Stochastic Processes and Diffusion on Spaces with Local Anisotropy

Sergiu I. Vacaru

Institute of Applied Physics, Academy of Sciences of Moldova,
5 Academy str., Chișinău 277028, Republic of Moldova

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Abstract

The purpose of this work is to extend the formalism of stochastic calculus to the case of spaces with local anisotropy (modeled as vector bundles with compatible nonlinear and distinguished connections and metric structures and containing as particular cases different variants of Kaluza-Klein and generalized Lagrange and Finsler spaces). We shall examine nondegenerate diffusions on the mentioned spaces and theirs horizontal lifts.

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I. INTRODUCTION

Probability theorists, physicists, biologists and financiers are already familiar with classical and quantum statistical and geometric methods applied in various branches of science and economy [1-6]. We note that modeling of diffusion processes in nonhomogeneous media and formulation of nonlinear thermodynamics in physics, or of dynamics of evolution of species in biology, requires a more extended geometrical background than that used in the theory of stochastic differential equations and diffusion processes on Riemann and Lorentz manifolds [7-10].

Our aim is to formulate the theory of diffusion processes on spaces with local anisotropy. As a model of such spaces we choose vector bundles on space-times provided with nonlinear and distinguished connections and metric structures [11,12]. Transferring our considerations on tangent bundles we shall formulate the theory of stochastic differential equations on generalized Lagrange spaces which contain as particular cases Lagrange and Finsler spaces [13-17].

The plan of the presentation is as follow: In Section II we give a brief summary of the geometry of locally anisotropic spaces. Section III is dedicated to the formulation of the theory of stochastic differential equations in vector bundle spaces. This section also concerns the basic aspects of stochastic calculus in curved spaces. In Section IV the heat equations in bundle spaces are analyzed. The nondegenerate diffusion on spaces with local anisotropy is defined in Section V. We shall generalize in Section VI the results of Section IV to the case of heat equations for distinguished tensor fields in vector bundles with (or not) boundary conditions. Section VII contains concluding remarks and a discussion of the obtained results. We present a brief introduction into the theory of stochastic differential equations and diffusion processes on Euclidean spaces in the Appendix.
II. NONLINEAR AND DISTINGUISHED CONNECTIONS AND METRIC STRUCTURES IN VECTOR BUNDLES

As a preliminary to the discussion of stochastic calculus on locally anisotropic spaces we summarize some modern methods of differential geometry of vector bundles, generalized Lagrange and Finsler spaces. This Section serves the twofold purpose of establishing the geometrical background and deriving equations which will be used in the next two Sections. In general lines we follow conventions from [11,12].

Let $\mathcal{E} = (E, \pi, F, Gr, M)$ be a locally trivial vector bundle, v-bundle, where $F = \mathbb{R}^m$ is a real $m$ dimensional vector space, $\mathbb{R}$ denotes the real number field , $Gr = GL(m, \mathbb{R})$ is the group of automorphisms of $\mathbb{R}^m$, base space $M$ is a differentiable (class $C^\infty$) manifold of dimension $n$, $\dim M = n$, $\pi : E \to M$ is a surjective map and the differentiable manifold $E, \dim E = n + m = q$, is called the total space of v-bundle $\mathcal{E}$ . We locally parametrize space $\mathcal{E}$ by coordinates $u^a = (x^i, y^a)$, where $x^i$ and $y^a$ are, correspondingly, local coordinates on $M$ and $F$. We shall use Greek indices for cumulative coordinates on $\mathcal{E}$ , Latin indices $i, j, k, ... = 0, 1, ..., n − 1$ for coordinates on $M$ and $a, b, c, ... = 1, 2, ..., m$ for coordinates on $F$. Changings of coordinates $(x^k, y^a) \to (x'^k, y'^a)$ on $\mathcal{E}$ are written as

$$x'^k = x'^k(x^k), y'^a = M'^a_a(x)y^a,$$

$$\text{rank} \left( \frac{\partial x'^k}{\partial x^k} \right) = n, M'^a_a(x) \in Gr.$$  \hspace{1cm} (2.1)

A nonlinear connection, N-connection, in v-bundle $\mathcal{E}$ is defined as a global decomposition into horizontal $\mathcal{H}\mathcal{E}$ and vertical $\mathcal{V}\mathcal{E}$ subbundles of the tangent bundle:

$$\mathcal{T}\mathcal{E} = \mathcal{H}\mathcal{E} \oplus \mathcal{V}\mathcal{E}.$$  \hspace{1cm} (2.2)

N-connection defines a corresponding covariant derivation in $\mathcal{E}$:

$$\nabla_Y A = Y^i \left\{ \frac{\partial A^a}{\partial x^i} + N^a_i (s, A) \right\} s_a,$$  \hspace{1cm} (2.3)
where $s_a$ are local linearly independent sections of $E$, $A = A^a s_a$ and $Y = Y^i s_i$ is a vector field decomposed on local basis $s_i$ on $M$. Differentiable functions $N^a_i$ from (2.3), written as functions on $x^i$ and $y^a$, i.e. as $N^a_i(x, y)$, are called the coefficients of N-connection.

In v-bundle $E$ we can define a local frame (basis) adapted to a given N-connection:

$$\delta_\alpha = \frac{\delta}{\delta u^\alpha} = (\delta_i = \frac{\delta}{\delta x^i} = N^a_i(x, y) \frac{\partial}{\partial y^a}, \delta_a = \frac{\delta}{\delta y^a} = \partial_a = \frac{\partial}{\partial y^a}).$$

(2.4)

The dual basis is written as

$$\delta^\alpha = \delta u^\alpha = (\delta^i = \delta x^i = dx^i, \delta^a = \delta y^a = dy^a + N^a_i(x, y) dx^i).$$

(2.5)

By using adapted bases one introduces the algebra of tensorial distinguished fields (d-fields or d-tensors) on $E$, $T = \bigoplus T^p_r q_s$, which is equivalent to the tensor algebra on v-bundle $E_d$ defined as

$$\pi_d : \mathcal{H}E \oplus VE \rightarrow E.$$

An element $t \in T^p_r q_s$, d-tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, can be written in local form as

$$t = \delta^{i_1} \otimes ... \otimes \delta^{i_p} \otimes d^{j_1} \otimes ... \otimes d^{j_q} \otimes \partial^{a_1} \otimes ... \otimes \partial^{a_r} \otimes \delta^{b_1} \otimes ... \otimes \delta^{b_s}.$$

In addition to d-tensors we can consider d-objects with various group and coordinate transforms adapted to a global splitting (2.2). For our further considerations the linear d-connection structure, defined as a linear connection $D$ in $E$ conserving under parallelism the Whitney sum $\mathcal{H}E \oplus VE$ associated to a fixed N-connection in $E$ will play an important role. Components $\Gamma^\alpha_{\beta \gamma}$ of d-connection $D$ are introduced as

$$D_\gamma \delta_\beta = D_\beta \delta_\gamma = \Gamma^\alpha_{\beta \gamma} \delta_\alpha.$$

Torsion $T^\alpha_{\gamma \delta}$ and curvature $R^\alpha_{\beta \gamma \delta}$ of d-connection $\Gamma^\alpha_{\beta \gamma}$ are defined in a standard manner:

$$T(\delta_\gamma, \delta_\beta) = T^\alpha_{\beta \gamma} \delta_\alpha,$$

$$R^\alpha_{\beta \gamma \delta} = R^\alpha_{\beta \gamma \delta} \delta_\alpha.$$
where

\[ T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} + w^\alpha_{\beta\gamma}, \]  

(2.6)

and

\[ R(\delta_\tau, \delta_\gamma, \delta_\beta) = R^\alpha_{\beta\gamma\tau} \delta_\alpha, \]

where

\[ R^\alpha_{\beta\gamma\tau} = \delta_\tau \Gamma^\alpha_{\beta\gamma} - \delta_\gamma \Gamma^\alpha_{\beta\tau} - \Gamma^\varphi_{\beta\gamma} \Gamma^\alpha_{\varphi\tau} - \Gamma^\varphi_{\beta\tau} \Gamma^\alpha_{\varphi\gamma} + \Gamma^\alpha_{\beta\varphi} w^\varphi_{\gamma\tau}. \]  

(2.7)

In formulas (2.6) and (2.7) we have used nonholonomy coefficients \( w^\alpha_{\beta\gamma} \) of the adapted frames (2.3)

\[ [\delta_\alpha, \delta_\beta] = \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma_{\alpha\beta} \delta_\beta. \]  

(2.8)

Another important geometric structure on \( E \) which will be considered is the metric structure on \( G \), defined as a nondegenerate, second order, with constant signature (in this work positive), tensor field \( G_{\alpha\beta} \) on \( E \). We can associate a map

\[ G(u) : T_u E \times T_u E \rightarrow R \]

to a metric

\[ G = G_{\alpha\beta} du^\alpha \otimes du^\beta \]

parametrized by a symmetric, \( \text{rank} G = q \), matrix

\[ \begin{pmatrix} G_{ij} & G_{ia} \\ G_{aj} & G_{ab} \end{pmatrix}, \]

where \( G_{ij} = G(\partial_i, \partial_j), G_{ia} = G(\partial_i, \partial_a), G_{ab} = G(\partial_a, \partial_b) \).

We shall be interested by such a concordance of N-connection and metric structures when \( G(\delta_i, \partial_a) = 0 \), or, equivalently,

\[ N^a_i(x, y) = G_{ib}(x, y)G^{ba}(x, y), \]  

(2.9)
where matrix $G^{ba}$ is inverse to matrix $G_{ab}$. In this case metric $G$ on E is defined by two independent d-tensors, $g_{ij}$ of type $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $h_{ab}$ of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, and can be written with respect to bases (2.4) and (2.5) as

$$G = g_{ij}(x, y)dx^i \otimes dx^j + h_{ab}(x, y)\delta y^a \otimes \delta y^b \quad (2.10)$$

We shall call a metric $G$ compatible with a d-connection $D$ if conditions

$$D_\alpha G_{\beta\gamma} = 0 \quad (2.11)$$

are satisfied.

**Definition 1.** A vector bundle $\mathcal{E}$ on base $M$ provided with compatible $N$-connection, $d$-connection and metric structures (when conditions (2.9) and (2.11) are satisfied) is called a space with local anisotropy, (la-space) and denoted as $\mathcal{E}_N$.

**Remarks.**

1. For the case when instead of v-bundle $\mathcal{E}$ the tangent bundle $TM$ is considered the conditions (2.9), (2.11) and the requirement of compatibility of compatibility of $N$-connections with the almost Hermitian structure on $TM$ lead to the equality [11,12] (see metric (2.10) on $TM$ )

$$h_{ij}(x, y) = g_{ij}(x, y).$$

The metric field $g_{ij}(x, y)$ is the most general form of metric structure with local anisotropy, considered in generalized Lagrange geometry.

2. Metrics on a Lagrange space $(M, \mathcal{L})$ can be considered as a particular case of metrics of type (2.10) on $TM$ for which there is a differentiable function $\mathcal{L} : TM \rightarrow \mathcal{R}$ with the property that d-tensor

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \quad (2.12)$$

is nondegenerate.
3. We obtain a model of Finsler space \((M, L)\) if \(L = L^2\), where \(L\) is a Finsler metric [11-12].

We emphasize that in this work all geometric constructions and results on stochastic calculus will be formulated for the general case of la-spaces.

On a la-space \(E\) we can consider arbitrary compatible with metric \(G\) d-connections \(\Gamma_{\alpha \beta \gamma}\), which are analogous of the affine connections on locally isotropic spaces (with or not torsion). On \(E\) it is defined the canonical d-structure \(\Gamma_{\beta \gamma}^\alpha\) with coefficients generated by components of metric and N-connection

\[
\Gamma_{jk}^i = L_{jk}^i, \quad \Gamma_{ja}^i = C_{ja}^i, \quad \Gamma_{ab}^i = 0, \quad \Gamma_{ia}^i = 0,
\]

\[
\Gamma_{jk}^a = 0, \quad \Gamma_{jb}^a = 0, \quad \Gamma_{bk}^a = L_{bk}^a, \quad \Gamma_{bc}^a = C_{bc}^a.
\]  

(2.13)

where

\[
L_{jk}^i = \frac{1}{2} g^{ip} (\delta_k g_{pj} + \delta_j g_{pk} - \delta_p g_{jk}),
\]

\[
L_{ba}^i = \partial_b N_i^a + \frac{1}{2} h^{ac} (\delta_i h_{bc} - \partial_b N_i^d h_{dc} - \partial_c N_i^d h_{db}),
\]

\[
C_{jc}^i = \frac{1}{2} g^{ik} \partial_b g_{jk},
\]

\[
C_{bc}^a = \frac{1}{2} h^{ad} (\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}).
\]

In formulas (2.13) we have used matrices \(g^{ij}\) and \(h^{ab}\) which are respectively inverse to matrices \(g_{ij}\) and \(h_{ab}\).

We also present the explicit formulas for unholonomy coefficients \(w_{\beta \gamma}^\alpha\) of the adapted frame basis (2.4):

\[
w_{ij}^k = 0, \quad w_{aj}^k = 0, \quad w_{ia}^k = 0, \quad w_{ab}^k = 0, \quad w_{ab}^c = 0,
\]

\[
w_{ij}^a = R_{ij}^a, \quad w_{ai}^b = -\partial_a N_i^b, \quad w_{ia}^b = \partial_a N_i^b.
\]

(2.14)

Putting (2.13) and (2.14) into, correspondingly, (2.6) and (2.7) we can computer the components of canonical torsion \(T_{\beta \gamma}^\alpha\) and curvature \(R_{\beta \gamma \delta}^\alpha\) with respect to the locally adapted bases (2.4) and (2.5) (see [10,11]).
Really, on every la-space $\mathcal{E}_N$ a linear multiconnection d-structure is defined. We can consider at the same time some ways of local transports of d-tensors by using, for instance, an arbitrary d-connection $\Gamma^\alpha_{\beta\gamma}$, the canonical one $\Gamma^\alpha_{\beta\gamma}$, or the so-called Christoffel d-symbols defined as

$$\left\{ \frac{\alpha}{\beta\gamma} \right\} = \frac{1}{2} G^{\alpha\tau}(\delta_{\gamma}G_{\beta\tau} + \delta_{\beta}G_{\gamma\tau} - \delta_{\tau}G_{\beta\gamma}).$$

(2.15)

Every compatible with metric d-connection $\Gamma^\alpha_{\beta\gamma}$ can be characterized by a corresponding deformation d-tensor with respect, for simplicity, to $\left\{ \frac{\alpha}{\beta\gamma} \right\}$:

$$P^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \left\{ \frac{\alpha}{\beta\gamma} \right\}$$

(2.16)

(the deformation of the canonical d-connection is written as $\Gamma^\alpha_{\beta\gamma} - \left\{ \frac{\alpha}{\beta\gamma} \right\}$).

