ON CERTAIN GENERALIZATIONS OF ONE FUNCTION AND RELATED PROBLEMS

SYMON SERBENYUK

Abstract. The present article is devoted to the generalized Salem functions, the generalized shift operator, and certain related problems. A description of further investigations of the author of this article is given. These investigations (in terms of various representations of real numbers) include the generalized Salem functions and generalizations of the Gauss-Kuzmin problem.

1. Introduction

Nowadays it is well known that functional equations and systems of functional equations are using widely in mathematics and other sciences. Modeling of functions with complicated local structure by systems of functional equations is a shining example of their applications in function theory.

Note that a class of functions with complicated local structure consists of singular (for example, [16, 33, 13, 22]), continuous nowhere monotonic [18, 28] and nowhere differentiable functions (for example, [4, 20], etc.).

Now researchers are trying to find simpler examples of singular functions. Interest in such functions is explained by their connection with modeling of real objects, processes, and phenomena (in physics, economics, technology, etc.) and with different areas of mathematics (for example, see [3, 1, 9, 14, 30, 31, 32]). A brief historical remark on singular functions is given in [2].

One of the simplest examples of singular functions was introduced by Salem. In [16], Salem modeled the function

\[ s(x) = \beta_{a_1} + \sum_{n=2}^{\infty} \left( \beta_{a_n} \prod_{i=1}^{n-1} q_i \right) = y = \Delta^{Q_2}_{a_1 a_2 \ldots a_n}, \]

where \( q_0 > 0, q_1 > 0, \) and \( q_0 + q_1 = 1. \) This function is a singular function. However, generalizations of the Salem function can be non-differentiable functions or do not have the derivative on a certain set.

Note that many researches are devoted to the Salem function and its generalizations (for example, see [2, 8, 17, 18, 28] and references in these papers).

Describing the present investigations, a certain generalization of the \( q \)-ary representation is considered and certain properties of generalized shift operator defined in terms of some of these representations are studied. Also, several related further researches of the author of this paper are noted. The main attention is given to modelling some generalization of the Salem function by certain systems of functional equations and by using the generalized shift operator.

2. Some generalizations of \( q \)-ary expansions of real numbers

Let us consider the following representation introduced by G. Cantor in [5] in 1869.
Let $Q \equiv (q_k)$ be a fixed sequence of positive integers, $q_k > 1$, $\Theta_k$ be a sequence of the sets $\Theta_k \equiv \{0, 1, \ldots, q_k - 1\}$, and $i_k \in \Theta_k$. Then

\[ [0, 1] \ni x = \Delta_{i_{12} \ldots i_{kn}}^Q \equiv \frac{i_1}{q_1} + \frac{i_2}{q_1 q_2} + \cdots + \frac{i_n}{q_1 q_2 \cdots q_n} + \cdots, \tag{1} \]

It is easy to see that the last expansion is the $q$-ary expansion

\[ \frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \cdots + \frac{\alpha_n}{q^n} + \cdots \equiv \Delta_{\alpha_1 \alpha_2 \ldots \alpha_n}^q, \tag{2} \]

of numbers from the closed interval $[0, 1]$ whenever the condition $q_k = q$ holds for all positive integers $k$. Here $q$ is a fixed positive integer, $q > 1$, and $\alpha_n \in \{0, 1, \ldots, q - 1\}$.

Let us note that certain numbers from $[0, 1]$ have two different representations by series \text{\textit{(1)}}, i.e.,

\[ \Delta_{i_{12} \ldots i_{m-1}i_m000 \ldots}^Q = \Delta_{i_{12} \ldots i_{m-1}[i_m-1][q_m+1-1][q_m+2-1] \ldots}^Q = \sum_{k=1}^{m} \frac{i_k}{q_1 q_2 \cdots q_k}. \]

Such numbers are called $Q$-\textit{rational}. The other numbers in $[0, 1]$ are called $Q$-\textit{irrational}.

Let $c_1, c_2, \ldots, c_m$ be an ordered tuple of integers such that $c_j \in \{0, 1, \ldots, q_j - 1\}$ for $j = 1, m$.

A cylinder $\Delta_{c_1c_2 \ldots c_m}^Q$ of rank $m$ with base $c_1c_2 \ldots c_m$ is the following set

\[ \Delta_{c_1c_2 \ldots c_m}^Q \equiv \{ x : x = \Delta_{c_1c_2 \ldots c_m i_m+1 \ldots i_m+k}^Q \}. \]

That is, any cylinder $\Delta_{c_1c_2 \ldots c_m}^Q$ is a closed interval of the form

\[ \left[ \Delta_{c_1c_2 \ldots c_m000 \ldots}, \Delta_{c_1c_2 \ldots c_m [q_m+1-1][q_m+2-1] \ldots}^Q \right]. \]

By analogy, in the case of representation \text{\textit{(2)}}, we get

\[ \Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^q = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_{m-1} \alpha_m-1[q-1][q-1] \ldots}^q = \sum_{k=1}^{m} \frac{\alpha_k}{q_1 q_2 \cdots q_k}. \]

Also, an arbitrary cylinder $\Delta_{c_1c_2 \ldots c_m}^Q$ is a closed interval of the form

\[ \left[ \Delta_{c_1c_2 \ldots c_m000 \ldots}, \Delta_{c_1c_2 \ldots c_m [q-1][q-1] \ldots}^q \right]. \]

3. Shift operators

In this section, the shift operator is described and the generalized shift operator is studied for the cases of $q$-ary expansions and of expansions of numbers in series \text{\textit{(1)}}.

