END SUMS OF IRREDUCIBLE OPEN 3-MANIFOLDS

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1. Introduction

An end sum is a non-compact analogue of a connected sum. Suppose we are given two connected, oriented \(n\)-manifolds \(M_1\) and \(M_2\). Recall that to form their connected sum one chooses an \(n\)-ball in each \(M_i\), removes its interior, and then glues together the two \(S^{n-1}\) boundary components thus created by an orientation reversing homeomorphism. Now suppose that \(M_1\) and \(M_2\) are also open, i.e. non-compact with empty boundary. To form an end sum of \(M_1\) and \(M_2\) one chooses a halfspace \(H_i\) (a manifold homeomorphic to \(\mathbb{R}^{n-1} \times [0, \infty)\)) embedded in \(M_i\), removes its interior, and then glues together the two resulting \(\mathbb{R}^{n-1}\) boundary components by an orientation reversing homeomorphism. In order for this space \(M\) to be an \(n\)-manifold one requires that each \(H_i\) be end-proper in \(M_i\) in the sense that its intersection with each compact subset of \(M_i\) is compact. Note that one can regard \(H_i\) as a regular neighborhood of an end-proper ray (a 1-manifold homeomorphic to \([0, \infty)\)) \(\gamma_i\) in \(M_i\).

The concept of an end sum was introduced by Gompf in [1], where he developed a smooth version of it to use in the study of exotic \(\mathbb{R}^4\)’s. He showed that it induces a monoid structure on the set of oriented diffeomorphism classes of smooth 4-manifolds which are homeomorphic to \(\mathbb{R}^4\). He proved that it is well defined by showing that all end-proper rays in such a 4-manifold are ambient isotopic. In general, the choice of \(\gamma_i\) determines an end of \(M_i\). One can informally regard the process of forming an end sum as gluing together an end of \(M_1\) and an end of \(M_2\). Simple examples show that different choices of these ends may yield non-homeomorphic \(n\)-manifolds. However, Gompf’s arguments generalize to show that for \(n \geq 4\) the result depends only on the choice of the end-proper homotopy classes of the rays \(\gamma_i\). (An end-proper map is a map under which preimages of compact sets are compact; an end-proper homotopy is a homotopy which is an end-proper map.)

This paper examines the case \(n = 3\). It compares the resulting theory with both that of connected sums of 3-manifolds and of higher dimensional end sums. Recall that in general the connected sum of 3-manifolds depends on the orientations of the

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summands. The standard examples are the pairs of connected sums of certain lens spaces with themselves obtained by choosing the opposite orientation on one of the summands. (See [3] or [4].) A similar phenomenon occurs for end sums. Given any two connected, oriented, one-ended, irreducible open 3-manifolds one can choose the halfspaces so that a change of orientation in one of the summands yields a non-homeomorphic 3-manifold (Theorem 5.1(b)). One can also ensure that neither end sum admits an orientation reversing homeomorphism. In particular this can be done when both manifolds are homeomorphic to $\mathbb{R}^3$.

Unlike the higher dimensional theory it turns out that end sums of 3-manifolds do not depend merely on the choice of end-proper homotopy classes of rays. In fact, given any two open 3-manifolds as above one can form their end sum in uncountably many non-homeomorphic ways along rays in any fixed end-proper homotopy classes (Theorem 5.1(c)). Again note that this applies in particular to end sums of $\mathbb{R}^3$ with itself.

The dependence of an end sum on the choice of halfspaces raises difficulties for any attempt to inductively define an end sum with more than two summands. We take a different approach to defining such multiple end sums which has the advantage of allowing infinitely many summands. It is patterned after Scott’s work [7] on infinite connected sums of 3-manifolds. First suppose that we have a countable tree $\Gamma$ to each vertex $v_i$ of which we have associated a connected, oriented, non-compact 3-manifold $V_i$ whose boundary is a non-empty disjoint union of planes. Suppose further that whenever $v_i$ and $v_j$ are joined by an edge we have associated to that edge a boundary plane of $V_i$ and a boundary plane of $V_j$. The result of gluing each pair of associated planes together via an orientation reversing homeomorphism is a connected, oriented, non-compact 3-manifold $M$ such that $\partial M$ is either empty or a disjoint union of planes. $M$ is called the plane sum of the $V_i$ along $\Gamma$.

Now suppose that we are given a countable tree $\Gamma$ to each vertex $v_i$ of which we have associated a connected, oriented, open 3-manifold $M_i$. Suppose further that to each edge of $\Gamma$ incident with $v_i$ we have associated an end of $M_i$ and have chosen an end-proper halfspace determining that end. The same end may be associated to different edges, but we assume that the halfspaces associated to each edge are distinct and disjoint and that their union is end-proper in $M_i$. We then remove the interiors of these halfspaces from each $M_i$ to obtain a 3-manifold $V_i$ and form the plane sum of these $V_i$ as above. The 3-manifold $M$ so obtained is called an end sum of the $M_i$ along $\Gamma$.

In the theory of connected sums of 3-manifolds primeness and irreducibility play a key role, and so one would like appropriate analogues for the theory of end sums. Recall that $M$ is prime if whenever it is a connected sum of two 3-manifolds one of the summands must be $S^3$; it is irreducible if every 2-sphere in $M$ bounds a 3-ball in $M$. Irreducible 3-manifolds are prime; every prime, oriented 3-manifold is either irreducible or homeomorphic to $S^1 \times S^2$.

Given the dependence of an end sum on the choice of halfspace and the fact that one can obtain non-trivial end sums all of whose summands are $\mathbb{R}^3$ it seems reasonable to define an open 3-manifold $M$ to be end-prime if whenever it is an end sum of two 3-manifolds one of the manifolds must be $\mathbb{R}^3$ and the halfspace in this $\mathbb{R}^3$
must be the standard halfspace $\mathbb{R}^2 \times [0, \infty)$. We say that a non-compact 3-manifold $M$ is $\mathbb{R}^2$-irreducible if it is irreducible and every end-proper plane in $M$ bounds an end-proper halfspace in $M$. Then it is clear that open $\mathbb{R}^2$-irreducible 3-manifolds are end-prime. However there are uncountably many connected, orientable, irreducible, end-prime 3-manifolds which fail to be $\mathbb{R}^2$-irreducible (Theorem 5.8).

Every compact, oriented 3-manifold is either prime or admits a decomposition, unique up to homeomorphism, as a connected sum of prime 3-manifolds. (See [3] or [4].) Scott gave an example [7] of an open 3-manifold which is not prime and is not a connected sum, finite or infinite, of prime 3-manifolds. His example is simply connected but has uncountably many ends. In this paper we give an example, inspired by that of Scott, of an open 3-manifold which is not end-prime and is not an end sum, finite or infinite, of end-prime 3-manifolds. Moreover, it is eventually end-irreducible and, unlike Scott’s example, is irreducible and contractible (hence is one-ended). See Theorem 6.2.

The family of 2-spheres arising in a connected sum $M$ has certain non-triviality properties. One of the summing 2-spheres bounds a 3-ball in $M$ if and only if some component of the 3-manifold $M'$ obtained by splitting $M$ along all the summing 2-spheres is a 3-ball. Two summing 2-spheres are parallel in $M$ if and only if some component of $M'$ either is $S^2 \times [0, 1]$ or becomes $S^2 \times [0, 1]$ upon adding some 3-balls in $M$ bounded by some of its boundary 2-spheres. Thus it is very easy to characterize such “degenerate” connected sums. For end sums and plane sums we regard the sum as being **degenerate** if a summing plane is **trivial** (bounds an end-proper $\mathbb{R}^2 \times [0, \infty)$ in $M$), or two such planes are **parallel** (cobound an end-proper $\mathbb{R}^2 \times [0, 1]$ in $M$), or, in the case of plane sums, a summing plane is $\partial$-**parallel** (cobounds an end-proper $\mathbb{R}^2 \times [0, 1]$ with a plane in $\partial M$).

Degenerate finite plane sums of irreducible 3-manifolds are easily characterized. They must have an $\mathbb{R}^2 \times [0, \infty)$ or $\mathbb{R}^2 \times [0, 1]$ summand (Theorem 3.1.) For infinite plane sums the situation is more delicate, and degenerate sums can arise in surprising ways. However, these can also be completely characterized (Theorem 3.2.) Moreover the degeneracies can sometimes be eliminated. Any irreducible open 3-manifold which is an end sum of end-prime 3-manifolds is either end-prime or can be expressed as a non-degenerate end sum of end-prime 3-manifolds (Theorem 6.1.) This is a key step in proving that certain 3-manifolds do not have end-prime decompositions.

The bad behavior of some of our examples when considered as end sums is detectable because of their good behavior when considered as plane sums. In particular the sums in these examples are non-degenerate, and so the summing planes are non-trivial and non-parallel. Moreover, they have the very strong property that every proper non-trivial plane is ambient isotopic to a summing plane. We give some general criteria on the plane summands (“strong aplanarity” and “anannularity at infinity”) which ensure that we obtain such a “strong” sum (Theorem 4.1). Define a 3-manifold $V$ to be **aplanar** if every proper plane in $V$ is either trivial or $\partial$-parallel. This result enables us to prove that if a 3-manifold $M$ has a decomposition as a strong plane sum, then every decomposition of $M$ as a non-degenerate plane sum of aplanar 3-manifolds along a locally finite tree is ambient isotopic to
the given strong plane sum (Theorem 4.3). This has implications for the mapping class group of \( M \) (Corollary 4.4).

2. Preliminaries

In this section we state some basic definitions and some technical results from [6]. We then investigate these properties for the cases of \( \mathbb{R}^2 \times [0, \infty) \) and \( \mathbb{R}^2 \times [0, 1] \).

We shall work throughout in the PL category. An \( m \)-manifold \( M \) may or may not have boundary but is assumed to be second countable. \( \partial M \) and \( int M \) denote the manifold theoretic boundary and interior of \( M \), respectively. Let \( A \) be a subset of \( M \). The topological boundary, interior, and closure of \( A \) in \( M \) are denoted by \( Fr_A M \), \( \text{Int}_A M \), and \( Cl_A M \), respectively, with the subscript deleted when \( M \) is clear from the context. All isotopies of \( A \) in \( M \) will be ambient. \( A \) is bounded if \( Cl \ A \) is compact. \( M \) is open if \( \partial M = \emptyset \) and no component of \( M \) is compact.

