THE COLLAPSIBILITY OF SOME CAT(0) SIMPLICIAL COMPLEXES OF DIMENSION 3

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Abstract. We study the collapsibility of finite simplicial complexes of dimension 3 endowed with a CAT(0) metric. Our main result states that, under an additional hypothesis, finite simplicial 3-complexes endowed with a CAT(0) metric collapse to a point through CAT(0) subspaces.

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1. Introduction

In this paper we find a sufficient condition for the collapsibility of a particular class of finite simplicial complexes of dimension 3. Namely, we show that the existence of a CAT(0) metric guarantees the collapsibility of those complexes which satisfy a so called Property A. Roughly, Property A refers to preserving the strongly convex metric on a subcomplex obtained by performing an elementary collapse on a finite CAT(0) 3-complex. Property A imposes restrictions only when deleting a 3-simplex by starting at its free face. A similar restriction is not encountered when deleting a 2-simplex by starting at its free face.

The collapsibility of finite simplicial complexes was studied before. In [Whi70] it is shown that finite, strongly convex simplicial complexes of dimension 2 are collapsible, whereas in dimension 3 such complexes collapse to a 2-dimensional spine. It is the paper’s object to show that in dimension 3 a stronger metric condition given by the CAT(0) metric, ensures, under additional assumptions, collapsibility not only to a spine of dimension 2, but even to a point.

Using discrete Morse theory (see [For98]), Crowley proved in 2008, under a technical condition, that nonpositively curved simplicial complexes of dimension 3 or less endowed with the standard piecewise Euclidean metric, collapse to a point (see [Cro08]). She constructed a CAT(0) triangulated disk by endowing it with the standard piecewise Euclidean metric and requiring that each of its interior vertices has degree at least 6. The naturally associated standard piecewise Euclidean metric on the disk became then CAT(0).

Adiprasito and Benedetti extended Crowley’s result to all dimensions (see [AB13], Theorem 3.2.1). Namely, they proved using discrete Morse theory that every complex that is CAT(0) with a metric for which all vertex stars are convex, is collapsible. It is important to note that, although the 3-complexes in our paper are also CAT(0) spaces, they are no longer necessarily endowed with the standard piecewise
Euclidean metric like the ones in Crowley’s and Adiprasito and Benedetti’s papers. Still, they can also be collapsed to a point.

In \cite{BL14} we show further, using again discrete Morse theory, that systolic simplicial complexes (see \cite{JS06}) are also collapsible. Moreover, we prove that both systolic and CAT(0) locally finite simplicial complexes possess an arborescent structure. The collapsibility of systolic simplicial complexes is also proven by Chepoi and Osajda in \cite{CO15} (see Corollary 4.3).

It is known that in dimension 2 the CAT(0) metric guarantees the collapsibility, through CAT(0) subspaces, of finite simplicial complexes, not necessarily endowed with the standard piecewise Euclidean metric and whose interior vertices do not necessarily have degree at least 6 like the ones in Crowley’s paper (see \cite{Laz10}, chapter 3.1, page 35). In this paper we extend this result to dimension 3. Namely, we show that, in certain circumstances, finite, CAT(0) simplicial 3-complexes can be collapsed to a point through subspaces which are, at each step of the retraction, endowed with a CAT(0) metric. The result in dimension 3 works only under an additional Property A given below.

**Property A.** Let $K$ be a finite CAT(0) simplicial 3-complex and let $\sigma$ be a 3-simplex of $K$ with a free $n$-face $\alpha$, $1 \leq n \leq 2$. Let $K' = K \setminus \{\sigma, \alpha\}$ be the subcomplex obtained by performing an elementary collapse on $K$. Let $p, q$ be two points of $K$ which do not belong to $\sigma$ such that the geodesic segment $[p, q]$ intersects the interior of $\sigma$. Let $U$ be a small neighborhood of some vertex of $\sigma$ such that $\sigma$ is included in $U$. Let $U' = U \setminus \{\sigma, \alpha\}$. Then in $U'$ there do not exist two geodesic segments $\gamma_1, \gamma_2$ of equal length joining $p$ to $q$ such that $\gamma_1$ intersects one, while $\gamma_2$ intersects one or two of the three boundary edges of $\sigma$ which differ from any of the boundary edges of $\alpha$ (if $\alpha$ is 2-dimensional) or from $\alpha$ itself (if $\alpha$ is 1-dimensional).

Note that in general a finite CAT(0) 3-complex can not be simplicially collapsed to a point because once performing the first elementary collapse on the complex, the subcomplex we obtain does not inherit the strongly convex metric. This happens because the situation we exclude by imposing Property A on the complex, may in general occur. Our proof relies on the definition of an elementary collapse. It uses basic properties of CAT(0) spaces (see \cite{BH99}, \cite{BBI01}, \cite{Ale55}) and one of White’s results given in \cite{Whi70}. Namely, because CAT(0) spaces have a strongly convex metric, finite, CAT(0) 3-complexes have, according to White, a 3-simplex with a free face. One can therefore perform an elementary collapse on such complex. We show that the subcomplex obtained by performing an elementary collapse on a CAT(0) 3-complex enjoying Property A remains, at any step of the retraction, nonpositively curved. An important issue to solve will be to find the new geodesic segments in the neighborhood of each point of the subcomplex obtained by performing any step of the elementary collapse.

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2. Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

Let $(X, d)$ be a metric space. Let $a, b \in \mathbb{R}$ such that $[a, b]$ is a real interval. A geodesic path joining $x \in X$ to $y \in X$ is a path $c : [a, b] \to X$ such that $c(a) = x$, $c(b) = y$, $c([a, b]) = \overline{xy}$, and $d(c(t), c(s)) = |t - s|d(x, y)$ for all $t, s \in [a, b]$.
\[ c(b) = y \text{ and } d(c(t), c(t')) = |t - t'| \text{ for all } t, t' \in [a, b]. \] The image of \( c \) is called a \textit{geodesic segment} with endpoints \( x \) and \( y \). Since geodesic segments in \( \mathbb{R} \) are just closed intervals, this is a legitimate abuse of notation.

A \textit{geodesic metric space} \((X, d)\) is a metric space in which every pair of points can be joined by a geodesic segment. We denote any geodesic segment from a point \( x \) to a point \( y \) in \( X \), by \([x, y]\).

Given a path \( c : [0, 1] \rightarrow X \), its \textit{length} is defined by
\[
 l(c) = \sup \{ \sum_{i=1}^{n} d(c(t_{i-1}), c(t_i)) \},
\]
where the supremum is taken over all possible subdivisions of \([0, 1]\). \( 0 = t_0 < t_1 < \ldots < t_n = 1 \).

Let \((X, d)\) be a geodesic metric space. A \textit{geodesic triangle} in \( X \) consists of three distinct points \( x_1, x_2, x_3 \in X \), called \textit{vertices}, and a choice of three geodesic segments joining them, called \textit{sides}. Such a geodesic triangle is denoted by \( \triangle = \triangle(x_1, x_2, x_3) \). If a point \( a \in X \) lies in the union of \([x_1, x_2], [x_2, x_3] \) and \([x_3, x_1]\), then we write \( a \in \triangle \). A triangle \( \overline{\triangle} = \triangle(x_1, x_2, x_3) = \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3}) \) in \( \mathbb{R}^2 \) is called a \textit{comparison triangle} for \( \triangle \) if \( d(x_1, x_j) = d_{\mathbb{R}^2}(\overline{x_1}, \overline{x_j}) \), \( i, j \in \{1, 2, 3\} \). A point \( \overline{a} \in [\overline{x_1}, \overline{x_2}] \) is called a \textit{comparison point} for \( a \in [x_1, x_2] \) if \( d(x_1, a) = d_{\mathbb{R}^2}(\overline{x_1}, \overline{a}) \).

The interior angle of \( \overline{\triangle} \) at \( \overline{x_1} \) is called the \textit{comparison angle} between \( x_2 \) and \( x_3 \) at \( x_1 \). A \textit{tetrahedron} in \( X \) is the union of four geodesic triangles any two of which have exactly one side in common.

Let \( c, c', c'' \) be three geodesic paths in \( X \) issuing from the same point \( x \). The \textit{Aleksandrov angle} between \( c \) and \( c' \) at \( x \) is defined as
\[
 \angle(c, c') = \limsup_{s, t \to 0} \overline{\angle}(c(s), c'(t)) \in [0, \pi], \quad s, t \in [0, 1],
\]
where \( \overline{\angle}(c(s), c'(t)) \) is the angle at the vertex corresponding to \( x \) in a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(x, c(s), c(t)) \) in \( X \). The following inequality holds
\[
 \angle(c', c'') \leq \angle(c, c') + \angle(c, c''),
\]
(for the proof see [BH99], chapter I.1, page 10). Aleksandrov angles in \( \mathbb{R}^2 \) are the usual Euclidean angles.

Let \( \Delta = \Delta(p, q, r) \) be a geodesic triangle in a convex metric space \((X, d)\) and let \( \alpha, \beta, \gamma \) denote the Aleksandrov angles between the sides of \( \Delta \). We define the \textit{curvature of} \( \Delta \) by \( \omega(\Delta) = \alpha + \beta + \gamma - \pi \). Any geodesic triangle in \( X \) of curvature zero is isometric to its comparison triangle in \( \mathbb{R}^2 \) (for the proof see [Ale55], chapter V.6, page 218).

Let \((X, d)\) be a metric space. We call \( X \) a \textit{CAT(0) space} if it is a geodesic space of whose geodesic triangles satisfy the so-called CAT(0) inequality. Namely, for any geodesic triangle \( \Delta \subset X \) and for any \( x, y \in \Delta \),
\[
 d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}),
\]
where \( \overline{x}, \overline{y} \in \overline{\Delta} \) are the corresponding comparison points in the comparison triangle \( \overline{\Delta} \) of \( \Delta \) in \( \mathbb{R}^2 \). We call \( X \) \textit{nonpositively curved} if it is locally a CAT(0) space, i.e. for every \( x \in X \), there exists \( r_x > 0 \) such that the ball \( B(x, r_x) \), endowed with the induced metric, is a CAT(0) space.

A \textit{subembedding} in \( \mathbb{R}^2 \) of a 4–tuple of points \((x_1, y_1, x_2, y_2)\) in \( X \) is a 4–tuple of points \((\overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2})\) in \( \mathbb{R}^2 \) such that \( d_{\mathbb{R}^2}(\overline{x_1}, \overline{y_1}) = d(x_1, y_1), i, j \in \{1, 2\}, (x_1, x_2) \leq d_{\mathbb{R}^2}(\overline{x_1}, \overline{x_2}) \) and \( d_{\mathbb{R}^2}(\overline{y_1}, \overline{y_2}) \). We say \( X \) satisfies the \textit{CAT(0) 4–point condition} if every 4–tuple of points \((x_1, y_1, x_2, y_2)\) in \( X \) has a subembedding in \( \mathbb{R}^2 \).
A metric space is a CAT(0) space if and only if it is a geodesic space and if, for each of its geodesic triangles \( \triangle \), the Aleksandrov angle at any vertex of \( \triangle \) is not greater than the corresponding angle in its comparison triangle \( \overline{\triangle} \) in \( \mathbb{R}^2 \) (for the proof see [BH99], chapter II.1, page 161). Any complete, CAT(0) space satisfies the CAT(0) 4-point condition (for the proof see [BH99], chapter II.1, page 164). Any complete, simply connected, nonpositively curved space is a CAT(0) space (for the proof see [BH99], chapter II.4, page 194).

