A recursive enumeration of connected Feynman diagrams with an arbitrary number of external legs in the fermionic non-relativistic interacting gas

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In this work, we generalize a recursive enumerative formula for connected Feynman diagrams with two external legs. The Feynman diagrams are defined from a fermionic gas with a two-body interaction. The generalized recurrence is valid for connected Feynman diagrams with an arbitrary number of external legs and an arbitrary order. The recurrence formula terms are expressed in function of weak compositions of non-negative integers and partitions of positive integers in such a way that to each term of the recurrence correspond a partition and a weak composition. The foundation of this enumeration is the Wick theorem, permitting an easy generalization to any quantum field theory. The iterative enumeration is constructive and enables a fast computation of the number of connected Feynman diagrams for a large amount of cases. In particular, the recurrence is solved exactly for two and four external legs, leading to the asymptotic expansion of the number of different connected Feynman diagrams.

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I. INTRODUCTION

Enumeration of Feynman diagrams is currently an active research subject in quantum field[1][2][3] and many-body theoretical research[4][5][6]. The formal perturbative machinery flows into well-defined operations that unambiguously define the Feynman diagrams. The combinatorial character of this generative process is contained in two equivalent formalisms: the functional and the field operator approaches. The diagrams represent processes expressed commonly in terms of divergent integrals whose contribution is obtained afterwards by renormalization. Although the enumeration of the Feynman diagrams is independent of the integrals that represent the physical processes, when we take the total contribution of certain classes of diagrams, the global structure of the generative combinatorics is relevant. (This can be seen, for instance, in recent results[7], where the symmetry factor -or multiplicity- of the related Feynman diagrams appear explicitly in the integrals.)

The standard way to count Feynman diagrams is to define the theory in zero dimension[1][4][13]. Recently E.R Castro[8] used a uncomplicated principle for counting Feynman diagrams: in well-defined algebraic (multiplicative) relations between objects expressed as sums of Feynman diagrams associated with each object, the replacement of the sums by the explicit number of contractions that generates the specific diagrams in each order leads to recursive relations between the number of contractions associated with each object for each order of perturbation. In the generating functional terminology, this principle has straightforward interpretation. For example, consider the following multiplicative relation between \( \mathcal{G}_A(x) \), \( \mathcal{G}_B(x) \), \( \mathcal{G}_C(x) \) and \( \mathcal{G}_D(x) \),

\[
\mathcal{G}_A(x) = \mathcal{G}_B(x) \times \mathcal{G}_C(x) \times \mathcal{G}_D(x),
\]

where the \( \mathcal{G}_X(x) \) can be correlation functions, \( n \)-point functions etc, each one expressed as sum of diagrams for all the perturbation orders. In zero dimension, these functions take the form

\[
\mathcal{G}_X(x) = \sum_{m=0}^{\infty} n_m^{(X)} x^m,
\]

where \( n_m^{(X)} \) is the number of \( m \)-order Wick contractions present in \( \mathcal{G}_X \). Expression[2] is the generating functional of the number of \( m \)-order contractions and it induces in[1] the following sum, which associates the respective number of contractions for each order.
When the associated $m_i$-order Feynman diagrams have the same multiplicity, these relations determine the number of different Feynman diagrams. Particularly, this is the case in QED and in many-body theory. This simple counting principle was also used by F.B. Kugler in a generalized way to determine the number of different types of Feynman diagrams from well-defined many-body relations, leading to an efficient counting of a great variety of Feynman diagrams. There is vast literature dedicated to the counting of Feynman diagrams. See Ref.[10] for a brief introduction and, for an exhaustive study, see Refs.[11][12] and [13].

In this work, we generalize the previous recursive enumerative formula for connected Feynman diagrams with two external legs for a fermionic interacting gas to the case of an arbitrary number of external legs. The recursive enumerative formula was used in Ref.[8] to get an exact formula and find an equivalence with the Arquès formula for one-rooted maps (i.e., objects in algebraic topology)[14]. Particularly, equivalences between the counting of $N$-rooted maps and connected Feynman diagrams with $2N$ external legs have been established [15][16]. Exact formulas related to this algebraic curve topological theory have also been obtained. Other connections between Feynman diagrams and rooted maps can be seen in Ref.[17].

The derivation of our recurrence formula start with the Wick Theorem and has a possible interpretation in terms of elementary combinatorial theory. Based on the bijection found in Ref.[15], our counting also applies to the $N$-rooted map case and can be considered a different enumerative process.

This paper is organized as follows. In section II, from the set of possible Wick contractions, we establish the possible ways to construct an arbitrary disconnected Feynman diagram. By summing over all the possibilities, we obtain a set of recurrences, which relate the number of different connected Feynman diagrams for different orders and the number of external legs. In section III, we enormously simplify the recurrence, reducing it to a form that allows an easy computation of the number of different connected Feynman diagrams. Section IV exactly solves the recurrences for $N = 1$ and $N = 2$ (two and four external legs, respectively) and from these exact values we find many terms of the respective asymptotic expansions. We also we determine the existence of a new type of asymptotic contribution (negligible in respect to the main contribution) which is not present in the conventional literature. Section V contains the final considerations of this work.

II. WICK THEOREM AND FACTORIZATION IN TERMS OF CONNECTED FEYNMAN DIAGRAMS

As we see in Ref.[8], the object that generates the $m$-order Feynman diagrams with two external legs is the following expectation value,

$$
\langle \phi_0|T[H_I(t_1) \cdots H_I(t_m)]\hat{\psi}_\alpha(x)\hat{\psi}_\beta^\dagger(y)|\phi_0\rangle,
$$

where $H_I$ is the two-body interaction in second-quantization format

$$
H_I = \frac{1}{2} \sum_{\lambda_i, \mu_i, \nu_i} \int d^3 z_i d^3 z_i' \hat{\psi}_{\lambda_i}^\dagger(z_i) \hat{\psi}_{\mu_i}^\dagger(z_i') U(z_i, z_i') \hat{\psi}_{\lambda_i}(z_i) \hat{\psi}_{\mu_i}(z_i'),
$$

with $i \in \{1, 2, \cdots, m\}$. Considering this interaction as of the Coulomb type, the associated system would be a non-relativistic interacting gas of identical particles. In the fermionic case, the precise rules for the construction of the Feynman diagram are given, for example, in Chapter 3, section 9 of Ref.[18]. We will only consider the Feynman diagrams in the fermionic case. The $2m$ variables $\{z_i, z_i'\}$ correspond to the $2m$ internal vertices. The interaction $U(z_i, z_i')$ corresponds to a wavy line edge joining the vertices $z_i$ and $z_i'$. The fermionic directed leg starting at $z_a$ and ending in $z_b$ corresponds to the Wick contraction association

$$
\hat{\psi}(z_a)\hat{\psi}_\beta^\dagger(z_b).
$$

The incoming (outcoming) external legs are obtained (respectively) by the substitution $z_a \to x$ ($z_b \to y$) in the previous expression.

