Solution of a class of one-dimensional reaction-diffusion models in disordered media

M. Mobilia‡ and P.-A. Bares†
Institute of Theoretical Physics, Swiss Federal Institute of Technology Lausanne, CH-1015 Lausanne EPFL, Switzerland
(November 3, 2018)

We study a one-dimensional class of reaction-diffusion models on a 10−parameters manifold. The equations of motion of the correlation functions close on this manifold. We compute exactly the long-time behaviour of the density and correlation functions for quenched disordered systems. The quenched disorder consists of disconnected domains of reaction. We first consider the case where the disorder comprises a superposition, with different probabilistic weights, of finite segments, with periodic boundary conditions. We then pass to the case of finite segments with open boundary conditions: we solve the ordered dynamics on a open lattice with help of the Dynamical Matrix Ansatz (DMA) and investigate further its disordered version.

PACS number(s): 64.60.Cn, 02.50.-r, 75.10.Jm, 05.50.+q

I. INTRODUCTION

In this work we investigate in detail some exact properties, in ordered and disordered media, of a wide class of one-dimensional single-species reaction-diffusion models, which play a central role in the description of interacting many-particle systems in physics, chemistry, biology, etc. We consider systems on the 10−parameter manifold on which (in arbitrary dimensions) the equations of motion of correlation functions close, and the dynamics is soluble.

So far, most theoretical studies on Reaction-Diffusion models consider infinite, ordered media (see e.g. and references therein). We show that for times that exceed the typical diffusion time needed to span the size of the sample, i.e. in the regime where \( L^2/t \ll 1 \), (where \( L \) denotes the size of the lattice chain and \( t \) the time, see below), finite-size effects dramatically affect the dynamics. A (quenched-)disordered system consisting of a random collection of reaction-domains (each segment, of length \( \mathcal{L} \) weighted with a probability \( \omega(\mathcal{L}) \)) of varying sizes, when \( \omega(\mathcal{L}) \) is a decreasing exponential, the density exhibits a stretched exponential decay in the diffusion-limited pair annihilation and diffusion-limited coagulation models. This exponential decay also applies to the autocorrelation function of symmetric exclusion process (SEP). The picture is realistic for media, where disorder fractures the medium into disconnected reaction zones. The stretched exponential decay of the density of the photogenerated excitons in (MX) and (TMMC) chains is an experimentally established phenomenon. For other distributions \( \omega(\mathcal{L}) \) and other reaction-diffusion systems the situation is more involved. However, in the regime where \( L^2/t \ll 1 \), the quenched-disorder always affects the dynamics. Below, we study the density and the two-point non-instantaneous correlation functions for a class of reaction-diffusion systems in presence of quenched disorder.

This paper is organized as follows: in the first part we study reaction-diffusion processes occuring on the soluble 10−parametric manifold on which the equations of motion of the correlation functions are closed. In section II we determine this manifold as well as the equations of motions of the correlation functions. In section III.A we obtain the density and non-instantaneous two-point correlation functions for a finite and periodic chain. In section III.B, via the Dynamical Matrix Ansatz (DMA), we generalize previous works by Stinchcombe and Schütz on the soluble 10−parametric manifold and obtain original results. The approach developed allows the exact computation of correlation functions on an open chain of arbitrary length, with injection and evacuation of particles at the boundaries. In the thermodynamic limit we compute the exact density and the non-instantaneous correlation functions. As a by-product we generalize the “Bethe Ansatz” equations previously obtained by Stinchcombe and Schütz for the isotropic chain with symmetry-breaking boundary fields. In section IV, we introduce the disordered systems which we consider in the sequel. In section V, we take advantage of the results of section III.A to study the dynamics of quenched-disordered systems, where the quenched-disorder consist in a superposition of periodic segments with probabilistic weight \( \omega(\mathcal{L}) \). We consider the cases where \( \omega(\mathcal{L}) \) is exponential and where it is an algebraic function. In section VI we use the original results obtained in section III.B to study the dynamics of quenched-disordered systems, where the

*To appear in Physical Review B
†email:mauro.mobilia@epfl.ch
‡email:pierre-antoine.bares@epfl.ch

arXiv:cond-mat/0101253v2 [cond-mat.stat-mech] 8 Apr 2001
quenched-disorder consist in a probabilistic superposition of open segments with probabilistic weight $\omega(\mathcal{L})$. In section VII, we illustrate previous discussions by considering two physically motivated examples.

II. THE FORMALISM

It is known that models of stochastic hard-core particles are soluble on some manifold on which the equations of motion of their correlation functions close\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}. Consider an hypercubic lattice of dimension $d$ with $N$ sites ($N = L^d$), where $L$ represents the linear dimension of the hypercube. On the lattice, local bimolecular reactions between single-species particle $A$, with hard-core, take place. Each site can be empty (denoted by the symbol 0) or occupied at most by a particle of type $A$ denoted in the following by the index 1. The reactions are specified by the transition rates $\Gamma_{\alpha\beta}$, where $\alpha, \beta, \gamma, \delta = 0, 1$:

\[ \forall(\alpha, \beta) \neq (\gamma, \delta), \Gamma_{\alpha\beta}^\gamma = \alpha + \beta \rightarrow \gamma + \delta. \]

Probability conservation implies $\Gamma_{\alpha\beta}^{\gamma\delta} = -\sum_{(\alpha', \beta') \neq (\gamma', \delta')} \Gamma_{\alpha'\beta'}^{\gamma'\delta'}$ and $\Gamma_{\alpha\beta}^{\gamma\delta} \geq 0$, $\forall(\alpha, \beta) \neq (\gamma, \delta)$ The state of the system is represented by the ket $|P(t)\rangle = \sum_{\{n\}} P(\{n\}, t)|\{n\}\rangle$, where the sum runs over the $2^N$, $N = L^d$ configurations. At site $i$ the local state is specified by the ket $|n_i\rangle = (1 0)^T$ if the site $i$ is empty, $|n_i\rangle = (0 1)^T$ if the site $i$ is occupied by a particle of type $A$ (1).

We define the left vacuum $\langle \tilde{\chi} | \equiv \sum_{\{n\}} |\{n\}\rangle$, which local representation is $\langle \tilde{\chi} | = (1 1) \otimes (1 1)$. It is by now well established that a master equation can be rewritten formally as an imaginary time Schrödinger equation: $\frac{d}{dt} |P(t)\rangle = -H |P(t)\rangle$, where $H$ is the Stochastic Hamiltonian which governs the dynamics of the system. In general, it is neither hermitian nor normal. Its construction from the master equation is a standard procedure (see e.g.\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}). The evolution operator $H$ acts locally on two adjacent sites, $H = \sum_m H_{m,m+1}$ and because of probability conservation, we have $\langle \tilde{\chi} |H| = 0$. The explicit form of the $H$ considered here can be found, e.g., in Ref.\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}. Below we shall assume an initial state $|P(0)\rangle$ and investigate the expectation value of an operator $O(\langle (t)\rangle = \langle \tilde{\chi} | O e^{-\hat{H} t} | P(0)\rangle$). For general single-species bimolecular reaction-diffusion systems, there are 12 independent parameters. If one imposes to these parameters the 2 following constraints:

\[ \Gamma_{00}^{10} + \Gamma_{11}^{11} - (\Gamma_{00}^{00} + \Gamma_{11}^{01}) = \Gamma_{00}^{01} + \Gamma_{11}^{10} - (\Gamma_{00}^{00} + \Gamma_{11}^{11}) \quad ; \quad \Gamma_{00}^0 + \Gamma_{00}^1 - (\Gamma_{00}^{00} + \Gamma_{11}^{01}) = \Gamma_{01}^0 + \Gamma_{10}^1 - (\Gamma_{00}^{01} + \Gamma_{11}^{11}), \]  

the equations of motion of the correlation functions close and the system is formally soluble (in arbitrary dimensions). It is useful to point out that these solvability constraints allow, through a similarity transformation\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}, to transform the equivalent quantum chain, into another quantum chain sharing the same eigenspectra of an XXZ quantum chain with surface fields, of the type solved in Ref.\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}.

