Towards Reconciliation between Bayesian and Frequentist Reasoning

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Abstract

A theory of quantitative inference about the parameters of sampling distributions is constructed deductively by following very general rules, referred to as the Cox-Pólya-Jaynes Desiderata. The inferences are made in terms of probability distributions that are assigned to the parameters. The Desiderata, focusing primarily on consistency of the plausible reasoning, lead to unique assignments of these probabilities in the case of sampling distributions that are invariant under Lie groups. In the scalar cases, e.g. in the case of inferring a single location or scale parameter, the requirement for logical consistency is equivalent to the requirement for calibration: the consistent probability distributions are automatically also the ones with the exact calibration and vice versa. This equivalence speaks in favour of reconciliation between the Bayesian and the frequentist schools of reasoning.

1 Introduction

A theory of quantitative inference about the parameters of sampling distributions is formulated with special attention being paid to the consistency of the theory and to its ability to make verifiable predictions. In the present article only basic concepts of the theory and their most important applications are presented while details can be found elsewhere [1].

Let $p(x_1|\theta I)$ be the probability for a random variate $x$ to take the value $x_1$ (to take a value in an interval $(x_1, x_1 + dx)$ in the case of a continuous variate), given the family $I$ of sampling distributions, and the value $\theta$ of the parameter that specifies a unique distribution within the family (for example, a sampling distribution from the exponential family $I$, $\tau^{-1}\exp\{-x/\tau\}$, is uniquely determined by the value of the parameter $\tau$). An inference about the parameter is made by specifying a real number, called (degree of) plausibility, $(\theta|x_1x_2\ldots I)$, to represent our degree of belief in the value of the
(continuous) parameter to be within an interval \((\theta, \theta + d\theta)\). Every such plausibility is conditioned upon the information that consists of measured value(s) \(x_1, x_2, \ldots\) of the sampling variate and of the specified family \(I\) of sampling distributions.

We assume all considered plausibilities to be subjects to very general requirements, referred to as the Cox-Pólya-Jaynes (CPJ) Desiderata [2,1], focusing primarily on consistency of the plausible reasoning. The requirement of consistency can be regarded as the first of the requirements to be satisfied by every theoretical system, be it empirical or non-empirical. As for an empirical system, however, besides being consistent, it must also be falsifiable [3]. We therefore added a Desideratum to CPJ Desiderata, requiring that the predictions of the theory must be verifiable so that, in principle, they may be refuted.

It should be stressed that in this way the list of basic rules is completed. That is, the entire theory of inference about the parameters is built deductively from the aforementioned Desiderata: in order not to jeopardize the consistency of the theory no additional \textit{ad hoc} principles are invoked.

2 Cox’s and Bayes’ Theorems

Richard Cox showed [4] that a system for manipulating plausibilities is either isomorphic to the probability system or inconsistent (i.e. in contradiction with CPJ Desiderata). Without any loss of generality we therefore once and for all choose probabilities \(p(\theta|x_1I)\) among all possible plausibility functions \((\theta|x_1I)\) to represent our degree of belief in particular values of inferred parameters. In this way the so-called inverse probabilities, \(p(\theta|x_1I)\), and the so-called direct (or sampling) probabilities \(p(x_1|\theta I)\), become subjects to identical rules.

Transformations of probability distributions that are induced by variate transformations are also uniquely determined by the Desiderata. Let \(f(x|\theta I)\) be the probability density function (pdf) for a continuous random variate \(x\) so that its probability distribution is expressible as

\[
p(x|\theta I) = f(x|\theta I) \, dx.
\]

Then, if the variate \(x\) is subject to a one-to-one transformation \(x \rightarrow y = g(x)\), the pdf for \(y\) reads:

\[
f(y|\theta I') = f(x|\theta I) \left| \frac{dy}{dx} \right|^{-1}
\]

