On the Schwarzschild field

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Abstract
General relativity is a non-linear theory with the distinguishing feature that gravitational field energy also acts as gravitational charge density. In the well-known Schwarzschild solution describing field of an isolated massive body at rest, the scalar function $\phi$ characterising the field acts as a gravitational potential as well as it curves space part of spacetime. We demonstrate explicitly that it is the latter property that accounts for the non-linear (gravity as its own source) aspect which is not explicit in usual derivations. It is worth noting that the Einstein vacuum equations ultimately reduce to the Laplace equation and its first integral which fixes zero of $\phi$ at infinity. Thus the Schwarzschild field alongwith its asymptotic flat character is completely determined without application of any boundary condition by the field equations themselves. That means non-zero constant value of $\phi$ will have non-vacuous effect. It in fact produces stresses exactly of the form required to represent a global monopole. By retaining freedom of choosing zero of $\phi$, which will break asymptotic flatness, we can obtain the Schwarzschild black hole with global monopole charge. It is the non-linear aspect responsible for “curving” space, which has no Newtonian analogue, survives even when $\phi$ is constant but not zero.

PACS numbers: 0420, 9880.

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1. Introduction

In the Newtonian theory (NT), gravitational field is entirely determined by the equation $\nabla^2 \phi = 4\pi \rho$ and motion of particle in the field is given by $\ddot{r} = -\nabla \phi$. Here we have set $G = 1$, and $\rho$ is the matter density. In absence of matter $\rho = 0$, we have the Laplace equation, which for a radially symmetric field has the well-known solution $\phi = k - M/r$. It describes gravitational field of a body of mass $M$ at rest at the origin of the coordinate system. The arbitrary constant $k$ can be chosen freely to set the zero of the potential $\phi$. Normally one sets it to zero to have it vanish at infinity. The main criticism of the above equation is, one it does not prescribe propagation of field in space and two it cannot take into account the distinguishing feature of gravity, gravitational field being its own source. To incorporate these aspects we will have to go to general relativity (GR).

The latter property would demand that in matter free region the equation should be modified to

$$\nabla^2 \phi = \frac{1}{2} (\nabla \phi)^2$$  \hspace{1cm} (1.1)



Let us consider the field of a mass point in GR to see whether do we really solve the above equation? It turns out that we still solve the Laplace equation rather than eqn.(1.1). Then how does GR take care of this aspect? Eqn.(1.1) would be referred to flat spacetime while in GR spacetime is curved and it is the curvature of spacetime that describes gravitational field. That opens up an interesting possibility that one can retain the Laplace equation and non-linear aspect of the field (being its own source) is taken care by curvature of space. It is remarkable that this is what exactly happens.

One of the main aims of this paper is to demonstrate this aspect explicitly and clearly, which is generally neither duly emphasized nor appreciated in the usual derivation of the Schwarzschild solution. Secondly GR field equations for vacuum ultimately reduce to two equations of which one is the good old Laplace equation
and the other is its first integral, which determines the free parameter $k = 0$ [1]. Thus asymptotic flatness of the Schwarzschild field is not due to a proper choice of boundary condition but instead is entirely determined by the field equations themselves. We have no choice to make $\phi$ zero anywhere else than infinity. This is what is reflected in the fact that the Schwarzschild solution is the unique spherically symmetric vacuum solution. This aspect though known [2,3] but not well-known enough and certainly not without surprise for many. It is well-known that the Schwarzschild spacetime is fully written in terms of the Newtonian potential. In both the theories there is only one scalar function that describes the field and hence it should arouse some surprise that GR equations determine this scalar function absolutely while NT offers freedom of additive constant to fix its value at the boundary. It is to be noted that GR does not offer this freedom.

