ON RIESZ TYPE INEQUALITIES, HARDY-LITTLEWOOD TYPE THEOREMS AND SMOOTH MODULI

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ABSTRACT. The purpose of this paper is to develop some methods to study Riesz type inequalities, Hardy-Littlewood type theorems and smooth moduli of holomorphic, pluriharmonic and harmonic functions in high-dimensional cases. Initially, we prove some sharp Riesz type inequalities of pluriharmonic functions on bounded symmetric domains. The obtained results extend the main results in (Trans. Amer. Math. Soc. 372 (2019) 4031–4051). Furthermore, some Hardy-Littlewood type theorems of holomorphic and pluriharmonic functions on John domains are established. Additionally, we also discuss the Hardy-Littlewood type theorems and smooth moduli of holomorphic, pluriharmonic and harmonic functions. Consequently, we improve and generalize the corresponding results in (Acta Math. 178 (1997) 143–167) and (Adv. Math. 187 (2004) 146–172).

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1. INTRODUCTION

Denote by \( \mathbb{C}^n \) the complex space of dimension \( n \), where \( n \) is a positive integer. We can also interpret \( \mathbb{C}^n \) as the real \( 2n \)-space \( \mathbb{R}^{2n} \). For \( z = (z_1, \ldots, z_n) \), \( w = \ldots \)

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\((w_1, \ldots, w_n) \in \mathbb{C}^n\), the standard Hermitian scalar product on \(\mathbb{C}^n\) and the Euclidean norm of \(z\) are given by

\[
\langle z, w \rangle := \sum_{k=1}^{n} z_k \overline{w_k} \quad \text{and} \quad |z| := \langle z, z \rangle^{1/2},
\]

respectively. For \(a \in \mathbb{C}^n \ (a \in \mathbb{R}^m \ \text{resp.})\), we use \(B^n(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}\) \((B^m(a, r) = \{x \in \mathbb{R}^m : |x - a| < r\} \ \text{resp.})\) to denote the (open) ball of radius \(r > 0\) with center \(a\), where \(m \geq 2\) is a positive integer.

Let \(S^{m-1}(a, r) := \partial B^m(a, r)\) be the boundary of \(B^m(a, r)\). Then the boundary of \(\mathbb{B}^n(a, r)\) is \(\partial \mathbb{B}^n(a, r) = S^{2m-1}(a, r)\). In particular, let \(\mathbb{B}^n := \mathbb{B}^n(0, 1)\) \((\mathbb{B}^m := \mathbb{B}^m(0, 1) \ \text{resp.})\) and \(S^{m-1} := S^{m-1}(0, 1)\).

A two times continuously differentiable real-valued function \(u\) defined in an open subset \(\mathcal{M}\) of \(\mathbb{R}^m\) is said to be harmonic if \(\Delta u(x) = 0\) for all \(x = (x_1, \ldots, x_m) \in \mathcal{M}\), where

\[
\Delta := \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}
\]

is a Laplace operator (see [1]). In particular, a two times continuously differentiable complex-valued function \(f\) defined in an open subset \(\mathcal{D}\) of \(\mathbb{C}^n\) is said to be pluriharmonic if it satisfies the \(n^2\) partial differential equations

\[
\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} = 0
\]

for \(j, k \in \{1, \ldots, n\}\) (see [8, 35, 37, 41]). In [36], Rudin showed that a two times continuously differentiable complex-valued function \(f\) defined in \(\mathbb{B}^n\) is pluriharmonic if and only if \(f\) satisfies

\[
\Delta f = 0 \text{ and } \tilde{\Delta} f = 0,
\]

where

\[
\tilde{\Delta} = (1 - |z|^2) \left( \Delta f - 4 \sum_{j, k=1}^{n} z_j \overline{z}_k \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} \right)
\]

is the Laplace-Beltrami operator. In a simply connected domain \(\mathcal{D} \subset \mathbb{C}^n\), a pluriharmonic function \(f : \mathcal{D} \to \mathbb{C} := \mathbb{C}^1\) has a representation \(f = h + g\), where \(h\) and \(g\) are holomorphic in \(\mathcal{D}\) (see [3, 4, 8, 41]); this representation is unique up to an additive constant. Furthermore, from this representation, it is easy to know that pluriharmonic functions are a broader class of functions than holomorphic functions. In particular, if \(n = 1\), then the pluriharmonic function is equivalent to the harmonic function.

Throughout of this paper, we use the symbol \(C\) to denote the various positive constants, whose value may change from one occurrence to another.
2. Preliminaries and main results

2.1. The Riesz type inequalities and the Hardy-Littlewood type Theorems.

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain with origin and Bergman-Silov boundary $b$. Denote by $\Gamma$ the group of holomorphic automorphisms of $\Omega$, and denote by $\Gamma_0$ the isotropy group of $\Gamma$. It is well known that $\Omega$ is circular and star-shaped with respect to 0 and that $b$ is circular. The group $\Gamma_0$ is transitive on $b$ and $b$ has a unique normalized $\Gamma_0$-invariant measure $\sigma$ with $\sigma(b) = 1$ (see [2, 15, 18, 24, 29, 38, 42]). Obviously, the unit polydisk and the unit ball $B^n$ are bounded symmetric domains. For $p \in (0, \infty]$, the pluriharmonic Hardy space $PH_p(\Omega)$ consists of all those pluriharmonic functions $f : \Omega \to \mathbb{C}$ such that, for $p \in (0, \infty)$,

$$\|f\|_p := \sup_{r \in [0,1)} M_p(r, f) < \infty,$$

and, for $p = \infty$,

$$\|f\|_\infty := \sup_{r \in [0,1)} M_\infty(r, f) < \infty,$$

where

$$M_p(r, f) = \left( \int_b |f(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}}$$

and

$$M_\infty(r, f) = \sup_{\zeta \in b} |f(r\zeta)|.$$

Suppose that $\omega \in L^1_{loc}(\mathbb{R}^m)$ with $\omega > 0$ a.e.. For $p \in (0, \infty)$, we denote by $L^p(E, \omega)$ the space of all measurable functions $f$ with

$$\|f\|_{p,E,\omega} = \left( \int_E |f(x)|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} < \infty,$$

where $E \subset \mathbb{R}^m$ is a $m$-dimensional Lebesgue measurable set and $d\mu$ is the Lebesgue measure on $\mathbb{R}^m$. For the $m$-dimensional Lebesgue measurable set $E$ with

$$0 < \int_E \omega(x) d\mu(x) < \infty,$$

let

$$f_{E,\omega} = \frac{\int_E f(x) \omega(x) d\mu(x)}{\int_E \omega(x) d\mu(x)}$$

be the average, where $f$ is a measurable function in $E$. In particular, we write $\|f\|_{p,E} := \|f\|_{p,E,\omega}$ and $f_E := f_{E,\omega}$ when $\omega \equiv 1$. We say that $\omega$ is a Muckenhoupt weight, $\omega \in A^q_M(\Omega)$, where $q \in (1, \infty)$ and $M \in [1, \infty)$, if $\omega > 0$ a.e. on $\Omega$, and

$$\|\omega\|_{1,Q} \leq M|Q|^{\frac{1}{q}} \|\omega\|_{1/(1-q),Q}$$

for each cube or ball $Q$ contained in the domain $\Omega \subset \mathbb{R}^m$.

Let’s recall one of the celebrated results on holomorphic functions in $PH^p(D)$ by Riesz, where $D := \mathbb{D}$ and $p \in (1, \infty)$.

**Theorem A.** (M. Riesz) If a real-valued harmonic function $u \in PH^p(D)$ for some $p \in (1, \infty)$, then its harmonic conjugate $v$ with $v(0) = 0$ is also of class $PH^p(D)$. Furthermore, there is a constant $C$, depending only on $p$, such that
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\[ M_p(r, v) \leq CM_p(r, u) \]
for all \( r \in [0, 1) \).

By replacing \( \| \cdot \|_p \) with \( \| \cdot \|_{p,D} \) in Theorem A, Hardy and Littlewood [16] proved the following result for \( p \in (0, \infty) \).

**Theorem B.** For each \( p \in (0, \infty) \), there exists a constant \( C \), depending only on \( p \), such that

\[ \| u - u(0) \|_{p,D} \leq C \| v \|_{p,D} \]
for all real valued harmonic functions \( u \) and \( v \) in the unit disk \( D \) such that \( u + iv \) is holomorphic in \( D \), where \( d\mu \) is the Lebesgue area measure on \( D \).

In [33], Pichorides improved (2.1) into the following sharp form.

\[ \| v \|_p \leq \cot \frac{\pi}{2p^*} \| u \|_p, \]
where \( p^* = \max\{p, p/(p-1)\} \). Later, Verbitsky [40] further improved (2.2) and obtained the following sharp inequalities

\[ \frac{1}{\cos \frac{\pi}{2p^*}} \| v \|_p \leq \| f \|_p \leq \frac{1}{\sin \frac{\pi}{2p^*}} \| u \|_p, \]
where \( f = u + iv \) is holomorphic. As an analogy of Theorem A, Kalaj [21] established the following sharp Riesz type inequalities for complex-valued harmonic functions in \( \mathcal{H}^p(D) \) with \( p \in (1, \infty) \).

**Theorem C.** (Theorems 2.1 and 2.3) Let \( p \in (1, \infty) \) be a constant and \( f = h + g \in \mathcal{H}^p(D) \), where \( h \) and \( g \) are holomorphic functions in \( D \).

(\( \mathcal{A}_1 \)) If \( \text{Re}(g(0)h(0)) \geq 0 \), then,

\[ \left( \int_0^{2\pi} \left( |h(e^{i\theta})|^2 + |g(e^{i\theta})|^2 \right) \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} \leq \frac{1}{C_1(p)} \| f \|_p, \]
where \( C_1(p) = \sqrt{1 - \left| \cos \frac{\pi}{p} \right|} \) and “\( \text{Re} \)” denotes the real part of a complex number.

(\( \mathcal{A}_2 \)) If \( \text{Re}(g(0)h(0)) \leq 0 \), then,

\[ \| f \|_p \leq C_2(p) \left( \int_0^{2\pi} \left( |h(e^{i\theta})|^2 + |g(e^{i\theta})|^2 \right) \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}, \]
where \( C_2(p) = \sqrt{2} \max \left\{ \sin \frac{\pi}{2p}, \cos \frac{\pi}{2p} \right\} \).

