Symmetries of Differential Equations in Cosmology

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Abstract

The purpose of the current article is to present a brief albeit accurate presentation of the main tools used in the study of symmetries of Lagrange equations for holonomic systems and subsequently to show how these tools are applied in the major models of modern cosmology in order to derive exact solutions and deal with the problem of dark matter/energy. The key role in this approach are the first integrals of the field equations. We start with the Lie point symmetries and the first integrals defined by them, that is the Hojman integrals. Subsequently we discuss the Noether point symmetries and the well known method for deriving the Noether integrals. By means of the Inverse Noether Theorem we show that to every Hojman quadratic first integral one is possible to associate a Noether symmetry whose Noether integral is the original Hojman integral. It is emphasized that the point transformation generating this Noether symmetry need not coincide with the point transformation defining the Lie symmetry which produces the Hojman integral. We discuss the close connection between the Lie point and the Noether point symmetries with the collineations of the metric defined by the kinetic energy of the Lagrangian. In particular the generators of Noether point symmetries are elements of the homothetic algebra of that metric. The key point in the current study of cosmological models is the introduction of the mini superspace, that is the space which is defined by the physical variables of the model, which is not the spacetime where the model evolves. The metric in the mini superspace is found from the kinematic part of the Lagrangian and we call it the kinetic metric. The rest part of the Lagrangian is the effective potential. We consider coordinate transformations of the original mini superspace metric in order to bring it to a form where we know its collineations that is, the Killing vectors, the homothetic vector etc. Then we write the field equations of the cosmological model and we use the connection of these equations with the collineations of the mini superspace metric to compute the first integrals and subsequently to obtain analytic solutions for various allowable potentials and finally draw conclusions about the problem of dark energy. We consider the $\Lambda$CDM cosmological model, the scalar field cosmology, the Brans Dicke cosmology, the $f(R)$ gravity, the two scalar fields cosmology with interacting scalar fields and the Galilean cosmology. In each case we present the relevant results in the form of Tables for easy reference. Finally we discuss briefly the higher order symmetries (the contact symmetries) and show
how they are applied in the cases of scalar field cosmology and in the $f(R)$ gravity

Keywords: Lie symmetries; Noether symmetries; Dynamical systems; Integrability; Conservation laws; Invariants; Dark Energy; Modified theories of gravity; Cosmology

1 Introduction

In order to understand properly the role of symmetries in Cosmology we have to make a short detour into General Relativity and the relativistic models in general. General Relativity associates the gravitational field with the geometry of spacetime as this is specified by the metric of the Riemannian structure. Concerning the matter this is described in terms of various dynamical fields which are related to the matter via Einstein equations $G_{ab} = T_{ab}$ (we consider that the Einstein’s gravitational constant is $k=1$). Einstein equations are not equations, in the sense that they equate known quantities in terms of unknown ones, that is there is no point to look for a ”solution” of them in this form. These equations are rather generators of equations which result after one introduces certain assumptions according to the model required. These assumptions are of two kinds: a) Geometric assumptions; b) Other non-geometric assumptions among the physical fields which we call equations of state. The first specify the metric to a certain degree and are called collineations [1] (or ”symmetries” which is in common use and is possible to create confusion in our discussion of symmetries in Cosmology). The collineations are the familiar Killing vectors (KV), the conformal Killing vectors (CKV), etc. The collineations affect the Einstein tensor which is expressed in terms of the metric. For example for a KV $X$ one has $L_X g_{ab} = L_X G_{ab} = 0$.

Through Einstein field equations collineations pass over to $T_{ab}$ and restrict its the possible forms, therefore the types of matter it can be described in the given geometry-model. For example if one considers the symmetries of the Friedmann Robertson Walker (FRW) model, which we shall discuss in the following, then the collineations restrict the $T_{ab}$ to be of the form

$$T_{ab} = \rho u_a u_b + p h_{ab}$$

where $\rho, p$ are two dynamical scalar fields the density and the isotropic pressure, $u^a$ are the comoving observers $u^a (u^a u_a = -1)$ and $h_{ab} = g_{ab} + u_a u_b$ is the spatial projection operator. An equation of state is a relation between the dynamical variables $\rho, p$.

Once one specifies the metric by the considered collineations (and perhaps some additional requirements of geometric nature) of the model and consequently the dynamical fields in the energy momentum tensor then Einstein equations provide a set of differential equations which describe the defined relativistic model. What it remains is the solution of these equations and the consequent determination of the Physics of the model. At this point one introduces the equations of state which simplify further the resulting field equations.

When one has the final form of the field equations enters a second use of the concept of ”symmetry” which is the main objective of the current work. Let us refer briefly some history.

In the late of 19th century Sophus Lie in a series of works [2-4] with the title “Theory of transformation groups” introduced a new method for the solution for differential equations via the concept of ”symmetry”. In particular Lie defined the concept of symmetry of a differential equation by the requirement the point transformation leaves invariant the set of solution curves of the equation, that is, under the action of the transformation a point form one solution curve is mapped to a point in another solution curve. Subsequently Lie introduced a simple algebraic algorithm for the determination of this type of symmetries and consequently on the solution of differential equations. Since then, symmetries of differential equations is one of the main
methods which is used for the determination of solutions for differential equations. Some important works which established the importance of symmetries in the scientific society are those of Ovsiannikov [5], Bluman and Kumei [6], Ibragimov [7], Olver [8], Crampin [9], Kalotas [10] and many others; for instance see [11–20].

As it is well known an important fact in the solution of a differential equation are the first integrals. Inspired by the work of Lie in the early years of the 20th century, Emmy Noether required another definition of symmetry, which is known as Noether symmetry, which concerns Lagrangian dynamical systems and it is defined by the requirement that the action integral under the action of the point transformation changes up to a total derivative so that Lagrange equation(s) remain the same [21]. She established a connection between the Noether symmetries and the existence of a first integral which she expressed by a simple mathematical formula. In addition, Noether’s work except of its simplicity had a second novelty by alloying the continuous transformation to depend also on the derivatives of the dependent function, which was the first generalization of the context of symmetry from point transformations to higher-order transformations. Symmetries play an important role in various theories of physics, from analytical mechanics [22], to particle physics [23, 24], and gravitational physics [11–25].

In the following we shall present briefly the approach of Lie and Noether symmetries and will show how the symmetries of differential equations are related to the collineations of the metric in superspace. We shall develop an algorithm which indicates how one should work in order to get the analytical solution of a cosmological model. The various examples will demonstrate the application of this algorithm.

In conclusion, by symmetries in Cosmology we mean the work of Lie and Noether applied to the solution of the field equations of a given cosmological model - scenario.

2 Point transformations

On a manifold $M$ with coordinates $(t, q^a)$ one defines the jet space $J^m(M)$ of order $m$ over $M$ to be a manifold with coordinates $t, q^a, \frac{dq^a}{dt}, \ldots, \frac{d^m q^a}{dt^m}$. Let $X = \xi (t, q, \ldots, q^{[m]}) \frac{\partial}{\partial t} + \eta^{a[A]} (t, q, \ldots, q^{[m]}) \frac{\partial q^{[a]}[A]}{\partial q^{[m]}}$ where $A = 1, \ldots, m$ be a vector field on $J^m(M)$ which generates the infinitesimal point transformation

\begin{align}
    t' &= t + \varepsilon \xi + O^2 (\varepsilon^2) + \cdots \\
    q'^a &= q^a + \varepsilon \eta^a[1] + O^2 (\varepsilon^2) + \cdots \\
    \cdots \\
    q'^{[m]} &= q^{[m]} + \varepsilon \eta^{[m]} + O (\varepsilon^2) + \cdots
\end{align}

Well behaved infinitesimal point transformations form a group under the operation of composition of transformations. If the infinitesimal point transformation depends on many parameters, that is,

$$q'^i = q'^i (q^i, q^{(m)}, E)$$

where $E = \varepsilon^\beta \partial_\beta$, is a vector field in $\mathbb{R}^\kappa$, $\beta = 1...\kappa$ with the same properties as in the case of the one parameter infinitesimal point transformations, then the point transformation is called a multi-parameter point transformation. These transformations are generated by $\kappa$–vector fields which form a (finite or infinite dimensional) Lie algebra. That is, if the vector fields $X, Y$ are generators of a multi-parameter point transformation so is the commutator $Z = [X, Y]$.
2.1 Prolongation of point transformations

A differential equation \( H(t, q, q_1, \ldots, q^{(m)}) = 0 \) where \( q^i(t) \) and \( q^{(m)} = \frac{d^m q^i}{dt^m} \) may be considered as a function on the jet space \( J^m(M) \). In order to study the effect of a point transformation in the base manifold \( M(t, q) \) to the differential equation one has to prolong the transformation to the space \( J^m(M) \). To do that we consider in \( J^m(M) \) the induced point transformation

\[
\bar{t} = t + \varepsilon \xi, \quad \bar{q}^i = x + \varepsilon \eta
\]

\[
\bar{q}^{(1)} = q^{(1)}(1) + \varepsilon X^1, \quad \ldots
\]

\[
\bar{q}^{(n)} = q^{(n)} + \varepsilon X^n
\]

where \( X[k] \) \( k = 1, 2, \ldots, m \) are the components of a vector field

\[
W = \xi \partial_x + \eta \partial_y + X^{[1]} \partial_{y^{(1)}} + \ldots + X^{[m]} \partial_{y^{[m]}}.
\]

(4)

in \( J^m(M) \) called the lift of the vector field \( X = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q} \) of \( M \).

There are many ways to lift a vector field from the base manifold to a vector field in the bundle space \( J^m(M) \) depending on the geometric properties one wants to preserve.

One type of lift, the complete lift or prolongation, is defined by the requirement that the tangent to the vector field \( X \) in \( M \) goes over to the tangent of the vector field \( W \) in \( J^m(M) \) at the corresponding lifted point. Equivalently one may define the prolongation by the requirement that under the action of the point transformation the variation of the variables equals the difference of the derivatives before and after the action of the one parameter transformation. For example for \( \eta^{[1]} \) we have

\[
\eta^{[1]} = \lim_{\varepsilon \to 0} \left[ \frac{1}{\varepsilon} \left( \bar{q}^{(1)} - q^{(1)} \right) \right] = \frac{d\eta}{dt} - q^{(1)} \frac{d\xi}{dq}.
\]

(5)

For the \( k - \text{th} \) prolongation \( \eta^{[k]} \) follows the recursive formula.

\[
\eta^{[k]}(t, q, q^{(1)}, \ldots, q^{(m)}) = \frac{d\eta^{[k-1]}}{dt} - q^{(k)} \frac{d\xi}{dq} = \frac{d^k}{dq^k} \left( \eta - q^{(1)} \xi \right) + q^{(k+1)} \xi.
\]

(6)

Two important observations for the prolongation \( \eta^{[n]} \) are, (a) \( \eta^{[n]} \) is linear in \( q^{(n)} \), and (b) \( \eta^{[n]} \) is a polynomial in the derivatives \( q^{(1)}, \ldots, q^{(n)} \) whose coefficients are linear homogeneous in the functions \( \xi(x, y), \eta(x, y) \) up to \( n \)th order partial derivatives.

Concerning the general vector \( W \) on \( J^m(M) \) one defines its components by the requirement

\[
X^{[m]}(t, q, q^{(1)}, \ldots, q^{(m)}) = \eta^{[k]}(t, q, q^{(1)}, \ldots, q^{(m)}) + \phi^m
\]

where \( \phi^m(t, q, q^{(1)}, \ldots, q^{(m)}) \) are some functions on \( J^m(M) \) which will be defined by additional requirements. For example the complete lift is defined by the requirement \( \phi^i = 0 \).

In the case we have a manifold with \( n \) independent variables \( \{x^i : i = 1 \ldots n\} \) and \( m \) dependent variables \( \{u^A : A = 1 \ldots m\} \), we consider the one parameter point transformation in the jet space \( \{x^i, u^A\} \) of the form

\[
\bar{x}^i = \Xi^i(x^i, u^A, \varepsilon), \quad \bar{u}^A = \Phi^A(x^i, u^A, \varepsilon)
\]

The infinitesimal generator is

\[
X = \xi^i(x^k, u^A) \partial_i + \eta^A(x^k, u^A) \partial_A
\]

(7)
The zero order invariant is indeed invariant under the action of $X$,

$$\xi^i (x^k, u^A) = \frac{\partial \xi^i}{\partial x^i} (x^j, u^A, \varepsilon) \bigg|_{\varepsilon \to 0} , \quad \eta^A (x^k, u^A) = \frac{\partial \eta^A}{\partial x^i} (x^j, u^A, \varepsilon) \bigg|_{\varepsilon \to 0}.$$  

In a similar way, the prolongation vector is calculated to be

$$X^{[n]} = X + \eta^A_i \partial_{x^i} + ... + \eta^A_{i_j ... i_n} \partial_{u_{i_j ... i_n}}$$

where now

$$\eta^A_i = D_i \eta^A - u^A_j D_i \xi^j,$$

and the operator $D_i$ is defined as

$$D_i = \frac{\partial}{\partial x^i} + u^A_i \frac{\partial}{\partial u^A} + u^A_j \frac{\partial}{\partial u^A_j} + ... + u^A_{i_j ... i_n} \frac{\partial}{\partial u^A_{i_j ... i_n}}.$$  

In terms of the partial derivatives of the components $\xi^i (x^k, u^A), \eta^A_i (x^k, u^A)$ of the generator \( \xi \), the first and the second extension of the symmetry vector are given as follows

$$X^{[1]} = X + \left( \eta^A_i + u^A_i \eta^A_B - \xi^j u^A_j - u^A_{i} u^B_{j} \xi^j \right) \partial_{u^A_i}$$

$$X^{[2]} = X^{[1]} + \left[ \begin{array}{c}
\eta^A_{i_j} + 2u^A_{i} u^B_{j} \eta^A_B - \xi^k u^A_k + \eta^A_{i_j} u^B_{j} u^C_{i} - 2\xi^k u^A_k + \\
-\xi^k u^A_k u^B_{j} u^A_{i} + \eta^A_{i_j} u^B_{j} - 2\xi^k u^A_k u^B_{j} \xi^j + -\xi^k u^A_k u^B_{j} + \right] \partial_{u_{i_j}}.$$  

Similar expressions can be considered for the general vector field in the jet space \( \{x^i, u^A\} \).

### 2.2 Invariance of functions

A differentiable function $F (q^i, q^{|m|})$ on $J^m(M)$ is said to be invariant under the action of $X$ iff

$$X (F) = 0,$$  

or, equivalently, iff there exists a function $\lambda$ such that

$$X (F) = \lambda F , \text{mod} F = 0.$$  

In order to determine the invariant functions of a given infinitesimal point generator $X$ one has to solve the associated Lagrange system

$$\frac{dt}{\xi} = \frac{dq^i}{\eta^i} = ... = \frac{dq^{|m|}}{\eta^{|m|}}.$$  

The characteristic function or zero order invariant $w$ of the vector $X$ is defined as follows

$$dw = \frac{dt}{\xi} - \frac{dq^i}{\eta^i}.$$  

The zero order invariant is indeed invariant under the action of $X$, that is $X (w) = 0$. Therefore, any function of the form $F = F (w)$ satisfies the symmetry condition \( \xi \) and it is invariant under the action of $X$. The higher order invariants $v_{(b)}$ are given by the expression

$$v_{(m)} = \frac{dt}{\xi} \frac{dq^{(m)}}{\eta^{(m)}}.$$  

5
3 Symmetries of differential equations

Consider an \( m \)th order system of differential equations defined on \( J^m(M) \) of the form \( q^a[m] = F(t, q, ..., q^{|m|}) \).

We say that the point transformation (2) in \( J^m(M) \) generated by the vector field \( X = \xi \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a} \) is a symmetry of the system of equations if it leaves the set of solutions of the system the same. Equivalently if we consider the function \( G = q^a[m] - F(t, q, ..., q^{|m|}) = 0 \) on \( J^m(M) \) then a symmetry of the differential equation is a vector field leaving \( G \) invariant.

