Fractal mesh refinement, rare events and type Ia supernova

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For the study of rare events, we propose a method of fractal mesh refinement, allowing high levels of strongly focused resolution. The method is proposed to assess the extreme events generated by multifractal turbulent nuclear deflagration. Such events, in a white dwarf type Ia supernova progenitor, are assumed to lead to a deflagration to detonation transition, which produces the observed type Ia supernova.

Fractal mesh refinement enables mesh refinement regimes to the Gibson scale and beyond. Conventional multifractal subgrid models have cost considerations which also lead to local mesh refinement, but more importantly, the application of these models to compressible turbulent deflagration fronts is a topic for future research, while subgrid models tuned to this complex physics lack a multifractal search focus, and for cost reasons do not allow the large search volumes required to find the sought for rare event to trigger a DDT.

Here we propose methods to resolve fine scales and locate rare possible DDT trigger events within large volumes while addressing multifractal issues at a feasible computational cost. We are motivated by the goal of confirming or assessing the hypothesis of a delayed detonation in type Ia supernova and to assessing the delay in this event if it is found to occur.

I. INTRODUCTION

Type Ia super nova provide a major method to assess the distribution of mass and relative velocities at large distances in the universe \[4\]. They are formed from a white dwarf which becomes unstable to nuclear synthesis in a deflagration front while accreting mass. Due to the very large amounts of energy released, turbulence ensues and the deflagration fronts become unstable. Available evidence suggests a delayed deflagration to detonation transition (DDT). Triggering the DDT event may be intense fluctuations in the turbulence intensity and flame front complexity occurring over a critical volume. Such a trigger is a rare event. It is likely to be located in a fractal subset of dimension \(D < 4\) of space time, making the cost of a search for the trigger prohibitively expensive in the absence of a focused search strategy. Definitive confirmation of the DDT possibility is missing.

The phenomena of Ia supernova are summarized in recent simulation studies \[2, 7\] from which further references can be traced. Questions remain concerning the type of progenitor system, the nature of the burning front and its location within the white dwarf progenitor, the detailed mechanism leading to a transition for deflagration to detonation (DDT) and whether this is even possible within common models for white dwarf progenitors.

The detailed mechanism for DDT is presumed to be pressure waves arising from some local combustion or reaction event of extreme intensity within a localized region. The pressure waves generated by this event lower the ignition temperature of the region they reach, and if the thermodynamic conditions are very close to ignition, a wide spread ignition and an explosion results according to the Zeldovich theory \[2\].

The trigger for DDT is presumed to be a burning front of high turbulent intensity embedded in a larger region which is close to the ignition temperature. The burning intensity is primarily determined by the length of the burning front, so that the trigger can be postulated as a local extreme of the burning front length. The burning front is unstable to wrinkling at scales above the Gibson scale, and the instability has been estimated to lead to a fractal burning front with a spatial dimension \(D_f \sim 2.3 > 2\). The local intensity should increase when multifractal (multiple simultaneous length scales for wrinkling) are considered, leading to a smaller fractal dimension and a more intense localized combustion hot spot.

Simulation studies depend sensitively on subgrid models, which introduce subgrid properties of the turbulent mixing of burned and unburned stellar material, the turbulent adjusted flame speed and the turbulent deflagration. It is to the subgrid modeling that this short note is addressed. A major complication to subgrid modeling is the juxtaposition of multifractal turbulence coupled to complex physical processes. This subgrid complication is common to a wide range of problems in computational physics and engineering.

The multifractal physics of turbulence \[2\] is associated with isolated or rare events of turbulent intensity due to clustering of turbulence volatility. Not only do regions of high turbulent intensity cluster, but there is a compound clustering, so that the clusters themselves cluster. This compound clustering extends to all orders, with a clustering of clusters of clusters, etc. This clustering hierarchy of turbulent intensity is called multifractal turbulence. Its overall strength and importance is measured by the energy dissipation rate \(\epsilon\), which is large in the supernova context. The decrease in fractal dimension of these compound clustering events saturates at some limiting dimension \(D_\infty > 2\), and is nearly constant beyond multiple clustering of order 8 or so.