A map \( f : M \rightarrow N \) of manifolds is \( \partial \)-proper if \( f^{-1}(\partial N) = \partial M \). It is end-proper if preimages of compact sets are compact. It is proper if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property.

An exhaustion \( C = \{ C_n \} \) for a connected, non-compact \( m \)-manifold \( M \) is a sequence \( C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \) of compact, connected \( m \)-submanifolds of \( M \) whose union is \( M \) such that \( C_n \cap \partial M \) is either empty or an \((m - 1)\)-manifold, \( C_n \subseteq \text{Int} C_{n+1} \), and \( M - C_n \) has no bounded components. Connected non-compact \( m \)-manifolds always have exhausting. A sequence \( V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \) of open subsets of \( M \) is an end sequence associated to \( C \) if each \( V_n \) is a component of \( M - C_n \). Two end sequences \( \{ V_n \} \) and \( \{ W_p \} \) associated to exhaustions \( C \) and \( K \) for \( M \) are cofinal if for every \( n \) there is a \( p \) such that \( V_n \supseteq W_p \) and for every \( p \) there is a \( q \) such that \( W_p \supseteq V_q \). Cofinality is an equivalence relation on end sequences of \( M \). The equivalence classes are called the ends of \( M \). The set of all ends of \( M \) is denoted by \( \varepsilon(M) \). An end-proper map \( M \rightarrow N \) induces a well defined function \( \varepsilon(M) \rightarrow \varepsilon(N) \). If \( \partial M \) has no compact components, then the inclusion map induces a well defined bijection \( \varepsilon(\text{Int} M) \rightarrow \varepsilon(M) \).

We refer to [3] and [4] for basic 3-manifold topology, including the definitions of irreducibility, incompressibility, parallel, \( \partial \)-parallel, etc., and of splitting a manifold along a codimension one submanifold.

We shall need two technical results from [6].

Proposition 2.1. Let \( M \) be a connected, irreducible, non-compact 3-manifold which is not homeomorphic to \( \mathbb{R}^3 \). Let \( \mathcal{P} \) and \( \mathcal{Q} \) be proper surfaces in \( M \) which are in general position. Let \( \mathcal{J} \) be a union of simple closed curve components of \( \mathcal{P} \cap \mathcal{Q} \). Assume that the following conditions are satisfied.

1. No component of \( \mathcal{P} \) or of \( \mathcal{Q} \) is a 2-sphere.
2. Each component \( J \) of \( \mathcal{J} \) bounds a disk \( D(J) \) on \( \mathcal{P} \) and a disk \( G(J) \) on \( \mathcal{Q} \).
3. There is no infinite sequence \( \{ J_m \} \) of distinct components of \( \mathcal{J} \) such that either \( D(J_m) \subseteq \text{Int} D(J_{m+1}) \) for all \( m \) or \( G(J_m) \subseteq \text{Int} G(J_{m+1}) \) for all \( m \), i.e. there is no infinite nesting on \( \mathcal{P} \) or on \( \mathcal{Q} \) among the components of \( \mathcal{J} \).
Then there is an ambient isotopy of $P$ in $M$, fixed on $\partial M$, which takes $P$ to a surface $P'$ such that $P'$ and $Q$ are in general position and $(P' \cap Q) \subseteq (P \cap Q) - J$. Moreover, the isotopy is fixed on $P' \cap Q$.

Proof. This is Proposition 2.1 of [6]. □

Lemma 2.2. Let $M$ be a connected, irreducible, non-compact 3-manifold. A proper plane $P$ in $M$ is trivial if and only if there exist sequences $\{D_n\}$ and $\{D'_n\}$ of disks in $M$ such that $\{D_n\}$ is an exhaustion for $P$, $D'_n \cap P = \partial D_n$, and $\cup D'_n$ is end-proper in $M$.

Proof. This is Lemma 1.1 (1) of [6]. □

Lemma 2.3. $\mathbb{R}^3$ is $\mathbb{R}^2$-irreducible.

Proof. This follows from the result of Harrold and Moise [2] that a topologically embedded 2-sphere in $S^3$ which is wild at at most one point bounds a 3-ball on at least one side. □

Lemma 2.4. $\mathbb{R}^2 \times [0, \infty)$ is $\mathbb{R}^2$-irreducible.

Proof. Let $\{B_n\}$ be an exhaustion for $\mathbb{R}^2$ be concentric disks. Let $C_n = B_n \times [0, n + 1]$ and $F_n = \partial C_n - \text{int} B_n$. Suppose $P$ is a proper plane in $\mathbb{R}^2 \times [0, \infty)$. Let $\mathcal{P} = P$ and $\mathcal{Q} = \cup F_n$. Since $P$ is proper and $\{C_n\}$ is an exhaustion there must be an infinite sequence $\{K_i\}$ of components of $\mathcal{P} \cap \mathcal{Q}$ which is nested on $P$. We may assume this sequence is maximal in the sense that if $J$ is a component of $\mathcal{P} \cap \mathcal{Q}$ which is not in the sequence then $D(J)$ does not contain any of the $K_i$. Let $\mathcal{J} = (\mathcal{P} \cap \mathcal{Q}) - \cup K_i$. Apply Proposition 2.1 to eliminate $\mathcal{J}$ from the intersection. $\mathcal{P} \cap \mathcal{Q}$ is now an infinite subsequence of $\{K_i\}$. By passing to a further subsequence we may assume that the disks $D'_i$ on $Q$ bounded by the $K_i$ are disjoint. Lemma 2.2 now implies that $P$ is trivial. □

It should be noted that $\mathbb{R}^2 \times [0, \infty)$ contains proper planes which are not $\partial$-parallel. See the discussion in section 3.

Lemma 2.5. $\mathbb{R}^2 \times [0, 1]$ is aplanar.

Proof. Let $\{B_n\}$ be as in Lemma 2.4. We identify $B_n$ with $B_n \times \{0\}$. Let $C_n = B_n \times [0, 1]$ and $F_n = (\partial B_n) \times [0, 1]$.

Suppose $P$ can be isotoped off $C_0$. Then for each $n > 0$ we may assume that either $P \cap F_n$ is empty or consists of simple closed curves which bound disks on $F_n$. It then follows from Lemma 2.2 that $P$ is trivial.

So assume this cannot be done. Let $D$ be a disk in $P$ containing $P \cap C_0$ in its interior. Choose $n_0 > 0$ such that $D \subseteq \text{Int} C_{n_0}$. Let $J_0$ be the component of $P \cap F_{n_0}$ such that $D \subseteq \text{Int} D(J_0)$ and $J_0$ is innermost on $P$ among all such curves. Suppose $J$ is a component of $D(J_0) \cap F_{n_0}$ other than $J_0$ which is innermost on $P$ among such curves. Then $D(J) \cap F_{n_0} = J$, $D(J) \cap D = \emptyset$, and so $D(J)$ lies in $(\mathbb{R}^2 - \text{int} B_0) \times [0, 1]$. Since $F_{n_0}$ is incompressible in this irreducible 3-manifold there is an isotopy of $D(J_0)$, fixed on $J_0$, which removes $J$ from $D(J_0) \cap F_{n_0}$ and adds no new components. Continuing in this fashion we get $D(J_0) \cap F_{n_0} = J_0$. In
a similar way we can isotop \( P \) so as to remove all those components \( J \) of \( P \cap F_{n_0} \) such that \( D(J) \) does not contain \( D(J_0) \).

Now let \( M = (\mathbb{R}^2 - \text{int } B_{n_0}) \times [0,1], \mathcal{P} = P \cap M, \) and \( Q = \cup_{n > n_0} F_n \). Then \( \mathcal{P} \) consists of a half-cylinder \( S^1 \times [0, \infty) \) and possibly some annuli. These surfaces are incompressible in \( M \) since otherwise the incompressibility of \( F_{n_0} \) would imply that \( D(J_0) \) could be isotoped off \( C_0 \). We apply Proposition 2.1 to isotop \( \mathcal{P} \) so that afterwards \( \mathcal{P} \cap Q \) consists of simple closed curves which do not bound disks on \( \mathcal{P} \) or \( Q \). It follows that these curves are concentric on \( P \) about \( J_0 \). Denote them by \( J_k, k \geq 1, \) where \( D(J_k) \subseteq \text{int } D(J_{k+1}) \).

Choose \( k_0 > 0 \) such that \( (P \cap C_{n_0}) \subseteq D(J_{k_0}) \) and \( n_1 > n_0 \) such that \( D(J_{k_0}) \subseteq \text{int } C_{n_1} \). Next choose \( k_1 > k_0 \) such that \( (P \cap C_{n_1}) \subseteq \text{int } D(J_{k_1}) \) and \( n_2 > n_1 \) such that \( D(J_{k_1}) \subseteq \text{int } C_{n_2} \). Continuing in this fashion we define sequences \( \{k_i\} \) and \( \{n_i\} \) such that \( (P \cap C_{n_i}) \subseteq \text{int } D(J_{k_i}) \) and \( D(J_{k_i}) \subseteq \text{int } C_{n_{i+1}} \). We assume that \( k_i \) and \( n_i \) are chosen to be the minimal such indices satisfying these conditions. Then \( J_{k_i} \subseteq F_{n_i} \).

