Numerical Field Theory on the Continuum

Stephen C. Hahn and G. S. Guralnik
Department of Physics, Brown University,
Providence RI 02912–1843

Abstract

An approach to calculating approximate solutions to the continuum Schwinger–Dyson equations is outlined, with examples for $\phi^4$ in $D = 1$. This approach is based on the source Galerkin methods developed by García, Guralnik and Lawson. Numerical issues and opportunities for future calculations are also discussed briefly.

1 Introduction

The now-conventional technique of numerical calculation of quantum field theory involves evaluating the path integral

$$Z(J) = \int \mathcal{D}\phi \exp \left[ -S(\phi) + J\phi \right]$$

on a spacetime lattice using Monte Carlo integration methods. Monte Carlo methods have been successful for an interesting class of problems; however, these techniques have had less success in evaluating theories with actions that are not manifestly positive definite or which have important effects from the details of fermionic interactions beyond the quenched approximation. In these cases, we have either a basic algorithmic difficulty or an inadequacy of compute power.

We are developing an alternative computational method which works both on the lattice and the continuum and which handles fermions as easily as bosons. Furthermore, our “source Galerkin” method is less restrictive as to the class of allowed actions. Source Galerkin tends to use significantly less compute time than Monte Carlo methods but can
consume significantly larger amounts of memory. This talk is confined to continuum applications; examples of lattice calculations have been given elsewhere [2, 7, 8]. While it is not clear that Source Galerkin can replace Monte Carlo techniques, it appears that it will be able to solve some problems which are currently inaccessible.

Our approach begins with the differential equations satisfied by the vacuum functional $Z$ for a quantum field theory with external sources. For the sake of simplicity, in this talk we will mostly confine our attention to $\phi^4$ interactions. To date, we have also studied non-linear sigma models and four-fermion interactions and have gauge theory calculations in progress.

The vacuum persistence function $Z$ for a scalar field $\phi$ with interaction $g\phi^4/4$ coupled to a scalar source $J(x)$ satisfies the equation:

$$\left(-\partial^2_D + M^2\right) \frac{\delta Z}{\delta J(x)} + g \frac{\delta^3 Z}{\delta J^3(x)} - J(x)Z = 0 \quad (2)$$

The source Galerkin technique is designed to directly solve functional differential equations of this type. Before we proceed to outline a solution technique, it is essential to point out that this equation by itself does not uniquely specify a theory [3]. This is dramatically illustrated by considering the special case of the above equation limited to one degree of freedom (zero dimensions).

$$M^2 \frac{dZ}{dJ} + g \frac{d^3 Z}{dJ^3} - JZ = 0 \quad (3)$$

This is a third order differential equation, and therefore possesses three independent solutions. It is easy to see that one of the solutions for small $g$ asymptotically approaches the perturbation theory solution, while the second asymptotically approaches the loop expansion “symmetry breaking” solution for small $g$ and the third solution has an

---

1This can happen because the iterative process for improving a source Galerkin calculation involves successively higher-point Green functions, whereas calculation of these correlations in a Monte Carlo scenario is optional.
essential singularity as $g$ becomes small. The situation becomes much more interesting for finite dimensions where the infinite class of solutions coalesces or becomes irrelevant in a way which builds the phase structure of the field theory. Any numerical study must be cognizant of the particular boundary conditions and hence solution of the class of solutions that is being studied. Care must be taken to stay on the same solution as any iterative technique is applied. The solutions discussed in this talk will be the usual ones which correspond to the symmetry preserving solutions obtained from evaluating a path integral with real axis definitions for the regions of integration. These are the solutions that are regular in the coupling, $g$, as it approaches zero.

We, for convenience, write (2) in the form:

$$\hat{E}_j Z(j) = 0$$

(4)

The source Galerkin method is defined by picking an approximation $Z^*(j)$ to the solution $Z(j)$ such that

$$\hat{E}_j Z^*(j) = R$$

(5)

where $R$ is a residual dependent on $j$. We pick the parameters of our approximation to make this residual as small as possible on the average. To give this statement a meaning, we define an inner product over the domain of $j$: \(i.e. (A, B) \equiv \int d\mu(j) A(j)B(j)\). In addition we assume we have a collection of test functions which are members of a complete set) \(\{\varphi_i(j)\}\) The source Galerkin minimization of the residual $R$ is implemented by setting the parameters of our test function $Z^*(j)$ so that projections of test functions against the residual vanish so that $||Z^* - Z||_2 \to 0$ as the number of test functions $\to \infty$.

