Retracing some paths in categorical semantics: From process-propositions-as-types to categorified real numbers and monoidal computers

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Abstract

The logical parallelism of propositional connectives and type constructors extends beyond the static realm of predicates, to the dynamic realm of processes. Understanding the logical parallelism of process propositions and dynamic types was one of the central problems of the semantics of computation, albeit not always clear or explicit. It sprung into clarity through the early work of Samson Abramsky, where the central ideas of denotational semantics and process calculus were brought together and analyzed by categorical tools, e.g. in the structure of interaction categories. While some logical structures borne of dynamics of computation immediately started to emerge, others had to wait, be it because the underlying logical principles (mainly those arising from coinduction) were not yet sufficiently well-understood, or simply because the research community was more interested in other semantical tasks. Looking back, it seems that the process logic uncovered by those early semantical efforts might still be starting to emerge and that the vast field of results that have been obtained in the meantime might be a valley on a tip of an iceberg.

In the present paper, I try to provide a logical overview of the gamut of interaction categories and to distinguish those that model computation from those that capture processes in general. The main coinductive constructions turn out to be of this latter kind, as illustrated towards the end of the paper by a compact category of all real numbers as processes, computable and uncomputable, with polarized bisimulations as morphisms. The addition of the reals arises as the biproduct, real vector spaces are the enriched bicompletions, and linear algebra arises from the enriched kan extensions. At the final step, I sketch a structure that characterizes the computable fragment of categorical semantics.

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Personal introduction

I first learned about Samson Abramsky’s work from his invited plenary lecture at the International Category Theory Meeting in Montreal in 1991. It was the golden age of category theory, and Montreal was at the heart of it, and I got to be a postdoc there. Just a few years earlier, I was a dropout freelance programmer, but had become a mathematician, and was uninterested in computers. I was told, however, that Abramsky had constructed some categories that no one had seen before, so I came to listen to his talk. I also had a talk to give later that day, but for some reason, I do not recall how that went. At the end of Abramsky’s plenary lecture, Saunders Mac Lane stood up, one of the two fathers of category theory, high up near the ceiling of the amphitheater, and spoke for a long time. He criticized computer science in general. After that, Bill Lawvere stood up, and provided some friendly comments, suggesting directions for progress and improvement.

Two years later, I became an "EU Human Capital Mobility" fellow within the Theory Group, led by Samson Abramsky at the Imperial College in London. I started learning computer science and spent a lot of time trying to understand Samson’s interaction categories [3]. In the meantime, he had constructed more categories that no one had seen before. My fellowship ended after a year or two, and the human capital mobility turned out to be much greater than anyone could imagine, but I continued to think about interaction categories for years. Here I try to summarize some of the outcomes of that process.

1 Introduction: On categorical logics and propositions-as-types

The category of sets or types. This is a paper about categorical semantics. It is written for a collection intended for logicians. If you are reading this, then you are presumed to be interested in categorical logic, although you may not be interested in categories in general. To ease this tension, I will avoid abstract categories, and mostly stick with the category $S$ of sets and functions. It is presented, however, as a universe of types, by specifying which type constructors are used in each construction. Initially we just need the cartesian products, but later constructions require more. Naive set theory used to be presented incrementally. Nowadays most mathematicians think of types as sets, and most programmers think of sets as types, so it seems reasonable for logicians and computer scientists to identify the two. To keep the naive-set-theory flavor, we usually call the type inhabitants elements, where type theorists use the term terms.

When a set is constructed as a type, then it can also be construed as a proposition: its elements are some constructions, and they can be viewed as proofs [64]. Such interpretations originate from logic, where the idea of propositions-as-types was first traced along the path of proofs-as-constructions [27] [49] [63]. We retrace these paths first, and proceed throughout with propositions-as-types, types-as-sets, terms-as-elements, elements-as-morphisms [38] [56].

1.1 Logics of types

Bertrand Russell proposed his ramified theory of types [95] as a logical framework for paradox prevention. Alonzo Church and Stephen Kleene advanced type theory into a model of computation
Dana Scott adopted type theory as the foundation for a mathematical approach to the semantics of computation [96]. The semantics of programming languages were built steadily upon that foundation [41, 93]. Process semantics also arose from that foundation [66], but had to undergo a substantive conceptual evolution before the types could extend in time, and capture dynamics. I followed these developments through Samson Abramsky’s work.

The propositions-as-types paradigm was discovered many times. In logic and computer science, it is attributed to Haskell Curry and William Howard [97, 38, Ch. 3]. Howard got the idea from Georg Kreisel [106], and Kreisel’s goal was to formalize Brouwer’s concept of proofs-as-constructions [50]. An early formalization of Brouwer’s concept goes back to Kolmogorov [49].

The structural reason why propositions and types obey analogous laws was offered by Lawvere [59], who pointed out that the propositional and the typing rules are instances of analogous categorical adjunctions, and that the proof constructions and the term derivations arise from the adjunction units and counits. This gave rise to the idea of categorical proof theory, pursued by Lambek [52, 54, 55], and to the basic structures of categorical semantics, succinctly described in [57] and the references therein]. In the preface to his seminal report [96], Dana Scott explained that

"a category represents the 'algebra of types’, just as abstract rings give us the algebra of polynomials, originally understood to concern only integers or rationals. One can of course think only of particular type systems, but, for a full understanding, one needs also to take into account the general theory of types, and especially translations or interpretations of one system in another.”

Samson Abramsky spearheaded the efforts towards expanding the categorical semantics of program abstraction, as formalized in type theory, and merge it with a categorical semantics of process abstraction and interaction, as formalized in the theory of concurrency and process calculi. This led to interaction categories [3, 28, 75, 77], specification structures [11, 87], and a step further to geometry of interaction [13] and game semantics [5, 12, 14, 15, and many other publications]. As the realm of program abstraction expanded, e.g. into quantum computation and protocols, the semantical apparatus also expanded [7, 8, 10], the tree branched [30, 82], some branches crossed [1]. In the present paper, however, we are only concerned with the root.

1.2 Categorical proof theory

Proofs-as-constructions. The Curry-Howard isomorphism is one of the conceptual building blocks of type theory, built deep into the foundation of computer science and functional programming [38, ch. 3]. The fact that it is an isomorphism means that the type constructors on one side obey the same laws as the propositional connectives on the other side; and these laws are expressed as a bijection between the terms and the proofs.

---

1E.g., [81] used the methods of [87] to expand the models of [10].
1.2.1 Entailments as morphisms

In categorical proof theory, logical sequents are treated as arrows in a category \([52, 54, 55, 59]\). The reflexivity and the transitivity of the entailment relation then correspond to the main categorical structures: the identities and the composition.

\[
\begin{align*}
A \vdash A & \quad \text{1} \\
\end{align*}
\]

\[
\begin{align*}
A \vdash B & \quad B \vdash C \\
\sum & \quad \sum \\
A \vdash C & \quad \sum \\
\end{align*}
\]

But while there is at most one sequent \(A \vdash B\) for given \(A\) and \(B\), there can be many arrows between \(A\) and \(B\) in a category. Categorical semantics of the logical entailment must therefore be imposed by equations:

\[
\begin{align*}
S(A, B) & \quad S(A, B) \\
\langle \text{id}, \text{id} \rangle & \quad \langle \text{id}, \text{id} \rangle \\
S(A, A) \times S(A, B) & \quad S(A, B) \\
\text{id} & \quad \text{id} \\
S(A, B) & \quad S(A, B) \\
\langle \langle \text{id}, \text{id} \rangle \rangle & \quad \langle \langle \text{id}, \text{id} \rangle \rangle \\
S(A, B) \times S(B, B) & \quad S(A, B) \times S(B, B) \\
\end{align*}
\]

\[
\begin{align*}
S(A, B) \times S(B, C) \times S(C, D) & \quad \text{id} \langle \langle \text{id}, \text{id} \rangle \rangle \\
\langle \langle \text{id}, \text{id} \rangle \rangle & \quad \langle \langle \text{id}, \text{id} \rangle \rangle \\
S(A, B) \times S(B, D) & \quad S(A, B) \times S(B, D) \\
\end{align*}
\]

\[
\begin{align*}
S(A, C) \times S(C, D) & \quad \langle \langle \text{id}, \text{id} \rangle \rangle \\
\langle \langle \text{id}, \text{id} \rangle \rangle & \quad \langle \langle \text{id}, \text{id} \rangle \rangle \\
S(A, D) & \quad S(A, D) \\
\end{align*}
\]
1.2.2 Conjunction and disjunction as product and coproduct

Algebraically, the conjunction and the disjunction are the meet and the join in the proposition lattice. Categorically, they are the product and the coproduct:

\[
\begin{array}{c}
X \vdash A & X \vdash B \\
\hline
X \vdash A \land B
\end{array}
\]

\[
S(X, A) \times S(X, B)
\]

\[
\gamma(\langle - \rangle \circ -) \circ \langle \pi_{A \circ -}, \pi_{B \circ -} \rangle
\]

\[
S(X, A \times B)
\]

\[
\begin{array}{c}
A \vdash X & B \vdash X \\
\hline
A \lor B \vdash X
\end{array}
\]

\[
S(A, X) \times S(B, X)
\]

\[
\gamma(\langle - \rangle \circ -) \circ \langle \sigma_{A \lor -}, \sigma_{B \lor -} \rangle
\]

\[
S(A + B, X)
\]

The difference between the algebraic and the categorical view, is that in the first case there is at most one entailment \(X \vdash A\), whereas in the second case there can be many arrows \(X \to A\), usually labelled, and viewed as functions in the category \(S\). The mapping in (2) on the right establishes the bijection between the proofs or functions \(X \to A \times B\) and the pairs of proofs or functions \(X \to A\) and \(X \to B\). The proof transformations thus become function manipulations. If the elements of sets, or entries of data types, are thought of as witnesses of the corresponding propositions, then the data services, such as the logical operations on propositions realized by the data services, such as copying or pairing of data entries. It often comes as a surprise that such simple-minded analogies become effective tools in functional programming [86]. They also have far-reaching logical consequences, some of which are pursued in this paper.

1.2.3 Logic of abstraction: Implication as exponent

The fact that the conjunction \(A \land (-)\) is the right adjoint to the implication \(A(-)\) [59] means that the implication introduction and elimination can be expressed as the reversible rule on the left.

\[
\begin{array}{c}
(A \land X) \vdash B \\
\hline
X \vdash (A \supset B)
\end{array}
\]

\[
S(A \times X, B)
\]

\[
\gamma(A \supset -) \circ \langle \pi_{A \times -} \circ \epsilon \circ \pi_{X \circ (A \supset -)} \rangle
\]

\[
S(X, (A \supset B))
\]

The reversibility of the logical rule on the right was the first example of the propositions-as-types phenomenon, i.e. the first proof-as-term, noticed back in the 1930s by Haskell Curry. But while the reversibility of the logical rule of the left captures the one-to-one correspondence of proofs, the categorical adjunction on the right also captures the fact that the correspondence is \textit{natural} with respect to \(X\), i.e. that it is preserved under all \(f \in S(X, Y)\), in the sense that the following square
commutes.\[ S(A \times Y, B) \xrightarrow{-\circ f} S(A \times X, B) \\]
\[ (A \Rightarrow B) \xrightarrow{\eta} \] \[ (A \Rightarrow B) \xrightarrow{\epsilon} \]
\[ S(Y, (A \Rightarrow B)) \xrightarrow{-\circ f} S(X, (A \Rightarrow B)) \] (5)

In fact, there are two squares in this diagram: one formed by $\eta$s, the other by $\epsilon$s. Their commutativity formally establishes categorically that the correspondence is polymorphic, i.e. valid for all $X$, and under all of its transformations. The same polymorphic correspondence is expressed type-theoretically by the familiar conversion rules:

\[ A \times (A \Rightarrow (A \times X)) \]
\[ \xrightarrow{A \times g \Rightarrow} \]
\[ A \times X \xrightarrow{id} A \times X \]
\[ A \Rightarrow X \xrightarrow{id} A \Rightarrow X \]
\[ (A \Rightarrow X) \xrightarrow{\eta} \]
\[ A \Rightarrow (A \times (A \Rightarrow X)) \]

where the function application $\epsilon : (A \Rightarrow B) \times A \rightarrow B$ is abbreviated to $g \cdot a = \epsilon(g, a)$.

1.3 Modalities as monads and comonads

1.3.1 Possibility and side-effects

A possibility modality can be introduced by the rules on the left.

\[ A \vdash \Diamond A \]
\[ A \land B \vdash \Diamond C \]
\[ \Diamond A \land B \vdash \Diamond C \]
\[ S(A \times B, MC) \]
\[ \# \xrightarrow{\text{monad}} S(MA \times MB, MC) \]

Each of the logical rules corresponds to one of the categorical transformations on the right, where the mapping up is the precomposition $A \times B \xrightarrow{\eta} M(A \times B) \rightarrow MC$, the mapping down is the operation $\#$ lifting $A \times B \rightarrow C$ to $MA \times MB \rightarrow MC$, and the triple $(M, \eta, \#)$ is a monad [23, 57, 61, 62]. If sequents are viewed as morphisms, or labelled, then the derivation rules on the left become reversible by imposing the following equations on the operations on the right:

\[ \eta^\#_A = \text{id}_{MA} \]
\[ f^\# \circ \eta_{A \times B} = f \]
\[ (f^\# \circ \text{id})^\# = f^\# \circ \text{id}^\# \]

\[ \text{The third component of the monad signature in the first three references is different from the one that we use here, but equivalent if B is omitted. The present one, without the B component, originates from the fourth reference. It is more convenient for programming and in type theory. The B component is a succinct way to also impose the commutativity requirement on the monad. More in Sec. 1.4.} \]
The third equation defines the composition in the *Kleisli* category

\[
|S_M| = |S| \\
S_M(A, B) = S(A, MB)
\]

A morphism in the form \(A \rightarrow MB\) can be thought of as a function that produces not just the outputs of type \(B\), but also some *side-effects*, modeled by the monad \(M\). The idea that computations do not just consume inputs and produce outputs, but also cause side-effects, that must be taken into account in process theory, goes back to [66]. E.g., the fact that computations may not terminate means that they implement functions in the form \(A \rightarrow B_\perp\) where

\[
(-)_\perp : S \rightarrow S \\
X \mapsto X \cup \{\perp\}
\]

(7)

where \(\perp\) is a fresh element, denoting the divergence. This is the *maybe* monad. The category \(S_\perp\) is easily seen to be equivalent to the category of sets and partial functions.

Some computations may depend on the states of the computer, which may depend on the environment. Running the same program on the same inputs of type may therefore produce different outputs at different times, for no unobservable reason. Such computations implement functions in the form \(A \rightarrow \wp B\) from elements of \(A\) to *sets of* elements of \(B\). The type constructor

\[
\wp : S \rightarrow S \\
X \mapsto \{V \subseteq X\}
\]

(8)

is the powerset functor. It also maps to every function \(X \xrightarrow{g} Y\) the function \(\wp X \xrightarrow{\wp g} \wp Y\), which takes subsets to their images along \(g\). This is the *nondeterminism* (or *powerset*) monad. For reasons discussed in Appendix A, it satisfies

\[
S(A, \wp B) \simeq S(B, \wp A)
\]

which makes the category \(S_\wp\) of nondeterminisic functions self-dual, equipping it with the natural bijection \(S_\wp(A, B) \cong S_\wp(B, A)\). The idea is that, given a nondeterministic function \(A \rightarrow \wp B\), i.e. knowing all possible \(B\)-outputs for each \(A\)-input allows us to extract all possible \(A\)-inputs for each \(B\)-output, which yields just another nondeterministic function \(B \rightarrow \wp A\). See Appendix A for more.

**Notation.** Since they will play leading roles, the above categories of functions with effects will be called:

- \(S_\perp = P\) — category of partial functions, and
- \(S_\wp = R\) — category of relations.
Background. The observation that the type constructors $M$ capturing functions with effects $A \rightarrow MB$ carry the monad structure $(M, \eta, \#)$ goes back to [70]. E.g., the powerset functor forms a monad $(\wp, \eta, \#)$ with the units $X \mapsto \wp X$ mapping the elements of $X$ to singleton sets $\eta(x) = \{x\}$, and lifting the functions $A \rightarrow \wp B$ to $\wp A \rightarrow \wp B$ where $f^\#(V) = \bigcup_{v \in V} f(v)$. Nowadays, monads as tools for encapsulating computational effects, are at least as popular in programming practices as in semantical theories. Mathematically, they are in tools for encapsulating algebraic theories [62]. Any algebraic theory induces a monad $M$, where $M B$ is the free algebra generated by $B$. E.g., $\wp B$ is the free semilattice over $B$, and $B_\bot$ is the free algebra over $B$ for the algebraic theory with a single constant and no other operations or equations. The other way around, any monad corresponds to an algebraic theory, albeit with infinitary operations. The tacit assumption is thus that the side-effects of computations can always be captured by some algebraic operations.

1.3.2 Necessity and reductions

Dually, a necessity modality can be introduced by

\[ \Box A \vdash A \quad \Box A \vdash B \lor C \quad \Box A \vdash \Box B \lor \Box C \]

\[ S(GA, B + C) \quad \Box(\epsilon_{B+C} \circ f^\#) = f \quad (f \circ t^\#)^\# = f^\# \circ t^\# \]

This time the triple $(G, \epsilon, \#)$ is made into a comonad by the equations:

\[ e_A^\# = \text{id}_{GA} \]

\[ \epsilon_{B+C} \circ f^\# = f \]

\[ (f \circ t^\#)^\# = f^\# \circ t^\# \]

The third equation defines the composition in the Kleisli category

\[ |S_G| = |S| \]

\[ S_G(A, B) = S(GA, B) \]

Computational interpretations of comonads are less standard, but overviews can be found in [24, 99]. We will need a history comonad to capture the time extension of processes in Sec. 2.3.1. For the moment, let us just mention the indexing comonads

\[ A \times (-) : S \rightarrow S \]

\[ X \mapsto A \times X \]

which exist for each $A \in S$, with the counits $A \times X \xrightarrow{\epsilon} X$ realized by the projections, and the lifting $A \times X \rightarrow Y + Z$ defined to be $A \times X \xrightarrow{(\text{id}_A, h)} A \times (Y + Z) \cong (A \times Y) + (A \times Z)$. The Kleisli category $S_{A\times}$ freely adjoins an indeterminate arrow $1 \rightarrow A$ to $S$, and plays the role of the polynomial extension

\[ 3^\text{The monad signature used there, and in most other earlier presentations, has a cochain map \( \mu \) as the third component. The equivalent version with \( \# \) seems more convenient for program derivations.} \]
Like any Kleisli category, \( S_A \) provides a *resolution* of its comonad, in the sense that it factors through the functors

\[
A \times (-) = \left( S \xrightarrow{\text{ref} \circ #} S_A \xrightarrow{\Pi} S \right)
\]

as displayed in (9). While the Kleisli resolution is *initial* among the resolutions of the comonad \( A \times (-) \), some of the constructions in this paper are built upon the fact that the resolution

\[
A \times (-) = \left( S \xrightarrow{\Pi} S/A \xrightarrow{\text{Dom}} S \right)
\]

is *final* among all resolutions. Here \( S/A \) is the category of \( S \)-morphisms into \( A \), the functor \( \Pi \) maps \( X \) to the projection \( A \times X \xrightarrow{\pi_A} A \), whereas the Dom functor Dom takes the \( S/A \)-objects, which are the \( S \)-morphisms with the codomain \( A \), to their domains \( \text{Dom}(X \rightarrow A) = X \).

