Stochastic Graph Neural Networks

Zhan Gao†, Elvin Isufi‡ and Alejandro Ribeiro†

Abstract—Graph neural networks (GNNs) model nonlinear representations in graph data with applications in distributed agent coordination, control, and planning among others. Current GNN architectures assume ideal scenarios and ignore link fluctuations that occur due to environment, human factors, or external attacks. In these situations, the GNN fails to address its distributed task if the topological randomness is not considered accordingly. To overcome this issue, we put forth the stochastic graph neural network (SGNN) model: a GNN where the distributed graph convolution module accounts for the random network changes. Since stochasticity brings in a new learning paradigm, we conduct a statistical analysis on the SGNN output variance to identify conditions the learned filters should satisfy for achieving robust transference to perturbed scenarios, ultimately revealing the explicit impact of random link losses. We further develop a stochastic gradient descent (SGD) based learning process for the SGNN and derive conditions on the learning rate under which this learning process converges to a stationary point. Numerical results corroborate our theoretical findings and compare the benefits of SGNN robust transference with a conventional GNN that ignores graph perturbations during learning.

Index Terms—Graph neural networks, graph filters, distributed learning

I. INTRODUCTION

Graph neural networks (GNNs) are becoming the predominant tool to learn representations for network data [2], [3] with resounding success in rating prediction [4], [5], distributed agent coordination [6], [7], and learning molecular fingerprints [8], [9]. Inspired by CNNs in structured Euclidean domains (e.g., time and image), GNNs replace the temporal or the spatial filters with graph filters [10]–[12], which are tools that generalize the operation of convolution to the irregular graph domain [13], [14]. By encapsulating the filtered features into a nonlinearity, GNNs can model complex data relationships by adapting the graph filter to the application of interest [15]. One key property of GNNs inherited from graph filters is their distributed implementation. This property makes GNNs suitable candidates for distributed learning over networks, where the sensing, computing, and acting are local operations [16]–[18]. By allowing nodes to communicate only with their direct neighbors, GNNs have found applications in distributed agent coordination [6], smart grid failure detection [19], and control [20].

While seminal to establish distributed learning with GNNs, the above works consider the underlying graph fixed and deterministic. However, in applications involving sensor, communication road, and smart grid networks the graph structure changes randomly over time [21]–[23]. In a robot coordination network, for instance, the communication pattern may change due to agent malfunctions or communication links that fall with a certain probability. These random topological changes lead to a random graph filter [24], hence, to a random GNN output. Thus, the GNN will be implemented over random time varying topologies mismatched with the deterministic topology over which it was trained, therefore, degrading the performance. In this work, we hypothesize this mismatch between testing and training phases should be addressed by developing a GNN architecture that is trained on stochastic graphs representing the practical distributed setting.

We propose the stochastic graph neural network (SGNN) model to account for random link losses during training. The SGNN incorporates the random graph realizations into the architecture and makes each node rely on its neighbors with uncertainty. This neighborhood uncertainty makes the learned parameters robust to topological fluctuations; hence, it endows the SGNN with the ability to transfer more robustly its learning function to topologies affected by link perturbations. To characterize the robust transference, we analyze theoretically the SGNN variance output and highlight the role played by the different actors such as the filter Lipschitz properties, the link loss probability, and the architecture width and depth. More in detail, our three main contributions are as follows.

(i) Stochastic graph neural networks (Section IV): We define the SGNN as a similar layered architecture to GNN, but where stochastic graph filters are employed during learning. These filters account for the underlying topological variations in the node data exchanges to build random higher-level features in the graph convolutional layer.

(ii) Variance analysis (Section V): We prove the variance of the SGNN output is upper bounded by a factor that is quadratic in the link sampling probability. By developing the concept of generalized filter frequency response over a succession on graphs, we conducted this statistical analysis by requiring no more than Lipschitz continuity of the frequency response function. The variance bound indicates also the effects the architecture depth and width as well as the nonlinearities have on the SGNN performance [Theorem 1].

(iii) Convergence analysis (Section VII): We postulate the SGNN learning problem that accounts for the graph stochasticity in the cost function. We develop a stochastic gradient descent algorithm for this learning procedure and derive conditions on the learning rate under which a stationary point is achieved. We prove the convergence rate is proportional to the inverse square root of the number of iterations [Theorem 2].

Numerical results on source localization and robot swarm control corroborate our model in Section VIII. The conclusions are drawn in Section VII. All proofs are collected in the...
IEEE TRANSACTIONS ON SIGNAL PROCESSING (SUBMITTED)

II. RANDOM EDGE SAMPLING GRAPH MODEL

Consider symmetric unweighted graph $G = (V, E)$. The vertex set contains $N$ nodes $V = \{1, \ldots, N\}$ and the edge set contains $M$ undirected edges $(i, j) = (j, i) \in E$. The adjacency matrix $A \in \mathbb{R}^{N \times N}$ has entries $[A]_{ij} > 0$ if node $i$ is connected to node $j$ (i.e. $(i, j) \in E$) and $[A]_{ij} = 0$ otherwise, and the Laplacian matrix is $L = \text{diag}(A \mathbf{1}) - A$. To keep discussion general, introduce the shift operator $S \in \mathbb{R}^{N \times N}$ as a stand-in for either the adjacency or Laplacian matrix of $G$. Symmetry of the graph implies symmetry of the shift operator, $S = S^\top$.

Our interest is in settings where edges can drop randomly, such as the case in communication networks with packet drops. This results in stochastic graph topologies that we model with the random edge sampling (RES) model which we define here for reference [24].

**Definition 1** (Random Edge Sampling Graph). For a given graph $G = (V, E)$ and edge inclusion probability $p$, we define $\text{RES}(G, p)$ a random graph with realizations $G_k = (V, E_k)$ such that edge $(i, j)$ is in $E_k$ with probability $p$.

$$\Pr[(i, j) \in E_k] = p, \quad \text{for all } (i, j) \in E. \quad (1)$$

Edge inclusions in $E_k$ are drawn independently.

As per Definition 1, realizations $G_k = (V, E_k)$ have edges $(i, j) \in E_k$ independently drawn from the edges of $G$ with probability $p$. The graph realization $G_k$ induces a realization of an adjacency matrix $A_k$. If we let $B_k$ be a symmetric matrix with independently drawn Bernoulli entries $b_{k, ij} = b_{k, ji}$ we can write the adjacency of graph $G_k$ as the Hadamard product

$$A_k = B_k \circ A. \quad (2)$$

The Laplacian of $G_k$ is $L_k = \text{diag}(A_k \mathbf{1}) - A_k$. We will use $S_k$ to denote either. We emphasize that the choice of $S$ and $S_k$ are compatible. We either have $S = A$ and $S_k = A_k$ or $S = L$ and $S_k = L_k$. For future reference define the expected shift operator as $S = \mathbb{E}[S]$ and the expected graph as $\mathbb{G}$ – the one with matrix representation $\mathbb{S}$. We remark that it is ready to have edge drop probabilities depend on nodes or edges. We make them equal to simplify expressions.

A. Stochastic Graph Neural Network

A stochastic graph neural network (SGNN) on $G$ is a graph neural network (GNN) run on a sequence of random realizations of a $\text{RES}(G, p)$ random graph. To be precise, let $x = [x_1, \ldots, x_N]^\top \in \mathbb{R}^N$ be a graph signal supported on $G$ in the sense that entry $x_i$ is associated to vertex $i$. Further consider the sequence of shift operators $S_k$ associated to graphs $G_k$ drawn independently as per Definition 1. The diffusion sequence is a corresponding collection of signals $x^{(k)}$ expressed recursively as $x^{(k)} := S_k x^{(k-1)}$ with $x^{(0)} = x$. Agreeing by convention that $S_0 = \mathbf{1}$ and unrolling the recursion, the signals in the diffusion sequence are

$$x^{(k)} = S_k x^{(k-1)} = (S_k \ldots S_0) x := S_{k:0} x, \quad (3)$$

where we have defined $S_{k:0} := S_k \ldots S_0$ in the last equality. The graph signal $x^{(K-1)}$ is diffused over graph $G_K$ to produce the signal $x^{(K)}$. This means the signal $x^{(k)}$ is a diffused version of $x = x^{(0)}$ over the random sequence of graphs $S_0, \ldots, S_K$.

The main motivation for this randomly time varying diffusion process is a communication network in which $E$ represents a set of possible links and $E_k$ a set of links that are activated at time index $k$. The model also applies to a social network where contacts are activated at random. See Section VII.

We use the diffusion sequence in (3) to define a graph convolutional filter over the random $\text{RES}(G, p)$ graph. Fix a filter order $K$ and introduce $K + 1$ coefficients $h_k$ with $k = 0, \ldots, K$. The graph convolution of $x$ with coefficients $h_k$ is a linear combination of the entries of the diffusion sequence in (3) modulated by coefficients $h_k$,

$$u = \sum_{k=0}^K h_k x^{(k)} = \sum_{k=0}^K h_k S_{k:0} x := H(S_{K:0}) x. \quad (4)$$

where we defined the graph filter $H(S_{K:0}) := \sum_{k=0}^K h_k S_{k:0}$ in which $S_{K:1} = (S_{K:0}, \ldots, S_1)$ represents the sequence of shift operators that appear in the filter. This expression generalizes graph convolutional filters to settings where the topology changes between shifts. We shall refer to $H(S_{K:0})$ as a stochastic graph filter – see Figure 1.

To build an SGNN relying on the graph filters in (4), we consider the composition of a set of $L$ layers. The first layer $\ell = 1$ consists of a bank of $F$ filters $H_{1}^{f} (S_{K:0})$ with coefficients $h_{1:0}^{f}$ each of which produces the output graph signal $u_{1}^{f}$. These filter outputs are passed through a pointwise nonlinear function $\sigma(\cdot)$ to produce a collection of $F$ features $x_{1}^{f}$ that constitute the output of layer 1,

$$x_{1}^{f} = \sigma[u_{1}^{f}] = \sigma[H_{1}^{f} (S_{K:0}) x] = \sigma[\sum_{k=0}^{K} h_{1:0}^{f} S_{1:0:0} k x]. \quad (5)$$

The notation $\sigma[u_{1}^{f}]$ signifies the vector $[\sigma(u_{1:0}^{f}); \ldots; \sigma(u_{F:0}^{f})]$ where the function $\sigma$ is applied separately to each entry of $u_{1}^{f}$. We further emphasize that shift sequences $S_{1:0:0}^{f}$ are specific to the feature $f$ and drawn independently from the random graph $\text{RES}(G, p)$.

