Periodic orbits of Euler vector fields on 3-manifolds

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Abstract

In this paper we study periodic orbits in the flow of non-singular steady Euler fields $X$ on closed 3-manifolds, that is $X$ is a solution of time independent Euler equations. We show that when $X$ is $C^2$ the flow always possess a periodic orbit, unless the manifold is a torus bundle over the circle. Moreover, we show that if the ambient manifold is $S^3$, there exist an unknotted periodic orbit. These results generalize previous results of J. Etnyre and R. Ghrist by weakening the real analytic hypothesis to $C^2$.

Keywords: Euler equation, periodic orbit, geodesible vector field, Seifert’s conjecture

1 Introduction and results

The results on this paper give a new sufficient hypothesis for the existence of a periodic orbit of the flow of a non-singular vector field on closed 3-manifolds. The problem of determining when the flow of a non-singular vector field has a periodic orbit has a long history. On one side, there are examples of vector fields whose flow has no periodic orbits on any closed 3-manifold. These were first constructed by P. A. Schweitzer for $C^1$ vector fields [15], and then by K. Kuperberg in the $C^\infty$ category, and even real analytic [10]. Schweitzer’s construction was then achieved in the volume preserving category by G. Kuperberg, giving $C^1$ volume preserving vector fields without periodic orbits [9].

On the other side, results by H. Hofer and C. H. Taubes guarantee the existence of a periodic orbit for the flow of a Reeb vector field on any 3-manifold [5] [17]. Recently, the existence of a periodic orbit was stated for volume preserving geodesible vector fields, also known as Reeb vector fields of stable Hamiltonian structures, on 3-manifolds that are not torus bundles over the circle [3] [13]. A vector field is geodesible if there exist a Riemannian metric making its orbits geodesics. Volume preserving geodesible vector fields form a particular set of solutions to the steady Euler equations.

We will deal with vector fields that satisfy the steady Euler equations of an incompressible fluid, to which we will simply refer as Euler equations, on closed oriented Riemannian 3-manifolds. Vector fields arising from suspensions of diffeomorphisms of surfaces satisfy these equations, and hence we cannot expect an Euler vector field to always have periodic orbits. The main result that we are going to prove is

**Theorem 1.1** Let $X$ be a $C^2$ vector field satisfying the steady Euler equations on a closed oriented Riemannian 3-manifold $(M, g)$ endowed with a volume form $\mu$. Assume that $M$ is not a torus bundle over the circle, then $X$ has a periodic orbit.
For real analytic vector fields, Theorem 1.1 was proved by J. Etnyre and R. Ghrist [2]. A vector field that is a solution to the Euler equations is either a Reeb vector field or it admits a first integral (that may be equal to zero, but otherwise non-constant). For Reeb vector fields the existence of a periodic orbit is given by C. H. Taubes’ theorem [17]. On the other case, assuming that the first integral is not zero, the real analyticity allows for study the critical levels of such a function and find a periodic orbit (for the details we refer to [2]). When the first integral is identically zero, the vector field admits a global section: A closed 2-manifold transverse to \( X \). Euler characteristic arguments prove that if this 2-manifold is not a torus then \( X \) possess a periodic orbit. The difference for the \( C^2 \) case is when analyzing the behaviour of \( X \) near the critical levels of the first integral.

Once the existence of a periodic orbit is proved, we can ask ourselves about the knots and links that can arise as periodic orbits. In [8], H. Hofer, K. Wysocki and E. Zehnder prove the existence of an unknotted periodic orbit for Reeb vector fields on \( S^3 \). A periodic orbit is unknotted in \( S^3 \) if it bounds an embedded disc. The same statement was proved for \( C^\omega \) solutions to the Euler equations in [2], strongly using the real analyticity. Here we will prove the existence of such an orbit for Euler vector fields.

**Theorem 1.2** Let \( X \) be a steady state \( C^2 \) solution of the Euler equations on \( S^3 \), then \( X \) has an unknotted periodic orbit.

The article is organized as follows: In Section 2 we will describe the steady Euler equations on 3-manifolds and the types of vector fields that we talked about in this introduction. We will then give the proof of Theorem 1.1 in Section 3 and of Theorem 1.2 in Section 4.

## 2 Types of vector fields and useful results

In this section we will define the main three types of vector fields that appeared in the introduction, and that will appear along the proofs of the Main Theorems, these are: Euler vector fields (Section 2.1), volume preserving geodesible vector fields and Reeb vector fields associated to contact structures (Section 2.2).

### 2.1 Euler equations on 3-manifolds

In this section we briefly describe Euler equations on manifolds, but we refer to [11] for a detailed treatment of the subject.

Consider a Riemannian 3-manifold \((M, g)\), equipped with an arbitrary distinguished volume form \( \mu \). We adopt the following definition of the curl of a vector field \( X \). The curl of \( X \) is the unique vector field \( \text{curl}(X) \) satisfying equation

\[
\iota_{\text{curl}(X)} \mu = d\iota_X g.
\]

Taking the curl with respect to an arbitrary volume form makes the subsequent results valid for a more general class of fluids: For example \textit{basotropic flows}, these are compressible for the Riemannian volume and incompressible for a rescaled volume form. We refer the reader to section VI.2.A of V. I. Arnold and B. A. Khesin’s book [1]. When \( \mu \) is the Riemannian volume the curl assumes the more common form

\[
\text{curl}(X) = \psi(\ast d_{X} g),
\]

where \( \ast \) is the Hodge star operator, and \( \psi \) is the isomorphism between vector fields and differential 1-forms derived from \( g \). In this situation, the curl depends only on the Riemannian metric.
Definition 2.1 The Euler equations of an ideal steady incompressible fluid on a Riemannian manifold \((M, g)\) endowed with a volume form \(\mu\), is

\[
\nabla_X X = -\operatorname{grad}(p) \\
L_X \mu = 0,
\]

where the velocity vector field is \(X\), and \(\nabla_X\) is the covariant derivative along \(X\). In the second equation \(L_X\) is the Lie derivative along \(X\), hence this equation implies that \(X\) preserves the volume form \(\mu\).

Let \(\alpha\) be the 1-form associated to \(X\) by the Riemannian metric \(g\), that is \(\alpha = \iota_X g\). As in Lemma-Definition 7.21 of [1], we can write the first Euler equation in Definition 2.1 as

\[
L_X \alpha = -dP. \tag{1}
\]

In fact, the 1-form associated to the vector field \(\nabla_X X\) is \(L_X \alpha - \frac{1}{2} d(\alpha(X))\), and hence,

\[
L_X \alpha = -d \left( p - \frac{1}{2} \alpha(X) \right) = -dP \\
\iota_X d\alpha = -d \left( p + \frac{1}{2} \alpha(X) \right) = -df,
\]

and hence \(L_X f = 0\) meaning that \(f\) is invariant under the flow of \(X\). Using Cartan’s formula we get that

\[
L_X d\alpha = d(\iota_X d\alpha) = d(-df) = 0,
\]

meaning that \(d\alpha\) is preserved by \(X\). The function \(f\) is known as the Bernoulli function of \(X\). We will call Euler vector fields to the solutions to the equations in Definition 2.1.

Through the proofs, we will always make a distinction between the vector fields that are parallel to their curl and those who are not. The first ones are known as Beltrami vector fields in hydrodynamics. In magnetodynamics these vector fields are known as force-free vector fields.

