Alternative algebras admitting derivations with invertible values and invertible derivations

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Abstract. We prove an analogue of the Bergen–Herstein–Lanski theorem for alternative algebras: describe all alternative algebras that admit derivations with invertible values. We also prove an analogue of Moens’ theorem for alternative algebras (a finite-dimensional alternative algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible Leibniz derivation).

Keywords: derivation, alternative algebra, nilpotent algebra.

§ 1. Introduction

The notion of a derivation with invertible values as a derivation of a ring with identity that takes only invertible or zero values was introduced in [1], where Bergen, Herstein and Lanski described the structure of associative rings that admit derivations with invertible values. Their results were generalized in the associative case in [2]–[6].

Another interesting class of derivations are the invertible derivations. The definition of an invertible derivation as a derivation which is an invertible linear map was introduced in [7], where the nilpotency of any finite-dimensional Lie algebra admitting an invertible derivation was proved. Research on this theme was continued in [8], [9].

The study of non-associative algebras and superalgebras with derivations is now of great interest. For example, [10], [11] determine the structure of differentiably simple alternative and Jordan algebras while [12]–[19] give a description of generalizations of derivations of simple and semisimple alternative, Jordan and structurable algebras and superalgebras. Nevertheless, the problem of characterizing those algebras in the classical non-associative varieties (such as alternative, Jordan, structurable and other algebras) that admit derivations with invertible values and invertible derivations was not considered. Our aim here is to fill this gap.

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§ 2. Basic definitions and identities

We use standard notation:

\[(x, y, z) := (xy)z - x(yz)\]

is the associator of \(x, y, z\), and

\[[x, y] := xy - yx\]

is the commutator of \(x, y\).

An algebra \(A\) is said to be alternative if the following identities hold in \(A\):

\[(x, x, y) = 0, \quad (x, y, y) = 0\]

(see [20] for more information on alternative algebras). It is easy to check that in every alternative algebra the associator is a skew-symmetric function of its arguments, and the flexible identity \(x(yx) = (xy)x\) holds. It is also well known ([20], Russian p. 35) that every alternative algebra satisfies the middle Moufang identity \((xy)(zx) = x(yz)x\).

The nucleus of an algebra \(A\) is the set

\[N(A) = \{n \in A \mid (n, A, A) = (A, n, A) = (A, A, n) = (0)\},\]

the commutative centre of \(A\) is the set

\[K(A) = \{k \in A \mid [k, A] = [A, k] = (0)\},\]

and the centre of \(A\) is the set \(Z(A) = N(A) \cap K(A)\).

A derivation \(d\) is said to be inner if it lies in the smallest subspace (of the space of all linear operators on \(A\)) that contains all operators of right and left multiplication by elements of \(A\) and is closed under commutation. Otherwise \(d\) is said to be outer.

The most important class of alternative algebras is the class of Cayley–Dickson algebras. The definition and properties of Cayley–Dickson algebras and the Cayley–Dickson process can be found, for example, in [20]. It is known that every Cayley–Dickson algebra \(C\) over a field \(F\) is 8-dimensional, non-associative, alternative, simple and has an identity element. Moreover, \(C\) is quadratic over \(F\), that is, for every \(x \in C\) we have

\[x^2 - t(x)x + n(x) = 0,\]

where \(t(x), n(x) \in F\), \(t(x)\) is an \(F\)-linear map, and \(n(x)\) is a strictly non-degenerate quadratic form satisfying \(n(xy) = n(x)n(y)\) for all \(x, y \in C\). Hence every Cayley–Dickson algebra is equipped with a symmetric bilinear non-degenerate form \(f(x, y) = n(x + y) - n(x) - n(y)\). Given a subset \(M \subseteq C\), we write \(M^\perp\) for the orthogonal complement to \(M\) with respect to \(f\).

A Cayley–Dickson algebra containing divisors of zero is said to be split. It is known ([20], Russian p. 43) that an element \(x\) of a split Cayley–Dickson algebra is invertible if and only if \(n(x) \neq 0\). It is also known ([20], Russian p. 46) that every split Cayley–Dickson algebra over a field \(F\) is isomorphic to the Cayley–Dickson matrix algebra \(C(F)\) consisting of all matrices of the form \(a = (\begin{array}{cc} \alpha & u \\ v & \beta \end{array})\), where \(\alpha, \beta \in F\), \(u, v \in F^3\).
Addition and scalar multiplication of elements of $C(F)$ is the ordinary addition and scalar multiplication of matrices. However, multiplication of elements of $C(F)$ is given by the following matrix multiplication:

$$
\begin{pmatrix}
\alpha & u \\
v & \beta
\end{pmatrix}
\begin{pmatrix}
\gamma & t \\
w & \delta
\end{pmatrix} =
\begin{pmatrix}
\alpha \delta + (u, w) & \alpha t + \delta u - v \times w \\
\gamma v + \beta w + u \times t & \beta \gamma + (v, t)
\end{pmatrix},
$$

where

$$(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

is the dot product of vectors $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in F^3$, and

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

is their cross product. In this notation we have $t(a) = \alpha + \beta, n(a) = \alpha \beta - (u, v)$.

