Fine Selmer groups of congruent Galois representations

Meng Fai Lim∗  Ramdorai Sujatha†

Abstract
In this paper, we study the fine Selmer groups of two congruent Galois representations over an admissible $p$-adic Lie extension. We show that under appropriate congruence conditions, if the dual fine Selmer group of one is pseudo-null, so is the other. Our results also compare the $\pi$-primary submodules of the two dual fine Selmer groups. We then apply our results to compare the structure of Galois group of the maximal abelian unramified pro-$p$ extension of an admissible $p$-adic Lie extension and the structure of the dual fine Selmer group over the said admissible $p$-adic Lie extension.

Keywords and Phrases: Fine Selmer groups, admissible $p$-adic Lie extensions, pseudo-nullity, congruence of Galois representations.

Mathematics Subject Classification 2010: 11R23, 11R34, 11F80, 16S34.

1 Introduction
Throughout the paper, $p$ will always denote a fixed prime. Let $F$ be a number field. If $p = 2$, we assume that $F$ is totally imaginary. We write $F^{\text{cycl}}$ for the cyclotomic $\mathbb{Z}_p$-extension of $F$, whose Galois group $\text{Gal}(F^{\text{cycl}}/F)$ is in turn denoted $\Gamma$. Denote by $K(F^{\text{cycl}})$ the maximal unramified pro-$p$ extension of $F^{\text{cycl}}$ in which every prime of $F^{\text{cycl}}$ above $p$ splits completely. In [18], Iwasawa proved that $\text{Gal}(K(F^{\text{cycl}})/F^{\text{cycl}})$ is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module, and he further conjectured that this Galois group is finitely generated over $\mathbb{Z}_p$ (see also [17]). Throughout this article, we shall call this conjecture the Iwasawa $\mu$-conjecture. On the other hand, it has been known that this finite generation property does not hold for the dual of the classical Selmer group of an abelian variety over the cyclotomic $\mathbb{Z}_p$-extension in general (see [27, §10, Example 2]). It was only about a decade ago that Coates and the second named author [6, 36] gave a correct formulation of the analogue of the Iwasawa $\mu$-conjecture for an elliptic curve. Namely, they considered a smaller group, called the fine Selmer group, which is a subgroup of the classical Selmer group, and they conjectured that the Pontryagin dual of this fine Selmer group over $F^{\text{cycl}}$ is finitely generated over $\mathbb{Z}_p$ [6, Conjecture A]. Since then, analogues of this conjecture have been formulated for fine Selmer groups attached to more general Galois representations (see [2, 19, 20, 24, 26]). In this paper,
we shall collectively (and loosely) address these conjectures as Conjecture A. A striking observation is that, besides being a natural analogue of the Iwasawa $\mu$-conjecture, Conjecture A is related to the latter conjecture in a very precise manner (see [6, Theorem 3.4], [24, Theorem 3.5], [26, Theorem 5.5], [36, Theorem 4.5] or [39, Section 8]; also see Theorem 3.1 below).

In their paper [6], Coates and the second author also studied the structure of the fine Selmer group over extensions of $F$ whose Galois group $G = \text{Gal}(F_\infty/F)$ is a $p$-adic Lie group of dimension larger than 1. There they formulated an important conjecture on the structure of the Pontryagin dual of the fine Selmer group of an abelian variety which predicts that the said module is pseudo-null over the Iwasawa algebra $\mathbb{Z}_p[G]$ (see [6, Conjecture B]). To some extent, their conjecture can be thought as an analogue of a conjecture of Greenberg, which we now briefly describe. Recall that a Galois extension $F_\infty$ of $F$ is said to be a strongly admissible, pro-$p$, $p$-adic Lie extension of $F$ if (i) $G = \text{Gal}(F_\infty/F)$ is a compact pro-$p$, $p$-adic Lie group without $p$-torsion, (ii) $F_\infty$ contains the cyclotomic $\mathbb{Z}_p$ extension $F^{\text{cyc}}$ of $F$ and (iii) $F_\infty$ is unramified outside a finite set of primes. We denote by $H$ the Galois group $\text{Gal}(F_\infty/F^{\text{cyc}})$. Let $K(F_\infty)$ denote the maximal unramified abelian pro-$p$ extension of $F_\infty$ in which every prime above $p$ splits completely. When $F_\infty$ is the composite of all the $\mathbb{Z}_p$-extensions of $F$, Greenberg [11] conjectured that $\text{Gal}(K(F_\infty)/F_\infty)$ is pseudo-null over $\mathbb{Z}_p[G]$. (Actually, to be more precise, Greenberg’s original conjecture is concerned with the pseudo-nullity of a slightly bigger Galois group.) For a general $F_\infty$, the validity of the pseudo-nullity of $\text{Gal}(K(F_\infty)/F_\infty)$ is not guaranteed, and this was first observed by Hachimori and Sharifi in [13], where they constructed a class of extensions $F_\infty$ whose Galois group $\text{Gal}(K(F_\infty)/F_\infty)$ is not pseudo-null. Despite these constructions of Hachimori and Sharifi, Coates and the second named author have expressed optimism that the corresponding assertion for the dual fine Selmer group of an elliptic curve should hold, for strongly admissible extensions $F_\infty$ (see [6, Section 4]).

Since then, the question of the pseudo-nullity of the dual fine Selmer group over a general, strongly admissible $p$-adic Lie extension has been the subject of much study (see [3, 19, 23, 24, 30]). In [19], Jha formulated an analogue of this conjecture for a Hida family and its specialization. In his work, Jha was able to show that if the dual fine Selmer group of one specialization of the Hida family is pseudo-null, then the dual fine Selmer groups for all but finitely many specializations of the Hida family are also pseudo-null (see [19, Theorem 10]). Of course, in view of the conjectures of Coates-Sujatha and Jha, one expects that every (arithmetic) specialization has a pseudo-null dual fine Selmer group, and therefore, the theorem of Jha gives a very strong evidence to this. A careful examination of Jha’s proof actually yields a criterion of determining which arithmetic specialization has a pseudo-null dual fine Selmer group, and the determining criterion relies on the structure of the central torsion submodule of the dual fine Selmer group of the big Galois representation. In view of Jha’s result, it is then natural to investigate the preservation of pseudo-nullity of the fine Selmer groups of congruent Galois representations in general. More precisely, motivated by Jha’s theorem, the following natural question is of interest: Suppose that the dual fine Selmer group of one of the congruent Galois representations is pseudo-null, can one deduce the pseudo-nullity of the fine Selmer group of the other representation via a criterion on the structure of the dual fine Selmer group of the initial representation? The primary goal of this paper is to develop such
a criterion. As will be seen below, our criterion depends on the \( p \)-primary submodule of the dual fine Selmer group of the initial representation (see Theorems 3.7 and 3.5). We emphasise that these dual fine Selmer groups are expected to have trivial \( \mu_{O[H]} \)-invariants and hence their \( p \)-primary submodules are pseudo-null as \( O[G] \)-modules. However, as \( O[H] \)-modules, the structures of these \( p \)-primary submodules are not known and it is not clear what to expect of them. Our results (Theorems 3.7 and 3.5) will therefore consist of considering the situations when the dual fine Selmer group of the initial representation has a trivial \( \mu_{O[H]} \)-invariant and when it does not.

We then apply our criterion to study the relation between \( \text{Gal}(K(F_\infty)/F_\infty) \) and the dual fine Selmer groups. Motivated by the relation between the Iwasawa \( \mu \)-conjecture and Conjecture A, one may ask whether there is an analogous relationship between the pseudo-nullity of \( \text{Gal}(K(F_\infty)/F_\infty) \) and the pseudo-nullity of the dual fine Selmer groups. Of course, the constructions of Hachimori-Sharifi tell us that such an analogue does not hold on the nose. Nevertheless, we can still ask the question of deducing the pseudo-nullity of the dual fine Selmer groups from the knowledge of the pseudo-nullity of the Galois group of the maximal abelian unramified pro-\( p \) extension (see Question B' in Section 4). Some partial result in this direction has been obtained by the first named author in [23, Theorem 2.3]. In this paper, we give a refinement of these results (see Propositions 4.1 and 4.2). We also relate the \( p \)-primary submodule of \( \text{Gal}(K(F_\infty)/F_\infty) \) and the \( \pi \)-primary submodule of the dual fine Selmer group (as \( Z_p[H] \)-modules).

Finally, we return to the situation of an elliptic curve \( E \) with good ordinary reduction at all primes above \( p \). We deduce a relation between Conjecture A and the structure of the Selmer group of the said elliptic curve (see Theorem 5.2). This relation can be thought of as the mod-\( p \) analogue of Mazur’s conjecture, which states that the dual Selmer group over the cyclotomic \( Z_p \)-extension is a torsion module over the Iwasawa algebra \( Z_p[\Gamma] \). It is well known that this statement is equivalent to the defining sequence for the Selmer group being short exact and the validity of an appropriate version of the Weak Leopoldt conjecture. In fact, the corresponding equivalence between the dual Selmer group being torsion over the associated Iwasawa algebra and the defining sequence for the Selmer group being short exact (modulo an appropriate version of the Weak Leopoldt conjecture) is true for more general strongly admissible \( p \)-adic Lie extensions (see [11, Theorem 4.12] or [22, Proposition 3.3]). In our situation, Theorem 5.2 asserts that the dual strict Selmer group of \( E[p] \) being torsion over the associated Iwasawa algebra over \( \mathbb{F}_p \) is equivalent to the defining sequence for the strict Selmer group of \( E[p] \) being short exact and the validity of Conjecture A. (Here the strict Selmer group is in the sense of Greenberg [10].) This result provides conceptual clarity to a result of Vatsal and Greenberg (see [12, Page 18, Statement A]). In particular, we shall apply Theorem 5.2 to give a proof of the said result of Vatsal and Greenberg (see Corollary 5.4).

We now give a brief description of the layout of the paper. In Section 2, we recall certain algebraic notions which will be used subsequently in the paper. In Section 3, we introduce the fine Selmer groups. It is also here that we establish our main results. In Section 4, we then apply the criterion developed in Section 3 to relate the Iwasawa module theoretical structure of Galois groups and fine Selmer groups. In Section 4, we then apply the criterion developed in Section 3 to relate the Iwasawa module theoretical structure of Galois groups and fine Selmer groups. In Section 5, we revisit the situation of an elliptic
curve.

