Input-Feedforward-Passivity-Based Distributed Optimization Over Jointly Connected Balanced Digraphs

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Abstract—In this paper, a distributed optimization problem is investigated via input feedforward passivity. First, an input-feedforward-passivity-based continuous-time distributed algorithm is proposed. It is shown that the error system of the proposed algorithm can be interpreted as output feedback interconnections of a group of input feedforward passive (IFP) systems. Based on this IFP framework, convergence conditions of a suitable coupling gain are derived for the IFP-based algorithm over directed and uniformly jointly strongly connected (UJSC) weight-balanced topologies. Second, a novel distributed derivative feedback algorithm is proposed based on the passivation of IFP systems. While most works on directed topologies require the knowledge of the smallest nonzero eigenvalue of the graph Laplacian, the derivative feedback algorithm is fully distributed, namely, it is robust against randomly changing weight-balanced digraphs with any positive coupling gain and without knowing any graph information. Finally, numerical examples are presented to demonstrate the proposed distributed algorithms.

Index Terms—Continuous-time algorithms, input feedforward passivity, weight-balanced digraphs, uniformly jointly strongly connected topologies, derivative feedback.

I. INTRODUCTION

Distributed optimization over multi-agent systems has been widely investigated in recent years, due to its broad applications in various aspects including wireless networks, smart grids, and machine learning. In addition to the discrete-time algorithms (e.g., [2]–[4]), a variety of continuous-time distributed algorithms have been proposed to solve distributed optimization problems [5]–[8]. Continuous-time algorithms can be implemented in hardware devices like analog circuits [9], achieving tasks such as motion coordination of multi-agent systems [10]. Studying optimization in the continuous-time scheme enjoys the benefit of numerous control techniques for stability analysis and also opens up the possibility to address commonly encountered problems in large-scale networks, such as disturbance rejection, robustness to delays or uncertainties, or channel constraints [11]. However, most of the proposed algorithms are only for undirected topologies and not applicable to directed topologies [5]–[8]. To deal with this difficulty, some parameters in the original algorithm can be tuned to stabilize gradient dynamics [12], while some variants of the gradient dynamics are proposed in [13], [14]. However, these methods often employ coordinate transformation and complicated Lyapunov functions in convergence analysis, and hence the eigenvalues of the graph Laplacian should be known to design some parameters [12]–[15]. Compared with these methods, a more systematic approach is needed for this problem.

It is well known that dissipativity (as well as its special case, passivity) is a useful tool for stability analysis and control design [16]–[18]. Recently, there emerged some continuous-time passivity-based algorithms on distributed optimization under some communication constraints [11], [19]–[21]. However, these passivity-based algorithms can only be applied over undirected graphs, while it is shown that output consensus can be achieved over directed graphs through simple output feedback interconnections of passive systems [16], [17]. Motivated by these works, we aim to study distributed algorithms over directed graphs via dissipativity/passivity techniques. On one hand, we conjecture that it is in general difficult to directly construct a distributed algorithm that can be interpreted as output feedback interconnections of passive systems. On the other hand, works in [22]–[24] point out that output consensus can be achieved over directed graphs even among IFP (or passivity-short) systems. Therefore, if a distributed algorithm inherits input feedforward passivity, it can also be directly applied to weight-balanced digraphs through output feedback interconnections. As a byproduct of having the IFP properties, the distributed algorithm can also be applicable in uniformly jointly strongly connected (UJSC) topologies, while the effort in constructing complicated candidate Lyapunov functions is greatly reduced in convergence analysis. This feature is remarkable since it greatly reduces communication costs, and hence is more practical in large-scale networks. Though the problem of UJSC switching topologies has been considered in discrete-time algorithms [3], [4], to the best of our knowledge, it has never been addressed in the continuous-time scheme, due to the difficulties of stability analysis under the time-varying nature and lack of connectedness of topologies.

In this paper, we investigate the distributed optimization problem via input feedforward passivity. First, we propose an IFP-based distributed algorithm which is interpreted as output feedback interconnections of a group of IFP systems. Based on this IFP framework, we study the distributed algorithm over directed and UJSC weight-balanced topologies and derive convergence conditions of a suitable coupling gain for the IFP-based algorithm. Second, we propose a novel distributed
derivative feedback algorithm based on the passivation of IFP systems. While most works on directed topologies in the literature require the knowledge of the smallest nonzero eigenvalue of the graph Laplacian \([12]–[15]\), we show that the derivative feedback algorithm is fully distributed, namely, it is robust against randomly changing weight-balanced digraphs with any positive coupling gain and without knowing any graph information. In other words, the derivative feedback algorithm is applicable for gossip-like balanced digraphs \([23]\), reducing communication costs. Moreover, the passivation also provides an insight into how the widely used derivative feedback affects the system’s properties. The challenges in our work lie in the construction of a group of verifiable nonlinear IFP systems that solve the distributed optimization problem, the design of the fully distributed algorithm, and the convergence analysis of the proposed algorithms.

A preliminary version of this work appeared in \([1]\), where only the IFP-based algorithm has been proposed. In this work, we propose the IFP-based algorithm with a possibly time-varying coupling parameter and construct more practical conditions that are easier to verify in a distributed sense. Moreover, a fully distributed algorithm is proposed.

The rest of this paper is organized as follows. In Section \([\text{II}]\) some background knowledge of convex analysis, graph theory, and passivity is reviewed and the problem formulation is given. In Section \([\text{III}]\) an IFP-based distributed algorithm is proposed and studied over weight-balanced UIJSC topologies. In Section \([\text{IV}]\) a fully distributed algorithm over weight-balanced UIJSC digraphs is proposed. In Section \([\text{V}]\) numerical examples are presented to demonstrate the effects of the two algorithms. Finally, the paper is concluded in Section \([\text{VI}]\).

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Notation

Let \(\mathbb{C}, \mathbb{R}, \) and \(\mathbb{Z}\) be the set of complex, real, and integer numbers, respectively. The Kronecker product is denoted as \(\otimes\). Let \(\|\cdot\|\) denote the 2-norm of a vector and also the induced 2-norm of a matrix. Given a symmetric matrix \(M \in \mathbb{R}^{m \times m}\), the notation \(M > 0\) \((M \geq 0)\) means that \(M\) is positive definite (positive semi-definite). Denote the eigenvalues of \(M\) in ascending order as \(s_1(M) \leq s_2(M) \leq \ldots \leq s_m(M)\). Let \(I\) and \(0\) denote the identity matrix and zero matrix (or vector) of proper dimensions, respectively. \(1_m := (1, \ldots, 1)^T \in \mathbb{R}^m\) denotes the vector of \(m\) ones. \(\text{col}(v_1, \ldots, v_m) := (v_1^T, \ldots, v_m^T)^T\) denotes the column vector stacked with vectors \(v_1, \ldots, v_m\). The notation \(\text{diag}\{a_i\}\) denotes a (block) diagonal matrix with its \(i\)th diagonal element (block) being \(a_i\). The notation \(C^k\) is used to denote a \(k \in \mathbb{Z}_{\geq 1}\) times continuously differentiable function.

