Fractal Structures of Quantum Gravity in Two Dimensions

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ABSTRACT

Recent numerical results on the fractal structure of two-dimensional quantum gravity coupled to $c = -2$ matter are reviewed. Analytic derivation of the fractal dimensions based on the Liouville theory and diffusion equation is also discussed. Excellent agreements between the numerical and theoretical results are obtained. Some problems on the non-universal nature of the fractal structure in the continuum limit are pointed out.

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1. Introduction

One of the most important recent developments in gravity theory is that we obtained a well defined regularization of quantum gravity in two dimensions. This recognition comes from the fact that the continuum formulation[1] and the dynamical triangulation[2] are equivalent. The dynamical triangulation can be identified as a lattice regularization of quantum gravity and can be analyzed analytically by the matrix model[3] and numerically by the computer simulation.[4,5] It has become clear numerically that the fractal structure of space time is the fundamental nature of the quantum gravity.[4,5] Analytical investigations by Liouville quantum gravity support the numerical results.[6,7,8] It is an interesting question if the dynamical triangulation works as a regularization scheme of three- and four-dimensional quantum gravity, which may be tested only by numerical simulations at this moment.[9,10]

In this manuscript we will summarize the recent numerical simulations of two-dimensional quantum gravity coupled to $c = -2$ matter.[5,8] We also show that our analytical investigations by Liouville theory[8] have excellent agreements with the numerical estimation of a fractal dimension and the mean squared distance of gravitational random walks treated by a diffusion equation.

Our recent analytic investigation on the fractal structure of $c = 0$ model[11] suggests that we must be careful about non-universal behaviors of the fractal structures in the continuum limit of dynamical triangulation.

2. The shape of Typical Quantum Gravitational Space-Time and the Fractal Structure

In the dynamical triangulation of two-dimensional gravity, the metric integration in the sense of path integral is replaced by a summation of all the different types of surface configuration with a given number of triangles and a
given topology. Here we specify the space-time topology as a sphere. Formally the continuum partition function is given by

\[ Z(A) = \int \mathcal{D}g \, \delta(\int dx \sqrt{g} - A) \, Z_m[g], \]  

where \( Z_m[g] \) is a matter part of the partition function with gravitational background and \( A \) is the total area. Reguralized counterpart of the above partition function by dynamical triangulation is

\[ Z_{\text{reg}}(A) = \sum_G Z_m[G] \delta_{N a^2, A} \sim Z_m[G_0], \]  

where \( N \) is the number of equilateral triangles and \( a^2 \) is the area of the triangle. \( G \) denotes a triangulation and \( G_0 \) is the typical triangulation which we select from the huge set of triangulations. The last approximate equality in Eq.(2) is valid up to the normalization factor and if the selection of the typical surface is carried out by a correct procedure which we explain later. Since the path integration of the metric is carried out after the selection of the typical surface, \( G_0 \) carries the information of the quantum fluctuation of space time effectively.

A natural question now is: “How does the typical surface look like?” Since the typical surface carries the information of quantum gravitational fluctuation, it may look quite different from the classical space time. Some of the possible surfaces we can imagine beforehand may be; 1) smooth surface, 2) spiky surface, 3) branched surface, .... The next question is: “How do we parametrize the shape of the typical surface?”

Suppose we obtain the typical surface by the recursive sampling method which we explain later, the actual measurements for the typical surface are carried out as follows. We first fix a marking site on the triangulated surface, from which we measure the internal geodesic distance \( r \). We count the number of triangles \( (V_t(r)) \), links \( (V_l(r)) \), and sites \( (V_s(r)) \) within \( r \) geodesic steps.
$V_t(r)$ is essentially the two-dimensional volume in radius $r$. The number of the disconnected boundaries $N_b(r)$ can be counted by using the following relation: $\chi = 2 - N_b(r)$ and $\chi = V_s(r) - V_l(r) + V_t(r)$, where $\chi$ is the Euler number of sphere with $N_b(r)$ boundary holes. $L_b(r) = V_t(r + 1) - V_t(r)$ is essentially a derivative of $V_t(r)$ with respect to the geodesic distance and is the total length of the $N_b(r)$ disconnected boundaries at the distance $r$ from the marking site.

In the actual calculations we define the geodesic distance on the triangulated lattice. To be precise, the region of geodesic distance $r$ from a site is defined as a thin connected area composed of all the triangles that are attached to the links located at $r$-steps from the marking site. The number of triangles of this area and the number of disconnected boundaries precisely correspond to $L_b(r)$ and $N_b(r)$ respectively. It is also possible to define the geodesic distance on the dual lattice.

