New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals

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Abstract
This paper gives some novel generalizations by considering the generalized conformable fractional integrals operator for reverse Minkowski type and reverse Hölder type inequalities. Furthermore, novel consequences connected with this inequality, together with statements and confirmation of various variants for the advocated generalized conformable fractional integral operator, are elaborated. Moreover, our derived results are provided to show comparisons of convergence between old and modified operators towards a function under different parameters and conditions. The numerical approximations of our consequence have several utilities in applied sciences and fractional integro-differential equations.

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1 Introduction
Fractional calculus, generally referred to as the calculus of non-integer order, was a trademark outgrowth of traditional definitions of calculus integral and derivative. The concept of fractional calculus has provoked a host of researchers and was comprehensively studied in the literature for the last few decades. A continuous effort has been made on an enormous scale and everybody has been stimulated by its different aspects. In the present century, the exceptional idea has been described by several mathematicians with a slightly distinct technique in different time scales; see, for instance, the Liouville, Riemann, Grunwald, Letnikov, Hadamard, Weyl, Riesz, Marchaud, Kober and Caputo fractional integrals (see [1–11]). Most of these researchers first of all added fractional integrals, on the concept of which the associated fractional derivative and other associated results had been produced. Recently, Khalil et al. [2] and Abdeljawad [1] introduced fractional operators known as fractional conformable derivatives and integrals. Jarad et al. [12] established the fractional conformable integral operators. Meanwhile in [13], Anderson and Ulness introduced the concept of local derivatives for upgrading the concept of the fractional
Conformable derivatives are nonlocal fractional derivatives. They can be called fractional since we can take derivatives up to arbitrary order. However, since in the community of fractional calculus, nonlocal fractional derivatives only are to be called fractional, we prefer to replace conformable fractional by conformable (as a type of local fractional). Conformable derivatives and other types of local fractional derivatives or modified conformable derivatives in [13] can gain in importance by the ability to use them to generate more generalized nonlocal fractional derivatives with singular kernels (see [15–22]).

Integral inequalities have potential application in several areas of science: technology, mathematics, chemistry, plasma physics, among others; especially we point out initial value problems, the stability of linear transformation, integral differential equations, and impulse equations [23–33]. Variants regarding fractional integral operators are of use in significant strategies amongst researchers and accumulate fertile functional applications in various areas of science; see [34–45]. On account of their potential results to be utilized for the presence of nontrivial and positive solutions of a distinct kind of fractional differential equations, our findings concerning fractional integrals are appreciable and essential.

An enormous heft of present literature comprises generalizations of several variants by fractional integral operators and their applications [46–52]. We state some of them, that is, the variants of Minkowski, Hardy, Opial, Hermite–Hadamard, Grüss, Lyeng, Wirtinger, Ostrowski, Čebyšev and Pólya–Szegö [53–59]. Such applications of fractional integral operators compelled us to show the generalization of the reverse Minkowski inequality [43,44,53] involving generalized conformable fractional integrals operators.

The article is composed thus: in Sect. 2 we demonstrate the notations and primary definitions of our newly introduced operator generalized conformable fractional integrals. Also, we present the results concerning the reverse Minkowski inequality. In Sect. 3, we advocate essential consequences such as the reverse Minkowski inequality via the generalized conformable fractional integral operators. In Sect. 4, we show the associated variants using this fractional integral.

2 Preliminaries

This section is dedicated to some recognized definitions and results associated with the generalized conformable fractional integral operators and their generalization related to the generalized conformable fractional integral operators. Set et al. in [60] proved the Hermite–Hadamard, and reverse Minkowski inequalities for Riemann–Liouville fractional integrals. Additionally, Hardy’s type and reverse Minkowski inequalities were supplied by Bougoffa in [38]. The subsequent consequences concerning the reverse Minkowski inequalities are of significance for the classical integrals.
Theorem 2.1 ([60]) For $p \geq 1$ and let there be two positive functions $f_1$ and $f_2$ on $[0, \infty)$. If $0 < \theta_1 \leq \frac{\theta_2}{\theta_2+1} \leq \theta_2$, $y \in [r_1, r_2]$, then
\[
\left( \int_{r_1}^{y} f_1^p(y) \, dy \right)^{\frac{1}{p}} + \left( \int_{r_1}^{y} f_2^p(y) \, dy \right)^{\frac{1}{p}} \leq \frac{1 + \theta_2(\theta_1 + 2)}{(\theta_1 + 1)(\theta_2 + 1)} \left( \int_{r_1}^{y} (f_1 + f_2)^p(y) \, dy \right)^{\frac{1}{p}}.
\]

Theorem 2.2 ([60]) For $p \geq 1$ and let there be two positive functions $f_1$ and $f_2$ on $[0, \infty)$. If $0 < \theta_1 \leq \frac{\theta_2}{\theta_2+1} \leq \theta_2$, $y \in [r_1, r_2]$, then
\[
\left( \int_{r_1}^{y} f_1^p(y) \, dy \right)^{\frac{1}{p}} + \left( \int_{r_1}^{y} f_2^p(y) \, dy \right)^{\frac{1}{p}} \geq \left( \frac{(1 + \theta_2)(\theta_1 + 1)}{\theta_2} - 2 \right) \left( \int_{r_1}^{y} f_1^p(y) \, dy \right)^{\frac{1}{p}} \left( \int_{r_1}^{y} f_2^p(y) \, dy \right)^{\frac{1}{p}}.
\]

In [44], Dahmani used the Riemann–Liouville fractional integral operators to prove the subsequent reverse Minkowski inequalities.

