PRESENTATIONS OF PRINCIPAL SUBSPACES OF HIGHER LEVEL STANDARD $A_2^{(2)}$-MODULES

CORINA CALINESCU, MICHAEL PENN AND CHRISTOPHER SADOWSKI

Abstract. We study the principal subspaces of higher level standard $A_2^{(2)}$-modules, extending earlier work in the level one case, by Calinescu, Lepowsky and Milas. We prove natural presentations of principal subspaces and also of certain related spaces.

1. Introduction

The principal subspaces of standard (integrable highest weight) modules introduced in [FS1]-[FS2] have been studied by several authors from different standpoints. Our approach is based on vertex operator algebra theory ([B], [FLM2], [FHL], [LL]). Algebraic and combinatorial properties, such as presentations, combinatorial bases and graded dimensions, of the principal subspaces of certain modules for untwisted affine Lie algebras were proved in [CLM1]-[CLM2], [CalLM1]-[CalLM3], [Bu1]-[Bu3] and other works. Analogous results in the case of twisted affine Lie algebras appear in in [CalLM4], [CMP], [PS1]-[PS2], [BS], and in [MP], [P], [PSW] for lattice vertex operator algebras. There are also “commutative” principal subspaces studied in [Pr], [Je], [T1]-[T2], etc., and the quantum case was studied in [Ko].

In this paper, a continuation of [CalLM4], we switch our attention to level $k$ standard modules for $A_2^{(2)}$ and their principal subspaces, where $k$ is an integer with $k \geq 1$. In this case, new spaces, which we call virtual subspaces, arise naturally. Denote by $\mathfrak{n}$ the Lie subalgebra of $A_2$ spanned by the root vectors for the positive roots, and by $\mathfrak{n}[\hat{\nu}]$ an appropriate affinization of $\mathfrak{n}$. The virtual subspaces, denoted by $W_{k,i}$, are defined as $W_{k,i} = U(\mathfrak{n}[\hat{\nu}]) \cdot v_{k,i}$, where $v_{k,i}$ are certain highest weight vectors. With our definitions, we have that $W_{k,0}$ is the principal subspace of the level $k$ standard $A_2^{(2)}$-module. The virtual subspaces are analogous to the principal-like subspaces in the untwisted case introduced in [CalLM3]. These subspaces were implicitly used in the proof of all other twisted results, [CalLM4], [CMP], [PS1]-[PS2], and [PSW], but the increase in the complexity of the current setting requires that these spaces to be worked with explicitly.

The virtual subspaces play a role similar to that of the principal subspaces of level $k$ non-vacuum modules for $A_2^{(1)}$, denoted by $W((k-i)\Lambda_0+i\Lambda_1)$ in [CLM2] and [CalLM2], where $\Lambda_0, \Lambda_1$ are fundamental weights. It was proved in [CalLM2] that the principal subspaces $W((k-i)\Lambda_0+i\Lambda_1)$ have a presentation given by an ideal generated by a single family of degree $k+1$ terms with an additional generator of

---

C.C was partially supported by the Simons Foundation Collaboration Grant for Mathematicians, and by PSC-CUNY Research Awards.
$\alpha^-1 (k-i+1)$, where $\alpha$ is the positive simple root. In this work we prove that our virtual subspaces satisfy

$$W_{k,i} \cong U(\mathfrak{g}[\hat{\nu}])/I_{k,i}$$

where the ideal $I_{k,i}$ is generated by $k+2$ different families of degree $k+1$ terms and a family of degree $k-i+1$ monomials constructed by using $x_{\alpha_1}^\nu (-\frac{1}{4})$ and $x_{\alpha_1+\alpha_2}^\nu (-1)$, where $\alpha_1, \alpha_2$ are positive simple roots. These presentations of $W_{k,i}$ are given in Theorem 5.1, the main result of our paper. The proof of this theorem adapts strategies from [S1] and other results involving principal subspaces into a nested inductive argument. The outer induction descends through the virtual subspaces ending at the principal subspace $W_{k,0}$, while the inner induction homogenizes the extra terms in the ideals, $x_{\alpha_1}^\nu (-\frac{1}{4})^m x_{\alpha_1+\alpha_2}^\nu (-1)^n$, by iteratively decreasing the power of $x_{\alpha_1}^\nu (-\frac{1}{4})$, until the appropriate extra term is a power of $x_{\alpha_1+\alpha_2}^\nu (-1)$. In our proof, we make use of certain operators we call $Y_i$, which play the role of the constant terms of intertwining operators used in [CalLM1]-[CalLM3].

The paper is organized as follows. We recall the vertex operator constructions of $A_2^{(2)}$ and higher level standard modules for $A_2^{(2)}$ in sections 2 and 3. These results are standard and mostly taken from [L1], [FLM1] (see also [CalLM4]). Section 4 gives certain maps that play an important role in proving the presentations of principal subspaces. In Section 5 we prove the main result of this paper, Theorem 3.2. We note that the maps used to prove the presentations do not give exact sequences of maps among these virtual subspaces, as it is the case in [CLM1]-[CLM2], [CalLM1]-[CalLM4], [PS1]-[PS2], [PSW], and [CMP]. This shows that the theory in our setting is much more subtle than in the untwisted case or the case of the level one modules for the twisted affine Lie algebra $A_2^{(2)}$.

2. Vertex operator construction of $A_2^{(2)}$

In this section we recall from [CalLM4] (which follows [L1] and [FLM1], [FLM2] and [L2]) the vertex operator construction of $A_2^{(2)}$ using the lattice vertex algebra construction. Let $\mathfrak{g} = sl(3, \mathbb{C})$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Fix a root system $\Delta \subset \mathfrak{h}^*$ and take $\{\alpha_1, \alpha_2\}$ to be a choice of simple roots. Identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via a suitable symmetric, invariant bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that

$$\langle (\alpha_i, \alpha_j) \rangle_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Now consider the root lattice of $\mathfrak{g}$

$$L = \mathbb{Z}\Delta = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset \mathfrak{h},$$

equipped with the bilinear form $\langle \cdot, \cdot \rangle$. Take $\nu \in \text{Aut} \ L$ to be the isometry of $L$ given by the folding of the Dynkin diagram

$\begin{tikzpicture}[baseline=(current bounding box.center)]
  \draw (0,0) node[above] {$\alpha_1$} -- (1,0) node[above] {$\alpha_2$};
\end{tikzpicture}$

in other words, we have

$$\nu(ra_1 + sa_2) = ra_2 + sa_1$$

for any integers $r$ and $s$. Following the construction of twisted modules for lattice vertex algebras, we let $l$ be a positive integer such that $\nu^l = \text{id}_L$ and

$$\langle \nu^{l/2} \alpha, \alpha \rangle \text{ for } \alpha \in L.$$
In our setting this amounts to setting \( l = 4 \), even though \( \nu^2 = \text{id}_L \). This doubling of the period was seen in [CalLM4], [CMP], and handled more generally in [PSW]. Now that we have set the period of the isometry to be 4 we fix a primitive fourth root of unity, \( \eta \), which we take \( \eta = i \).

We consider two central extensions of the \( L \) by the cyclic group \( \langle i \rangle \cong \mathbb{Z}/4\mathbb{Z} \) which we denote by \( \hat{L} \) and \( \hat{L}_\nu \) with associated commutator maps
\[
C_0 : L \times L \to \mathbb{C}^x \\
(\alpha, \beta) \mapsto (-1)^{\langle \alpha, \beta \rangle}
\]
and
\[
C : L \times L \to \mathbb{C}^x \\
(\alpha, \beta) \mapsto -(-1)^{\langle \nu \alpha, \beta \rangle},
\]
respectively. We also have normalized cocycles \( \epsilon_{C_0} \) and \( \epsilon_C \) and normalized sections \( (\alpha \mapsto e_\alpha) \) associated to these central extensions so that
\[
e_\alpha e_\beta = \epsilon_{C_0}(\alpha, \beta)e_{\alpha+\beta} \text{ in } \hat{L}.
\]
For concreteness we can take
\[
\epsilon_{C_0}(m\alpha_1 + n\alpha_2, r\alpha_1 + s\alpha_2) = (-1)^{nr},
\]
where \( m, n, r, s \in \mathbb{Z} \).

We also recall the affine Lie algebras
\[
\hat{h} = h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k
\]
and
\[
\hat{h}[\nu] = \bigsqcup_{n \in \mathbb{Z}} h(kn) \otimes t^n \oplus \mathbb{C}k
\]
with their usual brackets. We refer the reader to [CalLM4] for further details regarding these constructions.

