QUASI-PROJECTIVE VARIETIES WITH ORBIFOLD FUNDAMENTAL GROUPS

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Abstract. In this work we study smooth complex quasi-projective varieties whose fundamental group is a free product of cyclic groups. In particular, we prove the existence of a (maybe irrational) pencil from the quasi-projective variety to a Riemann surface (endowed with an orbifold structure) which induces an isomorphism of (orbifold) fundamental groups. The orbifold structure is given by the multiple fibers of the pencil. Associated with this result, we prove addition-deletion Lemmas of fibers in the pencil which explain how these operations affect the fundamental group of the quasi-projective variety. Our methods also allow us to produce curves whose fundamental group of their complement is a free product of cyclic groups, generalizing classical results on $C_{p,q}$ curves and torus type projective sextics, and showing how general this phenomenon is.

1. Introduction

This paper is devoted to the general problem of describing the topology of smooth complex quasi-projective varieties. From the point of view of first homotopy groups, using Lefschetz-type theorems it is enough to focus on complements of curves in smooth projective surfaces. A classical tool to describe this topology is the existence of morphisms onto Riemann curves. This is described in Castelnuovo-de Franchis’ theorem for the existence of morphisms onto Riemann surfaces of genus $g \geq 2$, in Arapura’s structure theorem [2], as well as its orbifold version [5]. If $X$ denotes a smooth projective surface and $D \subset X$ a reduced curve, the latter describes properties of the fundamental group from the existence of (orbifold) dominant morphisms from the quasi-projective surface $X \setminus D$ to a Riemann surface $S$. If $X$ is simply connected, then $S = P^1$ with an additional orbifold structure. Since the image of this morphism is an open Riemann surface whose higher order homotopy groups are trivial, a quotient of the fundamental group of $P^1$; the extremal case being when this morphism produces an isomorphism. In this paper we give conditions for $\pi_1(X \setminus D)$ to be an open orbifold group of $S$, namely a free product of cyclic groups. Note that the connection between the projective case ($D = \emptyset$) and (closed) orbifold groups is studied by Arapura in [3] and Catanese in [8]. The quasi-projective case where $\pi_1(X \setminus D)$ is free is also considered by Bauer [6] and Catanese in [8]. In this paper we study the general case of smooth quasi-projective varieties whose fundamental group is an open orbifold group.

The simplest example of a curve such that the fundamental group of its complement is an open orbifold group of $P^1$ is due to Zariski [28] who proved the existence of a sextic $D$ in $P^2$ satisfying $\pi_1(P^2 \setminus D) = \mathbb{Z} \ast \mathbb{Z}$. In the 70’s, Oka proved his classical result on $C_{p,q}$ curves in [23] (also see Dimca [13] Prop. §4(4.16)),

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exhibiting a family of irreducible curves
\[ C_{p,q} = \{ (x^p + y^q)^2 + (y^q + z^p)^p = 0 \} \]
for any \( p, q > 1 \) coprime such that \( \pi_1(\mathbb{P}^2 \setminus C_{p,q}) \cong \mathbb{Z}_p \ast \mathbb{Z}_q \). This was generalized in the 80’s by Némethi in [20].

One of the main results of this paper provides geometric conditions for the complement of a curve \( D \) in a smooth projective surface \( X \) to have an infinite open orbifold fundamental group. These conditions include the existence of an admissible map to a Riemann surface \( (\Sigma, \pi) \), where \( \pi \) is a curve. Suppose that \( \pi \) is a Riemann surface \( \pi \) is a curve. Let \( \text{Theorem 1.1.} \)

- **(i)** \( F \) induces an orbifold morphism
  \[ F : X \setminus D \rightarrow S_{(n+1,m)}, \]
  where \( S_{(n+1,m)} \) is maximal with respect to \( F \), \( n \geq 0 \) and \( m = (m_1, \ldots, m_s) \).
- **(ii)** \( F_* : \pi_1(X \setminus D) \rightarrow \pi_1^{\text{orb}}(S_{(n+1,m)}) \) is an isomorphism.
- **(iii)** \( D = D_f \cup D_t \), where
  - \( D_f \) is a fibered-type curve which is the union of the \( n + 1 \) fibers above the distinguished points \( \Sigma_0 \subset S \) with \( n = r - 2g_S \) and \( m \) represents the orbifold structure on \( s \) points of \( S \setminus \Sigma_0 \) (see Section 2.4 for the relevant definitions).

**Theorem 1.1.** Let \( X \) be a smooth connected projective surface, and let \( D \subset X \) be a curve. Suppose that \( \pi_1(X \setminus D) \cong \mathbb{Z}_r \ast \mathbb{Z}_m \ast \ldots \ast \mathbb{Z}_m \) is infinite. Then, there exists a Riemann surface \( S \) of genus \( g_S \) and an admissible map \( F : X \rightarrow S \) such that:

- \( F \) is a typical fiber or not. We also prove the
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In particular, the maximality condition of part (ii) implies that \( F : X \setminus D \rightarrow S \) is a surjective algebraic map with exactly \( s \) multiple fibers of multiplicities \( (m_1, \ldots, m_s) \), determined by the torsion of \( \pi_1(X \setminus D) \). Theorem 1.1 is proved in Section 6.

Theorem 1.1 is extended in Section 6.2 to the case where \( \pi_1(X \setminus D) \) is finite as long as \( X \) is simply-connected, under some extra assumptions that are always satisfied if \( X = \mathbb{P}^2 \). As a result, we prove a refinement of Theorem 1.1 for the case \( X = \mathbb{P}^2 \) in Section 6.3 (Corollary 6.12) and show that in this case \( \pi_1(X \setminus D) \cong \mathbb{Z}_r \ast \mathbb{Z}_m \ast \mathbb{Z}_q \) for some \( p, q \in \mathbb{Z}_{\geq 1} \) coprime and \( r \in \mathbb{Z}_{\geq 0} \), and that \( D = D_f \) is a union of unordered fibers of a pencil \( F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \). The case \( X = \mathbb{P}^2 \) was in fact the original motivation for this paper. In particular, Corollary 6.12 has the following form in the case of curves whose complements have the same fundamental group as Oka’s \( C_{p,q} \) curves.

**Corollary 1.2.** If a curve complement in \( \mathbb{P}^2 \) has fundamental group \( \mathbb{Z}_p \ast \mathbb{Z}_q \), with \( p, q \in \mathbb{Z}_{\geq 1} \) coprime integers, then the curve is necessarily given by a polynomial of the form \( f_p^p + f_q^q \), for some \( f_p \) and \( f_q \) homogeneous polynomials in \( \mathbb{C}[x,y,z] \) of degrees \( p \) and \( q \) with no common factors, and such that neither of them is a \( k \)-th power of another polynomial for any \( k \geq 2 \).

The second type of main results are referred to as Addition-Deletion Theorems in Section 3. Consider \( U \) a smooth quasi-projective surface and \( F : U \rightarrow S \) an admissible map to a smooth projective curve \( S \), and define \( U_B = U \setminus F^{-1}(B) \) for any finite subset \( B \subset S \). In this context, we prove a Deletion Lemma 4.9 that describes the fundamental group of \( U_B \) if \( U_{B \cup(P)} \) has an open orbifold fundamental group. This is done regardless of whether \( P \in B \) or not, for \( B \neq \) the set of atypical values of \( F \), that is, whether \( F^{-1}(P) \) is a typical fiber or not. We also prove the following General Addition-Deletion Lemma in Section 4.

**Theorem 1.3 (Generic Addition-Deletion Lemma).** Let \( U \) be a smooth quasi-projective surface and \( F : U \rightarrow S \) be an admissible map to a smooth projective
curve $S$. Assume $B \subset S$, where $\#B = n \geq 1$, and let $P \in S \setminus (B_F \cup B)$. Consider $S_{(n+1,m)}$ (resp. $S_{(n,m)}$) the maximal orbifold structure of $S$ with respect to $F : U_{B \cup \{P\}} \to S \setminus (B \cup \{P\})$ (resp. $F : U_B \to S \setminus B$).

Then the following are equivalent:

- $F : \pi_1(U_B) \to \pi_1^{orb}(S_{(n,m)})$ is an isomorphism,
- $F : \pi_1(U_{B \cup \{P\}}) \to \pi_1^{orb}(S_{(n+1,m)})$ is an isomorphism.

Moreover, in that case,

$$\pi_1(U_{B \cup \{P\}}) \cong \mathbb{Z} \ast \pi_1(U_B).$$

We finally devote Section 5 to a number of applications of these results to the calculation of fundamental groups of complements of projective curves. In particular, in Section 5.1 we prove Theorem 1.4, which generalizes the aforementioned result on $C_{p,q}$-curves in several directions. First $(x^p + y^q)$ (resp. $(y^q + z^q)$) in 11 are replaced by any forms $f_p$ and $f_q$ of degrees $p$ and $q$ with no common factors and such that neither of them are a $k$-th power of another homogeneous polynomial for any $k \geq 2$. Also $C_{p,q}$ is allowed to be a union of $r + 1$ generic members in the pencil $\alpha f_p^\beta + \eta f_q^\gamma$, in which case $\pi_1(\mathbb{P}^2 \setminus C_{p,q}) \cong \mathbb{F}_r \ast \mathbb{Z}_p \ast \mathbb{Z}_q$. Finally, to our knowledge, the moreover part of this theorem is completely new in the literature. The key ingredients in the proof of this theorem are the Addition-Deletion results, which allow us to avoid any Zariski-Van Kampen calculations.

**Theorem 1.4.** Let $f_p$ (resp. $f_q$) be a homogeneous polynomial of degree $p$ (resp. $q$) with $\gcd(p, q) = 1$ such that $f_p$ (resp. $f_q$) is not an $n$-th power of a polynomial for any $n \geq 2$. Let $C_0, \ldots, C_r$ be $r + 1$ distinct generic fibers of $F = [f_p : f_q^\gamma]$, with $r \geq 0$. Let $C = \bigcup_{i=0}^r C_i$. Assume that

- $f_p$ and $f_q$ do not have any non-constant common factors, and
- The multiple fibers of $F$ lie over a subset of $\{[0 : 1], [1 : 0]\}$ (this always holds if $p, q \geq 2$).

Then,

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{F}_r \ast \mathbb{Z}_p \ast \mathbb{Z}_q.$$

Moreover, assume that $V(f_p)$ is irreducible and $\pi_1(\mathbb{P}^2 \setminus V(f_p)) \cong \mathbb{Z}_p$, then

$$\pi_1(\mathbb{P}^2 \setminus (C \cup V(f_p))) = \mathbb{F}_{r+1} \ast \mathbb{Z}_p.$$

Theorem 1.4 brings together several examples known in the literature. For instance, we apply it in Section 5.2 to provide a Zariski-Van Kampen-free proof of a result on the fundamental group of a union of lines and conics due to Amram-Teicher 1. Thm. 2.2, 2.5. As a last application, in 5.3 we generalize a classical result due to Oka-Pho in 25 on fundamental groups of maximal tame torus-type sextics.

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2. Preliminaries

For the purpose of completeness and to fix notation, a short exposition on the key topics of this paper will be given in this section.

2.1. **Admissible maps on $X$.** Following Arapura 2, we say that a surjective morphism $F : U \to S'$ from a smooth quasi-projective surface $U$ to a smooth quasi-projective curve $S'$ is admissible if it admits a surjective holomorphic enlargement $F : \bar{U} \to S$ with connected fibers, where $\bar{U}$ and $S$ are smooth compactifications of $U$ and $S'$ respectively.
If $U$ is a dense Zariski subset of a smooth projective surface $X$, then $F$ defines a rational map $F : X \dashrightarrow S$ which can be extended to an admissible map $F : X \setminus B \to S$ on a maximal open set $X \setminus B$, where $B$ is the finite subset of base points. Note that $F$ must be surjective; otherwise $S \setminus F(X \setminus B)$ is a finite set of points. The preimage of one of such points $P$ by its enlargement $F$ (given by blowing up the base points $B$) must be contained in the exceptional locus, which contradicts $(F^{-1}(P))^2 = 0$.

For convenience, this rational map will also be referred to as admissible.

Note that any fiber $F^{-1}(P)$ of $F : X \setminus B \to S$ is such that $B \subset F^{-1}(P) \subset X$. Also, since the indeterminacies of $F : X \setminus B \to S$ can be resolved by blow-ups which produce dicritical exceptional divisors, and such divisors cannot be mapped onto a smooth curve of positive genus, it follows that $B = \emptyset$ unless $S = \mathbb{P}^1$.

**Remark 2.1.** An admissible map $F : X \dashrightarrow S$ has connected fibers and hence it induces an epimorphism $F_\ast : \pi_1(X \setminus B) = \pi_1(X) \to \pi_1(S)$. In particular, $H_1(X; \mathbb{C}) = 0$ implies $S = \mathbb{P}^1$.

Given $P \in C$, the fiber $F^{-1}(P) \subset U$ defines an algebraic curve $C_P$. Assume $B \subset C$ is a finite set, we will denote $C_B = \cup_{P \in B} C_P$. Any such curve will be referred to as a fibered-type curve. It is well known that the minimal set of values $B_F$ for which $F : U \setminus C_B \to S' \setminus B_F$ is a locally trivial fibration is finite [27].

The points in $B_F$ are called atypical values of $F : U \to S'$. We will distinguish between $F^\ast(P)$ as the pulled-back divisor and $C_P$ as its reduced structure. Using this notation, one can describe the set of multiple fibers as

$$(2) \quad M_F = \{ P \in S' \mid F^\ast(P) = mD, m > 1 \text{, for some effective divisor } D \} \subset B_F.$$

Note that in general, the effective divisor $D$ in (2) need not be reduced. If $P \in S$ the multiplicity of $F^\ast(P)$ is defined as $m \geq 1$ if $F^\ast(P) = mD$ for some $D$ and whenever $F^\ast(P) = m'D'$, then $m' \leq m$.

**Remark 2.2.** If $X$ is a simply-connected surface and $F : X \dashrightarrow S$ is an admissible map, then $S = \mathbb{P}^1$ by Remark 2.1 and an analogous argument to the one given in the proof of [14, Prop. 2.8] shows that the number of multiple fibers of $F$ cannot exceed two.

From now on, we will use the following notation.

**Notation 2.3.** Let $F : U \to S'$ be an admissible map from a smooth quasi-projective surface $U$ to a smooth quasi-projective curve $S'$, and let $B \subset S'$ be a finite set. We denote by $U_B := U \setminus C_B$. Analogously, if $F : X \dashrightarrow S$ is an admissible rational map from a smooth projective surface $X$ to a smooth projective curve $S$ and $B \neq \emptyset$, one defines $X_B$ as $U_B$ for $U = X \setminus B$. Note that $X_B = X \setminus (\cup_{P \in B} F^{-1}(P))$.

### 2.2. Fundamental groups of quasi-projective varieties. Meridians.

Let $X$ be a quasi-projective variety and let $D = \cup_{i \in I} D_i$ be a curve in $X$, where $D_i$ are its irreducible components. For simplicity, we will also assume $X$ is simply-connected. When studying $\pi_1(X \setminus D, p)$ one has the following generating homotopy class of loops:

Take a regular point $p_i$ on $D_i$ and consider a disk $D_i \subset X$ transversal to $D_i$ at $p_i$ and such that $D_i \cap D = \{ p_i \}$. Let $\tilde{p}_i \in \partial D_i$ and consider $\tilde{\gamma}_i$ a loop based at $\tilde{p}_i$ around $\partial D_i$ travelled in the positive orientation. Define $\delta_i : [0, 1] \to X \setminus D$ a path in $X \setminus D$ starting at the base point $\delta_i(0) = p$ and ending at $\delta_i(1) = \tilde{p}_i$. Denote by $\tilde{\delta}_i$ the reversed path defined as usual as $\tilde{\delta}_i(t) := \delta_i(1 - t)$, $t \in [0, 1]$ starting at $p_i$ and ending at $p$. The following loop $\gamma_i := \delta_i \ast \tilde{\gamma}_i \ast \tilde{\delta}_i$ is based at $p$ and defines a homotopy class called a meridian around $D_i$. The following two results are well known.
Lemma 2.4. Let \( \gamma \) be a meridian around \( D_i \). A homotopy class \( \gamma' \) is a meridian around \( D_i \) if and only if \( \gamma' \) is in the conjugacy class of \( \gamma \) in \( \pi_1(X \setminus D) \).

Proof. See [22], also [10] Prop. 1.34] for a proof. \( \square \)

Lemma 2.5. Consider \( X_i := X \setminus (\cup_{j \in \Pi \setminus \{i\}} D_j) \) and the map \( (j_*) : \pi_1(X \setminus D) \to \pi_1(X_i) \) induced by the inclusion \( X \setminus D \to X_i \). Then \((j_*)_o\) is surjective, and \( \ker((j_*)_o) \) is the normal closure of any meridian \( \gamma_i \) in \( \pi_1(X \setminus D) \).

In particular, if \( X \) is simply connected, then the set \( \{ \gamma_i \}_{i \in I} \) normally generates \( \pi_1(X \setminus D) \).