Let \( (X, d) \) be a CAT(0) space. The distance function \( d : X \times X \to \mathbb{R} \) is convex (for the proof see [BH99], chapter II.2, page 176) and strongly convex (for the proof see [BH99], chapter II.1, page 161). Any CAT(0) space is contractible and hence simply connected (for the proof see [BH99], chapter II.1, page 160). The balls in \( X \) are convex spaces (for the proof see [BH99], chapter II.2, page 160). For every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that if \( m \) is the midpoint of a geodesic segment \([x, y] \subset X \) and if \( \max\{d(m, m'), d(y, m')\} \leq \frac{1}{2}d(x, y) + \delta \), then \( d(m, m') < \varepsilon \) (for the proof see [BH99], chapter II.1, page 160). For \( p, x, y \in X \), the geodesic segment \([x, y] \) is the union of the geodesic segments \([x, p] \) and \([p, y] \) if and only if \( \angle_p(x, y) = \pi \) (see [BH99], chapter II.1, page 163).

We will make frequent use of Aleksandrov’s Lemma given below (for the proof see [BH99], chapter I.2, page 25).

**Lemma 2.1.** Let \( a, b, c, d \) be points in \( \mathbb{R}^2 \) such that \( a \) and \( c \) are in different half-planes with respect to the line \( bd \). Consider a triangle \( \triangle(a', b', c') \) in \( \mathbb{R}^2 \) such that \( d_{R^2}(a, b) = d_{R^2}(a', b') \), \( d_{R^2}(b, c) = d_{R^2}(b', c') \), \( d_{R^2}(a, d) + d_{R^2}(d, c) = d_{R^2}(a', c') \) and let \( d' \) be a point on the segment \([a', c'] \) such that \( d_{R^2}(a, d) = d_{R^2}(a', d') \).

Then \( \angle_d(a, b) + \angle_d(b, c) < \pi \) if and only if \( d_{R^2}(b', d') < d_{R^2}(b, d) \). In this case, one also has \( \angle_{d'}(b', d') < \angle_d(a, d) \) and \( \angle_c(b', d') < \angle_d(b, d) \).

Furthermore \( \angle_d(a, b) + \angle_d(b, c) > \pi \) if and only if \( d_{R^2}(b', d') > d_{R^2}(b, d) \). In this case, one also has \( \angle_{d'}(b', d') > \angle_d(a, d) \) and \( \angle_c(b', d') > \angle_d(b, d) \).

Any one equality implies the others and occurs if and only if \( \angle_d(a, b) + \angle_d(b, c) = \pi \).

Let \( K \) be a simplicial complex and let \( \alpha \) be an \( i \)-simplex of \( K \). If \( \beta \) is a \( k \)-dimensional face of \( \alpha \) but not of any other simplex in \( K \), then we say there is an elementary collapse from \( K \) to \( K \setminus \{\alpha, \beta\} \). If \( K = K_0 \supseteq K_1 \supseteq \ldots \supseteq K_n = L \) are simplicial complexes such that there is an elementary collapse from \( K_{j-1} \) to \( K_j \), \( 1 \leq j \leq n \), then we say that \( K \) simplicially collapses to \( L \).

Let \( K \) be a finite, connected simplicial complex endowed with the standard piecewise Euclidean metric. We define the *standard piecewise Euclidean metric* on \(|K|\) by taking the distance between any two points \( x, y \) in \(|K|\) to be the infimum over all paths in \(|K|\) from \( x \) to \( y \). Each simplex of \( K \) is isometric with a regular Euclidean simplex of the same dimension with side lengths equal 1.

### 3. Collapsing Certain CAT(0) Simplicial Complexes of Dimension 3

In this section we prove that finite, CAT(0) simplicial 3-complexes satisfying Property A collapse to a point through CAT(0) subspaces. Our proof has two steps. Firstly, because CAT(0) spaces have a strongly convex metric, White’s result given in [Whi70] ensures that finite, 3-complexes endowed with a CAT(0) metric, have a 3-simplex with a free 2-dimensional (1-dimensional) face. So we may perform an elementary collapse on such complex. The second step is to investigate whether the
subcomplex obtained by performing an elementary collapse on a CAT(0) 3-complex remains, at each step of the retraction, nonpositively curved. We will be able to analyze whether such space still has locally a CAT(0) metric, only once we have found its new local geodesic segments.

We start by characterizing the curvature of a 2-simplex of a CAT(0) simplicial complex.

**Lemma 3.1.** Let $K$ be a simplicial complex. If $|K|$ admits a CAT(0) metric $d$, then any 2-simplex in $K$ is isometric to its comparison triangle in $\mathbb{R}^2$.

**Proof.** Let $\triangle(a,b,c)$ be a 2-simplex $\sigma$ of $K$ and let $d$ be a point on the edge $e = [b,c]$. Let $\triangle(a', b', d')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(a,b,d)$ in $|K|$ and let $\triangle(a', d', c')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(a, d, c)$ in $|K|$. We place the comparison triangles $\triangle(a', b', d')$ and $\triangle(a', d', c')$ in different half-planes with respect to the line $a'd'$ in $\mathbb{R}^2$.

Because any geodesic triangles in $|K|$ satisfies the CAT(0) inequality and $d \in [b,c]$, we have

$$\pi = \angle_d(b,c) \leq \angle_d(b,a) + \angle_d(a,c) \leq \angle_{d'}(b',a') + \angle_{d'}(a',c').$$

So, since $\angle_{d'}(b',a') + \angle_{d'}(a',c') \geq \pi$, Alexandrov’s Lemma implies

$$d_{\mathbb{R}^2}(a',d') \leq d(a,d).$$

But $\triangle(a', b', d')$ is a comparison triangle for the geodesic triangle $\triangle(a,b,d)$ in $|K|$ and therefore $d_{\mathbb{R}^2}(a',d') = d(a,d)$. Because one equality in Alexandrov’s Lemma implies the others, the following equalities hold $\angle_{d'}(b',a') + \angle_{d'}(a',c') = \pi$, $\angle_{b}(a,d) = \angle_{d'}(a',d')$, $\angle_{c}(a,d) = \angle_{d'}(a',d')$ and $\angle_{a}(b,d) + \angle_{a}(d,c) = \angle_{d'}(b',c').$ So the sum of the angles between the sides of $\sigma$ equals $\pi$. Therefore, because $|K|$ has a convex metric, the curvature of the 2-simplex $\sigma$ equals $\omega(\sigma) = \pi - \pi = 0$. So, since any 2-simplex in $K$ has curvature zero, any 2-simplex in $K$ is isometric to its comparison triangle in $\mathbb{R}^2$.

We shall use the following lemmas frequently.

**Lemma 3.2.** Let $(X,d)$ be a CAT(0) space. Then any path $c : [0,1] \rightarrow X$ in $X$ has a unique midpoint.

**Proof.** Let $t \in [0,1]$ be such that $l(c|_{[0,t]}) = l(c|_{[t,1]}) = \frac{1}{2}l(c|_{[0,1]})$. Because $X$ is a CAT(0) space, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that if

$$l(c|_{[0,t']}) = l(c|_{[t',1]}) = \frac{1}{2}l(c|_{[0,1]}) \leq \frac{1}{2}l(c|_{[0,1]}) + \delta,$$

$t' \in [0,1]$, then $d(c(t),c(t')) < \varepsilon$. So, because $d(c(t),c(t')) < \varepsilon$ for every $\varepsilon > 0$, $d(c(t),c(t')) = 0$. The path $c$ has therefore a unique midpoint.

**Lemma 3.3.** Let $(X,d)$ be a CAT(0) space and let $p,q,s,t$ be four distinct points in $X$ such that $\angle_s(p,t) + \angle_s(t,q) \geq \pi$. Then the following inequality holds $d(p,s) + d(s,q) < d(p,t) + d(t,q)$. 

Proof. Let $\triangle(p, t, s)$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, t, s)$ in $X$ and let $\triangle(q, t, s)$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(q, t, s)$ in $X$. We place the comparison triangles $\triangle(p, t, s)$ and $\triangle(q, t, s)$ in different half-planes with respect to the line $\overline{pq}$ in $\mathbb{R}^2$.

By the CAT(0) inequality,
\[
\angle_s(p, t) + \angle_s(t, q) \leq \angle_s(p, t) + \angle_s(t, q).
\]
So, by hypothesis, it follows that
\[
\angle_s(p, t) + \angle_s(t, q) \geq \pi.
\]

If in (1) we have equality, taking into account that $d_{\mathbb{R}^2}(t, s) \neq 0$, we get
\[
d_{\mathbb{R}^2}(p, q) = d_{\mathbb{R}^2}(p, s) + d_{\mathbb{R}^2}(s, q) < d_{\mathbb{R}^2}(p, t) + d_{\mathbb{R}^2}(t, q).
\]

Further
\[
d_{\mathbb{R}^2}(p, \overline{s}) + d_{\mathbb{R}^2}(\overline{s}, q) < \]
\[
d_{\mathbb{R}^2}(p, \overline{t}) + d_{\mathbb{R}^2}(\overline{t}, q) = d_{\mathbb{R}^2}(p, \overline{t}) + d_{\mathbb{R}^2}(\overline{t}, q).
\]

Hence
\[
d_{\mathbb{R}^2}(p, \overline{s}) + d_{\mathbb{R}^2}(\overline{s}, q) < d_{\mathbb{R}^2}(p, \overline{t}) + d_{\mathbb{R}^2}(\overline{t}, q).
\]
So the following inequality holds in $X$: 

\[
\angle_s(p, t) + \angle_s(t, q) \geq \pi.
\]
\[ d(p, s) + d(s, q) < d(p, t) + d(t, q). \]

\[ \square \]

**Lemma 3.4.** Let \( (X, d) \) be a CAT(0) space and let \( (s_n)_{n \in \mathbb{N}} \) and \( (t_n)_{n \in \mathbb{N}} \) be distinct sequences of points on a geodesic segment \( e \) in \( X \) such that \( \lim_{n \to \infty} d(s_n, t_n) = 0 \). Then for any point \( p \in X \) which does not lie on the geodesic segment \( e \), we have \( \lim_{n \to \infty} \angle_p(s_n, t_n) = 0 \).

**Proof.** We consider the comparison triangle \( \overline{\triangle}(p, s_n, t_n) \) in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, s_n, t_n) \) in \( X \). By hypothesis, it follows that

\[ \lim_{n \to \infty} d_{\mathbb{R}^2}(\overline{\triangle}(p, s_n, t_n)) = 0. \]

So

\[ \lim_{n \to \infty} \angle_p(s_n, t_n) = 0. \]

Because \( X \) is a CAT(0) space, we have

\[ 0 \leq \angle_p(s_n, t_n) \leq \overline{\angle}_p(s_n, t_n). \]

Hence

\[ \lim_{n \to \infty} \angle_p(s_n, t_n) = 0. \]

\[ \square \]

We fix, for the remainder of the paper, the following notations.

