LIE ALGEBRAS WITH ASSOCIATIVE STRUCTURES. APPLICATIONS TO THE STUDY OF 2-STEP NILPOTENT LIE ALGEBRAS

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Abstract. We investigate Lie algebras whose Lie bracket is also an associative or cubic associative multiplication to characterize the class of nilpotent Lie algebras with a nilindex equal to 2 or 3. In particular we study the class of 2-step nilpotent Lie algebras, their deformations and we compute the cohomology which parametrize the deformations in this class.

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1. Introduction

A finite dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero is $p$-nilpotent if for any $X \in \mathfrak{g}$ the operator $adX$ is $p$-nilpotent, that is, $(adX)^p = 0$. A very interesting class corresponds to $p = 2$, that is, the class of 2-step nilpotent Lie algebras. This class is especially studied in the geometrical framework. In fact there are numerous studies on left invariant structures on a Lie group $G$ with an associated Lie algebra $\mathfrak{g}$ which is 2-step nilpotent ([14]). Here we are mostly interested by the algebraic study of the family of $p$-step nilpotent Lie algebras for $p = 2$ and 3. We show that such an algebra is defined by a Lie bracket which is also associative or 3-associative. This leads to determine the properties of the corresponding operads. We show that the deformations of 2-step nilpotent Lie algebras are governed by the operad cohomolgy and we describe it. We compute this cohomology for two Lie algebras $\mathfrak{k}_{2p+1}$ and $\mathfrak{k}_{2p}$ corresponding to the odd or even dimensional cases and show that any 2-step nilpotent Lie algebra of dimension $2p + 1$ (respectively $2p$) with maximal characteristic sequence is a linear deformation of $\mathfrak{k}_{2p+1}$ (respectively $\mathfrak{k}_{2p}$). This permits a description of the class of 2-step nilpotent Lie algebras with maximal characteristic sequence.

2. Associative Lie multiplication

Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. If we denote by $[X, Y]$ the Lie bracket of $\mathfrak{g}$, it satisfies the following identities

$$
\left\{ \begin{array}{l}
[X, Y] = -[Y, X], \\
[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \quad \text{(Jacobi Identity)},
\end{array} \right.
$$

for any $X, Y, Z \in \mathfrak{g}$. We assume moreover that the Lie bracket is also an associative product, that is, it satisfies

$$
[[X, Y], Z] = [X, [Y, Z]],
$$

for any $X, Y, Z \in \mathfrak{g}$. The Jacobi Identity therefore implies

$$
[[Z, X]], Y] = 0.
$$
**Proposition 1.** The Lie bracket of the Lie algebra $\mathfrak{g}$ is an associative product if and only if $\mathfrak{g}$ is a two-step nilpotent Lie algebra.

In fact, the relation $[[Z,X],Y] = 0$ means that $[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] = \mathcal{C}^2(\mathfrak{g}) = 0$ where $\mathcal{C}^i(\mathfrak{g})$ denotes the ideals of the descending central sequence of $\mathfrak{g}$. The converse is obvious.

The classification of complex two-step nilpotent Lie algebras is known up to the dimension 7. In the following our notation and terminology will be based on [7].

Any two-step nilpotent complex non abelian and indecomposable Lie algebra of dimension less than 7 is isomorphic to one of the following algebras:

1. For the dimensions less than or equal to 3:
   - $n_3 = \mathfrak{h}_3 : [X_1, X_2] = X_3$.
2. In dimension 5:
   - $n_5^3 : [X_1, X_2] = X_3$, $[X_1, X_4] = X_5$;
   - $n_5^6$ is the Heisenberg algebra $\mathfrak{h}_2 : [X_1, X_2] = X_3$, $[X_4, X_5] = X_3$.
3. In dimension 6:
   - $n_6^{19} : [X_1, X_i] = X_{i+1}$, $i = 2, 4$, $[X_2, X_6] = X_5$;
   - $n_6^{20} : [X_1, X_i] = X_{i+1}$, $i = 2, 4$, $[X_2, X_4] = X_6$.
4. In dimension 7:
   - $n_7^{120} : [X_1, X_i] = X_{i+1}$, $i = 2, 4, 6$, $[X_2, X_4] = X_7$;
   - $n_7^{121} : [X_1, X_i] = X_{i+1}$, $i = 2, 4, 6$;
   - $n_7^{122} : [X_1, X_i] = X_{i+1}$, $i = 2, 4, 6$, $[X_2, X_4] = X_7$;
   - $n_7^{123} : [X_1, X_i] = X_{i+1}$, $i = 2, 4, 6$, $[X_2, X_4] = X_5$, $[X_4, X_6] = X_3$;
   - $n_7^{124} : [X_1, X_i] = X_{i+1}$, $i = 2, 4$, $[X_6, X_7] = X_5$, $[X_4, X_7] = X_3$;
   - $n_7^{125} : [X_1, X_i] = X_{i+1}$, $i = 2, 4$, $[X_6, X_7] = X_5$;
   - $n_7^{126} : [X_1, X_2] = X_3$, $[X_4, X_5] = X_3$, $[X_6, X_7] = X_3$.

To develop the operadic point of view, let us recall that an operad is a sequence $\mathcal{P} = \{\mathcal{P}(n), n \in \mathbb{N}^+\}$ of $\mathbb{K}[\Sigma_n]$-modules, where $\mathbb{K}[\Sigma_n]$ is the algebra group associated with the symmetric group $\Sigma_n$, with $\text{comp}_i$-operations (see [16]). The main example corresponds to the free operad $\Gamma(E) = \{\Gamma(E)(n)\}$ generated by a $\mathbb{K}[\Sigma_2]$-module. An operad $\mathcal{P}$ is called binary quadratic if there is a $\mathbb{K}[\Sigma_2]$-module $E$ and a $\mathbb{K}[\Sigma_3]$-submodule $R$ of $\Gamma(E)(3)$ such that $\mathcal{P}$ is isomorphic to $\Gamma(E)/R$ where $R$ is the operadic ideal generated by $R(3) = R$.

**Proposition 2.** There exists a binary quadratic operad, denoted by $2\text{Nilp}$, with the property that any $2\text{Nilp}$-algebra is a 2-step nilpotent Lie algebra.

In fact, we consider $E = sgn_2$, that is, the representation of $\Sigma_2$ by the signature, then $\Gamma(E)(3) = sgn_3 \oplus V_2$ where $V_2 = \{(x, y, z) \in \mathbb{K}^3, x + y + z = 0\}$. Let $R$ be the submodule of $\Gamma(E)(3)$ generated by $(x_i \cdot x_j \cdot x_k)$, $i, j, k$ all different. We deduce that $2\text{Nilp}(2)$ is the $\mathbb{K}[\Sigma_2]$-module generated by $x_1 \cdot x_2$ with the relation $x_2 \cdot x_1 = -x_1 \cdot x_2$ and it is a 1-dimensional vector space and $2\text{Nilp}(3) = \{0\}$.