Theorem 2.3 ([44]) Let $\varsigma > 0$ and $p \geq 1$, and let there be two positive functions $f_1$ and $f_2$ on $[0, \infty)$ such that, for all $y > 0$, $K_{r_1}^{\varsigma} f_1^p(y) < \infty$, $K_{r_2}^{\varsigma} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{\theta_2}{\theta_2+1} \leq \theta_2$, $\eta \in [r_1, y]$, then the following inequality holds:
\[
(K_{r_1}^{\varsigma} f_1^p(y))^{\frac{1}{p}} + (K_{r_2}^{\varsigma} f_2^p(y))^{\frac{1}{p}} \leq \frac{1 + \theta_2(\theta_1 + 2)}{(\theta_1 + 1)(\theta_2 + 1)} (K_{r_1}^{\varsigma} (f_1 + f_2)^p(y))^{\frac{1}{p}}.
\]

Theorem 2.4 ([44]) Let $\varsigma > 0$ and $p \geq 1$, and let there be two positive functions $f_1$ and $f_2$ on $[0, \infty)$ such that, for all $y > 0$, $K_{r_1}^{\varsigma} f_1^p(y) < \infty$, $K_{r_2}^{\varsigma} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{\theta_2}{\theta_2+1} \leq \theta_2$, $\eta \in [r_1, y]$, then the following inequality holds:
\[
(K_{r_1}^{\varsigma} f_1^p(y))^{\frac{1}{p}} + (K_{r_2}^{\varsigma} f_2^p(y))^{\frac{1}{p}} \geq \left( \frac{(1 + \theta_2)(\theta_1 + 1)}{\theta_2} - 2 \right) (K_{r_1}^{\varsigma} f_1^p(y))^{\frac{1}{p}} (K_{r_2}^{\varsigma} f_2^p(y))^{\frac{1}{p}}.
\]

Recall the definition of the generalized conformable fractional integral which is mainly due to [14].

Definition 2.5 ([14]) Let $f$ be a conformable integrable function on the interval $[r_1, r_2] \subseteq [0, \infty)$. The right-sided and left-sided generalized conformable fractional integrals $^{\tau}C_{r_1}^\varsigma f$ and $^{\varsigma}C_{r_2}^\varsigma f$ of order $\varsigma > 0$ are defined by
\[
^{\tau}C_{r_1}^\varsigma f(y) = \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} (y^{\tau+\varsigma} - \eta^{\tau+\varsigma})^{\varsigma-1} f(\eta) \frac{d\eta}{\eta^{1-\varsigma}}, \quad y > r_1,
\]
and
\[
^{\varsigma}C_{r_2}^\varsigma f(y) = \frac{1}{\Gamma(\varsigma)} \int_{y}^{r_2} (\eta^{\varsigma+\tau} - y^{\varsigma+\tau})^{\varsigma-1} f(\eta) \frac{d\eta}{\eta^{1-\varsigma}}, \quad y < r_2,
\]
where $\varsigma \in \mathbb{C}$, $\Re(\varsigma) > 0$, $\varsigma \in (0, 1]$, $\tau \in \mathbb{R}$ with $\tau + \varsigma \neq 0$, and $\Gamma$ is the well-known gamma function.
Remark 2.6 In Eqs. (2.1) and (2.2):

(i) If \( \tau = 0 \), then we attain the subsequent Riemann–Liouville type fractional conformable integral operators; see [12]:

\[
\varepsilon K_{r_1}^\varsigma f(y) = \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} \left( \frac{y^\theta - \eta^\theta}{\eta^\varrho} \right)^{\varsigma - 1} \frac{f(\eta)}{\eta^{1 - \varrho}} \, d\eta, \quad y > r_1,
\]

and

\[
\varepsilon K_{r_2}^\varsigma f(y) = \frac{1}{\Gamma(\varsigma)} \int_{y}^{r_2} \left( \frac{\eta^\theta - y^\theta}{y^\varrho} \right)^{\varsigma - 1} \frac{f(\eta)}{\eta^{1 - \varrho}} \, d\eta, \quad y < r_2,
\]

where \( \varsigma \in \mathbb{C}, \Re(\varsigma) > 0, \varrho \in (0, 1) \).

(ii) If \( \tau = 0 \) and \( \varrho = 1 \), then we attain the subsequent Riemann–Liouville type fractional integral operators; see [10, 15]:

\[
K_{r_1}^\varsigma f(y) = \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} (y - \eta)^{\varsigma - 1} f(\eta) \, d\eta, \quad y > r_1,
\]

and

\[
K_{r_2}^\varsigma f(y) = \frac{1}{\Gamma(\varsigma)} \int_{y}^{r_2} (\eta - y)^{\varsigma - 1} f(\eta) \, d\eta, \quad y < r_2,
\]

where \( \varsigma \in \mathbb{C}, \Re(\varsigma) > 0 \).

3 Reverse Minkowski inequalities via generalized conformable fractional integral operators

This section comprises our principal involvement of establishing the proof of the reverse Minkowski inequalities via generalized conformable fractional integral operators defined in (2.1) and (2.2) and an associated theorem insinuated as the reverse Minkowski inequalities.

