MEASURABLE SOLUTIONS FOR ELLIPTIC
AND EVOLUTION INCLUSIONS

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Abstract. This paper obtains existence of random variable solutions to elliptic and evolution inclusions. As a special case, surprising theorems are obtained for the quasistatic problems. A new existence theorem is also presented for evolution inclusions with set valued operators dependent on elements of a measurable space.

1. Introduction. This paper is on two typical problems, elliptic inclusions of the form

\[ f(\omega) \in A(u(\omega),\omega) \]

including variational inequalities, and evolution inclusions

\[ f(\omega) \in (By(\omega))' + A(y(\omega),\omega), \quad By(0) = By_0. \]

Here \( \omega \in \Omega \) where \( (\Omega, \mathcal{F}) \) is a measurable space and there are no monotonicity conditions placed on \( A(\cdot,\omega) \). The idea is to find solutions which are random variables or measurable functions of \( \omega \). For fixed \( \omega \), such problems are well understood at this time. Thus there are many theorems about existence of solutions \( u(\omega), y(\omega) \) for fixed \( \omega \) but this is about existence of measurable solutions.

The measurable space is completely arbitrary. It could be \(([0,T], \mathcal{F})\) where \( \mathcal{F} \) is the \( \sigma \)-algebra of Lebesgue measurable sets for example. If this is done in the first case, 1, one obtains a new interesting way to obtain existence of solutions to quasistatic problems by considering these as elliptic problems for each \( t \). These solutions are not linked to any \( L^p([0,T]; V) \) space.

The typical theorems for these problems pertain to a function \( u \to A(u,t) \) which is at least monotone as a map from \( L^p([0,T]; V) \) to its dual space \( L^{p'}([0,T]; V') \). However, the approach here allows consideration of the operators \( u \to A(u,t) \) as a map from \( V \) to \( 2^{V'} \) for each \( t \) and all Sobolev embedding theorems are available. Then if typical coercivity estimates are known for \( u \to A(u,t) \), one can obtain existence of solutions to the quasistatic problem which are in typical function spaces because from the measurability, one can take a time integral. This includes more exotic cases like having the measure space be \(([0,T] \times \Omega; \mathcal{P})\) where \( \mathcal{P} \) denotes the progressively measurable sets. Here \( (\Omega, \mathcal{P}, \mathcal{F}) \) is a probability space and a set \( E \subseteq \Omega \times [0,T] \) is progressively measurable if for each \( t \leq T, E \cap [0,t] \times \Omega \) is product...
measurable with respect to $B([0, t]) \times \mathcal{F}_t$ where \( \{ \mathcal{F}_t \} \) is a filtration. It is the same theory. One simply considers \( A(u, t, \omega) \) and obtains the existence of progressively measurable solutions. This has not been of much interest in the study of contact problems, but is a special case.

When one has a good theory of quasistatic problems, then the elliptic regularization method of Lions along with a suitable integration by parts formula can be used to generalize to the evolution inclusion. This requires further measurable selection results for weak limits in the function spaces \( L^p([0, T]; V) \).

The remainder of this paper is a summary what was done in [1], an example, and then a description of the main ideas for extending the result to the evolution inclusion case. Other extensions are possible.

2. The elliptic case. For the applications from contact mechanics, the operator \( A \) that appears in (1) is typically a sum of two multi-functions, with one that is constructed from boundary conditions and the other arising from a partial differential inclusion or equation. Part of the difficulty in establishing the existence of measurable solutions to such problems is in keeping track of measurability of selections of the two multi-functions. The main result for existence of approximate solutions in [1] is the following theorem in which there is a sum of two set valued pseudomonotone operators. However, the argument works with no significant change for any finite sum of pseudomonotone operators. It is based on obtaining measurable solutions to a piecewise linear approximation through the use of theorems which give existence of measurability of Brouwer fixed points and then a passage to a limit using a new measurable selection theorem designed to give a measurable representative in a set of weak limits of a sequence. To begin with is a technical result based on fixed point theorems for operators satisfying the following conditions for \( V \) a separable reflexive Banach space. This is a result on finite dimensional subspaces.

- **C₁** Values of \( A \)
  For each \( \omega \), \( A(\cdot, \omega) : V \rightarrow \mathcal{P}(V') \) has bounded, closed, nonempty, and convex values. \( A(\cdot, \omega) \) maps bounded sets to bounded sets.

- **C₂** Measurability
  For each \( u \in V \), there is a measurable selection \( z : \Omega \rightarrow V' \) for \( A \), i.e., \( z \) is a measurable function and for each \( \omega \),
  \[
  z(\omega) \in A(u, \omega).
  \]

- **C₃** Continuity
  For each \( \omega, u \rightarrow A(u, \omega) \) is upper-semicontinuous from the strong topology of \( V \) to the weak topology of \( V' \). That is, if \( u_n \rightarrow u \) in \( V \) strongly and if \( O \) is a weakly open set containing \( A(u, \omega) \), it follows that \( A(u_n, \omega) \in O \) for all \( n \) large enough.

**Theorem 2.1.** Let \( (\Omega, \mathcal{F}) \) be a measurable space, let \( V \) be a separable reflexive Banach space and let \( B : V \times \Omega \rightarrow \mathcal{P}(V') \) and \( C : V \times \Omega \rightarrow \mathcal{P}(V') \) satisfy conditions \((C₁)\) - \((C₃)\). Let \( K : \Omega \rightarrow \mathcal{P}(V) \) be a measurable multi-function which has convex closed bounded values. For each \( \omega \), let \( E(\omega) \) be an \( n \) dimensional subspace of \( V \) which has a basis \( \{ b₁(\omega), \ldots, bₙ(\omega) \} \) each of which is a measurable function into \( V \) and such that \( K(\omega) \subseteq E(\omega) \). Finally, let \( y : \Omega \rightarrow V' \) be a given measurable function. Then, there exist measurable functions \( w_B : \Omega \rightarrow V', w_C : \Omega \rightarrow V' \) and \( x : \Omega \rightarrow V \) with \( w_B(\omega) \in B(x(\omega), \omega), w_C(\omega) \in C(x(\omega), \omega), \) and \( x(\omega) \in K(\omega) \).
such that for all $z \in K(\omega)$,

$$\langle y(\omega) - (w_B(\omega) + w_C(\omega)), z - x(\omega) \rangle \leq 0.$$ 

In what follows $V$ denotes a reflexive separable Banach space with dual $V'$; $(\Omega, \mathcal{F})$ is a measurable space; and $A(\cdot, \omega) : V \to \mathcal{P}(V')$, for $\omega \in \Omega$, is a set valued operator. Assume the following on the operator $A$:

- **$H_1$ Measurability condition.** For each $u \in V$, there is a measurable selection $z(\omega)$ such that

$$z(\omega) \in A(u, \omega).$$

This is more general than saying $u \to A(u, \omega)$ is a measurable multi-function which would say there is a dense countable subset $\{y_i(\omega)\}_{i=1}^\infty$, each $y_i$ measurable such that the closure of these is $A(u, \omega)$.

