DYNAMICALLY DEFINED SEQUENCES
WITH SMALL DISCREPANCY

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Abstract. We study the problem of constructing sequences \((x_n)_{n=1}^{\infty}\) on \([0,1]\) in such a way that

\[ D_N^* = \sup_{0 \leq x \leq 1} \left| \frac{\# \{1 \leq i \leq N : x_i \leq x \}}{N} - x \right| \]

grows uniformly small. A result of Schmidt shows that necessarily \(D_N^* \gtrsim (\log N) N^{-1}\) for infinitely many \(N\) and there are several classical constructions attaining this growth. We describe a type of uniformly distributed sequence that seems to be completely novel: given \(\{x_1, \ldots, x_{N-1}\}\), we construct \(x_N\) in a greedy manner

\[ x_N = \arg \min_{x \in [0,1]} \min_{1 \leq k \leq N-9} \sum_{k=1}^{N-1} 1 - \log \left(2 \sin \left(\frac{\pi}{|x-x_k|}\right)\right). \]

We prove that \(D_N \lesssim (\log N) N^{-1/2}\) and conjecture that \(D_N \lesssim (\log N)^d N^{-1}\) for an analogous construction in higher dimensions and conjecture it to be \(D_N \lesssim (\log N)^d N^{-1}\).

1. Introduction

1.1. Introduction. Let \((x_n)_{n=1}^{\infty}\) be a sequence on \([0,1]\) and define the star discrepancy of the first \(N\) elements via

\[ D_N^* = \sup_{0 \leq x \leq 1} \left| \frac{\# \{1 \leq i \leq N : x_i \leq x \}}{N} - x \right|. \]

van der Corput asked in 1935 whether there was a sequence for which \(D_N \lesssim N^{-1}\). This was disproven by van Aardenne-Ehrenfest [1]. Roth [17] showed that \(D_N \gtrsim \sqrt{\log N} N^{-1}\) for infinitely many \(N\). The sharp result is due to Schmidt [18] who showed that there are infinitely many \(N\) for which

\[ D_N^* \gtrsim \frac{\log N}{N}. \]

Other proofs of Schmidt’s result were given by Bejian [4], Halasz [11] and Liardet [14], the best constant is due to Larcher & Puchhammer [13]. Several sequences attaining this growth have been constructed, we refer to the classical textbooks by Beck & Chen [3], Dick & Pillichshammer [9], Drmota & Tichy [10] and Kuipers & Niederreiter [12]. As soon as one generalizes the problem to sequences in higher dimensions \([0,1]^d\) (the supremum running over all axis-parallel hyperrectangles anchored in the origin), the problem is open. Roth [17] proved that any sequence...
\(\{x_n\}_{n=1}^{\infty}\) in \([0, 1]^d\) has

\[
D_N \gtrsim \frac{(\log N)^{d}}{N}
\]

for infinitely many \(N\).

A logarithmic improvement for \(d = 2\) is due to Beck \[2\]. The best known result is due to Bilyk, Lacey, Vagharshakyan \[6, 7\] and states that

\[
D_N \gtrsim \frac{(\log N)^{2+\varepsilon_d}}{N}
\]

for infinitely many \(N\) and some \(\varepsilon_d > 0\) depending only on \(d\). There is no consensus on what the sharp result should be: the two main conjectures (we refer to \[5\]) are that for any sequence there are infinitely many \(N\) such that

\[
D_N \gtrsim \frac{(\log N)^{d}}{N}
\]

or

\[
D_N \gtrsim \frac{(\log N)^{d+1}}{N}.
\]

Of course, both conjectures coincide for \(d = 1\). The first conjecture has the advantage of being structurally aligned with related conjectures in Harmonic Analysis and Probability Theory while the second conjecture has the advantage of being matched by the best known constructions. If the first conjecture were true, this would imply that in \(d \geq 2\) dimensions there are sequences more regular than anything we can currently construct. Many of these classical sequences attaining \(D_N \lesssim (\log N)^d N^{-1}\) exploit regular structures derived from Number Theory (irrational rotations on the torus, regularity in digit expansions), so one could try to understand whether it is possible to construct sequences with small discrepancy using a different viewpoint.