Let \( K \) be a finite simplicial 3-complex endowed with a CAT(0) metric \( d \) and satisfying Property A. Because \( K \) has a strongly convex metric, it has a 3-simplex \( \sigma \) with a free \( k \)-dimensional face \( a, k \in \{1, 2\} \). Let \( K' = K \setminus \{\alpha, \sigma\} \) be the subcomplex obtained by performing an elementary collapse on \( K \) and let \( d' \) be the induced metric on \( K' \). Let \( a, b, c \) and \( d \) be the vertices of the 3-simplex \( \sigma \). Let \( \tau_1, \tau_2 \) and \( \tau_3 \) be three 2-dimensional faces of \( \sigma \) different from the free face of \( \sigma \) (in case \( \sigma \) has a free 2-dimensional face). Let \( \tau_1 \cap \tau_2 = e_1 = [a, b] \), \( \tau_1 \cap \tau_3 = e_2 = [a, d] \) and \( \tau_2 \cap \tau_3 = e_3 = [a, c] \) be three edges of \( \sigma \) different from the free face of \( \sigma \) (in case \( \sigma \) has a free 1-dimensional face) or a face of the free face of \( \sigma \) (in case \( \sigma \) has a free 2-dimensional face). We denote by \( r = \max\{d(a, b), d(a, c), d(a, d), d(b, c), d(b, d), d(c, d)\} \). We consider in \( |K| \) a neighborhood of \( a \) homeomorphic to a closed ball of radius \( r \), \( U = \{x \in |K| \mid d(a, x) \leq r\} \). Note that \( U \) endowed with the induced metric is a CAT(0) space. Because \( U \) is complete and it has a strongly convex metric, any two points in \( U \) are joined by a unique geodesic segment which belongs to \( U \). So any geodesic triangle with vertices at any three points in \( U \) belongs to \( U \), and it satisfies the CAT(0) inequality. Furthermore, \( U \) being a CAT(0) space, any 2-simplex in \( U \) has curvature zero and it is therefore isometric to its comparison triangle in \( \mathbb{R}^2 \).

We consider in \( |K'| \) a neighborhood of \( a \) homeomorphic to a closed ball of radius \( r \), \( U' = \{x \in |K'| \mid d'(a, x) \leq r\} \). We note that \( U' = U \setminus \{\alpha, \sigma\} \). We consider in \( U \) two distinct points \( p \) and \( q \) that do not belong to \( \sigma \) such that the geodesic segment \( [p, q] \) intersects the interior of \( \tau_1 \) in \( p_1 \), and the interior of \( \tau_2 \) in \( q_1 \).

We study further whether \( K' \) still has locally a CAT(0) metric. Note that \( K' \) inherits Property A from \( K \). Namely, we will show that \( U' \) is a CAT(0) space. We consider only the case when \( K' \) is obtained by pushing in an entire 3-simplex with
The 3-simplex $\sigma$ in $K$ with the free $k$-dimensional face $\alpha$, $k \in \{1, 2\}$ intersected by the geodesic segment $[p, q]$

a free face, by starting at its free face. It is important to note, however, that the same result holds for any deformation retract of $|K|$ obtained by pushing in any tetrahedron $\delta$ in $|K|$ such that one face of $\delta$ belongs to the free face of $K$. We will be able to investigate whether any geodesic triangle in $U'$ satisfies the CAT(0) inequality, only once we have found the new geodesic segments in $U'$. We note that any two points in $U$ joined in $U$ by a segment which does not intersect the interior of $\sigma$, are joined in $U'$ by a segment that coincides with the segment that joins these points in $U$. So we still have to find the paths of shortest length in $U'$ joining those pair of points that are joined in $U$ by a segment that intersects the interior of $\sigma$. We shall be concerned with this problem in the following corollary and five lemmas.

**Lemma 3.5.** There exists a unique point $s$ on $c_1$ such that $\angle_s(a, p_1) = \angle_s(b, q_1)$ and $\angle_s(a, q_1) = \angle_s(b, p_1)$. For such point $s$ we have $\angle_s(p_1, t) + \angle_s(t, q_1) = \pi$, for any point $t$ on $c_1$ that differs from $s$. In particular, the following inequality holds $d(p_1, s) + d(s, q_1) < d(p_1, t) + d(t, q_1)$.

**Proof.** We show first the existence of such point $s$.

We consider the path $c_1 : [0, 1] \to U$, $c_1(0) = a, c_1(1) = b, c_1(t) \in e_1, \forall t \in (0, 1)$, i.e. the path $c_1$ is the edge $e_1$.

Note that for any $t \in (0, 1)$,

\[(2) \quad \angle_{c_1(t)}(a, p_1) + \angle_{c_1(t)}(p_1, b) = \angle_{c_1(t)}(a, q_1) + \angle_{c_1(t)}(q_1, b) = \pi.\]

So

\[(3) \quad \angle_{c_1(t)}(a, p_1) + \angle_{c_1(t)}(p_1, b) + \angle_{c_1(t)}(b, q_1) + \angle_{c_1(t)}(q_1, a) = 2\pi.\]

We call the points $c_1(t), t \in [0, 1]$ such that

\[\angle_{c_1(t)}(p_1, a) + \angle_{c_1(t)}(a, q_1) > \angle_{c_1(t)}(q_1, b) + \angle_{c_1(t)}(b, p_1),\]

points of type I. Relation (3) implies that for any point of type I we have: $\angle_{c_1(t)}(p_1, b) + \angle_{c_1(t)}(b, q_1) < \pi$ and $\angle_{c_1(t)}(q_1, a) + \angle_{c_1(t)}(a, p_1) > \pi$. Let $c_1(t_1), t_1 \in [0, 1]$ be the point of type I on $c_1$ such that $d(a, c_1(t_1)) < d(a, c_1(t_1'))$, for any point of type I $c_1(t_1'), t_1' \in [0, 1], t_1' \neq t_1$.

We call the points $c_1(t), t \in [0, 1]$ such that

\[\angle_{c_1(t)}(p_1, a) + \angle_{c_1(t)}(a, q_1) < \angle_{c_1(t)}(q_1, b) + \angle_{c_1(t)}(b, p_1),\]
points of type II. By (3), for any point of type II we have: \( \angle_{c_1(t)}(p_1, b) + \angle_{c_1(t)}(b, q_1) > \pi \) and \( \angle_{c_1(t)}(q_1, a) + \angle_{c_1(t)}(a, p_1) < \pi \). Let \( c_1(t_2), \ t_2 \in [0, 1] \) be the point of type II on \( c_1 \) such that \( d(b, c_1(t_2)) < d(b, c_1(t'_2)) \), for any point of type II \( c_1(t'_2), \ t'_2 \in [0, 1], t'_2 \neq t_2 \).

We call the points \( c_1(t), t \in [0, 1] \) such that
\[
\angle_{c_1(t)}(p_1, a) + \angle_{c_1(t)}(a, q_1) = \angle_{c_1(t)}(q_1, b) + \angle_{c_1(t)}(b, p_1),
\]
points of type III. Relation (3) implies that for any point of type III we have: \( \angle_{c_1(t)}(p_1, a) + \angle_{c_1(t)}(a, q_1) = \angle_{c_1(t)}(q_1, b) + \angle_{c_1(t)}(b, p_1) = \pi \). Note that, by (2), any point of type III fulfills the following \( \angle_{c_1(t)}(a, p_1) = \angle_{c_1(t)}(b, q_1) \) and \( \angle_{c_1(t)}(a, q_1) = \angle_{c_1(t)}(b, p_1) \).

Suppose that there are no points of type III on \( c_1 \). Any point on \( c_1 \) is therefore either a point of type I or a point of type II.

We define the mapping \( \text{mid} : c_1 \times c_1 \to c_1 \) by
\[
\forall t_1, t_2 \in [0, 1], \ \text{mid}(c_1(t_1), c_1(t_2)) = c_1(t), \ t \in [0, 1],
\]
where
\[
l(c_1|_{[t_1, t_2]}) = \frac{1}{2}l(c_1|_{[t_1, t_2]}).
\]
Because \( U \) is a CAT(0) space, Lemma 3.22 guarantees that the path \( c_1 \) has a unique midpoint. The mapping \( \text{mid} \) is therefore well-defined.

We define the sequence \((s_n)_{n \in \mathbb{N}}\) of tuples \((s'_n, s''_n)\) as follows:
- the elements \( s'_n \) are points of type I;
- the elements \( s''_n \) are points of type II;
- \( s_0 = (s'_0, s''_0) = (c_1(t_1), c_1(t_2)) \);
- \( s_1 = (s'_1, s''_1) = \begin{cases} (s'_0, \text{mid}(s'_0, s''_0)), & \text{if mid}(s'_0, s''_0) \text{ is a point of type II;} \\ (\text{mid}(s'_0, s''_0), s''_0), & \text{if mid}(s'_0, s''_0) \text{ is a point of type I;} \end{cases} \)
- \( \ldots \)
- \( s_n = (s'_n, s''_n) = \begin{cases} (s'_{n-1}, \text{mid}(s'_{n-1}, s''_{n-1})), & \text{if mid}(s'_{n-1}, s''_{n-1}) \text{ is a point of type II;} \\ (\text{mid}(s'_{n-1}, s''_{n-1}), s''_{n-1}), & \text{if mid}(s'_{n-1}, s''_{n-1}) \text{ is a point of type I.} \end{cases} \)

Let \( s'_n = c_1(t'_n) \) be a point of type I on \( c_1 \) and let \( s''_n = c_1(t''_n) \) be a point of type II on \( c_1, n \geq 1 \) such that the position of \( s'_n \) with respect to \( s''_n \) on the edge \( c_1 \) is as in the figure below. Because
\[
\begin{cases}
  l(c_1|_{[t'_n, s''_n]}) = \frac{1}{2}l(c_1|_{[0, 1]}), & \text{if } t'_n < t''_n, \\
  l(c_1|_{[t''_n, t'_n]}) = \frac{1}{2}l(c_1|_{[0, 1]}), & \text{if } t''_n \leq t'_n,
\end{cases}
\]
we have
\[
\begin{cases}
  \lim_{n \to \infty} l(c_1|_{[t'_n, s''_n]}) = 0, & \text{if } t'_n < t''_n, \\
  \lim_{n \to \infty} l(c_1|_{[t''_n, t'_n]}) = 0, & \text{if } t''_n \leq t'_n.
\end{cases}
\]
The segment \([p, q]\) that intersects the interior of \(\sigma\)

\(s'_n\) is a point of type I on \(e_1\)

\(s''_n\) is a point of type II on \(e_1\)

There exists a unique geodesic segment in \(U\) joining \(s'_n = c_1(t'_n)\) to \(s''_n = c_1(t''_n)\) whose length equals \(d(s'_n, s''_n)\). Because

\[
\begin{cases}
0 \leq d(s'_n, s''_n) \leq l(c_1|t'_n, t''_n|), & \text{if } t'_n < t''_n; \\
0 \leq d(s'_n, s''_n) \leq l(c_1|t''_n, t'_n|), & \text{if } t''_n \leq t'_n,
\end{cases}
\]

we get

\[
\lim_{n \to \infty} d(s'_n, s''_n) = 0.
\]

Hence, by Lemma 3.4,

\[\lim_{n \to \infty} \angle_{p_1}(s'_n, s''_n) = 0\]

and

\[\lim_{n \to \infty} \angle_{q_1}(s'_n, s''_n) = 0.\]

On the other hand, because the geodesic triangles \(\triangle(p_1, s'_n, s''_n)\) and \(\triangle(q_1, s'_n, s''_n)\) belong to 2-simplices of curvature zero, we have

\[
\angle_{p_1}(s'_n, s''_n) + \angle_{s'_n}(p_1, s''_n) + \angle_{s''_n}(s'_n, p_1) + \angle_{q_1}(s'_n, s''_n) + \angle_{s''_n}(q_1, s'_n) + \angle_{s'_n}(s''_n, q_1) = 2\pi.
\]

Because \(s'_n\) is a point of type I, while \(s''_n\) is a point of type II which lie one with respect to the other on the edge \(e_1\) as in the figure above, we have

\[
\angle_{s'_n}(p_1, s''_n) + \angle_{s''_n}(s'_n, q_1) < \pi
\]

and

\[
\angle_{s''_n}(p_1, s'_n) + \angle_{s'_n}(s''_n, q_1) < \pi.
\]

The above three relations imply that

\[
\angle_{p_1}(s'_n, s''_n) + \angle_{q_1}(s'_n, s''_n) > 0.
\]

So, since any Alexandrov angle is a value in the interval \([0, \pi]\), either

\[
\lim_{n \to \infty} \angle_{p_1}(s'_n, s''_n) \neq 0
\]

or
\[
\lim_{n \to \infty} \angle_{q_1}(s'_n, s''_n) \neq 0
\]
or
\[
\lim_{n \to \infty} \angle_{p_1}(s'_n, s''_n) \neq 0 \text{ and } \lim_{n \to \infty} \angle_{q_1}(s'_n, s''_n) \neq 0.
\]
Thus, according either to (4) or to (5) or to both, we have reached a contradiction. So there exist points of type III on \(e_1\).

