B_j + A_j + 1} \oplus \rho \mid - \frac{B_j - A_j}{2} \otimes S_{B_j + A_j + 1} \right) \oplus \phi_{\psi_{\mathcal{E}^-}}.
\]
Then applying Lemma 3.2, we get \( \phi_{\pi(\mathcal{E})} \geq_C \phi_{\psi_{\mathcal{E}}} \). This completes the proof of the theorem.

\section{New Algorithms for Determining Representation is of Arthur Type}

In this section, we extend the result in Proposition 4.3 and give new algorithms to determine whether a representation is of Arthur type. Comparing to the algorithms given in [Ato22, Algorithm 3.3] and [HLL22, Algorithm 7.9], the new algorithms do not need the computation of highest derivatives and the construction of non-tempered local Arthur packets, hence they have their own interests and theoretical importance.

**Definition 5.1.** Suppose \( \pi \in \Pi(G_n) \). Write
\[
\pi = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_f}[x_f, -y_f]; \pi(\phi, \varepsilon)).
\]

(1) We define \( \pi^{\rho_{-i}} \) to be the representation such that whose \( L \)-data is obtained by removing all copies of \( \Delta_{\rho}[x, -y] \) from the that of \( \pi \) such that
\[
x = \min \{ x_i \mid x_i - y_i = \min \{ x_j - y_j \mid \rho_j \cong \rho \} \},
\]
\[
y = x - \min \{ x_j - y_j \mid \rho_j \cong \rho \}.
\]

(2) We define \( \pi_{\rho_{-i}} \) to be the representation whose \( L \)-data is obtained by removing all \( \Delta_{\rho}[x_i, -y_i] \)'s from the that of \( \pi \), where \( \rho_i \cong \rho \) and \( x_i = \min \{ x_j \mid \rho_j \cong \rho \} \).

We determine whether \( \pi \) is of Arthur type using two different methods. The first method makes use of a reduction from \( \pi \) to \( \pi^{\rho_{-i}} \) (see Algorithm 5.12 below), while the second one makes use of a reduction from \( \pi \) to \( \pi_{\rho_{-i}} \) (see Algorithm 5.22 below).

\subsection{Reduction along \( \pi^{\rho_{-i}} \).}

In this subsection, we give an algorithm to determine whether \( \pi \) is of Arthur type from the information of \( \Psi(\pi^{\rho_{-i}}) \).

First, applying Proposition 4.3, we give a new algorithm to compute the \( L \)-data of \( \pi(\mathcal{E}) \). We can compute this \( L \)-data from the definition of \( \pi(\mathcal{E}) \) from [Ato20, §3.2, Theorem 3.7] combined with the algorithm for positive highest derivatives using [AM20, Theorem 7.1]. Atobe gives another algorithm for computing the \( L \)-data of \( \pi(\mathcal{E}) \) in the discussion following [Ato20, Theorem 1.5], which requires the algorithm for positive socles. In contrast, the new algorithm below does not need the computation of highest derivatives or socles, but uses the computation of \( \mathcal{E}^{[\max]} \) instead.

**Algorithm 5.2.** Given an extended multi-segment \( \mathcal{E} \in \overline{\text{Rep}} \), proceed as follows.

\textbf{Step 1:} Compute \( \mathcal{E}^{[\max]} = \cup_{\rho} \{(A_i, B_i, \rho, I, \eta_i) \}_{i \in \mathcal{I}(\rho, \eta)} \). If \( \mathcal{E}^{[\max]} \) satisfies \( (L) \) (see [HLL22, Definition 10.1]), then the \( L \)-data of \( \pi(\mathcal{E}) \) is given by [HLL22, Proposition 10.3]. Note that the condition that \( \mathcal{E} \) is positive is not necessary.

\textbf{Step 2:} If \( \mathcal{E}^{[\max]} \) does not satisfy \( (L) \), then choose \( \rho \) such that \( \mathcal{E}_0 \) (see [HLL22, Definition 3.8]) does not satisfy \( (L) \). Then construct \( \mathcal{E}^{[\max]}_\rho = \text{add}_j^{-1}(\mathcal{E}^{[\max]}_\rho) \) as in Proposition 4.3. Repeating the algorithm for \( \mathcal{E}^{[\max]}_\rho \), which is of smaller rank than \( \mathcal{E} \), we get the \( L \)-data of \( \pi((\mathcal{E}^{[\max]}_\rho)) \) and write
\[
\pi((\mathcal{E}^{[\max]}_\rho)) = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_f}[x_f, y_f]; \pi(\phi, \varepsilon)).
\]
Then
\[ \pi(\mathcal{E}) = L(\Delta_{\rho}[B_j, A_j], \Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_f}[x_f, y_f]; \pi(\phi, \varepsilon)) \] 

We remark that the repeating of Step 2 results in an extended multi-segment \( \mathcal{E}' \) such that \( \psi_{\mathcal{E}'} \) is tempered. Then \( \mathcal{E}' \) must satisfy (L) and hence the procedure terminates. Here is an example for the algorithm.

**Example 5.3.** To an extended multi-segment \( \mathcal{E} \), Atobe associated a symbol to \( \mathcal{E} \) in [Ato20, §3.1], which we use below. Let \( \rho \) be the trivial representation. Consider the following extended multi-segment of \( \text{Sp}_{10}(F) \)

\[ \mathcal{E} = \begin{pmatrix} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{pmatrix} = \mathcal{E}^{\text{max}}. \]

We have computed in [HLL22, Example 11.10.6] that

\[ \pi = L(\Delta_{\rho}[-3, -3], \Delta_{\rho}[-1, -2], \Delta_{\rho}[0, -1]; \pi(0^+)). \]

Applying Algorithm 5.2, we have

\[ \mathcal{E}_1 = \text{add}_1^{-1}(\mathcal{E}) = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{pmatrix}, \quad \mathcal{E}_1^{\text{max}} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{pmatrix}, \]

\[ \mathcal{E}_2 = \text{add}_1^{-1}(\mathcal{E}_1^{\text{max}}) = \begin{pmatrix} 0 & 1 \\ \circ & \circ \\ \circ & \circ \end{pmatrix}, \quad \mathcal{E}_2^{\text{max}} = \begin{pmatrix} 0 & 1 \\ \circ & \circ \\ \circ & \circ \end{pmatrix}, \]

\[ \mathcal{E}_3 = \text{add}_2^{-1}(\mathcal{E}_2^{\text{max}}) = \begin{pmatrix} 0 \\ \circ \\ \circ \end{pmatrix}. \]

Note that we perform a row exchange from \( \mathcal{E}_2 \) to \( \mathcal{E}_2' \) to satisfy the condition in the statement of Proposition 4.3. Then we have

\[ \pi(\mathcal{E}_3) = \pi(0^+), \]

\[ \pi(\mathcal{E}_2) = L(\Delta_{\rho}[0, -1]; \pi(0^+)), \]

\[ \pi(\mathcal{E}_1) = L(\Delta_{\rho}[-1, -2], \Delta_{\rho}[0, -1]; \pi(0^+)), \]

\[ \pi(\mathcal{E}) = L(\Delta_{\rho}[-3, -3], \Delta_{\rho}[-1, -2], \Delta_{\rho}[0, -1]; \pi(0^+)). \]

In this example, we also see that \( \psi_{\mathcal{E}_1} \in \Psi(\pi(\mathcal{E}_1)) \) does not satisfy the conclusion of Corollary 4.4 and Theorem 5.4 below since \( \psi_{\mathcal{E}_1} \neq \psi^{\text{max}}(\pi(\mathcal{E}_1)) \). Also, \( \pi(\text{add}_1^{-1}(\mathcal{E}_1)) \) vanishes, which demonstrates that Proposition 4.3 does not hold if we drop the absolute maximality condition.

In the example above, we observe that \( \pi = \pi(\mathcal{E}) \) is of Arthur type since \( \pi^{\rho_\tau} = \pi(\mathcal{E}_1) \) is of Arthur type, and there exists an \( \mathcal{E}_1 \) in the set \{ \( \mathcal{E}' \mid \pi(\mathcal{E}') = \pi^{\rho_\tau} \} \) which contains an extended segment of the form \([2, -2]_\rho, l, \eta \), so that we can perform \( \text{add}_3 \) to change it to \([3, -3]_\rho, l+1, \eta \) to recover \( \Delta_{\rho}[-3, -3] \) in the \( L \)-data of \( \pi \). That is, we can recover \( \mathcal{E} \) from the pair \( (\mathcal{E}_1, \Delta_{\rho}[-3, -3]) \). This motivates the new algorithm (Algorithm 5.12 below) to determine whether \( \pi \) is of Arthur type.

