On the Significance of Knowing the Arrival Order in Prophet Inequality

Tomer Ezra∗ Michal Feldman† Nick Gravin‡ Zhihao Gavin Tang‡

Abstract

In a prophet inequality problem, n boxes arrive online, each containing some value that is drawn independently from a known distribution. Upon the arrival of a box, its value is realized, and an online algorithm decides, immediately and irrevocably, whether to accept it or proceed to the next box. Clearly, an online algorithm that knows the arrival order may be more powerful than an online algorithm that is unaware of the order. Despite the growing interest in the role of the arrival order on the performance of online algorithms, the effect of knowledge of the order has been overlooked thus far.

Our goal in this paper is to quantify the loss due to unknown order. We define the order competitive ratio as the worst-case ratio between the performance of the best order-unaware and the best order-aware algorithms. We study the order competitive ratio for two objective functions, namely (i) max-expectation: maximizing the expected accepted value, and (ii) max-probability: maximizing the probability of accepting the box with the largest value. For the max-expectation objective, we’re golden: we give a deterministic order-unaware algorithm that achieves an order competitive ratio of the inverse of the golden ratio (i.e., $1/\phi \approx 0.618$). For the max-probability objective, we give a deterministic order-unaware algorithm that achieves an order competitive ratio of $\ln \lambda \approx 0.806$ (where $\lambda$ is the unique solution to $\frac{1}{\lambda^2} = \ln \frac{1}{\lambda}$). Both results are tight. Our algorithms are inevitably adaptive and go beyond single-threshold algorithms.

1 Introduction

Optimal stopping theory and the beautiful prophet inequality problem have become central in the study of market design in recent years due to their close connection to mechanism design and posted price mechanisms [Hajiaghayi et al. [2007]]. In the classic prophet inequality setting, there are n boxes, each contains a value $v_t$ drawn from a known distribution $F_t$. The boxes arrive in an online fashion, and upon the arrival of a box, its value is realized. An online algorithm needs to decide immediately and irrevocably whether to accept the realized value, in which case the game ends, or to reject it, in which case this value is lost forever and the game proceeds to the next box.

The performance of an online algorithm has been traditionally measured by the competitive ratio, defined as the worst-case ratio between the expected value accepted by the algorithm and the expected maximum value. The latter is referred to as a prophet who can see into the boxes, hence the term prophet inequality.

One can easily verify that an optimal algorithm that knows the arrival order uses backward induction to compute n thresholds, one for each box, where the value in stage t is accepted if and only if it exceeds the threshold, which is the expected value of the algorithm on the remaining
boxes. The question that was raised in Krengel and Sucheston [1977, 1978] is how well such an online algorithm performs relative to a prophet.

It is well known that no online algorithm (whether it knows the arrival order or not) can obtain a better competitive ratio than $1/2$. To see this, consider a game with two boxes. The first box has a deterministic value of 1, the second box has value $1/\epsilon$ with probability $\epsilon$ (and 0 otherwise), for some arbitrarily small $\epsilon$. The expected maximum value is roughly 2, while no online algorithm can get an expected value greater than 1; indeed, the expected value of both boxes is 1.

As it turns out, the backward induction algorithm is $1/2$-competitive. A more surprising result, which dates back to the 80’s Samuel-Cahn [1984], is that a much simpler algorithm, one that computes a single threshold and accepts the first value that exceeds it, also achieves a competitive ratio of $1/2$. There is more than one way to choose such a threshold. Examples include the median of the distribution Samuel-Cahn [1984], or half the expected maximum value Kleinberg and Weinberg [2012].

Interestingly, in stark contrast to the backward induction algorithm, which uses the arrival order quite heavily, a single-threshold algorithm is order-unaware — it needs no information about the arrival order. Order-unawareness is a desirable property of an online algorithm. First, in many real-life scenarios, we are unaware of the order in which outcomes will reveal themselves. Second, even in cases where we have a good sense of the order, an order-unaware algorithms is robust against unexpected changes in the order that may occur.

A natural question is therefore: what is the loss that is incurred due to unknown order? This is exactly the question that drives us in this paper. To quantify this loss, we measure the ratio between the performance of order-unaware and order-aware algorithms. That is, rather than using the standard prophet benchmark, we use the arguably more realistic benchmark of the best order-aware online algorithm.

This is part of a growing interest in alternative benchmarks to the prophet benchmark Kessel et al. [2021], Niazadeh et al. [2018], Papadimitriou et al. [2021]. For example, Niazadeh et al. [2018] quantify the loss due to single-threshold algorithms by the worst-case ratio between the best single-threshold algorithm and the best general online algorithm (single-threshold or not), both under a known order, and show that this $1/2$ ratio is tight. As another example, Papadimitriou et al. [2021] consider the problem of online matching in bipartite graphs. This problem is known to admit a $1/2$-competitive algorithm with respect to the prophet benchmark Feldman et al. [2015]. But Papadimitriou et al. [2021] propose to study the ratio between the optimal polynomial (order-aware) algorithm and the optimal computationally-unconstrained (order-aware) algorithm, and show that this ratio exceeds $1/2$. (Note that this question makes sense in the matching variant, where the optimal online algorithm for matching is computationally hard.)

It is quite surprising that despite the growing interest in the role of the arrival order on the performance of online algorithms, the question of the loss due to unknown order has been completely overlooked. Indeed, most attention was given to the effect of different arrival models (e.g., adversarial order, random order, best order) on the competitive ratio Arsenis et al. [2021], Esfandiari et al. [2017], Azar et al. [2018], Beyhaghi et al. [2021], Correa et al. [2021], Agrawal et al. [2020].

