On Beltrami fields with nonconstant proportionality factor on the plane

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Abstract

We consider the equation

\[ \text{rot} \overrightarrow{B} + \alpha \overrightarrow{B} = 0 \] (1)

on the plane with \( \alpha \) being a real-valued function and show that it can be reduced to a Vekua equation of a special form. In the case when \( \alpha \) depends on one Cartesian variable a complete system of exact solutions of the Vekua equation and hence of equation (1) is constructed based on L. Bers’ theory of pseudoanalytic formal powers.

1 Introduction

Solutions of the equation

\[ \text{rot} \overrightarrow{B} + \alpha \overrightarrow{B} = 0 \] (2)

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where $\alpha$ is a scalar function of space coordinates are known as Beltrami fields and are of fundamental importance in different branches of modern physics (see, e.g., [22], [18], [7], [21], [1], [9], [8], [11]). For simplicity, in this work we consider the real-valued proportionality factor $\alpha$ and real-valued solutions of (2), though the presented approach is applicable in a complex-valued situation as well with a considerable complication of mathematical techniques involved (instead of complex Vekua equations their bicomplex generalizations should be considered [5], [14]). We consider equation (2) on a plane of the variables $x$ and $y$, that $\alpha$ and $\vec{B}$ are functions of two Cartesian variables only. In this case as we show in section 3 equation (2) reduces to the equation

$$\text{div} \left( \frac{1}{\alpha} \nabla u \right) + \alpha u = 0.$$  \hspace{1cm} (3)

This second-order equation can be reduced (see [14]) to a corresponding Vekua equation (describing generalized analytic functions) of a special form. This reduction under quite general conditions allows us to construct a complete system of exact solutions of (3) explicitly (see [13] and [15]). For the reduction of (3) to a Vekua equation it is sufficient to find a particular solution of (3). In the present work (section 4) we show that in a very important for applications case of $\alpha$ being a function of one Cartesian variable a particular solution of (3) is always available in a simple explicit form. This situation corresponds to models describing waves propagating in stratified media (see, e.g., [16]). As a result in this case we are able to construct a complete system of solutions explicitly which for many purposes means a general solution. We give an example of such construction.

We show in this work that when $\alpha = \alpha(y)$ (of course in a similar way the case $\alpha = \alpha(x)$ can be considered) equation (3) and hence equation (2) reduce to the Vekua equation of the following form

$$\overline{\partial} W(x, y) = \frac{1}{2} \frac{f'(y)}{f(y)} \mathbf{W}(x, y)$$  \hspace{1cm} (4)

where

$$f = \frac{c_1}{\sqrt{\alpha}} \sin \mathcal{A} + \frac{c_2}{\sqrt{\alpha}} \cos \mathcal{A};$$

$\mathcal{A}$ is an antiderivative of $\alpha$ with respect to $y$, $c_1$ and $c_2$ are arbitrary real constants, $z = x + iy$ and $\overline{\partial} = \frac{1}{2} (\partial_x + i \partial_y)$. A complete (in a compact uniform convergence topology) system of exact solutions to (4) can be constructed
explicitly. The system represents a set of formal powers \[3, 6\] which generalize the usual analytic complex powers \((z - z_0)^n, n = 0, 1, 2, \ldots\) and in a sense give us a general solution of (4). Thus, in the case when \(\alpha\) is a function of one Cartesian variable the Vekua equation equivalent to (2) in a two-dimensional situation can be solved and a complete system of solutions of (2) is obtained.

2 Preliminaries

We need the following definition. Consider the equation

\[
\partial_x \varphi = \Phi \tag{5}
\]

on a whole complex plane or on a convex domain, where \(\Phi = \Phi_1 + i\Phi_2\) is a given complex valued function such that its real part \(\Phi_1\) and imaginary part \(\Phi_2\) satisfy the equation

\[
\partial_y \Phi_1 - \partial_x \Phi_2 = 0 \tag{6}
\]

then there exist real valued solutions of (5) which can be easily constructed in the following way

\[
\varphi(x, y) = 2 \left( \int_{x_0}^x \Phi_1(\eta, y) d\eta + \int_{y_0}^y \Phi_2(x_0, \xi) d\xi \right) + c \tag{7}
\]

where \((x_0, y_0)\) is an arbitrary fixed point in the domain of interest and \(c\) is an arbitrary real constant.