To introduce the new algorithm, we first establish a generalization of Corollary 4.4.
Theorem 5.4. Suppose $\pi \in \Pi_A(G_n)$ is of good parity. Write
\[ \pi = L(\Delta_\rho, x, y, \Delta_{\rho_1}, \ldots, \Delta_{\rho_r}; \pi(\phi, \varepsilon)), \]
where $x_i - y_i > x - y$ or $x_i > x$ if $\rho_i \equiv \rho$. Then $\psi^{\text{max}}(\pi)$ contains exactly $r$ copies of $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$. Moreover,
\[ \pi^{\rho^*} = L(\Delta_{\rho_1}, \ldots, \Delta_{\rho_r}; \pi(\phi, \varepsilon)) \]
is of Arthur type with $\pi^{\rho^*} \in \Pi_{\psi^-}$, where
\[ \psi^- := \psi^{\text{max}}(\pi) - (\rho \otimes S_{y+x+1} \otimes S_{y-x+1})^{\oplus r} + (\rho \otimes S_{y+x+1} \otimes S_{y-x+1})^{\oplus r}. \]

Again, we remark that the conclusions in above theorem do not hold for arbitrary $\psi \in \Psi(\pi)$. See Example 5.3 above.

Before proving this theorem, we first show that $\psi^{\text{max}}(\pi)$ contains no more than $r$ copies of $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$, which generalizes Proposition 4.3 as follows.

Lemma 5.5. Suppose $\mathcal{E} = \cup \rho \{([A_i, B_i \rho, l, \eta_i]) i \in (I_\rho) \in \mathcal{R}_a \}$ is absolutely maximal, and the admissible order on each $I_\rho$ satisfies that $A_i \geq A_j$ if $i > j$ and $B_i = B_j$. Fix $\rho$ such that $I_\rho \neq \emptyset$ and $b_\rho := \max(A_i - B_i + 1 | i \in I_\rho) > 1$. Take $j_1 = \min \{i \in I_\rho | A_i - B_i + 1 = b_\rho \}$ and denote
\[ \{j \in I_\rho | [A_j, B_j]_\rho = [A_{j_1}, B_{j_1}]_\rho \} = \{j_1, \ldots, j_r\}, \]
where $j_1 < \cdots < j_r$. Define $\mathcal{E}^- := \sum_{k=1}^r \text{add}_{j_k}^{-1}(\mathcal{E})$, which satisfies (P') by the assumptions. Then
(a) $\pi(\mathcal{E}^-) \neq \emptyset$.
(b) The $L$-data of $\pi(\mathcal{E}^-)$ can be obtained from that of $\pi(\mathcal{E})$ by removing $r$ copies of $\Delta_\rho[B_{j_1}, -A_{j_1}]$.

Proof. Following the same proof of Proposition 4.3, one can show that for $1 \leq s \leq r$, the extended multi-segment $\sum_{k=s}^r \text{add}_{j_k}^{-1}(\mathcal{E})$ gives nonzero representation. This proves (a).

For (b), Lemma 4.2 shows that for $1 \leq s \leq r$, the $L$-data of $\pi(\mathcal{E}_s)$ can be obtained from $\pi(\mathcal{E}_{s+1})$ by removing a copy of $\Delta_\rho[B_{j_1}, -A_j]$, where we set $\mathcal{E}_{r+1} := \mathcal{E}$. This completes the proof of the lemma.

Next, we show that $\psi^{\text{max}}(\pi)$ contains at least one copy of $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$.

Lemma 5.6. Suppose $\pi$ is a representation of $G_n$ of Arthur type and of good parity. If
\[ \pi = L(\Delta_\rho, x, y, \Delta_{\rho_1}, \ldots, \Delta_{\rho_r}; \pi(\phi, \varepsilon)), \]
then $\psi^{\text{max}}(\pi)$ contains at least one copy of $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$.

Proof. We prove by contradiction. Suppose $\psi^{\text{max}}(\pi)$ does not contain $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$. Write $\pi = \pi(\mathcal{E})$ where $\mathcal{E}$ is absolutely maximal and construct $\mathcal{E}^-$ as in Lemma 5.5. A key observation is that by the definition of each raising operator, $\psi^{\text{max}}(\pi(\mathcal{E}^-))$ also does not contain $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$, and $\Delta_\rho[x, -y]$ is still in the $L$-data of $\pi(\mathcal{E}^-)$. Repeat several times, we get a contradiction to Lemma 4.1.

Now we are ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let $\pi = \pi(\mathcal{E})$ where $\mathcal{E}$ is absolutely maximal and construct $\mathcal{E}^-$ as in Lemma 5.5. Then Lemma 5.6 implies that in the notation of Lemma 5.5, we have $A_{j_1} = y, B_{j_1} = x$, and hence $\psi^{\text{max}}(\pi)$ contains $s$ copies of $\rho \otimes S_{y+x+1} \otimes S_{y-x+1}$, where $1 \leq s \leq r$. However, if $r > s$, applying Lemma 5.6 on $\pi(\mathcal{E}^-)$ yields a contradiction. Therefore, $r = s$, and $\pi^{\rho^*} = \pi(\mathcal{E}^-), \psi^- = \psi_{\mathcal{E}^-}$. This completes the proof of the
Suppose that we are in the setting of Theorem 5.4 and let $\pi = \pi(\mathcal{E})$ where $\mathcal{E}$ is absolutely maximal. Lemma 5.5 constructs $\mathcal{E}^-$ from $\mathcal{E}$ such that $\pi(\mathcal{E}^-) = \pi^{0-}$. Next, we would like to find the inverse image of the map $\mathcal{E} \mapsto \mathcal{E}^-$. More precisely, given $\mathcal{E}$, we would like to construct an $\mathcal{E}^+$ such that $(\mathcal{E}^+)^- = \mathcal{E}$.

We first describe a special case.

**Lemma 5.7.** Suppose $\mathcal{E} = \cup_{\rho} \{([A_i,B_i]_{\rho}, l_i, \eta_i)\}_{i \in (I_{\rho}, \succ)} \in \text{Rep}^{(P')}$. We further assume for a fixed $\rho$, $b_{\rho} := \max\{A_i - B_i + 1 \mid i \in I_{\rho}\} \leq 2$. Let $r \in \mathbb{Z}_{\geq 0}$ and $x \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that $x - A_i \in \mathbb{Z}$ for any $i \in I_{\rho}$. We construct $\mathcal{E}^+$ by inserting $r$ copies of $([x,x-1]_{\rho}, 1, 1)$ in $\mathcal{E}$ with admissible order $\triangleright$ on $I_{\rho} \cup \{j_1, \ldots, j_r\}$, where $j_k$ corresponds to the $k$-th copy we inserted, given as follows:

$$\begin{cases} j_r \succ j_{r-1} \succ \cdots \succ j_1, \\ \alpha \succ \beta \iff \alpha \succ \beta \quad \text{for } \alpha, \beta \in I_{\rho}, \\ \alpha \succ j_k \iff B_\alpha > x - 1 \quad \text{for } \alpha \in I_{\rho}. \end{cases}$$

Then if $\psi_{\mathcal{E}}$ does not contain $\rho \circ S_2 \circ S_2$ and $A_i = B_i$ for all $i \in I_{\rho}$ such that $B_i \leq x - 2$, we have $\mathcal{E}^+ \in \text{Rep}^{(P')}$, and the $L$-data of $\pi(\mathcal{E}^+)$ can be obtained from that of $\pi(\mathcal{E})$ by inserting $r$ copies of $([x,x-1]_{\rho}, 1, 1)$, and applying Lemma 5.5 on $(\mathcal{E}^+)^{\max}$ gives the conclusion. This completes the proof of the lemma. □

Next, we need the following generalization of [HLL22, Lemma 4.3 (iii)] for the non-vanishing issue in the remaining cases.

**Lemma 5.8.** Let $\mathcal{E} = \cup_{\rho} \{([A_i,B_i]_{\rho}, l_i, \eta_i)\}_{i \in (I_{\rho}, \succ)} \in \text{Rep}^{(P')}$. Suppose there exists $j_1, \ldots, j_r \in I_{\rho}$ such that

- $j_1 < \cdots < j_r$ are adjacent under the admissible order on $I_{\rho},$
- $[A_{j_1},B_{j_1}]_{\rho} = \cdots = [A_{j_r},B_{j_r}]_{\rho},$
- $B_{j_r} \prec B_i$ and $A_{j_r} \geq A_i$ for all $i \geq j_r$.

Then $\mathcal{E}_r := \sum_{k=1}^r sh_{j_k}^{\rho} (\mathcal{E}) \in \text{Rep}^{(P')}$. "

Proof. Since $\pi(\mathcal{E}) \neq 0$, [HLL22, Theorem 3.19] implies that $\pi(\mathcal{E}_r) \neq 0$ if and only if $\pi(sh^d(\mathcal{E}_r)) \neq 0$ for any positive integer $d$. Therefore, we may assume $B_{j_1} > 0$.