We introduce the order competitive ratio to quantify the loss incurred due to order-unawareness. The order competitive ratio is defined as the worst-case ratio between the performance of the best order-unaware algorithm and the best order-aware algorithm. Note that this new measure uses the same benchmark as the one in Papadimitriou et al. [2021], Niazadeh et al. [2018], but unlike Papadimitriou et al. [2021], our algorithms are order-unaware, thus enjoy all the advantages and robustness of order-unawareness.
Figure 1: An example showing an upper bound of $1/\sqrt{2}$ on the order competitive ratio of deterministic algorithms.

To get a sense of the problem, let us go back to the classic prophet inequality result. As discussed above, there exists a single-threshold algorithm that achieves at least $1/2$ of the prophet. On the other hand, even the best order-aware algorithm cannot guarantee a higher ratio than $1/2$. One may wrongly interpret this result as saying that knowing the order is useless. This is inaccurate: it is only on this worst-case instance that knowing the order does not help the algorithm, but it may indeed be useful in other instances. To demonstrate the extra power that knowledge of the order provides, consider the following example.

Example 1.1. (see Figure 1) Suppose there are three boxes. Two boxes have deterministic values $\sqrt{2}$ and 1, respectively, and one box has value $1/\epsilon$ with probability $\epsilon$ (and 0 otherwise). Suppose the first observed value is $\sqrt{2}$, and an order-unaware algorithm $\text{ALG}$ needs to decide immediately whether to accept it. If $\text{ALG}$ accepts it, the next arriving value is the random one, followed by 1 (Figure 1(a)); else, the order flips, and 1 arrives next, followed by the random box (Figure 1(b)). Consider now the best online algorithm that knows the order. In the former case (where $\text{ALG}$ accepted $\sqrt{2}$), it rejects $\sqrt{2}$, and accepts the second value if it is positive, gaining an expected value of $\sim 2$. The order competitive ratio is then $1/\sqrt{2}$. In the latter case (where $\text{ALG}$ rejected $\sqrt{2}$), it accepts $\sqrt{2}$, while $\text{ALG}$ gets an expected value of 1, leading again to an order competitive ratio of $1/\sqrt{2}$.

Example 1.1 shows that the order competitive ratio cannot be better than $1/\sqrt{2} \approx 0.707$ (for deterministic algorithm). Thus, it lies somewhere in the interval $[0.5, 1/\sqrt{2}]$. (Clearly, the order competitive ratio is always weakly greater than the classic competitive ratio, hence it is at least 0.5.) We wish to give a tight bound on the order competitive ratio.

So far, we focused on the objective of maximizing the expected value, which is the standard objective in the prophet inequality literature. An additional well-motivated and well-studied objective is maximizing the probability to obtain the largest value. This “catching the max” objective resembles the famous secretary problem [Ferguson 1989], but with values drawn from known distributions, and under an adversarial order. This objective has been recently studied by [Esfandiari et al. 2020], who devise a single-threshold algorithm that catches the largest value with probability at least $1/e$. They also show that the ratio of $1/e$ is tight (up to lower-order terms). This tightness, however, is with respect to a prophet, who can look into the boxes, thus can always select the largest value. But can we do better relative to the (arguably more realistic) benchmark of the best online order-aware algorithm? In other words, what is the order competitive ratio with respect to the max-probability objective?

1.1 Our Results

We establish tight bounds for the order competitive ratios with respect to both objectives of (i) maximizing the expected value of the accepted box, and (ii) maximizing the probability of catching the largest value. For the first objective we show the following.

Main Result 1 (Section 3): There exists a deterministic online order-unaware algorithm that
obtains an expected value of at least $1/\phi \approx 0.618$ (where $\phi$ is the golden ratio) of the best online order-aware algorithm. Moreover, this is tight (with respect to deterministic algorithms).

In other words, the order competitive ratio for the objective of maximizing the expected value is the inverse of the golden ratio $\approx 0.618$ — a significant improvement over the competitive ratio of $1/2$ against the prophet.

As observed by Niazadeh et al. [2018], this result cannot be obtained by a single-threshold algorithm. Indeed, our algorithm is more complex and is inevitably adaptive.

We then move to the second objective and show the following.

**Main Result 2** (Section 4): There exists a deterministic online order-unaware algorithm that obtains the maximum value with probability at least $\ln \frac{1}{\lambda} \approx 0.806$ (where $\lambda$ is the unique solution to $x - \frac{1}{x} = \ln \frac{1}{x}$) of the best online order-aware algorithm. Moreover, this is tight (with respect to deterministic algorithms).

That is, the order competitive ratio for the objective of maximizing the probability of catching the largest value is $\approx 0.806$ — a huge improvement over the competitive ratio of $1/e$ against the prophet.

We also show that this result cannot be achieved by a single-threshold algorithm. Specifically, Claim 4.4 shows that no single-threshold algorithm can achieve a better order competitive ratio than $0.57$.

**Our Techniques.** Our algorithms for both max-expectation and max-probability objectives utilize adaptive thresholds, namely thresholds that depend on the partial arrival observed by the algorithm during its execution. Adaptive thresholds are necessary for our results. Adaptive thresholds have been also used for multi-choice prophet inequality [Kleinberg and Weinberg 2012]. However, the analysis is fundamentally different since in our case the value of the benchmark is unknown, and may change depending on the arrival order. Part of the challenge is to gradually learn the benchmark and adjust the algorithm accordingly. This is in contrast to the prophet benchmark which is known from the outset.

Our approach for the max-expectation objective includes several novel ideas. First, we combine two different static threshold policies. Namely, at each step of the algorithm we take the higher of two thresholds. Our thresholds are variants of known thresholds for the classic prophet inequality. The first threshold is $\tau_1 = \frac{1}{\phi} E[\max_i v_i]$; the second threshold is $\tau_2$ such that $E[\max_i (v_i - \phi \tau_2)] = \tau_2$. While the latter one is less known, a variant of it has already appeared in Samuel-Cahn [1984]. Second, since the benchmark is unknown, we use adaptive variants of the corresponding thresholds. Namely, at each stage we define the thresholds with respect to the remaining boxes only. We are not aware of prior work that uses the better of two strategies at every individual step of the algorithm.