In the next Section we shall first derive the Schwarzschild solution explicitly demonstrating, how field equations determine the field entirely without reference to any boundary condition and how non-linearity is taken care by “curving” space part of the metric. It would be shown that this part of curvature survives even when the scalar function determining the field is constant but not zero. In Section 3 we shall argue that the basic character of the Schwarzschild field will not be disturbed significantly even if we retain the free parameter $k$. As is clear that vacuum equations will then no longer remain satisfied. The metric will neither be asymptotically flat. In principle the Schwarzschild solution should not be asymptotically flat to permit presence of other matter-energy in the Universe. Realistically we should break asymptotic flatness. This is the only and perhaps the most harmless way of rendering asymptotic non-flat character to the Schwarzschild field. By breaking asymptotic flatness we give a generalization of the Schwarzschild solution which is equivalent to adding a global monopole to it [1,4]. Its effect on particle orbits and the Hawking radiation has been studied recently [5]. Finally we conclude with a discussion raising some points of principle and concept.
2. Schwarzschild’s solution

First we shall derive the Schwarzschild spacetime explicitly demonstrating how nonlinear aspect of gravity goes into curving space and there is no scope for application of any boundary condition. Asymptotic flatness of the solution is implied by the equations themselves. We shall carry out the analysis in curvature, Kerr-Schild and isotropic coordinates so as to indicate that it is not a coordinate dependent result.

2.1 The curvature coordinates:

Let us begin with

\[ ds^2 = B dt^2 - A dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (2.1)

where \( A \) and \( B \) are functions of \( r \) and \( t \). Note that the radial coordinate \( r \) defines the area of 2-sphere and hence it has proper physical meaning. The Ricci tensor for it reads as follows:

\[ R^0_0 = -\frac{1}{2AB} \left[ \nabla^2 B - \frac{B'}{2} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \ddot{A} + \frac{\dot{A}^2}{2A} + \frac{\dot{A} \dot{B}}{2B} \right] \] (2.2)

\[ R^1_1 = R^0_0 + \frac{1}{Ar} \left( \frac{A'}{A} + \frac{B'}{B} \right) \] (2.3)

\[ R^2_2 = R^3_3 = \frac{1}{r^2} \left[ 1 - \frac{1}{A} + \frac{r}{2A} \left( \frac{A'}{A} - \frac{B'}{B} \right) \right] \] (2.4)

\[ R_{01} = -\frac{\dot{A}}{Ar} \] (2.5)

where \( \nabla^2 \) is the Euclidian Laplacian, and an overhead prime and dot denote derivative w.r.t. \( r \) and \( t \).

From physical point of view it is obvious that field of a static body cannot depend upon \( t \) which is ensured by \( R_{01} = 0 \) in (2.5) which implies \( A = A(r) \). It is well-known that geodesic equation with \( B = 1 + 2\phi \) and \( A = 1 \) (space part of the metric being flat) incorporates the Newtonian equation of motion \( \ddot{r} = -\nabla \phi \) for slow motion and
weak field. Then $R_0^0 = 0$ will take the form (1.1), taking into account the red-shift factor for the field energy density. This indicates clearly the appearance of field energy density as gravitational charge density. If we now “curve” the space part by introducing $A$, then demanding $R_{01} = 0, R_0^0 = R_1^1$ from (2.2) and (2.3) that give

$$
\dot{A} = 0, \frac{A'}{A} + \frac{B'}{B} = 0 \Rightarrow AB = f(t) = 1. \tag{2.6}
$$

Note that this reduces $R_0^0 = 0$ to the Laplace equation indicating non-linear aspect is taken care of by curvature of space. Introduction of $A$ means curving space part of the metric, which cancels out contribution of field energy density from gravitational charge density leading to the Laplace equation again. It is this point which we wish to emphasise. Else one does not see how non-linear aspect of gravity is accounted for in the Schwarzschild solution.

Here $f(t)$ is absorbed by redefining $t$ which neither amounts to any loss of generality nor of invoking any boundary condition [2,3]. In the usual derivation it is here the asymptotic flatness is invoked to set $f(t) = 1$. Even if $f(t)$ is retained the field will be asymptotically flat and the solution with $f \neq 1$ is physically indistinguishable from the one with $f = 1$. So we apply no boundary condition to set $f = 1$. It only identifies the coordinate $t$ with that of the asymptotic observer. Hence $B = A^{-1} = 1 + 2\phi(r)$, say (Birkhoff’s theorem). Using this in (2.2) and (2.4) we are finally lead to the following two linear equations [1],

$$
R_0^0 = -\nabla^2 \phi = -\frac{1}{r} (r\phi)'' = 0 \tag{2.7}
$$

$$
R_2^2 = -\frac{2}{r^2} (r\phi)' = 0. \tag{2.8}
$$

Thus we have come back to the good old Laplace equation. Note that (2.8) is the first integral of (2.7) and hence we just need to integrate (2.8) to get to the Schwarzschild solution. As envisioned earlier the curvature of space (i.e. $A \neq 1$ and using (2.6) in (2.2)) just exactly cancels out the field energy density term in (2.2)
and synthesises its effect in the geometry of space. The former equation admits the well-known general solution

\[ \phi = k - \frac{M}{r} \]  

(2.9)

while the latter determines \( k = 0 \). Thus we obtain the Schwarzschild solution.

