Number Theory

Weyl’s law for the cuspidal spectrum of $\text{SL}_n$

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Abstract

Let $\Gamma$ be a principal congruence subgroup of $\text{SL}_n(\mathbb{Z})$ and let $\sigma$ be an irreducible unitary representation of $\text{SO}(n)$. Let $N^\Gamma_{\text{cus}}(\lambda, \sigma)$ be the counting function of the eigenvalues of the Casimir operator acting in the space of cusp forms for $\Gamma$ which transform under $\text{SO}(n)$ according to $\sigma$. In this Note we prove that the counting function $N^\Gamma_{\text{cus}}(\lambda, \sigma)$ satisfies Weyl’s law. In particular, this implies that there exist infinitely many cusp forms for the full modular group $\text{SL}_n(\mathbb{Z})$.

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Soit $G$ un groupe algébrique réductif connexe défini sur $\mathbb{Q}$ et soit $\Gamma \subset G(\mathbb{Q})$ un sous-groupe arithmétique de $G$. Un problème important dans la théorie des formes automorphes est la question de l’existence et de la construction de formes cuspidales pour $\Gamma$.

Dans cette Note, nous étudions le problème d’existence pour le groupe $G = \text{SL}_n$, $n \geq 2$. Soit $\Gamma$ un sous-groupe de congruence de $\text{SL}_n(\mathbb{Z})$. Soit $L^2(\text{cusp}(\Gamma \setminus \text{SL}_n(\mathbb{R})))$ la fermeture hilbertienne de l’espace engendré par les formes automorphes cuspidales. Soit $(\sigma, \text{V}_\sigma)$ une représentation irréductible unitaire de $\text{SO}(n)$. On pose

$$L^2(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma) = (L^2(\Gamma \setminus \text{SL}_n(\mathbb{R})) \otimes \text{V}_\sigma)^{\text{SO}(n)},$$

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et on définit $L^2_{\text{cus}}(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma)$ de manière similaire. Soit $\Omega \in \mathcal{Z}(\mathfrak{sl}(n, \mathbb{C}))$ l’élément de Casimir de $\text{SL}_n(\mathbb{R})$. Alors $-\Omega \otimes \text{Id}$ induit un opérateur auto-adjoint $\Delta_\sigma$, agissant sur l’espace de Hilbert $L^2(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma)$. Cet opérateur est borné inférieurement et la restriction de $\Delta_\sigma$ au sous-espace $L^2_{\text{cus}}(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma)$ est un opérateur à spectre ponctuel, formé de valeurs propres $\lambda_0(\sigma) < \lambda_1(\sigma) < \cdots$ de multiplicité finie. Soit $\mathcal{E}(\lambda_i(\sigma))$ l’espace propre associé à la valeur propre $\lambda_i(\sigma)$. Pour $\lambda \geq 0$ on pose
\[
N^\Gamma_{\text{cus}}(\lambda, \sigma) = \sum_{\lambda_i(\sigma) \leq \lambda} \dim \mathcal{E}(\lambda_i(\sigma)).
\]
Alors notre résultat principal est le théorème suivant.

**Théorème 0.1.** Pour $n \geq 2$, soit $X_n = \text{SL}_n(\mathbb{R})/\text{SO}(n)$. Soit $d_n = \dim X_n$. Alors pour tout sous-groupe de congruence principal $\Gamma$ de $\text{SL}_n(\mathbb{Z})$ et pour toute représentation irréductible unitaire $\sigma$ de $\text{SO}(n)$ tels que $\sigma|_{Z_{\Gamma}} = \text{Id}$, on a
\[
N^\Gamma_{\text{cus}}(\lambda, \sigma) \sim \dim(\sigma) \frac{\text{vol}(\Gamma \setminus X_n)}{(4\pi)^{d_n/2} \Gamma(d_n/2 + 1)} \lambda^{d_n/2}
\]
pour $\lambda \to \infty$.

La démonstration de ce théorème utilise la formule des traces d’Arthur combinée avec la méthode de l’équation de la chaleur.

1. Introduction

Let $G$ be a connected reductive algebraic group over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. An important problem in the theory of automorphic forms is the question of existence and the construction of cusp forms for $\Gamma$.

