Cooperative quantum jumps for three dipole-interacting atoms

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We investigate the effect of the dipole-dipole interaction on the quantum jump statistics of three atoms. This is done for three-level systems in a V configuration and in what may be called a D configuration. The transition rates between the four different intensity periods are calculated in closed form. Cooperative effects are shown to increase by a factor of two compared to two of either three-level systems. This results in transition rates that are, for distances of about one wavelength of the strong transition, up to 100% higher than for independent systems. In addition the double and triple jump rates are calculated from the transition rates. In this case cooperative effects of up to 170% for distances of about one wavelength and still up to 15% around 10 wavelengths are found. Nevertheless, for the parameters of an experiment with Hg$^+$ ions the effects are negligible, in agreement with the experimental data. For three Ba$^+$ ions this seems to indicate that the large cooperative effects observed experimentally cannot be explained by the dipole-dipole interaction.

I. INTRODUCTION

Cooperative effects due to the dipole-dipole interaction between atoms are of great importance in many fields, most recently in the study of possible quantum computers based on trapped ions or atoms, and therefore they have attracted considerable interest in the literature [1, 2]. A sensitive test for such cooperative effects can be provided by atoms showing macroscopic light and dark periods in their fluorescence. These can occur in a multilevel system if the electron is essentially shelved in a metastable state, thereby causing the photon emission to cease [2]. Two or three such systems accordingly show three or four periods of different intensity, namely one dark period and bright periods with once, twice, or three times the intensity of a single system’s bright period. The dipole-dipole interaction may alter the statistics of these periods. In an as yet unexplained experiment with two and three Ba$^+$ ions [3, 4] a large number of double and triple jumps, i.e., jumps by two or three intensity steps within a short resolution time, was observed, exceeding by far the value expected for independent atoms. The quantitative explanation of such large cooperative effects for distances of the order of 10 wavelengths of the strong transition has been found difficult [5, 6, 7, 8, 9, 10]. Experiments with other ions showed no observable cooperative effects [11, 12], in particular none were seen for Hg$^+$ for a distance of about 15 wavelengths [13]. More recently effects similar to Ref. [8] were found in an experiment with Ca$^+$ ions [14] in contrast to a comparable experiment [15]. A different method for observing the dipole-dipole interaction of two V systems was proposed in Ref. [16].

The effect of the dipole-dipole interaction for two V systems was investigated numerically in Ref. [17] and analytically in Ref. [18] and shown to be up to 30% in the double jump rate compared to independent systems. However, the systems used in the experimental setups of Refs. [3, 4, 15] cannot be described by a V system so that a direct comparison between theory and experiment was not possible. For this reason the present authors have investigated cooperative effects for two other systems [20], namely a D shaped system modeling the Hg$^+$ ions used in Ref. [13] and a four-level system modeling the Ba$^+$ ions of Refs. [3, 4]. For two D systems cooperative effects in the same order of magnitude as for the V systems were found for ion distances of a few wavelengths of the laser-driven transition. For larger distances practically no effects were found, in agreement with the experiments [13] and with the results of Ref. [21]. In contrast, only negligible effects for arbitrary ion-distances were found for two of the four level-systems. Although this result contradicts the findings of Refs. [3, 4] a direct quantitative comparison with the experiments was not possible since explicit experimental data were only provided for three Ba$^+$ ions.

The aim of this paper is to narrow this gap by investigating three dipole-interacting three-level systems in a V configuration and in a D configuration (see Figs. 1 and 2), respectively, and to compare the results with those for two such systems. For three system this becomes much more complicated since one has to deal with $729 \times 729$ matrices, and in order to do this we use group theoretical methods to exploit the symmetry of the problem.

We calculate the transition rates between the different intensity periods for both systems. Cooperative effects are found to increase by a factor of 2 in the first order terms in the interaction parameter $C_3$ when compared to two of either systems. This results in transition rates up to 100% higher than the rates for independent systems. We also calculate the double and triple jump rates for both systems. Here the cooperative effects are even larger.

A full description of the Ba$^+$ experiment [3, 4] would require the treatment of three of the four-level systems of Ref. [20]. However, here we will restrict ourselves to the three-level systems, since this reduces the complexity of the calculation considerably. Also, the similarities
between the results for the D system and the four-level system pointed out in Ref. 20 seem to allow to draw conclusions on the cooperative behavior of three four-level systems from the results presented here. Namely, the increase of cooperative effects is not strong enough to yield significant effects for three four-level systems.

Section II deals with the main assumptions of the models. In Section III the methods for the calculation of the transition rates first for the V systems and afterwards for the D system are explained. In Section IV the results of the calculations are presented, namely the transition rates between the different intensity periods. Finally in Section V the double and triple jump rates are calculated from the transition rates. The results are discussed and compared with those of two three-level systems.

