NONEXISTENCE OF SMOOTH SOLUTIONS FOR THE GENERAL COMPRESSIBLE ERICKSEN – LESLIE EQUATIONS IN THREE DIMENSIONS

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Abstract. We prove that the smooth solutions to the Cauchy problem for the compressible general three-dimensional Ericksen–Leslie system modeling nematic liquid crystal flow with conserved mass, linear momentum, and dissipating total energy, generally lose classical smoothness within a finite time.

Key words. general compressible three-dimensional Ericksen–Leslie system, nematic liquid crystal, the Cauchy problem, singularity formation

AMS subject classifications. 76A15

1. System, known results, and main problem. The liquid crystal state is often viewed as an intermediate state between liquid and solid. The molecules possess none or partial positional order but display a preferred orientation. The nematic phase is the simplest among all liquid crystal phases and is closest to the liquid state. The molecules float around as in a liquid, but have the tendency to align along a preferred direction due to their orientation. The hydrodynamic theory of liquid crystals, due to Ericksen and Leslie, was developed in the 1960’s [8, 9, 21, 22]. The full Ericksen–Leslie system consists of the following equations (cf. [10, 22, 23, 27, 19]):

\[
\frac{\partial}{\partial t} \rho + \text{div}(\rho u) = 0, \tag{1.1}
\]

\[
\frac{\partial}{\partial t} (\rho u) + \text{Div}(\rho u \otimes u) + \nabla p = -\partial_j \left( \rho \frac{\partial W}{\partial d_j}, \nabla d \right) + \text{Div}(\sigma), \tag{1.2}
\]

\[
J \rho \frac{D}{Dt} \omega = (h - g) \times d, \quad \omega = d \times \frac{D}{Dt} d, \tag{1.3}
\]

\[
p = A \rho^\gamma, \quad A > 0, \gamma > 1. \tag{1.4}
\]

Equations (1.1) - (1.3) are given in \( \mathbb{R} \times \mathbb{R}^3 \) and represent the conservation of mass, linear momentum, and angular momentum, respectively, with the anisotropic feature of liquid crystal materials exhibited in (1.3) and its nonlinear coupling in (1.2) (cf. [22, 27]).

Here, we consider the flow of a compressible isentropic material: \( \rho \) is the fluid density, \( p \) is a pressure, \( J \) is a positive inertial constant, \( \gamma \) is an adiabatic exponent,

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\( u = (u_1, u_2, u_3)^T \) is the velocity, \( d = (d_1, d_2, d_3)^T \) is the orientational order parameter representing the macroscopic average of the molecular directors, and \( \frac{\partial}{\partial t} := \frac{\partial}{\partial t} + (u, \nabla) \) denotes the material derivative.

The notations \( A = \frac{1}{2}(\nabla u + (\nabla u)^T) \), \( \Omega = \frac{1}{2}(\nabla u - (\nabla u)^T) \), \( N = \frac{\partial}{\partial t} d - \Omega d \) represent the rate of strain tensor, the skew-symmetric part of the strain rate (the vorticity of the fluid), the material derivative of \( d \) (transport of center of mass), and the rigid rotation part of the changing rate of the director by the fluid vorticity, respectively.

The Oseen-Zöcher-Frank free energy functional \( W \) for the equilibrium configuration of a unit director field for a nematic crystal is given by the sum of the splay, the bend, and the twist, i.e,

\[
W = K_1 \frac{1}{2} (\text{div } d)^2 + K_2 \frac{1}{2} |d \times (\text{curl } d)|^2 + K_3 \frac{1}{2} (d \cdot \text{curl } d)^2 \geq 0, \quad (1.5)
\]

and the vector field

\[
h = \rho \frac{\partial W}{\partial d} - \partial_i \left( \rho \frac{\partial W}{\partial d_i} \right) \quad (1.6)
\]
is the molecular field.

The kinematic transport \( g \) of the director \( d \) is defined by:

\[
g_i = \lambda_1 N_i + \lambda_2 d_j A_{ji}, \quad (1.7)
\]

and represents the effect of the macroscopic flow field on the microscopic structure. The material coefficients \( \lambda_1 \) and \( \lambda_2 \) reflect the molecular shape (Jeffrey’s orbit) and how slippery the particles are in the fluid.

