DENSITIES OF ULTRAPRODUCTS OF BOOLEAN ALGEBRAS

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ABSTRACT. We answer three problems by J. D. Monk on cardinal invariants of Boolean algebras. Two of these are whether taking the algebraic density \( \pi A \) resp. the topological density \( dA \) of a Boolean algebra \( A \) commutes with formation of ultraproducts; the third one compares the number of endomorphisms and of ideals of a Boolean algebra.

In set theoretic topology, considerable effort has been put into the study of cardinal invariants of topological spaces, see e.g. [Ju1] and [Ho], [Ju2]. In Monk’s book [Mo], similarly a systematic study of cardinal invariants of Boolean algebras is undertaken; in particular, the behaviour of these invariants with respect to algebraic constructions like taking subalgebras, quotients etc. is investigated. One of these is the ultraproduct construction, well known from model theory; cf. [ChK]. Many questions on ultraproducts are highly dependent on set theory; among the more recent results are those in Shelah’s pcf theory dealing with the possible cofinalities \( \text{cf}(\prod_{\alpha < \kappa} \lambda_{\alpha}/D) \) where the \( \lambda_{\alpha} \) are regular cardinals, hence well-ordered in a natural way, and the ultraproduct has the resulting linear order.

Monk’s book contains a list of 66 problems, three of which are answered (consistently) in this paper.

PROBLEM 9. Does there exist a system \((A_i)_{i \in I}\) of infinite Boolean algebras and an ultrafilter \( F \) on \( I \) such that \( d(\prod_{i \in I} A_i/F) < \prod_{i \in I} d(A_i)/F \)?

PROBLEM 12. Is it true that always \( \pi(\prod_{i \in I} A_i/F) = \prod_{i \in I} \pi(A_i)/F \)?

PROBLEM 60. Is there a Boolean algebra \( A \) such that \( |\text{End} A| < |\text{Id} A| \)?

Here \( \pi A \) and \( dA \) are the “algebraic” and the “topological” density of \( A \), defined by

\[
dA = \min\{|Y| : Y \text{ a dense subset of the Stone space of } A\}
\]

\[
\pi A = \min\{|X| : X \text{ a dense subset of } A\}
\]

(for more definitions and matters on cardinal functions, see [Mo]). Note that we are dealing only with infinite algebras and that, trivially, \( \omega \leq dA \leq \pi A \), \( d(\prod_{i \in I} A_i/F) \leq \prod_{i \in I} d(A_i)/F \) and \( \pi(\prod_{i \in I} A_i/F) \leq \prod_{i \in I} \pi(A_i)/F \).

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132
In Problem 60, End $A$ is the set of all endomorphisms, $\text{Id} A$ the set of all ideals of $A$.

In Section 1, we give a positive answer to Problem 12 under SCH. Here SCH is the Singular Cardinal Hypothesis: if $2^{\text{cf}\lambda} < \lambda$ (so $\lambda$ is singular), then $\lambda^{\text{cf}\lambda} = \lambda^+$. However, $\neg$SCH gives a negative answer to both Problems 9 and 12:

**THEOREM A.** Assume we have cardinals $\kappa$, $\mu$, and $(\lambda_\alpha)_{\alpha<\kappa}$ and an ultrafilter $D$ on $\kappa$ such that: $\kappa < \mu = \text{cf} \mu$, $\mu^{<\mu} < \lambda_\alpha = \text{cf} \lambda_\alpha$, and the cofinality of the ultraproduct $\prod_{\alpha<\kappa} \lambda_\alpha/D$ is less than its cardinality. Then there is a forcing notion $\mathbb{R}$ such that

(a) $\mathbb{R}$ is $\mu$-complete and satisfies the $(\mu^{<\mu})^+$-chain condition; hence forcing with $\mathbb{R}$ preserves all cardinalities and cofinalities outside the interval $[\mu^+, \mu^{<\mu})$

(b) for $K \subseteq \mathbb{R}$ $\mathbb{R}$-generic over $V$, the following holds in $V[K]$: there are Boolean algebras $(A_\alpha)_{\alpha<\kappa}$ such that $\lambda_\alpha = |A_\alpha| = \pi A_\alpha = dA_\alpha$, but for the ultraproduct $A = \prod_{\alpha<\kappa} A_\alpha/D$,

$$d(A) \leq \pi(A) = \text{cf} \left( \prod_{\alpha<\kappa} \lambda_\alpha/D \right) < \prod_{\alpha<\kappa} \lambda_\alpha/D = \prod_{\alpha<\kappa} \pi(A_\alpha)/D = \prod_{\alpha<\kappa} d(A_\alpha)/D.$$ 

Note that SCH is known to be independent from ZFC, modulo some large cardinal assumption (see [Ma]). And the assumption of Theorem A is a consequence of $\neg$SCH, as follows from pcf theory. A particularly easy case is the classical one for $\neg$SCH: assume $\lambda$ is strong limit and singular, $\kappa = \text{cf} \lambda$ satisfies $2^\kappa < \lambda$, but $\lambda^\kappa > \lambda^+$; let $\mu$ be regular such that $\kappa < \mu < \lambda$. Then there are (see [Sh, Chapter II, 1.5]) regular $\lambda_\alpha$ such that $\lambda = \sup_{\alpha<\kappa} \lambda_\alpha$, $\prod_{\alpha<\kappa} \lambda_\alpha/J^\text{bd}_\kappa$ has true cofinality $\lambda^+$ ($J^\text{bd}_\kappa$ the ideal of bounded subsets of $\kappa$), hence any uniform ultrafilter $D$ on $\kappa$ gives $\text{cf} ((\prod_{\alpha<\kappa} \lambda_\alpha/D) = \lambda^+ < \prod_{\alpha<\kappa} \lambda_\alpha/D].$

More generally if $\lambda$ violates SCH, i.e. for some $\kappa$, we have $2^\kappa < \lambda$ and $\lambda^\kappa > \lambda^+$, let $\lambda'$ be minimal such that $\lambda'^\kappa = \lambda^\kappa$ (i.e. $\lambda'^\kappa \geq \lambda$); so for every cardinal $\rho < \lambda'$, we have $\rho^\kappa < \lambda'$.

Now take $\mu = \kappa^+$ and find, by [Sh, Chapter II, 1.5], an appropriate family $(\lambda'_\alpha)_{\alpha<\kappa}$ with limit $\lambda'$ and $\text{cf} ((\prod_{\alpha<\kappa} \lambda'_\alpha/J^\text{bd}_\kappa) = \lambda'^\kappa$. Moreover we can replace $\lambda'^\kappa$ by any regular cardinal in the interval $[\lambda'^\kappa, \lambda'^\kappa]$; similarly for the strong limit case; see [Sh, Chapter VIII, §1].

Theorem 1.1 below and Theorem A show that the answer to Problem 12 is independent from ZFC. However, it has recently been shown in [RoSh 534, 2.6, 2.7] that Problem 9 has a positive answer even in ZFC.

Problem 60 is solved in Section 8 by

**THEOREM B.** Assume $\mu$ is a strong limit cardinal satisfying $\text{cf} \mu = \omega$ and $2^\mu = \mu^+$. Then there is a Boolean algebra $B$ such that $|B| = |\text{End} B| = \mu^+$ and $|\text{Id} B| = 2^\mu$.

The organization of Sections 2 to 7 is as follows. In Section 2, we introduce a first order theory $T$ for Boolean algebras with some extra structure which allows (e.g. in ultraproducts $A = \prod_{\alpha<\kappa} A_\alpha/D$ of models of $T$) to easily compute $\pi A$. In Section 3, we construct canonical models $A(p)$ of $T$ from what we call valuation functions $p$. In sections 4 to 6, we consider the forcing notion $\mathbb{P}$ of valuation functions, determine its completeness and chain conditions, and compute $dA$ and $\pi A$ for the canonical algebra $A = A(\mathbb{P})$ constructed from a generic valuation function $P$. In Section 7, we prove Theorem A.

