A CESÀRO AVERAGE FOR AN ADDITIVE PROBLEM WITH PRIME POWERS

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Abstract. In this paper we extend and improve our results on weighted averages for the number of representations of an integer as a sum of two powers of primes (the paper of the authors in Forum Math. 27 (2015), see also the paper of A.L., Riv. Mat. Univ. di Parma 7 (2016), Theorem 2.2). Let $1 \leq \ell_1 \leq \ell_2$ be two integers, $\Lambda$ be the von Mangoldt function and $r_{\ell_1,\ell_2}(n) = \sum_{m_1^{\ell_1} + m_2^{\ell_2} = n} \Lambda(m_1)\Lambda(m_2)$ be the weighted counting function for the number of representation of an integer as a sum of two prime powers. Let $N \geq 2$ be an integer. We prove that the Cesàro average of weight $k > 1$ of $r_{\ell_1,\ell_2}$ over the interval $[1, N]$ has a development as a sum of terms depending explicitly on the zeros of the Riemann zeta-function.

1. Introduction. We continue our recent work on the number of representations of an integer as a sum of primes. In [7] we studied the average number of representations of an integer as a sum of two primes, whereas in [8] we considered individual integers. In [10], see also Theorem 2.2 of [6], we studied a Cesàro weighted partial explicit formula for Goldbach numbers. Here we generalise and improve this last result by working on the Cesàro weighted counting function for the number of representation of an integer as a sum.
of two prime powers. We let $1 \leq \ell_1 \leq \ell_2$ be two integers and set
\[ r_{\ell_1,\ell_2}(n) = \sum_{m_1^{\ell_1} + m_2^{\ell_2} = n} \Lambda(m_1)\Lambda(m_2). \]

We also use the following convenient abbreviations for the various terms of the development:
\begin{align*}
\mathcal{M}_{1,k,\ell_1,\ell_2}(N) &= \frac{N^{1/\ell_1+1/\ell_2}}{\Gamma(k+1+1/\ell_1+1/\ell_2)} \frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1\ell_2}, \\
\mathcal{M}_{2,k}(N) &= \log^2(2\pi) \frac{\Gamma(1)}{\Gamma(k+1)}, \\
\mathcal{M}_{3,k,\ell}(N) &= -\log(2\pi) \frac{N^{1/\ell}}{\Gamma(k+1+1/\ell)} \frac{\Gamma(1/\ell)}{\ell + \log(2\pi) \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho/\ell)}} \frac{N^{\rho/\ell}}{\Gamma(k+1+\rho/\ell)}, \\
\mathcal{M}_{4,k,\ell_1,\ell_2}(N) &= -N^{1/\ell_2} \frac{\Gamma(1/\ell_2)}{\ell_2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+1/\ell_2+\rho/\ell_2)} \frac{N^{\rho_1/\ell_1+\rho_2/\ell_2}}{\Gamma(k+1+\rho_1/\ell_1+\rho_2/\ell_2)}. \\
\mathcal{M}_{5,k,\ell_1,\ell_2}(N) &= \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(k+1+\rho_1/\ell_1+\rho_2/\ell_2)} N^{\rho_1/\ell_1+\rho_2/\ell_2}. \\
\end{align*}

Here $\rho$, with or without subscripts, runs over the non-trivial zeros of the Riemann zeta-function $\zeta$ and $\Gamma$ is Euler’s function. The main result of the paper is the following theorem.

**Theorem 1.** Let $1 \leq \ell_1 \leq \ell_2$ be two integers, and $N$ be a positive integer. For $k > 1$ we have
\[
\sum_{n \leq N} r_{\ell_1,\ell_2}(n) \left(1 - n/N\right)^k = \mathcal{M}_{1,k,\ell_1,\ell_2}(N) + \mathcal{M}_{2,k}(N) + \mathcal{M}_{3,k,\ell_1}(N) + \mathcal{M}_{3,k,\ell_2}(N) + \mathcal{M}_{4,k,\ell_1,\ell_2}(N) + \mathcal{M}_{4,k,\ell_2,\ell_1}(N) + \mathcal{M}_{5,k,\ell_1,\ell_2}(N) + \mathcal{O}_{k,\ell_1,\ell_2}(N^{-1/2+1/\ell_1}).
\]

Clearly, depending on the size of $\ell_1, \ell_2$, some of the previous listed terms should be included in the error term. We remark that the double series over zeros in (3) converges absolutely for $k > 1/2$, and it seems reasonable to believe that the stated equality holds for the same values of $k$, possibly with a weaker error term, although the bound $k > 1$ appears in several places of the proof and it seems to be the limit of the method.

Theorem [1] generalises and improves our Theorem in [10], see also Theorem 2.2 of [6], which corresponds to the case $\ell_1 = \ell_2 = 1$. In fact in this case Theorem [1] leads to
\[
\sum_{n \leq N} r_G(n) \left(1 - n/N\right)^k = \frac{N^2}{\Gamma(k+3)} - 2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{\rho+1} - 2 \log(2\pi) \frac{N}{\Gamma(k+2)} + 2 \log(2\pi) \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{\rho} + \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(k+1+\rho_1+\rho_2)} N^{\rho_1+\rho_2} + \mathcal{O}(N^{1/2}), \tag{4}
\]
where $r_G(n) = r_{1,1}(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$, that is, we are now able to detect the term $\mathcal{M}_{3,k,1}$. Very recently Brüdern, Kaczorowski and Perelli [2] proved that (4) holds
for every \( k > 0 \). We point out that Theorem 1 covers other interesting and classical cases like the sum of two prime squares \((\ell_1 = \ell_2 = 2)\) or a prime and a prime square \((\ell_1 = 1, \ell_2 = 2)\).

We recall that our method is based on a formula due to Laplace \cite{12}, namely
\[
\frac{1}{2\pi i} \int_{(a)} v^{-s} e^{v} \, dv = \frac{1}{\Gamma(s)},
\]
where \( \Re(s) > 0 \) and \( a > 0 \), see, e.g., formula 5.4(1) on page 238 of \cite{3}. We will need the general case of (5), which can be found in de Azevedo Pribitkin \cite{1}, formulae (8) and (9):
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{iD u}}{(a + i u)^{s}} \, du = \begin{cases} 
D^{s-1} e^{-aD} / \Gamma(s) & \text{if } D > 0, \\
0 & \text{if } D < 0,
\end{cases}
\]
which is valid for \( \sigma = \Re(s) > 0 \) and \( a \in \mathbb{C} \) with \( \Re(a) > 0 \), and
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(a + i u)^{s}} \, du = \begin{cases} 
0 & \text{if } \Re(s) > 1, \\
1/2 & \text{if } s = 1,
\end{cases}
\]
for \( a \in \mathbb{C} \) with \( \Re(a) > 0 \). Formule (6)–(7) enable us to write averages of arithmetical functions by means of line integrals as we will see in \S 2 below.

The improvement we get in Theorem 1 follows using Lemma 1 below, which is a generalised and refined version of Lemma 4.1 of \cite{10}, see also Lemma 5.1 of \cite{6}. In fact Lemma 1 can be also used to generalise and improve our result in \cite{9} about the Hardy–Littlewood numbers to the \( p\ell + m^2, \ell \geq 1 \), problem; we will discuss this case in \cite{11}.