**Lemma 1.1** The domain functor \( \text{Dom} : S/A \rightarrow S \) is final among all functors \( F : C \rightarrow S \) which map the terminal object \( 1 \) into \( A \).

\[
\begin{align*}
\text{C} & \quad \forall F \quad \neg \exists F' \quad \text{Dom} \quad \downarrow \\
S & \quad \neg \exists F' \quad \downarrow \\
S/A & \quad \neg \exists F'
\end{align*}
\]

**Proof.** Given \( F \) with \( F1 = A \), the unique \( F' \) with \( \text{Dom} \circ F' = F \) is \( F'X = F(X \rightarrow 1) \). \( \Box \)

### 1.4 Labelled sequents, commutative monads, and surjections

In propositional logic, a sequent \( X \vdash Y \) transforms proofs of \( X \) into proofs of \( Y \). If there are several different ways to derive one from the other, the sequent \( X \vdash Y \) identifies them all. This leads to a mismatch within the propositions-as-types interpretation because it implies that there is at most one proof \( X \vdash A \supset B \), while there can be many different terms typed \( X \vdash (A \Rightarrow B) \). This mismatch is resolved by labelling the sequents, by writing \( X \vdash f \supset A \supset B \) for the former sequent. We use the symbol \( \vdash \) (and not \( \vdash \)) for labelled sequents, to be able to write \( X \vdash Y \) instead of \( X \vdash Y \) when the label \( f \) is irrelevant. The categorical proof theory originates from studies of labelled sequents in \([52, 54, 55]\). A non-categorical theory of labelled sequents was developed in \([36]\).

For a modality \( \Diamond \), the sequents between the propositions \( \Diamond A \land \Diamond B \) and \( \Diamond (A \land B) \) are derivable both ways, and the two are considered equivalent. The proposition \( \Diamond \top \) is also equivalent to the truth \( \top \). For a monad \( M \), the maps \( M(A \times B) \rightarrow MA \times MB \) and \( M1 \rightarrow 1 \) are derivable from the cartesian structure, and the maps \( MA \times MB \rightarrow M(A \times B) \) and \( 1 \rightarrow M1 \) are given by the monad structure. These maps both ways generally do not make their types isomorphic. This is in the first case justified since the side-effects of type \( M(A \times B) \) are different from the side-effects when \( MA \)
and $MB$ separately. On the other hand, the trivial outputs of type 1 should not cause nontrivial side effects of type $M1$. The type $M1$ should thus be trivial again, i.e. isomorphic with 1. If the monad $M$ is viewed as an algebraic theory, this requirement means that there should be no constants in the algebraic signature of $M$. This requirement is not satisfied either by the maybe monad, or by the nondeterminism monad, as the former gives the universe $P = S_\perp$ of sets and partial maps, the latter the universe $R = S_\wp$ of sets and relations. The former is the category of free algebras for the theory with a single constant $\perp$, and no other operations. The latter is the category of free join semilattices, where the lattice unit is a constant again.

Lemma [1.1] says that making 1 into the unit type (final object) in $R = S_\wp$ leads to the slice category $tR = R/1$, which boils down to

$$|tR| = \bigsqcup_{A \in |S|} \emptyset A$$

$$tR(S_{\subseteq A}, T_{\subseteq B}) = \{ R \in R(A, B) \mid (x \in S \iff \exists y \in T. xRy) \land$$

$$\land (y \in T \iff \exists x \in S. xRy) \} \quad (11)$$

Since the $\Rightarrow$-direction of each of the conjuncts in (11) implies the $\Leftarrow$-direction of the other conjunct, the requirement boils down to $\forall x \in S \exists y \in T. xRy$ and $\forall y \in T \exists x \in S. xRy$. The category $tR$ is thus equivalent to the subcategory of $R$ comprised of the relations that are total in both directions. Proceeding in a similar way to make 1 into the final type in the category $S_\perp = P$ leads to the slice category $tP = P/1$, which is equivalent to the subcategory of $S$ spanned by the surjective functions:

$$|tP| = \bigsqcup_{A \in |S|} \emptyset A$$

$$tP(S_{\subseteq A}, T_{\subseteq B}) = \{ f \in S(S, T) \mid y \in T \Rightarrow \exists x \in S. f(x) = y \} \quad (12)$$

**Remark for the category theorist.** The forgetful functor $tP \rightarrow tS$, where $tS$ is the category of sets and surjections, is an equivalence because it is surjective on the objects, and full and faithful on the morphisms. However, for each set $S \in S$ there is a proper class of sets $A$ such that $S_{\subseteq A} \in tP$ is mapped to $S \in tS$. Constructing the adjoint equivalence $tS \rightarrow tP$ thus involves a choice from these proper classes of objects.

## 2 Deriving process logics

### 2.1 Idea of process

The alignment of logics and type theory remains remarkably stable as long as the world is assumed to be stable for long enough, i.e. if true propositions remain true, and if data types remain static. The problems arise when processes need to be modeled, and the dynamic aspects need to be taken into account.
There are physical processes, chemical processes, mental processes, social processes. The common denominator seems to be that they evolve in time. In other words, they change state: a physical process changes the state of the matter; a mental process changes the state of mind. Computation is also a process. Although already a local execution of a program changes the local states of a computer, it seems that the crucial aspects of processes of computation arise from their interleaving with the processes of communication, from the resulting computational interactions, and only emerge into the light when the problem of concurrency is taken into account. That is why the semantics of computational processes, formalized in process calculi, initially forked off from the main branch of semantics of programming languages. The main part of Samson Abramsky’s work, which I am here trying to summarize in logical terms, was concerned with bringing the two branches together.

2.2 Process propositions and implications

2.2.1 Process sequents must be labelled

Process logics involve modeling states. There are many different ways to model states, but within a propositions-as-types framework, state spaces occur among the data types, and both are subject to propositional derivation rules. While we shall see in Sec. 2.4 that the two must be treated differently even on the logical side, they both require labelled sequents. For state spaces, this is clearly unavoidable. As mentioned in Sec. 1.4 an unlabelled sequent \( X \vdash Y \) identifies all different proofs that \( X \) entails \( Y \). In particular, there is just one entailment \( X \vdash X \), the trivial one. But if \( X \) is a state space, then modeling state transitions requires nontrivial sequents \( X \mid\xi \rightarrow X \). The labels allow distinguishing the nontrivial sequents, where the states change, from the trivial one, where they do not.

2.2.2 Process implications

A process implication \([A, B]\) asserts not just that \( A \) implies \( B \), but also that \( A \) implies \([A, B]\). Under the propositions-as-types interpretation, the type \([A, B]\) thus comes with two functions

- \( A \land [A, B] \mid\rightarrow B \) \hspace{1cm} (\( \nu^* \))
- \( A \land [A, B] \mid\rightarrow [A, B] \) \hspace{1cm} (\( \nu^\circ \))

The proposition \([A, B]\) thus asserts not just that \( B \) is true whenever \( A \) is, but it also asserts its own truth under the assumption that \( A \) is true. This is a typical impredicative logical construct. We are, of course, taught at high school that we should not use a proposition when we are proving that proposition. But the proposition \([A, B]\) only uses itself guarded by \( A \). This is the logical principle of guarded-induction, or coinduction, which turns out to be consistent, with most logical frameworks \([32, 80]\), and tacitly used in classical mathematics \([91, 88, 89]\). The idea is that, whenever a proposition \( X \), together with a proposition \( A \), entails a proposition \( B \), and moreover also itself, i.e. whenever \( X \) comes with the sequents

- \( A \land X \mid\rightarrow B \) \hspace{1cm} (\( [\llbracket \dashv \rrbracket]^* \))
• $A \land X \to X$ (Meta-

then $X$ also entails the process implication $[A, B]$. Putting it all together, we get the introduction rules for process implication:

$$
\begin{align*}
A \land [A, B] &\vdash \overset{\nu}{B \land [A, B]} \\
A \land X &\vdash B \land X \\
X &\to [A, B]
\end{align*}
$$

(13)

Terminology. A function in the form $\xi : A \times X \to \xi B \times X$ is often called a machine, and the set $X$ is construed as its state space. The induced description $\llbracket \xi \rrbracket : X \to [A, B]$ is called anamorphism.$^4$

Naturality. Comparing the $\llbracket \cdot \rrbracket$-rule with the $(\supset)$-rule in Sec. 12.3 shows the sense in which $[A, B]$ is a dynamic version of the implication $AB$. But note that the rule $(\supset)$ is reversible, whereas the rule $\llbracket \cdot \rrbracket$ is not; and that the $X$-natural bijection in (4) on the right boils down to a $X$-natural transformation on the right in (13). Moreover, since $X$ occurs on both sides of the sequent $A \land X \to B \land X$, and thus in both covariant and contravariant position in $S(A \times X, B \times X)$, the naturality of $\llbracket \cdot \rrbracket_X$ is not as simple as in (5), but it turns out to add more to the story. The naturality is this time in the form

$$
\begin{align*}
S(A \times Y, B \times Y) &\to S(A \times X, B \times X) \\
S(Y, [A, B]) &\to S(X, [A, B])
\end{align*}
$$

(14)

where $\Theta_{AB}$ is the functor

$$
\Theta_{AB} : S \to R
$$

(15)

where $R$ is the category of sets and relations, described in Appendix A. The arrow part of this functor transforms a function $f \in S(X, Y)$ into the relation $\Theta_{AB}f = (f) \subseteq S(A \times Y, B \times X) \times S(A \times X, B \times X)$ defined by

$$
\begin{align*}
A \times Y &\to A \times X \\
B \times Y &\to B \times X
\end{align*}
$$

(16)

$^4$Anamorphisms are the coalgebra homomorphisms into final coalgebras. The name is due, I believe, to Lambert Meertens. It seems to have caught on in functional programming without having been introduced in a publication. A machine $A \times X \rightarrow B \times X$ can be viewed as a coalgebra $X \rightarrow (A \Rightarrow (B \times X))$. 

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The relation $$(- \circ f)$$ in (14) is the arrow part of the functor

$$\Xi_{AB} : S^o \rightarrow R$$

where $$\Xi_{AB}f = (- \circ f) \subseteq S(Y, [A, B]) \times S(X, [A, B])$$ is defined

$$y(- \circ f)x \iff y \circ f = x$$

$$\Xi_{AB}$$ is, of course, just homming into $$[A, B]$$, i.e. a functor to $$S$$ extended along the inclusion $$S \hookrightarrow R$$ of functions as special relations, to allow expressing the naturality of $$\Theta_{AB} : \Pi \rightarrow \Xi_{AB}$$. Spelling out this naturality shows that $$\Pi$$ must preserve the machine (i.e. coalgebra) homomorphisms specified in (16). The concept of an $$AB$$-machine homomorphism has thus been reconstructed logically, from the properties of the dynamic implication $$[A, B]$$ in (13). Moreover, setting $$[A, B]$$ for $$Y$$ in (13) we get the outer square in

$$
\begin{array}{cccc}
S(A \times [A, B], B \times [A, B]) & \xrightarrow{\iota \iota} & S(A \times X, B \times X) \\
\downarrow & & \downarrow \\
S([A, B], [A, B]) & \xrightarrow{(- \Pi \iota \iota)} & S(X, [A, B])
\end{array}
$$

The inner square says that, if we bind together the two left-hand rules in (13) by requiring that

$$\Pi \iota \iota_{[A, B]} = \text{id}_{[A, B]}$$

then the naturality requirement in (13) implies that $$A \times [A, B] \xrightarrow{\iota} B \times [A, B]$$ is a final $$AB$$-machine.

### 2.2.3 Process propositions

A static proposition $$B$$ is equivalent with the static implication $$\top \supset X$$, where $$\top$$ is the true proposition. Propositions can thus be viewed as a special case of implications, namely the implications from the truth. A dynamic proposition $$[B]$$ can thus be defined in the form $$[B] = [\top, B]$$. Since the conjunctions $$\top \land X$$ are also equivalent with $$X$$, dynamic propositions can be defined by the rules

$$
\begin{array}{c}
\begin{array}{c}
[B] \xrightarrow{\iota} B \land [B] \\
X \xrightarrow{\beta} B \land X \\
X \xrightarrow{\Pi \iota \iota} [B]
\end{array}
\end{array}
$$

Retracing the analysis from Sec. 2.2.2 now presents a proposition $$[B]$$ with a structure map $$[B] \xrightarrow{\iota} B \times [B]$$, as final among all maps in the form $$X \rightarrow B \times X$$. The structure map is thus a pair $$\iota = (\iota^*, \iota^o)$$, where $$\iota^* : [B] \rightarrow B$$ gives an output of the process proposition, or an action, and $$\iota^o : [B] \rightarrow [B]$$ gives a resumption. It is thus a stream of elements in $$B$$. 

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2.3 Relating process implications and static implications

The static implication is defined by the rules and the correspondence in (4). The process implication is defined by the rules and the correspondence in (4). How are they related? Under which conditions are both sets of rules supported? Can the dynamic implication be derived from the static one by adding some feature capturing dynamics? Can the static implication be derived from the dynamic one by projection out that feature? — Prop. 2.3.3 answers these questions. We first define the structures involved in the answers.

2.3.1 History types

A process of $A$-histories over a state space $X$ is a pair of functions $\xi = (\xi^*, \xi^\circ)$ typed

$$A \overset{\xi^*}{\to} X \overset{\xi^\circ}{\leftarrow} A \times X$$

(20)

The idea is that,

- $\xi^*(a) \in X$ is the initial state of a process that starts with $a \in A$;
- $\xi^\circ(x, a) \in X$ is the next state of a process after the state $x \in X$ and event or action $a \in A$.

A history $a^n = (a_1 a_2 \cdots a_n)$ thus takes the process $\xi$ to the state

$$x_n = \xi^\circ(a_n, \xi^\circ(a_{n-1}, \ldots \xi^\circ(a_1, \xi^*(a_0)) \cdots))$$

Each string of $n$ actions, construed as an $A$-history is thus mapped to a unique element of $X$. If the histories $(a_1 \cdots a_n)$ are viewed as the elements of $A^n$, then the disjoint union (coproduct)

$$A^+ = \bigsqcup_{n=1}^{\infty} A^n$$

is the type of all $A$-histories. This is what we call a history type. For any process of $A$-histories $\xi$ over $X$ there is a unique function $A^+ \to X$, such that the following diagram commutes.

Hence the history type constructor, the functor

$$(-)^+: S \to S$$

(21)

$$A \mapsto A^+$$

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2.3.2 Retracts and idempotents

A **retract** of $A$ is a type $B$ together with a pair of maps $A \xrightarrow{q} B$ such that the following diagram commutes

$\begin{array}{ccc} A & \xrightarrow{q} & B \\ \downarrow & & \downarrow \\ A & \xrightarrow{i} & B \end{array}$

It is easy to see that $A \xrightarrow{q} B$ is a retract if and only if $\varphi = i \circ q$ is idempotent, i.e.

$$q \circ i = \text{id}_B \iff \varphi \circ \varphi = \varphi$$

We say that a universe has **retracts** if for every idempotent $A \xrightarrow{\varphi} A$ there is a retract $A \xrightarrow{q} B$ such that $\varphi = i \circ q$.

2.3.3 Proposition

Let $S$ be a cartesian category. Then

- a) $(A \Rightarrow B)$ with (4) induces $[A, B]$ with (13) if $S$ has history types;
- b) $[A, B]$ with (13) induces $(A \Rightarrow B)$ with (4) if $S$ has retracts.

The **proof** is given in the Appendix.

**Definition 2.1** A cartesian closed category with history types and retracts is called **process-closed**.

Dynamic function abstraction is function abstraction over history types. Prop. 2.3.3 says that a cartesian closed category with history types has final $AB$-machines for all types $A$ and $B$, and that their state spaces $[A, B] = (A+ \Rightarrow B)$ support rules (13). A final $AB$-state machine can be constructed as a final coalgebras for the functor

$$E_{AB} : S \rightarrow S$$

$$X \mapsto (A \Rightarrow B \times X)$$

i.e. as a limit of the tower in the form

$$1 \xleftarrow{1} (A \Rightarrow B) \xleftarrow{\varphi} (B \times (A \Rightarrow B)) \xleftarrow{\varphi} \ldots$$

$$\ldots \xleftarrow{E^n_{AB}(1)} E^{n+1}_{AB}(1) \xleftarrow{\varphi} \ldots \xleftarrow{E^n_{AB}(1)} E^{n+1}_{AB}(1) \xleftarrow{\varphi} \ldots$$

(22)
The dynamic implications \([A, B]\) are thus modeled together with the static implications \((A \Rightarrow B)\), and both sets of rules (4) and (13) are supported. Processes can thus be modeled as machines. This was indeed the starting idea of process semantics [66]. However, early on along this path, it becomes clear that many different machines implement indistinguishable processes, and the problem of process equivalence arises [67]. The input and the output types \(A\) and \(B\) of a process are observable, but the state space \(X\) may not be. In fact, any observable behavior can be realized over many different, unobservable state spaces.

### 2.4 The problem of cut in process logics

The fact that a process model may not support process composition is not just a conceptual shortcoming, but a significant obstacle to applications. Engineering tasks are usually simplified by decomposing required processes into simpler components, and by composing the implemented components. The process component models should thus only display what the components can output, and not their internal structure. The models should, in a sense, not display any concrete implementation details, but should be fully abstract [66, Sec. 4].

In logical terms, this first of all means that the state spaces of state machines must be factored out. The reason is that the composition is logically modeled by a cut rule, something like (1). If process sequents modeled by state machines, and presented in the form \(X \times A \xrightarrow{\phi} X \times B\) and \(Y \times B \xrightarrow{\psi} Y \times C\), then the cut rule would be something like

\[
\frac{X \wedge A \xrightarrow{\phi} X \wedge B \quad Y \wedge B \xrightarrow{\psi} Y \wedge C}{Z \wedge A \xrightarrow{(\phi; \psi)} Z \wedge C}
\]

But how should we reconcile the mismatch of the state spaces \(X\) and \(Y\)? What should be the composite state space \(Z\)? In general, if processes are presented with explicit states, how should the states be passed from process to process?

The main conceptual difference between the data and the states is that the data are processed, whereas the states are the carriers of the processing. The main structural difference is that data can be copied and sent in messages, whereas states cannot be freely copied or communicated in general (although some states may be shared within a given scope). The problem of process composition is thus that the observable aspects of processes, that get passed in process composition from one process to another, need to be separated from the unobservable aspects, that remain hidden from the compositions. The same problem arises in applying processes as dynamic functions, on sources as dynamic elements. The latter are, of course, viewed as a special case of the former. The observable aspects are thus modeled as the data types, whereas the unobservable aspects are modeled as the state spaces.

Dispensing with the states, the process composition can thus be defined as a sequent in the form

\[
[A, B] \wedge [B, C] \xrightarrow{\gamma} [A, C]
\]
In static logics, such sequents that establish transitivity of the implication are equivalent with the cut rule like (1). In process logics, the sequents like (24) are the solution of the problem with the cut rule like (23). The fact that the process implications arise as final coalgebra, as established in Sec. 2.2.2 is thus the reason why the categories where processes are composed as morphisms have final coalgebras as hom-sets \([3, 11, 87, 51]\). The composition sequent in (24) can be derived as follows:

\[
A \land [A, B] \overset{\nu}{\to} B \land [A, B] \\
B \land [B, C] \overset{\nu}{\to} C \land [B, C]
\]

\[
\frac{A \land [A, B] \land [B, C] \overset{\alpha}{\to} B \land [A, B] \land [B, C]}{C \land [A, B] \land [B, C] \overset{\beta}{\to} \frac{\alpha; \beta}{\nu}}
\]

(25)

The task of composing processes, and applying them to sources, thus boils down to the task of interpreting process implications \([A, B]\), and process propositions \([A] = [\top, A]\).

### 3 Functions extended in time

#### 3.1 Elements extended in time as streams

The outputs of a machine \(a = (X, (a^\times, a^\circ)) : A \times X \to A \times X\) are observable as a stream \(a^\omega = (a_0, a_1, \ldots, a_n, \ldots)\). Starting from an initial state \(x_0 \in X\) the process

- outputs \(a_0 = a_{x_0}^\times\) and updates the state to \(x_1 = a_{x_0}^\circ\); then it
- outputs \(a_1 = a_{x_1}^\times\) and updates the state to \(x_2 = a_{x_1}^\circ\); after \(n\) steps, it
- outputs \(a_n = a_{x_n}^\times\) and updates the state to \(x_{n+1} = a_{x_n}^\circ\); and so on.

