We investigate numerically the influence of an homogeneous shear flow on the spinodal decomposition of a binary mixture by solving the Cahn-Hilliard equation in a two-dimensional geometry. Several aspects of this much studied problem are clarified. Our numerical data show unambiguously that, in the shear flow, the domains have on average an elliptic shape. The time evolution of the three parameters describing this ellipse are obtained for a wide range of shear rates. For the lowest shear rates investigated, we find the growth laws for the two principal axis $R_{1}(t) \sim \text{constant}$, $R_{2}(t) \sim t$, while the mean orientation of the domains with respect to the flow is inversely proportional to the strain. This implies that when hydrodynamics is neglected a shear flow does not stop the domain growth process. We investigate also the possibility of dynamic scaling, and show that only a non trivial form of scaling holds, as predicted by a recent analytical approach to the case of a non-conserved order parameter. We show that a simple physical argument may account for these results.

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**I. INTRODUCTION**

The study of phase ordering kinetics has a long history [1,3]. The canonical example is the coarsening process following the quench of a binary mixture A-B below its spinodal line. The properties of the resulting domain growth are rather well understood [1,3]. In the case where A and B are in equal concentration, an isotropic bicontinuous structure emerges, which is characterized by a typical length scale $L(t)$ growing as a power law of time $t$. Moreover, as $t$ increases, the structure evolves in a self-similar manner in the sense that its statistical properties are the same when space is rescaled by $L(t)$. Any function $C(r,t)$ that depends on space and time is then a function of the reduced variable $|r|/L(t)$ only: $C(r,t) \equiv \mathcal{C}(|r|/L(t))$. This property is termed ‘dynamic scaling’ [1,3].

The study of the spinodal decomposition in an homogeneous shear flow is of fundamental and practical interest [1,3], but despite an enormous amount of experimental [1-11], numerical [12-24] and analytical works [12,24,25], the problem is still not fully settled [26]. Early experiments [12,13] and simulations [19,20,24] have shown that isotropy is lost, the morphology of the bicontinuous structure being elongated along the flow direction. Hence, a single length scale can not describe the full structure. There are thus several questions which naturally arise. (1) How to characterize quantitatively the growing structure? (2) What is the time dependence of the different length scales, as compared to the un-sheared case? (3) Does the growth stop after some time? (4) Does a suitable generalization of the dynamic scaling property hold?

It is interesting to remark that in spite of the large number of works cited above, a definite answer to all these questions is still lacking. Several reasons make this problem non-trivial. First, the role of hydrodynamics is far from being understood. Simple physical arguments [4,9] show that it may become preponderant at large times, predicting a saturation of the domains at a typical size. This assumption is resolved neither at the experimental level (see, e.g., the opposite conclusions of Ref. [7] and Refs. [11]) nor at the numerical level, where powerful algorithms are needed to correctly deal with hydrodynamics. At present, this limits the numerical analysis to too small sizes to make any definite answers to questions (1)-(4) above, although much progress has been made recently [9,11].

Second, at the theoretical level, no analytical solution of a reasonable model of spinodal decomposition (even neglecting hydrodynamics) is available. One has then to make some predictions from the solution of solvable, but less realistic models, like the $O(n)$ model in the large-$n$ limit [24], or from the approximate solution in the case of a non-conserved order parameter [25]. A scaling argument based on the hypothesis that a generalization of dynamic scaling holds has also been developed in Ref. [12].

Third, there are, to our knowledge, no numerical simulations (neglecting hydrodynamics) validating these analytical predictions. Moreover, the scaling hypothesis on which is based the analytical argument of Ref. [12] has only been tested in Ref. [9], with negative results. Thus, the validity of the predicted growth laws may also be questioned. More crucially, there is up to now no consensus concerning the numerically measured growth laws: we discuss this point in a more detailed way in the last
In this work, we numerically study the spinodal decomposition process in a shear flow by solving the Cahn-Hilliard equation in two dimensions. All hydrodynamic effects are neglected. Although this involves a drastic reduction of the experimental situation, it is necessary, in our opinion, to have a good understanding of this ‘simple’ case, before studying more realistic problems. Our algorithm is different of but our study is technically comparable to the most recent one [12]. We shall however explore a wider range of shear rates, and this will lead us to a different interpretation of the numerical data.