**Proposition 3.** The operad $2\text{Nilp}$ is Koszul.
Recall some general definitions and results on the duality of a binary quadratic operad (see [16]). The generating function of a binary quadratic operad $P$ is

$$g_P(x) = \sum_{a \geq 1} \frac{1}{a!} \dim(P(a)) x^a.$$ 

Thus the generating function of the operad $2N\text{ilp}$ is the polynomial

$$g_{2N\text{ilp}}(x) = x + \frac{x^2}{2}.$$ 

The dual operad $P^!$ of the operad $P$ is the quadratic operad $P^! := \Gamma(E^\vee)/(R^\perp)$, where $R^\perp \subset \Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the pairing

$$\begin{align*}
\langle (x_i \cdot x_j) \cdot x_k, (x_{i'} \cdot x_{j'}) \cdot x_{k'} \rangle &= 0, \text{ if } \{i, j, k\} \neq \{i', j', k'\}, \\
\langle (x_i \cdot x_j) \cdot x_k, (x_{i'} \cdot x_{j'}) \cdot x_k \rangle &= (-1)^{\varepsilon(\sigma)}, \\
&\quad \text{with } \sigma = \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} \text{ if } \{i, j, k\} = \{i', j', k'\}, \\
\langle x_i \cdot (x_j \cdot x_k), (x_{i'} \cdot x_{j'}) \cdot x_{k'} \rangle &= 0, \text{ if } \{i, j, k\} \neq \{i', j', k'\}, \\
\langle x_i \cdot (x_j \cdot x_k), x_i \cdot (x_{j'} \cdot x_{k'}) \rangle &= -(-1)^{\varepsilon(\sigma)}, \\
&\quad \text{with } \sigma = \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} \text{ if } \{i, j, k\} = \{i', j', k'\}, \\
\langle (x_i \cdot x_j) \cdot x_k, x_i \cdot (x_{j'} \cdot x_{k'}) \rangle &= 0,
\end{align*}$$

and $(R^\perp)$ is the operadic ideal generated by $R^\perp$. We deduce

$$\dim(2N\text{ilp})^!(1) = 1, \quad \dim(2N\text{ilp})^!(2) = 1, \quad \dim(2N\text{ilp})^!(3) = 3, \quad \dim(2N\text{ilp})^!(4) = 15$$

and more generally, if we denote by $d_k$ the dimension of $(2N\text{ilp})^!(k)$, we have

$$\begin{align*}
d_{2k+1} &= \sum_{i=1}^{k} C_{2k+1}^{i} d_i d_{2k+1-i}, \\
d_{2k} &= \sum_{i=1}^{k-1} C_{2k}^{i} d_i d_{2k-i} + \frac{1}{2} C_{2k}^{k} d_k^2.
\end{align*}$$

In fact, the dual operad $(2N\text{ilp})^!$ is $\Gamma(\Pi)$ the free operad generated by a commutative operation. So the generating function of $2N\text{ilp}^!$ is

$$\sum_{k \geq 1} \frac{d_k}{k!} x^k.$$ 

If an operad $P$ is Koszul, then its dual $P^!$ is also Koszul and the generating functions are related by the functional equation

$$g_P(-g_{P^!}(-x)) = x.$$ 

It is known that $\Gamma(\Pi)$ is Koszul, so also $2N\text{ilp}$ and this implies the proposition. We can verify that the generating function $g_{2N\text{ilp}}$ of the operad $2N\text{ilp}$ satisfies the functional equation

$$g_{2N\text{ilp}}(-g_{2N\text{ilp}}(-x)) = x.$$ 

Remarks.
(1) Recall that the operad satisfies the Koszul property if the corresponding free algebra is Koszul, that is, its natural or operadic homology is trivial except in degree 0. If \( \mathcal{L}_r \) is the free Lie algebra of rank \( r \) (i.e. on \( k \) (free) generators), and if \( \mathcal{C}^3(\mathcal{L}_r) \) is the third part of its descending central series, thus the free two-step nilpotent Lie algebra of rank \( r \) is \( \mathcal{N}(2, r) = \mathcal{L}_r / \mathcal{C}^3(\mathcal{L}_r) \). If \( V_r \) is the \( r \)-dimensional vector space corresponding to the homogeneous component of degree 1 of \( \mathcal{L}_r \), thus \( \mathcal{N}(2, r) = V_r \oplus \bigwedge^2(V_r) \). For example, if \( r = 2 \), thus \( \mathcal{N}(2, 2) \) is a 3-dimensional Lie algebra with basis \( \{e_1, e_2, e_1 \wedge e_2\} \), where the last vector corresponds to \([e_1, e_2]\). If \( r = 3 \), thus \( \mathcal{N}(2, 3) \) is the 6-dimensional Lie algebra generated by \( \{e_1, e_2, e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\} \). We can define an homology of \( \mathcal{N}(2, r) \) using the standard complex \( (\bigwedge^*(\mathcal{N}(2, r)), \partial) \) where

\[
\partial_p : \bigwedge^p(\mathcal{N}(2, r)) \to \bigwedge^{p-1}(\mathcal{N}(2, r))
\]

is defined by

\[
\partial_p(x_1 \wedge x_2 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_p
\]

and \( \partial_p = 0 \) is \( p \leq 0 \). Let \( m_p \) be the dimension of the \( p \)-th homology space \( \text{Ker} \partial_p / \text{Im} \partial_{p+1} \). For example if \( r = 2 \), \( \dim \mathcal{N}(2, 2) = 3 \) and \( \mathcal{N}(2, 2) \) is generated by \( e_1, e_2, \{e_1, e_2\} = e_3 \). It is isomorphic to the Heisenberg algebra. We have \( m_0 = 1, m_1 = m_2 = 2, m_3 = 1 \). The general case was studied in [21]. The homology spaces are never trivial. For this complex, the free 2-step nilpotent Lie algebra is not Koszul.

(2) As a nilpotent Lie algebra is unimodular, we have the Poincaré duality. This implies that the second cohomology space of the free 2-step nilpotent algebra of rank \( r \) is trivial and this algebra is rigid in the variety of 2-step nilpotent Lie algebra of dimension \( r(r - 1)/2 \). It defines an open orbit and an algebraic component in this variety.

Remarks

(1) Let us consider an associative algebra \((A, \cdot)\) where \( x \cdot y \) denotes the multiplication in \( A \). Thus

\[
[x, y] = x \cdot y - y \cdot x
\]

is a Lie bracket. This Lie bracket is associative if and only if the multiplication of \( A \) satisfies

\[
(x \cdot y) \cdot z - (y \cdot x) \cdot z - (z \cdot x) \cdot y + (z \cdot y) \cdot x = 0.
\]

Let \( v \) be the vector of \( \mathbb{K} [\Sigma_3] \) \( v = \text{Id} - \tau_{12} + \tau_{13} - c^2 \) where \( \tau_{ij} \) is the transposition that exchanges the elements \( i \) and \( j \) and \( c \) the 3-cycle \((123)\). The orbit of \( v \) with respect to the action of the group \( \Sigma_3 \) generates a 2-dimensional vector space with basis \( \{v, \tau_{13}, v\} \). We deduce that these algebras can be considered as \( \mathcal{P} \)-algebras where the quadratic operad \( \mathcal{P} \) is defined by \( \mathcal{P}(2) = \text{sgn}_2 \) and \( \mathcal{P}(3) = \Gamma(E)(3)/R \) with \( R \) the submodule generated by \( v((x_1 x_2) x_3) \) and \( \tau_{13} \cdot v((x_1 x_2) x_3) \) with \( \tau((x_1 x_2) x_3) = ((x_{\tau(1)} x_{\tau(2)}) x_{\tau(3)}) \) for any \( \tau \in \Sigma_3 \). In particular \( \dim \mathcal{P}(3) = 4 \).