Theorem 3.1 For \( \varsigma > 0, \varrho \in (0, 1), \tau \in \mathbb{R} \) and \( \varrho + \tau \neq 0 \) with \( p \geq 1 \) and let there be two positive functions \( f_1, f_2 \) on \([0, \infty)\) such that, for all \( y > r_1, \varepsilon K_{r_1}^\varsigma f_1^p(y) < \infty \) and \( \varepsilon K_{r_2}^\varsigma f_2^p(y) < \infty \). If \( 0 < \vartheta_1 \leq \frac{\int_{r_1}^{\eta} K_{r_1}^\varsigma f_1^p(\eta) d\eta}{f_1(\eta)} \leq \vartheta_2 \) for \( \vartheta_1, \vartheta_2 \in \mathbb{R}^+ \) and for all \( x \in [r_1, y] \), then

\[
\left( \varepsilon K_{r_1}^\varsigma f_1^p(y) \right)^{\frac{1}{p}} + \left( \varepsilon K_{r_2}^\varsigma f_2^p(y) \right)^{\frac{1}{p}} \leq \frac{1 + \vartheta_2(\vartheta_1 + 2)}{(\vartheta_1 + 1)(\vartheta_2 + 1)} \left( \varepsilon K_{r_1}^\varsigma (f_1 + f_2)^p(y) \right)^{\frac{1}{p}}.
\]

Proof By the suppositions mentioned in Theorem 3.1, \( \frac{\int_{r_1}^{\eta} K_{r_1}^\varsigma f_1^p(\eta) d\eta}{f_1(\eta)} \leq \vartheta_2, r_1 \leq \eta \leq y \), we have

\[
(M + 1)^{p} f_1^p(\eta) \leq M^p (f_1(\eta) + f_2(\eta))^p.
\]
If we multiply both sides of (3.2) with \( \frac{1}{\Gamma(\zeta)\zeta_{1}^{r_{1}+\varphi}}(\zeta_{1}+\varphi)^{r_{1}} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_{1} \) to \( y \), we obtain

\[
\frac{(M + 1)^{p}}{\Gamma(\zeta)} \int_{r_{1}}^{y} \left( \frac{y^{\tau+\varphi} - \eta^{\tau+\varphi}}{\tau + \varphi} \right)^{\tau-1} f_{1}^{p}(\eta) \frac{d\eta}{s^{1-\tau+\varphi}} \leq \frac{M^{p}}{\Gamma(\zeta)} \int_{r_{1}}^{y} \left( \frac{y^{\tau+\varphi} - \eta^{\tau+\varphi}}{\tau + \varphi} \right)^{\tau-1} (f_{1}(\eta) + f_{2}(\eta))^{p} \frac{d\eta}{s^{1-\tau+\varphi}}.
\]

(3.3)

Similarly,

\[
\left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} \leq \frac{\theta_{2}}{\theta_{2} + 1} \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}}.
\]

(3.4)

In contrast, as \( mf_2(\eta) \leq f_1(\eta) \), it follows that

\[
\left( 1 + \frac{1}{\theta_{1}} \right)^{p} f_{2}^{p}(\eta) \leq \left( \frac{1}{\theta_{1}} \right)^{p} (f_{1}(\eta) + f_{2}(\eta))^{p}.
\]

(3.5)

Again, if we multiply both sides of (3.5) with \( \frac{1}{\Gamma(\zeta)\zeta_{1}^{r_{1}+\varphi}}(\zeta_{1}+\varphi)^{r_{1}} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_{1} \) to \( y \), we obtain

\[
\left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right)^{\frac{1}{p}} \leq \frac{\theta_{1}}{\theta_{1} + 1} \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right)^{\frac{1}{p}}.
\]

(3.6)

Thus adding (3.4) and (3.6) yields the desired inequality.

Inequality (3.1) is referred to as the reverse Minkowski inequality via generalized conformable fractional integrals.

**Theorem 3.2** For \( \zeta > 0, \varphi \in (0, 1], \tau \in \mathbb{R} \) and \( \varphi + \tau \neq 0 \) with \( p \geq 1 \) let there be two positive functions \( f_{1}, f_{2} \) on \( [0, \infty) \) such that, for all \( y > r_{1} \), \( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \leq \infty \) and \( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \leq \infty \). If \( 0 < \theta_{1} \leq \frac{\theta_{1}}{\theta_{2}} \leq \theta_{2} \) for \( \theta_{1}, \theta_{2} \in \mathbb{R}^{+} \) and for all \( \eta \in [r_{1}, y] \), then

\[
\left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} + \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right)^{\frac{1}{p}} \leq \left( \frac{\theta_{1} + 1}{\theta_{2} + 1} \right) \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} \leq \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}}.
\]

(3.7)

**Proof** The product of inequalities (3.4) and (3.6) yields

\[
\left( \frac{\theta_{1} + 1}{\theta_{2} + 1} \right) - 2 \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right)^{\frac{1}{p}} \leq \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}}.
\]

(3.8)

Now, utilizing the Minkowski inequality to the right hand side of (3.8), one obtains

\[
\left[ \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} \right]^{2} \leq \left[ \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) + \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right]^{2} \leq \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} + \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right)^{\frac{1}{p}} + 2 \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{1}^{p}}(y) \right)^{\frac{1}{p}} \left( \varphi^{\zeta_{1}} \varphi_{r_{1}f_{2}^{p}}(y) \right)^{\frac{1}{p}}.
\]

(3.9)

Thus, from inequalities (3.8) and (3.9), we obtain the inequality (3.7).
4 Certain associated inequalities via generalized conformable fractional integral operators (GCFI)

This section is dedicated to deriving certain associated variants regarding GCFI operator.

**Theorem 4.1** For $\zeta > 0$, $\phi \in (0, 1]$, $\tau \in \mathcal{R}$, $\phi + \zeta \neq 0$ with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, suppose that there are two positive functions $f_1, f_2$ on $[0, \infty)$ such that, for all $y > r_1$, $\zeta K_{r_1}^{\zeta} f_1^p (y) < \infty$ and $\zeta K_{r_1}^{\zeta} f_2^2 (y) < \infty$. If $0 < \theta_1 \leq \frac{f_1(y)}{f_2(y)} \leq \theta_2$ for $\theta_1, \theta_2 \in \mathcal{R}^+$ and for all $\eta \in [r_1, y]$, then

$$\left( \zeta K_{r_1}^{\zeta} f_1^p (y) \right)^{\frac{1}{p}} \left( \zeta K_{r_1}^{\zeta} f_2^2 (y) \right)^{\frac{1}{q}} \leq \left( \frac{\theta_2}{\theta_1} \right)^{\frac{1}{p}} \left( \zeta K_{r_1}^{\zeta} f_1 \left( \frac{1}{2} \right) (y) \right)^{\frac{1}{2}} \left( \zeta K_{r_1}^{\zeta} f_2 \left( \frac{1}{2} \right) (y) \right)^{\frac{1}{2}}.$$  (4.1)

