Tracy-Widom distributions in critical unitary random matrix ensembles and the coupled Painlevé II system

Shuai-Xia Xu∗ and Dan Dai†

Abstract We study Fredholm determinants of the Painlevé II and Painlevé XXXIV kernels. In certain critical unitary random matrix ensembles, these determinants describe special gap probabilities of eigenvalues. We obtain Tracy-Widom formulas for the Fredholm determinants, which are explicitly given in terms of integrals involving a family of distinguished solutions to the coupled Painlevé II system in dimension four. Moreover, the large gap asymptotics for these Fredholm determinants are derived, where the constant terms are given explicitly in terms of the Riemann zeta-function.

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Contents

1 Introduction .............................................................................................................. 2

1.1 Expressions of the P₂ and P₃₄ kernels .............................................................. 6

2 Statement of results .............................................................................................. 10

2.1 The coupled Painlevé II system and their asymptotics .................................... 10

2.2 Tracy-Widom type expressions for the Fredholm determinants ....................... 12

2.3 Large gap asymptotics for the Fredholm determinants .................................... 13

3 Riemann-Hilbert problem for the Fredholm determinant and differential identity .................................................................................................................... 15

3.1 Riemann-Hilbert problem for the Fredholm determinant and differential identity ................................................................................................................. 15

3.2 Model Riemann-Hilbert problem for Φ ............................................................... 16

∗Institut Franco-Chinois de l’Energie Nucléaire, Sun Yat-sen University, Guangzhou 510275, China. E-mail: xushx3@mail.sysu.edu.cn

†Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong. E-mail: dandai@cityu.edu.hk (Corresponding author)
4 Coupled Painlevé II equations
4.1 Lax pair ............................................ 20
4.2 Vanishing lemma .................................. 23
4.3 Bäcklund transformations .......................... 24

5 Asymptotics of $v_i(x)$ as $x \to +\infty$ 26
5.1 Nonlinear steepest descent analysis of $\Phi$ as $x \to +\infty$ .... 26
5.2 Proof of Theorem 1 ................................ 31

6 Proof of Theorem 2-3 32
6.1 Tracy-Widom formula for $P_{34}$ kernel: proof of Theorem 2 .... 32
6.2 Tracy-Widom formula for $P_{2}$ kernel: proof of Theorem 3 .... 33

7 Large gap asymptotics 34
7.1 Nonlinear steepest descent analysis of $\Phi$ as $s \to -\infty$ ........ 35
7.2 Large gap asymptotics: proof of Theorem 4 .................. 40
7.3 Large gap asymptotics: proof of Theorem 5 .................. 43

Appendix A Asymptotics of $v_i(x)$ as $x \to -\infty$ 43
A.1 Case I: $s < 0$ ........................................... 43
    A.1.1 Nonlinear steepest descent analysis of $\Phi$ as $x \to -\infty$ ... 43
    A.1.2 Asymptotics of $v_i(x)$ ............................ 46
A.2 Case II: $s > 0$ ....................................... 46
    A.2.1 Nonlinear steepest descent analysis of $\Phi$ as $x \to -\infty$ ... 47
    A.2.2 Asymptotics of $v_i(x)$ ............................ 52

1 Introduction

We consider the space of $n \times n$ Hermitian matrices $M$ with probability distribution

$$\frac{1}{Z_n} |\det M|^{2\alpha} e^{-nTV(M)} dM \quad \text{for } \alpha > -\frac{1}{2},$$

where $V : \mathbb{R} \to \mathbb{R}$ is a real analytic function and satisfies

$$\lim_{x \to \pm \infty} \frac{V(x)}{\log(1 + x^2)} = +\infty.$$  

Here $dM$ is the Lebesgue measure for Hermitian matrices and $Z_n$ is the normalization constant. It is well-known that the joint probability density function for the eigenvalues of $M$ is given by

$$p_n(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_n} \prod_{i=1}^{n} |\lambda_i|^{2\alpha} e^{-nV(\lambda_i)} \prod_{i<j}(\lambda_i - \lambda_j)^2,$$  

which can be put into a determinantal form

$$p_n(\lambda_1, \cdots, \lambda_n) = \frac{1}{n!} \det(K_n(\lambda_i, \lambda_j))_{i,j=1}^{n},$$
with the correlation kernel

\[ K_n(x, y) = |xy|^\alpha e^{-\frac{2}{\beta} V(x)} e^{-\frac{2}{\beta} V(y)} \sum_{k=0}^{n-1} P_k(x) P_k(y); \quad (1.5) \]

see for example [23] and [52]. The above kernel is so-called \textit{orthogonal polynomial kernel}, and \( P_k(x) \) is the \( k \)-th degree orthonormal polynomial with respect to the weight \( |x|^{2\alpha} e^{-n V(x)} \).

In the global regime, the limiting mean density of eigenvalues is

\[ \rho_V(x) = \lim_{n \to \infty} \frac{1}{n} K_n(x, x), \quad (1.6) \]

which depends on the exact potential \( V(x) \); see [23] and [27]. However, the local statistics of eigenvalues only rely on some general characteristics of the density function \( \rho_V(x) \) and satisfy fascinating universal behaviors. For example, given a general real analytic potential \( V(x) \) and \( \alpha = 0 \), the \textit{bulk university} holds for any point \( x^* \) in the bulk of the spectrum, which implies that the limiting correlation kernel is the sine kernel

\[ \lim_{n \to \infty} \frac{1}{n \rho_V(x^*)} K_n(x^* + \frac{x}{n \rho_V(x^*)}, x^* + \frac{y}{n \rho_V(x^*)}) = K_{\text{sin}}(x, y) = \frac{\sin \pi (x - y)}{\pi (x - y)}, \quad (1.7) \]

uniformly for \( x \) and \( y \) in compact subsets of \( \mathbb{R} \) whenever \( \rho_V(x^*) > 0 \). Moreover, the \textit{soft edge universality} holds at a right regular edge point \( b \) of the support of \( \rho_V(x) \). This means that, when \( \rho_V(x) \sim \frac{c}{\pi} (b - x)^{1/2} \) for \( c > 0 \) as \( x \to b^- \), the limiting correlation kernel is the Airy kernel

\[ \lim_{n \to \infty} \frac{1}{(cn)^{2/3}} K_n(b + \frac{x}{(cn)^{2/3}}, b + \frac{y}{(cn)^{2/3}}) = K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}, \quad (1.8) \]

uniformly for \( x \) and \( y \) in compact subsets of \( \mathbb{R} \). The above results for the bulk and soft edge universality were proved in [4, 28, 55].

Like other determinantal point processes, all information of unitary random matrix ensembles is contained in the correlation kernel \( K \). If we consider the \textit{gap probability} that there is no eigenvalue near the point \( x^* \) in the bulk of the spectrum, it is given in terms of a Fredholm determinant as follows

\[ \lim_{n \to \infty} \text{Prob} \left[ M \text{ has no eigenvalues in } (x^* - \frac{s}{n \rho_V(x^*)}, x^* + \frac{s}{n \rho_V(x^*)}) \right] = \det[I - K_{s}^{\text{sin}}], \quad (1.9) \]

where \( K_{s}^{\text{sin}} \) is the trace-class operator acting on \( L^2(-s, s) \) with the sine kernel \( K^{\text{sin}}(x, y) \) in (1.7) and the Fredholm determinant \( \det[I - K_{s}^{\text{sin}}] \) is given by the following series

\[ \det[I - K_{s}^{\text{sin}}] = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \int_{-s}^{s} \cdots \int_{-s}^{s} \det(K^{\text{sin}}(x_i, x_j))_{i,j=1}^{k} dx_1 \cdots dx_k, \quad (1.10) \]

We may also consider the gap probability near the rightmost edge point, i.e., the distribution of the largest eigenvalue. Let \( \lambda_n \) be the largest eigenvalue of the matrix \( M \) and \( b \)
be the rightmost regular edge point. Then, the limiting distribution of $\lambda_n$ is given by the following Fredholm determinant

$$\lim_{n \to \infty} \Pr[(cn)^{3/2}(\lambda_n - b) < s] = \det[I - K^A_s],$$

(1.11)

where $K^A_s$ is the trace-class operator acting on $L^2(s, \infty)$ with the Airy kernel $K^A(x, y)$ in (1.8) and the Fredholm determinant $\det[I - K^A_s]$ has a similar series expansion as that in (1.9).

In [60], Tracy and Widom discovered that the Fredholm determinant $\det[I - K^A_s]$ has a more explicit form as follows

$$\det[I - K^A_s] = F_{TW}(s) := \exp \left( - \int_s^{+\infty} (x - s) y^2(x; 0) dx \right),$$

(1.12)

where $y(x; 0)$ is the Hastings-McLeod solution to the homogeneous Painlevé II (P$_2$) equation ($\alpha = 0$)

$$y''(x; \alpha) = xy(x; \alpha) + 2y^3(x; \alpha) - \alpha,$$

(1.13)

satisfying the following asymptotic behaviors

$$y(x; 0) \sim \text{Ai}(x), \quad \text{as } x \to +\infty;$$

(1.14)

$$y(x; 0) \sim \sqrt{-\frac{x}{2}} \left( 1 + \frac{1}{8x^2} + O(x^{-6}) \right), \quad \text{as } x \to -\infty.$$

(1.15)

It is remarkable that the Tracy-Widom distribution $F_{TW}(s)$ appears not only in random matrices, but also in random permutations [2], totally asymmetric simple exclusion process [46] and many other areas.

**Painlevé II universality**

For the unitary ensembles (1.1), when the limiting density function $\rho_V(x)$ vanishes quadratically at an interior point $x^*$, the Painlevé II universality emerges; see [4, 5, 18, 20]. For example, consider the unitary ensemble (1.1) with the following quartic potential

$$V(x) = \frac{x^4}{4} + \frac{g}{2}x^2.$$

(1.16)

When $g_{cr} = -2$, the limiting density function is

$$\rho_V(x) = \frac{x^2}{2\pi} \sqrt{4 - x^2}, \quad \text{for } x \in [-2, 2].$$

And there is a one-cut to two-cut transition near the point $x = 0$ when the parameter $g$ varies in the neighbourhood of $g_{cr}$. This type of phase transition is described by the Painlevé II kernels. More precisely, if $n \to \infty$ in the way such that $2^{-1/3}n^{1/3}(g + 2) \to t$, the double scaling limit of the correlation kernel near the origin is given by

$$\lim_{n \to \infty} \frac{2^{2/3}}{n^{1/3}} K_n \left( \frac{x}{2-2/3n^{1/3}}, \frac{y}{2-2/3n^{1/3}} \right) = K^P_\alpha(x, y; t),$$

(1.17)
uniformly for $x$ and $y$ in compact subsets of $\mathbb{R}$; see [5, 20]. The limiting kernel is constructed out of the $\psi$-functions associated with the Hastings-McLeod solution to the $P_2$ equation (1.13). The precise description of the $P_2$ kernel will be given later.

Similar to (1.9), once the limiting kernel is obtained, the gap probability near the origin is given as follows

$$\lim_{n \to \infty} \text{Prob} \left[ M \text{ has no eigenvalues in } \left( -\frac{s}{2^{2/3}n^{1/3}}, \frac{s}{2^{2/3}n^{1/3}} \right) \right] = \det [I - K_{\alpha,s}^{P_2}],$$

where $K_{\alpha,s}^{P_2}$ is the trace-class operator acting on $L^2(-s, s)$ with the $P_2$ kernel in (1.17).

\textbf{Painlevé XXXIV universality}

Now we turn to effect of the algebraic singular term $|\det M|^{2\alpha}$ in (1.1) near the soft edge. Although this singular term does not change the eigenvalue distributions in the global regime, it modifies the local eigenvalue statistics. Indeed, if there is a potential $V(x)$ such that the origin is a right regular edge point, then instead of the Airy kernel in (1.8), the limiting eigenvalue correlation kernel becomes the Painlevé XXXIV (P$_{34}$ for short) kernel; see Its, Kuijlaars and Östensson [40]. Later, a more general P$_{34}$ kernel with two parameters was obtained in the critical situation where a Fisher-Hartwig singularity of both root and jump types appears near the soft edge of a perturbed Gaussian unitary ensemble (GUE). More precisely, the joint probability density function for the eigenvalues in this model is given by

$$p_n(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_{n,\alpha,\omega}} \prod_{i=1}^{n} |\lambda_i - \mu|^{2\alpha} \chi(\lambda_i - \mu) e^{-2n\lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \quad \text{for } \alpha > -\frac{1}{2},$$

where $\chi(\lambda) = \begin{cases} \omega, & \lambda > 0 \\ 1, & \lambda < 0 \end{cases}$ and $\omega \in \mathbb{C} \setminus (-\infty, 0)$; see [61] and [6, 63].

When $\alpha \in \mathbb{N}$ and $\omega \in [0, 1]$, it is interesting to note that the above model can be interpreted as a thinned and conditioned GUE. More precisely, let us consider a thinned process for the GUE by removing each eigenvalue independently with probability $\omega \in [0, 1]$; see [7, 8]. Then, the eigenvalue distribution under the conditions that $\mu$ is an eigenvalue with multiplicity $\alpha$ in GUE and all other thinned eigenvalues are smaller than $\mu$ is given by (1.19). Recently, the thinning and conditioning models have appeared in many situations. For example, gap and conditional probabilities for the thinned unitary ensembles are derived in [6, 14]; the asymptotic behavior of mesoscopic fluctuations in the thinned CUE is studied in [3]; the transition between the Tracy-Widom distribution and the Weibull distribution as the probability $\omega \downarrow 0$ is considered in [10]; see also [11] for another interesting transition. A nice application in the study of the Riemann zeros can be found in [9].

Now let us consider the limiting kernel and take $n \to \infty$ in a way such that

$$\lim_{n \to \infty} 2n^{2/3}(\mu_n - 1) = t.$$
The double scaling limit of the correlation kernel near the soft edge $\lambda = 1$ is given by
\[
\lim_{n \to \infty} \frac{1}{2n^{2/3}} K_n(\mu_n + x, \mu_n + y) = K_{P^{34}}(x, y; t),
\]
(1.20)
uniformly for $x$ and $y$ in compact subsets of $\mathbb{R} \setminus \{0\}$, with
\[
K_{P^{34}}(x, y; t) = \frac{\psi_2(x; t)\psi_1(y; t) - \psi_1(x; t)\psi_2(y; t)}{2\pi i (x - y)},
\]
(1.21)
where $(\psi_1(x; t), \psi_2(x; t))^T$ satisfies the Lax pair associated with the $P^{34}$ equation. (The detailed information about the functions $(\psi_1(x; t), \psi_2(x; t))^T$ will be provided later in Section 1.1.) For $\alpha = 0$ and $\omega = 1$, as the density function $p_n(\lambda_1, \cdots, \lambda_n)$ in (1.19) is reduced to that of GUE, the $P_{34}$ kernel becomes the shifted Airy kernel accordingly
\[
K_{0,1}^{P^{34}}(x, y; t) = \frac{Ai(x + t)Ai'(y + t) - Ai'(x + t)Ai(y + t)}{x - y};
\]
(1.22)
see [40, Eq. (1.11)]. So, the $P_{34}$ kernel furnishes as a generalization of the Airy kernel. Moreover, the distribution of the largest eigenvalue in (1.11) is replaced by
\[
\lim_{n \to \infty} \text{Prob}[2n^{2/3}(\lambda_n - 1) < s] = \text{det}[I - K_{P^{34}}^{\alpha,\omega,s}],
\]
(1.23)
where $K_{P^{34}}^{\alpha,\omega,s}$ is the trace-class operator acting on $L^2(s, \infty)$ with the $P_{34}$ kernel in (1.21).

In the past a few years, various Painlevé kernels have been adopted to characterize new universality classes in different critical random matrix models; for example, see [1, 17, 18, 20, 62, 64]. However, there are very few results about the Tracy-Widom type formulas for these kernels or their large gap asymptotics. To the best of our knowledge, the only higher-order analogues of the Tracy-Widom formula for the Fredholm determinant associated with the Painlevé I hierarchy was obtained by Claeys, Its and Krasovsky in [16], where the density function $\rho_V(x)$ in (1.6) vanishes with the order $2k + \frac{1}{2}, k \in \mathbb{N}$ at an endpoint of its support. Besides the large gap asymptotics in [16], the only other asymptotic result is obtained by Bothner and Its [12] in the study of the $P_2$ kernel in (1.17) with the parameter $\alpha = 0$. However, an analogous expression of the Tracy-Widom formula for the $P_2$ kernel is still to be discovered.

In the present paper, we study the Fredholm determinant of the $P_2$ and $P_{34}$ kernels with general parameters. We aim to find analogous expressions of the Tracy-Widom formula for these determinants and evaluate their large gap asymptotics. We also plan to study the $P_3$ kernel at the hard edge with pole singularities in the potential in a forthcoming publication [22].

1.1 Expressions of the $P_2$ and $P_{34}$ kernels

Before the statement of our main results, let us first give the specific representation of the $P_{34}$ kernel in (1.21). The $P_2$ kernel will be provided through its relation with the $P_{34}$ kernel.
The functions $\psi_1(x; t)$ and $\psi_2(x; t)$ in the $P_{34}$ kernel (1.21) appear as solutions of the Lax pair associated with the $P_{34}$ equation. In addition, it is convenient to characterize this kernel in terms of the following Riemann-Hilbert (RH) problem; see Its, Kuijlaars and Östensson [40].

![Contour Diagram](image)

**Figure 1:** Contours for the RH problem for $\Psi$: $\Sigma_1 = \{\arg \zeta = 0\}$, $\Sigma_2 = \{\arg \zeta = \frac{2}{3}\pi\}$, $\Sigma_3 = \{\arg \zeta = \pi\}$ and $\Sigma_4 = \{\arg \zeta = -\frac{2}{3}\pi\}$.

**RH problem for $\Psi$:**

(a) $\Psi(\zeta) := \Psi(\zeta; t; \alpha, \omega)$ is analytic for $\zeta \in \mathbb{C} \cup \bigcup_{i=1}^4 \Sigma_i$, where the contours $\Sigma_i$ are depicted in Fig. 1.

(b) $\Psi(\zeta)$ satisfies the following jump conditions

\[
\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Sigma_1, \tag{1.24}
\]

\[
\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 \\ \epsilon^{2\pi i a} \end{pmatrix}, \quad \zeta \in \Sigma_2, \tag{1.25}
\]

\[
\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in \Sigma_3, \tag{1.26}
\]

\[
\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 \\ \epsilon^{-2\pi i a} \end{pmatrix}, \quad \zeta \in \Sigma_4; \tag{1.27}
\]

(c) As $\zeta \to \infty$, the asymptotic behavior of $\Psi(\zeta)$ is given by

\[
\Psi(\zeta) = \left( \begin{pmatrix} 1 & 0 \\ -im_2(t) & 1 \end{pmatrix} \right) \left[ I + \frac{1}{\zeta} \begin{pmatrix} m_1(t) & im_2(t) \\ -im_3(t) & -m_1(t) \end{pmatrix} \right] + O(\zeta^{-2}) \left( \frac{I + i\sigma_1 e^{-\theta(\zeta, t)\sigma_3}}{\sqrt{2}} \right) \zeta^{-\frac{1}{2}\sigma_3} e^{\theta(\zeta, t)\sigma_3},
\]

where $\theta(\zeta, t) = \frac{2}{3}\zeta^2 + t\zeta^2$, $\arg \zeta \in (-\pi, \pi)$ and $m_i(t) := m_i(t; \alpha, \omega), i = 1, \cdots, 3$.

