Upper bound for lifespan of solutions to certain semilinear parabolic, dispersive and hyperbolic equations via a unified test function method

Masahiro Ikeda* and Motohiro Sobajima†

Abstract. This paper is concerned with the blowup phenomena for initial-boundary value problem

\[
\begin{align*}
\tau \partial_t^2 u(x, t) - \Delta u(x, t) + a(x)\partial_t u(x, t) &= \lambda |u(x, t)|^p, & (x, t) \in C_\Sigma \times (0, T), \\
u(x, t) &= 0, & (x, t) \in \partial C_\Sigma \times (0, T), \\
u(x, 0) &= \epsilon f(x), & x \in C_\Sigma, \\
\tau \partial_t u(x, 0) &= \tau \epsilon g(x), & x \in C_\Sigma,
\end{align*}
\]

(0.1)

where \(C_\Sigma\) is a cone-like domain in \(\mathbb{R}^N\) \((N \geq 2)\) defined as \(C_\Sigma := \text{int} \left\{ \rho \omega \in \mathbb{R}^N : \rho \geq 0, \omega \in \Sigma \right\}\) with a connected open set \(\Sigma\) in \(S^{N-1}\) with smooth boundary \(\partial \Sigma\). If \(N = 1\), then we only consider two cases \(C_\Sigma = (0, \infty)\) and \(C_\Sigma = \mathbb{R}\). Here \(a(x)\) is a non-zero coefficient of \(\partial_t u\) which could be complex-valued and space-dependent, \(\lambda \in \mathbb{C}\) is a fixed constant, and \(\epsilon > 0\) is a small parameter. The constants \(\tau = 0, 1\) switch the parabolicity and hyperbolicity of the problem (0.1). The result proposes a unified treatment of estimates for lifespan of solutions to (0.1) by test function method. The Fujita exponent \(p = 1 + 2/N\) appears as a threshold of blowup phenomena for small data when \(C_\Sigma = \mathbb{R}^N\), but the case of cone-like domain with boundary the threshold changes and explicitly given via the first eigenvalue of corresponding Laplace–Beltrami operator with Dirichlet boundary condition as in Levine–Meier [24].

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1 Introduction

In this paper, we discuss the blow-up phenomena of the following initial-boundary value problem

\[
\begin{align*}
\tau \partial_t^2 u(x, t) - \Delta u(x, t) + a(x)\partial_t u(x, t) &= \lambda |u(x, t)|^p, & (x, t) \in C_\Sigma \times (0, T), \\
u(x, t) &= 0, & (x, t) \in \partial C_\Sigma \times (0, T), \\
u(x, 0) &= \epsilon f(x), & x \in C_\Sigma, \\
\tau \partial_t u(x, 0) &= \tau \epsilon g(x), & x \in C_\Sigma,
\end{align*}
\]

(1.1)

where \(C_\Sigma\) is a cone domain in \(\mathbb{R}^N\) \((N \geq 2)\) defined as

\[C_\Sigma := \text{int} \left\{ \rho \omega \in \mathbb{R}^N : \rho \geq 0, \omega \in \Sigma \right\}\]

with a connected open set \(\Sigma\) in \(S^{N-1}\) with smooth boundary \(\partial \Sigma\). If \(N = 1\), then we only consider two cases \(C_\Sigma = (0, \infty)\) and \(C_\Sigma = \mathbb{R}\). Here \(a(x)\) is a non-zero coefficient of \(\partial_t u\) satisfying

\[|a(x)| \leq a_0(x)^{-\sigma}\]

(1.2)

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with $\alpha \in [0, 1]$; note that $a(x)$ could be complex-valued and space-dependent, $\lambda \in \mathbb{C}$ is a fixed constant, and $\epsilon > 0$ is a small parameter. The initial data $(f, \tau g)$ at least belongs to the following class:

$$(f, \tau g) \in H_0^1(C_\Sigma) \cap L^2(C_\Sigma), \quad (\tau g + a f)|x|^\gamma \in L^1(C_\Sigma)$$

for some $\gamma \geq 0$. Finally, the constants $\tau = 0, 1$ switch the parabolicity and hyperbolicity of the problem (1.1).

The problem (1.1) is a unified form of several partial differential equations of parabolic, dispersive and hyperbolic type. For example, if $\tau = 0$, $a(x) \equiv 1$, $C_\Sigma = \mathbb{R}^N$ (that is, $\Sigma = S^{N-1}$), and $\lambda = 1$ with $f \geq 0$, then (1.1) becomes the usual nonlinear heat equation of Fujita type:

$$\begin{cases}
\partial_t u(x, t) - \Delta u(x, t) = u(x, t)^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) = \epsilon f(x), & x \in \mathbb{R}^N.
\end{cases} \tag{1.3}$$

If $\tau = 0$, $a(x) \equiv -i$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $C_\Sigma = \mathbb{R}^N$, then (1.1) becomes the nonlinear Schrödinger equation without gauge invariance:

$$\begin{cases}
i \partial_t u(x, t) + \Delta u(x, t) = -\lambda |u(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) = \epsilon f(x), & x \in \mathbb{R}^N.
\end{cases} \tag{1.4}$$

If $\tau = 0$, $a(x) \equiv e^{-i\xi}$, $\lambda = e^{i(\eta-\xi)}$ and $C_\Sigma = \mathbb{R}^N$, then (1.1) becomes the complex Ginzburg–Landau equation without gauge invariance:

$$\begin{cases}
\partial_t u(x, t) - e^{i\xi} \Delta u(x, t) = e^{i\eta |u(x, t)|^p}, & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) = \epsilon f(x), & x \in \mathbb{R}^N.
\end{cases} \tag{1.5}$$

Finally, if $\tau = 1$, $a(x) > 0$, $\lambda = 1$ and $C_\Sigma = \mathbb{R}^N$, then (1.1) becomes the nonlinear wave equation with space-dependent damping

$$\begin{cases}
\partial^2_t u(x, t) - \Delta u(x, t) + a(x) \partial_t u(x, t) = |u(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) = \epsilon f(x), & x \in \mathbb{R}^N, \\
\partial_t u(x, 0) = \epsilon g(x), & x \in \mathbb{R}^N.
\end{cases} \tag{1.6}$$

Moreover, we can treat halved space $C_\Sigma = \mathbb{R}^k \times \mathbb{R}^{N-k}$ when we take $\Sigma = \{(\omega_i) \in S^{N-1}; \omega_i > 0 \ (i = 1, \ldots, k)\}$. Therefore the corn-like domain $C_\Sigma$ is a generalization of domains with scale-invariance (see also Levine–Meier [24]).

The study of blowup phenomena for solutions to the respective equations has a long history. For the semilinear heat equation (1.3), the blowup solutions were found in Fujita [7] when $p < 1 + \frac{2}{N}$; the exponent $p_F = 1 + 2/N$ is well-known as the “Fujita exponent”. Then in the critical case $p = p_F$ blowup phenomena were shown by Hayakawa [11], Sugitani [41] (including fractional Laplacian) and Kobayashi–Shirao–Tanaka [22]. The sharp upper and lower estimates for lifespan of solutions to (1.3) was established in Lee–Ni [23] by using the structure of the heat kernel and the maximum principle as

$$\text{LifeSpan}(u) \sim \begin{cases} e^{-(\frac{1}{2} - \frac{p}{N})^{-1}}, & \text{if } p < 1 + \frac{2}{N}, \\
\exp(C\epsilon^{-(p-1)}), & \text{if } p = 1 + \frac{2}{N}.
\end{cases}$$

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Later, the further profile of blowup solutions including sign-changing solutions are considered by some mathematicians (see, e.g., Mizoguchi–Yanagida [32, 33], Fujishima–Ishige [5, 6] and their references therein).

