HODGE SPECTRUM OF HYPERPLANE ARRANGEMENTS

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Abstract. In this article there are two main results. The first result gives a formula, in terms of a log resolution, for the graded pieces of the Hodge filtration on the cohomology of a unitary local system of rank one on the complement of an arbitrary divisor in a smooth projective complex variety. The second result is an application of the first. We give a combinatorial formula for the spectrum of a hyperplane arrangement. M. Saito recently proved that the spectrum of a hyperplane arrangement depends only on combinatorics. However, a combinatorial formula was missing. The formula is achieved by a different method.

1. Introduction

In this article there are two main results. The first result, Theorem 3.5, is concerned with the computation of the Hodge filtration on the cohomology of local systems on the complement $U$ of an arbitrary divisor $D$ in a smooth complex projective variety $X$. For a unitary local system of rank one $\mathcal{V}$ on $U$, we give a formula in terms of a log resolution of $(X, D)$ for the graded pieces $\text{Gr}_{F}^{p} H^{m}(U, \mathcal{V})$. This formula is related to the multiplier ideals of $(X, D)$ and generalizes [B06] - Proposition 6.4 and part of [DS] - Theorem 2. It is also related to [B06]- Theorems 1.3, 1.4, and [L07] - Theorem 2.1. Hodge numbers of local systems on the complements of planar divisors and of isolated non normal crossings divisors, in both local and global case, were discussed in [L83], [L01], [L03], [L04].

The second result is an application of Theorem 3.5 and concerns the Hodge spectrum of a hyperplane arrangement. For any closed subscheme $D$ of $X$, the spectrum, as the multiplier ideals and the b-function, is a measure of the complexity of the singularities of $D$. When $D$ is a hypersurface with an isolated singularity, the spectrum enjoys a semicontinuity property which has been very useful for the deformation theory of such singularities (see [Ku] and references there). Less is known about the spectrum for arbitrary singularities. All three types of invariants (spectrum, multiplier ideals, b-functions) are notoriously difficult to compute, see [S07b]. Despite implementations of algorithms in programs like Macaulay 2, Singular, and Risa/Asir, computation in the cases when the dimension of $X$ is $\geq 3$ is very expensive. For the class of varieties defined by monomial ideals it can be said that there are satisfying formulas for all three notions in terms of combinatorics, making the computations faster ([Ho], [DMS], [BMSa], [BMSb]). For the class
of hyperplane arrangements, some invariants (such as the ring $H^*(X - D, \mathbb{Z})$, see [OS]) turned out to depend only on combinatorics. Hence it is natural to ask if the information from multiplier ideals, spectra, and $b$-functions for this class is combinatorially determined. M. Mustață [Mu] (see also [Te]) gave a formula for multiplier ideals of a hyperplane arrangement. However, it was not clear that the jumping numbers (the most basic numerical invariants that come out of the multiplier ideals) admit a combinatorial formula. M. Saito [S07a] then proved that the spectrum and the jumping numbers of a hyperplane arrangement depend only on combinatorics. However, a combinatorial formula was missing. A different proof and a combinatorial formula for the jumping numbers and for the beginning piece of the spectrum was given in [B08]. In this article we give a combinatorial formula and a different proof of the combinatorial invariance for the spectrum of a hyperplane arrangement, see Theorem 5.9. The $b$-function for hyperplane arrangements remains yet to be determined. See [S06] for the latest advances.

The structure of the article is the following. In section 2 we fix notation and review the multiplier ideals, Hodge spectrum, and intersection theory. In section 3 we prove Theorem 3.5 on the Hodge filtration for local systems, based on the geometrical interpretation of rank one unitary local systems from [B06]. In section 4, we recall first how the cohomology of the Milnor fiber of a homogeneous polynomial can be understood in terms of local systems. Then we apply the result of the previous section to reduce the computation of the spectrum of a homogeneous polynomial to intersection theory on a log resolution. In section 5, we prove Theorem 5.9 on the combinatorial formula for the spectrum of hyperplane arrangement by making use of the explicit intersection theory on the canonical log resolution. In section 6 we give some examples showing how Theorem 5.9 works.

We thank M. Saito for sharing with us the preprint [S07a] which was the inspiration for this article. We also thank: A. Dimca, A. Libgober, L. Maxim, and T. Shibuta for useful discussions.

2. Notation and Review

We fix notation and review basic notions, which we need later, about multiplier ideals, Hodge spectrum, and intersection theory.

**Notation.** By a *variety* we will mean a complex algebraic variety, reduced, and irreducible. For a smooth variety $X$, the canonical line bundle is denoted $\omega_X$ and we always fix a canonical divisor, $K_X$, such that $\mathcal{O}_X(K_X) = \omega_X$. Let $\mu : Y \to X$ be a proper birational morphism. The exceptional set of $\mu$, denoted by $Ex(\mu)$, is the set of points $\{y \in Y\}$ where $\mu$ is not biregular. For a divisor $D$ on $X$ with support $\text{Supp}(D)$, we say that $\mu$ is a *log resolution* of $(X, D)$ if $Y$ is smooth and $\mu^{-1}(\text{Supp}(D)) \cup Ex(\mu)$ is a divisor with simple normal crossings. Such a resolution always exists, by Hironaka. The *relative canonical divisor* of $\mu$ is $K_{Y/X} = K_Y - \mu^*(K_X)$. If $D = \sum_{i \in S} \alpha_i D_i$ is a divisor on $X$ with real coefficients, where $D_i$ are the irreducible components of $D$ for $i \in S$, and $\alpha_i \in \mathbb{R}$, the round down of $D$ is the integral divisor $\lfloor D \rfloor = \sum_{i} \lfloor \alpha_i \rfloor D_i$. Here, $\lfloor \cdot \rfloor$ is the round-down of
a real number. We also use \( \{.\} \) to mean the fractional part of a real number. For \( \alpha \in \mathbb{R}^S \) and a reduced effective divisor \( D = \bigcup_{i \in S} D_i \), we frequently use the notation \( \alpha \cdot D \) to mean the \( \mathbb{R} \)-divisor \( \sum_{i \in S} \alpha_i D_i \).

**Multiplier ideals.** See [La]- Chapter 9 for more on multiplier ideals. Let \( X \) be a smooth variety. Let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( X \). Let \( \mu : Y \to X \) be a log resolution of \((X, D)\). The **multiplier ideal** of \( D \) is the ideal sheaf 
\[
\mathcal{J}(D) := \mu_* \mathcal{O}_Y(K_Y/X - \mu^* D) \subset \mathcal{O}_X.
\]

The choice of a log resolution does not matter in the definition of \( \mathcal{J}(D) \). Equivalently, \( \mathcal{J}(D) \) can be defined analytically to consist, locally, of all holomorphic functions \( g \) such that 
\[
|g|^2 / \prod_{i \in S} |f_i|^{2\alpha_i}
\]

is locally integrable, where \( f_i \) are local equations of the irreducible components of \( D \) and \( \alpha_i \) their multiplicities. For the following see [La]- Theorems 9.4.1, 9.4.9.

**Theorem 2.1.** With the notation as above, \( R^j \mu_* \mathcal{O}_Y(K_Y/X - \mu^* D) = 0 \), for \( j > 0 \). Assume in addition that \( X \) is projective. Let \( L \) be any integral divisor such that \( L - D \) is nef and big. Then \( H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X J(D)) = 0 \) for \( i > 0 \).