The main reason for studying the symmetries of a system of differential equations is to find first integrals and/or invariant solutions. Both these items facilitate the solution and the geometric / physical interpretation of the system of equations.

In the following we shall be interested in systems of second order differential equations (SODE) of the form \( \ddot{q}^a - K^a(t, q, \dot{q}) = 0 \), therefore we shall work on the jet space \( J^1(M) \) which is essentially the space \( \mathbb{R} \times TM \). In this case the infinitesimal point transformation (2) is written

\[
\begin{align*}
t' & = t + \varepsilon \xi + O(\varepsilon^2) + \cdots \\
q^i' & = q^i + \varepsilon \eta^i + O(\varepsilon^2) + \cdots \\
\dot{q}^i' & = \dot{q}^i + \varepsilon \left( \dot{\eta}^i - \dot{\xi} \dot{q}^i + \phi^i \right) + O(\varepsilon^2) + \cdots \tag{18}
\end{align*}
\]

and it is generated by the vector field

\[
X^W = X^{[1]} + \phi^i \frac{\partial}{\partial q^i}
\]

where \( \phi^i(t, q, \dot{q}) \) are some general (smooth) functions and \( X^{[1]} \) is the prolonged vector field:

\[
X^{[1]} = \xi(t, q, \dot{q}) \frac{\partial}{\partial t} + \eta^a(t, q, \dot{q}) \frac{\partial}{\partial q^a} + X^{[1]}_a \frac{\partial}{\partial \dot{q}^a}
\]

where

\[
X^{[1]}_a = \frac{d\eta_a}{dt} - \dot{q}^a \frac{d\xi}{dt} \tag{21}
\]

If \( \xi(t, q), \eta^i(t, q) \) then \( X = \xi \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a} \) is defined on the base manifold \( M \) and \( X^{[1]} \) is called the first prolongation of \( X \) in \( TM \).

4 The conservative holonomic dynamical system

Consider the conservative holonomic system (CHS) with Lagrangian \( L = \frac{1}{2} \gamma_{ij} \dot{q}^i \dot{q}^j - V(t, q) \), where \( V(t, q) \) is the potential of all conservative forces, whose equations of motion are

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. \tag{22}
\]

These are written

\[
E^i(L) = 0 \tag{23}
\]

where \( E^i \) is the Euler vector field in the jet space \( J^1(t, q, \dot{q}) \)

\[
E_i = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}.
\]

Replacing the Lagrangian in (22) we find

\[
\ddot{q}^i = \omega^i \tag{24}
\]
where
\[
\omega^i(t, q, \dot{q}) = -V^i - \Gamma^i_{jk} \dot{q}^j \dot{q}^k
\]  
(25)

The CHS defines two important geometric quantities in the jet space \( J^1(t, q, \dot{q}) \)

a. The kinetic metric \( \gamma_{ij} = \frac{\partial^2 L}{\partial q^i \partial q^j} \) (we assume the Lagrangian to be regular, that is \( \det \frac{\partial^2 L}{\partial q^i \partial q^j} \neq 0 \) so that the kinetic metric is non-degenerate) which is essentially the kinetic energy of the dynamical system. This metric is different from the metric of the space where motion occurs. It is a positive definite metric of dimension depending on the degrees of freedom of the dynamical system.

b. The Hamiltonian vector field \( \Gamma \)
\[
\Gamma = \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial q^i} \dot{q}^i + \omega^i \frac{\partial}{\partial q^i}.
\]  
(26)

Obviously the Hamiltonian vector field is characteristic of the dynamical equations (24) and can be defined in all cases irrespective of the Lagrangian function.

In the following we shall restrict our considerations to the symmetries of second order differential equations of the form (24).

5 Types of symmetries

There are various types of multi parameter point transformations in \( J^1(M) \) which generate a symmetry of a system of second order differential equations (SODE). In cosmology - at least at the current status - it appears that two of them are of importance, that is, the Lie symmetries and the Noether symmetries. Both type of symmetries lead to first integrals hence providing ways to solve the considered cosmological equations and shall be discussed in the following. In case the components \( \xi, \eta^a \) of the generators are functions of \( t, q^a \) only the symmetries are called point symmetries.

The requirements for each type of symmetry lead to a set of conditions which when solved give the generators of the corresponding point transformation and consequently the way to determine first integrals. As it has been shown the generators of these symmetries for autonomous conservative holonomic systems are related to the collineations of the kinetic metric. Specifically it has been shown [26] that the Lie point symmetries are elements of the special projective algebra and the Noether point symmetries elements of the homothetic algebra [27]. Concerning the partial differential equations the generators are related with the conformal group of the kinetic metric [28, 29].

5.1 Lie symmetries

Definition: The vector field \( X^w \) on the jet space \( J^1(M) \) is a (dynamical) Lie symmetry of the equations (24) if it is a symmetry of the Hamiltonian vector field; that is if the following condition is satisfied
\[
L_{X^w} \Gamma = \lambda \Gamma
\]  
(27)

where \( \lambda(t, q, \dot{q}) \) is a function to be defined.

Geometrically a Lie symmetry preserves the set of solutions of the equations (24), in the sense, that under the action of the point transformation generated by \( X^w \) a point of a solution curve is transformed to a point of another solution curve of (24).
The symmetry condition (27) gives:

\[
[X^W, \Gamma] = -\Gamma(\xi) \frac{\partial}{\partial t} + (X^W(q^a) - \Gamma(\eta^a)) \frac{\partial}{\partial q^a} + (X^W(F^a) - \Gamma(X^a + \phi^a)) \frac{\partial}{\partial \dot{q}^a} = \lambda \Gamma.
\]  

(28)

Condition (28) is equivalent to the following system of equations:

\[-\Gamma(\xi) = \lambda \Gamma(\eta^a) - \Gamma(\xi) \dot{q}^a - \phi^a = X^a
\]

(29)

The second condition gives \(\phi^a = 0\) so that \(X^W = X^{[1]}\) where

\[
X^{[1]} = \xi(t, q^k, \dot{q}^k) \frac{\partial}{\partial t} + \eta^a(t, q^k, \dot{q}^k) \frac{\partial}{\partial q^a} + [\Gamma(\eta^a) - \Gamma(\xi) \dot{q}^a] \frac{\partial}{\partial \dot{q}^a}
\]

is given by (21). Then the third condition (30) becomes

\[
\xi \frac{\partial \omega^a}{\partial t} + \eta^b \frac{\partial \omega^a}{\partial q^b} + (\Gamma(\eta^a) - \Gamma(\xi) \dot{q}^a) \frac{\partial \omega^a}{\partial q^b} + \Gamma(\xi) F^a = \Gamma(X^a + \phi^a).
\]

(30)

The second condition gives \(\phi^a = 0\) so that \(X^W = X^{[1]}\) where

\[X^{[1]} = \xi(t, q^k, \dot{q}^k) \frac{\partial}{\partial t} + \eta^a(t, q^k, \dot{q}^k) \frac{\partial}{\partial q^a} + [\Gamma(\eta^a) - \Gamma(\xi) \dot{q}^a] \frac{\partial}{\partial \dot{q}^a}
\]

is given by (21). Then the third condition (30) becomes

\[
\xi \frac{\partial \omega^a}{\partial t} + \eta^b \frac{\partial \omega^a}{\partial q^b} + (\Gamma(\eta^a) - \Gamma(\xi) \dot{q}^a) \frac{\partial \omega^a}{\partial q^b} + \Gamma(\xi) \omega^a = \Gamma(X^a + \phi^a).
\]

(31)

We note that the functions \(\phi^a\) do not take part into the conditions of Lie symmetries and can be omitted. The exact form of the Lie symmetry conditions depends on the functional dependence of the functions \(\xi(t, q, \dot{q}), \eta^a(t, q, \dot{q})\) and of course on the form of the Hamiltonian field. In the following we consider two important cases of Lie symmetries.

5.2 Lie point symmetries

In case \(\xi(t, q), \eta^a(t, q)\) the Lie symmetry is called a Lie point symmetry. For the special class of differential equations of the form

\[\ddot{q} + \Gamma^i_{jk} \dot{q}^j \dot{q}^k + V^i(t, q^k) = 0\]

(32)

the Lie symmetry condition (30) leads to the following system of covariant conditions (30)

\[L_\eta V^i + \eta^a_{,tt} + 2V^i_{,\xi} \xi_{,t} + \xi_{,t} = 0\]

(33)

\[2\eta^a_{,bt} - \delta^a_b \xi_{,tt} + (2\delta^a_c \xi_{,b} + \delta^a_b \xi_{,c}) V_{,c} = 0\]

(34)

\[L_\eta \Gamma^a_{bc} = 2\delta^a_{(bc)} \xi_{,t}\]

(35)

\[\delta^a_{(bc)} = 0\]

(36)

The use of an algebraic computing program does not reveal directly the Lie symmetry conditions in this geometric form. Equation (36) implies that \(\xi_{,t}\) is a gradient KV of the kinetic metric. Equation (35) means that \(\eta^i\) is a special projective collineation of the metric with projective function \(\xi_{,t}\). The remaining two equations (33) and (34) are constraint conditions, which relate the components \(\xi, n^i\) of the Lie point symmetry vector with the potential function \(V(t, q)\).
Conditions (33) - (36) can be obtained as special cases from known results. Indeed in [30] it has been shown that the conditions for the point Lie symmetry of the dynamical equations

$$\ddot{q}^a + \Gamma^i_{jk} \dot{q}^j \dot{q}^k + P^i (t, q^k) = 0$$

are the following:

$$L_q P^i + 2 \xi^i_t P^i + \xi P^i_{tt} + \eta^i_{tt} = 0$$

(38)

$$\left( \xi_{,k} \delta^i_j + 2 \xi_{,j} \delta^i_k \right) P^k + 2 \eta^i_{,tij} - \xi_{,tt} \delta^i_j = 0$$

(39)

$$L_{\eta} \Gamma^i_{jk} - 2 \xi_{,tij} \delta^i_k = 0$$

(40)

$$\xi_{(i | k} \delta^i_{d)} = 0.$$  

(41)

In order to obtain (33) - (36) one simply replaces in (38) - (41) $P^i = V^i (t, q^k)$.

5.3 Lie point symmetries and first integrals

It is possible that a Lie point symmetry leads to a first integral. These integrals have been called Hojman integrals and belong to the class of non-Noetherian first integrals [31]. As shall be discussed below by means of the Inverse Noether Theorem one is able to associate a Noether symmetry to a given quadratic first integral. In this sense as far as the first integrals are concerned, Noether symmetries are the prevailing ones. Concerning the Hojman symmetries we have the following [31]

**Proposition:** i. Necessary and sufficient condition that the point transformation $q^i(t) = q^i + \varepsilon q^i (t, q^k, \dot{q}^k)$ is a Lie point symmetry of the (SODE) $\ddot{q}^i = \omega^i (t, q^k, \dot{q}^k)$ is that the generator $\eta^i$ satisfies the conditions

$$\Gamma(\Gamma \eta^i) - \eta^{i[1]} (\omega^j) = 0 \text{ or } \bar{\eta}^j - \eta^{i[1]} (\omega^j) = 0$$

(42)

where $\bar{\eta}^j = \Gamma (\eta^j)$ and $\eta^{i[1]} = \eta^i \frac{\partial}{\partial q^j} + \bar{\eta}^j \frac{\partial}{\partial \dot{q}^j}$.

ii. The scalar

$$I_2 = \frac{1}{\gamma} \left( \frac{\partial}{\partial q^j} (\gamma \eta^i) \right) + \frac{\partial \bar{\eta}^j}{\partial \dot{q}^j}$$

is a first integral of the ODE $\ddot{q}^i = \omega^i$ iff

a. The vector $\eta^i$ is a Lie symmetry of the SODE $\ddot{q}^i = \omega^i$

b. The function $\gamma (t, q, \dot{q})$ is defined by the condition

$$\text{Trace} \left( \frac{\partial \omega^i}{\partial \dot{q}^j} \right) = \frac{\partial \omega^j}{\partial \dot{q}^j} = - \frac{d}{dt} \ln \left( \gamma (q^k) \right).$$

One solution for all $\eta^i$ is $\omega^j = a q^j + \omega^j (t, q^j)$ where $a =$ const. For this solution we have that the SODE has the generic form:

$$\ddot{q}^j - a \dot{q}^j = \omega^j (t, q^j).$$

which is the equation for forced motion with with linear dumping $a$.  

9
5.4 Noether point symmetries

Noether point symmetries concern Lagrangian dynamical systems and are defined as follows.

**Definition:** Suppose that \( A(q^i, \dot{q}^i) \) is the functional (the action integral)
\[
A(q^i, \dot{q}^i) = \int_{t_1}^{t_2} L(t, q^i, \dot{q}^i) \, dt.
\]
(43)

The vector field \( X^W = X^1 + \phi^i \frac{\partial}{\partial \dot{q}^i} \) in the jet space \( J^1(M) \) generating the point transformation (18) is said to be a Noether symmetry of the dynamical system with Lagrangian \( L \) if

a. The action integral under the action of the point transformation transforms as follows
\[
A'(q^i', \dot{q}^i') = A(q^i, \dot{q}^i) + \epsilon \int_{t_1}^{t_2} \frac{df(t, q^i, \dot{q}^i)}{dt} \, dt.
\]
(44)

where \( f(t, \epsilon) \) is a smooth function.

b. The infinitesimal transformation causes a zero end point variation (i.e. the end points of the integral remain fixed).

Noether symmetries use the fact that when we add a perfect differential to a Lagrangian the equations of motion do not change. Hence the set of solutions remains the same. This is another view of the "invariance" of the dynamical equations (24).

The condition for a Noether symmetry under the action of the point transformation (18) is
\[
X^1(L) + \phi^i \frac{\partial L}{\partial \dot{q}^i} + L(t, q^i, \dot{q}^i) \dot{\xi}^i = \dot{f}
\]
(45)

Eqn. (45) is called the weak Noether condition and it is also known as the First Noether theorem.

5.5 First integral defined by a Noether symmetry

Next we determine the conditions which a Noether symmetry must satisfy in order to lead to a first integral of Lagrange equations. Expanding the Noether condition (45) and making use of Lagrange equations (24) we find
\[
\phi^i \frac{\partial L}{\partial \dot{q}^i} = \frac{d}{dt} \left( f - L \dot{\xi} - \frac{\partial L}{\partial \dot{q}^i} (\eta^i - \xi \dot{q}^i) \right).
\]
(46)

This leads to what is known as the Second Noether Theorem.

**Proposition:** The quantity
\[
I = f - L \dot{\xi} - \frac{\partial L}{\partial \dot{q}^i} (\eta^i - \xi \dot{q}^i)
\]
(47)

is a first integral for the holonomic system defined by (22) provided the functions \( \phi^i \) (i.e. the variations along the fibers) vanish. In this case the weak Noether condition (45) becomes
\[
X^1(L) + L(t, q^i, \dot{q}^i) \dot{\xi}^i = \dot{f}
\]
(48)

and the Noether symmetry reduces to a special Lie symmetry (that is a Lie symmetry which in addition satisfies the Noether condition). The function \( f \) is called the Noether or the gauge function.

We remark that contrary to what is generally believed a Noether point symmetry for an autonomous conservative holonomic system does not lead necessarily to a first integral of the equations of motion. Indeed the weak Noether symmetry condition (45) is more general than the standard Noether condition (48) because it holds for general \( \phi^a \) whereas the latter holds only for \( \phi^a = 0 \).
In the following by a Noether symmetry we shall mean a point Noether symmetry which leads to a first integral, that is the quantities \( \phi^i = 0 \). These Noether symmetries are special Lie symmetries which satisfy condition \((48)\). It has been remarked already above \([27]\) the Lie point symmetries are the elements of the special projective collineations of the kinetic metric and the Lie point symmetries which are Noether point symmetries are elements of the homothetic subalgebra.