2. Settings. Let \( \ell \geq 1, 1 \leq \ell_1 \leq \ell_2 \) be integer numbers and
\[
\tilde{S}_\ell(z) = \sum_{m \geq 1} \Lambda(m) e^{-m \ell z},
\]
where \( z = a + i y \) with \( y \in \mathbb{R} \) and real \( a > 0 \). Moreover let us define the density of the problem as
\[
\lambda = 1/\ell_1 + 1/\ell_2.
\]

We recall that the Prime Number Theorem (PNT) is equivalent to the statement
\[
\tilde{S}_\ell(a) \sim \frac{\Gamma(1/\ell)}{\ell a^{1/\ell}} \quad \text{for } a \to 0+.
\]
By (8) we have
\[
\tilde{S}_{\ell_1}(z)\tilde{S}_{\ell_2}(z) = \sum_{n \geq 1} r_{\ell_1, \ell_2}(n) e^{-nz}.
\]
Hence, for \( N \in \mathbb{N} \) with \( N > 0 \) and \( a > 0 \) we have
\[
\frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}_{\ell_1}(z)\tilde{S}_{\ell_2}(z) \, dz = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \sum_{n \geq 1} r_{\ell_1, \ell_2}(n) e^{-nz} \, dz.
\]
Since
\[
\sum_{n \geq 1} |r_{\ell_1, \ell_2}(n) e^{-nz}| = \tilde{S}_{\ell_1}(a)\tilde{S}_{\ell_2}(a) \asymp_{\ell_1, \ell_2} a^{-\lambda}
\]
by \([10]\), where \(f \asymp g\) means \(g \ll f \ll g\), we can exchange the series and the line integral in \((11)\) provided that \(k > 0\). In fact, if \(z = a + iy\), taking into account the estimate
\[
|z|^{-1} \asymp \begin{cases} a^{-1} & \text{if } |y| \leq a, \\ |y|^{-1} & \text{if } |y| \geq a, \end{cases} \tag{12}
\]
we have
\[
|e^{Nz}z^{-k-1}| \asymp e^{Na} \begin{cases} a^{-k-1} & \text{if } |y| \leq a, \\ |y|^{-k-1} & \text{if } |y| \geq a, \end{cases}
\]
and hence, recalling \([10]\), we obtain
\[
\int_{(a)} |e^{Nz}z^{-k-1}| \sum_{n \geq 1} r_{\ell_1, \ell_2}(n) e^{-nz} \, |dz| \ll a^{-\lambda}e^{Na} \left[ \int_{0}^{a} a^{-k-1} \, dy + \int_{a}^{+\infty} y^{-k-1} \, dy \right],
\]
which is \(\ll k a^{-\lambda-\kappa}e^{Na}\), but the rightmost integral converges only for \(k > 0\). Using \([6]\) for \(n \neq N\) and \([7]\) for \(n = N\), we see that for \(k > 0\) the right-hand side of \((11)\) is
\[
= \sum_{n \geq 1} r_{\ell_1, \ell_2}(n) \left( \frac{1}{2\pi i} \int_{(a)} e^{(N-n)z}z^{-k-1} \, dz \right) = \sum_{n \leq N} r_{\ell_1, \ell_2}(n) \frac{(N-n)^k}{\Gamma(k+1)}.
\]

**Remark.** As in \([10]\) the previous computation reveals that we cannot get rid of the Cesàro weight in our method since, for \(k = 0\), it is not clear whether the integral on the right-hand side of \((11)\) converges absolutely or not.

Summing up, for \(a > 0\) and \(k > 0\) we have
\[
\sum_{n \leq N} r_{\ell_1, \ell_2}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz}z^{-k-1} \tilde{S}_{\ell_1}(z) \tilde{S}_{\ell_2}(z) \, dz,
\]
where \(N \in \mathbb{N}\) with \(N > 0\). This is the fundamental relation for the method.

### 3. Inserting zeros.

In this section we need \(k > 1\). By Lemma \([1]\) below we have
\[
\tilde{S}_{\ell}(z) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma \left( \frac{\rho}{\ell} \right) - \log(2\pi) + E(a, y, \ell) = M(\ell, z) + E(a, y, \ell),
\]
say, where \(E(a, y, \ell)\) satisfies \((16)\). Hence
\[
\tilde{S}_{\ell_1}(z) \tilde{S}_{\ell_2}(z) = M(\ell_1, z)M(\ell_2, z) + E(a, y, \ell_1)E(a, y, \ell_2) + E(a, y, \ell_2)M(\ell_1, z) + E(a, y, \ell_1)M(\ell_2, z).
\]

We have \(|M(\ell, z)| = |\tilde{S}_{\ell}(z) - E(a, y, \ell)| \leq \tilde{S}_{\ell}(a) + |E(a, y, \ell)| \ll a^{-1/\ell} + |E(a, y, \ell)|\) by \((10)\) again, so that
\[
\tilde{S}_{\ell_1}(z) \tilde{S}_{\ell_2}(z) = M(\ell_1, z)M(\ell_2, z) + O_{\ell_1, \ell_2} \left( |E(a, y, \ell_1)E(a, y, \ell_2)| \right) + O_{\ell_1, \ell_2} \left( |E(a, y, \ell_2)|a^{-1/\ell_1} + |E(a, y, \ell_1)|a^{-1/\ell_2} \right), \tag{13}
\]
choosing $0 < a \leq 1$, since $1 \leq \ell_1 \leq \ell_2$. Recalling (12) and taking into account (16), for $k > 1$ we have

$$
\int_{(a)} |E(a, y, \ell_1)E(a, y, \ell_2)| |e^{Nz}| |z|^{-k-1} |dz|
\ll \ell_1, \ell_2 e^{Na} \int_0^a a^{-k} dy + e^{Na} \int_a^{+\infty} y^{-k}(1 + \log^2(y/a))^2 dy
\ll_k, \ell_1, \ell_2 e^{Na} a^{-k+1} + e^{Na} a^{-k+1} \int_1^{+\infty} v^{-k}(1 + \log^2 v)^2 dv \ll_k, \ell_1, \ell_2 e^{Na} a^{-k+1}.
$$

If we choose $a = 1/N$, the error term is $\ll_k, \ell_1, \ell_2 N^{k-1}$ for $k > 1$. For $a = 1/N$, by (12) and (16), the second remainder term in (13) for $k > 1/2$ is

$$
\ll_k, \ell_1, \ell_2 N^{1/\ell_1} \int_{(1/N)} |E(y, 1/N, \ell_2)| |e^{Nz}| |z|^{-k-1} |dz|
\ll_k, \ell_1, \ell_2 N^{1/\ell_1} \int_0^{1/N} N^{k+1/2} dy + N^{1/\ell_1} \int_{1/N}^{+\infty} y^{-k-1/2} \log^2(Ny) dy
\ll_k, \ell_1, \ell_2 N^{k-1/2+1/\ell_1} + N^{k-1/2+1/\ell_1} \int_1^{+\infty} v^{-k-1/2} \log^2 v dv \ll_k, \ell_1, \ell_2 N^{k-1/2+1/\ell_1}.
$$

Analogously, it is easy to see that the remaining term is $\ll_k, \ell_1, \ell_2 N^{k-1/2+1/\ell_2}$.

With a little effort we can give an explicit dependence on $k$ for the implicit constants in the last three estimates.

