Generalized quiver Hecke algebras

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Abstract

We generalize the methods of Varagnolo and Vasserot, [VV11b] and partially [VV11a], to generalized quiver representations introduced by Derksen and Weyman in [DW02]. This means we have a general geometric construction of an interesting class of algebras (the Steinberg algebras for generalized quiver-graded Springer theory) containing skew group rings of Weyl groups with polynomial rings, (affine) nil Hecke algebras and KLR-algebras (=quiver Hecke algebras). Unfortunately this method works only in the Borel case, i.e. all parabolic groups in the construction data of a Springer theory are Borel groups. Nevertheless, we try to treat also the parabolic case as far as this is possible here.

This is a short reminder of Derksen and Weyman’s generalized quiver representations from [DW02].

Definition 1. A generalized quiver with dimension vector is a triple \((G, G, V)\) where \(G\) is a reductive group, \(G\) is a centralizer of a Zariski closed abelian reductive subgroup \(H\) of \(G\), i.e. \(G = C_G(H) = \{g \in G \mid ghg^{-1} = h \ \forall h \in H\}\) (then \(G\) is also reductive, see lemma below) and \(V\) is a representation of \(G\) which decomposes into irreducible representations which also appear in \(G = \text{Lie}(G)\) seen as a \(G\)-module.

A generalized quiver representation is a quadruple \((G, G, V, G_v)\) where \((G, G, V)\) is a generalized quiver with dimension vector, \(v\) in \(V\) and \(G_v\) is the \(G\)-orbit.

Remark. Any such reductive abelian group is of the form \(H = A \times S\) with \(A\) finite abelian and \(S\) a torus, this implies that there exists finitely many elements \(h_1, \ldots, h_m\) such that \(C_G(H) = \bigcap_{i=1}^m C_G(h_i)\), see for example Humphreys’ book [Hum75], Prop. in 16.4, p.107.

We would like to work with the associated Coxeter systems, therefore it is sensible to assume \(G\) connected and replace \(G\) by its identity component \(G^0\). There is the following proposition

Proposition 1. Let \(G\) be a connected reductive group and \(H \subset G\) an abelian group which lies in a maximal torus. We set \(G := C_G(H)^0 = (\bigcap_{i=1}^m C_G(h_i))^0\). Then it holds

1. For any maximal torus \(T \subset G\), the following three conditions are equivalent:
   (i) \(T \subset G\).
   (ii) \(H \subset T\).
   (iii) \(\{h_1, \ldots, h_m\} \subset T\).

2. \(G\) is a reductive group.

3. If \(\Phi\) is the set of roots of \(G\) with respect to a maximal torus \(T\) with \(H \subset T\), then \(\Phi := \{\alpha \in \Phi \mid \alpha(h) = 1 \ \forall h \in H\}\) is the set of roots for \(G\) with respect to \(T\), its Weyl group is \(\langle s_\alpha \mid \alpha \in \Phi \rangle\) and for all \(\alpha \in \Phi\) the weight spaces are equal \(g_\alpha = G_\alpha\) (and 1-dimensional \(\mathbb{C}\)-vector spaces).

4. There is a surjection
   \[
   \{B \subset G \mid B \text{ Borel subgroup}, H \subset B\} \to \{B \subset G \mid B \text{ Borel subgroup}\} \\
   B \mapsto B \cap G
   \]
   If \(\Phi^+\) is the set of positive roots with respect to \((G, B, T)\) with \(H \subset T\), then \(\Phi^+ := \Phi \cap \Phi^+\) is the set of positive roots for \((G, G \cap B, T)\).
proof: Ad (1): This is easy to prove directly.

(2)-(4) are proven if $G = C_G(h)^o$ for one semisimple element $h \in G$ in Carters book [Car85], section 3.5. p.92-93. In general $G = (\bigcap_{i=1}^m C_G(h_i)^o)^o$ for certain $h_i \in H, 1 \leq i \leq m$. The result follows via induction on $m$. Set $G_1 := C_G(h_1)^o$. It holds $G = (\bigcap_{i=2}^m C_G(h_i)^o)^o = C_G(H)^o \subset G_1$ and $G_1$ is a connected reductive group. By induction hypothesis, all statements are true for $(G, G_1)$, so in particular $G$ is a reductive group. The other statements are then obvious. 

0.0.1 Notational conventions

We fix the ground field for all algebraic varieties and Lie algebras to be $\mathbb{C}$. For a Lie algebra $\mathfrak{g}$ we define the $k$-th power inductively by $\mathfrak{g}^1 := \mathfrak{g}, \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$. If we denote an algebraic group by double letters (or indexed double letters) like $G, B, U, ...$ (or $G', P_J$, etc.) we take the calligraphic letters for the Lie algebras, i.e. $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, ...$ (or $\mathfrak{g}', \mathfrak{p}_J$, etc.) respectively. If we denote an algebraic group by roman letters (or indexed roman letters) like $G, B, U, ...$ (or $G', P_J$, etc.) we take the small frakture letters for the Lie algebras, i.e. $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, ...$ (or $\mathfrak{g}', \mathfrak{p}_J$, etc.) respectively.

If we have a subgroup $P \subset G$ of a group and an element $g \in G$ we write $gP := gPg^{-1}$ for the conjugate subgroup.

We also recall the following.

Remark. Let $(W, S)$ be a Coxeter system, $J \subset S$. Then $(W_J := \langle J \rangle, J)$ is again a Coxeter system with the length function as the restriction of the length function of $(W, S)$ to elements in $W_J$. Then, the set $W^J$ of minimal length coset representatives $W^J \subset W$ for $W/W_J$ is defined via: An element $w$ lies in $W^J$ if and only if for all $s \in J$ we have $l(ws) > l(w)$. Also there is a factorization $W = W^J W_J$ and if $w = xy$ with $x \in W^J, y \in W_J$, their lengths satisfy $l(w) = l(x) + l(y)$. We will fix the bijection $c_J : W^J \to W/W_J, w \mapsto wW_J$. The Bruhat order of $(W, S)$ can be restricted to $W^J$ and transferred via the bijection to $W/W_J$.

For two subsets $K, J \subset S$ define $K^W^J := (W^K)^{-1} \cap W^J$, the projection $W \to W_K \setminus W/W_J$ restricts to a bijection $K^W^J \to W_K \setminus W/W_J$.

Let $(G, B, T)$ be a reductive group with Borel subgroup and maximal torus and $(W, S)$ be its associated Coxeter system. We fix for any element in $W$ a lift to the group $G$ and denote it by the same letter.

0.1 Generalized quiver-graded Springer theory

We define a generalized quiver-graded Springer theory for generalized quiver representations in the sense of Derksen and Weymann. Given $(G, P_J, U, H, V)$ (and some not mentioned $H \subset T \subset B \subset P_J$) with

* $G$ is a connected reductive group, $H \subset T$ is a subgroup of a maximal torus in $G$, we set $G = C_G(H)^o$ (then $G$ is also reductive with $T \subset G$ is a maximal torus in $G$).

* $T \subset B \subset G$ a Borel subgroup, then $B := B \cap G$ is a Borel subgroup of $G$.

We write $(W, S)$ for the Coxeter system associated with $(G, B, T)$ and $(W, S)$ for the one associated to $(G, B, T)$. Observe, that $W \subset W$. For any $J \subset S$ we set $P_J := P_J(B)B$ and call it a standard parabolic group.

* Now fix a subset $J \subset S$. We call a $P_J$-subrepresentation $U' \subset G = \text{Lie}(G)$ (of the adjoint representation which we denote by $(g, x) \mapsto g x, g \in G, x \in G$) suitable if

  * $(U')^T = \{0\}$,

  * $U' \cap sU'$ is $P_J$-stable for all $s \in S$.

Let $U = \bigoplus_{k=1}^l U^{(k)}$ a $P_J$-representation with each $U^{(k)}$ is suitable. (Examples of suitable $P_J$-representations are given by $U = U_J$, where $J \subset J' \subset S$, $U_J = \text{Lie}(U_{J'})$ with $U_J \subset P_J$ is the...
unipotent radical and \( U^j \), is the \( t \)-th power, \( t \in \mathbb{N} \). We define \( \mathbb{W}_J := \langle J \rangle \) and \( \mathbb{W}^J \) be the set of minimal coset representatives in \( \mathbb{W}/\mathbb{W}_J \), \( I_J := \mathbb{W} \setminus \mathbb{W}_J \subseteq \mathbb{W} \setminus \mathbb{W} \) and

\[
\bigcup_{J \subseteq \mathbb{G}} I_J
\]

We call \( I := I_\emptyset \) the set of **complete dimension filtrations**. Let \( \{ x_i \in \mathbb{W} \mid i \in I_J \} \) be a complete representing system of the cosets in \( I_J \). Every element of the Weyl groups \( \mathbb{W} \) (and \( W \)) we lift to elements in \( G \) (and \( G \)) and denote the lifts by the same letter. For every \( i \in I_J \) we set

\[
P_i := x_i P \cap G,
\]

Observe that \( H \subseteq T = wT \subset wP_J \) for all \( w \in \mathbb{W} \), therefore \( wP \cap G \) is a parabolic subgroup in \( G \) for any \( w \in \mathbb{W} \).  

* \( V = \bigoplus_{k=1}^{r} V^{(k)} \) with \( V^{(k)} \subset G \) is a \( G \)-subrepresentation.
* \( F_i = \bigoplus_{k=1}^{r} F_i^{(k)} \) with \( F_i^{(k)} := V^{(k)} \cap x_i U^{(k)} \) is a \( P_i \)-subrepresentation of \( V^{(k)} \).

We define

\[
E_i := G \times P_i F_i \quad \xleftarrow{\pi_i} V \quad \xrightarrow{\mu_i} G/P_i \quad \xleftarrow{(g, f)} (gP_i) \quad \xrightarrow{gP_i}. \]

Now, there are closed embeddings \( \iota_i : G/P_i \to G/P_J \), \( gP_i \mapsto gx_i P_J \) with for any \( i \neq i' \) in \( I_J \) it holds \( \text{Im} \iota_i \cap \text{Im} \iota_{i'} = \emptyset \). Therefore, we can see \( \bigsqcup_{i \in I_J} G_i P_i \) as a closed subscheme of \( G/P_J \). It can be identified with the closed subvariety of the fixpoints under the \( H \)-operation \( (G/P_J)^H = \{ gP_J \in G/P_J \mid hgP_J = gP_J \text{ for all } h \in H \} \).

\[
E_J := \bigsqcup_{i \in I_J} E_i \quad \xleftarrow{\pi_J} V \quad \xrightarrow{\mu_J} G/P_J. \]

We also set

\[
Z_{ij} := E_i \times_V E_j \quad \xleftarrow{p_{ij}} V \quad \xrightarrow{m_{ij}} G/P_i \times G/P_j \quad \xleftarrow{p_{ij}} V \quad \xrightarrow{m_{ij}} G/P_J \times G/P_J. \]

In an obvious way all maps are \( G \)-equivariant. We are primarily interested in the following **Steinberg variety**

\[
Z := Z_\emptyset.
\]

The equivariant Borel-Moore homology of a Steinberg variety together with the convolution operation (defined by Ginzburg) defines a finite dimensional graded \( \mathbb{C} \)-algebra. We set

\[
Z_G := H^*_e(Z)
\]

which we call **(G-equivariant) Steinberg algebra**. The aim of this section is to describe \( Z_G \) in terms of generators and relation (for \( J = \emptyset \)). This means all \( P_i \) are Borel subgroups of \( G \).

If we set

\[
H^G_{[p]}(Z) := \bigoplus_{i,j \in I} H^G_{e_{i+j-p}}(Z_{ij}), \quad e_i = \dim \mathbb{C} E_i
\]

then \( H^G_{[p]}(Z) \) is a graded \( H^*_G(pt) \)-algebra. Then, we denote the right \( \mathbb{W} \)-operation on \( I = W \setminus \mathbb{W} \) by \( (i, w) \mapsto iw, \ i \in I, w \in \mathbb{W} \). We prove the following.
Theorem 0.1. Let $J = \emptyset$. Then $Z_G \subset \text{End}_{\mathbb{C}[t]} \oplus_{i \in I} \mathcal{E}_i$, $\mathcal{E}_i = \mathbb{C}[t] = \mathbb{C}[x_i(1), \ldots, x_i(n)], i \in I$ is the $\mathbb{C}$-subalgebra generated by
\[ 1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = \text{rk}(T), i \in I, \quad \sigma_i(s), s \in S, i \in I \]
defined as follows for $k \in I, f \in \mathcal{E}_k$.
\[
1_i(f) := \begin{cases} f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}
\]
\[
z_i(t)(f) := \begin{cases} x_i(t)f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}
\]
\[
\sigma_i(s)(f) := \begin{cases} q_i(s)^{s(f)/\alpha_s}, & (\in \mathcal{E}_i) \text{ if } i = is = k, \\ q_i(s)s(f), & (\in \mathcal{E}_i) \text{ if } i \neq is = k, \\ 0, & \text{else.} \end{cases}
\]
where
\[
q_i(s) := \prod_{\alpha \in \Phi_U, s(\alpha) \notin \Phi_U, x_i(\alpha) \in \Phi_V} \alpha \quad \text{ in } \mathcal{E}_i.
\]
and $\Phi_U = \bigcup_k \Phi_{U(k)}$, $\Phi_{U(k)} \subset \text{Hom}_{\mathbb{C}}(t, \mathbb{C}) \subset \mathbb{C}[t]$ is the set of $T$-weights for $U^{(k)}$ and $\Phi_V = \bigcup_k \Phi_{V(k)}$, $\Phi_{V(k)} \subset \text{Hom}_{\mathbb{C}}(t, \mathbb{C})$ is the set of $T$-weights for $V^{(k)}$.
Furthermore, it holds
\[
\text{deg } 1_i = 0, \quad \text{deg } z_i(k) = 2, \quad \text{deg } \sigma_i(s) = \begin{cases} 2 \text{deg } q_i(s) - 2, & \text{if } is = i \\ 2 \text{deg } q_i(s), & \text{if } is \neq i \end{cases}
\]
where $\text{deg } q_i(s)$ refers to the degree as homogeneous polynomial in $\mathbb{C}[t]$.

The generality of the choice of the $U$ in the previous theorem is later used to understand the case of an arbitrary $J$ as an algebra of the form $e_J Z_G e_J$ for an associated Borel-case Steinberg algebra $Z_G$ and $e_J$ an idempotent element (this is content of a later article called \textit{parabolic Steinberg algebras}). For $J = \emptyset, U = \text{Lie}(U)^{\oplus r}$ for $U \subset B$ the unipotent radical we have the following result which generalizes KLR-algebras to arbitrary connected reductive groups and allowing quivers with loops.

Corollary 0.1. Let $J = \emptyset, U = \text{Lie}(U)^{\oplus r}, U \subset B$ the unipotent radical. Then
\[ Z_G \subset \text{End}_{\mathbb{C}[t]} \oplus_{i \in I} \mathcal{E}_i, \]
\[ \mathcal{E}_i = \mathbb{C}[t] = \mathbb{C}[x_i(1), \ldots, x_i(n)], i \in I \text{ is the } \mathbb{C}\text{-subalgebra generated by} \]
\[ 1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = \text{rk}(T), i \in I, \quad \sigma_i(s), s \in S, i \in I. \]
Let $f \in \mathcal{E}_k, k \in I, \alpha_s \in \Phi^+$ be the positive root such that $s(\alpha_s) = -\alpha_s$. It holds
\[
\sigma_i(s)(f) := \begin{cases} h_i(s)^{s(f)/\alpha_s}, & \text{if } i = is = k, \\ \alpha_i(s)^{s(f)/\alpha_s}, & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}
\]
where
\[
h_i(s) := \#\{k \in \{1, \ldots, r\} | x_i(\alpha_s) \in \Phi_{V(k)}\}
\]
where $V = \bigoplus_k V^{(k)}$ and $\Phi_{V(k)} \subset \Phi$ are the $T$-weights of $V^{(k)}$. 
(1) If \( Wx_i \neq Wx_is \) then
\[
h_i(s) = \# \{ k \mid V^{(k)} \subset \mathcal{R}, x_i(\alpha_s) \in \Phi_{V^{(k)}} \}.
\]
We say that this number counts arrows.

(2) If \( Wx_i = Wx_is \), then
\[
h_i(s) = \# \{ k \mid V^{(k)} \subset g, x_i(\alpha_s) \in \Phi_{V^{(k)}} \}.
\]
We say that this number counts loops.