The shift operator $\sigma$ of expansion \text{\textit{(1)}} is the following form

\[ \sigma(x) = \sigma \left( \Delta_{i_{12} \ldots i_{kn}}^Q \right) = \sum_{k=2}^{\infty} \frac{i_k}{q_2 q_3 \cdots q_k} = q_1 \Delta_{i_{12} \ldots i_{kn}}^Q. \]

It is easy to see that

\[ \sigma^n(x) = \sigma^n \left( \Delta_{i_{12} \ldots i_{kn}}^Q \right) = \sum_{k=n+1}^{\infty} \frac{i_k}{q_{n+1} q_{n+2} \cdots q_k} = q_1 \ldots q_n \Delta_{i_{n+1}i_{n+2} \ldots}^Q. \]

Therefore,

\[ x = \sum_{k=1}^{n} \frac{i_k}{q_1 q_2 \cdots q_k} + \frac{1}{q_1 q_2 \cdots q_n} \sigma^n(x). \tag{3} \]

Note that

\[ \sigma^n \left( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^q \right) = \sum_{k=n+1}^{\infty} \frac{\alpha_k}{q^{k-n}} = \Delta_{\alpha_{n+1} \alpha_{n+2} \ldots}^q. \]
In [21], the notion of the generalized shift operator was introduced in terms of series

$$x = \Delta_{-Q}^{i_1i_2...i_m...} = \frac{i_1}{-q_1} + \frac{i_2}{(-q_1)(-q_2)} + \cdots + \frac{i_n}{(-q_1)(-q_2)...(-q_n)} + \cdots$$  (4)

That is,

$$\sigma_m \left( \sum_{k=1}^{\infty} \frac{(-1)^k i_k}{q_1q_2\cdots q_k} \right) = \frac{i_1}{q_1} + \frac{i_2}{q_1q_2} - \frac{i_3}{q_1q_2q_3} + \cdots + \frac{(-1)^{m-1} i_{m-1}}{q_1q_2\cdots q_{m-1}q_m} + \frac{(-1)^m i_{m+1}}{q_1q_2\cdots q_{m-1}q_m q_{m+1}} + \cdots$$

The idea includes the following: any number from a certain interval can be represented by two fixed sequences (\(q_n\)) and (\(i_n\)). The generalized shift operator maps the preimage into a number represented by the following two sequences (\(q_1, q_2, \ldots, q_{m-1}, q_{m+1}, q_{m+2}, \ldots\)) and (\(i_1, i_2, \ldots, i_{m-1}, i_{m+1}, i_{m+2}, \ldots\)). In terms of certain encodings of real numbers, this number can belong to another interval.

Let us remark that, in this section, the main attention is given to generalized shift operator defined in terms of series [11] because this series is a generalization of a q-ary expansion and models (in the general case) a numeral system with a variable alphabet. Let us note that some numeral system is a numeral system with a variable alphabet whenever there exist at least two numbers \(k\) and \(l\) such that the condition \(A_k \neq A_l\) holds for the representation \(\Delta_{i_1i_2...i_m...}\) of numbers in terms of this numeral system, where \(i_k \in A_k\) and \(i_l \in A_l\), as well as \(k \neq l\).

Suppose a number \(x \in [0, 1]\) represented by series [11]. Then

$$\sigma_m(x) = \sum_{k=1}^{m-1} \frac{i_k}{q_1q_2\cdots q_k} + \sum_{l=m+1}^{\infty} \frac{i_l}{q_1q_2\cdots q_{m-1}q_{m+1}\cdots q_l}.$$

Denote by \(\zeta_{m+1}\) the sum \(\sum_{l=m+1}^{\infty} \frac{i_l}{q_1q_2\cdots q_{m-1}q_{m+1}\cdots q_l}\) and by \(\vartheta_{m-1}\) the sum \(\sum_{k=1}^{m-1} \frac{i_k}{q_1q_2\cdots q_k}\). Then \(\zeta_{m+1} = q_m(x - \vartheta_m)\) and

$$\sigma_m(x) = q_m x - (q_m - 1)\vartheta_{m-1} - \frac{i_m}{q_1q_2\cdots q_{m-1}}.$$  (5)

Let us remark that

$$\sigma(x) = \sigma_1(x) = \sum_{n=2}^{\infty} \frac{i_n}{q_2q_3\cdots q_n} = q_1\Delta_{i_2i_3...i_m...}^Q$$

and

$$\sigma_m(x) = \Delta_{i_1i_2...i_{m-1}000...}^Q + q_m\Delta_{i_1i_2...i_{m-1}0i_{m+1}i_{m+2}...}^Q = \Delta_{i_1i_2...i_{m-1}0i_{m+1}i_{m+2}...}^Q + (q_m - 1)\Delta_{i_1i_2...i_{m-1}0i_{m+1}i_{m+2}...}^Q.$$

**Lemma 1.** In the case of expansion [11], the generalized shift operator has the following properties:

- \(\sigma \circ \sigma_m^n(x) = \sigma^{m+1}(x)\).
- Suppose \((k_n)\) is a sequence of positive integers such that \(k_n = k_{n-1} + 1, n = 2, 3, \ldots\). Then

  $$\sigma^{k_{n+1}} \circ \sigma_{k_n} \circ \sigma_{k_{n-1}} \circ \cdots \circ \sigma_{k_1}(x) = \sigma^{k_n}(x).$$

- Suppose \((k_n)\) is an arbitrary finite subsequence of positive integers. Then

  $$\sigma^{k_{n-m}} \circ \sigma_{k_n} \circ \sigma_{k_{n-1}} \circ \cdots \circ \sigma_{k_1}(x) = \sigma^{k_n}(x).$$

- The mapping \(\sigma_m\) is continuous at each point of the interval (\(\inf \Delta_{c_1c_2...c_m}^Q, \sup \Delta_{c_1c_2...c_m}^Q\)). The endpoints of \(\Delta_{c_1c_2...c_m}^Q\) are points of discontinuity of the mapping.