**Proof** Under the given suppositions $\frac{f_1(y)}{f_2(y)} \leq \theta_2$, $r_1 \leq \eta \leq y$, therefore we have

$$f_2^\frac{1}{q} (\eta) \geq \theta_2 f_1^\frac{1}{p} (\eta).$$  (4.2)

Taking the product of both sides of (4.2) by $f_1^\frac{1}{p} (\eta)$, it follows that

$$f_1^\frac{1}{p} (\eta) f_2^\frac{1}{q} (\eta) \geq \theta_2^\frac{1}{p} f_1^\frac{1}{p} (\eta).$$  (4.3)

If we multiply both sides of (4.3) with $\frac{1}{\Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{x+\phi} - \eta^{x+\phi}}{\tau^{} + \phi} \right)^{\tau-1} f_1(\eta) d\eta$ and integrate the subsequent inequality with respect to $\eta$ from $r_1$ to $y$, we obtain

$$\frac{\theta_2}{\Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{x+\phi} - \eta^{x+\phi}}{\tau^{} + \phi} \right)^{\tau-1} f_1(\eta) f_2^\frac{1}{q} (\eta) d\eta \leq \frac{1}{\Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{x+\phi} - \eta^{x+\phi}}{\tau^{} + \phi} \right)^{\tau-1} f_1^\frac{1}{p} (\eta) f_2^\frac{1}{q} (\eta) d\eta.$$  (4.4)

Consequently, we have

$$\theta_2 \frac{1}{\Gamma(\zeta)} \left( \zeta K_{r_1}^{\zeta} f_1^p (y) \right)^{\frac{1}{p}} \leq \left( \zeta K_{r_1}^{\zeta} f_1 \left( \frac{1}{2} \right) (y) \right)^{\frac{1}{2}} \left( \zeta K_{r_1}^{\zeta} f_2 \left( \frac{1}{2} \right) (y) \right)^{\frac{1}{2}}.$$  (4.5)

In contrast, as $\theta_1 f_2(\eta) \leq f_1(\eta)$, we have

$$\theta_1 f_2^\frac{1}{p} (\eta) \leq f_1^\frac{1}{p} (\eta).$$  (4.6)

Again, if we multiply both sides of (4.6) by $f_2^\frac{1}{q} (\eta)$ and invoke the relation $\frac{1}{p} + \frac{1}{q} = 1$, it yields

$$\theta_1^\frac{1}{p} f_2^\frac{1}{p} (\eta) \leq f_1^\frac{1}{p} (\eta) f_2^\frac{1}{q} (\eta).$$  (4.7)

If we multiply both sides of (4.7) with $\frac{1}{\Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{x+\phi} - \eta^{x+\phi}}{\tau^{} + \phi} \right)^{\tau-1} f_1(\eta) f_2^\frac{1}{q} (\eta) d\eta$ and integrate the subsequent inequality with respect to $\eta$ from $r_1$ to $y$, we obtain

$$\theta_1^\frac{1}{p} \left( \zeta K_{r_1}^{\zeta} f_2(y) \right)^{\frac{1}{p}} \leq \left( \zeta K_{r_1}^{\zeta} f_1 \left( \frac{1}{2} \right) (y) \right)^{\frac{1}{2}} \left( \zeta K_{r_1}^{\zeta} f_2 \left( \frac{1}{2} \right) (y) \right)^{\frac{1}{2}}.$$  (4.8)

Multiplying (4.5) and (4.8), the required inequality (4.1) can be concluded. □
Theorem 4.2 For \( \xi > 0, \varrho \in (0, 1], \tau \in \mathcal{R}, \varrho + \tau \neq 0 \) with \( p, q \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Suppose that there are two positive functions \( f_1, f_2 \) on \([0, \infty)\) such that, for all \( y > r_1 \), \( \kappa^\xi f_1^p(y) < \infty \) and \( K^\xi f_2^q(y) < \infty \). If \( 0 < \theta_1 \leq k_1(\xi) \leq \theta_2 \) for \( \theta_1, \theta_2 \in \mathcal{R}^+ \) and for all \( \eta \in [r_1, y] \), then

\[
(\kappa^\xi f_1 f_2)(y) \leq \frac{2^{p-1} \theta_2^p}{p(\theta_2 + 1)^p} \left( \kappa^\xi (f_1^p + f_2^q)(y) \right) + \frac{2^{q-1}}{p(\theta_2 + 1)^q} \left( \kappa^\xi (f_1^p + f_2^q)(y) \right). \tag{4.9}
\]

Proof. By the given assumption \( k_1(\xi) < \theta_2 \), we have

\[
(\theta_2 + 1)^p f_1^p(\eta) \leq \theta_2^p f_1 f_2(\eta). \tag{4.10}
\]

If we multiply both sides of (4.10) with \( \frac{1}{\Gamma(\xi)} \frac{1}{\tau + q} e^{-y^{\xi q} / \tau q} \tau^{-1} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_1 \) to \( y \), we obtain

\[
\frac{(\theta_2 + 1)^p}{\Gamma(\xi)} \int_{r_1}^{y} \left( \frac{y^{\xi q} - \eta^{\xi q}}{\tau + q} \right)^{-1} f_1(\eta) d\eta \leq \frac{\theta_2^p}{\Gamma(\xi)} \int_{r_1}^{y} \left( \frac{y^{\xi q} - \eta^{\xi q}}{\tau + q} \right)^{-1} (f_1 + f_2)^p(\eta) d\eta \tag{4.11}
\]

