A central limit theorem for products of random matrices and GOE statistics for the Anderson model on long boxes

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Abstract

We consider products of random matrices that are small, independent identically distributed perturbations of a fixed matrix $T_0$. Focusing on the eigenvalues of $T_0$ of a particular size we obtain a limit to a SDE in a critical scaling. Previous results required $T_0$ to be a (conjugated) unitary matrix so it could not have eigenvalues of different modulus. From the result we can also obtain a limit SDE for the Markov process given by the action of the random products on the flag manifold. Applying the result to random Schrödinger operators we can improve some result by Valko and Virag showing GOE statistics for the rescaled eigenvalue process of a sequence of Anderson models on long boxes. In particular we solve a problem posed in their work.

Contents

1 Introduction and results
1.1 General SDE limits
1.2 Eigenvalue limits for random Schrödinger operators and the GOE
1.3 Correlations along different directions and SDE limit on the flag manifold
1.4 Jordan blocks and critical scaling

2 Evolution equation and estimates

3 Limit for the process $X_{\lambda,n}$

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4 Correlations between SDEs

4.1 Proof of Theorem 1.5

4.2 The action on the flag manifold

5 Application to random Schrödinger operators

5.1 Limit SDE

5.2 Limiting eigenvalue process

5.3 Limiting GOE statistics

A A limit theorem for Markov processes

1 Introduction and results

The goal of this paper is to study scaling limits of random matrix products

\[ T_{\lambda,n} T_{\lambda,n-1} \cdots T_{\lambda,1} \]

with \( \lambda \to 0 \) where the \( T_{\lambda,n} \) are perturbations of a fixed \( d \times d \) matrix of the form

\[ T_{\lambda,n} = T_0 + \lambda V_{\lambda,n} + \lambda^2 W_\lambda. \] (1.1)

Here, for every \( \lambda \), \( V_{\lambda,n} \) is a family of independent, identically distributed random matrices with \( \mathbb{E}(V_{\lambda,n}) = 0 \) and \( W_\lambda \) is a deterministic matrix, both of order one. In the simplest case, \( d = 1, T_0 = 1, W_\lambda = 0, \) and \( V_{\lambda,n} = V_n \) are independent centered random variables with variance one. Then, the classical Donsker’s central limit theorem applied to the logarithm of (1) shows that the product (for here denoted \( X_{\lambda,n} = T_{\lambda,n} T_{\lambda,n-1} \cdots T_{\lambda,1} \)) satisfies

\[ (X_{\lambda,t/\lambda^2}, t \geq 0) \Rightarrow (e^{B(t)-t/2}, t \geq 0) \]

as \( \lambda \to 0 \), where \( B(t) \) is a standard Brownian motion. Compared to the simplest case, the general case has extra interesting features.

- The matrix \( T_0 \) can have eigenvalues of different absolute values, so the product can grow exponentially at different rates in different directions.

- The matrix \( T_0 \) can have complex eigenvalues of the same absolute value that act as rotations; this can produce an averaging effect for the added drift and noise terms.

The main question that we resolve in this paper is the following.

**Question.** If the matrix \( T_0 \) have an eigenvalue of large absolute value, can one still understand the fine evolution of the product in the directions belonging to smaller eigenvalues?
Matrix products of this kind are used in the study of quasi-1-dimensional random Schrödinger operators, and the large eigenvalues are related to so-called hyperbolic channels. Indeed, the main motivating example is this case, which we will introduce in Section 1.2. When $T_0$ is (the multiple of) a unitary matrix this type of result has been established in that context, \[BR, VV1, VV2\] (see also \[SS2\]) and the limiting process is described by a stochastic differential equation (SDE). In \[KVV, VV1\] the SDE limit was used to study the limiting eigenvalue statistics of such random Schrödinger operators in a critical scaling $\lambda^2 n = t$. We can extend this result and obtain a limit for the rescaled eigenvalue process in the presence of hyperbolic channels as well (cf. Theorem 1.3). In particular, we solve Problem 3 raised in \[VV1\] and obtain limiting GOE statistics for the Anderson model on sequences of long boxes (cf. Theorem 1.2) with appropriate scalings. We essentially reduce the proof to a situation where it is left to analyze the same family of SDEs as in \[VV1\]. Deriving the GOE statistics then relies on the work of Erdős, Schlein, Yau and Yin \[ESYY, EYY\], but we do not repeat these steps that are done in \[VV1\]. The results are stated in Section 1.2.

Random matrix ensembles such as the Gaussian Orthogonal Ensemble were introduced by Wigner \[Wi\] to model the observed repulsion between eigenvalues in large nuclei. The local statistics is given by the Sine$_1$ kernel, see e.g. \[Me\]. This type of repulsion statistics is expected for many randomly disordered systems of the same symmetry class (time reversal symmetry), that have delocalized eigenfunctions. This is referred to as universality. Most models with rigorously proved universal bulk behavior are themselves ensembles of random matrices, e.g. \[DG, ESY, Joh, TV\]. Recently, T. Shcherbina proved universal GUE statistics (Gaussian Unitary Ensemble) for random block band matrix ensembles that in some sense interpolate between the classical matrix ensembles and Anderson models \[Shc\].

The Anderson model was introduced by P. W. Anderson to describe disordered media like randomly doped semi-conductors \[And\]. It is given by the Laplacian and a random independent identically distributed potential and has significantly less randomness than the matrix ensemble models. For large disorder and at the edge of the spectrum, the Anderson model in $\mathbb{Z}^d$ or $\mathbb{R}^d$ localizes \[FS, DLS, SW, CKM, AM, Aiz, Kio\] and has Poisson statistics \[Mi, Wa, CGK, GK\]. For small disorder in the bulk of the spectrum, localization and Poisson statistics appears in one and quasi-one dimensional systems \[GMP, KnS, CKM, Lac, KLS\] (except if prevented by a symmetry \[SS3\]) and is expected (but not proved) in 2 dimensions. Delocalization for the Anderson model was first rigorously proved on regular trees (Bethe lattices) \[KI\] and had been extended to several infinite-dimensional tree-like graphs \[KI, ASW, FHS, AW, KLM, FHH, KS, Sa2, Sa3, Sha\]. For 3 and higher dimensions one expects delocalized eigenfunctions (absolutely continuous spectrum) for small disorder and the eigenvalue statistics of large boxes should approximate GOE by universality. However, proving any of these statements in $\mathbb{Z}^d$ or $\mathbb{R}^d$ for $d \geq 3$ remains a great mathematical challenge.
In Theorem 1.3 we consider the limiting eigenvalue process of quasi-one dimensional models in a critical scaling limit \( \lambda^2 n = t = \text{constant} \) (at bandedges one has a different scaling as mentioned in Section 1.4). In this scaling limit, localization effects and Poisson statistics are not seen and the description through an SDE arises. As mentioned above, previous works \[VV1, BR\] had to modify the original Anderson model to avoid hyperbolic channels. We obtain that the hyperbolic channels only shift the eigenvalues but do not affect the local statistics, in particular we solve Problem 3 raised in \[VV1\]. In fact, fixing the width and base energy, the local eigenvalue statistics only depends on the number of so called elliptic channels. This can be seen as some universality statement by itself. Increasing the number of elliptic channels and choosing appropriate sequences of models, the GOE statistics arises.

As a byproduct of this work we solve some conjecture from \[Sa1\] showing that there is an SDE limit for the reduced transfer matrices in the presence of hyperbolic channels.

The papers \[BR, VV1, VV2\] are restricted to the subset of the important cases where all eigenvalues of \( T_0 \) had the same absolute value (and no Jordan blocks). The novelty of this work is to handle eigenvalues of different absolute value for \( T_0 \), the application to Schrödinger operators comes from applying Theorem 1.1 to the transfer matrices and following some calculations until we arrive at precisely the same family of SDEs as in \[VV1\].

Let us briefly explain why this is not a trivial extension. If \( T_0 \) (or \( AT_0A^{-1} \) for some matrix \( A \)) is unitary one simply has to remove the free evolution from the random products. To illustrate this, let for now \( X_{\lambda,n} = T_{\lambda,n}T_{\lambda,n-1} \cdots T_{\lambda,1} \). Then, \( T_0^{-n}X_{\lambda,n} = (1 + \lambda T_0^{-n}V_{\lambda,n}T_0^{n-1}) + \lambda^2 T_0^{-n}W_{\lambda,n}T_0^{n-1} \). The conjugations like \( T_0^{-n}(V_{\lambda,n}T_0^{-1})T_0^n \) simply lead to an averaging effect over the compact group generated by the unitary \( T_0 \) in the limit for the drift and diffusion terms. Adopting techniques by Strook and Varadhan \[SV\] and Ethier and Kurtz \[EK\] to this situation (cf. Proposition 23 in \[VV2\]) one directly obtains an SDE limit for \( T_0^{-n}X_{\lambda,n} \) in the scaling \( \lambda^2 n = t \). If \( T_0 \) has eigenvalues of different sizes then generically some entries of \( T_0^{-n}W_{\lambda,n}T_0^{-1} \) and the variance of some entries of \( T_0^nV_{\lambda,n}T_0^{n-1} \) will grow exponentially in \( n \). This destroys any hope of a limiting process. Now, instead one may then consider a process \( U^{-n}X_{\lambda,n} \) where \( U \) is a unitary just counteracting the fast rotations. But then one still has different directions growing at different exponential rates even for the free evolution, and simply projecting to some subspace, \( PU^{-n}T_0^{-n}X_{\lambda,n} \), does not work either!

The trick lies in finding a projection which cuts off the exponential growth of the free evolution and does not screw up the convergence of the random evolution to some drift and diffusion terms. The correct way to handle the exponential growing directions is choosing a Schur complement. The exponentially decreasing directions will tend to zero and not matter and the directions of size 1 will lead to a limit. The exponential growing directions have some non-trivial effect and lead to an additional drift term. As the Schur complement itself is not a Markov process, it will be better to consider it as part of a quotient of \( X_{\lambda,n} \) modulo a certain subgroup of GL(\( d \)). Then one still needs several estimates to handle the appearing inverses in the Schur complement and the error...
terms before one can apply some modification of Proposition 23 in [VV2] (cf. Proposition A.1).

Although we cannot take an SDE limit of the entire matrix as indicated above, it will be possible to describe the limit of its action on Grassmannians and flag manifolds. The limit processes live in certain submanifolds that are stable under the free, non-random dynamics of $T_0$. This result is related to the numerical calculations in [RS] who considered the action of the transfer matrices on the so called Lagrangian Planes, or Lagrangian Grassmannians (which is some invariant subspace of a Grassmannian). The limiting submanifold corresponds to the ‘freezing’ of some phases related to the hyperbolic channels. In the scaling limit, only a motion of the part corresponding to the so called elliptic channels can be seen and it is described by a SDE.

We will also study the case of non-diagonalizable Jordan blocks. These can be dealt with by a $\lambda$-dependent basis change which leads to a different critical scaling, see Section 1.4. In the Schrödinger case such Jordan blocks appear at band-edges and we give an example for a Jordan block of size $2d$ for general $d$.

First, in Section 1.1 we will explain the main theorem which is a limit SDE for products of random matrices as in (1.1). The proof will be done in Sections 2 and 3. In section 1.2 we will explain the consequences for random Schrödinger operators, our main application. Further details for the proofs are given in Section 5. Section 1.3 will explain the correlations for SDE limits corresponding to different sizes of eigenvalues of $T_0$ and the related limit for the action on the flag manifold, proofs are found in Section 4. Section 1.4 will explain how to handle Jordan blocks of $T_0$ and give some example of a finite range random Schrödinger operator where a Jordan block of size $2d$ appears at a bandedge.

1.1 General SDE limits

Without loss of generality we focus on the eigenvalues with absolute value 1 and assume that $T_0$ has no Jordan blocks for eigenvalues of size 1. Next, we conjugate the matrices $T_{\lambda,n}$ to get $T_0$ in Jordan form. We may write it as a block diagonal matrix of dimension $d_0 + d_1 + d_2$ of the form

$$T_0 = \begin{pmatrix} \Gamma_0 & U \\ U^* & \Gamma_2^{-1} \end{pmatrix}$$

(1.2)

where $U$ is a unitary, and $\Gamma_0$ and $\Gamma_2$ have spectral radius smaller than 1. The block $\Gamma_0$ corresponds to the exponential decaying directions and the block $\Gamma_2^{-1}$ to the exponential growing directions of $T_0$.

The only way the matrix product $T_{\lambda,n} \cdots T_{\lambda,1}$ can have a continuous limiting evolution is if we
compensate for the macroscopic rotations given by \( U \) (as in \([BR, VV1, VV2]\)). Hence define

\[
X_{\lambda,n} := R^{-n} T_{\lambda,n} T_{\lambda,n-1} \cdots T_{\lambda,1} X_0 \quad \text{where} \quad R = \begin{pmatrix} \mathbf{1}_{d_0} & U \\ \mathbf{0} & \mathbf{1}_{d_2} \end{pmatrix}
\] (1.3)

where \( X_0 \) is some initial condition and \( \mathbf{1}_d \) is the identity matrix of dimension \( d \).

In most of the following calculations we will use a subdivision in blocks of size \( d_0 + d_1 \) and \( d_2 \).

Let us define the Schur complement \( X_{\lambda,n} \) of size \((d_0 + d_1) \times (d_0 + d_1)\) by

\[
X_{\lambda,n} = A_{\lambda,n} - B_{\lambda,n} D_{\lambda,n}^{-1} C_{\lambda,n}, \quad \text{where} \quad X_{\lambda,n} = \begin{pmatrix} A_{\lambda,n} & B_{\lambda,n} \\ C_{\lambda,n} & D_{\lambda,n} \end{pmatrix}.
\] (1.4)

If \( X_{\lambda,n} \) and \( D_{\lambda,n} \) are both invertible, then

\[
X_{\lambda,n} = (P^* X_{\lambda,n} P)^{-1}
\] (1.5)

where \( P \) is the projection to the first \( d_0 + d_1 \) coordinates. Note that invertibility of \( D_{\lambda,n} \) is required to define \( X_{\lambda,n} \). Therefore, we demand the starting value \( D_0 \) to be invertible, where

\[
X_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \quad \text{and we define} \quad X_0 = A_0 - B_0 D_0^{-1} C_0.
\]

The first important observation, explained in Section 2, is that the pair

\[
(X_{\lambda,n}, Z_{\lambda,n}) \quad \text{where} \quad Z_{\lambda,n} = B_{\lambda,n} D_{\lambda,n}^{-1}
\] (1.6)

is a Markov process. Therefore, it will be more convenient to study this pair.

We need the following assumptions.

**Assumptions.** We assume that for some constants \( \epsilon > 0, \lambda_0 > 0 \) one has

\[
\sup_{0 < \lambda < \lambda_0} \mathbb{E}(\| V_{\lambda,n} \|^6 + \epsilon) < \infty.
\] (1.7)

Furthermore we assume that the limits of first and second moments

\[
\lim_{\lambda \to 0} W_\lambda, \quad \lim_{\lambda \to 0} \mathbb{E}(V_{\lambda,n} i,j (V_{\lambda,n}) k,l), \quad \lim_{\lambda \to 0} \mathbb{E}((V_{\lambda,n}^*) i,j (V_{\lambda,n}) t,k) \quad \text{exist.}
\] (1.8)

In order to state the main theorem, we need to subdivide \( V_{\lambda,n}, W_\lambda \) in blocks of sizes \( d_0, d_1, d_2 \). We denote the \( d_j \times d_k \) blocks by

\[
V_{\lambda,jk}, \quad \text{and} \quad W_{\lambda,jk} \quad \text{respectively.}
\] (1.9)

The covariances of the \( d_1 \times d_1 \) block \( V_{\lambda,11} \) will be important. A useful way to encode covariances of centered matrix-valued random variables \( A \) and \( B \) is to consider the matrix-valued linear functions
$M \mapsto \mathbb{E}(A^\top MB)$ and $M \mapsto \mathbb{E}(A^* MB)$. Choosing matrices $M$ with one entry 1 and all other entries zero one can read off $\mathbb{E}(A_{ij} B_{kl})$ and $\mathbb{E}(A_{ij}^* B_{kl})$ directly. Let us therefore define
\[ h(M) := \lim_{\lambda \to 0} \mathbb{E}(V_{\lambda,11}^\top MV_{\lambda,11}) , \quad \hat{h}(M) := \lim_{\lambda \to 0} \mathbb{E}(V_{\lambda,11}^* MV_{\lambda,11}) . \] (1.10)
Furthermore the lowest order drift term of the limit will come from the lowest order Schur-complement and hence contain some influence from the exponentially growing directions. Therefore, let
\[ W := \lim_{\lambda \to 0} W_{\lambda,11} - \mathbb{E}(V_{\lambda,12} \Gamma_2 V_{\lambda,21}) . \] (1.11)
By the assumption (1.8) above these limits exist.