In this paper we address the problem of existence for $G = \text{SL}_n$, $n \geq 2$. Let $\Gamma$ be a congruence subgroup of $\text{SL}_n(\mathbb{Z})$. Let $L^2_{\text{cus}}(\Gamma \setminus \text{SL}_n(\mathbb{R}))$ be the closure of the span of cusp forms for $\Gamma$. Let $(\sigma, V_\sigma)$ be an irreducible unitary representation of $\text{SO}(n)$. Set
\[
L^2(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma) = \left( L^2(\Gamma \setminus \text{SL}_n(\mathbb{R})) \otimes V_\sigma \right)^{\text{SO}(n)},
\]
and define $L^2_{\text{cus}}(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma)$ similarly. Let $\Omega \in \mathcal{Z}(\mathfrak{sl}(n, \mathbb{C}))$ be the Casimir element of $\text{SL}_n(\mathbb{R})$. Then $-\Omega \otimes \text{Id}$ induces a selfadjoint operator $\Delta_\sigma$ in the Hilbert space $L^2(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma)$ which is bounded from below. The restriction of $\Delta_\sigma$ to the subspace $L^2_{\text{cus}}(\Gamma \setminus \text{SL}_n(\mathbb{R}), \sigma)$ has a pure point spectrum consisting of eigenvalues $\lambda_0(\sigma) < \lambda_1(\sigma) < \cdots$ of finite multiplicity. Let $\mathcal{E}(\lambda_i(\sigma))$ be the eigenspace corresponding to the eigenvalue $\lambda_i(\sigma)$. For $\lambda \geq 0$ set
\[
N^\Gamma_{\text{cus}}(\lambda, \sigma) = \sum_{\lambda_i(\sigma) \leq \lambda} \dim \mathcal{E}(\lambda_i(\sigma)).
\]
Then our main result is the following theorem.

**Theorem 1.1.** For $n \geq 2$ let $X_n = \text{SL}_n(\mathbb{R})/\text{SO}(n)$. Let $d_n = \dim X_n$. For every principal congruence subgroup $\Gamma$ of $\text{SL}_n(\mathbb{Z})$ and every irreducible unitary representation $\sigma$ of $\text{SO}(n)$ such that $\sigma|_{Z_{\Gamma}} = \text{Id}$ we have
\[
N^\Gamma_{\text{cus}}(\lambda, \sigma) \sim \dim(\sigma) \frac{\text{vol}(\Gamma \setminus X_n)}{(4\pi)^{d_n/2} \Gamma(d_n/2 + 1)} \lambda^{d_n/2}
\]
as $\lambda \to \infty$. 

This is Weyl’s law for principal congruence subgroups of $\text{SL}_n(\mathbb{Z})$. For $n = 2$ this was proved by Selberg [13]. For $G = \text{SL}_3(\mathbb{Z})$ and $\sigma$ the trivial representation, Weyl’s law was proved by Miller [7]. It has been conjectured by Sarnak [12] and also by Müller [9] that Weyl’s law holds for every arithmetic subgroup of a reductive group $G$.

2. The adèlic version of Weyl’s law

Let $G = \text{GL}_n$ regarded as algebraic group over $\mathbb{Q}$ and let $A_G$ be the split component of the center of $G$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Denote by $\xi_0$ the trivial character of $A_G(\mathbb{R})^0$. Let $\Pi(G(\mathbb{A}), \xi_0)$ be the set of all irreducible unitary representations of $G(\mathbb{A})$ whose central character is trivial on $A_G(\mathbb{R})^0$ and let $\Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)$ be the subset of cuspidal automorphic representations in $\Pi(G(\mathbb{A}), \xi_0)$. Let $\mathbb{A}_f$ be the ring of finite adeles. Given an irreducible unitary representation of $G(\mathbb{A})$, write $\pi = \pi_\infty \otimes \pi_f$, where $\pi_\infty$ and $\pi_f$ are irreducible unitary representations of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$ be the Hilbert spaces of the representations $\pi_\infty$ and $\pi_f$, respectively. Let $K_f$ be an open compact subgroup of $G(\mathbb{A}_f)$. Denote by $\mathcal{H}_{\pi_f}^K$ the subspace of $K_f$-invariant vectors in $\mathcal{H}_{\pi_f}$. Let $G(\mathbb{R})^1$ be the subgroup of all $g \in G(\mathbb{R})$ with $|\det(g)| = 1$. Given $\pi \in \Pi(G(\mathbb{A}), \xi_0)$, denote by $\lambda_\pi$ the Casimir eigenvalue of the restriction of $\pi_\infty$ to $G(\mathbb{R})^1$. For $\lambda \geq 0$ let $\Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)_\lambda$ be the space of all $\pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)$ which satisfy $|\lambda_\pi| \leq \lambda$. Set $\varepsilon_{K_f} = 1$, if $-1 \in K_f$ and $\varepsilon_{K_f} = 0$ otherwise. Then we have