The paper is organized as follows. In the next Section, we define the model and describe the numerical procedure to solve it. Section III briefly recalls the results obtained when the shear flow is absent. Section IV describes the morphology of the domains under shear and its time evolution. Section V focuses on the problem of dynamic scaling. In the last Section, we compare our results with the relevant existing data in the literature and give a simple physical argument to explain them.

II. MODEL AND DETAILS OF THE SIMULATION

In this work we focus on the standard model for the spinodal decomposition of binary mixtures and solve numerically the Cahn-Hilliard equation [1–3]

\[ \frac{\partial \phi(r,t)}{\partial t} + v \cdot \nabla \phi(r,t) = \Gamma \nabla^2 \left( \frac{\delta F[\phi]}{\delta \phi(r,t)} \right) + \eta(r,t). \]  

(1)

In this expression, the order parameter \( \phi(r,t) \) is a scalar quantity which can be linked to the concentration \( c_A (c_B) \) of the component A (B) of the mixture by the relation \( \phi \equiv 1 - 2c_A = 2c_B - 1 \). Eq. (1) has the form of a continuity equation, which implies that the order parameter is a conserved quantity. The free energy \( F[\phi] \) is of the Ginzburg-Landau type

\[ F[\phi] = \int d^d r \left[ \frac{\xi^2}{2} |\nabla \phi|^2 + \frac{1}{4} \phi^4 - \frac{1}{2} \phi^2 \right], \]  

(2)

where the equilibrium correlation length \( \xi \) has been introduced. The noise term \( \eta \) is a random Gaussian variable, characterized the two moments \( \langle \eta(r,t) \rangle = 0 \) and \( \langle \eta(r,t)\eta(r',t') \rangle = -2T \delta(t-t')\nabla^2 \delta(r-r') \); \( T \) is the temperature of the thermal bath. All the simulations will be carried out at \( T = 0 \), since temperature is essentially irrelevant in this process [1]. It is also known to delay the onset of the asymptotic regime [2]. The second term in the left hand side of Eq. (1) results from the advection of the order parameter by the velocity field. The case of an homogeneously sheared system will be investigated. The flow is taken to be in the \( x \)-direction and the velocity field is then \( v = \gamma y \hat{e}_x \), which defines the shear rate \( \gamma \). We shall concentrate on the case of a constant shear rate.

We want then to solve numerically in two spatial dimensions the following equation:

\[ \frac{\partial \phi}{\partial t} = -\gamma y \frac{\partial \phi}{\partial x} - \Gamma \nabla^2 \left( \xi^2 \nabla^2 \phi - \phi^3 + \phi \right), \]  

(3)

where both space and time dependences have been removed for clarity. This is done by combining the numerical methods of Refs. [28,29]. A new frame \( (x',y') \) is first defined by \( 23 \quad x' \equiv x - S(t)y, \quad y' \equiv y \), where \( S(t) \equiv \int_0^t dt' \gamma \) is the strain. In the case of a constant shear rate, it is simply given by \( S(t) = \gamma t \). Defining further \( \phi(r,t) \equiv \phi(r',t) \), Eq. (3) becomes [23]

\[ \frac{\partial \phi(r,t)}{\partial t} = -\hat{\nabla}^2 \left( \hat{\nabla}^2 \phi - \hat{\phi}^3 + \hat{\phi} \right), \]  

(4)

with

\[ \hat{\nabla} = \left( \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'} - S(t) \frac{\partial}{\partial x} \right), \]  

\[ \hat{\nabla}^2 = \left( \frac{\partial}{\partial x'} \right)^2 + \left( \frac{\partial}{\partial y'} - S(t) \frac{\partial}{\partial x} \right)^2. \]  

(5)