Before passing to the next type of vector fields, let us state J. Etnyre and R. Ghrist’s result concerning the existence of periodic orbits for Euler vector fields, we refer to Theorem 1.1 of [2].

**Theorem 2.2 (Etnyre, Ghrist, Theorem 1.1 [2])** Any steady solution to the \(C^\omega\) Euler equations on \(S^3\) has an unknotted periodic orbit.

### 2.2 Geodesible and Reeb vector fields

Let us begin by defining geodesible and Reeb vector fields. We will see that the latter are always geodesible. In this section we will state some known results on the existence of periodic orbits.

**Definition 2.3** A non-singular vector field is said to be geodesible if there exists a Riemannian metric making its orbits geodesics.

Let \(X\) be a geodesible vector field and \(g\) a Riemannian metric making its orbits geodesics. Consider the 1-form \(\alpha = \iota_X g\). Modulo reparameterization, we can assume that \(\alpha(X) = 1\) and then the condition \(\nabla_X X = 0\) is translated to the language of differential forms as \(L_X \alpha = \iota_X d\alpha = 0\). In fact, it is not difficult to prove that a vector field on a 3-manifold is geodesible if and only if there exists a 1-form \(\alpha\) such that

\[
\alpha(X) = 1 \quad \text{and} \quad \iota_X d\alpha = 0.
\]
In such a situation, a Riemannian metric making the plane field defined by the ker(\(\alpha\)) perpendicular to \(X\) and \(X\) of constant length, makes the orbits geodesics.

Recalling the definition of the curl vector field, we conclude that a vector field is geodesible if and only if it is Beltrami \((\iota_X d\alpha = \iota_X (\text{curl}_X \mu) = 0)\). Hence, volume preserving geodesible vector fields satisfy Euler’s equations and are parallel to their curl.

Before continuing describing geodesible vector fields, we need to define Reeb vector fields of contact forms.

**Definition 2.4** On a 3-manifold \(M\) a 1-form \(\alpha\) is a contact form if \(\alpha \wedge d\alpha \neq 0\). The plane field \(\xi\) defined by the kernel of \(\alpha\) is non-integrable and is called a contact structure. The Reeb vector field of \(\alpha\) is given by the following equations

\[
\alpha(X) = 1 \quad \text{and} \quad \iota_X d\alpha = 0.
\]

Notice the analogy with geodesible vector fields, the difference is that for geodesible vector fields we do not need the condition \(\alpha \wedge d\alpha \neq 0\). Let us come back to geodesible vector fields. Observe that since \(X\) is not in the kernel of \(\alpha\) and is in the kernel of \(d\alpha\), the differential 3-form \(\alpha \wedge d\alpha\) is non zero whenever \(d\alpha \neq 0\). We can consider three subsets of geodesible vector fields on a 3-manifold \(M\):

- If \(d\alpha \neq 0\) then the 3-form \(\alpha \wedge d\alpha\) is a never zero. In other words, \(\alpha\) is a contact form and \(X\) is its Reeb vector field;
- If \(\alpha\) is a closed form, then D. Tischler’s theorem \[18\] implies that \(M\) is a fiber bundle over \(S^1\), and moreover \(X\) is transverse to the fibers. Hence, we can describe the flow of \(X\) as a suspension of an area preserving diffeomorphism of a closed surface.
- The mixed case, when \(d\alpha = 0\) on proper compact invariant sets.

A volume preserving geodesible vector field admits the function \(h\) given by \(\text{curl}(X) = hX\) as a first integral. This follows from

\[
0 = L_{\text{curl}(X)} \mu = L_{hX} \mu = h d\mu + dh \wedge \iota_X \mu.
\]

Since \(d\mu = 0\), we have that \(h\) is a first integral of \(X\).

Let us now state some results on the existence of periodic orbits for geodesible and Reeb vector fields. Beginning with Reeb vector fields, let us state C. H. Taubes’ theorem \[17\].

**Theorem 2.5 (Taubes, Theorem 1 \[17\])** If \(X\) is a Reeb vector field associated to a contact form on a closed oriented 3-manifold, it possesses a periodic orbit.

The proof of this theorem uses Seiberg-Witten invariants and Embedded Contact Homology. The existence of a periodic orbit was stated by H. Hofer \[5\] under some extra hypothesis. H. Hofer’s method is based on the use of pseudoholomorphic curves in a symplectisation of the ambient manifold. In \[6\] the existence of an unknotted periodic orbit is proved.

**Theorem 2.6 (Hofer, Wysocki, Zehnder, Theorem 1.1 \[6\])** If \(\alpha\) is any contact form on \(S^3\) then the associated Reeb vector field has an unknotted periodic orbit.

Let us now give some results for geodesible vector fields, we refer to the Main Theorems in \[8\] or \[13\].
Theorem 2.7 (Hutchings, Taubes; Rechtman [8, 13]) Let $X$ be a $C^2$ volume preserving geodesible vector field on a closed oriented 3-manifold $M$ that is not a torus bundle over the circle. Then $X$ has a periodic orbit.

As for Theorem 2.5 the proof in the general case uses Seiberg-Witten invariants and Embedded Contact Homology. If either $M = S^3$ or has non-trivial second homotopy group, Theorem 2.7 can be proved using H. Hofer’s techniques.

Let $T$ denote a two dimensional torus and $I$ a closed interval, following the lines of the proofs of Theorem 2.7 we obtain

Corollary 2.8 Let $X$ be a $C^2$ volume preserving geodesible vector field on a compact 3-manifold $N$ with boundary, and such that $X$ is tangent to $\partial N$. Then, if $X$ has no periodic orbits, $N$ is diffeomorphic to $T \times I$.

Both proofs of Theorem 2.7 analyze the levels of the function $h$ given by $\text{curl}(X) = hX$, that is invariant under the flow. Roughly speaking the proof goes as follows. If $h \neq 0$ the vector field $X$ is a Reeb vector field and the existence of a periodic orbit is given by Theorem 2.5. If $h \equiv 0$ the vector field is a suspension, and the result follows. The other case is when $h^{-1}(0)$ is a proper compact invariant set. When zero is a regular value, we can analyze the Reeb vector field in the connected components of $M \setminus h^{-1}(0)$ and conclude using Theorem 2.9 below. When zero is a critical value, it is necessary to analyze the structures near $h^{-1}(0)$ to find a neighborhood where $X$ is a suspension. This allows to finish the proof.

Theorem 2.9 (Hutchings, Taubes, Theorem 1.2 [8]) Let $N$ be a compact 3-manifold with boundary that is endowed with a Reeb vector field $X$ (of a contact form) tangent to $\partial N$. Then if $N$ is not diffeomorphic to $T \times I$, the vector field $X$ possesses a periodic orbit.

3 Existence of a periodic orbit for Euler vector fields

The aim of this section is to give a proof of Theorem 1.1, for which we will use Lemma 3.2 below. We will prove this lemma in Section 3.1.

Proof of Theorem 1.1. We know that the vector field $X$ is volume preserving and belongs to one of the following families:

- $X$ such that $\text{curl}(X) = hX$, for a function $h : M \to \mathbb{R}$;
- $\iota_X d\alpha = -df$, for a non-constant function $f : M \to \mathbb{R}$.