We prove the lemma by induction on $r$. The case that $r = 1$ is exactly [HLL22, Lemma 4.3 (iii)]. To prove the general case, we use the notation developed in [HLL22, §3.1]. Denote $A := A_{j_1}, B := B_{j_r}$, and write

$$\mathcal{E}_\rho = \mathcal{E}_{\rho,1} + \{([A,B]_{\rho}, *, *)\} + \{([A,B]_{\rho}, *, *)^{r-1}\} + \mathcal{E}_{\rho,2},$$

where the exponent $r - 1$ means $r - 1$ copies. The assumption implies that we may apply row exchanges on $\mathcal{E}_\rho$ to get

$$\mathcal{E}_\rho' = \mathcal{E}_{\rho,1} + \{([A,B]_{\rho}, *, *)\} + \mathcal{E}_{\rho,2}' + \{([A,B]_{\rho}, *, *)^{r-1}\}.$$
satisfies the non-vanishing criterion in [HLL22, Theorem 3.19 (ii)]. Also, the induction hypothesis indicates that 
\[ \mathcal{E}_{\rho,1} + \{([A,B]_{\rho}, \ast, \ast)\} + \mathcal{E}'_{\rho,2} + \{([A+1,B+1]_{\rho}, \ast, \ast)^{r-1}\} \]
satisfies the non-vanishing criterion as well. Therefore, one can see that 
\[ \mathcal{E}_{\rho,1} + \{([A+1,B+1]_{\rho}, \ast, \ast)\} + \mathcal{E}'_{\rho,2} + \{([A+1,B+1]_{\rho}, \ast, \ast)^{r-1}\} \]
also satisfies the non-vanishing criterion. By Lemma 2.15 (1), we conclude that \( \pi(\mathcal{E}_r) \neq 0 \). This completes the proof of the lemma.

Now we deal with the remaining cases.

**Lemma 5.9.** Let \( \mathcal{E} = \cup_{\rho}\{([A_i,B_i]_{\rho}, l_i, \eta_i)\}_{i \in (I,\succ)} \in \overline{\text{Rep}(P)} \). Suppose there exist \( j_1, \ldots, j_r \in I_\rho \) such that

(i) \( j_1 < \cdots < j_r \) are adjacent under the admissible order on \( I_\rho \),
(ii) \( [A_{j_1}, B_{j_1}]_{\rho} = \cdots = [A_{j_r}, B_{j_r}]_{\rho} \),
(iii) \( j_1 = \min\{i \in I_\rho \mid B_i = B_{j_1}\} \),
(iv) \( A_{j_i} - B_{j_i} + 3 \geq A_i - B_i + 1 \) for all \( i \in I_\rho \) and the equality does not hold for \( i < j_1 \).

Then \( \mathcal{E}^+ := \sum_{k=1}^r \text{add}_{j_k}(\mathcal{E}) \in \overline{\text{Rep}(P)} \), and the L-data of \( \pi(\mathcal{E}^+) \) can be obtained from that of \( \pi(\mathcal{E}) \) by inserting \( r \) copies of \( \Delta_{\rho}[B_{j_1} - 1, -A_{j_1} - 1] \).

**Proof.** Conditions (i), (ii), (iii) imply that the admissible order of \( \mathcal{E}^+ \) satisfies (P'). Then \( \pi(\mathcal{E}^+) \neq 0 \) if and only if \( \pi(\text{dual}(\mathcal{E}^+)) \neq 0 \). Note that 
\[ \text{dual}(\mathcal{E}^+) = \sum_{k=1}^r sh_{j_k}^1(\text{dual}(\mathcal{E})) \]
by Lemma 2.15(i).

Denote \( \succ' \) the admissible order on \( \text{dual}(\mathcal{E}) \). Condition (iii) and (iv) imply that for all \( i \succ' j_1 \), we have \( -B_i > -B_{j_1} \) and \( A_{j_1} \geq A_i \). Therefore, \( \pi(\sum_{k=1}^r sh_{j_k}^1(\text{dual}(\mathcal{E}))) \neq 0 \) by Lemma 5.8, and hence \( \pi(\mathcal{E}^+) \neq 0 \). The second part follows from the same argument for the second part of Lemma 5.7. This completes the proof of the lemma.

The following definition provides the notation for Theorem 5.11 and Algorithm 5.12.

**Definition 5.10.** Suppose \( \pi \) is a representation for \( G_n \) of good parity. Write 
\[ \pi = L(\Delta_\rho[x,-y], \ldots, \Delta_\rho[x,-y], \Delta_\rho_1[x_1,-y_1], \ldots, \Delta_\rho_f[x_f,-y_f]; \pi(\phi, \varepsilon)), \]
where \( x_i - y_i > x - y \) or \( x_i > x \) if \( \rho_i \cong \rho \).

1. We denote by \( \Psi(\pi^{\rho,-}; \Delta_\rho[x,-y], r) \) the set of local Arthur parameters \( \psi \) such that
   - \( \pi^{\rho,-} \in \Pi_\psi \),
   - If \( y - x - 1 > 0 \), then \( \psi \) contains \( r \) copies of \( \rho \otimes S_{x+y+1} \otimes S_{y-x-1} \),
   - Any summand of \( \psi \) of the form \( \rho \otimes S_a \otimes S_b \) satisfies \( b \leq y - x + 1 \), and \( a > x + y + 1 \) if \( b = y - x + 1 \).

2. For any \( \mathcal{E} \in \text{Rep}(P) \) such that \( \pi(\mathcal{E}) = \pi^{\rho,-} \) and \( \psi_\mathcal{E} \in \Psi(\pi^{\rho,-}; \Delta_\rho[x,-y], r) \), we define \( \mathcal{E}^+ \) as in Lemma 5.7 where we inserted \( r \) copies of \( ([y,x]_{\rho}, 1,1) \) if \( y - x - 1 = 0 \). If \( y - x - 1 > 0 \), we define \( \mathcal{E}^+ \) as in Lemma 5.9 with \( [A_{j_1}, B_{j_1}]_{\rho} = [y-1, x+1]_{\rho} \).
Now we state the main theorem of this section, which gives a criterion whether \( \pi \in \Pi(G_n) \) is of Arthur type in terms of conditions on \( \Psi(\pi^{\rho,-}) \), where \( \pi^{\rho,-} \in \Pi(G_m) \) for some \( m < n \).

**Theorem 5.11.** If \( \pi \) is a representation of \( G_n \) of good parity, then \( \pi \) is of Arthur type if and only if \( \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \) is non-empty. Moreover, for any \( \mathcal{E} \in \text{Rep}^{(P')} \) such that \( \pi(\mathcal{E}) = \pi^{\rho,-} \) and \( \psi \in \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \), we have \( \pi = \pi(\mathcal{E}+) \). In other words, for any \( \psi \in \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \), we have \( \psi^+ \in \Psi(\pi) \).

**Proof.** If \( \pi \) is of Arthur type, then \( \pi^{\rho,-} \in \Pi(\rho^-) \) in the notation of Theorem 5.4, where \( \psi^+ \in \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \).

Conversely, if \( \mathcal{E} \in \text{Rep}^{(P')} \) satisfies that \( \pi(\mathcal{E}) = \pi^{\rho,-} \) and \( \psi \in \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \), then \( \pi(\mathcal{E}+) \neq 0 \) and \( \pi(\mathcal{E}^+) )^{\rho,-} = \pi^{\rho,-} \) by Lemmas 5.7, and 5.9. Therefore, \( \pi = \pi(\mathcal{E}^+) \) is of Arthur type. Note that \( (\psi^+)^+ = \psi_{\mathcal{E}+} \). This completes the proof of the theorem. \( \square \)

Based on the above theorem, we give a new algorithm to determine whether a representation of good parity is of Arthur type. If the representation \( \pi \) is of Arthur type, then it outputs the set \( \{ \mathcal{E} \mid \pi(\mathcal{E}) = \pi \} / \text{(row exchanges)} \).

**Algorithm 5.12.** Given a representation \( \pi \) of \( G_n \) of good parity, proceed as follows:

**Step 1:** If \( \pi \) is tempered, then \( \pi \) is of Arthur type. Write \( \pi = (\phi, \varepsilon) \), where

\[
\phi = \bigoplus_{\rho \in I_\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{2a_i+1},
\]

where we equip \( I_\rho \) with a total order \( > \) such that \( a_i \) is non-decreasing. Then let

\[
\mathcal{E} = \cup_{\rho} \{ ([a_i, a_i], 0, \varepsilon(\rho, S_{2a_i+1})) \}_{i \in (I_\rho, >)};
\]

and output \( \Psi(\mathcal{E}) \) given by [HLL22, Theorem 7.4].

We regard the trivial representation of \( G_0 \) as a tempered representation, and output \( \{ \mathcal{E} \} \), where \( \mathcal{E} = \emptyset \).

**Step 2:** If \( \pi \) is not tempered, write

\[
\pi = L(\Delta_\rho[x, -y], \ldots, \Delta_\rho[x, -y], \Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_f}[x_f, -y_f]; \pi(\phi, \varepsilon)),
\]

where \( x_i - y_i > x - y \) or \( x_i > x \) if \( \rho_i \cong \rho \), and then apply the algorithm on \( \pi^{\rho,-} \) to see whether it is of Arthur type.