### 1.2 Open problems

Our model and results suggest natural problems for future research.

1. Our bounds are tight with respect to deterministic algorithms. Can randomized algorithms provide better ratios?

2. Study the order competitive ratio in combinatorial settings, where multiple elements can be accepted, subject to feasibility constraints. A clear candidate is matroid feasibility constraints, for which the competitive ratio of $1/2$ with respect to the prophet benchmark carries over [Kleinberg and Weinberg 2012]. Our bounds for the expected value objective
carry over to simple matroid settings, such as partition matroids (where a single element is chosen from each part). What is the order competitive ratio for general matroids?

3. More generally, we believe that the order competitive ratio is a meaningful measure, which captures the significance of knowing the arrival order in Bayesian online settings. It would be interesting to apply it to other Bayesian settings.

4. The algorithm proposed by [Papadimitriou et al., 2021] for online bipartite matching is order-aware. Can the same result be obtained by an order-unaware algorithm?

5. What order competitive ratio can be achieved by a single threshold algorithm for the max-probability objective?

1.3 Additional Related Work

Recent years have seen a growing interest in the effect of the arrival order on the expected value one can guarantee in prophet-like settings. Indeed, various arrival order models have been studied, ranging from (adaptively) adversarial order [Samuel-Cahn 1984], through random arrival order (a variant known as prophet secretary, as it combines the Bayesian assumption of prophet with the random order of secretary) [Esfandiari et al., 2017; Azar et al., 2018], all the way to “free-order” settings, where the algorithm can choose the arrival order [Beyhaghi et al., 2021; Agrawal et al., 2020]. The best ratio under adversarial order is 1/2. For prophet secretary it is at least 0.669 and no better than 0.732 [Correa et al., 2021]. No better bounds are known for the free order setting, except for some special cases, e.g., where the support of each distribution is of size at most 2 [Agrawal et al., 2020].

Interestingly, another recent study related to the arrival order in prophet inequality settings [Arsenis et al., 2021] has shown that for any arrival order π, the better of π and the reverse order of π achieves a competitive ratio of at least the inverse of the golden ratio, namely 1/φ ≈ 0.618.

To the best of our understanding, while our first main result obtains the exact same ratio, the two results are unrelated.

While the vast majority of studies on prophet inequalities concentrated on the objective of maximizing the expected value, [Gilbert and Mosteller 1966] and [Esfandiari et al., 2020] studied the objective of maximizing the probability of catching the maximum value. [Gilbert and Mosteller 1966] studied this problem for i.i.d. values, and devised an algorithm that achieves a competitive ratio of 0.5801. [Esfandiari et al., 2020] extended this study beyond i.i.d. values, and provided a single-threshold algorithm that gives a (tight) competitive ratio of 1/e for non-identical distributions.

A related line of work, initiated by [Kennedy 1985, 1987; Kertz 1986], extends the single choice optimal stopping problem to multiple-choice settings. More recent work extended it to additional combinatorial settings, including matroids [Kleinberg and Weinberg 2019; Azar et al., 2014; polymatroids [Dütting and Kleinberg 2015]), matching [Gravin and Wang 2019; Ezra et al., 2020], combinatorial auctions [Feldman et al., 2015; Dütting et al., 2020], and general downward closed feasibility constrains [Rubinstein 2016].

Prophet inequality problems have been also studied under limited information about the underlying distributions, where the emphasis is on the sample complexity of the problem [Azar et al., 2014; Correa et al., 2019, 2020; Ezra et al., 2018; Rubinstein et al., 2019].

2 Model and Preliminaries

Consider a setting with n boxes. Every box t contains some value v_t drawn from an underlying independent distribution F_t. The underlying distributions are known from the outset, but the
values are revealed sequentially in an online fashion. For convenience of notations, we assume that the boxes arrive in an order 1, 2, ..., n. I.e., at stage $t$, we observe the realized value $v_t = \theta_t$, where $v_t \sim F_t$. It will be clear from the context whether the order of arrival is assumed to be known. In any case, the identity of the arriving box is known. We denote by $F = \prod_t F_t$ the (product) distribution of the value profile $\vec{v} = (v_1, \ldots, v_n)$. Upon the arrival of value $v_t = \theta_t$, the algorithm needs to decide, immediately and irrevocably, whether to accept the box.

We consider two different objectives:

1. maximizing the expected value of the accepted box.
2. maximizing the probability of catching the box with the largest value.

An online algorithm is said to be order-aware if it knows the arrival order of the boxes from the outset, and is said to be order-unaware if it doesn’t.

Our goal is to measure the performance of order-unaware algorithms against that of the best online order-aware algorithm. Our order-unaware algorithm will be denoted by ALG, and the best order-aware algorithm by OPT.

Given an arrival order $\pi$ and a value profile $\vec{v}$, we denote by $\text{ALG}(\vec{v}, \pi)$ the value accepted by $\text{ALG}$ under values $\vec{v}$ and arrival order $\pi$, and by $\text{OPT}(\vec{v}, \pi)$ the value accepted by $\text{OPT}$ under $\vec{v}, \pi$.

We denote by $\text{ALG}(\pi)$ the performance of $\text{ALG}$. For the first objective, it is the expected accepted value, i.e., $\text{ALG}(\pi) = \mathbb{E}_{\vec{v} \sim F}[\text{ALG}(\vec{v}, \pi)]$. For the second objective, it is the probability of catching the maximum value, i.e., $\text{ALG}(\pi) = \mathbb{P}_{\vec{v} \sim F}[\text{ALG}(\vec{v}, \pi) = \arg\max_i v_i]$. When studying the objective of catching the maximum value, we assume a unique maximum value.