(d) The behavior of $\Psi(\zeta)$ at the origin is

\[
\Psi(\zeta) = \begin{pmatrix} O(\zeta^\alpha) \\ O(\zeta^\alpha) \end{pmatrix}, \quad \text{if} \quad -\frac{1}{2} < \alpha < 0 \tag{1.29}
\]
\[
\Psi(\zeta) = \begin{cases} 
\begin{pmatrix} O(\zeta^0) & O(\zeta^{-\alpha}) \\ O(\zeta^0) & O(\zeta^{-\alpha}) \end{pmatrix}, & \zeta \in \Omega_1 \cup \Omega_4, \text{ if } \alpha \geq 0. \\
\begin{pmatrix} O(\zeta^0) & O(\zeta^{-\alpha}) \\ O(\zeta^0) & O(\zeta^{-\alpha}) \end{pmatrix}, & \zeta \in \Omega_2 \cup \Omega_3.
\end{cases}
\]

Then, the \(P_{34}\) kernel
\[
K_{\alpha,\omega}^{P34}(x, y; t) = \frac{\psi_2(x; t)\psi_1(y; t) - \psi_1(x; t)\psi_2(y; t)}{2\pi i(x - y)},
\]
is given in terms of the functions
\[
\left( \begin{array}{c}
\psi_1(x; t) \\
\psi_2(x; t)
\end{array} \right) := \begin{cases}
\omega^{\frac{1}{2}}\Psi_+(x; t) \begin{pmatrix} 1 \\
0 \end{pmatrix}, & \text{if } x > 0,
\end{cases}
\]
\[
= \begin{cases}
e^{-\alpha\pi i}\Psi_+(x; t) \begin{pmatrix} 1 \\
e^{2\alpha\pi i} \end{pmatrix}, & \text{if } x < 0.
\end{cases}
\]

Obviously, from the behavior of \(\Psi(\zeta)\) at infinity in (1.28), the function \((\psi_1(x; t), \psi_2(x; t))^T\) satisfies the following behaviours:
\[
\left( \begin{array}{c}
\psi_1(x; t) \\
\psi_2(x; t)
\end{array} \right) = \frac{\omega^{\frac{3}{2}}}{\sqrt{2}} e^{-\left(\frac{3}{2}x^\frac{3}{2} + \frac{1}{4}\right)} \begin{pmatrix} x^{-\frac{1}{4}} + O(x^{-\frac{3}{4}}) \\
ix^{-\frac{1}{4}} + O(x^{-\frac{3}{4}}) \end{pmatrix},
\]
as \(x \to +\infty\); and
\[
\left( \begin{array}{c}
\psi_1(x; t) \\
\psi_2(x; t)
\end{array} \right) = \sqrt{2} \begin{pmatrix} |x|^{-\frac{1}{4}} \cos\left(\frac{\pi}{4} |x|^{\frac{3}{2}} - t \sqrt{|x|} - \alpha \pi - \pi/4\right) + O(|x|^{-\frac{3}{4}}) \\
-ix|^{-\frac{1}{4}} \sin\left(\frac{\pi}{4} |x|^{\frac{3}{2}} - t \sqrt{|x|} - \alpha \pi - \pi/4\right) + O(|x|^{-\frac{3}{4}}) \end{pmatrix},
\]
as \(x \to -\infty\). When \(\alpha = 0, \omega = 1\), the above RH problem for \(\Psi\) is indeed the RH problem for the Airy functions. Therefore, \(P_{34}\) kernel (1.31) is reduced to the shifted Airy kernel (1.22); see [40, Eq. (1.11)].

Until now, we have not explained how the kernel in (1.31) is related to the \(P_{34}\) equation. The following proposition reveals their relations.

**Proposition 1** ([40, 41]). For \(\alpha > -\frac{1}{2}, \omega \in \mathbb{C} \setminus (-\infty, 0)\) and \(t \in \mathbb{R}\), the model RH problem for \(\Psi\) is uniquely solvable. In addition, let \(m_2(t)\) be the function given in (1.28) and
\[
u(t; 2\alpha, \omega) = m_2(t) - \frac{t}{2},
\]
then \(u(t; 2\alpha, \omega) \) (\(u(t)\) for short) satisfies the \(P_{34}\) equation
\[
u''(t) = 4u^2(t) + 2tu(t) + \frac{u'(t)^2 - (2\alpha)^2}{2u(t)}.
\]
Furthermore, the solution $u(t; 2\alpha, \omega)$ is analytic on the real axis and uniquely determined by following asymptotic behaviors:

$$u(t; 2\alpha, \omega) = \frac{\alpha}{\sqrt{t}} \left( \sum_{k=0}^{n} \frac{a_k}{t^{3k/2}} + O(t^{-3(n+1)/2}) \right) + (e^{2\pi i \alpha} - \omega) \frac{\Gamma(2\alpha + 1)}{2^{2+6\alpha} \pi} t^{-(3\alpha + \frac{1}{2})} e^{-\frac{4}{3}t^{3/2}} (1 + O(t^{-1/4}))$$

(1.37)

as $t \to +\infty$, with $a_0 = 1$ and $a_1 = -\alpha$; and

$$u(t; 2\alpha, \omega) = \begin{cases} 
-t^2 + \frac{16\alpha^2 - 1}{8} t^{-2} + O(t^{-7/2}), & \text{if } \omega = 0, \\
\frac{1}{\sqrt{-t}} \left[ i\beta + \frac{1}{2} \frac{1}{\Gamma(1+\alpha-\beta)} e^{i\theta(t; \alpha, \beta)} + \frac{1}{2} \frac{1}{\Gamma(1+\alpha+\beta)} e^{-i\theta(t; \alpha, \beta)} \right] + O(t^{3|\Re \beta|-2}), & \text{if } \omega = e^{-2\pi i \beta},
\end{cases}$$

(1.38)

as $t \to -\infty$, with $|\Re \beta| < 1/2$ and $\theta(t; \alpha, \beta) = \frac{4}{3} t^{3/2} - \alpha \pi - 6i/\beta \ln 2 - 3i\beta \ln |t|$.

**REMARK 1.** The $P_{34}$ transcendent $u(t; 2\alpha, \omega)$ in the above Proposition satisfies the relation

$$2^{1/3} u(-2^{-1/3} t; 2\alpha, \omega) = y'(t; 2\alpha + \frac{1}{2}) + y^2(t; 2\alpha + \frac{1}{2}) + \frac{t}{2},$$

(1.39)

where $y(z; 2\alpha + \frac{1}{2})$ is one-parameter family of tronqué solutions of the $P_2$ equation (1.13) with the following asymptotics

$$y(t; \alpha) = -\sqrt{-t} - \frac{\alpha}{2t} + O(t^{-5/2}) + c(\omega)(t^{-3\alpha - 1} e^{-\frac{2\pi i}{3}(-1)^{\alpha} t^{3/2}} (1 + O(t^{-1/4}))),$$

(1.40)

as $t \to \infty$ and $\arg(-t) \in (-\frac{\pi}{3}, \frac{\pi}{3})$; see [37] (11.5.56). When $\omega = 0$, this solution reduces to the Hastings-McLeod solution for the $P_2$ equation (1.13), which is uniquely determined by the boundary conditions

$$y(t; \alpha) \sim \sqrt{-t}/2, \quad t \to -\infty, \quad y(t; \alpha) \sim \alpha/t, \quad t \to +\infty;$$

(1.41)

see [37] Remark 11.12).

Like the other Painlevé equations, the RH problem for $\Psi(\zeta)$ in (1.24)-(1.30) implies a Lax pair for the $P_{34}$ equation. More precisely, we have

$$\frac{\partial}{\partial \zeta} \Psi(\zeta) = \begin{pmatrix} \frac{u}{2\zeta} & i - i\frac{u}{2\zeta} \\
-i\zeta - i(u + t) - i\frac{(u+2^2 z - (2a)^2)}{4u\zeta} & -\frac{u}{2\zeta}
\end{pmatrix} \Psi(\zeta),$$

(1.42)

$$\frac{\partial}{\partial t} \Psi(\zeta) = \begin{pmatrix} 0 & i \\
-i\zeta - 2i(u + \frac{t}{2}) & 0
\end{pmatrix} \Psi(\zeta),$$

(1.43)

where $u(t)$ satisfies the $P_{34}$ equation (1.36); see [40] Theorem 1.5].
One can define the $P_2$ kernel via the RH problem in a similar way. On the other hand, the $P_2$ kernel in (1.17) can be determined by the $P_{34}$ kernel with the parameter $\omega = 0$ as follows:

$$K_{\alpha}^{P_2}(x, \pm y; t) = (xy)^{1/2} \left( K_{\omega = 0}^{P_{34}}(-2^{2/3}x^2, -2^{2/3}y^2; -2^{-1/3}t) + K_{\omega = 0}^{P_{34}}(-2^{2/3}x^2, -2^{2/3}y^2; -2^{-1/3}t) \right)$$

for $x > 0$. When $x < 0$, the $P_2$ kernel is obtained from the above formula and the symmetry relation

$$K_{\alpha}^{P_2}(x, y; t) = -K_{\alpha}^{P_2}(-x, -y; t);$$

see Claeys and Kuijlaars [19].

### 2 Statement of results

We will derive Tracy-Widom type formulas for Fredholm determinants of the $P_2$ and $P_{34}$ kernels, as well as their large gap asymptotics.

#### 2.1 The coupled Painlevé II system and their asymptotics

To express our Tracy-Widom type formulas, we need to introduce the following coupled $P_2$ systems in dimension four

$$\begin{align*}
\frac{dv_1}{dx} &= -\frac{\partial H}{\partial v_1} = 2(v_1 + v_2 + \frac{x}{2}) - w_1^2 \\
\frac{dv_2}{dx} &= \frac{\partial H}{\partial w_2} = 2v_1w_1 \\
\frac{dw_1}{dx} &= \frac{\partial H}{\partial v_1} = 2v_1w_1 \\
\frac{dw_2}{dx} &= \frac{\partial H}{\partial w_2} = 2v_2w_2 + 2\alpha
\end{align*}$$

(2.1)

where $v_i := v_i(x; s, 2\alpha)$, $w_i := w_i(x; s, 2\alpha + \frac{1}{2})$ and the Hamiltonian $H := H(v_1, v_2, w_1, w_2; x, s, 2\alpha)$ is given by

$$H(v_1, v_2, w_1, w_2; x, s, 2\alpha) = -(v_1 + v_2)^2 - (v_1 + v_2)x + v_1w_1^2 + v_2w_2^2 + sv_2 + 2\alpha w_2.$$  

(2.2)

With the transformations $v_i(x; s, 2\alpha) = 2^{-\frac{1}{2}}p_i(-2^{\frac{3}{4}}x; -2^{\frac{1}{2}}s, 2\alpha)$ and $w_i(x; s, 2\alpha + \frac{1}{2}) = -2^{\frac{1}{4}}q_i(-2^{\frac{3}{4}}x; -2^{\frac{1}{2}}s, 2\alpha + \frac{1}{2})$, the above Hamiltonian is equivalent to the one studied in Sasano [57] via the following simple relation

$$H^{Sasano}(p_1, p_2, q_1, q_2; x, s, 2\alpha) = -2^{-\frac{1}{2}}H(v_1, v_2, w_1, w_2; -2^{-\frac{1}{2}}x, -2^{-\frac{1}{2}}s, 2\alpha).$$

(2.3)

The above coupled $P_2$ system (2.1) was first introduced and studied by Sasano [57]. It is regarded as a fourth-order extension of the classical $P_2$ equation. The studies of other coupled Painlevé systems in dimension four can be found in [53, 58, 59]. In recent years, the program to classify the four-dimensional Painlevé-type equations has been carried out...
by Kawakami, Nakamura and Sakai. In [56], from the isomonodromic deformation theory of the Fuchsian equations, Sakai derived four source systems for 4-dimensional Painlevé type equations, namely, the Garnier system in two variables, the Fuji-Suzuki system, the Sasano system and the matrix Painlevé system. Later, the complete degeneration scheme of these four source systems was obtained in Kawakami, Nakamura and Sakai [50] and Kawakami [47, 48, 49]. The coupled P₂ system (2.1) appears in both of the degeneration schemes of the Garnier system in two variables [49, (3.5)-(3.7)] and the Sasano system [48, (3.22)-(3.23)]. Note that applications of the coupled P₂ system in the study of the Airy point process was discovered by Claeys and Doeraene [15] very recently.

Eliminating \( w_i \) and \( v_i \) from the Hamiltonian system (2.1), respectively, gives us the following nonlinear equations for \( v_i \):

\[
\begin{align*}
  v_{1xx} - \frac{v_{1x}^2}{2v_1} - 4v_1(v_1 + v_2 + \frac{x}{2}) &= 0 \\
  v_{2xx} - \frac{v_{2x}^2}{2v_2} - 4\alpha^2 v_2(v_1 + v_2 + \frac{x - s}{2}) &= 0
\end{align*}
\]  

(2.4)

and equations for \( w_i \):

\[
\begin{align*}
  w_{1xx} - 2w_1^3 + 2xw_1 + 4v_2(w_1 - w_2) - (4\alpha + 1) &= 0 \\
  w_{2xx} - 2w_2^3 + 2(x - s)w_2 - 4v_1(w_1 - w_2) - (4\alpha + 1) &= 0
\end{align*}
\]

(2.5)

The coupled equations (2.4) are similar to [37, (2.1)], which was obtained from similarity reduction of the Hirota-Satsuma system. Eliminating either one of the functions \( v_1 \) or \( v_2 \) from the above equation, one gets a fourth-order nonlinear differential equation. Besides, if we set \( v_1(x) = y^2(x) \), then the first equation becomes

\[
y'' - 2y^3 - xy = 2v_2 y.
\]

(2.6)

When \( \alpha = 0 \), taking the admissible solution \( v_2 = 0 \) in (2.4), then (2.6) is reduced to the standard P₂ equation (1.13).

On the way to our Tracy-Widom type formulas, we need a class of distinguished solutions to the couple P₂ system. The solutions satisfy the following properties.

THEOREM 1. For \( \alpha > -\frac{1}{2} \), \( \omega \in \mathbb{C} \setminus (-\infty, 0) \) and \( s \in \mathbb{R} \), there exist real analytic solutions \( v_i(x) \) to the coupled P₂ equations (2.4). Moreover, the solutions \( v_i(x, s; 2\alpha, \omega) \) satisfy the following asymptotic behaviors:

\[
v_1(x, s; 2\alpha, \omega) = \frac{1}{4\pi \sqrt{x}} e^{-\frac{4}{x} \frac{s}{24\alpha}} \left| \frac{s}{x} \right|^{2\alpha} \left( 1 + \alpha \frac{s}{x} + O\left( x^{-3/2} \right) \right), \quad \text{as } x \to +\infty,
\]

(2.7)

with the constant \( c = \begin{cases} \omega, & s > 0, \\ 1, & s < 0; \end{cases} \)

and

\[
v_2(x, s; 2\alpha, \omega) = \frac{\alpha}{\sqrt{x - s}} - \frac{\alpha^2}{(x - s)^2} + O(1/x^3), \quad \text{as } x \to +\infty.
\]

(2.8)
In addition, the functions \( w_1(x, s; 2\alpha + \frac{1}{2}, \omega) \) and the Hamiltonian \( H(x; s) := H(v_1, v_2, w_1, w_2; x, s, \alpha) \) in (2.1) satisfy the following asymptotic behaviors:

\[
\begin{align*}
  w_1(x, s; 2\alpha + \frac{1}{2}, \omega) &= -\sqrt{x} + O(\ln x), \quad \text{as } x \to +\infty, \\
  w_2(x, s; 2\alpha + \frac{1}{2}, \omega) &= -\sqrt{x-s} - \frac{\alpha + \frac{1}{2}}{x-s} + O(1/x^2), \quad \text{as } x \to +\infty, \\
  H(x; s) &= -2\alpha\sqrt{x-s} - \frac{\alpha^2}{x-s} + O(1/x^2), \quad \text{as } x \to +\infty.
\end{align*}
\]

**REMARK 2.** For \( \alpha = 0 \) and \( \omega = 1 \), then \( v_2 = 0 \) and the equation (2.6) is reduced to the \( P_2 \) equation (1.13). From the asymptotic behaviors in (2.7), one immediately sees that \( v_1(x) = y^2(x; 0) \), where \( y(x; 0) \) is the classical Hastings-McLeod solution to the \( P_2 \) equation given in (1.14)-(1.15).

**REMARK 3.** When \( s = 0 \), we have \( v_1 \equiv 0 \) and \( v_2(x, 0; 2\alpha, \omega) = u(x; 2\alpha, 0) \), where \( u(x; 2\alpha, 0) \) is the \( P_{34} \) transcendent given in Proposition 7.

### 2.2 Tracy-Widom type expressions for the Fredholm determinants

Now we have the following integral representations for the Fredholm determinants of the \( P_2 \) and \( P_{34} \) kernels.

**THEOREM 2.** For \( \alpha > -\frac{1}{2}, \omega \in \mathbb{C} \setminus (-\infty, 0) \) and \( s, t \in \mathbb{R} \), let \( K_{\alpha, \omega, s}^{P_{34}} \) be the trace-class operator acting on \( L^2(s, \infty) \) with the \( P_{34} \) kernel \( K_{\alpha, \omega}^{P_{34}}(x, y; t) \) in (1.31), then we have

\[
\det[I - K_{\alpha, \omega, s}^{P_{34}}] = \exp\left( -\int_{t}^{+\infty} (v_1(x + s) + v_2(x + s) - u(x))(x-t)dx \right),
\]

where \( v_i(x) = v_i(x, s; 2\alpha, \omega) \) are the smooth solutions to the coupled \( P_2 \) equations (2.4) with the properties specified in Theorem 1 and \( u(x) = u(x; 2\alpha, \omega) \) is the \( P_{34} \) transcendent given in Proposition 1.

**REMARK 4.** For \( \alpha = 0 \) and \( \omega = 1 \), we recover the celebrated Tracy-Widom distribution (1.12) from Theorem 1 and 2. Indeed, from the properties of the functions \( v_i(x) = v_i(x, s; 2\alpha, \omega) \) described in Remark 3, the integral representation (2.12) becomes

\[
\det[I - K_{s+t}^{\text{Ai}}] = \exp\left( -\int_{s+t}^{+\infty} (\tau - t - s)y^2(\tau; 0)d\tau \right),
\]

where \( K_{s+t}^{\text{Ai}} \) is the trace-class operator acting on \( L^2(s+t, \infty) \) with the Airy kernel in (1.8) and \( y(x; 0) \) is the Hastings-McLeod solution to the \( P_2 \) equation given in (1.14)-(1.15).
REMARK 5. When $\alpha \in \mathbb{N}$ and $\omega \in [0, 1]$, the Tracy-Widom type formula (2.12) is the largest eigenvalue distribution of the conditional GUE in (1.19). When $\alpha = 0$, it agrees with the results obtained by Claeys and Doeraene in [15, (2.8)]. Our result in (2.12) holds for general parameter $\alpha > -\frac{1}{2}$, which enables us to establish the Tracy-Widom type formula for the $P_2$ kernel below.

Next, we establish a relation between the Fredholm determinants for the $P_2$ and $P_{34}$ kernels.

LEMMA 1. Let $K_{\alpha,s}^{P_2}$ and $K_{\alpha,0,s'}^{P_{34}}$ be the trace-class operator acting on $L^2(-s, s)$ and $L^2(s', +\infty)$ with the $P_2$ kernel $K_{\alpha}^{P_2}(x, y; t)$ in (1.17) and $P_{34}$ kernel $K_{\frac{\alpha}{2} + \frac{1}{4}, 0}^{P_{34}}(x, y; -2^{-1/3}t)$ in (1.31), respectively, we have

$$\det[I - K_{\alpha,s}^{P_2}] = \det[I - K_{\frac{\alpha}{2} + \frac{1}{4}, 0,s'}^{P_{34}}] \det[I - K_{\frac{\alpha}{2} - \frac{1}{4}, 0,s'}^{P_{34}}],$$

(2.14)

where $s' = -2^{\frac{2}{3}}s^2$, $s > 0$ and $\alpha > -\frac{1}{2}$.

Then, Theorem 2 and the above lemma gives us the Tracy-Widom type formula for the Fredholm determinant of the $P_2$ kernel.

THEOREM 3. For $\alpha > -\frac{1}{2}$, $s > 0$ and $t \in \mathbb{R}$, let $K_{\alpha,s}^{P_2}$ be the trace-class operator acting on $L^2(-s, s)$ with the $P_2$ kernel $K_{\alpha}^{P_2}(x, y; t)$ given in (1.44), then we have

$$\det[I - K_{\alpha,s}^{P_2}] = \exp \left( - \int_{-\infty}^{t} (y^2(x; \alpha) - 2^{-2/3}w_2^2(-2^{-1/3}x + s', s'; \alpha))(x - t)dx \right),$$

(2.15)

where $s' = -2^{2/3}s^2$, $y(x; \alpha)$ is the Hastings-McLeod solution to the $P_2$ equation described in (1.41), and $w_2(x, s; \alpha)$ is the smooth solution to the equation (2.5) with the asymptotic behaviors given in (2.10).

2.3 Large gap asymptotics for the Fredholm determinants

From (1.33), it is easy to see that the $P_{34}$ kernel (1.31) satisfies the following asymptotic behavior as $x, y \to +\infty$

$$K_{\alpha,\omega}^{P_{34}}(x, y) = O(e^{-c(x^\frac{3}{2} + y^\frac{3}{2})}),$$

for certain constant $c > 0$. From the series expansion of the Fredholm determinant (for example, see the expansion for the sine kernel in (1.10)), we obtain

$$\ln \det[I - K_{\alpha,\omega,s}^{P_{34}}] = O(e^{-cs^\frac{3}{2}}), \quad \text{as} \quad s \to +\infty,$$

where $K_{\alpha,\omega,s}^{P_{34}}$ is the trace-class operator with kernel $K_{\alpha,\omega}^{P_{34}}$ acting on $L^2(s, \infty)$. The large gap asymptotics for the Fredholm determinant of the $P_{34}$ kernel as $s \to -\infty$ are much more involved and given in the following theorem.
THEOREM 4. For $\alpha > -\frac{1}{2}$, $\omega \in \mathbb{C} \setminus (-\infty, 0)$ and $t \in \mathbb{R}$, let $K_{\alpha,\omega,s}^{P34}$ be the trace-class operator acting on $L^2(s, \infty)$ with the kernel $K_{\alpha,\omega}^{P34}(x,y;t)$ given in (1.31), we have the asymptotic expansion for the Fredholm determinant as $s \to -\infty$

$$\ln \det[I - K_{\alpha,\omega,s}^{P34}] = -\frac{1}{12} |s + t|^3 + \frac{2}{3} \alpha |s|^2 - 2\alpha |t| - (\alpha^2 + \frac{1}{8}) \ln |s + t| - \frac{4}{3} \alpha \text{sgn}(t)|t|^2 + \alpha^2 \ln |t| + \int_{t}^{+\infty} (\tau - t) \left( u(\tau) - \frac{\alpha}{|\tau|^{\frac{3}{2}}} + \frac{\alpha^2}{\tau^2} \right) d\tau + c_0 + o(1),$$

(2.16)

where $u(x) = u(x; 2\alpha, \omega)$ is the $P_{34}$ transcendent given in Proposition 1. The constant $c_0$ in the above formula is given explicitly as

$$c_0 = \frac{1}{24} \ln 2 + \zeta'(1).$$

(2.17)

where $\zeta'(z)$ is the derivative of the Riemann zeta-function.

