NONCOMMUTATIVE MAXIMAL ERGODIC INEQUALITIES FOR AMENABLE GROUPS

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Abstract. We prove a pointwise ergodic theorem for actions of amenable groups on noncommutative measure spaces. To do so, we establish the maximal ergodic inequality for averages of operator-valued functions on amenable groups. The main arguments are a geometric construction of martingales based on the Ornstein-Weiss quasi-tilings and harmonic analytic estimates coming from noncommutative Calderón-Zygmund theory.

1. Introduction

This paper aims to establish the analogue of the following ergodic theorem due to Lindenstrauss [16] for actions on noncommutative measure spaces.

Theorem 1.1 (Lindenstrauss). Let $G$ be a second countable amenable group acting by measure preserving transformations on a measure space $X$. There exists a sequence $(F_n)_{n \in \mathbb{N}}$ of subsets of $G$ such that for any function $f \in L^p(X)$, $p \in [1, \infty)$,

$$|F_n|^{-1} \int_{F_n} g \cdot f \, dm(g) \rightarrow P(f) \quad \text{a.e.}$$

where the $P$ is the projection onto $G$-invariant functions which restricts to the orthogonal projection on $L_2(G) \cap L^p(G)$.

Lindenstrauss's theorem can be viewed as a generalisation to amenable groups of Birkhoff’s ergodic theorem. On the other hand, the study of noncommutative generalisations of Birkhoff ergodic theorem began with Lance [15] for bounded operators followed by Yeadon [22] for integrable operators. However, a number of classical techniques involving maximal functions was no longer available in the noncommutative setting, which prevented further progress in this field at the time. More than two decades later, Junge introduced a noncommutative analogue of Doob’s maximal inequalities in $L^p$ [13] in the context of martingale theory. This opened the possibility to study maximal inequalities for families of operators on $L^p$, and soon after Junge and Xu [14] extended Lance and Yeadon’s work to $L^p$, $(1 < p < \infty)$, finally obtaining a satisfactory noncommutative version of Birkhoff’s and Dunford-Schwartz’s ergodic theorems.

Since then, finding noncommutative analogues of pointwise ergodic theorems for general group actions becomes an open problem circulated in the community. In particular, the noncommutative analogue of Lindenstrauss’s theorem is amongst the major topics in this field. The first general result towards this direction was given recently by Hong, Liao and the second author [10], which investigated noncommutative extensions of Calderón’s pointwise ergodic theorem for groups of polynomial growth. But this latter work relies essentially on the analysis on doubling metric measure spaces, which does not apply to the general setting of amenable groups. In this current paper, we will introduce several new geometric and analytic methods, and thereby establish the desired noncommutative Lindenstrauss ergodic theorem for general amenable groups (Theorem 6.7).

In the remainder of the introduction, we get a bit more precise in order to state some of the main techniques of the paper. We refer to Section 2 for precise definitions of any unexplained notation. Let $G$ be a locally compact second countable amenable group equipped with a right-invariant Haar measure $m$, also often denoted by $|\cdot|$ for convenience. As shown by the transference principle in [10], the core of the study of maximal ergodic inequalities can be reduced to the analysis of
operator-valued functions on $G$. Let $F$ be a measurable subset of $G$ with $|F| < \infty$. Denote by $\hat{A}_F$ the averaging operator defined for any (potentially operator-valued) function $f$ and $x \in G$ by

\begin{equation}
\hat{A}_F(f)(x) := \frac{1}{|F|} \int_F f(xg) \, dm(g) = \frac{1}{|xF|} \int_{xF} f(g) \, dm(g).
\end{equation}

We will very often consider a sequence of subsets $(F_n)_{n \geq 0}$ of $G$ and denote the associated sequence of averaging operators $(\hat{A}_{F_n})_{n \geq 0}$ by $(\hat{A}_n)_{n \geq 0}$ for short. We aim to study the weak $(1,1)$ maximal inequality for the sequence $(\hat{A}_n)_{n \geq 0}$, which may be viewed as a noncommutative analogue of the classical weak maximal inequality of the form

$$\| \sup_{n \geq 0} |\hat{A}_n(f)| \|_{1,\infty} \lesssim \|f\|_1,$$

for any function $f \in L_1$. The interpolation principle of [14] then implies the maximal inequalities for $L_p$ for any $p \in (1, \infty)$. As in classical analysis, a standard Banach principle argument then yields the pointwise ergodic theorems on the corresponding spaces.

Classical proofs of maximal inequalities on groups often make use of covering lemmas which so far do not admit analogues in noncommutative harmonic analysis. A recent alternative approach in the field to circumvent the problem is to rely on martingale theory. This idea may date back to Mei [17, Chapter 3] in the study of operator-valued BMO spaces and was recently highlighted in noncommutative ergodic theory by [10], where they dominate the ergodic averages $\hat{A}_n$ by sums of several martingale averages. But the geometric requirements underlying the proof seem too strong to hold in any amenable group, and those dominations by martingales are not powerful enough to study other ergodic estimates such as variational inequalities. This leads the experts to find other related strategies; in particular, instead of dominating $\hat{A}_n$ by martingales averages, Hong and Xu [11, 21] considered the mappings

\begin{equation}
D := \sum_{n=0}^{\infty} \varepsilon_n D_n \quad \text{and} \quad D_n = A_n - E_n,
\end{equation}

where $\varepsilon_n$ is a sequence of independent Rademacher variables and the $E_n$’s are martingale averages. This operator allows to quantify the difference between ergodic and martingales averages and therefore to transfer boundedness properties from one to the other. This idea was inspired by the study of classical variational inequalities [11, 12], which are known to be much stronger than maximal inequalities. Our work is exactly based on this idea and the major part of the paper will be devoted to the construction of appropriate martingales and the boundedness of the resulting operator $D$ in the setting of amenable groups.

Let us summarize some key ideas and strategies around the study of this operator $D$ in our new setting. In the Euclidean case, the boundedness of $D$ associated with dyadic martingales was proved by Hong and Xu [11, 21] by exploiting the idea that $D$ is very close to being a Calderón-Zygmund operator. Its kernel does not enjoy smoothness properties but it possesses an (even better) pseudo-localizing behavior due to the fact that the $D_n$’s satisfy local estimates (see Subsection 5.1) which boil down to geometric consideration on the dyadic filtration. Various essential difficulties arise if we would like to extend this idea to the general setting of amenable groups. For instance, there are no standard choice of martingale structures for abstract groups; also, the sequence $(F_n)$ used in the averages $(\hat{A}_n)$ for general amenable groups is not necessarily associated with a translate-invariant metric, which yields additional complication if we want to transfer the Euclidean geometric consideration into the abstract setting.

To deal with these difficulties, we propose a more abstract geometric framework in Definition 3.5 which is described simply by group-theoretical languages without any metric structure. This framework indeed correctly reflects the key geometric data of the dyadic filtration that were implicitly used in the aforementioned local estimates of Hong and Xu. Roughly speaking, for any given increasing Følner sequence $(F_n)$, we will consider a filtration of partitions $\mathcal{P}$ so that at least a positive proportion of its atoms interact well with $(F_n)$ via some Følner invariant conditions and inclusion relations. These sequences will be referred to as regular filtered Følner sequences, and we will show that they always exist by using Ornstein and Weiss’s quasi-tiling construction [18].
We remark that at the early stage of our project, we indeed worked with the stronger geometric object called completely regular filtered Følner sequences, where, instead of a portion of atoms of \( \mathcal{P} \), all atoms of \( \mathcal{P} \) satisfy similar ideal geometric conditions. Such sequences do exist for all discrete groups due to exact tilings by \([7]\) instead of Ornstein and Weiss’s quasi-tilings, and many concrete examples can be found in \([6,8]\). It may be possible to obtain regular filtered sequences in any non-discrete amenable group as well; but from the viewpoint of harmonic analysis, it is nonetheless interesting to see that our technique is robust enough to work with less optimal martingales associated with regular filtered sequences.

We will then show that these geometric data are sufficient to yield nice martingales and derive suitable local estimates even in the general abstract setting, still following a scheme similar to that of \([11,21]\). It is worth mentioning that the Calderón-Zygmund arguments and local estimates in \([11,21]\) work with doubling metric measure spaces, so new adaptations are bound to be necessary in our general abstract setting. Our argument will rely on a Calderón-Zygmund decomposition associated with non-regular martingales given in \([4]\), where we replace again the Euclidean geometric behaviors in \([4]\) by purely group-theoretical ones. It turns out that every element of this decomposition happens to be localized in a suitable way with the desired local estimates, and combing all these estimates yields the desired boundedness property of \(\mathbb{D}\).

2. Basic facts from noncommutative integration

The readers familiar with noncommutative integration may skip this section where we recall standard facts and definitions. For a detailed presentation, see \([20]\) and references therein.

2.1. Noncommutative \(L_p\)-spaces. A noncommutative measure space is a couple \((\mathcal{M}, \tau)\) where \(\mathcal{M}\) is a von Neumann algebra and \(\tau\) is a semifinite normal faithful trace on \(\mathcal{M}\). If \(\mathcal{M}\) is commutative then there exists a measure space \((\Omega, \mu)\) such that \((\mathcal{M}, \tau) = (L_\infty(\Omega), \int d\mu)\). To \((\mathcal{M}, \tau)\) can be associated, for \(p \in (0,\infty]\), \(L_p\)-spaces denoted by \(L_p(\mathcal{M}, \tau)\) or more frequently \(L_p(\mathcal{M})\) since we very rarely consider multiple traces on the same algebra \(\mathcal{M}\). We have \(L_\infty(\mathcal{M}) = \mathcal{M}\) and for \(p < \infty\) and \(x \in L_p(\mathcal{M})\),

\[
\|x\|_p = \tau(|x|^p)^{1/p}.
\]

There is no notion of “point” underlying the construction of noncommutative \(L_p\)-spaces but orthogonal projections are reasonable replacements for measurable sets (more precisely, characteristic functions of measurable sets). In that spirit, we write

\[
\{|x| > \lambda\} := 1_{(\lambda, \infty)}(|x|),
\]

where the left hand side is just a notation corresponding to the projection well-determined on the right hand side by functional calculus (in the commutative case, both make sense and coincide). It is well-known that \(\tau(\{|x| > \lambda\}) = \tau(\{|x^*| > \lambda\})\). This enables us to also defined the weak \(L_1\)-norm

\[
\|x\|_{1,\infty} := \sup_{t > 0} t \tau(\{|x| > t\}),
\]

to which corresponds the space \(L_{1,\infty}(\mathcal{M})\). If \(\lambda_1 + \lambda_2 = \lambda\), we have

\[
(2.1)\quad \tau(\{|x + y| > \lambda\}) \leq \tau(\{|x| > \lambda_1\}) + \tau(\{|y| > \lambda_2\}).
\]

The tensor product construction allows to rigorously define operator-valued functions. Given a measure space \((\Omega, \mu)\), one can consider the algebra \(\mathcal{N} := \mathcal{M} \overline{\otimes} L_\infty(\Omega)\) which is a von Neumann algebra naturally equipped with the tensor product trace \(\varphi := \tau \otimes f\). The algebra \(\mathcal{N}\) can be thought as bounded \(\mathcal{M}\)-valued functions and the associated \(L_p\)-space, \(p < \infty\), verify

\[
L_p(\mathcal{N}) \approx L_p(\Omega, L_p(\mathcal{M})).
\]

For \(f \in L_1(\mathcal{N})\) the trace is concretely computed as follows

\[
\varphi(f) = \int_\Omega \tau(f(\omega)) d\omega = \tau \left( \int_\Omega f(\omega) d\omega \right).
\]
We have the following Khinchin inequality [3]:

\[ \| \int_{\Omega} f^* g \|_r \leq \left( \int_{\Omega} |f^* f| \right)^{1/2} \| f \|_p \left( \int_{\Omega} |g^* g| \right)^{1/2} \| q \]

where \( 0 < p, q, r \leq \infty \) be such that \( 1/r = 1/p + 1/q \), and \( f, g \) are operator-valued functions such that the norms at the right hand side are well-defined and finite (see e.g. [17, Proposition 1.1] for more details).