(2) A Pre-Lie algebra is a non associative algebra defined by the identity

\[
(xy)z - x(yz) = (xz)y - x(zy)
\]
for all \( x, y, z \). Assume that the Lie bracket of \( g \) satisfies also the Pre-Lie identity, that is,

\[
[[x, y], z] - [x, [y, z]] = [[x, z], y] - [x, [z, y]].
\]

Applying anticommutativity to this equation we obtain

\[
[[x, y], z] + [[y, z], x] + [[z, x], y] + [[y, z], x] = 0;
\]

and finally the Jacobi identity gives

\[
[[y, z], x] = 0.
\]

This shows that the Lie algebra is also 2-step nilpotent and the Lie bracket is an associative product.

(3) In [11], we have defined classes of non associative algebras including in particular Pre-Lie algebras, Lie-admissible algebras and more generally algebras with a non associative defining identity admitting a symmetry with respect to a subgroup of the symmetric group \( \Sigma_3 \). These algebras have been called \( G_i \)-associative algebras where \( G_i, i = 1, \ldots, 6 \) are the subgroups of \( \Sigma_3 \). More precisely, a \( G_1 = \{Id\} \)-associative algebra is an associative algebra, a \( G_2 = \{Id, \tau_{12}\} \)-associative algebra is a Vinberg algebra that is, its multiplication satisfies

\[
(xy)z - x(yz) = (yx)z - y(xz),
\]

a \( G_3 = \{Id, \tau_{23}\} \)-associative algebra is a Pre-Lie algebra, a \( G_4 = \{Id, \tau_{13}\} \)-associative algebra satisfies

\[
(xy)z - x(yz) = (zy)x - z(yx),
\]

a \( G_5 = \{Id, c, c^2\} \)-associative algebra satisfies

\[
(xy)z - x(yz) + (yz)x - y(zx) + z(xy) - z(xy) = 0,
\]

and a \( G_6 = \Sigma_3 \)-associative algebra is a Lie-admissible algebra [9]. While writing this paper we discover that this notion already appear in [19]. It is easy to see that if the Lie bracket of \( g \) satisfies the \( G_i \)-associativity for \( i = 1, 2, 3, \) or \( 4 \) then \( g \) is 2-step nilpotent and the Lie bracket is an associative multiplication. The defining equations associated to the cases \( i = 5 \) and \( 6 \) are always satisfied because the \( G_5 \)-conditions corresponds to the Jacobi identity and a Lie algebra is, in particular, a Lie-admissible algebra.

3. Cubic associative Lie multiplication

Let \( A \) be a \( \mathbb{K} \) associative algebra with binary multiplication \( xy \). The associativity which is the quadratic relation

\[
(xy)z = x(yz)
\]

implies six cubic relations

\[
\begin{cases}
((xy)z)t = (x(yz))t, \\
(xyz)t = x((yz)t), \\
x((yz)t) = x(y(zt)), \\
(x(yzt)) = (xy)(zt), \\
(xy)(zt) = ((xy)z)t.
\end{cases}
\]

Recall that these relations correspond to the edges of the Stasheff pentagon.
Definition 4. A binary algebra, that is, an algebra whose multiplication is given by a bilinear map, is called cubic associative if the multiplication satisfies the cubic relations (\(\ast\)).

We call these relations cubic because if we denote by \(\mu\), the multiplication, it occurs exactly three times in each term of the relations. For example, the first relation writes as
\[
\mu \circ (\mu \circ (\mu \otimes Id) \otimes Id) = \mu \circ (\mu \circ (Id \otimes \mu) \otimes Id)
\]
which is cubic in \(\mu\). It is the same thing for all other relations.

If \(\text{Ass} = \Gamma(E)/(R_{\text{Ass}})\) is the operad for associative algebras, the relations \((\ast)\) are the generating relations of \((R_{\text{Ass}})(4)\). But these relations are following from the relations defining \((R_{\text{Ass}})(3) = R_{\text{Ass}}\). In Definition 4, we do not assume that the algebra is associative. It is clear that \((\ast)\) do not implies associativity. From the relations \((\ast)\) we can define a binary cubic operad \(\text{AssCubic}\). This operad will be studied in the next paragraph.

Proposition 5. Let \(\mathfrak{g}\) be a Lie algebra. The Lie bracket is cubic associative if and only if \(\mathfrak{g}\) is 3-step nilpotent.

In fact, the first identity of \((\ast)\) becomes
\[
[[[X_1, X_2], X_3], X_4] = [[X_1, [X_2, X_3]], X_4] = -[[[X_2, X_3]], X_1], X_4
\]
and finally
\[
[[[X_1, X_2], X_3], X_4] + [[[X_2, X_3]], X_1], X_4 = [[[X_3, X_1]], X_2], X_4 = 0,
\]
which implies that \(\mathfrak{g}\) is 3-nilpotent. Conversely, if \(\mathfrak{g}\) is 3-nilpotent, all the relations of \((\ast)\) are satisfied.

The classification of the 3-step nilpotent Lie algebra of dimension less than 7 is the following:

**Dimension 4**
\[ n_4: [X_1, X_i] = X_{i+1}, \ i = 2, 3. \]

**Dimension 5**
\[ n_5: [X_1, X_i] = X_{i+1}, \ i = 2, 3, \ [X_2, X_5] = X_4; \]
\[ n_5: [X_1, X_i] = X_{i+1}, \ i = 2, 3, \ [X_2, X_3] = X_6. \]

**Dimension 6**
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_5, X_6] = X_4; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_5] = X_4; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_3] = X_6, \ [X_2, X_5] = X_6; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_3] = X_6 - X_6, \ [X_2, X_5] = X_6; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_5] = X_6, \ [X_5, X_6] = X_4; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_3] = X_4; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5; \]
\[ n_6: [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_5, X_6] = X_4; \]
Dimension 7

\(n_1^{77}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_2, X_5] = X_7;\)

\(n_1^{78}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_2, X_6] = X_4, \quad [X_2, X_5] = X_7;\)

\(n_1^{79}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_4, \quad [X_2, X_5] = X_7;\)

\(n_1^{80}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6;\)

\(n_1^{81}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_4;\)

\(n_1^{82}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_4, \quad [X_2, X_3] = X_7;\)

\(n_1^{83}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_7, \quad [X_2, X_3] = X_4 + X_7;\)

\(n_1^{84}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_7, \quad [X_2, X_3] = X_4;\)

\(n_1^{85}: \begin{cases} [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, & [X_3, X_5] = X_7, \quad [X_2, X_5] = X_4 + X_6, \\ [X_2, X_3] = X_4; & \end{cases}\)

\(n_1^{86}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_7, \quad [X_2, X_3] = X_7;\)

\(n_1^{87}: \begin{cases} [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, & [X_5, X_6] = X_7 + X_4, \quad [X_2, X_5] = X_4, \\ [X_2, X_3] = X_3; & \end{cases}\)

\(n_1^{88}: \begin{cases} [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, & [X_5, X_6] = X_4, \quad [X_3, X_5] = X_7, \\ [X_2, X_3] = X_4, \quad [X_2, X_6] = X_6; & \end{cases}\)

\(n_1^{89}: \begin{cases} [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, & [X_5, X_6] = X_4, \quad [X_2, X_3] = X_4, \\ [X_2, X_5] = X_7; & \end{cases}\)

\(n_1^{90}: \begin{cases} [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, & [X_2, X_3] = X_4, \quad [X_5, X_5] = X_7, \\ [X_2, X_5] = X_6; & \end{cases}\)

\(n_1^{91}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_5, X_6] = X_7;\)

\(n_1^{92}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, 6, \quad [X_2, X_3] = X_4, \quad [X_2, X_5] = X_7;\)

\(n_1^{93}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_5] = X_7;\)

\(n_1^{94}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_5] = X_4, \quad [X_2, X_3] = X_7;\)