**Theorem 1.1.** Let the assumptions (1.7) and (1.8) stand. Then, for $t > 0$ we have convergence in law, $Z_{\sqrt{n}, [tn]} \Rightarrow 0$ and
\[ X_{\sqrt{n}, \lfloor tn \rfloor} \Rightarrow X_t = \begin{pmatrix} 0_{d_0} & \Lambda_t \end{pmatrix} X_0 \quad \text{for } n \to \infty . \]
$\Lambda_t$ is a $d_1 \times d_1$ matrix valued process and the solution of
\[ d\Lambda_t = V\Lambda_t \, dt + dB_t \Lambda_t , \quad \Lambda_0 = 1 . \]
and $B_t$ is a complex matrix Brownian motion (i.e. $B_t$ is Gaussian) with covariances
\[ \mathbb{E}(B_t^\top MB_t) = g(M)t , \quad \mathbb{E}(B_t^* MB_t) = \hat{g}(M)t \] (1.12)
where
\[ V = \int_{\langle U \rangle} uWU^* u^\ast \, du \] (1.13)
\[ g(M) = \int_{\langle U \rangle} \pi U h(u^\top MU) U^* u^\ast \, du \] (1.14)
\[ \hat{g}(M) = \int_{\langle U \rangle} uU \hat{h}(u^* MU) U^* u^\ast \, du . \] (1.15)
Here, $\langle U \rangle$ denotes the compact abelian group generated by the unitary $U$, i.e. the closure of the set of all powers of $U$, and $du$ denotes the Haar measure on $\langle U \rangle$.

**Remark.** (i) The analogous theorem in the situation $d_2 = 0$ (no exponential growing directions) holds. In this case the matrices $B_{\lambda,n}, C_{\lambda,n}$ and $D_{\lambda,n}$ do not exist and one simply has $X_{\lambda,n} = A_{\lambda,n} = X_{\lambda,n}$. For this case one can actually simplify some of the estimates done for the proof, as one does not need to work with the process $B_{\lambda,n} D_{\lambda,n}^{-1}$ and no inverse is required.

(ii) In the case where $d_0 = 0$, i.e. no exponential decaying directions, the Theorem also works fine. In this case one simply has $X_t = \Lambda_t X_{11}$.
(iii) The Theorem does not hold for $t = 0$ and indeed it looks contradictory for small $t$. However, the exponentially decaying directions go to zero exponentially fast so that one obtains

$$X_{\sqrt{n} \{nt\}} \Rightarrow \begin{pmatrix} 0 \\ 1_{d_1} \end{pmatrix} X_0$$

for sufficiently small $\alpha$ which gives the initial conditions for the limiting process.

(iv) When defining the process $X_{\lambda,n}$ one may want to subtract some of the oscillating terms in the growing and decaying directions as well, i.e. one may want to replace $R$ in (1.3) by a unitary of the form $\hat{R} = \begin{pmatrix} U_0 & U_2 \\ U_3 & U_4 \end{pmatrix}$ written in blocks of sizes $d_0, d_1, d_2$, respectively. Then let

$$\hat{X}_{\lambda,n} = \hat{R}^{-1} R^n X_{\lambda,n} = \begin{pmatrix} \hat{A}_{\lambda,n} & \hat{B}_{\lambda,n} \\ \hat{C}_{\lambda,n} & \hat{D}_{\lambda,n} \end{pmatrix}$$

and define the corresponding Schur complement $\hat{X}_{\lambda,n} = \hat{A}_{\lambda,n} - \hat{B}_{\lambda,n} \hat{D}_{\lambda,n}^{-1} \hat{C}_{\lambda,n}$ as well as $\hat{Z}_{\lambda,n} = \hat{B}_{\lambda,n} \hat{D}_{\lambda,n}^{-1}$. Simple algebra shows that

$$\hat{X}_{\lambda,n} = \begin{pmatrix} U_0^{-n} & 1 \\ 1 & 1 \end{pmatrix} X_{\lambda,n}, \quad \hat{Z}_{\lambda,n} = \begin{pmatrix} U_0^{-n} & 1 \\ 1 & 1 \end{pmatrix} Z_{\lambda,n} U_2^n.$$

Hence, it is easy to see that for $n \to \infty$,

$$\hat{Z}_{\sqrt{\frac{n}{nt}}} \Rightarrow 0, \quad \hat{X}_{\sqrt{\frac{n}{nt}}} \Rightarrow X_t$$

where $X_t$ is the exact same process as in Theorem 1.1.

In Section 2 we will develop the evolution equations for the process $(X_{\lambda,n} Z_{\lambda,n})$, together with some crucial estimates. In Section 3 we will then obtain the limiting stochastic differential equations as in Theorem 1.1 using Proposition A.1 in Appendix A for convergence of Markov processes to SDE limits. The reader interested in the proofs can continue with Section 2 at this point.

1.2 Eigenvalue limits for random Schrödinger operators and the GOE

Let $Z_{n,d}$ be the adjacency matrix of the $n \times d$ grid, and let $V$ be a diagonal matrix with i.i.d. random entries of the same dimension. A fundamental question in the theory of random Schrödinger operators is how the eigenvalues of

$$Z_{n,d} + \lambda V$$

are distributed. Predictions from the physics literature suggest that in certain scaling regimes that correspond to the delocalized regime, random matrix behavior should appear. More precisely, the random set of eigenvalues, in a window centered at some energy $E$ and scaled properly, should converge to the Sine$_1$ point process. The latter process is the large-$n$ limit of the random set of eigenvalues of the $n \times n$ Gaussian orthogonal ensemble near 0.
A version of such predictions were proved rigorously \[\text{[VV1]}\] for subsequences of \(n_i \gg d_i \to \infty\), \(\lambda_i^2 n_i \to 0\) but only near energies \(E_i\) tending to zero. In a modified model where the edges in the \(d\) direction get weight \(r < 1\) the proof of \[\text{[VV1]}\] works for almost all energies in the range \((-2 + 2r, 2 - 2r)\). Proving such claims for almost all energies of the original model \[\text{(1.16)}\] presented a challenge, the main motivation for the present paper. For better comparison with \[\text{[VV1]}\] let us re-introduce the weight \(r\). It is natural to think of operators like \[\text{(1.16)}\] as acting on a sequence \(\psi = (\psi_1, \ldots, \psi_n)\) of \(d\)-vectors. So given the weight \(r\), let us define the \(nd \times nd\) matrix \(H_{\lambda,n,d}\) by

\[
(H_{\lambda,n,d}\psi)_k = \psi_{k+1} + \psi_{k-1} + (rZ_d + \lambda V_k)\psi_k
\]

with the notational convention that \(\psi_0 = \psi_{n+1} = 0\). Here, \(Z_d\) is the adjacency matrix of the connected graph of a path with \(d\) vertices and the \(V_k\) are i.i.d. real diagonal matrices, i.e.,

\[
Z_d = \begin{pmatrix} 0 & 1 \\ 1 & \cdots & \cdots \\ \cdots & \cdots & 1 \\ 1 & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}
\]

with

\[
\mathbb{E}(v_j) = 0, \quad \mathbb{E}(|v_j|^{6+\varepsilon}) < \infty, \quad \mathbb{E}(v_i v_j) = \delta_{ij}.
\]

Then we obtain the following:

**Theorem 1.2.** For any fixed \(r > 0\) and almost every energy \(E \in (-2 - 2r, 2 + 2r)\) there exist sequences \(n_k \gg d_k \to \infty, \sigma_k^2 := \lambda_k^2 n_k \to 0\), such that the process of eigenvalues of

\[n_k (H_{\lambda_k,n_k,d_k} - E)\]

converges to the Sine\(_1\) process. In particular the level statistics corresponds to GOE statistics in this limit.

Theorem \[\text{1.2}\] resolves Problem 3 posed in \[\text{[VV1]}\]. There one has \(r < 1\) and \(E \in (-2 + 2r, 2 - 2r)\) or \(r = 1\) and a sequence of energies converging to 0. (Note that this interval is smaller than the one in Theorem \[\text{1.2}\] and in fact empty for \(r \geq 1\).) Theorem \[\text{1.2}\] applies to the exact Anderson model \(r = 1\) with any fixed energy in the interior \((-4, 4)\) of the spectrum of the discrete two-dimensional Laplacian. It also applies in the case \(r > 1\). This is because hyperbolic channels can now be handled for the SDE limit. The exact definition of 'elliptic' and 'hyperbolic' channels will be given below. Overcoming this difficulty was the main motivation for this work.

Essentially, only the elliptic channels play a role in the eigenvalue process limit. It is thus important to have a sequence with a growing number of elliptic channels going to infinity. Indeed, one can obtain GOE statistics even for a sequence of energies \(E_k\) approaching the edge of the spectrum \(|E| = 2 + 2r\). For this, one needs that the sequence \(d_k\) grows fast enough, such that the number of elliptic channels at energy \(E_k\) grows.
In the sequel we will study the limiting eigenvalue process for $n \to \infty$ with $\lambda^2 n$ constant and $d$ fixed for more general random $nd \times nd$ matrices given by

$$(H_{\lambda,n} \psi)_k = \psi_{k+1} + \psi_{k-1} + (A + \lambda V_k) \psi_k.$$ (1.19)

Here, $A$ is a general Hermitian matrix, and the $V_k$ are general i.i.d. Hermitian matrices. We dropped the index $d$ now as $d$ will be fixed from now on and sometimes we may also drop the index $n$. Moreover, for simplicity, we can assume that $A$ is diagonal; indeed, this can be achieved by the change of basis $\psi_n \mapsto O^* \psi_n$ where $O^* AO$ diagonalize $A$ (and replaces $V_n$ by $O^* V_n O$).

The eigenvalue equation $H_{\lambda} \psi = E \psi$ is a recursion that can be written in the matrix form as follows.

$$
\begin{pmatrix}
\psi_{k+1} \\
\psi_k \\
\end{pmatrix} =
T_k
\begin{pmatrix}
\psi_k \\
\psi_{k-1} \\
\end{pmatrix}
\text{ where } T_k = T_{\lambda,k}^E =
\begin{pmatrix}
E 1 - A - \lambda V_k & -1 \\
1 & 0 \\
\end{pmatrix}.
$$ (1.20)

The $T_{\lambda,k}^E$ are called transfer matrices. For the limiting eigenvalue process we obtain the following:

**Theorem 1.3.** Let $E$ be an energy such that the unperturbed transfer matrix $T_{0,k}^E$ (which is independent of $k$) is diagonalizable and has $2d_e > 0$ eigenvalues of absolute value 1. (In the notions introduced below this means we have $d_e > 0$ elliptic, $d_h = d - d_e \geq 0$ hyperbolic and no parabolic channels.) Consider the process $\mathcal{E}_{\sigma,n}$ of eigenvalues of $n(H_{\frac{E}{\sqrt{n}},n} - E)$ and let $n_k$ be an increasing sequence such that $Z_{n_k}^* \to Z^*$ for $k \to \infty$ with $Z$ being the unitary, diagonal $d_e \times d_e$ matrix defined in (1.21). Then, $\mathcal{E}_{\sigma,n_k}$ converges to the zero process of the determinant of a $d_e \times d_e$ matrix,

$$
\mathcal{E}_{\sigma,n_k} \implies \text{zeros } \det
\begin{pmatrix}
\bar{Z}_* & Z_* \\
1_{d_e} & -1_{d_e} \\
\end{pmatrix}
\Lambda_{\sigma,\epsilon,1}
$$

where the $2d_e \times 2d_e$ matrix process $\Lambda_t = \Lambda_{\sigma,\epsilon,t}$ satisfies some SDE in $t$ (with $\sigma, \epsilon$ fixed) as given in Proposition 1.4.

Now, $E$ is an eigenvalue of $H_{\lambda,n}$ if there is a nonzero solution $(\psi_1, \psi_n)$ to

$$
\begin{pmatrix}
0 \\
\psi_n \\
\end{pmatrix} = T_n \cdots T_1
\begin{pmatrix}
\psi_1 \\
0 \\
\end{pmatrix}
$$
equivalently, when the determinant of the top left $d \times d$ block of $T_n \cdots T_1$ vanishes. So we can study the eigenvalue equation through the products

$$
T_{[1,k]} = T_k \cdots T_1
$$

which are the focus of Theorem 1.3. The matrices $T_k$ satisfy

$$
T^* J T = J \quad \text{where } J =
\begin{pmatrix}
0 & 1_d \\
-1_d & 0 \\
\end{pmatrix},
$$

10
the definition of elements of the hermitian symplectic group $\text{HSp}(2d)$. In particular, they are all invertible. The $T_k$ are all perturbations of the noiseless matrix

$$T_* := T_{0,1}^E.$$ 

This matrix is also block diagonal with $d$ blocks of size 2, and the eigenvalues of $T_{0,1}^E$ exactly the $2d$ solutions of the $d$ quadratics

$$z + z^{-1} = E - a_j, \quad a_j \text{ is an eigenvalue of } A.$$

so the solutions are on the real line or on the complex unit circle, depending on whether $|E - a_j|$ is less more more than two. We call the corresponding generalized eigenspaces of $T_* = T_{0,1}^E$ elliptic ($< 2$), parabolic ($= 2$) and hyperbolic ($> 2$) channels. Elliptic and hyperbolic channels correspond to two-dimensional eigenspaces, while parabolic channels correspond to a size 2 Jordan block. Traditionally, this notation refers to the solutions of the noiseless ($\lambda = 0$) recursion that are supported in these subspaces for every coordinate $\psi_n$.

Pick an energy $E$, such that there are no parabolic channels and at least one elliptic channel.

Suppose that $A$ is diagonalized so that $|E - a_j| > 2$ for $j = 1, \ldots, d_h$ and $|E - a_j| < 2$ for $j > d_h$. Correspondingly, we define the hyperbolic eigenvalues $\gamma_j$ and elliptic eigenvalues $z_j$ of $T_*$ by

$$\gamma_j + \gamma_j^{-1} = E - a_j, \quad |\gamma_j| < 1, \quad \text{for } j = 1, \ldots, d_h$$

$$z_j + z_j^{-1} = E - a_j + d_h, \quad |z_j| = 1, \text{Im}(z_j) > 0, \quad \text{for } j = 1 \ldots, d_e = d - d_h.$$

Furthermore we define the diagonal matrices

$$\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_{d_h}), \quad Z = \text{diag}(z_1, \ldots, z_{d_e}). \quad (1.21)$$

In order to complete the description of the limiting eigenvalue process, we need to consider a family of limiting SDE by varying the energy in the correct scaling. More precisely, define the $2d_e \times 2d_e$ unitary matrix $U$ and the $2d \times 2d$ matrix $Q$ by

$$U = \begin{pmatrix} \bar{Z} \\ Z \end{pmatrix}, \quad Q = \begin{pmatrix} \Gamma & \bar{Z} & Z & \Gamma^{-1} \\ 1_{d_h} & 1_{d_e} & 1_{d_h} & 1_{d_e} \end{pmatrix}. \quad (1.22)$$

so that $Q$ diagonalizes $T_*$ to a form as in (1.2) that is used for Theorem 1.1

$$T_* := Q^{-1} T_* Q = \begin{pmatrix} \Gamma & U \\ \Gamma^{-1} \end{pmatrix}.$$

Furthermore, let

$$T_k = T_{\varepsilon, \lambda, k} := Q^{-1} T_{\lambda, k}^E + 2 \varepsilon Q = T_* + \lambda V_k + \lambda^2 \varepsilon W, \quad T_{[1,k]} = T_{\varepsilon, \lambda, [1,k]} := T_k \cdots T_1 \quad (1.23)$$
with
\[ V_k = Q^{-1} \begin{pmatrix} -V_k & 0 \\ 0 & 0 \end{pmatrix} Q, \quad W = Q^{-1} \begin{pmatrix} 1_d & 0 \\ 0 & 0 \end{pmatrix} Q. \] (1.24)

The scaling \( \varepsilon \lambda^2 \) means that a unit interval of \( \varepsilon \) should contain a constant order of eigenvalues. In order to get limiting SDEs we consider a Schur complement and a projection of it as before, thus define the \( 2d_e \times 2d_e \) matrices
\[ \hat{T}_{\varepsilon, \lambda, n} = P_1^* \left( \mathcal{P}_{\leq 1}^{-1} \right)^{-1} P_1 \] with \( \mathcal{P}_{\leq 1} = \begin{pmatrix} 1_{d_h + 2d_e} & 0 \\ 0_{d_h \times (d_h + 2d_e)} & 1_{2d_e} \end{pmatrix} \) and \( P_1 = \begin{pmatrix} 0_{d_h \times 2d_e} & 1_{2d_e} \end{pmatrix} \).