- The mapping \(\sigma_m\) has a derivative almost everywhere (with respect to the Lebesgue measure). If the mapping has a derivative at the point \(x = \Delta_{e_1e_2...e_k...}^Q\), then \(\sigma_m'(x) = q_m\).
\[ x - \sigma_m(x) = \frac{i_m}{q_1 q_2 \cdots q_m} + \frac{\sigma^m(x)}{q_1 q_2 \cdots q_m} (1 - q_m). \]

**Proof.** All properties follow from the definition of \( \sigma_m \) and equality (5).

Let us consider a cylinder \( \Delta_{c_1 c_2 \ldots c_n}^Q \). It is easy to see that \( x_0 \to x_0 \sigma_m(x) = x_0 \) holds for any \( Q \)-irrational point from \( \Delta_{c_1 c_2 \ldots c_n}^Q \) and all \( Q \)-rational points whenever \( m \neq n \). If \( m = n \), then

\[ \lim_{x \to x_0} \sigma_m(x) = \sigma_m(x_0) = \sigma_m \left( \Delta_{i_1 i_2 \ldots i_{n-1} 0 0 0 \ldots}^Q \right), \]

\[ \lim_{x \to x_0} \sigma_m(x) = \sigma_m(x_0) = \sigma_m \left( \Delta_{i_1 i_2 \ldots i_{n-1} i_{n-1} i_{n-1} 0 0 0 \ldots}^Q \right), \]

and

\[ \sigma_m(x_0) - \sigma_m(x_0) = - \frac{1}{q_1 q_2 \cdots q_{m-1}}. \]

In addition,

\[ x - \sigma_m(x) = q_m x + \frac{\sigma^m(x)}{q_1 q_2 \cdots q_m} - q_m - \zeta_m + \frac{i_m}{q_1 q_2 \cdots q_m} + \frac{\sigma^m(x)}{q_1 q_2 \cdots q_m} (1 - q_m). \]

Let us consider expansion (6). In this case,

\[ \sigma_m(x) = \sigma_m \left( \sum_{n=1}^{\infty} \frac{(-1)^ni_n}{q_1 q_2 \cdots q_n} \right) = \sum_{k=1}^{m-1} \frac{(-1)^k i_k}{q_1 q_2 \cdots q_k} + \sum_{j=m+1}^{\infty} \frac{(-1)^{j-1} i_j}{q_1 q_2 \cdots q_{m-1} q_{m+1} \cdots q_j} \]

\[ = -q_m x + (1 + q_m) \sum_{k=1}^{m-1} \frac{(-1)^k i_k}{q_1 q_2 \cdots q_k} + \frac{(-1)^m i_m}{q_1 q_2 \cdots q_{m-1}}. \]

so, \( \sigma_m \) is a piecewise linear function since \( \sum_{k=1}^{m} \frac{(-1)^k c_k}{q_1 q_2 \cdots q_k} \) is constant for the set \( \Delta_{c_1 c_2 \ldots c_n}^Q \).

Let us note that, in the case of \( q \)-ary expansions of real numbers, properties of the generalized shift operator are similar with properties of the generalized shift operator for expansions (1). Really,

\[ \sigma_m \left( \Delta_{a_1 a_2 \ldots a_{n-1} 0}^Q \right) = q x - \frac{\alpha_m}{q^{m-1}} - (q - 1) \sum_{k=1}^{m-1} \frac{\alpha_k}{q^k}. \]

However, \( \sigma_m \left( \Delta_{a_1 a_2 \ldots a_{n-1} 0}^Q \right) = \Delta_{a_1 a_2 \ldots a_{m-1} 0 m+1 \ldots}^Q \).

In the paper [29], the generalized shift operator is investigated more detail. In the next articles of the author of this paper, the notion of the generalized shift operator will be investigated in more detail and applied by the author of the present article in terms of various representation of real numbers (e.g., positive and alternating Cantor series and their generalizations, as well as Luroth, Engel series, etc., various continued fractions).

Let us consider certain applications of the generalized shift operator. One can model generalizations of the Gauss-Kuzmin problem and generalizations of the Salem function.

## 4. Generalizations of the Gauss-Kuzmin Problem

This problem is one of the first and still one of the most important results in the metrical theory of continued fractions [11]. The problem was formulated by the Gauss and the first solution was received by Kuzmin [10]. The problem is investigated by a number of researchers for different types of continued fractions (for example, see [7] [11] [12] and references in the papers).

The Gauss-Kuzmin problem is to calculate the limit

\[ \lim_{n \to \infty} \lambda(\mathcal{E}_n(x)), \]
where $\lambda(\cdot)$ is the Lebesgue measure of a set and the set $E_n(x)$ is a set of the form

$$E_n = \{ z : \sigma^n(z) < x \}.$$ 

Here $z = \Delta_{i_1 i_2 \ldots i_k}$, i.e., $\Delta_{i_1 i_2 \ldots i_k}$ is a certain representation of real numbers, $\sigma$ is the shift operator.

Generalizations of the Gauss-Kuzmin problem are to calculate the limit

$$\lim_{k \to \infty} \lambda(\tilde{E}_{n_k}(x)),$$

for sets of the following forms:

- $\tilde{E}_{n_k}(x) = \{ z : \sigma_{n_k} \circ \sigma_{n_k-1} \circ \ldots \circ \sigma_{n_1} < x \}$

  including (here $(n_k)$ is a certain fixed sequence of positive integers) the cases when $(n_k)$ is a constant sequence.

- the set $\tilde{E}_{n_k}(x)$ under the condition that $n_k = \psi(k)$, where $\psi$ is a certain function of the positive integer argument.