II. DIPOLE-INTERACTING THREE-LEVEL SYSTEMS

In the following we investigate three dipole-interacting three-level systems both in a V-type and in a D-type configuration as shown in Figs. 1 and 2. For the V system the Rabi frequencies \( \Omega_2 \) and \( \Omega_3 \) and the Einstein coefficient \( A_3 \) satisfy

\[
\Omega_3, A_3 \gg \Omega_2
\]

so that the single system can show macroscopic light and dark periods. The D system exhibits the same property if the condition

\[
\Omega_3, A_3 \gg A_1, A_2
\]

for the Einstein coefficients and the Rabi frequency is fulfilled. We assume the three atoms to be at fixed positions forming an equilateral triangle, in agreement with the experimental setups. Furthermore, for simplicity, the direction of the laser beams are assumed to be perpendicular to the plane of this triangle.

The Bloch equation can be written in the form

\[
\dot{\rho} = -\frac{i}{\hbar} \left[ H_{\text{cond}} \rho - \rho H_{\text{cond}}^\dagger \right] + \mathcal{R}(\rho),
\]

where the conditional Hamiltonian \( H_{\text{cond}} \) and the reset operation \( \mathcal{R}(\rho) \) for a general three-level system are given by

\[
H_{\text{cond}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\hbar}{2i} A_j S_{ij}^+ S_{ij}^- + \sum_{i=1}^{3} \sum_{j=2}^{3} \frac{h}{2} [\Omega_j S_{ij}^+ + \text{h.c.}]
\]

\[
+ \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\hbar}{2i} C_{kl}^{(ij)} \left( S_{kj}^+ S_{ij}^- + S_{ij}^+ S_{kj}^- \right)
\]

and

\[
\mathcal{R}(\rho) = \sum_{i=1}^{3} \sum_{j=1}^{3} A_j S_{ij}^- \rho S_{ij}^+
\]

\[
+ \sum_{k=1}^{3} \sum_{j=1}^{3} \text{Re} C_{kl}^{(ij)} \left( S_{kj}^- \rho S_{ij}^+ + S_{ij}^- \rho S_{kj}^+ \right),
\]

with

\[
S_{i1}^+ = |2⟩_{ii}\langle 1|, \quad S_{22}^+ = |3⟩_{ii}\langle 2|,
\]

\[
S_{i3}^+ = |3⟩_{ii}\langle 1|, \quad \text{and} \quad S_{ij}^- = S_{ij}^+\dagger.
\]

Here,

\[
C_{kl}^{(ij)} = \frac{3A_j}{2} e^{i\alpha_{kl}^{(ij)}} \left[ \frac{1}{\hbar \omega_{kl}^{(ij)}} - \frac{1}{\hbar \omega_{kl}^{(ij)}} (1 - 3 \cos^2 \theta_{kl}) \right]
\]

is the coupling parameter which describes the dipole-dipole interaction between atom \( k \) and atom \( l \) for the transition connected with the Einstein coefficient \( A_j \), with \( \theta_{kl} \) being the angle between the dipole moments and the line connecting the atoms. The dimensionless parameter \( \alpha_{kl}^{(ij)} = 2\pi r_{kl} / \lambda_i \) is given by the interatomic distance \( r_{kl} \) multiplied by the wave number \( 2\pi / \lambda_i \) of this transition. The detunings of the lasers are taken as zero. By setting either \( A_1 = A_2 = c_{kl}^{(1)} = c_{kl}^{(2)} = 0 \) or \( \Omega_2 = 0 \) in Eqs. 4 and 5, the Hamiltonians and reset states for the V systems and the D systems, respectively, are obtained. For simplicity it would be preferable to have the

\[
\begin{align*}
\text{strong laser, } \Omega_3 &\Rightarrow A_3 \\
\text{weak laser, } \Omega_2 &\Rightarrow A_1, A_2
\end{align*}
\]
same coupling parameters for each pair of atoms (i.e., $C_{ij}^{(j)} \equiv C_j$). This would be the case if the angle between the dipole moments and the line connecting two atoms were the same for all pairs of atoms. However, the arrangement of the atoms in the trap makes this impossible, as is illustrated in Fig. 3. The atoms form an equilateral triangle (i.e., $\tau_{ij} = r$) with the laser beams perpendicular to the plane of this triangle and the dipole moments aligned by a magnetic field in a direction in this plane. In this situation, the same value of the coupling constants can only be achieved for two of the possible pairs of atoms. However, in spite of this we will assume $C_{ij}^{(j)} \equiv C_j$ because this case leads to maximal cooperative effects and can be seen as a limiting case for all other possible configurations. The reset state can then be written as a sum of density matrices of pure states

$$
\mathcal{R}(\rho) = \sum_{j=1}^{3} \left\{ (A_j + 2\text{Re} \, C_j) R_1^{(j)} \rho R_1^{(j)\dagger} + (A_j - \text{Re} \, C_j) \left[ R_2^{(j)} \rho R_2^{(j)\dagger} + R_3^{(j)} \rho R_3^{(j)\dagger} \right] \right\}, \quad (8)
$$

with

$$
R_1^{(j)} = \frac{1}{\sqrt{3}} \left( S_{ij}^+ + S_{ij}^- + S_{ij}^z \right), \\
R_2^{(j)} = \frac{1}{\sqrt{6}} \left( 2S_{ij}^+ - S_{ij}^- - S_{ij}^z \right), \\
R_3^{(j)} = \frac{1}{\sqrt{2}} \left( S_{ij}^- - S_{ij}^z \right). \quad (9)
$$