Whether there is a Riesz type theorem in high-dimensional space has always been a challenging problem (see [12, p.167-172]). Let \( u = (u_0, u_1, \ldots, u_n) \) be a vector valued harmonic function in \((n + 1)\)-dimensional upper half space \( \mathbb{R}^{n+1}_+ \) satisfying the Cauchy-Riemann system.
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\[ \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k}, \quad 0 \leq k, j \leq n. \]

By analogy with (2.3), Fefferman and Stein showed that \( u \in L^p \) if and only if \( u_0 \in L^p \). They also used the non-tangential maximal function to establish a Riesz type Theorem in \( \mathbb{R}^{n+1}_+ \) (see [12, Theorem 9] and [13]). The first aim of this paper is to establish some Riesz type inequalities of pluriharmonic functions on bounded symmetric domains, which extend Theorem C.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded symmetric domain with origin and Bergman-Silov boundary \( b \), and let \( p \in [1, \infty] \) be a given constant. Let \( f = h + \overline{g} \), where \( h \) and \( g \) are holomorphic functions in \( \Omega \). Then the following statements hold.

\( (B_1) \) If \( p \in (1, \infty) \), then \( f = h + \overline{g} \in \mathcal{P}\mathcal{H}^p(\Omega) \) if and only if \( h, g \in \mathcal{P}\mathcal{H}^p(\Omega) \); Moreover,

\[ \lim_{r \to 1^-} \left( \int_b (|h(r\zeta)|^2 + |g(r\zeta)|^2)^{\frac{2}{p}} d\sigma(\zeta) \right)^{\frac{1}{2}} \leq \frac{1}{C_1(p)} \|f\|_p, \]

where \( C_1(p) \) is the same as in Theorem C.

\( (B_1 (I)) \) If \( f = h + \overline{g} \in \mathcal{P}\mathcal{H}^p(\Omega) \) and \( \text{Re}(g(0)h(0)) \geq 0 \), then,

\[ \|f\|_p \leq C_2(p) \lim_{r \to 1^-} \left( \int_b (|h(r\zeta)|^2 + |g(r\zeta)|^2)^{\frac{2}{p}} d\sigma(\zeta) \right)^{\frac{1}{2}}, \]

where \( C_2(p) \) is the same as in Theorem C.

\( (B_2) \) If \( p = 1 \), then \( f = h + \overline{g} \in \mathcal{P}\mathcal{H}^1(\Omega) \) implies that \( h, g \in \mathcal{P}\mathcal{H}^q(\Omega) \) for all \( q \in (0, 1) \). Furthermore, if \( f = h + \overline{g} \in \mathcal{P}\mathcal{H}^1(\Omega) \), then

\[ M_1(r, h) = O \left( \log \frac{1}{1 - r} \right) \quad \text{and} \quad M_1(r, g) = O \left( \log \frac{1}{1 - r} \right) \]

as \( r \to 1^- \). In particular, if \( \Omega \) is the unit polydisk, then the estimates of (2.4) are sharp.

\( (B_3) \) If \( p = \infty \), then there are two unbounded holomorphic functions \( h \) and \( g \) such that \( f = h + \overline{g} \in \mathcal{P}\mathcal{H}^\infty(\Omega) \).

As in [21 Corollaries 2.2 and 2.5], we obtain the following corollary from Theorem 2.1 which generalizes the Riesz type inequalities (2.1), (2.2) and (2.3) to real valued pluriharmonic functions on bounded symmetric domains in \( \mathbb{C}^n \).

**Corollary 2.2.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded symmetric domain with origin and Bergman-Silov boundary \( b \), and let \( 1 < p < \infty \). If \( u \) and \( v \) are real pluriharmonic functions on \( \Omega \) with \( v(0) = 0 \) and \( g = u + iv \) is a holomorphic function on \( \Omega \), then, we have

\[ ||v||_p \leq \cos \frac{\pi}{2p} ||g||_p \]
and

\[
\|g\|_p \leq \frac{1}{\sin \frac{\pi}{2p}} \|u\|_p.
\]

**Remark 2.3.** If \( n = 1 \) and \( \Omega \) is the unit disk in Corollary 2.2, then the inequalities (2.5) and (2.6) are sharp (see [21, 40] and [33, Theorem 3.7]).

Let \( b^p(\mathbb{B}^n) \) denote the Bergman class of pluriharmonic functions \( f \) defined on \( \mathbb{B}^n \), satisfying the condition

\[
\|f\|_{b^p(\mathbb{B}^n)} := \left( \int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \right)^{1/p} < \infty.
\]

As in [21 Corollary 2.9], we also obtain the following corollary from Theorem 2.1.

**Corollary 2.4.** Let \( 1 < p < \infty \). If \( h \) and \( g \) are holomorphic functions on \( \mathbb{B}^n \), \( f = h + \overline{g} \in b^p(\mathbb{B}^n) \) and \( \text{Re}(g(0)h(0)) = 0 \), then we have

\[
\left( \int_{\mathbb{B}^n} (|h(z)|^2 + |g(z)|^2)^{p/2} d\mu(z) \right)^{\frac{1}{p}} \leq \frac{1}{C_1(p)} \left( \int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}},
\]

and

\[
\left( \int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} \leq C_2(p) \left( \int_{\mathbb{B}^n} (|h(z)|^2 + |g(z)|^2)^{p/2} d\mu(z) \right)^{\frac{1}{p}},
\]

where \( C_1(p) \) and \( C_2(p) \) are the same as in Theorem C.

**Definition 2.5.** Let \( \Omega \subset \mathbb{C}^n \) be a proper subdomain. We call \( \Omega \) a \( \delta \)-John domain, \( \delta > 0 \), if there is a point \( x_0 \in \Omega \) which can be joined with any point \( x \in \Omega \) by a continuous curve \( \gamma \subset \Omega \) which satisfies

\[
\delta|\xi - x| \leq d_\Omega(\xi)
\]

for all \( \xi \in \gamma \), where \( d_\Omega(\xi) \) is the Euclidean distance between \( \xi \) and the boundary of \( \Omega \).

Definition 2.5 is equivalent to the usual definition of a John domain. The \( \delta \)-John domains are wide class of domains including quasidisks and bounded uniform domains (cf. [30]). Note that a \( \delta \)-John domain is bounded in \( \mathbb{C}^n \) (see [19, p.273-p.274]).

Iwaniec and Nolder [19] established estimates similar to Theorem B for the components of a quasiregular mapping in domains in \( \mathbb{R}^n \) which satisfy certain geometric conditions. On John domains, Nolder [30] obtained a generalization of Theorem B to solutions of certain elliptic equations in divergence form, whose gradients have comparable local \( L^\alpha \) norms.

By analogy with Theorem B, we get the following result for pluriharmonic functions on John domains.
Theorem 2.6. Let \( f = h + \overline{g} \) be a pluriharmonic function in a domain \( \Omega \subset \mathbb{C}^n \), where \( h \) and \( g \) are holomorphic in \( \Omega \) and \( \Omega \) is a \( \delta \)-John domain. Then, for \( p \in (0, \infty) \), \( q \in (1, \infty) \) and \( M \in [1, \infty) \), there is a positive constant \( C \) depending only on \( p, q, n, \delta \) and \( M \) such that

\[
\inf_{a \in \mathbb{C}} \| h - a \|_{p, \Omega, \omega} + \inf_{b \in \mathbb{C}} \| g - b \|_{p, \Omega, \omega} \leq C \| f \|_{p, \Omega, \omega},
\]

where \( \omega \in A^q_M(\mathbb{C}^n) \).

The following result easily follows from Theorem 2.6.

Corollary 2.7. Let \( f = h + \overline{g} \) be a pluriharmonic function in a domain \( \Omega \subset \mathbb{C}^n \), where \( h \) and \( g \) are holomorphic in \( \Omega \) and \( \Omega \) is a \( \delta \)-John domain. Then, for \( p \in (0, \infty) \), \( q \in (1, \infty) \) and \( M \in [1, \infty) \), \( \| f \|_{p, \Omega, \omega} < \infty \) if and only if \( \| h \|_{p, \Omega, \omega} + \| g \|_{p, \Omega, \omega} < \infty \), where \( \omega \in A^q_M(\mathbb{C}^n) \).

By taking \( g = -h \) in Theorem 2.6 we obtain the following generalization of Theorem B to higher dimensions.

Corollary 2.8. Let \( h = u + iv \) be a holomorphic function in a domain \( \Omega \subset \mathbb{C}^n \), where \( u \) and \( v \) are real valued pluriharmonic functions in \( \Omega \) and \( \Omega \) is a \( \delta \)-John domain. Then, for \( p \in (0, \infty) \), \( q \in (1, \infty) \) and \( M \in [1, \infty) \), there is a positive constant \( C \) depending only on \( p, q, n, \delta \) and \( M \) such that

\[
\inf_{a \in \mathbb{R}} \| u - a \|_{p, \Omega, \omega} \leq C \| v \|_{p, \Omega, \omega},
\]

where \( \omega \in A^q_M(\mathbb{C}^n) \).

In the case \( p \geq 1 \), we further obtain the following results.

Theorem 2.9. Let \( p \in [1, \infty) \), \( q \in (1, \infty) \), \( M \in [1, \infty) \), and \( \omega \in A^q_M(\mathbb{C}^n) \). Let \( f = h + \overline{g} \) be a pluriharmonic function in a domain \( \Omega \subset \mathbb{C}^n \), where \( h, g \in L^p(\Omega, \omega) \) are holomorphic in \( \Omega \) and \( \Omega \) is a \( \delta \)-John domain. Then, there is a positive constant \( C \) depending only on \( p, q, n, \delta \) and \( M \) such that

\[
\| h - h_{\Omega, \omega} \|_{p, \Omega, \omega} + \| g - g_{\Omega, \omega} \|_{p, \Omega, \omega} \leq C \| f \|_{p, \Omega, \omega}.
\]

By taking \( g = -h \) in Theorem 2.9 we obtain the following result.

Corollary 2.10. Let \( p \in [1, \infty) \), \( q \in (1, \infty) \), \( M \in [1, \infty) \), and \( \omega \in A^q_M(\mathbb{C}^n) \). Let \( h = u + iv \in L^p(\Omega, \omega) \) be a holomorphic function in a domain \( \Omega \subset \mathbb{C}^n \), where \( u \) and \( v \) are real valued pluriharmonic functions in \( \Omega \) and \( \Omega \) is a \( \delta \)-John domain. Then, there is a positive constant \( C \) depending only on \( p, q, n, \delta \) and \( M \) such that

\[
\| u - u_{\Omega, \omega} \|_{p, \Omega, \omega} \leq C \| v \|_{p, \Omega, \omega}.
\]

Remark 2.11. Iwaniec and Nolder [19, Definition 1] introduced domains with a chain condition \( \Omega \in \mathcal{F}(\sigma, N) \) with \( \sigma > 1, N \geq 1 \). These classes contain many important types of domains in the Euclidean space such as cubes, balls and John domains. Every \( \Omega \in \mathcal{F}(\sigma, N) \) is bounded. Theorems 2.6, 2.9 and the above corollaries can be generalized to domains \( \Omega \in \mathcal{F}(\sigma, N) \) with \( \sigma > 1, N \geq 1 \).
2.2. The Hardy-Littlewood type theorems and smooth moduli. A continuous increasing function \( \psi : [0, \infty) \to [0, \infty) \) with \( \psi(0) = 0 \) is called a majorant if \( \psi(t)/t \) is non-increasing for \( t > 0 \) (see [9, 10]). For \( \delta_0 > 0 \) and \( 0 < \delta < \delta_0 \), we consider the following conditions on a majorant \( \psi \):

\[
(2.7) \quad \int_0^\delta \frac{\psi(t)}{t} \, dt \leq C \psi(\delta)
\]

and

\[
(2.8) \quad \delta \int_\delta^\infty \frac{\psi(t)}{t^2} \, dt \leq C \psi(\delta),
\]

where \( C \) denotes a positive constant. A majorant \( \psi \) is henceforth called fast (resp. slow) if condition (2.7) (resp. 2.8) is fulfilled. In particular, a majorant \( \psi \) is said to be regular if it satisfies the conditions (2.7) and (2.8) (see [9, 10, 31]).