6 Generalized Killing equations

We decompose the Noether condition along the vector \( \frac{df}{dt} \) and normal to it. In order to do that we expand the overdot terms and assume that \( \dot{q}^i \) are independent variables. The lhs is:

\[
L \left( \frac{\partial \xi}{\partial t} + \dot{q}^i \frac{\partial \xi}{\partial q^i} + \ddot{q}^i \frac{\partial \xi}{\partial q^i} \right) + \xi \frac{\partial L}{\partial t} + \eta^j \frac{\partial L}{\partial q^j} \\
+ \left( \frac{\partial \eta^i}{\partial t} + \dot{q}^i \frac{\partial \eta^i}{\partial q^i} + \ddot{q}^i \frac{\partial \eta^i}{\partial q^i} - \dot{q}^i \dot{q}^j \frac{\partial \xi}{\partial q^j} - \ddot{q}^i \dot{q}^j \frac{\partial \xi}{\partial q^j} - \dddot{q}^i \dddot{q}^j \frac{\partial \xi}{\partial q^j} \right) \frac{\partial L}{\partial q^i}
\]

\( = L \left( \frac{\partial \xi}{\partial q^i} + \dot{q}^i \frac{\partial \xi}{\partial q^i} \right) + \frac{\partial L}{\partial q^i} + \left( \eta^i \frac{\partial L}{\partial q^i} + \dot{q}^i \dot{q}^j \frac{\partial \eta^i}{\partial q^j} - \ddot{q}^i \dot{q}^j \frac{\partial \eta^i}{\partial q^j} - \dddot{q}^i \dddot{q}^j \frac{\partial \eta^i}{\partial q^j} \right) \frac{\partial L}{\partial q^i}
\]

\[ + \dot{q}^i \left( \frac{\partial \xi}{\partial \dot{q}^i} + \eta^j \frac{\partial \xi}{\partial \dot{q}^j} \right) \frac{\partial L}{\partial q^i} \]

the rhs gives:

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q}^i \frac{\partial f}{\partial q^i} + \ddot{q}^i \frac{\partial f}{\partial q^i}. \]

Therefore we obtain the following equivalent system of the equations:

\[
L \left( \frac{\partial \xi}{\partial t} + \dot{q}^i \frac{\partial \xi}{\partial q^i} \right) + \xi \frac{\partial L}{\partial t} + \eta^j \frac{\partial L}{\partial q^j} + \left( \frac{\partial \eta^i}{\partial t} + \dot{q}^i \frac{\partial \eta^i}{\partial q^i} - \ddot{q}^i \dot{q}^j \frac{\partial \xi}{\partial q^j} - \dddot{q}^i \dddot{q}^j \frac{\partial \xi}{\partial q^j} \right) \frac{\partial L}{\partial q^i} = \frac{\partial f}{\partial t} + \dot{q}^i \frac{\partial f}{\partial q^i} \quad (49)
\]

\[
\frac{\partial \xi}{\partial \dot{q}^i} L + \left( \frac{\partial \eta^i}{\partial \dot{q}^i} - \ddot{q}^i \frac{\partial \eta^i}{\partial q^i} \right) \frac{\partial L}{\partial q^i} = \frac{\partial f}{\partial q^i}. \quad (50)
\]

These equations have been called the generalized Killing equations (see eqns (17),(18) of Djukic in \([32]\)).

We note that the generalized Killing equations have \( 2n + 2 \) unknowns (the \( \xi, \eta^i, f, \phi^i \)) and are only \( n + 1 \) equations. Therefore there is not a unique solution \( X^W \) and we are free to fix \( n + 1 \) variables in order to get a solution. However this is not a problem because all these solutions admit the same first integral \( I \) of the dynamical equations (because all satisfy the Noether condition \((48)\)).

6.1 How to solve the generalized Killing equations

Suppose that by some method we have determined a quadratic first integral \( I \) of the dynamical equations. Our purpose is to determine a gauged Noether symmetry which will admit the given quadratic first integral. Assume the \( n + 1 \) gauge conditions \( \xi = 0, \phi^i = 0 \). Suppose \( L(t, q^i, \dot{q}^i) \) is the Lagrangian part of the dynamical equations. Let \( X^W = \eta^i(t, q^i, \dot{q}^i) \frac{\partial}{\partial q^i} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i} \) be the vector field generating a Noether symmetry which admits the first integral \( I = f - (\eta^i - \xi \dot{q}^i) \frac{\partial L}{\partial q^i} - L \xi \) where \( f(t, q^i, \dot{q}^i) \) is the Noether gauge function. We compute

\[ \frac{\partial I}{\partial \dot{q}^i} = \frac{\partial f}{\partial \dot{q}^i} - \dot{q}^j \frac{\partial L}{\partial \dot{q}^j} + \gamma_{ij} \eta^j \]

\(^1\)We only need the kinetic energy which will define the kinetic metric.
where \( \gamma_{ij} = \frac{\partial^2 I}{\partial q_i \partial q_j} \) is the kinetic metric determined by the Lagrangian \( L \). Then the second equation \( (50) \) gives that 

\[
\frac{\partial f}{\partial \dot{q}_i} - \frac{\partial \eta_j}{\partial \dot{q}_i} \frac{\partial L}{\partial \dot{q}_j} = 0
\]

and we find eventually the expression 

\[
\eta^i = \gamma_{ij} \frac{\partial I}{\partial \dot{q}_i}.
\]  

(51)

The vector \( X^W = \eta^i \frac{\partial}{\partial q_i} + \dot{\eta}^i \frac{\partial}{\partial \dot{q}_i} \) we have determined is not the only one possible. For example one may specify the gauge function \( f \) and assume a form for \( \xi(t, q) \) (while maintain the gauge \( \phi^i = 0 \)) and then use equation \( (50) \) to determine the solution \( X^W \) (see example below). However in all cases the first integral \( I \) is the same.

The first integral of a Noether point symmetry \( X^W = X^{[1]} \) is in addition an invariant of the Noether generator, that is 

\[
X^{[1]}(I) = 0
\]  

(52)

The result \( (52) \) means that a Noether point symmetry results in a twofold reduction of the order of Lagrange equations resulting from the given action integral. This is done as follows. The first integral \( I(t, q^i, \dot{q}^i) \) can be used to replace one of the second order equations by the first order ODE \( I(t, q^i, \dot{q}^i) = I_0 \), where \( I_0 \) is a constant fixed by the initial (or boundary) conditions. Property \( (52) \) says that this new equation admits the Lie symmetry \( X^W \) (because \( X^W (I - I_0) = 0 \)) therefore it can be used to integrate the equation once more, according to well known methods.

Noether symmetries are mainly applied to construct first integrals which are important to determine the solution of a given dynamical system. It is possible that there exist different (i.e. not differing by a perfect differential) Lagrangians describing the same dynamical equations. These Lagrangians have different Noether symmetries (see \[33–35\]). Therefore it is clear that when we refer to a Noether symmetry of a given dynamical system we should mention always the Lagrangian function we are assuming.

Consider the Emden - Fauler equation 

\[
t\ddot{q} + 2\dot{q} + a\nu q^{2\nu+3} = 0
\]

where \( a, \nu \) are arbitrary constants. This equation defines a conservative holonomic dynamical system with Lagrangian 

\[
L = \frac{1}{2} \left( t^2 \dot{q}^2 - \frac{a}{\nu + 2} \nu^{\nu+1} q^{2\nu+4} \right).
\]

Assume now that the function \( f(t, q, \dot{q}) = -AtL \) where \( A \) is some constant and assume further that \( \xi(t, q) \). Eqn. \( (50) \) gives 

\[
\frac{\partial \eta}{\partial \dot{q}} \frac{\partial L}{\partial q} = -At \frac{\partial L}{\partial \dot{q}} \Rightarrow \eta = -At \dot{q} + F(t, q)
\]

where \( F \) is an arbitrary function of its arguments. Replacing in \( (49) \) we find 

\[
L \left( \frac{\partial \xi}{\partial t} + \dot{q} \frac{\partial \xi}{\partial q} \right) + \eta \frac{\partial L}{\partial \dot{q}} + \xi \frac{\partial L}{\partial t} + \left( \frac{\partial \eta}{\partial t} + \dot{q} \frac{\partial \eta}{\partial q} - \dot{q} \frac{\partial \xi}{\partial t} - \ddot{q} \frac{\partial \xi}{\partial q} \right) \frac{\partial L}{\partial \dot{q}} = - \frac{\partial f}{\partial t} + \dot{q} \frac{\partial f}{\partial q}
\]

which provides the following system 

\[
\xi = \xi(t)
\]

\[
\left( \frac{\xi}{t} - \frac{1}{2} \xi, t \right) + F, q = -\frac{1}{2} A
\]

\[
F = F(q)
\]

\[
\frac{1}{2 \nu + 2} \left( \xi, t + \frac{\xi}{t} (\nu + 1) \right) + \frac{F}{q} = -\frac{1}{2} A.
\]
The solution of the latter system is
\[ \xi = -(2c_0 + A) t \quad \eta = -At\dot{q} + c_0q. \] (53)

Still we do not know the parameters \(A, c_0\). In order to compute them we turn to the first integral \(I = f - (\eta^i - \xi\dot{q}^i) \frac{\partial f}{\partial q^i} - L\xi\). Replacing we have:
\[
I = -\frac{1}{2} A \left( t^2\dot{q}^2 - \frac{a}{\nu + 2} t^{\nu + 1} q^{2\nu + 4} \right) t - (-At\dot{q} + c_0q + (2c_0 + A)t\dot{q}) t^2\dot{q} + \frac{1}{2} \left( t^2\dot{q}^2 - \frac{a}{\nu + 2} t^{\nu + 1} q^{2\nu + 4} \right) (2c_0 + A)t.
\]
\[
= -c_0 \left( t^2\dot{q}^2 + t^3\dot{q}^2 + \frac{a}{\nu + 2} t^{\nu + 2} q^{2(\nu + 2)} \right)
\]
Hence:
\[
I = t^3\dot{q}^2 + t^2\dot{q}\dot{q} + \frac{a}{2 + \nu} t^{\nu + 2} q^{2(\nu + 2)} = \text{const.}
\]
Having computed \(I\) we compute \(\eta^i\) from the relation \(\eta^i = \gamma^{ij} \frac{\partial I}{\partial q_j}\). The \(\gamma_{ij} = t^2\), that is, \(\gamma^{ij} = \frac{1}{t^2}\). Then
\[
\eta = \frac{1}{t^2}(2t^2\dot{q} + t^2\dot{q}) = 2t\dot{q} + q.
\]
Comparing this with what we have found we get \(A = -2\), \(c_0 = 1\). We note that for these values of \(A, c_0\) the \(\xi = 0\) as it is correct because the relation \(\eta^i = \gamma^{ij} \frac{\partial I}{\partial q_j}\) is valid only under the assumption \(\xi = 0\).

7 The Inverse Noether Theorem

One question which arises concerns the extend that the first integrals provided by different types of symmetry of a system of differential equations are independent. In this section we show that to any quadratic first integral one may associate a Noether symmetry which provides that integral as a Noether integral.

Suppose we have a quadratic first integral \(I\) of a Lagrangian system with non-degenerate kinetic metric. We define a vector \(\eta^i(t, q, \dot{q})\) and a function \(f(t, q, \dot{q})\) by the requirement
\[ I = f - \eta^i \dot{q}^i. \]
Because \(I\) is quadratic in the velocities \(\eta^i\) must be linear in the velocities and \(f\) must be at most quadratic in the velocities. We choose
\[ \eta_i = a_i(t, q) + b_{ij}(t, q)\dot{q}^j \quad \text{and} \quad f = \frac{1}{2} c_{ij}(t, q)\dot{q}^i \dot{q}^j + d_i(t, q)\dot{q}^i + e_i(t, q). \]
Then we have
\[ \eta_i = -\frac{\partial I}{\partial q^i} + \frac{\partial f}{\partial q^i} = -\frac{\partial I}{\partial q^i} + c_{ij}(t, q)\dot{q}^j + d_i(t, q) \]
ad replacing \(\eta_i\)
\[ a_i(t, q) + b_{ij}(t, q)\dot{q}^j = -\frac{\partial I}{\partial q^i} + c_{ij}(t, q)\dot{q}^j + d_i(t, q). \]
Let us assume that \(I\) has the general form
\[ I = \frac{1}{2} A_{ij}(t, q)\dot{q}^i \dot{q}^j + B_i(t, q)\dot{q}^i + C(t, q). \]
Then we have
\[ a_i(t, q) + b_{ij}(t, q)\dot{q}^j = A_{ij}(t, q)\dot{q}^i + B_i(t, q) + c_{ij}(t, q)\dot{q}^j + d_i(t, q) \]
from which follows
\[ b_{ij} = -A_{ij} + c_{ij} \]
\[ a_i = -B_i + d_i \]
This system has an infinite number of solutions. To pick up one solution we have to define $c_{ij}, d_i$. One choice is $c_{ij} = -A_{ij}$ which gives $b_{ij} = -2A_{ij}$ and $d_i = 0$ which implies $a_i = -B_i$. Therefore one answer is

\begin{align*}
\eta_i &= 2A_{ij}q^j + B_i \\
\eta_i &= \frac{1}{2}A_{ij}(t, q)q^i\dot{q}^j + e_i(t, q).
\end{align*}

Then the vector $X^W = \eta^i \frac{\partial}{\partial q^i} + \eta^{[1]} | \frac{\partial}{\partial q^i}$

\begin{equation}
X^W = \eta^i \frac{\partial}{\partial q^i} + \eta^{[1]} | \frac{\partial}{\partial q^i}
\end{equation}

generates a point transformation which is a gauged Noether symmetry (in the gauge $\xi = 0, \phi^i = 0$) of the conservative holonomic dynamical system admitting $I$ as a Noether integral.

Obviously all quadratic first integrals of SODEs correspond to a Noether symmetry which is computed as indicated above.

8 Symmetries of SODEs in flat space

Obviously an area where symmetries of ODEs play an important role is the cases in which the kinetic metric (not necessarily the spacetime metric) is flat. These cases cover significant part of Newtonian Physics where the kinetic energy is a positive definite metric with constant coefficients therefore there is always a coordinate transformation in the configuration space which brings the metric to the Euclidian metric. Similar remarks apply to Special Relativity and -as we shall see - to Cosmology.

The basic result in this cases is that the maximum number of Lie point symmetries a SODE can have is $n(n + 2)$ and the maximum number of Noether point symmetries $\frac{n(n + 1)}{2} + 1$. Moreover, the number of point symmetries which a SODE can posses is exactly one of 0, 1, 2, 3, or 8 \[36\]. Similar results exist for higher-order differential equations \[37\].

Lie has shown \[2\] the important result that “for all the second order ordinary differential equations which are invariant under the elements of the $sl(3, R)$, there exists a transformation of variables which brings the equation to the form $x'' = 0$ and vice versa” .

In current cosmological models of importance of interest is the case $n = 2$, therefore we shall restrict our attention to two cases

i. The case the of systems which admit the maximum number of Lie point symmetries which for $n = 2$ is eight and span the algebra $sl(3, R)$.

ii. The case that the Lie point symmetries span the algebra $sl(2, R)$.

8.1 The case of $sl(3, R)$ algebra

The prototype dynamical system which admits the $sl(3, R)$ algebra of eight Lie point symmetries is the Newtonian free particle moving in one dimension whose dynamical equation is

\begin{equation}
\ddot{x} = 0
\end{equation}

where $x = x(t)$ and a dot means differentiation with respect to the time parameter $t$.

