THE HILBERT SPACES
FOR STABLE AND UNSTABLE PARTICLES

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Abstract

The Hilbert spaces for stable scattering states and particles are determined by the representations of the characterizing Euclidean and Poincaré group and given, respectively, by the square integrable functions on the momentum 2-spheres for a fixed absolute value of momentum and on the energy-momentum 3-hyperboloids for a particle mass. The Hilbert spaces for the corresponding unstable states and particles are not characterized by square integrable functions. Their scalar products are defined by positive type functions for the cyclic representations of the time, space and spacetime translations involved. Those cyclic, but reducible translation representations are irreducible as representations of the corresponding affine operation groups which involve also the time, space and spacetime reflection group, characteristic for unstable structures.
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1 Introductory Remarks

In quantum theory, the operational structure of physics, e.g. the action of time and space translations, of rotations or of electrodynamical phase transformations, is experimentally observed in terms of transition amplitudes and probabilities. The related complex and positive numbers are formulated with the scalar product of Hilbert spaces acted upon with the operations groups, in general real Lie groups.

Each real Lie group determines ‘its’ characteristic complex Hilbert spaces. E.g., the real \((1 + 2s)\)-dimensional Heisenberg group \(H(s)\) with \(s\) position momentum pairs in the characterizing Lie algebra bracket \([x_a, p_b] = \delta_{ab}I\) determines - with the Stone-von Neumann theorem[11, 3, 20] - the square integrable function classes \(L^2(\mathbb{R}^s)\) as the Hilbert spaces for its faithful representations with characterizing invariant \(\hbar \neq 0\). The \(H(s)\)-Hilbert spaces are not universal. E.g. the irreducible Hilbert spaces for nonrelativistic scattering are proper subspaces of \(L^2(\mathbb{R}^3)\). In important cases, there are irreducible Hilbert spaces for an operation group which are not formalizable with square integrable functions. The Hilbert spaces for unstable structures are examples.

In general, the irreducible sets where a group \(G\) acts upon are \(G\)-orbits, isomorphic to subgroup classes \(G/H\) with the realization characterizing fix-group (‘little group’) \(H \subseteq G\). All elements of an orbit are cyclic \(G \cdot x \cong G/H\). Representations act - by definition - on vector spaces. A nondecomposable representation space is the linear span of the orbit of a cyclic vector - with the additional closure in the case of topological vector spaces. In the following without mentioning the contrary, all representations considered are in definite unitary groups, also called Hilbert representations.

Some basic representation theoretical concepts: Representations of Lie groups are direct sums of cyclic representations, i.e. of representations with a cyclic vector, and direct integrals of irreducible representations[12, 3, 7]. For compact Lie groups, cyclic representations are irreducible and finite dimensional - not necessarily for noncompact Lie groups where faithful representations have to be infinite dimensional.

2 The Hilbert Spaces for Stable Particles

According to Wigner[15], a particle is characterized by an irreducible unitary representation of the Poincaré group \(SO_0(1,3) \rtimes \mathbb{R}^4\) as semidirect product \(G = H \rtimes N\) (subgroup \(H\) acting upon normal subgroup \(N\)) of the homogeneous orthochronous Lorentz group with the spacetime translations. The Poincaré group acts upon an infinite dimensional Hilbert space with the invariant particle properties spin and mass and the eigenvalues 3rd spin component and energy-momenta for the eigenvectors.

The Poincaré group \(SL(\mathfrak{C}^2) \rtimes \mathbb{R}^4\) for special relativity, with covering group \(SL(\mathfrak{C}^2)\) of the homogeneous group \(SO_0(1,3) \cong SL(\mathfrak{C}^2)/\{\pm 1_2\}\), contains - according to each decomposition into time translations and position translations - direct products \(\times\) of time translations and Euclidean groups with rotations.
\[ \text{SO}(3) \cong \text{SU}(2)/\{\pm 1\} \] acting upon position translations

\[ \text{SO}_0(1, 3) \times \mathbb{R}^4 \supset \mathbb{R} \times [\text{SO}(3) \times \mathbb{R}^3] \]

Time and position translation representations, to be given first, are Lorentz compatibly composed to Poincaré group representations.

### 2.1 - for Energy Eigenstates with Time Translations

With Schur\cite{13}, the irreducible representations of the abelian time translation group are complex 1-dimensional. They are characterized by an invariant energy \( E \)

\[ \mathbb{R} \ni t \mapsto e^{iEt} \in \text{U}(1), \ E \in \mathbb{R} \]

The eigenvector \( |E\rangle \) for a stable state with energy \( E \) spans the 1-dimensional representation space \( \mathbb{C}|E\rangle \). It is acted upon with the time translations which describes its time development and gives the time orbit in the representation space - a circle in the complex plane

\[ \mathbb{R} \ni t \mapsto |E, t\rangle = e^{iEt}|E\rangle \in \mathbb{C}|E\rangle, \ |E, 0\rangle = |E\rangle \]

The translation dependence, here \( t \), is included in the ket and omitted for trivial translations, here for \( t = 0 \).

The time translation representing unitary group \( \text{U}(1) \) defines the scalar product of the representation space. The basic vector can be normalized (probability 1)

\[ \langle E|E \rangle = 1 \]

The time dependence of the scalar product (transition or probability amplitude) reproduces the time representation matrix element above

\[ \langle E, t_2|E, t_1\rangle = e^{iE(t_1-t_2)}\langle E|E \rangle = e^{iEt} \text{ with } t = t_1 - t_2 \]

Irreducible time translation representations with different characterizing invariant energies are Schur-orthogonal to each other as seen by integration with Haar measure \( dt \) over the parameters of the translation group

\[ \{ D^E \mid E \in \mathbb{R} \} \] with \( D^E(t) = e^{iEt} \)

\[ \{ D^{E_2} D^{E_1} \} = \int dt \ e^{-iE_2 t} e^{iE_1 t} = \delta(E_1 - E_2) \]

In general, the Schur scalar product\cite{13, 3, 6, 8} for the irreducible representation matrix elements (also called representation coefficients), denoted here with the ‘braced bra and ket’ \{\ldots|\ldots\}, has to be distinguished from the scalar product for the vectors of an irreducible representation space, denoted here with the ‘usual bra and ket’ \langle\ldots|\ldots\).

An easily accessible example for this distinction is the compact rotation group \( \text{SU}(2) \) with the scalar products for the finite dimensional irreducible representation spaces and the Schur scalar product for all \( \text{SU}(2)\)-functions
L²(SU(2)): In the Euler parametrized representations, e.g. the adjoint 3-dimensional one

\[ \text{SU}(2) \ni u(\chi, \varphi, \theta) \mapsto [2J](\chi, \varphi, \theta) \in U(1 + 2J) \]

\[ e^{i\chi e^{i\varphi} \cos^2 \frac{\theta}{2}} \]

\[ e^{-i\chi e^{i\varphi} \sin^2 \frac{\theta}{2}} \]

\[ e^{i\chi e^{i\varphi} \cos \theta} \frac{\sqrt{2}}{2} \]

\[ e^{i\chi e^{i\varphi} \sin \theta} \frac{\sqrt{2}}{2} \]

the (1 + 2J) columns constitute, for each group element \( u(\chi, \varphi, \theta) \), a spherical basis. These (1 + 2J) vectors can be chosen unitary with respect to the representation space scalar product

\[ \langle J; n' | J, n \rangle = \sum_m [J]_m^n(\chi, \varphi, \theta) [J]_m^{n'}(\chi, \varphi, \theta) = \delta_{nn'} \]

without any integration. Integrating over the group, all matrix elements are Schur-orthonormalized

\[ \{ [J]_{m'}^{n'} | [J]_{m}^{n} \} = \int_{-2\pi}^{2\pi} \frac{dx}{4\pi} \int_{0}^{2\pi} \frac{d\varphi}{2\pi} \int_{-1}^{1} \frac{d\cos \theta}{2} \cdot [J]_{m'}^{n'}(\chi, \varphi, \theta) [J]_{m}^{n}(\chi, \varphi, \theta) \]

\[ = \frac{1}{1+2J} \delta_{nn'} \delta_{mm'} \]

\[ e.g. \int_{-2\pi}^{2\pi} \frac{dx}{4\pi} \int_{0}^{2\pi} \frac{d\varphi}{2\pi} \int_{-1}^{1} \frac{d\cos \theta}{2} |e^{i\chi e^{i\varphi} \cos^2 \frac{\theta}{2}}| = \frac{1}{3} \]

### 2.2 - for Scattering States with Position Translations

The irreducible representations of the noncompact and nonabelian Euclidean group for position space, a semidirect product of rotations acting upon translations

\[ \text{SU}(2) \ltimes \mathbb{R}^3, \quad (O, \vec{x}) \in \text{SO}(3) \ltimes \mathbb{R}^3 \]

\[ (O_1, \vec{x}_1) \circ (O_2, \vec{x}_2) = (O_1 O_2, \vec{x}_1 + O_1 \vec{x}_2) \]

\[ \text{SO}(3) \ltimes \mathbb{R}^3 / \text{SO}(3) \cong \mathbb{R}^3 \] (no group isomorphism)

are more complicated than the 1-dimensional ones for the time translation group \( \mathbb{R} \).

With Schur again, the irreducible representations of translations in any dimension \( n = 1, 2, \ldots \) are 1-dimensional with the (energy-)momenta as invariant eigenvalues (dual group, characters)

\[ \mathbb{R}^n \ni x \mapsto e^{ipx} \in U(1), \quad p \in \mathbb{R}^n \]

There is a good method to give the representation of affine groups: With Wigner and Mackey[15, 9, 10, 3], the affine group representations can be induced from representations of fixgroups (Wigner’s little groups) which leave (energy-)momenta invariant.

