Gauge Field Production and Schwinger Reheating in Runaway Axion Inflation

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Abstract

In a class of (pseudoscalar) inflation, inflationary phase is followed by a kination phase, where the Universe is dominated by the kinetic energy of the inflaton that runs away in a vanishing scalar potential. In this class of postinflationary evolution of the Universe, reheating of the Universe cannot be achieved by the inflaton particle decay, which requires its coherent oscillation in a quadratic potential. In this study, we explore the U(1) gauge field production through the Chern-Simons coupling between the pseudoscalar inflaton and the gauge field during the kination era and examine the subsequent pair-particle production induced by the amplified gauge field known as the Schwinger effect, which can lead to reheating of the Universe. We find that with a rough estimate of the Schwinger effect for the Standard Model hyper U(1) gauge field and subsequent thermalization of the pair-produced particles, a successful reheating of the Universe can be achieved by their eventual domination over the kinetic energy of the inflaton, with some reasonable parameter sets. This can be understood as a concrete realization of the “Schwinger reheating”. Constraints from the later-time cosmology are also discussed.
I. INTRODUCTION

Inflation driven by a pseudoscalar or an axion-like particle (ALP), also often dubbed as natural inflation [1, 2], is one of the most well-motivated models of inflation. While one of the difficulties in constructing inflation models is how to realize a flat potential suitable for inflation against the quantum corrections, in axion inflation the flatness of the potential is guaranteed by the shift symmetry, which is the nature of the (pseudo) Nambu-Goldstone
bosons. On the one hand, it is difficult to drive inflation by the original QCD axion [3–8] due to the requirements on its potential as well as other interactions as the solution for the strong CP problem [9, 10]. On the other hand, ALPs that arise from, e.g., superstring theory compactifications [11], is allowed to have a non-trivial potential to drive the inflation that fits the observational data, as is seen in the axion monodromy [12, 13]. Moreover, ALPs can be applied for the model building of the inflation with an extremely flat potential such as $k$-inflation [14] or the quintessential inflation [15], since the shift symmetry can explain the flatness of the potential required in these models.

While inflation models with such an extremely flat potential and a resultant cosmic history have interesting phenomenologies, the connection to the hot Big Bang Universe is not clear. In such models, after inflation the inflaton runs away or continue to move in a vanishing potential. The energy density of the Universe is then dominated by the kinetic energy of the inflaton, which is often dubbed as the kination era [16, 17]. Since the inflaton does not oscillate coherently around the potential minimum, reheating of the Universe cannot be achieved by the inflaton particle decay. Instead, reheating is achieved by a small amount of radiation produced at a time after inflation that eventually dominates the energy density of the Universe. Note that the inflaton kinetic energy during the kination era is damped by the cosmic expansion as $\propto a^{-6}$, with $a$ being the scale factor, which is faster than that of radiation. Consequently, the production of the small amount of radiation is the key ingredient for the graceful exit in these models.

One of the most representative mechanisms to produce a small amount of radiation after inflation before or during kination is the gravitational particle production [18–20], where all the non-conformally coupled fields are produced due to the change of the vacuum caused by the change of the background spacetime at the end of inflation. This “gravitational reheating” is, however, not so efficient due to the weakness of gravity and often suffered from too much production of high-frequency gravitational waves, which are also produced gravitationally. The latter is constrained by the number of relativistic degrees of freedom at the Big Bang Nucleosynthesis (BBN) [21] and at the recombination with the observation of the cosmic microwave background (CMB) [22]. Although there are intensive studies to address this issue and to find successful scenarios in the gravitational reheating [23–35], it is important to explore other possibilities of reheating in the inflationary models with kination.

The difficulty in producing even small amount of radiation lies in the fact that the shift
symmetry forbids or at least suppresses ordinary couplings of the inflaton to other fields. However, once we identify that the inflaton is an ALP, a nontrivial coupling between the inflaton and other fields that respects the shift symmetry, that is, the Chern-Simons coupling, naturally arises. The Chern-Simons coupling is induced e.g., if the underlying Peccei-Quinn-like global symmetry is anomalous under some local gauge symmetries or by the Green-Schwarz mechanism of anomaly cancellation [36] in heterotic string theory. If the gauge symmetry is a U(1) symmetry, a coherent axion dynamics is known to produce the U(1) gauge fields copiously through their tachyonic instability [37–39]. These phenomena during inflation have been studied to constrain the strength of the coupling through the cosmological observations [40–46]. Moreover, if the U(1) gauge field is the one in the Standard Model of particle physics (SM), it can also explain the baryon asymmetry of the Universe [47–50] and the origin of the intergalactic magnetic fields [51–55] (See also the studies on the gauge field amplification during reheating [56–60]). It is natural that we expect that it would also lead to a successful reheating after kination, if the tachyonic instability of the U(1) gauge fields is sufficiently effective during kination.

In this article, we study the U(1) gauge field amplification triggered by the Chern-Simons coupling during kination. We take an inflation model with a “runaway”-type or a step-function-like potential as an example. In this kind of model, the gauge field amplification during kination is more efficient than during inflation since the ALP velocity takes its maximal value just at the onset of kination when almost all the potential energy is converted into the kinetic energy of the inflaton. We can easily see that due to the smallness of the kinetic energy during inflation, the gauge field amplification during inflation is negligibly small. Compared to the gauge field amplification from the ALP oscillation [56–58], that from this “runaway” ALP is distinctive in the point that purely only one helicity modes are amplified because the sign of the ALP velocity is unchanged, and the resultant gauge fields are maximally helical. We find that gauge field amplification during kination occurs when the mode that exited the horizon during inflation reenters it. It is contrasting to the one during inflation which occurs when the mode exits the horizon [37–39, 48]. The energy density of the amplified gauge fields is found to be typically much larger than that of gravitationally produced particles, which is an advantage for successful reheating.

To see if the gauge field amplification by the Chern-Simons coupling leads to the successful reheating, we need to examine the interactions of the SM particles. Indeed, even during the
gauge field amplification, strong electric fields would induce a pair production of charged particles [61–69], known as the Schwinger effect [70, 71], which also backreacts to the gauge-field dynamics. Unfortunately, the Schwinger effect caused by a dynamical gauge field is extremely difficult to give a precise estimate with the best of our present knowledge and technique, although there are several trials to challenge this problem [49, 72, 73]. In this article, we adopt the treatment developed in Ref. [49] to give a rough estimate. Although it would not give a precise estimate and the results in the present paper are not quantitatively correct, we believe that the estimate obtained in this way gives us the characteristic feature of the entire process of the gauge field amplification during kination. We find that the energy density of the pair-produced particles is typically as much as or even larger than that of the gauge fields and they eventually thermalized before dominating the energy density of the Universe. Therefore, the successful reheating through the Schwinger effect, or the “Schwinger reheating” [74] is realized. The electric fields are likely to be screened just after the saturation of the gauge field amplification, while the magnetic fields would decay slowly. The latter can lead to a Universe inconsistent with the present one due to the too much additional number of relativistic degrees of freedom, parameterized by the number of effective neutrino species, $N_{\text{eff}}$, constrained by the BBN [21]. It turns out that taking into account the magnetohydrodynamics (MHD) cascade decay, the energy density of the magnetic fields are diffused enough by the time of the BBN and $N_{\text{eff}}$ is sufficiently suppressed.

The rest of this paper is constructed in the following way. In Sec. II, we examine the gauge field production during the kination era in the ALP-photon system without other charged particles and derive the approximate formula for its energy density. In Sec. III, we include charged fermions to give a rough estimate for the Schwinger effect including the backreaction on the gauge field production. We also investigate the thermalization of the produced particles and screening of the electric field after the saturation of the gauge field amplification. In Sec. IV, we examine the late time evolution of magnetic fields and discuss cosmological constraints. Sec. V is devoted for conclusion and discussion.

II. GAUGE FIELD AMPLIFICATION IN RUNAWAY-TYPE INFLATION

Let us start with investigating the gauge field production in inflationary scenarios with the kination era without taking into account the backreaction from the Schwinger effect.
We focus on the gauge field dynamics by the motion of the ALP field, $\phi$, which acts as the inflaton with the potential $V(\phi)$. We consider the following action in the Friedmann-Robertson-Walker (FRW) background,

$$S = \int d^4x \sqrt{-g} \left[ \mathcal{L}_\phi + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_\psi \right],$$  

(1)

$$\mathcal{L}_\phi = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi),$$  

(2)

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$  

(3)

$$\mathcal{L}_{\text{CS}} = -\frac{\phi}{4\Lambda} F_{\mu\nu} \tilde{F}^{\mu\nu},$$  

(4)

$$\mathcal{L}_\psi = i \bar{\psi} \gamma^\mu D_\mu \psi = i \bar{\psi} \left[ a^{-1} \gamma^\mu (\partial_\mu + ig' QA_\mu) + \frac{3}{2} H \gamma^0 \right] \psi,$$  

(5)

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the (hypercharge) U(1) gauge field strength, $\tilde{F}_{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} / (2\sqrt{-g})$ is its dual, and $\gamma^\mu$ is the gamma matrix satisfying $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. $\mathcal{L}_{\text{CS}}$ is the Chern–Simons term, which is the key ingredient to amplify the gauge field. $\Lambda$ denotes an effective suppression scale, whose amplitude is related to the scale at which the anomalous coupling is generated. In this article, we take it as a free parameter. $\mathcal{L}_\psi$ with the charged fermion, $\psi$, is introduced for Sec. III where we will examine the particle production from the strong gauge field or the Schwinger effect. $g'$ is the (hyper) U(1) gauge coupling, and as the reference value we take it as 0.3. $Q$ is the (hyper)charge of the $\psi$ field. $H \equiv \dot{a} / a$ is the Hubble parameter with the dot being the derivative with respect to the physical time $t$ and $a(t)$ denoting the scale factor in the FRW metric $ds^2 = -dt^2 + a^2(t) dx^2 = -a^2(\eta)(d\eta^2 - dx^2)$ ($\eta$ is the conformal time). Since we have in mind the case where the mechanism works at a higher temperature than the electroweak scale, we identify the U(1) gauge field is the hyper U(1) gauge field in the standard model later. We however do not distinguish the hyper U(1) gauge field and the electromagnetic U(1) gauge field otherwise stated, since the discussion does not change.

We investigate a “runaway-type” inflation where the inflaton energy is converted mostly into the kinetic energy of the inflaton at the end of inflation so that the inflaton “runs away” in a flat direction of a vanishing potential. It is realized, for example, by a step-function like potential with a mild tilt where the potential has a flat region with a non-vanishing potential energy for slow-roll inflation and another flat region with a vanishing potential energy for
the runaway. For concreteness, we here have the following toy-model potential in mind,

$$V(\phi) = 3M_{pl}^2H_I^2(1 - \theta(\phi)), \quad (6)$$

where $\theta$ is the unit step function, which describes the steep cliff at $\phi = 0$. $H_I$ is the Hubble parameter during inflation, and $M_{pl}$ is the reduced Planck mass. Inflation takes place at $\phi < 0$ and the runaway reheating stage takes place at $\phi > 0$. We do not write the “inclination” term explicitly but assume that there is a very gentle slope enough for the inflaton to keep slow-rolling with $\dot{\phi} > 0$. Although we take a concrete model here, within our simplifications the essence of the phenomena we study do not depend on the detail of the potential. During inflation, the conformal time is given in terms of the scale factor as

$$\eta \simeq -\frac{1}{aH_I}. \quad (7)$$

After inflation, the Universe enters the so-called kination era [16, 17], where the energy density of the Universe is dominated by the kinetic energy of inflaton and decreases at a rate proportional to $a^{-6}$. The conformal time after inflation is then written as

$$\eta \simeq \frac{1}{2aH(\eta)} - \frac{3}{2a_{end}H_I} \quad (8)$$

so that at the end of inflation it is given by $\eta_{end} = -1/(a_{end}H_I)$ with $a_{end}$ being the scale factor at the end of inflation. Since the kinetic energy of the inflaton decays faster than that of radiation or matter, a small amount of them produced at, for example, the end of inflation eventually dominates the Universe, which is the reheating mechanism in this scenario. The production of such a small amount of radiation or matter after inflation in the runaway inflation scenario is the main topic of the present paper.