In the case of the previous corollary we call the Steinberg algebra \( Z_G \) generalized quiver Hecke algebra. It can be described by the following generators and relations. For a reduced expression \( w = s_1s_2 \cdots s_k \) we set
\[
\sigma_i(s_1s_2 \cdots s_k) := \sigma_i(s_1)\sigma_{is_1}(s_2) \cdots \sigma_{is_1s_2 \cdots s_{k-1}}(s_k)
\]
Sometimes, if it is understood that the definition depends on a particular choice of a reduced expression for \( w \), we write \( \sigma_i(w) := \sigma_i(s_1s_2 \cdots s_k) \). Furthermore, we consider
\[
\Phi: \bigoplus_{i \in I} \mathbb{C}[z_i(1), \ldots z_i(n)] \cong \bigoplus_{i \in I} \mathbb{C}[x_i(1), \ldots x_i(n)], \quad x_i(t) \mapsto z_i(t)
\]
as the left \( \mathcal{W} \)-module \( \text{Ind}_{\mathcal{W}}^{\mathcal{W}} \mathbb{C}[t] \), we fix the polynomials
\[
c_i(s, t) := \Phi(\sigma_i(s)(x_i(t))) \in \bigoplus_{i \in I} \mathbb{C}[z_i(1), \ldots z_i(n)], \quad i \in I, \; 1 \leq t \leq n, \; s \in \mathbb{S}.
\]

Now, we can describe under some extra conditions the relations of the generalized quiver Hecke algebras.

**Proposition 2.** Under the following assumption for the data \((G, \mathcal{B}, \mathcal{U} = (\text{Lie}(U))^\mathcal{B}, H, V), J = \emptyset:\)
Let \( \mathcal{S} \subset \mathcal{W} = \text{Weyl}(G, T) \) be the simple reflections, we assume for any \( s, t \in \mathcal{S} \)

(B2) If the root system spanned by \( \alpha_s, \alpha_t \) is of type \( B_2 \) (or \( stst = tstst \) is the minimal relation), then for every \( i \in I \) such that \( is = i = it \) it holds \( h_i(s), h_i(t) \in \{0, 1, 2\} \).

(G2) If the root system spanned by \( \alpha_s, \alpha_t \) is of type \( G_2 \) (or \( ststst = tstsst \) is the minimal relation), then for every \( i \in I \) such that \( is = i = it \) it holds \( h_i(s) = 0 = h_i(t) \).

Then the generalized quiver Hecke algebra for \((G, \mathcal{B}, \mathcal{U} = (\text{Lie}(U))^\mathcal{B}, H, V), J = \emptyset \) is the \( \mathbb{C} \)-algebra with generators
\[
1_i, \; i \in I, \; z_i(t), \; 1 \leq t \leq n = \text{rk}(T), \; i \in I, \; \sigma_i(s), \; s \in \mathcal{S}, \; i \in I
\]
and relations

(1) (orthogonal idempotents)
\[
1_i1_j = \delta_{i,j}1_i,
\]
\[
1_i z_i(t)1_i = z_i(t),
\]
\[
1_i \sigma_i(s)1_i = \sigma_i(s)
\]

(2) (polynomial subalgebras)
\[
z_i(t)z_i(t') = z_i(t')z_i(t)
\]

(3) (relation implied by \( s^2 = 1 \))
\[
\sigma_i(s)\sigma_is(s) = \begin{cases} 0 & \text{if } is = i, \text{ } h_i(s) \text{ is even} \\ -2\alpha_s h_i(s)^{-1} \sigma_i(s) & \text{if } is = i, \text{ } h_i(s) \text{ is odd} \\ (-1)^{h_is(s)}\alpha_s^{h_i(s)+h_is(s)} & \text{if } is \neq i \end{cases}
\]
Lemma 2. (a) Intersection with expression in is conjugated to a standard parabolic subgroup. The standard parabolic subgroups wrt a parabolic subgroup is a subgroup which contains a Borel subgroup, every parabolic subgroups for later on, we need to understand the relationship between parabolic subgroups in $G$ and in $G$. Recall that a parabolic subgroup is a subgroup which contains a Borel subgroup, every parabolic subgroups is conjugated to a standard parabolic subgroup. The standard parabolic subgroups wrt $(G, B, T)$ are in bijection with the set of subsets of $S$, via $J \mapsto B(J)B =: P_J$. As a first step, we need to study the relationship of the Coxeter systems $(W, S)$ and $(\mathbb{W}, S)$.

0.1.1 Relationship between parabolic groups in $G$ and $G$

For later on, we need to understand the relationship between parabolic subgroups in $G$ and in $G$. Recall that a parabolic subgroup is a subgroup which contains a Borel subgroup, every parabolic subgroups is conjugated to a standard parabolic subgroup. The standard parabolic subgroups wrt $(G, B, T)$ are in bijection with the set of subsets of $S$, via $J \mapsto B(J)B =: P_J$. As a first step, we need to study the relationship of the Coxeter systems $(W, S)$ and $(\mathbb{W}, S)$.

Lemma 1. It holds $G \cap \mathbb{W} = W$. It holds $W \cap S \subset S$. Let $l_S$ be the length function with respect to $(W, S)$ and $l_\mathbb{W}$ be the length function with respect to $(\mathbb{W}, S)$. For every $w \in W$ it holds $l_S(w) \leq l_\mathbb{W}(w)$.

proof: $N_G(T) \cap G = N_G(T)$ implies $G \cap \mathbb{W} = W$. The inclusion $\Phi^+ \cap s(-\Phi^+) \subset \Phi^+ \cap s(-\Phi^+)$ for any $s \in S$ implies $W \cap S \subset S$.

Let $w = t_1 \cdots t_r \in W, t_i \in S$ reduced expression and assume $l_\mathbb{W}(w) < r$. It must be possible in $\mathbb{W}$ to write $w$ as a subword of $t_1 \cdots t_i \cdots t_r$ for some $i \in \{1, \ldots, r\}$. But then $r = l_\mathbb{W}(w) \leq l_S(t_1 \cdots t_i \cdots t_r) < r$. $\square$

Definition 2. We call $J \subset S$. We say that $J$ is $S$-adapted if for all $s \in S$ with $s = s_1 \cdots s_r$ a reduced expression in $(\mathbb{W}, S)$ such that there exists $i \in \{1, \ldots, r\}$ with $s_i \in J$ then it also holds $\{s_1, \ldots, s_r\} \subset J$.

Lemma 2. (a) Intersection with $G$ defines a map

\[ \{P_J \mid J \subset S \text{ is } S \text{-adapted} \} \rightarrow \{P_J \mid J \subset S \} \]

\[ P_J \mapsto P_J \cap G = P_S \cap \mathbb{W}_J \]

(b) Let $G \cap \mathbb{W} = W$ is a Borel subgroup of $G$ with $B \subset G$ a Borel subgroup and $x \in \mathbb{W}$. Let $s \in S$, then it holds

1. If $Wxs \neq Wx$ then $G \cap xB = G \cap \mathbb{W}$.

2. If $Wxs = Wx$, then $x \in W$ and $G \cap xB = x [G \cap \mathbb{W}]$.

This gives an algorithm to find for any $x \in \mathbb{W}$ a $z \in W$ such that $G \cap \mathbb{W} = z[G \cap \mathbb{W}]$. Also, for every $J \subset S$ it then holds $G \cap \mathbb{W}_J = z[G \cap \mathbb{W}_J]$ and $W \cap \mathbb{W}_J = z[W \cap \mathbb{W}_J]$ where $x \in \mathbb{W}, z \in W$ as before and for every $S$-adapted $J \subset S$

\[ G \cap \mathbb{W}_J = z P_S \cap \mathbb{W}_J. \]
proof:

(a) It holds by the previous lemma \( G \cap W_J = W \cap W_J \) and because \( J \) is \( S \)-adapted it holds \( W_\cap W_J = (S \cap W_J) \), to see that:

Let \( w = t_1 \cdots t_r \in W_J \) with \( t_i \in S \) an \( S \)-reduced expression, we need to see \( t_i \in W_J, 1 \leq i \leq r \).

Wlog assume \( t_1 \notin W_J \). As \( J \) is \( S \)-adapted, there exists a \( S \)-reduced expression with elements in \( J \) of \( w \) which is a subword of \( t_2 \cdots t_r \). But this means a word of \( S \)-length \( r \) is a subword of a word of \( S \)-length \( r - 1 \), therefore \( t_1 \in W_J \).

Now, the following inclusion is obvious

\[
P_{S \cap W_J} = B(G \cap W_J)B \subset G \cap P_J.
\]

Because \( B \subset P_J \cap G \) there has to exist \( (W_J \cap S) \subset J' \subset S \) such that \( P_J \cap G = P_{J'} \), we need to see \( (S \cap W_J) = J' \). Let \( s \in J' \), then \( s \in P_J = B W_J B \) implies \( s \in W_J \).

(b) Let \( s \in S, s \notin W \), then \( \pm x(\alpha) \notin \Phi \) and this implies

\[
\Phi \cap x(s(\Phi)) = \Phi \cap [x(\Phi) \setminus \{x(\alpha)\} \cup \{-x(\alpha)\}] = \Phi \cap x(\Phi).
\]

Therefore, the Lie algebras of the Borel groups \( G \cap x^* B \) and \( G \cap x^* \mathbb{B} \) have the same weights for \( T \), this proves they are equal.

The point (2) is obvious.

\[ \square \]

Remark. In the setup of the beginning, we can always find unique representatives \( x_i \in W, i \in I \) for the elements in \( W \setminus W \) which fulfill

\[ B_i = G \cap x^* \mathbb{B} = G \cap \mathbb{B} = B. \]

This follows because for every \( i \in I \) there is a bijection

\[ W x_i \to \{ \text{ Borel subgroups of } G \text{ containing } T \} \]

\[ v x_i \mapsto v[G \cap x^* \mathbb{B}] \]

Then, there exists a unique \( v \in W \) such that \( v[G \cap x^* \mathbb{B}] = G \cap \mathbb{B} \), replace \( x_i \) by \( v x_i \) as a representative for \( W x_i \).

We will call these representatives minimal coset representatives\(^1\). Observe for \( s \neq i \) it holds \( x_is = x_is \) by lemma 2, (b), (2).

But since the images of \( G/B_i, i \in I \) inside \( G/\mathbb{B} \) are disjoint, we prefer not to identify all \( B_i, i \in I \).

In general, in the parabolic setup, it holds \( P_i \neq P_j \) for \( i \neq j \).

Lemma 3. (factorization lemma) Let \( J, K \subset S \) be \( S \)-adapted and set \( L := S \cap W_J, M := S \cap W_K \).

1. It holds \( W_L = W \cap W_J \) and for every element in \( w \in W \) the unique decomposition as \( w = w^J w_L, w^J \in W^J, w_L \in W_L \) fulfills \( w^J \in W_L = W \cap W_J, w_L \in W_L = W \cap W_J \).

2. It holds \( W^J w^K \cap W = L W^M \). In particular, every double coset \( W_J w W_K \) with \( w \in W \) contains a unique element of \( L W^M \).

\(^1\)if \( G \) is a Levi-group in \( S \) they are the minimal coset representatives, in this more general situation the notion is not defined.
proof:

(1) It holds $W_L(W \cap W_J) = W = W \cap W^J W_J \supset (W \cap W^J)(W \cap W^J)$, the uniqueness of the factorization in $W$ implies $(W \cap W^J) \subset W^L$.

Now take $a \in W^L$, we can factorize it in $W$ as $a = a^J a_J$ with $a^J \in W^J, a_J \in W_J$. We show that $a_J \in W$. Write $a = t_1 \cdots t_r$, $S$-reduced expression, assume $a_J \neq e$, then there exists a unique $i \in \{1, \ldots, r\}$ such that $a_J$ is a subword of $t_i \cdots t_r$ but no subword of $t_i+1 \cdots t_r$. Then, $t_i$ must have a subword contained in $W_J$, as $J$ is $S$-adapted we get $t_i \in W_J$. Continue with $t_i^{-1} a_J$ being a subword of $t_i+1 \cdots t_r$. By iteration you find $a_J = t_i \cdots t_k \in W$ for certain $i = i_1 < \cdots < i_k$, $i_j \in \{1, \ldots, r\}$. This implies $a_J = e$ and $a = a^J \in W \cap W^J$.

(2) By definition $J^W W^K \cap W = (W^J)^{-1} \cap W^K \cap W = (W^L)^{-1} \cap W^M = LW^M$. □

0.1.2 The equivariant cohomology of flag varieties

**Lemma 4.** (The (co)-homology rings of a point)

Let $G$ be a reductive group, $T \subset P \subset G$ with $P$ a parabolic subgroup and $T$ a maximal torus, we write $W$ for the Weyl group associated to $(G, T)$ and $X(T) = \text{Hom}_G(T, \mathbb{C}^*)$ for the group of characters. Let $ET$ be a contractible topological space with a free $T$-operation from the right.

(1) For every character $\lambda \in X(T)$ denote by

$$S_\lambda := ET \times^T \mathbb{C}_\lambda$$

the associated $T$-equivariant line bundle over $BT := ET/T$ to the $T$-representation $\mathbb{C}_\lambda$ which is $\mathbb{C}$ with the operation $t \cdot c := \lambda(t)c$. The first chern class defines a homomorphism of abelian groups

$$c: X(T) \rightarrow H^2(BT), \quad \lambda \mapsto c_1(S_\lambda).$$

Let $\text{Sym}_\mathbb{C}(X(T))$ be the symmetric algebra with complex coefficients generated by $X(T)$, it can be identified with the ring of regular function $\mathbb{C}[t]$ on $t = \text{Lie}(T)$ (with doubled degrees), where $X(T) \otimes_\mathbb{Z} \mathbb{C}$ is mapped via taking the differential (of elements in $X(T)$) to $t^* = \text{Hom}_{\mathbb{C} \text{-lin}}(t, \mathbb{C}) \subset \mathbb{C}[t]$ (both are the degree 2 elements).

The previous map extends to an isomorphism of graded $\mathbb{C}$-algebras

$$\mathbb{C}[t] \rightarrow H^*_T(pt) = H^*(BT)$$

In fact this is a $W$-linear isomorphism where the $W$-operation on $\mathbb{C}[t]$ is given by, $(w, f) \mapsto w(f), w \in W, f \in \mathbb{C}[t]$ with

$$w(f): t \mapsto c, t \mapsto f(w^{-1}tw).$$

We can choose $ET$ such that it also has a free $G$-operation from the right (i.e. $ET := EG$), then $BT = ET/T$ has an induced Weyl group action from the right given by $xT \cdot w := xwT$, $w \in W, x \in ET$. The pullbacks of this group operation induce a left $W$-operation on $H^*_T(pt)$.

(2) $H^*_T(pt) = H^*_T(pt), H^*_G(pt) = (H^*_T(pt))^W = (H^*_T(pt))^W = H^*_G(pt)$.

proof:

(1) For the isomorphism see for example and the explanation of the $W$-operation see (L. Tu; Characteristic numbers of a homogeneous space, axiv, [Tu03])

(2) Use the definition and Poincare duality for the first isomorphism, for the second also use the splitting principle.
Lemma 5. (The cohomology rings of homogeneous vector bundles over $G/P$)

Let $G$ be a reductive group, $T \subset B \subset P \subset G$ with $B$ a Borel subgroup, $P$ parabolic and $T$ a maximal torus.

1. For $\lambda \in X(T)$ we denote $L_\lambda := G \times^B C_\lambda$ the associated line bundle to the $B$-representation $C_\lambda$ given by the trivial representation when restricted to the unipotent radical and $\lambda$ when restricted to $T$. Let $\mu: E \to G/B$ be a $G$-equivariant vector bundle. Then, $\mu^*(L_\lambda)$ is a line bundle on $E$ and

$$K_\lambda := EG \times^G \mu^*(L_\lambda) \to EG \times^G E$$

is a line bundle over $EG \times^G E$. There is an isomorphism of graded $\mathbb{C}$-algebras

$$\mathbb{C}[t] \to H_G^*(E) = H^*(EG \times^G E)$$

with $\deg \lambda = 2$ for $\lambda \in X(T)$.

(By definition, equivariant Chern classes are defined as $c_i^G(\mu^* L_\lambda) := c_i(L_\lambda)$).