The synthetic reconstructions of turbulent velocity fields or velocity gradients which follow this multifractal statistics are called surrogate models (of multifractal turbulence), see \[2\] and references cited there. The sur-
rogate models serve as subgrid models to a hydro simulation, and introduce the intense turbulent effects as a stochastic sub grid scale (stochastic SGS) model. Such models also have a computational cost, when extended a few mesh levels beyond the feasible hydro mesh. A larger issue is the fact that, while well developed and validated for studies of single fluid incompressible turbulence, the extension of surrogate turbulence models to compressible turbulent deflagration does not yet exist.

Physics adapted subgrid modeling of turbulent compressible deflagration fronts [2] has been developed, but fractal and multifractal aspects of the modeling are not included.

Combining the multifractal incompressible single fluid stochastic subgrid models with the physically moti-vated compressible turbulent deflagration front modeling “head-on” is a major multi year intellectual enterprise, and is not attempted here. Rather we proceed in the spirit of importance sampling, commonly used in Monte Carlo studies of rare events.

We localize the search to the regions of most importance. Due to the narrower focus, the cost is reduced and so further levels of mesh refinement are feasible, probably to the Gibson scale and smaller. Of equal importance with the added resolution is the localization of the search within a promising fractal set, thereby circumventing the impossible requirement for large search volumes at high resolution.

We explore an aggressive mesh refinement program which allows the hydro solvers to achieve both objectives simultaneously, the high resolution and the search location restricted within a fractal set, with a control over the total cost of the simulation. The refined resolution of events of high turbulent intensity, those of high order multifractal clustering, are identified in a hierarchical manner.

As with importance sampling, control of computational cost is a central issue. Our strategy is a sub case of conventional automatic mesh refinement (AMR), which is to refine wherever solution accuracy indicates a need. We call the proposed method fractal mesh refinement (FMR).

Depending on parameters chosen for FMR, we can achieve an order of magnitude of improved resolution, to the Gibson scale, and finer. The Gibson scale, about $10^4$ cm, is the smallest scale at which the turbulence causes the flame front to wrinkle. However, the flame fronts may well contain closely spaced but smooth and nearly parallel regions of alternating burned and unburned material. For example, if a flame front is wound around a vortex spiral, the front may be relatively smooth (below the Gibson limit), but with narrow spacing between neighboring fronts. For this scenario, the separation is limited by the flame width, suggesting that turbulence and combustion phenomena below the Gibson scale could be important to DDT.

The search for intense turbulent fluctuation is sensitive to the volume over which the search occurs, and thus to the time of the search, as the laminar spherically symmetric flame from moves outward in the white dwarf. Thus the methods proposed have the potential not only to assess the delayed DDT, but to yield an estimate on the delay itself. It is evident that application of this methodology would not yield a consistent and deterministic (high probability) DDT event when applied at the very beginnings of the deflagration, due to the small volumes involved. Thus some level of delay appears to be required. The FMR method is applicable to other type Ia supernova scenarios. It can be used as a verification and calibration tool for conventional subgrid models in regions of intense turbulence. More broadly, FMR is applicable to problems which require assessing extreme events, assuming only that a knowledge of the nature of the event as observed at each length scale is known.

II. FMR PARAMETERS AND COSTS

AMR is a numerical algorithm and software tool that allows refinement of selected space time mesh cells in a regular rectangularly gridded domain [2]. Normally the refinement is by factors of 2. The power of the method is its application to successive levels of refinement, so that refined cells are further refined, as needed. There is a restriction that adjacent cells should differ at most by a single level of refinement. This tapering of the refinement levels introduces a major cost factor into our estimates. If an isolated cell is refined once, there is no additional refinements needed, but if one of its sub cells is refined again, any neighbor cell must be at least singly refined, if not refined already. For the single isolated double refined cell, there are $2^4 = 16$ singly refined cells (including the parent of the doubly refined cell) to be refined.

We start our cost estimates with a base simulation of cost $b$. As an FMR parameter, we choose a fraction $f$ of the cells at each stage to be refined. Applying this fraction to the base simulation, we label a fraction $f$ of the cells as refined (at level zero), and with a cost still $b$, so that the cost multiple is 1. At the first level of refinement, a fraction $f$ of the zero level refined cells are refined once. The number of cells is $f^3$ and the cost is proportional to 16, the number of space time sub cells produced by the refinement. We let $c_0$ be this proportionality factor and $\epsilon = 16 f$, so that the level one refinement cost is $b c_0 f^3$. At level one, there is no secondary refinement of cells to satisfy the adjacency requirement.