When $\text{char}(F) \neq 2$, $C$ can be obtained from $F$ by applying thrice the Cayley–Dickson process with the identity involution and parameters $\alpha, \beta, \gamma \in F$. Without going into full details of the Cayley–Dickson process, we give only a formula for the multiplication in the algebra $B = A + vA$ obtained by the Cayley–Dickson process from an algebra $A$ with involution $-$:

$$
(a_1 + vb_1)(a_2 + vb_2) = (a_1 a_2 + \gamma b_2 \overline{b_1}) + v(\overline{a_1} b_2 + a_2 b_1),
$$

where $a_i, b_i \in A$, $v^2 = \gamma \in F$.

We also need the following description [20] of all simple alternative non-associative algebras.

**Theorem 1.** Let $A$ be a simple non-associative alternative algebra. Then the centre of $A$ is a field and $A$ is a Cayley–Dickson algebra over its centre.

§ 3. Alternative algebras admitting derivations with invertible values

Let $A$ be an algebra with identity 1 over a field $F$. We denote the set of invertible elements of $A$ by $U$. In this section we consider only derivations with invertible values, that is, non-zero derivations $d$ such that for every $x \in A$ we have either $d(x) \in U$, or $d(x) = 0$.

In 1983 Bergen, Herstein and Lanski initiated a study whose purpose was to relate the structure of a ring and the special behaviour of one of the derivations of that ring. In [1] they described associative rings admitting derivations with invertible values. They proved that such a ring must be either a division ring, the ring of $2 \times 2$ matrices over a division ring, or a quotient of a polynomial ring over a division ring of characteristic 2. They also described division rings such that the rings of $2 \times 2$ matrices over them have inner derivations with invertible values. Associative rings with derivations with invertible values (and generalizations of them) were further discussed in [2]–[6]. The subject of [2] was the structure of semiprime associative rings with involution admitting a derivation with invertible values on the set of symmetric elements. In [3] Bergen and Carini described associative rings admitting a derivation with invertible values on some non-central Lie ideal. The structure of associative rings admitting $\alpha$-derivations with invertible
values (and their natural generalizations, \((\sigma, \tau)\)-derivations with invertible values) was studied in [4] and [5]. Associative rings admitting generalized derivations with invertible values were described in [6].

The purpose of this section is to generalize the results of Bergen, Herstein and Lanski in the case of alternative algebras.

Throughout the section, \(A\) is an alternative algebra with identity 1 and a derivation \(d\) with invertible values. The following lemmas were proved in [1] for associative algebras. They can be generalized with minor differences to the alternative case. For completeness we give their proofs.

**Lemma 2.** If \(d(x) = 0\), then either \(x = 0\), or \(x\) is invertible.

*Proof.* We note ([20], Russian p. 204) that the following identity holds in every alternative algebra:

\[
(a^{-1}, a, b) = 0. \tag{2}
\]

Using this identity, we easily see that the product of two invertible elements in an alternative algebra is also invertible. Indeed, if \(a\) and \(b\) are invertible, then by (2) we have

\[
(b^{-1}a^{-1})(ab) = a^{-1}((ab)b^{-1}) - (a^{-1}(ab))b^{-1} + (b^{-1}a^{-1})(ab)
\]

\[
= -(a^{-1}, ab, b^{-1}) + (b^{-1}a^{-1})(ab)
\]

\[
= -(b^{-1}, a^{-1}, ab) + (b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}(ab)) = 1.
\]

Assume that \(x \neq 0\). Since \(d \neq 0\), there is \(y \in A\) such that \(d(y) \in U\). Hence \(d(yx) = d(y)x \in U\) and \(d(y)^{-1}d(yx) = x\). Since \(d(y)\) and \(d(yx)\) are invertible, so is \(x\). \(\square\)

We now study the structure of the ideals of \(A\).

**Lemma 3.**

a) If \(L \neq 0\) is a one-sided ideal of \(A\), then \(d(L) \neq (0)\).

b) If \(I\) is a proper one-sided ideal of \(A\), then \(I\) is maximal and minimal.

c) If \(I\) is a proper ideal of \(A\), then \(I^2 = (0)\).

d) If \(\text{char}(A) \neq 2\), then \(A\) is simple.

*Proof.*

a) Since the assertion is obvious for \(L = A\), it suffices to consider the case when \(L \neq A\). If \(0 \neq a \in L\), then Lemma 2 yields that \(d(a) \neq 0\) since \(a\) is not invertible.

b) It suffices to show that every proper one-sided ideal of \(A\) is maximal. Let \(I \subset J\) be proper one-sided ideals of \(A\). It is easy to see that \(d(I) \cap I = (0)\) and \(I \oplus d(I)\) is also a one-sided ideal of \(A\). By a) we have \(d(I) \neq (0)\), whence \(d(I)\) contains invertible elements and, therefore, \(I \oplus d(I) = A\). For arbitrary \(j \in J\) we have \(j = a + d(b), a, b \in I\). Hence \(d(b) = j - a \in J \cap d(I) = (0)\) and, therefore, \(j = a \in I\).