Denote by \((V_L, Y)\) the vertex operator algebra associated with the root lattice \( L \). We have, in particular,
\[
Y(\iota(e_\alpha), x) = E^-(-\alpha, x)E^+(-\alpha, x)e_\alpha x^\alpha,
\]
where \( \iota : \hat{L} \to V_L \) is the obvious inclusion. Define the operators \( x_\alpha(n) \) such that
\[
Y(\iota(e_\alpha), x) = \sum_{n \in \mathbb{Z}} x_\alpha(n)x^{-n-1}.
\]
Extend \( \nu \) to an automorphism of \( V_L \) and call it \( \hat{\nu} \). Then \( \hat{\nu} \in \text{Aut} (V_L) \) and \( \hat{\nu}^4 = 1 \).

We have
\[
\hat{\nu}(e_{\alpha_1}) = i\nu(e_{\alpha_2}), \hat{\nu}(e_{\alpha_2}) = i\nu(e_{\alpha_1}), \text{ and } \hat{\nu}(e_{\alpha_1 + \alpha_2}) = i(e_{\alpha_1 + \alpha_2}).
\]

Denote by \((V_L^T, Y^\nu)\) the irreducible \( \hat{\nu} \)-twisted module for \( V_L \) on which \( \hat{h}[\nu] \) has a natural action. We have
\[
Y^\nu(\iota(e_\alpha), x) = 4^{-\frac{\langle \alpha, \alpha \rangle}{2}} \sigma(\alpha)E^-(-\alpha, x)E^+(-\alpha, x)e_\alpha x^{\alpha(0)} + \frac{(\alpha(0) - \alpha(\alpha))}{2} - \frac{\langle \alpha, \alpha \rangle}{2},
\]
where
\[
\sigma(\alpha) = (1 + i)^{\langle \nu \alpha, \alpha \rangle} 2^{\langle \alpha, \alpha \rangle/2}
\]
is a normalizing factor. For \( n \in (1/4)\mathbb{Z} \) define the operators \( x^n_\alpha(n) \) such that
\[
Y^n(e_\alpha), x = \sum_{n \in (1/4)\mathbb{Z}} x^n_\alpha(n)x^{-n} - \frac{(\alpha, \alpha)}{2}.
\]

Now we observe that the Lie algebra \( g \) may be realized as the vector space
\[
g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}x_\alpha
\]
with \( \mathfrak{h} = C\alpha_1 \oplus C\alpha_2 \) and
\[
[h, x_\alpha] = \langle h, \alpha \rangle = -[x_\alpha, h], [\mathfrak{h}, \mathfrak{h}] = 0
\]
\[
[x_\alpha, x_\beta] = \begin{cases} 
\epsilon_{\mathbb{C}0}(\alpha, -\alpha)\alpha & \text{if} \quad \langle \alpha, \beta \rangle = 0 \\
\epsilon_{\mathbb{C}0}(\alpha, \beta)x_{\alpha + \beta} & \text{if} \quad \langle \alpha, \beta \rangle = -1 \\
0 & \text{if} \quad \langle \alpha, \beta \rangle \geq 0
\end{cases}
\]

With our choice of cocycle the brackets of interest are
\[
[x_{\alpha_j}, x_{-\alpha_j}] = \alpha_j \quad \text{and} \quad [x_{\alpha_1}, x_{\alpha_2}] = x_{\alpha_1 + \alpha_2}
\]
for \( j \in \{1, 2\} \).

We now lift the isometry \( \nu : L \rightarrow L \), which may also be viewed as an automorphism of the Lie subalgebra \( \mathfrak{h} \subset g \) to an automorphism of \( g \) which we denote by \( \hat{\nu} \).

Explicitly we have
\[
\hat{\nu}x_{\alpha_1} = ix_{\alpha_2}, \hat{\nu}x_{\alpha_2} = ix_{\alpha_1}, \quad \text{and} \quad \hat{\nu}x_{\alpha_1 + \alpha_2} = x_{\alpha_1 + \alpha_2}
\]
with similar formulas for elements associated to the negative roots.

Define
\[
g_{(m)} = \{ x \in g | \hat{\nu}x = i^m x \}
\]
for \( m \in \mathbb{Z} \) and form
\[
g[\hat{\nu}] = \bigoplus_{n \in 1/4\mathbb{Z}} g_{(4n)} \otimes t^n \oplus \mathbb{C}c,
\]
the \( \hat{\nu} \)-twisted affine Lie algebra associated to \( g \) and \( \hat{\nu} \) with
\[
[x \otimes t^m, y \otimes t^n] = [x, y]t^{m+n} + (x, y)m\delta_{m+n, 0}c
\]
and
\[
[c, g[\hat{\nu}]] = 0
\]
for \( m, n \in 1/4\mathbb{Z}, x \in g_{(4m)}, \) and \( y \in g_{(4n)} \). Adjoining the degree operator to \( g[\hat{\nu}] \) gives a copy of the twisted affine Lie algebra \( A_2^{(2)} \) (cf. [K]).

We recall the following result:

**Theorem 2.1** ([L1] Theorem 9.1 and [PLM1] Theorem 3.1). The representation of \( \mathfrak{h}[\hat{\nu}] \) on \( V^\nu_L \) extends uniquely to a Lie algebra representation of \( g[\hat{\nu}] \) on \( V^\nu_L \) such that
\[
(x_\alpha)_{(4n)} \otimes t^n \mapsto x^n_\alpha(n)
\]
for all \( n \in 1/4\mathbb{Z} \). Moreover, \( V^\nu_L \) is an irreducible \( g[\hat{\nu}] \)-module.

The following structure results from [CalLM4] will be useful in this paper:
Lemma 2.1 ([CalLM4], Lemma 3.2). As operators on $V^T_L$, we have
\[ x^\nu_{\alpha_2}(m) = x^\nu_{\alpha_1}(m) \quad \text{if} \quad m \in \frac{1}{4} + \mathbb{Z} \]
and
\[ x^\nu_{\alpha_2}(m) = -x^\nu_{\alpha_1}(m) \quad \text{if} \quad m \in \frac{3}{4} + \mathbb{Z}. \]

Lemma 2.2 ([CalLM4], Lemma 3.3). As operators on $V^T_L$, we have
\[ [x^\nu_{\alpha_1}(m), x^\nu_{\alpha_1}(n)] = -\frac{i}{4}(i^{-4m} - (-i)^{-4m})x^\nu_{\alpha_1 + \alpha_2}(m+n) \quad \text{if} \quad m, n \in \frac{1}{4} + \frac{1}{2}\mathbb{Z} \]
and
\[ [x^\nu_{\alpha_1 + \alpha_2}(m), x^\nu_{\alpha_1}(n)] = 0 \quad \text{if} \quad m \in \mathbb{Z}, n \in \frac{1}{4}\mathbb{Z} \quad \text{and} \quad \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}. \]

Consider the $\hat{u}$-stable Lie subalgebra of $\mathfrak{g}$: $n = \mathbb{C}x_{\alpha_1} \oplus \mathbb{C}x_{\alpha_2} \oplus \mathbb{C}x_{\alpha_1 + \alpha_2}$, and its twisted affinization
\[ \hat{n}[\hat{u}] = \prod_{r \in (1/4)\mathbb{Z}} n_{(4r)} \otimes t^r. \]

We recall the definition of the principal subspace of $V^T_L$: \[ W^T_L = U(\hat{n}[\hat{u}]) : v_A \subset V^T_L. \]
We shall denote $v_A \in V^T_L$ by $1$: $v_A = 1 \in V^T_L$.

We recall the following result, from which we will derive the necessary presentation of our principal subspace:

Theorem 2.2 ([CalLM4], Theorem 5.3). On the standard $\mathfrak{g}[\hat{u}]$-module $V^T_L$ we have:
\[ \lim_{x_1^{1/4} \to x_2^{1/4}} (x_1^{1/2} + x_2^{1/2})Y^\nu(\iota(e_{\alpha_1}), x_2)Y^\nu(\iota(e_{\alpha_1}), x_1) = 0 \quad \text{for} \quad j = 1, 2, \]
\[ \lim_{x_1^{1/4} \to ix_2^{1/4}} (x_2^{1/2} - x_1^{1/2})Y^\nu(\iota(e_{\alpha_1}), x_2)Y^\nu(\iota(e_{\alpha_1}), x_1) = 0, \]
\[ Y^\nu(\iota(e_{\alpha_1 + \alpha_2}), x)^2 = 0, \]
and
\[ Y^\nu(\iota(e_{\alpha_1}), x)Y^\nu(\iota(e_{\alpha_1 + \alpha_2}), x) = 0 \quad \text{for} \quad j = 1, 2. \]

Certain truncations of the coefficients of the products of vertex operators in (2.23), (2.25) and (2.26), together with the highest weight vector relations, are all the relations of a presentation of the principal subspace $W^T_L$ (cf. Theorem 7.1 in [CalLM4]).