A dynamic\(^5\) element can thus be construed as a stream of outcomes of a repeated measurement or count. Such data streams arise in science, and modeling them is the subject of statistical inference \([33]\). If the possible outcomes boil down to yes-no statements, then such streams can be construed as \emph{process propositions}\(^6\), or with \emph{dynamic} truth values. When the frequencies are counted, then they are modeled as streams of random variables, or as stochastic processes. In information theory, they are called \emph{sources} \([20, \text{Ch. 6}]\).

#### 3.2 Functions extended in time as deterministic channels

A dynamic function from \(A\) to \(B\) is generated by a machine in the form \(f = (A \times X : (f^\times, f^\circ)) : B \times X \to A \times X\). Starting from an initial state \(x_0 \in X\) the process consists of the following data maps and state

\(^5\)We use the terms “\emph{dynamic}” and “\emph{extended in time}” interchangeably for the moment.

\(^6\)When no confusion with \emph{dynamic logic} seems likely, process propositions are also called \emph{dynamic} propositions.
updates:
\[
\begin{align*}
a_0 &\mapsto b_0 = f_{a_0}^*(a_0) & a_0 &\mapsto x_1 = f_{a_0}^o(a_0) \\
a_0 a_1 &\mapsto b_1 = f_{a_1}^*(a_1) & a_0 a_1 &\mapsto x_2 = f_{a_1}^o(a_1) \\
\cdots & \quad & \cdots & \\
a_0 a_1 \cdots a_n &\mapsto b_n = f_{a_n}^*(a_n) & a_0 a_1 \cdots a_n &\mapsto x_{n+1} = f_{a_n}^o(a_n) \\
\cdots & \quad & \cdots & \\
\end{align*}
\]

A dynamic function can thus be viewed as a stream of functions assignments in the form
\[
f^{\omega} = (f_0, f_1, \cdots, f_n, \cdots)
\]
where
\[
f_n = f_{a_n}^* : A^n \rightarrow B
\]

In information theory, such streams of functions are called deterministic channels [20, Sec. 3.2]. Their spaces will provide the propositions-as-types interpretation of process implication.

### 3.3 History monad and comonad

The history construction \((-)^+ : S \rightarrow S\), described in Sec. 2.3.1 is easily shown to be the semigroup monad, with the structure
\[
\begin{align*}
A \xrightarrow{\eta} A^+ & \quad & A^+ \xleftarrow{g^\circ} B^+ \\
\end{align*}
\]
where \(g \in S(B, A^+),\) and \(\cdot\) is the string concatenation, the semigroup operation in \(A^+.\) For our concerns, it is, however, more interesting that the same functor also forms a comonad, with the structure
\[
\begin{align*}
A &\xleftarrow{\varepsilon} A^+ \xrightarrow{f^\circ} B^+ \\
\end{align*}
\]
where \((a_1 a_2 \cdots a_n)\) can be thought of as histories. The cumulative functions \(f^\#\) are then extended in time. They allow capturing dynamic implications \([A, B]\) as types, or as objects in a category. We first capture them as the hom-sets of a category.

### 3.4 Category of functions extended in time

The category of free coalgebras for the comonad \((-)^+\) is
\[
|\text{S}_+| = |\text{S}| \\
\text{S}_+(A, B) = S(A^+, B)
\]
with the composition using $\#$

$$
\begin{align*}
A^+ & \xrightarrow{f} B \\
A^+ & \xrightarrow{f'} B^+ \\
B^+ & \xrightarrow{g} C
\end{align*}
$$

and the counit $A^+ \xrightarrow{\varepsilon} A$ playing the role of identity with respect to this composition. Note, however, that

$$
S_+(A, B) \cong S(1, [A, B])
$$

The category $S_+$, in a sense, externalizes processes from $S$, and makes them composable. Let us take a closer look at the compositions.

Since $A^+$ is the disjoint union of $\bigsqcup_{n=1}^{\infty} A^n$, a function $f : A^+ \rightarrow B$ can be viewed as the stream $f^\omega = (f_1 f_2 \cdots f_n \cdots)$ of functions $f_n : A^n \rightarrow B$, like in Sec. 3.2. The corresponding cumulative function $f^\#: A^+ \rightarrow B^+$ can then be viewed as the stream $f^\# = (f_1 f_2 \cdots f^n \cdots)$ of functions $f^n : A^n \rightarrow B^n$ which make the following diagram commute

$$
\begin{align*}
A & \xleftrightsquigarrow A^2 \xleftrightsquigarrow A^3 \xleftrightsquigarrow A^4 \xleftrightsquigarrow A^i \xleftrightsquigarrow \cdots \cdots \cdots \\
B & \xleftrightsquigarrow B^2 \xleftrightsquigarrow B^3 \xleftrightsquigarrow B^4 \xleftrightsquigarrow B^i \xleftrightsquigarrow \cdots \cdots \cdots
\end{align*}
$$

where each $\xleftrightsquigarrow$ projects away the rightmost component, and the functions $f^n$ are:

$$
\begin{align*}
f^1 &= f_1 \\
f^{i+1} &= f^i \circ \xleftrightsquigarrow, f_{i+1}
\end{align*}
$$

4 Partial functions extended in time

4.1 Output deletions and process deadlocks

Recall from Sec. 1.3.1(7) that the partiality monad $(-)_\bot : S \rightarrow S$ adjoins a fresh element $\bot$ to every type. A partial function $f : A \rightarrow B$ can be viewed as the total function $A \rightarrow B_\bot$, which sends to $\bot$ the elements where $f$ is undefined. There are two logically different ways to lift this to processes:

$$
\begin{align*}
A \wedge X & \rightarrow B_\bot \wedge X \\
X & \rightarrow [A, B_\bot] \\
A \wedge X & \rightarrow (B \wedge X)_\bot \\
X & \rightarrow [A, B]_\bot
\end{align*}
$$

On the left, the process may delete some of the outputs, but it always proceeds to the next state, whether if has produced the output or not. On the right, the process may deadlock and fail to
produce either the output or the next state. The meanings of the two implications \([A, B] \perp\) and \([A, B] \perp\) are captured, respectively, by the final coalgebras of the two functors

\[
D_{A, B} : S \to S \\
X \mapsto (A \Rightarrow (B \times X))
\]

\[
D_{A, B} : S \to S \\
X \mapsto (A \Rightarrow (B \times X))
\]

The state spaces of the final coalgebras of these two functors are then the hom-sets of the two categories of partial functions extended in time:

\[
|S|_{+} = |S| \\
S_{+}(A, B) = S(A^{+}, B^{\perp})
\]

\[
|S|_{+} = |S| \\
S_{+}(A, B) = \bigsqcup_{S \in S^{\perp}} S(S, B)
\tag{28}
\]

where \(\simeq A^{+}\) is the set of safety specifications in \(A\) \[3, 19, 87\]

\[
\simeq A^{+} = \{ S \subseteq A^{+} \mid \bar{x} \leq \bar{y} \in S \Rightarrow \bar{x} \in S \}
\tag{29}
\]

and where the prefix relation \(\bar{x} \leq \bar{y}\) means that there is \(\bar{z}\) such that \(\bar{x}z = \bar{y}\). An \(S_{+}\)-morphism is a ladder like \(30\), but with partial functions \(f_{i}\) as rungs. The commutativity requirement imples that \(f_{i}(\bar{x})\) must be defined whenever \(f_{i+1}(\bar{x}a)\) is defined for some \(a\). Hence \(S \in \simeq A^{+}\) in \(28\).

### 4.2 Safety and synchronicity

Since \(28\) for \(B = 1\) boils down to \(S_{+}(A, 1) \cong \simeq A^{+}\), the safety properties in \(\simeq A^{+}\) can be viewed as the objects of categories of safe dynamic functions. The morphisms may be synchronous or asynchronous, depending on whether the outputs may be deleted or not.

#### 4.2.1 Synchronous safe functions

The category \(SFun\) of safe dynamic functions has all safety specifications as its objects. Combining the ladders \(26\) in \(S_{+}\) with the surjections \(12\) of \(S_{+}/1\) shows that the safe dynamic functions are ladders in the form

\[
\begin{array}{cccccccc}
S_{1} & \leftarrow & \pi & S_{2} & \leftarrow & \pi & S_{3} & \leftarrow & \pi & S_{4} & \leftarrow & \pi & S_{i} & \leftarrow & \pi \\
| & \downarrow f_{1}^{\perp} & \downarrow \downarrow f_{2}^{\perp} & \downarrow \downarrow f_{3}^{\perp} & \downarrow \downarrow f_{4}^{\perp} & \downarrow \downarrow f_{i}^{\perp} \\
T_{1} & \leftarrow & \pi & T_{2} & \leftarrow & \pi & T_{3} & \leftarrow & \pi & T_{4} & \leftarrow & \pi & T_{i} & \leftarrow & \pi
\end{array}
\tag{30}
\]

where \(f_{i}^{\perp}\) are not just surjections, in the sense that for every history \(\bar{r} \in T\) there is a history \(\bar{s} \in S\) such that \(\bar{r} = f^{\#}(\bar{s})\), but they are surjections extended in time, in the sense that the prefixes of \(\bar{r}\) must have been the image of the prefixes of \(\bar{s}\), i.e. \(\pi(\bar{r}) = f^{\#}(\pi(\bar{s}))\). Categorically, this amounts to saying that the squares in \(30\) are weak pullbacks. Logically, the commutativity of \(30\) uncovers a general coinductive pattern:

\[
f^{\#}(\bar{s}) = \bar{r} \iff \forall b \in B \left(\bar{b} \in T \Rightarrow \exists a \in A. \bar{s}a \in S \land f^{\#}(\bar{s}a) = \bar{b}\right)
\tag{31}
\]
Such coinductive surjections lie at the heart of process theory as components of bisimulations, which we shall encounter in the next section. Before that, note that the dynamic surjections satisfying (31) must be synchronous, in the sense that they preserve the length of the histories: the time ticks steadily up the ladder. If there are silent actions, i.e. if functions may delete their outputs, this synchronicity may be breached.

4.2.2 Asynchronous safe functions

The functions extended in time asynchronously are obtained from $S_{⊥+}$. The element $⊥$ added to the outputs plays the role of the silent, unobservable action $43, 68$. Synchronous models tacitly assume global testing capabilities of the observer $1$. Asynchrony arises because some of the actions of the Environment may not be observable for the System. Capturing this leads to coarser process equivalences, where the Environment may perform several steps while the System observes just one. Viewed as channels, the asynchronous functions extended in time become deterministic deletion channels $69$. Combining both of the constructions (28) allows capturing both forms of the partiality in

\[ |S_{⊥+}| = |S| \]
\[ S_{⊥+}(A, B) = \bigcup_{S \in Y A^*} S(S, B_⊥) \]  \hspace{1cm} (32)

A function $f \in S(S, B_⊥)$ can be viewed as a stream of functions $f = (f_n : S_{≤n} \rightarrow B_⊥)_{n=1}^∞$, where $S_{≤n}$ are safe histories of length up to $n$, including the empty history, i.e.

\[ S_{≤n} = (S \cap A^{≤n}) + \{()\} \]  \hspace{1cm} (33)

where $A^{≤n}$ is the disjoint union (coproduct) $\bigsqcup_{i=0}^n A^i$. The cumulative form $f^\# = (f^{≤n} : S_{≤n} \rightarrow B^{≤n})_{n=1}^∞$ is now defined by

\[
\begin{align*}
  f^{≤1}() &= () \\
  f^{≤1}(a) &= \begin{cases} () & \text{if } f_1(a) = ⊥ \\ f_1(a) & \text{otherwise} \end{cases} \\
  f^{≤n+1}() &= () \\
  f^{≤n+1}(a\vec{x}) &= \begin{cases} f^{≤n}(\vec{x}) & \text{if } f_{n+1}(a\vec{x}) = ⊥ \\ f^{≤n}(\vec{x})::f_{n+1}(a\vec{x}) & \text{otherwise} \end{cases}
\end{align*}
\]

and its components are now the rungs of the ladder

\[
\begin{array}{cccccccc}
S_{≤1} & S_{≤2} & S_{≤3} & S_{≤4} & \cdots & S_{≤i} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots \\
B_{≤1} & B_{≤2} & B_{≤3} & B_{≤4} & \cdots & B_{≤i} & \cdots \\
\end{array}
\]  \hspace{1cm} (34)

where each $\pi$ again projects away the last component. The category $\text{ASFun} = S_{⊥+}/1$ of asynchronous safe functions has the safety specifications as its objects again, and a morphism $f \in$...
ASFun\((S \subseteq A, T \subseteq B)\) is a tower in the form

\[
\begin{array}{cccccccc}
S_{\leq 1} & \leftarrow & S_{\leq 2} & \leftarrow & S_{\leq 3} & \leftarrow & S_{\leq 4} & \leftarrow & \cdots & S_{\leq i} & \leftarrow & \cdots
\\
\downarrow & f^1 & \downarrow & f^2 & \downarrow & f^3 & \downarrow & f^4 & \downarrow & f^i & \downarrow & \cdots
\\
\downarrow & \pi & \downarrow & \pi & \downarrow & \pi & \downarrow & \pi & \downarrow & \pi & \downarrow & \cdots
\\
T_{\leq 1} & \leftarrow & T_{\leq 2} & \leftarrow & T_{\leq 3} & \leftarrow & T_{\leq 4} & \leftarrow & \cdots & T_{\leq i} & \leftarrow & \cdots
\end{array}
\] (35)

The difference from (34) is that the rungs of the squares are weak pullbacks, and that the rungs of the ladder are surjections. This tower says that the surjections extended in time asynchronously are the functions such that

\[
f^\#(s) = t \iff (\forall b \in B. \ \tilde{t}b \in T \Rightarrow \exists \tilde{a} \in A^+. \ \tilde{s}\tilde{a} \in S \land f^\#(\tilde{s}\tilde{a}) = \tilde{t}b)
\] (36)

The difference from (31) is that each step up the \(T\)-side by \(b \in B\) is followed on the \(S\)-side by a string of steps \(\tilde{a} \in A^+\), rather than a single step \(a \in A\).

5 Relations extended in time

5.1 External and internal nondeterminism

Nondeterminism is modeled using the powerset monad \(\wp : S \rightarrow S\), as mentioned in Sec. 1.3.1. The elements \(U \in \wp A\) are the subsets \(U \subseteq A\). The binary relations \(R \subseteq A \times B\) can thus be viewed as functions \(A \xrightarrow{R} \wp B\) and \(B \xrightarrow{\wp R} \wp A\). See Appendix A for more details. Nondeterminism affects processes in two ways again: internal nondeterminism affects the outputs, whereas the external also the states.

\[
\begin{array}{ccc}
A \times X & \xrightarrow{\xi} & (\wp B \times X)_\bot \\
X & \xrightarrow{\xi} & [A, \wp B]_\bot
\end{array}
\] \quad \begin{array}{ccc}
A \times X & \xrightarrow{\zeta} & \wp(B \times X) \\
X & \xrightarrow{\zeta} & [A, B]_{\wp}
\end{array}
\] (37)

External nondeterminism captures partiality as the empty set \(\emptyset \in \wp(B \times X)\). We add the partiality monad \((-)_\bot\) to the internal nondeterminism on the left explicitly, because nondeterministic processes that never deadlock appear artificial both conceptually and in models. An internal nondeterministic process \(\xi\) that does not deadlock at a state \(x \in X\) on an input \(a \in A\), then it determines a unique next state \(\xi^\circ(a, x) \in X\), and possibly produces an output from the set \(\xi^*(a, x) \in \wp B\). For an internally nondeterministic process \(\zeta\) on the right, both the outputs and the state transitions are impacted by the nondeterminism, and any pair from \(\zeta(a, x) \in \wp(B \times X)\) may be produced when the input \(a\) is consumed at state \(x\). The intended meanings of the two process implications \([A, \wp B]_\bot\)

\(^7\)In the regular case, the fact that the rungs are surjections follows from the fact that the starting component is a surjection, and that the squares are weak pullbacks.
and \([A, B]_\varnothing\) are captured, respectively, as the final coalgebras of the functors

\[
P_{AB} : S \to S \\
Q_{AB} : S \to S
\]

\[
X \mapsto (A \Rightarrow (\varnothing B \times X))_\perp \\
X \mapsto \varnothing(A \times B \times X)
\]

(38)

where we use \(\varnothing(A \times B \times X) \cong (A \Rightarrow \varnothing(B \times X))\). The state spaces of the final coalgebras of these two functors are quite different. We consider them separately, in the next two sections.

5.2 External nondeterminism

5.2.1 Synchronous safe relations

The state space of the final coalgebra of the functor \(P_{A\varnothing B}\) can be constructed within \(S\) as a limit of the tower like (22)

\[
1 \leftrightarrow (A \Rightarrow \varnothing B)_\perp \leftrightarrow (A \Rightarrow (\varnothing B \times (A \Rightarrow \varnothing B))_\perp \leftrightarrow \cdots
\]

\[
P_{AB}(1) \leftrightarrow P_{AB}(1) \leftrightarrow \cdots [A, \varnothing B]_\perp
\]

or presented simply as

\[
|S_{+\varnothing}| = |S|
\]

\[S_{+\varnothing}(A, B) = \bigcup_{S \in \mathcal{X} \mathcal{A}^+} S(S, \varnothing B)\]

(40)

A morphism from \(A\) to \(B\) in \(S_{+\varnothing}\) is thus a pair \(\langle S, R \rangle\), where \(S \subseteq A^+\) is a safety specification, and \(R\) is a stream of relations, presented as functions \(\bullet R = \left(S_n \xrightarrow{R_n} \varnothing B\right)_n\), where \(S_n = S \cap A^n\), or viewed cumulatively as

\[
\bullet R^\# = \left(S_n \xrightarrow{R_n} (\varnothing B)^n\right)_n
\]

The inductive definition is like at the end of Sec. 3. On any input \((a_1 a_2 \cdots a_n) \in S\) the \(n\)-th component of \(\bullet R^\#\) thus produces an \(n\)-tuple of subsets of \(B\):

\[
(a_1 a_2 \cdots a_n)R^n = \left\{a_1 R_1, (a_1 a_2)R_2, \ldots, (a_1 \cdots a_{n-1})R_{n-1}, (a_1 \cdots a_{n-1} a_n)R_n\right\}
\]

(41)

If each each function \(S_n \xrightarrow{R_n} (\varnothing B)^n\) is viewed as a relation \(S_n \xleftarrow{R_n} B^n\), then (41) says that they make the following tower commute

\[
S_1 \xleftarrow{\pi} S_2 \xleftarrow{\pi} S_3 \xleftarrow{\pi} S_4 \xleftarrow{\cdots} S_i \xleftarrow{\cdots}
\]

\[
R_1 \xleftarrow{\pi} R_2 \xleftarrow{\pi} R_3 \xleftarrow{\pi} R_4 \xleftarrow{\cdots} R_i \xleftarrow{\cdots}
\]

\[
B \xleftarrow{\pi} B^2 \xleftarrow{\pi} B^3 \xleftarrow{\pi} B^4 \xleftarrow{\cdots} B^i \xleftarrow{\cdots}
\]

(42)
To preclude nontrivial side-effects of processes with trivial outputs, we slice over the trivial type 1 again, and take the category of safe synchronous relations extended in time to be

\[ \text{SProc} = \text{S} \oplus_1 1 \]  

(43)

This is the original interaction category, introduced in [3], and further studied in [11, 87]. The descriptions were different, but it is easy to see that the objects coincide, since the morphisms \( S \in \text{S} \oplus_1 (A, 1) \) are the prefix-closed sets \( S \subseteq A^+ \). Reasoning like in Sec. 4.2.1, a morphism \( S \oplus_1 (\triangleleft A, 1) \rightarrow \rlap{\text{R}} T_\ominus B \) in \( \text{S} \oplus_1 1 \) is now reduced to a ladder of spans

\[
\begin{array}{cccc}
S_1 & \leftarrow & \pi & S_2 \\
& & \phi & S_3 \\
\rlap{\text{R}} & & \rlap{\text{R}} & \rlap{\text{R}} \rlap{\text{R}} \rlap{\text{R}} \rlap{\text{R}} \rlap{\text{R}} \rlap{\text{R}} \\
R_1 & \leftarrow & R_2 & R_3 \leftarrow R_4
\end{array}
\]

(44)

Like in (11), we have relations that are total in both directions, which means that the projections \( R \rightarrow S \) and \( R \rightarrow T \) are surjective, in this case componentwise. Like in (30), the surjections are extended in time, in the sense that all rhombi in (44) are weak pullbacks. Putting it all together, this tower says that \( R \) satisfies

\[
\exists a \in A \left( \exists b \in B. \bar{a} \in S \Rightarrow \exists b \in B. \bar{b} \in T \wedge \exists a \in A. \bar{a} \in S \land \exists a \in A. \bar{a} \in S \right) \wedge \left( \exists b \in B. \bar{b} \in T \Rightarrow \exists a \in A. \bar{a} \in S \land \exists a \in A. \bar{a} \in S \right)
\]

(45)

This condition means that \( S \oplus_1 (\triangleleft A, 1) \rightarrow \rlap{\text{R}} T_\ominus B \) is a strong or synchronous bisimulation relation [68, 74], as required in the original definition of \( \text{SProc} \) in [3].

