Abstract

We discuss the final state interaction effects at high energies via a multi-channel N/D method. We find that the 2 by 2 charge–exchange final state interactions typically contribute an enhancement factor of a few times $10^{-2}$ in the $B$ meson decay amplitudes, both for the real and the imaginary part.

We also make some discussions on the elastic rescattering effects.

PACS numbers: 11.55.Fv, 11.80.Gw, 13.20.Fc, 13.20.He
There have been increasing interests in recent years in studying the final state interaction (FSI) effects in B meson hadronic decays. In Ref. [1], a Regge pole model was suggested to estimate the FSI effects in B decays and it was found that, for $B \rightarrow DK$ decays the FSI effects are small in the sense that the corresponding partial wave $S$–matrix is close to unity. Especially the final state rescattering effects in charge-exchange processes are of order $O(10^{-2})$. These results imply that for the color non-suppressed charged final states, a tree level calculation to the B decay amplitude can be a good approximation [2]. Using a similar method as in Ref. [1], the authors of Ref. [3] observed that, as a consequence of Pomeron exchanges the FSI effects do not vanish as the center of mass energy square $s$ approaches infinity and therefore they suggest that there are sizable strong phases generated from FSIs. On the other hand, the authors of Ref. [4] pointed out that in above estimates the real part contributions of the rescattering amplitudes were not considered. They use a dispersion relation to estimate also the real part contributions of the charge–exchange processes ($B \rightarrow D\pi, \pi\pi$) and find that Reggeon contributions (in charge–exchange rescattering processes) are one order of magnitude larger than those estimated in Ref. [1,2]. Since this result, if correct, leads to important phenomenological consequences [4], it is therefore worthwhile to re-examine the problem by carefully studying the FSI effects, including the real part, of charge–exchange rescattering processes, which is what this paper mainly devote to. We will also discuss the FSI effects of elastic rescatterings although there exist some uncertainties related to such rescatterings in the scheme presented in this paper.

To correctly evaluate the FSI effects to the decay amplitudes, it is important to note that if the initial interactions are of short range the production amplitude and FSIs are actually factorized [6]. Assuming (quasi) two–particle unitarity we will use the multi-channel N/D method [7] to study the FSI effects. The inelasticity effects caused by those multi–particle states is presumably not large, since the pion multiplicity is rather low at $s = m_B^2$ as the multiplicity increases only as $\log(s)$ (see [4] and references therein) and at least part of the multi–particle final states can be classified as two–particle final states of resonances (accompanied by cascade resonance decays).
For a multi-channel system we can rewrite the partial wave T-matrix as,

\[ T = \frac{N}{D}, \]  

where the bold face represents a matrix and \( N \) contains only the left-hand singularities and \( D \) the right-hand singularities. In the physical region, \( T \) and the production amplitude, \( A \), satisfy the following unitarity relations,

\[ \text{Im}T = T^*\rho T, \quad \text{Im}A = A^*\rho T, \]  

which hold when the center of mass energy square, \( s \) takes any value along the positive real axis above the lightest threshold in the complex \( s \) plane, if there is no anomalous threshold. We will come back to discuss the influence of the anomalous thresholds on high energy FSIs later in this paper and will argue that their effects are negligible. The unitarity relations lead to the following simultaneous integral equations for \( N \) and \( D \),

\[ N = \frac{1}{\pi} \int_{L} \frac{b(s')D(s')}{s' - s} ds', \quad D = 1 - \frac{1}{\pi} \int_{R} \frac{\rho(s')N(s')}{s' - s} ds', \]  

where \( b \) is the discontinuity of \( T \) from the left-hand cut. Equation (3) may be solved by iteration method.