2. Let $\mu: E \to G/P$ be a $G$-equivariant vector bundle, then there is an isomorphism of graded $\mathbb{C}$-algebras

$$H^*_G(E) \to (H^*_G(pt))^W_L.$$

proof:

1. Arabia proved that $H^*_G(G/B) \cong H^*_G(pt)$ as graded $\mathbb{C}$-algebras (cp. [Ara85]), the composition with the isomorphism from the previous lemma gives an isomorphism

$$c: \mathbb{C}[t] \to H_G^*(G/B), \lambda \mapsto c_1(EG \times^G L_\lambda) =: c_1^G(L_\lambda)$$

Now, we show that for a vector bundle $\mu: E \to G/P$ with $P \subset G$ parabolic, the induced pullback map

$$\mu^*: H_G^*(G/P) \to H_G^*(E), \quad c_i^G(L_\lambda) \mapsto c_i^G(\mu^* L_\lambda)$$

is an isomorphism of graded $H_G^*(pt)$-algebras. We already know that it is a morphism of graded $H_G^*(pt)$-algebras, to see it is an isomorphism, apply the definition and Poincare duality to get a commutative diagram

$$\begin{array}{ccc}
H_G^2(G/P) & \xrightarrow{\mu^*} & H_G^2(E) \\
\cong & & \cong \\
H_G^{2 \dim G/P - k}(G/P) & \xrightarrow{\mu^*} & H_G^{2 \dim E - k}(E)
\end{array}$$

the lower morphism $\mu^*$ is the pullback morphism which gives the Thom isomorphism, therefore the upper $\mu^*$ is also an isomorphism.

2. By the last proof, we already know $H^*_G(E) \cong H^*_G(G/P)$. Then apply the isomorphism of Arabia see [Ara85], this gives $H^*_G(G/P) \cong H^*_G(pt)$. Now, $P$ homotopy-retracts on its Levy subgroup $L$, this implies $H^*_L(pt) = H^*_L(pt)$, together with the (2) in the previous lemma we are done.

Lemma 6. (The cohomology ring of the flag variety as subalgebra of the Steinberg algebra)

Let $G$ be a reductive group, $T \subset P \subset G$ with $P$ parabolic and $T$ a maximal torus. Let $V$ be a $G$-representation and $F \subset V$ be a $P$-subrepresentation, let $E := G \times^P F$ and $Z := E \times^V E$ be the associated Steinberg variety. The diagonal morphism $E \to E \times E$ factorizes over $Z$ and induces an isomorphism $E \to Z_e$ which induces an isomorphism of algebras

$$H^*_G(G/P) \to H^*_G(Z_e),$$

recall that the convolution product on $H^*_G(Z_e)$ maps degrees $(i,j) \mapsto i + j - 2 \dim E$.
Then, there are induced bijections $\rho^*: H^*_G(G/B) \to H^*_G(E) \cong H^*_G(Z_e) \to H^*_{\text{dim} E}(Z_e)$ where the last isomorphism is Poincare duality. But we need to see that this is a morphism of algebras where $H^*_G(Z_e)$ is the convolution algebra with respect to the embedding $Z_e \cong E \to E \times E$. This follows from [CG97], Example 2.7.10 and section 2.6.15.

We observe that the algebra $\mathbb{C}[t]$ with generators $t \in t^*$ in degree 2 plays three different roles in the last lemmata. It is the $T$-equivariant cohomology of a point, it is the $G$-equivariant cohomology of a complete flag variety $G/B$, it can be found as the subalgebra $H^*_G(Z_e) \subset H^*_G(Z)$.

### 0.1.3 Computation of fixed points

Recall the following result, for example see [Här99], satz 2.12, page 13.

**Lemma 7.** Let $T \subset P \subset G$ be a reductive group with a parabolic subgroup $P$ and a maximal torus $T$. Let $W$ be the Weyl group associated to $(G,T)$ and $\text{Stab}(P) := \{w \in W \mid wPw^{-1} = P\}$. For $w = x\text{Stab}(P) \in W/\text{Stab}(P)$ we set $wP := xP \in G/P$. Then, it holds

$$(G/P)^T = \{wP \in G/P \mid w \in W/\text{Stab}(P)\}$$

**Lemma 8.** Let $P_1, P_2 \subset G$ be a reductive group with two parabolic subgroup, $F_1, F_2 \subset V$ a $G$-representation with a $P_1$ and $P_2$-subrepresentation. Assume $(G_{F_1})^T = \{0\}$. We write $(E_i = G \times T, F_i, \mu_i; E_i \to G/P_i, \pi_i; E_i \to V)$ for the associated Springer triple and $Z := E_1 \times V E_2, m: Z \to (G/P_1) \times (G/P_2)$ for the Steinberg variety. Then, there are induced bijections $\mu^1_i: E^T_i \to (G/P_i)^T, m^T: Z^T \to (G/P_1)^T \times (G/P_2)^T$. More explicit we have

$$E^T_i = \{\phi_w := (0, wP_i) \in V \times G/P_i \mid w \in W/\text{Stab}(P_i)\} \subset E_i$$

$$Z^T = \{\phi_{x,y} := (0, xP_1, yP_2) \in V \times G/P_1 \times G/P_2 \mid x \in W/\text{Stab}(P_1), y \in W/\text{Stab}(P_2)\} \subset Z.$$

Furthermore, for any $w \in W/\text{Stab}(P_2)$ let $Z^w := m^{-1}(G \cdot (P_1, wP_2))$ and $m_w := m|_{Z^w}: Z^w \to G \cdot (P_1, wP_2)$ the induced map. There is an induced Bruhat order $\leq$ on $W/\text{Stab}(P_2)$ by taking the Bruhat order of minimal length representatives.

$$(Z^w)^T = \{\phi_{x,xw} := (0, xP_1, xwP_2) \in V \times G/P_1 \times G/P_2 \mid x \in W\}$$

$$Z^w^T = \{\phi_{x,xw} \mid x \in W, v \leq w\} = \bigcup_{v \leq w} (Z^v)^T$$

There is a bijection $W/(\text{Stab}(P_1) \cap ^w\text{Stab}(P_2)) \to (Z^w)^T, x \mapsto \phi_{x,xw}$. 

**proof** Obviously, they holds $E^T_i \subset V^T \times (G/P)^T = \{0\} \times (G/P)^T$. But we also have a zero section $s$ of the vector bundle $\pi: E_i \to G/P_1$ which gives the closed embedding $G/P_1 \to E_i \subset V \times (G/P_1), gP_1 \mapsto (0, gP_1)$.

It holds $Z^T \subset V^T \times (G/P_1)^T \times (G/P_2)^T = \{0\} \times (G/P_1)^T \times (G/P_2)^T$. But using the description of $Z = \{(v, gP_1, hP_2) \in V \times G/P_1 \times G/P_2 \mid (v, gP_1) \in E_1, (v, hP_2) \in E_2\}$, we see that $\{0\} \times (G/P_1)^T \times (G/P_2)^T \subset Z$ and these are obviously $T$-fixed points.

We have $(Z^w)^T \subset Z^w \cap Z^T = \{\phi_{x,xw} \mid x \in W\}$ and one can see the other inclusion, too. Also, we have $Z^w^T \subset (\bigcup_{v \leq w} Z^v)^T = \bigcup_{v \leq w} (Z^v)^T$. Consider the closed embedding $s: G/P_1 \times G/P_2 \to Z, (gP_1, hP_2) \mapsto (0, gP_1, hP_2)$.

Clearly $s(G(P_1, wP_2)) \subset Z^w \subset Z^w^T$, but since $s$ is a closed embedding we have

$$\bigcup_{v \leq w} (Z^v)^T \subset s(G(P_1, wP_2)) = s(G(P_1, wP_2)) \subset Z^w$$

which yields the other inclusion.

□
Lemma 9. Assume $J = \emptyset, \mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$. Let $x \in \mathbb{W}, s \in S$ we set 

$$h_\pi(s) := \# \{ k \in \{ 1, \ldots, r \} \mid x(\alpha_s) \in \Phi_{V(k)} \}$$

where $V = \bigoplus_{k=1}^r V(k)$ and $\Phi_{V(k)} \subset \Phi$ are the $T$-weights of $V(k)$. If $x = x^i x_i$ with $x^i \in W$, then $h_\pi(s) = h_{\pi^i}(s) =: h_i(s)$. It holds

$$F_{x_i}/F_{x_i x_is} = (G_{x_i(\alpha_s)})^{\oplus h_i(s)}.$$

(1) If $x^i x_is \notin W$ then

$$h_i(s) = \# \{ k \mid V(k) \subset R, \; x_i(\alpha_s) \in \Phi_{V(k)} \}.$$

(2) If $x^i x_is \in W$, then

$$h_i(s) = \# \{ k \mid V(k) \subset G, \; x_i(\alpha_s) \in \Phi_{V(k)} \}.$$
**proof:** Without loss of generality $V \subset \mathcal{G}, U = \text{Lie}(U)$, set $x := x_i$, we have a short exact sequence

$$0 \to V \cap \mathfrak{s}[U \cap \mathfrak{s}U] \to V \cap \mathfrak{s}U \to V \cap \mathcal{G}_{x(a_s)} \to 0$$

Now, $V \cap \mathcal{G}_{x(a_s)} = 0$ if and only if $x(a_s) \notin \Phi_V$.

1. If $\mathfrak{s}s \notin W$ then $x(a_s) \notin \Phi$ where $\Phi$ are the $T$-weights of $\mathfrak{g}$. That means, if $V \subset \mathfrak{g}$ we get $h_i(s) = 0$.

2. If $\mathfrak{s}s \in W$, then $x(a_s) \in \Phi$. This means, if $V \subset \mathcal{R}$ we get $h_i(s) = 0$.

$\square$

### 0.2 Relative position stratification

#### 0.2.1 In the flag varieties

Let $J \subset \mathbb{S}$, $w \in \mathcal{J}W^J$, $i, j \in I_J$. We define

$$C^w := \mathcal{G}_{\phi_{e,w}} \cap \left( \bigcup_{i \in I_J} G_i/P_i \times \bigcup_{i \in I_J} G_i/P_i \right)$$

$$C^\leq w := \mathcal{G}_{\phi_{e,w}} \cap \left( \bigcup_{i \in I_J} G_i/P_i \times \bigcup_{i \in I_J} G_i/P_i \right)$$

$$C^w_{i,j} := C^w \cap (G/P_i \times G/P_j)$$

$$C^\leq w_{i,j} := C^\leq w \cap (G/P_i \times G/P_j)$$

For an arbitrary $w \in \mathcal{W}$ there exists a unique $v \in \mathcal{J}W^J$ such that $\mathcal{W}_j w \mathcal{W}_j = \mathcal{W}_j v \mathcal{W}_j$, we set $C^w := C^w, C^w_{i,j} := C^w_{i,j}, C^\leq w := C^\leq w, C^\leq w_{i,j} := C^\leq w_{i,j}$. We remark that $C^\leq w, C^\leq w_{i,j}$ are closed (but not necessarily the closure of $C^w, C^w_{i,j}$, because it can happen that $C^w_{i,j} = \emptyset, C^\leq w_{i,j} \neq \emptyset$, see next lemma (3)).

Let $i, j \in I_J, C_{i,j} := \{C^w_{i,j} \mid w \in \mathcal{J}W^J, C^w_{i,j} \neq \emptyset\}$, $\text{Orb}_{i,j} := \{G\text{-orbits in } G/P_i \times G/P_j\}$, we have the following commutative diagram

$$\begin{array}{ccc}
\text{Orb}_{i,j} & \xrightarrow{\text{r}_p} & W \cap x^i \mathcal{W}_J \setminus W/W \cap x^i \mathcal{W}_J \\
\Phi \downarrow & & \downarrow \Psi \\
C_{i,j} & \xrightarrow{\text{r}_p \Psi} & \{(x^i \mathcal{W}_J)w(x^j \mathcal{W}_J) \mid w \in \mathcal{W}\}
\end{array}$$

defined as follows

$$\text{r}_p(G_{\phi_{x_i,w x_j}}) := (W \cap x^i \mathcal{W}_J)w(W \cap x^i \mathcal{W}_J),$$

$$\text{r}_p \Psi(C^w_{i,j}) := x^j \mathcal{W}_J \cap (G \phi_{x_i,w x_j})$$

$$\Psi([W \cap x^i \mathcal{W}_J]w(W \cap x^j \mathcal{W}_J)) := \{(x^i \mathcal{W}_J)w(x^j \mathcal{W}_J)\}$$

$r_p, r_p \Psi$ are bijections and $\Phi, \Psi$ are surjections. We will from now on assume that $\Phi, \Psi$ are bijections as well, i.e. for every nonempty $C^w_{i,j}$ there is a $w_0 \in W$ such that $\mathcal{W}_j x_i^{-1} w_0 x_j \mathcal{W}_J = \mathcal{W}_j w \mathcal{W}_J$ and $C^w_{i,j} = G_{\phi_{x_i,w x_j}} \subset G/P_i \times G/P_j$, this implies

$$C^w_{i,j} \cong G/(P_i \cap w_0 P_j \cap G).$$

**Lemma 10.** Let $J \subset \mathbb{S}, s \in \mathbb{S} \setminus J, i, j \in I_J$.

1. $C^\leq s$ is smooth, it equals $C^s \cup C^w$.  

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(2) $C^≤_{ij} = \emptyset$ unless $Wx_i W_j \in \{Wx_i W_j, Wx_i s W_j\}$.

(3) Assume that $Wx_i W_j \neq Wx_i s W_j$ and let $j \in I_j$ such that $x_isx_j^{-1} \in W$, then it holds

$$i_i(G/P_i) \neq i_j(G/P_j), \quad C^≤_{i,i} = C^≤_{i,j}, \quad C^≤_{i,i} = C^≤_{i,j}$$

and $G \cap x_i [P_j \cap s P_j] = G \cap x_i P_{J\cap j}, C^s_{i,j} = G/(G \cap x_i P_{J\cap j})$.

(4) Assume that $Wx_i W_j = Wx_i s W_j = Wx_j W_j$, then it holds $i = j$, in particular

$$i_i(G/P_i) = i_j(G/P_j), \quad C^w_{i,i} = C^w_{i,i}, \text{ for all } w$$

and the first equality implies $(x_i P_j) \cap G \neq (x_i s^P_j) \cap G$, there is an isomorphism of $G$-varieties

$$G \times P_i ((x_i P_j \cap (s) \cap G)) \times P_i \rightarrow C^≤_{i,i}, \quad (g, h P_i) \mapsto (g P_i, gh P_i).$$

**proof:**

(1) The variety $\bigcup_{i \in I_j} G_{i/P_i}$ is a smooth subvariety of $G/P_j$ because each $G/P_i, i \in I_j$ is smooth. It is known that $G_{\phi e.s} = G_{\phi e.s} G_{\phi e.c}$ is smooth in $G/P_j$, therefore its intersection (i.e. pullback) is smooth in $(G/P_j)^\mathfrak{I}$.

(2) Now, $C^≤_{i,j} = C^≤_{i,j} \cup C^≤_{i,j}$ and $C^≤_{i,j} \neq \emptyset$ iff it contains a $T$-fixed point $\phi_{x_i,vx_j}$ for a $v \in W$, that implies $x_i^{-1} vx_j P_j = \mathcal{P}_j$, i.e. there is an $f \in P_j$ such that $vx_j f = x_is_j$, therefore $f \in P_j \cap W = W_j$ and $Wx_i W_j = Wx_i s W_j$. Similar $C^≤_{i,i} \neq \emptyset$ iff $Wx_i W_j = Wx_i W_j$.

(3) The intersection $(G/P_i) \cap (G/P_j)$ is a $G$-equivariant subset of $G/P_j$, therefore it is nonempty iff it contains all $T$-fixed points $vx_i P_j = vx_j P_j$ with $v, w \in W$. This is equivalent to $Wx_i W_j = Wx_i W_j$.

As we have seen before $Wx_i s W_j = Wx_j W_j$ implies $C^≤_{i,j} = 0, C^s_{i,i} = \emptyset$ and therefore $C^≤_{i,i} = C^≤_{i,j} = C^≤_{i,i} = C^≥_{i,i}$.