We parametrize the above mentioned three quantities as follows:

(A) $N_b(r) \equiv \langle \text{number of boundaries at the step } r \rangle \sim r^\alpha$,

(B) $L_b(r) \equiv \langle \text{total length of boundaries at the step } r \rangle \sim r^\beta$,

(C) $V_t(r) \equiv A(r) \equiv \langle \text{number of triangles within } r \text{ steps} \rangle \sim r^\gamma$.

$\alpha$, $\beta$, and $\gamma$ may be called the fractal dimensions of internal space-time geometry if they become constant in the large $r$ asymptotic region. If the surface is smooth, these quantities behave as follows: $N_b(r) \sim 1$, $L_b(r) \sim r$, $V_t(r) \sim r^2$. Thus deviations from these behaviors signal the fractal nature of quantum gravitational fluctuation of space-time. In other words we expect that the space-time surface is smooth classically while the fractal nature of the space time is an essential feature of quantum gravity.
3. \( c = -2 \) Model and Recursive Sampling Method

In the numerical simulation of dynamical triangulation there are essentially two methods to generate the typical surface of two-dimensional quantum gravity: 1) flipping method by Monte Carlo and 2) recursive sampling method which is first proposed by Agishtein and Migdal\textsuperscript{[4]} for pure gravity (\( c = 0 \) model). The second method can generate much larger number of triangles than the first method. The second method, however, necessitates an analytic formula to generate a typical surface and is restricted to particular models such as \( c = 0 \) and \( c = -2 \) models. Here we investigate \( c = -2 \) model which has much simpler analytic formula than \( c = 0 \) model and is thus easier to simulate larger number of triangles: number of triangles = \( 5 \times 10^6 \) for \( c = -2 \) while \( 1.3 \times 10^5 \) for \( c = 0 \).

The \( c = -2 \) model or equivalently the two-dimensional gravity coupled to \( c = -2 \) matter was introduced and solved analytically in [12]. The partition function of two-dimensional gravity coupled to \( c = -2 \) matter is given by

\[
Z(N) = \sum_G \int D\bar{\psi}_i D\psi_i \exp \left\{ -\sum_{<ij>} (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \right\} \\
= \sum_{G} \det \Delta(G) \\
= \frac{1}{N + 2} \left( T_{N+1} R_{N/2+1} \right),
\]

where \( \psi_i \) and \( \bar{\psi}_i \) are fermion fields sitting at a site \( i \) of dynamically triangulated lattice \( G \). Here we fix the topology of the surface as sphere and the number of triangles as \( N \) and the area of the triangle is unity. The fermion integration leads to a random lattice version of Laplacian \( \Delta(G) \). In this particular model the determinant of the Laplacian and dual planar \( \phi^3 \) diagrams are related. \( T_{N+1} \) is the number of rooted dual tree diagrams with \( N + 2 \) external legs while \( R_{N/2+1} \) is a number of rainbow diagrams with \( N/2 + 1 \) lines. A rainbow diagram can be constructed by connecting the \( N + 2 \) external legs of a tree diagram in such a way
that none of the $N/2 + 1$ lines crosses over. The third equality in eq. (3) is a very useful one and can be established by an application of the well-known Kirchhoff theorem. The introduction of the fermion field effectively means an embedding of a surface into $c = -2$ dimension.

$T_n$ satisfies the Schwinger-Dyson equation: $T_n = \sum_{k=1}^{n-1} T_k T_{n-k}$, which can be solved by introducing a corresponding generating function of $T_n$ and leads to the solution: $T_n = (2n - 2)!/n!(n - 1)!$. $R_n$ satisfies the same type of relation as $T_n$, which leads to a relation: $R_{n-1} = T_n$.

The recursive sampling algorithm to generate a typical surface goes as follows. First of all one of the great advantages of $c = -2$ model is the factorization feature of $T_{N+1}$ and $R_{N/2+1}$ in Eq.(3). In order to generate a typical surface we independently generate a tree diagram and a rainbow diagram with a correct probability and then connect them. The branching probability to divide a tree diagram with $n + 1$ legs into two different tree diagrams with $k + 1$ external legs and $n - k + 1$ external legs is given by $W(n, k) = T_k T_{n-k}/T_n$. A typical tree diagram with $N$ vertices can be generated by applying $N$ times of the probability formula $W(n, k)$. A typical rainbow diagram can be generated similarly.