III. DENSITY AND NON-INSTANTANEOUS CORRELATION FUNCTIONS FOR ORDERED CASE, IN ONE-SPATIAL DIMENSION ON A FINITE LATTICE

For the sequel, in order to compute the sampling average of quantities such as the density and the non-instantaneous correlation functions we need to know the expression of these quantities on a finite lattice. In the following subsection we briefly derive the latter expressions on a periodic finite lattice. In the subsection III.B we compute, as original results, the same expressions on an open lattice with help of Dynamical Matrix Ansatz.

A. The periodic case

For the class of systems which are described on a 10-parameter manifold\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}, according to the solvability constraints\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}, the equation of motion of the density, for a periodic chain of $\mathcal{L} \leq L$ sites, in one-spatial dimension, reads:\footnote{This soluble 10–parametric manifold includes stochastic systems such as Glauber and voter models, the Symmetric Exclusion Process (SEP).}

\[ \frac{d}{dt} \langle n_m \rangle(t) = \frac{d}{dt} \langle \tilde{\chi} | n_m e^{-\hat{H} t} | P(0)\rangle = A \langle n_m \rangle(t) + B \langle n_{m+1} \rangle(t) + C \langle n_{m+1} \rangle(t) + D \langle n_{m-1} \rangle(t) \]

(2)
where we have defined:

\[
A \equiv 2\Gamma_{10}^{11} + \Gamma_{00}^{10} + \Gamma_{01}^{01}; \quad B \equiv -(2\Gamma_{10}^{11} + \Gamma_{00}^{10} + \Gamma_{01}^{01} + \Gamma_{10}^{00} + \Gamma_{11}^{01} + \Gamma_{00}^{11}) \\
C \equiv \Gamma_{01}^{11} + \Gamma_{10}^{11} - (\Gamma_{11}^{11} + \Gamma_{01}^{01}) ; \quad D \equiv \Gamma_{11}^{11} + \Gamma_{01}^{11} - (\Gamma_{11}^{11} + \Gamma_{01}^{01}) ; \quad E \equiv \pm \sqrt{CD}; \quad \mu \equiv \pm \sqrt{\frac{D}{C}} \tag{3}
\]

In this work we focus on the case where \( E > 0 \) (which also implies \( \mu > 0 \)).

Solving the equations of motion obtained from (3), we obtain in the ordered case (for integer \( m \) and \( m' \) and for periodic segments of length \( L \leq L' \)):

\[
\langle n_m(t) \rangle_L = \frac{2^{\frac{3m}{2}}}{L} \sum_{p \in 1.B.Z.} \left( \langle n_{m'}(0) \rangle_L - \langle n_{m'}(\infty) \rangle_L \right) \mu^{m-m'} \langle \text{sign}E \rangle^{m-m'} L \Gamma \left\{ p \left[ (m-m') \cos \left| p \left( m-m' \right) \right| e^{2|E|t} \cos \mu (B+C+D) \right] \right\}.
\]

where \( \langle n_{m'}(\infty) \rangle \equiv \phi(1-\delta_{A,0})\left( 1-\delta_{B+C+D,0} \right) \), with \( \phi \equiv -\frac{4}{B+C+D} \), and the sum \( m' \) runs over all the integers belonging to the segments of length \( L \) under consideration. The sum over \( p \) runs over the first Brillouin zone (1.B.Z.):

\[
p = \frac{2\pi n}{L}, \quad n = 0, \ldots, L - 1.
\]

With the same notations, we have for the non-instantaneous two-point correlation functions:

\[
\langle n_m(t)n_{m'}(0) \rangle_L = \frac{2^{\frac{3m}{2}}}{L} \sum_{p \in 1.B.Z.} \left( \langle n_{m'}(0) \rangle_L - \langle n_{m'}(\infty) \rangle_L \right) \mu^{m-m'} \langle \text{sign}E \rangle^{m-m'} L \Gamma \left\{ p \left[ (m-m') \cos \left| p \left( m-m' \right) \right| e^{2|E|t} \cos \mu (B+C+D) \right] \right\}.
\]

where \( \langle n_{m'}(\infty) \rangle \equiv \langle n_{m'}(\infty) \rangle \).

It is useful for the sequel to obtain the long-time behaviour of these quantities in the regime where \( L^2/|E|t \sim u = L^2/|E|t \sim 1 \), with \( L \approx L \gg 1 \) and \( |E|t \gg 1 \), for the density we have:

\[
\langle n_m(t) \rangle_L - \langle n_m(\infty) \rangle_L = \sum_{m'} \mu^{m-m'} \langle n_{m'}(0) \rangle_L - \langle n_{m'}(\infty) \rangle_L e^{(B+2|E|)t - \frac{(m-m')^2}{4|E|^t}} \frac{2\sqrt{\pi|E|t}}{2} \tag{4}
\]

The long-time behaviour \( L^2/|E|t \sim u = L^2/|E|t \sim 1 \), with \( L \approx L \gg 1 \) and \( |E|t \gg 1 \) of the two-point correlation functions reads

\[
\langle n_x(t)n_{x_0}(0) \rangle = \sum_y \frac{\mu^x-y e^{(B+2|E|)t - \frac{(x-y)^2}{4|E|^t}}}{2\sqrt{\pi|E|t}} \langle n_y(0)n_{x_0}(0) \rangle_L - \langle n_y(\infty)n_{x_0}(0) \rangle_L \tag{5}
\]

**B. Density and non-instantaneous correlation function for the ordered case with open boundary conditions**

It is instructive to consider the physically motivated case of a disordered lattice with open boundary conditions. To investigate this situation, we begin with an ordered lattice of length \( L \) and open boundary conditions: we assume that at site 1 particles can be injected in the system with rate \( \alpha \) and evacuated with rate \( \gamma \). At site 1 particles can be injected in the system with rate \( \alpha \) and evacuated with rate \( \gamma \). We apply the Dynamical Matrix Ansatz (DMA) introduced by Stinchcombe and Schütz to study the ordered Symmetric Exclusion Process (SEP) with open boundary conditions.

For two-states models, the Dynamical Matrix Ansatz assumes that the probability can be encoded by \( P(t) = \langle W | \{ \prod_{j=1}^L (E(t) + D(t)) \gamma \} | V \rangle Z_L \), where \( Z_L = \langle W | C_L^L | V \rangle \) is the normalization constant, \( C = E(t) + D(t) \), and \( \gamma_j \) denotes the usual Pauli matrix acting on site \( j \). We have introduced the vectors \( \langle W \rangle \) and \( \langle V \rangle \) which act the (finite and time-dependent) matrices \( E(t) + D(t) \) (not to be confused with the combination of rates \( \gamma \)).

To investigate the dynamics, we associate a spin-down to a particle and a spin-up to a vacancy. In so doing, the ferromagnetic ground-state \( | \gamma \rangle \) corresponds to an empty lattice.

In this formulation, correlation functions read:

\[
\langle n_{j_1}(t)n_{j_2}(t) \ldots n_{j_m}(t) \rangle = \langle W | C_{j_1}^{j_2} \ldots C_{j_2}^{j_3} \ldots C_{j_m}^{L-j_m} | V \rangle Z_L, \quad j_m > j_{m-1} > \ldots > j_1.
\]

The master equation, in its Hamiltonian formulation, reads \( \left( \frac{d}{dt} + H \right) P(t) = 0 \), where \( H = \sum_{j=1}^{L-1} H_{j,j+1} + b_1 + b_L \). The first term in this expression corresponds to the bulk contribution, while the terms \( b_1 + b_L \) correspond to the boundary at site 1 (injection with rate \( \alpha \) and evacuation with rate \( \gamma \) ) and \( L \) (injection with rate \( \beta \) and evacuation with rate \( \delta \) ), respectively.

