We present a gauge theory formulation of a two-dimensional quantum smectic and its relatives, motivated by their realizations in correlated quantum matter. The description gives a unified treatment of phonons and topological defects, respectively encoded in a pair of coupled gauge fields and corresponding charges. The charges exhibit subdimensional constrained quantum dynamics and anomalously slow highly anisotropic diffusion of disclinations inside a smectic. This approach gives a transparent description of a multi-stage quantum melting transition of a two-dimensional commensurate crystal (through an incommensurate crystal – a supersolid) into a quantum smectic, that subsequently melts into a quantum nematic and isotropic superfluids, all in terms of a sequence of Higgs transitions.

**Introduction.** A smectic state of matter, characterized by a uniaxial spontaneous breaking of rotational and translational symmetries is ubiquitous in classical liquid crystals of highly anisotropic molecules (e.g., classic 5CB). Its quantum realizations range from quantum Hall smectics of a two-dimensional electron gas at half-filled high Landau levels to the putative Fulder-Ferrel-Larkin-Ovchinnikov (FFLO) paired superfluids in imbalanced degenerate atomic gases and spin-orbit coupled Bose condensates.

A smectic can emerge from anisotropic partial melting of a two-dimensional (2D) crystal, understood in terms of a Kosterlitz-Thouless (KT)-like, single-species dislocation unbinding transition. However, such a 2D smectic is unstable to thermal fluctuations, driven into a nematic fluid at any nonzero temperature. In contrast, at zero temperature a 2+1D quantum smectic is a stable state of matter, whose studies have been limited to the simplest harmonic-phonons description, typically neglecting quantum effects of topological defects and of elastic nonlinearities (as an exception, see e.g., Refs. 11 and 12).

**Results.** In this Letter, we study a time-reversal invariant quantum smectic and transitions into it via two complementary approaches, summarized in Fig. 2. First, by dualizing a quantum smectic, we derive a 2+1D coupled U(1) vector gauge theory, with the dual Hamiltonian density,

\begin{equation}
\hat{H}_{\text{sm}} = \frac{1}{2} \kappa E^2 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} K e^2 + \frac{1}{2} (\nabla \times (a - \hat{x} \times A))^2 - A \cdot J - a \cdot j.
\end{equation}

supplemented by the generalized Gauss law constraints,

\begin{equation}
\nabla \cdot E = p - e \cdot \hat{x}, \quad \nabla \cdot e = n.
\end{equation}

The description is in terms of two coupled U(1) vector gauge fields with electric fields E and e, and corresponding canonically conjugate vector potentials A and
a, capturing smectic’s gapless phonon degrees of freedom. \( \textbf{p} \) and \( \textbf{J} \) are \( \hat{x} \)-dipole charge and current densities, representing \( \hat{y} \)-dislocations, while \( n \) and \( j \) are fractonic charge and current densities, corresponding to disclinations. The generalized gauge invariance of \( \mu \) gives coupled continuity equations for the densities

\[
\partial_t p + \nabla \cdot \textbf{J} = -j \cdot \hat{x}, \\
\partial_t n + \nabla \cdot j = 0,
\]

where dipole conservation is violated by a nonzero charge current, \( j \cdot \hat{x} \) along smectic layers.

These equations transparently encode the restricted mobility of the charges \( n \), with their mobility and diffusion coefficient vanishing along the \( \hat{x} \)-directed smectic layers, i.e., \( j \cdot \hat{x} = 0 \) in the absence of dislocation dipoles. We thus predict that disclination charges exhibit “anomalies.” The generalized gauge invariance of \( \mu \) gives a coupled constraint equations

\[
\Gamma_k = Dk^2 + \gamma k^4,
\]

as was also recently explored in Refs.\textsuperscript{26,28}

Alternatively, we utilize the coupled \( U(1) \) vector gauge theory dual of a 2+1D quantum crystal\textsuperscript{25}, “soften” it into a generalized Abelian-Higgs model, and drive it through a Higgs transition for one of the dipole species (\( \hat{y} \)-dipoles for coordinate choice in \( \text{Fig.1} \)). With the latter a dual description of an anisotropic quantum melting associated with a condensation of a single type of dislocations, we thereby obtain a dual gauge theory of a quantum smectic. As required by consistency, the resulting model is in full agreement with the first approach of a direct duality of a quantum smectic, as summarized by the flow chart in Fig\textsuperscript{1}. The electrostatic limit of the dual gauge theory efficiently reproduces the model and results of a 2D classical smectic.\textsuperscript{17,18}