These rules are easily generalized to the case of $2N$ external legs. In this case, the Feynman generator expectation value is

$$
\mathcal{N}_m^{(A)} = \sum_{m_1=0}^m \sum_{m_2=0}^m \sum_{m_3=0}^m \delta_{m_1+m_2+m_3=m} \mathcal{N}_{m_1}^{(B)} \mathcal{N}_{m_2}^{(C)} \mathcal{N}_{m_3}^{(D)}.
$$

(3)
\[ \langle \phi_0 | T [ \hat{H}_1(t_1) \cdots \hat{H}_I(t_m) \hat{\psi}_{\alpha_1}(x_1) \hat{\psi}_{\alpha_2}(x_2) \cdots \hat{\psi}_{\alpha_N}(x_N) \hat{\psi}_{\beta_1}^\dagger(y_1) \hat{\psi}_{\beta_2}^\dagger(y_2) \cdots \hat{\psi}_{\beta_N}^\dagger(y_N) ] | \phi_0 \rangle. \]  

(6)

For example, diagram (20), with four external legs, corresponds to

\[ \hat{\psi}(x_1) \hat{\psi}^\dagger(z_1) \times \hat{\psi}(z_1) \hat{\psi}^\dagger(y_1) U(z_1, z_2) \hat{\psi}(z_2) \hat{\psi}^\dagger(y_2) \times \hat{\psi}(y_2) \hat{\psi}^\dagger(z_2) . \]

The total number of possible contractions (i.e., possible association in pairs) \( \mathcal{N}^{(N)}_m \) in (6) is

\[ \mathcal{N}^{(N)}_m = (2^m + N)! . \]  

(7)

This is easy to see: In expression (6), there are \( 2^m + N \) annihilation field operators \( (N \) different \( \hat{\psi}_{\alpha_i}(x_i), m \) different \( \hat{\psi}_{\lambda_j}(z_j) \) and \( m \) different \( \hat{\psi}_{\mu_k}^\dagger(z'_k) \)) and \( 2^m + N \) creation field operators \( (N \) different \( \hat{\psi}_{\beta_i}^\dagger(y_i), m \) different \( \hat{\psi}_{\lambda'_j}^\dagger(z_j) \) and \( m \) different \( \hat{\psi}_{\mu'_k}^\dagger(z'_k) \).) Non-vanishing contractions only happen between creation and annihilation operators. Therefore, the total number of possible contractions is \( (2^m + N)! \). (All the field operators must be contracted. From now on, we omit the spinor indices, since we consider only the unpolarized spin contributions summing over the spins values.)

Also, it is easy to see that the total number of external legs is always even. For each internal vertex \( z_b \), there are only two associate field operators, \( \hat{\psi}(z_b) \) and \( \hat{\psi}^\dagger(z_b) \), which are contracted between them,

\[ \text{or with } \hat{\psi}^\dagger(z_c) \text{ and } \hat{\psi}(z_a), \text{ respectively,} \]

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Therefore, each vertex \( z_i \) belongs to a unique trail of fermion lines that is a closed cycle (with one or more vertices) or is a trail which begins in a unique \( x_a \) and ends in a unique \( y_b \). (The only field operators associated with \( x_a \) and \( y_b \) are \( \hat{\psi}(x_a) \) and \( \hat{\psi}^\dagger(y_b) \), respectively). An odd number of external legs implies the existence of at least one vertex with more than two associated field operators, which is a contradiction.

The wavy line \( U(z_j, z'_j) \) connects the vertices \( z_j \) and \( z'_j \), which may belong to the same fermionic trail or to different trails. For a given contraction, we have a set of trails and, by inserting the fixed interactions \( U(z_j, z'_j) \), we obtain the corresponding Feynman diagram. This diagram can be connected or disconnected. Suppose an arbitrary \( m \)-order disconnected Feynman diagram \( \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_{l+1} \)
which has \( l \) connected components, each of them with order \( m_1, m_2, \ldots, m_l \), respectively. The last component is not necessarily connected, its order is \( m_{l+1} \), and it only contains vacuum-bubble diagrams.

We have, then

\[
m = m_1 + m_2 + \cdots + m_{l+1}
\]

and

\[
N = n_1 + n_2 + \cdots + n_l.
\]

In how many ways can we choose the internal vertices of each component? The order of the first component is \( m_1 \), so there are \( \binom{m}{m_1} \) ways to choose the pairs \((z_i, z'_i)\). The order of the second component is \( m_2 \), and we then have \( \binom{m-m_1}{m_2} \) ways to choose the pairs, and so on. Thereby, the number of possible choices is

\[
\binom{m}{m_1} \binom{m-m_1}{m_2} \cdots \binom{m-m_1-\cdots-m_l}{m_{l+1}} = \frac{m!}{m_1!m_2!\cdots m_{l+1}!}.
\]

The external legs can also be chosen in different ways. There are \( N! \) ways to choose the incoming \( N \) legs. As we are not yet investigating the internal structure of each component, for now, it only matters to know the different forms to associate the outgoing legs with each component. The first component has \( n_1 \) outgoing legs. Therefore, the number of possibilities in choosing the outgoing legs in the first component, once the incoming legs are fixed, is \( \binom{N-n_1}{n_2} \). For the second component, there exist \( \binom{N-n_1}{n_2} \) possibilities, and so on. Once the incoming lines are fixed, the total number of possibilities is

\[
\binom{N}{n_1} \binom{N-n_1}{n_2} \cdots \binom{N-n_1-\cdots-n_{l-1}}{n_l} = \frac{N!}{n_1!n_2!\cdots n_l!}.
\]

If we have a different number of external legs for each component, the total number of possibilities is simply \( N! \) multiplied by the multinomial coefficient expressed in (14). This is an over-counting if there are components with equal number of external legs. In particular, if we have only \( r < l \) components with different number of external legs such that \( N = d_1n_1 + d_2n_2 + \cdots + d_rn_r \), where \( d_i \) is the number of components with the same number of external legs (and, evidently, \( l = d_1 + \cdots + d_r \)), the correct counting, in this case, is given by

\[
\frac{1}{d_1!d_2!\cdots d_r!} \times \frac{(N!)^2}{n_1!n_2!\cdots n_l!}.
\]

Now, it is time to study the internal structure of each component. Note that all the Feynman diagrams that satisfy (11) and (12), have the same external structure and, therefore, they carry the same counting as in (15). (I.e., the substitution of the component \( h \) by another with the same order \( m_h \) and the same number \( 2n_h \) of external legs leads to the same counting as expressed in (15).) Bearing that the diagram in (10) represents a product of \( l+1 \) integrals, it follows that the sum of all the different diagram contributions that satisfy (11) and (12) is factored in a product whose \( l+1 \) elements are the sum of all the possible components. The internal structure is considered by taking, instead of all the possible different components, the different contractions that have the respective order and number of external legs in each component. So, the total number of possible contractions satisfying (11) and (12) is

\[
\frac{1}{d_1!d_2!\cdots d_r!} \times \frac{(N!)^2}{n_1!n_2!\cdots n_l!} \times \frac{m!}{m_1!m_2!\cdots m_{l+1}!} \times T[F_1]_{n_1}^{m_1} \times \cdots \times T[F_l]_{n_l}^{m_l} \times T[F_{l+1}]_{0}^{m_{l+1}}
\]
with

\[ T[F_i]_{m_i}^{n_i} = \langle \phi_0 | T[H_i(t_1) \cdots H_i(t_{m_i}) \hat{\psi}(x_{e_1}) \hat{\psi}(x_{e_2}) \cdots \hat{\psi}(x_{e_n}) \hat{\psi}^\dagger(y_{e_1}) \hat{\psi}^\dagger(y_{e_2}) \cdots \hat{\psi}^\dagger(y_{e_n}) | \phi_0 \rangle_{\text{connected}}, \]  

where \( i \in \{1, 2, \cdots, l\} \), and

\[ T[F_{l+1}]_{0}^{m_{l+1}} = \langle \phi_0 | T[H_i(t_1) \cdots H_i(t_{m_{l+1}})] | \phi_0 \rangle. \]  

The index \( \text{connected} \) implies that we are only considering contractions that generate connected diagrams. Suppose that there is a total of \( N_{c,m_i}^{(n_i)} \) of such contractions. According to Ref. [8], by replacing \( T[F_i]_{m_i} \) by \( N_{c,m_i}^{(n_i)} \) and \( T[F_{l+1}]_{0}^{m_{l+1}} \) by \( \mathcal{D}_{m_{l+1}} = (2m_{l+1})! \), we obtain the total number of contractions that generate Feynman diagrams satisfying (11) and (12):

\[ \frac{1}{d_1!d_2! \cdots d_l!} \times \frac{(N!)^2}{n_1!n_2! \cdots n_l!} \times \frac{m!}{m_1!m_2! \cdots m_{l+1}!} N_{c,m_1}^{(n_1)} N_{c,m_2}^{(n_2)} \cdots N_{c,m_l}^{(n_l)} \mathcal{D}_{m_{l+1}}. \]  

Before continuing, let us verify what the minimal order possible of a connected diagram with 2\( N \) legs is. For \( N = 1 \), the diagram with minimal order possible is evidently the free propagator, with \( m = 0 \). For \( N = 2 \), the minimal order connected Feynman diagram is \( m = 1 \):

![Diagram](https://via.placeholder.com/150)

In diagram (20), we have two trails, each one with an internal vertex, and one wavy line connecting these internal vertices. For \( N = 3 \), we use the previous case to build the minimal order connected diagram. We must add another trail with two external legs and one internal vertex. To connect this trail, we simply add one internal vertex to one of the above trails and connect it to one additional wavy line. This construction is the minimal possible. So, for \( N = 3 \) we have \( m = 2 \). This construction can be generalized for all the other cases, obtaining \( m = N - 1 \) as the minimal possible order.