By \(\overline{\Phi}\) we denote the integral operator in (7):

\[
\overline{\Phi}[\Phi](x, y) = 2 \left( \int_{x_0}^x \Phi_1(\eta, y) d\eta + \int_{y_0}^y \Phi_2(x_0, \xi) d\xi \right) + c.
\]

Note that formula (7) can be extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve \(\Gamma\) leading from \((x_0, y_0)\) to \((x, y)\)

\[
\varphi(x, y) = 2 \left( \int_{\Gamma} \Phi_1 dx + \Phi_2 dy \right) + c.
\]

Thus if \(\Phi\) satisfies (6), there exists a family of real valued functions \(\varphi\) such that \(\partial_x \varphi = \Phi\), given by the formula \(\varphi = \overline{\Phi}[\Phi]\).
Let $f$ denote a given positive twice continuously differentiable function defined on a domain $\Omega \subset \mathbb{C}$. Consider the following Vekua equation

$$W_{\overline{z}} = \frac{f_{\overline{z}} W}{f} \quad \text{in } \Omega \quad (8)$$

where the subindex $\overline{z}$ means the application of the operator $\partial_{\overline{z}}$, $W$ is a complex-valued function and $\overline{W}$ is its complex conjugate function. As was shown in [12], [13], [14], [15], [17] equation (8) is closely related to the second-order equation of the form

$$(\text{div} \, p \, \text{grad} + q)u = 0 \quad \text{in } \Omega \quad (9)$$

where $p$ and $q$ are real-valued functions. In particular the following statements are valid.

**Theorem 1** [14] Let $u_0$ be a positive solution of (9). Assume that $f = \sqrt{p} u_0$ and $W$ is any solution of (8). Then $u = \frac{1}{\sqrt{p}} \text{Re} W$ is a solution of (9) and $v = \sqrt{p} \text{Im} W$ is a solution of

$$(\text{div} \, \frac{1}{p} \, \text{grad} + q_1)v = 0 \quad \text{in } \Omega \quad (10)$$

where

$$q_1 = -\frac{1}{p} \left( \frac{q}{p} + 2 \left\langle \nabla p, \nabla u_0 \right\rangle + 2 \left( \frac{\nabla u_0}{u_0} \right)^2 \right). \quad (11)$$

**Theorem 2** [14] Let $\Omega$ be a simply connected domain, $u_0$ be a positive solution of (9) and $f = p^{1/2} u_0$. Assume that $u$ is a solution of (9). Then a solution $v$ of (10) with $q_1$ defined by (11) such that $W = p^{1/2} u + ip^{-1/2} v$ is a solution of (8) is constructed according to the formula

$$v = u_0^{-1} \overline{A}(ipu_0^{2} \partial_{\overline{z}}(u_0^{-1} u)) \quad (12)$$

and vice versa, let $v$ be a solution of (11), then the corresponding solution $u$ of (9) such that $W = p^{1/2} u + ip^{-1/2} v$ is a solution of (8), is constructed according to the formula

$$u = -u_0 \overline{A}(ip^{-1} u_0^{-2} \partial_{\overline{z}}(u_0 v)). \quad (13)$$
Thus the relation between (8) and (9) is very similar to that between the Cauchy-Riemann system and the Laplace equation. Moreover, choosing $p \equiv 1, q \equiv 0$ and $u_0 \equiv 1$ we obtain that (12) and (13) become the well known formulas from the classical complex analysis for constructing conjugate harmonic functions.

For a Vekua equation of the form

$$W_\bar{z} = aW + b\bar{W}$$

where $a$ and $b$ are arbitrary complex-valued functions from an appropriate function space [20], a well developed theory of Taylor and Laurent series in formal powers was created (see [3], [4]) containing among others the expansion and the Runge theorems as well as more precise convergence results (see, e.g., [19]) and a general simple algorithm [15] for explicit construction of formal powers for the Vekua equation of the form (8).