For each \( i \geq 0 \) we have that \( P \cap C_{n_i} \) consists of a disk \( D_i \) with \( D(J_{k_{i-1}}) \subseteq D_i \subseteq D(J_{k_i}) \) and possibly a finite number of annuli. For \( i = 0 \) these annuli lie in \( C_{n_0} - C_0 \) and are each parallel in \( C_{n_0} - C_0 \) to an annulus in \( F_{n_0} \) which misses \( D_0 = D(J_0) \). There is an isotopy of \( P \cap (C_{n_i} - \text{int } C_0) \) in \( C_{n_i} - \text{int } C_0 \), fixed on \( \partial(C_{n_i} - \text{int } C_0) \), which carries it to a surface whose intersection with \( F_{n_0} \) is \( \partial D_0 \). Similarly for each even \( k > 0 \) these annuli lie in \( C_{n_k} - C_{n_{k-1}} \), each of them is parallel in \( C_{n_k} - C_{n_{k-1}} \) to an annulus in \( F_{n_i} \) which misses \( D_i \), and there is an isotopy of \( P \cap (C_{n_i} - \text{int } C_{n_{i-1}}) \) in \( C_{n_i} - \text{int } C_{n_{i-1}} \), fixed on \( \partial(C_{n_i} - \text{int } C_{n_{i-1}}) \), which carries it to a surface whose intersection with \( F_{n_i} \) is \( \partial D_i \). Since these isotopies have disjoint compact supports they define an ambient isotopy of \( P \) in \( \mathbb{R}^2 \times [0,1] \) after which \( P \cap F_{2p} \) is a single simple closed curve \( K_{2p} \) for each \( p \geq 0 \), and these curves are nested on \( P \).

We may assume \( D(K_0) = D_0 \). For \( p > 0 \) let \( A_{2p} = (B_{n_{2p}} - \text{int } B_{n_{2p-2}}) \) and \( A'_{2p} = D(K_{2p}) - \text{int } D(D_{2p-2}) \). For \( p \geq 0 \) let \( G_{2p} \) be the annulus in \( F_{n_{2p}} \) joining \( \partial B_{2p} \) and \( K_{2p} \). Since for \( p > 0 \) we have \( C_{n_{2p}} - \text{int } C_{n_{2p-2}} = A_{2p} \times [0,1], \) \( A'_{2p} \) is incompressible in \( A_{2p} \times [0,1], \) and \( A'_{2p} \cap (A_{2p} \times \{1\}) = \emptyset \) it follows that \( A'_{2p} \) is parallel in \( A_{2p} \times [0,1] \) to \( G_{2p} \cup A_{2p} \cup G_{2p-2} \). It follows that there is an embedding of \( A_{2p} \times [0,1] \) in \( \mathbb{R}^2 \times [0,1] \) with \( A_{2p} \times \{0\} = A_{2p}, (\partial A_{2p}) \times [0,1] = G_{2p} \cup G_{2p-2}, \) and \( A_{2p} \times \{1\} = A'_{2p} \). There is also an embedding of \( B_0 \times [0,1] \) in \( \mathbb{R}^2 \times [0,1] \) with \( B_{n_0} \times \{0\} = B_{n_0}, (\partial B_{n_0}) \times [0,1] = G_0, \) and \( B_{n_0} \times \{1\} = D_0 \). These embeddings can be chosen so as to agree on the \( G_{2p} \) and so define a parallelism from \( P' \) to \( \mathbb{R}^2 \times \{0\} \). \( \square \)

3. Degenerate Plane Sums

A plane sum \( M \) of 3-manifolds \( V_i \) along a tree \( \Gamma \) is **degenerate** if either (1) some summing plane \( E_j \) is trivial in \( M \), (2) some summing plane \( E_j \) is \( \partial \)-parallel in \( M \), or (3) some pair of distinct summing planes \( E_j \) and \( E_k \) are parallel in \( M \). In this section we give some conditions on the \( V_i \) which ensure that the plane sum is non-degenerate.

There are some obvious ways to obtain degenerate plane sums, such as having some summands homeomorphic to \( \mathbb{R}^2 \times [0, \infty) \) or \( \mathbb{R}^2 \times [0,1] \). We shall see below
that for a finite plane sum to be degenerate such summands must be present. For
infinite plane sums there are some sources of degeneracy which are only slightly less
obvious. For example, one can use summands which are homeomorphic to a closed
3-ball minus the complement of a disjoint union of open disks in its boundary to
build a degenerate plane sum having no \( \mathbb{R}^2 \times [0, \infty) \) or \( \mathbb{R}^2 \times [0, 1] \) summands.

Other sources of degeneracy are less obvious. We briefly describe one such ex-
ample. A non-compact 3-manifold is **almost compact** if it is homeomorphic to
a compact 3-manifold minus a closed subset of its boundary. In Example 1 of [8]
Scott and Tucker construct a 3-manifold with interior homeomorphic to \( \mathbb{R}^3 \) and
boundary homeomorphic to \( \mathbb{R}^2 \times \{0, 1\} \) which is not almost compact, hence is
not homeomorphic to \( \mathbb{R}^2 \times [0, 1] \), even though the complement of either boundary
plane is homeomorphic to \( \mathbb{R}^2 \times [0, 1] \). This example has an exhaustion by cylinders
\( C_n = D_n \times [0, 1] \), where \( C_n \) is embedded in \( C_{n+1} \) as a regular neighborhood of a
knotted arc joining the center of \( D_{n+1} \times \{0\} \) to that of \( D_{n+1} \times \{1\} \). Call this 3-
manifold \( V \) and its boundary planes \( E \) and \( E' \). The plane sum of \( V \) and \( \mathbb{R}^2 \times [0, \infty) \)
obtained by identifying \( E' \) with \( \mathbb{R}^2 \times \{0\} \) is homeomorphic to \( V - E' \) and hence
is homeomorphic to \( \mathbb{R}^2 \times [0, \infty) \). If one now takes copies \( (V_i, E_i, E_i') \) of \( (V, E, E') \)
for \( i \geq 0 \) and forms a plane sum \( W \) by identifying \( E_i' \) with \( E_{i+1} \), then given any
compact subset \( K \) of \( W \), there is an embedding of \( \mathbb{R}^2 \times [0, \infty) \) in \( W \) which takes
\( \mathbb{R}^2 \times \{0\} \) to \( E \) and whose image contains \( K \). It follows that \( W \) is homeomorphic to
\( \mathbb{R}^2 \times [0, \infty) \). One can then take a plane sum of \( W \) with an appropriate 3-manifold
to obtain an infinite plane sum \( M \) which has a trivial summing plane even though
there are no \( \mathbb{R}^2 \times [0, \infty) \) or \( \mathbb{R}^2 \times [0, 1] \) summands. In particular, extending the
construction of \( W \) to \( i < 0 \) expresses \( \mathbb{R}^3 \) as such a plane sum. (The author thanks B.
Winters for first pointing out this example to him.) Examples similar to \( V \) can
be constructed having more than two boundary components ([9] or [10]). These can
be used to build plane sums which have no almost compact summands but have
summing planes which violate conditions (2) or (3).

**Theorem 3.1.** A finite plane sum \( M \) of irreducible 3-manifolds is degenerate if
and only if it has an \( \mathbb{R}^2 \times [0, \infty) \) or \( \mathbb{R}^2 \times [0, 1] \) summand.

**Proof.** (1) Suppose \( E_j \) is trivial in \( M \). Then some component of \( M - E_j \) has closure
\( H \) which is homeomorphic to \( \mathbb{R}^2 \times [0, 1] \) with \( E_j = \mathbb{R}^2 \times \{0\} \). There is a summand
\( V_i \) contained in \( H \) such that \( \partial V_i \) is a single plane. By Lemma 2.4 \( \partial V_i \) is trivial in
\( H \), and it follows that \( V_i \) is homeomorphic to \( \mathbb{R}^2 \times [0, \infty) \).

(2) Suppose some \( E_j \) is \( \partial \)-parallel in \( M \). Then \( M - E_j \) has a component whose
closure \( Q \) is homeomorphic to \( \mathbb{R}^2 \times [0, 1] \), with \( E_j = \mathbb{R}^2 \times \{0\} \) and \( \mathbb{R}^2 \times \{1\} \) a
component of \( \partial M \). There must be a summand \( V_i \) contained in \( Q \) which either has
exactly one boundary component or has exactly two boundary components, one of
which is a component of \( \partial M \). In the first case it follows from Lemma 2.4 that \( V_i \) is
homeomorphic to \( \mathbb{R}^2 \times [0, \infty) \); in the second case it follows from Lemma 2.5 that
\( V_i \) is homeomorphic to \( \mathbb{R}^2 \times [0, 1] \).

(3) Suppose \( E_j \) and \( E_k, j \neq k \), are parallel in \( M \). Then \( M - (E_j \cup E_k) \) has three
components, one of which has closure \( Q \) which is homeomorphic to \( \mathbb{R}^2 \times [0, 1] \), with
\( E_j = \mathbb{R}^2 \times \{0\} \) and \( E_k = \mathbb{R}^2 \times \{1\} \). Let \( M' \) be the union of \( Q \) with the other
Theorem 3.2. Let component whose closure contains $E_j$. Then $E_j$ is $\partial$-parallel in $M'$ and the desired conclusion follows from (2). □

Suppose $M$ is a plane sum along a tree $\Gamma$ of irreducible 3-manifolds $V_i$. It will be convenient to adjoin to $\Gamma$ an edge for each component of $\partial M$; such edges have one vertex associated to a summand and another vertex which is not associated to a summand. We call such edges **boundary edges** and the regular edges **interior edges**. If a vertex $V_i$ meets an edge $E_j$ such that $E_j \cup \text{int } V_i$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$, then we say that $V_i$ is a **bad summand** with **bad boundary plane** $E_j$. If there are distinct edges $E_j$ and $E_k$ meeting $V_i$ such that $E_j \cup E_k \cup \text{int } V_i$ is homeomorphic to $\mathbb{R}^2 \times [0, 1]$, then $V_i$ is **doubly bad** with a **bad pair** $(E_j, E_k)$ of boundary planes.

A **branch** $\beta$ of $\Gamma$ is one of the two components of the graph obtained by deleting an interior edge $E_j$ of $\Gamma$. We associate to each vertex $V_k$ of $\beta$ a **leading edge** $E_\ell$ as follows. If $V_k$ meets $E_j$, then $E_\ell = E_j$. If $V_k$ does not meet $E_j$, then $E_\ell$ is the unique edge meeting $V_k$ which separates $V_k$ from $E_j$. A branch $\beta$ is **bad** if every vertex $V_i$ of $\beta$ is a bad summand whose leading edge is a bad boundary plane of $V_i$.