At subsequent intermediate layers $\ell = 2, \ldots, L - 1$ the output features $x_{\ell-1}^{f}$ of the previous layer, become inputs to a bank of $F^2$ filters with coefficients $h_{\ell:0}^{fg}$ each of which produces the output graph signal $u_{\ell}^{fg}$. To avoid exponential filter growth, the filter outputs derived from a common input feature $x_{\ell-1}^{g}$ are summed and the result is passed through a pointwise nonlinear function $\sigma(\cdot)$ to produce a collection of $F$ features $x_{\ell}^{f}$ that constitute the output of layer $\ell$

$$x_{\ell}^{f} = \sigma \left[ \sum_{g=1}^{F} u_{\ell}^{fg} \right] = \sigma \left[ \sum_{g=1}^{F} \sum_{k=0}^{K} h_{\ell:0}^{fg} S_{1:0:0} k x_{\ell-1}^{g-1} \right]. \quad (6)$$

The processing specified in (6) is repeated until the last layer $\ell = L$ in which we assume there is a single output feature $x_{L}^{f}$. The notation $S_{k:k'} = (S_{k}, \ldots, S_{k'})$ is a sequence of shift operators when it appears in an argument as in (4). It represents the product $S_{k:k'} := S_k \ldots S_{k'}$ when it is a term in an expression as in (3).
which we declare to be the output of the GNN. To produce this feature, we process each input feature $x_i^g$ with a graph filter with coefficients $h_{L,k}^g$, sum all features, and pass the result through the pointwise nonlinear function $\sigma(\cdot)$. This yields the GNN output

$$\Phi(x; S_{P;1}, \mathcal{H}) = \sigma \left[ \sum_{g=1}^{F} u_i^g \right] = \sigma \left[ \sum_{g=1}^{K} \sum_{k=0}^{3} h_{L,k}^g S_{L,k,0}^g x_i^g \right],$$

where the notation $\Phi(x; S_{P;1}, \mathcal{H})$ explicits that the SGNN output depends on the signal $x$, a sequence of $P = K(2F + (L - 2)F^2)$ independently chosen shift operators $S_{P;1}$, and the filter tensor $\mathcal{H}$ that groups filter coefficient $h_{L,k}^g$ for all layers $\ell$, orders $k$ and feature pairs $(f,g)$. The SGNN output in (7) is stochastic because the sequence of shift operators are realizations of the RES($\mathcal{G}$, $p$) graph in (1).

### B. Filter Tensor Training

To train the SGNN in (6) we are given a training set $T = \{(x_r, y_r)\}$ made up of $R$ input-output pairs $(x_r, y_r)$. We assume that both, input signals $x_r$ and output signals $y_r$ are graph signals supported on $S$. We are also given a cost function $c(y, \hat{y})$ to measure the cost of estimating output $\hat{y}$ when the actual output is $y$. Our interest is on the cost of estimating outputs $y_r$ with the SGNN in (6) averaged over the training set

$$C(S_{P;1}, \mathcal{H}) = \frac{1}{R} \sum_{r=1}^{R} c(y_r, \Phi(x_r; S_{P;1}, \mathcal{H})).$$

Given that the SGNN output $\Phi(x_r; S_{P;1}, \mathcal{H})$ is random, the cost $C(S_{P;1}, \mathcal{H})$ is random as well. We therefore consider the cost averaged over realizations of RES($\mathcal{G}$, $p$) and define the optimal filter tensor as the solution of the optimization problem

$$\mathcal{H}^* = \arg\min_{\mathcal{H}} \mathbb{E}_{S_{P;1}} \left( C(S_{P;1}, \mathcal{H}) \right).$$

In the expectation in (9), the shift operators $S_k$ in the sequence $S_{P;1}$ are drawn independently from the random RES($\mathcal{G}$, $p$) graph of Definition 1. Training to optimize the cost in (8) shall result in a SGNN with optimal average cost. To characterize the cost distribution, we study the variance of the SGNN output $\Phi(x; S_{P;1}, \mathcal{H})$ in Section III. We also show that the SGNN can be trained with a stochastic gradient descent based learning process over $T$ iterations and prove this learning process converges to a stationary solution of (9) with a rate of $O(1/\sqrt{T})$ in Section IV.

**Remark 1.** The SGNN defined by (5)-(7) admits a distributed implementation. This is because stochastic graph filters [cf. (4)] can be implemented in a distributed manner, which, in turn, is true because the diffusion sequence in (3) admits a distributable evaluation. To see that the latter, denote the entries of $S_k$ as $S_{k,ij}$ and observe the sparsity pattern of $S_k$ allows writing

$$x_i^{[k]} = \sum_{j:(j,i) \in E_k} S_{k,ij} x_j^{[k-1]}.$$  

Thus, the entry $x_i^{[k]}$ associated with node $i$ can be computed locally from entries $x_j^{[k-1]}$ received from nodes $j$ connected to $i$ in the graph realization $S_k$. Node $i$ can therefore evaluate all the $K$ entries $x_i^{[k]}$ through recursive exchanges of information with neighboring nodes. Node $i$ can then compute the $i$th entry of the filter output as $u_i = \sum_{k=0}^{K} h_{k} x_i^{[k]}$ [cf. (4)]. That the SGNN can be implemented in a distributed manner follows because the nonlinear function $\sigma(\cdot)$ is pointwise and therefore local. For instance, at layer $\ell = 1$, the output feature entries are $x_{i1}^{[1]} = \sigma[u_{i1}^{[1]}]$ where each of the filter outputs $u_{i1}^{[1]}$ is evaluated through local information exchanges. The same argument applies for filters and nonlinearities at subsequent layers so that node $i$ computes the SGNN output $\Phi(x; S_{P;1}, \mathcal{H})_i$ through recursive exchange of information with neighboring nodes.
III. VARIANCE ANALYSIS

The training optimizes the mean performance in (9), while it says little about the deviation of a single realization around this mean. We quantify the latter in this section by providing an upper bound on the SGNN output variance. For our analysis, we consider the variance over all nodes

\[
\text{var} [ \Phi(x; S_{P,1}, \mathcal{H}) ] = \sum_{i=1}^{N} \text{var} [ \Phi(x; S_{P,1}, \mathcal{H}) ]
\]

where \( [ \Phi(x; S_{P,1}, \mathcal{H}) ] \) is the \( i \)th component of \( \Phi(x; S_{P,1}, \mathcal{H}) \). We characterize the variance by pursuing in the graph spectral domain. This analysis is recently applied to deterministic GNNs [25], and shows the role of the graph topology on the corresponding outputs. We generalize this approach to the random graph scenario and aim to show how graph fluctuations influence the SGNN output variations.

A. Generalized Filter Frequency Response

Consider the graph filter \( \mathbf{H}(\mathbf{S}) \) [cf. (4) for \( p = 1 \)]. Since the shift operator \( \mathbf{S} \) is symmetric, it accepts the eigendecomposition \( \mathbf{S} = \mathbf{V} \Lambda \mathbf{V}^\top \) with eigenvector basis \( \mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_N] \) and eigenvalues \( \Lambda = \text{diag}(\{\lambda_1, \ldots, \lambda_N\}) \). We can expand signal \( x \) over \( \mathbf{V} \) as \( x = \sum_{i=1}^{N} \hat{x}_i \mathbf{v}_i \); an operation known as the graph Fourier expansion of \( x \) [14]. Vector \( \hat{x} = [\hat{x}_1, \ldots, \hat{x}_N]^\top \) contains the graph Fourier coefficients and it is called the graph Fourier transform (GFT) of \( x \). Substituting this expansion to the filter input \( \mathbf{u} = \mathbf{H}(\mathbf{S})x \), we can write

\[
\mathbf{u} = \sum_{k=0}^{K} h_k \mathbf{S}_k \sum_{i=1}^{N} \hat{x}_i \mathbf{v}_i = \sum_{i=1}^{N} \sum_{k=0}^{K} \hat{x}_i h_k \mathbf{v}_i \mathbf{k} \mathbf{v}_i.
\]

By further applying the Fourier expansion to the output \( \mathbf{u} = \sum_{i=1}^{N} \hat{u}_i \mathbf{v}_i \), we get the input-output spectral filtering relation \( \mathbf{u} = \mathbf{H}(\mathbf{A}) \hat{x} \). Here, \( \mathbf{H}(\mathbf{A}) \) is a diagonal matrix containing the filter frequency response on the main diagonal. For filters in the form (12), the frequency response has the analytic expression

\[
h(\lambda) = \sum_{k=0}^{K} h_k \lambda^k.
\]

The graph topology instantiates the variable \( \lambda \) to attain a value in the discrete set \( \lambda_1, \ldots, \lambda_N \) and allows representing (12) as a pointwise multiplication \( \hat{u}_i = h(\lambda_i) \hat{x}_i \) in the spectrum domain. The filter coefficients \( \{h_k\}_{k=0}^{K} \) determine the shape of the frequency response function \( h(\lambda) \). Figure 2 illustrates the latter concepts.

For the stochastic graph filter \( \mathbf{H}(\mathbf{S}_{K,0}) \) in (4), we have a deterministic shift operator \( \mathbf{S}_0 = \mathbf{I}_N \) and \( K \) random shift operators \( \mathbf{S}_1, \ldots, \mathbf{S}_K \). Since each \( \mathbf{S}_k \) for \( k = 1, \ldots, K \) is the shift operator of an undirected graph, it can be eigendecomposed as \( \mathbf{S}_k = \mathbf{V}_k \Lambda_k \mathbf{V}_k^\top \) with eigenvectors \( \mathbf{V}_k = [\mathbf{v}_{k1}, \ldots, \mathbf{v}_{kN}] \) and eigenvalues \( \Lambda_k = \text{diag}(\{\lambda_{k1}, \ldots, \lambda_{kN}\}) \). We can now use these shift eigenvectors to compute a chain of graph Fourier decompositions each with respect to a different shift operator \( \mathbf{S}_k \).

Starting from \( \mathbf{S}_0 = \mathbf{I}_N \), we can write the Fourier expansion of signal \( x \) on the identity matrix as \( x = \mathbf{S}_0 x = \sum_{i=1}^{N} \hat{x}_i \mathbf{v}_i \).

\[
\sum_{i=1}^{N} \mathbf{S}_0 \mathbf{v}_{i0}, \text{ where } \mathbf{v}_{i0} \text{ is the } i \text{th column eigenvector of } \mathbf{I}_N. \text{ When shifting the signal once over a RES(}G, p\text{) graph, we have}
\]

\[
x^{(1)} = \mathbf{S}_1 x = \sum_{i=1}^{N} \hat{x}_{i0} \mathbf{S}_1 \mathbf{v}_{i0}.
\]

We now treat each eigenvector \( \mathbf{v}_{i0} \) as a new graph signal and compute its graph Fourier decomposition with respect to shift operator \( \mathbf{S}_1 = \mathbf{V}_1 \Lambda_1 \mathbf{V}_1^\top \). This, in turn, allows writing \( x^{(1)} \) as \( x^{(1)} = \sum_{i=1}^{N} \hat{x}_{i10} \mathbf{v}_{i10} \). Substituting the latter into (14), we have

\[
x^{(2)} = \sum_{i=1}^{N} \mathbf{S}_2 \mathbf{v}_{i10} = \sum_{i=1}^{N} \sum_{i=1}^{N} \hat{x}_{i10} \mathbf{S}_2 \mathbf{v}_{i10}.
\]

Therefore, treating again each eigenvector \( \mathbf{v}_{i10} \) of \( \mathbf{S}_1 \) as a new graph signal and decomposing it on the Fourier basis of \( \mathbf{S}_2 = \mathbf{V}_2 \Lambda_2 \mathbf{V}_2^\top \), allows us to write (16) as

\[
x^{(2)} = \sum_{i=1}^{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \hat{x}_{i20} \mathbf{v}_{i20} \mathbf{S}_2 \mathbf{v}_{i20}.
\]

In (17), we have \( \{\mathbf{x}_{i20}\}_{i=1}^{N} \) additional \( N^2 \) Fourier coefficients introduced by the three chain shift operator \( \mathbf{S}_2 \). Notice that while only the eigenvectors \( \mathbf{V}_2 \) of the last seen shift operator \( \mathbf{S}_2 \) are explicit in (17), \( x^{(2)} \) is, nevertheless, influenced by all shift operators; especially, by their eigenspace alignment. The latter is captured by the GFT coefficients \( \{\mathbf{x}_{i10}\}_{i=1}^{N} \) and \( \{\mathbf{x}_{i21}\}_{i=1}^{N} \).