Let $X = \xi(t, x) \partial_t + \eta(t, x) \partial_x$ be the generator of a Lie point symmetry of \[54\]. The Lie point symmetries of \[54\] are given by the special projective vectors of $E^2$ (see condition \[33\]).
\[ X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t \partial_x, \quad X_4 = t^2 \partial_t + tx \partial_x, \quad X_5 = t \partial_t, \quad X_6 = x \partial_x, \quad X_7 = tx \partial_t + x^2 \partial_x, \quad X_8 = x \partial_t. \]

To show the validity of the aforementioned Lie’s result we consider the harmonic oscillator

\[ \ddot{x} + x = 0, \quad (55) \]

which also admits the eight Lie point symmetries \[38\]

\[ \bar{X}_1 = \partial_t, \quad \bar{X}_2 = \cos t \partial_x, \quad \bar{X}_3 = \sin t \partial_x, \quad \bar{X}_4 = \cos 2t \partial_t + x \cos 2t \partial_x, \quad \bar{X}_5 = \sin 2t \partial_t, \quad \bar{X}_6 = \cos 2t \partial_t - x \sin 2t \partial_x, \]

\[ \bar{X}_7 = x \sin t \partial_t + x^2 \cos t \partial_x, \quad \bar{X}_8 = x \cos t \partial_t - x^2 \sin t \partial_x, \]

which form another basis of the \( SL(3, R) \) Lie algebra. The transformation which relates the two different representations of the \( SL(3, R) \) algebra is

\[ t \rightarrow \arctan \tau, \quad x \rightarrow \frac{y}{\sqrt{1 + \tau^2}} \quad (56) \]

and it easy to show that under this transformation equation \[55\] becomes \( \ddot{y} = 0 \), which is equation \(54\).

In order to calculate the Noether point symmetries of \[55\] we have to define a Lagrangian. We recall that the Noether symmetries depend on the particular Lagrangian we consider. Let us assume the classical Lagrangian \( L_1 (t, x, \dot{x}) = \frac{1}{2} \dot{x}^2 \). Replacing in the Noether condition we find the associated Noether conditions

\[ \xi_{x,x} = 0, \quad \eta_{x,t} = 0 \]
\[ \eta_t - f_{,x} = 0, \quad f_{,t} = 0 \]

whose solution gives that the Noether point symmetries of \[55\] for the Lagrangian \( L_1 \) are the vector fields \( X_1, X_2, X_3, X_4 \) and \( X_N = 2X_5 + X_6 \), with corresponding non-constant Noether functions the \( f_3 = x \) and \( f_4 = \frac{1}{2} x^2 \).

Furthermore, from the second theorem of Noether the corresponding first integrals are calculated easily. The vector \( X_1 \), provides the conservation law of energy, \( X_2 \) the conservation law of momentum, while \( X_3 \) gives the Galilean invariance \[39\]. Finally the vector fields \( X_4 \) and \( X_N \) are also important because they can be used to construct higher-order conservation laws.

A more intriguing example is the slowly lengthening pendulum whose equation of motion in the linear approximation is

\[ \ddot{x} + \omega^2 (t) x = 0, \quad (59) \]

which also admits 8 Lie point symmetries. According to Lie’s result there is a transformation which brings \[59\] to the form \(54\). In order to find this transformation one considers the Noether symmetries and shows that \(59\) admits the quadratic first integral \[40\]

\[ I = \frac{1}{2} \left\{ (\rho \dot{x} - \dot{\rho} x)^2 + \left( \frac{x}{\rho} \right)^2 \right\}, \quad (60) \]

\[ \text{The time dependence in the ‘spring constant’ is due to the length of the pendulum’s string increasing slowly} \[41\]
where $\rho = \rho(t)$, is a solution of the second-order differential equation

$$\ddot{\rho} + \omega^2(t) \rho = \frac{1}{\rho^3},$$

(61)

The first integral (60) is known as the Lewis invariant.

On the other hand, equation (61) is the well-known Ermakov-Pinney equation [42] whose solution has been given by Pinney [43] and it is

$$\rho(t) = \sqrt{A\upsilon_1^2 + 2B\upsilon_1\upsilon_2 + C\upsilon_2^2}$$

(62)

where $A$, $B$, $C$ are constants of which only two are independent, and the functions $\upsilon_1(t)$, $\upsilon_2(t)$, are two linearly independent solutions of (59).

Finally, the transformation which connects the time dependent linear equation (59) with (54) is the following

$$y = \frac{x}{\rho}, \quad P = \rho \dot{x} - \dot{\rho}x, \quad \tau = \int_{t_0}^t \rho^{-2}(\eta) \, d\eta,$$

(63)

where $\rho(t)$ is given by (62).

### 8.2 The case of the $sl(2, R)$ algebra

The prototype system in this case is the Ermakov system defined by equations (59) and (60) above whose Lie point symmetries span the $sl(2, R)$ algebra for arbitrary function $\omega(t)$, while any solution for a specific $\omega(t)$ can be transformed to a solution for another $\omega(t)$ by a coordinate transformation. The Ermakov system has numerous applications in diverting areas of Physics, see for instance [44–46].

Let us restrict our considerations to the autonomous case, with $\omega(t) = \mu^2$ a constant. Equation (61) becomes

$$\ddot{\rho} + \mu^2 \rho = \frac{1}{\rho^3},$$

(64)

while the elements of the admitted $sl(2, R)$ Lie algebra are

$$Z_1 = \partial_t, \quad Z_2 = 2t\partial_t + \rho\partial_{\rho}, \quad \text{and} \quad Z_3 = t^2\partial_t + t\rho\partial_{\rho}, \text{when } \mu = 0$$

and

$$Z_1 = \partial_t, \quad Z_2 = \frac{1}{\mu} e^{2\mu t} \partial_t + e^{2\mu t} \rho t \partial_{\rho}, \quad Z_3 = \frac{1}{\mu} e^{-2\mu t} \partial_t - e^{2\mu t} \rho \partial_{\rho}, \text{when } \mu \neq 0.$$ 

Concerning the Noether symmetries we consider the Lagrangian

$$L(t, \rho, \dot{\rho}) = \frac{1}{2} \dot{\rho}^2 - \frac{\mu^2}{2} \rho^2 - \frac{1}{2} \frac{1}{\rho^2},$$

(65)

and find that the Lie symmetries $Z_1, Z_2, Z_3$ satisfy the Noether condition, hence they are also Noether point symmetries, and lead to the quadratic first integral of energy

$$E = \frac{1}{2} \dot{\rho}^2 + \frac{\mu^2}{2} \rho^2 + \frac{1}{2} \frac{1}{\rho^2}$$

(66)

and the time dependent first integrals

$$I_1 = 2tE - \rho \dot{\rho}$$

(67)

$$I_2 = t^2E - t \rho \dot{\rho} + \frac{1}{2} \rho^2, \text{ when } \mu = 0$$

(68)
or

\[ I_+ = \frac{1}{2\mu} e^{2\mu t} E - e^{2\mu t} \rho \dot{\rho} + \mu e^{2\mu t} \rho^2 \] (69)

\[ I_- = \frac{1}{2\mu} e^{-2\mu t} E + e^{-2\mu t} \rho \dot{\rho} + \mu e^{-2\mu t} \rho^2, \text{ when } \mu \neq 0. \] (70)

While the first integrals \( I_1, I_2 \) and \( I_+, I_- \) are time-dependent, we can easily construct the time-independent Lewis invariant [47]. For instance \( I_+I_- \) is a time-independent first integral.

The two dimensional system with Lagrangian

\[ L \left(t, \rho, \theta, \dot{\rho}, \dot{\theta} \right) = \frac{1}{2} \dot{\rho}^2 + \frac{1}{2} \rho^2 \dot{\theta} - \frac{\mu^2}{2} \rho^2 \rho^2 - \frac{V(\theta)}{\rho^2} \] (71)

describes the simplest generalization of the Ermakov-Pinney system in two-dimensions. It can be shown that the Lie point symmetries \( Z_1, Z_2, Z_3 \) are Noether point symmetries of (71) with the same first integrals. Again with the use of the time dependent Noether integrals \( I_1, I_2 \) and \( I_+, I_- \) we are able to construct the autonomous conservation laws [48]

\[ \Phi = 4I_2E - I_1^2 = \rho^4 \dot{\phi}^2 + 2V(\phi) \] (72)

and

\[ \bar{\Phi} = E^2 - I_+I_- = \rho^4 \dot{\phi}^2 + 2V(\phi) \] (73)

As we shall see below the Ermakov-Pinney system and its generalizations are used in the dark energy models [47].

9 Symmetries of SODEs and the geometry of the underline space

As it has been mentioned the symmetries of a SODE concern the kinetic metric which is independent of the metric of the space where motion occurs. On the other hand one of the basic principles of General Relativity (and Newtonian Physics) is the Principle of Equivalence according to which the trajectories of free fall are the geodesics of the space where motion occurs. This means that the Principle of Relativity locks the Lie point symmetries of the geodesic equations (=free fall) with the collineations of the geometry (=metric) of space-time where the particle moves. The relation between the Lie and Noether point symmetries and the collineations of the space where motion occurs has been given not only for the case of geodesics but also for a general conservative dynamical system (see [26–29]).

Below, we briefly discuss the collineations of Riemannian manifolds and also the results of [27] because they are used in the construction of cosmological models.

9.1 Collineations

Consider a Riemannian manifold \( M \) of dimension \( n \) and metric \( g_{ij} \). Let \( A \) be a geometric object (not necessarily a tensor) defined in terms of the metric \( g_{ij} \), \( X \) a vector field in \( F_0(M) \) and \( B \) a tensor field on \( M \) which has the same number of indices as \( A \) and with the same symmetries of the indices. We say that \( X \) is a collineation of \( A \) if the following condition holds

\[ \mathcal{L}_X A = B \] (74)

This is not necessary but it is enough for the cases we consider below.
Table 1: Collineations of the metric and of the connection in a Riemannian space

| Collineation                  | A  | B  |
|------------------------------|----|----|
| Killing vector (KV)          | \(g_{ij}\) | 0  |
| Homothetic vector (HV)       | \(g_{ij}\) | \(\psi g_{ij}, \psi, i = 0\) |
| Conformal Killing vector (CKV)| \(g_{ij}\) | \(\psi g_{ij}, \psi, i \neq 0\) |
| Affine collineation          | \(\Gamma^i_{jk}\) | 0  |
| Projective collineation (PC) | \(\Gamma^i_{jk}\) | \(2\phi_{i,j} \delta^i_k, \phi, i \neq 0\) |
| Special Projective collineation (SPC) | \(\Gamma^i_{jk}\) | \(2\phi_{i,j} \delta^i_k, \phi, i \neq 0\) and \(\phi, jk = 0\) |

Table 2: The elements for the projective algebra of the Euclidean space

| Collineation                  | Gradient | Non-gradient |
|------------------------------|----------|--------------|
| Killing vectors (KV)          | \(S_I = \delta^i_j \partial_i\) | \(X_{IJ} = \delta^i_j \delta^i_k x_j \partial_i\) |
| Homothetic vector (HV)        | \(H = x^i \partial_i\) | |
| Affine Collineation (AC)      | \(A_{II} = x_I \delta^i_j \partial_i\) | \(A_{IJ} = x_J \delta^i_j \partial_i\) |
| Special Projective collineation (SPC) | \(P_I = S_I H\). | |

where \(\mathcal{L}_X\) denotes the Lie derivative. The collineations of a geometric object form a Lie algebra. The classification of the possible collineations in a Riemannian space can be found in [49]. The most important are the collineations given in Table 1.

Some general results concerning collineations of a Riemannian manifold \(M\) of dimension \(n\) are the following:

- \(M\) can have at most \(n(n+1)/2\) KVs and when this is the case \(M\) is called a maximally symmetric space. The curvature tensor of a maximally symmetric space is given by the expression

\[
R_{abcd} = R(g_{ac}g_{db} - g_{ad}g_{bc})
\]

where \(R\) the curvature scalar which is a constant. Flat space is a maximally symmetric space for which \(R = 0\).

- \(M\) can have at most one proper HV.

- \(M\) can have at most \(\frac{(n+1)(n+2)}{2}\) proper CKVs, \(n(n+1)\) proper ACs, and \(n(n+2)\) proper PCs. A 2-dimensional space has infinite CKVs.

- If the metric admits a SCKV then also admits a SPC, a gradient HV and a gradient KV [50]. Other properties of the collineations can be found in [1].

What shall be important in our discussions are the collineations of the \(n\)-dimensional flat space. These collineations we summarize in Table 2. It is important to note which collineations are gradient.

### 10 Motion and symmetries in a Riemannian space

The equation of motion of a particle moving in a Riemannian space is given by the SODE

\[\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = F^i.\] (75)
where $\Gamma_{jk}^i(x^r)$ are the connection coefficients and the field $F^i$ stands for the forces acting on the particle. The Lie symmetries of (75) are as follows [27]

$$L_\eta F^i + 2 \xi, t F^i + \xi F^i_{,t} + \eta^i_{,tt} = 0$$

(76)

$$\left(\xi, h \delta^j_i + 2 \xi, i \delta^j_k \right) F^k + 2 \eta^i_{,tij} - \xi, tt \delta^j_i = 0$$

(77)

$$L_\eta \Gamma^i_{(jk)} = 2 \xi, t (j \delta^i_k)$$

(78)

$$\xi, \delta^i_{(ij)} = 0.$$  

(79)

From (78) follows that the Lie point symmetries of the SODE (75) are generated from the special projective algebra of the space.

10.1 Lie point symmetries of (75)

In the case where the force is autonomous and conservative, that is, $F^i = g^{ij} V_j (x^k)$ and $V_j$ is not a gradient KV of the metric, the solution of the determining equations as also the generic Lie symmetry vector have been given in [27] and have as follows:

- **Case I** Lie point symmetries due to the affine algebra. The resulting Lie symmetries are

$$X = \left(1/2 \right) \left[ d_1 a_1 t + d_2 \right] \partial_t + a_1 Y^i \partial_i$$

(80)

where $a_1$ and $d_1$ are constants, provided the potential satisfies the condition

$$L_Y V^i + d_1 V^i = 0.$$  

(81)

- **Case IIa** The Lie point symmetries are generated by the gradient homothetic algebra and $Y^i \neq V^i$. The Lie point symmetries are

$$X = 2 \psi \int T(t) dt \partial_t + T(t) Y^i \partial_i$$

(82)

where the function $T(t)$ is the solution of the equation $T,tt = a_1 T$ provided the potential $V(x^i)$ satisfies the condition

$$\mathcal{L}_Y V^i + 4 \psi V^i + a_1 V^i = 0.$$  

(83)

- **Case IIb** The Lie point symmetries are generated by the gradient HV $Y^i = \kappa V^i$, where $\kappa$ is a constant. In this case the potential is the gradient HV of the metric and the Lie symmetry vectors are

$$X = \left( - c_1 \sqrt{\psi k} \cos \left(2 \sqrt{\psi k} t \right) + c_2 \sqrt{\psi k} \sin \left(2 \sqrt{\psi k} t \right) \right) \partial_t + \left( c_1 \sin \left(2 \sqrt{\psi k} t \right) + c_2 \cos \left(2 \sqrt{\psi k} t \right) \right) H^i \partial_i.$$  

(84)

- **Case IIIa** The Lie point symmetries due to the proper special projective algebra. In this case the Lie symmetry vectors are (the index $J$ counts the gradient KVs)

$$X_J = (C(t) S_J + D(t)) \partial_t + T(t) Y^i \partial_i$$

(85)

where the functions $C(t), T(t), D(t)$ are solutions of the system of simultaneous equations

$$D, t = \frac{1}{2} d_1 T \quad , \quad T,tt = a_1 T \quad , \quad T, t = c_2 C \quad , \quad D,tt = d_0 C \quad , \quad C, t = a_0 T.$$  

(86)
and in addition the potential satisfies the conditions

$$
\mathcal{L}_Y V^i + 2a_0 SV^i + d_1 V^i + a_1 Y^i = 0 \tag{87}
$$

$$
(S, k \delta^i_j + 2S, j \delta^i_k) V^k + (2Y^i, j - a_0 S \delta^i_j) c_2 - d_1 \delta^i_j = 0. \tag{88}
$$

- **Case IIIb** Lie point symmetries due to the proper special projective algebra and $Y^i_j = \lambda S_j V^i$, in which $V^i$ is a gradient HV, and $S^i_j$ is a gradient KV. The Lie symmetry vectors are

$$
X_J = (C(t) S_J + d_1) \partial_t + T(t) \lambda S_J V^i \partial_i \tag{89}
$$

where the functions $C(t)$ and $T(t)$ are computed from the relations

$$
T_{,tt} + 2C_{,t} = \lambda_1 T, \quad T_t = \lambda_2 C, \quad C_t = a_0 T \tag{90}
$$

and the potential satisfies the conditions

$$
\mathcal{L}_Y V^i + \lambda_1 S_J V^i = 0 \tag{91}
$$

$$
C \left( \lambda_1 S_J \delta^i_j + 2S_J, j V^i \right) + \lambda_2 \left( 2\lambda S_J V^i + \left( 2\lambda S_J - a_0 S \delta^i_j \right) \delta^i_j \right) = 0. \tag{92}
$$

The general case for time dependent conservative forces has been given in [51]. In the following only the autonomous case will be considered.