1.2. Results. This paper is a companion paper to \[20\] where we showed that minimizing a certain functional can decrease the discrepancy of point sets. Here we show that this functional also allows us to construct uniformly distributed sequences in a way that is very different from the usual constructions. Suppose we are given \(\{x_1, \ldots, x_{N-1}\} \subset [0, 1]\), we construct \(x_N\) in a greedy manner

\[
x_N = \arg \min_{x} \min_{k} |x - x_k| \geq N^{-10} \sum_{k=1}^{N-1} \left(1 - \log \left(2 \sin \left(\pi |x - x_k|\right)\right)\right).
\]

If the minimizer is not unique, any choice is admissible. The gap condition \(\min_{k} |x - x_k| \geq N^{-10}\) ensures that the new point \(x_N\) is not extremely close to any of the existing points. We could replace it by \(\min_{k} |x - x_k| \geq N^{-\ell}\) for any \(\ell \in \mathbb{N}\) without it affecting the main result (except for constants). One can start with any given set \(\{x_1, \ldots, x_m\} \subset [0, 1]\) and then obtain a sequence in this greedy manner.

**Theorem 1.** We have, for any sequence thus constructed,

\[
D_N \lesssim \frac{\log N}{\sqrt{N}},
\]

where the implicit constant depends only on the initial set.

This bound in itself is not impressive (random points behave in a similar manner) but it is interesting that the outcome of such a greedy algorithm can be controlled at all. However, we believe that a much stronger statement is true: we conjecture that (1) one can ignore the condition \(\min_{k} |x - x_k| \geq N^{-10}\) and (2) will obtain a low-discrepancy sequence.
Conjecture 1. For any initial set \( \{x_1, \ldots, x_k\} \subset [0,1] \), the greedy sequence arising out of
\[
    x_N = \arg \min_x \sum_{k=1}^{N-1} 1 - \log (2 \sin (\pi |x - x_k|))
\]
satisfies \( D_N \lesssim (\log N) N^{-1} \). A stronger conjecture would be that the implicit constant in \( D_N \lesssim (\log N) N^{-1} \) does not depend on the initial set as \( N \to \infty \).

If this statement were true, it would give rise to a large number of low-discrepancy sequences that are constructed by a technique very different from any of the usual techniques. One byproduct of our argument is as follows.

Theorem 2. Suppose we define a sequence in a greedy manner by picking \( x_N \) in such a way that
\[
    \sum_{n=1}^{N-1} \sum_{k=1}^{N} \frac{\cos(2\pi k(x_N - x_n))}{k} \leq 0,
\]
then
\[
    D_N \lesssim \frac{\log N}{\sqrt{N}}.
\]

We note that it is always possible to choose such a \( x_N \) since
\[
    \int_0^1 \sum_{n=1}^{N-1} \sum_{k=1}^{N} \frac{\cos(2\pi k(x - x_n))}{k} dx = 0.
\]

Theorem 2 is not very deep and might be close to optimal; presumably there are various different choices of \( x_N \) that are admissible and some of them might not be particularly good (though, as Theorem 2 states, they cannot be arbitrarily bad). The emphasis of our paper (as well as the numerical experimentation, see §1.3.) is that choosing the minimum may lead to very good behavior. In particular, an alternative sequence that may be interesting for further study could be
\[
    x_N = \arg \min_x \sum_{n=1}^{N-1} \sum_{k=1}^{N} \frac{\cos(2\pi k(x - x_n))}{k}
\]
but we do not pursue this further.

1.3. Basic Numerics. One of the main reasons why Conjecture 1 seems reasonable is that in numerical examples, the sequence performs extraordinarily well. Given the first \( N \) elements of a sequence \( (x_n)_{n=1}^N \), we will associate to it the set
\[
    X_N = \left\{ \left( \frac{n}{N}, x_n \right) : 1 \leq n \leq N \right\} \subset [0,1]^2
\]
and use the size of the star-discrepancy \( D^*_N(X_N) \) as a sign of quality (see Fig. 1).