In the first case, $X$ is geodesible and Theorem 2.7 guarantees the existence of a periodic orbit. Hence, we will concentrate in proving the theorem for the second case. The idea is the following: The function $f$ is a first integral, hence the inverse image of each one of the regular values is a finite collection of oriented surfaces, hence of tori. We need to analyze the behaviour near the critical levels of $f$. We will construct nice neighborhoods of the critical levels where $X$ preserves a transverse plane field, i.e. it is geodesible. Using the results in the previous section, we will conclude that $M$ has to be a torus bundle over the circle. Let us begin by the following result, for a proof we refer to Theorem 1.5 of [1].

Theorem 3.1 Every non critical level surface of $f$ is diffeomorphic to a 2-dimensional torus $T$. Let $J$ be an open interval of regular values of $f$. Then each connected component of $f^{-1}(J)$ is
Let us write \( \text{curl}(X) = hX + Y \) for a function \( h \) on \( M \) and a vector field \( Y \) in the kernel of \( \alpha \). We will now concentrate on the critical levels of \( f \). Observe that \( df = 0 \) when either \( \text{curl}(X) = 0 \) or \( \text{curl}(X) = hX \) for a function \( h \neq 0 \). Let \( \alpha = \iota_X \gamma \), then the two cases are equivalent to \( d \alpha = 0 \) and \( d \alpha = h_1 \cdot X \cdot \mu \), respectively. Since \( X \) is an Euler vector field we have that \( L_X d \alpha = 0 \), hence \( d \alpha \) and \( \text{curl}(X) \) are invariant under the flow of \( X \). This implies that if at a point \( p \in M \) we have \( \text{curl}(X) = 0 \) (respectively, \( \text{curl}(X) = hX \)) the same happens along the whole orbit of \( p \).

Assume that \( X \) has no periodic orbits. Denote by \( C \) the set of critical values of \( f \) and consider \( \Sigma \) the union of the connected components in \( \{ f^{-1}(c) \mid c \in C \} \) where \( df = 0 \). Then \( M \setminus \Sigma \) is foliated by invariant tori \( T \), and each connected component is of the form \( T \times I \) where \( X \) is a constant vector field tangent to each torus. In other words, on each connected component we can introduce coordinates as in Theorem 3.1 for which

\[
X = \tau_1(t) \frac{\partial}{\partial x} + \tau_2(t) \frac{\partial}{\partial y},
\]

where \( \frac{\tau_1(t)}{\tau_2(t)} \) is a constant irrational number depending only on the connected component of \( M \setminus \Sigma \).

Take now a connected component \( A \) of \( \Sigma \), and \( A_0 \subset A \) the set where the vector field \( Y \) is equal to zero. Then \( A_0 \) is invariant. We will use the following lemma.

**Lemma 3.2** Under the hypothesis above, there exists a closed neighborhood \( B \) of \( A \) where \( X \) is geodesible. Moreover, each boundary component of \( B \) is contained in a regular level of \( f \).

Let us finish the proof of Theorem 1.1 using Lemma 3.2. Recall that we assume that \( X \) has no periodic orbits. Lemma 3.2 implies that \( X \) restricted to \( B \) is a geodesible vector field. Then by Corollary 2.3 we have that \( B \) is diffeomorphic to \( T \times I \) and \( X \) is tangent to the boundary. We get that \( \Sigma \) has a neighborhood that is a finite collection of copies of \( T \times I \), and hence we conclude that \( M \) is a torus bundle over the circle. The contradiction we needed to prove Theorem 1.1.

Before passing to the proof of Lemma 3.2 let us make some remarks. First, we have the following corollary:

**Corollary 3.3** Let \( N \) be a closed 3-manifold with boundary and \( X \) an Euler vector field on \( N \) that is tangent to the boundary. If \( X \) does not have periodic orbits, then \( N \) is diffeomorphic to \( T \times I \).

Second, we used Theorem 2.9 that as we said is proved using Seiberg-Witten invariants and Embedded Contact Homology. A proof of Theorem 1.1 when \( M \) is either \( S^3 \) or has non trivial second homotopy group, that does not uses Seiberg-Witten invariants nor ECH, uses the following proposition (for a proof we refer to Section 1.5 of [12]):

**Proposition 3.4** Let \( \Delta \) be a finite collection of disjoint embedded tori in \( S^3 \) or a closed oriented 3-manifold \( M \) with \( \pi_2(M) \neq 0 \). Then there exists a connected component with closure \( B \) of \( S^3 \setminus \Delta \), respectively \( M \setminus \Delta \), such that either \( B \) is a solid torus or \( \pi_2(B) \neq 0 \).

H. Hofer’s method allow us to prove that a Reeb vector field on a manifold \( B \) with boundary that is tangent to \( \partial B \), has a periodic orbit if \( B \) is either a solid torus or \( \pi_2(B) \neq 0 \). For proof of
this fact we refer to Theorem 6.1 in [13] and the references therein. Using these results, we can prove (for details in the geodesic case we refer to section 6 of [13]):

Let $X$ be a $C^2$ Euler vector field on a closed 3-manifold $M$. Assume that $M$ is either diffeomorphic to $\mathbb{S}^3$ or has non-trivial second homotopy group, then $X$ has a periodic orbit.

### 3.1 Proof of Lemma 3.2

Assume that $A_0 \subset A \subset f^{-1}(c)$, where $c$ is a critical value of $f$. Consider $\epsilon > \delta > 0$ such that all the values in the intervals $[c - \delta, c - \epsilon]$ and $[c + \epsilon, c + \epsilon + \delta]$ are regular. Let $D \subset f^{-1}([c - \delta, c])$ and $B \subset f^{-1}([c - \epsilon, c + \epsilon])$ be the connected components that are neighborhoods of $A$. Modulo considering a connected component, assume that $D \setminus B$ is connected and that $f > c$ on $D \setminus B$.

In $D \setminus B$ we can introduce a coordinate system as in Theorem 3.1 such that

$$X = \tau_1(t) \frac{\partial}{\partial x} + \tau_2(t) \frac{\partial}{\partial y} \quad \text{curl}(X) = \gamma_1(t) \frac{\partial}{\partial x} + \gamma_2(t) \frac{\partial}{\partial y},$$

with $\frac{\tau_1(t)}{\tau_2(t)}$ constant and irrational. Hence in $D \setminus B$ the function $h$ depends only on $t$.

Since $\mu$ is $X$ and curl($X$) invariant, we can write $\mu = \beta(t) dx \wedge dy \wedge dt$ for a certain positive function $\beta$. Then we have on $D \setminus B$

$$\iota_Y \mu = \beta(t) (\gamma_1(t) - h(t) \tau_1(t)) dy - (\gamma_2(t) - h(t) \tau_2(t)) dx \wedge dt$$

$$= \beta(t) \gamma_1(t) dy - a_2(t) dx \wedge dt$$

$$= \beta(t) \gamma_1(t) dy - a_2(t) x dt$$

$$= d(\beta(t) \gamma_1(t) y - a_2(t) x dt)$$

$$= d\lambda,$$

where $a_i(t) = \gamma_i(t) - h(t) \tau_i(t)$ for $i = 1, 2$, and $\lambda = \beta(t)(a_1(t) y - a_2(t) x) dt$ is a differential 1-form defined on $D \setminus B$.

If $\iota_Y \mu$ is an exact form on $B$, then $d\alpha - \iota_Y \mu = d(\alpha - \lambda) = \iota_X \mu$, and we conclude that $X$ is geodesible on $B$. Hence we will assume that $\iota_Y \mu$ is not exact on $B$.