Proceeding as in Ref. [1], on the 10-parametric manifold \( \mathcal{L} \), we obtain the bulk contribution to the master equation:

\[
\frac{dD(t)}{dt} = (\sigma_1 + \sigma_4)C - (\sigma_3 + \sigma_5)D(t) + \sigma_2 CD(t)C^{-1} + \sigma_6 C^{-1} D(t) C
\]

(6)
\[ S(t) = (\sigma_1 - \sigma_4)C + (\sigma_5 - \sigma_3)D(t) + \sigma_2CD(t)C^{-1} - \sigma_6C^{-1}D(t)C \]

\[ \sigma_7D(t)^2(t) = \sigma_8CD(t) + \sigma_9D(t)C - \Gamma_{101}^1C^2 + \sigma_2CD(t)C^{-1}D(t) + \sigma_6D(t)C^{-1}D(t)C \]

Similarly, the boundary terms give rise to the following equations:

\[ 0 = \langle W \rangle \left\{ (2\sigma_4 - \tilde{\alpha})C - (2\sigma_5 - \tilde{\alpha} - \tilde{\gamma})D(t) + 2\sigma_6CD(t)C^{-1} \right\} = \left\{ (2\sigma_1 - \tilde{\delta})C - (2\sigma_3 - \tilde{\beta} - \tilde{\delta})D(t) + 2\sigma_2CD(t)C^{-1} \right\} |V\rangle, \]

where

\[ \sigma_1 \equiv \Gamma_{00}^{11} + \Gamma_{01}^{11} \geq 0 ; \sigma_2 \equiv \Gamma_{00}^{10} + \Gamma_{01}^{10} - \sigma_1 = C ; \sigma_3 \equiv \Gamma_{10}^{10} + \Gamma_{11}^{10} + \sigma_1 \geq 0 \]

\[ \sigma_4 \equiv \Gamma_{00}^{10} + \Gamma_{01}^{10} \geq 0 ; \sigma_5 \equiv \Gamma_{10}^{10} + \sigma_4 \geq 0 ; \sigma_6 \equiv \Gamma_{10}^{10} + \Gamma_{11}^{10} - \sigma_4 = D \]

\[ \sigma_7 \equiv \sigma_3 + \sigma_5 - (\Gamma_{10}^{11} + \Gamma_{01}^{11} + \Gamma_{11}^{11} + \Gamma_{01}^{11}) ; \sigma_8 \equiv \sigma_1 + \Gamma_{11}^{10} - \Gamma_{11}^{10} ; \sigma_9 \equiv \sigma_4 + \Gamma_{11}^{10} - \Gamma_{11}^{10} \]

with \( \sigma_2 = C, \sigma_6 = D, \sigma_1 = A \) and \( \sigma_3 + \sigma_5 = -B \), where \( A, B, C, D \) are the quantities defined in (3). For the SEP model, considered in (3), we have \( \Gamma_{10}^{10} = \Gamma_{11}^{10} = \Gamma_{01}^{11} = 0 \) and thus with \( \sigma_2 = \sigma_3 = \sigma_5 = \sigma_6 = \frac{3}{2} = \Gamma \) and \( \sigma_1 = \sigma_4 = \sigma_9 = 0 \), we recover previous results Ref. [21].

To study the dynamical equations (3) with boundary conditions (4), it is useful to introduce the following notation: \( \hat{D}_p(t) \equiv \hat{C}^{-1}D(t)\hat{C}^{-1} \), and so the correlation functions read \( \langle n_{j_1}(t)n_{j_2}(t)\ldots n_{j_m}(t) \rangle = \langle \langle W | \hat{D}_{p_1}(t)\hat{D}_{p_2}(t)\ldots \hat{D}_{p_m}(t)C^*|V\rangle \rangle \). \( j_m > j_{m-1} > \ldots > j_1 \). For the study of the dynamics of the system we can then introduce the Fourier transform of \( \hat{D}_p(t) \) (for \( p \neq 0 \)): \( \hat{D}_p(t) = \sum_{j} e^{ip\cdot j} D_j(t) \). Seeking a solution of the dynamical equation (3) (for \( p \neq 0 \)) of the form \( \hat{D}_p(t) = e^{-ip\cdot t} D_p(t) = 0 \), \( \exists \epsilon_p = (\sigma_3 + \sigma_5) - \sigma_2 e^{-ip\cdot t} - \sigma_6 e^{ip\cdot t} \). The equation (3) implies the relation: \( \hat{D}_{p_1}(t)\hat{D}_{p_2}(t) = S_{p_2,p_1} \hat{D}_{p_2}(t)\hat{D}_{p_1}(t) \), where \( S_{p_2,p_1} \) is the amplitude of the scattering matrix which plays a central role in the theory of integrable models. Here, we have

\[ S_{p_2,p_1} = \left( \frac{\sigma_6 + \sigma_2 e^{ip_1-p_2} - \sigma_7 e^{ip_2}}{\sigma_6 + \sigma_2 e^{ip_1-p_2} - \sigma_7 e^{ip_2}} \right), \quad (p_1, p_2) \neq (0, 0) \]

For the SEP model we recover\(^2\)\(^2\)\(^2\) the scattering amplitude of the isotropic Heisenberg chain.

With the definitions,

\[ 2\alpha = \frac{\tilde{\alpha} - 2\sigma_4}{2\sigma_6} ; 2\beta = 1 + \frac{\tilde{\beta} + 2(\sigma_1 - \sigma_4)}{2\sigma_2} ; 2\gamma = 1 + \frac{\tilde{\gamma} + 2(\sigma_4 - \sigma_5)}{2\sigma_6} ; 2\delta = \frac{\tilde{\delta} - 2\sigma_1}{2\sigma_2}, \]

it follows from the boundary conditions that

\[ T(p) \equiv \langle \langle W | \hat{D}_pC^*|V\rangle \rangle = -\frac{f_p(\alpha, \gamma)}{f_p(\beta, \delta)} e^{2ip \cdot T(-p)} = \frac{f_p(\beta, \delta)}{f_p(\beta, \delta)} e^{2ip \cdot T(-p)}, \quad (p \neq 0), \]

where \( f_p(a, b) \equiv 2(a + b) - 1 + e^{ip} \). These equations have as solution: \( T(p) = \sum_{l > 0} a_l \left( e^{ipl} - e^{i(2L-1)p} \right) f_p(\beta, \delta) \). Consistency implies the following relation which quantizes the momenta \( p \)'s, as in quantum Many-Body problems:

\[ B(p) \equiv \frac{f_p(\alpha, \gamma)f_{-p}(\beta, \delta)}{f_p(\alpha, \gamma)f_{-p}(\beta, \delta)} = e^{2ip(L-1)} \]

and more generally, for the \( m \)-points correlation functions, we have \( e^{2ip(L-1)} = B_{p_j} \prod_{k=1(k \neq j)}^{m} S(p_k, -p_k) \). Equations (3) and its \( m \)-points generalization are the analog of the Bethe Ansatz equations for quantum Many-Body problems\(^2\).

Correlation functions are then obtained as follows:

\[ \langle n_{j_1}(t)\ldots n_{j_m}(t)\rangle - \langle n_{j_1}(\infty)\ldots n_{j_m}(\infty)\rangle = \int \cdots \int \prod_{k=1}^{m} \left[ dp_k e^{-ip_k \cdot \tau_k} \right] T(p_1, \ldots, p_m), \]

\( j_m > \ldots > j_1 \); \( p_i \neq 0 \); \( 1 \leq i \leq m \). Where \( T(p_1, \ldots, p_m) \equiv \langle \langle W | \hat{D}_{p_1}(t)\ldots \hat{D}_{p_m}(0)C^*|V\rangle \rangle \) encodes the initial state of the system.

For finite systems the integration over the momenta, should be carried over \( p \)'s satisfying the relations (13). For infinite system (when \( L \to \infty \)) these relations (3) can always be fulfilled (there is a dense set of solutions of these
equations) and the integration over the $p$'s, runs over the first Brillouin zone (with exclusion of $p = 0$ which only contributes to the static part of the correlation functions).