### 2.2. Semi-commutative martingales

The notion of martingale admit a natural generalisation to noncommutative measure spaces which retains many interesting properties, making it a popular tool and subject of study among noncommutative analysts. This paper will not escape the trend. The general definition can be found in [13], only the semi-commutative case is of interest to us and will be presented here.

**Definition 2.1** (Atomic filtration). An atomic filtration of partitions \((P_n)_{n \geq 0}\) on \(\Omega\) is a sequence of partitions of \(\Omega\) such that

i. the atoms \(A \in P_n\) are measurable and have finite measure:

\[ \forall n \geq 0, \ \forall A \in P_n, \ |A| < \infty, \]

ii. the partitions are nested:

\[ \forall m \geq n \geq 0, \ \forall A \in P_n, \ \exists B \in P_m, \ A \subset B. \]

Denote by \(N_k\) the sub-algebra of \(N\) of functions which are constant on atoms of \(P_k\)

\[ N_k := \left\{ \sum_{A \in P_k} f_A 1_A : f_A \in M \right\}. \]

\(N_k\) is a noncommutative measure space equipped with the restriction of the trace \(\varphi\). The map

\[ E_k : f \mapsto \sum_{A \in P_k} \frac{1}{|A|} \left( \int_A f \right) 1_A \]

extends to a contraction from \(L_p(\Omega)\) to \(L_p(N_k)\) for all \(p \in [1, \infty]\), called a conditional expectation. It averages \(f\) on each atom \(A\) of the partition \(P_k\).

#### 2.3. Square functions and maximal inequalities

For a finite sequence \((f_k)\) in \(L_{1,\infty}(N)\), we define

\[ \| (f_k) \|_{L_{1,\infty}(N; \ell_2^2)} = \left\{ \left( \sum_k |f_k|^2 \right)^{1/2} \right\}_{1,\infty}, \quad \| (f_k) \|_{L_{1,\infty}(N; \ell_2^2)} = \left\{ \left( \sum_k |f_k|^2 \right)^{1/2} \right\}_{1,\infty}. \]

This yields quasi-norms on the space of finite sequences in \(L_{1,\infty}(N)\) and we denote their completions by \(L_{1,\infty}(N; \ell_2^2)\) and \(L_{1,\infty}(N; \ell_2^2)\) respectively. Define the sum space \(L_{1,\infty}(N; \ell_2^2) = L_{1,\infty}(N; \ell_2^2) + L_{1,\infty}(N; \ell_2^2)\), whose norm is a priori given by

\[ \| (f_k) \|_{L_{1,\infty}(N; \ell_2^2)} = \inf_{f_k = g_k + h_k} \{ \| (g_k) \|_{L_{1,\infty}(N; \ell_2^2)} + \| (h_k) \|_{L_{1,\infty}(N; \ell_2^2)} \}. \]

We have the following Khinchin inequality [3]:

\[ \| \sum_k \varepsilon_k f_k \|_{L_{1,\infty}(L_{\infty}(\Omega) \otimes N)} \approx \| (f_k) \|_{L_{1,\infty}(N; \ell_2^2)}, \]

where \(\varepsilon_n\) is a sequence of independent Rademacher variables on a probability space \(\Omega\). The following fact is standard to experts and may be intuitively thought of as the natural embedding of \(\ell_2\)-valued functions into \(\ell_\infty\)-valued ones.

**Lemma 2.2.** For a finite sequence \((f_k)\) in \(L_{1,\infty}(N)\), there exists a universal constant \(c > 0\) such that for all \(\lambda > 0\), we may find a projection \(e \in N\) with

\[ \sup_k \| e f_k e \|_\infty \leq \lambda \quad \text{and} \quad \lambda \varphi(1 - e) \leq c \| (f_k) \|_{L_{1,\infty}(N; \ell_2^2)}. \]
Proof. The proof is well-known to experts (see e.g. [11] Proof of Corollary 1.4]) and we only sketch the main idea. Take a decomposition $f_k = g_k + h_k$ so that
\[ \| (g_k) \|_{L_1, \infty (N; \ell^2)} + \| (h_k) \|_{L_1, \infty (N; \ell^2)} \leq \| (f_k) \|_{L_1, \infty (N; \ell^2)} + \varepsilon \]
as in the definition of $\| \cdot \|_{L_1, \infty (N; \ell^2)}$. Take $e = 1_{(0, \lambda)}(\sum_k |g_k|^2)^{1/2} \wedge 1_{(0, \lambda)}(\sum_k |h_k|^2)^{1/2})$. Then
\[ \lambda \varphi(1 - e) \leq \lambda \varphi(\{ \sum_k |g_k|^2 \}^{1/2} > \lambda) + \lambda \varphi(\{ \sum_k |h_k|^2 \}^{1/2} > \lambda) \leq \| (f_k) \|_{L_1, \infty (N; \ell^2)} + \varepsilon \]
and
\[ \| e f_k e \|_{\infty} \leq \| e g_k e \|_{\infty} + \| e h_k e \|_{\infty} \leq \| e |g_k|^2 e\|_{\infty}^{1/2} + \| e |h_k|^2 e\|_{\infty}^{1/2} \leq 2 \lambda. \]
\[ \square \]

3. Filtered Følner sequences

In this section, we discuss the geometric setting which will serve as a basis for our argument. Our general strategy is to use martingale theory through an atomic filtration on the group $G$ which should mimic some properties of the dyadic filtration on $\mathbb{R}^d$. It happens that the only properties we really require are invariance properties, which are precisely what is available in an amenable group. The setting we build is the one of regular filtered Følner sequences (see Definition 3.5). Roughly, we consider a sequence $(F_n)_{n \geq 0}$ and an atomic filtration $(P_k)_{k \geq 0}$ such that for $k \geq n$, $F_n$ have good invariance by the atoms of $P_k$ and for $n < k$, the atoms of $P_k$ have good $F_n$-invariant. The condition on $(F_n)$ ends up being stronger than the tempered condition employed by Lindenstrauss in [16]. The existence of such sequence for amenable groups will be proved using Ornstein and Weiss construction of quasi-tilings. It provides in general extremely lacunary sequences but in certain particular examples of groups which originally motivated this paper, the construction can be made explicit, with the $(F_n)$ growing exponentially.

3.1. Amenability and invariance. In this section, we recall the characterization of amenable groups through the existence of Følner sets and discuss two notions of invariance that both play an important part in the construction of filtered sequences and in the proof of local estimates in Section 5. A more detailed exposition of the same notions can be found in [18] Section I.

Let $G$ be a second countable locally compact group and $r$ a right-invariant Haar measure on $G$. For a measurable subset $E$ of $G$, we also denote $r(E)$ by $|E|$ for convenience. The most commonly used notion of invariance is the following. Let $K$ be a compact subset of $G$ and $\varepsilon > 0$. Assume that $|E| < \infty$.

**Definition 3.1.** We say that $E$ is $(\varepsilon, K)$-invariant if
\[ |E \cdot K \setminus E| \leq \varepsilon |E| \quad \text{and} \quad |E \cdot K^{-1} \setminus E| \leq \varepsilon |E|. \]

This notion of invariance allows to formulate the characterization of amenability that we will use in this paper. A locally compact second countable group is said to be amenable if it contains a sequence $(F_n)_{n \geq 0}$ such that for any $\varepsilon > 0$ and $K$ compact, $F_n$ is $(\varepsilon, K)$-invariant for sufficiently large $n$. Such a sequence will be called a Følner sequence.

A related but slightly different notion of invariance can be defined via the consideration of $K$-boundaries. In the following $E$ will still denote a subset of $G$ of finite measure and $K$ a compact subset of $G$. The $K$-boundary of $E$ denoted by $\partial_K(E)$ is defined as the union of the left translates of $K$: $g \cdot K$ with $g \in G$, such that $g \cdot K$ intersects both $E$ and $E^c := G \setminus E$. The $K$-interior and $K$-closure of $E$ are defined as
\[ \text{Int}_K(E) := E \setminus \partial_K(E) \quad \text{and} \quad \text{Cl}_K(E) := E \cup \partial_K(E). \]

**Definition 3.2.** We will say that $E$ is $(\varepsilon, K)$-boundary-invariant if
\[ |\partial_K(E)| \leq \varepsilon |E|. \]
Remark 3.3. Neither \((\varepsilon, K)\)-invariance implies \((\varepsilon, K)\)-boundary-invariant nor the opposite, while one can still deduce one from the other up to minor modifications of the subset \(K\). For convenience of exposition we keep both notions in this paper. Indeed, it can be checked that if \(E\) is \((\varepsilon, K \cup \{e\})\)-boundary-invariant then \(E\) is \((\varepsilon, K)\)-invariant; and if \(E\) is \((\varepsilon, (K^{-1}K)^2)\)-invariant then \(E \cdot K^{-1}K\) is \((\varepsilon, K)\)-boundary-invariant. The second verification is more subtle than the first but can be derived from the observations that \(\text{Cl}_K(E) = E \cdot K^{-1}K\) and \(E \subset \text{Int}_K(\text{Cl}_K(E))\). The important consequence of these claims is that a group \(G\) is amenable if and only if for any \(\varepsilon > 0\) and \(K\) compact, \(G\) contains an \((\varepsilon, K)\)-boundary-invariant subset.

Some definitions of the following subsections and constructions involve a number of different conditions of invariance of above types. To avoid the heavy non-essential notation, we will use the following terminology.

**Definition 3.4.** An *invariance condition* is any condition applicable to subsets of \(G\) that can be written as a finite conjunction of conditions of the type \((\varepsilon, K)\)-invariance or \((\varepsilon, K)\)-boundary-invariance with various \(\varepsilon > 0\) and compact subsets \(K\).

We will say that a set is *sufficiently invariant* if it satisfies an invariance condition for some suitable pairs \((\varepsilon, K)\) that we do not need to make explicit in the text. Similarly, we will say that a set is *sufficiently \(K\)-invariant* if it is \((\varepsilon, K)\)-invariant or \((\varepsilon, K)\)-boundary-invariant for a fixed compact subset \(K\) and for some sufficiently small \(\varepsilon\). Previous remarks imply that a group \(G\) is amenable if and only if for any invariance condition \(I\), there exists a subset of \(G\) satisfying \(I\).

Since we wish to include non unimodular groups in our construction, a quick warning is due about left and right invariance. We equipped \(G\) with a right-invariant measure \(r\) and consequently can only expect to have almost right-invariant sets. Indeed, by definition of the modular function \(\Delta : G \to \mathbb{R}_{>0}\), for any \(g \in G\) and \(E \subset G\), \(|gE| = \Delta(g) |E|\). Hence if \(\Delta(g) > 1\) and \(\varepsilon < \Delta(g) - 1\), there cannot exist any left-\((\varepsilon, \{g\})\)-invariant set.