It follows that

\[
(\kappa^\xi f_1^p)(y) \leq \frac{\theta_2^p}{(\theta_2 + 1)^p} (\kappa^\xi f_1^p f_2^q)(y). \tag{4.12}
\]

In contrast, using \( 0 < \theta_1 \leq k_1(\xi) \), \( r_1 < \eta < y \), we have

\[
(\theta_1 + 1)^q f_2^q(\eta) \leq (f_1 + f_2)^q(\eta). \tag{4.13}
\]

Again, if we multiply both sides of (4.13) with \( \frac{1}{\Gamma(\xi)} \frac{1}{\tau + q} e^{-y^{\xi q} / \tau q} \tau^{-1} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_1 \) to \( y \), we obtain

\[
(\kappa^\xi f_2^q)(y) \leq \frac{1}{(\theta_1 + 1)^q} (\kappa^\xi f_1^p f_2^q)(y). \tag{4.14}
\]

Now, taking into account Young’s inequality,

\[
f_1(\eta) f_2(\eta) \leq \frac{f_1^p(\eta)}{p} + \frac{f_2^q(\eta)}{q}. \tag{4.15}
\]

Now, if we multiply both sides of (4.15) with \( \frac{1}{\Gamma(\xi)} \frac{1}{\tau + q} e^{-y^{\xi q} / \tau q} \tau^{-1} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_1 \) to \( y \), we obtain

\[
(\kappa^\xi f_1 f_2)(y) \leq \frac{1}{p} (\kappa^\xi f_1^p f_2^q(y)) + \frac{1}{q} (\kappa^\xi f_1^p f_2^q(y)). \tag{4.16}
\]
With the aid of (4.12) and (4.14) with (4.16), one obtains

\[
\left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 f_2 \right)(y) \right) \\
\leq \frac{1}{p^{\frac{1}{p}}} \left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 \right)^p (y) \right) + \frac{1}{q^{\frac{1}{q}}} \left( \frac{\omega}{\rho} K_{r_1}^q \left( f_2 \right)^q (y) \right) \\
\leq \frac{\theta_2^p}{p (\theta_2 + 1)^p} \left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 + f_2 \right)^p (y) \right) + \frac{1}{q (\theta_1 + 1)^q} \left( \frac{\omega}{\rho} K_{r_1}^q \left( f_1 + f_2 \right)^q (y) \right).
\]

(4.17)

Using the inequality \((\mu + \nu)^s \leq 2^{s-1}(\mu^s + \nu^s), s > 1, \mu, \nu > 0,\) one can obtain

\[
\left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 + f_2 \right)^p (y) \right) \leq 2^{s-1} \left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 \right)^p (y) \right) \\
\]

(4.18)

and

\[
\left( \frac{\omega}{\rho} K_{r_1}^q \left( f_1 + f_2 \right)^q (y) \right) \leq 2^{s-1} \left( \frac{\omega}{\rho} K_{r_1}^q \left( f_1 \right)^q (y) \right).
\]

(4.19)

Hence, the proof of (4.9) can be concluded from (4.17), (4.18), and (4.19) collectively. \(\square\)

**Theorem 4.3** For \(\zeta > 0, \phi \in (0, 1], \tau \in \mathbb{R}, \phi + \tau \neq 0\) with \(p \geq 1\) and let there be two positive functions \(f_1, f_2\) on \([0, \infty)\) such that, for all \(y > r_1, \frac{\omega}{\rho} K_{r_1}^p f_1^p (y) < \infty\) and \(\frac{\omega}{\rho} K_{r_1}^q f_2^q (y) < \infty\). If \(0 < \lambda < \theta_1 \leq \frac{\phi(\eta)}{\phi(\omega)} \leq \theta_2\) for \(\theta_1, \theta_2 \in \mathbb{R}^*\) and for all \(\eta \in [r_1, y]\), then

\[
\frac{\theta_2 + 1}{\theta_2 - \lambda} \left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 (y) - \lambda f_2 (y) \right) \right) \leq \left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 \right)^p (y) \right)^{\frac{1}{p}} + \left( \frac{\omega}{\rho} K_{r_1}^q \left( f_2 \right)^q (y) \right)^{\frac{1}{q}} \\
\leq \frac{\theta_1 + 1}{\theta_1 - \lambda} \left( \frac{\omega}{\rho} K_{r_1}^p \left( f_1 (y) - \lambda f_2 (y) \right) \right)^{\frac{1}{p}}.
\]

(4.20)

**Proof** Under the given supposition \(0 < \lambda < \theta_1 \leq \frac{\phi(\eta)}{\phi(\omega)} \leq \theta_2\), we have

\[
\theta_1 \lambda \leq \theta_2 \lambda \quad \Rightarrow \quad \theta_1 \lambda + \theta_1 \leq \theta_1 \lambda + \theta_2 \leq \theta_2 \lambda + \theta_2 \\
\Rightarrow \quad (\theta_2 + 1)(\theta_1 - \lambda) \leq (\theta_1 + 1)(\theta_2 - \lambda).
\]

It follows that

\[
\frac{\theta_2 + 1}{\theta_2 - \lambda} \leq \frac{\theta_1 + 1}{\theta_1 - \lambda}.
\]

Also, we have

\[
\theta_1 - \lambda \leq \frac{f_1(\eta) - \lambda f_2(\eta)}{f_2(\eta)} \leq \theta_2 - \lambda,
\]

implying

\[
\frac{(f_1(\eta) - \lambda f_2(\eta))^p}{(\theta_2 - \lambda)^p} \leq \frac{(f_2(\eta))^p}{(\theta_1 - \lambda)^p}.
\]