Our sequence is actually comparable (or even superior) in quality to many of the classical constructions (see also [20]). We compare (see Table 1) the sequence with the Halton set (using base 2 and 3), the Hammersley sequence (using base 2) and the Kronecker-type set
\[
    \text{Kronecker}_{\sqrt{133}} = \left\{ \left( \frac{n}{N}, \frac{\sqrt{133} n}{N} \right) : 1 \leq n \leq N \right\}.
\]
The choice of \( \sqrt{133} \) is more or less at random but was selected to give somewhat nice behavior (except for \( N = 150 \), see Table 1). We observe that our sequence, starting with \( \{0.5, 0.95\} \) (which was also more or less chosen at random), is comparable or superior to the other examples.

| \( N \) | \( D_N(X_N) \) | \( D_N(\text{Halton}_{2,3}) \) | \( D_N(\text{Hammersley}_2) \) | Kronecker \( \sqrt{133} \) |
|---|---|---|---|---|
| 50 | 0.044 | 0.067 | 0.048 | 0.083 |
| 100 | 0.026 | 0.049 | 0.026 | 0.037 |
| 150 | 0.018 | 0.039 | 0.017 | 0.070 |
| 200 | 0.013 | 0.022 | 0.014 | 0.026 |
| 250 | 0.012 | 0.018 | 0.012 | 0.026 |

**Table 1.** Discrepancy \( D_N^*(X_N) \) for the sequence arising from \( \{0.5, 0.95\} \) and the value of classical sets of the same size.

This behavior seems quite robust under various initial conditions. We could try to intentionally 'break' the sequence by starting with a particularly bad initial configuration. We observe that the sequence auto-adjusts in a nice way. We illustrate this below for the sequence \( (x_n) \) starting with the initial set of points \( \{0.5, 0.51, 0.52, 0.53, 0.54\} \) (see Fig. 2).

| \( N \) | \( D_N(X_N) \) |
|---|---|
| 10 | 0.32 |
| 25 | 0.12 |
| 50 | 0.06 |
| 100 | 0.032 |
| 150 | 0.022 |
| 200 | 0.016 |

**Table 2.** Discrepancy \( D_N^*(X_N) \) for the sequence arising from the initial set \( \{0.5, 0.51, 0.52, 0.53, 0.54\} \).

In both cases we see that the newly added points initially avoid the clustered regions and then slowly return to it (though, initially, at a lower density, see Fig. 2). We refer to Table 2 for the behavior of their star discrepancy which is initially quite large (forced by the clustered initial points) and then stabilizes very quickly.
We observed numerically that certain initial conditions may be connected to variants of the van der Corput sequence in base 2. If we start with the set \{0.5, 1\}, then one admissible way of choosing minima leads to the sequence

\[
1, 1, 1, 3, 7, 3, 1, 5, 15, 7, 11, 3, 13, 5, 1, 2', 1', 4', 4', 8', 8', 8', 16', 16', 16', 16', 16', 16' \ldots
\]

Moreover, the van der Corput sequence in base 2 seems to be realizable as a sequence of this type (starting with \{1/2, 1\} and then picking minima appropriately whenever it is not unique). Can this be proven? Can admissible permutations of the van der Corput sequence, that can arise in this manner, be characterized?

1.4. Higher dimensions. The construction rule for sequences in \([0, 1]^d\) is slightly different: suppose we have constructed \(\{x_1, \ldots, x_{N-1}\} \subset [0, 1]^d\), then we want the next element

\[
x_N = (x_{N,1}, x_{N,2}, \ldots, x_{N,d})
\]

to satisfy

\[
\sum_{n=1}^{N-1} d \prod_{j=1}^{d} \left( 1 + \sum_{k=1}^{N} \cos \left( 2\pi k (x_{N,j} - x_{n,j}) \right) \right) \leq 1.
\]

Integrating over \([0, 1]^d\) shows that such a \(x_N\) always exists.

**Theorem 3.** Any such sequence satisfies

\[
D_N \lesssim \frac{(\log N)^d}{\sqrt{N}},
\]

where the implicit constant depends only on the initial set.