A. Gauge field amplification

Let us study the dynamics of the ALP and gauge field during the inflation and the kination era. Adopting the radiation gauge, $A_0 = \partial_i A^i = 0$, the physical electric and magnetic fields are given by

$$E_p = -\frac{1}{a^2} A^t, \quad B_p = \frac{1}{a^2} \nabla \times A, \quad (9)$$
where the prime denotes the derivative with respect to the conformal time $\eta$ and the subscript $p$ represents that the variables are evaluated in the physical frame. The quantization of the gauge fields are performed in the momentum space with the mode functions $A_{\pm}(k, \eta)$ as
\[
A_i(t, x) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} \epsilon^{ijk} \epsilon_i^{(\lambda)}(\hat{k}) \left[ a_{k}^{(\lambda)} A_{\lambda}(k, t) + a_{-k}^{(\lambda)\dagger} A_{\lambda}^{*}(k, t) \right].
\] (10)
Here $\lambda = \pm$ represents the circular polarization states and $\epsilon_i^{(\lambda)}(\hat{k})$ denotes the circular polarization vector that satisfies
\[
e^{(\lambda)}_i(\hat{k})e^{(\lambda')}_i(\hat{k}) = \delta_{\lambda\lambda'} \delta(3) (k - k').
\] (11)
The equation of motion for the homogeneous mode of the ALP, $\phi_0$, and the mode equation for the gauge fields derived from Eq. (1) are given as
\[
\ddot{\phi}_0(t) + 3H \dot{\phi}_0(t) + \frac{\partial V}{\partial \phi} = \frac{1}{\Lambda} \langle E_p \cdot B_p \rangle,
\] (12)
\[
\left( \partial^2_\eta + k^2 \mp 2k\xi aH \right) A_{\pm}(k, \eta) = 0,
\] (13)
where $\xi$ is the instability parameter defined as
\[
\xi = \frac{\dot{\phi}_0}{2\Lambda H}.
\] (14)
The angle bracket represents the quantum mechanical expectation value, which is identified as the classical ensemble average if it is exponentially amplified. For concreteness, we here take $\xi > 0$, but the gauge field amplification itself in the case with $\xi < 0$ can be investigated in the same way. The difference is that just the opposite helicity modes are amplified.

It would be desirable if we can simultaneously solve the equations of motion (Eqs. (12) and (13)) consistently, but it is practically difficult and model dependent. Instead we simplify the situation as follows. Since we consider the runaway type dynamics of the ALP as the inflaton, namely, the slow-roll inflation followed by the kination era, the ALP dynamics, $\dot{\phi}_0$, during inflation is suppressed by the slow roll parameter $\epsilon$ so that we can express as
\[
\dot{\phi}_0^2 = 2\epsilon M_{pl}^2 H^2 \quad \text{(during inflation)}.
\] (15)
During the kination era, on the other hand, the kinetic energy of the ALP, $\dot{\phi}_0^2/2$, dominates the energy density of the Universe so that we have

$$\dot{\phi}_0^2 = 2M_{\text{pl}}^2 H^2 \quad \text{(during kination)}.$$  \hfill (16)

The Hubble parameter approximately evolves as

$$H = \begin{cases} 
H_I, & \text{for } \eta < \eta_{\text{end}}, \\
H_I \left( \frac{a(\eta)}{a_{\text{end}}} \right)^{-3}, & \text{for } \eta > \eta_{\text{end}},
\end{cases}$$  \hfill (17)

with assuming the energy density of the amplified gauge fields (and possibly generated other particles) is negligible. The instability parameters (Eq. (14)) in each era are then given by

$$\xi_I = \epsilon \frac{1}{\sqrt{2}} \frac{M_{\text{pl}}}{\Lambda}, \quad \xi_K = \frac{1}{\sqrt{2}} \frac{M_{\text{pl}}}{\Lambda},$$  \hfill (18)

where the subscripts $I$ and $K$ mean that the quantities are evaluated during inflation and kination, respectively. One can see that we always have $\xi_I \ll \xi_K$. Note that the $\xi_I$ at the cosmic microwave background (CMB) scale is severely constrained by the observations. For example, a large non-Gaussianity on the temperature perturbation of the cosmic microwave background (CMB) induces a constraint as $|\xi_I| < 2.37$ [42, 44, 45]. In our setup, we can choose a relatively large $\xi_K$ while keeping $\xi_I$ within the acceptable range from the observational constraints, which makes the gauge field production more efficient.

Next we solve the mode equation for the gauge fields (Eq. (13)) on top of the background solutions with Eq. (18). During inflation, Eq. (13) is rewritten as

$$\left( \frac{\partial^2}{\partial \eta^2} + k^2 \pm 2 \frac{k \xi_I}{\eta} \right) A_{I, \pm}(k, \eta) = 0,$$  \hfill (19)

so that its general solution is given by

$$A_{I, \pm}(k, \eta) = C_{I, \pm} W_{\mp i \xi_I, 1/2}(\pm 2ik\eta) + C_{I, \pm} W_{\pm i \xi_I, 1/2}(-2ik\eta),$$  \hfill (20)

where $W_{\kappa, \mu}(x)$ is the Whittaker function and we have taken $\xi_I$ to be a constant. Strictly speaking $\xi_I$ is time-dependent, but it varies only slowly with time due to the slow-roll motion of the ALP field. For the practical purpose to solve Eq. (13), it is enough to take it as a constant. Requiring the mode functions are taken in the Bunch-Davies-like vacuum with the
asymptotic form \( \lim_{k\eta \to \infty} A_{I,\pm}(k, \eta) \sim \exp(-ik\eta)/\sqrt{2k} \), we obtain the positive frequency mode function with \( C_{I1}^\pm = e^{\pm \pi \xi_I/2}/\sqrt{2k} \) and \( C_{I2}^\pm = 0 \) as [39, 48]

\[
A_{I,\pm}(k, \eta) = \frac{1}{\sqrt{2k}} e^{\pm \pi \xi_I/2} W_{\pm i \xi_I/2} (+2i k \eta),
\] (21)

up to the phase factor. With \( \xi_I > 0 \), the positive helicity modes are amplified exponentially around the horizon crossing whereas the negative helicity modes remain oscillatory. This is because the positive helicity modes are tachyonic for \( k < 2\xi_I/\eta \) while the effective mass squared of the negative helicity modes is always positive.

The equation of motion during the kination era is rewritten as

\[
\left( \partial_{\eta K}^2 + k^2 \mp \frac{k \xi_K}{\eta_K} \right) A_{K,\pm}(k, \eta_K) = 0 \quad \text{with} \quad \eta_K \equiv \eta + \frac{3}{2 a_{\text{end}} H_I},
\] (22)

whose general solution is given by

\[
A_{K,\pm}(k, \eta_K) = C_{K1}^\pm W_{\pm i \xi_K/2,1/2} (+2i k \eta_K) + C_{K2}^\pm W_{\pm i \xi_K/2,1/2} (-2i k \eta_K).
\] (23)

The coefficients \( C_{K1}^\pm \) and \( C_{K2}^\pm \) are determined by the matching conditions to the solution during inflation Eq. (21) at the end of inflation as

\[
\begin{align*}
A_{I,\pm}(k, \eta_{\text{end}}) &= A_{K,\pm} \left( k, \eta_{\text{end}} + \frac{3}{2 a_{\text{end}} H_I} \right), \\
\partial_\eta A_{I,\pm}(k, \eta)|_{\eta = \eta_{\text{end}}} &= \partial_\eta A_{K,\pm} \left( k, \eta + \frac{3}{2 a_{\text{end}} H_I} \right)|_{\eta = \eta_{\text{end}}}. 
\end{align*}
\] (24)

Once more, for \( \xi_K > 0 \), regardless of the coefficients \( C_{K1}^\pm \) and \( C_{K2}^\pm \), only positive helicity modes are exponentially amplified while the negative ones remain oscillatory. Thus hereafter we focus on the positive helicity modes. Moreover, as we have seen \( \xi_I \ll \xi_K \) in our setup, in the following we approximate as \( \xi_I = 0 \) and investigate the case with \( \xi_K > 1 \).

From the matching conditions (24) we find

\[
C_{K1}^+ = -\frac{1}{8 \sqrt{2k}} (e^{\pi \xi_K} - 1) \Gamma \left( -\frac{i \xi_K}{2} \right) \left( 2 \log |u| + 2 \psi \left( \frac{i \xi_K}{2} \right) + 4 \gamma_E - i \pi \right) \left( 1 + O(u) \right), \quad (25)
\]

\[
C_{K2}^+ = \frac{-e^{\pi \xi_K/2}}{2 \sqrt{2k} \Gamma \left( -\frac{i \xi_K}{2} \right)} \left( 2 \log |u| + 2 \psi \left( -\frac{i \xi_K}{2} \right) + 4 \gamma_E + i \pi + \frac{4i}{\xi_K} \right) \left( 1 + O(u) \right), \quad (26)
\]

where \( u \equiv k/(a_{\text{end}} H_I) \), \( \psi(z) \) is the polygamma function and \( \gamma_E \) is the Euler’s gamma. For large \( \xi_K \), we find

\[
|C_{K1}| \approx |C_{K2}| \approx \frac{1}{2} \sqrt{\frac{\xi_K}{2\pi k}} e^{\frac{3\pi \xi_K}{4}},
\] (27)
for $k < (32/9\pi^2)\xi_K^{-1}a_{\text{end}}H_I$, and it is exponentially suppressed for larger $k$. Here we have omitted the logarithmic contributions. The exponential suppression for larger $k$ than $(32/9\pi^2)\xi_K^{-1}a_{\text{end}}H_I$ is due to the fact that such a mode has never exited the horizon and does not have time to be amplified exponentially. This threshold is also confirmed numerically. A non-zero $C_{K^2}^+$ means that gauge fields amplified in this mechanism include not only positive but also negative frequency modes. The asymptotic behavior of the Whittaker function tells that the exponential amplification of the gauge fields occurs at the horizon reentry and at a late time $k\eta_K \gg 1$, the mode function behaves as

$$A_+(k, \eta) \sim \frac{1}{2} \sqrt{\frac{\xi_K}{2\pi k}} e^{\pi(\xi_K/2)} \times \text{(oscillation with period } \Delta \eta \simeq k^{-1}), \quad \text{for } k < \frac{32}{9\pi^2} \frac{a_{\text{end}}H_I}{\xi_K} \tag{28}$$

up to the phase factor without any further amplifications. Here we have used the asymptotic behavior of the Whittaker function, $\lim_{x \to \infty} W_{\pm i\kappa,\mu}(\pm ix) = e^{\pi ix/2(\pm ix)^{\pm i\kappa}} = e^{-\pi \kappa/2} e^{(ix/2 - \kappa \log[x])}$. While a part of the exponential amplifications of the gauge fields during inflation [39, 48],

$$A_+(k, \eta \to 0) \simeq \frac{1}{\sqrt{\pi \xi_I}} \exp[\pi \xi_I]/\sqrt{2k}, \tag{29}$$

is explained by the asymptotic behavior of the Whittaker function at $k\eta \to -0$, $\lim_{x \to 0} W_{-i\kappa,\mu}(-ix) = 1/\Gamma[1 - i\kappa] \sim \exp[\pi \kappa/2]/\sqrt{\pi \kappa}$, those during kination is explained by the exponentially large coefficients in front of the Whittaker function, which overwhelms the exponential suppression of the asymptotic behavior of the Whittaker function at $k\eta \to \infty$. At the matching time $\eta = \eta_{\text{end}}$, the positive and negative frequency modes are canceled each other so that the gauge fields stay small.

To see explicitly the amplification mechanism described in the above, in Fig. 1 we show the typical evolution of the mode function of the gauge fields with positive helicity $A_+$ for $(\xi_I, \xi_K) = (0, 6)$ as well as $(3,0)$ for comparison with Eqs. (21) and (23) together with the matching condition Eq. (24). Here we take $k = 0.01 k_c$ with $k_c$ being the horizon scale at the end of inflation, $k_c \equiv a_{\text{end}}H_I$. The gauge field amplification during inflation (Eq. (29)) can be seen for the case with $\xi_I > 0$, in which the gauge field is amplified at horizon exit. On the other hand, for the case $\xi_K > 0$ the gauge field is amplified at the horizon reentry. This is the unique characteristics of the magnetogenesis during the kination era. In realistic models of inflation, $\xi_I$ becomes large at the end of inflation, and our treatment of the evolution of $\xi$ (Eq. (18)) does not hold. An exponential amplification with $\xi \gg 1$ can take place without
FIG. 1: Typical time evolutions of the gauge field normalized by the momentum, $\sqrt{2k}|A_{I,+}|$, are shown. The red and blue lines describe the case of $\xi_I = 3$, $\xi_K = 0$ and $\xi_I = 0$, $\xi_K = 6$, respectively. In the former case, the gauge field is amplified at the horizon exit during inflation, while in the latter case the amplification occurs at horizon reentry during the kination era. The red and blue dotted lines represent the approximate formula of gauge field amplification given in Eq. (28) and Eq. (29), respectively.

the kination era, but only for the modes that exit the horizon around the end of inflation. One may think that practically the amplification in Eq. (28) is not a significant amplification and the late-time observables are not different compared to realistic inflation models (with the instant reheating). In this case, however, the exponential amplification $\sim \exp[\pi\xi_K/2]$ takes place not only for the modes that exit the horizon around the inflation end but also for the modes that exited the horizon much before the end of inflation and reenter the horizon during the kination era, which shows the significant difference to the instant reheating case. Indeed, the momentum of the most amplified mode is almost the same, as we will see, but the shape of the spectrum is different, which we will not explore in depth.