Following this recursion, we can write the shifting of the graph signal \( x \) over \( k \) consecutive \( \text{RES}(G, p) \) realizations \( \mathbf{S}_0, \ldots, \mathbf{S}_k \) as

\[
x^{(k)} = \mathbf{S}_k 0 x = \sum_{i=1}^{N} \sum_{i=1}^{N} \mathbf{S}_k 0 i \hat{x}_{i0} \cdots \hat{x}_{k_{i-1}0} \prod_{j=0}^{k} \mathbf{v}_{i0}.
\]
Aggregating then the $K+1$ shifted signals $x^{(0)},\ldots,x^{(K)}$, we can write the stochastic graph filter output $u = \mathbf{H}(S_{K,1})x$ as

$$u = \sum_{i_0=1}^N \ldots \sum_{i_K=1}^N \sum_{k=0}^K \hat{x}_{0i_0} \ldots \hat{x}_{Ki_{K+1}} h_k \prod_{j=0}^k \lambda_{ij} v_{Ki_k}$$  \hspace{1cm} (19)$$

where $\{\hat{x}_{0i_0}\}$ and $\{\hat{x}_{Ki_{K+1}}\}_{k=0}^{K-1}$ are Fourier coefficients of expanding $x$ in the $(K+1)$ chain of shift operators $S_0,\ldots,S_K$ and $\{v_{Ki_k}\}_{i_k=1}$ are eigenvectors of the last seen shift operator $S_K$. The output in (19) is similar to (12). In fact, for $p = 1$ when all shift operators are the same, deterministic, and their eigenspaces align perfectly—(19) reduces to (12). Therefore, we can consider (19) as a graph filtering operation over a chain of different shift operators. Considering then a generic eigenvector $\lambda = [\lambda_1,\ldots,\lambda_K]^\top$ (or graph frequency vector) in which $\lambda_k$ is the frequency variable for the shift operator $S_k$, we define the $K$-dimensional analytic generalized frequency response

$$h(\lambda) = \sum_{k=0}^K h_k \lambda_k \cdots \lambda_1 \lambda_0 = \sum_{k=0}^K h_k \lambda_k$$  \hspace{1cm} (20)$$

where $\lambda_k := \lambda_k \cdots \lambda_1 \lambda_0$ is defined for convenience and $\lambda_0 = 1$ by definition (i.e., $S_0 = \mathbf{I}_N$). The chain of shift operators $S_1,\ldots,S_K$ instantiates the generic vector $\lambda$ to specific eigenvalues in each dimension to evaluate the analytic function $h(\lambda)$, while the coefficients $\{h_k\}_{k=0}^K$ determine the $K$-dimensional surface of the generalized frequency response $h(\lambda)$.

Both the frequency response $h(\lambda)$ and the generalized frequency response $h(\lambda)$ are characterized by the filter coefficients $\{h_k\}_{k=0}^K$, while the specific shift operators only instantiate their variables. As such, by focusing directly on properties of $h(\lambda)$ and $h(\lambda)$, we can analyze the filter performance and further the SGNN performance independently on specific shift operators (or graphs).

**B. Variance of the Stochastic Graph Filter**

For mathematical tractability, we start by characterizing the variance of the stochastic graph filter. Consider the stochastic graph filter output $u = \mathbf{H}(S_{K,1})x$ in (4). The variance of $u$ over all nodes is $\text{var}[u] = \text{tr}[(\mathbb{E}[uu^\top]) - \bar{u}\bar{u}^\top]$ where $\bar{u} = \mathbb{E}[u] = \mathbf{H}(S)x$ is the expected filter output and $\text{tr}(\cdot)$ is the trace. Our goal is to upper bound the variance for any underlying shift operator $S$ under the RES($G,p$) model. For this, we pursue the spectral domain analysis to be independent on specific eigenvalues and need the following assumptions.

**Assumption 1.** For a set of filter coefficients $\{h_k\}_{k=0}^K$ and graph frequencies $\Lambda$ in a finite set $\Lambda$, the filter frequency response $h(\lambda)$ in (13) is bounded. I.e., there exists a constant $C_U$ such that for all $\lambda \in \Lambda$ the frequency response satisfies

$$|h(\lambda)| \leq C_U.$$  \hspace{1cm} (21)$$

**Assumption 2.** For a set of filter coefficients $\{h_k\}_{k=0}^K$ and graph frequency vectors $\Lambda$ in a finite space $\Lambda^K$, the generalized filter frequency response $h(\lambda)$ in (20) is Lipschitz. I.e.,

$$|h(\lambda_1) - h(\lambda_2)| \leq C_g ||\lambda_1 - \lambda_2||_2.$$  \hspace{1cm} (22)$$

Assumption 1 is commonly used in graph signal processing and states the filter frequency response in (13) evolves within certain finite margins. Assumption 2 indicates the stochastic graph filters are Lipschitz likewise the deterministic graph filters in (23) which are the particular case for $p = 1$. In other words, Assumption 2 implies the generalized frequency response $h(\lambda)$ does not change faster than linear in any frequency direction of $\lambda$. With above preliminaries in place, the following proposition states the variance of the stochastic graph filter is upper bounded by a factor that is quadratic in the link sampling probability.

**Proposition 1.** Consider the stochastic graph filter $\mathbf{H}(S_{K,0})$ of order $K$ in (4) over a RES($G,p$) graph model [cf. Def. 7] of $M$ edges. Let $S$ be the underlying shift operator, $S$ the expected shift operator, and $\{h_k\}_{k=0}^K$ the filter coefficients. Let also the generalized frequency response $h(\lambda)$ in (20) satisfies Assumption 2 with constant $C_g$.

Then, for any input signal $x$, the variance of filter output $\mathbf{H}(S_{K,0})x$ is upper bounded as

$$\text{var}[\mathbf{H}(S_{K,0})x] \leq p(1-p)C||x||_2^2 + O(p^2(1-p)^2)$$  \hspace{1cm} (23)$$

with constant $C = 2\alpha MKC_g^2$ and scalar $\alpha$ that depends on the shift operator.

**Proof.** See Appendix A.

Proposition 1 shows the variance of the filter output does not diverge and varies within a finite range that depends on the link sampling probability $p$. The bound represents the influence of $\alpha$ for example, $\alpha = 1$ if $S$ is the adjacency matrix or $\alpha = 2$ if $S$ is the graph Laplacian.
random link fluctuations on the filter variance. When links are stable \((p \to 1)\) or links are highly unstable \((p \to 0)\), the variance is small indicating the filter output varies close to the expected value. For the extreme cases \(p = 0\) or \(p = 1\), the bound reduces to zero because all RES\((G, 0)\) or RES\((G, 1)\) realizations are deterministic graphs. The maximum variance is achieved for \(p = 0.5\), corresponding to the most uncertain case about the presence of links. Constant \(C\) represents the role of the graph and filter: \(M\) is the number of edges; \(K\) and \(C_g\) are the filter order and the Lipschitz constant of generalized frequency response, respectively.

C. Variance of the SGNN

The variance of the SGNN is directly influenced by the variance of the stochastic graph filter with additional effects of the nonlinearity and the layered architecture. Before claiming the main result, we require an assumption for the nonlinearity.

**Assumption 3.** The nonlinearity \(\sigma(\cdot)\) satisfying \(\sigma(0) = 0\) is Lipschitz and variance non-increasing. I.e., there exists a constant \(C_\sigma\) such that for all \(x, y \in \mathbb{R}\) the nonlinearity satisfies

\[
|\sigma(x) - \sigma(y)| \leq C_\sigma |x - y|, \quad \text{var}[\sigma(x)] \leq \text{var}[x].
\]

The proof of Lemma 1 is in the supplementary material. The following theorem then formally quantifies the variance of the SGNN output and details the role of the SGNN architecture on the variance.

**Theorem 1.** Consider the SGNN in (6) of \(L\) layers and \(F\) features, over a RES\((G, \rho)\) graph model [cf. Def. 7] of \(M\) edges. Let \(S\) be the underlying shift operator, \(\Phi\) the expected shift operator, and \(H\) the SGNN filter tensor. Let the stochastic graph filters be of order \(K\) with frequency response (13) and generalized frequency response (20) satisfying Assumption 7 with constant \(C_U\) and Assumption 2 with \(C_g\). Let also the nonlinearity \(\sigma(\cdot)\) satisfy Assumption 3 with constant \(C_\sigma\).

Then, for any input graph signal \(x\), the variance of the SGNN output is upper bounded as

\[
\text{var}[\Phi(x; S_{P:1}, H)] \leq p(1 - p) C|\|x\|^2 + O(p^2(1-p)^2)
\]

with constant \(C = 2\alpha M \sum_{l=1}^{L} F^{2l-3} C_\sigma^{2l-2} C_U^{2l-2} K C_g^2\) and scalar \(\alpha\) that depends on the shift operator.

**Proof.** See Appendix B

As it follows from (23), the SGNN variance bound has a similar form as the stochastic graph filter variance bound [cf. Proposition 1], therefore, the conclusions of Proposition 1 apply also here. However, there is a large difference between the two bounds within constant \(C\). In the SGNN, this constant is composed of three terms representing respectively the effect of the graph, filter, and neural network on the variance. The graph impact is captured by \(\alpha M\) which shows the role of shift operator type and that more connected graphs lead to a worse variance bound. The filter impact is captured by \(C_U^{2l-2} K C_g^2\) which is dictated by the filter response \(h(\lambda)\) and the generalized response \(\lambda(\lambda)\). Notice that while we might be tempted to consider filters with a small \(C_U\) [cf. Ass. 1] to have a smaller variance, it will lead to an information loss throughout the layers; hence to a lower performance. Our expectation is that filters with \(C_U\) close to one will be a good tradeoff.

The aforementioned factors are also our handle to design distributed SGNN architectures that are more robust to link losses.

IV. Convergence Analysis

In this section, we propose an explicit learning process to train the SGNN with stochasticity appropriately. This learning process consists of minimizing the cost in (8) by accounting for different RES\((G, \rho)\) realizations. The associated convergence analysis translates into identifying whether this learning process with stochasticity converges to a stationary point and if so under which conditions.