- **$H_2$ Values of $A$.** $A(\cdot, \omega) : V \to \mathcal{P}(V')$ has bounded, closed, nonempty, and convex values. $A(\cdot, \omega)$ maps bounded sets to bounded sets.

- **$H_3$ Limit conditions.**

If $u_n \to u$ and $\limsup_{n \to \infty} (z_n, u_n - u) \leq 0$, for $z_n \in A(u_n, \omega)$,

then for each $v$, there exists $z(v) \in A(u, \omega)$ such that

$$\lim_{k \to \infty} \inf z_n \leq (z(v), u - v).$$

These are the limit conditions for set valued bounded pseudomonotone operators.

Next is the main result. This depends on Theorem 2.1 which gives the existence of appropriate finite dimensional approximate problems. The complete proof is in [1]. This part consists of fairly standard arguments with the measurability thrown in and a measurable selection theorem for the set of weak limits.

**Theorem 2.2.** Let $\Omega \ni \omega \to K(\omega)$ be a measurable set-valued function, where $K(\omega) \subset V$ is convex, closed and bounded. Let the operators $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ satisfy assumptions $H_1 - H_3$. Finally, let $\omega \to f(\omega)$ be measurable with values in $V'$.

Then, there exists a measurable function $\omega \to u(\omega) \in K(\omega)$ such that the two functions $\omega \to w^A(\omega)$, and $\omega \to w^B(\omega)$ are such that $w^A(\omega) \in A(u(\omega), \omega)$ and $w^B(\omega) \in B(u(\omega), \omega)$, and

$$\langle f(\omega) - (w^A(\omega) + w^B(\omega)), z - u(\omega) \rangle \leq 0,$$

for all $z \in K(\omega)$.

If it is only known that $K(\omega)$ is closed and convex, the same conclusion holds true if it can be shown that for some $z(\omega) \in K(\omega)$ the operator $A(\cdot, \omega) + B(\cdot, \omega)$ is coercive, that is

$$\lim_{||v|| \to \infty} \inf \left\{ \frac{\langle z^*, v - z \rangle}{||v||} : z^* \in (A(v, \omega) + B(v, \omega)) \right\} = \infty.$$ 

Note that it follows from this theorem and various standard results that in the coercive case when, additionally, $K(\omega) = V$, if $f$ is measurable then there exists a measurable $u$ such that $f(\omega) \in A(u(\omega), \omega) + B(u(\omega), \omega)$. Specifically, there are measurable functions $w^A(\omega) \in A(u(\omega), \omega)$ and $w^B(\omega) \in B(u(\omega), \omega)$ such that $f(\omega) = w^A(\omega) + w^B(\omega)$. 
The next result, which is a corollary of Theorem 2.2, comes from specializing the 
measure space to \([0, T] \times \mathcal{F}\) where \(\mathcal{F}\) is the Lebesgue measurable sets. It provides an 
existence result for a broad class of quasistatic variational inequalities and, therefore, 
is presented as a theorem. A host of such problems can be found in the mechanics of 
solids, especially when contact is involved, see e.g., [12, 13, 14]. Indeed, the theorem 
resolves the last open question, that of the existence of measurable solutions without 
the assumptions of monotonicity or uniqueness on the problem operators for each 
fixed \(t\).

**Theorem 2.3.** Let \([0, T] \ni t \to f(t)\) be a measurable function into \(V'\) and let \(K(t)\) 
be a closed, bounded and convex subset of \(V\), such that \(t \to K(t)\) is a measurable 
multi-function.

Then, there exists a solution \((u, w)\), with \(u(t) \in K(t)\), for all \(t\), of the variational 
inequality

\[
\langle f(t) - w(t), z - u(t) \rangle \leq 0, \quad w(t) \in A(u(t), t), \tag{3}
\]

for all \(z \in K(t)\). Moreover, both \(u\) and \(w\) are measurable.

Furthermore,

(i) If \(u \to A_u(u, t)\) is coercive, then the sets \(K(t)\) need only be closed and convex.

(ii) If \(A\) is the sum of two operators \(B, C\) and \(u \to B(u, t)\) and \(u \to C(u, t)\) 
are set-valued bounded pseudomonotone operators, \(t \to B(u, t)\), \(t \to C(u, t)\) having 
measurable selections, then \(w(t) = w_B(t) + w_C(t)\), where both terms are measurable 
and \(w_B(t) \in B(u(t), t)\), \(w_C(t) \in C(u(t), t)\). If suitable estimates hold and \(f \in 
L^p([0, T] ; V')\), then \(w \in L^p([0, T] ; V')\) and \(u \in L^p([0, T] ; V)\).

### 2.1. An example

Here is an example of the use of our theory to a quasistatic 
problem in which there is no coercivity of the nonlinear operator valid for all \(t\), no 
underlying function space \(L^2([0, T] ; V)\), a forcing function which is not in any of 
the \(L^p([0, T] ; H)\) spaces, and in which the convex set is a measurable multi-function 
of \(t\), different for each \(t\). This is a more detailed explanation of an example of [1], 
including some items not shown there. Let \(\sigma(r, t)\) be a continuous function of \(r\) 
which satisfies

\[
\begin{align*}
    r & \to \sigma(r, t) \text{ is continuous, } t \to \sigma(r, t) \text{ is measurable,} \\
    0 & < \delta(t) \leq \sigma(r, t) \leq 1/\delta(t) \text{ for all } r
\end{align*}
\]

There is no uniform lower bound needed for \(\delta(t)\). Let

\[
V \equiv \{ u \in H^1(\Omega) : \gamma u = 0 \text{ on } \Sigma_0 \}
\]

where \(\Sigma_0\) has positive surface measure. Here \(\Sigma_0\) is a subset of the boundary of \(\Omega\), 
a bounded open set having Lipschitz boundary.

**Claim:** For fixed \(t \in [0, T]\), \(A(\cdot, t) : V \to V'\) given by

\[
\langle A(u, t), v \rangle \equiv \int_{\Omega} \sigma(u, t) \nabla u \cdot \nabla v
\]

is pseudomonotone.