Given a majorant \( \psi \) and a proper subdomain \( \Omega^* \) of \( \mathbb{R}^m \) or \( \mathbb{C}^n \), a function \( f \) from \( \Omega^* \) into \( \mathbb{C} \) or \( \mathbb{R} := (-\infty, \infty) \) is said to belong to the Lipschitz space \( L_\psi(\Omega^*) \) if there is a positive constant \( C := C(f) \) such that

\[
(2.9) \quad |f(z) - f(w)| \leq C \psi(|z - w|), \quad z, w \in \Omega^*.
\]

Note that if \( f \in L_\psi(\Omega^*) \), then \( f \) is continuous on \( \overline{\Omega^*} \) and (2.9) holds for \( z, w \in \overline{\Omega^*} \) (see [10]). In particular, we say that a function \( f \) belongs to the local Lipschitz space \( L_\psi(\Omega^*) \) if (2.9) holds, with a fixed positive constant \( C \), whenever \( z \in \Omega^* \) and \( |z - w| < \frac{1}{2} d_{\Omega^*}(z) \) (cf. [10, 14, 26]). Moreover, \( \Omega^* \) is called an \( L_\psi \)-extension domain if \( L_\psi(\Omega^*) = \text{loc}\, L_\psi(\Omega^*) \). On the geometric characterization of \( L_\psi \)-extension domains, see [14]. In [26], Lappalainen generalized the characterization of [14], and proved that \( \Omega^* \) is a \( L_\psi \)-extension domain if and only if each pair of points \( z, w \in \Omega^* \) can be joined by a rectifiable curve \( \gamma \subset \Omega^* \) satisfying

\[
(2.10) \quad \int_\gamma \frac{\psi(d_{\Omega^*}(\zeta))}{d_{\Omega^*}(\zeta)} \, ds(\zeta) \leq C \psi(|z - w|)
\]

with some fixed positive constant \( C = C(\Omega^*, \psi) \), where \( ds \) stands for the arc length measure on \( \gamma \). Furthermore, Lappalainen [26, Theorem 4.12] proved that \( L_\psi \)-extension domains exist only for fast majorants \( \psi \). In particular, \( \mathbb{R}^n \) is an \( L_\psi \)-extension domain of \( \mathbb{C}^n \) for fast majorant \( \psi \) (see [10]).

Given two sets \( E_1 \) and \( E_2 \) in \( \mathbb{R}^m \) or \( \mathbb{C}^n \), we write \( L_\psi(E_1, E_2) \) for the class of those continuous functions \( f \) in \( E_1 \cup E_2 \) which satisfy (2.9), with some positive constant \( C = C(f) \), whenever \( z \in E_1 \) and \( w \in E_2 \) (see [10]).

In [9], Dyakonov gave some characterizations of the holomorphic functions of the class \( L_\psi(\mathbb{D}) \) in terms of their moduli. Let’s recall one of the important findings in [9] as follows (see also [31, Theorem A]).

**Theorem D.** Let \( \psi \) be a regular majorant and \( f \) be a holomorphic function in \( \mathbb{D} \). Then \( f \in L_\psi(\mathbb{D}) \) if and only if \( |f| \in L_\psi(\mathbb{D}) \).
Later, Pavlović [31] came up with a relatively simple proof of Theorem D. By using Pavlović’s method, in conjunction with other techniques, Dyakonov [10] generalized Theorem D into the following form.

**Theorem E.** ([10] Theorem 1) Let $\psi$ be a fast majorant, and let $\mathcal{E}$ be an $\mathcal{L}_\psi$-extension domain of $\mathbb{C}^n$. Suppose that $f$ is a holomorphic function in $\mathcal{E}$. Then $f \in \mathcal{L}_\psi(\mathcal{E}) \iff |f| \in \mathcal{L}_\psi(\mathcal{E} \cup \partial \mathcal{E})$.

If $f = u + iv$ is a holomorphic function in $\mathbb{D}$, and if $u = \text{Re}(f) \in \mathcal{L}_\psi(\mathbb{D})$, then so does $v$ (and hence $f$), with the same $\alpha \in (0, 1)$, where $\psi_\alpha(t) = t^\alpha$ ($t \geq 0$). This fact, proved in [17], is known as the Hardy-Littlewood theorem on conjugate functions. We remark, however, that in the case $\alpha \in (0, 1)$ an earlier-and essentially equivalent-result of Privalov [34] tells us that harmonic conjugation preserves the Lipschitz class $\mathcal{L}_\psi(\mathbb{T})$ of the circle $\mathbb{T} : = \partial \mathbb{D} = \mathbb{S}^0$. In [10], Dyakonov extended the Hardy-Littlewood theorem to the high-dimensional case as follows (see [10], Ineq. (2.3)).

**Theorem F.** Let $\psi$ be a fast majorant, and let $\mathcal{E}$ be an $\mathcal{L}_\psi$-extension domain of $\mathbb{C}^n$. If $u + iv$ is a bounded holomorphic function in $\mathcal{E}$, then $u \in \mathcal{L}_\psi(\mathcal{E}) \Rightarrow v \in \mathcal{L}_\psi(\mathcal{E})$.

Hardy and Littlewood [17] proved the following theorem for holomorphic functions with respect to the majorant $\psi_\alpha(t) = t^\alpha$ ($t \geq 0$) as follows, where $\alpha \in (0, 1]$. For relevant research on harmonic functions, see [25, 28, 32].

**Theorem G.** ([17] Theorem 40) Let $f$ be a holomorphic function in $\mathbb{D}$ and continuous up to $\mathbb{D} : = \mathbb{D} \cup \mathbb{T}$. Let $\psi_\alpha(t) = t^\alpha$ ($t \geq 0$). Suppose that there exists $\alpha \in (0, 1]$ and a positive constant $C$ such that

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq C\psi_\alpha(|\theta_1 - \theta_2|)$$

for $0 \leq \theta_1, \theta_2 < 2\pi$.

Then

$$|f'(z)| \leq C\frac{\psi_\alpha(1)}{\psi_\alpha(d) \left(\frac{d}{d}(z)\right)}$$

for $z \in \mathbb{D}$.

In fact, there is an essential connection among Theorems E, F and G. By using new techniques, in conjunction with some methods of Dyakonov and Pavlović, we remove “the boundedness condition” in Theorem F and establish extensions of Theorems E, F and G and also a connection among Theorems E, F and G as follows.

**Theorem 2.12.** Let $\psi$ be a fast majorant, and let $\mathcal{E}$ be an $\mathcal{L}_\psi$-extension domain of $\mathbb{C}^n$. If $f = u + iv$ is holomorphic in $\mathcal{E}$, then the following statements are equivalent, where $u$ and $v$ are real-valued pluriharmonic functions in $\mathcal{E}$.

$(C_1) \quad u \in \mathcal{L}_\psi(\mathcal{E})$;

$(C_2) \quad$ There is a positive constant $C$ such that

$$|\nabla u(z)| \leq C\frac{\psi_\alpha(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)} \quad z \in \mathcal{E},$$

where $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$ and

$$\nabla u(z) = (u_{x_1}, u_{y_1}, \ldots, u_{x_n}, u_{y_n}).$$
(C₃) \(|u| \in \mathcal{L}_\psi(\mathcal{E})\);

(C₄) If \(n \geq 2\), then, for a given \(p \geq \frac{2n-2}{2n-1}\), there is a positive constant \(C\) such that

\[
I_{u,p}(z) \leq C \frac{\psi(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)}, \quad z \in \mathcal{E},
\]

where \(I_{u,p}(z) := \|B^n(z,d_{\mathcal{E}}(z)/2)^{-1/p}\|\psi\|_{p,B^n(z,d_{\mathcal{E}}(z)/2)}\). Moreover, if \(n = 1\), then, for a given \(p > 0\), there is a positive constant \(C\) such that \((2.11)\) holds;

(C₅) \(v \in \mathcal{L}_\psi(\mathcal{E})\);

(C₆) There is a positive constant \(C\) such that

\[
|\nabla v(z)| \leq C \frac{\psi(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)}, \quad z \in \mathcal{E};
\]

(C₇) \(|v| \in \mathcal{L}_\psi(\mathcal{E})\);

(C₈) If \(n \geq 2\), then, for a given \(p \geq \frac{2n-2}{2n-1}\), there is a positive constant \(C\) such that

\[
I_{v,p}(z) \leq C \frac{\psi(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)}, \quad z \in \mathcal{E}.
\]

Moreover, if \(n = 1\), then, for a given \(p > 0\), there is a positive constant \(C\) such that \((2.12)\) holds;

(C₉) \(f \in \mathcal{L}_\psi(\mathcal{E})\);

(C₁₀) \(|f| \in \mathcal{L}_\psi(\mathcal{E})\);

(C₁₁) \(|f| \in \mathcal{L}_\psi(\mathcal{E}, \partial \mathcal{E})\).

Let \(f\) be a holomorphic function or a quasiconformal mapping. It follows from \([9, 10, 11]\) that the implication

\[
|f| \in \mathcal{L}_\psi(\Omega_1) \Rightarrow f \in \mathcal{L}_\psi(\Omega_1)
\]

depends heavily on the geometry of the proper subdomain \(\Omega_1\) of \(\mathbb{C}^n\) or \(\mathbb{R}^m\). One might still look for a universal theorem, relating the properties of \(|f|\) to those of \(f\), that should work for all domains \(\Omega_1\) of \(\mathbb{C}^n\) and all holomorphic functions \(f\) in \(\Omega_1\). In order to produce such a statement, as in Dyakonov \([10]\), we modify the Lipschitz condition by using a suitable “internal distance” in \(\Omega_1\), where \(\Omega_1\) a proper subdomain of \(\mathbb{C}^n\) or \(\mathbb{R}^m\). For \(z_1, z_2 \in \Omega_1\), let

\[
d_{\psi,\Omega_1}(z_1, z_2) := \inf_{\gamma} \int_{\gamma} \frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)} ds(z),
\]

where \(\psi\) is a majorant and the infimum is taken over all rectifiable curves \(\gamma \subset \Omega_1\) joining \(z_1\) to \(z_2\). Further, we write \(F \in \mathcal{L}_\psi, \text{int}(\Omega_1)\) to mean that there is a positive constant \(C = C(F)\) such that

\[
|F(z_1) - F(z_2)| \leq C d_{\psi,\Omega_1}(z_1, z_2)
\]

whenever \(z_1, z_2 \in \Omega_1\), where \(F\) is a function of \(\Omega_1\) into \(\mathbb{C}\) or \(\mathbb{R}\).