As in the one-dimensional case, we have the following

**Conjecture 2.** The greedy algorithm

\[
x_N = \arg \min_x \sum_{\ell=1}^{N-1} d \prod_{j=1}^{d} (1 - \log (2 \sin (\pi |x_{n,j} - x_{\ell,j}|))).
\]

leads to a sequence with \(D_N \lesssim (\log N)^d N^{-1}\).
2. Proofs

2.1. A Lemma. We start by proving a regularity statement for minimizers of the functional. When we apply it to prove the main results, one of the relevant quantities is inside a logarithm. As a consequence, it is not tremendously important whether we prove Lemma 1 and Lemma 2 with bounds at scale $N^{-5}$ or $N^{-500}$ and thus have not tried to optimize the arguments. However, a much stronger version of Lemma 1, in particular showing that the minimum is for many terms along the sequence actually at scale $\sim -\log N$, could possibly improve the main result.

**Lemma 1.** Let $\{x_1, \ldots, x_N\} \subset [0,1]$. Then there exists $0 < x < 1$ such that

$$
\sum_{k=1}^{N} -\log (2 \sin (\pi |x-x_k|)) \leq -\frac{1}{N^2} \quad \text{and} \quad \min_{1 \leq k \leq N} \|x-x_k\| \gtrsim N^{-4}.
$$

**Proof.** We introduce a one-parameter family of functions for $t \geq 0$ via

$$f_t(x) = \sum_{k \in \mathbb{Z}, k \neq 0} e^{-4\pi^2 k^2 t} \frac{e^{2\pi i k x}}{k}$$

and note that

$$f_0(x) = -\log (2 \sin (\pi |x|)).$$

$f_t$ is the solution of the heat equation starting with $f_0$, in particular the maximum principle for parabolic equations is telling us that for any $t > 0$

$$\min_x \sum_{n=1}^{N} f_t(x-x_n) \geq \min_x \sum_{n=1}^{N} f_0(x-x_n).$$

Moreover, by construction, for every $0 < y < 1$

$$\int_0^1 f_t(x-y)dx = \int_0^1 f_0(x-y)dx = 0.$$

We now establish a series of bounds on $f_t$. We will work at scale $t \sim N^{-2}$ but this is not important at this point. We first observe that

$$\left\| \sum_{n=1}^{N} f_t(x-x_n) \right\|_{L^2}^2 = \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}, k \neq 0} e^{-4\pi^2 k^2 t} \frac{e^{2\pi i k (x-x_n)}}{k}^2$$

$$= \sum_{k \in \mathbb{Z}, k \neq 0} \frac{e^{-8\pi^2 k^2 t}}{k^2} \left| \sum_{n=1}^{N} e^{-2\pi i k x_n} \right|^2_{L^2}$$

We now use a basic Lemma of Montgomery [15, 16] (see also [8]) ensuring that

$$\sum_{|k| \leq 100 N} e^{-2\pi i k x_n} \gtrsim N^2.$$
and obtain
\[ \left\| \sum_{n=1}^{N} f_t(x - x_n) \right\|_{L^2}^2 \geq e^{-800 \pi^2 N^2 t}. \]

We next observe that
\[ \left\| \frac{d}{dx} f_t \right\|_{L^\infty} = \left\| \sum_{k \in \mathbb{Z}, k \neq 0} e^{-4 \pi^2 k^2 t} e^{2 \pi ikx} \frac{1}{k} \right\|_{L^\infty} \lesssim \sum_{k \in \mathbb{Z}, k \neq 0} e^{-4 \pi^2 k^2 t} \lesssim \frac{1}{\sqrt{t}} \]
and thus
\[ \left\| \frac{d}{dx} \sum_{n=1}^{N} f_t(x - x_n) \right\|_{L^\infty} \lesssim \frac{N}{\sqrt{t}}. \]

Altogether, abbreviating
\[ g(x) = \sum_{n=1}^{N} f_{N-2}(x - x_n), \]
we have shown that
\[ \int_0^1 g(x)dx = 0, \quad \|g\|_{L^2} \gtrsim 1 \quad \text{and} \quad \|g\|_{L^\infty} \lesssim N^2. \]

