Erdős-Gallai-type results for the rainbow disconnection number of graphs*

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Abstract

Let $G$ be a nontrivial connected and edge-colored graph. An edge-cut $R$ of $G$ is called a rainbow cut if no two edges of it are colored with a same color. An edge-colored graph $G$ is called rainbow disconnected if for every two distinct vertices $u$ and $v$ of $G$, there exists a $u-v$ rainbow cut separating them. For a connected graph $G$, the rainbow disconnection number of $G$, denoted by $rd(G)$, is defined as the smallest number of colors that are needed in order to make $G$ rainbow disconnected. In this paper, we will study the Erdős-Gallai-type results for $rd(G)$, and completely solve them.

Keywords: rainbow cut, rainbow disconnection coloring (number), Erdős-Gallai-type result

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a nontrivial connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ and $N_G[v]$ denote the open neighbour of $v$ and the closed neighbour of $v$ in $G$, respectively. For any notation or terminology not defined here, we follow those used in [2].

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Throughout this paper, we use $K_n$ to denote a complete graph of order $n$. A $k$-factor of $G$ is a $k$-regular spanning subgraph of $G$, and $G$ is $k$-factorable if there are edge-disjoint $k$-factors $H_1, H_2, \ldots, H_n$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_n$. A subset $M$ of $E$ is called a matching of $G$ if any two edges of $M$ do not share a common vertex of $G$.

Let $G$ be a graph with an edge-coloring $c : E(G) \to [k] = \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. When adjacent edges of $G$ receive different colors under $c$, the edge-coloring $c$ is called proper. The chromatic index of $G$, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper coloring of $G$. By a famous theorem of Vizing [14], one has

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for every nonempty graph $G$.

A path is called rainbow if no two edges of it are colored the same. An edge-colored graph $G$ is called rainbow connected if every two distinct vertices of $G$ are connected by a rainbow path in $G$. An edge-coloring under which $G$ is rainbow connected is called a rainbow connection coloring. Clearly, if a graph is rainbow connected, it must be connected. For a connected graph $G$, the rainbow connection number of $G$, denoted by $rc(G)$, is the smallest number of colors that are needed to make $G$ rainbow connected. Rainbow connection was introduced by Chartrand et al. [5] in 2008. For more details on rainbow connection, see the book [10] and the survey papers [9,11].

In this paper, we investigate a new concept that is somewhat reverse to rainbow connection. This concept of rainbow disconnection of graphs was introduced by Chartrand et al. [4] very recently in 2018.

An edge-cut of a connected graph $G$ is a set $S$ of edges such that $G - S$ is disconnected. The minimum number of edges in an edge-cut is defined as the edge-connectivity $\lambda(G)$ of $G$. We have the well-known inequality $\lambda(G) \leq \delta(G)$. For two vertices $u$ and $v$, let $\lambda(u, v)$ denote the minimum number of edges in an edge-cut $S$ such that $u$ and $v$ lie in different components of $G - S$. The so-called upper edge-connectivity $\lambda^+(G)$ of $G$ is defined by

$$\lambda^+(G) = \max\{\lambda(u, v) : u, v \in V(G)\}.$$ 

$\lambda^+(G)$ is the maximum local edge-connectivity of $G$, while $\lambda(G)$ is the minimum global edge-connectivity of $G$.

An edge-cut $R$ of an edge-colored connected graph $G$ is called a rainbow cut if no two edges in $R$ are colored the same. A rainbow cut $R$ is said to separate two vertices $u$ and $v$ if $u$ and $v$ belong to different components of $G - R$. Such rainbow cut is
called a $u - v$ rainbow cut. An edge-colored graph $G$ is called rainbow disconnected if for every two distinct vertices $u$ and $v$ of $G$, there exists a $u - v$ rainbow cut in $G$. In this case, the edge-coloring $c$ is called a rainbow disconnection coloring of $G$. Similarly, we define the rainbow disconnection number of a connected graph $G$, denoted by $rd(G)$, as the smallest number of colors that are needed in order to make $G$ rainbow disconnected. A rainbow disconnection coloring with $rd(G)$ colors is called an $rd$-coloring of $G$.

The Erdős-Gallai-type problem is an interesting problem in extremal graph theory, which was studied in [7, 8, 12] for rainbow connection number $rc(G)$; in [6] for proper connection number $pc(G)$; in [3] for monochromatic connection number $mc(G)$, and many other parameter of graphs in literature. We will study the Erdős-Gallai-type results for the rainbow disconnection number $rd(G)$ in this paper.