Perhaps, it is more convenient to consider physical models and geometric constructions with respect to the torsionless d-connection $\left\{ \frac{\alpha}{\beta\gamma} \right\}$. The more general ones will be obtained by using deformations of connections of type (2.16). But sometimes it is possible to write out d-covariant equations on $\mathcal{E}_N$, having significance, by changing respectively components $\left\{ \frac{\alpha}{\beta\gamma} \right\}$ on $\Gamma^\alpha_{\beta\gamma}$. This holds for definition of stochastic differential equations on la-spaces (see, in particular, [1,33] on diffusion on Finsler spaces) and in this paper we shall use the last way.

Let suppose that $\mathcal{E}_N$ is locally trivial and $\sigma$-compact. In this case $\mathcal{E}_N$ is a paracompact manifold and has a countable open base. We denote as $F(\mathcal{E}_N)$ the set of all real $C^\infty$-functions on $\mathcal{E}_N$ and as $F_0(\mathcal{E}_N)$ the subclass of $F(\mathcal{E}_N)$ consisting from functions with compact carriers. $F_0(\mathcal{E}_N)$ and $F(\mathcal{E}_N)$ are algebras on the field of real numbers $\mathcal{R}$ with usual operations $f + q, fq$ and $\lambda f$ ($f, q \in F(\mathcal{E}_N)$ or $F_0(\mathcal{E}_N), \lambda \in \mathcal{R}$).

Vector fields on $\mathcal{E}_N$ are defined as maps

$$V : u \in \mathcal{E}_N \rightarrow V(u) \in T_u(\mathcal{E}_N).$$

Vectors $\left( \partial_{\alpha} \right)_u, (\alpha = 0, 1, 2, ..., m + n - 1)$, form a local linear basis in $T_u(\mathcal{E}_N)$. We shall also use decompositions on locally adapted basis (2.4), $\left( \delta_{\alpha} \right)_u$, and denote by $X(\mathcal{E}_N)$ the set of $C^\infty$-vector fields on $\mathcal{E}_N$. 

8
Now, let introduce the bundle of linear adapted frames \( GL(E_N) \) on \( E_N \). As a linear adapted frame \( e = [e_\alpha], (\alpha = 0, 1, \ldots, m+n-1) \), in point \( u \in E_N \) we mean a linear independent system of vectors \( e_\alpha \in T_u(E_N) \) obtained as a linear distinguished transform of local adapted basis (2.4), i.e.\( e_\alpha = e^\alpha \delta_\alpha \), where \( e^\alpha \in GL^d(\mathcal{R}) = GL(n, \mathcal{R}) \oplus GL(m, \mathcal{R}) \). Then \( GL(E_N) \) is defined as the set of all linear adapted frames in all points \( u \in E_N \):

\[
GL(E_N) = \{ r = (u, e), u \in E_N \text{ and } e \text{ is a linear adapted frame in the point } u \}.
\]

Local coordinate charts on \( GL(E_N) \) are defined as \( (\tilde{U}_\alpha, \tilde{\varphi}_\alpha) \), where \( \tilde{U}_\alpha = \{ r = (u, e) \in GL(E_N), u \in U_\alpha \subset E_N, \text{ and } e \text{ is a linear adapted frame in the point } u \} \), \( \tilde{\varphi}_\alpha (r) = (\varphi_\alpha (u) = (u^\alpha), e^\alpha) \) and \( e_\alpha = e^\alpha \delta_\alpha |_u \). So \( GL(E_N) \) has the structure of \( C^\infty \)-manifold of dimension \( n + m + n^2 + m^2 \). Elements \( a \in GL^d(\mathcal{R}) \) act on \( GL(E_N) \) according the formula \( T_a(u, e) = (u, ea) \), where \( (ea)_\alpha = e^\beta \delta_\beta \). The surjective projection \( \pi : GL(E_N) \to E_N \) is defined in a usual manner by the equality \( \pi(u, e) = u \).

Every vector field \( L \in X(E_N) \) induces a vector field \( \tilde{L} \) on \( GL(E_N) \). Really, for \( f \in X(E_N) \) we can consider

\[
(\tilde{L}f) (r) = \frac{d}{dt} f ((\exp tL)u, (\exp tL)_se) |_{t=0},
\]

where \( r = (u, e) \) and

\[
(\exp tL)_se = [(\exp tL)_se_1, (\exp tL)_se_2, \ldots, (\exp tL)_se_{m+n}],
\]

is the differential (an isomorphism \( T_u(E_N) \to T_{(\exp tL)_u}E_N \) for every \( u \in E_N \)) of \( \exp tL \) and the local diffeomorphism \( u \to v(t, u) \) is defined by differential equations

\[
\frac{dv_\alpha}{dt}(t, u) = a^\alpha(v(t, u)),
\]

\[
(L = a^\alpha(u) \delta_\alpha), v(0, u) = u.
\]

Let \( L \in X(E_N) \) and introduce functions \( f_L^\alpha(r) \in F(GL(E_N)) \) for every \( \alpha = 0, 1, 2, \ldots, m+n-1 \) by the equalities

\[
f_L^\alpha(r) = (e^{-1})^\alpha a^\alpha(u)
\]
written in locally adapted coordinates \( r = (u^\alpha, e = (e^\alpha_\alpha)) \) on the manifold \( GL(\mathcal{E}_N) \), where \( L = a^\alpha(u)\delta_\alpha \) and \( e^{-1} \) is the matrix inverse to \( e \). Because the equality (2.20) does not depend on local coordinates, we have defined a global function on \( GL(\mathcal{E}_N) \). It’s obvious that for \( L(1), L(2) \in X(\mathcal{E}_N) \) we have

\[
(\tilde{L}(1)f^\alpha_{L(2)})(r) = f^\alpha_{[L(1)L(2)]}(r),
\]

where \( \tilde{L}(1) \) and \( \tilde{L}(2) \) are constructed similarly to operator (2.18), and

\[
[L(1), L(2)] = L(1)L(2) - L(2)L(1).
\]

A distinguished connection \( \Gamma^\alpha_{\beta\gamma} \) defines the covariant derivation of d-tensors in \( \mathcal{E}_N \) in a usual manner. For example, we can introduce a d-covariant derivation \( DB \) of a d-tensor field \( B(u) = B^\alpha_{\beta_1\beta_2...\beta_q}(u) \) in the form

\[
D_\gamma B^\alpha_{\beta_1\beta_2...\beta_q}(u) = \delta^\gamma_\gamma B^\alpha_{\beta_1\beta_2...\beta_q}(u) + \sum_{\gamma=1}^p \Gamma^\gamma_\gamma(u) B^\alpha_{\beta_1\beta_2...\beta_q}(u) - \sum_{\tau=1}^q \Gamma^\delta_\tau(u) B^\alpha_{\beta_1\beta_2...\beta_q}(u),
\]

(2.21)
or the covariant derivative \( D_Y B \) in the direction \( Y = Y^\alpha\delta_\alpha \in X(\mathcal{E}_N) \),

\[
(D_Y B)^\alpha_{\beta_1\beta_2...\beta_q}(u) = Y^\delta B^\alpha_{\beta_1\beta_2...\beta_q}(u)
\]

(2.22)
and the parallel transport along a (piecewise) smooth curve \( c : \mathcal{R} \supset I = (t_1, t_2) \ni t \to c(t) \) (considering \( B(t) = B(c(t)) \))

\[
\frac{d}{dt} B^\alpha_{\beta_1\beta_2...\beta_q}(t) + \sum_{\gamma=1}^p \Gamma^\gamma_\gamma(c(t)) B^\alpha_{\beta_1\beta_2...\beta_q}(c(t)) \frac{dc^\gamma}{dt} - \sum_{\tau=1}^q \Gamma^\delta_\tau(c(t)) B^\alpha_{\beta_1\beta_2...\beta_q}(c(t)) \frac{dc^\gamma}{dt} = 0.
\]

(2.23)

For every \( r \in GL(\mathcal{E}_N) \) we can define the horizontal subspace

\[
H_r = \{ U = a^\alpha\delta_\alpha \mid u - \Gamma^\alpha_{\beta\gamma}(u) e^\gamma_\beta a^\beta \frac{\partial}{\partial e^\gamma}, a^\alpha \in \mathcal{R}^{m+n} \}
\]

of \( T_u(GL(\mathcal{E}_N)) \). Vector \( U \in H_r \) is called horizontal. Let \( \xi \in T_u(\mathcal{E}_N) \), then \( \tilde{\xi} \in T_r(GL(\mathcal{E}_N)) \) is a horizontal lift of \( \xi \) if the vector \( \tilde{\xi} \) is horizontal, i.e. \( \pi(r) = u \) and
If \( r \) is given as to satisfy \( \pi (r) = u \) the \( \tilde{\xi} \) is uniquely defined. So, for given \( U \in X (E_u) \) there is a unique \( \tilde{U} \in X (GL (E_u)) \), where \( \tilde{U}_r \) is the horizontal lift \( U_{\pi(r)} \) for every \( r \in GL (E_u) \). \( \tilde{U} \) is called the horizontal lift of vector field \( U \). In local coordinates

\[
\tilde{U} = U^\alpha (u) \delta_\alpha - \Gamma^\delta_{\alpha \beta} (u) U^\alpha (u) e^\beta_{\delta/\gamma} \quad \text{if} \quad U = U^\alpha (u) \delta_\alpha.
\]

In a similar manner we can define the horizontal lift

\[
\tilde{c}(t) = (c(t), e(t)) = [e_0(t), e_1(t), ..., e_{q-1}(t)] \in GL (E_u)
\]
of a curve \( c(t) \in E_u \) with the property that \( \pi (\tilde{c}(t)) = c(t) \) for \( t \in I \) and \( \frac{d \tilde{c}}{dt}(t) \) is horizontal. For every \( \alpha = 0, 1, ..., q - 1 \) there is a unique vector field \( \tilde{L}_\alpha \in X (GL (E_u)) \), for which \( \left( \tilde{L}_\alpha \right)_r \) is the horizontal lift of vector \( e_\alpha \in T_u (E_u) \) for every \( r = (u, e = [e_0, e_1, ..., e_{q-1}]) \). In coordinates \( \left( u^\alpha, e^\beta_\gamma \right) \) we can express

\[
\tilde{L}_\alpha = e^\alpha_2 \delta_\alpha - \Gamma^\alpha_{\beta \gamma} e^\beta_\gamma \frac{\partial}{\partial e^\alpha_\gamma}.
\]

Vector fields \( \tilde{L}_\alpha \) form the system of canonical horizontal vector fields.

Let \( B (u) = B_{\beta_1 \beta_2 ... \beta_s}^{\alpha_1 \alpha_2 ... \alpha_p} (u) \) be a \((p,s)\)-tensor field and define a system of smooth functions

\[
F_B (r) = \left( F_{B_{\beta_1 \beta_2 ... \beta_s}^{\alpha_1 \alpha_2 ... \alpha_p}} (r) = B_{\beta_1 \beta_2 ... \beta_s}^{\gamma_1 \gamma_2 ... \gamma_p} (u) e_{\gamma_1}^{\alpha_1} e_{\gamma_2}^{\alpha_2} ... e_{\gamma_p}^{\alpha_p} e^\beta_1 e^\beta_2 ... e^\beta_s \right)
\]

(the scalarization of the \( d \)-tensor field \( B (r) \) with the respect to the locally adapted basis \( e \))
on \( GL (E_u) \), where we consider that

\[
B (u) = F_{B_{\beta_1 \beta_2 ... \beta_s}^{\alpha_1 \alpha_2 ... \alpha_p}} (u) e_{\alpha_1} \otimes e_{\alpha_2} \otimes ... e_{\alpha_p} \otimes e^\beta_1 \otimes e^\beta_2 ... \otimes e^\beta_s,
\]

the matrix \( e^\beta_\gamma \) is inverse to the matrix \( e_{\alpha_\gamma} \), basis \( e_\gamma \) is dual to \( e \) and \( r = (u, e) \). It is easy to verify that

\[
\tilde{L}_\alpha (F_{B_{\beta_1 \beta_2 ... \beta_s}^{\alpha_1 \alpha_2 ... \alpha_p}} (r)) = (F_{\nabla_B})_{\beta_1 \beta_2 ... \beta_s}^{\alpha_1 \alpha_2 ... \alpha_p} (r)
\]

(2.25)

where the covariant derivation \( \nabla_\alpha A^\beta_\gamma = A^\beta_\gamma \) is taken by using connection

\[
\Gamma^\alpha_{\beta \gamma} = e^\alpha_\delta e^\beta_\gamma e^\gamma_\alpha + e^\alpha_\gamma e^\beta_\delta e^\delta_\alpha.
\]
in $GL(\mathcal{E}_N)$ (induced from $\mathcal{E}_N$).