Recall (5.18) from [CalLM4] with $j = 1$ and $t = 1/2$:
\[ R_{1,1/2} = x^\nu_{\alpha_1} \left( -\frac{3}{4} \right) x^\nu_{\alpha_1} \left( -\frac{1}{4} \right) + x^\nu_{\alpha_1} \left( -\frac{1}{4} \right) x^\nu_{\alpha_1} \left( -\frac{3}{4} \right) + 2x^\nu_{\alpha_1} \left( -\frac{1}{4} \right)^2, \]
which implies
\[ x^\nu_{\alpha_1} \left( -\frac{3}{4} \right) x^\nu_{\alpha_1} \left( -\frac{1}{4} \right) + x^\nu_{\alpha_1} \left( -\frac{1}{4} \right) x^\nu_{\alpha_1} \left( -\frac{3}{4} \right) \cdot 1 = 0. \]
As a consequence of Lemmas 2.1 and 2.2 and (2.27) we note that

\[(2.28) \quad x^{\tilde{\nu}}_{\alpha_1 + \alpha_2}(-1) \cdot 1 = c x^{\tilde{\nu}}_{\alpha_1} \left( -\frac{1}{4} \right) x^{\tilde{\nu}}_{\alpha_1} \left( -\frac{1}{4} \right) \cdot 1, \]

where \( c \) is a nonzero constant.

Finally, as in [CalLM4], [CMP], etc. we have a tensor product grading on \( V_T \) given by the eigenvalues of \( \tilde{\nu}(0) \), where \( \tilde{\nu}(\omega, z) = \sum_{m \in \mathbb{Z}} L^{\tilde{\nu}}(m) z^{-m-2} \), which we call the weight grading. We also have a grading by charge, given by the eigenvalues of \( \gamma = (\alpha_1 + \alpha_2)(0) \). We note that these gradings are compatible, and refer the reader to [CalLM4] for more details.

3. Vertex operator construction of higher level standard modules for \( A_2^{(2)} \)

Let \( k \) be a nonnegative integer. We will use \( k \) to denote the level of representations in this paper. Consider the vector space

\[ V_L^{\otimes k} = V_L \otimes \cdots \otimes V_L \]

and the vertex operator

\[ Y^{\otimes k}(v_1 \otimes \cdots \otimes v_k, x) = Y(v_1, x) \otimes \cdots \otimes Y(v_k, x). \]

Define the vectors

\[(3.1) v_{k,0} = 1 \otimes \cdots \otimes 1, \]

and, more generally, for \( 0 \leq i \leq k \),

\[(3.2) v_{k,i} = 1 \otimes (k-i) \otimes \underbrace{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}}_{k-i \text{ times}}. \]

It is known that \((V_L^{\otimes k}, Y^{\otimes k})\) is a vertex operator algebra (cf. [LL]). Define the operators \( x_{\alpha}(m) \) on \( V_L^{\otimes k} \) such that

\[ Y^{\otimes k}(x_{\alpha}(-1) \cdot v_{k,0}, x) = \sum_{m \in \mathbb{Z}} x_{\alpha}(m)x^{-m-1}. \]

Denote by \((V_L^T)^{\otimes k}\) the tensor product of \( k \) copies of \( V_L^T \):

\[ (V_L^T)^{\otimes k} = V_L^T \otimes \cdots \otimes V_L^T. \]

Consider the automorphism \( \hat{\nu} \otimes \cdots \otimes \hat{\nu} \) of \( V_L^{\otimes k} \), and denoted it by \( \hat{\nu} \). Then \( \hat{\nu}^4 = 1 \). Let \( Y^{\hat{\nu}, \otimes k} \) be the vertex operator

\[ Y^{\hat{\nu}, \otimes k} = Y^{\hat{\nu}} \otimes \cdots \otimes Y^{\hat{\nu}}. \]

Since \( V_L^T \) is an irreducible \( \hat{\nu} \)-twisted \( V_L \)-module, then \(((V_L^T)^{\otimes k}, Y^{\hat{\nu}, \otimes k})\) is an irreducible \( \hat{\nu} \)-twisted module for \( V_L^{\otimes k} \). Define the operators \( x^{\hat{\nu}}_{\alpha}(m) \) on \((V_L^T)^{\otimes k}\) as follows:

\[ Y^{\hat{\nu}, \otimes k}(x_{\alpha}(-1) \cdot v_{k,0}, x) = \sum_{m \in \mathbb{Z}} x^{\hat{\nu}}_{\alpha}(m)x^{-m-1}. \]
By \cite{Li} we know that $L^\hat{\nu}(k\Lambda_0) := U(\hat{\mathfrak{g}}[\hat{\nu}]) \cdot v_{k,0} \subset (V_L^T)^\otimes k$ is a level $k$ standard $\hat{\mathfrak{g}}[\hat{\nu}]$-module with the tensor product action (diagonal action)

\begin{equation}
(3.3) \quad x^\rho_\alpha(n) \cdot v_{k,0} = x^\rho_\alpha(n) \cdot 1 \otimes 1 \cdots 1 + \cdots + 1 \otimes 1 \cdots 1 \otimes x^\rho_\alpha(n) \cdot 1.
\end{equation}

We also note here that our gradings on $V_L^T$ by weight and charge naturally extend to $(V_L^T)^\otimes k$ (cf. \cite{BS} for more details).

Define the vector spaces

\begin{equation}
(3.4) \quad W_{k,i} = U(\bar{n}[\hat{\nu}]) \cdot v_{k,i}
\end{equation}

for any $0 \leq i \leq k$ (recall \cite{EM} and \cite{EM2}), which we call virtual subspaces. Note that

\begin{equation}
(3.5) \quad W_{k,0} = U(\bar{n}[\hat{\nu}]) \cdot v_{k,0}
\end{equation}

is the principal subspace of the level $k$ standard module $L^\hat{\nu}(k\Lambda_0)$. These virtual subspaces are analogous to the principal-like subspaces introduced in \cite{CalLM3}.

**Theorem 3.1.** On the $\hat{\mathfrak{g}}[\hat{\nu}]$-module $L^\hat{\nu}(k\Lambda_0)$ we have

\begin{equation}
(3.6) \quad \left( \prod_{1 \leq i < j \leq k+1-r} \lim_{x_i \rightarrow x, x_j \rightarrow x} \left( x_i^{1/2} + x_j^{1/2} \right) \right)^{k+1-r} Y^\hat{\nu} \otimes \delta(\epsilon(e_\alpha), x_1) \cdots Y^\hat{\nu} \otimes \delta(\epsilon(e_\alpha), x_{k+1-r}) \left( Y^\hat{\nu} \otimes \delta(\epsilon(e_{\alpha+2}), x) \right)^r = 0
\end{equation}

for $0 \leq r \leq k - 1$ and

\begin{equation}
(3.7) \quad Y^\hat{\nu} \otimes \delta(\epsilon(e_\alpha), x) Y^\hat{\nu} \otimes \delta(\epsilon(e_{\alpha+2}), x)^k = 0
\end{equation}

for $l = 1, 2$. We also have

\begin{equation}
(3.8) \quad Y^\hat{\nu} \otimes \delta(\epsilon(e_{\alpha+2}), x)^{k+1} = 0.
\end{equation}

**Proof.** Recall that

\[ Y^\hat{\nu} \otimes \delta(\epsilon(e_\alpha), x) \]

\[ = Y^\hat{\nu}(\epsilon(e_\alpha), x) \otimes 1_V \otimes \cdots \otimes 1_V + \cdots + 1_V \otimes \cdots \otimes Y^\hat{\nu}(\epsilon(e_\alpha), x), \]

where $\alpha$ is any root and $1_V$ is the identity operator. Now the statement follows from the corresponding statement for the level 1 case (Theorem 5.3 in \cite{CalLM4}).

