Bäcklund transformation
for
non-relativistic Chern-Simons vortices

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Abstract.
A Bäcklund transformation yielding the static non-relativistic Chern-Simons vortices of Jackiw and Pi is presented.

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1. Introduction
In the non-relativistic ‘Chern-Simons’ version of the Abelian Higgs model of Jackiw and Pi [1-2-3], the scalar field, $\Psi$, is described by the gauged, planar non-linear Schrödinger equation

\begin{equation}
    i\partial_t \Psi = \left[ -\frac{1}{2}(\vec{\nabla} - i\vec{A})^2 + A^0 - g \Psi^* \Psi \right] \Psi.
\end{equation}

(We work in our units where $\hbar = m = e = 1$). Here the ‘electromagnetic’ field, associated with the vector potential $(A^0, \vec{A})$, is assumed to satisfy the Chern-Simons field-current identity

\begin{equation}
    \mathbf{B} = \epsilon_{ij} \partial_i A_j = \frac{1}{\kappa} \varrho, \quad \mathbf{E}^i = -\partial_i A^0 - \partial_t A^i = \frac{1}{\kappa} \epsilon^{ij} J^j,
\end{equation}

\begin{equation}
    (i, j = 1, 2), \text{ where } \varrho = \Psi^* \Psi \text{ and } \vec{J} = (-i/2) [\Psi^* \vec{D} \Psi - \Psi (\vec{D} \Psi)^*] \text{ denote the particle density and the current, respectively. Explicit multivortex solutions have only been found so far for the special value of } g = \pm 1/\kappa; \text{ for static fields, the second-order field equations above can be reduced to the first-order ‘self-dual’ system } (D_2 \pm i D_2)\Psi = 0. \text{ In a suitable gauge and away from the zeros of } \varrho, \text{ this becomes the Liouville equation}
\end{equation}

\begin{equation}
    \Box \log \varrho = \mp \frac{2}{\kappa} \varrho.
\end{equation}

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Regular solutions arise by choosing the upper (resp. lower) sign for $\kappa < 0$ (resp. for $\kappa > 0$),

$$q = 4|\kappa| \frac{|f'|^2}{[1 + |f|^2]^2},$$

where $f \equiv f(z)$ is a meromorphic function on the complex plane. The remaining fields are expressed in terms of the particle density as

$$A^0 = \frac{1}{2|\kappa|} q, \quad \vec{A} = \frac{1}{2} \vec{\nabla} \times \log q + \vec{\nabla} \omega, \quad \Psi = \sqrt{q} e^{i\omega},$$

where $\omega$ is chosen so that $\vec{A}$ is regular at the zeros of $q$.

For $g = \pm 1/\kappa$, all static solutions are self-dual. This can be proved using the conformal invariance of the system [2].

The problem of integrability of the full, time-dependent second-order system (1.1-2) was examined by Lévy et al. [4] and by Knecht et al. [5], who found that it was not in general integrable. In [5] it was shown in particular, that the full system fails to pass the Painlevé test, as extended to partial differential equations by Weiss, Tabor and Carnevale (WTC) [6].

The point is that the WTC method — when it works — has the additional bonus to provide Bäcklund transformations for generating solutions. In this paper, we take advantage of this to construct, in the static case and for $g = \pm 1/\kappa$, Bäcklund transformations allowing us to rederive all static solutions.

2. A Bäcklund transformation

Write $\Psi = \sqrt{q} e^{i\omega}$ and introduce, following Knecht et al. [5], the new variables

$$q = |\kappa|^3 f^2, \quad A^0 - \partial^0 \omega = -\frac{\kappa^2}{2} w$$

$$A_1 - \partial_1 \omega = -\kappa u, \quad A_2 - \partial_2 \omega = -\kappa v,$$

$$x_1 = \frac{x}{|\kappa|}, \quad x_2 = \frac{y}{|\kappa|},$$

in terms of which the static CS Eqns read (1)

$$u_y - v_x = -f^2,$$

$$w_x = 2vf^2, \quad w_y = -2u f^2,$$

$$f_{xx} + f_{yy} = -2\epsilon^{-1} f^3 - f(w - u^2 - v^2),$$

where $\epsilon = 1/g|\kappa|$. The WTC method [6], [5] amounts to developing the fields into a generalized Laurent series,

$$u = \sum_{k=0}^{\infty} u_k \Phi^{k-p_u}, \quad v = \sum_{k=0}^{\infty} v_k \Phi^{k-p_v},$$

$$w = \sum_{k=0}^{\infty} w_k \Phi^{k-p_w}, \quad f = \sum_{k=0}^{\infty} f_k \Phi^{k-p_f},$$

where $\Phi = 0$ is the ‘singular manifold’. Inserting these expressions into the eqns. of motion fixes the values of the $p_i$’s and provides us with recursion relations, except for some particular values in $k$ called ‘resonances’.

