Kinematics of evaporating black holes

G.A. Vilkovisky

Lebedev Physical Institute, and Lebedev Research Center in Physics,
Leninsky Prospect 53, 119991 Moscow, Russia.
E-mail: vilkov@sci.lebedev.ru

Abstract

The correspondence principle and causality divide the spacetime of a macroscopic collapsing mass into three regions: classical, semiclassical, and ultraviolet. The semiclassical region covers the entire evolution of the black hole from the macroscopic to the microscopic scale if the latter is reached. It is shown that the metric in the semiclassical region is expressed purely kinematically through the Bondi charges. The only quantum calculation needed is the one of radiation at infinity. The ultraviolet ignorance of semiclassical theory is irrelevant. The metric with arbitrary Bondi charges is obtained and studied.
1 Introduction

The problem of backreaction of the Hawking radiation [1] remains unsolved. In the general setting, it consists in obtaining the solution of the expectation-value equations

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi T^{\mu\nu}, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{source}} + T^{\mu\nu}_{\text{vac}} \]  

(1.1)

for the gravitational field with the collapse initial conditions. Here \( T^{\mu\nu}_{\text{source}} \) is the energy-momentum tensor of a given matter source, and \( T^{\mu\nu}_{\text{vac}} \) is a certain retarded functional of the curvature: the energy-momentum tensor of the in-vacuum [2]. Although there has been a number of studies (for their discussion and the list of references see the book [3]), no consistent approach to the problem was proposed. Meanwhile, the Hawking effect is within well-established physics and is unambiguously described by semiclassical field theory. Semiclassical theory is incapable of giving the full solution of eq. (1.1) because it fails at small scale but, if the matter source has macroscopic parameters, there is a region of the expectation-value spacetime in which semiclassical theory is valid. Moreover, this region is causally complete (see below) and covers the entire evolution of the black hole from the macroscopic to the microscopic scale if the latter is reached. This is the first thing shown in the present paper. The solution of the backreaction problem should, therefore, be derivable from semiclassical field theory.

Immediately, two questions emerge. First, the semiclassical \( T^{\mu\nu}_{\text{vac}} \) is a sum of the vacuum loops and is not calculable exactly. Then what approximations are to be used? Second, in semiclassical theory, \( T^{\mu\nu}_{\text{vac}} \) is even theoretically calculable only up to terms local in the curvature and proportional to the quantum constant. This arbitrariness is the expression of the ultraviolet ignorance of semiclassical theory. Owing to their locality, the indefinite terms of \( T^{\mu\nu}_{\text{vac}} \) do not affect the radiation at infinity [2] but how can it be that they do not affect the metric in the compact domain? The present paper gives an economical answer to both questions: the semiclassical \( T^{\mu\nu}_{\text{vac}} \) need not be calculated in the compact domain. Up to negligible corrections, the metric in the semiclassical region is expressed through the Bondi charges by a set of kinematic equations. These equations will close as soon as
the Bondi charges will be expressed through the metric in the semiclassical region. For that, $T_{\mu\nu}^{\text{vac}}$ needs to be calculated at infinity and nowhere else. Thereby, the "backreaction of radiation" acquires a literal meaning. The indefinite terms of $T_{\mu\nu}^{\text{vac}}$ do not affect the metric in the semiclassical region because, in this region, they are negligible altogether. The latter conclusion conforms to the significance of these terms and is conditioned by their two distinctive properties: locality, and proportionality to the quantum constant.

In brief outline, the contents of the paper is seen from the titles of its sections: (2) Correspondence principle; (3) Correspondence principle (continued); (4) Solution in the region of strong field; (5) Field equations; (6) Solution in the region of weak field; (7) Global solution; (8) Global solution (continued); (9) Summary. The correspondence principle is considered carefully in sections 2 and 3 because it provides the initial conditions for the expectation-value equations and gives a key to establishing the bounds of the semiclassical region. The concept of strong field is introduced in section 4, and it is shown that, in the region of strong field, the gravity equations close purely kinematically leaving the arbitrariness only in the data functions. For the equations to close in the region of weak field, it suffices that at least one component of $T_{\mu\nu}^{\text{vac}}$ in a certain basis remain microscopic. This is discussed in detail in section 5. The solution in the region of weak field is obtained in section 6 in terms of arbitrary data functions. The solutions in the regions of strong and weak field are next sewn together, and the data functions for these regions are related. In this way in sections 7 and 8 the global solution is obtained containing two arbitrary data functions. These functions are the Bondi charges at the future null infinity.

It is assumed below that the matter source in eq. (1.1) has a compact spatial support and is spherically symmetric. The correspondence principle will make it possible to limit the consideration to the vacuum region. Then the only relevant parameter of the source is its mass $M$ which is also the ADM mass of the expectation-value spacetime [2]. The principal condition assumed in the present study is

$$\lambda \ll 1$$  \hspace{1cm} (1.2)

where

$$\lambda = \frac{\mu}{M},$$ \hspace{1cm} (1.3)
\( \mu \) is the Planckian mass. An observable that in the units of \( M \) (to the appropriate power) vanishes as \( \lambda \to 0 \) will be denoted as \( \mathcal{O} \). The dimension of \( \mathcal{O} \) in terms of \( M \) may or may not be pointed out explicitly. With this notation, inequalities of the form \( X > |\mathcal{O}| \) assume any \( \mathcal{O} \), and equalities of the form \( Y = \mathcal{O} \) assume some \( \mathcal{O} \). Since \( \lambda \) is the only quantum parameter in the problem, these specifications signify that \( X \) is a macroscopic quantity, and \( Y \) is a microscopic quantity. The notation \( O(Z) \) will have its usual meaning, i.e., \( O(Z) = ZO(1) \), and \( |O(1)| \) is bounded from above, but it will be added that \( |O(1)| < 1/|\mathcal{O}| \). The \( O(1) \) will be considered dimensionless.

For a general spherically symmetric spacetime, the Lorentzian subspace referred to below is its section at fixed angles, \( 4\pi r^2 \) is the area of the symmetry orbit passing through a given point, \( v \) and \( u \) are the advanced and retarded times with past-directed gradients, labelling the radial past and future light cones respectively. Assuming the asymptotic flatness, \( v \) is normalized at the past null infinity \( (I^-) \) as

\[
(\nabla r, \nabla v)_{|I^-} = 1 . \tag{1.4}
\]

For one choice of \( u \), \( u = u^+ \), a similar normalization holds at the future null infinity \( (I^+) \):

\[
(\nabla r, \nabla u^+)_{|I^+} = -1 . \tag{1.5}
\]

Some other choices are considered below. The additive normalizations of \( v \) and \( u \) are left arbitrary. The partial derivatives \( \partial_u \) and \( \partial_v \) are defined as the derivatives along the lines \( v = \text{const.} \) and \( u = \text{const.} \) respectively. The mapping on the 2-dimensional Lorentzian subspace enables one to speak of points and lines instead of 2-spheres and spherically symmetric hypersurfaces.

The curvature tensor of a spherically symmetric spacetime can be reduced to two independent scalars. With \( \triangle \) the D’Alembert operator in the Lorentzian subspace, they can be introduced as

\[
E = \frac{r}{2} \left( 1 - (\nabla r)^2 \right) , \quad B = r \triangle r . \tag{1.6}
\]

The Riemann tensor (of the full 4-dimensional metric) is expressed through these scalars in a differential manner, and their vanishing in a domain is necessary and sufficient for the spacetime in this domain to be flat. The function \( E \) determines the ADM and Bondi
masses as, respectively, its limits at the spatial and null infinities, and the (black) apparent horizon as the hypersurface

\[(\nabla r)^2 = 0, \quad (\nabla r, \nabla v) \neq 0.\] (1.7)

For the consideration of the Bondi charges see section 6.

A spherically symmetric $T^{\mu\nu}$ has four algebraically independent components with two differential constraints imposed on them by the conservation equation. The constraints are solved explicitly by expressing $T^{\mu\nu}$ through $E$ and $B$. For the components of $T^{\mu\nu}$, the following basis is introduced:

\[
A = 4\pi r^2 (T^{\mu\nu} \nabla_\mu u \nabla_\nu v) \left( \frac{(\nabla r, \nabla v)}{(\nabla u, \nabla v)} \right)^2,
\]

(1.8)

\[
D = 4\pi r^2 (T^{\mu\nu} \nabla_\mu v \nabla_\nu v) \frac{1}{(\nabla r, \nabla v)}^2,
\]

(1.9)

\[
T_1 = 4\pi r^2 (T^{\mu\nu} \nabla_\mu u \nabla_\nu v) \frac{2}{(\nabla u, \nabla v)} ,
\]

(1.10)

\[
T_1 + T_2 = 4\pi r^2 T^{\mu\nu} g_{\mu\nu}.
\]

(1.11)

Here $T_1$ is $4\pi r^2$ times the trace of $T^{\mu\nu}$ in the Lorentzian subspace, $T_2$ is $4\pi r^2$ times the trace of $T^{\mu\nu}$ in the complementary subspace, $A$ and $D$ govern the speed of expansion of the radial light cones (eqs. (4.8) and (4.81) below).

The expressions for $T^{\mu\nu}$ through $E$ and $B$ are

\[
B = 1 - (\nabla r)^2 + T_1 ,
\]

(1.12)

\[
(\nabla r, \nabla v) \partial_v E = A - \frac{1}{4} T_1 (\nabla r)^2 ,
\]

(1.13)

\[
(\nabla r, \nabla u) \partial_u E = \frac{1}{4} D \left( (\nabla r)^2 \right)^2 - \frac{1}{4} T_1 (\nabla r)^2 ,
\]

(1.14)

\[
T_2 (\nabla r)^2 = BD (\nabla r)^2 + (\nabla r, \nabla v) (r \partial_v D) (\nabla r)^2 + (\nabla r, \nabla u) (r \partial_u T_1)
= 4AD + 4 \left( \frac{\nabla r, \nabla u}{(\nabla r)^2} \right) (r \partial_u A) + (\nabla r, \nabla v) (r \partial_v T_1) .
\]

(1.15)
2 Correspondence principle

The correspondence principle for the collapse problem under condition (1.2) can be formulated in terms of any congruence of falling observers or falling light. It concerns the observables of geometry which are, generally, scalar functions $I(x)$ of a spacetime point $x$, and functionals of the geometry. The correspondence principle is the assertion that, under certain limitations to be discussed below, the values of observables as measured in the units of $M$ by a given observer at a given instant of his proper time differ from the classically predicted values by $O$. Both the proper time and the parameters identifying the observer are supposed to be measured in the units of $M$.

Under the presently considered symmetry, it is convenient to choose for the congruence of falling observers the family of radial past light cones $v = \text{const}$. The proper time of the observer is then replaced by the affine time $\tau$ along $v = \text{const}$, normalized at $\mathcal{I}^-$ as

$$\left( \frac{d}{d\tau} r \bigg|_{v=\text{const.}} \right)_{\mathcal{I}^-} = -1 .$$

With this normalization, the respective exact equation is of the form

$$\frac{d}{d\tau} r \bigg|_{v=\text{const.}} = -(\nabla r, \nabla v) .$$

The observers’ $v$ and $\tau$ specify the points $x$ of the Lorentzian subspace. For functions $I(x)$ on this subspace the assertion of the correspondence principle is

$$I(v, \tau) = I_{\text{class}}(v, \tau) + O$$

where $I_{\text{class}}(v, \tau)$ is the value of $I$ predicted for the given $v$ and $\tau$ by classical theory.

