CHARGED BLACK HOLES AND UNUSUAL WORMHOLES IN SCALAR-TENSOR GRAVITY

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We consider static, spherically symmetric, electrically or/and magnetically charged configurations of a minimally coupled scalar field with an arbitrary potential \( V(\phi) \) in general relativity. Using the inverse problem method, we obtain a four-parameter family of asymptotically dS, flat and AdS solutions, including those with naked singularities and both extreme and non-extreme black-hole (BH) solutions. The parameters are identified as the asymptotic cosmological constant, an arbitrary length scale, mass and charge. In all asymptotically flat BH solutions, the potential \( V(\phi) \) is partly negative, in accord with Bekenstein and Mayo’s no-hair theorem. The well-known conformal mapping extends the BH solutions to Jordan’s pictures of a general class of scalar-tensor theories (STT) of gravity under the condition that the nonminimal coupling function \( f(\phi) \) is everywhere positive. Relaxing the latter condition and assuming \( f = 0 \) at some value of \( \phi \), we obtain wormhole solutions in a particular subclass of STT. In such solutions, the double horizon of an extreme BH in Einstein’s picture maps into the second spatial infinity in Jordan’s. However, at this second infinity, the effective gravitational coupling infinitely grows.

1. Introduction

Black holes and wormholes are strong field configurations in which curvature manifests itself in the global properties of space-time. While black holes (BHs) are believed to inevitably result from gravitational collapse of sufficiently heavy bodies and are for many years an object (or at least a goal) of many astrophysical observations, wormholes have only recently appeared in the focus of active discussion. A reason is that (traversable, Lorentzian) wormholes need for their existence (at least in the framework of general relativity) some unusual matter able to violate the null energy condition \( T_{\mu\nu}w^\mu w^\nu \geq 0 \) where \( T_{\mu\nu} \) is the stress-energy tensor and \( w^\mu \) is any null vector.

Now, with the discovery of the accelerated expansion of the Universe, its hypothetic source, the so-called dark energy (DE) seems to be a more or less plausible material for wormhole construction. More precisely, the null energy condition is violated if the pressure-to-density ratio \( w = p/\varepsilon < -1 \). By modern observations, \( w \) for DE belongs to a range including \(-1\), but values a little smaller than \(-1\) seem to be the most favourable [1]. Various models of DE, their theoretical and observational properties are described in the review [2].

Wormholes are imagined as regular bridges, or tunnels, connecting large or infinite regions of space-time, belonging to the same universe or to different universes. If they can be sufficiently large and stable, they can serve as shortcuts between remote parts of the Universe or as time machines. Moreover [3], they can lead to many observable astrophysical effects. It is thus important to know if really existing kinds of matter, including DE candidates, can produce and support such objects. By now, there are quite a number of various wormhole solutions, many of them obtained with different kinds of phantom matter (e.g., phantom scalar fields with negative kinetic energy [4–6]), by virtue of macroscopic quantum effects [7,8] and in the frameworks of generalized, e.g., multidimensional theories of gravity [9,10]; see Refs. [11] for recent reviews. A search for realistic wormhole solutions with more usual and maybe more viable sources still remains topical.

Ref. [12] has studied the possible wormhole existence in a class of scalar-tensor theories (STT) of gravity, in particular, those theories which, being non-phantom by nature (i.e., with positive kinetic energy of the scalar field in the Einstein picture), are able to produce a phantom-like (\( w < -1 \)) behaviour in a certain epoch in cosmology [13]. It has been shown [12] that, even in the presence of electric or magnetic fields, if the non-minimal coupling function \( f(\Phi) \) is everywhere positive, wormhole solutions are absent, and this conclusion holds in both Einstein and Jordan pictures. It also turned out [12] that if \( f \) remains non-negative but is allowed to reach zero at some value of the scalar field \( \Phi \), wormholes in Jordan’s picture are not excluded though require a very specific kind of STT and severe fine tuning.