At level 2, $f^3$ cells are refined. The cost is $b f(c_0 \epsilon)^2$. At level $n$, the cost is $b f(c_0 \epsilon)^n$. $c_0$ is not small, so that the series is divergent, and only a finite number of levels of refinement are allowed. But the factor $\epsilon$ is normally chosen to be small, so that the cost per refinement level is substantially reduced below that of AMR. The result is a significant increase in the number of refinement levels allowed.

The cost of FMR relative to AMR depends on the fraction of cells to be refined in AMR at each level of refine-
ment. As this fraction is problem dependent, no universal estimate is possible. However, with the ability to choose \( f \) as small, for example \( f = .01 \) or \( f = .001 \), and with the possibility of applying this construction within the normal AMR sequence as well as as an extension of it, it is likely that FMR will allow a significant increase in the number of allowed levels of refinement. Beyond that, the refined cells, once resolved, are far more likely to be located in a designated fractal subspace that can serve as a trigger for DDT.

III. MULTIFRACTAL FMR AND TURBULENT DEFLAGRATION

A selection criteria based on turbulent intensity alone yields a fractal construction. Turbulence is a fractal set with dimension \( D < 3 \) and the fractal construction will yield refined subgrid cells within it. Multifractal constructions, and compound clustering of intensities, clusters of clusters, occur on smaller fractal sets, which mix turbulent intensities with disparate length scales. The construction of a multifractal algorithm is a simple modification of Sec. [1] We only need to change the definition of the filter function \( f \) from intensities of turbulence to intensities of multifractal turbulence. This is conveniently done through structure functions of order \( n \). We then define \( f \) to selects extreme values of this structure function.

The \( n \)th structure function \( S_n \) is a measure of high order statistical fluctuations in the velocity field \( u \), reflecting compound clustering or intermittancy of turbulent intensity. This clustering is concentrated on a fractal set of dimension \( \zeta_n \) according to the formula

\[
S_n = \langle (u(x + r) - u(x))^n \rangle = |r|^{\zeta_n}.
\]

Both the fractal and the multifractal definitions are turbulence centric and are better replaced with a combustion centric analysis. The heat release from a deflagration front is primarily proportional to the length of the flame front. Above the Gibson scale, the flame front is wrinkled and is itself a fractal, thus having a fractal dimension \( D_f > 2 \). By a modification of the criteria used to define the filter function \( f \), we ensure that the mesh refined regions lie on this fractal set, with its enhanced rate of heat release. The wrinkled flame front itself may have a multifractal structure, so that compound short wave length flame front convolutions and longer wave length ones may be superimposed, and with the contributions from multiple wave lengths of fluctuations in the compound clustering of multifractal fluctuations. The final proposal is to choose \( f \) to be located in the fractal set defined by a high order of multifractal wrinkling.

Fractal (but not multifractal) analysis of flame fronts is examined both experimentally and in simulation [1]. Fractal dimensions for the flame front in the range 2.18 to 2.35 are found. A more turbulent flame front regime of a broken, not wrinkled flame front is cited, but not further explored.

The multifractal nature of the deflagration front and the multifractal nature of the turbulence are related, as the turbulent fluctuations drive the convolutions of the deflagration front.

Which of these fractal sets (or some new one to be delineated) will actually contain a trigger for DDT remains to be explored. Such a test will contribute to assessing whether the conventional view of DDT is actually correct.

IV. CONCLUSIONS

We propose a method of fractal mesh refinement, (FMR), allowing high levels of mesh resolution of extreme events. We have proposed events that could lead to a trigger for the initiatation of DDT in a type Ia pregenitor. We propose its use to assess a hypothesized scenario for DDT in a type Ia supernova as well as other type Ia supernova scenarios. Calibration of conventional subgrid models in regions of intense turbulence is another application. FMR is applicable more generally to computational physics and applied science problems which require assessing extreme events.

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