The limitations on the validity of eq. (2.3) concern both the observers and the observables. One expects that, as the test light ray $v = \text{const}$ reaches the values of $r$ as small as $r = O$, the correspondence principle may cease being valid. However, this is not the only limitation. Let us confine ourselves to the region $r > |O|$, and let $v_0$ be some value of $v$ for which, in this region, the correspondence principle is valid. It will be valid as well
for any value of $v$ differing from $v_0$ by a finite multiple of $M$ but one cannot guarantee that it will remain valid for $v$ as large as $v_0 + M/|\mathcal{O}|$ because, at the classical limit, for being able to emit such a test light ray from $\mathcal{I}^-$, one should live for an infinitely long time. This aspect of the correspondence principle has been emphasized in ref. [4]. There may exist a critical value of $v$ defined with accuracy $O(M)$:

$$v_{\text{crit}}(\lambda) - v_0 = \frac{M}{|\mathcal{O}|} + O(M)$$

such that, for $v > v_{\text{crit}}$, the correspondence principle is no longer valid. Even at large $r$ but $v$ also large, the geometry will be nonclassical if the light cone $v = \text{const.}$ is crossed by radiation.

The latter limitation on the validity of the correspondence principle has its own limitation. A significant vacuum radiation can occur only at sufficiently late $u$, not earlier than the red shift will become large:

$$\frac{du^+}{du^-} \gg 1.$$  \hspace{1cm} (2.5)

Here $u^-$ is the retarded time counted out by an early falling observer:

$$u^- = 2\tau\Big|_{u=v_0}.$$  \hspace{1cm} (2.6)

In eq. (2.6), one can replace $v_0$ with any value $v_0 + O(M)$. This will alter $du^+/du^-$ only by a factor of order 1.

To establish the bound in $u$ critical for the correspondence principle, consider an outgoing light ray $u = \text{const.}$ that crosses the line $v = v_0$ at some $r > |\mathcal{O}|$. If, on its way to $v = v_{\text{crit}}$, this ray does not meet with small $r$, one can use the classical geometry to calculate what will be its $r$ at $v = v_{\text{crit}}$. Its classical law of motion is

$$u = \text{const.}: \quad \frac{r}{2M} + \ln \left| \frac{r}{2M} - 1 \right| = \frac{v - v_0}{4M} + \left( \frac{r}{2M} + \ln \left| \frac{r}{2M} - 1 \right| \right) \bigg|_{v=v_0}.$$  \hspace{1cm} (2.7)

Hence one finds that if

$$u = \text{const.}: \quad r\bigg|_{v=v_0} - 2M > |\mathcal{O}|,$$  \hspace{1cm} (2.8)

then $r$ along the ray $u = \text{const.}$ grows with $v$, so that eq. (2.7) can be used indeed, and, at $v = v_{\text{crit}}$, this ray comes to be already in the asymptotically flat region:

$$u = \text{const.}: \quad r\bigg|_{v=v_{\text{crit}}} = \frac{M}{|\mathcal{O}|}.$$  \hspace{1cm} (2.9)
At \( v < v_{\text{crit}} \) it passes across the region of classical geometry, and at \( v > v_{\text{crit}} \) it passes across the asymptotically flat region where the geometry may differ from the classical one only if the red shift is already large. The red-shift factor in eq. (2.5) may be written as a product of two

\[
\frac{du^+}{du^-} = \frac{du^+}{du^*} \frac{du^*}{du^-} \quad (2.10)
\]

with

\[
u^* = 2\tau|_{v=v_{\text{crit}}}. \quad (2.11)
\]

The first factor in this product involves only the asymptotically flat region. The red shift cannot accumulate in this region since the curvature in it is everywhere small:

\[
\frac{du^+}{du^*} = 1 + \mathcal{O}. \quad (2.12)
\]

The second factor involves only the region of classical geometry and, therefore, can be calculated:

\[
u = \text{const.} : \quad \frac{du^*}{du^-} = \frac{1 - (2M/r)|_{v=v_{\text{crit}}}}{1 - (2M/r)|_{v=v_0}}. \quad (2.13)
\]

Under condition (2.8) one obtains

\[
\frac{du^*}{du^-} < \frac{1}{|\mathcal{O}|}. \quad (2.14)
\]

Thus, for \( u \) satisfying condition (2.8), the red shift is still moderate, and, therefore, the line \( u = \text{const.} \) with this value of \( u \) lies entirely in the region of classical geometry.

Denote as \( u_0 \) the value of \( u \) for which

\[
u = u_0 : \quad r|_{v=v_0} = 2M(1 + \mathcal{O}_0) \quad (2.15)
\]

with some chosen \( \mathcal{O}_0 \). We found two (overlapping) regions in the vacuum in which the correspondence principle is valid, CL.I and CL.II:

- **CL.I**: \( r > |\mathcal{O}|, \quad v < v_{\text{crit}} \)
- **CL.II**: \( u^- < u^-_{\text{crit}} - |\mathcal{O}| \)

Their union will be denoted as CL and called classical region. Note that the classical region is *causally complete* in a sense that it contains all of its causal past (in the vacuum). For
CL.I, this follows from the fact that, by the classical geometry, the line \( r = |\mathcal{O}| \) in the vacuum is spacelike. It should be emphasized that for local or retarded equations with data in the past, as the expectation-value equations are in any approximation, the causal completeness of the domain of validity of the approximation is a necessary condition for obtaining the solution.

As mentioned above, the limitations on the validity of the correspondence principle concern also the observables \( I(x) \). If the point \( x \) is in the classical region but the dependence of \( I(x) \) on the geometry is nonlocal, then \( I(x) \) may involve not only the classical region. The correspondence principle is valid deliberately only for local and retarded observables, i.e., the functions \( I(x) \) that depend on the geometry only at \( x \) and in the past of \( x \). Important examples are the scalars \((\nabla r, \nabla v)\) and \((\nabla r, \nabla u^+)\). The former is a retarded observable whereas the latter is an advanced one. Therefore, even in the classical region, \((\nabla r, \nabla u^+)\) may differ drastically from its classical value (see below). On the other hand, by the correspondence principle,

\[
\text{CL : } (\nabla r, \nabla v) = 1 + \mathcal{O} . \tag{2.18}
\]

Eq. (2.2) then integrates to give

\[
\text{CL : } \tau = -r + f(v) . \tag{2.19}
\]

Using this relation in eq. (2.3) one obtains the final formulation of the correspondence principle. This is the assertion

\[
I(v, r) = I_{\text{class}}(v, r) + \mathcal{O} \tag{2.20}
\]

valid for all local and retarded observables \( I \) in the union of regions (2.16) and (2.17).

By the correspondence principle, in the classical region, in an appropriate vector basis, \( T_{\text{vac}}^{\mu\nu} \) should be small. For being appropriate, the vector basis must satisfy two requirements: (i) it should be nonsingular in the classical geometry, and (ii) the basis vectors may depend on the metric only in a local or retarded manner. Then, since \( T_{\text{vac}}^{\mu\nu} \) is retarded, its projections on the basis vectors will be retarded observables. The following vector basis in the Lorentzian subspace meets these requirements:

\[
\nabla_\mu v , \quad (\nabla v, \nabla u)^{-1} \nabla_\mu u . \tag{2.21}
\]
Note that the second vector is independent of the normalization of \( u \). The respective coordinate basis is the one in which the spacetime points are labelled with the values of \( v \) and \( \tau|_{v=\text{const.}} \). This is the labelling accomplished by the presently chosen congruence of falling observers. The vectors (2.21) are retarded. Dividing the first of them and multiplying the second by \((\nabla r, \nabla v)\), one can build a purely local basis satisfying the same requirements:

\[
(\nabla r, \nabla v)^{-1} \nabla_{\mu} v, \quad (\nabla r, \nabla v)(\nabla v, \nabla u)^{-1} \nabla_{\mu} u \equiv (\partial_{\mu} r) \nabla_{\mu} u . \tag{2.22}
\]

This is the vector basis used in eqs. (1.8)-(1.11). Therefore, by the correspondence principle,

\[
\text{CL}: \quad A, D, T_1, T_2 = O , \tag{2.23}
\]

and for the basic curvature scalars in eq. (1.6) one has

\[
\text{CL}: \quad E = M(1 + O) , \quad B = \frac{2M}{r} + O . \tag{2.24}
\]

The projections of covariant derivatives of \( T_{\text{vac}}^{\mu\nu} \) on the vectors (2.22) also are retarded observables and, by the correspondence principle, should also be \( O \) in the classical region. By the dimension of the coupling constant, the differential order of the expectation-value spacetime is at least \( C^4 \), as distinct from the classical \( C^2 \). In particular, the local terms of \( T_{\text{vac}}^{\mu\nu} \) contain at least the second-order derivatives of the curvature. Therefore, one disposes of conditions on the first and second derivatives of \( T_{\text{vac}}^{\mu\nu} \). These conditions are obtained, as one can check, by acting with linear and quadratic combinations of the operators

\[
M \partial_{\nu} \quad \text{and} \quad M(\partial_{\mu} r)^{-1} \partial_{\nu} \tag{2.25}
\]
on the scalars in eq. (2.23), and, in CL, equating the results to \( O \).
3  Correspondence principle (continued)

Consider now the spacetime region complementary to the classical one. It is hardly possible that, at \( r = \mathcal{O} \), the curvature in the units of \( M \) does not become as large as \( 1/\mathcal{O} \). Then, at \( r = \mathcal{O} \), not only classical theory is invalid. In a region where the curvature is \( 1/\mathcal{O} \), the indefinite local terms of \( T_{\text{vac}}^{\mu\nu} \) cease being small. This invalidates semiclassical theory as well. However, one does not expect that the curvature becomes that large at \( r > |\mathcal{O}| \) including at \( v > v_{\text{crit}} \). This is potentially the region of validity of semiclassical theory. The ultraviolet problem bears apparently no relation to it. The reserves "potentially" and "apparently" are made because the region \( r > |\mathcal{O}| \) may be not causally complete. If there are small \( r \) in its causal past in the vacuum, one will not be able to use semiclassical theory in this region. The region \( r = \mathcal{O} \) together with its causal future ought to be, rightfully, called ultraviolet region and excluded from the consideration. Semiclassical region (denoted below as SCL) is, by definition, the region of validity of semiclassical theory.

To establish the bounds set by causality, consider again an outgoing light ray \( u = \) const. that, at \( v = v_0 \), has \( r > |\mathcal{O}| \). Before this ray reaches \( v = v_{\text{crit}} \) or meets with small \( r \), the law of its motion is the one in eq. (2.7). One finds that if

\[
\left. u = \text{const.} \right| \quad r \left|_{u=v_0} \right. -2M < -|\mathcal{O}| ,
\]

then \( r \) along this ray decreases with \( v \) and reaches the value \( |\mathcal{O}| \) at

\[
\left. u = \text{const.} \right| \quad v \left|_{r=|\mathcal{O}|} \right. = v_0 + |\mathcal{O}(M)| .
\]

The future of such a light ray is not predictable either by classical or by semiclassical theory. Its future beyond the value of \( v \) in eq. (3.2) is in the ultraviolet region, and its past is in the classical region.