Following \cite{CalLM1}, \cite{CalLM4}, \cite{CMP}, etc. and using Theorem 3.1 we introduce the following formal infinite sums indexed by $t \in \frac{1}{4}Z$:

\begin{equation}
(3.9) \quad R(\alpha_1, \alpha_1 + \alpha_2, r | t)
\end{equation}

\[ = \sum_{i_1, \ldots, i_{k+1-r} \in \mathbb{Z}} \left( \prod_{j=1}^{k+1-r} x_{\alpha_1}^\rho(m_j + i_j) \right) x_{\alpha_1 + \alpha_2}^\rho(m_{k+2-r}) \cdots x_{\alpha_1 + \alpha_2}^\rho(m_{k+1}); \]

\begin{equation}
(3.10) \quad R(\alpha_1, \alpha_1 + \alpha_2, k | t) = \sum_{m_1, \ldots, m_k \in \frac{1}{4}Z, m_{k+1} \in \mathbb{Z}} x_\alpha^\rho(m_1) x_{\alpha_1 + \alpha_2}^\rho(m_2) \cdots x_{\alpha_1 + \alpha_2}^\rho(m_{k+1})
\end{equation}
and

\[(3.11) \quad R(\alpha_1 + \alpha_2 | t) = \sum_{m_1, \ldots, m_{k+1} \in \mathbb{Z}} x_{\alpha_1+\alpha_2}(m_1) \cdots x_{\alpha_1+\alpha_2}(m_{k+1}).\]

As in \[\text{CalLM1}, \text{CalLM2}, \text{CMP}, \text{PS1} - \text{PS2}, \text{etc.}\] we may write

\[(3.12) \quad R(\alpha_1, \alpha_1 + \alpha_2, r | t) = R^0(\alpha_1, \alpha_1 + \alpha_2, r | t) + a \]

where \(a \in U(\mathbb{\hat{\nu}} | \mathbb{\hat{\nu}})_+\) and \(R^0(\alpha_1, \alpha_1 + \alpha_2, r | t)\) is the finite sum associated to \(R(\alpha_1, \alpha_1 + \alpha_2, r | t)\), for \(0 \leq r \leq k\), \(m_1, \ldots, m_{k+1-r} \in \left(\frac{1}{2} + \frac{1}{2} \mathbb{Z}\right) < 0\), \(m_{k+2-r}, \ldots, m_{k+1} \in \mathbb{Z}_{<0}\), and all the other conditions on indices remain the same, and \(U(\mathbb{\hat{\nu}} | \mathbb{\hat{\nu}})_+\) is an appropriate completion of \(U(\mathbb{\hat{\nu}} | \mathbb{\hat{\nu}})_+\) (see the Appendix in \[\text{S2}\] for a formal construction of this completion). Similarly, for any \(t \in \mathbb{Z}\) we have

\[(3.13) \quad R(\alpha_1 + \alpha_2 | t) = R^0(\alpha_1 + \alpha_2 | t) + a,\]

where \(a \in U(\mathbb{\hat{\nu}} | \mathbb{\hat{\nu}})_+\) and \(R^0(\alpha_1 + \alpha_2 | t)\) is the finite sum indexed by the integers \(m_1, \ldots, m_{k+1} < 0\).

We now note importantly that our relations may be rewritten in a simpler form:

**Lemma 3.1.** We have

\[(3.14) \quad R^0(\alpha_1, \alpha_1 + \alpha_2, r | t) = \sum_{n=0}^{k+1-r} \sum_{p \in \left(\frac{1}{2} \mathbb{Z}\right)_{<n}} a_p x_{\alpha_1}(p_1) \cdots x_{\alpha_1}(p_{k+1-r-2n}) x_{\alpha_1+\alpha_2}(p_{k+2-r-2n}) \cdots x_{\alpha_1+\alpha_2}(p_{k+1-n}) \]

where the \(a_p \in \mathbb{C}\) are constants related to reordering of the terms.

**Proof.** Consider any monomial in \(R^0(\alpha_1, \alpha_1 + \alpha_2, r | t)\)

\[x_{\alpha_1}(m_1+i_1)x_{\alpha_1}(m_2+i_2) \cdots x_{\alpha_1}(m_{k+1-r+i_1+i_2}) x_{\alpha_1+\alpha_2}(m_{k+1-r}) \cdots x_{\alpha_1+\alpha_2}(m_{k+1})\]

(recall \[3.9\] and \[3.12\]). Choosing the term with the smallest \(m_j + i_j\) for \(j = 1, \ldots, k+1-r\), we move this term to the left of our monomial using the commutation relations \[2.20\]. Namely, we have that:

\[x_{\alpha_1}(m_1+i_1)x_{\alpha_1}(m_2+i_2) \cdots x_{\alpha_1}(m_{k+1-r+i_1+i_2}) x_{\alpha_1+\alpha_2}(m_{k+1-r}) \cdots x_{\alpha_1+\alpha_2}(m_{k+1})\]

for some constant \(c \in \mathbb{C}\), which is 0 in the case that \(m_2 + m_j + i_\ell + i_j \not\in \mathbb{Z}\). At each step, this creates a new monomial with either the same number of \(x_{\alpha_1}(\cdot)\) terms, or introduces a \(x_{\alpha_1+\alpha_2}(\cdot)\) term at the expense of two \(x_{\alpha_1}(\cdot)\) terms. One repeats the above process by moving the term \(x_{\alpha_1}(p)\) with the smallest \(p\), to the left of its corresponding monomial. After doing this we will have created monomials which have at most \(\left\lfloor \frac{k+1-r}{2} \right\rfloor \) \(x_{\alpha_1+\alpha_2}(q)\) terms in the monomials. We also use \[2.21\] to order the terms \(x_{\alpha_1+\alpha_2}(q)\). Now \[3.14\] follows.

**Remark 3.1.** Of importance in the proof of our upcoming main result, Theorem 5.1, will be the variance of the entries in the summands of \[3.14\]. The unique longest term of \[3.14\] with the smallest such variance will be called the most “balanced” term of the expression.
Set
\[(3.15)\]
\[J = \sum_{r=0}^{k} \left( \sum_{t \geq \frac{k+3r}{4}} U(\mathfrak{m}[\hat{\nu}])R^0(\alpha_1, \alpha_1 + \alpha_2, r|t) \right) + \sum_{t \geq k+1} U(\mathfrak{m}[\hat{\nu}])R^0(\alpha_1 + \alpha_2|t).\]

Define the ideals
\[(3.16)\]
\[I_{k,0} = J + U(\mathfrak{m}[\hat{\nu}])\mathfrak{m}[\hat{\nu}]_+\]
and
\[(3.17)\]
\[I_{k,i} = I_{k,0} + \sum_{t \geq k+1} U(\mathfrak{m}[\hat{\nu}])x_{\alpha_1 + \alpha_2}^{\hat{\nu}} \left( -\frac{1}{4} \right)^{\ell} x_{\alpha_1 + \alpha_2}^{\hat{\nu}} (-1)^{k+1-i-\ell} \]
for \(0 \leq i \leq k\).

Recall the virtual subspaces (3.4). There are natural surjective maps
\[(3.18)\]
\[f_{k,i} : U(\mathfrak{m}[\hat{\nu}]) \rightarrow W_{k,i}, \quad a \mapsto a \cdot v_{k,i}.\]
The kernel of these maps will be called the \textit{presentation} of \(W_{k,i}\). Our main goal in this work is to prove that the presentations \(\text{Ker} f_{k,i}\) are equal to the ideals \(I_{k,i}\) above, a result analogous to results found in [CalLM1]–[CalLM4] and related works.

4. Shifting Maps

Continuing to follow [CalLM4], [CMP], etc. we define certain maps acting on \(U(\mathfrak{m}[\hat{\nu}])\) and \((V_L^T)^{\otimes k}\), which are needed in the proof of the presentation of the virtual subspaces \(W_{k,i}\) for \(0 \leq i \leq k\).

Let
\[\gamma = \frac{1}{2}(\alpha_1 + \alpha_2)(= \alpha_1(0)) \in \mathfrak{h}(0),\]
and
\[\theta : L \rightarrow C^\times,\]
defined by
\[\theta(\alpha_1) = -i \text{ and } \theta(\alpha_2) = i,\]
and extend it linearly. Consider the Lie algebra homomorphism on \(\mathfrak{m}[\hat{\nu}]\) defined by
\[x_{\alpha}^{\hat{\nu}}(m) \mapsto \theta(\alpha)x_{\alpha}^{\hat{\nu}}(m + \langle \alpha(0), \gamma \rangle)\]
which extends to an automorphism of \(U(\mathfrak{m}[\hat{\nu}])\), which we will denote by \(\tau_{\gamma,\theta}\). In particular, we have
\[(4.1)\]
\[\tau_{\gamma,\theta}(x_{\alpha_1}(m_1) \cdots x_{\alpha_1}(m_r)x_{\alpha_1 + \alpha_2}(n_1) \cdots x_{\alpha_1 + \alpha_2}(n_s))\]
\[= (-i)^s x_{\alpha_1}^{\hat{\nu}} \left( m_1 + \frac{1}{2} \right) \cdots x_{\alpha_1}^{\hat{\nu}} \left( m_r + \frac{1}{2} \right) x_{\alpha_1 + \alpha_2}^{\hat{\nu}}(n_1 + 1) \cdots x_{\alpha_1 + \alpha_2}^{\hat{\nu}}(n_s + 1)\]
for \(m_1, \ldots, m_r \in \frac{1}{2} \mathbb{Z}\) and \(n_1, \ldots, n_s \in \mathbb{Z}\). We also have
\[(4.2)\]
\[\tau_{\gamma,\theta}^{-1} = \tau_{-\gamma,\theta}^{-1}.\]