Because a smectic is a condensate of \( \hat{x} \)-dislocations, it is necessarily accompanied by a liquid of vacancies and interstitials. Thus, the dual gauge theory \( \mu \) is implicitly understood to be coupled (via axion-like \( \mathcal{E} - B \), \( B - E \) couplings) to a conventional \( U(1) \) gauge theory with fields \( \mathcal{E}, B \)- a dual to a liquid of vacancies and interstitials.\textsuperscript{21,22} Since vacancies and interstitials consist of a pair of oppositely-charged dislocations, a condensate of the latter necessarily drives a condensation of the former, requiring a smectic to be an incommensurate “super-smectic” (in contrast to a possibility of two distinct - Mott insulating “normal” (commensurate) and supersolid (incommensurate) crystals.\textsuperscript{21,23}

**Smectic duality.** We formulate a 2+1D quantum smectic in terms of a phonon (layer displacement) \( u = u(r)\hat{y} \) and the unit-normal (layer orientation) \( \hat{n}(r) = -\hat{x}\sin \theta + \hat{y}\cos \theta \equiv \hat{y} + \delta \hat{n} \) field operators, and the corresponding canonically conjugate angular and linear momentum fields, \( \pi(r) \) and \( L(r) \), with the Hamiltonian density,

\[
\mathcal{H}_{sm} = \frac{1}{2} \pi^2 + \frac{1}{2} L^2 + \frac{1}{2} \kappa (\nabla u + \delta \hat{n})^2 + \frac{1}{2} K (\nabla \hat{n})^2,
\]

where \( \kappa, K \) are elastic constants.

It is convenient to work with a phase-space path-integral formulation, corresponding to the Lagrangian density,

\[
\mathcal{L}_{sm} = \frac{1}{2} \partial_t u + L \partial_t \theta - \frac{1}{2} \kappa^2 - \frac{1}{2} L^2 + \frac{1}{2} \kappa \pi (\nabla u + \delta \hat{n})^2 + \frac{1}{2} K (\nabla \hat{n})^2,
\]

where we neglected \( \theta \) nonlinearities and took the \( x \)-axis to be along the smectic layers. Functionally integrating over the smooth, single-valued parts of the phonon \( u \) and orientation \( \theta \) fields, we obtain coupled constraint equations

\[
\partial_t \pi - \sigma \cdot \nabla \theta = 0, \\
\partial_t L - \nabla \cdot \tau = \dot{x} \cdot \sigma,
\]

Newton’s laws encoding the linear and angular momenta conservation.

As in electrodynamics, these equations (corresponding to the analog of Faraday law) are readily solved by expressing densities in terms of gauge fields,

\[
\tau = \dot{\hat{z}} \cdot (\nabla \times \mathbf{A}), \\
\sigma = \dot{\hat{x}} \cdot (\partial_t \mathbf{A} + \nabla A_0),
\]

\[
L = \dot{\hat{z}} \cdot (\nabla \times \mathbf{a} - \hat{x} \times \mathbf{A}), \\
\pi = \dot{\hat{x}} \cdot (\partial \mathbf{a} + \nabla a_0 - \hat{x} A_0).
\]

Reformulating the Lagrangian density \( \mathcal{L}_{sm} \) in terms of these Goldstone-mode encoding gauge fields, we obtain the Maxwell part of the smectic dual Lagrangian,

\[
\mathcal{L}_{M}^{sm} = \frac{1}{2} \left( \partial_t \mathbf{A} + \nabla A_0 \right)^2 - \frac{1}{2} (\nabla \times \mathbf{A})^2
\]

\[
+ \frac{1}{2K} \left( \partial_t \mathbf{a} + \nabla a_0 - A_0 \mathbf{a} \right)^2 - \frac{1}{2} (\nabla \times \mathbf{a} - \hat{x} \times \mathbf{A})^2,
\]

displaying a nontrivial “minimal” coupling between the translational and orientational gauge fields, which encodes semi-direct product of spatial translations and rotations. The Lagrangian exhibits a generalized gauge invariance under transformations,

\[
(A_0, \mathbf{A}) \to A_\mu \rightarrow (A_0 - \partial_t \phi, A + \nabla \phi), \quad (10a)
\]

\[
(a_0, \mathbf{a}) \to a_\mu \rightarrow (a_0 - \partial_t \chi, a + \nabla \chi - \hat{x} \phi). \quad (10b)
\]

The six gauge field degrees of freedom \( A_\mu, a_\mu \) reduce to two physical ones (corresponding to coupled phonon \( u \) and orientation \( \theta \) Goldstone modes) after gauge fixing \( \phi, \chi \) and implementing two Gauss law constraints \( \mu \).