If \( \pi^{\rho,-} \) is not of Arthur type, then neither is \( \pi \) and the procedure ends.

If \( \pi^{\rho,-} \) is of Arthur type, then \( \pi \) is of Arthur type if and only if there exists an \( \mathcal{E} \in \{ \mathcal{E}' \mid \pi(\mathcal{E}') = \pi^{\rho,-} \} \) such that \( \psi \in \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \) (see Definition 5.10). Then, we output \( \{ \mathcal{E}' \mid \pi(\mathcal{E}') = \pi(\mathcal{E}^+) \} \) / (row exchanges) given by [HLL22, Theorem 7.4].

**Remark 5.13.** (1) Note that repeatedly taking \( \pi^{\rho,-} \) results in a tempered representation, so the algorithm terminates.

(2) In comparison with [Ato22, Algorithm 3.3] and [HLL22, Algorithm 7.9], the new algorithm does not require the construction of a possibly non-tempered local Arthur packet, and it does not rely on the computation of highest derivatives.

At the end of this subsection, we apply the new algorithm to [HLL22, Example 7.11]. We omit the computation of \( \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r) \), but only list an element \( \psi \) in this set when it is non-empty. Again, we refer to [Ato20, §3.1] for the symbol associated to an extended multi-segment \( \mathcal{E} \).
Example 5.14. 1. Consider

$$\pi = L \left( \Delta_\rho \left[ \frac{-1}{2}, \frac{-5}{2} \right], \Delta_\rho \left[ \frac{-1}{2}, \frac{-1}{2} \right], \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right]; \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2} \right) \right).$$

Let

$$\pi_1 := L \left( \Delta_\rho \left[ \frac{-1}{2}, \frac{-1}{2} \right], \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right]; \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2} \right) \right),$$

$$\pi_2 := L \left( \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right]; \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2} \right) \right),$$

$$\pi_3 := \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2} \right).$$

$\pi_3$ is tempered, and hence it is of Arthur type. Next, we have $\pi_3 = \pi(\mathcal{E}_3)$, $\psi_{\mathcal{E}_3} \in \Psi(\pi_3; \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right], 1)$, and $\pi_2 = \pi(\mathcal{E}_3^+)$, where

$$\mathcal{E}_3 = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \oplus & \oplus & \oplus \end{array} \right), \quad \mathcal{E}_3^+ = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \downarrow & \downarrow & \downarrow \end{array} \right).$$

Thus $\pi_2$ is of Arthur type. Similarly, we have $\pi_2 = \pi(\mathcal{E}_2)$, $\psi_{\mathcal{E}_2} \in \Psi(\pi_2; \Delta_\rho \left[ \frac{-1}{2}, \frac{1}{2} \right], 1)$, and $\pi_1 = \pi(\mathcal{E}_2^+)$, where

$$\mathcal{E}_2 = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \oplus & \oplus & \oplus \end{array} \right), \quad \mathcal{E}_2^+ = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \downarrow & \downarrow & \downarrow \end{array} \right).$$

Thus $\pi_1$ is of Arthur type. Finally, applying [HLL22, Theorem 7.4], the set $\{ \mathcal{E} \mid \pi(\mathcal{E}) = \pi(\mathcal{E}_2^+) = \pi_1 \}$/(row exchanges) consists of four extended multi-segments

$$\left( \begin{array}{ccc} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \oplus & \oplus & \oplus \end{array} \right), \quad \left( \begin{array}{ccc} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \oplus & \oplus & \oplus \end{array} \right), \quad \left( \begin{array}{ccc} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \oplus & \oplus & \oplus \end{array} \right), \quad \left( \begin{array}{ccc} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \oplus & \oplus & \oplus \end{array} \right).$$

Thus, $\Psi(\pi_1; \Delta_\rho \left[ \frac{-1}{2}, \frac{-5}{2} \right], 1)$ is empty, and we conclude that $\pi$ is not of Arthur type.

2. Consider

$$\pi = L \left( \Delta_\rho \left[ \frac{-1}{2}, \frac{-5}{2} \right], \Delta_\rho \left[ \frac{-1}{2}, \frac{-1}{2} \right], \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right]; \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^+}{2} \right) \right).$$

Let

$$\pi_1 := L \left( \Delta_\rho \left[ \frac{-1}{2}, \frac{-1}{2} \right], \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right]; \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^-}{2} \right) \right),$$

$$\pi_2 := L \left( \Delta_\rho \left[ \frac{3}{2}, \frac{-5}{2} \right]; \pi \left( \frac{1^+}{2}, \frac{3^+}{2}, \frac{5^-}{2} \right) \right),$$

$$\pi_3 := \pi \left( \frac{1^-}{2}, \frac{3^+}{2}, \frac{5^-}{2} \right).$$
Following the same computation in the previous example, we get $\pi_1$ is of Arthur type and $\pi_1 = \pi(\mathcal{E}_2^+)$, where

$$\mathcal{E}_2^+ = \left(\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\
\varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing
\end{array}\right)_{\rho}.$$

Then we see $\psi_{\mathcal{E}_1} \in \Psi(\pi_1; \Delta_{\rho}[\frac{1}{2}, \frac{5}{2}], 1)$ and $\pi = \pi(\mathcal{E}_1^+)$, where

$$\mathcal{E}_1 = \left(\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\
\varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing
\end{array}\right), \quad \mathcal{E}_1^+ = \left(\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\
\varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing
\end{array}\right)_{\rho}.$$

Thus $\pi$ is of Arthur type.

5.2. Reduction along $\pi_{\rho,-}$. In this subsection, we give another algorithm to determine whether $\pi$ is of Arthur type from the information of $\Psi(\pi_{\rho,-})$. This is inspired by a related question from Atobe for which we thank him, and we include it here for the convenience of future references.

**Definition 5.15.** Let $\mathcal{E} = \cup_{\rho}\{\{[A_i, B_i], l_i, \eta_i]\}_{i \in (I_\rho, >)} \in \text{Rep}^{(P)}$. If there exists an $i \in I_\rho$ such that $A_i \neq B_i$, then after row exchanges if necessary, there exists a unique $B \in \frac{1}{2}\mathbb{Z}$ and a decomposition $I_\rho = I_{\rho,1} \sqcup I_{\rho,2} \sqcup I_{\rho,3}$ such that for any $i_1 \in I_{\rho,1}, i_2 \in I_{\rho,2}, i_3 \in I_{\rho,3}$, we have the following:

- $A_{i_1} = B_{i_1} \leq B$,
- $B = B_{i_2} < A_{i_2}$,
- $B < B_{i_3}$.

If $l_{i_2} \geq 1$ for any $i_2 \in I_{\rho,2}$, then we define

$$\mathcal{E}_{\rho,-} := \sum_{i_2 \in I_{\rho,2}} \text{add}_{i_2}^{-1}(\mathcal{E}).$$

The condition that $\pi(\mathcal{E}_{\rho,-}) \neq 0$ is equivalent to $u_{i,j}^{-1}$, dual $\circ u_{i,j} \circ$ dual or dual$^-$ not being applicable for $i, k \in I_{\rho,2}$. As a consequence, if $\mathcal{E}$ is absolutely maximal, then either $A_i = B_i$ for any $i \in I_\rho$ or $\pi(\mathcal{E}_{\rho,-}) \neq 0$.

The following proposition relates the $L$-data of $\pi(\mathcal{E})$ with $\pi(\mathcal{E}_{\rho,-})$, provided that the latter representation is nonzero.

**Proposition 5.16.** If $\pi(\mathcal{E}_{\rho,-}) \neq 0$, then the $L$-data of $\pi(\mathcal{E})$ can be recovered from that of $\pi(\mathcal{E}_{\rho,-})$ by inserting $\{\Delta_{\rho}[B_{i_2}, -A_{i_2}]\}_{i \in I_{\rho,2}}$. Moreover, $\pi(\mathcal{E}_{\rho,-}) = \pi(\mathcal{E})$ (see Definition 5.1 (2)).

**Proof.** We follow the notation in [HLL22, Definition 3.8] to write $\mathcal{E}_\rho = \mathcal{E}_{\rho,1} + \mathcal{E}_{\rho,2} + \mathcal{E}_{\rho,3}$ where

$$\mathcal{E}_{\rho,j} := \{([A_{i,j}, B_{i,j}], l_{i,j}, \eta_{i,j}]_{i_{j} \in (I_{\rho,j}, >)}\},$$

for $j = 1, 2, 3$. To describe the relation between $\pi(\mathcal{E})$ and $\pi(\mathcal{E}_{\rho,-})$, we apply induction on the size of $I_{\rho,2}$, say $r$. When $r = 0$, nothing to prove.