For definitions and results on set theory, see [Je]; for Boolean algebras, [Ko].
1. **Problem 12 under SCH.** We give here a positive answer to Monk’s Problem 12 under SCH. Given an ultraproduct $A = \prod_{i \in K} A_i/D$ of infinite Boolean algebras, we let $\lambda_i = \pi A_i$, so $\omega \leq \lambda_i$. For simplicity of notation, we will denote, in this section, by $\prod_{i \in K} \lambda_i/D$ both the ultraproduct of the $\lambda_i$ and its cardinality.

Note first that the answer is easy if $\lambda_i \leq 2^\kappa$ for $D$-almost all $i \in \kappa$ (i.e. if $\{i \in \kappa : \lambda_i \leq 2^\kappa\}$ is in $D$) and $D$ is regular. For in this case, each $A_i$ has an infinite set of pairwise disjoint elements, so $A$ has cellularity at least $2^\kappa$ and, on the other hand, $\prod_{i \in K} \lambda_i/D \leq 2^\kappa$, hence $2^\kappa \leq \pi A \leq \prod_{i \in K} \lambda_i/D \leq 2^\kappa$. Thus Theorem 1.1 covers the interesting case: $2^\kappa < \lambda_i$ for $D$-almost all $i$.

**Theorem 1.1 (SCH).** Assume $2^\kappa < \lambda_i = \pi A_i$ for all $i \in \kappa$ and $D$ is an ultrafilter on $\kappa$; let $A = \prod_{i \in K} A_i/D$. Then $\pi A = \prod_{i \in K} \lambda_i/D$.

**Proof.** We know that $\pi A \leq \prod_{i \in K} \lambda_i/D$. Let

$$\lambda = D - \lim(\lambda_i : i \in \kappa),$$

i.e. $\lambda$ is the least cardinal $\rho$ such that $\lambda_i \leq \rho$ holds for all $D$-almost all $i$. Without loss of generality, $\lambda_i \leq \lambda$ holds for all $i \in \kappa$.

**Claim 1.** If $\theta < \lambda$, then $\theta^\kappa \leq \lambda$.

To see this, pick $i$ such that $\theta < \lambda_i$. Now if $\theta \leq 2^\kappa$, then $\theta^\kappa = 2^\kappa < \lambda_i \leq \lambda$. Otherwise, $\kappa < 2^\kappa < \theta < \theta^\kappa < \lambda_i$, $(\theta^\kappa)^\kappa = \theta^\kappa$ by SCH, so $\theta^\kappa < \theta^\kappa < \lambda_i \leq \lambda$.

**Claim 2.** $\pi A \geq \lambda$.

Otherwise pick a dense subset $Y$ of $A$ of size $\rho$, where $\rho < \lambda$, say $Y = \{y_\alpha/D : \alpha < \rho\}$ with $y_\alpha = (y_\alpha(i))_{i \in K}$ in $\prod_{i \in K} A_i$ and $y_\alpha(i) \neq 0$. Since $\rho < \lambda$, we may assume without loss of generality that $\rho < \lambda_i$ for all $i$. So we can pick, for $i \in \kappa$, $a_i \in A_i \setminus \{0\}$ satisfying $y_\alpha(i) \notin a_i$ for all $\alpha < \rho$. The sequence $a = (a_i)_{i \in K}$ is such that $y_\alpha/D \notin a/D$ for $\alpha < \rho$, a contradiction.

The theorem now follows immediately from the next three claims.

**Claim 3.** If $\pi A \geq \lambda^+$, then the assertion of the theorem holds.

For in this case, $\lambda^+ \leq \pi A \leq \prod_{i \in K} \lambda_i/D \leq \lambda^\kappa/D \leq \lambda^\kappa \leq \lambda^+$, where the last inequality follows from SCH and $2^\kappa < \lambda$.

**Claim 4.** If $\pi A = \lambda$, then every function $f \in \prod_{i \in K} \lambda_i/D$ is bounded below $\lambda$, modulo $D$.

For the proof, work as in Claim 2: fix a dense subset $Y$ of $A$, $Y = \{y_\alpha/D : \alpha < \lambda\}$, $y_\alpha = (y_\alpha(i))_{i \in K}$, $y_\alpha(i) \neq 0$. Given $f \in \prod_{i \in K} \lambda_i$, we know that $Y_i = \{y_\alpha(i) : \alpha < f(i)\}$ cannot be dense in $A_i$, since $|Y_i| \leq |f(i)| < \lambda_i = \pi A_i$. So pick $a = (a_i)_{i \in K}$ where $a_i \in A_i \setminus \{0\}$ is such that $y_\alpha(i) \notin a_i$, for all $\alpha < f(i)$. Since $Y$ is dense in $A$, pick $\alpha < \lambda$ such that $y_\alpha/D \leq a/D$. It follows that: $y_\alpha(i) \leq a_i$, for $D$-almost all $i$; $\alpha \notin f(i)$ for these $i$, so $f(i) < \alpha$; i.e. $f(i) \leq \alpha$ for $D$-almost all $i$. Thus $f$ is bounded by $\alpha < \lambda$.

**Claim 5.** If $\pi A = \lambda$, then the assertion of the theorem holds.

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For Claim 4 says that for every $f \in \prod_{i \in \kappa} \lambda_i, f/D = f'/D$ for some $f': \kappa \to \nu$ and some $\nu < \lambda$. By Claim 1, $\prod_{i \in \kappa} \lambda_i/D \leq \sum_{\nu < \kappa} |\nu|^\kappa \leq \lambda$. It now follows from Claim 2 that $\lambda \leq \pi A \leq \prod_{i \in \kappa} \lambda_i/D \leq \lambda$.

2. The theory $T$. We sketch here a first order theory $T$. Its relevance for solving Problem 12 of [Mo] lies in the fact that the models $\mathcal{M}$ of $T$ are enlargements $(A,\ldots)$ of a Boolean algebra $A$; the extra structure of $\mathcal{M}$ allows to easily compute $\pi(A)$—see Remark 2.1. below. Since every ultraproduct $\mathcal{U} = (U,\ldots)$ of models of $T$ is again a model of $T$, we can then similarly compute $\pi(U)$.

Let $T$ be the first order theory (in an appropriate language) saying that, for every model $\mathcal{M} = (A,+,\cdot,-,0,1,L,\leq,\sim,\lor,\land)$ of $T$, the following hold true.

(a) $(A,+,\cdot,-,0,1)$ is a Boolean algebra.
(b) $L \subseteq A$ is totally ordered by $\leq_L$ and has no greatest element. (We do not require any connection between $\leq_L$ and the Boolean partial order of $A$, except the one stipulated by (e) below.)
(c) $\lor$ is a map from $L$ to $A$; for $l \in L$, $A_l = \{a \in A : \lor(a) <_L l\}$ is a subalgebra of $A$.
(d) $\sim$ is an equivalence relation on $L$ and its equivalence classes are convex, with respect to $\leq_L$.
(e) $\land$ is a map from $L$ into $A$ (we write $x_i$ for $x(i)$) such that $i < l$ implies $x_i \not\leq x_l$.

Moreover for $l \in L$, the set $\{x_i : i \sim l\}$ is dense for $A_l$ in the sense that for every $a \in A_l \setminus \{0\}$ there is some $i \sim l$ satisfying $0 < x_i < a$. (Hence $\{x_i : i \in L\}$ is a dense subset of $A$.)