The FSI contribution to the production amplitude obeys a simple formula,

\[ A = \frac{A_0}{D}, \]  

where \( A \) is the full decay amplitude and \( A_0 \) is analytic in a region surrounding the positive real \( s \) axis. In the present physical situation, it implies \( A_0 \) is a smooth function of \( s \) in a large domain surrounding \( s = m^2_B \) in the presence of inelasticity. It should be noticed that the dispersion integrals in Eq.(3) for those components of \( N \) and \( D \) in which Pomeron exchanges involve should receive one subtraction. In potential scattering theory \( D \) is normalized to unity when \( s \) equals to infinity. Since the total cross sections of hadron–hadron scatterings are continuously rising as a function of \( s \), it is impossible to have such a normalization condition for high energy hadron interactions. \( A_0 \) in Eq. (4) is the Born term in potential
scattering theory and be constant. In the present case there is an uncertainty in relating $A_0$ to the bare amplitude. This is related to the multiplicative ambiguity in N/D method and the lacking of a natural normalization point in D. However, as will be seen below, this ambiguity can actually be avoided in calculating charge–exchange rescatterings.

To extract the information on charge–exchange processes from Eq. (4) we may rewrite $D$ as,

$$ D = \overline{D} - D^c, \quad (5) $$

where $\overline{D}$ contains only channels dominated by Pomeron exchanges and $D^c$ represents charge–exchange processes. Assuming $D^c$ to be a small quantity and that $\overline{D}$ does not deviate far from unity (these assumptions will be justified by the results given later), we can make an expansion in the denominator of Eq. (4) in powers of $D^c$ and keep only the leading term to obtain,

$$ A_{\text{charge–exchange}} = A_{el} D^c / \overline{D}, \quad (6) $$

where $A_{el}$ is the physical amplitude in the limit of vanishing charge–exchange rescatterings:

$$ A_{el} = A_0 / \overline{D}, \quad (7) $$

which can be a good approximation to the physical amplitudes for decay channels with large(larger) bare amplitude. Equation (6) can be written in an explicit form after a little algebra,

$$ A_{\text{ch.–ex.}} = A_{el} \left\{ \frac{1}{\pi} \int \frac{\rho(s') T^c(s') D^P(s')}{s' - s} ds' + \frac{1}{\pi} \int \frac{\rho(s') T^P(s') D^c(s')}{s' - s} ds' \right\}, \quad (8) $$

where $s = m_B^2$ and,

$$ D^P(s') = \frac{\overline{D}(s')}{\overline{D}(s)}, \quad D^c(s') = \frac{D^c(s')}{\overline{D}(s)}. \quad (9) $$

It is straightforward to check that $D^P$ satisfies the following integral equation,

$$ D^P(s') = 1 - \frac{s' - s}{\pi} \int \frac{\rho \ T^P(s'') D^P(s'') ds''}{(s'' - s')(s'' - s)}. \quad (10) $$
and Eqs. (3), (8) and (9) define an integral equation for $D_c$, provided $T_P$, $T_c$ and $D_P$ are known. The above equations are valid to the leading order in the expansion in powers of $T_c$ and to all orders of Pomeron exchanges.

In order to calculate the FSI contributions in charge–exchange processes it is required to determine $D_P(s')$ through Eq. (10). It is a difficult task since for a process $i \rightarrow j$ under consideration there can be many channels ($j' \rightarrow j$) dominated by Pomeron exchange exist contributing to $i \rightarrow j$ through $i \rightarrow j' \rightarrow j$. To proceed let’s look at the elastic process ($j' = j$) first and approximate the solution of Eq. (10) by first order iteration result, one obtains,

$$D_{el}(s') = 1 - \frac{s' - s}{\pi} \int \frac{\rho T_{el}^P(s'') ds''}{(s'' - s')(s'' - s)} , \quad (11)$$

where

$$T_{el}^P(s) = \frac{i\beta^P}{16\pi s_0(\lambda + \alpha' \log(s/s_0) - i\pi\alpha'/2)} , \quad (12)$$

where $\beta^P$ is the Pomeron coupling to matter and $\lambda = 2.82$ is the parameter characterizing the form-factor of the Pomeron coupling, also we set $\alpha_0^P = 1$ in above. From Eq.(11) we find that $D_P(s)$ is a smooth function of $s$ and deviates very little from unity for any reasonable value of $s$. Even for $s$ being as large as $W$ boson mass square, one still has,