Acknowledgments. Some part of the research of this article took place when the first author was visiting the National Center for Theoretical Sciences, the Institute of Mathematics of Academia Sinica, the University of Toronto and the Ganita Lab. The first author would like to thank these institutes for their hospitality. The authors also thank Manfred Kolster and Romyar Sharifi for many interesting conversations on the subject of $p$-primary submodules of Iwasawa modules. Finally, the first author’s research is supported by the National Natural Science Foundation of China under The Research Fund for International Young Scientists (Grant No: 11550110172), and the second author gratefully acknowledges support from NSERC (Discovery Grant No: 402071).

2 Algebraic Preliminaries

As before, $p$ will denote a prime number. Let $\mathcal{O}$ be the ring of integers of a fixed finite extension of $\mathbb{Q}_p$. For a compact $p$-adic Lie group $G$, the completed group algebra of $G$ over $\mathcal{O}$ is defined by

$$\mathcal{O}[G] = \lim_{\leftarrow} \mathcal{O}[G/U],$$

where $U$ runs over the open normal subgroups of $G$ and the inverse limit is taken with respect to the canonical projection maps. Throughout the paper, we usually work under the assumption that our group $G$ is pro-$p$ and has no $p$-torsion. In this setting, it is known that $\mathcal{O}[G]$ is an Auslander regular ring with no zero divisors (cf. [37, Theorem 3.26] and [29]). Therefore, the ring $\mathcal{O}[G]$ admits a skew field $\mathcal{Q}(G)$ which is flat over $\mathcal{O}[G]$ (see [31, Chapters 6 and 10] or [21, Chapter 4, §9 and §10]). The $\mathcal{O}[G]$-rank of a finitely generated $\mathcal{O}[G]$-module $M$ is then defined to be

$$\text{rank}_{\mathcal{O}[G]}(M) = \dim_{\mathcal{Q}(G)} (\mathcal{Q}(G) \otimes_{\mathcal{O}[G]} M).$$

We say that the $\mathcal{O}[G]$-module $M$ is torsion if $\text{rank}_{\mathcal{O}[G]} M = 0$. We frequently make use of the following well-known equivalent characterization for a torsion $\mathcal{O}[G]$-module without further comment, namely: $\text{Hom}_{\mathcal{O}[G]}(M, \mathcal{O}[G]) = 0$ (for instance, see [24, Lemma 4.2]). A finitely generated torsion $\mathcal{O}[G]$-module $M$ is then said to be pseudo-null if $\text{Ext}^1_{\mathcal{O}[G]}(M, \mathcal{O}[G]) = 0$.

Let $\pi$ denote a fixed local parameter for $\mathcal{O}$ and let $k$ denote the residue field of $\mathcal{O}$. The completed group algebra of $G$ over $k$ is denoted $k[G]$ and we have, as before,

$$k[G] = \lim_{\leftarrow} k[G/U],$$

where $U$ runs over the open normal subgroups of $G$ and the inverse limit is taken with respect to the canonical projection maps. For a compact $p$-adic Lie group $G$ without $p$-torsion, it follows from [37, Theorem 3.30(ii)] (or [24, Theorem A.1]) that $k[G]$ is an Auslander regular ring. Furthermore, if $G$ is pro-$p$, then the ring $k[G]$ has no zero divisors (cf. [1, Theorem C]). Therefore, one can define the notion
of $k[G]$-rank as above when $G$ is pro-$p$ without $p$-torsion. Similarly, we say that a finitely generated $k[G]$-module $N$ is a torsion module if $\text{rank}_{k[G]} N = 0$.

For a given finitely generated $\mathcal{O}[G]$-module $M$, we denote by $M(\pi)$ the $\mathcal{O}[G]$-submodule of $M$ which consists of elements of $M$ that are annihilated by some power of $\pi$. We shall call $M(\pi)$ the $\pi$-primary submodule of $M$. A finitely generated $\mathcal{O}[G]$-module is then said to be $\pi$-primary if $M = M(\pi)$. Since the ring $\mathcal{O}[G]$ is Noetherian, the $\pi$-primary submodule $M(\pi)$ of $M$ is also finitely generated over $\mathcal{O}[G]$. Therefore, one can find an integer $r \geq 0$ such that $\pi^r$ annihilates $M(\pi)$. Following [37, Formula (33)], we define the $\mu$-invariant

$$
\mu_{\mathcal{O}[G]}(M) = \sum_{i \geq 0} \text{rank}_{k[G]} (\pi^i M(\pi)/\pi^{i+1}).
$$

(For another alternative, but equivalent, definition, see [37, Definition 3.32].) By the above discussion and our definition of $k[G]$-rank, the sum on the right is a finite one. It is immediate from the definition that $\mu_{\mathcal{O}[G]}(M) = \mu_{\mathcal{O}[G]}(M(\pi))$.

Now suppose that $G$ is pro-$p$ without $p$-torsion. Then as seen in the discussion above, both rings $\mathcal{O}[G]$ and $k[G]$ are Auslander regular and have no zero divisors. For a finitely generated $\mathcal{O}[G]$-module $M$, it then follows from either [10, Proposition 1.11] or [37, Theorem 3.40] that there is a $\mathcal{O}[G]$-homomorphism

$$
\varphi : M(\pi) \longrightarrow \bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i},
$$

whose kernel and cokernel are pseudo-null $\mathcal{O}[G]$-modules, and where the integers $s$ and $\alpha_i$ are uniquely determined. We call $\bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i}$ the elementary representation of $M(\pi)$. In fact, in the process of establishing the above, one also has that $\mu_{\mathcal{O}[G]}(M) = \sum_{i=1}^s \alpha_i$ (see loc. cit.). We will set

$$
\theta_{\mathcal{O}[G]}(M) := \max_{1 \leq i \leq s} \{\alpha_i\}.
$$

For convenience, we recall some of the necessary results from [25] that will we need here.

**Proposition 2.1.** Let $M$ and $N$ be two finitely generated $\mathcal{O}[G]$-modules such that $M$ is a torsion $\mathcal{O}[G]$-module and such that $\mu_{\mathcal{O}[G]}(M/\pi^i) = \mu_{\mathcal{O}[G]}(N/\pi^i)$ for every $1 \leq i \leq \theta_{\mathcal{O}[G]}(M) + 1$.

Then $N$ is torsion over $\mathcal{O}[G]$ and we have the equality $\theta_{\mathcal{O}[G]}(M) = \theta_{\mathcal{O}[G]}(N)$. Furthermore, $M(\pi)$ and $N(\pi)$ have the same elementary representations as $\mathcal{O}[G]$-modules.

We actually require the following pseudo-null analogue of the above proposition.

**Proposition 2.2.** Let $H$ be a closed normal subgroup of $G$ with $G/H \cong \mathbb{Z}_p$. Let $M$ and $N$ be two finitely generated $\mathcal{O}[G]$-modules which are also finitely generated over $\mathcal{O}[H]$. Suppose that $M$ is a pseudo-null $\mathcal{O}[G]$-module and that $\mu_{\mathcal{O}[H]}(M/\pi^i) = \mu_{\mathcal{O}[H]}(N/\pi^i)$ for every $1 \leq i \leq \theta_{\mathcal{O}[H]}(M) + 1$.

Then $N$ is pseudo-null over $\mathcal{O}[G]$ and we have the equality $\theta_{\mathcal{O}[H]}(M) = \theta_{\mathcal{O}[H]}(N)$. Furthermore, $M(\pi)$ and $N(\pi)$ have the same elementary representations as $\mathcal{O}[H]$-modules.
Proof. If $M$ is an $\mathcal{O}[G]$-module which is finitely generated over $\mathcal{O}[H]$, it then follows from a well-known result of Venjakob [38] that $M$ is a pseudo-null $\mathcal{O}[G]$-module if and only if it is a torsion $\mathcal{O}[H]$-module. The conclusion of the proposition is now immediate consequence of Proposition 2.4. \qed

Remark 2.3. Note that we are concerned with the $\mu_{\mathcal{O}[H]}$-invariants in the proposition which may not be zero even if the modules in question are pseudo-null over $\mathcal{O}[G]$. We end this section with two more useful lemmas. For a closed subgroup $U$ of a compact $p$-adic Lie group $H$ and a compact $\mathcal{O}[U]$-module $M$, we denote by $\text{Ind}_U^H(M) := M \hat{\otimes}_{\mathcal{O}[U]} \mathcal{O}[H]$ the compact induction of $M$ from $U$ to $H$. If $M$ is a compact $k[U]$-module, we can also view it as a compact $\mathcal{O}[U]$-module and we have the identification $\text{Ind}_U^H(M) = M \hat{\otimes}_{k[U]} k[H]$.

Lemma 2.4. Let $H$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion and $U$ a closed subgroup of $H$ of dimension at least 1. Let $N$ be a finitely generated $\mathcal{O}[U]$-module which is finitely generated over $\mathcal{O}$, and hence torsion as an $\mathcal{O}[U]$-module. Then $\text{Ind}_U^H(N)$ is a finitely generated torsion $\mathcal{O}[H]$-module with $\mu_{\mathcal{O}[H]}(\text{Ind}_U^H(N)) = 0$.

Proof. By [31] Lemma 5.5, we have an isomorphism

$$\text{Hom}_{\mathcal{O}[H]}(\text{Ind}_U^H(N), \mathcal{O}[H]) \cong \text{Ind}_U^H \left( \text{Hom}_{\mathcal{O}[U]}(N, \mathcal{O}[U]) \right).$$

Since $U$ has dimension at least 1 and $N$ is finitely generated over $\mathcal{O}$, it follows that $N$ is torsion over $\mathcal{O}[U]$. Thus, the term on the right of the isomorphism is zero, and by the isomorphism, this in turns implies that $\text{Ind}_U^H(N)$ is a finitely generated torsion $\mathcal{O}[H]$-module. Again, using the fact that $U$ has dimension at least 1 and that $N$ is finitely generated over $\mathcal{O}$, we have that $\pi^n N(\pi)/\pi^{n+1}$ is torsion over $k[U]$. Since $\text{Ind}_U^H(-)$ is an exact functor (cf. [31] Lemma 6.10.8]), it follows from a straightforward argument that

$$\pi^n \left( (\text{Ind}_U^H N)(\pi) \right) / \pi^{n+1} \cong \text{Ind}_U^H \left( \pi^n N(\pi)/\pi^{n+1} \right).$$

Via a similar argument as above, we have that $\pi^n \left( (\text{Ind}_U^H N)(\pi) \right) / \pi^{n+1}$ is a torsion $k[H]$-module. Combining this observation with the definition of the $\mu_{\mathcal{O}[H]}$-invariant will yield that $\mu_{\mathcal{O}[H]}(\text{Ind}_U^H(N)) = 0$. \qed

The next lemma is a refinement of [25] Lemma 2.4.3.