Eqs. (2.8) and (3.1) leave for the semiclassical region only the values of \( u \) for which

\[
\left. r \right|_{u=v_0} = 2M(1 \pm |\mathcal{O}|) ,
\]

11
i.e., only the interval

\[ u^- = u_0^- \pm |\mathcal{O}|, \quad (3.4) \]

\[ v > v_{\text{crit}} \quad (3.5) \]

with some range of $\mathcal{O}$ in eq. (3.4). It will be emphasized that, in particular, the calculation of the radiation at $\mathcal{I}^+$, whatever semiclassical technique is used, the effective action or WKB, is valid only in the interval (3.4) of $u$. Along $v = v_0$, this interval is a microscopic neighbourhood of $r = 2M$. An early falling observer crosses it without noticing because the whole of this interval is within the quantum uncertainty of measuring of his proper time. However, because of a large red shift, an interval microscopic in $u^-$ may be macroscopic and even infinite in $u^+$. The later falls the observer, the longer is, for him, this interval. For the incoming light signal with $v = v_{\text{crit}}$, the length of this interval in $\tau$ and $r$ may already equal units of $M$, and, possibly, this interval covers the whole of the future of $\mathcal{I}^+$.

In the semiclassical region, $T_{\mu\nu}^{\text{vac}}$ and its derivatives may be not small any more but, as a matter of principle, their projections on the basis vectors (2.22) remain bounded. Specifically,

\[ A, \ D, \ T_1, \ T_2 = O(1), \quad (3.6) \]

\[ M(\partial_u r)^{-1} \partial_u A = O(1), \quad (3.7) \]

\[ M(\partial_u r)^{-1} \partial_u D = O(1), \quad (3.8) \]

\[ M(\partial_u r)^{-1} \partial_u B = O(1), \quad (3.9) \]

\[ M^2 \left( (\partial_u r)^{-1} \partial_u \right) \left( (\partial_u r)^{-1} \partial_u \right) A = O(1), \quad (3.10) \]

\[ M^2 (\partial_u r)^{-1} \partial_u \partial_v B = O(1). \quad (3.11) \]

Here use is made of the operators (2.25), and the conditions for $B$ are obtained from the conditions for $T_1$. In addition, with a certain reserve one may use that

\[ |\mathcal{O}| < (\nabla r, \nabla v), \quad \frac{E}{M}, \quad \frac{r B}{2E} < \frac{1}{|\mathcal{O}|} \quad (3.12) \]
because, in the classical region, these quantities equal 1. In the semiclassical region, they may eventually turn into zero or infinity but, before that, there will be an evolution. Obtaining this evolution is one’s goal.

By the correspondence principle, the apparent horizon (AH) enters the vacuum region CL.I with

\[ \frac{dr_{AH}}{dv} \geq 0 \]  

and, in this region, has \( r_{AH} = 2M(1 + O) \) at all \( v \). Therefore, the AH initially gets into the interval (3.4) of \( u \). As it evolves, it cannot go out of this interval to smaller \( u \) for, otherwise, it will get to the region CL.II at variance with the correspondence principle. It can go out of this interval to greater \( u \) but only when having already \( r_{AH} = O \). Indeed, a greater-\( u \) outgoing ray is the one in eqs. (3.1), (3.2). Along it, \( r \) decreases monotonically down to \( r = O \). If, instead of getting to a singularity, this ray crosses the AH, then only at \( r = O \). It follows that the semiclassical region (3.4) covers the whole of the evolution of the AH from \( r_{AH} = 2M \) to \( r_{AH} = O \). The latter value may be not reached. Then the AH stays in the semiclassical region always.

Suppose that eq. (3.13) holds throughout the interval (3.4). Then one of the lines \( u = \text{const.} \) in this interval is an event horizon hiding a black hole of mass greater than or equal to the ADM mass. This is at variance with the radiation of positive energy. Therefore, there should be a point of the AH at which the derivative in eq. (3.13) changes the sign. At this point, the AH is exactly null and tangent to one of the lines \( u = \text{const.} \) in the interval (3.4). It is natural to assume that this point is in CL.I where the deviation of the AH from a null line is within the quantum uncertainty. One can then choose the \( O_0 \) in eq. (2.15) and shift \( v_0 \) by \( O(M) \) so that this point be \( (u_0, v_0) \). To the past from this point in the advanced time, the AH is spacelike, and to the future timelike. The outgoing rays with \( u < u_0 \) never cross it, and the ones with \( u > u_0 \) cross it twice. The first crossing occurs in the support of \( T_{\mu \nu}^{\text{source}} \) or in the Planckian neighbourhood of this support \([5]\). The tangency point \( (u_0, v_0) \) is in the band of quantum uncertainty around the support of \( T_{\mu \nu}^{\text{source}} \). Therefore, when considering the vacuum region, one may confine oneself to \( v \geq v_0 \). This limitation is implied below. Respectively, unless the context assumes otherwise, the discussion of the AH below refers to its second, in the order in
which the light rays $u = \text{const.}$ cross it, branch. The behaviour of the first branch in the vacuum is subject to a different physics: creation of the virtual pairs as opposed to the real ones. The effect of the local vacuum polarization is considered in ref. [5]. The AH is shown in Fig. 1 for not very late $u$. The point 0 in Fig. 1 is $(u_0, v_0)$.

Even after crossing the AH the second time, the light rays $u = \text{const.}$ might not go out to $\mathcal{I}^+$ but, if the second branch of the AH is caused by radiation, they do. Then the chart $u^+$ extends to the AH. Consequently,

\[
(\nabla v, \nabla u^+)_{\text{AH}} \neq 0, \pm \infty, \quad (3.14)
\]

\[
(\nabla r, \nabla u^+)|_{\text{AH}} = 0 \quad (3.15)
\]

including at the point 0 where the geometry is "most classical". Eq. (3.15) is at no variance with the fact that the classical value of the \textit{advanced} observable $(\nabla r, \nabla u^+)$ is $-1$.

The observable $-(\nabla v, \nabla u^+)/2$ is the red-shift factor that the outgoing light signal with the current value of $u$ accumulates while passing from the current value of $v$ to $\mathcal{I}^+$:

\[
- \frac{1}{2} (\nabla v, \nabla u^+) = \frac{d u^+}{d u_{\text{current } v}},
\]

\[
u_{\text{current } v} = 2 \tau|_{\text{current } v}.
\]

The full red-shift factor is

\[
- \frac{1}{2} (\nabla v, \nabla u^+)|_{v = v_0} = \frac{d u^+}{d u_{\text{current } v}}.
\]

In the classical geometry, the observable (3.16) turns into infinity at the AH because the red shift becomes infinite. The advanced observable

\[
\frac{\partial r}{\partial u^+} = \frac{1}{2} (\nabla r, \nabla u^+) = \frac{\nabla r, \nabla v}{(\nabla v, \nabla u^+)} \quad (3.18)
\]

differs from $(\nabla v, \nabla u^+)^{-1}$ only by the coefficient $(\nabla r, \nabla v)$ of order 1 (in CL, just 1). Therefore, in the classical geometry, it vanishes at the AH. In the expectation-value geometry, the observable (3.16) remains finite at the AH by virtue of eq. (3.14). As a consequence, the observable (3.18) does not vanish:

\[
\frac{\partial r}{\partial u^+}|_{\text{AH}} = \frac{d}{d u^+} r_{\text{AH}} = 2 \frac{d}{d u^+} E_{\text{AH}} \neq 0 \quad (3.19)
\]
and determines the law by which $r$ and $E$ vary along the AH. It does not vanish, in particular, at the point 0 where the AH is null. At this point one has

$$
(\partial_v r)_{|0} = 0 , \quad (\partial^2_{vv} r)_{|0} = 0 , \quad (\partial_v E)_{|0} = 0 ,
$$

$$
\frac{du^+_{AH}}{dv}_{|0} = 0 , \quad \frac{dr_{AH}}{dv}_{|0} = 0 , \quad \frac{dr_{AH}}{du^+} \neq 0 . \quad (3.20)
$$

The semiclassical region is in the future domain of dependence of the classical region. With the classical region included, it is causally complete. Therefore, the correspondence principle plays the role of the initial condition for the expectation-value equations in the semiclassical region. Now I go over to the question where these equations will come from if $T_{\nu\mu}^\text{vac}$ is not to be calculated except at $I^+$. Below, only the semiclassical region is considered.
4 Solution in the region of strong field

The key point is that, in the region where the outgoing light signals acquire a large red shift:

\[ \frac{\partial r}{\partial u^+] = \mathcal{O}, \]  

(4.1)
i.e., in the region of strong gravitational field, one does not need field equations. The conditions of boundedness of the curvature (3.6)-(3.11) take the place of the field equations in this region. First note that, since, by eq. (3.12), \((\nabla r, \nabla v) > 0\) and \(B > 0\), one has

\[ \frac{\partial r}{\partial u} < 0, \quad \partial_v \left| \frac{\partial r}{\partial u} \right| > 0. \]  

(4.2)
Furthermore, for \((\nabla r)^2 > |\mathcal{O}|\) one has

\[ (\nabla r)^2 > |\mathcal{O}|: \quad (\nabla r, \nabla u^+) = O(1) \]  

(4.3)
and hence

\[ (\nabla r)^2 > |\mathcal{O}|: \quad \left| \frac{\partial r}{\partial u^+} \right| > |\mathcal{O}|. \]  

(4.4)
Eq. (4.3) is a consequence of the boundedness condition \(A = O(1)\), and its proof repeats with an obvious modification the derivation of eq. (6.26) below. Eq. (4.4) then follows from the identity (3.18). Therefore, condition (4.1) implies \((\nabla r)^2 \leq |\mathcal{O}|\), and it is natural to assume that the apparent horizon \((\nabla r)^2 = 0\) is in the region (4.1):

\[ \left. \frac{\partial r}{\partial u^+} \right|_{\text{AH}} = \mathcal{O}. \]  

(4.5)
Then, outside the AH, i.e., at \((\nabla r)^2 > 0\), condition (4.1) holds as long as \((\nabla r)^2 = |\mathcal{O}|\), and, inside the AH, it holds as far as the rays \(u = \text{const.}\) extend by virtue of eqs. (4.2) and (4.5). It follows that, in the chart \(u^+\), the region (4.1) covers the union

\[ \left( \left( \nabla r \right)^2 = |\mathcal{O}| \right) \cup \left( (\nabla r)^2 < 0 \right). \]  

(4.6)
Below it will be shown that, at \((\nabla r)^2 < 0\), the chart \(u^+\) does not extend beyond \((\nabla r)^2 = -|\mathcal{O}|.\)
In the region (4.6), one can replace $M$ in eq. (3.9) with $r$ replacing first $M$ with $E$ on the basis of condition (3.12) and next $E$ with $r$ on the basis of eq. (4.6). This gives the equation

$$r \partial_u B = O(1) \frac{\partial r}{\partial u},$$

(4.7)

and the following equations hold identically:

$$r \partial_u \ln(\nabla r, \nabla v) = -D \frac{\partial r}{\partial u},$$

(4.8)

$$r \partial_u \ln r = \frac{\partial r}{\partial u},$$

(4.9)

$$r \partial_v \ln \left| \frac{\partial r}{\partial u} \right| = \frac{B}{2(\nabla r, \nabla v)}.$$  

(4.10)

With any choice of $u$ for which $\partial r/\partial u = O$, not necessarily $u = u^+$, eqs. (4.7)-(4.10) close to lowest order in $\partial r/\partial u$. This is a consequence of two boundedness conditions: eq. (3.9) and $D = O(1)$, which thus play the role of the field equations.