**Hodge spectrum.** See [Ku]-II.8 for more on Hodge spectrum. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be the germ of a non-zero holomorphic function. Let \( M_f \) be the Milnor fiber of \( f \) defined as 
\[
M_f = \{ z \in \mathbb{C}^n \mid |z| < \epsilon \text{ and } f(z) = t \}
\]

for \( 0 < |t| \ll \epsilon \ll 1 \). It will not matter which \( t \) is chosen. The cohomology groups \( H^*(M_f, \mathbb{C}) \) carry a canonical mixed Hodge structure such that the semisimple part \( T_s \) of the monodromy acts as an automorphism of finite order of these mixed Hodge structures (see [St77] for \( f \) with an isolated singularity, [Na] and [S91] for the general case). Define for \( \alpha \in \mathbb{Q} \), the **spectrum multiplicity** of \( f \) at \( \alpha \) to be 
\[
n_\alpha(f) := \sum_{j \in \mathbb{Z}} (-1)^j \dim \text{Gr}^{n-\alpha-j}_F \tilde{H}^{n-1+j}(M_f, \mathbb{C})_{e^{-2\pi i \alpha}},
\]

where \( F \) is the Hodge filtration, and \( \tilde{H}^*(M_f, \mathbb{C})_\lambda \) stands for the \( \lambda \)-eigenspace of the reduced cohomology under \( T_s \). The **Hodge spectrum** of the germ \( f \) is the fractional Laurent polynomial 
\[
\text{Sp}(f) := \sum_{\alpha \in \mathbb{Q}} n_\alpha(f) t^\alpha.
\]

It was first defined by Steenbrink ([St77], [St87]). We are using however a slightly different definition, as in [B03], [BS].

**Proposition 2.2.** ([BS]-Proposition 5.2.) For \( \alpha \notin (0, n) \), \( n_\alpha(f) = 0 \)

**Corollary 2.3.**
\[
\text{Sp}(f) = \sum_{\alpha \in (0,n) \cap \mathbb{Q}} \left( \sum_{j \in \mathbb{Z}} (-1)^j \dim \text{Gr}^{n-\alpha-j}_F \tilde{H}^{n-1+j}(M_f, \mathbb{C})_{e^{-2\pi i \alpha}} \right) t^\alpha.
\]
Proof. In other words, we can use usual, instead of reduced, cohomology in the definition of \( n_\alpha(f) \), provided with restrict to the range \( \alpha \in (0, n) \). We have

\[
(1) \quad \tilde{H}^j(M_f, \mathbb{C})_\lambda = \begin{cases} 
H^j(M_f, \mathbb{C})_\lambda, & \text{if } j \neq 0 \text{ or } \lambda \neq 1, \\
coker(H^0(\text{point}, \mathbb{C}) \rightarrow H^0(M_f, \mathbb{C})), & \text{if otherwise,}
\end{cases}
\]

where the last map is induced by a constant map \( X \rightarrow \text{point} \in X \) (see e.g. \cite[p.106]{D92}). Hence if \( \alpha \notin \mathbb{Z} \), we are in the first case of (1) and we can replace \( \tilde{H}^* \) by \( H^* \) in the definition of \( \text{Sp}(f) \). Assume \( \alpha \in \mathbb{Z} \). We have \( \text{Gr}_F^2H^0(\text{point}, \mathbb{C}) \) is 0 if \( j \neq 0 \) and is \( \mathbb{C} \) if \( j = 0 \). By the second case of (1), we only need to worry about the case when \( \downarrow n - \alpha \uparrow = n - \alpha \) is exactly 0. But this is ruled out by Proposition 2.2. \( \square \)

**Intersection theory.** See \cite{F} for more on intersection theory. Let \( X \) be a smooth projective variety of dimension \( n \). For a vector bundle, or locally free \( \mathcal{O}_X \)-module, of finite rank \( r \), \( \mathcal{E} \) on \( X \), we denote by \( c_i(\mathcal{E}) \) the image of the \( i \)-th Chern class of \( \mathcal{E} \) in \( H^{2i}(X, \mathbb{Z}) \). For \( i \neq \{0, \ldots, r\} \), \( c_i(\mathcal{E}) = 0 \), and \( c_0(\mathcal{E}) = 1 \). We have the following definitions:

\[
(2) \quad \begin{align*}
    c(\mathcal{E}) &= \sum_i c_i(\mathcal{E}) \quad \text{(total Chern class)}, \\
    c_r(\mathcal{E}) &= \sum_i c_i(\mathcal{E})t^i \quad \text{(Chern polynomial)}, \\
    x_i \text{ formal symbols} &\colon \prod_{1 \leq i \leq r} (1 + x_i t) = c_i(\mathcal{E}) \quad \text{(Chern roots)}, \\
    ch(\mathcal{E}) &= \sum_{1 \leq i \leq r} \exp(x_i) \quad \text{(Chern character)}, \\
    Q(x) &= x/(1 - \exp(-x)) \quad \text{(Todd class)}, \\
    td(\mathcal{E}) &= \prod_{1 \leq i \leq r} Q(x_i) \quad \text{(Todd class of } \mathcal{E}), \\
    c(X) &= c(T_X) \quad \text{(total Chern class of } X), \\
    Td(X) &= Td(T_X) \quad \text{(Todd class of } X).
\end{align*}
\]

The meaning of the Chern roots is the following. The coefficients of powers of \( t \) in \( \prod_{1 \leq i \leq r} (1 + x_i t) \) are elementary symmetric functions in \( x_1, \ldots, x_r \) and they are set to equal the Chern classes \( c_j(\mathcal{E}) \). Any other symmetric polynomial, such as the homogeneous terms of fixed degree in the Taylor expansion of \( ch(\mathcal{E}) \) or \( td(\mathcal{E}) \), can be expressed in terms of elementary symmetric functions, hence in terms of the \( c_j(\mathcal{E}) \). See \cite{F} - Examples 3.2.3, 3.2.4.

Let \( 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0 \) be an exact sequence of vector bundles, let \( x_1, \ldots, x_r \) be the Chern roots of \( \mathcal{E} \), and let \( \mathcal{F} \) another vector bundle with Chern roots \( y_1, \ldots, y_s \). Then (\cite{F} - Section 3.2):

\[
(3) \quad \begin{align*}
    c_i(\mathcal{E}') &= (-1)^i c_i(\mathcal{E}), \\
    c_i(\Lambda^p \mathcal{E}) &= \prod_{i_1 < \ldots < i_p} (1 + (x_{i_1} + \ldots + x_{i_p}) t), \\
    c_i(\mathcal{E} \otimes \mathcal{F}) &= \prod_{i,j} (1 + (x_i + y_j) t), \\
    ch(\mathcal{E}' \otimes \mathcal{F}) &= ch(\mathcal{E}') \cdot ch(\mathcal{E}'').
\end{align*}
\]

For every element of the Grothendieck group of coherent sheaves on \( X \) there are well defined Chern classes. For an algebraic class \( \xi \) in the ring \( H^*(X, \mathbb{Z}) \), let \( \xi_j \in H^{2j}(X, \mathbb{Z}) \) denote the degree 2\( j \) part of \( \xi \), such that \( \xi = \sum_j \xi_j \).