A **trail** $\alpha$ of $\Gamma$ joining vertices $V_i$ and $V_j$ is the unique reduced edge path between them. If one chooses edges $E_p$ and $E_q$ meeting $V_i$ and $V_j$, respectively, which are not edges of the trail and one also chooses an orientation for $\alpha$ such that $V_i$ and $V_j$ are, respectively, the first and last vertex of $\alpha$, then to each vertex $V_k$ of $\alpha$ we associate a **leading edge** $E_\ell$ and a **lagging edge** $E_m$ as follows. If $k = i$, then $E_\ell = E_p$ and $E_m$ is the first edge in the trail. If $k = j$, then $E_\ell$ is the last edge in the trail and $E_m = E_q$. If $k \neq i, j$, then the trail enters $V_k$ through $E_\ell$ and exits through $E_m$. The other edges of $\Gamma$ meeting $V_k$ are called **side edges**. Each of them determines a unique **side branch** which does not contain $V_k$. A trail $\alpha$ is **bad** if every vertex of $\alpha$ is a doubly bad summand whose leading and lagging edges are a bad pair of boundary planes and every side branch of $\alpha$ is bad.

**Theorem 3.2.** Let $M$ be a plane sum of irreducible 3-manifolds along a tree $\Gamma$.

(1) A summing plane $E_j$ is trivial in $M$ if and only if one of the branches determined by deleting $E_j$ from $\Gamma$ is a bad branch which contains no boundary edges.

(2) A summing plane $E_j$ is parallel to a boundary component $E$ of $M$ if and only if $E_j$ is the leading edge of a vertex $V_p$ which is joined by a bad trail to a vertex $V_q$ having $E$ as its lagging edge and none of the side branches contains a boundary edge.

(3) Two distinct summing planes $E_j$ and $E_k$ are parallel in $M$ if and only if $E_j$ is the leading edge of a vertex $V_p$ which is joined by a bad trail to a vertex $V_q$ having $E_k$ as its lagging edge and none of the side branches contains a boundary edge.

**Corollary 3.3.** A plane sum of irreducible 3-manifolds having no bad summands is non-degenerate. □

**Proof.** In each case the sufficiency of the conditions is clear, so we prove only their necessity.
(1) Suppose $E_j$ is trivial in $M$. Then $M - E_j$ has a component whose closure $G$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$ with $E_j = \mathbb{R}^2 \times \{0\}$. The corresponding branch clearly has no boundary edges. Let $E_{k}$ be the leading edge of a vertex $V_{k}$ of the corresponding branch. By Lemma 2.2 $E_{k}$ is trivial in $G$ and so $G - E_{k}$ has a component whose closure $H$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$ with $E_{k} = \mathbb{R}^2 \times \{0\}$.

Let $X = E_{k} \cup \operatorname{int} V_{k}$. Suppose $K$ is a compact, connected subset of $X$ such that $K \cap E_{k} \neq \emptyset$. Then there is a 3-ball $B$ in $H$ such that $B \cap E_{k}$ is a disk and $K$ lies in $\operatorname{Int} H B$. Let $D$ be the closure of $(\partial B) - (B \cap E_{k})$. Isotop $D$ in $H - K$ so that it is in general position with respect to the union of all the summing planes other than $E_{k}$. Let $D'$ be an innermost disk on $D$ bounded by one of the components of the intersection. Then $D'$ lies in some $V_{r}$ and $\partial D' = \partial D''$ for a disk $D''$ in a component $E_{s}$ of $\partial V_{r}$. By the irreducibility of $V_{r}$ we have that $D' \cup D''$ bounds a 3-ball $B'$ in $V_{r}$. Since $K$ is connected, lies in $V_{k}$, and meets $E_{k}$ one has that $B' \cap K = \emptyset$. Thus an isotopy fixed on $K$ can be performed to reduce the number of intersection components. Continuing in this fashion there is an isotopy fixed on $K$ which carries $B$ into $V_{k}$. It follows that $X$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$ and thus $V_{k}$ is a bad summand with bad boundary plane $E_{k}$.

(2) Suppose $E_j$ is parallel in $M$ to a component $E$ of $\partial M$. Then $M - E_j$ has a component whose closure $Y$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$ with $E_j = \mathbb{R}^2 \times \{0\}$ and $E = \mathbb{R}^2 \times \{1\}$. Let $V_p$ and $V_q$ be the vertices of the corresponding branch determined by $E_j$ such that $E_j$ is in $\partial V_p$ and $E$ is in $\partial V_q$. There is a unique trail $\alpha$ joining $V_p$ and $V_q$. Suppose $\beta$ is a side branch of $\alpha$ determined by the edge $E_{\ell}$. Since $E_{\ell}$ does not separate $E_j$ from $E$ it follows from Lemma 2.4 that $E_{\ell}$ is trivial in $Y$. Hence by part (1) $\beta$ is a bad branch which contains no boundary edges. Now suppose $V_{k}$ is a vertex of $\alpha$ with leading edge $E_{\ell}$ and lagging edge $E_m$. These two planes cannot be trivial in $Y$, and so by Lemma 2.4 they must be $\partial$-parallel in $Y$. It follows that they must be parallel to each other. Since any other components of $\partial V_{k}$ determine side edges they must be trivial in $Y$, from which it follows that $V_{k}$ is a doubly bad summand with bad pair $(E_{\ell}, E_m)$ of boundary planes. Thus $\alpha$ is a bad trail.

(3) Suppose $E_j$ and $E_k$, $j \neq k$, are parallel in $M$. Then we apply part (2) to the obvious manifold $M' \subseteq M$ such that $E_k$ is a component of $\partial M'$.

\[ \square \]

4. Strong Plane Sums

Suppose $V$ is a connected, non-compact, irreducible 3-manifold whose boundary is either empty or has each component a plane. A partial plane is a non-compact, simply connected 2-manifold with non-empty boundary. $V$ is strongly aplanar if it is aplanar and has the property that given any proper surface $\mathcal{P}$ in $V$ each component of which is a partial plane, there exists a collar on $\partial V$ containing $\mathcal{P}$. $V$ is annular at infinity if for every compact subset $K$ of $V$ there is a compact subset $L$ of $V$ containing $K$ such that $V - L$ is annular, i.e. every proper incompressible annulus in $V - L$ is $\partial$-parallel. We emphasize that in this definition one takes $V - L$, not the closure of the complement of a regular neighborhood of $L$.

By Lemma 2.4 we have that $\mathbb{R}^2 \times [0, \infty)$ is $\mathbb{R}^2$-irreducible and hence aplanar; it is clearly annular at infinity. It is not strongly aplanar: Let $\mathcal{P} = \cup P_n$, where
$P_n = \{(x, y, z) \mid x^2 + z^2 = n^2\}$. By Lemma 2.5 we have that $\mathbb{R}^2 \times [0, 1]$ is aplanar. It is not strongly aplanar: Let $\mathcal{P} = \{(x, 0, z) \mid -\infty < x < \infty, 0 \leq z \leq 1\}$. It is also not anannular at infinity since any compact subset $L$ is contained in a 3-ball of the form $D \times [0, 1]$ for some disk $D$ in $\mathbb{R}^2$, and $(\partial D) \times [0, 1]$ is not $\partial$-parallel in the complement of $L$.

A strong plane sum is a non-degenerate plane sum of irreducible, strongly aplanar 3-manifolds each of which is anannular at infinity. In this section we prove that a strong plane sum has the property that all of its expressions as a non-degenerate plane sum of aplanar 3-manifolds along a locally finite tree are unique up to ambient isotopy. We treat a slightly more general situation (which arises in the next section) by allowing non-separating planes.

**Theorem 4.1.** Let $M$ be a connected, irreducible, non-compact 3-manifold whose boundary is either empty or has each component a plane. Let $\mathcal{E}$ be a proper surface in $M$ each component of which is a plane. Suppose no component of $\mathcal{E}$ is trivial or $\partial$-parallel and that no two distinct components are parallel in $M$. Suppose each component $V_i$ of the manifold $M'$ obtained by splitting $M$ along $\mathcal{E}$ is strongly aplanar and anannular at infinity. Then any proper plane $P$ in $M$ which is neither trivial nor $\partial$-parallel in $M$ is ambient isotopic to a component of $\mathcal{E}$ via an isotopy fixed on $\partial M$.

**Proof.** Put $P$ in general position with respect to $\mathcal{E}$. If $P \cap \mathcal{E} = \emptyset$, then $P$ lies in some $V_i$ and we are done, so assume the intersection is non-empty. Let $P'$ be the surface obtained by splitting $P$ along $P \cap \mathcal{E}$. We shall denote the two planes in $\partial M'$ which are identified to obtain $E_j$ by $E_j'$ and $E_j''$.

**Case 1:** There is no infinite nesting on $P$ among the components of $P \cap \mathcal{E}$. Suppose that there is infinite nesting on $\mathcal{E}$ among the components of $P \cap \mathcal{E}$. Then there is infinite nesting on some component $E_j$ of $\mathcal{E}$ among the components of $P \cap E_j$. Let $\{\alpha_n\}$ be a maximal nested sequence on $E_j$ of components of $P \cap E_j$. Since there is no infinite nesting on $P$ we may pass to a subsequence whose elements bound disjoint disks on $P$. Since $P$ is proper it then follows from Lemma 2.2 that $E_j$ is trivial in $M$, a contradiction. Thus this situation cannot occur.

By Proposition 2.1 we can remove all the compact components of $P \cap \mathcal{E}$. Then each component $P_k'$ of $P'$ is a partial plane. Suppose $P$ meets the component $E_j$ of $\mathcal{E}$. Then $E_j$ lies in $V_i \cap V_m$ for some components $V_i$ and $V_m$ of $M'$, where possibly $i = m$. We may assume that $E_j' \subseteq V_i$ and $E_j'' \subseteq V_m$. Since $V_i$ is strongly aplanar the union of the $P_k'$ it contains must lie in a collar on $\partial V_i$; a similar statement holds for those $P_k'$ contained in $V_m$. Thus a $P_k'$ cannot meet distinct components of $\partial V_i$ or distinct components of $\partial V_m$. It follows that $E_j$ is the only component of $\mathcal{E}$ meeting $P$. Thus $P$ lies in $V_i \cup V_m$ and in fact must lie in a regular neighborhood of $E_j$. Thus $P$ can be isotoped off $\mathcal{E}$, and we are done.

