Abstract. We present a new method for computation of the Korteweg–de Vries hierarchy via heat invariants of the 1-dimensional Schrödinger operator. As a result new explicit formulas for the KdV hierarchy are obtained. Our method is based on an asymptotic expansion of resolvent kernels of elliptic operators due to S. Agmon and Y. Kannai.

1. Introduction and main results

1.1. Heat asymptotics of Schrödinger operator and the KdV hierarchy. Consider the 1-dimensional Schrödinger (or Sturm-Liouville) operator

\[ L = \frac{\partial^2}{\partial x^2} + u(x). \]  

(1.1.1)

Its heat kernel \( H(t, x, y) \) is a fundamental solution of the heat equation

\[ \left( \frac{\partial}{\partial t} - L \right) f = 0. \]

The heat kernel has the following asymptotics on the diagonal as \( t \to 0^+ \) (see [3]):

\[ H(t, x, x) \sim \frac{1}{\sqrt{4\pi t}} \sum_{n=0}^{\infty} h_n[u] t^n, \]  

(1.1.2)

where \( h_n[u] \) are some polynomials in \( u(x) \) and its derivatives. The coefficients \( h_n[u] \) are called heat invariants of the 1-dimensional Schrödinger operator.

Computation of heat invariants of self-adjoint elliptic operators is a well-known problem in spectral theory which has many applications, in particular to geometry and theoretical physics (see [3, 4, 6]). Heat asymptotics of the 1-dimensional Schrödinger operator are of particular interest due to their relation to the Korteweg-de Vries (KdV) hierarchy which is one of the basic objects in the theory of integrable systems.
(see [9, 10]). Namely, the KdV hierarchy is defined by ([2]):
\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} G_n[u],
\]
where
\[
G_n[u] = \frac{(2n)!}{2^n \cdot n!} h_n[u], \quad n \in \mathbb{N}.
\]
Set \(u_0 = u\), \(u_n = \partial^n u / \partial x^n\), \(n \in \mathbb{N}\), where \(u_n, n \geq 0\) are formal variables. The sequence of polynomials \(G_n[u] = G_n[u_0, u_1, u_2, \ldots]\) starts with ([2]):
\[
G_1[u] = u_0, \quad G_2[u] = u_2 + 3u_0^2, \quad G_3[u] = u_4 + 10u_0u_2 + 5u_1^2 + 10u_3^0, \ldots
\]
In particular, substituting \(G_2[u]\) into (1.1.3) we obtain the familiar Korteweg-de Vries equation ([9, 10, 5, 2]):
\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x}.
\]

1.2. Explicit formulas for the KdV hierarchy and the heat invariants. There are many ways to compute the polynomials \(G_n[u]\) and \(h_n[u]\). Originally it was done through a recursive relation between the heat invariants (see [5, 8]). However, in order to understand the structure of heat invariants and the KdV hierarchy closed expressions are desirable. This might also make actual computations easier. Such expressions were first obtained by R. Schimming and I. Avramidi ([13, 2]), all of them having a very complicated combinatorial structure.

In this paper we present a method for computation of heat invariants of the 1-dimensional Schrödinger operator (see [11] for a similar method for the Laplacian on a 2-dimensional Riemannian manifold). As a result we obtain simpler explicit formulas for \(h_n[u]\) and hence for the KdV hierarchy. Our method is based on an asymptotic expansion of resolvent kernels of elliptic operators due to S. Agmon and Y. Kannai ([1, 11]).

We introduce the following notation. Consider \(L^k[x^{2k-2n}]|_{x=0}, k \geq n \geq 1\). This is a polynomial in \(u(0), u'(0), \ldots, u^{(2k-2)}(0)\). Let us make a formal change of variables: \(u^{(i)}(0) \rightarrow u_i, i = 0, \ldots, 2k - 2\), and denote the obtained polynomial by \(P_{kn}[u]\).