Lemma 2.5. Suppose that $H$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $\varphi : M \rightarrow N$ be a homomorphism of finitely generated $\mathcal{O}[H]$-modules, whose kernel and cokernel are finitely generated torsion $\mathcal{O}[H]$-modules with trivial $\mu_{\mathcal{O}[H]}$-invariants. Then $M(\pi)$ and $N(\pi)$ have the same elementary representations as $\mathcal{O}[H]$-modules.

Proof. The statement will follow if it holds in the two special cases of exact sequences

$$0 \rightarrow P \rightarrow M \rightarrow N \rightarrow 0,$$

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0,$$
where $P$ is a finitely generated $\mathcal{O}[H]$-module with trivial $\mu_{\mathcal{O}[H]}$-invariant. We will prove the second case, the first case has a similar argument. Choose a sufficiently large $n$ such that $\pi^n$ annihilates $M(\pi)$ and $N(\pi)$. Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\
\downarrow{\pi^n} & & \downarrow{\pi^n} & & \downarrow{\pi^n} & & \downarrow{\pi^n} & & \downarrow{0} \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\
\end{array}
$$

with exact rows, and the vertical maps are given by multiplication by $\pi^n$. From this, we obtain an exact sequence

$$
0 \longrightarrow M(\pi) \xrightarrow{\varphi} N(\pi) \longrightarrow P(\pi).
$$

Since $\mu_{\mathcal{O}[H]}(P) = 0$, it follows from [37, Remark 3.33] that $P(\pi)$ is a pseudo-null $\mathcal{O}[H]$-module. This gives the required conclusion in the lemma. Let

$$
f : N(\pi) \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}[G]/\pi^{\alpha_i}
$$

be a homomorphism of $\mathcal{O}[H]$-modules, whose kernel and cokernel are pseudo-null $\mathcal{O}[H]$-modules. Then

$$
f \circ \varphi : M(\pi) \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}[G]/\pi^{\alpha_i}
$$

is a homomorphism of $\mathcal{O}[H]$-modules, whose kernel and cokernel are pseudo-null. Therefore, $M(\pi)$ and $N(\pi)$ have the same elementary representations.

3 Fine Selmer groups

We now move to arithmetic. As before, $p$ denotes a prime. Let $F$ be a number field. If $p = 2$, assume further that the number field $F$ has no real primes. Let $K$ be a fixed finite extension of $\mathbb{Q}_p$, whose ring of integers is denoted $\mathcal{O}$. We shall also fix a choice of local parameter $\pi$ for $\mathcal{O}$. Suppose that we are given a finite dimensional $K$-vector space $V$ with a continuous $\text{Gal}(\overline{F}/F)$-action which is unramified outside a finite set of primes. By a standard compactness argument, $V$ contains a $\text{Gal}(\overline{F}/F)$-stable $\mathcal{O}$-lattice which we will fix once and for all, and denote it by $T$. Write $A = V/T$.

Let $S$ denote a finite set of primes of $F$ which contains all the primes above $p$, the ramified primes of $A$ and the infinite primes. We then denote by $F_S$ the maximal algebraic extension of $F$ unramified outside $S$. For any algebraic (possibly infinite) extension $L$ of $F$ contained in $F_S$, we shall write $G_S(L) = \text{Gal}(F_S/L)$.

Let $v$ be a prime in $S$. For each finite extension $L$ of $F$ contained in $F_S$, we define

$$
K^i_v(A/L) = \bigoplus_{w|v} H^i(L_w, A) \quad (i = 0, 1),
$$
where \( w \) runs over the (finite) set of primes of \( L \) above \( v \). If \( L \) is an infinite extension of \( F \) contained in \( F_S \), we define
\[
K_w^1(A/L) = \lim_{\rightarrow} K_w^1(A/L),
\]
where the direct limit is taken over all finite extensions \( L \) of \( F \) contained in \( L \) under the restriction maps on the cohomology.

For any algebraic (possibly infinite) extension \( L \) of \( F \) contained in \( F_S \), the fine Selmer group of \( A \) over \( L \) (with respect to \( S \)) is defined to be
\[
R_S(A/L) = \ker \left( H^1(G_S(L), A) \rightarrow \bigoplus_{v \in S} K^1_w(A/L) \right).
\]
We shall write \( Y_S(A/L) \) for the Pontryagin dual \( R_S(A/L)^\vee \) of the fine Selmer group. The following conjecture, first formally stated in [6], which is also implicit in [17, 18] (see discussion below), is now folklore.

**Conjecture A.** For any number field \( F \), \( Y_S(A/F^{\text{cyc}}) \) is a finitely generated \( \mathcal{O} \)-module.

We take this opportunity to highlight the following observation. For any extension \( L \) of \( F^{\text{cyc}} \) contained in \( F_S \), let \( K(L) \) be the maximal unramified pro-\( p \) extension of \( L \) where every prime of \( L \) above \( p \) splits completely. It then follows that every finite prime of \( L \) splits completely in \( K(L) \). Therefore, in the case when \( V = Q_p \) and \( A = Q_p/Z_p \), the dual of the fine Selmer group \( Y_S(A/F^{\text{cyc}}) \) is precisely \( \text{Gal}(K(F^{\text{cyc}})/F^{\text{cyc}}) \). In this context, Conjecture A (for \( A = Q_p/Z_p \)) is equivalent to the conjecture made by Iwasawa [17, 18] which we shall call the Iwasawa \( \mu \)-conjecture for \( F^{\text{cyc}} \).

For the purpose of the paper, we will require another equivalent formulation of Conjecture A. Write \( T^* = \text{Hom}_{cts}(A, \mu_{p\infty}) \), where \( \mu_{p\infty} \) denotes the group of all \( p \)-power roots of unity. (One should not confuse this with \( T \).) In fact, it’s not difficult to show that \( T^* = \text{Hom}_{\mathcal{O}}(T, \mathcal{O}(1)) \), where ‘(1)’ means Tate twist. The relationship between \( T, T^* \) and \( A \) is illustrated by the following diagram
\[
\begin{array}{ccc}
T & \xrightarrow{\text{Hom}_{\mathcal{O}}(-, \mathcal{O}(1))} & T^* \\
\downarrow & & \downarrow \\
\oplus_{\mathcal{O}} K/A & \xrightarrow{\text{Hom}_{cts}(-, \mu_{p\infty})} & A
\end{array}
\]
By an application of the Poitou-Tate duality (see [6] Eqn. (45)), we have the following exact sequence
\[
0 \rightarrow Y_S(A/L) \rightarrow H^2_{Iw}(\mathcal{L}/F, T^*) \rightarrow \left( \bigoplus_{v \in S} K_v^0(A/L) \right)^\vee,
\]
where \( H^2_{Iw}(\mathcal{L}/F, T^*) = \lim_{\rightarrow} H^2(G_S(L), T^*) \). By [6] Lemma 3.2 (or see [24] Lemma 3.4), Conjecture A holds for \( A \) over \( F^{\text{cyc}} \) if and only if \( H^2_{Iw}(F^{\text{cyc}}/F, T^*) \) is finitely generated over \( \mathcal{O} \). We will frequently make use of this equivalent formulation of Conjecture A without further comment.
As mentioned in the introductory section, Conjecture A is intimately related to the Iwasawa \( \mu \)-conjecture. This was observed by Coates and the second author in the context of an elliptic curve (see [6, Theorem 3.4]). Subsequently, this observation has been generalized to more general \( p \)-adic representations (for instance, see [20, Theorem 8] and [24, Theorem 3.5]). We now recall this result. Let \( \bar{\rho} : G_S(F) \rightarrow \text{Aut}_\mathbb{A}(E[\pi]) \) be the representation induced by the Galois action of \( G_S(F) \) on \( E[\pi] \). Denote by \( F(E[\pi]) \) the subextension of \( F_S \) which is fixed by the kernel of \( \bar{\rho} \). Note that this is a finite Galois extension of \( F \). The following is a slight refinement of the observation of Coates and the second author.

**Theorem 3.1.** Let \( F_{\text{cyc}} \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \). Suppose that \( F(E[\pi]) \) is a finite \( p \)-extension of \( F \). Then Conjecture A holds for \( E \) over \( F_{\text{cyc}} \) if and only if the Iwasawa \( \mu \)-conjecture holds for \( F_{\text{cyc}} \).

There are a number of approaches towards proving the above theorem (see [6, Theorem 3.4], [20, Theorem 8], [24, Theorem 3.5], [26, Theorem 5.5] and [36, Theorem 4.5]). We will give a proof of the above theorem along the lines of [20, Theorem 8] and [24, Theorem 3.5]. To prepare for the proof, we need to introduce another Galois representation. Let \( V' \) be another finite dimensional \( K \)-vector space with a continuous \( \text{Gal}(\bar{F}/F) \)-action which is unramified outside a finite set of primes. Write \( B = V'/T' \) for a (fixed) Galois invariant lattice \( T' \) of \( V' \). From now on, our (finite) set \( S \) of primes is always assumed to contain the primes above \( p \), the ramified primes of \( A \) and \( B \), and the infinite primes. We can now state the following.

**Proposition 3.2.** Suppose that there is an isomorphism \( A[\pi] \cong B[\pi] \) of \( G_S(F) \)-modules. Then Conjecture A holds for \( Y_S(A/F_{\text{cyc}}) \) if and only if Conjecture A holds for \( Y_S(B/F_{\text{cyc}}) \).