**Proof of claim:** Suppose that \(u_n \to u\) weakly in \(V\) and

\[
\lim_{n \to \infty} \langle A(u_n, t), u_n - u \rangle \leq 0
\]
Does the lim inf condition hold? If not, then there exists a subsequence and \( v \in V \) such that

\[
\lim_{n \to \infty} \langle A(u_n, t), u_n - v \rangle = \lim_{n \to \infty} \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v) < \langle A(u, t), u - v \rangle = \int_{\Omega} \sigma(u, t) \nabla u \cdot (\nabla u - \nabla v)
\]

By compactness of the embedding of \( V \) into \( L^2(\Omega) \), there is a further subsequence still denoted with \( n \) such that \( u_n \to u \) strongly in \( L^2(\Omega) \) and pointwise. Consider

\[
\int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v)
\]

Now by the dominated convergence theorem,

\[
\int_{\Omega} |\sigma(u_n, t) - \sigma(u, t)|^2 \to 0
\]

and so in fact \( \sigma(u_n, t) \nabla u_n \to \sigma(u, t) \nabla u \) weakly in \( L^2(\Omega)^3 \). Then

\[
\int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v) = \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla u) + \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u - \nabla v)
\]

\[
\geq \int_{\Omega} \sigma(u_n, t) \nabla u \cdot (\nabla u_n - \nabla u) + \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u - \nabla v)
\]

The second term in the above converges to \( \int_{\Omega} \sigma(u, t) \nabla u \cdot (\nabla u - \nabla v) \).

Consider the first term after \( \geq \). It equals

\[
\int_{\Omega} (\sigma(u_n, t) - \sigma(u, t)) \nabla u \cdot (\nabla u_n - \nabla u) + \int_{\Omega} \sigma(u, t) \nabla u \cdot (\nabla u_n - \nabla u) \quad (*)
\]

The second of these terms converges to 0 because of weak convergence of \( u_n \) to \( u \).

As to the first, if the measure of \( E \) is small enough, then

\[
\left( \int_E |\nabla u|^2 \right)^{1/2} < \delta
\]

By Egoroff’s theorem, there is a set \( E \) having measure this small such that off this set, \( \sigma(u_n(x), t) - \sigma(u(x), t) \to 0 \) uniformly for \( x \notin E \). Thus an application of Holder’s inequality shows that

\[
|\int_{E^c} (\sigma(u_n, t) - \sigma(u, t)) \nabla u : (\nabla u_n - \nabla u)| \leq \delta
\]

whenever \( n \) is sufficiently large thanks to the weak convergence of \( u_n \) to \( u \) which implies that \( \nabla u_n - \nabla u \) is bounded in \( L^2(\Omega)^3 \). As to the integral over \( E \), the fact that \( \sigma \) is bounded for fixed \( t \) implies the existence of a constant \( C \) independent of \( n \) such that

\[
\left| \int_E (\sigma(u_n, t) - \sigma(u, t)) \nabla u \cdot A_x \cdot (\nabla u_n - \nabla u) \right| \leq C \left( \int_E |\nabla u|^2 \right)^{1/2} < C\delta
\]

Thus the first term in * has absolute value no larger than \((C + 1)\delta\) provided \( n \) is sufficiently large. Since \( \delta \) is arbitrary, the limit of the first term in * is 0. Thus,

\[
\lim_{n \to \infty} \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v) \geq \int_{\Omega} \sigma(u, t) \nabla u \cdot (\nabla u - \nabla v)
\]

This is a contradiction. Thus the lim inf condition must hold.
Of course this argument will not work if one tries it on the space $L^2([0,T];V)$ and its dual space $L^2([0,T];V')$. It makes essential use of the compactness of the embedding. It only works because it is used for fixed $t$, but in addition, the operator $Au (t) = A(u (t), t)$ is neither coercive on $L^2([0,T];V)$ nor bounded as a map to $L^2([0,T];V')$.

Next consider another operator. Let $\Sigma_1$ be in $\partial \Omega$ but not in $\Sigma_0$ and have positive surface measure. Let $r \to a(r,t)$ be lower semicontinuous and $r \to b(r,t)$ be upper semicontinuous. Let $0 < \delta (t) \leq a(r,t) \leq b(r,t) \leq \frac{1}{\delta (t)}$. Also let there be a measurable function $t \to c(t)$ between these two functions for each fixed $r$. Now $\gamma : V \to L^2(\Sigma_1)$ and so $\gamma^* : L^2(\Sigma_1) \to V'$ defined in the usual way. Then $z \in B(u, t)$ will mean $z = \gamma^* w$ for some $w \in L^2(\Sigma_1)$ with

$$w (x) \in [a(\gamma u (x), t), b(\gamma u (x), t)]$$

for a.e. $x$ such that

$$\langle z, v \rangle = \int_{\Sigma_1} w (x) \gamma v (x)$$

Then $B$ is also a pseudomonotone set valued operator having the right measurability.

For the remainder of the example, here is a technical lemma.

**Lemma 2.4.** Let $V$ be a closed subset of $W^{1,p} (\Omega), p > 1$ and let $k \in V$. Then $\max (k, u) \in V$ and if $u_n \to u$ in $V$, then $\max (u_n, k) \to \max (u, k)$ in $V$.

Consider $k \in C ([0, T]; V)$. Let

$$K(t) \equiv \{ u \in V : u (x) \geq k(t, x) \text{ a.e. } x \}$$

This is clearly a convex subset of $V$. Is it closed and convex? Is $t \to K(t)$ a set valued measurable function?

**Claim:** $K(t)$ is closed and convex.

**Proof:** It is obvious it is convex. Suppose $u_n \to u$ in $V, u_n \in K(t)$. Then there is a subsequence, still denoted as $u_n$ such that $u_n(x) \to u(x)$ a.e. Hence $K(t)$ is closed.

**Claim:** $t \to K(t)$ is a measurable multifunction.

**Proof:** Consider the subset of $C ([0, T]; V)$ defined by

$$\{ u \in C ([0, T]; V) : \text{ for a.e. } t, u(t, x) \geq k(t, x) \text{ a.e. } x \}$$

This is a subset of the completely separable set $C ([0, T]; V)$ and so it is also separable. Let $\{ d_i \}_{i=1}^\infty$ be a dense subset of $C ([0, T]; V)$. Then let $\{ b_i \}_{i=1}^\infty$ be defined by $b_i (t, x) \equiv \max (k(t, x), d_i (t, x))$. Thus the functions $x \to b_i (t, x)$ are each in $K(t)$ because of the above Lemma. They are also measurable into $V$ when considered as functions of $t$ because $k, d_i \in C ([0, T]; V)$. Is $\{ b_i (t, \cdot) \}_{i=1}^\infty$ dense in $K(t)$? Suppose $u \in K(t)$. Then $t \to v (t, x) \equiv u(x)$ is in $C ([0, T]; V)$ and so there is a subsequence denoted by $d_i$, which converges pointwise to $u$ in $C ([0, T]; V)$. Therefore, there exists a subsequence such that by the above lemma, $\max (k (t, \cdot), d_i (t, \cdot)) \to u (\cdot)$ in $V$.

