Short Note on the Riemann Hypothesis

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Abstract Robin criterion states that the Riemann hypothesis is true if and only if the inequality \( \sigma(n) < e^\gamma n \log \log n \) holds for all natural numbers \( n > 5040 \), where \( \sigma(n) \) is the sum-of-divisors function of \( n \) and \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant. Let \( q_1 = 2, q_2 = 3, \ldots, q_m \) denote the first \( m \) consecutive primes, then an integer of the form \( \prod_{i=1}^{m} q_i^{a_i} \) with \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \) is called an Hardy-Ramanujan integer. If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers \( n > 5040 \) such that Robin inequality does not hold and we prove that \( n \left( 1 - \frac{0.6253}{\log \log n} \right) < N_m \), where \( N_m = \prod_{i=1}^{m} q_i \) is the primorial number of order \( m \) and \( q_m \) is the largest prime divisor of \( n \). In addition, we show that \( q_m \) will not have an upper bound by some positive value for these counterexamples and therefore, the value of \( q_m \) tends to infinity as \( n \) goes to infinity.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part \( \frac{1}{2} \) [4]. Let \( N_m = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times q_m \) denotes a primorial number of order \( m \) such that \( q_m \) is the \( m^{th} \) prime number [3]. As usual \( \sigma(n) \) is the sum-of-divisors function of \( n \) [1]:

\[
\sum_{d \mid n} d
\]
where \( d \mid n \) means the integer \( d \) divides \( n \) and \( d \nmid n \) means the integer \( d \) does not divide \( n \). Define \( f(n) \) to be \( \frac{\sigma(n)}{n} \). Say Robins\((n)\) holds provided

\[
f(n) < e^\gamma \times \log \log n.
\]

The constant \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant and \( \log \) is the natural logarithm. The importance of this property is:

**Theorem 1.1** If the Riemann hypothesis is false, then there are infinitely many natural numbers \( n > 5040 \) such that Robins\((n)\) does not hold [4].

We recall that an integer \( n \) is said to be square free if for every prime divisor \( q \) of \( n \) we have \( q^2 \nmid n \) [1]. Let \( q_1 = 2, q_2 = 3, \ldots, q_m \) denote the first \( m \) consecutive primes, then an integer of the form \( \prod_{i=1}^{m} q_i^{a_i} \) with \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \) is called an Hardy-Ramanujan integer [1]. Based on the theorem 1.1, we know this result:

**Theorem 1.2** If the Riemann hypothesis is false, then there are infinitely many natural numbers \( n > 5040 \) which are an Hardy-Ramanujan integer and Robins\((n)\) does not hold [1].

We prove if the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers \( n > 5040 \) such that Robins\((n)\) does not hold and \( n \left( 1 - \frac{0.6253}{\log \log n} \right) < N_m \), where \( N_m = \prod_{i=1}^{m} q_i \) is the primorial number of order \( m \) and \( q_m \) is the largest prime divisor of \( n \). Furthermore, we show that \( q_m \) will not have an upper bound by some positive value for these counterexamples and thus, the value of \( q_m \) tends to infinity as \( n \) goes to infinity.

### 2 Known Results

These are known results:

**Theorem 2.1** [1]. For \( n > 1 \):

\[
f(n) < \prod_{q \mid n} \frac{q}{q - 1}.
\]

**Theorem 2.2** Let \( \frac{\pi^2}{6} \times \log \log n' \leq \log \log n \) for some \( n > 5040 \) such that \( n' \) is the square free kernel of the natural number \( n \). Then Robins\((n)\) holds [7].

**Theorem 2.3** Robins\((n)\) holds for all natural numbers \( n > 5040 \) when a prime \( q \leq 1771559 \) complies with \( q \nmid n \) [7].