For the semilinear Schrödinger equation without gauge invariance (1.4), blowup phenomena are discovered by Ikeda–Wakasugi [16] when \( p \leq p_F = 1 + 2/N \). Later the estimates of lifespan of solutions to (1.4) was found in Fujiwara–Ozawa when \( p < p_F \). The similar analysis in view of stochastic aspect can be found in Oh–Okamoto–Pocovnicu [36]. We have to remark that the estimates of lifespan in the critical case \( p = p_F \) left as an open problem in \( L^1 \)-initial data.

For the complex Ginburg–Landau equation without gauge invariance (1.5), blowup and lifespan of solutions to (1.5) in one-dimensional torus is studied by Ozawa–Yamazaki [38]. Of course the complex Ginburg–Landau equation with nonlinear term \( (\kappa + i\beta)|u|^{p-1}u \) (with gauge invariance) has been considered (see for existence, e.g., Ginible–Velo [10], Okazawa–Yokota [37], and for blowup phenomena, e.g., Masmoudi–Zaag [29], Cazenave–Dickstein–Weissler [3] and Cazenave–Correia–Dickstein–Weissler [2]).

For the nonlinear damped wave equation without gauge invariance (1.6), the blowup phenomena and estimates of the lifespan of solutions to (1.6) have been intensively studied for almost 20 years. First result should be Li–Zhou [27] and they proved blowup and upper bound of lifespan of solutions of (1.6) for \( 1 < p \leq p_F \) when \( a(x) = 1 \) and \( N = 1, 2 \). Then the same question for the case of \( N = 3 \) is answered by Nishihara [34]. For general, but subcritical case \( 1 < p < p_F \) with \( a(x) = 1 \), Todorova–Yordanov [42] established blowup phenomena of (1.6) for arbitrary dimensions. In the critical case \( p = p_F \) for general dimensions Zhang [47] obtained the same conclusion. Then the interest goes to the case of time-dependent or space-dependent damping. For time dependent case, we refer the study of Lin–Nishihara–Zhai [28], Ikeda–Wakasugi [17] and Ikeda–Ogawa [14] and the reference therein. In the case of space-dependent damping, Ikeda–Todorova–Yordanov [21] found that the threshold for global existence of global solutions with small initial data and blowup for arbitrary small initial data for (1.6) with \( a(x) \sim \langle x \rangle^{-\alpha} \) and \( \alpha \in [0, 1) \). We point out that the threshold shifts from the Fujita exponent \( p_F \) to \( p_F(\alpha) = 1 + \frac{2}{N-\alpha}\). Very recently, Lai–Zhou [26] succeeded in proving the sharp estimate of lifespan of solutions to (1.6) when \( a(x) = 1 \) and \( p = p_F \) by applying the consideration in [23].

The similar study of respective problems for halved space \( \mathbb{R}^k \times \mathbb{R}^{N-k} \) has been done separately in literature (see e.g., Meier [30, 31], Levine–Meier [24, 25] and Ikehata [18, 19, 20]). In particular, Levine–Meier [24, 25] considered the nonlinear heat equation in \( C_2 \) by using the explicit representation for heat kernel on the cone-like domain and found the corresponding threshold for blowup phenomena.

We would summarize the situation of study of blowup phenomena that although the detailed analysis has been done for respective equations in the respective cases, but many open problems are posed separately.

The purpose of the present paper is to give a unified treatment for proving the upper bound of lifespan of solutions by using test function method to the general problem (1.1) in cone-like domain including all the respective critical situations for respective equations. The crucial idea is mainly in the proof of Lemma 3.10 (see also Remark 3.3 below).

The paper is organized as follows. In Section 2, we demonstrate our technique for simple three cases \( \partial_t u - \Delta u = u^p, \partial_t^2 u - \Delta u + \partial_t u = |u|^p \) and \( i\partial_t u + \Delta u = |u|^p \) in \( \mathbb{R}^N \) to explain what is a crucial view point in our argument. In Section 3, we state some basic fact of selfadjointness of the Laplacian on \( C_2 \) endowed with Dirichlet boundary condition for treating linear equations of the respective equation, the solvability of (1.1) for each case \( \tau = 0 \) and \( \tau = 1 \) and prepare an important lemma via the unified test function in the proof of the upper bound of lifespan. Then Section 4 is devoted to give main results of the present paper and to prove them.
2 Test function method for the simple cases in whole space

The purpose of this section is to explain our test function method by using well-understood model. To do this, we begin with the following semilinear heat equation of Fujita type:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= u^p, \quad (x, t) \in \mathbb{R}^N \times (0, T), \\
\quad u(x, 0) &= \epsilon f(x), \\
\end{align*}
\]

Here we assume \( f \in C_0^\infty(\mathbb{R}^N) \) and \( f \geq 0 \). In this case, by the standard argument for semilinear equations, we can construct a unique local-in-time classical nonnegative solution \( u_\epsilon \) of \((2.1)\). Therefore we define the lifespan of solutions \( u_\epsilon \) as follows:

\[ \text{LifeSpan}(u_\epsilon) = \{ T > 0 ; \text{there exists a classical solution of } (2.1) \text{ in } [0, T) \}. \]

The following assertion was given by [23].

**Proposition 2.1.** Assume that \( f \in C_0^\infty(\mathbb{R}^N), f \geq 0 \) and \( f \not= 0 \). Let \( u_\epsilon \) be the unique classical solution of \((2.1)\). If \( 1 < p \leq 1 + \frac{2}{N} \), then \( \text{LifeSpan}(u_\epsilon) \) is well-defined. Moreover, \( \text{LifeSpan}(u_\epsilon) \) has the following upper bound: there exist constants \( \epsilon_0 > 0 \) and \( C \geq 1 \) such that for every \( \epsilon \in (0, \epsilon_0] \),

\[
\text{LifeSpan}(u_\epsilon) \leq \left\{ \begin{array}{ll}
C \epsilon^{-\left(\frac{1}{p-1} - \frac{2}{N}\right)} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\exp(C \epsilon^{-(p-1)}) & \text{if } p = 1 + \frac{2}{N}.
\end{array} \right.
\]

**Proof.** Set \( r_0 := \max(|x| : x \in \text{supp } f) \). Without loss of generality, we may assume \( R_0 := 2r_0^2 < T_\epsilon \). Put the following functions

\[
\eta(s) = \begin{cases} 
1 & \text{if } s \in [0, 1/2) \\
\text{decreasing} & \text{if } s \in (1/2, 1) \\
0 & \text{if } s \in [1, \infty), \\
\end{cases}
\]

(\( \eta \in C^\infty([0, \infty)) \)) and for \( R > 0 \), define the cut-off functions

\[
\psi_R(x, t) := \left[ \eta\left(\frac{|x|^2 + t}{R}\right) \right]^{2p'} \quad \psi_R^*(x, t) := \left[ \eta\left(\frac{|x|^2 + t}{R}\right) \right]^{2p'}.
\]

Then by the equation in \((2.1)\), we see from integration by parts that for every \( R \in [R_0, T_\epsilon) \),

\[
\int_{\mathbb{R}^N} u_\epsilon(x, t)^p \psi_R(x, t) \, dx = \frac{d}{dt} \int_{\mathbb{R}^N} u_\epsilon(x, t) \psi_R(x, t) \, dx - \int_{\mathbb{R}^N} \left( u_\epsilon(x, t) \frac{\partial \psi_R}{\partial t} + u_\epsilon(x, t) \Delta \psi_R \right) \psi_R \, dx
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^N} u_\epsilon(x, t) \psi_R(x, t) \, dx - \int_{\mathbb{R}^N} u_\epsilon(x, t) \left( \frac{\partial \psi_R}{\partial t} + \Delta \psi_R \right) \, dx.
\]