For position translations \( \vec{x} \in \mathbb{R}^3 \) with nontrivial momentum eigenvalues

\[ \vec{p} \in \mathbb{R}^3 \text{ with } \vec{p}^2 = P^2 > 0 \]

the fixgroup consists of the axial rotations \( \text{SO}(2) \) around the momentum direction \( \frac{\vec{p}}{P} \). The \( \text{SO}(2) \)-representations together with the position translation representations induce all representations of the Euclidean group \( \text{SU}(2) \ltimes \mathbb{R}^3 \).
They collect all translation representations \( \mathbb{R}^3 \mapsto e^{-i\vec{p} \cdot \vec{x}} \) for momenta \( \vec{p} \) on the momentum sphere \( \Omega^2 \cong \text{SO}(3)/\text{SO}(2) \) with invariant radius \( P \). E.g., the scalar matrix elements arise with the trivial \( \text{SO}(2) \)-representation

\[
\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3p}{2\pi^2} \delta(\vec{p}^2 - P^2)e^{-i\vec{p} \cdot \vec{x}} = \frac{\sin Pr}{Pr} = j_0(Pr)
\]

for \( P > 0 \) with \( r = \sqrt{\vec{x}^2} \), supported by \( \vec{p}^2 = P^2 \).

They are normalized for the neutral position translation \( \vec{x} = 0 \) ('here'). The value of the spherical Bessel function \( j_0(Pr) \) is a matrix element of an infinite dimensional representation. It has to be seen in analogy to \( e^{iEt} \) as matrix element of a 1-dimensional time translation representation (more below).

As a translation \( \mathbb{R}^3 \)-representation, the matrix element \( j_0(Pr) \) belongs to a cyclic representation. It is not decomposable into a direct sum of irreducible \( \mathbb{R}^3 \)-representations, but into a direct integral. As an \( \text{SO}(3) \times \mathbb{R}^3 \)-representation it is irreducible as seen from the nontrivial rotation behavior for the irreducible components \( e^{-i\vec{p} \cdot \vec{x}} \mapsto e^{-i\vec{p} \cdot (O \cdot \vec{x})} \). With the homogeneous group acting upon the translations, all momenta on a 2-sphere have to be included as eigenvalues of the translation representation \( \{e^{i(O \cdot \vec{p}) \cdot \vec{x}} | O \in \text{SO}(3)\} \).

Starting from flat position space \( \mathbb{R} \) with trivial rotations \( \text{SO}(1) = \{1\} \), the \( \text{SO}(3) \)-rotation degrees of freedom use - via the \( \Omega^2 \)-derivative \( \frac{\partial}{\partial \vec{x}} \) - the 2-sphere spread of a 1-dimensional translation representation matrix element

\[
\int d^3p \, \delta(\vec{p}^2 - 1)e^{-i\vec{p} \cdot \vec{x}} = -i\frac{\partial}{\partial \vec{x}} \int dp \, \delta(p^2 - 1)e^{-ipr} = -i\frac{\partial}{\partial \vec{x}} \cos r = 2\pi j_0(r)
\]

Matrix elements for a nontrivial representation of the homogeneous rotation group \( \text{SO}(3) \) are obtained by position derivations \( \frac{\partial}{\partial \vec{x}} = \frac{\vec{x}}{\vec{x}} \cdot \frac{\partial}{\partial \vec{x}} = 2\vec{x} \frac{\partial}{\partial \vec{x}} \), e.g. with angular momentum \( L = 1 \) involving the spherical harmonics \( Y_{1,2,3}^1(\varphi, \theta) \sim \frac{\vec{x}}{r} \)

multiplied with the matching spherical Bessel function \( j_1 \)

\[
\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3y}{2\pi^2} \, i\frac{y}{r^2} \delta(\vec{y}^2 - P^2)e^{-i\vec{y} \cdot \vec{x}} = \frac{\vec{x}}{r} j_1(Pr), \quad j_1(r) = \frac{\sin r - r \cos r}{r^2}
\]

The matching products \( \vec{x} \mapsto Y_{m}^L(\varphi, \theta)j_L(Pr) \) of spherical harmonics and spherical Bessel functions are familiar e.g. from the planar wave decomposition into matrix elements of irreducible \( \text{SO}(3) \times \mathbb{R}^3 \)-representations

\[
e^{i\vec{p} \cdot \vec{x}} = \sum_{L=0}^{\infty} (1 + 2L)P^L(\cos \theta)i^Lj_L(Pr), \quad P^L(\cos \theta) = \sqrt{\frac{4\pi}{1 + 2L}} Y_{0}^L(\varphi, \theta)
\]

Nonrelativistic scattering theory is described with the irreducible \( \text{SU}(2) \times \mathbb{R}^3 \)-representations. The Hilbert spaces induced by a trivial or faithful representation \( \text{SO}(2) \) on \( W \cong \mathbb{C}^n \), \( n = 1, 2 \), have a measure related distributive basis with generalized (distributive) eigenstates

\[
\begin{align*}
& \text{for } J = 0 : \quad \{|P^2, 0; \vec{\omega}, h\} \, | \, \vec{\omega} \in \Omega^2, \, h = 0 \\
& \text{for } J = \frac{1}{2}, 1, \ldots : \quad \{|P^2, J; \vec{\omega}, h\} \, | \, \vec{\omega} \in \Omega^2, \, h = \pm 1
\end{align*}
\]

By abuse of language since not a Hilbert space vector, an element of the distributive basis \( |P^2, J; \vec{\omega}, h\rangle \) is called a scattering ‘eigenstate’ with momentum \( \vec{p} \) of absolute value \( P \) (translation invariant), direction \( \vec{\omega} \) (translation
eigenvalues) and polarization \( J = 0, \frac{1}{2}, 1, \ldots \) (rotation invariant) with \( \text{SO}(2) \)-eigenvalues \( h \). The notation for an ‘eigenstate’ shows the representation characterizing invariants, here \( (P^2, J) \), before the semicolon and the eigenvalues in this representations, here \( (\vec{\omega}, h) \), behind the semicolon.

The distributive basis is acted upon with the inducing \( \text{SO}(2) \times \mathbb{R}^3 \)-representation

\[
(e^{\pm ix}, \vec{x}) \cdot |P^2, J; \vec{\omega}, h\rangle = e^{iJx}e^{-iP\vec{\omega} \vec{x}}|P^2, J; \vec{\omega}, h\rangle
\]

The momentum direction on the sphere \( \vec{\omega}/|\vec{\omega}| = \vec{\omega} \in \Omega^2 \cong \text{SU}(2)/\text{SO}(2) \) is the axis for the fixgroup \( \text{SO}(2) \) rotations

\[
\Omega^2 \ni \vec{\omega} = \vec{\omega} = \left( \frac{\sin \theta \cos \varphi}{\sin \theta \sin \varphi}, \frac{\sin \theta \sin \varphi}{\cos \theta} \right), \quad e^{-i\vec{p} \vec{x}} = e^{-i\vec{p} \vec{\omega} \vec{x}}, \quad \int d^3p = \int_0^\infty |p|^2 d|p| \int d^2\omega
\]

\[
f d^2\omega = f_0^\infty d\varphi \int_0^\pi d\cos \theta, \quad \delta(\vec{\omega}) = \frac{1}{\sin \theta} \delta(\theta) \delta(\varphi)
\]

The scalar product distribution is the product of the two scalar products for the semidirect factors

\[
\langle P^2, J; \vec{\omega}', h' | P^2, J; \vec{\omega}, h \rangle = \delta_{hh'} \delta(\frac{\vec{\omega} - \vec{\omega}'}{4\pi})
\]

\[
\int \frac{d^2\omega}{4\pi} |P^2, J; \vec{\omega}, h\rangle \langle P^2, J; \vec{\omega}, h| \cong 1_n \text{id}_{L^2(\Omega^2 \times \mathbb{R}^n)} = \text{id}_{L^2(\Omega^2 \times \mathbb{R}^n)}, \quad n = 1, 2
\]

With respect to the cyclic, but reducible position translation representations \( \mathbb{R}^3 \ni \vec{x} \mapsto j_0(Pr) = \int d^2\omega e^{-iP\vec{\omega} \vec{x}} \) the scalar product involves the positive and orthogonal Dirac distribution on the 2-sphere \( \delta(\vec{\omega} - \vec{\omega}') \). With respect to the fixgroup representations \( \text{SO}(2) \longrightarrow \text{SU}(n) \) the Kronecker \( \delta_{hh'} \) shows the scalar product in \( \mathfrak{C}^n, n = 1, 2 \).

The Hilbert space consists of 2-sphere square integrable momentum wave packets \( L^2(\Omega^2, \mathfrak{C}^n) \) valued in \( \mathfrak{C}^n, n = 1, 2 \)

\[
f \in L^2(\Omega^2, \mathfrak{C}^n) : \left\{ \begin{array}{l}
|P^2, J; f\rangle = \int \frac{d^2\omega}{4\pi} f(\vec{\omega})h|P^2, J; \vec{\omega}, h\rangle \\
\langle P^2, J; f_2 | P^2, J; f_1 \rangle = \int \frac{d^2\omega}{4\pi} f_2(\vec{\omega})h f_1(\vec{\omega})h
\end{array} \right.
\]

The transformation behavior of the Hilbert space vectors is built by that of the distributive basis. The \( \text{SO}(3) \)-representations are induced by the fixgroup \( \text{SO}(2) \)-representations[20].