3 Reduction of (2) to a Vekua equation

We consider equation (2) where both $\alpha$ and $B$ are supposed to be dependent on two Cartesian variables $x$ and $y$. Then equation (2) can be written as the following system

$$\partial_y B_3 + \alpha B_1 = 0 \quad (14)$$
$$- \partial_x B_3 + \alpha B_2 = 0 \quad (15)$$
$$\partial_x B_2 - \partial_y B_1 + \alpha B_3 = 0.$$

Solving this system for $B_3$ leads to the equation

$$\Delta B_3 - \left\langle \frac{\nabla \alpha}{\alpha}, \nabla B_3 \right\rangle + \alpha^2 B_3 = 0 \quad (16)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product of two vectors.

Note that

$$\alpha \text{ div} \left( \frac{1}{\alpha} \nabla B_3 \right) = \Delta B_3 - \left\langle \frac{\nabla \alpha}{\alpha}, \nabla B_3 \right\rangle$$

and hence (16) can be rewritten as follows

$$\text{div} \left( \frac{1}{\alpha} \nabla B_3 \right) + \alpha B_3 = 0. \quad (17)$$
Thus equation (2) reduces to an equation of the form (9) with $p = 1/\alpha$ and $q = \alpha$.

Let us notice that (see, e.g., [14])

$$\text{div} \frac{1}{\alpha} \nabla + \alpha = \frac{1}{\sqrt{\alpha}} (\Delta - r) \frac{1}{\sqrt{\alpha}}$$

where

$$r = -\frac{1}{2} \frac{\Delta \alpha}{\alpha} + \frac{3}{4} \left( \frac{\nabla \alpha}{\alpha} \right)^2 - \alpha^2. \quad (18)$$

That is $B_3$ is a solution of (17) iff the function $f = B_3/\sqrt{\alpha}$ is a solution of the stationary Schrödinger equation

$$(-\Delta + r) f = 0 \quad (19)$$

with $r$ defined by (18). As was explained in section 2, given its particular solution this equation reduces to the Vekua equation (8). Unfortunately, in general we are not able to propose a particular solution of (17). Nevertheless in an important special case when $\alpha$ depends on one Cartesian variable, a particular solution of (17) is always available in explicit form. We give this result in the next section.

4 Solution in the case when $\alpha$ is a function of one Cartesian variable

Let us consider equation (19) where $\alpha = \alpha(y)$. We assume that $\alpha$ is a nonvanishing function and look for a solution of the corresponding ordinary differential equation

$$\frac{d^2 f_0}{dy^2} + \left( \frac{1}{2} \frac{\alpha''}{\alpha} - \frac{3}{4} \left( \frac{\alpha'}{\alpha} \right)^2 + \alpha^2 \right) f_0 = 0.$$ 

Its general solution is known (see [10] 2.162 (14)) and is given by the expression

$$f_0(y) = \frac{c_1}{\sqrt{\alpha(y)}} \sin \mathcal{A}(y) + \frac{c_2}{\sqrt{\alpha(y)}} \cos \mathcal{A}(y) \quad (20)$$

where $\mathcal{A}$ is an antiderivative of $\alpha$ and $c_1, c_2$ are arbitrary real constants.
Choosing, e.g., \( c_1 = 1, c_2 = 0 \) and calculating the coefficient \( (\partial_z f_0) / f_0 \) we arrive at the following Vekua equation which is equivalent to (2) in the case under consideration (and which is considered in any simply connected domain where \( \sin \mathcal{A}(y) \) does not vanish),

\[
\partial_z W(x, y) = \frac{i}{2} \left( \alpha(y) \cot \mathcal{A}(y) - \frac{\alpha'(y)}{2\alpha(y)} \right) W(x, y).
\]

Note that \( F = f_0 = \frac{\sin \mathcal{A}(y)}{\sqrt{\alpha(y)}} \) and \( G = \frac{i}{f_0} = \frac{i\sqrt{\alpha(y)}}{\sin \mathcal{A}(y)} \) represent a generating pair for this Vekua equation (see [13], [15]) and hence if \( W \) is its solution, the corresponding pseudoanalytic function of the second kind \( \omega = \frac{1}{f_0} \text{Re} W + if_0 \text{Im} W \) satisfies the equation

\[
\omega_\tau = \frac{1 - f_0^2}{1 + f_0^2} \omega_\tau \tag{21}
\]

which can be written in the form of the following system

\[
\phi_x = \frac{1}{f_0^2} \psi_y, \quad \phi_y = -\frac{1}{f_0^2} \psi_x
\]

where \( \phi = \text{Re} \omega \) and \( \psi = \text{Im} \omega \).