Then putting \( C = C_1 + C_2 / R_0 \) and integrating it over \((0, T_\epsilon)\), we have for every \( R \in [R_0, T_\epsilon) \)

\[
\int_{\mathbb{R}^N} f(x) \, dx + \int_0^{T_\epsilon} \int_{\mathbb{R}^N} u_\epsilon(x, t)^p \psi_R^*(x, t) \, dx \, dt \leq \frac{C}{R} \int_0^{T_\epsilon} \int_{\mathbb{R}^N} u_\epsilon(x, t) \psi_R^*(x, t) \, dx \, dt
\]

\[
\leq CR^{-\left(\frac{1}{p-1} - \frac{2}{N}\right)} \int_0^{T_\epsilon} \int_{\mathbb{R}^N} u_\epsilon(x, t)^p \psi_R^*(x, t) \, dx \, dt, \quad (2.2)
\]
where we have used $\psi_R(\cdot, 0) \equiv 1$ on supp $f$. At this moment, we put new functions $y \in C(0, T_e)$ and $Y \in C^1(0, T_e)$ as follows:

$$Y(R) := \int_0^R y(r) r^{-1} \, dr, \quad y(r) := \int_0^T \int_{\mathbb{R}^N} u(x, t)^p \psi_r(x, t) \, dx \, dt. \quad (2.3)$$

Then we have

$$\int_0^R y(r) r^{-1} \, dr = \int_0^R \left( \int_{r_0}^T \int_{\mathbb{R}^N} u_r(x, t)^p \left[ \eta^\prime \left( \frac{|x|^2 + t}{r^2} \right) \right]^{2p'} \, dx \, dt \right) r^{-1} \, dr$$

$$= \int_0^T \int_{\mathbb{R}^N} u_r(x, t)^p \left( \int_0^R \left[ \eta^\prime \left( \frac{|x|^2 + t}{r^2} \right) \right]^{2p'} \, dr \right) \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^N} u_r(x, t)^p \left( \int_{|x|^2 + t}^\infty \left[ \eta^\prime (s) \right]^{2p'} s^{-1} \, ds \right) \, dx \, dt.$$

Noting that the inequality

$$\int_\sigma^\infty [\eta^\prime (s)]^{2p'} s^{-1} \, ds \leq \log 2 \left[ \eta(\sigma) \right]^{2p'}, \quad \sigma \geq 0$$

can be verified by the decreasing property of $\eta$, we deduce

$$Y(R) \leq \log 2 \int_0^T \int_{\mathbb{R}^N} u_r(x, t)^p \psi_R(x, t) \, dx \, dt.$$

By using the function $Y$, (2.2) can be rewritten by

$$\left( \varepsilon + \frac{Y(R)}{\log 2} \right)^p \leq C R^{-(\frac{1}{p-1} - \frac{n}{2}(p-1)+1)} Y'(R), \quad R \in (R_0, T_e).$$

Therefore we obtain

$$0 \leq \limsup_{R \to T_e} \left( \varepsilon + \frac{Y(R)}{\log 2} \right)^{1-p} \leq \left( \varepsilon \|f\|_{L^1(\mathbb{R}^N)} + \frac{Y(R_0)}{\log 2} \right)^{1-p} - \frac{p-1}{C \log 2} \int_{R_0}^{T_e} r^{\frac{1}{p-1} - \frac{n}{2}(p-1)-1} \, dr$$

$$\leq \left( \varepsilon \|f\|_{L^1(\mathbb{R}^N)} \right)^{1-p} - \frac{p-1}{C \log 2} \int_{R_0}^{T_e} r^{\frac{1}{p-1} - \frac{n}{2}(p-1)-1} \, dr.$$

This implies the desired upper bound for $T_e$. \hfill \Box

Remark 2.1. The crucial point is to introduce the function $Y$. In fact, the inequality including integral for $t$ enables us to treat as a differential inequality by virtue of the use of $Y$. This consideration will be summarised in Lemma 3.10 below.

This argument is also applicable to the semilinear problem of damped wave equation

$$\begin{align*}
\partial_t^2 u_e(x, t) - \Delta u_e(x, t) + \partial_t u_e(x, t) &= |u_e(x, t)|^p, \quad (x, t) \in \mathbb{R}^N \times (0, T), \\
u_e(x, 0) &= \varepsilon f(x), \quad x \in \mathbb{R}^N, \\
\partial_t u_e(x, 0) &= \varepsilon g(x), \quad x \in \mathbb{R}^N,
\end{align*}$$

(2.4)
where we assume that \( f, g \in C_c^\infty(\mathbb{R}^N) \) with

\[
\int_{\mathbb{R}^N} f(x) + g(x) \, dx > 0.
\]

In this case existence of weak solutions to (2.4) is proved for \( 1 < p < \infty \) when \( N = 1, 2 \) and \( 1 < p < \frac{N+2}{N-2} \) when \( N \geq 3 \). The definition of lifespan is changed as follows:

\[
\text{LifeSpan}(u_\varepsilon) = \{ T > 0 \mid \text{there exists a weak solution of (2.4) in [0, T]} \}.
\]

As in the proof of Proposition 2.1, we can find the following estimate

\[
\varepsilon \int_{\mathbb{R}^N} f(x) + g(x) \, dx + \int_0^T \int_{\mathbb{R}^N} |u_\varepsilon(x, t)|^p \psi_R(x, t) \, dx \, dt \leq CR^{-\left(\frac{1}{p} - \frac{2}{N}\right)} \int_0^T \int_{\mathbb{R}^N} |u_\varepsilon(x, t)|^p \psi_R(x, t) \, dx \, dt
\]

for \( R \in (R_0, T_\varepsilon) \). Therefore by the use of the function \( Y \), we can easily prove the upper bound of lifespan of \( u_\varepsilon \).

**Proposition 2.2.** Assume that \( f, g \in C_c^\infty(\mathbb{R}^N) \), \( f \geq 0 \) and \( \int_{\mathbb{R}^N} f(x) + g(x) \, dx > 0 \). Let \( u_\varepsilon \) be the unique weak solution of (2.4). If \( 1 < p \leq 1 + \frac{2}{N} \), then \( \text{LifeSpan}(u_\varepsilon) < \infty \). Moreover, \( \text{LifeSpan}(u_\varepsilon) \) has the following upper bound: there exist constants \( \varepsilon_0 > 0 \) and \( C \geq 1 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \),

\[
\text{LifeSpan}(u_\varepsilon) \leq \begin{cases} CE^{-\left(\frac{1}{p} - \frac{2}{N}\right)}^{-1}, & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\exp(C \varepsilon^{-(p-1)}), & \text{if } p = 1 + \frac{2}{N}. \end{cases}
\]

**Remark 2.2.** The critical case \( p = 1 + \frac{2}{N} \) of Proposition 2.2 has already been proved by Lai–Zhou [26]. It is worth noticing that the proof of Proposition 2.2 is much simpler than that of [26].

Furthermore, by the same argument we can also treat semilinear Schrödinger equation without gauge invariance:

\[
\begin{cases}
\frac{\partial u_\varepsilon(x, t)}{\partial t} + \Delta u_\varepsilon(x, t) = |u_\varepsilon(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) = \varepsilon f(x), & x \in \mathbb{R}^N,
\end{cases}
\]

with \( f \in C_c^\infty(\mathbb{R}^N) \). The existence of weak solutions to (2.5) is proved for \( 1 < p < \infty \) when \( N = 1, 2 \) and \( 1 < p < \frac{N+2}{N-2} \) when \( N \geq 3 \). The definition of lifespan is changed as follows:

\[
\text{LifeSpan}(u_\varepsilon) = \{ T > 0 \mid \text{there exists a weak solution of (2.5) in [0, T]} \}.
\]

For simplicity, we suppose \( \varepsilon f(x) \in [0, \infty) \) and \( f \neq 0 \). Then multiplying \( \psi_R \) to the equation and taking the real part, we have

\[
\varepsilon \int_{\mathbb{R}^N} f(x) \, dx + \int_0^T \int_{\mathbb{R}^N} |u_\varepsilon(x, t)|^p \psi_R(x, t) \, dx \, dt \leq CR^{-\left(\frac{1}{p} - \frac{2}{N}\right)} \int_0^T \int_{\mathbb{R}^N} |u_\varepsilon(x, t)|^p \psi_R(x, t) \, dx \, dt.
\]

This gives us the fact that the essential point is completely the same as the previous cases (2.1) and (2.4). Consequently, we can obtain the following assertion.