The equations defining the quantum field theory are differential equations in the field sources and spacetime. While it is straightforward to deal with the spacetime
problem by resorting to a lattice, we can remain in a continuum formulation by tak-
ing advantage if our knowledge of functional integration. We know how to evaluate
Gaussian functional integrals on the continuum:

\[
\int [dj] \exp \left[ \int_{xy} j(x)A(x,y)j(y) + \int_x j(x)\beta(x) \right]
= \frac{1}{\sqrt{\det A}} \exp \left[ \int_{xy} \beta^*(x)A^{-1}(x,y)\beta(y) \right]
\]  \quad (6)

Consequently, we can evaluate integrals of the form:

\[
I = \int [dj] \exp \left[ -j^2(x)/\epsilon^2 \right] P(j)
\]  \quad (7)

Using this we can define an inner product of sources on the continuum as follows:

\[
(j(x_1) \cdots j(x_n), j(y_1) \cdots j(y_m))_j = \begin{cases} 
\epsilon^{n+m}\delta_+\{x_1 \cdots x_n y_1 \cdots y_m\} & n + m \text{ even} \\
0 & \text{otherwise}
\end{cases}
\]  \quad (8)

where we have absorbed a factor of 2 by redefining \(\epsilon\). \(\delta_+\) is defined by

\[
\delta_+\{x\alpha \beta \cdots\} = \delta(x - \alpha)\delta_+\{\beta \cdots\} + \delta(x - \beta)\delta_+\{\alpha \cdots\} + \cdots, 
\]  \quad (9)

\[
\delta_+\{x\alpha\} = \delta(x - \alpha).
\]  \quad (10)

In addition to this inner product definition, we need good guesses for approximate
form for \(Z^*\) and numerical tools to calculate, symbolically or numerically, various func-
tions and their integrals, derivatives, and so on. For most of our calculations we have
found it very useful to choose a lesser known class of functions, with very suitable
properties for numerical calculation, known as Sinc functions. We take our notation
for the Sinc functions from Stenger \(\text{[8]}\):

\[
S(k, h)(x) = \frac{\sin(\pi(x - kh)/h)}{\pi(x - kh)/h}
\]  \quad (11)

4
Some of the identities that Sinc approximations satisfy are given below:

\[ S(k, h)(lh) = \delta_{kl} \]  
(12)

\[ \int_x S(k, h)(x)S(l, h)(x) = \delta_{kl} \]  
(13)

\[ \int_x F(x) \approx h \sum_{k=-N}^{N} F(kh) \]  
(14)

\[ F(x) \approx \sum_{k=-N}^{N} F(kh)S(k, h)(x) \]  
(15)

\[ F'(x) \approx \sum_{k=-N}^{N} F(kh)S'(k, h)(x) \]  
(16)

\[ F^{(n)}(lh) \approx \sum_{k=-N}^{N} F(kh)\delta_{k-l}^{(n)} \]  
(17)

These properties, which are proven and expanded upon greatly in [8], make these functions very easy to use for Galerkin methods, collocation, integration by parts, and integral equations.

With the definition of a norm and set of expansion functions, we can postulate an ansatz for \( \mathcal{Z} \)

\[ \mathcal{Z}^* = \exp \left[ \sum \int_{xy} j(x)G_2(x-y)j(y) + \cdots \right] \]  
(18)

where each Green function, \( G_n \), is represented by a \( d \)-dimensional Sinc expansion. For \( d = 4 \):

\[ G_2(x-y) = \sum G_{ijkl}^{ijkl} \]

\[ S(i, h)(x^0 - y^0)S(j, h)(x^1 - y^1)S(k, h)(x^2 - y^2)S(l, h)(x^3 - y^3) \]  
(19)

It is easy to examine this expansion in the case of a free field where we limit the approximation to the terms quadratic in \( j(x) \). While the example is trivial, it shows
Figure 1: Mass, $m^*$, versus distance, $x$, for two-dimensional free scalar field. Note breakdown near origin (approximation of $\delta$-function) and at large distance (spatial truncation). $m_0 = 0.5$.

that, as always, numerical approximations must be handled with care. Results of this calculation are shown in Figure 1.