REMARK 2.1. Let $\mathcal{M} = (A,\ldots)$ be a model of $T$, $\rho$ the cofinality of the linear order $(L,\leq_L)$ and assume that all equivalence classes in $L$ have cardinality at most $\rho$. Then $\pi(A) = \rho$.

PROOF. To see that $\pi(A) \leq \rho$, fix a cofinal subset $M$ of $L$ of size $\rho$. The set $\{x_i : i \sim m, \text{ for some } m \in M\}$ has size $\rho$ and is dense in $A$, by (e). Assume for contradiction that $A$ has a dense subset $X$ of size less than $\rho$. Without loss of generality, $X \subseteq \{x_i : i \in L\}$; pick $l \in L$ such that $x_i \in X$ implies $i < l$. $X$ being dense in $A$, there is $x_i \in X$ such that $0 < x_i \leq a$. So $i < l$ which is impossible by (e).

In Sections 3 and 4, we will construct “standard” models $\mathcal{M} = (A,\ldots)$ of $T$ which will roughly look like this, for some regular cardinal $\lambda$; $|A| = \lambda$, so without loss of generality, $\lambda \subseteq A$; we let $L = \lambda$ and $\leq_L$ its natural well-ordering. $A$ will be generated by a sequence $(x_i)_{i \in \lambda}$; we then let $A_l$ be the subalgebra of $A$ generated by $\{x_i : i \leq l\}$ and define $\lor(a)$ to be the least $i$ such that $a \in A_{i+1}$. Finally we will have an infinite cardinal $\mu < \lambda$ and define $i \sim l$ iff $i \leq l < i+\mu$ and $l \leq i < l+\mu$ (ordinal addition); thus the equivalence classes will have size $\mu$. Satisfaction of condition (e) above will be guaranteed by a careful choice of the generators $x_i$—see Proposition 5.1. In particular, $\pi A$ will be $\lambda = |A|$.
3. Valuation functions. We construct Boolean algebras $A(p)$ from certain functions $p$, the so-called valuation functions. Later the Boolean algebras $A(P)$, where $P$ will be a generic valuation function, provide the counterexample for Problems 9 and 12 in [Mo] looked for.

We denote the three-element set consisting of the symbols $\geq$, $\perp$, $u$ — “undefined” by $3$. For any set $w$ with some linear order on it (later $w$ will be a subset of some cardinal $\lambda$, hence well-ordered), recall that $[w]^2 = \{(i,j) : i < j \text{ in } w\}$.

Given a Boolean algebra $A$ and a family $(x_i)_{i \in w}$ indexed by $w$ in $A \setminus \{0\}$, we can assign to $(x_i)_{i \in w}$ the function $p : [w]^2 \to 3$ defined by

$$p(i,j) = \begin{cases} 
\geq & \text{if } x_i \geq x_j \\
\perp & \text{if } x_i \perp x_j, \text{ i.e. } x_i \cdot x_j = 0 \\
u & \text{otherwise.}
\end{cases}$$

Clearly $p$ has then the following properties:

1. If $p(i,j) = \geq$ and $p(j,k) = \geq$ then $p(i,k) = \geq$ (where $i < j < k$)

2. If $i < j < k$ and $\{p(i,j), p(i,k)\} = \{\perp, \geq\}$, then $p(j,k) = \perp$; similarly if $i < j < k$ and $p(i,j) = \perp$, $p(j,k) = \geq$, then $p(i,k) = \perp$.

Let us call a function $p$ satisfying (1) and (2) above a valuation function and $w$ its domain.

Conversely, given a valuation function $p : [w]^2 \to 3$, we construct a Boolean algebra $A = A(p)$ from $p$ as follows. Let $Fr w$ be the free Boolean algebra on the set $\{u_i : i \in w\}$ of independent generators and let $N(p)$ be the ideal in $Fr w$ generated by the elementary products $u_j \cdot u_i$ where $p(i,j) = \perp$ resp. $u_j \cdot -u_i$ where $p(i,j) = \geq$. Let then $A(p)$ (or $A$, for short) be the quotient algebra $Fr w / N(p)$ and let $c : Fr w \to A(p)$ be the canonical homomorphism. Setting $x_i = c(u_i)$, for $i \in w$, we find that the $x_i$ generate $A$. By the very choice of the ideal $N(p)$, $p(i,j) = \geq$ implies that $x_i \geq x_j$ and $p(i,j) = \perp$ implies that $x_j \perp x_i$. To see that no other relations than those imposed by $p$ hold for the $x_i$, note the following general principle on construction of Boolean algebras via generators with prescribed relations.

**Remark 3.1.** Let $E$ be a set of finite partial functions from $w$ to $\{0,1\}$ and let, for $e \in E$, $q_e$ be the elementary product $\Pi_{e(i)=1} u_i \cdot \Pi_{e(i)=0} -u_i$ in $Fr w$. Assume $N$ is the ideal of $Fr w$ generated by the $q_e$, $e \in E$. Then for any function $g : w \to \{0,1\}$, there is an ultrafilter of $Fr w / N$ including $\{x_i : g(i) = 1\} \cup \{-x_i : g(i) = 0\}$ (i.e. $\{x_i : g(i) = 1\} \cup \{-x_i : g(i) = 0\}$ has the finite intersection property) iff no $e \in E$ is extended by $g$.

This gives the following properties of the $x_i$ in $A = A(p)$, where $p$ is a valuation function.

**Remark 3.2.** $x_i$ is not in the ideal generated by $\{x_j : j > i\}$. In particular, $x_i \neq 0$, the $x_i$ are pairwise distinct, and $i < j$ implies that $x_i \not\leq x_j$.

To see this, consider the function $g : w \to \{0,1\}$ such that $g(k) = 1$ iff $k = i$ or ($k < i$ and $p(k,i) = \geq$). By Remark 3.1, let $s$ be the ultrafilter of $A$ induced by $g$. Thus $x_i \in s$ but, for $j > i$, $x_j \not\in s$, which shows the claim.
REMARK 3.3. $x_t$ is not in the subalgebra of $A$ generated by $\{x_j : j < i\}$.

For consider the functions $g$ and $h$ from $w$ to $\{0, 1\}$ where $g$ is defined as in the proof of Remark 3.2, $h(k) = g(k)$ for $k \neq i$, but $h(i) = 0$. Let $s$ and $t$ be the corresponding ultrafilters of $A$, $\phi$ and $\psi$ the homomorphisms from $A$ to the two-element algebra corresponding to $s$ and $t$. Now $\phi$ and $\psi$ coincide on $x_j$ for all $j < i$, but not on $x_i$.

4. The partial order of valuation functions. For the next sections, fix infinite cardinals $\lambda$ and $\mu$ such that $\mu^\lambda = \mu$, $\mu < \lambda$, and $\lambda$ is regular. We shall choose $\lambda$ and $\mu$ somewhat more carefully in Section 7. Let $P(\lambda, \mu)$ (or $P$, for short) be the notion of forcing

$$P = \{p : p \text{ is a valuation function and } \text{dom } p \subseteq \lambda \text{ has size less than } \mu\}$$

ordered by reverse inclusion.

REMARK 4.1. $P$ is $\lambda$-closed.