Thus $\{ b_i (t, \cdot) \}_{i=1}^\infty$ is dense in $K (t)$ and so $t \to K(t)$ is a measurable multifunction.

As an example, one could simply take $k$ to be the restriction to $\Omega \times [0,T]$ of a smooth function.

Let $h(t) \in H = L^2(\Omega)$ for each $t$ and let $h$ be Lebesgue measurable. Similarly, let $\beta(t) \in L^2(\Sigma_1)$ with $\beta$ Lebesgue measurable into $L^2(\Sigma_1)$. Then there exist
$w(t) \in B(u(t), t)$ and $t \to u(t)$ is measurable, $t \to w(t)$ measurable, such that
$u(t) \in K(t)$ and for each $t$, and $v \in K(t)$,
\[
\left( h(t), v - u(t) \right) + \beta(t) (w(t), \gamma v - \gamma u(t))_{L^2(\Omega)} \leq 0
\]
This gives existence of Lebesgue measurable solutions to a quasistatic obstacle problem
having a moving obstacle, without any assumption of coercivity with respect to
$L^2([0, T]; V)$ on the elliptic operator, which is also not monotone.

Of course, if one assumes $\delta(t) > \delta > 0$ for all $t$ and if $0 \in K(t)$, this results in $u$
in $L^2([0, T]; V)$ if it were also the case that $h \in L^2([0, T]; H), \beta \in L^2([0, T]; \Sigma)$
because, other than routine estimates, the issue is measurability of the solution. In case $0 \in K(t)$, the measurable solution is in a weighted $L^2$ space, also more general
than usual for these kinds of problems.

Unfortunately this theory does not adapt well to some quasistatic problems which
are memory dependent, including many elastic visco-plastic problems. These still
are best studied using additional assumptions of monotonicity and fixed point argu-
ments. It can be used easily to obtain solutions to approximate problems involving
a delay in the solution, but difficulties arise in passing to a limit unless the mem-
ory involves only lower order terms or comes from linear operators and in this case,
the above theory might allow one to consider the problem for each $t$ and pass to a
limit getting a measurable solution. Then routine estimates could be used if there
were suitable coercivity conditions.

3. Evolution inclusions. This section is an application of the above theory to
the case of evolution inclusions. It is like what was done above in the sense that
it is based on existence of measurable solutions to elliptic inclusions which are
regularizations coming from the evolution inclusions, but here the Banach space
of interest is $L^p([0, T]; V)$ rather than $V$. This necessitates a much more difficult
measurable selection theorem, Theorem 3.2 below.

Let $V$ and $W$ be two reflexive separable Banach spaces such that $V \subseteq W$ and $V$
is dense in $W$, so that $W' \subseteq V'$, and denote by $\langle \cdot , \cdot \rangle$ the duality pairing of $W$ and
$W'$. Next, for $p > 1$, let
\[
V \equiv L^p([0, T]; V),
\]
and it follows from the Riesz representation theorem that $V' = L^{p'}([0, T]; V')$, where $p'$ is the conjugate of $p$, i.e., $1/p + 1/p' = 1$. Assume here that $B : W \to W'$
satisfies
\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle , \quad B \neq 0.
\]
If $B = 0$, the situation is typically contained in the quasistatic case above.

The following well known theorem [11] is used in what follows. It is stated here in
the more general setting in which a Hölder condition is assumed rather than a
bound on the weak derivatives.

**Theorem 3.1.** Let $E \subseteq F \subseteq G$, where the injection map is continuous from $F$
to $G$ and compact from $E$ to $F$. Let $p \geq 1$, let $q > 1$, $C$ and $R$ be two positive
constants, and define
\[
S = S_{CR} \equiv \{ u \in L^p([a, b], E) : \| u(t) - u(s) \|_G \leq C |t - s|^{1/q}, \| u \|_{L^p([a, b], E)} \leq R \}.
\]
Then, $S$ is bounded in $L^p([a, b], E)$ and precompact in $L^p([a, b], F)$. Thus, if
$\{ u_n \}_{n=1}^\infty \subseteq S$, it has a subsequence $\{ u_{n_k} \}$ that converges in $L^p([a, b], F)$. 
The same conclusion holds true when the Hölder condition is replaced with the condition that \( \|u^\prime\|_{L^1([0,b];G)} \) is bounded.

Obtaining measurable solutions is dependent on a harder measurable selection theorem than the one used for elliptic problems. This is proved in [7]. It gives product measurability of the limit of subsequences. Recall that the product measurable sets \( B([0,T]) \times F \) consist of those in the smallest \( \sigma \) algebra which contains measurable rectangles \( A \times B \) where \( A \) is Bore measurable and \( B \) is in \( F \).

**Theorem 3.2.** Let \( V \) be a reflexive separable Banach space with dual \( V' \), and let \( p,p' \) be such that \( p > 1 \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Let the functions \( t \to u_n(t,\omega) \), for \( n \in \mathbb{N} \), be in \( L^p([0,T];V) \equiv V \) and \( (t,\omega) \to u_n(t,\omega) \) be \( B([0,T]) \times F \equiv \mathcal{P} \) measurable into \( V \). Suppose

\[
\|u_n(\cdot,\omega)\|_V \leq C(\omega),
\]

for all \( n \). Then, there exists a product measurable function \( u \) such that \( t \to u(t,\omega) \) is in \( V \) and for each \( \omega \) a subsequence \( u_{n(\omega)} \) such that \( u_{n(\omega)}(\cdot,\omega) \to u(\cdot,\omega) \) weakly in \( V \).

Note that by the weak compactness in \( V \), for each \( \omega \) every subsequence of \( \{u_n\} \) has a further subsequence that converges weakly in \( V \) to some \( v(\cdot,\omega) \in V \), however, there is no guarantee that \( v \) is \( \mathcal{P} \) measurable. The constants \( C(\omega) \) are not necessarily uniformly bounded and it is also not know whether they are integrable. Indeed, it is not even necessary to have a measure associated with \( \omega \).

The following lemma describes what is meant for a function to be measurable into \( V \) or \( V' \). Simply put, these functions have product measurable representatives.