Expression (19) is only one particular case of (11) and (12). If we add all other possible cases, we obtain \( \mathcal{Y}_m^{(N)} \), namely, the number of total contractions in (10),

\[ \mathcal{Y}_m^{(N)} = \sum_{(n_1, \cdots, n_l) \in \mathcal{P}_N} \left[ \sum_{m_1=n_1-1}^{m} \cdots \sum_{m_l=n_l-1}^{m} \sum_{m_{l+1}=0}^{m} \delta_{m_1+\cdots+m_{l+1},m} \frac{1}{d_1!d_2! \cdots d_l!} \times \frac{(N!)^2}{n_1!n_2! \cdots n_l!} \times \frac{m!}{m_1!m_2! \cdots m_{l+1}!} N_{c,m_1}^{(n_1)} N_{c,m_2}^{(n_2)} \cdots N_{c,m_l}^{(n_l)} \mathcal{D}_{m_{l+1}} \right], \]  

where \( \mathcal{P}_N \) is the numerical partition set of \( N \). The index \( l \) depends on each partition. The Kronecker delta guarantees that, for each partition of \( N \), we have a sum over the weak compositions of \( m \), with \( N_i - 1 \leq m_i \leq m \) for \( i \in \{1, \cdots, l\} \), and \( 0 \leq m_{l+1} \leq m \). (Compositions are partitions where the order of the addends matters and, for weak compositions, the zero addend is allowed.)

From equation (21), it is possible to recursively find the values of \( N_{c,m}^{(N)} \). (Particularly, it allows a different recurrence for each \( N \).) Let us write these recurrences for \( N = 1, 2 \) and 3.

For \( N = 1 \) (two external legs), we have a unique partition \( 1=1 \). Therefore, \( l = 1 \) and

\[ \mathcal{Y}_m^{(1)} = \sum_{m_1=0}^{m} \sum_{m_2=0}^{m} \delta_{m_1+m_2,m} \frac{m!}{m_1!m_2!} N_{c,m_1}^{(n_1)} \mathcal{D}_{m_2}. \]  

(See this recurrence in Ref. [8].)

For \( N = 2 \) (four external legs), the partitions are \( (1+1) \) and \( (2) \), with \( l = 2 \) and \( l = 1 \), respectively. So, we have
\[ \mathcal{R}_m^{(2)} = 2 \sum_{m_1=0}^{m} \sum_{m_2=0}^{m} \sum_{m_3=0}^{m} \delta_{m_1+m_2+m_3,m} \frac{m!}{m_1!m_2!m_3!} \mathcal{N}_{c,m_1}^{(1)} \mathcal{N}_{c,m_2}^{(1)} \mathcal{D}_{m_3} \]
\[ + 2 \sum_{m_1=1}^{m} \sum_{m_2=0}^{m} \delta_{m_1+m_2,m} \frac{m!}{m_1!m_2!} \mathcal{N}_{c,m_1}^{(2)} \mathcal{D}_{m_2}. \]  

(23)

For \( N = 3 \) (six external legs), the partitions are \((1+1+1), (2+1)\) and \((3)\), with \( l = 3, l = 2 \) and \( l = 1 \), respectively. So, we have

\[ \mathcal{R}_m^{(3)} = 6 \sum_{m_1=0}^{m} \sum_{m_2=0}^{m} \sum_{m_3=0}^{m} \delta_{m_1+m_2+m_3,m} \frac{m!}{m_1!m_2!m_3!} \mathcal{N}_{c,m_1}^{(1)} \mathcal{N}_{c,m_2}^{(1)} \mathcal{D}_{m_3} \]
\[ + 18 \sum_{m_1=1}^{m} \sum_{m_2=0}^{m} \sum_{m_3=0}^{m} \delta_{m_1+m_2+m_3,m} \frac{m!}{m_1!m_2!m_3!} \mathcal{N}_{c,m_1}^{(2)} \mathcal{N}_{c,m_2}^{(1)} \mathcal{D}_{m_3} \]
\[ + 6 \sum_{m_1=2}^{m} \sum_{m_2=0}^{m} \delta_{m_1+m_2,m} \frac{m!}{m_1!m_2!} \mathcal{N}_{c,m_1}^{(3)} \mathcal{D}_{m_2}. \]  

(24)

Recurrence (22) determines the numbers \( \mathcal{N}_{c,m}^{(1)} \), which can be used in the recurrence (23) to find the numbers \( \mathcal{N}_{c,m}^{(2)} \), and so on.

III. RECURRENCE SIMPLIFICATION

Recurrence (21) has an uncomplicated interpretation in terms of a discrete convolution in the number of contractions that generates the arbitrary component associated with the perturbative order of each component. This was correctly noticed in the recurrences obtained in Ref.\[9\]. Also, some care must be taken when converting diagramatic expressions like (10) into numerical convolutions, because, as in (10), combinatorial weights can be involved in the discrete convolution.

In our case, the terms of the discrete convolutions are indexed by weak compositions \[19\]. For a precise recurrence computation, this can be a problem, since this would require evaluating a huge number of possibilities. Fortunately, expression (21) can be greatly simplified. In particular, we have

\[ \mathcal{R}_m^{(N)} = N \sum_{j=0}^{m} \binom{m}{j} \mathcal{N}_{c,j}^{(1)} \mathcal{R}_{m-j}^{(N-1)} + N \sum_{i=2}^{N} \frac{(N-1)^2(N-2)^2 \cdots (N-i+1)^2}{(i-1)!} \sum_{j=i-1}^{m} \binom{m}{j} \mathcal{N}_{c,j}^{(i)} \mathcal{R}_{m-j}^{(N-i)}, \]  

(25)

where \( \mathcal{R}_0 = \mathcal{D}_l = (2l)! \). To prove this from expression (21), note that in (21) we have a sum indexed over the possible partitions of \( N \). Consider the set of all the partitions \( \mathcal{A}_{n_{i}} \subset \mathcal{P}_N \) that contain the arbitrary number \( n_i \). Since \( N = n_i + (N-n_i) \), the number of such partitions is identical to the total number of partitions in \( \mathcal{P}_{N-n_i} \). (That is to say, there exists a bijection between \( \mathcal{A}_{n_{i}} \) and \( \mathcal{P}_{N-n_{i}} \).) Considering an arbitrary partition of \( N \) that contain \( n_i \), the redefinition \( n_i \rightarrow n_a \) and \( n_{j+1} \rightarrow n_j \) for \( i \leq j \leq l \), the corresponding term in (21) can be written as

\[ \sum_{m_a=n_a-1}^{m} \frac{m!}{m_a!(m-m_a)!} \mathcal{N}_{c,m_a}^{(n_a)} \left[ \sum_{m_1=n_a-1}^{m-m_a} \cdots \sum_{m_{l-1}=n_{l-1}-1}^{m-m_{l-1}} \delta_{m_1+\cdots+m_{l-1}+m_a,m} \frac{1}{d_1!d_2!\cdots d_l!} \right] \times \frac{(N!)^2}{(n_1!n_2!\cdots n_{l-1}!n_a)!} \times \frac{(m-m_a)!}{m_1!m_2!\cdots m_l!} \mathcal{N}_{c,m_1}^{(n_1)} \mathcal{N}_{c,m_2}^{(n_2)} \cdots \mathcal{N}_{c,m_{l-1}}^{(n_{l-1})} \mathcal{D}_{m_l}, \]  

