AN APPLICATION OF FUNCTIONAL EQUATIONS FOR GENERATING $\varepsilon$-INVARIANT MEASURES

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Abstract. Let $(X, \mathcal{A}, \mu)$ be a probability space and let $S : X \to X$ be a measurable transformation. Motivated by the paper of K. NIKODEM [Czechoslovak Math. J. 41(116) (4) (1991) 565–569], we concentrate on a functional equation generating measures that are absolutely continuous with respect to $\mu$ and $\varepsilon$-invariant under $S$. As a consequence of the investigation, we obtain a result on the existence and uniqueness of solutions $\varphi \in L^1([0, 1])$ of the functional equation

$$\varphi(x) = \sum_{n=1}^{N} |f_n'(x)| \varphi(f_n(x)) + g(x),$$

where $g \in L^1([0, 1])$ and $f_1, \ldots, f_N : [0, 1] \to [0, 1]$ are functions satisfying some extra conditions.

1. Introduction

The aim of this paper is to study the problem of the existence of solutions $\varphi \in L^1(X)$ of the following equation

$$(1.1) \quad \varphi = P\varphi + g,$$

where $g \in L^1(X)$ and $P : L^1(X) \to L^1(X)$ are given. In the rest of the introduction, we let $X = [0, 1]$ be equipped with the Borel $\sigma$-algebra and the Lebesgue measure. The motivation for studying such a problem is twofold. An original impulse for our investigation came from the paper [7], where integrable solutions of the equation

$$(1.2) \quad \varphi(x) = \frac{1}{2}\varphi\left(\frac{x}{2}\right) + \frac{1}{2}\varphi\left(\frac{x+1}{2}\right) + g(x)$$

were investigated in connection with $\varepsilon$-invariant measures under the 2-adic transformation. The next section contains more details on $\varepsilon$-invariant measures and on functional equations associated with them. Let us note here only that equation (1.2) is a very particular case of

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an interesting functional equation of the form

\begin{equation}
\varphi(x) = \sum_{n=1}^{N} |f'_n(x)| \varphi(f_n(x)) + g(x).
\end{equation}

We will always assume that \( N \geq 2 \).

The second inspiration to study integrable solutions of equation (1.3), and hence also of (1.1), is strictly connected with a problem posed by Janusz Matkowski in [3] and a question posed during the 47th International Symposium on Functional Equations by Jacek Wesolowski in connection with probability measures investigated in [4]. Namely, assume that \( f_1, \ldots, f_N : [0,1] \rightarrow [0,1] \) are strictly increasing contractions satisfying

\begin{equation}
(f_n((0,1)) \cap f_m((0,1)) = \emptyset \quad \text{for all } n \neq m
\end{equation}

and consider the class \( \mathcal{C} \) consisting of all increasing and continuous functions \( \phi : [0,1] \rightarrow [0,1] \) such that \( \phi(0) = 0, \phi(1) = 1 \) and

\[ \phi(x) = \sum_{n=1}^{N} \phi(f_n(x)) - \sum_{n=1}^{N} \phi(f_n(0)). \]

Wide classes of singular functions belonging to the class \( \mathcal{C} \) were constructed in [5, 6]. So far we have some idea how singular functions from the class \( \mathcal{C} \) look like, but we do not know much about absolutely continuous functions from \( \mathcal{C} \). However, we know that under a weak assumption on the functions \( f_n \), each function belonging to the class \( \mathcal{C} \) can be expressed as a convex combination of absolutely continuous and singular functions from \( \mathcal{C} \). Finally, observe that (again under weak assumptions on the functions \( f_n \)) absolutely continuous functions belonging to the class \( \mathcal{C} \) are in one-to-one correspondence with densities satisfying (1.3) with \( g = 0 \).

2. Preliminaries

In this section we explain more precisely our impulse for studying integrable solutions of equation (1.3), as well as of its generalization (1.1). We begin with recalling some definitions and results useful in the main part of this paper.

Throughout this paper we assume that \((X, \mathcal{A}, \mu)\) is a probability space and \( S : X \rightarrow X \) is a measurable transformation.

In the case where \( X \) is a Borel subset of \( \mathbb{R} \) we assume that \( \mathcal{A} \) is the \( \sigma \)-algebra \( \mathcal{B}(X) \) of all Borel subsets of \( X \) and \( \mu \) is the Lebesgue measure restricted to \( \mathcal{B}(X) \).

We say that \( S \) is \textit{nonsingular} if \( \mu(S^{-1}(A)) = 0 \) for every \( A \in \mathcal{A} \) such that \( \mu(A) = 0 \). We say that \( S \) is \textit{measure preserving} if \( \mu(S^{-1}(A)) = \mu(A) \) for every \( A \in \mathcal{A} \); we will alternately say that the measure \( \mu \) is \textit{invariant} under \( S \) if \( S \) is measure preserving. Observe that every measure preserving transformation is nonsingular.
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Fix a real number $\varepsilon \geq 0$. A probability measure $\nu$ defined on $\mathcal{A}$ is said to be $\varepsilon$-invariant under $S$ if

$$|\nu(S^{-1}(A)) - \nu(A)| \leq \varepsilon \mu(A)$$

for every $A \in \mathcal{A}$. It is clear that every measure that is 0-invariant under $S$ is invariant under $S$, and so the concept of measures $\varepsilon$-invariant under $S$ generalizes the notion of measures invariant under $S$.