**Lemma 4.1.** We have
\[(4.3)\]
\[\tau_{\gamma,\theta}(I_{k,k}) = I_{k,0}.\]
Proof. Notice that for any $0 \leq r \leq k$ and $t \in (1/4)\mathbb{Z}$ we have

$$\tau_{\gamma,\vartheta} \left( R^0(\alpha_1, \alpha_1 + \alpha_2, r|t) \right) = R^0 \left( \alpha_1, \alpha_1 + \alpha_2, r|t - \frac{1}{2}(k + 1 + r) \right) + a$$

for some $a \in U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+$ and for $t \in \mathbb{Z}$ we have

$$\tau_{\gamma,\vartheta} \left( R^0(\alpha_1, \alpha_1 + \alpha_2|t) \right) = R^0(\alpha_1, \alpha_1 + \alpha_2|t - (k + 1)) + a,$$

for some $a \in U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+$, up to nonzero constants due to the character $\vartheta$. It is obvious that

$$\tau_{\gamma,\vartheta} \left( U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+ \right) \subset U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+$$

and

$$x_{\alpha_1,\alpha_2}^\vartheta(0) = \tau_{\gamma,\vartheta}(x_{\alpha_1,\alpha_2}^\vartheta(-1)), \quad x_{\alpha_1}^\vartheta \left( \frac{1}{4} \right) = \tau_{\gamma,\vartheta} \left( x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) \right).$$

Now (4.3) follows. \hfill \Box

We recall the map

$$\psi_{\gamma,\vartheta} : U(\mathbb{P}[\overline{\nu}]) \to U(\mathbb{P}[\overline{\nu}])$$

$$a \mapsto \tau_{\gamma,\vartheta}^{-1}(a)x_{\alpha_1}^\vartheta \left( \frac{1}{4} \right)$$

from [CalLM3]. Then for any $a \in U(\mathbb{P}[\overline{\nu}])$ we have

$$\psi_{\gamma,\vartheta} \tau_{\gamma,\vartheta}(a) = ax_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right).$$

Lemma 4.2. We have

$$\psi_{\gamma,\vartheta} \tau_{\gamma,\vartheta}(I_{k,k}) \subset I_{k,k-1}.$$ 

Proof. Need to show that $I_{k,k}x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) \subset I_{k,k-1}$. Repeated applications of (2.20) imply that

$$\prod_{j=1}^s x_{\alpha_1}^\vartheta(n_j), x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) = \sum_{j=1}^s c_j \left( \prod_{r \neq j} x_{\alpha_1}^\vartheta(n_r) \right) x_{\alpha_1,\alpha_2}^\vartheta \left( n_j - \frac{1}{4} \right)$$

for nonzero constants $c_j$. Then it follows that for $0 \leq r \leq k$ we have

$$R^0(\alpha_1, \alpha_1 + \alpha_2, r|t)x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) = x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) R^0(\alpha_1, \alpha_1 + \alpha_2, r|t) + a$$

$$R^0(\alpha_1, \alpha_1 + \alpha_2, r|t)\tau_{\gamma,\vartheta} \left( \frac{1}{4} \right) = x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) R^0(\alpha_1, \alpha_1 + \alpha_2|t).$$

Thus

$$\psi_{\gamma,\vartheta} \tau_{\gamma,\vartheta}(J) \subset J + U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+.$$ 

One can see easily that

$$\psi_{\gamma,\vartheta} \tau_{\gamma,\vartheta} \left( U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+ \right) \subset U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+ + U(\mathbb{P}[\overline{\nu}])x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right) \subset U(\mathbb{P}[\overline{\nu}])\mathbb{P}[\overline{\nu}]_+ + U(\mathbb{P}[\overline{\nu}])x_{\alpha_1}^\vartheta \left( -\frac{1}{4} \right)^2.$$ 

Therefore, we have

$$\psi_{\gamma,\vartheta} \tau_{\gamma,\vartheta}(I_{k,k}) \subset I_{k,k-1}.$$
We also have the following result:

**Lemma 4.3.** For all $s, i \geq 0$ with $0 \leq i + s \leq k$ we have:

\[ I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i-s} x_{\alpha_1 + \alpha_2}^\nu (-1)^s \subset I_{k,i}. \]

**Proof.** We first prove that we have

\[ I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} \subset I_{k,i} \]

for all $0 \leq i \leq k$. One can see that

\[
R^0(\alpha_1, \alpha_1 + \alpha_2, r|t) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} = \sum_{j=0}^{k-i} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^j R^0(\alpha_1, \alpha_1 + \alpha_1 + \alpha_2, r + k - i - j|t + \frac{(k - i - j)(k - i)}{4}) + a,
\]

where $a \in U(\mathfrak{p}[\nu])\mathfrak{p}[\nu]_+$, and

\[
R^0(\alpha_1, \alpha_1 + \alpha_2|t) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} = x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} R^0(\alpha_1 + \alpha_2|t),
\]

and

\[
R^0(\alpha_1 + \alpha_2|t) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} = x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} R^0(\alpha_1 + \alpha_2|t).
\]

Then

\[
I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} \subset J + U(\mathfrak{p}[\nu])\mathfrak{p}[\nu]_+ + U(\mathfrak{p}) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k+1-i} + U(\mathfrak{p}) x_{\alpha_1 + \alpha_2}^\nu (-1) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} \subset I_{k,i}.
\]

Now let $i, s \geq 0$ such that $0 \leq i + s \leq k$. By (4.14) we have

\[ I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i-s} \subset I_{k,i+s}. \]

Hence,

\[
I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i-s} x_{\alpha_1 + \alpha_2}^\nu (-1)^s \subset I_{k,i+s} x_{\alpha_1 + \alpha_2}^\nu (-1)^s
\]

\[
= I_{k,0} x_{\alpha_1 + \alpha_2}^\nu (-1)^s + \sum_{\ell=0}^{k+1-i-s} U(\mathfrak{n}[\nu]) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{\ell} x_{\alpha_1 + \alpha_2}^\nu (-1)^s R^0(\nu, \nu|\alpha) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k+1-i-s-\ell} x_{\alpha_1 + \alpha_2}^\nu (-1)^s
\]

\[
\subset I_{k,0} + \sum_{\ell=0}^{k+1-i-s} U(\mathfrak{n}[\nu]) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{\ell} x_{\alpha_1 + \alpha_2}^\nu (-1)^s R^0(\nu, \nu|\alpha) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k+1-i-s-\ell} x_{\alpha_1 + \alpha_2}^\nu (-1)^s
\]

\[
\subset I_{k,i},
\]

where we use the fact that $x_{\alpha_1 + \alpha_2}^\nu (-1)$ commutes with all elements of $I_{k,0}$. □
Consider the linear map
\[(4.15) \quad e_{\alpha_1} : V^T_L \rightarrow V^T_L,\]
and its restriction to the principal subspace \(W^T_L\)
\[(4.16) \quad e_{\alpha_1} : W^T_L \rightarrow W^T_L.\]
Then
\[(4.17) \quad e_{\alpha_1}(a \cdot 1) = A^\sigma(\cdot, \theta(\cdot)) \psi_{\gamma, \theta}(a) \cdot 1,
\text{ where } a \in U(\overline{\mathfrak{g}}) \text{ and } A^\sigma(\cdot, \theta(\cdot)) \text{ is a nonzero constant depending on the maps } C(\cdot, \cdot),
\sigma(\cdot) \text{ and } \theta(\cdot).\]
Now consider the linear map
\[(4.18) \quad e_{\alpha_1} \otimes_k : (V^T_L)^\otimes_k \rightarrow (V^T_L)^\otimes_k.\]
Then we have
\[
(4.19) \quad e_{\alpha_1} \otimes_k : W_{k,0} \rightarrow W_{k,k},
\]
where
\[(4.20) \quad e_{\alpha_1} \otimes_k (a \cdot v_{k,0}) = \tau_{\gamma, \theta}^{-1}(a) \cdot v_{k,k},\]
up to certain nonzero constants.
Let \(\lambda_1\) be the fundamental weight of \(\mathfrak{g}\) defined by:
\[
\lambda_1 = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2.
\]
We recall from \([\text{CalLM3}]\) the operators
\[(4.21) \quad \Delta^T(\lambda_1, -x) = i^{2\lambda_1(0)} x^{\lambda_1(0)} E^+ (-\lambda_1, x) \in (\text{End } V^T_L)[[x^{1/4}, x^{-1/4}]],
\text{ where } \lambda_1(0) = \frac{1}{2}(\alpha_1 + \alpha_2). \text{ Denote by } \Delta^T_x(\lambda_1, -x) \text{ the constant term of } \Delta^T(\lambda_1, -x).\]
Recall also
\[
\Delta^T_x(\lambda_1, -x) : W^T_L \rightarrow W^T_L, \quad a \cdot 1 \mapsto \tau_{\gamma, \theta}(a) \cdot 1.
\]
We now define the following operators:
\[(4.22) \quad \mathcal{Y}_i = 1_V \otimes \cdots \otimes 1_V \otimes e_{\alpha_1} \circ \Delta^T_x(-\lambda_1, x) \otimes 1_V \cdots \otimes 1_V : (V^T_L)^\otimes_k \rightarrow (V^T_L)^\otimes_k,
\text{ where the non-identity operator occurs at the } i\text{-th slot. Using the operators } \mathcal{Y}_{k-i}
\text{ one can show that}
\[(4.23) \quad \ker f_{k,i} \subset \ker f_{k,i+1}
\text{ for any } 0 \leq i \leq k - 1.
\]
5. Presentations of the subspaces \(W_{k,i}\)
We now have the necessary ingredients to prove a presentation of the spaces \(W_{k,i}\).