\footnote{One actually gets one more relation, which corresponds to the continuity equation and appears here as a consistency condition.}
when consistency conditions have to be satisfied. In detail, for \( k = 0 \) we find, consistently with Knecht et al. [5], that \( p_u = p_v = p_f = 1, p_w = 2 \) and

\[
\begin{align*}
    u_0 &= -\epsilon \Phi_y, \\
    v_0 &= \epsilon \Phi_x, \\
    w_0 &= \epsilon^2 (\Phi_x^2 + \Phi_y^2), \\
    f_0^2 &= -\epsilon (\Phi_x^2 + \Phi_y^2).
\end{align*}
\]  

(2.4)

Resonances occur for \( k = 1, 2 \) and 4. For \( k = 1 \) we get

\[
\begin{align*}
    2f_0^2 u_1 + \Phi_x w_1 + 4v_0 f_0 f_1 &= w_0 x, \\
    -2f_0^2 v_1 + \Phi_y w_1 - 4u_0 f_0 f_1 &= v_0 y, \\
    -2f_0 w_0 u_1 - 2f_0 v_0 v_1 + f_0 w_1 - 6(\Phi_x^2 + \Phi_y^2) f_1 &= (\Phi_{xx} + \Phi_{yy}) f_0 + 2(\Phi_x f_{0x} + \Phi_y f_{0y}).
\end{align*}
\]  

(2.5)

From the first of these equations we deduce, using Eq. (2.4)

\[
\begin{align*}
    f_1^2 &= -\epsilon \frac{(\Delta \Phi)^2}{4(\Phi_x^2 + \Phi_y^2)}.
\end{align*}
\]  

(2.6)

The remaining system of three equations has a vanishing determinant. Consistency requires hence

\[
\epsilon^2 = 1 \quad \text{i.e.} \quad g = \frac{1}{|\kappa|}.
\]  

(2.7)

Then \( u_1 \) and \( v_1 \) can be expressed as a function of \( w_1 \), which is arbitrary.

\( k = 2 \) is again a resonance value: the l.h.s. of

\[
\begin{align*}
    \Phi_y u_2 - \Phi_x v_2 + 2f_0 f_2 &= -u_1 y + v_1 x - f_1^2, \\
    2f_0^2 v_2 + 4v_0 f_0 v_2 &= w_1 x - 2(f_1^2 v_0 + 2v_1 f_0 f_1), \\
    -2f_0^2 u_2 - 4u_0 f_0 u_2 &= w_1 y + 2(f_1^2 u_0 + 2u_1 f_0 f_1), \\
    -6(\Phi_x^2 + \Phi_y^2) f_2 &= -(f_{0xx} + f_{0yy}) - 6\epsilon^{-1} f_1^2 f_0 - w_1 f_1 \\
    &+ f_0 (u_1^2 + v_1^2) + 2f_1 (u_0 u_1 + v_0 v_1).
\end{align*}
\]  

(2.8)

has vanishing determinant. Consistency requires hence the same to be true for the r.h.s., which only happens when

\[
w_1 = -2\epsilon f_0 f_1 = -\Delta \Phi.
\]  

(2.9)

The arisal of the constraint (2.9) shows already that even the static system (2.2) fails to pass the Painlevé test of WTC, which would allow an arbitrary \( w_1 \). We can, nevertheless, continue our search for finding solutions constrained to satisfy (2.9). Then \( u_2, v_2 \) and \( w_2 \) are expressed using the arbitrary function \( f_2 \). (Condition (2.9) is below related to self-duality).

Inserting \( w_1 \) into (2.5) we get, with the help of (2.4) and (2.6)

\[
\begin{align*}
    2(\Phi_x^2 + \Phi_y^2) v_1 - \epsilon \Delta \Phi \Phi_x &= -\epsilon (\Phi_x^2 + \Phi_y^2) x, \\
    2(\Phi_x^2 + \Phi_y^2) u_1 + \epsilon \Delta \Phi \Phi_y &= \epsilon (\Phi_x^2 + \Phi_y^2) y.
\end{align*}
\]  

(2.10)
while the last equation is identically satisfied. Thus

\begin{align*}
  u &= -\epsilon (\log \Phi)_y + u_1 + u_2 \Phi + \ldots, \\
  v &= \epsilon (\log \Phi)_x + v_1 + v_2 \Phi + \ldots, \\
  w &= -\Delta \log \Phi + w_2 + w_3 \Phi + \ldots, \\
  f^2 &= \epsilon \Delta \log \Phi + (2f_0f_2 + f_1^2) + (2f_1f_2 + 2f_0f_3)\Phi + \ldots.
\end{align*}