4. Numerical Results on Fractal Structures of $c = -2$ Model

We show the numerical results of the fractal dimensions $\alpha$, $\beta$ and $\gamma$ with the fittings of (A) $N_b(r) \sim r^{\alpha}$, (B) $L_b(r) \sim r^{\beta}$, and (C) $V_t(r) \equiv A(r) \sim r^{\gamma}$ in Figs.1, 2, and 3, respectively. We have measured those quantities for the following number of triangles: (1) $8 \times 10^3$, (2) $4 \times 10^4$, (3) $2 \times 10^5$, (4) $10^6$, and (5) $5 \times 10^6$. The values $\alpha$, $\beta$ and $\gamma$ are approaching to constant values $2.55 \pm 0.1$, $2.53 \pm 0.1$ and $\sim 3.5$, respectively, and thus show clear fractal behaviors. We obtain an approximate relation $\gamma - \beta \simeq 1$ numerically, which is expected from the relation $dV_t(r)/dr \simeq L_b(r)$. What is interesting and unexpected is that the two-dimensional quantum gravitational space time is very much branching. For
example the space-time surface with \( c = -2 \) matter splits into \( \sim 6000 \) branches after \( \sim 60 \) geodesic steps in the case of \( 5 \times 10^6 \) triangles. See Fig.1(a).

It happens that the fractal dimension of the number of boundaries \( (\alpha) \) is approximately same as that of the total length of the boundaries \( (\beta) \). To estimate the average perimeter length of a branch we have measured the ratio \( (D): L_b(r)/N_b(r) \) in Fig.4. It shows clear constant behavior as a function of the geodesic distance. The numerical value is approximately \( \sim 15 \) in the case of \( 10^6 \) triangles. It looks as if all the branches have similar perimeter length. In order to see if this is the case we have measured \( (E): \) the number of boundaries \( P_r(l) \) with a given boundary length \( l \) as a function of the boundary length \( l \) for a given geodesic steps \( r \) measured from a marking point. The results are shown in Fig.5. As we can see in the figures, different sizes of boundary length are uniformly distributed with roughly a power law behavior of \( \sim l^{-1.8} \). In other words there is no particular preference of the perimeter length even though the average perimeter length is \( \sim 15 \).

We can now imagine how the two-dimensional quantum gravitational space time looks like. It has many branches whose perimeter length varies small size to large size. Number of the branches with large perimeter size is small but there exists such branches with a certain probability. If we look into this quantum gravitational universe microscopically, we feel like being in a dense jungle and the universe looks like complete chaos. The universe, however, have beautiful fractal structures and thus has an order as a whole. We have given here a numerical results of \( c = -2 \) model. In \( c = 0 \) model Agishtein and Migdal found a similar branching behavior of the space-time surface, but didn’t observe the fractal behavior.\(^4\) This suggests that the branching behavior of the quantum gravitational space time is quite universal phenomena irrespective of the choice of matter.

The geodesic distance so far has been measured on the triangulated lattice.
Geodesic distance on the dual lattice or equivalently on the dual planar $\phi^3$ diagram is defined as a number of minimum steps located between two vertices. It is one of interesting questions whether the numerical results of the fractal structure change depending on the different definitions of the geodesic distance. We show the numerical result of the fractal dimension of boundary length ($F$):

$$\beta(r) \equiv d \log L_b(r)/d \log r$$

measured with the geodesic distance on the dual lattice in Fig. 6, which should be compared with the corresponding result in Fig. 2. As we can see, the behavior to approach the stationary value is very slow in the case of geodesic distance on the dual lattice. This may be understood by recognizing the fact that the geodesic distance on the dual lattice generally takes much more steps than those on the triangulated lattice to measure the same distance. It is, however, natural to expect that $\beta$ defined on the dual lattice will eventually approach to the same value as that defined on the triangulated lattice.