**Proof.** For \( Z = A, B \), by a similar argument to that in [36, Proposition 4.6], one can show that Conjecture A is equivalent to the assertion that \( H^2(G_S(F_{\text{cyc}}), Z[\pi]) = 0 \). The conclusion is now immediate from the congruence condition.

We now give the proof of Theorem 3.1.

**Proof of Theorem 3.1** Suppose, for now, that \( L' \) is a finite \( p \)-extension of \( L \). By a classical argument (cf. [17, Theorem 3]), we have that the Iwasawa \( \mu \)-conjecture holds for \( L'_{\text{cyc}} \) if and only if the Iwasawa \( \mu \)-conjecture holds for \( L_{\text{cyc}} \). Via a similar argument to that in [26, Theorem 5.5], we also have that Conjecture A holds for \( Y_S(A/L_{\text{cyc}}) \) if and only if Conjecture A holds for \( Y_S(A/L'_{\text{cyc}}) \). Hence it suffices to prove the equivalence in the theorem over \( F(A[\pi]) \). In particular, without loss of generality, we may assume that \( F = F(A[\pi]) \). It then follows from this that \( A[\pi] \) is a trivial \( G_S(F) \)-module and that \( A[\pi] \cong (O/\pi)^d \cong (\mathbb{Z}/p)^{jd} \) as \( G_S(F) \)-modules. Here \( d = \text{corank}_O(A) \) and \( f = [K : \mathbb{Q}_p] \). By Proposition 3.2 Conjecture A holds for \( Y_S(A/F_{\text{cyc}}) \) if and only if Conjecture A holds for \( Y_S((\mathbb{Q}_p/\mathbb{Z}_p)^{jd}/F_{\text{cyc}}) \). But the latter clearly holds if and only if the Iwasawa \( \mu \)-conjecture holds for \( F_{\text{cyc}} \).

We now turn to the situation of a higher dimensional \( p \)-adic Lie extension. Recall that a Galois extension \( F_{\infty} \) of \( F \) is said to be a strongly \( S \)-admissible, pro-\( p \), \( p \)-adic Lie extension of \( F \) if (i) \( \text{Gal}(F_{\infty}/F) \)
Lemma 3.3. Let $F_\infty$ be a strongly $S$-admissible pro-$p$, $p$-adic Lie extension of $F$. Then the following statements are equivalent.

(a) $Y_S(A/F^{\text{cyc}})$ is a finitely generated $\mathcal{O}$-module.

(b) $H^2_{\text{Iw}}(F^{\text{cyc}}/F,T^*)$ is a finitely generated $\mathcal{O}$-module.

(c) $Y_S(A/F_\infty)$ is a finitely generated $\mathcal{O}[[H]]$-module.

(d) $H^2_{\text{Iw}}(F_\infty/F,T^*)$ is a finitely generated $\mathcal{O}[[H]]$-module.

Proof. The proof is entirely similar to that in [6, Lemma 3.2].

By a well-known result of Venjakob [38], we have that an $\mathcal{O}[[G]]$-module $M$ which is finitely generated over $\mathcal{O}[[H]]$, is a pseudo-null $\mathcal{O}[[G]]$-module if and only if it is a torsion $\mathcal{O}[[H]]$-module. In view of this, we may now pose the following question.

Question B: Let $F_\infty$ be a strongly $S$-admissible, pro-$p$, $p$-adic Lie extension of $F$ of dimension $> 1$, and suppose that Conjecture A holds for $Y_S(A/F^{\text{cyc}})$. Is $Y_S(A/F_\infty)$ a pseudo-null $\mathcal{O}[[G]]$-module, or equivalently a torsion $\mathcal{O}[[H]]$-module?

When $A$ is the (discrete) quotient module of a Galois representation coming from arithmetic, the assertion of the question is conjectured to always hold (see [6, 13, 24, 30]). When $A = \mathbb{Q}_p/\mathbb{Z}_p$, the dual fine Selmer group is precisely $\text{Gal}(K(F_\infty)/F_\infty)$, where $K(F_\infty)$ is the maximal unramified pro-$p$ extension of $F_\infty$ at which every prime of $F_\infty$ above $p$ splits completely. In this case, it is expected that the pseudo-nullity property should hold for admissible $p$-adic Lie extensions “coming from algebraic geometry” (see [13, Question 1.3] for details, and see also [34, Conjecture 7.6] for a related assertion and [35] for positive results in this direction). However, we should mention that Hachimori and Sharifi [13] constructed a class of admissible $p$-adic Lie extensions $F_\infty$ of $F$ of dimension $> 1$ such that $\text{Gal}(K(F_\infty)/F_\infty)$ is not pseudo-null. Note that their extensions come from CM fields, where it is generally expected that these fields cannot be realized as admissible $p$-adic Lie extensions which are carved out by algebraic geometrical objects (see [13, Page 570]).

We now record a lemma which relates the module theoretical structure of the dual fine Selmer groups and the second Iwasawa cohomology groups over certain strongly $S$-admissible, pro-$p$, $p$-adic Lie extensions.

Lemma 3.4. Let $F_\infty$ be a strongly $S$-admissible, pro-$p$, $p$-adic Lie extension of $F$ of dimension $> 1$. Suppose that the following conditions are satisfied.
(i) Conjecture A holds for \( Y_S(A/F^{\text{cyc}}) \).

(ii) For every \( v \in S \), the decomposition group of \( G \) at \( v \) has dimension \( \geq 2 \).

Then \( Y_S(A/F_\infty) \) is a pseudo-null \( \mathcal{O}[G] \)-module if and only if \( H^2_{w}(F_\infty/F,T^*) \) is a pseudo-null \( \mathcal{O}[G] \)-module. Furthermore, \( Y_S(A/F_\infty)(\pi) \) and \( H^2_{w}(F_\infty/F,T^*)(\pi) \) have the same elementary representations as \( \mathcal{O}[H] \)-modules.

Proof. By the Poitou-Tate sequence (see [6] Eqn. (45)), we have the following exact sequence

\[
0 \longrightarrow Y_S(A/F_\infty) \longrightarrow H^2_{w}(F_\infty/F,T^*) \longrightarrow \left( \bigoplus_{v \in S} K^0_v(A/F_\infty) \right)^{\vee}.
\]

In view of assumption (i) and Lemma 3.3, the first two terms in the exact sequence are finitely generated \( \mathcal{O}[H] \)-modules. Therefore, by virtue of Lemma 2.5, the assertions of the lemma will follow once we can show that the last term in the exact sequence is a finitely generated torsion \( \mathcal{O}[H] \)-module with trivial \( \mu_{\mathcal{O}[H]} \)-invariant. For each \( v \in S \), fix a prime \( w \) of \( F_\infty \) above \( v \). By abuse of notation, we shall also denote the prime of \( F^{\text{cyc}} \) below \( w \) by \( w \). Denote by \( H_w \) the decomposition group of \( H \) at \( w \). Then one can check that \( K^0_v(A/F_\infty)^{\vee} \) is isomorphic to a finite sum of terms of the form

\[
\text{Ind}^{H_w}_{H} \left( A(F_\infty,w)^{\vee} \right),
\]

where \( A(F_\infty,w) = \text{Gal}(F_w/F_\infty,w) \). Since \( (A(F_\infty,w)^{\vee} \) is finitely generated over \( \mathcal{O} \) and \( H_w \) has dimension at least 1 by hypothesis (ii), we may apply Lemma 2.4 to conclude that \( \text{Ind}^{H_w}_{H} (A(F_\infty,w)^{\vee}) \) is finitely generated over \( \mathcal{O}[H] \) with trivial \( \mu_{\mathcal{O}[H]} \)-invariant. Thus, the lemma is proven. \( \square \)

We now come to the main theme of the paper which is to study the preservation of the pseudo-nullity property under congruences. We first mention that under the assumption of the validity of Conjecture A, it follows from Lemma 3.3 that \( Y_S(A/F_\infty) \) is a finitely generated \( \mathcal{O}[H] \)-module. By a standard result of Howson [15] Lemma 2.7, this in turn implies that \( \mu_{\mathcal{O}[G]}(Y_S(A/F_\infty)) = 0 \). By [37] Remark 3.33, it then follows that \( Y_S(A/F_\infty)(\pi) \) is pseudo-null as an \( \mathcal{O}[G] \)-module. However, being a finitely generated \( \mathcal{O}[H] \)-module, the structure of \( Y_S(A/F_\infty)(\pi) \) is a more subtle issue, and this is precisely the point of our next two theorems when considering the preservation of the pseudo-nullity of the dual fine Selmer groups under congruences.

Theorem 3.5. Let \( F_\infty \) be a strongly \( S \)-admissible pro-p \( p \)-adic Lie extension of \( F \) of dimension \( > 1 \). Suppose that the following conditions are satisfied.

(a) There is an isomorphism \( A[\pi^{\theta_A+1}] \cong B[\pi^0\theta_A+1] \) of \( G_S(F) \)-modules, where \( \theta_A := \theta_{\mathcal{O}[H]}(Y_S(A/F_\infty)) \).

(b) Conjecture A holds for \( Y_S(A/F^{\text{cyc}}) \) (and hence for \( Y_S(B/F^{\text{cyc}}) \) by Proposition 3.2).

(c) \( Y_S(A/F_\infty) \) is a pseudo-null \( \mathcal{O}[G] \)-module.

(d) For each \( v \in S \), the decomposition group of \( G \) at \( v \) has dimension \( \geq 2 \).
Then $Y_S(B/F_\infty)$ is a pseudo-null $\mathcal{O}[G]$-module. Furthermore, $Y_S(A/F_\infty)(\pi)$ and $Y_S(B/F_\infty)(\pi)$ have the same elementary representations as $\mathcal{O}[H]$-modules.