2 Preliminary results

For given integers $k$ and $n$ with $1 \leq k \leq n - 1$, the authors in [4] determined the minimum size of a connected graph $G$ of order $n$ with $rd(G) = k$. So, this brings up the question of determining the maximum size of a connected graph $G$ of order $n$ with $rd(G) = k$. The authors of [4] conjectured and we determined in [1] the maximum size of a connected graph $G$ of order $n$ with $rd(G) = k$, for odd integer $n$. But for even integer $n$, it was left without solution. Now we consider the question of determining the maximum size of a connected graph $G$ of even order $n$ with $rd(G) = k$ and we get the following result.

**Theorem 2.1** Let $k$ and $n$ be integers with $1 \leq k \leq n - 1$ and $n$ be even. Then the maximum size of a connected graph $G$ of order $n$ with $rd(G) = k$ is $\left\lfloor \frac{(k+1)(n-1)}{2} \right\rfloor$.

Before we give the proof of Theorem 2.1, some useful lemmas are stated as follows.

**Lemma 2.2** [4] If $G$ is a nontrivial connected graph, then

$$\lambda(G) \leq \lambda^+(G) \leq rd(G) \leq \chi'(G) \leq \Delta(G) + 1.$$ 

**Lemma 2.3** [4] Let $G$ be a nontrivial connected graph. Then $rd(G) = 1$ if and only if $G$ is a tree.

**Lemma 2.4** [4] For each integer $n \geq 4$, $rd(K_n) = n - 1$. 

3
Remark 1: For any integer \( n \geq 2 \), \( rd(K_n) = n - 1 \) since it is easy to verify that \( rd(K_2) = 1 \) and \( rd(K_3) = 2 \).

Lemma 2.5 \[13\] Let \( G \) be a graph of order \( n \) (\( n \geq k + 2 \geq 3 \)). If \( e(G) > \frac{k+1}{2}(n-1) - \frac{1}{2}\sigma_k(G) \), where \( \sigma_k(G) = \sum_{x \in V(G)} (k - d(x)) \), then \( \lambda^+(G) \geq k + 1 \).

Lemma 2.6 If \( n \) is even, then there exists a \( k \)-regular graph \( G \) of order \( n \), where \( 1 \leq k \leq n - 2 \), that satisfies both of the following conditions:

(i) \( G \) is 1-factorable.

(ii) there exists a vertex \( u \) of \( G \) such that \( G[N(u)] \) can add \( \lfloor \frac{k}{2} \rfloor \) matching edges.

Proof. Let \( G_1 = K_{2n} \). Since \( K_{2n} \) is 1-factorable, \( K_{2n} \) has \( 2n-1 \) edge-disjoint 1-factors. First, we remove a 1-factor from \( K_{2n} \), namely \( n \) matching edges. Let \( e_1, e_2, \ldots, e_n \) be \( n \) removing matching edges of \( G_1 \) and let \( v_{i,1} \) and \( v_{i,2} \) be the two end vertices of \( e_i \). Let \( u = v_{n,1} \) be a vertex of \( G_1 \). Then the remaining graph \( G_2 \) is a \((2n-2)\)-regular graph with \( N_{G_2}(u) = \{v_{i,1}, v_{i,2}|1 \leq i \leq n - 1\} \), and \( \{e_i|1 \leq i \leq n - 1\} \) are matching edges which can be added to \( G_2[N(u)] \) and the number of matching edges is \( \lfloor \frac{2n-2}{2} \rfloor \).

Second, we remove the 1-factor from \( G_2 \) which contains the edge \( uv_{n-1,1} \), and denote remaining graph by \( G_3 \). Obviously, \( G_3 \) is a \((2n-3)\)-regular graph and \( N_{G_3}(u) \) is \( N_{G_2}(u) \setminus v_{n-1,1} \), and \( \{e_i|1 \leq i \leq n - 2\} \) are matching edges which can be added to \( G_3[N(u)] \) and the number of matching edges is \( \lfloor \frac{2n-3}{2} \rfloor \).

Third, we remove the 1-factor from \( G_3 \) which contains the edge \( uv_{n-1,2} \), and denote remaining graph by \( G_4 \). Obviously, \( G_4 \) is a \((2n-4)\)-regular graph and \( N_{G_4}(u) \) is \( N_{G_3}(u) \setminus v_{n-1,2} \), and \( \{e_i|1 \leq i \leq n - 2\} \) are matching edges which can be added to \( G_4[N(u)] \) and the number of matching edges is \( \lfloor \frac{2n-4}{2} \rfloor \).