B. Convex Analysis

A differentiable function \(f : \mathbb{R}^m \to \mathbb{R}\) is convex over a convex set \(\mathcal{X} \subset \mathbb{R}^m\) if and only if \[\nabla f(x) - \nabla f(y))^T (x - \frac{ym}{x} \geq 0, \forall x, y \in \mathcal{X}\) and \(x \neq y\), and is strictly convex if and only if the strict inequality holds. It is \(\mu\)-strongly convex if and only if \[\nabla f(x) - \nabla f(y))^T (x - \frac{ym}{x} > 0, \forall x, y \in \mathcal{X}\) and \(x \neq y\). An equivalent condition for the strong convexity is the following: \(\| f(y) \| \geq f(x) + \nabla f(x)^T (x - y) + \frac{\mu}{2}\|x - y\|^2, \forall x, y \in \mathcal{X}\) and \(x \neq y\). An operator \(f : \mathbb{R}^m \to \mathbb{R}^m\) is \(\ell\)-Lipschitz continuous over a set \(\mathcal{X} \subset \mathbb{R}^m\) if \[\| f(x) - f(y)\| \leq \ell\|x - y\|, \forall x, y \in \mathcal{X}\]

C. Graph Theory

The information exchanging network is represented by a graph \(G = (\mathcal{N}, \mathcal{E}, \mathcal{A})\), where \(\mathcal{N} = \{1, \ldots, N\}\) is the node set of all agents, \(\mathcal{E} \subset \mathcal{N} \times \mathcal{N}\) is the edge set, and \(\mathcal{A}\) is the adjacency matrix. The edge \((i, j) \in \mathcal{E}\) means that agent \(i\) can obtain information from agent \(j\), and \(j \in \mathcal{N}_i\), where \(\mathcal{N}_i = \{(i, j) \in \mathcal{E}\}\) is agent \(i\)’s neighbor set. The graph \(G\) is said to be undirected if \((i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}\) and directed otherwise.

A sequence of successive edges \(\{(i, p), (p, q), \ldots, (v, j)\}\) is a directed path from agent \(i\) to agent \(j\). \(G\) is said to be strongly connected if there exists a directed path between any two agents. A time-varying graph \(G(t)\) is said to be uniformly jointly strongly connected (UIJSC) if there exists a \(T > 0\) such that for any \(t_k\), the union \(\cup_{t_e \in [t_k, t_k + T]} G(t)\) is strongly connected. The adjacency matrix is defined as \(\mathcal{A} = [a_{ij}]\), where \(a_{ii} = 0; a_{ij} > 0\) if \((i, j) \in \mathcal{E}\), and \(a_{ij} = 0\), otherwise. The in-degree and out-degree of the \(i\)th agent are \(d_{in} = \sum_{j=1}^{N} a_{ij}\) and \(d_{out} = \sum_{j=1}^{N} a_{ij}\), respectively. The graph \(G\) is said to be weight-balanced if \(d_{in} = d_{out}\), \(\forall i \in \mathcal{N}\). The in-degree matrix is \(W_{in} = \text{diag}\{d_{in}\}\). The Laplacian matrix of \(G\) is defined as \(L = W_{in} - A\).

D. Passivity

Consider a nonlinear dynamics described by

\[
\begin{align*}
\dot{x} &= F(x, u) \\
y &= H(x, u)
\end{align*}
\]

where \(x \in \mathcal{X} \subset \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m\) and \(y \in \mathcal{Y} \subset \mathbb{R}^m\) are the state, input and output, respectively, and \(\mathcal{X}, \mathcal{U}\) and \(\mathcal{Y}\) are the state, input and output spaces, respectively. The functions \(F : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n, H : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^m\) represent system and output dynamics, respectively, and are assumed to be sufficiently smooth.

Let us first give the definition of passivity and input feedforward passivity for a nonlinear system based on \([26] \text{ Definition 6.3}], [27] \text{ Definition 2.12}\).

**Definition 1.** System \((1)\) is said to be passive if there exists a continuously differentiable positive semi-definite function \(V(x)\), called the storage function, such that \(\dot{V} \leq u^T y, \forall (x, u) \in \mathcal{X} \times \mathcal{U}\).

Moreover, it is said to be input feedforward passive (IFP) if \(\dot{V} \leq u^T y - \nu u^T y, \forall (x, u) \in \mathcal{X} \times \mathcal{U}, \nu \in \mathbb{R}, \text{ denoted as IFP}(\nu)\).

The sign of the IFP index \(\nu\) denotes an excess or shortage of passivity. Particularly, when \(\nu > 0\), the system is said to be input strictly passive (ISP). When \(\nu < 0\), the system is said to be input feedforward passivity-short (IFPS). If we define a new output as \(\tilde{y} = y - \nu u\), then the IFP system becomes passive with respect to \(u\) and \(\tilde{y}\).

Throughout this paper, we consider the storage function to be positive definite and radially unbounded.
E. Problem Formulation

Let us formulate the problem and give some necessary assumptions in this subsection. Consider the convex distributed optimization problem among a group of agents with the node set \( N = \{1, \ldots, N\} \),

\[
\min_{\mathbf{x}} \sum_{i \in N} f_i(x)
\]

(3)

where \( x \in \mathbb{R}^m \) and each local objective function \( f_i : \mathbb{R}^m \mapsto \mathbb{R} \) satisfies the following assumption.

**Assumption 1.** Each \( f_i(x) \) is \( C^2 \) and \( \mu_i \)-strongly convex, with its gradient \( \nabla f_i(x) \) being \( l_i \)-Lipschitz continuous.

This assumption also implies that \( \|\nabla f_i(x) - \nabla f_i(x')\| \leq l_i \|x - x'\| \) and \( \mu_i I \leq \nabla^2 f_i(x) \leq l_i I \), \( \forall x, x' \in \mathbb{R}^m \). Note that Assumption 1 is widely adopted in the literature, see, e.g., [13], [28]. It is required in this paper to ensure IFP properties and to estimate IFP indices of agents. In addition, it is shown later that the Lipschitz requirement can be relaxed by selecting proper parameters in the algorithms.

Problem (3) is equivalent to (12)

\[
\min_{x, i \in N} f(x) = \sum_{i \in N} f_i(x_i)
\]

subject to \( i \in N \)

(4)

where \( x_i \in \mathbb{R}^m \) is the local decision variable for the \( i \)-th agent and \( x = \text{col}(x_1, \ldots, x_N) \). Under Assumption 1 the optimal solution to problem (4) should satisfy

\[
\sum_{i \in N} \nabla f_i(x_i) = \mathbf{0}, \quad i \in N
\]

(5)

This is because if the constraints in (4) are satisfied, then \( x_i \in N \) in (5) can be denoted by the same variable \( x \), i.e.,

\[
\sum_{i \in N} \nabla f_i(x) = \mathbf{0}
\]

(6)

which is exactly the optimality condition for (3) [29, Section 5.5.3].

Consider the distributed optimization over UIJC weight-balanced digraphs. To the best of our knowledge, this problem has never been addressed in the continuous-time scheme.

**Assumption 2.** The agents interact with each other through a sequence of UIJC digraphs \( \mathcal{G}(t) \), where \( \mathcal{G}(t) \) is weight-balanced pointwise in time and \( L(t) \neq \mathbf{0}, \forall t \geq 0 \).