(26)

where, in the new definition, the Kronecker delta guarantee that \( m_1+\cdots+m_l = m-m_a \). Using \( N = d_1n_1+\cdots+d_en_r \), we have

\[ \frac{1}{d_1!d_2!\cdots d_l!} \times \frac{(N!)^2}{(n_1!n_2!\cdots n_{l-1}!n_a)!} = \frac{1}{d_1!d_2!\cdots d_l!} \times \frac{N[(N-1)!]^2}{(n_1!n_2!\cdots n_{l-1}!n_a)!} \times (d_1n_1+\cdots+d_en_r). \]  

(27)
From the previous equation, it is obvious that we have \( r \) different choices for the index \( n_a \) associated with the partition related to \( (26) \). Therefore, we can decompose \( (26) \) in \( r \) terms. The term associated with \( n_a \) has the weight

\[
\frac{1}{d_1! \cdots d_a! \cdots d_l!} \times \frac{N! [(N - 1)!]^2}{(n_1! n_2! \cdots n_{l-1})! n_a!} \times d_a n_a = \frac{N(N - 1)^2 \cdots (N - n_a + 1)^2}{(n_a - 1)!}
\]

\[
\times \left[ \frac{1}{d_1! \cdots (d_a - 1)! \cdots d_l!} \times \frac{[(N - n_a)!]^2}{n_1! n_2! \cdots n_{l-1}!} \right]. \tag{28}
\]

So, according to expression \( (26) \), we have \( r \) terms associated in the following format

\[
\frac{N(N - 1)^2 \cdots (N - n_a + 1)^2}{(n_a - 1)!} \sum_{m_a=n_a-1}^{m} \binom{m}{m_a} \mathcal{N}_{cm_a}^{(n_a)} \times \sum_{m_1=n_1-1}^{m-m_a} \cdots \sum_{m_l-1=n_l-1}^{m-m_{l-1}} \sum_{m_{l-1}=n_{l-1}}^{m-m_{l-1}} \delta_{m_1+\cdots+m_{l-1},m-m_{l-1}}
\]

\[
\times \frac{1}{d_1! \cdots (d_a - 1)! \cdots d_l!} \times \frac{[(N - n_a)!]^2}{n_1! n_2! \cdots n_{l-1}!}
\]

\[
\times \left[ \frac{(m - m_a)!}{m_1! m_2! \cdots m_l!} \times \mathcal{N}_{cm_1}^{(n_1)} \mathcal{N}_{cm_2}^{(n_2)} \cdots \mathcal{N}_{cm_{l-1}}^{(n_{l-1})} \mathcal{O}_{m_l} \right]. \tag{29}
\]

The important fact about these \( r \) terms associated with the initial arbitrary partition of \( N \) is that all of them have the same form, and we can consider \( (29) \) as a generic term. The factor in the square bracket of \( (29) \) corresponds to a partition of \( N - n_a \) and is identical to the associated term in \( \mathcal{O}_{m-m_{a}}^{(N-n_{a})} \). (See expression \( (21) \).) From the bijection \( \mathcal{A}_{n_a} \leftrightarrow \mathcal{P}_{N-n_a} \), we exhaust all the possibilities for the other partitions of \( N \) that contain the element \( n_a \), getting all the partitions in \( \mathcal{P}_{N-n_a} \) and therefore generating all the terms of \( \mathcal{O}_{m-m_{a}}^{(N-n_{a})} \).

\[
\frac{N(N - 1)^2 \cdots (N - n_a + 1)^2}{(n_a - 1)!} \sum_{m_a=n_a-1}^{m} \binom{m}{m_a} \mathcal{N}_{cm_a}^{(n_a)} \mathcal{O}_{m-m_{a}}^{(N-n_{a})}. \tag{30}
\]

For all the other possible values of \( n_a \), we repeat the process. Since \( (21) \) is associated with all the partitions of \( N \), the decomposition \( (27) \) and the bijection \( \mathcal{A}_{n_a} \leftrightarrow \mathcal{P}_{N-n_a} \) guarantee the validity of relation \( (25) \). For \( n_a = 1 \), it is clear that the factor \( (N - 1)^2 \cdots (N - n_a + 1)^2/(n_a - 1)! \) does not appear.

Now, remember that the number \( \mathcal{N}_{cm}^{(N)} \) is the total number of contractions that generate \( m \)-order connected Feynman diagrams with \( 2N \) external legs. Some of these contractions generate the same Feynman diagram. In particular, every \( m \)-order Feynman diagram has multiplicity (i.e., different equivalent contractions) equal to \( 2^m \times m! \) \( \square \) or the same symmetry factor. Therefore, the number of different \( m \)-order connected Feynman diagrams with \( 2N \) external legs \( h_m^{(N)} \) is

\[
h_m^{(N)} = \frac{\mathcal{N}_{cm}^{(N)}}{2^m m!}. \tag{31}
\]

Table \( 1 \) shows the initial series of values for \( h_m^{(N)} \).

**Table I: Initial series of values for \( h_m^{(N)} \).**

| \( m \) | \( h_m^{(1)} \) | \( h_m^{(2)} \) | \( h_m^{(3)} \) | \( h_m^{(4)} \) | \( h_m^{(5)} \) | \( h_m^{(6)} \) | \( h_m^{(7)} \) |
|------|--------|--------|--------|--------|--------|--------|--------|
| 0    | 1      | 0      | 0      | 0      | 0      | 0      | 0      |
| 1    | 2      | 1      | 0      | 0      | 0      | 0      | 0      |
| 2    | 10     | 13     | 6      | 0      | 0      | 0      | 0      |
| 3    | 74     | 165    | 172    | 72     | 0      | 0      | 0      |
| 4    | 706    | 2273   | 3834   | 3438   | 1320   | 0      | 0      |
| 5    | 8162   | 34577  | 81720  | 115008 | 91968  | 32760  | 0      |
| 6    | 110410 | 581133 | 1775198| 3432864| 4227840| 3082080| 1028160|
| 7    | 1708394| 10749877| 40320516| 99431808| 166020720| 184019040| 124126560|
The sequence $h_m^{(1)}$ corresponds to the OEIS sequence A000698.

The numerical solutions of recurrences \[ \text{(25)} \] are constructive, that is to say, they are solved by beginning with case $N = 1$ until finite order $m$. Then, all these values are used to solve case $N = 2$ until finite order $m$, and so on. For example, using the program MATHEMATICA, we have calculated the exact values of $\mathcal{N}_{c,m}^{(N)}$ up to $m = 3000$ for the cases $N = 1, 2, \cdots, 7$ in a few minutes.