To include dislocations and disclinations we allow for the nonsingle-valued component of \( u \) and \( \theta \), respectively defined by

\[
p = \dot{\hat{z}} \cdot \nabla \times \nabla u, \\
\mathbf{J} = \dot{\hat{z}} \times (\nabla \partial_t u - \partial_t \nabla u),
\]

\[
n = \dot{\hat{z}} \cdot \nabla \times \nabla \theta, \\
\mathbf{j} = \dot{\hat{z}} \times (\nabla \partial_t \theta - \partial_t \nabla \theta).
\]

This together with \( \mathcal{L}_{M}^{sm}(A_\mu, a_\mu) \) gives the dual Lagrangian density for the quantum smectic,

\[
\mathcal{\tilde{L}}_{sm} = \mathcal{L}_{M}^{sm}(A_\mu, a_\mu) + A \cdot \mathbf{J} + A_0 p + a \cdot \mathbf{j} - a_0 n.
\]

(12)
corresponding to the Hamiltonian \([1]\) and Gauss laws \([2]\). Requiring this dual Lagrangian to be gauge invariant under \([10]\), immediately leads to coupled continuity equations \([3]\) for the densities. The dipole (dislocation) continuity equation is violated by a nonzero charge (disclination) current \(j_x\) along smectic layers. Thus, in the absence of gapped dipoles \(p\) (\(y\)-dislocations), we find \(j_x = 0\), i.e., a motion of isolated lineon charges (disclinations) is restricted to be transverse to the smectic layers, as moving along the layers requires \(x\)-dipoles (\(y\)-dislocations), a nonlocal operation of an insertion of a half-layer of atoms, illustrated in Fig\(\text{[3]}\) that are gapped inside the smectic ground state.

Subdiffusive dynamics. At finite densities coupled continuity equations \([3]\) also lead to anomalous subdiffusive dynamics of disclination charges. To see this, we note that a smectic \(x\)-dipole (a dislocation \(b\), i.e., a dipole \(p\)) in a quantum smectic, forbidding their separation, that corresponds to a nonlocal process of adding a smectic half-layer per lattice constant of charge separation.

Let \(\dot{\psi}_x = (p, \mathbf{J})\) to obtain
\[
\dot{\mathcal{L}}_{\text{sm}} = \frac{g_0}{2} (\partial t \phi_x - A_0)^2 + g \cos(\nabla \phi_x - \mathbf{A}) + \mathcal{L}_{\text{nem}}^\text{M}(A_\mu, a_\mu),
\]
\[
= \frac{J}{2} (\partial_x - iA_\mu)\psi_x)^2 - V(|\psi_x|) + \mathcal{L}_{\text{nem}}^\text{M}(A_\mu, a_\mu) \quad (14)
\]
The Lagrangian of a relativistic form, encoding dipole neutrality of a stress-free smectic. Above, \(\phi_x\) is the phase of the \(x\)-dipole (\(y\)-dislocation) field \(\psi_x\) with a Fourier transform \(\hat{U}(\mathbf{q}) = \kappa K_q^2/(\kappa q_x^2 + K_q^2)\). The \(\psi_x \neq 0\) ("superconducting" dipole condensate) Higgs phase, corresponds to a condensed plasma of unbound dislocations, that gaps out the translational gauge field \(A_\mu\), which can therefore be safely integrated out. This reduces the model to a conventional Maxwell form for the rotational gauge field \(A_\mu\), with \(\mathcal{L}_{\text{nem}}^\text{M}(A_\mu) \approx \mathcal{L}_{\text{nem}}^\text{M}(A_\mu, a_\mu) = \frac{1}{2} K^{-1} e^2 - \frac{1}{2} (\nabla \times a)^2\), that is a dual to the quantum \(xy\)-model of the nematic, \(\mathcal{L}_{\text{nem}} = \frac{1}{2} (\partial_\mu \theta)^2 - \frac{1}{2} K(\nabla \theta)^2\). Fluctuation corrections lead to an anisotropic stiffness and subdominant higher order gradients. As with the conventional U(1) Higgs (normal-superconductor) transition, mean-field approximation breaks down for \(d+1 \leq 4\), and may be driven first-order by translational gauge-field, \(A_\mu\) fluctuations. We leave the analysis of the resulting non-mean-field criticality of the quantum smectic-nematic transition for a future study.
where \( \psi_{k=x,y} \) correspond to \( \hat{x} \) - and \( \hat{y} \) -oriented dipoles (\( \hat{y} \)- and \( \hat{x} \)-dislocations), \( A^k_x, a_y \) gauge fields capture the \( k = x, y \) phonons and bond orientational Goldstone modes, \( V(\{\psi_k\}) \) the Landau potential, and

\[
\mathcal{L}^\text{ct}_M = \frac{1}{2} \kappa^{-1}(\partial_k A^k + \nabla A^\tau_0)^2 - \frac{1}{2} (\nabla \times A^k)^2
\]