Consider a $C^\infty$ bump function $p : [c + \epsilon, c + \epsilon + \delta] \to [0, 1]$ such that $p(t) = 1$ for $t$ near $c + \epsilon$, $p(t) = 0$ for $t$ near $c + \epsilon + \delta$ and $p(t) \leq 0$. On $D \setminus B$, let $\tilde{\lambda} = p(t) \lambda$ that coincides with $\lambda$ near $B \cap \{f^{-1}(c + \epsilon)\}$. Set $\tilde{\lambda} = p(t) \lambda$ and $d\tilde{\lambda} = p(t) \iota_Y \mu$. Hence $d\tilde{\lambda}$ can be extended by $\iota_Y \mu$ over $B$. That is, we obtain a differential 2-form $\omega$ on $M$ such that

$$\omega = \begin{cases} 0 & \text{in } M \setminus D \\ p(t) \iota_Y \mu & \text{in } D \setminus B \\ \iota_Y \mu & \text{in } B. \end{cases}$$

We claim that given any constant $C_0 > 0$, we can assume that $\|\omega\| < C_0$. For this, take $E$ the connected component of $f^{-1}(c - \epsilon - 2\delta, c + \epsilon + 2\delta)$ containing $B$ and assume that $f$ has no critical points on $E \setminus B$. Then the previous analysis can be extended to $E \setminus B$ and we can consider a volume form $\nu$ that is equal to $C_1 \mu$ on $D$ and to $C_2 \mu$ on $M \setminus E$, with a convenient bump function of $t$ defined on $[\delta, 2\delta]$ and constants $C_1, C_2$. Then if $Z$ is the curl of $X$ with respect to $\nu$, we have that $Z = \frac{1}{C_1}(hX + Y)$ on $D$. Hence we can assume that $\|\iota_Y \mu\| < C_0$ on $D$.

**The cohomology class of $\omega$ on $M$.**

Consider the exact sequence of homologies with real coefficients

$$\cdots \rightarrow H_1(M \setminus A_0) \rightarrow H_1(M) \rightarrow H_1(M, M \setminus A_0) \rightarrow \cdots$$
and a finite collection of embedded curves $\sigma_1, \sigma_2, \ldots, \sigma_n$ in $M \setminus A_0$ such that they form a basis for the kernel of the map $H_1(M) \to H_1(M, M \setminus A_0)$. These curves are at positive distance from $A_0$, then for $\epsilon$ small enough we can assume that the $\sigma_i$ are at positive distance from $B$.

Using the duality of Poincaré (see for example chapter 26 of [4]) we have that $H_1(M) \cong H^2(M)$ and hence for every $i = 1, 2, \ldots, n$ we get a 2-form $\omega_i$, the dual of $\sigma_i$, whose support is contained in a tubular neighborhood of $\sigma_i$ contained in $M \setminus B$.

**Lemma 3.5** For $\epsilon$ small enough there are unique real numbers $r_1, r_2, \ldots, r_n$ such that

$$[\omega] = \sum_{i=1}^n r_i [\omega_i]$$

in $H^2(M)$. Moreover, there exists a constant $C_3$ independent of $\epsilon$ such that $|r_i| \leq C_3 \epsilon$ for every $i$.

**Proof.** Denote by

$$\phi_1 : H_1(M) \to H_1(M, M \setminus A_0)$$

$$\phi_2 : H^2(M) \to H^2(A_0).$$

Using the isomorphism given by the duality of Poincaré we have a map $\ker(\phi_1) \to \ker(\phi_2)$ that is injective. Then to prove the existence and uniqueness of the numbers $r_i$ we need to prove that the map between the kernels is surjective.

Take an element $\Omega$ in the kernel of $\phi_2$. It can be represented by a form whose support is in $M \setminus A_0$, then $[\Omega] \in H^2(M \setminus A_0)$ (since it has compact support). The dual of this class is an homology class $\sigma \in H_1(M \setminus A_0)$ satisfying that for every element $S \in H_2(M \setminus A_0, \partial A_0)$

$$\sigma \cdot S = \int_S \sigma.$$

Using the inclusion $i : M \setminus A_0 \to M$, we get

$$i_* \sigma \cdot S = \int_S \sigma,$$

for all $S \in H_2(M)$. Then $i_* \sigma \in H_1(M)$ is the dual of $[\Omega] \in H^2(M)$, and $\phi_1(i_* \sigma) = 0$. Then the map is surjective.

Since $\omega$ is a non-zero element of the kernel of $\phi_2$, we obtain $\sigma \in \ker(\phi_1)$ that can be written as a combination of the $\sigma_i$ above with coefficients $r_i$. We need to prove now that the $r_i$ are bounded. For $j = 1, 2, \ldots, n$ fix an oriented embedded surface $S_j$ in $M$ that intersects the $\sigma_i$. Then

$$r_j = \int_{S_j} \sum_{i=1}^n r_i \omega_i = \int_{S_j} \omega_i.$$

Using the bound on $\iota_Y \mu$ we get a constant $C_3 > 0$ that is independent of $\epsilon$ and such that $|r_j| \leq C_3$ for all $j$.

\[\Box\]

The differential 2-form $\Gamma = \omega - \sum_{i=1}^n r_i \omega_i$ is closed and exact in $M$. The next step is to find a primitive of $\Gamma$ that is bounded. The bounds above imply that $\|\Gamma\| \leq C_4$ for a positive constant $C_4$ independent of $\epsilon$. We need to find a primitive $\gamma$ whose norm is bounded by the norm of $\Gamma$. The existence of such a primitive is given by combining the main result of F. Laudenbach’s paper [11] and Theorem 1.1 of J.-C. Sikorav’s paper [16]. The first one gives a method to find a primitive and the second one a bound for it. We get,
Lemma 3.6 There exists a 1-form $\gamma$ such that $d\gamma = \Gamma$ and $\|\gamma\| \leq C_5\|\Gamma\|$, where $C_5$ is a constant depending only on $M$.

The constant $C_5$ depends on a fixed triangulation of the manifold $M$. Then, using the previous bounds we have $\|\gamma\| \leq C_5C_4$. Thus, for an adequate choice of $C_5$, the 1-form $\alpha - \gamma$ satisfies that $d(\alpha - \gamma) = h\xi X\mu + \nu Y\mu - \omega + \sum_{i=1}^n r_i\omega_i$, is equal to $h\xi X\mu$ in $B$, since the support of the $\omega_i$ is contained in $M \setminus B$. Moreover, $(\alpha - \gamma)(X) > 0$ as a consequence of the bounds. Then, in $B$, the vector field $X$ is in the kernel of the form $d(\alpha - \gamma)$ and hence it is geodesible. This finishes the proof of Lemma 3.2.

4 Existence of an unknotted periodic orbit

In this section we will prove Theorem 1.2. We will divide the proof into several steps. First, in Section 4.1, we will study Reeb vector fields in certain manifolds with boundary. Assuming that such a manifold is embedded in $S^3$, we will use Theorem 2.6 and contact surgery, to prove that the Reeb vector field has an unknotted periodic orbit: That is a periodic orbit that is the boundary of an embedded disc in $S^3$. Second, we will prove that given an Euler vector field on $S^3$ that is not Reeb, there exists an invariant 3-manifold such that therein the vector field is a Reeb vector field. In this step we will first analyze geodesible vector fields (Section 4.2), and then Euler vector fields (Section 4.3).