**3.2. Filtrations and admissible atoms.** As mentioned previously, we will consider the ergodic averages associated to some Følner sequence and the martingale averages associated to a suitable atomic filtration of partitions. The essence of this section is to find conditions on a filtered sequence that are both sufficiently weak to be realized in any amenable group and sufficiently strong for the associated ergodic and martingale averages to be comparable. These conditions are described in the following definitions.

**Definition 3.5.** A *filtered sequence* is a triple \(S = ((F_n)_{n \geq 0}, (B_n)_{n \geq 0}, (P_n)_{n \geq 0})\) where \((F_n)_{n \geq 0}\) and \((B_n)_{n \geq 0}\) are sequences of subsets of \(G\) and \((P_n)_{n \geq 0}\) is an atomic filtration of partitions. If \((F_n)_{n \geq 0}\) is a Følner sequence, we say that \(S\) (or simply \((F_n)_{n \geq 0}\) if we do not need to specify the auxiliary information on \((B_n)_{n \geq 0}\) and \((P_n)_{n \geq 0}\)) is a filtered Følner sequence.

1. (Admissible atoms and functions) For \(k \geq 0\), we say that \(A \in \mathcal{P}_k\) is an *admissible atom* if
   a) there exists \(g \in G\) such that \(g \cdot A \subset B_k\),
   b) for any \(n < k\), \(A\) is \((2^{n-k}, F_n)\)-invariant.

   For any \(k \geq 0\), denote by \(\mathcal{P}_k^a\) the set of admissible atoms in \(\mathcal{P}_k\) and write
   \[
   \text{Adm}(S) := \bigcap_{k \geq 0} \bigcup_{A \in \mathcal{P}_k^a} A.
   \]

   We say that a function on \(G\) is *admissible* if it is supported in \(\text{Adm}(S)\).

2. (Regular filtered sequence) Let \(c \in (0, 1)\). We say that \(S\) is *\(c\)-regular* (or simply *regular*) if
   a) for any \(n \geq k\), \(F_n\) is \((2^{k-n}, B_n)\)-boundary-invariant,
   b) there exists a Følner sequence \((D_n)_{n \geq 0}\) such that for any \(n \geq 0\),
   \[
   |D_n \cap \text{Adm}(S)| \geq c|D_n|.
   \]

   If moreover \(\text{Adm}(S) = G\), we say that \(S\) is *completely regular*.

The model example for a (completely) regular filtered sequence comes from the euclidean case \(G = \mathbb{R}^d\) and the triple given by \(F_n = B(0, 2^n), B_n = [0, 1]^n\) and \((P_n)_{n \geq 0}\) the dyadic filtration. The various conditions imposed in the definitions above are meant to emulate, in a group in which we
potentially have little control over the geometry, the geometrical regularity of the dyadic filtration and dyadic balls. In the end, the reason why this strategy works in amenable groups is that we choose to retain very little of this regularity: only the relative invariance properties between balls and cubes of different generations. Thankfully, this is enough to carry out the arguments of Section 3.

The remainder of this section is devoted to proving the following theorem.

**Theorem 3.6.** Let $G$ be a second countable locally compact amenable group and $c \in (0, 1)$. Then any Følner sequence in $G$ admits a $c$-regular filtered Følner subsequence.

For countable groups, a construction of Downarowicz, Huczak and Zhang provides a stronger result (see [7, Theorem 5.2]).

**Proposition 3.7.** If $G$ is countable discrete then it admits a completely regular filtered Følner sequence.

3.3. **Quasi tilings.** In this section we recall the construction of Ornstein and Weiss [14] of so-called quasi tilings of $G$ (see Definition 3.8). We provide complete proofs both for the paper to be self-contained and to include the case of non unimodular groups. Compared to [14], the only adaptation required to account for non unimodularity is in the proof of Lemma 3.10 where left and right invariant measures come into play. Otherwise, we mainly follow [14].

We say that a finite number of finite subsets $\{E_i\}_{i \in I}$ of $G$ are $\varepsilon$-disjoint if they can be ordered $\{E_i\}_{i \in I} = \{E_1, \ldots, E_k\}$ so that for any $n \in \{1, \ldots, k\},$

$$\tag{3.1} \left| E_n \setminus \left( \bigcup_{m<n} E_m \right) \right| \geq (1 - \varepsilon) |E_n|.$$  

**Definition 3.8** ($\varepsilon$-quasi tiling property). A family of subsets of $G$, $\{S_1, \ldots, S_n\}$ $\varepsilon$-quasi tiles $G$ if for any compact $D \subset G$ which is sufficiently invariant, one can find sets $C_i$, $i \leq n$ (which will be called sets of centers) such that

1. for any $i$, the sets $g \cdot S_i, g \in C_i$ are $\varepsilon$-disjoint,
2. for $i \neq j$, $C_i \cdot S_i$ and $C_j \cdot S_j$ are disjoint,
3. for any $i$, $C_i \cdot S_i \subset D$,
4. $\left| \bigcup_{i \leq n} C_i \cdot S_i \cap D \right| \geq (1 - \varepsilon) |D|$.

**Proposition 3.9.** Let $\varepsilon > 0$ and $S_1, \ldots, S_N$ be compact subsets of $G$. Assume that

- $(1 - \varepsilon)^N \leq \varepsilon$,
- for any $j < i$, $S_i$ is $(\varepsilon^3, S_j)$-invariant.

Then the family $\{S_1, \ldots, S_N\}$ $4\varepsilon$-quasi tiles $G$.

The proof relies on the following key lemma. Its proof will make use of the left-invariant measure $l$ on $G$ defined by $l(E) = r(E^{-1})$. It is easily checked that for any $E \subset G$,

$$\tag{3.2} |E| = \int_E \Delta(x) dl(x).$$

**Lemma 3.10.** Let $D$ be an $(\varepsilon, K)$-boundary-invariant set and $A \subset D$. Then there exists $x \in G$ such that $xK \subset D$ and

$$|xK \cap A| \leq \frac{|A|}{(1 - \varepsilon)|D|} |xK|.$$  

**Proof.** Denote $I := \operatorname{Int}_K(D)K^{-1}$. By $(\varepsilon, K)$-boundary-invariant of $D$, $|I| \geq |\operatorname{Int}_K(D)| \geq (1 - \varepsilon) |D|$. One can also check that for any $x \in I$, $xK \subset D$.

In this proof, we consider the left-invariant measure $l$ on $G$. By [32], on one hand we have

$$\int_G |xK \cap A| dl(x) = \int_{x \in G} \int_{y \in A} \Delta(y) 1_A(y) 1_{xK}(y) dl(y) dl(x).$$
Hence, by Fubini’s theorem
\[
\int_I |xK \cap A| \, dl(x) \leq \int_{y \in A} \Delta(y) \left( \int_{x \in G} 1_{yK^{-1}}(x) \, dl(x) \right) \, dl(y) = |A| |K|.
\]
On the other hand, we have
\[
\int_I |xK| \, dl(x) = \int_{x \in I} \Delta(x) |K| \, dl(x) = |I| |K| \geq (1 - \varepsilon) |D| |K|.
\]
Combining the two, we obtain the existence of an \( x_0 \in I \) such that
\[
\frac{|xK \cap A|}{|xK|} \geq \frac{|A| |K|}{(1 - \varepsilon) |D| |K|} = \frac{|A|}{(1 - \varepsilon) |D|}.
\]
\[\blackdiamond\]

**Lemma 3.11.** Let \( \varepsilon < 1/2 \). Assume once again that \( D \) is \((\varepsilon, K)\)-boundary-invariant and compact. Then there exists a finite set of centers \( C = \{ x_i \}_{0 \leq i \leq n} \subset \Int_K(D) \) such that the \( x_i \cdot K \) are \( 2\varepsilon \)-disjoint and \(|C \cdot K| \geq \varepsilon |D|\).

**Proof.** We construct the \( x_i \)'s inductively using Lemma 3.10. Assume that we have constructed \( x_0, \ldots, x_{i-1} \) in \( \Int_K(D) \) so that the \( x_i \cdot K \) are \( 2\varepsilon \)-disjoint. Set \( C_i := \{ x_0, \ldots, x_{i-1} \} \). If \(|C_i \cdot K| > \varepsilon |D|\) we can stop the induction and set \( n = i - 1 \). Otherwise, Lemma 3.10 provides an element \( x_i \Int_K(D) \) such that
\[
|x_i K \cap C_i \cdot K| \leq \frac{|C_i \cdot K|}{(1 - \varepsilon) |D|} |x_i K| \leq 2\varepsilon |x_i K|.
\]
Hence, the \( 2\varepsilon \)-disjointness condition remains verified. Furthermore,
\[
|C_i \cdot K \cup x_i K| \geq |C_i \cdot K| + (1 - 2\varepsilon) |x_i K|.
\]
Therefore, together with the \( 2\varepsilon \)-disjointness of \( \{ x_n K \}_{n \leq i - 1} \) in the induction hypothesis, we get that \(|C_{i+1} \cdot K| \geq (1 - 2\varepsilon) \sum_{n=0}^{i} |x_n K| \geq i(1 - 2\varepsilon)c|K|\), where \( c := \min_{x \in D} \Delta(x) > 0 \) since \( D \) is compact. This guarantees that in a finite number of induction steps, we may obtain a set \( C = C_m \) with \(|C \cdot K| \geq (m - 1)(1 - 2\varepsilon)c|D| \geq \varepsilon |D|\). \[\blackdiamond\]

**Lemma 3.12.** Let \( \varepsilon, \delta > 0 \). Let \( A_0, \ldots, A_n \) be \( \varepsilon \)-disjoint, \((\delta, K)\)-boundary-invariant compact subsets of \( G \). Then \( \bigcup_{i \leq n} A_i \) is \((\delta(1 - \varepsilon)^{-1}, K)\)-boundary-invariant.

**Proof.** Let us argue by induction on \( n \). Let \( B_{n-1} := \bigcup_{i \leq n} A_i \) be \((\delta(1 - \varepsilon)^{-1}, K)\)-boundary-invariant and set \( B_{n} = \bigcup_{i \leq n} A_i \). By the induction hypothesis,
\[
|\partial_K(B_{n-1})| \leq \frac{\delta}{1 - \varepsilon} |B_{n-1}|.
\]
Note that by the \( \varepsilon \)-disjointness, we have \(|B_n| \geq |B_{n-1}| + (1 - \varepsilon) |A_n|\). It follows that
\[
|\partial_K(B_n)| \leq |\partial_K(B_{n-1})| + |\partial_K(A_n)| \leq \frac{\delta}{1 - \varepsilon} |B_{n-1}| + \delta |A_n| \leq \frac{\delta}{1 - \varepsilon} |B_n|.
\]
\[\blackdiamond\]

**Lemma 3.13.** Let \( \varepsilon > 0 \). Let \( A \subset B \) and \( K \) be compact sets. Assume that \(|A| \leq (1 - \varepsilon) |B|\), \( A \) is \((\varepsilon^2, K)\)-boundary-invariant and \( B \) is \((\varepsilon^2, K)\)-boundary-invariant then \( B \setminus A \) is \((2\varepsilon, K)\)-boundary-invariant.