**A. Learning Process**

Consider the SGNN has a fixed sequence of RES\((G, \rho)\) shift operator realizations \(S_{P:1}\) in (6) when processing the input. We shall refer to the latter fixed architecture as an SGNN realization. The learning process of the SGNN follows that of the conventional GNN but now with stochastic graph filter coefficients \(H\) learned through descent algorithms. The tensor \(H\) is updated iteratively and each iteration \(t\) comprises a forward and a backward phase. In the forward phase, the SGNN has the tensor \(H_t\) and a fixed architecture realization (i.e., the shift operator realizations \(S_{P:1}\) are fixed), processes all input signals \(\{x_r\}_{r=1}^L\) obtains the respective outputs \(\{\Phi(x_r; S_{P:1}, H_t)\}_{r=1}^L\), and computes the cost \(C(S_{P:1}, H_t)\) as per (8). In the backward phase, the tensor \(H_t\) gets updated with a gradient descent algorithm with step-size \(\alpha_t\) – see Algorithm 1.

This learning procedure incorporates the graph stochasticity in each gradient descent iteration \(t\) by fixing an SGNN realization. The latter mimics the network randomness caused by practical link losses; hence, it makes the cost function \(C(S_{P:1}, H_t)\) at each iteration random. To understand the proposed SGNN learning process, we consider the following stochastic optimization problem

\[
\min_{H_t} C(H) = \min_{H_t} \mathbb{E}[C(S_{P:1}, H)]
\]
where the expectation $E[\cdot]$ is with respect to the graph stochasticity. The stochastic gradient descent (SGD) is a canonical tool to solve problems of the form (27) [26], [27]. SGD approximates the true gradient $\nabla_{\mathcal{H}} C(\mathcal{H})$ with a realization $\nabla_{\mathcal{H}} C(S_{P:1}, \mathcal{H})$ at each iteration and uses this approximation to update $\mathcal{H}$. The complete SGD method for (27) is summarized in Algorithm 1. The cost function $C(S_{P:1}, \mathcal{H})$ is entirely determined by the SGNN realization $\Phi(\cdot; S_{P:1}, \mathcal{H})$. This, in turn, indicates that fixing a random SGNN realization during learning is equivalent to randomly picking a cost function $C(S_{P:1}, \mathcal{H})$ in the SGD for (27). The latter implies that steps 3-4 in Algorithm 1 are equivalent to step 3 in Algorithm 2. Put simply, we conclude the learning process of the SGNN boils down to running the SGD on (27).

The graph stochasticity during training renders the learned tensor more robust to link losses since each node does not rely certainly on information from all its neighbors. This robustness matches the distributed network behavior during the test phase for which it was trained for. However, the stochasticity also makes the proposed learning process random. As such, it is unclear if the SGNN learning process converges. We analyze this aspect next and prove that under conventional mild conditions, the SGNN reaches a stationary point.

### B. Convergence of SGNN Learning Process

Given the equivalence between Algorithms 1 and 2, we analyze the convergence of the SGNN learning process by proving the convergence of the SGD counterpart; i.e., we prove there exists a sequence of tensors $\{\mathcal{H}_t\}$ generated by the SGD that approaches a stationary point $\mathcal{H}^*_{\alpha}$ of (27). Since problem (27) is nonconvex due to the nonlinear nature of the SGNN, we can no longer use the metric $C(\mathcal{H}) - C(\mathcal{H}^*)$ or $\|\mathcal{H} - \mathcal{H}^*\|_2^2$ as a convergence criterion. We instead use the gradient norm $\|\nabla_{\mathcal{H}} C(\mathcal{H})\|_2^2$, which is a typical surrogate to quantify stationarity and has also a similar order of magnitude as the above two quantities [28]. To render this analysis tractable, we assume the following.

**Assumption 4.** The expected cost $\bar{C}(\mathcal{H})$ in (27) is Lipschitz continuous. I.e., there exists a constant $C_L$ such that

$$\|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_1) - \nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_2)\|_2 \leq C_L \|\mathcal{H}_1 - \mathcal{H}_2\|_2$$

for any $\mathcal{H}_1$ and $\mathcal{H}_2$.

Theorem 2 states that with an appropriate choice of step-size $\alpha_t$, Algorithm 1 converges to a stationary point for tensor $\mathcal{H}_t$. The step-size $\alpha_t$ in (30) depends on the total number of iterations $T$; a more practical way is to set $\alpha_t \propto 1/t$ or $1/\sqrt{T}$. Due to the non-convexity of the SGNN, the learning process has guaranteed convergence to a local stationary minima. The latter can be extended to a better (potentially global) minimum with standard approaches such as training the SGNN multiple times.

Overall, we defined the stochastic graph neural network in (6) as an GNN architecture composed of stochastic graph filters. The SGNN output is random due to the stochastic architecture and thus the training shall optimize the SGNN with respect to average performance. In Section III, we quantified how far from the mean a SGNN realization can behave and the effects played by different factors such as the shift operator.
the filter type and the architecture structure. We also developed the SGD based learning process for the SGNN and the result shows that the learning process is sufficient to reach a local minima in expectation.

V. NUMERICAL SIMULATIONS

We corroborate the proposed model and compare it with the conventional GNN on source localization (Sec. V-A) and robot swarm control (Sec. V-B). To train the architectures, we used the ADAM optimizer with decaying factors $\beta = 0.9$ and $\beta_2 = 0.999$ [30]. In the test phase, we assume all links may fall according to the RES($G, p$) model.

A. Source Localization

We consider a signal diffusion process over a stochastic block model (SBM) graph of $N = 40$ nodes divided equally into $C = 4$ communities, with inter- and intra-block edge probability of 0.8 and 0.2 respectively. The goal is for a single node to find out distributively which community is the source of a given diffused signal. The initial source signal is a Kronecker delta $\delta_c \in \mathbb{R}^N$ centered at the source node $\{n_c\}_{c=1}^C$ and diffused at time $\tau$ as $x_{rc} = S^{T} \delta_c + n$ with $S = A/\lambda_{\text{max}}(A)$ and $n \in \mathbb{R}^N$ a zero-mean Gaussian noise.

We considered a one-layer SGNN with 32 parallel filters of order $K = 10$ and ReLU nonlinearity. The learning rate is $\alpha = 10^{-3}$ with the mini-batch size of 1,000 samples. The training set comprises $10^4$ tuples $\{(x_{rc}, n_c)\}$ picked uniformly at random for $r \in \{0, \ldots, 40\}$ and $n_c \in \{1, \ldots, 40\}$; the validation set contains 2,400 of these tuples; the test set contains 1,000 tuples. Our results are averaged over ten different data and ten different graph realizations for a total of 100 Monte-Carlo runs.

We first corroborate the convergence analysis in Section IV and show the SGNN approaches a stationary point. Figure 4 shows the learning process of the SGNN with link sampling probabilities $p = 0.9$, $p = 0.7$, and $p = 0.5$. These values correspond to stable, relatively stable, and vulnerable scenarios, respectively. The cost value decreases with the number of iterations, leading to a convergent result in all cases. When $p = 0.9$, the SGNN exhibits the best behavior and converges to a lower value. This is because of the higher link stability. As $p$ decreases indicating more graph randomness, the convergent value increases accordingly as observed the lines corresponding to $p = 0.7$ and $p = 0.5$. The convergent values of $p = 0.7$ and $p = 0.5$ also have larger varying errors, which can be explained by the increasing stochastic error induced by the increased network randomness. These errors can be further reduced by either decreasing the step-size or training the network longer.

Next, we compare the impact of link losses on the SGNN with the GNN. Figure 5 (a) shows the classification test accuracy over relatively stable topologies, i.e., $p \in [0.85, 0.95]$. The SGNN exhibits a higher accuracy and lower variance, which are emphasized for lower values of $p$. That is, the more links are lost, the more the SGNN outperforms the GNN. This highlights the importance of accounting for the topological randomness during training. When $p$ approaches one, the GNN shows a comparable test accuracy, indicating the GNN is a valid choice only for highly stable topologies. Figure 5 (b) compares the two architectures for $p \in [0.3, 0.8]$, where links are lost often and the network varies dramatically. The SGNN maintains a good performance even in this severe scenario. Instead, the GNN losses entirely its discriminative power and yields a performance similar to a random classifier. We attribute the latter to the fact that the ideal fixed graph in the GNN training deviates substantially from practical random graphs encountered in the test phase.

Finally, we compare the SGNN with the GNN under different graph settings. We train the SGNN on networks divided equally into $C \in \{2, 3, 4, 5, 6\}$ communities with each community containing 10 nodes. The link sampling probability is $p = 0.7$. Figure 5 (c) illustrates the SGNN outperforms the GNN in all scenarios. The GNN degrades towards a random classifier because the problem becomes more challenging as the number of communities $C$ increases. However, the SGNN maintains a good performance, which highlights the importance of robust transference.

B. Robot Swarm Control

The goal of this experiment is to learn a distributed controller for robot swarms to fly together and avoid collision [6]. We consider a network of $N$ agents, where each agent $i$ is described by its position $z_i \in \mathbb{R}^2$, velocity $v_i \in \mathbb{R}^2$, and acceleration $u_i \in \mathbb{R}^2$. The problem has an optimal centralized solution on accelerations

$$u_i^* = -\sum_{j=1}^{N}(v_i - v_j) - \sum_{j=1}^{N}\rho(z_i, z_j)$$  \hspace{1cm} (32)

that assigns each agent’s velocity to the mean velocity. Here, $\rho(z_i, z_j)$ is the collision avoidance potential. The centralized controller is, however, not practical because it requires velocity and position information of all agents at each agent $i$. We aim to learn a distributed controller with GNN by relying only on local neighborhood information.
We consider agent $i$ communicates with agent $j$ if their distance $|z_i - z_j| \leq r$ is within the communication radius $r$. The communication graph $G = (\mathcal{V}, \mathcal{E})$ involves the node set $\mathcal{V} = \{1, \ldots, N\}$ as agents, the edge set $\mathcal{E}$ as available communication links, and $\mathbf{S}$ as the associated graph shift operator. The graph signal $x$ is the relevant feature designed with the agent position $z$ and velocity $v$. We measure the controller performance with the variance of velocities for a trajectory, which quantifies how far the system is from consensus in velocities.

As baseline, we assume $N = 50$ agents and a communication radius $r = 3.0m$. The agents are distributed randomly in a circle with a minimum separation of 0.1m and initial velocities sampled uniformly in the interval $[-3.0m/s, +3.0m/s]$. We consider a one-layered SGNN with 32 parallel filters of order $K = 3$ and use tangent nonlinearity like in [6]. We use imitation learning to train the SGNN over a training set of 1000 trajectories, each containing 100 time steps. The validation and test sets contain each 100 extra trajectories. We train the SGNN for 30 epochs with batch size of 20 samples and learning rate $\alpha = 3 \cdot 10^{-4}$. Our results are averaged over 10 simulations.