Now we are ready to evaluate the characteristic properties of the produced gauge fields, that is, the energy density and coherent length. The physical energy density of the gauge field, which can be divided into that of the electric and magnetic fields, is estimated as

$$
\rho_{EE}(\eta) \equiv \frac{1}{2} \langle E^2_p(\eta) \rangle = \frac{1}{2a^4} \int \frac{d^3k}{(2\pi)^3} |\partial_\eta A_+(k,\eta)|^2,
$$

and

$$
\rho_{BB}(\eta) \equiv \frac{1}{2} \langle B^2_p(\eta) \rangle = \frac{1}{2a^4} \int \frac{d^3k}{(2\pi)^3} k^2 |A_+(k,\eta)|^2.
$$
Note that this integral formally suffers from the UV divergence and requires an appropriate renormalization \[75\]. Here we shall evaluate them simply by setting the UV (and IR) cutoff, which corresponds to the mode that has experienced the tachyonic instability during the kination era in a similar way adopted in Ref. \[48\], which is consistent with the results in Ref. \[75\]. Namely, we set the UV and IR cutoffs, \(k_{\text{max}}\) and \(k_{\text{min}}\), as \(k_{\text{min}} \sim 1/\eta_K\) and \(k_{\text{max}} = \xi_K/\eta_K^{\text{end}} = 2\xi_K a_{\text{end}} H_I\), respectively, with \(\eta_K^{\text{end}} \equiv \eta_{\text{end}} + 3/(2a_{\text{end}} H_I) = 1/(2a_{\text{end}} H_I)\). \(k_{\text{min}}\) is the mode which reenters the horizon at \(\eta_K\), and \(k_{\text{max}}\) is the mode which experiences the tachyonic instability only at the end of inflation. Note that the mode function has the intrinsic cutoff at \((32/9\pi^2)\xi_K^{-1} a_{\text{end}} H_I\), which is smaller than the UV cutoff for the evaluation of the integral set in the above. Using the approximation formula in Eq. (28), the asymptotic behaviors of \(\rho_{\text{EE}}\) and \(\rho_{\text{BB}}\) at later times are analytically evaluated with Eqs. (30) and (31) as

\[
\rho_{\text{EE}}(\eta) = \frac{H_I^4}{32\pi^3} \left( \frac{a_{\text{end}}}{a(\eta)} \right)^4 \frac{e^{\pi\xi_K}}{\xi_K^2} \int_{2\xi_K\eta_K^{\text{end}}/\eta_K}^{2\xi_K} \frac{d\xi x}{\sqrt{a_{\text{end}} H_I x/2\pi e^{\pi\xi_K/2}}} \left. \frac{\partial A_+ (\eta, x)}{A_+ (\eta, x) \xi_K e^{\pi\xi_K/2} / 2\sqrt{2\pi a_{\text{end}} H_I x}} \right|_{\partial A_+ (\eta, x)}^2
\]

\[
\rho_{\text{BB}}(\eta) = \frac{H_I^4}{32\pi^3} \left( \frac{a_{\text{end}}}{a(\eta)} \right)^4 \frac{e^{\pi\xi_K}}{\xi_K^3} \int_{2\xi_K\eta_K^{\text{end}}/\eta_K}^{2\xi_K} \frac{d\xi x}{\sqrt{a_{\text{end}} H_I x/2\pi e^{\pi\xi_K/2}}} \left. \frac{A_+ (\eta, x)}{\xi_K e^{\pi\xi_K/2} / 2\sqrt{2\pi a_{\text{end}} H_I x}} \right|_{A_+ (\eta, x)}^2
\]

for \(\eta_K > 2\xi_K\eta_K^{\text{end}}\), where we have introduced the dimensionless conformal time \(x = \xi_K k/(a_{\text{end}} H_I)\) normalized by the momentum. Note that from Eq. (28) we have estimated \(\partial_\eta A_+ \simeq k A_+ = (a_{\text{end}} H_I/\xi_K) x A_+\). At a sufficiently late time, \(\eta_K > (9\pi^2/16)\xi_K\eta_K^{\text{end}}\), when the lower bound of the integral becomes smaller than the unity, the integral becomes a constant and independent of \(\xi_K\) or \(H_I\) due to the intrinsic cutoff or the peak of the integrand at \(x \simeq 32/9\pi^2 \sim 1\). This means that the amplification of the total energy density of the gauge fields terminates at \(\eta_K \simeq (9\pi^2/16)\xi_K\eta_K^{\text{end}}\) and dilutes after that according to the cosmic expansion. Since the gauge fields propagate almost freely after their amplification terminates, the energy density of the electric and magnetic fields equilibrate, \(\rho_{\text{EE}} \simeq \rho_{\text{BB}}\), during the kination era until the charged particles start to screen the electric fields. We will discuss the screening effect in Sec. III.

While Eqs. (32) and (33) show the approximate parameter dependence of the energy densities of the gauge fields analytically, we need to perform the numerical integration to
FIg. 2: Typical time evolution of energy density of the gauge fields with $\xi_I = 0$ and $\xi_K > 0$ are shown. We numerically evaluate the energy density of the electric and magnetic field in Eqs. (30) and (31). The energy densities are given in the comoving quantities, $a^4 \rho_{EE}$ and $a^4 \rho_{BB}$, for red and blue lines, respectively. Solid, dotted, and dashed lines denote the cases $\xi_K = 4$, 6, and 8, respectively.

obtain the quantitative estimates. By performing the integration in Eqs. (32) and (33), we obtained $\rho_{EE}$ and $\rho_{BB}$ as the function of the scale factor $a$ after inflation. We show the typical gauge field amplifications with $\xi_K = 4$, 6, and 8 in Fig. 2. We can see that the gauge field amplification in the comoving values is saturated around

$$a = a_{\text{sat}} \simeq \frac{3 \pi \sqrt{\xi_K}}{4} a_{\text{end}},$$

and becomes constant after that, which is consistent with the estimates Eqs. (32) and (33). With these numerical calculations, quantitatively we find

$$\rho_{EE}(\eta) = 1 \times 10^{-2} H_I^4 \left( \frac{a_{\text{end}}}{a(\eta)} \right)^4 e^{\pi \xi_K} \frac{\xi_K^3}{\xi_K^3},$$

$$\rho_{BB}(\eta) = 1 \times 10^{-2} H_I^4 \left( \frac{a_{\text{end}}}{a(\eta)} \right)^4 e^{\pi \xi_K} \frac{\xi_K^3}{\xi_K^3},$$

which are the main results of the present paper. In the following we use these fitting formula for the investigation of the consequence of this mechanism.

Some more comments follows. We can see that the magnetic field is amplified at first, and the electric field catches up it. This is because in the gauge field amplification during
the kination era, the gauge fields are amplified when they turn from the superhorizon mode to the subhorizon mode and hence the magnetic fields, which are the spatial derivative of the vector field, grows faster than the electric fields, which are the time derivative of the vector field. This is the opposite behavior to the gauge field amplification during inflation, where the gauge fields are amplified when they exit the horizon. In this case, the electric fields are amplified faster and stay larger than magnetic fields in the instant reheating approximation.

While in the inflation with the instant reheating case, there will be no free propagation of the gauge fields and there are no equilibration between the electric and magnetic fields, in the kination case, after the amplification ends and the relevant modes enter subhorizon, the gauge fields freely oscillate and end up with equal energy densities between the electric and magnetic fields. See Ref. [76] for a similar discussion on the equilibration of the electric and magnetic fields during the reheating era.

The coherence length of the gauge fields is defined by

$$\lambda_{\text{phys}}(\eta) \equiv \frac{1}{\rho_{BB}} \frac{1}{2a^4} \int \frac{d^3k}{k} \frac{2\pi a(\eta)}{(2\pi)^3} k^2 |A_+(k, \eta K)|^2$$

$$= \frac{2\pi \xi K}{H_I} \left( \frac{a(\eta)}{a_{\text{end}}} \right) \left( \frac{\int_{2\xi K a_{\text{end}}/\eta K}^{2\xi K} x^2 dx |A_+(x, \eta)K/(\xi K e^{\pi \xi K/2}/2)^2 |2}{\int_{2\xi K a_{\text{end}}/\eta K}^{2\xi K} x^3 dx |A_+(x, \eta)K/(\xi K e^{\pi \xi K/2}/2)^2 |2} \right).$$

(37)

Once more, with the numerical calculation we find the quantitative fitting formula for the coherence length as

$$\lambda_{\text{phys}}(\eta) = 0.13 \frac{2\pi}{H_I} \left( \frac{a(\eta)}{a_{\text{end}}} \right) \xi K,$$

(38)

which is similar to the one obtained in the gauge field amplification at the end of inflation with the instant reheating approximation, $\lambda_{\text{phys}} \simeq 0.6(2\pi/H)\xi I$ [48].

For later purpose we also evaluate the comoving helicity density,

$$h_c \equiv a^2 \langle \mathbf{A} \cdot \mathbf{B}_p \rangle = \int \frac{d^3k}{(2\pi)^3} k |A^k_+|^2 = a^3 \rho_{BB} \lambda_{\text{phys}} \frac{\pi}{\xi K^2},$$

(39)

which is quantitatively estimated by using Eqs. (36) and (38) as

$$h_c(\eta) = 3 \times 10^{-3} a_{\text{end}}^3 H_I^3 \frac{e^{\pi \xi K}}{\xi K^2}.$$

(40)

Since only the plus mode is amplified, the gauge fields are maximally helical and the electric and magnetic fields effectively runs parallel. Note that the helicity density is related to the
cross correlation of $E_p$ and $B_p$, $\rho_{EB} \equiv \langle E_p \cdot B_p \rangle$, as
\[
\frac{d}{d\eta} h_e = -2a^4 \langle E_p \cdot B_p \rangle = -2a^4 \rho_{EB}.
\] (41)

Comparing with $\rho_{EE}$ and $\rho_{BB}$, we roughly estimate $\rho_{EB}$ as
\[
\rho_{EB} \simeq 1 \times 10^{-2} H_I \left( \frac{a_{end}}{a(\eta)} \right)^4 \frac{e^{\pi \xi_K}}{\xi_K^4},
\] (42)

which is used to evaluate the backreaction on the ALP dynamics from the gauge field amplification in the next subsection.

**B. Constraints from backreaction**

The explosive amplification of gauge fields causes the backreaction on the cosmic expansion and the ALP dynamics Eq. (12). If the backreaction is too large, the investigation in the previous subsection is spoiled. For the inflation and the kination era, we define the following two quantities that describe the back reaction, namely, (1) the ratio of the energy density of the hypergauge fields to that of the ALP on the Friedman equation and (2) the ratio of the source term to the Hubble friction term on the Klein-Gordon equation Eq. (12), represented $\delta_F$ and $\delta_K$, respectively, as [48]
\[
\delta_F \equiv \frac{\rho_{EE} + \rho_{BB}}{3H^2 M_{pl}^2}, \quad \delta_K \equiv \left| \frac{\rho_{EB}/\Lambda}{3H^2} \right| = \frac{\xi_K}{3} \left| \frac{\rho_{EB}}{H^2 M_{pl}^2} \right|.
\] (43)

If $\delta_F \gg 1$, the assumption that the Universe is dominated by the inflaton is broken down. Note that it may corresponds to a realization of reheating if it occurs during kination. However, in order to see if the Universe really becomes the thermal radiation dominated, one needs to investigate the particle production from the gauge fields, or the Schwinger effect, which we will discuss it in the next section. In such a case, it is difficult to perform a consistent analysis. If $\delta_K \gg 1$, additional friction term on the ALP dynamics overwhelms the Hubble friction, which would make the instability parameter $\xi$ much smaller and hence the gauge amplification is suppressed. For the consistency of our analysis, therefore, we require $\delta_F, \delta_K < 1$ until the saturation of gauge field amplification.

Let us first consider the backreaction on the cosmic expansion $\delta_F$. Since we have estimated the gauge field amplification saturates at $a_{sat}$ without backreaction, we require $\delta_F < 1$ at $a = a_{sat}$. It is evaluated as
\[
\delta_F(a_{sat}) \simeq \frac{2}{3} \times 10^{-2} \left( \frac{H_I}{M_{pl}} \right)^2 \left( \frac{a_{sat}}{a_{end}} \right)^2 \frac{e^{\pi \xi_K}}{\xi_K^2} \simeq 3.7 \times 10^{-2} \left( \frac{H_I}{M_{pl}} \right)^2 \frac{e^{\pi \xi_K}}{\xi_K^2}.
\] (44)
For $H_I \simeq 10^{13}$ GeV, $\delta_F < 1$ is satisfied for $\xi_K < 10$. On the backreaction on the ALP dynamics, $\delta_K$, at the saturation of the gauge field amplification, it is evaluated as

$$
\delta_K(a_{\text{sat}}) \simeq \frac{\xi_K}{3} \times 10^{-2} \left( \frac{H_I}{M_{\text{pl}}} \right)^2 \left( \frac{a_{\text{sat}}}{a_{\text{end}}} \right)^2 \frac{e^{\pi \xi_K}}{\xi_K^3} \simeq 1.9 \times 10^{-2} \left( \frac{H_I}{M_{\text{pl}}} \right)^2 \frac{e^{\pi \xi_K}}{\xi_K},
$$

where we have used Eq. (18). For $H_I \simeq 10^{13}$ GeV, the condition $\delta_K(a_{\text{sat}}) < 1$ is satisfied for $\xi_K \lesssim 10$.