## 10.2 Noether point symmetries of (75)

In case the force is autonomous and conservative the SODE (75) follows from the regular Lagrangian

$$
L \left( t, x^k, \dot{x}^k \right) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V \left( x^k \right). \tag{93}
$$

The Noether point symmetries of Lagrangian [93] as well as the corresponding first integrals have been given in [27] and are generated by the homothetic algebra of the metric $g_{ij}$ as follows:

- **Case I.** The HV satisfies the condition:

$$
V, k Y^k + 2\psi Y V + c_1 = 0. \tag{94}
$$

The Noether point symmetry vector is

$$
X = 2\psi Y t \partial_t + Y^i \partial_i, \quad f = c_1 t, \tag{95}
$$

where $T(t) = a_0 \neq 0$ and the corresponding first integral is

$$
\Phi = 2\psi Y t E - g_{ij} Y^i \dot{x}^j + c_1 t \tag{96}
$$

- **Case II.** When the metric admits the gradient KVs $S_J$, the gradient HV $H^i$ and the potential satisfies the condition

$$
V, k Y^k + 2\psi Y V = c_2 Y + d. \tag{97}
$$

In this case the Noether point symmetry vector and the Noether function are

$$
X = 2\psi Y \int T(t) dt \partial_t + T(t) S^i_j \partial_i, \quad f \left( t, x^k \right) = T_i S_j \left( x^k \right) + d \int T dt. \tag{98}
$$
and the functions $T(t)$ and $K(t)$ ($T_t \neq 0$) are computed from the relations
\[ T_{tt} = c_2 T, \quad K_t = d \int T dt + \text{constant} \] (99)
where $c_2$ is a constant with first integral
\[ \Phi = \psi Y \int T dt - T g_{ij} \dot{x}^i \dot{x}^j + T_t H + d \int T dt. \] (100)

In addition to the above cases there is also the Noether point symmetry $\partial_t$ whose first integral is the energy $E$.

From the above results it is clear that in order a given dynamical system to admit Lie / Noether point symmetries, the underlying space must admit collineations. This means that by studying the collineations of the underlying geometry we can infer important information for the existence or not and also compute the Lie/Noether point symmetries. In this respect we may say that geometry determines the evolution of the dynamical systems.

### 10.3 Point symmetries of constrained Lagrangians

The previous analysis holds for regular dynamical systems while when the dynamical system is constrained extra conditions are introduced.

Consider the constrained Lagrangian
\[ \bar{L} (t, x^k, \dot{x}^k, N) = \frac{1}{2N} g_{ij} \dot{x}^i \dot{x}^j - NV (x^k) \] (101)
where $N = N (t)$ is a singular degree of freedom with Euler-Lagrange equation $\frac{\partial L}{\partial N} = 0$. The latter equation is a constraint of the system. Lagrangian functions of the form of (101) are provided in cosmological studies.

#### 10.3.1 Lie point symmetries of (101)

The generic Lie point symmetry vector for the Euler-Lagrange equations of (101) is
\[ X_L = X_N - 2a_3 \partial_N \]
(102)
where
\[ X_N = \alpha_2 \chi (t) \partial_t + (2\alpha_1 \tau (x^k) + \alpha_2 \chi, t) N \partial_N - \alpha_1 \eta^i \partial_i , \]
(103)
and where $\eta^i$ is a CKV of the metric $g_{ij}$ with conformal factor $\tau (x^k)$ which is related with the potential by the following condition/constraint:
\[ \mathcal{L}_\eta V (x^k) + 2 (\tau (x^k) + a_3) V (x^k) = 0. \] (104)

We note that the Lie point symmetries of the constrained Lagrangian are generated by the elements of the conformal algebra, whereas for regular systems these symmetries are generated by the elements of the special projective algebra. Except that, there are also differences in the constraint condition for the potential.
10.3.2 Noether point symmetries of (101)

In order to compute the Noether point symmetries of (101) we consider the Noether condition (48). We find one Noether point symmetry which is again the vector $X_N$ [52]; the potential satisfies the following condition / constraint:

$$\mathcal{L}_\eta V (x^k) + 2 \tau (x^k) V (x^k) = 0.$$  

(105)

The first integral defined by the Noether point symmetry $X_N$ is given by the formula [52]

$$\Phi^* = \frac{1}{N} g_{ij} \dot{x}^i \dot{x}^j - \chi (t) \left( \frac{\partial L}{\partial N} \right) \simeq \frac{1}{N} g_{ij} \dot{x}^i \dot{x}.$$

(106)

The function $\Phi^*$ is a “weak” conservation law in the sense that someone has to impose the constraint condition $\left( \frac{\partial L}{\partial N} \right) = 0$, in order $\frac{d\Phi}{dt} = 0$; this is because $\frac{d\Phi}{dt} = 2 \frac{\partial L}{\partial N}$. Furthermore we note that there is a difference in the Noether point symmetries between the regular and the constrained Lagrangian. For instance, while the regular Lagrangian $L (t, x, \dot{x}) = \frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2$ possesses five Noether point symmetries [53], for the singular Lagrangian $\bar{L} (t, x, \dot{x}, N) = \frac{1}{2} N \dot{x}^2 - \frac{1}{2} N x^2$ we found only the $X_N$ Noether symmetry.

We note that if $a_3 = 0$ then there is only one Lie point symmetry which is also a Noether point symmetry.

11 Symmetries in Cosmology

The nature of the source which drives the late-acceleration phase of the universe is an important problem of modern cosmology. Currently the late-acceleration phase of the universe is attributed to a perfect fluid with negative equation of state parameter, which has been named the dark energy. The simplest dark energy candidate is the cosmological constant model leading to the $\Lambda$CDM cosmology. In this model the gravitational field equations can be linearized and one is able to write the analytic solution in closed-form. However in spite of its simplicity, the $\Lambda$CDM cosmological model suffers from two major problems, the fine tuning problem and the coincidence problem [54–56]. In order to overpass these problems cosmologists introduced dynamical evolving dark energy models. In these models the dark energy fluid can be an exotic matter source like the Chaplygin gas, quintessence, $k$–essence, tachyons or it can be of a geometric origin provided by a modification of Einstein’s General Relativity [57–70]. The dark energy components introduce new terms in the gravitational field equations which are nonlinear or increase the degrees of freedom. The reafter, the linearization process applied in the case of the $\Lambda$CDM model fails and other mathematical methods must be applied in the study of integrability of the field equations and the construction of analytic solutions. An alternative approach is the polytropic dark matter models which can also describe the recent acceleration of the universe through a polytropic process [71, 72].

Two different groups, de Ritis et al. [73] and Rosquist et al. [74] applied independently the symmetries of differential equations in order to construct first integrals in scalar field cosmology. In particular, they determined the forms of the scalar field potential, which drives the dynamics of the dark energy, in order the field equations to admit Noether point symmetries. The classification scheme is based on an idea proposed by Ovsiannikov [5]. Since then, the classification scheme has been applied to various dark energy models and modified theories of gravity. Some of these classifications are complete while some others lack mathematical completeness leading to incorrect results. The purpose of the current review is to present the application of symmetries of differential equations in modern cosmology.
A cosmological model is a relativistic model therefore requires two assumptions: a. A specification of the metric, which is achieved mainly by the collineations for the comoving observers we discussed above and b. Equations of state which specify (compatible with the assumed collineations defining the metric!) the matter of the model universe. The latter is done by the introduction of a potential function in the action integral from which the field equations follow. One important class of cosmological models are the ones in which spacetime brakes in 1+3 parts, that is, the cosmic time and the spatial universe respectively. The latter is realized geometrically by the three dimensional spacelike hypersurfaces which are generated by the orbits of KVs of a three dimensional Lie algebra. In 1898, Luigi Bianchi classified all possible real three dimensional real Lie algebras in nine types. Each Lie algebra leads to a (hypersurface orthogonal) cosmological model called a Bianchi Spatially homogeneous cosmological model. These nine models have been studied extensively in the literature over the years and have resulted in many important cosmological solutions.

The principal advantage of Bianchi cosmological models is that, due to the geometric structure of spacetime the physical variables depend only on the time thus reducing the Einstein and the other governing equations to ordinary differential equations. Although, the gravitational field equations in General Relativity for the Bianchi cosmologies are ordinary second-order differential equations, due to the existence of nonlinear terms, exact solutions have been determined only for a few of them, while there was a debate few years ago on the integrability or not of the Mixmaster universe (Bianchi type IX model). In order to get detailed information on these alternative models one has to find an analytical solution of the field equations. This can be a formidable task depending on the form of the potential function and the free parameters that it has. The standard method to find an analytical solution is to use Noether symmetries and compute first integrals of the field equations. Indeed the application of symmetries of differential equations in the dark energy models started with the use of the first Noether theorem and with the consideration of the second Noether theorem. Both approaches are equivalent. Since then, Noether symmetries have been applied to a plethora of models for the determination of first integrals, and consequently analytical solutions.

We refer the reader to some of them. It is important to note that some of the published results are mathematically incorrect. For instance, in the authors used the Noether conditions in order to solve the dynamical equations of the model, and posteriori they determined the symmetries of the field equations. This is not correct because Noether symmetries are imposed by the requirement that they transform the Action Integral in a certain way and not as extra conditions to the Euler-Lagrange equations.

The difficulty with the above approach is that one has to work in spacetime where the geometry is not simple and the field equations are rather complex. To bypass that difficulty a new scenario has been developed in which one transforms the problem to a minisuperspace defined by the dynamical variables through a Lagrangian which produces the field equations in that space. Then one considers the Lagrangian in two parts. The kinematic part which defines the kinetic metric and the remaining part which defines the effective potential. If one knows the homothetic algebra of the kinetic metric then the application of the results of Section provide the Noether symmetries and the corresponding Noether first integrals of the field equations in mini superspace. Therefore the solution of the field equations is made possible and by the inverse transformation one finds the solution of the original field equations in the original dynamic variables in spacetime. This approach brought new results in various dark energy models and modified theories of gravity. Some of these results are discussed in the following.

Before we enter into detailed discussion it is useful to state the action summary of this method of work in order to provide a working tool to new cosmologists not experience in this field.
11.0.1 Method of work - scenario

1. Consider the Action Integral of the model in spacetime and produce the field equations.
2. Change variables and give a new set of field equations in the minisuperspace of dynamical variables in a convenient form.
3. Define a Lagrangian for the field equations in the mini superspace.
4. Read form the Lagrangian the kinetic metric and the effective potential. The new variables must be such that the kinetic metric will be flat or at least one for which one knows already the homothetic algebra. This defines the phrase "convenient form" stated in step 2 above.
5. Apply the results of section 3 to get a classification of Noether symmetries and compute the corresponding first integrals of the field equations in mini superspace.
6. Solve these equations for the various cases of the effective potential and other possible parameters.
7. Apply the inverse transformation and get the solution of the original field equations in terms of the original dynamical variables in spacetime.

In the sections which follow we apply this scenario to the major cosmological models proposed so far and give the detailed results in each case.

11.1 FRW spacetime and the ΛCDM cosmological model

The FRW spacetime is a decomposable 1+3 spacetime in which the three dimensional hypersurfaces are maximally symmetric spacelike hypersurfaces of constant curvature which are normal to the time coordinate. The metric of a FRW spacetime is specified modulo a function of time, the scale factor \( a(t) \). In comoving coordinates \( \{t,x,y,z\} \) it has the form

\[
ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right) .
\]

In the Bianchi classification it is a Type IX spacetime. This spacetime in classical General Relativity for comoving observers

\[
u^a = \frac{2}{3} (u^a u_a = -1)
\]

can support matter which is a perfect fluid, that is the energy momentum tensor is

\[
T_{ab} = \rho u_a u_b + p h_{ab}
\]

where \( \rho, p \) are the energy density and the isotropic pressure of matter as measured by the comoving observers. \( h_{ab} = g_{ab} + u_a u_b \) is the tensor projecting normal to the vector \( u^a \). This spacetime has been used in the early steps of relativistic cosmology.

The first cosmological model using this spacetime was the ΛCDM cosmology which was a vacuum spatially flat FRW spacetime with matter generated by a non-vanishing cosmological constant \( \Lambda \). For this model Einstein field equations are

\[
-3a \ddot{a} + 2a^3 \Lambda = 0,
\]

(108)

and

\[
\ddot{a} + \frac{1}{2a} \dot{a}^2 - a \Lambda = 0.
\]

(109)

whose solution is the well-known de Sitter solution \( a(t) = a_0 e^{\sqrt{3\Lambda} t} \) which is a maximally symmetric spacetime (not only the maximally symmetric 3d hypersurfaces). According to earlier comments there exists a coordinate

\[
\text{For non-comovig observers can support all types of matter.}
\]
transformation which brings the system to the linear equation (110) \[125\]. Indeed if we introduce the new variable \(r(t) = a(t)^{3/2}\) the field equation (109) becomes

\[-\frac{1}{2} r'^2 + \frac{3}{2} \Lambda r^2 = 0 , \quad \ddot{r} - \frac{3}{2} \Lambda r = 0.\]  

(110)

which is the one dimensional hyperbolic linear oscillator which admits 8 Lie point symmetries.

Let us demonstrate the geometric scenario mentioned above in this simple case. For this we need to find the maximum number of Noether point symmetries admitted by the field equation (109). We choose the variables \(\{t, a\}\) and have a two-dimensional mini superspace. A Lagrangian for equation (109) in the minisuperspace \(\{t, a\}\) is

\[L(t, a, \dot{a}) = 3a\dot{a}^2 + 2\Lambda a^3.\]  

(111)

from where we read the kinetic metric \(s^2(t) = 3a\dot{a}^2\) and the effective potential \(V(a) = -2\Lambda a^3\). The Noether condition for the Lagrangian (111) gives that it admits five Noether point symmetries, which is the maximum number for admitted Noether symmetries for a two-dimensional system. These Noether symmetries are

\[X_{\Lambda(1)} = \partial_a , \quad X_{\Lambda(2)} = \frac{e^{\frac{3\sqrt{6}t}{\Lambda}}}{\sqrt{a}} \partial_a , \quad X_{\Lambda(3)} = \frac{e^{\frac{-3\sqrt{6}t}{\Lambda}}}{\sqrt{a}} \partial_a \]

\[X_{\Lambda(4)} = e^{\frac{3\sqrt{6}t}{\Lambda}} \left(3\sqrt{\frac{3}{\Lambda}} \partial_t + \sqrt{6a} \partial_a\right) , \quad X_{\Lambda(5)} = e^{\frac{-3\sqrt{6}t}{\Lambda}} \left(3\sqrt{\frac{3}{\Lambda}} \partial_t - \sqrt{6a} \partial_a\right).\]

This simple application shows how the geometry of the kinetic metric can be used to recognize the equivalence of well-known systems of classical mechanics with dark energy models.