REMARK 6. For $\alpha = 0$, we recover the large gap asymptotics for the Fredholm determinant associated with the Airy kernel as $s + t \to -\infty$

$$\ln \det[I - K_{s+t}^{\text{Ai}}] = -\frac{1}{12} |s + t|^3 - \frac{1}{8} \ln |s + t| + c_0 + o(1),$$

(2.18)

where $K_{s+t}^{\text{Ai}}$ is the trace-class operator with the Airy kernel acting on $L^2(s + t, +\infty)$ and $c_0$ is given in (2.17); see [24, 60].

We also have the large gap asymptotics for the Fredholm determinant of the $P_2$ kernel.

THEOREM 5. For $\alpha > -\frac{1}{2}$, let $K_{\alpha}^{P2}$ be the trace-class operator acting on $L^2(-s, s)$ with the kernel $K_{\alpha}^{P2}(x,y;t)$ given in (1.44), we have the asymptotic expansion for the Fredholm determinant as $s \to +\infty$

$$\ln \det[I - K_{\alpha,s}^{P2}] = -\frac{2}{3} (s^2 + \frac{t}{2})^3 + \frac{4}{3} \alpha s^3 + 2\alpha st - (\alpha^2 + \frac{3}{4}) \ln s + \frac{2\sqrt{2}}{3} \alpha \text{sgn}(t)|t|^\frac{3}{2} + \int_{-\infty}^{t} (\tau - t) \left( y^2(\tau; \alpha) + \frac{\tau}{2} - \frac{\alpha}{\sqrt{2\tau}} + \frac{\alpha^2}{2\tau^2} + \frac{1}{8} \right) d\tau + c_1 + o(1),$$

(2.19)

where $y(x; \alpha)$ is the Hastings-McLeod solution to the $P_2$ equation described in (1.41) and the constant $c_1$ is given explicitly as

$$c_1 = -(\alpha^2 + \frac{5}{24}) \ln 2 + 2\zeta'(-1).$$

(2.20)

REMARK 7. For $\alpha = 0$, the above large gap asymptotics can be reduced to that in Bothner and Its [12]

$$\ln \det[I - K_{0,s}^{P2}] = -\frac{2}{3} s^6 - s^4 t - \frac{1}{2} (st)^2 - \frac{3}{4} \ln s + \int_{t}^{+\infty} (\tau - t) y^2(\tau; 0)d\tau + c_2 + o(1),$$

(2.21)
where \( y(x; 0) \) is the Hastings-McLeod solution to the \( P_2 \) equation and the constant term 
\[
c_2 = -\frac{1}{6} \ln 2 + 3\zeta'(-1).
\]

In order to reduce (2.19) to (2.21), one needs the following total integral of the Hastings-McLeod solution \( y(x; 0) \) to the \( P_2 \) equation
\[
\int_t^\infty (\tau - t) y^2(\tau; 0) d\tau + \int_{-\infty}^t (\tau - t)(y^2(\tau; 0) + \frac{\tau}{2} + \frac{1}{8\tau^2}) d\tau = -\frac{t^3}{12} + \frac{1}{8} \ln |t| - c_0, \tag{2.22}
\]
where \( c_0 \) is defined in (2.17). Note that the above formula directly follows from (1.12) and (2.18).

The rest of the paper is organized as follows. In Section 3, we provide a representation of the Fredholm determinant in terms of a RH problem and derive a differential identity for the Fredholm determinant. Then, we transform this RH problem to a model one to facilitate our future study. In Section 4, a Lax pair is derived from the model RH problem obtained in the Section 3. The compatibility condition of the Lax pair is described by the coupled \( P_2 \) equations. The Bäcklund transformations for the coupled \( P_2 \) system are also studied. In Section 5, we study the asymptotic behaviors of the functions in the Hamiltonian system (2.1) as \( x \to +\infty \) and prove Theorem 1. Section 6 is then devoted to the proof of Theorems 2-3. In Section 7, we evaluate the large gap asymptotics for the Fredholm determinants. Finally, for the sake of completeness and possible interests, the asymptotics of the functions \( v_i(x) \) in Theorem 1 as \( x \to -\infty \) are derived in Appendix A.

### 3 Riemann-Hilbert problem for the Fredholm determinant and differential identity

#### 3.1 Riemann-Hilbert problem for the Fredholm determinant and differential identity

Let \( K_{P_2}^{\alpha,\omega,s} \) be the trace-class operator acting on \( L^2(s, \infty) \) with the kernel \( K_{P_2}^{\alpha,\omega} \) in (1.31), the Fredholm determinant \( \det[I - K_{P_2}^{\alpha,\omega,s}] \) can be characterized in terms of the solution of certain RH problems; see Deift, Its and Zhou [26] and Its et al. [38].

**RH problem for \( Y \):**

(a) \( Y(z) \) is analytic for \( z \in \mathbb{C} \setminus (s, \infty) \);

(b) \( Y(z) \) satisfies the jump condition on \( (s, \infty) \)

\[
Y_+(x) = Y_-(x)J_Y(x), \quad J_Y(x) = \begin{pmatrix} 1 + \psi_1(x)\psi_2(x) & -\psi_1(x)^2 \\ \psi_2(x)^2 & 1 - \psi_1(x)\psi_2(x) \end{pmatrix}, \tag{3.1}
\]

where \( \psi_i(x) \) are defined in (1.32).
(c) The asymptotic behavior of \( Y(z) \) at infinity:

\[
Y(z) = I + \frac{Y^{-1}}{z} + O(1/z^2) \quad \text{as } z \to \infty;
\]

\[ (3.2) \]

(d) At possible endpoints \( z = 0 \) and \( z = s \), \( Y(z) \) is square integrable.

**LEMMA 2** ([20]). Let \( K_{\alpha,\omega,s}^{P34} \) be the trace-class operator acting on \( L^2(s, \infty) \) with the kernel \( K_{\alpha,\omega}^{P34} \) in (1.31) and assume \( (I - K_{\alpha,\omega,s}^{P34})^{-1} \) exists, then the solution for the above RH problem is given by

\[
Y(z) = I - \frac{1}{2\pi i} \int_s^{+\infty} \frac{M(x)}{x-z} \, dx, \quad M(x) = \begin{pmatrix} -F_1(x)\psi_2(x) & F_1(x)\psi_1(x) \\ -F_2(x)\psi_2(x) & F_2(x)\psi_1(x) \end{pmatrix},
\]

\[ (3.3) \]

where \( \psi_i(x) \) are defined in (1.32) and \( F_i(x) = (I - K_{\alpha,\omega,s}^{P34})^{-1}\psi_i(x) \). Conversely, the functions \( F_i(x) \) can be expressed in terms of \( Y \) as follows

\[
\begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} = Y_+(x) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad x > s, x \neq 0.
\]

\[ (3.4) \]

From the above RH problem for \( Y \), we derive a differential identity for the Fredholm determinant of the \( P_{34} \) kernel as follows.

**PROPOSITION 2.** For \( \alpha > -\frac{1}{2} \), \( \omega \in \mathbb{C} \setminus (-\infty, 0) \) and \( s, t \in \mathbb{R} \), we have

\[
\frac{d}{dt} \ln \det[I - K_{\alpha,\omega,s}^{P34}] = i(Y^{-1})_{12},
\]

\[ (3.5) \]

where \( Y^{-1} \) is the coefficient of the \( 1/z \) term as \( z \to \infty \); cf. (3.2).

**Proof.** From (1.31) and (1.43), we have

\[
\frac{d}{dt} K_{\alpha,\omega}^{P34}(x, y; t) = -\frac{1}{2\pi} \psi_1(x)\psi_1(y).
\]

\[ (3.6) \]

Using properties of trace-class operators, we get

\[
\frac{d}{dt} \ln \det[I - K_{\alpha,\omega,s}^{P34}] = -\text{tr}((I - K_{\alpha,\omega,s}^{P34})^{-1} \frac{dK_{\alpha,\omega,s}^{P34}}{dt}) = \frac{1}{2\pi} \int_s^{+\infty} ((I - K_{\alpha,\omega,s}^{P34})^{-1}\psi_1)(x)\psi_1(x) \, dx.
\]

\[ (3.7) \]

Then, (3.5) follows from (3.3) and (3.4). \( \square \)

### 3.2 Model Riemann-Hilbert problem for \( \Phi \)

Next, we transform the original RH problem for \( Y \) into a new one with constant jumps. Observe that the jump matrix \( J_Y(x) \) in (3.1) can be factorized as follows:

\[
J_Y(x) = \Psi_+(x) \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix} \Psi_+^{-1}(x) \quad \text{for } x > 0
\]

\[ (3.8) \]
\[ J_Y(x) = \Psi_+(x) \begin{pmatrix} 1 & 0 \\ e^{2\pi i \alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & -e^{-2\pi i \alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{2\pi i \alpha} & 1 \end{pmatrix} \Psi_+(x)^{-1} \text{ for } x < 0. \quad (3.9) \]

This evokes us to introduce the following transformation
\[ \tilde{\Phi}(z) = Y(z)\Psi(z), \quad \text{if } s \geq 0 \quad (3.10) \]
and
\[
\tilde{\Phi}(z) = \begin{cases} 
Y(z)\Psi(z), & z \in \Omega^R_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega^R_4 \\
Y(z)\Psi(z) \begin{pmatrix} 1 & 0 \\ e^{2\pi i \alpha} & 1 \end{pmatrix}, & z \in \Omega^L_1, \\
Y(z)\Psi(z) \begin{pmatrix} 1 & 0 \\ -e^{-2\pi i \alpha} & 1 \end{pmatrix}, & z \in \Omega^L_4,
\end{cases}
\]
where the regions are indicated in Fig. 2. Then, \( \tilde{\Phi} \) satisfies a RH problem as follows.

![Figure 2: Regions and contours for \( \tilde{\Phi} \) (\( \hat{\Sigma}_1 = [0, s] \) for \( s > 0 \) and \( \hat{\Sigma}_1 = [s, 0] \) for \( s < 0 \)).](image)

**RH problem for \( \tilde{\Phi} \):**

(a) \( \tilde{\Phi}(z) := \tilde{\Phi}(z; x, s; \alpha, \omega) \) is analytic for \( z \in \mathbb{C} \setminus \hat{\Sigma}_i \); see Fig. 2

(b) \( \tilde{\Phi}(z) \) satisfies the jump condition
\[ \tilde{\Phi}_+(z) = \tilde{\Phi}_-(z)J_1(z), \quad \text{for } z \in \hat{\Sigma}_i \quad (3.12) \]
with
\[
J_1(z) = \begin{cases} 
\begin{pmatrix} e^{2\pi i \alpha} & 0 \\ 0 & e^{-2\pi i \alpha} \end{pmatrix}, & \text{if } s < 0, \\
\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, & \text{if } s > 0,
\end{cases} \\
J_2(z) = \begin{pmatrix} 1 & 0 \\ e^{2\pi i \alpha} & 1 \end{pmatrix}, \quad J_3(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_4(z) = \begin{pmatrix} 1 & 0 \\ e^{-2\pi i \alpha} & 1 \end{pmatrix};
\]
(c) The asymptotic behavior of $\Phi(z)$ at infinity:

$$\Phi(z) = \left( \begin{array}{cc} 1 & 0 \\ -im_2(t) & 1 \end{array} \right) \left( I + O \left( \frac{1}{z^2} \right) \right) z^{-\frac{1}{2} \sigma_3} I + i\sigma_1 e^{-\theta(z,t)\sigma_3}, \quad \text{as } z \to \infty,$$

where $\theta(z,t) = \frac{2}{3} z^2 + tz^\frac{1}{2}$, arg $z \in (-\pi, \pi)$;

(d) The asymptotic behavior of $\Phi(z)$ at the node point $s$:

$$\Phi(z) = \Phi(s)(z) \left\{ \begin{array}{ll}
1 & \frac{\omega \ln(z-s)}{2\pi i}, \quad \text{if } s > 0 \\
0 & 1, \quad \text{as } z \to s, \\
1 & \frac{\ln(z-s)}{2\pi i}, \\
0 & 1, \quad \text{if } s < 0
\end{array} \right\} C, \quad \text{as } z \to s, \quad (3.14)$$

where $\Phi(s)(z) = \Phi_0(s)(I + \Phi_1(s)(z-s) + \cdots)$ is analytic at $z = s$ and the constant matrix $C$ is

$$C = \begin{cases}
(I - \sigma_-) e^{\pi i \alpha \sigma_3} & z \in \Omega_2 \\
(I + \sigma_-) e^{-\pi i \alpha \sigma_3} & z \in \Omega_3 \\
e^{\pi i \alpha \sigma_3} & z \in \Omega_1^L \\
e^{-\pi i \alpha \sigma_3} & z \in \Omega_4^L
\end{cases}; \quad (3.15)$$

(e) The asymptotic behavior of $\Phi(z)$ at $z = 0$:

$$\Phi(z) = O(1) \Psi(z), \quad s > 0, \quad \Phi(z) = \Phi^{(0)}(z) z^{\alpha \sigma_3}, \quad s < 0, \quad \text{as } z \to 0, \quad (3.16)$$

where $\Phi^{(0)}(z)$ is analytic at $z = 0$.

To facilitate our future derivation of the associated Lax pair, let us introduce one more transformation:

$$\Phi(z) = \left( \begin{array}{cc} 1 & 0 \\ \eta & 1 \end{array} \right) \Phi(s + z). \quad (3.17)$$

Then, $\Phi(z)$ solves the following model RH problem.

**RH problem for $\Phi$:**

(a) $\Phi(z) := \Phi(z; x, s; \alpha, \omega)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_i$; see Fig. 3.

(b) $\Phi(z)$ satisfies the same jump conditions as $\Phi(s + z)$ on $\Sigma_i$.

(c) The asymptotic behavior of $\Phi(z)$ at infinity:

$$\Phi(z) = \left( \begin{array}{cc} 1 & 0 \\ -ir_2 & 1 \end{array} \right) \left( I + \Phi_{-1} + \Phi_{-2} \frac{1}{z^2} + \Phi_{-3} \frac{1}{z^3} + O \left( \frac{1}{z^4} \right) \right) z^{-\frac{1}{2} \sigma_3} I + i\sigma_1 e^{-\theta(z,x)\sigma_3}, \quad (3.18)$$
Figure 3: The contour for $\Phi$ (left: $s < 0$; right: $s > 0$).

where $\theta(z, x) = \frac{2}{3}z^\frac{3}{2} + xz^\frac{1}{2}$, $\arg z \in (-\pi, \pi)$ and

$$
\Phi_{-1} = \begin{pmatrix} r_1(x) & i r_2(x) \\ -i r_3(x) & -r_1(x) \end{pmatrix}, \quad \Phi_{-2} = \begin{pmatrix} k_1(x) & i k_2(x) \\ -i k_3(x) & -k_1(x) \end{pmatrix},
$$

$x = t + s$ and the pre-factor in (3.17) is chosen to be $\eta = i m_2(t) - i r_2 + \frac{s(2x - s)}{4}i$ to simplify the differential equations for $\Phi$;

(d) The asymptotic behavior of $\Phi(z)$ at $z = 0$:

$$
\Phi(z) = \Phi^{(0)}(z) \begin{cases} 
1 & \frac{\omega \ln z}{2\pi i} \\
0 & 1 \\
1 & \frac{\ln z}{2\pi i} \\
0 & 1 
\end{cases}, \quad \text{if } s > 0
$$

$$
\Phi(z) = \Phi^{(0)}_{-1} C, \quad \text{as } z \to 0,
$$

where $\Phi^{(0)} = \Phi^{(0)}_0 (I + \Phi^{(0)}_1 z + \cdots)$ is analytic at $z = 0$ and the constant matrix $C$ is defined in (3.15);

(e) The asymptotic behavior of $\Phi(z)$ as $z \to -s$

$$
\Phi(z) = O(1) \Psi(z + s) \quad \text{for } s > 0,
$$

$$
\Phi(z) = \Phi^{(-s)}_0 (I + \Phi^{(-s)}_1 (z + s) + O(z + s)^2) (z + s)^{\alpha_3} \quad \text{for } s < 0,
$$

where the behavior of $\Psi$ near origin is given in (1.29) and (1.30).

For $s = 0$, the jump conditions for $\Phi(z; t, 0; \alpha, \omega)$ in (3.17) is the same as that of $\Psi(z; t; \alpha, 0)$. From (1.29) and (3.3), one can see that $\Phi(z; t, 0; \alpha, \omega) \Psi(z; t; \alpha, 0)^{-1} = Y(z) \Psi(z; t; \alpha, \omega) \Psi(z; t; \alpha, 0)^{-1}$ is analytic in the complex plane with a removable singularity at the origin and tends to the identity matrix as $z \to \infty$. Thus, we have the following remark.

**REMARK 8.** For $s = 0$, we have

$$
\Phi(z; t, 0; \alpha, \omega) = \Psi(z; t; \alpha, 0),
$$

where $\Psi(z; t; \alpha, \omega)$ is the solution to the RH problem associated with the $P_{34}$ equation given in Section [1.1].
4 Coupled Painlevé II equations

In this section, we derive a Lax pair from the model RH problem for Φ, which is a Garnier system in two variables. The compatibility condition of the Lax pair gives us the coupled P₂ equations. We also show an important relation between the Hamiltonian \( H \) in (2.1) and the RH problem for Φ. The solvability of the model RH problem for Φ and the existence of real analytic solution to the P₂ equations are justified. The Bäcklund transformations for the coupled P₂ system are also studied.

4.1 Lax pair

We derive a Lax pair from the RH problem for Φ. It furnishes as a generalization of the Lax pair for the P₃4 equation given in (1.42).

**PROPOSITION 3.** We have the following Garnier system in two variables

\[
\Phi_z(z; x, s) = \left( \begin{array}{c}
\frac{u_{1z}}{z^2} + \frac{u_{2z}}{2(z+s)} \\
-i(z + x + v_1 + v_2 + \frac{u_{1z}^2}{4v_{1z}} + \frac{u_{2z}^2}{4v_{2z}(z+s)}) - \frac{v_{1z}}{2z} - \frac{v_{2z}}{2(z+s)}
\end{array} \right) \Phi(z; x, s), \quad (4.1)
\]

\[
\Phi_x(z; x, s) = \left( \begin{array}{cc}
0 & i \\
-iz - 2i(v_1 + v_2 + \frac{z}{2}) & 0
\end{array} \right) \Phi(z; x, s), \quad (4.2)
\]

and

\[
\Phi_s(z; x, s) = \left( \begin{array}{c}
\frac{u_{2z}}{2(z+s)} \\
iv_2 - i\frac{u_{2z}^2 - 4\alpha^2}{4v_{2z}(z+s)} - \frac{v_{2z}}{2(z+s)}
\end{array} \right) \Phi(z; x, s). \quad (4.3)
\]

Moreover, the compatibility conditions \( \Phi_{zx} = \Phi_{xz} \) and \( \Phi_{xs} = \Phi_{sx} \) gives us the couple P₂ equations (2.4) and

\[
v_{2x} = -(v_{1z} + v_{2z}), \quad (4.4)
\]

respectively.