Proof. See [22]. Also, as a consequence of [26 Lemma 2.3]. \( \square \)

2.3. Homology of the complement. Consider \( X \) a smooth projective surface and let \( D = D_0 \cup \cdots \cup D_r \subset X \) be the decomposition of a curve \( D \) into its irreducible components. The homology exact sequence of \((X, X \setminus D)\) results in

\[
H_2(X; \mathbb{Z}) \to H_2(X, X \setminus D; \mathbb{Z}) \to H_1(X \setminus D; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \to 0.
\]

Using excision and Lefschetz duality one has

\[
H_2(X, X \setminus D; \mathbb{Z}) \cong H_2(TD, \partial TD; \mathbb{Z}) \cong H^2(TD; \mathbb{Z}) \cong H^2(D; \mathbb{Z}),
\]

where \( TD \) is a regular neighborhood of \( D \). \( H^2(D; \mathbb{Z}) \cong \mathbb{Z}^{r+1} \) generated by the cohomology classes of each \( D_i \) irreducible component of \( D \). Hence, if \( H_1(X; \mathbb{Z}) = 0 \) (resp. if \( H_1(X; \mathbb{Q}) = 0 \)), the previous exact sequence becomes

\[
H_2(X; \mathbb{Z}) \xrightarrow{j} \mathbb{Z}^{r+1} \to H_1(X \setminus D; \mathbb{Z}) \to 0
\]

(resp. with \( \mathbb{Q} \)-coefficients), where \( j(C) = \sum_{i=0}^{r} (C, D_i)_X D_i \) (see for instance [7]). In particular,

\[
H_1(X \setminus D; \mathbb{Z}) = \mathbb{Z}^{r+1} / \text{Im} \ j \quad \text{(resp. } H_1(X \setminus D; \mathbb{Q}) = \mathbb{Q}^{r+1} / \text{Im} \ j).\]

Example 2.6. If \( X = \mathbb{P}^2 \), then \( \text{Im} \ j \) is generated by \( j(\ell) = \sum_{i=0}^{r} d_i D_i \), where \( \ell \) is a line in \( \mathbb{P}^2 \) and \( d_i \) is the degree of the component \( D_i \) of \( D \). Thus

\[
H_1(X \setminus D; \mathbb{Z}) \cong \mathbb{Z}^{r+1} / \mathbb{Z} d, \quad \text{where } d = \gcd(d_i).
\]

Example 2.7. If \( X \) is a hypersurface in \( \mathbb{P}^3 \), then \( X \) can be seen as a hyperplane section by the Veronese embedding. By the Lefschetz hyperplane Theorem, \( X \) is simply connected. For instance, if \( X \) is a smooth cubic surface and \( D \) is a union of \( r+1 \) generic hyperplane sections, then \( \text{Im} \ j \) is generated by \( j(\ell) = 3D, \) where \( \ell \) is a line in \( X \), thus

\[
H_1(X \setminus D; \mathbb{Z}) \cong \mathbb{Z}^{r+1} / \mathbb{Z} 3.
\]

Example 2.8. If \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) a quadric surface, then \( \text{Im} \ j \) is generated by \( j(\ell_1) = \sum_{i=0}^{r} d_1 D_i \) and \( j(\ell_2) = \sum_{i=0}^{r} d_2 D_i \), where \( (d_1, d_2) \) is the bidegree of the component \( D_i \) of \( D \). Thus

\[
H_1(X \setminus D; \mathbb{Z}) \cong \mathbb{Z}^{r+1} \times \mathbb{Z} d_1 \times \mathbb{Z} d_2,
\]

where \( d_i \) is the \( i \)-th invariant of the \( 2 \times (r+1) \) representation matrix of \( \text{Coker} \ j \), with the convention that \( \mathbb{Z} 0 = \mathbb{Z} \).

The following condition on the irreducible components of a curve allows for a particularly simple description of the homology of \( X \setminus D \). In this paper, whenever we refer to a divisor as a curve, the divisor is meant to have a reduced structure.

Condition 2.9. The curve \( D \) decomposes into irreducible components as \( D = \bigcup_{i=0}^{r} D_i \), and the irreducible components are such that,

\[
m_i D_i \equiv n_i D_j \quad \text{for some } n_0, \ldots, n_r \in \mathbb{Z}_{\geq 0},
\]

where \( \equiv \) here means numerical equivalence.

Remark 2.10. These are typical sources of examples satisfying Condition 2.9.
• For any $D$, if the surface $X$ is such that $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X) = \mathbb{Z}$.

• For any $X$, if there exist $m_i \geq 1$ such that $m_i D$ is a (multiple if $m_i > 1$) fiber of an admissible map from $X$ onto a curve.

• For any $X$, whenever $D$ is irreducible, since $\text{Im}(j) \subset \mathbb{Z} D$.

**Proposition 2.11.** If $X$ is a smooth projective surface such that $H_1(X; \mathbb{Z})$ is torsion, and $D$ is a curve satisfying Condition 2.9, then

$$H_1(X \setminus D; \mathbb{Q}) \cong \mathbb{Q}^r.$$ 

Moreover, if $H_1(X; \mathbb{Z}) = 0$, then

$$H_1(X \setminus D; \mathbb{Z}) \cong \mathbb{Z}^r \times \mathbb{Z}_d,$$

where $d \in \mathbb{Z}_{>0}$ is determined by the components $D_i$.

**Proof.** The result follows from (4) since Condition 2.9 implies that $\text{Im} j$ is a principal ideal $d \mathbb{Z}$ for some $d \in \mathbb{Z}_{>0}$ if using $\mathbb{Z}$-coefficients, or a $k$-dimensional vector space for $k = 0$ or $k = 1$ if using $\mathbb{Q}$-coefficients. Using the definition of the map $j$ in (3), we see that if $d = 0$ or $k = 0$, if and only if every component $D_i$ is numerically equivalent to 0. Since no curve in a smooth projective surface is numerically equivalent to 0, we conclude that $d > 0$ and $k = 1$. \hfill \square

**Remark 2.12.** In particular, if $X = \mathbb{P}^2$ and $D \subset \mathbb{P}^2$ is a curve with $r + 1$ irreducible components. Then,

$$H_1(X \setminus D; \mathbb{Z}) \cong \mathbb{Z}^r \times \mathbb{Z}_d,$$

where $d$ is the greatest common divisor of the degrees of the irreducible polynomials defining each of the components of $D$ (see (3) §4 Prop. (1.3)).

2.4. Orbifold fundamental groups and orbifold admissible maps. A free group can be realized as the fundamental group of the complement of a finite number of points in $\mathbb{P}^1$. A generalization of free groups in this context of quasi-projective varieties is orbifold fundamental groups. One can consider a Riemann surface $S$ of genus $g$ with an orbifold structure by choosing $\varphi : S \rightarrow \mathbb{C} \mathbb{P}^1$ such that $\varphi(P) \neq 1$ only for a finite number of points, say $\Sigma = \Sigma_0 \cup \Sigma_+ \subset S$ for which $\varphi(P) = 0$ if $P \in \Sigma_0$ and $\varphi(Q) = m_Q > 1$ if $Q \in \Sigma_+$. This structure will be denoted by $S_{(n+1, m)}$, where $n + 1 = \# \Sigma_0$, and $m$ is a $(\# \Sigma_+)$-tuple whose entries are the corresponding $m_Q$'s. The orbifold fundamental group $\pi_1^{\text{orb}}(S_{(n+1, m)})$ has the following presentation

$$\pi_1^{\text{orb}}(S_{(n+1, m)}) = \pi_1(S \setminus \Sigma)/(\mu_P, P \in \Sigma),$$

where $\mu_P$ is a meridian in $S \setminus \Sigma$ around $P \in \Sigma$. Note that $\pi_1^{\text{orb}}(S_{(n+1, m)})$ is hence generated by

$$(a_i, b_i)_{i=1, \ldots, g} \cup \{\mu_P\}_{P \in \Sigma}$$

and presented by the following relations

$$(6) \quad \mu_P^{m_P} = 1, \quad \text{for } P \in \Sigma_+, \quad \text{and} \quad \prod_{P \in \Sigma} \mu_P = \prod_{i=1, \ldots, g} [a_i, b_i].$$

In the particular case where $\Sigma_0 \neq \emptyset$, (4) and (5) show that $\pi_1^{\text{orb}}(S_{(n+1, m)})$ is a free product of cyclic groups as follows

$$\pi_1^{\text{orb}}(S_{(n+1, m)}) \cong \pi_1(S \setminus \Sigma_0) \ast \left( \ast_{P \in \Sigma_+} \left( \mathbb{Z} / m_P \mathbb{Z} \right) \right) \cong \mathbb{F}_r \ast \mathbb{Z}_{m_1} \ast \cdots \ast \mathbb{Z}_{m_s},$$

where $r = 2g - 1 + \# \Sigma_0 = 2g + n$, $s = \# \Sigma_+$, and $m = (m_1, \ldots, m_s)$. 

Definition 2.13. We say a group $G$ is an orbifold group if $G \cong \pi^\text{orb}_1(S_{n+1,m_1})$ for some orbifold structure on a Riemann surface $S$. We refer to an open orbifold group when the orbifold structure on $S$ is such that $\Sigma_0 \neq \emptyset$, or equivalently, $n \geq 0$.

Definition 2.14. The orbifold Euler characteristic of an orbifold $S_{n+1,m_1}$ is given as

$$\chi^\text{orb}(S_{n+1,m_1}) := 2 - 2g - (n + 1) - \sum_i \left(1 - \frac{1}{m_i}\right) = 1 - (s + 2g + n) + \sum_i \frac{1}{m_i}.$$ 

Definition 2.15. Let $X$ be a smooth algebraic variety. A dominant algebraic morphism $F : X \to S_{n+1,m_1}$ defines an orbifold morphism if for all $P \in S$ for $\varphi(P) > 0$, the divisor $F^*(P)$ is a $\varphi(P)$-multiple. The orbifold $S_{n+1,m_1}$ is said to be maximal (with respect to $F$) if $F(X) = S \setminus \Sigma_0$ and no divisor $F^*(P)$, $P \in F(X)$ is an $n$-multiple for $n > \varphi(P)$.

The following result is well-known (see for instance [4, Prop. 1.4]).

Remark 2.16. Let $F : X \to S_{n+1,m_1}$ be an orbifold morphism. Then, $F$ induces a morphism

$$F_* : \pi_1(X) \to \pi^\text{orb}_1(S_{n+1,m_1}).$$

Moreover, if the generic fiber of $F$ is connected, then $F_*$ is surjective.

The following result extends [21].

Lemma 2.17. Consider $G = \mathbb{Z}_{m_1} \ast \cdots \ast \mathbb{Z}_{m_s}$, with $s \geq 2$, $m_i > 1$, and $m := \text{lcm}(m_i, i \in I)$, for $I = \{1, \ldots, s\}$. Let $\pi_1 : G \to \mathbb{Z}_m$ be the natural epimorphism of $G$ onto its maximal cyclic factor. Then $\ker(\pi_1) \cong \mathbb{F}_p$, a free group of rank

$$\rho = 1 - m + m \sum_{i \in I} \left(1 - \frac{1}{m_i}\right) = 1 - m\chi^\text{orb}(\mathbb{Z}_{(1,m)}).$$

Proof. We will proceed by induction over the number of distinct prime factors $n$ of $m$. Suppose $n = 1$, then $m = p^k$ and consider the slightly more general situation $\pi_1 : G = \mathbb{F}_p \ast \mathbb{Z}_{p^{k_1}} \ast \cdots \ast \mathbb{Z}_{p^{k_s}} \to \mathbb{Z}_{p^k}$, where $k_1 \leq \cdots \leq k_s = k$. Then one has that $\ker(\pi_1) \cong \mathbb{F}_p(\pi_1)$, with

$$(7) \quad \rho(\pi_1) = rp^k + p^k \sum_{i=1}^{s-1} \left(1 - \frac{1}{p^{k_i}}\right) = rp^k + 1 - p^k + p^k \sum_{i=1}^{s} \left(1 - \frac{1}{p^{k_i}}\right).$$

This can easily be checked using the Reidemeister-Schreier theorem. One can choose a set-theoretical section of $\pi_1$ by $\sigma(j) = \gamma^j$, $j = 0, \ldots, p^k - 1$, where $\gamma$ is a generator of the cyclic subgroup $\mathbb{Z}_{p^k}$ of $G$. The generators of $\ker(\pi_1)$ are elements $g_{i,j} := \gamma^j g_i \gamma^{-(i+jp^{k-i})}$, for $i = 1, \ldots, s+r-1$, and $j = 0, \ldots, p^k - 1$, where $g_i$ is a generator of the cyclic group $\mathbb{Z}_{p^k}$, a subgroup of $G$ if $i = 1, \ldots, s-1$ and $\{g_s, \ldots, g_{s+r-1}\}$ is a set of generators of the $\mathbb{F}_p$ free factor of $G$. The only relations are given by rewriting $\gamma^i g_i \gamma^{-(i+jp^{k-i})}$ for $i = 1, \ldots, s-1$ as $\gamma^i g_i \gamma^{-(i+jp^{k-i})} = \prod_{t=0}^{p^k - 1} g_{i,j+tp^{k-i}} = 1$, which allows one to eliminate the generators $g_{i,j}$ for $i = 1, \ldots, s-1$ and $j = 0, \ldots, p^k - 1 - i$. Hence (7) follows.

For the induction step we consider the map $\pi_1 : G = \mathbb{Z}_{m_1} \ast \cdots \ast \mathbb{Z}_{m_s} \to \mathbb{Z}_{mp^k}$, where $p$ is prime, $m := \text{lcm}(m_i, i \in I)$, gcd$(m, p) = 1$, and $k := k_s \geq k_{s-1} \geq \cdots \geq k_1$. Denote by $\pi_1,m : G_m = \mathbb{Z}_{m_1} \ast \cdots \ast \mathbb{Z}_{m_s} \to \mathbb{Z}_m$ the epimorphism onto its maximal cyclic factor. Note that $\ker(\pi_1)$ can be obtained as follows. Consider $\pi_{1,p}$ the morphism $G \to \mathbb{Z}_m$ resulting from composing $\pi_1 : G \to \mathbb{Z}_{mp^k}$ with the projection $\mathbb{Z}_{mp^k} \to \mathbb{Z}_m$ and denote by $K_p := \ker(\pi_{1,p})$. Then $\ker(\pi_1) = \ker(K_p \to \mathbb{Z}_{mp^k})$. Using the Reidemeister-Schreier theorem as above one can easily check that

$$(8) \quad K_p \cong \mathbb{F}_p(\pi_{1,m}) \ast \mathbb{Z}_{p^{k_1}}^{\text{orb}} \ast \cdots \ast \mathbb{Z}_{p^{k_s}}^{\text{orb}}.$$
Hence, using (7), one obtains
\[
\rho(\pi_1) = p^k\rho(\pi_{1,m}) + 1 - p^k + p^k \sum_{i=1}^s \sum_{j=1}^m \left(1 - \frac{1}{p^r}\right)
\]
\[
= p^k \left(1 - m + m \sum_{i \in I} \left(1 - \frac{1}{m_i}\right) + \left(1 - p^k + p^k \sum_{i \in I} \frac{m_i}{m_i} \left(1 - \frac{1}{p^r}\right)\right)\right)
\]
\[
= 1 - mp^k + mp^k \sum_{i \in I} \left(1 - \frac{1}{p^r}\right),
\]
which ends the proof. □

The following result is a generalization of [25, Lemma 4].

**Lemma 2.18.** Let \(G\) be an open orbifold group. Then, \(G\) is a Hopfian group, i.e.
every endomorphism of \(G\) which is an epimorphism is an isomorphism.

**Proof.** Consider \(G = F_r \ast Z_{m_1} \ast \cdots \ast Z_{m_s}\), where \(m_i \geq 2\) for any \(i \in I = \{1, \ldots, s\}\). Let \(J = F_r\) and \(H = Z_{m_1} \ast \cdots \ast Z_{m_s}\). Finitely generated free groups are Hopfian, so \(J\) is Hopfian. \(H\) is a free product of finitely many finite groups, so it is virtually free and thus residually finite. Finitely generated residually finite groups are Hopfian, so \(H\) is Hopfian. By [12], the free product of two finitely generated Hopfian groups is Hopfian. □

### 2.5. Fundamental groups of complements of fibered-type curves

In this section, \(F : U \to S\) is going to be an admissible map from a smooth quasi-projective variety to a smooth projective curve \(S\). Following [17] we say the generic fiber \(F^{-1}(P)\) of an admissible map is of type \((g_F, s_F)\) if \(F^{-1}(P)\) is homeomorphic to a Riemann surface of genus \(g_F\) with \(s_F\) points on its boundary. More generally, we will denote the fundamental group of a Riemann surface of genus \(g\) and \(s\) punctures by

\[
\Omega_{(g,s)} = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, x_0, \ldots, x_{s-1} : \Pi_{i=1}^g [a_i, b_i] = \Pi_{j=0}^{s-1} x_j \rangle,
\]
where \(x_0, \ldots, x_{s-1}\) are meridians around the punctures of the Riemann surface.

As above, consider the admissible map \(F : U \to S\), \(B = \{P_0, P_1, \ldots, P_n\} \subset S\), \(n \geq 0\). Let \(\Gamma_S = \{\gamma_1, \ldots, \gamma_r\}\), \(r = 2g_S + n\) \((g_S\) the genus of \(S\)) be a set of loops in \(\pi_1(U_B)\) such that:

1. \(\{F_*(\gamma_k)\}_{k=1}^r\) generates \(\pi_1(S \setminus B) \cong F_r\) for all \(k = 1, \ldots, r\),
2. the loops \(\gamma_k \in \pi_1(U_B)\) result from lifting a meridian around \(P_k \in S\), for \(k = 1, \ldots, n\),
3. the product \(\tilde{\gamma} = \prod_{j=1}^{g_S} [\gamma_{n+2j-1}, \gamma_{n+2j}] (\prod_{i=1}^{s} \gamma_i)^{-1}\) is such that \(F_*(\tilde{\gamma})\) is a meridian around \(P_0\),
4. For every \(P_k \in B\) such that \(F^*(P_k)\) is not a multiple fiber, \(\gamma_k\) is a product of meridians (positively or negatively oriented) about irreducible components of \(C_{P_k} \subset C_B\) for all \(k = 1, \ldots, n\). In the particular case where \(C_{P_k}\) is irreducible, \(\gamma_k\) is a positively oriented meridian about \(C_{P_k}\).