We now build up some machinery for constructing elements of $P$ with prescribed properties. Given a set $r$ of relations of the form $x_i \geq x_j$, $x_i \perp x_j$ (where $i, j \in \lambda$; the relations may be thought of as being formulas in some formal language in the variables $x_i, i \in \lambda$), we define when a relation $\rho$ can be derived from $r$ and we write $r \vdash \rho$: if $\rho$ has the form $x_k \geq x_i$, $r \vdash \rho$ iff there are $i_1, \ldots, i_m \in \lambda$ such that the relations $x_k \geq x_{i_1}, x_{i_1} \geq x_{i_2}, \ldots, x_{i_m} \geq x_i$ are all in $r$ (in particular, $r \vdash x_i \geq x_i$); if $\rho$ has the form $x_k \perp x_i$, $r \vdash \rho$ iff there are $\alpha, \beta \in \lambda$ such that $x_\alpha \perp x_\beta$ is in $r$ and $r \vdash x_\alpha \geq x_\beta, r \vdash x_\beta \geq x_i$.

Call $r$ consistent if no relation of the form $x_j \geq x_i$ where $i < j$ and no relation of the form $x_k \perp x_i$ is derivable from $r$. Given $p \in P$, define $\text{rel } p$, the relevant part of $p$, by

$$\text{rel } p = \{x_i \geq x_j : p(i, j) = \geq\} \cup \{x_i \perp x_j : p(i, j) = \perp\}.$$

PROPOSITION 4.2. If $|r| < \mu$, then $r$ is consistent iff $r \subseteq \text{rel } p$ for some $p \in P$.

PROOF. Assume first that $p \in P$ and $r \subseteq \text{rel } p$ where $\text{dom } p = w \subseteq \lambda$. Then in the Boolean algebra $A(p)$ constructed in Section 3, all relations in $r$ and hence all relations derivable from $r$ are satisfied by the canonical generators $\{x_i : i \in w\}$; moreover, these generators are non-zero. Hence no relation $x_k \perp x_k$ and no relation of the form $x_j \geq x_i$, $i < j$, can be derived from $r$.

Conversely, if $r$ is consistent, let $w$ be any subset of $\lambda$ such that $|w| < \mu$ and $i \in \lambda : x_i$ occurs in $r \subseteq w$. Define $p : [w]^2 \rightarrow 3$ by

$$p(i, j) = \begin{cases} \geq & \text{iff } r \vdash x_i \geq x_j \\
\perp & \text{iff } r \vdash x_i \perp x_j \\
\text{u} & \text{otherwise.} \end{cases}$$

$p$ is a well-defined function (i.e., $r$ does not derive both $x_i \geq x_j$ and $x_i \perp x_j$, for $i < j \in w$) since otherwise, $r \vdash x_j \perp x_j$, contradicting the consistency of $r$. By the above definition of derivability from $r$, $p$ is a valuation function.
For further reference, call \( p \in \mathbb{P} \) defined from a consistent set \( r \) and \( w \subseteq \lambda \) as in the proof above the canonical extension of \( r \) over \( w \).

We give one trivial and one not-so-trivial application of this machinery. If \( G \subseteq \mathbb{P} \) is \( \mathbb{P} \)-generic over our universe \( V \) of set theory, then clearly \( P_G = \bigcup G \) is a valuation function with \( \text{dom} P_G = \bigcup_{p \in G} \text{dom} p \).

**Remark 4.3.** If \( G \) is generic, then \( \text{dom} P_G = \lambda \).

To see this, we have to make sure that, for \( i \in \lambda \), the set \( D_i = \{ p \in \mathbb{P} : i \in \text{dom} p \} \) is dense in \( \mathbb{P} \). But given \( q \in \mathbb{P} \), let \( w \subseteq \lambda \) be such that \( |w| < \mu \) and \( \text{dom} q \cup \{ i \} \subseteq w \). Now by Proposition 4.2, \( \text{rel} q \) is consistent; let \( p \) be the canonical extension of \( \text{rel} q \) over \( w \). Then \( p \in D_i \) and \( q \subseteq p \).

**Proposition 4.4.** If \( p, q \in \mathbb{P} \) coincide on \( \text{dom} p \cap \text{dom} q \), then they are compatible in \( \mathbb{P} \).

**Proof.** This follows from a number of claims. We write \( p \vdash \cdots \) instead of \( \text{rel} p \vdash \cdots \) and we say that a relation, e.g. \( x_i \geq x_j \), is in \( p \) if \( p(i, j) = \geq \) etc.

**Claim 1.** If \( p \vdash x_i \geq x_j \) where \( i < j \), then \( i, j \in \text{dom} p \) and the relation \( x_i \geq x_j \) is in \( p \).

Similarly for \( q \) and for relations of the form \( x_i \perp x_j \).—The claim holds because \( \text{rel} p \), for \( p \in \mathbb{P} \), is closed under derivations.

By Proposition 4.2 we are left with showing that the set

\[
\mathcal{R} = \text{rel} p \cup \text{rel} q
\]

is consistent.

**Claim 2.** If \( \mathcal{R} \vdash x_i \geq x_j \), then \( p \vdash x_i \geq x_j \) or \( q \vdash x_i \geq x_j \) or, for some \( \alpha \), \( p \vdash x_i \geq x_\alpha \) and \( q \vdash x_\alpha \geq x_j \) or, for some \( \alpha \), \( q \vdash x_i \geq x_\alpha \) and \( p \vdash x_\alpha \geq x_j \).

**Claim 3.** If \( \mathcal{R} \vdash x_i \perp x_j \), then \( p \vdash x_i \perp x_j \) or \( q \vdash x_i \perp x_j \) or, for some \( \alpha \), \( p \vdash x_i \perp x_\alpha \) and \( q \vdash x_\alpha \geq x_j \) or, for some \( \alpha \), \( q \vdash x_i \perp x_\alpha \) and \( p \vdash x_\alpha \geq x_j \) (or similarly with \( i \) interchanged with \( j \)).

**Claim 4.** If \( \mathcal{R} \vdash x_i \geq x_j \) and \( i, j \in \text{dom} p \), then \( p \vdash x_i \geq x_j \). Similarly for \( q \) and for relations of the form \( x_i \perp x_j \).

The proofs are easy but require consideration of a number of cases. We give two typical examples. In Claim 3, assume e.g. that \( p \vdash x_\gamma \perp x_\delta \), \( q \vdash x_\gamma \geq x_i \) and \( q \vdash x_\delta \geq x_j \). Then \( \gamma \) and \( \delta \) are in \( \text{dom} p \cap \text{dom} q \), \( x_\gamma \perp x_\delta \) is (by Claim 1) in \( p \), hence in \( q \), because \( p \) and \( q \) coincide on \( \text{dom} p \cap \text{dom} q \), and \( q \vdash x_j \perp x_j \).

Similarly in Claim 4, assume e.g. that \( p \vdash x_i \geq x_\alpha \) and \( q \vdash x_\alpha \geq x_j \) where \( i, j \in \text{dom} p \).

Since \( \alpha \) is in \( \text{dom} p \cap \text{dom} q \), it follows that \( x_\alpha \geq x_j \) is in \( p \), hence \( p \vdash x_i \geq x_j \).

**Claim 5.** \( \mathcal{R} \) is consistent.—For otherwise by Claim 3, we may assume that, e.g., for some \( \alpha \), \( p \vdash x_k \perp x_\alpha \) and \( q \vdash x_\alpha \geq x_k \). Then \( k \) and \( \alpha \) are in \( \text{dom} p \cap \text{dom} q \), \( x_\alpha \geq x_k \) is in \( q \) and \( x_k \perp x_k \) is in \( p \), a contradiction.
PROPOSITION 4.5. \( P \) satisfies the \( \mu^+ \)-chain condition.

PROOF. If \( X \) is a subset of \( P \) of size \( \mu \), then by \( \mu < \mu = \mu \) and the \( \Delta \)-lemma there are \( p \) and \( q \) in \( X \) coinciding on \( \text{dom } p \cap \text{dom } q \). So we are finished by Proposition 4.4.