Let $Wx_i W_j \neq Wx_i s W_j$, we need to show that $G \cap x_i [P_j \cap s P_j] = G \cap x_i P_{J\cap j}$. Let $\Phi = \Phi_+ \cup -\Phi_-$ be the set of roots for $(G, B, T)$ decomposing as positive and negative roots, let $\Delta_j \subset \Phi_+$ be the simple roots corresponding to $J \subset S$ and let $\Phi$ be the roots for $(G, T)$. It is enough to prove that the $T$-weights on $\text{Lie}(G \cap x_i [P_j \cap s P_j])$ equal the $T$-weights on $\text{Lie}(G \cap x_i P_{J\cap j})$.

Now, $Wx_i W_j \neq Wx_i s W_j$ implies $x_is \notin W$ or equivalently $x_i(\alpha_s) \notin \Phi$ where $\alpha_s \in \Phi_+$ is the simple root negated by $s$. We have the $T$-weights of $\text{Lie}(x_i^s P_j)$ are $\{x_i(\alpha) | \alpha \in \Phi_+ \cup -\Delta_j \}$, the $T$-weights of $\text{Lie}(x_i^s P_j)$ are $\{x_i(\alpha) | \alpha \in \Phi_+ \cup -\Delta_j \cup \{-\alpha_s\}\}$.

It follows that the $T$-weights of $\text{Lie}(G \cap x_i^s P_j \cap x_i^s P_j)$ are

$$\{x_i(\alpha) | \alpha \in \Phi_+ \cup -\Delta_j \cup \{-\alpha_s\}\} \cap \Phi$$

and these are the $T$-weights of $\text{Lie}(G \cap x_i^j P_j \cap P_j)$. (4) The first part is by definition. Assume $Wx_i W_j = Wx_i s W_j$ implies $x_isx_j^{-1} = ab$ with $a \in W, b \in x_i W_j$. Now $x_i^s P_j \cap G$ is a parabolic subgroup of $G$ conjugated to $P_{J\cap S}$, therefore

$$x_i^s P_j \cap G = (x_isx_j^{-1} (x_i^s P_j)) \cap G = a(x_i^s P_j) \cap G = a((x_i^s P_j) \cap G)$$

and assume that this is equal $x_i^s P_j \cap G$ that implies $a \in x_i(J \cap S)x_i^{-1}$, then $x_isx_j^{-1} = ab \in x_i^s W_j$ that implies $s \in J$ contradicting our assumption $s \notin J$.

Finally, consider the closed embedding $G \times P_i ((x_i^s P_j \cap G) \cap P_j) \rightarrow G \times P_i G/P_i$ and compose it with the $G$-equivariant isomorphism $G \times P_i G/P_i \rightarrow G/P_i x G/P_i, (g, h P_i) \mapsto (g P_i, gh P_i)$. The image is precisely $C^≤_{i,i} \cup C^≥_{i,i}$.
0.2.2 In the Steinberg variety

Let \( w \in W_J \), \( i, j \in I_J \), recall that we have a map \( m_J : Z_J \to G/P_J \).

\[
Z_{i,j}^w : = m_{i,j}^{-1}(C_{i,j}^w)
\]

\[Z^w = \bigcup_{i,j \in I_J} Z_{i,j}^w \]

\[Z_{i,j}^{\leq w} = \bigcup_{v \leq w, v \in W_J} Z_J^v \]

\[Z_{i,j}^{\leq w} : = \bigcup_{v \leq w, v,w \in W_J} Z_{i,j}^v \]

**Lemma 11.** (a) If \( C_{i,j}^w \neq \emptyset \), the restriction \( m_{i,j} : Z_{i,j}^w \to C_{i,j}^w \) is a vector bundle with fibres isomorphic to \( F_i \cap x_iw_{i,j}^{-1} F_j \), it induces a bijection on \( T \)-fixed points. In particular, all nonempty \( Z_{i,j}^w \) are smooth.

(b) For any \( s \in S \) the restriction \( m : \overline{Z^s} \to C^{\leq s} \) is a vector bundle over its image, in particular \( \overline{Z^s} \) is smooth. More precisely, it is a disjoint union \( \overline{Z_{i,i}^s} \to C^{\leq s}_{i,i} \) with

1. \( \overline{Z_{i,j}^s} \neq \emptyset \) implies \( Wx_iW_J = Wx_iW_J \).
2. Assume that \( Wx_iW_J \neq Wx_iW_J \), then \( \overline{Z_{i,j}^s} = Z_{i,j}^s \) and \( \overline{Z_{i,i}^s} = \emptyset \).
3. Assume that \( Wx_iW_J = Wx_iW_J \), then it holds \( \overline{Z_{i,i}^s} \to C^{\leq s}_{i,i} \) is a vector bundle.

**proof:**

(a) As \( C_{i,j}^w \) is assumed to be a diagonal \( G \)-orbit in \( G/P_i \times G/P_j \), it is a homogeneous space and the statement easily follows from a wellknown lemma, cp. [Slo80], p.26, lemma 4.

(b) (1) If \( \overline{Z_{i,j}^s} \neq \emptyset \), then \( C_{i,j}^s \neq \emptyset \) and by the proof of the previous lemma 10, (2), the claim follows.

(2) If \( Wx_iW_J \neq Wx_iW_J \), then by lemma 10, (3), \( C_{i,j}^{\leq s} = C_{i,j}^s \) is already closed, therefore \( Z_{i,j}^s \) is closed as well. Also, \( C_{i,i}^{\leq s} = C_{i,i}^s \) is already closed, therefore \( Z_{i,i}^s \) is closed as well.

(3) If \( Wx_iW_J = Wx_iW_J \), then \( C_{i,i}^{\leq s} \) is the closure of the \( G \)-orbit \( C_{i,i}^s \) and by lemma 10, (4) we have \( G \times F_i \left( (x_iP_j \cup \{s\} \cap G)/P_i \right) \to C_{i,i}^{\leq s} \), \((g, hP_i) \to (gP_i, ghP_i)\) is an isomorphism. We set \( X := \{(g, gP_i, ghP_i) \in G(F_i \cap x_i^s F_i) \times G/P_i \times G/P_i \mid g \in G, f \in F_i \cap x_i^s F_i, h \in x_iP_j \cup \{s\} \cap G\} \) and we claim \( \overline{Z_{i,i}^s} = X \). First, observe that \( X \subset Z_{i,i}^s \) because \( gf = gh(h^{-1}f) \) with \( h^{-1}f \in F_i \cap x_i^{-1} F_i \). One can easily check the following steps.

(*) \( X \to C_{i,i}^{\leq s} \) is a vector bundle with fibre over \( F_i \cap x_i^{-1} F_i \). In particular, we get that \( X \) is smooth irreducible and \( \dim X = \dim Z_{i,i}^s \).

(*) \( Z_{i,i}^s \subset X \).

(*) \( X \) is closed in \( Z_{i,i}^s \) because we can write it as \( X = p^{-1}(G(F_i \cap x_i^{-1} F_i)) \cap m^{-1}(C_{i,i}^{\leq s}) \).

Since \( F_i \cap x_i^{-1} F_i \) is (by definition) \( B_i \)-\( B \)-stable, we get \( G(F_i \cap x_i^{-1} F_i) \) is closed in \( V \). This implies \( X \) is closed.
1 A short lamentation on the parabolic case

From the next section on we assume that all $P_i = B_i$ are Borel subgroups. What goes wrong with the more general assumption (which we call the parabolic case)?

(1) We do not know whether $C^w_{i,j}$ (see previous section) is always a $G$-orbit. That is relevant for Euler class computation in Lemma 14.

(2) The cellular fibration property has to be generalized because $C^w := \{ gP, gwP' \mid g \in G \} \subset G/P \times G/P' \xrightarrow{pt.} G/P$ is not a vector bundle (its fibres are unions of Schubert cells). This complicates Lemma 12.

(3) We do not know what is the analogue of lemma 13, i.e. what can we say about $Z^{\leq x} \ast Z^{\leq y}$?

(4) The cycles $[Z_{i,j}^\pm]$ are not in general multiplicative generators. If we try to understand more generally $[Z_{i,j}^\pm]$, the multiplicity formula does not give us as much information as for $[Z_{i,j}^\pm]$ because $Z_{i,j}^\pm$ is even smooth. Also understanding the $[Z_{i,j}^\pm]$ is not enough, since they do not give a basis as a free $E$-module because the rank is wrong (cp. failing of cellular fibration lemma).

The point (4) is the biggest problem. Even for $H^G_e(G/P \times G/P)$ we do not know a set of generators and relations (see next chapter).

So, from now on we assume $J = \emptyset$.

1.1 Convolution operation on the equivariant Borel-Moore homology of the Steinberg variety

Definition 3. Let $H \in \{ pt, T, G \}$ with $T \subset G$ where $T$ is a maximal torus.

We define the $H$-equivariant algebra of a point to be $H^H_*(pt)$ with product equals the cup-product, we will always identify it with $H^*_H(pt) := H^*_H(pt)$. It is a graded $\mathbb{C}$-algebra concentrated in negative even degrees.

We define the $H$-equivariant Steinberg algebra to be the $H$-equivariant Borel-Moore homology algebra of the Steinberg variety, the product is the convolution product, see [CG97], [VV11b].

We say ($H$-equivariant) company algebra to the $H$-equivariant cohomology algebra of $E$, the product is the cup-product.

\[ \Lambda_H := H^H_*(pt) \] for the $H$-equivariant algebra of a point,

\[ Z_H := H^H_*(Z) \] for the $H$-equivariant Steinberg algebra,

\[ \mathcal{E}_H := H^H_*(E) \] for the $H$-equivariant company algebra.

For $H = pt$ we leave out the adjective $H$-equivariant and leave out the index $H$.

Recall, that $Z_H$ and $\mathcal{E}_H$ are left graded modules over $\Lambda_H$. Furthermore, $\mathcal{E}_H$ is a left module over $Z_H$. This follows from considering $M_1 = M_2 = M_3 = E$ smooth manifolds and $Z \subset M_1 \times M_2, E = E \times \{ e, 0 \} \subset M_2 \times M_3$. Then the set-theoretic convolution gives $Z \circ E = E$, which implies the operation.

Also, $Z_H$ is a left module over $\mathcal{E}_H$. This follows from considering $M_1 = M_2 = M_3 = E$ smooth manifolds ($\dim_k E = : e$) and $E \equiv M_1 \times M_2$ diagonally, $Z \subset M_2 \times M_3$, then the set-theoretic convolution gives $E \circ Z = Z$, that implies that we have a map

\[ H^H_{2e_i-p}(E_i) \times H^H_{e_i+e_j-q}(Z_{i,j}) \to H^H_{e_i+e_j-(p+q)}(Z_{i,j}) \]

Using Poincare duality we get $H^H_{2e_i-p}(E) \cong H^P_{p}(E)$ and the grading $H^H_*(Z) := \bigoplus_{i,j} H^H_{e_i+e_j-q}(Z)$ the previous map gives an operation of the $H^H_*(E)$ on $H^H_*(Z)$ which is $H^H_*(pt)$-linear. We denote the operations by

\[ * : Z_H \times \mathcal{E}_H \to \mathcal{E}_H \]

\[ \circ : \mathcal{E}_H \times Z_H \to Z_H \]
Furthermore, there are forgetful algebra homomorphisms
\[
\begin{align*}
\Lambda_G &\to \Lambda_T \to \Lambda, \\
Z_G &\to Z_T \to Z, \\
\mathcal{E}_G &\to \mathcal{E}_T \to \mathcal{E}.
\end{align*}
\]

Let us investigate some elementary properties of the convolution operations. From [VV11a], section 5, p.606, we know that the operation of \( Z_G \) on \( \mathcal{E}_G \) is faithful, i.e. we get an injective \( \mathbb{C} \)-algebra homomorphism
\[
Z_G \hookrightarrow \text{End}_{\mathbb{C}-\text{alg}}(\mathcal{E}_G).
\]
We have the following cellular fibration property. We choose a total order \( \leq \) refining Bruhat order on \( \mathbb{W} \). For each \( i, j \in I \) we get a filtration into closed \( G \)-stable subsets of \( Z_{i,j} \) by setting \( Z_{i,j}^\leq := \bigcup_{v \leq w} Z_{i,j}^v, \ w \in \mathbb{W} \). Via the first projection \( \pi_1 : C_{i,j}^v \to G/B_i \) is a \( G \)-equivariant vector bundle with fibre \( B_i v B_j / B_j \), we call its (complex) dimension \( d_{i,j}^v \), also \( Z_{i,j}^v \to C_{i,j}^v \) is a \( G \)-equivariant vector bundle, we define the complex fibre dimension \( f_{i,j}^v \). By the \( G \)-equivariant Thom isomorphism (applied twice) we get
\[
H^G_m(Z_{i,j}^v) = H^G_{m-2d_{i,j}-2f_{i,j}}(G/B_i).
\]
In particular, it is zero when \( m \) is odd and \( H^G_{*}(Z_{i,j}^v) \) is a free \( H^G_{*}(pt) \)-module with basis \( b_x, \ x \in W, \deg b_x = 2\dim(B_i x B_i) / B_i + 2d_{i,j}^v + 2f_{i,j}^v \).

Using the long exact localization sequence in \( G \)-equivariant Borel-Moore homology for every \( v \in \mathbb{W} \), we see that \( Z_{i,j}^v \) is open in \( Z_{i,j}^\leq \) with an closed complement \( Z_{i,j}^{>v} \). We conclude inductively using the Thom isomorphism that \( H^G_{\text{odd}}(Z_{i,j}^{<v}) = 0 \) and that \( H^G_{*}(Z_{i,j}^{\leq v}) = \bigoplus_{v \leq w} H^G_{*}(Z_{i,j}^w) \). We observe, that \( \# \{ w \in \mathbb{W} \mid Z_{i,j}^w \neq \emptyset \} = \# W \) for every \( i, j \in I \). It follows that \( H^G_{*}(Z_{i,j}) \) is a free \( H^G_{*}(pt) \)-module of rank \( \#(W \times W) \), and that every \( H^G_{*}(Z_{i,j}^{<v}) \to H^G_{*}(Z_{i,j}) \) is injective.

We can strengthen this result to the following lemma.

**Lemma 12.** Let \( \leq \) be a total order refining Bruhat order on \( \mathbb{W} \). For any \( w \in \mathbb{W} \) set \( Z_{<w} := m^{-1}(\bigcup_{v \leq w} C_{i,j}^v) = \bigcup_{v \leq w} Z_{i,j}^v \). The closed embedding \( i : Z_{<w} \to Z \) gives rise to an injective morphism of \( H^G_{*}(E) \)-modules \( i_* : Z_{<w}^G \to H^G_{*}(Z_{<w}) \to Z_G \). We identify in the following \( Z_{<w}^G \) with its image in \( Z_G \). For all \( v \in W \) we have
\[
\begin{align*}
Z_{<w}^G &= \bigoplus_{v \leq w} \mathcal{E}_G \circ [Z_{i,j}^v] \quad \text{as } \mathcal{E}_G \text{-module} \\
1_i * Z_{<w}^G * 1_j &= \bigoplus_{v \leq w} \mathcal{E}_i \circ [Z_{i,j}^v] \quad \text{as } \mathcal{E}_i \text{-module}
\end{align*}
\]
where \( \mathcal{E}_i = H^*_i(E_i) \). Each \( [Z_{i,j}^v] \) is nonzero (and not necessarily a homogeneous element). In particular, \( Z_G \) (as ungraded module) is a free left \( \mathcal{E}_G \)-module of rank \( \#\mathbb{W} \).

**proof:** Now first observe that set-theoretically we have \( E \circ Z^v = Z^v \) (where we use the diagonal embedding for \( E \) again). This implies that the direct sum decomposition \( H^G_{*}(Z) = \bigoplus_{v \in \mathbb{W}} H^G_{*}(Z^v) \) is already a decomposition of \( H^G_{*}(E) \)-modules.