**Remark 3.6.** Note that it follows from assumption (b) that $Y_S(A/F_\infty)(\pi)$ and $Y_S(B/F_\infty)(\pi)$ have the same elementary representations as $\mathcal{O}[G]$-modules. In fact, as seen in the discussion before Theorem 3.5, they are pseudo-null as $\mathcal{O}[G]$-modules. **The point of our theorem is that we can relate the $\mathcal{O}[H]$-module structure of the $\pi$-primary submodules of the two dual fine Selmer groups.**

**Proof of Theorem 3.5.** For every $i \geq 0$ and every open subgroup $G_0$ of $G$, one has an isomorphism

$$\text{Ext}_i^{\mathcal{O}[G]}(M, \mathcal{O}[G]) \cong \text{Ext}_i^{\mathcal{O}[G_0]}(M, \mathcal{O}[G_0])$$

for any finitely generated $\mathcal{O}[G]$-module $M$ (cf. [28 Proposition 5.4.17]). Therefore, replacing $F$ if necessary, we may assume that $H$ and $G$ are uniform pro-$p$ groups. By Lemma 3.4, we may work with second Iwasawa cohomology groups which we shall do for the proof of the theorem. By Proposition 2.2 it remains to show that

$$\mu_{\mathcal{O}[H]}(H^2_{Iw}(F_\infty/F, T_A^*/\pi^n)) = \mu_{\mathcal{O}[H]}(H^2_{Iw}(F_\infty/F, T_B^*/\pi^n))$$

for $1 \leq n \leq \theta_H(A) + 1$. Fix such an arbitrary $n$. We now note that it follows from [24 Lemma 2.1] that $H^2_{Iw}(F_\infty/F, T_A^*/\pi^n) \cong H^2_{Iw}(F_\infty/F, T_B^*/\pi^nT_B^*)$ for $Z = A, B$. By the congruence condition (a) of the hypotheses, we have an isomorphism $T_A^*/\pi^nT_A^* \cong T_B^*/\pi^nT_B^*$ of $G_S(F)$-modules which in turn induces an isomorphism

$$H^2_{Iw}(F_\infty/F, T_A^*/\pi^nT_A^*) \cong H^2_{Iw}(F_\infty/F, T_B^*/\pi^nT_B^*)$$

of $\mathcal{O}[H]$-modules. The required equality of $\mu_{\mathcal{O}[H]}$-invariants is now an immediate consequence of this. Thus, we have proven the theorem.

When $\mu_{\mathcal{O}[H]}(Y_S(A/F_\infty)) = 0$, we can prove the following theorem which has slightly weaker requirement on the decomposition groups of the $p$-adic Lie extension. Due to this weaker assumption, we cannot employ Lemma 3.4 and so we have to work with the dual fine Selmer groups directly.

**Theorem 3.7.** Let $F_\infty$ be a strongly $S$-admissible pro-$p$ $p$-adic Lie extension of $F$ of dimension $> 1$. Suppose that the following conditions are satisfied.

(a) There is an isomorphism $A[\pi] \cong B[\pi]$ of $G_S(F)$-modules.

(b) Conjecture A holds for $Y_S(A/F^{\text{cyc}})$ (and hence for $Y_S(B/F^{\text{cyc}})$ by Proposition 3.2).

(c) We have $\mu_{\mathcal{O}[H]}(Y_S(A/F_\infty)) = 0$, and hence $Y_S(A/F_\infty)$ is a pseudo-null $\mathcal{O}[G]$-module.

(d) For each $v \in S$, either one of the following holds.

(i) The decomposition group of $G$ at $v$, denoted by $G_v$, has dimension $\geq 2$.

(ii) For every prime $w$ of $F_\infty$ above $v$, $B(F_{\infty, w})$ is a divisible $\mathcal{O}$-module.
Then $Y_S(B/F_\infty)$ is a pseudo-null $O[G]$-module. Moreover, we have $\mu_{O[H]}(Y_S(B/F_\infty)) = 0$.

**Proof.** It follows from Lemma [5.3] and assumptions (b) and (c) that $Y_S(A/F_\infty)$ is a finitely generated torsion $O[H]$-module. By [13 Corollary 1.10] (or [24 Proposition 4.12]), we then have

$$0 = \text{rank}_{O[H]}(Y_S(A/F_\infty)) = \text{rank}_{k[H]}(Y_S(A/F_\infty)/\pi) - \text{rank}_{k[H]}(Y_S(A/F_\infty)[\pi]).$$

On the other hand, by an application of [13 Proposition 1.6, Corollary 1.7], we have

$$\text{rank}_{k[H]}(Y_S(A/F_\infty)[\pi]) = \mu_{O[H]}(Y_S(A/F_\infty)[\pi]),$$

and the latter quantity is zero as a consequence of assumption (c). Hence we may conclude that $Y_S(A/F_\infty)/\pi$ is a torsion $k[H]$-module. To continue, we need to introduce the $\pi$-fine Selmer groups. For a prime $v$ in $S$, and for each finite extension $L$ of $F$ contained in $F_\infty$, we set

$$K_v^1(A[\pi]/L) = \bigoplus_{w \mid v} H^1(L_w, A[\pi]),$$

where $w$ runs over the (finite) set of primes of $L$ above $v$. The $\pi$-fine Selmer group (with respect to $S$) is then defined to be

$$R_S(A[\pi]/L) = \ker \left( H^1(G_S(L), A[\pi]) \rightarrow \bigoplus_{v \in S} K_v^1(A[\pi]/L) \right).$$

Set $R_S(A[\pi]/F_\infty) = \varprojlim_l R_S(A[\pi]/L)$, where $L$ runs through all finite subextensions of $F_\infty/F$. We then have the following diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & R_S(A[\pi]/F_\infty) & \rightarrow & H^1(G_S(F_\infty), A[\pi]) & \rightarrow \bigoplus_{v \in S} K_v(A[\pi]/F_\infty) \\
& & \downarrow f_A & & \downarrow g_A & \downarrow h_A \\
0 & \rightarrow & R_S(B/F_\infty)[\pi] & \rightarrow & H^1(G_S(F_\infty), B[\pi]) & \rightarrow \bigoplus_{v \in S} K_v(B[\pi]/F_\infty) \\
& & \downarrow f_B & & \downarrow g_B & \downarrow h_B \\
0 & \rightarrow & R_S(B/F_\infty)[\pi] & \rightarrow & H^1(G_S(F_\infty), B)[\pi] & \rightarrow \bigoplus_{v \in S} K_v(B/F_\infty)[\pi]
\end{array}
$$

with exact rows. The long exact sequence in cohomology arising from $0 \rightarrow A[\pi] \rightarrow A \rightarrow A \rightarrow 0$ shows that $\ker g_A$ is finite. Thus, $\ker f_A$ is also finite. Since $H$ has dimension $\geq 1$, the Pontryagin dual of $\ker f_A$ is a torsion $k[H]$-module. Therefore, we have a $k[H]$-homomorphism

$$Y_S(A/F_\infty)/\pi \rightarrow Y_S(A[\pi]/F_\infty)$$

with cokernel which is a torsion $k[H]$-module. Since we have already shown above that $Y_S(A/F_\infty)/\pi$ is a torsion $k[H]$-module, it follows that $Y_S(A[\pi]/F_\infty)$ is also a torsion $k[H]$-module. By assumption (a), this in turn implies that $Y_S(B[\pi]/F_\infty)$ is a torsion $k[H]$-module. Now consider the following diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & R_S(B[\pi]/F_\infty) & \rightarrow & H^1(G_S(F_\infty), B[\pi]) & \rightarrow \bigoplus_{v \in S} K_v(B[\pi]/F_\infty) \\
& & \downarrow f_B & & \downarrow g_B & \downarrow h_B \\
0 & \rightarrow & R_S(B/F_\infty)[\pi] & \rightarrow & H^1(G_S(F_\infty), B)[\pi] & \rightarrow \bigoplus_{v \in S} K_v(B/F_\infty)[\pi]
\end{array}
$$
with exact rows. By a similar cohomological argument as above, the map $g_B$ has finite kernel and trivial cokernel. We shall now show that $\ker h_B$ is a cofinitely generated torsion $k[H]$-module. Write $h_B = \oplus_v h_{B,v}$, where $v$ runs over the (finite) set of primes of $F^{cy}$ above $S$. Denote by $H_v$ the decomposition group of $F_\infty/F^{cy}$ corresponding to a fixed prime of $F_\infty$, which we also denote by $v$, above $w$. Then we have $\ker h_{B,v} = \text{Coind}^H_B (B(F_\infty,w)/\pi)$, where $v$ is the prime of $F$ below $w$. If $v$ satisfies assumption (d)(ii), then $\ker h_{B,v} = 0$. Now suppose that $v$ satisfies assumption (d)(i), a similar argument to that in Lemma 3.4 shows that $\ker h_{B,v}$ is a cotorsion $k[H]$-module. In conclusion, we have shown that $\ker h_B$ is a cofinitely generated torsion $k[H]$-module. By a diagram chasing argument, one then has that the cokernel of the map $f_B$ is a cofinitely generated torsion $k[H]$-module. Hence we have a $k[H]$-homomorphism

$$Y_S(B/F_\infty)/\pi \longrightarrow Y_S(B[\pi]/F_\infty)$$

whose kernel and cokernel are torsion $k[H]$-modules. Combining this with the above observation that $Y_S(B[\pi]/F_\infty)$ is torsion over $k[H]$, we have that $Y_S(B/F_\infty)/\pi$ is torsion over $k[H]$. By [15, Corollary 1.10] (or [24, Proposition 4.12]), we have

$$\text{rank}_{O[H]} (Y_S(B/F_\infty)) = \text{rank}_{k[H]} (Y_S(B/F_\infty)/\pi) - \text{rank}_{k[H]} (Y_S(B/F_\infty)[\pi]).$$

From these, it follows that $Y_S(B/F_\infty)$ is a finitely generated torsion $O[H]$-module and $Y_S(B/F_\infty)[\pi]$ is a finitely generated torsion $k[H]$-module. The latter is equivalent to $\mu_{O[H]}(Y_S(B/F_\infty)) = 0$ by Remark 3.33.

\section{Comparing Galois groups and fine Selmer groups}

As before, $p$ denote a prime and $F$ a number field. If $p = 2$, assume further that $F$ has no real primes. Let $O$ be the ring of integers of a fixed finite extension $K$ of $Q_p$. Let $A$ denote the quotient module of a finite dimensional $K$-vector space, which is endowed with a continuous $G_S(F)$-action for a finite set $S$ of primes. We shall also assume that the set $S$ contains all the primes above $p$, the ramified primes of $A$ and the infinite primes. Inspired by the relation between the Iwasawa $\mu$-conjecture and Conjecture $A$, the first author is led to ask the following question (see [23]).