IV. EXACT SOLUTION FOR CASES $N = 1$ AND $N = 2$, ASYMPTOTIC EXPANSION

Recurrences \[ \text{(25)} \] can be solved exactly for cases $N = 1$ and $N = 2$. This allows the calculation of many terms in the asymptotic expansion ($m \to \infty$) of $h_m^{(N)}$. In Ref.\[8\], an explicit formula for $\mathcal{N}_{c,m}^{(1)}$ is obtained:

\[
\mathcal{N}_{c,m}^{(1)} = \sum_{n=1}^{m} C_n^m \left( \mathcal{A}_n^{(1)} - \mathcal{D}_n \right),
\]

with

\[
C_n^m = \sum_{i=1}^{m-n} (-1)^i \sum_{a_1, \cdots, a_i=1}^{\infty} \delta_{a_1+\cdots+a_i,m-n} \left( \frac{m-a_1}{m-a_1-a_2} \cdots \left( \frac{m-a_1-\cdots-a_{i-1}}{m-a_1-\cdots-a_{i-1}-a_i} \right) \prod_{j=1}^{i} \mathcal{D}_{a_j} \right),
\]

for $n < m$, and $C_n^m = 1$ for $m \in \mathbb{N}$. The above formula can be written as

\[
C_n^m = -\frac{m!}{n!} \sum_{i=0}^{m-n-1} (-1)^i \sum_{a_1, \cdots, a_{i+1}=1}^{\infty} \delta_{a_1+\cdots+a_{i+1},m-n} \prod_{j=1}^{i+1} \frac{(2a_j)!}{a_j!} \cdot
\]

If we compare the expression \[ \text{(34)} \] to the Arquès-Walsh sequence formula \[8\], we obtain

\[
C_n^m = -2(m-n) \left( \frac{m}{n} \right) \mathcal{N}_{c,m-n-1}^{(1)}.
\]

For arbitrary and finite $m$, the above formula simplifies the calculation of the symbols $C_n^m$ to the case $n \leq m$. In this case, it is only necessary to know the first values of $\mathcal{N}_{c,m}^{(1)}$, which are obtained iterating \[ \text{(25)} \] for $N = 1$. In particular,

\[
C_{m-1}^m = -2m
\]

\[
C_{m-2}^m = -8m(m-1)
\]

\[
C_{m-3}^m = -80m(m-1)(m-2)
\]

\[
C_{m-4}^m = -1184m(m-1)(m-2)(m-3)
\]

\[
C_{m-5}^m = -22592m(m-1)(m-2)(m-3)(m-4)
\]

\[
C_{m-6}^m = -522368m(m-1)(m-2)(m-3)(m-4)(m-5)
\]

\[
C_{m-k}^m = -\frac{2k}{k!} \mathcal{N}_{c,k-1} m(m-1)(m-2)(m-3)(m-4) \cdots (m-k+1).
\]

We are interested in an asymptotic expansion for $h_m^{(1)}$, when $m \to \infty$. Using $\mathcal{A}_m - \mathcal{D}_m = (2m)(2m)!$, the number of different connected Feynman diagrams with two external legs from \[ \text{(32)} \] is
\[
h_m^{(1)} = \frac{N_m^{(1)} - \mathcal{D}_m}{2m!} \left[ 1 + \sum_{n=1}^{m-1} c_m^{(n)} \frac{N_m^{(1)} - \mathcal{D}_m}{2^{m-1} m!} \right] = \frac{m!m}{2^{m-1} (2m)} \left[ 1 + \sum_{k=1}^{m-1} \frac{(m-k)(2[m-k])!}{m(2m)!} c_{m-k}^{(m-k)} \right]. \tag{43}
\]

In the last step, we use \( k = m - n \).

Now, let’s focus on the square bracket term in (43). It is easy to notice, by Using (34), that each term in the sum, is a quotient of polynomials in \( m \). To see this, choose a fixed value for \( k \) and use (42). The terms in question are proportional to

\[
(m - k) \frac{1}{(2m)(2m-1)(2m-3)(2m-5) \cdots (2m - (2k-1))}
\]

for \( k < m \) and \( k \in \mathbb{N} \). By adding the first \( \ell \) terms, it is not hard to see that

\[
\sum_{k=1}^{\ell} \frac{(m-k)(2[m-k])!}{m(2m)!} c_{m-k}^{(m-k)} = \frac{A^{(\ell)}m^{\ell} + B^{(\ell)}m^{\ell-1} + \cdots + X^{(\ell)}m^2 + Y^{(\ell)}m + Z^{(\ell)}}{(2m)(2m-1)(2m-3)(2m-5) \cdots (2m - (2\ell-1))}
\]

where the numbers \( A^{(\ell)}, B^{(\ell)}, \cdots, Z^{(\ell)} \) are integers generated by the usual algebraic operations when we factor the terms in the sum. In the complex plane \((m \to z)\), and this is done just to calculate the asymptotic expansion, which is afterwards restricted to the positive integer \( m \), the quotient of polynomials (45) has uncomplicated analytical properties. For example, it has at least \( \ell \) poles of order 1. (Considering the possibility that the numerator polynomial has a zero that “cancels” some pole, this number is strictly smaller. For the first values of \( \ell \), this possibility does not happen. If the cancellation does not occur, the \( \ell \) poles are equally distributed in the points \( 1/2, 3/2, 5/2, \ldots, l-1/2 \) of the positive real axis). What is important in this case is that the point in the infinity is regular, and the complex form of (45) is analytical for \(|z| > \ell - 1/2\). Therefore, this function admits an analytical Taylor expansion in \( z = \infty \). By making the transformation \( w = 1/z \), we see that the convergence radius of the Taylor expansion in \( \omega = 0 \) is \( 2/(2\ell - 1) \) (with the assumption that, for all \( \ell \), \( z_0 = \ell - 1/2 \) is a pole). For increasing values of \( \ell \), more and more poles appear in the real positive axis and the convergence radius of the Taylor series in \( \omega = 0 \) tends to zero.

Fortunately, this does not prevent the asymptotic analysis in \( m \to \infty \). Note that, for fixed \( \ell \), only the first \( a \) left-hand-side terms of (45) contribute in the first \( a \) Taylor-series terms of the right hand side, when \( m > \ell - 1/2 \). This last condition guarantees that \( m \) is inside the convergence radius, and the Taylor expansion of the left-hand-side terms can be added term by term. In particular the left hand side terms of (45) have the next Taylor expansion in the infinity

\[
\frac{2k}{\ell!} \mathcal{N}_{\ell, k-1} \frac{(m-k)}{(2m)(2m-1)(2m-3)(2m-5) \cdots (2m - (2k-1))} = \frac{a_1}{m^k} + \frac{a_2}{m^{k+1}} + \cdots, \quad m > k - \frac{1}{2}.
\tag{46}
\]

From this expression, it becomes clear that we only need to sum up the \( a \) initial Taylor terms of the \( a \) initial left-hand-side terms of (45) to get the first \( a \) Taylor-series terms of the right-hand-side. For \( k > a \), the Taylor terms of (46) are of order \( O(1/m^{a+n}) \), with \( n \in \mathbb{N} \), and do not contribute to the first \( a \) Taylor terms of the right-hand-side of (45). This analysis is valid for \( m \) arbitrarily large and finite, and we can interpret the Taylor-series in the infinity with convergent radius zero as the asymptotic expansion of \( h_m^{(1)} \).

In the expansion (43) for arbitrarily large \( m \), we only analyze the contribution of the terms \( k \ll m \). We now show that the terms \( k \leq m \) are also present in the contribution to the first asymptotic terms of the expansion. Using (34), we have the following term, for \( k = m - 1 \)

\[
\frac{(m-1)!2}{0!(2m!)} \sum_{i=0}^{m-2} (-1)^{i+1} \sum_{a_1, \ldots, a_{i+1} = 1}^{m-1} \delta_{a_1 + \cdots + a_{i+1}, m-1} \prod_{j=1}^{i+1} \frac{(2a_j)!}{a_j!}.
\tag{47}
\]

For a fixed \( i \), we see that the term of (47) represent compositions of \( m - 1 \) with \( i + 1 \) elements. (The elements are the \( a_k \) coefficients.) The first terms are
In particular, this term is of order $O(1/m^5)$ is

$$\frac{(m-1)!2!}{0!2m!} \left[ \frac{[2(m-1)]!}{(m-1)!} - 2 \frac{[2(m-2)]!}{(m-2)!} \frac{2!}{1!} - 2 \frac{[2(m-3)]!}{(m-3)!} \frac{4!}{2!} + \cdots \right]$$

$$+ 3 \frac{[2(m-3)]!2!2!}{(m-3)!1!1!1!} + 6 \frac{[2(m-4)]!4!2!}{(m-4)!2!1!} + 3 \frac{[2(m-5)]!4!4!}{(m-5)!2!2!} \cdots$$

$$- 4 \frac{[2(m-4)]!2!2!}{(m-4)!1!1!1!} - 12 \frac{[2(m-5)]!4!2!2!}{(m-5)!2!1!1!} \cdots \right]. \quad (48)$$