- $\tilde{E}_{n_k}(x) = \{ z : \sigma_{n_k} \circ \sigma_{n_k-1} \circ \ldots \circ \sigma_{n_1}(z) < x \}$

  where $\varphi$ is a certain function and $m, c$ are some parameters (if applicable). That is, for example,

  $$\tilde{E}_{n_k}(x) = \{ z : \sigma_m \circ \sigma_m \circ \ldots \circ \sigma_m(z) < x \},$$

  where $k > c$ and $c$ is a fixed positive integer, or

  $$\tilde{E}_{n_k}(x) = \{ z : \sigma_m \circ \sigma_m \circ \ldots \circ \sigma_m(z) < x \},$$

  where $k \equiv 1 \pmod{c}$ and $c > 1$ is a fixed positive integer.

- In the general case,

  $$\tilde{E}_{n_k}(x) = \{ z : \sigma_{\psi(m,k,c)} \circ \ldots \circ \sigma_{\psi(1)}(z) < x \},$$

  where $\psi(n)$ is a certain function of the positive integer argument.

In addition, one can formulate such problems in terms of the shift operator. For example, one can formulate the Gauss-Kuzmin problem for the following sets:

$$\tilde{E}_{n_k}(z) = \{ z : \sigma^{n_k}(z) < \sigma^{k_0}(z) \},$$

where $k_0$, $(n_k)$ are a fixed number and a fixed sequence.

$$\tilde{E}_{n_k}(x) = \{ z : \sigma^{n_k}(z) < \sigma^{k_0}(x) \},$$

$$\tilde{E}_n(x) = \{ z : \sigma^{\psi(n)}(z) < x \},$$

where $\psi(n)$ is a certain function of the positive integer argument.

In addition,

$$\tilde{E}_n(z) = \{ z : \sigma^{\psi(n)}(z) < \sigma^{\phi(n)}(z) \},$$

$$\tilde{E}_n(x) = \{ z : \sigma^{\psi(n)}(z) < \sigma^{\phi(n)}(x) \},$$

where $\psi, \varphi$ are certain functions of the positive integer arguments.

It is easy to see that similar problems can be formulated for the case of the generalized shift operator.
In next articles of the author of the present article, such problems will be considered by the author of this article in terms of various numeral systems (with a finite or infinite alphabet, with a constant or variable alphabet, positive, alternating, and sign-variable expansions, etc.).

5. Generalizations of the Salem function

Let us consider certain functions whose argument represented by the $q$-ary expansion.

Suppose $(n_k)$ is a fixed sequence of positive integers such that $n_i \neq n_j$ for $i \neq j$ and such that for any $n \in \mathbb{N}$ there exists a number $k_0$ for which the condition $n_{k_0} = n$ holds.

**Theorem 1.** Let $P_q = \{p_0, p_1, \ldots, p_{q-1}\}$ be a fixed tuple of real numbers such that $p_i \in (-1, 1)$, where $i = 0, q - 1$, $\sum p_i = 1$, and $0 = \beta_0 < \beta_1 = \sum_{j=0}^{i-1} p_j < 1$ for all $i \neq 0$. Then the finite system of functional equations

$$f \left( \sigma_{n_{k-1}} \circ \sigma_{n_{k-2}} \circ \ldots \circ \sigma_{n_1}(x) \right) = \beta_{\alpha_{n_k}} f \left( \sigma_{n_k} \circ \sigma_{n_{k-1}} \circ \ldots \circ \sigma_{n_1}(x) \right),$$

(7)

where $x = \Delta^q_{n_1n_2\ldots n_k}$, has the unique solution

$$g(x) = \beta_{\alpha_{n_1}} + \sum_{k=2}^{\infty} \left( \beta_{\alpha_{n_k}} \prod_{j=1}^{k-1} p_{\alpha_{n_j}} \right)$$

in the class of determined and bounded on $[0, 1]$ functions.

**Proof.** Since the function $g$ is a determined on $[0, 1]$ function, using system (7), we get

$$g(x) = \beta_{\alpha_{n_1}} + p_{\alpha_{n_1}} g(\sigma_{n_1}(x))$$

$$= \beta_{\alpha_{n_1}} + p_{\alpha_{n_1}} (\beta_{\alpha_{n_2}} + p_{\alpha_{n_2}} g(\sigma_{n_2} \circ \sigma_{n_1}(x))) = \ldots$$

$$\cdots = \beta_{\alpha_{n_1}} + \beta_{\alpha_{n_2}} p_{\alpha_{n_1}} + \beta_{\alpha_{n_3}} p_{\alpha_{n_1}} p_{\alpha_{n_2}} + \ldots + \beta_{\alpha_{n_k}} \prod_{j=1}^{k} p_{\alpha_{n_j}} + \left( \prod_{t=1}^{k} p_{\alpha_{n_t}} \right) g(\sigma_{n_k} \circ \cdots \circ \sigma_{n_2} \circ \sigma_{n_1}(x)).$$

So,

$$g(x) = \beta_{\alpha_{n_1}} + \sum_{k=2}^{\infty} \left( \beta_{\alpha_{n_k}} \prod_{j=1}^{k-1} p_{\alpha_{n_j}} \right)$$

since $g$ is a determined and bounded on $[0, 1]$ function and

$$\lim_{k \to \infty} g(\sigma_{n_k} \circ \cdots \circ \sigma_{n_2} \circ \sigma_{n_1}(x)) \prod_{t=1}^{k} p_{\alpha_{n_t}} = 0,$$

where

$$\prod_{t=1}^{k} p_{\alpha_{n_t}} \leq \left( \max_{0 \leq i \leq q-1} p_i \right)^k \to 0, \quad k \to \infty.$$

\[\square\]

**Theorem 2.** The following properties hold:

- The function $g$ is continuous at any $q$-irrational point of $[0, 1]$.
- The function $g$ is continuous at $q$-rational point

$$x_0 = \Delta^q_{\alpha_1\alpha_2\ldots \alpha_{m-1}000\ldots} = \Delta^q_{\alpha_1\alpha_2\ldots \alpha_{m-1}[\alpha_{m-1}][q-1][q-1]\ldots}$$

whenever a sequence $(n_k)$ is such that the conditions $k_0 = \max \{ k : n_k \in \{ 1, 2, \ldots, m \} \}$ and $n_{k_0} = m$ hold. In the other case, a $q$-rational point $x_0$ is a point of discontinuity.
- The set of all points of discontinuities of the function $g$ is a countable, finite, or empty set.