**Theorem 2.4.** (Hirzebruch-Riemann-Roch, \cite{F} - Corollary 15.2.1) Let \( \mathcal{E} \) be a vector bundle on a smooth projective variety \( X \) of dimension \( n \). Then \( \chi(X, \mathcal{E}) \) is the intersection number \( (ch(\mathcal{E}) \cdot Td(X))_n \).
3. Hodge filtration for local systems

Recall (e.g. from [Di04]) that a complex local system $V$ on a complex manifold $X$ is a locally constant sheaf of finite dimensional complex vector spaces. The rank of $V$ is the dimension of a fiber of $V$. Local systems of rank one on $X$ are equivalent with representations $H_1(X, \mathbb{Z}) \to \mathbb{C}^\ast$. Unitary local systems of rank one on $X$ correspond to representations $H_1(X, \mathbb{Z}) \to S^1$, where $S^1$ is the unit circle in $\mathbb{C}$. For a smooth variety $X$, local systems are defined on the corresponding complex manifold.

Let $X$ be a smooth projective variety of dimension $n$. Let $D$ be a reduced effective divisor on $X$ with irreducible decomposition $D = \bigcup_{i \in S} D_i$, for a finite set of indices $S$. Let $U = X - D$ be the complement of $D$ in $X$. Rank one unitary local systems on $U$ have the following geometric interpretation. Define first the group of realizations of boundaries of $X$ on $D$

$$\text{Pic}^r(X, D) := \left\{ (L, \alpha) \in \text{Pic}(X) \times [0, 1)^S : c_1(L) = \sum_{i \in S} \alpha_i \cdot [D_i] \in H^2(X, \mathbb{R}) \right\},$$

where the group operation is

$$(L, \alpha) \cdot (L', \alpha') = (L \otimes L' \otimes \mathcal{O}_X(-\sum \alpha_i + \alpha' \cdot D)), \{\alpha + \alpha'\}.$$ 

Here $\alpha \cdot D$ means the divisor $\sum_{i \in S} \alpha_i D_i$, and $\sum$ (resp. $\{\} \}$) is taking the round-down (resp. fractional part) componentwise. Note that the inverse of $(L, \alpha)$ is $(M, \beta)$ where $M = L^\vee \otimes \mathcal{O}_X(\sum_{\alpha_i = 0} D_i)$, and $\beta_i$ is 0 if $\alpha_i = 0$ and is $1 - \alpha_i$ otherwise.

**Theorem 3.1.** ([B06] - Theorem 1.2.) Let $X$ be a smooth projective variety, $D$ a divisor on $X$, and let $U = X - D$. There is a natural canonical group isomorphism

$$\text{Pic}^r(X, D) \cong \text{Hom}(H_1(U, \mathbb{Z}), S^1)$$

between realizations of boundaries of $X$ on $D$ and unitary local systems of rank one on $U$.

Fix a log resolution $\mu : Y \to X$ of $(X, D)$ which is an isomorphism above $U$. Let $E = Y - U$ with irreducible decomposition $E = \bigcup_{j \in S'} E_j$.

**Proposition 3.2.** ([B06] - Proposition 3.3.) The map $\text{Pic}^r(X, D) \to \text{Pic}^r(Y, E)$ given by $(L, \alpha) \mapsto (\mu^* L - \sum e \cdot, \{e\})$ is an isomorphism, where $e \in \mathbb{R}^{S'}$ is given by $\mu^*(\alpha \cdot D) = e \cdot E$.

For a unitary local system $V$ on $U$, denote by $\nabla$ the vector bundle on $Y$ given by the canonical Deligne extension of $V$ to $Y$ (see [De]). The relation between canonical Deligne extensions and realizations of boundaries, and the explicit isomorphism of Theorem 3.1 in the case of the complement of a simple normal crossings divisor is the following:

**Lemma 3.3.** ([B06] - Proof of Theorem 1.2 and Remark 8.2 (a).) With the notation as in Proposition 3.2, let $V \in \text{Hom}(H_1(U, \mathbb{Z}), S^1)$ be a rank one unitary local system on $U$. Then $V$ corresponds to $(M, \beta) \in \text{Pic}^r(Y, E)$ where $M = \nabla \otimes \mathcal{O}_Y(\sum_{\beta_j \neq 0} E_j)$ and $\beta_j \in [0, 1)$ is such that the monodromy of $V$ around a general point of $E_j$ is multiplication by $\exp(2\pi i \beta_j)$. 


Unitary local systems admit a canonical Hodge filtration $F$ on cohomology such that:

**Theorem 3.4.** ([11]) 2nd Theorem, part (a.) With notation as in Proposition 3.3, let $V$ be a unitary local system on $U$. Then

$$\dim \text{Gr}_F^p H^{p+q}(U, V) = h^q(Y, \Omega_Y^p(\log E) \otimes \overline{V}).$$

The main result of this section describes the pieces of the Hodge filtration on the cohomology of unitary rank one local systems on complements to arbitrary divisors. It generalizes [B06] - Proposition 6.4 and part of [DS] - Theorem 2. It is also related to [B06] - Theorems 1.3, 1.4, and [L07] - Theorem 2.1.

**Theorem 3.5.** Let $X$ be a smooth projective variety of dimension $n$, $D$ a divisor on $X$, and $U = X - D$. Let $V \in \text{Hom}(H_1(U, \mathbb{Z}), S^1)$ be a rank one unitary local system on $U$ corresponding to $(L, \alpha) \in \text{Pic}^c(X, D)$. Let $\mu : (Y, E) \to (X, D)$ be a log resolution which is an isomorphism above $U$. Then:

(a) $$\dim \text{Gr}_F^p H^{p+q}(U, V^\vee) = h^n(Y, \Omega_Y^n(\log E)^\vee \otimes \omega_Y \otimes \mu^* L \otimes \Omega_Y(-\mu^* (\alpha \cdot D)), \omega_Y)$$

(b) if $\alpha_i \neq 0$ for all $i \in S$, then

$$\dim \text{Gr}_F^p H^{p+q}(U, V) = h^q(Y, \Omega_Y^{n-p}(\log E)^\vee \otimes \omega_Y \otimes \mu^* L \otimes \Omega_Y(-\mu^* ((\alpha - \epsilon) \cdot D)), \omega_Y)$$

for all $0 < \epsilon \ll 1$.

In particular, in terms of multiplier ideals:

(c) $\dim \text{Gr}_F^p H^q(U, V^\vee) = h^n((X, \omega_X \otimes L \otimes \mathcal{J}(\alpha \cdot D)));$

(d) if $\alpha_i \neq 0$ for all $i \in S$, for all $0 < \epsilon \ll 1$, then

$$\dim \text{Gr}_F^p H^{n+q}(U, V) = h^q((X, \omega_X \otimes L \otimes \mathcal{J}(\alpha - \epsilon) \cdot D)),$$

which is $0$ if $q \neq 0$.