**Theorem 2.1.** Let $G = \text{GL}_n$ and let $d_n = \dim \text{SL}_n(\mathbb{R})/SO(n)$. Let $K_f$ be an open compact subgroup of $G(\mathbb{A}_f)$ and let $(\tau, V_\tau)$ be an irreducible unitary representation of $O(n)$ such that $\sigma(-1) = \text{Id}$ if $-1 \in K_f$. Then

$$
\sum_{\pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)} \dim(\mathcal{H}_{\pi_f}^K) \dim(\mathcal{H}_{\pi_f}^K \otimes V_\tau) d_n^{O(n)} \\
\sim \dim(\tau) \frac{\text{vol}(G(\mathbb{Q}) A_G(\mathbb{R})^0 \backslash G(\mathbb{A}) / K_f)}{(4\pi)^{d_n/2} \Gamma(d_n/2 + 1)} (1 + \varepsilon_{K_f}) \lambda_\pi^{d_n/2}
$$

(1)
as $\lambda \to \infty$.

Let $N \in \mathbb{N}$ and let $N = \prod_p p^{r_p}$, $r_p \geq 0$, be the prime factor decomposition of $N$. Put $K_p(N) = \{k \in \text{GL}_n(\mathbb{Z}/p^\infty) \mid k \equiv 1 \text{ mod } \mathbb{Z}/p^\delta \}$. Then $K(N) = \prod_{p < \infty} K_p(N)$ is an open compact subgroup of $G(\mathbb{A}_f)$ and as an $\text{SL}_n(\mathbb{R})$-module, $L^2(G(\mathbb{Q}) A_G(\mathbb{R})^0 \backslash G(\mathbb{A}) / K(N))$ is isomorphic to the direct sum of $|\mathbb{Z}/N\mathbb{Z}|$ copies of $L^2(\Gamma(N) \backslash \text{SL}_n(\mathbb{R}))$, where $\Gamma(N)$ is the principal congruence subgroup of $\text{SL}_n(\mathbb{Z})$ of level $N$. Using this fact, Theorem 1.1 is an immediate consequence of Theorem 2.1.

The proof of Theorem 2.1 is based on Arthur’s trace formula combined with the heat equation method. Let $G(\mathbb{A})^1$ be the subgroup of all $g \in G(\mathbb{A})$ satisfying $|\det(g)| = 1$. The noninvariant trace formula of Arthur [1] is an identity

$$
\sum_{x \in X} J_x(f) = \sum_{\sigma \in O} J_\sigma(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),
$$

(2)

between distributions on $G(\mathbb{A})^1$. The left-hand side is the spectral side $J_{\text{spec}}(f)$ and the right-hand side the geometric side $J_{\text{geo}}(f)$ of the trace formula.