We compare the performance of the SGNN with the GNN under different link sampling probabilities. Figure 6 (a) shows the cost value of the two architectures for $p \in [0.3, 0.9]$. The SGNN achieves both a lower mean and variance compared with the GNN. This improved SGNN performance is more visible for lower link sampling probabilities $p$, i.e., as links become more unstable. We again attribute this behavior to the robust transference of the SGNN, since it accounts for link instabilities during training. However, notice the SGNN also degrades when the graph varies dramatically (small $p$). This is because the information loss induced by link fluctuations leads to inevitable errors, which cannot be resolved by training.

Figures 6 (b) and 6 (c) depict the performance of the SGNN and the GNN in different coordination scenarios; namely, different communication radius $r$ and different number of agents $N$, respectively. The link sampling probability is $p = 0.7$. In general, the SGNN outperforms the GNN in all scenarios. Figure 6 (b) shows the SGNN performance improves as the communication radius $r$ increases. This follows our intuition because a larger communication radius increases the node exchanges, which helps contrasting some of the link losses. On the other hand, a denser graph with larger communication radius magnifies the impact of link losses by introducing more randomness, such that the decreasing rate of cost value reduces seen from $r = 3m$ to $r = 4m$. The performance improvement gained from the SGNN increases in this case, which emphasizes the importance of accounting for the graph randomness during training.

In Figure 6 (c), we see the cost decreases as the number of agents increases, which can be explained by the increased information exchanges in large networks. This result indicates the SGNN is capable of handling large-scale robot swarms while retaining a good performance. In addition, the SGNN improvement becomes more visible for larger networks since topological randomness has more effects in these cases and thus robust transference plays a more important role.

VI. Conclusions

We propose a distributed stochastic graph neural network that can operate over random time varying topologies. The architecture is similar to the conventional GNNs but substitutes the convolutional graph filters with stochastic graph filters to account for the topological randomness during training. A detailed mathematical analysis characterized the output variance of the SGNN, which is upper bounded by a factor that is quadratic in the link sampling probability, indicating the impact of link losses to the SGNN output. We further formulated a learning process that accounts for the randomness in the cost function and leveraged stochastic gradient descent to prove this learning process converges to a stationary point. Numerical results corroborated the proposed theoretical model on distributed source localization and decentralized robot swarm control, showing superior performance compared with the GNN that ignores link losses. In near future, we plan to corroborate these findings with other topology variation models and in other distributed applications, such as power outage prediction in smart grids.
APPENDIX A

PROOF OF PROPOSITION 1

We need the following lemma, whose proof is in the supplementary material.

Lemma 2. Consider the underlying graph $G$ with the shift operator $S$ and let $S_k$ be the shift operator of $k$th RES($G, p$) realization of $S$. $S = \mathbb{E}[S_k]$ the expected shift operator, and $D$ the diagonal degree matrix with $d_i$ the degree of node $i$. Then, it holds that

$$\mathbb{E}[S_k^2] = \begin{cases} S^2 + p(1-p)D, & \text{if } S = A, \\ S^2 + 2p(1-p)S, & \text{if } S = L \end{cases}$$

(33)

with $A$ the adjacency matrix and $L$ the Laplacian matrix.

Proof of Proposition 1. Let $u = H(S_{k:0})x$ and $\bar{u} = H(S)x$ be the output and the expected output of stochastic graph filter. Substituting the filter expression into the variance, we get

$$\text{var}[u] = \mathbb{E} \left[ \text{tr} \left( uu^H - \bar{u}\bar{u}^H \right) \right] = \sum_{k, \ell = 0}^{K} h_k h_\ell \left( \mathbb{E} \left[ \text{tr} \left( T(k, \ell) \right) \right] - \mathbb{E} \left[ \text{tr} \left( \bar{T}(k, \ell) \right) \right] \right),$$

(34)

where $T(k, \ell) = S_{k:0} xx^H S_{\ell:0}$ and $\bar{T}(k, \ell) = \bar{S}^k xx^H \bar{S}^\ell$. To further simplify notation, we denote with $[k \ell] = \max(k, \ell)$ and $[k \ell] = \min(k, \ell)$. Expression (34) holds because of the linearity of the trace and expectation and the symmetry of the shift operators $S_k$ and $S$. Since $S_k$ is a realization, we represent it as $S_k = \bar{S} + E_k$ where $E_k$ is the deviation of $S_k$ from the mean $S$. Substituting this result into $T(k, \ell)$ and since $\mathbb{E}[E_k] = 0$, we have

$$\mathbb{E}[T(k, \ell)] = \mathbb{E} \left[ (\bar{S} + E_k) \cdots xx^\top \cdots (\bar{S} + E_\ell) \right]$$

(35)

$$= \bar{S}^k xx^\top \bar{S}^\ell + \mathbb{E} \left[ \sum_{r=1}^{[k \ell]} \bar{S}^{k-r} E_r S^{r-1} \bar{S}^{\ell-r} E_r \right] = \mathbb{E} \left[ C_{k \ell} \right].$$

The first term in (35) yields from the maximum powers of $\bar{S}$ of the products; the second term captures all cross-products where we should note that for $k \neq \ell$ we have $\mathbb{E}[E_k E_\ell] = \mathbb{E}[E_k] \mathbb{E}[E_\ell] = 0$ due to independence and also the terms $r > [k \ell]$ are null due to the presence of a single expectation $\mathbb{E}[E_r]$; the third term $C_{k \ell}$ collects the sum of the remaining terms. By substituting $\text{tr} \left( \mathbb{E} \left[ T(k, \ell) \right] \right) = \text{tr} \left( \bar{S}^k \bar{x} x^\top \bar{S}^\ell \right)$ and (35) into (34), we have

$$\text{var}[u] = \sum_{k, \ell = 0}^{K} h_k h_\ell \text{tr} \left( \mathbb{E} \left[ C_{k \ell} \right] \right)$$

(36)

$$+ \sum_{k, \ell = 1}^{K} h_k h_\ell \text{tr} \left( \mathbb{E} \left[ \sum_{r=1}^{[k \ell]} \bar{S}^{k-r} E_r S^{r-1} \bar{S}^{\ell-r} E_r \right] \right).$$

We now analyze the two terms in (36) separately. For this analysis, we will need the inequality

$$\text{tr}(AB) \leq \frac{\|A + A^\top\|_2}{2} \text{tr}(B) \leq \|A\|_2 \text{tr}(B)$$

(37)

that holds for any square matrix $A$ and positive semi-definite matrix $B$. [32]

Second term. By bringing the trace inside the expectation (due to their linearity) and leveraging the trace cyclic property $\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)$, we can write

$$\mathbb{E} \left[ \text{tr} \left( \sum_{k, \ell = 1}^{K} h_k h_\ell \sum_{r=1}^{[k \ell]} \bar{S}^{k-r} E_r S^{r-1} \bar{S}^{\ell-r} E_r \right) \right]$$

(38)

$$= \mathbb{E} \left[ \sum_{k, \ell = 1}^{K} \text{tr} \left( \sum_{r=1}^{K} h_k h_\ell E_r S^{k+\ell-2r} E_r S^{r-1} \bar{S}^{r-1} \right) \right].$$

Notice that in (38) we also rearranged the terms to change the sum limits. Since both matrices $\sum_{k, \ell = r}^{K} h_k h_\ell E_r S^{k+\ell-2r} E_r = (\sum_{k=r}^{K} h_k S^{k-r} E_r)^\top (\sum_{k=r}^{K} h_k S^{k-r} E_r)$ and $S^{r-1} xx^\top S^{r-1}$ are positive semi-definite, we can use the Cauchy-Schwarz inequality $\text{tr}(AB) \leq \text{tr}(A) \text{tr}(B)$ [33] to upper bound (38) by

$$\mathbb{E} \left[ \sum_{k, \ell = r}^{K} h_k h_\ell \text{tr} \left( E_r S^{k+\ell-2r} E_r \right) \text{tr} \left( S^{r-1} xx^\top S^{r-1} \right) \right].$$

(39)
We now proceed by expressing the graph signal $x$ in the frequency domain of the expected graph. Let $S = \mathbf{V A V}^\top$ be the eigendecomposition of $S$ with eigenvectors $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_N]^\top$ and eigenvalues $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$. Substituting the graph Fourier expansion $x = \sum_{i=1}^{N} \hat{x}_i \mathbf{v}_i$ into $tr(S^{r-1}xx^\top S^{-1})$, we get
\[
tr(S^{r-1}xx^\top S^{-1}) = \sum_{i=1}^{N} \hat{x}_i^2 \lambda_i^{-2r+2} tr(\mathbf{v}_i \mathbf{v}_i^\top) = \sum_{i=1}^{N} \hat{x}_i^2 \lambda_i^{-2r+2} \quad (40)
\]
where $tr(\mathbf{v}_i \mathbf{v}_i^\top) = 1$ for $i = 1, \ldots, N$ due to the orthonormality of eigenvectors. By substituting (40) into (39), we get
\[
\sum_{i=1}^{N} \hat{x}_i^2 \sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{-2r+2} tr(E_r S^{k+\ell-2r} E_r) = \sum_{i=1}^{N} \hat{x}_i^2 \sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{-2r+2} tr(E_r S^{k+\ell-2r} E_r) \quad (41)
\]
Using again the trace cyclic property to write $tr(E_r S^{k+\ell-2r} E_r) = tr(S^{k+\ell-2r} E_r^\top E_r)$ and the linearity of the expectation, we have
\[
\sum_{i=1}^{N} \hat{x}_i^2 \sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{-2r+2} tr(E_r S^{k+\ell-2r} E_r) = \sum_{i=1}^{N} \hat{x}_i^2 \sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{-2r+2} \quad (42)
\]
From Lemma [2] we have $E[E_r^2] = \alpha p(1-p) E$ with $\alpha = 1$ and $E = \mathbf{D}$ if $S = \Lambda$ and $\alpha = 2$ and $E = \mathbf{S}$ if $S = \mathbf{L}$. By substituting this result into (42) and using inequality (37) since $E$ is positive semi-definite, we have
\[
\sum_{i=1}^{N} \hat{x}_i^2 \sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{-2r+2} \quad (43)
\]
with $tr(E) = \sum_{i=1}^{N} d_i = 2M$ and $M$ the number of edges.