**Lemma 3.3.** Let \( f(\cdot,\omega) \in V' \) and assume that \( \omega \to f(\cdot,\omega) \) is measurable into \( V' \). Then, for each \( \omega \), there exists a representative \( \hat{f}(\cdot,\omega) \in V' \), \( \hat{f}(\cdot,\omega) = f(\cdot,\omega) \) in \( V' \), such that \( (t,\omega) \to \hat{f}(t,\omega) \) is \( B([0,T]) \times F \) product measurable. If \( f(\cdot,\omega) \in V' \) and \( (t,\omega) \to f(t,\omega) \) is product measurable, then \( \omega \to f(\cdot,\omega) \) is measurable into \( V' \). The same statement holds true when \( V' \) is replaced with \( V \).

In this paper, \( B \) is time independent, but a time dependent version is available in which \( B' \) is assumed continuous. This is currently in preparation.

Let \( Lu = (Bu)' \).

\[
u \in D(L) \equiv \hat{X} \equiv \left\{ u \in L^p(0,T,V) : Lu \equiv (Bu)' \in L^{p'}(0,T,V') \right\}
\]

\[
\|u\|_{\hat{X}} \equiv \max \left( \|u\|_{L^p(0,T,V)}, \|Lu\|_{L^{p'}(0,T,V')} \right)
\]

(4)

Since \( L \) is closed, this \( \hat{X} \) is a Banach space.

The proof of the main result is dependent on the following integration by parts result.

**Proposition 1.** Let \( p \geq 2 \) if \( B \) is time dependent, and \( p > 1 \) otherwise. Let \( u,v \in \hat{X} \), then the following hold:

1. \( t \to \langle B(t)u(t),v(t) \rangle \) equals a.e. an absolutely continuous function, denoted by \( (Bu,v)(\cdot) \);
2. \( (Lu(t),u(t)) = \frac{1}{2} \left[ (Bu,u')(t) + \langle B'(t)u(t),u(t) \rangle \right] \) a.a.t, where \( Lu \equiv (Bu) \);
3. \( |(Bu,v)(t)| \leq C \|u\|_{\hat{X}} \|v\|_{\hat{X}} \) for some \( C > 0 \) and all \( t \in [0,T] \);
4. \( t \to B(t)u(t) \) equals a.e. a function in \( C(0,T;W') \) denoted by \( Bu(\cdot) \);
5. \( \text{sup}\{||Bu(t)||_{W'} : t \in [0,T]\} \leq C||u||_{X'} \) for some \( C > 0 \). Assume, next, that the operator \( K : X \rightarrow X' \) is given by

\[
\langle Ku, v \rangle_{X', X} = \int_0^T \langle Lu(t), v(t) \rangle + \langle Bu, v \rangle(0),
\]

then:

6. \( K \) is linear, continuous and weakly continuous;

7. \( \langle Ku, u \rangle = \frac{1}{2} \left[ \langle Bu, u \rangle(T) + \langle Bu, u \rangle(0) \right] + \frac{1}{2} \int_0^T \langle B'(t) u(t), u(t) \rangle dt; \)

8. If \( Bu(0) = 0 \), for \( u \in X \), there exists \( u_n \rightarrow u \) in \( X \) such that \( u_n(t) \) is 0 near 0. A similar conclusion could be deduced at \( T \) if \( Bu(T) = 0 \).

Also of use is the following generalization of the Gram-Schmidt theorem. The last claim is available, but is not used here. Here in this paper \( B \) is independent of \( \omega \). In fact, it is usually the case that \( B = I \) and one is considering an evolution inclusion.

**Theorem 3.4.** Suppose \( V \) and \( W \) are separable Banach spaces, such that \( V \) is dense in \( W \) and \( B \in \mathcal{L}(W, W') \) satisfies

\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle, \quad B \neq 0.
\]

Then, there exists a countable set \( \{e_i\}_{i=1}^{\infty} \) of vectors in \( V \) such that

\[
\langle Be_i, e_j \rangle = \delta_{ij},
\]

and for each \( x \in W \),

\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i, \quad \text{and} \quad \langle Bx, x \rangle = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle^2.
\]

The series converges in \( W' \). In the case \( B = B(\omega) \), where \( \omega \rightarrow B(\omega) \) is measurable into \( \mathcal{L}(W, W') \), these vectors \( e_i \) depend on \( \omega \) and are measurable functions of \( \omega \).

Let \( U \) be a separable reflexive Banach space that is dense in \( V \), with compact embedding. Actually, \( U \) can be assumed a Hilbert space and in the applications of most interest to us, it is usually obtained by the Sobolev embedding theorems. Let \( r > \max (2, p) \) and \( \mathcal{U}_t = L^r([0,T]; U) \). Also, for \( I = [0, \hat{T}] \), \( \hat{T} < T \), denote by \( \mathcal{V}_I \) the space \( L^p(I; V) \) with similar notation in other spaces. If \( u \in \mathcal{V} = L^p([0,T]; V) \), then always consider \( u \in \mathcal{V}_I \) by simply restricting it to \( I \). With this convention, it is clear that if \( u \) is measurable into \( \mathcal{V} \), then it is also measurable into \( \mathcal{V}_I \).

The conditions on the operator \( A : \mathcal{V}_I \rightarrow \mathcal{P} (\mathcal{V}_I') \) are: \( A(u, \omega) \) is a convex and closed set in \( \mathcal{V}_I' \), whenever \( u \in \mathcal{V}_I \) and

- **\( H_1 \) Growth estimate** For \( u \in \mathcal{V}_I \),

\[
\sup \left\{ \||u^*||_{\mathcal{V}_I'} : u^* \in A(u, \omega) \right\} \leq a(\omega) + b(\omega) ||u||^{p-1}_{\mathcal{V}_I}, \quad (5)
\]

where \( a(\omega) \) and \( b(\omega) \) are nonnegative numbers which need not be uniformly bounded in \( \omega \).

- **\( H_2 \) Coercivity estimate** For each \( t \leq T \) and for some \( \lambda(\omega) \geq 0 \),

\[
\inf \left( \int_0^t (u^*, u) + \lambda(\omega) \langle Bu, u \rangle ds : u^* \in A(u, \omega) \right) \geq \delta(\omega) \int_0^t ||u||^2_{\mathcal{V}} ds - m(\omega), \quad (6)
\]
where $m(\omega)$ is a nonnegative constant for fixed $\omega$, and $\delta(\omega) > 0$. No uniformity in $\omega$ is assumed for the $\delta(\omega)$, neither a strictly positive lower bound that is independent of $\omega$.