The important point to note is that we had no freedom to use any boundary condition, in particular asymptotic flat behaviour of the solution is implied by the field equations themselves. That is the solution is fully determined by the theory leaving no scope for boundary conditions. Secondly it is very insightful to see how non-linear aspect is incorporated by “curving” space. These points are not brought forth emphatically in the usual derivations.

Note that eqns. (2.7) and (2.8) are the exact Einstein’s equations and \( \phi \) satisfies the Laplace equation (which is \( R_0^0 = 0 \)) without any approximation. Since it is \( R_0^0 = 0 \) that is supposed to be the analogue of the Newtonian Laplace equation which defines gravitational potential, hence \( \phi \) in (2.7) would define relativistic gravitational potential (an analogue of the Newtonian potential taken over to GR) exactly at least in the coordinates used without any approximation of weak field and slow motion. The first point we wish to make is that unlike the Newtonian theory, GR determines scalar function \( \phi \) describing the field absolutely. That means \( \phi = k \neq 0 \) is not a solution of Einstein’s equation \( R_{ik} = 0 \) as is clear from (2.8) and it produces non-zero curvature. Thus constant \( \phi \) attains non-trivial physical meaning. This is a very curious and unique feature of GR.

At this stage it may be instructive to have a look at the Riemann curvatures of the metric (2.1) with \( B = A^{-1} = 1 + 2\phi(r) \),

\[ R_{01}^{01} = \phi'', \quad R_{02}^{02} = R_{12}^{12} = \frac{\phi'}{r}, \quad R_{23}^{23} = \frac{2\phi}{r^2}. \]  

(2.10)

It is clear that \( \phi = \text{const.} \) does not make \( R_{2323} = 0 \). This is because \( \phi \) appears as it is in curvature imparting physical meaning to itself. This is in contrast to usual
concept of potential in classical physics. It is purely a relativistic feature arising from non-linearity of the theory.

Recall that we have argued above that $B$ was responsible for the Newtonian acceleration in the geodesic equation while $A$ brought in the non-linear effect, gravity as its own source. When $\phi$ is constant, both $A$ and $B$ are constants. Clearly gravitational acceleration vanishes and $B$ can be absorbed by redefining $t$. But $A$ which represents non-linear aspect cannot be got rid of unless $\phi = 0$, and it persists even when $\phi$ is constant. Since the Newtonian theory was free of the non-linear aspect and space was flat, that is why constant potential was physically inert. But it is not so in GR for relativistic potential $\phi$, an analogue of the Newtonian potential, produces curvature in space. This is the crucial point that in GR, $\phi$ not only determines field and motion of test particles but also curves space. The latter property is sensitive to the absolute value of $\phi$. This is how $\phi$ attains physical meaning. In GR the field equations not only regulate the behaviour of the field but they also determine the spacetime itself. Hence they provide a more constrained system than that in NT.

It is well-known that metric potentials $A$ and $B$ contribute equally in the Schwarzschild solution for bending of light ray, that is half the contribution comes from gravitational potential, $B$ and the half from curvature of space through $A$, caused by non-linear aspect of the field. To gain some more insight into their working let us consider their effects separately.

(i) Let $B = 1 - 2M/r$ and $A = 1$, which will generate the stresses,

$$T_0^0 = 0, \quad 8\pi T_1^1 = -\frac{2M/r^3}{1 - 2M/r}, \quad 8\pi T_2^2 = \frac{(M/r^3)(1 - M/r)}{(1 - 2M/r)^2}.$$  

Note that energy density $T_0^0 = 0$ but gravitational charge density, $4\pi \rho_c = (M^2/r^4)(1 - 2M/r)^{-2} \neq 0$. The proper red shifted acceleration of a free particle relative to infinity is given by $\alpha \nabla (\ln \alpha)$ where $\alpha = B^{1/2}$, the lapse function, which will give the Newtonian acceleration $M/r^2$. It is $\rho_c$ that is responsible for this acceleration. Note that vanishing of proper acceleration implies vanishing of $\rho_c$.  

