ON SOME PROPERTIES OF THE MANIFOLDS WITH SKEW-SYMMETRIC TORSION AND HOLONOMY SU(N) AND SP(N)

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Abstract In this paper we provide examples of hypercomplex manifolds which do not carry HKT structure, thus answering a question in [16]. We also prove that the existence of HKT structure is not stable under small deformations. Similarly we provide examples of compact complex manifolds with vanishing first Chern class which do not admit a Hermitian structure with restricted holonomy of its Bismut connection in $SU(n)$, thus providing a counter-example of the conjecture in [18]. Again we prove that such property is not stable under small deformations.

1. Introduction

Let $(M, J, g)$ be a Hermitian manifold. Then there is a unique Hermitian connection $\nabla$ with torsion $T$ such that $g(X, T(Y, Z))$ is totally skew-symmetric. This connection was used by Bismut to prove a local index formula for the Dolbeault operator when the manifold is not Kähler. Metric connections with such torsion have also applications in type II string theory, supersymmetric $\sigma$-models and the geometry of black hole moduli spaces - see e.g. [20] for more references. For some applications the holonomy of the connection is restricted to a proper subgroup of $U(n)$. We concentrate here on $SU(n)$ and $Sp(n)$ as a possible (restricted) holonomy groups.

In (4,0) supersymmetric $\sigma$-models with Wess-Zumino term the target manifold has a metric connection with skew-symmetric torsion with holonomy in $Sp(n)$. Such manifolds carry a triple of anticommuting complex structures called hypercomplex structure compatible with the metric and are known in the physics literature as HKT manifolds [19]. A lot of information about their geometry is known (from a mathematical viewpoint see [16]). In particular it is known that there is a local existence result, reduction theory, non-trivial Dolbeault cohomology properties and many examples. So a natural question is (see [16]) Do all hypercomplex manifolds admit HKT structure? Moreover for some basic examples as $SU(3)$ and $S^1 \times S^{3n-1}$ there is also a local deformation rigidity - any small deformation of the invariant hypercomplex structure there admits HKT structure [16] [17]. This was generalized by M. Verbitsky [28] to all manifolds for which $\partial$-closed (2,0)-forms are (2,0)-components of a closed forms. It rises also the question Whether every small hypercomplex deformation of HKT structure is HKT? One of the purposes of this paper is to provide negative answer to both questions.
When the holonomy group of connection with skew-symmetric torsion on Hermitian manifold is contained in $SU(n)$ then the manifold has vanishing first Chern class. Such models in string theory has been considered first by A. Strominger [27] and C. Hull [20]. On a large class of manifolds however, the models suggested in [27] are degenerate (with vanishing torsion) [21, 13]. More recently $SU(3)$-structures on non-Kähler manifolds have attracted attention as a more general models for string compactifications and many examples have been found: [3, 4, 6, 14, 15]. It was conjectured in [18] that any compact complex manifold with vanishing first Chern class admits a Hermitian metric and connection with totally skew-symmetric torsion and (restricted) holonomy in $SU(n)$. The other aim of this paper is to provide counter-example of this conjecture. Moreover we show that, as in the hypercomplex case, there is no stability under small deformations.

The examples we provide are all compact quotients of nilpotent Lie groups. To show non-existence, we use the ”symmetrization” process and reduce the problem to non-existence of invariant structures with the given property. Then previous works by Dotti, Fino, Salamon and Parton [7, 10] lead to the results.

The paper is organized as follows: In Section 2 we provide the material about the symmetrization of geometric structures on compact quotients of Lie groups. In Section 3 we provide a 2-step nilmanifold example of hypercomplex manifolds which do not admit HKT structure and a one parameter family of hypercomplex structures, such that for one value of the parameter it admits HKT structure, while for the other values it does not. Finally, in Section 4, we first prove that a compact Hermitian manifold with holomorphically trivial canonical bundle and vanishing Ricci tensor of its Bismut connection is (globally) conformally balanced. Then, we use the results from Section 2 to show that in a particular 2-parameter family of complex structures on the Iwasawa manifold for all values of the parameter except one it does not admit balanced metrics.