The discussion in this section does not limited to the SM U(1) gauge fields but is also applicable to the any dark U(1) gauge fields. In the next section, we will investigate phenomena inherent in the SM gauge field or in the presence of the charged particles, namely, the Schwinger effect.

### III. SCHWINGER EFFECT DURING KINATION AND REHEATING

Thus far, we have studied the dynamics of the system only with the runaway ALP and the hypergauge field. Once we take into account the matter field in the system, there arises an inevitable effect on the dynamics of the system. Namely, the amplified hypergauge fields induce pair production of particle and antiparticle charged under the hypergauge interaction [70, 77], which is known as the Schwinger effect. This effect hinders the amplification of the gauge field due to the following two reasons: (1) The pair production of charged particles acts as the friction term for the gauge field amplification, and (2) the produced charged particles screen the (hyper) electric field. In particular, the produced charged particles are eventually thermalized, which opens up the possibility to complete the reheating of the Universe in the scenario with the kination era, which is also dubbed as “Schwinger reheating” [74]. In this section, we discuss these two consequences of the Schwinger effect.

Indeed, the evaluation of the Schwinger effect with the dynamical gauge field background is quite involved, and to our best of knowledge even a method to analyze the system consistently has not been established. Therefore we adopt several simplification and assumptions, mainly following the analysis in Ref. [49], to get a rough estimate for the consequence of the Schwinger effect and to show the possible realization of the Schwinger reheating\(^1\).

\(^1\) See also recent study on the Schwinger effect in axion inflation with the gradient expansion formalism [73], which however requires model-dependent numerical analysis.
Hereafter we assume that the Higgs field acquires a sufficiently large induced mass through an appropriate non-minimal coupling to gravity or spectator fields, which keeps the electroweak symmetry unbroken during the period of interest and suppresses the production of the Higgs fields, and focus on the production of the (massless hyper U(1) charged) fermions. This assumption has also an advantage to avoid the Higgs vacuum instability into the unwanted true AdS vacuum.

A. Review of the Schwinger effect and its application to gauge field amplification during kination

We first briefly review the structure of the fermion spectrum in the presence of the gauge field with introducing the Landau levels and how we can estimate the number density of the fermions pair produced by the Schwinger effect. Let us consider a massless Dirac fermion $\psi$ with a hypercharge $Q$ with the Lagrangian Eq. (5). The equation of motion for the fermion in the presence of a “background” gauge field $A_\mu$ is given as

$$ \left[ a^{-1} \gamma^\mu (\partial_\mu + ig'QA_\mu) + \frac{3}{2} H \gamma^0 \right] \psi = 0. $$

(46)

Since massless fermions and gauge fields are conformal, we can eliminate $a$ and $H$ by rescaling $\tilde{\psi} \equiv a^{3/2} \psi, \tilde{A}_\mu \equiv A_\mu, \tilde{A}^\mu \equiv a^2 A^\mu$. Eq. (46) is then rewritten as

$$ \gamma^\mu \left( \partial_\mu + ig'Q\tilde{A}_\mu \right) \tilde{\psi} = 0. $$

(47)

It is desirable if we could solve the quantum evolution equation for both the gauge fields and the fermions simultaneously, but it is technically difficult. Instead, we take the gauge field amplified by the ALP dynamics as the background field and examine the spectrum and dynamics of the fermions with the consistency conditions. To make the analytic estimate possible, we employ an approximation that the gauge fields are uniformly distributed while maximally helical to grasp the properties of the gauge field of our interest. Namely, we write the background gauge field as $\tilde{A}_\mu = (0, 0, B_c x, -E_c \eta)$ (z-axis is the direction of the electromagnetic field.) so that the electric and magnetic fields are parallel as discussed in Sec. II. This approximation corresponds to a treatment that we stop the gauge field amplification at a certain time and take one patch within its correlation length. We also ignore the cosmic expansion and substitute $a = 1$, which means that we replace $\eta$ with
This also suggests that the hyper electric and magnetic fields we discuss here are the comoving ones, but not the physical ones discussed in the previous section. We added the subscript $c$ to indicate that explicitly. These assumptions are justified if the following two conditions are satisfied. First, the time scale of the fermion production is not much slower than that of the gauge field time evolution and the Hubble time scale. Second, the gauge field coherence length and the Hubble length is much larger than the one that corresponds to the typical energy scale of the produced fermions. We will take the time scale of the fermion production as the Hubble time at $a = a_{\text{sat}}$, which is the same order as the time scale of the gauge field evolution (see Fig. 2), so that the former condition is satisfied. We will confirm the latter condition later.

Let us now investigate the Schwinger mechanism. Eq. (47) is rewritten as

$$[\partial_t + s \nabla \cdot \sigma - i s g' Q(B_c x \sigma_y - E_c t \sigma_z)] \tilde{\psi}_{L/R} = 0,$$

(48)

where the Dirac fermion is decomposed into the chiral (Weyl) components $\tilde{\psi}_{L/R}$. $s$ takes $+1$ and $-1$ for left- and right-handed component, respectively. By introducing an auxiliary field $\Psi_{L/R}$ as

$$\tilde{\psi}_{L/R} \equiv [-\partial_t + s \nabla \cdot \sigma - i s g' Q(B_c x \sigma_y - E_c t \sigma_z)] \Psi_{L/R},$$

(49)

Eq. (48) becomes

$$[-\partial_t^2 + \nabla^2 - 2 ig' Q(B_c x \partial_y - E_c t \partial_z) - g^2 Q^2 (B_c^2 x^2 + E_c^2 t^2) + g' Q(B_c + i s E_c) \sigma_z] \Psi_{L/R} = 0.$$

(50)

By performing the (partial) Fourier transformation of $\Psi_{L/R}$ with respect to the spatial coordinates $y$ and $z$ as

$$\Psi_{L/R}(t, x) = \int \frac{dk_y dk_z}{(2\pi)^2} e^{i(k_y y + k_z z)} \Psi_{L/R}(t, x, k_y, k_z),$$

(51)

Eq. (50) is rewritten as

$$[-\partial_t^2 + \partial_x^2 - (g' Q B_c x - k_y)^2 - (g' Q E_c t + k_z)^2 + g' Q(B_c + i s E_c) \sigma_z] \Psi_{L/R} = 0.$$

(52)

By redefining the coordinate as $X \equiv \sqrt{g' |Q| B_c x - \varsigma k_y} / \sqrt{g' |Q| B_c}$, where $\varsigma$ is the sign of $Q$, Eq. (52) turns into the form that can be solved by the method of the separation of variables with decomposing the field as $\Psi_{L/R}(t, x, k_y, k_z) \equiv h_n(X) g_{L/R}(t, n, k_z) \chi_\varsigma$, where $\sigma_z \chi_\varsigma = \varsigma \chi_\varsigma$. 19
It is clear that the $X$-dependent part of the equation of motion, \( (\partial_X^2 - X^2) h_n(X) = -(2n + 1) h_n(X) \), is the same as that of the harmonic oscillator and it has a solution with discretized energy levels labeled by a non-negative integer \( n = 0, 1, 2, \ldots \), as

\[
h_n(X) = \frac{1}{\sqrt{2^n n!}} \left( \frac{g'|Q|B_c}{\pi} \right)^{1/4} e^{-X^2/2} H_n(X),
\]

where \( H_n(X) \) is the Hermite polynomial. Here we have normalized the $X$-dependent part so that it satisfies \( \int dx |h_n(X)|^2 = 1 \). The equation of motion for the $X$-independent part reads

\[
[\partial_t^2 + (g'Q E_c t + k_z)^2 - g'|Q|(B_c + isE_c)] g_{L/R} = -(2n + 1) g'|Q|B_c g_{L/R}.
\]

Note that $X$-independent part \( g_{L/R} \) depends on the energy level \( n \) but not \( k_y \) since \( X \) already contains \( k_y \).

To see the spectrum of the system, we first clarify the case with vanishing electric field \( E_c = 0 \). The solution of Eq. (54) is then just a plane wave,

\[
g_{L/R} = \frac{1}{\sqrt{2\omega_{L/R}}} e^{i\omega_{L/R} t}
\]

with the dispersion relation

\[
\omega_{L/R} = \begin{cases} 
\pm \sqrt{k_z^2 + 2ng'|Q|B_c} & (n = 1, 2, \ldots) \\
-s\kappa k_z & (n = 0)
\end{cases}.
\]

This discrete energy level is nothing but the relativistic Landau level. The discretization of the energy levels can be understood by the consequence that the uniform magnetic fields restrict the transverse motion of the charged particles by the Lorentz force. We can see that the \( n = 0 \) level called the lowest Landau level (LLL) has a unique dispersion relation where the negative frequency is continuously connected to the positive frequency in an opposite way for left- and right-handed fermions (with the same sign of charge). This is because the LLL is understood as a fermion moving in the direction of the magnetic field with aligning its spin (anti)parallel to the same direction. On the other hand, the \( n \geq 1 \) levels called the higher Landau levels (HLL) have the same structure for the right- and left-handed fermions. As a result, while the HLL contribute to the particle production without chirality, the LLL contribute to the chiral charge of the pair-produced particles.
Let us now examine the particle production when we apply the electric field in the system where the Landau levels are formed. Since the states are not well-defined under a non-vanishing electric field due to the explicit time dependence of the action, we consider the case where we apply a constant electric field parallel to the magnetic field as described in the above only for a certain time duration $0 < t < \tau_{\text{prod}}$, with the initial condition where the system is in the vacuum state so that the fermion states are filled up to $\omega_{L/R} = 0$. For the LLL, the fermions are accelerated up to $k_z = g'Q E_c \tau_{\text{prod}}$ along the dispersion relation for the LLL, which means that the pair-created particles are a right-handed fermion and a left-handed antifermion for $Q > 0$. Similar arguments apply for $Q < 0$. The (comoving) number density of the produced particles at the LLL is then evaluated as [49]

$$n_{\psi}^{\text{LLL}} = 2 \times \frac{1}{V} \int d^3x \int \frac{dk_y dk_z}{(2\pi)^2} [h_0(X)]^2 \Theta(-k_z) \Theta(k_z + g'|Q|E_c \tau_{\text{prod}})$$

$$= \frac{g'^2 |Q|^2}{4\pi^2} E_c B_c \tau_{\text{prod}},$$

(57)

where $V$ denotes the volume of the system. The prefactor 2 counts the right-handed fermion and the left-handed antifermion. One can see that this process is consistent with the chiral anomaly. This is not just a coincidence but the consequence of the chiral anomaly as is seen in the discussion of Nielsen and Ninomiya [78]. On the other hand, for the HLL, due to the nonzero electric field, the positive and the negative frequency modes for given $n$ are mixed and the fermion and the antifermion for both left and right-handed particles are produced, which is understood as the quantum tunneling process between the energy gap. Then the (comoving) number density of the produced particles is evaluated as

$$n_{\psi}^{(n)} = 4 \times \frac{1}{V} \int d^3x \int \frac{dk_y dk_z}{(2\pi)^2} [h_0(X)]^2 \Theta(-k_z) \Theta(k_z + g'|Q|E_c \tau_{\text{prod}}) e^{-2\pi n B_c / E_c}$$

$$= \frac{g'^2 |Q|^2}{2\pi^2} E_c B_c \tau_{\text{prod}} e^{-2\pi n B_c / E_c}.$$  

(58)

The prefactor 4 counts both the particle and the antiparticle of the left and right-handed fermions.