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The metric potential therefore produces acceleration as well as tidal acceleration for radial and non-radial motion.

(ii) When \( B = 1 \) and \( A = (1 - 2M/r)^{-1} \), we have

\[
T^0_0 = 0 = T^0_\alpha - T^\alpha_\alpha = \rho_c, T^1_1 = M/4\pi r^3, T = 0.
\]

Here not only energy density \( T^0_0 \) but \( \rho_c \) and \( T \) also vanish. Since \( B = 1 \), which means vanishing of acceleration for free particles as well as gravitational charge density \( \rho_c \). Radially falling particles will experience neither acceleration nor tidal acceleration. The curvature of space produced by \( A \), will manifest only in tidal acceleration for non-radial motion.

Roughly speaking \( B \) accounts for the usual Newtonian gravity while \( A \) brings in the Einsteinian aspect of non-linearity of gravitational field. When the two are synthesized together, then the Schwarzschild solution follows.

**2.2 The Kerr-Schild coordinates**

This is the another coordinate system in which like the curvature coordinates the gravitational potential can be defined exactly. Here we write the metric in the form,

\[
ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + 2\phi(dt + dr)^2
\]

with

\[
\nabla^2 \phi = 0
\]

which admits the general solution as given in (2.9). However the vacuum equation \( R_{ik} = 0 \) will again demand \( k = 0 \), which implies asymptotic flatness (the Schwarzschild solution). Note that here again we have the Laplace equation for \( R^0_0 = 0 \). It is legitimate to ask what does the classically trivial solution of this equation, \( \phi = \text{const.} \) correspond to?
2.3 The isotropic coordinates:

At first sight it may appear that we can just transform from curvature to isotropic coordinates. The important point to note is that $k$ plays a non-trivial role in the transformation; i.e. the character of transformed metric becomes radically different from the one when $k$ is zero. In contrast to the above two cases, there does not exist a natural choice for potential in this case. In the above cases, it was $R_0^0 = 0$ that could be written as the Laplace equation while here it happens for $G_0^0 \equiv R_0^0 - \frac{1}{2} R = 0$. But $R_0^0 = 0$ cannot be cast as the Laplace equation without approximation. In the Newtonian limits of weak field and slow motion, $R_0^0 \simeq G_0^0$. However if we take the special relativistic limit for which field is weak but motion is relativistic, then $R_0^0 \not\simeq G_0^0$ and it is $R_0^0 = 0$ that should approximate to the Laplace equation and define potential. Here this does not happen and hence we cannot define the analogue of $\phi$ except writing it in the isotropic coordinates. This is because the isotropic $r$ is not the physical radial coordinate as it does not define correctly the area of a sphere of radius $r$.

By writing the Ricci tensor in the isotropic coordinates and solving the vacuum equations we shall once again demonstrate that the solution fixes one of the free parameters which is equivalent to $k = 0$ and that implies asymptotic flatness. We write the metric in the isotropic form as

$$ds^2 = c^2 dt^2 - a^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2). \quad (2.13)$$

We shall now consider the metric functions to be functions of $r$ alone, because the validity of the Birkhoff’s theorem is not in question. The non-zero Ricci components read as follows:

$$R_0^0 = -\frac{1}{a^2} \left[ \frac{c''}{c} + \frac{c'}{c} \left( \frac{a'}{a} + \frac{2}{r} \right) \right] \quad (2.14)$$

$$R_1^1 = -\frac{1}{a^2} \left[ \frac{2a''}{a} - \frac{2a'^2}{a^2} + \frac{2a'}{ar} + \frac{c''}{c} - \frac{a'c'}{ac} \right] \quad (2.15)$$
\[ R_2^2 = -\frac{1}{a^2} \left[ \frac{a''}{a} + \left( \frac{a'}{c} + \frac{1}{r} \right) + \frac{3a'}{ar} \right]. \]  
\[ (2.16) \]

The first two equations for \( R_0^0 = R_1^1 = 0 \) yield the following first integral and a second order non-linear differential equation,

\[ 1 + \frac{a'}{ar} = k_1 c \]  
\[ (2.17) \]

\[ \left( \frac{a'}{ar} \right)'ar^2 = k_1 k_2. \]  
\[ (2.18) \]

This equation admits the general solution as given by

\[ a = r^n \left( 1 + \frac{M}{2r^{n+1}} \right)^2 \]  
\[ (2.19) \]

which in view of (2.17) fully determines the metric by giving

\[ c = \frac{1 - M/2r^{n+1}}{1 + M/2r^{n+1}} \]  
\[ (2.20) \]

where \( M = k_2/k_1 \) and \( n = k_1 - 1 \).

