The spectrum of multi-flavor $QCD_2$ and the non-Abelian Schwinger equation

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Abstract

Massless $QCD_2$ is dominated by classical configurations in the large $N_f$ limit. We use this observation to study the theory by finding solutions to equations of motion, which are the non-Abelian generalization of the Schwinger equation. We find that the spectrum consists of massive mesons with $M^2 = \frac{e^2 N_f}{2\pi}$ which correspond to Abelian solutions. We generalize previously discovered non-Abelian solutions and discuss their interpretation. We prove a no-go theorem ruling out the existence of soliton solutions. Thus the semi-classical approximation shows no baryons in the case of massless quarks, a result derived before in the strong-coupling limit only.

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1 Introduction

Since the seminal work of ’t Hooft[1], two-dimensional QCD ($QCD_2$) serves us as a theoretical laboratory of the real world four-dimensional QCD. Questions concerning the IR nature of QCD, such as the spectrum and confinement can be answered in this framework (For recent reviews see [2, 3]).

During the past years, there has been a lot of progress in this field. Apart from ’t Hooft large $N_c$ limit, some other limits were investigated. In particular, the model with adjoint matter attracted a lot of attention lately[4, 5, 6, 7, 8]. This model, due to universality[7], is equivalent, in the massless case, to a model with fundamental fermions and $N_f = N_c$. Another interesting limit which was applied in the past, is the massive strongly coupled model ($e \gg m$), in which the low lying hadronic spectrum was found[10]. The question of confinement versus screening in massless and massive $QCD_2$ has been also thoroughly investigated in recent years[11, 12, 13, 14].

In this paper we discuss the problem of massless $QCD_2$ with large number of flavors. It seems natural to analyze the theory in this limit since large $N$ ($N_c$ or $N_f$) simplifies the problem. The space of massless theories, in the $N_f, N_c$ plane is described in Fig.1

![Figure 1: Massless QCD2 - The flavor-color plane](image)

The diagonal line represents theories with $N_f = N_c$, or adjoint fermions. The arc represents theories with large $N$. The flow along this arc seems to be interesting. Starting from small $N_f$ and large $N_c$ (with $e^2N_c$ fixed), the spectrum of the theory consists of a single Regge trajectory. Moving along the arc, we arrive to the adjoint fermions model, where the spectrum seems to
contain infinitely many Regge trajectories. We claim that when we approach the large \( N_f \) regime, all these trajectories collapse to a few massive mesons. Demonstrating this phenomena is the purpose of this paper.

The main idea that is used in the present paper is that in the bosonized description of massless \( QCD_2 \) the color degrees of freedom can be separated from the flavor ones, and the “colored” action is multiplied by an overall factor \( N_f \). Thus, \( 1/N_f \) plays the role of \( \hbar \) and in the limit of large \( N_f \) the system is classical and therefore the physical states are determined by solutions of the equations of motion\[12\].

The study of the equations of motion transformed into a form of “non-Abelian Schwinger equation”\[12\]. We investigate three classes of solutions of this equation: (i) Abelian plane wave solutions, (ii) non-Abelian plane wave solutions, and (iii) soliton solutions. We show that the first class corresponds to mesons with mass of \( M^2 = \frac{e^2 N_f}{2\pi} \) and a degeneracy of \( N_c^2 - 1 \). The \( SU(2) \) non-Abelian solutions are generalized to \( SU(N_c) \) solutions and are shown to relate (on the real line) to constant solutions. As for soliton solutions, in \[12\] it was argued that there are no such solutions. Here we present a proof of this no-go theorem.

The paper is organized as follows: In the second section we describe the model and derive the equations of motion. In the third section we show that the equations of motion, in a particular gauge, can be derived from a Hamiltonian. The trivial Abelian solutions are also described in this section. Next, in the fourth and fifth section, we describe non-Abelian solutions to the equations of motion. In the sixth section, we show that apart from the Abelian solution and a constant solution (space time independent), there are no other solutions which can be interpreted as particles in the quantum theory. We arrive to this conclusion by showing that there are no solitons solutions. Because any solution which describes a particle with well defined momentum can be Lorentz transformed into a particle at rest with a time independent energy-momentum tensor, no solitons implies that there are no other single particle states in this theory.

Next, we show how mass term can affect the spectrum of the theory. The last section is devoted to a summary.