Hence, by (9) and (11) we have

$$
\sum_{n \leq N} r_{\ell_1, \ell_2}(n) \frac{(N - n)^k}{\Gamma(k + 1)}
= \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} M(\ell_1, z)M(\ell_2, z) dz + O_{k, \ell_1, \ell_2}(N^{k-1/2+1/\ell_1})
= I_1(N; \ell_1, \ell_2, k) + I_2(N; k) + I_3(N; \ell_1, k) + I_3(N; \ell_2, k)
+ I_4(N; \ell_1, \ell_2, k) + I_4(N; \ell_1, \ell_1, k) + I_5(N; \ell_1, \ell_2, k) + O_{k, \ell_1, \ell_2}(N^{k-1/2+1/\ell_1}),
$$

say, where

$$
I_1(N; \ell_1, \ell_2, k) = \frac{1}{2\pi i} \frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1\ell_2} \int_{(1/N)} e^{Nz} z^{-k-1-\lambda} dz,
I_2(N; k) = \frac{\log^2(2\pi)}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} dz,
I_3(N; \ell, k) = \frac{\log(2\pi)}{2\pi i} \left\{ -\frac{\Gamma(1/\ell)}{\ell} \int_{(1/N)} e^{Nz} z^{-k-1-\ell} dz
+ \int_{(1/N)} e^{Nz} z^{-k-1} \sum_\rho z^{-\rho/\ell} \Gamma(\frac{\rho}{\ell}) dz \right\},
I_4(N; \ell_1, \ell_2, k) = -\frac{1}{2\pi i} \frac{\Gamma(1/\ell_1)}{\ell_1} \int_{(1/N)} e^{Nz} z^{-k-1-\ell_1} \sum_\rho z^{-\rho/\ell_2} \Gamma(\frac{\rho}{\ell_2}) dz,
$$
We also recall that, uniformly for a > 0, we have
\[ e^{Nz}z^{-k-1} \sum_{\rho_1} \sum_{\rho_2} z^{-\rho/\ell_1 - \rho/\ell_2} \Gamma\left(\frac{\rho_1}{\ell_1}\right) \Gamma\left(\frac{\rho_2}{\ell_2}\right) dz. \]
The evaluation of the integrals \( I_j \) is a straightforward application of (5) with \( s = Nz \), except that the interchange of the series with the integrals needs to be justified: see §3.7 for a proof that this is in fact permitted when \( k > 1 \). The proof that the double sum over zeros converges absolutely for \( k > 1/2 \) is given in §3 below. Combining the resulting expressions and dividing through by \( N^k \) we get Theorem 1.

4. Lemmas. We recall some basic facts in complex analysis. First, if \( z = a + iy \) with \( a > 0 \), we see that for complex \( w \) we have
\[ z^{-w} = |z|^{-w} \exp(-iw \arctan(y/a)) = |z|^{-\Re(w) - i\Im(w)} \exp((-i\Re(w) + \Im(w)) \arctan(y/a)), \]
so that
\[ |z^{-w}| = |z|^{-\Re(w)} \exp(\Im(w) \arctan(y/a)). \] (14)
We also recall that, uniformly for \( x \in [x_1, x_2] \), with \( x_1 \) and \( x_2 \) fixed, and for \( |y| \to +\infty \), by the Stirling formula (see, e.g., Titchmarsh [14 §4.42]) we have
\[ |\Gamma(x + iy)| \sim \sqrt{2\pi e^{-\pi|y|^2}} |y|^{-1/2}. \] (15)

The following lemma generalises and improves Lemma 4.1 of [10], see also Lemma 5.1 of [6]. The improvement depends on the fact that the constant term \( \log(2\pi) \) is now explicit since we realised that, in the application, this term leads, in some cases, to a non-trivial contribution in the final result. We follow the line of the proof in [10], but, in some cases, the integration path has to be changed; for clarity we repeat the whole argument.

**Lemma 1.** Let \( \ell \geq 1 \) be an integer, \( z = a + iy \), where \( a > 0 \) and \( y \in \mathbb{R} \). Then
\[ \tilde{S}_\ell(z) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma\left(\frac{\rho}{\ell}\right) \log(2\pi) + E(a, y, \ell), \]
where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \) and
\[ E(a, y, \ell) \ll \ell |z|^{1/2} \begin{cases} 1 & \text{if } |y| \leq a, \\ 1 + \log^2(|y|/a) & \text{if } |y| > a. \end{cases} \] (16)

**Proof.** Following the line of Hardy and Littlewood, see [4] §2.2, [5] Lemma 4] and of §4 in Linnik [13], we have
\[ \tilde{S}_\ell(z) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma\left(\frac{\rho}{\ell}\right) - \frac{\ell^4}{\zeta(0)} - \sum_{m=1}^{\ell/4} \Gamma\left(-\frac{2m}{\ell}\right) z^{2m/\ell} \]
\[ - \frac{1}{2\pi i} \int_{L_\ell} \frac{\zeta'(w)}{\zeta(w)} \Gamma(w) z^{-w} dw, \] (17)
where \( L_\ell \) is the vertical line \( \Re(w) = -1/2 \) if \( 4 \nmid \ell \) and it is \( \{-1/2 + it: |t| > C\} \cup \{-1/2 + it: 1/\ell \leq |t| \leq C\} \cup \gamma_\ell \) otherwise, \( C > 1/\ell \) is an absolute constant to be chosen later and \( \gamma_\ell \) is the right half-circle centred in \(-1/2\) of radius \( 1/\ell \).
Now we estimate the integral in (17). Assume $4 \nmid \ell$. Writing $w = -1/2 + it$, we have $|\zeta'/\zeta(\ell w)| \ll \log(|t| + 2)|z^{1/2}| \geq |z|^{1/2} \exp(t \arctan(y/a))$ by (14) and, for $|t| > C$, $\Gamma(w) \ll |t|^{-1}\exp(-\frac{\pi}{2}|t|)$ by (15). Letting $L_C = \{-1/2 + it: |t| > C\}$ we have

$$
\int_{L_C} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w} \, dw \ll_{\ell} |z|^{1/2} \int_{L_C} \frac{\log|t|}{|t|} \exp\left(-\frac{\pi}{2}|t| + t \arctan(y/a)\right) \, dt.
$$

If $ty \leq 0$ we call $\eta$ the quantity $\frac{\pi}{2} + |\arctan(y/a)| \in [\pi/2, \pi)$. If $|y| \leq a$ we define $\eta$ as $\frac{\pi}{2} - \arctan(y/a) > \frac{\pi}{2} - \arctan(1) = \frac{\pi}{4}$. In the remaining case ($|y| > a$ and $ty > 0$) we set $\eta = \arctan(a/|y|) \gg a/|y|$. Now fix $C$ such that $C\eta < 1$ (e.g., $C = 1/\pi$ is allowed). Letting $u = \eta t$, we get

$$
\int_{L_C} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w} \, dw \ll_{\ell} |z|^{1/2} \int_{C\eta}^{+\infty} e^{-ut} \frac{\log u}{u} \, du
$$

$$
= |z|^{1/2} \int_{C\eta}^{+\infty} e^{-ut} \frac{\log u}{u} \, du + |z|^{1/2} \log(1/\eta) \int_{C\eta}^{+\infty} e^{-ut} \, du = J_1 + J_2. \tag{18}
$$