**Proposition 2.3.** Assume that \( f \in C_c^\infty(\mathbb{R}^N) \), \( f(x) \in [0, \infty) \) and \( f \neq 0 \). Let \( u_\varepsilon \) be the unique weak solution of (2.5). If \( 1 < p \leq 1 + \frac{2}{N} \), then \( \text{LifeSpan}(u_\varepsilon) < \infty \). Moreover, \( \text{LifeSpan}(u_\varepsilon) \) has the following upper bound: there exist constants \( \varepsilon_0 > 0 \) and \( C \geq 1 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \),

\[
\text{LifeSpan}(u_\varepsilon) \leq \begin{cases} CE^{-\left(\frac{1}{p} - \frac{2}{N}\right)}^{-1}, & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\exp(C \varepsilon^{-(p-1)}), & \text{if } p = 1 + \frac{2}{N}. \end{cases}
\]
Remark 2.3. The critical case $p = 1 + \frac{2}{N}$ of Proposition 2.3 has not been proved so far. The assertion of 2.3 can be regarded as a refinement of the results of Ikeda–Wakasugi [16] and Fujiwara–Ozawa [8]. It is worth noticing that the technique by Lai–Zhou [26] seems to be difficult to apply to (2.5) because they use the positivity of heat kernel for heat equations. Despite of this difficulty, we could prove the blowup phenomena and lifespan estimates by using only the positivity of the nonlinear term.

3 Preliminaries for general cases

To generalize the argument in Section 2 into certain problems in corn-like domains stated in the introduction, we prepare some technical tools to indicate the existence of corresponding problems.

3.1 Corresponding linear equations in $C_\Sigma$

First we state the assertion for the first eigenvalue and eigenfunction of the Laplace–Beltrami operator in $\Sigma$ endowed with Dirichlet boundary condition (see [43, Chapter IX] for detail).

Lemma 3.1. The Laplace–Beltrami operator $-\Delta_\Sigma$ in $L^2(\Sigma)$ endowed with domain $H^2(\Sigma) \cap H^1_0(\Sigma)$ is selfadjoint and all resolvent operator of $-\Delta_\Sigma$ are compact. The first eigenvalue $\lambda_\Sigma$ is nonnegative and simple, and the corresponding eigenfunction $\varphi_\Sigma$ is positive in $\Sigma$. Moreover, $\lambda_\Sigma$ is positive if and only if $\Sigma \neq S^{N-1}$.

Remark 3.1. In the case $C_\Sigma = R^k \times R^{N-k}$, $\varphi_\Sigma$ and $\lambda_\Sigma$ are explicitly given by $\varphi_\Sigma(\omega) = \omega_1 \omega_2 \cdots \omega_k$ and $\lambda_\Sigma = k(N-2+k)$.

Here we define $\gamma$ as a smallest root of the following:

$$\gamma = \begin{cases} 
0 & \text{if } N = 1, C_\Sigma = R \\
1 & \text{if } N = 1, C_\Sigma = R_+ \\
\text{the positive root of } \gamma^2 + (N-2)\gamma - \lambda_\Sigma = 0 & \text{if } N \geq 2.
\end{cases}$$

Then the following assertion holds.

Lemma 3.2. Set

$$\Phi(x) = \begin{cases} 
0 & \text{if } N = 1, C_\Sigma = R \\
x & \text{if } N = 1, C_\Sigma = R_+ \\
|x|^{\gamma} \varphi_\Sigma \left( \frac{x}{|x|} \right), & x \in C_\Sigma.
\end{cases}$$

Then $\Phi$ satisfies

$$\begin{cases} 
\Delta \Phi(x) = 0 & x \in C_\Sigma, \\
\Phi(x) > 0 & x \in C_\Sigma, \\
\Phi(x) = 0 & x \in \partial C_\Sigma, \\
x \cdot \nabla \Phi(x) = \gamma \Phi(x) & x \in C_\Sigma.
\end{cases}$$

Remark 3.2. In the case $C_\Sigma = R^k \times R^{N-k}$, we can easily see that $\Phi(x) = x_1 x_2 \cdots x_k$.

Proof of Lemma 3.2. By Lemma 3.1 we can directly verify the desired properties of $\Phi$. □
Next we consider the properties of Dirichlet Laplacian in \( \Sigma \). Let \( A_{\min} \) be defined as follows:

\[
\begin{align*}
A_{\min} u &= -\Delta u, \\
D(A_{\min}) &= \{ u \in C^0_c(\mathbb{R}^N \setminus \{0\}) ; u = 0 \text{ on } \partial \Sigma \}.
\end{align*}
\]

We first prove the Hardy inequality in \( \Sigma \).

**Proof.**

We use integration by parts for the second term. Noting that

\[
\text{for every } u \in D(A_{\min}),
\]

we obtain (3.1).

**Lemma 3.3.** For every \( u \in D(A_{\min}) \),

\[
\left( \frac{N - 2}{2} + \gamma \right) \int_{C_\Sigma} \frac{|u(x)|^2}{|x|^2} dx \leq \int_{C_\Sigma} |\nabla u(x)|^2 dx.
\]

**Proof.** Since \( u \in D(A_{\min}) \) can be approximated by functions belonging to \( C^0_c(\Sigma) \) in \( H^1(\Sigma) \)-topology, it suffices to show (3.1) for \( u \in C^0_c(\Sigma) \).

Let \( u \in C^0_c(\Sigma) \) and set \( Q(r\omega) = r^{-\frac{N-2}{2}} \varphi_\Sigma(\omega) \). Setting \( v = Q^{-1} u \in C^0_c(\Sigma) \), we have

\[
\int_{C_\Sigma} |\nabla u(x)|^2 dx = \int_{C_\Sigma} Q(x)^2|\nabla \varphi_\Sigma(x)|^2 dx + 2 \int_{C_\Sigma} Q(x)\nabla Q(x) \cdot \text{Re}(\overline{\varphi_\Sigma(x)} \nabla \varphi_\Sigma(x)) dx + \int_{C_\Sigma} |\nabla Q(x)|^2|\varphi_\Sigma(x)|^2 dx
\]

where we used integration by parts for the second term. Noting that

\[
\Delta Q(r\omega) = - \left( \frac{N - 2}{2} \right)^2 r^{-\frac{N-2}{2}} \varphi_\Sigma(\omega) + r^{-\frac{N-2}{2}} \Delta \varphi_\Sigma(\omega)
\]

we obtain (3.1).

Here we prove the essential selfadjointness of \( A_{\min} \) under the condition \( \gamma \geq \frac{4N}{2} \). If \( \Sigma = \mathbb{R}^N \) and \( \gamma = 1 \), therefore this condition becomes \( N \geq 4 \) which is equivalent to that of essential selfadjointness of \( -\Delta \) with domain \( C^0_c(\mathbb{R}^N \setminus \{0\}) \) (see e.g., Reed–Simon [39, Theorems X.11 and X.30]).