The stress tensor \( \sigma \) has the following form:

\[
\sigma_{ij} = \mu_1 d_k A_{kp} d_p d_i d_j + \mu_2 N_i N_j + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 A_{ik} d_k d_j + \mu_6 d_i A_{jk} d_k. \quad (1.8)
\]

The independent coefficients \( \mu_1, \ldots, \mu_6 \), which may depend on the material and temperature, are called Leslie coefficients.

The following relations are frequently introduced in the literature (cf. [22, 23]):

\[
\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \quad (1.9)
\]

\[
\mu_2 + \mu_3 = \mu_6 - \mu_5. \quad (1.10)
\]

The identities (1.9) are necessary conditions to satisfy the equation of motion identically (cf. [22], Section 6). The identity (1.10) is called Parodi’s relation (cf. [22]), which is derived from Onsager’s reciprocal relations expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium (cf. [31]). Under the assumption of Parodi’s relation, we see that the dynamics of an incompressible nematic liquid crystal flow involve five independent Leslie coefficients.

As was proved in [19], \( |d(0)| = 1 \) implies \( |d(t)| = 1 \) and \( (d(0), \omega(0)) = 0 \) implies \( (d(t), \omega(t)) = 0 \) for all time.

When \( d \) is a constant vector field, the system (1.1) – (1.4) becomes the compressible Navier-Stokes equations describing the motion of a compressible fluid.

Since the mathematical structure of the Ericksen–Leslie system is quite complicated, the study of the full model presents several mathematical difficulties. Almost all existing investigations were restricted to simplified versions. For the incompressible
model in [24]. Lin introduced a simplification of the general Ericksen–Leslie system that keeps many of the mathematical difficulties of the original system by using a Ginzburg–Landau approximation to relax the nonlinear constraint $d \in S^2$. Namely, instead of the restriction $|d| = 1$ he added the penalty term $\frac{1}{\epsilon^2}(|d|^2 - 1)^2$ to the free energy functional $W$ and neglected the inertial constant $J$. Later in [25], Lin and Liu showed the global existence of weak solutions and smooth solutions for that approximation. In [35], a very simple proof of local well-posedness for this coupled system was provided using a contraction mapping argument. It was proved that this system is globally well-posed and has compact global attractors in 2D. For more results on the approximative system see [26, 28, 30, 36, 12, 2]. Further, taking the limit of $\epsilon \to 0$ in the penalty term, one can get a coupling between the compressible Navier–Stokes equations and a transported heat flow of harmonic maps into $S^2$ [27]. It is a macroscopic continuum description of the evolution for liquid crystals of nematic type under the influence of both the flow field $u$, and the macroscopic description of the microscopic orientation configurations $d$ of (rod-like) liquid crystals. Recently, Hong [14] and Lin-Liu-Wang [29] showed independently the global existence of weak solution of an incompressible model in two dimensional space. Moreover, in [29], the regularity of solutions, except for a countable set of singularities whose projection on the time axis is a finite set, has been obtained (see also [15]). In [27], Wang established a global well-posedness theory for the incompressible liquid crystals for rough initial data, provided that $\|u_0\|_{BMO} + \|d_0\|_{BMO} < \epsilon_0$ for some $\epsilon_0 > 0$. In [4], regularity and uniqueness for solutions to density dependent nematic liquid crystals systems in the Ginzburg–Landau approximation were established for a bounded domain: in 2D the system has a global classical solution, in contrast to the 3D case. A family of exact solutions with finite energy to the incompressible liquid crystals in two dimensions was constructed in [7].

