A NOTE ON THE ALGEBRA OF OPERATIONS FOR HOPF COHOMOLOGY AT ODD PRIMES

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Abstract. Let $p$ be any prime, and let $B(p)$ be the algebra of operations on the cohomology ring of any cocommutative $F_p$-Hopf algebra. In this paper we show that when $p$ is odd (and unlike the $p = 2$ case), $B(p)$ cannot become an object in the Singer category of $F_p$-algebras with coproducts, if we require that coproducts act on the generators of $B(p)$ coherently with their nature of cohomology operations.

1. Introduction

After noticing that the algebra $B(p)$ of Steenrod operations on $\text{Ext}_A(Z_p, Z_p)$, the cohomology of a graded cocommutative Hopf algebra $A$ over $F_p$, is (not even only for $p = 2$) neither a Hopf algebra nor a bialgebra, William B. Singer introduced in [25] the notions, one dual to the other, of a $k$-algebra with coproducts and $k$-coalgebra with products, for any commutative ring $k$, arguing that this is the right categorial setting to study $B(2)$ and its dual. Further examples of $k$-algebras with coproducts appeared in literature in the last decade. For instance the third author studied in [21] those arising as invariants of finitely generated $F_2$-polynomial algebras under the action of the general linear groups and their upper triangular subgroups.

More recently [7], the authors have taken into account $B(p)$, when $p$ is an odd prime. Such algebra has been described in terms of generators and relations by Liulevicius in [18]. The important role of $B(p)$ for stable homotopy computations is well and long established (see for example [1], [2], [3], [17], [18]). Furthermore a relevant subalgebra of $B(p)$ is a quotient of the universal Steenrod algebra $Q(p)$, introduced in [24] and broadly examined by the authors ([1], [16], [19], [20], [22]). Along the spirit of [25], in [7] the authors equipped $B(p)$ with a suitable collection of $F_p$-linear mappings that made it the underlying set of an object in the Singer category of $F_p$-algebras with coproducts. Yet, in [7], the chosen coproduct acting on the Bockstein operator $\beta$ has little to do with its nature of cohomology operation.

Our Theorem 2.3 states that $B(p)$ does not admit a structure of $F_p$-algebra with coproducts consistent with (2.10), and hence with the geometric meaning of all its generators.

A comparison between Theorems 2.2 and 2.3 shows that the non-primitivity of the Bockstein operator stands as unavoidable obstruction. This is an interesting phenomenon that deserves to be further investigated. In fact it suggests that Singer’s notion of algebra with coproducts in [25] needs to be refined, or perhaps that the algebra $B(p)$ is a deformation of a certain geometrically significant (and yet-to-be-determined) algebraic object.

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2. A Theorem of Non-existence

Let \( p \) be an odd prime. We recall that the algebra \( \mathcal{B}(p) \) of Steenrod operations on the cohomology ring of any cocommutative Hopf \( \mathbb{F}_p \)-algebra \( \Lambda \) is generated by

\[
P^k : \text{Ext}^{q,t}_{\Lambda}(\mathbb{Z}_p, \mathbb{Z}_p) \to \text{Ext}^{q+2k(p-1)\cdot t}_{\Lambda}(\mathbb{Z}_p, \mathbb{Z}_p) \quad (k, q, t \geq 0),
\]

and

\[
\beta : \text{Ext}^{q,t}_{\Lambda}(\mathbb{Z}_p, \mathbb{Z}_p) \to \text{Ext}^{q+1,pt}_{\Lambda}(\mathbb{Z}_p, \mathbb{Z}_p) \quad (q, t \geq 0)
\]

subject to the following relations (see [18]):

\[
\beta^2 = 0,
\]

\[
P^a P^b = \sum_{t=0}^{\lfloor \frac{a}{p} \rfloor} A(b, a, t) P^{a+b-t} P^t \quad \text{when} \ a < pb,
\]

\[
P^a \beta P^b = \sum_{t=0}^{\lfloor \frac{a}{p} \rfloor} B(b, a, t) \beta P^{a+b-t} P^t + \sum_{t=0}^{\lfloor \frac{a-1}{p} \rfloor} A(b, a-1, t) P^{a+b-t} \beta P^t \quad \text{when} \ a \leq pb.
\]

Coefficients in the several sums of (2.4) and (2.5) read as follows:

\[
A(k, r, j) = (-1)^{r+j} \left( \frac{(p-1)(k-j) - 1}{r - pj} \right)
\]

and

\[
B(k, r, j) = (-1)^{r+j} \left( \frac{(p-1)(k-j)}{r - pj} \right).
\]

Unlike the element with the same name in the ordinary Steenrod algebra \( \mathcal{A}_p \), \( P^0 \) in \( \mathcal{B}(p) \) is not the identity.