**Proof.** Observe that
\[
\partial_K(B \setminus A) \subset \partial_K(B) \cup \partial_K(A).
\]
Consequently,
\[
|\partial_K(B \setminus A)| \leq \varepsilon^2 |B| + \varepsilon^2 |A| \leq 2\varepsilon^2 |B| \leq 2\varepsilon |B \setminus A|.
\]
\[\blackdiamond\]
Proof of Proposition. Choose $S_1, \ldots, S_N$ as in Proposition. Let $D \subset G$ be a compact subset which is $(\varepsilon^2, S)$-invariant for any $i \in \{1, \ldots, N\}$. Recall that we want to construct sets of centers $C_i$ as in Definition. Using Lemma we can choose $C_N \subset \text{Int}_{S_N}(D)$ such that $(xS_N)_{x \in C_N}$ are $2\varepsilon^2$-disjoint and

$$|D \setminus C_N \cdot S_N| \leq (1 - \varepsilon)|D|.$$ 

If $|C_N \cdot S_N| \geq (1 - \varepsilon)|D|$, then we obtain an $\varepsilon$-quasi tiling of $D$ as desired and the proof is complete. Otherwise, we have $|D \setminus C_N \cdot S_N| \geq \varepsilon|D|$ and we pursue further constructions. Recall that $(xS_N)_{x \in C_N}$ are $2\varepsilon^2$-disjoint and $(\varepsilon^3, S_{N-1})$-boundary-invariant, so by Lemma $C_N \cdot S_N$ is $(\varepsilon^3, S_{N-1})$-boundary-invariant. Hence, by Lemma $D \setminus C_N \cdot S_N$ is $(2\varepsilon, S_{N-1})$-boundary-invariant.

Now assume by induction that we have chosen $C_N, \ldots, C_{N-j}$ such that $(xS_N)_{x \in C_{N-i}}$ are $4\varepsilon$-disjoint, $C_kS_k$ and $C_lS_l$ are disjoint for $k \neq l$, and $D \setminus (\bigcup_{i \leq j} C_{N-i})_{i \leq N-j}$ is $(2\varepsilon, S_{N-j-1})$-boundary-invariant with

$$|D \setminus (\bigcup_{i \leq j} C_{N-i}D_{N-i})| \leq (1 - \varepsilon)^{N+1-j}|D|.$$ 

Let us show that we may add one more subset $C_{N-j-1}$ such that $C_N, \ldots, C_{N-j-1}$ satisfy similar properties. If $|\bigcup_{i \leq j} C_{N-i}D_{N-i}| \geq (1 - \varepsilon)|D|$, then we obtain an $\varepsilon$-quasi tiling of $D$ as desired and the proof is complete by taking $C_{N-j} = \emptyset$ for $j > i$. Otherwise, we have $|D \setminus (\bigcup_{i \leq j} C_{N-i}D_{N-i})| \geq \varepsilon|D|$ and we may repeat the previous argument to construct $C_{N-j-1}$.

More precisely, by Lemma and the $(2\varepsilon, S_{N-j-2})$-boundary-invariance of $D \setminus (\bigcup_{i \leq j} C_{N-i}D_{N-i})$, we may choose $C_{N-j-1} \subset \text{Int}_{S_{N-j-1}}(D \setminus (\bigcup_{i \leq j} C_{N-i}D_{N-i}))$ such that $(xS_{N-j-1})_{x \in C_{N-j-1}}$ are $4\varepsilon$-disjoint and $|C_{N-j-1}D_{N-j-1}| \geq 2\varepsilon|D \setminus (\bigcup_{i \leq j} C_{N-i}D_{N-i})|$, which in particular implies

$$|D \setminus \bigcup_{i \leq j+1} C_i \cdot S_i| \leq (1 - \varepsilon)^{N-j}|D|.$$ 

As before, Lemma yields that $\bigcup_{i \leq j+1} C_i \cdot S_i$ is $(\varepsilon^2, S_{N-j-2})$-boundary-invariant, and Lemma implies that $D \setminus (\bigcup_{i \leq j+1} C_i \cdot S_i)$ is $(2\varepsilon, S_{N-j-2})$-boundary-invariant.

This inductive procedure therefore constructs the subsets $\{C_0, \ldots, C_N\}$ satisfying (1)-(3) with parameter $4\varepsilon$ in Definition and moreover at each step

$$|D \setminus \bigcup_{j=i}^N C_i \cdot S_i| \leq \min\{\varepsilon, (1 - \varepsilon)^{N+1-i}\}|D|.$$ 

In particular, if $(1 - \varepsilon)^N \leq 4\varepsilon$, we obtain an $4\varepsilon$-quasi tiling of $D$. \hfill \Box

3.4. Quasi-partitions. In the following subsections, we will make the connection between the quasi-tilings of Ornstein and Weiss and regular filtered partitions. This subsection is devoted in particular to removing the intersections between the tiles, the resulting objects will be called quasi-partitions (see Definition).
The proof then boils down to proving that
\[ \partial_K(I) \subset \partial_{K^{-1}}(E). \]
Let \( x \in \partial_K(I) \). By definition, there exists \( g \) such that \( x \in gK \) and \( gK \) intersects both \( I \) and \( I^c \). Pick \( y \in gK \setminus I \). We necessarily have \( y \in E \) but \( y \notin I \) so \( y \in \partial_K(E) \). This means that we can find \( h \) such that \( hK \) contains \( y \) and intersects \( E^c \). Finally, choose \( z \in hK \setminus E \). Since \( y \in hK \) and \( y \in gK \), \( h \) and \( g \) belong to \( yK^{-1} \). Hence \( z \) and \( x \) belong to \( yK^{-1}K \) which implies that \( x \in \partial_{K^{-1}}(E) \).

The following lemma will guarantee that the atoms of \( \varepsilon \)-partitions that we construct from quasi-tilings retain some invariance properties.

**Lemma 3.16.** Let \( \varepsilon, \delta \) be positive real numbers with \( \varepsilon < \delta < 1/4 \). Let \( K, L \) be compact subsets of \( G \) and \( \{ E_i \}_{i \leq k} \) be a family of \( \varepsilon \)-disjoint, \((\varepsilon, K^{-1}K)\)-boundary-invariant, \((\delta, L)\)-invariant subsets of \( G \). Then the sets
\[ I_i = \text{Int}_K(E_i) \setminus \left( \bigcup_{i<j} \text{Int}_K(E_j) \right) \]
are \((4\varepsilon, K)\)-boundary-invariant, \((6\delta, L)\)-invariant, disjoint and \( |I_i| \geq (1 - 2\varepsilon)|E_i| \) for all \( i \leq k \).

**Proof.** The disjointedness is guaranteed by the definition of \( I_i \). The \( \varepsilon \)-disjointedness of \( E_i \) means that
\[ |E_i \cap \left( \bigcup_{m<i} E_m \right)| < \varepsilon|E_i|. \]
Then together with Lemmas 3.14, we have
\[ |\partial_K(I_i)| \leq |\partial_K(\text{Int}_K(E_i))| + |E_i \cap \left( \bigcup_{m<i} E_m \right)| \]
\[ \leq \varepsilon|\text{Int}_K(E_i)| + \varepsilon|E_i|. \]
On the other hand, by Lemma 3.15 and the invariance condition of \( E_i \),
\[ |I_i| \geq |\text{Int}_K(E_i)| - |E_i \cap \left( \bigcup_{m<i} E_m \right)| \geq (1 - 2\varepsilon)|E_i|. \]
Hence,
\[ |\partial_K(I_i)| \leq \frac{2\varepsilon}{1 - 2\varepsilon}|I_i|, \]
which yields the desired \((4\varepsilon, K)\)-boundary-invariant condition. Moreover, together with the \((\delta, L)\)-invariance and the above inequality \( |I_i| \geq (1 - 2\varepsilon)|E_i| \), we obtain
\[ |I_i \cdot L \setminus I_i| \leq |E_i \cdot L \setminus E_i| + |E_i \setminus I_i| \leq \delta|E_i| + 2\varepsilon|E_i| \leq \frac{3\varepsilon}{1 - 2\varepsilon}|I_i| \leq 6\delta|I_i|. \]

**Definition 3.17** \(((\varepsilon, B, \mathcal{I})\text{-partition})\). Let \( \varepsilon > 0 \), \( B \subset G \) and \( \mathcal{I} \) be an invariance condition. Let \( D \) be a subset of \( G \) with finite measure. We say that a set \( \mathcal{P} \) of subsets of \( D \) is an \((\varepsilon, B, \mathcal{I})\text{-partition} \) of \( D \) if
\begin{enumerate}
\item for any \( A \neq C \in \mathcal{P} \), \( A \cap C = \emptyset \),
\item \( |\cup_{A \in \mathcal{P}} A| \geq (1 - \varepsilon)|D| \),
\item for any \( A \in \mathcal{P} \), there exists \( \gamma \in G \) such that \( A \subset \gamma B \),
\item any \( A \in \mathcal{P} \) satisfies \( \mathcal{I} \).
\end{enumerate}
An \((\varepsilon, B, \mathcal{I})\text{-partition} \) will be also called a \textit{quasi-partition} in the sequel if we do not need to specify the data \((\varepsilon, B, \mathcal{I})\).

**Proposition 3.18.** Let \( \mathcal{I} \) be an invariance condition, and \( \varepsilon > 0 \). There exists a compact set \( B \) such that for any sufficiently \( B \)-invariant set \( D \), \( D \) admits an \((\varepsilon, B, \mathcal{I})\text{-partition} \). Moreover for any compact subset \( K \), the subset \( B \) can be chosen large enough so that \( K \subset B \).
Proof. Let $\delta > 0$ be sufficiently small and $\mathcal{T}'$ be an invariance condition. Note that given a fixed compact subset $K$, we may always choose large Følner subsets in $G$ which contains $K$. In particular, we may choose $S_1, \ldots, S_n$ which contain $K$ and satisfy the assumption of Proposition 3.21 as well as the invariance condition $\mathcal{T}'$. Then $S_1, \ldots, S_n$ $\delta$-quasi tile $G$. Set

$$B := \bigcup_{i=1}^{n} S_n.$$ 

Let $D$ be sufficiently $B$-invariant, we can choose sets of centers $C_1, \ldots, C_n$ as in Definition 3.28 to $\delta$-quasi tile $D$.

Applying Lemma 3.18 to the subsets $\{gS_i\}_{g \in C_i}$, which are $\delta$-disjoint as required in Definition 3.28, we see that if $\mathcal{T}'$ was chosen sufficiently strong compared to $\mathcal{T}$, one can find a $(2\delta, S_i, \mathcal{T})$-partition of $C_i \cdot S_i$ which we denote by $\mathcal{P}_i$. Setting $\mathcal{P} = \bigcup_{i \leq n} \mathcal{P}_i$, one obtains a $(\varepsilon, B, \mathcal{T})$-partition of $D$. \hfill $\square$

For simplicity, we say in the sequel that $B$ $(\varepsilon, \mathcal{T})$-partitions $G$ if any sufficiently $B$-invariant subset admits an $(\varepsilon, B, \mathcal{T})$-partition as discussed in the previous proposition.

3.5. Construction of almost regular filtered sequences. In this section, we prove that any locally compact second countable amenable group contains an almost regular filtered Følner sequence (Theorem 3.6). Compared to Proposition 3.18, we have to mainly address two issues: covering “all of $G$” and obtain a sequence of nested quasi-partitions, not a single quasi-partition.