Concerning the compressible 3D case, local existence and uniqueness of strong solutions for the coupling between the compressible Navier-Stokes equations and a transported heat flow of harmonic maps was proved (see [13]), provided that the initial data $\rho_0, u_0, d_0$ are sufficiently regular and satisfy a natural compatibility condition. A criterion for possible breakdown of such a local strong solution at finite time was given in terms of blow up of the $L^\infty$-norms of $\rho$ and $\nabla d$. Alternative blow-up criteria were derived in terms of the $L^\infty$-norms of $\nabla u$ and $\nabla d$ in [17] and in terms of integral of $L^\infty$-norms of $\nabla u$ and the BMO-norm of $\nabla d$, in [3]. The global existence of weak solutions with large initial data is still an outstanding open problem for dimensions larger then or equal to 3. So far, only results in one space dimension have been obtained, for instance, we refer to [13, 14]. In [16], the existence and uniqueness of global strong solutions to the Cauchy problem is proved for 3D in critical Besov spaces provided that the initial data is close to an equilibrium state.

The overview of numerical methods used for the nematic liquid crystal flows can be found in [1].

At the same time, up to now there are no results on the classical solvability for the main initial and initial-boundary problems for the full system (1.1) - (1.4). In the present paper we prove that a global classical solution to the Cauchy problem does not exists, and that this phenomenon is due to the presence of the viscosity term. The nature of loss of smoothness is similar to the case of the compressible Navier-Stokes equation and the anisotropic features do not influence this phenomenon.
2. A general blowup result. In this paper we study the system (1.1) - (1.4) with the initial condition
\[(\rho, u, d) \bigg|_{t=0} = (\rho_0, u_0, d_0), \quad d_0 \in S^2. \tag{2.1}\]

We introduce the following natural functionals defined on the solution of the system (1.1) – (1.4): mass
\[m(t) = \int_{\mathbb{R}^3} \rho \, dx,\]
linear momentum
\[P(t) = \int_{\mathbb{R}^3} \rho u \, dx,\]
and total energy
\[E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \rho |u|^2 \, dx + \frac{1}{\gamma - 1} \int_{\mathbb{R}^3} p \, dx + \frac{1}{2} \int_{\mathbb{R}^3} J |\omega|^2 \, dx + \int_{\mathbb{R}^3} \rho W \, dx = E_k(t) + E_i(t) + E_{p,1}(t) + E_{p,2}(t) \geq 0, \tag{2.2}\]
where \(E_k(t), E_i(t), \) and \(E_{p,i}, i = 1, 2,\) are the kinetic, internal, and potential components of energy, respectively.

**Definition 2.1.** A solution \((\rho, u, d)\) to the Cauchy problem (1.1) – (1.4), (2.1) belongs to the class \(K\) if the solution is classical, \(\rho \geq 0,\) the mass \(m(t),\) linear momentum \(P(t),\) and total energy are finite for all \(t \geq 0,\) and, in addition, the mass and linear momentum are conserved, i.e., \(m(t) = m = \text{const}, P(t) = P = \text{const}.\)

Thus, if the solution belongs to the class \(K,\) then
\[
\rho \in L^1 \cap L^\gamma, \quad \sqrt{\rho} u \in L^2, \quad \sqrt{\rho} \omega \in L^2, \quad \nabla d \in L^2, \quad (d, \text{curl} \, d) \in L^1, \quad d \times \text{curl} \, d \in L^1, \quad (d, d^T A d) \in L^2, \quad d \in L^2.
\]
where \(L^\gamma = L^\gamma(\mathbb{R}^3).\)

Further, if we additionally assume
\[u \in H^1, \quad d^T A d \in L^2, \quad N \in L^2. \tag{2.3}\]
A direct calculation with smooth solution \((\rho, u, d)\) to the system (1.1) – (1.4) yields (cf. [27], Theorem 2.1),
\[
\frac{d}{dt} E(t) = - \int_{\mathbb{R}^3} \left( \mu_1 |d^T A d|^2 + \frac{\mu_4}{2} |\nabla u|^2 + (\mu_5 + \mu_6) |Ad|^2 \right) \, dx
+ \lambda_1 \int_{\mathbb{R}^3} |N|^2 \, dx + (\lambda_2 - \mu_2 - \mu_3) \int_{\mathbb{R}^3} (N, Ad) \, dx. \tag{2.4}\]