After transformation, Eq. (4) is formally identical to the Cahn-Hilliard equation without shear, which is solved by the implicit spectral algorithm developed in Ref. [29]. Space is measured in units of the correlation length \( \xi \) (also the interface width) and time in units of \( \xi^3 / \Gamma \). This microscopic time scale represents the typical time it takes to create a well-defined domain wall. Periodic boundary conditions are imposed in the deformed frame. The single parameter of the simulation is then the shear rate \( \gamma \), which introduces a time scale \( \gamma^{-1} \). The choice of the parameters for the discretization has been discussed in Ref. [24] and the values \( \Delta t = 0.5, \Delta x = \Delta y = 0.5 \) are used throughout the simulation. For each shear rate, the system size has been carefully checked to be large enough so that the reported growth laws are unaffected by the boundaries. Since the growth is strongly anisotropic, a rectangular simulation box has been chosen with sizes up to \( L_y = 512 \) and \( L_x = 8192 \). The shear rates investigated in this paper are \( \gamma = 0.04, 0.02, 0.01, 0.005, 0.0025 \) and 0.00125. This corresponds to a time scale \( \gamma^{-1} \) in the range [25,800]. We wish to emphasize that the condition \( \gamma^{-1} \gg \xi^3 / \Gamma \) has to be fulfilled, since we are interested in the scaling regime where well-defined domains coarsen. This remark will become important for the interpretation of the numerical results. An alternative solution would be to apply the shear flow after an initial transient, so that large domains have already grown. Such initial conditions are discussed at the end of the paper.

III. ZERO SHEAR CASE

Although the \( \gamma = 0 \) case has been extensively studied [1–3], we briefly consider this well-known situation with
three aims. These results are presented to (i) validate our numerical procedure, (ii) present the quantities of interest and above all (iii) make comparisons to the sheared case easier.

The domain growth takes place typically as in Fig. 1, where an isotropic bicontinuous structure coarsens with time. This coarsening process is well characterized by the two-point correlation function defined by

\[ C(r, t) \equiv \frac{1}{V} \int d^2x \phi(x, t)\phi(x + r, t), \tag{6} \]

which is nothing but the Fourier transform of the structure factor, experimentally measured through light scattering experiments. A typical two-point function is represented in Fig. 2 which shows the isotropy of this surface. The average shape of the domains of Fig. 1 may be extracted from this plot by taking the intersection of this surface with a horizontal plane \( z = \text{constant} \). This allows to measure the time dependence of the length scale \( R_x(t) \) \( (R_y(t)) \) in direction \( x \) \( (y) \). Both length scales are represented in the inset of Fig. 2 and have the expected power law behavior \( R_x \sim R_y \propto t^{1/3} \)\(^2\).\(^3\).

The dynamic scaling hypothesis is tested in the main frame of Fig. 3, where the two-point function is circularly averaged and plotted as a function of \(|r|/L(t)\). This works perfectly well.

We are thus confident in our numerical setup, and we shall now address the question of the influence of the shear flow on the spinodal decomposition.

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**IV. ANISOTROPIC GROWTH OF THE DOMAINS UNDER SHEAR**

**A. Basic observations**

The time evolution of the domains when a shear flow is applied is followed in Fig. 4. This evolution is the basic result of all previous numerical works \(^2\)\(^3\)\(^4\)\(^5\)\(^6\)\(^7\)\(^8\)\(^9\)\(^10\) and shows that the domain growth is essentially unaffected for times \( t \lesssim \gamma^{-1} \) since the first snapshot is very similar to those in Fig. 1. At intermediate times, the domains begin to have an anisotropic shape: an average direction of the domains is clearly apparent. This direction rotates and becomes more aligned with the flow when the strain increases. At large times, \( t \gg \gamma^{-1} \), domains are nearly aligned with the flow, and have a strongly anisotropic shape.

These features are also clearly discernible in Fig. 5, where we have plotted the two-point correlation function for a strain \( S(t) = 5 \) at \( \gamma = 0.01 \). The surface is clearly stretched in the flow direction, compare with the Fig. 1. In particular, it becomes impossible to perform a circular average as in the unsheared case: isotropy is lost.

The average shape of the domains is again recorded through the intersection of the two-point correlation function with a horizontal plane. For definiteness, we take \( z = 0.5 \), but the value 0.5 is unimportant. It is found numerically that this intersection is very well represented by an ellipse. Computing the parameters of this ellipse gives then access to two typical length scales, \( R_L(t) \) (large axis) and \( R_S(t) \) (small axis), and to the mean orientation of the domains, \( \theta(t) \) (angle between the large axis and the \( x \)-direction). We present the typical time evolution of the elliptic shape of the domains in Fig. 6.
where data are also fitted to an elliptic form, in a very satisfactory way.