When $r \geq 1$, we apply induction on $n$, the rank of the group. Observe that the non-vanishing of $\pi(\mathcal{E}_{\rho,-})$ implies that

$$\mathcal{E}_{\rho}^{[\text{max}]} = \mathcal{E}_{\rho,1} + \mathcal{E}_{\rho,2} + (\mathcal{E}_{\rho,3})^{[\text{max}]}.$$
and hence we may assume $\mathcal{E}_{\rho,3} = (\mathcal{E}_{\rho,3})^{\max}$. Let $j$ be the index chosen in Proposition 4.3. If $j \in I_{\rho,2}$, then the conclusion follows from Proposition 4.3 and the induction hypothesis for $|I_{\rho,2}| = r - 1$. If $j \in I_{\rho,3}$, let

\[
\mathcal{E}_1 := \mathcal{E}^\rho \cup (\mathcal{E}_{\rho,1} + add_j^{-1}(\mathcal{E}_{\rho,3})),
\]

\[
\mathcal{E}_2 := \mathcal{E}^\rho \cup (\mathcal{E}_{\rho,1} + add^{-1}(\mathcal{E}_{\rho,2}) + add_j^{-1}(\mathcal{E}_{\rho,3})).
\]

We have $\pi(\mathcal{E}_1) \neq 0$ by Proposition 4.3, $\pi(\mathcal{E}_{\rho,-}) \neq 0$ by assumption, and then one can show that $\pi(\mathcal{E}_2) \neq 0$ by an argument similar to the proof of Lemma 5.8, which we omit here.

For brevity, we denote $\pi + \{\Delta_\rho[x_j, y_j]\}_{j \in J}$ the representation whose $L$-data is obtained from the $L$-data of $\pi$ by inserting $\{\Delta_\rho[x_j, y_j]\}_{j \in J}$. Then Proposition 4.3 and the induction hypothesis on rank $n$ imply that

\[
\pi(\mathcal{E}) = \pi(\mathcal{E}_1) + \{\Delta_\rho[B_j, -A_j]\}
\]

\[
= \pi(\mathcal{E}_2) + \{\Delta_\rho[B_{i_2}, -A_{i_2}]\}_{i \in I_{\rho,2}} + \{\Delta_\rho[B_j, -A_j]\}
\]

\[
= \pi(\mathcal{E}_{\rho,-}) + \{\Delta_\rho[B_{i_2}, -A_{i_2}]\}_{i \in I_{\rho,2}}.
\]

Finally, by applying Algorithm 5.2 on $\mathcal{E}_{\rho,-}$, one can see that any segment $\Delta_\rho[x, y]$ in the $L$-data of $\pi(\mathcal{E}_{\rho,-})$ must satisfy that $x > B$. Therefore, $\pi(\mathcal{E}_{\rho,-}) = \pi(\mathcal{E})_{\rho,-}$. This completes the proof of the proposition.

When $I_{\rho,1}$ is empty, [Ato20, Theorem 5.1 (2)] says that

\[
\pi(\mathcal{E}) = S_{\rho,|y_1| \cdots \rho,|y_s|}(\pi(\mathcal{E}_{\rho,-})),
\]

where

\[
\bigcup_{i \in I_{\rho,2}} [B_i, -A_i]_\rho = \{\rho \cdot |y_1, \ldots, \rho \cdot |y_s\},
\]

and $y_1 \geq \cdots \geq y_s$. Then Proposition 5.16 says that the effect of the composition of socles in (5.1) is to insert $\{\Delta_\rho[B_{i_2}, -A_{i_2}]\}_{i \in I_{\rho,2}}$ in the $L$-data of $\pi(\mathcal{E}_{\rho,-})$.

We have the following algorithm to compute $\pi(\mathcal{E})$, which is an analogue of Algorithm 5.2. Here, we use Proposition 5.16 as the analogue for Proposition 4.3.

**Algorithm 5.17.** Given an extended multi-segment $\mathcal{E} \in \text{Rep}$, proceed as follows.

**Step 1:** Compute $\mathcal{E}^{\max} = \cup_{\rho} \{(A_{i}, B_{j}) \mid \rho, |l_i|, \rho, |l_j|\} \in (I_{\rho,>})$. If $\mathcal{E}^{\max}$ satisfies (L) (see [HLL22, Definition 10.1]), then the $L$-data of $\pi(\mathcal{E})$ is given by [HLL22, Proposition 10.3]. Note that the condition that $\mathcal{E}$ is positive is not necessary.

**Step 2:** If $\mathcal{E}^{\max}$ does not satisfy (L), then choose $\rho$ such that $\mathcal{E}_{\rho}$ (see [HLL22, Definition 3.8]) does not satisfy (L). Construct $(\mathcal{E}^{\max})_{\rho,-}$ as in Definition 5.15. Repeating the algorithm for $(\mathcal{E}^{\max})_{\rho,-}$, which is of smaller rank than $\mathcal{E}$, we get the $L$-data of $\pi((\mathcal{E}^{\max})_{\rho,-})$. Then the $L$-data of $\pi(\mathcal{E})$ is given by inserting $\{\Delta_\rho[B_{i_2}, -A_{i_2}]\}_{i \in I_{\rho,2}}$ in that of $\pi((\mathcal{E}^{\max})_{\rho,-})$.

**Remark 5.18.** We may replace the $\mathcal{E}^{\max}$ in above algorithm by $\mathcal{E}^*$ given in [Ato20, Algorithm 5.6], with a modification that we need to replace the definition of $B_{\rho}^{\min}$ and $I_{\rho}^{\min}$ in Step 1 of [Ato20, Algorithm 5.6] by $B_{\rho}^{\min} = \min\{B_j \mid j \in I_{\rho}, A_j \neq B_j\}$ and $I_{\rho}^{\min} = \{j \in I_{\rho} \mid B_j = B_{\rho}^{\min}, A_j \neq B_j\}$. Note that if $\{B_j \mid j \in I_{\rho}, A_j \neq B_j\} = \emptyset$, then we can compute the contribution of $\mathcal{E}_{\rho}$ to the $L$-data of $\pi(\mathcal{E})$ explicitly already.

Algorithms 5.2, 5.17 are identical on Example 5.3 since $(\mathcal{E}^{\max})^- = (\mathcal{E}^{\max})_{\rho,-}$ for these extended multi-segments. We give the following example to demonstrate the difference between two algorithms.
Example 5.19. Let
\[
E = \begin{pmatrix}
0 & 1 & 2 & 3 \\
\ll & \gg & \ll & \gg \\
\ll & \gg & \ll & \gg \\
\end{pmatrix}_\rho = E^{[\max]}.
\]
We set
\[
E_1 = \begin{pmatrix}
1 & 2 & 3 \\
\ll & \gg & \ll & \gg \\
\ll & \gg & \ll & \gg \\
\end{pmatrix}_\rho, E_2 = \begin{pmatrix}
0 & 1 & 2 & 3 \\
\ll & \gg & \ll & \gg \\
\end{pmatrix}_\rho, E_3 = \begin{pmatrix}
2 \\
\gg \\
\end{pmatrix}_\rho.
\]
Then \(E_1 = E_{\rho,-}, E_2 = E^-, \text{ and } E_3 = (E_{\rho,-})_{\rho,-} = (E^-)^-\). Algorithm 5.2 gives
\[
\pi(E_3) = \pi(2^+), \quad \pi(E_2) = L(\Delta [0, -1]; \pi(2^+)), \quad \pi(E) = L(\Delta [1, -3], \Delta [0, -1]; \pi(2^+)).
\]
Algorithm 5.17 gives
\[
\pi(E_3) = \pi(2^+), \quad \pi(E_1) = L(\Delta [1, -3]; \pi(2^+)), \quad \pi(E) = L(\Delta [1, -3], \Delta [0, -1]; \pi(2^+)).
\]
Thus, the representations involved in the intermediary steps of Algorithms 5.2 and 5.17 are different.

Next, we provide some notation which is used in Theorem 5.21 and Algorithm 5.22 below to determine whether a representation is of Arthur type.

Definition 5.20. Suppose \(\pi\) is a representation of \(G_n\) of good parity. Write
\[
\pi = L(\Delta_{\rho_1} [x_1, -y_1], \ldots, \Delta_{\rho_f} [x_f, -y_f]; \pi(\phi, \epsilon)).
\]
We denote
\[
\begin{align*}
x_{\rho,-} & := \min\{x_i \mid \rho_i \cong \rho\}, \\
\Delta_{\rho,-} & := \{\Delta_{\rho_i} [x_i, -y_i] \mid 1 \leq i \leq f, \rho_i \cong \rho, x_i = x_{\rho,-}\}, \\
\psi_{\rho,-} & := \sum_{i \in \Delta_{\rho,-}} \rho \otimes S_{x_i+y_i+1} \otimes S_{y_i-x_i+1}, \\
\psi_{\rho,+} & := \sum_{i \in \Delta_{\rho,-}} \rho \otimes S_{x_i+y_i+1} \otimes S_{y_i-x_i+1}.
\end{align*}
\]
Note that \(\Delta_{\rho,-}\) is a multi-set.