**Proof.** (a) By Theorem 3.4

$$\dim \text{Gr}_F^p H^{p+q}(U, V^\vee) = h^q(Y, \Omega_Y^p(\log E) \otimes \overline{V^\vee}).$$

By Proposition 3.3, $V$ corresponds to $(\mu^* L \otimes \mathcal{O}_Y(-\mu^* (\alpha \cdot D)), \beta)$ in $\text{Pic}^c(Y, E)$, where $\beta_j$ is the fractional part of the coefficient of $E_j$ in $\mu^* (\alpha \cdot D)$. By calculating the inverse in the group $\text{Pic}^c(Y, E)$, the dual local system $V^\vee$ corresponds to $(\mu^* L^\vee \otimes \mathcal{O}_Y(-\mu^* (\alpha \cdot D)) + \sum_{\beta_j \neq 0} E_j, \gamma)$ in $\text{Pic}^c(Y, E)$, where $\gamma_j$ is $0$ if $\beta_j = 0$ and is $1 - \beta_j$ if $\beta_j \neq 0$. Hence, by Lemma 3.3

$$\overline{V^\vee} = \mu^* L^\vee \otimes \mathcal{O}_Y(-\mu^* (\alpha \cdot D)) + \sum_{\beta_j \neq 0} E_j - \sum_{\gamma_j \neq 0} E_j$$

$$= \mu^* L^\vee \otimes \mathcal{O}_Y(-\mu^* (\alpha \cdot D)).$$

The conclusion follows by Serre duality.

(b) We have

$$\dim \text{Gr}_F^p H^{p+q}(U, V) = h^q(Y, \Omega_Y^p(\log E) \otimes \overline{V}).$$
By Lemma \ref{lem:1}, \( \overline{V} = \mu^* L \otimes \mathcal{O}_Y(-n \mu^*(\alpha \cdot D) - \sum_{\beta_j \neq 0} E_j) \). Also, we use the isomorphism \( \Omega^p_Y(\log E) \cong \Omega^{p-q}_Y(\log E)^\vee \otimes \omega_Y \otimes \mathcal{O}_Y(\sum E_j) \) (see \cite{EV}-6.8 (b)). If \( \alpha_i \neq 0 \) for all \( i \), then \( V \) is not the restriction to \( U \) of a local system over a larger open subset of \( X \), then the coefficients of \( E_j \) in \( \mu^*(\alpha \cdot D) \) are nonzero. Hence
\[
-\nu \mu^*(\alpha \cdot D) + \sum_{\beta_j = 0} E_j = -\nu \mu^*((\alpha - \epsilon) \cdot D),
\]
for all \( 0 < \epsilon \ll 1 \). The conclusion follows.

(c) We let \( p = 0 \) in (a) and use the definition of multiplier ideals. The identification of \( H^{n-q}_Y \) of the \( \mathcal{O}_Y \)-module from (a) with \( H^{n-q}_X \) of the \( \mathcal{O}_X \)-module \( \omega_X \otimes L \otimes \mathcal{J}(\alpha \cdot D) \) is due to the triviality of the Leray spectral sequence that follows from the projection formula \( \cite{Ha-III.8 Ex. 8.3} \) and the first part of Theorem \ref{thm:2.1}.

(d) We let \( p = n \) in (b), then proceed as in (c). The vanishing for \( q \neq 0 \) follows from the second part of Theorem \ref{thm:2.1} \( \square \).

\textbf{Remark 3.6}. (i) Part (a) of Theorem \ref{thm:3.5} generalizes part of \cite{DS}- Theorem 2. They proved that when \( X = \mathbb{P}^n \) and \( D \) is a hypersurface in \( X \),
\[
F^n H^n(U, \mathbb{C}) = H^0(X, V^0(\omega_X(D))).
\]
The sheaf \( V^0(\omega_X(D)) \) has a description in terms of multiplier ideals via a result of \cite{BS}. More precisely,
\[
V^0(\omega_X(D)) = \omega_X \otimes \mathcal{O}_X(D) \otimes \mathcal{J}((1 - \epsilon)D),
\]
for all \( 0 < \epsilon \ll 1 \). To see that this indeed follows from part (a) of Theorem \ref{thm:3.5}, we have isomorphisms:
\[
F^n H^n(U, \mathbb{C}) = \text{Gr}_F^n H^n(U, \mathbb{C}) = H^n(Y, \Omega^p_Y(\log E)^\vee \otimes \omega_Y) = H^n(Y, \omega_Y \otimes \mathcal{O}_Y(E)) = H^0(Y, \omega_Y \otimes \mu^*(\mu^*(1 - \epsilon)D)) = H^0(X, V^0(\omega_X(D))).
\]

(ii) Examples of Hodge numbers of local systems can be found in \cite{B06}- Example 6.6, \cite{L01}, \cite{L07}- Section 6.

\section{4. Milnor fibers and local systems}

We recall first how the cohomology of the Milnor fiber of a homogeneous polynomial can be understood in terms of local systems. Then we apply the result of the previous section to reduce the computation of the spectrum of a homogeneous polynomial to intersection theory on a log resolution.

Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a homogeneous polynomial of degree \( d \). Let \( f = \prod_{i \in S} f_i^{m_i} \) be the irreducible decomposition of \( f \), and \( d_i \) be the degree of \( f_i \). Denote by \( D \) (resp. \( D_i \)) the hypersurface defined by \( f \) (resp. \( f_i \)) in \( X := \mathbb{P}^{n-1} \). Let \( U = X - D \).

The \textit{global Milnor fiber} of \( f \) is \( M := f^{-1}(1) \subset \mathbb{C}^n \). The \textit{geometric monodromy} is the map \( h : M \to M \) given by \( a \mapsto e^{2\pi i/d} \cdot a \). It is known that \( H^i(M, \mathbb{C}) = H^i(M_f, \mathbb{C}) \), where \( M_f \) is the Milnor fiber of the germ of \( f \) at \( 0 \in \mathbb{C}^n \), such that \( h^* \)
on $H^i(M, \mathbb{C})$ corresponds to the monodromy action $T$ on $H^i(M_f, \mathbb{C})$. Hence the monodromy $T$ is diagonalizable and the eigenvalues are $d$-th roots of unity. See e.g. [Di92]-p.72. Also, the Hodge filtration $F$ on $H^i(M_f, \mathbb{C})$ is induced by the one on $H^i(M, \mathbb{C})$. This fact seems well known to experts, and it has been used for example in [St87]- Theorem 6.1. However we could not find a reference. Since a proof of this fact would not be elementary and take us too far, we take as the definition of $F$ the canonical Hodge filtration of $H^i(M, \mathbb{C})$.

The group $G := < h > = \mathbb{Z}/d\mathbb{Z}$ acts on $M$ freely and the quotient $M/G$ can be identified with $U$. Let $p : M \to U$ be the covering map. Write

$$p_* \mathbb{C}_M = \oplus_{k=1}^d \mathcal{V}_k,$$

where $\mathcal{V}_k$ is the rank one unitary local system on $U$ given by the $e^{-2\pi i k/d}$-eigenspaces of fibers of the local system $p_* \mathbb{C}_M$. Then, since $p$ is finite, by Leray spectral sequence one has for $1 \leq k \leq d$ (see also [CS]-Theorem 1.6):

$$H^i(M, \mathbb{C}) e^{-2\pi i k/d} = H^i(U, \mathcal{V}_k).$$

This isomorphism preserves the Hodge filtration by the functoriality of the Hodge filtration for unitary local systems (see [Ti]-§6). Thus, in the case of homogeneous polynomials, the computation of the Hodge filtration on the cohomology of the Milnor fiber is reduced to the computation of the Hodge filtration on the cohomology of unitary rank one local systems on the complement of the projective hypersurface.