These three quantities depend both on the time $t$, and on the shear rate $\gamma$ and we successively study in the following subsections the time evolution of $\theta(t)$ and of the length scales $R_{\parallel}(t)$ and $R_{\perp}(t)$.

![FIG. 4. Snapshots of size $L_y = 512$ and $L_x = 2048$ (parts of a $512 \times 4096$ system) for a shear rate $\gamma = 0.01$ and strains $S(t) = 1$, 5, 10 and 50 (from top to bottom). Each color represents one phase of the mixture.](image)

![FIG. 5. Two-point correlation function for $\gamma = 0.01$ and $S(t) = 5$. The surface is stretched in the $x$-direction. Note, in particular, that the $x$ and $y$ ranges are different in this figure.](image)

Before performing this analysis, a remark has to be made about the identification of the relevant length scales. It is quite clear from the above analysis that 3 parameters are needed to fully characterize the growing structure under shear. This feature was already noted in experiments [3,4]. In some of the previous numerical works, only two parameters were studied, namely the length scales in the $x$ and $y$ directions. Although these length scales should well represent the structure at long times, it is physically preferable to study instead the domain size in the directions defined by the angle $\theta(t)$. In Ref. [13], the direction of the domains’ shape was recorded, but there was no attempt at a quantitative analysis of its behavior. We note last that this quantitative analysis of the domain morphology in two dimensions naturally arises from the analytical work of Ref. [25], in the case of a non-conserved order parameter.

![FIG. 6. Time evolution of the intersection of the two-point correlation with a horizontal plane, for strains $S(t) = 5$ (squares), 10 (circles) and 20 (triangles) and shear rate $\gamma = 0.01$. The points are the data, while the lines are fits to an elliptic shape. Note, in particular, that the $x$ and $y$ ranges are different in this figure.](image)

**B. The mean orientation $\theta(t)$**

The first effect of the shear flow is to give the domains an anisotropic shape and hence to create a preferred direction in the system. Physically, it can be expected that for a given shear rate $\gamma$, the mean orientation of the domains is inversely proportional to the strain $S(t)$, as would be the case for a rigid rod advected by the shear flow:

$$\theta(t) \simeq \frac{\theta_0(\gamma)}{\gamma t}.$$  \hfill (7)

This relation is tested in Fig. 7, where $\theta_0(\gamma)$ is used as an adjustable parameter. This figure shows that the relation (7) is well satisfied in the whole range of shear rates investigated.

The parameter $\theta_0$ is found to be a slowly increasing function of the shear rate. One finds $\theta_0(0.00125) \simeq 0.975$.
and \( \theta_0(0.04) \approx 2.0 \). By definition, this angle corresponds to the mean orientation of the domains when the strain is 1, \( \theta_0 \equiv \theta(t = \gamma^{-1}) \), and its variation may be understood by the following argument. For times \( t \leq \gamma^{-1} \), the domain growth has been mainly unaffected by the flow and thus the larger the shear rate, the smaller the domains at time \( t \sim \gamma^{-1} \). Since large domains are more easily deformed than small ones (because of the surface tension), it is expected that at strain \( S(t) \sim 1 \), large domains (small \( \gamma \)) are more deformed than small ones (large \( \gamma \)). Hence, the smaller the shear rate, the smaller \( \theta_0 \).

\[
R(t) \equiv \theta(t = \gamma^{-1}) \approx \frac{1}{\gamma t}.
\]