So far we have not used \( R_2^2 = 0 \), which had previously fixed \( k = 0 \), it will now determine the arbitrary constant, \( n = 0, -2 \), from

\[ R_2^2 = -\frac{n(n + 2)}{a^2 r^2}. \]  
\[ (2.21) \]

Both these values however yield the same spacetime. Hence fixing \( n \) here is equivalent to fixing \( k \) in the curvature coordinates and the two are related as \( 2k = n(n + 2) \). As a matter of fact it can be verified that if we transform from the curvature coordinates to the isotropic coordinates with \( k \neq 0 \), we shall end up with (2.19) and (2.20). The constant \( \phi \) in the curvature coordinates reflects in a very different way in the isotropic coordinates. This is because, here the true radial coordinate (the one that defines the area of a sphere) is \( ar \) and not \( r \) itself. Clearly in the isotropic coordinates, it is not obvious to see the association of the parameter.
with constant potential. It is however true that its being different from zero and
−2 only makes $R_2^2$ alone non-zero.

Let us see how $\phi$ is expressed in terms of the isotropic $r$,

$$2\phi = n(n + 2) - \frac{2M/r^{n+1}}{(1 + M/2r^{n+1})^2}$$

(2.22)

and then we can also write

$$a = \frac{2M/r}{n(n + 2) - 2\phi}, \quad c = [1 - n(n + 2) + 2\phi]^{1/2}.$$  

(2.23)

Note that $\phi = \text{const.}$ does not make $a = \text{const.}$, and hence spacetime does not
become flat. Obviously $\phi$ does not satisfy the Laplace equation in these coordinates.

In the curvature and the Kerr-Schild coordinates, the Schwarzschild solution can
be written in terms of the potential which arises as the solution of $\nabla^2 \phi = 0$
corresponding to $R_0^0 = 0$, and putting that equal to constant will give the spacetime
corresponding to $\phi = k$ in (2.9). It is however not so transparent in the isotropic
coordinates because the isotropic $r$ is not the proper radial coordinate. This is also
reflected in the fact that in the former cases (2.9) was the solution of $R_0^0 = 0$, which
could be written as the Laplace equation by using $R_0^0 = 0$ and $R_1^1 = 0$, while in the
latter case this doesn’t happen (what does take the required form is $G_0^0 = 0$). This is
why in the isotropic coordinates the point we wish to raise does not become visible
directly but one has to go through the derivation of the Schwarzschild solution in
these coordinates to see how the free parameter $n$ is determined.

The point that emerges without any ambiguity from the above discussion is that
asymptotic flat character of the Schwarzschild solution is dictated by the field
equations themselves leaving no scope for any boundary condition to operate. The
choice different from $k = 0$ or $n = 0, -2$ implies $R_2^2 \neq 0$ and asymptotic non-
flatness. This means the field equations cannot admit an asymptotically non-flat
spherically symmetric vacuum solution. This is exactly what the uniqueness of the
Schwarzschild solution signifies. The amazing thing is that we have no choice to
impliment any boundary condition.
In NT we had the general solution given by (2.9), where we were free to choose \( k \) to fix zero of the solution. This is a different question that the canonical choice is \( \phi = 0 \) at infinity, but we had the freedom to make another choice, which is not available in GR. Now the question arises what happens if we have the true analogue of the Newtonian situation with freedom to choose the constant \( k \)? This is what we take up in the next Section.