Next result is well known to experts. We give a proof since we could not find a reference.

**Lemma 4.1.** With notation as above, the monodromy of $\mathcal{V}_k$ around a general point of $D_i$ is given by multiplication by $e^{2\pi i km_i/d}$.

**Proof.** The $\mathcal{V}_k$, with tensor product, form a group isomorphic to $G$ (e.g. [Bo6]-§5). So it is enough to prove the lemma for $k = 1$.

Fix $i$ and denote $m_i$ by $m$. Let $P$ be a general point of $D_i$. We consider a small loop $\tau = \{Q_\theta \in U \mid \theta \in [0, 1]\}$ around $P$, with $Q_0 = Q_1 = : Q$. We need to look at the action $T_i$ of going along $\tau$ counterclockwise on the fiber $(\mathcal{V}_1)_Q$.

First, $(p_* \mathbb{C}_M)_Q = \oplus_{1 \leq j \leq d} \mathbb{C}v_{\xi^j x}$, where $x \in M \subset \mathbb{C}^n$ is fixed and such that $p(x) = Q$, $\xi = e^{-2\pi i/d}$, and $v_{\xi^j x}$ are linearly independent. Here $\{\xi^j x, \ldots, \xi^d x\}$ is $p^{-1}(Q)$. The action induced by $h$ on $(p_* \mathbb{C}_M)_Q$ is given by $v_{\xi^j x} \mapsto v_{\xi^{j+m} x}$. Hence, its $\xi$-eigenspace is

$$(\mathcal{V}_1)_Q = \{ a \cdot \sum_{1 \leq j \leq d} \xi^j v_{\xi^j x} \mid a \in \mathbb{C} \}.$$  

We will show that $T_i v_{\xi x} = v_{\xi^{i+m} x}$. This implies that $T_i$ acts on $(\mathcal{V}_1)_Q$ via multiplication by $\xi^{-m}$, which is what we wanted to show.

By considering the loop $\tau$ lying in a (real) plane, we can simplify the computation. To this end, after linear change of coordinates, we can assume the following. First, we can assume $i = 1$. Let $V = \{x_3 = \ldots = x_n = 0\} \subset \mathbb{C}^n$ be transversal to all the hyperplanes $\{f_i = 0\}$. Define $M' := M \cap V$, $U' := U \cap PV \subset PV = \mathbb{P}^1$, and denote $\mathcal{V}$ by $\mathcal{V}'$.
Proof. By Corollary 2.3 and (4), \( \alpha, k, p, f \) polynomial at the origin. By above discussion and Corollary 2.3, the only rational

For each \( \theta \) fix \( a_\theta \) such that \( a_\theta^d = g(e^{2\pi i \theta}, 1)^{-1} \). We can assume \( a_0 = a_1 \). Then

Fix \( j \) and let \( x = x_{j,0} \). Starting at \( x \), going counterclockwise along the inverse image by \( p' \) of \( \tau \), we end up at \( x_{j,1} = e^{-2\pi i d/j} \). This shows that \( T_i v_{\xi,j} = v_{\xi,j+m} \). \( \square \)

**Lemma 4.2.** With notation as in Lemma 4.1, let \( (L^{(k)}, \alpha^{(k)}) \in \text{Pic}^r(X, D) \) correspond to \( V_k \) under the isomorphism of Theorem 7.1. Then:

\[
\alpha_i^{(k)} = \left\{ \frac{km_i}{d} \right\}, \quad L^{(k)} = \mathcal{O}_X \left( \sum_{i \in S} \alpha_i^{(k)} d_i \right).
\]

Proof. By Proposition 3.2 and Lemma 3.3, \( \alpha^{(k)} \in [0, 1) \) is given by the monodromy of \( V_k \) around a general point of (the proper transform of) \( D_i \). The conclusion for \( \alpha_i^{(k)} \) then follows from Lemma 4.1. The condition that \( (L^{(k)}, \alpha^{(k)}) \in \text{Pic}^r(X, D) \) is that the degree of \( L^{(k)} \) equals \( \sum_{i \in S} \alpha_i^{(k)} d_i \).

Alternatively, one can prove Lemma 4.2 using [B06]-Corollary 1.10.

Now we draw some conclusions about the spectrum \( \text{Sp}(f) \) of a homogeneous polynomial at the origin. By above discussion and Corollary 2.3, the only rational numbers which can have nonzero multiplicity in \( \text{Sp}(f) \) are of the type

\[
(5) \quad \alpha = \frac{k}{d} + \rho \in (0, n), \quad \text{with } k, \rho \in \mathbb{Z}, \ 1 \leq k \leq d, \ 0 \leq \rho < n.
\]

Let \( \mu : (Y, E) \to (X, D) \) be a log resolution which is an isomorphism above \( U \). With \( \alpha, k, \rho, \) as in \( (5) \), define

\[
\beta_i^{(k)} := \left\{ -\frac{km_i}{d} \right\}, \quad M^{(k)} := \mathcal{O}_X \left( \sum_{i \in S} \beta_i^{(k)} d_i \right),
\]

\[
\mathcal{E}_\alpha := \mathcal{O}_Y^{n-p-1}(\log E)^\vee \otimes \omega_Y \otimes \mu^* M^{(k)} \otimes \mathcal{O}_Y \left( -\mu^* (\beta^{(k)} \cdot D_{\text{red}}) \right),
\]

where in the last sheaf the tensor products are over \( \mathcal{O}_Y \).

**Proposition 4.3.** Let \( \alpha \) be as in \( (5) \). The multiplicity of \( \alpha \) in \( \text{Sp}(f) \) is

\[
n_{\alpha}(f) = (-1)^{n-p-1} \chi(Y, \mathcal{E}_\alpha).
\]

Proof. By Corollary 2.3 and (4),

\[
n_{\alpha}(f) = \sum_{j \in \mathbb{Z}} (-1)^j \dim \text{Gr}_F^{n-p-1} H^{n-1+j}(U, V_k) = \ast.
\]
Now, \((M^{(k)}, \beta^{(k)})\) is an element of \(\text{Pic}^r(X, D)\) and it can be checked by using Lemma 4.2 that it corresponds to the dual \(\mathcal{V}_k^\vee\). In fact, \(\mathcal{V}_k^\vee\) equals \(\mathcal{V}_{d-k}\) if \(k \neq d\), and equals \(\mathcal{V}_d\) if \(k = d\). Since \((\mathcal{V}_k^\vee)^r = \mathcal{V}_k\), by applying Theorem 3.3(a), we have

\[
* = \sum_{j \in \mathbb{Z}} (-1)^j h^n - j - p^1(Y, \mathcal{E}_\alpha),
\]

which is equivalent to what we claimed. 