For \( f_0 \) being representable in a separable form \( f_0(x, y) = X(x)Y(y) \) the formulas for constructing corresponding formal powers explicitly were presented already by L. Bers and A. Gelbart (see [3] and [6]). Using them we obtain the following representation for the formal powers corresponding to (21)

\[
\ast Z^{(n)}(a, z_0; z) = a_1 \sum_{k=0}^{n} \binom{n}{k} (x - x_0)^{(n-k)}i^k Y^k + ia_2 \sum_{k=0}^{n} \binom{n}{k} (x - x_0)^{(n-k)}i^k \tilde{Y}^k
\]

(we preserve the notations from [3]) where \( z_0 = x_0 + iy_0 \) is an arbitrary point of the domain of interest, \( a \) is an arbitrary complex number: \( a = a_1 + ia_2 \), \( Y^k \) and \( \tilde{Y}^k \) are constructed as follows

\[
Y^{(0)}(y_0, y) = \tilde{Y}^{(0)}(y_0, y) = 1
\]
and for $n = 1, 2, \ldots$

$$Y^{(n)}(y_0, y) = n \int_{y_0}^{y} Y^{(n-1)}(y_0, \eta) f_{0}^{2}(\eta) d\eta \quad n \text{ odd}$$

$$Y^{(n)}(y_0, y) = n \int_{y_0}^{y} Y^{(n-1)}(y_0, \eta) \frac{d\eta}{f_{0}^{2}(\eta)} \quad n \text{ even}$$

$$\tilde{Y}^{(n)}(x_0, x) = n \int_{x_0}^{x} \tilde{Y}^{(n-1)}(x_0, \eta) f_{0}^{2}(\eta) d\eta \quad n \text{ odd}$$

$$\tilde{Y}^{(n)}(x_0, x) = n \int_{x_0}^{x} \tilde{Y}^{(n-1)}(x_0, \eta) \frac{d\eta}{f_{0}^{2}(\eta)} \quad n \text{ even.}$$

The system \( \{ \star Z^{(n)}(1, z_0; z), \star Z^{(n)}(i, z_0; z) \} \) represents a complete (in a compact uniform convergence topology [2]) system of solutions of (21) that means that any solution \( \omega \) of (21) in a simply connected domain \( \Omega \) can be represented as a series

$$\omega(z) = \sum_{n=0}^{\infty} \star Z^{(n)}(a_n, z_0; z) = \sum_{n=0}^{\infty} \left( a'_n \star Z^{(n)}(1, z_0; z) + a''_n \star Z^{(n)}(i, z_0; z) \right)$$

where \( a'_n = \text{Re} a_n, \ a''_n = \text{Im} a_n \) and the series converges normally (uniformly on any compact subset of \( \Omega \)). Consequently the system of functions

$$\{ f_0(y) \text{Re}(\star Z^{(n)}(1, z_0; z)), f_0(y) \text{Re}(\star Z^{(n)}(i, z_0; z)) \} \quad \{ \sqrt{\alpha(y)} f_0(y) \text{Re}(\star Z^{(n)}(1, z_0; z)), \sqrt{\alpha(y)} f_0(y) \text{Re}(\star Z^{(n)}(i, z_0; z)) \}$$

where \( \alpha(y) \) is a complete system of solutions of (21). Thus in the case under consideration any solution \( B_3 \) of (17) can be represented in the form

$$B_3(x, y) = \sum_{n=0}^{\infty} \left( a_n \sin \mathcal{A}(y) \text{Re}(\star Z^{(n)}(1, z_0; z)) + b_n \sin \mathcal{A}(y) \text{Re}(\star Z^{(n)}(i, z_0; z)) \right)$$
where \( a_n \) and \( b_n \) are real constants.

The other two components of the vector \( \overrightarrow{B} \) are obtained from (14) and (15):

\[
B_1 = -\frac{1}{\alpha} \partial_y B_3 \quad \text{and} \quad B_2 = \frac{1}{\alpha} \partial_x B_3
\]

(23)

that gives us a complete system of solutions of (2) in the case under consideration. On the following example we explain how this procedure works.