In the case of two systems, it was convenient to use a Dicke basis, i.e., a basis consisting of the symmetric and antisymmetric linear combinations of the product states. Generally speaking, this means using a basis which is adapted with respect to the symmetry group $S_2$ of permutations of two atoms. The symmetric and antisymmetric states correspond to the irreducible representations of this group. For three three-level systems, we therefore use a basis that is adapted to the symmetry group $S_3$ of permutations of three particles. On the subspace spanned by the product states with all three atoms in different states the irreducible representations of the $S_3$ are the two one-dimensional representations mentioned above and another two equivalent two-dimensional representations. This leads to the states

$$
|s_{123}\rangle = \frac{1}{\sqrt{6}} \left( |1\rangle|2\rangle|3\rangle + |2\rangle|3\rangle|1\rangle + |3\rangle|1\rangle|2\rangle \right), \quad (10a)
$$

$$
|a_{123}\rangle = \frac{1}{\sqrt{6}} \left( |1\rangle|2\rangle|3\rangle + |2\rangle|3\rangle|1\rangle + |3\rangle|1\rangle|2\rangle \right) - |1\rangle|3\rangle|2\rangle - |2\rangle|1\rangle|3\rangle - |3\rangle|2\rangle|1\rangle \right), \quad (10b)
$$

$$
|b_{123}\rangle = \frac{1}{\sqrt{12}} \left( 2|1\rangle|2\rangle|3\rangle - |2\rangle|3\rangle|1\rangle - |3\rangle|1\rangle|2\rangle \right) + 2|1\rangle|3\rangle|2\rangle - |2\rangle|1\rangle|3\rangle - |3\rangle|2\rangle|1\rangle \right), \quad (10c)
$$

$$
|c_{123}\rangle = \frac{1}{2} \left( |2\rangle|3\rangle|1\rangle - |3\rangle|1\rangle|2\rangle \right) - (|2\rangle|1\rangle|3\rangle + |3\rangle|2\rangle|1\rangle \right), \quad (10d)
$$

$$
|d_{123}\rangle = \frac{1}{\sqrt{12}} \left( 2|1\rangle|2\rangle|3\rangle - |2\rangle|3\rangle|1\rangle - |3\rangle|1\rangle|2\rangle \right) - 2|1\rangle|3\rangle|2\rangle + |2\rangle|1\rangle|3\rangle + |3\rangle|2\rangle|1\rangle \right), \quad (10e)
$$

$$
|e_{123}\rangle = \frac{1}{2} \left( |2\rangle|3\rangle|1\rangle - |3\rangle|1\rangle|2\rangle \right) + (|2\rangle|1\rangle|3\rangle - |3\rangle|2\rangle|1\rangle \right) \quad (10f)
$$

in the case where all three atoms are in different states. For the remaining states one then easily gets for $i, j = 1, 2, 3, i \neq j$,

$$
|s_{ijj}\rangle = \frac{1}{\sqrt{3}} \left( |i\rangle|j\rangle|j\rangle + |j\rangle|i\rangle|j\rangle + |j\rangle|j\rangle|i\rangle \right), \quad (11a)
$$

$$
|b_{ijj}\rangle = \frac{1}{\sqrt{6}} \left( 2|i\rangle|j\rangle|j\rangle - |j\rangle|i\rangle|j\rangle - |j\rangle|j\rangle|i\rangle \right), \quad (11b)
$$

$$
|c_{ijj}\rangle = \frac{1}{\sqrt{2}} \left( |j\rangle|i\rangle|j\rangle - |j\rangle|i\rangle|j\rangle \right) \quad (11c)
$$

if two atoms are in the same state and

$$
|g_{ij}\rangle = |1\rangle|1\rangle|1\rangle, \quad |e_{22}\rangle = |2\rangle|2\rangle|2\rangle, \quad |e_{33}\rangle = |3\rangle|3\rangle|3\rangle \quad (12)
$$

if all three atoms are in the same state.

### III. TRANSITION RATES

For the calculation of the transition rates, we carry over the methods that have already been used for the description of two dipole-interacting V systems and D systems, respectively [13, 20].