**Bisimulations are intrinsic.** Bisimulations were originally motivated by the intended semantical identifications of processes, and imposed as a requirement. Here, they are not imposed, but arise as a property of morphisms in a category. The category is, however, built by applying the nondeterminism monad \( \varphi \), the history comonad \((-)^+ \), and then it is sliced over 1. The notion of bisimulation is thus a logical property of nondeterministic processes, provided that the processes with trivial outputs have trivial side-effects.

### 5.2.2 Asynchronous safe relations

Capturing unobservable, silent actions leads to asynchronicity, and to the notion of weak or observational bisimulation [43, 68]. Proceeding like in Sec. 4.2.2, we consider the final coalgebras of the functors

\[
P_{AB} : S \rightarrow S \\
X \mapsto (A \Rightarrow (\varphi(B_\perp) \times X))_\perp
\]

(46)
as the hom-sets of the category
\[
|S_{\varphi \perp}| = |S| \quad \text{(47)}
\]
\[
S_{\varphi \perp}(A, B) = \bigcup_{S \in \mathcal{Y} A^+} S(S, \varphi(B_\perp))
\]

The morphism tower is like (42), but with each $S_n$, $R_n$ and $B^n$ replaced with $S_{\leq n}$, $R_{\leq n}$ and $B_{\leq n}$, as in (33) and (34). The category of *safe asynchronous relations extended in time* is now
\[
\text{ASProc} = S_{\varphi \perp}/1
\]

and the morphism tower is like (44), with the same modification of the subscripts and the superscripts. This modified tower characterizes the following logical property of the *asynchronous* relation $R$ extended in time:
\[
\vec{s} R \vec{t} \iff \forall a \in A \left( \vec{s}a \in S \Rightarrow \exists \vec{b} \in B \left( \vec{t}b \in T \land \vec{s}a R \vec{t}b \right) \right) \land \forall \vec{b} \in B \left( \vec{t}b \in T \Rightarrow \exists a \in A \left( \vec{s}a \in S \land \vec{s}a R \vec{t}b \right) \right)
\]
\[
\left(48\right)
\]

This characterizes the *weak* or *observationsl* bisimulations of [43, 68]. The category ASProc is equivalent to the one introduced and studied under the same name in [3, 77, 87].

### 5.3 Internal nondeterminism

#### 5.3.1 Synchronous dynamic relations

The state space of the final coalgebra of the functor $Q_{AB}$ from (38) should again come with a tower like
\[
1 \xleftarrow{\emptyset} \varphi(A \times B) \xleftarrow{\varphi(A \times B \times 1)} \varphi(A \times B \times \varphi(A \times B)) \xleftarrow{\varphi^2(A \times B)} \ldots
\]
\[
\text{(49)}
\]

The trouble is that such a tower never stabilizes within a universe of sets, since there is no set $X$ such that $X \cong \varphi X$. If we take $A = B = 1$, the tower boils down to
\[
1 \xleftarrow{\emptyset} \varphi 1 \xleftarrow{\varphi} \varphi \varphi 1 \xleftarrow{\varphi} \varphi \varphi \varphi 1 \xleftarrow{\varphi} \varphi \varphi \varphi \varphi 1 \xleftarrow{\varphi} \ldots \left[1, 1\right]_\rho = \mathcal{Y}
\]
\[
\left(50\right)
\]

where the coinductive fixpoint $\mathcal{Y}$ is the class of *hypersets*, or *non-wellfounded sets* [16]. It is dual to von Neumann’s class of well-founded sets [105, 107], which arises as the inductive fixpoint $\mathfrak{B}$ along the tower
\[
\emptyset \xrightarrow{\epsilon} \varphi \emptyset = 1 \xrightarrow{\epsilon} \varphi \varphi 1 \xrightarrow{\epsilon} \varphi \varphi \varphi 1 \xrightarrow{\epsilon} \varphi \varphi \varphi \varphi 1 \xrightarrow{\epsilon} \ldots \mathfrak{B}
\]
\[
\left(51\right)
\]
Von Neumann, of course, did not draw categorical diagrams, but specified his construction in terms of transfinite induction

\[
V_0 = \emptyset \quad V_\beta = \bigcup_{\alpha < \beta} \wp(V_\alpha) \quad \wp = \bigcup_{\alpha \in \text{Ord}} V_\alpha
\]  

(52)

The class \( \text{Ord} \) of ordinals is assumed to be given, so the construction actually provides an inner model of set theory within a given universe of sets and classes [16], or equivalently within a universe with an inaccessible cardinal, which can then play the role of the class \( \text{Ord} \) [21]. In any case, reach a fixpoint within a given universe, the constructor \( \wp \) must be restricted to stay within a smaller universe. Early on, Gödel restricted it to the subsets definable in the language of set theory, and constructed the universe \( \mathcal{U} \) of constructible sets, proving the independence of the Continuum Hypothesis, and launching the whole industry of the independence proofs [39]. Inner models of set-theory in categories of topological spaces, or abstract spaces, have been constructed by restricting to open subspaces [45]. Although set theorists often explicitly exclude \( \aleph_0 \) from the definition of inaccessible cardinals, the fact that the inequalities \( 2^n < \aleph_0 \) and \( \cup n < \aleph_0 \) are satisfied for all for all \( n < \aleph_0 \) makes \( \aleph_0 \) inaccessible from the universe \( \mathcal{F} \) of finite sets. Formally, it is the subcategory of \( S \) spanned by \( U \in S \) such that \( \#U < \aleph_0 \), where \( \#U \) denotes the cardinality of \( U \). Since computation is mostly concerned with finite sets, \( \mathcal{F} \) is often taken to be the universe of ”small sets”, and \( S \) is interpreted as the universe of ”classes”. The powerset construction \( \wp : S \to S \) where \( \wp X = \{ U \subset X \} \) is then replaced with \( \mathcal{P} : S \to S \) where

\[
\mathcal{P} X = \wp_{<\omega} X = \{ U \subset X \mid \#U < \aleph_0 \}
\]

(53)

which restricts to \( \mathcal{P} : \mathcal{F} \to \mathcal{F} \). The tower (50) for \( \wp \) instead of \( \wp \) thus lies in \( \mathcal{F} \), and reaches a fixpoint \( \mathcal{H} \cong \mathcal{P}^\omega \mathcal{H} \) in \( S \) after countably many steps. Since \( \mathcal{P} \) does not preserve limits, the tower does not stabilize at its limit. It turns out to stabilize at a retract of its limit [17, 21, 53, 80]. The projections from the fixpoint down the tower are still jointly monic, and thus still allow inductive reasoning about \( \mathcal{H} = [1, 1]_p \) and \( [A, B]_p \).

Continuing with the workflow from the preceding sections, we use the dynamic implications defined in (49) and define the universe of sets with synchronous dynamic relations:

\[
|S^p| = |S| \quad S^p(A, B) = [A, B]_p
\]

(54)

Like before, we factor out any nontrivial side-effects of processes with trivial outputs by slicing over the trivial type 1 again and define the category dynamic synchronous relations

\[
\text{DProc} = S^p/1
\]

(55)

But now something new happens. When nondeterminism is internalized and the coinductive process accumulates states along the towers (49) by reapplying the powerset constructor \( \wp \) to

---

8A universe with sets and classes can be viewed as a model of the \( \text{NBG} \) set theory, whereas the one with an inaccessible cardinal can be interpreted in terms of the \( \text{ZFC} \) axioms [65, Ch. 4].
the previously accumulated state spaces, then that the label sets $A, B \in S$ turn out to be superfluous, and dispensable. They were used in all constructions so far to identify actions across different processes. Now the actions can be identified by their histories, recorded in their structure. When nondeterminism is internalized, the coinductive process if building process becomes self-contained.

5.3.2 Internalising the labels

All process universes presented up to so far have been built starting from a given universe $S$ of labels. The coinductive construction leading to $\text{DProc}$ has a novel feature that it can be built starting from nothing: the role of the label sets $A \in S$ can be played by structures arising from the construction itself. The role of the labels $a \in A$ is to identify the same action when it occurs in different observations, or safety specifications $S$ or $T$. This is assured by modeling them as subsets $S, T \subseteq A^*$. The upshot is that there can be at most one label-preserving function $S \to T$, namely the inclusion $S \hookrightarrow T$.

When all actions arise in a cumulative hierarchy, by iterating the constructor $\mathcal{P}$, be it inductively (51) or coinductively (50), they are always given as sets with the element relation $\epsilon$, which records the elements of each set, their elements, and so on. The axiom of extensionality

$$a = b \iff (\forall x. x \epsilon a \iff x \epsilon b)$$

(56)

says that this $\epsilon$-structure completely determines the identity of each set: two sets with the same elements are the same set. In the cumulative hierarchy, the elements are also sets, so the same elements are also the sets with the same elements. If such hereditary $\epsilon$-relations are unfolded into trees, then the extensionality axiom means that these trees must be irredundant: they have no nontrivial automorphisms. In other words, they cannot contain isomorphic subtrees at the same level [75]. The $\epsilon$-structures that arise from the cumulative processes in (51) and (50) are extensional, thus irredundant, because the powerset constructors impose $\{a, a, b, c, \ldots\} = \{a, b, c, \ldots\}$. The other way around, Mostowski’s Collapse Lemma [72] says that every well-founded extensional relation corresponds to the $\epsilon$-structure of a set somewhere in $\mathfrak{B}$. Aczel’s crucial observation in [16] is that the well-foundedness assumption can be dropped: any extensional relation, including non-wellfounded, can be reconstructed as the $\epsilon$-relation of a hyperset, somewhere in $\mathfrak{S}$, or for finite sets somewhere in $\mathcal{H}$. The upshot is that any two hypersets $S, T \in \mathcal{H}$, there is at most one $\epsilon$-preserving function $S \rightarrow T$, or else nontrivial automorphisms arise. The role of the label sets can now be played by the $\epsilon$-structures.

Lemma 5.1 For every countable $A \in S$, i.e. such that $\# A \leq \aleph_0$, there are dynamic relations $A \xrightarrow{m} 1$ and $1 \xrightarrow{e} A$ in $S^\mathcal{P}$ which make $A$ into a retract of $1$, i.e. their composite in $S^\mathcal{P}$ is

$$\text{id}_A = \left( A \xrightarrow{e} 1 \xrightarrow{m} A \right)$$

A proof is sketched in Appendix C. To a category theorist, Lemma 5.1 says that the subcategory $S^\mathcal{P}_{\leq \aleph_0} \hookrightarrow S^\mathcal{P}$ spanned by the countable sets is the idempotent completion within $S^\mathcal{P}$ of the endomorphism monoid $H = S^\mathcal{P}(1, 1)$. For the categories

$$\text{dProc} = H/1 \quad \text{DProc}_{\leq \aleph_0} = S^\mathcal{P}_{\leq \aleph_0}/1$$

(57)

29
we have the following corollary, proved in Appendix D.

**Corollary 5.2**  The inclusion

\[ \text{dProc} \leftrightarrow \text{DProc}_{\leq \aleph_0} \]  \hfill (58)

is an equivalence of categories.

**Remark.** The equivalence in the preceding corollary means that the embedding is full and faithful, and essentially surjective, i.e., that every type in \( \text{DProc}_{\leq \aleph_0} \) is isomorphic to a type in the image of \( \text{dProc} \). This notion of equivalence allows finding an adjoint functor in the opposite direction provided that the axiom of choice is given, in this case for classes. The equivalence is thus far from effective globally. Locally, however, any structure present in \( \text{DProc} \) can be found in \( \text{dProc} \), as long as we do not need uncountable sets of labels. In the rest of this paper, we elide the labels.

### 5.3.3 Synchronous dynamic relations as hypersets

The objects of the category \( \text{dProc} \) boil down to the elements of the universe of finite \( \mathcal{H} \), that arises as the coinductive fixpoint of the tower like (50), but with \( \mathcal{P} = \mathcal{P}_{\leq \aleph_0} \). Since \( \mathcal{H} \cong \mathcal{P}\mathcal{H} \), an element of \( \mathcal{H} \) can also be viewed as its finite subset, which unfolds it into a tower

\[ S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow S_4 \leftarrow \cdots \leftarrow S_n \leftarrow \cdots \leftarrow S \]

\[ \mathcal{P}_1 \leftarrow \mathcal{P}_2 \leftarrow \mathcal{P}_3 \leftarrow \mathcal{P}_4 \leftarrow \cdots \leftarrow \mathcal{P}_n \leftarrow \cdots \leftarrow \mathcal{H} \]  \hfill (59)

where all \( S_n \) and \( \mathcal{P}_n \) are in \( \mathcal{F}_S \). This seems like the most convenient presentation of the objects of \( \text{dProc} \). A tower corresponding to a morphism \( R \in \text{dProc}(S, T) \) looks just like (44) in Sec. 5.2.1 except that the projections \( \pi \) are replaced by the set-theoretic operation \( \cup \). The bisimulation condition (45) now becomes

\[ s R t \iff \forall s' \in s \exists t' \in t. s' R t' \land \forall t' \in t \exists s' \in s. s' R t' \]  \hfill (60)

### 5.3.4 Asynchronous dynamic relations

So far, asynchrony has been modeled by adding a silent action label \( \perp \), which allowed waiting. When the actions are modeled using the element relation \( \in \), i.e., by choosing an element from a set, then waiting can be enabled by making the relation \( \in \) reflexive, i.e., by assuming \( x \in x \) for all \( x \). The objects of the category \( \text{aProc} \) of *asynchronous dynamic relations* are now the reflexive finite hypersets, conveniently viewed as the towers of finite subsets

\[ S_{\leq 1} \leftarrow S_{\leq 2} \leftarrow S_{\leq 3} \leftarrow \cdots \leftarrow S_{\leq n} \leftarrow \cdots \leftarrow S \]

\[ \mathcal{P}_{\leq 1} \leftarrow \mathcal{P}_{\leq 2} \leftarrow \mathcal{P}_{\leq 3} \leftarrow \cdots \leftarrow \mathcal{P}_{\leq n} \leftarrow \cdots \leftarrow \mathcal{H} \]  \hfill (61)
where $\mathcal{P}^X = \coprod_{i=0}^n \mathcal{P}^i X$, and $\mathcal{H}^\cup$ is the universe of reflexive finite hypersets. A morphism $R \in \mathfrak{aProc}(S, T)$ is now a reflexive hyperset relation, satisfying the following property

$$s R t \iff \forall s' \in s \ (\exists t' \in t. s' R t' \lor \exists s'' \in s'. s'' R t') \land \forall t' \in t (\exists s' \in s. s' R t' \lor \exists t'' \in t. s R t'')$$

(62)

While this simulation strategy arises from the mathematical structure of final coalgebras again, its computational meaning was studied in [101, 102].

6 Integers, interactions, and real numbers

Counting generates ordinals [105], but the integers arise from the duality of counting up and down. Geometric and algebraic transformations generate monoids, but capturing the symmetries requires groups. The interactions between the system and the environment generate the process universes in the preceding sections, but the dual interactions between the environment and the system were not captured. The duality inherent in process interactions was noted, albeit in passing, very early on in process theory:

”The whole meaning of any computing agent [would be that it is] a transducer, whose input sequence consists of enquiries by, or responses from, its environment, and whose output sequence consists of enquiries of, or responses to, its environment” [66, p. 160].

A similar vision of dual interactions between the system and the environment as an ongoing question-answer protocol re-emerged in linear logic [37]. It was formalized categorically in [13], and retraced in [4]. The mathematical underpinning turned out to be the Int-construction, the free construction of compact structure from traced monoidal structure [46]. The name Int does not refer to the interaction interpretation, but to the integers. Applied to the additive monoid $\mathbb{N}$ of natural numbers, viz. a discrete monoidal category, the construction gives rise to the additive group $\mathbb{Z}$ of integers, viewed as a discrete compact category. In particular, the set of integers is defined as the quotient

$$\mathbb{Z} = \text{Int}_\mathbb{N} = \mathbb{N}_- \times \mathbb{N}_+ / \sim$$

where the equivalence relation $\sim$ is:

$$\langle m_-, m_+ \rangle \sim \langle n_-, n_+ \rangle \iff m_- + n_+ = n_- + m_+$$

The two components of the product are annotated for convenience, e.g. as $\mathbb{N}_- = \{”-”\} \times \mathbb{N}$ and $\mathbb{N}_+ = \{”+”\} \times \mathbb{N}$. The general form of this Int-construction, applicable to suitable monoidal categories, is outlined in Appendix E. The crucial feature of the monoid $\mathbb{N}$ which allows reducing the equivalence classes in $\text{Int}_\mathbb{N}$ to the representatives in the form $\langle n, 0 \rangle$ or $\langle 0, n \rangle$ is that

$$m + k = n + k \implies m = n$$

The fact that interactions and integers are captured by the same structure is a nice example of the workings of category theory.
The crucial feature needed for lifting the Int-construction to monoidal categories is the trace operation. We now describe how the trace operation arises in categories of relations, including the relations extended in time, thus allowing the applications of the Int-construction to the interaction categories.

As the categories of relations, as described in Appendix [A], are self-dual, the coproducts + generating the universe of sets $S$ give rise to biproducts $\oplus$, not only in the category $R$ of static relations, but also in the dynamic cases $SProc$, $ASProc$, $dProc$ and $aProc$. The biproducts are by definition both products and coproducts. The unit of the monoidal structure that they form is the coproduct unit 0. For every type $X$, the biproduct structure consists of

- a monoid $0 \xrightarrow{!} X \xleftarrow{[id,id]} X \oplus X$, and
- a comonoid $0 \xleftarrow{!} X \xrightarrow{([id,id])} X \oplus X$,

which are natural for all morphisms into and out of $X$. The projections $X \xleftarrow{\pi} X \oplus Y \xrightarrow{\pi'} Y$ and the injections $X \xrightarrow{\iota} X \oplus Y \xrightarrow{\iota'} Y$ are derived from the comonoid counits and from the monoid units respectively. A propositions-as-types interpretation of biproducts is tenuous but a process category with the biproducts and the hom-sets $[A, B]$ supporting a coinductive rule

$$A \oplus X \xrightarrow{\xi} B \oplus X$$
$$X \xrightarrow{[\xi]} [A, B]$$

comes with the trace structure $Tr$ derived by

$$A \xrightarrow{\iota} A \oplus Y \quad A \oplus Y \oplus [A \oplus Y, B \oplus Y] \xrightarrow{\iota \cup \pi} B \oplus Y \oplus [A \oplus Y, B \oplus Y] \quad B \oplus Y \xrightarrow{\pi} B$$

$$A \oplus [A \oplus Y, B \oplus Y] \xrightarrow{\iota \cup \pi} B \oplus [A \oplus Y, B \oplus Y]$$

$$[A \oplus Y, B \oplus Y] \xrightarrow{Tr\parallel \iota \cup \pi} [A, B]$$

Each of the categories of relations, $R$, $SProc$, $dProc$, etc., is easily seen to give rise to the trace structure in this way. See Appendix [E] for more.