The order-competitive ratio of an order-unaware algorithm $\text{ALG}$ measures the loss in performance due to unknown order. It is defined as the worst-case ratio of the performance of $\text{ALG}$ and the performance of OPT, over all arrival orders.

**Definition 2.1.** The order competitive ratio of an order-unaware algorithm $\text{ALG}$ is

$$\Gamma(\text{ALG}) = \min_{\pi} \frac{\text{ALG}(\pi)}{\text{OPT}(\pi)}$$

### 3 Maximizing the Expected Value

Our main result in this section is a deterministic order-unaware algorithm that obtains a tight order competitive ratio of the inverse of the golden ratio (i.e., $\frac{1}{\phi} \approx 0.618$) with respect to the objective of maximizing the expected value.

**An order-unaware algorithm.** For the convenience of notations, we assume the boxes arrive in a specific order from 1, 2, ..., n. I.e., at stage $t$, we observe $v_t = \theta_t$ where $v_t \sim F_t$. It will be clear from the description of our algorithm that it is order-unaware.

We define the following series of random variables

Prophet in the future: $y_t \overset{\text{def}}{=} \max_{s > t} v_s$.

At each step $t \in [n]$, our algorithm will use the larger of the following two thresholds

$$\alpha_t \overset{\text{def}}{=} \frac{1}{\phi} \cdot \mathbb{E}[y_t] \quad \beta_t \overset{\text{def}}{=} \text{solution to the equation } \mathbb{E}\left[(y_t - \phi \cdot x)^+\right] = x, \quad (1)$$

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$^1$E.g., when the supports of the distributions are disjoint.
where $\phi = \frac{\sqrt{5} + 1}{2}$ is the golden ratio. When $t = n$, $\alpha_t = \beta_t = 0$. Note that the equation defining $\beta_t$ has a unique solution, since its LHS is a strictly decreasing continuous function in $x$ that starts from a non-negative number when $x = 0$ and goes to 0 for $x = \infty$, and the RHS is a strictly increasing continuous function in $x$ that starts from 0 when $x = 0$ and goes to $\infty$ when $x = \infty$.

Our algorithm $\text{ALG}$ stops at box $t$ if and only if the realized value $\theta_t$ of $v_t$ satisfies

$$\theta_t \geq \tau_t \overset{\text{def}}{=} \max(\alpha_t, \beta_t).$$

Note that $\text{ALG}$ is order-unaware, since it does not need to know the arrival order of the remaining boxes in order to calculate $\alpha_t, \beta_t$. Denote the expected value obtained by this algorithm as $\text{ALG}$ and the one obtained by the best order-aware algorithm as $\text{OPT}$.

Our main theorem in this section is the following.

**Theorem 3.1.** For every arrival order $\pi$, $\text{ALG}(\pi) \geq \frac{1}{\phi} \text{OPT}(\pi)$, where $\phi$ is the golden ratio.

**Proof.** Fix an order $\pi$. Let $\text{ALG}_t$ denote the expected value of $\text{ALG}$ when run only on the boxes from $t$ to $n$. We first establish the following useful bound on the performance of $\text{ALG}_t$ relative to the thresholds $\alpha_t, \beta_t$.

**Lemma 3.2.** It holds that $\text{ALG}_{t+1} \geq \beta_t \geq \frac{1}{\phi} \mathbb{E}[y_t] = \frac{1}{\phi} \alpha_t$ for any $t \in [n-1]$.

**Proof.** According to the definition of $\beta_t$, $\beta_t = \mathbb{E}[(y_t - \phi \cdot \beta_t)^+] \geq \mathbb{E}[y_t] - \phi \cdot \beta_t$. Hence, $\beta_t \geq \frac{1}{\phi} \mathbb{E}[y_t] = \frac{1}{\phi} \mathbb{E}[y_t]$, which concludes the proof of the second inequality.

We next prove the first inequality by induction on the number of remaining boxes. We use $t = n$ as the base case of our induction, which is satisfied since $\alpha_n = \beta_n = 0$. Assume that $\text{ALG}_{t+1} \geq \beta_t$. We need to prove that $\text{ALG}_t \geq \beta_{t-1}$. First, observe that

$$\text{ALG}_t = \mathbb{E}_{v_t} \left[ 1(v_t > \tau_t) \cdot v_t + 1(v_t \leq \tau_t) \cdot \text{ALG}_{t+1} \right]$$

$$\geq \mathbb{E}_{v_t} \left[ 1(v_t > \tau_t) \cdot v_t + 1(v_t \leq \tau_t) \cdot \beta_t \right] \geq \beta_t,$$

where the first inequality holds by the induction hypothesis, and to obtain the second inequality we observe that $\tau_t \geq \beta_t$.

Consider the difference function $f(x) \overset{\text{def}}{=} \mathbb{E}[(y_{t-1} - \phi \cdot x)^+] - x$ (see Equation [1]). $f$ is strictly decreasing, with $f(x) = 0$ for $x = \beta_{t-1}$, by definition of $\beta_{t-1}$. Therefore, to prove that $\text{ALG}_t \geq \beta_{t-1}$ it is sufficient to prove that $f(\text{ALG}_t) \leq 0$. We have

$$f(\text{ALG}_t) = \mathbb{E}_{y_{t-1}} \left[(y_{t-1} - \phi \cdot \text{ALG}_t)^+] - \text{ALG}_t \right]$$

$$\leq \mathbb{E}_{y_{t-1}} \left[(y_{t-1} - \phi \cdot \beta_t)^+] - \text{ALG}_t \right]$$