Now we know that we have by the Thom-isomorphism algebra isomorphisms
\[
H^G_{*}(E) \cong H^G_{*}(\bigsqcup_{i \in I} G/B_i) \cong H^G_{*}(Z^v),
\]
using that \( \# \{ (i, j) \mid Z_{i,j}^v \neq \emptyset \} = \# I \). Now, Poincare duality is given by \( H^G_{*}(Z_{i,j}^v) \to H^G_{2 \dim Z_{i,j}^v - q}(Z_{i,j}^v) \), \( \alpha \mapsto \alpha \cdot [Z_{i,j}^v] \) the composition gives
\[
H^G_{*}(E_i) \to H^G_{2 \dim Z_{i,j}^v - q}(Z_{i,j}^v), \quad c \mapsto c \cdot [Z_{i,j}^v].
\]
\[\square\]
Lemma 13. For each $x, y \in \mathbb{W}$ with $l(x) + l(y) = l(xy)$ we have
\[
Z_{G}^{\leq x} \ast Z_{G}^{\leq y} \subset Z_{G}^{\leq xy}
\]

proof: By definition of the convolution product, it is enough to check that for all $w \leq x, v \leq y$ it holds for the set theoretic convolution product
\[
Z_{i,j}^{w} \ast Z_{j,k}^{v} \subset \begin{cases} \emptyset, \\ Z_{i,k}^{\leq xy}, \\ j = j' \end{cases}
\]
for $i, j, j', k \in I$, because by definition $Z_{i}^{\leq x} \ast Z_{j}^{\leq y} = \bigcup_{x \leq v, y \leq v} Z^{w} \ast Z^{v}$. Now, the case $j \neq j'$ follows directly from the definition. Let $j = j'$. Let $C_{w} := G(B, wB) \subset G/B \times G/B$. According to Hinrich, Joseph [HJ05], 4.3 it holds $C_{w} \circ C_{v} \subset C_{wv}$ for all $v, w \in \mathbb{W}$. Now, we can adapt this argument to prove that $C_{i,j} \circ C_{j,k} \subset C_{i,k}$ as follows:

Since $C_{i,j} \neq \emptyset, C_{j,k} \neq \emptyset$ we have that $w_{0} = x_{i}w_{v}x_{j}^{-1} \in W, v_{0} = x_{j}v_{w}x_{k}^{-1} \in W$ and $C_{i,j} = G(B_{i}, w_{0}B_{j}), C_{j,k} = G(B_{j}, v_{0}B_{k})$. We pick $M_{1} = G/B_{i}, M_{2} = G/B_{j}, M_{3} = G/B_{k}$ for the convolution and get
\[
p_{13}(p_{12}^{-1}C_{i,j}^{w} \cap p_{23}^{-1}C_{j,k}^{w}) = \{g(B_{i}, w_{0}B_{j}) | g \in G, b \in B_{j}\}.
\]

Now since the length are adding one finds $B_{i}w_{0}B_{j}v_{0}B_{k} = B_{i}(w_{0}v_{0}B_{k})$, as follows
\[
w_{0}B_{j}v_{0}B_{k} = x_{i}[w(t_{i}^{-1}G \cap B)v(t_{j}^{-1}G \cap B)]x_{k}^{-1} \subset x_{i}[wBvB]x_{k} \cap G \subset x_{i}[BwB]x_{k}^{-1} \cap G
\]
\[
= [t_{i}B(x_{i}wvx_{k}^{-1}B)] \cap G = B_{i}w_{0}v_{0}B_{k}
\]

For the last equality, clearly $B_{i}w_{0}v_{0}B_{k} \subset [t_{i}B(x_{i}wvx_{k}^{-1}B)] \cap G$. Assume $[t_{i}B(x_{i}wvx_{k}^{-1}B)] \cap G = \bigcup B_{i}t_{j}B_{k}$ for certain $t \in W$, then clearly $B_{i}t_{j}B_{k} \subset [t_{i}B(x_{i}wvx_{k}^{-1}B)] \cap G \cap [t_{i}Bt^{\pm}B] \cap G$ as this intersection is empty if $t \neq (x_{i}wvx_{k}^{-1})$, the last equality follows.

Then using $Z_{i,j}^{w} = \{g(f_{i} = w_{0}f_{j}, B_{i}, w_{0}B_{j}) \in V \times G/B_{i} \times G/B_{j} | g \in G, f_{i} \in F_{i}, f_{j} \in F_{j}\}$ one concludes by definition that $Z_{i,j}^{w} \ast Z_{j,k}^{v} \subset Z_{i,k}^{sw}$

We have the following corollary whose proof we have to delay until we have introduced the localization to the $T$-fixed point.

Corollary 1.1. For $s \in S, w \in \mathbb{W}$ with $l(sw) = l(w) + 1,$
\[
[Z_{i}^{s}] \ast [Z_{j}^{w}] = [Z_{i,k}^{sw}] \text{ in } Z_{G}^{\leq sw}/Z_{G}^{\leq sw}.
\]

Since $[Z_{i}^{w}] = \sum_{s,t \in I} [Z_{s,t}^{w}]$ for all $v \in \mathbb{W}$, this is equivalent to $i, j, l, k \in I$ we have
\[
[Z_{i,j}^{w}] \ast [Z_{l,k}^{w}] = \delta_{i,l} [Z_{i,k}^{sw}] \text{ in } Z_{G}^{sw}/Z_{G}^{\leq sw}.
\]

1.1.1 Computation of some Euler classes

Definition 4. (Euler class) Let $M$ be a finite dimensional complex $t = Lie(T)$-represenation. Then, we have a weight space decomposition
\[
M = \bigoplus_{\alpha \in Hom_{C}(t, \mathbb{C})} M_{\alpha}, \quad M_{\alpha} = \{m \in M | tm = \alpha(t)m\}.
\]

We define
\[
eu(M) := \prod_{\alpha \in Hom_{C}(t, \mathbb{C})} \alpha^{\dim M_{\alpha}} \in \mathbb{C}[t] = H_{T}^{*}(pt)
\]
For a $T$-variety $X$ and a $T$-fixed point $x \in X$, we define the **Euler class** of $x \in X$ to be

$$eu(X, x) := eu(T_x X),$$

where the $t$-operation on the tangent space $T_x X$ is the differential of the natural $T$-action. Observe, that $eu(T_x^* X) = (-1)^{\dim T_x X} eu(T_x X)$.

Recall from an earlier section the notation $Z^w := m^{-1}(C^w)$. We are particularly interested in the following Euler classes, let $w = w^k x_k, x = x^i x_i, y = y^j x_j \in W, w^k, x^i, y^j \in W$

$$\Lambda_w := eu(E, \phi_w) = eu(T_{\phi^w x_k} E_k), \quad \in H_1^T(pt)$$

$$eu(Z^w, \phi_{x,y}) = (eu(T_{\phi^v x_i y_j} Z^w))^{-1}, \quad \in K := Quot(H_1^T(pt))$$

Remember $F_w := \mu^{-1}(\phi_w) = \mu_k^{-1}(\phi_{w^k x_k}) = w^k F_k, \ F_{x,y} := m^{-1}(\phi_{x,y}) = x^i F_i \cap y^j F_j = F_x \cap F_y$. In particular, we can see them as $t$-representations. We also consider the following $t$-representations

$$n_w := T_{w^k P_k} G/P_k = g \cap u^- = w^k [g \cap x_k U^-]$$

$$m_{x,y} := \frac{n_x}{n_x \cap n_y} = g \cap \frac{x U^-}{x U^- \cap U^-}$$

where $U^- := \text{Lie}(U^+)$ with $U^- \subset B^- := w_0 B$ is the unipotent radical where $w_0 \in W$ is the longest element. Some properties can easily be seen.

1. $n_x = \prod_{\alpha \in \Phi \cap x^{-1}g^-} \alpha$.
2. If $s \in S, x \in W$ such that $x s \in W$, then
   $$eu(n_x) = - eu(n_{xs}), \quad eu(m_{xs,x}) = - eu(m_{xs,x}) = x(\alpha_s)$$
3. If $s \in S, x \in W$ such that $x s \not\in W$, then
   $$n_x = n_{xs}, \quad eu(m_{xs}) = eu(m_{xs,x}) = 0$$

Furthermore, for $s \in S, x \in W, i \in I$ we write set as a shortage

$$Q_x(s) := eu(F_x/F_{x, s}), \quad Q_i(s) := Q_x(s), \quad q_i(s) := \prod_{\alpha \in \Phi_{t, s}(\alpha) \notin \Phi_{t, x}(\alpha) \in \Phi_V} \alpha.$$

for $x = x^i x_i$ with $x^i \in W$ it holds $Q_x(s) = x^i Q_i(s), Q_i(s) = x_i q_i(s)$, i.e.

$$Q_x(s) = x(q_i(s))$$

**Lemma 14.** Let $J = \emptyset$, it holds

1. for $w \in W$
   $$\Lambda_w = eu(F_w \oplus n_w)$$
2. If $s \in S, x \in W, \alpha_s \in \Phi^+$ with $s(\alpha_s) = - \alpha_s$ and $x s \in W$
   $$eu(Z^w, \phi_{x, xs}) = eu(F_{x, xs} \oplus n_x \oplus m_{x, xs}) = x(\alpha_s) Q_x(s)^{-1} \Lambda_x$$
   $$eu(Z^w, \phi_{x, x}) = eu(F_{x, x} \oplus n_x \oplus m_{x, x}) = - eu(Z^w, \phi_{x, xs}).$$
3. If $s \in S, x \in W$ and $x s \not\in W$
   $$eu(Z^w, \phi_{x, xs}) = eu(F_{x, xs} \oplus n_x) = Q_x(s)^{-1} \Lambda_x$$
4. Let $x, w \in W$. Then
   $$eu(Z^w, \phi_{x, sxw}) = eu(F_{x, sxw} \oplus n_x \oplus m_{x, sxw})$$

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proof

(1) We know $\mu_k: E_k \rightarrow G/B_k$, $B_k = G \cap \mathfrak{s}_k \mathfrak{B}$ is a vector bundle, therefore we have a short exact sequence of tangent spaces

$$0 \rightarrow T_{\phi_k} \mathfrak{u}_k = 1 (w^k B_k) \rightarrow T_{\phi_k} E_k \rightarrow T_{w^k B_k} G/B_k \rightarrow 0$$

which is a split sequence of $T$-representations implying the first statement.

ad (3.2) Let $i,j \in I_J$ such that $x^i := xx_i^{-1}, y^j := xsx_j^{-1} \in W$.

(2) If $s^i \in W$ we have that $i = j$ and $Z_{i,i}^s \rightarrow C_{i,i}^S \cong G \times B_i (G \cap \mathfrak{s}_i \mathfrak{P} / \mathfrak{P}_i) / B_i$ is a vector bundle. For $x' \in \{x, xs\}$ we have a short exact sequence on tangent spaces

$$0 \rightarrow F_{x,xs} \rightarrow T_{\phi_{x,xs}^s} Z_{i,i}^s \rightarrow T_{\phi_{x,xs}^s} C_{i,i}^S \rightarrow 0$$

Using the isomorphism $G \times B_i [(x_i \mathfrak{P} / \mathfrak{P}_i) \cap G / B_i] \rightarrow C_{i,i}^S$, $(g, hB_i) \mapsto (gB_i, ghB_i)$ we get

$$\text{eu}(T_{\phi_{x,xs}^s} C_{i,i}^S) = \left\{ \begin{array}{ll}
\text{eu}(T_{(x_i \mathfrak{P} / \mathfrak{P}_i)} G / B_i) & = \text{eu}(n_x) \cdot \text{eu}(m_{x,xs}), \quad x' = x \\
\text{eu}(T_{(x_i \mathfrak{P} / \mathfrak{P}_i)} G / B_i) & = \text{eu}(n_x) \cdot \text{eu}(m_{x,xs}), \quad x' = xs
\end{array} \right.$$ 

It follows $\text{eu}(Z^s, \phi_{x,xs}) = \text{eu}(F_{x,xs}) \cdot \text{eu}(n_x) \cdot \text{eu}(m_{x,xs})$ and $\text{eu}(Z^s, \phi_{x,xs}) = \text{eu}(F_{x,xs} \oplus n_x \oplus m_{x,xs})$.

(3) If $s^i \notin W$ we get $i \neq j$ and $Z_{i,j}^s$ is closed and a vector bundle over $C_{i,j}^s = G / (G \cap \mathfrak{s} \mathfrak{B})$, we get a short exact sequence on tangent spaces

$$0 \rightarrow F_{x,xs} \rightarrow T_{\phi_{x,xs}} Z_{i,j}^s \rightarrow T_{\phi_{x,xs}} C_{i,j}^s \rightarrow 0.$$ 

We obtain $\text{eu}(Z^s, \phi_{x,xs}) = \text{eu}(F_{x,xs}) \text{eu}(n_x)$.

(4) Pick $i,j \in I$ such that $x \in Wx_i, xw \in Wx_j$. We have the short exact sequence

$$0 \rightarrow F_{x,xw} \rightarrow T_{\phi_{x,xw}} Z_{i,j}^w \rightarrow T_{\phi_{x,xw}} C_{i,j}^w \rightarrow 0$$

Then, recall the isomorphism

$$C_{i,j}^w = G_{\phi_{x,xw}} \rightarrow G / (G \cap \mathfrak{s} \mathfrak{B} \cap \mathfrak{xw} \mathfrak{B})$$

$$\phi_{x,xw} \mapsto \mathfrak{U} := e(G \cap \mathfrak{s} \mathfrak{B} \cap \mathfrak{xw} \mathfrak{B})$$

Again we have a short exact sequence

$$0 \rightarrow T_r (G \cap \mathfrak{s} \mathfrak{B}) / (G \cap \mathfrak{s} \mathfrak{B} \cap \mathfrak{xw} \mathfrak{B}) \rightarrow T_r G / (G \cap \mathfrak{s} \mathfrak{B} \cap \mathfrak{xw} \mathfrak{B}) \rightarrow T_r G / (G \cap \mathfrak{s} \mathfrak{B}) \rightarrow 0$$

Together it implies $\text{eu}(Z_{i,j}^w, \phi_{x,xw}) = \text{eu}(F_{x,xw}) \text{eu}(n_x / (n_x \cap n_{xw})) \text{eu}(n_x)$.

\[ \square \]

Corollary 1.2. Let $J = \emptyset, U = \text{Lie}(U) \oplus T$, it holds

(1) If $s \in S, x \in W$ and $s^i \in W$, then $h_{\mathfrak{P}}(s) = h_{\mathfrak{P}}(s)$ and

$$\Lambda_x = (-1)^{1+h_{\mathfrak{P}}(s)} \Lambda_{xs}$$

$$\text{eu}(Z^s, \phi_{x,xs}) = (x(\alpha_s))^{-1-h_{\mathfrak{P}}(s)} \Lambda_x$$

(2) If $s \in S, x \in W$ and $s^i \notin W$

$$\text{eu}(Z^s, \phi_{x,xs}) = x(\alpha_s)^{-h_{\mathfrak{P}}(s)} \Lambda_x$$
proof: This follows from $q_x(s) = x(\alpha_s)^{h_{\mathcal{T}}(s)}$ and if $s \in W$ we have that $i = j$ and $h_{\mathcal{T}}(s) = h_{\mathcal{T}}(s)$. Therefore we get
\[
\begin{align*}
eu(F_x) &= x(\alpha_s)^{h_{\mathcal{T}}(s)} \neu(F_{x,s}) \\
&= (-1)^{h_{\mathcal{T}}(s)(x(s))^{h_{\mathcal{T}}(s)}} \neu(F_{x,s,x}) \\
&= (-1)^{h_{\mathcal{T}}(s)} \neu(F_{x,s})
\end{align*}
\]
Using that $\neu(n_x) = -\neu(n_{xs})$ we obtain $\Lambda_x = (-1)^{1+h_{\mathcal{T}}(s)} \Lambda_{xs} \square$

1.1.2 Localization to the torus fixed points

Now, we come to the application of localization to $T$-fixed points. We remind the reader that $Z$ is a cellular fibration and $E$ is smooth, therefore in both cases the odd ordinary (=singular) cohomology groups vanish for $Z$ and $E$. This implies in particular that $E, Z$ are equivariantly formal, which is (in the case of finitely $T$-fixed points) equivalent to $Z_G$ and $E_G$ are free modules over $H^*_{G}(pt)$.

If we denote by $K$ the quotient field of $H^*_G(pt)$ and for any $T$-variety $X$
\[
H^*_T(X) \to \mathcal{H}_*(X) := H^*_T(X) \otimes H^*_T(pt) K, \quad \alpha \mapsto \alpha \otimes 1.
\]

Lemma 15. (1) $\mathcal{H}_*(E) = \bigoplus_{w \in W} K\psi_w, \quad \mathcal{H}_*(Z) = \bigoplus_{x,y \in W} K\psi_{x,y}$
where $\psi_w = [\{\phi_w\}] \otimes 1, \psi_{x,y} = [\{\phi_{x,y}\}] \otimes 1$.