For example, the second term in the second line represents the composition $m - 1 = (m - 4) + 2 + 1$, and the multiplicative factor 6 represents the possible permutations of these three coefficients. The same analysis is, then, performed for $k = m - 2$, $k = m - 3$, etc. By a similar analysis, performed beforehand, of equation (46), only a finite number of these terms contribute to the first $a$ asymptotic terms of the entire expression (43). In Appendix A, we explicitly write all the terms that contribute to $a = 6$. The expansion until $O(1/m^5)$ is

$$\left[ 1 + \sum_{k=1}^{m-1} \frac{(m-k)(2[m-k])!}{m(2m)!} \binom{m}{m-k} \right] \sim 1 - \frac{1}{2m} - \frac{3}{4m^2} - \frac{19}{8m^3} - \frac{191}{16m^4} - \frac{2551}{32m^5} - \frac{41935}{64m^6} \cdots \quad (49)$$

and the expansion of the central binomial coefficient until order six [20] is

$$\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}} \left[ 1 - \frac{1}{8m} + \frac{1}{128m^2} + \frac{5}{1024m^3} - \frac{21}{32768m^4} - \frac{399}{262144m^5} + \frac{869}{4194304m^6} \cdots \right]. \quad (50)$$

By using both these expansions in [43], we obtain the expansion for $h_m^{(1)}$:

$$h_m^{(1)} \sim 2^{m!m^4} \frac{2m}{\sqrt{\pi}} 12^m \left[ 1 - \frac{5}{8m} - \frac{87}{128m^2} - \frac{2335}{1024m^3} - \frac{381733}{32768m^4} - \frac{20512763}{262144m^5} - \frac{2706890307}{4194304m^6} \cdots \right]. \quad (51)$$

The first two coefficients match with the ones given in Ref. [1], which use the functional approach. (In this reference, the variable used is $m' = 2m$.) For $m = 1000, 2000$ and $3000$, our asymptotics match with the exact value in the first 17, 19 and 20 digits, respectively.

The excellent numerical matching suggests that we have obtained the first six terms ($a \leq 6$) of the asymptotic expansion. The other terms ($a \geq 6$) can be found by the same process. (The calculation for growing $a$ gets more complicated, since it involves more and more terms of [43] and [44].) In particular, the terms of [43] for $k \leq m$ can be written as quotients of multinomial coefficients. For example, consider the last term of [43],

$$12 \frac{(m-1)!2!(m-5)!2!2!}{(2m)!2!(m-5)!2!1!1!1!} = 12 \frac{\binom{m-1}{m-5,2,1,1}}{\binom{2m}{m-10,4,2,2,2}}. \quad (52)$$

All the terms in (48), and also for the cases $k = m - 2$, $m - 3$, etc., can be written as quotients of multinomial coefficients. In the limit $k \leq m$ studied before, we only consider non-centered elements of the corresponding multinomial distributions. Every one of these terms is also represented by a numerical partition, which have a unique element depending on $m$.

Other cases not analyzed correspond to centered elements of the multinomial distribution or to numerical partitions with more than one coefficient depending on $m$. Do these terms present asymptotic contribution in $m \to \infty$? To analyze this, let us assume that $m$ is even. A specific example in the case $k = m - 1$ would be

$$12 \frac{(m-1)!(m-10)!2!(m + 4)!2!2!}{(2m)!2!(m/2 - 5!)(m/2 + 2)!1!1!1!} = 12 \frac{\binom{m-1}{m/2-5,m/2+2,2,1,1}}{\binom{2m}{m-10,m+4,2,2,2}}. \quad (53)$$

In particular, this term is of order

$$\sim \frac{12\sqrt{2}}{2^m m^4} \quad (54)$$
This asymptotic behavior is very different from those found in (46). However, we can estimate and compare respect to the asymptotics terms in (49). Expression (53) can be re-written as

$$\frac{12 (m-1)! [2(m-b-5)]! 2!(2(b+2))! 2!}{(2m)!(m-b-5!)(b+2)!1!1!}.$$  

The first asymptotic term of this expression, for finite $b$, is

$$\frac{3}{2^{5+m}} \times \frac{[2(2+b)]!}{(2+b)!} \times \frac{1}{m^{6+b}} \xrightarrow{b \to m/2} \frac{3}{2^{5+m}} \times \frac{(m+4)(m+3)}{m^6} \times \left[ \frac{(m-1+3)}{m} \frac{(m-2+3)}{m} \cdots \frac{(m-m/2+3)}{m} \right].$$  

By calling $f(m)$ the term within the big square brackets, we have

$$\frac{1}{2m/2} < f(m) \ll 1.$$  

We see that, for big $m$,

$$\frac{3}{2^{5+m}} \times \frac{(m+4)(m+3)}{m^6} \sim \frac{12\sqrt{2}}{2^m m^4}.$$  

Thereby, $f(m)$ can be seen as an asymptotic deviation of this centered multinomial term, respect to the non-centered terms expressed in (55).

These centered multinomial contributions are well defined, provided that we consider elements represented by partitions of $b$, such that

$$\frac{b}{2} + \frac{b}{2}; \frac{b}{2} - 1 + \frac{b}{2} + 1; \frac{b}{2} - 2 + \frac{b}{2} + 1 + 1; \frac{b}{2} - 1 + \frac{b}{2} - 1 + 1 + 1 \cdots$$  

(58)

Partitions like $(\frac{b}{2} - n) + (\frac{b}{2} + n)$ for finite $n$ generate indeterminate coefficients on the asymptotic expansion (since these partitions have identical first term for $n \in \mathbb{N}$). Therefore, we only consider partitions like (58). In this case, $b$ is considered even. However, for an arbitrary $b$ we consider

$$b = \left\lfloor \frac{b}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil,$$  

(59)

with $[r] \ (\lceil r \rceil)$ the nearest integer to rational $r$ from below (above).

The 11 (13, if $m$ is odd) contributing terms in (43) for the first four asymptotic terms in the binomial-centered case are in $k = m - 1$, $k = m - 2$ and $k = m - 3$. In particular, following the same procedure in (51) for these terms, if $m$ is odd, the contribution for $h_m^{(1)}$ is

$$\sim \frac{2}{\sqrt{\pi}} m! m^{\frac{1}{2}} \left[ \frac{\sqrt{2}}{m^2} + \frac{25}{2\sqrt{2} m^3} + \frac{2737}{16\sqrt{2} m^4} + \cdots \right]$$  

(60)

and if $m$ is even,

$$\sim \frac{2}{\sqrt{\pi}} m! m^{\frac{1}{2}} \left[ \frac{2\sqrt{2}}{m^2} + \frac{21}{\sqrt{2} m^3} + \frac{2005}{8\sqrt{2} m^4} + \cdots \right]$$  

(61)

The trinomial-centered contribution can be calculated from the partition

$$b = \left\lceil \frac{b}{3} \right\rceil + \left\lfloor \frac{b}{3} \right\rfloor + \left\lceil \frac{b}{3} \right\rceil.$$
for $b$ odd and

$$b = \left\lfloor \frac{b}{3} \right\rfloor + \left\lceil \frac{b}{3} \right\rceil + \left\lfloor \frac{b}{3} \right\rfloor$$

for $m$ even, and so on.