$$= \mathbb{E}_{y_{t-1}, v_t} \left[(\max(y_{t-1}, v_t) - \phi \cdot \beta_t)^+] - \text{ALG}_t \right]$$

$$\leq \mathbb{E}_{y_{t-1}, v_t} \left[(\max(y_{t-1}, \tau_t) - \phi \cdot \beta_t)^+] + (v_t - \tau_t)^+] - \text{ALG}_t \right]$$

$$\leq \mathbb{E}_{y_{t-1}} \left[(\max(y_{t-1}, \tau_t) - \phi \cdot \beta_t)^+] + \mathbb{E}_{v_t} \left[(v_t - \tau_t)^+] - \mathbb{E}_{v_t} \left[1(v_t > \tau_t) \cdot v_t + 1(v_t \leq \tau_t) \cdot \beta_t \right] \right]$$

$$= \mathbb{E}_{y_{t-1}} \left[(\max(y_{t-1}, \tau_t) - \phi \cdot \beta_t)^+] - \mathbb{E}_{v_t} \left[1(v_t > \tau_t) \cdot \tau_t + 1(v_t \leq \tau_t) \cdot \beta_t \right] \right]$$

$$\leq \mathbb{E}_{y_{t-1}} \left[(\max(y_{t-1}, \tau_t) - \phi \cdot \beta_t)^+] - \beta_t \right],$$

where the first inequality follows since $\text{ALG}_t \geq \beta_t$ by Equation [2]; the second inequality holds since $(\max(a, b) - c)^+ \leq (\max(a, d) - c)^+ + (b - d)^+$ for any $a, b, c, d \in \mathbb{R}$; to get the third inequality,
we use the first part of Equation (2); the last inequality follows not only in expectation over \( v_t \) but for any fixed value \( \theta_t \) of \( v_t \), as \( \tau_t = \max(\alpha_t, \beta_t) \geq \beta_t \).

Furthermore,

\[
\begin{align*}
\frac{f(\text{ALG}_t)}{E_{y_t}[(\max(y_t, \tau_t) - \phi \cdot \beta_t)^+] - \beta_t} &= E_{y_t} \left[ \max \left\{ (y_t - \phi \cdot \beta_t)^+ , (\tau_t - \phi \cdot \beta_t)^+ \right\} \right] - \beta_t \\
&\leq E_{y_t} [(y_t - \phi \cdot \beta_t)^+ + (\tau_t - \phi \cdot \beta_t)^+] - \beta_t \\
&= \left( E_{y_t} [(y_t - \phi \cdot \beta_t)^+] - \beta_t \right) + E_{y_t} [(\tau_t - \phi \cdot \beta_t)^+] \\
&= 0,
\end{align*}
\]

where the first inequality is precisely (3); the second inequality follows by observing that \( \max(a, b) \leq a + b \) for any \( a, b \in \mathbb{R}_+ \); the last equality follows by observing that both terms equal 0. The first term \( (E_{y_t} [(y_t - \phi \cdot \beta_t)^+] - \beta_t) \) equals 0 by the definition of \( \beta_t \), and the second term \( (E_{y_t} [(\tau_t - \phi \cdot \beta_t)^+]) \) equals 0 by recalling that \( \tau_t = \max(\alpha_t, \beta_t) \) and by the fact that \( \beta_t \geq \frac{1}{\phi} \alpha_t \) proved above. Thus \( f(\text{ALG}_t) \leq 0 \) and \( \text{ALG}_t \geq \beta_t - 1 \), which concludes the proof of the induction step. \( \square \)

We are now ready to prove Theorem 3.1. We prove the statement of the theorem by induction on the total number of boxes \( n \). For the base case \( n = 1 \), \( \alpha_1 = \beta_1 = 0 \) and \( \text{ALG}_n = \text{OPT}_n \).

Suppose that the statement of the theorem holds for any \( n - 1 \) boxes. Let \( \text{OPT}_t \) and \( \text{ALG}_t \) denote the expected value of the optimal order-aware algorithm and our order-unaware algorithm on the boxes \( t, \ldots, n \). We shall prove the induction step that \( \text{ALG}_1 = \text{ALG} \geq \frac{1}{\phi} \text{OPT} = \frac{1}{\phi} \text{OPT}_1 \) for the case of \( n \) boxes. By the induction hypothesis we have \( \text{ALG}_2 \geq \frac{1}{\phi} \text{OPT}_2 \). We consider four cases based on the realized value \( \theta_1 \) of the first box. We denote by \( \text{ALG}(\theta_1) \) and \( \text{OPT}(\theta_1) \) the respective expected values of our algorithm and the optimal order-aware algorithm, given that the value in the first box is \( v_1 = \theta_1 \). We show that \( \text{ALG}(\theta_1) \geq \frac{1}{\phi} \text{OPT}(\theta_1) \) for any \( \theta_1 \).

**Case 1** Both \( \text{ALG} \) and \( \text{OPT} \) stop and take value \( \theta_1 \). Then, \( \text{ALG}_1(\theta_1) = \text{OPT}_1(\theta_1) = \theta_1 \).

**Case 2** \( \text{ALG} \) takes value \( \theta_1 \) but \( \text{OPT} \) doesn’t. Then, \( \text{ALG}(\theta_1) \geq \frac{1}{\phi} \text{OPT}_1(\theta_1) = \frac{1}{\phi} \text{OPT}_2 = \frac{1}{\phi} \text{OPT}(\theta_1) \), since \( \text{OPT}_2 \) cannot do better than the prophet on boxes \( t \in \{2, \ldots, n\} \).