For \( G_j \) (\( j \geq 2 \)), if \( j \) is even, then we remove the 1-factor from \( G_j \) which contains the edge \( uv_{n-[\frac{j}{2}],1} \); if \( j \) is odd, then we remove the 1-factor from \( G_j \) which contains the edge \( uv_{n-[\frac{j}{2}],2} \). Obviously, the remaining graph \( G_{j+1} \) is a \((2n-j-1)\)-regular graph and \( N_{G_{j+1}}(u) \) is \( N_{G_j}(u) \setminus v_{n-[\frac{j}{2}],i} \) (\( i = 1 \) or \( 2 \)), and \( \{e_i|1 \leq i \leq n - \lfloor \frac{j}{2} \rfloor - 1\} \) are matching edges which can be added to \( G_{j+1}[N(u)] \) and the number of matching edges is \( \lfloor \frac{2n-j-1}{2} \rfloor \).

Repeating the above process, we can get a \( k \)-regular graph \( G_{2n-k} \) of order \( n \) which is 1-factorable. Furthermore, \( \{e_i|1 \leq i \leq n - \lfloor \frac{2n-k}{2} \rfloor\} \) are matching edges which can be added to \( G_{2n-k}[N(u)] \) and the number of matching edges is \( \lfloor \frac{k}{2} \rfloor \). \( \square \)

Now we are ready to give a proof to Theorem 2.1.
Proof of Theorem 2.1. It is easy to see that the graphs $G$ of maximum size with order $n$ and $rd(G) = k$ is not more than $\frac{(k+1)(n-1)}{2}$. Otherwise, $rd(G) \geq k + 1$ by Lemmas 2.2 and 2.5. Now we show that the graphs $G$ of maximum size with even order $n$ and $rd(G) = k$ is $\frac{(k+1)(n-1)}{2}$ for $1 \leq k \leq n - 1$. For $k = n - 1$, let $G = \bar{K}_n$. Note that $|E(G)| = \left\lceil \frac{(k+1)(n-1)}{2} \right\rceil$ and $rd(G) = n - 1$ by Remark 1. For $k = 1$, let $G$ be a tree. Note that $|E(G)| = \left\lceil \frac{(k+1)(n-1)}{2} \right\rceil$ and $rd(G) = 1$ by Lemma 2.3. Now we construct a graph $G$ for $2 \leq k \leq n - 2$ as follows. Let $H_{k-1}$ be a $(k-1)$-regular graph of order $n$. For $k \geq 2$, $H_{k-1}$ can be selected so that it is 1-factorable and there exists a vertex $u$ of $H_{k-1}$ such that $\left\lfloor \frac{k-1}{2} \right\rfloor$ matching edges can be added to $N_{H_{k-1}}(u)$ by Lemma 2.6. Let $G$ be a graph by adding $\left\lfloor \frac{k-1}{2} \right\rfloor$ matching edges to $N_{H_{k-1}}(u)$ and adding $n - k$ edges of $\{uw|w \in V(H_{k-1}) \setminus N_{H_{k-1}}[u]\}$ in $H_{k-1}$. Thus, $G$ is a graph of order $n$ with $|E(G)| = \frac{(k-1)n}{2} + \frac{k-1}{2} + n - k = \left\lceil \frac{(k+1)(n-1)}{2} \right\rceil$. Since $\chi'(H_{k-1}) = k - 1$, we obtain a proper edge-coloring $c_0$ of $H_{k-1}$ using colors from $[k-1]$. We may extend $c_0$ to an edge-coloring $c$ of $G$ by assigning a fresh color $k$ to all newly added edges in $H_{k-1}$. Note that the set $E_x$ of edges incident with $x$ in $G$ is a rainbow set for each vertex $x \in V(G) \setminus u$. Let $p$ and $q$ be two vertices of $G$. Then at least one of $p$ and $q$ is not $u$, say $p \neq u$. Since $E_p$ is a $p - q$ rainbow cut, $c$ is a rainbow disconnection coloring of $G$ using at most $k$ colors. Therefore, $rd(G) \leq k$. On the other hand, $E(G) = \left\lceil \frac{(k+1)(n-1)}{2} \right\rceil > \frac{k(n-1)}{2}$ since $n \geq 3$, it follows from Lemmas 2.2 and 2.5 that $rd(G) \geq k$. \hfill \Box

3 Erdős-Gallai-type results for $rd(G)$

Now we consider the following two kinds of Erdős-Gallai-type problems for $rd(G)$.

Problem A. Given two positive integers $n$ and $k$ with $1 \leq k \leq n - 1$, compute the maximum integer $g(n,k)$ such that for any graph $G$ of order $n$, if $|E(G)| \leq g(n,k)$, then $rd(G) \leq k$.

Problem B. Given two positive integers $n$ and $k$ with $1 \leq k \leq n - 1$, compute the minimum integer $f(n,k)$ such that for any graph $G$ of order $n$, if $|E(G)| \geq f(n,k)$ then $rd(G) \geq k$.