2. INVARIANT STRUCTURES ON COMPACT QUOTIENTS OF LIE GROUPS

In this section we prove some general facts about ”symmetrizing” geometric structures on compact quotients of a Lie group. We start with the following theorem, which is a consequence of [25, Lemma 6.2] where it is shown that any simply-connected Lie group which admits a discrete subgroup with compact quotient is unimodular and in particular admits a bi-invariant volume form $d\mu$. For such manifolds we have:

**Theorem 2.1.** Suppose that $M = \Gamma \backslash G$ is a compact quotient of a simply-connected Lie group with a Riemannian metric $g$. Denote by $d\mu$ a bi-invariant volume form and define a new (left-invariant) metric by

$$\overline{g}(A, B) = \int_M g_m(A_m, B_m)d\mu.$$ 

Then we have the following:
(a) \( \overline{\varphi}(\nabla AB, C) = \int_{m \in M} g_m(\nabla AB|_m, C_m) d\mu \), where \( A, B, C \) are projections of left-invariant vector fields from \( G \) to \( M \).

Similarly, if \( B_1, ..., B_k \) are projections of left-invariant vector fields from \( G \) to \( M \) and \( \omega(B_1, ..., B_k) = \int_{m \in M} \omega_m(B_1|_m, ..., B_k|_m) d\mu \) where \( \omega \) is a \( k \)-form on \( M \), then we have

\[
\int_{m \in M} g_m(\nabla A B|_m, C |_m) d\mu = \int_{m \in M} \omega_m(B_1|_m, ..., B_k|_m) d\mu.
\]

\( (b) \) \( d\overline{\omega}(B_1, ..., B_{k+1}) = \int_{m \in M} d\omega_m(B_1|_m, ..., B_{k+1}|_m) d\mu. \)

\( \text{Proof.} \) We follow the lines of \([5]\). First note that

\[
\int_{M} Xf d\mu = \int_{M} \mathcal{L}_X(f) d\mu = \int_{M} d(i_X f) d\mu = 0,
\]

for any function \( f \) and left-invariant vector field \( X \). Next, we remind the formula for the Levi-Civita connection:

\[
g(\nabla_X Y, Z) = \frac{1}{2}(Y g(X, Z) - Z g(X, Y) + X g(Y, Z) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X)).
\]

Now, using the above identity for left-invariant fields \( A, B, C \), we have:

\[
\int_{M} g_m(\nabla AB|_m, C_m) d\mu = \int_{M} g_m([A, B]|_m, C_m) + g_m([C, A]|_m, B_m) - g_m([B, C]|_m, A_m) d\mu
\]

\[
= \frac{\varphi([A, B], C)}{\varphi(\nabla A B, C)} + \frac{\varphi([C, A], B)}{\varphi(\nabla A B, C)} - \frac{\varphi([B, C], A)}{\varphi(\nabla A B, C)}
\]

which proves (a). Using the definition for the exterior differential on \( k \)-forms, we obtain (b) in a similar fashion. \( \square \)

As a consequence, we have the following:

**Theorem 2.2.** Suppose that \( M \) is as in Theorem 2.1 and admits a left-invariant complex structure \( J \). Suppose moreover that \( F \) is a Kähler form of a non-invariant Hermitian metric \( g \). Then the left-invariant form defined as:

\[
\alpha(A_1, ..., A_{2n-2}) = \int_{M} F^{n-1}|_m(A_1|_m, ..., A_{2n-2}|_m) d\mu,
\]

for left-invariant vector fields, is equal to \( \overline{F}^{n-1} \) for some Kähler form of a left-invariant Hermitian metric. Moreover if \( dF^{n-1} = 0 \) then \( d\overline{F}^{n-1} = 0 \).