This straightforward expansion works for interacting theories but with more than $G_2$, the computational costs and storage costs become overwhelming: $G_{2n}$ requires $N^{(2n-1)d}$ storage units. There are many ways that storage costs can be reduced, but in general these approaches are difficult, not particularly elegant, and eventually reaches a limit due to the exponential growth in the number of coefficients of the representation.

We can use our knowledge of the spectral representations of field theory and graphical approaches to introduce a much more beautiful and intuitive approach to producing candidates for $Z^*$. We introduce regulated Lehmann representations. These build in the appropriate spacetime Lorentz structure into our approximations and remove the growth of operational cost with spacetime dimension shown by our previous naïve decomposition into complete sets of functions. Any exact two-point function can be represented as a sum over free two-point functions. We choose as the basis of our
Table 1: Convergence of Sinc approximation to integral. (Exact integral calculated using Maple V, with 40 digits of precision.) \( x^2 = 10, m = 1, \Lambda^2 = 10. \)

Numerical solutions, a regulated Euclidean propagator structure:

\[
\Delta(m; x) \equiv \int (dp) \frac{e^{ip \cdot x - p^2/\Lambda^2}}{p^2 + m^2} = \int (dp) \int_0^{\infty} ds \, e^{ip \cdot x - p^2/\Lambda^2 - s(p^2+m^2)}
\]

\[
= \frac{1}{(2\pi)^d} \int_0^{\infty} ds \left[ \frac{\pi}{s + 1/\Lambda^2} \right]^{d/2} e^{-sm^2 - x^2/4(s + 1/\Lambda^2)}
\]  

This regulation assures that we never have to deal with infinities in any calculated amplitude as long as we keep the cutoff finite.

This integral can be approximated using Sinc methods

\[
\Delta(m; x) \approx \frac{h}{(2\pi)^d} \sum_{k=-N}^{N} \frac{1}{e^{kh}} \left[ \frac{\pi}{zk + 1/\Lambda^2} \right]^{d/2} \exp \left[ -zk^2m^2 - \frac{x^2}{4(zk + 1/\Lambda^2)} \right],
\]

The example in Table 1 demonstrates that we can have as many digits as are necessary for the calculation, with the associated increase in compute time. For practical purposes, 80 terms is appropriate for most hardware floating-point representations. Thus we have a form for a two-point scalar Green function, regulated by the scale \( \Lambda^2 \) with constant computational cost regardless of spacetime dimension. We can take derivatives explicitly or by construction:

\[
\partial^2 \Delta(m; x) = m^2 \Delta(m; x) - \bar{\delta}(x)
\]

where \( \bar{\delta}(x) = e^{-x^2\Lambda^2/4} \)
From this representation, we can directly construct a fermion two-point function:

$$S(m; x) = (\gamma \cdot \partial - m)\Delta(m; x)$$ (24)

These representations mean that free scalar and free fermion results are exact and immediate in any Galerkin evaluation of these trivial cases. Furthermore, because of this simplicity, we have the basis for a complete numerical approach to conventional perturbation theory.

2 Results with Lehmann representation: $\phi^4$

We itemize some results obtained using a regulated single propagator with parameters set by the Source Galerkin method. At lowest order, our ansatz for the generating functional is

$$Z^* = \exp \int \frac{1}{2} j_x G_{xy} j_y.$$ (25)

Results for this ansatz are given in Figure 2. These results are strikingly accurate and
Figure 3: Additional connector-based ansätze for the four-point function, $H$, in $\lambda \phi^4$. In the bottom row, we have two contact ansätze on the left, followed by two mediated ansätze.

can matched up essentially exactly with results of Monte Carlo calculations in two and higher dimensions.