**Lemma 3.4.** If \( \gamma \geq \frac{4N}{2} \), then \( A_{\min} \) is essentially selfadjoint in \( L^2(\Sigma) \).

**Proof.** To prove the essential selfadjointness of \( A_{\min} \), it suffices to show that

\[
\nu \in L^2(\Sigma), \quad \int_{C_\Sigma} (u + A_{\min} u)\nu dx = 0 \quad \forall u \in D(A_{\min})
\]

implies \( \nu = 0 \) a.e. on \( \Sigma \). Assume (3.2), noting that since the operator \( A_{\min} \) does not have pure imaginary coefficient, we may assume without loss of generality that \( \nu \) is real. By elliptic regularity we have \( C^{\infty}(\Sigma \setminus \{0\}) \) and \( \nu = 0 \) on \( \partial C_\Sigma \setminus \{0\} \). Then integration by parts yields

\[
\int_{C_\Sigma} uv + \nabla u \cdot \nabla v dx = 0 \quad \forall u \in D(A_{\min}).
\]


Fix $\zeta \in C_0^\infty(\mathbb{R})$ with $\zeta \equiv 1$ on $[-1, 1]$. For $R > 1$, set

$$\chi_R(x) = \frac{|x|}{\sqrt{1 + |x|^2}} \zeta \left(\frac{\log |x|}{R}\right), \quad u(x) = [\chi_R(x)]^2 v(x) \in D(A_{\min}).$$

Then (3.3) can be rewritten by

$$\int_{C\xi} \zeta_R^2 v^2 \, dx + \int_{C\xi} |\nabla (\chi_R v)|^2 \, dx = \int_{C\xi} |\nabla \chi_R|^2 v^2 \, dx.$$ 

Using Lemma 3.3 and computing $\nabla \chi_R$ explicitly, we have

$$\int_{C\xi} \left(\frac{|x|^2}{1 + |x|^2} + \frac{N - 2}{2} \right) \zeta \left(\frac{\log |x|}{R}\right)^2 v^2 \, dx \leq \int_{C\xi} \frac{1}{(1 + |x|^2)^2} \zeta \left(\frac{\log |x|}{R}\right) \left(\frac{\log |x|}{R}\right) v^2 \, dx.$$ 

Since all coefficient of $v^2$ are bounded and have limits, dominated convergence theorem gives

$$\int_{C\xi} \left(\frac{|x|^2}{1 + |x|^2} + \frac{N - 2}{2} \right) \zeta \left(\frac{\log |x|}{R}\right)^2 v^2 \, dx \leq \int_{C\xi} \frac{1}{(1 + |x|^2)^3} v^2 \, dx.$$ 

Therefore by $\frac{N - 2}{2} + \gamma \geq 1$ we obtain $\nu = 0$ a.e. on $C\xi$. \hfill $\square$

In view of Lemma 3.4, we denote $A$ as a closure of $A_{\min}$, that is, $A$ is selfadjoint in $L^2(C\xi)$. Noting that form domain $D(A^{1/2})$ of $A$ coincides with $H_0^1(C\xi)$, we see from the Gagliardo–Nirenberg–Sobolev inequalities that

**Lemma 3.5.** Assume $\gamma \geq \frac{4-N}{2}$. Then $D(A^{1/2})$ is continuously embedded into

$$\begin{aligned} &L^\infty(C\xi) \quad N = 1, \\
\begin{cases} &L^q(C\xi) \quad (2 < q < \infty) \quad N = 2, \\
\end{cases} &L^{2N/3}(C\xi) \quad N \geq 3. 
\end{aligned}$$

Combining all lemmas as above, by the standard argument we obtain the wellposedness of local-in-time weak solutions to (1.1) with $\tau = 0$ and $\tau = 1$. We omit both proof of propositions stated below.

**Proposition 3.6.** Assume that $\tau = 0, a(x) = e^{i\zeta}, \zeta \in [-\pi/2, \pi/2]$ and $\gamma \geq \frac{4-N}{(N-2)_+}$, and for $f \in H_0^1(C\xi)$, there exist $T = T(||f||_{H^1(C\xi)}, \varepsilon) > 0$ and a unique weak solution $u$ of (1.1) in $[0, T)$ in the following sense:

$$u \in C([0, T); H_0^1(C\xi)) \cap L^p_{loc} \overline{C\xi} \times [0, T))$$

with $u(x, 0) = \varepsilon f(x)$ and for every $\psi \in C^1([0, T); D(A))$ with supp $\psi \subset \subset \overline{C\xi} \times [0, T)$

$$\varepsilon e^{i\zeta} \int_{C\xi} f(x) \psi(x, 0) \, dx + \lambda \int_0^T \int_{C\xi} |u(x, t)|^p \psi(x, t) \, dx \, dt$$

$$= \int_0^T \int_{C\xi} \left(\nabla u(x, t) \cdot \nabla \psi(x, t) - e^{i\zeta} u(x, t) \partial_t \psi(x, t)\right) \, dx \, dt.$$
Proposition 3.7. Assume that \( \tau = 1, \lambda = 1 \) and \( 0 < \gamma < \frac{4N}{N-2} \). Then for \( 1 < p < \frac{N}{N-2} \) and for \((f,g) \in H_0^1(\Sigma) \times L^2(\Sigma)\), there exist \( T = T(\|f\|_{H^1(\Sigma)}, \|g\|_{L^2(\Sigma)}), \varepsilon > 0 \) and a unique strong solution \( u \) of (1.1) in \([0, T]\) in the following sense:

\[
  u \in C([0, T); H_0^1(\Sigma)) \cap C^1([0, T); L^2(\Sigma)) \cap L^p_{\text{loc}}(\Sigma \times [0, T])
\]

with \( u(x, 0) = \varepsilon f(x) \) and for every \( \psi \in C^2([0, T); D(A)) \) with \( \text{supp} \psi \subset \subset \Sigma \times [0, T) \)

\[
  \varepsilon \int_{\Sigma} g(x) \psi(x, 0) \, dx + \int_0^T \int_{\Sigma} |u(x, t)|^p \psi(x, t) \, dx \, dt \leq \int_0^T \int_{\Sigma} \left( \nabla u(x, t) \cdot \nabla \psi(x, t) - \delta_t u(x, t) \partial_t \psi(x, t) + a(x) \partial_t u(x, t) \psi(x, t) \right) \, dx \, dt.
\]

To the end of this subsection we state the wellposedness of (1.1) with a singular damping coefficient \( V_0|x|^{-1} (V_0 > 0) \) in \( \mathbb{R}^N \). The proof of following proposition is given in Ikeda–Sobajima [15].

Proposition 3.8. Let \( N \geq 3, \tau = 1, \lambda = 1, \Sigma = S^{N-1}, a(x) = V_0|x|^{-1} (V_0 \geq 0) \) and

\[
  \begin{aligned}
    1 < p < \infty & \quad \text{if } N = 3, 4 \\
    1 < p < \frac{N-2}{N-4} & \quad \text{if } N \geq 5.
  \end{aligned}
\]

For every \((f,g) \in H^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \) and \( \varepsilon > 0 \), there exist \( T = T(\|f\|_{H^2}, \|g\|_{H^1}, \varepsilon) > 0 \) and a unique strong solution of (1.1) in the following class:

\[
  u \in C^2([0, T); L^2(\mathbb{R}^N)) \cap C^1([0, T); H^1(\mathbb{R}^N)) \cap C([0, T); H^2(\mathbb{R}^N)).
\]

### 3.2 The unified choice of test functions

Although we already gave the same functions in Section 2, we repeat the argument for the reader’s convenience.