**Theorem 5.1.** For \(i = 0, \ldots, k\) we have
\[(5.1) \quad \ker f_{k,i} = I_{k,i}.\]
Proof. Let $0 \leq i \leq k$. Since the inclusion $I_{k,i} \subset \text{Ker} f_{k,i}$ is obvious, the remainder of the proof will show that $\text{Ker} f_{k,i} \subset I_{k,i}$ for $i = 0, \ldots, k$. Suppose that for some $\ell = 0, \ldots, k$ we have $\text{Ker} f_{k,\ell} \not\subset I_{k,\ell}$, and consider the set

$$\{ a \in U(\mathfrak{m}[\nu]) | a \in \text{Ker} f_{k,\ell} \setminus I_{k,\ell} \text{ for some } \ell = 0, \ldots, k \}.$$  

We may and do assume that elements of this set have positive weight. Among the elements of this set, we look at those with lowest charge. Among the elements of lowest charge, we choose an element of lowest weight and call it $a$.

We first show that $\ell \neq k$. Suppose $a \in \text{Ker} f_{k,k} \setminus I_{k,k}$. Then, we have that

$$(5.2) \quad a \cdot (e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}) = 0,$$

and thus by (4.18) we have

$$(5.3) \quad e_{\alpha_1}^{\otimes k} (\tau_{\gamma,\nu}(a)(1 \otimes \cdots \otimes 1)) = 0.$$  

Now the injectivity of $e_{\alpha_1}^{\otimes k}$ implies that

$$(5.4) \quad \tau_{\gamma,\nu}(a)(1 \otimes \cdots \otimes 1) = 0.$$  

Since $\tau_{\gamma,\nu}(a)$ has lower weight than $a$, we have that $\tau_{\gamma,\nu}(a) \in I_{k,0}$, and so applying Lemma 4.1 we have that $a \in I_{k,k}$, a contradiction.

We now proceed by induction and assume that we have shown that $\ell \neq k, k-1, \ldots, i+1$ for some $0 \leq i \leq k-1$. We will show that this implies $\ell \neq i$ as well. We suppose our minimal element $a \in \text{Ker} f_{k,i} \setminus I_{k,i}$.

We first show, using an inductive proof, that we can write

$$(5.5) \quad a = b + cx_{\alpha_1+\alpha_2} (-1)^{k-i}$$

with $b \in I_{k,i}$ and $c \in U(\mathfrak{n}[\nu])$. Since $a \in \text{Ker} f_{k,i} \subset \text{Ker} f_{k,i+1} = I_{k,i+1}$ we may write

$$(5.6) \quad a = b^{(0)} + \sum_{j=0}^{k-i} c_j^{(0)} x_{\alpha_1}^{\rho} \left( -\frac{1}{4} \right)^{k-i-j} x_{\alpha_1+\alpha_2} (-1)^{j},$$

where $b^{(0)} \in I_{k,0}$ and $c_j^{(0)} \in U(\mathfrak{n}[\nu])$ for $0 \leq j \leq k-i$. We will proceed inductively towards (5.5) by first showing that

$$(5.7) \quad c_0^{(0)} x_{\alpha_1}^{\rho} \left( -\frac{1}{4} \right)^{k-1-i} \in I_{k,i} + \sum_{m+n \geq 1, m+n=k-i} U(\mathfrak{n}[\nu]) x_{\alpha_1}^{\rho} \left( -\frac{1}{4} \right)^{m} x_{\alpha_1+\alpha_2} (-1)^{n}.$$  

Using the decomposition (5.6), we have

$$(5.8) \quad (a - b^{(0)})(1^{\otimes (k-i)} \otimes e_{\alpha_1}^{\otimes i}) = 0,$$

and thus

$$(5.9) \quad \sum_{j=0}^{k-i} c_j^{(0)} x_{\alpha_1}^{\rho} \left( -\frac{1}{4} \right)^{k-i-j} x_{\alpha_1+\alpha_2} (-1)^{j} (1^{\otimes (k-i)} \otimes e_{\alpha_1}^{\otimes i}) = 0.$$  

Using the diagonal action we can write

$$(5.10) \quad \sum_{j=0}^{k-i} c_j^{(0)} \left( x_{\alpha_1}^{\rho} \left( -\frac{3}{4} \right) e_{\alpha_1} \right)^{\otimes j} \otimes e_{\alpha_1}^{\otimes (k-j)} + w_0 = 0$$

and so

$$\sum_{j=0}^{k-i} c_j^{(0)} \left( x_{\alpha_1}^{\rho} \left( -\frac{3}{4} \right) e_{\alpha_1} \right)^{\otimes j} \otimes e_{\alpha_1}^{\otimes (k-j)} + w_0 = 0.$$
where $w_0$ is a sum of tensor factors all of which have $x_{\alpha_1}^\nu \left( -\frac{3}{4} \right) e_{\alpha_1}$ as at least one of the first $k-i$ entries or it is zero. We now write

\begin{equation}
(5.11) \quad e_{\alpha_1}^{\otimes k} \left( \sum_{j=0}^{k-i} \tau_{\gamma,\theta} (c_j^{(0)}) \left( e_{\alpha_1}^{\otimes j} \otimes 1^{\otimes (k-j)} + w_0' \right) \right) = 0
\end{equation}

where $e_{\alpha_1}^{\otimes k} (w_0') = w_0$ and thus $w_0'$ is a sum of tensor factors all of which have $e_{\alpha_1}$ as one of the first $k-i$ entries or is zero. By the injectivity of $e_{\alpha_1}^{\otimes k}$ we have

\begin{equation}
(5.12) \quad \sum_{j=0}^{k-i} \tau_{\gamma,\theta} (c_j^{(0)}) \left( e_{\alpha_1}^{\otimes j} \otimes 1^{\otimes (k-j)} + w_0' \right) = 0
\end{equation}

We now apply the composition operators $\mathcal{Y}_1 \circ \mathcal{Y}_2 \circ \cdots \circ \mathcal{Y}_{k-i}$ which collapses (5.12) to

\begin{equation}
(5.13) \quad \tau_{\gamma,\theta} \left( c_0^{(0)} \right) \left( e_{\alpha_1}^{\otimes k-i} \otimes 1^{\otimes i} \right) = 0.
\end{equation}

So $\tau_{\gamma,\theta} \left( c_0^{(0)} \right) \in \text{Ker} f_{k,k-i}$. Since $\tau_{\gamma,\theta} \left( c_0^{(0)} \right)$ has lower charge than $a$, then

\begin{equation}
(5.14) \quad \tau_{\gamma,\theta} \left( c_0^{(0)} \right) \in I_{k,k-i} = I_{k,0} + \sum_{m \geq 0, n \geq 0 \atop m+n=i+1} U(\hat{\mathcal{A}} x_1^\nu) x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^m x_{\alpha_1+\alpha_2} (2)^n.
\end{equation}

So we have, using Lemma 4.1

\begin{equation}
(5.15) \quad c_0^{(0)} \in I_{k,k} + \sum_{m \geq 0, n \geq 0 \atop m+n=i+1} U(\hat{\mathcal{A}} x_1^\nu) x_{\alpha_1}^\nu \left( -\frac{3}{4} \right)^m x_{\alpha_1+\alpha_2} (-2)^n
\end{equation}

Now putting this into the context of (5.6) we have

\begin{equation}
(5.16) \quad c_0^{(0)} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} \in I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} + \sum_{m \geq 0, n \geq 0 \atop m+n=i+1} U(\hat{\mathcal{A}} x_1^\nu) x_{\alpha_1}^\nu \left( -\frac{3}{4} \right)^m x_{\alpha_1+\alpha_2} (-2)^n
\end{equation}