From now on, for a nonsingular $S$ we denote by $P_S$ the corresponding Frobenius–Perron operator, i.e. $P_S : L^1(X) \to L^1(X)$ is the operator uniquely defined by the equation

$$\int_A P_S f(x) d\mu(x) = \int_{S^{-1}(A)} f(x) d\mu(x) \quad \text{for every } A \in \mathcal{A}. \quad (2.1)$$

The operator $P_S$ is linear and continuous. If $S$ is nonsingular, then every $m$-th iterate $S^m$ of $S$ is also nonsingular and the Frobenius-Perron operator corresponding to $S^m$ is the $m$-th iterate $P^m_S$ of $P_S$. Here and throughout we adopt the convention that $P^0 = \text{id}_X$ for every operator $P : L^1(X) \to L^1(X)$. In the case where $X = [0, 1]$ and $\mu$ is the one-dimensional Lebesgue measure the Frobenius–Perron operator corresponding to $S$ can be written explicitly as follows

$$P_S f(x) = \frac{d}{dx} \int_{S^{-1}([0,x])} f(y) dy \quad (2.2)$$

(see [2, Formula 1.2.7]).

A useful tool for studying measures $\varepsilon$-invariant under nonsingular $S$ that are absolutely continuous with respect to $\mu$ reads as follows.

**Theorem 2.1** (see [7, Theorem 1]). A finite measure $\nu$ that is absolutely continuous with respect to $\mu$ is $\varepsilon$-invariant under a nonsingular $S$ if and only if the Radon–Nikodym derivative $f$ of $\nu$ with respect to $\mu$ satisfies

$$|P_S f(x) - f(x)| \leq \varepsilon \quad \text{for } \mu\text{-almost all } x \in X. \quad (2.1)$$

In the case where $\varepsilon = 0$ in Theorem 2.1 we recognize the well known fact saying that an absolutely continuous measure $\nu$ with respect to $\mu$ is invariant under a nonsingular $S$ if and only if the Radon-Nikodym derivative $f$ of $\nu$ with respect to $\mu$ is a fixed point of the Frobenius–Perron operator $P_S$ (see [2, Theorem 4.1.1]).

From Theorem 2.1 we see that to find measures $\varepsilon$-invariant under a nonsingular transformation $S$ it is enough to solve, in the space $L^1(X)$, equation (1.1) with $P = P_S$ and $|g| \leq \varepsilon$. Therefore, we are interested in finding solutions $\varphi \in L^1(X)$ of the following special case of equation (1.1)

$$\varphi = P_S \varphi + g. \quad (2.3)$$

We will do it for a wide class of important transformations. But before we introduce the class, observe that according to (2.1) a necessary
condition for \( g \in L^1(X) \) in order that equation (2.3) has a solution \( \varphi \in L^1(X) \) is

\[
\int_X g(x) d\mu(x) = 0. \tag{2.4}
\]

A measure preserving \( S \) such that \( S(A) \in \mathcal{A} \) for every \( A \in \mathcal{A} \) is said to be exact if \( \lim_{m \to \infty} \mu(S^m(A)) = 1 \) for every \( A \in \mathcal{A} \) such that \( \mu(A) > 0 \). The following characterization of exactness is well known.

**Theorem 2.2** (see [2, Corollary 4.4.1]). A measure preserving \( S \) is exact if and only if for every \( f \in L^1(X) \) the sequence \( (P^m_S f)_{m \in \mathbb{N}} \) converges in \( L^1(X) \) to \( \int_X f(x) d\mu(x) \).

### 3. A generalization of a Nikodem result

We begin this section with a result on the existence of integrable solutions of equation (2.3), whose proof is a direct trace of the proof of Theorem 2 from [7], but for the convenience of the readers we repeat it.

**Theorem 3.1.** Assume that \( g \in L^1(X), S \) is exact and \( P_S 1 = 1 \). Then equation (2.3) has a solution in \( L^1(X) \) if and only if the series \( \sum_{m=0}^{\infty} P^m_S g \) converges in \( L^1(X) \). Moreover, every solution \( \varphi \in L^1(X) \) of equation (2.3) is of the form

\[
\varphi = \sum_{m=0}^{\infty} P^m_S g + c,
\]

where \( c \) is a real constant.

**Proof.** Assume first that the series \( \sum_{m=0}^{\infty} P^m_S g \) converges in \( L^1(X) \). Fix a real constant \( c \) and set \( \varphi = \sum_{m=0}^{\infty} P^m_S g + c \). The linearity and continuity of \( P_S \) jointly with the equality \( P_S 1 = 1 \) imply

\[
P_S \varphi + g = \sum_{m=0}^{\infty} P^m_S g + P_S c + g = \sum_{m=0}^{\infty} P^m_S g + c = \varphi.
\]

Assume now that \( \varphi \in L^1(X) \) satisfies (2.3). Then, by the linearity of \( P_S \), we have

\[
P_S^k \varphi = P_S^{k+1} \varphi + P_S^k g
\]

for every \( k \in \mathbb{N} \). Adding the above equation over \( k = 0, \ldots, m \) leads to

\[
\sum_{k=0}^{m} P_S^k g = \varphi - P_S^{m+1} \varphi.
\]

for every \( m \in \mathbb{N} \). Finally, passing with \( m \) to \( \infty \) and making use of Theorem 2.2 we conclude that the series \( \sum_{k=0}^{\infty} P_S^k g \) converges in \( L^1(X) \) and that

\[
\sum_{k=0}^{\infty} P_S^k g = \varphi - \int_X \varphi(x) d\mu(x),
\]
which completes the proof.