In this paper, we give explicit analytic examples of such static, spherically symmetric wormhole solutions.
To do that, it proves necessary to build an extreme BH solution with a scalar field in Einstein’s picture. The latter is impossible if the scalar field is the only source of gravity, as follows from the global structure theorem [14] for scalar fields with arbitrary potentials in general relativity, but can exist if a radial electric or magnetic field is present. To find them, in Sec. 2, we consider static, spherically symmetric, minimally coupled scalar (with an arbitrary potential $V(\phi)$) and electromagnetic fields in general relativity. Using the inverse problem method, we obtain a family of solutions, including those with naked singularities and both extreme and non-extreme charged BH solutions. They seem to be the first explicit examples of charged asymptotically flat BH solutions with scalar and electromagnetic fields interacting only via gravity\(^2\). In the BH solutions found here, the field potential $V(\phi)$ is partly negative, in accord with the no-hair theorem [18] stating that charged asymptotically flat BHs cannot possess scalar fields with non-negative potential via gravity. In the BH solutions found here, the potential $V(\phi)$ is partly negative, in accord with the no-hair theorem [18] stating that charged asymptotically flat BHs cannot possess scalar fields with non-negative potentials outside their event horizons.

In Sec. 3, using the appropriate conformal mappings, we extend the BH solutions of Sec. 2 to a general class of STT under the condition that $f(\phi) > 0$ everywhere. Then we obtain and briefly discuss wormhole solutions in some particular, specially designed STT in which $f = 0$ at some value of $\phi$. In these solutions, the double horizon of an extreme BH in Einstein’s picture maps into the second spatial infinity (another mouth of the wormhole) in Jordan’s.

2. Charged BHs in Einstein’s picture

Consider general relativity with minimally coupled scalar ($\phi$) and electromagnetic ($F_{\mu\nu}$) fields as sources. The Lagrangian is\(^3\)

$$L_E = \frac{1}{2} [R + g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) - F_{\mu\nu} F^{\mu\nu}],$$

(1)

where $R$ is the Ricci scalar and $V(\phi)$ is the scalar field potential.

Let us assume static spherical symmetry, considering the theory (1) in a space-time with the metric

$$ds_E^2 = A(\rho) dt^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho) d\Omega^2,$$

(2)

Maxwell fields compatible with spherical symmetry are radial electric ($F_{01} F^{10} = q_e^2 / r^4$) and magnetic ($F_{23} F^{23} = q_m^2 / r^4$) fields, where the constants $q_e$ and $q_m$ are the electric and magnetic charges, respectively.

The scalar field equation and three independent combinations of the Einstein equations read

$$A(r^2 \phi')' = r^2 dV/d\phi,$$

$$A(r^2)' = - 2r^2 V + 2q^2 / r^2;$$

$$2r'' / r = - \phi^2;$$

$$A(r^2)^{\prime\prime} - r^2 A'' = 2 - 4q^2 / r^2,$$

(3)-(6)

where the prime denotes $d/d\rho$ and $q^2 = q_e^2 + q_m^2$. Eq. (3) follows from (4)–(6), which, given a potential $V(\phi)$, form a determined set of equations for the unknowns $r(\rho)$, $A(\rho)$, $\phi(\rho)$. The scalar field is normal, with positive kinetic energy, whence, due to (5), we have $r'' \leq 0$, which forbids wormhole throats (minima of $r(\rho)$), to say nothing of wormholes as global configurations.

To obtain examples of BH solutions, we will use the inverse problem method. Namely, we choose the function $r(\rho)$ and then, consecutively, find $A(\rho)$ from Eq. (6), $\phi(\rho)$ from (5) and $V(\rho)$ from (4). If $r'' < 0$ everywhere, the function $\phi(\rho)$ is strictly monotonic, and therefore the potential $V(\phi)$ obtained from $\phi(\rho)$ and $V(\rho)$ is well-defined.

So, we make the simple choice

$$r(\rho) = (\rho^2 - b^2)^{1/2},$$

(7)

where $b$ is an arbitrary constant. Then the field equations give:

$$A(\rho) = B_0 \rho^2 + 1 + 3m \left\{ - \rho + \frac{r^2}{2b^2} \ln \frac{\rho + b}{\rho - b} \right\}$$

$$- \frac{q^2}{b^4} \left[ b^2 - b \ln \frac{\rho + b}{\rho - b} + \frac{r^2}{4} \ln^2 \frac{\rho + b}{\rho - b} \right],$$

(8)

$$\phi(\rho) = \phi_0 + \frac{\sqrt{7}}{2} \ln \frac{\rho + b}{\rho - b},$$

(9)

$$V(\rho) = - \frac{B_0}{\rho} \left( 3 \rho^2 - b^2 \right) + \frac{9m \rho}{b^2} \rho^2 + \frac{q^2 (3 \rho^2 - 2 b^2)}{2 b^4 \rho^4}$$

$$- \frac{3m (3 \rho^2 - b^2)}{2 b^4 \rho^2} \ln \frac{\rho + b}{\rho - b}$$

$$+ \frac{q^2 (3 \rho^2 - b^2)}{4 b^4 \rho^2} \ln^2 \frac{\rho + b}{\rho - b},$$

(10)

where $B_0$, $m$ and $\phi_0$ are integration constants. As $\rho \to \infty$, the metric becomes flat or (anti-)de Sitter according to the value of $B_0 = - V(\infty) / 3$, where $V(\infty)$ plays the role of a cosmological constant at large $\rho$.