### 11.2 Scalar-field cosmology

In the case of classical General Relativity with a minimally coupled scalar field (quintessence or phantom) the Action Integral in spacetime is

\[S_M = \int dx^4 \sqrt{-g} \left[R + \frac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j - V(\phi)\right].\]  

(112)

Assuming a spatially flat FLRW background and comoving observers the field equations are

\[-3a\dot{a}^2 + \frac{\varepsilon}{2} a^2 \dot{\phi}^2 + a^3 V(\phi) = 0\]  

(113)

\[\ddot{a} + \frac{1}{2a} \dot{a}^2 + \frac{\varepsilon}{4} \dot{\phi}^2 - \frac{1}{2} a V = 0\]  

(114)

\[\ddot{\phi} + \frac{3}{a} \dot{a} \dot{\phi} + \varepsilon V,\phi = 0.\]  

(115)

We consider the mini superspace defined by the dynamic variables \(\{a, \phi\}\). A point-like Lagrangian in the mini superspace of the for field equations (114) and (115) is

\[L(t, a, \dot{a}, \dot{\phi}) = -3a\dot{a}^2 + \frac{\varepsilon}{2} a^2 \dot{\phi}^2 - a^3 V(\phi).\]  

(116)

To bring the Lagrangian in the "convenient form" we consider the coordinate transformation \(\{a, \phi\}\) to \(\{r, \theta\}\)

\[r = \sqrt{\frac{8}{3}} a^{3/2} , \quad \theta = \sqrt{\frac{3\varepsilon}{8}} \phi.\]  

(117)
and again \( \{r, \theta\} \) to \( \{x, y\} \) where
\[
x = r \cosh \theta, \quad y = r \sinh \theta,
\]
and the new variables have to satisfy the following inequality: \( x \geq |y| \). In the coordinates \( \{x, y\} \) the scale factor becomes
\[
a = \left[ \frac{3(x^2 - y^2)}{8} \right]^{1/3}.
\]
Under the coordinate transformation \( \{a, \phi\} \rightarrow \{x, y\} \) the point-like Lagrangian takes the simpler form
\[
L(t, x, \dot{x}, y, \dot{y}) = \frac{1}{2} \left( \dot{y}^2 - \dot{x}^2 \right) - V_{\text{eff}}(x, y)
\]
in which the metric in the coordinates \( \{x, y\} \) is the Lorentzian 2d metric \( \text{diag}(-1, 1) \) which is the metric of a flat space while the effective potential is
\[
V_{\text{eff}}(x, y) = \frac{3}{8} (x^2 - y^2) \tilde{V}(\theta).
\]
Application of the previous analysis gives the following classification of Noether symmetries of the model for various forms of the effective potential \[120\]

- For arbitrary potential \( V_{\text{eff}}(x, y) \), Lagrangian \[120\] admits the Noether point symmetry \( \partial_t \) with first integral the constraint equation \[113\].
- For constant potential \( V(\theta) = V_0 \), the system admits the extra Noether symmetry \( x\partial_y - y\partial_x \) with first integral the angular momentum \( r^2 \dot{\theta} = \text{const.} \).
- For the exponential potential \( V_{\text{eff}}(x, y) = r^2 e^{-d\theta} \), Lagrangian \[120\] admits an extra Noether symmetry provided by the proper HV of the two dimensional flat space, that is,
\[
X_{(\phi)1} = 2t\partial_t + \left( x + \frac{4}{d}y \right) \partial_x + \left( y + \frac{4}{d}x \right) \partial_y,
\]
with first integral
\[
\Phi_{(\phi)1} = \left( x + \frac{4}{d}y \right) \dot{x} - \left( y + \frac{4}{d}x \right) \dot{y}
\]
while when \( d = 2 \), the Lagrangian admits the additional symmetry \( \partial_x + \partial_y \), with corresponding Noetherian first integral
\[
\Phi_{(\phi)2} = \dot{x} - \dot{y}.
\]
- Finally, when \( V_{\text{eff}}(x, y) = \frac{1}{2} (\omega_1 x^2 - \omega_2 y^2) \), that is, \( \tilde{V}(\theta) = \frac{1}{2} (\omega_1 \cosh^2(\theta) - \omega_2 \sinh^2(\theta)) \), the dynamical system admits four extra Noether point symmetries
\[
X_{(\phi)2} = \sinh(\sqrt{\omega_1}t) \partial_x, \quad X_{(\phi)3} = \cosh(\sqrt{\omega_1}t) \partial_x,
\]
\[
X_{(\phi)4} = \sinh(\sqrt{\omega_2}t) \partial_y, \quad X_{(\phi)5} = \cosh(\sqrt{\omega_2}t) \partial_y,
\]
with corresponding first integrals
\[
I_{n_2} = \sinh(\sqrt{\omega_1}t) \dot{x} - \sqrt{\omega_1} \cosh(\sqrt{\omega_1}t) x,
\]
\[
I_{n_3} = \cosh(\sqrt{\omega_1}t) \dot{x} - \sqrt{\omega_1} \sinh(\sqrt{\omega_1}t) x,
\]
\[
I_{n_4} = \sinh(\sqrt{\omega_2}t) \dot{y} - \sqrt{\omega_2} \cosh(\sqrt{\omega_2}t) y,
\]
\[
I_{n_5} = \cosh(\sqrt{\omega_2}t) \dot{y} - \sqrt{\omega_2} \sinh(\sqrt{\omega_2}t) y,
\]
The latter dynamical hyperbolic dynamical system reduces to that of the unified dark matter potential (UDM) when $\omega_1 = 2\omega_2$. It is noticeable the amount of information one receives by the direct application of the geometric symmetries of the kinetic metric.

To find the solution in the original dynamical variables $\{a, \phi\}$ we apply the inverse transformation. The result is

$$a^3 (t) = \frac{3}{8} \left[ \sinh^2 (\sqrt{\omega_1} t + \theta_1) - \frac{\omega_1}{\omega_2} \sinh^2 (\sqrt{\omega_2} t + \theta_2) \right], \quad \text{(124)}$$

$$\phi (t) = \sqrt{\frac{8}{3\varepsilon}} \arctan \left( \frac{\sqrt{\omega_1} \sinh (\sqrt{\omega_1} t + \theta_1)}{\sqrt{\omega_2} \sinh (\sqrt{\omega_1} t + \theta_1)} \right), \quad \text{(125)}$$

### 11.3 Brans-Dicke Cosmology

The Brans-Dicke action is

$$S_{NM} = \int dt dx^3 \sqrt{-g} \left[ F_0 \psi^2 R - \frac{1}{2} \tilde{g}_{ij} \psi^i \psi^j + \tilde{V} (\psi) \right] \quad \text{(126)}$$

where $F_0$ is related to the Brans-Dicke parameter.

In the case of the spatially flat FRW background and comoving observers, the point-like Lagrangian in the mini superspace defined by the variables $\{a, \psi\}$ which describes the gravitational field equations is

$$L \left( t, a, \dot{a}, \psi, \dot{\psi} \right) = 6F_0 \psi^2 a \ddot{a}^2 - 12F_0 \psi a^2 \dot{\psi} - \frac{1}{2} a^3 \dot{\psi}^2 + a^3 V (\psi). \quad \text{(127)}$$

If one performs the coordinate transformation $\{a, \psi\}$ to $\{r, \theta\}$ by the equations

$$a \simeq r^2, \quad \theta \simeq \ln \psi, \quad \text{(128)}$$

Lagrangian (127) becomes

$$L \left( t, r, \dot{r}, \theta, \dot{\theta} \right) = e^{k\theta} (-dr^2 + r^2 d\theta^2) - r^2 V (\theta) \quad \text{(129)}$$

from which we have that the kinetic metric of the minisuperspace is the conformally flat Lorentzian 2d metric $e^{k\theta} (-dr^2 + r^2 d\theta^2)$ whose symmetry algebra depends on the values $|k| \neq 1 |k| = 1$ while the effective potential is $V_{effec.} = -r^2 V (\theta)$.

We consider cases.

#### 11.3.1 Case \(|k| \neq 1\).

For $|k| \neq 1$ the homothetic algebra of the minisuperspace consists of the gradient KVs

$$K^1 = \frac{e^{(1-k)\theta} r^k}{N_0^2} \left( -\partial_r + \frac{1}{r} \partial_\theta \right) \quad \text{(130)}$$

$$K^2 = \frac{e^{-(1+k)\theta} r^{-k}}{N_0^2} \left( \partial_r + \frac{1}{r} \partial_\theta \right) \quad \text{(131)}$$

the non gradient KV

$$K^3 = r \partial_r - \frac{1}{k} \partial_\theta \quad \text{(132)}$$

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and the gradient HV

\[ H^i = \frac{1}{N_0^2 (k^2 - 1)} \left( -r \partial_r + k \partial_\theta \right) , \quad H (r, \theta) = \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2 - 1} . \]  

(133)

The symmetry classification provides the following results:

- For arbitrary potential \( V (\theta) \), the dynamical system admits the Noether point symmetry \( \partial_t \).
- For \( V (\theta) = V_0 e^{2\theta} \) there are two additional Noether symmetries \( K^1, tK^1 \) with first integrals

\[ I_1 = \frac{d}{dt} \left( \frac{r^{1+k} e^{(1+k)\theta}}{k+1} \right) , \quad I_2 = \frac{d}{dt} \left( \frac{r^{1+k} e^{(1+k)\theta}}{k+1} - \frac{r^{1+k} e^{(1+k)\theta}}{k+1} \right) . \]  

(134)

- For \( V (\theta) = V_0 e^{2\theta} - \frac{mN_0^2}{2(k^2 - 1)} e^{2\theta} \) there are two additional Noether symmetries \( e^{\pm \sqrt{m} t} K^1 \), with corresponding first integrals

\[ I'_\pm = e^{\pm \sqrt{m} t} \left[ \frac{d}{dt} \left( \frac{r^{1+k} e^{(1+k)\theta}}{k+1} \right) \mp \sqrt{m} \left( \frac{r^{1+k} e^{(1+k)\theta}}{k+1} \right) \right] . \]  

(135)

- For \( V (\theta) = V_0 e^{-2\theta} \), we have the extra Noether symmetries \( K^2, tK^2 \) with Noether first integrals

\[ J_1 = \frac{d}{dt} \left( \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right) , \quad J_2 = \frac{d}{dt} \left( \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} - \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right) . \]  

(136)

- For \( V (\theta) = V_0 e^{2k\theta} \) the additional symmetry is the vector field \( K^3 \) with first integrals as given by the second theorem of Noether

\[ I_3 = \frac{r e^{2k\theta}}{k} (k\dot{r} + \dot{r}) . \]  

(138)

- For \( V (\theta) = V_0 e^{-2\left(\frac{k^2 - 2}{k^2 - 1}\right)} \), \( k^2 - 2 \neq 0 \) the extra Noether symmetries are \( 2t \partial_\theta + H^i, t^2 \partial_\theta + tH^i \) with first integrals

\[ I_{H_1} = -\frac{d}{dt} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2 - 1} \right) , \quad I_{H_2} = -\frac{t}{dt} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2 - 1} \right) + \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2 - 1} . \]  

(139)

We note that in this case the system is the Ermakov-Pinney dynamical system (because it admits the Noether symmetry algebra the \( sl(2, R) \), hence the Lie symmetry algebra is at least \( sl(2, R) \)).

- For \( V (\theta) = V_0 e^{-2\left(\frac{k^2 - 2}{k^2 - 1}\right)} - \frac{N_0^2}{2(k^2 - 1)} e^{2k\theta} \), \( k^2 - 2 \neq 0 \) we have the Noether symmetries \( \frac{2}{\sqrt{m}} e^{\pm \sqrt{m} t} \partial_t \pm e^{\pm \sqrt{m} t} H^i \), \( m = \text{constant} \) with Noether first integrals

\[ I_{+,-} = e^{\pm 2\sqrt{m} t} \left[ \pm \frac{d}{dt} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2 - 1} \right) + 2\sqrt{m} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2 - 1} \right) \right] . \]  

(140)

For this potential the Noether symmetries form the \( sl(2, R) \) Lie algebra, i.e. the dynamical system is the two dimensional Kepler-Ermakov system.

- The case \( V (\theta) = 0 \) corresponds the two dimensional free particle in flat space and the dynamical system admits seven additional Noether symmetries.
11.3.2 Case $|k| = 1$

We have to consider two cases i.e. $k = 1$ and $k = -1$. It is enough to consider the case $k = 1$, because the results for $k = -1$ are obtained directly from those for $k = 1$ if we make the substitution $\theta_{(k=-1)} = -\bar{\theta}$.

For $k = 1$, the homothetic algebra of the minisuperspace is given by the vector fields $K^{1,2}_{k=1}$ of \[130\, 131\] and the vector field

$$K^3_{k=1} = -r \left( \ln \left( re^{-\theta} \right) - 1 \right) \partial_r + \ln \left( re^{-\theta} \right) \partial_{\theta}.$$  \hspace{1cm} (141)

Hence, the symmetry classification provides the following cases

- For arbitrary potential $V(\theta)$, the dynamical system admits the Noether symmetry $\partial_t$.

- All the rest cases admit additional symmetries.

- If $V(\theta) = V_0 e^{2\theta}$ we have the extra Noether symmetries $K^1$, $tK^1$ with Noether first integrals the \[134\] with $k = 1$.

- If $V(\theta) = V_0 e^{2\theta} - \frac{m}{2} \theta e^{2\theta}$ we have the Noether symmetries $e^{\pm \sqrt{mt}} K^1$ with Noether first integrals the \[135\] with $k = 1$.

- Noether symmetries generated by the KV $K^2$.

- If $V(\theta) = V_0 e^{-2\theta}$ then we have the Noether symmetries $K^2$, $tK^2$ with Noether first integrals

$$I'_2 = \frac{d}{dt} \left( \theta - \ln r \right), \hspace{0.5cm} I'_2 = t \left[ \frac{d}{dt} \left( \theta - \ln r \right) \right] - \left( \theta - \ln r \right).$$

- If $V(\theta) = V_0 e^{-2\theta} - \frac{1}{4} pe^{2\theta}$ then we have the Noether symmetries $K^2$, $tK^2$ with Noether first integrals

$$I'_1 = \frac{d}{dt} \left( \theta - \ln r \right) - pt, \hspace{0.5cm} I'_2 = t \left[ \frac{d}{dt} \left( \theta - \ln r \right) \right] - \left( \theta - \ln r \right) - \frac{1}{2} pt^2.$$  

- If $V(\theta) = 0$ then the system becomes the free particle and admits seven extra Noether symmetries.

The exact solutions of the models and their physical properties can be found in \[123\]. The results from the classification analysis are presented in Tables 3 and 4. For the notation of the admitted Lie algebra we follow the Mubarakzyanov Classification Scheme \[129\, 131\].

11.4 $f(R)$-gravity

$f(R)$-gravity (in the metric formalism) is a fourth-order theory where the Action Integral in spacetime is

$$S = \int d^4x \sqrt{-g} f(R)$$ \hspace{1cm} (142)

In the case of FRW background and comoving observers the resulting field equations follow from the Lagrangian \[132\]

$$L \left( t, a, \dot{a}, R, \ddot{R} \right) = 6a f' \dot{a}^2 + 6a^2 f'' \ddot{a}R + a^3 \left( \dot{f}' R - f \right) - 6Kaf'$$ \hspace{1cm} (143)

where $K = 0, \pm 1$ is the spatial curvature of the FRW spacetime, and a prime denotes derivative with respect to the dynamical parameter $R$, that is $f' = \frac{df}{dR}$. Since $f(R)$ theory can be written as a special case of Brans-Dicke theory, the so-called O’Hanlon gravity \[132\], the results will be common with that of the previous analysis. However for completeness we present them below.