We turn now to the time evolution of the two length scales \( R_\perp(t) \) and \( R_\parallel(t) \). These quantities are studied for a very broad range of shear rates from \( \gamma = 0.04 \) (\( \gamma^{-1} = 25 \)) to \( \gamma = 0.00125 \) (\( \gamma^{-1} = 800 \)). All our results are summarized in Fig. 7, where \( R_\parallel \) and \( R_\perp \) are represented for each value of the shear rate as functions of the strain. We obtain numerically that the growth laws are well represented at large strains by the following algebraic forms,

\[
R_\parallel(t) \approx R_\parallel^0 \cdot (\gamma t)^{\alpha_\parallel}, \quad R_\perp(t) \approx R_\perp^0 \cdot (\gamma t)^{\alpha_\perp}, \quad (8)
\]

which define the growth exponents \( \alpha_\parallel \) and \( \alpha_\perp \). This was first obtained in a simulation by Padilla and Toxvaerd \[22\] and subsequently in similar systems in Refs. \[12\,18\]. In Fig. 7 we have fitted our data using the forms \[8\]. These fits deserve some comments. The growth law for \( R_\parallel \) becomes a nice straight line in a log-log plot at large strains \( S(t) \gtrsim 10 \). Only the data points for such large strains have been used to compute the exponent \( \alpha_\parallel \). Concerning the time behavior of \( R_\perp(t) \), two different fits have been tested. First, a horizontal line corresponding to \( \alpha_\perp = 0 \) has been compared to the data. Second, following the available analytical results \[12\,22\,23\], we also tried to fit the data with the ansatz

\[
\alpha_\perp = \alpha_\parallel - 1.
\]

Note that both fits are equivalent when \( \alpha_\parallel = 1 \), which is nearly the case for the two smallest shear rates. Several comments are in order.

1 - The algebraic fits are clearly a very good representation of the data. Some logarithmic corrections that come out from the analytical study of the \( O(n) \) model and of the non-conserved case may be present, but we do not expect to be able to determine them numerically.

2 - Although the relation \[3\] reasonably accounts for the data, the value \( \alpha_\perp = 0 \) is also possible and works well for all the shear rates investigated. This means that it could be possible to rescale all the curves for \( R_\perp(t) \) by plotting \( R_\perp(t)/R_{\perp,0} \) as a function of the strain for different shear rates, using \( R_{\perp,0} \) as a fitting parameter. This parameter is found to be a decreasing function of the shear rate, which means that the smaller the shear rate, the wider the domains. This rescaling is performed in Fig. 8 and the data are indeed compatible with this hypothesis. Let us add the remark that \( R_\perp(t) \) varies (at most) by a factor 2 in all the simulations. This strongly supports the hypothesis that there is in fact no growth in the perpendicular direction. Of course, once again, logarithmic corrections can not be numerically dismissed.

3 - We do not see any evidence for the oscillations reported in Refs. \[12\,18\]. We do not have a clear explanation for that, but an hypothesis is that these oscillations are a preasymptotic artifact of the measure process itself. The definition of the length scales used in Ref. \[12\] leads in some cases to a ratio \( R_\parallel(t)/R_\parallel(t) \approx 10 \) for a strain \( S(t) \approx 1 \) \[12\]. For this strain, the domains are still nearly circular and in our simulations, this ratio is never above the value 1.5. However, this argument is made weaker by the observation that Qiu et al. \[9\] have used the same measurement procedure as Corberi et al. \[12\], at a lower shear rate \( \gamma = 0.01 \) and do not observe any oscillations.

4 - An important point is the fact that the exponents apparently depend on the value of the shear rate. More precisely, we find that \( \alpha_\parallel \) decreases from the value \( \alpha_\parallel \approx 1.35 \) at \( \gamma = 0.04 \) to \( \alpha_\parallel \approx 1.0 \) at \( \gamma = 0.00125 \). This means that we are in fact measuring effective exponents and that the true asymptotic behavior has not been reached in some of the cases studied here.

The problem is then to determine which exponent is the right one! As emphasized in the Introduction, well-defined interfaces exist only if \( \gamma \ll 1 \). This indicates that the ‘true’ asymptotic behavior is reached for the smallest shear rates investigated, and favors the value \( \alpha_\parallel = 1 \) found for \( \gamma = 0.0025 \) and 0.00125. Together with the behavior of \( \alpha_\perp \) and the observation that the domains are wider at lower shear rates, we are led to the conclusion
that the growth exponents are given by

\[ \alpha_\parallel = 1, \quad \alpha_\perp = 0. \]

The analysis of dynamic scaling properties in the following section will reinforce this conclusion.