For both types of systems, the configuration decouples into four independent subspaces if one neglects the small parameters (i.e., $\Omega_2 = 0$ for the V systems and $A_1 =$...
in analogy to the case of two of either systems. In a
subspace and then the rate of build-up of population in
be calculated by using a density matrix in one particular
t
short in comparison to the length of the intensity periods,

\[ A_3^{-1} \Omega_3^{-1} \ll \Delta t \ll \Omega_2^{-1} \] (V system),

\[ A_3^{-1} \Omega_3^{-1} \ll \Delta t \ll A_1^{-1}, A_2^{-1} \] (D system) . (14)

For the calculation the Bloch equation is written in a
Liouvillean form

\[ \dot{\rho} = \mathcal{L}_0 \rho = \{ \mathcal{L}_0(A_3, C_3, \Omega_3) + \mathcal{L}_1 \} \rho , \] (15)

where \( \mathcal{L}_1 \) serves as the perturbation depending on \( \Omega_3 \) or
\( A_1, A_2, C_1, \) and \( C_2, \) respectively. We then get

\[ \rho(t_0 + \Delta t; \rho_{ss,i}) = \rho_{ss,i} + \int_{t_0}^{t_0 + \Delta t} \frac{d\tau}{\tau} \mathcal{L}_0 \mathcal{L}_1 \rho_{ss,i} , \] (16)

where \( \rho_{ss,i} \) is the quasisteady state in subsystem \( \mathcal{S}_i \). As
a Liouvillean of Bloch equations, \( \mathcal{L}_0 \) has an eigenvalue 0
corresponding to the quasisteady states. The other
eigenvalues have negative real parts of the order of \( \Omega_3 \) and \( A_3 \).
While \( \mathcal{L}_1 \rho_{ss,i} \) is a superposition of just the eigenstates for
nonzero eigenvalues of \( \mathcal{L}_0 \) in the case of three V systems
this is not true for three D systems, which makes it necessary
to discuss the two cases separately.

A. Three V systems

For the V systems, \( \mathcal{L}_1 \rho_{ss,i} \) consists only of coher-
ences between the subspace \( \mathcal{S}_i \) and the neighbouring sub-
spaces, since \( \mathcal{L}_1 \) describes the coupling due to the weak
laser (with Rabi frequency \( \Omega_2 \)) in this case. The zero-
eigenvalue subspace of \( \mathcal{L}_0 \), on the other hand, is spanned
by the quasisteady states \( \rho_{ss,i} \). Therefore, \( \mathcal{L}_1 \rho_{ss,i} \) has no
components in the zero eigenvalue subspace of \( \mathcal{L}_0 \) in the
case of V systems. The other eigenvalues all have negative
real parts of the order of \( A_3 \) and \( \Omega_3 \). Therefore the integrand in Eq. (16) is rapidly damped which allows us
to extend the upper integration limit to infinity. This

\[ \rho(t_0 + \Delta t; \rho_{ss,i}) = \rho_{ss,i} + (\epsilon - \mathcal{L}_0)^{-1} \mathcal{L}_1 \rho_{ss,i} , \] (17)

independent of \( \Delta t \).

From the Bloch equations we get the exact relations

\[ \frac{d}{dt} \langle x_i | x_i \rangle = \Omega_2 \text{Im} \left[ 2 \langle s_{112} | \rho | s_{122} \rangle - \langle b_{112} | \rho b_{122} \rangle - \langle c_{112} | \rho c_{122} \rangle - \sqrt{3} \langle s_{122} | \rho | e_{2} \rangle + \sqrt{3} \langle s_{223} | \rho | s_{223} \rangle \right] 
- \frac{1}{\sqrt{2}} \left( \langle b_{223} | \rho b_{223} \rangle + \langle c_{223} | \rho c_{223} \rangle \right) + \sqrt{3} \left( \langle d_{223} | \rho d_{223} \rangle - \langle e_{223} | \rho e_{223} \rangle \right) \right] \frac{d}{dt} \langle e_2 | e_2 \rangle , \] (18a)

\[ \frac{d}{dt} \sum_{x_i \in \mathcal{S}_1} \langle x_i | x_i \rangle = \Omega_2 \text{Im} \left[ 2 \langle s_{112} | \rho | s_{122} \rangle - \langle b_{112} | \rho b_{122} \rangle - \langle c_{112} | \rho c_{122} \rangle - \sqrt{3} \langle s_{122} | \rho | e_{2} \rangle + \sqrt{3} \langle s_{223} | \rho | s_{223} \rangle \right] 
- \frac{1}{\sqrt{2}} \left( \langle b_{223} | \rho b_{223} \rangle + \langle c_{223} | \rho c_{223} \rangle \right) + \sqrt{3} \left( \langle d_{223} | \rho d_{223} \rangle - \langle e_{223} | \rho e_{223} \rangle \right) \right] \frac{d}{dt} \sum_{x_i \in \mathcal{S}_1} \langle x_i | x_i \rangle , \] (18b)