5. Diffusion Equation on the Dynamically Triangulated Lattice and the Continuum Limit

In this section we try to investigate the fractal structure of quantum gravity analytically by diffusion equation. We first define adjacency matrix $K_{ij}$. For a given typical surface $G_0$ we number the sites of the triangulated lattice. Then the $(i,j)$ component of the adjacency matrix $K_{ij}$ is one if $i$-th site and $j$-th site are connected by a link as next neighboring sites, and zero if they are not connected by a single link. It is interesting to note that $(n,n_0)$ component of $(K^T)_{n,n_0}$ counts the number of possible random walks reaching from a marking site $n_0$ to a site $n$ after $T$ steps. The Laplacian defined on the dynamically triangulated lattice is given by

$$\Delta_L = 1 - S, \quad S_{ij} = \frac{1}{q_i} K_{ij},$$

where $q_i$ is called coordination number and denotes a number of links connected.
to the site $i$. $S_{ij}$ is thus a probability of one step random walk from the site $j$ to the neighboring site $i$. The diffusion equation on a given surface $G_0$ with $N$ triangles is now given by

$$
\partial_T \Psi_N^{(G_0)}(T; n, n_0) = \Delta_L(G_0) \Psi_N^{(G_0)}(T; n, n_0),
$$

(5)

where $\partial_T$ is a difference operator in $T$ direction and $\Psi_N^{(G_0)}(T; n, n_0)$ is a wave function of the diffusion equation and denotes the probability of finding the random walker at the site $n$ after $T$ steps from the starting site $n_0$. A solution of the diffusion equation can be easily obtained as $\Psi_N^{(G_0)}(T; n, n_0) = e^{T\Delta_L(G_0)}(\delta_{n,n_0})$, where $(\delta_{n,n_0})$ is $N$-component vector with unit $n_0$ entry.

We now consider the continuum limit of this diffusion equation. First of all we recover the lattice constant $a$. In taking continuum limit, the total physical area $A = a_i^2 N_i$ is fixed and $a_i \to 0 (N_i \to \infty)$ is taken, where $N_i$ is the number of triangles and $a_i^2$ is the area of a triangle. In each step of the limiting process we select a typical surface $G_i$ for the given number of triangles $N_i$, on which the lattice Laplacian $\Delta_L(G_i)$ of Eq.(4) is defined. Now the lattice version of the diffusion equation (5) can be rewritten as

$$
\frac{1}{a_i^2} \{ \Psi_A^{(G_i)}(T + a_i^2; x, x_0) - \Psi_A^{(G_i)}(T; x, x_0) \} = \frac{1}{a_i^2} \Delta_L(G_i) \Psi_A^{(G_i)}(T; x, x_0),
$$

(6)

where the location of the site $x$ is measured with respect to the lattice constant $a_i$. Thus we identify the dimension of $T$ as that of area: $\text{dim}[T] = \text{dim}[A]$. In the continuum limit the solution of the diffusion equation (6) is expected to approach the continuum wave function: $\Psi_A^{(G_i)}(T; x, x_0) \to \Psi_A^{(G_\infty)}(T; x, x_0)$. Numerically we approximate the limiting surface as the typical surface($G_0$) of the maximum size: $G_\infty \simeq G_0$ As we have already noted in Eq.(2), the metric integration is effectively carried out for the equation (6) since we have chosen a typical surface. This means that the quantum effect is included for the wave function of Eq.(6).
On the other hand the solution of the continuum counterpart of the diffusion equation: \( \partial_\tau \Psi(\tau; x, x_0) = \Delta(g)\Psi(\tau; x, x_0) \) is still background metric dependent in general. Furthermore the dimensions of \( T \) and \( \tau \) may not necessarily be equal.

5. Fractal Dimensions by Liouville Theory

An analytical treatment of fractal dimensions by Liouville theory has been first given by Kawai and Ninomiya.\(^6\) In their treatment fermion is introduced as a test particle to derive the fractal dimension. It has been recognized that there are several ways of defining fractal dimensions, which may explain the discrepancy between the theoretical and numerical results.\(^5\) Here we derive fractal dimensions by investigating the gravitational random walks with the help of diffusion equation and Liouville theory. In this section we briefly sketch the derivation of the fractal dimension while the details of the derivation will be given elsewhere.\(^8\)

Let us now define the comeback probability of random walk on the triangulated lattice and relate it with the continuum expression of Liouville theory as follows:

\[
G(T) \equiv \Psi^{(G_0)}_A(T; x_0, x_0) \sim \frac{\langle dx \sqrt{g} \Psi(\tau; x, x) \rangle_A}{\langle dx \sqrt{g} \rangle_A} = \frac{1}{A} \sim \frac{1}{T},
\]

where \( \langle O \rangle_A \) is an expectation value of \( O \) with the partition function given by Eq.(1). We should remind of the fact that the metric integration is effectively carried out since we have chosen the typical surface \( G_0 \) for the wave function of the comeback probability. The initial wave function can be formally written as \( \Psi(0; x, x) = \lim_{y \to x} \delta(y - x) \frac{1}{\sqrt{g}} \) and need to be regularized. The first similarity relation of Eq.(7) can be understood from the relation: \( \int dy \sqrt{g} \Psi(\tau; y, x) = 1 \), which is obvious from the provability interpretation of the wave function. The
dimensional arguments coming from the equation (6) supports the last similarity relation. Here comes a prediction of comeback provability (G): $G(T)T \sim \text{const.}$