We use the following conventions: \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \) and the Lie algebra generators are normalized such that \( tr \ T^a T^b = \frac{1}{2} \delta^{ab} \).
2 Multi-flavor QCD$_2$

Massless multi-flavor QCD$_2$ with fermions in the fundamental representation of $SU(N_c)$ is described by the following action

$$S = \int d^2x \ tr\left(-\frac{1}{2e^2}F^2_{\mu\nu} + i\bar{\Psi} \slashed{D}\Psi\right)$$

where $\Psi = \Psi^i_a, i = 1 \ldots N_c, a = 1 \ldots N_f$.

It is natural to bosonize this system, since bosonization in the $SU(N_c) \times SU(N_f) \times U_B(1)$ scheme decouples color and flavor degrees of freedom (in the massless case). The bosonized form of the action of this theory is given by [2]

$$S_{\text{bosonized}} = (2) N_f S_{\text{WZW}}(h) + N_c S_{\text{WZW}}(g) - \int d^2x \ tr\left(\frac{1}{2e^2}F^\mu F^\mu + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi\right)$$

$$-\frac{N_f}{2\pi} \int d^2x \ tr(ih^\dagger \partial_+ h A_- + ih \partial_- h^\dagger A_+ + A_+ h A_- h^\dagger - A_+ A_-)$$

where $h$ is a unitary $N_c \times N_c$ matrix which belongs to the color gauge group, $g \in SU(N_f), \phi$ is the baryon number and $S_{\text{WZW}}$ stands for the Wess-Zumino-Witten action, which for complex fermions reads

$$S_{\text{WZW}}(g) = \frac{1}{8\pi} \int_\Sigma d^2x \ tr(\partial_\mu g \partial^\mu g^{-1}) +$$

$$\frac{1}{12\pi} \int_B d^3y e^{ijk} tr(g^{-1} \partial_i g)(g^{-1} \partial_j g)(g^{-1} \partial_k g),$$

Since we are interested in the massive spectrum of the theory and the flavor degrees of freedom are entirely decoupled from the system and they are massless, we can put aside the $g$ and $\phi$ fields (There is a residual interaction of the zero modes of the $g, h$ and $\phi$ fields, but it is not crucial to our discussion[9]).

We arrive at the following action

$$S = N_f\{S_{\text{WZW}}(h) - \int d^2x \ tr\left(\frac{1}{8\pi\alpha} F^\mu F^\mu\right)$$

$$-\frac{1}{2\pi} \int d^2x \ tr(ih^\dagger \partial_+ h A_- + ih \partial_- h^\dagger A_+ + A_+ h A_- h^\dagger - A_+ A_-)\},$$
where $\tilde{\alpha} = \frac{e^2 N_f}{4\pi}$. Since the action is multiplied by an overall $N_f$ factor, the partition function is dominated by the classical configurations in the $N_f \to \infty$ limit. It is equivalent to the $\hbar \to 0$ limit. Thus an investigation of the equations of motion in this limit, captures the quantum behavior of this theory.

The equations of motion for the dual of the gauge field $F = \frac{1}{2} \epsilon^{\mu\nu} F_{\mu\nu}$ are\cite{15,16}

$$ (D^\mu D_\mu + 2\tilde{\alpha}) F = 0, \quad (3) $$

where $D_\mu = \partial_\mu - i[A_\mu, \cdot]$. This equation is the non-Abelian generalization of the Schwinger equation for the massive gluon. Finding solutions to this equation is the main goal of this paper.

### 3 The Hamiltonian and the Abelian solution

The equation of motion (3) can also be derived from an effective Hamiltonian. Starting with the action (2), fixing the gauge $A^- = 0$ and integrating over $A^+$, we obtain the following action

$$ S = N_f \left\{ S_{WZW}(h) + \pi \tilde{\alpha} \int d^2 x \, tr \, \tilde{j}^+ \frac{1}{\partial^-} \tilde{j}^+ \right\}, \quad (4) $$

Where $\tilde{j}^+ = \frac{e}{2\pi} h \partial_- h^+$. The light-cone energy of this system is

$$ P^- = -\frac{e^2}{4} \int dx^- \, tr \, j \frac{1}{\partial^2} j, \quad (5) $$

where $j = N_f \tilde{j}^+ = \sqrt{2} \Psi \gamma^+ \Psi$.