### 6.1 Games as labelled polarized relations extended in time

The biproducts in $ASProc$ are in the form

$$(S \oplus T)_{\leq 1} \xleftarrow{\iota} (S \oplus T)_{\leq 2} \xleftarrow{\iota} (S \oplus T)_{\leq 3} \xleftarrow{\iota} \cdots \xleftarrow{\iota} (S \oplus T)_{\leq i} \xleftarrow{\iota} \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$(A + B)_{\leq 1} \xleftarrow{\iota} (A + B)_{\leq 2} \xleftarrow{\iota} (A + B)_{\leq 3} \xleftarrow{\iota} \cdots \xleftarrow{\iota} (A + B)_{\leq i} \xleftarrow{\iota} \cdots$$

where $(S \oplus T)_{\leq A+B}$ are all shuffles of $S_{\leq A}$ and $T_{\leq B}$.

$$(S \oplus T)_{\leq i} = \left\{ \vec{x} \in (A + B)^{\leq i} \mid \vec{x} \uparrow A \in S \land \vec{x} \uparrow B \in T \right\}$$

32
The trace structure of categories of relations with respect to the biproducts as the monoidal structure was analyzed already in the final section of [46], and explained in more detail for the interaction categories in [4]. The analysis presented in that paper suggests that the AJM-games \([12, 15, 14]\) should be construed in terms of the \(\text{Int}\)-construction. The AJM-games are, of course, one of the crowning achievements of the quest for fully abstract models of PCF, and a tool of many other semantical results. They appeared in many different semantical contexts \([5, 14, 100]\), with many refinements and different presentation details. A crude common denominator can be obtained by applying the \(\text{Int}\)-construction from Appendix E to the category \(\text{ASProc}\), leading to

\[
|\text{Gam}| = |\text{ASProc}|_\times |\text{ASProc}|_+ \\
\text{Gam}(S, T) = \text{ASProc}(S_+ T_+ T_- S_+) 
\]

Some of the crucial features of game semantics, such as the copycat strategy, and the various switching and starting conditions, arise in such reconstructions as abstract mathematical properties, like the notions of bisimulations arose before.

### 6.2 Polarized dynamics

Since \(\mathcal{P}(A + B) \equiv \mathcal{P}A \times \mathcal{P}B\), applying the powerset constructor on polarized sets \(X_+ X_+\) leads to the functor

\[
Q: S \rightarrow S \\
X \mapsto \mathcal{P}X_\times \mathcal{P}_+X 
\]

where the subscripts are still just annotations, and we can take, e.g., \(\mathcal{P}X_\equiv \{\text{"}_-, \text{"}_+\} \times PX\) and \(\mathcal{P}_+X = \{\text{"}+\}\times PX\).

#### 6.2.1 Synchronous case

The universe of signed finite hypersets can be constructed just like the universe of hypersets in Sec. 5.3, just bifurcating at each step:

\[
1 \longleftrightarrow \mathcal{P}1 \times \mathcal{P}1 \longrightarrow \mathcal{P}_-(\mathcal{P}1 \times \mathcal{P}1) \times \mathcal{P}_+(\mathcal{P}1 \times \mathcal{P}1) \longleftrightarrow \\
\mathcal{H}_+ 
\]

The final coalgebra structure still maps each hyperset to its elements, but this time they can be positive or negative

\[
\mathcal{H}_\pm \leftarrow \mathcal{H}_\pm \mathcal{P}_\times \mathcal{P}_\pm \mathcal{H}_\pm 
\]
Notation. Given \( s \in \mathcal{H}_\pm \), we write \( s^- = \varphi_-(s) \) for the negative part and \( s^+ = \varphi_+(s) \) for the positive part. We often tacitly identify \( \mathcal{H}_\pm \) with \( \mathcal{P}_- \mathcal{H}_\pm \times \mathcal{P}_+ \mathcal{H}_\pm \), in which case \( s \in \mathcal{H}_\pm \) becomes a pair \( s = (s^-, s^+) \), where \( s^- = \varphi_-(s) \) and \( s^+ = \varphi_+(s) \). We follow \([31]\) and denote a generic element of \( s^- \) by \( s_- \), and a generic element of \( s^+ \) by \( s_+ \), and abbreviate \( s_- \in \varphi_-(s) \) and \( s_+ \in \varphi_+(s) \) to \( s_- \), \( s_+ \in s \). Writing \( s = \{s_- | s_+\} \) instead of \( s = \langle s^-, s^+ \rangle \) is yet another well-established notational abuse, used to great effect used by John Conway in \([31]\). E.g., instead of \( \cup s = \langle \cup s^-, \cup s^+ \rangle \), the unions in \([65]\) can be written in the form

\[
\cup s = \{s_- \cup s_+ | s_-, s_+\}
\]

and other coinductive definitions become even simpler,

\[
\oplus s = \{\oplus s_+ | \oplus s_-\} \quad s \oplus t = \{s_- \oplus t, s_+ \oplus t_- | s_+ \oplus t, s \oplus t_+\}
\]

Synchronous hypergames. The objects of the category \( \text{gam} \) are the signed finite hypersets from the universe \( \mathcal{H}_\pm \). The final coalgebra structure \([65]\) separates their elements into a negative and a positive part. In game semantics, this is interpreted as separating a game \( s \in \mathcal{H}_\pm \) into a pair \( s = \langle s^-, s^+ \rangle \), where \( s^- = \{s_- \in s\} \in \mathcal{P}_- (\mathcal{H}_\pm) \) are the moves available to the player \( - \), whereas \( s^+ = \{s_+ \in s\} \in \mathcal{P}_+ (\mathcal{H}_\pm) \) are the moves available to the player \( + \). The projections \( \mathcal{H}_\pm \overset{q_i}{\longrightarrow} \mathcal{Q}^i \) down the tower \([64]\) represent each game \( s \in \mathcal{H}_\pm \) as a stream \([s^1, s^2, s^3, \ldots, s^{n+1}, \ldots]\), where \( s^{n+1} = q_{n+1} \in Q^{n+1} = \mathcal{P}_-(\mathcal{Q}^1) \times \mathcal{P}_+(\mathcal{Q}^1) \), and thus \( s^n = \langle s_{n+1}^-, s_{n+1}^+ \rangle \), where \( s_{n+1}^-, s_{n+1}^+ \subseteq \mathcal{Q}^n \).

A morphism \( R \in \text{gam}(s, t) \) should be a synchronous hyperstrategy. It is a hyperstrategy because the players \( - \) and \( + \) play two games \( s \) and \( t \), or more if at least one these are already a composite game, or less if one of them is empty. The dual goals of the two players makes a hyperstrategy into a polarized version of synchronous bisimulations \([60]\). But the polarization separates the two simulation tasks, and each player is tasked with one:

\[
s \; R \; t \iff \forall s_- \in s \exists t_- \in t. \ s_- \; R \; t_- \quad \land \quad \forall t_+ \in t \exists s_+ \in s. \ s_+ \; R \; t_+.
\]

(67)

The player \( - \) is tasked with simulating every \( s \)-step by a \( t \)-step, whereas the player \( + \) is tasked with simulating every \( t \)-step by an \( s \)-step.

6.2.2 Asynchronous case

Using the functor \( \overline{\mathcal{Q}} : S \to S \) where \( \overline{\mathcal{Q}}X = X + QX \), the tower in \([64]\) becomes

\[
1 \leftarrow Q^{n+1}(1) \overset{\overline{\mathcal{Q}}}{\leftarrow} Q^{n+2}(1) \overset{\ldots}{\leftarrow} Q^{n+1}(1) \overset{\overline{\mathcal{Q}}}{\leftarrow} Q^{n+1}(1) \overset{\ldots}{\leftarrow} R
\]

where \( Q^{n+1}(1) = \bigcup_{i=0}^{n} Q^i(1) \). The final coalgebra structure is thus

\[
\mathcal{R} \overset{\bigoplus}{\leftarrow} \mathcal{R} + \mathcal{P}_- \mathcal{R} \times \mathcal{P}_+ \mathcal{R}
\]

(69)

The coalgebra structure \( \varphi \) maps \( s = \langle s^-, s^+ \rangle \) to \( s = \varphi(s) \) if \( s^- \in s^- \) and \( s^+ \in s^+ \). Otherwise it unfolds its elements into \( s^- = \varphi_p s \) and \( s^+ = \varphi_+ s \) like before. A straightforward induction along the tower gives the following.
Lemma 6.1 Every $s \in \mathcal{R}$ is $\epsilon$-transitive, in the sense that for all $s_-, s_+ \in s$ holds

$$s_- \subseteq s^- \subseteq s^+_-$ $s^+_+ \subseteq s^+ \subseteq s^+$$

(70)

The elements of the universe $\mathcal{R}$ of transitive finite signed hypersets can be thought of as asynchronous hypergames. They are the objects of the category $\mathcal{R}$. An asynchronous hyperstrategy $R \in \mathcal{R}(s, t)$ resembles a branching bisimulation from (62), except that the two simulation tasks are again separated, like in (67), and assigned to the two players:

$$s R t \iff \forall s_- \in s. (\exists t_- \in t. s_- R t_- \lor \exists s_+ \in s s_- R t) \land \forall t_+ \in t. (\exists s_+ \in s. s_+ R t_+ \lor \exists t_- \in t s R t_-)$$

(71)

Lemma 6.1 makes the relations induced by the coalgebra structure on $\mathcal{R}$ into hyperstrategies. Remember that $s_- \epsilon s$ abbreviates $s_- \in \mathcal{P}_- s$, whereas $s_+ \epsilon s$ abbreviates $s_+ \in \mathcal{P}_+ s$.

Lemma 6.2 For any $s \in \mathcal{R}$ and all $s_-, s_+ \epsilon s$, the relations $s_- \epsilon s$ and $s \epsilon s_+$ satisfy (71).

Proof. $s_- \subseteq s^-$ implies that for every $s_-$ there is $s'_-$ with $s_- \epsilon s'_-$, $s^+ \subseteq s^+_+$ implies that for every $s_+$ there is some $s_+$ with $s_+ \epsilon s_+$. Hence (71) for $s_- \epsilon s$. $s^+ \subseteq s^+ _-$ implies that for every $s_-$ there is $s_- s_+$. Hence (71) for $s \epsilon s_+$.

Remark. The property in (71) is not self-dual under the relational converse, but under the polarity change in (66). In game semantics, this duality corresponds to switching the roles of the player and the opponent. A winning strategy of the player becomes winning for the opponent, and vice versa. The game-theoretic equilibrium strategies, where both players play their best responses, seems to correspond to reimposing the bisimulation requirement: that the same relation is a simulation both ways. The equilibrium strategies would support two dualities: not just the polarity change (i.e. switching the players), but also the relational converse (i.e. playing in the opposite direction). While the two dualities are in general not independent, in the situations when they commute, they would induce the dagger-compact structure, akin to the adjunction vs. transposition over complex vector spaces, in theory of modules, and in many other areas of geometry. This structure was not used in game semantics, but it emerged in the Abramsky-Coecke models of quantum protocols, and has been explored in other areas of semantics of computation [10, 29, 81].

6.3 A category of real numbers

In closing this section, we encounter a remarkable fact: that the posetal collapse of the category $\mathcal{R}$ boils down to the ordered field $\mathbb{R}$ of the real numbers. On one hand, this may not be surprising, since John Conway reconstructed numbers from games a long time ago [31], and game semantics was informed by his constructions since early on [12]. On the other hand, game semantics has been developed as semantics of computational processes.
6.3.1 Coalgebra of reals

We adapt the alternating dyadics from [88, Sec. 3.2] to present the real numbers. Consider the alphabet \( \Sigma = \{-, +\} \), and denote by \( \Sigma^\circ \) the set of finite and infinite strings over it. It comes with the coalgebra structure

\[
\Sigma^\circ \xrightarrow{\chi} 1 + \Sigma \times \Sigma^\circ
\]

where \( \chi \) maps the empty string (\( () \)) into 1 and each nonempty strings into its head symbol and the tail string. Equivalently, this coalgebra can be written in the form

\[
\Sigma^\circ \xrightarrow{\kappa} 1 + \Sigma^- + \Sigma^+
\]

Where the product \( \Sigma \times \Sigma^\circ \), which is \( \{-, +\} \times \Sigma^\circ \) is expanded into \( \{-\} \times \Sigma^\circ + \{+\} \times \Sigma^\circ \), and the products with the singletons are abbreviated as subscripts. The structure map \( \kappa \) now maps the empty string into 1, and the strings in the form \( \pm : \vec{x} \) as \( \vec{x} \) into \( \Sigma^\circ_2 \), whereas the components \( h_- \) and \( h_+ \) add \(-\) and \(+\) as the head, while \( o \) maps the singleton from 1 into the empty string.

Each \( \Sigma \)-string encodes a unique real number. The idea is that we count the first string of \(-s\) or \(+s\) in the unary, and after that proceed in the alternating dyadics, e.g.

\[
++---+--- \mapsto +1 + 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64}
\]

\[
---+-++... \mapsto -1 - 1 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} \ldots
\]

Since the infinite strings of \(-s\) and of \(+s\) encode the two infinities, we will have a map into the extended reals \( \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\} \). The bijection \( \Sigma^\circ \cong \overline{\mathbb{R}} \) is described in Appendix F. We henceforth identify the two, and use both names interchangeably, since \( \Sigma^\circ \) refers to the encoding, and \( \overline{\mathbb{R}} \) says what is encoded.

**Ordering.** The usual ordering of the reals in \( \overline{\mathbb{R}} \) corresponds to the lexicographic ordering of \( \Sigma^\circ \). When the finite strings are padded by 0s, the symbol ordering is \(- < 0 < +\).

6.3.2 Numbers extended in time: Conway’s version of Dedekind cuts

**Theorem 6.3** There are functors

\[
\overline{\mathbb{R}} \xrightarrow{\Gamma} \mathbb{R}
\]

which make the extended continuum \( \overline{\mathbb{R}} \) into the posetal collapse of the category \( \mathbb{R} \) of asynchronous hypergames. In particular,

\[\text{[89, 91] for a broader context.}\]
• for every real number \( \varsigma \in \mathbb{R} \) holds \( \Upsilon \Gamma(\varsigma) = \varsigma \);

• for every asynchronous hypergame \( s \in \mathcal{R} \) there are natural hyperstrategies
  \[ s \xrightarrow{\eta} \Gamma \Upsilon(s) \quad \text{and} \quad \Gamma \Upsilon(s) \xrightarrow{\xi} s \]

**Proof** (sketch). The functor \( \Gamma \) can be obtained from the anamorphism \( \llbracket \kappa \rrbracket \)
\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\kappa} & \mathbb{R} + \mathcal{P}\mathbb{R} \times \mathcal{P}_+\mathbb{R} \\
\llbracket \kappa \rrbracket & & \llbracket \kappa \rrbracket + \mathcal{P}\mathbb{R} \times \mathcal{P}_+\mathbb{R} \\
\downarrow & & \downarrow \\
\mathcal{R} & \xrightarrow{\gamma} & \mathcal{R} + \mathcal{P}\mathcal{R} \times \mathcal{P}_+\mathcal{R}
\end{array}
\]

where \( \kappa \) is derived from (73), by mapping the empty string to the empty string, the \( \Sigma \)-strings in the form \((- :: \varsigma)\) to the pair \( \langle \{ \varsigma \}, \emptyset \rangle \), and the strings in the form \((+ :: \varsigma)\) to \( \langle \emptyset, \{ \varsigma \} \rangle \). Setting \( \Gamma \varsigma = \llbracket \kappa \rrbracket \varsigma \), the functoriality of \( \Gamma \) boils down to the observation that the lexicographic order \( \varsigma \leq \vartheta \) on \( \Sigma^\circ \) lifts to a relation \( s \leq t \) on \( s = \Gamma \varsigma \) and \( t = \Gamma \vartheta \) which satisfies (71), i.e.
\[
s \leq t \iff \forall s_- \in s (\exists t_- \in t. \ s_- \leq t_- \lor \exists s_{+} \in s_. \ s_{+} \leq t) \land \forall t_+ \in t (\exists s_+ \in s. \ s_+ \leq t_+ \lor \exists t_{+} \in t+. \ s \leq t_{+})
\]
(75)

As long as \( \varsigma \) and \( \vartheta \) are unpadded by 0s, their lexicographic ordering leads to \( s = \Gamma \varsigma \) and \( t = \Gamma \vartheta \) satisfying the synchronous comparison clauses \( s_- \leq t_- \) and \( s_+ \leq t_+ \) of (75). If \( \vartheta \) is padded by 0s, then (75) is satisfied because the lexicographic ordering induces \( s_{+} \leq t \). If \( \varsigma \) is padded by 0s, then it induces \( s \leq t_{+} \). This completes the definition of \( \Gamma \).

The functor \( \Upsilon \) arises from Conway’s *simplicity theorem* [31, Thm. 11]. It picks the simplest representatives of the equivalence classes of the posetal collapse of \( \mathcal{R} \), where the simplicity is measured in [31] by the “birthday ordinal”, which for our finite hypersets, signed or not, boils down each element’s position on its coinduction tower. The simplicity theorem plays a central role in all presentations of surreal numbers, and suitable versions have been proved in detail in [18, 40]. The arrow part of \( \Upsilon \) collapses the \( \mathcal{R} \)-morphisms to the lexicographic order on \( \Sigma^\circ \). Conway shortcuts his proof of the simplicity theorem by imposing the posetal collapse directly signed hypersets by
\[
s \leq t \iff \forall s_- \in s \forall t_+ \in t. \ \ t \not\leq s_- \land \ t_+ \not\leq s
\]
(76)

Instantiating this definition to \( t \leq s_- \) and to \( t_+ \leq s \) (76) gives
\[
t \not\leq s_- \iff \exists t_- \in t. \ s_- \leq t_- \lor \exists s_{+} \in s_. \ s_{+} \leq t
\]
\[
t_+ \not\leq s \iff \exists t_{+} \in t+. \ s \leq t_{+} \lor \exists s_+ \in s. \ s_+ \leq t_+
\]
and shows that (76) implies (75). The converse, just like the proofs of the simplicity theorem in [18, 40], involve routinely but extensive case reasoning. The equivalence classes of the posetal quotient of \( \mathcal{R} \) are thus ordered by (76), which on \( \Sigma^\circ \) boils down to the lexicographic order. \( \square \)
Remarks. Conway’s proof of the simplicity theorem demonstrates coinduction in action, not only at the formal level in (76), but also at the meta-level. In order to define the \( R \)-ordering of the minimal representatives of the equivalence classes of his games, reduced to numbers, he imposes the sought ordering as a preorder on the arbitrary representatives, and then uses that preorder as a shortcut to prove the existence of the minimal representatives. Lemma 6.2 also shows how the simplicity follows from the coinductive construction, as it implies \( \Upsilon(s_-) \leq \Upsilon(s) \leq \Upsilon(s_+) \), and steers the coinductive descent towards the simplest representative.

6.3.3 Real numbers as processes

Thm. 6.3 says that real numbers can be viewed as processes, and the other way around, that asynchronous, polarized, reflexive processes boil down to real numbers, and that the simulations between them are consistent with the real number ordering. If these processes are thought of as observations, then the reals are the outcomes of the measurements. On the other hand, computations with the reals always involve some embedding into a universe where multiple processes correspond to each number. For irredundant representations, where each real number corresponds to a unique stream of digits, there are always basic arithmetical operations where no finite prefix of an input suffices to determine a finite prefix of the output [25]. Such operations clearly cannot be computable.