By Hirzebruch-Riemann-Roch (Theorem 2.4, Proposition 4.3) is useful when the topology of a log resolution is known:

**Corollary 4.4.** Let \(\alpha\) be as in \(\square\). The multiplicity of \(\alpha\) in \(\text{Sp}(f)\) is the intersection number

\[
n_\alpha(f) = (-1)^{n-p-1} (ch(\mathcal{E}_\alpha) \cdot Td(Y))_{n-1}.
\]

5. **Spectrum of hyperplane arrangements**

A **central hyperplane arrangement** in \(\mathbb{C}^n\) is a finite set \(\mathcal{A}\) of vector subspaces of dimension \(n-1\). The **intersection lattice** of \(\mathcal{A}\), denoted \(L(\mathcal{A})\), is the set of subspaces of \(\mathbb{C}^n\) which are intersections of subspaces \(V \in \mathcal{A}\) (see [OT]). For \(V \in \mathcal{A}\), let \(f_V\) be the linear homogeneous equation defining \(V\), and let \(m_V \in \mathbb{N} - \{0\}\). Let \(f = \prod_{V \in \mathcal{A}} f_V^{m_V} \in \mathbb{C}[x_1, \ldots, x_n]\) be a homogeneous polynomial of degree \(d = \sum_{V \in \mathcal{A}} m_V\). Denote by \(D\) the hypersurface defined by \(f\) in \(X := \mathbb{P}^{n-1}\). Let \(U = X - D\). Let \(\mathcal{G}' \subset L(\mathcal{A}) - \{\mathbb{C}^n\}\) be a building set (see [DP]-2.4 or [Te]-Definition 1.2). Let \(\mathcal{G} = \mathcal{G}' \cup \{0\}\). For simplicity, one can stick with the following example for the rest of the article: \(\mathcal{G} = L(\mathcal{A}) \cup \{0\} - \{\mathbb{C}^n\}\), when \(\mathcal{G}'\) is chosen to be \(L(\mathcal{A}) - \{\mathbb{C}^n\}\). The advantage of considering smaller building sets is that computations might be faster (see [Te]-Example 1.3-(c)). For any vector space \(V\) of \(\mathbb{C}^n\), we denote by \(\delta(V)\) (resp. \(r(V)\)) the dimension (resp. codimension) of \(V\).

**The canonical log resolution.** We consider the canonical log resolution \(\mu : (Y, E) \rightarrow (X, D)\) of \((X, D)\) obtained from successive blowing-ups of the (disjoint) unions of (the proper transforms) of \(\mathbb{P}(V)\) for \(V \in \mathcal{G} - \{0\}\) of same dimension. This is the so-called wonderful model of [DP]-section 4. More precisely, \(\mu\) and \(Y\) are constructed as follows (see also [B08]-Section 4, [S07a]). Let \(X_0 = X\). Let \(C_0\) be the disjoint union of \(\mathbb{P}(V)\) for \(V \in \mathcal{G} - \{0\}\) with \(\delta(V) = 1\). Let \(\mu_0 : X_1 \rightarrow X_0\) be the blow up of \(C_0\). Then \(\mu_i\) and \(X_{i+1}\) are constructed inductively as follows. Let \(C_i \subset X_i\) be the disjoint union of the proper transforms, under the map \(\mu_{i-1}\), of \(\mathbb{P}(V)\) for \(V \in \mathcal{G} - \{0\}\) with \(\delta(V) = i + 1\). Let \(\mu_i : X_{i+1} \rightarrow X_i\) for \(0 \leq i < n - 2\) be the blow up of \(C_i\). Define \(Y = X_{n-2}\) and \(\mu\) as the composition of the \(\mu_i\).

For \(V \in \mathcal{G} - \{0\}\) with \(\delta(V) = i + 1\), let \(E_V\) be the proper transform of the exceptional divisor in \(X_{i+1}\) corresponding to (the proper transform of) \(\mathbb{P}(V)\) (in \(X_i\)). Also let \(E_0\) denote the proper transform in \(Y\) of a general hyperplane of \(X\). Denote by \([E_V]\) the cohomology class of \(E_V\) on \(Y\) \((V \in \mathcal{G})\), where it will be clear from context what coefficients (integral, rational) we are considering. \(E\) denotes the union of \(E_V\) for \(V \in \mathcal{G} - \{0\}\).
Intersection theory on the canonical log resolution. Let $I \subset \mathbb{Z}[c_V]_{V \in \mathcal{G}}$ be the ideal generated by two types of polynomials:

\[(6) \prod_{V \in \mathcal{H}} c_V \]

if $\mathcal{H} \subset \mathcal{G}$ is not a nested subset, and by

\[(7) \prod_{V \in \mathcal{H}} c_V \left( \sum_{W' \subset W} c_{W'} \right)^{d_{\mathcal{H},W}}, \]

where $\mathcal{H} \subset \mathcal{G}$ is a nested subset, $W \in \mathcal{G}$ is such that $W \subsetneq V$ for all $V \in \mathcal{H}$, and $d_{\mathcal{H},W} = \delta(\cap_{V \in \mathcal{H}} V) - \delta(W)$. In (7), one considers $\mathcal{H} = \emptyset$ to be nested, in which case (7) is defined for every $W \in \mathcal{G}$ by setting $\delta(\emptyset) = n$. Here $I$ is the ideal of \[DP\]-5.2, for the projective case. $I$ depends only on $\mathcal{G}$ and $\mathbb{Z}[[c_V]]_{V \in \mathcal{G}}/I$ is isomorphic to the cohomology ring of the canonical log resolution:

**Theorem 5.1.** (\[DP\]) With notation as above, there is an isomorphism

\[(8) \quad \mathbb{Z}[c_V]_{V \in \mathcal{G}}/I \xrightarrow{\sim} H^*(Y, \mathbb{Z}) \longleftrightarrow \mathbb{Z}[[c_V]]_{V \in \mathcal{G}}/I \]

1 $\mapsto [Y]$, 
$c_V \mapsto [E_V]$ if $V \neq 0$, 
$c_0 \mapsto -[E_0]$.

Theorem 5.1 follows from \[DP\]-5.2 Theorem, \[DP\]-4.1 Theorem, part (2), and \[DP\]-4.2 Theorem, part (4); see also \[B08\]-Remark 4.3. Remark that the degree $n - 1$ homogeneous part of $\mathbb{Z}[[c_V]]_{V \in \mathcal{G}}/I$ can be identified with $\mathbb{Z} \cdot (-c_0)^{n-1}$.

For every $V \in \mathcal{G} - \{0\}$ define a formal power series $F_V \in \mathbb{Z}[[c_V]]_{V \in \mathcal{G}}$ by

\[F_V := (1 - \sum_{W \subsetneq V \atop W \in \mathcal{G}} c_W)^{-r(V)}(1 + c_V)(1 - \sum_{W \subset V \atop W \in \mathcal{G}} c_W)^{r(V)}.\]

Also, set $F_0 = (1 - c_0)^n$ and define $F := \prod_{V \in \mathcal{G}} F_V$.

**Proposition 5.2.** (\[B08\]-Proposition 4.7.) The total Chern class $c(Y)$ is the image in $H^*(Y, \mathbb{Z})$ of $F$ under the map (8).

Let $Q(x)$ be as in (2). For every $V \in \mathcal{G} - \{0\}$ define a formal power series $G_V \in \mathbb{Q}[[c_V]]_{V \in \mathcal{G}}$ by

\[G_V := Q(-\sum_{W \subsetneq V \atop W \in \mathcal{G}} c_W)^{-r(V)}Q(c_V)Q(-\sum_{W \subset V \atop W \in \mathcal{G}} c_W)^{r(V)}.\]

Also, set $G_0 = Q(-c_0)^n$ and define $G := \prod_{V \in \mathcal{G}} G_V$.