1. We denote by \(\Psi(\pi_{\rho,-}; \Delta_{\rho,-})\) the set of local Arthur parameters \(\psi\) such that
   \begin{itemize}
   \item \(\pi_{\rho,-} \in \Pi_\psi\),
   \item \(\psi \trianglerighteq \psi_{\rho,-}\),
   \item any summand \(\rho \otimes S_a \otimes S_b\) of \(\psi\) with \(a - b \leq 2x_{\rho,-}\) satisfies that \(b = 1\).
\end{itemize}
   For any \(\psi \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})\), we define \(\psi_+ := \psi - \psi_{\rho,-} + \psi_{\rho,+}\).

2. For any \(E = \cup_{\rho} \{[A_i, B_i], l_i, \eta_i \mid \rho \in I_{\rho,i} > \} \in \text{Rep}^{(\Pi)}\) such that \(\pi(E) = \pi_{\rho,-}\) and \(\psi_\rho \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})\), there exists a unique decomposition \(I_\rho = I_{\rho,1} \cup I_{\rho,2} \cup I_{\rho,3}\) such that for any \(i_1 \in I_{\rho,1}, i_2 \in I_{\rho,2}, i_3 \in I_{\rho,3}\),
   \begin{itemize}
   \item \(A_{i_1} = B_{i_1} \leq x_{\rho,-}\),
   \item \(x_{\rho,-} + 1 = B_{i_2}\),
   \item \(B_{i_3} > x_{\rho,-}\).
\end{itemize}
   Let \(m\) be the multiplicity of \(\Delta_{\rho} [x_{\rho,-}, -x_{\rho,-} - 1]\) in the \(L\)-data of \(\pi\). We define
   \[
   E_+ := E^0 \cup (E_{\rho,1} + \{(x_{\rho,-} + 1, x_{\rho,-}), (1, 1)^m\} + \text{add}^1 (E_{\rho,2} + E_{\rho,3}),
   \]
in the notation of [HLL22, §3.1].

The following theorem gives a criterion for whether \(\pi\) is of Arthur type from \(\Psi(\pi_{\rho,-})\).
Theorem 5.21. If π is a representation of $G_n$ of good parity, then π is of Arthur type if and only if $\Psi(\pi_{\rho,-}; \Delta_{\rho,-})$ is non-empty. Moreover, for any $E \in \text{Rep}^{(P)}$ such that $\pi(E) = \pi_{\rho,-}$ and $\psi_E \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})$, we have $\pi = \pi(E)$. In other words, for any $\psi \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})$, we have $\psi_+ \in \Psi(\pi)$.

Proof. Suppose π is of Arthur type and take $E$ such that $\pi = \pi(E)$. Then $E^{\text{max}}$ satisfies all conditions in Proposition 5.16, and hence $\pi_{\rho,-} = \pi(E_{\rho,-})$ is of Arthur type. It follows from the definition that $\psi_{\pi_{\rho,-}}$ is in $\Psi(\pi_{\rho,-}; \Delta_{\rho,-})$.

Conversely, if $E \in \text{Rep}^{(P)}$ satisfies that $\psi_E \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})$ and $\pi(E) = \pi_{\rho,-}$. We have that $\pi(E_{\rho,+}) \neq 0$ by the dual version of [HLL22, Lemma 4.3(i)]. Since $(E_{\rho,+})_{\rho,-} = E$, Proposition 5.16 implies that $\pi = \pi(E_{\rho,+})$ is of Arthur type. Note that $(\psi_E)_+ = \psi_{\pi_{\rho,+}}$. This completes the proof of the theorem. □

We obtain an algorithm similar to Algorithm 5.12 applying the above theorem.

Algorithm 5.22. Given a representation $\pi$ of $G_n$ of good parity, proceed as follows:

**Step 1:** If $\pi$ is tempered, then $\pi$ is of Arthur type. Write $\pi = (\phi, \epsilon)$, where

$$\phi = \bigoplus_{\rho, i \in I_\rho} \rho \otimes S_{2a_i + 1},$$

where we equip $I_\rho$ with a total order $>$ such that $a_i$ is non-decreasing. Then let

$$E = \cup_{\rho} \{ ([a_i, a_i], 0, \epsilon(\rho \otimes S_{2a_i + 1})) | i \in (I_\rho, >) \},$$

and output $\Psi(E)$ given by [HLL22, Theorem 7.4].

We regard the trivial representation of $G_0$ as a tempered representation, and output $\{E\}$, where $E = \emptyset$.

**Step 2:** If $\pi$ is not tempered, then apply the algorithm on $\pi_{\rho,-}$ (see Definition 5.1) to see whether it is of Arthur type.

If $\pi_{\rho,-}$ is not of Arthur type, then neither is $\pi$ and the procedure ends.

If $\pi_{\rho,-}$ is of Arthur type, then $\pi$ is of Arthur type if and only if there exists an $E \in \{E' | \pi(E') = \pi_{\rho,-}\}$ such that $\psi_E \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})$ (see Definition 5.20). Then, we output $\{E' | \pi(E') = \pi(\pi_{\rho,-})\}$/(row exchanges) given by [HLL22, Theorem 7.4].

We apply Algorithm 5.22 on the same representations in Example 5.14.

Example 5.23. 1. Consider

$$\pi = L\left(\Delta_{\rho} \left[ \begin{array}{cc} -1 & -5 \\ 2 & 2 \end{array} \right], \Delta_{\rho} \left[ \begin{array}{cc} 1 & -1 \\ 2 & 2 \end{array} \right], \Delta_{\rho} \left[ \begin{array}{cc} 3 & -5 \\ 2 & 2 \end{array} \right]; \pi \left( \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right) \right).$$

Then

$$\pi_{\rho,-} = L\left(\Delta_{\rho} \left[ \begin{array}{cc} 3 & -5 \\ 2 & 2 \end{array} \right]; \pi \left( \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right) \right),$$

$$\Delta_{\rho,-} = \left\{ \Delta_{\rho} \left[ \begin{array}{cc} -1 & -5 \\ 2 & 2 \end{array} \right], \Delta_{\rho} \left[ \begin{array}{cc} -1 & -1 \\ 2 & 2 \end{array} \right] \right\}.$$

By the same computation in Example 5.14, we see $\pi_{\rho,-}$ is of Arthur type, and the set $\{E | \pi(E) = \pi_{\rho,-}\}$/(row exchanges) consists of two extended multi-segments

$$\begin{pmatrix} \oplus & \oplus & \oplus \\ \ominus & \ominus \end{pmatrix}_{\rho}, \quad \begin{pmatrix} \ominus & \ominus & \ominus \\ \ominus \end{pmatrix}_{\rho}.$$
Thus $\Psi(\pi_{\rho,-}; \Delta_{\rho,-})$ is empty and $\pi$ is not of Arthur type.

2. Consider

$$\pi = L \left( \Delta_{\rho} \left[ -\frac{1}{2}, -\frac{5}{2} \right], \Delta_{\rho} \left[ -\frac{1}{2}, -\frac{1}{2} \right], \Delta_{\rho} \left[ -\frac{3}{2}, -\frac{5}{2} \right] \right).$$

Then

$$\pi_{\rho,-} = L \left( \Delta_{\rho} \left[ -\frac{1}{2}, -\frac{5}{2} \right], \pi \left( -\frac{1}{2}, \frac{3}{2}, -\frac{5}{2} \right) \right),$$

$$\Delta_{\rho,-} = \left\{ \Delta_{\rho} \left[ -\frac{1}{2}, -\frac{5}{2} \right], \Delta_{\rho} \left[ -\frac{1}{2}, -\frac{1}{2} \right] \right\}.$$

By the same computation in Example 5.14, we see $\pi_{\rho,-}$ is of Arthur type. Since $\pi_{\rho,-} = \pi(\mathcal{E})$ where

$$\mathcal{E} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \odot & \oplus & \ominus \end{pmatrix}_\rho,$$

and $\psi_\mathcal{E} \in \Psi(\pi_{\rho,-}; \Delta_{\rho,-})$, we see $\pi$ is of Arthur type and $\pi = \pi(\mathcal{E}_+)$, where

$$\mathcal{E}_+ = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \odot & \ominus & \oplus \end{pmatrix}_\rho.$$

6. Enhanced Shahidi’s Conjecture

Let $G$ be a quasi-split connected reductive group over $F$. We assume that there is a theory of local Arthur packets for $G$ as conjectured in [Art89]. In this section, we show that Conjecture 1.14, along with certain assumptions (Working Hypotheses 6.6), implies that a local Arthur packet of $G$ is tempered if it has a generic member, which is an essential part of the Enhanced Shahidi’s Conjecture. Then, we verify these assumptions for symplectic and split odd special orthogonal groups. Combining with Theorem 1.15, this gives a new proof of the Enhanced Shahidi’s Conjecture (Conjecture 1.17) in these cases.