We show further that there exists a unique point of type III on \(e_1\). Suppose, on the contrary, there exist two points of type III, say \(s_1\) and \(s_2\), on \(e_1\). We assume that the position of \(s_1\) with respect to \(s_2\) on the edge \(e_1\) is as in the figure below.

There exists a unique point of type III on \(e_1\)

So
\[
\angle_{s_1}(p_1, s_2) + \angle_{s_1}(s_2, q_1) = \pi
\]
and
\[
\angle_{s_2}(p_1, s_1) + \angle_{s_2}(s_1, q_1) = \pi.
\]
Note that, since that the geodesic triangles \(\triangle(p_1, s_1, s_2)\) and \(\triangle(q_1, s_1, s_2)\) belong to 2-simplices of curvature zero, we have
\[
\angle_{p_1}(s_1, s_2) + \angle_{s_1}(p_1, s_2) + \angle_{s_2}(p_1, s_1) + \angle_{q_1}(s_1, s_2) + \angle_{s_1}(q_1, s_2) + \angle_{s_2}(q_1, s_1) = 2\pi.
\]
The above three relations imply that:
\[
\angle_{p_1}(s_1, s_2) = 0
\]
and
\[
\angle_{q_1}(s_1, s_2) = 0.
\]
Because the points \(s_1\) and \(s_2\) belong to 2-simplices that are isometric to geodesic triangles in \(\mathbb{R}^2\), these relations ensure that \(s_1 = s_2\). So there exists a unique point, say \(s\), on \(e_1\) such that \(\angle_{s}(p_1, t) + \angle_{s}(t, q_1) = \pi\), for any point \(t\) on \(e_1\) that differs from \(s\). Then, according to Lemma 3.3, the following inequality holds
\[
d(p_1, s) + d(s, q_1) < d(p_1, t) + d(t, q_1).
\]
The aim of the following lemma is to show that a relation similar to the one proven in the lemma above for the pair \( p_1, q_1 \), holds for the pair of points \( p, q \) as well. We prove this by showing that such relation holds, in fact, for any pair of points on the geodesic segment \([p, q]\) such that one point of the pair lies on \([p_1, p]\), while the other one lies on \([q_1, q]\). The lemma follows due to the fact that the points \( p, p_1, q_1 \) and \( q \) lie, in this order, on the same geodesic segment in a CAT(0) space. It is important to keep in mind that any Alexandrov angle is a value in the interval \([0, \pi]\).

**Lemma 3.6.** Let \( s \) be a point on \( e_1 \) such that \( \angle_s(a, p_1) = \angle_s(b, q_1) \) and \( \angle_s(a, q_1) = \angle_s(b, p_1) \). Then \( \angle_s(p, t) + \angle_s(t, q) \geq \pi \) for any point \( t \) on \( e_1 \) that differs from \( s \). In particular the following inequality holds: \( d(p, s) + d(s, q) < d(p, t) + d(t, q) \).

**Proof.** By Lemma 3.5, the point \( s \) exists, it is unique and it fulfills the following relation

\[
\angle_s(p_1, t) + \angle_s(t, q_1) = \pi
\]

for any point \( t \) on \( e_1 \) that differs from \( s \). In particular,

\[
(6) \quad \angle_s(p_1, a) + \angle_s(a, q_1) = \pi.
\]

We construct a sequence of points \( (p_n^*)_{n \in \mathbb{N}} \) such that \( p_0^* = p, p_n^* \in [p, p_1], \)

\[
\lim_{n \to \infty} d(p_1, p_n^*) = 0.
\]

Lemma 3.4 implies

\[
(7) \quad \lim_{n \to \infty} \angle_s(p_1, p_n^*) = 0.
\]

Similarly, we construct a sequence of points \( (q_n^*)_{n \in \mathbb{N}} \) such that \( q_0^* = q, q_n^* \in [q, q_1], \)

\[
\lim_{n \to \infty} d(q_1, q_n^*) = 0.
\]

Lemma 3.4 implies

\[
(8) \quad \lim_{n \to \infty} \angle_s(q_1, q_n^*) = 0.
\]
Note that, since \( p_n^* \in [p, p_1] \) and \( U \) is a CAT(0) space, we have
\[
\angle_{p_1^*}(p, p_1) = \pi.
\]
Also note that
\[
\angle_s(b, p_1) \leq \angle_s(b, p_n^*).
\]
Thus, since
\[
\angle_s(b, p_1) + \angle_s(p_1, a) = \pi,
\]
while
\[
\angle_s(b, p_n^*) + \angle_s(p_n^*, a) = \pi,
\]
it follows that
\[
\angle_s(p_n^*, a) \leq \angle_s(p_1, a).
\]
Hence, since
\[
\angle_s(p_n^*, a) \leq \angle_s(p_1, a) \leq \angle_s(p_1, p_n^*) + \angle_s(p_n^*, a),
\]
by (7), we have
\[
\lim_{n \to \infty} \angle_s(p_n^*, a) = \angle_s(p_1, a).
\]
Similarly, relation \( \text{8} \) implies that
\[
\lim_{n \to \infty} \angle_s(a, q_n^*) = \angle_s(a, q_1).
\]
So
\[
\lim_{n \to \infty} (\angle_s(p_n^*, a) + \angle_s(a, q_n^*)) = \angle_s(p_1, a) + \angle_s(a, q_1).
\]
Suppose that for any \( n \),
\[
\angle_s(p_n^*, a) + \angle_s(a, q_n^*) < \pi.
\]
The relations \( \text{8} \) and \( \text{9} \) imply in this case a contradiction. So there exists \( m_0 \in \mathbb{N} \) such that \( p_{m_0}^* \in [p_0^*, p_1] \), \( q_{m_0}^* \in [q_0^*, q_1] \) and
\[
\angle_s(p_{m_0}^*, a) + \angle_s(a, q_{m_0}^*) \geq \pi.
\]
We argue by induction on \( m_k \). The case \( k = 0 \) is discussed above. Replacing the pair \( p_1, q_1 \) by the pair \( p_{m_0}^*, q_{m_0}^* \) and arguing as above, it follows that there exists \( m_1 \in \mathbb{N}^* \) such that \( p_{m_1}^* \in [p, p_{m_0}^*], q_{m_1}^* \in [q, q_{m_0}^*] \) and
\[
\angle_s(p_{m_1}^*, a) + \angle_s(a, q_{m_1}^*) \geq \pi.
\]
We proceed with the second step of the induction. Suppose there exists \( m_k \in \mathbb{N}^* \), \( k \in \mathbb{N}^* \) such that \( p^*_{m_k} \in [p, p^*_{m_{k-1}}] \), \( q^*_{m_k} \in [q, q^*_{m_{k-1}}] \) and
\[
\angle_s(p^*_{m_k}, a) + \angle_s(a, q^*_{m_k}) \geq \pi.
\]
Replacing the pair \( p^*_{m_{k-1}}, q^*_{m_{k-1}} \) by the pair \( p^*_{m_k}, q^*_{m_k} \) and arguing as for the case \( k = 0 \), it similarly follows that there exists \( m_{k+1} \in \mathbb{N}^* \) such that \( p^*_{m_{k+1}} \in [p, p^*_{m_k}] \), \( q^*_{m_{k+1}} \in [q, q^*_{m_k}] \) and
\[
\angle_s(p^*_{m_{k+1}}, a) + \angle_s(a, q^*_{m_{k+1}}) \geq \pi
\]
which concludes the second step of the induction.

Note that, since \( (p^*_{m_k})_{k \in \mathbb{N}} \) is a sequence of points on \([p, p_1]\) such that \( p^*_{m_k} \in [p, p^*_{m_{k-1}}] \), we have
\[
\lim_{k \to \infty} d(p^*_{m_k}, p) = 0.
\]
Similarly note that \( (q^*_{m_k})_{k \in \mathbb{N}} \) is a sequence of points on \([q, q_1]\) such that
\[
\lim_{k \to \infty} d(q^*_{m_k}, q) = 0.
\]
Lemma 3.4 further implies that
\[
\lim_{k \to \infty} \angle_s(p^*_{m_k}, p) = 0
\]
and
\[
\lim_{k \to \infty} \angle_s(q^*_{m_k}, q) = 0.
\]
Note that
\[
\angle_s(p^*_{m_k}, a) \leq \angle_s(p^*_{m_k}, p) + \angle_s(p, a),
\]
while
\[
\angle_s(q^*_{m_k}, a) \leq \angle_s(q^*_{m_k}, q) + \angle_s(q, a).
\]
So, by (11) and (12), we get
\[
\lim_{k \to \infty} (\angle_s(p^*_{m_k}, a) + \angle_s(a, q^*_{m_k})) \leq \angle_s(p, a) + \angle_s(a, q).
\]
Hence, by (10),
\[
\angle_s(p, a) + \angle_s(a, q) \geq \pi.
\]
Arguing similarly, one can show that
\[
\angle_s(p, t) + \angle_s(t, q) \geq \pi
\]
for any \( t \) on \( e_1 \) that differs from \( s \). Lemma 5.30 ensures that
\[
d(p, s) + d(s, q) < d(p, t) + d(t, q).
\]
\[\square\]
Remark. Note that in the proof of the previous lemma, the fact that the point \( t \) lies on the edge \( e_1 \) does not influence the proof in any way. So a similar result holds for the case when \( t \in |U|, t \notin e_1 \). Moreover, according to the hypothesis of Lemma 3.6, \( s \) is the unique point on \( e_1 \) such that \( \angle_s(a, p_1) = \angle_s(b, q_1) \) and \( \angle_s(a, q_1) = \angle_s(b, p_1) \) and hence \( \angle_s(p_1, t) + \angle_s(t, q_1) = \pi \). Note that a slightly modified hypothesis in Lemma 3.6, namely \( \angle_s(p_1, t) + \angle_s(t, q_1) \geq \pi \), would imply the same result. Furthermore, note that the particular choice of the point \( s \) does not influence the proof of Lemma 3.6 either. Hence, for any \( l \in e_1 \) for whom \( \angle_l(p_1, t) + \angle_l(t, q_1) \geq \pi \) holds, Lemma 3.6 ensures the following corollary.