\[ \frac{d}{dt} \sum_{x_i \in \mathcal{S}_2} \langle x_i | x_i \rangle = \Omega_2 \text{Im} \left[ \frac{1}{\sqrt{2}} \left( \langle b_{123} | \rho b_{123} \rangle + \langle c_{123} | \rho c_{123} \rangle \right) + \sqrt{3} \left( \langle d_{123} | \rho d_{123} \rangle - \langle e_{123} | \rho e_{123} \rangle \right) \right] \frac{d}{dt} \sum_{x_i \in \mathcal{S}_2} \langle x_i | x_i \rangle , \] (18c)

\[ \frac{d}{dt} \sum_{x_i \in \mathcal{S}_3} \langle x_i | x_i \rangle = \Omega_2 \text{Im} \left[ \frac{1}{\sqrt{2}} \left( \langle b_{113} | \rho b_{113} \rangle + \langle c_{113} | \rho c_{113} \rangle \right) + \sqrt{3} \left( \langle b_{311} | \rho d_{123} \rangle - \langle c_{113} | \rho c_{123} \rangle - \sqrt{3} \langle g | \rho s_{211} \rangle \right) \right] \frac{d}{dt} \sum_{x_i \in \mathcal{S}_3} \langle x_i | x_i \rangle . \] (18d)
Together with Eq. (17) this allows us to calculate the transition rates as
\[ p_{ij} = \frac{d}{dt} \sum_{x_k \in S_j} \langle x_k | \rho | x_k \rangle |_{\rho = \rho(t_0 + \Delta t; \rho_{ss,i})} , \]  
(19)
Note that \( p_{ij} = 0 \) for \( |i - j| \geq 2 \) so that no direct, i.e., instantaneous, double jumps occur.

**B. Three D systems**

In the case of D systems, \( L_1 \) describes spontaneous emission due to the Einstein coefficients \( A_1 \) and \( A_2 \). Therefore \( L_1 \rho_{ss,i} \) consists of density matrix elements \( \langle x_i | \rho | x_j \rangle \) where both states \( |x_i\rangle \) and \( |x_j\rangle \) lie in the same subspace \( S_i \). It is thus a superposition of eigenstates of \( L_0 \) with zero as well as nonzero eigenvalues. We write
\[ L_1 \rho_{ss,i} = \sum_{j=0}^{3} \alpha_{ij} \rho_{ss,j} + \hat{\rho} , \]  
(20)
where \( \hat{\rho} \) contains the contributions from the eigenstates for nonzero eigenvalues of \( L_0 \). The coefficients \( \alpha_{ij} \) are calculated by means of the dual eigenstates \( \rho_{ss,j}^{\dagger} \) \cite{21}.
\[ \alpha_{ij} = \text{Tr}(\rho_{ss,j}^{\dagger} L_1 \rho_{ss,i}) . \]  
(21)
Inserting Eq. (20) into Eq. (16) one obtains
\[ \rho(t_0 + \Delta t) = \rho_{ss,i} + \sum_{j=0}^{3} \alpha_{ij} \rho_{ss,j} \Delta t + (\epsilon - L_0)^{-1} \hat{\rho} . \]  
(22)
The last term is much smaller than the preceding term and can be neglected \cite{21}. The coefficients \( \alpha_{ij} \) can then be interpreted as the transition rates between the subspaces \( S_i \) and \( S_j \),
\[ p_{ij} = \alpha_{ij} . \]  
(23)