5. Computing \( \pi(A(P)) \). In this and the following section, let \( G \) be a \( P \)-generic filter over \( V \) and \( P \) the resulting generic valuation function (see Remark 4.3). Write \( A \) for \( A(P) \). We prove condition (e) of Section 2 for \( A \), thus being able to compute \( \pi(A) \) in \( V[G] \).

PROPOSITION 5.1. The following holds in \( V[G] \). Let \( \alpha < \lambda \) be an ordinal, \( \alpha \subseteq \alpha \) finite, \( e: \alpha \rightarrow \{0, 1\} \) and

\[
y = \prod_{e(i)=1} x_i \cdot \prod_{e(i)=0} -x_i > 0 \quad (\text{in } A).
\]

Then there is \( i^* \in [\alpha, \alpha + \mu) \) (ordinal addition) such that \( x_{i'} \leq y \). In particular, the set \( \{x_i : i^* \in [\alpha, \alpha + \mu)\} \) is dense for the subalgebra of \( A \) generated by \( \{x_i : i < \alpha\} \).

PROOF. We do not distinguish notationally between elements of \( V[G] \) and their \( P \)-names; in particular since \( a \) and \( e \), being finite, are in the ground model. Pick \( p \in G \) such that

\[
p \forces y = \prod_{e(i)=1} x_i \cdot \prod_{e(i)=0} -x_i > 0;
\]

it suffices to prove that

\[
D = \{t \in P : t \leq p, \text{ and } t \forces x_{i'} < y \text{ for some } i^* \in [\alpha, \alpha + \mu)\}
\]

is dense below \( p \). To this end, let \( q \leq p \) be arbitrary. By Remark 4.3, we can fix \( r \leq q \) such that \( a \subseteq \text{dom } r \). Then fix \( i^* \in [\alpha, \alpha + \mu) \setminus \text{dom } r \); this is possible by \( |\text{dom } r| < \mu \).

We define a function \( s \) with domain \( a \cup \{i^*\} \) by putting

\[
s|a|^2 = r|a|^2
\]

\[
s(i, i^*) = \begin{cases} 
\geq & \text{if } i \in a \text{ and } e(i) = 1 \\
\perp & \text{if } i \in a \text{ and } e(i) = 0.
\end{cases}
\]

CLAIM. \( s \in P \), i.e. \( s \) is a valuation function.

Let us check just one case. Note that, for \( u \in P \), \( u(i, j) = \geq \) implies that \( u \vdash x_i \geq x_j \) and similarly for \( \perp \) instead of \( \geq \) since for any generic \( H \subseteq P \) containing \( u, u \subseteq P_H \) and thus \( x_i \geq x_j \) will hold in \( A(P_H) \). Assume e.g. \( i < j \in a \), \( s(i, j) = \geq \) and \( s(j, i^*) = \geq \); we have to show that \( s(i, i^*) = \geq \). The assumptions say that \( r(i, j) = \geq \) (since \( i, j \in a \) and \( e(j) = 1 \)); we have to show that \( e(i) = 1 \). But if \( e(i) = 0 \), then: \( p \vdash 0 \neq -x_i \cdot x_j \) (because \( p \vdash 0 < y \leq -x_i \cdot x_j \), \( r \vdash 0 \neq -x_i \cdot x_j \) (since \( r \leq p \)), \( r \vdash x_i \geq x_j \) (by the above assumption), \( r \vdash -x_i \cdot x_j = 0 \), a contradiction. Now \( r \) and \( s \) coincide on \( a = \text{dom } r \cap \text{dom } s \), so by Proposition 4.4, pick \( t \in P \) extending both \( r \) and \( s \). Then \( t \leq q \) and \( s \vdash x_{i'} \leq y \), by the very definition of \( s \) above, so \( t \in D \).
Corollary 5.2. \( \pi(A) = \lambda \) (in \( V[G] \)).

Proof. This follows from Remark 2.1 and the sketch of the model \( \mathcal{M} = (A, \ldots) \models T \) following it, plus Proposition 5.1. Let us remark that Theorem 6.1 gives another proof, since \( dA = \lambda, dA \leq \piA \) holds for all Boolean algebras and \( \piA \leq |A| = \lambda \).

Example 5.3. Our construction of \( A = A(P) \) and Proposition 5.1 above give a counterexample to the assertion in Theorem 4.1 of [Mo], in \( V[G] \). For this, let \( A_\alpha \) be the subalgebra of \( A \) generated by \( \{x_i : i < \alpha\} \); so if \( \alpha \in I = \{\alpha < \lambda : cf \alpha = \mu\} \), then by Remark 2.1 and Proposition 5.1 above, we have \( \piA_\alpha = \mu \). Moreover \( A = \bigcup_{\alpha \in I} A_\alpha \) and \( \piA = \lambda \) where \( \lambda \) can be larger than \( \mu^+ \).—In fact, the argument given in [Mo, 4.1] depends on the assumption that the chain \( (A_\alpha)_{\alpha \in I} \) is continuous which is not the case here.

6. Computing \( d(A(P)) \). Our single theorem here is the following.

Theorem 6.1. In \( V[G] \), \( A = A(P) \) satisfies \( d(A) = \lambda \).

Proof. Otherwise, the cardinal \( \sigma = d(A)^{V[G]} \) is less than \( \lambda \). There are a \( P \)-name \( u \) and a condition \( p \in P \) (in fact, \( p \in G \)) such that

\[
p \not\vdash u \text{ is a sequence } (u_v)_{v < \sigma}, \text{ each } u_v \text{ is an ultrafilter of } A, \text{ and } A \setminus \{0\} = \bigcup_{v < \sigma} u_v.
\]

For \( \alpha < \lambda \), fix \( p_\alpha \in P \) and \( \nu_\alpha < \sigma \) such that \( p_\alpha \leq p \) and

\[
p_\alpha \vdash x_\alpha \in u_{\nu_\alpha}
\]

(\( x_\alpha \) the (name of the) \( \alpha \)-th generator of \( A \)). In the next steps, we construct stationary subsets \( S_1 \supseteq S_2 \supseteq S_3 \supseteq S_4 \) of \( \lambda \).

Step 1. \( S_1 = \{\alpha \in \lambda : cf \alpha = \mu\} \) is stationary in \( \lambda \) because \( \mu < \lambda \) and \( \lambda \) is regular.

Step 2. Since \( \sigma < \lambda = cf \lambda \), there are \( \nu^* < \sigma \) and a stationary \( S_2 \subseteq S_1 \) such that \( \nu_\alpha = \nu^* \), for all \( \alpha \in S_2 \).

Step 3. Write \( w_\alpha = \text{dom} p_\alpha \), for \( \alpha \in \lambda \). We find \( \alpha^* \in \lambda \) and a stationary \( S_3 \subseteq S_2 \) such that for all \( \alpha \in S_3 \), \( \alpha^* < \alpha \) and \( w_\alpha \cap \alpha \subseteq \alpha^* \) hold. To this end, note that \( cf \alpha = \mu \) for \( \alpha \in S_3 \) and \( |w_\alpha \cap \alpha| < \mu \); so pick \( j_\alpha < \alpha \) satisfying \( w_\alpha \cap \alpha \subseteq j_\alpha \). Apply Fodor’s theorem to obtain \( S_3 \).

Step 4. We find a stationary set \( S_4 \subseteq S_3 \) such that \( \alpha < \beta \) in \( S_4 \) implies \( w_\alpha \subseteq \beta \). To do this, define by induction \( f : \lambda \rightarrow \lambda \) strictly increasing and continuous such that, for all \( \alpha, \bigcup_{\nu < \alpha} w_\nu \subseteq f(\alpha) \) and let \( S_4 = S_3 \cap C \) where \( C = \{\alpha : f(\alpha) = \alpha\} \) is closed unbounded. Then \( S_4 \) is stationary and, for \( \alpha < \beta \) in \( S_4 \), we have \( w_\alpha \subseteq f(\beta) = \beta \).