\(n_1^{95}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5 \quad [X_2, X_3] = X_7;\)

\(n_1^{96}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_5] = X_7, \quad [X_2, X_6] = X_4;\)

\(n_1^{97}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_5] = X_4, \quad [X_3, X_5] = -X_4, \quad [X_2, X_5] = X_7;\)

\(n_1^{98}: \begin{cases} [X_1, X_i] = X_{i+1}, i = 2, 3, 5, & [X_2, X_5] = X_4, \quad [X_3, X_5] = -X_4, \\ [X_2, X_5] = X_7, \quad [X_5, X_6] = X_4; & \end{cases}\)

\(n_1^{99}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_7] = X_6, \quad [X_2, X_3] = X_4;\)

\(n_1^{100}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_7] = X_4, \quad [X_5, X_7] = X_6;\)

\(n_1^{101}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_7] = X_6, \quad [X_5, X_7] = X_4;\)

\(n_1^{102}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_5, X_7] = X_4;\)

\(n_1^{103}: [X_1, X_i] = X_{i+1}, i = 2, 3, 5, \quad [X_2, X_7] = X_4;\)


\( n^{104} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_5, X_7] = X_4, \ [X_2, X_3] = X_4; \)

\( n^{105} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_7] = X_4, \ [X_2, X_3] = X_4; \)

\( n^{106} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_7] = X_4, \ [X_5, X_6] = X_4; \)

\( n^{107} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_5, X_7] = X_4, \ [X_6, X_7] = X_4; \)

\( n^{108} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_7] = X_6, \ [X_2, X_3] = X_6; \)

\( n^{109} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_7] = X_6, \ [X_5, X_7] = X_4; \)

\( n^{110} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_3] = X_6, \ [X_5, X_7] = X_4; \)

\( n^{111} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_7] = X_6, \ [X_2, X_3] = X_4; \)

\( n^{112} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_3] = X_6, \ [X_2, X_7] = X_4; \)

\( n^{113} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_5, X_7] = X_6, \ [X_5, X_6] = X_4; \)

\( n^{114} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_7] = X_4, \ [X_5, X_6] = X_4, \ [X_5, X_7] = X_6; \)

\( n^{115} : [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_3] = X_4, \ [X_5, X_7] = X_3, \ [X_6, X_7] = X_4; \)

\( n^{116} : \begin{cases} [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_3, X_5] = -X_4, \ [X_2, X_6] = X_4, \\ [X_5, X_7] = -X_4; \end{cases} \)

\( n^{117} (\alpha) : \begin{cases} [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_5] = X_7, \ [X_2, X_7] = X_4, \\ [X_5, X_6] = X_4, \ [X_5, X_7] = \alpha X_4; \end{cases} \)

\( n^{118} : \begin{cases} [X_1, X_i] = X_{i+1}, \ i = 2, 3, 5, \ [X_2, X_5] = X_7, \ [X_2, X_6] = X_4, \\ [X_3, X_5] = -X_4, \ [X_5, X_7] = -\frac{1}{2} X_4; \end{cases} \)

\( n^{119} : [X_1, X_2] = X_3, \ [X_1, X_3] = X_4, \ [X_2, X_5] = X_4, \ [X_6, X_7] = X_4. \)

Remarks.

(1) We can generalize this process to define \((n - 1)\)-associative (binary) algebras: we consider the relations defining the \(\Sigma_n\)-module \(\mathcal{A}ss(n)\) of the quadratic operad \(\mathcal{A}ss\) and define, as above, an algebra with a multiplication which is a bilinear map \(\mu\) (nonassociative), satisfying the previous relations where \(\mu\) occurs \(n - 1\) times in each term of the relations. This algebra will be called \((n - 1)\)-associative (binary) algebra. If the Lie bracket of a algebra \(\mathfrak{g}\) is also \((n - 1)\)-associative, we prove a similar way than for the cubic associative case that \(\mathfrak{g}\) is a nilpotent Lie algebra of nilindex \(n - 1\).

(2) There exits another notion of associativity for \(n\)-ary algebras (an \(n\)-ary algebra is a vector space with a multiplication which is an \(n\)-linear map), the total associativity. For example, a totally associative 3-ary algebra has a ternary multiplication, denoted \(xyz\), satisfying the relation:

\[(xyz)tu = x(yzt)u = xy(ztu)\]

for any \(x, y, z, t, u\). The corresponding operad is studied in [12], [13], [17] and [15]. Let \(\mathfrak{g}\) be a Lie algebra. We have the notion of Lie triple product given by \([[x, y], z]\). If we consider the vector space \(\mathfrak{g}\) provided with the 3-ary product given by the Lie triple product, then \(\mathfrak{g}\) is a 3-Lie algebra ([6] or [4]). Let us suppose now that the Lie triple bracket of \(\mathfrak{g}\) is a totally associative product. This implies

\[ [[[X, Y], Z], T], U] = [[X, [[Y, Z], T]], U] = [[X, Y], [[Z, T], U]]. \]
LIE ALGEBRAS WITH ASSOCIATIVE STRUCTURES. APPLICATIONS TO THE STUDY OF 2-STEP NILPOTENT LIE

But
\[
[[X,Y],[[Z,T],U]] = -[[[Z,T],U],X]
= [X,[Y,[[Z,T],U]]] + [Y,[[[Z,T],U],X]]
= [[[Z,T],U],Y],X] - [[[Z,T],U],X],Y]
= [[[Z,T],U],X] - [[Z,[[T,U],[X],Y]].
\]

We deduce
\[
[[X,[[Y,Z],T]],U]] = [[Z,[[T,U],Y]],X] - [[Z,[T,U]],X],Y]
= 2^5[[X,[[Y,Z],T]],U] - 2^5[[X,[[Y,Z],U]],T].
\]

Then
\[
(2^5 - 1)[[X,[[Y,Z],T]],U] = 2^5[[X,[[Y,Z],U]],T].
\]

This implies
\[
[[X,[[Y,Z],T]],U] = 0 = [[[X,Y],Z],T],U]] = [[X,Y],[[Z,T],U]].
\]

The Lie algebra is 4-step nilpotent.

4. CUBIC OPERADS

Let E be a \( \mathbb{K}[\Sigma_2] \)-module and \( \Gamma(E) \) the free operad generated by E. Consider a \( \mathbb{K}[\Sigma_4] \)-submodule \( R \) of \( \Gamma(E)(4) \). Let \( R \) the ideal of \( \Gamma(E) \) generated by \( R \). We have
\[
\mathcal{R} = \{ \mathcal{R}(n), n \in \mathbb{N}^* \}
\]
with \( \mathcal{R}(1) = \{0\}, \mathcal{R}(2) = \{0\}, \mathcal{R}(3) = \{0\}, \mathcal{R}(4) = R. \)

**Definition 6.** We call cubic operad generated by E and defined by the relations \( R \subset \Gamma(E)(4) \), the operad \( \mathcal{P}(E, R) \) given by
\[
\mathcal{P}(E, R)(n) = \frac{\Gamma(E)(n)}{\mathcal{R}(n)}.
\]

The operad \( \text{AssCubic} \) is the cubic operad generated by \( E = \mathbb{K}[\Sigma_2] \) and the \( \mathbb{K}[\Sigma_4] \)-submodule of relations \( R \) generated by the vectors
\[
\begin{align*}
&\{ (x_1x_2)x_3 - (x_1(x_2x_3))x_4, (x_1(x_2x_3))x_4 - x_1((x_2x_3)x_4), x_1((x_2x_3)x_4) - x_1(x_2(x_3x_4)), \\
&x_1(x_2(x_3x_4)) - (x_1x_2)(x_3x_4), (x_1x_2)(x_3x_4) - ((x_1x_2)x_3)x_4. \\
\end{align*}
\]

Thus we have \( \text{AssCubic}(2) = \mathbb{K}[\Sigma_2], \text{AssCubic}(3) = \mathbb{K}[\Sigma_3], \) and
\[
\text{AssCubic}(4) = \frac{\Gamma(E)(4)}{\mathcal{R}(4)}
\]
is the 24-dimensional \( \mathbb{K} \)-vector space generated by \{\((x_\sigma(1)x_\sigma(2))x_\sigma(3))x_\sigma(4), \sigma \in \Sigma_4\}.