At this point, we proceed to upper bound the filter matrix norm in (43). A standard procedure to bound the spectral norm of a matrix $A$, is to upper bound the norm of $\|AA^\top\|$ as $\|AA^\top\| \leq A[|a|]_2$ for any vector $a$ [34]. In this instance, $A$ is the upper bound for the norm of $\Lambda$. Following this rationale, we consider the GFT expansion of a vector $a$ on the expected graph $a = \sum_{j=1}^{N} \hat{a}_j \mathbf{v}_j$ where $\{\mathbf{v}_j\}_{j=1}^{N}$ are orthonormal. Then, we have
\[
\sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{2r-2} \lambda_j^{k+\ell-2r} \quad (44)
\]
Consider now the expression inside the absolute value in (44). This expression is linked to the partial derivative of the generalized frequency response $h(\Lambda)$ in (20). To detail this, we introduce the first-order partial derivative of the generalized frequency response $h(\Lambda)$ w.r.t. the $r$th entry $\lambda_i$ of $\Lambda$
\[
\frac{\partial h(\Lambda)}{\partial \lambda_r} = \sum_{k=1}^{K} h_k \lambda_k \Lambda(i, r+1) \Lambda(r, 1)-1, \text{ for all } r = 1, \ldots, K \quad (45)
\]
where $\lambda_K(i+1) = \lambda_K \cdots \lambda_{r+1}$ and $\lambda(r, 1)-1 = \lambda_{r-1} \cdots \lambda_1$. Let us then consider $K$ specific eigenvalue vectors of dimensions $K \times 1$: $\lambda_{ij}^k = [\lambda_j, \ldots, \lambda_j]^\top$, $\hat{\lambda}_{ij}^k = [\lambda_i, \lambda_j, \ldots, \lambda_j]^\top$, $\ldots$, $\hat{\lambda}_{ij}^K = [\hat{\lambda}_i, \ldots, \hat{\lambda}_j]^\top$ for two eigenvalues $\lambda_i$ and $\lambda_j$ of $S$, and their respective generalized frequency responses $(h(\lambda_j), \ldots, h(\lambda_j))$ [cf. (13)]. The $r$th first-order partial derivative $\frac{\partial h(\Lambda)}{\partial \lambda_r}$ of the generalized frequency response instantiated on $\lambda_{ij}^k$ is
\[
\frac{\partial h(\lambda_{ij}^k)}{\partial \lambda_r} = \sum_{k} h_k \hat{\lambda}_i^{k-1} \lambda_j^{r-k}, \text{ for all } r = 1, \ldots, K. \quad (46)
\]
We then observe the expression inside the absolute value in (44) can be represented as the sum of $K$ first-order partial derivatives; i.e., we can write it in the compact form
\[
\sum_{i=1}^{N} \hat{x}_i^2 \sum_{k, \ell} K \sum_{r=1}^{K} h_k h_{\ell} \lambda_i^{-2r+2} \lambda_j^{k+\ell-2r} = \sum_{r=1}^{K} \left( \frac{\partial h(\lambda_{ij}^r)}{\partial \lambda_r} \right)^2. \quad (47)
\]
From Assumption [2] the generalized frequency responses are Lipschitz with constant $C_g$. Thus we can upper bound (47) as
\[
\left| \sum_{k=1}^{K} \sum_{\ell=1}^{K} h_k h_{\ell} \lambda_j^{k+\ell-2r} \right|^2 \leq K^2 C_g^4 \quad (48)
\]
which implies the norm of the filter matrix in (44) is upper bounded by $K C_g^2$. By substituting this norm bound into (44) and altogether into (38), we have
\[
E \left[ \sum_{k, \ell=1}^{K} h_k h_{\ell} \sum_{r=1}^{K} \left( S^{k-r} E_r S^{-1} xx^\top S^{-1} E_r S^{\ell-r} \right) \right] \leq 2\alpha M K C_g^2 \sum_{r=1}^{N} \hat{x}_i^2 p(1-p) = 2\alpha M K C_g^2 \|x\|_2^2 p(1-p). \quad (49)
\]
First term. Matrix $C_{k\ell}$ comprises the sum of the remaining expansion terms. Each of these terms is a quadratic form in the error matrices $E_k, E_r$ with $k \neq r$; i.e., it is of the form $f_1(S, h_k) E_k f_2(S, h_k) E_k f_3(S, h_k) E_k f_4(S, h_k) E_k f_5(S, h_k) E_k f_6(S, h_k) E_k$ for some functions $f_1(\cdot), \ldots, f_6(\cdot)$ that depend on the expected shift operator and filter coefficients. Each of these double-quadratic terms can be bounded by a factor containing at least two terms $tr(\mathbf{E}E_k^2)$ and $tr(\mathbf{E}E_r^2)$. Since the frequency response $h(\lambda)$ is bounded from Assumption [1] also the coefficients $\{h_k\}_{k=0}^{K}$ are bounded. Further since $\|S\|_2$ is bounded and $\mathbf{E}E_k^2 = \alpha p(1-p) E$ from Lemma [2] we can write the first term in (35) as
\[
E \left[ \sum_{k, \ell=0}^{K} h_k h_{\ell} C_{k\ell} \right] \leq O(p^2(1-p)^2) \|x\|_2^2 \quad (50)
\]
Finally, substituting the results for the first term (50) and second term (49) into (35), we have the variance bound
\[
\text{var}[u_s] \leq 2\alpha M K C_g^2 \|x\|_2^2 p(1-p) + O(p^2(1-p)^2) \quad (51)
\]
completing the proof.  

\[\Box\]

*The generalized frequency response $h(\Lambda)$ in (13) is an analytic function of the vector variable $\Lambda = [\lambda_1, \ldots, \lambda_K]$ such that $\Lambda$ can take any value.*
APPENDIX B
PROOF OF THEOREM 1

In the proof, we need the following lemma with the proof in the supplementary material that shows the bound on the filter output.

Lemma 3. Consider the graph filter $H(S)$ [cf. (4) for $p=1$] with coefficients $\{h_{k}\}_{k=0}^{K}$ and let $S$ be the graph shift operator. Let the frequency response $\{G_{k}\}$ satisfy Assumption 7 with constant $C_{U}$. Then, the norm of the graph filter is upper bounded as

$$\|H(S)\|_{2} \leq C_{U}. \tag{52}$$

Proof of Theorem 4. From the SGNN definition in (6) and Assumption 3, the variance can be upper bounded as

$$\text{var}[\Phi(x; S_{P-1}, H)] = \sqrt{\left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right) \left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right)^{T}} \leq \sqrt{\sum_{f=1}^{F} u_{f}^{L}_{-1}} \leq \sqrt{\sum_{f=1}^{F} u_{f}^{L}_{-1}} \tag{53}$$

where $\text{var}[]$ is defined in (11). By exploiting the relation between the trace of covariance matrix and the variance, we can rewrite (53) as

$$\text{var}\left[\sum_{f=1}^{F} u_{f}^{L}_{-1}\right] = \text{tr}\left(\sqrt{\left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right) \left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right)^{T}}\right) \tag{54} \leq \text{tr}\left(\sqrt{\left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right) \left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right)^{T}}\right).$$

Denote $u_{f}^{L}_{-1} = H_{f}^{L} x_{f-1}^{L}$ and $u_{f}^{L}_{-1} = H_{f}^{L} x_{f-1}^{L}$ as concise notations of stochastic graph filter output $H_{f}^{L}(S_{K,0}) x_{f-1}^{L}$ and expected graph filter output $E\left[H_{f}^{L}(S_{K,1}) x_{f-1}^{L}\right]$. By expanding (54), we get

$$\text{var}\left[\sum_{f=1}^{F} u_{f}^{L}_{-1}\right] = \text{tr}\left(\sum_{f=1}^{F} u_{f}^{L}_{-1} \left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right)^{T}\right) \leq \text{tr}\left(\sum_{f=1}^{F} u_{f}^{L}_{-1} \left(\sum_{f=1}^{F} u_{f}^{L}_{-1}\right)^{T}\right).$$

By adding and subtracting $H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}$ inside the first expectation, (55) becomes

$$\sum_{f=1}^{F} \sum_{g=1}^{F} \text{tr}\left(\left[H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}\right] - \left[H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}\right]\right) = \text{tr}\left(\left[H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}\right] - \left[H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}\right]\right).$$

Expression (56) is composed of two groups of terms shown there in the two separate lines.

First term. For the first term, when $f \neq g$ such that filters $H_{f}^{L}(S_{K,0})$ and $H_{g}^{L}(S_{K,0})$ are independent, we have

$$\text{tr}\left(\left[H_{f}^{L} x_{f-1}^{L} \left(H_{g}^{L} x_{f-1}^{L}\right)^{T}\right] - \left[H_{f}^{L} x_{f-1}^{L} \left(H_{g}^{L} x_{f-1}^{L}\right)^{T}\right]\right) = 0. \tag{57}$$

We then use (57) to derive the upper bound

$$\sum_{f=1}^{F} \sum_{g=1}^{F} \text{tr}\left(\left[H_{f}^{L} x_{f-1}^{L} \left(H_{g}^{L} x_{f-1}^{L}\right)^{T}\right] - \left[H_{f}^{L} x_{f-1}^{L} \left(H_{g}^{L} x_{f-1}^{L}\right)^{T}\right]\right) = \sum_{f=1}^{F} \text{tr}\left(\left[H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}\right] - \left[H_{f}^{L} x_{f-1}^{L} \left(H_{f}^{L} x_{f-1}^{L}\right)^{T}\right]\right) \leq \Delta \sum_{f=1}^{F} \left\|x_{f-1}^{L}\right\|^{2} (1-p) + O(p^{2}(1-p)^{2}) \tag{58}$$

where $\Delta = 2\alpha MKC_{g}^{2}$ and the last inequality holds from Proposition 1. For the norm of $x_{f-1}^{L}$, we observe that

$$\left\|x_{f-1}^{L}\right\|^{2} = \text{tr}\left(\sum_{g=1}^{F} u_{g}^{f-2} u_{g}^{f-2}\right) \leq C_{w}^{2} \text{tr}\left(\sum_{g=1}^{F} u_{g}^{f-2} u_{g}^{f-2}\right) \tag{59}$$

where Assumption 3 and the triangle inequality are used in the last inequality. By further representing $\left\|u_{g}^{f-2}\right\|^{2}$ with the trace $\text{tr}(u_{g}^{f-2} u_{g}^{f-2})$ and expanding the latter as in (39), we have

$$\text{tr}\left(\sum_{k,t=0}^{K} h_{k}(L-2) h_{t}(L-2) S_{k} S_{t}^{\top} \left[H_{f}^{L}(S_{K,2}) x_{f-2}^{L}\right] \right) \leq \left\|H_{f-2}^{L} \right\|_{2}^{2} \text{tr}\left(\left[H_{f-2}^{L}(S_{K,2}) x_{f-2}^{L}\right] \right) \leq C_{w}^{2} \left\|x_{f-2}^{L}\right\|^{2} \tag{60}$$

with $x_{f-2}^{L} = x_{f-2}^{L}(x_{f-2}^{L})^{\top}$. For the first term, by using the cyclic property of trace and the inequality (37), we can bound it as

$$\text{tr}\left(\sum_{k,t=0}^{K} h_{k}(L-2) h_{t}(L-2) S_{k} S_{t}^{\top} \left[H_{f}^{L}(S_{K,2}) x_{f-2}^{L}\right] \right) \leq \left\|H_{f-2}^{L} \right\|_{2}^{2} \text{tr}\left(\left[H_{f-2}^{L}(S_{K,2}) x_{f-2}^{L}\right] \right) \leq C_{w}^{2} \left\|x_{f-2}^{L}\right\|^{2} \tag{61}$$