c) If \(I \neq A\) is an ideal of \(A\), then

\[
d(I^2) \subset d(I)I + Id(I) \subset I.
\]

It follows by a) that \(I^2 = (0)\) since the product of two ideals in an alternative algebra is also an ideal ([20], Russian p. 115) and \(I\) contains no invertible elements.
Suppose that $2A \neq 0$ and $I \neq (0)$. Then by a) there is a $b \in I$ such that $d(b) \in U$. Since $b^2 = 0$, we have

$$0 = d^2(b^2) = d^2(b)b + 2d(b)^2 + bd^2(b)$$

and, therefore, $2d(b)^2 \in I$. Since $d(b)$ is invertible, $d(b)^2$ is also invertible and $2d(b)^2 = 0$, whence $2 = 0$, a contradiction. □

We denote the set of all derivations of an algebra $A$ by $\text{Der}(A)$. Fix a subset $D \subset \text{Der}(A)$. An ideal $I$ is called a $D$-ideal if $\partial(x) \in I$ for all $\partial \in D$, $x \in I$. An algebra $A$ is said to be $D$-simple if $A^2 \neq 0$ and $A$ has no proper $D$-ideals (for more detailed information on $D$-simple algebras see [10], [11] and the references therein).

The following lemma is an immediate consequence of Lemma 3, a).

**Lemma 4.** If an alternative algebra $A$ admits a derivation $d$ with invertible values, then $A$ is $d$-simple.

Now, if $\text{char}(A) \neq 2$, then we conclude from Lemma 3, d) and Theorem 1 that $A$ is either an associative algebra or a Cayley–Dickson algebra over its centre. The following lemma examines the remaining case when $A$ is non-simple and non-associative.

**Lemma 5.** If $A$ is non-simple and non-associative, then $A = C[x]/(x^2)$, where $C$ is a Cayley–Dickson algebra over its centre $Z(C)$, $C$ is a division algebra, $\text{char}(C) = 2$, $d(C) = 0$, $d(x) = 1 + ax$ for some $a \in Z(C)$, and $d$ is an outer derivation.

**Proof.** Combining parts b) and d) of Lemma 3, we see that $\text{char}(A) = 2$, $I^2 = (0)$ for every proper ideal $I$ of $A$, and all proper one-sided ideals of $A$ are simultaneously minimal and maximal. Hence $A$ contains a unique (left, right, two-sided) ideal $M$, and $M^2 = 0$. Arguing as in the proof of Lemma 3, b), we have $A = M \oplus d(M)$.

Hence for every $a \in A$ there are $m, n \in M$ such that $d(a) = m + d(n)$. Therefore $m = d(a - n) \in M \cap d(A) = (0)$ and, putting $C = \ker(d)$, we have $A = C + M$.

By Lemma 2, $C$ is a division algebra, whence $A = C \oplus M$. We define linear maps $\lambda : M \to C$ and $\mu : M \to M$ by the formula $d(m) = \lambda(m) + \mu(m)$ for all $m \in M$.

For all $a \in C$ and $b \in M$ we easily see that

$$a\mu(b) + a\lambda(b) = ad(b) = d(ab) = \mu(ab) + \lambda(ab),$$

where $a\mu(b), \mu(ab) \in M$ and, therefore, $a\lambda(b) = \lambda(ab) \in \lambda(M)$. Similarly, $\lambda(ba) = \lambda(b)a \in \lambda(M)$. Hence $\lambda(M)$ is an ideal of $C$.

Since $C$ is simple and $\lambda(M) \neq (0)$, it follows that $C$ is isomorphic to $M$ as a left $C$-module. Putting $x = \lambda^{-1}(1)$, we have $A = C \oplus Cx$. Since $\lambda$ is a module isomorphism, we easily see that $[x, C] = 0$. On the other hand, the identity

$$3(k, x, y) = 3(y, k, x) = 3(x, y, k) = [xy, k] - x[y, k] - [x, k]y = 0$$

holds for all $k \in K(B)$ and $x, y \in B$ in any alternative algebra $B$ ([20], Russian p. 136). Using this identity and taking the structure of $A$ into account, we obtain that $x \in Z(A)$. Therefore $A \cong C[x]/(x^2)$. Since $A$ is non-associative, it follows from Theorem 1 that $C$ is a Cayley–Dickson algebra over its centre $Z(C)$. We easily
see that \( \mu(x) = ax \) for some \( a \in C \). Since \( x \in Z(A) \) and \( \text{char}(A) = 2 \), for arbitrary \( c \in C \) we have

\[
0 = d(cx + xc) = c(1 + ax) + (1 + ax)c = cax + axc = (ca + ac)x.
\]

Since \( C \) is a division algebra, we have \( ca + ac = 0 \). Hence \( a \in Z(C) \).

Finally, every ideal of \( A \) is invariant under any inner derivation. But we have \( x \in M \) and \( d(x) \notin M \). Therefore \( d \) is not inner. \( \square \)

**Theorem 6.** Let \( A \) be an alternative algebra with identity 1 admitting a derivation \( d \) with invertible values. Then two cases may occur.