In our further considerations we shall also use the bundle of orthonormalized adapted frames on $\mathcal{E}_N$, defined as a subbundle of $GL(\mathcal{E}_N)$ satisfying conditions:

$$O(\mathcal{E}_N) = \{ r = (u, e) \in GL(\mathcal{E}_N), e \text{ is a orthonormalized basis in } T_u(\mathcal{E}_N) \}.$$ 

So $r = (u^\alpha, e^\alpha_\underline{\alpha} \in O(\mathcal{E}_N))$ if and only if

$$G^{\alpha\beta} e^\alpha_\underline{\alpha} e^\beta_\underline{\beta} = \delta_{\underline{\alpha}\underline{\beta}}$$

or, equivalently,

$$\sum_{\alpha=0}^{q-1} e^\alpha_\underline{\alpha} e^\beta_\underline{\alpha} = G^{\alpha\beta},$$

where the matrix $G^{\alpha\beta}$ is inverse to the matrix $G_{\alpha\beta}$ from (2.10).

### III. STOCHASTIC DIFFERENTIAL EQUATIONS IN VECTOR BUNDLE SPACES

In this Section we assume that the reader is familiar with the concepts and basic results on stochastic calculus, Brownian motion and diffusion processes (an excellent presentation can be found in [7-9, 18-20], see also a brief introduction into the mentioned subjects in the Appendix of this paper). The purpose of the Section is to extend the theory of stochastic differential equations on Riemannian spaces [7-9] to the case of spaces with general anisotropy, defined in the previous Section as $v$-bundles.

Let $A_0, A_1, ..., A_r \in X(\mathcal{E}_N)$ and consider stochastic differential equations

$$dU(t) = A_\alpha \circ dB^\alpha(t) + A_0(U(t)) dt,$$  

(3.1)

where $\hat{\alpha} = 1, 2, ..., r$ and $\circ$ is the symmetric Q-product (see Appendix A1). We shall use the point compactification of space $\mathcal{E}_N$ and write $\widehat{\mathcal{E}}_N = \mathcal{E}$ or $\widehat{\mathcal{E}}_N = \mathcal{E} \cup \{\Delta\}$ in dependence of that if $\mathcal{E}_N$ is compact or noncompact. By $\widehat{\mathcal{W}}(\mathcal{E}_N)$ we denote the space of paths in $\mathcal{E}_N$, defined as

$$\widehat{\mathcal{W}}(\mathcal{E}_N) = \{ w : w \text{ is a smooth map } [0,\infty) \to \widehat{\mathcal{E}}_N \text{ with the property that } w(0) \in \mathcal{E}_N \text{ and } w(t) = \Delta, w(t') = \Delta \text{ for all } t' \geq t \}$$
and by $\mathcal{B} \left( \hat{\mathcal{W}} (\mathcal{E}_N) \right)$ the $\sigma$-field generated by Borel cylindrical sets.

The explosion moment $e(w)$ is defined as

$$e(w) = \inf \{ t, w(t) = \Delta \}$$

**Definition 3.1.** The solution $U = U(t)$ of equation (3.1) in $v$-bundle space $\mathcal{E}_N$ is defined as such a $(\mathcal{F}_t)$-compatible $\hat{\mathcal{W}} (\mathcal{E}_N)$-valued random element (i.e. as a smooth process in $\mathcal{E}_N$ with the trap $\Delta$), given on the probability space with filtration $(\mathcal{F}_t)$ and $r$-dimensional $\mathcal{F}_t$-Brownian motion $B = B(t)$, with $B(0) = 0$, for which

$$f(U(t)) - f(U(0)) = \int_0^t A_\alpha(t)(U(s)) \delta B_\alpha(s) + \int_0^t (A \circ f)(U(s)) \, ds$$

for every $f \in F_0(\mathcal{E}_N)$ (we consider $f(\Delta) = 0$), where the first term is understood as a Fisk-Stratonovich integral.

In (3.2) we use $\delta B_\alpha(s)$ instead of $dB_\alpha(s)$ because on $\mathcal{E}_N$ the Brownian motion must be adapted to the N-connection structure.

In a manner similar to that for stochastic equations on Riemannian spaces [7] we can construct the unique strong solution to the equations (3.1). To do this we have to use the space of paths in $\mathcal{R}^r$ starting in point 0, denoted as $W_0^r$, the Wiener measure $\mathcal{P}^W$ on $W_0^r$, $\sigma$-field $\mathcal{B}_t \left( W_0^r \right)$-generated by Borel cylindrical sets up to moment $t$ and the similarly defined $\sigma$-field.

**Theorem 3.1.** There is a function $F : \mathcal{E}_N \times W_0^r \to \hat{\mathcal{W}} (\mathcal{E}_N)$ being $\bigcap_\mu \mathcal{B} (\mathcal{E}_N) \times \mathcal{B}_t (W_0^r)^{\mu \times \mathcal{P}^W} / \mathcal{B}_t (\hat{\mathcal{W}} (\mathcal{F}_t))$-measurable (index $\mu$ runs all probabilities in $(\mathcal{E}_N, \mathcal{B}(\mathcal{E}_N))$) for every $t \geq 0$ and having properties:

1) For every $U(t)$ and Brownian motion $B = B(t)$ the equality $U = F(U(0), B)$ a.s. is satisfied.

2) For every $r$-dimensional $(\mathcal{F}_t)$-Brownian motion $B = B(t)$ with $B = B(0)$, defined on the probability space with filtration $\mathcal{F}_t$, and $\mathcal{E}_N$-valued $\mathcal{F}_t$-measurable random element $\xi$,
the function \( U = F(\xi, B) \) is the solution of the differential equation (3.1) with \( U(0) = \xi \), a.s.

**Sketch of the proof.** Let take a compact coordinate vicinity \( V \) with respect to a locally adapted basis \( \delta_\alpha \) and express \( A_\alpha = \sigma_\alpha(u) \delta_\alpha \), where functions \( \sigma_\alpha(u) \) are considered as bounded smooth functions in \( \mathcal{R}^{m+n} \), and consider on \( V \) the stochastic differential equation

\[
dU_\alpha^\alpha_t = \sigma_\alpha^\alpha(U_\alpha^\alpha_t) \circ dB_\alpha^\alpha(t) + \sigma_0^\alpha(U_t) dt,
\]

\( U_0^\alpha = u^\alpha, (\alpha = 0, 1, ... q - 1) \).

Equations (3.3) are equivalent to

\[
dU_\alpha^\alpha_t = \sigma_\alpha^\alpha(U_\alpha^\alpha_t) dB_\alpha^\alpha(t) + \sigma_0^\alpha(U_t) dt,
\]

where \( \sigma_0^\alpha(u) = \sigma_0^\alpha(u) + \frac{1}{2} \sum_{\alpha=1}^n \left( \frac{\delta \sigma_\alpha^\alpha(u)}{\delta u^\alpha} \right) \sigma_\beta^\beta(u) \). It’s known [7] that (3.3) has a unique strong solution \( F : \mathcal{R}^{n+m} \times W_0 \rightarrow \hat{W}^{n+m} \) or \( F(u, w) = (U(t, u, w)) \). Taking \( \tau_V(w) = \inf\{t : U(t, u, w) \in V \} \) we define

\[
U_V(t, u, w) = U(t \wedge \tau_V(w), u, w)
\]

In a point \( u \in V \cap \hat{V} \), where \( \hat{V} \) is covered by local coordinates \( u^{\tilde{\alpha}} \), we have to consider transformations

\[
\sigma_\alpha^{\tilde{\alpha}}(\tilde{u}(u)) = \sigma_\alpha^\alpha(u) \frac{\partial u^{\tilde{\alpha}}}{\partial u^\alpha},
\]

where coordinate transforms \( u^{\tilde{\alpha}}(u^\alpha) \) satisfy the properties (2.1). The global solution of (3.3) can be constructed by gluing together functions (3.4) defined on corresponding coordinate regions covering \( \mathcal{E}_N \). ♣ (end of the proof)

Let \( P_u \) be a probability law on \( \hat{W}(\mathcal{E}_N) \) of a solution \( U = U(t) \) of equation (3.1) with initial conditions \( U(0) = u \). Taking into account the uniqueness of the mentioned solution we can prove that \( U = U(t) \) is a \( \mathcal{A} \)-diffusion and satisfy the Markov property [7] (see also Appendix A3). Really, because for every \( f \in F_0(\mathcal{E}_N) \)
\[ df(U(t)) = (A_\alpha f)(U(t)) \circ dw^\alpha + (A_0 f)(U(t)) \, dt = \]

\[ (A_\alpha f)(U(t)) \, dw^\alpha + (A_0 f)(U(t)) \, d + \frac{1}{2} d(A_\alpha f)(U(t)) \cdot dw^\alpha(t) \]

and

\[ d(A_\beta f)(U(t)) = A_\alpha \left( A_\beta f \right)(U(t)) \circ dw^\alpha(t) + (A_0 A_\beta f)(U(t)) \, dt, \]

we have

\[ d(A_\alpha f)(U(t)) \cdot dw^\alpha(t) = \sum_{\hat{\alpha}=1}^{r} A_\alpha (A_\alpha f)(U(t)) \, dt. \]

Consequently, it follows that

\[ df(U(t)) = (A_\alpha f)(U(t)) \, dw^\alpha(t) + (A f)(U(t)) \, dt, \]

i.e. the operator \((Af)\), defined by the equality

\[ Af = \frac{1}{2} \sum_{\hat{\alpha}=1}^{r} A_\alpha (A_\alpha f) + A_0 f \] \hspace{1cm} (3.5)

generates a diffusion process \(\{P_u\}, u \in \mathcal{E}_N\).

The above presented results are summarized in this form:

**Theorem 3.2.** A second order differential operator \(Af\) generates a \(A\)-diffusion on \(\tilde{W}(\mathcal{E}_N)\) of a solution \(U = U(t)\) of the equation (3.5) with initial condition \(U(0) = u\).

Using similar considerations as in flat spaces [7] on carts covering \(\mathcal{E}_N\), we can prove the uniqueness of \(A\)-diffusion\(\{P_u\}, u \in \mathcal{E}_N\) on \(\tilde{W}(\mathcal{E}_N)\).

**IV. HEAT EQUATIONS IN BUNDLE SPACES AND FLOWS OF DIFFEOMORPHISMS**

Let \(v\)-bundle \(\mathcal{E}_N\) be a compact manifold of class \(C^\infty\). We consider operators

\[ A_0, A_1, \ldots, A_r \in X(\mathcal{E}_N) \]
and suppose that the property

$$E[\sup_{t \in [0,1]} \sup_{u \in \mathcal{U}} |D^\omega \{f(U(t, u, w))\}|] < \infty$$

is satisfied for all $f \in F_0(\mathcal{E}_N)$ and every multiindex $\alpha$ in the coordinate vicinity $\mathcal{U}$ with $\overline{\mathcal{U}}$ being compact for every $T > 0$. The heat equation in $F_0(\mathcal{E}_N)$ is written as

$$\frac{\partial \nu}{\partial t}(t,u) = A\nu(t,u),$$

(4.1)

$$\lim_{t \downarrow 0, u \to u} \nu(t, u) = f(u),$$

where operator $A$ acting on $F(F_0(\mathcal{E}_N))$ is defined in (3.5).

We denote by $C^{1,2}([0, \infty) \times F_0(\mathcal{E}_N))$ the set of all functions $f(t, u)$ on $[0, \infty) \times \mathcal{E}_N$ being smoothly differentiable on $t$ and twice differentiable on $u$.

The existence and properties of solutions of equations (4.1) are stated according the theorem:

**Theorem 4.1.** The function

$$\zeta(t, u) = E[f(U(t, u, w))] \in C^\infty([0, \infty) \times \mathcal{E}_N)$$

(4.2)

$f \in F_0(\mathcal{E}_N)$ satisfies heat equation (4.1). Inversely, if a bounded function $\nu(t, u) \in C^{1,2}([0, \infty) \times \mathcal{E}_N)$ solves equation (4.1) and satisfies the condition

$$\lim_{k \uparrow \infty} E[\nu(t - \sigma_k, U(\sigma_k, u, w)) : \sigma_k \leq t] = 0$$

(4.3)

for every $t > 0$ and $u \in \mathcal{E}_N$, where $\sigma_k = \inf\{t, U(t, u, w) \in D_k\}$ and $D_k$ is an increasing sequence with respect to closed sets in $\mathcal{E}_N, \bigcup_k D_k = \mathcal{E}_N$.

Sketch of the proof. The function $\zeta(t, u)$ is a function on $\mathcal{E}_N$, because $u \to f(U(t, u, w)) \in C^\infty$ and in this case the derivation under mathematical expectation symbol is possible. According to (3.2) we have

$$f(U(t, u, w)) - f(u) = \text{martingale} + \int_0^t (At)(U(s, u, w)) \, ds$$

for every $u \in \mathcal{E}_N$, i.e.
\[ \zeta(t, u) = f(u) + \int_0^t E[(Af)(U(s, u))] \, ds. \] (4.4)

Because \( A^n f \in F_0(\mathcal{E}_N), (n = 1, 2, \ldots) \), we can write
\[
\zeta(t, u) = f(u) + t(Af)(u) + \int_0^t dt_1 \int_0^{t_1} E \left[ (A^2 f)(U(t_2, u, w)) \right] dt_2 = \\
f(u) + t(Af)(u) + \frac{t^2}{2} (A^2 f)(u) + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} E[(A^3 f)(U(t_3, u, w))] dt_3 = \\
f(u) + t(Af)(u) + \frac{t^2}{2} (A^2 f)(u) + \ldots + \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} E[(A^n f)(U(t_n, u, w))] dt_n
\]
from which it is clear that
\[ \zeta(t, u) \in C^\infty([0, \infty) \times \mathcal{E}_N). \]

In Chapter V, Section 3 of the monograph [7] it is proved the equality
\[ (A\zeta_t)(u) = E[(Af)U(t, u, w)] \] (4.5)
for every \( t \geq 0 \), where \( \zeta_t(u) = \zeta(t, u) \).

From (4.4) and (4.5) one follows that
\[ \zeta(t, u) = f(u) + \int_0^t (A\zeta)(s, u) \, ds \]
and
\[ \frac{\partial \zeta}{\partial t}(t, u) = A\zeta(t, u), \]
i.e. \( \zeta = \zeta(t, u) \) satisfies the heat equation (4.1).