**Theorem 1.2.1.** The polynomials \(G_n[u], n \in \mathbb{N}\), are given by:
\[
G_n[u] = \sum_{m=n}^{2n} \sum_{k=n}^{m} \frac{(-1)^{n+k} m! (m)}{2^{2m-2n+1} (2n)! (2k-2n)!} P_{kn}[u],
\]
Expanding the expression for $L^k$ (see [12]) we get:

**Theorem 1.2.2.** The polynomials $G_n[u]$, $n \in \mathbb{N}$ are equal to:

\[
G_n[u] = \sum_{m=n}^{2n} \sum_{k=n}^{m} \frac{(-1)^n + k}{2} \binom{m}{n} \binom{m}{m} \sum_{p=1}^{k} \sum_{j_1, \ldots, j_p \in 2(n-p)} C_{j_1, \ldots, j_p} u_{j_1} \cdots u_{j_p},
\]

where

\[
C_{j_1, \ldots, j_p} = \sum_{0 \leq l_0 \leq l_1 \leq \cdots \leq l_{p-1} = k-p \atop 2l_0 \geq j_1 + \cdots + j_{p-1}, i = 0, \ldots, p-1} \binom{2l_0}{j_1} \binom{2l_1 - j_1}{j_2} \cdots \binom{2l_{p-1} - j_1 - \cdots - j_{p-1}}{j_p}.
\]

Expressions for the heat invariants $h_n[u]$ can be easily deduced from (1.2.1) and (1.2.2) using (1.1.4).

Theorems 1.2.1 and 1.2.2 are proved in sections 3.1 and 3.2.

### 2. Computation of Heat Invariants

#### 2.1. A modification of Agmon-Kannai expansion.

The original Agmon–Kannai theorem ([1]) deals with asymptotic behaviour of resolvent kernels of elliptic operators. In [11] we have obtained a concise reformulation of this theorem suitable for computation of heat invariants. Before presenting it we introduce some notations.

Let $H$ be a self-adjoint elliptic differential operator of order $s$ on a Riemannian manifold $M$ of dimension $d < s$ and let $H_0$ be the operator obtained by freezing the coefficients of the principal part $H'$ of the operator $H$ at some point $x \in M$: $H_0 = H'(x)$. Denote by $R_\lambda(x, y)$ the kernel of the resolvent $R_\lambda = (H - \lambda)^{-1}$, and by $F_\lambda(x, y)$ — the kernel of $F_\lambda = (H_0 - \lambda)^{-1}$.

**Theorem 2.1.1.** ([11]). The resolvent kernel $R_\lambda(x, y)$ has the following asymptotic representation on the diagonal as $\lambda \to \infty$:

\[
R_\lambda(x, x) \sim \rho(x)^{-1} \sum_{m=0}^{\infty} X_m F^{m+1}_\lambda(x, x),
\]

where $\rho(x) dx$ is the volume form on the manifold $M$ and the operators $X_m$ are defined by:

\[
X_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} H^k H_0^{m-k}, \quad m \geq 0.
\]
2.2. From the resolvent to the heat kernel. Let us return to our particular case of 1-dimensional Schrödinger operator. We have

\[ M = \mathbb{R}^1, \quad d = 1, \quad s = 2, \quad H \equiv -L, \quad H_0 \equiv -\frac{d^2}{dx^2}, \quad \rho(x) \equiv 1. \]

Note that (2.1.2) is an expansion in powers of \(-\lambda\) due to the following formula ([1]):

\[
\frac{d^\alpha}{dx^\alpha} F_{\lambda}^{m+1}(x, x) = (-\lambda)^{\frac{\alpha - 2m - 1}{2}} \frac{(-1)^\frac{\alpha}{2}}{2\pi} \int_{-\infty}^{\infty} \frac{\xi^\alpha d\xi}{(\xi^2 + 1)^{m+1}}. \tag{2.2.1}
\]

Collecting terms with the same powers of \(-\lambda\) in (2.1.2) one receives the standard asymptotic expansion of the kernel of \(R_\lambda = (-L - \lambda)^{-1}\) on the diagonal as \(\lambda \to \infty\) (see [1]):

\[
R_\lambda(x, x) \sim (-\lambda)^{-1/2} \sum_{n=0}^{\infty} r_n[u] (-\lambda)^{-n/2}, \tag{2.2.2}
\]

where \(r_n[u]\) are polynomials in \(u(x)\) and its derivatives.

For each \(n \in \mathbb{N}\) the polynomials \(r_n[u]\) and the heat invariants \(h_n[u]\) differ by a constant multiplier (cf. [11]):

\[
r_n[u] = \frac{(2n)!}{2^{2n+1} n!} h_n[u], \tag{2.2.3}
\]

3. Proofs of main theorems

3.1. Proof of Theorem 1.2.1. Due to (1.1.4) and (2.2.3),

\[
G_n[u] = 2^{2n} r_n[u]. \tag{3.1.1}
\]

Therefore in order to compute \(G_n[u]\) we need to find the coefficient \(r_n[u]\) in (2.2.2).