**Proof.** Since \( \alpha \) is a strictly positive \((n-1, n-1)\)-form, by a linear algebra argument (see \([21]\), p.279) it can be written as \( \overline{F}^{n-1} \), where \( \overline{F} \) is a strictly positive \((1, 1)\)-form. Then it is a Kähler form of an invariant metric and the last assertion follows from Theorem 2.1. \( \square \)
3. EXAMPLES OF HYPERCOMPLEX MANIFOLDS WHICH DO NOT ADMIT HKT METRIC

A hypercomplex structure on a manifold $M$ is a triple of complex structures \( \{ J_i \}_{i=1,2,3} \) satisfying the quaternions relations $J_i^2 = -I$, $i = 1, 2, 3$, $J_1 J_2 = -J_2 J_1 = J_3$. Let $g$ be a Riemannian metric on $M$. Then $M$ is said to be hyper-Hermitian if it is Hermitian with respect to every $J_i$, $i = 1, 2, 3$.

A hyper-Hermitian manifold $(M, \{ J_i \}_{i=1,2,3})$ is an HKT manifold if there is a connection $\nabla$ such that

$$\nabla g = 0, \quad \nabla J_i = 0, \quad c(X, Y, Z) = g(X, T(Y, Z)) \text{ a 3-form.}$$

If such connection exists it is unique. A hyper-Hermitian manifold $M$ will admit a HKT connection if and only if $J_1 dF_1 = J_2 dF_2 = J_3 dF_3$, where $F_i$ is the Kähler form associated to $(J_i, g)$.

A hypercomplex structure on a Lie algebra $\mathfrak{g}$ is a triple of complex structures \( \{ J_i \}_{i=1,2,3} \) satisfying the quaternions relations $J_i^2 = -I$, $i = 1, 2, 3$, $J_1 J_2 = -J_2 J_1 = J_3$.

The hypercomplex structure will be called abelian if

$$[J_i X, J_i Y] = [X, Y], \quad X, Y \in \mathfrak{g}.$$

Let $g$ a inner product compatible with the hypercomplex structure.

By \[ \{ \mathfrak{g} \} \ (\{ J_i \}_{i=1,2,3}, g) \text{ is an HKT structure on } \mathfrak{g} \text{ if and only if } g \text{ satisfy the extra condition} \]

$$g([J_i X, J_i Y], Z) + g([J_i Y, J_i Z], X) + g([J_i Z, J_i X], Y) =$$
$$g([J_2 X, J_2 Y], Z) + g([J_2 Y, J_2 Z], X) + g([J_2 Z, J_2 X], Y) =$$
$$g([J_3 X, J_3 Y], Z) + g([J_3 Y, J_3 Z], X) + g([J_3 Z, J_3 X], Y),$$

for any $X, Y, Z \in \mathfrak{g}$.

When the hypercomplex structure is abelian the previous condition is always satisfied for any inner product $g$ compatible with the hypercomplex structure.

We say that a Lie group $G$ has an invariant HKT structure $(\{ J_i \}_{i=1,2,3}, g)$ if the hypercomplex structure $\{ J_i \}_{i=1,2,3}$ and the metric $g$ arise from corresponding left-invariant tensors. In \[ \mathfrak{N} \] it is shown that, if $G$ is a 2-step nilpotent Lie group, the hypercomplex structure of an invariant HKT structure is abelian. Then one has the following:

**Lemma 3.1.** A 2-step nilpotent Lie group $G$ with a non abelian hypercomplex structure admits no invariant HKT metric compatible with such hypercomplex structure.

If we restrict to the 8-dimensional case all simply connected nilpotent Lie groups which carry invariant abelian hypercomplex structures are described in \[ \mathfrak{N} \]. There are three such groups and they are central extensions of the real, complex or quaternionic Heisenberg group respectively:

$$N_1 = \mathbb{R}^3 \times H_5, \quad N_2 = \mathbb{R}^2 \times H_3^C, \quad N_3 = \mathbb{R} \times H_7.$$
As remarked in [8], $N_1$ and $N_2$ can only carry abelian hypercomplex structures, but $N_3$ has a non abelian hypercomplex structure so it admits no invariant HKT metric.