$$D_{el}(M_W^2) = 0.99 + 6.1 \times 10^{-2}i \quad (13)$$

in $D\pi$ case. The $D$ matrix elements of inelastic diffractive channels are expected to be the same order as $D_{el} - 1 \sim 10^{-2}$ but with different phases. Large cancellation is natural when summing over all the intermediate states $j'$ which ensures terms like $T_{ij}^c T_{j'j}^P$ in Eq. (8) remains to be second order comparing with $T_{ij}^c$. Therefore we can approximate Eq. (8) by,

$$\mathbf{A}_{ch.-ex.} = \mathbf{A}_{el} \left\{ \frac{P}{\pi} \int \frac{\rho(s') T_{c}^c(s') ds'}{s' - s} + i\rho T_{c}^c \right\} , \quad (14)$$

which is, in simple words, obtained by neglecting all the diffractive scattering effects in
charge–exchange rescatterings.\footnote{This equation was used in Ref. [8] without any justification.} Equation (14) is the basic formula for numerical calculations in the following.\footnote{For $\pi\pi$ rescatterings, $D_{el}(M_W^2) = 0.95 \pm 0.20i$, which is not as close to unity as in the $D\pi$ case. This is not surprising as the diffractive FSI effects are stronger for $\pi\pi$ rescatterings. However, Eq. (14) should still be good enough to work with in an order of magnitude estimate.}

In order to compare with previous work, we now focus on the process, $B \to D^0\pi^0$ through a $D^+\pi^-$ intermediate state. The $D^+\pi^- \to D^0\pi^0$ scattering amplitude is,

$$T_{D^+\pi^- \to D^0\pi^0} = 2\sqrt{2}\rho , \quad (15)$$

where the $\rho$–Reggeon contribution is parameterized as \footnote{This equation was used in Ref. [8] without any justification.} \footnote{For $\pi\pi$ rescatterings, $D_{el}(M_W^2) = 0.95 \pm 0.20i$, which is not as close to unity as in the $D\pi$ case. This is not surprising as the diffractive FSI effects are stronger for $\pi\pi$ rescatterings. However, Eq. (14) should still be good enough to work with in an order of magnitude estimate.}

$$\rho = \beta^R(t) \left( \frac{s}{s_0} \right)^{\alpha^R(t)} \frac{1 - e^{-i\pi\alpha^R(t)}}{\sin\pi\alpha^R_0} , \quad (16)$$

in the small negative $t$ region. We take $s_0 \simeq 2\alpha' - 1$ and also neglect the $t$ dependence of $\beta^R$, according to Ref. \footnote{This equation was used in Ref. [8] without any justification.}. The Reggeon coupling adopted in Ref. \footnote{This equation was used in Ref. [8] without any justification.} is obtained by relating $\beta_R$ to the on-shell vector meson coupling in the $t$–channel physical process which however over-estimates the magnitude comparing with the one directly extracted from high energy scattering data. We take the value used in Ref. \footnote{This equation was used in Ref. [8] without any justification.} for numerical calculation and will come back to this difference later.

The approximation made above will be self-consistent if the final results on FSI effects are small which is indeed the case as we find,

$$\frac{A(B \to D^+\pi^- \to D^0\pi^0)}{A(B \to D^+\pi^-)} = (-6.9 + 4.4i) \times 10^{-2} . \quad (17)$$

Similarly for the $\pi^+\pi^- \to \pi^0\pi^0$ rescattering, we have,

$$T_{\pi^+\pi^- \to \pi^0\pi^0} = 4\rho , \quad (18)$$

in which $s_0 = \alpha' - 1$ is taken and the numerical result is,
\[
\frac{A(B \rightarrow \pi^+\pi^- \rightarrow \pi^0\pi^0)}{A(B \rightarrow \pi^+\pi^-)} \simeq (-10. + 5.5i) \times 10^{-2}.
\] (19)

These results are significantly smaller in magnitude than the one obtained in Ref. [4]. Still, a more careful fit to the high energy scattering data suggest that the \(\rho\)-Reggeon coupling is further suppressed by a factor about 2.5\(^2\) comparing with the previously used. With this suppression factor the \(\beta_R\) parameter is close to the value used in Ref. [1]. We therefore suggest that Reggeon contributions to FSIs are typically of order of a few times \(10^{-2}\), both for the real parts and the imaginary parts.