Inversely, let \( \nu(t, u) \in C^{1,2}([0, \infty) \times \mathcal{E}_N) \) be a bounded solution of the equation (4.1). Taking into account that \( P(e[U(\cdot, u, w))] = \infty) = 1 \) for every \( u \in \mathcal{E}_N \) and using the Ito formula (see (A6) in the Appendix) we obtain that for every \( t_0 > 0 \) and \( 0 \leq t \leq t_0 \).

\[
E[\nu(t_0 - t \wedge \sigma_n, U(t \wedge \sigma_n, u, w))] - \nu(t_0, u) = \\
E\left[ \int_0^{t \wedge \sigma_n} \{(A\nu)(t_0 - s, U(s, u, w)) - \frac{\partial \nu}{\partial t}(t_0 - s, U(s, u, w))\} \, ds \right].
\] (4.6)
Supposing that conditions (4.3) are satisfied and considering \( n \uparrow \infty \) we obtain

\[
E = \{ \nu (t_0 - t, U(t, u, w)) ; \epsilon [U(\cdot, u, w)] > t \} = \nu (t_0, u) .
\]

For \( t \uparrow t_0 \) we have \( E[f(U(t_0, u, w))] = \zeta (t_0, u) \), i.e. \( \nu (t, u) = \zeta (t, u) \).

Remarks; 1. The conditions (4.3) are necessary in order in order to select a unique solution of (4.1).

2. Defining

\[
\zeta (t, u) = E[\exp \left\{ \int_0^t C(U(s, u, w)) ds \right\} f(U(t, u, w))]
\]

instead of (4.2) we generate the solution of the generalized heat equation in \( E_N \):

\[
\frac{\partial \nu}{\partial t} (t, u) = (Av)(t, u) + C(u) \nu (t, u),
\]

\[
\lim_{t \downarrow 0, u \to u} \nu (t, \overline{u}) = f(u).
\]

For given vector fields \( A(\alpha) \in X(E_N), (\alpha) = 0, 1, ..., r \) in Section III we have constructed the map

\[
U = (U(t, u, w)) : E_N \times W_0^r \ni (u, w) \mapsto U(\cdot, u, w) \in \hat{W}(E_N),
\]

which can be constructed as a map of type

\[
[0, \infty) \times E_N \times W_0^r \ni (u, w) \mapsto U(t, u, w) \in \hat{E}_N.
\]

Let us show that map \( u \in E_N \to U(t, u, w) \in \hat{E}_N \) is a local diffeomorphism of the manifold \( E_N \) for every fixed \( t \geq 0 \) and almost every \( w \) that \( \in E_N \).

We first consider the case when \( E_N \cong \mathcal{R}^{n+m}, \sigma (u) = (\sigma^\alpha (u)) \in \mathcal{R}^{n+m} \otimes \mathcal{R}^{n+m} \) and \( b(u) = (b^\alpha (u)) \in \mathcal{R}^{n+m} \) are given smooth functions (i.e. \( C^\infty \)-functions) on \( \mathcal{R}^{n+m} \), \( \| \sigma (u) \| + \| b(u) \| \leq K (1 + |u|) \) for a constant \( K > 0 \) and all derivations of \( \sigma^\alpha \) and \( b^\alpha \) are bounded. It is known [7] that there is a unique solution \( U = U(t,u,w) \), with the property that \( E[[U(t)]^p] < \infty \) for all \( p > 1 \), of the equation

\[
dU^\alpha_t = \sigma^\alpha (U_t) dw^\alpha (t) + b^\alpha (U_t) dt,
\]

\[
U_0 = u, (\alpha = 1, 2, ..., m + n - 1),
\]

(4.7)
defined on the space \( (W^r_0, P^W) \) with the flow \( (\mathcal{F}^0_t) \).

In order to show that the map \( u \rightarrow U(t, u, w) \) is a diffeomorphism of \( \mathcal{R}^{n+m} \) it is more convenient to use the Fisk-Stratonovich differential (see, for example, the Appendix A1) and to write the equation (4.7) equivalently as

\[
dU_t^\alpha = \sigma^\alpha_\alpha (U_t) \circ \delta w^\alpha (t) + \tilde{b}^\alpha (U_t) \, dt, \tag{4.8}
\]

\[
U_0 = u,
\]

by considering that

\[
\tilde{b}^\alpha (u) = \sigma^\alpha (u) + \frac{1}{2} \sum_{\alpha=1}^r \left( \delta_\beta \sigma^\alpha_\alpha \right) \sigma^\beta_\alpha (u).
\tag{4.9}
\]

We emphasize that for solutions of equations of type (4.8) one holds the usual derivation rules as in mathematical analysis.

Let introduce matrices

\[
\sigma'_{\alpha} = \left( \sigma'(u)^{\alpha}_{\alpha\beta} = \frac{1}{\delta \omega} \sigma^\alpha_\alpha (u) \right), \quad b' (u) = \left( b'(u)^{\alpha}_{\beta} = \frac{\delta b^\alpha}{\delta \omega} \right), \quad I = \delta^\alpha_\alpha
\]

and the Jacobi matrix

\[
Y (t) = \left( Y^\alpha_\beta (t) = \frac{dU^\alpha_t}{d\omega} (t, u, w) \right),
\]

which satisfy the matrix equation

\[
Y (t) = I + \int_0^t \sigma'_{\alpha} (U(s)) Y (s) \circ dw^\alpha (s) + \int_0^t b' (U(s)) Y (s) \, ds. \tag{4.10}
\]

As a modification of a process \( U(t, u, w) \) one means a such process \( \hat{U} (t, u, w) \) that

\[
P^W \{ \hat{U} (t, u, w) = U (t, u, w) \text{ for all } t \geq 0 \} = 1 \text{ a.s.}
\]

It is known this result for flows of diffeomorphisms of flat spaces [24-26,7]:

**Theorem 4.2.** Let \( U(t, u, w) \) be the solution of the equation (4.8) (or (4.7)) on Wiener space \( (W^r_0, P^W) \). Then we can choose a modification \( \hat{U} (t, u, w) \) of this solution when the map \( u \rightarrow U(t, u, w) \) is a diffeomorphism \( \mathcal{R}^{n+m} \) a.s. for every \( t \in [0, \infty) \).

Process \( u = \hat{U} (t, u, w) \) is constructed by using equations

\[
dU_t^\alpha = \sigma^\alpha_\alpha (U_t) \circ \delta w^\alpha (t) - b^\alpha (U_t) \, dt,
\]

\[
U_0 = u.
\]

Then for every fixed \( T > 0 \) we have

\[
U (T - t, u, w) = \hat{U} (t, U (T, u, w), \hat{w})
\]
for every $0 \leq t \leq T$ and $u$ $P^W$-a.s., where the Wiener process $\hat{w}$ is defined as $\hat{w}(t) = w(T - t) - w(T), 0 \leq t \leq T$.

Now we can extend the results on flows of diffeomorphisms of stochastic processes to $v$-bundles. The solution $U(t, u, w)$ of the equation (3.1) can be considered as the set of maps $U_t: u \rightarrow U(t, u, w)$ from $E_N$ to $E_N = E_N \cup \{\triangle\}$.

**Theorem 4.3.** A process $|U|(t, u, w)$ has such a modification, for simplicity let denote it also as $U(t, u, w)$, that the map $U_t: u \rightarrow U(t, u, w)$ belongs to the class $C^\infty$ for every $f \in F_0(E_N)$ and all fixed $t \in [0, \infty)$ a.s. In addition, for every $u \in U$ and $t \in [0, \infty)$ the differential of map $u \rightarrow U(t, u, w)$,

$$U(t, u, w)_*: T_u(U(t, u, w)) \rightarrow T_{U(t,u,w)}(E_N),$$

is an isomorphism, a.s., in the set $\{w: U(t, u, w) \in E_N\}$.

**Proof.** Let $u_0 \in E_N$ and fix $t \in [0, \infty)$. We can find a sequence of coordinate carts $U_1, U_2, ..., U_p \subset E_N$ that for almost all $w$ that $U(t, u_0 \subset w) \in E_N$ there is an integer $p > 0$ that $\{U(s, u_0, w) : s \in [(k - 1)t/p, kt/p]\} \subset U_\parallel, (k = 1, 2, ..., p)$. According to the theorem 4.2 we can conclude that for every coordinate cart $U$ and $\{U(s, u_0, w) : s \in [0, 1]\} \subset U$ a map $v \rightarrow U(t_0, v, w)$ is a diffeomorphism in the neighborhood of $v_0$. The proof of the theorem follows from the relation $U(t, w_0, w) = [U_{t/p}(\theta_{(p-1)t/p}w \circ ... \circ U_{t/p}(\theta_{t/p}w) \circ U_{t/p}^{-1}(u_0)],$ where $\theta_t: W_0^r \rightarrow W_0^r$ is defined as $(\theta_tw)(s) = w(t + s) - w(t)$.\)

Let $A_0, A_1, ..., A_r \in X(E_N)$ and $U_t = (U(t, u, w))$ is a flow of diffeomorphisms on $E_N$. Then $\tilde{A}_0, \tilde{A}_1, ..., \tilde{A}_r \in X(GL(E_N))$ define a flow of diffeomorphisms $r_t = (r(t, r, w))$ on $GL(E_N)$ with $(r(t, r, w)) = (U(t, u, w), e(t, u, w))$, where $r = (u, e)$ and $e(t, r, u) = U(t, u, w)_*e$ is the differential of the map $u \rightarrow U(t, u, w)$ satisfying the property $U(t, u, w)_*e = [U(t, u, w)_*e_0, U(t, u, w)_*e_1, ..., U(t, u, w)_*e_{q-1}]$. In local coordinates $A_\alpha^\alpha(u) = \sigma_\alpha^\alpha, (\alpha = 1, 2, ..., r), A_0(u) = b^\alpha(u)\delta_\alpha, e_\alpha^\alpha(t, u, w) = Y_\alpha^\alpha(t, u, w)\epsilon_\alpha^\alpha$, where $Y_\alpha^\alpha(t, u, w)$ is defined from (4.10). So we can construct flows of diffeomorphisms of the bundle $E_N$. \)

20
V. NONDEGENERATE DIFFUSION IN BUNDLE SPACES

Let a v-bundle $E \nabla$ be provided with a positively defined metric of type (2.10) being compatible with a d-connection $D = \{\Gamma_{\beta\gamma}^\alpha\}$. The connection $D$ allows us to roll $E \nabla$ along a curve $\gamma(t) \subset \mathcal{R}^{n+m}$ in order to draw the curve $c(t)$ on $E \nabla$ as the trace of $\gamma(t)$. More exactly, let $\gamma: [0, \infty) \ni t \mapsto \gamma(t) \subset \mathcal{R}^{n+m}$ be a smooth curve in $\mathcal{R}^{n+m}$, $r = (u, e) \in O(E \nabla)$. We define a curve $\tilde{c}(t) = (c(t), e(t))$ in $O(E \nabla)$ by using the equalities

$$
\frac{dc^\alpha(t)}{dt} = e^\alpha_\alpha(t) \frac{d\gamma^\alpha}{dt},
$$

$$
\frac{dc^\alpha(t)}{dt} = -\Gamma^\alpha_{\beta\gamma}(c(t)) e^\gamma_\alpha(t) \frac{dc^\beta}{dt},
$$

(5.1)

$$
c^\alpha(0) = u^\alpha, e^\alpha_\alpha(0) = e^\alpha_\alpha.
$$

Equations (5.1) can be written as

$$
\frac{d\tilde{c}(t)}{dt} = \tilde{L}_\alpha(\tilde{c}(t)) d\gamma^\alpha,
$$

$$
\tilde{c}(0) = r,
$$

where $\{\tilde{L}_\alpha\}$ is the system of canonical horizontal vector fields (see (2.21)). Curve $c(t) = \pi(\tilde{c}(t))$ on $E \nabla$ depends on fixing of the initial frame $p$ in a point $u$; this curve is parametrized as $c(t) = c(t, r, \gamma)$, $r = r(u, e)$.

Let $w(t) = (w^\alpha(t))$ is the canonical realization of a $n+m$-dimensional Wiener process. We can define the random curve $U(t) \subset E \nabla$ in a similar manner. Consider $r(t) = (r(t, r, w))$ as the solution of stochastic differential equations

$$
\frac{dr(t)}{dt} = \tilde{L}_\alpha(r(t)) \circ \delta w^\alpha(t),
$$

(5.2)

$$
r(0) = r,
$$

where $r(t, r, w)$ is the flow of diffeomorphisms on $O(E \nabla)$ corresponding to the canonical horizontal vector fields $\tilde{L}_1, \tilde{L}_2, ..., \tilde{L}_{q-1}$ and vanishing drift field $\tilde{L}_0 = 0$. In local coordinates the equations (5.2) are written as

$$
\frac{dU^\alpha(t)}{dt} = e^\alpha_\alpha(t) \circ \delta w^\alpha(t),
$$

21
\[ de^\alpha_\alpha (t) = -\Gamma^\alpha_{\beta\gamma} (U (t)) e^\gamma_\alpha \circ \delta u^\beta, \]

where \( r (t) = (U^\alpha (t), e^\alpha_\alpha (t)) \). It is obvious that \( r (t) = (U^\alpha (t), e^\alpha_\alpha (t)) \in O (E_N) \) if \( r (0) \in O (E_N) \) because \( \tilde{L}_\alpha \) are vector fields on \( O (E_N) \). The random curve \( \{ U^\alpha (t) \} \) on \( E_N \) is defined as \( U (t) = \pi [r (t)] \). We point out that \( aw = (aw (t)) \) is another \((n+m)\)-dimensional Wiener process and as a consequence the probability law \( U (\cdot, r, w) \) does not depend on \( a \in O (n + m) \). It depends only on \( u = \pi (r) \). This law is denoted as \( P_w \) and should be mentioned that it is a Markov process because a similar property has \( r (\cdot, r, w) \).