We remark that $0 \leq u^{-1} \log u \leq e^{-1}$ for $u \geq 1$, since the maximum of $u^{-1} \log u$ is attained at $u = e$. Since

$$
0 \leq \int_{1}^{+\infty} e^{-ut} \frac{\log u}{u} \, du \leq e^{-1} \int_{1}^{+\infty} e^{-ut} \, du \ll 1
$$

and

$$
\left| \int_{C\eta}^{1} e^{-ut} \frac{\log u}{u} \, du \right| \leq \int_{C\eta}^{1} - \frac{\log u}{u} \, du = \left[-\frac{1}{2} \log^2 u\right]_{C\eta}^{1} \ll \log^2(1/\eta)
$$

we have $J_1 \ll |z|^{1/2} \log^2(1/\eta)$. For $J_2$ it is sufficient to remark that

$$
0 \leq J_2 \leq |z|^{1/2} \log(1/\eta) \left( \int_{C\eta}^{1} \frac{du}{u} + \int_{1}^{+\infty} e^{-ut} \, du \right) \ll |z|^{1/2} \log^2(1/\eta).
$$

Inserting the last two estimates in (18), recalling the definition of $\eta$, remarking that the integration over $|t| \leq C$ gives immediately a contribution $\ll_{\ell} |z|^{1/2}$, we get

$$
\int_{L_{\ell}} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w} \, dw \ll_{\ell} |z|^{1/2} \begin{cases} 1 & \text{if } |y| \leq a \\ 1 + \log^2(|y|/a) & \text{if } |y| > a. \end{cases}
$$

provided that $4 \nmid \ell$. Recalling $(\zeta'/\zeta)(0) = \log(2\pi)$ and remarking that

$$
\sum_{m=1}^{\ell/4} \Gamma\left(-\frac{2m}{\ell}\right) z^{2m/\ell} \ll_{\ell} |z|^{1/2}, \tag{19}
$$

we see that the case $4 \nmid \ell$ of the lemma is proved.

Assume now that $4 \mid \ell$. The computation over $L_C$ can be performed as in the previous case; we can also choose $C = 1/\pi$ as we did before. On the vertical segments $S$ given by $\Re(w) = -1/2$, $|\Im(w)| \in [1/\ell, C]$, we exploit the boundedness of the $\Gamma$-function and the estimate $|z^{-w}| \ll |z|^{1/2}$ which holds on $S$ since the argument of $z$ is bounded there. This gives

$$
\frac{1}{2\pi i} \int_{S} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w} \, dw \ll_{\ell} |z|^{1/2}.
$$
It remains to consider the contribution over $\gamma_\ell$; on this path we can again make use of the boundedness of the $\Gamma$-function and that $|z^{-w}| \ll |z|^{1/2}$ since the argument of $z$ is bounded on $\gamma_\ell$. This leads to
\[
\frac{1}{2\pi i} \int_{\gamma_\ell} \frac{\zeta'(w)}{\zeta} (\ell w) \Gamma(w) z^{-w} \, dw \ll |z|^{1/2}.
\]
Summing up, for $4 | \ell$ we see that the integral in (17) is dominated by the right hand side of (16) and this, together with (19) and $(\zeta'/\zeta)(0) = \log(2\pi)$, proves this case of the lemma.

We remark that, at the cost of some other complications in the details, Lemma 1 can be extended to the case $\ell \in \mathbb{R}$, $\ell > 0$.

In the next sections we will need to perform several times a set of similar computations; we collected them in the following two lemmas, which extend Lemmas 4.2 and 4.3 in [10].

**Lemma 2.** Let $\ell \geq 1$ be an integer, let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function and $\alpha > 1$ be a parameter. For any fixed $c \geq 0$ the series
\[
\sum_{\rho: \gamma > 0} \left(\frac{\gamma}{\ell}\right)^{\beta/\ell - 1/2} \int_{1}^{+\infty} (\log u)^c \exp\left(-\frac{\gamma}{\ell} \arctan \frac{1}{u}\right) \frac{du}{u^{\alpha + \beta/\ell}}
\]
converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge.

**Proof.** Setting $y = \arctan(1/u)$, for any real $\gamma > 0$ we have
\[
\int_{1}^{+\infty} \exp\left(-\frac{\gamma}{\ell} \arctan \frac{1}{u}\right) \frac{du}{u^{\alpha + \beta/\ell}} = \int_{0}^{\pi/4} \exp\left(-\frac{\gamma y}{\ell}\right) \frac{(\sin y)^{\alpha + \beta/\ell - 2}}{(\cos y)^{\alpha + \beta/\ell}} \, dy
\]
\[
\ll \alpha \int_{0}^{\pi/4} \exp\left(-\frac{\gamma y}{\ell}\right) y^{\alpha + \beta/\ell - 2} \, dy = \left(\frac{\gamma}{\ell}\right)^{1 - \alpha - \beta/\ell} \int_{0}^{\pi\gamma/(4\ell)} \exp(-w) w^{\alpha + \beta/\ell - 2} \, dw
\]
\[
\ll \alpha, \ell \left(\frac{\gamma}{\ell}\right)^{1 - \alpha - \beta/\ell} \left(\Gamma(\alpha - 1) + \Gamma(\alpha + 1/\ell - 1)\right),
\]
since $0 < \beta < 1$. This shows that the series over $\gamma$ converges for $\alpha > 3/2$. For $\alpha = 3/2$ essentially the same computation shows that the integral is $\gg \gamma^{-2 - \beta/\ell}$ and it is well known that in this case the series over zeros diverges.

**Lemma 3.** Let $\ell \geq 1$ be an integer, $\alpha > 1$, $z = a + iy$, $a \in (0, 1)$ and $y \in \mathbb{R}$. Let further $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function. We have
\[
\sum_{\rho} \left|\frac{\gamma}{\ell}\right|^{{\beta/\ell - 1/2}} \int_{Y_1 \cup Y_2} \exp\left(\frac{\gamma}{\ell} \arctan \frac{y}{a} - \frac{\pi}{2} \frac{\gamma}{\ell}\right) \frac{dy}{|z|^{\alpha + \beta/\ell}} \ll \alpha, \ell a^{1 - \alpha - 1/\ell},
\]
where $Y_1 = \{y \in \mathbb{R}: y\gamma \leq 0\}$ and $Y_2 = \{y \in [-a, a]: y\gamma > 0\}$. The result remains true if we insert in the integral a factor $(\log(|y|/a))^c$, for any fixed $c \geq 0$.

**Proof.** We first work on $Y_1$. By symmetry, we may assume that $\gamma > 0$. For $y \in (-\infty, 0]$ we have $(\gamma/\ell) \arctan(y/a) - \frac{\pi}{2} |\gamma/\ell| \leq -\frac{\pi}{2} |\gamma/\ell|$ and hence the quantity we are estimating
we may assume that $k > 1/2$ in this section. We need to establish the convergence of

$$\sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \exp\left(-\frac{\pi}{2} \frac{\gamma}{\ell} \right) \left[ e^{Nz} \right]_{Nz} \left| e^{Nz} \right|_{-k-1} \left| z^{\rho/\ell} \right| |dz|. \quad (20)$$

By (14) and the Stirling formula (15), we are left with estimating

$$\sum_{\rho: \gamma > 0} \left| \frac{\gamma}{\ell} \right|^{\beta/\ell - 1/2} \int_{\mathbb{R}} \exp\left(\frac{\gamma}{\ell} \arctan(Ny) - \frac{\pi}{2} \frac{\gamma}{\ell} \right) \left| z^{y^{k+1}/\ell} \right| \frac{dy}{|z|^{k+1 + \beta/\ell}}. \quad (21)$$

We have just to consider the case $\gamma y > 0$, $|y| > 1/N$ since in the other cases the total contribution is $\ll_N y^{k+1/\ell}$ by Lemma 3 with $\alpha = k + 1$ and $a = 1/N$. By symmetry, we may assume that $\gamma > 0$. We see that the integral in (21) is

$$\ll \sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \int_{1/N}^{+\infty} \exp\left(-\frac{\gamma}{\ell} \arctan \frac{1}{Ny} \right) \left| y^{k+1}/\ell \right| \frac{dy}{y^{k+1 + \beta/\ell}}.$$

For $k > 1/2$ this is $\ll_N y^{k+1/\ell}$ by Lemma 2. This implies that the integrals in (21) and in (20) are both $\ll_N y^{k+1/\ell}$ and hence the exchange steps for $I_3$ are fully justified.