Proof of Theorem 3.7. We remark that in order to construct a regular filtered Følner sequence $((F_n)_{n \geq 0}, (B_n)_{n \geq 0}, (\mathcal{P}_n)_{n \geq 0})$, it suffices to construct the corresponding triple $((F_n)_{n \geq 0}, (B_n)_{n \geq 0}, (\mathcal{P}^a_n)_{n \geq 0})$ where the atoms in $(\mathcal{P}^a_n)_{n \geq 0}$ are admissible and cover a large proportion of volume of $G$. Indeed, the sets of admissible atoms can be completed with arbitrary nested atoms in order to form a usual atomic filtration of partitions.

Let us fix a $\varepsilon \in (0, 1)$ and a sequence $(\varepsilon_n)_{n \geq 0}$ such that $\Pi_{n > 0}(1 - \varepsilon_i) \geq 1 - \varepsilon$. First, let us recall exactly what we need to construct:

- a Følner sequence $(F_n)_{n \geq 0}$ with an auxiliary sequence $(B_n)_{n \geq 0}$, such that for any $n \geq 0$, $F_n$ is $(2^{k-n}, B_n)$-boundary-invariant for any $k \leq n$. Let us denote this invariance condition by $J_n$.
- a sequence of nested quasi-partitions $(\mathcal{P}_n)_{n \geq 0}$. The atoms of $\mathcal{P}_n$ must be $(2^{k-n}, F_k)$-invariant for any $k < n$. The latter invariance condition will be denoted by $\mathcal{I}_n$.
- A Følner sequence $(D_n)_{n \geq 0}$ such that for any $n \geq 0$,

$$\left| \bigcup_{A \in \mathcal{P}_n} A \cap D_n \right| \geq (1 - \varepsilon) |D_n|.$$

For the purpose of the construction, we impose additional conditions on $(D_n)_{n \geq 0}$:

(C1) for any $n \geq 0$, there exists $\gamma \in G$ such that $D_n \subset \gamma B_n$ and $D_n$ satisfies $\mathcal{I}_n$,
(C2) for $n \geq 1$, $D_n \setminus D_{n-1}$ is sufficiently $B_{n-1}$-boundary-invariant and $D_{n-1} \subset D_n$.

We choose the subsets $(D_n)_{n \geq 0}$, $(F_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ inductively as follows. At the $n$-th step of the induction, we start by picking $D_n$ sufficiently large and invariant so that it satisfies (C2) and $\mathcal{I}_n$. This is possible since

$$|\partial_{B_n}(D_n \setminus D_{n-1})| \leq |\partial_{B_{n-1}}(D_n)| + |\text{Cl}_{B_{n-1}}(D_{n-1})|.$$ 

So by taking $D_n$ sufficiently $B_{n-1}$-boundary invariant and $|\text{Cl}_{B_{n-1}}(D_{n-1})| << |D_n|$, we guarantee that $D_n \setminus D_{n-1}$ is sufficiently $B_{n-1}$-boundary-invariant.

Then, define a condition $\mathcal{T}'_n$ as the conjunction of $\mathcal{I}_n$ and sufficient $B_{n-1}$-invariance. By Proposition 3.18 we may choose a large enough subset $B_n$ which contains $D_n$ and $(\varepsilon_n, \mathcal{T}'_n)$-partitions $G$. Finally, by the Følner condition, we choose a subset $F_n$ satisfying $J_n$ from the given Følner sequence.

It remains to construct the quasi-partitions $(\mathcal{P}_n)_{n \geq 0}$. Let $k \geq 1$ and let us start by covering the set $D_k \setminus D_{k-1}$. Since $B_{k-1}$ $(\varepsilon_{k-1}, \mathcal{T}'_{k-1})$-partitions $G$ and $D_k \setminus D_{k-1}$ is sufficiently $B_{k-1}$-invariant, we can find an $(\varepsilon_{k-1}, \mathcal{T}'_{k-1}, B_{k-1})$-partition of $D_k \setminus D_{k-1}$ that we denote by $\mathcal{P}^{(k)}_{k-1}$. We then
construct partitions \( P^{(k)}_i \) for \( i < k - 1 \) by descending induction as follows. For any \( i \leq k - 1 \), set 
\[
\delta_i = 1 - \Pi_{j=i}^{k-1}(1 - \varepsilon_j).
\]
Assume that \( P^{(k)}_i \) has been constructed and is a \((\delta_i, \mathcal{I}_i, B_i)\)-partition of \( D_k \setminus D_{k-1} \). Let \( A \in \mathcal{P}^{(k)}_i \). Since \( A \) is sufficiently \( B_{i-1} \)-invariant by definition of \( \mathcal{I}_i \), there exists a
\((\varepsilon_{i-1}, \mathcal{I}_{i-1}, B_{i-1})\)-partition of \( A \) that we will denote by \( \mathcal{P}_A \). Set
\[
\mathcal{P}^{(k)}_{i-1} = \bigcup_{A \in \mathcal{P}^{(k)}_i} \mathcal{P}_A.
\]
By construction, \( \mathcal{P}^{(k)}_{i-1} \) covers a proportion at least \((1 - \varepsilon_{i-1})\) of what is covered by \( \mathcal{P}^{(k)}_i \) and hence is a \((\delta_{i-1}, \mathcal{I}_{i-1}, B_{i-1})\)-partition of \( D_k \setminus D_{k-1} \). Finally, note that by our choice of \( \varepsilon_i \)'s, \( \delta_i \leq \varepsilon \) for all \( i < k \). So to summarize, we have constructed a sequence \((\mathcal{P}^{(k)}_i)_{0 \leq i < k}\) of nested \((\varepsilon, \mathcal{I}, B_i)\)-partitions of \( D_k \setminus D_{k-1} \).

We can now conclude: set for any \( n \geq 0 \)
\[
\mathcal{P}_n = \bigcup_{k \geq n} \mathcal{P}^{(k)}_n \cup \{D_n\}.
\]
By (C1) and the construction of the \( \mathcal{P}^{(k)}_n \), the atoms \( A \in \mathcal{P}_n \) satisfy their required conditions \( \mathcal{I}_n \) and the containment \( \gamma A \subset B_n \) for some \( \gamma \in G \); in other words, they are admissible. It is also clear that \((\mathcal{P}_n)_{n \geq 0}\) is a nested sequence. And since \( \mathcal{P}^{(k)}_0 \) is an \((\varepsilon, B_0, \mathcal{I}_0)\)-partition of \( D_k \setminus D_{k-1} \) for any \( k \geq 1 \), by definition it covers a proportion at least \( 1 - \varepsilon \) of the volume of \( D_k \setminus D_{k-1} \). Thus \( \mathcal{P}_0 \) covers a proportion at least \( 1 - \varepsilon \) of \( D_k \) for any \( k \geq 0 \).

4. Calderón-Zygmund decomposition

In this subsection, we borrow ideas from \[4\] in which the noncommutative Calderón-Zygmund decomposition originally introduced by Parcet in \[19\] is refined in order to deal with non doubling measures and simplify its use. For the sake of self-containment, we provide a detailed description of the construction and complete proofs, though most of them can be found in \[4\], \[2\] or \[19\] and need surprisingly few adaptions to be transferred from the euclidean setting to amenable groups.

Let \( f \in \mathcal{L}_1(\mathcal{N}) \cap \mathcal{N}' \), positive and compactly supported. Consider an atomic filtration of partitions \((\mathcal{P}_k)_{k \geq 0}\), and we keep the notation of associated semi-commutative martingale structure introduced in Subsection \[22\]. Set \( f_k = \mathbb{E}_k f \). Let us recall the following lemma due to Cuculescu which so far has always been the starting point to define noncommutative Calderón-Zygmund decompositions.

**Lemma 4.1** (Cuculescu’s projections). Let \( \lambda > 0 \). There exists an increasing sequence of projections \((q_k)_{k \geq 0}\), determined inductively by the relation \( q_k = 1_{[0,\lambda]}(q_{k+1} f_k q_{k+1}) \), such that

1. \( \text{for all } k \geq 0, q_k f_k q_k \leq \lambda, \)
2. \( q_k \) commutes with \( q_{k+1} f_k q_{k+1} \),
3. \( q_k \in \mathcal{N}_k \),
4. \( \lambda \varphi(1 - q_0) \leq \|f\|_1. \)

Define, for all \( n \geq 0 \),
\[
p_n := q_{n+1} - q_n.
\]
Note that \( \sum_{k \geq 0} p_k = 1 - q_0 \). We may view \( p_k \in \mathcal{N}_k \) as an \( \mathcal{M} \)-valued function and write
\[
p_k = \sum_{A \in \mathcal{P}_k} p_A 1_A.
\]
We use the similar notation \( q_A \) for the projections \( q_k \).

The function \( f \) is decomposed into three parts
\[
f = g + h + b,
\]
a good part, a hybrid part (which appears when dealing with non-doubling filtrations) and a bad part. The bad part splits into diagonal terms and off-diagonal terms. The diagonal terms behaviour is similar the one of the bad part of the classical CZ decomposition. The off-diagonal
terms, however, are purely non-commutative: they would vanish in a commutative algebra. Parcet was the first to provide an estimate for those terms in [19] by introducing a pseudo-localisation principle for singular integrals. In this paper, we follow another strategy presented in [4] where off-diagonal terms are grouped differently and estimated through a more direct proof.

We collect, in the following lemmas, the construction and essential properties of each part.

**Lemma 4.2** (The good part). The good part is defined as

\[ g = g_0f_0g_0 + \sum_{k \geq 0} g_k \quad \text{with} \quad g_k = E_{k+1}(p_kf_kp_k) \]

and satisfies \( \|g\|_2^2 \leq 2\lambda \|f\|_1 \).