In what follows, to have a flat asymptotic as $\rho \to \infty$, we put $B_0 = 0$ and also, without loss of generality, take $\phi_0 = 0$. Then, at large $\rho$, $r \approx \rho$ and

$$A \approx 1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2} + \ldots,$$

(11)

so that the metric is approximately Reissner-Nordström, and $m$ is the Schwarzschild mass (in units of length).
The other extreme is \( \rho \to b \ (r \to 0 \), the singularity), where
\[
\begin{align*}
    r &\approx \sqrt{2b(\rho - b)}, & A &\approx \frac{q^2}{b^2} \ln \frac{2b}{\rho - b} \to \infty, \\
    V &\approx \frac{q^2}{r^4} \to \infty.
\end{align*}
\]
In the scalar-vacuum case \( q = 0 \), we recover the solution obtained in Ref. [19], where examples of BHs with a scalar field and a Schwarzschild-like global structure were obtained. Note that in this solution, though \( A \) tends to a finite limit as \( \rho \to b \), it is a singularity where the potential \( V \) is infinite:
\[
A \to 1 - 3m/b, \quad V \approx \frac{3m}{b r^2} \ln \frac{b - \rho}{2b} \to -\infty. \quad (13)
\]

Possible horizons are described by zeros of the function \( A(\rho) \), and for \( q \neq 0 \), just as in the Reissner-Nordström solution, the number of horizons may be 0, 1 (a double horizon) and 2, which correspond to configurations with a naked singularity, extreme BHs and non-extreme BHs, respectively.

The values of \( m \) and \( q \) corresponding to a double horizon \( \rho = \rho_h \), near which \( A(\rho) \sim (\rho - \rho_h)^2 \), can be found numerically. For this calculation let us introduce the dimensionless quantities
\[
\begin{align*}
    x &= \frac{\rho}{b}, & \tilde{r}(x) &= \frac{r}{b}, \\
    \tilde{m} &= \frac{m}{b}, & \tilde{q} &= \frac{q}{b}, & \tilde{V} &= Vb^2. \quad (14)
\end{align*}
\]
In what follows, we will use these quantities omitting the bars. Thus all dimensionful quantities are now measured in units equal to appropriate powers of the arbitrary length \( b \) (recall that due to \( c = \hbar = 1 \) all dimensions are represented as powers of length).

Then, for some values of \( m \), the charges \( q = q_2(m) \) providing a double horizon and the corresponding coordinate values \( x = x_h = \rho_h/b \) are as follows:
\[
\begin{align*}
    m = 0.35, & \quad q_2 \approx 0.08096, & x_h \approx 1.00116; \\
    m = 1, & \quad q_2 \approx 0.9222, & x_h \approx 1.324; \\
    m = 3, & \quad q_2 \approx 2.97409, & x_h \approx 3.1171; \quad (15) \\
    m = 10, & \quad q_2 \approx 9.9922, & x_h \approx 10.036; \\
    m = 40, & \quad q_2 \approx 39.9981, & x_h \approx 40.01.
\end{align*}
\]
Evidently, as \( m \) and \( q \) grow, these parameters more and more approach those of the Reissner-Nordström solution. And, in full similarity with the latter, given the mass \( m \), for \( q > q_2(m) \) the system has no horizon while for \( q < q_2(m) \) it possesses two simple horizons.

Figs. 1 and 2 show the behaviour of the functions \( A(\rho) \) and \( V(\rho) \) for some values of the solution parameters. Fig. 2 indicates that, as \( q \) grows, \( V(\phi) \) also grows, at least in the range of \( x \) shown. This may seem to question the validity of the no-hair theorem [18] that asserts the non-existence of charged BH solutions with \( V(\phi) \geq 0 \) for a class of theories including ours as a special case. An inspection shows that our BH solutions respect the theorem: at larger \( x \) there is always a range of \( x \) with negative (though tiny) values of \( V \), see an example in Fig. 3.