**Corollary A.** For any \( t \in |U| \) and for any \( l \in e_1 \), if \( \angle_l(p_1, t) + \angle_l(t, q_1) \geq \pi \) then \( \angle_1(p, t) + \angle_l(t, q) \geq \pi \).

We summarize the basic ideas behind the proof of the above results. For any \( t \in |U| \) and for any \( l \in e_1 \), the inequality \( \angle_l(p_1, t) + \angle_l(t, q_1) \geq \pi \) is fulfilled by the pair of points \( p_1, q_1 \) due to the fact that such points lie on 2-simplices that are isometric to their comparison triangles in Euclidean plane. Furthermore, such inequality is inherited by those pair of points on \( [p, q] \) for whom one point of the pair lies on \( [p, p_1] \), while the other point lies on \( [q, q_1] \).

**Lemma 3.7.** Let \( c : [0, 1] \to U \) be a path in \( U \) joining \( p \) to \( q \) that does not intersect \( \sigma \). Then there exists a point \( m \) on \( c \) such that neither the segment \([p, m] \) in \( U \) nor the segment \([q, m] \) in \( U \) intersects the interior of \( \sigma \).

**Proof.** We call the points \( c(t) \) such that the segment \([q, c(t)] \) intersects the interior of \( \sigma \) and the segment \([p, c(t)] \) does not intersect the interior of \( \sigma, t \in [0, 1], \) points of type I.

We call the points \( c(t) \) such that the segment \([p, c(t)] \) intersects the interior of \( \sigma \), points of type II. Notice that if \( c(t) \) is a point of type II, \( t \in [0, 1], \) then the segment \([q, c(t)] \) might also intersect the interior of \( \sigma \).

We call the points \( c(t) \) such that the segments \([q, c(t)] \) and \([p, c(t)] \) do not intersect the interior of \( \sigma, t \in [0, 1], \) points of type III.

Suppose that there are no points of type III on the path \( c \). Any point on \( c \) is therefore either a point of type I or a point of type II. Thus, for any \( t \in [0, 1], \) at least one of the segments \([q, c(t)] \) and \([p, c(t)] \) intersects the interior of \( \sigma \).

Considering \((s'_n, s''_n) = (p, q)\), we define as in Lemma 3.6 a sequence \((s_n)\) of tuples such that \( s'_n \) is a point of type I on \( c \) whereas \( s''_n \) is a point of type II on \( c \), \( n \geq 1 \). Assume that the position of \( s'_n \) with respect to \( s''_n \) on the path \( c \) is as in the figure below. Arguments similar to those in Lemma 3.6 ensure that

\[
\lim_{n \to \infty} d(s'_n, s''_n) = 0.
\]

We denote by \( p'_n \) the intersection point of \([p, s'_n] \) with \( \tau_1 \), and by \( q'_n \) the intersection point of \([p, s'_n] \) with \( \tau_2 \). We denote by \( p''_n \) the intersection point of \([q, s''_n] \) with \( \tau_1 \), and by \( q''_n \) the intersection point of \([q, s''_n] \) with \( \tau_2 \).

Because \( U \) is complete and a CAT(0) space, it satisfies the CAT(0) 4—point condition. So the 4-tuple of points \((p'_n, q'_n, s'_n, s''_n)\) in \( U \) has a subembedding \((\overline{p}_n, \overline{q}_n, \overline{s}_n, \overline{s''}_n)\) in \( \mathbb{R}^2 \). Thus

\[
d(s'_n, p'_n) \leq d_{\mathbb{R}^2}(\overline{s}_n, \overline{p}_n).
\]

Let \( Q \) denote the quadrilateral in \( \mathbb{R}^2 \) spanned by the vertices \( \overline{p}_n, \overline{q}_n, \overline{s}_n \) and \( \overline{s''}_n \).
The path $c$ that connects $p$ to $q$ in $U$ without intersecting $\sigma$

$s'_n$ is a point of type I on $c$

$s''_n$ is a point of type II on $c$

Suppose first that $Q$ is convex. Let $k$ denote the intersection point of its diagonals. By (13),

$$\lim_{n \to \infty} d_{\mathbb{R}^2}(s'_n, s''_n) = 0.$$  

Thus

$$\lim_{n \to \infty} \bar{Z}_k(s'_n, s''_n) = 0$$

and therefore

$$\lim_{n \to \infty} \bar{Z}_k(p'_n, q'_n) = 0.$$  

So

$$\lim_{n \to \infty} d_{\mathbb{R}^2}(p'_n, q'_n) = 0$$

and hence

(15)  \[ \lim_{n \to \infty} d(p'_n, q'_n) = 0. \]
On the other hand, the points $p'_n$ and $q'_n$ belong to the interior of some 2-simplices in $U$ both isometric to their comparison triangles in $\mathbb{R}^2$ and which do not coincide. So

$$\lim_{n \to \infty} d(p'_n, q'_n) \neq 0.$$ 

Hence, by (13), we have reached a contradiction. There exists therefore a point $m$ on the path $c$ such that neither the segment $[p, m]$ in $U$ nor the segment $[q, m]$ in $U$ intersects the interior of $\sigma$.

We analyze further the case when $Q$ is not convex. Suppose $\overline{p}_n$ is the vertex of $Q$ in the convex hull of the other three vertices of $Q$ (see the figure above). The other three cases can be handled similarly. Let $\triangle(\overline{s}_n, \overline{q}_n, \overline{s}'_n)$ be a geodesic triangle in $\mathbb{R}^2$ whose side lengths are equal to $d_{\mathbb{R}^2}(\overline{s}_n, \overline{q}_n)$, $d_{\mathbb{R}^2}(\overline{s}_n, \overline{s}'_n)$, and $d_{\mathbb{R}^2}(\overline{q}_n, \overline{s}'_n)$, respectively. Let $\overline{p}'_n$ be a point on $[\overline{q}_n, \overline{s}'_n]$ such that $d_{\mathbb{R}^2}(\overline{p}'_n, \overline{q}_n) = d_{\mathbb{R}^2}(\overline{q}_n, \overline{s}'_n)$. Because $d(s'_n, s''_n) = d_{\mathbb{R}^2}(\overline{s}_n, \overline{s}'_n) = d_{\mathbb{R}^2}(\overline{s}_n, \overline{s}'_n)$, by (13), we have

$$\lim_{n \to \infty} d_{\mathbb{R}^2}(\overline{s}_n, \overline{s}'_n) = 0.$$ 

So

$$\lim_{n \to \infty} \angle_{\overline{s}_n}(\overline{s}_n, \overline{s}'_n) = 0.$$ 

Hence

(16) $$\lim_{n \to \infty} d_{\mathbb{R}^2}(\overline{s}_n, \overline{p}'_n) = 0.$$ 

Note that $\angle_{\overline{s}_n}(\overline{q}_n, \overline{s}_n) + \angle_{\overline{s}_n}(\overline{s}_n, \overline{s}'_n) = \pi$, while $\angle_{\overline{s}_n}(\overline{q}_n, \overline{s}_n) + \angle_{\overline{s}_n}(\overline{s}_n, \overline{s}'_n) > \pi$ (this inequality holds because $Q$ is not convex and $\overline{p}'_n$ lies in the interior of the convex hull of the other three vertices of $Q$). Hence Alexandrov’s Lemma implies

$$d_{\mathbb{R}^2}(\overline{s}_n, \overline{p}'_n) \leq d_{\mathbb{R}^2}(\overline{s}_n, \overline{q}_n).$$

Relation (16) therefore ensures

$$\lim_{n \to \infty} d_{\mathbb{R}^2}(\overline{s}_n, \overline{p}'_n) = 0.$$
Thus, by (14),
\[ \lim_{n \to \infty} d(s'_n, p'_n) = 0. \]

On the other hand, the point \( p'_n \) lies in the interior of a 2-dimensional face of \( \sigma \) whereas the point \( s'_n \) lies on a path that does not intersect \( \sigma \). Hence
\[ \lim_{n \to \infty} d(s'_n, p'_n) \neq 0 \]
which implies, by (17), a contradiction. There exists therefore a point \( m \) on the path \( c \) such that neither the segment \([p, m]\) in \( U \) nor the segment \([q, m]\) in \( U \) intersects the interior of \( \sigma \).

\[ \Box \]

**Lemma 3.8.** Let \( s \) be a point on \( e_1 \) such that \( \angle_s(a, p_1) = \angle_s(b, q_1) \) and \( \angle_s(a, q_1) = \angle_s(b, p_1) \). Let \( c : [0, 1] \to U \) be a path in \( U \) joining \( p \) to \( q \) that does not intersect \( \sigma \). Let \( m \) be a point on \( c \) such that neither the segment \([p, m]\) in \( U \) nor the segment \([q, m]\) in \( U \) intersects the interior of \( \sigma \). Then, the following inequality holds in \( U' \):
\[ d'(p, s) + d'(s, q) < d'(p, m) + d'(m, q). \]

**Proof.** By Lemma 3.5 and 3.7 such points \( s \) and \( m \) exist. Moreover, such point \( s \) is unique. Let \( l \) be some point on \( e_1 \).

We denote by \( \Gamma_1 \) the union of the geodesic triangles \( \triangle(a, m, l) \) and \( \triangle(b, m, l) \). Note that \( \Gamma_1 \) intersects the boundary of \( \sigma \) along the edge \( e_1 = [a, b] \), i.e. along one common boundary edge \( \tau_1 \) and \( \tau_2 \) of (which are two 2-dimensional faces of \( \sigma \)). Further, note that
\[ \angle_l(a, m) + \angle_l(m, b) = \pi. \]

We denote by \( \Gamma_2 \) the union of the geodesic triangles \( \triangle(p_1, m, l) \) and \( \triangle(q_1, m, l) \). Note that \( \Gamma_2 \) intersects the boundary of \( \sigma \) along two sides of the geodesic triangle \( \triangle(p_1, q_1, l) \), i.e. along the interior of \( \tau_1 \) and \( \tau_2 \) which are also two 2-dimensional faces of \( \sigma \). Hence relation (15) guarantees that
\[ \angle_l(p_1, m) + \angle_l(m, q_1) > \pi. \]

Because \( l \in e_1 \) while \( m \in U \), \( m \notin e_1 \), Corollary A ensures that
Thus, by Lemma 3.3, we have
\[ d(p, l) + d(l, q) < d(p, m) + d(m, q). \]
According to Lemma 3.6, it follows that
\[ d(p, s) + d(s, q) < d(p, t) + d(t, q), \]
for any \( t \) on \( e_1 \) that differs from \( s \). The above relations ensure that
\[ d(p, s) + d(s, q) < d(p, m) + d(m, q). \]
Because the segments \([p, s], [s, q], [p, m] \) and \([m, q] \) in \( U \) do not intersect the interior of \( \sigma \), the same inequality holds in \( U' \):
\[ d'(p, s) + d'(s, q) < d'(p, m) + d'(m, q). \]

We find further the geodesic segments in \( U' \) joining those pairs of points that are joined in \( U \) by a segment that intersects the interior of \( \sigma \). Because \( K \) satisfies Property A, there are no geodesic segments \([p, q] \) in \( U \) such that the points \( p, q \) are joined in \( U' \) by two geodesic segments \( \gamma_1, \gamma_2 \) of equal length such that \( \gamma_1 \) intersects one, while \( \gamma_2 \) intersects one or two of the boundary edges of \( \alpha \) (if \( \alpha \) is 2-dimensional) or from \( \alpha \) itself (if \( \alpha \) is 1-dimensional).