**Case 2:** There is infinite nesting on $P$ among the components of $P \cap \mathcal{E}$. Choose a maximal nested sequence $\{\alpha_n\}$ on $P$ of components of $P \cap \mathcal{E}$. Let $\mathcal{J}$ be the union of the remaining components. If there is infinite nesting on some component $E_j$ of $\mathcal{E}$ among the components of $\mathcal{J}$, then as in the previous case we may pass to a subsequence and apply Lemma 2.2 to conclude that $E_j$ is trivial in
$M$, a contradiction. We can therefore apply Proposition 2.1 to eliminate $\langle J \rangle$ from $P \cap \mathcal{E}$. If this now has only finitely many components we may use irreducibility to perform a finite sequence of isotopies which pushes $P$ off $\mathcal{E}$, and we are done. So assume that $P \cap \mathcal{E}$ is now a nested infinite sequence $\{\alpha_n\}$. Let $\alpha'_n$, $\alpha''_n$ denote the preimages of $\alpha_n$ in $E'_j$, $E''_j$ respectively.

**Lemma 4.2.** There is an $N \geq 0$ and a $j$ such that for all $n \geq N$ one has that $\alpha_n$ lies in $E_j$. Moreover, $\{\alpha_n\}_{n \geq N}$ can be re-indexed so as to form a nested sequence on $E_j$. If $E'_j$ and $E''_j$ lie in the same component $V_j'$ of $M'$, then no component of $P'$ with boundary in $\cup_{n \geq N} (\alpha'_n \cup \alpha''_n)$ meets both $E'_j$ and $E''_j$.

**Proof.** If $P$ meets infinitely many $E_j$ then choosing an innermost $\alpha_n$ on each of these $E_j$ yields an end-proper disjoint union of disks to which we may apply Lemma 2.2 to conclude that $P$ is trivial. Therefore $P$ meets only finitely many components of $\mathcal{E}$.

Consider a plane $E_j$ which meets $P$ in infinitely many components. Suppose infinitely many of these components bound disks on $E_j$ whose interiors miss $P$. Then, as above, there is a subsequence of $\{\alpha_n\}$ which one may use to contradict the non-triviality of $P$. Thus there are only finitely many such components, and so after deleting finitely many curves the rest can be renumbered so as to form a nested sequence on $E_j$.

Suppose $P$ meets each of two distinct planes $E_j$ and $E_k$ infinitely often. We may assume that $E'_j$ and $E'_k$ are both boundary components of some $V_i$. Then all but finitely many components of $P' \cap V_i$ are annuli. From these we may obtain a sequence $\{A_m\}$ of annuli each of which joins $E'_j$ to $E'_k$. The union of $A_m$ with the disks in $E'_j$ and $E'_k$ bounded by the components of $\partial A_m$ is a 2-sphere which bounds a 3-ball $B_m$ in $V_i$. Since $P$ is proper $\{B_m\}$ is, after renumbering, an exhaustion of $V_i$. Let $K = B_0$. Then for every compact subset $L$ of $V_i$ containing $K$ there is a $t > 0$ such that $L \subseteq \text{Int} B_t$. Then $A_t$ is a proper incompressible annulus in $V_i - L$. Since $A_t$ joins two different components of $\partial (V_i - L)$ it cannot be $\partial$-parallel in $V_i - L$. This contradicts the assumption that $V_i$ is an annular at infinity, and thus $P$ meets only one component of $\mathcal{E}$, say $E_j$, infinitely often.

A similar argument proves the assertion about $E''_j$ and $E''_j$ lying in the same $V_i$. □

Let $V_i$ be the component of $M'$ containing $E'_j$. Now $P - \text{int} D_N$ when split along its intersection with $\mathcal{E}$ meets $V_i$ in a family of proper annuli whose boundaries form a nested sequence on $E'_j$ (and on $E''_j$ if it also lies in $\partial V_i$; in this case none of these annuli meet both $E'_j$ and $E''_j$.)

We construct a sequence $\{A_m\}$ of certain of these annuli with $\partial A_m$ in $E'_j$ as follows. Let $\beta_0$ be the innermost of the $\alpha_n$, $n \geq N$. Let $A_0$ be the annulus having $\beta_0$ as one boundary component; let $\gamma_0$ be the other boundary component. Then $\beta_0 \cup \gamma_0 = \partial A'_0$ for an annulus $A'_0$ in $E_j$. Suppose $A_0, \ldots, A_m$ and $A'_0, \ldots, A'_m$ have been defined. Let $\beta_{m+1}$ be the innermost of the $\alpha_n$, $n \geq N$, which is not contained in $A'_0 \cup \ldots \cup A'_m$. Let $A_{m+1}$ be the annulus having $\beta_{m+1}$ as one boundary component; let $\gamma_{m+1}$ be the other boundary component. Then $\beta_{m+1} \cup \gamma_{m+1} = \partial A'_{m+1}$ for an annulus $A'_{m+1}$ in $E_j$. 11
Consider the torus $A_m \cup A'_m$. Compression of this torus along the disk in $E_j$ bounded by $\beta_m$ yields a 2-sphere which, by the irreducibility of $V_i$, bounds a 3-ball $B_m$ in $V_i$. The non-compactness and propriety of $E_j$ imply that $B_m$ contains the compressing disk. It follows that $A_m \cup A'_m$ bounds a compact 3-manifold $Q_m$ which either is a solid torus across which $A_m$ and $A'_m$ are parallel or is homeomorphic to the exterior of a non-trivial knot in $S^3$ for which $\beta_m$ is a meridian curve. By construction the $Q_m$ are disjoint.

Suppose there are infinitely many $Q_m$ which are homeomorphic to non-trivial knot exteriors. Let $K$ be a compact, connected subset of $V_i$ which meets the interior of the disk bounded by $\beta_0$. Since $V_i$ is annular at infinity there is a compact subset $L$ of $V_i$ such that $K \subseteq L$ and every $\partial$-proper incompressible annulus in $V_i - L$ is $\partial$-parallel. Since $P$ is proper in $W$ there are only finitely many $Q_m$ which meet $L$. So there is a $Q_p$ which misses $L$ and is homeomorphic to a non-trivial knot exterior. The fact that $Q_p$ is not a solid torus implies that $A_p$ is not $\partial$-parallel and so must be compressible in $V_i - L$. Let $D$ be a compressing disk, and let $D'$ be the disk on $E_j$ bounded by $\beta_p$. Then there is an annulus $A$ in $A_p$ such that $\partial A = \partial D \cup \partial D'$. The 2-sphere $D \cup D' \cup A$ bounds a 3-ball $B$ such that $B \cap \partial V_i = D'$. Moreover, $K \subseteq L \subseteq \text{Int}(B)$. This shows that $V_i$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$. As this contradicts the non-triviality of $E_j$ it follows that this situation cannot occur.

We may now assume, by choosing a larger $N$ and re-indexing, that all of the $Q_m$ are solid tori across which $A_m$ is parallel to $A'_m$. Perform an ambient isotopy with support in the union of the $Q_m$ to remove all $\alpha_n$, $n > N$, from $P \cap E_j$. Then $P \cap \mathcal{E}$ has only finitely many components. They can all be removed by irreducibility in the standard way, putting us in the case $P \cap \mathcal{E} = \emptyset$ of Case 1, and we are done. □

**Theorem 4.3.** Let $N$ be a non-degenerate plane sum of irreducible, aplanar 3-manifolds along a locally finite tree. Let $M$ be a strong plane sum. Let $\mathcal{P}$ and $\mathcal{E}$ be the unions of the respective sets of summing planes. Suppose $g : N \to M$ is a homeomorphism. Then $g$ is ambient isotopic rel $\partial N$ to a homeomorphism $h$ such that $h(\mathcal{P}) = \mathcal{E}$.

**Proof.** Choose a component $P_0$ of $\mathcal{P}$. Use Theorem 4.1 to isotop $g$ so that $g(P_0)$ is a component, say $E_0$, of $\mathcal{E}$. Now $N - P_0$ has two components. Denote their closures by $X_0$ and $Y_0$. Let $W_0$ be the summand of $N$ such that $W_0 \subseteq X_0$ and $P_0 \subseteq \partial W_0$. Use Theorem 4.1 finitely many times to perform an ambient isotopy of $g$ fixed on $Y_0$ after which $g(\mathcal{P} \cap W_0) \subseteq \mathcal{E}$. It then follows from the aplanarity of $W_0$ and the non-degeneracy of $M$ that $g(W_0)$ is a summand, say $V_0$, of $M$. Let $X_1$ be the closure of $X_0 - V_0$ in $X_0$. Apply the same argument to each component of $X_1$. Continue in this fashion with the successively smaller manifolds $X_{n+1}$ obtained by deleting one summand from each component of $X_n$ as above. Note that for a given summand there are at most finitely many isotopies which are not fixed on that summand, and so one can use the sequence of isotopies to define a single ambient isotopy defined on $X_0$. Then repeat this procedure on $Y_0$ to finally obtain the isotopy to the desired homeomorphism $h$. □

**Corollary 4.4.** Let $M$ be a strong plane sum along a locally finite tree; let $\mathcal{E}$ be the union of the set of summing planes. Suppose $g : M \to M$ is a homeomorphism.
Then $g$ is isotopic rel $\partial M$ to a homeomorphism $h$ such that $h(\mathcal{E}) = \mathcal{E}$. □

5. Strong End Sums

An end sum $M$ of 3-manifolds $M_i$ is **strong** if the halfspaces in the $M_i$ have been chosen so that the corresponding plane sum is strong. In this section we give conditions under which this can be done. We also use similar techniques to construct uncountably many end-prime 3-manifolds which are not $\mathbb{R}^2$-irreducible. Recall that the choice of an end-proper halfspace in $M_i$ is equivalent to the choice of an end-proper ray in $M_i$.