We now recall the definition of \( k \)-algebra with coproducts.

**Definition 2.1.** A \( k \)-algebra with coproducts is a bigraded unital algebra

\[
\mathcal{C} = \{ C_{n,s} \mid n, s \geq 0 \}
\]

together with degree preserving maps \( \epsilon_s : C_{n,s} \to k \) and \( \psi_s : C_{n,s} \to C_{n,s} \otimes C_{n,s} \) for each \( s \geq 0 \) such that:

(i) \( C_{n,s} \) is a graded coalgebra with counit \( \epsilon_s \) and coproduct \( \psi_s \), for each \( s \geq 0 \);

(ii) the algebra unit \( \eta : k \to C_{0,0} \) is a map of coalgebras;

(iii) the multiplication \( \mu : \otimes_{h+k=s} (C_{h,h} \otimes C_{k,k}) \to C_{n,s} \) preserves the coalgebra structure for each \( s \geq 0 \).

As already noted in [7], Item (iii) of Definition 2.1 makes sense: the category of graded coalgebras has tensor products and sums, and the category of graded algebras has tensor products and categorical products. Explicitly, given two graded algebras \( A \) and \( B \), on \( A \otimes B \) we assume defined the product

\[
(a \otimes b)(c \otimes d) = (-1)^{(\deg b)(\deg c)}(ac \otimes bd).
\]

It follows in particular that \( \mathcal{D}_{h,k} = C_{h,h} \otimes C_{k,k} \) is a coalgebra, and a comultiplication on \( \mathcal{E}_s = \otimes_{h+k=s} \mathcal{D}_{h,k} \) defined coordinatewise makes \( \mathcal{E}_s \) itself a coalgebra.

Note also that Item (iii) essentially says that each map in the family of maps \( \{ \psi_s \} \) is completely determined by ‘extending multiplicatively’ the action on the elements of a generating set of \( \mathcal{C} \).
In [7], we proved the following Theorem.

**Theorem 2.2.** Let $p$ be an odd prime. Once assigned the bidegree
\begin{equation}
|P^s| = (2s(p - 1), 1), \quad |\beta| = (1, 0),
\end{equation}
to its algebra generators, $\mathcal{B}(p)$ admits a unique structure as a $\mathbb{F}_p$-algebra with coproducts, where
\begin{equation}
\psi_0(\beta) = \beta \otimes 1 + 1 \otimes \beta \quad \text{and} \quad \psi_1(P^s) = \sum_{i+j=s} P^i \otimes P^j \quad (s \geq 0).
\end{equation}

The unique structure the statement referred to turned out to be the dual of a suitable $\mathbb{F}_p$-coalgebra with products in the sense of [25].

The proof uses the fact that the $\mathbb{F}_p$-vector space $\mathcal{B}(p)$, after suitably regrading its generators, admits a structure of $\mathbb{F}_p$-coalgebra with products in the sense of [25].

At a careful examination, Theorem 2.2 cannot be viewed as the odd $p$-counterpart of Theorem 1.2 in [25].

In fact the Bockstein operator $\beta$ acts on products in cohomology rings of Hopf algebras according to the formula
\begin{equation}
\beta(uv) = \beta(u) \cdot P^0(v) + (-1)^{|u|} P^0(u) \cdot \beta(v)
\end{equation}
(see Equation 3.2.5 in [18]). Consequently, a coproduct $\tilde{\psi}$ that would take such behaviour into account should satisfy
\begin{equation}
\tilde{\psi}(\beta) = \beta \otimes P^0 + P^0 \otimes \beta.
\end{equation}

It is quite natural to ask whether $\mathcal{B}(p)$, after suitably regrading its generators, admits a structure of $\mathbb{F}_p$-algebra with coproducts consistent with (2.10). Theorem 2.3 answers negatively to such question.