Dropping the infinite strings \(-\infty = ( \cdots - - - )\) and \(\infty = ( + + + \cdots )\) on the left-hand side of the retraction \( \mathbb{R} \xrightarrow{\ll} \mathcal{R} \) in (74), and the signed hypersets bisimilar to \(-\infty = \{-\infty\}\) and \(\infty = \{\infty\}\) on the right-hand side, we get the retraction \( \mathbb{R} \xrightarrow{\ll} \mathcal{R} \). It lifts to \( \mathbb{R}^p \xrightarrow{\ll} \mathcal{R}^p \), and makes real vector spaces into retracts of discrete functor categories. A real matrix \( L \in \mathbb{R}^{p \times q} \) becomes an \( \mathcal{R} \)-profunctor \( \Lambda = \left( p \xleftarrow{\Gamma L} q \right) \), and the linear operators \( \mathbb{R}^p \xrightarrow{L} \mathbb{R}^q \) and \( \mathbb{R}^q \xleftarrow{L^*} \mathbb{R}^p \) become the \( \mathcal{R} \)-extensions of \( \Lambda = \Gamma L \) along the Yoneda embeddings, in the enriched-category sense[1]

\[
\begin{array}{ccc}
  p & \xrightarrow{\varepsilon} & \mathbb{R}^p \\
  \downarrow & & \downarrow \Lambda^* \sim \Lambda_* \sim \\
  \Lambda & \sim & \mathcal{R} \\
  \downarrow & & \downarrow \\
  q & \xleftarrow{\Lambda} & \mathbb{R}^q \\
\end{array}
\]

The left Kan extension \( \Lambda^* \) maps the functor \( \alpha \in \mathbb{R}^p \) into the coend, which is the colimit along \( \alpha \) of its tensors with the left transpose of \( \Lambda \). The right Kan extension \( \Lambda_* \) maps the functor \( \beta \in \mathbb{R}^q \) into the end, which is the limit along \( \beta \) of its cotensors with the right transpose of \( \Lambda \). But since \( \alpha \) and \( \beta \) are discrete, the colimits boil down to coproducts, and the limits boil down to products. And since \( \mathcal{R} \) is self-dual, the products and the coproducts coincide as biproducts, and the tensors and the cotensors also coincide, and the extensions become

\[
\Lambda^*(\alpha) = \left( \bigoplus_{i=1}^{p} \alpha_i \otimes \Lambda_{ij} \right)_{j=1}^{q} \quad \Lambda_*^*(\beta) = \left( \bigoplus_{j=1}^{q} \Lambda_{ij} \otimes \beta_j \right)_{i=1}^{p} \]

11The reader unfamiliar with what any of this means is welcome to skip the next paragraph paragraph.
where $\oplus$ are the biproducts in $\mathcal{R}$, and $\otimes$ is a tensor, defined in [31] as the addition and the multiplication on the equivalence classes of Conway’s numbers, which in our framework amounts to $\Upsilon(s \oplus t) = \Upsilon s + \Upsilon t$ and $\Upsilon(s \otimes t) = \Upsilon s \cdot \Upsilon t$. The linear action of $\mathbb{R}$-matrices on $\mathbb{R}$-vectors are thus ”rediscovered” as the Kan extensions of $\mathbb{R}$-profunctors along the Yoneda embeddings as $\mathbb{R}$-completions.

6.4 Where is computation?

As exciting they are, the real numbers here suggest that we seem to have lost computation somewhere along the way, while retracing the path of categorical semantics of computation. The process universe $\mathcal{R}$ allows us to compute with the reals $\mathbb{R}$. Each real number $\varsigma \in \mathbb{R}$ is included as a process $\Gamma_\varsigma \in \mathcal{R}$. We can compute with the reals as in $\mathcal{R}$, and any outcome $s \in \mathcal{R}$ can be reduced to its irredundant representative $\Upsilon s \in \mathbb{R}$. The problem is that any real number can be obtained as an output of such computations. But most real numbers are not computable. An uncomputable real number can be constructed by a diagonal argument. If all computations can be encoded by programs, and programs are expressions in a countable language, then the computable reals can be enumerated. An uncomputable real number can be constructed by choosing its $n$-th digit to be different from the $n$-th digit in the $n$-th computable real number. Proceeding with quantifying how many such numbers there are shows that most real numbers are uncomputable, whichever way we quantify them. And we have all of them in $\mathcal{R}$. Everything any oracle can tell any computer is already there. On the path from propositions-as-types, through process propositions as types extended in time, to dynamic interactions, the idea of computability as programmability got overshadowed the process of computation.

In the final section, we retrace the path back to one of the original questions of categorical semantics: How can intensional computation be characterized semantically?

7 Categorical semantics as a programming language

A process is computable if it is programmable[12] In a universe of processes, types are used to specify requirements and to impose constraints. In a universe of computable processes, there is also a type $\mathbb{P}$ of programs. Since any Turing-complete language can encode its own interpreter, any model of a Turing-complete language must contain the type $\mathbb{P}$ of programs in that language.

A model of computable processes is extensional if it only describes the extensions of computations, i.e. their input-output functions, and does not say anything about the process of computation. Each computable function is thus assigned a unique ”program”. Type-theoretically, this unique ”program” is captured by the abstraction operation, which maps an $X$-indexed family of functions $f_\varsigma(a) : A \times X \to B$ to the corresponding $X$-indexed family of abstractions $\lambda a.f_\varsigma(a) : X \to (A \Rightarrow B)$.

[12]Network processes are sometimes also called computations, although they are not globally controllable, and thus not programmable. They can be steered by interacting programs and protocols, but that is a different story. The notion of computability was originally defined as computability by computers, and the term is still used in that sense.

[13]The tacit assumption is that a model of a programming language contains all types recognizable in that language.
The application operation applies an abstraction to its inputs and recovers the corresponding function. The bijection between the abstractions and their applications was displayed in (4). If a "program" tells not just which inputs go to which outputs, but also how some states change in computation, then it describes not just a computable function, but a computing machine, and the bijection between functions and their abstractions (4) becomes a mapping of machines to their anamorphisms in (13). Each machine \( \xi : A \times X \rightarrow B \times X \) is described by the induced anamorphism \( \llbracket \xi \rrbracket : X \rightarrow [A,B] \) in the sense that it assigns computational behaviors as meanings to the states in \( X \). It is just a one-way map now because there are in general some state assignments that do not describe any actual machines, and there are different machines that are assigned the same behaviors. The distinction between the static view (4), and the dynamic view (13) echoes the difference between the denotational and the operational approach to semantics [5, 9, 26]. In terms of the logical distinction between the extensional and the intensional models of meaning, going back to Frege, Carnap, Church and Martin-Löf [34], all models of computation with an operation that maps computations into programs fall squarely on the extensional side. Genuinely intensional models comprise operations to map programs to computations, but not the other way around.

7.1 Categorical semantics of computability

The logical schema of intensional computation therefore looks dual to (13), at least at the first sight:

\[
\begin{align*}
X \xrightarrow{\rho} P & \\
A \land X \xrightarrow{\rho} \square(B \land X) & \\
S(X, P) & \Downarrow \square \\
S_M(A \times X, B \times X) &
\end{align*}
\]

The duality of (77) and (13) is disturbed by a couple of details and a deep conceptual difference. First of all, the first clause of (13) is derivable in (77) as \( \nu = \llbracket \text{id}_P \rrbracket \), so that is not the issue. Adding \( \square \) and \( M \) to (13) would not change the subsequent analysis, so that is not the issue either. The feature that breaks the symmetry and points to the difference is the requirement that \( \llbracket \cdot \rrbracket \) in (77) must be a surjection. If there is a way to split this surjection, by an operation \( \sqrt{\cdot} \) that would choose for every computation \( c \) a program \( \sqrt{c} \) such that \( \llbracket \sqrt{c} \rrbracket = c \), then there is also a way to retract \( \llbracket \cdot \rrbracket \) into a bijection, and thus reduce (77) to (4), making the model extensional [42]. If no such operations can be constructed in the model, then it can be shown that there must be infinitely many programs \( p \) for each computation \( c \) such that \( c = \llbracket p \rrbracket \). In fact, such models always contain models of Turing machines [90]. In a slightly different presentation (with effect monads reduced to the monoidal structure that they induce), such models were introduced in [84] as monoidal computers. For a fixed universe \( S \) and a fixed effect monad \( M \) on it, the structure of a monoidal computer is essentially unique, in the sense that any two program types \( P \) with the structure from (77) must be isomorphic [85]. This means that this structure captures a property, like e.g. the structure of a lattice captures a property of the underlying poset: namely, the property that it contains the last upper bound and the greatest lower bound for every finite subset. If the structure of monoidal computer \( S \) thus presents a property, namely that all of its morphisms are programmable, then
this structure captures computability-as-programmability as an intrinsic property, in the sense of [9]. Computability of the functions in a monoidal computer can be tested and studied without any references to external structures, since the structure of a monoidal computer can be tested and studied without any such references, just like the structure of a group, or a topological space.

On the other hand, it was explained in [9, Sec. 1.2.3] that the notion of computability, as defined in the standard Church-Turing approach, is extrinsic, in the sense that a particular computable function is recognized as such only by referring to a particular external model, say a Turing machine or a definitional schema. This model describes a particular computation of the function, but it is not recorded or recognizable on the function itself. The model is external to the function in that precise sense. The fact that all such external models have been proved equivalent does not make them internal to the function. Moreover, no canonical choice of one model over another one can be given for a particular function. It is thus argued in [9] that the standard definitions do not specify computability as an intrinsic structure, even less as a property of a function.

In contrast, (77) expresses the idea of computability-as-programmability as a logical structure; and by the virtue of uniqueness of that structure, as a logical property. Whatever programming language $\mathcal{P}$ might be used to encode programs, its interpretation is always a mapping $\mathcal{S}(X, \mathcal{P}) \rightarrow \mathcal{S}_M(A \times X, B \times X)$ of programs into computational processes. (The relation between the $X$-indexing of the programs and the $X$-state updates of the executions will be clarified shortly.) Whichever Church-Turing model of computation might be used to define computability, its underlying execution model will map the process descriptions into the described processes, and this mapping will engender structure of a monoidal computer again. This structure thus provides a “canonical form witnessing computability”, sought in [9, Sec. 1.2.3]. Some of the goals stated at the end of that paper are also pursued in [85].

The naturality requirement implicit in (77) is dual, mutatis mutandis, to the one we spelled out for (13). More precisely, an $X$-indexed family $\{\vdash\}_{X}^{AB} : \mathcal{S}(X, \mathcal{P}) \rightarrow \mathcal{S}_M(A \times X, B \times X)$ is a natural transformation $\{\vdash\}_{X}^{AB} : \mathcal{P} \rightarrow \Theta_{AB}$ between the functors

\[
\begin{align*}
\mathcal{P} : & \mathcal{S} \rightarrow \mathcal{R} \\
X & \mapsto \mathcal{S}(X, \mathcal{P}) \\
\Theta_{AB} : & \mathcal{S} \rightarrow \mathcal{R} \\
X & \mapsto \mathcal{S}_M(A \times X, B \times X)
\end{align*}
\]

with the arrow parts like (16) and (18) in Sec. 2.2.2. The naturality requirement is dual to (14), so it implies that the diagram on the left commutes for every $p \in \mathcal{S}(X, \mathcal{P})$.

\[
\begin{array}{ccc}
\mathcal{S}((\mathcal{P}, \mathcal{P}) & \xrightarrow{(-, p)} & \mathcal{S}(X, \mathcal{P}) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\mathcal{S}_M(A \times \mathcal{P}, B \times \mathcal{P}) & \xrightarrow{(p)} & \mathcal{S}_M(A \times X, B \times X) \\
\end{array}
\quad
\begin{array}{ccc}
A \times X & \xrightarrow{\{p\}} & M(B \times X) \\
\downarrow & & \downarrow \\
A \times \mathcal{P} & \xrightarrow{\{\text{id}\}} & M(B \times \mathcal{P})
\end{array}
\]

The diagram on the right arises by chasing $\text{id} \in \mathcal{S}(\mathcal{P}, \mathcal{P})$ through the diagram on the left. The left-hand diagram says that $\{\text{id}\}_{\mathcal{P}}^{AB}$ and $\{p\}_{X}^{AB}$ are related under $\Theta_{AB}p$, which by the definition in (16) means that the right-hand square commutes. Since the naturality implies that for all $f \in \mathcal{S}(Y, X)$
and \( p \in S(X, \mathbb{P}) \) holds

\[
\|p \circ f\|_Y = \|p\|_X \circ (A \times f) = \|\text{id}\|_{\mathbb{P}} \circ (A \times pf)
\]

dropping the subscripts from \( \|\cdot\|_X \) seldom causes confusion. The surjectivity of \( \|\cdot\| \) moreover says that for every computation \( c \in S_M(A \times X, B \times X) \) there is an \( X \)-indexed program \( p_c \in S(X, \mathbb{P}) \) that makes right-hand square in (79) commute. Since this is true for all types \( A \) and \( B \), the claim is thus that \( \mathbb{P} \) is the state space of a weakly final \( AB \)-machine \( \|\text{id}\|_{\mathbb{P}} \in S_M(A \times \mathbb{P}, B \times \mathbb{P}) \) — for all types \( A \) and \( B \) in \( S \).

**Proposition 7.1** Let \( S \) be a process-closed category (see Def. 2.1). Let \( M : S \to S \) be a commutative monad and \( \mathbb{P} \) a fixed type. The following structures are equivalent:

a) for any \( A, B \in S \), a family of surjections \( \|\cdot\| : S(X, \mathbb{P}) \to S_M(A \times X, B \times X) \), called program executions, natural in \( X \) with respect to the functors in (78);

b) for any \( A, B, C \in S \)
   
   \begin{itemize}
   \item a universal program evaluator \( \varphi^{AC} \in S_M(A \times \mathbb{P}, C) \) and
   \item a partial program evaluator \( \sigma^{AB} \in S(A \times \mathbb{P}, \mathbb{P}) \)
   \end{itemize}

such that for any \( f \in S_M(A, C) \) there is \( p \in S(1, \mathbb{P}) \) such that

\[
\begin{align*}
  f &= \varphi^{AC} \circ (A \times p) \\
  \varphi^{(A \times B)C} &= \varphi^{AC} \circ (A \times \sigma^{AB})
\end{align*}
\]

\[\text{(80)}\]

**Proof** can be found in [85, 90].

**Upshot.** The proposition says that the structure of monoidal computer, displayed in (77) and formalized in condition (a), is a categorical version of the concept of acceptable enumeration, normally used in computability textbooks, e.g. [94]. The type \( \mathbb{P} \) of programs is used as the set of program indices for the enumeration. In the standard notation, the enumeration would thus be a sequence \( (\varphi^x_n)_{x \in \mathbb{P}} \), where \( x \) is the program index, and \( n \) is the arity of the computable function \( \varphi_x \). While the computable functions are usually modeled over natural numbers, and the arity \( n \) means that the function takes the inputs of type \( \mathbb{N}^n \), and always produces the outputs of type \( \mathbb{N} \), the categorical treatment is over abstract types, so we write \( \varphi^{AB} \) to specify the input type \( A \) and the output type \( B \). The basic constructions can be simply copied from textbooks. E.g., the fixpoint
The claim is that for every computable function \( g \in \mathcal{S}_M(A \times \mathcal{P}, B) \) there is a program \( \Gamma \in \mathcal{P} \) which describes the same function like \( g \) instantiated on it, i.e. \( g(a, \Gamma) = \varphi_{AB}^{\Gamma}(a) \), for all \( a \in A \). To construct \( \Gamma \), we precompose \( g \) with the partial evaluator \( \sigma \), set to evaluate programs on themselves, like in (81) on the left. The resulting computation is thus \( g(a, \sigma(x, x)) \). The first clause of (80) then gives a program \( p \) for this computation, which makes the square in (81) commute. Now we substitute \( p \) into \( \varphi(x, x) \) along the left-hand arrow in the bottom triangle of (81), and call the result \( \Gamma = \sigma(p, p) \). The bottom triangle commutes by the definition of the pairing. The right-hand triangle commutes by the second clause of (80). All of (81) is thus commutative, and the claim is proved. A more popular version of the Recursion Theorem says that for every effect-free \( \tau \in \mathcal{S}(\mathcal{P}, \mathcal{P}) \), thought of as a program transformer, there is a program \( T \) which computes the same as its transform, i.e. \( \{T\} = \{\tau T\} \). Diagram (81) is easily modified to capture this claim. The two versions are, in a suitable sense, geometrically equivalent. The claim is called "Kleene’s amazing theorem" and its many repercussions are discussed in [71]. Their intrinsic geometry surfaces in categorical semantics, be it in the form of commutative diagrams like (81), or in string diagrams [85, 84, 104]. They support a diagrammatic programming language that can be used to implement computable logic and arithmetic, program schemas, abstract metaprogramming concepts like compilation, super-compilation, synthesis, and to derive static, dynamic, and algorithmic complexity measures. The reconstruction is comparable with the use of the \( \lambda \)-calculus as a theoretical programming language, and as the foundation of the functional programming languages. The fundamental difference is that, as mentioned above, the mere presence of the operation of \( \lambda \)-abstraction implies that the underlying type system is essentially extensional, in the sense that it contains a canonical extensional retract, which precludes intrinsic intensional phenomena [42]. In any case, the idea of promoting semantical models into programming languages [92, 96] and studying semantical models of those [14, 44] has been an effective methodological principle of categorical semantics from the outset. It seems reasonable to look in that direction for an intrinsic view of intensional computation.
7.2 Computability-as-programmability

Computable and uncomputable functions and processes are housed in the same category \( \mathcal{S} \). For some observable computational effects \( M \), and for all types \( A \) and \( B \), we build some program evaluators \( \sigma_{AB} \in \mathcal{S}(A \times \mathcal{P}, B) \) and \( \varphi^{AB} \in \mathcal{S}_M(A \times \mathcal{P}, B) \). The partial evaluations of programs as the elements of type \( \mathcal{P} \) over a family of types \( X \in \mathcal{S} \), deemed computable, induce the indexed programs \( \mathcal{S}(X, \mathcal{P}) \subseteq \mathcal{S}(X, \mathcal{P}) \). The unrestricted universal and partial evaluations of the \( X \)-indexed programs induce the \( X \)-natural (polymorphic) program executions \( \mathcal{S}(X, \mathcal{P}) \to \mathcal{S}_M(A \times X, B \times X) \). The surjective images \( \mathcal{S}(X, \mathcal{P}) \twoheadrightarrow \mathcal{S}_M(A \times X, B \times X) \subseteq \mathcal{S}_M(A \times X, B \times X) \) determine the computable functions. They form a subcategory \( \mathcal{S} \hookrightarrow \mathcal{S} \), which carries the structure of a monoidal computer. Its structure supports logic and arithmetic, program evolution, and complexity measurement and limitations. It encodes any of the standard models of computation, much like they encode one another. The difference is that categorical semantics displays computability as an intrinsic property, in the sense of [9].