The Lie algebra $n_3$ of $N_3$ has structure equations

$$[e_1, e_2] = [e_3, e_4] = -e_6,$$
$$[e_1, e_3] = -[e_2, e_4] = -e_7,$$
$$[e_1, e_4] = [e_2, e_3] = -e_8,$$

and a non abelian hypercomplex structure is given by

$$J_1(e_1) = e_2, J_1(e_3) = e_4, J_1(e_5) = e_6, J_1(e_7) = e_8,$$
$$J_2(e_1) = e_3, J_2(e_2) = -e_4, J_2(e_5) = e_7, J_2(e_6) = -e_8.$$

In [9] it is shown that a 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ which admit a hypercomplex structure is 2-step, its first Betti number satisfies $b_1(\mathfrak{g}) \geq 4$ and there exists a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that

i) $\dim \mathfrak{g}_i = 4$;

ii) $\mathfrak{g}_i$ is invariant with respect to the hypercomplex structure;

iii) $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_2 \subset \mathfrak{z}$, ($\mathfrak{z}$: center of $\mathfrak{g}$).

If $b_1(\mathfrak{g}) = 4, 5$ the full description of all 8-dimensional nilpotent Lie algebras with a hypercomplex structure is given in [9]. If $b_1(\mathfrak{g}) = 6, 7$ the hypercomplex structure is abelian.

If $G$ is a simply-connected nilpotent Lie group and, if the structure equations of its Lie algebra are rational, then there exists a discrete subgroup $\Gamma$ of $G$ for which $M = \Gamma \backslash G$ is compact [23]. Any invariant HKT structure on $G$ will pass to a HKT structure on $M$.

**Theorem 3.1.** Any compact quotient $M = \Gamma \backslash G$ of a 2-step nilpotent Lie group $G$ with a non abelian left-invariant hypercomplex structure $\{J_i\}_{i=1,2,3}$ admits no HKT metric compatible with such hypercomplex structure.

**Proof.** Suppose that $M$ admits a (non invariant) HKT metric $g$ compatible with the non abelian left-invariant hypercomplex structure $\{J_i\}_{i=1,2,3}$. We will construct now, by integration, a HKT metric on $G$ which is left-invariant and so obtain a contradiction with the previous Lemma.

$G$ admits a bi-invariant volume element $d\mu$, which induces one on the compact quotient $M$ and possibly after rescaling, we can suppose that the volume of $M$ is equal to 1.

Let $\Omega_i$ the Kähler form associated to $(J_i, g)$, then the condition that $g$ is a HKT metric is equivalent to

$$J_1 d\Omega_1 = J_2 d\Omega_2 = J_3 d\Omega_3.$$

Define

$$F_i(A, B) := \int_{m \in M} (\Omega_i)_m(A_m, B_m) d\mu, \ i = 1, 2, 3,$$
where $A, B$ are projections on $M$ of left-invariant vectors fields on $G$. Then, since $J_i$ is left-invariant it is easy to check that $F_i$ is $J_i$-invariant, i.e. the Kähler form of a positive definite metric $g_0$ and that it is left-invariant and defined as in Theorem 2.1.

From Theorem 2.1 we conclude that

$$dF_i(A, B, C) = \int_M d\Omega_i(A, B, C)d\mu.$$ 

Then one has

$$J_1dF_1(A, B, C) = -dF_1(J_1A, J_1B, J_1C) = -\int_M d\Omega_1(J_1A, J_1B, J_1C)d\mu$$

$$= \int_M J_1d\Omega_1(A, B, C)d\mu = \int_M J_2d\Omega_2(A, B, C)d\mu = -\int_M d\Omega_2(J_2A, J_2B, J_2C)d\mu$$

$$= -dF_2(J_2A, J_2B, J_2C) = J_2dF_2(A, B, C),$$

so $g_0$ is a left-invariant HKT metric on $G$, which is impossible. □

Isomorphism classes of hypercomplex structures on 8-dimensional nilpotent Lie groups are studied in [22]. We show that there exists on a 2-step nilpotent Lie group a one parameter family of hypercomplex structures which is abelian for one value of the parameter and non-abelian for the others. Consider the family of Lie algebras $g_t$ with structure equations:

$$[e_1, e_2] = -te_6, [e_3, e_4] = (1 - t)e_6,$$
$$[e_1, e_3] = -te_7, [e_2, e_4] = (t - 1)e_7,$$
$$[e_1, e_4] = -te_8, [e_2, e_3] = (1 - t)e_8$$

and the hypercomplex structure

$$J_1e_1 = e_2, J_1e_3 = e_4, J_1e_5 = e_6, J_1e_7 = e_8,$$
$$J_2e_1 = e_3, J_2e_2 = -e_4, J_2e_5 = e_7, J_2e_6 = -e_8.$$

For any $t \neq 0, 1$, $g_t$ is isomorphic to the Lie algebra $n_3$ of $N_3$. Indeed, one has that $g_t = g_1 \oplus g_2$, with $g_2 = \mathfrak{z} = \text{span}\{e_5, e_6, e_7, e_8\}$ the center of $g$. Then if one considers the endomorphism $J_Z$ ($Z \in g_2$) of $g_1$, defined by

$$(1) \quad g(J_ZX, Y) = g([X, Y], Z),$$

where $g$ is the inner product on $g$, such that the basis $\{e_1, \ldots, e_8\}$ is orthonormal, one can check that $J_Z$ is invertible for any non zero $Z \in g_2$ (compare also [22, Example 7.4]).

The hypercomplex structure is abelian for $t = 1/2$ and non-abelian otherwise. Since the Lie algebras are isomorphic for all $t \neq 0, 1$, we obtain a one-parameter family of invariant hypercomplex structures on the compact quotient $M$ of $N_3$. Moreover they are not equivalent since one is abelian and the others are non-abelian. This proves the following:

**Theorem 3.2.** The HKT structure is not stable under deformations, i.e. there exists a small hypercomplex deformation of HKT structure which is not HKT.
Remark 3.1. M. Verbitsky informed us that stability of HKT structures under deformations holds for a large class of hypercomplex manifolds which include $S^1 \times S^{4n-1}$ and $SU(3)$. He showed that any compact HKT manifold for which $\Omega = F_2 + iF_3$ is $(2,0)$-part (with respect to $J_1$) of a closed 2-form is HKT stable under small deformations. This follows from his characterization of HKT manifolds as hypercomplex manifolds with strictly $q$-positive $\partial$-closed $(2,0)$-form. In fact strictly $q$-positivity is an open condition and if $\Omega = P(2,0)\alpha$ where $P(2,0)$ is the projection onto $(2,0)$-forms and $d\alpha = 0$, then $\Omega_t = P(2,0)t\alpha$ provides strictly $q$-positive $\partial_t$-closed form for deformed hypercomplex structure $J^t_1, J^t_2, J^t_3$ for small $t$.

4. Compact complex nilmanifolds which do not admit a metric with vanishing Ricci tensor of the Bismut connection

Let $(M, g, J)$ be a Hermitian manifold with Kähler form $F$. Then the Bismut connection is the only connection with torsion $T^B$ given by:

$$g(X, T^B(Y, Z)) = JdF(X, Y, Z) = -dF(JX, JY, JZ).$$

There is also a Chern connection with torsion $T^C$ uniquely determined by

$$g(X, T^C(Y, Z)) = dF(JX, Y, Z).$$

Both connections have Ricci forms, representing up to a constant the first Chern class of the complex structure. Then, as a consequence of formula (2.7.6) in [11], the relation between the two Ricci forms is:

$$(2) \quad Ric^B = Ric^C + d\delta F,$$

where $\delta F$ is the co-differential (the adjoint of the differential) of $F$. Moreover from (24) in [12] we have the formula for the conformal change $\tilde{g} = e^f g$:

$$d\tilde{\delta F} = d\delta F + dJd f,$$

where by definition $Jd f(X) = -d f(JX)$. For $d\delta F$ we also have:

Lemma 4.1. For a compact Hermitian manifold $(M, J, g)$ with Kähler form $F$ with co-differential $\delta F$, $d\delta F = 0$ iff $M$ is balanced (i.e. $\delta F = 0$ or equivalently $dF^{n-1} = 0$).