Then by Theorem 1.1 we obtain the correlated family (parameters \( \sigma, \varepsilon \)) of limiting processes
\[ U_{\varepsilon, 1}^{\lfloor tn \rfloor} \hat{T}_{\varepsilon, 1/\sqrt{n}, \lfloor tn \rfloor} \Rightarrow \Lambda_{\varepsilon, t} \] for \( n \to \infty \). (1.25)

These limiting processes correspond to the ones in Theorem 1.3 for \( \sigma = 1 \). For general \( \sigma \) one has to change \( V_K \) by \( \sigma V_k \) in (1.23).

**Remark.** For \( \varepsilon = 0 \), up to some conjugation, the matrix \( \hat{T}_{0, \lambda, n} \) corresponds to the reduced transfer matrix as introduced in [Sa1] for the scattering of a block described by \( H_{\lambda} \) of a finite length \( n \) inserted into a cable described by \( H_0 \) of infinite length (‘\( n = \infty \’ \)). Thus we obtain that in the limit \( \lambda^2 n = \text{const.}, n \to \infty \), the process of the reduced transfer matrix as defined in [Sa1] is described by a SDE, proving Conjecture 1 in [Sa1].

In order to express the limit SDEs explicitly we need to split the potential \( V_1 \) into the hyperbolic and elliptic parts, i.e. let
\[ V_1 = \begin{pmatrix} V_h & V_{he} \\ V_{he}^* & V_e \end{pmatrix} \text{ where } \ V_h \in \text{Mat}(d_h \times d_h), \ V_e \in \text{Mat}(d_e \times d_e). \]

Moreover, define
\[ Q = \int_{(Z)} z \mathbb{E}(V_{he}^* (\Gamma^{-1} - \Gamma)^{-1} V_{he}) \bar{z} \, dz, \quad S = \begin{pmatrix} (\bar{Z} - Z)^{-1} & 0 \\ 0 & (\bar{Z} - Z)^{-1} \end{pmatrix} \] (1.26)
where \( dz \) denotes the Haar measure on the compact abelian group \( (Z) \) generated by the diagonal, unitary matrix \( Z \). As we will see, \( S_h \) will give rise to a drift term coming from the hyperbolic channels. In fact, this is the only influence of the hyperbolic channels for the limit process. Moreover, to simplify expressions, we will be interested in one specific case.

**Definition.** We say that the matrix \( Z = \text{diag}(z_1, \ldots, z_{d_e}) \) with \( |z_j| = 1 \), \( \text{Im}(z_j) > 0 \) is chaotic, if all of the following apply for all \( i, j, k, l \in \{1, \ldots, d_e\} \),

\[ z_i z_j z_k z_l \neq 1, \quad \bar{z}_i \bar{z}_j z_k z_l \neq 1 \]
\[ \bar{z}_i \bar{z}_j z_k z_l \neq 1 \text{ unless } \{i, j\} = \{k, l\}. \]
Proposition 1.4. (i) The family of processes $\Lambda_{\sigma,\epsilon,t}$ as in equation (1.25) or Theorem 1.3 satisfy SDEs of the form

$$d\Lambda_{\sigma,\epsilon,t} = S \left( \varepsilon 1 - \sigma^2 Q - \varepsilon 1 + \sigma^2 Q \right) \Lambda_{\sigma,\epsilon,t} dt + \sigma^2 S \left( dA_t - dB_t \right) \Lambda_{\sigma,\epsilon,t}$$

with $\Lambda_{\sigma,\epsilon,0} = 1$ and $\sigma, \epsilon$ fixed, where $A_t, B_t, C_t$ are jointly Gaussian complex-valued $d_n \times d_n$ matrix Brownian motions, independent of $\epsilon$ and $\sigma$, with $A_t^\dagger = A_t^*, C_t^\dagger = C_t$ and certain covariances.

(ii) If $A$ and $V_n$ are real symmetric then we obtain

$$C_t = \overline{A_t} = A_t^\dagger \quad \text{and} \quad B_t^\dagger = B_t.$$

(iii) If $Z$ is chaotic then $B_t$ is independent of $A_t$ and $C_t$. Also, $A_t$ and $C_t$ have the same distribution. Moreover, with the subscript $t$ dropped, we have the following:

$$E[A_{ij}^2] = E[B_{ij}^2] = E[(V_e)_{ij}^2]$$

$$E(A_{ii}A_{kk}) = E((V_e)_{ii}(V_e)_{kk})$$

$$E(A_{ij}C_{ij}) = E(\overline{A_{ij}}C_{ji}) = E(B_{ij}\overline{B_{ji}}) = E((V_e)_{ij})^2$$

and whenever $\{i,j\} \neq \{k,l\}$ one finds

$$E(A_{ij}A_{kl}) = E(A_{ij}C_{kl}) = E(\overline{A_{ij}}B_{kl}) = 0$$

and for any $i,j,k,l$,

$$E(B_{ij}B_{kl}) = 0.$$

All other covariances are obtained from $A_t = A_t^*, C_t = C_t^*$.

1.3 Correlations along different directions and SDE limit on the flag manifold

Here we will use the same notations as in Section 1.1. When $T_0$ has eigenvalues of absolute value $c$ different from 1, and $T_0$ is diagonalized (or in Jordan form) so that the corresponding eigenspace are also the span of coordinate vectors and have no Jordan blocks, then we can apply Theorem 1.1 to the products of $T_{\lambda,n}/c$. Moreover, the convergence in law holds jointly for the processes corresponding to magnitudes 1 and $c$ (and in fact all magnitudes). To complete the picture, we just have to specify the covariance structure of the driving matrix-valued Brownian motions for the two processes. Towards this, we define

$$h_{1c}(M) := \lim_{\lambda \to 0} E(V_{\lambda,11}^TMV_{\lambda,11}^{(c)}), \quad \widehat{h}_{1c}(M) := \lim_{\lambda \to 0} E(V_{\lambda,11}^*M V_{\lambda,11}^{(c)})$$

where now $M$ is a $d_1(1) \times d_1(c)$ matrix, where $d_1(c)$ is the total dimension of all eigenspaces corresponding to eigenvalues of absolute value $c$. $V_{\lambda,11}^{(c)}$ denotes the corresponding $d_1(c) \times d_1(c)$ block of $V_{\lambda,n}$. Similarly, we define $h_{cc'}$ and $\widehat{h}_{cc'}$ for any two absolute values $c, c'$ (see also (1.10)).

As before, we also need the $d_1(c) \times d_1(c)$ unitaries $U_c$ (like $U$ in (1.2)) so that $T_0$ restricted to the eigenspaces of magnitude $c$ acts like $cU_c$. 

13
Theorem 1.5. The convergence of Theorem 1.1 holds jointly along all eigenspaces corresponding to absolute values $c$ of eigenvalues of $T_0$ that correspond to eigenspaces without Jordan block. We will denote the corresponding process for the magnitude $c$ by $\Lambda_t^{(c)}$. Then, the covariance of the driving Brownian motions $B, B'$ for the magnitudes $c, c'$ are given by

$$E(B_t^* M B'_t) = g_{cc'}(M)t, \quad E(B_t^* M B'_t) = \tilde{g}_{cc'}(M)t$$

(1.27)

where

$$g_{cc'}(M) = \frac{1}{cc'} \int_{(U_c, U_{c'})} \overline{u} U_c h_{cc'}(u^\top M v) U^*_c v^* d(u, v),$$

(1.28)

$$\tilde{g}_{cc'}(M) = \frac{1}{cc'} \int_{(U_c, U_{c'})} u U_c \tilde{h}_{cc'}(u^* M v) U^*_c v^* d(u, v).$$

(1.29)

Here, $(U_c, U_{c'})$ denotes the (block diagonal) compact abelian group generated by $(U_c, U_{c'})$, and $d(u, v)$ denotes the Haar measure on $\langle U_c, U_{c'} \rangle \ni (u, v)$.

If the eigenvalues of $T_0$ are of different absolute value, then the matrix product process grows at different directions at different exponential rates. Hence there is no hope to get a matrix limit of the process that captures all the directions and all the different SDE limits $\Lambda_t^{(c)}$ with their covariances at the same time.

First consider powers of the matrix $T_0$ in the case it is diagonalizable and all the eigenvalues are of different positive absolute value. Then high powers of $T_0$ take most vectors close to the direction of the top eigenspace. A natural way to understand the second eigenvector through typical behavior is through the action of $T_0$ on two-dimensional subspaces. A high power of $T_0$ will take two-dimensional eigenspaces into a two-dimensional space spanned by the top two eigenvectors of $T_0$.

A flag is a nested sequence of subspaces of all dimensions up to $d$. The set of all such flags forms a compact manifold. By the above argument, a high power of $T_0$ takes most flags close to the flag given by the nesting of the subspaces spanned by the top $k$ eigenvectors.

The above picture still holds when we add perturbations and consider the products $T_{\lambda, n} \cdots T_{\lambda, 1}$. So nothing interesting happens in this case. Things become more interesting when there are more than one eigenvalue of $T_0$ for a given absolute value. If this holds for the top one, then the direction of the action of a typical vector becomes dependent on the randomness, even in the limit. The deterministic dynamics only gives that the vector will be in the subspace spanned by the eigenvectors corresponding to the top absolute value. In this sense, the different exponential rates will still determine certain subspaces of the flag in the limit so that the limiting process will be in a specific submanifold that is invariant and attracting under the action of $T_0$.

Our next theorem shows how this happens. More precisely, we will consider a flag which is typical for the behavior of powers of $T_0$. This happens if the $k$-dimensional spaces of the flag do not include directions that are spanned by subsets of eigenvectors of $T_0$ corresponding to eigenvalues
of lower order. The matrix products applied to this flag will give a flag-valued process. This is described in Theorem 1.6.

As only invertible matrices act on a flag, suppose that for small $\lambda$ all $T_{\lambda,n}$ are invertible with probability one, i.e., there is $\lambda_0$ such that for all $0 \leq \lambda < \lambda_0$, $P(T_{\lambda,n} \text{ is invertible for all } n) = 1$. Suppose further that $T_0$ is diagonalizable and that we chose a basis such that

$$T_0 = \begin{pmatrix} c_1 U_{c_1} & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & c_k U_{c_k} & \end{pmatrix}, \quad \text{where } 0 < c_1 < c_2 < \ldots < c_k,$$

with the $U_{c_j}$ being unitary $d(c_j) \times d(c_j)$ matrices.

A flag can be represented by an invertible $d \times d$ matrix $F$ where the last $p$ column vectors, denoted by $F^{(p)}$, span the $p$-dimensional subspace. $F_1$ and $F_2$ represent the same flag if and only if $F_1 = F_2 M$ for an invertible lower triangular matrix $M$. This forms an equivalence relation and we denote the equivalence class of $F$ by $[F]$. Denoting the group of invertible, lower triangular $d \times d$ matrices by $\Delta(d)$ the flag manifold has

$$F = GL(d) / \Delta(d).$$

The stable submanifold $F^s$ is the set of all flags such that the $d(c_1) + \ldots + d(c_j)$ dimensional subspace is spanned by the last $d(c_1) + \ldots + d(c_j)$ vectors in the standard basis, i.e.

$$F^s = \left\{ \begin{pmatrix} a_1 & \ldots & 0 \\ 0 & \ldots & a_k \\ & \ldots & \\ & & \\ 0 & \ldots & a_k \end{pmatrix} : \text{for all } j, a_j \in GL(d(c_j)) \right\} \subset F.$$

This is an attractor by the deterministic dynamics given by the action of $T_0$ and the set of points in $F$ that is attracted is given by

$$F^a = \left\{ \begin{pmatrix} a_1 & \ldots & * \\ 0 & \ldots & a_k \\ & \ldots & \\ & & \\ 0 & \ldots & a_k \end{pmatrix} : \text{for all } j, a_j \in GL(d(c_j)), * \text{ arbitrary} \right\}.$$

To counteract all the rotations let

$$\widehat{R} = \begin{pmatrix} U_{c_1} & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & U_{c_k} & \end{pmatrix} \in U(d).$$

**Theorem 1.6.** Let $T_0$ be as in (1.30) and let $[F_0] \in F^s$ be represented in the form as described in (1.31). Furthermore let $F_{\lambda,n} = \widehat{R}^{-n} T_{\lambda,n} \cdots T_{1,\lambda} F_0$.

Then, for fixed $t > 0$ and $n \to \infty$ we have $[F_{1/\sqrt{n},[n]t}] \Rightarrow [F_t]$ in law with

$$F_t = \begin{pmatrix} \Lambda_{t}^{(c_1)} a_1 & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & \Lambda_{t}^{(c_k)} a_k & \end{pmatrix}.$$  

Here, $\Lambda_{t}^{(c_j)}$ are the correlated processes for the different magnitudes $c_j$ of eigenvalues of $T_0$ whose correlations are described in Theorem 1.5. Note that $[F_t] \in F^s$. 

15
The theorem is proved in Section 4.2.

**Remark.** If $\mathcal{T}_0$ cannot be brought into the structure as in (1.30) in general then one still obtains the SDE limits on the Grassmannians $G(p,d)$ for $d_2 < p \leq d_2 + d_1$ as in the proof. $G(p,d)$ denotes the space of $p$-dimensional subspaces of $\mathbb{C}^d$.

The interesting point in the theorem is that the process $\hat{\mathcal{R}}^{-n}\mathcal{T}_n \cdots \mathcal{T}_1$ does not have a distributional limit, its quotient by $\Delta(d)$ does.

**Question.** For which subgroups $\Delta'$ of $\text{GL}_d$ does the process $\hat{\mathcal{R}}^{-n}\mathcal{T}_n \cdots \mathcal{T}_1/\Delta'$ have a distributional limit?

When $\Delta'$ is algebraic and the quotient $\text{GL}_d/\Delta'$ is compact, then $\Delta'$ contains a conjugate of $\Delta(d)$, therefore, $\text{GL}_d/\Delta'$ can be seen as a quotient of the flag manifold itself and there is such a limit. In Section 2 we will see that the distributional limit of the pair $(X_{\lambda,n}, Z_{\lambda,n})$ can also be seen as the distributional limit on a certain quotient.