5. Interchange of summation over zeros with the line integral in $I_3$. We need $k > 1/2$ in this section. We need to establish the convergence of

$$\sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \exp\left(-\frac{\pi}{2} \frac{\gamma}{\ell} \right) \left[ e^{Nz} \right]_{Nz} \left| e^{Nz} \right|_{-k-1} \left| z^{\rho/\ell} \right| |dz|. \quad (20)$$

By (14) and the Stirling formula (15), we are left with estimating

$$\sum_{\rho: \gamma > 0} \left| \frac{\gamma}{\ell} \right|^{\beta/\ell - 1/2} \int_{\mathbb{R}} \exp\left(\frac{\gamma}{\ell} \arctan(Ny) - \frac{\pi}{2} \frac{\gamma}{\ell} \right) \left| z^{y^{k+1}/\ell} \right| \frac{dy}{|z|^{k+1 + \beta/\ell}}. \quad (21)$$

We have just to consider the case $\gamma y > 0$, $|y| > 1/N$ since in the other cases the total contribution is $\ll_N y^{k+1/\ell}$ by Lemma 3 with $\alpha = k + 1$ and $a = 1/N$. By symmetry, we may assume that $\gamma > 0$. We see that the integral in (21) is

$$\ll \sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \int_{1/N}^{+\infty} \exp\left(-\frac{\gamma}{\ell} \arctan \frac{1}{Ny} \right) \left| y^{k+1}/u \right| \frac{du}{u^{k+1 + \beta/\ell}}.$$

For $k > 1/2$ this is $\ll_N y^{k+1/\ell}$ by Lemma 2. This implies that the integrals in (21) and in (20) are both $\ll_N y^{k+1/\ell}$ and hence the exchange steps for $I_3$ are fully justified.

6. Interchange of summation over zeros with the line integral in $I_4$. We need $k > 1/2 - 1/\ell_2$ in this section. We need to establish the convergence of

$$\sum_{\rho} \sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \exp\left(-\frac{\pi}{2} \frac{\gamma}{\ell} \right) \left[ e^{Nz} \right]_{Nz} \left| e^{Nz} \right|_{-k-1/\ell_2} \left| z^{\rho/\ell_1} \right| |dz|. \quad (22)$$
and of the case in which \( \ell_1 \) and \( \ell_2 \) are interchanged. By (14) and the Stirling formula (15), we are left with estimating

\[
\sum_{\rho} \left| \frac{\gamma}{\ell_1} \right|^{\beta/\ell_1 - 1/2} \int_{\mathbb{R}} \exp \left( \frac{\gamma}{\ell_1} \arctan(Ny) - \frac{\pi}{2} \frac{\gamma}{\ell_1} \right) \frac{dy}{|y|^{k+1+1/\ell_2+\beta/\ell_1}}. \tag{23}
\]

We have just to consider the case \( \gamma y > 0, |y| > 1/N \) since in the other cases the total contribution is \( \ll_{k,\ell_1,\ell_2} N^{k+\lambda} \) by Lemma 3 with \( \alpha = k + 1 + 1/\ell_2 \) and \( a = 1/N \). By symmetry, we may assume that \( \gamma > 0 \). We have that the integral in (23) is

\[
\ll_{\ell_1} \sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell_1} \right)^{\beta/\ell_1 - 1/2} \int^{+\infty}_{1/N} \exp \left( -\frac{\gamma}{\ell_1} \arctan \frac{1}{Ny} \right) \frac{dy}{y^{k+1+1/\ell_2+\beta/\ell_1}} \\
= N^{k+1/\ell_2} \sum_{\rho: \gamma > 0} N^{\beta/\ell_1} \left( \frac{\gamma}{\ell_1} \right)^{\beta/\ell_1 - 1/2} \int^{+\infty}_{1} \exp \left( -\frac{\gamma}{\ell_1} \arctan \frac{1}{u} \right) \frac{du}{u^{k+1+1/\ell_2+\beta/\ell_1}}.
\]

For \( k > 1/2 - 1/\ell_2 \) this is \( \ll_{k,\ell_1,\ell_2} N^{k+\lambda} \) by Lemma 2. This implies that the integrals in (23) and in (22) are both \( \ll_{k,\ell_1,\ell_2} N^{k+\lambda} \) and hence the exchange step for \( I_4 \) is fully justified.

7. Interchange of the double summation over zeros with the line integral in \( I_5 \).

We need \( k > 1 \) in this section. Arguing as in Sections 5–6, we first need to establish the convergence of

\[
\sum_{\rho_1} \left| \Gamma \left( \frac{\rho_1}{\ell_1} \right) \right| \int_{(1/N)} \left| \sum_{\rho_2} \Gamma \left( \frac{\rho_2}{\ell_2} \right) z^{-\rho_2/\ell_2} \right| e^{Nz} |z|^{-k-1} |z^{-\rho_1/\ell_1}| |dz| \tag{24}
\]

Using the Prime Number Theorem and (16), we first remark that

\[
\sum_{\rho_2} \Gamma \left( \frac{\rho_2}{\ell_2} \right) z^{-\rho_2/\ell_2} \ll_{\ell_2} N^{1/\ell_2} + |z|^{1/2} \log^2(2N|y|). \tag{25}
\]

By symmetry, we may assume that \( \gamma_1 > 0 \). By (25), (12), (14) and (9), for \( y \in (-\infty, 0] \) we are first led to estimate

\[
\sum_{\rho_1: \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta/\ell_1 - 1/2} \exp \left( -\frac{\pi}{2} \frac{\gamma_1}{\ell_1} \right) \left( \int_{-1/N}^{0} N^{k+1+1/\ell_2+\beta/\ell_1} dy \right) \\
+ N^{1/\ell_2} \int_{-\infty}^{-1/N} \frac{dy}{|y|^{k+1+\beta_1/\ell_1}} + \int_{-\infty}^{-1/N} \log^2(2N|y|) \frac{dy}{|y|^{k+1+1/\ell_2+\beta_1/\ell_1}} \ll_{k,\ell_1,\ell_2} N^{k+\lambda}
\]

by the same argument used in the proof of Lemma 3 with \( \alpha = k + 1/2 \) and \( a = 1/N \). On the other hand, for \( y > 0 \) we split the range of integration into \((0, 1/N] \cup (1/N, +\infty)\). By (25), (12) and Lemma 3 with \( \alpha = k + 1 \) and \( a = 1/N \), on \([0, 1/N] \) we have

\[
N^{1/\ell_2} \sum_{\rho_1: \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta/\ell_1 - 1/2} \int_{0}^{1/N} \exp \left( \frac{\gamma_1}{\ell_1} \left( \arctan(Ny) - \frac{\pi}{2} \right) \right) \frac{dy}{|y|^{k+1+\beta_1/\ell_1}} \ll_{k,\ell_1,\ell_2} N^{k+\lambda}.
\]
By symmetry, we can consider using standard zero-density estimates, (12) and (9). On the other hand, for \( \gamma_j = 1 \)

This implies that the integral in (24) is \( \ll k, \ell_1, \ell_2 N^{k+\lambda} \) provided that \( k > 1 \) and hence we can exchange the first summation with the integral in this case.