**Proof.** Due to the fact that the jump matrices for Φ are all constants, then \( \Phi_z \Phi^{-1}, \Phi_x \Phi^{-1} \) and \( \Phi_s \Phi^{-1} \) are meromorphic functions with possible isolated singular points at \(-s\) and 0. Using the behavior of Φ at infinity, \(-s\) and 0, we have

\[
\Phi_z = \left( \begin{array}{cc}
a(z; x, s) & b(z; x, s) \\
c(z; x, s) & -a(z; x, s)
\end{array} \right) \Phi, \quad (4.5)
\]

\[
\Phi_x = \left( \begin{array}{cc}
0 & i \\
-iz + 2ir_1 - ir_2 - ir_{2x} & 0
\end{array} \right) \Phi, \quad (4.6)
\]

and

\[
\Phi_s = \left( \begin{array}{cc}
\frac{a_2(x; s)}{x+s} & \frac{iv_2(x; s)}{x+s} \\
-i\frac{r_2z}{x+s} + \frac{a_2(x; s)}{z+s} & -\frac{a_2(x; s)}{z+s}
\end{array} \right) \Phi, \quad (4.7)
\]
where

\[ a(z; x, s) = \frac{a_1(x; s)}{z} + \frac{a_2(x; s)}{z + s} \quad (4.8) \]

\[ b(z; x, s) = i - \frac{iv_1(x; s)}{z} - \frac{iv_2(x; s)}{z + s} \quad (4.9) \]

\[ c(z; x, s) = -iz - \frac{x}{2} + 2ir_1 - ir_2^2 + \frac{c_1(x; s)}{z} + \frac{c_2(x; s)}{z + s} \quad (4.10) \]

Making use of the large-$z$ expansion of $\Phi(z)$ in (3.18), we have from (4.6)

\[ \frac{1}{z} \left( \frac{d}{dx} \Phi_{-1} + i[\Phi_{-1}, \sigma_+] - i[\sigma_{-1}, \Phi_{-1}] - i[\Phi_{-2}, \sigma_-] \right) + O(1/z^2) = 0, \]

where $\sigma_{\pm}$ are the constant matrices given below

\[ \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (4.11) \]

Particularly, the (1, 2) entry of the above matrix equation gives us the useful relation

\[ r_1 = \frac{1}{2}(-r_{2x} + r_2^2). \quad (4.12) \]

Similarly, substituting large-$z$ expansion of $\Phi(z)$ into (4.7), we obtain

\[ r_{2s} = -v_2. \quad (4.13) \]

With the aid of the above two formulas, the equation (4.6) and the expression for $c(z; x, s)$ in (4.10) are simplified to

\[ \Phi_x = \begin{pmatrix} 0 & i \\ -iz - 2ir_{2x} & 0 \end{pmatrix} \Phi, \quad (4.14) \]

and

\[ c(z; x, s) = -iz - \frac{x}{2} - ir_{2x} + \frac{c_1(x; s)}{z} + \frac{c_2(x; s)}{z + s}. \quad (4.15) \]

Let us consider the equations (4.5) and (4.6). Their compatibility condition $\Phi_{xx} = \Phi_{xz}$ gives us the following equations:

\[ a = \frac{i}{2}b_x \quad (4.16) \]

\[ c = -(z + 2r_2)x + \frac{1}{2}b_{xx} \quad (4.17) \]

\[ c_x = -i - 2(iz + 2ir_{2x})a. \quad (4.18) \]

Moreover, from the behavior of $\Phi$ at 0, $-s$ in (3.20) and (3.21), we have

\[ \det \begin{pmatrix} a_1 & -iv_1 \\ c_1 & -a_1 \end{pmatrix} = 0 \quad \text{and} \quad \det \begin{pmatrix} a_2 & -iv_2 \\ c_2 & -a_2 \end{pmatrix} = -\alpha^2. \quad (4.19) \]
Substituting (4.9) and (4.15) into (4.17) yields
\[ r_{2x} = v_1 + v_2 + \frac{x}{2}. \] (4.20)

By (4.8), (4.13) and (4.19)-(4.20), we get the Lax pair (4.1)-(4.3). And the coupled P₄ equations (2.4) can be derived from (4.16), (4.17) and (4.19). Finally, by the compatibility condition \( \Phi_{x,s} = \Phi_{s,x} \) and formulas (4.13) and (4.20), we obtain (4.4).

Note that the Lax pair system for \( \Phi \) in (4.5)-(4.7) is equivalent to the one in [49, (3.5)-(3.7)] by elementary transformation. The Lax pair for a general couple P₂ system with \( k \) regular singular points and one irregular singular point can be found in [15].

The Hamiltonian \( H \) in (2.1) will play an important role in the derivation of the large gap asymptotics in Section 7. Besides its definition in (2.2), it also has the following simple relation with the RH problem for \( \Phi \).

**PROPOSITION 4.** Let \( H \) be the Hamiltonian defined in (2.2). We have
\[ H = \frac{x^2}{4} - r_2, \] (4.21)
where \( r_2(x; s) = -i(\Phi_{-1})_{12} \) and \( \Phi_{-1} \) is the coefficient of \( 1/z \) term in the large-\( z \) expansion of \( \Phi(z) \); see (3.19).

**Proof.** From (4.1), we have
\[
\Phi_z(z)\Phi^{-1}(z) = -iz\sigma_- + \left( \begin{array}{cc} 0 & i \\ -i(x + v_1 + v_2) & 0 \end{array} \right) + \frac{1}{z} \left( \begin{array}{cc} \frac{v_{1x} + v_{2x}}{4i} & -i(v_1 + v_2) \\ \frac{v^2_{1x}}{4iv_1} + \frac{v^2_{2x} - 4a^2}{4iv_2} & -\frac{v_{1x} + v_{2x}}{2} \end{array} \right) + O\left( \frac{1}{z^3} \right),
\] (4.22)
as \( z \to \infty \). On the other hand, from the large-\( z \) expansion of \( \Phi \) in (3.18), we get
\[
\Phi_z(z)\Phi^{-1}(z) = (I - ir_2\sigma_-) \{ -iz\sigma_- - i[\Phi_{-1}, \sigma_-] - \frac{i}{2} x\sigma_- + i\sigma_+ \\ + \frac{1}{z} (i[\Phi_{-1}, \frac{1}{2} x\sigma_- + \sigma_+ + \sigma_-\Phi_{-1}] - i[\Phi_{-2}, \sigma_-] + \frac{i}{2} x\sigma_+ - \frac{1}{4} \sigma_3) \\ + \frac{1}{z^2} (i[\Phi_{-1}, \frac{x}{2} \sigma_+ + \sigma_+\Phi_{-1}] + \frac{x}{2} \sigma_-\Phi_{-1} + \sigma_-\Phi_{-2} - \sigma_-\Phi_{-1}^2 + \frac{i}{4} \sigma_3) \\ + i[\Phi_{-2}, \frac{1}{2} x\sigma_- + \sigma_+ + \sigma_-\Phi_{-1}] - i[\Phi_{-3}, \sigma_-] - \Phi_{-1}^2) + O(1/z^3) \} (I + ir_2\sigma_-),
\] (4.23)
where the matrices \( \sigma_\pm \) are given in (4.11). Comparing the above two formulas, one obtains the following system of equations involving \( v_i \), \( r_i \) and \( k_i \) (the entries of \( \Phi_{-1} \) and \( \Phi_{-2} \); see (3.19)):
\[
\begin{cases}
  r^2_2 - 2r_1 = v_1 + v_2 + \frac{x}{2}, \\
  r_1r_2 - \frac{x}{2} r_2 = r_2(v_1 + v_2) + r_3 + \frac{1}{2}(v_{1x} + v_{2x}) - k_2 + \frac{1}{4} = 0, \\
  r^2_2(v_1 + v_2) - r_2(v_{1x} + v_{2x}) - r_2r_3 - 2r_1^2 + x r_1 + 2k_1 + \frac{v^2_{1x}}{4v_1} + \frac{v^2_{2x} - 4a^2}{4v_2} = 0, \\
  \frac{x}{2} r^2_2 - r_2r_3 + 2r_2k_2 + \frac{1}{2} r_2 - 2r_1^2 - x r_1 - 2k_1 + svv_2 = 0.
\end{cases}
\] (4.24)
Note that, in the above system, the first equation comes from the $O(1)$ term; the second and third one are from the $O(1/z)$ term; and the last one comes from the $(1,2)$ entry of the $O(1/z^2)$ term. Eliminating the functions $k_1, k_2$ and $r_1$, we obtain

$$r_2 = (v_1 + v_2 + \frac{x}{2})^2 - \frac{v_1^2}{4v_1} - \frac{v_2^2}{4v_2} - sv_2 = \frac{x^2}{4} - H,$$

where the definition of $H$ is given in (2.2). This finishes the proof of our proposition.

For later use, we derive two differential identities for the functions $v_2$ and $w_2$.

**PROPOSITION 5.** For $s < 0$ and $z \to s$, with the behavior of $\Phi(z)$ given in (3.21), we have

$$\frac{d}{dx} \ln \left( \Phi_0(-s) \right)_{11} (x) = w_2(x)$$

and

$$2\alpha \frac{d}{dx} \left( \Phi_1(-s) \right)_{11} (x) = -v_2(x).$$

**Proof.** From (4.2) in the Lax pair and (3.21), we have

$$\frac{d}{dx} \Phi_0(-s) = \begin{pmatrix} 0 & i \\ -2i(v_1 + v_2 + \frac{x-s}{2}) & 0 \end{pmatrix} \Phi_0(-s),$$

$$\frac{d}{dx} \Phi_1(-s) = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \Phi_0(-s),$$

and

$$\alpha \Phi_0(-s) \sigma_3 \left( \Phi_0(-s) \right)^{-1} = \begin{pmatrix} \frac{1}{2}v'^2_2 - 4\alpha^2 & -iv_2 \\ v'^2_2 - 4\alpha^2 & -\frac{1}{2}v'^2_2 \end{pmatrix}.$$

Then, the above three equations give us

$$\frac{d^2}{dx^2} \left( \Phi_0(-s) \right)_{11} = 2(v_1 + v_2 + \frac{x-s}{2}) \left( \Phi_0(-s) \right)_{11} = (w_2^2 + w_{2x}) \left( \Phi_0(-s) \right)_{11},$$

and

$$\frac{d}{dx} \left( \Phi_1(-s) \right)_{11} = -\frac{1}{2\alpha}v_2.$$ 

Thus, (4.26) and (4.27) immediately follow from the above two equations.

**4.2 Vanishing lemma**

**LEMMA 3.** Suppose that the homogeneous RH problem for $\tilde{\Phi}(z)$ shares the same jump conditions and boundary behaviors near 0 and $-s$ as $\Phi(z)$, but satisfies the following asymptotic behavior at infinity

$$\tilde{\Phi}(z) = O(z^{-\frac{1}{4}} e^{\frac{i}{2} \sqrt{2} e^{-(\frac{1}{2}z^{3/2} + x^2)}}),$$

Then, for $\alpha > -\frac{1}{2}, \omega \in \mathbb{C} \setminus (-\infty, 0)$ and $s, x \in \mathbb{R}$, the solution is trivial, that is $\tilde{\Phi}(z) \equiv 0$. 

23
Proof. The proof is similar to [40] and [63, Lemma 1]. The key point is that the entries in the jump matrices $J_i$ in (3.12) satisfy the following conjugate relations 

\[ (J_1)_{11} = (J_1)_{22} \]

and 

\[ (J_2)_{12} = (J_4)_{12}. \]

COROLLARY 1. For $\alpha > -\frac{1}{2}$, $\omega \in \mathbb{C} \setminus (-\infty, 0)$ and $s, x \in \mathbb{R}$, there is unique solution to the RH problem for $\Phi$. And there exist real analytic solutions $v_1(x; s)$ and $v_2(x; s)$ to the coupled $P_2$ equations [2.4] for real values of $x$. Moreover, the Hamiltonian $H$ in [4.21] is also real analytic for real values of $x$.

Proof. The solvability of the RH problem for $\Phi$ follows from Lemma 3, namely the vanishing Lemma; see the similar arguments in [28, 32, 33, 65]. Then, the functions $r_i(x)$ in (3.19) are all analytic for real values $x$. Therefore, the Hamiltonian $H$ in (4.21) is also real analytic for real values of $x$. Similarly, from (4.1), the functions $v_i(x)$ can be expressed as

\[
\begin{align*}
v_1(x) &= i \lim_{z \to 0} z(\Phi'(z)\Phi(z)^{-1})_{12}, \\
v_2(x) &= i \lim_{z \to -s} (z + s)(\Phi'(z)\Phi(z)^{-1})_{12},
\end{align*}
\]

where $'$ indicates the derivative with respect to $z$. Then, the analyticity of $v_1(x; s)$ and $v_2(x; s)$ also follows from the solvability of the RH problem for $\Phi$.

Moreover, once the solution exists, it is easy to prove its uniqueness with the boundary conditions given in the RH problem for $\Phi$. Let $\Upsilon := \begin{pmatrix} 1 & 0 \\ ir_2 & 1 \end{pmatrix}$ and note that $\sigma_3 \Upsilon \Phi(\bar{z}) \sigma_3$ also satisfies the RH problem for $\Upsilon \Phi$. From the uniqueness of the RH problem, we have

\[ \sigma_3 \Upsilon \Phi(\bar{z}) \sigma_3 = \Upsilon \Phi(z). \]

Finally, the above formula ensures that $v_1(x; s), v_2(x; s)$ are real for real values of $x$. \hfill \Box

4.3 Bäcklund transformations

From the model RH problem for $\Phi$, we have the following useful Bäcklund transformations for the coupled $P_2$ system. Similar arguments work for the Painlevé equations; for example, see [31] and [21, Sec. 6].

PROPOSITION 6. Let $v_i(x; 2\alpha + 1)$ and $v_i(x; 2\alpha)$ be the solutions to the coupled $P_2$ system given in Theorem 7, we have

\[
\begin{align*}
v_1(x; 2\alpha + 1) + v_2(x; 2\alpha + 1) - v_1(x; 2\alpha) - v_2(x; 2\alpha) &= -w_2'(x; 2\alpha + \frac{1}{2}), \\
w_2(x; 2\alpha + \frac{3}{2}) + w_2(x; 2\alpha + \frac{1}{2}) &= -\frac{2\alpha + 1}{v_2(x; 2\alpha + 1)},
\end{align*}
\]

where $'$ indicates derivative with respect to $x$.\hfill \Box
Proof. From the RH problem for \( \Phi \), we see that \( \Phi(z;2\alpha+1) \) and \( \sqrt{z+s} \sigma_3 \Phi(z;2\alpha)\sigma_3 \) satisfy the same jump conditions. According to the asymptotic behaviors of \( \Phi \) at infinity and the singular points \( z = 0, -s \), we find that \( \Phi(z;2\alpha)(\sqrt{z+s} \sigma_3 \Phi(z;2\alpha)\sigma_3)^{-1} \) is meromorphic with a simple pole at \( z = -s \). Moreover, using the behaviors of \( \Phi \) at infinity in (3.18), we have
\[
\Phi(z;2\alpha + 1) = \begin{pmatrix}
1 & 0 \\
-ir_2(z;2\alpha + 1) & 1
\end{pmatrix}
\begin{pmatrix}
-r_2(z;2\alpha + 1) & i \\
i(z+s) - i(r_1(x;2\alpha + 1) - r_1(x;2\alpha)) & -r_2(x;2\alpha)
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-ir_2(x;2\alpha) & 1
\end{pmatrix}
\frac{\sigma_3 \Phi(z;2\alpha)\sigma_3}{\sqrt{z+s}},
\] (4.38)
where \( r_i(x) \) are given in (3.19). Similarly, the local behavior of \( \Phi \) at \( z = -s \) gives us
\[
\lim_{z \to -s} \sqrt{z+s} \Phi(z;2\alpha+1)\sigma_3 \Phi(z;2\alpha)^{-1} \sigma_3 = \hat{\Phi}(-s;2\alpha + 1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \sigma_3 \hat{\Phi}(-s;2\alpha)^{-1} \sigma_3,
\] (4.39)
where \( \hat{\Phi} \) is the series part of the expansion of \( \Phi \) near \( z = -s \) in (3.21). From the equation (4.1), we have
\[
\hat{\Phi}(-s;2\alpha) = \sqrt{i v_2(x;2\alpha)/2\alpha} \left( \begin{array}{cc}
\frac{v'_2(x;2\alpha) - 2\alpha}{2v_2(x;2\alpha)} \\
\frac{v'_2(x;2\alpha) + 2\alpha}{2v_2(x;2\alpha)}
\end{array} \right) d_1^{r_2} \quad \text{for} \quad \alpha \neq 0
\] (4.40)
and
\[
\hat{\Phi}(-s;2\alpha) = \sqrt{-i v_2(x;2\alpha)} \left( \begin{array}{cc}
\frac{v'_2(x;2\alpha)}{2v_2(x;2\alpha) + 2\alpha} \\
0
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \quad \text{for} \quad \alpha = 0
\] (4.41)
with certain non-zero constants \( d_1 \) and \( d_2 \). Then, a combination of (4.38)-(4.41) yields the following equations
\[
\begin{align*}
\frac{r_2(z;2\alpha + 1) - r_2(z;2\alpha)}{2v_2(z;2\alpha)} &= \frac{v'_2(z;2\alpha) + 2\alpha}{2v_2(z;2\alpha)} ,
\frac{v'_2(x;2\alpha+1) + 2\alpha + 1}{2v_2(x;2\alpha+1)} + \frac{v'_2(x;2\alpha) - 2\alpha}{2v_2(x;2\alpha)} &= 0.
\end{align*}
\] (4.42)
Using (4.21), the first equation of the above formula gives us the Bäcklund transformation for the Hamiltonian \( H \):
\[
H(x;2\alpha + 1) - H(x;2\alpha) = \frac{v'_2(x;2\alpha) - 2\alpha}{2v_2(x;2\alpha)}.
\] (4.43)
Finally, (4.36) and (4.37) follow from (2.1), (4.20) and (4.42).

**REMARK 9.** In the case \( s = 0 \), we have \( v_1 = 0 \) and \( v_2(x;2\alpha,\omega) = u(x;2\alpha,0) \) satisfies the \( P_{34} \) equation (1.36) with parameter \( 2\alpha \) and \( w_2(x;2\alpha + \frac{1}{2}) = -2^{1/3}y(-2^{1/3}x;2\alpha + \frac{1}{2}) \) satisfies the \( P_2 \) equation (1.13) with parameter \( 2\alpha + \frac{1}{2} \). From (4.36) and (4.37), we recover the following Bäcklund transformations for \( P_2 \) and \( P_{34} \) equations
\[
u(x;2\alpha + 1,0) - u(x;2\alpha,0) = -2^{2/3}y'(-2^{1/3}x;2\alpha + \frac{1}{2}),
\] (4.44)
\[
y(x;2\alpha + \frac{3}{2}) + y(x;2\alpha + \frac{1}{2}) = \frac{2\alpha + 1}{y^2(x;2\alpha + \frac{1}{2}) + y'(x;2\alpha + \frac{1}{2}) + \frac{\alpha}{2};}
\] (4.45)
see [34] (32.7.1)-(32.7.2), [35] (3.23) and [36] (3.11).
5 Asymptotics of $v_i(x)$ as $x \to +\infty$

In the present section and Appendix A, we perform the Deift-Zhou nonlinear steepest descend method \cite{28,29,30} for the model RH problem of $\Phi$ as the parameter $x \to \pm \infty$. Then, based on the connection of $v_i(x)$ to $\Phi$ given in (4.33) and (4.34), we obtained their asymptotics as $x \to \pm \infty$. As the steepest descent analysis in each section is independent, we are going to use the same notation for the functions. We trust that this will not lead to any confusion.

5.1 Nonlinear steepest descent analysis of $\Phi$ as $x \to +\infty$

We rescale the variable and introduce the first transformation as follows:

$$A(z) = x^{\sigma_3/4} \begin{pmatrix} 1 & 0 \\ i r_2 & 1 \end{pmatrix} \Phi(xz). \quad (5.1)$$

To normalize the large-$z$ behavior of $A(z)$, we define the $g$-function

$$g_1(z) := \frac{2}{3}(z + 1)^{3/2}, \quad \arg(z + 1) \in (-\pi, \pi). \quad (5.2)$$

It is easy to see that $g_1(z) - \left(\frac{2^{3/2}}{3} + z^{1/2}\right) = \frac{1}{4^{1/2}} + O(z^{-3/2})$ as $z \to \infty$. The second transformation is devoted to a normalization at infinity and a shift of jump contours

$$B(z) = \begin{cases} A(z)e^{x^{3/2}g_1(z)\sigma_3}, & z \in I \cup III \cup IV, \\ A(z)e^{x^{3/2}g_1(z)\sigma_3} \begin{pmatrix} 1 & 0 \\ \pm e^{2x^{3/2}g_1(z)\sigma_3}e^{\pm 2\pi i \alpha} & 1 \end{pmatrix}, & z \in II \cup V, \end{cases} \quad (5.3)$$

where the regions are illustrated in Fig. 4. Then, $B(z)$ satisfies the following RH problem.

**Figure 4:** Regions and contours for $B$ ($c_1 = \min(0, -\frac{\pi}{2})$ and $c_2 = \max(0, -\frac{\pi}{2})$).

**RH problem 5.1.** The function $B(z)$ satisfies the following properties:

(a) $B(z)$ is analytic in $\mathbb{C}\setminus\{\Sigma_2 \cup \Sigma_4 \cup (-\infty, \max(0, -\frac{\pi}{2}))\}$;
(b) $B(z)$ satisfies the jump condition

$$B_+(z) = B_-(z)J_B(z), \quad (5.4)$$

where

$$J_B(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
1 & e^{2\pi i g_1(z)} e^{2\pi i \alpha} \\
0 & 1 \\
-1 & 0 
\end{pmatrix}, & z \in \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\
0 & e^{-2\pi i \alpha} \\
e^{2\pi i g_1(z)} e^{2\pi i \alpha} & 1 
\end{pmatrix}, & z \in \Sigma_4,
\end{cases}$$

and

$$J_B(z) = \begin{cases} 
e^{2\pi i \alpha} & z \in (0, -\frac{s}{x}) \text{ if } s < 0, \\
1 \omega e^{-2\pi \frac{3}{2} g_1(z)} & z \in (-\frac{s}{x}, 0) \text{ if } s > 0; 
\end{cases} \quad (5.7)$$

(c) The asymptotic behavior of $B(z)$ at infinity:

$$B(z) = \left( I + O \left( \frac{1}{z} \right) \right) z^{-\frac{1}{4} \sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} \quad \text{as } z \to \infty. \quad (5.5)$$

Note that

$$\text{Re } g_1(z) < 0, \quad \text{for } z \in \Sigma_2 \cup \Sigma_4, \quad \text{and } \quad \text{Re } g_1(z) > 0 \quad \text{for } z \in (-1, +\infty), \quad (5.6)$$

then, the off-diagonal entries of the jump matrices are exponentially small as $x \to +\infty$. Neglecting the exponential small terms, we arrive at the following outer parametrix.