### 1.4 Jordan blocks and critical scaling

Without loss of generality we will focus on the eigenvalues of size 1 of $\mathcal{T}_0$. Let us introduce the notation $J_k$ for the standard $k \times k$ Jordan block with eigenvalue 1, and $N_k$ for the standard Jordan block with eigenvalue 0, i.e.

$$J_k = \begin{pmatrix} 1 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix} = 1 + N_k$$

If a Jordan block of the form $e^{i\theta} J_k$ appears in (a possible conjugation of) $\mathcal{T}_0$ then we will do a $\lambda$-dependent conjugation. This trick was already used in [SS1] to analyze the Lyapunov exponent and density of states at a bandedge for a one-dimensional Schrödinger operator. The main point is the following observation. Define the $\lambda$-dependent, diagonal $k \times k$ matrices

$$S_{\lambda,\alpha,k} = \text{diag}(1, \lambda^\alpha, \ldots, \lambda^{(k-1)\alpha})$$

then

$$S_{\lambda,\alpha,k}^{-1} J_k S_{\lambda,\alpha,k} = 1_k + \lambda^\alpha N_k . \quad (1.32)$$

Now using blocks of sizes $d_0, d_1, d_2$ as before let

$$\mathcal{T}_0 = \begin{pmatrix} \Gamma_0 \\ e^{i\theta} J_{d_1} \\ \Gamma_2^{-1} \end{pmatrix} , \quad \mathcal{R} = \begin{pmatrix} 1 \\ e^{i\theta} 1 \\ 1 \end{pmatrix} , \quad S_{\lambda,\alpha} = \begin{pmatrix} 1_{d_0} \\ S_{\lambda,\alpha,d_1} \\ 1_{d_2} \end{pmatrix} \quad (1.33)$$
with $\Gamma_1$ and $\Gamma_2$ having spectral radius smaller than one as before. Conjugating $T_{\lambda,n}$ by $S_{\lambda,\alpha}$ will give a new drift term of order $\lambda^\alpha$ coming from (1.32), but it also brings a diffusion term of order $\lambda^{1-(d_1-1)\alpha}$ from conjugating $\lambda V_{\lambda,n}$. The diffusion has thus order $\lambda^{2-2(d_1-1)\alpha}$ and the most interesting SDE limit arises from balancing the new drift term and the diffusion term, i.e. $\alpha = 2 - 2(d_1 - 1)\alpha$, leading to $\alpha = \alpha(d_1) = 2/(2d_1 - 1)$. For smaller $\alpha$, the drift term dominates and for larger $\alpha$, the diffusion term dominates.

In fact, only the lower left corner entry of the middle $d_1 \times d_1$ block of $S_{\lambda,\alpha}^{-1} V_{\lambda,n} S_{\lambda,\alpha}$ will be of order $\lambda^{\alpha/2}$, all other terms from the conjugation will be at least of order $\lambda^{3\alpha/2}$. Hence, for the case as in (1.33) we find

$$S_{\lambda,\alpha(d_1)}^{-1} T_{\lambda,n} S_{\lambda,\alpha(d_1)} = \hat{T}_0 + \lambda^{1/(2d_1-1)} \hat{V}_{n} + \lambda^{2/(2d_1-1)} N + \lambda^{3/(2d_1-1)} \hat{V}_{\lambda,n}$$

where

$$\hat{T}_0 = \begin{pmatrix} \Gamma_0 & e^{i\theta} \mathbf{1} \\ e^{i\theta} \mathbf{1} & \Gamma_2^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} \mathbf{0} \\ e^{i\theta} N_{d_1} \end{pmatrix}, \quad \hat{V}_n = \begin{pmatrix} \mathbf{0} \\ V_{11,n} \end{pmatrix}. \quad (1.34)$$

Furthermore, $V_{11,n}$ has only one entry $v_n$ in the lower left corner, and the $v_n$ are i.i.d. random variables with mean zero,

$$V_{11,n} = \begin{pmatrix} \mathbf{0} & 0_{(d_1-1)\times(d_1-1)} \\ v_n & 0 \end{pmatrix}.$$

Therefore, application of Theorem 1.1 gives an SDE limit in the scaling $\lambda^{\alpha+n} = \lambda^{2/(2k-1)} N = t$.

**Theorem 1.7.** Let $T_{\lambda,n}$ be given as in (1.1) and let the assumptions as on page 6 and (1.33) be satisfied. Moreover let $S_{\lambda,\alpha}$ be defined as above with $\alpha = \alpha(d_1) = 2/(2d_1 - 1)$. Let

$$X_{\lambda,n} = \mathcal{R}^{-n} S_{\lambda,\alpha}^{-1} T_{\lambda,n} \cdots T_{\lambda_1} S_{\lambda,\alpha} X_0$$

with $X_0$ as before and let $X_{\lambda,n}$ be the corresponding Schur complement as before. Then

$$X_{\sqrt{\lambda N_{d_1}^{-1}}} = \begin{pmatrix} \mathbf{0} \\ \Lambda_t \end{pmatrix} X_0$$

$$d\Lambda_t = N_{d_1} \Lambda_t dt + \begin{pmatrix} \mathbf{0} \\ dB_t \end{pmatrix}, \quad \Lambda_0 = \mathbf{1} \quad (1.35)$$

where $B_t$ is a complex Brownian motion with covariances

$$\mathbb{E}(B_t^2) = e^{-2i\theta} \mathbb{E}(v_n^2), \quad \mathbb{E}(|B_t|^2) = \mathbb{E}(|v_n|^2).$$

1If the variance of that entry happens to be identically zero (no randomness) or of lower order in $\lambda$, then the diffusion term is of order $\lambda^{1-(k-2)\alpha}$ (or lower again). This may lead to other interesting scalings as for smaller Jordan blocks.
Note that for a vector \( x(t) = \Lambda_t x(0) \) equation (1.35) is equivalent to

\[
x_1^{(d_1)} = x_1 B'
\]

and \( x_{j+1} = x_1^{(j)} \), the \( j \)th derivative of \( x_1 \), and \( B' \) is the (distributional) derivative of the Brownian motion term.

**Remark.** The original drift term coming from \( \lambda^2 W \lambda \) is of too low order after the conjugation with \( S_{\lambda,\alpha} \) to matter in the limit. If one wants an additional drift term in (1.36) on the right hand side coming from an added term \( \lambda^\beta W \) then the conjugation \( S_{\lambda,\alpha}^{-1} \lambda^\beta W S_{\lambda,\alpha} \) needs to produce a term of order \( \lambda^{2d_1-1} = \lambda^\alpha \). If \( W \) is not zero in the lower left corner of the corresponding \( d_1 \times d_1 \) block for the SDE limit, then one needs \( \beta - (d_1 - 1)\alpha = \alpha \), i.e. \( \beta = d_1\alpha = 2d_1/(2d_1 - 1) \).

Jordan blocks do appear at so-called band-edges for transfer matrices of one-dimensional random Schrödinger operators with some finite range hopping. Similar as in Section 1.2 consider the random family of random real symmetric matrices \( H_{\lambda,n}^{(d)} \) acting on \( \mathbb{C}^n \ni \psi = (\psi_1, \ldots, \psi_n) \) given by

\[
(H_{\lambda,n}^{(d)} \psi)_k = \sum_{j=0}^{2d} (-1)^j \binom{2d}{j} \psi_{k-d+j} + \lambda v_k \psi_k ,
\]

where \( \psi_j = 0 \) for \( j < 1 \) and \( j > n \). We may sometimes drop the index \( n \). The \( v_k \) are independent, identically distributed real random variables with variance \( \mathbb{E}(v_k^2) = 1 \). Note that for \( d = 1 \) this operator corresponds to (1.19) with \( A = -2 \). The eigenvalue equation \( H_{\lambda,n}^{(d)} \psi = E \psi \) can be rewritten as

\[
\tilde{\psi}_k = (T + (E - \lambda v_k) S) \tilde{\psi}_{k-1}, \quad \text{where} \quad \tilde{\psi}_k = (\psi_{k+d}, \psi_{k+d-1}, \ldots, \psi_{k-d+1})^T
\]

and \( S \) and \( T \) are \( 2d \times 2d \) matrices given by: \( S_{1,d} = 1 \) and all other entries of \( S \) are zero; \( T_{1,k} = (-1)^{k+1} \binom{2d}{2d-k} \), \( T_{j,j-1} = 1 \) for \( j \geq 2 \) and all other entries of \( T \) are zero, i.e.

\[
T = \begin{pmatrix}
\binom{2d}{2d-1} & \cdots & \binom{2d}{1} & -\binom{2d}{0} \\
1 & 0 \\
\vdots & \\
1 & 0
\end{pmatrix}.
\]

For \( E = 0 \) and \( \lambda = 0 \) the transfer matrix \( T \) is equivalent to a Jordan block\(^2\) of maximum size for the eigenvalue 1. In order to bring it into the Jordan form, let us define the Pascal-triangle type matrix \( M \) by \( M_{jk} = \binom{2d-j}{k-1} \) for \( k + j \leq 2d + 1 \) and zero for all other entries, then one has

\(^2\)In fact \( E = 0 \) is at the edge of the spectrum of the operator \( H_{0}^{(d)} \) in the limit \( n \to \infty \); it is the upper edge for \( d \) odd and the lower edge for \( d \) even.
$M_{jk}^{-1} = (-1)^{j+k} (\frac{j-1}{2d-k})$ for $k + j \geq 2d + 1$ and all other entries zero, i.e.

$$M = \begin{pmatrix} 1 & 2d-1 & (2d-1) & \cdots & 1 \\ 1 & 2d-2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \\ \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 \\ 1 & -2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & -(2d-1) & 2d-1 & -1 \\ \end{pmatrix}.$$  

Then some calculation shows $M^{-1}TM = J_{2d}$ where $J_{2d}$ is the Jordan matrix as defined above. For the conjugation of the whole transfer matrix $T + E - \lambda v_k)S$ we also need to calculate $M^{-1}SM$. Its entries are given by $(M^{-1}SM)_{j,k} = M_{jk}^{-1}S_{1,d}M_{d,k}$ which is only not zero if $j = 2d$ and $k \leq d + 1$ in which case $(M^{-1}SM)_{2d,k} = \binom{d}{k-1}$, i.e.

$$M^{-1}SM = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \\ 1 & \cdots & \left(\frac{d}{2}\right) & 1 & 0 & \cdots & 0 \\ \end{pmatrix}.$$  

In particular the lower left corner has the entry 1. As above let $\alpha = \frac{2}{4d-1}$ and as in the remark scale energy differences by $E = \epsilon \lambda^{2d\alpha}$ to obtain

$$T_{\lambda,k} := S_{\lambda,\alpha}^{-1}M^{-1}(T + (\lambda^{2d\alpha} - \lambda v_k)S)M S_{\lambda,\alpha} = 1 - \lambda^2 v_kQ + \lambda^\alpha [\mathcal{N}_{2d} + \epsilon Q] + \mathcal{O}(\lambda^{3\alpha}).$$  

Then, for any vector $x \in \mathbb{C}^{2d}$ we find

$$T_{n-1/\alpha,|nt|} x \rightarrow x(t) \quad \text{with} \quad x(0) = x, \quad x_1^{(2d)} = x_1B' + \epsilon x_1, \quad x_{j+1} = x_1^{(j)}$$  

where $B'$ is the distributional derivative of a standard, real, one-dimensional Brownian motion.

Following the arguments of [KVV] or the arguments of the proof of Theorem 1.3 one could show that (along suitable subsequences so that the boundary conditions converge) the eigenvalue process of $n^{2d}H_{n-1/\alpha}^{(d)}$ with $\alpha = 2/(4d-1)$ converges to the process of eigenvalues of the random operator

$$\partial_x^{2d} - B'$$  

acting on the interval $[0,1]$ with appropriate boundary conditions. For periodic boundary conditions this is a generalization of the random Hill operator (at $d = 1$).

## 2 Evolution equation and estimates

Recall that $T_{\lambda,n} = T_0 + \lambda \mathcal{V}_{\lambda,n} + \lambda^2 \mathcal{W}_\lambda$ where the disordered part satisfies the assumptions [17] and [18]. For convenience we define $\mathcal{Y}_{\lambda,n} = \mathcal{V}_{\lambda,n} + \lambda \mathcal{W}_\lambda$, such that

$$T_{\lambda,n} = T_0 + \lambda \mathcal{V}_{\lambda,n} + \lambda^2 \mathcal{W}_\lambda = T_0 + \lambda \mathcal{Y}_{\lambda,n}.$$  

19
Then the assumptions imply that for small $\lambda$
\[
\mathbb{E}(|\mathcal{Y}_{\lambda,n}|^{6+\epsilon}) = \mathcal{O}(1), \quad \mathbb{E}(\mathcal{Y}_{\lambda,n}) = \lambda Y + o(1) \quad (2.1)
\]
For the proof of Theorem 1.1 we will fix some time $T > 0$ and obtain the SDE limit up to time $T > 0$ which is fixed but arbitrary. Let us first show the following.

**Proposition 2.1.** Without loss of generality we can assume
\[
\|\mathcal{Y}_{\lambda,n}\| < K \lambda^{s-1} \text{ for some } \frac{2}{3} < s < 1 \text{ and } K \lambda > 0. \quad (2.2)
\]

**Proof.** Assumption (1.7) and Markov’s inequality yield
\[
\mathbb{P}(\|\mathcal{Y}_{\lambda,n}\| \geq C) \leq \mathbb{E}(\|\mathcal{Y}_{\lambda,n}\|^{6+\epsilon}) \leq \frac{k}{C^{6+\epsilon}} \quad (2.3)
\]
for some fixed $k$, uniformly for small $\lambda$.

Now let $s$ be such that $1 - s$ lies between $2/(6 + \epsilon)$ and $1/3$ and define the truncated random variable $\tilde{\mathcal{Y}}_{\lambda,n}$ by
\[
\tilde{\mathcal{Y}}_{\lambda,n} = \begin{cases} \mathcal{Y}_{\lambda,n} & \text{if } \|\mathcal{Y}_{\lambda,n}\| < K \lambda^{1-s} \\ 0 & \text{else} \end{cases}
\]
By the choice of $s$, $(1 - s) > \frac{2}{6+\epsilon} > \frac{1}{3}$ and we obtain
\[
\mathbb{E}(\|\mathcal{Y}_{\lambda,n} - \tilde{\mathcal{Y}}_{\lambda,n}\|) = \int_{\|\mathcal{Y}_{\lambda,n}\| > K \lambda^{1-s}} \|\mathcal{Y}_{\lambda,n}\| d\mathbb{P} \leq \int_{K \lambda^{1-s}}^{\infty} C \cdot (6 + \epsilon) \frac{k}{C^{6+\epsilon}} dC 
\]
\[= \frac{(6 + \epsilon)k}{(5 + \epsilon) K^{5+\epsilon}} \lambda^{(5+\epsilon)(1-s)} = o(\lambda) \]
and similarly
\[
\mathbb{E}(\|\mathcal{Y}_{\lambda,n} - \tilde{\mathcal{Y}}_{\lambda,n}\|^2) \leq \frac{(6 + \epsilon)k}{(4 + \epsilon) K^{4+\epsilon}} \lambda^{(4+\epsilon)(1-s)} = o(1)
\]
for $\lambda \to 0$. Thus, using $\tilde{\mathcal{Y}}_{\lambda,n}$ instead of $\mathcal{Y}_{\lambda,n}$ in (1.13), (1.14) and (1.15) does not change the quantities $V$, $g(M)$ and $\hat{g}(M)$. Hence, the SDE limits mentioned in Theorem 1.1 for $\tilde{\mathcal{Y}}_{\lambda,n}$ and $\mathcal{Y}_{\lambda,n}$ are the same.

Let us assume that Theorem 1.1 is correct for $\tilde{\mathcal{Y}}_{\lambda,n}$ and obtain its validity for using $\mathcal{Y}_{\lambda,n}$ by showing that we obtain the same limit SDE. From (2.3)
\[
\mathbb{P}(\mathcal{Y}_{\lambda,n} \neq \tilde{\mathcal{Y}}_{\lambda,n}) \leq \frac{k}{K^{6+\epsilon} \lambda^{(6+\epsilon)(s-1)}} = c \lambda^{(6+\epsilon)(1-s)} = c \lambda^{2+\delta}
\]
where the last equations define $c > 0$ and $\delta > 0$. Hence,
\[
\mathbb{P}\left(\|\mathcal{Y}_{\lambda,n}\| > K \lambda^{s-1} \text{ for some } n = 1, 2, \ldots, \lfloor \lambda^{-2} T \rfloor \right) \leq T c \lambda^{\delta}
\]
which approaches zero for $\lambda \to 0$. Therefore, introducing a stopping time $T_\lambda := \min\{n : \mathcal{Y}_{\lambda,n} \neq \tilde{\mathcal{Y}}_{\lambda,n}\}$ and considering the stopped process $X_{\lambda,n\wedge T_\lambda}$ one obtains the same SDE limits. But the stopped processes coincide when using $\tilde{\mathcal{Y}}_{\lambda,n}$ instead of $\mathcal{Y}_{\lambda,n}$ and therefore lead to the same SDE limits. □
Thus we may assume equation (2.2) without loss of generality and we will do so from now on. Moreover, as the spectral radius of $\Gamma_0$ and $\Gamma_2$ are smaller than 1, using a basis change, we may assume:

$$\|\Gamma_0\| \leq e^{-\gamma}, \quad \|\Gamma_2\| \leq e^{-\gamma}, \quad \text{where } \gamma > 0.$$  \hspace{1cm} (2.4)

Before obtaining the evolution equations, we will first establish that the pair $(X_{\lambda,n}, Z_{\lambda,n})$ is a Markov process. Let us define the following subgroup of $\text{GL}(d, \mathbb{C})$.

$$\mathcal{G} = \left\{ \begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix} \in \text{Mat}(d, \mathbb{C}) : \text{where } D \in \text{GL}(d_2, \mathbb{C}) \right\}.$$  

Now let $\mathcal{X}_1$ and $\mathcal{X}_2$ be equivalent, $\mathcal{X}_1 \sim \mathcal{X}_2$, if $\mathcal{X}_1 = \mathcal{X}_2 Q$ for $Q \in \mathcal{G}$. As different representatives differ by multiplication from the right, multiplication from the left defines an action on the equivalence classes. Therefore, the evolution of the equivalence classes $[\mathcal{X}_{\lambda,n}]_\sim$ is a Markov process. As

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (2.5)

we see that the equivalence class $[\mathcal{X}_{\lambda,n}]_\sim$ is determined by the pair $(X_{\lambda,n}, Z_{\lambda,n})$.