Now we define a transformation $S$ whose Frobenius–Perron operator appears in equation (1.3). For this purpose put $X = [0, 1]$ and fix strictly monotone functions $f_1, \ldots, f_N : [0, 1] \to [0, 1]$ satisfying (1.4). Note that the functions $f_n$ are differentiable almost everywhere on $[0, 1]$ as they are monotone. Now define the announced transformation $S : [0, 1] \to [0, 1]$ by putting

$$ S(x) = \begin{cases} f_n^{-1}(x) & \text{for } x \in f_n((0, 1)) \text{ and } n \in \{1, \ldots, N\}, \\ 0 & \text{for } x \not\in \bigcup_{n=1}^N f_n((0, 1)). \end{cases} $$

Clearly, $S$ is well defined by (1.4). Moreover, it is nonsingular provided that all the functions $f_1, \ldots, f_N$ satisfy Luzin’s condition N (or equivalently all the inverses $f_1^{-1}, \ldots, f_N^{-1}$ are nonsingular) and

$$ \bigcup_{n=1}^N f_n([0, 1]) = [0, 1]. $$

Applying formula (2.2) we see that the Frobenius–Perron operator $P_S$ corresponding to a nonsingular $S$ defined by formula (3.1) is of the form

$$ P_S f = \sum_{n=1}^N |f_n'| (f \circ f_n). $$

Therefore, in the case where $S$ is defined by formula (3.1) equation (2.3) reduces to equation (1.3) and Theorem 3.1 implies the following result.

**Corollary 3.2.** Assume that $g \in L^1([0, 1])$ and $f_1, \ldots, f_N : [0, 1] \to [0, 1]$ are strictly monotone nonsingular functions satisfying (1.4) and

$$ \sum_{n=1}^N |f_n'(x)| = 1 \quad \text{for almost all } x \in [0, 1]. $$

If $S : [0, 1] \to [0, 1]$ defined by formula (3.1) is exact, then equation (1.3) has a solution in $L^1([0, 1])$ if and only if the series

$$ \sum_{m=1}^\infty \sum_{n_1, \ldots, n_m=1}^N \left( \prod_{k=1}^m |f_{n_k}' \circ f_{n_{k-1}} \circ \cdots \circ f_{n_1}| \right) g \circ f_{n_m} \circ f_{n_{m-1}} \circ \cdots \circ f_{n_1} $$

converges in $L^1([0, 1])$. Moreover, every solution $\varphi \in L^1([0, 1])$ of equation (1.3) is of the form

$$ \varphi = g + \sum_{m=1}^\infty \sum_{n_1, \ldots, n_m=1}^N \left( \prod_{k=1}^m |f_{n_k}' \circ f_{n_{k-1}} \circ \cdots \circ f_{n_1}| \right) g \circ f_{n_m} \circ f_{n_{m-1}} \circ \cdots \circ f_{n_1} + c, $$

where $c$ is a real constant.
There is a known criterion (being very close to a characterization) of exactness of nonsingular transformations (see [2, Proposition 5.6.2 and Remark 5.6.1 on p. 111]). A wide class of interesting examples of exact transformations defined by formula (3.1), including transformations studied by Rényi in [9] and by Rohlin in [10] (see also [11]), can be found in [2, Chapter 6.2]. Now we give one of the simplest possible realization of the assumptions of Corollary 3.2. For this purpose, fix non-zero real numbers $\alpha_1, \ldots, \alpha_N$ such that

$$\sum_{n=1}^{N} |\alpha_n| = 1 \quad (3.5)$$

and put

$$\beta_n = \sum_{k=1}^{n} |\alpha_k| - \frac{|\alpha_n| + \alpha_n}{2}, \quad (3.6)$$

for all $x \in [0, 1]$ and $n \in \{1, \ldots, N\}$. Clearly, (3.5) implies (3.4). Moreover, it is well known that in the considered case the transformation defined by formula (3.1) is exact (see [2, Theorem 6.2.1, Definition 5.6.2 and Proposition 5.6.2]). Thus by Corollary 3.2 we conclude that any solution $\varphi \in L^1([0, 1])$ of the equation

$$\varphi(x) = \sum_{n=1}^{N} |\alpha_n| \varphi(\alpha_n x + \beta_n) + g(x) \quad (3.7)$$

is of the form

$$\varphi(x) = \sum_{m=1}^{\infty} \sum_{n_1, \ldots, n_m=1}^{N} \left( \prod_{k=1}^{m} |\alpha_{n_k}| \right) g \left( \prod_{i=1}^{m} \alpha_{n_i} x + \sum_{i=1}^{m} \beta_{n_i} \prod_{j=i+1}^{m} \alpha_{n_j} \right) + g(x) + c, \quad (3.8)$$

where $c \in \mathbb{R}$; here we adopt the convention that $\prod_{j=m+1}^{m} a_j = 1$ for all $m \in \mathbb{N}$ and $a_j \in \mathbb{R}$.

Finally, note that Corollary 3.2 reduces to Theorem 2 from [7] in the case where $N = 2$ and $f_1, f_2$ are given by (3.6) with $\alpha_1 = \alpha_2 = \frac{1}{2}$.

4. Examples

In this section we give four examples of measures $\varepsilon$-invariant under a given nonsingular transformation. The first two examples will be constructed by hand, whereas in the next two we will apply Theorem 3.1 and Corollary 3.2.