This now implies that
\[ \min_{0<x<1} g(x) \leq -\frac{1}{N^2} \]
which we see as follows: clearly, from \( \|g\|_{L^2} \gtrsim 1 \) we observe that
\[ \max_{0<x<1} |g(x)| \gtrsim 1. \]

If the maximum is attained at a negative value, we are done. If it is attained at a positive value, then the bound on the derivative implies
\[ \int_0^1 g^+(x)dx \gtrsim \frac{1}{N^2} \]
which then, with the mean 0 condition, implies
\[ \min_x g(x) \leq \int_0^1 g^-(x)dx = -\int_0^1 g^+(x)dx \lesssim -\frac{1}{N^2}. \]

This implies the existence of a point \( 0 < x_0 < 1 \) such that
\[ \sum_{n=1}^{N} f_{N-2}(x_0 - x_n) \lesssim -\frac{1}{N^2}. \]

Moreover, since this is the solution of the heat equation, we can write (identifying the unit interval \([0,1]\) with the Torus \(T\))
\[ \sum_{n=1}^{N} f_{N-2}(x_0 - x_n) = \int_0^1 \left( \sum_{n=1}^{N} f_0(x_0 + y - x_n) \right) \theta_{N-2}(y)dy, \]
where
\[ \theta_t(x) = 1 + \sum_{k \in \mathbb{Z}, k \neq 0} e^{-4 \pi^2 k^2 t} e^{2 \pi ikx} \]
is the Jacobi $\theta$–function. The Jacobi $\theta$–function satisfies 
\[
\theta_t(x) \geq 0, \quad \int_0^1 \theta_t(x) dx = 1 \quad \text{and} \quad \theta_t(x) \lesssim \frac{1}{\sqrt{t}}.
\]
Using the easy estimate 
\[
\sum_{n=1}^{N} f_0(x - x_n) \geq -N
\]
and defining 
\[
A = \left\{ x : \min_k |x - x_k| \leq N^{-4} \right\} \quad \text{and} \quad m = \inf_{x \in A^c} \left( \sum_{n=1}^{N} f_0(x - x_n) \right),
\]
we can estimate 
\[
\int_0^1 \left( \sum_{n=1}^{N} f_0(x_0 + y - x_n) \right) \theta_{N^{-2}}(y) dy \geq -|A|N + \int_0^1 m \theta_{N^{-2}}(x) dx
\]
\[
= -|A|N + m \geq -2N^{-3} + m
\]
which implies $m \lesssim N^{-2}$ as desired. □

There is a technical step that could be slightly improved. We observe that 
\[
\left\| \frac{d}{dx} \sum_{k=1}^{N} f_t(x - x_n) \right\|_{L^\infty} = \left\| \frac{d}{dx} \sum_{k \in \mathbb{Z}, k \neq 0} e^{-4\pi^2 k^2 t} k e^{2\pi ikx} \sum_{n=1}^{N} e^{-2\pi ikx_n} \right\|_{L^\infty}
\]
The exponential cutoff localize the sum essentially at frequency scales $\sim N$ which shows that we can expect the derivative to be (possibly up to a logarithmic factor) at scale $\lesssim N$ as opposed to $\lesssim N^2$. However, this improvement would have no further impact on our main result.

2.2. An Error Bound. The second technical ingredient is straightforward.

**Lemma 2.** For all $N^{-4} < x < 1 - N^{-4}$ and all $M \geq N^{100}$, we have
\[
\left| \sum_{k=N^{100}}^{M} \frac{\cos (2\pi k x)}{k} \right| \leq \frac{1}{N^{96}}.
\]