Before proceeding, let us clarify how we shall apply these results to the case of our interest. When the hypergauge fields are amplified during kination, we have seen that their amplification is saturated at $a = a_{\text{sat}} \sim \frac{3\pi}{4} \sqrt{\xi_{\text{eff}}} a_{\text{end}} \gg a_{\text{end}}$ (see Fig. 2). With focusing on the last minute of the hypergauge field amplification, we take $\tau_{\text{prod}}$ as the Hubble time at $a = a_{\text{sat}}$, or $\eta_{\text{prod}} = 1/(2a_{\text{sat}}H_{\text{sat}})$ (see Eq. (8)), for the evaluation of the properties of the
pair-produced particles. As we have mentioned, this treatment justifies our approximation that the electric and magnetic fields are taken to be a constant in time. The (comoving) electric and magnetic fields can be expressed in terms of the physical ones as \( E_c = a^2 E_p \) and \( B_c = a^2 B_p \). In the case where there are multiple (Weyl) fermions in the system as in the SM, we can just replace \(|Q|^2\) in Eqs. (57) and (58) with an effective value, \( Q^2/2 \equiv \sum_i |q_i|^2/2 \), with \( q_i \) being the hypercharge of each (Weyl) fermion, \( |Q|^2 \rightarrow Q^2/2 \). Note that it is not the square of the sum of charges of each particle \( \sum_i |q_i|^2 \), and the factor \( 1/2 \) is multiplied due to the difference between the Weyl and Dirac representation. (For later convenience, we introduce the notation \( Q_n \equiv \sum_i |q_i|^n \) and distinguish it from \( Q^n_1 = (\sum_i |q_i|)^n \).) In summary, with Eqs. (57) and (58) (divided by the factor of \( a^3_{\text{sat}} \)), the physical number density of the LLL as well as HLL at the time of the saturation of the gauge field amplification is given by

\[
\begin{align*}
    n_{\psi}^{\text{LLL}}(a_{\text{sat}}) &= \frac{g^2 Q^2 E_p B_p}{16\pi^2 H_{\text{sat}}}, \\
    n_{\psi}^{(n)}(a_{\text{sat}}) &= \frac{g^2 Q^2 E_p B_p}{8\pi^2 H_{\text{sat}}} e^{-2\pi n B_p/E_p}.
\end{align*}
\]

Furthermore, we assume that the Schwinger pair production after the saturation of the hypergauge field amplification is negligibly small since the hypergauge field starts to oscillate and they are no longer constant, which violates our assumption for the particle production. The number density of the particles (Eqs. (57) and (58)) is assumed to be redshifted according to the cosmic expansion after that. In this point of view, we give a conservative estimate of the number density of the produced particles.

### B. Backreaction on gauge field amplification

Once the Schwinger effect becomes sufficiently effective, induced (hyper) electric current from the pair-produced charged particles is no longer negligible for the amplification of the hypergauge field. Once more, it is desirable if we could solve the evolution of the system simultaneously, but the equations of motion are highly non-linear and too difficult to solve with the best of our knowledge and technique. Instead, here we take into account the backreaction of the Schwinger effect on the gauge field dynamics by giving some assumptions, following Ref. [49].

The equation of motion for the physical energy density of the gauge fields in the presence
of the background ALP dynamics and the induced current reads
\[
\frac{d}{dt}(\rho_{EE} + \rho_{BB}) = -4H(\rho_{EE} + \rho_{BB}) + 2\xi_K H \langle E_p \cdot B_p \rangle - \left\langle E_p \cdot g' \sum_i q_i J_i \right\rangle ,
\]
(61)
where \( J_i \) is the induced matter current of the particle \( i \). By using the mode functions in the way described in the above, one can evaluate the induced current \( \langle J_i^z \rangle \equiv \langle \bar{\psi}_i \gamma^\alpha \psi_i \rangle \) in the presence of the homogeneous hyper electric and magnetic fields in \( z \) direction as \( g' \sum_i q_i \langle J_i^z \rangle \)\]
(62)

We will confirm that the scattering and thermalization is not effective at least during the
gauge field amplification and the estimate for the induced current in the above is valid.
By taking \( \langle E_p \cdot B_p \rangle = E_p B_p \) and \( \langle E_p \cdot g' \sum_i q_i J_i \rangle = g' \sum_i q_i E_p \langle J_i^z \rangle \), we can see that the
induced current from the Schwinger effect is formally removed in the evolution equation for
the energy density of the hypergauge fields (61) by replacing \( \xi_K \) with \( \xi_{\text{eff}} \), which is defined
as
\[
\frac{1}{a^3} g' \sum_i q_i \langle J_i^z \rangle \approx \frac{g'^3 Q_3}{4\pi^2} \coth \left( \frac{\pi B_p}{E_p} \right) \frac{E_p B_p}{H} .
\]
(62)
Thus once we specify the “background” (hyper) electric and magnetic field, we can take into
account the backreaction on the gauge field amplification from the Schwinger effect in terms
of the effective instability parameter \( \xi_{\text{eff}} \).

One might think that one can just solve the mode equation with \( \xi_{\text{eff}} \). However, we would
like to take the electric and magnetic fields amplified by the ALP dynamics themselves
as the “background” fields that cause the Schwinger effect, which appear in the effective
instability parameter \( \xi_{\text{eff}} \). Thus one cannot solve the mode equation consistently as if \( \xi_{\text{eff}} \)
is a constant, and hence further approximation is needed to determine the resultant hyper
electric and magnetic field strength with taking into account the backreaction from the
Schwinger effect. In Ref. [49], two ways of approximation were proposed. The one is to take
both side of Eq. (61) to be zero so that gauge field amplification and the dilution due to the
cosmic expansion is equilibrated. From the right-hand side, we can evaluate the relationship
between the electric and magnetic fields for this equilibrium solution and draw a contour
in the \( E-B \) plane, on which the electric and magnetic fields are “stable”. By looking at
this contour, we can determine the upper bound of the magnetic field strength, which is
identified as the “maximal” solution. The other way is to take the expression of the electric
and magnetic fields without the Schwinger effect, namely Eqs. (35) and (36), with replacing $\xi_K$ with $\xi_{\text{eff}}$ as

$$E_{\text{eff}}(\xi_{\text{eff}}) \approx B_{\text{eff}}(\xi_{\text{eff}}) = \sqrt{2\rho_B B} = 0.14H_2^2 I(\xi_{\text{eff}}) \left( \frac{a}{a_{\text{end}}} \right)^{-2},$$  

(64)

where $I(\xi_{\text{eff}}) \equiv e^{\pi\xi_{\text{eff}}/2}/\xi_{\text{eff}}^{3/2}$, and substitute Eq. (64) into the expression of $\xi_{\text{eff}}$ in Eq. (63).

By requiring the resultant $\xi_{\text{eff}}$ in the same as the input $\xi_{\text{eff}}$ at the time of the saturation of the gauge field amplification, $H = H_{\text{sat}}$,

$$\xi_{\text{eff}} = \xi_K - \frac{g'^3 Q_3}{8\pi^2} \cot \left( \frac{\pi B_{\text{eff}}}{E_{\text{eff}}} \right) \frac{E_{\text{eff}}}{H_{\text{sat}}^2},$$  

(65)

we can determine the self-consistent solution of $\xi_{\text{eff}}$ as a function of $\xi_K$, which is identified as the “equilibrium” solution.

The former solution is reasonable for the inflationary magnetogenesis, since the amplified magnetic fields are expected to be saturated and become constant. In the case of kination, which of our interest, however, the Hubble parameter decreases with time and the electric and magnetic fields are not expected to become a constant when the gauge field amplification saturates, as we have seen in the case without Schwinger effect Eqs. (35) and (36). Therefore, we take the latter, “equilibrium” solution, to be our rough estimate of the electric and magnetic field strength from the ALP dynamics during kination in the presence of the Schwinger effect. Here we evaluate the parameters at the time of gauge field saturation at $a = a_{\text{sat}}$. The numerical result of the “equilibrium” solution for the effective instability parameter $\xi_{\text{eff}}$ is depicted in Fig. 3, where we have taken $Q_3 = 41/12 \approx 3.4$, which is motivated from the particle contents in the standard model, and $g' = 0.3$ as well as $a_{\text{sat}} = \frac{3\pi}{4} \sqrt{\xi_{\text{eff}}} a_{\text{end}}$. We can see that while $\xi_{\text{eff}} \simeq \xi_K$ for $\xi_K \lesssim 2$, it is highly suppressed for $\xi_K > 2$ due to the backreaction from the Schwinger effect and it grows only logarithmically. The typical energy carried by the LLL fermion at the time of the gauge field saturation, $k/a_{\text{sat}} \approx g' Q E_c \tau_{\text{prod}}/a_{\text{sat}}$, is now estimated as $\sim 0.14g'QH_2 I(\xi_{\text{eff}})(a_{\text{sat}}/a_{\text{end}})$, which is larger than the Hubble scale as well as the coherence length of the hyperelectric fields at the time of the saturation of the gauge field amplification for $\xi_{\text{eff}} > 1$, and hence the approximation of the homogeneous electric field is justified. In the following discussion, we will take $\xi_{\text{eff}} = 4$ achieved by $\xi_K \sim 10$ as a reference value for the investigation of the dynamics of the system.
FIG. 3: “Equilibrium” solution of the effective instability parameter $\xi_{\text{eff}}$ for the hypergauge field amplification as a function of $\xi_K$ in Eq. (63). Here we take $Q_3 = 41/12$, $g' = 0.3$ (motivated by the standard model) and $a = a_{\text{sat}} = \frac{3\pi}{4}\sqrt{\xi_{\text{eff}}a_{\text{end}}}$. The dashed and the solid lines show the bare value $\xi_K$ and effective value $\xi_{\text{eff}}$, respectively. $\xi_{\text{eff}}$ is highly suppressed for $\xi_K > 2$ due to the backreaction of the induced current. Here, the horizontal line denotes $\xi_{\text{eff}} = 6.6$, below which the pair-produced particles are thermalized before they dominates the Universe for $H_I = 10^{13}$ GeV as discussed later in this section.

C. Scattering and thermalization of produced charged fermions and reheating

In the previous subsection, we have seen how the pair-produced particles suppress the efficiency of the gauge field amplification. However, we have not investigated their thermalization, which would be important for reheating as well as the late-time evolution of the gauge fields. In this subsection, we study the thermalization of the pair-produced particles and its effect on the dynamics of the background electric and magnetic fields as well as the screening of the electric fields, with examining the time-evolution of the pair-produced particles.

1. Non-thermalization of the pair-produced particles during the gauge field amplification

In this subsection, we shall see that the pair-produced particles are not thermalized until the saturation of the gauge field amplification. Let us first investigate the self-scattering of the LLL fermions and their (non-)thermalization. If the LLL fermions are accelerated by
the constant and homogeneous electric field $E_{\text{eff}}$ (together with $B_{\text{eff}}$) for a sufficiently long period such that $s_i \gg B_{\text{eff}}$, with $s_i$ denoting the center of mass energy squared of the fermion labeled by $i$, they are no longer confined along the magnetic field and their scattering rate is naively evaluated as

$$\Gamma_{\text{sc}}^{\text{LLL}} = \frac{g'^4}{12\pi s_i} n_{\psi_i}^{\text{LLL}}. \quad (66)$$

By taking the acceleration time as $\tau$ in the conformal time, their energy scale is evaluated as $s_i(\tau) = 2(g'q_i E_{\text{eff}} \alpha \tau)^2$, assuming that the acceleration is not disturbed by the scattering. At the same time, the physical number density of the LLL fermions is evaluated by the anomaly equation (see Eq. (57)) as $n_{\psi_i}^{\text{LLL}}(\tau) \sim (g'^2 q_i^2 / 4\pi^2) E_{\text{eff}} B_{\text{eff}} \alpha \tau$. The scatterings happen so often that the LLL fermions are thermalized if $\Gamma_{\text{sc}}^{\text{LLL}} \alpha \tau \gg 1$. We find that with the above estimates, it is evaluated as

$$\Gamma_{\text{sc}}^{\text{LLL}} \alpha \tau \simeq \frac{g'^4}{96\pi^3} \ll 1, \quad (67)$$

where we have used the condition $E_{\text{eff}} = B_{\text{eff}}$ (Eq. (64)). Apparently the LLL fermions are never thermalized. However, we need to take a special care in this situation. At the final stage of the gauge field amplification, the system is underoccupied or the typical momentum of the LLL becomes much larger than the inverse of the mean separation length, which can also be expressed as the condition $T_{\text{wb}}^{\text{LLL}} \ll \omega_{\psi_i}^{\text{LLL}}$, with $T_{\text{wb}}^{\text{LLL}}(\tau) \simeq ((30/\pi^2 g_*) n_{\psi_i}^{\text{LLL}}(\tau) \omega_{\psi_i}(\tau))^{1/4}$ and $\omega_{\psi_i} = \sqrt{s_i/2}$ being the “would-be temperature” and the energy of the LLL fermion, respectively. In such a stage the rate Eq. (66) is not directly applied, where large-angle scatterings are assumed. Instead, the main channel of the thermalization of the LLL fermions is turned out to be the multiple small-angle soft gauge boson scatterings [49] (see also [79–81]), which needs to be evaluated by taking into account the Landau-Pomeranchuk-Migdal (LPM) process [82, 83]. With these care, the thermalization rate (with non-Abelian gauge theories) for $T_{\text{wb}}^{\text{LLL}} \ll \omega_{\psi_i}^{\text{LLL}}$ case is found to be given by [84–87]

$$\Gamma_{\text{LPM}}^{\text{LLL}}(\tau) = \frac{g'^4}{16\pi^2} T_{\text{wb}}^{\text{LLL}}(\tau) \sqrt{\frac{T_{\text{wb}}^{\text{LLL}}(\tau)}{\omega_{\psi_i}^{\text{LLL}}(\tau)}}. \quad (68)$$

We find that $\Gamma_{\text{LPM}}^{\text{LLL}}(\tau) \alpha \tau$ is an increasing function of $\tau$ and

$$\Gamma_{\text{LPM}}^{\text{LLL}}(\tau) \alpha \tau \big|_{\tau = \eta_{\text{prod}}} \simeq 1.3 \times 10^{-4} \left( \frac{g_*}{106.75} \right)^{-\frac{3}{8}} \left( \frac{g'}{0.3} \right)^{\frac{37}{8}} |q_i|^\frac{5}{8} \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^{\frac{5}{8}} \left( \frac{a_{\text{sat}}}{\alpha} \right)^{\frac{5}{4}} \left( \frac{\xi_{\text{eff}}}{\xi_{\text{end}}} \right)^{\frac{5}{4}}, \quad (69)$$
which is smaller than unity typically for $\xi_{\text{eff}} < 13$. Note that such a large $\xi_{\text{eff}}$ causes the too large backreaction as we have seen in Sec. II B and is difficult to be obtained as can be seen in Fig. 3. This suggests that the thermalization condition is not satisfied during the gauge field amplification for the reasonable value of $\xi_{\text{eff}}$, even taking into account the LPM effect in the scattering rate. Therefore, we conclude that the LLL fermions would not be thermalized during the gauge field amplification.