All this can be seen as a distributional generalization of the finite dimensional case, e.g. from one basic vector with time dependence \( |E\rangle \mapsto e^{iEt}|E\rangle \) for the vector space \( \mathfrak{C} \) to a distributive basis \( \{ |P^2, J; \vec{\omega}, h\rangle \} \) for the infinite dimensional representation space \( L^2(\Omega^2, \mathfrak{C}^n) \). There are cyclic normalized vectors for the \( \text{SU}(2) \times \mathbb{R}^3 \)-representations, obtained by 2-sphere integration with the constant function \( 1 \in L^2(\Omega^2) \), \( f(\vec{\omega}) = 1 \) all elements of a distributive basis

\[
|P^2, J; 1, h\rangle = \int \frac{d^2\omega}{4\pi} |P^2, J; \vec{\omega}, h\rangle \in L^2(\Omega^2, \mathfrak{C}^n)
\]

\[
\langle P^2, J; 1, h'| P^2, J; 1, h \rangle = \delta_{hh'} j_0(0) = \delta_{hh'}
\]

The matrix element for time translation representations \( \langle E, t_2 | E, t_1 \rangle = e^{iEt} \), \( t = t_1 - t_2 \), has the analogue matrix element for the Euclidean position group representations on this cyclic vector

\[
f \frac{d^2\omega d^2\omega'}{(4\pi)^2} \langle P^2, J; \vec{\omega}_2, h_2, \vec{x}_2 | P^2, J; \vec{\omega}_1, h_1, \vec{x}_1 \rangle = \delta_{h_1 h_2} f \frac{d^2\omega}{4\pi} e^{-iP\vec{\omega} \vec{x}}
\]

\[
\langle P^2, J; 1, h_2, \vec{x}_2 | P^2, J; 1, h_1, \vec{x}_1 \rangle = \delta_{h_1 h_2} j_0(Pr), \quad \vec{x} = \vec{x}_1 - \vec{x}_2
\]
Hilbert spaces for different translation or rotation invariant \( \{P^2, L\} \) are orthogonal as seen in the Schur scalar product for the representation matrix elements, integrating with Haar measure \( d^3x \) over all translations

\[
\{P^2, L \mid P^2 > 0, \ L = 0, 1, \ldots \} \ 	ext{with} \quad \int d^3x \ D_{m^2}^{P^2, L}(\bar{x}) = \sqrt{\frac{4\pi}{1+2L}} \, \mathcal{Y}_m^L(\varphi, \theta) j^L Pr
\]

\[
\{D_{m^1}^{P^2, L_1} \mid D_{m^2}^{P^2, L_2} \} = \int d^3x \ D_{m^1}^{P^2, L_1}(\bar{x}) D_{m^2}^{P^2, L_2}(\bar{x})
\]

\[
= \frac{1}{1+2L} \delta_{L_1 L_2} \delta_{m_1 m_2} \int_0^\infty \frac{r^2 dr \, j_{L_1}(Pr) j_{L_1}(P_2r)}{4p_1^2 \delta \left(\frac{L_1 - 1}{2\pi}\right)}
\]

2.3 - for Stable Relativistic Particles with Spacetime Translations

The representations of the semidirect Poincaré group with the orthochronous Lorentz group acting on the spacetime translations

\[
\text{SL}(\mathbb{C}^2) \times \mathbb{R}^4, \ \ (\Lambda, x) \in \text{SO}_0(1, 3) \times \mathbb{R}^4
\]

\[
(\Lambda_1, x_1) \circ (\Lambda_2, x_2) = (\Lambda_1 \Lambda_2, x_1 + \Lambda_1 x_2)
\]

\[
\text{SO}_0(1, 3) \times \mathbb{R}^4/\text{SO}_0(1, 3) \cong \mathbb{R}^4 \text{ (no group isomorphism)}
\]

act - if nontrivial - on an infinite dimensional Hilbert space. There is Wigner’s classification[15] of these representations according to the fixgroup types for energy-momenta.

The representations for a stable massive particle arise by straightforward Lorentz compatible composition of time and position space structures: The fixgroup of energy-momenta \( p \in \mathbb{R}^4 \) with \( p^2 = m^2 > 0 \) is the rest system related position rotation group \( \text{SO}(3) \). The Poincaré group representation matrix elements collect the irreducible spacetime translation representations with eigenvalues on the forward and backward energy-momentum hyperboloids \( \mathcal{Y}_\pm^2 \cong \text{SO}_0(1, 3)/\text{SO}(3) \), e.g. for the Lorentz scalar representation matrix elements, relevant, e.g., for a stable pion

\[
\mathbb{R}^4 \ni x \longrightarrow \int \frac{d^4p}{2\pi^2 m^2} \delta(p^2 - m^2) e^{ipx} = \frac{\vartheta(-x^2)2\mathcal{K}_1(|mx|) - \vartheta(x^2)\pi N_{-1}(|mx|)}{|mx|}
\]

for \( m > 0 \) with \( |x| = \sqrt{x^2} \), supported by \( p^2 = m^2 \)

Starting from flat spacetime \( \mathbb{R}^2 \) with rotation free Poincaré group \( \text{SO}_0(1, 1) \times \mathbb{R}^2 \) and 1-dimensional energy-momentum hyperbolas \( \mathcal{Y}_\pm^1 \cong \text{SO}_0(1, 1) \) the embedding into 4-dimensional spacetime with rotation \( \text{SO}(3) \) degrees of freedom can be obtained with a 2-sphere spread by a Lorentz invariant derivative \( \frac{\partial}{\partial x} \)

\[
\int d^4p \delta(p^2 - 1) e^{ipx} = \int d^2p \delta(p^2 - 1) e^{ipx} \bigg|_{x=(t,r)}
\]

\[
= \int d^2p \vartheta(-x^2)2\mathcal{K}_0(|x|) - \vartheta(x^2)\pi N_0(|x|)
\]

The Neumann functions[5, 14] \( N_n \) for timelike translations \( \vartheta(x^2) \) and the MacDonald functions \( K_n \) for spacelike translations \( \vartheta(-x^2) \) integrate representation
matrix elements of 1-dimensional translations over the hyperboloid

\[
\mathbb{R} \ni t \mapsto -\pi N_0(t) = \int d\psi \cos(t \cosh \psi)
\]

\[
\mathbb{R} \ni r \mapsto 2K_0(r) = \int d\psi e^{-|r| \cosh \psi}
\]

Matrix elements of nontrivial representations of the Lorentz group \( \text{SO}_0(1, 3) \) are obtained by translation derivations \( \frac{\partial}{\partial x} = 2x \frac{\partial}{\partial r} \).

The Hilbert space for a stable massive particle with \( \text{SL}(\mathbb{C}^2) \times \mathbb{R}^4 \)-representation, induced by a finite dimensional representation of the spin-translation group on \( W \cong \mathbb{C}^{1+2J} \)

\[\text{SU}(2) \times \mathbb{R}^4 \longrightarrow \text{U}(1+2J) \]

\[ (u, x) \mapsto 2J(u)e^{ipx} \text{ with } p^2 = m^2 \]

has a distributive basis on the forward energy-momentum hyperboloid

\[ \{ |m^2, J; \vec{p}, a| \mid \vec{p} \in \mathbb{R}^3, c \in \mathbb{Y}^3, a = 1, \ldots, 1+2J \} \]

The hyperboloid is parametrizable with hyperbolic coordinates \( c \), appropriate for the Lorentz group action, or with the more familiar momentum coordinates \( \vec{p} \)

\[ \mathbb{Y}^3 \ni \vartheta(p^2) \theta(\pm p_0) \frac{\varphi}{|\varphi|} = \pm c = \left( \frac{\pm \cosh \psi}{\sinh \psi} \right), \quad p_0 = \sqrt{|p|^2 + \vec{p}^2} \]

\[ \vartheta(p^2) \theta(\pm p_0) e^{ipx} = e^{\pm i|p|x}, \quad \int d^4p \vartheta(\pm p_0) \theta(p^2) = \int_0^\infty |p|^3 d|p| \int_{-1}^1 d^3c \]

\[ f_+ d^3c = \int_0^\infty (\sinh \psi)^2 d\psi \int d^2\omega = \int \frac{d^4p}{2p_0} \delta(c) = \frac{1}{(\sinh \psi)^2} \delta(\psi) \delta(\bar{\psi}) \]

The elements of the distributive basis \( |m^2, J; \vec{p}, a| \) (no Hilbert space vectors) are called ‘eigenstates’ for a particle with the invariants mass \( m \), spin \( J \) and eigenvalues momentum \( \vec{p} \) and 3rd spin component \( a \).

The scalar product distribution for the ‘eigenstates’ is the product of the matrix elements of 1-dimensional translations over the hyperboloid

\[ \langle m^2, J; \vec{p}_2, a_2 | m^2, J; \vec{p}_1, a_1 \rangle = \delta_{a_1a_2} \delta(\vec{p}_1 - \vec{p}_2) \]

\[ \langle m^2, J; c_2, a_2 | m^2, J; c_1, a_1 \rangle = \delta_{a_1a_2} \delta(c_1 - c_2) \]

The representation space is the Hilbert space of the square integrable mappings on the forward energy-momentum hyperboloid. The completeness reads

\[ \int \frac{d^4p}{8\pi p_0} |m^2, J; \vec{p}, a\rangle \langle m^2, J; \vec{p}, a| = \int_+ \frac{d^3c}{4\pi} \delta(c) \delta(\vec{p}) = \mathbf{1}_{1+2J} \]
The spacetime translation action on the Hilbert space vectors is built by that on the distributive basis. The Lorentz $\text{SL}(\mathbb{C}^2)$-representations are induced by the rest system spin $\text{SU}(2)$-representations [15, 20].