(4.21)
Furthermore, we have
\[
\frac{1}{\theta_2} \leq \frac{f_2(\eta)}{f_1(\eta)} \leq \frac{1}{\theta_1} \quad \Rightarrow \quad \frac{\theta_1 - \lambda}{\lambda \theta_1} \leq \frac{f_1(\eta) - \lambda f_2(\eta)}{\lambda f_1(\eta)} \leq \frac{\theta_2 - \lambda}{\theta_2 \lambda}.
\]

It follows that
\[
\left( \frac{\theta_2}{\theta_2 - \lambda} \right)^p (f_1(\eta) - \lambda f_2(\eta))^p \leq (\frac{\theta_1}{\theta_1 - \lambda})^p (f_1(\eta) - \lambda f_2(\eta))^p. \tag{4.22}
\]

If we multiply both sides of (4.22) with \(\frac{1}{\Gamma(\zeta)^{\tau+\varrho}}(\lambda \tau + \lambda \varrho)^{\tau+\varrho} \) and then integrate the subsequent inequality with respect to \(\eta\) from \(r_1\) to \(y\), we obtain
\[
\frac{1}{(\theta_2 - \lambda)^p \Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau + \varrho} \right)^{\tau+\varrho} (f_1(\eta) - \lambda f_2(\eta))^p d\eta \leq \frac{1}{(\theta_1 - \lambda)^p \Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau + \varrho} \right)^{\tau+\varrho} (f_1(\eta) - \lambda f_2(\eta))^p d\eta.
\]

Accordingly, it can be written as
\[
\frac{1}{(\theta_2 - \lambda)^{\frac{1}{p}} \Gamma(\zeta)^{\frac{1}{p}}(r_1 + f_2(\eta))} \leq \frac{1}{(\theta_1 - \lambda)^{\frac{1}{p}} \Gamma(\zeta)^{\frac{1}{p}}(r_1 + f_2(\eta))} \leq \frac{1}{(\theta_1 - \lambda)^{\frac{1}{p}} \Gamma(\zeta)^{\frac{1}{p}}(r_1 + f_2(\eta))}. \tag{4.23}
\]

Adopting the same technique with (4.22), one obtains
\[
\frac{\theta_2}{\theta_2 - \lambda} (\frac{f_1(\eta) - \lambda f_2(\eta)}{f_1(\eta) - \lambda f_2(\eta)})^\frac{1}{p} \leq (\frac{\theta_1}{\theta_1 - \lambda} (\frac{f_1(\eta) - \lambda f_2(\eta)}{f_1(\eta) - \lambda f_2(\eta)})^\frac{1}{p}.
\]

Hence, by adding inequalities (4.23) and (4.24), we attain the inequality (4.20). \(\square\)

**Theorem 4.4** For \(\zeta > 0, \varrho \in (0, 1], \tau \in \mathbb{R}, \varrho + \tau \neq 0\) with \(p \geq 1\) and let there are two positive functions \(f_1, f_2\) on \([0, \infty)\) such that, for all \(y > r_1\), \(\varphi_{r_1} f_1(y) < \infty\) and \(\varphi_{r_1} f_2(y) < \infty\). If \(0 < h \leq f_1(\eta) \leq H\) and \(0 < f_2(\eta) \leq M\) for \(\theta_1, \theta_2 \in \mathbb{R}\) and for all \(\eta \in [r_1, y]\), then
\[
\left(\varphi_{r_1} f_1(\eta)^p\right)^\frac{1}{p} + \left(\varphi_{r_1} f_2(\eta)^p\right)^\frac{1}{p} \leq \frac{\mathcal{H}(h + M) + M(\mathcal{H} + m)}{(m + \mathcal{H})(h + M)} \left(\frac{\varphi_{r_1} (f_1 + f_2)^p(\eta)}{p}\right)^\frac{1}{p}. \tag{4.25}
\]

**Proof** Under the given suppositions, observe that
\[
\frac{1}{M} \leq \frac{1}{f_2(\eta)} \leq \frac{1}{m}. \tag{4.26}
\]

Conducting the product between (4.26) and \(0 \leq h \leq f_1(\eta) \leq H\), we have
\[
\frac{h}{M} \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \frac{H}{m}. \tag{4.27}
\]
From (4.27), we obtain

\[ f_2^n(\eta) \leq \left( \frac{M}{h+M} \right)^p (f_1(\eta) + f_2(\eta))^p \]  

(4.28)

and

\[ f_1^n(\eta) \leq \left( \frac{m + M}{m} \right)^p (f_1(\eta) + f_2(\eta))^p. \]  

(4.29)

If we multiply both sides of (4.28) with \( \frac{1}{\Gamma(\zeta) \int_{r_1}^{y} \left( \frac{y^{r+\varphi} - \eta^{r+\varphi}}{\tau + \varphi} \right)^{\tau-1} f_2^n(\eta) d\eta} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_1 \) to \( y \), we obtain

\[ \frac{1}{\Gamma(\zeta) \int_{r_1}^{y} \left( \frac{y^{r+\varphi} - \eta^{r+\varphi}}{\tau + \varphi} \right)^{\tau-1} f_2^n(\eta) d\eta} \]

\[ \leq \frac{M^p}{(h+M)^p \Gamma(\zeta)} \int_{r_1}^{y} \left( \frac{y^{r+\varphi} - \eta^{r+\varphi}}{\tau + \varphi} \right)^{\tau-1} (f_1(\eta) + f_2(\eta))^p(\eta) d\eta \]  

(4.30)

Accordingly,

\[ \left( \int_{r_1}^{y} K_1^\varphi f_2^n(\eta) d\eta \right)^\frac{1}{p} \leq \left( \frac{M}{h+M} \right)^p \int_{r_1}^{y} K_1^\varphi (f_1 + f_2)^p(\eta) d\eta \]  