- $H_3$ Limit conditions Let $U$ be a Banach space dense and compact in $V$. When $u_i \rightarrow u$ in $\mathcal{V}_I$ and $u^*_i \in A(u_i, \omega)$ with $(B u_i)' \rightarrow (B u)'$ weakly in $\mathcal{U}'_I$, and if
\[
\lim \sup_{i \rightarrow \infty} \langle u^*_i, u_i - u \rangle_{\mathcal{V}_I' \mathcal{V}_I} \leq 0,
\]
then, for each $v \in \mathcal{V}_I$, there exists $u^*(v) \in A u$ such that
\[
\lim \inf_{i \rightarrow \infty} \langle u^*_i, u_i - v \rangle_{\mathcal{V}_I' \mathcal{V}_I} \geq \langle u^*(v), u - v \rangle_{\mathcal{V}_I' \mathcal{V}_I}.
\]

Note that, typically, one obtains this from Theorem 3.1 when applied to lower order terms together with compactness of the embedding of $V$ into $W$, but it also can be obtained in case $B$ is one to one, along with pointwise pseudomonotone assumptions for $A$ [9].

- $H_4$ Measurability condition If $\omega \rightarrow u(\cdot, \omega)$ is measurable into $V$, then
\[
\omega \rightarrow A(\mathcal{X}_I u(\cdot, \omega), \omega)
\]
has a measurable selection into $\mathcal{V}_I$.

This condition means that there is a function $\omega \rightarrow u^*(\omega)$ that is measurable into $\mathcal{V}_I'$ such that $u^*(\omega) \in A(\mathcal{X}_I u(\cdot, \omega), \omega)$.

This is assured when the following standard measurability condition is satisfied for all open sets $O$ in $\mathcal{V}_I'$:
\[
\{ \omega : A(\mathcal{X}_I u(\cdot, \omega), \omega) \cap O \neq \emptyset \} \in \mathcal{F}.
\]

A sufficient condition for this is that $\omega \rightarrow A(u(\cdot, \omega), \omega)$ has a measurable selection into $\mathcal{V}'$ for any $\omega \rightarrow u(\cdot, \omega)$ measurable into $\mathcal{V}$ and if $u^* \in A(u(\cdot, \omega), \omega)$, then $\mathcal{X}_I u^* \in A(\mathcal{X}_I u(\cdot, \omega), \omega)$, and this is typical of what is considered, in which the values of $u^*$ are dependent on the earlier values of $u$ only.

Let $F$ be the duality map for $r, r = \max(p, 2)$ satisfying
\[
\langle Fu, u \rangle = ||u||^r, \quad ||Fu|| = ||u||^{r-1},
\]
and is demicontinuous. Let $X$ be the space of all $u \in \mathcal{U}_r$ such that $(Bu)' \in \mathcal{U}'_r$ with the norm given by $\max(||u||_{\mathcal{U}_r}, \|(Bu)\|_{\mathcal{U}'_r})$.

Using the measurable selection theorem and the same arguments in [10], which are generalizations of the method of elliptic regularization of Lions [11], one can obtain the following corollary. To do this, one uses the same passage to the limit argument of [10] on the special subsequence of Theorem 3.2, obtaining measurability of the limit function as well as this function being a solution to the evolution inclusion. The details of passing to the limit in such a way as to preserve measurability are in the process used in the argument following the Corollary. Then one uses Lemma 3.3 to obtain the product measurability claim.

**Corollary 1.** Let $A$ satisfy (5)–(9), let $f$ be measurable into $V'$, and let $u_0$ be measurable into $W$. Then, for each $\varepsilon > 0$, there exists a solution $u_\varepsilon$ to
\[
Lu_\varepsilon + \varepsilon Fu_\varepsilon + u^*_\varepsilon = f, \quad Bu_\varepsilon(0, \omega) = Bu_0(\omega),
\]
such that $Lu_\varepsilon, u^*_\varepsilon$ and $u_\varepsilon$ are measurable into $\mathcal{U}'_r, \mathcal{U}'_r$, and $\mathcal{U}_r$, respectively, and $u^*_\varepsilon(\omega) \in A(u_\varepsilon, \omega)$. In other words, for $v \in X = \{ u \in \mathcal{U}_r : Lu \in \mathcal{U}'_r \}$,
\[
\int_0^T \langle Lu_\varepsilon, v \rangle dt + \varepsilon \int_0^T \langle Fu_\varepsilon, v \rangle dt + \int_0^T \langle u^*_\varepsilon, v \rangle dt
\]
These functions $L_u, u^*_\varepsilon$ and $u_\varepsilon$ can be regarded as $\mathcal{B}([0,T]) \times \mathcal{F}$ measurable into $U', U'$, and $V$ respectively.

With the above corollary, one can pass to a limit as $\varepsilon \to 0$ to eliminate the term involving a duality map. This process is described next.

The reason for the $\varepsilon F$ term and the new space $\mathcal{U}$ is to include the case where $p < 2$. This case has been neglected in the past papers of this sort, including [10] and other similar versions. If $p \geq 2$, the above corollary is obtained as indicated, without the regularizing term, and with no reference to the new space using the measurable selection theorem Theorem 3.2, and the passing to a limit arguments of [10] applied to this sequence. In this case that $p \geq 2$, the same conclusion is obtained under the slightly more general condition in $H_3$ which involves replacing $(Bu_i) \to (Bu)$ weakly in $U'_I$ with $(Bu_i) \to (Bu)$ weakly in $V'_{pt}$.

Technically, this is a more general result because weak convergence in $U'_I$ is implied by weak convergence in $V'_{pt}$. However, it will typically not make any difference in applications involving the inclusion of lower order terms.

The main theorem for evolution inclusions in this paper is as follows.