We will now focus on the density, which reads (for large $L$):

$$
\langle n_x(t) \rangle - \langle n_x(\infty) \rangle = \sum_{l>0} \left( \langle n_l(\infty) \rangle - \langle n_l(0) \rangle \right) \int \frac{dp}{2\pi} e^{-ipx-\epsilon_{t,p}} \left( e^{ipL(0,\infty)} - e^{ip(2L-t)} f_p(0,\gamma) \right), \quad (p \neq 0)
$$

(14)

With the boundary conditions, we obtain for the stationary density:

i) For the SEP model, where $\sigma_3 + \sigma_5 = \sigma_2 + \sigma_6$ (i.e. $|B| = C + D$) with $\sigma_2 = \sigma_6 > 0$ and $\sigma_1 + \sigma_4 = A = 0$, we recover the results of Stinchcombe and Schütz [4].

ii) For the other cases on the 10–parameter manifold [1], for which $|B| \neq C + D$, i.e. $\sigma_3 + \sigma_5 \neq \sigma_2 + \sigma_6$, and thus $A = \sigma_1 + \sigma_4 \neq 0$:

$$
n_{1}(\infty) = \phi + \frac{-\sigma_2 \tau_4 \eta + \tau_2 (\tau_3 - \sigma_6 \eta^{L-2})}{(\eta \tau_2 + \tau_1)(\tau_3 - \sigma_6 \eta^{L-2} - \sigma_2 \sigma_6 \eta^{L-1})}; \quad n_{L}(\infty) = \phi + \frac{\eta \tau_4 - \sigma_6 \eta^{L-2}(\langle n_{1}(\infty) \rangle - \phi)}{\tau_3 + \sigma_6 \eta^{L-2}}
$$

$$
n_{x}(\infty) = (\langle n_1(\infty) \rangle + \langle n_{L}(\infty) \rangle - 2 \phi) \eta^{x-1} + \phi, \quad 1 < x < L,
$$

(15)

where, $\phi \equiv \frac{\sigma_1 + \sigma_4}{\sigma_3 + \sigma_5 - \sigma_2 - \sigma_6}$; $\eta \equiv \frac{\sigma_1 + \sigma_4 - \sqrt{(\sigma_1 + \sigma_4)^2 - 4 \sigma_2 \sigma_6}}{2 \sigma_2}$; $\tau_1 \equiv (1 - 2(\alpha + \gamma)) \sigma_6 - (\sigma_3 + \sigma_5)$; $\tau_2 \equiv 2\sigma_4 - \alpha + \phi [2(\sigma_6 - \sigma_5) + \alpha + \gamma]$; $\tau_3 \equiv (1 - 2(\beta + \delta)) \sigma_6 - (\sigma_3 + \sigma_5)$; and $\tau_4 \equiv 2\sigma_1 - \alpha + \phi \left[2(\sigma_2 - \sigma_3) + \beta + \delta\right]$.

The dynamical part of the density reads, according to [3], i) For the SEP model, where $B + C + D = \sigma_3 + \sigma_5 - \sigma_2 - \sigma_6 = 0$ and $A = \sigma_1 + \sigma_4 = 0$, we recover results obtained in references Ref [4].

ii) For the other processes on the 10–parameter manifold [1] (for which $\sigma_3 + \sigma_5 \neq \sigma_2 + \sigma_6$ with $A \neq 0$ and $\sigma_2 \sigma_6 \neq 0$), we have

$$
\langle n_x(t) \rangle - \langle n_x(\infty) \rangle = -e^{Bt} \sum_{y=1}^{L} \left( D \over C \right)^{y-\Delta} \left( \langle n_y(0) \rangle - \langle n_y(\infty) \rangle \right) (I_{x-y}(2E|t) + \Delta I_{x+y-1}(2E|t))
$$

$$
+ e^{Bt}(1 - \Delta^2) \sum_{y=1}^{L} \sum_{k=1}^{\Delta y-1} \left( D \over C \right)^{y-k-1} \left( \langle n_k(0) \rangle - \langle n_k(\infty) \rangle \right) I_{x+y-1}(2E|t),
$$

(16)

where $\Delta \equiv \sqrt{2(1 - 2(\alpha + \gamma)) - \frac{2\sigma_5 - (\alpha + \gamma)}{2\sqrt{\sigma_2 \sigma_6}}}$.

Similarly, we obtain the expression of the non-instantaneous two-point correlation functions $\langle n_x(t)n_{x_0}(0) \rangle$ for an initially translationally invariant system with initial density $\langle n_m(t = 0) \rangle = \rho(0), m = 1, \ldots, L$. In so doing, we have to replace in (16) $\langle n_j(0) \rangle$ with $\rho^2(0)(1 - \delta_{x,x_0}) + \rho(0)\delta_{x,x_0}$.

**IV. THE QUENCHED DISORDERED SITUATION**

In the subsequent sections we will exploit the results of the previous section to compute the density and non-instantaneous correlation functions of systems obeying (3) and in presence of quenched disorder. More specifically, we assume that the reaction-rates defined in (3) obey a specific probability distribution. We assume a distribution, which has already been considered in the study the Glauber-Ising Model and other diluted chains (see e.g. [5,4]), and more recently, of the symmetric one-dimensional single-species diffusion-limited reaction $A + 0 \leftrightarrow 0 + A$. The first distribution under consideration is the so-called “peaked distribution” which will describe a diluted chain: $\mathcal{P}(A,B,C,D) = \rho^4 L^2 \delta_{A,A} \delta_{B,B} \delta_{C,C} \delta_{D,D} + (1 - \rho^4 L^2) \delta_{A,0} \delta_{B,0} \delta_{C,0} \delta_{D,0}$, where $0 < q < 1$ and $A,B,C,D$ have been defined in (3). According to this distribution, the chain is broken into discontinuous segments of (finite) length $L$, with $L + 1$ sites. On these segments the reaction-rates are respectively $A_1, B_1, C_1, D_1$. Therefore, each realization has a probabilistic weight $\omega(L)$:

$$
\omega(L) = q^L \mathcal{L}^3, \quad \hat{\alpha}, \hat{\beta} > 0,
$$

(17)

if the segments under consideration have periodic boundary conditions and a probabilistic weight $\omega(L)$:
\[ \omega(\mathcal{L}) = \begin{cases} 
(1 - q^\alpha)^2 q^\alpha \delta^\beta, & \text{if } 1 \leq \mathcal{L} \leq L - 2 \\
(1 - q^\alpha)^2 q^\alpha \delta^\beta, & \text{if } \mathcal{L} = L - 1 
\end{cases} \]  
(18)

if the segments under consideration are open segments.

In Ref. 4, such a distribution \[ \mathcal{L} \] has been considered for the symmetric \( A + \emptyset \leftrightarrow \emptyset + A \) reaction, for an initially homogeneous and uncorrelated system. The authors used the relationship of their model to the Heisenberg chain and its symmetries to compute explicitly the autocorrelation function \[ \langle n_r(t) n_r(0) \rangle \].

For periodic segments, we also consider the algebraic disorder-distribution\[ \mathcal{L} \]

\[ \omega(\mathcal{L}) = \begin{cases} 
0, & \text{if } \mathcal{L} < b \\
(\gamma - 1) b^{\gamma - 1} L^{-\gamma}, & \text{if } \mathcal{L} > b; \text{ with } \gamma > 2 
\end{cases} \]  
(19)

In this section we want to extend Grynberg and Stinchcombe’s\[ \mathcal{L} \] results and calculate explicitly the density and the non-instantaneous two-point correlation functions on the 10-parameter space manifold \[ \mathcal{L} \] on which the equations of motion of the correlation functions are closed. This is first performed for disordered systems of periodic segments of length \( \mathcal{L} \) and then for disordered systems of open segments.