**C. Group theory**

For the calculation of the transition rates for both V systems and D systems it is necessary to calculate the quasisteady states \( \rho_{ss,i} \), i.e., to solve the linear equation
\[ L_0 \rho_{ss} = 0 . \]  
(24)
In addition, for V systems the first order term
\[ \rho_1^{(1)} = (\epsilon - L_0)^{-1} L_1 \rho_{ss,i} \]
of Eq. (17) must be calculated, which was done by solving
\[ L_0 \rho_1^{(1)} = L_1 \rho_{ss,i} . \]  
(25)
Eqs. (24) and (25) are linear equations for the 729 matrix elements of \( \rho_{ss,i} \) and \( \rho_1^{(1)} \), respectively. Luckily there are two different properties of \( L_0 \) that make it possible to restrict these equations to smaller subspaces, which reduces the calculation effort considerably. First, \( L_0 \) is independent of the small parameters \( (A_1, A_2 \text{ or } \Omega_2) \), which means that there is no coupling between the four subspaces of Eq. (13). Thus there exist 16 subspaces \( \mathcal{R}_{i,j} \), each consisting of the density matrix elements
\[ \langle x_i | \rho | y_j \rangle \text{ with } |x_i\rangle \in S_i \text{ and } |y_j\rangle \in S_j , \]  
(26)
respectively, which are invariant with respect to \( L_0 \). In addition the conditional Hamiltonian \( H_{\text{cond}} \) and the reset state \( \mathcal{R}(\rho) \) and therefore also \( L_0 \) are invariant under the exchange of atoms, as can be seen from Eqs. (11) and (14). Hence subspaces which consist of all density matrix elements which belong to a particular irreducible representation of \( S_3 \) are also invariant with respect to \( L_0 \). Since the density matrix elements form a representation of \( S_3 \) which is a tensor product of twice the representation spanned by the Dicke basis of Eq. (10), the new irreducible representations are easily found. The density matrix elements
\[ |s_a\rangle \langle s_b| , \quad |a_\alpha\rangle \langle a_\beta| , \quad \frac{1}{2} (|b_a\rangle \langle b_\beta| + |c_a\rangle \langle c_\beta|) , \quad \frac{1}{2} (|d_a\rangle \langle d_\beta| + |e_a\rangle \langle e_\beta|) , \]  
(27)
belong to the symmetric representation, the elements
\[ |s_a\rangle \langle a_\beta| , \quad |a_\alpha\rangle \langle s_b| , \quad \frac{1}{2} (|b_a\rangle \langle c_\beta| - |c_a\rangle \langle b_\beta|) , \quad \frac{1}{2} (|d_a\rangle \langle e_\beta| - |e_a\rangle \langle d_\beta|) , \]  
(28)
belong to the antisymmetric representation, and the remaining 24 possible linear combinations form two-dimensional representations. Here \( \alpha \) and \( \beta \) are one of the subscripts of the Dicke states. By transforming the Liouvillian \( L_0 \) into this new basis each of the 16 invariant subspaces \( \mathcal{R}_{i,j} \) is in itself decomposed into three invariant subspaces connected to the elements belonging to the symmetric, antisymmetric, and two-dimensional representations, respectively. For the calculation of both the quasisteady states \( \rho_{ss,i} \) and the transition rates for the V systems only the symmetric subspaces are needed. For the latter this can be seen from Eq. (18). With these two simplifications the dimension of the linear system of equations needed for the calculation reduces considerably (namely to a maximum of 20 for the calculation of \( p_{23} \) and \( p_{32} \)).

**IV. RESULTS**

The transition rates for the V systems can now be calculated according to Eqs. (19), (16), and (18). The result
are in a light period. Therefore the rates
to first order in \( p \) note that the single systems interact via
1, 2, or 3. For the first order terms it is important to
For distances of about a wavelength, 
\( C \) systems. This is not surprising as the quasisteady states
the same behavior as explained above for the three \( V \) systems
were identical and as the \( D \) systems also only interact via
3 \( p \). This surprising property is due to the simplicity of the quasisteady state \( \rho_{ss,3} \), namely all diagonal elements
of this state have the same first order dependence. Fig. 4 shows the transition rate \( p_{23} \) for three \( V \) systems to first and to second order in \( C_3 \). The first order rate becomes negative for distances of about one-half to three quarters of a wavelength of the strong transition. By looking at the second order rate one can see that is an artefact of the approximation. The rate with the dipole interaction included shows deviations of up to 100% from the rate for noninteracting atoms for distances of somewhat more than a wavelength \( \lambda_3 \).

By use of Eqs. (24), (21) and (26) the transition rates for three dipole interacting \( D \) systems were also calculated, with the result

\[
\begin{align*}
p_{01} &= 3A_1, \quad p_{12} = 2A_1, \quad p_{23} = A_1, \quad (30a) \\
p_{10} &= \frac{A_2\Omega_3^2}{\Omega_3^2 + 2\Omega_3^2} \quad (30b)
\end{align*}
\]

\[
\begin{align*}
p_{21} &= 2 \frac{A_2\Omega_3^2(A_3^2 + 2\Omega_3^2)}{(A_3^2 + 2\Omega_3^2)^2 + A_3^2(|C_3|^2 + 2A_3\text{Re}C_3)} = 2 \frac{A_2\Omega_3^2}{A_3^2 + 2\Omega_3^2} \left[ 1 - 2\text{Re}C_3 \frac{A_3^4}{(A_3^2 + 2\Omega_3^2)^2} \right] + \mathcal{O}(C_3^4), \quad (30c) \\
p_{32} &= \frac{3A_2\Omega_3^2}{(A_3^2 + 2\Omega_3^2)^2 + A_3^2(|C_3|^2 + 2A_3\text{Re}C_3)} \\
&= \frac{3A_2\Omega_3^2}{A_3^2 + 2\Omega_3^2} \left[ 1 - 4\text{Re}C_3 \frac{A_3^4}{(A_3^2 + 2\Omega_3^2)^2} \right] + \mathcal{O}(C_3^4). \quad (30d)
\end{align*}
\]

Compared to two \( D \) systems the transition rates show the same behavior as explained above for the three \( V \) systems. This is not surprising as the quasisteady states are identical and as the \( D \) systems also only interact via \( C_3 \) when they are in a light period. Fig. 4 shows the exact transition rate \( p_{32} \) compared to the interaction free case. For distances of about a wavelength, \( p_{32} \) deviates up to 75% from the rate without interaction. The first peak at about 0.7 wavelengths even reaches a maximum of seven times the rate for independent atoms. For such small distances, however, one would have to check the validity of the model (namely, that in a particular intensity period most of the population is in a specific subspace). Also one must keep in mind that all the experiments cited here were performed at greater ion distances.