FIG. 8. The two length scales \( R_\perp(t) \) and \( R_\parallel(t) \) are represented as a function of the strain. Each figure is labeled by the corresponding shear rate. The symbols are the data, and the full lines are algebraic fits with the exponents indicated near each fit. For the strongest shear rates, a horizontal dashed line has been added as a fit to \( \alpha_\perp = 0 \).
FIG. 9. Test of the relation (8) for the small axis of the ellipse. The full line represents the case \( \alpha = 0 \), whereas the dashed line is for \( \alpha = 1/3 \). The symbols are the same as in Fig. 7. The latter is clearly inconsistent with the numerical measurements for all the shear rates.

V. DYNAMIC SCALING

The last question we wish to address is the problem of dynamic scaling. We have recalled in Fig. 3 (where \( \gamma = 0 \)) that the two-point correlation function has the property that it can be rescaled in the form of a single variable function

\[
C(r, t) = C \left( \frac{|r|}{L(t)} \right) \tag{11}
\]

This indicates that \( L(t) \) in the only relevant length scale in the asymptotic regime characterized by \( \xi \ll L(t) \). This behavior is the basis of a scaling argument which, in the unsheared case, allows an elegant derivation of the growth laws [30].

In the shear flow, there are two relevant length scales and the scaling (11) can thus no longer be true. Two different generalizations have been suggested by analytical works. The solution of the \( O(n) \) model predicts the scaling form [24]

\[
C(r, t) = C \left( \frac{x}{R_x(t)} \right) \frac{y}{R_y(t)} \tag{12}
\]

where \( R_x \) and \( R_y \) are the typical sizes in the directions \( x \) and \( y \) respectively. A different scaling is expected from the non-conserved order parameter case, namely [25]

\[
C(r, t) = C \left( \frac{r_y}{R_y(t)} \right) \frac{r_{\perp}}{R_{\perp}(t)} \tag{13}
\]

where the subscripts refer to the rotating frame described previously. Corberi et al. [14] have used the form (13) as a starting point to generalize the argument of Bray [30] to the sheared case. It is thus important to see if this scaling behavior is detected in the simulation. Let us note that the tilt angle \( \theta(t) \) is very small in the long time regime we are interested in. Then, one might ask if the difference between the two forms (12)-(13) proposed above is relevant. Since the domains are very elongated in the \( x \)-direction, then even with a small angle there might be differences between the 'parallel' and the \( x \)-directions. Concerning the \( y \) and the 'perpendicular' directions, the support of the correlation function in these directions is very small, so that differences between the two are indeed not observable.

FIG. 10. Test of dynamic scaling in the \( x \)-direction for \( \gamma = 0.04 \) and 0.0025. In both cases, the collapse is not satisfying.

We now present our numerical results. In Fig. 10, we show for two different shear rates the attempt at the rescaling of the two-point function in the \( x \)-direction. In each case, we have considered a time window where the hypothetic scaling might hold (i.e. the growth law has reached its algebraic asymptotic form, \( S(t) \gtrsim 10 \)). We have also chosen \( R_x(t) \) in order to get the best collapse of the data. Clearly, the scaling (12) does not work, in
both cases of a high and a low shear rate.

We investigate the possibility for the second scaling form to hold in the direction defined by the tilt angle $\theta(t)$ in Fig. 11. There is here a clear qualitative difference between the shear rate $\gamma = 0.0025$ and $\gamma = 0.04$, since the collapse of the data is excellent for the small shear rate, whereas there is a clear systematic evolution for the highest shear rate.

In our opinion, these results are a very good indication that the scaling form (12) does not describe the asymptotic behavior of the Cahn-Hilliard equation. Moreover, they show that a self-similar asymptotic regime has been reached for the lowest shear rates investigated, characterized by the scaling (13) of the two-point function. This is also confirmed by the inspection of the scaling properties in the ‘parallel’ direction (which is equivalent to the $y$-direction). In Fig. 12, we show the results for this direction. Once again, the collapse is very good for the low shear rate $\gamma = 0.0025$, whereas it is clearly not satisfying for a higher shear rate $\gamma = 0.04$.