1) \( A \) is an associative algebra and one of the following conditions hold.
   a) \( A \) is a division algebra \( D \).
   b) \( A \) is the matrix algebra \( M_2(D) \) over a division algebra \( D \).
   c) \( A \) is the quotient \( D[x]/(x^2) \) of the polynomial algebra over a division algebra \( D \). Moreover, \( \text{char}(D) = 2 \), \( d(D) = 0 \) and \( d(x) = 1 + ax \) for some \( a \) in the centre of \( D \), and \( d \) is an outer derivation.

2) \( A \) is an alternative non-associative algebra and one of the following conditions holds.
   a) \( A \) is a Cayley–Dickson algebra over its centre \( Z(A) \).
   b) \( A \) is the quotient \( C[x]/(x^2) \) of the polynomial algebra over a Cayley–Dickson division algebra. Moreover, \( \text{char}(C) = 2 \), \( d(C) = 0 \) and \( d(x) = 1 + ax \) for some \( a \) in the centre of \( C \), and \( d \) is an outer derivation.

**Proof.** The associative case follows from [1]. The non-associative case follows from Theorem 1 and Lemmas 3, 5. \( \square \)

To complete the classification, we need only describe split Cayley–Dickson algebras admitting derivations with invertible values. This is done in the following lemma.

**Lemma 7.** An algebra \( C \) which is a split Cayley–Dickson algebra over its centre \( Z \) admits a derivation \( d \) with invertible values if and only if one of the following conditions holds.

I) \( C \) is obtained by means of the Cayley–Dickson process from an associative division subalgebra \( B \) of \( C \): \( C = B \oplus vB \), \( v^2 = \gamma \in Z \), \( \gamma \neq 0 \), where \( B = \ker(d) \) and \( \dim_Z B = 4 \). Moreover, in this case, any derivation \( d \) with invertible values is of the form \( d(a + vb) = v(bu) \), where \( a, b \in B \) and \( u \in B \) is a fixed element with \( t(u) = 0 \).

II) \( C \) is a direct sum of vector spaces \( C = B \oplus xB \), where \( t(x) = 0 \), \( B = \ker(d) \), \( B \) is a subfield of \( C \), \( B = B^\perp \) and \( \dim_Z B = 4 \). Moreover, in this case, any derivation \( d \) with invertible values is of the form \( d(a + xb) = b \), where \( a, b \in B \).

**Proof.** It is known (see, for example, [21]) that every derivation of \( C \) is inner. Therefore we easily see that \( Z \subseteq \ker(d) \) and \( d \) is a \( Z \)-linear map. Therefore we regard \( C \) as a \( Z \)-algebra. Suppose that \( C \) admits a derivation \( d \) with invertible values. Take a subspace \( V \subset C \) such that \( \dim_Z V = 4 \) and \( V \) contains no invertible elements. For example, taking into account that \( C \cong C(F) \) is the Cayley–Dickson
matrix algebra over $F$, we can put

$$V = \left\{ \begin{pmatrix} \alpha & u \\ 0 & 0 \end{pmatrix} \mid \alpha \in Z, u \in Z^3 \right\}.$$  

It follows from Lemma 2 that $\dim_Z d(V) = 4$ and $V \cap d(V) = (0)$, whence $C = V \oplus d(V)$. In particular, for every $x \in C$ there are $u, v \in V$ such that $d(x) = u + d(v)$. Therefore $u = d(x - v) \in V \cap d(A) = (0)$. Putting $B = \ker(d)$, we have $C = B + V$. Lemma 2 yields that $B$ is a division algebra, whence $C = B \oplus V$ and $\dim_Z B = 4$. Since $B$ is simple and $Z(C) \subseteq Z(B)$, we deduce from Theorem 1 that $B$ is an associative subalgebra of $C$. The following relation holds in $C$ by [20], Russian p. 39:

$$a \circ b - t(a) b - t(b) a - f(a, b) = 0. \quad (3)$$

Putting $b = d(a)$, we obtain

$$a \circ d(a) - t(a) d(a) - t(d(a)) a - f(a, d(a)) = 0. \quad (4)$$

Acting by $d$ on (1), we have

$$a \circ d(a) - t(a) d(a) = 0. \quad (5)$$

Subtracting (4) from (5), we obtain $t(d(a)) a + f(a, d(a)) = 0$. If $a$ and 1 are linearly independent over $Z$, then

$$f(a, d(a)) = 0. \quad (6)$$

If $a \in Z$, then $a \in \ker(d)$ and relation (6) is obvious. Linearizing (6), we obtain

$$f(a, d(b)) + f(d(a), b) = 0. \quad (7)$$

Since $B = \ker(d)$, it follows that we have $f(d(a), B) = -f(a, d(B)) = 0$ for all $a \in C$, whence $d(C) \subseteq B^\perp$. We consider two cases.