This assumption does not restrict the switching logic of \( \mathcal{G}(t) \) provided it is UIJC with respective to a finite \( T \). We will propose two algorithms in the following sections. The information of \( L(t) \) is required for the first algorithm, while it is not used at all for the second algorithm. Note that the time interval \( T \) is only imposed to ensure convergence performance, and our results in this work hold as long as \( \mathcal{G}(t) \) is strongly connected in a probability sense [25]. Here the trivial case of \( L(t) = \mathbf{0} \) is omitted.

III. IFP-BASED DISTRIBUTED ALGORITHM

In this section, we propose a distributed algorithm based on input feedforward passivity and study its stability over UIJC balanced topologies.

A. IFP-Based Distributed Algorithm

We propose an IFP-based distributed algorithm as follows.

**Algorithm 1** IFP-Based Distributed Algorithm

- **Initialization:**
  
  1) Choose constants \( \alpha > 0, \beta \in \mathbb{R}, \gamma > 0 \).
  
  2) Choose invertible matrices \( J_i, K_i \in \mathbb{R}^{m \times m} \) such that \( K_i J_i = C^T, \forall i \in N \).
  
  3) Choose any \( x_i(0) \in \mathbb{R}^m \), and \( \lambda_i(0) \in \mathbb{R}^m \) such that \( \sum_{i \in N} K_i \lambda_i(0) = \mathbf{0} \).

- **Design:** Design \( \sigma(t) > 0 \) based on chosen parameters.

- **Dynamics for agent \( i, \in N \):**

  \[
  \begin{align*}
  \dot{x}_i &= -\alpha \nabla f_i(x_i) - K_i \lambda_i + \beta u_i \quad (7a) \\
  \lambda_i &= -\gamma J_i u_i \quad (7b) \\
  u_i &= \sigma(t) \sum_{j \in N_i(t)} a_{ij}(t)(Cx_j - Cx_i) \quad (7c)
  \end{align*}
  \]

For the \( i \)-th agent, \( x_i, \lambda_i, u_i \in \mathbb{R}^m \) and \( u_i \in \mathbb{R}^m \) are local variables and input, respectively; \( J_i, K_i \in \mathbb{R}^{m \times m} \) are invertible matrices such that \( K_i J_i = C^T \) is a common matrix for all agents; \( \alpha > 0, \beta \in \mathbb{R} \) and \( \gamma > 0 \) are constant parameters and \( \sigma(t) > 0 \) is the coupling gain for the diffusive couplings (7). To ease the discussion on parameters, we assume that \( \alpha, \beta, \gamma, C, J_i, K_i, \forall i \in N \) are arbitrarily finite values while \( \sigma(t) \) is a finite and possibly time-varying coupling gain to be designed. Algorithm 1 is a distributed algorithm since each agent only exchanges information with neighboring agents.

Denote \( x = \text{col}(x_1, \ldots, x_N), \lambda = \text{col}(\lambda_1, \ldots, \lambda_N) \). All agents in Algorithm 1 are interconnected with diffusive coupling \( u_i, \forall i \in N \). Eliminating \( u_i \) and the compact form of the overall closed-loop system is written as

\[
\begin{align*}
\dot{x} &= -\alpha \nabla f(x) - K \lambda - \sigma(t)\beta L(t) C x \quad (8a) \\
\lambda &= \sigma(t)\gamma J L(t) C x \quad (8b)
\end{align*}
\]

where \( K = \text{diag}\{K_i\}, J = \text{diag}\{J_i\}, C = I_N \otimes C \) are block diagonal matrices, \( L(t) = L(t) \otimes I_m, \) and \( L(t) \) is the graph Laplacian of \( \mathcal{G} \).

**Lemma 1.** Under Assumptions 1 and 2, if there exists an equilibrium point \( (x^*, \lambda^*) \) to system (8) that satisfies \( \sum_{i \in N} K_i \lambda_i^* = \mathbf{0}, \) where \( x^* = \text{col}(x_1^*, \ldots, x_N^*) \), \( \lambda^* = \text{col}(\lambda_1^*, \ldots, \lambda_N^*) \), then \( (x^*, \lambda^*) \) is also unique with \( x_i^* \) being the optimal solution to problem (3).

**Proof.** The equilibrium point \( (x^*, \lambda^*) \) satisfies

\[
\begin{align*}
\dot{x}^* &= -\alpha \nabla f(x^*) - K \lambda^* = 0 \quad (9a) \\
\lambda^* &= \sigma(t)\gamma J L(t) C x^* = 0 \quad (9b)
\end{align*}
\]

where the term \(-\sigma(t)\beta L(t) C x^*\) in (9a) is zero and omitted since (9b) implies \( L(t) C x^* = 0 \). Since the graph is UIJC, \( \lambda^* = 0 \) for all \( t \) implies that \( C x_j^* = C x_i^* \), \( \forall i, j \in N \). Since \( K_i J_i = C^T \) and \( J_i, K_i \) are invertible, \( C \) is also invertible and thus \( x_i^* = x_j^*, \forall i, j \in N \). Next, multiplying (9a) by \((1_N \otimes I_m)^T\) from the left, one has,

\[
-(1_N \otimes I_m)^T \alpha \nabla f(x^*) - (1_N \otimes I_m)^T K \lambda^*
\]


which satisfies (5). Therefore, \( x^*_i \) is the optimal solution to problem (3). Besides, the strong convexity of \( f(x) \) by Assumption 1 implies that \( x^* \) is unique \([29], \text{Section 9.1.2}\). Since \( K \) is invertible, \( \lambda^* \) is unique as well. \( \square \)

Hereafter, we call \((x^*, \lambda^*)\) the optimal point. The convergence of Algorithm 1 will be addressed in Section III-C.

### B. Input Feedforward Passivity of the Error System

In this subsection, we show that the error subsystem of each agent inherits the input feedforward passivity, which is a crucial step before the convergence analysis over UJSC balanced digraphs, and the design of a passivated algorithm in next section.

By Lemma 1 for agent \( i \), one has

\[
\dot{x}^*_i = -\alpha \nabla f_i(x^*_i) - K_i \lambda^*_i \equiv 0 \quad (10a)
\]

\[
\dot{\lambda}^*_i \equiv 0. \quad (10b)
\]

Denote \( \Delta x_i = x_i - x^*_i \), \( \Delta \lambda_i = \lambda_i - \lambda^*_i \). Then, the group of error subsystems between (7) and (10), \( \forall i \in N \) is

\[
\Sigma_i : \begin{cases} 
\Delta \dot{x}_i = - \alpha \nabla f_i(x_i) - \nabla f_i(x^*_i)) - K_i \Delta \lambda_i + \beta u_i \\
\Delta \dot{\lambda}_i = - \gamma J_i u_i \\
y_i = C \Delta x_i 
\end{cases} \quad \forall i \in N \quad (11)
\]

where \( y_i \) is defined as the output of the \( i \)th error subsystem. Then the input \( u_i, \forall i \in N \) can be rewritten as

\[
u = \sigma(t) \sum_{j \in N_i} a_{ij}(t)(y_j - y_i), \quad \forall i \in N \quad (12)
\]

or compactly, as \( u = -\sigma(t)L(t)y \), where \( u = \text{col}(u_1, \ldots, u_N) \), \( y = \text{col}(y_1, \ldots, y_N) \). Assume that, corresponding to the real agents, there exist a group of virtual agents such that the \( i \)th virtual agent possesses the subsystem \( \Sigma_i \). Then, Algorithm 1 can be seen as output feedback interconnections of these virtual agents. In fact, no information of \((x^*_i, \lambda^*_i)\) is needed for communication since \( y_i - y_j = C\Delta x_i - C\Delta x_j = C(x_i - x_j) \). Then, each agent possesses the same information as its corresponding virtual agent.