The operad \( 3\text{Nilp} \) is the cubic operad defined by \( 3\text{Nilp}(2) = sgn_2, 3\text{Nilp}(3) = sgn_3 \oplus V_2/sgn_3 \) and \( 3\text{Nilp}(4) = \{0\}. \)

**Proposition 7.** The cubic operad \( 3\text{Nilp} \) is not Koszul.
Proof. In fact the Koszul operads has only quadratic relations.

Remark: The Jordan operad. There is a cubic operad which is really interesting, it is the operad $\mathcal{J}$ord corresponding to Jordan algebras. Recall that a $\mathbb{K}$-Jordan algebra is a commutative algebra satisfying the following identity

$$x(yx^2) = (xy)x^2.$$ 

Since $\mathbb{K}$ is of zero characteristic, linearizing this identity, we obtain

$$((x_2x_3)x_4)x_1 + ((x_3x_1)x_4)x_2 + ((x_1x_2)x_4)x_3 - (x_2x_3)(x_4x_1) - (x_3x_1)(x_4x_2) - (x_1x_2)(x_4x_3) = 0.$$ 

This relation is cubic. It is invariant by the permutations $\tau_{12}, \tau_{13}, \tau_{23}, c, c^2$ where $c$ is the cycle $(123)$. Thus the $\mathbb{K}[^2]$-module $\mathcal{R}(4)$ generated by the vector

$$((x_2x_3)x_4)x_1 + ((x_3x_1)x_4)x_2 + ((x_1x_2)x_4)x_3 - (x_2x_3)(x_4x_1) - (x_3x_1)(x_4x_2) - (x_1x_2)(x_4x_3)$$ 

is a vector space of dimension 4. Let us consider the cubic operad $\mathcal{J}$ord define by

$$\mathcal{J}$ord$(2) = \mathbb{I}$, $\mathcal{J}$ord$(3) = \mathbb{I} + V$, $\mathcal{J}$ord$(4) = \frac{\Gamma(\mathbb{I})(4)}{\mathcal{R}(4)}$$ 

where $\mathbb{I}$ is the identity representation of $\Sigma_2$. Since $\dim \Gamma(\mathbb{I})(4) = 15$, thus $\frac{\Gamma(\mathbb{I})(4)}{\mathcal{R}(4)}$ is of dimension 11.

If the Lie bracket of a Lie algebra $g$ is also a Jordan product, then $g$ is abelian. But maybe, it would be interesting to look Lie bracket satisfying the Jordan identity without the commutativity identity.

5. The variety of 2-step nilpotent Lie algebras

5.1. Chevalley-Hochschild cohomology of two-step nilpotent Lie algebras. Let $\mu_0$ be a Lie bracket of a two-step nilpotent Lie algebra $g_0$. If $\mu = \mu_0 + t\varphi$ is a linear deformation of $\mu_0$ then $\mu$ is a Lie bracket of a two-step nilpotent Lie algebra if and only if $\varphi$ satisfies the following conditions:

1. $\varphi \circ \varphi = 0$,
2. $\delta_{H,\mu_0}(\varphi) = 0$,
3. $\delta_{C,\mu_0}(\varphi) = 0$.

In the first condition, $\varphi$ is a skewsymmetric bilinear map and $\circ$ is the following trilinear map:

$$\varphi \circ \varphi(X, Y, Z) = \varphi(\varphi(X, Y), Z).$$

In the second condition since the Lie algebra $g_0$ is associative, $\delta_{H,\mu_0}$ is the coboundary operator of the Hochschild cohomology (20), that is,

$$\delta_{H,\mu_0}(\varphi)(X, Y, Z) = \mu_0(X, \varphi(Y, Z)) - \varphi(\mu_0(X, Y), Z) + \varphi(X, \mu_0(Y, Z)) - \mu_0(\varphi(X, Y), Z)$$

and in the third condition $\delta_{C,\mu_0}$ is the coboundary operator of the Chevalley operator (11), that is,

$$\delta_{C,\mu_0}(\varphi)(X, Y, Z) = \mu_0(\varphi(X, Y), Z) + \mu_0(\varphi(Y, Z), X) + \mu_0(\varphi(Z, X), Y) + \varphi(\mu_0(X, Y), Z) + \varphi(\mu_0(Y, Z), X) + \varphi(\mu_0(Z, X), Y).$$
Lemma 8. Let $\varphi$ be a skew-symmetric bilinear map on $\mathfrak{g}_0$ then $\varphi$ satisfies the conditions (2) and (3) if and only if we have
\[
\varphi(\mu_0(X,Y), Z) + \mu_0(\varphi(X,Y), Z) = 0
\]
for all $X,Y,Z$ in $\mathfrak{g}_0$.

Proof. In fact $\delta_{C,\mu_0}(\varphi)(X,Y,Z) + \delta_{H,\mu_0}(\varphi)(X,Y,Z) = \mu_0(\varphi(\mu_0(Z,X), Y) + \varphi(\mu_0(Z,X), Y)$ so $\delta_{C,\mu_0}(\varphi)(X,Y,Z) = \delta_{H,\mu_0}(\varphi)(X,Y,Z) = 0$ implies $\varphi(\mu_0(X,Y), Z) + \mu_0(\varphi(X,Y), Z) = 0$. Conversely, considering
\[
\delta_{H,C,\mu_0}(\varphi)(X,Y,Z) = \varphi(\mu_0(X,Y), Z) + \mu_0(\varphi(X,Y), Z)
\]
then
\[
\delta_{H,\mu_0}(\varphi)(X,Y,Z) = -\delta_{H,C,\mu_0}(\varphi)(X,Y,Z) - \delta_{H,C,\mu_0}(\varphi)(Y,Z,X)
\]
and
\[
\delta_{C,\mu_0}(\varphi)(X,Y,Z) = \delta_{H,C,\mu_0}(\varphi)(X,Y,Z) + \delta_{H,C,\mu_0}(\varphi)(Y,Z,X) + \delta_{H,\mu_0}(\varphi)(Z,X,Y)
\]
and $\delta_{H,\mu_0}(\varphi) = 0$ implies that $\delta_{C,\mu_0}(\varphi) = \delta_{H,\mu_0}(\varphi) = 0$.

In fact $\delta_{H,\mu_0}$ is a coboundary operator for the deformation cohomology of 2-step nilpotent Lie algebras in the algebraic variety $2\text{Nil}_n$ of $n$-dimensional 2-step nilpotent Lie algebras. We assume now that $\mathbb{K}$ is algebraically closed. Recall that the variety $2\text{Nil}_n$ is defined by the polynomial equations
\[
\sum_{i=1}^{n} C_{ij}^k C_{is}^t = 0
\]
for $1 \leq i,j,k,s \leq n$, an element $(C_{ij}^k)$ with $1 \leq i < j \leq n$ and $1 \leq k \leq n$ corresponds to a Lie algebra with constant structures $(C_{ij}^k)$ related to a fixed basis. Each element of this variety is an algebra on the operad $2\text{Nil}_p$. From Proposition 3 this operad is Koszul and the deformation cohomology is the classical operadic cohomology [16].