On the other hand, if \(P \in S \setminus (B_F \cup B)\), the typical fiber \(F^{-1}(P)\) is a Riemann surface of type \((g_F, s_F)\) and \(\pi_1(F^{-1}(P)) \cong \Omega_{(g_F,s_F)}\) as in (7). One can consider \(\Gamma_F\) the image induced by the inclusion, \(\iota : F^{-1}(P) \hookrightarrow U_B\) of such a set of generators as in (9).

**Definition 2.19.** In the construction above, we will refer to \(\Gamma_F\) (resp. \(\Gamma_S\)) as an adapted geometric set of fiber (resp. base) loops w.r.t. \(F\) and \(B\).

The following shows that adapted geometric sets of fiber (resp. base) loops exist for admissible maps.

**Lemma 2.20.** Let \(F : U \to S\) be an admissible map from a smooth quasi-projective surface \(U\) to a smooth projective curve \(S\) of genus \(g_S\). Consider \(B = \ldots\)
Using Bézout’s identity one can obtain a product of meridians \( \mu \). Moreover, \( \text{Lemma 2.21.} \)

Under the conditions above, if \( \pi_1(U_B) \) holds.

To see that we may choose \( \Gamma_S \) so that it also satisfies condition 4, note that there exists a meridian \( \mu \) around each irreducible component \( C \) of \( F^*(P_k) \) of multiplicity \( m = \text{mult}(C) \), such that \( F_*(\mu_C) = F_*(\gamma_C)^m \). Also note that

\[
m_k = \gcd\{\text{mult}(C) \mid C \text{ irreducible component in } F^*(P_k)\}.
\]

Using Bézout’s identity one can obtain a product of meridians \( \mu_k \) whose image is \( F_*(\gamma_C)^m \). In particular, \( \mu_k = \alpha \gamma_C^m \) for \( \alpha \in \ker F_* \). If \( F^*(P) \) is not multiple, then \( m_k = 1 \). Replacing \( \gamma_C \) by \( \alpha \gamma_C \) condition 4 follows, and conditions 1–3 still hold.

\[\text{Lemma 2.21. Under the conditions above, if } B \supset B_F \text{ contains the set } B_F \text{ of atypical values of } F, \text{ with } \#B \geq 1. \text{ Then } \pi_1(U_B) \text{ is a semidirect product of the form }\]

\[\pi_1(U_B) \cong \pi_1(F^{-1}(P)) \rtimes \pi_1(S \setminus B).\]

Moreover, \( \pi_1(U_B) \) has a presentation with generators \( \Gamma_F \cup \Gamma_S \) for

\[
\Gamma_F = \{a_i, b_i, x_j \mid i = 1, \ldots, g_F, j = 0, \ldots, s_F - 1\}, \quad \Gamma_S = \{\gamma_k \mid k = 1, \ldots, r\},
\]

where \( \Gamma_F \) (resp. \( \Gamma_S \)) is an adapted geometric set of fiber (resp. base) loops w.r.t. \( F \) and \( B \), and the following is a set of relations

\[
\begin{align*}
[\gamma_k, a_i] &= \alpha_{i,k}, \\
[\gamma_k, b_i] &= \beta_{i,k}, \\
[\gamma_k, \delta_{j,k}, x_j] &= 1, \\
\prod_j x_j &= \prod_i [a_i, b_i],
\end{align*}
\]

where \( i \in \{1, \ldots, g_F\}, j \in \{0, \ldots, s_F - 1\}, k \in \{1, \ldots, r\} \), and \( \alpha_{i,k}, \beta_{i,k}, \delta_{j,k} \) are words in the elements of \( \Gamma_F \).

\[\text{Proof.} \text{ The condition } B \supset B_F \text{ implies that } F : U_B \to S \setminus B \text{ is a locally trivial fibration, with fiber } F^{-1}(P). \text{ Let } \iota : F^{-1}(P) \hookrightarrow U_B \text{ be the inclusion. The long exact sequence of a fibration for homotopy groups yields }\]

\[
\pi_2(S \setminus B) \to \pi_1(F^{-1}(P)) \xrightarrow{\iota_*} \pi_1(U_B) \to \pi_1(S \setminus B) \to 1.
\]

Note that, since \( \pi_1(S \setminus B) \cong \mathbb{F}_r \), the epimorphism \( F_* \) splits. Since \( S \setminus B \) is homotopy equivalent to a wedge of circles, \( \pi_2(S \setminus B) = 1 \), which concludes the result.

The description of the semidirect product \( \pi_1(F^{-1}(P)) \rtimes \pi_1(S \setminus B) \) is given by considering the action of \( \gamma_k \) on the group \( \pi_1(F^{-1}(P)) \) generated by \( \Gamma_F \). For the generators \( a_i \) (resp. \( b_i \)) one can write \( \gamma_i^{-1} a_i \gamma_i = a_i \alpha_{i,k} \) for some \( \alpha_{i,k} \) in \( \pi_1(F^{-1}(P)) \).

\[\text{Corollary 2.22. Assume that } B \supset B_F, \text{ let } P \in S \setminus B, \text{ and let } \#B = n + 1 \geq 1, \text{ where } r := 2g_S + n \geq 1. \text{ Then } F_* : \pi_1(U_B) \to \mathbb{F}_r \text{ is an isomorphism if and only if } F^{-1}(P) \text{ is of type } (0,1) \text{ or } (0,0). \]

Moreover, if \( F : \mathbb{P}^2 \to \mathbb{P}^1 \), and \( U = \mathbb{P}^2 \setminus B \), then \( M_F = B_F \) and hence \( \#B_F \leq 2. \)
Proof. The first statement is an immediate consequence of Lemma 2.21. If $U$ is a Zariski open subset of $\mathbb{P}^2$, any pencil has at least a base point and thus any fiber $F^{-1}(P)$ of $F: \mathbb{P}^2 \to \mathbb{P}^1$ must be an open curve, so the fibers of $F|_U$ will be open curves as well. The moreover part follows from [17, Thm. 6.1] by direct inspection, since all the pencils of type $(0, 1)$ are classified.

Example 2.23. In particular, according to Corollary 2.22, the classification of all rational pencils on $\mathbb{P}^2$ of type $(0, 1)$ given in [17] provides a list of examples of curves whose complement have a free fundamental group.

Example 2.24. Lemma 2.21 can be applied to a pencil of plane smooth cubics in $\mathbb{P}^2$ intersecting the line at infinity at one point. Besides the line at infinity, which has multiplicity 3, such pencils have generically two nodal cubics as atypical fibers. Denote by $f : U = C^2 \to \mathbb{C}$ the affine map defining this pencil and denote by $B_f$ the set of atypical affine values for $f$. For instance, if $f(x, y) = x^3 + x^2 - y^2$, then $B_f = \{0, -\frac{4}{27}\}$. By Corollary 2.22, $\pi_1(U_B)$ is not free for any $B = \{\lambda_0 = 0, \lambda_1 = -\frac{4}{27}, \lambda_2, \ldots, \lambda_r\} \supset B_f$. In fact, according to Lemma 2.21, $\pi_1(U_B) = \pi_1(\mathbb{P}^2_{B_U(1:0)})$ is generated by $\{a, b\}$ (a set of free generators of the fundamental group of the generic fiber) and $\Gamma_{S} = \{\gamma_0, \gamma_1, \ldots, \gamma_r\}$, where $\gamma_k$ is a meridian around $C_{\lambda_k}$. A set of relations can be obtained as

$$\begin{align*}
[\gamma_0, a] &= a_0 = ab^{-1}a^{-1}, & [\gamma_1, a] &= a_1 = ba^{-1}, & [\gamma_k, a] &= 1, \\
[\gamma_0, b] &= b_0 = 1, & [\gamma_1, b] &= b_1 = b^{-1}a^{-1}, & [\gamma_k, b] &= 1,
\end{align*}$$

$k = 2, \ldots, r$. Note that $\pi_1(U_B)$ is not free by Corollary 2.22, however, factoring out $\pi_1(U_B)$ by either $\gamma_0$ or $\gamma_1$ results in the free group $\mathbb{F}_{r-1}$.

In Lemma 2.21, $B_F \subset B$. However, one can understand $\pi_1(U_B)$ for any non-empty finite set $B \subset S$ as follows.

Corollary 2.25. Assume $F : U \to S$ is an admissible map, and let $B \subset S$, with $\# B = n + 1 \geq 1$. Then, $\pi_1(U_B) \cong \pi_1(U_{B \cup B_F})/N$, where $N$ is the normal closure of meridians $\gamma \in \pi_1(U_{B \cup B_F})$ of the components of $C_{B \cup B_F} \setminus C_{B}$.

Proof. The result follows from Lemma 2.25.

This first result is well known in different settings (cf. [18,22,23]), but we include it here for the sake of completeness and in order to introduce some notation.

Corollary 2.26. Let $F : U \to S$ be an admissible map and let $B \subset S$ be a finite set, with $\# B = n + 1 \geq 1$. Let $S_{(n+1,m)}$ be the maximal orbifold structure on $S$ with respect to $F|_U$. Then,

$$\pi_1(F^{-1}(P)) \xrightarrow{\iota_*} \pi_1(U_B) \xrightarrow{F_*} \pi_1(S_{(n+1,m)}) \to 1$$

is an exact sequence.

Proof. Consider the commutative diagram:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(F^{-1}(P)) & \longrightarrow & \pi_1(U_{B \cup B_F}) & \longrightarrow & \pi_1(S \setminus (B \cup B_F)) & \longrightarrow & 1 \\
& & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
& & \pi_1(F^{-1}(P)) & \xrightarrow{\iota_*} & \pi_1(U_B) & \xrightarrow{F_*} & \pi_1(S_{(n+1,m)}) & & \\
\end{array}
$$

Here, the vertical arrows are all induced by inclusion, and the top row is exact by Lemma 2.21. The surjectivity of $F_*$ in the bottom row follows from the diagram (but also from Remark 2.16). Also, $\text{Im}(\iota_*) \subseteq \ker F_*$. Let us prove the other inclusion. Since $\text{Im}(\iota_*)$ is a quotient of a normal subgroup of $\pi_1(U_{B \cup B_F})$, it is a normal
subgroup. Using the exactness of the top row, Lemma 2.21 and Corollary 2.25 we see that the map

\[ \pi_1(U_B)/\text{Im}(\iota_*) \rightarrow \pi_1^{\text{orb}}(S_{(n+1,m)}) \]

has a splitting \( \sigma \). Let us see this. Let \( \gamma_k \in \Gamma_S \) (seen as an element of \( \pi_1(U_B \sqcup U_F) \)) correspond to a meridian around \( P_k \in M_F \setminus B \), and let \( m_k \) be the multiplicity of the fiber above \( P_k \). We need to make sure that \( \gamma_k^{m_k} \) is trivial in \( \pi_1(U_B)/\text{Im}(\iota_*) \).

As in the last paragraph of the proof of Lemma 2.20, there exists \( \mu_k \) a product of meridians (positively or negatively oriented) about irreducible components of \( C_{P_k} \) such that \( F_*(\mu_k) = F_*(\gamma_k^{m_k}) \). By the commutative diagram above, \( \mu_k \) and \( \gamma_k^{m_k} \) are the same element in \( \pi_1(U_B)/\text{Im}(\iota_*) \), but \( \mu_k \) is trivial in \( \pi_1(U_B) \). Hence, \( \pi_1(U_B)/\text{Im}(\iota_*) \rightarrow \pi_1^{\text{orb}}(S_{(n+1,m)}) \) has a splitting \( \sigma \).

Moreover, let \( P_k \in B_F \setminus (B \cup M_F) \). Then, \( \gamma_k \) is trivial in \( \pi_1(U_B) \) by Corollary 2.25 (recall condition 3 in Definition 2.19). Hence, \( \pi_1(U_B)/\text{Im}(\iota_*) \) has a presentation where the generators are the images by \( \sigma \) of a minimal set of generators of \( \pi_1^{\text{orb}}(S_{(n+1,m)}) \). Therefore, \( \sigma \) is an isomorphism, and thus \( \text{Im}(\iota_*) = \ker F_* \). \( \square \)

**Remark 2.27.** Suppose that \( X \) is a projective surface, \( S \) a compact Riemann surface, and \( F : X \rightarrow S \) a surjective holomorphic map with connected fibers. Let \( S_{(0,m)} \) be the maximal orbifold structure of \( S \) with respect to \( F : X \rightarrow S \). As mentioned in the proof of [13, Lemma 4.2], one also has an exact sequence like the one in Corollary 2.20 namely

\[ \pi_1(F^{-1}(P)) \rightarrow \pi_1(X) \rightarrow \pi_1^{\text{orb}}(S_{(0,m)}) \rightarrow 1 \]

where the first arrow is induced by the inclusion of a generic fiber \( F^{-1}(P) \) over \( P \in S \).

### 2.6. Characteristic Varieties

Characteristic varieties are invariants of finitely presented groups \( G \), and they can be computed using any connected topological space \( X \) (having the homotopy type of a finite CW-complex) such that \( G = \pi_1(X, x_0), x_0 \in X \) as follows. Let us denote \( H := H_1(X; \mathbb{Z}) = G/G' \). Note that the space of characters on \( G \) is a complex torus

\[ \mathbb{T}_G := \text{Hom}(G, \mathbb{C}^*) = \text{Hom}(H, \mathbb{C}^*) = H^1(X; \mathbb{C}^*) \]

\( \mathbb{T}_G \) can have multiple connected components, but it only contains one connected torus, which we will denote by \( \mathbb{T}_G^1 \).

Given \( \xi \in \mathbb{T}_G \), the following local system \( \mathbb{C}_\xi \) of coefficients over \( X \) can be constructed. Let \( \pi_{ab} : \tilde{X}_{ab} \rightarrow X \) be the universal abelian covering of \( X \). The group \( G \) acts freely (on the right) on \( \tilde{X}_{ab} \) by the deck transformations of the covering. The local system of coefficients \( \mathbb{C}_\xi \) is defined as the locally constant sheaf associated with:

\[ \pi_1 : \tilde{X}_{ab} \times G \mathbb{C} \rightarrow X \text{ where } \tilde{X}_{ab} \times G \mathbb{C} := \left( \tilde{X}_{ab} \times \mathbb{C} \right) / \langle (x,t) \sim (x^g, \xi(g^{-1})t) \rangle \]

**Definition 2.28.** The \( k \)-th characteristic variety of \( G \) is the subvariety of \( \mathbb{T}_G \), defined by:

\[ \mathcal{V}_k(G) := \{ \xi \in \mathbb{T}_G \mid \dim H^1(X, \mathbb{C}_\xi) \geq k \} \]

where \( H^1(X, \mathbb{C}_\xi) \) is classically called the twisted cohomology of \( X \) with coefficients in the local system \( \xi \in \mathbb{T}_G \).

It is also customary to use \( \mathcal{V}_k(X) \) for \( \mathcal{V}_k(G) \) whenever \( \pi_1(X) = G \).

For the sake of completeness, we include a result about the structure of characteristic varieties of orbifold groups of the form \( \mathbb{F}_r \ast \mathbb{Z}_{m_1} \ast \ldots \ast \mathbb{Z}_{m_s} \). In that case \( \mathbb{T}_G \) is a disjoint union of \( \mathbb{T}_G^1 \cong (\mathbb{C}^*)^r \) and translations \( \mathbb{T}_G^2 \) of \( \mathbb{T}_G^1 \) by every element \( \lambda \) of \( C = C_{m_1} \times \ldots \times C_{m_s} \), where \( C_m \) is the multiplicative group of \( m \)-th roots of
unity. Consider \( \lambda = (\lambda_1, \ldots, \lambda_n) \in C \). Denote by \( \ell(\lambda) \) the number of coordinates in \( \lambda \) that are different from 1.

Given an orbifold group \( G \), one can define \( \text{depth}(\rho) \) of a torsion element \( \rho \subset T_G \) as

\[
\text{depth}(\rho) := \max\{k \in \mathbb{Z}_{\geq 0} \mid T^0_G \in V_k(G)\}.
\]

**Proposition 2.29** ([5 Prop. 2.10]). Let \( G \cong \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s} \). Then,

\[
V_k(G) = \begin{cases} 
T_G & \text{if } 1 \leq k \leq r - 1 \\
\{1\} \cup \bigcup \{T^\lambda_G \mid \ell(\lambda) \geq 1\} & \text{if } k = r \\
\bigcup \{T^\lambda_G \mid \ell(\lambda) \geq k - r + 1\} & \text{if } r + 1 \leq k \leq r + s - 1 \\
\emptyset & \text{if } k \geq r + s.
\end{cases}
\]

is a decomposition in irreducible components of \( V_k(G) \).

**Remark 2.30.** As an immediate consequence of Proposition 2.29 if \( S = S_{(n+1,m)} \) one can check that \( V_1(\pi_1^{orb}(S)) \neq \emptyset \) if and only if \( \pi_1^{orb}(S) \) is not abelian, or equivalently, if \( \chi^{orb}(S) < 0 \), in which case \( S \) is called an orbifold of general type.

### 2.7. Iitaka’s (Quasi)-Albanese varieties.

**Definition 2.31** (Albanese). Let \( X \) be a smooth projective variety. Then, the Albanese variety is defined as

\[
\text{Alb}(X) := H^0(X, \Omega^1_X) \vee / \text{Free} H_1(X; \mathbb{Z}),
\]

where \( \vee \) denotes the dual as a \( \mathbb{C} \)-vector space, and \( \text{Free} H_1(X; \mathbb{Z}) \) denotes the torsion-free part of \( H_1(X; \mathbb{Z}) \). It is an abelian variety. Moreover, fixing a base point \( x_0 \in X \), the Albanese morphism \( \alpha_X \) is an algebraic morphism defined as follows:

\[
\alpha_X : \begin{array}{ccc}
X & \longrightarrow & \text{Alb}(X) \\
x & \longmapsto & \left(\omega \mapsto \int_{x_0}^{x} \omega \right).
\end{array}
\]

To generalize the Albanese to non-projective varieties, we first need the following generalization of abelian varieties.