Of course Regge parameterization can not always be correct when decreasing \(s\) down to thresholds even though we know from old experiences that it may indeed work well in charge–exchange processes at rather low energies. To estimate the violation of the Regge parameterization which we call \(\delta\), one has,

\[
\delta \simeq \frac{\bar{s} \rho T_c}{\pi m_B^2} < \frac{\bar{s}}{2\pi m_B^2},
\] (20)

where \(\rho T_c\) refers to the averaged value of the magnitude of \(\rho T_c\) in the range of \(\bar{s}\) which characterizes the region where Regge parameterization fails and the last inequality comes from unitarity. Taking \(\bar{s} = 10GeV^2\) we see that \(\delta\) is at best a few percent, in a very conservative estimate.

Strictly speaking, the present method only applies to a restricted class of \(T\) matrices which contain no anomalous threshold. An anomalous threshold comes from dynamical singularities which occurs only if a loosely bound composite system of hadronic constituents

\(^3\) When extracting \(\beta_R\) from high energy scattering data it is probably better not to include in the fit the nucleon–nucleon scattering data which contain strong SED violating contributions and can not be explained by a simple Regge pole model. Actually, the fact that meson–nucleon data looks better in favor of a simple Regge pole parameterization is the main reason to support the use of a simple Regge pole model for meson–meson scatterings. On the other hand, absorption effects, if exist, only reduce the low partial wave FSI effects.
is involved \[^2\]. The typical mass scale of an anomalous threshold is the scale of the masses of the light hadrons in the decay products \[^10\]. From a simple dimensional analysis we are convinced that these effects, if appear, are further suppressed in powers of \(1/s\) in high energy scatterings.

In above we have discussed the effects of charge–exchange FSIs, we obtained the relation between the rescattering amplitude and the physical amplitude in the limit of vanishing charge–exchange rescatterings (the bare amplitude renormalized by diffractive rescatterings). The ambiguity caused by the normalization of \(D\) is absorbed into \(A_{el}\). In discussing the elastic rescattering it is unavoidable to face such an ambiguity. As having been made clear in above discussions that the ambiguity comes from the fact that there is no natural normalization point for \(D\), unlike the case in potential scattering theory where \(D\) is normalized to 1 at \(s = \infty\). The only thing can be done in the present formalism is to compare the FSI strength between two energy scales. Here we define the FSI strength at \(s = m_B^2\) to be \(1/D\Lambda(s)\) where \(D\Lambda\) is normalized to unity at scale \(\Lambda\), and \(\Lambda\) is chosen somewhere below \(W\) boson mass and much larger than \(m_B\). From the previous estimation on \(D_{el}(M_W^2)\), recall that \(D(s_1)\) normalized at \(s_2\) is equal to the inverse of \(D(s_2)\) normalized at \(s_1\), we conclude that the FSI effects remain small even for elastic rescatterings at \(s = m_B^2\), at least in the \(D\pi\) case.\[^4\] Here we defined in some way (maybe rather artificial) the absolute value of the phase of the elastic channel, but we want to stress that what truly matters in a physical quantity (like the CP violation observables) is the difference between two elastic phases of different isospins which is of non-diffractive nature and should therefore be a small quantity.

\[^4\] See the discussion given in Ref. \[^11\].

\[^5\] Of course, a more careful numerical study calls for a calculation beyond the first order iteration in determining function \(D\), including inelastic effects which can only be done under various assumptions and approximations. The random S–matrix approach \[^11\] can be helpful in extracting out the essential feature of a system with many inelastic channels.
To conclude we find that charge–exchange rescatterings in B decays contribute typically an enhancement factor of order of a few times $10^{-2} \sim \lambda^2$ where $\lambda$ is the Wolfenstein parameter. For elastic rescatterings, the FSI effects most likely remain small. B decays contain rich phenomenology and an enhancement factor $\sim \lambda$ via charge–exchange rescatterings leads an important role [5]. In this sense the result of this paper, if correct, is in disfavor of an experimental extraction to the rich physical phenomena.

Acknowledgment: It is our pleasure to thank Valeri Markushin and Hong-An Peng for helpful discussions. The work of H.Z. is supported in part by China National Natural Science Foundations.
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