4.1 Reeb vector fields on manifolds with boundary

In this section we will perform contact surgery to prove Theorem 4.1 below. Dehn surgery is an important tool for constructing closed orientable 3-manifolds. In this section, we will briefly introduce Dehn surgery for certain contact manifolds and then use it to study Reeb vector fields.

Dehn Surgery.

The object of Dehn surgery is to drill out a solid torus from a 3-manifold and replace it with another solid torus, introducing some twisting. Let $L$ be a loop in $S^3$ having a tubular neighborhood $V_L$ diffeomorphic to $S^1 \times \mathbb{D}^2$, with $L$ as the core circle $S^1 \times \{0\}$. Choose cylindrical coordinates $(x, (y, t))$, where $t$ is the radius and $x, y$ are taken modulo $2\pi$, such that the curves $m = \{(0, y, 1)\}$ and $l = \{(x, 0, 1)\}$ correspond to a meridian and a longitude of $\partial V_L$, respectively. Denote by $\Psi : \partial V_L \to \partial V_L$ the diffeomorphism

\[
\Psi \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} q \\ p \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right),
\]

with $p, q, a, b$ integers such that $qa - pb = 1$. In other words, the matrix is in $PSL(2, \mathbb{Z})$. The $p/q$-Dehn surgery on $S^3$ along $L$ consists in removing $V_L$ and gluing it back via $\Psi$:

\[
M_L(p/q) = S^3 \setminus V_L \cup_\Psi V_L.
\]

The new manifold is completely determined by $L, p$ and $q$ (we refer to [14]).

Model contact form on $S^1 \times \mathbb{D}^2$. 

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When performing a Dehn surgery we will need to keep track of the Reeb vector field, and hence, of the contact form, on \( S^1 \times \mathbb{D}^2 \). Fix \( R > 0 \), and consider the 1-form

\[
\alpha_R = \frac{1}{\sin^2 t + \frac{R^2}{2} \cos^2 t} \left( \sin^2 t \, dt + \cos^2 t \, dx \right).
\]

Let \( \xi_R \) be the contact structure defined by the kernel of \( \alpha_R \), and \( X_R \) the Reeb vector field, then:

- The tori \( T_t = \{ t = \text{const.} \} \), for \( t > 0 \), are invariant under the flow generated by \( X_R \).
- On each torus \( T_t \), the vector field \( X_R \) is linear with slope \( \frac{2}{R^2} \). Hence, the circle \( \{ t = 0 \} \) is a periodic orbit for all \( R \), and for irrational values of \( R \), it is the unique periodic orbit.
- The characteristic foliation on each torus \( T_t \), given by the intersection of \( \xi_R \) with the tangent space of \( T_t \), is linear with slope \( -\frac{\tan^2 t}{\tan^2 t} \).

Reeb vector fields on manifolds with boundary.

In the following, we will need to find solid tori and 3-manifolds of the form \( T \times I \), equipped with a contact form such that the Reeb vector field and the characteristic foliation are linear of prescribed slope in the boundary. This will be done by choosing \( R \) and the boundary torus \( T_t \) in the previous model.

**Theorem 4.1** Let \( B \) be a 3-manifold with boundary embedded in \( S^3 \), satisfying one of the following:

(i) \( B \simeq \mathbb{T} \simeq S^1 \times \mathbb{D}^2 \) unknotted;
(ii) \( B \simeq S^3 \setminus V_K \), where \( V_K \) is a tubular neighborhood of a link \( K \) such that any of its components is a non-trivial knot;
(iii) \( B \simeq \mathbb{T} \setminus V_K \), with \( V_K \) as above and the solid torus \( \mathbb{T} \) unknotted.

Assume that \( B \) is endowed with a Reeb vector field tangent to the boundary with contact form \( \alpha \), such that near the boundary we can introduce coordinates \( (x, y, t) \) for which

\[
X = \frac{C}{\tau(t)} \frac{\partial}{\partial x} + \tau(t) \frac{\partial}{\partial y},
\]

\[
\alpha = \frac{\tau(t)}{2C} dx + \frac{1}{2\tau(t)} dy + A(t) dt,
\]

for \( \tau \) and \( A \) smooth functions and \( C \) a constant depending on the connected component of \( \partial B \). Then \( X \) has a periodic orbit in \( B \) that is unknotted in \( S^3 \).

**Proof.** We will prove the theorem case by case. Let us assume that \( X \) has no \( S^3 \)-unknotted periodic orbits. In particular, if the orbits in the boundary of \( B \) are closed, that is when the slope of \( X \) restricted to \( \partial B \) is a rational number, they have to be knotted. The slope of \( X \) in a boundary component is equal to \( \frac{C}{\tau(t)^2} \). If a boundary component is unknotted, and the orbits on it are periodic, the orbits describe torus knots which are non-trivial if \( \frac{C}{\tau(t)} \neq \frac{1}{n} \) for \( n \in \mathbb{Z} \) (we refer to [14]).

Let \( \xi_\alpha \) be the contact structure of \( \alpha \), then the characteristic foliation of \( \partial B \), denoted by \( (\partial B)_{\xi_\alpha} \), has slope \( -\frac{C}{\tau(t)} \).

**Proof for** \( B \simeq S^1 \times \mathbb{D}^2 \) unknotted.
The idea is to use two copies of $B$, plus a region $N \simeq T \times [r,s]$, to construct the lens space $L(1,q) \simeq S^3$ via surgery. Then, we will obtain $S^3$ equipped with a Reeb vector field without unknotted periodic orbits, giving us a contradiction to Theorem 2.6. The positive numbers $r$ and $s$ depend on the characteristic foliation of $\partial B$ and the surgery.

By a theorem J. Moser and A. Weinstein we know that: *Two contact structures that induce the same characteristic foliation on a surface are contactomorphic in a neighborhood of the surface* [7]. That is, the characteristic foliation of a surface defines the contact structure in a neighborhood of the surface.

Consider two copies $B_1$ and $B_2$ of $B$, and let $T_1 = B_1 \cup N$ and $T_2 = B_2$. Since $\partial N = T \times \{r,s\}$ we will paste $T_1$ to the boundary of $B_1$. We are going to form $L(1,p) = T_1 \cup \Psi T_2$, with $\Psi : \partial T_2 \to \partial T_1$ defined by a matrix
\[
\begin{pmatrix}
1 & b \\
p & a
\end{pmatrix} \in PSL(2,\mathbb{Z}).
\]
We have to find the appropriate values for $r, s, p, a$ and $b$.

Let us start by finding a suitable $N$ and a contact form $\lambda$ on it. We will assume that $C > 0$.