**Remark 4.1.** We can define \( r (t, r, w) \) as a flow of diffeomorphisms on \( GL (E_N) \) for every \( d \)-connection on \( E_N \). In this case \( \pi [r (\cdot, r, w)] \) does not depend only on \( u = \pi (t) \) and in consequence we do not obtain a Markov process by projecting on \( E_N \). The Markov property of diffusion processes on \( E_N \) is assumed by the conditions of compatibility of metric and \( d \)-connection (2.11) and of vanishing of torsion.

Now let us show that a diffusion \( \{ P_u \} \) on \( E_N \) can be considered as an A-diffusion process with the differential operator

\[ A = \frac{1}{2} \Delta_E + b, \quad (5.3) \]

where \( \Delta_E \) is the Laplace-Beltrami operator on \( E_N \),

\[ \Delta_E f = G^{\alpha\beta} \overrightarrow{D}_\alpha \overrightarrow{D}_\beta f = G^{\alpha\beta} \frac{\delta^2 f}{\delta u^\alpha \delta u^\beta} - \{ \frac{\alpha}{\gamma/\beta} \} \frac{\delta f}{\delta u^\alpha}, \quad (5.4) \]

where operator \( \overrightarrow{D}_\alpha \) is constructed by using Christoffel d-symbols (2.15) and \( b \) is the vector d-field with components

\[ b^\alpha = \frac{1}{2} G^{\beta\gamma} \left( \{ \frac{\alpha}{\beta/\gamma} \} - \Gamma^\alpha_{\beta\gamma} \right), \quad (5.5) \]

**Theorem 5.1.** The solution of stochastic differential equation (5.2) on \( O (E_N) \) defines a flow of diffeomorphisms \( r (t) = (r (t, r, w)) \) on \( O (E_N) \) and its projection \( U (t) = \pi (r (t)) \) defines a diffusion process on \( E_N \) corresponding to the differential operator (5.3).

Proof. Considering \( f (r) \equiv f (u) \) for \( r = (u, e) \) we obtain

\[ f (U (t)) - f (U (0)) = f (r (t)) - f (r (0)) = \int_0^t (\tilde{L}_\alpha f) (r (s)) \circ \delta u^\alpha = \]
\[
\int_0^t \tilde{L}_\alpha f(r(s)) \delta u^\alpha + \frac{1}{2} \int_0^t \sum_{\alpha=0}^{q-1} \tilde{L}_\alpha \left( \tilde{L}_\alpha f \right) (r(s)) \, ds.
\]

Let us show that \( \frac{1}{2} \sum_{\alpha=0}^{q-1} \tilde{L}_\alpha \left( \tilde{L}_\alpha f \right) = Af \). Really, because the operator (5.3) can be written as

\[
A = \frac{1}{2} G^{\alpha \beta} \overrightarrow{D}_\alpha \overrightarrow{D}_\beta = \frac{1}{2} \left( G^{\alpha \beta} \frac{\delta^2}{\delta u^\alpha \delta u^\beta} - \left\{ \frac{\alpha}{\gamma} \right\} \frac{\delta}{\delta u^\alpha} \right)
\]

and taking into account (2.25) we have

\[
\tilde{L}_\alpha \left( \tilde{L}_\alpha f \right) = \tilde{L}_\alpha (F_{\nabla f})_\alpha = (F_{\nabla \nabla f})_\alpha = (\nabla_\gamma \nabla_\delta f) e^\gamma_\alpha e^\delta_\beta.
\]

Now we can write

\[
\sum_{\alpha=0}^{q-1} \tilde{L}_\alpha \left( \tilde{L}_\alpha f \right) = \sum_{\alpha=0}^{q-1} (\overrightarrow{D}_\alpha \overrightarrow{D}_\beta f ) e^\alpha_\alpha e^\beta_\beta = G^{\alpha \beta} \overrightarrow{D}_\alpha \overrightarrow{D}_\beta f
\]

(see (2.26)), which complete our proof. ♦

**Definition 5.1.** The process \( r(t) = (r(t, r, w)) \) from the theorem 5.1 is called the horizontal lift of the A-diffusion \( U(t) \) on \( \mathcal{E}_N \).

**Proposition 5.1.** For every d-vector field \( b = b^\alpha (u) \delta_\alpha \) on \( \mathcal{E}_N \) provided with the canonical d-connection structure there is a d-connection \( D = \{ \Gamma^\alpha_{\beta \gamma} \} \) on \( \mathcal{E}_N \), compatible with \( d \)-metric \( G_{\alpha \beta} \), which satisfies the equality (5.5).

Proof. Let define

\[
\Gamma^\alpha_{\beta \gamma} = \left\{ \frac{\alpha}{\beta \gamma} \right\} + \frac{2}{q-1} \left( \delta^\gamma_\beta b_\gamma - G_{\beta \gamma} b^\alpha \right), \tag{5.6}
\]

where \( b_\alpha = G_{\alpha \beta} b^\beta \). By straightforward calculations we can verify that d-connection (5.6) satisfies the metricity conditions

\[
\delta_\gamma G_{\alpha \beta} - G_{\tau \beta} \Gamma^\tau_{\gamma \alpha} - G_{\alpha \tau} \Gamma^\tau_{\gamma \beta} = 0
\]

and that

\[
\frac{1}{2} G^{\alpha \beta} \left( \left\{ \frac{\gamma}{\alpha \beta} \right\} - \Gamma^\gamma_{\alpha \beta} \right) = b^\gamma. \Diamond
\]

We note that a similar proposition is proved in [7] for, respectively, metric and affine connections on Riemannian and affine connected manifolds: M. Anastasiei proposed [28]
to define Laplace-Beltrami operator (5.4) by using the canonical d-connection (2.13) in generalized Lagrange spaces. Taking into account (2.16) and (2.17) and a corresponding redefinition of components of d-vector fields (5.5), because of the existence of multiconnection structure on the space $H^{2n}$, we conclude that we can equivalently formulate the theory of d-diffusion on $H^{2n}$-space by using both variants of Christoffel d-symbols and canonical d-connection.

**Definition 5.2.** For $A = \frac{1}{2} \Delta_\mathcal{E}$ an A-diffusion $U(t)$ is called a Riemannian motion on $\mathcal{E}_N$.

Let an A-differential operator on $\mathcal{E}_N$ is expressed locally as

$$Af(u) = \frac{1}{2} a^{\alpha\beta}(u) \frac{\delta^2 f}{\delta u^\alpha \delta u^\beta}(u) + b^\alpha(u) \frac{\delta f}{\delta u^\alpha}(u),$$

where $f \in F(\mathcal{E}_N)$, matrix $a^{\alpha\beta}$ is symmetric and nonegatively defined. If $a^{\alpha\beta}(u) \xi_\alpha \xi_\beta > 0$ for all $u$ and $\xi = (\xi_\alpha) \in \mathcal{R}^q \setminus \{0\}$, then the operator $A$ is nondegenerate and the corresponding diffusion is called nondegenerate.

By using a vector d-field $b_\alpha$ we can define the 1-form

$$\omega(b) = b_\alpha(u) \delta u^\alpha,$$

where $b = b^\alpha \delta_\alpha$ and $b_\alpha = G^{\alpha\beta} b_\beta$ in local coordinates. According the de Rham-Codaira theorem [27] we can write

$$\omega(b) = dF + \tilde{\delta} \beta + \alpha \quad (5.7)$$

where $F \in F(\mathcal{E}_N)$, $\beta$ is a 2-form and $\alpha$ is a harmonic 1-form. The scalar product of p-forms $\Lambda_p(\mathcal{E}_N)$ on $\mathcal{E}_N$ is introduced as

$$(\alpha, \beta)_B = \int_{\mathcal{E}_N} <\alpha, \beta> \delta u,$$

where

$$\alpha = \sum_{\gamma_1 < \gamma_2 < \ldots < \gamma_p} \alpha_{\gamma_1 \gamma_2 \ldots \gamma_p} \delta u^{\gamma_1} \wedge \delta u^{\gamma_2} \wedge \ldots \wedge \delta u^{\gamma_p},$$

$$\beta = \sum_{\gamma_1 < \gamma_2 < \ldots < \gamma_p} \beta_{\gamma_1 \gamma_2 \ldots \gamma_p} \delta u^{\gamma_1} \wedge \delta u^{\gamma_2} \wedge \ldots \wedge \delta u^{\gamma_p},$$
\[
\beta^{\gamma_1\gamma_2\ldots\gamma_p} = G^{\gamma_1} G^{\gamma_2} \ldots G^{\gamma_p} \beta^{\tau_1\tau_2\ldots\tau_p},
\]
\[
< \alpha, \beta > = \sum_{\gamma_1 < \gamma_2 < \ldots < \gamma_p} \alpha_{\gamma_1\gamma_2\ldots\gamma_p} (u) \beta^{\gamma_1\gamma_2\ldots\gamma_p} (u),
\]
\[
\delta u = \sqrt{|\det G_{\alpha\beta}|} \delta u^0 \delta u^1 \ldots \delta u^{p-1}.
\]

The operator \( \hat{\delta} : \Lambda_p (\mathcal{E}_N) \to \Lambda_{p-1} (\mathcal{E}_N) \) from (5.7) is defined by the equality
\[
(\delta \alpha, \beta)_p = \left( \alpha, \hat{\delta} \beta \right)_{p-1}, \alpha \in \Lambda_{p-1} (\mathcal{E}_N), \beta \in \Lambda_p (\mathcal{E}_N).
\]

De Rham-Codaira Laplacian \( \Box : \Lambda_p (\mathcal{E}_N) \to \Lambda_p (\mathcal{E}_N) \) is defined by the equality
\[
\Box = - (d \hat{\delta} + \hat{\delta} d).
\]  

(5.8)

A form \( \alpha \in \Lambda_p (\mathcal{E}_N) \) is called as harmonic if \( \Box \alpha = 0 \). It is known that \( \Box \alpha = 0 \) if and only if \( d \alpha = 0 \) and \( \hat{\delta} \alpha = 0 \). For \( f \in F (\mathcal{E}_N) \) and \( U \in \mathbf{X} (\mathcal{E}_N) \) we can define the operators \( grad f \in \mathbf{X} (\mathcal{E}_N) \) and \( div U \in F (\mathcal{E}_N) \) by using correspondingly the equalities
\[
grad f = G^{\alpha\beta} \delta_{\alpha} \delta_{\beta} f
\]  

and
\[
div U = - \hat{\delta} \omega_U = \frac{1}{\sqrt{|\det G|}} \delta_{\alpha} \left( U^\alpha \sqrt{|\det G|} \right).
\]  

(5.10)

The Laplace-Beltrami operator (5.3) can be also written as
\[
\triangle_{\mathcal{E}} f = div (grad f) = - \hat{\delta} \hat{\delta} f
\]  

(5.11)

for \( F (M) \).

Let suggest that \( \mathcal{E}_N \) is compact and oriented and \( \{ P_u \} \) be the system of diffusion measures defined by a \( \Lambda \)-operator (5.3). Because \( \mathcal{E}_N \) is compact \( P_u \) is the probability measure on the set \( \hat{W} (\mathcal{E}_N) = W (\mathcal{E}_N) \) of all continuous paths in \( \mathcal{E}_N \).

**Definition 5.3.** The transition semigroup \( T_t \) of \( \Lambda \)-diffusion is defined by the equality
\[
(T_t f) (u) = \int_{W (\mathcal{E}_N)} f (w (t)) P_u (dw), f \in C (\mathcal{E}_N).
\]  

25
For a connected open region $\Omega \subset \mathcal{E}_N$ we define $\rho^\Omega w \in \hat{W}(\Omega), w \in \hat{W}(\mathcal{E}_N)$ by the equality

$$
(\rho^\Omega w)(t) = \begin{cases} w(t), & \text{if } t < \tau^\Omega(w), \\ \Delta, & \text{if } t \geq \tau^\Omega(w). 
\end{cases}
$$

where $\tau^\Omega(w) = \inf\{t : w(t) \notin \Omega\}$. We denote the image-measure $P^\Omega_u (u \in \Omega)$ on map $\rho^\Omega$ as $P^\Omega_{\Omega}u$; this way we define a probability measure on $\hat{W}(\Omega)$ which will be called as the minimal $A$-diffusion on $\Omega$. The transition group of this diffusion is introduced as

$$
(T_t^\Omega f)(u) = \int_{\hat{W}(\Omega)} f(w(t)) P^\Omega_u (dw) = 
\int_{\hat{W}(\mathcal{E}_N)} f(w(t)) I\{\tau^\Omega(w) > t\} P^\Omega_u (dw), f \in C_p(\Omega).
$$

**Definition 5.4.** The Borel measure $\mu(du)$ on $\mathcal{E}_N$ is called an invariant measure on $A$-diffusion $\{P_u\}$ if $\int T_t f(u) \mu(du) = \int f(u) \mu(du)$ for all $f \in C(\mathcal{E}_N)$.

**Definition 5.5.** An $A$-diffusion $\{P_u\}$ is called symmetrizable (locally symmetrizable) if there is a Borel measure $\nu(du)$ on $\mathcal{E}_N$ ($\nu^\Omega(du)$ on $\Omega$) that

$$
\int_{\mathcal{E}_N} T_t f(u) g(u) \nu(du) = \int_{\mathcal{E}_N} f(u) T_t g(u) \nu(du)
$$

for all $f, g \in C(\mathcal{E}_N)$ and

$$
\int_{\Omega} T_t^\Omega f(u) g(u) \nu^\Omega(du) = \int_{\Omega} f(u) T_t^\Omega g(u) \nu^\Omega(du)
$$

for all $f, g \in C(\Omega)$.