Let us further introduce the following commuting matrices of size $d_0 + d_1$,

$$R = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}, \quad S = \begin{pmatrix} \Gamma_0 & 0 \\ 0 & 1 \end{pmatrix}: \quad R, S \in \text{Mat}(d_0 + d_1, \mathbb{C}),$$  \hspace{1cm} (2.6)

Note that $R$ is unitary and that

$$\mathcal{R} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \in \text{U}(d).$$  \hspace{1cm} (2.7)

for $\mathcal{R}$ as defined in (1.3). As $\Gamma^n_0$ is exponentially decaying, we refer to the $d_0$ dimensional subspace corresponding to this matrix block as the decaying directions of $T^n_0$. Similarly, the $d_2$ dimensional subspace corresponding to the entry $\Gamma^{-n}_2$ are referred to as growing directions.

The evolution of $\mathcal{X}_{\lambda,n}$ is given by

$$\mathcal{X}_{\lambda,n} = \mathcal{R}^{-n} T_{\lambda,n}^{-1} \mathcal{R}^{n-1} \mathcal{X}_{\lambda,n-1}.$$  \hspace{1cm} (2.8)

Therefore, let

$$\mathcal{R}^{-n} T_{\lambda,n} \mathcal{R}^{n-1} = \begin{pmatrix} T^A_{\lambda,n} & T^B_{\lambda,n} \\ T^C_{\lambda,n} & T^D_{\lambda,n} \end{pmatrix}.$$  

Here, $A, B, C, D$ are used as indices to indicate that we use the same sub-division of the matrix as we did when defining $A_{\lambda,n}, B_{\lambda,n}, C_{\lambda,n}$ and $D_{\lambda,n}$.

\[3\text{even if } \Gamma_0 \text{ or } \Gamma_2 \text{ are not diagonalizable, one can make the norm smaller than one as one can make the off diagonal terms of Jordan blocks arbitrarily small.}\]
The action on the equivalence class of \( X_{\lambda,n-1} \sim \left( \begin{matrix} X_{\lambda,n-1} & Z_{\lambda,n-1} \\ 0 & 1 \end{matrix} \right) \) gives
\[
\begin{pmatrix}
T_A^{\lambda,n} & T_B^{\lambda,n} \\
T_C^{\lambda,n} & T_D^{\lambda,n}
\end{pmatrix}
\begin{pmatrix}
X_{\lambda,n-1} & Z_{\lambda,n-1} \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
T_A^{\lambda,n}X_{\lambda,n-1} + T_B^{\lambda,n}Z_{\lambda,n-1} & T_A^{\lambda,n}Z_{\lambda,n-1} + T_B^{\lambda,n}X_{\lambda,n-1} \\
T_C^{\lambda,n}X_{\lambda,n-1} + T_D^{\lambda,n}Z_{\lambda,n-1} & T_C^{\lambda,n}Z_{\lambda,n-1} + T_D^{\lambda,n}X_{\lambda,n-1}
\end{pmatrix}.
\]

Transforming the matrix on the right hand side into the form as in (2.6) we can read off the evolution equations
\[
Z_{\lambda,n} = (T_A^{\lambda,n}Z_{\lambda,n-1} + T_B^{\lambda,n}) \left( T_C^{\lambda,n}Z_{\lambda,n-1} + T_D^{\lambda,n} \right)^{-1}
\]
and
\[
X_{\lambda,n} = T_A^{\lambda,n}X_{\lambda,n-1} - (T_A^{\lambda,n}Z_{\lambda,n-1} + T_B^{\lambda,n}) \left( T_C^{\lambda,n}Z_{\lambda,n-1} + T_D^{\lambda,n} \right)^{-1} T_C^{\lambda,n}X_{\lambda,n-1}.
\]

For more detailed calculations, let
\[
Y_{\lambda,n} = V_{\lambda,n} + \lambda W_{\lambda} = \begin{pmatrix}
V_A^{\lambda,n} & V_B^{\lambda,n} \\
V_C^{\lambda,n} & V_D^{\lambda,n}
\end{pmatrix},
\]
then \( R^{-n}Y_{\lambda,n}R^{n-1} = \begin{pmatrix}
R^{-n}V_A^{\lambda,n}R^{n-1} & R^{-n}V_B^{\lambda,n} \\
V_C^{\lambda,n}R^{n-1} & V_D^{\lambda,n}
\end{pmatrix} \).

From (1.1), (1.2), (2.6) and (2.7) one finds
\[
T_A^{\lambda,n} = S + \lambda R^{-n}V_A^{\lambda,n}R^{n-1}, \quad T_B^{\lambda,n} = \lambda R^{-n}V_B^{\lambda,n}, \quad T_C^{\lambda,n} = \lambda V_C^{\lambda,n}R^{n-1}, \quad T_D^{\lambda,n} = \Gamma_2^{-1} + \lambda V_D^{\lambda,n}.
\]

We will first consider the Markov process \( Z_{\lambda,n} \) and denote the starting point by \( Z_0 = B_0D_0^{-1} \).

**Proposition 2.2.** For \( \lambda \) small enough and some constant \( K_Z \) we have the uniform bound
\[
\| Z_{\lambda,n} \| \leq K_Z(e^{-\gamma n/2} + \lambda^s) = \mathcal{O}(e^{-\gamma n/2}, \lambda^s)
\]
with \( \gamma \) as in (2.4). This implies \( Z_{\lambda,n} \Rightarrow 0 \) in law.

**Proof.** Take \( \lambda \) small enough, such that \( e^\gamma - K_Y\lambda^s(1 + \max(\|Z_0\|, 1)) > e^{\gamma/2} \) (with \( K_Y \) as in (2.2)) and
\[
\frac{(1 + K_Y\lambda^s) \max(\|Z_0\|, 1) + K_Y\lambda^s}{e^\gamma - K_Y\lambda^s(1 + \max(\|Z_0\|, 1))} \leq e^{-\gamma/2} \max(\|Z_0\|, 1).
\]

Then, using (2.9) we find for \( \| Z_{\lambda,n-1} \| \leq \max(\|Z_0\|, 1)\)
\[
\| Z_{\lambda,n} \| \leq \frac{(1 + K_Y\lambda^s) \max(\|Z_{\lambda,n-1}\|, 1) + K_Y\lambda^s}{e^\gamma - K_Y\lambda^s(1 + \max(\|Z_{\lambda,n-1}\|, 1))} \leq e^{-\gamma/2} \max(\|Z_0\|, 1) < \max(\|Z_0\|, 1).
\]

Hence, inductively, \( \| Z_{\lambda,n} \| < \max(\|Z_0\|, 1) \) for \( n \geq 1 \). Thus, using this equation and (2.14) again leads to
\[
\| Z_{\lambda,n} \| \leq e^{-\gamma/2} \| Z_{\lambda,n-1} \| + K_Y\lambda^s.
\]

By induction this yields the bound
\[
\| Z_{\lambda,n} \| \leq e^{-\gamma n/2} \| Z_0 \| + \frac{1 - e^{-\gamma n/2}}{1 - e^{-\gamma/2}} K_Y\lambda^s
\]
proving the proposition. \( \square \)
Remark. Note that the estimates show that $T_{\lambda,n}^C Z_{\lambda,n-1} + T_{\lambda,n}^D$ is invertible. Using $D_{\lambda,n} = T_{\lambda,n}^C B_{\lambda,n-1} + T_{\lambda,n}^D D_{\lambda,n-1} = (T_{\lambda,n}^C Z_{\lambda,n-1} + T_{\lambda,n}^D D_{\lambda,n-1}$ it follows inductively also that $D_{\lambda,n}$ is invertible. Hence, $X_{\lambda,n}$ and $Z_{\lambda,n}$ are always well defined for small $\lambda$ under assumption (2.2). Hence, under the assumptions of Theorem 1.1 they will be well defined up to $n = T \lambda^{-2}$ with probability going to one as $\lambda \rightarrow 0$.

Next, let the reminder term $\tilde{\Xi}_{\lambda,n}$ be given by

$$ (T_{\lambda,n}^D + T_{\lambda,n}^C Z_{\lambda,n-1})^{-1} = \Gamma_2 + \tilde{\Xi}_{\lambda,n}, \quad (2.15) $$

and define

$$ \Xi_{\lambda,n} := -T_{\lambda,n}^A Z_{\lambda,n-1} \Gamma_2 V_{\lambda,n}^C R_n^{-1} - (T_{\lambda,n}^A Z_{\lambda,n-1} + \lambda R_n^{-1} V_{\lambda,n}^B) \tilde{\Xi}_{\lambda,n} V_{\lambda,n}^C R_n^{-1}. \quad (2.16) $$

Furthermore let

$$ W_{\lambda,n} := V_{\lambda,n}^A - \lambda V_{\lambda,n}^B \Gamma_2 V_{\lambda,n}^C. \quad (2.17) $$

Then by (2.10) and (2.12) one obtains

$$ X_{\lambda,n} = SX_{\lambda,n-1} + \lambda R_n^{-1} W_{\lambda,n} R_n^{-1} X_{\lambda,n-1} + \lambda \Xi_{\lambda,n} X_{\lambda,n-1}. \quad (2.18) $$

The following estimates will be needed to obtain the SDE limit.

Lemma 2.3. Let $E_{X,Z}$ denote the conditional expectation given that $X_{\lambda,n-1} = X$ and $Z_{\lambda,n-1} = Z$.

(i) For small $\lambda$ one has the bounds

$$ E_{X,Z}(\Xi_{\lambda,n}) = E(\Xi_{\lambda,n} | Z_{\lambda,n-1} = Z) = O(\lambda^{2s-1} \| Z \|, \lambda^{3s-1}) \quad (2.19) $$

$$ E_{X,Z}(\| \Xi_{\lambda,n} \|^2) = O(\| Z \|^2, \lambda^2, \lambda \| Z \|) \quad (2.20) $$

$$ \Xi_{\lambda,n} = O(\| Z_{\lambda,n-1} \| \lambda^{s-1}, \lambda^{3s-1}). \quad (2.21) $$

(ii) $W_{\lambda,n}$ is independent of $Z_{\lambda,n-1}$ and $X_{\lambda,n-1}$ and there is a matrix $W_0$ and a constant $K_W$ such that

$$ E(W_{\lambda,n}) = \lambda W_0 + o(\lambda) \quad (2.22) $$

$$ W_{\lambda,n} = O(\lambda^{s-1}) \quad (2.23) $$

$$ E(\| W_{\lambda,n} \|^3) \leq K_W = O(1). \quad (2.24) $$

(iii)

$$ E_{X,Z} \left( X_{\lambda,n} X_{\lambda,n}^* \right) = SX^* S + \| X \|^2 \cdot O(\lambda^2, \lambda^{2s} \| Z \|). \quad (2.25) $$

Moreover, there is a function $K(T)$ such that

$$ E(\| X_{\lambda,n} \|^2) \leq K(T) \quad \text{for all } n < T \lambda^{-2}. \quad (2.26) $$
Proof. Note that (2.22) implies the uniform bounds
\[
V_{\lambda,n}^A = O(\lambda^{s-1}) , \quad V_{\lambda,n}^B = O(\lambda^{s-1}) , \quad V_{\lambda,n}^C = O(\lambda^{s-1}) , \quad V_{\lambda,n}^D = O(\lambda^{s-1}) .
\] (2.27)
As \( T_{\lambda,n}^D = \Gamma_2^{-1} + \lambda^n T_{\lambda,n}^A \), \( T_{\lambda,n}^A = S + \lambda R^{-n} V_{\lambda,n}^A R^{n-1} \) one finds
\[
\Xi_{\lambda,n} = O(\lambda^{s}) \quad \text{and} \quad (T_{\lambda,n}^A \lambda, n + \lambda R^{-n} V_{\lambda,n}^B \lambda, n \Xi_{\lambda,n}) R^{n-1} = O(\lambda^{3s-1}, \lambda^{2s-1}\| Z \|) .
\] (2.28)
Using (1.8) we see that \( E \) (Note that \( V_{\lambda,n}^A, V_{\lambda,n}^B \) and \( V_{\lambda,n}^C \) are independent of \( Z_{\lambda,n-1} \).) The moment condition (1.7) also yields \( E(\| T_{\lambda,n}^A, \lambda, n \Xi_{\lambda,n} \|^2) = O(\| Z \|^2) \). Combining this with (2.28), using Cauchy Schwarz in the form \( E(\| A + B \|^2) \leq \left( \sqrt{E(\| A \|^2)} + \sqrt{E(\| B \|^2)} \right)^2 \) and using \( O(\lambda^{3s-1}, \lambda^{2s-1}\| Z \|) \leq O(\lambda\| Z \|) \) we find for some constant \( K \) that \( E_{X,Z}(\| \Xi_{\lambda,n} \|^2) \leq K(\| Z \| + \lambda)^2 \) giving (2.20). Finally, (2.22) yields \( \| T_{\lambda,n}^A, \lambda, n \Xi_{\lambda,n} \|^2 = O(\| Z \|^{3s-1}) \) which combined with (2.28) gives (2.21).

To get (ii) note that equation (2.22) follows from (1.8), (2.27) yields (2.23) and the moment condition (1.7) implies (2.24).

For part (iii) note that by (2.18) one has
\[
E_X (X_{\lambda,n} X_{\lambda,n}^*) = \lambda^n E(W_{\lambda,n}) R^{n-1} X X^* X S + \lambda R^{-n} E(W_{\lambda,n}) R^{n-1} X X^* S + \lambda \left[ R^{-n} E(W_{\lambda,n}) R^{n-1} X X^* S \right]^* + \lambda E_{X,Z}(\Xi_{\lambda,n}) X X^* S + \lambda \left[ E(\Xi_{\lambda,n}) X X^* S \right]^* + O(\lambda^2 \| X \|^2 E_{X,Z}(\| W_{\lambda,n} \| + \| \Xi_{\lambda,n} \|^2)^2) .
\]
Using (2.19), (2.22) and \( E_X(\| W_{\lambda,n} \| + \| \Xi_{\lambda,n} \|^2) = O(1) \) one finally obtains equation (2.25). The latter estimate follows from (2.13), (2.21), (2.24) and Cauchy-Schwarz.