**Corollary 5.3.** (\[B08\]-Corollary 4.8.) The Todd class $Td(Y)$ is the image in $H^*(Y, \mathbb{Q})$ of $G$ under the map induced by (8) after $\otimes_{\mathbb{Z}} \mathbb{Q}$. 


For a power series $\xi \in \mathbb{Z}[[c_V]]_{V \in G}$, let $\xi_i$ denote the degree $i$ part, such that $\xi = \sum_i \xi_i$. Define a formal power series $H \in \mathbb{Z}[[c_V]]_{V \in G}$ by

$$H := \left( \sum_i (-1)^i F_i \right) \cdot \prod_{V \in G \setminus \{0\}} \frac{1}{1 - c_V}.$$

**Lemma 5.4.** The total Chern class $c(\Omega^1_Y(\log E))$ is the image in $H^*(Y, \mathbb{Z})$ of $H$ under the map (8).

**Proof.** $\Omega^1_Y(\log E)$ fits into a short exact sequence (see [EV] -2.3 Properties (a)):

$$0 \to \Omega^1_Y \to \Omega^1_Y(\log E) \to \bigoplus_{V \in G \setminus \{0\}} \mathcal{O}_{E_V} \to 0.$$

By (3), $c(\Omega^1_Y(\log E)) = c(\Omega^1_Y) \cdot \prod_{V \in G \setminus \{0\}} c(\mathcal{O}_{E_V})$. Now, by Proposition 5.2, $c_i(\Omega^1_Y) = (-1)^i c_i(T_X) = (-1)^i(F)_i$. Also, $c(\mathcal{O}_{E_V}) = 1/(1 - [E_V])$ since $E_V$ is a hypersurface in $Y$.

Fix $p \in \{0, \ldots, n - 1\}$. Denote by $e_i(x_1, \ldots, x_{n-1})$ the coefficient of $t^i$ in $\prod_{1 \leq i \leq n-1} (1 + x_i t)$. The coefficient of $t^i$ in $\prod_{1 \leq i < j < p \leq n-1} (1 + (x_i + \ldots + x_j) t)$ is $K_{p,i}(e_1, \ldots, e_{n-1})$ for some polynomial $K_{p,i}$ in $n-1$ variables over $\mathbb{Z}$. Here $K_{p,0} = 1$ if $i = 0$ and equals 0 if $i \neq 0$. Define

$$K_{p,i} := K_{p,i}(H_1, \ldots, H_{n-1}) \in \mathbb{Z}[[c_V]]_{V \in G},$$

where $H_j$ is the degree $j$ part of $H$.

**Lemma 5.5.** The Chern class $c_i(\Omega_Y(\log E))$ is the image in $H^*(Y, \mathbb{Z})$ of the polynomial $K_{p,i}$ under the map (8).

**Proof.** Since $\Omega^1_Y(\log E) = \bigwedge^p \Omega^1_Y(\log E)$, the claim follows from (3) and Lemma 5.4.

For $1 \leq p \leq n - 1$, the degree $j$ term in the Taylor expansion of $\sum_{1 \leq i \leq p} e^x_i$ is $P_{p,j}(e_1, \ldots, e_p)$ for some polynomial $P_{p,j}$ in $p$ variables over $\mathbb{Q}$. Let $P_{p,j} \in \mathbb{Q}[[c_V]]_{V \in G}$ be

$$P_{p,j} := P_{p,j}((-1)^{K_1}, \ldots, (-1)^{j}K_{p,i}, \ldots, (-1)^p K_{p,p}).$$

Define $P_p := \sum_i P_{p,i}$. For $p = 0$ set $P_0 = 1$. Then by (3) and Lemma 5.5 we have:

**Lemma 5.6.** The Chern character $\text{ch}(\Omega^1_Y(\log E)^\vee)$ is the image in $H^*(Y, \mathbb{Q})$ of $P_p$ under the map induced by (8) after $\otimes_{\mathbb{Z}} \mathbb{Q}$.

**Computation of spectrum.** Now we complete the computation of the Hodge spectrum $\text{Sp}(f)$ of $f$ at the origin. The only rational numbers $\alpha$ which can appear in $\text{Sp}(f)$ are of the type (5), i.e.

$$\alpha = \frac{k}{d} + p \in (0, n),$$

where $d$ is the degree of $f$. By Corollary 4.14, the multiplicity of $\alpha$ in $\text{Sp}(f)$ is the intersection number

$$n_\alpha(f) = (-1)^{n-p-1} (\text{ch}(\mathcal{E}_\alpha) \cdot Td(Y))_{n-1},$$

(9)
where \( \mathcal{E}_\alpha \) is defined as follows. For \( V \in \mathcal{A} \), let \( m_V \) be the multiplicity of the irreducible component \( V \) of \( f^{-1}(0) \). Let

\[
\beta^{(k)}_V := \left\{ -\frac{km_V}{d} \right\}, \quad M^{(k)} := \mathcal{O}_X \left( \sum_{V \in \mathcal{A}} \beta^{(k)}_V \right),
\]

\[
L^{(k)} := \omega_Y \otimes \mu^* M^{(k)} \otimes \mathcal{O}_Y \left( -\mu^* (\beta^{(k)} \cdot D_{\text{red}}) \right).
\]

Now define

\[
E_{\alpha} := \Omega^{n-p-1}_Y (\log E)^\vee \otimes L^{(k)}.
\]

We also need to fix some more notation. For \( \beta \in \mathbb{Q}^A \) and \( V \in \mathcal{G} \), let

\[
s_V(\beta) := \sum_{V \subset W \in \mathcal{A}} \text{mult}_{\mathbb{P}(W)}(\beta \cdot D_{\text{red}}),
\]

where the multiplicity of rational divisors is defined by linearity from the integral divisors. For \( k \) as above and \( V \in \mathcal{G} \) define

\[
a_{k,V} := r(V) - \sum_{W \in \mathcal{A}} s_V(\beta) E_W - \delta_{V,0},
\]

where \( \delta_{V,0} = 1 \) if \( V = 0 \) and is 0 if \( V \neq 0 \).

**Lemma 5.7.** \( L^{(k)} = \mathcal{O}_Y (-a_{k,0} E_0 + \sum_{V \in \mathcal{G} - \{0\}} a_{k,V} E_V) \).

**Proof.** First, \( K_Y = K_{Y/X} + \mu^* K_X \). We know \( \omega_X = \mathcal{O}_X (-n) \). Also,

\[
K_{Y/X} = \sum_{V \in \mathcal{G} - \{0\}} (r(V) - 1) E_V
\]

\[
\mu^*(\beta \cdot D_{\text{red}}) = \sum_{V \in \mathcal{G} - \{0\}} s_V(\beta) E_V, \quad \beta \in \mathbb{Z}^A,
\]

by \[\text{Lemma 2.1} \]. One can let \( \beta \in \mathbb{Q}^A \) in the last formula by multiplying with a scalar that clears denominators. Thus, writing \( L^{(k)} \) in divisor form, the coefficient of \( E_V \) (\( V \in \mathcal{G} \)) becomes

\[
\begin{cases}
  r(V) - 1 - \sum_{W \in \mathcal{A}} s_V(\beta^{(k)}) E_W & \text{if } V \neq 0, \\
  -n + \sum_{W \in \mathcal{A}} \beta^{(k)} & \text{if } V = 0.
\end{cases}
\]

This is equivalent to the claim. \( \square \)

**Lemma 5.8.** The Chern character \( ch(\mathcal{E}_\alpha) \) is the image in \( H^*(Y, \mathbb{Q}) \) of the formal power series

\[
R_\alpha := P_{n-p-1} \cdot e^{\sum_{V \in \mathcal{G}} a_{k,V} E_V} \in \mathbb{Q}[[c_V]]_{V \in \mathcal{G}}
\]

under the map induced by \[\text{Lemma 2.1} \] after \( \otimes \mathbb{Z} \mathbb{Q} \).