In order to derive to equation of motion, the canonical commutation relations should be specified. In this case it is the Kac-Moody algebra

$$ [j^a(x), j^b(y)] = i N_f \frac{\delta^{ab}}{\pi} \partial_- \delta(x - y) + 2i f^{abc} j^c(x) \delta(x - y) \quad (6) $$

The equation of motion are therefore

$$ \partial_+ j^a(x) = i[H, j^a(x)] = -\frac{e^2 N_f}{4\pi} \frac{1}{\partial_-} j^a(x) + \frac{e^2}{2} f^{abc} j^b(x) \frac{1}{\partial^2} j^c(x) \quad (7) $$

4
Substituting the relation $\partial_2^2 A_+ = -\frac{e^2}{2} j$ (which is the equation of motion of the gauge field) we obtain
\[ \partial_+ \partial_- A_+ - i[A_+, \partial_- A_+] + \tilde{\alpha} \partial_- A_+ = 0, \quad (8) \]
where now $[.,.]$ stands for the classical $SU(N_c)$ commutator. This equation is a total derivative. It is the form of (3), in the gauge $A_-=0$. Integrating this equation with zero boundary condition (by use of the residual gauge freedom) we arrive at
\[ \partial_+ \partial_- A_+ - i[A_+, \partial_- A_+] + \tilde{\alpha} A_+ = 0 \quad (9) \]
The above equation has a simple 'Abelian' solution. An Abelian solution is a solution in which $A_+$ points in some special direction in the algebra so that the commutator term in (9) vanishes.

Choosing it to be in the 1 direction, we get
\[ A^a = \delta^{a1} \exp i(k_- x_+ + k_+ x_-), \quad (10) \]
with $k_- k_+ = \tilde{\alpha}$. This solution is the non-Abelian analog of the Schwinger meson in the Abelian case. We argue that in the large $N_f$ limit, the spectrum of the theory contains physical asymptotic states which are mesons with $M^2 = 2\tilde{\alpha}$. Since we have $N_c^2 - 1$ possible directions for $A_+$, this plane wave meson solution appears with a degeneracy of $N_c^2 - 1$. These states were already formed using a different approach, in [10]. Whether these states are physical, namely asymptotic, at small $N_f$, has not been settled yet (see discussion in [16]). A recent suggestion [17] is that they correspond to massive poles in the gluon propagator. As a result, massless QCD$_2$ is in a screening phase.

Apart from the trivial Abelian solution, eq. (9) admits non-Abelian solutions. In the next section we review the special case of $SU(2)$.

4 Non-Abelian solutions in the case of $SU(2)$ and the energy-momentum tensor

A solution of (9) in the special case of $SU(2)$ was given in [12]. We expand $A_+$ in terms of the $SU(2)$ matrices $T^0 = \sigma_3$ and $T^\pm = \sigma_1 \pm i\sigma_2$. A solution for $A_+$ is
\[ A_+ = \frac{\tilde{\alpha} - k_- k_+ T^0}{k_-} \]
\[ + \sqrt{\frac{(k_- k_+ - \tilde{\alpha})\tilde{\alpha}}{4k_-^2}} T^+ \exp -i(k_- x_+ + k_+ x_-) \]
\[ + \sqrt{\frac{(k_- k_+ - \tilde{\alpha})\tilde{\alpha}}{4k_-^2}} T^- \exp i(k_- x_+ + k_+ x_-) \]

These solutions are truly non-Abelian, since there is no way to rotate (11) to the abelian solution. It is the non-Abelian analog of Abelian plane-waves.

Using \( \partial^2 A_+ = -i\bar{\alpha}h\partial_- h^\dagger \) we can compute the group element \( h \) which correspond to the above solution

\[ h = \exp -i k_- x_+(T^0 + q(T^+ + T^-)) \exp i(k_- x_+ + k_+ x_-)T^0, \]

where \( q = \frac{1}{2}\sqrt{\frac{k_- k_+}{\tilde{\alpha}}} - 1. \)

In order to find the mass of the non-Abelian solutions let us calculate the energy momentum tensor component. The mass of the states would be \( M^2 = 2P^+ P^- \).