We construct a special family of test functions $\tilde{\phi}_t^1 \in C_c^\infty(G(\mathbb{A})^1)$, $t > 0$, as follows. Let $\tau$ be an irreducible unitary representation of $O(n)$. Let $\tilde{E}_\tau \to G(\mathbb{R})^1/O(n)$ be the homogeneous vector bundle attached to $\tau$ and let $\tilde{\Delta}_\tau$ be the elliptic operator induced by $-\Delta \otimes \text{Id}$ in $C_c^\infty(\tilde{E}_\tau)$. Let $H^\tau_1 : G(\mathbb{R})^1 \to \text{End}(V_\tau)$ be the kernel of the heat operator $e^{-t \tilde{\Delta}_\tau}$. Set $h^\tau_1 = \text{tr} H^\tau_1$. We extend $h^\tau_1$ to a smooth function on $G(\mathbb{R})$ by $h^\tau_1(zg) = h^\tau_1(g)$, $g \in G(\mathbb{R})^1$, $z \in Z_{G(\mathbb{R})}$, the center of $G(\mathbb{R})$. Let $K_f$ be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\chi_{K_f}$ be the normalized characteristic function of $K_f$ in $G(\mathbb{A}_f)$. For $t > 0$ we define a smooth function $\phi_t$ on $G(\mathbb{A})$ by
Theorem 3.1. \( \phi_1(g) = h_1^2(g_\infty)\chi_K(g_f) \). \( g = g_\infty g_f \). Let \( \varphi \in C^\infty(\mathbb{R}) \) be such that \( \varphi(u) = 1 \), if \( |u| \leq 1/2 \), and \( \varphi(u) = 0 \), if \( |u| \geq 1 \). Given \( g_\infty \in G(\mathbb{R})^1 \), let \( r(g_\infty) \) be the Riemannian distance of the cosets in \( G(\mathbb{R})^1 / O(n) \) of \( g_\infty \) and \( e \), respectively. Put \( \varphi_t(g_\infty) = \varphi(r^2(g_\infty)/t^1/\lambda) \), \( g_\infty \in G(\mathbb{R})^1 \). Extend \( \varphi_t \) to a smooth function on \( G(\mathbb{R}) \) by \( \varphi_t(zg) = \varphi_t(g) \), \( g \in G(\mathbb{R})^1 \), \( z \in G(zg_\mathbb{R}) \), and then to a smooth function on \( G(\mathbb{A}) \) by multiplying \( \varphi_t \) by the characteristic function of \( K_f \). Put \( \varphi_t(g) = \varphi_t(g_\phi(g), g \in G(\mathbb{A}) \).

Let \( \tilde{\phi}_1 \) be the restriction of \( \tilde{\phi}_1 \) to \( G(\mathbb{A})^1 \). Then \( \tilde{\phi}_1 \in C^\infty_c(G(\mathbb{A})^1) \). To prove Theorem 2.1 we insert \( \tilde{\phi}_1 \) in the trace formula and compare the asymptotic behaviour of the left and right-hand side of the trace formula as \( t \to 0 \).

3. The spectral side of the Arthur trace formula

In this section we determine the asymptotic behaviour of \( J_{\text{spec}}(\tilde{\phi}_1) \) as \( t \to 0 \). By a parabolic subgroup of \( G \) we will always mean a parabolic subgroup which is defined over \( \mathbb{Q} \). Let \( M_0 \) be the Levi component of the standard minimal parabolic subgroup \( P_0 \) of \( G \). By a Levi subgroup we will mean a subgroup of \( G \) which contains \( M_0 \) and which is the Levi component of a parabolic subgroup of \( G \). Let \( L \) be the set of all Levi subgroups of \( G \). Given \( M \in L \), let \( L(M) \) be the set of Levi subgroups containing \( M \) and denote by \( \mathcal{P}(M) \) the set of parabolic subgroups with Levi component \( M \).

Let \( \mathcal{C}(G(\mathbb{A})^1) \) denote the space of integrable rapidly decreasing functions on \( G(\mathbb{A})^1 \) [10, §1.3]. By Theorem 0.1 of [11] the spectral side \( J_{\text{spec}}(f) \) of the trace formula is absolutely convergent for all \( f \in \mathcal{C}(G(\mathbb{A})^1) \) and can be written as a finite linear combination

\[
J_{\text{spec}}(f) = \sum_{M \in L} \sum_{L \in L(M)} \sum_{P \in \mathcal{P}(M)} \sum_{\tau \in W^L(\alpha_M)_{\text{reg}}} a_{M,s} J_{M,P}^1(f,s),
\]