I) If the restriction of the form $f$ to $B$ is non-degenerate, then Theorem 1 on Russian p. 32 of [20] yields that $C$ can be obtained from $B$ by means of the Cayley–Dickson process, that is, $C = B + vB$, $v^2 = \gamma \neq 0$, $B^\perp = vB$. In particular, $d(v) = vu$ for some $u \in B$. Therefore, for arbitrary $a, b \in B$ we have

$$d(a + vb) = d(v)b = (vu)b = v(bu).$$

By [20], Russian p. 26, for any $x, y, w \in C$ we have

$$n(x)f(y, w) = f(xy, zw).$$

Putting $x = v, y = 1, w = u$ and using (6), we obtain that

$$0 = f(v, vu) = n(v)t(u).$$

Since $v^2 = \gamma \in Z, \gamma \neq 0$, we have $n(v) \neq 0$ and $t(u) = 0$.

II) We now suppose that the restriction of $f$ to $B$ is degenerate. Hence there is a non-zero $b \in B$ such that $f(b, B) = 0$. In particular, $0 = f(b, b) = 2n(b)$. Since $b$ is invertible, $n(b) \neq 0$ and, therefore, $\text{char}(C) = 2$. The following relation holds in $C$ by [20], Russian p. 26:

$$f(x, z)f(y, w) = f(xy, zw) + f(xw, zy). \quad (7)$$
Substituting $x = b$, $z = a$, $y = b^{-1}c$, $w = 1$ into (7), where $a, c \in B$, we have

$$0 = f(b, a) f(b^{-1}c, 1) = f(c, a) + f(b, ab^{-1}c).$$

Since $a$ and $c$ are arbitrary, it follows that $f(B, B) = 0$, that is, $B \subseteq B^\perp$.

We claim that the opposite inclusion also holds. Indeed, suppose that there is an $x \in B^\perp$, $x \not\in B$. By (2) and the skew-symmetry of the associator, $\dim_Z xB = 4$ and $C = B \oplus xB$. Using (7), we have

$$f(a, xc) = f(a \cdot 1, xc) = -f(ac, x) + f(a, x) f(1, c) = 0$$

for all $a, c \in B$. Hence $xB \subset B^\perp$ and $C = B^\perp$, contrary to the non-degeneracy of $f$. We put $x = d^{-1}(1)$. Clearly, $x \not\in B$ and $C = B \oplus xB$. It follows from (6) that $0 = f(x, 1) = t(x)$. We need only prove that $B$ is a field. By definition of $f$ and since $f(B, B) = 0$, we have

$$0 = f(a, c) = n(a + c) - n(a) - n(c)$$

for arbitrary $a$, $c$ and, therefore, $n$ is a ring homomorphism from $B$ to $Z$. Since $B$ is simple and $n(1) = 1$, it follows that $\ker(n) = 0$. Thus we conclude that $B$ is a subfield of $Z$.

We now assume that condition I) or II) holds. We claim that $C$ admits a derivation with invertible values.

Suppose that condition I) holds, that is, $C$ is obtained from $B$ by means of the Cayley–Dickson process. Let $u \neq 0$ be an element of $B$ such that $t(u) = 0$. Consider the map $d: a + vb \mapsto v(bu)$, where $a, b \in B$. We claim that $d$ is a derivation. Indeed, for all $a_1, b_1, a_2, b_2 \in B$ we have

$$d(a_1 + vb_1)(a_2 + vb_2) + (a_1 + vb_1)d(a_2 + vb_2) = \gamma((b_2(vu + v) - u)) + v((a_2b_1 + ab_2)v)$$

$$= \gamma((b_2(vu + v) - u)) + v((a_2b_1 + ab_2)v) = d((a_1 + vb_1)(a_2 + vb_2)).$$

We also have

$$n(d(a + vb)) = n(v(bu)) = n(v)n(b)n(u) = -\gamma n(b)n(u) \neq 0$$

if $b \neq 0$ since $B$ is a division algebra. Hence $d(a + vb)$ is invertible for all $a \in B$ and $0 \neq b \in B$ and, therefore, $d$ is a derivation with invertible values.

We now assume that condition II) holds. Consider the map $d: a + xb \mapsto b$. We claim that $d$ is a derivation with invertible values. Indeed, since $B = B^\perp$, we have $t(a) = 0$ for all $a \in B$. Combining (3) and the equation $\text{char}(C) = 2$, we obtain

$$[x, a] = x \circ a = t(a)x + t(x)a + f(a, x) = f(a, x) \in Z. \quad (8)$$

In particular, $d([x, a]) = 0$. Substituting $x$ in (1), we deduce that $x^2 \in Z$. Using (8), we easily verify that the following identity holds for all $a, c \in B$:

$$(a, c, x) = af(c, x) + f(a, x)c + f(x, ac). \quad (9)$$
Therefore $d((a, c, x)) = 0$. We claim that $d$ is a derivation. For arbitrary $a, b, c, h \in B$ we have

$$d((ax + b(cx + h)) = d((ax)(cx) + (ax)h + b(cx) + bh)$$

$$= d((ax)(cx)) + d((ax)h) + d(b(cx)).$$

Consider the last two summands:

$$d((ax)h) = d((xa)h) = d(x(ah)) = ah, \quad d(b(cx)) = d((bc)x) = bc.$$ 

On the other hand,

$$d(ax + b)(cx + h) + (ax + b)d(cx + h) = a(cx + h) + (ax + b)c$$

$$= a(cx) + ah + (ax)c + bc.$$ 

Thus we must prove that $d((ax)(cx)) = a(cx) + (ax)c$. By making appropriate calculations, we have

$$a(cx) + (ax)c = (ac)x + (a, c, x) + a(xc) + (a, x, c)$$

$$= (ac)x + a(cx + f(c, x)) = (a, c, x) + af(c, x). \quad (10)$$

Using the middle Moufang identity, we obtain

$$d((ax)(cx)) = d((xa + f(a, x)cx) = d((xa)(cx)) + f(a, x)d(cx)$$

$$= d(x(ac)x) + f(a, x)c = d(x(x(ac) + f(x, ac))) + f(a, x)c$$

$$= d(x^2(ac)) + f(x, ac) + f(a, x)c = f(x, ac) + f(a, x)c \quad (11)$$

since $x^2 \in Z$ and $d(x^2(ac)) = n(x)d(ac) = 0$. Equating (10) and (11), we arrive at formula (9), which was shown above to hold identically. Therefore $d$ is a derivation of $C$. Since $d$ takes values in $B$ and $B$ is a field, it is clear that $d$ is a derivation with invertible values. □

**Example 8.** An example of a split Cayley–Dickson algebra $C$ having a subfield of dimension 4 was constructed in [22]. Consider an imperfect field $F$ of characteristic 2 and elements $\alpha, \beta \in F$ such that $\alpha, \beta, \alpha \beta$ are linearly independent over $F^2$. Let $B$ be the subalgebra of the matrix Cayley–Dickson algebra $C(F)$ generated by the elements

$$\begin{pmatrix} 0 & (\alpha, 0, 0) \\ (1, 0, 0) & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & (0, \beta, 0) \\ (0, 1, 0) & 0 \end{pmatrix}.$$ 

Then $B$ is a subfield of $C$ and $\dim_F B = 4$.

Note that the proof of our main result in this section (Theorem 6) resembles that in the associative case, but our argument does not work in the case of Jordan algebras since the product of two invertible elements of a Jordan algebra need not be invertible and the product of two ideals need not be an ideal. Hence the following problem arises naturally.

**Problem 9.** Classify all Jordan (and, more generally, structurable) algebras admitting derivations with invertible values.
§ 4. A characterization of nilpotent alternative algebras by invertible Leibniz derivations

Throughout the section we consider only finite-dimensional algebras over a field of characteristic zero. In 1955 Jacobson [7] proved that every finite-dimensional Lie algebra over a field of characteristic zero admitting a non-singular (invertible) derivation is nilpotent. The question of whether the converse holds remained open until the paper [23], which gives an example of a nilpotent Lie algebra all of whose derivations are nilpotent (and hence singular). Such Lie algebras are said to be characteristically nilpotent.

The study of derivations of Lie algebras leads to a natural generalization: the notion of a prederivation of a Lie algebra. This is a derivation of the Lie triple system induced by the algebra. It was proved in [8] that Jacobson’s result also holds for prederivations. Examples of nilpotent Lie algebras all of whose prederivations are nilpotent were given in [8], [24].

The notions of derivations and prederivations of Lie algebras were generalized in [9] to the notion of a Leibniz derivation of order $k$. Moens proved that a finite-dimensional Lie algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible Leibniz derivation. Then Fialowski, Khudoyberdiyev and Omirov [25] showed that the corresponding result for non-Lie Leibniz algebras does not hold if we use the definition of Leibniz derivation given in [9]. Namely, they gave an example of a non-nilpotent Leibniz algebra that admits an invertible Leibniz derivation. To extend the results of [9] to the case of Leibniz algebras, they introduced the notion of Leibniz derivations for Leibniz algebras (which agrees with the case of Leibniz derivations for Lie algebras) and proved that a Leibniz algebra is nilpotent of and only if it admits an invertible Leibniz derivation. Note that there are non-nilpotent finite-dimensional Filippov algebras ($n$-Lie algebras) with invertible derivations [26].

The purpose of this section is to prove an analogue of Moens’ theorem for alternative algebras.