It is especially remarkable that these two contributions are negligible respect to expression \eqref{eq:51}. Other centered multinomial expansions can be defined from \eqref{eq:49}. (In principle, one for each possible multinomial centered multinomial term.) In the same way, they do not seem to give a significant contribution to \eqref{eq:51}. Also, unlike the cases in which $k \ll m$, and the multinomial non-centered expansion cases in $k \lesssim m$, the expansions in this multinomial centered regime depend on whether $m$ is even or odd. The excellent numerical matching of \eqref{eq:51} with the exact values of $h^{(1)}_m$ for large $m$ suggests that, in expression \eqref{eq:43}, it is only necessary to take the limit $k \ll m$ and the multinomial non-centered cases in $k \lesssim m$ to obtain the conventional asymptotic expansion. However, an analysis of this multinomial centered regime and of the family of different asymptotic expansions associated with this regime can be interesting from a mathematical perspective.

**A. $N = 2$ case**

Recurrence \eqref{eq:25} for the case in which $N = 2$ is

$$g^{(2)}_m = 2 \sum_{j=0}^{m} \binom{m}{j} \Lambda^{(1)}_{c,j} g^{(1)}_{m-j} + 2 \sum_{j=1}^{m} \binom{m}{j} \Lambda^{(2)}_{c,j} D_{m-j}. \tag{62}$$

The exact solution of this recurrence for arbitrary $m$ is

$$\Lambda^{(2)}_{c,m} = \sum_{n=1}^{m} C^m_n \left( \frac{g^{(2)}_n}{2} - g^{(1)}_n \right)$$

$$+ \sum_{n=1}^{m} [2(m-n)-1] C^m_n \left( g^{(1)}_n - D_n \right). \tag{63}$$

The proof of \eqref{eq:63} is obtained by induction. (See Appendix B) The asymptotic expansion is obtained in a way similar to the case in which $N = 1$. Using

$$\frac{g^{(2)}_k}{2} - g^{(1)}_k = \left( k + \frac{1}{2} \right) \left( g^{(1)}_k - D_k \right), \tag{64}$$

we have

$$\Lambda^{(2)}_{c,m} = \left( \frac{g^{(2)}_m}{2} - g^{(1)}_m \right) \frac{2m - 1}{2m + 1} + \sum_{n=1}^{m-1} C^m_n \left( \frac{g^{(2)}_n}{2} - g^{(1)}_n \right) \frac{4m - 2n - 1}{2n + 1}. \tag{65}$$

Using \eqref{eq:31}, and after some manipulations, we get

$$h^{(2)}_m = \frac{mm!}{2^m (2m - 1)} \binom{2m}{m} \left[ 1 + \sum_{n=1}^{m-1} \frac{n (4m - 2n - 1) (2n)!}{m (2m - 1) (2m)!} C^m_n \right]$$

$$\quad = \frac{mm!}{2^m (2m - 1)} \binom{2m}{m} \left[ 1 + \sum_{k=1}^{m-1} \frac{(m-k) (2(m+k) - 1) (2m-k)!}{m (2m-k)!} C^m_{m-k} \right], \tag{66}$$

which serves to calculate, respectively, the different limits $n \ll m$ and $k \ll m$, for arbitrarily large $m$. For the same analysis used in case $N = 1$, and using expression \eqref{eq:50}, we see that the asymptotic expansion until $a = 6$ is
\[ h_m^{(2)} \sim \frac{1}{\sqrt{\pi}} m! \sqrt{m(2m-1)} 2^m \left[ 1 - \frac{5}{8m} - \frac{215}{128m^2} - \frac{4255}{1024m^3} - \frac{627749}{32768m^4} - \frac{32650491}{262144m^5} - \frac{4251341763}{4194304m^6} - \cdots \right]. \] (67)

In the same way as in case \( N = 1 \), the multinomial centered dilemma is presented. However, as in this previous case, it does not seem to have a significant contribution. Specifically, until \( a = 6 \), for \( m = 1000 \), \( m = 2000 \) and \( m = 3000 \), the approximation matches with the exact value in the first 17, 19 and 20 digits respectively.

We tried to find an explicit solution for the recurrence case \( N = 3 \), in terms of the symbols \( C_n^m \). Although it is possible to find a solution for the first orders, these solutions do not seem to have a simple form that can be generalized for an arbitrary order of \( m \), as in cases \( N = 1 \) and \( N = 2 \). (See expressions (32) and (63).) The same problem happens for \( N > 3 \), which makes it difficult to obtain a generalized solution of recurrence (25) for arbitrary \( N \).

V. DISCUSSION AND PERSPECTIVES

In this work, by using simple combinatorial arguments, we have proved a general recurrence formula for the number of different \( m \)-order Wick contractions that generates connected Feynman diagrams with an arbitrary number of external legs for the fermionic non-relativistic interacng gas. In this case, the recurrence determines the different number of connected Feynman diagrams. The recurrence is easy to process computationally, and it is possible to find exact numerical solutions for a large number of cases in only a few minutes.

An exact solution is obtained for the cases in which \( N = 1 \) and \( N = 2 \), enabling computation of many terms in the asymptotic expansion for the number of Feynman diagrams in large orders, in these specific cases. Enumeration of Feynman diagrams is generally understood within the QFT functional approach in zero dimension. (Here, the functional integral is transformed into a conventional integral.) The zero-dimensional theory can be understood as a toy model for the study of the formal mathematical machinery used in non-zero dimension quantum field theory. In particular, as a simplified model, exact results are possible and it could be extendable to non-zero-dimensional field theory. Perturbative field theory, despite of remarkable success in their results, presents certain inconsistencies in its foundation. (Haag theorem is a example.) Attempts to circumvent these inconsistencies in a new formalism that contains the perturbative approach exist. Exact results in the zero-dimensional field theory toy model could shed light on this matter.

Within this zero-dimensional approach, the asymptotic analysis is reasonably well understood. The contribution of our work is to provide an equivalent formulation, related to the field-operator approach, showing that this simpler machinery may also contribute with interesting insight. Our analysis shows that, apparently cumbersome expressions as (34) can contain relevant combinatorial information and, in particular, a relation with numerical partitions and compositions. Also, our asymptotic analysis for cases \( N = 1 \) and \( N = 2 \) comes from (34), in what we call the non-centered multinomial limit. Multinomial centered terms are other types of contributions which seem to be negligible respect to the non-centered contribution. However, we can define an expansion for every possible multinomial centered contribution. This defines a family of asymptotic expansions related to our problem. The non-centered limit studied here is of great interest as a new asymptotic method that enables the derivation of the same asymptotic expansion calculated by other methods up to the desired precision order. We hope that our work brings some new features and perspectives in the realm of zero-dimension QFT realm.

It is as well important to notice that, due to the bijection between Feynman diagrams and the \( N \)-rooted maps, our enumerative study is also valid for the \( N \)-rooted maps. In particular, it would be interesting to study whether our work bears some relation to the generalized catalan numbers used in Ref. in the context of the Eynard-Orantin topological recursion, which is another method for enumeration of Feynman diagrams.

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In this appendix, we write explicitly the terms of \( \text{(43)} \) and \( \text{(60)} \) that contribute to the asymptotic expansion until \( a = 6 \), for the case in which \( N = 1 \).