V. DOUBLE AND TRIPLE JUMP RATE

The physical quantity investigated in the experiments of Refs. 8,11,12,13 is the double jump rate. This is the rate at which jumps between periods of intensities that differ by twice the intensity of a single system occur within a small time interval. In Ref. 13 the double jump rate has been expressed in terms of the transition rates \( p_{ij} \) for two dipole-interacting \( V \) systems. The same will be done here for three systems. As one can calculate directly from Eqs. (19) and (26) there are no direct double jumps (i.e., \( p_{ij} = 0 \) for \( |i - j| > 1 \)). A double jump is therefore defined as two successive jumps in the same direction which occur within a time which is smaller than a time window \( T_m \) so that they cannot be resolved. As there are four periods of different intensity in the fluoro-
systems plotted versus the interatomic distance $r$ in units of the wavelength $\lambda_3$ of the strong transition. Solid line: $p_{23}$ up to second order in $C_3$. Dashed line: first order. Dotted line: independent systems. Parameter values are $A_3 = 2 \cdot 10^8 \text{s}^{-1}$, $\Omega_3 = 5 \cdot 10^7 \text{s}^{-1}$, and $\Omega_2 = 10^8 \text{s}^{-1}$.

FIG. 4: Transition rate $p_{32}$ for three dipole-interacting V systems. Dashed line: independent systems. Time window $T_m = 10^{-3} \text{s}$. Other parameter values as in Fig. 5.

The branching ratio for the following period to be of double intensity is $p_{12}/(p_{10} + p_{12})$. With the mean number of intensity periods $I_i$ per unit time denoted by $n_i$ the rate $n_{02}^{D_J}$ is given by

$$n_{02}^{D_J} = n_0 \frac{p_{12}}{p_{10} + p_{12}} \left( 1 - e^{-(p_{10} + p_{12})T_m} \right). \quad (32)$$

Analogously one finds

$$n_{31}^{D_J} = n_3 \frac{p_{21}}{p_{21} + p_{23}} \left( 1 - e^{-(p_{21} + p_{23})T_m} \right). \quad (33)$$

The remaining two rates are a little bit more complicated as the periods of intensity $I_1$ and $I_2$ can be followed by a period with either higher or lower intensity. The rates $n_{20}^{D_J}$ and $n_{13}^{D_J}$ have thus to be supplemented with the branching ratios $p_{21}/(p_{21} + p_{23})$ and $p_{12}/(p_{10} + p_{12})$, respectively, yielding

$$n_{13}^{D_J} = n_1 \frac{p_{12}}{p_{10} + p_{12}} \frac{p_{23}}{p_{21} + p_{23}} \left( 1 - e^{-(p_{21} + p_{23})T_m} \right)$$

and

$$n_{20}^{D_J} = n_2 \frac{p_{21}}{p_{21} + p_{23}} \frac{p_{10}}{p_{10} + p_{12}} \left( 1 - e^{-(p_{10} + p_{12})T_m} \right). \quad (35)$$

FIG. 5: Transition rate $p_{32}$ for three dipole-interacting D systems. Dashed line: independent systems. Parameter values are $A_1 = 1 \text{s}^{-1}$, $A_2 = 1 \text{s}^{-1}$, $A_3 = 2 \cdot 10^8 \text{s}^{-1}$, and $\Omega_3 = 10^7 \text{s}^{-1}$.

FIG. 6: Double jump rate $n_{02}$ for three dipole-interacting V systems. Solid line: $n_{02}$ up to second order in $C_3$. Dotted line: independent systems. Time window $T_m = 10^{-3} \text{s}$. Other parameter values as in Fig. 4.