Many languages of logic claim universality, and establish their universality on their own terms. E.g., set theory proves that it is the foundation of all mathematics, the first-order logic is the language of structures and predicates, etc. The statement that logic is tasked with discovering the universal laws of logic is a tautology, in a logic of logic. But a universal law should not be misunderstood as the last word about anything, but as the first word about something else. The idea that computability-as-programmability is a model-invariant, syntax-independent, device-free concept, and a property intrinsic to all computable objects and processes, is broader than any particular structure, albeit categorical, in which it may be expressed. The idea of computability-as-programmability, in a suitable formulation, is the conceptual content of Kolmogorov’s invariance theorem [60, Sec. 2.1]. Although recognizing a particular function as computable depends on encoding its computation in a particular model, as argued in [9], the invariance theorem implies that the computability of the function is an intrinsic property nevertheless, because the encodings of the function in the various models are not only computable but also programmable, and the program transformations that perform the re-encodings between the models are of constant lengths. Computability-as-programmability is not thus only testable by any of the equivalent models of computation, as claimed by the Church-Turing thesis, but it is also quantifiable, in Kolmogorov’s formulation by the length of programs, and the quantifications are equivalent, in the sense that the differences between the encodings, taken over all unbounded program descriptions, are bounded by a constant. Kolmogorov’s algorithmic complexity is thus the quantitative view of the intrinsic property of computability-as-programmability. By displaying programmability as a structure, categorical semantics provides the qualitative view of this property.

It should be noted that qualitative and quantitative views of computation as processing of programs, descriptions, or information in general come about in disguise in many corners of science. Although the search for programs that make a function computable-as-programmable is generally not a computable process, its average algorithmic complexity is an intrinsic quantity again: the Shannon entropy [73, 108]. Information theory as the theory of information processing, has thus been viewed as the theory of computation in microsystems, averaged out in thermodynamics. Along the same lines, domain theory can also be viewed as an abstract theory of intrinsic computability, this time factored out as approximation in suitable topologies [2, 98, Sec. 5.1]. A
natural path ahead for categorical semantics could be to bring such conceptual threads together, and close the loop.

8 Conclusion

In the propositions-as-types view, the extensional operations of abstraction and application, \(\text{viz}\) the structure of cartesian closed categories, correspond to the introduction and the elimination of the propositional implication:

\[
\begin{align*}
(A \land X) \vdash B \\
\hline
X \vdash (A \Rightarrow B)
\end{align*}
\]

\[
\begin{align*}
S(A \times X, B) \\
\hline
(A \Rightarrow) \circ \eta_X \downarrow \nabla_X \circ (A \times \cdot)
\end{align*}
\]

\[
\begin{align*}
S(X, (A \Rightarrow B))
\end{align*}
\]

In process logics, the process implication introduction rule corresponds to the coinductive interpretation of arbitrary states as process behaviors, captured in the final machine:

\[
\begin{align*}
A \land X \models \diamondsuit (B \land X) \\
\hline
X \models \llbracket \cdot \rrbracket \to [A, B]_\Diamond
\end{align*}
\]

\[
\begin{align*}
S_M(A \times X, B \times X) \\
\hline
\llbracket \cdot \rrbracket \downarrow \nabla_{\cdot} \downarrow
\end{align*}
\]

\[
\begin{align*}
S(X, [A, B]_M)
\end{align*}
\]

In terms of dynamic types, computation corresponds to program execution. In terms of process propositions, computability-as-programmability is thus an elimination rule, mapping programs, as intensional proofs of the universal proposition, the programming language, into computations as their extensions:

\[
\begin{align*}
X \models \rho \llbracket P \rrbracket \\
\hline
A \land X \models [\cdot] (B \land X)
\end{align*}
\]

\[
\begin{align*}
S(X, P) \\
\hline
\llbracket \cdot \rrbracket \downarrow \nabla_{\cdot}
\end{align*}
\]

\[
\begin{align*}
S_M(A \times X, B \times X)
\end{align*}
\]

Categorical semantics provides convenient and sometimes effective tools for reasoning about types and processes. Samson Abramsky led many of us through its vast landscape. I followed him to the best of my ability. The present paper is an attempt at a travel report. But the territory is largely uncharted, and there were times when I lost sight of Samson, probably somewhere far ahead. It is thus likely that the travel report is not just about what I learned from Samson, but also about what I misunderstood by getting lost, and maybe most of all about what I did not learn at all. Categorical semantics of computational processes is a computational process itself, and it is the nature of such processes that they may terminate, or not.
References

[1] Samson Abramsky. Observation equivalence as a testing equivalence. *Theor. Comput. Sci.*, 53(2-3):225–241, 1987.

[2] Samson Abramsky. Domain theory in logical form. *Ann. Pure Appl. Log.*, 51(1-2):1–77, 1991.

[3] Samson Abramsky. Interaction categories. In Geoffrey L. Burn, Simon J. Gay, and Mark Ryan, editors, *Theory and Formal Methods 1993*, Workshops in Computing, pages 57–69. Springer, 1993.

[4] Samson Abramsky. Retracing some paths in process algebra. In Ugo Montanari and Vladimiro Sassone, editors, *CONCUR*, volume 1119 of *Lecture Notes in Computer Science*, pages 1–17. Springer, 1996.

[5] Samson Abramsky. Semantics of interaction: an introduction to game semantics. In *Semantics and logics of computation*, volume 14 of *Publications of the Newton Institute*, pages 1–31. Cambridge University Press, 1997.

[6] Samson Abramsky. Abstract scalars, loops, and free traced and strongly compact closed categories. In J.L. Luiz Fiadeiro et al., editor, *Proceedings of the First International Conference on Algebra and Coalgebra in Computer Science (CALCO)*, volume 3629 of *Lecture Notes in Computer Science*, pages 1–29. Springer, 2005.

[7] Samson Abramsky. Coalgebras, chu spaces, and representations of physical systems. In *Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science, LICS 2010, 11-14 July 2010, Edinburgh, United Kingdom*, pages 411–420. IEEE Computer Society, 2010.

[8] Samson Abramsky. Big toy models: Representing physical systems as Chu spaces. *Synthese*, 186(3):697–718, 2012.

[9] Samson Abramsky. Intensionality, definability and computation. In Alexandru Baltag and Sonja Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, pages 121–142. Springer, 2014.

[10] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Proceedings of 19th IEEE Symposium on Logic in Computer Science (LICS 2004)*, 2004.

[11] Samson Abramsky, Simon J. Gay, and Rajagopal Nagarajan. Specification structures and propositions-as-types for concurrency. In Faron Moller and Graham M. Birtwistle, editors, *Logics for Concurrency (Proceedings of 8th Banff Higher Order Workshop)*, volume 1043 of *Lecture Notes in Computer Science*, pages 5–40. Springer, 1995.
[12] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic (extended abstract). In R. K. Shyamasundar, editor, *Foundations of Software Technology and Theoretical Computer Science, 12th Conference, New Delhi, India, December 18-20, 1992, Proceedings*, volume 652 of *Lecture Notes in Computer Science*, pages 291–301. Springer, 1992.

[13] Samson Abramsky and Radha Jagadeesan. New Foundations for the Geometry of Interaction. *Inf. Comput.*, 111(1):53–119, 1994.

[14] Samson Abramsky, Radha Jagadeesan, and Pasquale Malacaria. Full abstraction for PCF. *Information and Computation*, 163(2):409–470, 2000.

[15] Samson Abramsky, Pasquale Malacaria, and Radha Jagadeesan. Full abstraction for PCF. In Masami Hagiya and John C. Mitchell, editors, *TACS*, volume 789 of *Lecture Notes in Computer Science*, pages 1–15. Springer, 1994.

[16] Peter Aczel. *Non-well-founded Sets*. Center for the Study of Language and Information Publication Lecture Notes. Cambridge University Press, 1988.

[17] Jirí Adámek and Václav Koubek. On the greatest fixed point of a set functor. *Theor. Comput. Sci.*, 150(1):57–75, 1995.

[18] Norman L. Alling. *Foundations of Analysis over Surreal Number Fields*, volume 141 of *Notas de Matemática*. North-Holland, 1987.

[19] Bowen Alpern and Fred B. Schneider. Recognizing safety and liveness. *Distributed Comput.*, 2(3):117–126, 1987.

[20] Robert B. Ash. *Information Theory*. Dover Publications, 1990.

[21] Michael Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science*, 114(2):299 – 315, 1993.

[22] Michael Barr, Pierre A. Grillet, and Donovan H. van Osdol. *Exact Categories and Categories of Sheaves*, volume 236 of *Lecture Notes in Mathematics*. Springer Verlag, 1971.

[23] Michael Barr and Charles Wells. *Toposes, Triples, and Theories*. Number 278 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1985. Republished in: Reprints in Theory and Applications of Categories, No. 12 (2005) pp. 1-287.

[24] Stephen D. Brookes and Shai Geva. Computational comonads and intensional semantics. In M. P. Fourman et al., editor, *Applications of Categories in Computer Science*, volume 177 of *London Math. Society Lecture Note Series*, pages 1–44. Cambridge Univ. Press, 1992.

[25] L. E. J. Brouwer. Besitzt jede reelle Zahl eine Dezimalbruchentwicklung? *Mathematische Annalen*, 83(3):201–210, 1921.
[26] Roberto Bruni and Ugo Montanari. *Models of Computation*. Texts in Theoretical Computer Science. An EATCS Series. Springer International, 2017.

[27] Alonzo Church. A formulation of the simple theory of types. *The Journal of Symbolic Logic*, 5(2):56–68, 1940.

[28] J.Robin B. Cockett and D.A. Spooner. Categories for synchrony and asynchrony. *Electronic Notes in Theoretical Computer Science*, 1:66 – 90, 1995. MFPS XI, Mathematical Foundations of Programming Semantics, Eleventh Annual Conference.

[29] Bob Coecke, Éric Oliver Paquette, and Dusko Pavlovic. Classical and quantum structuralism. In Simon Gay and Ian Mackie, editors, *Semantical Techniques in Quantum Computation*, pages 29–69. Cambridge University Press, 2009.

[30] Bob Coecke and Dusko Pavlovic. Quantum measurements without sums. In G. Chen, L. Kauffman, and S. Lamonaco, editors, *Mathematics of Quantum Computing and Technology*, page 36pp. Taylor and Francis, 2007. arxiv.org/quant-ph/0608035.

[31] John H. Conway. *On numbers and games*. A K Peters, 2001. (2. ed.).

[32] Thierry Coquand. Infinite objects in type theory. In *International Workshop on Types for Proofs and Programs*, volume 806 of *Lecture Notes in Computer Science*, pages 62–78. Springer, 1993.

[33] Ronald A. Fisher. *Statistical methods and scientific inference*. Hafner Press, 1973.

[34] Melvin Fitting. Intensional logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2020 edition, 2020.

[35] Peter Freyd and Andre Scedrov. *Categories, Allegories*. Number 39 in Mathematical Library. North-Holland, 1990.

[36] Dov M. Gabbay. *Labelled Deductive Systems*. Labelled Deductive Systems. Clarendon Press, 1996.

[37] Jean-Yves Girard. Towards a geometry of interaction. In J.W. Gray and A. Scedrov, editors, *Categories in computer science and logic*, volume 92 of *Contemporary Mathematics*, pages 69–108. American Mathematical Society, 1989.

[38] Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types*. Number 7 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1989.

[39] Kurt Gödel. *The Consistency of the Axiom of Choice and of the Generalized Continuum-hypothesis with the Axioms of Set Theory*. Number 3 in Annals of Mathematics Studies. Princeton University Press, 1940.

[40] Harry Gonshor. *An Introduction to the Theory of Surreal Numbers*, volume 110 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1986.
[41] Carl Gunter. *Semantics of Programming Languages: Structures and Techniques*. Foundations of Computing. MIT Press, 1992.

[42] Susumu Hayashi. Adjunction of semifunctors: Categorical structures in nonextensional lambda calculus. *Theor. Comput. Sci.*, 41:95–104, 1985.

[43] Matthew Hennessy and Robin Milner. On observing nondeterminism and concurrency. In J. W. de Bakker and Jan van Leeuwen, editors, *Automata, Languages and Programming, 7th Colloquium, Noordwijkerhout, The Netherlands, July 14-18, 1980, Proceedings*, volume 85 of *Lecture Notes in Computer Science*, pages 299–309. Springer, 1980.

[44] J. Martin E. Hyland and C.-H. Luke Ong. On full abstraction for PCF: I, II, and III. *Inf. Comput.*, 163(2):285–408, 2000.

[45] André Joyal and Ieke Moerdijk. *Algebraic Set Theory*. Number 220 in London Mathematical Society Lecture Notes. Cambridge University Press, 1995.

[46] Andre Joyal, Ross Street, and Dominic Verity. Traced monoidal categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 119(3):447–468, 1996.

[47] G. Max Kelly and Manuel L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193 – 213, 1980.

[48] Stephen C. Kleene. A theory of positive integers in formal logic. *Amer. J. of Math.*, 57:153–173, 1935.

[49] Andrej Kolmogoroff. Zur Deutung der intuitionistischen Logik. *Mathematische Zeitschrift*, 35(1):58–65, 1932.

[50] Georg Kreisel. A survey of proof theory. *Journal of Symbolic Logic*, 33:321–388, 1968.

[51] Sava Krstić, John Launchbury, and Dusko Pavlović. Categories of processes enriched in final coalgebras. In Furio Honsell, editor, *Proceedings of FoSSaCS 2001*, volume 2030 of *Lecture Notes in Computer Science*, pages 303–317. Springer Verlag, 2001.

[52] Joachim Lambek. Deductive Systems and Categories I: Syntactic Calculus and Residuated Categories. *Mathematical Systems Theory*, 2:287–318, 1968.

[53] Joachim Lambek. A fixpoint theorem for complete categories. *Mathematische Zeitschrift*, 103:151–161, 1968.

[54] Joachim Lambek. Deductive Systems and Categories II: Standard Constructions and Closed Categories. In P. Hilton, editor, *Category Theory, Homology Theory and their Applications*, number 86 in Lecture Notes in Mathematics, pages 76–122. Springer-Verlag, 1969.
[55] Joachim Lambek. Deductive Systems and Categories III: Cartesian Closed Categories, Intuitionist Propositional Calculus, and Combinatory Logic. In F. William Lawvere, editor, *Toposes, Algebraic Geometry, and Logic*, number 274 in Lecture Notes in Mathematics, pages 57–82. Springer-Verlag, 1972.

[56] Joachim Lambek. From types to sets. *Adv. in Math.*, 36:113–164, 1980.

[57] Joachim Lambek and Philip Scott. *Introduction to Higher Order Categorical Logic*. Number 7 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1986.

[58] F. William Lawvere. An elementary theory of the category of sets. *Proceedings of the National Academy of Sciences of the United States of America*, 52:1506–1511, 1964. Reprinted in *Theory and Applications of Categories* 11(2005) 1–35.

[59] F. William Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969. Reprint in Theory and Applications of Categories, No. 16, 2006, pp.1–16.

[60] Ming Li and Paul M. B. Vitányi. *An introduction to Kolmogorov complexity and its applications* (2. ed.). Graduate texts in computer science. Springer, 1997.

[61] Saunders Mac Lane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag, 1971.

[62] Ernest Manes. *Algebraic Theories*. Number 26 in Graduate Texts in Mathematics. Springer-Verlag, 1976.

[63] Per Martin-Löf. An intuitionistic theory of types: Predicative part. In Harvey Rose and John Sheperdson, editors, *Logic Colloquium ’73*, number 80 in Studies in Logic and the Foundations of Mathematics, pages 73–118. North-Holland, 1975.

[64] Per Martin-Löf. *Intuitionistic Type Theory*. Bibliopolis, Naples, 1984.

[65] Elliott Mendelson. *Introduction to Mathematical Logic*. Discrete Mathematics and Its Applications. Taylor & Francis, 6 edition, 2015.

[66] Robin Milner. Processes: a mathematical model of computing agents. In *Proceedings of the Logic Colloquium. Bristol, July 1973*, volume 80 of *Studies in Logic and the Foundations of Mathematics*, pages 157–173. Elsevier, 1975.

[67] Robin Milner. *A Calculus of Communicating Systems*, volume 92 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, Heidelberg, 1982.

[68] Robin Milner. *Communication and concurrency*, volume 84 of *Series in Computer Science*. Prentice Hall, New York, 1989.

[69] Michael Mitzenmacher. A survey of results for deletion channels and related synchronization channels. *Probability Surveys*, 6:1–33, 2009.
[70] Eugenio Moggi. Notions of computation and monads. *Information and Computation*, 93(1):55 – 92, 1991. Selections from 1989 IEEE Symposium on Logic in Computer Science.

[71] Yiannis N. Moschovakis. Kleene’s amazing Second Recursion Theorem. *Bulletin of Symbolic Logic*, 16(2):189–239, 2010.

[72] Andrzej Mostowski. An undecidable arithmetical statement. *Fundamenta Mathematicae*, 36:143–164, 1949.

[73] Anatol Muchnik and Nikolai Vereshchagin. Shannon Entropy vs. Kolmogorov Complexity. In Dima et al. Grigoriev, editor, *Computer Science — Theory and Applications*, pages 281–291, Berlin, Heidelberg, 2006. Springer.

[74] David Park. Concurrency and automata on infinite sequences. In P. Deussen, editor, *Theoretical computer science*, number 104 in Lecture Notes in Computer Science, pages 167–183. Springer, 1981.

[75] Dusko Pavlovic. Convenient categories of processes and simulations I: modulo strong bisimilarity. In D. Pitt et al., editor, *Category Theory and Computer Science ‘95*, volume 953 of *Lecture Notes in Computer Science*, pages 3–24. Springer Verlag, 1995.

[76] Dusko Pavlovic. Maps I: relative to a factorisation system. *J. Pure Appl. Algebra*, 99:9–34, 1995.

[77] Dusko Pavlovic. Convenient categories of processes and simulations II: modulo weak and branching bisimilarities. In A. Edalat et al., editor, *Theory and Formal Methods of Computing 96*, pages 156–167. World Scientific, 1996.

[78] Dusko Pavlovic. Maps II: Chasing diagrams in categorical proof theory. *J. of the IGPL*, 4(2):1–36, 1996.

[79] Dusko Pavlovic. Categorical logic of names and abstraction in action calculus. *Math. Structures in Comp. Sci.*, 7:619–637, 1997.

[80] Dusko Pavlovic. Guarded induction on final coalgebras. *E. Notes in Theor. Comp. Sci.*, 11:143–160, 1998.

[81] Dusko Pavlovic. Relating toy models of quantum computation: comprehension, complementarity and dagger autonomous categories. *E. Notes in Theor. Comp. Sci.*, 270(2):121–139, 2011. arxiv.org:1006.1011.

[82] Dusko Pavlovic. Geometry of abstraction in quantum computation. *Proceedings of Symposia in Applied Mathematics*, 71:233–267, 2012. arxiv.org:1006.1010.

[83] Dusko Pavlovic. Tracing Man-in-the-Middle in monoidal categories. In Dirk Pattinson and Lutz Schroeder, editors, *Proceedings of CMCS 2012*, volume 7399 of *Lecture Notes in Computer Science*, pages 191–217. Springer Verlag, 2012. arXiv:1203.6324.
[84] Dusko Pavlovic. Monoidal computer I: Basic computability by string diagrams. *Information and Computation*, 226:94–116, 2013. arxiv:1208.5205.

[85] Dusko Pavlovic. *CatCom: Stories and Pictures of Computability and Complexity*. in progress; available on request, 2020.

[86] Dusko Pavlovic. Logic of fusion. In C. Talcott et al., editor, *Proceedings of the Symposium in Honor of Andre Scedrov*, Lecture Notes in Computer Science. Springer, 2020. to appear.

[87] Dusko Pavlovic and Samson Abramsky. Specifying interaction categories. In E. Moggi and G. Rosolini, editors, *Category Theory and Computer Science ’97*, volume 1290 of *Lecture Notes in Computer Science*, pages 147–158. Springer Verlag, 1997.

[88] Dusko Pavlovic and Vaughan Pratt. On coalgebra of real numbers. *E. Notes in Theor. Comp. Sci.*, 19:133–148, 1999.

[89] Dusko Pavlovic and Vaughan Pratt. The continuum as a final coalgebra. *Theor. Comp. Sci.*, 280(1-2):105–122, 2002.