We deduce that this cohomology is associated with the complex $(C^n(\mu_0, \mu_0), \delta^n_{H,C,\mu_0})$ where $C^n(\mu_0, \mu_0)$ is the vector space of skew $n$-linear map on $\mathfrak{g}_0$ with values in $\mathfrak{g}_0$ and $\delta^n_{H,C,\mu_0} : C^n(\mu_0, \mu_0) \rightarrow C^{n+1}(\mu_0, \mu_0)$ is the linear operator defined by
\[
\delta^n_{H,C,\mu_0} \psi(X_1, \cdots, X_{2n+1}) = \mu_0(X_1, \psi(X_2, \cdots, X_{2n+1})) + \sum_{i=1}^{n} \psi(X_1, \cdots, \mu_0(X_{2i}, X_{2i+1}), \cdots, X_{2n+1}),
\]
\[
\delta^{n-1}_{H,C,\mu_0} \psi(X_1, \cdots, X_{2n}) = \mu_0(X_1, \psi(X_2, \cdots, X_{2n})) + \sum_{i=1}^{n-1} \psi(X_1, \cdots, \mu_0(X_{2i+1}, X_{2i+2}), \cdots, X_{2n}).
\]
In particular $\delta_{H,C,\mu_0}$ corresponds to $\delta^2_{H,C,\mu_0}$. We will denote this cohomology by $H^n_{H,C}(\mathfrak{g}_0, \mathfrak{g}_0)$.

5.2. Rigidity and deformations in $2\text{Nil}_p$. The algebraic linear group $Gl(n, \mathbb{K})$ acts on the variety $2\text{Nil}_n$, the orbit of an element $\mu_0$ associated with this action corresponds to the set of Lie algebras isomorphic to $\mu_0$ (we identify a Lie algebra with its Lie product). Thus, if we denote by $\mathcal{O}(\mu_0)$ this orbit, $\mu \in \mathcal{O}(\mu_0)$ if and only if there is $f \in Gl(n, \mathbb{K})$ such that $\mu = f^{-1} \circ \mu_0 \circ (f \times f)$.

Definition 9. A Lie algebra $\mu \in 2\text{Nil}_n$ is called rigid in $2\text{Nil}_n$ if its orbit $\mathcal{O}(\mu)$ is open in $2\text{Nil}_n$ for the Zariski topology.
Theorem 10. Let $g_0$ be a $n$-dimensional 2-step nilpotent Lie algebra on $\mathbb{K}$. If $H^2_{H,C}(g_0, g_0) = 0$, then $g_0$ is rigid in $2\text{Nilp}_n$.

Proposition 11. The $(2p + 1)$-dimensional Heisenberg algebra $h_{2p+1}$ is rigid in $2\text{Nilp}_{2p+1}$.

Proof. The dimension of the algebra of derivations of $h_{2p+1}$ is given, for example, in [8]. We have $\dim B^2_{H,C}(h_{2p+1}) = p(2p + 1)$. Since the 2-cochains are skew-symmetric, to compute $\delta_{H,C,\mu_0}(\varphi) = 0$, it is sufficient to compute $\delta_{H,C,\mu_0}(\varphi)(X_i, X_j, X_k) = 0$ for $i < j$, where $\{X_1, \cdots, X_{2p+1}\}$ is a basis of $h_{2p+1}$ satisfying

$$[X_1, X_2] = \cdots = [X_{2i-1}, X_{2i}] = \cdots = [X_{2p-1}, X_{2p}] = X_{2p+1}.$$ 

If $\varphi(X_i, X_j) = \sum_{k=1}^{2p+1} a^k_{ij} X_k$, then

$$\delta_{H,C,\mu_0}(\varphi)(X_i, X_j, X_k) = \varphi(\mu_0(X_i, X_j), X_k) + \mu_0(\varphi(X_i, X_j), X_k) = 0$$

is equivalent to

$$\varphi(X_1, X_2) = \sum_{k=1}^{2p+1} a^1_{12} X_k;$$

$$\varphi(X_1, X_i) = a^2p+1_{1i} X_{2p+1}, 3 \leq i \leq 2p; \varphi(X_1, X_{2p+1}) = -a^1_{12} X_{2p+1};$$

$$\varphi(X_2, X_i) = a^{2p+1}_{2i} X_{2p+1}, 3 \leq i \leq 2p; \varphi(X_2, X_{2p+1}) = a^1_{12} X_{2p+1};$$

$$\vdots$$

$$\varphi(X_{2i-1}, X_{2i}) = \sum_{k=1}^{2p} a^k_{12} X_k + a^{2p+1}_{2i-1, 2i} X_{2p+1}, 2 \leq i \leq p;$$

$$\varphi(X_i, X_s) = a^{2p+1}_{is} X_{2p+1}, (l, s) \neq (2i - 1, 2i);$$

$$\varphi(X_{2l}, X_{2l+1}) = a^{2l-1}_{12} X_{2p+1}, l \leq p;$$

$$\varphi(X_{2l-1}, X_{2p+1}) = -a^{2l}_{12} X_{2p+1}, l \leq p.$$ 

We deduce that $\dim Z^2_{H,C}(h_{2p+1}, h_{2p+1}) = p(2p + 1)$. So $H^2_{H,C}(h_{2p+1}, h_{2p+1}) = 0$ and $h_{2p+1}$ is rigid in $2\text{Nilp}_{2p+1}$.

5.3. 2-step nilpotent Lie algebras with characteristic sequence $(2, \cdots, 2, 1)$. Recall that the characteristic sequence of a $n$-dimensional $\mathbb{K}$-Lie algebra $g$ is the invariant

$$c(g) = \max_{X \in g \setminus \{0\}} \{(c_1(X), \cdots, c_p(X), 1)\}$$

where $(c_1(X), \cdots, c_p(X), 1)$ is the decreasing sequence of the dimensions of the Jordan blocks of the nilpotent operator $ad(X)$, the maximum is computed with respect to the lexicographic order. This invariant as been introduced in [3] to study classification of nilpotent Lie algebras (see also [3] and [7]). For 2-step nilpotent Lie algebras this characteristic sequence is of type $(2, \cdots, 2, 1, \cdots, 1)$. For example the characteristic sequence of the Heisenberg algebra $h_{2p+1}$
is $(2, 1, \cdots, 1)$. Let $\mathfrak{g}_{2p+1}$ be the $(2p + 1)$-dimensional Lie algebras defined by the following brackets given in the basis $\{X_1, \cdots, X_{2p+1}\}$ by
\[
[X_1, X_{2i}] = X_{2i+1}, \quad 1 \leq i \leq p,
\]
the other non defined brackets are equal to zero. Its characteristic sequence is $(2, \cdots, 2, 1)$ where $2$ appears $p$ times.

**Lemma 12.** Any $(2p + 1)$-dimensional 2-step nilpotent Lie algebra with characteristic sequence $(2, \cdots, 2, 1)$ is isomorphic to a linear deformation of $\mathfrak{g}_{2p+1}$.