We show that each error subsystem \( \Sigma_i \) in (11) is IFP(\( \nu_i \)) with index \( \nu_i \leq 0 \).

**Lemma 2.** Under Assumption 1, each error subsystem \( \Sigma_i \) in (11) is IFP(\( \nu_i \)) with respect to input \( u_i \) and output \( y_i \).

**Proof.** Under Assumption 1 one has \( \nabla f_i(x_i) - \nabla f_i(x^*_i) = B_{x_i}(x_i - x^*_i) \), where \( B_{x_i} = \int_0^1 \nabla^2 f_i(x^*_i + \tau(x_i - x^*_i))d\tau \) is a positive definite matrix such that \( \mu_1 I \leq B_{x_i} \leq \mu_2 I \) (Lemma 1). Clearly, \( B_{x_i} \) is invertible and \( B_{x_i}^{-1} \) is also positive definite. Then, the \( i \)th subsystem in (11) can be written as

\[
\Delta \dot{x}_i = -\alpha B_{x_i} \Delta x_i - K_i \Delta \lambda_i + \beta u_i \\
\Delta \dot{\lambda}_i = -\gamma J_i u_i \\
y_i = C \Delta x_i.
\]

Since \( \dot{x}^*_i = \dot{\lambda}^*_i = 0 \), one has \( \dot{x}_i = \Delta \dot{x}_i \) and \( \lambda_i = \Delta \lambda_i \). Denote \( z_i = \alpha [\nabla f_i(x_i) - \nabla f_i(x^*_i)] + K_i \Delta \lambda_i \), or equivalently,

\[
z_i = \alpha B_{x_i} \Delta x_i + K_i \Delta \lambda_i \quad (13)
\]

then

\[
\dot{x}_i = \Delta \dot{x}_i = -z_i + \beta u_i. \quad (14)
\]

It follows from the gradient of \( f_i(x_i) \) that

\[
\dot{z}_i = \alpha \nabla^2 f_i(x_i) \Delta \dot{x}_i + K_i \Delta \dot{\lambda}_i. \quad (15)
\]

Let us consider the storage function

\[
V_i = \frac{\eta_i}{2} z_i^T z_i - \frac{1}{\gamma} \Delta x_i^T K_i \Delta \lambda_i + \frac{\alpha}{\gamma} [f_i(x_i) - f_i(x)] + \frac{\alpha}{\gamma} [\nabla f_i(x_i)^T \Delta x_i] \quad (16)
\]

where \( \eta_i \) is a positive parameter such that \( \eta_i > \frac{1}{\mu_1 \alpha \gamma} \). By the strong convexity of \( f_i \), one has

\[
f_i(x^*_i) \geq f_i(x_i) - \nabla f_i(x_i)^T \Delta x_i + \frac{\mu_1}{2} \Delta x_i^T \Delta x_i + \frac{\alpha}{\gamma} \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

\[
= f_i(x_i) - \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i) + \frac{\alpha}{\gamma} \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

Then

\[
V_i \geq \frac{\eta_i}{2} z_i^T z_i - \frac{1}{\gamma} \Delta x_i^T K_i \Delta \lambda_i + \frac{\alpha}{\gamma} \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i) - \frac{\alpha}{\gamma} \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

\[
+ \frac{\alpha}{\gamma} (\alpha B_{x_i} \Delta x_i)^T \mu_1 B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

\[
= \frac{\eta_i}{2} z_i^T z_i - (\alpha B_{x_i} \Delta x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i) - (\alpha B_{x_i} \Delta x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

\[
= \frac{\eta_i}{2} z_i^T z_i - \frac{1}{\gamma} \Delta x_i^T K_i \Delta \lambda_i + \frac{\alpha}{\gamma} \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

\[
= \frac{\eta_i}{2} z_i^T z_i - \frac{1}{\gamma} \Delta x_i^T K_i \Delta \lambda_i + \frac{\alpha}{\gamma} \nabla f_i(x_i)^T B_{x_i}^{-1} \alpha (\alpha B_{x_i} \Delta x_i)
\]

where \( R_i = \left[ \begin{array}{c} \eta_i I + \frac{\mu_1 B_{x_i}^{-1} \alpha}{2 \alpha \gamma} - \frac{B_{x_i}^{-1}}{\alpha \gamma} \frac{\eta_i I - B_{x_i}^{-1}}{2 \alpha \gamma} \\
\frac{\eta_i I - B_{x_i}^{-1}}{2 \alpha \gamma} \end{array} \right] \quad (16) \]

By the Schur complement \([30]\), \( R_i > 0 \) if and only if \( \frac{\eta_i}{2} > 0 \) and

\[
\frac{\eta_i}{2} + \frac{\mu_1 B_{x_i}^{-1}}{2 \alpha \gamma} - \frac{B_{x_i}^{-1}}{\alpha \gamma} > 0
\]

Select \( \eta_i \) such that \( \eta_i > \frac{1}{\mu_1 \alpha \gamma} \), then \( R_i > 0 \). Hence, \( V_i > 0 \) and \( V_i = 0 \) if and only if \((x_i, \lambda_i) = (x^*_i, \lambda^*_i)\).

Recall equations (13) to (15), the derivative of \( V_i \) gives

\[
\dot{V}_i = \frac{\eta_i}{2} z_i^T (-\alpha \nabla^2 f_i(x_i)(z_i - \beta u_i) - K_i \gamma J_i u_i)
\]

\[
= -\frac{1}{\gamma} \Delta x_i^T K_i (\gamma J_i u_i) + \frac{\alpha}{\gamma} \nabla f_i(x_i)^T (z_i - \beta u_i)
\]

\[
= -\Delta x_i^T K_i J_j u_j + \frac{1}{\gamma} z_i^T K_i \Delta \lambda_i - \beta u_i^T K_i \Delta \lambda_i
\]
where the first inequality follows from the strong convexity of $f_i$, the last inequality follows from the inequality of arithmetic and geometric means and $\eta_i > \frac{1}{\mu_i \gamma^2}$ and $\nu_i \leq -\frac{\|g_i\|}{(4\mu_i \eta_i - \frac{1}{2})} \leq 0$. Since parameters in $g_i$ and $\nabla^2 f_i(x_i)$ are bounded, given finite $\eta_i$, a constant $\nu_i$ can be obtained. Thus, the subsystem $\Sigma_i$ is IFP($\nu_i$).

As pointed out by [24], it is in general difficult to derive the exact IFP index for a nonlinear system, and only its lower bound can be obtained by specifying the storage function. With the storage function (16), the lower bound of the exact IFP index can be obtained locally by solving the minimax problem

$$\nu_i = -\min_{\eta_i} \max_{x_i} \frac{\|\eta_i \alpha \beta \nabla f_i(x_i) - \gamma C^T \| - \frac{\beta I}{\gamma}\|y_i\|^2}{4 \left( \mu_i \eta_i \alpha - \frac{1}{2} \right)}.$$ (18)

When each $f_i$ is quadratic, $\forall i \in N$, the error system (11) simply becomes a linear system. The exact IFP index for a linear system can be easily obtained by solving the LMI in [31, Lemma 2].