In contrast to position space with compact homogeneous rotation group $\text{SU}(2)$, the representation matrix elements of the Poincaré group with noncompact homogeneous Lorentz group $\text{SL}(\mathbb{C}^2)$ are not square integrable. The normalization of the integral of the basis distribution with the constant function $f(c) = 1$ is no element of $L^2(\mathbb{R}^3, \mathbb{C}^{1+2j})$ because of the infinite hyperboloid volume

$$|m^2, J; 1, a\rangle = \int dp \frac{dp}{4\pi} |m^2, J; c, a\rangle \notin L^2(\mathbb{R}^3, \mathbb{C}^{1+2j})$$

$$\langle m^2, J; 1, a_1 |m^2, J; 1, a_2\rangle = \delta_{a_1a_2} \int dp \frac{dp}{2\pi} \delta(p_0) \delta(p^2 - 1)$$

Its translation dependent scalar product involves an $\text{SO}(1, 3) \rtimes \mathbb{R}^4$-representation matrix element

$$|m^2, J; 1, a, x\rangle = \int dp \frac{dp}{4\pi} |m^2, J; c, a, x\rangle$$

$$\int \frac{dp}{(4\pi)^2} \langle m^2, J; c_1, a_2, x_2 |m^2, J; c_1, a_1, x_1\rangle = \delta_{a_1a_2} \int dp \frac{dp}{4\pi} e^{ipx}, \quad x = x_1 - x_2$$

$$= \langle m^2, J; 1, a_2, x_2 |m^2, J; 1, a_1, x_1\rangle = \delta_{a_1a_2} \int dp \frac{dp}{2\pi} \delta(p_0) \delta(p^2 - 1) e^{ipx}$$

The projections [19] to the nonrelativistic time and position ‘states’ is effected by the appropriate Dirac distributions

$$\int \frac{dp}{2\pi m^2} \delta(p - k) \delta(p^2 - m^2) e^{ipx} = \int \frac{dp_0}{2\pi m^2} \delta(p_0 - m^2 - k^2) e^{ip_0 - ikx}$$

$$= \frac{1}{2\pi m^2} \cos k\omega \cos k_0 \frac{1}{\sqrt{m^2 + k^2}}$$

for rest system $k = 0$

$$\int \frac{dp}{2\pi m^2} \delta(p_0 - E) \delta(p^2 - m^2) e^{ipx} = \int \frac{dp_0}{2\pi m^2} \delta(E^2 - p_0^2 - m^2) e^{ipE - ipx}$$

$$= e^{ipE} \frac{P}{m^2} \frac{P}{m^2} \sin \frac{p_0}{P} + P = \sqrt{E^2 - m^2}$$

Hilbert spaces for stable particles $\{m^2, J\}$ with different translation or rotation invariant (or also with different internal charge $z \in \mathbb{Z}$, e.g. for particle-antiparticle) are orthogonal. In the corresponding Schur scalar product there arises the infinite volume of the energy-momentum hyperboloid

$$D^{m^2}(x) = \int \frac{dp}{2\pi m^2} \delta(p^2 - m^2) e^{ipx}$$

$$\{D^{m^2_1} | D^{m^2_2}\} = \int \frac{dp}{4\pi} D^{m^2_1}(x) D^{m^2_2}(x) = \frac{1}{4m^2_1} \delta(m^2_1 - m^2_2) \int dp \delta(p^2 - 1)$$

### 2.4 Selfdual Translation Representations

Both for position and spacetime translation representations, there arise the trigonometric functions, as seen, e.g., in $t \mapsto (\cos \mu t, \sin \mu t)$ as matrix elements of selfdual time representations. They combine dual representations [2] of time translations

$$t \mapsto e^{\pm i\mu t}$$

which act upon a 2-dimensional representation space $\mathbb{C}[\mu] \oplus \mathbb{C}(\mu) \cong \mathbb{C}^2$ as the direct sum spanned by eigenvector (‘ket’) and dual eigenvector (‘bra’).
Such a selfdual time translation representation is used, e.g., in the position-momentum formulation of the harmonic oscillator with time development
\[
\mathcal{L} = \begin{pmatrix}
e^{i\mu t} & 0 \\
0 & e^{-i\mu t}
\end{pmatrix}
\Rightarrow \int dE \left( \begin{pmatrix}\delta(E-\mu) & 0 \\
0 & \delta(E+\mu)\end{pmatrix} \right) e^{iEt}
\]
supported by \( E = \pm \mu \)

where position and momentum are real and imaginary part \((x, -i \hat{p}) = \frac{u + u^*}{\sqrt{2}}\) of creation and annihilation operators with dual time development.

The basis distribution for forward and backward energy-momentum hyperboloid with creation and annihilation operators for the Poincaré group, acted upon with dual translation representations, can be seen explicitly in a field expansion, e.g. for a stable neutral pion
\[
\Phi(x) = \int \frac{dp}{4\pi p_0 m} \frac{e^{ipx} u(p) + e^{-ipx} u^*(p)}{\sqrt{2}}
\]
The distributive basis has the scalar product distribution, written as Fock state expectation value \( \langle \ldots \rangle \)
\[
\langle u^*(p_2) u(p_1) \rangle = 8\pi p_0 \delta(p_1 - p_2), \quad \langle u(p_2) u(p_1) \rangle = \langle u^*(p_1) u^*(p_2) \rangle = 0
\]
which leads to the scalar Poincaré group representation matrix element as Fock value of the anticommutator
\[
\langle \{ \Phi(x_2), \Phi(x_1) \} \rangle = \int \frac{dp}{2\pi m^2} \delta(p^2 - m^2) e^{ipx}, \quad x = x_1 - x_2
\]

The Feynman propagator contains, in addition to the Fock value of the anticommutator, the quantization commutator multiplied with the causal order function \(\epsilon(x_0)\)
\[
\epsilon(x_0)[\Phi(x_2), \Phi(x_1)] = -\int \frac{dp}{2\pi m^2} \frac{1}{p^2 + i\epsilon - m^2} e^{ipx}
\]
The \(\epsilon(x_0)\)-multiplied quantization is no representation matrix element of the Poincaré group[16] as visible in the off-shell contributions (P denotes the principal value integration).

3 The Hilbert Spaces for Unstable Particles

Unstable particles are described with complex translation invariants, starting with \(\mu + i\Gamma\) for an energy \(\mu\) with a nontrivial width \(\Gamma > 0\) for time translations. All representations for unstable states (particles), by abuse of language called unstable representations, are infinite dimensional, even for abelian time translations. For a first survey, only spinless unstable states (particles) are considered.
3.1 - for Unstable Energy States

An irreducible representation matrix element of time translations can be written with a Dirac energy distribution supported by the translation invariant \( \mu \in \mathbb{R} \), i.e. as a residual representation\(^{[20]}\)

\[
\mathbb{R} \ni t \mapsto e^{i\mu t} = \int dE \, \delta(E - \mu) e^{iEt}
\]

supported by \( E = \mu \)

Finite dimensional time translation representations with nontrivial width have to be indefinite unitary. They start with the 2-dimensional representations\(^{[17]}\) involving complex conjugate energies

\[
\mathbb{R} \ni t \mapsto e^{i\mu t}\left(\begin{array}{c}
\epsilon^{-\Gamma t} \\
0
\end{array}\right) \in U(1) \times SO_0(1, 1) \subset U(1, 1)
\]

The sum of the two matrix elements multiplied with the characteristic functions for future and past is a matrix element of an unstable Hilbert representation

\[
\mathbb{R} \ni t \mapsto \vartheta(t) e^{i(\mu + i\Gamma)t} + \vartheta(-t) e^{i(\mu - i\Gamma)t} = e^{i\mu t - \Gamma t}
\]

normalized for the neutral time translation \( t = 0 \) (present). The spectral function for the energies in the matrix element of unstable time translation representations

\[
\int \frac{dE}{\pi} \frac{\Gamma}{(E - \mu)^2 + \Gamma^2} e^{iEt} = \oint \frac{dE}{2\pi i} \frac{\vartheta(t)}{E - \mu - i\Gamma} - \frac{\vartheta(-t)}{E - \mu + i\Gamma} e^{iEt}
\]

has two complex conjugated poles as invariants. It is a Breit-Wigner function ('\( \Gamma \)-widened Dirac distribution'). The Dirac distribution is the imaginary part of the advanced and retarded distribution

\[
\delta(E - \mu) = \frac{1}{2\pi i} \frac{1}{E - \mu - io} - \frac{1}{E - \mu + io} = \frac{1}{\pi} \frac{o}{(E - \mu)^2 + o^2} \quad \text{with positive } o \rightarrow 0
\]

An unstable time translation representation for \( \Gamma > 0 \) is a cyclic, but reducible time translation representation. It is the direct integral over irreducible representations \( t \mapsto e^{iEt} \) for all energies \( E \in \mathbb{R} \), distributed with the positive function \( E \mapsto \frac{\Gamma}{(E - \mu)^2 + \Gamma^2} \). An unstable time representation is an irreducible representation of the semidirect product of the discrete time reflection group acting upon the time translations

\[
(\epsilon, t) \in O(1) \rtimes \mathbb{R}, \quad \epsilon \in \{\pm 1\} = O(1)
\]

\[
(\epsilon_1, t_1) \circ (\epsilon_2, t_2) = (\epsilon_1 \epsilon_2, t_1 + \epsilon_1 t_2)
\]

\[
O(1) \rtimes \mathbb{R}/O(1) \cong \mathbb{R} \quad \text{(no group isomorphism)}
\]

Faithful representations of \( O(1) \rtimes \mathbb{R} \) are inducible by \( \mathbb{R} \)-representations with trivial fixgroup. Irreducible reflection group representations are 1-dimensional
- either invariant (trivial) under time reflection $|+\rangle \mapsto |+\rangle$ or faithful $|-\rangle \mapsto -|-\rangle$.