**Theorem 2.3.** Let $p$ be an odd prime. There is no way to doubly filter $\mathcal{B}(p)$ in order to make it a $\mathbb{F}_p$-algebra with coproducts, if we require that coproducts are defined consistently with
\begin{equation}
\beta \mapsto \beta \otimes P^0 + P^0 \otimes \beta
\end{equation}
and
\begin{equation}
P^s \mapsto \sum_{i+j=s} P^i \otimes P^j \quad (s \geq 0)
\end{equation}

**Proof.** We argue by contradiction. Suppose there exists an $\mathbb{F}_p$-algebra with coproducts
\begin{equation}
(\tilde{\mathcal{B}} = \{\tilde{\mathcal{B}}_{n,s} \mid n, s \geq 0\}, \mu, \eta, \{\epsilon_s \mid s \geq 0\}, \{\tilde{\psi}_s \mid s \geq 0\}),
\end{equation}
where
\begin{equation}
\bigcup \tilde{\mathcal{B}}_{n,s} = \mathcal{B}(p),
\end{equation}
and the coproducts in $\{\tilde{\psi}_s \mid s \geq 0\}$ are consistent with (2.11) and (2.12).

Definition 2.1 in particular implies that $\tilde{\psi}_s(\mathcal{C}_{s,s}) \subseteq \mathcal{C}_{s,s} \otimes \mathcal{C}_{s,s}$. By (2.11) and (2.12) it follows that the Bockstein operator $\beta$ and the Steenrod powers $P^i$ ($i \geq 0$)
all belong to the same coalgebra $\mathcal{C}_{s,\bar{s}}$ for a suitable $\bar{s} \in \mathbb{N}_0$. Let $\bar{r}$ and $\bar{t}$ be the non-negative integers such that

$$\beta \in \mathcal{C}_{\bar{r},\bar{s}} \text{ and } P^0 \in \mathcal{C}_{\bar{t},\bar{s}}.$$ 

By Item (iii) of Definition 2.1 we get

$$\tilde{\psi}_{2s}(\beta^2) = \tilde{\psi}_{2s}(\beta)\tilde{\psi}_{2s}(\beta) = (\beta \otimes P^0 + P^0 \otimes \beta)(\beta \otimes P^0 + P^0 \otimes \beta).$$

The latter equality comes from (2.11). Recalling (2.6) and the fact that $\beta^2 = 0$ by (2.3), we obtain

$$(2.14) \quad \tilde{\psi}_{2s}(\beta^2) = \tilde{\psi}_{2s}(0) = (-1)^{\bar{t}^2} \beta P^0 \otimes P^0 \beta + (-1)^{\bar{t}^2} P^0 \beta \otimes \beta P^0.$$ 

Equation (2.14) contradicts the $F_p$-linearity of $\tilde{\psi}_{2s}$, in fact $\beta P^0$ and $P^0 \beta$ are both non-zero in $\mathcal{B}(p)$ and non-proportional. \qed

3. Toward Further Investigation

Theorem 2.3 does not foil the attempt to provide proper subalgebras of $\mathcal{B}(p)$ with a structure of an $F_p$-algebra with coproducts, as the next Proposition shows.

Corollary 3.1. Let $\mathcal{P}(p)$ be the subalgebra of $\mathcal{B}(p)$ generated by the set $\{ P^i \mid i \geq 0 \}$ of pure powers. Once assigned the bidegree

$$(3.1) \quad |P^s| = (2s(p-1), 1),$$

the algebra $\mathcal{P}(p)$ admits a unique structure as a $F_p$-algebra with coproducts, where

$$(3.2) \quad \psi_1(P^s) = \sum_{i+j=s} P^i \otimes P^j \quad (s \geq 0).$$

Proof. Once you input Theorem 2.2, the only relevant point is the absence of $\beta$ among the generating relations (2.4) in $\mathcal{P}(p)$. \qed

In [7], the authors took into account the subalgebra $\mathcal{C}(p)$ of $\mathcal{B}(p)$ generated by the set $\{ P^i, \beta P^i \mid i \geq 0 \}$. There is a good reason to believe that $\mathcal{C}(p)$ could be made an object in Singer’s category with coproducts consistent with (2.11) and (2.12). In fact, assigned the bidegree

$$(3.3) \quad |P^s| = (2s(p-1), 1), \quad |\beta P^s| = (2s(p-1) + 1, 2),$$

and set

$$\psi_2(\beta P^0) = \beta P^0 \otimes P^0 P^0 + P^0 P^0 \otimes \beta P^0,$$

doing that the map $\psi_2$ could be $F_p$-linear since the element $\beta P^0 \beta P^0 = 0$ would be mapped onto $(\beta P^0 \otimes P^0 P^0 + P^0 P^0 \otimes \beta P^0)^2$ which can be proved to vanish by (2.5) and (2.6). The proof that all relations are preserved will depend on the existence of an appropriate $F_p$-coalgebra with products: the dual to the required structure on $\mathcal{C}(p)$. 

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