We begin with the general observation that every measure invariant under a nonsingular transformation $S$ generates a large class of measures that are $\varepsilon$-invariant under $S$.

\footnote{There are actually two remarks numbered 5.6.1 in the book.}
Example 4.1. Assume that $S$ is nonsingular and $\mu$ is invariant under $S$. Fix sets $A_1, \ldots, A_m \in \mathcal{A}$ and real numbers $\varepsilon, \varepsilon_1, \ldots, \varepsilon_m > 0$ such that $\varepsilon = \sum_{i=1}^{m} \varepsilon_i$. Next define a finite measure $\nu$ on $\mathcal{A}$ by putting

$$\nu(A) = \sum_{i=1}^{m} \varepsilon_i \mu(A \cap A_i).$$

Then observe that for every $A \in \mathcal{A}$ we have

$$|\nu(S^{-1}(A)) - \nu(A)| \leq \sum_{i=1}^{m} \varepsilon_i |\mu(S^{-1}(A) \cap A_i) - \mu(A \cap A_i)|$$

$$\leq \sum_{i=1}^{m} \varepsilon_i \max \{\mu(S^{-1}(A) \cap A_i), \mu(A \cap A_i)\}$$

$$\leq \sum_{i=1}^{m} \varepsilon_i \max \{\mu(S^{-1}(A) \cap A_i), \mu(A)\} = \varepsilon \mu(A).$$

Note that if $x_1, \ldots, x_m$ are fixed points of a nonsingular $S$, then every convex combination of the Dirac measures $\delta_{x_1}, \ldots, \delta_{x_m}$ is a probability measure that is invariant under $S$. Thus Example 4.1 shows that any transformation defined by formula (3.1) with strictly monotone functions $f_1, \ldots, f_N$ satisfying (1.4), (3.2) and sending sets of measure zero to sets of measure zero has $\varepsilon$-invariant measures.

The next example is of similar type as the first one, however the difference is that we will not assume the existence of a measure that is invariant under $S$. It concerns transformations defined by formula (3.1) with $X = [0,1)$ and $f_n$ of the form (3.6).

Example 4.2. Fix positive real numbers $\alpha_1, \ldots, \alpha_N$ satisfying (3.5) and let for every $n \in \{1, \ldots, N\}$ the function $f_n$ be given by (3.6). Then put $X = [0,1)$ and consider the transformation $S: [0,1) \to [0,1)$ defined by (3.1). Obviously, $S$ is nonsingular.

For all $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \{1, \ldots, N\}$ we put

$$I_{n_k, \ldots, n_1} = \{f_{n_k} \circ \cdots \circ f_{n_1}(0), f_{n_k} \circ \cdots \circ f_{n_1}(1)\}.$$
Next for every \( k \in \mathbb{N} \) we put
\[
S_k = \{ I_{n_k, \ldots, n_1} : n_1, \ldots, n_k \in \{1, \ldots, N\} \} \quad \text{and} \quad S_0 = \{[0, 1)\}.
\]

It is easy to see that the family \( S = \bigcup_{k=0}^{\infty} S_k \cup \{\emptyset\} \) is a semi-algebra of subsets of the interval \([0, 1)\); i.e. \([0, 1) \in S \) and \( I, J \in S \) implies \( I \cap J \in S \) and \([0, 1) \setminus I \) can be expressed as a finite disjoint union of sets in \( S \) (see [3, Definition 1.4.1]). Note that the \( \sigma \)-algebra generated by the semi-algebra \( S \) coincides with the family \( B([0, 1)) \) of all Borel subsets of the interval \([0, 1)\).

Fix \( \varepsilon \in [0, 1] \), two different numbers \( p, q \in \{1, \ldots, N\} \) and define a function \( \xi : \{0, \ldots, N\} \to \mathbb{R} \) by putting
\[
\xi(n) = \begin{cases} 
\varepsilon \min\{\alpha_p, \alpha_q\} & \text{for } n = p, \\
-\varepsilon \min\{\alpha_p, \alpha_q\} & \text{for } n = q, \\
0 & \text{for } n \not\in \{p, q\}.
\end{cases}
\]

Clearly,
\[
(4.4) \quad \sum_{n=1}^{N} \xi(n) = 0.
\]

Now we want to define a probability measure \( \nu_0 \) on \( S \); i.e. a function \( \nu_0 : S \to [0, 1] \) such that \( \nu_0(\emptyset) = 0 \) and \( \nu_0(\bigcup_{n \in \mathbb{N}} J_n) = \sum_{n \in \mathbb{N}} \nu_0(J_n) \) for all pairwise disjoint elements \( (J_n)_{n \in \mathbb{N}} \) of \( S \) with \( \bigcup_{n \in \mathbb{N}} J_n \in S \) (see [3, Section 2.3]). We will do it inductively.

In the first step we define \( \nu_0 \) on \( S_0 \cup S_1 \) by putting
\[
\nu_0([0, 1)) = 1 \quad \text{and} \quad \nu_0(I_n) = \alpha_n + \xi(n).
\]

It is clear that \( \nu_0(I_n) \in [0, 1] \) for every \( I_n \in S_1 \). Moreover, \((3.5)\) and \((4.4)\) imply
\[
\nu_0([0, 1)) = 1 = \sum_{n=1}^{N} \alpha_n = \sum_{n=1}^{N} \nu_0(I_n),
\]

and we see (according to \((4.3)\) and \((4.1)\) with \( k = 1 \)) that \( \nu_0 \) is well defined on \( S_0 \cup S_1 \).