The problem of reducing this gap between the lower bound and the exact index of IFP remains open and leaves to the future work.

**Remark 1.** It is in general not difficult to obtain $\nu_i$ by solving (18) since local objective functions are usually of simple forms. Even when the local objective functions are complicated, problem (18) can be relaxed to

$$\nu_i \geq -\min_{\eta_i} \max_{x_i} \frac{\|\eta_i \alpha \beta \nabla f_i(x_i) - \gamma C^T \| + \| \eta_i \gamma \|}{4 \left( \mu_i \eta_i \alpha - \frac{1}{2} \right)}.$$ (19)

where $l_i$ is the Lipschitz index defined in Assumption 7. Here (19) can be easily solved, providing a lower bound of the exact IFP index, which we can denote as the new $\nu_i$. It can also be observed that when $\beta = 0$, (18) is reduced to $\nu_i = -\min_{\eta_i} \frac{\eta_i^2 \gamma^2 \|C^T\|^2}{4 \left( \mu_i \eta_i \alpha - \frac{1}{2} \right)}$. The IFP index of agent $i$ is only related to the strong convexity index $\mu_i$. In this case, the Lipschitz continuity of the gradients is not required.

**Remark 2.** Let $J_i = K_i = I$, and $\sigma(t) = 1$. When $\gamma = \alpha \beta$, Algorithm 7 reduces to the distributed algorithm in [13]. When $\alpha = \gamma = 1$, and $\beta = 0$, Algorithm 7 reduces to the simplified algorithm in [13]. Compared with algorithms in [13], Algorithm 7 includes more general cases whose convergence cannot be proved by methods in [13], e.g., when $\sigma(t)$ is time-varying, when $\beta$ is negative, and when $\gamma$ is independent of $\alpha, \beta$. Besides, agents in Algorithm 7 can exchange the information of $C x_i$ instead of $x_i$ thanks to extra matrices $J_i, K_i$. Moreover, it is shown later that Algorithm 7 is valid over UJSC topologies in addition to directed and strongly connected switching topologies [13].

**C. Algorithm Over UJSC Balanced Topologies**

In this subsection, we show that the IFP framework allows the study of distributed algorithms over directed and UJSC switching topologies. Meanwhile, the effort in constructing complicated candidate Lyapunov functions in convergence analysis is greatly reduced.

**Definition 2.** The group of agents $\Sigma_i, \forall i \in N$ is said to achieve output consensus if its outputs satisfy $\lim_{t \to \infty} \|y_i(t) - y_j(t)\| = 0, \forall i, j \in N$.

**Theorem 1.** Under Assumptions 1 and 2, Algorithm 1 will converge to the optimal point and solve problem (3) if $\sum_{i \in N} K_i \lambda_i(0) = 0$ and the coupling gain $\sigma(t)$ satisfies

$$0 < \sigma(t) < \frac{s_+ \left( L(t) + L^T(t) \right)}{2 \tilde{\nu} s_N \left( L^T(t) L(t) \right)}, \forall t > 0$$ (20)

where $\tilde{\nu} < 0$ is the smallest value of IFP index $\nu_i, i \in N$, $s_+(\cdot)$ denotes the nonzero smallest eigenvalue, and $s_N(\cdot)$ is defined in Section II-A.

It can be proved through the Lyapunov function $V = \sum_{i \in N} V_i$, where $V_i$ is defined in (16), and by the fact that $L(t) + L^T(t)$ and $L^T(t) L(t)$ have the same null space. The details of the proof with constant $\sigma$ can be found in the conference paper [1]. Condition (20) requires the calculation of eigenvalues, which may be difficult to verify in a large-scale network, thus, a more practical condition can be derived in a different manner, which is easier to verify or estimate for the design of the coupling gain in a distributed sense.

**Theorem 2.** Under Assumptions 1 and 2, Algorithm 7 with initial condition $\sum_{i \in N} K_i \lambda_i(0) = 0$ will converge to the optimal point and solve problem (5) if the coupling gain $\sigma(t)$ satisfies

$$0 < \sigma(t) < \frac{1}{2 \max_i \{d_i(t) | \nu_i | \}}$$ (21)

where $d_i(t)$ is the in/out degree of the $i$th agent.

**Proof.** Let $V = \sum_{i \in N} V_i$, where $V_i$ is defined in (16). Since $B_{z_i}, K_i$ are bounded, $\|\Delta \| \to \infty \Rightarrow V \to \infty$, and thus $V$ is radially unbounded. Suppose (21) holds, i.e.,
\[
\frac{1}{2} - \sigma(t)|\nu_i|d^i(t) > 0, \forall i \in N. \text{ Then, the derivative of } V \text{ gives }
\]
\[
\dot{V} \leq \sum_{i \in N} y_i^T u_i - \nu_i u_i^T u_i
\]
\[
= \sigma(t) \sum_{i \in N} \sum_{j \in N_i(t)} a_{ij}(t) (y_j - y_i) - \nu_i u_i^T u_i
\]
\[
= - \sigma(t) \left( \sum_{i \in N_j \in N(t)} a_{ij}(t) (y_j^T y_i - 2y_i^T y_j + y_j^T y_j) \right)
\]
\[
= - \sigma(t) \left( \sum_{i \in N} a_{ij}(t) (y_j - y_i) \right)^2
\]
\[
- \nu_i \left( \sum_{j \in N_i(t)} a_{ij}(t) (y_j - y_i) \right)^2
\]
\[
\leq - \sigma(t) \left( \sum_{j \in N_i(t)} a_{ij}(t) \right)^2 \sum_{j \in N_i(t)} a_{ij}(t) (y_j - y_i)^2
\]
\[
= - \sigma(t) \sum_{j \in N_i(t)} \left( \frac{1}{2} - \sigma(t)|\nu_i|d^i(t) \right) \sum_{j \in N_i(t)} a_{ij}(t) (y_j - y_i)^2
\]
\[
\leq 0
\]
where \( Y^T Y := \text{col}(y_1^T y_1, \ldots, y_N^T y_N) \), the fourth equality follows from \( (1_N^T \otimes I_m) L(t) = (1_N^T \otimes I_m) \otimes I_m = 0 \), the second inequality follows from the Cauchy-Schwarz inequality, and the last inequality follows from \([21]\).

Then \( \lim_{t \to +\infty} V(t) \) exists and is finite, \( \dot{V} \leq 0 \) implies that the states \( \Delta x, \Delta \lambda \) are bounded. The systems trajectories are bounded within the domain \( S_0 = \{ V(t) \leq V(0) \} \). By the first term in \([17]\) and the jointly connectedness of \( G(t) \), \( V \equiv 0 \) only if \( z_i = 0 \) and \( y_i = y_j, \forall i, j \in N \), where \( z_i \) is defined in \([13]\). Define the domain \( S_z := \{ z_i = 0, y_i = y_j, \forall i, j \in N \} \). Clearly, \( \{ S_z \} \) is bounded for any bounded \( \Delta x, \Delta \lambda \). Invoking the LaSalle’s Invariance Principle for nonautonomous systems \([32]\), we conclude that the system states ultimately reach the domain \( S_0 \cap S_z \). Then output consensus is achieved by Definition \([2]\). Recall \([12]\), one has \( u = 0 \) when output consensus is achieved. Then, by \([12, 14]\), \( \Delta x \to 0, \Delta \lambda \to 0 \), or equivalently, \( \dot{x} \to 0, \lambda \to 0 \) as \( t \to \infty \), i.e., the states of \([6]\) asymptotically converge to an equilibrium point.