**Proof.** Let $\mathfrak{g}$ be a $(2p + 1)$-dimensional 2-step nilpotent Lie algebra with characteristic sequence $(2, \cdots, 2, 1)$. There exists a basis $\{X_1, \cdots, X_{2p+1}\}$ such that the characteristic sequence is given by the operator $ad(X_1)$. If $\{X_1, \cdots, X_{2p+1}\}$ is the Jordan basis of $ad(X_1)$ then the brackets of $\mathfrak{g}$ write
\[
\left\{
\begin{aligned}
[X_1, X_{2i}] &= X_{2i+1}, \quad 1 \leq i \leq p, \\
[X_{2i}, X_{2j}] &= \sum_{k=1}^{p} a_{2i,2j}^{2k+1} X_{2k+1}, \quad 1 \leq i < j \leq p.
\end{aligned}
\right.
\]

The change of basis $Y_1 = X_1$, $Y_i = tX_i$ for $2 \leq i \leq 2p + 1$ shows that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_t$ whose brackets are
\[
\left\{
\begin{aligned}
[X_1, X_{2i}] &= X_{2i+1}, \quad 1 \leq i \leq p, \\
[X_{2i}, X_{2j}] &= t \sum_{k=1}^{p} a_{2i,2j}^{2k+1} X_{2k+1}, \quad 1 \leq i < j \leq p.
\end{aligned}
\right.
\]

If $\mu_t$ is the multiplication of $\mathfrak{g}_t$ and $\mu_0$ the multiplication of $\mathfrak{g}_{2p+1}$ we have $\mu_t = \mu_0 + t\phi$ with $\phi(X_{2i}, X_{2j}) = \sum_{k=1}^{p} a_{2i,2j}^{2k+1} X_{2k+1}, \quad 1 \leq i < j \leq p$ and $\phi(X_i, X_s) = 0$ for all the other cases with $l < s$. So $\mathfrak{g}$ is a linear deformation of $\mathfrak{g}_{2p+1}$.

Let us compute the second cohomological group $H^2_{H,C}(\mathfrak{g}_{2p+1}, \mathfrak{g}_{2p+1})$.

Since for any $f$ in $\text{End}(\mathfrak{g}_{2p+1})$ we have
\[
\delta f(X_1, X_i) = -f(X_{2i+1}) + a_{1i} X_{2i+1} + \sum_{k=1}^{p} a_{2k,2} X_{2k+1}
\]
with $f(X_i) = \sum_{s=1}^{2p+1} a_{st} X_s$, there exists in each class in $H^2_{H,C}(\mathfrak{g}_{2p+1}, \mathfrak{g}_{2p+1})$ a representant $\phi \in Z^2_{H,C}(\mathfrak{g}_{2p+1}, \mathfrak{g}_{2p+1})$ such that $\phi(X_1, X_{2i}) = 0$ for $1 \leq i \leq p$. This implies for such a cocycle that $0 = \delta \phi(X_1, X_{2i}, X_1) = \phi(X_{2i+1}, X_1)$ and this cocycle satisfies $\phi(X_1, Y) = 0$ for any $Y$ in $\mathfrak{g}_{2p+1}$. Moreover
\[
0 = \delta \phi(X_1, X_{2i}, X_l) = \phi(X_{2i+1}, X_l)
\]
for any $1 \leq l \leq 2p + 1$ and $1 \leq i \leq p$. If we put $\phi(X_{2i}, X_{2j}) = \sum_{k=1}^{2p+1} a_{2i,2j}^{2k} X_k$, the equations $\delta \phi(X_{2i}, X_{2j}, X_l) = 0$ for $l = 1, 2$ implies that $a_{2i,2j}^{2k} = a_{2i,2j}^{2k+1} = 0$ for $1 \leq k \leq p$. But
\[ \delta f(X_{2i}, X_{2j}) = a_{1,2i}X_{2j+1} - a_{1,2j}X_{2i+1}. \] We can choose the cocycle \( \varphi \) satisfying
\[
\begin{align*}
\varphi(X_2, X_4) &= \sum_{k=3}^{p} a_{24}^{2k+1}X_{2k+1}, \\
\varphi(X_2, X_{2i}) &= \sum_{k=2}^{p} a_{2,2i}^{2k+1}X_{2k+1}, \quad 3 \leq i \leq p, \\
\varphi(X_{2i}, X_{2j}) &= \sum_{k=1}^{p} a_{2i,2j}^{2k+1}X_{2k+1} \quad 2 \leq i < j \leq p,
\end{align*}
\]
if \( p \leq 3 \). If \( p = 2 \) the chosen cocycle \( \varphi \) is trivial and we obtain:

**Proposition 13.** Let \( \mathfrak{v}_{2p+1} \) the 2-step nilpotent Lie algebra defined by \([X_1, X_2] = X_{2i+1}, \ 1 \leq i \leq p \) then

- if \( p = 2 \), the algebra \( \mathfrak{v}_5 \) is rigid in \( 2\text{Nilp}_5 \),
- if \( p > 2 \), \( \dim(H^2_{H,C}(\mathfrak{v}_{2p+1}, \mathfrak{v}_{2p+1})) = \frac{p(p+1)(p-2)}{2} \).

We deduce that any 2-step nilpotent \((2p+1)\)-dimensional Lie algebra with characteristic sequence \((2, \cdots, 2, 1)\) is isomorphic to one of the following Lie algebras
\[
\begin{align*}
[X_1, X_{2i}] &= X_{2i+1}, \ 1 \leq i \leq p, \\
[X_2, X_4] &= \sum_{k=3}^{p} a_{24}^{2k+1}X_{2k+1}, \\
[X_2, X_{2i}] &= \sum_{k=2}^{p} a_{2,2i}^{2k+1}X_{2k+1}, \quad 3 \leq i \leq p, \\
[X_{2i}, X_{2j}] &= \sum_{k=1}^{p} a_{2i,2j}^{2k+1}X_{2k+1} \quad 2 \leq i < j \leq p.
\end{align*}
\]

Consider \( \mathfrak{g} \) a Lie algebra of this family \( \mathcal{F} \). Then the subspace \( \mathfrak{m} \) generated by \( \{X_2, \cdots, X_{2p+1}\} \) is a Lie subalgebra of \( \mathfrak{g} \). So the classification up to an isomorphism of the elements of the family \( \mathcal{F} \) corresponds to the classification of 2-step nilpotent \((2p)\)-dimensional Lie algebras. Moreover we can assume that \( X_2 \) is a characteristic vector of \( \mathfrak{m} \) that is \( c(\mathfrak{m}) \) is the characteristic sequence associated with \( \text{ad}(X_2) \). But \([X_2, X_3] = 0 \) and \( X_3 \not\in \text{Im}(\text{ad}X_2) \). We can assume that \( \{X_2, X_3, \cdots, X_{2p+1}\} \) is a Jordan basis of \( \text{ad}X_2 \). For example, if \( p = 3 \) we have \( c(\mathfrak{m}) = (2, 2, 1, 1) \) or \((2, 1, 1, 1, 1)\) or, in the abelian case, \((1, 1, 1, 1, 1, 1, 1)\). This corresponds to

- \([X_2, X_4] = X_7, [X_2, X_6] = X_5 \) if \( c(\mathfrak{m}) = (2, 2, 1, 1) \),
- \([X_2, X_4] = X_7 \) if \( c(\mathfrak{m}) = (2, 1, 1, 1, 1) \),
- \([X_2, X_4] = 0 \) if \( \mathfrak{m} \) is abelian.