The solution of eqs. (4.7)-(4.10) is given below, and its derivation will be found in section 7, but one obvious question should be answered right away. Eqs. (4.7)-(4.10) are invariant with respect to the choice of $u$, and there are choices for which $\partial r/\partial u$ is not small. Then what is the use of these equations? The answer is that the initial data to these equations break the invariance. An equation like (4.7) is usable only if its right-hand side contributes negligibly to the solution throughout the region (4.1). One possibility for that is to have

$$K > |O|$$

where

$$-K = \partial_u \ln \left| \frac{\partial r}{\partial u} \right|.$$  

This will prove to be the case below. However, $K$ is not invariant with respect to the choice of $u$. It is necessary that the condition $K > |O|$ hold with the same choice of $u$ with which the condition $\partial r/\partial u = O$ holds, i.e., $u = u^+$. This will be secured by the properties of the data functions.

Since the AH is in the region (4.1), the initial data to eqs. (4.7)-(4.10) can be taken at the AH. Four data functions are needed:

$$E_{\text{AH}} , B_{\text{AH}} , (\nabla r, \nabla v)|_{\text{AH}}, v_{\text{AH}}(u)$$  

(4.11)
where \( v = v_{\text{AH}}(u) \) or, conversely, \( u = u_{\text{AH}}(v) \) is the equation of the second branch of the AH. The data for \( r \) and \( \partial r / \partial u \) are expressed through \( E_{\text{AH}} \), and the following notation is introduced:

\[
\begin{align*}
\alpha &= (\nabla r, \nabla v)_{\text{AH}}, \\
\beta &= -\left. \frac{\partial r}{\partial u} \right|_{\text{AH}} = -2 \frac{dE_{\text{AH}}}{du}, \\
\kappa &= -\frac{d \ln \beta}{du} + \frac{B_{\text{AH}}}{4 \alpha E_{\text{AH}}} \frac{dv_{\text{AH}}}{du}.
\end{align*}
\]  

(4.12)  

(4.13)

Note that \( \beta \) and \( \kappa \) are not invariant with respect to the choice of \( u \). Throughout the present section, \( u \) is \( u^+ \) or differs from \( u^+ \) by a finite transformation. By eqs. (3.12) and (4.5),

\[
|\mathcal{O}| < \alpha, \ E_{\text{AH}} , \ B_{\text{AH}} < \frac{1}{|\mathcal{O}|}, \quad \beta = |\mathcal{O}|,
\]

(4.14)

and on the same grounds one may use that

\[
\frac{du_{\text{AH}}}{dv} < \frac{1}{|\mathcal{O}|}.
\]

(4.15)

It will additionally be assumed and next confirmed that

\[
\frac{dB_{\text{AH}}}{dv} = \mathcal{O}
\]

(4.16)

and

\[
|\mathcal{O}| < \kappa < \frac{1}{|\mathcal{O}|}, \quad \frac{d}{du} \frac{1}{\kappa} = \mathcal{O}.
\]

(4.17)

Conditions (4.16) and (4.17) on the data functions will be derived in section 7 when sewing together the solutions in the regions of strong and weak field. They are conditions of the existence of the global solution. The data functions taken at the second branch of the AH at the point with a given value of \( u \) will be denoted as \( E_{\text{AH}}(u) , \alpha(u) \), etc. The same functions taken at the point of the AH with a given value of \( v \) will be denoted as \( E_{\text{AH}}(v) , \alpha(v) \), etc. The data functions satisfy the identity following from eq. (4.13)

\[
\beta(u) e^{\Gamma_1} = \beta(v) e^{\Gamma_2} , \quad v \geq v_0
\]

(4.18)

where \( \Gamma_1 \) and \( \Gamma_2 \) are the integrals along the second branch of the AH

\[
\Gamma_1 = \int_{v_{\text{AH}}(u)}^{v} dv \frac{B_{\text{AH}}}{4 \alpha E_{\text{AH}}} , \quad \Gamma_2 = \int_{u}^{u_{\text{AH}}(v)} du \kappa .
\]

(4.19)
The values of the data functions at the point 0 will be denoted as \( \alpha_0, \beta_0, \kappa_0 \), etc.

With the data as above, the solution in the region (4.1) for \( v \geq v_0 \) is

\[
(1 + O) \frac{\partial r}{\partial u} = -\beta(u) e^{\Gamma_1} = -\beta(v) e^{\Gamma_2},
\]

(4.20)

\[
(1 + O)(\nabla r, \nabla v) = \alpha(v),
\]

(4.21)

\[
(1 + O)(\nabla u, \nabla v)^{-1} = \frac{\beta(v)}{\alpha(v)} e^{\Gamma_2},
\]

(4.22)

\[
r = 2E_{AH}(v) + \beta(v) \left( \frac{1}{\kappa(u)} e^{\Gamma_2} - \frac{1}{\kappa(v)} \right) (1 + O).
\]

(4.23)

The latter two equations give the metric in the null coordinates. One obtains

\[
(1 + O)(\nabla r)^2 = 2\beta(v) \frac{B_{AH}(v)}{4E_{AH}(v)} \left( \frac{1}{\kappa(u)} e^{\Gamma_2} - \frac{1}{\kappa(v)} \right),
\]

(4.24)

\[
(1 + O)(\nabla r, \nabla u) = \frac{B_{AH}(v)}{4E_{AH}(v)} \left( \frac{1}{\kappa(v)} e^{-\Gamma_2} - \frac{1}{\kappa(u)} \right),
\]

(4.25)

\[
E = E_{AH}(v) + \frac{1}{2} \beta(v) \left( 1 - B_{AH}(v) \right) \left( \frac{1}{\kappa(u)} e^{\Gamma_2} - \frac{1}{\kappa(v)} \right) (1 + O),
\]

(4.26)

\[
B = B_{AH}(v) + O,
\]

(4.27)

\[
T_1 = B_{AH}(v) - 1 + O,
\]

(4.28)

\[
A = \alpha(v) B_{AH}(v) \frac{dE_{AH}(v)}{dv} (1 + O)
\]

\[
= -\frac{1}{2} \beta(v) \alpha(v) \frac{du_{AH}(v)}{dv} B_{AH}(v) (1 + O).
\]

(4.29)

Only the curvatures \( D \) and \( T_2 \) are not obtained because they are contained in the approximation of higher order in \( \partial r/\partial u \). In eqs. (4.20)-(4.29), the left-hand sides are functions of the observation point \((u, v)\), and the coordinates of the observation point appear as the arguments of the data functions on the right-hand sides. For details of the derivation see section 7.

Under the restriction \( v \geq v_0 \) assumed in the solution above, the exterior and interior of the AH are respectively the regions

\[
(\nabla r)^2 > 0: \quad v > v_{AH}(u), \quad u < u_{AH}(v), \quad \Gamma_1 > 0, \quad \Gamma_2 > 0,
\]

(4.30)

\[
(\nabla r)^2 < 0: \quad v < v_{AH}(u), \quad u > u_{AH}(v), \quad \Gamma_1 < 0, \quad \Gamma_2 < 0.
\]

(4.31)
The solution has two immediate consequences. First, from eq. (4.25) one infers that the condition (4.3) extends to the whole of the exterior of the AH at \( v \geq v_0 \):

\[
(\nabla r)^2 > 0 : \quad (\nabla r, \nabla u) = O(1) .
\]  

(4.32)

Second, for the interior of the AH, eq. (4.24) yields the bound

\[
(\nabla r)^2 > -2\beta(v) \frac{B_{AH}(v)}{4E_{AH}(v)\kappa(v)}(1 + O)
\]

(4.33)
to be discussed below.

Another important consequence of the solution is that the curvature \( A \) proves to be \( O \). That \( A \) at the AH is \( O \) follows directly from the exact equations (1.12)-(1.14). Eqs. (1.12) and (1.14) imply

\[
(\partial_u E) \big|_{AH} = (1 - B_{AH}) \frac{dE_{AH}}{du} , \quad (\partial_v E) \big|_{AH} = B_{AH} \frac{dE_{AH}}{dv} .
\]  

(4.34)

and hence by eq. (1.13)

\[
A \big|_{AH} = \alpha B_{AH} \frac{dE_{AH}}{dv} .
\]  

(4.35)

Eq. (4.29) adds to this fact that \( A \) is constant along the lines \( v = \text{const.} \) and, therefore, is small throughout the region (4.1). The smallness of \( A \) and the boundedness of the second derivative of \( A \) (eq. (3.10)) imply the smallness of the first derivative of \( A \). The respective bound is obtained in the Appendix:

\[
M(\partial_u r)^{-1}\partial_u A = O(1)|A|^{1/2} = O(\beta^{1/2}(v)) .
\]  

(4.36)

The differential equations along the lines \( v = \text{const.} \), whose solutions are obtained above, have counterparts along the lines \( u = \text{const.} \). For the function \( r \), this is the identity

\[
r \partial_v \ln r = \frac{(\nabla r)^2}{2(\nabla r, \nabla v)} ,
\]

(4.37)

and, for the functions \( E \) and \( B \), these are eq. (1.13) and the second form of eq. (1.15) with \( T_1 \) expressed through \( B \). Note the distinction: the right-hand side of eq. (4.37) is \( O((\nabla r)^2) \) whereas the right-hand side of the analogous equation (4.9) is \( O(\partial r/\partial u) \). As explained below, this distinction is essential but, outside the AH where eq. (4.32) holds, there is no distinction:

\[
(\nabla r)^2 > 0 : \quad (\nabla r)^2 = O \left( \frac{\partial r}{\partial u} \right) .
\]  

(4.38)
Furthermore, outside the AH

\[(\nabla r)^2 > 0 : \quad \beta(v) < \left| \frac{\partial r}{\partial u} \right|\]  

(4.39)

as follows from eq. (4.20). Therefore,

\[(\nabla r)^2 > 0 : \quad \beta(v) = O\left(\frac{\partial r}{\partial u}\right), \quad \beta^{1/2}(v) = O\left(-\frac{\partial r}{\partial u}\right)^{1/2},\]  

(4.40)

and eqs. (1.13) and (1.15) take the form

\[(\nabla r)^2 > 0 : \quad \partial_v E = O\left(\frac{\partial r}{\partial u}\right), \quad r\partial_v B = O\left(-\frac{\partial r}{\partial u}\right)^{1/2}.\]  

(4.41)

Here use is made of eqs. (4.29) and (4.36). The differential equations (4.37)-(4.38) and (4.41) can be integrated along the lines \(u = \text{const.}\) with the aid of eqs. (4.10) and (3.12), e.g.,

\[
\int_{v_{AH}(u)}^v \frac{dv}{r} O\left(-\frac{\partial r}{\partial u}\right)^{1/2}\bigg|_{u=\text{const.}} = O\left(-\frac{\partial r}{\partial u}\right)^{1/2} - O\left(\beta^{1/2}(u)\right), \quad (4.42)
\]

and their solutions in the region of strong field are analogous to the solutions obtained above by integrating the equations along the lines \(v = \text{const.}\):

\[(\nabla r)^2 > 0 : \quad \begin{align*}
r &= 2E_{AH}(u)(1 + O), \\
E &= E_{AH}(u)(1 + O), \\
B &= B_{AH}(u) + O.
\end{align*}\]  

(4.43)-(4.45)

Then consider any point \((u, v)\) that belongs to the region (4.1) and is located outside the AH. In Fig. 1, this is point 1, and it defines points 2 and 3. It follows from eqs. (4.23)-(4.27) and (4.43)-(4.45) that, up to microscopic variations, the functions \(r, E,\) and \(B\) are constant in the triangle 123. As a consequence, with the same accuracy, the data functions \(E_{AH}\) and \(B_{AH}\) are constant in the sector 23 of the AH. Specifically, for the points 2 and 3 one obtains the relations

\[
E_{AH}(v) = E_{AH}(u)(1 + O), \quad B_{AH}(v) = B_{AH}(u) + O
\]

(4.46)

valid if \(u\) and \(v\) satisfy the conditions

\[
\left. \frac{\partial r}{\partial u} \right|_{\text{point (u,v)}} = O, \quad (\nabla r)^2 \bigg|_{\text{point (u,v)}} > 0. \]

(4.47)
For the exterior of the AH, this property of the data functions can be used in the solution (4.20)-(4.29).