**Lemma 3.9.** Let \( s \) be a point on \( e_1 \) such that \( \angle_s(a, p_1) = \angle_s(b, q_1) \) and \( \angle_s(a, q_1) = \angle_s(b, p_1) \). Let \( t \) be a point on \( e_2 \) such that \( \angle_t(c, d) = \angle_t(a, p_1) \) and \( \angle_t(a, c) = \angle_t(d, p_1) \). Let \( v \) be a point on \( e_3 \) such that \( \angle_v(c, q_1) = \angle_v(a, d) \) and \( \angle_v(a, q_1) = \angle_v(c, d) \). If \( d'(p, s) + d'(s, q) \leq d'(p, t) + d'(t, v) + d'(v, q) \), then the geodesic segment \([p, q] \) in \( U' \) with respect to \( d' \) is the union of the geodesic segments \([p, s] \) and \([s, q] \). Otherwise, the geodesic segment \([p, q] \) in \( U' \) with respect to \( d' \) is the union of the geodesic segments \([p, t], [t, v] \) and \([v, q] \).

**Proof.** Because \( U \) is a CAT(0) space, Lemma 3.3 guarantees that the points \( s, t \) and \( v \) exist and they are unique.

![The geodesic segment \([p, q] \) in \( U' \)](image)

In case \( d'(p, s) + d'(s, q) \leq d'(p, t) + d'(t, v) + d'(v, q) \), let \( c : [0, 1] \rightarrow U' \) denote the path obtained by concatenating the segments \([p, s] \) and \([s, q] \). Among all paths joining \( p \) to \( q \) in \( U' \) which pass through \( s \), the path \( c \) has the shortest length.
Suppose that there exists a path $c_0 : [0, 1] \to U'$ connecting $p$ to $q$ in $U'$ that does not pass through $s$ and whose length is less or equal to the length of the path $c$. Because the path $c_0$ does not intersect $\sigma$, there exists, according to Lemma 3.7, a point $m$ on $c_0$ such that the geodesic segments $[p, m]$ and $[m, q]$ in $U$ do not intersect the interior of $\sigma$. The geodesic segments $[p, m]$ and $[m, q]$ in $U$ belong therefore to $U'$. So

$$d'(p, m) + d'(m, q) \leq l(c_0) \leq l(c) = d'(p, s) + d'(s, q)$$

which is, by Lemma 3.8, a contradiction. Any path in $U'$ joining $p$ to $q$ and which does not pass through $s$, is therefore longer than $c$.

Altogether it follows that the geodesic segment joining $p$ to $q$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, s]$ and $[s, q]$.

In case $d'(p, s) + d'(s, q) > d'(p, t) + d'(t, v) + d'(v, q)$, one can similarly show, applying Lemma 3.8 twice, that the geodesic segment joining $p$ to $q$ in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, t], [t, v]$ and $[v, q]$. Namely, Lemma 3.8 will ensure that the geodesic segment joining $p$ to $v$ ($p$ to $q$) in $U'$ with respect to $d'$ is the union of the geodesic segments $[p, t]$ and $[t, v]$ ($[p, v]$ and $[v, q]$).

Using the CAT(0) inequality and Alexandrov’s Lemma, we show further that any geodesic triangle in $U'$ satisfies the CAT(0) inequality. Depending on the position of the vertices of such geodesic triangle with respect to the 3-simplex of $U$ with the free face, we must consider eight cases. We will find geodesic segments in $U'$ using, mostly without stating so explicitly, Lemma 3.8.

**Lemma 3.10.** Let $r$ be a point in $U$ such that the geodesic segments $[r, p]$ and $[r, q]$ do not intersect the interior of $\sigma$. Let $s$ be a point on $e_1$ such that $\angle_s(a, p_1) = \angle_s(b, q_1)$ and $\angle_s(a, q_1) = \angle_s(b, p_1)$. Let $t$ be a point on $e_2$ such that $\angle_t(c, d) = \angle_t(a, p_1)$ and $\angle_t(a, c) = \angle_t(d, p_1)$. Let $v$ be a point on $e_3$ such that $\angle_v(a, q_1) = \angle_v(a, d)$ and $\angle_v(a, q_1) = \angle_v(c, d)$. If $d'(p, s) + d'(s, q) < d'(p, t) + d'(t, v) + d'(v, q)$, then the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

**Proof.** Because $U$ is a CAT(0) space, according to Lemma 3.8 the point $s, t$ and $v$ exist and they are unique. Lemma 3.8 ensures that $d'(p, q) = d'(p, s) + d'(s, q)$.

\[
\begin{array}{c}
\text{The geodesic triangle } \triangle(p, q, r) \text{ in } U' \text{ satisfies the CAT(0) inequality.}
\end{array}
\]

Let $\triangle(p', q', r')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, q, r)$ in $U'$. Let $s' \in [p', q']$ be a comparison point for $s \in [p, q]$. 

Let $\triangle(p'', r'', s'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, r, s)$ in $U$ and let $\triangle(r'', s'', q'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, s, q)$ in $U$. We place the comparison triangles $\triangle(p'', r'', s'')$ and $\triangle(r'', s'', q'')$ in different half-planes with respect to the line $r''s''$ in $\mathbb{R}^2$.

![Comparison triangles in $\mathbb{R}^2$](image)

By the CAT(0) inequality, we have $\angle_r(p, s) \leq \angle_{r''}(p'', s'')$, $\angle_r(s, q) \leq \angle_{r''}(s'', q'')$, $\angle_p(r, s) \leq \angle_{p''}(r'', s'')$ and $\angle_q(r, s) \leq \angle_{q''}(r'', s'')$. Because $\angle_v(p', r') + \angle_v(r', q') = \pi$, Alexandrov’s Lemma further implies: $\angle_{r''}(p'', s'') \leq \angle_{r'}(p', s')$, $\angle_{r''}(s'', q'') \leq \angle_{r'}(s', q')$, $\angle_{p''}(r'', s'') \leq \angle_{p'}(r', s')$ and $\angle_{q''}(r'', s'') \leq \angle_{q'}(r', s')$. Altogether it follows that $\angle_r(p, q) \leq \angle_r(p, s) + \angle_r(s, q) \leq \angle_{r'}(p', s') + \angle_{r'}(s', q') = \angle_{r'}(p', q')$, $\angle_p(r, s) \leq \angle_{p'}(r', s')$ and $\angle_q(r, s) \leq \angle_{q'}(r', s')$. So the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

Lemma 3.11. Let $r$ be a point in $U$ such that the geodesic segments $[r, p]$ and $[r, q]$ do not intersect the interior of $\sigma$. Let $s$ be a point on $e_1$ such that $\angle_s(a, p_1) = \angle_s(b, q_1)$ and $\angle_s(a, q_1) = \angle_s(b, p_1)$. Let $t$ be a point on $e_2$ such that $\angle_t(c, d) = \angle_t(a, p_1)$ and $\angle_t(a, c) = \angle_t(d, p_1)$. Let $v$ be a point on $e_3$ such that $\angle_v(c, q_1) = \angle_v(a, d)$ and $\angle_v(a, q_1) = \angle_v(c, d)$. If $d'(p, t) + d'(t, v) + d'(v, q) < d'(p, s) + d'(s, q)$, then the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

Proof. Because $U$ is a CAT(0) space, by Lemma 3.3 the points $s, t$ and $v$ exist and they are unique. Lemma 3.3 further implies that $d'(p, q) = d'(p, t) + d'(t, v) + d'(v, q)$.

The geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.
Let $\triangle(p', q', r')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, q, r)$ in $U'$. Let $t' \in [p', q']$ be a comparison point for $t \in [p, q]$ and let $v' \in [p', q']$ be a comparison point for $v \in [p, q]$.

Let $\triangle(p'', r'', v'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, r, t)$ in $U$. Let $\triangle(r'', t'', v'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, t, v)$ in $U$. We place the geodesic triangles $\triangle(p'', r'', v'')$ and $\triangle(r'', t'', v'')$ in different half-planes with respect to the line $r''$ in $\mathbb{R}^2$. By the CAT(0) inequality, $\angle r(r, t) \leq \angle r''(r'', t'')$, $\angle r'(t, v) \leq \angle r''(t'', v'')$ and $\angle p(r, t) \leq \angle p''(r'', t'').$

Let $\triangle(p'''', r'''', v'''')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, r, v)$ in $U$. Let $t''' \in [p'''', v'''']$ be a comparison point for $t \in [p, v]$. Let $\triangle(r'''', v'''', q'''')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, v, q)$ in $U$. We place the comparison triangles $\triangle(p'''', r'''', v'''')$ and $\triangle(r'''', v'''', q'''')$ in different half-planes with respect to the line $r''''v''''$ in $\mathbb{R}^2$. By the CAT(0) inequality, $\angle q(r, v) \leq \angle q''''(r'''', v'''')$ and $\angle r(v, q) \leq \angle r''''(v'''', q'''')$.

Because $\angle r''''(p'''', r'''') + \angle v''''(r'''', v'''') = \pi$ and $\angle v''''(p'''', r'''') + \angle v''''(r'''', q'''') = \pi$, Alexandrov’s Lemma guarantees that $\angle r''''(r'''', t'''') \leq \angle p''''(r'''', t'''') \leq \angle p''''(r', t')$, $\angle r''''(p'''', v'''') \leq \angle r''''(p'''', v'''') \leq \angle r''''(p', v')$, while $\angle q''''(r'''', v'''') \leq \angle q(r', v')$. Altogether it follows that $\angle r(p, q) \leq \angle r''(r', q')$, $\angle q(r, q) \leq \angle q''(p', q')$ and $\angle r(p, q) \leq \angle r(r', q')$. So the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

\[ \square \]

**Lemma 3.12.** Let $r$ be a point in $U$ such that the geodesic segment $[r, q]$ does not intersect the interior of $\sigma$ whereas the geodesic segment $[p, r]$ intersects the interior of $\tau_1$ in $p_2$, and the interior of $\tau_2$ in $r_1$. Let $s$ be a point on $e_1$ such that $\angle s(a, p_1) = \angle s(b, q_1)$ and $\angle s(a, q_1) = \angle s(b, p_1)$. Let $t$ be a point on $e_1$ such that $\angle t(a, p_2) = \angle t(b, r_1)$ and $\angle t(a, r_1) = \angle t(b, p_2)$. If $d'(p, q) = d'(p, s) + d'(s, q)$ and $d'(p, r) = d'(p, t) + d'(t, r)$, then the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

**Proof.** Because $U$ is a CAT(0) space, Lemma 3.5 implies that the points $s$ and $t$ exist and they are unique.

The geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

Let $\triangle(p', q', r')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, q, r)$ in $U'$. Let $s' \in [p', q']$ be a comparison point for $s \in [p, q]$. Let $t' \in [p', r']$ be a comparison point for $t \in [p, r]$.