Now \( \mu^+ \leq \lambda \) and \( P \) satisfies the \( \mu^+ \)-chain condition. So we can find \( \alpha < \beta \) in \( S_4 \) such that \( p_\alpha \) and \( p_\beta \) are compatible in \( P \). Let \( r \) be the following set of relations:

\[
r = \text{rel}(p_\alpha) \cup \text{rel}(p_\beta) \cup \{x_\beta \perp x_\alpha\}
\]
DENSITIES OF ULTRAPRODUCTS OF BOOLEAN ALGEBRAS

Claim. \( r \) is consistent.

By the claim and Proposition 4.2, pick then \( q \in P \) such that \( r \subseteq \text{rel}(q) \). This \( q \) will force the following statements:

\[
x_\beta \perp x_\alpha
\]

\[
x_\alpha \in U_{p_\alpha} = U_{p^*} \quad \text{and} \quad x_\beta \in U_{p_\beta} = U_{p^*}
\]

\( U_{p^*} \) has the finite intersection property (being an ultrafilter), and this contradiction finishes the proof.

Proof of the Claim. Clearly no relation \( x_i \geq x_j \) where \( j < i \) can have a derivation from \( r \), since such a derivation would not use the relation \( x_\beta \perp x_\alpha \); hence \( x_i \geq x_j \) would be derivable from \( \text{rel}(p_\alpha) \cup \text{rel}(p_\beta) \), contradicting the compatibility of \( p_\alpha \) and \( p_\beta \).

Now assume \( r \vdash x_\beta \perp x_\alpha \). For some \( k \in \lambda \), a derivation witnessing this starts, without loss of generality, with the relation \( x_\beta \perp x_\alpha \). So in \( p_\alpha \cup p_\beta \) there are relations

\[
x_i \geq x_0, \ldots, x_{i-1} \geq x_i, \quad \text{where} \quad i_0 = \alpha, \ i_r = k
\]

\[
x_i \geq x_j, \ldots, x_{j-1} \geq x_j, \quad \text{where} \quad j_0 = \beta, \ j_k = k.
\]

Note that \( \alpha = i_0 < i_1 < \cdots < i_r = k \) (since if \( x_j \geq x_i \) is in \( p_\alpha \cup p_\beta \), then \( j < i \)); similarly, \( \beta = j_0 < j_1 < \cdots < j_k = k \).

We prove by induction on \( t \in \{0, \ldots, r\} \) that \( i_t \notin w_\beta = \text{dom} p_\beta \); for \( t = r \) this gives a contradiction because then \( k = i_r \notin w_\beta \), so \( k \in w_\alpha \) and \( k \geq \beta \), but \( w_\alpha \subseteq \beta \). First, \( i_0 \notin w_\beta \): otherwise, by Step 3, \( i_0 = \alpha \in w_\beta \cap \beta \subseteq \alpha^* \), contradicting \( \alpha^* < \alpha \) for \( \alpha \in S_3 \). If \( i_r \notin w_\beta \) but \( i_{r+1} \in w_\beta \), then the relation \( x_i \geq x_i \) must be in \( p_\alpha \). But then \( i_{r+1} \in w_\alpha \subseteq \beta \) and again \( i_{r+1} \in w_\beta \cap \beta \subseteq \alpha^* < \alpha \), a contradiction.

7. Proof of Theorem A.

7.1 Proof of Theorem A. Fix \( \kappa, \mu, \lambda_\alpha \) and \( D \) as given in the theorem; \( \mathcal{R} \) will be the iteration of two forcing notions. In the first step, collapse \( \mu < \kappa \) to \( \mu \) with \( Q = Fn(\mu, \mu < \kappa, < \mu) \) in Kunen’s notation ([Ku]). This forcing is \( \mu \)-closed and satisfies the \( (\mu < \kappa) + \text{-chain condition} \); in the resulting generic model \( V[H] \), \( \mu < \kappa \) holds. The notions of ultrafilters on \( \kappa \), the cartesian product \( U_{\alpha < \kappa} A_\alpha \) are absolute for this forcing by \( \mu \)-closedness of \( \kappa \) and \( \kappa < \mu \) thus all assumptions of the theorem continue to hold in \( V[H] \).

Working now in \( V[H] \), let for \( \alpha \in \kappa \), \( P_\alpha \) be the forcing notion \( P(\lambda_\alpha, \mu) \) defined in Section 4; let \( P \) be the full cartesian product \( P = \prod_{\alpha < \kappa} P_\alpha \) with the coordinate-wise partial order. For \( G \subseteq P \) generic over \( V \), \( G_\alpha = pr_{\alpha}^{-1}[G] \) is \( P_\alpha \)-generic over \( V[H] \) (\( pr_\alpha \) the \( \alpha \)-th projection). \( P \) is clearly \( \mu \)-closed, moreover, as in the proof of Proposition 4.5, the \( \Delta \)-lemma implies that \( P \) satisfies the \( \mu^* \)-chain condition since \( \mu < \mu < \mu \). Thus the assumptions of the theorem, as well as \( \mu < \mu = \mu \), continue to hold in \( V[H][G] \).

In \( V[H][G] \), \( P_\alpha = \bigcup G_\alpha : [\lambda_\alpha]^2 \rightarrow 3 \) is a generic valuation function. Let \( A_\alpha = A(P_\alpha) \) be its associated Boolean algebra; by Sections 5 and 6, \( \pi(A_\alpha) = d(A_\alpha) = \lambda_\alpha \). In the standard model \( \mathfrak{M}_\alpha = (A_\alpha, \ldots) \) of \( T \) (see Section 2), the predicate \( L \) is interpreted by \( \lambda_\alpha \) and the equivalence classes of \( \sim_L \) have size \( \mu \). So in the ultrapower \( \mathfrak{K} = \prod_{\alpha < \kappa} \mathfrak{M}_\alpha / D \), \( L \) is

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interpreted by $\prod_{\alpha < \kappa} \lambda_\alpha / D$ and the equivalence classes of $\sim_L$ have size $\leq |\mu^\kappa / D| = \mu$ (by $\kappa < \mu$ and $\mu^{< \aleph_\mu} = \mu$). Now Remark 2.1 says that $\pi(A) = \text{cf} \prod_{\alpha < \kappa} \lambda_\alpha / D$ and hence $d(A) \leq \pi(A) = \text{cf}(\prod_{\alpha < \kappa} \lambda_\alpha / D) < |\prod_{\alpha < \kappa} \lambda_\alpha / D| = |\prod_{\alpha < \kappa} \pi(A_\alpha) / D| = |\prod_{\alpha < \kappa} d(A_\alpha) / D|$. \hfill \qed

We can prove a little more:

**Remark 7.2.** In $V[H][G]$, let $A = \prod_{\alpha < \kappa} A_\alpha / D$ be the algebra constructed in subsection 7.1 and let $\lambda = \text{cf} \prod_{\alpha < \kappa} \lambda_\alpha / D$. Then $d(A) = \lambda$.