**Definition 2.32** (Semi-abelian varieties). A complex algebraic variety \( G \) is a semi-abelian variety if it is an algebraic variety which lies in a short exact sequence of algebraic groups

\[
1 \to T \to G \to A \to 1,
\]

where \( T \cong (\mathbb{C}^*)^n \) is a torus (for some \( n \geq 0 \)) and \( A \) is an abelian variety.

Iitaka [16] generalized the Albanese construction to smooth quasi-projective varieties as follows. For a detailed description, see [14].

**Definition 2.33** (Quasi-Albanese). Let \( U \) be a smooth quasi-projective variety, and let \( X \) be a smooth compactification of \( U \) such that \( D := X \setminus U \) is a simple normal crossing divisor. Then, the quasi-Albanese variety of \( U \) is defined as

\[
\text{Alb}(U) := H^0(X, \Omega^1_X(\log D)) \vee / \text{Free} H_1(U; \mathbb{Z}).
\]

where \( \vee \) denotes the dual as a \( \mathbb{C} \)-vector space, and \( \text{Free} H_1(U; \mathbb{Z}) \) denotes the torsion-free part of \( H_1(U; \mathbb{Z}) \). It is a semi-abelian variety. Moreover, fixing a base point \( u_0 \in U \), the Albanese morphism \( \alpha_U \) is an algebraic morphism defined as follows:

\[
\alpha_U : \begin{array}{ccc}
U & \longrightarrow & \text{Alb}(U) \\
u & \longmapsto & \left(\omega \mapsto \int_{u_0}^{u} \omega \right).
\end{array}
\]

**Remark 2.34.** Note that different choices of \( u_0 \in U \) give rise to the same Albanese morphism \( \alpha_U \), up to translation in the target.
The following result summarizes the facts about the (qua-
si)-Albanese varieties and morphisms that we will use in this paper.

**Theorem 2.35** ([4]). Let \( U \) be a smooth quasi-projective variety. Then, the fol-
lowing are true.

1. If \( U \) is a semi-abelian variety, \( \alpha_U \) is an isomorphism.
2. **Universal property of the Albanese:** If \( f : U \to G \) is an algebraic
    morphism to \( G \) a semi-abelian variety, then there exists a unique algebraic
    morphism \( \tilde{f} : \text{Alb}(U) \to G \) such that \( f = \tilde{f} \circ \alpha_U \).
3. **Functoriality of the Albanese:** Let \( f : U \to V \) a morphism between
    smooth quasi-projective varieties. Then, there exists a unique morphism of
    algebraic groups \( \text{Alb}(f) : \text{Alb}(U) \to \text{Alb}(V) \) such that the following diagram
    is commutative:
    \[
    \begin{array}{ccc}
    U & \xrightarrow{\alpha_U} & \text{Alb}(U) \\
    f \downarrow & & \downarrow \text{Alb}(f) \\
    V & \xrightarrow{\alpha_V} & \text{Alb}(V),
    \end{array}
    \]
    where \( \alpha_U \) and \( \alpha_V \) are constructed points \( u_0 \in U, \; v_0 \in V \) such that
    \( u_0 = f(v_0) \).
4. \( (\alpha_U)_* : H_1(U; \mathbb{Z}) \to H_1(\text{Alb}(U); \mathbb{Z}) \) is surjective, with kernel \( \text{Tors}_{\mathbb{Z}}H_1(U; \mathbb{Z}) \).
5. Let \( X \) be a smooth compactification of \( U \) such that \( D := X \setminus U \) is a simple
    normal crossing divisor, and let \( i : U \hookrightarrow X \) be the inclusion. Then, we have
    an exact sequence
    \[
    1 \to (\mathbb{C}^*)^r \to \text{Alb}(U) \xrightarrow{\text{Alb}(i)} \text{Alb}(X) \to 1,
    \]
    where \( r = \dim \mathcal{H}^0(X, \Omega_X^1(\log D)) - \dim \mathcal{C}^0(X, \Omega_X^1) \).

**Remark 2.36.** If \( f : X \to Y \) is a surjective morphism between smooth projective
varieties, then \( \text{Alb}(f) : \text{Alb}(X) \to \text{Alb}(Y) \) is a surjective group homomorphism.
This is a well known property that follows from the chain of inclusions \( \text{Im} \; \alpha_Y \subseteq \text{Im} \; \text{Alb}(f) \subseteq \text{Alb}(Y), \text{Im} \; \text{Alb}(f) \) being an abelian subvariety of \( \text{Alb}(Y) \), and the
fact that \( \text{Im} \; \alpha_Y \) generates \( \text{Alb}(Y) \) as an abelian variety by the universal property
(Theorem 2.33 [2]).

We end this section with two technical lemmas about Albanese varieties. These
results will be used in the proof of Theorems 3.1 and 3.2, which give necessary
geometric conditions for a curve to have an infinite group which is a free product
of cyclic groups as the fundamental group of its complement.

**Lemma 2.37.** Let \( U \) be a smooth quasi-projective surface such that \( \pi_1(U) \cong \mathbb{Z} \).
Then, \( \text{Alb}(U) \cong \mathbb{C}^* \), and \( \alpha_U : U \to \mathbb{C}^* \) is an admissible map with no multiple fibers
inducing isomorphisms in fundamental groups.

**Proof.** Let \( X \) be a smooth compactification of \( U \) such that \( X \setminus U \) is a simple normal
crossing divisor. Using its Hodge decomposition, the dimension of \( H_1(X; \mathbb{Q}) \) must
be even. Since \( \pi_1(X) \) is a quotient of \( \pi_1(U) \cong \mathbb{Z} \) (Lemma 2.35), \( \pi_1(X) \) must be a finite
group, and thus \( \text{Alb}(X) \) is a point. By Theorem 2.33 [2], \( \text{Alb}(U) \) is a complex
torus, and by Theorem 2.35 [1], it must be \( \mathbb{C}^* \) and \( \alpha_U : U \to \mathbb{C}^* \) induces
an isomorphism in fundamental groups.

Let us see that \( \alpha_U : U \to \mathbb{C}^* \) has connected fibers. Otherwise, using Stein
factorization on its enlargement and Remark 2.3, one would obtain a surjective
morphism \( f : U \to V \subseteq \mathbb{P}^1 \) with connected fibers which induces a surjection
on fundamental groups. Moreover, \( f_* : \pi_1(U) \to \pi_1(V) \) must be an isomorphism, since
\( (\alpha_U)_* \) is an isomorphism. This implies \( V = \mathbb{C}^* \). By Theorem 2.35 [2], \( f \) factors
through \( \alpha_U \), so \( \alpha_U \) has connected fibers and is surjective. Finally, by Remark 2.16 \( \alpha_U \) has no multiple fibers.

**Lemma 2.38.** Let \( S' \) be a smooth quasi-projective curve such that \( \pi_1(S') \cong \mathbb{F}_r \), for \( r \geq 1 \). Let \( U \) be a smooth quasi-projective surface and \( F : U \to S' \) be an admissible map such that \( \overline{F} : X \to S \) is a holomorphic extension of \( F \) with connected fibers, where \( X \) is a smooth projective compactification of \( U \), \( X \setminus U \) is a simple normal crossing divisor and \( S \) is a Riemann surface of genus \( g_S \). Let \( iv : U \hookrightarrow X \) be the inclusion.

Suppose that \( F_* : \pi_1(U) \to \pi_1(S') \) is an isomorphism.

1. If \( g_S = 0 \), then,
   (a) \( \alpha_S : S' \to \text{Alb}(S') \) is injective;
   (b) \( \text{Alb}(U) \cong (\mathbb{C}^*)^r \cong \text{Alb}(S') \);
   (c) \( \text{Alb}(F) : \text{Alb}(U) \to \text{Alb}(S') \) is an isomorphism;
   (d) up to isomorphism in the target, the map \( F \) coincides with the restriction of \( \alpha_U \) to its image, namely \( \alpha_U : U \to \text{Alb}(U) \).

2. If \( g_S \geq 1 \), then,
   (a) \( \alpha_S : S \to \text{Alb}(S) \) is injective;
   (b) \( \text{Alb}(\overline{F}) : \text{Alb}(X) \to \text{Alb}(S') \) is an isomorphism;
   (c) up to isomorphism in the target, the map \( F \) coincides with the restriction of \( \alpha_X \circ iv \) to its image, namely \( \alpha_X \circ iv : U \to \text{Alb}(X) \).

**Proof.** \( \text{Alb}(S) \) has (complex) dimension \( g_S \). Similarly, since \( X \) is a projective variety, the dimension of \( \text{Alb}(X) \) is half of the rank of \( H_1(X, \mathbb{Z}) \). Let us show that the rank of \( H_1(X, \mathbb{Z}) \) is \( 2g_S \). By Corollary 2.20 we have the following commutative diagram, where the vertical arrows are induced by inclusion, \( \overline{F}^{-1}(P) \) is a generic fiber of \( \overline{F} \), and the top map is trivial.

\[
\begin{array}{ccc}
\pi_1(F^{-1}(P)) & \longrightarrow & \pi_1(U) \\
\downarrow & & \downarrow \\
\pi_1(F^{-1}(P)) & \longrightarrow & \pi_1(X)
\end{array}
\]

This implies that the bottom arrow is trivial as well. Let \( S_{(0,m)} \) be the maximal orbifold structure of \( S \) with respect to \( \overline{F} : X \to S \). By Remark 2.27, \( \overline{F} : \pi_1(X) \to \pi_1^{\text{orb}}(S_{(0,m)}) \) is an isomorphism. Abelianizing, we obtain that the rank of \( H_1(X; \mathbb{Z}) \) is \( 2g_S \). In particular, if \( g_S = 0 \), \( \text{Alb}(X) \) (resp. \( \text{Alb}(S = \mathbb{P}^1) \)) is a point, and thus, by Theorem 2.25 \( \text{Alb}(U) \) (resp. \( \text{Alb}(S') \)) is a torus, which must necessarily have dimension \( r \) by Theorem 2.25. This concludes the proof of part (1a).

Assume that \( g_S = 0 \). Part (1a) is well known, the proof being as follows. Since \( r \geq 1 \), we have that \( S' = \mathbb{P}^1 \) with \( r + 1 \) points removed. We have an open immersion \( S' \hookrightarrow \mathbb{C}^* \), and since \( \mathbb{C}^* \) is a semiabelian variety, this injection must factor through \( \alpha_{S'} \) by the universal property of the Albanese (Theorem 2.25). Thus \( \alpha_{S'} : S' \to \text{Alb}(S') \) is injective.

Note that \( \text{Alb}(F) : \text{Alb}(U) \to \text{Alb}(S') \) is an algebraic map which is a homomorphism between \((\mathbb{C}^*)^r\) and itself and induces an isomorphism on fundamental groups by Theorem 2.25. By Cartier duality, \( \text{Alb}(F) \) is an isomorphism, and part (1a) is proved.

Part (1b) now follows both from the functoriality of the Albanese map (Theorem 2.25), and from parts (1a) and (1c).

Assume now that \( g_S \geq 1 \). Part (2a) is the well known Abel-Jacobi theorem. Let us prove part (2b). Note that \( \overline{F} : X \to S \) is surjective, so \( \text{Alb}(\overline{F}) : \text{Alb}(X) \to \text{Alb}(S) \) is a surjective group homomorphism by Remark 2.35. Hence, \( \text{Alb}(\overline{F}) \) must
be a fibration, and the dimension of the fiber is 0 when \( \text{Alb}(X) \) and \( \text{Alb}(S) \) have the same dimension, which we know equals \( 2g_S \) in both cases. Thus \( \text{Alb}(\overline{F}) : \text{Alb}(X) \to \text{Alb}(S) \) is a finite covering. Since the fibers of \( \overline{F} : X \to S \) are connected, the commutativity of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\overline{F}} & S \\
\downarrow{\alpha_X} & & \downarrow{\alpha_S} \\
\text{Alb}(X) & \xrightarrow{\text{Alb}(\overline{F})} & \text{Alb}(S)
\end{array}
\]

implies that \( \text{Alb}(\overline{F}) \) is an isomorphism. In particular, \( \alpha_X(X) \) is isomorphic to \( S \).

As in the \( g_S = 0 \) case, part (2a) now follows both from the functoriality of the Albanese, and from parts (2a) and (2b).

\[\square\]

3. Main theorem

Our purpose in this section is to give a necessary geometric condition for a curve to have an orbifold fundamental group \( \mathbb{F}_r \ast \mathbb{Z}_{m_1} \ast \ldots \ast \mathbb{Z}_{m_s} \) as the fundamental group of its complement. We will show that these curves come from admissible maps in most cases. We will do this in two stages. If \( r \geq 1 \) this is done in Theorem 3.1. If \( r = 0 \) and the group is of general type, this is done in Theorem 3.2. The strategy to prove Theorem 3.2 is to reduce it to Theorem 3.1 after taking a covering.

3.1. Proof of Theorem 1.1

Theorem 1.1 will be stated in two ways depending on whether or not the first Betti number of \( X \setminus D \) vanishes.

Theorem 3.1 (Main theorem, \( r \geq 1 \)). Theorem 1.1 holds if \( r \geq 1 \).

Proof. If \( \pi_1(X \setminus D) \cong \mathbb{Z} \), parts (i) and (ii) follow from Lemma 2.37 for \( S(n+1,m) = \mathbb{P}^1_{(1,\ldots,1)} \cong \mathbb{C} \).

Let us now prove parts (i) and (ii) assuming that the group \( G := \pi_1(X \setminus D) \cong \mathbb{F}_r \ast \mathbb{Z}_{m_1} \ast \ldots \ast \mathbb{Z}_{m_s} \) is non-abelian (i.e. \( \neq \mathbb{Z} \)). Using Remark 2.38 and Proposition 2.29, the characteristic variety of the group \( G \) has a positive dimensional irreducible component \( \mathbb{T}^1_G \) associated with the torsion character \( \lambda = (\xi_{m_1}, \ldots, \xi_{m_s}) \in \mathbb{T}_G \), where \( \xi_j \in \mathbb{C} \) is a \( j \)-th primitive root of 1. Since \( \mathbb{T}^1_G \) has dimension \( r \geq 1 \), by [3] Thm. 1, there exists an orbifold structure \( S(n',m') \), \( n' \geq 0 \) on a Riemann surface \( S \) of genus \( g_S \) and an admissible map \( F : X \to S \) which induces an orbifold morphism \( F_1 : X \setminus D \to S(n',m') \) such that \( S(n',m') \) is maximal with respect to \( F_1 \) and \( F_1^*(S(n',m')) = \mathbb{T}^1_G \) for some component \( V_{m'} = \mathbb{T}^1_{G_1} \) of the characteristic variety of the orbifold fundamental group \( G_1 = \pi_1^{\mathbb{F}_r} (S(n',m')) \). Since \( F \) has connected fibers, \( F_1 \) induces a surjection of orbifold groups \( \mathbb{F}_r : G_1 \to G_1 \) (cf. [5] Proposition 2.6) and hence an injection \( F^*: \mathcal{V}_k(G_1) \hookrightarrow \mathcal{V}_k(G) \) for all \( k \).

Since every component \( \mathbb{T}^1_G \), \( j = 1, \ldots, m - 1 \), for \( m = \text{lcm}\{m_i \mid i \in I\} \), is positive dimensional (note that in general \( j \neq m \), since \( \mathbb{T}^1_G = \mathbb{T}^1 = \{1\} \) if \( r = 1 \)), they all contain non-torsion characters. The proof of [5] Lemma 6.4] shows that the admissible map \( F \) is unique, that is, if \( \mathbb{T}^1_G^{\mathbb{F}_r} \) is positive dimensional, then there is a \( \rho' \in \mathbb{T}^{1}_{G_1} \), such that \( F^*(\mathbb{T}^{1}_{G_1}) = \mathbb{T}^{1}_{G} \) and \( \text{depth}(\rho) = \text{depth}(\rho') \).

Assume \( n' = 0 \), that is, \( S \) is a compact Riemann surface. According to the structure of its characteristic varieties (cf. [5] Proposition 2.11)], one has \( \text{depth}_{G_1}(1) = 2g_S - 2 = r - 1 \) (the depth of the positive component \( \mathbb{T}_G = \mathbb{T}_G^{\mathbb{F}_r} \), passing through the origin). Finally, \( 1 \in \mathcal{V}_{r+1}(G_1) \) but \( 1 \notin \mathcal{V}_{r+1}(G_1) \). This contradicts the inclusion of characteristic varieties for \( k = r + 1 \). Therefore \( n' = n + 1, n \geq 0 \) and \( S \) is an open Riemann surface, that is,

\[ \pi_1^{\mathbb{F}_r} (S(n+1,m')) \cong \mathbb{F}_{n'} \ast \mathbb{Z}_{m'_1} \ast \ldots \ast \mathbb{Z}_{m'_{n'}}. \]
where $n = r' - 2gs$. For $k = 0$, $V_{m'} = T_{G_1}^{n'}$ implies $r = r'$.