The classification scheme provides the following cases \[121\]:
### Table 3: Noether symmetry classification for the Brans-Dicke action in a spatially flat FLRW spacetime (I)

| $|k|$ | Potential | # Symmetries | Lie Algebra | Symmetries |
|------|-----------|--------------|-------------|------------|
| $\neq 1$ | $V(\theta)$ | 1 | $A_1$ | $\partial_t$ |
| $\neq 1$ | $V_0 e^{2\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $K^1$, $tK^1$ |
| $\neq 1$ | $V_0 e^{2\theta} - \frac{m N_2}{2(k^2-1)} e^{2k\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $e^{\pm \sqrt{m} t} K^1$ |
| $\neq 1$ | $V_0 e^{-2\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $K^2$, $tK^2$ |
| $\neq 1$ | $V_0 e^{2k\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $e^{\pm \sqrt{m} t} K^{21}$ |
| $\neq 1$ | $V_0 e^{-2\theta} \frac{1}{k^2} e^{2k\theta}$, $k^2 - 2 \neq 0$ | 3 | $Sl(2, R)$ | $\partial_t$, $t^2 \partial_t + H^i$, $t^2 \partial_t + tH^i$ |

### Table 4: Noether symmetry classification for the Brans-Dicke action in a spatially flat FLRW spacetime (II)

| $|k|$ | Potential | # Symmetries | Lie Algebra | Symmetries |
|------|-----------|--------------|-------------|------------|
| $= 1$ | $V_0 e^{2\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $K^1$, $tK^1$ |
| $= 1$ | $V_0 e^{2\theta} - \frac{m}{2} \theta e^{2\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $e^{\pm \sqrt{m} t} K^1$ |
| $= 1$ | $V_0 e^{-2\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $K^2$, $tK^2$ |
| $= 1$ | $V_0 e^{-2\theta} - \frac{1}{4} \theta e^{2\theta}$ | 3 | $A_1 \otimes_3 (2A_1)$ | $\partial_t$, $K^2$, $tK^2$ |
• For arbitrary function $f(R)$, there exists the autonomous symmetry $\partial_t$, which derives the constraint equation.

• For $f(R) = R^{2\xi}$, the theory admits the additional Noether symmetries

$$K_1 = 2t \partial_t + \frac{4}{3} a \partial_a - \frac{9f'}{2f''} \partial_R,$$

$$K_2 = \frac{1}{a^2} \frac{f'}{f''} \partial_R, \quad K_2^* = t \left( \frac{1}{a^2} \frac{f'}{f''} \partial_R \right),$$

with first integrals

$$\Phi_1 = 6a^2 \dot{a} \sqrt{R} + 6 \frac{a^3}{\sqrt{R}} \dot{R},$$

$$\Phi_2 = \frac{d}{dt} \left( a \sqrt{R} \right), \quad \Phi_2^* = t \frac{d}{dt} \left( a \sqrt{R} \right) - a \sqrt{R}.$$

• For $f(R) = R^{7/8}$ and $K = 0$, the theory admits the additional Noether symmetries

$$K_3 = 2t \partial_t + \frac{a}{2} \partial_a + \frac{1}{2} \frac{f'}{f''} \partial_R, \quad K_3^* = t^2 \partial_t + \left( \frac{a}{2} \partial_a + \frac{1}{2} \frac{f'}{f''} \partial_R \right),$$

with corresponding first integrals

$$\Phi_3 = \frac{d}{dt} \left( a^3 R^{-\frac{7}{8}} \right), \quad \Phi_3^* = t \frac{d}{dt} \left( a^3 R^{-\frac{7}{8}} \right) - a^3 R^{-\frac{7}{8}}.$$

• The power-law theory $f(R) = R^n$ (with $n \neq 0, 1, \frac{3}{2}, \frac{7}{8}$) and for $K = 0$, or with $K$ arbitrary and $n = 2$, the system admits the extra symmetry

$$K_1^* = 2t \partial_t + \left( \frac{4n}{3} - \frac{2}{3} \right) a \partial_a - 3 \frac{f'}{f''} \partial_R,$$

with first integral

$$\Phi_1^* = a^2 R^{n-1} \dot{a} (2-n) + \frac{1}{2} a^3 R^{n-2} \dot{R} (2n-1) (n-1).$$

• For $f(R) = (R - 2\Lambda)^{3/2}$ the extra Noether symmetries are

$$K_{(\pm)2} = e^{\pm \sqrt{m}t} \left( \frac{1}{a^2} \frac{f'}{f''} \partial_R \right),$$

with first integrals

$$\Phi_{(\pm)2} = e^{\pm \sqrt{m}t} \left( \frac{d}{dt} \left( a \sqrt{R - 2\Lambda} \right) \mp 9 \sqrt{ma} \sqrt{R - 2\Lambda} \right).$$

• Finally, when $f(R) = (R - 2\Lambda)^{7/8}$ the field equations admit the Noether symmetries

$$K_{(\pm)4} = \pm \frac{1}{\sqrt{m}} e^{\pm 2\sqrt{m}t} \partial_t + e^{\pm 2\sqrt{m}t} \left( \frac{a}{2} \partial_a + \frac{1}{2} \frac{f'}{f''} \partial_R \right),$$

with corresponding first integrals

$$\Phi_{(\pm)4} = \frac{d}{dt} \left( a^3 (R - 2\Lambda)^{-\frac{7}{8}} \right) \mp \frac{1}{2} \sqrt{ma} a^3 (R - 2\Lambda)^{-\frac{7}{8}}.$$

In Table 5 we collect the results of the classification scheme for $f(R)$-gravity.
Table 5: Noether symmetry classification for \( f(R) \) in FLRW spacetime

| Sp. Curv. \( K \) | \( f(\mathbf{R}) \) | \# Symmetries | Lie Algebra | Symmetries |
|------------------|-----------------|---------------|-------------|------------|
| \( = 0, \pm 1 \) | Arbitrary       | 1             | \( A_1 \)    | \( \partial_t \) |
| \( = 0, \pm 1 \) | \( R^{\frac{2}{3}} \) | 4             | \((2A_1) \otimes_s (2A_1)\) | \( \partial_t, K_1, K_2, K_2^* \) |
| \( = 0 \)       | \( R^{\frac{2}{3}} \) | 3             | \( Sl(2, R) \) | \( \partial_t, K_3, K_3^* \) |
| \( = 0 \)       | \( R^n \) (with \( n \neq 0, 1, \frac{3}{2}, \frac{7}{8} \)) | 2             | \( 2A_1 \)    | \( \partial_t, K_4^* \) |
| \( = 1 \)       | \( R^2 \)       | 2             | \( A_1 \otimes_s (2A_1) \) | \( \partial_t, K_{(\pm 2)} \) |
| \( = 0, \pm 1 \) | \( (R - 2\Lambda)^{3/2} \) | 3             | \( Sl(2, R) \) | \( \partial_t, K_{(\pm 4)} \) |
| \( = 0 \)       | \( (R - 2\Lambda)^{7/8} \) | 3             | \( Sl(2, R) \) | \( \partial_t, K_{(\pm 4)} \) |

11.5 Two-scalar field cosmology

We consider now a two-scalar field cosmological model in General Relativity with Action Integral

\[
S = \int dx^4 \sqrt{-g} \left( R - \frac{1}{2} g_{ij} \left( \Phi_C \right) \Phi^{A,i} \Phi^{B,j} + V \left( \Phi_C \right) \right),
\]  

(156)

where \( H_{AB} \) describes the coupling between the two scalar fields \( \Phi^A = (\phi, \psi) \) in the kinematic part. Moreover, we assume the metric tensor \( H_{AB} \) to be a maximally symmetric metric of constant curvature [124]. In such a scenario it is not possible to define new fields in order to remove the coupling in the kinematic part.

Assuming again a spatially flat FRW spacetime and comoving observers the field equations are

\[
-3a \ddot{a} + \frac{1}{2} a^3 H_{AB} \dot{\Phi}^A \dot{\Phi}^B + a \dot{V} \left( \Phi_C \right) = 0, \tag{157}
\]

\[
\dot{a} + \frac{1}{2a} \dot{a}^2 + \frac{a}{4} H_{AB} \dot{\Phi}^A \dot{\Phi}^B - \frac{1}{2} \dot{a} V = 0, \tag{158}
\]

\[
\ddot{\Phi}^A + \frac{3}{2a} \dot{\Phi}^A + \Gamma_{BC}^{A} \dot{\Phi}^B \dot{\Phi}^C + H^{AB} V_B = 0, \tag{159}
\]

where \( \Gamma_{BC}^{A} \) are the connection coefficients for the metric \( H_{AB} \left( \Phi_C \right) \).

In the mini superspace defined by the variables \( \{a, \Phi\} \) we introduce the new variables \( \{u, \phi, \psi\} \) by the requirements \( a = (\frac{1}{8})^{\frac{3}{2}} u^2 \), \( \text{diag} \left( 1, e^{2\phi} \right) = h_{AB} \dot{\Phi}^A \dot{\Phi}^B \) and the field equations become

\[
-\frac{1}{2} u^2 + \frac{1}{2} u^2 \left( \dot{\phi}^2 + e^{2\phi} \dot{\psi}^2 \right) + u^2 V (\phi, \psi) = 0, \tag{160}
\]

\[
\ddot{u} + u \dot{\phi}^2 + u e^{2\phi} \dot{\psi}^2 - 2u \dot{V} = 0, \tag{161}
\]

\[
\ddot{\phi} + \frac{2}{u} \dot{u} \dot{\phi} - e^{2\phi} \dot{\psi}^2 + V_{,\phi} = 0, \tag{162}
\]

\[
\ddot{\psi} + \frac{2}{u} \dot{u} \dot{\psi} + 2\dot{\phi} \dot{\psi} + e^{-2\phi} V_{,\psi} = 0. \tag{163}
\]

These follow form the point-like Lagrangian

\[
L \left( t, u, \dot{u}, \phi, \dot{\phi}, \psi, \dot{\psi} \right) = -\frac{1}{2} \dot{u}^2 + \frac{1}{2} u^2 \left( \dot{\phi}^2 + e^{2\phi} \dot{\psi}^2 \right) - u^2 V (\phi, \psi) \tag{164}
\]

which defines the 3d flat Lorentzian kinetic metric \( -\frac{1}{2} \ddot{u} + \frac{1}{2} u^2 \left( \dot{\phi}^2 + e^{2\phi} \dot{\psi}^2 \right) \) and the effective potential \( V_{eff} = u^2 V (\phi, \psi) \).
The kinetic metric admits a seven dimensional homothetic algebra consisting of the three gradient KVs (translations)

$$K^1 = -\frac{1}{2} \left( e^{\phi} \left( 1 + \psi^2 \right) + e^{-\phi} \right) \partial_u + \frac{1}{2u} \left( e^{\phi} \left( 1 + \psi^2 \right) - e^{-\phi} \right) \partial_\psi + \frac{1}{u} \psi e^{-\phi} \partial_\phi,$$

$$K^2 = -\frac{1}{2} \left( e^{\phi} \left( 1 - \psi^2 \right) - e^{-\phi} \right) \partial_u + \frac{1}{2u} \left( e^{\phi} \left( 1 - \psi^2 \right) + e^{-\phi} \right) \partial_\phi - \frac{1}{u} \psi e^{-\phi} \partial_\psi,$$

$$K^3 = -\psi e^\theta \partial_u + \frac{1}{u} \psi e^\phi \partial_\phi + \frac{1}{u} e^{-\phi} \partial_\psi,$$

with corresponding gradient functions $S_{(1-3)}$ given by

$$S_{(1)} = \frac{1}{2u} \left( e^{\phi} \left( 1 + \psi^2 \right) + e^{-\phi} \right),$$

$$S_{(2)} = \frac{1}{2u} \left( e^{\phi} \left( 1 - \psi^2 \right) - e^{-\phi} \right),$$

$$S_{(3)} = u \psi e^\theta,$$

the three non-gradient KVs (rotations) which span the $SO(3)$ algebra

$$X_{12} = \partial_\psi, \quad X_{23} = \partial_\phi + \psi \partial_\psi, \quad X_{13} = \psi \partial_\phi + \frac{1}{2} \left( \psi^2 - e^{2\phi} \right) \partial_\phi$$

and the gradient proper HV

$$H_V = u \partial_u, \quad \psi H_V = 1.$$

The classification of the Noether symmetries for the various potentials $V(\phi, \psi)$ is as follows [124]:

- For arbitrary potential $V(\phi, \psi)$, the field equations admit the Noether symmetry $\partial_t$ which provides the constraint equation of General Relativity.

- For $V(\phi, \psi) = 0$, the dynamical system is maximally symmetric and admits in total twelve Noether symmetries.

- For $V(\phi, \psi) = V(\phi)$, there exists the additional Noether symmetry, the vector field $X_{12}$, with conservation law the angular momentum on the two dimensional sphere, that is

$$\Phi_{12} = e^{2\phi} \psi.$$

- For $V_A(\phi, \psi) = \frac{\omega_0^2}{2} u^2 + \frac{\mu^2}{2(1-a_0^2)} \left( S_{(\mu)} + a_0 S_{(\nu)} \right)^2 - \frac{\omega_3^2}{2} S_{(\sigma)}^2, \quad a_0 \neq 1$, the system admits six additional Noether symmetries given by the vector fields

$$T_1(t) K^1, \ T_2(t) K^2, T_3(t) K^3$$

where

$$T^A_{\mu t} = \omega^\gamma_{\delta} T^\delta, \quad \omega^\gamma_{\delta} = \text{diag} \left( (\omega_1)^2, (\omega_2)^2, (\omega_3)^3 \right)$$

and $\mu, \nu, \sigma = 1, 2, 3$. The corresponding Noether integrals are expressed as follows

$$I^A_\gamma = T_{\gamma,\tau} \frac{d}{dt} S_{(\tau)} - T_{\gamma,\tau} S_{(\tau)}. \quad (165)$$

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However, when two constants $\omega_A$ are equal, for instance, $\omega_\mu = \omega_\nu$, then the dynamical system admits an extra Noether symmetry. That is, it admits the rotation normal to the plane defined by the axes $x^\mu, x^\nu$ given by the vector

$$X = x^\nu \partial_\mu - \varepsilon x^\mu \partial_\nu$$

where $\varepsilon = -1$ if $x^\nu/x^\mu = x$ and $\varepsilon = 1$ if $x^\nu/x^\mu \neq 1$.

- For the potential is $V_B (\phi, \psi) = \frac{\omega_1^2}{2} u^2 + \frac{\omega_2^2}{2(1-a_0^2)} (S(\mu) + a_0 S(\nu))^2 - \frac{\omega_3^2}{2} S(\sigma)$, $a_0 \neq 1$, the dynamical system admits the extra Noether symmetries

$$\bar{T} (t) (K^\mu + a_0 K^\nu) , T' (t) K^\sigma, T^* (t) (a_0 K^\mu + K^\nu)$$

where the functions $T, T'$ and $\bar{T}$ are given by the linear second-order differential equations

$$\bar{T},tt = (\mu^2 + \omega_0^2) \bar{T}, T_\sigma tt = (\omega_3^2 + \omega_0^2) T_\sigma, T^*_{tt} = \omega_0^2 T,$$

and $\mu, \nu, \sigma = 1, 2, 3$. Finally, the corresponding Noether integrals are expressed as follows

$$\Phi_{1u2} = \bar{T} \frac{d}{dt} (S(\mu) + a_0 S(\nu)) - \bar{T},t (S(\mu) + a_0 S(\nu)),$$

$$\Phi_3 = T_\sigma \frac{d}{dt} S(\sigma) - T_{\sigma,t} S(\sigma),$$

$$\Phi_{a12} = T^* \frac{d}{dt} (a_0 S(\mu) + S(\nu)) - T^*_{t} (a_0 S(\mu) + S(\nu)).$$

In both last cases, from the admitted algebras of Lie symmetries it is easy to recognize that the gravitational field equations can be linearized. Indeed for the potential $V_A (\phi, \psi)$ under the coordinate transformation

$$x = \frac{1}{2} u (e^\phi (1 + \psi^2) + e^{-\phi}),$$

$$y = \frac{1}{2} u (e^\phi (1 - \psi^2) - e^{-\phi}),$$

$$z = u \psi e^\phi,$$

the field equations become

$$\ddot{x} - (\omega_1)^2 x = 0,$$

$$\ddot{y} - (\omega_2)^2 y = 0,$$

$$\ddot{z} - (\omega_3)^2 z = 0,$$

$$- \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 + \frac{1}{2} \dot{z}^2 + \frac{\omega_1^2}{2} x^2 - \frac{\omega_2^2}{2} y^2 - \frac{\omega_3^2}{2} z^2 = 0.$$

which is the three-dimensional “unharmonic-oscillator”.