When $C < 0$, the proof is the same except we need to change the model contact form $\alpha_R$ for
\[
\tilde{\alpha}_R = \frac{1}{\sin^2 t + \frac{1}{R^2} \cos^2 t} \left( \sin^2 t dy - \cos^2 t dx \right).
\]
Hence under the hypothesis that $C > 0$, we need $\lambda$ to satisfy the following conditions:

(1) The slope of $(T_r,\xi_\lambda)$ has to coincide with the one of the characteristic foliation of $\partial B_1$. The latter foliation has slope $m_r = -\frac{C}{\tau(t)} < 0$, for the appropriate $t$, and hence we can find $r \in [0,\frac{\pi}{2})$ such that $m_r$ equal to $-\tan^2 r$. The Reeb vector field $X|_{T_r}$ has to have slope $-\frac{C}{\tau(t)} > 0$. We are going to take $\lambda$ near $T \times \{r\}$ equal to the model contact structure $\alpha_R$, hence we choose $R > 0$ such that $\frac{1}{R^2} = \frac{C}{\tau(t)}$. Since $X$ has no unknotted periodic orbits, there exists $n \in \mathbb{Z}$ such that
\[
\frac{1}{n+1} < \frac{1}{R^2} < \frac{1}{n}.
\]

(2) In order to perform surgery, we need that $\lambda$ induces the same characteristic foliation on $T_s$ as the pull back under $\Psi$ of $\alpha|_{\partial T_2}$. Hence, the slope of the characteristic foliation $(T_s,\xi_\lambda)$ has to be
\[
m_s = -\frac{1-pm_r}{b-am_r},
\]
and the slope of $X|_{T_s}$ has to be $\frac{p-R^2}{R^2b-a}$. We can easily choose $p, a$ and $b$ such that $m_s < m_r < 0$ and
\[
\frac{1}{n+1} < \frac{p-R^2}{R^2b-a} < \frac{1}{n}.
\]
Take $s \in (r,\frac{\pi}{2})$ such that $m_s = -\tan^2 s$ and near $T \times \{s\}$, we will take $\lambda$ equal to the model contact form $\alpha_S$ with
\[
\frac{1}{S^2} = \frac{p-R^2}{R^2b-a} > 0.
\]

(3) $X_\lambda$ without unknotted periodic orbits.
Let \( g : [r, s] \to [0, 1] \) such that \( g \) is equal to 1 for \( t \) near \( r \) and equal to zero for \( t \) near \( s \), and \( g' \leq 0 \). Define
\[
\lambda = \frac{1}{\sin^2 t + \left( \frac{g(t)}{R^2} + \frac{1-g(t)}{S^2} \right) \cos^2 t} \left( \sin^2 t dy + \cos^2 t dx \right),
\]
then
\[
d\lambda = \frac{\sin t \cos t}{\left[ \sin^2 t + \left( \frac{g(t)}{R^2} + \frac{1-g(t)}{S^2} \right) \cos^2 t \right]^2} \left[ \left( 2 + g'(t) \left( \frac{1}{R^2} - \frac{1}{S^2} \right) \sin t \right) dx \wedge dt - \left( 2 \left( \frac{g(t)}{R^2} + \frac{1-g(t)}{S^2} \right) - g'(t) \left( \frac{1}{R^2} - \frac{1}{S^2} \right) \sin t \cos t \right) dy \wedge dt \right].
\]
Hence \( \lambda \) is a contact form on \( N \), that is equal to \( \alpha_R \) near one boundary component and to \( \alpha_S \) near the other boundary component. Moreover, its Reeb vector field \( X_\lambda \) is tangent to the tori \( T_t \), for all \( r \leq t \leq s \) and its slope is strictly between \( \frac{1}{n+1} \) and \( \frac{1}{n} \). Then, the periodic orbits of \( X_\lambda \) are knotted.

Now we can use \( \Psi \), to paste \( T_1 \) and \( T_2 \), obtaining a contact form on \( S^3 \) such that all the periodic orbits are knotted. This gives us the contradiction proving case (i).

**Proof for** \( B \simeq S^3 \setminus V_K \).

Recall that \( B \) is embedded in \( S^3 \). Assume that \( X \) has no \( S^3 \)-unknotted periodic orbit. Consider a torus \( T \) in the boundary of \( B \), that in \( S^3 \) bounds a solid torus in the complement of \( B \).

There exist numbers \( R \) and \( r > 0 \) such that \( \mathbb{T}_r = \{(x, y, t) \in S^1 \times D^2 \mid t \leq r \} \) equipped with the contact form \( \alpha_R \) is such that the characteristic foliation of its boundary and the Reeb vector field coincide with those on \( T \). Hence, there is a contactomorphism of a neighborhood of \( \partial \mathbb{T}_r \) to a neighborhood of \( T \).

By hypothesis, all the periodic orbits on \( \mathbb{T}_r \) are \( S^3 \)-knotted, hence we obtain a contact form in \( B \) union the solid torus \( \mathbb{T}_r \) whose Reeb vector field has no unknotted periodic orbits. Observe that we can repeat this construction on every boundary component of \( B \), obtaining a Reeb vector field on \( S^3 \) without unknotted periodic orbits. Theorem 2.6 gives us the contradiction we are looking for.

**Proof for** \( B \simeq T \setminus V_K \).

The proof in this case is a combination of both cases above. First, we fill up \( V_K \) as in the proof of (ii) to obtain an unknotted solid torus satisfying the hypothesis of case (i). Then with two copies of the solid torus obtained we construct \( S^3 \) endowed with a Reeb vector whose periodic orbits are knotted, obtaining a contradiction.

This finishes the proof of Theorem 4.1. \( \square \)

### 4.2 Existence of an unknotted periodic orbit for geodesible vector fields

In this section we will prove Theorem 1.2 when \( X \) is a volume preserving geodesible vector field on \( S^3 \). The strategy is to prove that we can find an invariant manifold with boundary satisfying Theorem 4.1. Recall that when \( X \) is a geodesible vector field that preserves a volume there is a first integral \( h \), and we have three possible situations:

(I) \( h \neq 0 \) that implies that \( X \) is the Reeb vector field of the 1-form \( \alpha = \iota_X g; \)
(II) $h$ identically zero, this implies that $X$ is a suspension and $M$ fibers over $\mathbb{S}^1$. Hence, this case cannot occur in $\mathbb{S}^1$;

(III) $A = h^{-1}(0)$ non empty and different from $\mathbb{S}^3$.

H. Hofer, K. Wysocki and E. Zehnder’s Theorem guarantees the existence of an unknotted periodic orbit in case (I). So we will deal with case (III).

Proof of Theorem for case (III). Assume, by contradiction, that $X$ is a volume preserving geodesible vector field that has no unknotted periodic orbits. There are two cases we need to consider: When zero is a regular value and when zero is a critical value of $h$.

Let us start by picking an $\epsilon > 0$ such that $\pm \epsilon$ are regular values of the function $h$ and analyze $X$ and $\alpha$ near $h^{-1}(\pm \epsilon)$. Consider $0 < \delta < \epsilon$ such that all the values in the intervals $[-\epsilon - \delta, -\epsilon]$ and $[\epsilon, \epsilon + \delta]$ are regular and let

$$D = h^{-1}([-\epsilon - \delta, \epsilon + \delta]) \quad B = h^{-1}([-\epsilon, \epsilon]).$$

Hence $D \setminus B$ is foliated by invariant tori, let $D$ be a connected component of $D \setminus B$ and assume, without loss of generality, that $h_{|D} > 0$. We can introduce coordinates $(x, y, t)$ in $D$ such that $h(x, y, t) = \epsilon + t$, and denote each invariant tori by $T_t$.

Explicit expression for $X$ in $D$.

On each torus $T_t$ there exists a non-singular vector field $Z$ defined by the equations

$$\alpha(Z) = 0 \quad \text{and} \quad \iota_{Z \times \mu} = dh.$$

Observe that $Z$ is tangent to the tori and is in the plane field $\xi = \ker(\alpha)$. Since $X$ has no unknotted periodic orbits, $X|_{T_t}$ has no meridians as orbits and by geodesibility it has no tangent Reeb annuli (we refer to [3]). Hence, $X|_{T_t}$ admits a transverse circle that is the same on each torus of $D$, for $\delta$ small enough. For a proof of the following lemma we refer to Lemma 4.3 of [13].