We next consider how to accommodate the Weyl invariance into the diffusion equation of random walk by using DDK\textsuperscript{[1]} formulation of Liouville theory. Let us consider the following quantity by Liouville theory:

$$\langle \int dx \sqrt{g} \Psi(\tau; x, x) \rangle_A = \langle \int dx \sqrt{g} \Psi(0; x, x) \rangle_A + \tau \langle \int dx \sqrt{g} \Delta \Psi(0; x, x) \rangle_A + \cdots,$$

where the solution of the diffusion equation is expanded by $\tau$.

Taking a conformal gauge $g_{\mu\nu}(x) = \hat{g}_{\mu\nu} e^{\phi(x)}$ and introducing DDK arguments, we can rewrite the first and second terms of the Eq.(8) as

$$\langle \int dx \sqrt{g} \Psi(0; x, x) \rangle_A = \langle \int dx \sqrt{\hat{g}} \left[ \frac{1}{\sqrt{\hat{g}}} \delta(x - x_0) \right]_{x=x_0} \rangle_A,$$

$$\langle \int dx \sqrt{g} \Delta \Psi(0; x, x) \rangle_A = \langle \int dx \sqrt{\hat{g}} \left[ \hat{\Delta} e^{\alpha-1 \phi} \frac{1}{\sqrt{\hat{g}}} \delta(x - x_0) \right]_{x=x_0} \rangle_A,$$

where the term $e^{\alpha-1 \phi}$ is introduced to keep the Weyl invariance of the second term. The expectation value $\langle O(g) \rangle_A$ is now rewritten by using the well-known expression

$$\langle O(g) \rangle_A = \int \mathcal{D}\hat{g} \phi Z_{FP}[\hat{g}] Z_M[\hat{g}] \delta \left( \int dx \sqrt{\hat{g}} e^{\alpha_1 \phi} - A \right) O(\hat{g}, \phi) \exp \left( \frac{D - 25}{48\pi} S_L[\phi, \hat{g}] \right),$$

where $Z_{FP}$ is the Fadeev Popov contribution and $S_L[\phi, \hat{g}]$ is the Liouville action. $\alpha_n$ appeared in Eqs.(9) and (10) is given by

$$\alpha_n = \frac{2n}{1 + \sqrt{(25 - c - 24n)/(25 - c)}}.$$

Invariance of the expectation value under the translation of the conformal field
\( \phi \to \phi - \ln \lambda / \alpha_1 \) leads to the change:
\[
\delta \left( \int dx \sqrt{g} e^{\alpha_1 \phi} - A \right) \to \lambda \delta \left( \int dx \sqrt{g} e^{\alpha_1 \phi} - \lambda A \right),
\]
which is then interpreted as the scale change of the physical area \( A \to \lambda A \). To require the invariance of each term of Eqs. (8) and (9) under this scale transformation \( \tau \) should scale as \( \tau \to \lambda^{-\frac{\alpha_1}{\alpha - 1}} \tau \).

In other words,
\[
\dim \tau = \dim A^{-\frac{\alpha_1}{\alpha - 1}}.
\]

We now point out that the expectation value of the mean squared geodesic distance is evaluated by the standard continuum treatment
\[
\int dx \sqrt{g} \{ r(x, x_0) \}^2 \Psi(0; x, x) = -4\tau + O(\tau^2),
\]
which is now related with the quantum version of the mean squared geodesic distance in the small \( \tau \) region
\[
<r^2> \equiv \sum_x \{ r(x, x_0) \}^2 \Psi_A^{(G_0)}(T; x, x_0)
\sim \frac{< \int dx \sqrt{g} \int dx_0 \sqrt{g} \{ r(x, x_0) \}^2 \Psi(\tau; x, x) >_A}{< \int dx \sqrt{g} >_A}
\sim \tau \sim A^{-\frac{\alpha_1}{\alpha - 1}} \sim T^{-\frac{\alpha_1}{\alpha - 1}},
\]
where the geodesic distance on the dynamically triangulated lattice is same as that of the numerical simulation. The last three similarity relations are due to the dimensional arguments. We thus obtain an analytic prediction of the \( T \) dependence of the mean squared geodesic distance (H):
\[
<r^2> \sim T^{-\frac{\alpha_1}{\alpha - 1}} \equiv T^{\frac{2}{\gamma}}.
\]