In \([10\) Theorem 3\], by replacing the Euclidean distance with the internal distance, Dyakonov obtained the following result.
Theorem H. ([10, Theorem 3]) Let \( \psi \) be a fast majorant, and let \( f \) be a holomorphic function in a domain \( \Omega_1 \subset \mathbb{C}^n \). Then \( f \in \mathcal{L}_{\psi, \text{int}}(\Omega_1) \iff |f| \in \mathcal{L}_{\psi, \text{int}}(\Omega_1) \).

Remark 2.13. In fact, in view of Proposition 2.14, we should assume that \( \Omega_1 \) is bounded in Theorem H.

Proposition 2.14. There exists a fast majorant \( \psi_0 \) such that there does not exist a constant \( C > 0 \) such that
\[
\int_0^\delta \frac{\psi_0(t)}{t} \, dt = \frac{1}{e} \delta = \psi_0(\delta), \quad 0 < \delta < e.
\]

So, \( \psi_0 \) is a fast majorant. On the other hand, for \( r > e \), we have
\[
\int_0^r \frac{\psi_0(t)}{t} \, dt = 1 + \int_e^r \frac{\ln t}{t} \, dt = \frac{1}{2} + \frac{1}{2} (\ln r)^2 > \frac{1}{2} (\ln r) \psi_0(r).
\]

The proof of this proposition is finished. \( \square \)

By replacing the Euclidean distance with the internal distance, we establish extensions of Theorems F, G and H and also a connection among Theorems F, G and H as follows.

Theorem 2.15. Let \( \psi \) be a fast majorant, and let \( \Omega_1 \) be a bounded domain of \( \mathbb{C}^n \). If \( u \) and \( v \) are real-valued pluriharmonic functions in \( \Omega_1 \) with \( f = u + iv \) holomorphic on \( \Omega_1 \), then the following statements are equivalent.

\( (D_1) \) \( u \in \mathcal{L}_{\psi, \text{int}}(\Omega_1) \);
\( (D_2) \) There is a positive constant \( C \) such that
\[
|\nabla u(z)| \leq C \frac{\psi(d\Omega_1(z))}{d\Omega_1(z)}, \quad z \in \Omega_1;
\]
\( (D_3) \) \( |u| \in \mathcal{L}_{\psi, \text{int}}(\Omega_1) \);
\( (D_4) \) If \( n \geq 2 \), then, for a given \( p \geq \frac{2n-2}{2n-1} \), there is a positive constant \( C \) such that
\[
(2.13) \quad I_{u,p}(z) \leq C \frac{\psi(d\Omega_1(z))}{d\Omega_1(z)}, \quad z \in \Omega_1.
\]
Moreover, if \( n = 1 \), then, for a given \( p > 0 \), there is a positive constant \( C \) such that \( (2.13) \) holds;
\( (D_5) \) \( v \in \mathcal{L}_{\psi, \text{int}}(\Omega_1) \);
\( (D_6) \) There is a positive constant \( C \) such that
\[
|\nabla v(z)| \leq C \frac{\psi(d\Omega_1(z))}{d\Omega_1(z)}, \quad z \in \Omega_1;
\]
Let \( \Omega \) be an \( \mathcal{L}_\psi \)-extension domain of \( \mathbb{R}^n \), where \( n \geq 2 \). Let \( u \) be a real-valued harmonic function in \( \mathbb{R}^n \) and continuous on \( \overline{\mathbb{B}^n} \), where \( n \geq 2 \). Then, for \( \alpha \in (0, 1) \), the following conditions are equivalent:

1. \( |u(x) - u(y)| \leq C|x - y|^{\alpha} \), \( x, y \in \mathbb{B}^n \);
2. \( |u(y) - |u(\zeta)|| \leq C|y - \zeta|^{\alpha} \), \( y, \zeta \in S^{n-1} \);
3. \( |u(\zeta) - |u(r\zeta)|| \leq C|1 - r|^{\alpha} \), \( \zeta \in S^{n-1}, r \in (0, 1) \).

By using different proof methods from that in [32], we extend Theorems D, E, G, H and I to real-valued harmonic functions on an \( \mathcal{L}_\psi \)-extension domain of \( \mathbb{R}^n \) as follows, where \( n \geq 2 \).

**Theorem 2.16.** Let \( \psi \) be a fast majorant, and let \( \Omega_2 \) be an \( \mathcal{L}_\psi \)-extension domain of \( \mathbb{R}^n \), where \( n \geq 2 \). If \( u \) is a real-valued harmonic function in \( \Omega_2 \), then the following statements are equivalent.

1. \( u \in \mathcal{L}_\psi(\Omega_2) \);
2. There is a positive constant \( C \) such that \( |\nabla u(x)| \leq C \frac{\psi(d_{\Omega_2}(x))}{d_{\Omega_2}(x)}, \) \( x \in \Omega_2 \);
3. \( |u| \in \mathcal{L}_\psi(\Omega_2) \);
4. \( |u| \in \mathcal{L}_\psi(\Omega_2, \partial \Omega_2) \);
5. If \( n > 2 \), then, for a given \( p \geq \frac{n-2}{n} \), there is a positive constant \( C \) such that \( I_{u,p}(x) \leq C \frac{\psi(d_{\Omega_2}(x))}{d_{\Omega_2}(x)}, \) \( x \in \Omega_2 \)

where \( I_{u,p}(x) := |\mathbb{B}^n(x, d_{\Omega_2}(x)/2)|^{-1/p} \|
abla u\|_{L^p(\mathbb{B}^n(x, d_{\Omega_2}(x)/2))}. \) Moreover, if \( n = 2 \), then, for a given \( p > 0 \), there is a positive constant \( C \) such that \( I_{u,p}(x) \leq C \frac{\psi(d_{\Omega_2}(x))}{d_{\Omega_2}(x)} \) holds.

If the Euclidean distance is replaced by the internal distance in Theorem 2.16, then we obtain the following result. Here we omit the proof because it suffices to use (4.27), (4.28) and Theorems N, O instead of (4.2), (4.4) and Theorem M, respectively, and use arguments similar to those in the proof of Theorem 2.15.
Theorem 2.17. Let \( \psi \) be a fast majorant, and let \( \Omega_3 \) be a bounded domain of \( \mathbb{R}^n \), where \( n \geq 2 \). If \( u \) is a real-valued harmonic function in \( \Omega_3 \), then the following statements are equivalent.

\( (\mathcal{G}_1) \) \( u \in \mathcal{L}_\psi, \text{int}(\Omega_3) \);

\( (\mathcal{G}_2) \) There is a positive constant \( C \) such that

\[
|\nabla u(x)| \leq C \frac{\psi(d\Omega_3(x))}{d\Omega_3(x)}, \quad x \in \Omega_3;
\]

\( (\mathcal{G}_3) \) \( |u| \in \mathcal{L}_\psi, \text{int}(\Omega_3) \);

\( (\mathcal{G}_4) \) If \( n > 2 \), then, for a given \( p \geq \frac{n-2}{n-1} \), there is a positive constant \( C \) such that

\[
(2.16) \quad \tilde{I}_{u,p}(x) \leq C \frac{\psi(d\Omega_3(x))}{d\Omega_3(x)}, \quad x \in \Omega_3.
\]

Moreover, if \( n = 2 \), then, for a given \( p > 0 \), there is a positive constant \( C \) such that \( (2.16) \) holds.

The proofs of Theorems 2.1, 2.6 and 2.9 will be presented in section 3, and the proof of Theorems 2.12, 2.15 and 2.16 will be given in section 4.

3. The Riesz type inequalities and the Hardy-Littlewood type theorems

The following results will play an important role in the proof of Theorem 2.1

Lemma J. (cf. [6, Lemma 5]) Suppose that \( x, y \in [0, \infty) \) and \( \tau \in (0, \infty) \). Then

\[
(x + y)^\tau \leq 2^{\max\{\tau-1,0\}} (x^\tau + y^\tau).
\]

Theorem K. ([16, Theorem 7]) Suppose that \( f = u + iv \) is a holomorphic function in \( \mathbb{D} \) with \( v(0) = 0 \). If \( u \in \mathcal{P}\mathcal{H}^p(\mathbb{D}) \) for some \( 0 < p \leq 1 \), then \( v \) satisfies

\[
(3.1) \quad M_p(r, v) \leq C\|u\|_p + C\|u\|_p \left( \log \frac{1}{1-r} \right)^{\frac{1}{p}},
\]

where \( C \) is a positive constant depending only on \( p \).

Theorem L. ([29, Theorem 5]) Suppose that \( \Omega \subset \mathbb{C}^n \) is a bounded symmetric domain with origin and Bergman-Silov boundary \( b \). Let \( f = u + iv \) be a holomorphic function in \( \Omega \). If \( u \in \mathcal{P}\mathcal{H}^1(\Omega) \), then \( v \in \mathcal{P}\mathcal{H}^q(\Omega) \) for all \( q \in (0,1) \).