**Proof.** Follows by the multiplicativity of the Chern character, from Lemma 5.6 and Lemma 5.7. \( \square \)

**Theorem 5.9.** With \( \alpha \) as above, the multiplicity \( n_\alpha(f) \) of \( \alpha \) in \( Sp(f) \) is

\[
(10) \quad (-1)^{n-p-1} (R_\alpha \cdot G)_{n-1}
\]

where \[\text{Lemma 2.1} \] is viewed as a number via identification of the degree \( n-1 \) homogeneous part of \( \mathbb{Q}[[c_V]]_{V \in \mathcal{G}} / I \) with \( \mathbb{Q} \cdot (-c_0)^{n-1} \).
Proof. It follows immediately from \([9]\), Lemma \([5.8]\) and Corollary \([5.3]\). \(\square\)

6. Examples

The following examples illustrate how Theorem \([5.9]\) works.

(a) Consider the arrangement \(A\) of three lines in \(\mathbb{C}^2\) meeting at the origin. It is defined for example by the equation \(f = xy(x+y) \in \mathbb{C}[x,y]\). Let \(G = \{0, L_1, L_2, L_3\}\) where \(L_i\) are the lines. For \(V = L_i\), denote \(c_V\) by \(c_i\) \((i = 1, 2, 3)\). The ideal \(I \subset \mathbb{Z}[[c_V]]_{V \in G}\) is generated by \(c_0^2\) and \(c_0 + c_i\) \((i = 1, 2, 3)\). We have (skipping the terms of degree \(\geq 2\))

\[
F = 1 - 2c_0, \quad G = 1 - c_0, \quad H = 1 + 2c_0 + c_1 + c_2 + c_3, \quad K_{0,0} = K_{1,0} = 1, \quad K_{0,1} = 1, \quad K_{1,1} = 2c_0 + c_1 + c_2 + c_3, \quad P_0 = 1, \quad P_1 = 1 - (2c_0 + c_1 + c_2 + c_3).
\]

Also, passing directly to the quotient \(\mathbb{Z}[[c_V]]_{V \in G}/I\), we have

\[
R_{1/3} = 1 + c_0, \quad R_{4/3} = 1, \quad R_{2/3} = 1 + 2c_0, \quad R_{5/3} = 1 + c_0, \quad R_{3/3} = 1 + 3c_0,
\]

Then, denoting by \((\cdot)_1\) the coefficient of \(-c_0\), we have

\[
n_{1/3} = -(R_{1/3}G)_1 = -(1 - c_0^2)_1 = 0, \quad n_{2/3} = -(R_{2/3}G)_1 = -(1 + c_0)_1 = 1, \quad n_{1} = -(R_{3/3}G)_1 = -(1 + 2c_0)_1 = 2, \quad n_{4/3} = (R_{4/3}G)_1 = (1 - c_0)_1 = 1, \quad n_{5/3} = (R_{5/3}G)_1 = (1 - c_0^2)_1 = 0.
\]

Hence the spectrum of \(f\) is \(\text{Sp}(f) = t^{2/3} + 2t + t^{4/3}\), which is well-known.

(b) Consider the central hyperplane arrangements of degree 4 in \(\mathbb{C}^3\) given by

\[
f_1 = (x^2 - y^2)(x + z)(x + 2z), \quad f_2 = (x^2 - y^2)(x^2 - z^2).
\]

They are combinatorially equivalent. Here \(A = \{A_i \subseteq \mathbb{C}^3 \mid i = 1, \ldots, 4\}\), and \(G = L(A) - \{\mathbb{C}^3\}\) is given by

\(\{0, B_1, \ldots, B_6, A_1, \ldots, A_4\}\),

where \(B_j, A_i\) have codimension 2, resp. 1, and \(B_j \subset A_i\) if \((i, j)\) lies in \(M := \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 5), (2, 6), (3, 1), (3, 4), (3, 6), (4, 3), (4, 4), (4, 5)\}\).
The ideal $I$ is generated by $cA_i + \sum_{(i,j) \in M} cB_j + c_0, c_0 cC, cB_j cB_{j'}$ with $j \neq j', cB_j c_0$, and $c^2 B_j + c_0^2$. Then, modulo $I$, we have

$$F = 9c_0^2 - (cB_1 + \ldots + cB_6) - 3c_0 + 1,$$

$$G = c_0^2 - \frac{1}{2}(cB_1 + \ldots + cB_6) - \frac{3}{2}c_0 + 1,$$

$$H = c_0^2 - c_0 + 1,$$

$$P_0 = 1, \quad P_1 = -\frac{1}{2}c_0^2 + c_0 + 2, \quad P_2 = \frac{1}{2}c_0^2 + c_0 + 1,$$

$$R_{1/4} = \frac{1}{2}c_0^2 + c_0 + 1, \quad R_{7/4} = -\frac{1}{2}c_0^2 + 2(cB_1 + \ldots + cB_6) + 5c_0 + 2,$$

$$R_{2/4} = 2c_0^2 + 2c_0 + 1, \quad R_{8/4} = \frac{11}{2}c_0^2 + 2(cB_1 + \ldots + cB_6) + 7c_0 + 2,$$

$$R_{3/4} = \frac{3}{2}c_0^2 + (cB_1 + \ldots + cB_6) + 3c_0 + 1, \quad R_{9/4} = 1,$$

$$R_{4/4} = 5c_0^2 + (cB_1 + \ldots + cB_6) + 4c_0 + 1, \quad R_{10/4} = \frac{1}{2}c_0^2 + c_0 + 1,$$

$$R_{5/4} = -\frac{1}{2}c_0^2 + c_0 + 2, \quad R_{11/4} = -c_0^2 + (cB_1 + \ldots + cB_6) + 2c + 1,$$

$$R_{6/4} = \frac{3}{2}c_0^2 + 3c_0 + 2.$$

Then Theorem 5.9 gives

$$\text{Sp}(f_1) = \text{Sp}(f_2) = t^{3/4} + 3t + t^{6/4} - 3t^2 + t^{9/4}.$$ 

We used Macaulay 2 for some of the computations. The spectrum in this case can also be computed by Theorem 6.1 which treats the case of homogeneous polynomials with 1-dimensional critical locus. One can check that the outcome is the same as ours. Remark that there is a shift by multiplication by $t$ between the definition of spectrum of \textit{St87} and that of this article. Also, the beginning part of the spectrum, which is given by inner jumping numbers by \textit{B03}, can be computed via a different method, see \textit{B08} - Section 5, Example (b).

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