\textbf{Question B'}: Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of a number field $F$ of dimension $> 1$ with the property that $G_S(F_\infty)$ acts trivially on $A[\pi]$. Suppose that $\text{Gal}(K(F_\infty)/F_\infty)$ is a finitely generated $Z_p[H]$-module. Can one deduce that $Y_S(A/F_\infty)$ is a pseudo-null $O[G]$-module from the knowledge that $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $Z_p[G]$-module?

In the same paper, the first author gave a partial answer to the above question (see [23, Theorem 2.3]). We now apply the criterion in Section 3 to derive refinements of this result. As a start, we have the following.

\textbf{Proposition 4.1.} Suppose that $F$ contains $\mu_p$. Let $F_\infty$ be a strongly $S$-admissible pro-$p$ $p$-adic Lie extension of $F$ of dimension $> 1$. Suppose that the following conditions are satisfied.
(a) $G_S(F_\infty)$ acts trivially on $A[\pi]$.

(b) Conjecture A holds for $\text{Gal}(K(F_\infty)/F_\infty)$.

(c) $\mu_{Z_p[H]}(\text{Gal}(K(F_\infty)/F_\infty)) = 0$. In particular, $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $Z_p[[G]]$-module.

(d) For each $v \in S$, either one of the following holds.

(i) The decomposition group of $G$ at $v$, denoted by $G_v$, has dimension $\geq 2$.

(ii) For every prime $w$ of $F_\infty$ above $v$, $A(F_{\infty,w})$ is a divisible $\mathcal{O}$-module.

Then $Y_S(A/F_\infty)$ is a pseudo-null $\mathcal{O}[G]$-module.

Proof. Since $\mathcal{O}$ is free over $\mathbb{Z}_p$, we have an isomorphism

$$\text{Ext}^1_{\mathcal{O}[G]}(M, \mathcal{O}[[G]]) \cong \mathcal{O} \otimes_{\mathbb{Z}_p} \text{Ext}^1_{\mathbb{Z}_p[[G]]}(M, \mathcal{O}[[G]])$$

of $\mathcal{O}[[G]]$-modules for every $\mathcal{O}[[G]]$-module $M$. Therefore, it suffices to show that $Y_S(A/F_\infty)$ is a pseudo-null $\mathbb{Z}_p[[G]]$-module. Replacing $F$ by a larger extension if necessary, we may assume that $A[\pi]$ is a trivial $G_S(F)$-module. It then follows that $A[\pi] \cong (\mathcal{O}/\pi)^d \cong (\mathbb{Z}/p)^f$ as $G_S(F)$-modules. Here $d = \text{corank}_\mathcal{O}(A)$ and $f = [K : Q_p]$. Since $Y_S((\mathbb{Q}_p/Z_p)^f/F_\infty) = \text{Gal}(K(F_\infty/F_\infty)^f$, it follows from assumption (c) that $\mu_{Z_p[H]}(Y_S((\mathbb{Q}_p/Z_p)^f/F_\infty)) = 0$ and $Y_S((\mathbb{Q}_p/Z_p)^f/F_\infty)$ is a pseudo-null $\mathbb{Z}_p[[G]]$-module. The conclusion of the proposition now follows from an application of Theorem 3.1.

The next result considers the case when $\mu_{Z_p[H]}(\text{Gal}(K(F_\infty)/F_\infty)) \neq 0$. This result also gives a relation between the $\pi$-primary submodule of the dual fine Selmer group and the $p$-primary submodule of $\text{Gal}(K(F_\infty)/F_\infty)$.

**Proposition 4.2.** Let $F_\infty$ be a strongly $S$-admissible pro-$p$ $p$-adic Lie extension of $F$ of dimension $> 1$. Suppose that the following conditions are satisfied.

(a) $\text{Gal}(K(F_\infty)/F_\infty)$ is a finitely generated torsion $\mathbb{Z}_p[H]$-module.

(b) $G_S(F_\infty)$ acts trivially on $A[\pi^n]$, where

$$n = \left\lfloor \frac{\theta_{\mathcal{O}[H]}(\text{Gal}(K(F_\infty)/F_\infty)) + 1}{e} \right\rfloor$$

and $e$ is the ramification index of $\mathcal{O}/\mathbb{Z}_p$.

(c) For each $v \in S$, the decomposition group of $G$ at $v$ has dimension $\geq 2$.

Then $Y_S(A/F_\infty)$ is a finitely generated torsion $\mathcal{O}[H]$-module, and hence a pseudo-null $\mathcal{O}[G]$-module. Furthermore, $Y_S(A/F_\infty)(\pi)$ and $\text{Gal}(K(F_\infty)/F_\infty)(p)^f$ have the same elementary representations as $\mathbb{Z}_p[H]$-modules, and we have

$$e\theta_{\mathcal{O}[H]}(Y_S(A/F_\infty)) = \theta_{Z_p[H]}(\text{Gal}(K(F_\infty)/F_\infty)).$$

Here $d = \text{corank}_\mathcal{O}(A)$ and $f = [K : Q_p]$.
Proof of Proposition 4.2. Via a similar observation to that in Proposition 4.1, it suffices to show that $Y_S(A/F_\infty)$ is a pseudo-null $\mathbb{Z}_p[\text{Gal}(K/F_\infty)]$-module. Also, replacing $F$ if necessary, we may assume that $A[\pi^n] = A[p^n]$ is a trivial $G_{S}(F)$-module. It then follows that $A[p^n] \cong B[p^n]$ as $G_{S}(F)$-modules, where $B = (\mathbb{Q}_p/\mathbb{Z}_p)^{\text{id}}$. By assumption (b), we have that $ne \geq \theta_{\mathcal{O}(H)} \left( \text{Gal}(K(F_\infty)/F_\infty) \right) + 1$. In view of assumption (a), we may apply Theorem 3.5 to obtain the conclusion of the proposition.

5 Some further remarks

In this section, we mention a result which gives a relation between Conjecture A and the structure of the Selmer group of the residual representation. In the process of obtaining such a relation, we need to compare the Selmer group and the so-called strict Selmer group of Greenberg [10]. To prepare for this, we recall certain background material from [5]. Let $K$ be a finite extension of $\mathbb{Q}_p$ and $E$ an elliptic curve defined over $K$. Throughout our discussion, we shall always assume that our elliptic curve $E$ has good ordinary reduction. For every algebraic extension $L$ of $K$, we denote by $m_L$ the maximal ideal of the ring of integers of $L$. Let $\hat{E}(m_L)$ be the formal group of $E$ over the ring of integers of $L$. Kummer theory gives us the following commutative diagram

$$
\begin{array}{c}
0 \longrightarrow \hat{E}(m_K)/p \rightarrow H^1(K, \hat{E}(m_K)[p]) \longrightarrow H^1(K, \hat{E}(m_K)[p]) \rightarrow 0 \\
0 \longrightarrow E(K)/p \rightarrow H^1(K, E[p]) \longrightarrow H^1(K, E[p]) \rightarrow 0
\end{array}
$$

with exact rows. Let $K_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $K$. Then we have a similar commutative diagram as above for each intermediate field of the extension $K_\infty/K$. Taking direct limit of these diagrams and noting that $H^1(K_\infty, \hat{E}(m_K)) = 0$ (cf. [5 Corollary 3.2]), we obtain the following commutative diagram

$$
\begin{array}{c}
\hat{E}(m_{K_\infty})/p \cong H^1(K_\infty, \hat{E}(m_K))[p] \\
0 \longrightarrow E(K_\infty)/p \rightarrow H^1(K_\infty, E[p]) \longrightarrow H^1(K_\infty, E[p]) \rightarrow 0
\end{array}
$$

with exact row.

Lemma 5.1. With notation as above, we have a surjection

$$H^1(K_\infty, \hat{E}[p]) \rightarrow H^1(K_\infty, E)[p],$$

where $\hat{E}$ denotes the reduction of $E$ mod $p$.

Proof. From the commutative diagram before the lemma, we have that $\text{im} \lambda_p \subseteq \text{im} \kappa_p$ and this in turn induces a surjection

$$H^1(K_\infty, E[p])/(\text{im} \lambda_p) \rightarrow H^1(K_\infty, E[p])/(\text{im} \kappa) \cong H^1(K_\infty, E)[p].$$
It remains to show that $H^1(K_{\infty}, E[p]) / \text{im} \lambda_p \cong H^1(K_{\infty}, \hat{E}[p])$. But this is a consequence of the long exact sequence in cohomology arising from

$$0 \rightarrow \tilde{E}(\mathfrak{m}_K)[p] \rightarrow E[p] \rightarrow \tilde{E}[p] \rightarrow 0$$

which also gives the following exact sequence

$$H^1(K_{\infty}, \tilde{E}(\mathfrak{m}_K)[p]) \xrightarrow{\lambda_p} H^1(K_{\infty}, E[p]) \rightarrow H^1(K_{\infty}, \tilde{E}[p]) \rightarrow H^2(K_{\infty}, \tilde{E}(\mathfrak{m}_K)[p]),$$

and noting that the last term is zero by the fact that $\text{Gal}(\overline{K}/K_{\infty})$ has $p$-cohomological dimension $\leq 1$ (cf. [28, Theorem 7.1.8(i)]).