To obtain the density of the disordered systems, which we denote \[ \langle n_m(t) \rangle \], and of the non-instantaneous two-point correlation functions \[ \langle n_m(t) n_j(0) \rangle \], (where the symbol \[ \langle \ldots \rangle \] means the average over the time and the disorder), we need the density \[ \langle n_m(t) \rangle \] and the non-instantaneous two-point correlation functions \[ \langle n_m(t) n_j(0) \rangle \] for a finite and ordered system of length \( \mathcal{L} \). In this section we consider the disorder-averaging of an observable, say \( O \), as:

\[ \langle O(t) \rangle \equiv \frac{\sum_{\mathcal{L}} \omega(\mathcal{L}) \langle O(t) \rangle \mathcal{L}}{\sum_{\mathcal{L}} \omega(\mathcal{L}) \mathcal{L}}, \]  
(20)

where \( \sum_{\mathcal{L}} \omega(\mathcal{L}) \mathcal{L} \), and \( \langle O(t) \rangle \mathcal{L} \) denotes, respectively, the (time-)mean-value of the segment of length \( \mathcal{L} \) and the (time-)mean-value of an observable \( O \) on a lattice of size \( \mathcal{L} \). In \[ \mathcal{L} \], the averaging over the samples (i.e. the quenched-disorder) is carried out over segments of length \( \mathcal{L} \). It has to be stressed that this summation must be compatible with the nature of the observable \( O \). In fact, if one is interested in computing the density of particles at site \( m \), namely \[ \langle n_m(t) \rangle \], one should carry the summation over all the segments which include the site \( m \). In this case, a set \[ \left\{ \mathcal{L} \right\} \]

\[ \langle n_m(t) \rangle = \sum_{\mathcal{L} \geq m} \omega(\mathcal{L}) \mathcal{L} \langle n_m(t) \rangle \mathcal{L} \]  
(21)

In the sequel we will focus on the regimes \( u \sim 1 \) and \( u \ll 1 \) and omit the case \( u \gg 1 \).

We are interested in the long-time behaviour of statistical quantities such as density or correlation functions (see section II.A) and \[ \langle O(t) \rangle \]. It has been established (see e.g. Ref. 11) that the long-time behaviour of these quantities are governed by the small momenta \( p \), ergo by large segments, with \( 1 < \mathcal{L} \ll L \rightarrow \infty \). Here the observables \( O \) considered are the density operator at site \( j \), i.e. \( O = n_j \) and the operator \( n_j p_j \). For these operators, as discussed, the long-time behaviour on a finite segment of length \( \mathcal{L} \) follows from a small momentum expansion:\[ \langle O(t) \rangle \sim e^{(B + 2|E|)t} \sum_p \exp (-|E| t p^2 (1 + O(1/L^2))) \]  
(22)

The long-time behaviour \( (|E| t \gg 1) \) in the presence of quenched-disorder, in the regimes \( u \sim 1 \) and \( u \ll 1 \) follows as:

\[ \langle O(t) \rangle \sim \begin{cases} 
\sum_{\mathcal{L} > \sqrt{\pi t / (2B^2)}} \frac{\omega(\mathcal{L}) \langle O(t) \rangle \mathcal{L}}{\sum_{\mathcal{L}} \omega(\mathcal{L}) \mathcal{L}}, & \text{if } u \sim 1 \\
\sum_{\mathcal{L} > \sqrt{\pi t / (2B^2)}} \frac{\omega(\mathcal{L}) \langle O(t) \rangle \mathcal{L}}{\sum_{\mathcal{L}} \omega(\mathcal{L}) \mathcal{L}}, & \text{if } u \ll 1 
\end{cases} \]  
(23)

Let us note that the SEP model with free boundary conditions, i.e. \( \tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = \tilde{\delta} = 0 \), is special in the sense that in this case the quantization constraint \[ \mathcal{L} \] of the momenta \( p \)’s is always (for segments of arbitrary length \( \mathcal{L} \), and not only for the case \( \mathcal{L} \approx L \rightarrow \infty \)) fulfilled with \( p = \frac{2\pi}{\mathcal{L}}, j = 0, 1, \ldots, \mathcal{L} - 1 \).
V. QUENCHED DISORDER FOR PERIODIC SEGMENTS

In this section, we compute the density and the non-instantaneous two-point correlation functions on the soluble 10-parameteric manifold for systems with periodic boundary conditions.

For these periodic systems, we assume that quenched-disorder is characterized by two different distributions: the exponential distribution (19) and the power-law distribution (23).

1) We begin with the first distribution (17) and introduce the following notations: $\xi \equiv -\frac{1}{\ln q}$; $s \equiv \xi$; $\tau \equiv \frac{2E_0}{\xi}$; $A \equiv \hat{A}_\xi^2 \left( \frac{2\pi^2}{s^2} \right)^{\frac{\xi}{s^2}} + 2\pi^2 \xi^2 \left( \frac{\xi}{s^2} \right)^{-\frac{\xi}{s^2}}$; and $B \equiv \hat{A}_\xi \left( \frac{\xi}{s^2} \right)^{\frac{\xi}{s^2}} + 12\pi^2 \left( \frac{\xi}{s^2} \right)^{-\frac{\xi}{s^2}}$.

1.i) To study the density and the correlation functions in this regime, we first expand the expressions obtained in section III.A at small momenta (large $L$) and then proceed to the disorder-averaging according to (21) within a saddle point expansion.

In the regime where $u \ll 1$, we have for the density of a system with a non-uniform initial density profile: $\langle n_m(t = 0) \rangle = \langle n_m(t = \infty) \rangle (1 + \delta_m)$. With $r \equiv m - 1$, we have:

$$\langle n_m(t) \rangle - \langle n_m(\infty) \rangle \approx \frac{\sqrt{2\pi}(\xi^2 \frac{e^{\tau}}{s^2})}{\sqrt{B\xi^{\frac{\tau}{2}}}} \exp \left( -A \tau^{\frac{\tau}{2}} + (B + 2|E|) \frac{\tau^{\frac{\tau}{2}}}{2E} \right)$$

(22)

For the non-instantaneous correlation functions, we also consider a non-uniform initial system: $\langle n_y(0)n_x(0) \rangle = \langle n_y(\infty)n_x(0) \rangle (1 + \delta_{x,y})$ and $\langle n_{y}(0)\rangle \delta_{y,x}$. The expression of $\langle n_y(\infty)n_x(0) \rangle$ is obtained by replacing $\langle n_m(\infty) \rangle$ with $\mu^{\xi^{-1}} \langle n_x(\infty) \rangle + \mu^{1-\xi_0} \langle n_x(0) \rangle$.

1.ii) In the regime where $u \sim 1$, according to (21), we have for the density:

$$\langle n_m(t) \rangle - \langle n_m(\infty) \rangle \approx \left( \frac{\hat{\alpha}}{\xi^2} \right)^{\frac{2-\hat{\alpha}}{2}} \left( \frac{\pi^{2}|E|\tau}{s^2} \right)^{\frac{1}{2}} e^{\frac{\hat{\alpha}}{2}(\pi|E|\tau)^{\frac{1}{2}}} \left( \langle n_m(t) \rangle - \langle n_m(\infty) \rangle \right)$$

(23)

For the non-instantaneous correlation functions, according to (21), we replace in the r.h.s. of (23) with $\langle n_{r}(t)n_0(0) \rangle - \langle n_{r}(\infty)n_0(0) \rangle$ and $\langle n_{r}(t)n_0(0) \rangle - \langle n_{r}(\infty)n_0(0) \rangle$ have been obtained previously (2) and (5) for the (infinite) ordered periodic chain.

We now consider the second (algebraic) distribution (23) for the disorder average, with $\gamma > 2$.

2.i) For the system characterized by a non-uniform initial density: $\langle n_m(t = 0) \rangle = \langle n_m(\infty) \rangle (1 + \delta_m)$, with $r \equiv m - 1$

$$\langle n_m(t) \rangle - \langle n_m(\infty) \rangle \approx \frac{\langle n_m(\infty) \rangle e^{\tau}}{\xi^\gamma} \left( \frac{\gamma - 1}{2} \right)^{\frac{\gamma - 1}{2}} E^{-1} \left( \frac{e^{(B+2|E|)\tau}}{4\pi^2|E|^\gamma \tau^{\gamma - 1}} \right)$$

(24)

For systems with non-uniform initial states such that $\langle n_y(0)n_x(0) \rangle = \langle n_y(\infty)n_x(0) \rangle (1 + \delta_y) \langle n_y(0) \rangle \delta_{y,x}$, we have to replace in the r.h.s. of (23) $\langle n_m(\infty) \rangle$ with $\mu^{\xi^{-1}} \langle n_x(\infty) \rangle + \mu^{1-x_0} \langle n_x(0) \rangle - (1 + \delta_{x,0}) \langle n_x(\infty) \rangle$.