**Proof.** We use summation by parts. Summation by parts states that if $\{f_k\}, \{g_k\}$ are two sequences, then 
\[
\sum_{k=m}^{n} f_k(g_{k+1} - g_k) = (f_n g_{n+1} - f_m g_m) - \sum_{k=m+1}^{n} g_k(f_k - f_{k-1}).
\]
We set 
\[
g_k = \sum_{\ell=1}^{k} \cos (2\pi (\ell - 1) x) \quad \text{and} \quad f_k = \frac{1}{k}.
\]
Then 
\[
\left| \sum_{k=N^{100}}^{M} \frac{\cos (2\pi k x)}{k} \right| \lesssim \sup_k |g_k| \left( \frac{1}{N^{100}} + \frac{1}{M} + \sum_{k=N^{100}}^{m} \frac{1}{k^2} \right) \lesssim \frac{\sup_k |g_k|}{N^{100}}.
\]
It remains to estimate the supremum. We write
\[ g_k = \sum_{\ell=1}^{k} \cos(2\pi(\ell - 1)x) = \left| \sum_{\ell=1}^{k} e^{2\pi i(\ell - 1)x} \right| \lesssim \frac{1}{|e^{2\pi ix} - 1|} \lesssim N^4. \]

\[ \square \]

The main consequence of Lemma 1 and Lemma 2 can now be written as follows.

**Lemma 3.** Let \( \{x_1, \ldots, x_{N-1}\} \subset [0,1] \) be arbitrary and let
\[ x_N = \arg \min_{\min_k |x-x_k| \geq N^{-10}} \sum_{k=1}^{N-1} 1 - \log (2 \sin (\pi |x - x_k|)). \]

Then
\[ \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{\cos(2\pi k(x - x_n))}{k} \leq 0. \]

**Proof.** This follows from Lemma 1, Lemma 2 and the decomposition
\[ - \log (2 \sin (\pi |x|)) = \sum_{k=1}^{\infty} \frac{\cos (2\pi kx)}{k} = \sum_{k=1}^{N^{100}} \frac{\cos (2\pi kx)}{k} + \sum_{k=N^{100}+1}^{\infty} \frac{\cos (2\pi kx)}{k}. \]

\[ \square \]

2.3. **Proof of Theorem 1.**

**Proof.** Our derivation is motivated by the Erdős-Turan inequality bounding the discrepancy \( D_N \) of a set \( \{x_1, \ldots, x_N\} \subset [0,1] \) by
\[ D_N \lesssim \frac{1}{N^{100} + \sum_{k=1}^{N^{100}} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right|}, \]
where \( k \) is arbitrary. We can bound this from above by Cauchy-Schwarz
\[ \sum_{k=1}^{N^{100}} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right| \leq \left( \sum_{k=1}^{N^{100}} \frac{1}{k} \right)^{1/2} \left( \sum_{k=1}^{N^{100}} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right|^2 \right)^{1/2} \]
\[ \lesssim \sqrt{\log N} \left( \frac{1}{N^2} \sum_{k=1}^{N^{100}} \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right|^2 \right)^{1/2}. \]

We square the second term and decouple it into diagonal and off-diagonal terms
\[ \frac{1}{N^2} \sum_{k=1}^{N^{100}} \left( \frac{1}{k} \left| \sum_{n=1}^{N} e^{2\pi ikx_n} \right|^2 \right) = \frac{1}{N^2} \sum_{k=1}^{N^{100}} \frac{1}{k} \sum_{n,m=1}^{N} e^{2\pi ik(x_n - x_m)} \]
\[ \lesssim \frac{\log N}{N} + \frac{1}{N^2} \sum_{k=1}^{N^{100}} \frac{1}{k} \sum_{n,m=1}^{N} e^{2\pi ik(x_n - x_m)}. \]
Summing the points in pairs and exchanging the order of summation, we can simplify this expression to

$$
\sum_{k=1}^{N^{100}} \frac{1}{k} \sum_{m,n=1}^{N} \cos (2\pi k(x_m - x_n)) = \sum_{m,n=1}^{N^{100}} \sum_{k=1}^{N} \frac{\cos (2\pi k(x_m - x_n))}{k}.
$$

Altogether, this shows that

$$
D_N \lesssim \sqrt{\log N} \left( \frac{\log N}{N} + \frac{1}{N^2} \sum_{m,n=1}^{N} \frac{\cos (2\pi k(x_m - x_n))}{k} \right)^{1/2}
$$

$$
= \sqrt{\log N} \left( \frac{\log N}{N} + \frac{1}{N^2} \sum_{n=1}^{N-1} \sum_{m=1}^{N^{100}} \frac{\cos (2\pi k(x_m - x_n))}{k} \right)^{1/2}
$$