It depends on a sequence $(n_k)$. 


Proof. Let us note that a certain fixed function \( g \) is given by a fixed sequence \( (n_k) \) described above. One can write our mapping by the following:

\[
g : x = \Delta_{q_{\alpha_1\alpha_2...\alpha_k}} \rightarrow \beta_{\alpha_1} + \sum_{k=2}^{\infty} \left( \prod_{l=1}^{k-1} \beta_{\alpha_k} p_{\alpha_{n_l}} \right) = \Delta_{\alpha_1\alpha_2...\alpha_k} = g(x) = y.
\]

Let \( x_0 = \Delta_{q_{\alpha_1\alpha_2...\alpha_k}} \) be an arbitrary \( q \)-irrational number from \([0,1]\). Let \( x = \Delta_{\gamma_1...\gamma_k} \) be a \( q \)-irrational number such that the condition \( \gamma_{n_j} = \alpha_{n_j} \) holds for all \( j = 1, k_0 \), where \( k_0 \) is a certain positive integer. That is,

\[
x = \Delta_{\gamma_1...\gamma_{n_1}...\gamma_{n_2}...\gamma_{n_k}...} = \Delta_{\gamma_1...\gamma_{n_1}...\gamma_{n_2}...\gamma_{n_k}...} = \Delta_{\alpha_1\alpha_2...\alpha_k}.
\]

Then

\[
g(x_0) = \Delta_{\alpha_1\alpha_2...\alpha_k} \Rightarrow \beta_{\alpha_1} = \frac{1}{\prod_{j=1}^{k_0} p_{\alpha_{n_j}}},
\]

Since \( 0 \leq g(x) \leq 1 \), we have \( g(x) - g(x_0) = 0 \).

By the continuity of \( g \) at \( x \), we have

\[
g(x) = \Delta_{\alpha_1\alpha_2...\alpha_k} \Rightarrow \beta_{\alpha_1} = \frac{1}{\prod_{j=1}^{k_0} p_{\alpha_{n_j}}},
\]

and

\[
|g(x) - g(x_0)| \leq \prod_{j=1}^{k_0} p_{\alpha_{n_j}} \leq (\max\{p_0, \ldots, p_{q-1}\})^{k_0} \rightarrow 0 \quad (k_0 \rightarrow \infty).
\]

So, \( \lim_{x \rightarrow x_0} g(x) = g(x_0) \), i.e., the function \( g \) is continuous at any \( q \)-irrational point.

Let \( x_0 \) be a \( q \)-rational number, i.e.,

\[
x_0 = x_0^{(1)} = \Delta_{q_{\alpha_1\alpha_2...\alpha_m}} = \Delta_{q_{\alpha_1\alpha_2...\alpha_m}} = \Delta_{\alpha_1\alpha_2...\alpha_m} = x_0^{(2)}.
\]

Then there exist positive integers \( k^* \) and \( k_0^* \) such that

\[
y_1 = g(x_0^{(1)}) = \Delta_{g(x_0^{(1)})} = \Delta_{\alpha_1\alpha_2...\alpha_k} = \Delta_{\alpha_1\alpha_2...\alpha_k},
\]

\[
y_2 = g(x_0^{(2)}) = \Delta_{g(x_0^{(2)})} = \Delta_{\alpha_1\alpha_2...\alpha_k} = \Delta_{\alpha_1\alpha_2...\alpha_k}.
\]

Here \( n_{k^*} = m \), \( n_{k^*} \leq n_{k_0} \), and \( k_0 \) is the number such that \( \alpha_{n_{k_0}} \in \{\alpha_1, \ldots, \alpha_{m-1}, \alpha_m\} \) and \( k_0 \) is the maximum position of any number from \( \{1, 2, \ldots, m\} \) in the sequence \( (n_k) \).

Let us consider a fact (Section 2 in [28] and the paper [16], since such expansion of numbers is an analytic representation the Salem function) that a representation \( \Delta_{\alpha_1\alpha_2...\alpha_k} \) is the following whenever the conditions \( (n_k) = (k) \) and \( p_j > 0 \) for all \( j = 0, q-1 \), where \( k = 1, 2, \ldots \), hold:

\[
[0,1] \ni x = \Delta_{\alpha_1\alpha_2...\alpha_k} = \beta_{\alpha_1} + \sum_{k=2}^{\infty} \left( \prod_{l=1}^{k-1} p_{\alpha_l} \right) = \Delta_{\alpha_1\alpha_2...\alpha_k} = x_0^{(1)}.
\]

This representation is the \( q \)-ary representation whenever the condition

\[
0 < p_0 = p_1 = \cdots = p_{q-1} = \frac{1}{q}
\]

holds. Also, certain numbers have two different such representations, and the rest of the numbers have the unique such representation. That is,

\[
z_1 = \Delta_{\alpha_1\alpha_2...\alpha_m} = \Delta_{\alpha_1\alpha_2...\alpha_m} = \Delta_{\alpha_1\alpha_2...\alpha_m} = \Delta_{\alpha_1\alpha_2...\alpha_m} = z_2.
\]
Let us note that one can consider the intervals 
\[ \mu_1 \text{ and } \mu_2 \] 
and 
\[ k - \{ \text{ is easy to see that } \], \]

Since the Salem function is a strictly increasing function, conditions for holding \( x_1 < x_2 \) or \( x_1 > x_2 \) are identical in terms of the \( q \)-ary representation and of representation \( \mathbb{S} \).