Without loss of generality set \(x = 0\). We need to collect all terms in the sum (2.1.2) containing \((-\lambda)^{-n/2}\). Let us apply formula (2.2.1). Note that if \(\alpha\) is odd then (2.2.1) is zero identically. For the terms we are interested in we have \(\alpha = 2m - 2n\) and hence \(m \geq n\). On the other hand the order of the differential operator \(X_m\) defined by (2.1.3) is not greater than \(m\) (see Lemma 5.1, [1]). Hence \(\alpha = 2m - 2n \leq m\) which implies \(m \leq 2n\).

Denote by \(r'_n\) a polynomial in \(u(0), u'(0), u''(0), \ldots\) which is obtained from the polynomial \(r_n[u]\) by a formal change of variables \(u_i \to u^{(i)}(0), i \geq 0\).

Then due to (2.1.3) we have:

\[
r'_n = \sum_{m=n}^{2n} \sum_{k=n}^{m} (-1)^{m+k} \binom{m}{k} \Phi_{mn} L^k \frac{d^{2m-2k}}{dx^{2m-2k}} \left[ \frac{x^{2m-2n}}{(2m-2n)!} \right] \bigg|_{x=0},
\]
where
\[
\Phi_{mn} = \frac{(-1)^{m-n}}{2\pi} \int_{-\infty}^{\infty} \frac{\xi^{2m-2n} d\xi}{(\xi^2 + 1)^{m+1}} = \frac{(-1)^{(m-n)}}{2\pi} B \left( m - n + \frac{1}{2} ; n + \frac{1}{2} \right),
\]
and the integral above is computed using the tables ([7]).

Let us rewrite the \( B \)-function in terms of factorials:
\[
B \left( m - n + \frac{1}{2} ; n + \frac{1}{2} \right) = \frac{\Gamma \left( m - n + \frac{1}{2} \right) \Gamma \left( n + \frac{1}{2} \right)}{\Gamma (m + 1)} = (2n!) \cdot (2m-2n)! \cdot \pi^{2m} \cdot m! \cdot n! \cdot (m-n)!.
\]

Substituting this into the expression for \( \Phi_{mn} \) and simplifying \( r'_{n} \) we finally get:
\[
r_n[u] = \sum_{m=n}^{2n} \sum_{k=n}^{m} \frac{(-1)^{n+k}}{2^{m+1} \cdot (m-k)!} P_{kn}[u], \tag{3.1.2}
\]
where \( P_{kn}[u] = P_{kn}[u_0, u_1, \ldots, u_{2k-2}] \) are polynomials obtained from \( L^k[x^{2k-2n}] \) by a formal change of variables
\[
u(0) \rightarrow u_0, u'(0) \rightarrow u_1, \ldots, u^{(2k-2)}(0) \rightarrow u_{2k-2}.
\]

By (3.1.1), formula (3.1.2) implies (1.2.1) and this completes the proof of the theorem.

**Remarks.** The proof of Theorem 1.2.1 is similar to the proof of Theorem 1.4 in [11]. Formula (1.2.1) was programmed using Mathematica ([14]) and for \( 1 \leq n \leq 5 \) the results agreed with the already known ones (cf. (1.1.5), [5]).

### 3.2. Proof of Theorem 1.2.2.

In [12] an explicit formula for the powers of the 1-dimensional Schrödinger operator was obtained. We may rewrite it in the form:
\[
L^k = \frac{d^{2k}}{dx^{2k}} + \sum_{p=1}^{k} \sum_{j_0+j_1+\cdots+j_p=2(k-p)} C_{j_1,\ldots,j_p} u_{j_1}(x) \cdots u_{j_p}(x) \frac{d^{j_0}}{dx^{j_0}},
\]
where
\[
C_{j_1,\ldots,j_p} = \sum_{0 \leq l_0 \leq l_1 \leq \cdots \leq l_{p-1} = k-p \atop 2l_0 \geq j_1 + \cdots + j_{i+1}, i=0,\ldots,p-1} \binom{2l_0}{j_1} \binom{2l_1 - j_1}{j_2} \cdots \binom{2l_{p-1} - j_1 - \cdots - j_{p-1}}{j_p}.
\]

Setting \( j_0 = 2k-2n \) this allows to expand the expression for \( L^k[x^{2k-2n}] \) and hence for the polynomials \( P_{kn}[u] \). Substituting it into (3.1.2) we immediately receive (1.2.2) which completes the proof of the theorem.

**Acknowledgments.** This paper is a part of my Ph.D. research at the Department of Mathematics of the Weizmann Institute of Science.
I am very grateful to my Ph.D. advisor Professor Yakar Kannai for constant support and valuable remarks.

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