3. The generalized Schwarzschild field and global monopole

The main question we wish to address is what happens if we do not let \( k = 0 \) in (2.9) (which has no physical effect in NT). In that case \( R_{2}^{2} = -2k/r^2 \neq 0 \), however all other \( R_{ik} \) will be zero, and the spacetime will not be empty. Let us ask how much the empty space character of spacetime will be disturbed by this? In NT vanishing of matter density indicates emptiness. Its analogue in GR is the gravitational charge density defined by

\[
-4\pi \rho_c = R_{ik}u^iu^k \tag{3.1}
\]

where \( u^i \) is the timelike unit 4-velocity. That does however vanish even when \( k \neq 0 \). Clearly the parameter \( k \) will not have any physical effect at the Newtonian level. Though the spacetime is not strictly empty in the GR sense but it is empty enough in the Newtonian sense because \( \nabla^2 \phi = 0 \). The other way of looking at it would be to see what happens when stresses producing zero \( \rho_c \) are added to the Schwarzschild spacetime. That is all but \( R_{2}^{2} = -2k/r^2 \neq 0 \) vanish. That is a global monopole is added to a Schwarzschild field [4]. It may be noted that \( \rho_c = 0 \) indicates zero gravitational charge (mass).

The generalized Schwarzschild solution will read as

\[
ds^2 = (1 + 2k - \frac{2M}{r})dt^2(1 + 2k - \frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{3.2}
\]

because \( B = A^{-1} = 1 + 2\phi, \ \phi = k - M/r \). Similarly the Reissner-Nördstrom
and the de Sitter spacetime can be generalized by writing $\phi$ in (2.9) as $\phi = k - M/r + Q^2/2r^2 + \Lambda r^2/6$.

It is clear that parameter $k$ has no effect on radial acceleration for free particles. Hence the metric (3.2) is very nearly equivalent to the Schwarzschild field. Particle orbits and the Hawking radiation have been examined to study the effect of $k$ [5]. It turns out that existence, boundedness and stability of circular orbits scale up by the factor $(1 + 2k)^{-1}$, and the perihelion and the light bending by $(1 + 2k)^{-3/2}$ while the Hawking temperature scales down by $(1 + 2k)^2$ for a negative $k$. The presence of $k$ will only be felt when we consider geodesic deviation for transverse motion along $\theta$– or $\phi$– direction.

The metric gives rise to the stress system,

$$4\pi T^0_0 = -\frac{k}{r^2} = 4\pi T^1_1$$  \hspace{1cm} (3.3)

which exhibits tension in the radial direction is equal to energy density and the transverse stresses being zero. This is precisely the prescription for a global monopole with $k = -4\pi\eta^2$, where $\eta$ is the monopole charge [4]. This exotic object is supposed to occur when global $O(3)$ symmetry is spontaneously broken into $U(1)$ in phase transition in the very early Universe. It is interesting that the mundane situation of “constant $\phi$” shares the spacetime description with such an exotic object.

Further the spacetime (3.2) is not asymptotically flat instead it goes over to

$$ds^2 = dt^2 - (1 + 2k)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$  \hspace{1cm} (3.4)

where $t$ has been redefined to absorb the factor $(1 + 2k)$. This is indistinguishable from flat spacetime in all respects except for tidal acceleration for transverse motion. It produces the monopole stresses (3.3) which have vanishing gravitational charge density $\rho_c$. For the metric (3.4), only space part is curved and its curvature at a given $r(R_{23}^{23} = 2k/r^2)$ is in fact proportional to that of a sphere of radius $r$. This
is the only one non-zero curvature component which is an invariant for spherical symmetry preserving coordinate transformations [6].

We shall now generate the metric (3.4) by a geometric ansatz [7]. Consider 5-Minkowski spacetime

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - dw^2$$  \hspace{1cm} (3.5)

and now impose the restriction

$$x^2 + y^2 + z^2 + w^2 = K^2(x^2 + y^2 + z^2)$$  \hspace{1cm} (3.6)

where $K^2 = (1 + 2k)^{-1}$. Elimination of the extra variable leads to the metric (3.4). Note that the uniform $\phi$ metric has only its space part curved which is described by the metric induced on a cone, specified by the ansatz (3.6) in a 4-Euclidan space.