2.ii) In the regime where $u \sim 1$, we have for the density, according to (21):

$$\langle n_m(t) \rangle - \langle n_m(\infty) \rangle \approx \left( \frac{\gamma - 1}{2} \right)^{\gamma - 2} \left( \frac{4\pi^2|E|^\gamma \tau^{\gamma - 2}}{4\pi^2|E|^\gamma \tau^{\gamma - 1}} (\langle n_{m}(t) \rangle - \langle n_{m}(\infty) \rangle) \right)$$

(25)

For the non-instantaneous correlation functions, according to (21), we obtain:

$$\langle n_{r}(t)n_0(0) \rangle - \langle n_{r}(\infty)n_0(0) \rangle \approx \left( \frac{\gamma - 1}{2} \right)^{\gamma - 2} \left( \frac{4\pi^2|E|^\gamma \tau^{\gamma - 2}}{4\pi^2|E|^\gamma \tau^{\gamma - 1}} (\langle n_{r}(t)n_0(0) \rangle - \langle n_{r}(\infty)n_0(0) \rangle) \right)$$

In these expressions $\langle n_m(t) \rangle - \langle n_m(\infty) \rangle$ and $\langle n_{r}(t)n_0(0) \rangle - \langle n_{r}(\infty)n_0(0) \rangle$ have been obtained in section III.A.

\footnote{For segments (with periodic boundary conditions as well as for open segments) caution is required in order to preserve the parity of the processes when considering the quenched-disordered steady-states of the system. As an example of processes which preserve the parity of the number of particles, such as $A + \theta \leftrightarrow \emptyset + A$ and $A + A \leftrightarrow \emptyset + 2 \theta$ or the process $A + \theta \leftrightarrow \emptyset + A$ and $A + A \rightarrow \emptyset + \emptyset$, one has to take into account that for (finite and infinite) segments with initially odd number of particles there will always remain one particle on each segment (see Ref. 1). That is the reason for the steady-state (e.g. $\langle n_r(\infty) \rangle$) to be considered separately. In order to avoid the difficulties due to residual particles, we can restrict to the even subspace, i.e. $\frac{1}{2}(1 + \sum_{m} n_m^2)P(0)$. Therefore, we can, e.g., only take into account segments with even number of particles and the disordered steady-states correspond to $\langle ... \rangle (t = \infty) = \frac{1}{2} \sum_{L} \omega(E) \mathcal{L}(...)(t = \infty)$}
VI. QUENCHED-DISORDER FOR OPEN SEGMENTS

In this section (with the same notations as in sections V and VI), taking advantage of the results of the section IV, we compute density and correlation functions on the soluble 10-parametric manifold for systems with open boundary conditions.

We consider the effect of quenched-disorder for open segments. We assume that the system is initially uncorrelated, with density $\rho(0)$. For technical convenience, we assume the exponential distribution (18). In what follows, the case of (finite) segments with free-boundary conditions (without injection and evacuation of particles at the sites $1$ and $L$) corresponds to the case where $\tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = \tilde{\delta} = 0$ and thus $\alpha = \frac{\sigma_1 - \sigma_2}{2\sigma_0}, \beta = \frac{1}{2} + \frac{\sigma_1 + \sigma_2}{2\sigma_0}, \gamma = \frac{1}{2} + \frac{\sigma_1 - \sigma_2}{2\sigma_0}, \delta = -\frac{\sigma_1}{2\sigma_0}$.

We introduce the quantities $\xi_i = \xi_i(\tilde{\alpha}, \tilde{\beta}), i = 1, 2, 3$, where

$$\xi'_i \equiv \pm \ln \left( \frac{D}{C} \right), \quad \xi''_i \equiv \ln |\Delta| \pm \ln \left( \frac{D}{C} \right), \quad \xi'_3 \equiv \ln |\eta| \pm \ln \left( \frac{D}{C} \right)$$

(26)

where, as previously, we consider $D/C = \sigma_5/\sigma_2 > 0$ and assume $\Delta = \frac{2\sigma_5(\tilde{\alpha} + \tilde{\delta})}{\sigma_2} > 0$, for $\sigma_5 > 0$ and $DC = \sigma_6\sigma_2 > 0$, without further restrictions, we can always assume that $2\sigma_5 \geq \tilde{\alpha} + \tilde{\gamma} > 0$. In particular, this holds true for free-boundaries systems, where $\tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = \tilde{\delta} = 0$.

As in the previous section, we distinguish two regimes:

i) In the first one, $L^2/|E| t \ll 1$, the result is obtained via a saddle-point expansion.

To study the density and the correlation functions in this regime, we first expand the expressions (16) and its correlation functions for small momenta (large $L$) and then proceed to the disorder-averaging according to (16) within a saddle point expansion. In so doing, the saddle points $s_0 = s_0(\xi'_i), i = 1, 2, 3$ are solutions of the saddle-point equation: $\xi'_i - \tilde{\alpha}\tilde{\beta}(s_0\xi)^{\tilde{\beta}-1} + \frac{2\pi^2}{s_0^2} = 0, i = 1, 2, 3$ For the case $\tilde{\beta} = 1$, we have $s_0(\xi'_i) = \left( \frac{2\pi^2}{\alpha - \xi'_i} \right)^{1/3}$.

The long-time behaviour of the density after averaging over the time and the disorder reads (for $x \ll L$):

$$\langle n_x(t) \rangle - \langle n_x(\infty) \rangle \approx \nu(\xi'_i) \exp \left[ \frac{(B - 2|E|)\xi'^2}{2|E|} \max \{\xi'_i, \xi''_i, \xi'_3\} \right] \left( \xi'_i - \tilde{\alpha}\tilde{\beta} - s_0(\xi'_i)^{\tilde{\beta} - 1} - \frac{2\pi^2}{s_0(\xi'_i)^2} \right \}$$

(27)

where $\nu = \nu(\xi'_i)$ is an exponent which depends on the parameters $\xi'_i$. On the r.h.s. of (27), in the term $\max \{\xi'_i, \xi''_i, \xi'_3\}$, we select the maxima (over the $\xi'_i$) of the function $f(s_0(\xi'_i)) = \xi'_i s_0(\xi'_i) - \tilde{\alpha}\tilde{\beta} s_0(\xi'_i) - \frac{3}{2}\xi'_i s_0(\xi'_i) - \frac{2\pi^2}{s_0(\xi'_i)^2}$, where the $s_0(\xi'_i)$ are the solutions of the saddle-point equation.

We see from the formula (27) that the effect of the open boundary appears through the term $\xi'_i$ (26) and can affect the dynamics with respect to the case of segments with periodic boundary conditions.

For the case where $\tilde{\beta} = 1$ all the computations can be explicitly carried out as follows.