5.4. **2-step nilpotent Lie algebras with characteristic sequence** \((2, \cdots, 2, 1, 1)\). Let us denote by \( \mathfrak{v}_{2p} \) the \((2p)\)-dimensional Lie algebra given by the brackets
\[ [X_1, X_{2i}] = X_{2i+1}, \ 1 \leq i \leq p - 1, \]
other non defined brackets are equal to zero.
Lemma 14. Any 2-step nilpotent \((2p)\)-dimensional Lie algebra is isomorphic to a linear deformation of \(\mathfrak{k}_{2p}\).

Proof. It is similar to the proof of Lemma 12.

If \(\mu_t = \mu_0 + t\varphi\) is a linear deformation of the bracket \(\mu_0\) of \(\mathfrak{k}_{2p}\) then \(\varphi \in Z^2_{H,C}(\mathfrak{k}_{2p}, \mathfrak{k}_{2p})\) and it is also a bracket of a 2-step nilpotent Lie algebra. Let us determine these maps.

In \(\mathfrak{k}_{2p}\) we have

\[
\delta f(X_1, X_{2i}) = -f(X_{2i+1}) + a_{11}X_{2i+1} + \sum_{j=1}^{p-1} a_{2j,2i}X_{2j+1}
\]

for \(1 \leq i \leq p - 1\) and

\[
\delta f(X_1, X_{2p}) = \sum_{j=1}^{p-1} a_{2j,2p}X_{2j+1}.
\]

Thus any \(\tilde{\varphi} \in Z^2_{H,C}(\mathfrak{k}_{2p}, \mathfrak{k}_{2p})\) is cohomologous to a cocycle \(\varphi\) satisfying

\[
\begin{cases}
\varphi(X_1, X_{2i}) = 0, & i = 1, \ldots, p - 1, \\
\varphi(X_1, X_{2p}) = \sum_{k=1}^{p} a_{1,2p}^{2k}X_{2k}. 
\end{cases}
\]

For such a cocycle \(\varphi\) we have

\[
0 = \delta \varphi(X_1, X_{2p}, X_1) = [\varphi(X_1, X_{2p}), X_1]
\]

and \(a_{1,2p}^{2k} = 0\) for \(1 \leq k \leq p - 1\). This implies \(\varphi(X_1, X_{2p}) = a_{1,2p}^{2p}X_{2p}\). Since \(\mu_0 + t\varphi\) is a multiplication of a 2-step nilpotent Lie algebra, \(a_{1,2p}^{2p} = 0\) and \(\varphi(X_1, X_{2i}) = 0\) for \(1 \leq i \leq p\). This gives also \(0 = \delta \varphi(X_1, X_{2i+1}, X_1) = \varphi(X_{2i+1}, X_1)\) for \(1 \leq i \leq p - 1\) and \(\varphi(X_1, Y) = 0\) for any \(Y \in \mathfrak{k}_{2p}\). This implies \(0 = \delta \varphi(X_1, X_{2i}, X_j) = \varphi(X_{2i+1}, X_j)\) for \(1 \leq i \leq p - 1, 1 \leq j \leq 2p\). Thus this cocycle \(\varphi\) satisfies

\[
\varphi(X_{2i+1}, Y) = 0
\]

for any \(Y \in \mathfrak{k}_{2p}\) and for any \(1 \leq i \leq p - 1\).

We have also \(0 = \delta \varphi(X_{2i}, X_{2j}, X_1) = [\varphi(X_{2i}, X_{2j}), X_1]\) and \(\varphi(X_{2i}, X_{2j}) = \sum_{k=1}^{p-1} a_{2i,2j}^{2k+1}X_{2k+1} + a_{2i,2j}^{2p}X_{2p}\). But

\[
\begin{cases}
\delta f(X_2, X_{2i}) = a_{12}X_{2i+1} - a_{1,2i}X_3, & i = 1, \ldots, p - 1, \\
\delta f(X_2, X_{2p}) = -a_{1,2p}X_3
\end{cases}
\]
then we can assume that \( \varphi \) satisfies

\[
\begin{align*}
\varphi(X_1, X_i) &= 0, \varphi(X_{2j+1}, X_k) = 0, i = 1, \cdots, p, j = 1, \cdots, p - 1, k = 1, \cdots, 2p, \\
\varphi(X_2, X_4) &= \sum_{k=3}^{p-1} a_{2,4}^{2k+1} X_{2k+1} + a_{2,4}^{2p} X_{2p}, \\
\varphi(X_2, X_{2i}) &= \sum_{k=2}^{p-1} a_{2,2i}^{2k+1} X_{2k+1} + a_{2,2i}^{2p} X_{2p}, \quad i = 3, \cdots, p \\
\varphi(X_{2i}, X_{2j}) &= \sum_{k=1}^{p-1} a_{2i,2j}^{2k+1} X_{2k+1} + a_{2i,2j}^{2p} X_{2p}, \quad 2 \leq i < j \leq p.
\end{align*}
\]

By hypothesis the deformation \( \mu_0 + t\varphi \) is 2-step nilpotent. This is equivalent to say that the cocycle \( \varphi \) satisfies \( \varphi(\varphi(X, Y), Z) = 0 \) for any \( X, Y, Z \). This gives

\[
a_{2i,2j}^{2p} \varphi(X_{2k}, X_{2p}) = 0
\]

for any \( 1 \leq i, j, k \leq p \). In particular \( a_{2i,2p}^{2p} \varphi(X_{2i}, X_{2p}) = 0 \) which implies that \( a_{2i,2p}^{2p} = 0 \) for any \( 1 \leq i \leq p \). If there exists \((i, j)\) with \( a_{2i,2j}^{2p} \neq 0 \) then \( \varphi(X_{2k}, X_{2p}) = 0 \) for any \( 1 \leq k \leq p \). In this case the Lie algebra \( \mathfrak{g} \) is defined by the cocycle

\[
\begin{align*}
\varphi(X_1, X_i) &= 0, \varphi(X_{2j+1}, X_k) = 0, i = 1, \cdots, p, j = 1, \cdots, p - 1, k = 1, \cdots, 2p, \\
\varphi(X_2, X_4) &= \sum_{k=3}^{p-1} a_{2,4}^{2k+1} X_{2k+1} + a_{2,4}^{2p} X_{2p}, \\
\varphi(X_2, X_{2i}) &= \sum_{k=2}^{p-1} a_{2,2i}^{2k+1} X_{2k+1} + a_{2,2i}^{2p} X_{2p}, \quad i = 3, \cdots, p - 1 \\
\varphi(X_{2i}, X_{2j}) &= \sum_{k=1}^{p-1} a_{2i,2j}^{2k+1} X_{2k+1} + a_{2i,2j}^{2p} X_{2p}, \quad 2 \leq i < j \leq p - 1.
\end{align*}
\]

If \( a_{2i,2j}^{2p} = 0 \) for any \( 1 \leq i, j \leq p \) then \( \mathfrak{g} \) is defined by the cocycle

\[
\begin{align*}
\varphi(X_1, X_i) &= 0, \varphi(X_{2j+1}, X_k) = 0, i = 1, \cdots, p, j = 1, \cdots, p - 1, k = 1, \cdots, 2p, \\
\varphi(X_2, X_4) &= \sum_{k=3}^{p-1} a_{2,4}^{2k+1} X_{2k+1}, \\
\varphi(X_2, X_{2i}) &= \sum_{k=2}^{p-1} a_{2,2i}^{2k+1} X_{2k+1}, \quad i = 3, \cdots, p \\
\varphi(X_{2i}, X_{2j}) &= \sum_{k=1}^{p-1} a_{2i,2j}^{2k+1} X_{2k+1}, \quad 2 \leq i < j \leq p.
\end{align*}
\]

In particular these Lie algebras have been classified in dimension 8 in [22].
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