To exchange the second summation we have to consider

\[
\sum_{\rho_1: \gamma_1 > 0} \frac{\gamma_1}{\ell_1} \beta_{1/\ell_1 - 1/2} \int_{1/N}^{+\infty} \exp\left(-\gamma_1 \arctan \frac{1}{Ny} \right) \frac{N^{1/\ell_2} + y^{1/2} \log^2 (2Ny)}{y^{k+1+\beta_1/\ell_1}} \, dy
\]

\[
= N^k \sum_{\rho_1: \gamma_1 > 0} N^{\beta_1/\ell_1} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_{1/\ell_1 - 1/2}}
\]

\[
\times \int_{1}^{+\infty} \exp\left(-\gamma_1 \arctan \frac{1}{u} \right) \frac{N^{1/\ell_2} + u^{1/2} N^{-1/2} \log^2 (2u)}{u^{k+1+\beta_1/\ell_1}} \, du.
\]

Recalling Lemma 2 with \( \alpha = k + 1/2 \) shows that the last term is \( \ll k, \ell_1, \ell_2 N^{k+\lambda} \).

By symmetry, we can consider \( \gamma_1, \gamma_2 > 0 \) or \( \gamma_1 > 0, \gamma_2 < 0 \).

Assuming \( \gamma_1, \gamma_2 > 0 \), for \( y \leq 0 \) we have \( (\gamma_j/\ell_j) \arctan(Ny) - \frac{\pi}{2} (\gamma_j/\ell_j) \leq -\frac{\pi}{2} (\gamma_j/\ell_j) \), \( j = 1, 2 \), and, by (14), the corresponding contribution to (26) is \( \ll k, \ell_1, \ell_2 N^{k+\lambda} \) since

\[
\sum_{\rho_1: \gamma_1 > 0} \frac{\gamma_1}{\ell_1} \beta_{1/\ell_1 - 1/2} \exp\left(-\frac{\pi \gamma_1}{2 \ell_1} \right)
\]

\[
\times \sum_{\rho_2: \gamma_2 > 0} \frac{\gamma_2}{\ell_2} \beta_{2/\ell_2 - 1/2} \exp\left(-\frac{\pi \gamma_2}{2 \ell_2} \right) \left( \int_{-\infty}^{0} \frac{dy}{|z|^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}} \right)
\]

\[
\ll k N^{k+\lambda} \sum_{\rho_1: \gamma_1 > 0} \frac{\gamma_1}{\ell_1} \beta_{1/\ell_1 - 1/2} \exp\left(-\frac{\pi \gamma_1}{2 \ell_1} \right) \sum_{\rho_2: \gamma_2 > 0} \frac{\gamma_2}{\ell_2} \beta_{2/\ell_2 - 1/2} \exp\left(-\frac{\pi \gamma_2}{2 \ell_2} \right),
\]

using standard zero-density estimates, (12) and (9). On the other hand, for \( y > 0 \) we split the range of integration into \((0, 1/N] \cup (1/N, +\infty)\). On the first interval we have

\[
\sum_{\rho_1: \gamma_1 > 0} \frac{\gamma_1}{\ell_1} \beta_{1/\ell_1 - 1/2} \sum_{\rho_2: \gamma_2 > 0} \frac{\gamma_2}{\ell_2} \beta_{2/\ell_2 - 1/2}
\]

\[
\times \int_{0}^{1/N} \exp\left(\left(\frac{\gamma_1}{\ell_1} + \frac{\gamma_2}{\ell_2}\right) \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{dy}{|z|^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}
\]

\[
\ll \sum_{\rho_1: \gamma_1 > 0} \frac{\gamma_1}{\ell_1} \beta_{1/\ell_1 - 1/2} \sum_{\rho_2: \gamma_2 > 0} \frac{\gamma_2}{\ell_2} \beta_{2/\ell_2 - 1/2}
\]

\[
\times \exp\left(-\frac{\pi}{4} \left(\frac{\gamma_1}{\ell_1} + \frac{\gamma_2}{\ell_2}\right)\right) \int_{0}^{1/N} N^{k+1+\beta_1/\ell_1+\beta_2/\ell_2} \, dy
\]

\[
\ll k, \ell_1, \ell_2 N^{k+\lambda} \sum_{\rho_1: \gamma_1 > 0} \frac{\gamma_1}{\ell_1} \beta_{1/\ell_1 - 1/2} \exp\left(-\frac{\pi \gamma_1}{4 \ell_1} \right) \sum_{\rho_2: \gamma_2 > 0} \frac{\gamma_2}{\ell_2} \beta_{2/\ell_2 - 1/2} \exp\left(-\frac{\pi \gamma_2}{4 \ell_2} \right),
\]

which is also \( \ll k, \ell_1, \ell_2 N^{k+\lambda} \), by the same argument as above. With similar computations,
on the other interval we have
\[
\sum_{\rho_1 : \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_1/\ell_1 - 1/2} \sum_{\rho_2 : \gamma_2 > 0} \left( \frac{\gamma_2}{\ell_2} \right)^{\beta_2/\ell_2 - 1/2} 
\times \int_{1/N}^{+\infty} \exp \left( \frac{\gamma_1}{\ell_1} + \frac{\gamma_2}{\ell_2} \right) \left( \arctan(Ny) - \frac{\pi}{2} \right) \frac{dy}{y^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}} 
\]
\[
= N^k \sum_{\rho_1 : \gamma_1 > 0} N^{\beta_1/\ell_1} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_1/\ell_1 - 1/2} \sum_{\rho_2 : \gamma_2 > 0} N^{\beta_2/\ell_2} \left( \frac{\gamma_2}{\ell_2} \right)^{\beta_2/\ell_2 - 1/2} 
\times \int_{1}^{+\infty} \exp \left( - \left( \frac{\gamma_1}{\ell_1} + \frac{\gamma_2}{\ell_2} \right) \arctan \frac{1}{u} \right) \frac{du}{u^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}} .
\]

Arguing as in the proof of Lemma 2, we prove that the integral on the right is \( \asymp_{k,\ell_1,\ell_2} (\gamma_1 + \gamma_2)^{-k-\beta_1/\ell_1-\beta_2/\ell_2} \). The inequality
\[
\frac{\beta_1/\ell_1 - 1/2}{(\gamma_1 + \gamma_2)} \frac{\beta_2/\ell_2 - 1/2}{(\gamma_1 + \gamma_2)} \leq \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2}}
\]
shows, by using (9), that it is sufficient to consider
\[
N^k \sum_{\rho_1 : \gamma_1 > 0} \sum_{\rho_2 : \gamma_2 > 0} \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2} (\gamma_1 + \gamma_2)^k} N^{\beta_{k+\lambda}} \sum_{\rho_1 : \gamma_1 > 0} \sum_{\rho_2 : 0 < \gamma_2 \leq \gamma_1} \frac{1}{\gamma_2^{1/2}} \leq \asymp_{k,\ell_1,\ell_2} N^{k+\lambda} \sum_{\rho_1 : \gamma_1 > 0} \frac{\log \gamma_1}{\gamma_1^{k}}
\]
and the last series over zeros converges for \( k > 1 \).