Let $\triangle(p'', t'', s'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, t, s)$ in $U$. Let $\triangle(t'', s'', r'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(t, s, r)$ in $U$. Let $\triangle(r'', t'', v'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, t, v)$ in $U$. Let $t''' \in [p'''', v'''']$ be a comparison point for $t \in [p, v]$. Let $\triangle(r'''', v'''', q'''')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, v, q)$ in $U$. Let $\triangle(r'''', v'''', q'''')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, v, q)$ in $U$. We place the comparison triangles $\triangle(p'''', r'''', v'''')$ and $\triangle(r'''', v'''', q'''')$ in different half-planes with respect to the line $r''''v''''$ in $\mathbb{R}^2$. By the CAT(0) inequality, $\angle q(r, v) \leq \angle q''''(r'''', v'''')$ and $\angle r(v, q) \leq \angle r''''(v'''', q'''')$.

Because $\angle r''''(p'''', r'''') + \angle v''''(r'''', v'''') = \pi$ and $\angle v''''(p'''', r'''') + \angle v''''(r'''', q'''') = \pi$, Alexandrov’s Lemma guarantees that $\angle r''''(r'''', t'''') \leq \angle p''''(r'''', t'''') \leq \angle p''''(r', t')$, $\angle r''''(p'''', v'''') \leq \angle r''''(p'''', v'''') \leq \angle r''''(p', v')$, while $\angle q''''(r'''', v'''') \leq \angle q(r', v')$. Altogether it follows that $\angle r(p, q) \leq \angle r''(r', q')$, $\angle q(r, q) \leq \angle q''(p', q')$ and $\angle r(p, q) \leq \angle r(r', q')$. So the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

\[ \square \]
\( \triangle(t, s, r) \) in \( U \). We place the comparison triangles \( \triangle(p'', t'', s'') \) and \( \triangle(t'', s'', r'') \) in different half-planes with respect to the line \( t''s'' \) in \( \mathbb{R}^2 \). The CAT(0) inequality implies that \( \angle_p(t, s) \leq \angle_{p''}(t'', s'') \), \( \angle_r(t, s) \leq \angle_{r''}(t'', s'') \).

Let \( \triangle(p'''', r'''', s'''') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, r, s) \) in \( U \). Let \( t''' \in [p'''', r''''] \) be a comparison point for \( t \in [p, r] \). Let \( \triangle(q'''', r'''', s'''') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(q, r, s) \) in \( U \). We place the comparison triangles \( \triangle(p'''', r'''', s'''') \) and \( \triangle(q'''', r'''', s'''') \) in different half-planes with respect to the line \( r'''s''' \) in \( \mathbb{R}^2 \).

Because \( \angle_{p'''}(p'''', s'''') + \angle_{p'''}(s'''', r'''') = \pi \), by Alexandrov’s Lemma we have \( \angle_{p'''}(t'''', s'''') \leq \angle_{p'''}(t''', s''') \), \( \angle_{r'''}(t'''', s'''') \leq \angle_{r'''}(t''', s''') \).

Note that \( \angle_{s'}(p', r') + \angle_{s'}(r', q') = \pi \). So, by Alexandrov’s Lemma and the CAT(0) inequality, we have \( \angle_{s'}(p', r', q') \leq \angle_{p'''}(t'''', s'''') \), \( \angle_{r'}(r', s') \leq \angle_{r'''}(t'''', s'''') \). Thus the geodesic triangle \( \triangle(p, q, r) \) in \( U' \) satisfies the CAT(0) inequality.

**Lemma 3.13.** Let \( r \) be a point in \( U \) such that the geodesic segment \( [r, q] \) does not intersect the interior of \( \sigma \) whereas the geodesic segment \( [p, r] \) intersects the interior of \( \tau_1 \) in \( p_2 \), and the interior of \( \tau_3 \) in \( r_1 \). Let \( s \) be a point on \( e_1 \) such that \( \angle_s(a, p_1) = \angle_s(b, q_1) \) and \( \angle_s(a, q_1) = \angle_s(b, p_1) \). Let \( t \) be a point on \( e_2 \) such that \( \angle_t(a, p_2) = \angle_t(d, r_1) \) and \( \angle_t(a, r_1) = \angle_t(d, p_2) \). If \( d'(p, q) = d'(p, s) + d'(s, q) \) and \( d'(p, r) = d'(p, t) + d'(t, r) \), then the geodesic triangle \( \triangle(p, q, r) \) in \( U' \) satisfies the CAT(0) inequality.

**Proof.** Because \( U \) is a CAT(0) space, by Lemma 3.5 the points \( s \) and \( t \) exist and they are unique.

The geodesic triangle \( \triangle(p, q, r) \) in \( U' \) satisfies the CAT(0) inequality.

Let \( \triangle(p', q', r') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, q, r) \) in \( U' \). Let \( s' \in [p', q'] \) be a comparison point for \( s \in [p, q] \). Let \( t' \in [p', r'] \) be a comparison point for \( t \in [p, r] \).

Let \( \triangle(p''', t''', s''') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, t, s) \) in \( U \). Let \( \triangle(t''', s''', r''') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(t, s, r) \) in \( U \). We place the comparison triangles \( \triangle(p''', t''', s''') \) and \( \triangle(t''', s''', r''') \) in different half-planes with respect to the line \( t''s'' \) in \( \mathbb{R}^2 \). By the CAT(0) inequality we have \( \angle_p(t, s) \leq \angle_{p'''}(t'', s'') \), \( \angle_r(t, s) \leq \angle_{r'''}(t'', s'') \).
Lemma 3.14. Let $r$ be a point in $U$ such that the geodesic segment $[r, q]$ does not intersect the interior of $\sigma$ whereas the geodesic segment $[p, r]$ intersects the interior of $\tau_1$ in $p_2$, and the interior of $\tau_2$ in $r_1$. Let $s$ be a point on $e_1$ such that $\angle_s(a, p_1) = \angle_s(b, q_1)$ and $\angle_s(a, q_1) = \angle_s(b, p_1)$. Let $t$ be a point on $e_2$ such that $\angle_t(d, p_2) = \angle_t(a, c)$ and $\angle_t(a, p_2) = \angle_t(d, c)$. Let $v$ be a point on $e_3$ such that $\angle_v(c, d) = \angle_v(a, r_1)$ and $\angle_v(a, d) = \angle_v(b, r_1)$. If $d'(p, q) = d'(p, s) + d'(s, q)$ and $d'(p, r) = d'(p, t) + d'(t, v) + d'(v, r)$, then the geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

Proof. Because $U$ is a CAT(0) space, by Lemma 3.3, the points $s, t$ and $v$ exist and they are unique.

The geodesic triangle $\triangle(p, q, r)$ in $U'$ satisfies the CAT(0) inequality.

Let $\triangle(p'', q'', r'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, q, r)$ in $U'$. Let $v'' \in [p'', r'']$ be a comparison point for $v \in [p, r]$.

Let $\triangle(p'', t'', s'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, t, s)$ in $U$. Let $\triangle(t'', s'', v'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(t, s, v)$ in $U$. We place the comparison triangles $\triangle(p'', t'', s'')$ and $\triangle(t'', s'', v'')$ in different half-planes with respect to the line $t''s''$ in $\mathbb{R}^2$. The CAT(0) inequality ensures that $\angle_{p''}(t, s) \leq \angle_{p''}(t'', s'')$.

Let $\triangle(p''', v''', s''')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, v, s)$ in $U$. Let $\triangle(q''', v''', s''')$ be a comparison triangle in $\mathbb{R}^2$ for geodesic
triangle $\triangle (q, v, s)$ in $U$. We place the comparison triangles $\triangle (p'', v''', s'')$ and $\triangle (q'', v''', s''')$ in different half-planes with respect to the line $v''s'''$ in $\mathbb{R}^2$. Let $t''' \in [p''' , v''']$ such that $d_{\mathbb{R}^2}(p''', t''') = d(p,t)$.

Because $\angle (p''', s''') + \angle (s''', v''') = \pi$, Alexandrov’s Lemma implies that $\angle (p''', s''') \leq \angle (p''', s''')$.

Let $\triangle (p', v', q')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle (p, v, q)$ in $U$. Let $s' \in [p', q']$ be a comparison triangle in $\mathbb{R}^2$ for $s \in [p, q]$. Let $\triangle (v', q', r')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle (v, q, r)$ in $U$. We place the comparison triangles $\triangle (p', v', q')$ and $\triangle (v', q', r')$ in different half-planes with respect to the line $v'q'$ in $\mathbb{R}^2$.

Because $\angle (p', s') + \angle (s', q') = \pi$ and $\angle (p', q') + \angle (q', r') = \pi$, Alexandrov’s Lemma ensures that $\angle (p', s') \leq \angle (p', s')$ and $\angle (p', q') \leq \angle (p', q')$.

Altogether it follows that in $U'$ we have $\angle (t, s) = \angle (r, q) \leq \angle (r', q')$. One can similarly show that $\angle (p, q) \leq \angle (p', q')$, $\angle (q, p, r) \leq \angle (q', r')$. So the geodesic triangle $\triangle (p, q, r)$ in $U'$ satisfies the $\text{CAT}(0)$ inequality.

\textbf{Lemma 3.15.} Let $r$ be a point in $U$ such that the geodesic segment $[r, q]$ does not intersect the interior of $\sigma$ whereas the geodesic segment $[r, p]$ intersects the interior of $\tau_1$ in $p_2$, and the interior of $\tau_3$ in $r_1$. Let $s$ be a point on $e_1$ such that $\angle (a, p_1) = \angle (a, q_1)$ and $\angle (a, q_1) = \angle (a, b_1)$. Let $t$ be a point on $e_1$ such that $\angle (a, p_2) = \angle (a, b_2)$ and $\angle (a, c) = \angle (p_2, b)$. Let $v$ be a point on $e_3$ such that $\angle (b, c) = \angle (a, r_1)$ and $\angle (b, a) = \angle (e, r_1)$. If $d'(p, q) = d'(p, s) + d'(s, q) + d'(s, r)$, then the geodesic triangle $\triangle (p, q, r)$ in $U'$ satisfies the $\text{CAT}(0)$ inequality.

\textbf{Proof.} Because $U$ is a $\text{CAT}(0)$ space, by Lemma 3.5 the points $s, t$ and $v$ exist and they are unique.

Let $\triangle (p', q', r')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle (p, q, r)$ in $U'$. Let $s' \in [p', q']$ be a comparison point for $s \in [p, q]$ and let $v' \in [p', r']$ be a comparison point for $v \in [p, r]$.

The geodesic triangle $\triangle (p, q, r)$ in $U'$ satisfies the $\text{CAT}(0)$ inequality}

Let $\triangle (p'', s'', t'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle (p, s, t)$ in $U$. Let $\triangle (s'', t'', v'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle (s, t, v)$ in $U$. We place the comparison triangles $\triangle (p'', s'', t'')$ and $\triangle (s'', t'', v'')$
in different half-planes with respect to the line \( s''t'' \) in \( \mathbb{R}^2 \). The CAT(0) inequality implies \( \angle_p(t, s) \leq \angle_p'(t'', s'') \).

Let \( \triangle(p'', v'', s'') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, v, s) \) in \( U \). Let \( \triangle(q'', s'', v'') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(q, s, v) \) in \( U \). We place the comparison triangles \( \triangle(p'', v'', s'') \) and \( \triangle(q'', s'', v'') \) in different half-planes with respect to the line \( s''v'' \) in \( \mathbb{R}^2 \).