To show (i) it remains to show that $s = s'$ and $m_i = m'_i$ for all $i = 1, \ldots, s$. Using Proposition 2.29 and (12), one obtains $\text{depth}(\lambda) = s + r - 1$ and $\text{depth}(\lambda') = s' + r - 1$. Since both components are positive dimensional, $\text{depth}(\lambda) = \text{depth}(\lambda')$. This implies $s = s'$. Since $F^*$ is a group monomorphism and $F^*(\lambda') = \lambda$, the order of $\lambda'$ is $|\lambda'| = \text{lcm}(m'_i | i \in I |) = m$.

Also $\text{depth}(\lambda^j) = \text{depth}(F^*(\lambda')^j) = \text{depth}(\lambda^j)$, $j = 1, \ldots, m - 1$. Using Proposition 2.29 and (12), we obtain that

$$\# \{ i \in I \mid m_i \text{ divides } j \} = s + r - \text{depth}(\lambda') - 1,$$

where $I = \{1, \ldots, s\}$, and $j = 1, \ldots, m - 1$, and the same equality holds for $m'_i$. This uniquely determines the values of $m_i$, so $m_i = m'_i$ (after reordering) for all $i = 1, \ldots, s$. This concludes the proof of (i) if $G$ is non-abelian.

The fact that $F_\ast$ is an isomorphism follows from Corollary 2.26 and Lemma 2.18. This concludes the proof of (ii) if $G$ is non-abelian.

Let $B = S_0$ be the $n + 1$ points of $S$ of label 0 in the orbifold structure $S_{(n+1,m)}$. Note that, by the maximality of the orbifold structure of $S_{(n+1,m)}$ with respect to $F_1 : X \to S_{(n+1,m)}$, the extension $F : X \to S$ satisfies that $F^{-1}(B) = D_f \subset D$, hence $D = D_f \cup D_v$, where $D_f$ is the union of the irreducible components of $D$ which are not in $D_f$. We can further decompose $D_f$ as $D_v \cup D_h$, where $D_v$ is the union of the vertical components (irreducible components $C$ such that $F(C)$ is a point), and $D_h$ is the union of the horizontal components (irreducible components $C$ such that $F(C) = S$). Note that we can choose a meridian around any irreducible component of $D$ which is contained in a generic fiber of $F : X \to S \setminus B$. Hence, by Corollary 2.26 any meridian about any irreducible component of $D_h$ must be in the kernel of $F_\ast$, thus it must be trivial as a consequence of part (ii). Analogously, a meridian around an irreducible component of $D_v$ must also be in ker $F_\ast$ since its image is a power of a meridian around $P \in S \setminus B$ which is the boundary of a disk centered at $P$, and hence trivial. This concludes (iii).

**Theorem 3.2** (Main theorem, $r = 0$). Theorem 1.1 holds if $r = 0$ and $\pi_1(X \setminus D)$ is infinite.

**Proof.** Denote $G := \pi_1(X \setminus D) = \mathbb{Z}_{m_1} \ast \cdots \ast \mathbb{Z}_{m_s}$, where $m_i > 1$ for all $i \in I = \{1, \ldots, s\}$ and $s \geq 2$.

Consider the unramified covering associated with the maximal cyclic morphism $\nu : G \to \mathbb{Z}_m$, where $m := \text{lcm}(m_i | i \in I |)$. As above, given any meridian $\gamma_i$ around an irreducible component $D_i$ of $D$ such that $\nu(\gamma_i)$ has finite order $e_i$. By [19, Thm. 1.3.8], the (perhaps not reduced) divisor $E = \sum_{i} \mathbb{Z}_{m_i}$ with support $D$ is such that the unbranched covering $\tilde{Y} \to X \setminus D$ associated with $G$ extends to a branched covering $Y \to X$. This is defined by taking $H$ a divisor such that $mH$ is linearly equivalent to $E$. Consider $\sigma : X \dashrightarrow \mathcal{O}(E)$ a meromorphic section such that $\text{div}(\sigma) = E$. Then $Y = \{(x, v) \in \mathcal{O}(H) \mid \sigma(x) = v^m\}$ is a singular model of the required cyclic covering of degree $m$. Note that $\theta : U \to X \setminus D$ is an unbranched $m : 1$ cyclic cover, whereas $\theta : D' = Y \cap \{v = 0\} \to D$ is an isomorphism. The action of the deck group of $\theta : U \to X \setminus D$ on $U$ extends to an action on $Y$ by fixing $D'$ pointwise.

By Lemma 2.14 the kernel $\pi_1(U) \cong \ker \theta_\ast$ is the free group $\mathbb{F}_\rho$, where $\rho = 1 - m + m \sum_{i \in I} \left(1 - \frac{1}{m_i}\right) \geq 1$.

Denote by $\bar{Y}$ a projective surface such that $\bar{Y} \setminus \bar{D} = U$, where $\bar{D}$ is a normal-crossing divisor obtained by resolving the singularities of $D'$. We may assume that the action on $\bar{Y}$ (which on $U$ is the action by Deck transformations) extends to $\bar{Y}$. By Theorem 3.3 $U$ is induced by an admissible map $f' : U \to C'$ onto an
Consider \( \theta : C' \to C \) the quotient map by this action, where \( C \) is an open curve (non necessarily smooth). The map \( f' : U \to C' \) hence descends to a morphism \( f : X \setminus D \to C \subset \overline{C} \), where \( \overline{C} \) is a projective curve. Note that \( C \) (and \( \overline{C} \)) may not be smooth, but, by the universal property of the normalization, \( f \) lifts to \( \tilde{F} : X \setminus D \to \tilde{S} \), where \( \tilde{S} \) is the normalization of \( \overline{C} \) (a Riemann surface). Applying Stein factorization, we know that \( \tilde{F} = \pi \circ F \), where \( F : X \setminus D \to S \) is dominant and has connected fibers, and \( \pi : S \to \tilde{S} \) is a finite morphism. Note that \( F \) is an admissible map when restricted to its image.

Since the normalization \( \tilde{S} \to \overline{C} \) is a birational equivalence, a generic fiber of \( f \) is also a generic fiber of \( \tilde{F} \), which is a disjoint union of generic fibers of \( f \). Moreover, since the generic fiber of \( \tilde{F} \) is finite, the preimage through \( \theta \) of a generic fiber of \( f \) is a disjoint union of generic fibers of \( f' \). Restricting to connected components, one finds \( P \in S \) and \( P' \in C \) such that \( \theta : (f')^{-1}(P') \to F^{-1}(P) \) is a finite covering map, where \( (f')^{-1}(P') \) is a generic fiber of \( f' \) and \( F^{-1}(P) \) is a generic fiber of \( F \).

Let's check that the morphism \( F_* : \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(S) \) is an isomorphism, where \( S \) is endowed with the maximal orbifold structure with respect to \( F : X \setminus D \to S \). By Corollary 2.26, this is equivalent to showing that the image of \( \pi_1(F^{-1}(P)) \) in \( \pi_1(X \setminus D) \) is trivial. Since \( f' : U \to C' \) induces an isomorphism on fundamental groups, Corollary 2.26 tells us that the inclusion \( (f')^{-1}(P') \to U \) induces the trivial map on fundamental groups. Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_1((f')^{-1}(P)) & \longrightarrow & \pi_1(U) \\
\downarrow \varphi_* & & \downarrow \varphi_* \\
\pi_1(F^{-1}(P)) & \longrightarrow & \pi_1(X \setminus D) \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s},
\end{array}
\]

where the horizontal arrows are induced by inclusion. Note that the arrow on the left is a finite covering space, so its image is a finite index normal subgroup of \( \pi_1(F^{-1}(P)) \). The commutativity of the diagram implies that \( \varphi_* \) factors through \( \pi_1(F^{-1}(P))/\theta_* (\pi_1((f')^{-1}(P))) \). Hence, the image of \( \varphi_* \) is a finite subgroup of \( \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s} \). Moreover, by Corollary 2.26 this subgroup is normal. By the Kurosh subgroup theorem, the only finite normal subgroup of \( \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s} \) is the trivial subgroup. Hence, \( \varphi_* \) is the trivial morphism, and thus \( F_* : \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(S) \) is an isomorphism.

Note that if \( S \) is a Riemann surface such that \( \pi_1^{\text{orb}}(S) \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s} \), then \( S = \mathbb{P}^1 \), and the image of \( F \) must be \( \mathbb{P}^1 \) with one point removed. Indeed, after abelianizing the presentation of \( \pi_1^{\text{orb}}(S) \) from Section 2.4, it follows that \( S \) must have genus 0 (so \( S = \mathbb{P}^1 \)). Moreover, \( F(X \setminus D) \) is either \( \mathbb{P}^1 \) or \( \mathbb{C} \). Let us see that it is indeed the latter. Suppose that \( F(X \setminus D) = \mathbb{P}^1 \), so none of the irreducible components of \( D \) are fibers of \( F : X \to \mathbb{P}^1 \). Then, as in Theorem 3.1, the inclusion of \( X \setminus D \) to \( X \) induces an isomorphism in fundamental groups, and thus \( \pi_1(X) \) is isomorphic to a non-trivial free product. This is impossible by [15].

We have shown that, under the assumptions of this theorem, if \( \pi_1(X \setminus D) \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s} \), with \( m_i, s \geq 2 \) for all \( i = 1, \ldots, s \), then there exists an admissible map \( \tilde{F} : X \setminus D \to \mathbb{P}^1 \setminus D \) that induces an isomorphism \( F_* : \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1 \{1, m_0\}) \), where \( \mathbb{P}^1 \) is endowed with the maximal orbifold structure with respect to \( F : X \setminus D \to \mathbb{P}^1 \). Note that \( \tilde{m} = (m_1, \ldots, m_s) \), and those are the multiplicities of the multiple fibers. The remaining condition for \( D = D_f \cup D_t \) can be proved as in [3.1] \( \square \)
Moreover, if $D \leq s$ and $X$ is connected, we will see an example satisfying the hypotheses of Theorem 3.2 with a non-simply-connected surface. The simply-connectedness condition in Remark 3.3 is important.

Remark 3.4. The simply-connectedness condition in Remark 3.3 is important. We will see an example satisfying the hypotheses of Theorem 3.2 with a non-simply-connected surface $X$, a curve $D$ on $X$ such that $\pi_1(X \setminus D) \cong \mathbb{Z}_3 \ast \mathbb{Z}_3$, and a rational map realizing the isomorphism with the orbifold fundamental group of $\mathbb{P}^1((3,3))$.

Let $X$ be the double cover of $\mathbb{P}^2$ ramified along a generic sextic $C = \{f_3^2 + f_3^2 = 0\}$, where $f_i$ is a homogeneous polynomial in three variables of degree $i$. It is well known (cf. [28]) that $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_2 \ast \mathbb{Z}_3$ (also as a consequence of Theorem 1.4). A meridian around $C$ can be given as $xy$ where $x^3 = y^3 = 1$. Using the calculations in [3] for the proof of Lemma 2.17 one can show $\pi_1(X \setminus D) \cong \mathbb{Z}_3 \ast \mathbb{Z}_3$ generated by $y_0 := y$ and $y_1 := xyx$, where $D$ is the preimage of $C$. Note that the six-fold cover ramifies fully along $C$ and thus its preimage is irreducible. Since this cover factors through $D$, $D$ must also be irreducible and thus it satisfies Condition 2.9. The preimage of $(yx)^2 = y_0y_1$ becomes a meridian of $D$ and by Lemma 2.5 one has $\pi_1(X) = \pi_1(X \setminus D)/\text{NC}(y_0y_1) = H_1(X; \mathbb{Z}) = \mathbb{Z}_3$. Theorem 3.2 ensures the existence of an admissible map $F : X \dashrightarrow \mathbb{P}^1$ with two multiple fibers of multiplicity 3 and such that $D$ is the preimage of a point in $\mathbb{P}^1$.

Corollary 3.5. Under the notation of Theorem 1.1, suppose that an admissible map $F : X \dashrightarrow S$ satisfies conditions 2.7, 2.11 where $\pi_1(X \setminus D)$ is an open orbifold group (not necessarily infinite). If $H_1(X; \mathbb{Z})$ is torsion and $D$ satisfies Condition 2.9, then $S = \mathbb{P}^1$ and $r + 1$ is in fact the number of irreducible components of $D$. In particular, $D = D_f$ is a fibered-type curve.

Proof. The result follows from Remark 2.11 and Proposition 2.11.

Example 3.6. Let $Q$ be a smooth conic in $\mathbb{P}^2$ and $P \in Q$. Consider $\ell$ the tangent line to $Q$ through $P$ and $D$ a union of $\ell$ and $r$ lines through $P$. The quadric surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ can be defined as the 2:1 cover $\sigma : X \rightarrow \mathbb{P}^2$ ramified along $Q$. In particular, $X$ is simply connected. Also note that $\sigma^{-1}(D)$ is a union of $r + 2$ irreducible components, namely $r$ curves of bidegree $(1,1)$ and the rulings $\sigma^{-1}(\ell) = \ell_1 \cup \ell_2$, for $\ell_1$ (resp. $\ell_2$) of bidegree $(1,0)$ (resp. $(0,1)$) all of them passing through $\sigma^{-1}(P)$. The curve $\sigma^{-1}(D)$ has $r + 2$ irreducible components and does not satisfy Condition 2.9. Note that $\pi_1(\mathbb{P}^2 \setminus D) = F_r$ and one can check that also $\pi_1(X \setminus \sigma^{-1}(D)) = F_r$.

Theorem 1.1 gives necessary geometric conditions for a quasi-projective surface $X \setminus D$ to have an infinite open orbifold fundamental group. The curve $D$ need not be of fibered-type, but $D_f$ (which is a union of a non-empty subset of its irreducible components) is a fibered-type curve coming from an admissible map $F : X \dashrightarrow S$. The following result illustrates that $X \setminus D_f$ behaves exactly like $X \setminus D$ in Theorem 1.1.

Corollary 3.7. Under the conditions of Theorem 1.1 and using the notation therein, the inclusion induces an isomorphism $\pi_1(X \setminus D) \cong \pi_1(X \setminus D_f)$, and $F_* : \pi_1(X \setminus D_f) \rightarrow \pi^\text{orb}_{(n+1,m)}(S_{(n+1,m)})$ is an isomorphism.
Proof. The isomorphism $\pi_1(X \setminus D) \cong \pi_1(X \setminus D_j)$ follows from Lemma 2.6 and the fact that the meridians of $D_i$ are trivial in $\pi_1(X \setminus D)$. Using Corollary 2.26 for $U = X \setminus (D_i \cup B)$ and $U = X \setminus B$, we see that $F_1 : X \setminus D_j \rightarrow S \setminus \Sigma_0$ must also induce isomorphisms in (orbifold) fundamental groups, and that the maximal orbifold structure on $S$ with respect to $F_1 : X \setminus D_j \rightarrow S \setminus \Sigma_0$ must coincide with the one with respect to $F_1 : X \setminus D \rightarrow S \setminus \Sigma_0$. □

3.2. Extensions of the main theorem to finite open orbifold groups. Theorem 1.1 describes the geometry of a curve $D \subset X$ when $\pi_1(X \setminus D)$ is an infinite open orbifold group (i.e. non-abelian or $\mathbb{Z}$), where $X$ is a smooth projective surface. In this section, we give extra hypotheses under which similar results hold in the remaining abelian cases (the trivial group and finite cyclic groups).

Proposition 3.8 (Main theorem, trivial group case). Let $D \subset X$ be a curve in a smooth projective surface $X$. Assume that $\pi_1(X \setminus D)$ is trivial. If $D$ is a ample divisor, then there exists an admissible map $F : X \rightarrow \mathbb{P}^1$ as in Theorem 1.1 for $S_{(n+1,m)} = \mathbb{P}^1_{(1,-)}$ satisfying conditions (i)(iii) $\pi_1$.

Proof. Since $D$ is ample, $D$ defines an embedding $X \setminus D \hookrightarrow \mathbb{C}^k$ for some $k$. Projecting to a generic 1-dimensional subspace inside $\mathbb{C}^k$, we get a dominant morphism $F : X \setminus D \rightarrow \mathbb{C}$ with connected fibers, which can be extended to a rational map $F : X \dashrightarrow \mathbb{P}^1$. Using Remark 2.16 we have that $\pi_1^\text{orb}(\mathbb{P}^1_{(n+1,m)})$ is trivial, where $\mathbb{P}^1_{(n+1,m)}$ is the maximal orbifold structure with respect to $F : X \setminus D \rightarrow \mathbb{C}$. This implies that $n = 0$ (so in particular $F : X \setminus D \rightarrow \mathbb{C}$ is surjective) and that $m$ is the trivial orbifold structure. □

Remark 3.9. In order to clarify the hypothesis given in Proposition 3.8 we will exhibit an example where the result does not follow when $\pi_1(X \setminus D) = 1$ and $D$ is not an ample divisor.

Consider a line $L$ in a smooth cubic $X$. It is well known that $\pi_1(X) = \pi_1(X \setminus L) = 1$ and $L^2 = -1$. Thus $L$ cannot be the fiber of an admissible map $X \dashrightarrow \mathbb{P}^1$.

Proposition 3.10 (Main theorem, case $\mathbb{Z}_m$, $m > 1$). Let $D \subset X$ be a curve in a smooth projective surface $X$. Assume that $\pi_1(X \setminus D) \cong \mathbb{Z}_m$, for $m > 1$. If $X$ is simply connected, then there exists an admissible map $F : X \dashrightarrow \mathbb{P}^1$ as in Theorem 1.1 for $S_{(n+1,m)} = \mathbb{P}^1_{(1,m)}$ satisfying conditions (i)(iii). Moreover, $D = D_f$ is a fibered-type curve.