**RH problem 5.2.** The function $B^{(\infty)}(z)$ satisfies the following properties:

(a) $B^{(\infty)}(z)$ is analytic in $\mathbb{C} \setminus (-\infty, -\frac{s}{x}]$;

(b) $B^{(\infty)}(z)$ satisfies the jump condition

$$B^{(\infty)}_+(z) = B^{(\infty)}_-(z) \begin{cases} 
\begin{pmatrix} 0 & 1 \\
-1 & 0 
\end{pmatrix}, & z \in (-\infty, -1), \\
e^{2\pi i \alpha} & z \in (-1, -\frac{s}{x});
\end{cases} \quad (5.7)$$

(c) At infinity, $B^{(\infty)}(z)$ satisfies the same asymptotics as $B(z)$ in (5.5).
To construct a solution to the above RH problem, let us first introduce a scalar function $h(z)$ as follows:

$$h(z) := \left(\frac{\sqrt{z} - 1}{\sqrt{z} + 1}\right)^\alpha, \quad z \in \mathbb{C} \setminus (-\infty, 1],$$

(5.8)

where the power function $z^c, c \notin \mathbb{Z}$, takes the principle branch with the cut along $(-\infty, 0)$. Note that, $h(z)$ satisfies the following jump conditions

$$\begin{cases} h_+(x) = h_-(x)e^{2\pi i \alpha}, & x \in (0, 1) \\ h_+(x)h_-(x) = 1, & x \in (-\infty, 0). \end{cases}$$

(5.9)

Then, a solution to the RH problem for $B^{(\infty)}(z)$ is given explicitly as

$$B^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ 2\alpha i\sqrt{1 - z} & 1 \end{pmatrix} (z + 1)^{-\sigma_3/4} I + i\sigma_1 \frac{1}{\sqrt{2}} h_1(z)^{\sigma_3},$$

(5.10)

where

$$h_1(z) = h\left(\frac{z + 1}{1 - \frac{s}{x}}\right) = \left(\frac{\sqrt{z + 1} - \sqrt{1 - \frac{s}{x}}}{\sqrt{z + 1} + \sqrt{1 - \frac{s}{x}}}\right)^\alpha, \quad z \in \mathbb{C} \setminus \left(-\infty, -\frac{s}{x}\right].$$

(5.11)

The jump matrices of $B(z)B^{(\infty)}(z)^{-1}$ are not uniformly close to the unit matrix near the end-points $-1$, 0 and $-\frac{s}{x}$. Then, local parametrices have to be constructed in neighborhoods of the end-points.

Let $U_{-1} := \{z : |z + 1| < \delta\}$ for certain small $0 < \delta < 1/2$. The local parametrix near $-1$ should share the same jumps (5.4) with $B$ in the neighborhood $U_{-1}$, and match with $B^{(\infty)}(z)$ on $|z + 1| = \delta$. It is readily verified that such a parametrix can be represented as follows

$$B^{(-1)}(z) = E_0(z)Z_A(x(z + 1)) \begin{cases} e^{\left(\frac{x}{2}\right)^\frac{3}{2}(z + 1)^3 + \alpha \pi i)\sigma_3}, & \arg z \in (0, \pi), \\ e^{\left(\frac{x}{2}\right)^\frac{3}{2}(z + 1)^3 - \alpha \pi i)\sigma_3}, & \arg z \in (-\pi, 0), \end{cases}$$

(5.12)

where the pre-factor $E_0(z)$ is an analytic function in $U_{-1}$. Here, $Z_A(z)$ is the following solution to the well-known Airy model RH problem:

Figure 5: Regions and contours for $Z_A$. 

28
The function \( RH \) problem 5.3. indicated in Fig. 5; cf. \cite{29, (7.9)}.

Consider them together and look for the following local parametrix in

\[
\begin{bmatrix}
\text{Ai}(\lambda) & \text{Ai}(\omega^2 \lambda) \\
\text{Ai}'(\lambda) & \omega^2 \text{Ai}'(\omega^2 \lambda)
\end{bmatrix}
\]

\[
e^{-\frac{\pi i}{2} \sigma_3}, \quad \lambda \in I
\]

\[
\begin{bmatrix}
\text{Ai}(\lambda) & \text{Ai}(\omega^2 \lambda) \\
\text{Ai}'(\lambda) & \omega^2 \text{Ai}'(\omega^2 \lambda)
\end{bmatrix}
\]

\[
e^{-\frac{\pi i}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \lambda \in II
\]

\[
\begin{bmatrix}
\text{Ai}(\lambda) & -\omega^2 \text{Ai}(\omega^2 \lambda) \\
\text{Ai}'(\lambda) & -\text{Ai}'(\omega^2 \lambda)
\end{bmatrix}
\]

\[
e^{-\frac{\pi i}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \lambda \in III
\]

\[
\begin{bmatrix}
\text{Ai}(\lambda) & -\omega^2 \text{Ai}(\omega^2 \lambda) \\
\text{Ai}'(\lambda) & -\text{Ai}'(\omega^2 \lambda)
\end{bmatrix}
\]

\[
e^{-\frac{\pi i}{2} \sigma_3}, \quad \lambda \in IV,
\]

where \( \omega = e^{rac{2 \pi i}{x}} \), the constant matrix \( M_A = \sqrt{2\pi e^{rac{1}{2} \pi i}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \) and the regions are indicated in Fig. 5, cf. \cite{29, (7.9)}.

As \( x \to +\infty \), the other two endpoints \( 0 \) and \( -\frac{\pi}{x} \) are every close to each other. Then, we consider them together and look for the following local parametrix in \( U_0 := \{ z : |z| < \delta \} \).

**RH problem 5.3.** The function \( B^{(0)}(z) \) satisfies the following properties:

(a) \( B^{(0)}(z) \) is analytic in \( U_0 \setminus (-\infty, \max(0, -\frac{\pi}{x})) \);

(b) \( B^{(\infty)}(z) \) satisfies the jump condition

\[
B^{(0)}_+(z) = B^{(0)}_-(z) \begin{pmatrix} e^{2\pi i \sigma_3} & \cdot \cdot \\ e^{2\pi i \sigma_3} & \cdot \cdot \\ 0 & e^{-2\pi i \sigma_3} \end{pmatrix}
\]

\[
= \begin{pmatrix} e^{2\pi i \sigma_3} & \cdot \cdot \\ e^{2\pi i \sigma_3} & \cdot \cdot \\ 0 & e^{-2\pi i \sigma_3} \end{pmatrix}, \quad z \in (0, -\frac{\pi}{x}), \quad s < 0,
\]

and

\[
B^{(0)}_+(z) = B^{(0)}_-(z) \begin{pmatrix} e^{2\pi i \sigma_3} & \cdot \cdot \\ e^{2\pi i \sigma_3} & \cdot \cdot \\ 0 & e^{-2\pi i \sigma_3} \end{pmatrix}
\]

\[
= \begin{pmatrix} e^{2\pi i \sigma_3} & \cdot \cdot \\ e^{2\pi i \sigma_3} & \cdot \cdot \\ 0 & e^{-2\pi i \sigma_3} \end{pmatrix}, \quad z \in (-1, 0), \quad s < 0,
\]

and

(c) \( B^{(0)}(z) \) fulfils the following matching condition on \( \partial U_0 \):

\[
B^{(0)}(z)B^{(\infty)}(z)^{-1} = I + o(1), \quad \text{as} \quad x \to +\infty.
\]

A solution to the above RH problem is given explicitly as follows:

\[
B^{(0)}(z) = \left( \frac{1}{2\alpha i \sqrt{1 - \frac{\pi}{x}}} \right) (z+1)^{-\sigma_3/4} \frac{I + i \sigma_1}{\sqrt{2}} \begin{pmatrix} 1 & j(z) \end{pmatrix} h_1(z)^{\sigma_3},
\]

where

\[
j(z) = \begin{cases}
\frac{\omega}{2\pi i} \int_{-\frac{\pi}{x}}^{0} \frac{\exp(-\frac{3}{2} (\zeta + 1) \frac{\pi i}{2}) |h_1(z)|^2}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{-\frac{\pi}{x}}^{-\frac{3}{2}} \frac{\exp(-\frac{3}{2} (\zeta + 1) \frac{\pi i}{2}) |h_1(z)|^2}{\zeta - z} d\zeta, & s > 0 \\
\frac{1}{2\pi i} \int_{-\frac{\pi}{x}}^{0} \frac{\exp(-\frac{1}{2} (\zeta + 1) \frac{\pi i}{2}) |h_1(z)|^2}{\zeta - z} d\zeta, & s < 0,
\end{cases}
\]
for $|z| < \delta$ and $h_1(z)$ is defined in (5.11).

Now the final transformation is given by

$$C(z) = \begin{cases} B(z)B^{(\infty)}(z)^{-1}, & z \in \mathbb{C}(U_{-1} \cup U_0), \\ B(z)(B^{(0)}(z))^{-1}, & z \in U_0, \\ B(z)(B^{(-1)}(z))^{-1}, & z \in U_{-1}. \end{cases} \quad (5.17)$$

Then, $C(z)$ satisfies the following RH problem:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Contour $\Sigma_C$.}
\end{figure}

**RH problem 5.4.** The function $C(z)$ satisfies the following properties:

(a) $C(z)$ is analytic in $\mathbb{C} \setminus \Sigma_C$; see Fig. 6.

(b) $C(z)$ satisfies the jump condition $C_+(z) = C_-(z)J_C(z)$,

\begin{align*}
J_C(z) &= B^{(0)}(z)(B^{(\infty)}(z))^{-1}, \quad z \in \partial U_0, \\
J_C(z) &= B^{(-1)}(z)(B^{(0)}(z))^{-1}, \quad z \in \partial U_{-1}, \\
J_C(z) &= B^{(\infty)}(z)J_B(z)(B^{(\infty)}(z))^{-1}, \quad z \in \Sigma_C \setminus \partial(U_0 \cup U_{-1});
\end{align*}

(c) As $z \to \infty$, $C(z) = I + O(1/z)$.

It follows from the properties of the local parametrices that

$$J_C(z) = \begin{cases} I + O\left(x^{-3/2}\right), & z \in \partial U_{-1}, \\ I + O(e^{-ct}), & z \in \Sigma_R \setminus \partial U_{-1}, \end{cases} \quad (5.18)$$

where $c$ is a positive constant, and the error term is uniform for $z$ on the corresponding contours. Then, it follows from the above formula

$$C(z) = I + O(x^{-3/2}), \quad (5.19)$$

uniformly for $z$ in the complex plane; see [23, 29].
5.2 Proof of Theorem

The analyticity of \( v_1(x) \) for real values of \( x \) is proved in Corollary. Next, we compute their asymptotics.

Based on the transformations \( \Phi \mapsto A \mapsto B \) in (5.1) and (5.3), we have from (4.33) and (4.34)

\[
v_1(x) = i \lim_{z \to 0} (\Phi'(z)\Phi(z)^{-1})_{12} = \frac{i}{\sqrt{x}} \lim_{z \to 0} (B'(z)B(z)^{-1})_{12}, \quad (5.20)
\]

\[
v_2(x) = i \lim_{z \to -s} (z + s)(\Phi'(z)\Phi(z)^{-1})_{12} = \frac{i}{\sqrt{x}} \lim_{z \to -s/\sqrt{x}} ((z + \frac{s}{x})B'(z)B(z)^{-1})_{12}. \quad (5.21)
\]

By the transformation (5.17) and the approximation (5.19), we have for \( |z| < \delta \)

\[
B(z) = (I + O(x^{-3/2}))B^{(0)}(z). \quad (5.22)
\]

Recalling the expression of \( B^{(0)}(z) \) in (5.16), we find the asymptotics of \( v_1 \) from (5.20) and (5.22)

\[
v_1(x) = \left\{ \begin{array}{ll}
\frac{\omega}{4\pi\sqrt{x}} e^{-\frac{4\pi^2}{x} |h_1(0)|^2} (1 + O(x^{-\frac{3}{2}})), & s > 0, \\
\frac{1}{4\pi\sqrt{x}} e^{-\frac{4\pi^2}{x} |h_1(0)|^2} (1 + O(x^{-\frac{3}{2}})), & s < 0,
\end{array} \right. \quad as \ x \to +\infty, \quad (5.23)
\]

where

\[
|h_1(0)|^2 = \left| \frac{1 - \sqrt{1 - \frac{x}{2}}}{1 + \sqrt{1 - \frac{x}{2}}} \right|^{2\alpha} = \frac{1}{2^{4\alpha}} \left| \frac{s}{x} \right|^{2\alpha} \left( 1 + \alpha \frac{s}{x} + O \left( \frac{1}{x^2} \right) \right).
\]

Similarly, we get the asymptotics of \( v_2(x) \)

\[
v_2(x) = \frac{\alpha}{\sqrt{x - s}} - \frac{\alpha^2}{(x - s)^2} + O(x^{-3}), \quad as \ x \to +\infty, \quad if \ \alpha \neq 0. \quad (5.24)
\]

If \( \alpha = 0 \) and \( s < 0 \), the function \( \Phi(z) \) is analytic at the point \( z = -s \). By (5.21), we have \( v_2(x) = 0 \). If \( \alpha = 0 \) and \( s > 0 \), we have the following asymptotics

\[
v_2(x) = \frac{1 - \omega}{4\pi\sqrt{x}} e^{-\frac{4\pi^2}{x} (s-x)^2} (1 + O(x^{-3/2})). \quad (5.25)
\]

Recall the relations among \( v_i, w_i \) and \( H \) in (2.1) and (2.2). Then, the asymptotics of \( w_i \) in (2.9) and (2.10), as well as the Hamiltonian \( H \) in (2.11), are derived by direct computations. This completes the proof of Theorem.

To obtain the large gap asymptotics in Section, let us compute two more quantities, namely \( \Phi_0^{(-s)}(x, s) \) and \( \Phi_1^{(-s)}(x, s) \). When \( s < 0 \), recalling the expansion of \( \Phi(z) \) in (3.21), we have from (5.1), (5.3), (5.16) and (5.22)

\[
\Phi_0^{(-s)}(x, s) = 2^{-2\alpha - \frac{1}{2}} \exp(-\frac{2}{3}(x-s)^2)(x-s)^{-\alpha+\frac{1}{2}}(1+O(x^{-\frac{3}{2}})) \quad as \ x \to \infty \quad (5.26)
\]

and

\[
\Phi_1^{(-s)}(x, s) = -\sqrt{x-s} + O(1/x) \quad as \ x \to \infty. \quad (5.27)
\]
6 Proof of Theorem 2-3

6.1 Tracy-Widom formula for $P_{34}$ kernel: proof of Theorem 2

Before we use the differential identity in (3.5), let us first compute the coefficient $Y_{-1}$. Tracing back the transformation $Y \rightarrow \tilde{\Phi} \rightarrow \Phi$ in (3.10), (3.11) and (3.17), we obtain

$$Y(z) = \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix} \Phi(z-s)\Psi(z)^{-1}. \quad (6.1)$$

From the expansions of $\Psi$ and $\Phi$ at infinity in (1.28) and (3.18), the differential identity in (3.5) yields the following more explicit form

$$\frac{d}{dt} \ln \text{det}[I - K_{P_{34}}^{\alpha,\omega,s}] = -r_2(s+t; s) + m_2(t) - \frac{t^2}{4} + \frac{(s+t)^2}{4}, \quad (6.2)$$

where $m_2$ and $r_2$ are coefficients in the asymptotics of $\Psi$ and $\Phi$ near infinity in (1.28) and (3.18), respectively. Note that $x^2/4 - r_2(x)$ and $x^2/4 - m_2(x)$ are Hamiltonians for the coupled $P_2$ and the $P_2$ equations, respectively; see (4.21). The above formula and (4.20) give us

$$\frac{d^2}{dt^2} \ln \text{det}[I - K_{P_{34}}^{\alpha,\omega,s}] = u(t) - v_1(s+t) - v_2(s+t), \quad (6.3)$$

where $v_i(x) = v_i(x, s; 2\alpha, \omega)$ are the solutions to the coupled $P_2$ equations (2.4) with the properties stated in Theorem 1 and $u(x) = u(x; 2\alpha, \omega)$ is the $P_{34}$ transcendent with the boundary condition (1.37).

For fixed $s \in \mathbb{R}$, the limit below

$$\lim_{t \to +\infty} \ln \text{det}[I - K_{P_{34}}^{\alpha,\omega,s}] = 0 \quad (6.4)$$

follows from the estimation of functions $\psi_i$ in the $P_{34}$ kernel in (1.31)

$$|\psi_i(z, t)| = O(e^{-\frac{2}{3}(z+t)^{2/3}}|z|^\alpha), \quad \text{as } t \to +\infty, \quad (6.5)$$

uniformly for $z > c_0, c_0 \in \mathbb{R}$. The above estimation can be found in the derivation of the large-$t$ asymptotics of the $P_{34}$ transcendent in Its, Kuijlaars and Östensson [41]; see also Section 5 with the parameter $s = 0$ therein.

Denote $f(t, s) := \ln \text{det}[I - K_{P_{34}}^{\alpha,\omega,s}]$. From (6.3) and (6.4), an integration by parts gives us

$$\ln \text{det}[I - K_{P_{34}}^{\alpha,\omega,s}] = -\int_{t}^{+\infty} \frac{d}{dx} f(x, s) dx - \int_{t}^{+\infty} \frac{d}{dx} (v_1(x + s) + v_2(x + s) - u(x))(x - t) dx, \quad (6.6)$$

where the convergence of the above integral is guaranteed by the properties of $u$ in Proposition 1 and $v_i$ in Theorem 1. This completes the proof of Theorem 2.
6.2 Tracy-Widom formula for P₂ kernel: proof of Theorem 3

Let us first prove Lemma 1, which indicates the connection between the Fredholm determinants of the P₂ and the P₃₄ kernels.