**Definition 10** (after Moens). A *Leibniz derivation of order $n$* of an algebra $A$ is an endomorphism $\phi$ of $A$ satisfying the identity

$$\phi((\ldots(x_1x_2)\ldots)x_n) = \sum_{i=1}^{n}(\ldots((x_1x_2)\ldots\phi(x_i))\ldots)x_n.$$ 

Definition 10 can naturally be interpreted in terms of $n$-ary algebras. For every algebra $A$ one can endow its underlying vector space with the structure of an $n$-ary algebra $A_n$ with multiplication

$$[a_1, a_2, \ldots, a_n]_n = (\ldots(a_1a_2)\ldots)a_n.$$ 

As in the case of algebras with binary multiplication, we say that a linear map $d$ is a derivation of an $n$-ary algebra $B$ if it satisfies the identity

$$d([x_1, \ldots, x_n]) = \sum_{i=1}^{n}[x_1, \ldots, d(x_i), \ldots, x_n].$$
Hence $\phi$ is a Leibniz derivation of order $n$ of an algebra $A$ if and only if it is a derivation of $A_n$. The definition of an ideal is easily extended to the case of $n$-ary multiplication: a subspace $I \subseteq B$ is called an ideal if

$$[I, B, \ldots, B]_n \subseteq I, \quad \ldots, \quad [B, \ldots, B, I]_n \subseteq I.$$ 

An $n$-ary algebra $B$ is said to be soluble if its derived sequence

$$S_1 = B, \quad S_2 = [S_1, \ldots, S_1]_n, \quad \ldots, \quad S_{t+1} = [S_t, \ldots, S_t]_n, \quad \ldots$$

descends to zero. For every finite-dimensional $n$-ary algebra $C$ we denote the maximal soluble ideal of $C$ by $\text{rad}(C)$. This radical is well defined since the sum of any two soluble ideals of $C$ is again a soluble ideal of $C$.

**Theorem 11.** A finite-dimensional alternative algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz derivation.

**Proof.** Let $A$ be a finite-dimensional alternative algebra with an invertible Leibniz derivation $\phi$ of order $n$, and let $\beta(A)$ be the nilpotent radical of $A$ (it is known that in the finite-dimensional case $\beta(A)$ coincides with the soluble radical $\text{rad}(A)$ of $A$). It follows from [20], Russian p. 336, that $A/\beta(A)$ is the direct sum of its minimal ideals, each of which is either a full matrix algebra over a division ring or a Cayley–Dickson algebra over its centre. Hence the algebra $A/\beta(A)$ possesses an identity element $1$. We shall regard $A$ as a direct sum: $A = A_s + \beta(A)$, where $A_s$ is a semisimple alternative algebra isomorphic to $A/\beta(A)$ (the Wedderburn–Malcev decomposition). We use the idea of the proof in [9] to show that $\phi(\beta(A)) \subseteq \beta(A)$. Note that this was proved in [27] for all algebras with locally nilpotent radical in the case when $\phi$ is a derivation.

**Step 1.** We claim that the soluble radicals $\text{rad}(A_n)$ and $\text{rad}(A)$ of the algebras $A_n$ and $A$ coincide. Indeed, it is clear that $\text{rad}(A) \subseteq \text{rad}(A_n)$. Consider the natural homomorphism $\pi: A \to A^s$. We easily see that $\pi(\text{rad}(A_n))$ is a soluble ideal of $A^s$: applying $\pi$ to both sides of the relation

$$[A, \ldots, \text{rad}(A_n), \ldots, A]_n \subseteq \text{rad}(A_n)$$

and using the existence of an identity in $A^s$, we have

$$\pi(\text{rad}(A_n))A^s + A^s\pi(\text{rad}(A_n)) \subseteq \pi(\text{rad}(A_n)).$$

Consequently, since $A^s$ is semisimple, we have $\pi(\text{rad}(A_n)) = 0$.

**Step 2.** We now claim that $\phi(\beta(A)) \subseteq \beta(A)$. Indeed, put $\beta(A) = \tau = \tau_1 = \text{rad}(A_n)$ and $\tau_{t+1} = [\tau_1, \tau_t, \ldots, \tau_t]_n$. Then we have

$$\tau = \tau_1 \supsetneq \tau_2 \supsetneq \cdots \supsetneq \tau_p \supsetneq \tau_{p+1} = 0.$$ 

Since the product of two ideals of an alternative algebra is also an ideal, $\tau_t$ is an ideal of $A_n$ for every $t$.

We will show by induction over $t$ that $\phi^i(\tau_t) \subseteq \tau$ for all $i$. The inductive base is trivial for $t = p + 1$. Assume that $\phi^i(\tau_{t+1}) \subseteq \tau$ for all $i$. We shall show that $\phi^i(\tau_t) \subseteq \tau$ for all $i$. 
The set \( \tau + \phi(\tau_l) \) is a soluble ideal in \( A_n \) since

\[
[A, \ldots, A, \tau + \phi(\tau_l), A, \ldots, A]_n \subseteq \tau + \tau_l + \phi(\tau_l) = \tau + \phi(\tau_l),
\]

\[
[\tau + \phi(\tau_l), \ldots, \tau + \phi(\tau_l)]_n \subseteq \tau + [\phi(\tau_l), \ldots, \phi(\tau_l)]_n \subseteq \tau + \phi^n(\tau_{l+1}) \subseteq \tau.
\]