Proof. Let $D = \cup_{i=0}^r D_i$ be the decomposition of $D$ into irreducible components. Any meridian $\gamma_i$ around an irreducible component $D_i$ of $D$ has finite order $e_i$, dividing $m$. Consider the divisor $E = \sum \frac{m}{e_i} D_i$. Since $\pi_1(X)$ is the result of factoring out $\pi_1(X \setminus D)$ by the normal closure of the meridians around all the $D_i$’s, and $X$ is simply connected, note that $E$ is not a positive multiple of an effective divisor. By [19] Thm. 1.3.8, the unbranched universal covering $\tilde{Y} \rightarrow X \setminus D$ associated with $G$ extends to a branched covering $Y \rightarrow X$. This implies there exists an effective divisor $H$ in $X$ such that $E \sim mH$. The linear equivalence provides a morphism $F : X \dashrightarrow \mathbb{P}^1$ such that $F : X \setminus D \rightarrow \mathbb{C}$ (so in particular $D$ is a fibered-type curve). After applying Stein factorization, we may assume that $F : X \dashrightarrow \mathbb{P}^1$ is the composition of $\tilde{F} : X \dashrightarrow \mathbb{P}^1$ and $\beta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, where $\tilde{F}$ has connected fibers and $\beta$ is generically $k : 1$. In principle, $D$ is a union of $n + 1$ fibers of $\tilde{F}$ above points in $\mathbb{P}^1$, although it must be only one fiber, or else $\tilde{F}_* : \pi_1(X \setminus D) \rightarrow \pi_1^\text{orb}(\mathbb{P}^1_{(n+1,m)})$ wouldn’t be surjective, which would contradict Remark 2.16. Now, $F^*([1 : 0]) = E$ and $\beta^{-1}(1 : 0)$ is just a point, so $E$ must be $k$ times an effective divisor with support $D$, and hence $k = 1$. Thus, $F : X \dashrightarrow \mathbb{P}^1$ has connected fibers. We know that $F^*([0 : 1]) = mH$, so $F$ induces a surjective morphism $\pi_1(X \setminus D) \rightarrow \pi_1^\text{orb}(\mathbb{P}^1_{(1,m)})$. 


Since $Z_m$ is Hopfian, this morphism is an isomorphism, and $\mathbb{P}^1_{(1,(m))}$ is the maximal orbifold structure with respect to $F_I: X \setminus D \to \mathbb{C}$. □

**Remark 3.11.** In order to clarify the simply-connected hypothesis given in Proposition 3.10, we will exhibit an example where the result does not follow when $\pi_1(X) \neq 1$. Consider a surface $X \subset \mathbb{P}^n$ with finite cyclic fundamental group $\mathbb{Z}_m$ and containing a line $L \subset X$. Take a hyperplane $H \subset \mathbb{P}^n$ intersecting $X$ transversally such that $L \not\subset H$. Note that $D = H \cdot X$ defines a reduced irreducible divisor in $X$ and $L \cap D = L \cap H = \{P\}$. Hence, one can define a meridian $\gamma$ of $D$ around $P$ such that $\gamma \subset L$. Since $L$ is a rational curve, $\gamma$ is trivial in $X$. If there was a map $F: X \dashrightarrow S$ such that $D$ is a fiber of $F$, then $F_*: \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(S \setminus \{p\})$ would be surjective. Since $\pi_1(X \setminus D) = \mathbb{Z}_m$, this implies $g_S = 0$ and the orbifold structure of $S$ contains exactly one orbifold point of order $m > 1$. However, in that case $F_*(\gamma)$ would have order $m$, which is a contradiction since we have proved that $\gamma$ is trivial.

### 3.3. Main Theorem for curves in $\mathbb{P}^2$

We will pay a special attention to the case $X = \mathbb{P}^2$, in Theorem [11]. Recall that every curve in $\mathbb{P}^2$ satisfies Condition [2.9] and is ample. Hence, the extra hypotheses needed in the relevant results of Sections 3.1 and 3.2 are always satisfied for curves in $\mathbb{P}^2$, and thus, the following stronger version of the Main Theorem [1.1] holds.

**Corollary 3.12** (Main Theorem, curves in $\mathbb{P}^2$). Let $D$ be a curve in $\mathbb{P}^2$. Suppose that $\pi_1(\mathbb{P}^2 \setminus D)$ is an open orbifold group. Then, there exists $r \geq 0$ and $p \geq q \geq 1$ with $\gcd(p,q) = 1$ such that $\pi_1(\mathbb{P}^2 \setminus D) \cong \mathbb{F}_r \ast \mathbb{Z}_p \ast \mathbb{Z}_q$, and an admissible map $F: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that:

1. $F$ induces an orbifold morphism

$$F_I: \mathbb{P}^2 \setminus D \to \mathbb{P}^1_{(r+1,m)},$$

where $\mathbb{P}^1_{(r+1,m)}$ is maximal with respect to $F_I$, and $m = (p,q)$.

2. $F_*: \pi_1(\mathbb{P}^2 \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(r+1,m)})$ is an isomorphism.

3. $D = F^{-1}(\Sigma_0)$ is a fibered-type curve, which is the union of $r + 1$ irreducible fibers of $F$.

Moreover, after possibly a change of coordinates in $\mathbb{P}^1$, $F = [f_{d_1}^p : f_{d_2}^q]$, where $d_1 \geq d_2 \geq 1$ satisfy $\gcd(d_1,d_2) = 1$, $f_{d_i}$ is a homogeneous polynomial of degree $d_i$ for $i = 1, 2$ which is not a $k$-th power of another polynomial for any $k \geq 2$, $f_{d_1}$ and $f_{d_2}$ don’t have any components in common, and $M_F \subset \{[0 : 1],[1 : 0]\}$. More concretely,

1. If $p > q > 1$, then $d_1 = p$, $d_2 = q$ and the pencil $F = [f_{d_1}^p : f_{d_2}^q]$ has exactly two multiple fibers corresponding to $[0 : 1],[1 : 0] \not\in \Sigma_0$.

2. If $p > q \geq 1$, then $d_1 = p$, $[1 : 0] \in \Sigma_0$, and the pencil $F = [f_{d_1}^p : f_{d_2}^q]$ has at least one multiple fiber corresponding to $[0 : 1] \not\in \Sigma_0$.

3. If $q = p = 1$, then $F = [f_{d_1}^d : f_{d_2}^d]$ has at most two multiple fibers corresponding to $[0 : 1],[1 : 0] \in \Sigma_0$.

**Proof.** The existence of an admissible map $F: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ inducing an isomorphism $F_*: \pi_1(\mathbb{P}^2 \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(r+1,m)})$ follows from Theorem [11] and Remark [2.4] as well as from Propositions [3.8] and [3.10]. By Remark [3.3], $\pi_1(X \setminus D) \cong \mathbb{F}_r \ast \mathbb{Z}_p \ast \mathbb{Z}_q$. This concludes the proof of parts (i) and (ii). Part (iii) follows from Corollary [3.5].

By Remark [2.3], after possibly a change of coordinates in $\mathbb{P}^1$, we may assume $M_F \subset \{[0 : 1],[1 : 0]\}$. $F$ is of the form $F = [f_{k_1}^{p_1} : f_{k_2}^{p_2}]$ with where $f_{k_i}$ is a homogeneous polynomial of degree $k_i$ such that $d_1k_2 = d_2k_1$, $f_{k_1}$ and $f_{k_2}$ have no common factors, and $f_{k_i}$ is not a $l$-th power of another polynomial for any $l \geq 2$. 
for \(i = 1, 2\). The condition \(\gcd(d_1, d_2) = 1\) is necessary for the fibers of \(F\) to be connected. In particular, there exists \(t \in \mathbb{Z}_{>0}\) such that \(k_i = d_i t\) for \(i = 1, 2\). The rest of the proof is straightforward using Remark 2.12, including the fact that \(t = 1\) (and hence \(d_i = k_i\) for \(i = 1, 2\)) in each case \((1) \Rightarrow (2)\).

\[\square\]

4. Addition-Deletion Lemmas for orbifold groups

In Section 3, we have seen geometric conditions for a quasi-projective surface \(X \setminus D\) to have open orbifold fundamental group. Under the hypotheses and notation of Theorem 1.1, \(D = D_f \cup D_t\). Hence, if we let \(U = X \setminus (D_t \cup D_f)\), then \(X \setminus D = U \setminus D_f\) is the complement of a fibered-type curve inside a smooth quasi-projective surface given by an admissible map \(F : U \to S\) onto a smooth projective curve \(S\).

The purpose of this section is to prove addition-deletion results of fibers for complements of fibered-type curves inside smooth quasi-projective surfaces \(U\) which have open orbifold fundamental groups. Before we do that, we need some technical results regarding presentations of fundamental groups of complements of fibered-type curves.

4.1. Preparation Lemmas. The results in this section give presentations of fundamental groups of fibered-type curve complements, but they do not make use of any Zariski-Van Kampen type computations. Instead, they follow group-theoretical arguments from Lemma 2.21, Corollaries 2.25 and 2.26 and the results in Section 4.2.

In Lemma 2.21, we gave a presentation of \(\pi_1(U_B)\) in the case where \(B \supset B_F\). We now give a more explicit presentation for \(\pi_1(U_B)\) in the general case when \(B\) does not necessarily contain \(B_F\).

Throughout this section, \(F : U \to S\) will be an admissible map from a smooth quasi-projective surface \(U\) to a smooth projective curve \(S\). We will use the notation introduced in Section 2.5.

**Lemma 4.1.** Assume that \(F : U \to S\) is an admissible map. Let \(B \subset S\), with \(#B \geq 1\). Consider \(\Gamma_F\) (resp. \(\Gamma_S = \Gamma_S(B \cup B_F)\)) an adapted geometric set of fiber (resp. base) loops w.r.t. \(F\) and \(B\) (resp. \(B \cup B_F\)). Let \(B' = M_F \setminus B\), and let \(\Gamma_S' = \Gamma_S \setminus \{\gamma \mid P_k \in B_F \setminus (B \cup B')\}\).

Then, \(\Gamma_S'\) is an adapted geometric set of base loops w.r.t. \(F\) and \(B \cup B'\), and \(\pi_1(U_B)\) has a finite presentation with generators \(\Gamma_F \cup \Gamma_S'\) and relations of the form

\[
\begin{align*}
\text{(R1)} & \quad [\gamma_k, w] = 1, \text{ for any } w \in \Gamma_F \text{ and any } \gamma_k \in \Gamma_S' \text{ a lift of a meridian of } P_k \in B \setminus B_F, \\
\text{(R2)} & \quad [\gamma_k, w] = z_k, \text{ for the remaining } \gamma_k \in \Gamma_S', \text{ for any } w \in \Gamma_F, \text{ where } z_k \text{ is a word in } \Gamma_F, \text{ (that depends both on } \gamma_k \text{ and } w), \\
\text{(R3)} & \quad y = 1 \text{ for a finite number of words } y \text{ in } \Gamma_F, \\
\text{(R4)} & \quad z_k = \gamma_k m_k, \text{ any } \gamma_k \in \Gamma_S' \text{ a lift of a meridian of } P_k \in B', \text{ where } m_k \text{ is the multiplicity of the fiber } F^*(P_k), \text{ and } z_k \text{ is a word in } \Gamma_F, 
\end{align*}
\]

**Proof.** Note that \(B \cup B_F\) satisfies the conditions of Lemma 2.21 and hence \(\pi_1(U_{B \cup B_F})\) admits a presentation generated by \(\Gamma_F \cup \Gamma_S\), where \(\Gamma_F\) (resp. \(\Gamma_S\)) is an adapted geometric set of fiber (resp. base) loops w.r.t. \(F\) and \(B \cup B_F\). Note that \(\Gamma_F\) is an adapted geometric set of fiber loops w.r.t. \(F\) and \(B \cup B_F\) if and only if the same holds w.r.t. \(B\).

By Corollary 2.25, a presentation of \(\pi_1(U_B)\) can be obtained by factoring \(\pi_1(U_{B \cup B_F})\) by the normal closure of all the meridians about irreducible components of fibers above points in \(B_F \setminus B\). We abuse notation and see both the elements of \(\Gamma_F\) and \(\Gamma_S\) as elements of \(\pi_1(U_B)\), \(\pi_1(U_{B \cup B'})\) or \(\pi_1(U_{B \cup B_F})\) when no ambiguity seems likely to arise.
Let $\gamma_k$ be such that $F_\ast(\gamma_k)$ is a meridian around a point $P_k \in B_F \setminus (B \cup B')$. Since $F_\ast \ast(P_k)$ is not a multiple fiber, $\gamma_k$ is trivial in $\pi_1(U_B)$ by condition [4] in Definition 2.10. In particular, this proves that $\pi_1(U_B)$ can be generated by $\Gamma_F \cup \Gamma'_{S_F}$. Since $\Gamma_S$ is an adapted geometric set of base loops w.r.t $F$ and $B \cup B'$, then $\Gamma_{S_F}$ is an adapted geometric set of base loops w.r.t $F$ and $B \cup B'$. As in the proof of Lemma 2.20 there exists a meridian in $\pi_1(U_{B\cup B'})$ about any given irreducible component of $C_{P_k}$ of the form $w\gamma_k^m$ for some $m \geq 1$, and for $w \in \ker(F_{U_{B\cup B'}})_\ast$, which is a word in $\Gamma_F$ by Corollary 2.26. Hence, we have shown that the normal closure of the subgroup generated by the meridians about each irreducible component of $C_{P_k}$ is the normal closure of a subgroup of $\pi_1(U_{B\cup B'})$ generated by $\gamma_k$ and a finite number of words in $\Gamma_F$. This gives rise to relations of the form $[R_3]$.

Let $\gamma_k$ be such that $F_\ast(\gamma_k)$ is a meridian around a point in $B'$. Similarly as in the previous paragraph, we can choose a meridian in $\pi_1(U_{B\cup B'})$ about any of the irreducible components of $C_{P_k}$ which is of the form $w(\gamma_k^m)^m$ for some $m \geq 1$, and $w$ a word in $\Gamma_F$. Let $N_k$ be the normal closure of the subgroup generated by such choice of meridians about each of the irreducible components of $C_{P_k}$. Note that, by definition of $\gamma_k \in \pi_1(U_{B\cup B'})$ and Corollary 2.26 the equality $\gamma_k^m = z_k$ holds in $\pi_1(U_B)$ for some word $z_k$ in $\Gamma_F$ (an element of $\ker F_\ast$). Hence, the relations given by $w(\gamma_k^m)^m = 1$ coming from the chosen generators of $N_k$ (as a normal closure) together with the relation $\gamma_k^m = z_k$ (which we know holds in $\pi_1(U_B)$) are equivalent to a finite number of relations of the form $y = 1$ for words $y$ in the letters of $\Gamma_F$ (type $[R_3]$) and the relation $\gamma_k^m = z_k$ (all the relations of type $[R_4]$). Let $P_k \in B' \setminus B_F$, hence $F_\ast(\gamma_k) \in \pi_1(\{p_1\} \setminus (B \cup B'))$ induces the trivial monodromy morphism in the elements of $\pi_1(F^{-1}(P))$, since $P_k \notin B_F$. In other words, $\gamma_k \in \pi_1(U_B)$ commutes with any word in $\Gamma_F$.

The result now follows from adding the relations found in the previous paragraphs to the presentation of $\pi_1(U_{B\cup B'})$ given in Lemma 2.21, the relation $\prod x_j = \prod [a_i, b_i]$ is of type $[R_3]$, the monodromy relations of $\pi_1(U_{B\cup B'})$ corresponding to $P_k$ are the ones of type $[R_2]$ if $P_k \in (B_F \cap B \cup B')$, of type $[R_1]$ if $P_k \in B \setminus B_F$ (by the previous paragraph), or become of type $[R_5]$ after using that $\gamma_k$ is trivial in $\pi_1(U_B)$ if $P_k \in B_F \setminus (B \cup B')$. \hfill $\square$

Our next goal is to describe cases in which $\pi_1(U_B)$ has a presentation on generators $\Gamma_S'$. This will provide candidates for $B \subset S$ such that $F_\ast : \pi_1(U_B) \to \pi_1(\{p_1\} \setminus (B \setminus \{Q\}))$ is an isomorphism (and, in particular, such that $\pi_1(U_B)$ is an open orbifold group). This goal is achieved in Corollary 4.3 with the help of Lemma 4.2.

We follow notation from Section 2.5 and Lemma 4.1.

**Lemma 4.2.** Let $F : U \to S$ be an admissible map, $B = \{P_0, \ldots, P_n\} \subset S$ a non-empty set, and assume $Q \in B \setminus (B_F \cap B)$. Let $S_{(n+1,m)}$ (resp. $S_{(n,m)}$) be the maximal orbifold structure of $S$ w.r.t. $F$ : $U_B \to S \setminus B$ (resp. $F' : U_{B\setminus\{Q\}} \to S \setminus (B \setminus \{Q\})$). Consider $K := \ker F_\ast$ the kernel of $F_\ast : \pi_1(U_B) \to \pi_1^\orb(S_{(n+1,m)})$.

Moreover, assume that $F_\ast : \pi_1(U_{B\setminus\{Q\}}) \to \pi_1^\orb(S_{(n,m)})$ is an isomorphism, and furthermore, either
- $n \geq 1$ or
- $n = 0$ and $B' := M_F \setminus B \neq \emptyset$,

then $K$ is an abelian group. Furthermore, an adapted geometric set of base loops $\Gamma_S' = \Gamma_S(B \cup B')$ w.r.t $F$ and $B \cup B'$ can be chosen so that $K = N$, where $N$ is the normal closure of the subgroup $\langle \gamma_k^m \mid P_k \in B' \rangle$.

**Proof.** We use the presentation of $\pi_1(U_B)$ given in Lemma 4.1 and the notation therein.
Since $\gamma^{m_k}_k \in K$ for all $P_k \in B'$ and $K$ is normal, $N \leq K$ for every choice of $\Gamma'_S$ as in Lemma 4.1.2.