We now argue that the sum is negative because every inner sum is negative. Indeed, Lemma 3 implies that the choice

$$
x_n = \arg\min_x \sum_{m=1}^{n-1} - \log (2 \sin (\pi |x_m - x|))
$$

shows that

$$
\sum_{m=1}^{n-1} \sum_{k=1}^{N^{100}} \frac{\cos (2\pi k(x_m - x_n))}{k} \leq 0.
$$

This establishes the desired result. Rerunning the argument with $N$ instead of $N^{100}$ establishes Theorem 2.

We note that using this particular way of taking a limit to obtain a Fourier series was already hinted at in earlier work of the author [19].

2.4. Proof of Theorem 2.

Proof. We use the Erdős-Turan-Koksma inequality to bound the discrepancy of a set $\{x_1, \ldots, x_N\} \subset [0, 1]^d$ by

$$
D_N \lesssim_d \sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \frac{1}{N} \left| \sum_{\ell=1}^{N} e^{2\pi i (k, x_\ell)} \right|,
$$

where $r : \mathbb{Z}^d \to \mathbb{N}$ is given by

$$
r(k) = \prod_{j=1}^{d} \max \{1, k_j\}.
$$

The Cauchy-Schwarz inequality implies

$$
D_N \lesssim (\log N)^{\frac{d}{2}} \left( \sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \frac{1}{N^2} \left| \sum_{\ell=1}^{N} e^{2\pi i (k, x_\ell)} \right|^2 \right)^{1/2}.
$$
We rewrite the sum as
\[
\sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \sum_{\ell = 1}^{N} \left| e^{2\pi i (k, x_\ell)} \right|^2 = \frac{1}{N^2} \sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \sum_{\ell, m = 1}^{N} e^{2\pi i (k, x_\ell - x_m)}.
\]
However, this sum can also be written as (after additionally summing over \(k = 0\) and then subtracting the arising value \(N^2\))
\[
\sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \sum_{\ell, m = 1}^{N} e^{2\pi i (k, x_\ell - x_m)} = \sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \sum_{\ell, m = 1}^{N} \prod_{j = 1}^{d} e^{2\pi i k_j (x_{\ell,j} - x_{m,j})}
\]
\[
= -N^2 + \sum_{m, \ell = 1}^{N} \prod_{j = 1}^{d} \sum_{k = -N}^{N} \frac{1}{r(k)} e^{2\pi i k_j (x_{\ell,j} - x_{m,j})}
\]
\[
= -N^2 + \sum_{m, \ell = 1}^{N} \prod_{j = 1}^{d} \left( 1 + \sum_{k = 1}^{N} \cos \left( \frac{2\pi k (x_{\ell,j} - x_{m,j})}{k} \right) \right).
\]
We now remove the diagonal terms and see that
\[
\sum_{m, \ell = 1}^{N} \prod_{j = 1}^{d} \left( 1 + \sum_{k = 1}^{N} \cos \left( \frac{2\pi k (x_{\ell,j} - x_{m,j})}{k} \right) \right) \lesssim N (1 + \log N)^d
\]
\[
+ \sum_{m, \ell = 1}^{N} \prod_{j = 1}^{d} \left( 1 + \sum_{k = 1}^{N} \cos \left( \frac{2\pi k (x_{\ell,j} - x_{m,j})}{k} \right) \right).
\]
As in the proof of Theorem 1, we can reorder the sum and then use the fact that all the inner sums are negative. This shows that
\[
D_N \lesssim \left( \log N \right)^{\frac{d}{2}} \left( \sum_{0 < \|k\|_\infty \leq N} \frac{1}{r(k)} \sum_{\ell = 1}^{N} \left| e^{2\pi i (k, x_\ell)} \right|^2 \right)^{1/2} \lesssim \frac{(\log N)^d}{\sqrt{N}}.
\]
\[\square\]

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