For (2.20) note that the Hilbert-Schmidt norm is given by \( \| X \|_{HS}^2 = \text{Tr}(X X^*) \). Then (2.13) and (2.25) imply that for some constant \( K \) one finds
\[
E(\| X_{\lambda,n} \|_{HS}^2) \leq \begin{cases} E(\| X_{\lambda,n-1} \|_{HS}^2) (1 + K\lambda^2) & \text{for } n \leq s/\gamma \ln(\lambda^{-2}) \\ E(\| X_{\lambda,n-1} \|_{HS}^2) (1 + K\lambda^2) & \text{for } n > s/\gamma \ln(\lambda^{-2}) \end{cases}
\]
By induction, for small \( \lambda \) and \( n < T\lambda^{-2} \),
\[
E(\| X_{\lambda,n} \|_{HS}^2) \leq (1 + K\lambda^2)^{s/\gamma \ln(\lambda^{-2})} (1 + K\lambda^2)^T\lambda^{-2} \| X_0 \| < e^{KTK} \| X_0 \| .
\]
As all norms are equivalent, this finishes the proof. □

3 Limit for the process \( X_{\lambda,n} \)

We need to split up the \((d_0 + d_1) \times (d_0 + d_1)\) matrix \( X_{\lambda,n} \) into two blocks. Therefore, let
\[
P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Mat}(d_0 \times (d_0 + d_1)) , \quad P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Mat}(d_1 \times (d_0 + d_1))
\]
Then, using (2.6) one finds
\[
P_0 S = \Gamma_0 P_0 , \quad P_1 S = P_1 , \quad P_3 R^n = P_0 , \quad P_1 R^n = U^n P_1 .
\] (3.1)
Moreover, for any \((d_0 + d_1) \times (d_0 + d_1)\) matrix \(M\) one has

\[
M = \begin{pmatrix} P_0 M \\ P_1 M \end{pmatrix} = \begin{pmatrix} MP_0^* \\ MP_1^* \end{pmatrix}.
\] (3.2)

**Proposition 3.1.** There is a function \(K(T)\) such that for all \(n < T \lambda^{-2}\) one has

\[
\mathbb{E}\left(\|P_0 X_{\lambda,n}\|_{HS}^2\right) \leq e^{-2^\gamma n} \|P_0 X_0\|_{HS}^2 + K(T) \lambda^{2s}
\]

In particular, for any function \(f(n) \in \mathbb{N}\) with \(\lim_{n \to \infty} f(n) = \infty\) one has

\[
P_0 X_{\lambda,n} \xrightarrow{f(n)} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

**Proof.** Multiplying (2.25) by \(P_0\) from the left and \(P_0^*\) from the right, taking expectations and using the bound (2.26) gives

\[
\mathbb{E}(P_0 X_{\lambda,n} X_{\lambda,n}^* P_0^*) = \Gamma_0 \mathbb{E}(P_0 X_{\lambda,n-1} X_{\lambda,n-1}^* P_0^*) \Gamma_0^* + O(\lambda^{2s})
\]

which leads to

\[
\mathbb{E}\left(\|P_0 X_{\lambda,n}\|_{HS}^2\right) \leq e^{-2^\gamma} \mathbb{E}\left(\|P_0 X_{\lambda,n-1}\|_{HS}^2\right) + O(\lambda^{2s}).
\]

where the bound for the error term is uniform in \(n\) for \(n < \lambda^{-2}T\). Induction yields the stated result.

Finally, let us consider the part with an interesting limit. Multiplying (2.13) by \(P_1\) from the left and and using (3.1), (3.2) one finds

\[
P_1 X_{\lambda,n} = P_1 X_{\lambda,n-1} + \lambda U^{-n} P_1 W_{\lambda,n-1} (P_1^* U^{-1} P_1 X_{\lambda,n-1} + P_0^* P_0 X_{\lambda,n-1}) + \lambda P_1 \Xi_{\lambda,n} (P_1^* P_1 X_{\lambda,n-1} + P_0^* P_0 X_{\lambda,n-1})
\] (3.3)

We immediately obtain the following estimate.

**Proposition 3.2.** For \(n < \lambda^{-2}T\) one has uniformly

\[
\mathbb{E}(\|P_1 X_{\lambda,n} - P_1 X_0\|^2) \leq O(n \lambda^{2s}).
\]

This implies for any function \(f(n) \in \mathbb{N}\) with \(\lim_{n \to \infty} f(n) n^{-s} = 0\) that

\[
P_1 X_{\lambda,n} \xrightarrow{f(n)} P_1 X_0
\]

in law for \(\lambda \to 0\).

**Proof.** Using the estimates of Lemma 2.3 one finds similarly to (2.25) that

\[
\mathbb{E}_{X,Z}(P_1 X_{\lambda,n} - P_1 X_0)(P_1 X_{\lambda,n} - P_1 X_0)^* = P_1 XX^* P_1^* + \|X\|^2 O(\lambda^2, \lambda^{2s}\|Z\|).
\]

Using (2.20) and \(\|Z_{\lambda,n-1}\| \leq O(1)\) from (2.13) we find therefore that uniformly for \(n < \lambda^{-2}T\)

\[
\mathbb{E}((P_1 X_{\lambda,n} - P_1 X_0)(P_1 X_{\lambda,n} - P_1 X_0)^*) = \mathbb{E}((P_1 X_{\lambda,n-1} - P_1 X_0)(P_1 X_{\lambda,n-1} - P_1 X_0)^*) + O(\lambda^{2s})
\]

Taking traces (Hilbert-Schmidt norm) and induction yield the result. 

\[\Box\]
In order to use Proposition A.1, we need to consider stopped processes. So for any \( \lambda \), let \( T_K \) be the stopping time when \( \|P_1 X_{\lambda,n}\| \) is bigger then \( K \). We define the stopped process by
\[
P_1 X^K_{\lambda,n} := P_1 X_{\lambda,T_K \wedge n} , \quad P_0 X^K_{\lambda,n} := P_0 X_{\lambda,n} \cdot 1_{n \leq T_K} , \quad Z^K_{\lambda,n} := Z_{\lambda,n} \cdot 1_{n \leq T_K}
\]
where
\[
1_{n \leq T_K} = 1 \quad \text{for} \; n \leq T_K \quad \text{and} \quad 1_{n \leq T_K} = 0 \quad \text{for} \; n > T_K .
\]
As long as \( n \leq T_K \), (2.18) and Lemma 2.3 give \( \|P_0 X_{\lambda,n}\| \leq (e^{-\gamma} + O(\lambda^s))\|P_0 X_{\lambda,n-1}\| + O(\lambda^s) \). An induction similar as in Proposition 2.2 yields for any finite \( K \)
\[
\|P_0 X^K_{\lambda,n}\| < K P(e^{-\gamma/2n} + \lambda^s) , \quad \text{for some constant} \; K_P = K_P(K). \tag{3.4}
\]
For the limit, we will scale \( \lambda = 1/\sqrt{m} \) and \( n = \lfloor tm \rfloor \). First, define the good set
\[
G_m = G_m(K) := \{(X, Z) : \|Z\| < 2K_Z m^{-s/2}, \|P_0 X\| < 2K_P m^{-s/2}\}
\]
then by the estimates (2.13) and (3.4) one has
\[
(X^K_{1/\sqrt{m},n}, Z^K_{1/\sqrt{m},n}) \in G_m \quad \text{for} \; n > s/\gamma \ln(m) . \tag{3.5}
\]
For the variances in the SDE limit we need to recognize the connection to the matrix \( V \) and the functions \( g(M) \), \( \hat{g}(M) \) as defined in (1.13), (1.14) and (1.15). Using the notations as introduced in (1.9) combined with (2.11) and (2.17) one obtains
\[
P_1 W^*_{\lambda,n} P_1^* = (V_{\lambda,11} + \lambda W_{\lambda,11}) - \lambda(V_{\lambda,12} + \lambda W_{\lambda,12})I_2(V_{\lambda,21} + \lambda W_{\lambda,21}). \tag{3.6}
\]
Therefore, using the functions \( h, \hat{h} \) as defined in (1.10) and \( W \) as defined in (1.11) one finds
\[
E(P_1 W_{\lambda,n} P_1^*) = \lambda W + o(\lambda) \tag{3.7}
\]
\[
E((P_1 W_{\lambda,n} P_1^*)^T M P_1 W_{\lambda,n} P_1^*) = h(M) + o(1) \tag{3.8}
\]
\[
E((P_1 W_{\lambda,n} P_1^*)^* M P_1 W_{\lambda,n} P_1^*) = \hat{h}(M) + o(1) . \tag{3.9}
\]
Here the error terms \( o(\lambda) \) and \( o(1) \) are uniform in the limit \( \lambda \to 0 \). Note that \( W = P_1 W_0 P_1^* \) with \( W_0 \) as in (2.28).

Next we have to consider the conditional distribution of the differences \( Y_{\lambda,n} = Y_{\lambda,n}(X, Z) \) given that \( X_{\lambda,n-1} = X, Z_{\lambda,n-1} = Z, \) i.e. for Borel sets of matrices \( A \),
\[
\mathbb{P}(Y_{\lambda,n}(X, Z) \in A) := \mathbb{P}(P_1 X_{\lambda,n} - P_1 X \in A \mid X_{\lambda,n-1} = X, Z_{\lambda,n-1} = Z) , \tag{3.3}
\]
Using (3.3) one has
\[
Y_{\lambda,n} = \lambda U^{-n} P_1 W_{\lambda,n}(P_1^* U^{n-1} P_1 X + P_1^* P_0 X) + \lambda P_1(\Xi_{\lambda,n}|Z_{\lambda,n} = Z) X \tag{3.10}
\]
where \( (\Xi_{\lambda,n}|Z_{\lambda,n-1} = Z) \) is a random matrix variable distributed as \( \Xi_{\lambda,n} \) conditioned to \( Z_{\lambda,n-1} = Z \), this simply means that in (2.15) and (2.16) one replaces \( Z_{\lambda,n-1} \) by \( Z \).
Proposition 3.3. Assume \((X, Z) \in G_m, m = \lambda^{-2}\), thus \(P_0 X = O(\lambda^s)\) and \(Z = O(\lambda^s)\). Then one finds for \(Y_{\lambda,n} = Y_{\lambda,n}(X, Z)\) the uniform estimates (uniform in \(X, Z, n\))

\[
E(Y_{\lambda,n}) = \lambda^2 U^{-n} W U^{n-1} P_1 X + o(\lambda^2) \tag{3.11}
\]

\[
E(Y_{\lambda,n}^T M Y_{\lambda,n}) = \lambda^2 (P_1 X)^T U^{-n-1} h(U^n M U^{-n}) U^{n-1} P_1 X + o(\lambda^2) \tag{3.12}
\]

\[
E(Y_{\lambda,n}^* M Y_{\lambda,n}) = \lambda^2 (P_1 X)^* U^{-n-1} \hat{h}(U^n M U^{-n}) U^{n-1} P_1 X + o(\lambda^2) \tag{3.13}
\]

\[
E(\|Y_{\lambda,n}\|^2) \leq \lambda^3 K_1\|X\|^2 \quad \text{for some uniform } K_1 > 0. \tag{3.14}
\]

Note that any covariance of real and imaginary entries of \(Y_{\lambda,n}\) can be obtained by varying \(M\) in (3.12) and (3.13). Moreover, one obtains uniformly for \(0 < t < T\)

\[
\lim_{m \to \infty} \int_0^t m E\left(Y_{\lambda,n}^{(m)} (s)\right) ds = t V P_1 X \tag{3.15}
\]

\[
\lim_{m \to \infty} \int_0^t m E\left(Y_{\lambda,n}^{(m)} M Y_{\lambda,n}^{(m)} (s)\right) ds = t (P_1 X)^T g(M) P_1 X \tag{3.16}
\]

\[
\lim_{m \to \infty} \int_0^t m E\left(Y_{\lambda,n}^{(m)} M Y_{\lambda,n}^{(m)} (s)\right) ds = t (P_1 X)^* \hat{g}(M) P_1 X \tag{3.17}
\]

where \(V, g\) and \(\hat{g}\) are as in (1.13), (1.14) and (1.15).

Proof. Given \(X_{\lambda,n-1} = X, P_0 X = O(\lambda^s), Z_{\lambda,n-1} = Z = O(\lambda^s)\) and using the estimates (2.19) and (2.22) equation (3.10) yields

\[
E(Y_{\lambda,n}) = \lambda^2 E(U^{-n} P_1 W_{\lambda,n} P_1^* U^* U^n P_1 X) + O(\lambda^{3s}) = \lambda^2 U^{-n} W U^{n-1} P_1 X + o(\lambda^2)
\]

which implies (3.11). Using (2.23), (2.24) and \(Z_{\lambda,n-1} = Z = O(\lambda^s)\) we get

\[
Y_{\lambda,n} = \lambda U^{-n} P_1 W_{\lambda,n} P_1^* U^{n-1} P_1 X + O(\lambda^{2s}). \tag{3.18}
\]

Together with (3.8) and (3.9) this proves (3.12) and (3.13). Finally, (3.14) follows from (3.18) and (1.7).

Letting \(u = U^{-n} = U^{n*}\) we have the terms \(u W U^* u^*, \pi U^T h(u^T M u) U^* u^*\) and \(u \hat{h}(u^* M u) U^* u^*\) appearing in (3.11), (3.12) and (3.13), respectively. On the abelian compact group \(\langle U \rangle\) generated by the unitary \(U\), the functions

\[
u \mapsto u W u^*, \quad u \mapsto \pi U^T h(u^T M u) U^* u^*, \quad u \mapsto u U \hat{h}(u^* M u) U^* u^*
\]

are polynomials of the eigenvalues of \(u\) as \(h, \hat{h}\) are linear and all \(u \in \langle U \rangle\) are simultaneously diagonalizable. For any such polynomial \(p(u)\) one finds

\[
\lim_{m \to \infty} \int_0^t p(U^{-|m|}) ds = \lim_{m \to \infty} t \frac{1}{m} \sum_{k=1}^m p(U^{-k}) = t \int_{\langle U \rangle} p(u) du \tag{3.19}
\]

uniformly for \(t < T\), where \(du\) denotes the Haar measure on \(\langle U \rangle\). Applied to (3.11), (3.12), (3.13) this yields (3.15), (3.16) and (3.17). \(\square\)
Let $f(n) \in \mathbb{N}$ with $\lim_{n \to \infty} f(n) = \infty$ and $\lim_{n \to \infty} f(n)n^{-s} = 0$, then by Proposition 3.3 and 3.2 we find for large enough $K$ that $X_{1/\sqrt{n}}^K \Rightarrow \left( \begin{array}{cc} 0 & 0 \\ 0 & 1_{d_1} \end{array} \right) X_0$ where we used a subdivision in blocks of sizes $d_0$ and $d_1$. For sake of concreteness let us set $f(n) = \lfloor n^\alpha \rfloor$ with some $0 < \alpha < s$. From (3.3) we find for $m \to \infty$,

$$P\left((X_{1/\sqrt{n}}^K, Z_{1/\sqrt{n}}^K) \in G_m \text{ for all } T_m > n > f(m)\right) \to 1$$

Together with Proposition 3.3 we see that the stopped processes $(X_{1/\sqrt{n}}^K, Z_{1/\sqrt{n}}^K)$ for $n = 1, \ldots, mT$ satisfy the conditions of Proposition 3.1 with $X_n^m = P_1X_{1/\sqrt{n}}^K$, the good sets $G_m$ and $f(m) = \lfloor m^\alpha \rfloor$. Thus, with Proposition 3.1 it follows $X_{1/\sqrt{n}}^K \Rightarrow \left( \begin{array}{c} 0 \\ \Lambda_t^K \end{array} \right) X_0$, uniformly for $0 < t < T$, where $\Lambda_t^K = \Lambda_{t,T_k}$ denotes the stopped process of $\Lambda_t$ as described in Theorem 1.1 with stopping time $T_k$ when $\|P_1\Lambda_tX_0\| > K$. As we have this convergence for all such stopping times $T_k$, $\|P_1\Lambda_tX_0\|$ is almost surely finite and as the final time $T$ was arbitrary, one obtains $X_{1/\sqrt{n}}^K \Rightarrow \left( \begin{array}{c} 0 \\ \Lambda_t \end{array} \right) X_0$ for any $t > 0$. Together with Proposition 2.2 this finishes the proof of Theorem 1.1.

### 4 Correlations between SDEs

#### 4.1 Proof of Theorem 1.5

In this section we will obtain the correlations for the SDE limits for different sizes of eigenvalues of $T_\alpha$ and prove Theorem 1.5. We follow all calculations above for the limit processes $X_{1/\sqrt{n}}^c$ as in Corollary 3.3 (considering the matrices $\frac{1}{c}J_{\lambda,n}$ instead of $J_{\lambda,n}$) and define all the same objects as above with superscript $(c)$.