Since \( \lambda - \lambda(0) = \int_0^t \dot{\lambda}(\tau)d\tau \), given the initial condition \( \sum_{i \in N} K_i \lambda_i(0) = 0 \),
\[
(1_N \otimes I_m)^T K \lambda
\]
\[
= (1_N \otimes I_m)^T \left( \int_0^t \sigma(\tau) \gamma JL(\tau)Cx(\tau)d\tau + \lambda(0) \right)
\]
\[
= \gamma \int_0^t \sigma(\tau)(1_N \otimes I_m)^T (I_N \otimes C^T)(L(\tau) \otimes I_m)Cx(\tau)d\tau
\]
\[
+ \sum_{i \in N} K_i \lambda_i(0)
\]
\[
= \gamma \int_0^t \sigma(\tau)(1_N \otimes I_m)^T L(\tau) \otimes C^TCx(\tau)d\tau
\]
\[
= 0
\]
where the third equality follows from rules of the Kronecker product and the last follows from \( 1_N^T L(\tau) = 0 \). Then Lemma \([1]\) holds, the equilibrium point is the unique optimal point. Consequently, Algorithm \([4]\) will asymptotically converge to the optimal point.

The initial condition \( \sum_{i \in N} K_i \lambda_i(0) = 0 \) is required to ensure the optimality of the equilibrium according to Lemma \([1]\).

A simple initial choice can be \( \lambda_i(0) = 0, \forall i \in N \).

The case of fixed directed topologies can be seen as a special case of switching topologies, then the convergence is also guaranteed, which is stated by the following corollary. Readers can refer to our conference paper \([1]\) for more technical details.

**Corollary 1.** Suppose the communication digraph \( G \) is fixed, strongly connected and weight-balanced. Then, under Assumption \([\ref{T}]\), Algorithm \([\ref{alg}]\) with initial condition \( \sum_{i \in N} K_i \lambda_i(0) = 0 \) will converge to the optimal point if \([20\) or \(21\) holds.

**Remark 3.** Note that only weight-balanced graphs are considered here. The consensus over unbalanced graphs can be guaranteed similarly \([22, 24]\) with \( V = \sum_{i \in N} \xi_i V_i \), where \( \xi_i > 0 \) is the \( i \)th element of the left eigenvalue of \( L \). However, the sum of local objective functions will have a shift from global optimum \([7]\). Thus, some modification is needed. This problem may be solved by adding a state to estimate the left eigenvalues of \( L \) (e.g., \([15]\)), which we will leave to future work.

**D. Discussion on the coupling gain**

In this subsection, we proceed to discuss the parameters and the design of the coupling gain \( \sigma(t) \) for Algorithm \([\ref{alg}]\).

By Lemma \([2]\), the subsystem \( \Sigma_i \) is IFP regardless of values of \( \alpha, \beta, \gamma, K_i, \lambda_i \). Let \( \sigma_c \) be the threshold of \( \sigma(t) \). Clearly, \( \sigma_c > 0 \) by the above theorems, meaning that there always exists a small enough \( \sigma(t) \) to synchronize the outputs. Thus, \( \alpha, \beta, \gamma, K_i, \lambda_i \in N \) can be arbitrarily chosen within the range specified in Algorithm \([1]\). Intuitively, the larger \( \alpha, \beta, \gamma \), \( K_i, \lambda_i \in N \) are, the faster the convergence rate is, and the matrices \( J_i, K_i \) and \( C \) will affect the consensus speed for different elements of the output vectors. However, this choice of parameters will affect the IFP index by \([18]\), and hence affect the feasible range of \( \sigma(t) \). In fact, for proper parameters, there is usually a wide feasible range for the coupling gain. Let us take for instance the
By Lemma 1 the optimality is preserved. The above discussion is summarized as the following corollary.

**Corollary 2.** Under Assumptions 1 and 2, Algorithm 1 with initial condition \( \sum_{i \in \mathcal{N}_t} K_i \lambda_i(0) = 0 \) will converge to the optimal point and solve problem (3) if for \( k = 1, \ldots, q \), the coupling gain \( \sigma_i(t) \), for agent \( i \), \( i \in \mathcal{N}^k(t) \) satisfies

\[
0 < \sigma_i(t) < \frac{1}{2 \max_{j \in \mathcal{N}^k(t)} \{ d_j(t) / |v_j| \}}
\]

and \( \sigma_i(t) = \sigma_j(t) \), \( \forall i, j \in \mathcal{N}^k(t) \).

**IV. DISTRIBUTED DERIVATIVE FEEDBACK ALGORITHM**

Note that Algorithm 1 still depends on global graph information. In this section, we proposed a fully distributed derivative feedback algorithm based on the passivation of Algorithm 1.

**A. Passivation And Derivative Feedback**

The derivative feedback is widely used in distributed algorithms to ensure convergence or to modify algorithms for directed graphs [8, 14, 35, 36]. In this subsection, we design a new distributed algorithm and reveal that the input-feedforward passivation of IFPS agents through an internal feedforward loop is a form of derivative feedback.

Let us consider again each error subsystem \( \Sigma_i \) in (11). Suppose \( \Sigma_i \) is IFPS, \( \forall i \in \mathcal{N} \), we apply a passivation through feedforward of input. Define a new output as \( \tilde{y}_i \) for the \( i \)th subsystem. Let

\[
\tilde{y}_i = y_i - \nu_i u_i, \quad \forall i \in \mathcal{N}
\]

where \( \nu_i < 0 \) is the IFP index of agent \( i \). The transformation is shown in Figure 1.

![Block diagram of the input-feedforward passivation](image)

Fig. 1: Block diagram of the input-feedforward passivation of the \( i \)th virtual agent \( \Sigma_i \) in (11). The notation \( \Sigma_i \) denotes the transformed system after the input-feedforward passivation.

Obviously, the transformed system \( \Sigma_i \) is passive.

**Lemma 3.** Under Assumption 1, each subsystem \( \tilde{\Sigma}_i \), defined by (11) and (24), is passive from input \( u_i \) to output \( \tilde{y}_i \) with respect to the storage function (16).
then a novel distributed algorithm is constructed as follows.

Algorithm 2 Distributed Derivative Feedback Algorithm

- **Initialization:**
  1. Choose constants $\alpha > 0$, $\beta \in \mathbb{R}$, $\gamma > 0$.
  2. Choose invertible matrices $J_i, K_i \in \mathbb{R}^{m \times m}$ such that $K_i J_i = C^T$, $\forall i \in \mathcal{N}$.
  3. Choose any $x_i(0) \in \mathbb{R}^m$, and $\lambda_i(0) \in \mathbb{R}^m$ such that $\sum_{i \in \mathcal{N}} K_i \lambda_i(0) = 0$.