**Theorem 3.5.** Assume that $A$ satisfies conditions $(H_1)-(H_4)$ $(5)-(9))$. Let $u_0$ be measurable into $W$ and $f$ be measurable into $V'$. Let $B \in \mathcal{L}(W,V')$ be non-negative and self-adjoint as described above. Let $\sigma > 0$ be small. Then, there exist a pair of functions $(u,u^*)$, measurable into $V_{[0,T-\sigma]} \times V'_{[0,T-\sigma]}$, such that $u^*(\omega) \in A(X_{[0,T-\sigma]}u(\omega),\omega)$ for each $\omega$ and for $t \leq T-\sigma$,

$$Bu(t,\omega) - Bu_0(\omega) + \int_0^t u^*(s,\omega) \, ds = \int_0^t f(s,\omega) \, ds.$$  

*Proof.* Denoting by $u_\varepsilon$ the solution to Corollary 1, suppose that $Lu_\varepsilon \rightarrow Lu$ weakly in $U'_I$ with $Lu = (Bu_i) \in V'$ and $u_\varepsilon \rightarrow u$ weakly in $V$ and $u^*_\varepsilon \rightarrow u^*$ in $V'$, and $\varepsilon Fu_\varepsilon \rightarrow 0$ strongly in $U'_I$. By passing to the limit $\varepsilon \rightarrow 0$ in (11), one obtains that $Lu \in V'$. Next is to show that, indeed, $\varepsilon Fu_\varepsilon \rightarrow 0$ strongly in $U'_I$. It follows from (12) that the functions $u_\varepsilon$ satisfy,

$$\int_0^T \langle Lu_\varepsilon, v \rangle \, dt + \langle Bu_\varepsilon, v \rangle (0) + \int_0^T \langle u^*_\varepsilon, v \rangle \, dt + \varepsilon \int_0^T \langle Fu_\varepsilon, u_\varepsilon \rangle \, dt = \int_0^T \langle f, v \rangle \, dt + \langle Bu (0), u_0 \rangle , \quad (13)$$

for all $v \in X$, and so, since $v$ is arbitrary, $Bu_\varepsilon (0) = Bu_0$. Thus

$$Bu_\varepsilon (t) = Bu_0 + \int_0^t Lu_\varepsilon (s) \, ds. \quad (14)$$

The weak convergence of $Lu_\varepsilon$ implies that $Bu_\varepsilon (t) \rightarrow Bu(t)$ in $U'$. Thus,

$$Bu (t) = Bu_0 + \int_0^t Lu (s) \, ds, \quad (15)$$

and so $Bu (0) = Bu_0$. Then also $\langle Bu_\varepsilon, u_\varepsilon \rangle (0) = \langle Bu_0, u_0 \rangle$, $\langle Bu, u \rangle (0) = \langle Bu_0, u_0 \rangle$ thanks to Proposition 1. Indeed, $B(u - u_0) (0) = 0$ and so the above claim follows from (3) and (8) of that proposition.

There exist subsequences for which this convergence holds.
Using the equation to act on \( u \) in (11) or in (13), it follows from the assumed coercivity condition that for each fixed \( \omega \),
\[
\frac{1}{2} \langle Bu, u \rangle (t) - \frac{1}{2} \langle Bu, u \rangle (0) + \varepsilon \int_0^t \| u \|_{U'}^p \, ds + \delta (\omega) \int_0^t \| u \|_{V'}^p \, ds - m (\omega)
\leq \lambda (\omega) \int_0^t \langle Bu, u \rangle (s) \, ds + \int_0^t (f, u) (s) \, ds.
\]
Then, Gronwall’s inequality yields
\[
\langle Bu, u \rangle (t) + \varepsilon \int_0^T \| u \|_{U'}^p \, ds + \int_0^T \| u \|_{V'}^p \, ds \leq C (f, \omega),
\]
where the constant \( C \) depends only on the indicated data. This estimate and the definition of the duality map \( F \) show that if \( u_\varepsilon \) is the solution guaranteed in Corollary 1, then \( \varepsilon Fu_\varepsilon \rightarrow 0 \) strongly in \( U'_\varepsilon \). Indeed, from the above estimate,
\[
\| \varepsilon Fu_\varepsilon \|_{U'_\varepsilon} \leq C \varepsilon^1/r'
\]
ational text
Also, the estimates for \( A \) and the above estimate imply that \( Lu_\varepsilon \) is bounded in \( U'_\varepsilon \). Thus,
\[
\langle Bu_\varepsilon, u_\varepsilon \rangle (t) + \varepsilon \int_0^T \| u_\varepsilon \|_{U'}^p \, ds + \| u_\varepsilon \|_{V'}^p + \| Lu_\varepsilon \|_{U'_\varepsilon} + \| u_\varepsilon^* \|_{V'} \leq C (f, \omega).
\]
Also, each one of \( u_\varepsilon \) and \( u_\varepsilon^* \) is measurable into \( V \) and \( U'_\varepsilon \), respectively. By density considerations, \( u_\varepsilon^* \) is also measurable into \( V' \). It follows from Theorem 3.2 that there exists a pair \((u, u^*)\), which is measurable into \( V \times V' \), and a sequence \( \varepsilon (\omega) \), such that as \( \varepsilon (\omega) \rightarrow 0 \), \((u_{\varepsilon (\omega)} (\omega), u_{\varepsilon (\omega)}^* (\omega)) \rightarrow (u (\omega), u^* (\omega)) \) in \( V \times V' \). Then, passing to a further subsequence, one obtains, for fixed \( \omega \), the following convergences:
\[
\| \varepsilon Fu_\varepsilon \|_{U'_\varepsilon} \rightarrow 0
\]
\[
u_{\varepsilon (\omega)} (\omega) \rightarrow u (\omega) \text{ weakly in } V,
\]
\[
u_{\varepsilon (\omega)}^* (\omega) \rightarrow u^* (\omega) \text{ weakly in } V',
\]
\[
Lu_{\varepsilon (\omega)} \rightarrow Lu \text{ weakly in } U'_\varepsilon,
\]
These convergences continue to hold true when \( V \) and \( U'_\varepsilon \) are replaced with \( V_I \) and \( U'_{I,\varepsilon} \), by simply considering the restrictions of the functions to \( I \). However, it is not known that \( u \) is in \( U_\varepsilon \), so this requires some difficult considerations. The above convergences applied to (11) yield
\[
Lu (\omega) + u^* (\omega) = f
\]
Let \( \sigma > 0 \), then there exists \( \hat{T} (\omega) > T - \sigma \) such that for each \( \varepsilon (\omega) \) in the sequence,
\[
\langle Bu_{\varepsilon (\omega)}, u_{\varepsilon (\omega)} \rangle (\hat{T}) = \langle B \left(u_{\varepsilon (\omega)} (\hat{T})\right), u_{\varepsilon (\omega)} (\hat{T})\rangle, \quad Bu_{\varepsilon (\hat{T})} = B \left(u_{\varepsilon (\hat{T})}\right),
\]
for all \( \varepsilon (\omega) \) in the sequence converging to 0 and also
\[
Bu (\hat{T}) = B \left(u (\hat{T})\right), \langle Bu, u \rangle (\hat{T}) = \langle B \left(u (\hat{T})\right), u (\hat{T})\rangle.
\]
Now, let \( \{ e_i \} \) be the vectors in \( U \) constructed in Theorem 3.4. Then, at \( \hat{T}(\omega) \),
\[
\langle Bu_\varepsilon, u_\varepsilon \rangle (\hat{T}) = \sum_{i=1}^{\infty} \langle B(u_\varepsilon(\hat{T})), e_i \rangle^2.
\]
It follows from Fatou’s lemma that
\[
\liminf_{\varepsilon \to 0} \langle Bu_\varepsilon, u_\varepsilon \rangle (\hat{T}) = \liminf_{\varepsilon \to 0} \sum_{i=1}^{\infty} \langle B(u_\varepsilon(\hat{T})), e_i \rangle^2
\]
\[
\geq \sum_{i=1}^{\infty} \liminf_{\varepsilon \to 0} \langle B(u_\varepsilon(\hat{T})), e_i \rangle^2
\]
\[
= \sum_{i=1}^{\infty} \langle Bu(\hat{T}), e_i \rangle^2
\]
\[
= \langle Bu(\hat{T}), u(\hat{T}) \rangle = \langle Bu, u(\hat{T}) \rangle.
\]
(18)
Then, by (13), we obtain,
\[
\frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (\hat{T}) - \frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (0) + \int_0^{\hat{T}} \varepsilon \langle Fu_\varepsilon, u_\varepsilon \rangle dt
\]
\[
+ \int_0^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon \rangle = \int_0^{\hat{T}} \langle f, u_\varepsilon \rangle.
\]
(19)
Since \( u^*, f \in V' \), then \( Lu \in V' \). As noted above, \( \langle Bu_\varepsilon, u_\varepsilon \rangle (0) = \langle Bu_0, u_0 \rangle \) and, \( \langle Bu, u \rangle (0) = \langle Bu_0, u_0 \rangle \) also. Then, using integration by parts on (17) yields
\[
\frac{1}{2} \langle Bu, u \rangle (\hat{T}) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^{\hat{T}} \langle u^*, u \rangle dt = \int_0^{\hat{T}} \langle f, u \rangle dt,
\]
which shows that
\[
\int_0^{\hat{T}} \langle u^*, u \rangle dt = \int_0^{\hat{T}} \langle f, u \rangle dt - \frac{1}{2} \langle Bu, u \rangle (\hat{T}) + \frac{1}{2} \langle Bu_0, u_0 \rangle.
\]
Then, (19) and the lower-semicontinuity, shown in (18), imply that
\[
\limsup_{\varepsilon \to 0} \int_0^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon \rangle \leq \int_0^{\hat{T}} \langle f, u \rangle dt + \frac{1}{2} \langle Bu_0, u_0 \rangle
\]
\[
- \liminf_{\varepsilon \to 0} \frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (\hat{T})
\]
\[
\leq \int_0^{\hat{T}} \langle f, u \rangle dt + \frac{1}{2} \langle Bu_0, u_0 \rangle - \frac{1}{2} \langle Bu, u \rangle (\hat{T})
\]
\[
= \int_0^{\hat{T}} \langle u^*, u \rangle dt.
\]
Thus, \( u_\varepsilon \to u \) weakly in \( V_I \) and \( (Bu_\varepsilon)' \to (Bu)' \) weakly in \( U'_I \), and
\[
\limsup_{\varepsilon \to 0} \int_0^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon - u \rangle \leq \int_0^{\hat{T}} \langle u^*, u \rangle - \int_0^{\hat{T}} \langle u^*, u \rangle = 0.
\]
Theorem 3.6. Consider the setting of Theorem 3.5, and let included such an additive noise term in the context of the Navier Stokes equations. A stopping time argument. Such problems are like the early result in [2] which measurable function into $V$ rate inclusion has an added stochastic integral denoted as $v_t$. Therefore, by the limit condition (3), for each $t,ω$. In particular, this holds for $u$ and so, in fact, $\int_0^T \langle u^*_t, u_t - u \rangle \to 0$. Thus,