On the other hand, for the potential $V_B (\phi, \psi)$, we perform the additional transformation

$$x = (w + v), \quad y = \frac{1}{a_0} (w - v), \quad z = z,$$

and the field equations are linearized as follows

$$\ddot{w} - (\mu^2 + \omega_0^2) w = 0,$$
\[ \ddot{v} + \frac{a_0^2 + 1}{a_0^2 - 1} \mu^2 w - \omega_0^2 \nu = 0, \quad (179) \]
\[ \ddot{z} - (\omega_3^2 + \omega_2^2) \nu = 0, \quad (180) \]
\[ 0 = \frac{1}{2} \left( \left( \frac{1}{a_0^2} - 1 \right) \dot{w}^2 + \left( \frac{1}{a_0^2} + 1 \right) dwdv + \left( \frac{1}{a_0^2} - 1 \right) dv^2 + \frac{1}{2} z^2 \right) + \]
\[ - \frac{2 \mu^2}{(a_0^2 - 1)} \dot{w}^2 - \frac{1}{2} \left( \omega_3^2 + \omega_2^2 \right) \dot{z}^2 + \frac{\omega_0^2}{2} \left( (w + v)^2 - \frac{1}{a_0^2} (w - v)^2 \right). \quad (181) \]

### 11.6 Galilean cosmology

The cubic Galilean cosmological model in a spatially flat FRW spacetime with comoving observers is defined by the Lagrangian [133]

\[ L(a, \dot{a}, \phi, \dot{\phi}) = 3a \ddot{a} - \frac{1}{2} a^3 \dot{\phi}^2 + a^3 V(\phi) + g(\phi)a^2 \dot{a} \dot{\phi}^3 - \frac{g'(\phi)}{6} a^3 \dot{\phi}^4. \quad (182) \]

From the symmetry condition we should determine two functions, \( V(\phi) \) and \( g(\phi) \). Indeed, we find that when \[ V_0 = 0 \]

\[ V(\phi) = V_0 e^{-\lambda \phi} \quad \text{and} \quad g(\phi) = g_0 e^{\lambda \phi} \quad (183) \]

Lagrangian (182) admits the Noether point symmetries

\[ X_1 = \partial_t, \quad X_2 = t \partial_t + \frac{a}{3} \partial_a + \frac{2}{\lambda} \partial_\phi, \quad (184) \]

which form the 2A_1 Lie algebra.

Noether symmetry \( X_1 \) provides as first integral the constraint equation, while \( X_2 \) gives the first integral

\[ \Phi_2 = - \left( 2a^2 \dot{a} - \frac{2}{\lambda} a^3 \dot{\phi} \right) + g_0 e^{\lambda \phi} a^3 \dot{\phi}^3 - \frac{6}{\lambda} g_0 a^2 e^{\lambda \phi} \dot{a} \dot{\phi}^2. \quad (185) \]

Furthermore, the same first integral exists in the limit in which \( V_0 = 0 \). In addition, we remark that when the universe is dominated by the potential of the scalar field then \( g(\phi) \to 0 \), and the model reduces to that of a minimally coupled scalar field.

### 12 Higher-Order Symmetries in Cosmology

In the previous section we presented classification of cosmological models based on point transformations. However, these are not the only cases where first integrals are used. Indeed one is possible to extend the classification scheme by applying non-point symmetries such as the contact symmetries.

In particular for Lagrangians of the form [133] it has been found that the vector field \( X = K_j^i (t, q^k) \dot{x}^i \partial_t \) is a contact symmetry for the Action Integral iff the following conditions are satisfied [10]

\[ K_{(ij;ik)} = 0, \quad (186) \]
\[ K_{ij,t} = 0, \quad f_{,t} = 0, \quad (187) \]
\[ K^{ij} V_j + f_{,i} = 0, \quad (188) \]
The latter conditions follow directly from the application of the weak Noether condition. From the symmetry condition (187) it follows that $K^i_j = K^j_i (q^k)$ and $f = f (q^k)$. Furthermore, the second-theorem of Noether provides the first integral

$$I = K_{ij} \dot{x}^i \dot{x}^j - f (x^i).$$

Condition (186) means that the second rank tensor $K^i_j (q^k)$ is a Killing tensor of order 2 of the metric $g_{ij}$.

Condition (188) is a constraint relating the potential with the Killing tensor $K_{ij}$ and the Noether function $f$. Application of contact symmetries in cosmological studies can be found in [74, 134–137].

From the following results we shall see that the classification scheme according to the admitted contact symmetries, include the dynamical systems which admit point symmetries, as also, provide new integrable dynamical systems. More specifically, as we let more freedom in the admitted symmetry vectors then we can find new integrable dynamical systems.

As far as concerns the contact symmetries, their application in classical mechanic is important and they can explain the Runge-Lenz vector for the Kepler problem, the Lewis invariant for the Ermakov-Pinney system and many others. As far as concerns the application in cosmology from previous studies [134–137] new integrable models and new analytic solutions for the evolution of the universe have been determined.

In the following we present the results for the classification of contact symmetries in scalar-field cosmology and $f (R)$-gravity.

### 12.1 Scalar-field cosmology from contact symmetries

In the polar coordinates (117) the Lagrangian of the field equations in scalar field cosmology becomes

$$L (r, \theta, \dot{r}, \dot{\theta}) = -\frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 - r^2 V (\theta)$$

while the Killing tensors of rank two for two-dimensional flat space in Cartesian coordinates $\{x, y\}$ are

$$K_{ij} = \begin{pmatrix}
  c_1 y^2 + 2c_2 y + c_3 & c_6 - c_1 y x - c_2 x - c_4 y \\
  c_6 - c_1 y x - c_2 x - c_4 y & c_1 x^2 + 2c_4 x + c_5
\end{pmatrix}.$$ (191)

Thus condition (188) provides the following cases [134]:

**Case A:** For the hyperbolic Potential $V (\theta) = c_1 + c_2 \cosh^2 \theta$ the field equations admit the contact symmetry

$$X_1 = - \left( \cosh^2 \theta \dot{r} + \frac{1}{2} r \sinh (2\theta) \dot{\theta} \right) \partial_r + \frac{1}{r} \left( \frac{1}{2} \sinh (2\theta) \dot{r} + r \sinh \theta \dot{\theta} \right) \partial_{\theta}$$

with corresponding Noether Integral

$$I_1 = \left( \cosh \theta \dot{r} + r \sinh \theta \dot{\theta} \right)^2 - 2r^2 (c_1 + c_2) \cosh^2 \theta.$$ (193)

In the special case where $c_2 = 3c_1$ the field equations admit the second contact symmetry

$$\bar{X} = - r^2 \sinh \theta \dot{\theta} \partial_r + \left( \sinh \theta \dot{r} + 2r \cosh \theta \dot{\theta} \right) \partial_{\theta}$$

with corresponding Noether Integral

$$\bar{I}_1 = \left( r^2 \sinh \theta \dot{r} + r^3 \cosh \theta \dot{\theta} \right)^2 + 2c_1 r^3 \cosh \theta \sinh^2 \theta,$$ (195)
Case B: For the potential $V(\theta) = c_1 (1 + 3 \cosh^2 \theta) + c_2 (3 \cosh \theta + \cosh^3 \theta)$ there exists the contact symmetry

$$I_2 = \left( r^2 \sinh \theta \dot{r} \dot{\theta} + r^3 \cosh \theta \dot{\theta}^2 \right) + r^3 \sinh^2 \theta \left( 2c_1 \cosh \theta + c_2 \left( 1 + \cosh^2 \theta \right) \right)$$ (196)

with Noether Integral

$$I_2 = \left( r^2 \sinh \theta \dot{r} \dot{\theta} + r^3 \cosh \theta \dot{\theta}^2 \right) + r^3 \sinh^2 \theta \left( 2c_1 \cosh \theta + c_2 \left( 1 + \cosh^2 \theta \right) \right)$$ (197)

Case C: For potential $V(\theta) = c_1 \left( 1 - 3 \sinh^2 \theta \right) + c_2 \left( 3 \sinh \theta - \sinh^3 \theta \right)$ the admitted contact symmetry is

$$\dot{X}_2 = -r^2 \cosh \theta \dot{\theta} \partial_r + \left( \cosh \theta \dot{r} + 2r \sinh \theta \dot{\theta} \right) \partial_\theta$$ (198)

with corresponding Noether Integral

$$\bar{I}_2 = \left[ \cosh \theta \dot{r} + r \sinh \theta \dot{\theta} \right] r^2 \dot{\theta} - r^3 \cosh^2 \theta \left( 2c_1 \sinh \theta - c_2 \left( 1 - \sinh^2 \theta \right) \right).$$ (199)

This potential is equivalent to case B under the transformation $\theta = \tilde{\theta} + i \frac{\pi}{2}$.

Case D: For $V(\theta) = c_1 + c_2 e^{2\theta}$ the admitted contact symmetry is

$$X_3 = -e^{2\theta} \left( \dot{r} + r \dot{\theta} \right) \partial_r + \frac{e^{2\theta}}{r} \left( \dot{r} + r \dot{\theta} \right) \partial_\theta$$ (200)

with corresponding Noether Integral

$$I_3 = e^{2\theta} \left( \left( \dot{r} + r \dot{\theta} \right) \right)^2 - 2r^2 c_1).$$ (201)

Moreover, when $c_1 = 0$ the dynamical system admits the additional contact symmetry

$$X_3 = -r^2 e^\theta \dot{\theta} \partial_r + e^\theta \left( \dot{r} + 2r \dot{\theta} \right) \partial_\theta$$ (202)

with corresponding Noether Integral

$$\bar{I}_3 = r^2 e^\theta \left( \dot{\theta} \right) + \frac{2}{3} c_2 r^3 e^{3\theta}.$$ (203)

Case E: Finally, for the potential

$$V(\theta) = c_1 e^{2\theta} + c_2 e^{3\theta}$$ (204)

the field equations admit the first integral

$$I_3 = r^2 e^\theta \left( \dot{\theta} \right) + r^3 e^{3\theta} \left( \frac{2}{3} c_1 + c_2 e^\theta \right)$$ (205)

generated by the contact symmetry. It is important to note that in all cases the results remain the same under the transformation $\theta \to -\theta$.

### 12.2 $f(R)$-gravity from contact symmetries

Without loss of generality we define $\phi = f'(R)$, where now $f(R)$-gravity can be written in its equivalent form as a Brans-Dicke scalar field cosmological model. Specifically the Lagrangian of the field equations is written equivalently as

$$L(a, \dot{a}, \phi, \dot{\phi}) = 6a\dot{\phi}\dot{a}^2 + 6a^2\dot{\phi} + a^3 V(\phi),$$ (206)
Table 6: Analytic forms of f(R) theory where the field equations admits contact symmetries

| Potential V(φ) | Function f(R) |
|----------------|---------------|
| V_I(φ)         | (R - V_1)^2   |
| V_{II}(φ)      | (R - V_1)^2   |
| V_{III}(φ)     | R^2 - V_1     |
| V_{IV}(φ), V_1 = 0 | R^4     |
| V_{V}(φ)       | R^2          |
| V_{V}(φ), V_1 = ±4V_2\sqrt{β} | ±√βR + R^4 |

where

\[ V(φ) = (f'R - f) \text{ or } V(f'(R)) = (f'R - f). \]  \hspace{1cm} (207)

The classification in terms of the contact symmetries provides the following five cases for the potential \( V(φ) \) and the corresponding first integrals.

Case A: For \( V_I(φ) = V_1φ + V_2φ^3 \), the field equations admit the quadratic first integral

\[ I_I = 3(φ\dot{a} + a\dot{φ})^2 - V_1 a^2 φ^2 \]  \hspace{1cm} (208)

generated by the KT \( K^{ij}_{22} \).

Case B: For \( V_{II}(φ) = V_1φ - V_2φ^7 \), the field equations admit the quadratic first integral

\[ I_{II} = 3a^4(φ\dot{a} - a\dot{φ})^2 + 4V_2a^6φ^{-6}, \]  \hspace{1cm} (209)

Case C: For \( V_{III}(φ) = V_1 - V_2φ^{-\frac{2}{3}} \), the field equations admit the quadratic first integral

\[ I_{III} = 6a^3\dot{a}\left(φ\dot{φ} - φ\dot{a}\right) - a^5\left(\frac{3}{5}V_1 - V_2φ^{-\frac{2}{3}}\right). \]  \hspace{1cm} (210)

Case D: For \( V_{IV}(φ) = V_1φ^3 + V_2φ^4 \), the field equations admit the quadratic first integral

\[ I_{IV} = 12a^2(φ^2\dot{φ}^2 - \dot{φ}^2a^2) + (aφ)^4(3V_1 + 4V_2φ). \]  \hspace{1cm} (211)

Case E: For \( V_{V}(φ) = V_1(φ^3 + βφ) + V_2(φ^4 + 6βφ^2 + β^2) \), the field equations admit the quadratic first integral

\[ I_V = 12a^2[(β - φ^2)a^2 + a^2\dot{φ}^2] + \]
\[ - a^4(β - φ^2)(V_1(β + 3φ^2) + 4V_2(3βφ + φ^3)) \]  \hspace{1cm} (212)

However in order to derive the function \( f(R) \) one has to solve the Clairaut equation [207]. For the above cases, Clairaut equation has a closed-form solution only for some particular forms of \( V(φ) \). The analytic forms of \( f(R) \) functions which admit contact symmetries are presented in Table 6.
13 Conclusions

In this work we discussed the dark energy problem using the classification of the cosmological models which are based on the FRW background for comoving observers using the tool of Noether symmetries. We discussed the definitions for the Lie and Noether symmetries (point and generalized) for conservative holonomic dynamical systems. Moreover, we established the relation of Lie and Noether symmetries with the properties of the underlying geometry for singular and regular dynamical systems. In particular for regular dynamical systems we found that the generators of Noether symmetries are the elements of the homothetic algebra of the mini superspace defined by the dynamical variables of the system whereas for the singular systems the generators of Noether symmetries are constructed by the CKVs of the mini superspace.

These geometric results have been used to develop a geometric scenario based on the admitted Noether symmetries of the mini superspace metric which leads to the classification scheme of the dark energy models. We demonstrated the application of this scenario to the most well known cosmological models including the modified theories of gravity and derived in many of them analytical cosmological solutions. This scenario is not limited to the case of cosmological models and can be applied to other areas of study of dynamical equations especially in the case of general holonomic systems.

However, a direct question is, how the integrable cosmological models determined by the application of symmetries describe the real universe? The integrable cosmological models are not sensitive on the initial conditions and they can be used as toy models for the study of various phases in the evolution of the universe. For instance, different potentials in scalar-field cosmology, or different functions in $f(R)$-gravity provides different cosmological evolution which can describe various eras in the evolution of the universe.

For instance, the $f_V(R)$ model which determined in Section (12.2), for some specific initial conditions and when $V_1 > 4V_2\sqrt{\beta}$, it has been found that the Hubble function is given by \( H_0 \),

\[
\left( \frac{H(a)}{H_0} \right)^2 \simeq \Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{\Lambda} \tag{213}
\]

which corresponds to a universe filed with radiation, dust fluid and cosmological constant. In a similar way it was found that the majority of the integrable models can describe various eras of special interests [103-117, 134-135, 138].

Various cosmological constraints for the integrable models which discussed before have been performed in [103-117, 134-135] and references therein. In particular, from the data analysis have been found that the integrable models describe well the late-time acceleration phase of the universe.

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