In particular we define the random variables $Y_{\lambda,n}^{(c)}(X^{(c)}, Z^{(c)})$ being distributed as $P_1^{(c)}X_{\lambda,n}^{(c)}$, given that $X_{\lambda,n}^{(c)} = X^{(c)}$, $Z_{\lambda,n}^{(c)} = Z^{(c)}$ with $X^{(c)}$, $Z^{(c)}$ in the corresponding good sets. Analogously we define $Y_{\lambda,n}^{(c)}(X^{(c)}, Z^{(c)})$. Then as in (3.18) combining with (3.4) one obtains

$$Y_{\lambda,n}^{(c)} = \frac{1}{c} \lambda U_n^{-n} Y_{\lambda,11}^{(c)} U_n^* U_n P_1^{(c)} X^{(c)} + \mathcal{O}(\lambda^2)$$

with $U_n$ and $V_{\lambda,11}^{(c)}$ defined as in Section 1.8. The factor $\frac{1}{c}$ comes from the fact that we have to consider $\frac{1}{c}T_{\lambda,n}$ in the calculations above to define $Y_{\lambda,n}^{(c)}$. Similar to (3.12) and (3.13) one obtains

$$E\left((Y_{\lambda,n}^{(c)})^\top M Y_{\lambda,n}^{(c)}\right) = \lambda^2 (P_1^{(c)} X^{(c)})^\top U_n^{T_n-1} \rho_{n}(U_n^* M U_n^*) U_n^{T_n-1} P_1^{(c)} X^{(c)}) + o(\lambda^2)$$

$$E\left((Y_{\lambda,n}^{(c)})^* M Y_{\lambda,n}^{(c)}\right) = \lambda^2 (P_1^{(c)} X^{(c)})^* U_n^{T_n-1} \rho_{n}(U_n^* M U_n^*) U_n^{T_n-1} P_1^{(c)} X^{(c)} + o(\lambda^2)$$
Hence, by (3.19) this leads to
\[
\lim_{m \to \infty} \int_0^t m \mathbb{E} \left( \frac{Y^{(c)}(s)}{\sqrt{m}} MA^{(c)}_{\lambda,n} \right) ds = t P_t(\hat{c}) X_t(c) \quad (\text{for } c > 0)
\]
\[
\lim_{m \to \infty} \int_0^t m \mathbb{E} \left( \frac{Y^{(c)}(s)}{\sqrt{m}} MB^{(c)}_{\lambda,n} \right) ds = t P_t(\hat{c}) X_t(c) \quad (\text{for } c < 0)
\]
for the functions \( g_{cc'} \) as in (1.28) and (1.29). Using Proposition A.1, this gives the correlation between the limit processes \( X_t^{(c)} \) and \( X_t^{(c')} \) as described in Theorem 1.5.

### 4.2 The action on the flag manifold

In this subsection we prove Theorem 1.6. Recall we have
\[
\mathcal{F}_0 = \begin{pmatrix} a_1 & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & \cdots & a_k \end{pmatrix}, \quad \text{for all } j, a_j \in \text{GL}(d(c_j)), \text{ * arbitrary}
\]
so that \( [\mathcal{F}_0] \in \mathbb{R}^n \).

Let \( G(p, d) \) denote the Grassmannian manifold of \( p \)-dimensional subspaces of \( \mathbb{C}^d \). Note that \( \mathcal{F}^{(p)} \in G(p, d) \). As \( \mathcal{F} \) can be seen as a submanifold of \( \prod_{p=1}^d G(p, d) \) it will be sufficient to prove \( \mathcal{F}^{(p)} \Rightarrow \mathcal{F}_t^{(p)} \) in \( G(p, d) \) jointly for any (fixed) \( p \).

As the action of \( T \) and \( cT \) on \( \mathcal{F} \) or \( G(p, d) \) is the same, we may for fixed \( p \) scale the matrices such that \( d_2 < p \leq d_2 + d_1 \) in the sense of the definitions of \( d_1, d_2 \) in the Section 1.1 (Note that this basically means \( c_j = 1 \) for some \( j \), \( d_2 = d(c_1) + d(c_2) + \ldots + d(c_{j-1}) \) and \( d_1 = d(c_j) \)). Now for \( \mathcal{F}_1, \mathcal{F}_2 \in \text{GL}(d) \) one finds that
\[
\mathcal{F}_1^{(p)} = \mathcal{F}_2^{(p)} \quad \text{if and only if} \quad \mathcal{F}_1 = \mathcal{F}_2 \begin{pmatrix} M_1 & 0 \\ * & M_2 \end{pmatrix}, \quad M_2 \in \text{GL}(p), \ M_1 \in \text{GL}(d - p).
\]

Using blocks of size \( d_0 + d_1 \) and \( d_2 \) and representing \( [\mathcal{F}_0] \in \mathbb{R}^n \) as above we find
\[
\mathcal{F}_0 = \begin{pmatrix} A_0 & B_0 \\ 0 & D_0 \end{pmatrix} \quad \text{with} \quad A_0 = \begin{pmatrix} a_{00} & a_{01} \\ 0 & a_{11} \end{pmatrix}
\]
where \( D_0, a_{00} \) and \( a_{11} \) are invertible. Note that in fact \( a_{11} = a_j \) for some \( j \) as in the notations above and that \( a_{00} \) contains the \( a_k \) for \( k > j \) and \( D_0 \) contains the \( a_k \) for \( k < j \). So we can choose \( \lambda_0 = \mathcal{F}_0 \) and consider the processes \( X_{\lambda,n} \) as above. Then clearly \( \mathcal{F}_n^{(p)} = X_{\lambda,n}^{(p)} \) and in terms of representatives in \( G(p, d) \) they are equivalent to
\[
\begin{pmatrix} A_{\lambda,n} & B_{\lambda,n} \\ C_{\lambda,n} & D_{\lambda,n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -D_{\lambda,n}^{-1} C_{\lambda,n} & D_{\lambda,n}^{-1} \end{pmatrix} = \begin{pmatrix} X_{\lambda,n} & Z_{\lambda,n} \\ 0 & \mathbb{1}_{d_2} \end{pmatrix}.
\]

Note that from the proof of Theorem 1.1 the inverse \( D_{\lambda,n}^{-1} \) exists for small \( \lambda \) (with sufficiently high probability) and therefore, as we consider invertible matrices here, we also find that \( X_{\lambda,n} \) is invertible. As \( X_{\lambda,n}^{-1} \mid_{\{0\}} \Rightarrow (0 \mid_{\Lambda t D_0}) \) with \( \Lambda t \) invertible, we find for \( n \sim \lambda^{-2} \) and large...
n that \((\begin{smallmatrix} 1_{d_0} & x_{\lambda,n} & 0 \\ 0 & 1_{d_1} \end{smallmatrix})\) is invertible. Hence, the right hand side of (4.3) represents the same \(p\)-dimensional subspace as
\[
\begin{pmatrix} X_{\lambda,n} & Z_{\lambda,n} \\ 0 & 1_{d_2} \end{pmatrix} \begin{pmatrix} X_{\lambda,n}^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & X_{\lambda,n} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] (4.4)

Therefore by Theorem 1.1 we find
\[
F^{(p)}_{\sqrt{n}, \lfloor \varepsilon_n \rfloor} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = F^{(p)}_t. \quad (4.5)
\]

The last equation is easy to see if one realizes that the last \(p\) column vectors end somewhere inside the \(a_{11}\) term and therefore span indeed the same \(p\)-dimensional subspace as \(F_t\). Clearly, looking at this convergence jointly in \(p\) we obtain the correlations as in Theorem 1.5.

5 Application to random Schrödinger operators

In this section we will show the results from Section 1.2. and use the notations introduced there. Recall the following definitions and equations (1.22), (1.23) and (1.24):
\[
\begin{align*}
U &= \begin{pmatrix} \bar{Z} & 0 \\ 0 & Z \end{pmatrix}, \\
Q &= \begin{pmatrix} \Gamma & 0 & 0 \\ 0 & \bar{Z} & Z \\ 1_{d_h} & 0 & 0 \\ 0 & 1_{d_e} & 1_{d_e} & 0 \end{pmatrix},
\end{align*}
\] (5.1)

\[
T_k = T_{\varepsilon, \lambda, k} := Q^{-1} T_{\lambda,n}^{E+\lambda^2 \varepsilon} Q = T_0 + \lambda V_n + \lambda^2 \varepsilon W, \quad T_{\varepsilon, \lambda, [1,n]} = T_{\varepsilon, \lambda, n} \cdots T_{\varepsilon, \lambda, 1}, \quad (5.2)
\]

where
\[
T_0 = \begin{pmatrix} \Gamma & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & \Gamma^{-1} \end{pmatrix}, \quad V_n = Q^{-1} \begin{pmatrix} -V_n & 0 \\ 0 & 0 \end{pmatrix} Q, \quad W = Q^{-1} \begin{pmatrix} 1_d & 0 \\ 0 & 0 \end{pmatrix} Q. \quad (5.3)
\]

5.1 Limit SDE

In this subsection we prove Proposition 1.4 and stick to the case \(\sigma = 1\). The general case is a trivial modification where \(V_1\) given below is replaced by \(\sigma V_1\). In order to describe the limit SDE, we split the potentials into the corresponding hyperbolic and elliptic blocks. Recall the definition
\[
V_1 = \begin{pmatrix} V_h & V_{he} \\ V_{he} & V_e \end{pmatrix} \quad \text{where} \quad V_h \in \text{Mat}(d_h \times d_h), \quad V_e \in \text{Mat}(d_e \times d_e). \quad (5.4)
\]
Next, note that
\[ Q^{-1} = S \begin{pmatrix} 1_{d_h} & 0 & -\Gamma^{-1} & 0 \\ 0 & 1_{d_a} & 0 & -Z \\ 0 & -1_{d_a} & 0 & \bar{Z} \\ -1_{d_h} & 0 & \Gamma & 0 \end{pmatrix} \] (5.5)

where
\[ S_{\Gamma,Z} = \begin{pmatrix} -S_{\Gamma} & 0 & 0 & 0 \\ 0 & S_{\Gamma} & 0 & 0 \\ 0 & 0 & S_{\Gamma} & 0 \\ 0 & 0 & 0 & -S_{\Gamma} \end{pmatrix} \], \quad S_{\Gamma} = (\Gamma^{-1} - \Gamma)^{-1}, \quad S_{Z} = (\bar{Z} - Z)^{-1} \] (5.6)

we chose the sign on \( S_{\Gamma} \) this way, so that \( S_{\Gamma} > 0 \) is a positive diagonal matrix. This leads to
\[ \nu = S_{\Gamma,Z} \begin{pmatrix} -V_{h}\Gamma & -V_{he}Z & -V_{he}\Gamma^{-1} \\ -V_{he}\Gamma & -V_{e}Z & -V_{he}\Gamma^{-1} \\ V_{he}\Gamma & V_{e}\bar{Z} & V_{e}Z \\ V_{he}\Gamma & V_{he}\bar{Z} & V_{he}Z \end{pmatrix}, \quad \nu = S_{\Gamma,Z} \begin{pmatrix} \Gamma & 0 & 0 & \Gamma^{-1} \\ 0 & \bar{Z} & Z & 0 \\ 0 & -Z & -Z & 0 \\ -\Gamma & 0 & 0 & \Gamma^{-1} \end{pmatrix} \] (5.7)

In the notations as introduced in Section 1 and used for Theorem 1.1 we have \( \Gamma_{2} = \Gamma \) and
\[ V_{\lambda_{11}}U^* = S \begin{pmatrix} -V_{e} & -V_{e} \\ V_{e} & V_{e} \end{pmatrix} + \lambda e \begin{pmatrix} 1_{d_a} & 1_{d_a} \\ -1_{d_a} & -1_{d_a} \end{pmatrix} \] (5.8)
\[ V_{\lambda_{12}}\Gamma V_{\lambda_{21}}U^* = S \begin{pmatrix} V_{he}^{*}S_{\Gamma}V_{he} & V_{he}^{*}S_{\Gamma}V_{he} \\ -V_{he}^{*}S_{\Gamma}V_{he} & -V_{he}^{*}S_{\Gamma}V_{he} \end{pmatrix} \] (5.9)

with
\[ S = \begin{pmatrix} S_{Z} & 0 \\ 0 & S_{Z} \end{pmatrix} = \begin{pmatrix} (\bar{Z} - Z)^{-1} & 0 \\ 0 & (\bar{Z} - Z)^{-1} \end{pmatrix} \] (5.10)

as in (1.26). Also recall from (1.26) that we defined the positive, Hermitian matrix
\[ Q = \int_{(Z)} z \mathbb{E}(V_{he}^{*}S_{\Gamma}V_{he}) \bar{z} \, dz, \] (5.11)

where \( dz \) describes the Haar measure on \( (Z) \), the group of diagonal, unitary matrices generated by \( Z \). In order to calculate the drift term, note that
\[ \int_{(Z)} \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mathbb{E} \left( \begin{pmatrix} e1 - V_{he}^{*}S_{\Gamma}V_{he} & e1 - V_{he}^{*}S_{\Gamma}V_{he} \\ -e1 + V_{he}^{*}S_{\Gamma}V_{he} & -e1 + V_{he}^{*}S_{\Gamma}V_{he} \end{pmatrix} \begin{pmatrix} z \ 0 \\ 0 \ z \end{pmatrix} \right) dz = \begin{pmatrix} e1 - Q & 0 \\ 0 & -e1 + Q \end{pmatrix} \]

where we used that for any \( d_{e} \times d_{e} \) matrix \( M \) one finds
\[ \int_{(Z)} z M z \, dz = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Z^{k}M_{ij}Z^{k} = \left( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (z_{i}z_{j})^{k} M_{ij} \right)_{ij} = 0 \]
which follows as we have \(|z_iz_j|=1\) and \(\text{Im}(z_i) > 0\), \(\text{Im}(z_j) > 0\) which implies that \(z_i z_j \neq 1\) for any \(i, j \in \{1, \ldots, d_e\}\). Therefore, application of Theorem 1.4 gives

\[
d\Lambda_{\epsilon,t} = S \begin{pmatrix} \varepsilon I - Q & 0 \\ 0 & -\varepsilon I + Q \end{pmatrix} \Lambda_{\epsilon,t} dt + S Z \begin{pmatrix} dA_t & dB_t \\ -dB_t^* & -dC_t \end{pmatrix} \Lambda_{\epsilon,t}
\]

(5.12)

where \(A_t, B_t, C_t\) are matrix Brownian motions with \(A_t = A_t^*, C_t = C_t^*\). In order to express the covariances as described by (1.12) in more detail recall \(Z = \text{diag}(z_1, \ldots, z_{d_e})\), \(|z_j| = 1\), leading to

\[
\int_{\Omega} \prod_{j=1}^{d_e} z_{jj}^{n_j} \, dz = \chi \left( \prod_{j=1}^{d_e} z_{jj}^{n_j} \right) \quad \text{with} \quad \chi(z) = \begin{cases} 1 & \text{for} \quad z = 1 \\ 0 & \text{else} \end{cases}
\]

(5.13)

where \(z_{jj}\) is the \(j\)-th diagonal entry of the diagonal matrix \(z \in \Omega\), and \(n_j\) are integers. This leads to the following covariances,

\[
\mathbb{E}((A_t)_{ij}(A_t)_{kl}) = \mathbb{E}((A_t^*)_{ji}(A_t)_{kl}) = \mathbb{E}((C_t)_{ij}(C_t)_{kl}) = \mathbb{E}((C_t^*)_{ji}(C_t)_{kl})
\]

\[
= t \mathbb{E}((V_e)_{ij}(V_e)_{kl}) \chi(\tilde{z}_i \tilde{z}_j \tilde{z}_k \tilde{z}_l);
\]

\[
\mathbb{E}((B_t)_{ij}(B_t)_{kl}) = t \mathbb{E}((V_e)_{ij}(V_e)_{kl}) \chi(z_i z_j z_k z_l);
\]

\[
\mathbb{E}((B_t^*)_{ij}(B_t)_{kl}) = t \mathbb{E}((V_e)_{ij}(V_e)_{kl}) \chi(\tilde{z}_i \tilde{z}_j \tilde{z}_k \tilde{z}_l).
\]

The correlations between the Brownian motions are given by

\[
\mathbb{E}((A_t)_{ij}(C_t)_{kl}) = \mathbb{E}((A_t^*)_{ji}(C_t^*)_{kl}) = t \mathbb{E}((V_e)_{ij}(V_e)_{kl}) \chi(z_i \tilde{z}_j z_k \tilde{z}_l);
\]

\[
\mathbb{E}((A_t^*)_{ij}(B_t^*)_{kl}) = \mathbb{E}((A_t^*)_{ji}(B_t)_{kl}) = t \mathbb{E}((V_e)_{ij}(V_e)_{kl}) \chi(\tilde{z}_i \tilde{z}_j z_k \tilde{z}_l);
\]

\[
\mathbb{E}((C_t)_{ij}(B_t^*)_{kl}) = \mathbb{E}((C_t^*)_{ji}(B_t)_{kl}) = t \mathbb{E}((V_e)_{ij}(V_e)_{kl}) \chi(z_i \tilde{z}_j z_k \tilde{z}_l).
\]

This shows part (i) of Proposition 1.4 for \(\sigma = 1\). Changing \(V_e \), to \(\sigma V_1\) immediately gives the general case. If \(V_e\) is almost surely real, which is the case if \(O^* V_1 O\) is almost surely real, then one has \(C_t = A_t\) and \(B_t = B_t^*\) giving part (ii). Part (iii) of Proposition 1.4 follows from using the chaoticity assumption in the equations for the covariances.