Lemma 4.2 There are $C^\infty$ functions $a_1, a_2, a_3, a_4$ defined on $[0, \delta]$ such that the vector fields

$$a_1(t)X + a_2(t)Z \quad \text{and} \quad a_3(t)X + a_4(t)Z,$$

are linearly independent on $T_t$ and all their orbits are periodic of period one.

Assume that the circles $\{x = \text{const.}\}$ are the meridians, then

$$X = \tau_1(t)\frac{\partial}{\partial x} + \tau_2(t)\frac{\partial}{\partial y},$$

with $\tau_2(t) \neq 0$. If $\frac{\tau_1(t)}{\tau_2(t)}$ is rational, all the orbits are periodic and give us torus knots.

Explicit expression for $\alpha$ in $D$.

Let $\alpha = A_1 dx + A_2 dy + A_3 dt$ for some functions $A_1, A_2$ and $A_3$ on $D$. Since the kernel of $\alpha$ is generated by $Z$ and the gradient of $h$ in $D$, we conclude that $A_1, A_2$ are functions depending only on $t$, and $A_3 = 0$. Hence, we can write

$$\alpha = \frac{dx}{2\tau_1(t)} + \frac{dy}{2\tau_2(t)}.$$

Since $L_X \alpha = 0$ we get that

$$0 = L_X \left( \alpha \left( \frac{\partial}{\partial \ell} \right) \right) = -\alpha \left( \tau_1'(t)\frac{\partial}{\partial x} + \tau_2'(t)\frac{\partial}{\partial y} \right) = -\frac{\tau_1'(t)}{2\tau_1(t)} - \frac{\tau_2'(t)}{2\tau_2(t)},$$

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then $\tau_1(t)\tau_2(t) = C$ a nonzero constant. For simplicity in the rest of the proof we are going to write $\tau(t) = \tau_2(t)$, and

$$
\alpha = \frac{\tau(t)dx}{2C} + \frac{dy}{2\tau(t)}.
$$

**Explicit expression for $\iota_X\mu$.**

In this coordinate system we can write $\mu = \beta(x, y, t)dx \wedge dy \wedge dt$, with $\beta$ a positive function. From the identity $d\alpha = \iota_X\mu$ we get that $\beta$ is a function depending only on $t$, and

$$
\tau'(t) \left( \frac{dy}{2\tau(t)^2} - \frac{dx}{2C} \right) \wedge dt = (\epsilon + t)\beta(t) \left( \frac{C}{\tau(t)^2}dy - \tau(t)dx \right) \wedge dt,
$$

implying that $\frac{\tau''(t)}{\tau(t)} = 2C(\epsilon + t)\beta(t)$.

**Case where zero is a regular value of $h$.**

Since zero is a regular value, the set $A = h^{-1}(0)$ is a finite family of disjoint tori. Consider $\epsilon > 0$ such that all the values in $(-\epsilon, \epsilon]$ are regular, then each connected component of $B = \{h^{-1}([-\epsilon, \epsilon])\}$ is diffeomorphic to $T \times I$. Observe that $X$ restricted to any connected component of $S^3 \setminus B$ is a Reeb vector field tangent to the boundary. The following proposition gives us a connected component satisfying the hypothesis of Theorem 4.3, implying the existence of an unknotted periodic orbit of $X$.

**Proposition 4.3** Let $\Delta \subset S^3$ be a finite family of two by two disjoint two dimensional tori. Then there exists a connected component of $S^3 \setminus \Delta$ with closure $B$ that is diffeomorphic to one of the following:

- (i) $T \simeq S^1 \times D^2$ unknotted;
- (ii) $S^3 \setminus V_K$, where $V_K$ is a tubular neighborhood of a link $K$ such that any of its components is a non-trivial knot;
- (iii) $T \setminus V_K$, with $V_K$ as above and the solid torus $T$ unknotted.

**Proof.** We will proceed by induction on $|\Delta|$. If $|\Delta| = 1$ the torus is either knotted or not. In the first case we get two connected components of type (i), and in the second case we obtain (ii).

Let us assume that the proposition is true for $|\Delta| = k$, and prove it for $k + 1$. Assume first that there is $T_0 \in \Delta$ unknotted and let $\Delta' = \Delta \setminus T_0$. By hypothesis, there exist $B \in S^3 \setminus \Delta'$ satisfying the proposition. If $B \cap T_0 = \emptyset$ we are done, if not let us analyze each case:

- (i) When $B$ is an unknotted solid torus. Since $S^3 \setminus T_0$ is composed by two solid tori, one is contained in $B$, hence there exists a connected component $A$ of $S^3 \setminus \Delta$ that is an unknotted solid torus.
- (ii), (iii) When $B \simeq S^3 \setminus V_K$ or $B \simeq T \setminus V_K$. Let $A_1$ and $A_2$ be the two connected components of $B \setminus T_0$. If one of them is such that $A_i \cap \Delta' = \emptyset$ we obtain a connected component $A \simeq S^1 \times D^2$ unknotted. If not either $A_1$ or $A_2$ have its boundary composed by $T_0$ and knotted tori, hence $S^3 \setminus \Delta$ has a connected component of the third type.

We have to consider now the case where all the tori in $\Delta$ are knotted. Take one torus $T_0$ and define $\Delta'$ as before. By hypothesis there is a connected component $B$ of $S^3 \setminus \Delta'$ whose closure is diffeomorphic to $S^3 \setminus V_K$ and $T_0$ is the boundary of a knotted solid torus $T_0$. Let $A = B \setminus T_0$, then $A$ is the sphere minus the tubular neighborhood of a link $K$ as in (ii).
Case where zero is a critical value of \(h\): Construction of a closed 1-form in \(B\) transverse to \(X\).

With the local expressions for the forms \(\alpha\) and \(\iota_X\mu\), we will construct a closed 1-form in \(B\) transverse to \(X\). This will imply that \(X\) restricted to \(B\) admits a section and \(B\) fibers over \(S^1\). Before finding such a 1-form, let us explain the reason why this is enough to prove Theorem 1.2 in one of the following types:

- (i) \(\mathbb{T} \simeq S^1 \times \mathbb{D}^2\) unknotted;
- (ii) \(S^3 \setminus V_K\), where \(V_K\) is a tubular neighborhood of a link \(K\), such that any of its components is a non-trivial knot;
- (iii) \(T \setminus V_K\), with \(V_K\) as above and the solid torus \(\mathbb{T}\) unknotted.

If \(X|_B\) is a Reeb vector field, the existence of an \(S^3\) unknotted periodic orbit is guaranteed by Theorem 4.1. If \(X|_B\) is not a Reeb vector field, then we have a closed 1-form in \(B\) that is transverse to \(X\), in other words \(X\) is obtained by a suspension of a diffeomorphism of a disc and we are in case (i). By Brouwer’s fixed point theorem, we have a periodic orbit that is \(S^3\)-unknotted since the solid torus \(B\) is unknotted.

Hence to finish the proof Theorem 1.2 in the geodesible case, we have to construct the closed 1-form in \(B\). The procedure is similar to the one we used to prove Lemma 3.3.