To do the inductive step we assume that for a fixed \( k \in \mathbb{N} \) we have defined \( \nu_0 \) on \( \bigcup_{n=0}^{k} S_k \). Now we define \( \nu_0 \) on \( S_{k+1} \) by putting
\[
\nu_0(I_{n_k+1, \ldots, n_1}) = (\alpha_{n_{k+1}} + \xi(n_{k+1})) \prod_{i=1}^{k} \alpha_{n_i}.
\]

(Note that adopting the convention that \( \prod_{i=1}^{0} \alpha_{n_i} = 1 \) the above formula for \( \nu_0 \) coincides with that from the first step.) Applying \((3.5)\) we obtain
\[
(4.5) \quad \sum_{n=1}^{N} \nu_0(I_{n_k, \ldots, n_1, n}) = (\alpha_{n_k} + \xi(n_k)) \prod_{i=1}^{k-1} \alpha_{n_i} \sum_{n=1}^{N} \alpha_n = \nu_0(I_{n_k, \ldots, n_1})
\]
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for every $I_{n_k,\ldots,n_1} \in S_k$, which means (according to (4.2) and (4.1)) that
$\nu_0$ is well defined on $\bigcup_{n=0}^{k+1} S_k$.

Finally, putting $\nu_0(\emptyset) = 0$ we have defined $\nu_0$ on $S$.

It remains to prove that $\nu_0$ is $\sigma$-additive. For this purpose fix a
pairwise disjoint sequence $(J_m)_{m \in \mathbb{N}}$ of elements of the semi-algebra $S$
such that $\bigcup_{m \in \mathbb{N}} J_m = J \in S$. It simplifies the argument, and causes
no loss of generality, to assume $J = [0, 1)$. Thus we need to show that
$\sum_{m \in \mathbb{N}} \nu_0(J_m) = 1$.

Define a nondecreasing sequence $(k_m)_{m \in \mathbb{N}}$ of integers in such a way
that $\{J_1, \ldots, J_m\} \subset \bigcup_{k=1}^{k_m} S_k$ for every $m \in \mathbb{N}$. Note that for all $m \in \mathbb{N}$,
$l \in \{1, \ldots, m\}$ and $I \in S_{k_m}$, we have either $J_l \cap I = \emptyset$ or $J_l \cap I = I$.

Next for every $m \in \mathbb{N}$ put
$$D_m = \left\{ I \in S_{k_m} : I \subset \bigcup_{l=1}^{m} J_l \right\} \quad \text{and} \quad d_m = \sum_{I \in S_{k_m} \setminus D_m} \nu_0(I).$$

Making use of (4.2), (4.5) and (4.3) we conclude that
$$\sum_{l=1}^{m} \nu_0(J_l) = \sum_{I \in D_m} \nu_0(I) = \sum_{I \in S_{k_m}} \nu_0(I) - d_m = 1 - d_m.$$

Now it is enough to show that
(4.6) \[ \lim_{m \to \infty} d_m = 0. \]

By the definition of $\nu_0$ it is easy to see that for every $I \in S$ we have
$\nu_0(I) \leq 2l(I)$;

here and later on the symbol $l$ denotes the Lebesgue measure on the
real line. This jointly with (4.1) yields
$$d_m \leq 2 \sum_{I \in S_{k_m} \setminus D_m} l(I) = 2l \left( \bigcup_{I \in S_{k_m} \setminus D_m} I \right) = 2l \left( \bigcup_{l=m+1}^{\infty} J_l \right).$$

Passing with $m$ to $\infty$ we get (4.6).

Thus we have proved that $\nu_0$ is a probability measure defined on the
semi-algebra $S$.

Extend $\nu_0$ to a probability measure $\nu : \mathcal{B}([0, 1)) \to [0, 1]$; such an
extension exists and it is unique (see [8, Corollary 2.4.9 and Proposition 2.5.1]).

Now we will show that the measure $\nu$ is $\varepsilon$-invariant under $S$.

For this purpose fix $I_{n_k,\ldots,n_1} \in S_k$. Then
$$S^{-1}(I_{n_k,\ldots,n_1}) = \bigcup_{n=1}^{N} I_{n,n_k,\ldots,n_1},$$
which jointly with (4.1), (3.5) and (4.4) implies

\[ \nu(S^{-1}(I_{n_k,\ldots,n_1})) = \sum_{n=1}^{N} \nu_0(I_{n,n_k,\ldots,n_1}) = \prod_{i=1}^{k} \alpha_{n_i} \sum_{n=1}^{N} (\alpha_n + \xi(n)) \]

\[ = \prod_{i=1}^{k} \alpha_{n_i} = \nu(I_{n_k,\ldots,n_1}) - \xi(n_k) \prod_{i=1}^{k-1} \alpha_{n_i}. \]

In consequence,

\[ |\nu(S^{-1}(I_{n_k,\ldots,n_1})) - \nu(I_{n_k,\ldots,n_1})| \leq |\xi(n_k)| \prod_{i=1}^{k-1} \alpha_{n_i} \leq \varepsilon l(I_{n_k,\ldots,n_1}). \]