We define $\xi'_i$ according to $\{\hat{\alpha} - \xi'_i\}^{2/3} = \max_{\xi'_i, \xi''_i, \xi'_3} (\hat{\alpha} - \xi'_i)^{2/3}$. To compute the density, we then distinguish two cases (with $\hat{\alpha} \neq \xi'_i$). First, if $\xi'_i \neq 0$ (or $\xi'_i = 0$ and $\Delta = 1$), as for the SEP model,

$$\langle n_x(t) \rangle - \langle n_x(\infty) \rangle \approx \frac{2\pi^2}{3} \left( \frac{4\pi^2}{(\hat{\alpha} - \xi'_i)^2} \right)^{1/6} \exp \left( -\frac{3(\hat{\alpha} - \xi'_i)^{2/3}}{2(4\pi^2\tau)^{1/3}} + \frac{(B + 2|E|)}{2|E|} \right) \xi'^2 \tau$$

(28)

If $\xi'_i = 0$, $\Delta \neq 1$ and $\hat{\beta} = 1$, for the situation considered here (systems initially uncorrelated with initial density $\rho(0) \neq 0$), we have

$$\langle n_x(t) \rangle - \langle n_x(\infty) \rangle \approx -\frac{\hat{\alpha}^{1/3}}{2\pi^2} \exp \left( \frac{3}{2}(4\pi^2\hat{\alpha}^{2/3})^{1/3} + \frac{(B + 2|E|)}{2|E|} \right) \xi'^2 \tau$$

(29)

ii) In the other one, where $L^2 \sim |E| t$, at $\hat{\beta} = 1$, with $\hat{\alpha} \neq \xi'_i$, according to (21), we have

$$\langle n_x(t) \rangle - \langle n_x(\infty) \rangle \approx \frac{\hat{\alpha}}{\xi L} (\pi^2 |E| t \tau)^{1/2} \exp \left[ -\frac{\hat{\alpha}}{\xi} (\pi^2 |E| t \tau)^{1/2} \right] \langle (n_x(t)) - \langle n_x(\infty) \rangle \rangle$$

(30)

In this expression, $\langle n_x(t) \rangle - \langle n_x(\infty) \rangle$ is given in (11), for the case $|B| \neq C + D$.

Similar computations can be performed for $\langle n_x(t)n_{x_0} \rangle - \langle n_x(\infty)n_{x_0} \rangle$. For systems initially uncorrelated and homogeneous with initial density $\rho(0)$ and for the case $\hat{\beta} = 1$, the long-time behaviour ($|E| t \gg$
1, u = L^2/|E|t ≪ 1) of the two-point correlation functions is the following: \( \langle n_x(t)n_{x_0} \rangle - \langle n_x(\infty)n_{x_0} \rangle \approx r^\nu(\xi') \exp \left( -\frac{3(\hat{\alpha}-\xi')}{2} \right) \left( 4\pi^2 \tau \right)^{1/3} + \frac{B + 2|E|}{2|E|} \xi^2 \tau \), where \( \xi' \) is the quantity defined above. We see that in absence of initial correlations, the two-point correlation function has, in this regime, the same stretched-exponential long-time behaviour as the density \( \langle n_x \rangle \). The exponent \( \nu(\xi') \) depends on \( \xi' \). For the cases considered here (\( \hat{\beta} = 1 \)) its value is \( \nu = \frac{1}{2} \) (as for the SEP model with free boundary conditions, see below and Ref.\[\text{[1]}\]) or \( \nu = \frac{1}{3} \).

In the other regime, the long-time behaviour \( (|E|t \gg 1, u = L^2/|E|t \sim 1) \) of the two-point correlation functions follows as in \( \langle \xi \rangle \): \( \langle n_x(t)n_{x_0} \rangle - \langle n_x(\infty)n_{x_0} \rangle \approx \frac{\hat{\alpha}}{\xi} \left( 4\pi^2 \right)^{1/3} \exp \left[ -\frac{\hat{\alpha}}{\xi} \left( 4\pi^2 \right)^{1/3} \right] \left( \langle n_x(t)n_{x_0}(0) \rangle - \langle n_x(\infty)n_{x_0}(0) \rangle \right) \).

In the presence of initial correlations, the situation changes radically: one should take into account the initial correlation in the disorder-averaging. The latter give rise to results which would be very different from the ones obtained for the density.

VII. ILLUSTRATION: TWO EXAMPLES

As a first example we consider the symmetric exclusion model (SEP) \( A + \theta \leftrightarrow \theta + A \), with rates \( \Gamma_{10}^0 = \Gamma_{01}^0 = \Gamma > 0 \), on a diluted chain (periodic and with free boundaries) with distribution probability: \( \mathcal{P}(\Gamma, \Sigma) = q\delta_{\Gamma,\Gamma_1} + (1-q)\delta_{\Gamma,0} \).

With the results of the section III.A, III.B, V and VI, setting \( \hat{\alpha} = \hat{\beta} = 1 \), in the expression of \( A \) and \( B \) (see section V), in the regime \( \Gamma t \gg L^2 \), we recover the result of Gryenberg and Stinchcomb\[\text{[2]}\]:

$$
\langle n_x(t)n_{x_0} \rangle - \langle n_x(\infty)n_{x_0} \rangle \approx \rho(0)(1 - \rho(0)) \sqrt{\frac{2\pi}{3}} \left( 4\pi^2 \tau \right)^{1/3} \exp \left[ -\frac{3}{2} \left( 4\pi^2 \right)^{1/3} \right]$$

(31)

This result is valid for open segments as well as periodic segments.

As second illustration, we consider the dynamics of the system described by the following processes: \( A + A \leftrightarrow \theta + \theta \), with rates \( \Gamma_{11}^0 = \Gamma_{00}^0 = \Sigma \) and \( A + \theta \leftrightarrow \theta + A \), with rates \( \Gamma_{10}^1 = \Gamma_{01}^1 = \Gamma \).

This model is relevant to the description of dimer adsorption and desorption (see Ref.\[\text{[1]}\], and references therein).

Contrary to the SEP model, here the effect of open boundaries is real. We assume that \( \Gamma_1 > \Sigma_1 \) and that there is neither injection nor evacuation of particles at the boundary (free boundary conditions) of a segment (i.e. \( \hat{\alpha} = \hat{\beta} = \hat{\gamma} = \hat{\delta} = 0 \)).

This model preserves the parity of the number of particles, a property which has to be taken into account when computing the steady states (see footnote 2).

Consider a diluted chain described by the distribution probability: \( \mathcal{P}(\Gamma, \Sigma) = q\delta_{\Gamma,\Gamma_1} \delta_{\Sigma,\Sigma_1} + (1-q)\delta_{\Gamma,0}\delta_{\Sigma,0} \), \((0 < q < 1)\).

Thus, for segments with periodic boundary conditions, the disorder-distribution simply reads \( \omega(L) = q\xi^L \).

If we assume segments with free boundary conditions, the disorder-distribution reads:

$$
\omega(L) = \left\{ \begin{array}{l}
(1 - q)^\xi^L, \quad \text{if } 0 \leq L \leq L - 2 \\
(1 - q)^\xi^L, \quad \text{if } L = L - 1 
\end{array} \right.
$$

It is useful to introduce the notation: \( \phi = \frac{1}{2} ; \quad 0 < \eta = \frac{\sqrt{\Gamma_1 - \Gamma_{10}^0}}{\sqrt{\Gamma_1 - \Sigma_1}} \leq 1 ; \quad \Delta = 1 + \frac{2\Sigma_1}{\Gamma_1} > 0 ; \quad B = -2(\Gamma_1 + \Sigma_1) ; \quad E = \Gamma_1 - \Sigma_1 ; \quad \xi'_1 = 0 ; \quad \text{and } \xi'_3 = \ln \eta , \quad \sigma_1 = \sigma_3 = \Sigma_1 ; \quad \sigma_2 = \sigma_6 = \Gamma_1 - \Sigma_1 ; \quad \sigma_4 = \sigma_5 = \Gamma_1 + \Sigma_1 ; \quad \sigma_7 = 2\Gamma_1 ; \quad \sigma_9 = \sigma_0 = 2\Sigma_1.