Proof of Lemma 1. By definition, we have the gap probability of the random matrices (1.16) near zero

\[
\text{Prob}[M \text{ has no eigenvalues in } (-s(n), s(n))] = \frac{1}{Z_n} \int_I^n \prod_{k=1}^n |x_k|^{2\alpha} e^{-nV(x_k)} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \prod_{k=1}^n dx_k, \quad (6.7)
\]

where \( I := (-\infty, -s(n)) \cup (s(n), +\infty) \), \( s(n) = \frac{2}{\sqrt{\pi} n^{1/4}} \), \( V(x) = \frac{x^4}{4} + \frac{\beta}{2} x^2 \) and the constant \( Z_n = \int_{\mathbb{R}^n} \prod_{k=1}^n |x_k|^{2\alpha} e^{-nV(x_k)} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \prod_{k=1}^n dx_k \). The multiple integrals can be written in a form of the following Hankel determinants

\[
\text{Prob}[M \text{ has no eigenvalues in } (-s(n), s(n))] = \det \left[ \frac{\int_I |x|^{2\alpha} e^{-nV(x)} x^i_j dx}{\int_{\mathbb{R}^n} |x|^{2\alpha} e^{-nV(x)} x^i_j dx} \right]_{i,j=0}^{n-1}. \quad (6.8)
\]

Using the fact that the entries in the Hankel determinant vanish when \( i \) and \( j \) have different parity, we rearrange the rows and columns in the Hankel determinant and obtain

\[
\det D = \det \left[ \frac{\int_I |x|^{2\alpha} e^{-nV(x)} x^i_j dx}{\int_{\mathbb{R}^n} |x|^{2\alpha} e^{-nV(x)} x^i_j dx} \right]_{i,j=0}^{n-1} = \det \left[ \begin{array}{cc} \left[ \frac{\int_I |x|^{2\alpha} e^{-nV(x)} x^{2(i+j)} dx}{\int_{\mathbb{R}^n} |x|^{2\alpha} e^{-nV(x)} x^{2(i+j)} dx} \right]_{i,j=0}^{(n+1)/2-1} & 0 \\ 0 & \left[ \frac{\int_I |x|^{2\alpha} e^{-nV(x)} x^{2(i+j+1)} dx}{\int_{\mathbb{R}^n} |x|^{2\alpha} e^{-nV(x)} x^{2(i+j+1)} dx} \right]_{i,j=0}^{n/2-1} \end{array} \right].
\]

see similar arguments in Forrester [34]. From the above formula, we have

\[
\text{Prob}[M \text{ has no eigenvalues in } (-s(n), s(n))] = \mathbb{P}_{\frac{\alpha}{2} + \frac{1}{4}} \mathbb{P}_{\frac{\alpha}{2} - \frac{1}{4}}, \quad (6.9)
\]

where

\[
\mathbb{P}_{\frac{\alpha}{2} \pm \frac{1}{4}} = \frac{\det \left[ \int_{s(n)^2}^{+\infty} |x|^{\alpha \pm \frac{1}{2}} e^{-nV(\sqrt{x})} x^{i+j} dx \right]_{i,j=0}^{l_{\pm}-1}}{\det \left[ \int_0^{+\infty} |x|^{\alpha \pm \frac{1}{2}} e^{-nV(\sqrt{x})} x^{i+j} dx \right]_{i,j=0}^{l_{\pm}-1}}, \quad (6.10)
\]

with \( l_+ = \lfloor n/2 \rfloor \) and \( l_- = \lfloor (n+1)/2 \rfloor \). Then \( \mathbb{P}_{\frac{\alpha}{2} \pm \frac{1}{4}} \) describe the gap probability of the following unitary ensembles of positive definite Hermitian matrices of size \( n/2 \)

\[
\frac{1}{Z_{n/2}} |\det(M)|^\alpha \frac{1}{2^n} e^{-\frac{n}{2} \text{Tr}(M^2/4 + gM)} dM, \quad (6.11)
\]

where \( n \) is even. For \( g_{cr} = -2 \), the density of the limiting eigenvalue distribution is given by \( \frac{1}{2\pi} \sqrt{x(4-x)} \), \( x \in [0, 4] \). At the origin, the soft edge and the hard edge coalesce. In
the critical regime \( g = -2 + \frac{2^{1/3}}{t^{2/3}} \), the limiting eigenvalue correlation kernel is the \( P_{34} \) kernel \( K_{\frac{2}{2} + \frac{1}{4}, 0}^{P_{34}}(x, y; -2^{-1/3}t) \) in (1.31); see \[19\]. Then, the gap probability of these unitary ensembles is given by
\[
\lim_{n \to \infty} \mathbb{P}_{\frac{2}{2} + \frac{1}{4}} = \det[I - K_{\frac{2}{2} + \frac{1}{4}, 0, s'}^{P_{34}}],
\]
(6.12)
where \( K_{\frac{2}{2} + \frac{1}{4}, 0, s'}^{P_{34}} \) is the trace-class operator acting on \( L^2(s', +\infty) \) with \( s' = -2^{2/3} s^2 \). Meanwhile, the large-\( n \) limit of the gap probability for the unitary ensemble with quartic potential is expressed as the Fredholm determinant of the \( P_2 \) kernel in (1.18). Then, the uniqueness of the limit of (6.9) as \( n \to \infty \) implies (2.14). \( \square \)

Now, we are ready to derive the Tracy-Widom formula for the \( P_2 \)-kernel determinant.

**Proof of Theorem 3.** By Theorem 2 and Lemma 1 we have for real values \( t \) and \( s \geq 0 \)
\[
\det[I - K_{\alpha, s}^{P_2}] = \exp\left(-\int_{t'}^{+\infty} \sum_{i=1}^{2} v_i(\tau + s'; \alpha - \frac{1}{2}, 0) - u(\tau; \alpha - \frac{1}{2}, 0))(\tau - t')d\tau\right)
\]
\[
\exp\left(-\int_{t'}^{+\infty} \sum_{i=1}^{2} v_i(\tau + s'; \alpha + \frac{1}{2}, 0) - u(\tau; \alpha + \frac{1}{2}, 0))(\tau - t')d\tau\right),
\]
where \( s' = -2^{2/3} s^2 \), \( t' = -2^{-1/3} t \), \( v_i(x) = v_i(x; s; \alpha, \omega) \) are the solutions to the coupled \( P_2 \) equations (2.4) with the properties stated in Theorem 1 and \( u(t) = u(t; \alpha, \omega) \) is the \( P_{34} \) transcendent determined by the boundary condition (1.37). By the Hamiltonian systems of equations (2.1) and the Bäcklund transformations (4.36), we have
\[
\sum_{i=1}^{2} (v_i(x; \alpha + \frac{1}{2}) + v_i(x; \alpha - \frac{1}{2})) = w_2^2(x; \alpha) - (x - s).
\]
(6.13)
Similarly, using (4.44) and the relation between the \( P_2 \) and \( P_{34} \) transcendents in (1.39), we get
\[
u(x; \alpha + \frac{1}{2}, 0) + u(x; \alpha - \frac{1}{2}, 0) = 2^{2/3} y^2(-2^{1/3} x; \alpha) - x,
\]
(6.14)
where \( y(x; \alpha) \) is the Hastings-McLeod solution to the \( P_2 \) equation described in (1.41). Combining the above three formulas, we arrive at (2.15). This completes the proof of Theorem 3. \( \square \)

### 7 Large gap asymptotics

Finally, we apply the Deift-Zhou nonlinear steepest descent method to the RH problem for \( \Phi \) and obtain the asymptotics of \( \Phi(z; s + t, s) \) as \( s \to -\infty \). Then, we evaluate the large gap asymptotics for the Fredholm determinants as \( s \to -\infty \).
7.1 Nonlinear steepest descent analysis of $\Phi$ as $s \to -\infty$

As we need the asymptotics of $\Phi(z; s + t, s)$ for $t$ fixed and $s \to -\infty$, let us focus on the case $s < 0$. In the first transformation, we rescale the variable and define:

$$S(\xi) = \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{ir_2} & 1 \end{array} \right) \Phi(|s|\xi)e^{\frac{1}{2} |s|^{1/2} \lambda(\xi) \sigma_3}, \quad (7.1)$$

where

$$\lambda(\xi) := \frac{2}{3} \xi^{3/2} + \frac{s + t}{|s|} \xi^{1/2} = \xi^{1/2} \left( \frac{2}{3} \xi - 1 - \frac{t}{s} \right).$$

Then $S$ satisfies the following RH problem.

![Figure 7: Contours for the RH problem S.](image)

**RH problem 7.1.** The function $S(\xi)$ satisfies the following properties:

(a) $S(\xi)$ is analytic in $\mathbb{C}\setminus\hat{\Sigma}_{S,i}$, where $\hat{\Sigma}_{S,i}$ are indicated in Fig. 7.

(b) $S(\xi)$ satisfies the jump condition

$$S_+(\xi) = S_-(\xi) J_{S,i}(\xi) \quad \text{for} \ \xi \in \hat{\Sigma}_{S,i} \quad (7.2)$$

with

$$J_{S,1}(\xi) = \begin{pmatrix} e^{2\pi i \alpha} & 0 \\ 0 & e^{-2\pi i \alpha} \end{pmatrix}, \quad J_{S,2}(\xi) = \begin{pmatrix} 1 & 0 \\ e^{2|s|^{1/2} \lambda(\xi)} e^{2\pi i \alpha} & 1 \end{pmatrix},$$

$$J_{S,3}(\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_{S,4}(\xi) = \begin{pmatrix} 1 & 0 \\ e^{2|s|^{1/2} \lambda(\xi)} e^{-2\pi i \alpha} & 1 \end{pmatrix};$$

(c) The asymptotic behavior of $S(\xi)$ at infinity

$$S(\xi) = \left( I + O \left( \frac{1}{\xi} \right) \right) (|s| \xi)^{-\frac{1}{2} |\sigma_3 I + i \sigma_1 \sqrt{2}} \quad \text{as} \ \xi \to \infty; \quad (7.3)$$

(d) The asymptotic behavior of $S(\xi)$ at $\xi = 0, 1$ is the same as that of $\Phi(z)$ at $z = 0, |s|$, given in (3.20) and (3.21).
Note that
\[ \text{Re } \lambda(\xi) < 0, \quad \text{for } \xi \in \hat{\Sigma}_{S,2} \cup \hat{\Sigma}_{S,4}, \] (7.4)
the off-diagonal entries of the jump matrices are exponentially small as \( |s| \to +\infty. \)
Neglecting the exponential small terms, we arrive at the following outer parametrix.

**RH problem 7.2.** The function \( S^{(\infty)}(\xi) \) satisfies the following properties:
(a) \( S^{(\infty)}(\xi) \) is analytic in \( \mathbb{C}\setminus(-\infty, 1] \);
(b) \( S^{(\infty)}(\xi) \) satisfies the jump condition
\[
S_+^{(\infty)}(\xi) = S_-^{(\infty)}(\xi) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \xi \in (-\infty, 0),
\]
\[ e^{2\pi i \sigma_3}, \quad \xi \in (0, 1); \] (7.5)
(c) At infinity, \( S^{(\infty)}(\xi) \) satisfies the same asymptotics as \( S(\xi) \) in (7.3).

Similar to **RH problem 5.2**, the solution is given explicitly as
\[
S^{(\infty)}(\xi) = \left( \begin{pmatrix} 1 & 0 \\ 2\alpha i |s|^{1/2} & 1 \end{pmatrix} \right) (|s|\xi)^{-1/2} \frac{I + i\sigma_1}{\sqrt{2}} h(\xi)^{\sigma_3},
\] (7.6)
where \( h(\xi) = \left( \frac{\sqrt{\xi - 1}}{\sqrt{\xi + 1}} \right)^{\alpha} \) is defined in (5.8) and satisfies the following expansion
\[
h(\xi) = 1 - 2\alpha \frac{1}{\xi^{1/2}} + 2\alpha^2 \frac{1}{\xi} - \frac{2}{3}(\alpha + 2\alpha^3) \frac{1}{\xi^{3/2}} + \cdots, \quad \text{as } \xi \to \infty. \] (7.7)

For later use, we also compute the following refined expansion
\[
S^{(\infty)}(\xi) = \left( I + \frac{1}{\xi} \left( -\frac{2}{3}(\alpha - 4\alpha^3)|s|^{1/2} - 2\alpha^2 \right) + O \left( \frac{1}{\xi^{3/2}} \right) \right) (|s|\xi)^{-1/2} \frac{I + i\sigma_1}{\sqrt{2}}.
\] (7.8)

To construct the local parametrix near \( \xi = 0 \), we take the conformal mapping
\[
\lambda^2(\xi) = \xi(1 + t - \frac{2}{3} \xi^{1/2})^2, \quad |\xi| \leq \frac{1}{2}. \] (7.9)
The local parametrix is constructed in terms of the Bessel functions as follows
\[
S^{(0)}(\xi) = E(\xi) Z_0(|s|^3 \lambda(\xi)^2) \begin{cases} e^{(|s|^3 \lambda(\xi) + \alpha \pi i) \sigma_3}, & \text{arg } \xi \in (0, \pi), \\ e^{(|s|^3 \lambda(\xi) - \alpha \pi i) \sigma_3}, & \text{arg } \xi \in (-\pi, 0), \end{cases}
\] (7.10)
where \( E(\xi) \) is an analytic function in the disk \( |\xi| \leq 1/2 \), and the function \( Z_0(z) \) is explicitly given in terms of the modified Bessel functions as follows:
\[
Z_0(z) = \pi^{3/2} \sigma_3 \begin{cases} \begin{pmatrix} I_0(\sqrt{z}) & i \pi K_0(\sqrt{z}) \\ \pi i \sqrt{z} I_0'(\sqrt{z}) & -\sqrt{z} K_0'(\sqrt{z}) \end{pmatrix}, & \text{for } z \in I, \\ \begin{pmatrix} I_0(\sqrt{z}) & i \pi K_0(\sqrt{z}) \\ \pi i \sqrt{z} I_0'(\sqrt{z}) & -\sqrt{z} K_0'(\sqrt{z}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } z \in II, \\ \begin{pmatrix} I_0(\sqrt{z}) & i \pi K_0(\sqrt{z}) \\ \pi i \sqrt{z} I_0'(\sqrt{z}) & -\sqrt{z} K_0'(\sqrt{z}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in III, \end{cases}
\] (7.11)
for arg $z \in (-\pi, \pi)$. It is well-known that, the above function $Z_0(z)$ satisfies the following model RH problem; see [51].

**RH problem 7.3.** $Z_0(z)$ satisfies the following properties:

(a) $Z_0(z)$ is analytic in $\mathbb{C}\setminus \Sigma_i$, where the contours $\Sigma_i$ are illustrated in Fig. 8.

(b) $Z_0(z)$ satisfies the jump condition

$$Z_{0+}(z) = Z_{0-}(z)J_i(z), \quad z \in \Sigma_i, \quad i = 2, 3, 4,$$

where
$$J_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

(c) The asymptotic behavior of $Z_0(z)$ at infinity

$$Z_0(z) = z^{-\frac{1}{4} \sigma_3} \frac{I + i \sigma_1}{\sqrt{2}} \left( I + \frac{1}{8\sqrt{z}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} + O \left( \frac{1}{z} \right) \right) e^{\sqrt{z} \sigma_3} \quad \text{as } z \to \infty.$$

To match the local parametrix $S^{(0)}(\xi)$ with the outer parametrix $S^{(\infty)}(\xi)$ in (7.6) on $|\xi| = \frac{1}{2}$, we choose the analytic pre-factor $E(\xi)$ in (7.10) as

$$E(\xi) = S^{(\infty)}(\xi) \left\{ \begin{array}{ll} e^{-\pi i \alpha \sigma_3} \frac{I - i \sigma_1}{\sqrt{2}} (|s|^3 \lambda(\xi))^\frac{1}{4} \sigma_3, & \text{arg } \xi \in (0, \pi), \\ e^{\pi i \alpha \sigma_3} \frac{I - i \sigma_1}{\sqrt{2}} (|s|^3 \lambda(\xi))^\frac{1}{4} \sigma_3, & \text{arg } \xi \in (-\pi, 0). \end{array} \right.$$  

(7.14)

Then, $E(\xi)$ is analytic in the disk $|\xi| \leq 1/2$. Moreover the following matching condition is fulfilled

$$S^{(0)}(\xi) = (I + O(1/s))S^{(\infty)}(\xi), \quad |\xi| = \frac{1}{2}.$$

(7.15)

For later use, we compute $E(0)$ and $E'(0)$. Let

$$\hat{E}(\xi) = \begin{pmatrix} 1 & 0 \\ -2\alpha i |s|^{1/2} & 1 \end{pmatrix} E(\xi),$$

37
then we get from (7.14)

$$
\hat{E}(\xi) = \frac{1}{2} (\hat{h}(\xi) + \hat{h}(\xi)^{-1}) (s^2 \lambda(\xi)^2 / \xi)^{1/2} + \frac{1}{2} (|s| \xi)^{-4/3} (\hat{h}(\xi) - \hat{h}(\xi)^{-1}) \sigma_2 (|s|^3 \lambda(\xi)^2)^{1/2},
$$

(7.16)

where

$$
\hat{h}(\xi) := \left( \frac{1 - \sqrt{\xi}}{1 + \sqrt{\xi}} \right)^{\alpha} = 1 - 2\alpha \xi^{1/2} + 2\alpha^2 \xi - \frac{2}{3}(\alpha + 2\alpha^3) \xi^{3/2} + O(\xi^2)
$$

(7.17)
as $\xi \to 0$. Thus, we find

$$
\hat{E}(0) = \begin{pmatrix}
|s|^{\frac{1}{2}} (1 + \frac{t}{s})^{\frac{1}{2}} & 2\alpha i |s|^{-1} (1 + \frac{t}{s})^{\frac{1}{2}} \\
0 & |s|^{-\frac{1}{2}} (1 + \frac{t}{s})^{-\frac{1}{2}}
\end{pmatrix},
$$

(7.18)

and

$$
\hat{E}^\rho_{21}(0) = -2\alpha i |s|(1 + \frac{t}{s})^{1/2}.
$$

(7.19)

In the final transformation, we define

$$
R(\xi) = \begin{cases}
S(\xi)(S^{(\infty)}(\xi))^{-1}, & |\xi| > 1/2 \\
S(\xi)(S^{(0)}(\xi))^{-1}, & |\xi| < 1/2.
\end{cases}
$$

(7.20)

Then, $R$ satisfies the following RH problem.

**RH problem 7.4.** The function $R(\xi)$ defined in (7.20) satisfies the following properties:

(a) $R(\xi)$ is analytic in $\mathbb{C} \setminus \Sigma_R$;

(b) $R(\xi)$ satisfies the jump condition $R_+(\xi) = R_-(\xi) J_R(\xi)$,

\[
J_R(\xi) = \begin{cases}
S^{(0)}(\xi) (S^{(\infty)}(\xi))^{-1}, & |\xi| = \frac{1}{2}, \\
S^{(\infty)}(\xi) J_S (S^{(\infty)}(\xi))^{-1}, & \xi \in \Sigma_R \setminus \{|\xi| = \frac{1}{2}\};
\end{cases}
\]

(c) As $\xi \to \infty$, $R(\xi) = I + O(1/\xi)$.

The jump $J_R - I$ is exponentially small for $\xi \in \Sigma_R \setminus \{|\xi| = \frac{1}{2}\}$. On the circle $|\xi| = \frac{1}{2}$, from asymptotics of the Bessel parametrix in (7.13), we have

$$
J_R = |s|^{-\frac{1}{2}\sigma_3} \left( I + |s|^{-3/2} \begin{pmatrix} 1 & 0 \\ 2\alpha i & 1 \end{pmatrix} J_1(\xi) \begin{pmatrix} 1 & 0 \\ -2\alpha i & 1 \end{pmatrix} + O((-s)^{-5/2}) \right) |s|^{\frac{1}{2}\sigma_3},
$$

(7.21)

where $J_1(\xi)$ is given by

$$
J_1(\xi) = \begin{pmatrix}
\frac{\hat{h}(\xi)^2 + \hat{h}(\xi)^{-1} - 2}{8\xi^{1/2}(1 + \frac{4}{5} - \frac{2}{3}\xi)} & -i((\hat{h}(\xi) + \hat{h}(\xi)^{-1})^2 - 3) \\
-\frac{\hat{h}(\xi)^2 - \hat{h}(\xi)^{-2}}{8(1 + \frac{4}{5} - \frac{2}{3}\xi)} & \frac{\hat{h}(\xi)^2 - \hat{h}(\xi)^{-2}}{8\xi^{1/2}(1 + \frac{4}{5} - \frac{2}{3}\xi)}
\end{pmatrix}
$$

(7.22)
with \( \hat{h}(\xi) \) defined in (7.17). Note that, the functions \( \hat{h}(\xi) + \hat{h}(\xi)^{-1}, (\hat{h}(\xi) - \hat{h}(\xi)^{-1})/\xi^{1/2} \) are analytic near the origin and satisfy
\[
\hat{h}(\xi) + \hat{h}(\xi)^{-1} = 2 + 4\alpha^2\xi + O(\xi^2),
(\hat{h}(\xi) - \hat{h}(\xi)^{-1})/\xi^{1/2} = -4\alpha - 4/3(\alpha + 2\alpha^3)\xi + O(\xi^2).
\]
(7.23)

From the expansion of the jump \( J_R \) in (7.21), we get the following expansion
\[
R(\xi) = |s|^{-\frac{4}{3}\sigma_3} \left( I + \frac{1}{|s|^{3/2}} \begin{pmatrix} 1 & 0 \\ 2\alpha i & 1 \end{pmatrix} R_1(\xi) \begin{pmatrix} 1 & 0 \\ -2\alpha i & 1 \end{pmatrix} + O(s^{-3}) \right) |s|^{\frac{1}{3}\sigma_3},
\]
(7.24)
where \( R_1(\xi) \) satisfies a RH problem as follows.