Using the case of a \( q \)-ary irrational number, let us consider the limits 
\[
\lim_{x \to x_0^+} g(x) = \lim_{x \to x_0^{(1)}} g(x) = y_1, \quad \lim_{x \to x_0^-} g(x) = \lim_{x \to x_0^{(2)}} g(x) = y_2.
\]

Whence \( y_1 = y_2 \) whenever a sequence \( (n_k) \) is such that the conditions \( n_{k_0} = m \) and \( k_0 = \max \{ k : n_k \in \{1, 2, \ldots, m\} \} \) hold.

So, the set of all points of discontinuities of the function \( g \) is a countable, finite, or empty set. It depends on a sequence \( (n_k) \).

Suppose \( (n_k) \) is a fixed sequence and \( c_{n_1}, c_{n_2}, \ldots, c_{n_r} \) is a fixed tuple of numbers \( c_{n_j} \in \{0, 1, \ldots, q - 1\} \), where \( j = 1, r \) and \( r \) is a fixed positive integer.

Let us consider the following set 
\[
S_{q,(c_{n_r})} \equiv \left\{ x : x = \Delta^n_{a_1 a_2 \ldots a_n \ldots} \right\},
\]

where \( k = 1, 2, \ldots, c_{n_j} \in \{c_{n_1}, c_{n_2}, \ldots, c_{n_r}\} \) for all \( j = 1, r \). This set has non-zero Lebesgue measure (for example, similar sets are investigated in terms of other representations of numbers in [21]). It is easy to see that \( S_{q,(c_{n_r})} \) maps to
\[
g(S_{q,(c_{n_r})}) \equiv \left\{ y : y = \Delta^n_{c_{n_1} c_{n_2} \ldots c_{n_r} \ldots} \right\}
\]
under \( g \).

For a value \( \mu_g \left( S_{q,(c_{n_r})} \right) \) of the increment, the following is true.

\[
\mu_g \left( S_{q,(c_{n_r})} \right) = g \left( \sup S_{q,(c_{n_r})} \right) - g \left( \inf S_{q,(c_{n_r})} \right),
\]

where 
\[
\Delta^n_{a_1 a_2 \ldots a_n \ldots} = \begin{cases}
\sum_{j=1}^r p_{c_{n_j}} & \text{if } a_j = 1 \\
0 & \text{if } a_j = 0
\end{cases}
\]
and
\[
\mu_g \left( S_{q,(c_{n_r})} \right) = \mu_g \left( [\inf S_{q,(c_{n_r})}, \sup S_{q,(c_{n_r})}] \right) = \prod_{j=1}^r p_{c_{n_j}}.
\]

So, one can formulate the following statements.

**Theorem 3.** The function \( g \) has the following properties:

1. If \( p_j \geq 0 \) or \( p_j > 0 \) for all \( j = 0, q - 1 \), then:
   - \( g \) does not have intervals of monotonicity on \([0, 1]\) whenever the condition \( n_k = k \) holds for no more than a finite number of values of \( k \);
   - \( g \) has at least one interval of monotonicity on \([0, 1]\) whenever the condition \( n_k \neq k \) holds for a finite number of values of \( k \);
• \( g \) is a monotonic non-decreasing function (in the case when \( p_j \geq 0 \) for all \( j = 0, q - 1 \)) or is a strictly increasing function (in the case when \( p_j > 0 \) for all \( j = 0, q - 1 \)) whenever the condition \( n_k = k \) holds for \( k \in \mathbb{N} \).

(2) If there exists \( p_j = 0 \), where \( j = 0, q - 1 \), then \( g \) is a constant almost everywhere on \([0, 1]\).

(3) If there exists \( p_j < 0 \) (other \( p_j \) are positive), where \( j = 0, q - 1 \), and the condition \( n_k = k \) holds for almost all \( k \in \mathbb{N} \), then \( g \) does not have intervals of monotonicity on \([0, 1]\).

Let us note that the last statements follow from (3).

Let us consider a cylinder \( \Delta^q_{c_1c_2...c_n} \). We obtain

\[
\mu_g (\Delta^q_{c_1c_2...c_n}) = \Delta^q_{\xi_{n1}...\xi_{nq-1}} = \frac{1}{(q-1)!} \prod_{j=1}^{q-1} (q - 1 - j)^{c_j} \xi_{n1}...\xi_{nq-1},
\]

where \( c_j \in \{1, 2, ..., q\} \) and \( (c_j) \) is a certain sequence of numbers from \( \mathbb{N} \cup \{0\} \).

So, differential properties of \( g \) depend on a sequence \( (n_k) \) and the set of numbers \( P_q = \{p_0, p_1, ..., p_{q-1}\} \).

**Statement.** The function \( g \) can be a singular or non-differentiable function. It depends on a sequence \( (n_k) \) and \( P_q = \{p_0, p_1, ..., p_{q-1}\} \).

Differential properties including special partial cases will be considered in the next articles of the author of this paper since the technique of proofs introduced by Salem in [10] is not suitable for proving statements in our general case.

In addition, let us note the following.

**Lemma 2.** Let \( \eta \) be a random variable defined by the following form

\[
\eta = \Delta^q_{\xi_{n1}...\xi_{nq}},
\]

where \( k = 1, 2, 3, ..., q \), the digits \( \xi_{nk} \) are random and taking the values 0, 1, ..., \( q - 1 \) with probabilities \( p_0, p_1, ..., p_{q-1} \). That is \( \xi_{nk} \) are independent and \( P(\xi_{nk} = \alpha_{nk}) = p_{\alpha_{nk}}, \alpha_{nk} \in \{0, 1, ..., q - 1\} \). Here \( (n_k) \) is a sequence of positive integers such that \( n_i \neq n_j \) for \( i \neq j \) and such that for any \( n \in \mathbb{N} \) there exists a number \( k_0 \) for which the condition \( n_{k_0} = n \) holds.