### 3. Charged black holes and wormholes in Jordan’s picture

The Lagrangian of a general STT of gravity in a Jordan-frame manifold \( M_J \) with the metric \( g_{\mu\nu} \) and the Maxwell field \( F_{\mu\nu} \) may be written as
\[
2L = f(\Phi)\bar{R} + h(\Phi)\bar{g}^{\mu\nu}F_{\mu\nu} - 2U(\Phi) - F^{\mu\nu}F_{\mu\nu}, \quad (16)
\]
where \( \bar{R} \) is the Ricci scalar corresponding to \( \bar{g}_{\mu\nu} \), while \( f(\Phi) \) (the nonminimal coupling function), \( h(\Phi) \) and

![Figure 1: The metric function \( A(x) \) in the solutions (7)–(10). The curves correspond to the following parameter values: 1 — \( m = 1, q = 0 \) (scalar-vacuum solution); 2 — \( m = 1, q = 0.7 \) (with two horizons); 3 — \( m = 1, q = 0.9222 \) (with a double horizon); 4 — \( m = 1, q = 1.3 \) (without a horizon); 5 — \( m = 3, q = 2.97409 \) (with a double horizon).](image1)

![Figure 2: The potential \( V(x) \) in the solutions (7)–(10). The curve labels refer to the same parameter values as in Fig. 1.](image2)
The Brans-Dicke theory is then the special case \( \omega = 0 \). The potential \( U(\Phi) \) are arbitrary functions, changing from one particular STT to another. Suppose (as is usually done)

\[
f \geq 0, \quad l(\Phi) := fh + \frac{3}{2} \left( \frac{df}{d\Phi} \right)^2 \geq 0, \tag{17}
\]
then the well-known conformal mapping

\[
\tilde{g}_{\mu\nu} = f(\Phi)g_{\mu\nu}, \tag{18}
\]
from \( \mathcal{M}_J \) to another manifold \( \mathcal{M}_E \) with the metric \( g_{\mu\nu} \) converts the theory to another formulation called the Einstein frame (or picture). The Lagrangian (16) is transformed (up to a full divergence) to (1), where the scalars \( \Phi \) and \( \phi \) and the potentials \( U \) and \( V \) are connected by the relations

\[
\frac{d\phi}{d\Phi} = \frac{\sqrt{l(\Phi)}}{f(\Phi)}, \quad V(\phi) = f^{-2}U(\Phi). \tag{19}
\]

The Lagrangian (16) admits reparametrization of the field \( \Phi \), and without loss of generality we may use the so-called Brans-Dicke parametrization in which

\[
f(\Phi) = \Phi, \quad h(\Phi) = \frac{\omega(\Phi)}{\Phi}, \quad l(\Phi) = \omega(\Phi) + \frac{3}{2}, \tag{20}
\]
(The Brans-Dicke theory is then the special case \( \omega = \text{const.} \).

Now, any solution to the field equations due to (1), including the BH solutions of Sec. 2, may be treated as solutions of an arbitrary STT (16) in its Einstein picture. If the conformal factor \( f \) is everywhere positive, then spatial infinity in one picture maps to spatial infinity in another, a horizon maps to a horizon of the same order, and a centre to a centre. Therefore our BH solutions have counterparts in Jordan pictures of all such STT. A good example is the Brans-Dicke theory with \( \omega > -3/2 \), for which the conformal factor is

\[
f = \Phi = \exp \left( \frac{\phi}{\sqrt{\omega + 3/2}} \right). \tag{21}
\]

Of interest for us is also the case of a possible conversion of an extreme BH in Einstein’s picture to a traversable wormhole in Jordan’s. As shown in Ref. [12], this is only possible for such STT that, in the parametrization (20), \( l(\Phi) \sim \Phi \) where \( \Phi \rightarrow 0 \) corresponds to a double horizon in \( \mathcal{M}_E \) and to the second spatial infinity in \( \mathcal{M}_J \). Evidently, in all such cases one must have \( f(\Phi) \sim (x - x_h)^2 \) as \( x \rightarrow x_h \) to obtain a finite value of \( \tilde{g}_{tt} \).