(2)
(by using the symbol $I'$ instead of $I$ on the left-hand side of (2) it is stressed that the above transformations may in general alter the form of the sampling distribution). Since the direct and the inverse probabilities are subjects to the same rules, the transformation of the pdf for the inferred parameter, $f(\theta|xI)$, under a one-to-one transformation $\theta \rightarrow \nu = \bar{g}(\theta)$ is analogous to the transformation of the sampling pdf:

$$f(\nu|xI) = f(\theta|xI) \left| \frac{d\nu}{d\theta} \right|^{-1}. \quad (3)$$

Once the probabilities are chosen, the usual product and sum rules \[2\] become the fundamental equations for manipulating the probabilities, while many other equations follow from the repeated applications of the two. In this way, for example, Bayes’ Theorem for updating the probabilities can be obtained:

$$f(\theta|x_1x_2I) = \frac{f(\theta|x_1I) p(x_2|\theta x_1I)}{\int f(\theta'|x_1I) p(x_2|\theta' x_1I) d\theta'}. \quad (4)$$

Here $f(\theta|x_1I)$ denotes the pdf for $\theta$ based on $x_1$ and $I$ only (i.e. prior to taking datum $x_2$ into account), $p(x_2|\theta x_1I)$ is the probability for $x_2$ (the so-called likelihood) given values $\theta$ and $x_1$, while the integral in the denominator on the right-hand side ensures appropriate normalization of the updated pdf $f(\theta|x_1x_2I)$ for $\theta$ (i.e. the pdf for $\theta$ posterior to taking $x_2$ into account).

Bayes’ Theorem \[3\] allows only for updating pdf’s $f(\theta|x_1I)$ that were already assigned prior to their updating. Consequently, the existing applications of our basic rules must be extended in order to allow for assignment of probability distributions to the parameters, with such assignments representing natural and indispensable starting points in every sequential updating of probability distributions.

### 3 Consistency Theorem

According to the CPJ Desiderata, the pdf for $\theta$ should be invariant under reversing the order of taking into account two independent measurements of the sampling variate $x$. This is true if and only if the pdf that is assigned to $\theta$ on the basis of a single measurement of $x$, is directly proportional to the likelihood for that measurement,

$$f(\theta|xI) = \frac{\pi(\theta) p(x|\theta I)}{\int \pi(\theta') p(x|\theta' I) d\theta'}, \quad (5)$$

where $\pi(\theta)$ is the consistency factor while the integral in the denominator on the right-hand side of \[5\] again ensures correct normalization of $f(\theta|xI)$. 

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There is a remarkable similarity between the Bayes’ Theorem (4), applicable for updating the probabilities, and the Consistency Theorem (5), applicable for assigning the probability distributions to the values of the inferred parameters, but there is also a fundamental and very important difference between the two. While $f(\theta|x_1 I)$ in the former represents the pdf for $\theta$ prior to taking datum $x_2$ into account, $\pi(\theta)$ in the latter is (by construction of the Consistency Theorem [1]) just a proportionality coefficient between the pdf for $\theta$ and the appropriate likelihood $p(x|\theta I)$, so that no probabilistic inference is ever to be made on the consistency factor alone, nor can $\pi(\theta)$ be subject to the normalization requirement that is otherwise perfectly legitimate in the case of prior pdf’s.

The form of the consistency factor depends on the only relevant information that we possess before the first datum is collected, i.e. it depends on the specified sampling model. Consequently, when assigning probability distributions to the parameters of the sampling distributions from the same family $I$, this must be made according to the Consistency Theorem by using the consistency factors of the forms that are identical up to (irrelevant) multiplication constants.

### 4 Consistency Factor

According to (4) and (5) combined, the consistency factors $\pi(\theta)$ for $\theta$ and $\tilde{\pi}(\tilde{g}(\theta))$ for the transformed parameter $\tilde{g}(\theta)$ are related as

$$\tilde{\pi}(\tilde{g}(\theta)) = k \pi(\theta) |\tilde{g}'(\theta)|^{-1},$$  

where $k$ is an arbitrary constant (i.e. its value is independent of either $x$ or $\theta$), while $\tilde{g}'(\theta)$ denotes the derivative of $\tilde{g}(\theta)$ with respect to $\theta$. However, for the parameters of sampling distributions with the form $I$ that is invariant under simultaneous transformations $g_a(x)$ and $\tilde{g}_a(\theta)$ of the sample and the parameter space,

$$f(g_a(x)|\tilde{g}_a(\theta) I') = f(x|\theta I) |g_a'(x)|^{-1} = f(g_a(x)|\tilde{g}_a(\theta) I)$$  

(i.e. when $I' = I$), $\tilde{\pi}$ and $\pi$ must be identical functions up to a multiplication constant, so that (6) reads:

$$\pi(\tilde{g}_a(\theta)) = k(a) \pi(\theta) |\tilde{g}_a'(\theta)|^{-1}.$$  

Index $a$ in the above expressions indicates parameters of the transformations and $k$, in general, can be a function of $a$. In the case of multi-parametric
transformation groups the derivative \( g_a'(\theta) \) is to be substituted by the appropriate Jacobian.