Fix a set \( B \in \mathcal{B}([0,1]) \), a number \( \delta > 0 \) and choose a countable family \( \{ F_j : j \in J \} \) of pairwise disjoint elements of the semi-algebra \( S \) such that \( \bigcup_{j \in J} F_j \subset B \),

\[ |\nu(B) - \nu\left( \bigcup_{j \in J} F_j \right)| \leq \delta \quad \text{and} \quad |\nu(S^{-1}(B)) - \nu\left( S^{-1} \left( \bigcup_{j \in J} F_j \right) \right)| \leq \delta; \]

such a family exists, because on any complete separable metric space any finite Borel measure is regular (see [1, Theorem 7.1.4]). Then making use of (4.7) we get

\[ |\nu(S^{-1}(B)) - \nu(B)| \leq \sum_{j \in J} |\nu(S^{-1}(F_j)) - \nu(F_j)| + 2\delta \]

\[ \leq \varepsilon l(F_j) + 2\delta \leq \varepsilon l(B) + 2\delta. \]

Finally, tending with \( \delta \) to 0 we conclude that the measure \( \nu \) is \( \varepsilon \)-invariant under \( S \).

To give the next two examples we need the following observation.

**Remark 4.3.**

(i) If \( \varphi \) solves (2.3), then \( \int_X g(x) d\mu(x) = 0 \).

(ii) If \( P_S g = 0 \), then \( \int_X g(x) d\mu(x) = 0 \).

(iii) If \( P_S g = 0 \), then \( P_m S g = 0 \) for every \( m \in \mathbb{N} \).

**Proof.** The first two statements are an immediate consequence of (2.1), whereas the third one follows from the linearity of \( P_S \).

Remark 4.3 helps us to apply Theorem 3.1. Indeed, assertion (i) says that to find an integrable solution of (2.3) we must assume that \( g \) has integral equals zero over \( X \). In view of assertion (ii) it can be realized by choosing \( g \) in such a way that \( P_S g = 0 \). Having chosen such a \( g \), assertion (iii) implies the convergence of the series \( \sum_{m=0}^{\infty} P_m S g \) to \( g \). Concluding, if we fix \( \varepsilon \in [0,1] \) and \( g \in L^1(X) \) such that \( |g| \leq \varepsilon \) and
\( P_S g = 0 \), then \( g + 1 \) is the density of a probability measure that is \( \varepsilon \)-invariant under \( S \), i.e. the formula

\[
(4.8) \quad \nu(A) = \mu(A) + \int_A g(x) d\mu(x) \quad \text{for every } A \in \mathcal{A}
\]
defines a probability measure that is \( \varepsilon \)-invariant under \( S \).

To see that in many cases \( g \) can be fixed in such a way that \( |g| \leq \varepsilon \) and \( P_S g = 0 \), we consider in the next two examples the case where \( X = [0,1] \) and \( S \) is defined by formula \( (3.1) \).

**Example 4.4.** Fix strictly monotone functions \( f_1, \ldots, f_N : [0,1] \to [0,1] \) satisfying \( (1.4), (3.2), (3.4) \). We assume that \( f_N(0) \geq f_1(1) \) for \( n \leq N - 1 \) and further that \( |f'_N| \geq \frac{1}{2} \). Then choose an integrable function \( g_0 : [0,f_X(0)] \to [0,\varepsilon] \) and extend it to an integrable function \( g : [0,1] \to [0,\varepsilon] \) by putting \( g = -\frac{1}{|f'_N|} \sum_{n=1}^{N-1} |f'_n|(g_0 \circ f_n \circ f_N^{-1}) \) on \( (f_N(0),1) \). Since in the considered case \( P_S \) is of the form \( (3.3) \), we have \( P_S g = \sum_{n=1}^{N} |f'_n|(g \circ f_n) = 0 \).

The last example shows not only how to choose a \( g \) with \( |g| \leq \varepsilon \) and \( P_S g = 0 \), but how to choose such a \( g \) to calculate the integral in \( (1.8) \).

**Example 4.5.** Put \( X = [0,1] \) and let \( S \) be defined by formula \( (3.1) \) with the \( f_n \) given by \( (3.3) \), where \( \alpha_1, \ldots, \alpha_N \) are non-zero real numbers satisfying \( (3.5) \). Fix \( \varepsilon \in (0,1] \) and real numbers \( \gamma_1, \ldots, \gamma_N \in [-\varepsilon,\varepsilon] \) such that \( \sum_{n=1}^{N} |\alpha_n| \gamma_n = 0 \). Choose \( g \) to be a constant equal to \( \gamma_n \) on every interval \( I_n = (\min\{f_n(0), f_n(1)\}, \max\{f_n(0), f_n(1)\}) \). Then \( P_S g = \sum_{n=1}^{N} |\alpha_n| \gamma_n = 0 \). Finally, according to \( (1.8) \) we conclude that the formula

\[
\nu(A) = \sum_{n=1}^{N} (1 + \gamma_n) l(A \cap I_n)
\]
defines a Borel probability measure that is \( \varepsilon \)-invariant under \( S \).

5. Further results

We begin this section with a generalization of Theorem \( 3.1 \) To formulate the result, we recall some definitions.