**Proof.** Our proof will closely follow that of Theorem 6.1.

Fix a sequence $(f_\gamma)_{\gamma \in \Lambda}$ in $\prod_{\alpha < \kappa} \lambda_\alpha$ such that $(f_\delta / D)_{\gamma \in \Lambda}$ is strictly increasing and cofinal in the ultraproduct $\prod_{\alpha < \kappa} \lambda_\alpha / D$. By [Sh, Chapter II], the set

$$S = \{ \gamma \in \Lambda : \text{cf} \gamma = \mu^+, \text{and there is } g \in \prod_{\alpha < \kappa} \lambda_\alpha \text{ such that } g / D \text{ is the least upper bound of } \{ f_\delta / D : \delta < \gamma \} \text{ and cf } g(\alpha) = \mu^+ \text{ for all } \alpha \in \kappa \}$$

is stationary; so we may assume that, for $\gamma \in S$, $f_\gamma$ satisfies the requirements for $g$ above.

Now note that, in $V[H][G]$, $d(A) \leq \pi(A) = \lambda$ as shown in the proof of subsection 7.1; so assume for contradiction that $d(A) < \lambda$. Thus, in $V[H][G]$ there are a $\mathbb{P}$-name $u, \sigma < \lambda$ and $p \in \mathbb{P}$ such that

$$p \forces u = (u_\nu)_{\nu < \sigma}$$

is a sequence of ultrafilters of $A$ covering $A \setminus \{0\}$.

For $\gamma \in S$, fix $p_\gamma \geq p$ and $\nu_\gamma \in \sigma$ such that

$$p_\gamma \forces y_\gamma / D \in u_{\nu_\gamma},$$

where $y_\gamma$ is a (a $\mathbb{P}$-name for) $(x_{f_\gamma(\alpha)})_{\alpha < \kappa} / D$ and $x_\gamma$ is a (a $\mathbb{P}$-name for) the $i$-th canonical generator of $A_\alpha$, for $i < \lambda_\alpha$. There is a stationary subset $S_1$ of $S$ such that $\nu_\gamma$ is some fixed $\nu^\ast$, for $\gamma \in S_1$ (because $\nu_\gamma < \sigma < \lambda$ and $\lambda$ is regular). As in Step 3 in the proof of Theorem 6.1, there exists $\beta^\ast < \gamma$ such that, for $D$-almost all $\alpha$, $\text{dom} p_{\gamma}(\alpha) \cap f_\gamma(\alpha) \subseteq f_{\beta^\ast}(\alpha)$.

Without loss of generality (i.e. by passing to a stationary subset), $\beta^\ast$ is some fixed $\beta^\ast$, for all $\gamma \in S_1$. Now $K_{\gamma} = \{ \alpha \in \kappa : \text{dom} p_{\gamma}(\alpha) \cap f_\gamma(\alpha) \subseteq f_{\beta^\ast}(\alpha) \} \in D$, for $\gamma \in S_1$; since $2^\kappa < \lambda$, we may assume without loss of generality that $K_{\gamma}$ is some fixed $\vec{K}_{\gamma} \in D$, for $\gamma \in S_1$.

As in Step 4 of the proof of Theorem 6.1, we may assume that $\gamma < \delta$ in $S_1$ implies that

$$K_{\gamma\delta} = \{ \alpha \in \kappa : \text{dom} p_{\gamma}(\alpha) \subseteq f_{\delta}(\alpha) \} \in D$$

because $(f_{\delta} / D)_{\delta < \kappa}$ is cofinal in $\prod_{\alpha < \kappa} \lambda_\alpha / D$.

Now $\mathbb{P}$ satisfies the $\mu^+$-chain condition and $S_1$ has size $\lambda \geq \mu^+$; so fix $\gamma < \delta$ in $S_1$ such that $p_\gamma$ and $p_\delta$ are compatible in $\mathbb{P} = \prod_{\alpha < \kappa} \mathbb{P}_\alpha$, i.e. $p_\gamma(\alpha)$ and $p_\delta(\alpha)$ are compatible in $\mathbb{P}_\alpha$, for all $\alpha \in \kappa$.

We conclude as in Theorem 6.1: for all $\alpha \in K^* \cap K_{\gamma\delta}$, the set

$$r_\alpha = \text{rel} p_{\gamma}(\alpha) \cup \text{rel} p_{\delta}(\alpha) \cup \{ x_{f_{\delta}(\alpha)} \downarrow x_{f_{\gamma}(\alpha)} \}$$

is consistent; so pick $q_{\alpha} \in \mathbb{P}_\alpha$ satisfying $r_\alpha \subseteq \text{rel} q_{\alpha}$. Choose $q \in \mathbb{P}$ having $\alpha$-th coordinate $q_{\alpha}$, for $\alpha \in K^* \cap K_{\gamma\delta}$; then $q$ forces that: $y_\gamma / D \downarrow y_\delta / D, y_\gamma / D \in u_{\nu_\gamma} = u_{\nu^\ast}$ and $y_\delta / D \in u_{\nu_\delta} = u_{\nu^\ast}$, $u_{\nu^\ast}$ is an ultrafilter. This gives a contradiction. \hfill \qed
8. Proof of Theorem B. To abbreviate the main body of the proof, we state in advance two easy lemmas. The proofs are left to the reader.

**Lemma 8.1.** Assume \( h: C \to D \) is a homomorphism of Boolean algebras, \( \{c_n : n \in \omega \} \) is a partition of unity in \( C \), and also \( \{h(c_n) : n \in \omega \} \) is a partition of unity in \( D \). Then, if \( x_n \in C \) are such that \( \sum_{n \in \omega} x_n \cdot c_n \) exists, we have \( h(\sum_{n \in \omega} x_n \cdot c_n) = \sum_{n \in \omega} h(x_n \cdot c_n) \).

Given a subalgebra \( C \) of \( D \) and \( x \in D \), let \( I_C(x) = \{ c \in C : c \cdot x = 0 \} \), an ideal of \( C \). Call \( x, y \in D \) equivalent over \( C \) (and write \( x \sim_C y \)) if both \( I_C(x) = I_C(y) \) and \( I_C(-x) = I_C(-y) \) hold, i.e. if \( x \) and \( y \) realize the same quantifier-free type over \( C \).

**Lemma 8.2.** If \( x, y \in D \) are equivalent over \( C \), then there is no \( c \in C \setminus \{0\} \) disjoint from \( x + -y \).

We break up the proof of Theorem B into eight preparatory steps in which certain objects are constructed or notation is fixed, plus four claims. Let \( C \leq D \) denote that \( C \) is a subalgebra of \( D \); \( \hat{A} \) is the completion of \( A \).

**Step 1.** Take \( \mu \) as assumed in the theorem, fix a set \( U \) of cardinality \( \mu \), and let \( A = Fr U \), the free Boolean algebra over \( U \). Since \( |A| = \mu^+ \geq 2^\mu \), we have \( |\hat{A}| = \mu^+ \). The algebra \( B \) promised in the theorem will be a subalgebra of \( \hat{A} \), generated by \( A \) and pairwise distinct elements \( b_i \) of \( \hat{A} \), \( i < \mu^+ \). So \( |B| = \mu^+ \) and we know in advance that \( \mu^+ \leq |\text{End } B| \) and \( |\text{Id } B| \leq 2^\mu \).

**Step 2.** Fix an enumeration \( \{g_i : j < \mu^+ \} \) of all homomorphisms from \( A \) into \( \hat{A} \). This is possible since \( |A| = \mu \) and \( |\hat{A}| = \mu^+ = (2^\mu)^\mu \).

**Step 3.** Fix a sequence \( (\mu_n)_{n \in \omega} \) of cardinals such that \( \mu = \sup_{n \in \omega} \mu_n \) and \( 2^{\mu_n} < \mu_{n+1} \).

**Step 4.** For each ordinal \( i < \mu^+ \), fix subsets \( S_{in} \) of \( i \) such that \( i = \bigcup_{n \in \omega} S_{in}, S_{in} \subseteq S_{in+1} \) and \( |S_{in}| \leq \mu_n \). This is possible since \( |i| \leq \mu \).