The above functional equation has a unique solution for the transformations \( g_a(\theta) \) with the continuous range of admissible values \( a \), i.e. if the set of admissible transformations \( g_a(\theta) \) forms a Lie group. If a sampling distribution for \( x \) is invariant under a Lie group, then it is necessarily reducible (by separate one-to-one transformations of the sampling variate \( x \to y \) and of the parameter \( \theta \to \mu \)) to a sampling distribution that can be expressed as a function of a single variable \( y - \mu \), \( f(y|\mu I) = \phi(y - \mu) \). Sampling distributions of the form \( \psi(x/\sigma) \) are examples of such distributions: by substitutions \( y = \ln x \) and \( \mu = \ln \sigma \) they transform into \( \phi(y - \mu) = \exp\{y - \mu\} \psi(\exp\{y - \mu\}) \) (the scale parameters \( \sigma \) are reduced to location parameters \( \mu \)).

It is therefore sufficient to determine the form of consistency factors for the location parameter \( \mu \) since we can always make use of (9) to transform \( \tilde{\pi}(\mu = g(\theta)) \) into the appropriate consistency factor \( \pi(\theta) \) for the original parameter \( \theta \). Sampling distributions of the form \( \phi(x - \mu) \) are invariant under simultaneous translations \( x \to x + a \) and \( \mu \to \mu + a; \forall a \in (-\infty, \infty) \), and the functional equation (8) in the case of the translation group reads

\[
\pi(\mu + a) = k(a) \pi(\mu), \tag{9}
\]

implying the consistency factor for the location parameters to be \( \pi(\mu) \propto \exp\{-q\mu\} \), with \( q \) being an arbitrary constant. Accordingly, \( \pi(\sigma) \propto \sigma^{-\frac{q+1}{2}} \) is the appropriate form of the consistency factor for the scale parameters.

The value of \( q \) is then uniquely determined by recognizing the fact that sampling distributions of the forms \( \phi(x - \mu) \) and \( \sigma^{-1} \psi(x/\sigma) \) are just special cases of two-parametric sampling distributions

\[
f(x|\mu\sigma I) = \frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right), \tag{10}
\]

with \( \sigma \) being fixed to unity and with \( \mu \) being fixed to zero, respectively. The consistency factor \( \pi(\mu) \) therefore corresponds to assigning pdf’s \( f(\mu|\sigma x I) \) while \( \pi(\sigma) \) is to be used when assigning \( f(\sigma|x I) \). When neither \( \sigma \) nor \( \mu \) is fixed, however, the pdf (10) is invariant under a two-parametric group of transformations, \( x \to ax + b, \mu \to a\mu + b \) and \( \sigma \to a\sigma; \forall a \in (0, \infty) \) and \( \forall b \in (-\infty, \infty) \), and the functional equation (8) for the consistency factor \( \pi(\mu, \sigma) \) for assigning \( f(\mu,\sigma|x I) \) reads

\[
\pi(a\mu + b, a\sigma) = \frac{k(a, b)}{a^2} \pi(\mu, \sigma), \tag{11}
\]
so that $\pi(\mu, \sigma)$ is to be proportional to $\sigma^{-r}$, $r$ being an arbitrary constant. According to the product rule, $f(\mu\sigma|xI)$ can be factorized as

$$f(\mu\sigma|xI) = f(\mu|\sigma xI) f(\sigma|xI) = f(\sigma|\mu xI) f(\mu|xI),$$

where $f(\sigma|xI)$ and $f(\mu|xI)$ are the marginal pdf’s, e.g.