The fundamental properties of $A$-diffusion measures are satisfied by the following theorem and corollary:

**Theorem 5.2.** a) An $A$-diffusion is symmetrizable if and only if $\hat{\delta}\beta = \alpha = 0$ (see (5.8)); this condition is equivalent to the condition that $b = \text{grad}F, F \in F(\mathcal{E}_N)$ and in this case the invariant measures are of type $C \exp[2F(u)]du$, where $C = \text{const}$.

b) An $A$-diffusion is locally symmetrizable if and only if $\hat{\delta}\beta = 0$ (see (5.8)) or, equivalently, $dw(b) = 0$. 

26
c) A measure $cdu$ (constant $c > 0$) is an invariant measure of an $A$-diffusion if and only if $dF = 0$ (see (5.8)) or, equivalently, $\hat{\delta}w(b) = -\text{div}b = 0$.

**Corollary 5.1.** An $A$-diffusion is symmetric with respect to a Riemannian volume $du$ (i.e. is symmetrizable and the measure $\nu$ in (5.14) coincides with $du$) if and only if it is a Brownian motion on $\mathcal{E}_\mathcal{N}$.

We omit the proofs of the theorem 5.2 and corollary 5.1 because they are similar to those presented in [7] for Riemannian manifolds. In our case we have to change differential forms and measures on Riemannian spaces into similar objects on $\mathcal{E}_\mathcal{N}$.

**VI. HEAT EQUATIONS FOR DISTINGUISHED TENSOR FIELDS IN VECTOR BUNDLES**

To generalize the results presented in Section IV to the case of $d$-tensor fields in $\mathcal{E}_\mathcal{N}$ we use the Ito idea of stochastic parallel transport [27,30] (correspondingly adapted to transports in vector bundles provided with $N$-connection structure).

**A. Scalarized Tensor $d$-fields and Heat Equations**

Consider a compact bundle $\mathcal{E}_\mathcal{N}$ and the bundle of orthonormalized adapted frames on $\mathcal{E}_\mathcal{N}$ denoted as $O(\mathcal{E}_\mathcal{N})$. Let $\{\tilde{L}_0, \tilde{L}_1, ..., \tilde{L}_{q-1}\}$ be the system of canonical horizontal vector fields on $O(\mathcal{E}_\mathcal{N})$ (with respect to canonical $d$-connection $\Gamma^\alpha_{\beta\gamma}$. The flow of diffeomorphisms $r(t) = r(t, r, w)$ on $O(\mathcal{E}_\mathcal{N})$ is defined through the solution of equations

$$dr(t) = \tilde{L}_\alpha (r(t)) \circ \delta w^\alpha(t),$$

$$r(0) = r,$$

and this flow defines a diffusion process, the horizontal Brownian motion on $O(\mathcal{E}_\mathcal{N})$, which corresponds to the differential operator

$$\frac{1}{2} \Delta_{O(\mathcal{E}_\mathcal{N})} = \frac{1}{2} \sum_\alpha \tilde{L}_\alpha (\tilde{L}_\alpha).$$

(6.1)
For a tensor d-field $S(u) = S_{\alpha_1\beta_2...\beta_q}(u)$ we can define its scalarization $F_S(r) = F_{S_{\alpha_1\beta_2...\beta_q}}(r)$ (a system of smooth functions on $O(\mathcal{E}_N)$) similarly as we have done in Section II, but in our case by using frames satisfying conditions (2.23) in order to deal with bundle $O(\mathcal{E}_N)$.

The action of Laplace-Beltrami operator on d-tensor fields is defined as

$$(\Delta T)^{\alpha_1\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q} = G^{\alpha\beta} \left( \vec{D}_\alpha \left( \vec{D}_\beta T \right) \right)^{\alpha_1\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q} = G^{\alpha\beta} T^{\alpha_1\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q} \Gamma^\alpha_{\beta\gamma},$$

where $\vec{D}T$ is the covariant derivation with respect to $\Gamma^\alpha_{\beta\gamma}$. We can calculate (by putting formula (2.21) into (6.1)) that

$$\Delta_O(\mathcal{E}_N)(F_{S_{\alpha_1\beta_2...\beta_q}}) = (F\Delta S)^{\alpha_1\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q}.$$

For a given d-tensor field $S = S(u)$ let defined this system of functions on $[0, \infty) \times O(\mathcal{E}_N)$:

$$V^{\alpha_1\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q}(t, r) = E\left[ F_{S_{\alpha_1\beta_2...\beta_q}}(r(t, r, w)) \right].$$

According to the theorem 4.1 $V^{\alpha_1\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q}$ is a unique solution of heat equation

$$\frac{\partial V}{\partial t} = \frac{1}{2} \Delta_O(\mathcal{E}_N) V, \quad (6.2)$$

$$V|_{t=0} = F_{S_{\alpha_1\beta_2...\beta_q}}.$$

In a similar manner we can construct unique solutions of heat equations (6.2) for the case when instead of differential forms one considers $\mathcal{R}^{m+n}$-tensors (see [7] for details concerning Riemannian manifolds). We have to take into account the torsion components of the canonical d-connection on $\mathcal{E}_N$.

**B. Boundary Conditions**

We analyze the heat equations for differential forms on a bounded space $\mathcal{E}_N$:

$$\frac{\partial \alpha}{\partial t} = \frac{1}{2} \square \alpha, \quad (6.3)$$

$$\alpha|_{t=0} = f,$$
\( \alpha_{\text{norm}} = 0, (d\alpha)_{\text{norm}} = 0 \) \hspace{1cm} (6.4)

where \( \Box \) is the de Rham-Codaira Laplacian (5.8),
\[
\alpha_{\text{norm}} = \theta_{q-1}(u) du^{q-1}, \quad (d\alpha)_{\text{norm}} = \sum_{\gamma=1}^{q} \left( \frac{\delta \alpha}{\delta u^{\gamma-1}} - \frac{\delta \alpha_{q-1}}{\delta u^{\gamma}} \right) \delta u^{q-1} \wedge \delta u. \]

We consider the boundary of \( \mathcal{E}_{\mathcal{N}} \) to be a manifold of dimension \( q = m+n \) and denote by \( \tilde{\mathcal{E}}_{\mathcal{N}} \) the interior part of \( \mathcal{E}_{\mathcal{N}} \) and as \( \partial \mathcal{E}_{\mathcal{N}} \) the boundary of \( \mathcal{E}_{\mathcal{N}} \). In the vicinity \( \tilde{\mathcal{U}} \) of the boundary we introduce the system of local coordinates \( u = \{(u^\alpha), u^{q-1} \geq 0\} \) for every \( u \in \mathcal{U} \) and \( u \in \mathcal{U} \cap \partial \mathcal{E}_{\mathcal{N}} \) if and only if \( u^{q-1} = 0 \).

The scalarization of 1-form \( \alpha \) is defined as
\[
[F_\alpha]_\beta (r) = \theta_\beta (u) e^\beta_\beta, \quad r = (u^\beta, e^\beta_\beta) \in \mathcal{O}(\mathcal{E}_{\mathcal{N}}). 
\]

Conditions (6.4) are satisfied if and only if
\[
ed_{q-1} [F_\alpha]_\omega (r) = 0
\]
and
\[
ed_\beta \frac{\delta}{\delta u^{q-1}} [F_\alpha]_\beta (r) = 0,
\]
\( \omega=0,1,2,...,q-1, \) where \( e^\beta_\beta \) is inverse to \( e_\beta^\beta \).

Now we can formulate the Cauchy problem for differential 1-forms (6.3) and (6.4) as a corresponding problem for \( \mathcal{R}^{n+m} \)-valued equivariant functions \( V_\omega (t, r) \) on \( \mathcal{O}(\mathcal{E}_{\mathcal{N}}) \) :
\[
\frac{\partial V_\omega}{\partial r} (t, r) = \frac{1}{2} \{ \Delta_{\mathcal{O}(\mathcal{E}_{\mathcal{N}})} V_\omega (t, r) + R^\beta_\omega (r) V_\beta (t, r), \quad \text{(6.5)}
\]
\[
V_\omega (0, r) = (F_f)_\omega (r), \quad (\beta = 0, 1, ..., q-2), \quad (\omega, \beta = 0, 1, ..., q-1),
\]
\[
ed_\beta \frac{\delta}{\delta u^{q-1}} V_\beta (t, r) \mid_{\partial \mathcal{O}(\mathcal{E}_{\mathcal{N}})} = 0, \quad f^\beta_{q-1} V_\beta (t, r) \mid_{\partial \mathcal{O}(\mathcal{E}_{\mathcal{N}})} = 0,
\]
where \( R^\beta_\omega (r) \) is the scalarization of the Ricci d-tensor and \( \partial \mathcal{O}(\mathcal{E}_{\mathcal{N}}) = \{ r = (u, e) \in \mathcal{O}(\mathcal{E}_{\mathcal{N}}), u \in \partial \mathcal{E}_{\mathcal{N}} \} \).
The Cauchy problem (6.5) can be solved by using the stochastic differential equations for the process \((U(t), c(t))\) on \(R^{n+m} \times R^{(n+m)^2}\):

\[
dU^\alpha_t = e_\beta^\alpha(t) \circ dB^\beta(t) + \delta_{q-1} \delta \varphi(t),
\]

\[
dc^\alpha(t) = -\Gamma^\alpha_{\beta \gamma}(U(t)) e_\delta^\gamma(t) \circ dB^\delta(t) - \Gamma^\alpha_{q-1 \tau}(U(t)) e^\tau_{\beta}(t) \delta \varphi(t),
\]

\[\left(\hat{\beta}, \hat{\tau} = 1, 2, ..., q - 1\right)\]

(6.6)

where \(B^\alpha(t)\) is a \((n+m)\)-dimensional Brownian motion, \(U(t)\) is a nondecreasing process which increase only if \(U(t) \in \partial E_N\). In [7] (Chapter IV,7) it is proved that for every Borel probability measure \(\mu\) on \(R^{n+m} \times R^{(n+m)^2}\) there is a unique solution \((U(t), c(t))\) of equations (6.6) with initial distribution \(\mu\). Because if

\[
G_{\alpha \beta}(U(0)) e^\alpha_\alpha(0) e^\beta_\beta(0) = \delta_{\alpha \beta},
\]

then for every \(t \geq 0\)

\[
G_{\alpha \beta}(U(t)) e^\alpha_\alpha(t) e^\beta_\beta(t) = \delta_{\alpha \beta} \text{ a.s.}
\]

(this is a consequence of the metric compatibility criterions (2.10)) we obtain a diffusion process \(r(t) = (U(t), c(t))\) on \(O(E_N)\). This process is called the horizontal Brownian motion on the bundle \(O(E_N)\) with a reflecting bound. Let introduce the canonical horizontal fields (as in (2.21))

\[\left(\tilde{L}_\alpha F\right)(r) = e^\alpha_\alpha \frac{\partial F(r)}{\partial u^\alpha} - \Gamma^\alpha_{\beta \gamma}(u) e^\gamma_\alpha e^\beta_\gamma \frac{\partial F(r)}{\partial c^\alpha}, r = (u, c),\]

define the Bochner Laplacian as

\[
\Delta_{O(E_N)} = \sum_{\alpha=1}^{q} \tilde{L}_\alpha \left(\tilde{L}_\alpha\right)
\]

and put

\[
\alpha^{q-1q-1}(r) = G^{q-1q-1}(u), \alpha^{q-1\beta} = -e^\tau_{\beta} \Gamma^\beta_{q-1 \tau}(u) G^{q-1q-1}.\]
Theorem 6.1. Let \( r(t) = (U(t), c(t)) \) be a horizontal Brownian motion with reflecting bound giving as a solution of equations (6.6). Then for every smooth function \( S(t, r) \) on \([0, \infty) \times O(E_N)\) we have

\[
dS(t, r(t)) = \hat{L} S(t, r(t)) \delta B + \left\{ \frac{1}{2} \left( \Delta_{O(E_N)} S(t, r(t)) + \frac{\partial S}{\partial t}(t, r(t)) \right) dt + \left( \hat{U}_{q-1} S \right)(t, r(t)) \delta \varphi(t) \right\},
\]

where \( \hat{U}_{q-1} \) is the horizontal lift of the vector field \( U_{q-1} = \frac{\delta}{\delta u_{q-1}} \) defined as

\[
\left( \hat{U}_{q-1} S \right)(t, r) = \frac{\delta S}{\delta u_{q-1}}(t, r) + \frac{\alpha_{q-1}^\beta}{\alpha_{q-1}^q}(r) \frac{\partial S}{\partial e_\beta}(t, r)
\]

and

\[
\delta U_{q-1}(t) \delta U_{q-1}(t) = \alpha_{q-1}^q(r(t)) dt, \delta U_{q-1}(t) \delta e_\beta(t) = \alpha_{q-1}^\beta(r(t)) dt.
\]

The proof of this theorem is a straightforward consequence of the Ito formula (see (a6) in the Appendix) and of the property that \( \sum e_\alpha(t) e_\beta(t) = G_{\alpha\beta}(U(t)) \) (see (2.26)).

Finally, in this subsection, we point out that for diffusion processes we are also dealing with the so-called (A,L)-diffusion for bounded manifolds (see, for example, [7] and formula (a12)) which is defined by second order operators \( A \) and \( L \) given correspondingly on \( E_N \) and \( \partial E_N \).

VII. DISCUSSION

In the present paper we have given a geometric evidence for a generalization of stochastic calculus on spaces with local anisotropy. It was possible a consideration rather similar to that for Riemannian manifolds by using adapted to nonlinear connection lifts to tangent bundles [31] and restricting our analysis to the case of v-bundles provided with compatible N-connection, d-connection and metric structures. We emphasize that in the so-called almost Hermitian model of generalized Lagrange geometry [11,12] this condition is naturally satisfied. As a matter of principle we can construct diffusion processes on every space \( E_N \).
provided with arbitrary d-connection structure. In this case we can formulate all results with respect to an auxiliary convenient d-connection, for instance, induced by the Christoffel d-symbols (2.15), and then by using deformations of type (2.16) (or (2.17)) we shall find the deformed analogous of stochastic differential equations and theirs solutions.