A linear operator \( P : L^1(X) \to L^1(X) \) is said to be a Markov operator if \( Pf \geq 0 \) and \( \| Pf \| = \| f \| \) for every \( f \in L^1(X) \) such that \( f \geq 0 \). It is easy to see that every Frobenius–Perron operator is a special type of Markov operator. We say that a sequence \( (f_m)_{m \in \mathbb{N}} \) of functions from \( L^1(X) \) is weakly Cesàro convergent to a function \( f \in L^1(X) \) if

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} \int_X f_k(x) h(x) d\mu(x) = \int_X f(x) h(x) d\mu(x)
\]
for every \( h \in L^\infty(X) \). A Markov operator \( P : L^1(X) \to L^1(X) \) such that \( P1 = 1 \) is said to be ergodic if for every density \( f \in L^1(X) \) the sequence \( (P^m f)_{m \in \mathbb{N}} \) is weakly Cesàro convergent to \( 1 \).
Note that if \( P : L^1(X) \to L^1(X) \) is a Markov operator and \( \varphi \in L^1(X) \) is a solution of equation (1.1), then
\[
\int_X g(x) d\mu(x) = \int_X \varphi(x) d\mu(x) - \int_X P\varphi(x) d\mu(x) = 0.
\]
Thus condition (2.4) is necessary for \( g \in L^1(X) \) in order that equation (1.1) has a solution in \( L^1(X) \).

**Theorem 5.1.** Assume (2.4) and let \( P : L^1(X) \to L^1(X) \) be an ergodic Markov operator. Then equation (1.1) has a solution in \( L^1(X) \) if and only if the sequence \( \left( \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g \right)_{m \in \mathbb{N}} \) converges in \( L^1(X) \). Moreover, every solution \( \varphi \in L^1(X) \) of equation (1.1) is of the form
\[
\varphi = \lim_{m \to \infty} \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g + c,
\]
where \( c \) is a real constant.

**Proof.** Since \( P \) is an ergodic Markov operator, it follows that 1 is the unique density such that \( P1 = 1 \); indeed, assuming that there exists another density \( f \in L^1(X) \) such that \( Pf = f \), we would have
\[
\int_X f(x) h(x) d\mu(x) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} \int_X P^k f(x) h(x) d\mu(x) = \int_X h(x) d\mu(x)
\]
for every \( h \in L^\infty(X) \), which is impossible in the case where \( f \neq 1 \). Now from [2, Theorem 5.2.2] (see also Proposition 5.2.1 in the same source) we conclude that for every density \( f \in L^1(X) \) the sequence \( \left( \frac{1}{m} \sum_{k=1}^{m} P^k f \right)_{m \in \mathbb{N}} \) converges to 1 in \( L^1(X) \), and by the linearity of \( P \) we deduce that for every \( f \in L^1(X) \) the sequence \( \left( \frac{1}{m} \sum_{k=1}^{m} P^k f \right)_{m \in \mathbb{N}} \) converges to \( \int_X f(x) d\mu(x) \) in \( L^1(X) \). In particular, making use of (2.4), that is that the integral of \( g \) over \( X \) vanishes, we obtain
\[
(5.1) \quad \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} P^k g = 0.
\]

Assume first that the sequence \( \left( \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g \right)_{m \in \mathbb{N}} \) converges in \( L^1(X) \). Fix a real constant \( c \) and set \( \varphi = \lim_{m \to \infty} \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g + c \). The linearity and continuity of \( P \) jointly with the equality \( P1 = 1 \) and (5.1) imply
\[
P\varphi + g = \lim_{m \to \infty} \sum_{k=0}^{m-1} \frac{m-k}{m} P^{k+1} g + Pc + g
\]
\[
= \lim_{m \to \infty} \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g + \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} P^k g + c = \varphi.
\]
Assume now that \( \varphi \in L^1(X) \) satisfies (1.1). Then, by the linearity of \( P \), we have

\[
\frac{m-k}{m} P^k \varphi = \frac{m-k}{m} P^{k+1} \varphi + \frac{m-k}{m} P^k g
\]

for all \( m \in \mathbb{N} \) and \( k \in \{0, \ldots, m - 1\} \). Adding the above equation over \( k = 0, \ldots, m - 1 \) with fixed \( m \), leads to

\[
\varphi - \frac{1}{m} \sum_{k=1}^{m} p^k \varphi = \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g
\]

for every \( m \in \mathbb{N} \). Finally, passing with \( m \) to \( \infty \) and making use of the fact that the sequence \( \left( \frac{1}{m} \sum_{k=1}^{m} P^k \varphi \right) \) converges to \( \int_X \varphi(x) d\mu(x) \) in \( L^1(X) \), we conclude that the sequence \( \left( \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g \right) \) converges in \( L^1(X) \) and that

\[
\varphi - \int_X \varphi(x) d\mu(x) = \lim_{m \to \infty} \sum_{k=0}^{m-1} \frac{m-k}{m} P^k g,
\]

which completes the proof.

Theorem 5.1 generalizes Theorem 3.1 in two directions, because there are Markov operators that are not Frobenius–Perron operators and there are transformations \( S \) that are not exact, but the corresponding Frobenius–Perron operators are ergodic. For example, it is easy to see that the operator defined in Section 3 by formula (3.3) can be ergodic, but not exact. Moreover, it fails to be a Frobenius–Perron operator in the case where at least one of the functions \( f_n \) does not satisfy the Luzin’s condition N, but it is still a Markov operator in such a case.

The second result of this section shows that it can happen that equation (1.1) has exactly one integrable solution; note that in such a case we must have \( P^1 \neq 1 \).