By Lemma 4.3 we have $I_{k,k} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} \subset I_{k,i}$. Now we focus on the terms $x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^m x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i} x_{\alpha_1+\alpha_2} (-2)^n$ such that $m + n = i + 1$. Each of these is a summand (with non-zero coefficient) of the expression $R^B (\alpha_1, \alpha_1 + \alpha_2, n \| m+n+i+1 \|$ and is the most “balanced” term of this expression. That is, by Lemma 8.3 every other summand of this expression is either an element of $I_{k,i}$ or a $U(\hat{\mathcal{A}} x_1^\nu)$-multiple of

$$x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i-j} x_{\alpha_1+\alpha_2} (-1)^j$$

for $1 \leq j \leq k-i$ which verifies (5.7). Using (5.6) we have

\begin{equation}
(5.17) \quad a = b^{(1)} + \sum_{j=1}^{k-i} c_j^{(1)} x_{\alpha_1}^\nu \left( -\frac{1}{4} \right)^{k-i-j} x_{\alpha_1+\alpha_2} (-1)^j,
\end{equation}

for $b^{(1)} \in I_{k,i}$ and $c_j^{(1)} \in U(\hat{\mathcal{A}} x_1^\nu)$.
Using the diagonal action we can write

\[ \tau \left( \frac{1}{4} \right)^{k-i-j} x_{\alpha_1 + \alpha_2} (-1)^j \]

for some \( b^{(s)} \in I_{k,i} \) and \( c_j^{(s)} \in U(\mathfrak{n}[\hat{\nu}]) \), where \( 1 \leq s < k - i \). We show that

\[ a = b^{(s+1)} + \sum_{j=s+1}^{k-i} c_j^{(s+1)} x_{\alpha_1} \left( -\frac{1}{4} \right)^{k-i-j} x_{\alpha_1 + \alpha_2} (-1)^j, \]

where \( b^{(s+1)} \in I_{k,i} \) and \( c_j^{(s+1)} \in U(\mathfrak{n}[\hat{\nu}]). \)

Using the decomposition (5.18), we have

\[ (a - b^{(s)})(1^{\otimes (k-i)} \otimes e_{\alpha_1}^{(i)}) = 0 \]

and thus

\[ \sum_{j=s}^{k-i} c_j^{(s)} x_{\alpha_1} \left( -\frac{1}{4} \right)^{k-i-j} x_{\alpha_1 + \alpha_2} (-1)^j (1^{\otimes (k-i)} \otimes e_{\alpha_1}^{(i)}) = 0 \]

Using the diagonal action we can write

\[ \sum_{j=s}^{k-i} c_j^{(s)} \left( x_{\alpha_1} \left( -\frac{3}{4} \right)_i \otimes e_{\alpha_1}^{(i)} + w_s' \right) = 0 \]

where \( w_s \) is a sum of tensor factors all of which have \( x_{\alpha_1} \left( -\frac{4}{4} \right) c_{\alpha_1} \) as at least one of the entries between the \( s + 1 \) and \( k - i \) position or is zero. We now write

\[ e_{\alpha_1}^{(k)} \left( \sum_{j=s}^{k-i} \tau_{\gamma,\theta}(c_j^{(s)}) \left( e_{\alpha_1}^{(j)} \otimes 1^{\otimes (k-j)} + w_s' \right) \right) = 0 \]

where \( e_{\alpha_1}^{(k)w_s'} = w_s \) and thus \( w_s' \) is a sum of tensor factors all of which have \( e_{\alpha_1} \) as at least one of the entries between the \( s + 1 \) and \( k - i \) position or is zero. By the injectivity of \( e_{\alpha_1}^{(k)} \) we have

\[ \sum_{j=s}^{k-1} \tau_{\gamma,\theta}(c_j^{(s)}) \left( e_{\alpha_1}^{(j)} \otimes 1^{\otimes (k-j)} + w_s' \right) = 0 \]

Now apply the composition of operators \( \mathcal{Y}_{k+1} \circ \mathcal{Y}_{k+2} \circ \cdots \circ \mathcal{Y}_{k-i} \) which collapses (5.24) to

\[ \tau_{\gamma,\theta}(c_s^{(k)}) \left( e_{\alpha_1}^{(k-i)} \otimes 1^{\otimes i} \right) = 0. \]

So \( \tau_{\gamma,\theta}(c_s^{(k)}) \in \text{Ker} \mathcal{I}_{k,k-i} \) and has lower charge than \( a \), thus

\[ \tau_{\gamma,\theta}(c_s^{(k)}) \in I_{k,k-i} = I_{k,0} + \sum_{m \geq 0, n \geq 0 \atop m+n=i+1} U(\mathfrak{n}[\hat{\nu}]) x_{\alpha_1} \left( -\frac{1}{4} \right)^m x_{\alpha_1 + \alpha_2} (-1)^n. \]

Using Lemma\[4.1\] again, we have

\[ c_s^{(k)} \in I_{k,k} + \sum_{m \geq 0, n \geq 0 \atop m+n=i+1} U(\mathfrak{n}[\hat{\nu}]) x_{\alpha_1} \left( -\frac{3}{4} \right)^m x_{\alpha_1 + \alpha_2} (-2)^n. \]
Now putting this into the context of \((5.18)\) we have
\[(5.28)\]
\[
\left(\frac{1}{4}\right) x_{\alpha_1}^\rho \left(-\frac{1}{4}\right)^{k-i-j} a_{\alpha_1+\alpha_2} \left(-1\right)^s \in I_{k,k} x_{\alpha_1}^\rho \left(-\frac{1}{4}\right)^{k-i-j} a_{\alpha_1+\alpha_2} \left(-1\right)^s + \sum_{m \geq 0, n \geq 0, m + n = i + 1} U(n) x_{\alpha_1}^\rho \left(-\frac{3}{4}\right)^m a_{\alpha_1+\alpha_2} \left(-1\right)^s .
\]

By Lemma \(4.3\) we have \(I_{k,k} x_{\alpha_1}^\rho \left(-\frac{1}{4}\right)^{k-i-j} a_{\alpha_1+\alpha_2} \left(-1\right)^s \subseteq I_{k,i}\) so we focus on the terms \(x_{\alpha_1}^\rho \left(-\frac{1}{4}\right)^m a_{\alpha_1+\alpha_2} \left(-1\right)^s \) such that \(m + n = i + 1\). Each of these is a summand (with non-zero coefficient) of the expression \(R^0 (\alpha_1, \alpha_1 + \alpha_2, n + s) \left(\alpha_1, \alpha_1 + \alpha_2, n + s\right)\) and is the most “balanced” term of this expression. That is, by Lemma \(4.3\) every other summand of this expression is either an element of \(I_{k,i}\) or a \(U(n)\)-multiple of
\[
I_{k,i} 
\]
which finishes the inductive argument that verifies the claim described by \(5.5\).

Then we write
\[(5.29)\]
\[
\left(\frac{1}{4}\right) + c x_{\alpha_1+\alpha_2} \left(-1\right)^{k-i}
\]
with \(b \in I_{k,i}\) and \(c \in U(n)\). Now since
\[(5.30)\]
\[
(a - b) \left(1 \otimes^{k-i} e_{\alpha_1} \right) = 0
\]
we have
\[(5.31)\]
\[
c x_{\alpha_1+\alpha_2} \left(-1\right)^{k-i} \left(1 \otimes^{k-i} e_{\alpha_1} \right) = 0
\]
and thus
\[(5.32)\]
\[
c \left(\left(x_{\alpha_1}^\rho \left(-\frac{3}{4}\right)^m a_{\alpha_1+\alpha_2} \left(-1\right)^s \right) \otimes^{k-i} e_{\alpha_1} \right) = 0
\]
which implies
\[(5.33)\]
\[
e_{\alpha_1} \left(\tau_{\gamma, \theta}(c) e_{\alpha_1} \otimes 1 \otimes^{i} 1 \right) = 0
\]
and by the injectivity of \(e_{\alpha_1} \) we have
\[(5.34)\]
\[
\tau_{\gamma, \theta}(c) e_{\alpha_1} \otimes 1 \otimes^{i} 1
\]
so \(\tau_{\gamma, \theta}(c) \) in \(\text{Ker} f_{k,k-i}\). Since \(\tau_{\gamma, \theta}(c) \) has lower charge than \(a\), then \(\tau_{\gamma, \theta}(c) \in I_{k,k-i}\).