**Case 3** \( \text{OPT} \) takes \( \theta_1 \), but \( \text{ALG} \) doesn’t. It holds that \( \text{ALG}_1(\theta_1) = \text{ALG}_2 \geq \max(\beta_1, \frac{1}{\phi} \alpha_1) \geq \frac{1}{\phi} \max(\beta_1, \alpha_1) \geq \frac{1}{\phi} \text{OPT}_1 \), where the first inequality follows by Lemma 3.2 and the last inequality holds since \( \text{ALG} \) rejected \( \theta_1 < \tau_1 \), whereas \( \text{OPT} \) selected it (thus \( \text{OPT}(\theta_1) = \frac{1}{\phi} \text{OPT}(\theta_1) \)).

**Case 4** Neither \( \text{ALG} \) nor \( \text{OPT} \) takes \( \theta_1 \). Then, \( \text{ALG}(\theta_1) = \text{ALG}_2 \) and \( \text{OPT}(\theta_1) = \text{OPT}_2 \), and the claim holds by the induction hypothesis.

Therefore, \( \text{ALG} = E_{v_1} [\text{ALG}(v_1)] \geq E_{v_1} [\text{OPT}(v_1)] = \frac{1}{\phi} \text{OPT} \). This concludes the proof. \( \square \)

We next show that the above bound is tight, namely that no order-unaware deterministic algorithm may achieve an order competitive ratio better than the golden ratio \( \frac{1}{\phi} = \frac{2}{\sqrt{5} + 1} \).

**Theorem 3.3.** For the objective of maximizing the expected value, no deterministic order-unaware algorithm achieves a better order competitive ratio than \( \frac{1}{\phi} \) in the worst case.
Proof. Consider an instance that consists of a set of boxes with deterministic values $\phi, \phi - \varepsilon, \phi - 2\varepsilon, \ldots, 1$, and a single random box $HV$ (we call it a high variance box) with value $1/\varepsilon$ realized with probability $\varepsilon$ and value 0 realized otherwise. Let $ALG$ be any given order-unaware deterministic algorithm. Let $\pi$ be an arrival order where the deterministic boxes arrive first, in decreasing order: $\phi, \phi - \varepsilon, \phi - 2\varepsilon, \ldots, 1$, followed by the $HV$ box.

Case 1 $ALG$ accepts some deterministic value $x > 1$. Consider now another arrival order $\pi_x$ which is the same as $\pi$ up to the deterministic $x$ box, but with the $HV$ box arriving immediately after it, and followed by the remaining deterministic boxes $x - \varepsilon, \ldots, 1$ in any order. Then, $ALG$ achieves value $x$ (we slightly abuse notations, and denote it $ALG(\pi_x) = x$), whereas $OPT$ for $\pi_x$ achieves $OPT(\pi_x) = \varepsilon \cdot 1/\varepsilon + (1 - \varepsilon)(x - \varepsilon)$, by waiting for the $HV$ box and taking it when its realized value is $1/\varepsilon$ (otherwise $OPT$ takes $x - \varepsilon$). As $\varepsilon$ goes to 0, $ALG/OPT$ goes to $\frac{x}{1 + x} \leq \frac{\phi}{1 + \phi} = \frac{1}{\phi}$, where the inequality follows since $x \leq \phi$.

Case 2 $ALG$ waits for the last deterministic item or the $HV$ box. Then, $ALG(\pi) = 1$, whereas $OPT(\pi)$ could select the first item (with value $\phi$), leading to an order competitive ratio of $\frac{1}{\phi}$.

\[\square\]

4 Maximizing the Probability of Catching the Maximum Value

Our main result in this section is a deterministic order-unaware algorithm that obtains an order competitive ratio of $\ln \frac{1}{\lambda} \approx 0.806$, where $\lambda \approx 0.4464$ is the unique solution to $\frac{x}{1 + x} = \ln \frac{1}{\lambda}$, with respect to the objective of maximizing the probability to catch the maximum value.

To prove this result, we consider a slightly more general game: let there be an extra number $\theta$ given in advance, and our objective is to maximize the probability of catching the box with the largest value that exceeds $\theta$. If all boxes have values less than $\theta$, no algorithm wins.

From now on, we shall work on this variant of the problem. Observe that the original problem is a special case where $\theta = 0$.

An order-unaware algorithm. At round $t \in [n]$, let $v_s = \theta_s$ be the realized values for each $s \leq t$. Let $\theta_0 = \theta$. We accept the current box $v_t = \theta_t$ if it satisfies the following condition:

$$\theta_t = \max_{0 \leq s \leq t} \theta_s \quad \text{and} \quad \Pr[v_{t+1}, \ldots, v_n \mid \max_{1 \leq s \leq n} v_s < \theta_t] \geq \lambda,$$

where $\lambda$ is the unique solution to $\frac{x}{1 + x} = \ln \frac{1}{\lambda}$.

Note that our algorithm is order-unaware since calculating the probability $\Pr[\max_{t+1 \leq s \leq n} v_s < \theta_t]$ requires no information on the order of remaining boxes. Indeed, it is the probability that all remaining boxes have value less than $\theta_t$. We use $ALG(\pi)$ to denote the probability of catching the maximum value that exceeds $\theta$ by our order-unaware algorithm and $OPT(\pi)$ to denote the winning probability of the best order-aware algorithm when the actual arrival order is $\pi$.

Theorem 4.1. For every arrival order $\pi$, the probability of catching the maximum value of the algorithm satisfies $ALG(\pi) \geq \ln \frac{1}{\lambda} \cdot OPT(\pi)$.

Proof. For simplicity, we omit $\pi$ and write $ALG$ and $OPT$ instead of $ALG(\pi)$ and $OPT(\pi)$, respectively. We prove the statement by induction on the number of boxes. The base case when $n = 1$ is trivial, since both our algorithm and the optimal algorithm would accept the first box with value $v_1$ if and only if $v_1 > \theta_0$. Suppose the statement is correct for $n - 1$ boxes and consider
the case for \( n \) boxes. We shall prove that for any realized value of \( v_1 \) the winning probability of our algorithm is at least \( \ln \frac{1}{\lambda} \) times the winning probability of the optimal algorithm.