Proof. If $<, >$ is the extension of the metric $g$ on 2-forms, then

$$0 = \int_M < d\delta F, F > \ vol = \int_M < \delta F, \delta F > \ vol$$

and the result follows.

From here there is the following:
Theorem 4.1. Suppose that a compact complex manifold \((M, J)\) admits a holomorphic non-vanishing \((n, 0)\)-form. Then, if the Ricci tensor of the Bismut connection of some Hermitian metric \(g\) vanishes, \((M, J, g)\) is conformally balanced and in particular admits a balanced metric.

Proof. First notice that, for the Ricci curvature of the Chern connection on \(M\), we have the well known formula

\[
\text{Ric}^C = dJD\log \det(g_{\alpha\overline{\beta}}).
\]

Let \(\sigma\) denotes the holomorphic \((n,0)\)-form. It is locally represented as

\[
\sigma = f dz^1 \wedge \ldots \wedge dz^n,
\]

for a holomorphic function \(f\). Then \(|\sigma|^2 = |f|^2\det(g_{\alpha\overline{\beta}})\) and

\[
dJd\log |f|^2\det(g_{\alpha\overline{\beta}}) = dJd\log \det(g_{\alpha\overline{\beta}})
\]

so

\[
\text{Ric}^C = -dJd\log |\sigma|^2.
\]

Now from (2) above it follows that \(d\delta F = dJdlog|\sigma|^2\) and after a conformal change \(F \to |\sigma|^2F\) we can make \(d\delta F\) equal to 0.

Then from the Lemma above the new metric is balanced. \(\square\)

Note that on any \(2n\)-dimensional compact quotient of a nilpotent Lie group with an invariant complex structure there exists an invariant closed \((n, 0)\)-form. For the next examples we need to use the definition of Lie algebra in terms of structure equations for the differentials of invariant 1-forms. It is equivalent to the standard one giving the commutators of vector fields. Similarly we can define complex structures as endomorphisms of the space of invariant 1-forms. Consider now the family of Lie algebras \(g_{s,t}\) defined by:

\[
\begin{align*}
    de^i &= 0, \quad i = 1, \ldots, 4 \\
    de^5 &= se^1 \wedge e^2 + 2se^3 \wedge e^4 + te^1 \wedge e^3 - te^2 \wedge e^4, \\
    de^6 &= te^1 \wedge e^1 + te^2 \wedge e^3
\end{align*}
\]

and the complex structure \(J\)

\[
Je^1 = e^2, \quad Je^3 = e^4, \quad Je^5 = e^6.
\]

For the corresponding \((1, 0)\)-forms

\[
\omega_1 = e^1 + ie^2, \quad \omega_2 = e^3 + ie^4, \quad \omega_3 = e^5 + ie^6
\]

one has that

\[
\begin{align*}
    d\omega_1 &= 0, \quad i = 1, 2, \\
    d\omega_3 &= \frac{1}{2}i\overline{s} \omega_1 \wedge \overline{\omega}_1 + i\overline{s} \omega_2 \wedge \overline{\omega}_2 + t \omega_1 \wedge \omega_2
\end{align*}
\]

and thus \(d(\omega_1 \wedge \omega_2 \wedge \omega_3) = 0\), so there is a holomorphic \((3,0)\)-form on the corresponding simply connected Lie group.
For any \( t \neq 0 \), the Lie algebras \( g_{s,t} \) are isomorphic to the real Lie algebra underlying the complex Heisenberg group

\[
G = \left\{ \begin{pmatrix}
1 & z^1 & z^3 \\
0 & 1 & z^2 \\
0 & 0 & 1
\end{pmatrix} : z^i \in \mathbb{C} \right\}
\]

with structure equations

\[
\begin{align*}
de^i &= 0, \quad i = 1, \ldots, 4, \\
de^5 &= e^1 \wedge e^3 - e^2 \wedge e^4, \\
de^6 &= e^1 \wedge e^4 + e^2 \wedge e^3,
\end{align*}
\]