Next we examine the scattering of the HLL fermions. In the situation of our interest, the inverse of the mean separation length of the HLL fermions is found to also be always much smaller than their typical momentum. Thus we shall evaluate their scattering rate with the LPM one, $\Gamma^{\text{HLL}(n)}_{\text{LPM}}$. In the same way in the case of the LLL fermions, we find that $\Gamma^{\text{HLL}(n)}_{\text{LPM}} a\tau$ is a monotonically increasing function of time, and for example, for $n = 1$ HLL fermions, it is estimated as

$$\Gamma^{\text{HLL}(1)}_{\text{LPM}}(\tau) a\tau \bigg|_{\tau = \eta_{\text{prod}}} \simeq 1.6 \times 10^{-5} \left( \frac{g_*}{106.75} \right)^{-\frac{2}{3}} \left( \frac{g'}{0.3} \right)^{\frac{2}{3}} |q_i|^\frac{2}{3} \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^{\frac{5}{2}} \left( \frac{a_{\text{sat}}}{\frac{3\pi}{4} \sqrt{\xi_{\text{eff}} a_{\text{end}}}} \right)^{\frac{5}{4}} \left( \frac{\xi_{\text{eff}}}{4} \right)^{\frac{5}{32}}. \tag{70}$$

Here we take the typical energy of the $n$-th HLL fermions as $\omega_{\psi_i}^{(n)}(\tau) = \sqrt{(g'|q_i| E_{\text{eff}} a\tau)^2 + 2ng'|q_i| B_{\text{eff}}}$ as indicated by the dispersion relation (56), and the would-be temperature as $T_{\text{wb}}^{(n)}(\tau) = \left( (30/\pi^2 g_*) n_{\psi_i}^{(n)}(\tau) \omega_{\psi_i}(\tau) \right)^{1/4}$ with the physical number density of the $n$-th HLL fermions $n_{\psi_i}^{(n)}(\tau) \simeq (g'^2 |q_i|^2/2\pi^2) E_{\text{eff}} B_{\text{eff}} a\tau e^{-2\pi n B_{\text{eff}}/E_{\text{sat}}}$. We also find that $\Gamma^{\text{HLL}(n)}_{\text{LPM}}(\tau) a\tau$ at $\tau = \eta_{\text{prod}}$ is even more suppressed for $n > 1$ level. Thus as long as $\xi_{\text{eff}} < 14$, $\Gamma^{\text{HLL}(n)}_{\text{LPM}}(\tau) a\tau$ is always much smaller than the unity and it is unlikely that the HLL fermions are thermalized before the saturation of the gauge field amplification. In other words, the validity of the estimate of the Schwinger effect in the previous subsection, where we do not take into account the scattering of the produced particles, is now confirmed.

2. Screening of the electric field

Next, let us investigate the screening of the (hyper) electric field by the pair-produced particles. After the gauge field amplification saturates, the energy injection from the background ALP into the hyper electric field becomes negligible. As a consequence, the hyper electric field begins to decay due to the screening by the pair-produced particles. The screening by the charged particles is characterized by the electric conductivity. Strictly speaking, it would be desirable if we could evaluate it by using Kubo formula with taking into account
the accurate phase space distribution, which is practically difficult. Before the thermal-
ization (See Eq (76)), we instead evaluate it by adopting the Drude model. The electric
conductivity carried by the particles labeled by $i$ is evaluated as

$$
\sigma_i \sim \frac{n_{\psi_i} g_i^2}{\omega_{\psi_i} \Gamma_{\mathrm{LPM}}} = \frac{16 \pi^2}{g^2} \left( \frac{\pi^2 g_*}{30} \right)^{\frac{3}{8}} n_{\psi_i}^{\frac{5}{8}} \omega_{\psi_i}^{-\frac{7}{8}},
$$

(71)

where we take the characteristic time scale as the inverse of the LPM scattering rate
(Eq. (68)). Since the plasma is dominated by the LLL fermions, by substituting Eq. (59)
and $\omega_{\psi_i}^{\mathrm{LLL}} = \sqrt{s_i/2}$ into Eq. (71) and taking into account the redshift, we obtain

$$
\sigma_i \sim 7.4 \times 10 H_I \left( \frac{g_*}{106.75} \right)^{\frac{3}{8}} \left( \frac{g'}{0.3} \right)^{-\frac{13}{8}} |q_i|^\frac{3}{2} \left( \frac{I(\xi_{\mathrm{eff}})}{I(4)} \right)^{\frac{3}{4}} \left( \frac{\alpha_{\mathrm{sat}}}{\sqrt{\xi_{\mathrm{eff}} a_{\mathrm{end}}}} \right)^{-\frac{3}{4}} \left( \frac{\alpha}{\alpha_{\mathrm{sat}}} \right)^{-1}.
$$

(72)

We can see that it is already much larger than the Hubble rate $H = H_I (a/a_{\mathrm{end}})^{-3}$ at $a = a_{\mathrm{sat}}$. Since the electric conductivity $\sigma$ means that the electric fields are screened in a time scale of $\sigma^{-1}$, the hyper electric field is immediately screened just after the gauge field amplification saturates. The energy of the hyper electric fields is converted to the one carried by the LLL fermions. This would change the estimate of their thermalization and reheating slightly, but quantitatively it is not significant in the parameter space we are interested in. On the other hand, the hyper magnetic fields are not screened for a considerably long time $\sim \sigma \lambda_{\mathrm{phys}}^2$ with $\lambda_{\mathrm{phys}} \gg \sigma^{-1}$, and we assume that they evolve as long-range non-oscillatory stochastic fields after that. Their evolution would affect the thermal history of the early Universe, on which we will discuss in the next section.

3. Eventual thermalization of the pair-produced particles and reheating

Let us now investigate how the produced particles are eventually thermalized and domi-
nate the energy density of the Universe. The physical energy density of the LLL as well as
the HLL fermions at the saturation of the gauge field amplification, $\tau = \eta_{\mathrm{prod}} = (2a_{\mathrm{sat}} H_{\mathrm{sat}})^{-1}$
or \( a = a_{\text{sat}} \), is approximated as

\[
\rho_{\psi}^{\text{LLL}}(\eta_{\text{prod}}) = \sum_i \frac{1}{a_{\text{sat}}^4 V} \int d^3 x \int \frac{dk_y dk_z}{(2\pi)^2} [h_0(X)]^2 \Theta (-k_z) \Theta (k_z + g|q_i|E_c \eta_{\text{prod}}) \\
\times (k_z + g'|q_i|E_c \eta_{\text{prod}})
\]

\[
= \frac{g^3}{4\pi^2 a_{\text{sat}}^4} Q_3 B_c E_c^2 \eta_{\text{prod}}^2
\]

\[
\simeq 0.48 \times \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^3 \left( \frac{g'}{0.3} \right)^3 \left( \frac{Q_3}{41/12} \right) H_i^{4},
\]

\[
\rho_{\psi}^{(n)}(\eta_{\text{prod}}) = \sum_i \frac{2}{a_{\text{sat}}^4 V} \int d^3 x \int \frac{dk_y dk_z}{(2\pi)^2} [h_0(X)]^2 \Theta (-k_z) \Theta (k_z + g|q_i|E_c \eta_{\text{prod}}) \\
\times (k_z + g'|q_i|E_c \eta_{\text{prod}}) e^{-2\pi n B_c/E_c}
\]

\[
= \frac{g^3}{2\pi^2 a_{\text{sat}}^4} Q_3 B_c E_c^2 \eta_{\text{prod}}^2 e^{-2\pi n B_c/E_c}
\]

\[
\simeq 0.96 \times \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^3 \left( \frac{g'}{0.3} \right)^3 \left( \frac{Q_3}{41/12} \right) H_i^{4} e^{-2\pi n},
\]

where we have used \( E_c = B_c = a^2 E_{\text{eff}} = a^2 B_{\text{eff}} \) and Eq. (64). We can see that the energy density of the HLL fermions is suppressed by a factor of \( e^{-2\pi n} \) and the total energy density of the produced particles is dominated by the LLL fermions. We also find that the energy density of the LLL fermions is typically larger than that of the electric and magnetic fields,

\[
\frac{\rho_{\psi}^{\text{LLL}}}{\rho_{EE} + \rho_{BB}} = 2.6 \times \left( \frac{g'}{0.3} \right)^3 \left( \frac{Q_3}{41/12} \right) \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right) \left( \frac{a_{\text{sat}}}{\sqrt{\xi_{\text{eff}} a_{\text{end}}}} \right)^4 \left( \frac{\xi_{\text{eff}}}{4} \right)^2
\]

where we have used Eqs. (35) and (36) and taken \( \xi_{\text{eff}} = 4 \) as a reference value.

As has been discussed in the previous section, the hyperelectric fields are screened just after the saturation of gauge field amplification. Thus we assume that thereafter they evolve adiabatically, \( n_{\psi} \propto a^{-3} \) and \( \omega_{\psi} \propto a^{-1} \). Under this assumption, the LPM scattering rate evolves as \( \Gamma_{\text{LPM}}^{\text{LLL}} \propto a^{-1} \) and eventually becomes larger than the Hubble rate, which is proportional to \( a^{-3} \) during kination. Once it becomes \( H \sim \Gamma_{\text{LPM}}^{\text{LLL}} \), we expect that the LLL fermions are thermalized. The scale factor at that time is estimated as

\[
a = a_{\text{th}} = 3.3 \times 10^2 \left( \frac{g_\star}{106.75} \right)^{\frac{1}{16}} \left( \frac{g'}{0.3} \right)^{-\frac{37}{16}} |q_i|^{-\frac{1}{16}} \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^{-\frac{7}{16}} \left( \frac{a_{\text{sat}}}{\sqrt{\xi_{\text{eff}} a_{\text{end}}}} \right)^{-\frac{1}{4}} \left( \frac{\xi_{\text{eff}}}{4} \right)^{-\frac{1}{8}} a_{\text{end}}.
\]

\[29\]
The relevant quantities at the time of LLL fermion thermalization are given as

\[ H_{th} = \left( \frac{a_{end}}{a_{th}} \right)^3 H_I \simeq 2.8 \times 10^5 \text{GeV} \left( \frac{g_*}{106.75} \right)^{-\frac{2}{11}} \left( \frac{g'}{0.3} \right)^{\frac{11}{16}} \left( \frac{I(\xi_{eff})}{I(4)} \right)^{\frac{15}{16}} \]

\times \left( \frac{a_{sat}}{\sqrt{\xi_{eff} d_{end}}} \right)^{\frac{3}{2}} \left( \frac{\xi_{eff}}{4} \right)^{\frac{3}{8}} \left( \frac{H_I}{10^{13} \text{GeV}} \right)^{\frac{17}{16}}, \quad (77)

\[ T_{th} = \left( \frac{30 \rho_{LLL}(a_{th})}{\pi^2 g_*} \right)^{1/4} \simeq 4.9 \times 10^{10} \text{GeV} \left( \frac{g_*}{106.75} \right)^{-\frac{2}{11}} \left( \frac{g'}{0.3} \right)^{\frac{40}{11}} \left( \frac{Q_3}{41/12} \right)^{1/4} \left( \frac{I(\xi_{eff})}{I(4)} \right)^{\frac{17}{16}} \]

\times \left( \frac{a_{sat}}{\sqrt{\xi_{eff} d_{end}}} \right)^{\frac{5}{2}} \left( \frac{\xi_{eff}}{4} \right)^{\frac{7}{8}} \left( \frac{H_I}{10^{13} \text{GeV}} \right)^{\frac{17}{16}}, \quad (78)

\[ B_{p,th} = B_{eff}(a_{th}) \simeq 8.6 \times 10^{21} \text{GeV}^2 \left( \frac{g_*}{106.75} \right)^{-\frac{5}{11}} \left( \frac{g'}{0.3} \right)^{\frac{37}{11}} \left( \frac{I(\xi_{eff})}{I(4)} \right)^{-\frac{5}{11}} \]