The distribution function \( \hat{F}_\eta \) of the random variable \( \eta \) can be represented by

\[
\hat{F}_\eta(x) = \begin{cases} 
0, & x < 0 \\
\beta_{\alpha_{n1}}(x) + \sum_{k=2}^{\infty} \left( \beta_{\alpha_{nk}}(x) \prod_{\tau=1}^{k-1} p_{\alpha_{n\tau}}(x) \right), & 0 \leq x < 1 \\
1, & x \geq 1,
\end{cases}
\]

where \( x = \Delta^q_{\alpha_{n1}...\alpha_{nq}} \).

A method of the corresponding proof is described in [18].

**Theorem 4.** The Lebesgue integral of the function \( g \) can be calculated by the formula

\[
\int_0^1 g(x)dx = \frac{1}{q - 1} \sum_{j=0}^{q-1} \beta_j.
\]

**Proof.** By \( A \) denote the sum \( \sum_{j=0}^{q-1} \beta_j \) and by \( B \) denote the sum \( \sum_{j=0}^{q-1} p_j \). Since

\[
x = \frac{1}{q} \sigma_m(x) + (q - 1) \sum_{k=1}^{m-1} \frac{\alpha_k}{q^k} + \frac{\alpha_m}{q^{m-1}},
\]

\[
\int_0^1 g(x)dx = \frac{1}{q - 1} \sum_{j=0}^{q-1} \beta_j.
\]
and

\[ dx = \frac{1}{q} d(\sigma_m(x)), \]

we have

\[ \int_0^1 g(x)dx = \sum_{j=0}^{q-1} \int_{\frac{j}{q}}^{\frac{j+1}{q}} g(x)dx = \sum_{j=0}^{q-1} \int_{\frac{j}{q}}^{\frac{j+1}{q}} (\beta_j + p_j g(\sigma_n(x))) dx \]

\[ = \frac{1}{q} \sum_{j=0}^{q-1} \beta_j + \frac{1}{q} \left( \sum_{j=0}^{q-1} p_j \right) \int_0^1 g(\sigma_n(x))d(\sigma_n(x)) \]

\[ = \frac{1}{q} \sum_{j=0}^{q-1} \beta_j + \frac{1}{q} \left( \sum_{j=0}^{q-1} p_j \right) \left( \sum_{j=0}^{q-1} \int_{\frac{j}{q}}^{\frac{j+1}{q}} (\beta_j + p_j g(\sigma_n(x))) d(\sigma_n(x)) \right) \]

\[ = A + B \left( A + B \int_0^1 g(\sigma_n(x))d(\sigma_n(x)) \right) \]

\[ = A + AB + B^2 \left( \sum_{j=0}^{q-1} \int_{\frac{j}{q}}^{\frac{j+1}{q}} (\beta_j + p_j g(\sigma_n(x))) d(\sigma_n(x)) \right) \]

\[ = A + AB + B^2 \left( A + B \int_0^1 g(\sigma_n(x))d(\sigma_n(x)) \right) \]

\[ = A + AB + B^2 + B^3 \left( A + B \int_0^1 g(\sigma_n(x))d(\sigma_n(x)) \right) \]

\[ = A + AB + \cdots + A + B^{k-1} + B^k \left( A + B \int_0^1 g(\sigma_n(x))d(\sigma_n(x)) \right) \]

Since

\[ B^{k+1} = \left( \frac{1}{q} \sum_{j=0}^{q-1} p_j \right)^{k+1} \]

\[ = \left( \frac{1}{q} \right)^{k+1} \rightarrow 0 \text{ as } k \rightarrow \infty, \]

we obtain

\[ \int_0^1 g(x)dx = \lim_{k \to \infty} \left( \sum_{l=0}^{k} AB^l + B^{k+1} \int_0^1 g(\sigma_{n_k+1} \circ \sigma_{n_k} \circ \cdots \circ \sigma_n(x))d(\sigma_{n_k+1} \circ \sigma_{n_k} \circ \cdots \circ \sigma_n(x)) \right) \]

\[ = \sum_{k=0}^{\infty} AB^k = \left( \sum_{j=0}^{q-1} \beta_j \right) \left( \sum_{k=0}^{\infty} \frac{1}{q^{k+1}} \right) = \frac{1}{q-1} \sum_{j=0}^{q-1} \beta_j. \]

\[ \square \]

In the next articles of the author of this paper, generalizations and properties of solutions of system (7) of functional equations will be investigated for the cases of various numeral systems (with a finite or infinite alphabet, with a constant or variable alphabet, positive, alternating, and sign-variable expansions, etc.).

**Remark 1.** It can be interesting to consider the case when \((n_k)\) is an arbitrary fixed sequence (finite or infinite) of positive integers. Then the function \(g\) can be a constant function, a linear function, or a function having pathological (complicated) structure, etc. It depends on \((n_k)\). Such problems will be investigated in the next papers of the author of this article.
REFERENCES

[1] E. de Amo, M.D. Carrillo and J. Fernández-Sánchez, On duality of aggregation operators and k-negations, *Fuzzy Sets and Systems*, 181 (2011), 14–27.

[2] E. de Amo, M.D. Carrillo and J. Fernández-Sánchez, A Salem generalized function, *Acta Math. Hungar.* 151 (2017), no. 2, 361–378. https://doi.org/10.1007/s10474-017-0690-x

[3] L. Berg and M. Kruppel, De Rham’s singular function and related functions, *Z. Anal. Anwendungen.*, 19(2000), no. 1, 227–237.