**Proof.** Recall that by Cuculescu’s construction (Lemma 4.1) \( q_k \) and \( q_kf_kq_{k+1} \) commute. Hence, \( p_kf_kp_k = q_{k+1}f_{k+1}q_k - q_kf_kq_k \) and

\[ g_k = q_{k+1}f_{k+1}q_k - E_{k+1}(q_kf_kq_k) \]

We will estimate the 2-norm of \( g \) by computing its martingale differences.

\[ \Delta_k(g) = \Delta_k(q_0f_0g_0) + \sum_{i=0}^{k-1} \Delta_k(g_i) \]

\[ = \Delta_k(q_0f_0g_0) + \sum_{i=0}^{k-1} \Delta_k(q_{i+1}f_{i+1}q_{i+1}) - \Delta_k(q_i,f_iq_i) \]

\[ = \Delta_k(q_0f_0g_0) + \Delta_k(q_kf_kq_k) - \Delta_k(q_0f_0g_0) = \Delta_k(q_kf_kq_k). \]

We have obtained the following:

\[ \|\Delta_k(g)\|^2 = \|\Delta_k(q_kf_kq_k)\|^2 = \|q_kf_kq_k\|^2 - \|E_{k+1}(q_kf_kq_k)\|^2. \]

The terms can be rearranged as follows

\[ \|g\|^2 = \sum_{k \geq 0} \|q_kf_kq_k\|^2 - \|E_{k+1}(q_kf_kq_k)\|^2 \]

\[ = \|q_0f_0g_0\|^2 + \sum_{k \geq 1} \|q_kf_kq_k\|^2 - \|E_k(q_{k-1}f_{k-1}q_{k-1})\|^2. \]

And now let us note that

\[ \|q_kf_kq_k\|^2 - \|E_k(q_{k-1}f_{k-1}q_{k-1})\|^2 = \varphi((q_kf_kq_k - E_k(q_{k-1}f_{k-1}q_{k-1}))(q_kf_kq_k + E_k(q_{k-1}f_{k-1}q_{k-1}))) \]

\[ \leq \|q_kf_kq_k - E_k(q_{k-1}f_{k-1}q_{k-1})\|_1 \|q_kf_kq_k + E_k(q_{k-1}f_{k-1}q_{k-1})\|_\infty. \]

By (4.1),

\[ \|q_kf_kq_k - E_k(q_{k-1}f_{k-1}q_{k-1})\|_1 = \|E_k(p_{k-1}f_{k-1}p_{k-1})\|_1 = \varphi(p_{k-1}f). \]

And by the construction of Cuculescu (Lemma 4.1),

\[ \|q_kf_kq_k + E_k(q_{k-1}f_{k-1}q_{k-1})\|_\infty \leq 2\lambda. \]

Note also that \( \|q_0f_0g_0\|^2 \leq \varphi(q_0f)\lambda. \) Putting the pieces back together we get

\[ \|g\|^2 \leq \lambda \varphi(q_0f) + \sum_{k \geq 1} 2\lambda \varphi(p_{k-1}f) \leq 2\lambda \|f\|_1. \]

\[ \square \]

**Lemma 4.3** (The hybrid part). The hybrid part is defined as

\[ h = \sum_{k \geq 0} h_k \quad \text{with} \quad h_k = p_kf_kp_k - g_k \]

and satisfies

1. \( E_{k+1}(h_k) = 0, \)
2. \( h_k \in \mathcal{N}_k, \)
3. \( \sum_{k \geq 0} \|h_k\|_1 \leq 2\|f\|_1. \)
Proof. The first two properties are straightforward consequences of the construction. The third is checked as follows. First note that for any \( k \geq 0 \),
\[
\|h_k\|_1 \leq \|pk, f_kp_k\|_1 + \|E_{k+1}(p_k, f_kp_k)\|_1 = 2\varphi(p_kf).
\]
Then, summing up over \( k \) we get
\[
\sum_{k \geq 0} \|h_k\|_1 \leq 2 \sum_{k \geq 0} \varphi(p_kf) \leq 2\varphi((1 - q_k)f) \leq 2\|f\|_1.
\]
\[\square\]

**Lemma 4.4 (The bad part).** The bad part is defined as follows
\[
b = \sum_{k \geq 0} b_d^k + b_{\text{off}}^k \quad \text{with} \quad b_d^k = p_k(f - f_k)p_k \quad \text{and} \quad b_{\text{off}}^k = p_kfq_k + q_kfp_k.
\]
It satisfies
\[
(1) \quad \|b_d^k\|_1 \leq 2\varphi(p_kf),
\]
\[
(2) \quad \text{let } E \text{ be a union of } \mathcal{P}_k\text{-atoms and } K \subset E,
\]
\[
\|\int_K b_{\text{off}}^k\|_1 \leq \lambda 2\varphi(1_E p_k f).
\]
\[
(3) \quad E_k(b_d^k) = E_k(b_{\text{off}}^k) = 0.
\]

**Proof.** We leave the first and third properties to the reader and concentrate on the second. By the triangle inequality, it suffices to consider the case where \( E = A \) is an atom of \( \mathcal{P}_k \). By definition, \( b_{\text{off}}^k = p_kfq_k + q_kfp_k \). For convenience, we only write the argument of \( p_k, fq_k \), the other summand admits the same estimate by adjunction. By the Hölder type inequality stated in (2.2), we have
\[
\|\int_K p_k, fq_k\|_1 \leq \left( \left( \int_K p_k, fp_k \right)^{1/2} \right) \|p_k, fq_k\|^{1/2} \|\int_K \|_\infty.
\]
We estimate each factor separately. Note that \( 1_K p_k = 1_K p_A \) by our assumption \( E = A \). Using once again Hölder’s inequality, we get
\[
\left( \left( \int_K p_k, fp_k \right)^{1/2} \right) \|_1 \leq \left( \left( \int_A p_A, fp_A \right)^{1/2} \right) \leq \tau(p_A)^{1/2} \varphi(1_A p_k f)^{1/2}.
\]
And since \( q_A, f_A, q_A \leq \lambda \) we obtain
\[
\|\int_K q_k, fq_k\|^1/2 \|_\infty \leq \|\int_A q_k, fq_k\|^1/2 \leq (|A|\lambda)^{1/2}.
\]
Since \( p_k, fq_k \geq \lambda p_k \), we have \( |A|\tau(p_A)\lambda = \varphi(1_A p_k \lambda) \leq \varphi(1_A p_k f) \). Hence, combining the estimates for both factors,
\[
\|\int_K p_k, fq_k\|_1 \leq \tau(p_A)^{1/2} \varphi(1_A p_k f)^{1/2} (|A|\lambda)^{1/2} \leq \varphi(1_A p_k f).
\]
\[\square\]

**Remark 4.5.** If \( f \) is admissible then for any \( k \geq 0 \), \( f_k \), \( p_k \) and \( q_k \) are supported on admissible atoms of \( \mathcal{P}_k \). This can be directly inferred from Cuculescu’s constructions and definitions of these functions.

**Lemma 4.6 (Dilated support).** Let \( S = (\mathcal{F}_n)_{n \geq 0}, (A_n)_{n \geq 0}, (\mathcal{P}_n)_{n \geq 0} \) be a filtered Følner sequence. Set
\[
\zeta = (\vee_{k \geq 1} \vee_{A \in \mathcal{P}_k} p_A 1_{A:(\cup_{n < k} A_n) - 1} )^1 \in \mathcal{N}.
\]
We have:
\[
(1) \quad \lambda \varphi(1 - \zeta) \lesssim \|f\|_1,
\]
\[
(2) \quad \text{for any } x \in G \text{, } n < k \text{ and } y \in xF_n \text{ if } f \text{ is admissible then,}
\]
\[
\zeta(x)b_d^k(y)\zeta(x) = \zeta(x)b_{\text{off}}^k(y)\zeta(x) = 0.
\]
Proof. (1) Recall that any admissible atom $A \in \mathcal{P}^a_k$ is by definition $(2^{n-k}, F_n)$-invariant for $n < k$. We have

$$\varphi(1 - \zeta) \leq \sum_{k \geq 1} \sum_{A \in \mathcal{P}^a_k} \tau(p_A) |A \cdot (\cup_{n < k} F_n)^{-1}| \leq \sum_{k \geq 1} \sum_{A \in \mathcal{P}^a_k} \tau(p_A) \left( |A| + \sum_{n < k} |AF_n^{-1} \setminus A| \right)$$

$$\leq 2 \sum_{k \geq 1} \sum_{A \in \mathcal{P}^a_k} \tau(p_A) |A| = 2 \sum_{k \geq 1} \varphi(p_k) = 2 \varphi(1 - q_0)$$

$$\leq 2 \|f\| / \lambda.$$  

(2) If $f$ is admissible, then by Remark 13, the functions $f_k$, $p_k$ and $q_k$ are supported on admissible atoms of $\mathcal{P}_k$. In particular $b^d_k$ and $b^off_k$ are also supported on admissible atoms of $\mathcal{P}_k$. Now if $y \in A$ for some $A \in \mathcal{P}^a_k$, then we have $x \in AF_n^{-1}$ and therefore $\zeta(x) = 0$ by definition. Thus the assertion is verified. \(\square\)

5. The difference operator

We work in the setting developed in Section 3. Let $(F_n)_{n \geq 0}$ be a Følner sequence, $(B_n)_{n \geq 0}$ a subordinate sequence and $(\mathcal{P}_n)_{n \geq 0}$ an atomic filtration. Recall that we denote by $\mathbb{A}_n$ the averaging operator associated to $F_n$ and $\mathbb{E}_n$ the conditional expectation associated to $\mathcal{P}_n$.

We are interested in linking properties of $\mathbb{A}_n$ and properties of $\mathbb{E}_n$ in order to be able to use martingale techniques to obtain results on standard averages. Theorem 5.6 should be considered as the main result of this note though more striking corollaries can be found in Section 6.

Concretely, we show boundedness properties for the difference operator

$$\mathbb{D} : f \mapsto \sum_{n \geq 0} \varepsilon_n \mathbb{D}_n,$$

where $\mathbb{D}_n = \mathbb{A}_n - \mathbb{E}_n$ and $(\varepsilon_n)_{n \geq 0}$ is a sequence of independent Rademacher variables. By the Khinchin inequality, this corresponds to estimating the norm of the square function associated to the sequence $(\mathbb{D}_n(f))_{n \geq 0}$. We divide the work into two subsections: in the first, we show local estimates i.e. estimates for individual $\mathbb{D}_n$ which boil down to simple geometric considerations. In the second, we prove our main theorem on the weak $(1, 1)$ boundedness of $\mathbb{D}$ by combining local estimates and the noncommutative Calderón-Zygmund decomposition. The reader will find that at this point, most of the technical work has already been done and we simply need to glue the (many) pieces back together.

5.1. Local estimates.

Lemma 5.1. If $E$ is $(\varepsilon, K)$-invariant then

\begin{equation}
\|1_E - \mathbb{A}_K(1_E)\|_1 \leq 2 \varepsilon |E|.
\end{equation}

Proof. For any $y \in K$, $|Ey\Delta E| = 2 |EY \setminus E| \leq 2 \varepsilon |E|$. Then, (5.1) follows from the computation:

$$\|1_E - \mathbb{A}_K(1_E)\|_1 = \int_G \left| 1_E(x) - |K|^{-1} \int_K 1_E(xy) dm(y) \right| dm(x)$$

$$= |K|^{-1} \int_G \left| \int_K 1_E(x) - 1_E(xy) dm(y) \right| dm(x)$$

$$\leq |K|^{-1} \int_G \int_K |1_E(x) - 1_E(xy)| dm(y) dm(x)$$

$$= |K|^{-1} \int_K |Ey\Delta E| dm(y)$$

$$\leq 2 |K|^{-1} \int_K \varepsilon |E| dm(y) = 2 \varepsilon |E|.$$  \(\square\)

Proposition 5.2. Let $f \in L_p(\mathcal{N})$ be an admissible function and $n, k \geq 0$ and $p \in [1, \infty)$. 

(1) if $n < k$ and $f \in L_p(\mathcal{N}_k)$, then $\|D_n(f)\|_p \lesssim 2^{(n-k)/p}\|f\|_p$,
(2) if $n \geq k$ and $E_k(f) = 0$, then $\|D_n(f)\|_p \lesssim 2^{k-n}\|f\|_p$.