The distinction between the equations along $v = \text{const.}$ and $u = \text{const.}$ is essential inside the AH because, there, $\Gamma_2$ is negative and may become as large as $-1/|\mathcal{O}|$. This makes condition (4.32) invalid at $(\nabla r)^2 < 0$: inside the AH, the function $(\nabla r, \nabla u)$ may grow up to $1/|\mathcal{O}|$. Then eqs. (4.38)-(4.40) will no longer be valid, and solutions (4.43)-(4.45) will not apply. It is entirely owing to this fact that $E$ and $B$ can undergo macroscopic variations along the AH. Indeed, consider any line $u = \text{const.}$ crossing the AH. In Fig. 1, it is shown as passing through the points 4 and 5. Since

$$\partial_v \Gamma_2 = \frac{du_{AH}(v)}{dv}\kappa(v) \geq 0,$$  \hspace{1cm} (4.48)

$\Gamma_2$ decreases along this line from point 4 where it is zero to point 5 where it has a minimum. If, at this minimum, $\Gamma_2$ is $O(1)$, the solutions (4.43)-(4.45) apply, and one has $E_4 = E_5(1 + \mathcal{O})$, $B_4 = B_5 + \mathcal{O}$. On the other hand, by eqs. (4.26) and (4.27), $E_5 = E_0(1 + \mathcal{O})$, $B_5 = B_0 + \mathcal{O}$. As a result, for all $u$ for which

$$\int_{u_0}^{u} du \kappa < \frac{1}{|\mathcal{O}|},$$  \hspace{1cm} (4.49)

one obtains

$$E_{AH}(u) = M(1 + \mathcal{O}), \quad B_{AH}(u) = 1 + \mathcal{O}.$$  \hspace{1cm} (4.50)

This is a specific case of eq. (4.46). If eq. (4.46) were valid for any point $(u, v)$ inside the AH, the result (4.50) would hold for all $u$.

The remedy is in the fact that $\Gamma_2$ decreases also along the lines $v = \text{const.}$ including the line $v = v_0$, and, as distinct from the previous case, this decrease is unbounded. Indeed, since $u$ is $u^+$ or differs from $u^+$ by a finite transformation, it may take arbitrarily large values. As it becomes $u = u_{AH}(v) + M/|\mathcal{O}|$, one obtains owing to conditions (4.17)

$$u = u_{AH}(v) + \frac{M}{|\mathcal{O}|}; \quad -\Gamma_2 = \frac{1}{|\mathcal{O}|}, \quad (\nabla r, \nabla u) = \frac{1}{|\mathcal{O}|}.$$  \hspace{1cm} (4.51)

In particular,

$$u = u_0 + \frac{M}{|\mathcal{O}|}; \quad \int_{u_0}^{u} du \kappa = \frac{1}{|\mathcal{O}|}.$$  \hspace{1cm} (4.52)
This allows $E_{AH}(u)$ and $B_{AH}(u)$ to differ macroscopically from their values in eq. (4.50). If the points 4 and 5 are at that large $u$, then point 6 is already outside the region (4.1).

The above raises the issue of the range of $u$ for which the presently considered solution is valid. In the region of strong field, the reserve accompanying eq. (3.12) transfers to the conditions (4.14) and (4.15) for the data at the AH. It may turn out that these conditions hold only up to a certain point of the AH with a finite value of $u^+$. Then the present solution is valid only in the causal past of this point. Where the matter stands is, however, unknown and will be known only at the final stage when the data functions will be obtained. Therefore, it makes sense to drive the solution to the limit $u^+ \to \infty$ within the present assumptions. It will then be easy to cut it off at any value of $u^+$.

Eq. (4.51) explains the nature of the bound (4.33). At $v$ fixed, and $u$ as large as in eq. (4.51), this bound is already almost saturated. It is saturated at the limit $u \to \infty$, $v = \text{const.}$, if the conditions (4.17) for $\kappa$ hold up to $u \to \infty$. One may even admit that $\kappa(u)$ decreases as $u \to \infty$ but the law of decrease is restricted to the condition

$$v = \text{const.}, \quad u \to \infty : \quad -\Gamma_2 \to \infty , \quad \frac{1}{\kappa(u)} e^{\Gamma_2} \to 0 . \quad (4.53)$$

It suffices that

$$\kappa(u) > O \left( \frac{1}{u} \right) , \quad u \to \infty \quad (4.54)$$

and even that

$$\kappa(u) \geq \frac{\text{const.}}{u} , \quad \text{const.} > 1 , \quad u \to \infty . \quad (4.55)$$

Then, at the limit $u \to \infty$ along $v = \text{const.}$, one obtains

$$v = \text{const.}, \quad u \to \infty : \quad -(\nabla u, \nabla v) \to \infty , \quad \frac{\partial r}{\partial u} \to 0 . \quad (4.56)$$

It follows that the bound (4.33) is the end of the chart $u^+$, and the line

$$\mathbb{E}H : \quad (\nabla r)^2 = -2\beta(v) \frac{B_{AH}(v)}{4E_{AH}(v)\kappa(v)} (1 + O) \quad (4.57)$$

is the event horizon. It is easy to check that this line is null ($u = u_{\text{EH}} = \text{const.}$), and eq. (4.56) implies that, as $u \to u_{\text{EH}}$, the red shift becomes infinite. At the event horizon,

$$r_{\text{EH}}(v) = r_{\text{AH}}(v) \left( 1 - \frac{\beta(v)}{2E_{AH}(v)\kappa(v)} (1 + O) \right) \quad (4.58)$$
whence
\[ u^-_{\text{EH}} - u^-_0 = \frac{2\beta_0}{\kappa_0} (1 + O) , \quad u^+_{\text{EH}} = \infty \]  \hspace{1cm} (4.59)

Of the light rays \( u = \text{const.} \), only the ones in the interval of \( u \)
\[ u^-_{\text{EH}} > u^- > u^-_0 \]  \hspace{1cm} (4.60)
cross the AH twice and go out to \( I^+ \). The interval (4.60) is a subinterval of the semiclassical interval (3.4).

Note that
\[ \frac{d\ln \beta}{du} \bigg|_{u \to u_0} > 0 , \quad \frac{d\ln \beta}{du} \bigg|_{u \to \infty} < 0 . \]  \hspace{1cm} (4.61)

Here the first inequality follows from the fact that, by eq. (4.13),
\[ \frac{d\ln \beta}{du} \bigg|_0 = +\infty . \]  \hspace{1cm} (4.62)
The second follows from the assumption that conditions (4.14) hold up to \( u \to \infty \). Then, at \( u \to \infty \), \( E_{\text{AH}} \) remains finite, and, therefore,
\[ \beta \bigg|_{u \to \infty} = +0 . \]  \hspace{1cm} (4.63)

Eq. (4.61) suggests that there is a point of the AH where \( \beta \) has a maximum. Call it point I.

point I: \[ \frac{d\beta}{du} = 0 , \quad \beta = \beta_{\text{max}} . \]  \hspace{1cm} (4.64)

By eq. (4.13),
\[ v < v_1 : \quad \alpha \frac{du_{\text{AH}}}{dv} < \frac{B_{\text{AH}}}{4E_{\text{AH}} \kappa} , \]  \hspace{1cm} (4.65)
\[ v = v_1 : \quad \alpha \frac{du_{\text{AH}}}{dv} = \frac{B_{\text{AH}}}{4E_{\text{AH}} \kappa} , \]  \hspace{1cm} (4.66)
\[ v > v_1 : \quad \alpha \frac{du_{\text{AH}}}{dv} > \frac{B_{\text{AH}}}{4E_{\text{AH}} \kappa} . \]  \hspace{1cm} (4.67)

This helps to complete the spacetime diagram of the semiclassical region.

Along every ray \( u = \text{const.} \) that crosses the AH twice, \( (\nabla r)^2 \) must have a minimum at some negative value of \( (\nabla r)^2 \): \( (\nabla r)^2 = (\nabla r)^2_{\text{min}} \). These minima make a line passing in the interior of the AH. From the identity
\[ r \partial_v (\nabla r)^2 = (\nabla r, \nabla v)^{-1} \left( -2A + \frac{1}{2} B (\nabla r)^2 \right) \]  \hspace{1cm} (4.68)
and the solution above one obtains the equation of the line of minima:

\[(\nabla r)^2 = (\nabla r)^2_{\text{min}}: \quad (\nabla r)^2 = -2\beta(v)\alpha(v)\frac{d\nu_{\text{AH}}(v)}{dv}(1 + \mathcal{O}). \tag{4.69}\]

Since \((\nabla r)^2_{\text{min}} \leq 0\), one finds that all minima are at \(v \geq v_0\). On the other hand, comparing expression (4.69) with the bound (4.33) and using eqs. (4.65)-(4.67) one infers that all minima are at \(v < v_1\). At \(v = v_1\), the line of minima crosses the event horizon. Then, along the event horizon, \((\nabla r)^2\) first decreases and next, upon passing through the minimum at \(v = v_1\), increases up to the value

\[(\nabla r)^2_{\text{EH}} \to -0, \quad v \to \nu_{\text{AH}}\bigg|_{u=\infty} \tag{4.70}\]

reached at a finite or infinite value of \(v\)

\[\nu_{\text{AH}}(\infty) \overset{\text{def}}{=} \nu_{\text{AH}}\bigg|_{u=\infty}. \tag{4.71}\]

This follows from eqs. (4.57) and (4.63). At \(v = \nu_{\text{AH}}(\infty)\), the event horizon meets with the apparent horizon. Condition (4.15) implies that \(\nu_{\text{AH}}(\infty)\) is infinite. Then the apparent horizon is asymptotically tangent to the event horizon. However, condition (4.15) may not hold up to \(u = \infty\), and then \(\nu_{\text{AH}}(\infty)\) may be finite. In this case, the apparent horizon either crosses the event horizon or originates its new, third, branch. In general, the AH may in the chart \(u^+\) have any even number of branches.