We can enhance these results by including additional 4 source terms in $Z^*$. Some simple additional terms that we include with weights and masses to be calculated using the Source Galerkin technique are the terms of the forms given in Figure 3. The effect of adding a fourth order term is shown in Figure 4.

In addition to the illustrations given here, we have examined $(\bar{\psi}\psi)^2$ in the mean field in two dimensions and have found rapid convergence to the known results from large-$N$ expansions. It therefore appears that, at least for the simple cases studied, we have produced a numerical method which draws on the structural information already known in general through symmetry and spectral representations which when combined with Galerkin averaging to set parameters converges with very simple guesses for the vacuum amplitude to known correct answers produced through other methods of solution including Monte Carlo methods. More complicated gauge problems are under study.
Figure 4: Comparison of four-$H$ approximation with lowest order and exact answer, in one-dimension ($\Lambda^2 = 70$).
3 Numerical issues

In this very brief presentation we have avoided discussion of many of the difficult numerical issues involved in constructing this approach. We note some of these issues here without discussion to have on record:

- Both interpolative and spectral problems result in medium- to large-scale non-linear systems; systems solvable using many variable Newton’s method

- Finite storage is the key constraint for interpolative representations, which must be constrained to two-point “connectors”, particularly in high dimensions

- Storage is a non-issue for spectral representations (memory use entirely for caching calculated quantities)

- Both methods are also time-bound to connector-based representations for higher point functions

- Time cost from internal loops; however, algorithm can be made parallel via partitioning of sums

- Resolution of elementary pole structures (i.e. differentiating between $\delta$ and $(p^2 + m^2)^{-1}$) may be addressable with arbitrary precision libraries

- Arbitrary precision may also be useful for regulated perturbation calculations.

These numerical issues, and related general numerical techniques for source Galerkin are discussed in greater detail in [1, 5, 4].
4 Conclusions

We have discussed a method for numerical calculations for field theories on the continuum; this method being based on the source Galerkin technique introduced in [2]. The direct approach, using Sinc functions for interpolation, is effective but will ultimately be limited by the finite nature of current computational resources. The Lorentz invariant regulated representation derived from Lehmann representation does not suffer from these limitations, and is applicable to both perturbation and mean-field-based ansätze. This approach has the computational advantages of minimal memory utilization and parallelizable algorithms and also allows direct representation of fermionic Green functions. Finally, a number of useful peripheral calculations can be made using this approximate representation: one can calculate diagrams in a regulated perturbation theory, as well as calculating dimensionally regularized loops numerically. In general, this technique of evaluating field theories takes advantage of the symmetries of the Lorentz group; future work includes the extension of the method to more general internal groups, such as gauge groups or supersymmetry.

Acknowledgments

This work was supported in part by U. S. Department of Energy grant DE-FG09-91-ER-40588—Task D. The authors have been the beneficiaries of many valuable conversations with S. García, Z. Guralnik, J. Lawson, K. Platt, and P. Emirdağ. Certain results in this work were previously published in Hahn [4]. Computational work in support of this research was performed at the Theoretical Physics Computing Facility at Brown University.

References
[1] S. García. *A new numerical method for quantum field theory.* PhD thesis, Brown University, 1993.

[2] S. García, G. S. Guralnik, and J. W. Lawson. A new approach to numerical quantum field theory. *Physics Letters*, B333:119, 1994.

[3] S. García, Z. Guralnik, and G. S. Guralnik. Theta vacua and boundary conditions of the Schwinger–Dyson equations. *hep-th/9612079*, 1996.

[4] S. C. Hahn. *Functional methods of weighted residuals and quantum field theory.* PhD thesis, Brown University, 1998.

[5] J. W. Lawson. *Numerical method for quantum field theory.* PhD thesis, Brown University, 1994.

[6] J. W. Lawson and G. S. Guralnik. New numerical method for fermion field theory. *Nuclear Physics*, B459:612, 1996. *hep-th/9507131*.

[7] J. W. Lawson and G. S. Guralnik. Source Galerkin calculations in scalar field theory. *Nuclear Physics*, B459:589, 1996. *hep-th/9507130*.

[8] F. Stenger. *Numerical Methods Based on Sinc and Analytic Functions.* Springer Series in Computational Mathematics. Springer–Verlag, 1993.