**Theorem 5.2.** Assume that the operator \( P : L^1(X) \to L^1(X) \) is linear and continuous such that for every density \( f \in L^1(X) \) the sequence \( (P^m f)_{m \in \mathbb{N}} \) converges to the trivial function in \( L^1(X) \). Then equation (1.1) has a solution in the space \( L^1(X) \) if and only if the series \( \sum_{m=0}^{\infty} P^m g \) converges in \( L^1(X) \). Moreover, every solution \( \varphi \in L^1(X) \) of equation (1.1) is of the form

\[
\varphi = \sum_{m=0}^{\infty} P^m g.
\]

**Proof.** By the linearity of \( P \) it is easy to see that the sequence \( (P^m f)_{m \in \mathbb{N}} \) converges to the trivial function in \( L^1(X) \) for every \( f \in L^1(X) \).

Assume first that the series \( \sum_{m=0}^{\infty} P^m g \) converges in \( L^1(X) \). Setting \( \varphi = \sum_{m=0}^{\infty} P^m g \) and applying the linearity and continuity of \( P \) we
obtain

\[ P \varphi + g = \sum_{m=0}^{\infty} P^{m+1} g + g = \sum_{m=0}^{\infty} P^m g = \varphi. \]

Assume now that \( \varphi \in L^1(X) \) satisfies (1.1). By the linearity of \( P \) we have \( P^k \varphi = P^{k+1} \varphi + P^k g \) and hence

\[ \sum_{k=0}^{m} P^k g = \varphi - P^{m+1} \varphi \]

for every \( m \in \mathbb{N} \). Passing with \( m \) to \( \infty \) we deduce that the series \( \sum_{k=0}^{\infty} P^k g \) converges in \( L^1(X) \) and that (5.2) holds. \( \square \)

To give an example of a realization of the assumptions of Theorem 5.2 fix, to the end of this paper, strictly monotone functions \( f_1, \ldots, f_N : [0,1] \to [0,1] \) satisfying condition (1.4) and consider the operator \( P_0 : L^1([0,1]) \to L^1([0,1]) \) defined by

\[ P_0 f = \sum_{n=1}^{N} |f'_n|(f \circ f_n). \]

Obviously, \( P_0 \) is linear. To see that \( P_0 \) is continuous note that (1.4) yields

\[ \int_A P_0 f(x) dx = \sum_{n=1}^{N} \int_A |f'_n(x)| f(f_n(x)) dx = \int_{\bigcup_{n=1}^{N} f_n(A)} f(y) dy \]

for all nonnegative \( f \in L^1([0,1]) \) and Lebesgue measurable sets \( A \subset [0,1] \).

Assume now that the family \( \{f_1, \ldots, f_N\} \) forms an iterated function system and let \( A_* \) be its attractor, i.e.

\[ A_* = \bigcap_{m \in \mathbb{N}} A_m, \]

where \( A_0 = [0,1] \) and \( A_m = \bigcup_{n=1}^{N} f_n(A_{m-1}) \) for every \( m \in \mathbb{N} \). Fix a nonnegative \( f \in L^1([0,1]) \). According to (5.3) we have

\[ \|P_0^m f\| = \int_{A_m} f(y) dy \]

for every \( m \in \mathbb{N} \), and as the sequence \( (A_m)_{m \in \mathbb{N}} \) is descending we get

\[ \lim_{m \to \infty} \|P_0^m f\| = \int_{A_*} f(y) dy. \]

In consequence, we have proved the following lemma.

**Lemma 5.3.** If the family \( \{f_1, \ldots, f_N\} \) forms an iterated function system with the attractor of Lebesgue measure zero, then for every non-negative \( f \in L^1([0,1]) \) the sequence \( (P_0^m f)_{m \in \mathbb{N}} \) converges to the trivial function in \( L^1([0,1]) \).
As an immediate consequence of Theorem 5.2 and Lemma 5.3, we obtain the following result.

**Corollary 5.4.** Assume that the family \( \{f_1, \ldots, f_N\} \) forms an iterated function system and let its attractor have Lebesgue measure zero. Then equation (1.3) has a solution in \( L^1([0,1]) \) if and only if the series \( \sum_{m=0}^{\infty} P_m g \) converges in \( L^1([0,1]) \). Moreover, every solution \( \varphi \in L^1([0,1]) \) of equation (1.3) is of the form

\[
\varphi = \sum_{m=0}^{\infty} P_m g.
\]

Observe that assumptions of Corollary 5.4 are satisfied if the \( f_n \) are defined by (3.6) with real numbers \( \alpha_1, \ldots, \alpha_N \) and non-zero real numbers \( \beta_1, \ldots, \beta_N \) such that

\[
0 \leq \min\{\beta_1, \alpha_1 + \beta_1\} < \max\{\beta_1, \alpha_1 + \beta_1\} \leq \min\{\beta_2, \alpha_2 + \beta_2\} < \max\{\beta_2, \alpha_2 + \beta_2\} \leq \cdots \leq \min\{\beta_N, \alpha_N + \beta_N\} < \max\{\beta_N, \alpha_N + \beta_N\} \leq 1
\]

and

\[
\bigcup_{n=1}^{N} [\min\{\beta_n, \alpha_n + \beta_n\}, \max\{\beta_n, \alpha_n + \beta_n\}] \neq [0,1].
\]

It is easy to see that the family \( \{f_0, \ldots, f_N\} \) forms an iterated function system and its attractor \( A_* \) has Lebesgue measure zero. Thus by Corollary 5.4 we conclude (in contrast to the counterpart case from Section 3) that now equation (3.7) has exactly one solution \( \varphi \in L^1([0,1]) \) and it is of the form (3.8) with \( c = 0 \).

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