**Theorem 5.1.** Let $\{M_i\}$ be a countable collection (with at least two elements) of connected, oriented, irreducible open 3-manifolds each of which has only finitely many ends. Suppose $\Gamma$ is a countable, locally finite tree whose vertices $v_i$ correspond bijectively to the $M_i$. Suppose that to each edge $e_j$ of $\Gamma$ incident with $v_i$ we have associated an end-proper ray $\gamma_{i,j}$ in $M_i$. Suppose the $\gamma_{i,j}$ are disjoint and that each end of $M_i$ is determined by at least one such ray. Then we have:

(a) The union $\gamma$ of the $\gamma_{i,j}$ is end-proper homotopic to an embedded 1-manifold $\gamma'$ such that the corresponding end sum $M$ along $\Gamma$ is a strong end sum.

(b) $\gamma'$ can be chosen so that $M$ admits no homeomorphisms which reverse orientation or send one summing plane to another or send one summand to another. Moreover, if the orientation is changed on any one of the summands then the resulting end sum $M^*$ is not homeomorphic to $M$.

(c) There are uncountably many choices of $\gamma'$ yielding pairwise non-homeomorphic such $M$.

This will be deduced from Theorem 4.3 and the following result.

**Proposition 5.2.** Let $U$ be a connected, orientable, irreducible, open 3-manifold with $\mu < \infty$ ends. For each $1 \leq m \leq \mu$ let $1 \leq \nu_m < \infty$. Suppose $\beta$ is an end-proper 1-manifold in $U$ whose components are rays $\beta^{m,p}$, where $1 \leq m \leq \mu$, $1 \leq p \leq \nu_m$, and $\beta^{m,p}$ determines the $m$th end of $U$. Then $\beta$ is end-proper homotopic to a 1-manifold $\alpha$ having the following properties.

1. The 3-manifold $V$ obtained by removing the interiors of disjoint regular neighborhoods $H^{m,p}$ of the $\alpha^{m,p}$ is irreducible, strongly aplanar, and anannular at $\infty$.

2. If $\tilde{V}$ is formed by re-attaching, for each end, some, but not all, of the halfspaces $H^{m,p}$ to that end, then $\tilde{V}$ has all the properties listed in (1).

3. Each $\tilde{V}$ admits no homeomorphisms which reverse orientation or take one component of $\partial \tilde{V}$ to another; distinct $\tilde{V}$ are non-homeomorphic.

4. There are uncountably many choices of the $\alpha$ yielding pairwise non-homeomorphic $V$ with properties (1), (2), and (3); moreover the $\tilde{V}$ are pairwise non-homeomorphic.

**Proof of Theorem 5.1.** We apply Proposition 5.2 to $M_i$. We get strength from statement (1) along with statement (2), which implies that there are no bad summands.
and hence by Corollary 3.3 that the sum is non-degenerate. (4) allows us to choose distinct \( V_i \) to be non-homeomorphic; there are uncountably many such choices for the set of all \( V_i \). By Theorem 4.3 any homeomorphism between two such sums or a sum and itself can be isotoped so as to carry one set of summing planes to the other, from which it then follows that it must carry summand to corresponding summand. The remainder of the theorem then follows from (3) and (4). □

The existence of an \( \alpha \) satisfying (1)–(4) and having the appropriate distribution of its ends among the ends of \( U \) was proven in Theorems 6.1, 6.5, and 6.8 of [6]. We will briefly outline that construction, modifying it so that \( \alpha \) is end-proper homotopic to \( \beta \). We first list some technical tools that we shall use.

**Proposition 5.3.** Let \( V \) be a connected, irreducible, orientable, non-compact 3-manifold which has a finite number \( \mu \) of ends and whose boundary consists of a finite number of disjoint planes. Suppose \( V \) has an exhaustion \( \{ C_n \} \) such that \( C_n \cap \partial V \) consists of a single disk in each component of \( \partial V \), \( C_{n+1} - \text{Int} C_n \) is irreducible, \( \partial \)-irreducible, and anannular, each component of \( \text{Fr} C_n \) has negative Euler characteristic and positive genus, and \( V - \text{Int} C_n \) has \( \mu \) components for all \( n \geq 0 \). Then \( V \) is strongly aplanar and anannular at \( \infty \).

*Proof.* This follows from Lemma 1.3, Theorem 3.4, and Theorem 5.3 of [6]. □

A compact, connected, 3-manifold \( X \) which is not a 3-ball is called **excellent** if it is \( \mathbb{P}^2 \)-irreducible and \( \partial \)-irreducible, contains a 2-sided, proper, incompressible surface, and every connected, proper, incompressible surface of zero Euler characteristic in \( X \) is \( \partial \)-parallel. The closure of the complement of a regular neighborhood of a submanifold \( A \) of a manifold \( Q \) is called the **exterior** of \( A \) in \( Q \). A proper 1-manifold \( \lambda \) in a compact 3-manifold \( Q \) is called **excellent** if its exterior in \( Q \) is excellent. We say that \( \lambda \) is **poly-excellent** if every non-empty union of the components of \( \lambda \) is excellent.

**Proposition 5.4.** Let \( Q \) be a compact, connected, orientable 3-manifold whose boundary is non-empty and contains no 2-spheres. Suppose \( \kappa \) is a proper arc in \( Q \). The \( \kappa \) is homotopic rel \( \partial \kappa \) to an excellent arc \( \lambda \) in \( Q \).

*Proof.* This follows from Theorem 1.1 of [5]. □

An \( n \)-tangle \( \tau \) is an \( n \) component proper 1-manifold embedded in a 3-ball such that each component of \( \tau \) is an arc.

**Proposition 5.5.** For each \( n \geq 1 \) poly-excellent \( n \)-tangles exist.

*Proof.* This is Theorem 6.3 of [6]. □

**Lemma 5.6.** Let \( R \) be a compact, connected 3-manifold. Let \( S \) be a compact, proper, 2-sided surface in \( R \). Let \( R' \) be the 3-manifold obtained by splitting \( R \) along \( S \). Let \( S' \) and \( S'' \) be the two copies of \( S \) in \( \partial \) \( R' \) which are identified to obtain \( R \). If each component of \( R' \) is excellent, \( S' \cup S'' \) and \( \partial \) \( R' \) – \( \text{int} \) \( (S' \cup S'') \) are incompressible in \( R' \), and each component of \( S \) has negative Euler characteristic, the \( R \) is excellent.

*Proof.* This is Lemma 2.1 of [5]. □
Proof of Proposition 5.2. Let \{K_n\} be an exhaustion for \(U\). We may assume that each \(U - \text{int } K_n\) has \(\mu\) components \(U_n^m\), \(1 \leq m \leq \mu\). Let \(Y_{n+1}^m = U_{m+1}^m \cap (K_{n+1} - \text{int } K_n)\). By attaching 1-handles to \(K_n\) inside \(U - \text{int } K_n\) and then passing to a subsequence we may assume that \(Y_{n+1}^m\) and \(G_n^m = K_n \cap Y_{n+1}^m\) are each connected and that each \(G_n^m\) has genus at least two. Put \(\beta\) in general position with respect to \(\cup \partial K_n\). We may assume that \(\beta \cap K_0 = \partial \beta\). By attaching 1-handles to \(K_n\) whose cores are compact components of \(\partial \cap (U - \text{int } K_n)\) and passing to a subsequence we may further assume that each component of \(\beta \cap (U - \text{int } K_n)\) is non-compact and thus that \(\beta_{m,p} \cap (K_{n+1} - \text{int } K_n) = \beta_{m,p} \cap Y_{n+1}^m\) is an arc \(\beta_{n+1}^{m,p}\) joining \(G_n^m\) to \(G_{n+1}^m\). Let \(D_n^m\) be a disk in \(G_n^m\) whose interior contains \(\beta \cap G_n^m\). Let \(\beta_{n+1}^{m,1} = \beta_{n+1}^{m,1} \cup \cdots \cup \beta_{n+1}^{m,\nu_m}\). Let \(N_{n+1}^m\) be a regular neighborhood of \(D_n^m \cup D_{n+1}^m \cup \beta_{n+1}^m\) in \(Y_{n+1}^m\), chosen so that \(N_{n+1}^m = N_{n+2}^m \cap G_{n+2}^m\). By Proposition 5.4 there is an excellent proper arc \(\eta_{m,n}^m\) in \(Y_{n+1}^m - \text{int } N_{n+1}^m\) such that \(\partial \eta_{m,n}^m\) lies in \((\partial N_{m,n+1}^m) \cap \text{int } Y_{n+1}^m\). Let \(T_{n+1}^m\) be the union of \(N_{n+1}^m\) and a regular neighborhood of \(\eta_{n+1}^m\) in \(Y_{n+1}^m - \text{int } N_{n+1}^m\). \(T_{n+1}^m\) is a cube with \(\nu_m\) handles whose exterior \(L_{n+1}^{m,n+1}\) in \(Y_{n+1}^m\) is excellent. Moreover, \(\beta_{n+1}^m\) is a proper 1-manifold in \(T_{n+1}^m\) consisting of unknotted, unlinked arcs. We shall homotop \(\beta_{n+1}^m\) in \(T_{n+1}^m\) relative to its boundary to obtain a 1-manifold \(\alpha_{n+1}^{m}\) such that the union of the \(\alpha_{n+1}^{m}\) over all \(m\) and \(n\) gives the desired union of rays \(\alpha\).

A classical knot space \(Q\) is a 3-manifold homeomorphic to the exterior of a non-trivial knot in \(S^3\). We say that \(Q\) is incompressibly embedded in a 3-manifold \(R\) if \(Q \subseteq R\) and \(\partial Q\) is incompressible in \(R\).

Lemma 5.7. Let \(T\) be a cube with \(g\) handles. Let \(J_1, \ldots, J_{\nu}\) be excellent knots in \(S^3\). Then there are disjoint classical knot spaces \(Q_1, \ldots, Q_{\nu}\) in \(T\) and disjoint proper arcs \(p_1, \ldots, p_{\nu}\) in \(T - \text{int } (Q_1 \cup \cdots \cup Q_{\nu})\) such that \(Q_p\) is homeomorphic to the exterior of \(J_p\) in \(S^3\), there are disjoint 3-balls \(B_p\) in \(T\) such that \(Q_p \subseteq B_p\) and \(B_p \cap \rho_q = \emptyset\) for \(p \neq q\), and for every non-empty subset \(\{p_1, \ldots, p_k\}\) of \(\{1, \ldots, \nu\}\)

(i) the exterior \(R\) of the 1-manifold \(\rho_{p_1} \cup \cdots \cup \rho_{p_k}\) in \(T\) is \(P^2\)-irreducible, \(\partial\)-irreducible, and anannular,

(ii) \(\partial Q_{p_{\nu}}\) is incompressible in \(R\), and

(iii) given any classical knot space \(Q\) incompressibly embedded in \(T\) there is an ambient isotopy of \(Q\) in \(R\), fixed on \(\partial R\), which takes \(Q\) to some \(Q_{p_{\nu}}\).