\( T^{++} \) is given by the Sugawara construction

\[ T^{++} = \frac{1}{N_f + N_c} \pi \ tr : j^+ j^+ : , \]

where the \( N_c \) contribution in the denominator is due to normal ordering. Since our treatment is classical (and \( N_f \gg N_c \)), we will use the classical expression

\[ T^{++} = \frac{\pi}{N_f} tr(j^+ j^+) = \frac{N_f}{4\pi\tilde{\alpha}^2} tr(\partial_-^2 A_+)^2 \]

The second relevant component can be read from the action

\[ T^{--} = N_f \pi \tilde{\alpha} tr(\frac{1}{\partial_-} j^+)^2 = \frac{N_f}{4\pi\tilde{\alpha}} tr(\partial_- A_+)^2 \]

One can check that the conservation of the energy-momentum tensor follows from the equation of motion (9).
Substituting the $SU(2)$ solution we obtain

$$T^{++] = \frac{1}{2\pi} N_f k_2 \frac{k_+ k_-}{\tilde{\alpha}} (1 - \frac{\tilde{\alpha}}{k_+ k_-})$$

(16)

$$T^{--} = \frac{1}{2\pi} N_f k_+ k_- (1 - \frac{\tilde{\alpha}}{k_+ k_-})$$

(17)

Since $T^{++}$ and $T^{--}$ are space independent, $P^+ = \int dx^- T^{++}$ and $P^- = \int dx^- T^{--}$ diverge. This is also the case in the Abelian solution. In the later case we can construct wave-packets with finite energy. For the non-Abelian solution a linear combination is not a solution of the equations of motion and there is no room for a wave-packet construction. Therefore the interpretation of this solution is not of a single particle state, but a coherent state of infinite particles, like a constant electric field in the real 4D world.

However, one may also adopt a DLCQ approach where the light-cone space coordinate is taken to be discretized. In this case the possible values of $k_\pm$ have to obey $k_\pm L = 2\pi n$. If on top of that one requires that $h(x^-) = h(x^- + L)$ and hence $\sqrt{\frac{k_+ k_-}{\tilde{\alpha}}} k_- L = 2\pi m$, then the expression of $M^2 = 2P^+ P^-$ takes the following form

$$M^2 = 2\tilde{\alpha} N_f^2 (\frac{m^2}{n} - n)^2$$

(18)

In the continuum limit of $L \to \infty$ for finite $n$ we have $k_- \to 0$. However, in the limit of $k_+ \to \infty$ such that $k_+ k_-$ is finite, $M^2$ is finite. It will be interesting to look for such states in the DLCQ studies which are done at finite $N_f$.

The states in (18) decouple in the large $N_f$ limit, except possibly those with $m = n$, as they are massless.

### 5 $SU(N)$ Non-Abelian solutions

Let us find $SU(N)$, non-Abelian solutions to equation(3). An ansatz for such solutions is constant $A_+$ and $A_-$. The equation of motion for such solutions takes the following form

$$- [A_+,[A_-, [A_+, A_-]]] + \tilde{\alpha}[A_+, A_-] = 0$$

(19)
A convenient choice of generators which spans the $SU(N)$ algebra space is

$$(T_j^i)_b = \delta_b^i \delta_j^a - \frac{1}{N} \delta_j^i \delta_b^a$$  \hspace{1cm} (20)$$

The $N^2$ matrices $T_j^i$ are not linear independent since $\sum_i T_i^i = 0$. In addition, $T_j^i$ obey the following relations:

$$tr \ T_j^i = 0$$  \hspace{1cm} (21)
$$tr \ T_j^i T_l^k = \delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k$$  \hspace{1cm} (22)
$$[T_j^i, T_l^k] = \delta_l^i T_j^k - \delta_j^i T_l^k$$  \hspace{1cm} (23)

Recall we can always bring $A_-$ say to diagonal form by a constant gauge transformation. Let us restrict ourselves here to a special $A_-$ so that

$$A_- = T_1^1$$  \hspace{1cm} (24)
$$A_+ = \alpha_j^i T_i^j,$$  \hspace{1cm} (25)

where $\alpha$ is a constant Hermitian matrix.

The most general solution can be brought to the form

$$A_+ = \hat{\alpha} T_1^1 + \sum_{i=2}^N v_i(T_i^1 + T_i^i),$$  \hspace{1cm} (26)

by use of $SU(N-1)$ rotation in the directions $i = 2, ..., N$. $v_i$ are arbitrary real numbers.