We now have to show that \( \tau + \phi^k(\tau_l) \) is a soluble ideal of \( A_n \) for every \( k \). Using the induction hypothesis, we have

\[
[A, \ldots, A, \tau + \phi^k(\tau_l), A, \ldots, A]_n \]

\[
\subseteq \tau + \phi([A, \ldots, A, \phi^{k-1}(\tau_l), A, \ldots, A]_n)
\]

\[
+ \sum [A, \ldots, \phi(A), \ldots, A, \phi^{k-1}(\tau_l), A, \ldots, A]_n \subseteq \ldots
\]

\[
\cdots \subseteq \tau + \sum_{a_0 + \cdots + a_{n-1} = k, a_i \geq 0} \phi^{a_0}([\phi^{a_1}(A), \ldots, \phi^{a_{l-1}}(A), \tau_l, \phi^{a_l}(A), \ldots, \phi^{a_{n-1}}(A)]_n)
\]

\[
+ \phi^k([A, \ldots, A, \tau_l, A, \ldots, A]_n)
\]

\[
\subseteq \tau + \sum_{i=0}^{k-1} \phi^i(\tau_l) + \phi^k(\tau_l) = \tau + \tau_l + \phi^k(\tau_l) = \tau + \phi^k(\tau_l),
\]

\[
[\tau + \phi^k(\tau_l), \ldots, \tau + \phi^k(\tau_l)]_n \subseteq \tau + [\phi^k(\tau_l), \ldots, \phi^k(\tau_l)]_n \subseteq \tau + \phi^{kn}(\tau_{l+1}) \subseteq \tau.
\]

Therefore, \( \phi^i(\tau_l) \subseteq \tau \) and \( \phi(\tau) \subseteq \tau \).

**Step 3.** Since \( \phi \) is an invertible derivation (that is, its kernel is trivial), we have \( \phi(A/\beta(A)) = A/\beta(A) \). This contradicts the fact that \( A/\beta(A) \) is unital because \( \phi(1) = n\phi(1) \) and \( \dim(\phi(A/\beta(A))) < \dim(A/\beta(A)) \). The resulting contradiction shows that \( A = \beta(A) \), that is, \( A \) is nilpotent.

**Step 4.** The opposite also holds. Indeed, to see that any nilpotent alternative algebra \( A \) with nilpotency index \( s \) has an invertible Leibniz derivation of order \( n = \lceil s/2 \rceil + 1 \), it suffices to consider the sum \( A = W + A^n \) of vector spaces and define a linear map \( \phi \) by the formula

\[
\phi(x) = \begin{cases} x & \text{if } x \in W, \\ nx & \text{if } x \in A^n. \end{cases}
\]

We easily see that \( \phi \) is a Leibniz derivation of \( A \) of order \( n \). \( \square \)

**Proposition 12.** Over fields of positive characteristic there are nilpotent alternative algebras possessing only trivial derivations.

**Proof.** Non-associative alternative algebras of dimension at most 7 were classified in [28]. Using this classification, we define a 7-dimensional algebra \( A \) with basis \( \{e_1, e_2, e_3, u_1, u_2, v, w\} \) over a field of characteristic 3 by the following multiplication table (all other products are equal to zero):

\[
e_1^2 = u_1, \quad e_2^2 = u_2, \quad e_2e_3 = e_3e_2 = -v, \quad e_3e_1 = u_2, \quad e_1e_3 = u_2,
\]

\[
e_1u_1 = u_1e_1 = v, \quad e_2u_2 = u_2e_2 = w, \quad e_1v = ve_1 = u_1^2 = w.
\]
It is easy to see that $A^2 = \langle u_1, u_2, v, w \rangle$, $A^3 = \langle v, w \rangle$, $A^4 = \langle w \rangle$. Then for every derivation $D$ of $A$ we have

$$D(v) = D((e_1 e_1) e_1) \in A^4, \quad D(w) = D(((e_1 e_1) e_1) e_1) \in A^4.$$ 

Hence there is an $x \neq 0$ such that $D(x) = 0$. □

Thus the restriction on the characteristic of the field in Theorem 11 is essential.

Remark 13. For every alternative algebra over a field of positive characteristic $p$, the identity map is a Leibniz derivation of order $p + 1$.

The requirement for the algebra to be finite-dimensional is also essential.

Assertion 14. The free alternative algebra with $n$ generators admits an invertible derivation and is non-nilpotent.

Proof. It suffices to consider a derivation which acts as the identity on the generators of the algebra. □

Remark 15. Theorem 11, Remark 13 and Assertion 14 also hold in the case of associative algebras.

It turns out that Herstein’s result [7] holds in the case of Jordan algebras.

Assertion 16. Every finite-dimensional Jordan algebra over a field of characteristic zero admitting an invertible derivation is nilpotent.

Proof. Indeed, it follows from Slin’ko’s work [27] that $d(\beta(J)) \subseteq \beta(J)$ for every derivation $d$ of a Jordan algebra $J$ over a field of characteristic zero. Repeating verbatim Step 3 in the proof of Theorem 11, we obtain the desired result. □

As shown above and in [25], the analogue of Moens’ theorem holds not only for algebras in well-studied varieties (such as the varieties of associative algebras and Lie algebras) but also for some algebras that generalize them (alternative algebras and Leibniz algebras). In view of Remark 15 we can pose the following problem.

Problem 17. Does Moens’ theorem hold in the classes of Jordan algebras and Malcev algebras?

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