- **Design:** Calculate $\nu_i$, $\forall i \in \mathcal{N}$ based on chosen parameters.

- **Dynamics for agent $i$, $i \in \mathcal{N}$:**
  
  \begin{align}
  \dot{x}_i &= -\alpha \nabla f_i(x_i) - K_i \lambda_i + \beta u_i \\
  \dot{\lambda}_i &= -\gamma J_i u_i \\
  y_i &= C x_i - \nu_i u_i \\
  u_i &= \sigma(t) \sum_{j \in \mathcal{N}(i)} a_{ij}(t)(\hat{y}_j - \hat{y}_i)
  \end{align}

(27a) (27b) (27c) (27d)

Eliminating $\hat{y}_i$ with (27c) and eliminating $u_i$ with (27b), Algorithm 2 can be rewritten in a compact form

\begin{align}
\dot{x} &= -\alpha \nabla f(x) - K \lambda - \frac{\beta}{\gamma} J^{-1} \dot{\lambda} \\
\dot{\lambda} &= \sigma(t) \gamma J L(t) C x + \sigma(t) J L(t) \nu J^{-1} \dot{\lambda}
\end{align}

(28a) (28b)

where $\nu = diag(\nu_i) \otimes I_m$. Since $J$ is a block diagonal matrix, $J^{-1} = diag(J_i^{-1})$ is also a block diagonal matrix and $J_i^{-1}$ is a local matrix for agent $i$, each agent only requires information from neighboring agents. Thus, Algorithm 2 is a distributed algorithm.

Before proceeding to the next step, let us note that the diffusive couplings of the new outputs of passivated subsystems bring in algebraic loops [37] Section 8.3 into the overall closed-loop system. Thus, we have to check whether the feedback interconnection is well-posed.

The equation (28b) can be written as

\begin{align}
(I - \sigma(t) J L(t) \nu J^{-1}) \dot{\lambda} &= \sigma(t) \gamma J L(t) C x
\end{align}

(29)

Notice that $(I - \sigma(t) J L(t) \nu J^{-1})$ should be nonsingular, then the system in Algorithm 2 can be written in the explicit form, ensuring the well-posedness of the feedback interconnection [38].

The equation (28a) can be written as

\begin{align}
\dot{x} &= -\alpha \nabla f(x) - K \lambda - \sigma \beta J^{-1} (I - \sigma J L(t) \nu J^{-1})^{-1} J L C x
\end{align}

(30a)
$\sigma(t) > 0$, thus $-1$ is not an eigenvalue of $-\sigma(t)JL(t)\nu J^{-1}$. By Lemma 5, $(I - \sigma(t)JL(t)\nu J^{-1})$ is invertible and hence nonsingular. \hfill \Box

B. Algorithm Over UJSC Balanced Topologies

Next, we derive the following theorem that Algorithm 2 is applicable over UJSC weight-balanced digraphs with any positive coupling gain $\sigma(t)$. Theorem 3. Under Assumptions 1 and 2, Algorithm 2 with initial condition $\sum_{i \in N} K_i \lambda_i(0) = 0$ will converge to the optimal point and solve problem (3) given any coupling gain $\sigma(t) > 0$.

Proof. When $\lambda = 0$, system (28) reduces to system (8), meaning that the derivative term does not affect the equilibrium set of system (7). Besides, given the initial condition $\sum_{i \in N} K_i \lambda_i(0) = 0$, $$(1_N \otimes I_m)^T K \lambda = (1_N \otimes I_m)^T K \lambda = \left( \int_0^t \left( \sigma(\tau) \gamma JLCx + \sigma(\tau)JL\nu J^{-1} \lambda \right) d\tau + \lambda(0) \right)$$

$$= \gamma \int_0^t \sigma(\tau)(1_N \otimes I_m)^T (1_N \otimes C^T)(L(\tau) \otimes I_m)Cx(\tau) d\tau + \int_0^t \sigma(\tau)(1_N \otimes I_m)^T (1_N \otimes C^T)(L(\tau) \otimes I_m)\nu J^{-1} \lambda(\tau) d\tau$$

$$+ \sum_{i \in N} K_i \lambda_i(0)$$

$$\int_0^t \sigma(\tau)(1_N^T L(\tau) \otimes C^T) \left( \gamma Cx(\tau) + \nu J^{-1} \lambda(\tau) \right) d\tau$$

$$= 0$$

where the third equality follows from rules of the Kronecker product and initial conditions, the last follows from $1_N^T L(\tau) = 0$. It can also be shown by using the explicit expression (30b) of $\lambda$ that $(1_N \otimes I_m)^T K \lambda = 0$, satisfying Lemma 1. Thus, the equilibrium point of Algorithm 2 with the initial condition $\sum_{i \in N} K_i \lambda_i(0) = 0$ is still the optimal point to the distributed optimization problem (3).

The information of $(x_t^i, \lambda_t^i)$ is not required for exchange. Then Algorithm 2 can be implemented by output feedback interconnections of virtual agents $\Sigma_i, \forall i \in N$. Since $\Sigma_i$ is passive with respect to input $u_i$ and output $\tilde{y}_i$ by Lemma 3, the consensus analysis among passive agents is similar to that among IFP agents with IFP indices being zero. Specifically, let $V = \sum_{i \in N} V_i$, where $V_i$ is defined in (16). By (25),

$$V = \sum_{i \in N} \tilde{y}_i^T u_i$$

$$- \sigma(t) \sum_{i \in N} \sum_{j \in N(N_i)} a_{ij}(t) (\tilde{y}_i^T \tilde{y}_j - \tilde{y}_i)$$

$$= - \frac{\sigma(t)}{2} \sum_{i \in N} \sum_{j \in N(N_i)} a_{ij}(t) (\tilde{y}_i^T \tilde{y}_j - \tilde{y}_i + \tilde{y}_j^T \tilde{y}_j)$$

$$- \frac{\sigma(t)}{2} \sum_{i \in N} \sum_{j \in N(N_i)} a_{ij}(t) (\tilde{y}_i^T \tilde{y}_i - \tilde{y}_i^T \tilde{y}_j) \leq 0$$

where $\tilde{y}^T \tilde{y} = col(\tilde{y}_1^T, \ldots, \tilde{y}_N^T)$. Following similar lines of the proof of Theorem 1, Algorithm 2 with initial condition $\sum_{i \in N} K_i \lambda_i(0) = 0$ will asymptotically converge to the optimal point. \hfill \Box

Similar to Corollary 1, Theorem 3 can be directly applied to fixed weight-balanced strongly connected digraphs as a special case of UJSC topologies, as stated in the following corollary.

Corollary 3. Suppose the communication digraph $\mathcal{G}$ is fixed, strongly connected and weight-balanced. Then, under Assumption 1 Algorithm 2 with initial condition $\sum_{i \in N} K_i \lambda_i(0) = 0$ will converge to the optimal point for any $\sigma(t) > 0$.

C. Discussion on the Derivative Feedback Algorithm

Though Algorithm 2 requires agents to exchange with each other more information like derivatives of states, its advantages are significant.