[4] K. A. Bush, Continuous functions without derivatives, *Amer. Math. Monthly* 59 (1952), 222–225.

[5] G. Cantor, Uber die einfachen Zahlensysteme, *Z. Math. Phys.* 14 (1869), 121–128. (German)

[6] S. Ito and T. Sadahiro, Beta-expansions with negative bases *Integers* 9 (2009), no. 1, 227–237.

[7] Sofia Kalpazidou, On a problem of Gauss-Kuzmin type for continued fraction with odd partial quotients, *Pacific J. Math.* 123 (1986), no. 1, 103–114. https://projecteuclid.org/euclid.pjm/1102701402

[8] Kiko Kawamura, The derivative of Lebesgue’s singular function, *Real Analysis Exchange* Summer Symposium 2010, pp. 83–85.

[9] M. Kruppel, De Rham’s singular function, its partial derivatives with respect to the parameter and binary digital sums, *Rostock. Math. Kolloq.* 64 (2009), 57–74.

[10] R.O. Kuzmin, On a problem of Gauss, *Dokl. Akad. Nauk SSSR Ser. A* (1928) 375-380. [Russian; French version in *Att. Congr. Internaz. Mat. (Bologna, 1928), Tomo VI (1932) 83-89. Zanichelli, Bologna*].

[11] Dan Lascu, A Gauss-Kuzmin-type problem for a family of continued fraction expansions, *Journal of Number Theory* 133 (2013), no. 7, 2153–2181. https://doi.org/10.1016/j.jnt.2012.12.007

[12] Dan Lascu, A Gauss-Kuzmin Theorem for Continued Fractions Associated with Nonpositive Integer Powers of an Integer \( m \geq 2 \), *The Scientific World Journal* 2014 (2014), Article ID 984650, 8 pages. http://dx.doi.org/10.1155/2014/984650

[13] H. Minkowski, Zur Geometrie der Zahlen. In: Minkowski, H. (ed.) Gesammeine Abhandlungen, Band 2, pp. 5051. Druck und Verlag von B. G. Teubner, Leipzig und Berlin (1911)

[14] T. Okada, T. Sekiguchi, and Y. Shiota, An explicit formula of the exponential sums of digital sums, *Japan J. Indust. Appl. Math.* 12 (1995), 425–438.

[15] A. Rényi, Representations for real numbers and their ergodic properties, *Acta. Math. Acad. Sci. Hungar.* 8 (1957), 477–493.

[16] R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.* 53 (1943), 423–439.

[17] S. O. Serbenyuk, Functions, that defined by functional equations systems in terms of Cantor series representation of numbers, *Naukovi Zapysky NaUKMA* 165 (2015), 34–40. (Ukrainian), available at https://www.researchgate.net/publication/292606546

[18] S. O. Serbenyuk, Continuous Functions with Complicated Local Structure Defined in Terms of Alternating Cantor Series Representation of Numbers, *Journal of Mathematical Physics, Analysis, Geometry* 13 (2017), No. 1, 57–81. https://doi.org/10.15407/mag13.01.057

[19] S. Serbenyuk, Nega-Q-representation as a generalization of certain alternating representations of real numbers, *Bull. Taras Shevchenko Natl. Univ. Kyiv Math. Mech.* 1 (35) (2016), 32–39. (Ukrainian), available at https://www.researchgate.net/publication/308273000

[20] S. Serbenyuk, On one class of functions with complicated local structure, *Siauliai Mathematical Seminar* 11 (19) (2016), 75–88.

[21] S. Serbenyuk, Representation of real numbers by the alternating Cantor series, *Integers* 17 (2017), Paper No. A15, 27 pp.

[22] S. Serbenyuk, On one fractal property of the Minkowski function, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 112 (2018), no. 2, 555–559, doi:10.1007/s13398-017-0396-5

[23] S. O. Serbenyuk Non-Differentiable functions defined in terms of classical representations of real numbers, *Zh. Mat. Fiz. Anal. Geom.* 14 (2018), no. 2, 197–213. https://doi.org/10.15407/mag14.02.197

[24] Symon Serbenyuk, Representation of real numbers by the alternating Cantor series, slides of talk (2013) (Ukrainian). Available from: https://www.researchgate.net/publication/303720347

[25] Symon Serbenyuk, Representation of real numbers by the alternating Cantor series, preprint (2013) (Ukrainian). Available from: https://www.researchgate.net/publication/31678375

[26] Serbenyuk S. On some generalizations of real numbers representations, [arXiv:1602.07929v1](https://arxiv.org/abs/1602.07929) (in Ukrainian)

[27] Symon Serbenyuk, Generalizations of certain representations of real numbers, [arXiv:1801.10540v3](https://arxiv.org/abs/1801.10540), 8 pp.
[28] Symon Serbenyuk, On one application of infinite systems of functional equations in function theory, *Tatra Mountains Mathematical Publications* 74 (2019), 117-144. https://doi.org/10.2478/tmmp-2019-0024

[29] Symon Serbenyuk, Generalized shift operator of certain encodings of real numbers, arXiv:1911.12140v1, 6 pp.

[30] H. Sumi, Rational semigroups, random complex dynamics and singular functions on the complex plane, *Sugaku* 61 (2009), no. 2, 133–161.

[31] H. Takayasu, Physical models of fractal functions, *Japan J. Appl. Math.* 1 (1984), 201–205.

[32] S. Tasaki, I. Antoniou, and Z. Suchanecki, Deterministic diffusion, De Rham equation and fractal eigenvectors, *Physics Letter A* 179 (1993), no. 1, 97–102.

[33] T. Zamfirescu, Most monotone functions are singular, *Amer. Math. Mon.* 88 (1981), 47–49.

45 Shchukina St., Vinnytsia, 21012, Ukraine

*E-mail address: simon6@ukr.net*