Let us give an example using the extreme BH from Sec. 2 and assuming the conformal factor \( f \) depending on the coordinate \( x \) as

\[
f(\Phi) = \Phi = (1 - x_h/x)^2. \tag{22}
\]

Flat spatial infinity at \( x = \infty \) is then provided in both manifolds \( \mathcal{M}_E \) and \( \mathcal{M}_J \). The metric in \( \mathcal{M}_J \) is now

\[
ds_J^2 = \frac{x^2}{(x - x_h)^2} \left[ A(x)dt^2 - \frac{b^2ds^2}{A(x)} - b^2x^2d\Omega^2 \right], \tag{23}
\]
where \( r^2 = x^2 - 1 \); \( A(x) \) and \( V(x) \) are given by (8) and (10) with the following substitutions according to our asymptotic flatness assumption and transition to dimensionless quantities:

\[
B_0 = 0, \quad b \rightarrow 1, \quad \rho \rightarrow x.
\]

The STT itself is characterized, in the Brans-Dicke parametrization, by the functions

\[
\omega(\Phi) = \frac{3}{2}, \quad U(\Phi) = \frac{(x - x_h)^4}{x^4}V(x), \tag{24}
\]
where \( x = \coth(\phi/\sqrt{2}) = x_h/(1 - \sqrt{\Phi}) \).

In \( \mathcal{M}_J \) the solution is defined between \( x = \infty \) (spatial infinity) and \( x = x_h > 1 \) (another spatial infinity) and describes a wormhole, the manifold \( \mathcal{M}_J \) being geodesically complete. In \( \mathcal{M}_E \), the solution is defined in a larger range of \( x \), \( 1 < x < \infty \) where \( x = 1 \) is the central singularity. Thus, according to the definition of Refs. [20], we have a conformal continuation from \( \mathcal{M}_J \) to \( \mathcal{M}_E \), and the transition surface \( S_{\text{trans}} \) in \( \mathcal{M}_E \), i.e., the double horizon, corresponds to the second spatial infinity in \( \mathcal{M}_J \). It is a novel feature since in the previous examples of conformal continuations [20,21] the regular transition surfaces \( S_{\text{trans}} \) were obtained by conformal mappings from different kinds of singularities.

It should be noted that the second spatial infinity in \( \mathcal{M}_J \), \( x = x_h \), is generally non-flat but rather possesses a solid angle excess or deficit, analogously to the well-known feature of global monopoles [22]. Indeed, near \( x = x_h \), we have \( A(x) \approx A_2(x - x_h)^2 \), where \( A_2 = A''(x_h) \) (the prime denotes \( d/dx \)). The solid angle deficit \( \mu \) (or excess if \( \mu < 0 \)) in \( \mathcal{M}_J \) at \( x = x_h \) is determined from the relation

\[
1 - \mu = \frac{1}{2} b^2 A_2(x_h^2 - 1), \tag{25}
\]
where $\tilde{r} = \sqrt{-g_{rr}}$ is the radius of a coordinate sphere $t = \text{const}$, $x = \text{const}$ in $M_{1}$. Numerical estimates for the same parameter values as in (15) give

\[
\begin{align*}
&m = 1, \quad A_2 \approx 3.344, \quad 1 - \mu \approx 1.25898, \\
&m = 3, \quad A_2 \approx 0.234791, \quad 1 - \mu \approx 1.02538, \\
&m = 10, \quad A_2 \approx 0.020105, \quad 1 - \mu \approx 1.00246, \quad (26) \\
&m = 40, \quad A_2 \approx 0.001250, \quad 1 - \mu \approx 0.99987.
\end{align*}
\]

(Note that, at large $m$, the solution approaches the Reissner-Nordström one in which a double horizon occurs at $r_{h} = m = q$ and a formal substitution to (25) leads to $1 - \mu = (m^2 - 1)/m^2 < 1$.) Since, by (26), $\mu > 0$ at smaller $m$ and $\mu < 0$ at larger $m$, we conclude by continuity that there is (at least one) set of parameters $(m, q)$ for which $\mu = 0$ and the wormhole is twice asymptotically flat: both $m$ and $q$ for it must be slightly smaller than 40.

This confirms the inference of Ref. [12] that the existence of such twice asymptotically flat wormholes requires severe fine tuning. Moreover, as follows from (22) and (24), at $x = x_h$ we have in $M_{1}$ an infinite effective gravitational constant $G_{\text{eff}}$ since (see, e.g., [13])

\[
G_{\text{eff}} = G \frac{\omega + 2}{\Phi \omega + 3/2} \sim \Phi^{-2} \quad \text{as} \quad x \to x_h, \quad (27)
\]

where $G$ is the initial gravitational constant. Thus such wormholes, even if they exist, cannot connect different parts of our Universe but can only be bridges to other universes (if any) with very unusual physics.

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