As demonstrated earlier curvature of (3.4) is due to non-linear character of GR, which persists even when $\phi$ is constant. This is the remarkable and purely relativistic effect directly arising from non-linearity of the theory. Here “constant $\phi$” attains physical meaning. This can be seen geometrically as follows:

Consider flat space 3-metric, $dl^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$. The metric (3.4) corresponds to $\phi = k$ which amounts to “curving” this metric by introducing a const. $\neq 1$ as a coefficient for $dr^2$. It will now have curvature proportional to a sphere (isocurvature spheres). The $\phi$ of (2.9) incorporates gravitational acceleration in geodesic as well as “curves” 3-space. The latter is, as argued earlier, the non-linear effect which is not anulled out even when $\phi$ is constant. This is how constant $\phi$ produces curvature and consequently attains physical meaning.

For measuring this effect we should do exactly what we do to measure curvature of a sphere. Let two particles move on a $r = \text{const.}$ surface and then measure how they come closer. Consider a spaceship in an orbit which will be freely falling and hence free of gravity. Let two particles propagate across the ship and the amount of
their convergence will measure the curvature of the metric (3.2). That will measure the constant $k$.

The curvature of spacetime goes as $r^{-2}$ and hence it will diverge as $r \to 0$. It makes curvature singular. The important point to note is how do we measure curvature? Only through the geodesic deviation, does that diverge? Remember that particles need to propagate transversely because only $R_{23}^{23}$ alone is non-zero. At $r = 0$, there cannot be any transverse motion. It is the same situation as curvature of sphere diverges as $r \to 0$. Hence it is not a physically realisable singularity and is rather innocuous. This is the asymptotic limit of the Schwarzschild global monopole spacetime (3.2). Thus all physical measurements in (3.2) should be referred to it.

The spacetime (3.4) could in some sense be thought of as “minimally” curved because, (i) it has zero gravitational mass ($\rho_c = 0$), (ii) it is free of radial acceleration and tidal acceleration occurs only for transverse motion and (iii) its curvature is purely generated by constant relativistic potential $\phi$.

4. Discussion

We have demonstrated above that relativistic $\phi$, that specifies the Schwarzschild field, is determined absolutely by the Einstein vacuum equations. That is, its non-zero constant value has non-trivial physical effect. This has happened because the field equations also dictate the boundary condition leaving no freedom to relate this field with any other at the boundary. The spacetime has to be asymptotically flat if it were to be vacuum. This is rather very extraordinary and unique feature. Usually one has freedom to choose boundary conditions to relate the situation under consideration with other field at the boundary.

In a real Universe an isolated body cannot be described strictly by the Schwarzschild solution because the body does not exist all alone in the Universe. There are other bodies. And gravity can never be screened off. A good approximation would be that a body is sitting at the centre of an empty spherical cavity surrounded by homogeneous and isotropic matter-energy distribution, representing rest of the
Universe (ROU). As a matter principle the Schwarzschild solution must admit existence of non-empty ROU to accord with the real Universe. But it cannot as we have shown above because asymptotic flatness, which cannot permit presence of any other matter, is inherent in the solution. Giving up asymptotic flatness implies giving up vacuum as well. In practice the Schwarzschild solution is very successful but we are raising a question of principle that it must be asymptotically non-flat to be consistent with non-empty ROU.

Can this be achieved without disturbing the basic character of the field so as to continue to enjoy the observational support? The answer is yes, as we have shown in Section 3. That is the field is described by a scalar function, as before, with restoration of freedom to choose its zero. It is the denial of this freedom that led to asymptotic flatness. Thus the field should remain basically undisturbed for the key equation in the problem is the Laplace equation that continued to hold good. The additional parameter $k$ in the generalized metric (3.2) will now relate to the constant gravitational potential produced by ROU in the interior of the cavity. The picture now becomes identical with the Newtonian picture with the basic difference that the field is now sensitive to absolute value of the scalar function defining it. In NT, since field was not sensitive to absolute value of potential and hence constant potential produced by ROU in interior of cavity did not make any physical difference. In GR, on the other hand, non-zero constant potential due to non-empty ROU will now make a non-trivial contribution to spacetime in the interior of the cavity. But it does not alter the basic character of the field. This is the only and the most harmless way to make the Schwarzschild field asymptotically non-flat so as to be consistent with the realistic setting.