From the dimensional arguments of Eq. (14), we obtain the following relation:
\[
dim[A] = dim[r^{-2\alpha_1/\alpha - 1}] \equiv dim[r^{\gamma(c)}].
\]
In the numerical simulation (C): \( V_t(r) \equiv A(r) \sim r^7 \), the fractal dimension \( \gamma \) for \( c = -2 \) is estimated from the measurement of the two-dimensional volume of space time. It is then natural to expect that
the dimension of $A(r)$ and that of the total area $A$ is same. We then obtain the analytic evaluation of the fractal dimension

$$
\gamma(c) = -2 \frac{\alpha_1}{\alpha_{-1}} = 2 \times \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}},
$$

(15)

where some of the typical values of the fractal dimension are: $\gamma(1) = 2(1 + \sqrt{2})$, $\gamma(0) = 4$, $\gamma(-2) = \frac{1}{2}(3 + \sqrt{17}) = 3.56 \cdots$, $\gamma(-\infty) = 2$. We show the $c$ dependence of the fractal dimension $\gamma(c)$ in Fig.7. As we can see from Fig.7, the fractal dimension varies smoothly from the classical value $\gamma(-\infty) = 2$ to $\gamma(1) = 2(1 + \sqrt{2})$ and gets imaginary for $c > 1$.

We now summarize the analytic predictions:

(G) $G(T)T \sim \text{const.}$,

(H) $< r^2 > = T^{\frac{2}{\gamma(-2)}} \simeq T^{-0.56}$,

(I) $\gamma(-2) = \frac{1}{2}(3 + \sqrt{17}) = 3.56 \cdots$.

First of all the analytic prediction of $\gamma(-2) = 3.56 \cdots$ should be compared with the numerical value of $\gamma \sim 3.5$. The theoretical value (0.561) has an excellent agreement with the experimental value. We show the numerical evaluations of (G): $G(T)T$ and the mean squared distance of random walks with gravitational quantum fluctuations (H): $< r^2 >$ in Fig.8 and Fig.9, respectively. As we can see, $G(T)T$ in Fig.8 show the clear constancy and thus excellent agreement with the prediction (G). In Fig.9 the numerical value of the power of $T$ is approaching to the theoretical value and slightly away in the large $T$ region where finite size effects may be important. If we consider the accuracy of the vertical measure we may conclude that the agreement with the theoretical prediction is excellent again.
6. Non-universal Nature of the Fractal Structure in the Continuum Limit

In our recent investigation\[11\] we have obtained analytic formulation to evaluate the fractal structure of two dimensional quantum gravity without matter \((c = 0)\). In this analysis we have used the results of matrix model and introduced a combinatorial consideration and then succeeded to derive a transfer matrix. The notion of the geodesic distance explained in this manuscript played an important role in the formulation.

We briefly summarize the main conclusion of this investigation.\[11\] Let us first define a continuum function \(\rho(L; D)\) which is a function of the boundary length \(L\) of a loop located at the continuum geodesic distance \(D\) measured from a marking point. \(\rho(L; D)dL\) counts the number of boundaries whose boundary lengths lie between \(L\) and \(L + dL\). It is evaluated by taking the continuum limit from the transfer matrix and disk amplitude of dynamical triangulation. The functional form of \(\rho(L; D)\) for \(c = 0\) model is given by

\[
\rho(L; D)D^2 = \frac{3}{7\sqrt{\pi}}(x^{-5/2} + \frac{1}{2}x^{-3/2} + \frac{14}{3}x^{1/2})e^{-x}, \tag{16}
\]

where \(x = L/D^2\) is a scaling parameter. Surprising fact is that the function \(\rho(L; D)D^2\) is a universal function with respect to the scaling parameter \(x\). This quantity \(\rho(L; D)D^2\) for \(c = 0\) model has recently been measured numerically and gets excellent agreement with the theoretical result (16)\[13\]. Fig.5 (a) and (b) are the lattice counterparts of the multiplicity function of \(c = -2\) model; (a) \(\rho(L, r = 18)\) and (b) \(\rho(L, r = 40)\), respectively, where continuum geodesic distance \(D\) is replaced by the lattice geodesic step \(r\). In Fig.5 only the small \(x\) region is shown.