3.1. The proof of Theorem 2.1. We first prove \( (\mathcal{R}_1) \). Let \( f = h + \overline{g} = u + iv \in \mathcal{P}\mathcal{H}^p(\Omega) \), where \( h = u_1 + iv_1 \) and \( g = u_2 + iv_2 \). Then \( u, v \in \mathcal{P}\mathcal{H}^p(\Omega) \). Set \( F = h + g \) and \( v^* = \text{Im}(F) \), where “\( \text{Im} \)” denotes the imaginary part of a complex number. It is not difficult to know that \( \text{Re}(F) = \text{Re}(f) = u \in \mathcal{P}\mathcal{H}^p(\Omega) \) and by using Fubini’s theorem, we obtain

\[
M_p^p(r, u) = \int_b |u(r\zeta)|^p d\sigma(\zeta) = \int_b M_{p,\zeta}^p(r, u)d\sigma(\zeta),
\]
where

\begin{equation}
M_{p,\zeta}^p (r, u) := \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta.
\end{equation}

Since $|u|^p$ is plurisubharmonic on $\Omega$ (see [41]), $|U(z)|^p := |u(z\zeta)|^p$ is subharmonic with respect to $z \in \mathbb{D}$. Thus, by (3.2), we see that $M_{p,\zeta}^p (r, u)$ is increasing on $r \in [0, 1)$ for all $\zeta \in \partial$, which, together with Lebesgue’s monotone convergence Theorem, yields that

\begin{equation}
\|u\|_p = \left( \int_{\partial} \lim_{r \to 1^-} M_{p,\zeta}^p (r, u) d\sigma (\zeta) \right)^{\frac{1}{p}} < \infty.
\end{equation}

Without loss of generality, we assume that $v_1 (0) = v_2 (0) = 0$. Since $V(z) := v^* (z\zeta)$ is the harmonic conjugate of $U(z)$ on $z \in \mathbb{D}$, by (2.3), we see that

\begin{equation}
M_{p,\zeta}^p (r, v^*) \leq \cot \frac{\pi}{2p^*} M_{p,\zeta}^p (r, u)
\end{equation}

for all $r \in (0, 1)$ and $\zeta \in \partial$. By raising both sides of (3.4) to the $p$-th power and integrating over $\partial$, we have

\begin{equation}
\|v^*\|_p = \sup_{r \in (0,1)} M_p (r, v^*) \leq \cot \frac{\pi}{2p^*} \sup_{r \in (0,1)} M_p (r, u) = \cot \frac{\pi}{2p^*} \|u\|_p < \infty.
\end{equation}

It follows from Lemma J that

\[ |v_2|^p = |\text{Im}(g)|^p = \frac{|v^* - v|^p}{2^p} \leq \frac{|v^*|^p + |v|^p}{2}, \]

which, together with (3.5), yields that

\[ \|v_2\|_p = \sup_{r \in (0,1)} M_p (r, v_2) \leq 2 - \frac{1}{p} (\|v^*\|_p + \|v\|_p) < \infty. \]

For $\xi \in \Omega$, let

\[ G(\xi) = -ig(\xi) + iu_2(0) = v_2(\xi) + i(u_2(0) - u_2(\xi)). \]

By using similar reasoning as in the proof of (3.5), we have

\[ \|\text{Im}(G)\|_p \leq \cot \frac{\pi}{2p^*} \|v_2\|_p < \infty, \]

which implies that $g \in \mathcal{P}\mathcal{H}^p (\Omega)$. Another desired conclusion $h \in \mathcal{P}\mathcal{H}^p (\Omega)$ follows from the following inequality

\[ |h|^p = |f - g|^p \leq 2^{p-1} (|f|^p + |g|^p). \]

Now we prove ($\mathcal{A}_1 (1)$). Since

\[ |\mathcal{Q}(z)|^p := (|h(z\zeta)|^2 + |g(z\zeta)|^2)^{\frac{p}{2}}, \]
is subharmonic with respect to $z \in \mathbb{D}$ (see e.g. [21] Lemma 2.13), we see that $M^p_{\gamma,\zeta}(r, (|h|^2 + |g|^2)^{\frac{1}{2}})$ is increasing on $r \in [0, 1]$ for all $\zeta \in b$, which, together with (B1), Lemma J and Lebesgue’s monotone convergence Theorem, yields that

\[ (3.6) \sup_{r \in [0, 1]} \left( \int_{b} M^p_{\gamma,\zeta}(r, Q) d\sigma(\zeta) \right)^{\frac{1}{p}} = \left( \int_{b} \lim_{r \to 1^-} M^p_{\gamma,\zeta}(r, Q) d\sigma(\zeta) \right)^{\frac{1}{p}} < \infty. \]

By Theorem C (A1), we have

\[ (3.7) \quad M^p_{\gamma,\zeta}(r, Q) \leq \frac{1}{C_1(p)} M^p_{\gamma,\zeta}(r, f) \]

for all $r \in (0, 1)$ and $\zeta \in b$. Raise both sides of (3.7) to the $p$-th power and integrate over $b$, which, together with (3.6), gives that

\[ \lim_{r \to 1^-} \left( \int_{b} (|h(r\zeta)|^2 + |g(r\zeta)|^2)^{\frac{2}{p}} d\sigma(\zeta) \right)^{\frac{1}{p}} \leq \frac{1}{C_1(p)} \|f\|_p. \]

The conclusion of (B1(II)) follows from the similar reasoning as in the proof of (B1(II)).

Next, we prove (B2). Since $f = h + \overline{g} = u + iv \in \mathcal{P} \mathcal{H}^1(\Omega)$, we see that $u, v \in \mathcal{P} \mathcal{H}^1(\Omega)$. By Theorem L, we know that $v^* = \text{Im}(F) = \text{Im}(h + g) \in \mathcal{P} \mathcal{H}^2(\Omega)$ for all $q \in (0, 1)$. Consequently,

\[ |v_2|^q = |\text{Im}(g)|^q = \frac{|v^* - v|^q}{2^q} \leq \frac{|v^*|^q + |v|^q}{2^q}, \]

which, together with

\[ \sup_{r \in [0, 1]} \int_b |v(r\zeta)|^q d\sigma(\zeta) \leq \sup_{r \in [0, 1]} \left( \left( \int_b |v(r\zeta)| d\sigma(\zeta) \right)^q \left( \int_b d\sigma(\zeta) \right)^{1-q} \right) < \infty, \]

gives that $v_2 \in \mathcal{P} \mathcal{H}^q(\Omega)$ for all $q \in (0, 1)$. Let $F_1 = h - g$ and $u^* = \text{Re}(F_1)$. Then

\[ \text{Re}(-iF_1) = \text{Im}(F_1) = \text{Im}(f) \in \mathcal{P} \mathcal{H}^1(\Omega), \]

which, together with Theorem L, implies that

\[ -u^* = \text{Im}(-iF_1) \in \mathcal{P} \mathcal{H}^q(\Omega) \]

for all $q \in (0, 1)$. Hence

\[ |u_2|^q = |\text{Re}(g)|^q = \frac{|u - u^*|^q}{2^q} \leq \frac{|u|^q + |u^*|^q}{2^q}, \]

which, together with

\[ \sup_{r \in [0, 1]} \int_b |u(r\zeta)|^q d\sigma(\zeta) \leq \sup_{r \in [0, 1]} \left( \left( \int_b |u(r\zeta)| d\sigma(\zeta) \right)^q \left( \int_b d\sigma(\zeta) \right)^{1-q} \right) < \infty, \]

yields that $u_2 \in \mathcal{P} \mathcal{H}^q(\Omega)$ for all $q \in (0, 1)$. Combining the results $u_2, v_2 \in \mathcal{P} \mathcal{H}^q(\Omega)$ for all $q \in (0, 1)$ and Lemma J gives that $g = u_2 + iv_2 \in \mathcal{P} \mathcal{H}^q(\Omega)$ for
all \( q \in (0, 1) \). Thus, by Lemma J, we see that \( h = f - \overline{\eta} \) also belongs to \( \mathcal{P}\mathcal{H}^q(\Omega) \) for all \( q \in (0, 1) \).

Now we prove the second part of (\( \mathcal{P}_2 \)). Using the above notations, we may assume that \( u_1(0) = u_2(0) = v_1(0) = v_2(0) = 0 \). Since \( M_{1,\zeta}(r, u) \) is increasing on \( r \in [0, 1) \) for all \( \zeta \in b \), by (3.3), we see that \( M_{1,\zeta}(1, u) < \infty \) for almost all \( \zeta \in b \). Hence it follows from (3.1) that there is an absolute constant \( C > 0 \) such that, for all \( r \in [0, 1) \) and almost all \( \zeta \in b \),

\[
(3.8) \quad M_{1,\zeta}(r, v^*) \leq CM_{1,\zeta}(1, u) \left(1 + \log \frac{1}{1 - r}\right).
\]

By integrating both sides of (3.8) over \( b \), we obtain

\[
(3.9) \quad M_1(r, v^*) \leq C\|u\|_1 \left(1 + \log \frac{1}{1 - r}\right) \leq C\|f\|_1 \left(1 + \log \frac{1}{1 - r}\right).
\]

Since

\[
|v_2| = |\text{Im}(g)| = \frac{|v^* - v|}{2} \leq \frac{|v^*| + |v|}{2},
\]

by (3.9), we see that

\[
(3.10) \quad M_1(r, v_2) \leq \frac{1}{2} \left(\int_b |v^*(r\zeta)|d\sigma(\zeta) + \int_b |v(r\zeta)|d\sigma(\zeta)\right)
\]

\[
\leq \frac{1}{2} \left(M_1(r, v^*) + \|f\|_1\right)
\]

\[
\leq \frac{\|f\|_1}{2} \left(1 + C + C\log \frac{1}{1 - r}\right).
\]

Let \( F_1^* = -i(h - g) \) and \( \tilde{u} = \text{Im}(F_1^*) \). Then

\[
\text{Re}(F_1^*) = \text{Im}(f) = v \in \mathcal{P}\mathcal{H}^1(\Omega),
\]

which, together with the similar reasoning as in the proof of \( M_1(r, v^*) \), implies that there is an absolute constant \( C > 0 \) such that

\[
(3.11) \quad M_1(r, \tilde{u}) \leq C\|f\|_1 \left(1 + \log \frac{1}{1 - r}\right).
\]

Elementary calculations lead to

\[
|u_2| = |\text{Re}(g)| = \frac{|u + \tilde{u}|}{2} \leq \frac{|u| + |\tilde{u}|}{2},
\]
which, together with (3.11), gives that
\[
M_1(r, u_2) \leq \frac{1}{2} \left( \frac{1}{2} \right) \left( \int_b |\tilde{u}(r\zeta)|d\sigma(\zeta) + \int_b |u(r\zeta)|d\sigma(\zeta) \right) \\
\leq 2^{-1}M_1(r, \tilde{u}) + 2^{-1}\|f\|_1 \\
\leq 2^{-1}C\|f\|_1 \left( 1 + \log \frac{1}{1-r} \right) + 2^{-1}\|f\|_1.
\]
From (3.10) and (3.12), we conclude that there is an absolute constant $C > 0$ such that
\[
M_1(r, g) \leq M_1(r, u_2) + M_1(r, v_2) \leq C\|f\|_1 \left( 1 + \log \frac{1}{1-r} \right) + \|f\|_1.
\]
On the other hand, by $h = f - g$ and (3.13), we see that there is an absolute constant $C > 0$ such that
\[
M_1(r, h) \leq M_1(r, f) + M_1(r, g) \leq 2\|f\|_1 + C\|f\|_1 \left( 1 + \log \frac{1}{1-r} \right).
\]
Next, we prove the sharpness part of (2.4). Let $\Omega$ be the unit polydisk. Consider the functions
\[
h(z) = g(z) = \frac{1}{1-z_{j_0}^n}, \quad z \in \Omega,
\]
where $j_0 \in \{1, \ldots, n\}$. It is not difficult to know that $f = h + \overline{g} = 2\text{Re}(h) \in P\mathcal{H}^1(\Omega)$ (see also [7, Chapter 3] or [16]). However, by [5, Theorem E], we have
\[
M_1(r, h) = \left( \frac{1}{2\pi} \right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} |h(re^{i\theta_1}, \ldots, re^{i\theta_n})|d\theta_1 \cdots d\theta_n \\
= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta_{j_0}}{|1-re^{i\theta_{j_0}}|} = \sum_{k=1}^{\infty} \left( \frac{\Gamma\left(n + \frac{n}{2}\right)}{n!\Gamma\left(\frac{n}{2}\right)} \right)^2 r^{2n} \\
\sim \log \frac{1}{1-r}.
\]
At last, we prove (B3). Suppose that $z^0 \in \partial \Omega$ be fixed such that
\[
h_0(z^0, z^0) = \max_{z \in \Omega} h_0(z, z),
\]
where $h_0$ denotes the Bergman metric of $\Omega$ at $0$. Then, for $z \in \Omega$, let
\[
f(z) = h(z) + g(z),
\]
where
\[
h(z) = g(z) = i\log \frac{h_0(z^0, z^0) + h_0(z, z^0)}{h_0(z^0, z^0) - h_0(z, z^0)}.
\]
By [27, Theorem 6.5], $z^0 \in b$. Also, it is not difficult to know that $h, g \notin P\mathcal{H}^\infty(\Omega)$ and $f \in P\mathcal{H}^\infty(\Omega)$. The proof of this theorem is finished. \qed
3.2. The proof of Theorem 2.6  