Let \( t'' \in [p'', v''] \) be a comparison point for \( t \in [p, v] \). Because \( \angle_{\nu'}(p'', s'') + \angle_{\nu'}(s'', v'') = \pi \), Alexandrov’s Lemma ensures that \( \angle_{\nu'}(t'', s'') \leq \angle_{\nu'}(t'', v'') \).

Let \( \triangle(p', v', q') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, v, q) \) in \( U \). Let \( \triangle(v', q', r') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(v, q, r) \) in \( U \). We place the comparison triangles \( \triangle(p', v', q') \) and \( \triangle(v', q', r') \) in different half-planes with respect to the line \( v'q' \) in \( \mathbb{R}^2 \). Let \( s' \in [p', v'] \) be a comparison point for \( s \in [p, v] \). Because \( \angle_{s'v'}(p', v') + \angle_{s'v'}(v', q') = \pi \) and \( \angle_{s'v'}(q', q') + \angle_{s'v'}(q', r') = \pi \), Alexandrov’s Lemma further implies \( \angle_{\nu'}(s'', v'') \leq \angle_{\nu'}(q', v') \).

Thus, \( \angle_{\nu'}(q, r) \leq \angle_{\nu'}(q', r') \). One can similarly show that \( \angle_{\nu'}(r, p) \leq \angle_{\nu'}(r', p') \) and \( \angle_{\nu'}(p, q) \leq \angle_{\nu'}(p', q') \). So the geodesic triangle \( \triangle(p, q, r) \) in \( U' \) satisfies the CAT(0) inequality.

\[ \square \]

**Lemma 3.16.** Let \( r \) be a point in \( U \) such that the geodesic segment \([r, q]\) does not intersect the interior of \( \sigma \) whereas the geodesic segment \([p, r]\) intersects the interior of \( \tau_1 \) in \( p_2 \), and the interior of \( \tau_2 \) in \( r_1 \). Let \( s \) be a point on \( e_3 \) such that \( \angle_s(a, q_1) = \angle_s(b, c) \) and \( \angle_s(a, b) = \angle_s(c, q_1) \). Let \( t \) be a point on \( e_3 \) such that \( \angle_t(a, p_1) = \angle_t(c, d) \) and \( \angle_t(a, d) = \angle_t(d, p_1) \). Let \( u \) be a point on \( e_3 \) such that \( \angle_u(a, p_2) = \angle_u(c, a) = \angle_u(d, p_2) \). Let \( v \) be a point on \( e_3 \) such that \( \angle_v(d, c) = \angle_v(a, r_1) \) and \( \angle_v(d, a) = \angle_v(c, r_1) \).

If \( d'(p, q) = d'(p, t) + d'(t, s) + d'(s, q) \) and \( d'(p, r) = d'(p, u) + d'(u, v) + d'(v, r) \), then the geodesic triangle \( \triangle(p, q, r) \) in \( U' \) satisfies the CAT(0) inequality.

**Proof.** Because \( U \) is a CAT(0) space, by Lemma 3.5, the points \( s, t, u \) and \( v \) exist and they are unique.

![Diagram](image)

The geodesic triangle \( \triangle(p, q, r) \) in \( U' \) satisfies the CAT(0) inequality.

Let \( \triangle(p', q', r') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, q, r) \) in \( U' \).

Let \( \triangle(p'', t'', u'') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle \( \triangle(p, t, u) \) in \( U \). Let \( \triangle(t'', u'', v'') \) be a comparison triangle in \( \mathbb{R}^2 \) for the geodesic triangle...
△(t, u, v) in U. We place the comparison triangles △(p′′, t′′, u′′) and △(t′′, w′′, v′′) in different half-planes with respect to the line t′′u′′ in R². By the CAT(0) inequality we have ϖ_{p′′}(t, u) ≤ ϖ_{p′′}(t′′, u′′).

Let △(p′′′, t′′′, v′′′) be a comparison triangle in R² for the geodesic triangle △(p, t, v) in U. Let △(t′′′, w′′′, s′′′) be a comparison triangle in R² for the geodesic triangle △(t, v, s) in U. We place the comparison triangles △(p′′′, t′′′, v′′′) and △(t′′′, w′′′, s′′′) in different half-planes with respect to the line v′′′t′′′ in R². Alexandrov’s Lemma implies that ϖ_{p′′′}(t′′′, u′′′) ≤ ϖ_{p′′′}(t′′′, v′′′).

Let △(p′, t′, v′) be a comparison triangle in R² for the geodesic triangle △(p, s, v) in U. Let △(s′, v′, q′) be a comparison triangle in R² for the geodesic triangle △(s, v, q) in U. We place the comparison triangles △(p′, t′, v′) and △(s′, v′, q′) in different half-planes with respect to the line s′v′ in R². Alexandrov’s Lemma implies ϖ_{p′}(t′, u′) ≤ ϖ_{p′}(s′, v′).

Let △(p′′, q′′, v′′) be a comparison triangle in R² for the geodesic triangle △(p, q, v) in U. Let △(r′′, q′′, v′′) be a comparison triangle in R² for the geodesic triangle △(r, q, v) in U. We place the comparison triangles △(p′′, q′′, v′′) and △(r′′, q′′, v′′) in different half-planes with respect to the line q′′v′′ in R². Alexandrov’s Lemma ensures that ϖ_{p′′}(s′′, v′′) ≤ ϖ_{p′′}(q′′, v′′) ≤ ϖ_{p′′}(q′′, r′′).

Altogether we have ϖ_{p}(q, r) ≤ ϖ_{p′}(q′, r′). One can show similarly that ϖ_{q}(p, r) ≤ ϖ_{q′}(p′, r′) and ϖ_{r}(p, q) ≤ ϖ_{r′}(p′, q′). So the geodesic triangle △(p, q, r) in U′ satisfies the CAT(0) inequality.

Lemma 3.17. Let r be a point in U such that the geodesic segment [r, q] intersects the interior of τ₂ in r₂ and the interior of τ₂ in q₂ whereas the geodesic segment [p, r] intersects the interior of τ₁ in p₂ and the interior of τ₁ in r₁. Let s be a point on e₁ such that ϖ₁(a, q₁) = ϖ₁(b, p₁) and ϖ₁(a, p₁) = ϖ₁(b, q₁). Let t be a point on e₂ such that ϖ₂(a, p₂) = ϖ₂(d, r₁) and ϖ₂(a, r₂) = ϖ₂(d, p₂). Let u be a point on e₃ such that ϖ₃(a, r₃) = ϖ₃(c, q₂) and ϖ₃(a, q₂) = ϖ₃(c, r₂). If d′(p, q) = d′(p, s) + d′(s, q), d′(p, r) = d′(p, t) + d′(t, r) and d′(r, q) = d′(r, u) + d′(u, q), then the geodesic triangle △(p, q, r) in U′ satisfies the CAT(0) inequality.

Proof. Because U is a CAT(0) space, Lemma 5.5 implies that the points s, t and u exist and they are unique.

The geodesic triangle △(p, q, r) in U′ satisfies the CAT(0) inequality

Let △(p′, q′, r′) be a comparison triangle in R² for the geodesic triangle △(p, q, r) in U′. Let s′ ∈ [p′, q′] be a comparison point for s ∈ [p, q].
Let $\triangle(p', r'', s'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, r, s)$ in $U$. Let $\triangle(r'', s'', q'')$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, s, q)$ in $U$. We place the comparison triangles $\triangle(p', r'', s'')$ and $\triangle(r'', s'', q'')$ in different half-planes with respect to the line $r's''$ in $\mathbb{R}^2$. The CAT(0) inequality implies that $\angle_p(r, s) \leq \angle_p(r'', s'')$. Because $\angle_x(p', r') + \angle_x(r', q') = \pi$, Alexandrov’s Lemma implies $\angle_y(p', r') \leq \angle_y(r', s')$. So $\angle_p(r, s) \leq \angle_y(p', r')$. One can similarly show that $\angle_q(p, r) \leq \angle_q(p', r')$ and $\angle_r(p, q) \leq \angle_r(p', q')$. Hence the geodesic triangle $\triangle(p, r, q)$ in $U'$ satisfies the CAT(0) inequality.

The previous eight lemmas imply the following proposition.

**Proposition 3.18.** The subcomplex $K'$ obtained by performing an elementary collapse on a finite, CAT(0) simplicial 3-complex $K$ satisfying Property A, is nonpositively curved.

**Proof.** We must show that every point in $|K'|$ has a neighborhood which is a CAT(0) space.

Let $u, v, w$ be three distinct points in $U$ chosen such that they do not belong to the interior of $\sigma$ and such that the geodesic segments $[u, v], [u, w]$ and $[v, w]$ in $U$ do not intersect the interior of $\sigma$. Let $y$ be a point in $|K|$ that does not belong to the interior of $\sigma$. Let $U_y$ be a neighborhood of $y$ homeomorphic to a closed ball of radius $r_y$, $U_y = \{ x \in |K| \mid d(y, x) \leq r_y \}$. The radius $r_y$ is chosen small enough such that $U_y$ does not intersect $\sigma$. For any $y$ in $|K'|$ that does not belong to $\tau_1, \tau_2$ or $\tau_3$, we consider a neighborhood $U'_y$ that coincides with $U_y$. $U'_y$ is thus a CAT(0) space.

So, because every point in $|K'|$ has a neighborhood which is a CAT(0) space, $|K'|$ is nonpositively curved.

The main result of the paper is an immediate consequence of the above proposition.

**Corollary B.** Any finite, CAT(0) simplicial 3-complex $K$ that fulfills Property A, collapses to a point through CAT(0) subspaces.

**Proof.** Because $K$ has a strongly convex metric, it has, by [Whi70], a 3-simplex $\sigma$ with a free 2-dimensional (1-dimensional) face. We fix a point $y$ in the interior of a 3-simplex of $K$. We define the mapping $R : |K| \times [0, 1] \to |K|$ which associates for any $x \in |K|$ and for any $t \in [0, 1]$, to $(x, t)$ the point a distance $t \cdot d(y, x)$ from $x$ along the geodesic segment $[y, x]$. We note that $R$ is a continuous retraction of $|K|$ to $y$. $R(|K| \times [0, 1])$ is therefore contractible and then simply connected. Let $m, n, s, t$ be the vertices of a tetrahedron $\delta$ in $R(|K| \times [0, 1])$ such that the segment $[m, n]$ either belongs to a 1-simplex (2-simplex) that is the face of a single 3-simplex in the complex or it is itself the 1-dimensional face of a single 3-simplex in the complex. For each tetrahedron $\delta$, we deformation retract $R(|K| \times [0, 1])$ by pushing in $\delta$ starting at $[m, n]$. We obtain each time a subspace $|K'| = R(|K| \times [0, 1])$ which remains simply connected and, by Proposition 3.18, nonpositively curved. So $|K'|$ is a CAT(0) space. Any two points in $|K'|$ are therefore joined by a unique geodesic
segment in $[K']$. If at a certain step we delete the point $y$, we fix another point in the interior of a 3-simplex of $K'$, define the mapping $R$ as before and retract $[K']$ by CAT(0) subspaces further. Because $K$ is finite we reach, after a finite number of steps, a 2-dimensional spine $L$ which is also a CAT(0) space. So, by [Laz10] (Theorem 3.1.10), $L$ can be collapsed further through CAT(0) subspaces to a point. Note that Property A refers only to segments intersecting 3-simplices in $U$. The 2-dimensional spine $L$ does therefore no longer fulfill Property A. Hence we may indeed apply [Laz10], Theorem 3.1.10.

□

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