When the results of this paper have been communicated during the Iasi Academic Days, Romania, October 1994 [32] Academician R. Miron and Professor M. Anastasiei pointed our attention to the pioneer works on the theory of diffusion on Finsler manifolds with applications in biology by P.L. Antonelli and T.J. Zastavniak [1,33]. Here we remark that because on Finsler spaces the metric in general is not compatible with connection the definition of stochastic processes is very sophisticated. Perhaps, the incompatible metric and connection structures are more convenient for modeling of stochastic processes in biology and this is successfully exploited by the mentioned authors in spite of the fact that in general it is still unclear the possibility and manner of definition of metric relations in biology. As for formulation of physical models of diffusion in anisotropic media and on locally anisotropic spaces we have to pay a due attention to the mutual concordance of the laws of transport (i.e. of connections) and of metric properties of the space, which in physics plays a crucial role. This allows us to define the Laplace-Beltrami, gradient and divergence operators and in consequence to give the mathematical definition of diffusion process on la-spaces in a standard manner.

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APPENDIX A: STOCHASTIC EQUATIONS AND DIFFUSION PROCESSES ON EUCLIDEAN SPACES

We summarize the results necessary for considerations in the original part of this paper. Details on stochastic calculus and diffusion can be found, for example, in works [7,18-23].

1. Basic Concepts and Notations

Let consider the probability space \((\Omega, \mathcal{F}, P)\), where \((\Omega, \mathcal{F})\) is a measurable space, called the sample space, on which a probability measure \(P\) can be placed. A stochastic process is a collection of random variables \(X = \{X_t; 0 \leq t < \infty\}\) on \((\Omega, \mathcal{F})\), which take values in a second measurable space \((S, \mathcal{B})\), called the state space. For our purposes we suggest that \((S, \mathcal{B})\) is locally a \(d\)-dimensional Euclidean space equipped with a \(\sigma\)-field of Borel sets, when we have the isomorphism \(S \cong \mathbb{R}^d\) and \(\mathcal{B} \cong B(\mathbb{R}^d)\), where \(B(U)\) denotes the smallest \(\sigma\)-field containing all open sets of a topological space \(U\). The index \(t \in [0, \infty)\) of the random variables \(X_t\) will admit a convenient interpretation as time.

We equip the sample space \((\Omega, \mathcal{F})\) with a filtration, i.e. we consider a nondecreasing family \(\{\mathcal{F}_t, t \geq 0\}\) of sub \(\sigma\)-fields of \(\mathcal{F}: \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}\) for \(0 \leq s < t < \infty\). We set \(\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F})\).

One says that a sequence \(X_n\) converges almost surely (one writes in brief a.s.) to \(X\) if for all \(\omega \in \Omega\), excepting subsets of zero probability, one holds the convergence

\[
\lim_{n \to \infty} X_n(\omega) = X(\omega).
\]

A random variable \(X_t\) is called \(p\)-integrable if

\[
\int_\Omega |X(\omega)|^p \, P(d\omega) < \infty, \, p > 0, \, \omega \in \Omega, \text{a.s.} \tag{a1}
\]

\((X_t\) is integrable if (a1) holds for \(p = 1\)). For an integrable variable \(X\) the number

\[
E(X) = \int_\Omega X(\omega) \, P(d\omega)
\]
is the mathematical expectation of $X$ with respect to the probability measure $P$ on $(\Omega, \mathcal{F})$.

Using a sub-$\sigma$-field $\mathcal{G}$ of $\sigma$-field $\mathcal{F}$ we can define the value

$$E(X, \mathcal{G}) = \int \limits_{\mathcal{G}} X(\omega) \, d\omega$$

called as the conditional mathematical expectation of $X$ with respect to $\mathcal{G}$.

Smooth random processes are modeled by the set of all smooth functions $w : [0, \infty) \ni t \to w(t) \in \mathcal{R}^r$, denoted as $W^r = C([0, \infty) \to \mathcal{R}^r)$. Set $W^r$ is complete and separable with respect to the metric

$$\rho(w_1, w_2) = \sum_{n=1}^{\infty} 2^{-n} \left[ \left( \max_{0 \leq t \leq n} |w_1(t) - w_2(t)| \right) \wedge 1 \right],$$

$w_1, w_2 \in W^r$, where $a \wedge 1 = \min\{a, 1\}$.

Let $\mathcal{B}$ ($W^r$) be a topological $\sigma$-field. As a Borel cylindrical set we call a set $B \subset W^r$, defined as $B = \{w : (w(t_1), w(t_2), \ldots, w(t_n)) \}$ and $E \subset \mathcal{B}(\mathcal{R}^{nr})$. We define as $\mathcal{C} \subset \mathcal{B}$ ($W^r$) the set of all Borel cylindrical sets.

**Definition A1.** A process $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is said to be a submartingale (or supermartingale) if for every $0 \leq s < t < \infty$, we have $P$–a.s. $E(X_t|\mathcal{F}) \geq X_s$ (or $E(X_t|\mathcal{F}) \leq X_s$). We shall say that $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is a martingale if it is both a submartingale and a supermartingale.

Let the function $p(t, x), t > 0, x \in \mathcal{R}^r$ is defined as

$$p(t, x) = (2\pi t)^{-\frac{r}{2}} \exp \left[ -\frac{|x|^2}{2t} \right],$$

and $X = (X_t)_{t \in [0, \infty)}$ is a r-dimensional process that for all $0 < t_1 < \ldots < t_m$ and $E_i \in \mathcal{B}(\mathcal{R}^r), i = 1, 2, \ldots, m$,

$$P\{X_{t_1} \in E_1, X_{t_2} \in E_2, \ldots, X_{t_m} \in E_m\} =
\int \limits_{\mathcal{R}^r} \mu(dx) \int \limits_{E_1} p(t_1, x_1 - x) \, dx_1 \int \limits_{E_2} p(t_2 - t_1, x_2 - x_1) \, dx_2 \ldots \int \limits_{E_m} p(t_m - t_{m-1}, x_m - x_{m-1}) \, dx_m,$$  

(a2)
where \( \mu \) is the probability measure on \((\mathbb{R}^r, B(\mathbb{R}^r))\).

**Definition A2.** A process \( X = (X_t) \) with the above stated properties is called a \( r \)-dimensional Brownian motion (or a Wiener process with initial distribution \( \mu \)). A probability \( P^X \) on \((W^r, B(W^r))\), where \( P\{w : w(t_1) \in E_1, w(t_2) \in E_2, \ldots, w(t_m) \in E_m\} \) is given by the right part of (a.2) is called a \( r \)-dimensional Wiener distribution with initial distribution \( \mu \).

Now, let us suggest that on the probability space \((\Omega, \mathcal{F}, P)\) a filtration \((\mathcal{F}_t)_{t \in [0, \infty)}\) is given. We can introduce a \( r \)-dimensional \((\mathcal{F}_t)\)-Brownian motion as a \( d \)-dimensional smooth process \( X = (X_t)_{t \in [0, \infty)} \), \((\mathcal{F}_t)\)-coordinated and satisfying condition

\[
E[\exp[i < \xi, X_t - X_s>] | \mathcal{F}_s] = \exp[-(t-s)|\xi|^2/2] \text{ a.s.}
\]

for every \( \xi \in \mathbb{R}^r \) and \( 0 \leq s < t \).

The next step is the definition of the Ito stochastic integral [21,7,18-20]. Let denote as \( L_2 \) the space of all real measurable processes \( \Phi = \{\Phi(t, u)\}_{t \geq 0} \) on \( \Omega, (\mathcal{F}_t) \)-adapted for every \( T > 0 \),

\[
\| \Phi \|^2_{2,T} \doteq E\left[ \int_0^T \Phi^2(s, \omega) \, ds < \infty \right],
\]

where "\( \doteq \)" means "is defined to be". For \( \Phi \in L_2 \) we write

\[
\| \Phi \|_2 \doteq \sum_{n=1}^{\infty} 2^{-2} \left( \| \Phi \|_{2,n} \wedge 1 \right).
\]

We restrict our considerations to processes of type

\[
\Phi(t, \omega) = f_0(\omega) I_{\{t=0\}} + \sum_{i=0}^{\infty} f_i(\omega) I_{\{t=t_{i+1} \}}(t),
\]

where \( I_A(B) = 1 \), if \( A \subset B \) and \( I_A(B) = 0 \), if \( A \subsetneq B \).

Let denote \( M_2 = \{X = (X_t)_{t \geq 0} : X \text{ is a quadratic integrable martingale on } (\Omega, \mathcal{F}, P) \text{ referring to } (\mathcal{F}_t)_{t \geq 0} \text{ and } X_0 = 0 \text{ a.s.} \} \) and \( M_2^c = \{X \in M_2; \ t \to X \text{ is smooth a.s.} \} \). For \( X \in M_2 \) we use denotations

\[
|X|_T \doteq E \left[ X_T^2 \right]^{1/2}, T > 0,
\]
and $|X| = \sum_{n=1}^{\infty} 2^{-n} (|X|_n \wedge 1)$.

Now we can define stochastic integral on $(\mathcal{F}_t)$-Brownian motion $B(t)$ on $(\Omega, \mathcal{F}, P)$ as a map

$$L_2 \ni \Phi \mapsto I(\Phi) \in \mathcal{M}_2^c.$$

For a process of type (a3) we postulate

$$I(\Phi)(t, \omega) = \sum_{i=0}^{n-1} [f_i(\omega)(B(t_{i+1}, \omega) - B(t_i, \omega)) + f_n(\omega)(B(t, \omega) - B(t_n, \omega))].$$

for $t_n \leq t \leq t_{n+1}, n = 0, 1, 2, \ldots$.

Process $I(\Phi) \in \mathcal{M}_2^c$ defined by (a4) is called the stochastic integral of $\Phi \in L_2$ on Brownian motion $B(t)$ and is denoted as

$$I(\Phi)(t) = \int_0^t \Phi(s, \omega) dB(s, \omega) = \int_0^t \Phi(s) dB(s).$$

It is easy to verify that the integral (a5) satisfies properties:

$$|I(\Phi)|_T = \| \Phi \|_{2,T} = \| \Phi \|_2,$$

$$E\left(I(\Phi)(t)^2\right) = \sum_{i=0}^{\infty} E\left[f_i^2\left(t \wedge t_{i+1} - t \wedge t_i\right)\right] = E\left[\int_0^t \Phi^2(s, w) \, ds\right]$$

and

$$I(\alpha \Phi + \beta \Psi)(t) = \alpha I(\Phi)(t) + \beta I(\Psi)(t),$$

for every $\Phi, \Psi \in L_2, (\alpha, \beta \in \mathbb{R})$ and $t \geq 0$.

Consider a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $(\mathcal{F}_t)$. A random time $T$ is a stopping time of this filtration if the event $\{T \leq t\}$ belongs to the $\sigma$-field $(\mathcal{F}_t)$ for every $t \geq 0$. A random time is an optional time of the given filtration if $\{T \leq t\} \in (\mathcal{F}_t)$ for every $t \geq 0$. A real random process $X = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a local $(\mathcal{F}_t)$-martingale if it is adapted to $(\mathcal{F}_t)$ and there is a sequence of stopping $(\mathcal{F}_t)$-moments $\sigma_n$ with $\sigma_n < \infty, \sigma_n \uparrow \infty$ and $X_n = (X_n(t))$ is a $(\mathcal{F}_t)$-martingale for every $n = 1, 2, \ldots$, where
$X_n = X(t \wedge \sigma_n)$. If $X_n$ is a quadratic integrable martingale for every $n$, than $X$ is called a local quadratic integrable ($\mathcal{F}_t$)-martingale.

Let denote $\mathcal{M}_2^{loc} = \{X = (X_t)_{t \geq 0}, X$ is a locally quadratic integrable ($\mathcal{F}_t$)-martingale and $X_0 = 0$ a.s.$\}$ and $\mathcal{M}_2^{c,loc} = \{X \in \mathcal{M}_2^{loc} : t \to X_t$ is smooth a.s.$\}$. In a similar manner with the Brownian motion we can define stochastic integrals for processes $\Phi \in L^2$ and $\Phi \in L^{loc,2}$ on $\mathcal{M} \subset \mathcal{M}_2^{loc}$ (we have to introduce $M(t_j, \omega)$ instead of $B(t_j, \omega)$ respectively for $t_j = t_{i+1}, t_j = t_i, t_i = t_n, t_j = 1$ in formulas (a4)). In this case one denotes the stochastic integral as

$$I^M(\Phi)(t) = \int_0^t \Phi(s) dM(s).$$

A great part of random processes can be expressed as a sum of a mean motion and fluctuations (Ito processes)

$$X(t) = X(0) + \int_0^t f(s) ds + \int_0^t g(s) dB(s),$$

where $\int_0^t f(s) ds$ defines a mean motion, $\int_0^t g(s) dB(s)$ defines fluctuations and $\int dB$ is a stochastic integral on Brownian motion $B(t)$. In general such processes are sums of processes with limited variations and martingales. Here we consider the so-called smooth semimartingales introduced on a probability space with a given filtration $(\mathcal{F}_t)_{t \geq 0}$, $M^i(t) \in \mathcal{M}_2^{c,loc}$ and $A^i(t)$ being smooth ($\mathcal{F}_t$)-adapted processes with trajectories having a limited variation and $A^i(0) = 0$ (i=1,2,...,r). So a smooth r-dimensional semimartingale can be written as

$$X^i(t) = X^i(0) + M^i(t) + A^i(t).$$

For processes of type (a6) one holds the Ito formula $[21,7,18-20]$ which gives us a differential-integral calculus for paths of random processes:

$$F(X(t)) - F(X(0)) = \int_0^t F'_i(X(s)) dM^i(s) + \frac{1}{2} \int_0^t F''_{ij}(X(s)) d<M^i M^j>(s),$$

where $F'_i = \frac{\partial F}{\partial x^i}, F''_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}, <M^i M^j>$ is the quadratic covariation of processes $M^i, M^j \in \mathcal{M}_2$, which really represents a random process $A = A_t$, parametrized as a difference