Here we fix two kinds of functions \( \eta \in C^\infty([0, \infty)) \) and \( \eta^* \in L^\infty((0, \infty)) \) as follows, which will be used in the cut-off functions:

\[
\eta(s) \begin{cases} 
  = 1 & \text{if } s \in [0, 1/2] \\
  \text{is decreasing} & \text{if } s \in (1/2, 1) \\
  = 0 & \text{if } s \not\in [1, \infty), 
\end{cases} \quad \eta^*(s) \begin{cases} 
  = 0 & \text{if } s \in [0, 1/2), \\
  \eta(s) & \text{if } s \in [1/2, \infty).
\end{cases}
\]

**Definition 3.1.** For \( p > 1 \), we define for \( R > 0 \),

\[
  \psi_R(x, t) = [\eta(s_R(x, t))]^{2p}, \quad (x, t) \in C_\Sigma \times [0, \infty),
\]

\[
  \psi^*_R(x, t) = [\eta^*(s_R(x, t))]^{2p}, \quad (x, t) \in C_\Sigma \times [0, \infty).
\]

with

\[
  s_R(x, t) = R^{-1} \left( \langle x \rangle^{2-\alpha} + t \right),
\]

where \( \alpha \) is the constant in (1.2). We also set

\[
  P(R) = \left\{ (x, t) \in C_\Sigma \times [0, \infty) : \langle x \rangle^{2-\alpha} + t \leq R \right\}.
\]
Lemma 3.9. Let \( \psi_R \) and \( \psi_R^* \) be as in Definition 3.1. Then \( \psi_R \) satisfies the following properties:

(i) If \( (x, t) \in P(R/2) \), then \( \psi_R(x, t) = 1 \), and if \( (x, t) \notin P(R) \), then \( \psi_R(x, t) = 0 \).

(ii) There exists a positive constant \( C_1 \) such that for every \( (x, t) \in P(R) \),

\[
|\partial_t \psi_R(x, t)| \leq C_1 R^{-1}[\psi_R^*(x, t)]^{\frac{1}{p}}.
\]

(iii) There exists a positive constant \( C_2 \) such that for every \( (x, t) \in P(R) \),

\[
|\partial^2_t \psi_R(x, t)| \leq C_2 R^{-2}[\psi_R^*(x, t)]^{\frac{1}{p}}.
\]

(iv) There exists a positive constant \( C_3 \) such that for every \( (x, t) \in P(R) \),

\[
|\Delta \psi_R(x, t)| \leq C_3 R^{-1}(x)^{-\alpha}[\psi_R^*(x, t)]^{\frac{1}{p}}.
\]

**Proof.** In view of the definition of \( \psi_R \) and \( \psi_R^* \), the assertions are verified by direct calculations. \qed

### 3.3 Key Lemma for Estimates of Lifespan

**Lemma 3.10.** Let \( \delta > 0 \), \( C_0 > 0 \), \( R_1 > 0 \), \( \theta \geq 0 \) and \( 0 \leq w \in L^1_{\text{loc}}([0, T); L^1(C_\Sigma)) \) for \( T > R_1 \). Assume that for every \( R \in [R_1, T) \),

\[
\delta + \iint_{P(R)} w(x, t) \psi_R(x, t) \, dx \, dt \leq C_0 R^{-\frac{2}{p} + \eta} \left( \iint_{P(R)} w(x, t) \psi_R^*(x, t) \, dx \, dt \right)^{\frac{1}{p}}.
\]

Then \( T \) has to be bounded above as follows:

\[
T \leq \begin{cases} 
    \left( R_1^{p-1} + (\log 2) C_0^{\theta} \delta^{-(p-1)} \right)^{\frac{1}{1-\theta}} & \text{if } \theta > 0, \\
    \exp\left( \log R_1 + (\log 2)(p-1)^{-1} C_0^\eta \delta^{-\frac{p-1}{2}} \right) & \text{if } \theta = 0.
\end{cases}
\]

Although the upper bound of \( T \) for \( \theta > 0 \) can be verified by the simple way via Young inequality, we give a proof different from that via a different view point. This view point enables us to treat not only the subcritical case \( \theta > 0 \) but also the critical case \( \theta = 0 \).

**Proof of Lemma 3.10.** We define

\[
y(r) := \iint_{P(r)} w(x, t) \psi^*_R(x, t) \, dx \, dt, \quad r \in (0, T),
\]

Then we have

\[
\int_0^R y(r)r^{-\frac{1}{p}} \, dr \leq \int_0^R \left( \iint_{P(R)} w(x, t) \left[ \eta^* \left( \frac{s_r(x, t)}{r} \right) \right]^{2p'} \, dx \, dt \right) r^{-\frac{1}{p}} \, dr
\]

\[
= \iint_{P(R)} w(x, t) \left( \int_0^R \left[ \eta^* \left( \frac{(x+r)}{r} \right) \right]^{2p'} \, r^{-\frac{1}{p}} \, dr \right) \, dx \, dt
\]

\[
= \iint_{P(R)} w(x, t) \left( \int_{\{x+r\}^2} \left[ \eta^* (s) \right]^{2p'} s^{-\frac{1}{p}} \, ds \right) \, dx \, dt.
\]
On the other hand, by the definition of $\eta$ and $\eta^*$, for every $\sigma \geq 1$,

$$
\int_0^\infty [\eta^*(s)]^{2^{p'}} s^{-1} ds = 0,
$$

and for every $\sigma \in (0, 1)$

$$
\int_0^\infty [\eta^*(s)]^{2^{p'}} s^{-1} ds = \int_{\max[1/2, \sigma]}^1 [\eta(s)]^{2^{p'}} s^{-1} ds
= [\eta(\sigma)]^{2^{p'}} \int_{1/2}^1 s^{-1} ds
= (\log 2) [\eta(\sigma)]^{2^{p'}} ,
$$

where we have used the non-increasing property of $\eta$. Therefore we deduce from (3.4) that for $R \in (R_1, T)$,

$$
\delta + \frac{1}{\log 2} \int_0^{R} y(r)r^{-1} \, dr \leq \delta + \int_{\mathcal{P}(R)} w\psi_R \, dx \, dt
\leq C_0 R^{\frac{p}{p'}} \left( \int_{\mathcal{P}(R)} w\psi_R^* \, dx \, dt \right)^{\frac{1}{p'}}
\leq C_0 R^{\frac{p}{p'}} (y(R))^{\frac{1}{p'}} .
$$

Taking

$$
Y(R) = \int_0^{R} y(r)r^{-1} \, dr, \quad \rho \in (R_1, T),
$$

we have

$$
(\log 2)\delta + Y(R) \leq (\log 2)^p C_1 R^{1-(p-1)\theta} Y'(R).
$$

Taking

$$
Y(R) = Z \left( \int_{R_1}^{R} r^{(p-1)\theta-1} \, dr \right), \quad 0 < \rho < \rho_T = \int_{R_1}^{T} r^{(p-1)\theta-1} \, dr.
$$

This gives

$$
\frac{d}{dp}(\log 2)\delta + Z(\rho_2)^{1-p} \leq -(p-1)(\log 2)^{-p} C_1^{-p}, \quad \rho \in (0, \rho_T). \quad (3.5)
$$

Integrating it over $[\rho_1, \rho_2] \subset (0, \rho_T)$, we have

$$
\left( (\log 2)\delta + Z(\rho_2) \right)^{1-p} \leq \left( (\log 2)\delta + Z(\rho_1) \right)^{1-p} - (p-1)(\log 2)^{-p} C_1^{-p}(\rho_2 - \rho_1). \quad (3.6)
$$

Then we obtain

$$
\rho_2 < \rho_1 + (p-1)^{-1}(\log 2)C_1^p \delta^{-(p-1)}.
$$

Letting $\rho_2 \uparrow \rho_T$ and $\rho_1 \downarrow 0$, we find

$$
\int_{R_1}^{T} r^{(p-1)\theta-1} \, dr \leq (p-1)^{-1}(\log 2)C_1^p \delta^{-(p-1)}.
$$

This is nothing but the desired upper bound of $T$. \hfill \Box

**Remark 3.3.** The crucial idea in the present paper is to regard the inequality (3.4) as a differential inequality of $y(r)$ or $Y(R)$ in the proof. This idea with the choice of cut off functions in Definition 3.1 enable us to treat not only the case $\theta > 0$ but the critical case $\theta = 0$. We can find not only the upper bound of $T$ (which will be the lifespan) but also a lower estimate for $Y(R)$ by using (3.6).
4 Blowup phenomena and upper bound of lifespan for several equations

In this section we prove blowup phenomena for several equations which can be written by the form (1.1). To simplify the situation, we split the case of the problem with \(\tau = 0\) and that with \(\tau = 1\).