Take a \(C^\infty\) function \(p : [0, 1] \to [0, 1]\) such that \(p(s) = 1\) for \(s < \frac{1}{3}\), for \(s > \frac{2}{3}\) we set \(p(s) = 0\), and \(p'(s) \leq 0\). Define a 1-form \(\tilde{\alpha}\) in a connected component \(D\) of \(\mathcal{D} \setminus \mathcal{B}\) as

\[
\tilde{\alpha} = \frac{1}{2C} \left[ \tau(0) + p \left( \frac{t}{\delta} \right) \left( \tau(t) - \tau(0) \right) \right] dx + \frac{1}{2} \left[ \frac{1}{\tau(0)} + p \left( \frac{t}{\delta} \right) \left( \frac{1}{\tau(t)} - \frac{1}{\tau(0)} \right) \right] dy,
\]

for \(t \in [0, \delta]\). We can define this form in each component of \(\mathcal{D} \setminus \mathcal{B}\) and extended it by \(\alpha\) in \(\mathcal{B}\), since \(\alpha = \tilde{\alpha}\) when \(t \sim 0\). We have that

\[
d\tilde{\alpha} = \beta(t) \left( \frac{Ch_1(t)}{\tau(t)} dy - h_2(t) \tau(t) dx \right) \wedge dt
\]

\[
h_1(t) = \frac{1}{2} \left( \frac{1}{C \delta \beta(t) \tau(t)} p' \left( \frac{t}{\delta} \right) \left( \tau(t) - \tau(0) \right) + 2p \left( \frac{t}{\delta} \right) \left( \epsilon + t \right) \right)
\]

\[
h_2(t) = \frac{1}{2} \left( \frac{1}{C \delta \beta(t) \tau(t)} p' \left( \frac{t}{\delta} \right) \left( \tau(t) - \tau(0) \right) + 2p \left( \frac{t}{\delta} \right) \left( \epsilon + t \right) \right).
\]
Observe that $h_1(0) = \epsilon = h_2(0)$ and $h_1(\delta) = 0 = h_2(\delta)$. We claim that there are positive constants $C_1$ and $C_2$ such that $|h_1| \leq C_1 \epsilon$. First observe that in the region $D$ we have that

$$p \left( \frac{t}{\delta} \right) (\epsilon + t) < 2\epsilon.$$ 

If we choose $\delta$ small enough we can assume that $\beta(s) \tau(s) \leq 2\beta(t) \tau(t)$ for every $s \in [0, t]$ and $t \leq \delta$. Then

$$|\tau(t) - \tau(0)| = 2C \left| \int_0^t (\epsilon + s) \beta(s) \tau(s) ds \right| \leq 4C |\beta(t) \tau(t) \left( et + \frac{t^2}{2} \right) | \leq 6C |\beta(t) \tau(t)| \epsilon \delta,$$

proving the claim.

Recall that we are looking for a 1-form in $B$ such that it is closed and transverse to $X$. We will extend $d \tilde{\alpha}$ to a 2-form $\omega$ on $M$ such that on $D$ we have $\|\omega\| \leq D \epsilon$ for a constant $D > 0$, and study the cohomology class of $\omega$ to find a 1-form different from $\tilde{\alpha}$ and such that its derivative is equal to $\omega$ in $B$.

**The cohomology class of $\omega$ on $M$.**

Consider the exact sequence of homologies with real coefficients

$$\cdots \to H_1(M \setminus A) \to H_1(M) \to H_1(M, M \setminus A) \to \cdots$$

where $A = \{ h^{-1}(0) \}$. Consider a finite collection of embedded curves $\sigma_1, \sigma_2, \ldots, \sigma_n$ in $M \setminus A$ such that they form a basis for the kernel of the map $H_1(M) \to H_1(M, M \setminus A)$. These curves are at positive distance from $A$, then for $\epsilon$ small enough we can assume that the $\sigma_i$ are at positive distance from $B$.

Using the duality of Poincaré, for every $i = 1, 2, \ldots, n$, we can find a 2-form $\omega_i$ that is the dual of $\sigma_i$ and whose support is contained in a tubular neighborhood of $\sigma_i$ contained in $M \setminus B$.

**Lemma 4.4** For $\epsilon$ small enough there are unique real numbers $r_1, r_2, \ldots, r_n$ such that

$$[\omega] = \sum_{i=1}^{n} r_i [\omega_i]$$

in $H^2(M)$. Moreover, there exists a constant $C'$ independent of $\epsilon$ such that $|r_i| \leq C' \epsilon$ for every $i$.

**Proof.** For $\epsilon$ small we can assume that $B$ does not intersect the supports of the forms $\omega_i$. Denote by

$$f_1 : H_1(M) \to H_1(M, M \setminus B)$$
$$f_2 : H^2(M) \to H^2(B).$$

Using the isomorphism given by the duality of Poincaré we have a map $\ker(f_1) \to \ker(f_2)$ that is bijective as we proved in Lemma 3.5. Recall that $\omega$ is exact and non-zero in $B$, hence we obtain the existence and uniqueness of the numbers $r_i$.

For $i = 1, 2, \ldots, n$ fix an oriented embedded surface $S_i$ in $M$ that intersects the $\sigma_j$. Then

$$r_i = \int_{S_i} \sum_{j=1}^{n} r_j \omega_j = \int_{S_i} \omega.$$
Using the bound on $\omega$ we get a constant $C'$ that is independent of $\epsilon$ and such that $|r_i| \leq C' \epsilon$.

The differential 2-form given by $\gamma = \omega - \sum_{i=1}^{n} r_i \omega_i$ is closed and exact in $M$. Lemma 3.6 implies that there exists a 1-form $\lambda$ such that $d\lambda = \gamma$ and $\|\lambda\| \leq \hat{C}\|\gamma\| \leq \hat{C}C'\epsilon$, where $\hat{C}$ is a constant independent of $\epsilon$. Thus the 1-form $\alpha - \lambda$ satisfies that $d(\alpha - \lambda) = h(X_{\mu} - \omega + \sum_{i=1}^{n} r_i \omega_i)$, is equal to zero in $B$, and $(\alpha - \lambda)(X) > 0$ as a consequence of the bounds we found and the fact that they are independent of $\epsilon$. Then this is the 1-form we were looking for: A closed 1-form in $B$ that is transverse to $X$. This finishes the proof of the theorem.

4.3 Existence of an unknotted periodic orbit for Euler vector fields

In this section we will sketch the proof of Theorem 1.2 for non-geodesible Euler vector fields. By hypothesis, $\iota(\text{curl}(X))X_{\mu} = df$, for a non-constant function $f$ on $S^3$. As in Section 3, let $\mathcal{C}$ be the set of critical values of $f$ and consider $\Sigma$ the union of the connected components in $\{f^{-1}(c) | c \in \mathcal{C}\}$ where $df = 0$. First, Lemma 3.2 gives us the existence of neighborhood $B$ of $\Sigma$ where $X$ is geodesible.

Moreover, $\partial B$ is a finite collection of invariant tori and the connected components of $S^3 \setminus B$ are all diffeomorphic to $T \times I$.

Assume that $X$ restricted to $S^3 \setminus B$ has no unknotted periodic orbits. Proposition 4.3 implies that one of the connected components of $B$ is of one of the types listed. Let us call such a connected component $B$. If $X|_B$ is a Reeb vector field we finish, hence assume that $X|_B$ is not a Reeb vector field. Then $X|_B$ is a volume preserving geodesible vector field, and repeating the arguments in Section 4.2 we conclude that $X|_B$ has an $S^3$-unknotted periodic orbit.

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