**Step 5.** Fix a sequence \( (A_n)_{n \in \omega} \) of subalgebras of \( A \) such that \( A = \bigcup_{n \in \omega} A_n, A_n \subseteq A_{n+1} \) and \( |A_n| \leq \mu_n \).

**Step 6.** Define a tree \( T = \bigcup_{n \in \omega} T_n \) with \( n \)th level \( T_n = \mu_0 \times \cdots \times \mu_{n-1} \) where \( t \leq s \) in \( T \) means that \( s \) extends \( t \); so \( |T| = \mu \). The cartesian product \( F = \prod_{n \in \omega} \mu_n \) has size \( \mu^+ = \mu^+ \); fix a one-one enumeration \( \{f_i : i < \mu^+ \} \) of \( F \).

Split \( U \subseteq A = Fr U \) (cf. Step 1) into two disjoint subsets \( X \) and \( Z \) such that \( |X| = |Z| = \mu \); then split both \( X \) and \( Z \) into pairwise disjoint subsets \( X_t, Z_t \in T, \) \( t \in T \), such that \( |X_t| = \mu \) and \( Z_t \neq \emptyset \).

**Step 7.** Here we define, for \( i < \mu^+ \), the elements \( b_i \) of \( \hat{A} \) and then let \( B \) be the subalgebra of \( \hat{A} \) generated by \( A \cup \{b_i : i < \mu^+ \} \). \( b_i \) is constructed out of certain elements \( x_{in}, y_{in}, z_{in}, n \in \omega, \) of \( U \) by putting
\[
\begin{align*}
s_{in} &= x_{in} + y_{in} \\
d_{in} &= s_{in} - \prod_{m < n} s_{im} \\
b_i &= \sum_{n \in \omega} z_{in} \cdot d_{in}
\end{align*}
\]
To choose the \( x_{in}, y_{in}, z_{in} \), fix \( i < \mu^+ \) and \( n \in \omega \); thus
\[
i = f_i \upharpoonright n
\]
is an element of the tree \( T \). Pick \( z_{in} \in Z_i \) (see Step 6) arbitrarily. \( x_{in} \) and \( y_{in} \) are chosen much more carefully: we want them to be distinct elements of \( X_i \) satisfying
\[
(*) \quad \text{for all } j \in S_{in}, \; g_j(x_{in}) \sim_{A_n} g_j(y_{in})
\]
(cf. Steps 4, 2, 5, and the definition of \( \sim_{A_n} \) before Lemma 8.2). This is possible since:
\[
|A_n| \leq \mu_n
\]
there are at most \( 2^\mu_n \) equivalence classes in \( \tilde{A} \), with respect to \( \sim_{A_n} \), since there are at most \( 2^\mu_n \) ideals in \( A_{in} \).

\[
|S_{in}| \leq n
\]
the set \( \{(g_j(x) / \sim_{A_n})_{j \in S_{in}} : x \in X_i \} \) has size at most \( 2^\mu_n \).

\[
2^{\mu_n} < \mu = |X_i|.
\]

**Step 8 (Remark).** For \( b \in A \), let us denote by \( \text{supp } b \) (the support of \( b \)) the smallest subset of \( U \) generating \( b \). Now for \( i < \mu^+ \), the supports \( \{\text{supp } s_{in} : n \in \omega \} \) are pairwise disjoint and thus \( \sum s_{in} = 1 \). It follows that the pairwise disjoint set \( \{d_{in} : n \in \omega \} \) is a partition of unity in \( \tilde{A} \) and all \( d_{in} \) are non-zero.—Similarly, for any homomorphism \( g : A \to \tilde{A} \), the sets \( \{g(d_{in}) : n \in \omega \} \) and \( \{g(s_{in}) : n \in \omega \} \) have the same upper bounds in \( A \) resp. \( \tilde{A} \).

**Claim 1.** If \( j < i < \mu^+ \), then \( \{g_j(d_{in}) : n \in \omega \} \) is a partition of unity (in \( \tilde{A} \)).—Otherwise, assume \( a \in A^+ \) and \( a \cdot g(s_{in}) = 0 \) for all \( n \) (cf. Step 8). Pick \( n \) so large that \( a \in A_n \) and \( j \in S_{in} \). Then \( a \cdot g(x_{in} + y_{in}) = 0 \), so \( a \cdot (g_j(x_{in}) + g_j(y_{in})) = 0 \), contradicting (*) and Lemma 8.2.

**Claim 2.** Let \( g \) be an endomorphism of \( B \), say \( g^1 A = g_j \) (see Step 2). Then for all \( i > j \), \( g(b_i) = \sum A g_j(z_{in}) \cdot g_j(d_{in}) \) holds. Hence \( g \) is uniquely determined by its action on \( A \cup \{b_i : i \leq j\} \).—This follows from Claim 1 and Lemma 8.1.

**Claim 3.** \( |\text{End } B| \leq \mu^+ \).—To completely describe some \( g \in \text{End } B \), we have only \( \mu^+ \) choices for \( g^1 A \) (Step 2) and, for \( j < \mu^+ \), at most \( (\mu^+)^{|i|} \leq 2^\mu = \mu^+ \) choices for \( (g(b_i))_{i \in J} \), so we are finished by Claim 2.

**Claim 4.** The generators \( \{b_i : i < \mu^+ \} \) are ideal-independent; hence \( |\text{Id } B| = 2^{\mu^+} \).—We prove that, for \( i \in \mu^+ \) and \( J \) a finite subset of \( \mu^+ \setminus \{i\} \), \( b_i \not\in \sum_{j \in J} b_j \). (It follows that the ideals \( I_k \) generated by \( \{b_i : i \in K\} \) for \( K \subseteq \mu^+ \), are all distinct, so \( B \) has \( 2^{\mu^+} \) ideals.)

The argument is elementary but a little tedious and we give it in some detail. Assume for contradiction that \( b_i \leq \sum_{j \in J} b_j \).

For arbitrary \( n \in \omega \), we have the following situation. \( d_{in} \) is non-zero and for \( j \in J \), \( \{d_{jm} : m \in \omega \} \) is a partition of unity; hence there are elements \( m(j) \in \omega \), for \( j \in J \), such that \( p = d_{in} \cdot \prod_{j \in J} d_{jm(j)} \) is non-zero. Now \( b_i \cdot d_{in} \leq z_{in} \) and thus \( b_i \cdot p \leq z_{in} \); similarly \( b_j \cdot p \leq z_{jm(j)} \) holds for \( j \in J \). It follows from \( b_i \leq \sum_{j \in J} b_j \) that \( z_{in} \cdot p \leq b_i \cdot p \leq \sum_{j \in J} z_{jm(j)} \).
But supp \( p \subseteq X \) and \( z_{in} \), \( z_{jm(j)} \) are in \( Z \); hence \( z_{in} \leq \sum_{j \in J} z_{jm(j)} \). So \( z_{in} = z_{jm(j)} \), for some \( j \in J \), since \( Z \subseteq U \) is independent. Since \( z_{in} \) was chosen in Step 7 from \( Z_t \), where \( t = f_i \upharpoonright n \), and \( (Z_t)_{t \in T} \) was a disjoint family, it follows that \( n = m(j) \) and \( f_i \upharpoonright n = f_j \upharpoonright n \).

We have thus shown that for every \( n \in \omega \), there is some \( j \in J \) satisfying \( f_i \upharpoonright n = f_j \upharpoonright n \). But then \( f_i \in \{ f_j : j \in J \} \) and \( i \in J \) (since the enumeration \( \{ f_i : i < \mu^+ \} \) in Step 6 was one-one), a contradiction.

\[ \square \]

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