In order to examine the scaling property of the fractal structure, it is conve-
nient to introduce the following quantities:

\[ < L^n > = \int_0^\infty dL \ L^n \rho(L; D). \]  \hspace{1cm} (17)

We can then derive the fractal scaling behaviors of \( c = 0 \) model on the following quantities

\[ < L^0 > \equiv N_b(D) \sim const \times D^3 a^{-3/2} + const \times Da^{-1/2} + const \times D^0, \]
\[ < L^1 > \equiv L_b(D) \sim const \times D^3 a^{-1/2} + const \times D^2, \]
\[ < L^n > \sim const \times D^{2n} \hspace{1cm} (n \geq 2), \]  \hspace{1cm} (18)

where \( a \) is the lattice constant and \( N_b(D) \) and \( L_b(D) \) are the same quantities as those defined in (A): \( N_b(r) \sim r^\alpha \) and (B): \( L_b(r) \sim r^\beta \) in the numerical simulations except that the argument is now the continuum geodesic distance \( D \).

As we can see from Eq.(18), \( N_b(D) \) and \( L_b(D) \) include the inverse power of the lattice constant dependent part as a dominant contribution. For this \( c = 0 \) model the fractal dimension \( \gamma = \beta + 1 = 4 \) obtained from the \( D \) dependence of \( L_b(D) \) coincides with that of \( \gamma(0) = 4 \) obtained from the formula (15). What is unexpected is that the coefficient of this power dependent term is lattice constant dependent and thus has non-universal nature. It should also be noted that the fractal dimension obtained from \( < L^2 > \sim D^4 \), which has the same dimension as \( A(D) \sim D^\gamma \), happens to reproduce the same fractal dimension \( \gamma = 4 \) discussed above. This fractal dimension derived from the \( < L^2 > \) does not have lattice constant dependence and thus should have a universal nature. It is a crucial question if \( \gamma(0) = 4 \) obtained from the Liouville theory is equivalent either with the one derived from \( < L^1 > = L_b(D) \) or with another one derived from \( < L^2 > \).

The analytic result of \( c = 0 \) model given in Eq.(18) suggests that \( N_b(D) \) and \( L_b(D) \) include non-universal part as a dominant contribution which become irrelevant in the continuum limit but show a fractal nature in the microscopic level.
Agishtein and Migdal have carried out the numerical simulation for \( c = 0 \) model\(^4\) and measured the quantity \( \gamma(r) = d \log A(r)/d \log r \) with the maximum number of triangles \( \simeq 1.3 \times 10^5 \). The observed fractal dimension at the maximum number of triangles is roughly 3 and still far below the analytically expected value 4. We believe that \( \gamma(r) \) will approach to the analytic value for much larger number of triangles.

7. Conclusion and Discussions

In this manuscript we have reported the following numerical results and analytic predictions and corresponding figures:

\((A)\) \( N_b(r) \equiv \langle \text{number of boundaries at the step} \ r \rangle \sim r^\alpha \) —— Fig.1,

\((B)\) \( L_b(r) \equiv \langle \text{total length of boundaries at the step} \ r \rangle \sim r^\beta \) —— Fig.2,

\((C)\) \( V_t(r) \equiv A(r) \equiv \langle \text{number of triangles within} \ r \ \text{steps} \rangle \sim r^\gamma \) —— Fig.3,

\((D)\) \( L_b(r)/N_b(r) \) —— Fig.4,

\((E)\) The number of boundaries \( P_r(l) \) with a given boundary length \( l \) as a function of the boundary length \( l \) for a given geodesic step \( r \) measured from a marking point —— Fig.5,

\((F)\) \( L_b(r) \sim r^\beta \), where \( r \) is defined on the dual lattice —— Fig.6,

\((G)\) \( G(T)T \sim \text{const.} \) —— Fig.8,

\((H)\) \( < r^2 > = T^{\frac{2}{3+\sqrt{17}}} \simeq T^{-0.56} \) —— Fig.9,

\((I)\) \( \gamma(-2) = \frac{1}{2}(3 + \sqrt{17}) = 3.56 \cdots \) —— Fig.3.