**Theorem 2.4** [6]. For \( q_m \geq 20000 \), we have

\[
\log q_m < \log \log N_m + \frac{0.1253}{\log q_m}.
\]
Theorem 2.5 [5]. For $x \geq 286$:

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times (\log x + \frac{1}{2 \times \log(x)}).$$

Theorem 2.6 [2]. For $x > -1$:

$$\frac{x}{x+1} \leq \log(1+x).$$

3 Proof of Main Theorem

Theorem 3.1 If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robins$(n)$ does not hold and $n \left(1 - \frac{\sqrt{6253}}{\log q_m}\right) < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order $m$ and $q_m$ is the largest prime divisor of $n$. In addition, $q_m$ will not have an upper bound by some positive value for these counterexamples and therefore, the value of $q_m$ tends to infinity as $n$ goes to infinity.

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of some natural number $n > 5040$ as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. The primes $q_1 < \cdots < q_m$ must be the first $m$ consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since the natural number $n > 5040$ could be an Hardy-Ramanujan integer. We assume that Robins$(n)$ does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as $n > 5040$ when the Riemann hypothesis is false according to the theorem 1.2. From the theorem 2.3, we know that necessarily $q_m \geq 1771559$. So,

$$e^\gamma \times \log \log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^\gamma \times (\log q_m + \frac{1}{2 \times \log(q_m)}),$$

because of the theorems 2.1 and 2.5. Hence,

$$\log \log n < \log q_m + \frac{0.5}{\log(q_m)}.$$

From the theorem 2.4, we have that

$$\log \log n < \log \log N_m + \frac{0.1253}{\log(q_m)} + \frac{0.5}{\log(q_m)}.$$

That is the same as

$$\log \log n - \log \log N_m < \frac{0.6253}{\log(q_m)}.$$
Then,
\[
\log \log n - \log \log N_m = \log \left( \log N_m + \log \left( \frac{n}{N_m} \right) \right) - \log \log N_m
\]
\[
= \log \left( \log N_m \times \left( 1 + \frac{\log \left( \frac{n}{N_m} \right)}{\log N_m} \right) \right) - \log \log N_m
\]
\[
= \log \log N_m + \log \left( 1 + \frac{\log \left( \frac{n}{N_m} \right)}{\log N_m} \right) - \log \log N_m
\]
\[
= \log \left( 1 + \frac{\log \left( \frac{n}{N_m} \right)}{\log N_m} \right).
\]

In addition, we know that
\[
\log \left( 1 + \frac{\log \left( \frac{n}{N_m} \right)}{\log N_m} \right) \geq \log \left( \frac{\log \left( \frac{n}{N_m} \right)}{\log n} \right)
\]
using the theorem 2.6 since \(\frac{\log \left( \frac{n}{N_m} \right)}{\log N_m} > -1\). Certainly, we will have that
\[
\log \left( 1 + \frac{\log \left( \frac{n}{N_m} \right)}{\log N_m} \right) \geq \log \left( \frac{\log \left( \frac{n}{N_m} \right)}{\log n} \right)
\]
\[
\geq \frac{\log \left( \frac{n}{N_m} \right)}{\log n} + \log N_m = \frac{\log \left( \frac{n}{N_m} \right)}{\log n}.
\]

In this way, we have that
\[
\frac{\log \left( \frac{n}{N_m} \right)}{\log n} < 0.6253
\]
which is equivalent to
\[
\log \left( \frac{n}{N_m} \right) < \log (n^{0.6253})
\]
and thus
\[
\frac{n}{N_m} < n^{0.6253}.
\]

Finally, we obtain that
\[
\frac{n}{N_m} < n^{0.6253}.
\]

Moreover, we know that \(q_m\) will not have an upper bound by some positive value for these counterexamples because of the theorem 2.2. Certainly, if there is a possible upper bound for \(q_m\), then it cannot exist infinitely many Hardy-Ramanujan integers \(n > 5040\) such that \(\text{Robins}(n)\) does not hold as a consequence of the theorem 2.2.

Acknowledgments

The author would like to thank his mother, maternal brother and his friend Sonia for their support.
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