By the last inequality we used Lemma 3 and $\text{tr}(E[x_{f-2}^{L}]) = \left\|x_{f-2}^{L}\right\|^{2}$. For the second term and the third term, we use the result (49) and (50) to write

$$\left\|H_{f-2}^{L} \right\|_{2}^{2} \text{tr}\left(\left[H_{f-2}^{L}(S_{K,2}) x_{f-2}^{L}\right] \right) \leq \left\|H_{f-2}^{L} \right\|_{2}^{2} \text{tr}\left(\left[H_{f-2}^{L}(S_{K,2}) x_{f-2}^{L}\right] \right) \leq C_{w}^{2} \left\|x_{f-2}^{L}\right\|^{2} \tag{62}$$

Solving recursion (63) with the initial condition $\left\|x_{0}^{L}\right\|^{2} = \left\|x_{0}^{L}\right\|^{2}$ yields

$$\left\|x_{f-1}^{L}\right\|^{2} \leq C_{w}^{2} \left\|x_{f-2}^{L}\right\|^{2} + O(p(1-p)). \tag{64}$$

$$\left\|x_{f-2}^{L}\right\|^{2} \leq C_{w}^{2} \left\|x_{f-3}^{L}\right\|^{2} + O(p(1-p)). \tag{65}$$
By substituting (64) into (58), we bound (58) by
\[ \Delta F^{2L-3} C_2^{2L-2} C_2^{2L-2} \|x\|^2 p(1-p) + O(p^2(1-p)^2). \] (65)

**Second term.** For the second term in (56), we have
\[ \sum_{j=1}^{F} \sum_{g=1}^{F} \text{tr} \left[ E \left[ H_j^T x_{L-1} (H_j^T x_{L-1})^\top \right] - E \left[ H_j^T x_{L-1} \right] E \left[ H_j^T x_{L-1}^\top \right] \right] \]
\[ = \sum_{j=1}^{F} \text{var} \left[ H_j^T x_{L-1} \right] + \sum_{j \neq g} F \text{cov} \left[ H_j^T x_{L-1}, H_g^T x_{L-1} \right]. \] (66)

By using the property of covariance [35]
\[ \text{cov}[x, y] \leq \sqrt{\text{var}[x]} \sqrt{\text{var}[y]} \leq \frac{\text{var}[x] + \text{var}[y]}{2} \] (67)
for two random variables $x$ and $y$, we can bound (66) by
\[ F \sum_{j=1}^{F} \text{var} \left[ H_j^T x_{L-1} \right] = F \sum_{j=1}^{F} \text{tr} \left[ H_j^T H_j \Sigma_{L-1}^f \right] \] (68)
where $\Sigma_{L-1}^f = E \left[ (x_{L-1} - E[x_{L-1}]) (x_{L-1} - E[x_{L-1}])^\top \right]$ is positive semi-definite matrix for all $f = 1, \ldots, F$. And $H_j^T H_j$ is square matrix. We then refer to inequality (37) to bound (68) as
\[ \|H_L^2\|_2^2 F \sum_{j=1}^{F} \text{tr} \left[ \Sigma_{L-1}^f \right] \leq C_2^2 \sum_{j=1}^{F} \text{var} \left[ x_{L-1} \right] \] (69)
where the last inequality is due to Lemma 3. By substituting bounds (65) and (69) into (56) and then altogether into (54), we observe a recursion where the variance of the $L$th layer output depends on the variance of the $(L-1)$th layer output as well as the bound in (65). Therefore, we have
\[ \text{var}[x_L] = \text{var}[\Phi(x; S_{P-1}, \mathcal{H})] \leq C_2^2 \sum_{j=1}^{F} \text{var} \left[ x_{L-1} \right] \] (70)
\[ + \Delta F^{2L-3} C_2^{2L-2} C_2^{2L-2} \|x\|^2 p(1-p) + O(p^2(1-p)^2). \]

**Unrolling.** By expanding this recursion until the input layer, we have
\[ \text{var}[\Phi(x; S_{P-1}, \mathcal{H})] \leq \Delta \sum_{t=2}^{L} \sum_{i=1}^{T} \sum_{j=1}^{F} \text{var} \left[ x_{L-1} \right] \] (71)
where the first term in (71) is the accumulated sum of all second terms in (70) during the recursion. Since $x_1 = H_1^T x$ and $x_1^T = H_2^T x$ and $H_1$ and $H_2$ are independent, if $f \neq g$, we have $\sum_{f \neq g} \text{cov}[x_1^T, x_1^T] = 0$. Therefore, (71) becomes
\[ \text{var}[\Phi(x; S_{P-1}, \mathcal{H})] \leq 2 \alpha M \sum_{l=1}^{F} \sum_{i=1}^{T} \sum_{j=1}^{F} \text{var} \left[ x_{L-1} \right] \] (72)
\[ + \Delta F^{2L-4} C_2^{2L-2} C_2^{2L-2} \|x\|^2 p(1-p) + O(p^2(1-p)^2). \]

This completes the proof.

---

**Appendix C**

**Proof of Theorem 2**

\[ E\left[ \bar{C}(\mathcal{H}_{t+1}) \right] = E\left[ \bar{C}(\mathcal{H}_t) + \nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t) \nabla H \mathcal{H}_t + \frac{1}{2} \right] (\mathcal{H}_{t+1} - \mathcal{H}_t) \]
\[ + \frac{1}{2} \left( \mathcal{H}_{t+1} - \mathcal{H}_t \right) \nabla \mathcal{H}_t \mathcal{H}_t + \frac{1}{2} \|\mathcal{H}_{t+1} - \mathcal{H}_t\|^2 \] (73)

where $\mathcal{H}_t$ is in the line segment joining $\mathcal{H}_{t+1}$ and $\mathcal{H}_t$, and we substituted $\mathcal{H}_t$ in the Hessian term $\nabla \mathcal{H}_t \mathcal{H}_t$ of (73) for the series truncation. Since for any vector $a$ and matrix $A$, we have the inequality $a^T A a \leq \lambda_{\max}(A) \|a\|^2$ where $\lambda_{\max}(A)$ is the largest eigenvalue of $A$. With this result and the Lipschitz continuity $\|\nabla \mathcal{H}_t \mathcal{H}_t\|_2 \leq C_L$ in Assumption 4, we get
\[ E\left[ \bar{C}(\mathcal{H}_{t+1}) \right] \leq E\left[ \bar{C}(\mathcal{H}_t) \right] + \lambda_{\max}(A) \|\mathcal{H}_{t+1} - \mathcal{H}_t\|^2 \] (74)

By substituting the SGNN update rule $\mathcal{H}_{t+1} = \mathcal{H}_t - \alpha_t \nabla_{\mathcal{H}} C(S_{P-1}, \mathcal{H}_t)$ with random cost realization, we get
\[ E\left[ \bar{C}(\mathcal{H}_{t+1}) \right] \leq E\left[ \bar{C}(\mathcal{H}_t) \right] + \lambda_{\max}(A) \|\mathcal{H}_{t+1} - \mathcal{H}_t\|^2 \] (75)

Exploiting the linearity of expectation and the identity $E[\nabla_{\mathcal{H}} C(S_{P-1}, \mathcal{H}_t)] = E[\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)]$, we can write (75) as
\[ E\left[ \bar{C}(\mathcal{H}_{t+1}) \right] \leq E\left[ \bar{C}(\mathcal{H}_t) \right] - \alpha_t E\left[ \|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)\|^2 \right] \]
\[ + \frac{1}{2} \lambda_{\max}(A) \|\mathcal{H}_{t+1} - \mathcal{H}_t\|^2. \] (76)

Subsequently, using the gradient bound in Assumption 5, we can upper bound the third term of (76) as
\[ \frac{1}{2} \lambda_{\max}(A) \|\mathcal{H}_{t+1} - \mathcal{H}_t\|^2. \] (77)

Since $\|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)\|^2 \leq \epsilon$ is the convergence criterion for nonconvex problems, we focus on the second term in (76). We first move it to the left side, and then move $E\left[ \bar{C}(\mathcal{H}_{t+1}) \right]$ on the right side, and finally define the difference between two successive costs as
\[ \Delta \bar{C}_{t:t+1} = \bar{C}(\mathcal{H}_t) - \bar{C}(\mathcal{H}_{t+1}). \] (78)

With these arithmetic steps, we can rewrite (76) as
\[ \Delta \bar{C}_{t:t+1} \leq \frac{1}{\alpha_t} E[\|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)\|^2] \]
\[ + \frac{1}{\alpha_t} E[\|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)\|^2]. \] (79)

By considering a constant step $\alpha_t = \alpha$ and summing up all terms in (79) – recall (79) should hold for all $t = 0, \ldots, T-1$ – we get
\[ \sum_{t=0}^{T-1} E[\|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)\|^2] \leq \frac{1}{\alpha_t} E[\Delta \bar{C}_{0:T}] + \frac{T_{\alpha} C_L^2 C_B^2}{2}. \] (80)

For the optimal tensor $\mathcal{H}^*$, we have $\bar{C}(\mathcal{H}^*) \leq \bar{C}(\mathcal{H}_T)$. By substituting this result into $\Delta \bar{C}_{0:T}$, we bound (80) as
\[ \min \|\nabla_{\mathcal{H}} \bar{C}(\mathcal{H}_t)\|^2 \leq \frac{1}{T_{\alpha}} \left( \bar{C}(\mathcal{H}_0) - \bar{C}(\mathcal{H}_T) + \frac{1}{\alpha_t} C_L^2 C_B^2 \right). \] (81)
By further setting the constant step-size as
\[
\alpha = \sqrt{\frac{2(C(H_0) - C(H^*))}{TC_LC_B^2}} \tag{82}
\]
and substituting it into (81), we have
\[
\min t E[\|\nabla_H C(H_t)\|^2] \leq C \sqrt{t} \tag{83}
\]
with constant \(C = \sqrt{2(C(H_0) - C(H^*))C_LC_B}\). Therefore, the bound decreases to zero with the rate \(O(1/\sqrt{t})\). This completes the proof. \(\Box\)