These solutions are directly related to the $SU(2)$ solutions which were described in the previous section. A gauge transformation along the $T_1^1$ direction, which depends on $x^-$ only $\hat{U} = \exp i x^- T_1^1$ can be used to eliminate $A_-$. The resulting $\tilde{A}_+$ is a solution which can be written in the following way

$$\tilde{A}_+ = \hat{U}^\dagger(x^-) A_+ \hat{U}(x^-)$$  \hspace{1cm} (27)$$

Thus $\tilde{A}_+$ can be written as a plane-wave, similarly to (11) (The $SU(2)$ solution was given as a rotation of a constant matrix in [12]). This is a truly $SU(N)$ plane wave.

The discussion at the end of section 4 regarding divergence of energy on the line, and the situation in the DLCQ case applies here too.
6 No-Go theorem: There are no solitons

In this section we would like to prove that there are no solitons solutions for eq. (3). Let’s start with the Abelian case. For the Schwinger equation, assuming static solution leads to

\[-\partial_1^2 + \frac{e^2}{\pi} F = 0, \tag{28}\]

which, clearly, has no static solution with finite energy.

In the non-Abelian case, the problem is harder. For a soliton like solution, only gauge invariant quantities should be time independent. Thus \(A_\mu\) and \(F\) may in principle be time dependent. Our definition for solitons is solutions in which the Hamiltonian and momentum densities are time independent

\[\partial_0 T^{++} = 0 \tag{29}\]
\[\partial_0 T^{+-} = 0 \tag{30}\]

In addition, the total energy of the solution should be finite.

Conditions (29) and (30) together with equations (14) and (15) implies

\[(\partial_+ + \partial_-)tr \left\{ \frac{1}{2} (\partial_- F)^2 \right\} = 0 \tag{31}\]

\[(\partial_+ + \partial_-)tr \left\{ \frac{1}{2} \tilde{\alpha} F^2 \right\} = 0 \tag{32}\]

Now, let us assume that solitons solutions exists.

**Lemma 1:** For solitons \(tr \left\{ \frac{1}{2} (\partial_- F)^2 \right\} = tr \left\{ \frac{1}{2} \tilde{\alpha} F^2 \right\}.

**Proof:** Multiplying the equations of motion (8) by \(\partial_- F\) (note that \(F = \partial_- A_+\)) and taking the trace we obtain

\[tr \partial_- F \partial_+ \partial_- F + tr \tilde{\alpha} F \partial_- F = 0 \tag{33}\]

from which

\[\partial_+ tr \left\{ \frac{1}{2} (\partial_- F)^2 \right\} + \partial_- tr \left\{ \frac{1}{2} \tilde{\alpha} F^2 \right\} = 0 \tag{34}\]

Using (34) we obtain

\[tr \left\{ \frac{1}{2} (\partial_- F)^2 - \frac{1}{2} \tilde{\alpha} F^2 \right\} = f(x^+) \tag{35}\]
Since \( f(x^+) \) is \( x^- \) independent, we can evaluate its value at \( x^- = \infty \). But the L.H.S. of (35) at infinity is zero, since the field \( F \) should vanish at this point (otherwise, the energy of the system diverges). Thus \( f(x^+) = 0 \). QED.

**Lemma 2:** Solitons obey \( tr \left( \frac{1}{2} F^2 + \frac{1}{2} \tilde{\alpha} A_+^2 \right) \).

**Proof:** The proof is similar to the proof of Lemma 1. Multiplying (9) by \( F \) and taking the trace we obtain

\[
tr \ F \partial_+ F + tr \ \tilde{\alpha} A_+ \partial_- A_+ = 0 \quad (36)
\]

Using (32) we arrive to

\[
tr \ \left( \frac{1}{2} F^2 - \frac{1}{2} \tilde{\alpha} A_+^2 \right) = g(x^+) \quad (37)
\]

Now, let us show that \( g(x^+) = 0 \). The L.H.S. of (37) is zero at \( x^- = \infty \) since both \( F \) and \( tr A_+^2 \) should vanish at this point. To see that \( tr A_+^2 = 0 \) at infinity we multiply equation (4) by \( A_+ \) and take the trace. we obtain

\[
tr \ A_+ \partial_+ A_+ + tr \ \tilde{\alpha} A_+ A_+ = 0 \quad (38)
\]

Since \( F = \partial_- A_+ = 0 \) at \( x^- = \infty \) and \( A_+ \) is bounded, \( tr A_+ A_+ \) is also zero. Hence \( g(x^+) = 0 \). QED.