Assume now that $F_*$ is an isomorphism and either $n \geq 1$ or $n = 0$ and $B' \neq 0$. We will show that $K$ is abelian and $K \leq N$ for some choice of $\Gamma'_S$. Let $\gamma \in \pi_1(U_B)$ be any positively oriented meridian about the irreducible fiber $C_Q$, such that $F_*(\gamma)$ is a positively oriented meridian around $Q$ in $S \setminus (B \cup B_F)$. Note the following:

1. Since $C_Q$ is irreducible, $\text{NCl}(\gamma)$ is independent of the choice of the meridian $\gamma$ by Lemma 4.1.
2. $K \leq \text{NCl}(\gamma) \leq \pi_1(U_B)$ is a subgroup of the normal closure of $\langle \gamma \rangle$ in $\pi_1(U_B)$. To see this consider $w \in K$. The projection of $w$ in $\pi_1(U_B \setminus \{Q\})$ is trivial by Corollary 2.26, so $w$ is an element of $\text{NCl}(\gamma)$.
3. $K \leq \pi_1(U_B)$ is abelian. To see this note that any meridian around the typical fiber $C_Q$ commutes with $K = \iota_*(F^{-1}(P))$ by Lemma 4.1.(R1). By Lemma 2.26 this means $\text{NCl}(\gamma)$ is contained in the centralizer of $K$. By (3) $K \leq \text{NCl}(\gamma)$, in particular, $K$ is an abelian subgroup.
4. There exists an adapted geometric set of base loops $\Gamma'_S$ w.r.t. $F$ and $B \cup B'$ such that one such positively oriented meridian $\gamma$ can be written as a word in $\Gamma'_S$.

If $n \geq 1$, we may assume $Q = P_1$ and $\gamma = \gamma_1$ in $\Gamma'_S$, which concludes (4) in the case $n \geq 1$.

Suppose that $n = 0$ and $b' = \#B' \geq 1$. In this case, $r = 2g$. Let $\Gamma'_S = \Gamma_S(B \cup B')$ be as in Lemma 4.1.1. By definition of $\Gamma'_S$, if $\tilde{\gamma} = \prod_{i=1}^{g_{b'}} [\gamma_{b'+2i-1}, \gamma_{b'+2i}] \cdot \prod_{P_k \in B'} \gamma_k^{m_k}$ (see condition (3) in Definition 2.19), then $F_*(\tilde{\gamma})$ is a meridian around $Q$ in $\pi_1(S \setminus (B \cup B'))$. By Corollary 2.26 there exists a word $z$ in the letters $\Gamma'_F$ such that $\gamma = \tilde{\gamma}z$ is a meridian around $C_Q$ whose image by $F_*$ is a meridian around $Q$. After replacing $\gamma_1$ by $z^{-1}\gamma_1$ in $\Gamma'_S$ one can assume $\gamma = \tilde{\gamma}$. This concludes (4) in the case $n = 0$ and $b' \geq 1$.

Finally, let’s show $K \leq N$, where $N$ is defined using the $\Gamma'_S$ found in observation 4.2 above. By 4.1, $\gamma = w(\gamma_1, \ldots, \gamma_{r+b'})$ is a word in $\Gamma'_S$. Let $\varphi : \mathbb{F}_{r+b'} = \langle \delta_1, \ldots, \delta_{r+b'} \rangle \to \pi_1(U_B)$ be the group homomorphism given by $\delta_i \mapsto \gamma_i$ for all $i \in \{1, \ldots, r+b'\}$. Let $\alpha$ be an element in $\text{NCl}(\gamma)$ and write $\alpha$ as a product $\prod_{i=1}^{g} g_i^{-1} \gamma_i^{d_i} g_i$, where $g_i$ is an element of $\pi_1(U_B)$. By Lemma 4.1 applied to $\pi_1(U_B)$, $g_i$ in $\pi_1(U_B)$ can be written as $g_i = w_1 h_i$, where $w_i$ is in $\Gamma_F$ (so $w_i \in K$) and $h_i$ is a word in $\Gamma'_S$. Since $\gamma$ commutes with the elements of $K \leq \pi_1(U_B)$, $\alpha$ can be written as the product $\prod_{i=1}^{g} h_i^{-1} \gamma_i^{d_i} h_i$. In other words, we have shown that $\text{NCl}(\gamma) = \varphi(\text{NCl}(\alpha))$, where $\text{NCl}(\alpha)$ is the normal closure in $\mathbb{F}_{r+b'}$ of the subgroup generated by $a$, where $a = w(\delta_1, \ldots, \delta_{r+b'})$ (the word $w$ for $\gamma$ in the new letters $\delta_1, \ldots, \delta_{r+b'}$). In particular, one has $K \leq \varphi(\text{NCl}(\alpha))$.

A similar argument, this time using that $K$ is abelian, shows that

$$N = \varphi(\text{NCl}((\delta_k^{m_k} | P_k \in B')))$$

Now, the composition $F_* \circ \varphi$ induces an isomorphism in the quotients given by the composition of

$$\mathbb{F}_r \ast \left( \bigast_{P_k \in B'} \mathbb{Z}_{m_k} \right) \cong \mathbb{F}_{r+b'} / \text{NCl}((\delta_k^{m_k} | P_k \in B')) \to \pi_1(U_B)/N$$

with

$$\pi_1(U_B)/N \to \pi_1^{\text{orb}}(S_{n+1,m_i}) \cong \mathbb{F}_r \ast \left( \bigast_{P_k \in B'} \mathbb{Z}_{m_k} \right).$$
This implies that \( \ker(F_r) \cap \text{Im}(\phi)/N \) is trivial. Recall that \( K = \ker(F_r) \) and that, \( K \leq \phi(\text{NCl}(a)) \). Hence \( K = \ker(F_r) \cap \text{Im}(\phi) \) and thus we arrive at the equality \( K = N \). This concludes the proof.

**Corollary 4.3.** Under the same notation as Lemma 4.1 and the same hypotheses and choice of \( \Gamma'_S \) as Lemma 4.2 every element of \( K \) can be written as a word in the letters \( \Gamma'_S \). In particular, the presentation of \( \pi_1(U_B) \) from Lemma 4.2 can be transformed to a presentation on generators \( \Gamma'_S \).

**Proof.** Consider the presentation of \( \pi_1(U_B) \) given in Lemma 4.1. The generators of \( \pi_1(U_B) \) in \( \Gamma_F \) are elements of \( K \), which equals \( N \) by Lemma 4.2. Equation (13) in the proof of Lemma 4.2 implies that the elements of \( N \) are products of elements of the form \( v_1^{m_1} \cdots v_k^{m_k} w^{-1} \), where \( v \) is a word in \( \Gamma'_S \), and \( P_k \in B' \). Using this, one can eliminate the generators of \( \pi_1(U_B) \) in \( \Gamma_F \) in the presentation given in Lemma 4.1. \( \square \)

**Remark 4.4.** Suppose that \( n \geq 1 \), and the hypotheses, choice of \( \Gamma'_S \) and notation of Lemma 4.2 in the proof of Lemma 4.2 \( \Gamma'_S \) was chosen so that \( \gamma = \gamma_1 \in \Gamma'_S \). Since \( \text{NCl}(\gamma) \) is contained in the centralizer of \( N = K \) by (3) in the proof of Lemma 4.2 we can assume that the elements of \( N \) are products of elements of the form \( v_1^{m_1} \cdots v_k^{m_k} w^{-1} \), where \( v \) is a word in \( \Gamma'_S \setminus \{\gamma_1\} \), i.e. the letter \( \gamma_1 = \gamma \) does not appear in \( v \).

**Remark 4.5.** Under the hypotheses, choice of \( \Gamma'_S \) and notation of Lemma 4.2, suppose moreover that \( U \) is simply-connected, \( B = \{Q\} \) (\( Q \notin B_F \)), and \( M_F \neq \emptyset \). Recall that in this case \( r = n = 0 \) (Remark 2.1) and \( B' = M_F \). In this setting, Corollary 2.2 implies \( \text{NCl}(\gamma) = \pi_1(U_B) \). By (3) in the proof of Lemma 4.2 one has \( K = N \) is contained in the center of \( \pi_1(U_B) \). In particular, any subgroup generated by elements in \( K \) is normal, and thus \( N = \langle \gamma_1^{m_k} | P_k \in M_F \rangle \).

The following two corollaries pertain to the case \( n \geq 1 \) (Corollary 4.6) and \( n = 0 \), \( B' = M_F \setminus B = \emptyset \) (Corollary 4.7) in Lemma 4.2 respectively, and give useful presentations of \( \pi_1(U_B) \). Note that in Corollary 4.7 \( S = \mathbb{P}^1 \) and \( U \) are both assumed to be simply connected.

**Corollary 4.6.** Assume \( F : U \to S \) is an admissible map. Let \( B \subset S \) be such that \#B = \#B + 1 \geq 2. Let \( S_{\{n+1, n\}} \) (resp. \( S_{\langle n, n\rangle} \)) be the maximal orbifold structure of \( S \) with respect to \( F : U_B \rightarrow S \setminus B \) (resp. \( F : U_B \setminus \{P_1\} \rightarrow S \setminus (B \setminus \{P_1\}) \)).

Suppose that \( F_n : \pi_1(U_B) \rightarrow \pi_1(S_{\{n+1, n\}}) \) is an isomorphism, and that \( P_1 \notin B_F \). Let \( B' = M_F \setminus B \), and let \( \Gamma'_S = \Gamma'_S(B \cup B') \) be an adapted geometric set of base loops w.r.t. \( F \) and \( B \cup B' \) as in Remark 4.4. Then \( \pi_1(U_B) \) has a finite presentation

\[
\pi_1(U_B) = \langle \Gamma'_S \setminus \{\gamma_1\} | \gamma_j, w_i \rangle \text{ for all } j \in J \text{ and } \gamma_i = \gamma_1 w_i \text{ for all } i \in I, \text{ where } w_i \text{ is a word in } \Gamma'_S \setminus \{\gamma_1\}.
\]

**Proof.** Consider the presentation of \( \pi_1(U_B) \) explained in the proof of Corollary 4.3 on generators \( \Gamma'_S \), which arises from the presentation in Lemma 4.1. Using Remark 4.4 we see that all of the relations appearing in our presentation are either of type \( R_i \) (relations (R1)) in Lemma 4.1 for \( k = 1 \) or of type \( R_j \) (rest of the relations in Lemma 4.1). \( \square \)

**Corollary 4.7.** Assume \( F : U \to \mathbb{P}^1 \) is an admissible map, where \( U \) is a simply-connected quasi-projective surface. Let \( Q \in \mathbb{P}^1 \setminus B_F \), and let \( \mathbb{P}^1_{\{1, n\}} \) be the maximal orbifold structure of \( \mathbb{P}^1 \) with respect to \( F : U_{\{Q\}} \rightarrow \mathbb{P}^1 \setminus \{Q\} \). Assume that \( M_F \neq \emptyset \).
Let $\Gamma_S$ be an adapted geometric set of base loops w.r.t. $F$ and $\{Q = P_0\} \cup M_F$ given by Lemma 4.2. Then, $\pi_1(U_B)$ has a finite presentation

$$\pi_1(U_B) = \langle \Gamma_5' : \{R_j\}_{j \in J}, \{[\gamma_{k}, \gamma_{n}^m]\}_{P_k, P \in M_F} \rangle,$$

where $m$ is the multiplicity of the fiber $F^*(P_k)$ and $R_j = \prod_{P_k \in M_F} [\gamma_{k}, \gamma_{n}^m]^{m_j}$ for some $m_j, j \in J$.

**Remark 4.8.** By Remark 4.2, note that $\#M_F = 1$ or 2 in Corollary 4.7.

**Proof of Corollary 4.7.** Since $U$ is simply-connected and $F$ is an admissible map, the group $\pi_1(U_B)$ must be trivial by Remark 2.10. Hence, $F_+: \pi_1(U) \to \pi_1(U_B)$ is trivially an isomorphism. Thus the hypotheses of Lemma 4.2 are satisfied.

By Remark 4.5, the elements of $K = N$ (an abelian group) are all of the form $\prod_{P_k \in M_F} (\gamma_{n}^m)^{m_k}$, where $m_k \in \mathbb{Z}$.

Since Remark 4.5 says that $N$ is in the center of $\pi_1(U_B)$, we can add the relations $[\gamma_{k}, \gamma_{n}^m]$ for all $P_k, P \in M_F$ to the presentation of $\pi_1(U_B)$ of Corollary 4.3 without changing the group. After that, note that we can transform the relations already appearing in the presentation of Corollary 4.3 (coming from (R2) (R4) in Lemma 4.1) to elements of $K = N$, and hence, as relations of type $R_j$.

**4.2. Deletion Lemma.** The following results describe the basic operations of addition and deletion of fibers that preserve the structure of orbifold group for the fundamental group of the complement of a curve of pencil type.

**Theorem 4.9 (Deletion Lemma).** Let $U$ be a smooth quasi-projective surface and $F: U \to S$ be an admissible map to a compact Riemann surface $S$. Assume $B \subset S$ is such that $\#B = n \geq 1$ and $r := 2g_S + n$. Consider $P \in S \setminus B$. Let $S_{(n+1,m)}$ (resp. $S_{(n,m')}\}$) be the maximal orbifold structure of $S$ with respect to $F_+: U_{B \cup\{P\}} \to S \setminus (B \cup \{P\})$. If $F_+: \pi_1(U_{B \cup\{P\}}) \to \pi_1(U_B)$ is an isomorphism, then

$$F_+: \pi_1(U_B) \to \pi_1(U_{B \cup\{P\}})$$

is an isomorphism.

Moreover, if $\pi_1(U_{B \cup\{P\}}) \cong F_r \ast \mathbb{Z}_{m_1} \cdots \mathbb{Z}_{m_s}$ and $p \geq 1$ denotes the multiplicity of $F^*(P)$, then

$$\pi_1(U_B) \cong F_{r-p} \ast \mathbb{Z}_{m_1} \cdots \mathbb{Z}_{m_s}.$$

**Proof.** By hypothesis, $\pi_1(U_{B \cup\{P\}}) \cong \pi_1(S_{(n+1,m)}) \cong F_r \ast \mathbb{Z}_{m_1} \cdots \mathbb{Z}_{m_s}$ for integers $m_i \geq 2$, $i \in I = \{1, \ldots, s\}$. Also, by Corollary 2.20, the inclusion of the generic fiber (over a point $Q \in S \setminus (B \cup \{P\} \cup B_F)$) induces the trivial morphism $\pi_1(F^{-1}(Q)) \to \pi_1(U_{B \cup\{P\}})$. Since $U_{B \cup\{P\}} \subset U_B$, the inclusion of the generic fiber also induces the trivial morphism $\pi_1(F^{-1}(Q)) \to \pi_1(U_B)$. By Corollary 2.20

$$F_+: \pi_1(U_B) \to \pi_1(U_{B \cup\{P\}})$$

is an isomorphism since $n > 0$.

For the moreover part, note that if $\bar{m} = (m_1, \ldots, m_s)$, then

$$\bar{m}' := \begin{cases} \bar{m} & \text{if } p = 1, \\ (p, m_1, \ldots, m_s) & \text{if } p > 1. \end{cases}$$

**Remark 4.10.** The Deletion Lemma also holds in the case $n = 0$ in the following way: if $F_+: \pi_1(U_{\{P\}}) \to \pi_1(S_{(1,m)})$ is an isomorphism, then $F_+: \pi_1(U) \to \pi_1(S_{(0,m')})$ is also an isomorphism. A more subtle proof can be given using the presentation of $U_{\{P\}}$ from Lemma 4.1 taking into account that the elements of $\Gamma_F$ are trivial in $\pi_1(U_{\{P\}})$. Since the result is not needed for the purpose of this paper, we omit it.
4.3. Proof of the Generic Addition-Deletion Lemma.

Proof of Theorem 1.3. The “if” as well as the “moreover” parts of the statement are a particular case of the Deletion Lemma 4.9.

Let us show the “only if” part. Let \( P_1 = P, B \cup \{ P \} = \{ P_0, \ldots, P_n \} \). By Corollary 4.10 (and using the notation therein) applied to \( B \) and \( B \cup \{ P \} \), we have that \( \pi_1(U_{B \cup \{ P \}}) \) has a presentation of the form

\[
\pi_1(U_{B \cup \{ P \}}) = \langle \Gamma'_{S} : \{ R_j \}_{j \in J}, \{ \bar{R}_i \}_{i \in I} \rangle
\]

where \( R_j \) are words in \( \Gamma'_S \setminus \{ \gamma_1 \} \), and \( \bar{R}_i = [\gamma_1, w_i] \), where \( w_i \) is a word in \( \Gamma'_S \setminus \{ \gamma_1 \} \).

Let \( H = \langle \Gamma'_S \setminus \{ \gamma_1 \} : \{ R_j \}_{j \in J} \rangle \), and let \( \varphi : F_{r+b'} = \langle \delta_1, \ldots, \delta_{r+b'} \rangle \to \pi_1(U_B) \) be the epimorphism sending \( \delta_i \) to \( \gamma_i \) for all \( i = 1, \ldots, r + b' \). We have that \( \varphi \) factors through \( \tilde{\varphi} : Z * H \to \pi_1(U_{B \cup \{ P \}}) \), where the \( Z \) free factor is generated by the letter \( \gamma_1 \). In particular \( \tilde{\varphi} \) is an epimorphism.