The fact that the line of minima crosses the event horizon means that the event horizon is not the border of the semiclassical region. Having obtained the metric in the chart \(u^+\), one should be able to analytically continue it along the rays \(v = \text{const.}\) until these rays reach the ultraviolet region. The case of infinite \(\nu_{\text{AH}}(\infty)\) is easy to complete. Consider the function

\[\gamma = -2\alpha \frac{dE_{\text{AH}}}{dv} = \beta\alpha \frac{d\nu_{\text{AH}}}{dv}. \tag{4.72}\]

If \(\nu_{\text{AH}}(\infty)\) is infinite, one has

\[\gamma\bigg|_{u \to \infty} = +0 \tag{4.73}\]

They are, therefore, within the validity of the solution. In view of this fact, eqs. (4.69) or (4.33) prove the claim made above that, at \((\nabla r)^2 < 0\), the chart \(u^+\) does not extend beyond \((\nabla r)^2 = -|\mathcal{O}|\).
for the same reason for which eq. (4.63) holds. Then there should be a point of the AH where \( \gamma \) has a maximum. Call it point II.

\[
\text{point II: } \quad \frac{d\gamma}{du} = 0, \quad \gamma = \gamma_{\text{max}}. \tag{4.74}
\]

Point II is in the future of point I. This is seen from eqs. (4.65)-(4.67): at point I, \( \gamma \) is still growing. It follows from eq. (4.69) that, at \( v = v_{\text{II}} \), the line of minima is tangent to \( u = \text{const.} \), and, at \( v = \infty \), it meets with the apparent horizon. At \( v > v_{\text{II}} \), this line is already the line of maxima of \((\nabla r)^2\). The rays \( u = \text{const.} \) with \( u > u_{\text{EH}} \) cross it twice (or never cross it). Along these rays, \((\nabla r)^2\) remains negative and, after the second crossing, decreases already incessantly pulling these rays to the ultraviolet region \((\nabla r)^2 = -1/|\mathcal{O}|\).

At \( u < u_0 \), the line of minima recedes into the support of \( T_{\text{source}}^{\mu \nu} \) and, there, extends to \( u = -\infty \). All rays \( u = \text{const.} \) that go out to \( \mathcal{I}^+ \) cross this line. The completed spacetime diagram showing the event horizon and the line of extrema of \((\nabla r)^2\) is given in Fig. 2 for the case of infinite \( v_{\text{AH}}(\infty) \).

One more question of interest can be answered with the solution above. The light rays \( u = \text{const.} \) cannot cross the AH the second time if the energy current through \( u = \text{const.} \), \( \partial_v E \), is nonnegative. At least around the second branch of the AH, there should be a band in which \( \partial_v E < 0 \) and, thereby, the dominant energy condition is violated. Inserting the solution above in eq. (1.13), one obtains at the AH, at the line of minima, and at the line \( v = v_0 \)

\[
(\partial_v E)|_{\text{AH}} = -\frac{1}{2} \beta(v) B_{\text{AH}}(v) \frac{du_{\text{AH}}(v)}{dv}, \tag{4.75}
\]

\[
(\partial_v E)|_{(\nabla r)^2=(\nabla r)^2_{\text{min}}} = -\frac{1}{2} \beta(v) \frac{du_{\text{AH}}(v)}{dv} (1 + \mathcal{O}), \tag{4.76}
\]

\[
(\partial_v E)|_{v=v_0} = \beta_0 \mathcal{O}. \tag{4.77}
\]

The latter equation follows from the fact that, at \( v = v_0 \), \( B = 1 + \mathcal{O} \). The boundary of the band of negative \( \partial_v E \) is

\[
\partial_v E = 0: \quad (\nabla r)^2 = 2\beta(v)\alpha(v) \frac{du_{\text{AH}}(v)}{dv} \frac{B_{\text{AH}}(v)}{1 - B_{\text{AH}}(v) + \mathcal{O}}. \tag{4.78}
\]

One finds that, at the line of minima and at the second branch of the AH, \( \partial_v E \) is negative, and at the first branch positive as it should. Inside the AH, the boundary
passes in CL.I, i.e., in the $O(M)$ neighbourhood of the line $v = v_0$ where the deviation of $\partial_v E$ from zero is within the quantum uncertainty. Therefore, at $v = v_0 + M/|O|$, the boundary $\partial_v E = 0$ cannot already get inside the AH. This has an important consequence for the data at the second branch of the AH:

$$B_{\text{AH}} \leq 1 + O.$$  \hspace{1cm} (4.79)

Indeed, for $v < v_{\text{crit}}$, the equality $B_{\text{AH}} = 1 + O$ holds by the correspondence principle, and, for $v > v_{\text{crit}}$, condition (4.79) follows from the fact that $(\nabla r)^2$ in eq. (4.78) cannot be negative.

Eq. (4.78) is valid only when the boundary that it defines is in the region of strong field. Outside the AH it can be used if, at sufficiently large $v$, $B_{\text{AH}}$ becomes macroscopically distinct from 1. The band of negative $\partial_v E$ is then such as shown in Fig. 2. At $I^+$, the positivity of $\partial_v E$ restores but this does not mean that the energy dominance restores in full. In fact, the violation of the dominant energy condition in the region of strong field occurs with certainty for only one null projection of $T_{\text{vac}}^{\mu\nu} : A$. Indeed, from eqs. (4.29), (4.28), and (4.79) one finds for $v > v_0$

$$A < 0, \quad T_1 \leq O.$$  \hspace{1cm} (4.80)

Nonpositivity of $T_1$ is what the dominant energy condition requires, and the behaviour of $\partial_v E$ is a consequence of eq. (4.80). If, outside the AH, $(\nabla r, \nabla u)$ decreases along $u = \text{const.}$ monotonically, then the negativity of $A$ persists up to $I^+$. This follows from the identity

$$r \partial_v (\nabla r, \nabla u) = - A (\nabla r, \nabla v)^{-1} \left( \frac{\partial r}{\partial u} \right)^{-1}$$  \hspace{1cm} (4.81)

analogous to (4.8).

Of the four data functions (4.11), two are responsible for the choice of $u$ and $v$. Since eqs. (4.7)-(4.10) are invariant with respect to this choice, the arbitrariness of this choice in the solution above is restricted only to the conditions

$$\frac{\partial r}{\partial u} = O, \quad |O| < (\nabla r, \nabla v) < \frac{1}{|O|}.$$  \hspace{1cm} (4.82)

Fixing $u$ as $u^+$, and $v$ as in eq. (1.4) will determine two of the data functions but, for that, the data at the AH should be related to the data at $I^+$ and $I^-$. This requires knowing
the solution globally and not only in the region of strong field. The remaining two data functions are the basic curvature scalars $E$ and $B$ at the AH. They too should be related to the data at the asymptotically flat infinity. Thus one still needs field equations but, remarkably, only for the region of weak field.
5 Field equations

Although, in the semiclassical region, $T_{\text{vac}}^{\mu\nu}$ need not be small, some of its projections may remain small. One such projection one does have. Since it suffices to consider the massless vacuum particles, the full trace of $T_{\text{vac}}^{\mu\nu}$ is zero or, at most, has an anomaly. It does not matter whether there is no anomaly or there is one, and what is its specific form. It is only important that the trace anomaly is local and proportional to the quantum constant. This is sufficient for the equation

$$T_1 + T_2 = \mathcal{O}$$  \hspace{1cm} (5.1)

to hold throughout the semiclassical region. It can then be used as one of the field equations.

However, under spherical symmetry, one needs two field equations. This raises the question if any other projection of $T_{\text{vac}}^{\mu\nu}$ can be assumed small throughout the semiclassical region. The answer is that any such assumption is correct provided that it brings to a solution, and the solution confirms the assumption. This follows from the fact that the initial values of all the projections are small. The problem is to find an assumption of this kind that would stand its own dynamics. Fortunately, there is a hint.

The hint is that no assumption may be made about the data at the AH because these data will subsequently be determined by the data at infinity, and the data at infinity are subject to the quantum dynamics. At the kinematical level, the data functions should remain arbitrary. Keeping the data at the AH arbitrary, one can trace $T_{\text{vac}}^{\mu\nu}$ on going out of the region of strong field to see if any its projection becomes small. Only such a projection can remain small in the region of weak field. A convenient way of going out of the region of strong field is moving away from the AH towards earlier $u$ along the lines $v = \text{const}$.

Right away one arrives at an important inference that the projection $T_1$ cannot be assumed small. Indeed, if one admits that the data function $B_{\text{AH}}$ may become macroscopically distinct from 1, then $T_1$ at the AH is not small. Moreover, $T_1$ is conserved along
the lines $v = \text{const.}$ throughout the region of strong field and, therefore, does not become small on going out of this region.

Another example that is worth mentioning is the energy current $\partial_v E$. As distinct from $T_1$, $\partial_v E$ at the AH is small but, on going out of the region of strong field, it grows and tends to a macroscopic value along with $T_1$. Therefore, in the region of weak field, $\partial_v E$ cannot be assumed small either. The projection $T_2$ is expressed through $T_1$ by eq. (5.1) and is not small if $T_1$ is not small. About the projection $D$, there is no information except its boundedness. There remains to be considered the projection $A$, and here one has good luck. This projection is small at the AH and is conserved along the lines $v = \text{const.}$ throughout the region of strong field. It is, therefore, small on going out of this region and can be assumed small outside this region.

To summarize, there is only one viable candidate for the second field equation:

$$A = O .$$

(5.2)

Of course, at some initial stage of the evolution, all projections of $T^\mu_\nu_{\text{vac}}$ remain small. This stage may be called the epoch of small vacuum currents but, if $B_{\text{AH}}$ breaks away from 1, this epoch has an end.² The field equation (5.2) deliberately covers this epoch and has a chance to outlast it. The bounds to the validity of this equation are set by conditions (4.14) and (4.15) for the data functions. As long as these conditions are valid, $A|_{\text{AH}}$ in eq. (4.35) is small. As soon as it will cease being small, the end will come to the validity of eq. (5.2) as well.

Eq. (5.2) implies that also the first derivatives of $A$ are $O$. Specifically, eq. (3.7) gets

²The equation $T_1 = O$ (combined with eq. (5.1)) is necessary and sufficient for all projections of $T^\mu_\nu_{\text{vac}}$ to be $O$. The sufficiency follows from the second form of eq. (1.15). Indeed, if $T_1$ is $O$, then its first derivative $\partial_v$ is also $O$ because its second derivative is bounded (the proof repeats the one in the Appendix). Besides, the behaviours of $T_1$ and $T_2$ at infinity should be taken into account (see the next section). Then the identity (1.15) takes the form of the following equation for $A$:

$$\frac{d}{dr} \left( \frac{A}{(\nabla r, \nabla v)^2} \right)_{v=\text{const.}} = \frac{O}{r^2} .$$

With the initial condition $A|_{\text{I}} = 0$, its solution is $A = O$. Finally, as shown below, the equation $A = O$ combined with eq. (5.1) implies $D = O$. 

30
This follows from the boundedness of the second derivatives of $A$ (see the Appendix).
6 Solution in the region of weak field

It is not that the field equations (5.1) and (5.2) are valid only in the region of weak field. They are valid globally but the accuracy with which they are given enables one to use them only in the region of weak field. The condition defining this region is

\[-\frac{\partial r}{\partial u^+} > |\mathcal{O}|.\] (6.1)

In the chart $u^+$, it is equivalent to the condition

\[(\nabla r)^2 > |\mathcal{O}|.\] (6.2)

In the region of weak field, the initial data to the field equations cannot be taken at the AH but can be taken at the asymptotically flat infinity. For the metric at the future null infinity, one writes the general analytic expansion

\[(\nabla r)^2 \bigg|_{\mathcal{I}^+} = 1 - \frac{2\mathcal{M}(u)}{r} + \frac{Q^2(u)}{r^2} + \cdots,\] (6.3)

\[(\nabla r, \nabla u^+) \bigg|_{\mathcal{I}^+} = -1 - \frac{c_1(u)}{r} - \frac{c_2(u)}{r^2} + \cdots.\] (6.4)

(The coefficient $Q^2(u)$ is not necessarily positive.) Similar expansions hold for the metric at the past null infinity and spatial infinity ($i^0$). The coefficients of these expansions will be called charges because they represent the strengths of long-range fields having their sources in a compact domain. The coefficients of the expansions at $\mathcal{I}^+$ and $\mathcal{I}^-$ (the Bondi charges) describe respectively the emission and absorption of charges by an isolated system. Thus, the Bondi mass $\mathcal{M}(u)$ is the amount of the gravitational charge that, in the process of emission, remains in the compact domain by the instant $u$ of retarded time. Other coefficients may involve matter charges. For example, if the total $T^{\mu\nu}$ is the energy-momentum tensor of a system of electric charges and their electromagnetic field, then $Q^2(u)$ is positive, and $Q(u)$ is the Bondi electric charge of this system.