37
of two natural integrable processes with the property that $M_tN_t - A_t$ is a $(\mathcal{F}_t)$-martingale. Here we remark that a process $Q = Q_t$ is an increasing integrable process if it is $(\mathcal{F}_t)$-adapted, $Q_0 = 0$, the map $t \to Q_t$ is left-continuous, $Q_t \geq 0$ and $E(Q_t) < \infty$ for every $t \in [0, \infty)$. A process $Q$ is called natural if for every bounded martingale $M = (M_t)$ and every $t \in [0, \infty)$

$$E \left[ \int_0^\tau M_s dA_s \right] = E \left[ \int_0^t M_s dA_s \right].$$

There is another way of definition of stochastic integration instead of Ito integral, the so-called Fisk-Stratonovich integral, which obeys the usual rules of mathematical analysis. Let introduce denotations: $\mathcal{A}$ is the set of all such smooth $(\mathcal{F}_t)$-adapted processes $A = (A_t)$ that $A_0 = 0$ and $t \to A_t$ is a function with limited variation on every finite integral a.s.; $\mathcal{B}$ is the set of all such $(\mathcal{F}_t)$-predictable processes $\Phi = (\Phi_t)$ that with the probability one the function $t \to \Phi_t$ is a bounded function on every finite interval and $(t, \omega) \to X_t(\omega)$ is $\mathcal{C}/\mathcal{B}(\mathcal{R}^r)$-measurable; $\mathcal{O}$ is the set of quasimartingales (for every $X \in \mathcal{O}$ we have the martingale part $M_X$ and the part with limited variation). For every $\Phi \in \mathcal{B}$ and $X \in \mathcal{O}$ one defines the scalar product

$$(\Phi \circ X) = X(0) + \int_0^t \Phi(s, w) dM_X(s) + \int_0^t \Phi(s, w) dA_X(s), t \geq 0,$$

as an element of $\mathcal{O}$. One introduces an element $\Phi \circ dX \in d\mathcal{O}$ as

$$\Phi dX = \Phi \circ dX = d(\Phi \circ X)$$

in order to define the symmetric $Q$-product:

$$Y \circ dX = Y dX + \frac{1}{2} dX dY$$

for $dX \in d\mathcal{O}$ and $Y \in \mathcal{O}$. The stochastic integral $\int_0^t Y \circ dX$ is called the symmetric Fisk-Stratonovich integral.
2. Stochastic Differential Equations

Let denote as $A_{c,r}$ the set of functions satisfying the conditions: \( \alpha(t, w) : [0, \infty) \times W^r \to \mathcal{R}^r \times \mathcal{R}^r \) are \( B((0, \infty)) \times B(\mathcal{R}^r \otimes \mathcal{R}^r) \)-measurable, for every \( t \in [0, \infty) \) a function \( W^r \ni w \to \alpha(t, w) \in R^r \times R^r \) are \( B_t(W^r) / B(R^d \otimes R^c) \)-measurable, where \( R^r \times R^c \) denotes the result of identification of \( R^r \times R^c \) with \( rc \)-dimensional Euclidean space.

Suppose that values \( \alpha \in A_{r,c} \) and \( \beta \in A_{r,1} \) are given and consider the next stochastic differential equations for a \( r \)-dimensional smooth process \( X = (X(t))_{t \geq 0} : \)

\[
dX^\epsilon_t = \sum_{j=1}^{r} \alpha^\epsilon_j(t, X) dB^\gamma(t) + \beta^\epsilon(t, X) dt, \quad (\epsilon = 1, 2, ..., r), \tag{a8}
\]

for simplicity also written as

\[
dX_t = \alpha(t, X) dB(t) + \beta(t, X) dt.
\]

As a weak solution (with respect to a \( c \)-dimensional Brownian motion \( B(t), B(0) = 0 \) a.s.) of the equations (a8) we mean a \( r \)-dimensional smooth process \( X = (X(t))_{t \geq 0} \), defined on the probability space \( (\Omega, \mathcal{F}, P) \) with such a filtration of \( \sigma \)-algebras \( (\mathcal{F}_t)_{t \geq 0} \), that \( X = X(t) \) is adapted to \( (\mathcal{F}_t)_{t \geq 0} \), i.e. a map \( \omega \in \Omega \to X(\omega) \in W^r \) is defined and for every \( t \in [0, \infty) \) this map is \( \mathcal{F}_t / B_1(W^r) \)-measurable; we can define processes \( \Phi^\delta(t, \omega) = \alpha^\delta(t, X(\omega)) \subset \mathcal{L}^{loc}_2 \) and \( \Psi^\delta(t, \omega) = \beta^\delta(t, X(\omega)) \subset \mathcal{L}^{loc}_1 \); values \( X(t) = (X^1(t), X^2(t), ..., X^r(t)) \) and \( B(t) = (B^1(t), B^2(t), ..., B^c(t)) \) satisfy equations

\[
X^i(t) - X^i(0) = \sum_{\beta=1}^{c} \int_{0}^{t} \alpha^\delta_{\beta}(s, X) dB^\beta(s) + \int_{0}^{t} \beta^\delta(s, X) ds, \quad (a9)
\]

with the unit probability, where the integral on \( dB^\beta(s) \) is considered as the Ito integral (a7).

The first and the second terms in (a9) are called correspondingly as the martingale and drift terms.

Let \( \sigma(t, x) = \sigma^j(t, x) \) be a Borel function \( (t, x) \in [0, \infty) \times \mathcal{R}^r \to \mathcal{R}^r \otimes \mathcal{R}^r \) and \( b(t, x) = \{b^i(t, x)\} \) be a Borel function \( (t, x) \in [0, \infty) \times \mathcal{R}^r \to \mathcal{R}^r \). Then \( \alpha(t, w) = \sigma(t, w(t)) \subset A^{r,c} \)
and $\beta(t, w) = b(t, w(t)) \in A^{r,1}$. In this case the stochastic differential equations (a8) are of the Markov type and can be written in the form

$$dX^i(t) = \sum_{k=1}^{r} \sigma^i_k(t, X(t)) dB^k(t) + b^i(t, X(t)) dt. \quad (a10)$$

If $\sigma$ and $b$ depend only on $x \in \mathcal{R}^r$ we obtain a equation with homogeneous in time $t$ coefficients.

Function $\Phi (x, w) : \mathcal{R}^r \times W_0^c \to W^c, W_0^c = \{ w \in \mathcal{C} ([0, \infty) \to \mathcal{R}^r); w(0) = 0 \}$ is called $\tilde{\mathcal{E}} (\mathcal{R}^r \times W_0^c)$-measurable if for every Borel probability measure $\mu$ on $\mathcal{R}^r$ there is a function $\tilde{\Phi}_\mu (x, w) : \mathcal{R}^r \times W_0^c \to W^c$, which is $\mathcal{B}(\mathcal{R}^r \times W_0^c)^{\mu \times P^W} / \mathcal{B} (W^c)$-measurable for all $x(\mu), \Phi (x, w) = \tilde{\Phi}_\mu (x, w)$ and $P^W$-almost all $w$ ($P^W$ is the c-dimensional Wiener measure on $W_0^c$, i.e. a distribution $B$).

A solution $X = (X(t))$ of the equations (a8) with a Brownian motion $B = B(t)$ is called a strong solution if there is a function $F (x, w) : \mathcal{R}^r \times W_0^c \to W^c$, which is $\tilde{\mathcal{E}} (\mathcal{R}^r \times W_0^c)$-measurable for every $x \in \mathcal{R}^r$, $w \to F (x, w)$ is $\mathcal{B}(\mathcal{R}^r \times W_0^c)^{P^W} / \mathcal{B}(W^c)$-measurable for every $t \geq 0$ and $X = F (X(0), B)$ a.s.

We obtain a unique strong solution if for every r-dimensional($\mathcal{F}_{\cup}$)-Brownian motion $B = B(t)$ ($B(0) = 0$) on the probability space with the filtration ($\mathcal{F}_t$) and arbitrary ($\mathcal{F}_0$)-measurable $\mathcal{R}^r$-valued random vector $X = F (\xi, B)$ is a solution of (a8) on this space with $X(0) = \xi$ a.s. So, a strong solution can be considered as a function $F (x, w)$ which generates a solution $X$ of equation (a8) if and only if we shall fix the initial value $X(0)$ and Brownian motion $B$.

3. Diffusion Processes

As the diffusion processes one names the class of processes which are characterized by the Markov property and smooth paths (see details in [22,23,.7]). Here we restrict ourselves with the definition of diffusion processes generated by second order differential operators on $\mathcal{R}^r$:
\[
Af (x) = \frac{1}{2} \sum_{i,j=1}^{r} a^{ij} (x) \frac{\partial^2 f}{\partial x^i \partial x^j} (x) + \sum_{i=1}^{r} b^i (x) \frac{\partial f}{\partial x^i} (x),
\]

where \( a^{ij} (x) \) and \( b^i (x) \) are real smooth functions on \( \mathcal{R}^r \), matrix \( a^{ij} (x) \) is symmetric and positively defined. Let denote by \( \hat{\mathcal{R}}^r = \mathcal{R}^r \cup \{ \Delta \} \) the point compactification of \( \mathcal{R}^r \). Every function \( f \) on \( \mathcal{R}^r \) is considered as a function on \( \hat{\mathcal{R}}^r \) with \( f (\Delta) = 0 \). The region of definition of the operator \((a11)\) is taken the set of doubly differentiable functions with compact carrier, denoted as \( C^2_\mathcal{K}(\mathcal{R}^r) \). Let \( \mathcal{B} (\hat{\mathcal{W}}^r) \) be the \( \sigma \)-field generated by the Borel cylindrical sets, where \( \hat{\mathcal{W}}^r = \{ w : [0, \infty) \ni t \to w (t) \in \hat{\mathcal{R}}^r \text{ is smooth and if } w (t) = \Delta, \text{ then } w (t') = \Delta \text{ for all } t' \geq t \} \). The value \( e (w) = \inf \{ t; w (t) = \Delta, w \in \hat{\mathcal{W}}^r \} \) is called the explosion time of the path \( w \).

**Definition A3.** A system of Markov probability distributions \( \{ P_x, x \in \mathcal{R}^r \} \) on \( (\hat{\mathcal{W}}^r, \mathcal{B} (\hat{\mathcal{W}}^r)) \), which satisfy conditions: \( P_X \{ w : w (0) = x \} = 1 \) for every \( x \in \mathcal{R}^r \); \( f (w (t)) - f (w (0)) - \int_0^t (Af) (w (s)) \, ds \) is a \( \{ P_x, \mathcal{B}_t (\hat{\mathcal{W}}^r) \} \)-martingale for every \( f \in C^2_\mathcal{K}(\mathcal{R}^r) \) and \( x \in \mathcal{R}^r \) defines a diffusion measure generated by an operator \( A \) (or \( A \)-diffusion).

**Definition A4.** A random process \( X = (X (t)) \) on \( \mathcal{R}^r \) is said to be a diffusion process, generated by the operator \( A \) (or simply a \( A \)-diffusion process) if almost all paths \( [t \to X (t)] \in \hat{\mathcal{W}}^r \) and probability law of the process \( X \) coincides with \( P_\mu (\cdot) = \int_{\mathcal{R}^r} P_x (\cdot) \mu (dx) \), where \( \mu \) is the diffusion measure generated by the operator \( A \) and \( \{ P_x \} \) is the probability law of \( X (0) \).

To a given \( A \)-diffusion we can associate a corresponding stochastic differential equation. Let the matrix function \( \sigma (x) = (\sigma^i_j (x)) \in \mathcal{R}^r \times \mathcal{R}^r \) defines \( a^{ij} (x) = \sum_{k=1}^{r} \sigma^i_k (x) \sigma^j_k (x) \) and consider the equations
\[
dX^i (t) = \sum_{k=1}^{r} \sigma^i_k (X (t)) \, dB^k (t) + b^i (X (t)) \, dt.
\]

There is an extension of \((\Omega, \mathcal{F}, P)\) with a filtration \( (\mathcal{F}_t) \) of the probability space
\[
(\hat{\mathcal{W}}^r, \mathcal{B} (\hat{\mathcal{W}}^r), P_X)
\]
and with a filtration \( \mathcal{B}_t (\hat{\mathcal{W}}^r) \) and a \( (\mathcal{F}_t) \)-Brownian motion \( B (t) \) (see [7] and the previous subsections in this Appendix) that putting \( X (t) = w (t) \) and \( e = e (w) \) one obtains for
\[ t \in [0, e) \]
\[ X^i(t) = x^i + \sum_{k=1}^{r} \int_0^t \sigma^i_k(X(s)) dB^k(s) + \int_0^t b^i(X(s)) ds. \]

So \((X(t), B(t))\) is the solution of the equations (a12) with \(X(0) = x\).

If bounded regions are considered, diffusion is described by second order partial differential operators with boundary conditions. Let denote \(D = \mathcal{R}_r^f = \{x = (x^1, x^2, \ldots, x^r); x^r \geq 0\}\), \(\partial D = \{x \in D; x^r = 0\}\), \(D^0 = \{x \in D; x^r > 0\}\). The Wentzell bound operator is defined as a map from \(C^2_K(L)\) to the space of smooth functions on \(\partial D\) of this type:

\[ Lf(x) = \frac{1}{2} \sum_{i,j=1}^{r-1} \alpha^{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{r-1} \beta^i(x) \frac{\partial f}{\partial x^i}(x) + \mu(x) \frac{\partial f}{\partial x^r}(x) - \rho(x) Af(x), \quad (a13) \]

where \(x \in \partial D, \alpha^{ij}(x), \beta^i(x), \mu(x)\) and \(\rho(x)\) are bounded smooth functions on \(\partial D, \alpha^{ij}(x)\) is a symmetric and nondegenerate matrix, \(\mu(x) \geq 0\) and \(\rho(x) \geq 0\).

A diffusion process defined by the operators (a11) and (a13) is called a \((A, L)\)-diffusion.
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