Moreover, according to the presentation of \( \pi_1(U_{B \cup \{ P \}}) \) above,

\[
\pi_1(U_B) \cong \pi_1(U_{B \cup \{ P \}}) / \text{NCl}(\gamma_1) \cong H
\]

Hence, we have found an epimorphism

\[
\tilde{\varphi} : Z * \pi_1(U_B) \to \pi_1(U_{B \cup \{ P \}}).
\]

Let \( F_* : \pi_1(U_{B \cup \{ P \}}) \to \pi_1(\text{orb}(S_{n+1, m})) \). Since \( \tilde{\varphi} \) is an epimorphism and \( F_* \) is also an epimorphism by Remark 2.10, \( F_* \circ \tilde{\varphi} \) is an epimorphism from the open orbifold fundamental group \( Z * \pi_1(U_B) \) to \( \pi_1(\text{orb}(S_{n+1, m})) \), which is isomorphic \( Z * \pi_1(U_B) \cong Z * \pi_1(\text{orb}(S_{n+1, m})). \) Hence, \( F_* \circ \tilde{\varphi} \) is an isomorphism by Lemma 2.12. In particular, \( \tilde{\varphi} \) and \( F_* \) are both isomorphisms, and

\[
\pi_1(U_{B \cup \{ P \}}) \cong Z * \pi_1(U_B). \quad \square
\]

Corollary 4.11. The fundamental group of the complement of \( r \) generic fibers of a primitive polynomial map \( f : \mathbb{C}^2 \to \mathbb{C} \) is free of order \( r \).

Proof. Consider \( F : \mathbb{P}^2 \to \mathbb{P}^1, F(x, y, z) = [f(x, y, z) : z^d] \), where \( \overline{f(x, y, z)} \) is the homogenization of \( f(x, y) \) and \( d = \deg(f) \). Since \( f \) is primitive (it has connected fibers), \( F \) is an admissible map. Let \( B \) be the base points of \( F \). Consider \( U = \mathbb{P}^2 \setminus B, F_1 : U \to \mathbb{P}^1 \). The restriction \( F_1 : \mathbb{C}^2 \to \mathbb{C} \) induces an isomorphism of (trivial) fundamental groups. Note that \( F_1 : \mathbb{C}^2 \to \mathbb{C} \) doesn’t have multiple fibers, or else \( F_1 \) would factor through a surjection from \( \pi_1(\mathbb{C}^2) \) to \( \pi_1(\mathbb{C}) \) for an orbifold on \( \mathbb{C} \), which is non-trivial. Consider \( r \) generic fibers of \( F_1 \). Then the result follows from the Generic Addition-Deletion Lemma 1.3 to \( F_1 : U \to \mathbb{P}^1 \) and \( B = \{ [1 : 0] \}. \quad \square \)

Example 4.12. Other examples of complements of curves with free fundamental groups include the following. Consider \( f(x, y) \) polynomial such that \( \pi_1(\mathbb{C}^2 \setminus C_1) = \mathbb{Z} \) for \( C_1 = V(f) = \mathbb{C}^2 \) (for instance, if \( C_1 \) is irreducible and only has nodal singularities, including at infinity). We have that \( f_* : \pi_1(\mathbb{C}^2 \setminus C_1) \to \pi_1(\mathbb{C}^*) \) is an epimorphism from \( \mathbb{Z} \) to itself, so it is an isomorphism. Note that \( f \) doesn’t have multiple fibers, or else \( f_* \) would factor through a surjection from \( \pi_1(\mathbb{C}^2 \setminus C_1) \cong \mathbb{Z} \) to \( \pi_1(\mathbb{C}^*) \) for an orbifold of general type on \( \mathbb{C}^* \), which is a non-abelian group. Consider \( C_2, \ldots, C_r \) generic fibers of \( f \). Then the Generic Addition-Deletion Lemma 1.3 yields \( \pi_1(\mathbb{C}^2 \setminus C) = F_r \) for \( C = C_1 \cup \cdots \cup C_r \).

Analogously, if \( f_p \) (resp. \( f_q \)) is a form of degree \( p \) (resp. \( q \)) with \( \gcd(p, q) = 1 \) and \( \pi_1(\mathbb{P}^2 \setminus C_1) = \mathbb{Z} \) for \( C_1 = V(f_q) \cup V(f_p) \subset \mathbb{P}^2 \) (for instance, if \( f_p \) and \( f_q \) are irreducible and \( C_1 \) has only nodal singularities). Consider \( C_2, \ldots, C_r \) generic fibers of \( F = [f_p : f_q] \). Then we are under the hypothesis of the Generic Addition-Deletion Lemma 1.3 and hence \( \pi_1(\mathbb{P}^2 \setminus C) = F_r \) for \( C = C_1 \cup \cdots \cup C_r \).
4.4. A base case for the Addition Lemma. Recall Notation 4.3. In light of Example 4.12 one might wonder if other pencils $F : X = \mathbb{P}^2 \rightarrow \mathbb{P}^1$ give rise to curves whose fundamental group of their complement is isomorphic to an open orbifold group. The following result provides a base case for the Addition Lemma in the case $M_F \neq \emptyset$.

**Theorem 4.13.** Let $X$ be a simply-connected smooth projective surface, let $F : X \rightarrow \mathbb{P}^1$ be an admissible map, and let $P \in \mathbb{P}^1$ be such that $F^{-1}(P)$ is a typical fiber. Suppose that $M_F \neq \emptyset$.

Let $\mathbb{P}^1_{(1,m)}$ be the maximal orbifold structure of $\mathbb{P}^1 \setminus \{P\}$ with respect to $F : X_{(P)} \rightarrow \mathbb{P}^1 \setminus \{P\}$, where $m = (p,q)$, $p \geq q \geq 1$, and $\text{gcd}(p,q) = 1$. Then the following statements are equivalent:

1. $H_1(X_{(P)}) = \mathbb{Z}_{pq}$
2. $F_* : \pi_1(X_{(P)}) \rightarrow \pi_1^{\text{orb}}(\mathbb{P}^1_{(1,m)}) \cong \mathbb{Z}_p \times \mathbb{Z}_q$ is an isomorphism.

**Proof.** After an isomorphism in $\langle \pi \rangle$ and the epimorphism (recall Remark 2.16) $\mathbb{P}^1 \rightarrow \mathbb{P}^1$, we may assume that $M_F \subset \{0 : 1\}, \{1 : 0\}$ (see Remark 4.8), the fiber above $[0 : 1]$ has multiplicity $p$, the fiber above $[1 : 0]$ has multiplicity $q$, and and $P \notin \{0 : 1\}, \{1 : 0\}$.

(2) $\Rightarrow$ (1) is trivial. Let us prove (1) $\Rightarrow$ (2).

Assume that $H_1(X_{(P)}) = \mathbb{Z}_{pq}$. Note that

- $\#M_F = 2$ if and only if $q > 1$, and
- $\#M_F = 1$ if and only if $q = 1$.

Corollary 4.7 applied to $U = \mathbb{P}^2 \setminus B$ and Remark 4.8 say that $\pi_1(X_{(P)})$ has a presentation of the form

$$
\pi_1(X_{(P)}) = \langle \gamma_1, \gamma_2 \mid \{\gamma_1^{k_j}, q_j^{l_j} \}_{j \in J}, [\gamma_1, \gamma_2], [\gamma_2, \gamma_1] \rangle
$$

where $k_j, l_j \in \mathbb{Z}$ for all $j \in J$. Indeed, this is clear if $\#M_F = b' = 2$ (i.e. $q > 1$), but it is also true if $b' = 1$ (i.e. $q = 1$), picking $k_1 = 0$, $l_1 = 1$. Hence, from now on we assume that $\pi_1(X_{(P)})$ has a finite presentation as in equation (14).

Let $k = \text{gcd}_{j \in J}(k_j)$, and $l = \text{gcd}_{j \in J}(l_j)$, with the convention that the greatest common divisor of various 0’s is 0. Note that this group has a quotient

$$
\langle \gamma_1, \gamma_2 \mid \gamma_1^{k}, \gamma_2^{l}, [\gamma_1, \gamma_2] \rangle,
$$

so the quotient map induces an epimorphism on the abelianizations

$$
\mathbb{Z}_{pq} \twoheadrightarrow \mathbb{Z}_{pl} \times \mathbb{Z}_{ql}.
$$

Hence, $k = l = 1$. Using that $\gamma_1^p$ and $\gamma_2^q$ commute, we can modify the presentation of $\pi_1(X_{(P)})$ in equation (14) to get

$$
\langle \gamma_1, \gamma_2 \mid \gamma_1^{p}, \gamma_2^{q}, a^b, [\gamma_1, \gamma_2], [\gamma_2, \gamma_1] \rangle,
$$

where $b \geq 1$, $a \in \{0, \ldots, b-1\}$ and $\text{gcd}(a,b) = 1$. Thus, a presentation matrix of the abelianization of $\pi_1(X_{(P)})$ as a $\mathbb{Z}$-module is given by

$$
M = \begin{pmatrix}
 p & 0 \\
 qa & qb
\end{pmatrix}
$$

By hypothesis, we know that the matrix $M$ is equivalent (over $\mathbb{Z}$) to the diagonal $2 \times 2$ matrix with diagonal $(p,q)$, since both matrices present the same abelian group and have the same dimensions and rank. In particular, both matrices have the same determinant, so $b = 1$, and thus $a = 0$. Plugging that data back in for the presentation in equation (15), we get that

$$
\pi_1(X_{(P)}) = \langle \gamma_1, \gamma_2 \mid \gamma_1^p, \gamma_2^q \rangle,
$$

and the epimorphism (recall Remark 4.10)

$$
F_* : \pi_1(X_{(P)}) \rightarrow \pi_1^{\text{orb}}(S_{(1,m)})
$$
is in fact an isomorphism by Lemma 2.18

5. Applications

5.1. $C_{p,q}$-curves revisited. In this subsection we prove a generalization of the classical $C_{p,q}$ Theorem.

Proof of Theorem 1.4 Note that $F$ is an admissible map from $X = \mathbb{P}^2$ to $\mathbb{P}^1$. If $p = q = 1$, the result is trivial. Suppose that $p \geq 1$ or $q > 1$, which implies that $M_F \neq \emptyset$. Let $P \in \mathbb{P}^1$ be such that $F^{-1}(P)$ is a typical fiber. In particular, $F^{-1}(P)$ is given by the zeros of an irreducible degree $pq$ polynomial, and by Example 2.3 $H_1(X_P) \cong \mathbb{Z}$. In other words, (2) in Theorem 4.4.4 holds, and the result for a finite union of generic fibers is proved for $r = 0$. The claim for $r > 0$ follows from the Generic Addition-Deletion Lemma applied to $U = \mathbb{P}^2 \setminus B$.

5.2. Fundamental group of a union of conics. Another instance where our results apply is given in a collection of conics in a pencil. We provide a new proof of Theorems 2.2 and 2.5 in [1] which does not depend on braid monodromy results. Another instance where our results apply is given in a collection of conics in a pencil. We provide a new proof of Theorems 2.2 and 2.5 in [1] which does not depend on braid monodromy results.

Theorem 5.2. Let $F = [f_2 : f_3^2]$ be a pencil generated by a smooth conic $C_0 = V(f_2)$ and a double line $l = V(f_1)$. Consider $C = C_0 \cup \cdots \cup C_r$ a union of $r + 1$ smooth conics of $F$, then

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{F}_r * \mathbb{Z}_2.$$  

Proof. This result can be obtained from Theorem 1.4 for $p = 2, q = 1$.  

5.3. Fundamental group of tame torus-type sextics. In a remarkable paper, Oka-Pho [24] describe the fundamental group of the complement of irreducible sextics in a pencil of type $F = [f_2^i : f_3^j]$, where $f_i$ is a homogeneous form of degree $i$, whose set of singular points are base points of the pencil, that is, $\text{Sing} V(f) = V(f_2) \cap V(f_3)$. The term torus type refers to the former property and the term tame refers to the latter. According to the authors, any such a sextic $C$ satisfies $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_2 * \mathbb{Z}_3$ except for the particular case where $C$ has four singular points: one of type $C_{3,9}$ and three of type $A_2$. Type $C_{3,9}$ singularities have a local equation $f(x, y) = x^3 + y^9 + x^2 y^2 \in \mathbb{C}[x, y]$, and $A_2$ singularities are ordinary cusps of local equation $f(x, y) = x^2 + y^3 \in \mathbb{C}[x, y]$.

It is enough to check the result on maximal irreducible tame torus-type sextics, that is, those with maximal total Milnor number (either 19 or 20). According to [25] there are seven types of such irreducible curves, which can be described by the configuration $\Sigma$ of singularities (see Table 1), since their moduli space is connected for each configuration. Also, by the maximality of the total Milnor number, the multiple fibers $C_2 := V(f_2)$ and $C_3 := V(f_3)$ are uniquely determined by the moduli space of the curve $V(f_2) \cup V(f_3)$, that is, they depend only on the singularities of $C_2$ and $C_3$ and the topological type of their intersection. Such moduli spaces are connected in all cases.
Theorem 5.3 ([25]). Let \( C = \{ f_2^3 + f_3^3 = 0 \} \) be an irreducible maximal tame torus sextic of type (2,3) whose configuration of singularities \( \Sigma_C \neq \{ [C_{3,9}, 3A_2] \} \) (see Table 1). Then

\[ \pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_2 \ast \mathbb{Z}_3. \]

Proof. The idea of the proof is to show that any irreducible maximal tame torus sextic of type (2,3) except for exceptional case \( \Sigma_C \neq \{ [C_{3,9}, 3A_2] \} \) (family (7) in Table 1) is a generic member of a primitive pencil satisfying the conditions in Theorem 4.1.

We will do it in detail for \( C \in \mathcal{M}([C_{3,15}]). \) Table 1 gives a possible equation \( C = \{ f_2^3 + f_3^3 = 0 \} \) for such a curve as a member of a pencil generated by a smooth conic \( C_2 = \{ f_2 = 0 \} \) and a nodal cubic \( C_3 = \{ f_3 = 0 \} \) whose node \( P \in C_2 \) is such that \( (C_2, C_3)_P = 6, \) that is, \( C_2 \cap C_3 = \{ P \} \) (see [24] Thm. 1). To see that \( C \) is in fact a generic member one can obtain the resolution of indeterminacies is shown in Figure 1 where \( \tilde{F}^*([0 : 1]) = 3C_2 + E_{2,1} + 2E_{2,2} + 3E_{2,3} + E_{2,4}, \) \( \tilde{F}^*([1 : 0]) = 2C_3 + E_{3,1} + E_{3,2}. \)

The dicritical divisors \( D_1 \) and \( D_2 \) define 1:1 morphisms \( \hat{F} : D_1 \rightarrow \mathbb{P}^1 \) so there is no degeneration of fibers on the dicritical divisors. One can check that \( C \) defines a tame curve, that is, there are no singularities outside the base point \( P. \) This implies that \( C \) is a generic fiber. \( \square \)

Remark 5.4. Let \( C = \{ f_2^3 + f_3^3 = 0 \} \) be the curve in the moduli space \( \mathcal{M}([C_{3,9}, 3A_2]) \) given by the equation corresponding to family (7) in Table 1. \( C \) is not a generic sextic in the pencil \([f_2^3 : f_3^3]. \) If it were, Theorem 4.1 would contradict \( \pi_1(\mathbb{P}^2 \setminus C) \neq \)
$\mathbb{Z}_2 \ast \mathbb{Z}_3$ (Oka-Pho). Nonetheless, one can check directly that $C$ is not a typical fiber, since this pencil is of type $(1,6)$, but $C$ is a rational curve. This is a consequence of $C$ being irreducible and $\bar{\delta}(C_3,9) + 3\delta(A_2) = 7 + 3 = 10$ (see Table 1).

**Theorem 5.5.** Let $F = \left[ f_3^2 : f_2^3 \right] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be a pencil such that $C = \{ f_3^2 + f_2^3 = 0 \}$ is an irreducible maximal tame torus sextic of type $(2,3)$ and a generic member of the pencil $F$. Let $\Sigma_C$ be its configuration of singularities. Consider $B \subset \mathbb{P}^1$ be a collection of $r + 1$ typical values, $C_B = \bigcup_{\lambda \in B} C_\lambda$, and $C_j = V(f_j)$ for $j = 2,3$.

1. If $\Sigma_C \neq \{ B_{3,10}, A_2 \}, \{ B_{3,8}, E_6 \}, \{ C_{3,9}, 3A_2 \}$, namely, $C$ is a curve in a family $(1) - (4)$, then
   $$\pi_1(\mathbb{P}^2 \setminus C_B \cup C_3) \cong \mathbb{Z}_3.$$ 

2. If $\Sigma_C \neq \{ Sp_1, A_2 \}, \{ C_{3,9}, 3A_2 \}$, namely, $C$ is a curve in a family $(1) - (3),(5)$ or $(6)$, then
   $$\pi_1(\mathbb{P}^2 \setminus C_B \cup C_2) \cong \mathbb{Z}_2.$$ 

**Proof.** For the proof of part 1, note that these are the only families where $C_3$ is irreducible. As mentioned before Table 1 the curves $C_3$ are well defined. In families $(1) - (3), C_3$ is a nodal cubic, and in family $(4)$, it is a cuspidal cubic transversal to the line at infinity. In both of these cases, $\pi_1(\mathbb{P}^2 \setminus C_3) \cong \mathbb{Z}_3$. The result now follows from the Generic Addition-Deletion Lemma 1.3.

For the proof of part 2, note that these are the families where $C_2$ are irreducible, and hence a smooth conic. Note that family (7) also has irreducible $C_2$, but it still does not satisfy the hypothesis, since $C$ is not a typical fiber in that pencil (Remark 5.4). Since $C_2$ is smooth, $\pi_1(\mathbb{P}^2 \setminus C_2) \cong \mathbb{Z}_2$. The result now follows from the Generic Addition-Deletion Lemma 1.3. 

\[\Box\]

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