It is remarkable that the above proposed generalization is equivalent to ascribing a global monopole charge to the Schwarzschild particle. It is clearly demonstrated that existence of non-vacuous ROU is equivalent to adding a global monopole charge. The spacetime (3.4) generated by constant $\phi$ is the same as the global monopole metric with $M = 0$. This is very interesting and may be a manifestation
of a deeper relationship between ROU and global monopole charge. Note that
constant potential due to ROU is in fact the measure of global monopole charge.
Global monopole is caused by spontaneous breaking of global $O(3)$ symmetry into
$U(1)$ and is viewed as a topological defect [4]. The other thing to note is that this
feature is purely relativistic as it is shown in Section 2 that it arises from curvature
of space part of the metric which is responsible for non-linear aspect of gravity (it
being its own source). It is this that lends physical meaning to “constant $\phi$”. This
is how constant $\phi$ which is physically inert in NT becomes physically active in GR.

What we mean by potential here is a scalar function that fully determines the field.
It corresponds to the Newtonian potential in the limit. In the standard curvature
coordinates, we do ultimately come to $R^0_0 = -\nabla^2 \phi = 0$, hence it satisfies the Laplace
equation corresponding to $R^0_0 = 0$. Any scalar function, not only being a solution
of the Laplace equation given by $R^0_0 = 0$, that fully determines the field and radial
acceleration for free particle should be entitled to be termed potential. In either
theory, field of a body at rest can be described by only one scalar function. The point
that disturbs people is the fact that the vacuum equations determine this scalar
absolutely and so constant but non-zero value of it is physically distinguishable
from the zero value. This is new and unique feature of GR and is directly related
to its non-linear character.

Leaving the question aside for the moment whether $\phi$ in (2.9) represents potential
in GR or not, it is clear that it completely describes the field of a Schwarzschild
particle. Irrespective of whether it is NT or GR, field of a mass point can require no
more than one quantity to describe it. All what we say is that the Einstein vacuum
equations determine this quantity absolutely while NT offers freedom to choose its
zero arbitrarily. In order to incorporate the feature that gravitational field energy
has gravitational charge, the Newtonian Laplace equation needs to be modified to
take the non-linear form (1.1). This would however be referred to flat spacetime
background. In GR, on the other hand background spacetime is curved that permits
us to retain the Laplace equation as it is with non-linear aspect being taken care
of by “curving” space. The price we pay for it is that $\phi$ now gets determined absolutely. This happens because now $\phi$ has also to “curve” space appropriately to exactly counteract field energy density on the right of (1.1). This it could do only by fixing one of the two free parameters $k$ and $M$ in (2.9). $M$ cannot be specified as it represents mass of the body that should remain free. Thus $k$, which is physically inert in classical physics, can only be fixed. This is how $\phi$, the solution of the Laplace equation is determined absolutely in GR. Its absolute zero value is given by asymptotic flat spacetime. This is the trade off for incorporating the distinguishing non-linear aspect of the theory.

Finally we wish to say that the Einstein vacuum equations determine the relativistic $\phi$ absolutely and its zero being defined by the asymptotic flat spacetime. Retaining the free parameter $k$ in (2.9) is the only way to generalize the Schwarzschild field to render it asymptotically non-flat to be in principle consistent with the realistic setting as it actually obtains in the Universe. At the same time the basic physical character of the field that has strong observational support is not significantly disturbed.

**Acknowledgement:**

Over a period of past one year I have benefitted from discussions and criticism from several friends, and they included amongst others Sailo Mukherjee, Jayant Narlikar, Reza Tavakol, Jose Senovilla and Malcolm MacCallum. This however should not be taken to mean that they all share this view. Their criticism was nonetheless very useful for me to gain insight and sharpen some of the arguments and I hope that they would perhaps be happier with the final version. I thank them all warmly.
References:

1. N. Dadhich, GR-14 Abstracts, A.98 (1995).

2. H. Stephani, General Relativity (Cambridge University Press, 1990), p.99.

3. R.A. D’Inverno, Introducing Einstein’s Relativity (Oxford University Press, 1992) p.187.

4. M. Barriola and A. Vilenkin, Phys. Rev. Lett., 63, 341 (1989).

5. N. Dadhich, K. Narayan and U. Yajnik, Schwarzschild black hole with global monopole charge, submitted.

6. N. Dadhich, Ph.D. thesis (Poona University, 1970) unpublished.

7. N. Dadhich and K. Narayan, An ansatz for spacetimes of zero gravitational mass : global monopoles and textures, submitted.