If \( \|f\|_{p,\Omega,\omega} = \infty \), then it is obvious. Without loss of generality, we assume that \( \|f\|_{p,\Omega,\omega} < \infty \). Let \( f = h + \overline{g} = u + iv \), where \( h = u_1 + iv_1 \) and \( g = u_2 + iv_2 \). Set \( F = h + g \). Then \( \text{Re}(F) = \text{Re}(f) \).

Since \( \|f\|_{p,\Omega,\omega} < \infty \), we see that both \( \|u\|_{p,\Omega,\omega} \) and \( \|v\|_{p,\Omega,\omega} \) are finite. This gives \( \|\text{Re}(F)\|_{p,\Omega,\omega} = \|u\|_{p,\Omega,\omega} \leq \|f\|_{p,\Omega,\omega} < \infty \). We have

\[
\inf_{a \in \mathbb{R}} \|\text{Re}(F) - a\|_{p,\Omega,\omega} \leq \|\text{Re}(F)\|_{p,\Omega,\omega} \leq \|f\|_{p,\Omega,\omega} < \infty.
\]

It follows from the Cauchy Riemann equations and [30, Theorem 3.1] that there is a positive constant \( C \) depending only on \( p, q, n, \delta \) and \( M \) such that

\[
\inf_{a \in \mathbb{R}} \|\text{Im}(F) - a\|_{p,\Omega,\omega} \leq C \inf_{a \in \mathbb{R}} \|\text{Re}(F) - a\|_{p,\Omega,\omega} \leq C \|f\|_{p,\Omega,\omega} < \infty.
\]

Without loss of generality, let \( \|\text{Im}(F) - a_0\|_{p,\Omega,\omega} = \inf_{a \in \mathbb{R}} \|\text{Im}(F) - a\|_{p,\Omega,\omega} \).

Since \( \text{Im}(g) = v_2 = \frac{\text{Im}(F) - v}{2} \),

we see that

\[
\left\| \text{Im}(g) - \frac{a_0}{2} \right\|_{p,\Omega,\omega} = \frac{1}{2} \left( \int_{\Omega} |\text{Im}(F(z)) - a_0 - v(z)|^p \omega(z) d\mu(z) \right)^{\frac{1}{p}} \leq \frac{1}{2} C_0 (\|\text{Im}(F) - a_0\|_{p,\Omega,\omega} + \|v\|_{p,\Omega,\omega}) \leq \frac{1}{2} C_0 (C + 1) \|f\|_{p,\Omega,\omega},
\]

where \( C_0 = 2^{\max\{1 - \frac{1}{p}, \frac{1}{p} - 1\}} \). Using the similar method, we obtain that there exists a constant \( a_1 \in \mathbb{R} \) such that

\[
\|\text{Re}(g) - a_1\|_{p,\Omega,\omega} \leq C \left| \|\text{Im}(g) - \frac{a_0}{2}\|_{p,\Omega,\omega} \leq \frac{1}{2} C_0 C(C + 1) \|f\|_{p,\Omega,\omega}.
\]

Consequently, by Lemma J, we have

\[
(3.14) \quad \left\| g - a_1 - \frac{a_0}{2} \right\|_{p,\Omega,\omega} \leq C_0 \left( \|\text{Re}(g) - a_1\|_{p,\Omega,\omega} \right. \\
+ \left. \|\text{Im}(g) - \frac{a_0}{2}\|_{p,\Omega,\omega} \right) \leq \frac{1}{2} C_0^2 (C + 1)^2 \|f\|_{p,\Omega,\omega}.
\]

Since \( h = f - \overline{g} \), by (3.14) and Lemma J, we see that
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\[ \left\| h + a_1 - \frac{a_0}{2} \right\|_{p,\Omega,\omega} \leq C_0 \left( \| f \|_{p,\Omega,\omega} + \| g - a_1 - \frac{a_0}{2} \|_{p,\Omega,\omega} \right) \]
\[ \leq C \| f \|_{p,\Omega,\omega}. \]

Combining (3.14) and (3.15) gives the desired result. \( \square \)

3.3. The proof of Theorem 2.9. We use the following lemma to prove Theorem 2.9.

Lemma 3.1. Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) and let \( \omega \in L^1_{\text{loc}}(\mathbb{C}^n) \) with \( \omega > 0 \) a.e.. Assume that \( p \in [1, \infty) \). Then for complex valued functions \( f \in L^p(\Omega,\omega) \), we have
\[ |f_{\Omega,\omega}| < \infty \]
and
\[ \| f - f_{\Omega,\omega} \|_{p,\Omega,\omega} \leq 2^{2-\frac{1}{p}} \inf_{a \in \mathbb{C}} \| f - a \|_{p,\Omega,\omega}. \]

Also, for real valued functions \( u \in L^p(\Omega,\omega) \), we have
\[ |u_{\Omega,\omega}| < \infty \]
and
\[ \| u - u_{\Omega,\omega} \|_{p,\Omega,\omega} \leq 2^{2-\frac{1}{p}} \inf_{a \in \mathbb{R}} \| u - a \|_{p,\Omega,\omega}. \]

Proof. We give a proof for \( |f_{\Omega,\omega}| < \infty \) and that for (3.16). The proof for the second part is similar. Since \( f \in L^p(\Omega,\omega) \), \( p \in [1, \infty) \), \( \Omega \) is bounded and \( \omega \in L^1_{\text{loc}}(\mathbb{C}^n) \) with \( \omega > 0 \) a.e., we obtain \( |f_{\Omega,\omega}| < \infty \).

Without loss of generality, let
\[ \| f - a_0 \|_{p,\Omega,\omega} = \inf_{a \in \mathbb{C}} \| f - a \|_{p,\Omega,\omega}. \]

Then, by Lemma 1, we have
\[ \| f - f_{\Omega,\omega} \|_{p,\Omega,\omega} = \| (f - a_0) + (a_0 - f_{\Omega,\omega}) \|_{p,\Omega,\omega} \]
\[ \leq 2^{1-\frac{1}{p}} (\| f - a_0 \|_{p,\Omega,\omega} + \| a_0 - f_{\Omega,\omega} \|_{p,\Omega,\omega}). \]

Since
\[ \| a_0 - f_{\Omega,\omega} \|_{p,\Omega,\omega} = |a_0 - f_{\Omega,\omega}| \left( \int_{\Omega} \omega(z) d\mu(z) \right)^{\frac{1}{p}} \]
\[ \leq \left( \int_{\Omega} \omega(z) d\mu(z) \right)^{\frac{1}{p} - 1} \]
\[ \leq \| a_0 - f \|_{p,\Omega,\omega}, \]
where the last inequality in the case \( p > 1 \) follows from the H"older inequality. Thus, we have
\[ \| f - f_{\Omega,\omega} \|_{p,\Omega,\omega} \leq 2^{2-\frac{1}{p}} \| f - a_0 \|_{p,\Omega,\omega}. \]

This completes the proof. \( \square \)

Theorem 2.9 follows from Theorem 2.6 and Lemma 3.1. \( \square \)
4. THE HARDY-LITTLEWOOD TYPE THEOREMS AND SMOOTH MODULI

Let’s begin this section with the following result.

**Theorem M.** ([20 Theorem 2.3]) Let \( u \) be a pluriharmonic function of \( \mathbb{B}^n \) into \((-1, 1)\). Then, for \( z \in \mathbb{B}^n \), the following sharp inequality holds:

\[
|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - u^2(z)}{1 - |z|^2}.
\]

4.1. The proof of Theorem 2.12

We first prove \((C_1) \iff (C_2)\). Let’s begin to prove \((C_1) \Rightarrow (C_2)\). For any fixed \( z = (z_1, \ldots, z_n) \in \mathcal{E} \), let \( r = 2d_{\mathcal{E}}(z)/3 \). By the assumption, for \( w \in \mathbb{B}^n(z, r) \), we see that there is a positive constant \( C \) such that

\[
|u(z) - u(w)| \leq C\psi(|z - w|) \leq C\psi(r).
\]

For \( w = (w_1, \ldots, w_n) \in \mathbb{B}^n(z, r) \), it follows from the invariant Poisson integral formula for real valued pluriharmonic functions that

\[
u(w) = \int_{\partial \mathbb{B}^n} P_r(w - z, \xi)u(z + r\xi)d\sigma(\xi),
\]

where \( P_r(w - z, \xi) = \frac{(r^2 - |w - z|^2)^n}{|r - (w - z, \xi)|^n} \). For \( j \in \{1, \ldots, n\} \), elementary calculations lead to

\[
\left| \frac{\partial}{\partial w_j} P_r(w - z, \xi) \right| \leq n \frac{|w_j - z_j| (r^2 - |w - z|^2)^{n-1}}{|r - (w - z, \xi)|^{2n}}
\]

\[
+ \frac{n|w_j - z_j| (r^2 - |w - z|^2)^{n-1}}{|r - (w - z, \xi)|^{2n+2}}
\]

\[
\leq n \frac{|w_j - z_j| (r^2 - |w - z|^2)^{n-1}}{|r - (w - z, \xi)|^{2n}}
\]

\[
+ \frac{n (r^2 - |w - z|^2)^n |r - (w - z, \xi)|}{|r - (w - z, \xi)|^{2n+2}}.
\]