FIG. 7: Double jump rate $n_{02}$ for three dipole-interacting D systems. Dashed line: independent systems. Time window $T_m = 5 \cdot 10^{-3} \text{s}$. Other parameter values as in Fig. 4.
Using the relations
\[ n_0 = \frac{p_{10}}{p_{10} + p_{12}} n_1, \quad n_3 = \frac{p_{23}}{p_{21} + p_{23}} n_2 \] (36a)
and
\[ n_2 = \frac{p_{12}}{p_{10} + p_{12}} n_1 + n_3, \quad n_1 = n_0 + \frac{p_{21}}{p_{21} + p_{23}} n_2 \] (36b)
the double jump rates can be simplified to
\[ n_{DJ}^{02} = n_{DJ}^{20} = n_1 \frac{p_{10} p_{12}}{(p_{10} + p_{12})^2} \left( 1 - e^{-(p_{10} + p_{12}) T_m} \right) \] (37)
and
\[ n_{DJ}^{13} = n_{DJ}^{31} = n_1 \frac{p_{12} p_{23}}{(p_{21} + p_{23})(p_{10} + p_{12})} \left( 1 - e^{-(p_{21} + p_{23}) T_m} \right). \] (38)

We denote the mean durations of the intensity periods by \( T_i \) and note that
\[ T_0 = \frac{1}{p_{01}}, \quad T_1 = \frac{1}{p_{10} + p_{12}}, \quad T_2 = \frac{1}{p_{21} + p_{23}}, \quad T_3 = \frac{1}{p_{32}}. \] (39)

In addition they fulfill
\[ \sum_{i=0}^{3} n_i T_i = 1. \] (40)

The averaging window \( T_m \) is much smaller than the mean durations of the intensity periods. Therefore the exponential can be expanded and with Eq. (44) one gets
\[ n_{DJ} = 2 n_1 \frac{p_{12} (p_{10} + p_{23})}{p_{10} + p_{12}} T_m. \] (41)

Using Eqs. (39), (40), and (41) we finally obtain
\[ n_{DJ} = 2 \frac{p_{01} p_{21} p_{32}}{p_{21} p_{32} (p_{01} + p_{10}) + p_{01} p_{12} (p_{23} + p_{32})} T_m \] (42)
as the double jump rate for three of either three-level systems. A similar calculation yields for the triple jump rate
\[ n_{TJ} = 2 \frac{p_{01} p_{10} p_{12} p_{21} p_{23} p_{32}}{p_{21} p_{32} (p_{01} + p_{10}) + p_{01} p_{12} (p_{23} + p_{32})} T_m^2. \] (43)

Note that the defining time window \( T_m \) enters quadratically in this case. Figs. 6 and 7 show plots of \( n_{DJ} \) for the V systems and the D systems, respectively, whereas Figs. 8 and 9 show plots of the triple jump rate \( n_{TJ} \) for both systems. For the D systems the exact values for the \( p_{ij} \) are used whereas for the V systems only the expanded expressions up to second order in \( C_3 \) are used since \( p_{23} \) and \( p_{23} \) could not be calculated exactly for the V systems. For the V systems there are cooperative effects of up to 110% for the double jump rate \( n_{DJ} \) and 170% for the triple jump rate \( n_{TJ} \) for distances of somewhat more than a wavelength of the strong transition. For the same distance range the D system shows cooperative effects of up to 150% for both \( n_{DJ} \) and \( n_{TJ} \). The first peak at three quarters of a wavelength reaches 16 times the value for independent systems for both rates. For distances of about 10 wavelengths cooperative effects of 15% are still present for both systems. In the case of the D system, which models the level configuration of the \( \text{Hg}^+ \) ions used in the experiments of Refs. 13, 26, large cooperative effects only appear if the Rabi frequency \( \Omega_3 \) is smaller than the Einstein coefficient \( A_3 \). So, for the experimental parameters (i.e., \( \Omega_3 > A_3 \) and \( r/\lambda_3 \approx 15 \)) the effects are negligible, in agreement with the experimental results.

VI. CONCLUSIONS

We have investigated the effect of the dipole-dipole interaction on three three-level systems showing macroscopic light and dark periods in their fluorescence. This was done for the V and the D configuration, respectively. The latter models the effective level configuration of the \( \text{Hg}^+ \) ions in the experiments of Refs. 13, 26. We have ex-
plicitly calculated the transition rates between the different intensity periods for both configurations. In addition, the double and triple jump rates have been derived from these transition rates. Both systems show the same first order dependency on the coupling parameter $C_3$, leading to an enhancement in the cooperative effects by a factor of 2 for the transition rate $p_{32}$. This leads to cooperative effects of about 100\% compared to the value for independent systems for interatomic distances of somewhat more than a wavelength of the strong transition. For the double and triple jump rates even larger cooperative effects can be seen. For three D systems the first peak at about three quarters of a wavelength is seven times higher for $p_{32}$ and 16 times higher for $n_{DJ}$ and $n_{TJ}$ than for independent atoms.

Although we did not treat the four-level system of Ref. [20] here, which models the Ba$^+$ ions of Refs. [3, 4], it is still possible to arrive at some conclusions on this experiment from our results for the three level systems. As was pointed out in Ref. [20], the results for two D systems and two four-level systems are very similar, in particular in their first order term in $C_3$. It is therefore very likely that the cooperative effects for three four-level systems are also only enhanced by a factor of about 2, and since the effects for two four-level systems were already negligibly small one can expect a similar behavior also for three of such systems.

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