Compared with most works on directed topologies, Algorithm 2 is robust against randomly changing weight-balanced digraphs with any positive coupling gain and independent of any graph information. Since the time interval $T$ for UJSC graphs is not used in the proofs, $\mathcal{G}(t)$ can be relaxed to be strongly connected in a probability sense, namely, it is applicable for gossip-like balanced digraphs [25]. It can be observed from Figure 1 that this modified algorithm can be easily realized by adding a local input feedforward loop to each subsystem $\Sigma_i$. Since the input $u_i$ of the $i$th virtual agent is the same as the input of the real agent $i$, the input feedforward of virtual agents is actually the same as the input feedforward of real agents. Also note that the passivation is achieved by each agent locally, no global information is needed beforehand. Thus, Algorithm 2 is a fully distributed algorithm.

V. Numerical Examples

Example 1

We present a numerical example to demonstrate the effect of Algorithm 1 over directed and switching topologies in this example. Consider a network of 4 agents possessing the following local objective functions: $f_i : \mathbb{R} \mapsto \mathbb{R}, \ i = 1, 2, 3, 4$, respectively.

$$f_1(x) = 0.4x^2 - x$$

$$f_2(x) = \ln(e^{-0.3x} + e^{0.5x}) + 0.6x^2$$

$$f_3(x) = x^2 + \cos x$$

$$f_4(x) = \frac{x^2}{\sqrt{x^2 + 1}} + 0.9x^2.$$
By calculation, we obtain that $\mu_1 = l_1 = 0.8; \mu_2 = 1.20, l_2 = 1.36; \mu_3 = 1, l_3 = 3; \mu_4 = 1.76, l_4 = 3.8$. Let $\alpha = \beta = \gamma = 1$, and $J_i = \frac{1}{7}, K_i = i$. Then, we obtain that each subsystem in (11) is IFP with $\nu_1 = -0.31, \nu_2 = -0.49, \nu_3 = -1, \nu_4 = -0.68$. Next, we consider two cases of topologies.

Case 1: the agents are connected through a ring graph that is strongly directed and weight-balanced, as shown in Figure 2.

![Diagram of a ring graph](image)

**Fig. 2:** The communication graph for the 4 agents.

The corresponding graph Laplacian is $L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$.

Case 2: for every 0.1 second, the graph $G(t)$ switches randomly among three modes as shown in Figure 3.

![Diagram of a switching communication graph](image)

**Fig. 3:** The switching communication graph for the 4 agents.

The corresponding graph Laplacians are $L_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, L_3 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and $L_4 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$, respectively.

The threshold coupling gains are obtained as $\sigma_x = 0.50$ in (21) for both cases. We implement Algorithm 1 in MATLAB over these two cases with $x_i(0) \in [0, 1], \lambda(0) = 0$ satisfying the initial condition. Let $\sigma(t) = 0.35 + 0.1 \cos(t) < \sigma_x$ in Case 1. To demonstrate Corollary 2, we adopt different coupling gains for different disjoint subgraphs in Case 2. Specifically, let $\sigma_1(t) = \sigma_2(t) = 0.3 + \sin(t)$ and $\sigma_3(t) = \sigma_4(t) = 0.35 + \cos(t)$ for Mode 1; $\sigma_2(t) = \sigma_3(t) = 0.3 + \sin(t)$ and $\sigma_1(t) = 0.35 + \cos(t)$ for Mode 2; $\sigma_2(t) = \sigma_3(t) = 0.3 + \sin(t)$ and $\sigma_1(t) = 0.35 + \cos(t)$ for Mode 3.

The convergence results are shown in Figure 4. It can be observed that the trajectories of $x_i, \forall i \in N$ in Algorithm 1 over a weight-balanced digraph with a time-varying coupling gain.

![Trajectories of agents](image)

**Fig. 4:** The trajectories of $x_i, \forall i \in N$ in Algorithm 1 over a weight-balanced digraph with a UJSC graph with different time-varying couplings for disjoint subgraphs.

**Example 2**

We present another example to compare the effects of the two distributed algorithms in this example. Consider a network of 4 agents interconnected through the same graph as Figure 2.

The local objective functions are:

$$f_i(x) = 0.025(i + 1)(x - i)^2, \; x \in \mathbb{R}, \; i = 1, 2, 3, 4.$$}

Let $\alpha = \beta = \gamma = 1$ and $J_i = K_i = I$. By solving the LMI in [31] Lemma 2 with the YALMIP Toolbox [39], we obtain...
that the agents are IFPS with \( \nu_1 = -89.96, \nu_2 = -37.77, \nu_3 = -20.00 \) and \( \nu_4 = -12.00 \). Then by \( (21) \), the coupling gain threshold is obtained as \( \sigma_e = 0.0056 \). According to Corollary 1 when \( \sigma < \sigma_e \), the trajectories of Algorithm 1 will converge to the optimal point. Then, we implement the two distributed algorithms in MATLAB with \( x_i(0) \in [2, 3] \), \( \lambda(0) = 0 \) satisfying the initial condition, and \( \sigma = 0.005 \in (0, \sigma_e) \). The trajectories of the two algorithms asymptotically converge to the optimal solution \( x_i^* = 2.857, i = 1, 2, 3, 4 \), as shown in Figures 5(a) and 5(b).

When the coupling gain is outside the feasibility range, Algorithm 1 is not guaranteed to converge. The error system (11) is a linear system:

\[
\begin{bmatrix}
\Delta \dot{x} \\
\Delta \lambda
\end{bmatrix} =
\begin{bmatrix}
-F - \sigma L & -I \\
\sigma L & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda
\end{bmatrix}
\]

where \( F = \text{diag}\{0.05(i + 1)\} \). Clearly, it is unstable when \( \sigma \in [0.1, 0.14] \), which accords with our discussion in Section III-D. On the other hand, Algorithm 2 should be valid with any positive \( \sigma \) by Theorem 3. To show this, we compare the two distributed algorithms with \( x_i(0) \in [2, 3] \), \( \lambda(0) = 0 \) satisfying the initial condition, and \( \sigma = 0.1 \notin (0, \sigma_e) \). It can be observed from Figures 6(a) and 6(b) that Algorithm 1 is unstable while the trajectories of \( x_i, \forall i \in \mathcal{N} \) in Algorithm 2 asymptotically converge to the optimal solution. Here the IFP indices are different and agents are passivated locally.

VI. Conclusion

This paper has investigated a distributed optimization problem via input feedforward passivity. An input-feedforward-passivity framework has been adopted to construct a distributed algorithm that is applicable over weight-balanced digraphs. Moreover, a novel distributed derivative feedback algorithm, which is fully distributed, has been proposed via the input-feedforward passivation. The proposed algorithms have been studied over directed and uniformly strongly connected balanced topologies. Convergence conditions of a suitable coupling gain for the IFP-based distributed algorithm have been derived, while it has been shown that the distributed derivative feedback algorithm is robust against randomly changing weight-balanced digraphs with any positive coupling gain and without knowing any graph information.

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Fig. 6: The trajectories of $x_i$, $\forall i \in \mathcal{N}$ with $\sigma = 0.1$ in the two distributed algorithms. The initial conditions are $x_i(0) \in [2, 3]$, and $\lambda(0) = 0$, while the other parameters are chosen as $\alpha = \beta = \gamma = 1$, $J_i = K_i = I$.

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