**RH problem 7.5.** The function \( R_1(\xi) \) satisfies the following properties:
(a) \( R_1(\xi) \) is analytic in \( \mathbb{C}\backslash\{|\xi| = 1/2\} \);
(b) \( R_1^+(\xi) - R_1^-(\xi) = J_1(\xi), \) for \( |\xi| = 1/2; \)
(c) \( \text{As } \xi \to \infty, R_1(\xi) = O(1/\xi). \)

Performing a residue computation, we get the solution to the above RH problem
\[
R_1(\xi) = -\left( \begin{array}{c}
\frac{-\hat{h}(\xi)^2 + \hat{h}(\xi)^{-2}}{8\xi^{1/2}(1 + \frac{1}{s} - \frac{2}{\xi})} \\
-\frac{i((\hat{h}(\xi) + \hat{h}(\xi)^{-1})^2 - 1)}{8\xi^2(1 + \frac{1}{s} - \frac{2}{\xi})} \\
\frac{-i((\hat{h}(\xi) + \hat{h}(\xi)^{-1})^2 - 1)}{8\xi^2(1 + \frac{1}{s} - \frac{2}{\xi})} \\
\frac{i((\hat{h}(\xi) + \hat{h}(\xi)^{-1})^2 - 1)}{8\xi^2(1 + \frac{1}{s} - \frac{2}{\xi})}
\end{array} \right)
\]
for \( |\xi| < \frac{1}{2} \) (7.25)

and
\[
R_1(\xi) = \begin{pmatrix} 0 & \frac{1}{8\xi(1 + \frac{1}{s})} \\ 0 & 0 \end{pmatrix}
\]
for \( |\xi| > \frac{1}{2} \). (7.26)

Particularly, we get by substituting (7.23) into (7.25)
\[
(R_1'(0))_{21} = i(2\alpha^2(1 + \frac{t}{s})^{-1} + \frac{1}{4}(1 + \frac{t}{s})^{-2}).
\]
(7.27)

Tracing back the transformation \( \Phi \to S \to R \) gives
\[
\Phi(|s|\xi) = \begin{pmatrix} 1 & 0 \\ -i\tau_2 & 1 \end{pmatrix} R(\xi)S(\xi)e^{-|s|^{3/2}\lambda(\xi)\sigma_3}.
\]
(7.28)

By (7.24) and (7.26), we get
\[
R(\xi) = I + \frac{1}{8(1 + \frac{1}{s})\xi} \begin{pmatrix} -2\alpha|s|^{-3/2} & -i|s|^{-2} \\ -4\alpha^2i|s|^{-1} & 2\alpha|s|^{-3/2} \end{pmatrix} + O(|s|^{-5/2}),
\]
(7.29)
for \( |\xi| > 1/2. \)

From the steepest descent analysis done above, we are able to derive the following asymptotics for the functions \( v_i, w_i \) and the Hamiltonian \( H \). These results will be used in the derivation of the large gap asymptotics for the Fredholm determinants in the next subsection.
PROPOSITION 7. For fixed $t$, we have the following asymptotics, as $s \to -\infty$

$$v_1(s + t; s) = -\frac{s + t}{2} + O(s^{-1/2}), \quad (7.30)$$

$$v_2(s + t; s) = \alpha \frac{1}{\sqrt{|s|}} + O(1/s), \quad (7.31)$$

$$w_1(s + t; s) = \frac{1}{2(s + t)} + O(s^{-3/2}), \quad (7.32)$$

$$w_2(s + t; s) = -\sqrt{|s|} + O(s^{-1/2}), \quad (7.33)$$

$$H(s + t; s) = \frac{(s + t)^2}{4} - 2\alpha \sqrt{|s|} - \frac{1}{8} s + t + O(s^{-3/2}), \quad (7.34)$$

**Proof.** Recall that the outer parametrix $S^{(\infty)}(\xi)$ is given explicitly in (7.6). From the relation (7.1), the asymptotics of $\Phi(z)$ at $z = -s$ are obtained from $S^{(\infty)}(1)$. More precisely, we have

$$\ln \left(\Phi_0^{(-s)}\right)_{11} (s + t; s) = \frac{1}{3} |s|^{3/2} - t \sqrt{|s|} - (\alpha + \frac{1}{4}) \ln |s| - (2\alpha + \frac{1}{2}) \ln 2 + O(1/s), \quad (7.35)$$

and

$$\left(\Phi_1^{(-s)}\right)_{11} (s + t; s) = -\frac{1}{2} \sqrt{|s|} - \frac{t}{2} \sqrt{|s|} - \frac{\alpha}{2} \frac{1}{|s|} + O(s^{-3/2}). \quad (7.36)$$

Then, the expansions for $v_2$ in (7.31) and $w_2$ in (7.33) follow from the above formulas and the differential identities in (4.26) and (4.27). Based on the relations among $v_i, w_i$ and $H$ in (2.1) and (2.2), the other asymptotic expansions follow directly. \qed

### 7.2 Large gap asymptotics: proof of Theorem 4

In Theorem 2, we have successfully expressed the Fredholm determinant of the $P_{34}$ kernel as an integral of solutions to the coupled $P_2$ equations (2.4). This important representation can be further rewritten in terms of the tau function for the coupled P2 system. Quite recently, in [13, 42, 43, 44], the asymptotics of the tau functions for the classical Painlevé equations have been successfully evaluated including the constant terms. In this section, we will derive the large gap asymptotics by evaluating the asymptotic of the tau function for the coupled P2 system.

From (4.20) and (4.21), we have

$$H'(x) = -v_1(x) - v_2(x). \quad (7.37)$$

Then, the Tracy-Widom formula (2.12) obtained in Theorem 2 can be written as

$$\ln \det[I - K_{\alpha,\omega,s}^{P_{34}}] = \int_t^{+\infty} \left( H'(x + s) + \frac{\alpha}{\sqrt{x}} - \frac{\alpha^2}{x^2} \right) (x-t) dx + \int_t^{+\infty} \left( u(x) - \frac{\alpha}{\sqrt{x}} + \frac{\alpha^2}{x^2} \right) (x-t) dx.$$
Note that both integrals above are convergent due to the asymptotics of \(u(x)\) in (1.37) and \(H(x)\) in (2.11). Moreover, an integration by parts of the first integral gives us

\[
\ln \det[I-K_{\alpha,\omega,s}^{P34}] = -\int_{s+t}^{\infty} \left( H(\tau) + 2\alpha \sqrt{|\tau - s|} + \frac{\alpha^2}{\tau - s} \right) d\tau + \int_{t}^{\infty} (\tau - t) \left( u(\tau) - \frac{\alpha}{\sqrt{|\tau - t|}} + \frac{\alpha^2}{\tau - s} \right) d\tau.
\]

(7.38)

Note that, according to the theory of isomonodromic tau-functions in the sense of Jimbo-Miwa-Ueno [45], the tau function can be defined as \(d_x \ln \tau = H(x)dx\), where \(H\) is the Hamiltonian. Now, the only task for us is to compute the asymptotics of the first integral as \(s \to -\infty\).

From the Hamiltonian system (2.1), we have

\[
H = \frac{1}{3}(v_1 w_1 + v_2 w_2 + 2xH)_x + 2\alpha w_2 + \frac{2}{3} sv_2 - (v_1 w_{1x} + v_2 w_{2x} + H).
\]

(7.39)

Next, let us integrate both sides of the above formula from \(s + t\) to \(+\infty\). To ensure the convergence, we need to add a few terms according to the asymptotics of the functions \(u_i, w_i\) and \(H\) in Theorem 1. More precisely, we get

\[
-\int_{s+t}^{\infty} \left( H(\tau) + 2\alpha \sqrt{|\tau - s|} + \frac{\alpha^2}{\tau - s} \right) d\tau = -2\alpha I_1(s + t; \alpha, \omega) - \frac{2s}{3} I_2(s + t; \alpha, \omega) + I_3(s + t; \alpha, \omega) + \frac{1}{3} \left( v_1 w_1 + v_2 w_2 + \alpha + 2(s + t)H + 4\alpha s \sgn(t) \sqrt{|t|} + 4\alpha \sgn(t)|t|^{3/2} + 2\alpha^2 \right),
\]

(7.40)

where

\[
I_1(s + t; \alpha, \omega) = \int_{s+t}^{\infty} \left( w_2(\tau) + \sqrt{|\tau - s|} + \frac{\alpha + \frac{3}{4}}{\tau - s} \right) d\tau
\]

\[
= -\frac{2}{3} \sgn(t)|t|^{3/2} - (\alpha + \frac{1}{4}) \ln |t| - (2\alpha + \frac{1}{2}) \ln 2 - \ln \left( \Phi_{10}^{(-s)} \right)_{11}(s + t; s),
\]

(7.41)

\[
I_2(s + t; \alpha, \omega) = \int_{s+t}^{\infty} \left( v_2(\tau) - \frac{\alpha}{\sqrt{|\tau - s|}} \right) d\tau = 2\alpha \left( \left( \Phi_{10}^{(-s)} \right)_{11} + \sgn(t) \sqrt{|t|} \right),
\]

(7.42)

and

\[
I_3(s + t; \alpha, \omega) = \int_{s+t}^{\infty} \left( v_1(\tau) w_{1x}(\tau) + v_2(\tau) w_{2x}(\tau) + H(\tau) + 2\alpha \sqrt{|\tau - s|} + \frac{\alpha(2\alpha + 1)}{2(\tau - s)} \right) d\tau.
\]

(7.43)

Here, to obtain the explicit expressions of \(I_1(s + t; \alpha, \omega)\) in (7.41) and \(I_2(s + t; \alpha, \omega)\) in (7.42), we use the differential identity in (4.26) and (4.27), as well as the asymptotics of \(\Phi_{10}^{(-s)}\) and \(\Phi_{11}^{(-s)}\) in (5.26) and (5.27).

Although the exact expression of the integral \(I_3(s + t; \alpha, \omega)\) is unavailable now, we may consider its derivative with respect to the parameter \(\alpha\). From the Hamiltonian system (2.1) and (2.2), we have

\[
(v_1 w_{1x} + v_2 w_{2x} + H)_{\alpha} = (v_1 w_{1\alpha} + v_2 w_{2\alpha})_x + 2w_2.
\]

(7.44)
The above formula implies
\[
\frac{\partial}{\partial \alpha} I_3(s + t; \alpha, \omega) = - (v_1(s + t) w_{1a}(s + t) + v_2(s + t) w_{2a}(s + t)) + 2I_1(s + t; \alpha, \omega). \tag{7.45}
\]
For \( s < 0 \), the function \( \Phi(z; x, s) \) is independent of the parameter \( \omega \); see the model Riemann-Hilbert problem for \( \Phi \) in Sec. 3.2. Then, \( I_3(s; \alpha, \omega) \) is also independent of the parameter \( \omega \) for negative \( s \). For \( \alpha = 0 \) and \( \omega = 1 \), we have \( v_2 = 0 \) and \( v_1(x) = y^2(x; 0) \), where \( y(x; 0) \) is the classical Hastings-McLeod solution to \( P_2 \) equation; see Remark 2.

Moreover, the Hamiltonian \( H \) in (2.2) is reduced to the Hamiltonian \( \mathcal{H} \) for the \( P_2 \) equation. By (7.40), we have
\[
I_3(x; 0, 1) = - \int_0^\infty \mathcal{H}(\tau) d\tau + \frac{1}{3} (v_1(x) w_1(x) + 2x \mathcal{H}(x))
\]
Note from (7.37), \( \mathcal{H}'(x) = -v_1(x) \). An integration by parts gives us
\[
- \int_0^\infty \mathcal{H}(\tau) d\tau = - \int_0^\infty (\tau - x) \mathcal{H}'(\tau) d\tau = \int_0^\infty (\tau - x) y^2(\tau; 0) d\tau,
\]
which is exactly the exponent of the Tracy-Widom distribution in (1.12). Therefore, using (2.18), as well as (7.30) and (7.32), we have
\[
I_3(x; 0, 1) = - \frac{|x|^3}{12} - \frac{1}{8} \ln |x| + \chi(-1) + \frac{1}{24} \ln 2 + \frac{1}{6} + o(1), \quad \text{as} \; x \to -\infty. \tag{7.47}
\]

Recalling the approximations in (7.30)-(7.36), we have asymptotics of the integrals in (7.41), (7.42) and (7.45)
\[
I_1(s + t; \alpha, \omega) = - \frac{1}{3} |s|^{3/2} + t \sqrt{|s|} + (\alpha + \frac{1}{4}) \ln |s| - 2 \frac{3}{3} \sgn(t) |t|^{3/2} - (\alpha + \frac{1}{4}) \ln |t| + O(1/s), \tag{7.48}
\]
\[
I_2(s + t; \alpha, \omega) = 2\alpha \left( - \frac{1}{2} \sqrt{|s|} + \sgn(t) \sqrt{|t|} - \frac{1}{2} \sqrt{|s|} - \frac{1}{2} s + \frac{1}{4} s + O(s^{-3/2}) \right), \tag{7.49}
\]
\[
\frac{\partial}{\partial \alpha} I_3(s + t; \alpha, \omega) = - \frac{2}{3} |s|^{3/2} + 2t \sqrt{|s|} + 2(\alpha + \frac{1}{4}) \ln |s| - 4 \frac{3}{3} \sgn(t) |t|^{3/2} - 2(\alpha + \frac{1}{4}) \ln |t| + o(1). \tag{7.50}
\]

Integrating the above formula about \( \alpha \), we have
\[
I_3(s + t; \alpha, \omega) = I_3(s + t; 0, 0) - \frac{2\alpha}{3} |s|^{3/2} + 2\alpha t \sqrt{|s|} + (\alpha^2 + \alpha + \frac{1}{2}) \ln |s| - 4 \frac{3}{3} \sgn(t) |t|^{3/2} - (\alpha^2 + \alpha + \frac{1}{2}) \ln |t| + o(1). \tag{7.51}
\]

Substituting the approximations (7.47)-(7.49) and (7.51) into (7.40), we obtain the asymptotics
\[
- \int_{s + t}^\infty \left( H(\tau) + 2\alpha \sqrt{\tau - s} + \frac{\alpha^2}{\tau - s} \right) d\tau = - \frac{1}{12} |s + t|^3 + \frac{1}{3} \alpha |s|^{3/2} - 2\alpha |s|^{1/2} t
\]
\[
- (\alpha^2 + \frac{1}{8}) \ln |s + t| + \frac{4}{3} \alpha \sgn(t) |t|^{3/2} + \alpha^2 \ln |t| + c_0 + o(1), \tag{7.52}
\]
where the constant \( c_0 \) is given in (2.17).

This completes the proof of Theorem 4.
7.3 Large gap asymptotics: proof of Theorem 5

From Theorem 4 and the relation (2.14), we obtain the following asymptotic expansion for the Fredholm determinant of the $P_2$ kernel as $s \to +\infty$

$$\ln \det[I - K^{P_2}_{\alpha,s}] = -\frac{2}{3}(s^2 + \frac{t}{2})^3 + \frac{4}{3}\alpha s^3 + 2\alpha st - (\alpha^2 + \frac{3}{4}) \ln s + \frac{2\sqrt{2}}{3}\alpha \sgn(t)|t|^{3/2} + \left(\frac{\alpha^2}{2} + \frac{1}{8}\right) \ln |t| + c_1 + I^* + o(1), \quad (7.53)$$

where $c_1$ is given in (2.20), and the integral $I^*$ is given by

$$I^* = \int_{-2^{-1/4}t}^{+\infty} (\tau + 2^{-1/4}t) \left( u(\tau, \alpha - \frac{1}{2}, 0) + u(\tau, \alpha + \frac{1}{2}, 0) - \frac{\alpha}{\sqrt{\tau}} + \frac{\alpha^2}{2} + \frac{1}{8} \right) d\tau$$

$$- \int_{-\infty}^{t} (\tau - t) \left( \frac{u(-2^{-1/4}\tau, \alpha - \frac{1}{2}, 0)}{2^{\frac{3}{4}}} + \frac{u(-2^{-1/4}\tau, \alpha + \frac{1}{2}, 0)}{2^{\frac{3}{4}}} - \frac{\alpha}{\sqrt{2\tau}} + \frac{\alpha^2}{2} + \frac{1}{8} \right) d\tau.$$

Then, the above formula and (6.14) gives us the integral in (2.19).

This completes the proof of Theorem 5.

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Appendix A Asymptotics of $v_i(x)$ as $x \to -\infty$

In this appendix, we derive the following asymptotics of $v_i(x)$ as $x \to -\infty$. Similar to Section 5, the Deift-Zhou nonlinear steepest descend method is applied to obtain the asymptotics. Depending on the sign of $s$, we divide our computations into two parts.

A.1 Case I: $s < 0$

A.1.1 Nonlinear steepest descent analysis of $\Phi$ as $x \to -\infty$

We first remove the exponential term in the large-$z$ expansion $e^{-\theta(z,x)\sigma_3}$ of $\Phi(z)$ in (3.18) by introducing the following transformation:

$$A(z) = \begin{pmatrix} 1 & 0 \\ ir_2 & 1 \end{pmatrix} \Phi(z)e^{\theta(z,x)\sigma_3}, \quad (A.1)$$

Then, we have a RH problem as follows.
RH problem A.1. The function \( A(z) \) satisfies the following properties:
(a) \( A(z) \) is analytic in \( \mathbb{C} \setminus \Sigma_A \), where \( \Sigma_A = \bigcup_{i=1}^{4} \hat{\Sigma}_i \) is indicated in Fig. 9;
(b) \( A(z) \) satisfies the jump condition
\[
A_+(z) = A_-(z)J_{A,i}(z) \quad \text{for } z \in \hat{\Sigma}_i, \tag{A.2}
\]
with
\[
J_{A,1} = \begin{pmatrix}
e^{2\pi i \alpha} & 0 \\
0 & e^{-2\pi i \alpha}
\end{pmatrix}, \quad J_{A,2} = \begin{pmatrix}1 & 0 \\
e^{2\theta(z,x)} e^{2\pi i \alpha} & 1
\end{pmatrix},
J_{A,3} = \begin{pmatrix}0 & 1 \\
-1 & 0
\end{pmatrix}, \quad J_{A,4} = \begin{pmatrix}1 & 0 \\
e^{2\theta(z,x)} e^{-2\pi i \alpha} & 1
\end{pmatrix};
\]
(c) The asymptotic behavior of \( A(z) \) at infinity:
\[
A(z) = \left(I + O\left(z^{-\frac{1}{4}}\sigma_3 \frac{I + i\sigma_1}{\sqrt{2}}\right)\right) z^{-\frac{1}{4} \sigma_3} \quad \text{as } z \to \infty; \tag{A.3}
\]
(d) \( A(z) \) has the same behavior near \( z = 0, -s \) as \( \Phi(z) \), given in (3.20) and (3.21).

Because
\[
\text{Re} \theta(z,x) = \text{Re} \left(\frac{2}{3}z^\frac{3}{2} + xz^\frac{1}{2}\right) < 0, \quad \text{for } z \in \mathbb{C}^\pm, \tag{A.4}
\]
as \( x \to -\infty \), the jumps \( J_{A,i}(z) \) are exponentially close to the identity matrix except the ones on the real line. Neglecting the exponential small terms, we arrive at the following outer parametrix.

RH problem A.2. The function \( A^{(\infty)}(z) \) satisfies the following properties:
(a) \( A^{(\infty)}(z) \) is analytic in \( \mathbb{C} \setminus (-\infty, |s|] \);
(b) \( A^{(\infty)}(z) \) satisfies the jump condition
\[
A_+^{(\infty)}(z) = A_-^{(\infty)}(z) \begin{cases}
\begin{pmatrix}0 & 1 \\
-1 & 0
\end{pmatrix}, & z \in (-\infty, 0), \\
e^{2\pi i \alpha \sigma_3}, & z \in (0, |s|);
\end{cases} \tag{A.5}
\]
(c) At infinity, $A^{(\infty)}(z)$ satisfies the same asymptotics as $A(z)$ in (A.3).

Similar to RH problem 5.2 a solution to the above RH problem can be constructed explicitly as

$$A^{(\infty)}(z) = \left( \begin{array}{cc} 1 & 0 \\ 2\alpha i |s|^{1/2} & 1 \end{array} \right) z^{-\frac{1}{4} \sigma_3} I + i \frac{\sigma_1}{\sqrt{2}} h(z/|s|) \sigma_3,$$  

(A.6)

where $h(z)$ is defined in (5.8).

Then, we turn to the local parametrix near the origin. Let $\theta(z,x)^2 = z(\frac{2}{3}z + x)^2$ be a conformal mapping in the neighbourhood of the origin. Then, similar to $S^{(0)}(\xi)$ in (7.10) of Section 7.1, the local parametrix $A^{(0)}(z)$ is given explicitly as follows:

$$A^{(0)}(z) = E_1(z) Z_0 \left( \theta(z,x)^2 \right) \begin{cases} e^{\theta(z,x) \sigma_3} e^{\pi i \sigma_3}, & \text{arg } z \in (0, \pi), \\
\sigma_1 e^{\theta(z,x) \sigma_3} e^{-\pi i \sigma_3}, & \text{arg } z \in (-\pi, 0), \end{cases} \quad \text{for } z \in U_0.$$  

(A.7)

Then, the following matching condition is fulfilled

$$A^{(0)}(z) = \left( I + O\left( \frac{1}{x} \right) \right) A^{(\infty)}(z) \quad \text{as } x \to -\infty,$$  

(A.9)

uniformly for $z \in \partial U_0$.