Definition 4.1. We denote \(\text{LifeSpan}(u)\) as the maximal existence time of solutions to (1.1) in the sense of Propositions 3.6, 3.7 and 3.8, respectively. Namely,

\[
\text{LifeSpan}(u) = \sup\{T > 0 ; u \text{ is a unique weak (strong) solution of (1.1) in } [0, T]\}.
\]

The statements of the main results are the following:

Theorem 4.1. Assume that \(\tau = 0, a(x) = e^{i\xi}, \xi \in [-\pi/2, \pi/2]\) and \(\gamma \in [0, 1] \cup \mathbb{N}\) and \(1 < p < \frac{N}{N-2}\). Let \(u\) be the unique solution of (1.1) with \(f \in H^1_0(C_\Sigma)\) satisfying \(f \Phi \in L^1(C_\Sigma)\) and

\[
\int_{C_\Sigma} f(x) \Phi(x) \, dx \notin \{-\rho \lambda e^{-\xi} \in \mathbb{C}; \rho \geq 0\}.
\]

If \(1 < p \leq 1 + \frac{2}{N+\gamma}\), then \(\text{LifeSpan}(u) < \infty\). Moreover one has

\[
\text{LifeSpan}(u) \leq \begin{cases} 
\exp\left( C e^{-\gamma (p-1)} \right) & \text{if } p = 1 + \frac{2}{N+\gamma}, \\
C e^{-\left( p \cdot \frac{N+\gamma}{2} \right) - \gamma} & \text{if } 1 < p < 1 + \frac{2}{N+\gamma}.
\end{cases}
\]

Theorem 4.2. Assume that \(\tau = 1, \lambda = 1\) and \(\gamma \in [0, 1] \cup \mathbb{N}\) and \(1 < p < \frac{N}{N-2}\) and for \((f, g) \in H^1_0(C_\Sigma) \times L^2(C_\Sigma)\). Let \(a(x)\) be real-valued with (1.2) and let \(u\) be the unique solution of (1.1) in Propositions 3.7. Further assume that \(g \Phi, f \Phi \in L^1(C_\Sigma)\) with

\[
\int_{C_\Sigma} (g(x) + a(x)f(x)) \Phi(x) \, dx > 0.
\]

If \(1 < p \leq 1 + \frac{2}{N+\gamma}\), then \(\text{LifeSpan}(u) < \infty\). Moreover, there exists a constant \(\varepsilon_0 > 0\) such that for every \(0 < \varepsilon \leq \varepsilon_0\)

\[
\text{LifeSpan}(u) \leq \begin{cases} 
\exp\left( C e^{-\gamma (p-1)} \right) & \text{if } p = 1 + \frac{2}{N+\gamma-a}, \\
C e^{-\left( p \cdot \frac{N+\gamma}{2} \right) - \gamma} & \text{if } 1 + \frac{a}{N+\gamma-a} < p < 1 + \frac{2}{N+\gamma-a}, \\
C e^{-\gamma (p-1)} (\forall \delta > 0) & \text{if } p = 1 + \frac{2}{N+\gamma-a}, \\
C e^{-\gamma (p-1)} & \text{if } 1 < p < 1 + \frac{2}{N+\gamma-a}.
\end{cases}
\]

If \(u\) be the unique solution of (1.1) in Proposition 3.8 (for \(N \geq 2\) and \(a(x) = V_0 |x|^{-1}\) in \(\mathbb{R}^N\)), then

Moreover, there exists a constant \(\varepsilon_0 > 0\) such that for every \(0 < \varepsilon \leq \varepsilon_0\)

\[
\text{LifeSpan}(u) \leq \begin{cases} 
\exp\left( C e^{-\gamma (p-1)} \right) & \text{if } p = \frac{N+1}{N-1}, \\
C e^{-\left( p \cdot \frac{N}{N-1} \right) - \gamma} & \text{if } \frac{N}{N-1} < p < \frac{N+1}{N-1}.
\end{cases}
\]

Remark 4.1. Since the upper bounds in Theorem 4.1 with \(C_\Sigma = \mathbb{R}^N\) coincides with that in Lee–Ni [23] when we consider the nonlinear heat equation of Fujita-type and the upper bounds in Theorem 4.2 with \(C_\Sigma = \mathbb{R}^N\) matches that in Li–Zhou [27], Nishihara [34] and also Lai–Zhou [26]. Moreover, Theorems 4.1 and 4.2 give the lifespan of solutions even when the equation in the cone-like domain \(C_\Sigma\) has a critical nonlinearity which depends on the shape of \(\Sigma\). In particular, we could obtain the the lifespan of solutions to nonlinear Schrödinger equation in \(\mathbb{R}^N\) with the critical nonlinearity \(p = p_F\).
4.1 Proof of Theorem 4.1

We remark that the solution \( u \in C([0, T); H^1(\Omega)) \cap C([0, T); H^1(\Omega)) \) satisfies

\[
e e^{i\xi} \int_{\Omega} f(\xi) \, dx + \lambda \int_0^T \int_{\Omega} |u(t)|^p \psi(t) \, dx \, dt \\
= \int_0^T \int_{\Omega} \left( \nabla u(t) \cdot \nabla \psi(t) - e^{i\xi} u(t) \partial_t \psi(t) \right) \, dx \, dt.
\]

Fix \( \xi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) such that

\[
\text{Im} \left( e^{i(\xi+\eta)} \lambda^{-1} \int_{\Omega} f(x) \Phi(x) \, dx \right) > 0.
\]

Multiplying \( \mu = \lambda^{-1} e^{i\xi} \) with \( \xi \in (-\pi/2, \pi/2) \), we see that

\[
e \lambda^{-1} e^{i(\xi+\xi)} \int_{\Omega} f(\xi) \, dx + \lambda \int_0^T \int_{\Omega} |u(t)|^p \psi(t) \, dx \, dt \\
= -\mu \int_0^T \int_{\Omega} u(t) \left( \Delta \psi(t) + e^{i\xi} \partial_i \psi(t) \right) \, dx \, dt,
\]

where we used integration by parts which is verified by the regularity of test function \( \psi(x) \in D(A) \).

Here we choose \( \psi(x, t) = \Phi(x) \psi_R(x, t) \) with \( \alpha = 0 \). Since \( \psi(x, t) = 0 \) on \( (\partial \Omega \setminus \{0\}) \times (0, \infty) \) and

\[
\Delta \psi(x, t) = 2\nabla \Phi(x) \cdot \nabla \psi_R(x, t) + \Phi(x) \Delta \psi_R(x, t)
\]

is a compactly supported bounded function, this choice is reasonable. Noting that

\[
\lim_{R \to \infty} \left( \int_{\Omega} f(x) \Phi(x) \psi_R(x, 0) \, dx \right) = \int_{\Omega} f(x) \Phi(x) \, dx,
\]

we can choose \( R_0 > 0 \) and \( c_0 > 0 \) such that for every \( R \geq R_0 \),

\[
\text{Re} \left( e^{i(\xi+\eta)} \lambda^{-1} \int_{\Omega} f(x) \Phi(x) \psi_R(x, 0) \, dx \right) \geq c_0 > 0.
\]

Now we assume \( R_0 < \text{LifeSpan}(u) \). Taking real part of (4.1), we have for \( R \in (R_0, \text{LifeSpan}(u)) \),

\[
c_0 e + c_0 \xi \int_{P(R)} |u(t)|^p \Phi \psi_R^*(t) \, dx \, dt \leq \int_{P(R)} |u(s)| \left( |\Delta \psi(t)| + |\partial_i \psi(t)| \right) \, dx \, dt \\
\leq \frac{C}{R} \int_{P(R)} |u(t)| \Phi[\psi_R^*(t)] \, dx \, dt \\
\leq \frac{C}{R} \left( \int_{P(R)} \Phi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{P(R)} |u(t)|^p \Phi \psi_R^*(t) \, dx \, dt \right)^{\frac{1}{p}} \\
\leq C' R^{\theta} \left( \int_{P(R)} |u(t)|^p \Phi \psi_R^*(t) \, dx \, dt \right)^{\frac{1}{p}}
\]

with

\[
\theta = \frac{1}{p-1} - \frac{N + \gamma}{2}.
\]