REFERENCES
[1] Z. Gao, E. Isufi, and A. Ribeiro, “Stochastic graph neural networks,” in International Conference on Acoustics, Speech, and Signal Processing (ICASSP), 2020.
[2] F. Scarselli, M. Gori, A. Tsoi, M. Hagenbuchner, and G. Monfardini, “The graph neural network model,” IEEE Transactions on Neural Networks, vol. 20, no. 1, pp. 61–80, 2009.
[3] Z. Wu, S. Pan, F. Chen, G. Long, C. Zhang, and P. S. Yu, “A comprehensive survey on graph neural networks,” IEEE Transactions on Neural Networks and Learning Systems, pp. 1–21, 2020.
[4] R. Ying, R. He, K. Chen, P. Eksombatchai, W. L. Hamilton, and J. Leskovec, “Graph convolutional networks for web-scale recommender systems,” in International Conference on Knowledge Discovery and Data Mining (KDDM), 2018.
[5] S. Wu, Y. Tang, Y. Zhu, L. Wang, X. Xie, and T. Tan, “Session-based recommendation with graph neural networks,” in Association for the Advancement of Artificial Intelligence (AAAI), 2019.
[6] E. Tolstaya, F. Gama, J. Paulos, G. Pappas, V. Kumar, and A. Ribeiro, “Learning decentralized controllers for robot swarms with graph neural networks,” in arXiv preprint arXiv:1903.10527, 2019.
[7] T. Wang, R. Liao, J. Ba, and S. Fidler, “Nervenet: Learning structured policy with graph neural networks,” in International Conference on Learning Representations (ICLR), 2018.
[8] D. K. Duvenaud, D. Maclaurin, J. Iparraguirre, R.Bombarell, T. Hrizel, A. Aspuru-Guzik, and R. P. Adams, “Convolutional networks on graphs for learning molecular fingerprints,” in International Conference on Neural Information Processing Systems (NIPS), 2015.
[9] A. Fout, J. Byrd, B. Shariat, and A. Ben-Hur, “Protein interface prediction using graph convolutional networks,” in International Conference on Neural Information Processing Systems (NIPS), 2017.
[10] J. Bruna, W. Zaremba, A. Szlam, and Y. LeCun, “Spectral networks and deep locally connected networks on graphs,” in International Conference on Learning Representations (ICLR), 2013.
[11] G. Fernando, E. Isufi, and A. Ribeiro, “Graphs, convolutions, and neural networks,” in arXiv preprint arXiv:2003.03777, 2020.
[12] M. Defferrard, X. Bresson, and P. Vandergheynst, “Convolutional neural networks on graphs with fast localized spectral filtering,” in International Conference on Neural Information Processing Systems (NIPS), 2016.
[13] A. Sandryhaila and J. M. F. Moura, “Discrete signal processing on graphs,” IEEE Transactions on Signal Processing, vol. 61, no. 7, pp. 1644–1656, 2013.
[14] A. Ortega, P. Frossard, J. Kovavei, J. M. F. Moura, and P. Vandergheynst, “Graph signal processing: Overview, challenges and applications,” Proceedings of the IEEE, vol. 106, no. 5, pp. 808–828, 2018.
[15] E. Isufi, F. Gama, and A. Ribeiro, “Edgenets: Edge varying graph neural networks,” in arXiv preprint arXiv:2001.07620, 2020.
[16] D. I. Shuman, P. Vandergheynst, D. Kressner, and P. Frossard, “Distributed signal processing via chebyshev polynomial approximation,” IEEE Transactions on Signal and Information Processing over Networks, vol. 65, no. 4, pp. 736–751, 2018.
[17] A. Gavili and X. Zhang, “On the shift operator, graph frequency, and optimal filtering in graph signal processing,” IEEE Transactions on Signal Processing, vol. 65, no. 23, pp. 6303–6318, 2017.
[18] E. Isufi, A. Loukas, A. Simonetto, and G. Leus, “Autoregressive moving average graph filtering,” IEEE Transactions on Signal Processing, vol. 64, no. 2, pp. 274–288, 2016.
[19] O. Overko, F. Gama, and A. Ribeiro, “Predicting power outages using graph neural networks,” in IEEE Global Conference on Signal and Information Processing (GlobalSIP), 2018.
Proof of Lemma 2: \textbf{Absolute value.} Denote with $\sigma(\cdot) = |\cdot|$ the nonlinearity, $x$ the random variable and $y = \sigma(x)$ the output. Since $y^2 = |x|^2 = x^2$, we have
\begin{equation}
\var[y] = \mathbb{E}[y^2] - \mathbb{E}[y]^2 = \int_{-\infty}^{\infty} x^2 p(x) dx - \mathbb{E}[y]^2 = \int_{-\infty}^{\infty} x^2 p(x) dx - \mathbb{E}[y]^2
\end{equation}
where $p(x)$ is the probability density function of $x$. From Jensen’s inequality [35] $\mathbb{E}[x^2] \leq \mathbb{E}[|x|^2] = \mathbb{E}[y]^2$ and $\var[y] \geq 0$, we get
\begin{equation}
\var[y] = \int_{-\infty}^{\infty} x^2 p(x) dx - \mathbb{E}[y]^2 \leq \int_{-\infty}^{\infty} x^2 p(x) dx - \mathbb{E}[x]^2 = \var[x].
\end{equation}

\textbf{ReLU.} Now denote with $\sigma(\cdot) = \max(0,\cdot)$ the nonlinearity. Similarly, we have
\begin{equation}
\var[y] = \mathbb{E}[y^2] - \mathbb{E}[y]^2 = \int_{-\infty}^{\infty} \max(x,0)^2 p(x) dx - \mathbb{E}[y]^2 = \int_{0}^{\infty} x^2 p(x) dx - \mathbb{E}[y]^2
\end{equation}
and $\var[x] = \int_{-\infty}^{\infty} x^2 p(x) dx - \mathbb{E}[x]^2$. Thus, the difference of $\var[x]$ and $\var[y]$ is given by
\begin{equation}
\var[x] - \var[y] = \int_{-\infty}^{0} x^2 p(x) dx - \mathbb{E}[x]^2 + \mathbb{E}[y]^2
\end{equation}
\begin{equation}
\int_{-\infty}^{\infty} x^2 p(x) dx + (\mathbb{E}[y] + \mathbb{E}[x])(\mathbb{E}[y] - \mathbb{E}[x]).
\end{equation}
Defining the integrals $A = \int_{-\infty}^{0} x|p(x) dx$ and $B = \int_{0}^{\infty} x p(x) dx$, we have
\begin{equation}
\mathbb{E}[x] = B - A, \quad \mathbb{E}[y] = B.
\end{equation}
By substituting (88) into (87), we get
\begin{equation}
\var[x] - \var[y] = \int_{0}^{\infty} x^2 p(x) dx - A^2 + 2AB
\end{equation}
with $2AB \geq 0$ by definition. Consider the term $\int_{0}^{\infty} x^2 p(x) dx - A^2$ and by defining the variable $z = \min(0, x)$, we have
\begin{equation}
\int_{-\infty}^{0} x^2 p(x) dx - A^2 = \int_{-\infty}^{0} x^2 p(x) dx - \left(\int_{-\infty}^{0} x|p(x) dx\right)^2
\end{equation}
\begin{equation}
= \int_{-\infty}^{0} z^2 p(z) dz - \left(\int_{-\infty}^{0} z p(z) dz\right)^2 = \var[z] \geq 0
\end{equation}
where $z \in (-\infty, 0]$. By using (90) in (89), we prove $\var[x] - \var[y] \geq 0$ completing the proof. \hfill $\Box$

Proof of Lemma 2: \textbf{Adjacency.} For the adjacency matrix $S = A$, the $(i,j)$th entry of $S$ is $S_{ij} = s_{ij}$ where $s_{ii} = 0$ for all $i$. The $(i,j)$th entry of the $\text{RES}(G,p)$ realization $S_k$ can be represented as $[S_k]_{ij} = \delta_{ij}s_{ij}$ where $\delta_{ij}$ is a Bernoulli variable that is one with probability $p$ and zero with probability $1-p$. By exploiting the matrix multiplication for $S_iS_k$ and $SS_i$, the $(i,j)$th entries of $E[S_k^2]$ and $S^2$ are respectively given by
\begin{equation}
E[S_k^2]_{ij} = \sum_{n=1}^{N} s_{in}s_{nj}E[\delta_{in}\delta_{nj}], \quad S^2_{ij} = \sum_{n=1}^{N} s_{in}s_{nj}p^2
\end{equation}
where in the second equality we also substituted $S = pS$. The Bernoulli variables $\{\delta_{ij}\}$ are independent except for $\delta_{ij} = \delta_{ji}$ since $S_k$ is symmetric. Thus, we get
\begin{equation}
E[\delta_{in}\delta_{nj}] = \begin{cases} p^2, & \text{if } i \neq j, \\ p, & \text{if } i = j. \end{cases}
\end{equation}
By substituting (92) into (91), we have
\begin{equation}
E[S_k^2]_{ij} = \begin{cases} \sum_{n=1}^{N} s_{in}s_{nj}p^2, & \text{if } i \neq j, \\ \sum_{n=1}^{N} s_{in}s_{nj}p^2, & \text{if } i = j. \end{cases}
\end{equation}
Since $\sum_{n=1}^{N} s_{in}s_{ni} = d_i$ is the degree of node $i$ and from (91) and (93), we can write
\begin{equation}
E[S_k^2] = S^2 + (1-p)D.
\end{equation}
\textbf{Laplacian.} For the adjacency matrix $S = L$, we have $s_{ii} = -\sum_{n \neq i} \delta_{in}s_{in}$ for all $i = 1, \ldots, N$. In this case, $E[S_k^2]$ is
\begin{equation}
E[S_k^2]_{ij} = \begin{cases} \sum_{n \neq i,j} s_{in}s_{nj}p^2 + E[\delta_{ij}s_{ii}s_{ij} + \delta_{ij}s_{ij}s_{jj}], & \text{if } i \neq j, \\ \sum_{n \neq i} s_{in}s_{nj}p^2 + E[s_{ii}^2], & \text{if } i = j. \end{cases}
\end{equation}
For $S^2$: if $i \neq j$, we have
\begin{equation}
S^2_{ij} = \sum_{n \neq i,j} s_{in}s_{nj}p^2 + \sum_{n \neq i} s_{in}s_{ij}p^2 - \sum_{n \neq j} s_{ij}s_{jn}p^2;
\end{equation}
if $i = j$, we have
\begin{equation}
S^2_{ij} = \sum_{n \neq i} s_{in}s_{nj}p^2 + \sum_{n \neq i} s_{in}s_{in}p^2.
\end{equation}
Now consider the terms $E[\delta_{ij}s_{ii}s_{ij} + \delta_{ij}s_{ij}s_{jj}]$ and $E[s_{ii}^2]$. By expanding $s_{ii}$ and $s_{ij}$ and from (92), we have
\begin{equation}
E[s_{ii}\delta_{ij}s_{ij}] = -\sum_{n \neq i,j} s_{in}s_{ij}p^2 - s_{ij}^2, \quad E[\delta_{ij}s_{ij}s_{jj}] = -\sum_{n \neq i,j} s_{ij}s_{jn}p^2 - s_{ij}^2, \quad E[s_{ii}^2] = \sum_{n \neq i} \sum_{n \neq i} s_{in}s_{ij}p^2 + \sum_{n \neq i} s_{in}s_{in}p^2.
\end{equation}
Substitute (98) into (95) and by comparing (95) with (96) and (97) and using the fact $s_{ij} = -1$ for $i \neq j$, we get
\begin{equation}
E[S_k^2] = S^2 + 2p(1-p)S
\end{equation}
which proves the lemma. \hfill $\Box$
Proof of Lemma 3. For an input signal $x$, we can write the $i$th entry of the filter output $u = H(S)x$ in the GFT domain $\hat{u}$ as $\hat{u}_i = h(\lambda_i)\hat{x}_i$ where $\hat{x}_i$ is the $i$th coefficient of $x$. From the energy conservation and Assumption $|h(\lambda_i)| \leq C_U$ for $i = 1, \ldots, N$, we have
\[
\|u\|_2^2 = \|\hat{u}\|_2^2 = \sum_{i=1}^{N} h(\lambda_i)^2\hat{x}_i^2 \leq C_U^2\|x\|_2^2.
\]
(100)