We now turn to the number field context. Let $F$ be a number field. From now on, $E$ will denote an elliptic curve defined over $F$. Throughout our discussion, we always assume that $E$ has good ordinary reduction at all primes of $F$ above $p$. The classical $p^n$-Selmer group of $E$ is defined to be

$$S(E[p^n]/F) = \ker \left( H^1(F, E[p^n]) \rightarrow \bigoplus_v H^1(F_v, E[p^n]) \right),$$

where $v$ runs through all the primes of $F$. Let $S$ be a finite set of primes of $F$ which contains the primes above $p$, the bad reduction primes of $E$ and the infinite primes. By [7, p. 8], we have the following equivalent description of the $p^n$-Selmer group

$$S(E[p^n]/F) = \ker \left( H^1(G_S(F), E[p^n]) \rightarrow \bigoplus_{v \in S} H^1(F_v, E[p^n]) \right).$$

We set $S(E/F) = \lim_{n \to \infty} S(E[p^n]/F)$ which is precisely the classical $p$-primary Selmer group. Let $F_{\infty}$ be a strongly admissible pro-$p$, $p$-adic Lie extension of $F$. Write $G = \text{Gal}(F_{\infty}/F)$ and $H = \text{Gal}(F_{\infty}/F_{\text{cycl}})$. Similarly, we can define $S(E[p^n]/L)$ and $S(E/L)$ for each finite extension $L$ of $F$ which is contained in $F_{\infty}$. We then set $S(E[p^n]/F_{\infty}) = \lim_{L \uparrow F_{\infty}} S(E[p^n]/L)$ and $S(E/F_{\infty}) = \lim_{L \uparrow F_{\infty}} S(E/L)$. Writing $S(F_{\infty})$ for the set of primes of $F_{\infty}$ above $S$, it is not difficult to verify that

$$S(E[p^n]/F_{\infty}) = \ker \left( H^1(G_S(F_{\infty}), E[p^n]) \rightarrow \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, E[p^n]) \right)$$

and

$$S(E/F_{\infty}) = \ker \left( H^1(G_S(F_{\infty}), E[p^n]) \rightarrow \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, E[p^n]) \right).$$

Furthermore, we have that $S(E[p^n]/F_{\infty}) = \lim_{L \uparrow F_{\infty}} S(E[p^n]/L_{\text{cycl}})$ and $S(E/F_{\infty}) = \lim_{L \uparrow F_{\infty}} S(E/L_{\text{cycl}})$.

We now introduce the strict Selmer group of Greenberg [10]. For each $v \in S$, we define the strict $p^n$-Selmer group of $E$ over $F$ to be

$$S^{\text{str}}(E[p^n]/F) = \ker \left( H^1(F, E[p^n]) \rightarrow \bigoplus_{v \in S} H^1(F_v, D_v[p^n]) \right),$$

17
where $D_v$ is taken to be $E[p^n]$ or $\widetilde{E}_v[p^\infty]$ according as $v$ does not or does divide $p$, and here $\widetilde{E}_v$ denotes the reduction of $E \mod v$. We then set $S^{str}(E[p^n]/F) = \lim\limits_{\rightarrow} S^{str}(E[p^n]/F)$. For each intermediate subfield $L$ of $F_{\infty}/F$, we have analogous definition for $S^{str}(E[p^n]/L)$, and we set $S^{str}(E[p^n]/F_{\infty}) = \lim\limits_{\rightarrow} S^{str}(E[p^n]/L)$. The strict $(p^\infty)$-Selmer group of $E$ over $F_{\infty}$ is then given by $\lim\limits_{\rightarrow} S^{str}(E[p^n]/F_{\infty})$. It is a straightforward exercise to verify that

$$S^{str}(E[p^n]/F_{\infty}) = \ker \left( H^1(G_S(F_{\infty}), E[p^n]) \longrightarrow \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, D_w[p^n]) \right)$$

and

$$S^{str}(E/F_{\infty}) = \ker \left( H^1(G_S(F_{\infty}), E[p^\infty]) \longrightarrow \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, D_w[p^\infty]) \right).$$

As before, we also have $S^{str}(E[p^n]/F_{\infty}) = \lim\limits_{\rightarrow} S^{str}(E[p^n]/L_{cyc})$ and $S^{str}(E/F_{\infty}) = \lim\limits_{\rightarrow} S^{str}(E/L_{cyc})$.

It is well-known that $H^1(F_{\infty,w}, D_w[p^\infty]) = H^1(F_{\infty,w}, E)[p^\infty]$. (This can be easily verified when $w$ does not divide $p$; in the event that $w$ divides $p$, this follows from [5, Proposition 4.8].) As a consequence, we have $S(E/F_{\infty}) = S^{str}(E/F_{\infty})$. Write $X(E/F_{\infty})$ for the Pontryagin dual of $S(E/F_{\infty})$. On the other hand, for the $p$-Selmer groups, we have the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Sel}^{str}(E[p]/F_{\infty}) & \longrightarrow & H^1(G_S(F_{\infty}), E[p]) & \longrightarrow & \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, D_w[p]) \\
& & \alpha & & \psi & & \\
0 & \longrightarrow & \text{Sel}(E[p]/F_{\infty}) & \longrightarrow & H^1(G_S(F_{\infty}), E[p]) & \longrightarrow & \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, E[p])
\end{array}
$$

with exact rows and the map $\varphi$ is surjective.

We can now state the main result of this section.

**Theorem 5.2.** Let $E$ be an elliptic curve defined over a number field $F$ which has good ordinary reduction at all primes above $p$. Let $F_{\infty}$ be a strongly admissible pro-$p$, $p$-adic Lie extension of $F$. Suppose that $X(E/F_{\infty})$ is torsion over $\mathbb{Z}_p[G]$. Then the following statements are equivalent.

1. $X(E/F_{\infty})$ is finitely generated over $\mathbb{Z}_p[H]$.
2. We have $H^2(G_S(F_{\infty}), E[p]) = 0$ and there is a short exact sequence

$$0 \longrightarrow S^{str}(E[p]/F_{\infty}) \longrightarrow H^1(G_S(F_{\infty}), E[p]) \longrightarrow \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, D_w[p]) \longrightarrow 0.$$

**Remark 5.3.** Note that the assertion “$H^2(G_S(F_{\infty}), E[p]) = 0$” in statement (2) is equivalent to Conjecture A being valid for $Y(E/F_{cyc})$ (see [56, Proposition 4.6]). (Actually, the said assertion is only proved in [56, Proposition 4.6] for the cyclotomic situation but it is not difficult to check that the same argument carries over to the general case.)
Theorem 5.2 can be viewed as the mod $p$ analogue of the following result, namely: the dual Selmer over an admissible $p$-adic Lie extension is torsion if and only if the defining sequence for the Selmer group is short exact and $H^2(G_S(F_{\infty}), E[p^{\infty}]) = 0$. We now apply Theorem 5.2 to give another different, and slightly more conceptual, proof of a result of Vatsal and Greenberg (see [12, Page 18, Statement A]).

**Corollary 5.4.** Let $E$ and $E'$ be two elliptic curves defined over a number field $F$ which have good ordinary reduction at all primes above $p$. Suppose that $E[p] \cong E'[p]$ as $\text{Gal}(\overline{F}/F)$-modules. Let $F_{\infty}$ be a strongly admissible pro-$p$, $p$-adic Lie extension of $F$. Then $X(E/F_{\infty})$ is finitely generated over $\mathbb{Z}_p[H]$ if and only if $X(E'/F_{\infty})$ is finitely generated over $\mathbb{Z}_p[H]$.

**Proof.** The corollary follows from combining Theorem 5.2 with the hypothesis that $E[p] \cong E'[p]$. □

We record another interesting corollary of Theorem 5.2. Namely, we give a sufficient condition which ensures that the defining sequence for the classical Selmer group of $E[p]$ is surjective.

**Corollary 5.5.** Let $E$ be an elliptic curve defined over a number field $F$ which has good ordinary reduction at all primes above $p$. Let $F_{\infty}$ be a strongly admissible pro-$p$, $p$-adic Lie extension of $F$. Suppose that $X(E/F_{\infty})$ is finitely generated over $\mathbb{Z}_p$. Then we have a short exact sequence

\[
0 \longrightarrow S(E[p]/F_{\infty}) \longrightarrow H^1(G_S(F_{\infty}), E[p]) \longrightarrow \bigoplus_{w \in S(F_{\infty})} H^1(F_{\infty,w}, E[p]) \longrightarrow 0.
\]

**Proof.** This follows from Theorem 5.2 and the diagram before Theorem 5.2. By loc. cit., the map $\psi_s$ is surjective. It then follows from the commutativity of the rightmost square that $\psi$ is also surjective which gives the required short exact sequence. □

It therefore remains to prove Theorem 5.2. We shall first require a lemma. Set $S^*(E[p]/F_{\infty}) = \varprojlim_L \left(S^{str}(E[p]/L)\right)$.

**Lemma 5.6.** We have an injection

\[
S^*(E[p]/F_{\infty}) \hookrightarrow \text{Hom}_{F_p[G]}(S^{str}(E[p]/F_{\infty})^\vee, F_p[G]).
\]

In particular, if $S^{str}(E[p]/F_{\infty})^\vee$ is torsion over $F_p[G]$, then $S^*(E[p]/F_{\infty}) = 0$.

**Proof.** The proof is quite similar to that in [14 Proposition 7.1]. For the convenience of the reader, we shall give a detailed proof here. For each finite extension $L$ of $F$ contained in $F_{\infty}$, we write $G_L = \text{Gal}(F_{\infty}/L)$. The restriction maps on cohomology induces a map on Selmer groups

\[
r_L : S^{str}(E[p]/L) \longrightarrow S^{str}(E[p]/F_{\infty})^{G_L},
\]

whose kernel is contained in $H^1(G_L, E(F_{\infty})[p])$. From this, we have an exact sequence

\[
0 \longrightarrow \varprojlim_m \ker(r_L) \longrightarrow S^*(E[p]/F_{\infty}) \longrightarrow \varprojlim_L S^{str}(E[p]/L)^{G_L}.
\]
Here the inverse limit is taken with respect to corestriction for the second term, and for the last term, the inverse limit is taken with respect to the map induced by the following map

\[ S^{str}(E[p]/F_\infty)^{G_{L'}} \rightarrow S^{str}(E[p]/F_\infty)^{G_L}, \quad x \mapsto \sum_{\sigma \in \text{Gal}(L'/L)} \sigma(x) \]

for \( L' \supseteq L \). Now for sufficiently large enough \( L \), we have \( H^1(G_L, E(F_\infty)[p]) = E(F_\infty)[p] \). Therefore, for these \( L \), the group \( G_L \) acts trivially on \( H^1(G_L, E(F_\infty)[p]) \) and hence \( \ker(r_L) \). It then follows that the corestriction map from \( \ker(r_{L'}) \) to \( \ker(r_L) \) is precisely multiplication by \([L' : L]\) for \( L' \supseteq L \). Since these groups are killed by \( p \), the said map is the zero map. Thus, we have \( \lim_{L} \ker(r_L) = 0 \). Consequently, we have an injection

\[ S^*(E[p]/F^{\text{cyg}}) \rightarrow \lim_{L} S^{str}(E[p]/F_\infty)^{G_{L'}}. \]