We first consider the case of segments with non-uniform initial conditions. For a system with non-uniform initial conditions: \( \langle n_{m_0}(0) \rangle = \langle n_{m(\infty)} \rangle \langle 1 + \delta_{m,1} \rangle = \frac{1}{2} \langle 1 + \delta_{m,1} \rangle \), we obtain (\( r = m - 1 \)), in the regime where \( \xi_1(\Gamma_1 - \Sigma_1)t \gg 1 \), and \( \tau \equiv \frac{(\Gamma_1 - \Sigma_1)t}{\xi_1^2} \):

$$
\langle n_r(t) \rangle - \langle n_r(\infty) \rangle \approx \langle n_{m(\infty)} \rangle \left( \frac{4\pi^2 \tau}{\hat{\alpha}^4} \right)^{1/6} \exp \left( -\frac{3}{2} \left( 2\pi^2 \right)^{1/3} - \frac{4\Sigma_1 \xi_2^2 \tau}{\Gamma_1 - \Sigma_1} \right)$$

(32)

Let us now pass to an open system, with free boundary conditions, for which the initial state is uncorrelated with density \( \langle n_{m_0}(0) \rangle = \rho(0) \), (with \( \rho(0) \neq 0 \)), we have:

In the regime where \( |\Gamma_1|t, |\Sigma_1|t \gg L^2 \gg 1 \), with \( \xi'_1 = \ln \Delta > 0 \), \( \langle \hat{\alpha} \neq \xi' \rangle \):

$$
\langle n_x(t) \rangle - \langle n_x(\infty) \rangle \approx \left\{ \begin{array}{l}
\left( \frac{4\pi^2 \tau}{(\hat{\alpha}^2 \ln \Delta)^4} \right)^{1/6} \exp \left( -\frac{3(\hat{\alpha} - \xi' \ln \Delta)^2}{2} \left( 4\pi^2 \tau \right)^{1/3} - \frac{4\Sigma_1 \xi_2^2 \tau}{\Gamma_1 - \Sigma_1} \right) \quad \text{if } \Delta < e^{2\hat{\alpha}/\xi'} \\
-\hat{\alpha}^{1/3} \sqrt{\frac{2\pi^2}{4\Sigma_1}} \exp \left[ -\frac{3}{2} \left( 2\pi^2 \right)^{1/3} - \frac{4\Sigma_1 \xi_2^2 \tau}{\Gamma_1 - \Sigma_1} \right] \quad \text{if } \Delta \geq e^{2\hat{\alpha}/\xi'}
\end{array} \right.
$$

(33)

We note on the basis of this simple example, the effects of the boundary conditions. The SEP model, where the long-time dynamics does not depend on the boundary conditions (see Ref.\[\text{[1]}\]) of the segments is a special case.
VIII. SUMMARY AND CONCLUSION

We have considered the single-species one-dimensional models characterized by rates lying on a 1-parametric manifold. Such reaction-diffusion processes are (in arbitrary dimensions) formally completely soluble because the equations of motion of correlation functions are closed. After having generalized a previous approach to the dynamics of (ordered) systems with open boundaries, with help of the Dynamical Matrix Ansatz, we investigated the dynamics of this class of processes in the presence of quenched disorder (consisting in a random distribution of the reaction of periodic and open segments). We have computed the density and the two-point non-instantaneous correlation functions for various initial states. The study of the dynamics in the presence of quenched disorder leads to various regimes. The most remarkable effect of the disorder appears when time is much larger than the typical diffusion time. In this case the finite-size effect radically changes the dynamics: for the SEP model, the non-instantaneous correlation functions no longer decay algebraically but follow rather a stretched exponential. For the other models, in this regime, the quenched disorder also affects the dynamics but less dramatically. The decay of the density and two-point correlation functions is a combination of the ordered exponential (corrected by subdominant terms arising from the disorder average) and a power-law arising from the quenched disorder average. Similar effects of disorder have been reported in recent experiments. We also pointed out the non-trivial effect of the boundary conditions on the disorder average. We have illustrated our results by considering two physically relevant examples.

The support of Swiss National Fonds is gratefully acknowledged.

---

1 “Nonequilibrium Statistical Mechanics in One Dimension”, edited by V. Privman (Cambridge University Press, Cambridge, 1997); J. Marro, R. Dickman, “Phase Transitions in Lattice Systems” (Cambridge University Press, Cambridge, 1998); B. Chopard, M. Droz, “Cellular Automata Modelling of Physical Systems’ (Cambridge University Press, Cambridge, 1998); D.C. Mattis and M.L. Glasser, Rev. Mod. Phys. 70, 979 (1998);

2 G.M. Schütz, Exactly solvable models for many-body systems far from equilibrium, to appear in Phase Transitions and Critical Phenomena, C. Domb and J. Lebowitz (eds.) (Academic Press, London, 2000)

3 M.D. Grynberg, T.J. Newman and R.B. Stinchcombe, Phys. Rev. E 50, 957 (1994); M.D. Grynberg and R.B. Stinchcombe, Phys. Rev. Letter 74, 1242 (1995); M.D. Grynberg and R.B. Stinchcombe, Phys. Rev. E 52, 6013 (1995); M.D. Grynberg and R.B. Stinchcombe, Phys. Rev. Letter 76, 851 (1996); G.M. Schütz, J. Phys. A 28, 3405 (1995); G.M. Schütz, Phys. Rev. E 53, 1475 (1996); P.-A. Bares, M. Mobilia Phys. Rev E 59, 1996 (1999)

4 F.C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg, Ann. Phys. (N.Y.) 230, 250 (1994)

5 C. Mandache and D. ben-Avraham, J. Chem. Phys. 112, 7735-7740 (2000). We emphasize that in this work the computations were performed for the diffusion-coagulation (and diffusion-annihilation) processes on an infinite lattice, in the continuum limit and thus, the boundary conditions for the segments of length $L < L$ were not specified.

6 M.D. Grynberg and R.B. Stinchcombe, Phys. Rev. E 61, 324 (2000)

7 N. Kuroda, Y. Wakabayashi, M. Nishida, N. Wakabayashi, M. Yamashita, and N. Matsushita, Phys. Rev. Lett. 79, 2510 (1997); H. Okamoto, Y. Oka, T. Mitani, and M. Yamashita, Phys. Rev. B 55, 6330 (1997); R. Kroon and R.Sprik, in Nonequilibrium Statistical Mechanics in One Dimensions, V. Privman, ed., (Cambridge University Press, 1997). N. Kuroda, Y. Tabata, M. Nishida, and M. Yamashita, Phys. Rev. B 59, 12973 (1999); N. Kuroda, M. Nishida, Y. Tabata, Y. Wakabayashi and K. Sasaki, Phys. Rev. B 61, 11217 (2000)

8 R.B. Stinchcombe and G.M. Schütz, Euro. Phys. Lett. 29, 663 (1995); R.B. Stinchcombe and G.M. Schütz, Phys. Rev. Lett. 75, 143 (1995)

9 G.M. Schütz, J. Stat.Phys. 79, 243 (1995); Y. Fujii and M. Wadati, J. Phys. Soc. of Japan 66, 3770 (1997); M. Mobilia and P.-A. Bares, Phys. Rev. E 63 036121 (2001); A. Aghamohammadi, A.M. Fatollahi, M. Khorrami and A. Shariati, Phys. Rev. E 60 4642 (2000)

10 F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, J.Phys. A 20, 6397 (1987)

11 G.M. Schütz Euro. Phys. J. B 5, 277 (1998)

12 T. Sosomoto and M. Wadati, J. Phys. Soc. Japan 66, 279 (1997)

13 M. Droz, J. Kampphorst Leal da Silva, and A. Malaspina, Phys. Lett. A 115, 448 (1986)

14 R.B. Stinchcombe, Diluted Magnetism in Phase Transitions and Critical Phenomena, edited by C. Domb and J.L. Lebowitz (Academic Press, London 1983)

15 We have seen that for open systems the $p$’s are quantized according to the special constraints [13]. In the thermodynamic limit this constraint is fulfilled, e.g., by $p = \pm \frac{\pi j}{L}$, $j = 0, 1, \ldots, L - 1$. The “phase-shift” $\frac{\pi j}{L}$ appears in the expression [16] of the density and the correlation functions (for $L \ll L \to \infty$) through the parameter $\Delta$. 

16
The solution of this transcendental equations at small $p$ leads to $p = \pm \frac{\pi}{L} + O(L^{-2})$. Therefore, it is sufficient to assume $p = \pm \frac{\pi}{L}, j = 0, 1, \ldots$ in the large $L$ expansion of the terms contributing to the density and correlation functions.