6.1. On the Enhanced Shahidi’s Conjecture. We recall the Enhanced Shahidi’s Conjecture (Conjecture 1.17) as follows.

**Conjecture 6.1** ([LS22, Conjecture 1.5], Enhanced Shahidi’s Conjecture). For any quasi-split reductive group $G$, a local Arthur packet is tempered if and only if it has a generic member.

Shahidi’s Conjecture ([Sha90, Conjecture 9.4]) states that if $\phi$ is tempered, then $\Pi_\phi$ contains a generic representation. This generic representation is expected to be unique once the Whittaker data being fixed. Notice that if $\phi$ is tempered, then $\phi$ is also a local Arthur parameter and the local $L$-packet is expected to coincide with the corresponding local Arthur packet. Shahidi’s Conjecture has been proved for many cases (for examples, see [Kon02, JS03, JS04, Liu11, JS12, MW12, Wal12, Art13, JL14, KMSW14, Mok15, BP16, Var17, Ato17, JL22]). The new content of the Enhanced Shahidi’s Conjecture is that if the local Arthur packet $\Pi_\psi$ contains a generic representation, then $\psi$ is a tempered local Arthur parameter.

Fix an infinitesimal parameter $\lambda : W_F \to \mathfrak{g}$ and let $V_\lambda$ be the Vogan variety. Let $\phi_0$ be the local $L$-parameter of $G$ corresponding to the zero orbit in $V_\lambda$. Conjecture 1.14 has a direct corollary as follows.
Corollary 6.2. Assume that Conjecture 1.14 holds for \( G \) and that \( \pi \in \Pi_{\phi_0} \). If \( \phi_0 \) is of Arthur type, namely, \( \phi_0 = \phi_{\psi_0} \) for an Arthur parameter of \( \psi_0 \), then \( \Psi(\pi) = \{ \psi_0 \} \).

**Proof.** By Conjecture 1.14, if \( \pi \in \Pi_\psi \) for some local Arthur parameter \( \psi \), then \( \phi_0 \geq C \phi_\psi \). But \( \phi_0 \) corresponds to the zero orbit in \( V_\lambda \) and hence this can only occur if \( \psi = \psi_0 \). \( \square \)

It is known that the local \( L \)-parameter for an unramified representation is closed, the above result is consistent with [Art13, Theorem 1.5.1]. For the ABV-packets defined in [CFMMX22], the analogue of Corollary 6.2 is known for general \( p \)-adic connected reductive groups since the analogue of Conjecture 1.14 has been proved there for these packets using geometric tools. See [CFZ21, CFZ22] for the analogous results for unipotent representations of the exceptional group \( G_2 \).

For a fixed \( \lambda \), the zero orbit is clearly the unique minimal orbit of \( V_\lambda \) under the closure ordering. It turns out that there is also a unique maximal orbit under this ordering.

**Lemma 6.3 ([CFMMX22, Proposition 5.6]).** The Vogan variety \( V_\lambda \) contains a unique open dense orbit.

Following [CFMZ22], a local \( L \)-parameter \( \phi \) is called open if \( C_\phi \) is open in \( V_{\lambda_\phi} \). The following result describes the basic properties of open parameters.

**Proposition 6.4 ([CFMZ22]).** Let \( \phi \) be a local \( L \)-parameter of \( G \) and let \( \lambda = \lambda_\phi \) be the infinitesimal parameter associated with \( \phi \).

1. The parameter \( \phi \) is tempered if and only if \( \phi \) is both of Arthur type and open.
2. The parameter \( \phi \) is open if and only if \( L(s, \phi, \text{Ad}) \) is regular at \( s = 1 \).

In the following, given an infinitesimal parameter \( \lambda \), we usually write the unique open local \( L \)-parameter (up to conjugacy) as \( \phi^0 \) and we don’t distinguish a local \( L \)-parameter and the orbit it represents. The closed orbit \( \phi_0 \) and the open orbit \( \phi^0 \) are related as follows.

**Lemma 6.5.** If \( V_\lambda \) contains a local \( L \)-parameter of Arthur type, then \( \phi^0 \) and \( \phi_0 \) are also of Arthur type. If we denote the corresponding local Arthur parameters by \( \psi^0 \) and \( \psi_0 \), respectively, then we have \( \hat{\psi}^0 = \psi_0 \).

**Proof.** Take \( \psi \) a local Arthur parameter such that \( \lambda = \lambda_{\phi_\psi} \). Then we consider the local \( L \)-parameter \( \psi^{\Delta} \) defined by

\[
\psi^{\Delta}(w, x) := \psi(w, x, x),
\]

which is tempered since \( \psi|_{W_P} \) has bounded image. It follows from direct computation that \( \lambda_{\phi^0} = \lambda_{\phi_\psi} = \lambda \), and hence \( \psi^{\Delta} = \phi^0 \) by Proposition 6.4 (1). Therefore, \( \phi^0 \) is of Arthur type and the local Arthur parameter \( \psi^0 \) is given by

\[
\psi^0(w, x, y) := \psi(w, x, x).
\]

Next, we show that \( \phi_{\psi^0} \) gives the minimal element in \( V_\lambda \). Indeed, since \( \hat{\psi}^0 \) is trivial on \( \text{SL}_2^D(\mathbb{C}) \), \( \phi_{\psi^0} \) is trivial on \( \text{SL}_2(\mathbb{C}) \). Therefore, \( C_{\phi_{\psi^0}} \) is the zero orbit, which is minimal by definition. \( \square \)

Given a local Arthur parameter \( \psi \), the local Arthur packet \( \Pi_\psi \) and \( \Pi_{\hat{\psi}} \) are expected to be related by the Aubert-Zelevinsky duality as follows. Let \( \pi \) be an irreducible representation of \( G \). In [Aub95], Aubert showed that there exists \( \varepsilon \in \{ \pm 1 \} \) such that

\[
\hat{\pi} := \varepsilon \sum_P (-1)^{\dim(A_P)}[\text{Ind}_P^G(\text{Jac}_P(\pi))].
\]
gives an irreducible representation. Here the sum is over all standard parabolic subgroups $P$ of $G$ and $A_P$ is the maximal split torus of the center of the Levi subgroup of $P$. We say $\hat{\pi}$ is the Aubert-Zelevinsky dual or Aubert-Zelevinsky involution of $\pi$.

**Working Hypotheses 6.6.** Let $G$ be a quasi-split connected reductive group over $F$. Suppose that there is a theory of local Arthur packets for $G$ as conjectured in [Art89]. We assume that the following hold.

1. Given a local Arthur parameter $\psi$ of $G$, we have
   \[ \Pi_\hat{\psi} = \{ \hat{\pi} \mid \pi \in \Pi_\psi \}. \]

2. Let $\pi$ be an irreducible generic representation of $G$ of Arthur type, then $\hat{\pi} \in \Pi_{\phi_0}$, where $\phi_0$ is the local $L$-parameter which corresponds to the zero orbit in $V_\lambda$ with $\lambda = \lambda_{\phi_\pi}$.

In the following, we show that the new content of the Enhanced Shahidi’s Conjecture 6.1 is a dual version of Corollary 6.2 under Working Hypotheses 6.6, and thus is a consequence of Conjecture 1.14.

**Theorem 6.7.** Let $G$ be a quasi-split connected reductive group over $F$. Suppose that there is a theory of local Arthur packets for $G$ as conjectured in [Art89]. Assume Conjecture 1.14 and the Working Hypotheses 6.6 hold for $G$. If a local Arthur packet $\Pi_\psi$ of $G$ contains a generic representation, then $\psi$ is tempered.

**Proof.** By Working Hypotheses 6.6 (1), $\pi \in \Pi_\psi$ if and only if $\hat{\pi} \in \Pi_\hat{\psi}$, which amounts to a bijection
\[ \Psi(\pi) \to \Psi(\hat{\pi}), \]
\[ \psi \mapsto \hat{\psi}. \]

Suppose that $\pi$ is a generic representation and $\pi \in \Pi_\psi$ for a local Arthur packet $\psi$ of $G$. Let $\lambda$ be the infinitesimal parameter associated with $\phi_\psi$. Let $\psi_0$ (resp. $\psi^0$) be the local Arthur parameter of $\phi_0$ (resp. $\phi^0$) as in Lemma 6.5. By Working Hypotheses 6.6 (2), we have $\hat{\pi} \in \Pi_{\phi_0}$. Conjecture 1.14 implies that $\Psi(\hat{\pi}) = \{ \psi_0 \}$, see Corollary 6.2. This implies that $\Psi(\pi) = \{ \psi_0 \} = \{ \psi^0 \}$. Thus $\psi = \psi^0$, which is tempered by Proposition 6.4. \qed

### 6.2. A new proof of the Enhanced Shahidi’s Conjecture for $\text{Sp}_{2n}$ and split $\text{SO}_{2n+1}$

In this subsection, we discuss certain known cases regarding Working Hypotheses 6.6 and give a new proof of Conjecture 6.1 for $\text{Sp}_{2n}$ and split $\text{SO}_{2n+1}$.