The full, or ADM charges appearing as coefficients of the expansion at $i^0$ are the limits as $u^+ \to -\infty$ of the Bondi charges at $\mathcal{I}^+$. Because $T^{\mu\nu}_{\text{source}}$ is assumed to have a compact
spatial support, and $T_{\mu\nu}^{\text{vac}}$ is retarded, there is no flux of charges through $I^-$. Therefore, the Bondi charges at $I^-$ are constant and equal to the respective ADM charges. The ADM charges are conserved, and, initially, the only nonvanishing macroscopic charge is the ADM mass $M$ but it would be absurd to exclude a possibility for the source to have microscopic amounts of other charges, e.g., a microscopic electric charge. Therefore,

$$\mathcal{M}(-\infty) = M, \quad Q^2(-\infty) = O, \quad \ldots$$

$$c_1(-\infty) = O, \quad c_2(-\infty) = O, \quad \ldots .$$

(6.5) \hspace{2cm} (6.6)

A limitation on the behaviour of the metric at $I^+$ stems from the locality of the trace of $T_{\mu\nu}^{\text{vac}}$. From eqs. (6.3), (6.4) one can calculate

$$T_1\bigg|_{I^+} = \frac{c_1}{r} + \cdots , \quad T_2\bigg|_{I^+} = -\frac{dc_1}{du^+} + \cdots .$$

(6.7)

On the other hand, with these behaviours, all possible local invariants in the trace anomaly decrease in such a way that $T_1 + T_2$ vanishes at $I^+$. Hence

$$\frac{dc_1}{du} = 0 ,$$

(6.8)

and thus, with regard for the initial condition in eq. (6.6),

$$c_1(u) = \text{const.} = O .$$

(6.9)

With constant $c_1$ one calculates the traces anew and obtains

$$T_1\bigg|_{I^+} = \frac{1}{r} c_1 + \frac{1}{r^2} \left( 2c_2 - c_1^2 - 2\mathcal{M}c_1 - Q^2 \right) + \cdots ,$$

(6.10)

$$T_2\bigg|_{I^+} = -\frac{1}{r} \left( c_1 + 2 \frac{dc_2}{du^+} \right) + \cdots$$

(6.11)

but the trace anomaly decreases faster:

$$(T_1 + T_2)\bigg|_{I^+} = \frac{O}{r^3} + \cdots .$$

(6.12)

Hence

$$\frac{dc_2}{du} = 0 ,$$

(6.13)

$$c_2(u) = \text{const.} = O ,$$

(6.14)
and therefore
\[ T_1 \bigg|_{\mathcal{I}^+} = \frac{c_1}{r} - \frac{Q^2(u) + \mathcal{O}}{r^2} + \cdots. \] (6.15)

Eqs. (6.12) and (6.15) are the final results for \( T_1 \) and \( T_2 \) at \( \mathcal{I}^+ \).

To summarize, the behaviours of the basic curvature scalars at \( \mathcal{I}^+ \) are
\[ E \bigg|_{\mathcal{I}^+} = \mathcal{M}(u) - \frac{1}{2} \frac{Q^2(u)}{r} + \cdots, \] (6.16)
\[ B \bigg|_{\mathcal{I}^+} = \frac{2\mathcal{M}(u) + c_1}{r} - \frac{2Q^2(u) + \mathcal{O}}{r^2} + \cdots. \] (6.17)

The behaviours at \( \mathcal{I}^- \) and \( i^0 \) are similar
\[ E \bigg|_{\mathcal{I}^-}, i^0 = M + \frac{\mathcal{O}}{r^2} + \cdots, \] (6.18)
\[ B \bigg|_{\mathcal{I}^-}, i^0 = \frac{2M + c_1}{r} + \frac{\mathcal{O}}{r^2} + \cdots \] (6.19)
but with all the coefficients constant.

Eqs. (4.81) and (6.9) have the following consequence for the curvature \( A \):
\[ rA \bigg|_{\mathcal{I}^+} = \mathcal{O}. \] (6.20)

Using the asymptotic behaviours (6.18)-(6.19) in eq. (1.13), one infers that \( A \) has the same property at \( \mathcal{I}^- \) and \( i^0 \):
\[ rA \bigg|_{\mathcal{I}^-}, i^0 = \mathcal{O}. \] (6.21)

This property thus holds as \( r \to \infty \) in any direction. Therefore, the field equation (5.2) can be strengthened as
\[ rA = \mathcal{O}. \] (6.22)

In the region of weak field, it can now be integrated. For that, rewrite the identity (4.81) as
\[ (\partial_v r)^{-1} \partial_v \ln |(\nabla r, \nabla u)| = -\frac{4A}{r (\nabla r)^2} \] (6.23)

\[ ^3 \text{Survival of the constant } c_1 \text{ has an alerting consequence at } i^0. \text{ Namely, the radial acceleration of a freely falling particle depends on its velocity even at the Newtonian limit:} \]
\[ \frac{d^2 r}{ds^2} \bigg|_{i^0} = -\frac{1}{r^2} \left( (M + c_1) + c_1 \left( \frac{dr}{ds} \right)^2 \right) + \cdots. \]

Here \( s \) is the proper time of the particle. However, since \( c_1 = \mathcal{O} \), only a particle of enormous energy can feel this dependence. Therefore, \textit{apriori} this consequence cannot be used to rule \( c_1 \) out.
and integrate it along \( u = \text{const.} \) with the initial condition at \( I^+ \):

\[
\ln |(\nabla r, \nabla u^+)| = \int_r^\infty {\frac{dr}{r}} \frac{4A}{(\nabla r)^2} \left|_{u=\text{const.}} \right. \tag{6.24}
\]

Insertion of eq. (6.22) and use of condition (6.2) give

\[
(\nabla r)^2 > |O|: \quad \ln |(\nabla r, \nabla u^+)| = \int_r^\infty {\frac{dr}{r}} \frac{O}{r} = O. \tag{6.25}
\]

Hence one obtains the solution in the region of weak field:

\[
(\nabla r)^2 > |O|: \quad (\nabla r, \nabla u^+) = -1 + O. \tag{6.26}
\]

As a consequence, one has

\[
(\nabla r)^2 > |O|: \quad \frac{\partial r}{\partial u^+} = -\frac{1}{2}(\nabla r)^2 (1 + O). \tag{6.27}
\]

As the next step, rewrite the identity (1.15) (the second form) as

\[
r(\partial_u r)^{-1} \partial_u T_1 + 2T_1 = \frac{2}{(\nabla r)^2} \left[ (T_1 + T_2)(\nabla r)^2 - 4AD - 2r(\partial_u r)^{-1} \partial_u A \right]. \tag{6.28}
\]

Here equation (5.3) is to be used. It follows from the asymptotic behaviours above that this equation can be strengthened as

\[
r(\partial_u r)^{-1} \partial_u A = O. \tag{6.29}
\]

Use eqs. (5.1), (5.2), (6.29), and the boundedness of \( D \). The result is that, in the region of weak field, eq. (6.28) closes as a differential equation along \( u = \text{const.} \):

\[
(\nabla r)^2 > |O|: \quad r(\partial_u r)^{-1} \partial_u T_1 + 2T_1 = O. \tag{6.30}
\]

Introduce the notation

\[
Y = r(\partial_u r)^{-1} \partial_u \left( T_1 - \frac{c_1}{r} \right) + 2 \left( T_1 - \frac{c_1}{r} \right). \tag{6.31}
\]

Inserting in eq. (6.31) the analytic expansion of \( T_1 \) at \( I^+ \), one finds that the expansion of \( Y \) at \( I^+ \) begins with \( 1/r^3 \) because the term of order \( 1/r^2 \) drops out. Then

\[
T_1 - \frac{c_1}{r} = \frac{1}{r^2} \left( a(u) - \int_r^\infty dr \frac{rY}{|_{r=\text{const.}}} \right). \tag{6.32}
\]
where \( a(u) \) is an arbitrary function of \( u \), and the integral with \( Y \) converges. By eqs. (6.30) and (6.31),

\[
(\nabla r)^2 > |O| : \quad Y = O - \frac{c_1}{r} = O
\]

because \( c_1 = O \). Hence, using eq. (6.15) to identify \( a(u) \) with the data at \( I^+ \), one obtains the solution:

\[
(\nabla r)^2 > |O| : \quad T_1 = \frac{c_1}{r} - \frac{Q^2(u) + O}{r^2}.
\]

(6.34)

As the last step, rewrite the identity (4.68) as

\[
r(\partial_v r)^{-1} \partial_v \left( 1 - (\nabla r)^2 \right) + \left( 1 - (\nabla r)^2 \right) = Z
\]

with

\[
Z = \frac{4A}{(\nabla r)^2} - T_1.
\]

(6.36)

Inserting in eq. (6.35) the analytic expansion of \((\nabla r)^2\) at \( I^+ \), one finds that the expansion of \( Z \) at \( I^+ \) begins with \( 1/r^2 \). Then

\[
1 - (\nabla r)^2 = \frac{1}{r} \left( b(u) - \int \frac{dr}{r} Z \big|_{u=\text{const.}} \right)
\]

(6.37)

where \( b(u) \) is an arbitrary function of \( u \), and the integral with \( Z \) converges. By eqs. (6.36), (6.34), (6.9), and (5.2), in the region of weak field

\[
(\nabla r)^2 > |O| : \quad Z = \frac{Q^2(u)}{r^2} + O.
\]

(6.38)

Hence, using eq. (6.3) to identify \( b(u) \) with the data at \( I^+ \), one obtains

\[
(\nabla r)^2 > |O| : \quad (\nabla r)^2 = 1 - \frac{2M(u)}{r} + \frac{Q^2(u) + O}{r^2}.
\]

(6.39)

Eqs. (6.39) and (6.26) give the solution for the metric in the region of weak field. For the basic curvature scalars one obtains

\[
(\nabla r)^2 > |O| : \quad E = M(u) - \frac{1}{2} \frac{Q^2(u) + O}{r},
\]

(6.40)

\[
(\nabla r)^2 > |O| : \quad B = \frac{2M(u) + c_1}{r} - \frac{2Q^2(u) + O}{r^2}.
\]

(6.41)
There remains to be calculated the curvature $D$. For that, rewrite the identity (1.14) as

$$D (\nabla r)^2 \frac{\partial r}{\partial u} = 2 \partial_u E + T_1 \frac{\partial r}{\partial u}. \quad (6.42)$$