The Hilbert space for an unstable state is infinite dimensional. It has a distributive basis involving all energies

$$\Gamma > 0: \quad \{ |\mu, \Gamma; E \rangle \mid E \in \mathbb{R} \}, \quad |\mu, \Gamma; E, t \rangle = e^{iEt}|\mu, \Gamma; E \rangle$$

By abuse of language $|\mu, \Gamma; E \rangle$ is called an unstable ‘eigenstate’ with time translation eigenvalue $E$ for the invariant $\mu$ with width $\Gamma$. The Hilbert space vectors for the unstable time translation representations

$$|\mu, \Gamma; f \rangle = \int dE \ f(E)|\mu, \Gamma; E \rangle, \quad |\mu, \Gamma; f, t \rangle = \int dE \ f(E)|\mu, \Gamma; E, t \rangle$$

are defined with Fourier transformable elements $f(E) = \int dt \ \tilde{f}(t)e^{iEt}$ from the convolution Banach algebra $L^1(\mathbb{R})$ (group algebra) with the absolute integrable function classes $\int dt \ |f(t)| < \infty$ on the time translation group. Their scalar product for $\Gamma > 0$ is not defined by square integrability, but by a positive type function classes $\sup_{t} |\tilde{f}(t)| < \infty$. The unstable state scalar product is defined exactly by the matrix element $\{ t \mapsto D_{\mu,\Gamma}(t) \} \in L^\infty(\mathbb{R})$ of the cyclic time translation representation

$$\langle \mu, \Gamma; f_2 | \mu, \Gamma; f_1 \rangle = \int \frac{dE}{\pi} \frac{f_2(E)}{(E-\mu)^2+t^2} f_1(E)$$

$$= \int dt_1 dt_2 \ \tilde{f}_2(t_2) \ D_{\mu,\Gamma}(t_1-t_2) \ \tilde{f}_1(t_1)$$

The Hilbert space vectors are the finite norm functions $\langle \mu, \Gamma; f | \mu, \Gamma; f \rangle < \infty$. The scalar product involves the positive scalar product distribution for the distributive basis

$$\langle \mu, \Gamma; E' | \mu, \Gamma; E \rangle = \frac{1}{\pi} \frac{\Gamma}{(E-\mu)^2+1^2} \delta\left(\frac{E-E'}{2\pi}\right)$$

It is important to realize that - according to Gel’fand and Raikov [4] - cyclic representations and scalar product inducing positive type functions are uniquely related to each other.

Integrating with the constant function $f(E) = 1$ over all energies, one obtains a cyclic vector for the unstable time representation

$$|\mu, \Gamma; 1 \rangle = \int dE \ |\mu, \Gamma; E \rangle, \quad \langle \mu, \Gamma; 1 | \mu, \Gamma; 1 \rangle = 1$$

$$|\mu, \Gamma; 1, t \rangle = \int dE \ e^{iEt}|\mu, \Gamma; E \rangle, \quad \langle \mu, \Gamma; 1 | \mu, \Gamma; 1, t \rangle = D_{\mu,\Gamma}(t)$$

The associated Fourier transform $\int \frac{dE}{2\pi} e^{-iEt} = \delta(t) = \delta_0(t)$ is no element of the time translation group algebra $\delta_0 \notin L^1(\mathbb{R})$. The distribution $\delta_0$ embeds the trivial time translation $t = 0$. It can be approximated by $L^1(\mathbb{R})$-elements (approximate identity[3]).

The stable case for the Hilbert space functions is rediscovered as limit $\Gamma \to 0$. It reduces the distributive eigenvector basis for all energies $E \in \mathbb{R}$ to one basic vector for one energy $E = \mu$

$$\Gamma \to 0: \quad \langle \mu, \Gamma; f_2 | \mu, \Gamma; f_1 \rangle \to \int dE \ \tilde{f}_2(E) \ \delta(E-\mu) \ f_1(E)$$

$$= \int dt_1 dt_2 \ \tilde{f}_2(t_2) \ e^{i\mu(t_1-t_2)} \ \tilde{f}_1(t_1)$$

$$= \tilde{f}_2(\mu) \ f_1(\mu) \text{ with } |\mu; f \rangle = f(\mu)|\mu \rangle$$
The Schur product for the unstable time translation representations
\[ \{ D^{\mu, \Gamma} \mid \mu \in \mathbb{R}, \Gamma \geq 0 \} \text{ with } D^{\mu, \Gamma}(t) = e^{i\mu t - \Gamma |t|} \]
\[ \{ D^{\mu_2, \Gamma_2} \mid D^{\mu_1, \Gamma_1} \} = \int dt \, e^{i(\mu_1 - \mu_2)t - (\Gamma_1 + \Gamma_2)|t|} = 2 \frac{\Gamma_1 + \Gamma_2}{(\mu_1 - \mu_2)^2 + (\Gamma_1 + \Gamma_2)^2} \]
is not orthogonal. It gives nontrivial transition elements for unstable representations. Unstable particles can constitute collectives with other particles - stable or unstable (more below).

### 3.2 - for Unstable Scattering States

The transition from stable to unstable scattering states in position space \( \mathbb{R}^3 \) with complex translation invariant is obtained by widening - with width \( \gamma = \frac{\pi}{2} \) - the Dirac distribution on the momentum 2-sphere to a Breit-Wigner function

\[ \delta(p^2 - P^2) \rightarrow \delta_\gamma(p^2 - P^2) = \frac{1}{2\pi} \left[ \frac{1}{|p^2 - (P + i\gamma)^2|} - \frac{1}{|p^2 - (P - i\gamma)^2|} \right] \]

selfdual invariant singularities at \( |p| = \pm (P \pm i\gamma) \)

The unstable representations of the Euclidean position group involve representations of the position space reflection group

\[ \text{O}(3) \times \mathbb{R}^3, \quad \text{O}(3) \cong \text{O}(1) \times \text{SO}(3) \]
\[ \text{O}(3) \times \mathbb{R}^3/\text{O}(3) \cong \mathbb{R}^3 \text{ (no group isomorphism)} \]

It has the scalar matrix elements, normalized for the neutral position translation \( \vec{x} = 0 \)

\[ \mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3p}{2\pi^3} \delta_\gamma(p^2 - P^2)e^{-ip\vec{x}} \]
\[ = -\frac{1}{P} \frac{\partial}{\partial P} \int dp \delta_\gamma(p^2 - P^2)e^{-ipr} = -\frac{1}{P} \frac{\partial}{\partial P} \left[ \frac{e^{ipr}}{2(P + i\gamma)} + \frac{e^{-ipr}}{2(P - i\gamma)} \right] \]
\[ = \sin^2 \frac{Pr}{2\gamma} e^{-\gamma r} = D^{P_2, \gamma^2}(\vec{x}) \text{ with } P > 0, \gamma \geq 0 \]

The distributive basis for the expansion of the Hilbert space vectors from \( L^1(\mathbb{R}^3) \) uses all momenta

\[ \gamma > 0: \quad \{ |P^2, \gamma; \vec{p}\rangle \mid \vec{p} \in \mathbb{R}^3 \}, \quad |P^2, \gamma; \vec{p}, \vec{x}\rangle = e^{-i\vec{p}\vec{x}} |P^2, \gamma; \vec{p}\rangle \]
\[ |P^2, \gamma; f\rangle = \int \frac{d^3p}{2\pi} f(\vec{p}) |P^2, \gamma; \vec{p}\rangle, \quad |P^2, \gamma; f, \vec{x}\rangle = \int \frac{d^3p}{2\pi} f(\vec{p}) |P^2, \gamma; \vec{p}, \vec{x}\rangle \]

Again, \( |P^2, \gamma; \vec{p}\rangle \) is called an unstable scattering ‘eigenstate’ with position translation eigenvalue \( \vec{p} \) for the invariant \( P^2 \) with width \( \gamma^2 \).

The instability characteristic structure is the scalar product which is not defined by square integrability, but exactly by the positive momentum spectral function which occurs in the scalar representation matrix elements

\[ \langle P^2, \gamma; \vec{p}_2 | P^2, \gamma; \vec{p}_1 \rangle = \frac{1}{P} \delta_\gamma(p^2 - P^2) 2\pi \delta(\vec{p}_1 - \vec{p}_2) \]
\[ \langle P^2, \gamma; f_2 | P^2, \gamma; f_1 \rangle = \int \frac{d^3p}{2\pi} f_2(\vec{p}) \delta_\gamma(p^2 - P^2) f_1(\vec{p}) \]
The scalar product can also be written with the positive type function in the position translation parametrization

\[
\langle P^2, \gamma; f_2 | P^2, \gamma; f_1 \rangle = \int d^3x_1 d^3x_2 \overline{f_2(\vec{x}_2)} D^{P^2, \gamma^2}(\vec{x}_1 - \vec{x}_2) f_1(\vec{x}_1)
\]

for \( f(\vec{p}) = \int d^3x \, \tilde{f}(\vec{x}) e^{-i\vec{p}\cdot\vec{x}} \)

A cyclic vector can be constructed with an approximate identity in \( L^1(\mathbb{R}^3) \)

\[
|P^2, \gamma; 1\rangle = \int \frac{d^3\vec{p}}{2\pi} | P^2, \gamma; \vec{p}\rangle, \quad |P^2, \gamma; 1, \vec{x}\rangle = \int \frac{d^3\vec{p}}{2\pi} e^{-i\vec{p}\cdot\vec{x}} | P^2, \gamma; \vec{p}\rangle
\]

\[
\langle P^2, \gamma; 1 | P^2, \gamma; 1, \vec{x}\rangle = D^{P^2, \gamma^2}(\vec{x})
\]