With the outer and local parametrices constructed explicitly in (A.6) and (A.7), we introduce the final transformation as follows:

$$B(z) = \begin{cases} A(z)(A^{(\infty)}(z))^{-1}, & |z| > \delta, \\
A(z)(A^{(0)}(z))^{-1}, & |z| < \delta. \end{cases}$$  

(A.10)

By the matching condition (A.9), one can verify that the jump of $B(z)$ is

$$J_B(z) = I + O\left( \frac{1}{x} \right) \quad \text{as } x \to -\infty,$$  

which implies

$$B(z) = I + O\left( \frac{1}{x} \right) \quad \text{as } x \to -\infty,$$  

(A.11)

uniformly for $z$ in the complex plane; see the similar analysis in RH problem 5.4.
A.1.2 Asymptotics of $v_i(x)$

Recall the transformation (A.1) and the representations of $v_i(x)$ in (4.33) and (4.34), we have

$$v_1(x) = i \lim_{z \to 0} z (A'(z)A(z)^{-1})_{12}, \quad (A.12)$$

$$v_2(x) = i \lim_{z \to -s} (z + s)(A'(z)A(z)^{-1})_{12}. \quad (A.13)$$

By the transformation (A.10), we obtain

$$A(z) = B(z)A(0)(z) \text{ for } |z| < \delta. \quad (A.14)$$

Note that $B(z)$ is analytic at the origin and satisfies the approximation (A.11). Moreover, by (A.7), (7.11) and (A.8) in the expression of $A(0)(z)$, the pre-factor $E_1(z)$ is analytic at the origin and the Bessel parametrix satisfies the following relation

$$\lim_{z \to 0} z Z_0'(z) Z_0(z)^{-1} = \begin{pmatrix} 0 & \frac{1}{2i} \\ 0 & 0 \end{pmatrix}. \quad (A.16)$$

Thus, we have from (A.12) and (A.14)

$$v_1(x) = \left( B(0)E_1(0) \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} E_1(0)^{-1}B(0)^{-1} \right)_{12} = \frac{1}{2}(E_1(0))_{11}^2(1 + O(\frac{1}{x})). \quad (A.15)$$

From (A.6) and (A.8), we obtain

$$(E_1(0))_{11}^2 = |x|. \quad (A.16)$$

Finally, we get the asymptotics

$$v_1(x) = -\frac{x}{2} (1 + O(1/x)), \quad x \to -\infty. \quad (A.17)$$

Similarly, by using the relation (A.10), (A.13), the definition of $A(0)$ in (A.6) and the approximation (A.11), we have

$$v_2(x) = \frac{\alpha}{\sqrt{|s|}} + O(1/x), \quad x \to -\infty. \quad (A.18)$$

A.2 Case II: $s > 0$

For $s > 0$ and $\omega = 0$, it is easy to see that $\Phi(z)$ is analytic at $z = s$ and $v_1(x) = 0$. Therefore, in this section, we consider the case $\omega = e^{-2\pi i \beta}$ with $|\text{Re}\beta| < \frac{1}{2}$. 

46
A.2.1 Nonlinear steepest descent analysis of $\Phi$ as $x \to -\infty$

When $s < 0$, the function $A(z)$ defined in (A.1) satisfies the following RH problem.

**RH problem A.3.** The function $A(z)$ satisfies the following properties:

(a) $A(z)$ is analytic in $\mathbb{C} \setminus \hat{\Sigma}_i$, where the contours $\hat{\Sigma}_i$ are indicated in Fig. 10;

(b) $A(z)$ satisfies the jump condition

$$A_+(z) = A_-(z) J_{A,i}(z) \quad \text{for } z \in \hat{\Sigma}_i$$

with

$$J_{A,1} = \begin{pmatrix} e^{g_0+(z) - g_0-(z)} & e^{-2\pi i\beta} \\ 0 & e^{g_0-(z) - g_0+(z)} \end{pmatrix}, \quad J_{A,2} = \begin{pmatrix} 1 & 0 \\ e^{2\theta(z,x)} e^{2\pi i\alpha} & 1 \end{pmatrix},$$

$$J_{A,3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_{A,4} = \begin{pmatrix} 1 & 0 \\ e^{2\theta(z,x)} & e^{-2\pi i\alpha} \end{pmatrix};$$

(c) The asymptotic behavior of $A(z)$ at infinity

$$A(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\frac{1}{4} \sigma_3} I + i \sigma_1 \sqrt{2} \quad \text{as } z \to \infty;$$

(d) The asymptotic behavior of $A(z)$ at $z = 0, -s$ is the same as that of $\Phi(z)$.

Based on the factorization

$$\begin{pmatrix} e^{g_0+(z) - g_0-(z)} & e^{-2\pi i\beta} \\ 0 & e^{g_0-(z) - g_0+(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e^{2\pi i\beta} e^{2\theta_0-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & e^{-2\pi i\beta} \\ e^{2\pi i\beta} e^{2\theta_0+(z)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2\pi i\beta} & 1 \end{pmatrix},$$

we introduce the second transformation

$$B(z) = \begin{cases} A(z) \begin{pmatrix} 1 & 0 \\ e^{2\pi i\beta} e^{2\theta(z,x)} & 1 \end{pmatrix}, & \text{z in the lower lens-shaped region;} \\
A(z) \begin{pmatrix} 1 & 0 \\ -e^{2\pi i\beta} e^{2\theta(z,x)} & 1 \end{pmatrix}, & \text{z in the upper lens-shaped region;} \\
A(z), & \text{otherwise.} \end{cases}$$

Then, $B(z)$ satisfies the following RH problem.
RH problem A.4. The function $B(z)$ satisfies the following properties:

(a) $B(z)$ is analytic in $\mathbb{C}\setminus \Sigma_{B,i}$; see Fig. 11 for the contours $\Sigma_{B,i}$;

(b) $B(z)$ satisfies the jump condition

\[ B_+(z) = B_-(z)J_{B,i}(z) \quad \text{for } z \in \Sigma_{B,i} \]  \hspace{1cm} (A.22)

with

\[ J_{B,1} = \begin{pmatrix} 0 & e^{-2\pi i \beta} \\ -e^{2\pi i \beta} & 0 \end{pmatrix}, \quad J_5 = J_6 = \begin{pmatrix} 1 & 0 \\ e^{2\pi i \beta} e^{2\theta(z,x)} & 1 \end{pmatrix}, \]

\[ J_{B,2} = \begin{pmatrix} 1 & 0 \\ e^{2\theta(z,x)} e^{2\pi i \alpha} & 1 \end{pmatrix}, \quad J_{B,3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_{B,4} = \begin{pmatrix} 1 & 0 \\ e^{2\theta(z,x)} e^{-2\pi i \alpha} & 1 \end{pmatrix}; \]

(c) At infinity, $B(z)$ satisfies the same asymptotics as $A(z)$ in (A.20);

(d) $B(z)$ has the same behavior near $z = 0, -s$ with $\Phi(z)$ given in (3.20) and (3.21).

Due to (A.4), as $x \to -\infty$, the jumps are close to the identity matrix except the ones on the real line. So, we arrive at the following outer parametrix.

RH problem A.5. The function $B^{(\infty)}(z)$ satisfies the following properties:

(a) $B^{(\infty)}(z)$ is analytic in $\mathbb{C}\setminus (-\infty, 0]$;

(b) $B^{(\infty)}(z)$ satisfies the following jump condition

\[ B_+^{(\infty)}(z) = B_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -s), \]

\[ B_+^{(\infty)}(z) = B_-^{(\infty)}(z) \begin{pmatrix} 0 & e^{-2\pi i \beta} \\ -e^{2\pi i \beta} & 0 \end{pmatrix}, \quad z \in (-s, 0); \]  \hspace{1cm} (A.23)

(c) At infinity, $B^{(\infty)}(z)$ satisfies the same asymptotics as $A(z)$ in (A.20);
To construct the outer parametrix, let us define the following scalar function $h_2(z)$ by

$$ h_2(z) = \left( \frac{\sqrt{z} + i\sqrt{s}}{\sqrt{z} - i\sqrt{s}} \right)^\beta, \quad z \in \mathbb{C} \setminus (-\infty, 0], $$

where the power function $z^c, c \notin \mathbb{Z}$, takes the principle branch with the branch cut along $(-\infty, 0)$. Then, $h_2(z)$ satisfies the following jump condition

$$ h_{2+}(x)h_{2-}(x) = \begin{cases} e^{2\pi i \beta}, & x \in (-s, 0) \\ 1, & x \in (-\infty, -s). \end{cases} $$

With the function $h_2(z)$, one solution to the above RH problem is explicitly given by

$$ B^{(\infty)}(z) = \left( \begin{array}{cc} 1 & 0 \\ 2\beta\sqrt{s} & 1 \end{array} \right) z^{-\frac{1}{4}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} h_2(z)^{\sigma_3}. $$

In the neighborhood of the origin, the local parametrix is similar to (A.7) and given explicitly as follows:

$$ B^{(0)}(z) = E_2(z)Z_0(\theta(z), x^2)e^{(\frac{3}{4}z^2 + xz\frac{1}{2})\sigma_3}e^{\pi i \beta \sigma_3}, $$

where $Z_0$ is defined in terms of the Bessel functions (7.11). The analytic pre-factor $E_2(z)$ is given by

$$ E_2(z) = B^{(\infty)}(z) \begin{cases} e^{-\pi i \beta \sigma_3} \frac{I - i\sigma_1}{\sqrt{2}z^{\frac{1}{4}\sigma_3}} \left( |x| - \frac{2}{3}z \right)^{\frac{3}{2}\sigma_3}, & \arg z \in (0, \pi), \\ e^{-\pi i \beta \sigma_3} \frac{I - i\sigma_1}{\sqrt{2}z^{\frac{1}{4}\sigma_3}} \left( |x| - \frac{2}{3}z \right)^{\frac{3}{2}\sigma_3}, & \arg z \in (-\pi, 0), \end{cases} $$

for $|z| < \delta$. With the properties of the Bessel functions, one can verify the following matching condition

$$ B^{(0)}(z) = (I + O(1/x))B^{(\infty)}(z), $$

uniform for $|z| = \delta$ as $x \to -\infty$.

In the neighborhood of $-s$, we seek a local parametrix of the following form

$$ B^{(1)}(z) = E_3(z)\hat{B}^{(1)}(f_0(z))e^{\frac{1}{2}\pi i \beta \sigma_3}e^{\theta(z)x\sigma_3}, $$

where $f_0(z)$ is the conformal mapping near $-s < 0$:

$$ f_0(z) := (|x|z^\frac{1}{4} - \frac{2i}{3}z^\frac{3}{4}) + (|x|s^\frac{1}{4} + \frac{2}{3}s^\frac{3}{4}) $$

$$ = (s^{1/2} + \frac{1}{2}|x|s^{-1/2})(z + s) + O((z + s)^2), $$

with $\arg z \in (0, 2\pi)$ and $E_3(z)$ is analytic for $|z + s| < \frac{s}{2}$. Let

$$ \hat{B}^{(1)}(z) = \hat{B}^{(1)}(z) \begin{cases} e^{-\pi i \alpha \sigma_3}, & \arg z \in (0, \frac{\pi}{2}) \\ e^{\pi i \alpha \sigma_3}, & \arg z \in (\frac{3\pi}{2}, 2\pi) \\ I, & \arg z \in (\frac{\pi}{2}, \frac{3\pi}{2}). \end{cases} $$

Then, $\hat{B}^{(1)}$ satisfies the following RH problem.
Figure 12: Regions and contours for $\tilde{B}$.

**RH problem A.6.** The function $\tilde{B}^{(1)}(z)$ satisfies the following properties:

(a) $\tilde{B}^{(1)}(z)$ is analytic in $\mathbb{C}\setminus\Sigma_i$, where $\Sigma_i$ are indicated in Fig. 12.

(b) $\tilde{B}^{(1)}(z)$ satisfies the following jump condition

$$\tilde{B}^{(1)}_+(z) = \tilde{B}^{(1)}_-(z)J_i(z), \quad \text{for } z \in \Sigma_i \quad (A.33)$$

with

$$J_1 = \begin{pmatrix} 0 & e^{-\pi i \beta} \\ -e^{\pi i \beta} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ e^{\pi i (\beta - 2\alpha)} & 1 \end{pmatrix}, \quad J_3 = J_7 = e^{\pi i \sigma_3},$$

$$J_4 = \begin{pmatrix} 1 & 0 \\ e^{-\pi i (\beta - 2\alpha)} & 1 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & e^{\pi i \beta} \\ -e^{\pi i \beta} & 0 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 1 & 0 \\ e^{-\pi i (\beta + 2\alpha)} & 1 \end{pmatrix},$$

and $J_8 = \begin{pmatrix} 1 & 0 \\ e^{\pi i (\beta + 2\alpha)} & 1 \end{pmatrix}$.

(c) As $z \to 0$,

$$\tilde{B}^{(1)}(z) = \begin{pmatrix} O(|z|^\alpha) & O(|z|^{-|\alpha|}) \\ O(|z|^\alpha) & O(|z|^{-|\alpha|}) \end{pmatrix}, \quad \alpha \neq 0,$$

and

$$\tilde{B}^{(1)}(z) = \begin{pmatrix} O(1) & O(\ln z) \\ O(1) & O(\ln z) \end{pmatrix}, \quad \alpha = 0.$$

The above RH problem is the same as the one in [25, (4.25)-(4.31)] up to a rotation.

The solution $\tilde{B}^{(1)}(z)$ can be constructed explicitly in terms of the confluent hypergeometric functions $\psi(a, b, z)$

$$\tilde{B}^{(1)}(z) = C_0 \begin{pmatrix} (2e^{\pi i/2}z)^\alpha \psi(\alpha + \beta, 1 + 2\alpha, 2e^{\pi i/2}z)e^{i\pi(\alpha + 2\beta)}e^{-iz} \\ -\frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)}(2e^{\pi i/2}z)^{-\alpha}\psi(1 - \alpha + \beta, 1 - 2\alpha, 2e^{\pi i/2}z)e^{i\pi(-3\alpha+\beta)}e^{-iz} \\ -\frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha+\beta)}(2e^{\pi i/2}z)^{\alpha}\psi(1 + \alpha - \beta, 1 + 2\alpha, 2e^{-\pi i/2}z)e^{i\pi(\alpha+\beta)}e^{iz} \\ (2e^{\pi i/2}z)^{-\alpha}\psi(-\alpha - \beta, 1 - 2\alpha, 2e^{-\pi i/2}z)e^{-i\pi\alpha}e^{iz} \end{pmatrix},$$

with $C_0$ being a constant.
We take the analytic pre-factor $E_0$ in the sector $\arg z \in (0, \pi/2)$. The expression of the solution in the other sectors is then determined by using the jump conditions. Note that, from the properties of the confluent hypergeometric functions $\psi(a, b, z)$, the asymptotics of $\tilde{B}^{(1)}(z)$ as $z \to \infty$ is given by

$$\tilde{B}^{(1)}(z) = \left[ I + \frac{1}{z} \begin{pmatrix} -i(\alpha^2 - \beta^2)/2 & -i2^{2\beta-1}\Gamma(1+\alpha-\beta)/\Gamma(\alpha-\beta)e^{-i\pi(\alpha-\beta)} & 0 \\ i2^{-2\beta-1}\Gamma(1+\alpha+\beta)/\Gamma(\alpha-\beta)e^{-i\pi(\alpha-\beta)} & -i2^{2\beta-1}\Gamma(1+\alpha-\beta)/\Gamma(\alpha+\beta)e^{i\pi(\alpha-\beta)}/i(\alpha^2 - \beta^2)/2 \end{pmatrix} \right] z^{-\beta\sigma_3} e^{-iz\sigma_3},$$

in the region $\arg z \in (0, \pi/2)$. Moreover, $\tilde{B}^{(1)}(z)$ satisfies the following differential equation

$$\frac{d}{dz} \tilde{B}^{(1)}(z) = \begin{pmatrix} -i\sigma_3 + \frac{1}{z} \begin{pmatrix} -\beta & 2^{2\beta}\Gamma(1+\alpha-\beta)/\Gamma(\alpha+\beta)e^{i\pi(\alpha-\beta)} \\ 2^{-2\beta}\Gamma(1+\alpha+\beta)/\Gamma(\alpha-\beta)e^{-i\pi(\alpha-\beta)} & -\beta \\ \end{pmatrix} \\ 0 \end{pmatrix} \tilde{B}^{(1)}(z).$$

We take the analytic pre-factor $E_3(z)$ in (A.30) as

$$E_3(z) = \begin{cases} \begin{pmatrix} B^{(\infty)}(z)e^{-(\alpha+\beta/2)\pi i\sigma_3}e^{i[\frac{3}{2}|s|^2 - x|s|^2]|\sigma_3 f_0(z)|^2] \end{pmatrix} & \text{Im } z > 0, \\ \begin{pmatrix} B^{(\infty)}(z) & 0 \\ 0 & 1 \end{pmatrix} e^{-(\alpha+\beta/2)\pi i\sigma_3}e^{i[\frac{3}{2}|s|^2 - x|s|^2]|\sigma_3 f_0(z)|^2] & \text{Im } z < 0, \end{cases}$$

with $\arg f_0(z) \in (0, 2\pi)$. Particularly, we have

$$E_3(-s) = s^{-\frac{1}{2}}\sigma_3 \begin{pmatrix} 1 & 0 \\ 2\beta & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-(\alpha+\beta/2+1)\pi i\sigma_3}e^{i[\frac{3}{2}|s|^2 + |x|s|^2]|\sigma_3 (2\sqrt{s}|x-2s)|^2\beta s_3}. (A.37)$$

Using the expressions of $B^{(\infty)}(z)$ in (A.26), the asymptotics of $\tilde{B}^{(1)}(z)$ in (A.34) and the definition of $B^{(1)}(z)$ in (A.30), we have the following matching condition

$$B^{(1)}(z)B^{(\infty)}(z)^{-1} = I + x^{-Re\beta}\alpha O(1/x)x^{Re\beta}\sigma_3 = I + O(x^{2Re\beta}|-1). (A.38)$$

In the final transformation, we define

$$C(z) = \begin{cases} B(z)(B^{(0)}(z))^{-1}, & \text{if } |z| < \delta, \\ B(z)(B^{(1)}(z))^{-1}, & \text{if } |z + s| < \delta, \\ B(z)(B^{(\infty)}(z))^{-1}, & \text{otherwise.} \end{cases} (A.39)$$

By the matching conditions (A.29) and (A.38), one can verify that the jump of $C(z)$ is

$$J_C(z) = I + O(x^{2Re\beta-1}),$$

where $2|Re\beta| - 1 < 0$. Therefore, we have

$$C(z) = I + O(x^{2Re\beta-1}), \quad (A.40)$$

uniformly for $z$ in the complex plane; see the similar analysis in RH problem 5.4.
A.2.2 Asymptotics of $v_i(x)$

Recall the transformations (A.1) and (A.21), and the representations of $v_i(x)$ in (4.33) and (4.34), we have

$$v_1(x) = i \lim_{z \to 0^+} z(B'(z)B(z)^{-1})_{12},$$

(A.41)

and

$$v_2(x) = i \lim_{z \to -s} (z + s)(B'(z)B(z)^{-1})_{12}.$$  

(A.42)

From the transformation (A.39), we get

$$B(z) = C(z)B'(0)(z), \quad \text{for } |z| < \delta.$$  

(A.43)

Thus, the asymptotics of $v_1(x)$ follows from (A.27) and the approximation (A.40)

$$v_1(x) = -\frac{x}{2}(1 + O(x^{2\Re\beta - 1})), \quad x \to -\infty.$$  

(A.44)

Combining (A.30), (A.35) and (A.36), we get from (A.42)

$$v_2(x) = i \left( C(-s)E_3(-s) \left( -\beta \frac{2\beta\Gamma(1+\alpha-\beta)}{\Gamma(\alpha-\beta)} e^{-i\pi(\alpha-\beta)} + \frac{\Gamma(1+\alpha+\beta)}{2\Gamma(\alpha+\beta)} e^{-i\pi(\alpha+\beta)} \right) \right) E_3^{-1}(-s)C^{-1}(-s)_{12}.$$  

(A.45)

Some straightforward computations give us

$$v_2(x) = \frac{1}{\sqrt{s}} \left( i\beta + \frac{1}{2} \frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)} e^{i\tilde{\vartheta}(x,s,\alpha,\beta)} + \frac{1}{2\Gamma(\alpha-\beta)} e^{-i\vartheta(x,s,\alpha,\beta)} \right) (1 + O(x^{2\Re\beta - 1})), $$  

(A.46)

where $\tilde{\vartheta}(x, s, \alpha, \beta) = -2xs^{1/2} + 4s^{3/2} - \alpha\pi - 6i\beta \ln 2 - i\beta \ln(s) - 2i\beta \ln \left| \frac{x}{2} - s \right|.$

Finally, summarizing the asymptotics of $v_1(x)$ as $x \to -\infty$ in (A.17), (A.18), (A.44) and (A.46), we obtain the following results.

**THEOREM 6.** Under the same conditions as in Theorem 7, the asymptotic behaviors of $v_i(x)$ to the coupled $P_2$ equations (2.4) are given by

$$v_1(x, s; 2\alpha, \omega) = -\frac{x}{2} \left( 1 + o(1) \right),$$  

(A.47)

as $x \to -\infty$; and

$$v_2(x, s; 2\alpha, \omega) = \begin{cases} 
\frac{1}{\sqrt{s}} \left( i\beta + \frac{\Gamma(1+\alpha-\beta)}{2\Gamma(\alpha+\beta)} e^{i\tilde{\vartheta}(x,s,\alpha,\beta)} \right) (1 + O(x^{2\Re\beta - 1})), & \text{if } s > 0, \\
\frac{\alpha}{\sqrt{|s|}} + O(1/x), & \text{if } s < 0,
\end{cases}$$  

(A.48)

as $x \to -\infty$, where $\omega = e^{-2\pi i} \beta$, $|\Re \beta| < 1/2$ and $\tilde{\vartheta}(x, s, \alpha, \beta) = -2xs^{1/2} + 4s^{3/2} - \alpha\pi - 6i\beta \ln 2 - i\beta \ln(s) - 2i\beta \ln \left| \frac{x}{2} - s \right|.$

52
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