**Example 3** Let us consider the following relatively simple situation in which the corresponding integrals are not difficult to evaluate. Let

\[
\alpha(y) = \frac{1}{\sqrt{1 - y^2}}
\]

(24)

and \( \Omega \) be an open unitary disk with a center in the origin. We take in (20) \( c_1 = 0 \) and \( c_2 = 1 \). Then it is easy to verify that

\[
f_0(y) = (1 - y^2)^{\frac{3}{2}}.
\]

The first three formal powers with a centre in the origin can be calculated as follows

\[
*_Z^{(1)}(1, 0; z) = x + iy \left( \frac{y(1 - y^2)^{\frac{3}{2}}}{4} + \frac{3y(1 - y^2)^{\frac{1}{2}}}{8} + \frac{3}{8} \arcsin y \right)
\]

\[
*_Z^{(1)}(i, 0; z) = -\frac{y}{(1 - y^2)^{\frac{1}{2}}} + ix,
\]

\[
*_Z^{(2)}(1, 0; z) = x^2 - \frac{1}{4} y^2 - \frac{3}{4} y \arcsin y
\]

\[
+ 2ix \left( \frac{y(1 - y^2)^{\frac{3}{2}}}{4} + \frac{3y(1 - y^2)^{\frac{1}{2}}}{8} + \frac{3}{8} \arcsin y \right),
\]

\[
*_Z^{(2)}(i, 0; z) = -\frac{2xy}{(1 - y^2)^{\frac{1}{2}}} + i \left( x^2 - y^2 - \frac{1}{2} y^4 \right),
\]
\[ Z^{(3)}(1, 0; z) = x^3 - 3x \left( \frac{1}{4} y^2 + \frac{3}{4} y \arcsin y \right) \]
\[ + 3ix^2 \left( \frac{y(1 - y^2)^{\frac{3}{2}}}{4} + \frac{3y(1 - y^2)^{\frac{1}{2}}}{8} + \frac{3}{8} \arcsin y \right) \]
\[ - i\left( - \frac{3}{24} y(1 - y^2)^{\frac{3}{2}} + \frac{3}{96} y(1 - y^2)^{\frac{1}{2}} + y(1 - y^2)^{\frac{1}{2}} (\frac{51}{128} - \frac{9}{64} y^2) \right) \]
\[ - \frac{9}{16} (1 - y^2)^{\frac{3}{2}} \arcsin y + \frac{33}{128} \arcsin y, \]
\[ Z^{(3)}(i, 0; z) = - \frac{3x^2y}{(1 - y^2)^{\frac{3}{2}}} + \frac{3y(1 + y^2)}{4 (1 - y^2)^{\frac{3}{2}}} \frac{3}{4} \arcsin y + ix \left( x^2 - 3 \left( y^2 - \frac{1}{2} y^4 \right) \right). \]

Now taking the real parts of these formal powers and multiplying them by the factor \( \sqrt{\alpha f_0} \) (see (22)) we obtain the first elements of the complete system of solutions of (17), that is any solution \( B_3 \) of (17) in a simply connected domain can be represented as an infinite linear combination of the functions

\[ \{(1 - y^2)^{\frac{3}{2}}, \ x(1 - y^2)^{\frac{1}{2}}, \ -y, \ (1 - y^2)^{\frac{1}{2}} \left( x^2 - \frac{1}{4} y^2 - \frac{3}{4} y \arcsin y \right) \}, \]
\[ -2xy, \ (1 - y^2)^{\frac{3}{2}} \left( x^3 - 3x \left( \frac{1}{4} y^2 + \frac{3y \arcsin y}{4 (1 - y^2)^{\frac{1}{2}}} \right) \right), \]
\[ -3x^2y + \frac{3}{4} y(1 + y^2) - \frac{3}{4} (1 - y^2)^{\frac{3}{2}} \arcsin y, \ldots \}

and the corresponding series converges normally.