The scalar product defining Breit-Wigner function \( \vec{p} \mapsto \delta_\gamma(\vec{p}^2 - P^2) \) describes a \( \gamma \)-shell around the momentum 2-sphere with radius \( \mathcal{P} \). For the stable case \( \gamma = 0 \) the support of the Hilbert space functions is restricted to the momentum 2-sphere. In this limiting case, i.e. for stable states, the functions have to be square integrable

\[
\gamma \to 0 : \quad \langle P^2, \gamma; f_2 | P^2, \gamma; f_1 \rangle \to \int \frac{d^3\vec{p}}{2\pi} \overline{\tilde{f}_2(\vec{p})} \delta(\vec{p}^2 - P^2) f_1(\vec{p}) = \int \frac{d^3\vec{\omega}}{4\pi} \tilde{f}_2(\vec{\omega}) f_1(\vec{\omega})
\]

\[
= \int d^3x_1 d^3x_2 \overline{f_2(\vec{x}_2)} \frac{\sin P|\vec{x}_1 - \vec{x}_2|}{P|\vec{x}_1 - \vec{x}_2|} \tilde{f}_1(\vec{x}_1)
\]

The change from square integrable translation \( \mathbb{R}^n \)-representations using the full (energy-)momentum space \( \mathbb{R}^n \) and Plancherel unitarity

\[
f \in L^2(\mathbb{R}^n) : \quad \int \frac{d^np}{(2\pi)^n} \overline{\tilde{f}_2(p)} f_1(p) = \int d^nx \overline{\tilde{f}_2(x)} \tilde{f}_1(x)
\]

to stable representations of the Euclidean group \( SU(2) \times \mathbb{R}^3 \) shows up in the restriction to the momentum 2-sphere \( L^2(\Omega^2) \) and the position spherical Bessel function. For unstable representations there is the restriction with \( e^{-\gamma r} \) to \( L^1(\mathbb{R}^3) \)-functions.

### 3.3 - for Unstable Relativistic Particles

The embedding of unstable time and position ‘states’ into a Lorentz compatible spacetime framework uses the Poincare group with reflection

\[
SO(1, 3) \times \mathbb{R}^4, \quad SO(1, 3) \cong O(1) \times SO_0(1, 3)
\]

\[
SO(1, 3) \times \mathbb{R}^4/\mathbb{SO}(1, 3) \cong \mathbb{R}^4 \text{ (no group isomorphism)}
\]

It contains the groups \( O(1) \times \mathbb{R} \) for time and \( O(3) \times \mathbb{R}^3 \) for space translations. The factor acting upon the spacetime translations is the special Lorentz group which is itself a semidirect group with a 2-elementic reflection group \( O(1) \) acting on the orthochronous Lorentz group

\[
(\epsilon, \Lambda) \in SO(1, 3) : \quad (\epsilon_1, \Lambda_1) \circ (\epsilon_2, \Lambda_2) = (\epsilon_1 \epsilon_2, \Lambda_1 \epsilon_1 \Lambda_2 \epsilon_1)
\]

The reflection group is not Lorentz invariant. It can be realized e.g. with time reflections \( \epsilon \cong \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) or with space reflections \( \epsilon \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).
The transition from stable to unstable particles is obtained by widening the Dirac distribution on the backward and forward energy-momentum hyperboloid to a Breit-Wigner function with a width $\gamma = \frac{E}{2}$

$$\delta(p^2 - m^2) \rightarrow \delta_\gamma(p^2 - m^2) = \frac{1}{2\pi} \left| \frac{1}{p^2 - (m+i\gamma)^2} - \frac{1}{p^2 - (m-i\gamma)^2} \right|$$

$$= \frac{1}{\pi} \left( \frac{m}{p^2 - m^2 + \gamma^2} \right)$$

$$m\gamma > 0$$

Therewith the unstable scalar Poincaré group representation matrix elements are

$$\mathbb{R}^4 \ni x \mapsto \int \frac{d^4p}{(2\pi)^4} \delta_\gamma(p^2 - m^2)e^{ipx}$$

$$= 2\frac{m}{(2\pi)^2} \int d^2p \delta_\gamma(p^2 - m^2)e^{ipx}|_{x=(t,r)}$$

$$= D^{m^2,\gamma^2}(x) \text{ with } m > 0, \gamma \geq 0$$

Since no longer only ‘on-shell’ $\delta(p^2 - m^2)$, the Hilbert space vectors need an expansion with a distributive basis for all energy-momenta

$$\gamma > 0: \{ |m^2, \gamma; p\rangle | p \in \mathbb{R}^4 \}, \quad |m^2, \gamma; p, x\rangle = e^{ipx}|m^2, \gamma; p\rangle$$

$$|m^2, \gamma; f\rangle = \int \frac{d^2p}{(2\pi)^2} f(p)|m^2, \gamma; p\rangle, \quad |m^2, \gamma; f, x\rangle = \int \frac{d^2p}{2\pi} f(p)|m^2, \gamma; p, x\rangle$$

Again, $|m^2, \gamma; p\rangle$ is called an unstable particle ‘eigenstate’ with spacetime translation eigenvalue $p$ for the invariant $m^2$ with width $\gamma^2$.

The characterizing scalar product of the energy-momentum functions for an unstable particle is defined exactly by the positive energy-momentum spectral function of the representation. It formalizes a $\gamma$-shell around the stable particle mass forward and backward hyperboloid $\mathcal{V}_\pm^3(m)$. Only for the stable limiting case the Hilbert space vectors are square integrable functions supported by the two shell energy-momentum hyperboloid

$$\langle m^2, \gamma; p_1 | m^2, \gamma; p_2 \rangle = \frac{m}{m^2} \delta_\gamma(p_1^2 - m^2)2\pi \delta(p_1 - p_2)$$

$$\langle m^2, \gamma; f_1 | m^2, \gamma; f_2 \rangle = \int \frac{d^2p}{(2\pi)^2} f_2(p)\delta_\gamma(p^2 - m^2)f_1(p)$$

for $\gamma \rightarrow 0$:

$$\rightarrow \int \frac{d^2p}{(2\pi)^2} f_2(p)\delta(p^2 - m^2)f_1(p)$$

$$\rightarrow \int \frac{d^2p}{4\pi p_0} f_2(p)f_1(p)$$

The projection[19] on matrix elements for unstable energy ‘states’ defines sharp momenta systems

$$\int \frac{d^4p}{2\pi m^2} \delta(\vec{p} - \vec{k})\delta_\gamma(p^2 - m^2)e^{ipx} = \frac{1}{2\pi m^2} \int dp \delta_\gamma(p^2 - E^2)e^{ipt-i\vec{k} \vec{x}}$$

$$= \frac{1}{2\pi m^2} \left[ \frac{e^{iEt}}{2(E+i\gamma_k)} + \frac{e^{-iEt}}{2(E-i\gamma_k)} \right]e^{-\gamma_k |t|}e^{-i\vec{k} \vec{x}}$$

$$\vec{k} = 0 \Rightarrow E + i\gamma_k = m + i\gamma :$$

$$= \frac{1}{2\pi m^2} \frac{m \cos m\gamma + \sin E \gamma |t|}{m^2 + \gamma^2}e^{-\gamma |t|}$$

where the complex invariants $(E \pm i\gamma_k)^2$ arise from

$$p_0^2 - \vec{k}^2 - (m + i\gamma)^2 = p_0^2 - (E + i\gamma_k)^2$$

$$\Rightarrow E + i\gamma_k = \sqrt{\vec{k}^2 + m^2 - \gamma^2 + 2im\gamma}$$

$$= \sqrt{\vec{k}^2 + m^2 + \frac{im^2}{m^2 - \gamma^2}}$$
The projection on matrix elements for unstable scattering ‘states’ defines sharp energy systems
\[ \int \frac{d^4p}{2\pi m^2} \delta(p_0 - E) \delta_{ij}(p^2 - m^2) e^{ipx} \]
\[ = - \frac{P_c^2}{m^2} \int \frac{d^4p}{2\pi m^2} \delta(p^2 - P^2) e^{iEt - ip^x} \]
where the complex invariants \((P \pm i\gamma E)^2\) arise from
\[ E^2 - \bar{p}^2 - (m - i\gamma)^2 = -[\bar{p}^2 - (P + i\gamma E)^2] \]
\[ \Rightarrow P + i\gamma E = \sqrt{E^2 - m^2 + \gamma^2 + 2imT} \gamma = \frac{\sqrt{E^2 - m^2}(1 + \frac{my}{E^2 - m^2} + …)}{2} \]
Also mixed projections are possible, e.g. systems with both nontrivial energy and momentum spread.

4 Scalar Products for Hilbert Spaces

Unstable translation representations involve positive type functions as an essential structure. It is useful to illustrate the general structure of the related scalar products first on finite dimensional vector spaces.

4.1 Dual Bases and Scalar Product Related Bases

A vector space \( V \cong \mathbb{C}^n \) has a basis \( \{ e_j \mid j = 1, \ldots, n \} \). The dual vector space \( V^T \cong \mathbb{C}^n \), i.e. the linear forms of \( V \), has the dual basis \( \{ \epsilon_j \mid j = 1, \ldots, n \} \) leading to a decomposition of the identity transformation

\[ \text{dual product}: \langle \epsilon_k, e^j \rangle = \delta_{k}^{j}, \quad \text{id}_V = e^j \otimes \epsilon_j \]

With dual bases, a \( V \)-endomorphism \( S : V \longrightarrow V \) can be written as tensor product with its matrix components \( S = S_j^k e^j \otimes \epsilon_k, S_j^k \in \mathbb{C} \). All this is purely algebraic - without any metrical structure. There is no natural isomorphism between \( V \) and its dual \( V^T \).