Assume now \( \gamma_1 > 0, \gamma_2 < 0 \). For \( y \leq 0 \) we have \( \gamma_2 \arctan(Ny) - \frac{\pi}{2} \gamma_2 \leq - \frac{\pi}{2} \gamma_2 \), by (12) and (9), the corresponding contribution to (26) is
\[
\asymp_{k,\ell_1,\ell_2} \sum_{\rho_1 : \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_1/\ell_1 - 1/2} \exp \left( - \frac{\pi}{2} \frac{\gamma_1}{\ell_1} \right) 
\times \left\{ \sum_{\rho_2 : \gamma_2 < 0} \left| \frac{\gamma_2}{\ell_2} \right|^{\beta_2/\ell_2 - 1/2} \exp \left( - \frac{\pi}{4} \frac{\gamma_2}{\ell_2} \right) \right\} \int_{1/N}^{+\infty} \frac{dy}{y^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}} 
\]
\[
+ \int_{-\infty}^{-1/N} \exp \left( - \frac{\gamma_2}{\ell_2} \left( \arctan(Ny) + \frac{\pi}{2} \right) \right) \frac{dy}{|y|^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}} \} 
\]
\[
\asymp_{k,\ell_1,\ell_2} N^{k+\lambda} \sum_{\rho_1 : \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_1/\ell_1 - 1/2} \exp \left( - \frac{\pi}{2} \frac{\gamma_1}{\ell_1} \right) \sum_{\rho_2 : \gamma_2 < 0} \left| \frac{\gamma_2}{\ell_2} \right|^{\beta_2/\ell_2 - 1/2} \exp \left( - \frac{\pi}{4} \frac{\gamma_2}{\ell_2} \right) 
\]
\[
+ N^{k+\lambda} \sum_{\rho_1 : \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_1/\ell_1 - 1/2} \exp \left( - \frac{\pi}{2} \frac{\gamma_1}{\ell_1} \right) \sum_{\rho_2 : \gamma_2 < 0} \left| \frac{\gamma_2}{\ell_2} \right|^{\beta_2/\ell_2 - 1/2} 
\times \int_{1}^{+\infty} \exp \left( - \frac{\gamma_2}{\ell_2} \arctan \frac{1}{u} \right) \frac{du}{u^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}} 
\]
\[
\asymp_{k,\ell_1,\ell_2} N^{k+\lambda} + N^{k+\lambda} \sum_{\rho_1 : \gamma_1 > 0} \left( \frac{\gamma_1}{\ell_1} \right)^{\beta_1/\ell_1 - 1/2} \exp \left( - \frac{\pi}{2} \gamma_1 \right) \asymp_{k,\ell_1,\ell_2} N^{k+\lambda}
\]
for \( k > 1/2 \), by Lemma 2 and standard zero-density estimates.
On the other hand, the case $\gamma_1 > 0$, $\gamma_2 < 0$ and $y > 0$ can be estimated in a similar way essentially exchanging the role of $\gamma_1$ and $\gamma_2$ in the previous argument.

This implies that the integral in (26) is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ provided that $k > 1$. Combining the convergence conditions for (24)–(26), we see that we can exchange both summations with the integral provided that $k > 1$.

8. Convergence of the double sum over zeros. In this section we prove that the double sum on the right of (3) converges absolutely for every $k > 1/2$; the other series in (1) and (2) clearly converge for $k > 0$ or better. We need (15) uniformly for $x \in [0, k + 3]$ and $|y| \geq T$, where $T$ is large but fixed: this provides both an upper and a lower bound for $|\Gamma(x + iy)|$. Let

$$
\Sigma = \sum_{\rho_1} \sum_{\rho_2} \left| \frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(\rho_1/\ell_1 + \rho_2/\ell_2 + k + 1)} \right|,
$$

so that, by the symmetry of the zeros of the Riemann zeta-function, we have

$$
\Sigma = 2 \sum_{\rho_1: \gamma_1 > 0} \sum_{\rho_2: \gamma_2 > 0} \left| \frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(\rho_1/\ell_1 + \rho_2/\ell_2 + k + 1)} \right| + 2 \sum_{\rho_1: \gamma_1 > 0} \sum_{\rho_2: \gamma_2 > 0} \left| \frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(\rho_1/\ell_1 + \rho_2/\ell_2 + k + 1)} \right|
$$

$$
= 2(\Sigma_1 + \Sigma_2),
$$
say. It is clear that if both $\Sigma_1$ and $\Sigma_2$ converge, then the double sum on the right-hand side of (3) converges absolutely. In order to estimate $\Sigma_1$ we choose a large $T$ and let

$$
D_0 = \{(\rho_1, \rho_2): (\gamma_1, \gamma_2) \in [0, 2T]^2\}, \quad D_3 = \{(\rho_1, \rho_2): \gamma_2 \geq T, T \leq \gamma_1 \leq \gamma_2\},
$$

$$
D_1 = \{(\rho_1, \rho_2): \gamma_1 \geq T, T \leq \gamma_2 \leq \gamma_1\}, \quad D_4 = \{(\rho_1, \rho_2): \gamma_2 \geq T, 0 \leq \gamma_1 \leq T\},
$$

so that $\Sigma_1 \leq \Sigma_{1,0} + \Sigma_{1,1} + \Sigma_{1,2} + \Sigma_{1,3} + \Sigma_{1,4}$, say, where $\Sigma_{1,j}$ is the sum with $(\rho_1, \rho_2) \in D_j$. Now, $D_0$ contributes a bounded amount, that depends only on $T$, and, by symmetry again, $\Sigma_{1,1} = \Sigma_{1,3}$ and $\Sigma_{1,2} = \Sigma_{1,4}$. We also recall the inequality (27) which is valid for all couples of zeros considered in $\Sigma_1$. Hence

$$
\ll_{\ell_1,\ell_2} \sum_{\rho_1: \gamma_1 \geq T} \sum_{\rho_2: \gamma_2 \leq \gamma_1} \frac{e^{-\pi(\gamma_1/\ell_1+\gamma_2/\ell_2)/2(\gamma_1/\ell_1)^{1/2}(\gamma_2/\ell_2)^{1/2}}}{e^{-\pi(\gamma_1/\ell_1+\gamma_2/\ell_2)/2(\gamma_1/\ell_1+\gamma_2/\ell_2)^{1/2}}} \ll_{\ell_1,\ell_2} \sum_{\rho_1: \gamma_1 \geq T} \sum_{\rho_2: \gamma_2 \leq \gamma_1} \frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}(\gamma_1+\gamma_2)^{k+1/2}}
$$

$$
\ll_{\ell_1,\ell_2} \sum_{\rho_1: \gamma_1 \geq T} \sum_{\rho_2: \gamma_2 \leq \gamma_1} \frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}} \ll_{\ell_1,\ell_2} \sum_{\rho_1: \gamma_1 \geq T} \frac{\log \gamma_1}{\gamma_1^{k+1/2}}.
$$