\[
+ \frac{1}{2} K^{-1}(\partial_k a_k + \partial_k a_0 - A^\tau_0)^2 - \frac{1}{2} (\nabla \times a + A^\tau_0)^2
\]

is the Maxwell part of the crystal dual Lagrangian, with \( A_\mu = e_\mu A^k \). The crystal-smectic partial melting transition corresponds to a condensation of one of the dipole charges, that according to Figs.\[12\] we take to be \( \hat{y} \)-dipoles, \( \psi_y \neq 0 \). This Higgs transition gaps out \( A^\tau_0 \), which then can be safely integrated out. To lowest order it corresponds to \( A^\tau_0 \approx 0 \), reducing the crystal’s Maxwell Lagrangian to that of a smectic [14], with

\[
\mathcal{L}^\text{sm}_M(A^\mu_\tau, a_\mu) \approx \mathcal{L}^\text{ct}_M(A^\mu_\tau, A^\mu_\mu \approx 0, a_\mu).
\]

**Vacancies and interstitials.** So far, we have neglected vacancies and interstitials. As discussed in detail in Refs.\[21\]\[23\] at zero temperature two qualitatively distinct - commensurate and incommensurate quantum crystals can appear, respectively distinguished by Mott insulating and superfluid vacancies and interstitials. In the former the \( U(1) \) symmetry-enriching constraint imposes a glide-only motion of dislocation, that is broken in the latter, where dipole dislocation motion is unconstrained. In contrast, we observe that since a smectic is a condensate of \( \hat{x} \)-dislocations (created by \( b^k_\mu \)) and vacancies and interstitials (created by \( a^k_\mu \)) consist of pairs of oppositely-charged dislocations (i.e., are disclination charge quadrupoles), the allowed coupling \( \tilde{a}^k_\mu \tilde{b}^k_\mu \) drives a condensation of vacancies and interstitials. Thus, a smectic is necessarily an incommensurate “super-smectic”, and its dual gauge theory [1] is implicitly understood to be coupled (via axion-like \( \mathcal{E} \sim B, B \sim E \) couplings) to a conventional \( U(1) \) gauge theory with fields \( \mathcal{E}, B \) - a dual to a liquid of vacancies and interstitials.\[21\] [22]

**External perturbations.** Repeating the analysis in the presence of an incommensurate substrate, we find that it reduces the orientational sector (\( e, a \)) from a compact \( U(1) \) to \( Z_s \) gauge theory (coupled to a noncompact \( U(1) \) translational sector \( (E, A) \)), compactifying the corresponding flux energy

\[
\frac{1}{2} (\nabla \times a - \hat{x} \times A)^2 \rightarrow -\cos [2\pi(\nabla \times a - \hat{x} \times A)s]/s,
\]

where \( s \) is an integer characterizing orientational commensurability. For \( s = 1 \), the orientational sector is confined, reducing the model to a conventional noncompact \( U(1) \) gauge theory of the translational sector \( (E, A) \). Latter can also get confined by a translationally commensurate substrate, corresponding to the gapping out of the smectic phonon.

A smectic can also be subjected to an external stress. Unlike a crystal, there is no smectic’s resistance to a static shear strain, \( \partial_x u \), as it corresponds to a reorientation of the smectic order that spontaneously breaks a rotationally invariant fluid. A compressive stress, \( \sigma_y \), couples to the strain \( \partial_y u \), that in a gauge theory induces an electric field \( E_x \) across the dual “dielectric”. For \( \sigma_y \) above a critical value, a dielectric breakdown takes place, corresponding to a proliferation of \( \hat{y} \)-dislocations under a super-critical compressive stress.

**Summary.** Utilizing quantum elasticity duality\[20\] [25], we presented a formulation of a quantum smectic in terms of a fractonic vector gauge theory. It gives a unified description of phonons and topological defects and transparently captures the subdimensional constrained dynamics of the disclination charges, predicting their subdiffusive hydrodynamics. The dual coupled vector gauge theory exhibits a unifying global phase diagram of a quantum crystal (that comes in distinct commensurate and incommensurate - supersolid forms), smectic, nematic, and isotropic superfluid phases and describes quantum phase transitions between them in terms of various Higgs transition. The electrostatic limit of the theory gives an efficient description and reproduces results of the corresponding classical phase transitions.\[17\] [18] [22] We expect that this dual description will be of value for more detailed exploration of these phases and corresponding phase transitions.

**Note Added:** After this work was completed we became aware of an interesting but somewhat orthogonal work by Gromov, and by Gromov and Moroz, where quantum smectic also appears.\[32\] [33]

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