\times \left( \frac{a_{sat}}{\sqrt{\xi_{eff} d_{end}}} \right)^{-\frac{1}{2}} \left( \frac{\xi_{eff}}{4} \right)^{\frac{5}{7}} \left( \frac{H_I}{10^{13} \text{GeV}} \right)^{-\frac{5}{11}}, \quad (79)

\[ \lambda_{phys,th} = \lambda_{phys}(a_{th}) = 1.1 \times 10^{-10} \text{GeV}^{-1} \left( \frac{g_*}{106.75} \right)^{\frac{3}{11}} \left( \frac{g'}{0.3} \right)^{-\frac{37}{11}} \left( \frac{I(\xi_{eff})}{I(4)} \right)^{\frac{5}{11}} \]

\times \left( \frac{a_{sat}}{\sqrt{\xi_{eff} d_{end}}} \right)^{-\frac{1}{2}} \left( \frac{\xi_{eff}}{4} \right)^{\frac{5}{7}} \left( \frac{H_I}{10^{13} \text{GeV}} \right)^{-\frac{5}{11}}, \quad (80)

where we have taken \(|q_i| = 1\) to determine \(a_{th}\). Note that we here assume that the thermalization occurs during the kination era, which is satisfied for the threshold \(\xi_{eff} \lesssim 6.6\) when \(H_I = 10^{13} \text{ GeV}\) and for larger threshold of \(\xi_{eff}\) with smaller \(H_I\). For \(\xi_{eff}\) larger than the threshold, radiation, mainly composed of the LLL fermions, dominates the Universe before thermalization. In this case, the scale factor at the thermalization is larger than the estimate of Eq. (76). As we can see from Fig. 3, however, larger \(\xi_{eff}\) requires much larger \(\xi_K\). Such an inflation model is difficult to construct with avoiding the strong coupling problem or too large backreaction problem. Hereafter we thus focus on the case for smaller \(\xi_{eff}\) than the threshold.

As mentioned, the energy density of the thermal LLL fermions eventually dominates the energy density of the Universe. The Hubble parameter and the (would-be) temperature at the time when the energy density of radiation, \(\rho_{\text{rad}}\), dominates the Universe (namely,
reheating temperature) are given as

\[ H_{re} = 0.5 \text{GeV} \left( \frac{g'}{0.3} \right)^{9/2} \left( \frac{Q_3}{41/12} \right)^{3/2} \left( \frac{I(\xi_{eff})}{I(4)} \right)^{9/2} \left( \frac{\xi_{eff}}{4} \right)^3 \left( \frac{H_I}{10^{13} \text{GeV}} \right)^4, \]

(81)

\[ T_{re} = 6 \times 10^8 \text{GeV} \left( \frac{g_*}{106.75} \right)^{-1/4} \left( \frac{g'}{0.3} \right)^{9/4} \left( \frac{Q_3}{41/12} \right)^{3/4} \left( \frac{I(\xi_{eff})}{I(4)} \right)^{9/4} \left( \frac{\xi_{eff}}{4} \right)^3 \left( \frac{H_I}{10^{13} \text{GeV}} \right)^2, \]

(82)

where we have used Eqs. (64) and (73). Note that this estimate for the cosmic expansion does not depend on the condition if the LLL fermions are thermalized or not (The “would-be” temperature is the temperature if the relativistic components are thermalized.). Here we take \( \rho_{rad} = \rho_{LLL}^{\psi} \) and omitted other relativistic components such as \( \rho_{EE} \) and \( \rho_{BB} \). As we can see from Eq. (75), they are comparable to the LLL fermion or can be a bit larger, but the estimate does not change much. Thus we conclude that reheating well before the electroweak symmetry breaking or the BBN is realized for sufficiently large Hubble parameter during inflation and appropriate value of the effective instability parameter \( \xi_{eff} \), which is much more efficient than the usual gravitational reheating [18–20, 28]. For example, for \( \xi_{eff} = 4 \), reheating takes place before the electroweak symmetry breaking for \( H_I > 10^{10} \) GeV and before the BBN for \( H_I \gg 10^7 \) GeV. It can be also said that this is a realization of the Schwinger reheating [74] without a dark sector.

**IV. LATE-TIME EVOLUTION OF MAGNETIC FIELDS AND COSMOLOGICAL CONSEQUENCES**

In the previous section we have seen that the fermions produced by the Schwinger effect can be successfully thermalized and dominate the energy density of the Universe well before the electroweak symmetry breaking. However, it is not enough to conclude that the Universe that experienced this mechanism is consistent with the present Universe, since the relics might cause some unwanted phenomena. Namely, while we have seen that the hyper electric fields are immediately damped, the hyper magnetic fields can last for a relatively long time, which should pass the constraints of the number of additional relativistic degrees of freedom especially from the Big Bang Nucleosynthesis (BBN) [88–94] as well as the baryon overproduction/baryon isocurvature perturbation [95–101]. To check if the (hyper) magnetic fields are harmless or not, we should examine their evolution. In this section, we examine
the evolution of the magnetic fields with the magnetohydrodynamics (MHD), focusing on
the former problem. Indeed, we will see that the baryon overproduction is unavoidable for
the electroweak crossover in the light of the 125 GeV Higgs [100] unless the chiral plasma
instability [102–106] completely cancels the chirality and helicity [50]. We discuss the way
out of that in Sec. IV B.

A. Evolution of the magnetic fields and the BBN constraint

After the saturation of the gauge field amplification, the (hyper) magnetic fields are
diffused and damped exponentially by the conducting fluid in a relatively slow time scale
$\sigma \lambda^2_{\text{phys}}$, in the absence of the bulk velocity fields and the chiral magnetic effect (CME) [107,
108]. Once the LLL fermions are thermalized, the electric conductivity gets to be evaluated
by the one for the high-temperature thermal plasma, $\sigma \simeq (1/g^2 \ln g^{-1})T = c_\sigma T$ with
$c_\sigma \simeq 10^2$ [109, 110]. At the same time, the bulk velocity fields $v$ are excited, obeying the
Navier-Stokes equation,

$$\frac{\partial}{\partial \eta} v + v \cdot \nabla v = \nu_c \nabla^2 v + \frac{1}{\rho_c + p_c} (\nabla \times B_c) \times B_c - \frac{1}{\rho_c + p_c} \nabla p,$$

(83)

where $\rho_c$ and $p_c$ are the comoving energy density and pressure of the fluid, respectively,
and $\nu_c \sim (1/g^4 \ln g^{-1})/(aT)^3/(\rho_c + p_c) = c_\nu/(aT)$ with $c_\nu \simeq 10$ [110, 111] is the viscosity
normalized by $\rho_c + p_c$. Note that we have come back to the comoving frame and assumed the
incompressibility ($\nabla \cdot v = 0$). As a result, magnetic fields evolve according to the Maxwell
equation with the MHD approximation,

$$\frac{\partial}{\partial \eta} B_c = \frac{1}{\sigma_c} \nabla^2 B_c + \nabla \times (v \times B_c) + \frac{g^2}{2\pi^2 \sigma_c} \mu_5 \nabla \times B_c,$$

(84)

where $\sigma_c \equiv a \sigma \simeq c_\sigma aT$ is the comoving electric conductivity and the last term is the CME
induced current. If the velocity fields become strong enough, the magnetic fields no longer
simply decay with the diffusion but enter the cascade regime. In this regime, magnetic fields
do not decay exponentially but show a power-law decay. If the magnetic fields are maximally
helical, they evolve according to the so-called inverse cascade.

Though the appropriate numerical MHD simulation is needed to determine precisely in
which case the system enters the cascade regime, it is useful to introduce the Reynolds
numbers to give the criteria. The magnetic Reynolds number is defined as

$$R_m \equiv \sigma_c \lambda_c v,$$

(85)
where $v$ is the typical amplitude of the velocity field. Here the quantities are defined in the comoving frame. It gives the typical ratio between advection term (second term of Eq. (84)) and the diffusion term (first term of Eq. (84)). We can set the criteria that the magnetic fields enter the cascade regime in an eddy-turnover time, $\sim \lambda_c/v$, if the magnetic Reynolds number is larger than unity so that the diffusion is less efficient.

The typical amplitude of the velocity fields can be estimated by investigating the kinetic Reynolds number, defined as

$$R_e \equiv \frac{v\lambda_c}{\nu_c}.$$  

(86)

It gives the typical ratio between the advection term (second term of the left-hand side of Eq. (83)) and the diffusion term (first term of the right-hand side of Eq. (83)). If both the magnetic and kinetic Reynolds number is much larger than the unity, the advection term is comparable to the Lorentz force term (second term of the right-hand side of Eq. (83)), which suggests that the equipartition is reached, $\rho_c v^2/2 \simeq B_c^2/2$. In this case, the evolution of the system is fully nonlinear and such a regime is called as the “turbulence” regime. Note that we always get $R_m \gg R_e$ from the relation between the electric conductivity and viscosity. On the other hand, for $R_m \gg 1 > R_e$, the advection term in Eq. (83) is negligible and the diffusion term would be balanced to the Lorenz force term, $\nu_p v/\lambda_c \simeq B_p^2/\rho_c$, or $\rho_c v^2/2 \simeq R_e B_c^2/2$. This case is called “viscous” regime. To summarize, the rough estimate of the velocity fields excited by the magnetic fields is given as

$$v \simeq \begin{cases} \frac{B_c}{\sqrt{\rho_c}} = \frac{B_p}{\sqrt{\rho_p}} = \left( \frac{30}{\pi^2 g_*} \right)^{1/2} \frac{B_p}{T^2}, & \text{for } R_m \gg R_e > 1, \\ \frac{B_c^2 \lambda_c}{\nu_c (\rho_c + p_c)} = \frac{B_p^2 \lambda_{\text{phys}}}{\nu_p (\rho_p + p_p)} \sim \frac{3}{\pi^2 g_*} \left( \frac{c_v}{10} \right)^{-1} \frac{B_p^2 \lambda_{\text{phys}}}{T^3}, & \text{for } R_m \gg 1 > R_e, \end{cases}$$

(87)

where we have rewritten in terms of the physical quantities. From the helicity conservation, $h_c \sim B_c^2 \lambda_c = \text{const.}$ and the condition that the coherence length of the magnetic fields is roughly equal to the eddy turnover scale $\sim v\eta$ (or by solving the Maxwell equation (Eq. (84)))

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\[ \text{Footnote: Here we assume that the CME (third term of Eq. (84)) is subdominant. It can be important at a later time to cause the chiral plasma instability at smaller scales [50], but it does not affect the estimate.} \]
only with the advection term [50]), we obtain the cascade law as
\[ B_c \propto \eta^{-1/3}, \quad \lambda_c \propto \eta^{2/3}, \quad v \propto \eta^{-1/3}, \quad \text{for } R_m \gg R_e > 1, \quad (88) \]
\[ B_c \propto \eta^{1/2}, \quad \lambda_c \propto \eta, \quad v = \text{const.}, \quad \text{for } R_m \gg 1 > R_e, \quad (89) \]

after the eddy-turnover time. Note that here we have assumed that the chiral plasma instability does not occur. If it takes place to erase total magnetic helicity, the decay of the magnetic fields can be stronger. In our present purpose it is enough not to consider such a case, because the case without this phenomena is more problematic for the BBN due to larger amount of remaining magnetic fields.