Therefore applying Lemma 3.10 with \( w = |u|^p \Phi \), we have the desired upper bound of \( \text{LifeSpan}(u) \). \( \square \)
4.2 Proof of Theorem 4.2

Note that the solution $u$ satisfies
\[ u \in C([0, T); H^1_0(C_\Sigma)) \cap C^4([0, T); L^2(C_\Sigma)) \cap L^p_{\text{loc}}(C_\Sigma \times [0, T)) \]
with $u(x, 0) = e f(x)$ and for every $\psi \in C^2([0, T); D(A))$ with $\text{supp} \psi \subset C_\Sigma \times [0, T)$
\[ \epsilon \int_{C_\Sigma} g \psi(0) \, dx + \int_0^T \int_{C_\Sigma} |u(t)|^p \psi(t) \, dx \, dt 
   = \int_0^T \int_{C_\Sigma} \left( \nabla u(t) \cdot \nabla \psi(t) - \partial_t u(t) \partial_t \psi(t) + a \partial_t u(t) \psi(t) \right) \, dx \, dt. \]

By integration by parts for variable $x$ and $t$, we have
\[ \epsilon \int_{C_\Sigma} g \psi(0) - f \partial_t \psi(0) + a f \psi(0) \, dx \n \int_0^T \int_{C_\Sigma} |u(x, t)|^p \phi(x, t) \, dx \, dt 
   = \int_0^T \int_{C_\Sigma} u(t) \left( \partial_t^2 \psi(t) - \Delta \psi(t) - a \partial_t \psi(t) \right) \, dx \, dt \]

Here noting that
\[ \lim_{R \to \infty} \left( \int_{C_\Sigma} (g \psi_R(0) - f \partial_t \psi_R(0) + a f \psi_R(0)) \Phi \, dx \right) = \int_{C_\Sigma} (g + a f) \Phi \, dx > 0, \]

Then we see that there exist $R_0 > 0$ and $c_0 > 0$ such that for every $R \geq R_0$,
\[ \int_{C_\Sigma} (g \psi_R(0) - f \partial_t \psi_R(0) + a f \psi_R(0)) \Phi \, dx \geq c_0. \]

Now we assume that $\text{LifeSpan}(u) > R_0$. Since $\Phi$ is independent of $t$, it follows from Lemmas 3.2 and 3.1 that
\[ \partial_t^2 (\Phi \psi_R) - \Delta (\Phi \psi_R) - \partial_t (\alpha(x) \Phi \psi_R) 
   = \Phi \partial_t^2 \psi_R - 2 \nabla \Phi \cdot \nabla \psi_R - \Phi \Delta \psi_R - \alpha(x) \Phi \partial_t \psi_R 
   \leq \frac{C_2}{R^2} \Phi [\psi_R^+)^\frac{p}{2} + \frac{4p'}{R} \Phi \cdot x(1 - \alpha) [\psi_R^+]^\frac{p}{2} + \frac{C_3}{R} (1 - \alpha) \Phi [\psi_R^+]^\frac{p}{2} + \frac{C_4}{R} (1 - \alpha) \Phi [\psi_R^+]^\frac{p}{2} 
   \leq \left( \frac{C_2}{R^2} + \frac{4p' \gamma + C_3}{R} (1 - \alpha) \psi_R^+ \right) \Phi [\psi_R^+]^\frac{p}{2}. \]

Therefore choosing the test function $\psi(\cdot, t) = \Phi(\cdot) \psi_R(\cdot, t) \in D(A)$ implies that
\[ c_0 \epsilon + \int_{P(R)} |u(t)|^p \Phi \psi_R(t) \, dx \, dt \]
\[ \leq \int_{P(R)} u(t) \left( \partial_t^2 (\Phi \psi_R(t)) - \Delta (\Phi \psi_R(t)) - \partial_t (\alpha \Phi \psi_R(t)) \right) \, dx \, dt \]
\[ \leq C_4 \int_{P(R)} u \left( \frac{1}{R^2} + \frac{1}{R} (1 - \alpha) \right) \Phi [\psi_R^+]^\frac{p}{2} \, dx \, dt \]
\[ \leq C_4 \left( \int_{P(R)} \frac{1}{R} + (1 - \alpha) \Phi \, dx \, dt \right)^\frac{p}{2} \left( \int_{P(R)} |u(t)|^p \Phi \psi_R(t) \, dx \, dt \right)^\frac{1}{2}. \]
Noting that for $\beta = 0, \alpha$,

$$\int \int_{P(R)} \langle x \rangle^{-\beta p'} \Phi dx dt \leq \int_0^R \int_{B(0,R^{1/2})} \langle x \rangle^{-\beta p'} \Phi dx dt$$

$$= \int_\Sigma \varphi_\Sigma(\omega) d\omega \int_0^R dt \int_0^{R^{1/2}} (1 + r^2)^{-\beta p'/2} r^{N+\gamma-1} dr$$

$$\leq \begin{cases} CR^{1+\frac{\gamma p}{2-N\gamma}} & \text{if } p > 1 + \frac{\beta}{N+\gamma-a} \\ CR \log R & \text{if } p = 1 + \frac{\beta}{N+\gamma-a} \\ CR & \text{if } p < 1 + \frac{\beta}{N+\gamma-a} \end{cases}$$

we deduce

$$c_0 \epsilon + \int \int_{P(R)} |u(t)|^p \Phi \psi_R(t) dx dt \leq C_5 q(R)^{1/p'} \left( \int \int_{P(R)} |u(t)|^p \Phi \psi_R(t) dx dt \right)^{\frac{1}{p'}}$$

with

$$q(R) = \begin{cases} R^{-\frac{2\gamma}{p-1} \left( \frac{N-\gamma-a}{p-1} \right)} & \text{if } p > 1 + \frac{\epsilon}{N+\gamma-a} \\ R^{-\frac{1}{p-1} (\log R)} & \text{if } p = 1 + \frac{\epsilon}{N+\gamma-a} \\ R^{-\frac{1}{p-1}} & \text{if } 1 < p < 1 + \frac{\epsilon}{N+\gamma-a}. \end{cases}$$

Therefore applying Lemma 3.10 with $w = |u|^p \Phi$, we have

$$T_{\max} \leq \begin{cases} \exp \left( C \epsilon \left( p-1 \right) \right) & \text{if } p = 1 + \frac{2}{N+\gamma-a} \\ C \epsilon \left( p-1 \right) & \text{if } 1 + \frac{\epsilon}{N+\gamma-a} < p < 1 + \frac{2}{N+\gamma-a} \\ C \epsilon \left( p-1 \right) - \delta (\forall \delta > 0) & \text{if } p = 1 + \frac{2}{N+\gamma-a} \\ C \epsilon \left( p-1 \right) & \text{if } 1 < p < 1 + \frac{2}{N+\gamma-a}. \end{cases}$$

The part of proof for the solution in Proposition 3.7 is complete.

Finally, we only give a comment for the proof of upper bound for solution in Proposition 3.8. If we consider the case $\alpha(x) = V_0|x|^{-1}$ and $C_\Sigma = \mathbb{R}^N$, that is, $\gamma = 0$ and $\alpha = 1$, then we can deduce the same upper bound for the lifespan of $u$ as above only when $\frac{N}{N+1} < p \leq \frac{N+1}{N-1}$. The crucial point for that restriction is due to the integrability of

$$\int \int_{P(R)} |x|^{-p'} dx dt.$$

The proof is complete. \square

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