Now, let us look at the quantity

\[
\int_{-\infty}^{\infty} dx^- tr \ (\partial_- F + \tilde{\alpha} A_+)^2 \quad (39)
\]

By opening the brackets and integration by parts we arrive to

\[
\int_{-\infty}^{\infty} dx^- tr \ ( (\partial_- F)^2 + \tilde{\alpha}^2 A_+^2 - 2\tilde{\alpha} \partial_- A_+ F ) = \quad (40)
\]

\[
\int_{-\infty}^{\infty} dx^- tr \ ( (\partial_- F)^2 + \tilde{\alpha}^2 A_+^2 - 2\tilde{\alpha} F^2 ) \quad (41)
\]

But according to Lemma 1 and Lemma 2 this quantity is zero. Now (39) is an integral of positive contributions. In addition,

\[
tr \ (\partial_- F + \tilde{\alpha} A_+)^2 = \frac{1}{2} \sum_a (\partial_-^2 A_+ + \tilde{\alpha} A_+^a)(\partial_-^2 A_+ + \tilde{\alpha} A_+^a) \quad (42)
\]

Hence

\[
\partial_-^2 A_+^a + \tilde{\alpha} A_+^a = 0 \quad (43)
\]
The solution of this equation is

\[ A_+ = e^{i\sqrt{\tilde{\alpha}}x} M(x^+) + e^{-i\sqrt{\tilde{\alpha}}x} M^\dagger(x^+), \tag{44} \]

where \( M(x^+) \) is a matrix.

From the equation of motion (9) we get

\[ M = Ce^{i\sqrt{\tilde{\alpha}}x}, \tag{45} \]

where \( C \) is some arbitrary matrix.

Thus the solution is

\[ A_+ = Ce^{i\sqrt{\tilde{\alpha}}x^0} \tag{46} \]

which is, obviously, not a soliton (It carries infinite energy).

We conclude that since there are no solitons in this theory, there are no asymptotic fermionic states. Furthermore, apart from the trivial Abelian solutions, there are no other solutions which can be interpreted as single-particle asymptotic states. See also discussion at the end of section 4.

7 Adding a mass term

In this section we would like to comment on the spectrum of the theory in the presence of a massive fermions with bare mass \( m_q \). The form of the colored part of the action, in the large \( N_f \) limit is given by (see for example [13])

\[
S = N_f \left\{ S_{WZW}(h) - \int d^2 x \ tr \frac{1}{8\pi\tilde{\alpha}} F_{\mu\nu} F^{\mu\nu} \\
- \frac{1}{2\pi} \int d^2 x \ tr (ih^\dagger \partial_+ h A_- + ih \partial_- h^\dagger A_+ + A_+ h A_- h^\dagger - A_+ A_-) + \\
e^{\gamma\sqrt{\tilde{\alpha}} m_q} \int d^2 x \ tr (h + h^\dagger) \right\},
\]

where \( \gamma \) is the Euler number. The coefficient of the last term depends on the assumed normal ordering scale. Fixing the gauge, \( A_- = 0 \) and integrating over \( A_+ \), we obtain the following action

\[
S = N_f \left\{ S_{WZW}(h) + \pi\tilde{\alpha} \int d^2 x \ tr \ j^+ j^+ + e^{\gamma\sqrt{\tilde{\alpha}} m_q} \int d^2 x \ tr (h + h^\dagger) \right\}, \tag{47}
\]
Using the relation between $h$ and $j^+$ namely, $j^+ = \frac{i}{2\pi} h \partial_- h^\dagger$, we can expand $h$ in powers of $j^+$ and $\frac{1}{\partial_-} j^+$ as

$$h = 1 - \frac{2\pi}{i} \frac{1}{\partial_-} j^+ + \left( \frac{2\pi}{i} \right)^2 \frac{1}{\partial_-} (j^+ (\frac{1}{\partial_-} j^+)) + \ldots$$

(48)

The dominant contribution to the partition function, in large $N_f$, would be obtained by small values of the field $\frac{1}{\partial_-} j^+$ and therefore

$$S = \int d^2 x \ 	ext{tr} \left\{ S_{WZW}(h) + 2\pi\bar{\alpha} \int d^2 x \ j^+ \frac{1}{\partial_-} j^+ - 4\pi^2 e^\gamma \sqrt{\bar{\alpha} m_q} \int d^2 x \ tr \left( \frac{1}{\partial_-} j^+ \right)^2 \right\}$$

(49)

Hence the effect of adding a mass term to the action translates (approximately) to a redefinition of the coupling constant $\bar{\alpha}' = \bar{\alpha} + 2\pi e^\gamma \sqrt{\bar{\alpha} m_q}$. Thus the theory includes mesons with $M^2 = 2\bar{\alpha}'$. Note that in this section we included only the colored part of the theory. In principle, there are other sectors of the theory that couple to the color sector, but here we focus only at special solutions of the equations of motion in which the other sectors do not appear. In particular, another massless state exists. This state is responsible to the confining nature of the massive theory. A discussion of this state as a solution of the equations of motion in the large $N_f$ is given in [13].