Consider the four cases depending on the behavior of our algorithm and the optimal algorithm on the realization of the first box \( v_1 \).

**Case 1** both ALG and OPT accept. In this case, \( ALG = OPT \).

**Case 2** both ALG and OPT reject. In this case, we update the current maximum to \( \theta_1 = \max(\theta_0, v_1) \) and apply the induction hypothesis to conclude the proof. We remark that this is the place where we need the generalized version of the problem.

**Case 3** ALG accepts and OPT rejects. Then we have

\[
ALG = \Pr[\max_{s \geq 2} v_s < \theta_1] \geq \lambda \quad \text{and} \quad \OPT \leq \Pr[\max_{s \geq 2} v_s \geq \theta_1] \leq 1 - \lambda,
\]
due to the second condition of our algorithm. Therefore, \( ALG \geq \frac{\lambda}{1 - \lambda} \cdot \OPT \).

**Case 4** ALG rejects and OPT accepts. In this case, we must have \( \theta_1 \geq \theta_0 \) and \( \Pr[\max_{s \geq 2} v_s < \theta_1] < \lambda \). Let \( 2 \leq t \leq n \) be the index such that

\[
\Pr[\max_{s \geq t} v_s < \theta_1] \geq \lambda \quad \text{and} \quad \Pr[\max_{s \geq t} v_s < \theta_1] < \lambda. \tag{4}
\]

We define \( p_s = \Pr[v_s \geq \theta_1] \) for all \( s \geq 2 \). Since the optimal algorithm takes the first box with realized value of \( \theta_1 \), its winning probability is

\[
\OPT = \Pr[\max_{s \geq 2} v_s < \theta_1] = \prod_{2 \leq s \leq n} (1 - p_s). \tag{5}
\]

Next, we analyze our algorithm by studying the following events \( A_s \) for \( t \leq s \leq n \):

\[
A_s \overset{\text{def}}{=} \{ v_s \geq \theta_1 \text{ and } v_k < \theta_1, \forall k \neq s \}.
\]

**Claim 4.2.** Our algorithm wins if \( A_s \) happens for any \( t \leq s \leq n \).

**Proof.** It suffices to show that our algorithm accepts box \( s \). Indeed, \( ALG \) does not stop before box \( s \) since all other boxes have values smaller than \( \theta_1 \), violating the first condition of our algorithm. The second stopping condition of \( ALG \) is satisfied for box \( s \) as

\[
\Pr[\max_{k > s} v_k < \theta_1] \geq \Pr[\max_{k > s} v_k < \theta_1] \geq \Pr[\max_{k > s} v_k < \theta_1] \geq \lambda,
\]

where the last inequality is by the first inequality of (4). \( \square \)

Finally, we conclude the proof of the case:

\[
ALG \geq \Pr[\bigcup_{t \leq s \leq n} A_s] = \sum_{t \leq s \leq n} p_s \cdot \prod_{2 \leq k \leq n, k \neq s} (1 - p_k) = \prod_{2 \leq s \leq n} (1 - p_s) \cdot \left( \sum_{t \leq s \leq n} \frac{p_s}{1 - p_s} \right)
\]

\[
\geq \prod_{2 \leq s \leq n} (1 - p_s) \cdot \left( \sum_{t \leq s \leq n} \ln \frac{1}{1 - p_s} \right) = \prod_{2 \leq k \leq n} (1 - p_k) \cdot \ln \left( \frac{1}{\prod_{2 \leq s \leq n} (1 - p_s)} \right) \geq \OPT \cdot \ln \frac{1}{\lambda}.
\]

The first inequality follows by Claim 4.2. The second inequality holds since \( \frac{1}{1 + x} \geq \ln \frac{1}{1 + x} \) for any \( x \in [0, 1] \). The last inequality follows from Equations (4) and (5).
This finishes the proof of the inductive step.

We next show that the above result is tight, namely that no order-unaware deterministic algorithm may achieve an order-competitive ratio better than \( \ln \frac{1}{\lambda} \approx 0.806 \), where \( \lambda \) is the unique solution to \( x - x = \ln \frac{1}{x} \).

**Theorem 4.3.** For the objective of catching the maximum value, no deterministic order-unaware algorithm achieves a better order competitive ratio than \( \ln \frac{1}{\lambda} \approx 0.806 \).

**Proof.** Consider an instance with \( n + 1 \) boxes. One of the box has a deterministic value of \( \frac{1}{2} \). Among the remaining \( n \) boxes, the \( i \)-th box (for \( i = 1, \ldots, n \)) has value \( v_i = i \) with probability \( \epsilon \), and value 0 otherwise, where \( n \to \infty \), \( \epsilon \) is a small number that satisfies \( (1 - \epsilon)^n = \lambda \). Let the deterministic box comes first.

- Suppose the algorithm accepts the deterministic box. Then the algorithm wins when all the remaining boxes have value 0, i.e. \( \text{ALG} = (1 - \epsilon)^n = \lambda \). Then, the remaining boxes arrive in decreasing order, i.e., \( n, n - 1, \ldots, 1 \). The optimal algorithm that knows the arrival order would reject the deterministic box and accept the first box that has non-zero value. This algorithm wins when at least one of the randomized boxes has non-zero value, i.e. \( \text{OPT} = 1 - (1 - \epsilon)^n = 1 - \lambda \). This gives us an order competitive ratio of \( \frac{\lambda}{1 - \lambda} \).