[90] Dusko Pavlovic and Muzamil Yahia. Monoidal computer III: A coalgebraic view of computability and complexity. In Corina Cîrstea, editor, *Coalgebraic Methods in Computer Science (CMCS) 2018 — Selected Papers*, volume 11202 of *Lecture Notes in Computer Science*, pages 167–189. Springer, 2018. arxiv:1704.04882.

[91] Duško Pavlović and Martín Escardó. Calculus in coinductive form. In Vaughan Pratt, editor, *Proceedings. Thirteenth Annual IEEE Symposium on Logic in Computer Science*, pages 408–417. IEEE Computer Society, 1998.

[92] Gordon Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–255, 1977.

[93] John Reynolds. *Theories of Programming Languages*. Cambridge University Press, 1998.

[94] Hartley Rogers, Jr. *Theory of recursive functions and effective computability*. MIT Press, Cambridge, MA, USA, 1987.

[95] Bertrand Russell. Mathematical logic based on the theory of types. *American Journal of Mathematics*, 30:222–262, 1908. Reprinted in [103], pages 150–182.

[96] Dana S. Scott. A type-theoretical alternative to ISWIM, CUCH, OWHY. *Theoretical Computer Science*, 121(1-2):411–440, 1993. technical report written in 1969.

[97] Jonathan P. Seldin and J. Roger Hindley, editors. *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*. Academic Press, London, 1980.

[98] Michael B. Smyth. Topology. In S. Abramsky and D. Gabbay, editors, *Handbook of Logic in Computer Science (Vol. 1). Background: Mathematical Structures*, pages 641–761. Oxford University Press, Inc., USA, 1993.
References

[99] Tarmo Uustalu and Varmo Vene. Comonadic notions of computation. *Electronic Notes in Theoretical Computer Science*, 203(5):263–284, 2008.

[100] Matthijs Vákár, Radha Jagadeesan, and Samson Abramsky. Game semantics for dependent types. *Inf. Comput.*, 261(Part):401–431, 2018.

[101] Rob J. van Glabbeek. The Linear Time – Branching Time Spectrum II. The semantics of sequential processes with silent moves. In Eike Best, editor, *Proceedings of CONCUR ’93*, volume 715 of *Lecture Notes in Computer Science*, pages 66–81. Springer, 1993.

[102] Rob J. van Glabbeek and W. P. Weijland. Branching time and abstraction in bisimulation semantics. *J. ACM*, 43(3):555–600, 1996.

[103] Jan van Heijenoort, editor. *From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931*. Harvard University Press, 1967. Reprinted 1971, 1976.

[104] Jaap van Oosten. *Realizability: An Introduction to Its Categorical Side*, volume 152. Elsevier Science, 2008.

[105] Johann von Neumann. Zur Einführung der transfiniten Zahlen. *Acta litt. Acad. Sc. Szeged*, X(1):199–208, 1923. English translation, “On the introduction of transfinite numbers” in [103], pages 393–413.

[106] Philip Wadler. Propositions as types. *Commun. ACM*, 58(12):75–84, November 2015.

[107] Ernst Zermelo. Über Grenzzahlen Und Mengenbereiche: Neue Untersuchungen Über Die Grundlagen der Mengenlehre. *Fundamenta Mathematicae*, 16:29–47, 1930.

[108] Wojciech H. Zurek. Algorithmic randomness and physical entropy. *Physical Review A*, 40(8):4731, 1989.

Appendices

A Category R of sets and relations

Relations $A \leftarrow R \rightarrow B$ arise in two ways:

a) as subsets $R \ni A \times B$, so that

$$aRb \iff \exists x \in X. a = r_A(x) \land r_B(x) = b$$

b) as a nondeterministic functions $A \ni \wp B$ and $B \ni \wp A$, so that

$$aRb \iff \wp(a) \ni b \iff a \in \wp(b)$$

where $\wp : S \rightarrow S$ is the powerset monad.
The equivalence between the two views lies at the heart of the elementary structure of topos \([23, 35, 57]\), which can be defined in terms of the correspondence between the subsets \(R \to A \times B\) and the elements \(\chi_R \in \wp(A \times B)\), and the natural bijections

\[
S(X \times A, \wp B) \cong S(X, \wp(A \times B)) \cong S(X \times B, \wp A)
\]  

(82)

A relational calculus can, however, be developed entirely in terms of subobjects \(R \to A \times B\), in type universes without the powerset monad. Process relations are presented from this angle. The universe \(S\) only needs to be regular \([22, 76]\). In addition to the cartesian structure, it is thus also assumed to have the equalizers (i.e., the subsets characterized by equations), which induce the pullback squares. The final assumption, crucial for the relational calculus, is that every function \(f : A \to B\) has an epi-mono (surjective-injective) factorization: it can be decomposed in the form \(f = (A \to A' \to B)\), where \(e_f \in E\) and \(m_f \in M\). The family \(E\) can be thought of as the quotient maps (coequalizers), whereas \(M\) are all monics. The family \(E\) is required to be stable under the pullbacks. The category of relations in \(S\) is then defined to be

\[
|R| = |S|
\]

\[
R(A, B) = M_{\approx}/(A \times B)
\]

(83)

where \(M_{\approx}\) is the set of the equivalence classes modulo the relation

\[
\begin{array}{c}
R \\
\downarrow \approx \\
\downarrow \cong
\end{array}
\]

\[
\begin{array}{c}
R' \\
\downarrow m \\
\downarrow m'
\end{array}
\]

\[
\begin{array}{c}
X
\end{array}
\]

Without this quotienting, \(R(A, B)\) would in general be a proper class. The composition of relations \(A \xleftarrow{R} B\) and \(B \xleftarrow{S} C\), viewed as the \(M\)-monics \(R \to A \times B\) and \(S \to B \times C\), is defined using the pullback \(R \times S\) and the factorization in the following diagram.

The identity \(A \leftrightarrow A\) in \(R_S\) is the diagonal \(A \to A \times A\) in \(S\). More general categories of relations can be defined in more general situations using technically different but conceptually similar constructions \([76, 78]\). If \(S\) has the coproducts \(+\), they become biproducts in \(R\). The products \(\times\) from \(S\) induce a canonical monoidal structure in \(R\), with the compact structure \(\eta : 1 \leftrightarrow A \leftrightarrow A \times A\) and \(\epsilon : A \times A \leftrightarrow A \leftrightarrow 1\) on every \(A\) \([47]\).
B Proof of Prop. 2.3.3

a) Suppose that $S$ is a cartesian closed category with the exponents written $(A \Rightarrow B)$, and with an initial $F_A$-algebra

$$A + (A \times A^+) \xrightarrow{[\varepsilon : \cdot]} A^+$$

The intuition is that $A^+$ is the type of nonempty lists of elements of $A$, i.e. the free semigroup generated by $A$. The initial $F_A$-algebra structure consists of the inclusion $A \hookrightarrow A^+$, and the operation $A \times A^+ \xrightarrow{\cdot} $ which can be thought of as prepending a symbol $a \in A$ to the list $\alpha \in A^+$, to construct the list $a :: \alpha$.

The final machine with the inputs from $A$ and the outputs in $B$ is in the form

$$[A, B] \times A \xrightarrow{(\xi_0, \xi_1)} [A, B] \times B$$

(84)

where $\xi_0$ is derived from prepending ($::$) and the closed structure by

$$[A, B] \times A \xrightarrow{(=B) \times A} (A \Rightarrow B) \times A \xrightarrow{\varepsilon} B$$

$$[A, B] \times A \xrightarrow{\xi_0} [A, B]$$

whereas $\xi_1$ is just the evaluation restricted along $\iota$

$$[A, B] \times A \xrightarrow{(=B) \times A} (A \Rightarrow B) \times A \xrightarrow{\varepsilon} B$$

$$[A, B] \times A \xrightarrow{\xi_1} B$$

To show that (84) is a final machine, note first that every machine $X \times A \xrightarrow{(\xi_0, \xi_1)} X \times B$ induces an $F_A$-algebra over $(X \Rightarrow B)$ by transposing

$$X \times (A + (A \times (X \Rightarrow B))) \equiv (X \times A) + (X \times A \times (X \Rightarrow B)) \xrightarrow{(X \Rightarrow A) \times (X \Rightarrow B)} X \times (A + (X \Rightarrow B)) \equiv (X \times A) + ((X \Rightarrow B) \times B) \xrightarrow{[x, \varepsilon]} B$$

where $\overline{\kappa}$ is the composite

$$X \times (A + (A \times (X \Rightarrow B))) \equiv (X \times A) + (X \times A \times (X \Rightarrow B)) \xrightarrow{(X \Rightarrow A) \times (X \Rightarrow B)} X \times (A + (X \Rightarrow B)) \equiv (X \times A) + ((X \Rightarrow B) \times B) \xrightarrow{[x, \varepsilon]} B$$

The $F_A$-algebra $\kappa$ now induces the catamorphism $\langle \kappa \rangle$, which induces the anamorphism $\llbracket x \rrbracket$

$$A + (A \times A^+) \xrightarrow{[\varepsilon : \cdot]} A^+$$

$$A + (A \times (X \Rightarrow B)) \xrightarrow{\kappa} (X \Rightarrow B)$$

$$[A, B] \times A \xrightarrow{\xi} [A, B] \times B$$
by the tranposition
\[ A^+ \xrightarrow{\{i\}} (X \Rightarrow B) \]
\[ X \xrightarrow{\{e\}} [A, B] \]
The diagram chase showing that the commutativity and uniqueness of the catamorphism on the left
induce the commutativity and uniqueness of the anamorphism on the right is lengthy but straight-
forward.

b) The assumption is that \( S \) has final machines
\[ [A, B] \times A \xrightarrow{\langle \pi_0, \xi_1 \rangle} [A, B] \times B \]
so that the machine \( [A, B] \times A \xrightarrow{\langle \pi_0, \xi_1 \rangle} [A, B] \times B \) induces the anamorphism \( \llbracket \pi_0, \xi_1 \rrbracket \), as displayed on
the following diagram.

Since it is easy to see that \( \llbracket \pi_0, \xi_1 \rrbracket \) is an endomorphism on the machine \( [A, B] \times A \xrightarrow{\langle \pi_0, \xi_1 \rangle} [A, B] \times B \),
the uniqueness of \( \llbracket \pi_0, \xi_1 \rrbracket \) as an anamorphism implies
\[ \llbracket \pi_0, \xi_1 \rrbracket \circ \llbracket \pi_0, \xi_1 \rrbracket = \llbracket \pi_0, \xi_1 \rrbracket \]
Using the assumption that the idempotents in \( S \) split, we now define the exponent \((A \Rightarrow B)\) as the
splitting \( [A, B] \xrightarrow{q} ([A \Rightarrow B]) \xrightarrow{m} [A, B] \) of \( \llbracket \pi_0, \xi_1 \rrbracket \). The morphism \((A \Rightarrow B) \times A \xrightarrow{e} B\), induced
by the splitting in the above diagram, is the counit of the adjunction \((-) \times A : (A \Rightarrow -)\), for the
transposition operation \( \lambda \) from Table ?? defined
\[ S(X \times A, B) \xrightarrow{\lambda} S(X, (A \Rightarrow B)) \]
\[ f \mapsto \lambda f = q \circ \llbracket \pi_0, f \rrbracket \]
To show that \( e \circ (\lambda f \times A) = f \) holds, chase the following diagram:
C  Proof sketch for Lemma 5.1

Since \( \#A \leq \aleph_0 \), there is an ordinal number \( \kappa \leq \omega \) large enough to support an retraction \( \mathcal{P}(A \times A) \rightarrow \mathcal{P}^\kappa(A) \rightarrow \mathcal{P}(A \times A) \), and thus also \( Q_{A1} \rightarrow Q_{A1}^\kappa \rightarrow Q_{A1} \) for the functor \( Q \) defined in (38). Hence the tower of retractions:

The symmetry \( A \times 1 \cong 1 \times A \) lifts to a similar retraction

\[
[A, A]_P \overset{m_1}{\longrightarrow} [1, A]_P \overset{e_1}{\longrightarrow} [A, A]_P
\]

With these retractions, the proof boils down to showing the commutativity of the following diagram

where \( \llbracket \cdot; \cdot \rrbracket \) are the enriched compositions, constructed like in (25) (or see [51] for more details), whereas \( \llbracket \text{id} \rrbracket \) is the enriched identity, constructed as the anamorphism (final coalgebra homomorphism) from the identity machine \( A \times 1 \rightarrow \mathcal{P}(A \times 1) \), where \( \eta \) is the unit of the monad \( \mathcal{P} \). This diagram says that \( j = m_0 \llbracket \text{id} \rrbracket \in \mathcal{S}^P(A, 1) \) and \( r = m_1 \llbracket \text{id} \rrbracket \in \mathcal{S}^P(1, A) \) display \( A \) as a retract of \( 1 \) in \( \mathcal{S}^P \), i.e. that they compose to

\[
\text{id}_A = \left( A \xrightarrow{j=m_0\llbracket \text{id} \rrbracket} 1 \xrightarrow{r=m_1\llbracket \text{id} \rrbracket} A \right)
\]

\[ (85) \]

D  Proof of Corollary 5.2

Since the embedding (58) is full and faithful by definition, we only need to prove that it is essentially surjective: for an arbitrary object \( S \in \text{dProc}_{\leq \aleph_0} \) we must find \( S' \in \text{dProc} \) such that \( S \cong S' \) in
DProc_{≤ℵ₀}. An object of DProc_{≤ℵ₀} is a dynamic relation \( A \xleftarrow{S} 1 \) in \( S^p \), where \( #A ≤ ℵ₀ \). An object of dProc is a hyperset \( S' \), viewed as a dynamic relation \( 1 \xleftarrow{S'} 1 \) in \( S^p \). By Lemma 5.1 there are the relations \( j ∈ S^p(A, 1) \) and \( r ∈ S^p(1, A) \) such that \( (j; r) = id_A \). Setting

\[
S' = \left( 1 \xleftarrow{r} A \xleftarrow{S} 1 \right)
\]

assures that the inner triangle in the following diagram commutes.

\[
\begin{array}{ccc}
1 & \xrightarrow{j} & A \\
\downarrow & & \downarrow \\
S' & \xrightarrow{r} & S \\
\downarrow & & \downarrow \\
1 & \xrightarrow{} & 1
\end{array}
\]

The outer triangle commutes because \( S' ◦ j = S ◦ r ◦ j = S \) by (85). So we have the morphisms \( r ∈ DProc_{≤ℵ₀}(S', S) \) and \( j ∈ DProc_{≤ℵ₀}(S, S') \). They form an isomorphism because \( r ◦ j = id_S \) by (85) again, and \( j ◦ r ∈ DProc_{≤ℵ₀}(S', S') \) must be an identity because \( S' \) is a subobject of the terminal object in DProc_{≤ℵ₀}. □

E   **Traces and the \( \mathbb{Int} \)-construction**

The *trace* operation on a symmetric (or braided) monoidal category \((C, ⊗, I)\) is typed by the rule

\[
\begin{array}{c}
A ⊗ Y \xrightarrow{f} B ⊗ Y \\
\hline
A \xrightarrow{\text{Tr}_I(f)} B
\end{array}
\]

The equations for this operation, with some examples and explanations can be found in \([6, 46, 83]\).

The free compact category over any traced monoidal \( C \)

\[
|\text{Int}_C| = |C|_- × |C|_+
\]

\[
\text{Int}_C(A, B) = C(A_- ⊗ B_+, B_- ⊗ A_+)
\]

where \( X_- = \{-\} × X \) and \( X_+ = \{+\} × X \). The composition of \( \text{Int}_C(A, B) × \text{Int}_C(B, C) \xrightarrow{\cdot} \text{Int}_C(A, C) \) is defined by

\[
\begin{array}{c}
A_- ⊗ B_+ \xrightarrow{f} B_- ⊗ A_+ \\
\hline
A_- ⊗ C_+ ⊗ B_- ⊗ B_+ \xrightarrow{\sigma} A_- ⊗ B_+ ⊗ B_- ⊗ C_+ \xrightarrow{f \otimes g \circ \sigma} B_- ⊗ A_+ ⊗ C_- ⊗ B_+ \xrightarrow{\sigma} C_- ⊗ A_+ ⊗ B_- ⊗ B_+ \xrightarrow{\sigma} C_- ⊗ A_+
\end{array}
\]

\[
g \cdot f = \left( A_- ⊗ C_+ \xrightarrow{\text{Tr}_{B_-, B_+}} C_- ⊗ A_+ \right)
\]

58
The extended reals as alternating dyadics

Recall from Sec. 6.3.1 that \( \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\} \) is the extended real continuum, and that \( \Sigma^\otimes = \bigsqcup_{i=0}^{\infty} \Sigma^i \) is the set of finite or infinite (countable) strings of symbols from \( \Sigma = \{-, +\} \), which are treated in (87) as \( \{\pm 1\} \).

Define the value of the function \( \Phi : \Sigma^\otimes \rightarrow \overline{\mathbb{R}} \) on an arbitrary string \( \varsigma = (\varsigma_0, \varsigma_1, \varsigma_2, \ldots) \) to be

\[
\Phi(\varsigma) = z \cdot \varsigma_0 + \sum_{i=z+1}^{\infty} \frac{\varsigma_i}{2^{i-z}}
\]

(87)

where \( z = \mu n. \varsigma_n \neq \varsigma_{n+1} \) is the length of the initial segment before the sign flips. If \( \varsigma \) is the infinite string of either one sign or the other, then \( z \) is infinite, and the value of \( \Phi(\varsigma) \) is either \( \infty \) or \( -\infty \).

Leaving the two infinities aside, \( \Phi \) establishes a bijection between the remaining \( \Sigma \)-strings, where the sign eventually flips, and the finite real numbers from \( \mathbb{R} \). For an arbitrary \( x \in \mathbb{R} \), the string \( \xi \in \Sigma^\otimes \) such that \( x = \Phi(\xi) \) can be constructed as follows:

- Decompose the real line as the disjoint union of the closed-open and open-closed intervals

\[
\mathbb{R} = \bigsqcup_{n=1}^{\infty} [-n, -n+1) + \{0\} + \bigsqcup_{n=1}^{\infty} (n-1, n]
\]

leaving the 0 on its own. Then there are 3 cases:

- (0) If \( x = 0 \) then \( \xi \) is the empty string (\( \cdot \)).
- (-) If \( x \in [-n_0, -n_0+1) \), then \( \xi \) begins with \( \underline{- - \cdots -} \)
- (+) If \( x \in [n_0-1, n_0) \), then \( \xi \) begins with \( \underline{+ + \cdots +} \).

- In case (-), find
  - the smallest \( n_1 \) such that \( x \leq -n_0 + \sum_{i=1}^{n_1} \frac{1}{2^n} \) and append \( \underline{+ + \cdots +} \) to \( \xi \);
  - the smallest \( n_2 \) such that \( x \geq -n_0 + \sum_{i=1}^{n_1} \frac{1}{2^n} - \sum_{i=1}^{n_2} \frac{1}{2^{n_1+n}} \) and append \( \underline{- \cdots -} \) to \( \xi \);
  - the smallest \( n_3 \) such that \( x \leq \cdots \), etc.

- In case (+), find
  - the smallest \( n_1 \) such that \( x \geq n_0 - \sum_{i=1}^{n_1} \frac{1}{2^n} \) and append \( \underline{- \cdots -} \) to \( \xi \);
  - the smallest \( n_2 \) such that \( x \leq \cdots \), etc.

- If you ever reach a sum equal to \( x \), then halt and leave \( \xi \) finite. Otherwise \( \xi \) is infinite.

In any case, it is easy to see that \( \Phi(\xi) = x \) and that \( \Phi(\xi) = \Phi(\zeta) \) implies \( \xi = \zeta \). So \( \Phi \) is an injection.

And we have just shown that it is a surjection by constructing for an arbitrary \( x \in \overline{\mathbb{R}} \) a \( \xi \in \Sigma^\otimes \) such that \( x = \Phi(\xi) \). The function \( \Phi \) defined by (87) is thus the claimed bijection.