$$f(\sigma|xI) = \int f(\mu'|\sigma xI) d\mu'.$$

The equalities (12) are achieved if and only if $q = 0$ and $r = 1$, i.e. if the three consistency factors, determined uniquely up to arbitrary multiplication constants, read:

$$\pi(\mu) = 1 \text{ and } \pi(\sigma) = \pi(\mu, \sigma) = \sigma^{-1}.$$  

5 Calibration

In order to exceed the level of a mere speculation, the theory of probabilistic inference about the parameters must be able to make predictions that can be verified (or falsified) by experiments. Let therefore a random variate $x$ be subject to a family of sampling distributions $I$ and let several independent values $x_i$ of the variate be recorded. The predictions of the theory are made in probabilities

$$P(\theta \in (\theta_{i,1}, \theta_{i,2}) | x_i I) = \int_{\theta_{i,1}}^{\theta_{i,2}} f(\theta' | x_i I) d\theta' = \delta$$

that given measured value $x_i$ of the sampling variate, an interval $(\theta_{i,1}, \theta_{i,2})$ contains the actual value of the parameter $\theta$ of the sampling distribution. For the sake of simplicity, the intervals are chosen in such a way that the probabilities $\delta$ are equal in each of the assignments. The predictions are then verifiable at long term relative frequencies: our probability judgments (15) are said to be calibrated if the fraction of inferences with the specified intervals containing the actual value of the parameter, coincides with $\delta$.

An exact calibration of an inference about a parameter $\theta$ is ensured if the assigned pdf $f(\theta|xI)$ is related to the (cumulative) distribution function $F(x, \theta)$ of the sampling variate as

$$f(\theta|xI) = \left| \frac{\partial}{\partial \theta} F(x, \theta) \right|,$$

(16)
and the consistency factors $\pi(\mu)$ and $\pi(\sigma)$ do meet the above requirement. Furthermore, if besides of being calibrated, the pdf for $\theta$ is to be assigned according to the Consistency Theorem, the distribution of the sampling variate $x$ is necessarily reducible to a distribution of the form $\phi(y - \mu)$ [5]. But exactly the same necessary condition was obtained by requiring invariance of the sampling distribution under a Lie group, with such an invariance being indispensable for determination of consistency factors solely by imposing consistency to the assignment of pdf’s. Imposing logical consistency to the theory is thus equivalent to imposing calibration to its predictions: every probabilistic inference about a parameter of a sampling distribution that we are sure is consistent will thus at the same time also be calibrated and, vice versa, every calibrated inference, based on a posterior pdf that is factorized according to (5), will simultaneously be logically consistent, too. The equivalence of the two requirements speaks in favour of reconciliation between the (objective) Bayesian and the frequentist schools of reasoning, the former paying attention primarily to logical consistency and the latter stressing the importance of verifiable predictions.

6 Consistency Lost and Regained

Numerous examples can be found with the sampling distributions lacking invariance under Lie groups: there are sampling distributions for continuous random variates (e.g. the Weibull distribution) that are not invariant under continuous groups of transformations, the symmetry can be broken by imposing constraints to parameter spaces of otherwise invariant sampling distributions, or the sampling space may be discrete (e.g. in counting experiments), just to name three of the most common ones. No consistent qualitative parameter inference is possible in such cases, but under very general conditions the remedy is just to collect more data relevant to the estimated parameters. Then, according to the Central Limit Theorem, the discrete sampling distributions approach their dense (Gaussian) limits, the constraints of the parameter spaces become more and more irrelevant, and the sampling distributions of the maximum likelihood estimates of the inferred parameter $\theta$ gain Gaussian shapes with $\theta$ being the location parameters of the latter, so the ability of making consistent inferences is regained.
7 Consistency Preserved

Consistency factors are determined exclusively by utilizing the tools such as the product rule \((12)\) and marginalization \((13)\), that are deducible directly from the basic Desiderata: in order to preserve consistency of inference it is crucial to refrain from using \textit{ad hoc} shortcuts on the course of inference. For regardless how close to our intuitive reasoning these \textit{ad hoc} procedures may be, how well they may have performed in some other previous inferences, and how respectable their names may sound (e.g. the principle of insufficient reason or its sophisticated version - the principle of maximum entropy, the principle of group invariance, the principle of maximum likelihood, and the principle of reduction), they are all found in general to lead to inferences that are neither consistent nor calibrated.

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