(2.11)

Now we try to \textit{truncate} these infinite series by only keeping terms of order less or equal to zero,

\begin{align*}
  u_k &= v_k = w_{k+1} = f_k \equiv 0 \quad \text{for} \quad k \geq 2.
\end{align*}

(2.12)

Inserting these relations into the first equation of (2.8), we find that the r. h. s. vanishes. Hence

\begin{align*}
  u_1 y - v_1 x &= -f_1^2.
\end{align*}

(2.13)

From the other eqns. of (2.8) we get, for the Ansatz (2.12) and using the constraint (2.9),

\begin{align*}
  f_1 x &= -\epsilon v_1 f_1, \quad \text{and} \quad f_1 y = \epsilon u_1 f_1
\end{align*}

and

\begin{align*}
  w_2 &= -\epsilon f_1^2.
\end{align*}

(2.15)

Note that if \( f_1 \) is not identically zero, then Eq. (2.14) implies

\begin{align*}
  \left( \frac{f_1 x}{f_1} \right)_x + \left( \frac{f_1 y}{f_1} \right)_y &= -\epsilon f_1^2
\end{align*}

(2.16)

which is the \textit{Liouville equation}. Requiring the constraint (2.9) means therefore reducing the second-order equation (2.2) to a first-order system.

So far we have only considered the terms \( k = 0, 1, 2 \). The consistency of our procedure follows from the verification, using the formulæ given in Ref. [5], that all remaining equations, including the compatibility condition for the resonance value \( k = 4 \), are identically satisfied. The case \( k = 3 \) we show in particular that the fields \( f_1, u_1, v_1, w_2 \) satisfy the equations (2.2) we started with; they provide us therefore with a “seed solution” in our Bäcklund transformation.

Collecting our results and returning to the physical variables, cf. Eq. (2.1), we have proved the following. Let \( \rho \sim f^2_1, \vec{a} \sim (u_1, v_1), a^0 \sim w_2 \) be any “seed solution”,

\begin{align*}
  \kappa \vec{\nabla} \times \vec{a} &= -\rho, \quad \vec{\nabla} \times \rho = -2\epsilon (\text{sign} \kappa) \vec{a} \rho, \quad a^0 = \epsilon \frac{1}{2|\kappa|} \rho,
\end{align*}

such that

\begin{align*}
  (\Delta \Phi)^2 &= -4 \frac{\epsilon}{|\kappa|} \rho [ (\partial_1 \Phi)^2 + (\partial_2 \Phi)^2 ], \\
  2 (\text{sign} \kappa) [ (\partial_1 \Phi)^2 + (\partial_2 \Phi)^2 ] \epsilon_{ij} a_j + \epsilon \Delta \Phi \partial_i \Phi &= \epsilon \partial_i [ (\partial_1 \Phi)^2 + (\partial_2 \Phi)^2 ]
\end{align*}

(2.17) Then the Bäcklund transformation

\begin{align*}
  \dot{\rho} &= \epsilon |\kappa| \Delta \log \Phi + \rho, \\
  \dot{\vec{A}} &= \epsilon (\text{sign} \kappa) \vec{\nabla} \times \log \Phi + \vec{a} + \vec{\nabla} \omega, \\
  \dot{A^0} &= \frac{1}{2} \Delta \log \Phi + a^0
\end{align*}

(2.18)

(2.19)

(where \( \rho \sim f^2, \vec{A} \sim (u, v), A^0 \sim w \)) provides us with a new set of solutions for (1.1-2).

This follows either from the proof above, or can be directly verified.
If $\rho \neq 0$, the first two equations in (2.17) reduce to the Liouville equation for $\rho$, while $\vec{a}$ and $a^0$ are expressed as in (1.5). The seed solution is hence necessarily self-dual.

3. The construction of solutions

The seed solution may not be a physical one: a judicious choice may simplify a great deal the equations to be solved. Below we obtain in fact the general solution (1.4) by choosing $\rho$ to vanish.