(4.31)

Adopting the same technique as (4.29), one obtains

\[ \left( \int_{r_1}^{y} K_1^\varphi f_1^n(\eta) d\eta \right)^\frac{1}{p} \leq \left( \frac{m + M}{m} \right)^p \int_{r_1}^{y} K_1^\varphi (f_1 + f_2)^p(\eta) d\eta \]  

(4.32)

Hence, by adding (4.31) and (4.32), we obtain the inequality (4.25).

\[ \square \]

\textbf{Theorem 4.5} For \( \zeta > 0, \varphi \in (0,1], \tau \in \mathcal{R}, \varphi + \tau \neq 0 \) with \( p \geq 1 \) let there be two positive functions \( f_1, f_2 \) on [0, \infty) such that, for all \( y > r_1 \), \( \int_{r_1}^{y} K_1^\varphi f_2^n(\eta) < \infty \) and \( \int_{r_1}^{y} K_1^\varphi f_1^n(\eta) < \infty \). If \( 0 < h \leq f_1(\eta) \leq H \) and \( 0 < m \leq f_2(\eta) \leq M \) for \( \theta_1, \theta_2 \in \mathcal{R} \) for all \( \eta \in [r_1, y] \), then

\[ \frac{1}{\theta_2} \left( \int_{r_1}^{y} K_1^\varphi f_2^n(\eta) f_2(\eta) d\eta \right) \leq \frac{1}{(\theta_1 + 1)(\theta_2 + 1)} \left( \int_{r_1}^{y} K_1^\varphi (f_1(\eta) + f_2(\eta))^2 d\eta \right)^2 \]

\[ \leq \frac{1}{\theta_1} \left( \int_{r_1}^{y} K_1^\varphi f_1^n(\eta) f_2(\eta) d\eta \right). \]  

(4.33)

\textbf{Proof} Under the given suppositions, \( 0 < \theta_1 \leq f_1(\eta) \leq \theta_2 \), it follows that

\[ f_2(\eta)(\theta_1 + 1) \leq f_1(\eta) + f_2(\eta) \leq f_2(\eta)(\theta_2 + 1). \]  

(4.34)

Additionally, we have \( \frac{1}{\theta_2} \leq \frac{f_2(\eta)}{f_1(\eta)} \leq \frac{1}{\theta_1} \), which yields

\[ \left( \frac{\theta_1 + 1}{\theta_2} \right) f_1(\eta) \leq f_1(\eta) + f_2(\eta) \leq \left( \frac{\theta_1 + 1}{\theta_1} \right) f_1(\eta). \]  

(4.35)

The product of (4.34) and (4.35) gives

\[ \frac{f_1(\eta)f_2(\eta)}{\theta_2} \leq \frac{(f_1(\eta) + f_2(\eta))^2}{(\theta_1 + 1)(\theta_2 + 1)} = \frac{f_1(\eta)f_2(\eta)}{\theta_1}. \]  

(4.36)
Now, if we multiply both sides of (4.36) with \( \frac{1}{\Gamma(\xi, t^{\alpha})} \left( \frac{t^{\alpha}}{\xi} \right)^{\tau-1} \) and then integrate the subsequent inequality with respect to \( \eta \) from \( r_1 \) to \( y \), we obtain

\[
\frac{1}{\theta_2} \int_{r_1}^{y} \left( \frac{y^{\tau_1} - \eta^{\tau_1}}{\tau + \phi} \right)^{\tau-1} f_2(\eta) f_3(\eta) \, d\eta \\
\leq \frac{1}{\Gamma(\xi, (\theta_1 + 1)(\theta_2 + 1))} \int_{r_1}^{y} \left( \frac{y^{\tau_1} - \eta^{\tau_1}}{\tau + \phi} \right)^{\tau-1} \left( f_1(\eta) + f_2(\eta) \right)^2 \, d\eta \\
\leq \frac{1}{\theta_1} \int_{r_1}^{y} \left( \frac{y^{\tau_1} - \eta^{\tau_1}}{\tau + \phi} \right)^{\tau-1} f_1(\eta) f_2(\eta) \, d\eta.
\]

One observes that

\[
\frac{1}{\theta_2} \left( \mathcal{K}_r \mathcal{F}(y) \mathcal{F}(y) \right) \leq \frac{1}{(\theta_1 + 1)(\theta_2 + 1)} \left( \mathcal{K}_r \mathcal{F}(y) \mathcal{F}(y) \right)^2 \leq \frac{1}{\theta_1} \left( \mathcal{K}_r \mathcal{F}(y) \mathcal{F}(y) \right),
\]

which is the desired result. \( \square \)

**Theorem 4.6** For \( \xi > 0, \phi \in (0, 1], \tau \in \mathbb{R}, \phi + \tau \neq 0 \) with \( \rho \geq 1 \) and let there are two positive functions \( f_1, f_2 \) on \([0, \infty)\) such that, for all \( y > r_1 \), \( \mathcal{K}_r f_1(y) < \infty \) and \( \mathcal{K}_r f_2(y) < \infty \). If \( 0 < h \leq f_1(\eta) \leq \mathcal{H} \) and \( 0 < \theta_1 \leq f_2(\eta) \leq \theta_2 \) for \( \theta_1, \theta_2 \in \mathbb{R}^+ \) and for all \( \eta \in [r_1, y] \), then

\[
(\mathcal{K}_r f_1(y))^\frac{1}{p} + (\mathcal{K}_r f_2(y))^\frac{1}{q} \leq 2(\mathcal{K}_r f_1(y) f_2(y))^{\frac{1}{p}} + 2(\mathcal{K}_r f_1(y) f_2(y))^{\frac{1}{q}},
\]

where \( \mathcal{U}(f_1(\eta), f_2(\eta)) = \max[\theta_1(1 + \frac{\theta_2}{\theta_1}) f_1(\eta) - \theta_2 f_2(\eta), \frac{\theta_1 + \theta_2 f_1(\eta) - f_2(\eta)}{\theta_1}] \).