The numerical results \((A), (B)\) and \((C)\) show clear fractal structure of two-dimensional quantum gravitational space-time. In particular the space time is violently branching. The results \((D)\) and \((E)\) show that the perimeter length of the branches varies from small to large sizes but the average size is independent
of the geodesic distance measured from a marking point. This suggests an existence of some rule even for \( c = -2 \) model based on the fractal nature of the branching behavior. In fact for pure gravity \( (c = 0) \) we have analytically derived the \( c = 0 \) counterpart of the multiplicity function (E) of Fig.5. The result (F) in Fig.6 shows that the geodesic distance defined of the dual lattice enforces the very slow approach of the fractal dimension to the asymptotic value. The analytic results (G) and (H) have excellent agreement with the numerical results of corresponding figures Fig.8 and Fig.9, respectively. Thus the two-dimensional quantum gravity with dynamical triangulation can be treated by the diffusion equation of random walk with quantum gravitational fluctuations. The lattice version of the wave function and the continuum counterpart of diffusion equation are related by Eq.(7), which makes it possible to accommodate the Liouville theory. The fractal dimension (I): \( \gamma(-2) \) obtained from the formula (15) excellently agrees with the numerical results of \( c = -2 \) model obtained from Fig.2 and Fig.3 while the \( \gamma(0) \) obtained from (15) coincides with the analytic results derived from (18). Therefore we tend to believe that the formula (15) derived from the Liouville theory provides correct fractal dimensions of two-dimensional quantum gravity with matter fields.

As we have mentioned in the last section, \( N_b(D) \) and \( L_b(D) \) of \( c = 0 \) model include non-universal lattice constant dependent part as a dominant power contribution of geodesic distance which becomes irrelevant in the continuum limit. This may also be the case for \( c = -2 \) model. If \( N_b(D) \) and \( L_b(D) \) include non-universal part even for \( c = -2 \) model, we are puzzled why the fractal dimension \( \gamma \simeq \beta + 1 \simeq 3.5 \) obtained from the numerical results of \( L_b(r) \) and \( A(r) \) excellently agrees with the analytic result \( \gamma(-2) = 3.56 \cdots \) obtained from the formula (15) which correctly reproduces the \( c = 0 \) analytic result. This kind of question may be partially answered by numerical measurements for the models other than \( c = 0 \). Recently we have measured \( \rho(L; D) \) and \( \langle L^n \rangle \) for \( c = -2 \) model numerically and observed the similar scaling behavior as \( c = 0 \) model.\(^{[14]}\)
The problems are not yet completely settled at this moment since we have not yet obtained the analytic results of $< L^n >$ for the model other than $c = 0$. There are still several other issues to be cleared up: in particular the relation between the derivations of fractal dimensions by Liouville theory and the exact treatment.

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FIGURE CAPTIONS

Fig. 1a (A): $N_b(r) \sim r$ dependence, where $N_b(r)$ is the number of boundaries at the step $r$ for various number of triangles of sphere topology: (1) $8 \times 10^3$, (2) $4 \times 10^4$, (3) $2 \times 10^5$, (4) $10^6$, and (5) $5 \times 10^6$.

Fig. 1b (A): $\alpha(r) \equiv d\log N_b(r)/d\log r$ for various number of triangles, where $\alpha(r)$ is the fractal dimension parametrized in (A).

Fig. 2 (B): $\beta(r) \equiv d\log L_b(r)/d\log r$ for various sizes of triangles, where the fractal dimension $\beta(r)$ is parametrized in (B).

Fig. 3 (C): $\gamma(r) \equiv d\log V_t(r)/d\log r$ for various sizes of triangles, where $\gamma(r)$ is parametrized in (C).

Fig. 4 (D): $L_b(r)/N_b(r)$ for the case of $10^6$ triangles.

Fig. 5 (E): The number of boundaries $P(l)$ with a given boundary length $l$ as a function of the boundary length $l$ for the geodesic distance (a) $r = 18$ and (b) $r = e = 40$, where the number of triangles is $10^6$.

Fig. 6 (F): $\beta(r) \equiv d\log L_b(r)/d\log r$ for various sizes of triangles, where the geodesic distance here is defined on the dual lattice.

Fig. 7 The fractal dimension $\gamma(c)$ given by Eq.(15) as a function of the matter central charge $c$.

Fig. 8 (G): $G(T)T$ as a function of geodesic step $T$, where $G(T)$ is the comeback probability of random walk given by Eq.(7) for the case of $10^6$ triangles.

Fig. 9 (H): $\delta \equiv d\log <r^2>/d\log T$ as a function of geodesic step $T$, where $<r^2>$ is the mean squared geodesic distance. The theoretical value 0.561 is shown as a solid line in the figure.