### 5.2 Limiting eigenvalue process

In this subsection we will prove Theorem 1.3 and restrict without loss of generality to the case \(\sigma = 1\). We won’t need the precise form of the limit SDE but it is important how we obtain this SDE. Therefore we need to look at the matrix parts giving the Schur complement as in the proof of Theorem 1.4. Hence, using \(U\) and \(T_{\epsilon,\lambda,[1,n]}\) as above let

\[
\mathcal{R} = \begin{pmatrix} 1_{d_h} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1_{d_h} \end{pmatrix}, \quad \mathcal{X}_0 = \begin{pmatrix} 1_{d_h} & 0 & -1_{d_h} \\ 0 & 1_{2d_e} & 0 \\ 0 & 0 & 1_{d_h} \end{pmatrix}, \quad \mathcal{X}_{t,\lambda,n} = \mathcal{R}^{-n} T_{\epsilon,\lambda,[1,n]} \mathcal{X}_0.
\]

(5.14)
Using blocks of sizes $d_h + 2d_e$ and $d_h$, let

$$X_{\varepsilon,\lambda,n} = \begin{pmatrix} A_{\varepsilon,\lambda,n} & B_{\varepsilon,\lambda,n} \\ C_{\varepsilon,\lambda,n} & D_{\varepsilon,\lambda,n} \end{pmatrix}$$

and define

$$X_{\varepsilon,\lambda,n} = A_{\varepsilon,\lambda,n} - B_{\varepsilon,\lambda,n} D_{\varepsilon,\lambda,n}^{-1} C_{\varepsilon,\lambda,n}. \quad (5.15)$$

Then by Theorem 1.1, $X_{\varepsilon,1/\sqrt{n},\lfloor tn \rfloor} \xrightarrow{\text{in law}} \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{\varepsilon,t} \end{pmatrix}$, with the process $\Lambda_{\varepsilon,t} = \Lambda_{1,\varepsilon,t}$ as in Theorem 1.3 and in (1.25). Let us define

$$\Theta_0 := \Lambda_0^{-1} Q^{-1} \begin{pmatrix} 1_d \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \Gamma^{-1} - \Gamma \\ (\bar{Z} - Z) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1_d & 0 \\ -1_d & 0 \\ 0 & 1_{d_h} \end{pmatrix} \quad (5.16)$$

as well as

$$M_{\varepsilon,\lambda,n} = \begin{pmatrix} 1_d \\ -D_{\varepsilon,\lambda,n}^{-1} C_{\varepsilon,\lambda,n} \begin{pmatrix} 0 \\ 1_d \end{pmatrix} \\ D_{\varepsilon,\lambda,n}^{-1} \end{pmatrix} \in \text{GL}(d) \quad (5.17)$$

Then, for $t > 0$,

$$X_{\varepsilon,1/\sqrt{n},\lfloor tn \rfloor} \Theta_0 M_{\varepsilon,1/\sqrt{n},\lfloor tn \rfloor} \xrightarrow{\text{in law}} \begin{pmatrix} 0 & 0 \\ 0 & 1_{d_h} \end{pmatrix} \quad (5.18)$$

Let us also define

$$\Theta_n^* := \begin{pmatrix} 0 & 1_d \\ \Gamma^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1_d \\ 0 \end{pmatrix} Q R_n = \begin{pmatrix} 0 & \bar{Z}^{n+1} & Z^{n+1} & 0 \\ 1_{d_h} & 0 & 0 & 1_{d_h} \end{pmatrix}. \quad (5.19)$$

An energy $E + \lambda^2 \varepsilon$ is an eigenvalue of $H_{\lambda,n}$, precisely if there is a solution to the eigenvalue equation with $\psi_0 = 0$ and $\psi_{n+1} = 0$, i.e. if and only if

$$\det \begin{pmatrix} 1_d & 0 \\ \Gamma^{-1} & 0 \end{pmatrix} T_{\lambda,[1,n]}^{E + \lambda^2 \varepsilon} \begin{pmatrix} 1_d \\ 0 \end{pmatrix} = 0. \quad (5.20)$$

As $T_{\lambda,[1,n]}^{E + \lambda^2 \varepsilon} = Q R_n^* X_{\varepsilon,\lambda,n} X_{\lambda,n}^{-1} Q^{-1}$, this is equivalent to

$$\det (\Theta_n^* X_{\varepsilon,\lambda,n} X_{\lambda,n}^{-1} \Theta_0 M_{\varepsilon,\lambda,n}) = 0 \quad (5.21)$$

Along a subsequence $n_k$ of the positive integers where $Z^{n_k+1}$ converges to $Z_*$, we find that for $\lambda_k = 1/\sqrt{n_k}$

$$\Theta_{n_k}^* X_{\varepsilon,\lambda_k,n_k} \Theta_0 M_{\varepsilon,\lambda_k,n_k} \xrightarrow{\text{in law}} \begin{pmatrix} \bar{Z}_* & Z_* \end{pmatrix} \Lambda_{\varepsilon,1} \begin{pmatrix} 1_d \\ -1_d \end{pmatrix} \begin{pmatrix} 0 \\ 1_{d_h} \end{pmatrix} \quad (5.22)$$

in law, uniform on compact sets in $\varepsilon$. As the determinant is a holomorphic function, we find that the zero processes of the determinant also converge in law. This proves Theorem 1.3 for $\sigma = 1$. A proper rescaling immediately implies the result for general $\sigma$. 33
5.3 Limiting GOE statistics

In this subsection we will prove Theorem 1.2 by reduction to the work in [VV1]. Without loss of generality we focus on energies $E$ smaller than 0 and consider $r = 1$. The more general case needs some more care and notations in the subdivision into elliptic and hyperbolic channels, but the main calculations remain the same. We need to consider the SDE limit as described above a bit more precisely for this particular Anderson model as in (1.19) with $A = Z_d$ and $V_n$ as in (1.17).

In Proposition 1.4 especially for the definitions of $V_h$, $V_e$ and $V_{he}$ it was assumed that $A$ is diagonal. So in order to use these calculations we need to diagonalize $Z_d$ and see how this unitary transformation changes $V_n$.

We let $d \geq 2$, then $Z_d$ is diagonalized by the orthogonal matrix $O$ given by

$$O_{jk} = \sqrt{2/(d+1)} \sin(\pi jk/(d+1)).$$

(5.23)

The corresponding eigenvalue of $Z_d$ with eigenvector being the $j$-th column vector of $O$ is given by

$$a_j = 2 \cos(\pi j/(d+1)), \quad j = 1, \ldots, d.$$ (5.24)

For $-2 < E < 0$ there is $d_h < d$ such that

$$2 \cos(\pi j/(d+1)) - E > 2 \quad \text{for} \quad j = 1, \ldots, d_h \quad \text{and}$$

$$-2 < 2 \cos(\pi j/(d+1)) - E < 2 \quad \text{for} \quad j = d_h + 1, \ldots, d.$$ (5.25) (5.26)

So we have $d_h$ hyperbolic and $d_e = d - d_h$ elliptic channels and the upper $d_h \times d_h$ block of $O^T Z_d O$ corresponds to the hyperbolic channels. Using (1.17) and the notations as in (5.4) we have

$$\begin{pmatrix} V_h & V_{he} \\ V_{he}^* & V_e \end{pmatrix} = O^T \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_d \end{pmatrix} O, \quad E(v_j) = 0, \quad E(v_j v_k) = \delta_{jk}. \quad (5.27)$$

Let $E$ be such that $Z$ is chaotic, then by Proposition 1.4 (iii) we need to consider the following the covariances

$$E(\langle |V_e|_{ij} \rangle^2) = E(\langle |O^T V_1 O|_{i+d_h,j+d_h} \rangle^2 = \langle |O_{i+d_h}|^2, |O_{j+d_h}|^2 \rangle$$

(5.28)

$$E(\langle V_e \rangle_{ii} \langle V_e \rangle_{jj}) = E(\langle (O^T V_1 O)_{i+d_h,i+d_h} (O^T V_1 O)_{j+d_h,j+d_h} \rangle) = \langle |O_{i+d_h}|^2, |O_{j+d_h}|^2 \rangle.$$ (5.29)

Here, by $|O_i|^2$ we denote the vector $(|O_{k,i}|^2)_{k=1,\ldots,d}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product. As stated in [VV1], one finds

$$(d+1) \langle |O_i|^2, |O_j|^2 \rangle = \begin{cases} 3/2 & \text{for} \quad i = j \\ 1 & \text{for} \quad i \neq j. \end{cases} \quad (5.30)$$

Let us further calculate the drift contribution $Q$ from the hyperbolic channels as introduced above. Using chaoticity, it is not hard to see from (5.11) that $Q$ is diagonal. Moreover one has

$$Q_{jj} = E(V_{he}^* S \Gamma V_{he})_{jj} = \sum_{k=1}^{d_h} (S \Gamma)_{kk} E(\langle |V_{he}|_{kj} \rangle^2) = \sum_{k=1}^{d_h} \frac{\langle |O_k|^2, |O_{j+d_h}|^2 \rangle}{\gamma_k - \gamma_j}. \quad (5.31)$$

34
It follows that $Q$ is a multiple of the unit matrix, more precisely

$$Q = q 1 \quad \text{with} \quad q = \frac{1}{d + 1} \sum_{k=1}^{d} (\gamma_k^{-1} - \gamma_k)^{-1}. \quad (5.32)$$

Thus, using Proposition 1.4 we obtain the following SDE limits,

$$d\Lambda_{\sigma,\varepsilon,t} = S(\varepsilon - \sigma^2 q) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda_{\sigma,\varepsilon,t} + \sigma^2 S \begin{pmatrix} dA_t & dB_t \\ -dB_t^T & -dA_t \end{pmatrix} \Lambda_{\sigma,\varepsilon,t} \quad (5.33)$$

where $A_t$ and $B_t$ are independent matrix Brownian motions, $A_t$ is Hermitian, $B_t$ complex symmetric, i.e.

$$A_t^* = A_t, \quad B_t^T = B_t \quad (5.34)$$

with covariance structure

$$\mathbb{E}((B_t)_{ij}^2) = \mathbb{E}((A_t)_{ij}^2) = \mathbb{E}((A_t)_{ii}(A_t)_{jj}) = \mathbb{E}((A_t)_{ii}) = \begin{cases} \frac{q}{t} & \text{for } i=j \\ t & \text{for } i \neq j. \end{cases} \quad (5.35)$$

Except for the additional drift $\sigma^2 q$ which can be seen as a shift in $\varepsilon$, this is the exact same SDE as it appears in [VV1]. In fact, the matrix $S$ here corresponds to $iS^2$ as in [VV1] and the process there corresponds to the process above conjugated by $|S|^{1/2}$.

Thus, from now on the proof to obtain the Sine1 kernel and GOE statistics follows precisely the arguments as in [VV1].

## A A limit theorem for Markov processes

We need the following variation of Proposition 23 in [VV2].

**Proposition A.1.** Fix $T > 0$, and for each $m \geq 1$ consider a Markov chain

$$(X_n^m \in \mathbb{R}^d, n = 1 \ldots \lfloor mT \rfloor).$$

as well as a sequence of “good” subsets $G_m$ of $\mathbb{R}^d$. Let $Y_n^m(x)$ be distributed as the increment $X_{t+1}^m - x$ given $X_n^m = x \in G_m$. We define

$$b^m(t,x) = m\mathbb{E}[Y_{mt}^m(x)], \quad a^m(t,x) = m\mathbb{E}[Y_{mt}^m(x)Y_{mt}^m(x)^T].$$

Let $d' \leq d$, and let $\tilde{x}$ denote the first $d'$ coordinates of $x$. These are the coordinates that will be relevant in the limit. Also let $\tilde{b}^m$ denote the first $d'$ coordinates of $b^m$ and $\tilde{a}^m$ be the upper left $d' \times d'$ sub-matrix of $a^m$.

Furthermore, let $f$ be a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ with $f(m) = o(m)$, i.e $\lim_{m \to \infty} f(m)/m = 0$.

Suppose that as $m \to \infty$ for $x, y \in G_n$ we have

$$|\tilde{a}^m(t,x) - \tilde{a}^m(t,y)| + |\tilde{b}^m(t,x) - \tilde{b}^m(t,y)| \leq c|\tilde{x} - \tilde{y}| + o(1) \quad (A.1)$$

and

$$\sup_{x \in G_m,n} \mathbb{E}[|\tilde{Y}_n^m(x)|^2] \leq cm^{3/2} \quad \text{for all } n \geq f(m), \quad (A.2)$$
and that there are functions $a,b$ from $\mathbb{R} \times [0,T]$ to $\mathbb{R}^{d',2}$, respectively with bounded first and second derivatives so that uniformly for $x \in G_m$,

$$\sup_{x \in G_m,t} \left| \int_0^t \tilde{a}^m(s,x) \, ds - \int_0^t a(s,\tilde{x}) \, ds \right| \to 0$$ \quad (A.3)

$$\sup_{x \in G_m,t} \left| \int_0^t \tilde{b}^m(s,x) \, ds - \int_0^t b(s,\tilde{x}) \, ds \right| \to 0.$$ \quad (A.4)

Suppose further that

$$\tilde{X}^m_{f(m)} \implies X_0.$$

and that $\mathbb{P}(X^m_n \in G_m \text{ for all } n \geq f(m)) \to 1$. Then $(\tilde{X}^m_{\lfloor mt \rfloor}, 0 < t \leq T)$ converges in law to the unique solution of the SDE

$$dX = b \, dt + a \, dB, \quad X(0) = X_0.$$

**Proof.** This is essentially Proposition 23 in [VV2]. The first difference is that the coordinates $d' + 1, \ldots, d$ of the $X^m$ do not appear in the limiting process. A careful examination of the proof of that Proposition shows that it was not necessary to assume that all coordinates appear in the limit, as long as the auxiliary coordinates do not influence the variance and drift asymptotics.

The second difference is the introduction of the “good” set $G_m$, possibly a proper subset of $\mathbb{R}^d$. Since we assume that the processes $X^m$ stay in $G_m$ with probability tending to one, we can apply the Proposition 23 of [VV1] to $X^m$ stopped when it leaves this set. Then, the probability that the stopped process is different from the original tends to zero, completing the proof.

The third difference is the weak convergence of $\tilde{X}^m_{f(m)}$ instead of $\tilde{X}^m_0$ and that we have the bound in (A.2) only for $m \geq f(m)$. Note that for the Markov family $\tilde{X}^m_l = \tilde{X}^m_{\max(l,f(m))}$ all the same conditions apply with $f(m) = 0$ and the initial conditions converge weakly. Moreover, for any fixed $t > 0$ and $m$ large enough one has $\tilde{X}^m_{\lfloor mt \rfloor} = X^m_{\lfloor mt \rfloor}$. \qed

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