(2) For every $i \in I, w \in Wx_i$ we have a map $w: \mathcal{E}_i := H^*_G(E_i) \to \mathbb{C}[t]$, via taking the forgetful map composed with the pullback map under the closed embedding $i_w: \{\phi_w\} \to E_i$
\[
\mathcal{E}_i = H^*_G(E_i) \to H^*_T(E_i) \to H^*_T(pt) = \mathbb{C}[t],
\]
we denote the map by $f \mapsto w(f), f \in \mathcal{E}_i, w \in W$. Furthermore, composing the forgetful map with the map from before we get an injective algebra homomorphism
\[
\Theta_i: \mathcal{E}_i \to H^*_T(E_i) \to H^*_T(E_i) \otimes K \cong \bigoplus_{w \in Wx_i} K\psi_w
\]
\[
c \longmapsto \sum_{w \in Wx_i} w(c)\Lambda_{w}^{-1}\psi_w.
\]
We set $\Theta = \bigoplus_{i} \Theta_i: \mathcal{E}_G \to \bigoplus_{w \in W} K\psi_w$.

proof:

(1) This is GKM-localization theorem for $T$-equivariant cohomology, for a source also mentioning the GKM-theorem for $T$-equivariant Borel-Moore homology see for example [Bri00], Lemma 1.

(2) This is [EG98], Thm 2, using the equivariant cycle class map to identify $T$-equivariant Borel-Moore homology of $E$ with the $T$-equivariant Chow ring.
The $\mathcal{W}$-operation on $\mathcal{E}_G$: Recall that the ring of regular functions $\mathbb{C}[t]$ on $t = \text{Lie}(T)$ is a left $W$-module and a left $\mathcal{W}$-module with respect to $w \cdot f(t) = f(w^{-1}tw)$, $w \in \mathcal{W}(\supset W)$. The from $W$ to $\mathcal{W}$ induced representation is given by

$$\text{Ind}_{W}^{\mathcal{W}} \mathbb{C}[t] = \bigoplus_{i \in I} x^{-1}_i \mathbb{C}[t],$$

for $w \in \mathcal{W}, i \in I$ the operation of $w$ on $x^{-1}_i \mathbb{C}[t]$ is given by

$$x^{-1}_i \mathbb{C}[t] \rightarrow x^{-1}_{iw^{-1}} \mathbb{C}[t]$$

$$x^{-1}_i f \rightarrow wx^{-1}_i f$$

where we use that $wx^{-1}_i W = x^{-1}_{iw^{-1}} W$.

Now, we identify $\mathcal{E}_G = \bigoplus_{i \in I} \mathcal{E}_i$ with the left $\mathcal{W}$-module $\text{Ind}_{W}^{\mathcal{W}} \mathbb{C}[t]$ via $\mathcal{E}_i = x^{-1}_i \mathbb{C}[t]$.

Furthermore, we have the (left) $\mathcal{W}$-representation on $\bigoplus_{x \in \mathcal{W}} K(\Lambda^{-1}_x \psi_x)$ defined via

$$w(k(\Lambda^{-1}_x \psi_x)) := k(\Lambda^{-1}_{xw^{-1}} \psi_{xw^{-1}}), \quad k \in K, w \in \mathcal{W}.$$

**Lemma 16.** The map $\Theta : \mathcal{E}_G \rightarrow \bigoplus_{x \in \mathcal{W}} K(\Lambda^{-1}_x \psi_x)$ is $\mathcal{W}$-invariant.

**proof:** Let $w \in \mathcal{W}$, we claim that there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}_G & \xrightarrow{\Theta} & \bigoplus_{x \in \mathcal{W}} K(\Lambda^{-1}_x \psi_x) \\
\downarrow{w} & & \downarrow{w} \\
\mathcal{E}_G & \xrightarrow{\Theta} & \bigoplus_{x \in \mathcal{W}} K(\Lambda^{-1}_x \psi_x)
\end{array}
$$

$c \rightarrow \sum_{x \in \mathcal{W}} i^*_x(c) \Lambda^{-1}_x \psi_x$

$$w \cdot c \rightarrow \sum_{x \in \mathcal{W}} i^*_x(c) \Lambda^{-1}_x \psi_x$$

We need to see $i^*_x(w \cdot c) = i^*_x(c)$. Let $xw \in WX_i$, $x \in WX_{i, w}^{-1}$ This means that the diagram

$$
\begin{array}{ccc}
\mathcal{E}_i & \xrightarrow{w} & \mathcal{E}_{i, w}^{-1} \\
\downarrow{i^*_x} & & \downarrow{H^*_T(pt)} \\
\mathcal{E}_i & \xrightarrow{w} & \mathcal{E}_{i, w}^{-1}
\end{array}
$$

is commutative. But it identifies with

$$
\begin{array}{ccc}
x^{-1}_i \mathbb{C}[t] & \xrightarrow{w} & x^{-1}_{iw^{-1}} \mathbb{C}[t] \\
\downarrow{xw} & & \downarrow{xw} \\
\mathbb{C}[t] & \xrightarrow{x} & \mathbb{C}[t]
\end{array}
$$

The diagram is commutative. $\Box$

**Remark.** From now on, we use the following description of the $\mathcal{W}$-operation on $\mathcal{E}_G$. We set $\mathcal{E}_i = \mathbb{C}[t], i \in I$. Let $w \in \mathcal{W}\,$

$$w(\mathcal{E}_i) = \mathcal{E}_{i, w}^{-1}, \quad \mathcal{E}_i = \mathbb{C}[t] \ni f \mapsto w \cdot f \in \mathbb{C}[t] = \mathcal{E}_{i, w}^{-1}.$$

The isomorphism $p := \bigoplus_{i \in I} p_i$ defined by

$$p_i : \mathbb{C}[t] \rightarrow x^{-1}_i \mathbb{C}[t]$$

$$f \mapsto x^{-1}_i(f)$$

gives the identification with the induced representation $\text{Ind}_{W}^{\mathcal{W}} \mathbb{C}[t]$ which we described before.
1.1.3 Calculations of some equivariant multiplicities

In some situation one can actually say something on the images of algebraic cycle under the GKM-localization map, recall the

**Theorem 1.1.** (multiplicity formula, [Bri00], section 3) Let $X$ equivariantly formal $T$-variety with a finite set of $T$-fixpoints $X^T$, by the localization theorem,

$$[X] = \sum_{x \in X^T} \Lambda^X_x \{x\} \in H^*_T(X) \otimes K$$

where $\Lambda^X_x \in K$. If $X$ is rationally smooth in $x$, then $\Lambda^X_x \neq 0$ and $(\Lambda^X_x)^{-1} = eu(x, x) \in H^*_T(X)$, $n = \dim_\mathbb{C}(X)$.

**Remark.** It holds for any $w \in \mathbb{W}

$$[Z^w] = \sum_{i,j \in I} [Z^w]_{i,j}.$$ 

Especially $1 = [Z^e] = \sum_{i \in I} [Z^e]_{i,i}$ is the unit and $1_i = [Z^e]_{i,i}$ are idempotent elements, $1_i \ast 1_j = 0$ for $i \neq j$, $[Z_{i,j}] = 1_i \ast \{Z\} \ast 1_j$. In particular, for $s \in S$ by lemma 11, we have

$$[Z^s] = \sum_{i,j \in I: i = i} [Z^s]_{i,i} + \sum_{i,j \in I: j \neq i} [Z^s]_{i,j}.$$ 

By the multiplicity formula we have

$$[Z^w]_{i,j} = \begin{cases} \sum_{x \in W} \Lambda^w_{xx,xx,xx} \psi_{xx,xx,xx}, & \text{if } i = i \\ \sum_{x \in W} \Lambda^w_{xx,xx,xx} \psi_{xx,xx,xx}, & \text{if } i \neq i \end{cases}$$

with $\Lambda^w_{y,z} = (eu(Z^w_{i,j}, \phi_{y,z}))^{-1}$, for all $y, z \in \mathbb{W}$ as above

$$[Z^w]_{i,j} = \begin{cases} \sum_{x \in W} \Lambda^w_{xx,xx,xx} \psi_{xx,xx,xx}, & \text{if } iw = j \\ 0, & \text{if } iw \neq j \end{cases}$$

with $\Lambda^w_{xx,xx,xx} = (eu(Z^w_{i,iw}, \phi_{xx,xx,xx}))^{-1}$ for all $x \in W$.

1.1.4 Convolution on the fixed points

The following key lemma on convolution products of $T$-fixed points

**Lemma 17.** For any $w, x, y \in \mathbb{W}$ it holds

$$\psi_{x, w} \ast \psi_{y} = \Lambda_{w} \psi_{x}, \quad \psi_{x, w} \ast \psi_{y, w} = \Lambda_{w} \psi_{x, y}$$

**proof:** We take $M_1 = M_2 = M_3 = E$ and $Z_{1,2} := \{\phi_{x, w} = ((0, xB), (0, wB)) \subset E \times E, Z_{2,3} := \{\phi_{w', y} \subset E \times E$, then the set theoretic convolution gives

$$\{\phi_{x, w} \circ \phi_{w'}\} = \begin{cases} \{\phi_{x, y}\}, & \text{if } w = w' \\ \emptyset, & \text{if } w \neq w' \end{cases}$$

Similar, take $M_1 = M_2 = E, M_3 = pt$, $Z_{1,2} := \{\phi_{x, w}\}, Z_{2,3} = \phi_{w'} \times pt$, then

$$\{\phi_{x, w} \circ \phi_{w}\} = \begin{cases} \{\phi_{x}\}, & \text{if } w = w' \\ \emptyset, & \text{else} \end{cases}$$

To see that we have to multiply with $\Lambda_{w}$, we use the following proposition
Proposition 3. (see [CG97], Prop. 2.6.42, p.109) Let $X_1 \subset M, i = 1, 2$ be two closed (complex) submanifolds of a (complex) manifold with $X := X_1 \cap X_2$ is smooth and $T_x X_1 \cap T_x X_2 = T_x X$ for all $x \in X$. Then, we have

$$[X_1] \cap [X_2] = e(T) \cdot [X]$$

where $T$ is the vector bundle $T_xM/(T_x X_1 + T_x X_2)$ on $X$ and $e(T) \in H^*(X)$ is the (non-equivariant) Euler class of this vector bundle, $\cap : H^*_B(X_1) \times H^*_B(X_2) \to H^*_B(X)$ is the intersection pairing (cp. Appendix, or [CG97], 2.6.15) and $\cdot$ on the right hand side stands for the $H^*(X)$-operation on the Borel-Moore homology (introduced in [CG97], 2.6.40).

Set $E_T := E \times T_{ET}, (\phi_x)_{ET} := \{\phi_x\} \times T_{ET}(\cong E \times T = BT)$. We apply the proposition for $M = E_T^0$, $X_1 := (\phi_x)_{ET} \times (\phi_w)_{ET} \times E_T, X_2 := E_T \times (\phi_x)_{ET} \times (\phi_w)_{ET}, X_1 \cap X_2 \cong \{\phi_{x,w}\} (\cong BT)$, then $T = (\phi_x)_{ET} \times T_{ET}$ and (the non-equivariant) Euler class is the top Chern class of this bundle which is the $T$-equivariant top Chern class of the constant bundle $T_{\phi_w} E$ on the point $\{\phi_{x,w}\}$. Since $T_{\phi_w} E = \bigoplus \lambda C_\lambda$ for one-dimensional $T$-representations $C_\lambda$ with $t \cdot c := \lambda(t)c, t \in T, c \in C = C_\lambda$. It holds

$$c^T_{\text{top}}(T_{\phi_w} E) = \prod \lambda c_1^T(C_\lambda) = \prod \lambda = \Lambda_w.$$}

Secondly, apply the proposition with $M = E_T^0 \times (pt)T, X_1 := (\phi_x)_{ET} \times (\phi_w)_{ET} \times (pt)T, X_2 := E_T \times (\phi_x)_{ET} \times (pt)T$, to see again $e(T) = \Lambda_w$.

Now we can give the missing proof of Corollary 1.1.

**Proof of Corollary 1.1:** By the lemma 13 we know that there exists a $c \in \mathcal{E}_G$ such that $[Z_{i,j}^s]^* [Z_{j,k}^w] \in \mathcal{Z}_{G}^{sw}/\mathcal{Z}_{G}^{\leq sw}$. We show that $c = 1$. We pass with the forgetful map to $T$-equivariant Borel-Moore homology and tensor over $K = \text{Quot}(H^*_T(pt))$ and write $[Z_{i,j}^T], x \in W, s, t \in I$ for the image of the same named elements. Let $i,j,k \in I$ with $x_jw k^{-1} \in W$.

$$[Z_{i,j}^s]^* [Z_{j,k}^w] = \left( \sum_{x \in W} \Lambda_{x,x,x,s}^s \psi_{x,x,x,s} + \Lambda_{x,x,x,x,s}^s \psi_{x,x,x,s} \right)^*$$

$$= \left( \sum_{x \in W} \Lambda_{x,x,x,s}^w \psi_{x,x,x,w} \right)^*$$

Now, this has to be equal to $c \sum_{x \in W} \Lambda_{x,x,x,s}^s \psi_{x,x,x,s}$ in $\mathcal{Z}_{G}^{sw}/\mathcal{Z}_{G}^{\leq sw}$. Comparing coefficients at $x$ gives

$$c = \frac{\text{eu}(E_j, \phi_{x,x,s}) \text{eu}(Z_{i,k}^s, \phi_{x,x,x,s})}{\text{eu}(Z_{i,j}^s, \phi_{x,x,x,s})} \frac{\text{eu}(Z_{j,k}^w, \phi_{x,x,x,s})}{\text{eu}(Z_{i,j}^s, \phi_{x,x,x,s})}$$

$$= \frac{\text{eu}(g \cap x, U^*g \cap x, U^*g \cap x, U)}{\text{eu}(g \cap x, U^*g \cap x, U^*g \cap x, U^*g \cap x, U)} \cdot \prod_{l=1}^{r} \frac{\text{eu}(V(l), \phi_{x,x,s}^l \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x)}{\text{eu}(V(l), \phi_{x,x,s}^l \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x)}$$

$$= \frac{\text{eu}(g \cap x, U^*g \cap x)}{\text{eu}(g \cap x, U^*g \cap x)} \cdot \prod_{l=1}^{r} \frac{\text{eu}(V(l), \phi_{x,x,s}^l \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x)}{\text{eu}(V(l), \phi_{x,x,s}^l \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x, \phi(l) \cap x)}$$

$$= 1.$$
That for each $x$ and each $l \in \{1, \ldots, r\}$ the big two fraction in the product are equal to 1 is a consequence of the following lemma. \hfill □

**Lemma 18.** Let $T \subset \mathcal{B} \subset \mathbb{G}$ a maximal torus in a Borel subgroup in a reductive group (over $\mathbb{C}$), $F \subset \text{Lie}(\mathbb{G}) = \mathcal{G}$ a $\mathcal{B}$-subrepresentation. Let $(\mathcal{W}, \mathcal{S})$ be the Weyl group for $(\mathbb{G}, T)$. Let $w \in \mathcal{W}, s \in \mathcal{S}$ such that $l(sw) = l(w) + 1$, then it holds for any $x \in \mathcal{W}$

$$x(\frac{sF}{F} \oplus \frac{wF}{F}) \cong x(\frac{swF}{F} \cap \frac{swF}{F}).$$

In particular, this holds also for $F = u^\cdot$.

**proof:** Let $\Phi_F := \{\alpha \in \text{Hom}(t, \mathbb{C}) \mid F\alpha \neq 0\} \subset \Phi$, $\Phi^+(y) := \Phi^+ \cap y(\Phi^-), \Phi_F^-(y) := \Phi_F \cap \Phi^-(y), y \in \mathcal{W}$ where $\Phi, \Phi^+, \Phi^-$ are the set of roots (of $T$ on $\mathcal{G}$), positive roots, negative roots respectively.