\[
\int_0^T \langle u^*_t, u_t - u \rangle = \lim_{\varepsilon \to 0} \int_0^T \langle u^*_t, u_t - u \rangle \\
\geq \liminf_{\varepsilon \to 0} \left( \int_0^T \langle u^*_t, u_t - u \rangle + \int_0^T \langle u^*_t, u_t - v \rangle \right) \\
\geq \int_0^T \langle u^*_t, u_t - v \rangle, \text{ some } u^*_t \in A(u,ω).
\]

Since $v$ is arbitrary, this shows from separation theorems that $u^*(ω) \in A(u(ω),ω)$ in $V'_0\{0,T\}$.

The following is an interesting generalization of Theorem 3.5, where the degenerate inclusion has an added stochastic integral denoted as $t \to q(t,ω)$. It involves a stopping time argument. Such problems are like the early result in [2] which included such an additive noise term in the context of the Navier Stokes equations.

**Theorem 3.6.** Consider the setting of Theorem 3.5, and let $q(t,ω)$ be a product measurable function into $V$, a stochastic integral, such that $t \to q(t,ω)$ is continuous and $q(0,ω) = 0$.

Then, for each $\sigma > 0$ (small), there exists a solution $u$ of the integral equation

\[
Bu(t,ω) + \int_0^t u^*(s,ω) ds = \int_0^t f(s,ω) ds + Bu_0(ω) + Bq(t,ω), \quad t \leq T - \sigma,
\]

where $(t,ω) \to u(t,ω)$ is product measurable. Moreover, for each $ω$,

\[
Bu(t,ω) = B(u(t,ω))
\]

for a.e. $t$ and $u^*(.,ω) \in A(u(.,ω),ω)$ for a.e. $t$, $u^*$ is product measurable into $V'$. Also, for each $a \in [0,T - \sigma]$,

\[
Bu(t,ω) + \int_a^t u^*(s,ω) ds = \int_a^t f(s,ω) ds + Bu(a,ω) + Bq(t,ω) - Bq(a,ω).
\]

One can obtain general stochastic inclusions in which the stochastic term has values in $W$ not just $V$ but it is not yet clear that there are good examples for which there is no uniqueness for fixed $ω$. So far, the inclusion of multiplicative noise in which the stochastic integral’s integrand depends on the unknown function, seem to require some sort of monotonicity of the nonlinear operator for fixed $ω$.

One can also consider the case of periodic boundary conditions with similar arguments. Instead of an initial condition, one uses the condition that $Bu(0,ω) = Bu(T,ω)$. This is in preparation.

As an application, consider any existence result which comes from the main theorem of [10] and let the forcing function and operators have the measurability described above. Then this results in measurable solutions. This allows integration of solutions of the evolution inclusion with respect to $ω$ in case the measurable space is endowed with a measure.
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Received August 2019; revised December 2019.

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