It follows from (4.3) that, for \( w \in \mathbb{B}^n(z, 3r/4) \) and \( j \in \{1, \ldots, n\} \),

\[
\left| \frac{\partial}{\partial w_j} P_r(w - z, \xi) \right| \leq n \left( \frac{3}{4}r^{2n-1} (r^2 - |w - z|)^{2n} + \frac{7}{4}r^{2n+1} (r - |w - z|)^{2n+2} \right)
\]

\[
\leq \frac{115 \cdot 4^{2n-1}n}{r}.
\]
Combining (4.1) and (4.4) gives
\[
|u_w(w)| \leq \left( \frac{\sum_{j=1}^{n} \left( \int_{\partial B_n} \frac{\partial}{\partial w_j} P_{r}(w - z, \xi) \left| u(z + r\xi) - u(z) \right| d\sigma(\xi) \right)^2}{2} \right)^{\frac{1}{2}}
\]
\[
\leq \frac{115 \cdot 4^{2n-1} n^{\frac{n}{r}}}{r} \int_{\partial B_n} \left| u(z + r\xi) - u(z) \right| d\sigma(\xi)
\]
\[
\leq 115 \cdot 4^{2n-1} n^{\frac{n}{r}} C^\frac{n}{r}
\]
and
\[
|u_{\bar{w}}(w)| = |u_w(w)| \leq 115 \cdot 4^{2n-1} n^{\frac{n}{r}} C^\frac{n}{r},
\]
where \( w \in B^n(z, 3r/4) \), \( u_w = (u_{w_1}, \ldots, u_{w_n}) \) and \( u_{\bar{w}} = (u_{\bar{w}_1}, \ldots, u_{\bar{w}_n}) \). Taking \( w = z \) in (4.5), we obtain
\[
|\nabla u(z)| = |u_w(z)| + |u_{\bar{w}}(z)| \leq C_1 \psi(\frac{r}{r}) = \frac{3C_1}{2} \psi(\frac{2d_\mathcal{E}(z)}{3})
\]
\[
\leq \frac{3C_1}{2} \psi(d_\mathcal{E}(z)),
\]
where \( C_1 = 230 \cdot 4^{2n-1} n^{\frac{n}{r}} C \).

Now, we prove (\( E_2 \))\( \Rightarrow \) (\( E_1 \)). Since \( \mathcal{E} \) is an \( \mathcal{L}_\psi \)-extension domain, we see that for all \( z_1, z_2 \in \Omega \), by using (2.10), there is a positive constant \( C \) which does not depend on \( z_1, z_2 \) and a rectifiable curve \( \gamma \subset \Omega \) joining \( z_1 \) to \( z_2 \) such that
\[
|u(z_1) - u(z_2)| \leq \int_{\gamma} (|u_w(\zeta)| + |u_{\bar{w}}(\zeta)|) d\sigma(\zeta)
\]
\[
\leq C_1 \int_{\gamma} \frac{\psi(d_\mathcal{E}(\zeta))}{d_\mathcal{E}(\zeta)} d\sigma(\zeta)
\]
\[
\leq C \psi(|z_1 - z_2|).
\]

We come to prove (\( E_1 \))\( \Leftrightarrow \) (\( E_3 \)). Since (\( E_1 \))\( \Rightarrow \) (\( E_3 \)) is obvious, we only need to prove (\( E_3 \))\( \Rightarrow \) (\( E_1 \)). Without loss of generality, we assume that \( \sup_{z \in \mathcal{E}} |u(z)| > 0 \). Let \( z \in \mathcal{E} \) be fixed. Without loss of generality, let \( \zeta_1 \in \partial \mathcal{E} \) such that \( d_\mathcal{E}(z) = |z - \zeta_1| \). Since \( \psi(t)/t \) is non-increasing for \( t > 0 \), we see that
\[
\psi(2t) = \frac{\psi(2t)}{t} \leq \frac{\psi(t)}{t}.
\]
It follows from (4.7) that, for all \( w \in B^n(z, d_\mathcal{E}(z)) \), there is a positive constant \( C \) such that
\[
|u(w)| - |u(z)| \leq |u(z)| - |u(\zeta_1)| + |u(\zeta_1)| - |u(w)| \leq C \psi(2d_\mathcal{E}(z)) + C \psi(d_\mathcal{E}(z)) \leq 3C \psi(d_\mathcal{E}(z)).
\]
Consequently,

\begin{equation}
M_z - |u(z)| = \sup_{w \in \mathbb{B}^n(z,d_{\mathcal{E}}(z))} (|u(w)| - |u(z)|) \leq 3C\psi(d_{\mathcal{E}}(z)),
\end{equation}

where \( M_z := \sup\{|u(\xi)| : |\xi - z| < d_{\mathcal{E}}(z)\} \). By (4.8) and the identity Theorem, we have \( 0 < M_z < \infty \). Let

\begin{equation}
\mathcal{U}(w) = \frac{u(z + d_{\mathcal{E}}(z)w)}{M_z}, \quad w \in \mathbb{B}^n.
\end{equation}

By Theorem M, we have

\begin{equation}
d_{\mathcal{E}}(z)|\nabla u(z)| = |\nabla \mathcal{U}(0)| \leq \frac{4}{\pi} (1 - |\mathcal{U}(0)|^2) \leq \frac{8}{\pi} (1 - |\mathcal{U}(0)|) = \frac{8}{\pi} \left(1 - \frac{|u(z)|}{M_z}\right).
\end{equation}

Consequently, we have

\begin{equation}
d_{\mathcal{E}}(z)|\nabla u(z)| \leq \frac{8}{\pi}(M_z - |u(z)|),
\end{equation}

which, together with (4.8), yields that

\begin{equation}
|\nabla u(z)| \leq \frac{24C}{\pi} \psi(d_{\mathcal{E}}(z)).
\end{equation}

Since \((\mathcal{C}_2)\) implies \((\mathcal{C}_1)\), by (4.9), we see that \( u \in \mathcal{L}_\psi(\mathcal{E}) \).

Next, we prove \((\mathcal{C}_1) \iff (\mathcal{C}_4)\). We first show \((\mathcal{C}_1) \implies (\mathcal{C}_4)\). Let \( z \in \mathcal{E} \) be a fixed point. For \( w \in \mathbb{B}^n(z,d_{\mathcal{E}}(z)/2) \), it follows from (4.5) that

\begin{equation}
|\nabla u(w)| = |u_w(w)| \leq \frac{3C_1}{2} \frac{\psi(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)},
\end{equation}

where \( C_1 \) is the same as in (4.6). Raise both sides of (4.10) to the \( p \)-th power and integrate over \( \mathbb{B}^n(z,d_{\mathcal{E}}(z)/2) \). This gives

\begin{equation}
I_{u,p}(z) = \frac{3C_1}{2} \frac{\psi(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)}.
\end{equation}

Now we begin to prove \((\mathcal{C}_4) \implies (\mathcal{C}_1)\). Since \( p \geq \frac{2n-2}{2n-1} \) for \( n \geq 2 \) and \( p > 0 \) for \( n = 1 \), for any given point \( z \in \mathcal{E} \), it follows from Theorem A that \( |\nabla u|^p \) is subharmonic in \( \mathbb{B}^n(z,d_{\mathcal{E}}(z)/2) \). Then

\begin{equation}
|\nabla u(z)|^p \leq \frac{1}{|\mathbb{B}^n(z,d_{\mathcal{E}}(z)/2)|} \int_{\mathbb{B}^n(z,d_{\mathcal{E}}(z)/2)} |\nabla u(w)|^p d\mu(w),
\end{equation}

which implies that there is a positive constant \( C \) such that

\begin{equation}
|\nabla u(z)| \leq I_{u,p}(z) \leq C \frac{\psi(d_{\mathcal{E}}(z))}{d_{\mathcal{E}}(z)}.
\end{equation}

Since \((\mathcal{C}_2)\) implies \((\mathcal{C}_1)\), by (4.11), we see that \((\mathcal{C}_4)\) implies \((\mathcal{C}_1)\).
By using similar reasoning as in the proof of “($E_1$)⇔($E_2$)⇔($E_3$)⇔($E_4$)”, we conclude that ($E_6$)⇔($E_7$)⇔($E_8$). Since $|\nabla u| = |\nabla v|$, we see that ($E_2$) is equivalent to ($E_6$). Therefore,

(4.12) \( (E_1) \iff (E_2) \iff (E_3) \iff (E_4) \iff (E_5) \iff (E_6) \iff (E_7) \iff (E_8). \)

At last, we show that

\[
(\xi_1) \iff (\xi_6).
\]

Since \( (\xi_6) \iff (\xi_1) \) is obvious, we only need to prove \( (\xi_1) \iff (\xi_9) \). Let \( u \in L_\psi(E) \). Then \( v \in L_\psi(E) \), which implies that there is a positive constant \( C \) such that

\[
|f(z_1) - f(z_2)| \leq |u(z_1) - u(z_2)| + |v(z_1) - v(z_2)| \\
\leq C \psi(|z_1 - z_2|) + C \psi(|z_1 - z_2|) \\
= 2C \psi(|z_1 - z_2|)
\]

for \( z_1, z_2 \in E \). Consequently, \( (\xi_9) \) holds. Combining (4.12), (4.13) and Theorem E gives

\[
(\xi_1) \iff (\xi_2) \iff (\xi_3) \iff (\xi_4) \iff (\xi_5) \iff (\xi_6) \iff (\xi_7) \iff (\xi_8) \iff (\xi_9) \\
\iff (\xi_{10}) \iff (\xi_{11}).
\]

The proof of this theorem is finished. \( \square \)

4.2. The proof of Theorem 2.15

Let’s prove \( \mathcal{P}_1 \iff \mathcal{P}_2 \). We first prove \( \mathcal{P}_1 \iff \mathcal{P}_2 \). For any fixed \( z = (z_1, \ldots, z_n) \in \Omega_1 \), let \( r = 2d_\Omega(z)/3 \). It follows from (4.2) and (4.4) that, for \( w \in B^n(z, 3r/4) \),

\[
|u_w(w)| \leq \frac{C(n)}{r} \int_{\partial B^n}|(u(z + r\xi) - u(z))|d\sigma(\xi),
\]

where \( C(n) = 115 \cdot 4^{n-1}n\sqrt{n} \). Next, we estimate \( |u(w_z) - u(z)| \) for \( \xi \in \partial B^n \), where \( w_z = z + r\xi \). By the assumption, we see that there is a positive constant \( C \) such that

\[
|u(w_z) - u(z)| \leq C d_\psi,\Omega_1(w_z, z) \leq C \int_{[z, w_z]} \psi(d_\Omega^1(\zeta)) \frac{d\Omega_1(\zeta)}{d_\Omega^1(\zeta)} ds(\zeta),
\]

where \( [z, w_z] \) is the straight segment with endpoints \( z \) and \( w_z \). Since \( [z, w_z] \subset B^n(z, r) \subset \Omega_1 \) and \( \psi(t)/t \) is non-increasing in \((0, \infty)\), we see that

\[
\frac{\psi(d_\Omega^1(\zeta))}{d_\Omega^1(\zeta)} \leq \frac{\psi(d_{B^n}(z, r)(\zeta))}{d_{B^n}(z, r)(\zeta)}
\]

for \( \zeta \in [z, w_z] \). Note that

\[
d_{B^n}(z, r)(\zeta) = r - |\zeta - z|.
\]