It is worth mentioning that the two parameters $f(n,k)$ and $g(n,k)$ are equivalent to another two parameters. Let $t(n,k) = \min\{|E(G)| : |V(G)| = n, rd(G) \geq k\}$ and $s(n,k) = \max\{|E(G)| : |V(G)| = n, rd(G) \leq k\}$. It is easy to see that $g(n,k) = t(n,k+1) - 1$ and $f(n,k) = s(n,k-1) + 1$.

We first state two lemmas, which will be used to determine the values of $f(n,k)$ and $t(n,k)$.
Lemma 3.1  For integers $k$ and $n$ with $1 \leq k \leq n - 1$, the minimum size of a connected graph of order $n$ with $rd(G) = k$ is $n + k - 2$.

Note that the following result from [1] is also true for $n = 1, 3$. So we can state it as follows, without $n \geq 5$.

Lemma 3.2  Let $k$ and $n$ be integers with $1 \leq k \leq n - 1$ and $n$ be odd. Then the maximum size of a connected graph $G$ of order $n$ with $rd(G) = k$ is $\frac{(k+1)(n-1)}{2}$.

Using Lemma 3.1, we first solve Problem A.

Theorem 3.3  $g(n,k) = n + k - 2$ for $1 \leq k \leq n - 1$.

Proof. It follows from Lemma 3.1 that $t(n,k) = n + k - 2$ for $1 \leq k \leq n - 1$. Thus, $g(n,k) = t(n,k+1) - 1 = n + k - 2$. $\square$

Now we come to the solution for Problem B, we get the following result.

Theorem 3.4  $f(n,k) = \left\lceil \frac{k(n-1)}{2} \right\rceil + 1$ for $1 \leq k \leq n - 1$.

Proof. If $n$ is odd, then $s(n,k) = \frac{(k+1)(n-1)}{2}$ for $1 \leq k \leq n - 1$ from Lemma 3.2. Thus, $f(n,k) = s(n,k-1) + 1 = \frac{k(n-1)}{2} + 1 = \left\lceil \frac{k(n-1)}{2} \right\rceil + 1$ for $1 \leq k \leq n - 1$.

If $n$ is even, then $s(n,k) = \left\lfloor \frac{(k+1)(n-1)}{2} \right\rfloor$ for $1 \leq k \leq n - 1$ from Theorem 2.1.

Thus, $f(n,k) = s(n,k-1) + 1 = \left\lfloor \frac{k(n-1)}{2} \right\rfloor + 1$ for $1 \leq k \leq n - 1$ for $n$ is even. $\square$

References

[1] X. Bai, R. Chang, X. Li, More on rainbow disconnection in graphs, arXiv:1810.09736 [math.CO].

[2] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 224, Springer, 2008.

[3] Q. Cai, X. Li, D. Wu, Erdős-Gallai-type results for colorful monochromatic connectivity of a graph, J. Comb. Optim. 33(2017), 123–131.

[4] G. Chartrand, S. Devereaux, T.W. Haynes, S.T. Hedetniemi, P. Zhang, Rainbow disconnection in graphs, Discuss. Math. Graph Theory 38(2018), 1007–1021.

[5] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem. 133(2008), 85–98.
[6] F. Huang, X. Li, S. Wang, Upper bounds of proper connection number of graphs, \textit{J. Comb. Optim.} \textbf{34}(2017), 165–173.

[7] H. Li, X. Li, Y. Sun, Y. Zhao, Note on minimally $d$-rainbow connected graphs, \textit{Graphs & Combin.} \textbf{30}(2014), 949–955.

[8] X. Li, M. Liu, Schiermeyer, Rainbow connection number of dense graphs, \textit{Discuss Math Graph Theory} \textbf{33}(2013), 603–611.

[9] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, \textit{Graphs & Combin.} \textbf{29}(2013), 1–38.

[10] X. Li, Y. Sun, Rainbow Connections of Graphs, \textit{SpringerBriefs in Math.}, Springer, New York, 2012.

[11] X. Li, Y. Sun, An updated survey on rainbow connections of graphs - a dynamic survey, \textit{Theo. Appl. Graphs} \textbf{0}(1) (2017), Art. 3, 1–67.

[12] A. Lo, A note on the minimum size of $k$-rainbow-connected graphs, \textit{Discete Math.} \textbf{331}(2015), 20–21.

[13] W. Mader, Ein Extremalproblem des Zusammenhangs von Graphen, \textit{Math. Z.} \textbf{131}(1973), 223–231.

[14] V.G. Vizing, On an estimate of the chromatic class of a $p$-graph, \textit{Diskret. Anal.} \textbf{3}(1964), 25–30, in Russian.