Then
\[(5.35)\]
\[
c \in I_{k,k} + \sum_{m \geq 0, n \geq 0, m + n = i + 1} \left(U(n) x_{\alpha_1}^\rho \left(-\frac{3}{4}\right)^m a_{\alpha_1+\alpha_2} \left(-1\right)^s \right),
\]
and so
\[(5.36)\]
\[
c x_{\alpha_1+\alpha_2} \left(-1\right)^{k-i} \in I_{k,k} x_{\alpha_1+\alpha_2} \left(-1\right)^{k-i} + \sum_{m \geq 0, n \geq 0, m + n = i + 1} \left(U(n) x_{\alpha_1}^\rho \left(-\frac{3}{4}\right)^m a_{\alpha_1+\alpha_2} \left(-1\right)^s \right).
\]
By Lemma 3.3 we have $I_{k,i} x^\alpha_{1+i} (-1)^{k-i} \subset I_{k,i}$, so we focus on the terms $x^\alpha_{1+i} (-\frac{3}{4})^m x^\alpha_{1+2} (-2)^n x^\alpha_{1+2} (-1)^{k-i}$ for $m + n = i + 1$. Each of these is a summand (with non-zero coefficient) of the expression $R^0 (\alpha_1, \alpha_1 + \alpha_2, n + k - i \frac{3m + 8n + 4k - 4i}{4})$ and is the most “balanced” term of this expression. That is, every other summand of this expression is and element of $I_{k,i}$. So we have $cx^\alpha_{1+i} (-1)^{k-i} \in I_{k,i}$ and thus $a \in I_{k,i}$, a contradiction which finishes the proof.

Remark 5.1. We would like to point out that the maps used in this paper to prove the presentation of the principal subspace $W_{k,0}$, and more generally, virtual subspaces $W_{k,i}$, do not give exact sequences of maps among these virtual subspaces, as it is the case in [CalLM3, CalLM4, CLM1, CLM2, CMP, FS1, PS1, PS2, and PSW]. This proves once again the uniqueness and subtlety of these principal subspaces of higher level standard $A_2^{(2)}$-modules. The difficulty of constructing exact sequences in the higher level case appears also in [C and S2], where a limited set of exact sequences among principal subspaces of higher level standard modules for $A_2^{(1)}$ is given, but the general case is open.

References

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA 83 (1986), 3068–3071.
[Bu1] M. Butorac, Combinatorial bases of principal subspaces for the affine Lie algebra of type $B_2^{(1)}$, J. Pure Appl. Algebra 218 , (2014), 424-447.
[Bu2] M. Butorac, Quasi-particle bases of principal subspaces for the affine Lie algebras of types $B_2^{(1)}$ and $C_1^{(1)}$, Glas. Mat. Ser. III 51 (2016), 59-108.
[Bu3] M. Butorac, Quasi-particle bases of principal subspaces of the affine Lie algebra of type $G_2^{(1)}$, Glas. Mat. Ser. III 52 (2017), 79-98.
[BS] M. Butorac and C. Sadowski, Combinatorial bases of principal subspaces of modules for twisted affine Lie algebras of types $A_2^{(1)}$, $D_4^{(1)}$, $E_6^{(1)}$ and $D_4^{(3)}$, preprint.
[C] C. Calinescu, Principal subspaces of higher-level standard $\mathfrak{sl}(3)$-modules, J. Pure Appl. Algebra 2 (2010), 2007, 559-575.
[CalLM1] C. Calinescu, J. Lepowsky and A. Milas, Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$-modules, I: level one case, Internat. J. Math. 19 (2008), 71–92.
[CalLM2] C. Calinescu, J. Lepowsky and A. Milas, Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$-modules, II: higher level case, J. Pure Appl. Algebra 212 (2008), 1928–1950.
[CalLM3] C. Calinescu, J. Lepowsky and A. Milas, Vertex-algebraic structure of the principal subspaces of level one modules for the untwisted affine Lie algebras of types $A$, $D$, $E$, J. Algebra 323 (2010), 167–192.
[CalLM4] C. Calinescu, J. Lepowsky and A. Milas, Vertex-algebraic structure of principal subspaces of standard $A_2^{(2)}$-modules, I, Internat. J. Math. 25 (2014).
[CLM1] S. Capparelli, J. Lepowsky and A. Milas, The Rogers-Ramanujan recursion and intertwining operators, Comm. in Contemp. Math. 5 (2003), 947–966.
[CLM2] S. Capparelli, J. Lepowsky and A. Milas, The Rogers-Selberg recursions, the Gordon-Andrews identities and intertwining operators, The Ramanujan Journal 12 (2006), 379–397.
[CMP] C. Calinescu, A. Milas and M. Penn, Vertex-algebraic structure of principal subspaces of basic $A_2^{(2)}$-modules J. Pure Appl. Algebra 220 (2016), 1752-1784.
[FS1] B. Feigin and A. Stoyanovsky, Quasi-particles models for the representations of Lie algebras and geometry of flag manifold, [arXiv:hep-th/9308079].
[FS2] B. Feigin and A. Stoyanovsky, Functional models for representations of current algebras and semi-infinite Schubert cells (Russian), Funktsional Anal. i Prilozhen. 28 (1994), 68–90; translation in: Funct. Anal. Appl. 28 (1994), 55–72.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Memoirs American Math. Soc. 104, 1993.

[FLM1] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator calculus, in: Mathematical Aspects of String Theory, Proc. 1986 Conference, San Diego, ed. by S.-T. Yau, World Scientific, Singapore, 1987, 150–188.

[FLM2] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., Vol. 134, Academic Press, 1988.

[Je] M. Jerkovic, Recurrences and characters of Feigin-Stoyanovsky’s type subspaces. Vertex operator algebras and related areas, Contemp. Math. 497 (2009), 113–123.

[K] V. Kac, Infinite Dimensional Lie Algebras, 3rd edition, Cambridge University Press, 1990.

[Ko] S. Kozić, Principal subspaces for quantum affine algebra $U_q(A_1^{(1)})$, J. Pure Appl. Algebra 218 (2014), 2119–2148.

[L1] J. Lepowsky, Calculus of twisted vertex operators, Proc. Nat. Acad. Sci. USA 82 (1985), 8295–8299.

[L2] J. Lepowsky, Perspectives on vertex operators and the Monster, in: Proc. 1987 Symposium on the Mathematical Heritage of Hermann Weyl, Duke Univ., Proc. Symp. Pure Math., American Math. Soc. 48 (1988), 181–197.

[LL] J. Lepowsky and H. Li, Introduction to Vertex Operator Algebras and Their Representations, Progress in Mathematics, Vol. 227, Birkhäuser, Boston, 2003.

[Li] H.-S. Li, Local systems of twisted vertex operators, vertex superalgebras and twisted modules, Contemp. Math. 193 (1996), 203–236.

[MP] A. Milas and M. Penn, Lattice vertex algebras and combinatorial bases: general case and W-algebras, New York J. Math. 18 (2012), 621–650.

[P] M. Penn, Lattice Vertex Superalgebras I: Presentation of the Principal Subspace, Communications in Algebra, Volume 42, Issue 3 (2014), 933-961.

[PS1] M. Penn and C. Sadowski, Vertex-algebraic structure of principal subspaces of basic $D^{(3)}$-modules, The Ramanujan Journal, 43:4 (2017), 571-617.

[PS2] M. Penn and C. Sadowski, Vertex-algebraic structure of principal subspaces of basic modules for twisted affine Kac-Moody Lie algebras of type $A^{(2)}_{2n+1}$, $D^{(2)}_n$, $E^{(2)}_6$, J. Algebra Volume 496, (2018), pp. 242-291

[PSW] M. Penn, C. Sadowski, and G. Webb Twisted Modules of Principal Subalgebras of Lattice Vertex Algebras, 40 pgs, submitted

[Pr] M. Primc, $(k, r)$-admissible configurations and intertwining operators, Contemp. Math 422 (2007), 425-434.

[S1] C. Sadowski, Presentations of the principal superspaces of higher level $\widehat{sl}(3)$-modules, J. Pure Appl. Algebra, 219 (2015), 2390-2345.

[S2] C. Sadowski, Principal subspaces of $\mathfrak{sl}(n)$-modules, Int. J. Math., Vol. 26 No. 08, 1550063 (2015).

[T1] G. Trupcevic, Combinatorial bases of Feigin-Stoyanovsky’s type subspaces of level one standard modules for $\mathfrak{sl}(l+1, \mathbb{C})$, Comm. Algebra 38 (2010), 3913-3940.

[T2] G. Trupcevic, Combinatorial bases of Feigin-Stoyanovsky’s type subspaces of higher-level standard $\mathfrak{sl}(l+1, \mathbb{C})$-modules, J. Algebra 322 (2009), 3744-3774.
Department of Mathematics and Computer Science, Ursinus College, Collegeville, PA 19426
E–mail address: csadowski@ursinus.edu