A scalar product of \( V \), sesquilinear and positive definite

\[ d: V \times V \longrightarrow \mathbb{C}, \quad \begin{cases} d(v, w) = d(w, v), & d(v, \alpha w) = \alpha d(v, w), \quad \alpha \in \mathbb{C} \\ v \neq 0 \iff d(v, v) > 0 \\ d(e^j, e^k) = \delta^{jk} \end{cases} \]

endows \( V \) with a Hilbert space structure.

Since nondegenerate, \( \det d^{jk} \neq 0 \), it defines an antilinear isomorphism between \( V \) and its dual \( V^T \). Such a dual antilinear isomorphism is the origin of Dirac’s bra-ket notation

\[ V \leftrightarrow \mathbb{C}^n, \quad e^j = |e^j\rangle \leftrightarrow \langle e^j| = d^{jk} \epsilon_k, \quad d^{jk} = \langle e^j| e^k \rangle \]

With the scalar product induced isomorphism there is an antilinear isomorphy \( \cong \) also for the tensor representation of the endomorphisms, e.g. for the decomposition of the identity which becomes antilinear isomorphic to the inverse scalar product

\[ S = S_j^k e^j \otimes \epsilon_k \cong S_j^k |e^j\rangle d^{kl}(e^l) = S_{jl} |e^j\rangle \langle e^l|, \quad S_{jl} = S_j^k d_{kl} \]

\[ e.g. \quad \text{id}_V = e^j \otimes \epsilon_j \cong |e^j\rangle d_{jl}(e^l) \]
For a non-diagonal scalar product the bra-ket decomposition of the identity is non-diagonal too.

The positivity of a scalar product can be expressed for the characterizing matrix by the $C^*$-algebra positivity

$$\text{complex } (n \times n)\text{-matrix } d \succeq 0 \iff d = \xi \circ \xi$$

$$\iff d = d^* \text{ and } \text{spec } d \subset \mathbb{R}_+$$

The hermitian $d$ can be unitarily diagonalized with positive diagonal elements (positive spectral values)

$$d = u^* \circ \text{diag } d \circ u, \quad u \in \text{SU}(n)$$

For an orthogonal basis with individual normalization factors one obtains

$$(\text{diag } d)^{jk} = |d(j)|^2 \delta^{jk} \text{ (no } j\text{-sum}) \Rightarrow \text{id}_V \cong \sum_j \langle e^j | \frac{1}{|d(j)|^2} \langle e^j | e^j \rangle$$

By renormalization, an orthonormal basis can be used

$$d = u^* \circ \text{diag } d \circ u = \xi^* \circ 1_n \circ \xi \text{ with } \xi = \sqrt{\text{diag } d \circ u}$$

The scalar product defining positive type functions above, e.g. for unstable energy ‘eigenstates’

$$\langle \mu, \gamma; E' | \mu, \gamma; E \rangle = \frac{1}{\pi} \frac{\gamma}{(E - \mu)^2 + \gamma^2} \delta\left(\frac{E - E'}{2\pi}\right)$$

are the distributional analogue to an orthogonal, but not normalized basis. For particle collectives (below) - in a basis with the translation ‘eigenstates’ - the scalar product is even non-diagonal.

### 4.2 Positive Type Functions

Positive type functions are the continuous generalization of scalar products for finite dimensional spaces.

In contrast to compact groups, Hilbert representations of a locally compact noncompact group $G$ with Haar measure have not to act on square integrable functions. The group algebra $L^1(G)$ contains the absolute integrable function classes on $G$. All function properties hold Haar measure almost everywhere. A Hilbert metric for $L^1(G)$ uses the group algebra dual, the continuous linear forms $L^\infty(G)$ with the essentially bounded function classes. $L^\infty(G)$ has the functions of positive type $d \in L^\infty(G)_+$ as a cone. Positive type functions or, also, scalar product inducing functions are defined by the property to give a positive linear form of the group algebra

$$\text{for all } f \in L^1(G) : \langle \hat{f} \ast f \rangle_d = \int_{G \times G} dg_1 dg_2 \hat{f}(g_1) \overline{f(g_1^{-1} g_2)} f(g_2) \geq 0$$

Positive type functions have not to be positive functions. A continuous essentially bounded group function $d \in L^\infty(G)$ is a positive type function if it gives rise to finite scalar product matrices

$$d \in L^\infty(G)_+ \iff d(g_j^{-1} g_k)_{j,k=1}^n \succeq 0 \text{ for all } n = 1, 2, \ldots, g_{j,k} \in G$$
It follows the conjugation invariance of \( d \) and the absolute value restriction by the value at the neutral element

\[
\begin{align*}
n & = 1 : \quad d(e) \geq 0 \\
n & = 2 : \quad (g_1, g_2) = (e, g), \quad \left( \frac{d(e)}{d(g^{-1})}, \frac{d(g)}{d(e)} \right) \geq 0 \Rightarrow \left\{ \begin{array}{l}
\frac{d(g^{-1})}{d(g)} = \overline{d(g)} \\
|d(g)| \leq d(e)
\end{array} \right.
\end{align*}
\]

There is a bijection between the positive type functions and the cyclic Hilbert representations[4], i.e. representations with a cyclic vector \(|c\rangle\). All positive type functions are representation matrix elements with a cyclic vector \(d(g) = \langle c|D(g)c \rangle\). In general, the algebra \( L^1(G) \) contains no unit, i.e. \( \delta_e \notin L^1(G) \) for the Dirac distribution. The class of an approximate identity \( \delta_e \) leads to a cyclic vector whose representation matrix element is exactly the positive type function.

A state is a normalized positive type function, i.e. \( d(e) = 1 \). There is a bijection between the pure states (extremal normalized positive type functions) and the irreducible Hilbert representations. All pure states are irreducible representation matrix elements with a cyclic vector (more exact formulation and more details in [3, 6]).

In the former sections cyclic representations of translations \( \mathbb{R}^n \) have been used which are irreducible representations of affine subgroups \( H \times \mathbb{R}^n \subseteq GL(\mathbb{R}^n) \times \mathbb{R}^n \). Positive type functions for cyclic translation representations have the properties

\[
d \in L^\infty(\mathbb{R}^n)_+, \quad x \in \mathbb{R}^n \Rightarrow \quad d(-x) = \overline{d(x)}, \quad |d(x)| \leq d(0)
\]

The positivity of spectral values (diagonal elements) for finite matrices has its continuous counterpart - with Bochner’s theorem [1] - in the positivity of the Fourier transformed positive type functions, i.e. in positive spectral functions (positive Radon measures) for the translation invariants (energy-momenta) \( p \in \text{spec} \mathbb{R}^n \cong \mathbb{R}^n \)

\[
d(x) = \int d^n p \overline{\tilde{d}(p)} e^{ipx} \text{ with positive distribution } \tilde{d}
\]

With the energy-momentum distribution the Hilbert space can be defined via energy-momentum functions

\[
\langle f_1 | f_2 \rangle = \langle \hat{f}_1 \ast f_2 \rangle_d = \int d^n x_1 d^n x_2 \overline{\hat{f}_1(x_1)} d(x_2 - x_1) f_2(x_2) = \int d^n p \overline{\hat{f}_1(p)} \tilde{d}(p) \hat{f}_2(p)
\]

The examples above for stable and unstable states (particles) are summarized in the following table with positive widths \( \Gamma, \gamma > 0 \):

| Irreducible for group | \( d \in L^\infty(\mathbb{R}^n)_+ \), \( x \mapsto d(x) \) | \( d(x) = \int d^n p \tilde{d}(p) e^{ipx} \) |
|----------------------|---------------------------------|-------------------------------|
| \( \mathbb{R} \)     | \( t \mapsto e^{ipt} \)         | \( \int d^n \psi \delta(E - \mu) e^{iE t} \) |
| \( \text{SO}(3) \times \mathbb{R}^3 \) | \( \vec{x} \mapsto \frac{\sin \vec{p} \vec{x}}{i20m^2\vec{x}^2} \) | \( \int \frac{d^3 p}{2\pi^2} \delta(p^2 - P^2)e^{-ip\vec{x}} \) |
| \( \text{SO}_3(1,3) \times \mathbb{R}^4 \) | \( x \mapsto \frac{1}{2\sqrt{2m^2 \vec{x}^2}} e^{-\gamma \vec{x}^2} \) | \( \int \frac{d^4 p}{2\pi^2} \delta(p^2 - m^2)e^{ip\gamma \vec{x}} \) |
| \( \text{O}(1) \times \mathbb{R} \)     | \( t \mapsto e^{it\gamma} \) |
| \( \text{O}(3) \times \mathbb{R}^3 \)     | \( \vec{x} \mapsto \frac{\sin \vec{p} \vec{x}}{i20m^2\vec{x}^2} e^{-\gamma \vec{x}^2} \) | \( \int \frac{d^3 p}{2\pi^2} \frac{1}{\sqrt{(p^2 - m^2)^2 + 4\gamma^2} e^{-ip\vec{x}}} \) |
| \( \text{SO}(3) \times \mathbb{R}^4 \)     | \( x \mapsto \frac{1}{2\sqrt{2m^2 \vec{x}^2}} e^{-\gamma \vec{x}^2} \) | \( \int \frac{d^4 p}{2\pi^2} \frac{1}{\sqrt{(p^2 - m^2 + \gamma^2)^2 + 4\gamma^2\vec{x}^2} e^{-ip\gamma \vec{x}}} \) |
The Dirac energy-momentum distributions for stable structures lead to Hilbert spaces with square integrable functions whereas the Breit-Wigner distributions define the scalar products for the not square integrable functions used for unstable structures. The irreducible translation representation matrix elements \( \mathbb{R}^n \ni x \mapsto e^{ipx} \) are pure states, i.e. extremal normalized positive type functions.