Now if \( M \) is a finite group killed by \( p \), we have a natural non-degenerate pairing \( M \times M^\vee \rightarrow \mathbb{F}_p \) which induces an isomorphism

\[ M \xrightarrow{\cong} \text{Hom}_{\mathbb{F}_p}(M^\vee, \mathbb{F}_p). \]

Applying this to each \( S^{str}(E[p]/F_\infty)^{G_{L'}} \) and taking inverse limit, we have

\[ \lim_{L} S^{str}(E[p]/F_\infty)^{G_{L'}} \cong \lim_{L} \text{Hom}_{\mathbb{F}_p} \left( \left( S^{str}(E[p]/F_\infty)^\vee \right)^{G_{L'}} , \mathbb{F}_p \right). \]

On the other hand, for a finitely generated \( \mathbb{F}_p[G] \)-module \( N \), we have an isomorphism

\[ \text{Hom}_{\mathbb{F}_p}(N_{G_L}, \mathbb{F}_p) \cong \text{Hom}_{\mathbb{F}_p[G]}(N, \mathbb{F}_p[G/G_L]). \]

Applying this isomorphism to each \( \left( S^{str}(E[p]/F_\infty)^\vee \right)^{G_{L'}} \) and taking limit, we have

\[ \lim_{L} \text{Hom}_{\mathbb{F}_p} \left( \left( S^{str}(E[p]/F_\infty)^\vee \right)^{G_{L'}} , \mathbb{F}_p \right) \cong \lim_{m} \text{Hom}_{\mathbb{F}_p[G]} \left( S^{str}(E[p]/F_\infty)^\vee , \mathbb{F}_p[G/G_L] \right) \]

\[ \cong \text{Hom}_{\mathbb{F}_p[G]} \left( S^{str}(E[p]/F_\infty)^\vee , \mathbb{F}_p[G] \right). \]

Combining these observations, we obtain the required injection. \( \square \)

**Remark 5.7.** When \( F_\infty = F^{\text{cyg}} \), one can even show that the injection in Lemma 5.6 is an isomorphism.

We can now give the proof of Theorem 5.2.

**Proof of Theorem 5.2.** Suppose that statement (1) holds. Being a quotient of \( X(E/F_\infty) \), \( Y(E/F_\infty) \) is therefore finitely generated over \( \mathbb{Z}_p[H] \). By \cite[Proposition 4.6]{5} (and see Remark 5.3), we then have \( H^2(G_S(F_\infty), E[p]) = 0 \). In view of this, the Poitou-Tate sequence then gives us the following exact sequence

\[ 0 \rightarrow S^{str}(E[p]/F_\infty) \rightarrow H^1(G_S(F_\infty), E[p]) \rightarrow \bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, D_w[p]) \rightarrow S^*(E[p]/F_\infty)^\vee \rightarrow 0. \]
By virtue of statement (1), $S(E[p]/F_\infty)^\vee$ is finitely generated over $\mathbb{F}_p[H]$, and hence torsion over $\mathbb{F}_p[G]$.

By Lemma 5.6, this implies that $S^*(E[p]/F_\infty) = 0$ which in turn gives the required short exact sequence.

Conversely, suppose that $H^2(G_S(F_\infty), E[p]) = 0$ and that one has a short exact sequence

$$0 \rightarrow S^{str}(E[p]/F_\infty) \rightarrow H^1(G_S(F_\infty), E[p]) \rightarrow \bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, D_w[p]) \rightarrow 0.$$ 

A standard $\mathbb{F}_p[\Gamma]$-corank calculation will show that both $H^1(G_S(F_\infty), E[p])$ and $\bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, D_w[p])$ have $\mathbb{F}_p[G]$-corank $[F : \mathbb{Q}]$ (see [14, Theorem 7.1] and [32, Theorem 4.1]). It then follows that $S^{str}(E[p]/F_\infty)^\vee$ has zero $\mathbb{F}_p[G]$-rank. From this, it follows that $X(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$. □

References

[1] K. Ardakov and K. A. Brown, Primeness, semiprimeness and localisation in Iwasawa algebras, *Trans. Amer. Math. Soc.* 359(4) (2007) 1499-1515.

[2] C. S. Aribam, On the $\mu$-invariant of fine Selmer groups, *J. Number Theory* 135 (2014) 284-300.

[3] A. Bhave, Analogue of Kida’s formula for certain strongly admissible extensions, *J. Number Theory* 122 (2007) 100-120.

[4] J. Coates, Fragments of the $GL_2$ Iwasawa theory of elliptic curves without complex multiplication, in *Arithmetic Theory of Elliptic Curves*, ed. C. Viola, Lecture Notes in Math. 1716 (Springer, Berlin, 1999), pp. 1-50.

[5] J. Coates and R. Greenberg, Kummer theory for abelian varieties over local fields, *Invent. Math.* 124 (1996) 129-174.

[6] J. Coates and R. Sujatha, Fine Selmer groups of elliptic curves over $p$-adic Lie extensions, *Math. Ann.* 331(4) (2005) 809-839.

[7] ———, *Galois Cohomology of Elliptic Curves*, 2nd Ed., Tata Institute of Fundamental Research Lectures on Mathematics, 88. Published by Narosa Publishing House, New Delhi; for the Tata Institute of Fundamental Research, Mumbai, 2010.

[8] J. Dixon, M. P. F. Du Sautoy, A. Mann and D. Segal, *Analytic Pro-p Groups*, 2nd edn, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, UK, 1999.

[9] K. R. Goodearl and R. B. Warfield, *An introduction to non-commutative Noetherian rings*, London Math. Soc. Stud. Texts 61, Cambridge University Press, 2004.

[10] R. Greenberg, Iwasawa theory for $p$-adic representations, in *Algebraic Number Theory— in honor of K. Iwasawa*, ed. J. Coates, R. Greenberg, B. Mazur and I. Satake, Adv. Std. in Pure Math. 17, 1989, pp. 97-137.

[11] ————, Iwasawa theory—past and present, in: *Class field Theory—its centenary and prospect*, Adv. Std. in Pure Math. 30, 2001, pp. 335-385.

[12] R. Greenberg and V. Vatsal, On the Iwasawa invariants of elliptic curves, *Invent. Math.* 142 (2000) 17-63.

[13] Y. Hachimori and R. Sharifi, On the failure of pseudo-nullity of Iwasawa modules, *J. Alg. Geom.* 14(3) (2005) 567-591.

[14] Y. Hachimori and O. Venjakob, Completely faithful Selmer groups over Kummer extensions. Kazuya Kato’s fiftieth birthday. *Doc. Math.* 2003, Extra Vol., 443-478.

[15] S. Howson, Euler characteristic as invariants of Iwasawa modules, *Proc. London Math. Soc.* 85(3) (2002) 634-658.

[16] ————, Structure of central torsion Iwasawa modules, *Bull. Soc. Math. France* 130(4) (2002) 507-535.

[17] K. Iwasawa, On the $\mu$-invariants of $Z_l$-extensions, in: *Number Theory, Algebraic Geometry and Commutative Algebra, in honour of Yasuo Akizuki*, Kinokuniya, Tokyo, 1973, 1-11.

[18] ————, On $Z_l$-extensions of algebraic number fields, *Ann. of Math.* 98 (1973) 246-326.
[19] S. Jha, Fine Selmer group of Hida deformations over non-commutative $p$-adic Lie extensions, *Asian J. Math.* 16(2) (2012) 353-366.

[20] S. Jha and R. Sujatha, On the Hida deformations of fine Selmer groups, *J. Algebra* 338 (2011) 180-196.

[21] T. Y. Lam, *Lectures on Modules and Rings*, Grad. Texts in Math. 189, Springer 1999.

[22] M. F. Lim, A remark on the $\mathfrak{M}_H(G)$-conjecture and Akashi series, *Int. J. Number Theory* 11(1) (2015) 269-297.

[23] __________, On the pseudo-nullity of the dual fine Selmer groups, *Int. J. Number Theory* 11(7) (2015) 2055-2063.

[24] __________, Notes on the fine Selmer groups, accepted for publication in *Asian J. Math.*

[25] __________, Comparing the $\pi$-primary submodules of the dual Selmer groups, accepted for publication in *Asian J. Math.*

[26] M. F. Lim and V. K. Murty, The growth of fine Selmer groups, *J. Ramanujan Math. Soc.* 31(1) (2016) 79-94.

[27] B. Mazur, Rational points of abelian varieties in towers of number fields, *Invent. Math.* 18 (1972) 183-266.

[28] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, 2nd Ed., Grundlehren Math. Wiss. 323, Springer 2008

[29] A. Neumann, Completed group algebras without zero divisors, *Arch. Math.* 51(6) (1988) 496-499.

[30] Y. Ochi, A remark on the pseudo-nullity conjecture for fine Selmer groups of elliptic curves, *Comment. Math. Univ. St. Pauli* 58(1) (2009) 1-7.

[31] Y. Ochi and O. Venjakob, On the structure of Selmer groups over $p$-adic Lie extensions, *J. Alg. Geom.* 11(3) (2002) 547-580.

[32] __________, On the ranks of Iwasawa modules over $p$-adic Lie extensions, *Math. Proc. Camb. Phil. Soc.* 135 (2003) 25-43.

[33] L. Ribes and P. Zalesskii, *Profinite Groups*, Second edition, Ergeb. Math. Grenzgeb. 40, Springer, 2010.

[34] R. Sharifi, Massey products and ideal class groups, *J. reine angew. Math.* 603 (2007) 1-33.

[35] __________, On Galois groups of unramified pro-$p$ extensions, *Math. Ann.* 342 (2008) 297-308.

[36] R. Sujatha, Elliptic curves and Iwasawa’s $\mu = 0$ conjecture, in: *Quadratic forms, linear algebraic groups, and cohomology* 125-135, Dev. Math., 18, Springer, New York, 2010.

[37] O. Venjakob, On the structure theory of the Iwasawa algebra of a $p$-adic Lie group, *J. Eur. Math. Soc.* 4(3) (2002) 271-311.

[38] __________, A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory, *J. reine angew. Math.* 559 (2003) 153-191.

[39] C. Wuthrich, Iwasawa theory of the fine Selmer group, *J. Alg. Geom.* 16 (2007) 83-108.