Working Hypotheses 6.6 (1) essentially says that the Aubert-Zelevinsky involution should be compatible with endoscopic transfer, and it is known to be true for many groups.

**Theorem 6.8 ([Xu17, §A]).** Let $G = \text{Sp}_{2n}$ or quasi-split $\text{SO}_n$ and let $\psi$ be a local Arthur parameter of $G$. Then
\[ \Pi_{\hat{\psi}} = \{ \hat{\pi} \mid \pi \in \Pi_\psi \}. \]

Regarding Working Hypotheses 6.6 (2), inspired by [CFMZ22], we expect that it is true without the assumption that $\pi$ is of Arthur type.

**Conjecture 6.9.** Let $\pi$ be a generic representation of $G$. Then $\hat{\pi} \in \Pi_{\phi_0}$, where $\phi_0$ is the local $L$-parameter corresponding to the zero orbit in $V_\lambda$ with $\lambda = \lambda_{\phi_\pi}$.

Note that the above conjecture implies the following.

**Conjecture 6.10 ([CFMZ22]).** A local $L$-parameter $\phi$ of $G$ is generic if and only if $\phi$ is open.
By Proposition 6.4 (2), the above Conjecture 6.10 is equivalent to the following conjecture of Gross-Prasad and Rallis.

**Conjecture 6.11 (GP92, Conjecture 2.6).** A local $L$-parameter $\phi$ of $G$ is generic if and only if $L(s, \phi, \text{Ad})$ is regular at $s = 1$.

Conjecture 6.11 was proved in [JS04, Theorem C, (3)] for $SO_{2n+1}$, in [Liu11, Theorem 1.2] for $Sp_{2n}$, and in [JL14, Theorem 1.5] for $SO_n$. In a more general setting, Conjecture 6.11 was proved by Gan-Ichino [GI16, Proposition B.1] under some hypothesis [GI16, Sect. B.2], which are known to be true for general linear groups and classical groups. Thus Conjecture 6.10 and Conjecture 6.11 hold for general linear groups and classical groups.

In the geometric setting [CFMMX22], the local Langlands correspondence gives a bijection between irreducible smooth representations of $G$ with infinitesimal parameter $\lambda$ and simple objects of the category of $H_\lambda$-equivariant perverse sheaves on $V_\lambda$, which is denoted by $\text{Per}_{H_\lambda}(V_\lambda)$. For an irreducible representation $\pi$, we denote by $P(\pi)$ the corresponding perverse sheave. A Fourier transform operator $\hat{F} : \text{Per}_{H_\lambda}(V_\lambda) \to \text{Per}_{H_\lambda}(V_\lambda)$ is defined in [CFMMX22] and it is expected that $\hat{F}(P(\pi)) = P(\tilde{\pi})$. Moreover, if $\pi$ is an irreducible representation corresponding to $(\phi^0, 1)$ under the local Langlands correspondence, where 1 is the trivial representation of $A_{\phi^0}$, then it is known that $\hat{F}(P(\pi))$ corresponds to a representation with local $L$-parameter $\phi_0$. See [CFMZ22] for more details. Thus, the geometric analogue of Conjecture 6.9 follows from Conjecture 6.10 or Conjecture 6.11.

In the following, we show that Working Hypotheses 6.6 (2) holds for $Sp_{2n}$ and split $SO_{2n+1}$.

**Lemma 6.12.** Let $G$ be $Sp_{2n}$ or split $SO_{2n+1}$. Let $\pi$ be an irreducible representation of $G$. If $\pi$ is generic and of Arthur type, then $\tilde{\pi} \in \Pi_{\phi_0}$.

**Proof.** Suppose that $\pi$ is generic. By Proposition 6.4 (2) and Conjecture 6.11, which were known in these cases by [Liu11] and [JS04], or [GI16], the local $L$-parameter $\phi_\pi$ of $\pi$ must be open in $V_\lambda$. Consequently, $\phi_\pi = \phi^0$ by Lemma 6.3. By Lemma 6.5, $\phi^0$ is of Arthur type and there exists an Arthur parameter $\psi^0$, which must be tempered by Proposition 6.4 (1), such that $\phi^0 = \phi_{\psi^0}$. We then have $\pi \in \Pi_{\psi^0} = \Pi_{\phi_0}$. Note that $\phi_0 = \phi_{\psi^0}$ since it is trivial on $SL_2(\mathbb{C})$.

Next we use extended multi-segments to check that $\tilde{\pi}$ is indeed in the local $L$-packet $\Pi_{\phi_0}$ associated with $\psi_0$. By Theorem 2.4, we may write $\pi = \tau_{\psi_1} \times \pi(\mathcal{E})$, where

$$\psi_1 \oplus \psi_\mathcal{E} \oplus \psi_1^\vee = \psi^0,$$

which is tempered. Then $\tilde{\pi}$ is in $\Pi_{\psi^0}$ and hence $\tilde{\pi} = \tau_{\psi_1} \times \pi(\mathcal{E})$. To show $\tilde{\pi} \in \Pi_{\phi_0}$, it remains to show that $\pi(\mathcal{E})$ is in the local $L$-packet associated with $\tilde{\psi}_\mathcal{E}$.

Since $\pi$ is tempered and generic, $\mathcal{E}$ is of the form

$$\mathcal{E} = \bigcup_{\rho} \{([A_i, A_j], 0, 1)\}_{i \in \{l_{\rho}, >\}},$$

Then it follows from the definition of the operator dual (see [Ato20, Definition 6.1], or [HLL22, Definition 3.27]) that

$$\text{dual}(\mathcal{E}) = \bigcup_{\rho} \{([A_i, -A_j], [A_j], 1)\}_{i \in \{l_{\rho}, \leq\}},$$

which satisfies the conditions in [HLL22, Theorem 10.4]. Hence,

$$\pi(\mathcal{E}) = \pi(\text{dual}(\mathcal{E})) \in \Pi_{\phi_{\tilde{\psi}_\mathcal{E}}}.$$

This completes the proof of the lemma. \qed
Note that a key ingredient in the above proof is Conjecture 6.11. We expect that Conjecture 6.11 implies Working Hypotheses 6.6 (2) in general.

Since we have verified the Working Hypotheses 6.6 for $\text{Sp}_{2n}$ and split $\text{SO}_{2n+1}$, we can give a new proof of the Enhanced Shahidi’s Conjecture 6.1 in these cases.

**Theorem 6.13.** The Enhanced Shahidi’s Conjecture 6.1 holds for $\text{Sp}_{2n}$ and split $\text{SO}_{2n+1}$.

**Proof.** If $\psi$ is tempered, then it is well-known that $\Pi_{\psi}$ contains a generic representation. Thus it suffices to show that if $\psi$ is an Arthur packet and $\Pi_{\psi}$ contains a generic representation, then $\psi$ must be tempered. This follows from Theorem 6.7 directly. Notice that in this case, Conjecture 1.14 is Theorem 1.15, Working Hypotheses 6.6 (1) is Theorem 6.8 and Working Hypotheses 6.6 (2) is Lemma 6.12. □

The Enhanced Shahidi’s Conjecture 6.1 for $\text{Sp}_{2n}$ and split $\text{SO}_{2n+1}$ has been proved in [HLL22, Theorem 9.5], whose proof relies on the construction of representations of Arthur type, while the new proof above relies on the geometry of $L$-parameters. More precisely, in [HLL22, Theorem 9.5], the main tool to show that a generic representation of Arthur type is tempered is the classification of irreducible generic representations in terms of their Langlands classification. Then the uniqueness of a local Arthur packet that contains a given generic tempered representation was proved by showing that none of the operators (see §2.4) are applicable. In contrast, in Theorem 6.13, to show that a generic representation of Arthur type is tempered, we use the geometric description of a generic parameter. We obtain the uniqueness by taking the Aubert-Zelevinsky dual of the generic representation, which lies in the $L$-packet of an $L$-parameter of Arthur type corresponding to the zero orbit, and then the uniqueness follows from the closure ordering (see the proof of Theorem 6.7).

**Remark 6.14.** Proposition 6.4 implies that open parameters (or generic parameters assuming Conjecture 6.11) are generalizations of tempered parameters. Since $ABV$-packets defined in [CFMMX22] are expected to be generalizations of the notion of Arthur packets, the $ABV$-version of Conjecture 6.1 is that $\Pi_{\phi}^{ABV}$ contains a generic representation if and only if $\phi$ is open, see [CMFZ22]. In view of the fact that $\Pi_{\phi} \subset \Pi_{\phi}^{ABV}$ and Proposition 6.4, it is also a generalization of Conjecture 6.11. The main result of [CMFZZ22] is that the above $ABV$-version conjecture holds for quasi-split classical groups, which inspires our new proof of the Enhanced Shahidi’s Conjecture for $\text{Sp}_{2n}$ and split $\text{SO}_{2n+1}$.

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