Now we can examine the evolution of the system of our interest by evaluating the magnetic and kinetic Reynolds number. Evaluating the parameters at the time of the LLL thermalization, from Eqs. (78), (79), (80), and (87), we find that the kinetic Reynolds number is smaller than the unity for
\[ \xi_{\text{eff}} \lesssim 7 \] as
\[ R_e = \frac{1}{\rho_c + p_c} \left( \frac{B_c \lambda_c}{\nu_c} \right)^2 = \frac{30}{\pi^2 g_* c_v T} \left( \frac{B_p \lambda_p}{c_v T} \right)^2 \]
\[ \simeq 1 \times 10^{-1} \left( \frac{g_*}{106.75} \right)^{-\frac{1}{2}} \left( \frac{g'}{0.3} \right)^{-\frac{3}{2}} \left( \frac{c_v}{10} \right)^{-2} \]
\[ \times \left( \frac{Q_3}{41/12} \right)^{-\frac{1}{4}} \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^{\frac{1}{2}} \left( \frac{a_{\text{sat}}}{\frac{3\pi}{4} \sqrt{\xi_{\text{eff}} a_{\text{end}}}} \right)^{-2} \left( \frac{\xi_{\text{eff}}}{4} \right). \quad (90) \]

Then the magnetic Reynolds number is evaluated as
\[ R_m = \sigma_c \frac{B_c^2 \lambda_c^2}{\nu_c (\rho_c + p_c)} = \frac{30}{\pi^2 g_* c_v} \left( \frac{B_p \lambda_p}{c_v T} \right)^2 \]
\[ \simeq 1 \times 10^2 \left( \frac{g_*}{106.75} \right)^{-\frac{1}{2}} \left( \frac{g'}{0.3} \right)^{-\frac{3}{2}} \left( \frac{c_v}{10} \right)^{-3} \left( \frac{c_v}{10^2} \right) \]
\[ \times \left( \frac{Q_3}{41/12} \right)^{-\frac{1}{4}} \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^{\frac{1}{2}} \left( \frac{a_{\text{sat}}}{\frac{3\pi}{4} \sqrt{\xi_{\text{eff}} a_{\text{end}}}} \right)^{-2} \left( \frac{\xi_{\text{eff}}}{4} \right), \quad (91) \]
which is larger than the unity for \( \xi_{\text{eff}} > 1 \). Thus we determine that in the parameters of our interest, \( 1 \lesssim \xi_{\text{eff}} \lesssim 6.6 \), the magnetic fields would evolve with the inverse cascade in the viscous regime (Eq. (89)). The eddy-turnover time (in the comoving frame) is evaluated as
\[ \sigma_c \lambda_c^2 \simeq 5.7 \times 10^{-8}\text{GeV}^{-1} a^{-1} \left( \frac{g_*}{106.75} \right)^{-\frac{1}{16}} \left( \frac{g'}{0.3} \right)^{-\frac{25}{16}} \left( \frac{c_v}{10^2} \right) \left( \frac{Q_3}{41/12} \right)^{\frac{1}{4}} \]
\[ \times \left( \frac{I(\xi_{\text{eff}})}{I(4)} \right)^{\frac{9}{16}} \left( \frac{a_{\text{sat}}}{\frac{3\pi}{4} \sqrt{\xi_{\text{eff}} a_{\text{end}}}} \right)^{\frac{3}{4}} \left( \frac{\xi_{\text{eff}}}{4} \right)^{\frac{11}{4}} \left( \frac{H_I}{10^{13}\text{GeV}} \right)^{-1}, \quad (92) \]
which suggests that the magnetic fields starts to evolve according to the inverse cascade at

\[ H = H_{\text{IC}} \simeq 0.9 \times 10^7 \text{GeV} \left( \frac{g_*}{106.75} \right)^{\frac{1}{20}} \left( \frac{g'}{0.3} \right)^{\frac{25}{20}} \left( \frac{c_\sigma}{10^2} \right)^{-1} \left( \frac{Q_3}{41/12} \right)^{-\frac{1}{4}} \times \left( \frac{\mathcal{I}(\xi_{\text{eff}})}{\mathcal{I}(4)} \right)^{-\frac{7}{20}} \left( \frac{a_{\text{sat}}}{\frac{2\pi}{3} \sqrt{\xi_{\text{eff}} a_{\text{end}}} \frac{\xi_{\text{eff}}}{4}} \right)^{-\frac{3}{4}} \left( \frac{\xi_{\text{eff}}}{4} \right)^{-\frac{11}{8}} \left( \frac{H_{I}}{10^{13} \text{GeV}} \right). \]  

(93)

With the estimation in the above, we can examine the constraints on the abundance of the magnetic fields from the BBN, which gives the upper bound of the magnetic fields. Since the magnetic fields act as additional relativistic degrees of freedom, their abundance is characterized by the effective number of neutrino flavors, \( N_{\text{eff}} \), as

\[ \frac{\rho_{\text{BB}}}{\rho_{\text{rad}}} = \frac{7\Delta N_{\text{eff}}}{22 + 7N_{\text{eff}}} \]  

with \( \Delta N_{\text{eff}} \equiv N_{\text{eff}} - 3 \). Recent constraints on \( N_{\text{eff}} \) from the BBN \[21\] tells \( \Delta N_{\text{eff}} < 0.16 \). Thus the energy density of magnetic fields at the BBN should be constrained as

\[ \frac{\rho_{\text{BB}}}{\rho_{\text{rad}}} < 2.5 \times 10^{-2}. \]  

(95)

Although at the time of the LLL thermalization, the radiation and magnetic energy density is the same order, the latter becomes to decay faster than the former once they enter the cascade regime. The ratio between the energy density of magnetic fields and that of the radiation in the viscous regime decays in proportion to \( \eta^{-1} \) and hence \( H^{2/3} \) during the kination era and \( \eta^{-1} \) and hence \( H^{1/2} \) during the radiation dominated era. Comparing to the Hubble parameters at the onset of the cascade (Eq. (93)), reheating (Eq. (81)), and the BBN (\( H \sim 10^{-24} \text{GeV} \)), we find that the duration of the cascade evolution of the magnetic fields are long enough to satisfy the constraints from the BBN on the magnetic field energy density.

B. Comment on baryogenesis

Since the magnetic fields are generated before the electroweak symmetry breaking as the hypermagnetic fields and maximally helical, baryon asymmetry is generated at the time of the electroweak symmetry breaking \[95–100\]. As has been shown in Ref. \[100\] (see also Ref. \[101\] for the simple derivation), the resultant baryon-to-entropy ratio is evaluated as

\[ \eta_B \equiv \frac{n_B}{s} \sim 4 \times 10^{-4} \frac{h_c}{s_c}. \]  

(96)
where $s_c = (4\pi^2/45)g_{*s}(aT)^3$ is the comoving entropy density. Since the helicity ($h_c = \lambda_c B_c^2$) is conserved during the cascade process, one can see that the resultant baryon asymmetry $\eta_B$ is inevitably much larger than the observed value, $\eta_B \approx 9 \times 10^{-11}$. Note that from Eqs. (78), (79), and (80), we find $h_c/s_c \approx 0.7(I(\xi_{\text{eff}})/I(4))^{-1/4}(\xi_{\text{eff}}/4)^{3/8}$, unless the chiral plasma instability completely erases the hypermagnetic helicity and chirality well before the electroweak symmetry breaking. One might think that if the chiral plasma instability erases the helicity, the magnetic fields are harmless. However, even if it is the case so that the net baryon asymmetry vanishes, the magnetic fields would remain and they are inconsistent with the constraint from the inhomogeneous BBN from the baryon isocurvature perturbation generated from the hypermagnetic fields [101]. Therefore, in the present setup with the standard nature of the electroweak symmetry breaking, the baryon overproduction or too large baryon isocurvature perturbation is unavoidable, and the resultant Universe is inconsistent with ours.

A way out from this constraint is to introduce additional entropy production by, e.g., gravitational heavy particle production [28]. However, in a simple model construction, it is difficult to have a sufficiently large entropy production due to the ineffectiveness of the gravitational particle production. In this case, the gravitational particle production is responsible for the reheating and it is no longer the realization of the Schwinger reheating. Another way out from this constraint is to assume the change of the electroweak symmetry breaking. The reason why we have an efficient conversion from the magnetic helicity to the baryon asymmetry is that the sphaleron freezeout takes place earlier than the completion of the electroweak symmetry breaking in terms of the effective weak mixing angle [112, 113]. If a new physics lies just above the electroweak scale to modify the nature of the electroweak symmetry breaking so that the change of the effective weak mixing angle takes place much earlier than the sphaleron freezeout, the resultant baryon asymmetry can be suppressed, in a similar way studied in Ref. [99]. In such a case, if the cascade decay lasts until recombination, the magnetic fields would remain as the intergalactic magnetic fields today. Since they evolve with the MHD cascade, their coherence length would be relatively small, but their field strength is relatively large so that they might be able to explain the blazar observations [51–55]. The late-time magnetic field evolution, especially with the chiral magnetic effect, is extremely complicated, and the precise estimate is left for future study.
V. CONCLUSION AND DISCUSSION

Graceful exit in inflationary models where the inflaton continuously runs away after the inflationary phase, of which period is known as kination, is one of the most important issue for their realistic model building, albeit their phenomenologically interesting features. In such models, a coherent inflaton oscillation phase does not follow the inflationary era, and reheating of the Universe cannot be achieved by the inflaton particle decay. In this article, we pointed out that the Chern-Simons coupling between the inflaton and the gauge fields is allowed by the shift symmetry behind the model with kination, if we identify the inflaton is an ALP. We examined the U(1) gauge field production through the tachyonic instability during kination caused by the Chern-Simons coupling to see if it can lead to successful reheating in these models.

First we investigated the dynamics of the U(1) gauge fields with the Chern-Simons coupling during ALP kination without other fields. We found that the energy density of the gauge fields are enhanced by a factor of \( \exp[\pi \xi_K]/\xi_K^3 \) with \( \xi_K (> 1) \) being the instability parameter, similar to the gauge field amplification during inflation [37–39], which is much more efficient than the gravitational particle production [18, 19]. Since the sign of the ALP velocity does not change during the process, only one helicity mode is amplified and the generated gauge fields are completely helical. While the gauge fields are amplified when the mode exited the horizon in the case of the one during inflation, they are once more amplified when they reenter the horizon (the mode should have exited the horizon once during inflation). Unless \( \xi_K \) is very large (as large as 10), the energy density of the gauge fields are much smaller than that of the kinetic energy density of the inflaton at the time of the saturation of their amplification even without considering the backreaction, but they eventually dominate it since the kinetic energy of the inflaton decreases much faster than that of gauge fields, \( \rho_{\text{kin}} \propto a^{-6} \). The results obtained here are not limited to the U(1) gauge field in the SM but also can be applied for any dark U(1) gauge fields.

We then took into account the charged fermions with the SM particle species in mind and examined the Schwinger pair-particle production and its backreaction on the gauge field dynamics. In principle, the consistent treatment of the Schwinger effect in the dynamical gauge fields and its backreaction on the gauge field dynamics is a conceptually difficult problem, and a precise estimate is almost impossible at the best of our knowledge and
technique. In this regard, we here adopted the way to give its rough estimate proposed in Ref. [49], where the backreaction is characterized by the effective instability parameter \( \xi_{\text{eff}} \) obtained from the consistency equation for the original instability parameter \( \xi_K \). In this treatment, the amplified gauge field spectrum as well as that of the pair-produced particles are described by \( \xi_{\text{eff}} \). The backreaction from the Schwinger effect becomes significant for \( \xi_K \gtrsim 2 \). We found that the energy density of the pair-produced (LLL) fermions are typically comparable to or larger than that of the gauge fields and the LLL fermions is thermalized well before they dominate the energy density of the Universe for \( \xi_{\text{eff}} \lesssim 6.6 \). The electric fields are screened soon after the saturation of the gauge field amplification. For sufficiently large Hubble parameter during inflation, the reheating of the Universe (when the energy density of radiation dominates over that of the inflaton) occurs well before the BBN and also the electroweak symmetry breaking for \( 2 \lesssim \xi_{\text{eff}} \lesssim 6 \). Note that \( \xi_{\text{eff}} \simeq 4-5 \) corresponds to \( \xi_K \simeq O(10) \), as can be seen in Fig. 3. This process is a concrete realization of the “Schwinger reheating”. If the magnetic fields survive for a too long time, it acts as additional relativistic degrees of freedom at the BBN, which spoils its success, but we confirmed that the MHD cascade decay is sufficiently efficient to decrease the energy density of the magnetic fields before the BBN.

Our results strongly suggests that the reheating of the Universe can be successfully achieved in the inflation models followed by the kination era with the aid of the tachyonic instability of the gauge field caused by the Chern-Simons term and the Schwinger effect. However, our analytic treatment, which is based on the one developed in Ref. [49], relies on several approximations and assumptions. Namely,

1. We adopted the Schwinger effect for the static and homogeneous gauge fields, while they have finite coherence length and time dependence in our setup.

2. We estimated the backreaction on the amplification of the gauge fields by requiring the consistency condition for their evolution equation but did not solve it simultaneously with the evolution equation for fermions.

We admit that the estimate is much less precise than that for the inflationary magnetogenesis [37–39, 48] where the system is expected to reach a static configuration eventually. Nevertheless, at the present knowledge and technique, our treatment would be the optimal one. To give more precise estimate, however, the technique to solve the dynamical co-evolution
of the gauge fields and charged particles should be developed further. We also assumed a simplified background cosmic expansion, with a instant connection between the inflation (with a constant Hubble parameter) and kination to make the analytic calculations possible. However, as has also been discussed in Ref. [28] for the gravitational particle production, a smooth connection between the inflationary phase and the kination phase can make a difference in the gauge field amplification (especially for high momentum modes) and the time of reheating to a certain extent. Numerical studies with a concrete model of the background dynamics of inflation and the following kination era are essential to compare the mechanism with the cosmological observations, which strongly depends on the time when the Universe becomes radiation dominated, and to identify the inflation model. Nevertheless, we believe that the investigation in the present study is the first step for the concrete realization of the application of the Schwinger effect associated with the inflationary magnetogenesis to the reheating of the Universe, worth investigating in more depth.

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