A similar argument proves that

$$
\ll_{k,T,\ell_1,\ell_2} \sum_{\rho_1: \gamma_1 \geq T} \frac{1}{\gamma_1^{k+1/2}}.
$$
since $\Gamma(\rho_2)$ is uniformly bounded, in terms of $T$, for $(\rho_1, \rho_2) \in D_2$. Summing up, we have

$$\Sigma_1 \ll_{k, T, \ell_1, \ell_2} 1 + \sum_{\rho_1: \gamma_1 \geq T} \frac{\log \gamma_1}{k+1/2},$$

which is convergent provided that $k > 1/2$. In order to estimate $\Sigma_2$ we use a similar argument. Choose a large $T$ and for $\{i, j\} = \{1, 2\}$ set

$$E_0(i, j) = \left\{ (\rho_1, \rho_2): \left( \frac{\gamma_i}{\ell_i}, \frac{\gamma_j}{\ell_j} \right) \in [0, 2T]^2 \right\},$$

$$E_1(i, j) = \left\{ (\rho_1, \rho_2): \frac{\gamma_i}{\ell_i} \geq 2T, 0 \leq \frac{\gamma_j}{\ell_j} \leq T \right\},$$

$$E_2(i, j) = \left\{ (\rho_1, \rho_2): \frac{\gamma_i}{\ell_i} \geq 2T, T \leq \frac{\gamma_j}{\ell_j} \leq \frac{\gamma_i}{\ell_i} - T \right\},$$

$$E_3(i, j) = \left\{ (\rho_1, \rho_2): \frac{\gamma_i}{\ell_i} \geq 2T, \frac{\gamma_i}{\ell_i} - T \leq \frac{\gamma_j}{\ell_j} \leq \frac{\gamma_i}{\ell_i} \right\},$$

so that $\Sigma_2 \leq \Sigma_0(1, 2) + \Sigma_1(1, 2) + \Sigma_2(1, 2) + \Sigma_3(1, 2) + \Sigma_3(2, 1) + \Sigma_2(2, 1) + \Sigma_1(2, 1)$, say, where $\Sigma_r(i, j)$ is the sum with $(\rho_1, \rho_2) \in E_r(i, j)$. Now, $E_0$ contributes a bounded amount, that depends only on $T$, $\ell_1$ and $\ell_2$. We remark that similar arguments apply when dealing with $\Sigma_1(1, 2)$ and $\Sigma_1(2, 1)$; $\Sigma_2(1, 2)$ and $\Sigma_2(2, 1)$; $\Sigma_3(1, 2)$ and $\Sigma_3(2, 1)$ respectively. Again we use (15) as above; hence

$$\Sigma_2(1, 2) \ll_{\ell_1, \ell_2} \left( \sum_{(\rho_1, \rho_2) \in E_2(1, 2)} \sum_{\gamma_2 \leq \gamma_1^{1/2}} + \sum_{(\rho_1, \rho_2) \in E_3(1, 2)} \sum_{\gamma_2 > \gamma_1^{1/2}} \right) \frac{(\gamma_1/\ell_1)^{\beta_1/\ell_1 - 1/2} (\gamma_2/\ell_2)^{\beta_2/\ell_2 - 1/2} e^{-\pi \gamma_2/\ell_2}}{(\gamma_1/\ell_1 - \gamma_2/\ell_2)^{\beta_1/\ell_1 + \beta_2/\ell_2 + k + 1/2}}.$$

We bound the first sum by a further subdivision of the zeros $\rho_2$, treating differently those with $\beta_2 < \ell_2/2$ and the other ones, if any. The first sum is

$$\ll_{\ell_1, \ell_2} e^{-\pi T} \sum_{\gamma_1 \geq 2T \ell_1} \gamma_1^{\beta_1/\ell_1 - 1/2} \sum_{\gamma_2 \in [T \ell_2, \gamma_1^{1/2}]} \frac{\gamma_2^{\beta_2/\ell_2 - 1/2}}{\gamma_1^{\beta_1/\ell_1 + \beta_2/\ell_2 + k + 1/2}},$$

$$\ll_{T, \ell_1, \ell_2} \sum_{\gamma_1 \geq 2T \ell_1} \frac{1}{k+3/2} \left( \sum_{\gamma_2 < \ell_2/2} + \sum_{\gamma_2 \geq \ell_2/2} \right) \frac{\gamma_2^{\beta_2/\ell_2 - 1/2}}{\gamma_1^{\beta_1/\ell_1 + \beta_2/\ell_2 + k + 1/2}},$$

$$\ll_{T, \ell_1, \ell_2} \sum_{\gamma_1 \geq 2T \ell_1} \frac{1}{k+3/2} \left( \sum_{\beta_2 < \ell_2/2} \left( \frac{\gamma_1}{T} \right)^{1/2 - \beta_2/\ell_2} + \gamma_1^{1/2} \log \gamma_1 \right),$$

$$\ll_{T, \ell_1, \ell_2} \sum_{\gamma_1 \geq 2T \ell_1} \frac{\log \gamma_1}{\gamma_1^{k+1/2}}.$$
The rightmost series over zeros plainly converges for $k > 1/2$. The second sum is

$$
\ll T, \ell_1, \ell_2 \sum_{\gamma_1 \geq 2T \ell_1} \gamma_1^{\beta_1/\ell_1 - 1/2} e^{-\pi \gamma_1^{1/2}/\ell_2} \times \sum_{\gamma_2 \in [\gamma_1^{1/2}, (\gamma_1/\ell_1 - T) \ell_2]} \gamma_2^{\beta_2/\ell_2 - 1/2} (\gamma_1/\ell_1 - \gamma_2/\ell_2)^{\beta_1/\ell_1 + \beta_2/\ell_2 + k + 1/2}
$$

$$
\ll T, \ell_1, \ell_2 \sum_{\gamma_1 \geq 2T \ell_1} \gamma_1^{\beta_1/\ell_1 - 1/2} e^{-\pi \gamma_1^{1/2}/\ell_2} (\gamma_1 \log \gamma_1) T^{-(\beta_1/\ell_1 + k + 1/2)} \gamma_1^{1/2},
$$

which is very small. The contribution of zeros in $E_1(1, 2)$ is treated in a similar fashion, using the uniform upper bound $\Gamma(\rho_2) \ll T$, and is also small. We now deal with $\Sigma_3(1, 2)$: we have

$$
\Sigma_3(1, 2) \ll \ell_1, \ell_2 \sum_{(\rho_1, \rho_2) \in E_3} e^{-\pi \gamma_1/(2\ell_1)} \gamma_1^{\beta_1/\ell_1 - 1/2} e^{-\pi \gamma_2/(2\ell_2)} \gamma_2^{\beta_2/\ell_2 - 1/2} \left( \prod_{0 \leq t \leq T} \left| \Gamma(x + it) \right| \right)^{-1}
$$

$$
\ll k, T, \ell_1, \ell_2 \sum_{\rho_1 : \gamma_1 \geq 2T \ell_1} e^{-\pi \gamma_1/\ell_1} \gamma_1^{\beta_1/\ell_1 + 1/\ell_1} \log(\gamma_1 + T),
$$

provided that $T$ is large enough. Here we are using Theorem 9.2 of Titchmarsh [15] with $T$ large but fixed. The series at the extreme right is plainly convergent.

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