Proof. The case \(g = 1\) is Lemma 6.7 of [6]. Choose disjoint proper disks \(D_1, \ldots, D_g\) in \(T\) which split it to a 3-ball \(B\). Let \(f : B \rightarrow T\) be the identification map. Let \(Z_1, \ldots, Z_{\nu}\) be disjoint disks in \(\text{int } D_g\). Let \(T_p\), \(1 \leq p \leq \nu\), be disjoint regular neighborhoods of \(\partial Z_p\) in \(T\), chosen so that \(A_p = D_g \cap T_p\) is a regular neighborhood of \(\partial Z_p\) in \(D_g\). Then \(f^{-1}(T_p)\) is the union of two solid tori \(T'_p\) and \(T''_p\). Let \(B^* = B - \text{Int}(U_{p=1}^{\nu} T'_p \cup T''_p)\). Then \(f(\partial B^*) \cap (D_1 \cup \cdots \cup D_g)\) consists of \(D_1, \ldots, D_{g-1}\), together with disks \(E_1, \ldots, E_{\nu}\), and a disk with \(\nu\) holes \(P\) contained in \(D_g\). Whenever \(S\) is one of these surfaces, \(f^{-1}(S)\) consists of two surfaces \(S'\) and \(S''\). We let \(T^*\) denote \(T - \text{int } (T_1 \cup \cdots \cup T_{\nu})\).

By Proposition 5.5 \(B^*\) contains a poly-excellent \(2(g+1)\nu\)-tangle. We denote its components by \(p_{p,j}\), \(1 \leq p \leq \nu\), \(1 \leq j \leq 2(g+1)\). Isotop this tangle so that \(p_{p,j}\) runs from \(\text{int } f^{-1}(\partial T)\) to \(\text{int } D'_1\), \(p_{p,2i}\) runs from \(\text{int } D''_i\) to itself for \(1 \leq i \leq g - 1\),
\( \rho_{p,2i+1} \) runs from \( \text{int } D_i' \) to \( \text{int } D_{i+1}' \) for \( 1 \leq i \leq g-2 \). \( \rho_{p,2g-1} \) runs from \( \text{int } D_{g-1}' \) to \( \text{int } E_{p}' \), \( \rho_{p,2g} \) runs from \( \text{int } E_{p}'' \) to itself, \( \rho_{p,2g+1} \) runs from \( \text{int } E_{p}' \) to \( P' \), and \( \rho_{p,2g+2} \) runs from \( P'' \) to \( \text{int } f^{-1}(\partial T) \). We further require that the endpoints of the \( \rho_{p,j} \) match up in such a way that \( f(\cup_{j=1}^{2g+2} \rho_{p,j}) \) is a proper arc \( \rho_p \) in \( T^* \).

We now glue the exteriors \( Q_p \) of the knots \( J_p \) to \( T^* \) so that a meridian of \( J_p \) is identified with \( \partial E_p \) and let \( B_p \) be the union of \( Q_p \) and a regular neighborhood of \( E_p \) in \( T^* \). Then \( B_p \) is a 3-ball, from which it follows that this space is again a cube with \( g \) handles, which we denote again by \( T \).

Now suppose that we have a non-empty subset \( \{p_1, \ldots, p_k\} \) of \( \{1, \ldots, \nu\} \). It follows from Lemma 5.6 that \( \rho_{p_1} \cup \cdots \cup \rho_{p_k} \) is excellent in \( T - \text{int } (Q_{p_1} \cup \cdots \cup Q_{p_k}) \).

Standard general position and isotopy arguments now show that the exterior \( R \) of this 1-manifold is \( \mathbb{P}^2 \)-irreducible, \( \partial \)-irreducible, and anannular, that each \( \partial Q_{p_i} \) is incompressible in \( R \) and that every incompressible torus in \( R \) is isotopic to one of these tori. The result then follows. \( \Box \)

We now complete the proof of Proposition 5.2. Let \( \alpha_{m+1}^m \) be a proper 1-manifold in \( T_{n+1}^m \) consisting of \( \nu_m \) arcs having the properties stated in Lemma 5.7. Denote the classical knot spaces involved by \( Q_{n+1}^{m,p} \). Since \( \pi_1(\partial T_{n+1}^m) \to \pi_1(T_{n+1}^m) \) is onto we may isotop \( \alpha_{n+1}^m \) so that \( \partial \alpha_{n+1}^m = \partial \beta_{n+1}^m \) and \( \alpha_{n+1}^m \) and \( \beta_{n+1}^m \) are homotopic relative to this common boundary. Thus the union \( \alpha \) of the \( \alpha_{n+1}^m \) is end-proper homotopic to \( \beta \).

Now for each \( m, 1 \leq m \leq \mu \), choose a non-empty subset of \( \{1, \ldots, \nu_m\} \). By property (i) of Lemma 5.7 the exterior \( R_{n+1}^m \) in \( T_{n+1}^m \) of the corresponding union of components of \( \alpha_{n+1}^m \) is irreducible, \( \partial \)-irreducible, and anannular. Since the same is true of the exterior \( L_{n+1}^m \) of \( T_{n+1}^m \) in \( Y_{n+1}^m \) standard general position and isotopy arguments show that these properties hold for \( X_{n+1}^m = L_{n+1}^m \cup R_{n+1}^m \). Let \( X_{n+1} = X_{n+1}^1 \cup \cdots \cup X_{n+1}^\mu \), \( C_0 = K_0 \), and \( C_{n+1} = K_0 \cup X_1 \cup \cdots \cup X_{n+1} \). Then \( \{C_n\} \) is an exhaustion for the exterior \( \tilde{V} \) of the corresponding components of \( \alpha \) in \( U \).

The application of Proposition 5.3 to \( \{C_n\} \) now implies properties (1) and (2) of Proposition 5.2.

The proof of properties (3) and (4) of Proposition 5.2 is identical to that of Theorem 6.8 of [6], with Lemma 6.7 of that paper replaced by Lemma 5.7 of this paper. For the sake of completeness we briefly recall the construction, referring the reader to [6] for details. We choose a countably infinite family of excellent knots in \( S^3 \) whose exteriors admit no orientation reversing homeomorphisms. We index this family by quadruples \( (m, p, n, q) \), where \( 1 \leq m \leq \mu, 1 \leq p \leq \nu_m, n \geq 0, \) and \( q \in \{0, 1\} \). We choose some function \( q = \varphi(m, p, n) \) and then carry out our construction with the knot space \( Q(m, p, n, q) \) associated to the arc \( \alpha_{n+1}^{m,p} \) as in Lemma 5.7; this produces a manifold \( V[\varphi] \). Given a collection \( E^{m,p} \) of boundary planes which includes at least one plane from each end and any compact subset of \( V[\varphi] \) meeting exactly these boundary planes, it turns out that there is a larger compact subset meeting exactly these boundary planes such that the incompressibly embedded knot spaces in the complement of this subset are precisely the corresponding \( Q(m, p, n, q) \) which it contains. This property, together with the fact that the knot spaces admit no orientation reversing homeomorphisms and the fact that the set of functions \( \varphi \)
is uncountable, yields properties (3) and (4). □

**Theorem 5.8.** There are uncountably many connected, irreducible, open 3-manifolds which are end-prime but not $\mathbb{R}^2$-irreducible.

*Proof.* Let $U$ be any connected, oriented, irreducible, open 3-manifold with one end. By Proposition 5.2 we may choose two disjoint proper halfspaces in $U$ such that the 3-manifold $V$ obtained by removing their interiors is irreducible, strongly aplanar, and anannular at infinity. Let $M$ be the 3-manifold obtained by gluing the two components of $\partial V$ together via an orientation reversing homeomorphism. Since the plane $E$ in $M$ which is the image of $\partial V$ is non-separating $M$ cannot be $\mathbb{R}^2$-irreducible. Suppose it were not end-prime. Then $M$ would be the plane sum of 3-manifolds $V_1$ and $V_2$ each having boundary a plane but neither being homeomorphic to $\mathbb{R}^2 \times [0, \infty)$. Thus the plane $P$ along which the sum is taken is non-trivial in $M$. Then by Theorem 4.1 we have that $P$ is ambient isotopic to $E$, which cannot happen since $P$ separates $M$. □

6. End-prime Decompositions

In this section we show that there are irreducible open 3-manifolds which are not end-prime and do not admit decompositions via end sum into end-prime 3-manifolds. We first show that one can reduce to the case of non-degenerate end sum decompositions.

**Theorem 6.1.** Suppose $M$ is an end sum of connected, irreducible, end-prime open 3-manifolds $M_i$ along a tree $\Gamma$. Then either $M$ is end-prime or can be expressed as a non-degenerate end sum of connected, irreducible, end-prime open 3-manifolds.

*Proof.* Let $\{V_i\}$ and $\{E_j\}$ denote the vertices and edges of the corresponding plane sum.

Suppose some of the summing planes are trivial. By Theorem 3.2 each such plane determines a bad branch. Partially order the bad branches by inclusion. If there are no maximal bad branches, then $M$ is the monotone union of a sequence $\{\beta_j\}$ of bad branches associated with a sequence $\{E_j\}$ of summing planes. Since the union of all summing planes is proper in $M$ it follows that every compact subset of $M$ lies in the interior of some branch, and hence $M$ is homeomorphic to $\mathbb{R}^3$ and so is end-prime.