Proof. (1) It is clear that $\|D_n : \mathcal{N}_k \to \mathcal{N}_k\| \lesssim 1$. So by complex interpolation, it suffices to treat the case $f \in L_1(\mathcal{N}_k)$. In this case we may assume that $f = f_E1_E$ for an atom $E \in \mathcal{F}_k$ and that $f_E$ is positive. Note that $f$ is assumed to be admissible, which in particular means that $E$ is admissible and hence $(2^{n-k}, \mathcal{F}_n)$-invariant. Note that $E \in \mathcal{F}_k$ with $n < k$, which in particular implies that $1_E \in \mathcal{N}_k$ and hence

$$D_n(f) = f_E(E_n(1_E) - E_n(1_E)) = f_E(E_n(1_E) - 1_E).$$

Using (5.1) and the $(2^{n-k}, \mathcal{F}_n)$-invariance, we get

$$\|D_n(f)\|_1 = \|f_E\|_1\|E_n(1_E) - 1_E\|_1 \lesssim 2^{n-k}\|f\|_1.$$

(2) Assume now that $n \geq k$. Note that $E_k(f) = 0$, so $\int_A f = 0$ for all $A \in P_k$ and $E_n(f) = E_nE_k(f) = 0$. Therefore for $x \in G$,

$$D_n(f)(x) = A_n(f)(x) = \frac{1}{|xF_n|} \int_{xF_n} f = \frac{1}{|xF_n|} \int_{\partial\rho_k(xF_n) \cap xF_n} f =: D_n,k(f),$$

where $\partial\rho_k(xF_n) := \{A \in P_k : A \cap xF_n \neq \emptyset, A \not\subset xF_n\}$ and we used the assumption that $f$ is admissible. Note that by admissibility $A \in P_k$ is contained in a left-translate of $B_k$ and therefore $\partial\rho_k(xF_n) \subset \partial B_k(xF_n)$.

Let us show that $\|D_{n,k} : L_p(\mathcal{N}) \to L_p(\mathcal{N})\| \lesssim 2^{k-n}$. For a positive function $\rho \in L_p(\mathcal{N})$, we have for $x \in G$,

$$D_{n,k}(\rho)(x) \leq \frac{1}{|xF_n|} \int_{\partial B_k(xF_n)} \rho = \frac{|\partial B_k(xF_n)|}{|xF_n|} A_{\partial B_k}(F_n)(\rho)(x).$$

Note that $\partial B_k(xF_n) = x\partial B_k(F_n)$ and $F_n$ is by assumption $(2^{k-n}, B_k)$-invariant. We therefore obtain

$$D_{n,k}(\rho) \leq \frac{|\partial B_k(F_n)|}{|F_n|} A_{\partial B_k}(F_n)(\rho) \leq 2^{k-n} A_{\partial B_k}(F_n)(\rho).$$

Since $A_{\partial B_k}(F_n)$ is a contraction on $L_p(\mathcal{N})$ for all $p \in [1, \infty]$, this concludes the proof.

Let us collect two more local estimates which are consequences of specific properties of the Calderón-Zygmund decomposition presented in the previous section.

Lemma 5.3. Let $f \in L_1(\mathcal{N})$ be an admissible function. For $n < k$,

$$\zeta D_n(b^d_k)\zeta = \zeta D_n(b^\text{off}_k)\zeta = 0.$$

Proof. Note that by Lemma 4.6 for $x \in G$,

$$\zeta(x)A_n(b^d_k)(x)\zeta(x) = |F_n|^{-1} \int_{F_n} \zeta(x)b^d_k(xy)\zeta(x)dm(y) = 0.$$

Similar formulas hold for $E_n(b^d_k)$. Indeed, by the definition of $\zeta$ in Lemma 4.6 we see that $\zeta^+ \geq p_k$. But $p_k \in \mathcal{N}_k \subset \mathcal{N}_n$ since $n < k$. So together with the definition of $b^d_k$, we have $E_n(b^d_k) = E_n(p_kb^d_k) = p_kE_n(b^d_k)$. Hence,

$$\zeta E_n(b^d_k)\zeta = \zeta p_kE_n(b^d_k)\zeta = 0.$$

The same argument applies for $b^\text{off}_k$. \hfill \square

Lemma 5.4. Let $f \in L_1(\mathcal{N})$ be an admissible function. For $n \geq k$, $\|D_n(b^\text{off}_k)\|_1 \lesssim 2^{k-n}\varphi(p_kf)$.

Proof. Since $E_n(b^\text{off}_k) = 0$ (see Lemma 4.4), using the notation and arguments introduced in the proof of Proposition 5.2, we have again

$$D_n(b^\text{off}_k) = |xF_n|^{-1} \int_{xF_n \cap \partial\rho_k(xF_n)} b^\text{off}_k,$$
Then, by (1.2) in Lemma 4.4
\[ \|D_n(b_{k})\|_1 \leq |xF_n|^{-1} \varphi \left( \int_{\partial B_k(xF_n)} p_k f p_k \right). \]

Recall that as in the proof of Proposition 5.2, we have \( \partial \mathbb{P}_k(xF_n) \subset \partial B_k(xF_n) = x\partial B_k(F_n) \) and that \( F_n \) is \((2^{k-n}, B_k)\)-invariant. Thus the above inequality yields that
\[ \|D_n(b_{k})\|_1 \leq |xF_n|^{-1} \varphi \left( \int_{\partial B_k(xF_n)} p_k f p_k \right) = \frac{\|\partial B_k(F_n)\|}{|F_n|} \|\hat{A}_{\partial B_k}(F_n)(p_k f)\|_1 \leq 2^{k-n} \varphi(p_k f). \]

**5.2. Weak (1,1) boundedness.** Now we are ready to prove the desired boundedness property of the difference operator \( D \). We start by establishing the \( L_2 \)-boundedness of \( D \).

**Proposition 5.5.** Let \( f \in L_2(\mathcal{N}) \) be an admissible function. Then
\[ \|D(f)\|_2 \lesssim \|f\|_2. \]

**Proof.** We decompose into a sum of martingale differences \( f = \sum_{k \geq 0} df_k \) with \( df_k = f_k - f_{k+1} \). Note that \( \mathbb{E}_{k+1}(df_k) = 0 \) and \( df_k \in \mathcal{N}_k \) so that by the two local estimates stated in Proposition 5.2 for any \( k, n \geq 0 \), we have
\[ \|D_n(df_k)\|_2^2 \lesssim 2^{-|k-n|} \|df_k\|_2^2. \]

By changement of index \( s = k - n \), we may write
\[ \sum_n \|D_n(df_{n+s})\|_2^2 \lesssim 2^{-|n|} \|f\|_2^2. \]

But
\[ \|D(f)\|_2 = \left( \sum_n \|D_n f\|_2^2 \right)^{1/2} = \left( \sum_n \| \sum_k D_n(df_k) \|_2^2 \right)^{1/2} \]
\[ = \left( \sum_n \| \sum_s D_n(df_{n+s}) \|_2^2 \right)^{1/2} \leq \sum_s \left( \sum_n \|D_n(df_{n+s})\|_2^2 \right)^{1/2} \]

So combining the two above inequalities we obtain
\[ \|D(f)\|_2 \lesssim \sum_s 2^{-|n|/2} \|f\|_2 \lesssim \|f\|_2. \]

Now we arrive at the main result of this section.

**Theorem 5.6.** Let \( f \in L_1(\mathcal{N}) \) be an admissible function. Then
\[ \|D f\|_{1,\infty} \lesssim \|f\|_1. \]

**Proof.** Let \( f \in L_1(\mathcal{N})^+ \), bounded and compactly supported. Let \( \lambda > 0 \). We aim to show that
\[ \lambda \varphi(\{|D(f)| > \lambda\}) \lesssim \|f\|_1. \]

By (2.1), it suffices to deal with each part of the Calderón-Zygmund decomposition separately. Namely to show that
\[ \lambda \varphi(\{|D(u)| > \lambda\}) \lesssim \|f\|_1 \]

for \( u = g, h \) and \( b \).

**The good part.** By Lemma 4.2 \( \|g\|_2^2 \lesssim \|f\|_1. \) Using Tchebychev’s inequality as well as the \( L_2 \)-boundedness of \( D \) (Proposition 5.5) we get
\[ \varphi(\{|D(g)| > \lambda\}) \lesssim \frac{\|D(g)\|_2^2}{\lambda^2} \lesssim \frac{\|g\|_2^2}{\lambda^2} \lesssim \frac{\|f\|_1}{\lambda}. \]
The hybrid part. By Lemma 4.3 we have $\mathcal{E}_{k+1}(h_k) = 0$ and $h_k \in \mathcal{N}_k$ so by the local estimates of Proposition 5.2 for any $k, n \geq 0$

$$\|D_n(h_k)\|_1 \lesssim 2^{-|k-n|} \|h_k\|_1.$$ 

Summing up those inequalities over $n$ and $k$, we get

$$\|D(h)\|_1 \lesssim \sum_{k \geq 0} \|h_k\|_1.$$ 

It follows, using the norm estimate in Lemma 4.3 that

$$\|D(h)\|_1 \lesssim \|f\|_1.$$ 

We conclude once again using Tchebychev’s inequality.

The bad part. For the bad part, it is not possible to get norm estimates right away. However, Lemma 5.3 tells us that $D$ (and in general CZ-operators) behaves better outside of the projection $\zeta$ i.e. away from the support of $b$.

Concretely, by (2.1) we may write

$$\varphi(\{|D(b)| > \lambda\}) \leq \varphi(\{|\zeta b\| > \lambda\}) + \varphi(\{|\zeta D(b)\| > \lambda\}) \lesssim \lambda^{-1} \|f\|_1 + \lambda^{-1} \|\zeta D(b)\|_1,$$

where we used Lemma 4.0 to estimate the first two terms and Tchebychev’s inequality for the last one. Hence, it remains to prove that

$$\|\zeta D(b)\|_1 \lesssim \|f\|_1.$$ 

For the diagonal part, by Lemma 4.1 Lemma 5.3 and Proposition 5.2 (2), we have for any $n$ and $k$

$$\|\zeta D_n(b_k^d)\|_1 \lesssim 2^{-|n-k|} \|b_k^d\|_1 \lesssim 2^{-|n-k|} \lambda \varphi(p_k).$$

Similarly, by Lemma 5.4 and Lemma 5.3

$$\|D_n(b_k^\text{off})\|_1 \lesssim 2^{-|n-k|} \lambda \varphi(p_k).$$

Now note that by Lemma 4.1

$$\lambda \sum_{k \geq 0} p_k = \lambda \varphi(1 - q_0) \leq \|f\|_1,$$

which, as seen for the hybrid part, concludes the proof.

6. Ergodic theorems

In order to establish the weak $(1, 1)$ maximal ergodic inequality on the whole $L_1$-space, we first deduce the inequality for admissible functions from previous results.

**Lemma 6.1.** Let $(F_n)_{n \geq 0}$ be a filtered Følner sequence on $G$. Let $f \in L_1(\mathcal{N})$ be an admissible function. Then for any $\lambda > 0$ there exists a projection $e \in \mathcal{N}$ satisfying

$$\sup_{n \geq 1} \|e_k(f) e\|_\infty \leq \lambda \quad \text{and} \quad \lambda \varphi(1 - e) \lesssim \|f\|_1.$$ 

**Proof.** Combining Theorem 5.6 Equation (2.3) and Lemma 2.2 we see that there exists a projection $e' \in \mathcal{N}$ satisfying

$$\sup_{n \geq 1} \|e'_n(f) e'\|_\infty \leq \lambda/2 \quad \text{and} \quad \lambda \varphi(1 - e') \lesssim \|f\|_1.$$ 

The Cuculescu inequality in Lemma 4.1 asserts that there is a projection $q_0 \in \mathcal{N}$ satisfying

$$\sup_{n \geq 1} \|q_0 e_n(f) q_0\|_\infty \leq \lambda/2 \quad \text{and} \quad \lambda \varphi(1 - q_0) \lesssim \|f\|_1.$$ 

Then taking $e = e' \wedge q_0$ yields the lemma. □
Now the desired maximal inequality for operator-valued functions is in reach.