of distributions \( J_{M,P}^1(f,s) \), where \( W^L(\alpha_M)_{\text{reg}} \) is a certain set of Weyl group elements. The main ingredients of the distribution \( J_{M,P}^1(f,s) \) are generalized logarithmic derivatives of intertwining operators \( M_Q(P) : \mathcal{A}(P) \to \mathcal{A}(Q) \), \( P, Q \in \mathcal{P}(M) \), \( \lambda \in \alpha_M^\times \), acting between spaces of square-integrable automorphic forms attached to \( P \) and \( Q \), respectively. For a detailed description of \( J_{M,P}^1(f,s) \) see [11].

Let \( \phi_1 \) denote the restriction of \( \phi_1 \) to \( G(\mathbb{A})^1 \). Then \( \tilde{\phi}_1 \) belongs to \( C^1(G(\mathbb{A})^1) \) and it follows from the proof of the absolute convergence of the spectral side [11], [10] that

\[
|J_{\text{spec}}(\tilde{\phi}_1) - J_{\text{spec}}(\phi_1)| \leq C e^{-c\sqrt{t}}
\]
as \( t \to 0 \). Thus it suffices to determine the asymptotic behaviour of \( J_{\text{spec}}(\phi_1) \) as \( t \to 0 \).

Let \( \xi_0 \) be the trivial character of \( A_G(\mathbb{R})^0 \) and let \( \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0) \) be the set of all irreducible unitary representations of \( G(\mathbb{A}) \) which are equivalent to a subrepresentation of the regular representation of \( G(\mathbb{A}) \) in \( L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A})) \). Given \( \tau \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0) \), let \( m(\tau) \) denote the multiplicity with which \( \tau \) occurs in \( L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A})) \). Let \( \tau \in \Pi(\Omega(n)) \).

Theorem 3.1. We have

\[
J_{\text{spec}}(\phi_1) = \sum_{\tau \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\tau) \dim(H_{\pi_\tau}^{K_f}) \dim(H_{\pi_\tau \otimes V_\tau}^{\infty(n)}) e^{i\lambda + s} + O(t^{-d_0 - 1/2}),
\]
as \( t \to 0^+ \), and the series on the right-hand side is convergent for all \( t > 0 \).

The proof of this theorem is based on (3). We evaluate the distributions \( J_{M,P}^1 \) at \( \phi_1 \). If \( M = L = G \), then \( s = 1 \) and \( J_{G,G}^1(\phi_1,1) \) equals the series on the right-hand side of (4). The proof is completed by showing that for all proper Levi subgroups \( M \in L \), all \( L \in L(M) \), \( P \in \mathcal{P}(M) \) and \( s \in W^L(\alpha_M)_{\text{reg}} \) we have

\[
J_{M,P}^1(\phi_1, s) = O(t^{-d_0 - 1/2})
\]
as $t \to 0$. This is the key result. The proof of (5) relies on estimations of generalized logarithmic derivatives of the intertwining operators $M_{Q|P}(\lambda)$, $P, Q \in \mathcal{P}(M)$, on $\lambda \in \mathfrak{a}_{M}^{*}$. Given $\pi \in \Pi_{\text{dis}}(M(\lambda), \xi_{0})$, let $M_{Q|P}(\pi, \lambda)$ be the restriction of the intertwining operator $M_{Q|P}(\lambda)$ to the subspace $\mathcal{A}_{\mathbb{A}}^{\infty}(P)$ of automorphic forms of type $\pi$. The intertwining operators can be normalized by certain meromorphic functions $r_{Q|P}(\pi, \lambda)$ on $\mathfrak{a}_{M}^{*}$. Given $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}), \xi_{0})$, let $M_{Q|P}(\pi, \lambda)$ be the restriction of the intertwining operator $M_{Q|P}(\lambda)$ to the subspace $\mathcal{A}_{\mathbb{A}}^{\infty}(P)$ of automorphic forms of type $\pi$.