$$(3.1) \quad \rho \equiv 0$$

In detail, let us assume that (3.1) holds. Then Eq. (2.17) is satisfied by $a^0 \equiv 0$ and $\vec{a}$ any curl-free vector potential. Similarly, the upper equation in (2.18) now requires that $\Phi$ solve the Laplace equation

$$(3.2) \quad \triangle \Phi = 0.$$ Introducing the complex notations $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $\partial = \frac{1}{2} (\partial_1 - i \partial_2)$, $\bar{\partial} = \frac{1}{2} (\partial_1 + i \partial_2)$, the solution of (3.2) is given by $\Phi(z, \bar{z}) = f(z) + g(\bar{z})$, where $f(z)$ is analytic and $g(\bar{z})$ is anti-analytic. With this choice, $\rho$ is seen to be real if $g(\bar{z}) = 1/f(z)$. Hence

$$(3.3) \quad \Phi(z, \bar{z}) = f(z) + \frac{1}{f(z)}.$$ Positivity of $\rho$ requires finally to set $\epsilon = 1$. In conclusion, the particle density is

$$(3.4) \quad \rho = |\kappa| \triangle \log \Phi = |\kappa| \triangle \log \left[1 + |f|^2\right],$$

i.e., the general solution (1.4) of the Liouville eqn.

The second Eqn. in (2.18) allows us to express the vector potential of the seed solution as

$$(3.5) \quad \vec{a} = -\frac{1}{2} (\text{sign} \kappa) \vec{\nabla} \times \log \left[ (\partial_1 \Phi)^2 + (\partial_2 \Phi)^2 \right] = -\frac{1}{2} (\text{sign} \kappa) \vec{\nabla} \times \log \frac{|f'|^2}{f^2},$$

whose curl indeed vanishes, as required for consistency. Note that $\vec{a}$ combines with the first term in the $\vec{A}$-equation of (2.19) to yield the curl of $\log \rho$, cf. Eq. (1.5). Finally, $A^0 = \rho/2|\kappa|$ follows from (2.17) and (2.19).

For example, for each fixed $0 \neq c \in \mathbb{C}$,

$$(3.6) \quad \Phi = \left(\frac{z}{c}\right)^N + \left(\frac{\bar{z}}{\bar{c}}\right)^{-N}$$

is a rotationally invariant solution of the Laplace equation, which yields the well-known radial solution

$$(3.7) \quad \rho = \frac{4N^2|\kappa|}{r^2} \left( \left( \frac{r}{r_0} \right)^N + \left( \frac{r_0}{r} \right)^N \right)^{-2}$$

($r_0 = |c|$ [1-3]). The first term in the $\vec{A}$-equation of (2.19) is

$$(3.8) \quad (\text{sign} \kappa) \frac{2N}{\bar{z}} \frac{1}{1 + |z|^{2N}},$$

while the seed solution is

$$(3.9) \quad a \equiv a_1 + ia_2 = -(\text{sign} \kappa) \frac{N + 1}{\bar{z}}. $$
At the origin, the sum of these terms behaves as

\[ (\text{sign } \kappa) i \frac{(N - 1)}{\bar{z}}, \]

so that the singularities are avoided if the phase \( \omega (\partial_t \omega = 0) \) is chosen to be

\[ \omega = (-\text{sign } \kappa) (N - 1) \arg z. \]

The magnetic charge \( Q \equiv \int B \, d^2 \bar{r} \) is conveniently calculated as

\[ Q = \oint_S \bar{A} \cdot d\bar{l}, \]

where \( S \equiv S_\infty \) denotes the circle at infinity. At infinity (3.8) falls off, so that only the seed and \( \omega \) terms contribute. We find

\[ Q = - (\text{sign } \kappa) 2\pi(N + 1) - (\text{sign } \kappa) 2\pi(N - 1) = - (\text{sign } \kappa) 4\pi N, \]

as expected. More generally,

\[ \Phi = \prod_{i=1}^N (z - z_i) \frac{P(z)}{\prod_{i=1}^N (z - z_i)} \]

where the \( z_i \)'s are arbitrary complex numbers and \( P(z) \) is a polynomial of \( z \) of degree at most \( N - 1 \) \((P(z_i) \neq 0)\), provides us with a \( 4N \)-parameter family of solutions with magnetic charge \( Q = (-\text{sign } \kappa) 4\pi N \). This can be shown along the same lines as above.

**4. Discussion**

Our formulæ are equivalent to the expression in Refs. [1-3], [7]. To see this, observe that \( \Phi \) is manifestly invariant with respect to the transformation \( f \rightarrow 1/\bar{f} \). But \( \varphi \) is also invariant with respect to complex conjugation \( f \rightarrow \bar{f} \) (cf. (3.4)), and thus also with respect to \( f \rightarrow 1/f \). But for (3.13), \( 1/f \) is decomposed into partial fractions as

\[ \frac{1}{f} = \sum_{i=1}^N \frac{d_i}{z - z_i}, \]

which is the standard choice [1-3], [7].

A Bäcklund transformation for the Liouville equation has been constructed before by D’Hoker and Jackiw [8]. Their approach also involves a solution of the Laplace equation. They need, however, to solve an additional system of coupled, first-order differential equations. Our formulæ are hence different from theirs.

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