**VI. DISCUSSION**

We have investigated in this paper the ordering kinetics of a binary mixture quenched below its spinodal line in a homogeneous shear flow, through the numerical solution of the Cahn-Hilliard equation in 2 spatial dimensions.

We have found that the bicontinuous coarsening structure is well described by three parameters. Since the average shape of the domains is elliptic, it is sufficient to give the lengths $R_\perp(t)$ and $R_\parallel(t)$ of the two principal axes of the ellipse, and the orientation of the large axis with respect to the flow, $\theta(t)$. We have measured the time evolution of these three quantities, defined from the two-point correlation function, and all our results may be summarized by the following relations, which have been
shown to hold in the asymptotic regime of large times
\[ R_{||}(t) \simeq \gamma t, \]
\[ R_{\perp}(t) \simeq \text{constant}, \]
\[ \theta(t) \simeq \frac{1}{\gamma t}, \]
\[ C(r,t) \simeq C \left( \frac{r_{||}}{R_{||}(t)}, \frac{r_{\perp}}{R_{\perp}(t)} \right). \]

In particular, this asymptotic regime could not be reached for the largest shear rates we have investigated. Moreover, our attempts at detecting (in the cases \( \gamma = 0.04 \) and 0.02) a crossover from a preasymptotic regime with \( \alpha_{||} > 1 \) and \( \alpha_{\perp} > 0 \) towards the regime (14) were unsuccessful, because this would require too large a system size \( L_x \). These results also demonstrate that the domain growth does not stop under the shear flow, in the case where hydrodynamics is neglected and answer then questions (1)-(4) of the introduction.

We now compare our results with previous ones. It is interesting to note that, even though this simulation was not intended to be able to reproduce experiments since hydrodynamics has been neglected, our results reproduce quantitatively the experiments of Chan et al. [4,11] on a mixture of water and isobutyric acid and of Qiu et al. [21] on a polymer blend of polystyrene and poly-(vinyl methyl ether), for the three quantities describing the morphology of the domains. These experiments are probably done in a regime where hydrodynamic effects are negligible, since other experiments by Hashimoto and collaborators [10] report a saturation of the domains to a \( \alpha \)-dependent size. This stationary steady state has been termed `string phase’ and has also been observed by Hobbie et al. [3] in a strong shear rate regime.

Our numerical findings can be compared with previous works neglecting hydrodynamics, and our results for the growth laws are similar to those found by Qiu et al. [21] and Corberi et al. [12]. Note that in this last reference [12], the growth laws are compatible with the regime (14), although they are compared with the behavior \( R_{||}(t) \sim t^{1/3} \) and \( R_{\perp}(t) \sim t^{1/3} \). The authors admit however that the latter regime was not reached within the numerical time-window. Earlier simulations were not able to be very quantitative [19,21,22,23], although Padilla and Toxvaerd [22] have suggested that the anisotropic domain growth was well-described by algebraic laws. This is, to our knowledge, the first time that the elliptic average shape of the domains is systematically investigated in a numerical experiment. Concerning the scaling of the two-point correlation function, Qiu et al. [21] have shown that the form (12) is not appropriate, but did not investigate the form (14).

Analytically, the known results in the case of a conserved order parameter are for the \( O(n) \) model, which has been solved with shear in the large-\( n \) limit [24] (see also [18]). The second result stems from a scaling argument developed by Corberi et al. [12]. Both predict that the growth laws should be \( R_{||}(t) \sim t^{4/3} \) and \( R_{\perp}(t) \sim t^{1/3} \), with a scaling form for \( C(r,t) \) as in Eq. (12). Our results do not corroborate these predictions. A reason may be that these results do not take into account the elliptic shape of the domains and we have clearly demonstrated this could be a key point for understanding the scaling properties of the system. One might also question the validity of an RG-type argument in the case where the exponent \( \alpha_{\perp} \) is zero, since upon space rescaling, domains become thinner and thinner. Recently [23], an approximate solution of the case of a non-conserved order parameter has been found, predicting the scaling (13) in two spatial dimensions. However, extending this solution to the conserved case is certainly a very hard task, since already in the unsheared case the conservation law makes the calculations very involved [8].