- Suppose algorithm rejects the deterministic box. Then, the remaining boxes arrive in the increasing order, i.e., 1, 2, \ldots, \( n \). We study the best online algorithm that knows the order afterwards. It is straightforward to check that 1) the optimal algorithm only accepts non-zero boxes; 2) if the optimal algorithm accepts the \( i \)-th box when \( v_i \neq 0 \), it should also accept the \( j \)-th box for all \( j \geq i \) when \( v_j \neq 0 \). I.e., such algorithm can be described by a single parameter \( s \in [n] \) and it would simply accept the first non-zero box after the \( s \)-th box. Its winning probability is

\[
\text{ALG} = \Pr \left[ \text{exactly one randomized box of } \{s, s + 1, \ldots, n\} \text{ has non-zero value} \right] \\
= (n - s + 1) \cdot \epsilon(1 - \epsilon)^{n-s} \leq n\epsilon(1 - \epsilon)^{n-1}.
\]

The last expression approaches \( \lambda \ln \frac{1}{\lambda} \) as \( n \to \infty \). On the other hand, the optimal order-aware algorithm \( \text{OPT} \) would simply accept the deterministic box and win with probability \( \lambda \) (if all remaining boxes have value 0). Again, this leads to an order competitive ratio of \( \frac{\lambda \ln \frac{1}{\lambda}}{\lambda} = \ln \frac{1}{\lambda} \).

Finally, we show that the above order competitive ratio cannot be achieved with a single threshold algorithm.

**Claim 4.4.** No single threshold algorithm can guarantee an order competitive ratio better than 0.57 with respect to the objective of maximizing the probability of catching the maximum value.

**Proof.** Consider the following instance with \( n \) boxes, where box \( i \) has value \( i \) with probability \( \frac{1}{\sqrt{n}} \), and 0 otherwise.

Consider an arbitrary single threshold algorithm and let \( T \) be its threshold. Consider an arrival order \( \pi \), where

\[
\pi_i = \begin{cases} 
\text{Box } T + i & \text{if } 0 \leq i < \lfloor \frac{n-T}{2} \rfloor \\
\text{Box } n + \lfloor \frac{n-T}{2} \rfloor - i & \text{if } \lfloor \frac{n-T}{2} \rfloor \leq i \leq n - T \\
\text{Box } n - i & \text{if } n - T < i < n.
\end{cases}
\]
Figure 2: Upper bound for single threshold algorithm with threshold $T$.

That is, the arrival order $\pi$ is partitioned into three periods. In the first period, the boxes are ordered in an increasing order from $T$ to $T + \lfloor \frac{n-T}{2} \rfloor - 1$. In the second period, the boxes are ordered in a decreasing order from $n$ to $T + \lfloor \frac{n-T}{2} \rfloor$. In the third period, the boxes are ordered in a decreasing order from $T - 1$ to $1$. For $i \in \{1, 2, 3\}$, let $X_i$ be the random variable denoting the number of non-zero values in period $i$. The single threshold algorithm with threshold $T$ selects the first non-zero value if it belongs to either period 1 or period 2. It wins if and only if $(X_1 = 1$ and $X_2 = 0)$ or $(X_1 = 0$ and $X_2 > 0)$ or $(X_1 = X_2 = X_3 = 0)$. Suppose that the number of boxes $n$ goes to infinity. Thus, the winning probability of the single threshold algorithm with threshold $T$ satisfies:

$$\text{ALG} = \Pr [X_1 = 1 \land X_2 = 0] + \Pr [X_1 = 0 \land X_2 > 0] + \Pr [X_1 = X_2 = X_3 = 0]$$

$$\approx \frac{n-T}{2} \cdot \frac{1}{\sqrt{n}} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{n-T}{2}} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{n-T}{2}} + \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{n-T}{2}} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{n-T}{2}} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{n-T}{2}},$$

where the approximation is due to ignoring the floor and noticing that the probability that $X_1 = X_2 = X_3 = 0$ approaches 0 as $n$ goes to infinity. Let us use $\alpha$ to denote $\frac{n-T}{\sqrt{n}}$. We get

$$\text{ALG} \approx \frac{\alpha}{2} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^{\alpha\sqrt{n}} + \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{\alpha\sqrt{n}}{2}} - \left(1 - \frac{1}{\sqrt{n}}\right)^{\alpha\sqrt{n}}$$

$$\approx \frac{\alpha}{2} \cdot e^{-\alpha} + e^{-\frac{\alpha}{2}} - e^{-\alpha},$$

(6)

We next lower bound the performance of the best order aware algorithm $\text{OPT}$ by the performance of an order aware algorithm that discards all values in period 1, and selects the first non-zero value afterwards. This algorithm wins if and only if $X_2 > 0$ or $X_1 = X_2 = 0$. We get

$$\text{OPT} \geq \Pr [X_2 > 0] + \Pr [X_1 = X_2 = 0]$$

$$\approx 1 - \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{\alpha\sqrt{n}}{2}} + \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{\alpha\sqrt{n}}{2}} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{n-T}{2}}$$

$$= 1 - \left(1 - \frac{1}{\sqrt{n}}\right)^{\frac{\alpha\sqrt{n}}{2}} + \left(1 - \frac{1}{\sqrt{n}}\right)^{\alpha\sqrt{n}}$$

$$\approx 1 - e^{-\frac{\alpha}{2}} + e^{-\alpha}. \quad (7)$$

By combining Equations (6) and (7), the best threshold corresponds to the value of $\alpha$ that maximizes the ratio

$$\frac{\frac{\alpha}{2} \cdot e^{-\alpha} + e^{-\frac{\alpha}{2}} - e^{-\alpha}}{1 - e^{-\frac{\alpha}{2}} + e^{-\alpha}}.$$  

This expression is maximized at $\alpha \approx 1.12324$, which gives a bound of 0.57.  

[Diagram: Box $T$, Box $T + \frac{n-T}{2}$, Box $n$, Box $T + \frac{n-T}{2}$, Box $T - 1$, Box 1, Period 1, Period 2, Period 3]
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