Therefore, we cannot conclude that the redefinition of the coupling constant is the only change, when a mass term is added. The full theory, in the large $N_f$, contains a richer spectrum than the massless theory, and including solitons in the color-flavor sector [18]. Also, by expanding $h$ and keeping only one term, we loose the soliton (like in the Sine-Gordon case).

8 Summary

In this paper we analyzed multi flavor $QCD_2$ with $N_f \gg N_c$. We have shown that there are Abelian solutions that can be interpreted as mesons with mass $M^2 = \frac{e^2 N_f}{2\pi}$. In addition there are truly non-Abelian solutions, that cannot have a single particle interpretation (See also discussion at the end of section 4). Apart from these solutions, we have shown that there are no non-Abelian solitonic solutions. This means that the spectrum of the theory consists of the Abelian mesons only.
This observation was made earlier by a different approach[19]. The Hilbert space of the massless theory can be spanned by the vacuum state, \(|0\rangle\) and currents that act on this state, \(|\Psi\rangle = \text{tr} J^n |0\rangle\). The spectrum of the theory can be found by diagonalizing \(M^2 = 2P^+ P^-\). Both \(P^+\) and \(P^-\) can be expressed in terms of currents (for details see [4] and [19]). The algebra which governs this calculation is the Kac-Moody algebra(6) in momentum space

\[
[J^a_m, J^b_n] = N_f \delta^{ab} \delta_{m+n} + i f^{abc} J^c_{m+n}
\]  

(50)

In the large \(N_f\) limit, the dominant contribution will be from the first term. Thus, the non-Abelian terms are suppressed and the spectrum should be similar to that of a set of \(N^2_c - 1\) multi-flavor massless Schwinger models.

Two remarks are in order. (i) It is important to emphasize that in spite of the fact that \(A^a\) carries an index of the color adjoint representation it is a gauge singlet, as we fixed the gauge completely. However, the \(N^2_c - 1\) degeneracy may be related to the screening nature of the massless theory[11]. After all in a confining theory we do not expect to see any remnant of the number of colors in the low energy spectrum. (ii) We do not have conclusive evidence that these states survive at finite \(N_f\). A BRST analysis of the spectrum[12] hinted that they may. On the other hand, numerical studies, in particular for \(N_f = N_c[4]\), do not find them. It is hard for us to imagine that the states at large \(N_f\) will disappear from the spectrum with smaller number of flavors. This question deserves further investigation.

We would like to add a remark concerning solving (nonlinear) differential equations for matrices. Associating such an equation with a non-Abelian gauge fixed Lagrangian enables one to find solutions by choosing a different gauge for which the equations of motion are simplified, and gauge transforming their solutions. We have used this procedure by solving the easier equation (19). Gauge transformation of these solutions (27) produced solutions to the more elaborated equation (19).

Thus it seems that the large \(N\) (where \(N\) can be either \(N_f\) or \(N_c\)) two-dimensional massless gauge theories have a rich and interesting spectrum. The \(N_f = 1, N_c \to \infty\) model (the ’t Hooft model) consists of a single Regge trajectory. The \(N_f \to \infty\) model consists of few mesons with \(N^2_c - 1\) degeneracy. An interesting limit is the adjoint fermions model which can be viewed a model of fundamental fermions with both \(N_f \to \infty, N_c \to \infty\) and \(N_f = N_c\). This model seems to contain infinite set of Regge trajectories[8, 19]. When
we decrease the number of flavors from infinity to one we obtain the ’t Hooft model. It is interesting that when the number of colors is decreased from infinity to few, all the infinite tower collapses to a few mesons. Since we believe that such an orbit exists in the space of \((N_f, N_c)\), it would be interesting to find it, maybe with the aid of a numerical study.

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