**Proof** Under the given suppositions \( 0 < \theta_1 \leq \frac{f_2(y)}{f_1(y)} \leq \theta_2, r_1 \leq \eta \leq y \), we have

\[
0 < \theta_1 \leq \theta_2 \leq \frac{f_1(\eta)}{f_2(\eta)} \quad (4.38)
\]

and

\[
\theta_2 + \theta_1 - \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_1 \quad (4.39)
\]

From (4.38) and (4.39), one obtains

\[
f_2(\eta) < \frac{(\theta_2 + \theta_1 f_2(\eta) - f_1(\eta)}{\theta_1} \leq \mathcal{U}(f_1(\eta), f_2(\eta)),
\]

where \( \mathcal{U}(f_1(\eta), f_2(\eta)) = \max[\theta_2(1 + \frac{\theta_2}{\theta_1}) f_1(\eta) - \theta_2 f_2(\eta), \frac{\theta_1 + \theta_2 f_1(\eta) - f_2(\eta)}{\theta_1}] \). Also, from the given supposition \( 0 < \frac{1}{\theta_2} \leq \frac{f_2(y)}{f_1(y)} \leq \frac{1}{\theta_1} \), one has

\[
\frac{1}{\theta_2} \leq 1 + \frac{1}{\theta_1} - \frac{f_2(\eta)}{f_1(\eta)} \quad (4.40)
\]

and

\[
\frac{1}{\theta_2} + \frac{1}{\theta_1} - \frac{f_2(\eta)}{f_1(\eta)} \leq \frac{1}{\theta_1}.
\]
From (4.41) and (4.42), we get
\[
\frac{1}{\theta_2} \leq \frac{\left( \frac{1}{\theta_1} + \frac{1}{\theta_2} \right) f_1(\eta) - f_2(\eta)}{f_1(\eta)} \leq \frac{1}{\theta_1}, \tag{4.43}
\]
implying
\[
f_1(\eta) \leq \theta_2 \left( \frac{1}{\theta_1} + \frac{1}{\theta_2} \right) f_1(\eta) - \theta_2 f_2(\eta)
= \frac{\theta_2 (\theta_1 + \theta_2) f_1(\eta) - \theta_2 \theta_1 f_2(\eta)}{\theta_1 \theta_2}
= \left( \frac{\theta_2}{\theta_1} + 1 \right) f_1(\eta) - \theta_2 f_2(\eta)
\leq \theta_2 \left[ \left( \frac{\theta_2}{\theta_1} + 1 \right) f_1(\eta) - \theta_2 f_2(\eta) \right]
\leq \mathcal{U}(f_1(\eta), f_2(\eta)). \tag{4.44}
\]

From (4.40) and (4.44), we have
\[
f_1^p(\eta) \leq \mathcal{U}^p(f_1(\eta), f_2(\eta)) \tag{4.45}
\]
and
\[
f_2^p(\eta) \leq \mathcal{U}^p(f_1(\eta), f_2(\eta)). \tag{4.46}
\]

If we multiply both sides of (4.45) with \(\frac{1}{\Gamma(\varsigma)\varsigma^{1-t-\rho}} \left( \frac{y^{t+\rho} - \eta^{t+\rho}}{\tau + \rho} \right)^{t-1} \) and then integrate the subsequent inequality with respect to \(\eta\) from \(r_1\) to \(y\), we obtain
\[
\frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} \left( \frac{y^{t+\rho} - \eta^{t+\rho}}{\tau + \rho} \right)^{t-1} f_1^p(\eta) \, d\eta \leq \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} \left( \frac{y^{t+\rho} - \eta^{t+\rho}}{\tau + \rho} \right)^{t-1} \mathcal{U}^p(f_1(\eta), f_2(\eta)) \, d\eta. \tag{4.47}
\]

Accordingly,
\[
\left( \mathcal{J}_1^\varsigma_{r_1} f_1^p(y) \right)^{\frac{1}{p}} \leq \left( \mathcal{J}_1^\varsigma_{r_1} \mathcal{U}^p(f_1(\eta), f_2(\eta)) \right)^{\frac{1}{p}}. \tag{4.48}
\]

Adopting the same technique for (4.46), we have
\[
\left( \mathcal{J}_1^\varsigma_{r_1} f_2^p(y) \right)^{\frac{1}{p}} \leq \left( \mathcal{J}_1^\varsigma_{r_1} \mathcal{U}^p(f_1(\eta), f_2(\eta)) \right)^{\frac{1}{p}}. \tag{4.49}
\]

Hence, by adding (4.48) and (4.49), we obtain the inequality (4.37).

\[\square\]

5 Concluding remarks

This paper begins with a compact evaluation of fractional integrals in the sense of Riemann–Liouville and Riemann–Liouville type conformable fractional integral operators in addition to a new fractional integral operator according to Khan et al. [14]. We
Rashid et al. Journal of Inequalities and Applications (2020) 2020:177 Page 13 of 15

[Image]

generalize the reverse Minkowski inequalities via generalized conformable fractional integrals; specifically, the inequality concerning fractional integrals in the Riemann–Liouville sense is given [44]. The associated significant variants regarding generalized conformable fractional integrals are demonstrated. Numerous variants can be established for the application of several defined fractional integral operators. One of the well-known inequalities is the Chebyshev inequality lately derived in [38]. Finally, this concept can be extended in the form of a $K$-analogue for deriving similar types of results and these are also helpful for establishing the refinements of several existing results in the literature.

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