Thus we may assume that maximal bad branches exist. Suppose two such branches $\beta_j$ and $\beta_k$ intersect. If $\beta_j \cap \beta_k$ is a summing plane, then again $M$ is homeomorphic to $\mathbb{R}^3$. So we may assume that $E_j$ lies in the interior of $\beta_k$. Since $\beta_k$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$, which is $\mathbb{R}^2$-irreducible, $E_j$ is trivial in $\beta_k$, from which it follows that $M$ is homeomorphic to $\mathbb{R}^3$.

Hence we may assume that distinct maximal bad branches are disjoint. All the trivial summing planes are contained in the union of the maximal bad branches. Suppose $V_i$ is a plane summand which is not contained in a bad branch. If $V_i$ meets a bad branch $\beta_j$ in a summing plane $E_j$, then $\beta_j$ is maximal. Let $V_i'$ be the union of $V_i$ with all such $\beta_j$. Then $V_i$ and $V_i'$ each have interior homeomorphic to $M_i$. We can thus eliminate all those $M_k$ whose corresponding $V_k$ lie in a bad branch.
If only one summand remains, then $M$ is end-prime. If more than one summand remains, the $M$ is expressed as an end sum of end-prime 3-manifolds in which no summing plane is trivial.

Now suppose some pairs of distinct summing planes are parallel. By Theorem 3.2 they determine a bad trail. Since there are no trivial summing planes there are no bad branches and hence this trail has no side branches. Each of its vertices is therefore homeomorphic to $\mathbb{R}^2 \times [0, 1]$. Partially order the bad trails by inclusion. If there are no maximal elements then there is an infinite nested sequence of bad trails in $M$ and therefore a trivial summing plane. Thus maximal bad trails exist and clearly distinct such arcs are disjoint. Fix a summand $V$ in a bad trail. Let $\{x_i\}$ be an exhaustion to inductively define a new end sum in this manner. The set of end summands is a subset of the original set and is clearly non-degenerate. \hfill \Box

A connected, non-compact 3-manifold is \textbf{eventually end-irreducible} if it admits an exhaustion $\{C_n\}$ such that $FrC_n \cup FrC_{n+1}$ is incompressible in $C_{n+1} - IntC_n$ for all $n \geq 0$.

**Theorem 6.2.** There exists an irreducible, eventually end-irreducible, contractible open 3-manifold $M$ which is not end-prime and does not admit a decomposition via end sum into end-prime 3-manifolds.

\textbf{Proof.} By Theorems 4.3 and 6.1 it suffices to construct an irreducible, eventually end-irreducible, contractible open 3-manifold $M$ which is homeomorphic to a strong end sum of itself with itself. We shall construct such an $M$ having an exhaustion $\{C_n\}$ where $C_n$ is a cube with $2^{n+1}$ handles. We shall present $M$ as the direct limit of a sequence $K_0 \xrightarrow{g_0} K_1 \xrightarrow{g_1} K_2 \xrightarrow{g_2} \cdots$, where $K_n$ is a cube with $2^{n+1}$ handles and $g_n$ embeds $K_n$ into the interior of $K_{n+1}$. The image of $K_n$ in the direct limit will be $C_n$.

We regard $K_n$ as two copies $W_n^+$ of a cube with $2^n$ handles joined by a 1-handle $L_n = D_n \times [-1, 1]$, where $D_n \times \{\pm 1\}$ are disks lying in $\partial W_n^\pm$, respectively. We denote $D_n \times \{0\}$ by $D_n$ and the center of $D_n$ by $x_n$. We let $L_n^-$ and $L_n^+$ denote $D_n \times [-1, 0]$ and $D_n \times [0, 1]$, $D_n^\pm$ denote $D_n \times \{\pm 1\}$, and $x_n^\pm$ denote $\{x_n\} \times \{\pm 1\}$, respectively. We let $p_n : D_n \times [-1, 1] \rightarrow D_n$ be projection onto the first factor. Let $r_n$ be an orientation preserving involution of $K_n$ which interchanges $W_n^+$ and $W_n^-$ as well as $L_n^+$ and $L_n^-$; we assume that the restriction of $r_n$ to $L_n$ has the form $r_n(x, t) = (s_n(x), -t)$, where $s_n$ is reflection in a diameter of $D_n$. Thus $r_n$ fixes $x_n$ and interchanges $x_n^+$ and $x_n^-$. We also choose an orientation preserving homeomorphism $f_n : K_n \rightarrow W_{n+1}^+$. We next describe the embeddings $g_n : K_n \rightarrow K_{n+1}$.

Let $h_0 : W_0^+ \rightarrow int W_1^+$ be any null-homotopic embedding. We define $g_0$ to be $h_0$ on $W_1^+$. By Proposition 5.4 there is an excellent arc $\alpha_0$ in $W_1^+ - int g_0(W_0^+)$ joining $x_1^+$ to $\partial g_0(W_0^+)$. We extend $g_0$ over $D_0 \times [\frac{1}{2}, 1]$ by sending it to a regular
neighborhood $N_0$ of $\alpha_0$ in $W^+_1 - \text{int} \ g_0(W^+_0)$. We require that $g_0$ take $D_0 \times \{\frac{1}{2}\}$ to a disk in the interior of $D^+_1$ whose image under $p_1$ is invariant under $r_1$. We then extend $g_0$ over $D_0 \times [0, \frac{1}{2}]$ by setting $g_0(x, t) = (p_1(g_0(x, \frac{1}{2})), 2t)$. Finally, we extend $g_0$ over $W^-_0 \cup L^-_0$ by defining it to be $r_1 g_0 r_0$. Thus we have $g_0 : K_0 \to K_1$. Note that the image is invariant under $r_1$ and that $r_1 g_0 = g_0 r_0$.

We now define $g_1 : K_1 \to K_2$. The basic idea is to embed $W^+_1$ in $W^+_2$ in the same fashion in which $K_0$ is embedded in $K_1$, then choose an excellent arc joining $x^+_2$ to the boundary of the image of this embedding, and then extend the embedding to all of $K_1$ as in the previous step. More precisely, we let $h_1 = f_1 g_0 f_0^{-1} : W^+_1 \to W^+_2$ be our initial embedding. We define $g_1$ on $W^+_1$ to be $h_1$. Let $\alpha_1$ be an excellent arc in $W^+_2 - \text{int} g_1(W^+_1)$ joining $x^+_2$ to $\partial g_1(W^+_1)$. We extend $g_1$ over $D_1 \times [\frac{1}{2}, 1]$ by sending it to a regular neighborhood $N_1$ of $\alpha_1$ in $W^+_2 - \text{int} g_1(W^+_1)$, requiring that $p_2(g_1(D_1 \times \{\frac{1}{2}\}))$ be invariant under $r_2$. We extend $g_1$ over $D_1 \times [0, \frac{1}{2}]$ by setting $g_1(x, t) = (p_2(g_1(x, \frac{1}{2})), 2t)$. Finally, we extend $g_1$ over $W^-_1 \cup L^-_1$ by defining it to be $r_2 g_1 r_1$. Thus we have $g_1 : K_1 \to K_2$, an embedding whose image is invariant under $r_2$ such that $r_2 g_1 = g_1 r_1$.

Continuing in this manner we construct sequences of embeddings $g_n : K_n \to K_{n+1}$, $h_n : W^+_n \to W^+_{n+1}$, where $h_n = f_n g_{n-1} f_{n-1}$, $g_n, W^+_n = h_n, g_n(D_n \times [\frac{1}{2}, 1])$ is a regular neighborhood $N_n$ of an excellent arc $\alpha_n$ joining $x^+_n$ to $\partial g_n(W^+_n)$, $g_n(x, t) = (p_n+1(g_n(x, \frac{1}{2})), 2t)$ for $(x, t) \in D_n \times [0, \frac{1}{2}]$, and letting $g_n$ be $r_{n+1} g_n r_n$ on $W^-_n \cup L^-_n$.

We then let $M$ be the direct limit of the sequence $K_0 \xrightarrow{g_0} K_1 \xrightarrow{g_1} K_2 \xrightarrow{g_2} \cdots$ and denote the image of $K_n$ in $M$ by $C_n$. We let $V_n$ be the image of $W^+_n$ in $M$, and let $V$ be the union of the $V_n$. Note that the $r_n$ induce an involution $r$ of $M$, $M = V \cup r(V)$, and $V \cap r(V) = \partial V = \partial r(V)$, which is a plane $E$ invariant under $r$.

**Lemma 6.3.** $M$ is irreducible, eventually end-irreducible, and contractible. $V$ is irreducible, strongly aplanar, and anannular at $\infty$.

**Proof.** $M$ is irreducible because it is a monotone union of cubes with handles. $C_{n+1} - \text{int} C_n$ is homeomorphic to $K_{n+1} - \text{int} g_n(K_n)$, which is the union of $W^+_{n+1} - \text{Int} (g_n(W^+_n) \cup N_n)$ and its image under $r_{n+1}$. Since $\alpha_n$ is an excellent arc these manifolds are irreducible, $\partial$-irreducible, and anannular. They intersect in an incompressible annulus, and so by a standard general position and isotopy argument their union is irreducible and $\partial$-irreducible. Since $h_0 : W^+_0 \to W^+_1$ is null-homotopic we have that $g_0 : K_0 \to K_1$ is null-homotopic, and so $h_1 = f_1 g_0 f_0^{-1} : W^+_1 \to W^+_2$ is null-homotopic. Thus by induction we have that all the $h_n$ and $g_n$ are null-homotopic, and so $M$ is contractible.

The irreducibility of $V$ follows from that of $M$ and the fact that $E$ is a plane. The strong aplanarity and anannularity at $\infty$ of $V$ follow from Proposition 5.3 and the anannularity of $W^+_{n+1} - \text{Int} (g_n(W^+_n) \cup N_n)$. □

**Lemma 6.4.** $\text{Int} V$ is homeomorphic to $M$.

**Proof.** $\text{Int} V$ is the direct limit of the sequence $W^+_1 \xrightarrow{h_1} W^+_2 \xrightarrow{h_2} W^+_3 \xrightarrow{h_3} \cdots$. Since the homeomorphisms $f_n : K_n \to W^+_{n+1}$ satisfy $f_{n+1} g_n = h_n f_n$ we have that they induce a homeomorphism $f : M \to \text{Int} V$. □
This completes the proof of Theorem 6.2. □

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