From (23) it is easy to calculate the corresponding components \( B_1 \) and \( B_2 \) respectively,

\[ \{y, \ xy, \ (1 - y^2)^{\frac{1}{2}}, \ \frac{3}{4} (1 - y^2)^{\frac{1}{2}} \arcsin y + y \left( x^2 - \frac{3}{4} y^2 + \frac{5}{4} \right), \}
\[ 2x(1 - y^2)^{\frac{1}{2}}, \ \frac{9}{4} x(1 - y^2)^{\frac{1}{2}} \arcsin y + y \left( x^3 - \frac{9}{4} xy^2 + \frac{15}{4} x \right), \]
\[ - \left( \frac{9}{4} y^2 - 3x^2 \right) (1 - y^2)^{\frac{1}{2}} - \frac{3}{4} y \arcsin y, \ldots \}

and

\[ \{0, \ (1 - y^2), \ 0, \ 2x(1 - y^2), \ -2y(1 - y^2)^{\frac{1}{2}}, \}
\[ (3x^2 - \frac{3}{4} y^2)(1 - y^2) - \frac{9}{4} y(1 - y^2)^{\frac{1}{2}} \arcsin y, \ -6xy(1 - y^2)^{\frac{1}{2}}, \ldots \}. \]
Thus, we obtain the following complete system of solutions of (2) with the proportionality factor $\alpha$ defined by (24),

$$
\vec{B}_0 = \begin{pmatrix} y \\ 0 \\ (1 - y^2)^{\frac{1}{2}} \end{pmatrix}, \quad \vec{B}_1 = \begin{pmatrix} xy \\ (1 - y^2) \frac{1}{2} \\ x(1 - y^2)^{\frac{1}{2}} \end{pmatrix}, \quad \vec{B}_2 = \begin{pmatrix} (1 - y^2)^{\frac{1}{2}} \\ 0 \\ -y \end{pmatrix},
$$

$$
\vec{B}_3 = \begin{pmatrix} \frac{3}{4}(1 - y^2)^{\frac{1}{2}} \arcsin y + y(x^2 - \frac{3}{4}y^2 + \frac{3}{4}) \\ 2x(1 - y^2) \\ (1 - y^2)^{\frac{1}{2}} \left(x^2 - \frac{1}{4}y^2 - \frac{3}{4y\arcsin y}{(1-y^2)^{\frac{1}{2}}} \right) \end{pmatrix}, \quad \vec{B}_4 = \begin{pmatrix} 2x(1 - y^2)^{\frac{1}{2}} \\ -2y(1 - y^2)^{\frac{1}{2}} \end{pmatrix},
$$

$$
\vec{B}_5 = \begin{pmatrix} \frac{9}{4}x(1 - y^2)^{\frac{1}{2}} \arcsin y + y \left(x^3 - \frac{9}{4}xy^2 + \frac{15}{4}x \right) \\ (3x^2 - \frac{3}{4}y^2)(1 - y^2) - \frac{3}{4}y(1 - y^2)^{\frac{1}{2}} \arcsin y \\ (1 - y^2)^{\frac{1}{2}} \left(x^3 - 3x \left(\frac{1}{4}y^2 + \frac{3}{4y\arcsin y}{(1-y^2)^{\frac{1}{2}}} \right) \right) \end{pmatrix},
$$

$$
\vec{B}_6 = \begin{pmatrix} -\frac{9}{4}y^2 - 3x^2)(1 - y^2)^{\frac{1}{2}} - \frac{3}{4}y \arcsin y \\ -6xy(1 - y^2)^{\frac{1}{2}} \\ -3x^2y + \frac{3}{4}y(1 + y^2) - \frac{3}{4}(1 - y^2)^{\frac{1}{2}} \arcsin y \end{pmatrix},
$$

...
Of course not always the integrals involved in the construction of the complete system of solutions are sufficiently easy to evaluate explicitly as in the example \textsuperscript{3}. Nevertheless our numerical experiments confirm that in general the formal powers and hence the solutions of (2) can be calculated with a remarkable accuracy. For example, the vector $\vec{B}_{40}$ (see notations in the example \textsuperscript{3}) in the Matlab 7 package on a usual PC can be calculated with a precision of the order $10^{-4}$. Thus, the use of formal powers for numerical solution of boundary value problems corresponding to (2) and more generally to equations of the form (9) is really promising. The work in this direction will be reported elsewhere.

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