since \( g_{s,t} = g_1 \oplus g_2 \) with

\[
\begin{align*}
g_1 &\cong \mathbb{R}^4 = \text{span}\{e_1, e_2, e_3, e_4\}, \\
g_2 &\cong \mathbb{R}^2 = \text{span}\{e_5, e_6\},
\end{align*}
\]

and the endomorphism \( J_Z \) defined by (1) of \( g_1 \) is non-singular, for any non zero \( Z \in g_2 \) (compare also [22, Example 6.1 and 6.5]).

Now for any invariant Hermitian metric its Kähler form is of the form:

\[
F = x_1 i \omega_1 \wedge \overline{\omega}_1 + (x_2 + ix_3) \omega_1 \wedge \overline{\omega}_2 + (x_2 - ix_3) \overline{\omega}_1 \wedge \omega_2 \\
+ (x_4 + ix_5) \omega_1 \wedge \overline{\omega}_3 + (x_4 - ix_5) \overline{\omega}_1 \wedge \omega_3 + x_6 i \omega_2 \wedge \overline{\omega}_2 \\
+ (x_7 + ix_8) \omega_2 \wedge \overline{\omega}_3 + (x_7 - ix_8) \overline{\omega}_2 \wedge \omega_3 + x_9 i \omega_3 \wedge \overline{\omega}_3,
\]

where \( x_i, i = 1, \ldots, 9 \), are real numbers such that the restriction of \( F \) to any complex line is non zero. In particular one has that \( x_1, x_6, x_9 > 0 \). The invariant Hermitian metric is balanced if and only if \( F \) is orthogonal to the image of \( d \) in \( \Lambda^2 g^* \) (see [10]).

The condition that \( F \) is orthogonal to \( d\omega_3 \) becomes \( s(x_1 + \frac{1}{2} x_6) = 0 \). Then, if \( s \neq 0 \), there is no invariant balanced Hermitian metric on \( g_{s,t} \). If \( s = 0 \) the complex structure is the bi-invariant complex structure \( J_0 \) on the complex Heisenberg group. In this way the Iwasawa manifold \( M = \Gamma \bs G \) (compact quotient of the complex Heisenberg group by the discrete subgroup \( \Gamma \) for which \( z^i \) are Gaussian integers) comes equipped with a family \( J_{s,t} \) of invariant complex structures, for \( t \neq 0, s, t \) real and we have:

**Theorem 4.2.** The Iwasawa manifold \( (M, J_{s,t}) \) admits a metric with vanishing Ricci tensor of the Bismut connection if and only if \( s = 0 \).

**Proof.** Suppose that \( s \neq 0 \). If there exists a metric with vanishing Ricci tensor of the Bismut connection, then by Theorem 4.1 there exists a balanced metric, since there is a holomorphic \((3,0)\)-form. Now, by Theorem 2.2, there exists an invariant balanced metric. However this is impossible as we showed earlier. Now when \( s = 0 \), there is an invariant balanced metric \( g \). For any such metric \( \text{Ric}^C = \text{Ric}^B = dd^c \log|\sigma|^2 \) as in the proof of Theorem 4.1. Then the conformal change \( \tilde{g} = e^{\sigma} g \) (up to a constant) provides a metric with vanishing Ricci tensor of the Bismut connection. The result also follows from Proposition 6.1 in [10]. \( \square \)
Remark 4.1. With the above result we provide a counter-example of the Conjecture 1.1 in [18]. Moreover it shows that the property "vanishing Ricci tensor of the Bismut connection" is not stable under small deformations.

Remark 4.2. Similar deformation appears in [2] where the authors prove that the existence of balanced metric is not stable under small deformations. The arguments presented here however are different.