Here and throughout, we always assume that
\[
\lambda_1 < 0, \quad \mu_5 + \mu_6 \geq 0, \quad \mu_1 \geq 0, \quad \mu_4 > 0. \tag{2.5}\]
These assumptions are necessary conditions for the dissipation of the director field [11, 23].
Indeed, as was shown in [38] by means of the Hölder inequality, if (2.5) holds, under the additional assumption
\[ |\lambda_2 - (\mu_2 + \mu_3)| \leq 2 \sqrt{-\lambda_1(\mu_5 + \mu_6)} \] (2.6)
one has the following energy inequality: the basic energy law (without Parodi’s relation) holds
\[ \frac{d}{dt} E(t) \leq - \int_{\mathbb{R}^3} \left( \mu_1 |d^T A d|^2 + \frac{\mu_1}{2} |\nabla u|^2 \right) dx \leq 0. \] (2.7)

Our main result is the following theorem.

**Theorem 2.2.** Suppose that \( \gamma \geq \frac{6}{5} \) and inequalities (2.5) and (2.6) hold. Then there is no global in time solution to the Cauchy problem (1.1) – (1.4), (2.1) in the class \( K \) satisfying (2.3).

The proof is similar to [33], [34], where a blow-up result was proved for the compressible Navier-Stokes system and the compressible magnetohydrodynamics system.

The next lemma is a key technical point of the proof of the theorem.

**Lemma 2.3.** Let \( \gamma \geq \frac{6}{5} \) and \( u \in H^1(\mathbb{R}^3) \). If \( |P| \neq 0 \), then there exists a positive constant \( C \) such that for the solutions from the class \( K \) the following inequality holds:
\[ \int_{\mathbb{R}^3} |Du|^2 dx \geq C E_i^{-\frac{\gamma-1}{\gamma}}(t). \] (2.8)

**Proof.** First, from the inequality
\[ \left( \int_{\mathbb{R}^n} |u|^\frac{2}{\gamma} dx \right) \frac{\gamma-2}{\gamma} \leq C_1 \int_{\mathbb{R}^n} |Du|^2 dx, \] (2.9)
where the constant \( C_1 > 0 \) depends on \( n, n \geq 3 \), [13, p.22] we get \( H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3) \). Then, using \( \gamma \geq \frac{6}{5} \), the Hölder inequality gives
\[ |P| = \int_{\mathbb{R}^n} \rho u dx \leq \left( \int_{\mathbb{R}^n} \rho^\frac{2}{\gamma} dx \right)^\frac{\gamma}{n-2} \left( \int_{\mathbb{R}^n} |u|^6 dx \right)^\frac{1}{n}. \] (2.10)
Then, by the Jensen inequality (e.g., [20], Theorem 8.1.3), we have for \( \gamma \geq \frac{6}{5} \),
\[ \left( \frac{1}{m} \int_{\mathbb{R}^n} \rho^\frac{2}{\gamma} dx \right)^{5(\gamma-1)-1} \leq \int_{\mathbb{R}^n} \rho^\gamma dx \leq \frac{(\gamma-1)E_i(t)}{mA}. \] (2.11)
Thus, (2.10) and (2.11) imply
\[ |P| \leq C_2 \left( E_i(t) \right)^{\frac{1}{5(\gamma-1)-1}} \left( \int_{\mathbb{R}^n} |u|^6 dx \right)^\frac{1}{n}, \] (2.12)
with the constant \( C_2 = m^\frac{1}{n} \left( \frac{\gamma-1}{mA} \right)^\frac{1}{5(\gamma-1)-1} > 0 \). Thus, the inequality (2.8) follows from (2.9), (2.12), with the constant \( C = \frac{|P|^2}{C_1^2} \).
Proof of Theorem 2.2: Since $E_i(t) \leq E(t) \leq E(0)$ by (2.7), from (2.6) and (2.8) we conclude
\[
\frac{d}{dt} E(t) \leq -\mu_4 C(E_i(t))^{\frac{4}{3}} \leq -\mu_4 C(E(0))^{\frac{4}{3}}.
\]
which contradicts the non-negativity of $E(t)$ for all $t > 0$. This concludes the proof of the theorem. □

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