In the limit where \( n \lesssim m \) (or \( k \ll m \)), only six terms contribute on the left-hand side of \( \text{(45)} \). They are

\[
\begin{align*}
\frac{-2(m-1)}{2m(2m-1)} & \quad \frac{-4(m-2)}{2m(2m-1)(2m-3)} & \quad \frac{-20(m-3)}{2m(2m-1)(2m-3)(2m-5)} & \quad \frac{-148(m-4)}{2m(2m-1)(2m-3)(2m-5)(2m-7)} \\
\frac{-2m(2m-1)(2m-3)(2m-5)}{1412(m-5)} & \quad \frac{-2m(2m-1)(2m-3)(2m-5)(2m-7)(2m-9)}{2m(2m-1)(2m-3)(2m-5)(2m-7)(2m-9)(2m-11)}.
\end{align*}
\]

In the limit where \( k \lesssim m \), the cases in which \( k = m-1, m-2, \ldots, m-5 \) are the only ones that present a contribution. For \( k = m-1 \), all the contributions are given in \( \text{(48)} \). Similar expressions are found for the other cases. Summing them up, the total contribution in this regime is

\[
\begin{align*}
\frac{-2}{2m(2m-1)} & \quad \frac{-8}{2m(2m-1)(2m-3)} & \quad \frac{60}{2m(2m-1)(2m-3)(2m-5)} & \quad \frac{592}{2m(2m-1)(2m-3)(2m-5)(2m-7)} \\
& \quad \frac{7060}{2m(2m-1)(2m-3)(2m-5)(2m-7)(2m-9)}.
\end{align*}
\]

For the binomial centered contribution expansion in \( N = 1 \), the cases in which \( k = m-1, m-2, m-3 \) present contribution in the first three asymptotics terms of \( \text{(60)} \). In particular, when \( m \) is odd, by adding all the contributions we have

\[
\begin{align*}
\frac{2[(m-1)!!]^{3}}{(2m)!\left(\frac{m-1}{2}\right)!^{2}} + \frac{24(m-3)![(m-1)!!]^{2}}{(2m)!(\frac{m-1}{2})!(\frac{m-3}{2})!} + \frac{384[(m-1)!!]^{2}(m-5)!}{(2m)!(\frac{m-1}{2})!(\frac{m-5}{2})!} + \frac{192(m-1)!(m-5)!!^{2}}{(2m)!(\frac{m-3}{2})!^{2}}.
\end{align*}
\]

For the case in which \( N = 2 \), we almost have the same contribution terms. The square bracket terms of \( \text{(66)} \) and \( \text{(43)} \) only differ by the multiplicative factor

\[
\frac{2(m+k)-1}{2m-1}
\]

and contribute to exactly the same terms with the new corresponding multiplicative factor.
Appendix B: Proof of the recurrence solution in case $N = 2$

For $m = 1, 2,$ and $3$, it is easy to see that (63) is satisfied. Suppose that (63) is valid for $k \leq m$. From (62), we have that, for $m + 1$,

$$
\mathcal{N}_{c_{m+1}}^{(2)} = \left(\frac{\mathcal{N}_{m+1}^{(2)}}{2} - \mathcal{N}_{m+1}^{(1)}\right) - \sum_{j=1}^{m} \left(\frac{m+1}{j}\right) \mathcal{N}_{m+1-j}^{(1)} - \sum_{j=1}^{m} \left(\frac{m+1}{j}\right) D_{m+1-j} \mathcal{N}_{c_{j}}^{(2)} - \mathcal{N}_{c_{m+1}}^{(1)}.
$$

(B1)

In the third sum, $1 \leq j \leq m$. Therefore, we can insert (63) in $\mathcal{N}_{c_{j}}^{(2)}$, obtaining the next two terms:

$$
\mathcal{N}_{c_{m+1}}^{(2)} = \left[\left(\frac{\mathcal{N}_{m+1}^{(2)}}{2} - \mathcal{N}_{m+1}^{(1)}\right) - \sum_{j=1}^{m} \sum_{n=1}^{j} \left(\frac{m+1}{m+1-j}\right) D_{m+1-j} C_{n}^{(1)} \left(\frac{\mathcal{N}_{n}^{(2)}}{2} - \mathcal{N}_{n}^{(1)}\right)\right]
$$

$$
+ \left[\sum_{j=1}^{m} \sum_{n=1}^{j} \left[2(j-n) - 1\right] m+1 \sum_{n=1}^{j} \left[n \mathcal{N}_{m+1-j} C_{n}^{(1)} \left(\frac{\mathcal{N}_{n}^{(2)}}{2} - \mathcal{N}_{n}^{(1)}\right) - D_{n}\right] - \mathcal{N}_{c_{m+1}}^{(1)}\right].
$$

(B2)

The procedure used in Ref.[8] to prove case $N = 1$ (see formulas (22), (23) and (24) in this reference) can be used for the first term of (B2), which is identical to

$$
\sum_{n=1}^{m+1} C_{n+1}^{m+1} \left(\frac{\mathcal{N}_{n}^{(2)}}{2} - \mathcal{N}_{n}^{(1)}\right).
$$

(B3)

Let us now focus on the second term of (B2). A careful appreciation lets us find that the first double sum can be rewritten as

$$
-\sum_{k=1}^{m} \left[\mathcal{N}_{k}^{(1)} - D_{k}\right] \sum_{n=k}^{m} \left[2(n-k) - 1\right] m+1 \left(\frac{m+1}{n}\right) D_{m+1-n} C_{k}^{n}.
$$

(B4)

On the other hand, the second double sum can be rewritten as

$$
-\sum_{k=1}^{m} \left[\mathcal{N}_{k}^{(1)} - D_{k}\right] \sum_{n=k}^{m} \left[2(m+1-n) + 1\right] m+1 \left(\frac{m+1}{n}\right) D_{m+1-n} C_{k}^{n}.
$$

(B5)

By Adding the two previous equations, we obtain

$$
-\sum_{k=1}^{m} \left[2(m+1-k)\right] \left[\mathcal{N}_{k}^{(1)} - D_{k}\right] \sum_{n=k}^{m} \left(m+1\right) m+1 \left(\frac{m+1}{n}\right) D_{m+1-n} C_{k}^{n}.
$$

(B6)

Note that the dependence of $n$ on factor $2(m+1-k)$ disappeared and the same procedure used in Ref.[8] is valid. (Formulas (23) and (24) of this reference.) Therefore, the previous expression is

$$
\sum_{k=1}^{m} \left[2(m+1-k)\right] C_{k+1}^{m+1} \left(\mathcal{N}_{k}^{(1)} - D_{k}\right)
$$

(B7)

and, by using (32) for $\mathcal{N}_{c_{m+1}}^{(1)}$ in (B2), we have
\[ N_{cm+1}^{(2)} = \sum_{n=1}^{m+1} C_n^{m+1} \left( \frac{N_{cm}^{(2)}}{2} - \gamma_n^{(1)} \right) + \sum_{n=1}^{m+1} [2(m + 1 - n) - 1] C_n^{m+1} \left( \gamma_n^{(1)} - D_n \right), \]  

which proves (63).

**Appendix C: Generating functions and recurrences**

An alternative way to derive relations (21) and (25) is using generating functions. Let

\[ F(x, y) = \sum_{N=0}^{\infty} \sum_{m=0}^{\infty} \frac{N_{cm}^{(N)}}{(N!)^2} x^N y^m \]  

and

\[ G(x, y) = \text{Log} \left( \sum_{m=0}^{\infty} D_m y^m \right) + \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{N_{cm}^{(n)}}{n!m!} x^n y^m \]

be the generating functions of \( N_{cm}^{(N)} \) and \( N_{cm}^{(n)} \), respectively. Connected and disconnected generating functions of Feynman diagrams are easily related. This relation is maintained in zero-dimensional field theory. In particular,

\[ F(x, y) = \exp \left( G(x, y) \right) \]

By redefining the sum indices on the right side, using the multinomial theorem, and carefully comparing term by term, we obtain expression (21).

Formula (25) is obtained by differentiating (C3) consecutively and evaluating in \( x = 0 \), defining

\[ \frac{dF}{dx}(x, y) = F'(x, y) \]

From expression (C3), we get all the special cases of expression (25). In particular, we obtain case \( N = 1 \) from

\[ F'(0, y) = F(0, y)G'(0, y), \]

\( N = 2 \) from

\[ F''(0, y) = F(0, y)G''(0, y) + F'(0, y)G'(0, y), \]

\( N = 3 \) from

\[ F'''(0, y) = F(0, y)G'''(0, y) + 2F'(0, y)G''(0, y) + F''(0, y)G'(0, y), \]

and so on.