5 Probability Collectives of Unstable Particles

Unstable particles come in collectives\([18, 19]\) as familiar from the system with the two unstable neutral kaons which - via nontrivial transition elements - ‘share one identity’. The scalar product can involve nontrivial (non-diagonal) transitions between unstable states (particles).

5.1 Nonorthogonal Transition Probabilities

A Hilbert space operator \( H \), e.g. a Hamiltonian, with eigenvectors

\[
\langle E_2 | H | E_1 \rangle = E_1 \langle E_2 | E_1 \rangle = \overline{E_2} \langle E_2 | E_1 \rangle \\
\langle E_2 | H^* | E_1 \rangle = \overline{E_1} \langle E_2 | E_1 \rangle = E_2 \langle E_2 | E_1 \rangle
\]

has - if hermitian - only real eigenvalues and orthogonal eigenvectors for different eigenvalues

\[
H = H^* \Rightarrow \begin{cases} E = \overline{E} \\ E_1 \neq E_2 \iff \langle E_2 | E_1 \rangle = 0 \end{cases}
\]

If there arise complex energy eigenvalues, i.e. \( \text{spec} \ H \not\subset \mathbb{R} \), the Hamiltonian cannot be hermitian, \( H \neq H^* \). Then, different eigenvectors have not to be orthogonal. Several eigenvectors may constitute a probability collective with more than one dimension, i.e. a Hilbert subspace, whose scalar product is not decomposable with eigenvectors. The transition from a translation eigenstate basis \( \{ |E\rangle \} \) to an orthogonal basis \( \{ |U\rangle \} \) defines a Hilbert-bein for the probability collective, a non-unitary Hilbert space automorphism \( \xi \not\in \text{U}(n) \), as a representative from the Hilbert-bein manifold \( \text{GL}(\mathbb{C}^n)/\text{U}(n) \)

\[
|E\rangle = \xi |U\rangle, \quad \langle U | U \rangle = 1_n \Rightarrow \langle E | E \rangle = \xi^* \circ \xi
\]

The scalar product matrix \( \xi^* \circ \xi \) is positive definite, but not diagonal.

An example for an unstable particle collective is constituted by the short and long lived neutral kaon \( \{ |E\rangle \} = \{ K^0_{S,L} \} = \{ |S\rangle, |L\rangle \} \)

\[
m_S - m_L \sim 35 \times 10^{-13} \text{MeV} \quad \text{e}^2, \quad \begin{cases} \Gamma_S \sim 72 \times 10^{-13} \text{MeV} \quad \text{e}^2 \\ \Gamma_L \sim 0.13 \times 10^{-13} \text{MeV} \quad \text{e}^2 \end{cases}
\]

which are related to the orthonormal CP- or time reflection eigenstates \( \{ |U\rangle \} = \{ K^0_\pm \} = \{ |+\rangle, |-\rangle \} \). A 2×2-Hilbert-bein \( \xi \in \text{GL}(\mathbb{C}^2)/\text{U}(2) \) can be parametrized
with two complex numbers
\[
\left( \begin{array}{c} |S\rangle \\ |L\rangle \end{array} \right) = \xi \left( \begin{array}{c} e^{i\epsilon} \\ 1 \end{array} \right), \quad \xi = \frac{1}{N \sqrt{1+|\epsilon|^2}} \left( \begin{array}{c} 1 \\ e^{i\epsilon} \end{array} \right), \quad \epsilon, N \in \mathbb{C}
\]
\[
\xi^* \circ \xi = \left( \begin{array}{cc} |S\rangle \langle S| & |S\rangle \langle L| \\ |L\rangle \langle S| & |L\rangle \langle L| \end{array} \right) = \frac{1}{|N|^2} \left( \begin{array}{cc} 1 & \delta \\ -\delta & 1 \end{array} \right). \quad \text{spec } \xi^* \circ \xi = \{ \frac{i+\delta}{|N|^2} \}
\]
\[
\delta = \frac{\epsilon+|\epsilon|}{1+|\epsilon|^2} \approx 0.327 \times 10^{-2} \text{ (experimental value)} \Rightarrow \xi \notin \mathbb{U}(2)
\]

For an understanding of the actual representations used and their Schur-nonorthogonality the representations of the affine group \( \mathbb{O}(1) \times \mathbb{R} \) as embedded in a relativistic spacetime structure has to be considered. That will not be done here. The related invariants (masses and widths) and the nonorthogonal scalar product matrices are assumed as experimentally given.

5.2 Collectives of Unstable Energy States

The time translation matrix elements for a collective with \( n \) unstable states come in \((n \times n)\)-matrices

\[
\mathbb{R} \ni t \longmapsto \mathbb{E}(t) = \xi^* \circ \text{diag} \int \frac{dE}{\pi} \frac{\Gamma_j}{(E-\mu_j)^2+\Gamma_j^2} e^{iEt} \circ \xi = \xi^* \circ \text{diag} e^{i\mu_j t-\Gamma_j |t|} \circ \xi
\]

\[
\text{diag} \frac{1}{\pi} \frac{\Gamma_j}{(E-\mu_j)^2+\Gamma_j^2} = \frac{1}{\pi} \left( \begin{array}{cccc} \frac{\Gamma_S}{(E-m_S)^2+\Gamma_S^2} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma_L}{(E-m_L)^2+\Gamma_L^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma_L}{(E-m_L)^2+\Gamma_L^2} \end{array} \right)
\]

A Hilbert-bein \( \xi \notin \mathbb{U}(n) \) relates to each other an energy-eigenstate basis and an orthogonal one. The time translation matrix elements for the neutral translation \( t = 0 \) give the scalar product matrix \( \xi^* \xi \).

E.g., the explicit matrix for the kaon collective with invariants \( m_{S,L} \pm i\Gamma_{S,L} \) looks as follows

\[
\int \frac{dE}{\pi} \frac{\Gamma_S}{(E-m_S)^2+\Gamma_S^2} + |\epsilon|^2 \frac{\Gamma_L}{(E-m_L)^2+\Gamma_L^2} e^{imLt-\Gamma_L |t|} + \frac{\Gamma_S}{(E-m_S)^2+\Gamma_S^2} + |\epsilon|^2 \frac{\Gamma_L}{(E-m_L)^2+\Gamma_L^2} e^{imLt-\Gamma_L |t|} e^{iEt} \equiv \frac{1}{\pi} \left( \begin{array}{cc} e^{im Lt-\Gamma_L |t|} & + |\epsilon|^2 e^{im Lt-\Gamma_L |t|} \\ e^{im Lt-\Gamma_L |t|} & + |\epsilon|^2 e^{im Lt-\Gamma_L |t|} \end{array} \right)
\]

As in general for unstable 'states', a distributive basis needs all energies for all 'eigenstates'

\[
\Gamma_j > 0 : \quad \{ |\mu_j, \Gamma_j ; E \rangle \mid E \in \mathbb{R}, j = 1, \ldots, n \}
\]

The scalar product distribution is the positive representation matrix function

\[
\langle \mu_l, \Gamma_l ; E' | \mu_k, \Gamma_k ; E \rangle = \xi^* \circ \frac{1}{\pi} \text{diag} \frac{\Gamma_j}{(E-\mu_j)^2+\Gamma_j^2} \circ \xi \delta(\frac{E-E'}{2\pi})
\]

The Hilbert space vectors uses functions for all 'eigenstates'

\[
|\mu, \Gamma; f \rangle = \sum_{j=1}^{n} \int dE \ f_j(E) |\mu_j, \Gamma_j ; E \rangle
\]
5.3 Collectives of Unstable Particles

The Lorentz group compatible extension for scalar particle collectives

\[ \mathbb{R}^4 \ni x \mapsto \xi^* \circ \text{diag} \int \frac{d^4 p}{2\pi} \frac{1}{m_j^2} \delta_{\gamma_j}(p^2 - m_j^2) e^{ipx} \circ \xi \]

\[ \text{diag} \frac{1}{m_j^2} \delta_{\gamma_j}(p^2 - m_j^2) = \frac{1}{\pi} \begin{pmatrix} \frac{2\gamma_j}{(p^2 - m_1^2 + \gamma_1^2)^2 + 4m_1^2 \gamma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{2\gamma_n}{(p^2 - m_n^2 + \gamma_n^2)^2 + 4m_n^2 \gamma_n} \end{pmatrix} \]

leads to a Hilbert space with a distributive basis with 'eigenstates' of mass \( m_j^2 \) and widths \( \gamma_j \) for all energy-momenta

\[ \gamma_j > 0 : \{ |m_j^2, \gamma_j; p\rangle \mid p \in \mathbb{R}^4, j = 1, \ldots, n \} \]

The scalar product distribution involves the positive representation matrix function

\[ \langle m_l^2, \gamma_l; p_2 | m_k^2, \gamma_k; p_1 \rangle = \xi^* \circ \frac{1}{\pi} \text{diag} \frac{2\gamma_j}{(p^2 - m_j^2 + \gamma_j^2)^2 + 4m_j^2 \gamma_j} \circ \xi \ 2\pi \delta(p_1 - p_2) \]

The Hilbert space vectors uses functions for all 'eigenstates' in a nondecomposable collective

\[ |m^2, \gamma; f\rangle = \sum_{j=1}^n \int \frac{d^4 p}{2\pi} f_j(p) |m_j^2, \gamma_j; p\rangle \]

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