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References

[1] E. Abbena, S. Garbiero, S. Salamon, Almost Hermitian geometry of 6-dimensional nilmanifolds, Ann. Sc. Norm. Sup. 30 (2001), 147-170.
[2] L. Alessandrini, G. Bassanelli, Small deformations of a class of compact non-Kähler manifolds, Proc. Am. Math. Soc. 109 (1990), no. 4, 1059–1062.
[3] K. Becker, K. Dasgupta, Heterotic strings with torsion, hep-th/0209077 to appear in JHEP.
[4] K. Becker, M. Becker, K. Dasgupta, P. Green, Compactifications of Heterotic Theory on non-Kähler complex manifolds, hep-th/0301161.
[5] F. A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1-40.
[6] G.L. Cardoso, G. Curio, G. Dall’Agata, D. Lust, P.Manousselis, G.Zoupanos, Non-Kaehler string backrounds and their five torsion classes, hep-th/0211118.
[7] I. Dotti, A. Fino, Hyperkähler torsion structures invariant by nilpotent Lie groups, Classical Quantum Gravity 19 (2002), 551-562.
[8] I. Dotti, A. Fino, Abelian hypercomplex 8-dimensional nilmanifolds, Ann. Glob. Anal. and Geom. 18 (2000), 47-59.
[9] I. Dotti, A. Fino, Hypercomplex 8-dimensional nilpotent Lie groups, to appear in J. Pure Applied Algebra.
[10] A. Fino, M. Parton, S. Salamon, Families of strong KT structures in six-dimensions, to appear in Comm. Math. Helv.
[11] P. Gauduchon, Hermitian connections and Dirac operators, Boll. Un. Mat. Ital. B (7) 11 (1997), suppl., 257-288.
[12] P. Gauduchon, La 1-forme de torsion d’une varieté hermitienne compacte, Math. Ann. 267 (1984), 495-518.
[13] J. Gauntlet, D. Martelli, S.Pakis, D.Waldram, G-Structures and warped NS5-branes, hep-th/0205050.
[14] J. Gauntlet, D. Martelli, D. Waldram, Superstrings with intrinsic torsion, hep-th/0302158.
[15] E. Goldstein, G. Prokushkin, Geometric model for complex non-Kähler mani-
folds with $SU(3)$ structure, hep-th/0212307.
[16] G. Grantcharov, Y. S. Poon, Geometry of Hyper-Kähler connections with tor-
sion, Comm. Math. Phys. 213 (2000), 19-37.
[17] G. Grantcharov, G. Papadopoulos, Y. S. Poon, Reduction of HKT structures,
J. Math. Phys. 43(7) (2002), 3766-3783.
[18] J. Gutowski, S. Ivanov, G. Papadopoulos, Deformations of generalized cali-
brations and compact non-Kähler manifolds with vanishing first Chern class,
math.DG/0205012, to appear in Asian J. Math.
[19] P. Howe, G. Papadopoulos, Twistor spaces for HKT manifolds, Phys.Lett.B 379
(1996) 80-86.
[20] C. M. Hull, Compactification of the heterotic superstrings, Phys.Lett. B 178
(1986), 357-364.
[21] S. Ivanov, G. Papadopoulos, A no-go theorem for string wrapped compactifica-
tions, Phys.Lett.B 497 (2001), 309-316.
[22] J. Lauret, Geometric structures on nilpotent Lie groups: On their classification
and a distinguished compatible metric, math.DG/0210143.
[23] A. I. Malcev, On a class of homogeneous spaces, reprinted in Amer. Math. Soc.
Translations, Series 1, 9 (1962), 276-307.
[24] M. L. Michelsohn, On the existence of special metrics in complex geometry, Acta
Math. 149 (1982), 261-295.
[25] J. Milnor, Curvature of left invariant metrics on Lie groups, Advances in Math.
21 (1976), 293-329.
[26] G. Papadopoulos, KT and HKT geometries in strings and black hole moduli
spaces, hep-th/0201111.
[27] A. Strominger, Strings with torsion, Nucl.Phys. B 274 (1986), 253-284.
[28] M. Verbitsky, private correspondance.
[29] M. Verbitsky, Hyperkähler manifolds with torsion obtained from hyperholomor-
phic bundles, math.DG/0303129.

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