The assumption $l(sw) = l(w) + 1$ implies $\Phi_F^+(sw) = s\Phi_F^+(w) \cup \Phi_F^-(s)$ and for $\Phi_F^-(y) := -\Phi_F^+(y)$,

$\Phi_F(y) := \Phi_F^+(y) \cup \Phi_F^-(y) = \Phi_F \setminus (\Phi_F \cap y(\Phi_F))$ it holds $\Phi_F(sw) = s\Phi_F(w) \cup \Phi_F(s)$ and for any $x \in \mathcal{W}$ it holds $x\Phi_F(sw) = x(s\Phi_F(w) \cup \Phi_F(s))$. Now, the weights of $x(\frac{sF}{F} \oplus \frac{wF}{F})$ are $x\Phi_F(sw)$, the weights of $x(\frac{swF}{F} \cap \frac{swF}{F})$ are $x(s\Phi_F(w) \cup \Phi_F(s))$. \hfill □

**1.2 Generators for $\mathcal{Z}_G$**

Let $J = \emptyset$. Recall, we denote the right $\mathcal{W}$-operation on $I = W \setminus \mathcal{W}$ by $(i, w) \mapsto iw, i \in I, w \in \mathcal{W}$. For $i \in I$ we set $\mathcal{E}_i := H^i_G(E_i) = \mathbb{C}[t] = \mathbb{C}[x_i(1), \ldots, x_i(m)],$ we write

$$w(\alpha_s) = w(\alpha_s(x_{iw-1}(1), \ldots, x_{iw-1}(m))) \in \mathcal{E}_{iw-1}$$

for the element corresponding to the root $w(\alpha_s), s \in \mathcal{S}, w \in \mathcal{W}$ without mentioning that it depends on $i \in I$.

We define a collection of elements in $\mathcal{Z}_G$

$$1_i := [Z_{i,i}^e],$$

$$z_i(t) := x_i(t) \in \mathcal{Z}_{G}^{\leq e} < \mathcal{Z}_G$$

$$\sigma_i(s) := [Z_{i,i}^s] \in \mathcal{Z}_{G}^{\leq s},$$

where we use that $\mathcal{E}_i \subset \mathcal{Z}_{G}^{\leq e} \subset \mathcal{Z}_G$ and the degree of $x_i(t)$ is 2 in $H^G_{[e]}(Z)$, see Lemm 6 and the definition of the grading (just before theorem 2.1). It is also easy to see that $1_i \in H^G_{[0]}(Z)$ because $\text{deg} 1_i = 2e_i - 2 \dim Z_{i,i}^e = 0$. Furthermore, the degree of $\sigma_i(s)$ is

$$e_{is} + e_i - 2 \dim Z_{i,is}^s = \begin{cases} 2 \deg q_i(s) - 2, & \text{if } is = i \\ 2 \deg q_i(s), & \text{if } is \neq i. \end{cases}$$

Recall $\mathcal{Z}_G \hookrightarrow \text{End}(\mathcal{E}_G) = \text{End}(\bigoplus_{i \in I} \mathcal{E}_i)$ from [VV11a], remark after Prop.3.1, p.12. Let us denote by $1_i, z_i(t), \sigma_i(s)$ the images of $1_i, z_i(t), \sigma_i(s)$.  

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Proposition 4. Let $k \in I$, $f \in \mathcal{E}_k$, $\alpha_s \in \Phi^+$ be the positive root such that $s(\alpha_s) = -\alpha_s$. It holds

$$
\tilde{\sigma}_i(f) := \frac{1}{\alpha_s}[s] - \frac{f}{\alpha_s} \quad \text{for } f \in \mathcal{E}_k \ni f \rightarrow \sum_{w \in W_{x_i}} w(f)\Lambda^{-1}_w \psi_w
$$

We write $\delta_i := \frac{s(1) - s(0)}{\alpha_s}$, it is the BGG-operator from [Dem73], i.e. for $s = i, f \in \mathcal{E}_i$, $\sigma_i(s)(f) = q_i(s)\delta_i(f)$.

**proof:** Consider the following two maps

- $\Theta : \mathcal{E}_G \rightarrow \mathcal{E}_T \rightarrow \mathcal{E}_T \otimes K \rightarrow \bigoplus_{w \in W} K \psi_w$

- $\Theta : \mathcal{E}_G \rightarrow \mathcal{E}_T \rightarrow \mathcal{E}_T \otimes K \rightarrow \bigoplus_{w \in W} K \psi_w

$C : \bigoplus_{w \in W} K \psi_w \rightarrow \bigoplus_{w \in W} K \psi_w$

$\psi_w \mapsto \sum_{w \in W_{x_i}} [Z^s_{x_i}]* \psi_w$

To calculate $\sum_{w \in W_{x_i}} [Z^s_{x_i}]* \psi_w$ it is enough to calculate $\sum_{w \in W_{x_i}} [Z^s_{x_i}]* \Theta(f) = C(\Theta(f))$ because $\Theta$ is an injective algebra homomorphism.

$$C \Theta(f) = \left\{ \begin{array}{ll}
\delta_{i,s,k} \sum_{w \in W_{x_i}} [w(f)] \Lambda_w^s \psi_w, & \text{if } i = is \\
\delta_{i,s,k} \sum_{w \in W_{x_i}} [w(s)] \Lambda_w^s \psi_w, & \text{if } i \neq is
\end{array} \right.$$

Now, recall,

1. If $i = is = k$

$$C \Theta(f) = \sum_{w \in W_{x_i}} w[q_i(s)] \frac{s(f) - f}{\alpha_s} \Lambda_w^{-1} \psi_w

= \Theta(q_i(s)) [s(f) - f]/\alpha_s \right)$$

Once we identify $\mathcal{E}_k = \mathbb{C}[t]$, $k \in I$, we see that $\sigma_i(s) : \mathcal{E}_G \rightarrow \mathcal{E}_G$ is the zero map on the $k$-th summand, $k \neq i$ and on the $i$-th summand

$\mathbb{C}[t] \rightarrow \mathbb{C}[t]$

$f \mapsto q_i(s) \frac{s(f) - f}{\alpha_s}$
If the root system spanned by $\alpha_s, \alpha_t$ is of type $B_2$ (i.e. $\alpha_t \alpha_s = \alpha_s \alpha_t$ is the minimal relation), then for every $i \in I$ such that $i = i = 1 = 2$, it holds $\delta_i(s), h_i(t) \in \{0, 1, 2\}$.

(G2) If the root system spanned by $\alpha_s, \alpha_t$ is of type $G_2$ (i.e. $\alpha_t \alpha_s = \alpha_s \alpha_t$ is the minimal relation), then for every $i \in I$ such that $i = i = 1 = 2$ it holds $h_i(s) = 0 = h_i(t)$.

Then the generalized quiver Hecke algebra for $(\mathbb{G}, \mathbb{B}, U = \text{Lie}(U)^{\oplus r}, H, V)$ is the $\mathbb{C}$-algebra with generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = rk(T), i \in I, \quad \sigma_i(s), s \in S, i \in I$$

and relations

$$(1) \ (\text{orthogonal idempotents})$$

$$1_i 1_j = \delta_{i,j} 1_i,$$

$$1_i z_i(t) 1_i = z_i(t),$$

$$1_i \sigma_i(s) 1_{i_S} = \sigma_i(s)$$
(2) (polynomial subalgebras)
\[ z_i(t)z_i(t') = z_i(t')z_i(t) \]

(3) (relation implied by \( s^2 = 1 \))
\[ \sigma_i(s)\sigma_{is}(s) = \begin{cases} 0, & \text{if } i = i, h_i(s) \text{ is even} \\ -2\alpha_s h_i(s)^{-1}\sigma_i(s), & \text{if } i = i, h_i(s) \text{ is odd} \\ (-1)^{h_{is}(s)}\alpha_s h_i(s) + h_{is}(s), & \text{if } i \neq i \end{cases} \]

(4) (straightening rule)
\[ \sigma_i(s)z_i(t) - s(z_i(t))\sigma_i(s) = \begin{cases} c_i(s,t), & \text{if } i = i \\ 0, & \text{if } i \neq i. \end{cases} \]

(5) (braid relations)
Let \( s, t \in S, st = ts, \) then
\[ \sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s) \]

Let \( s, t \in S \) not commuting such that \( x := sts \cdots = tst \cdots \) minimally, \( i \in I. \) There exists explicit polynomials \( \{Q_w\}_{w<x} \) in \( \alpha_s, \alpha_t \in \mathbb{C}[t] \) such that
\[ \sigma_i(sts \cdots) - \sigma_i(tst \cdots) = \sum_{w<x} Q_w \sigma_i(w) \]

(observe that for \( w < x \) there exists just one reduced expression).

**proof:** For the convenience of the reader who wants to check the relations for the generators of \( \mathbb{Z}_G, \)
we include the detailed calculations. (1), (2) are clear. Let always \( f \in \mathbb{C}[t] \cong \mathcal{E}_{is}. \) We will use as shortcut \( \delta_s(f) := \frac{sf(t) - f(t)}{\alpha_s} \) and use that these satisfy the usual relations of BGG-operators (cp. [Dem73]).

(3) If \( i = i, \) then
\[ \sigma_i(s)\sigma_i(s)(f) = \alpha_s^{h_i(s)}\delta_s(\alpha_s^{h_i(s)}\delta_s(f)) = \alpha_s^{h_i(s)}\delta_s(\alpha_s^{h_i(s)}\delta_s(f)) = \left[(-1)^{h_i(s)} - 1\right]\alpha_s^{h_i(s)^{-1}}\sigma_i(s)(f). \]

If \( i \neq i, \) then
\[ \sigma_i(s)(f)\sigma_{is}(s) = \alpha_s^{h_i(s)}s(\alpha_s^{h_i(s)}s(f))s(f) = (-1)^{h_{is}(s)}\alpha_s^{h_i(s) + h_{is}(s)}f. \]

(4) (straightening rule)
The case \( i \neq i \) is clear by definition. Let \( i = i, \) then the relation follows directly from the product rule for BGG-operators, which states \( \delta_s(xf) = \delta_s(x)f + s(x)\delta_s(f), \) \( x, f \in \mathbb{C}[t]. \)

(5) (braid relations)
\( s, t \in S, st = ts, f \in \mathbb{C}[t], \) to prove
\[ \sigma_i(s)\sigma_{is}(t)(f) = \sigma_i(t)\sigma_{it}(s)(f) \]
we have to consider the following four cases. We use the following: \( t(\alpha_s) = \alpha_s, s(\alpha_t) = \alpha_t, h_i(s) = h_{it}(s), h_i(t) = h_{is}(t), \delta_s(\alpha_t^{h_i(t)}) = 0 = \delta_t(\alpha_s^{h_i(s)}). \)

1. \( i = i, it = i, \) use \( \delta_i \delta_t = \delta_t \delta_i \)
\[ \sigma_i(t)\sigma_i(s)(f) = \alpha_t^{h_i(t)}\delta_i(\alpha_s^{h_i(s)}\delta_s(f)) = \alpha_s^{h_i(s)}\alpha_t^{h_i(t)}\delta_i(f) = \alpha_s^{h_i(s)}\alpha_t^{h_i(t)} \delta_i\delta_s(f) = \alpha_s^{h_i(s)}\delta_i(\alpha_t^{h_i(t)}\delta_i(f)) = \sigma_i(s)\sigma_i(t)(f) \]
2. \(i = i, \text{ it } \neq i\), use \(\delta_s t = t \delta_s\)

\[
\sigma_i(t)\sigma_{it}(s)(f) = \alpha_t^{h_i(t)} t(\alpha_s^{h_it(s)} \delta_s(f)) = \alpha_t^{h_i(t)} \alpha_s^{h_is(s)} t \delta_s(f) \\
= \alpha_t^{h_i(t)} \alpha_s^{h_is(s)} t \delta_s(t(f)) \\
= \alpha_t^{h_s(s)} \alpha_s^{h_is(t)} t(f) = \sigma_i(s) \sigma_{it}(t)(f)
\]

3. \(i \neq i, \text{ it } = i\), follows by symmetry from the last case.

4. \(i \neq i, \text{ it } \neq i\).

\[
\sigma_i(t)\sigma_{it}(s)(f) = \alpha_t^{h_i(t)} t(\alpha_s^{h_is(s)} s(f)) = \alpha_t^{h_i(s)} s(\alpha_t^{h_is(t)} t(f)) \\
= \sigma_i(s) \sigma_{it}(t)(f).
\]

Let \(st \neq ts\). There are three different possibilities, either

(A) \(sts = tst\)

(B) \(stst = tsts\)

(C) \(stst = tsts\)

We write \(Stab_i := \{ w \in \langle s, t \rangle \mid iw = i \}\). For each case we go through the subgroup lattice to calculate explicitly the polynomials \(Q_w\).

(A) \(sts = tst\): \(\langle s, t \rangle \cong S_3, s(\alpha_t) = t(\alpha_s) = \alpha_s + \alpha_t\). We have five (up to symmetry between \(s\) and \(t\)) subgroups to consider. Always, it holds

\[
h_{ist}(t) = h_{it}(s), h_{ist}(s) = h_i(t), h_{its}(t) = h_i(s)
\]

which implies an equality which we use in all five cases

\[
\alpha_s^{h_is(s)}(\alpha_t^{h_is(t)}) s t(\alpha_s^{h_is(t)}) = \alpha_s^{h_is(s)}(\alpha_s + \alpha_t)^{h_{ist}(s)} \alpha_t^{h_{ist}(s)} \\
= \alpha_t^{h_i(t)} t(\alpha_s^{h_is(t)}) t s(\alpha_t^{h_{ist}(t)})
\]

A1. \(Stab_i = \langle s, t \rangle\), this implies \(h_i(s) = h_i(t) = h\) by definition \((x_i(\alpha_s) \in \Phi_{V(k)}\) if and only if \(x_{it}(\alpha_s) = x_i(\alpha_t + \alpha_t) = x_{is}(\alpha_t) \in \Phi_{V(k)} \Leftrightarrow x_i(\alpha_t) \in \Phi_{V(k)}\) and as a consequence we get

\[
\alpha_t^{h_i(t)} t(\alpha_s^{h_is(t)}) = 0.
\]

This simplifies the equation to

\[
\sigma_i(s) \sigma_i(t)(s) - \sigma_i(t) \sigma_{it}(s) = \delta_s(\alpha_t^{h_i(t)} \delta(\alpha_t^{h_is(t)})) \sigma_i(s) - \delta_t(\alpha_t^{h_i(t)} \delta(\alpha_t^{h_is(t)})) \sigma_i(t)
\]

note that \(Q_s := \delta(\alpha_t^{h_i(t)} \delta(\alpha_t^{h_is(t)}))\), \(Q_t := -\delta_t(\alpha_t^{h_i(t)} \delta(\alpha_t^{h_is(t)}))\) are polynomials in \(\alpha_s, \alpha_t\).

A2. \(Stab_i = \langle s \rangle\) (analogue \(Stab_i = \langle t \rangle\)). It holds \(ist = its\). We use in this case

\[
h_{ist}(t) = h_i(t), h_{ist}(s) = h_{it}(s), h_{its}(t) = h_{i}(s).
\]

\[
\sigma_i(s) \sigma_i(t)(s) s(\alpha_t^{h_i(t)} t(t) - \sigma_i(t) \sigma_{it}(s)) s(I(t))(f) \\
= \alpha_s^{h_is(s)} s(\alpha_t^{h_is(t)} s(t)) = \alpha_s^{h_is(s)} s(\alpha_t^{h_is(t)} s(t)) \\
= -\alpha_t^{h_i(t)} t(\alpha_s^{h_is(t)} s(t)) = 0
\]

Since \(st \delta_s = \delta_s t\) and \(\delta_s(\alpha_t^{h_i(t)} t(\alpha_t^{h_is(t)})) = 0\).