We are then faced with the problem of having numerical results that cannot be understood within an existing framework. We now give a simple physical argument leading to Eqs. (14), inspired by the original argument given by Huse [4] to describe the zero-shear case. The spinodal decomposition under a shear flow results basically from two competing effects.

(i) The advection of the order parameter, which becomes efficient for times \( t \gtrsim \gamma^{-1} \), deforms the nearly circular domains existing at time \( t \sim \gamma^{-1} \). It is very easy to compute that, with the advection only, a circular domain is deformed into an elliptic shape with principal axes scaling at large strain as \( R_{||}(t) \sim (\gamma t)^{-1} \) and \( R_{\perp}(t) \sim t \), with a tilt angle \( \theta(t) \sim (\gamma t)^{-1} \).

(ii) The domain growth arises because of the existence of a gradient of the chemical potential \( \mu \equiv \delta F/\delta \phi \). This force gives rise to currents which make an interface of curvature \( R \) move with a velocity \( \partial R/\partial t \propto 1/R^2 \). This interface motion results, in the absence of shear, to a coarsening of the domain structure [12]. Here, we modify the argument by taking into account the fact that the structure is no longer isotropic. While the domain growth does not affect the tilt angle \( \theta(t) \), it leads to two different interface velocities, namely \( dR_{||}/dt \sim -1/R_{\perp}^2 \) and \( dR_{\perp}/dt \sim +1/R_{||}^2 \).

With the strong assumption that a balance can be made between these two effects, this leads to the following equations for the the three parameters of the ellipse

\[ \frac{dR_{||}(t)}{dt} = -\frac{1}{\gamma t^2} + \frac{1}{R_{||}(t)^2}, \]
\[ \frac{dR_{\perp}(t)}{dt} = +\gamma - \frac{1}{R_{\perp}(t)^2}, \]
\[ \theta(t) = \frac{1}{\gamma t}, \]

which indeed imply Eqs. (14). Although simple and heuristic, this argument captures in fact the essence of the coarsening process under a shear flow. It should correctly describe the domain growth when the domains are quite large. Otherwise, the notion of interface velocity is
meaningless. It is quite clear however that subtleties, such as logarithmic corrections that exits in more involved computations \cite{23}, will not be captured by such a naive argument.

Let us note that when it is applied to the non-conserved case, this argument leads to the growth laws $R_\perp \sim \text{constant}$ and $R_\parallel \sim t$, at leading order. This not so far from the result $R_\perp \sim (\ln t)^{-1/4}$ and $R_\parallel \sim t(\ln t)^{1/4}$ found in Ref. \cite{24}.

The second point we want to notice is that these equations may be used to understand the following experiment. We have mentioned above the possibility of applying the shear flow a certain time after the quench, in order to have large circular domains as initial conditions. From Eqs. (13)-(17), we expect the following behavior. For a relatively small strain, since the domains are quite large, the second term in the rhs of Eqs. (13) is negligible and $R_\perp$ first decreases. Thus the second term in the rhs of Eq. (14) becomes important and the growth of $R_\parallel$ is slower than for a direct quench in the flow. This initial behavior is qualitatively different from the previous case and after this unusual transient, the growth becomes similar to the one studied previously. This predicted behavior agrees satisfactorily with the corresponding numerical experiment, as can be seen in Fig. \ref{fig13}. Note, in particular, that the slope of the curve $R_\parallel(t)$ is minimum when $R_\perp(t)$ reaches its smallest value, as can also be deduced from Eq. (16).

![Graph](image)

FIG. 13. Time evolution of the two length scales as a function of the strain when the system is first allowed to grow without shear during a time $t = 400$ and then submitted to a shear flow with $\gamma = 0.04$. We have added as a visual guide a horizontal dashed line and the data for $R_\parallel(t)$ for a direct quench in the shear flow (lowest $R_\parallel$ curve).

In conclusion, we hope this work has clarified several issues concerning the spinodal decomposition in a shear flow. It would be very interesting to perform the same measurements in three dimensions, since the analytical approach of Ref. \cite{25} shows that dynamic scaling properties might be different in 2 and 3 dimensions. Of course, the next (big) problem is to have a better understanding of the effect of hydrodynamics on the phase separation in a shear flow.

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