The intertwining operators can be normalized by certain meromorphic functions $r_{Q|P}(\pi, \lambda)$ on $\mathfrak{a}_{M}^{*}$. Using Arthur’s theory of $(G, M)$-families [2], our problem can be reduced to the estimation of derivatives of the normalized intertwining operators $N_{Q|P}(\pi, \lambda)$ and the normalizing factors $r_{Q|P}(\pi, \lambda)$ on $\mathfrak{a}_{M}^{*}$. The derivatives of $N_{Q|P}(\pi, \lambda)$ can be estimated using Proposition 0.2 of [11]. The normalizing factors are defined in terms of the Rankin–Selberg $L$-functions $L(s, \pi_{i} \otimes \pi_{j})$. So the problem is reduced to the estimation of the logarithmic derivatives of Rankin–Selberg $L$-functions on the line $\text{Re}(s) = 1$. Estimates are derived using the analytic properties of the Rankin–Selberg $L$-functions together with standard methods of analytic number theory.

4. The geometric side of the Arthur trace formula

To study the asymptotic behaviour of the geometric side $J_{\text{geo}}(\tilde{\varphi}_{1}^{t})$ of the trace formula, we use the fine $\alpha$-expansion [4]

$$
J_{\text{geo}}(f) = \sum_{M \in \mathcal{L}} \sum_{\gamma \in (M(\mathbb{Q}_{S}))_{M,S}} a^{M}(S, \gamma) J_{M}(f, \gamma), \quad f \in C_{c}^{\infty}(G(\mathbb{A})),
$$

which expresses the distribution $J_{\text{geo}}(f)$ in terms of weighted orbital integrals $J_{M}(\gamma, f)$. Here $S$ is a finite set of places of $\mathbb{Q}$, and $(M(\mathbb{Q}_{S}))_{M,S}$ is a certain set of equivalence classes in $M(\mathbb{Q}_{S})$. This reduces our problem to the investigation of weighted orbital integrals. The key result is that

$$
\lim_{t \to 0} r_{M}(\tilde{\varphi}_{1}^{t}, \gamma) = 0,
$$

unless $M = G$ and $\gamma = \pm 1$. The contributions to (6) of the terms where $M = G$ and $\gamma = \pm 1$ are easy to determine. Set $\varepsilon_{Kf} = 1$, if $-1 \in Kf$ and $\varepsilon_{Kf} = 0$ otherwise. Using the behaviour of the heat kernel $h_{t}^{\nu}(\pm 1)$ as $t \to 0$, it follows that

$$
J_{\text{geo}}(\tilde{\varphi}_{1}^{t}) \sim \text{dim}(\tau) \frac{\text{vol}(G(\mathbb{Q}) \setminus G(\mathbb{A})^{1}/K_{f})}{(4\pi)^{d/2}}(4\pi)^{d/2}(1 + \varepsilon_{Kf}) t^{-d_{\mathbb{A}}/2}
$$

as $t \to 0$.

5. Proof of the main theorem

By the trace formula (2) we have $J_{\text{spec}}(\tilde{\varphi}_{1}^{t}) = J_{\text{geo}}(\tilde{\varphi}_{1}^{t})$, $t > 0$. Using (4) and (8), it follows that

$$
\sum_{\pi \in \Pi_{\text{dis}}(G(\lambda), \xi_{0})} m(\pi) \text{dim}(\mathcal{H}_{\pi_{\mathbb{A}}}) \text{dim}(\mathcal{H}_{\pi_{\infty}} \otimes V_{\tau})^{0(n)} e^{\lambda_{s}}
$$

$$
\sim \text{dim}(\tau) \frac{\text{vol}(G(\mathbb{Q}) \setminus G(\mathbb{A})^{1}/K_{f})}{(4\pi)^{d/2}}(1 + \varepsilon_{Kf}) t^{-d_{\mathbb{A}}/2}
$$

as $t \to 0$. Using [5] and [8] it follows that in (9) one can replace $\Pi_{\text{dis}}(G(\lambda), \xi_{0})$ by $\Pi_{\text{cus}}(G(\lambda), \xi_{0})$ and the same asymptotic formula remains true. Then Theorem 2.1 is an immediate consequence of Karamata’s theorem [6, p. 446]. As explained above, Theorem 2.1 implies Theorem 1.1.
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