A3. \(Stab_i = \langle s, t \rangle\), then \(ist = is, its = it\).

\[
\sigma_i(s) \sigma_{is}(s)(f) - \sigma_i(t) \sigma_{it}(s)(f) \\
= \alpha_s^{h_is(s)} s(\alpha_t^{h_is(t)} s(t)) = \alpha_s^{h_is(s)} s(\alpha_t^{h_is(t)} s(t)) \\
= \alpha_t^{h_i(t)} s(\alpha_s^{h_is(s)} s(t)) = \alpha_t^{h_i(t)} s(\alpha_s^{h_is(s)} s(t))
\]

using \(t \delta_s t = s \delta_s t\).
A4. \( Stab_i = \{1\} \) (and the same for \( Stab_i = \{st\} \))

\[
\begin{align*}
\sigma_i(s)\sigma_i(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_i(s)\sigma_{ists}(t) &= \alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) sts - \alpha_t^{h(t)} t(\alpha_s^{h(s)}) tst(\alpha_t^{h(t)}) tst \\
&= 0
\end{align*}
\]

(B) \( stst = tsts: \quad < s, t > \cong D_4(\text{order is 8}), \)

\[
\begin{align*}
t(\alpha_s) &= \alpha_s + \alpha_t, \quad st(\alpha_s) = \alpha_s + \alpha_t, \quad tst(\alpha_s) = \alpha_s \\
s(\alpha_t) &= 2\alpha_s + \alpha_t, \quad ts(\alpha_t) = 2\alpha_s + \alpha_t, \quad sts(\alpha_t) = \alpha_t.
\end{align*}
\]

Here we have to consider ten different cases because \( D_4 \) has ten subgroups. It always holds the following

\[
h_{stst}(s) = h_i(s), h_{ists}(t) = h_{is}(t), h_{ist}(s), h_{tist}(t) = h_i(t)
\]

which implies

\[
\alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) s(\alpha_t^{h(t)}) = \alpha_t^{h(t)} t(\alpha_s^{h(s)}) ts(\alpha_t^{h(t)}) ts(\alpha_t^{h(t)}) st(\alpha_s^{h(s)})
\]

This will be used in all cases, it is particular easy to see that for

\[
Stab_i = \{1\}, \quad Stab_i = \{ts, st, stst\}, \quad Stab_i = \{1, stst\}
\]

we obtain that the difference is zero from the above equality. Let us investigate the other cases. Furthermore, the following is useful to notice

\[
\delta_s(t(\alpha_s)^s) = 0, \quad \delta_t(s(\alpha_t)^t) = 0
\]

B1. \( Stab_i = \{s, t\} \). We prove the following

\[
\begin{align*}
\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)(f) &= \Psi_{stst}(\sigma_i(s)\sigma_i(t)) + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) t(\alpha_t^{h(t)}) ts(\alpha_t^{h(t)}) tsts(\alpha_t^{h(t)}) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) \delta_{tsts}(\alpha_t^{h(t)}) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) t(\alpha_s^{h(s)}) ts(\alpha_t^{h(t)}) \delta_{stst}(\alpha_t^{h(t)}) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) \delta_{stst}(\alpha_t^{h(t)})
\end{align*}
\]

with \( Q_{st} = \delta_s(\alpha_t^{h(t)}) \delta_t(\alpha_s^{h(s)}) s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) t(\alpha_t^{h(t)}) ts(\alpha_t^{h(t)}) = Q_{ts} \) is a polynomial in \( \alpha_s, \alpha_t \). By a long direct calculation (applying the product rule for the \( \delta_s \) several times

\[
\begin{align*}
\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)(f) &= \alpha_t^{h(t)} \delta_t(\alpha_s^{h(s)} s(\alpha_t^{h(t)}) t(\alpha_s^{h(s)}) tsts(\alpha_t^{h(t)})) \delta_t(f) \\
&\quad + [\alpha_s^{h(s)} s(\alpha_t^{h(t)}) t(\alpha_s^{h(s)}) \delta_t(\alpha_s^{h(s)} s(\alpha_t^{h(t)}))] \delta_t(f) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) t(\alpha_s^{h(s)}) ts(\alpha_t^{h(t)}) \delta_{stst}(f) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) \delta_{stst}(f)
\end{align*}
\]

We have a look at the polynomials occurring in front of the \( \delta_w \):

\[
w = t: \quad \text{by the product rule}
\]

\[
\begin{align*}
\alpha_s^{h(s)} \delta_s(\alpha_t^{h(t)}) \delta_t(\alpha_s^{h(s)} s(\alpha_t^{h(t)})) &= \alpha_s^{h(s)} \delta_s(\alpha_t^{h(t)} t(\alpha_s^{h(s)} s(\alpha_t^{h(t)}))) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) \delta_t(\alpha_s^{h(s)} s(\alpha_t^{h(t)})) + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) \delta_t(\alpha_s^{h(s)} s(\alpha_t^{h(t)})) \\
&\quad + \alpha_s^{h(s)} s(\alpha_t^{h(t)}) st(\alpha_s^{h(s)}) \delta_{sts}(\alpha_t^{h(t)})
\end{align*}
\]

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\[ w = st : \]
\[
\alpha_s^{h_i(s)}(\alpha_t^{h_i(t)}) s\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})) = \\
\alpha_s^{h_i(s)}(\alpha_t^{h_i(t)}) s\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + \\
\alpha_s^{h_i(s)} (\alpha_t^{h_i(t)})\delta_s(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) = \\
\alpha_s^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)}) s(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + \\
= \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + s(\alpha_t^{h_i(t)})\delta_s(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})) + t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}))]
\]
\[
= \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}))
\]
using \( s(\delta_s(\alpha_t^{h_i(t)})) = \delta_s(\alpha_t^{h_i(t)}) \) and \( s(\delta_s(\alpha_t^{h_i(t)})) = -\delta_s(\alpha_t^{h_i(t)}) \).

\[ w = tst : \]
\[
\alpha^{h_i(s)}(\alpha_t^{h_i(t)}) s\delta_t(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}))) = 0
\]
Now, look at \( \sigma_i(s)\sigma_i(t)(f) = \alpha^{h_i(s)}(\alpha_t^{h_i(t)})\delta_t(f) + \alpha^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}))) \), which implies
\[
\alpha^{h_i(s)}(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) = Q_s st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) = 0
\]
replace the previous expression and compare coefficients in front of \( \delta_t(f) \) again gives the polynomial
\[
\alpha^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}))) = \alpha^{h_i(s)}(\alpha_t^{h_i(t)})\delta_s(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})))
\]
We conclude
\[
\sigma_i(s)\sigma_i(t)\sigma_i(t) - \sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s) = Q_s st(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) = 0
\]
\[
= \sigma_i(s)\sigma_i(t)\sigma_i(t)\sigma_i(t) - \sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)
\]
Since \( \delta_{sts}(\alpha_t^{h_i(t)}) = 0 = \delta_{tst}(\alpha_t^{h_i(t)}) \) for \( h, k \in \{0, 1, 2\} \) since the maps \( \delta_{sts}, \delta_{tst} \) map polynomials of degree \( d \) to polynomials of degree \( d - 3 \) or to zero, the claim follows. In general, if we localize to \( \mathbb{C}[t][\alpha_t^{-1}, \alpha_t^{-1}] \) we could still have the analogue statement.

B2. \( \text{Stab}_i = \langle s \rangle \) (analogue \( \text{Stab}_i = \langle t \rangle \)) and use \( \delta_s(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}))) = 0 \) to see
\[
\sigma_i(s)\sigma_i(s)\sigma_i(t)\sigma_i(t) st(\alpha_t^{h_i(t)} s(\alpha_s^{h_i(s)} st(\alpha_t^{h_i(t)}))) = 0
\]
because \( st\delta_s = \delta_{sts} \).

B3. \( \text{Stab}_i \{ 1, s t s \} \) (analogue \( \text{Stab}_i \{ 1, t s t \} \)). It holds \( it s = it s, i s = ist \). We have
\[
[\sigma_i(s)\sigma_i(s)\sigma_i(t)\sigma_i(t) st(\alpha_t^{h_i(t)} s(\alpha_s^{h_i(s)} st(\alpha_t^{h_i(t)})))]
\]
\[
= \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}))) st(\alpha_t^{h_i(t)})
\]
\[
= \alpha_s^{h_i(s)}(\alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}))) st(\alpha_t^{h_i(t)})
\]
using \( st\delta_s = ts\delta_s \) and \( st(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})) = \alpha_t^{h_i(t)} st(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})) \)
B4. $\text{Stab}_i = \{1, s, tst, stst\}$ (analogous $\text{Stab}_i = \{1, t, sts, stst\}$). It holds $i = is, it = its, ist = sts, itst = itsts$.

$$\begin{align*}
[\sigma_i(s)\sigma_{its}(t)\sigma_{stst}(s)\sigma_{itst}(t) - \sigma_i(t)\sigma_{its}(s)\sigma_{itst}(t)\sigma_{stst}(s)](f) & = a_s^{h_i(s)}\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(f)) - a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(f)) \\
& = [a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))\delta_s\sigma_i(s)(f) \\
& + [f(a_t^{h_i(t)})\delta_s(a_t^{h_i(t)}(t)\delta_s(a_t^{h_i(t)}(f)) - a_t^{h_i(t)}(t)\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f)](f)
\end{align*}$$

using $\delta_s t\delta_s t = t\delta_s t\delta_s$.

This finishes the investigation of the ten possible cases. We also like to remark that in the example in chapter 5 the case B4 only occurs for $\text{Stab}_i = \{1, t, sts, stst\}$, i.e. the other stabilizer never occurs.

(C) $\text{ststst} = \text{tststst}$: $\langle s, t \rangle \cong D_6$,

$t(\alpha_s) = \alpha_s + \alpha_t$, $st(\alpha_s) = 2\alpha_s + \alpha_t$, $tst(\alpha_s) = st(\alpha_s)$, $ts(\alpha_t) = 3\alpha_s + 2\alpha_t$, $sts(\alpha_t) = ts(\alpha_t)$.

It holds

$h_{its}stst(s) = h_{is}(s)$, $h_{its}stst(t) = h_{is}(t)$, $h_{its}stst(s) = h_{ist}(s)$

$h_{its}(t) = h_{ists}(t)$, $h_{it}(s) = h_{istst}(s)$, $h_{it}(t) = h_{ists}(t)$.

this implies

$$\begin{align*}
\alpha_t^{h_i(t)}(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f) & = \alpha_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f) \\
& + [f(a_t^{h_i(t)})\delta_s(a_t^{h_i(t)}(t)\delta_s(a_t^{h_i(t)}(f)) - a_t^{h_i(t)}(t)\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f)](f)
\end{align*}$$

Now, $D_6$ has 13 subgroups. In the following cases the above equality directly implies that $\sigma_i(ststst) = \sigma_i(tststst) = 0$:

$\text{Stab}_i = \{1\}$, $\text{Stab}_i = \{1, t\}$, $\text{Stab}_i = \{1, ts\}$, $\text{Stab}_i = \{1, ststst\}$, $\text{Stab}_i = \{s\}$.

C1. $\text{Stab}_i = \langle s, t \rangle$. By assumption we have $h_i(s) = 0 = h_i(t)$ in this case, therefore

$$\begin{align*}
\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t) - \sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s) = \delta_s \delta_t \delta_s \delta_t \delta_s \delta_t = 0
\end{align*}$$

because that is known for the divided difference operators, cp [Dem73].

C2. $\text{Stab}_i = \{1, s\}$ (analogous $\text{Stab}_i = \{1, t\}$). Then, $is = i, itst = itstst$.

$$\begin{align*}
[\sigma_i(s)\sigma_{ists}(t)\sigma_{stst}(s)\sigma_{itst}(t)\sigma_{stst}(t) - \sigma_i(t)\sigma_{ists}(s)\sigma_{itst}(t)\sigma_{stst}(t)\sigma_{itst}(t)\sigma_{stst}(t)](f) & = a_s^{h_i(s)}\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f) \\
& - a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f) \\
& = a_s^{h_i(s)}\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))\sigma_i(s)(f) \\
& = 0
\end{align*}$$

using $\delta_s tstst = tstst\delta_s$ and $\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)})) = 0$ because

$$\begin{align*}
s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)})) & = a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}(a_s^{h_i(s)})\delta_s(a_t^{h_i(t)}))
\end{align*}$$

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C3. \( \text{Stab}_i = \{1, tst\} \) (analogue \( \text{Stab}_i = \{1, ststs\} \)). Then \( its = itst, ists = istst \).

\[
\begin{align*}
&\{\sigma_i(s)\sigma_{ts}(t)\sigma_{ist}(s)\sigma_{iststs}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itssts}(t)\sigma_{iststs}(s)\}^f \nonumber
= \alpha_s^{h_i(s)}\delta_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
&- \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
= \alpha_s^{h_i(s)}\delta_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
+ \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
- \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
- \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
= [\delta_s^{h_i(s)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f)]\sigma_i(s)(f) \\
&\text{using } ts\delta sts = st\delta st.
\end{align*}
\]

C4. \( \text{Stab}_i = \{1, s, tst, stsst\} \) (analogue \( \text{Stab}_i = \{1, t, ststs, ststst\} \)). Then \( is = i, itst = its \). Observe, in this case

\[ h_i(t) = h_{it}(t), \text{ and } h_{it}(s) = h_{its}(s) \]

and it holds

\[
\begin{align*}
&\{\sigma_i(s)\sigma_{ts}(t)\sigma_{ist}(s)\sigma_{iststs}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itssts}(t)\sigma_{iststs}(s)\}^f \\
= \alpha_s^{h_i(s)}\delta_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
- \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
- \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
- \alpha_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f) \\
= [\alpha_s^{h_i(s)}\delta_s^{h_i(t)}(\alpha_s^{h_i(s)})ts(\alpha_s^{h_i(s)})sts(\alpha_s^{h_i(s)})s(\alpha_s^{h_i(s)})st(f)]\sigma_i(s)(f) \\
&\text{using } \delta_t st \delta st = ts\delta t st\delta s.
\end{align*}
\]

C5. \( \text{Stab}_i = \{1, ststs, ststst\} \). Then \( is = ist, it = its \). Observe, in this case

\[ h_i(s) = h_{is}(s), \text{ and } h_{it}(t) = h_{it}(t) \]
and it holds
\[ |\sigma_t(s)\sigma_{ts}(t)\sigma_{ist}(s)\sigma_{iststs}(s)\sigma_{iststs}(t) - \sigma_t(s)\sigma_{ts}(t)\sigma_{ist}(s)\sigma_{iststs}(s)\sigma_{iststs}(t)\sigma_{iststs}(s)| (f) \]
\[ = \alpha_s^h(s)\sigma_t^h(t)\alpha_t^h(s)\alpha_t^h(s)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})s(f) \]
\[ = \alpha_s^h(s)\sigma_t^h(t)\alpha_t^h(s)\alpha_t^h(s)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)}) \cdot f \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\sigma_t^h(t)st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(s(f)) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\sigma_t^h(t)st\delta_t(f) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sta(\alpha_t^{\text{histsts}(s)})ststs(\alpha_t^{\text{histsts}(s)})st\delta_t\sigma_t^h(t)st\delta_t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)}) \cdot f \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)}) \cdot f \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)ststs(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)}) \cdot f \]
\[ = Pf + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)ststs(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ = P(f + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)ststs(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ = P(f + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)ststs(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]

using \( s(t) \delta(t) \) where \( P(f + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)ststs(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ = P(f + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ - \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)st(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ + \alpha_t^h(t)\alpha_t^h(s)\sigma_t^h(t)\alpha_t^h(s)ststs(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]

and analogously
\[ s(t) \delta(t) \]
\[ = \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ = \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]
\[ = \alpha_s^h(s)\sigma_t^h(t)\sigma_t^h(s)sts(\alpha_t^{\text{histsts}(s)})stst(\alpha_t^{\text{histsts}(s)})st\delta_t(\alpha_t^{\text{histsts}(s)})st\delta_t(f) \]

Then a simple substitution gives that the difference above is of the form
\[ Q(f + Q_{sts}(s)\sigma_{ts}(t)\sigma_{ist}(s)(f) + Q_{tst}(s)\sigma_{ts}(t)\sigma_{ist}(s)(t) \]
\[ for some polynomials Q, Q_{sts}, Q_{tst} in \alpha_s, \alpha_t. \]

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Now, let \( A \) be the algebra given by generator \( \tilde{1}_i, \tilde{z}_i(t), \tilde{\sigma}_i(s) \) subject to relations (1)-(5). Then, by the straightening rule and the braid relation it holds that if \( w = s_1 \cdots s_k = t_1 \cdots t_k \) are two reduced expressions then
\[
\sigma(t_1 \cdots t_k) \in \sum_{v \leq s_1 \cdots s_k} \mathcal{E} * \sigma(v).
\]
Therefore, once we have fixed one (any) reduced expression for each for \( w \in W \), it holds
\[
A = \sum_{w \in W} \mathcal{E} * \sigma(w).
\]
Since the generators of \( Z_G \) fulfill the relations (1)-(5), we have a surjective algebra homomorphism
\[
A \to Z_G
\]
mapping \( \tilde{1}_i \mapsto 1_i, \tilde{z}_i(t) \mapsto z_i(t), \tilde{\sigma}_i(s) \mapsto \sigma_i(s) \). Since \( Z_G = \bigoplus_{w \in W} \mathcal{E} * \sigma(w) \) and the map is by definition \( \mathcal{E} \)-linear it follows that \( A = \bigoplus_{w \in W} \mathcal{E} * \sigma(w) \) and the map is an isomorphism. \( \square \)

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