Let $\mathcal{S} = ((F_n)_{n \geq 0}, (B_n)_{n \geq 0}, \mathcal{P})$ be a filtered sequence. For $g \in G$, we denote by $g \cdot \mathcal{P}$ the filtration determined by

$$(g \cdot \mathcal{P})_k := \{ g \cdot A : A \in \mathcal{P}_k \}, \quad k \geq 0.$$ 

Note that the admissible atoms in $g \cdot \mathcal{P}$ are given by $g \cdot \text{Adm}(\mathcal{S})$.

Recall that $\mathcal{N} = \mathcal{M} \overline{\otimes} L_\infty(G)$ with $\mathcal{M}$ a noncommutative measure space.

Lemma 6.2. Let $\mathcal{S} = ((F_n)_{n \geq 0}, (B_n)_{n \geq 0}, \mathcal{P})$ be a regular filtered sequence. Let $f \in L_1(\mathcal{N})$ of compact support (as a function from $G$ to $L_1(\mathcal{M})$). Then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $L_1(\mathcal{N})$, a sequence $(g_n)_{n \geq 0} \subset G$ and a constant $\alpha > 0$ such that

- for any $n \geq 0$, $f_n$ is admissible for the filtration $g_n \cdot \mathcal{P}$,
- for any $n \geq 0$, $\|f_n\|_1 \leq \alpha^{-n}\|f\|_1$,
- $\sum_{n=0}^{\infty} f_i = f$.

Proof. Let $E = \text{Adm}(\mathcal{S})$. A function is admissible for the filtration $g \cdot \mathcal{P}$ if and only if its support is in $g \cdot E$. Let $K$ be the support of $f$. Assume without loss of generality that $f$ is positive. For any $\varepsilon > 0$, we can find an $(\varepsilon, K^{-1})$-invariant compact subset $D$ of $G$. We may also assume by the definition of a regular filtered sequence that $|E \cap D| \geq c|D|$ for some $c \in (0, 1)$. In particular, for any $y \in K$,

$$|Ey^{-1} \cap D \cdot K^{-1}| \geq |(E \cap D)y^{-1}| \geq c|D|.$$ 

We aim to find an element $x_0$ of $G$ such that $\|f1_{x_0^{-1}E}\|_1 \geq c'\|f\|_1$ for some $c' \in (0, 1)$. To do so, we compute the average value of $\|f1_{x_0^{-1}E}\|_1$ for $x \in D \cdot K^{-1}$

$$\frac{1}{|D \cdot K^{-1}|} \int_{x \in D \cdot K^{-1}} \|f1_{x^{-1}E}\|_1 \, dx = \frac{1}{|D \cdot K^{-1}|} \left( \int_{y \in K} \left( \int_{x \in D \cdot K^{-1}} 1_{Ey^{-1}}(x) \, dx \right) \, dy \right) = \frac{1}{|D \cdot K^{-1}|} \int_{y \in K} |D \cdot K^{-1} \cap Ey^{-1}| \, dy \geq \frac{c|D|}{|D \cdot K^{-1}|} \|f\|_1 \geq \frac{c}{1 + \varepsilon} \|f\|_1,$$

which implies there exists $x_0 \in D \cdot K^{-1}$ such that

$$\|f1_{x_0^{-1}E}\|_1 \geq \frac{c}{1 + \varepsilon} \|f\|_1.$$ 

Set $f_0 = f1_{x_0^{-1}E}$ and $f' = f - f_0$. Note that $\|f'\|_1 \leq \frac{1 + \varepsilon}{1 + \varepsilon} \|f\|_1$. Repeat the argument, replacing $f$ by $f'$ to construct $f_1$. The result follows by induction.

\[\Box\]

Theorem 6.3. Let $\mathcal{S} = ((F_n)_{n \geq 0}, (B_n)_{n \geq 0}, \mathcal{P})$ be a regular filtered sequence on $G$. Let $f \in L_1(\mathcal{N})$. For any $\lambda > 0$ there exists a projection $e \in \mathcal{N}$ satisfying

$$\sup_{n \geq 1} \|e\mathcal{A}_n(f)e\|_\infty \leq \lambda \quad \text{and} \quad \lambda \varphi(1 - e) \leq \|f\|_1.$$ 

Proof. By Theorem 3.6 we can write $f = \sum_{i \geq 0} f_i$ with each $f_i$ admissible for a certain filtration $x_i\mathcal{P}$ and $\|f_i\|_1 \leq \alpha^{-i}\|f\|_1$. Using Lemma 6.1 for $\lambda_i = \alpha^{-i/2}\lambda$ we obtain projections $e_i$ satisfying

$$\sup_{n \geq 0} \|e_i\mathcal{A}_n(f_i)e_i\|_\infty \leq \alpha^{-i}\lambda \quad \text{and} \quad \lambda \varphi(1 - e_i) \leq \alpha^{-i/2}\|f\|_1.$$ 

Set $e = \bigwedge_{i \geq 0} e_i$. We check that for any $n \geq 0$,

$$\|e\mathcal{A}_n(f_i)e\|_\infty \leq \sum_{i \geq 0} \|e\mathcal{A}_n(f_i)e\|_\infty \leq \sum_{i \geq 0} \|e_i\mathcal{A}_n(f_i)e_i\|_\infty \leq 2\lambda$$

and

$$\lambda \varphi(1 - e) = \lambda \varphi \left( \bigvee_{i \geq 0} 1 - e_i \right) \leq \lambda \sum_{i \geq 0} \varphi(1 - e_i) \leq 2\|f\|_1.$$

\[\Box\]
Combining the noncommutative Marcinkiewicz interpolation theorem [14, Theorem 3.1] and the transference principle in [10, Theorem 3.1 and Theorem 3.3], we deduce the following maximal ergodic theorem for general actions. By abuse of notation, for a \( \text{w}^* \)-continuous action of \( G \) on the von Neumann algebra \( M \) by \( \tau \)-preserving automorphisms, we still denote \( \mathbb{A}_n(x) = |F_n|^{-1} \int_{F_n} \alpha_g x dm(g) \) for \( x \in \mathcal{M} \). It is well-known and easy to see that \( \mathbb{A}_n \) extends to contractions on \( L_p(M) \) for all \( p \in [1, \infty] \).

**Theorem 6.4.** Let \( (F_n) \) be a regular filtered Følner sequence on \( G \). If \( \alpha \) is a \( \text{w}^* \)-continuous action of \( G \) on a semifinite von Neumann algebra \( M \) by \( \tau \)-preserving automorphisms, then for any \( x \in L_1(\mathcal{M}) \) and \( \lambda > 0 \), there exists a projection \( e \in \mathcal{M} \) satisfying

\[
\sup_{n \geq 1} \| e \mathbb{A}_n(x) e \|_\infty \leq \lambda \quad \text{and} \quad \lambda \tau(1 - e) \lesssim \| x \|_1;
\]

and for any \( x \in L_p(M)_+ \) and \( p \in (1, \infty) \), there is an \( a \in L_p(M)_+ \) satisfying

\[
\mathbb{A}_n(x) \leq a \quad \text{and} \quad \| a \|_p \lesssim \| x \|_p, \quad \forall n \geq 0.
\]

Now let us derive the pointwise ergodic theorem. We recall the following well-known notion of (bilaterally) almost uniform convergence considered in Lance’s work [15], which gives a suitable replacement for almost everywhere convergence in the noncommutative context.

**Definition 6.5** (Almost uniform convergence). Let \( \mathcal{M} \) be a von Neumann algebra and \( \tau \) a trace on \( \mathcal{M} \). A sequence of operators \( (x_n) \) is said to converge bilaterally almost uniformly (abbreviated as b.a.u.) to \( x \) if for any \( \varepsilon > 0 \) there exists a projection \( e \in \mathcal{M} \) such that \( \tau(1 - e) \leq \varepsilon \) and

\[
\| e(x_n - x) e \|_\infty \to 0.
\]

It is said to converge almost uniformly (abbreviated as a.u.) to \( x \) if for any \( \varepsilon > 0 \) there exists a projection \( e \in \mathcal{M} \) such that \( \tau(1 - e) \leq \varepsilon \) and

\[
\| (x_n - x) e \|_\infty \to 0.
\]

As in the classical setting, we have the following Banach principle building a bridge between maximal inequalities and almost uniform convergence.

**Lemma 6.6** ([5, Theorem 3.1]). Let \( 1 \leq p < 2 \) (resp. \( 2 \leq p < \infty \)) and \( (\Phi_n)_{n \geq 0} \) be a sequence of positive linear maps on \( L_p(\mathcal{M}) \). Assume that for any \( x \in L_p(\mathcal{M}) \) and \( \lambda > 0 \), there exists a projection \( e \in \mathcal{M} \) satisfying

\[
\sup_{n \geq 1} \| e \Phi_n(x) e \|_\infty \leq \lambda \quad \text{and} \quad \lambda \tau(1 - e)^{1/p} \lesssim \| x \|_p.
\]

Then the space of the elements \( x \in L_p(\mathcal{M}) \) such that \( \Phi_n(x) \) converges b.a.u. (resp. a.u.) is closed in \( L_p(\mathcal{M}) \).

Now we may conclude our final main result.

**Theorem 6.7.** Let \( G \) be a locally compact second countable amenable group and \( (F_n) \) be a regular filtered Følner sequence on \( G \) (which always exists according to Theorem 3.6). If \( \alpha \) is a \( \text{w}^* \)-continuous action of \( G \) on a semifinite von Neumann algebra \( M \) by \( \tau \)-preserving automorphisms, then for all \( x \in L_p(\mathcal{M}) \) with \( 1 \leq p < 2 \) (resp. \( 2 \leq p < \infty \)),

\[
\frac{1}{m(F_n)} \int_{F_n} \alpha_g x dm(g) \to P x \quad \text{b.a.u. (resp. a.u.)},
\]

where \( P \) is a bounded projection onto the space of fixed points of \( \alpha \).

**Proof.** As explained in [10, Subsection 2.2], the bounded projection \( P \) onto the fixed point subspace always exists, and the span of following subspaces

\[
\{ x \in L_p(\mathcal{M}) \mid \alpha_g x = x, \forall g \in G \} \quad \text{and} \quad \text{span}\{ x - \alpha_g x \mid x \in L_p(\mathcal{M}) \cap \mathcal{M}, g \in G \}
\]

is dense in \( L_p(\mathcal{M}) \). It is easy to see from the Følner property of \( (F_n) \) that the desired almost uniform convergence holds on this subspace. Thus by Theorem 6.4 and Lemma 6.6 we obtain the theorem. \( \square \)
Remark 6.8. Our method is also useful for the study of some other more general or stronger form of ergodic theorems, for instance, the square estimates \((A_n - A_{n-1})_n \geq 0\) and differential transforms similar to \([11][21]\), maximal and individual ergodic theorems for group actions on noncommutative \(L_p\)-spaces with a fixed \(p \in (1, \infty)\) as in \([10]\). Also our previous results for noncommutative \(L_p\)-spaces with \(p \in (1, \infty)\) holds true as well for general non-tracial von Neumann algebras up to standard adaptations as in \([14][9]\). We leave the details to interested readers.

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