Combining (2.7), (4.15), (4.16) and (4.17) gives that there is a positive constant \( C \) such that
where the last inequality follows from the assumptions that $\psi$ is a fast majorant and $\Omega_1$ is bounded. Since $\psi$ is a continuous increasing function, by (4.14) and (4.18), we conclude that

\[
(4.19)
\left| u_w(w) - u(z) \right| \leq \frac{C \cdot C(n)}{r} \int_{\partial B^0} \psi(r) d\sigma = C \cdot C(n) \frac{\psi(r)}{r}
\]

Taking $w = z$ in (4.19), we obtain

\[
|\nabla u(z)| = |u_w(z)| + |u_{\Gamma}(z)| = 2|u_w(z)| \leq 3C \cdot C(n) \frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)}.
\]

Now, we prove (P2) $\Rightarrow$ (P1). By the assumption, we see that, for $z_1, z_2 \in \Omega_1$,

\[
|u(z_1) - u(z_2)| \leq \inf_{\gamma} \int_{\gamma} |\nabla u(\zeta)| d\sigma(\zeta) \leq C \cdot \inf_{\gamma} \int \frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)} d\sigma(z)
\]

where the infimum is taken over all rectifiable curves $\gamma \subset \Omega_1$ joining $z_1$ to $z_2$. Consequently, $u \in \mathcal{L}_{\psi, \text{int} \Omega_1}$.

Next, we prove (P1) $\Rightarrow$ (P3). Since (P1) $\Rightarrow$ (P3) is obvious, we only need to prove (P3) $\Rightarrow$ (P1). Without loss of generality, we assume that $\sup_{z \in \Omega_1} |u(z)| > 0$. Let $z \in \Omega_1$ be fixed. For $w \in B^n(z, d_{\Omega_1}(z))$, there is a positive constant $C$ such that

\[
(4.20)
|u(w)| - |u(z)| \leq C \cdot d_{\psi, \Omega_1}(w, z) \leq C \int_{[w, z]} \frac{\psi(d_{\Omega_1}(\zeta))}{d_{\Omega_1}(\zeta)} d\sigma(\zeta),
\]

where $[w, z]$ denotes the straight segment with endpoints $w$ and $z$. We observe that if $\zeta \in [w, z]$, then one has

\[
[w, z] \subset B^n(z, d_{\Omega_1}(z)) \subset \Omega_1
\]

and therefore

\[
d_{\Omega_1}(\zeta) \geq d_{B^n(z, d_{\Omega_1}(z))}(\zeta).
\]

This gives
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\[(4.21)\]

\[
\psi(d_{\Omega_1}(\zeta)) \leq \frac{\psi(d_{\mathbb{B}^n}(z,d_{\Omega_1}(z)))(\zeta)}{d_{\mathbb{B}^n}(z,d_{\Omega_1}(z))}(\zeta).
\]

Since \(\psi\) is fast and \(\Omega_1\) is bounded, by (4.20) and (4.21), we see that there is a positive constant \(C\) such that

\[
|u(w)| - |u(z)| \leq C \int_{|w,z|} \psi(d_{\mathbb{B}^n}(z,d_{\Omega_1}(z)))(\zeta) d\zeta.
\]

For this, we obtain that

\[
M_z - |u(z)| \leq C\psi(d_{\Omega_1}(z)),
\]

where \(M_z := \sup\{|u(\xi)| : |\xi - z| < d_{\Omega_1}(z)\}\). By (4.22) and the identity principle, we have \(0 < M_z < \infty\). Let

\[
U(\eta) = \frac{u(z + d_{\Omega_1}(z)\eta)}{M_z}, \quad \eta \in \mathbb{B}^n,
\]

From the proof of Theorem 2.12 we have

\[
(4.23) \quad d_{\Omega_1}(z)|\nabla u(z)| \leq \frac{8}{\pi}(M_z - |u(z)|).
\]

Combining (4.22) and (4.23) yields that there is a positive constant \(C\) such that

\[
(4.24) \quad |\nabla u(z)| \leq C\frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)}.
\]

Since \((\mathcal{D}_2)\) implies \((\mathcal{D}_1)\), by (4.24), we see that \((\mathcal{D}_3)\) implies \((\mathcal{D}_1)\).

We prove \((\mathcal{D}_1) \Leftrightarrow (\mathcal{D}_4)\). We first show \((\mathcal{D}_1) \Rightarrow (\mathcal{D}_4)\). Let \(z \in \Omega_1\) be a fixed point. For \(w \in \mathbb{B}^n(z,d_{\Omega_1}(z)/2)\), it follows from (4.19) that

\[
(4.25) \quad |\nabla u(w)| = |u_w(w)| + |u_{\overline{w}}(w)| \leq \frac{3C_1}{2} \frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)},
\]

where \(C_1 = C \cdot C(n)\) is the same as in (4.19). Raise both sides of (4.25) to the \(p\)-th power and integrate over \(\mathbb{B}^n(z,d_{\Omega_1}(z)/2)\). This gives

\[
I_{u,p}(z) \leq \frac{3C_1}{2} \frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)}.
\]

Now we begin to prove \((\mathcal{D}_4) \Rightarrow (\mathcal{D}_1)\). For any given point \(z \in \Omega_1\), it follows from [39, Theorem A] that \(|\nabla u|^p\) is subharmonic in \(\mathbb{B}^n(z,d_{\Omega_1}(z)/2)\). Then
\[ |\nabla u(z)|^p \leq \frac{1}{|B^n(z,d_{\Omega_1}(z)/2)|} \int_{B^n(z,d_{\Omega_1}(z)/2)} |\nabla u(w)|^p d\mu(w), \]

which implies that there is a positive constant \( C \) such that

\[ |\nabla u(z)| \leq I_{u,p}(z) \leq C \frac{\psi(d_{\Omega_1}(z))}{d_{\Omega_1}(z)}. \]

Since \((\mathcal{R}_2)\) implies \((\mathcal{R}_1)\), by \((4.26)\), we see that \((\mathcal{R}_4)\) implies \((\mathcal{R}_1)\).

Since the remaining proof of this theorem is similar to the proof of Theorem 2.12, we omit it here. The proof of this theorem is finished. \(\square\)

**Theorem N.** ([22, Theorem 1.8]) Let \( u : B^2 \to (-1, 1) \) be a harmonic function. Then

\[ |\nabla u(x)| \leq \frac{4}{\pi} \frac{1 - |u(x)|^2}{1 - |x|^2}, \quad x \in B^2. \]

In addition, this inequality is sharp.

**Theorem O.** ([23, Theorem 2.5]) Let \( n \geq 3 \) and \( u : B^n \to (-1, 1) \) be a harmonic function. Then

\[ |\nabla u(x)| \leq \frac{n}{2} \frac{1 - |u(x)|^2}{1 - |x|^2}, \quad x \in B^n. \]

In addition, this inequality is strict for \( n \geq 4 \).

### 4.3. The proof of Theorem 2.16

We first prove \((\mathcal{F}_1)\Rightarrow(\mathcal{F}_2)\). For any fixed \( x \in B^n \), let \( r = 2d_{\Omega_2}(x)/3 \). For \( y \in B^n(x,r) \), it follows from the Poisson integral formula that

\[ u(y) = \int_{S^{n-1}} P_r(y - x, \zeta) u(r\zeta + x) d\sigma(\zeta), \]

where \( P_r(y - x, \zeta) = r^{n-2} (r^2 - |y - x|^2) / |y - x - r\zeta|^n \) is the Poisson kernel for \( B^n(x,r) \). By subtracting \( u(x) \) from both sides and differentiating them with respect to \( y \in B^n(x,r) \), we have

\[ \nabla u(y) = \int_{S^{n-1}} \nabla P_r(y - x, \zeta) (u(r\zeta + x) - u(x)) d\sigma(\zeta). \]

If \( y \in B^n(x,3r/4) \), then there is a positive constant \( C \) depending only on \( n \) such that

\[ |\nabla P_r(y - x, \zeta)| \leq \frac{C}{r} \text{ for } \zeta \in S^{n-1}. \]

From the assumption, we see that there is a positive constant \( C \) such that

\[ |u(r\zeta + x) - u(x)| \leq C\psi(r). \]

Combining \((4.27)\), \((4.28)\) and \((4.29)\) yields that there is a positive constant \( C \) such that
Taking $y = x$ in (4.30), we have
\[ |\nabla u(x)| \leq \frac{3C}{2} \psi(d_{\Omega_2}(x)). \]

The proof of $(\mathcal{F}_2) \Rightarrow (\mathcal{F}_1)$ is similar to the proof of Theorem 2.12. Also, if we replace Theorem M by Theorems N and O in the proof of Theorem 2.12, then we can prove $(\mathcal{F}_1) \Leftrightarrow (\mathcal{F}_3)$. The proof of $(\mathcal{F}_1) \Leftrightarrow (\mathcal{F}_4)$ is similar to the proof $(\mathcal{F}_1) \Leftrightarrow (\mathcal{F}_3)$.

At last, we show $(\mathcal{F}_1) \Leftrightarrow (\mathcal{F}_5)$. We first prove $(\mathcal{F}_2) \Rightarrow (\mathcal{F}_1)$. Since $p \geq \frac{n-2}{n-1}$ for $n > 2$ and $p > 0$ for $n = 2$, for any given point $x \in \Omega_2$, it follows from [39, Theorem A] that $|\nabla u|^p$ is subharmonic in $B^n(x, d_{\Omega_2}(x)/2)$. Then
\[ |\nabla u(x)|^p \leq \frac{1}{|B^n(x, d_{\Omega_2}(x)/2)|} \int_{B^n(x, d_{\Omega_2}(x)/2)} |\nabla u(x)|^p d\mu(x), \]
which implies that there is a positive constant $C$ such that
\[ (4.32) \quad |\nabla u(x)| \leq \hat{I}_{u,p}(x) \leq C \frac{\psi(d_{\Omega_2}(x))}{d_{\Omega_2}(x)}. \]

Since $(\mathcal{F}_2)$ implies $(\mathcal{F}_1)$, by (4.32), we see that $(\mathcal{F}_5)$ implies $(\mathcal{F}_1)$. The proof of this theorem is finished. \qed

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