The integral form of this identity is

$$2(M - E) = \int_r^\infty dr \left( D(\nabla r)^2 - \frac{c_1}{r} \right) \bigg|_{v=\text{const.}} - \int_r^\infty dr \left( T_1 - \frac{c_1}{r} \right) \bigg|_{v=\text{const.}}. \quad (6.43)$$

Insertion of the solution above in eq. (6.42) yields the result for $D$ in the region of weak field:

$$(\nabla r)^2 > |\mathcal{O}|: \quad D (\nabla r)^2 \frac{\partial r}{\partial u} = 2 \left( \frac{dM(u)}{du} - \frac{1}{2r} \frac{dQ^2(u)}{du} \right) + \mathcal{O} \frac{\partial r}{\partial u}. \quad (6.44)$$

It is shown in the next section that, with the choice $u = u^+$, the derivatives of both data functions $M(u)$ and $Q^2(u)$ are

$$\frac{dM(u)}{du^+} = \mathcal{O}, \quad \frac{dQ^2(u)}{du^+} = \mathcal{O}. \quad (6.45)$$

Then, in view of eq. (6.27),

$$(\nabla r)^2 > |\mathcal{O}|: \quad D = \mathcal{O}. \quad (6.46)$$

Eq. (6.46) is analogous to $A = \mathcal{O}$ but the equation analogous to (6.22) is not true:

$$rD \neq \mathcal{O}. \quad (6.47)$$

Indeed, from eq. (6.42) and the asymptotic conditions for $T_1$ one finds that

$$D \bigg|_{I^-, \, \, i^o} = \frac{c_1}{r} + \cdots, \quad c_1 = \mathcal{O} \quad (6.48)$$

but

$$D \bigg|_{I^+} = -4 \frac{dM(u)}{du^+} + \cdots, \quad (6.49)$$

i.e., $D$ vanishes not in all directions as $r \to \infty$. Owing to this fact, $E$ can differ macroscopically from its value at $I^-$ even in the epoch of small vacuum currents, and one can answer the question where on the incoming light ray $v = \text{const.}$ does the difference (6.43) accumulate. It accumulates at large $r$, $r = M/|\mathcal{O}|$, but not in the asymptotic region of
$I^-$. It accumulates only on very late rays, $v - v_0 = M/|O|$, because, after having passed the region of $I^-$, these rays get into the region of validity of eq. (6.49), i.e., into the asymptotic region of $I^+$. In this region, their $r$ is of order

$$r = MO \left( \frac{dM(u)}{du^+} \right)^{-1},$$

and the deficit of $E$ that accumulates when passing across this region is

$$M - E = M - M(u) + O.$$

When the ray $v = \text{const.}$ goes out into the region $r = O(M)$, the variation of $E$ along it can continue only if $Q^2(u)$ is already macroscopic. Otherwise, this variation ceases, and it will be recalled that no such variation occurs when approaching the AH. On the line $v = \text{const.}$, the AH is a regular point. The variation of $E$ along $v = \text{const.}$ is an effect of weak field but this effect is responsible for the distinction of $E_{\text{AH}}(v)$ from $M$.

Eq. (6.47) is also the cause for which the calculation of $(\nabla r, \nabla v)$ is more involved than the calculation of $(\nabla r, \nabla u)$ in eqs. (6.23)-(6.26). Integrating eq. (4.8) with the initial condition at $I^-$, one obtains

$$\ln(\nabla r, \nabla v) = \int_r^\infty \frac{dr}{r} D \bigg|_{v=\text{const.}},$$

and insertion of expression (6.44) yields

$$(\nabla r)^2 > |O|: \quad \ln(\nabla r, \nabla v) = 2 \int_r^\infty \frac{dr}{r} \frac{dM(u)}{du^+} \left( (\nabla r)^2 \frac{\partial r}{\partial u^+} \right)^{-1} \bigg|_{v=\text{const.}} + O \frac{r}{r}. \quad (6.53)$$

Since $dM/du^+ = O$, the sector of the integration path at which $r = O(M)$ contributes $O$:

$$(\nabla r)^2 > |O|: \quad \ln(\nabla r, \nabla v) = O + 2 \int_{M/|O|}^{\infty} \frac{dr}{r} \frac{dM(u)}{du^+} \left( (\nabla r)^2 \frac{\partial r}{\partial u^+} \right)^{-1} \bigg|_{v=\text{const.}}. \quad (6.54)$$

In the remaining integral, one can put $(\nabla r)^2 = 1 + O$ to obtain

$$\int_{M/|O|}^{\infty} \frac{dr}{r} \frac{dM(u)}{du^+} \left( \frac{\partial r}{\partial u^+} \right)^{-1} \bigg|_{v=\text{const.}} = \frac{|O|}{M} \int_{M/|O|}^{\infty} \frac{dr}{r} \frac{dM(u)}{du^+} \left( \frac{\partial r}{\partial u^+} \right)^{-1} \bigg|_{v=\text{const.}}$$

$$= \frac{|O|}{M} (M - M) = O. \quad (6.55)$$
This calculation can be done more rigorously by dividing the integration interval in eq. (6.52) into three: \((\nabla r)^2 = \mathcal{O}, |\mathcal{O}| < (\nabla r)^2 < 1 - |\mathcal{O}|, \) and \((\nabla r)^2 = 1 - |\mathcal{O}|\) with appropriately chosen border points. The main thing is that the integral (6.52) contains an extra \(1/r\) as compared to the analogous integral in eq. (6.43). The result is

\[(\nabla r)^2 > |\mathcal{O}|: \quad (\nabla r, \nabla v) = 1 + \mathcal{O} . \quad (6.56)\]

The specific form of the \(\mathcal{O}\) in expression (6.56) is of interest only at infinity. To obtain the behaviour of \((\nabla r, \nabla v)\) at \(I^+\), introduce in eq. (6.53) the integration variable \(u\) and go over to the limit of \(I^+\) in the integrand using that, by eq. (4.37),

\[r \bigg| _{I^+} = \frac{v}{2} + \cdots . \quad (6.57)\]

The resultant behaviour is

\[\left. (\nabla r, \nabla v) \right| _{I^+} = 1 + 2 \frac{M - \mathcal{M}(u) + \mathcal{O}}{r} + \cdots , \quad (6.58)\]

and the behaviours

\[\left. (\nabla r, \nabla v) \right| _{I^-} = 1 + \frac{c_1}{r} + \cdots \quad (6.59)\]

follow from eqs. (6.52) and (6.48).
7 Global solution

By the consideration above, the quantity (6.22) is uniformly bounded with some $O$:

\[
\frac{r}{M}|A| < A, \quad A = \text{const.} = O. \tag{7.1}
\]

Then, inspecting the right-hand sides of eqs. (6.23), (6.28), (6.35), and using the result in the Appendix, one infers that the condition of validity of the weak-field solution is

\[
(\nabla r)^2 > O(\sqrt{A}). \tag{7.2}
\]

In all the equations (6.25)-(6.44) one can replace the condition $(\nabla r)^2 > |O|$ with condition (7.2). This implies that there exists a region:

**OVERLAP:** \[ O = (\nabla r)^2 > O(\sqrt{A}), \tag{7.3} \]

or, equivalently,

**OVERLAP:** \[ O = -\frac{\partial r}{\partial u^+} > O(\sqrt{A}) \tag{7.4} \]

in which both the weak-field and strong-field solutions are valid.

The fact that the regions of weak field and strong field overlap will now be used to relate the data at the AH to the data at infinity. The data functions $\beta$ and $\kappa$ in eqs. (4.12) and (4.13) depend on the choice of $u$ for the strong-field solution. The choice will now be made as $u = u^+$. The functions $\beta$ and $\kappa$ below refer to this choice. As far as the data at infinity are concerned, the consistency requirements in the asymptotic domain bring to no limitations on the functions $\mathcal{M}(u)$ and $Q^2(u)$. Only their initial values are fixed as in eq. (6.5). These values enable one to use the conditions

\[
|O| < \mathcal{M}(u) < \frac{1}{|O|}, \quad |Q^2(u)| < \frac{1}{|O|} \tag{7.5}
\]

on the same grounds as conditions (3.12). It will be shown below that additional limitations on the data functions $\mathcal{M}(u)$ and $Q^2(u)$ stem from the requirement of boundedness of the curvature in the compact domain.
First note that eqs. (4.10) and (3.12) enable one to integrate any equation of the form

$$r \partial_v X = O \left( \frac{\partial r}{\partial u} \right) .$$  \hspace{1cm} (7.6)

With the data for $X$ on an arbitrary line

$$L:\quad v = f(u) ,$$  \hspace{1cm} (7.7)

one obtains

$$X = X \bigg|_L - O \left( \frac{\partial r}{\partial u} \bigg|_L \right) + O \left( \frac{\partial r}{\partial u} \right) .$$  \hspace{1cm} (7.8)

The solution in this form is valid globally but is useful only in the region of strong field. Introduce the function

$$K = - (\partial_a r)^{-1} \partial_a \frac{\partial r}{\partial u^+} .$$  \hspace{1cm} (7.9)

Using eqs. (4.10) and (4.8), one can calculate

$$r \partial_v K = - \frac{1}{2(\nabla r, \nabla v)} \left[ (\partial_a r)^{-1} \partial_a B + \frac{B(D - 1)}{r} \right] \frac{\partial r}{\partial u^+} ,$$  \hspace{1cm} (7.10)

and hence, by the boundedness conditions,

$$r \partial_v K = \frac{O(1)}{M} \frac{\partial r}{\partial u^+} = \frac{1}{M} O \left( \frac{\partial r}{\partial u^+} \right) .$$  \hspace{1cm} (7.11)

This equation is of the form (7.6). Therefore, the solution (7.8) applies and, for the region of strong field, yields

$$- \frac{\partial r}{\partial u^+} = O: \quad K = K \bigg|_L - \frac{1}{M} O \left( \frac{\partial r}{\partial u^+} \bigg|_L \right) + \frac{1}{M} O \left( \frac{\partial r}{\partial u^+} \right) .$$  \hspace{1cm} (7.12)

On the other hand, in the region of weak field one has eq. (6.27). Combining it with the identity

$$r \partial_u (\nabla r)^2 = (B - D(\nabla r)^2) \frac{\partial r}{\partial u} ,$$  \hspace{1cm} (7.13)

one obtains in the region of weak field

$$- \frac{\partial r}{\partial u^+} > O(\sqrt{A}): \quad K = \frac{1}{2r} \left( B - D(\nabla r)^2 \right)$$  \hspace{1cm} (7.14)

and, hence, in the region of overlap

\text{OVERLAP}: \quad K = \frac{B}{4E} + O .  \hspace{1cm} (7.15)
Eq. (7.15) provides the initial condition for the region of strong field. Choosing the line $L$ as passing in the region of overlap, for example,

$$ L: \quad -\frac{\partial r}{\partial u^+} = \mathcal{A}^\varepsilon, \quad 0 < \varepsilon < \frac{1}{2}, \quad \varepsilon = \text{const}, \quad (7.16) $$

one obtains from eqs. (7.12) and (7.15)

$$ -\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad K = \frac{B}{4E} \bigg|_L + \mathcal{O}. \quad (7.18) $$

Hence

$$ -\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad |\mathcal{O}| < K < \frac{1}{|\mathcal{O}|} \quad (7.19) $$
in virtue of eq. (3.12).

Next, calculate the action of the operator $\partial/\partial u^+$ on the quantity (7.10), and use condition (7.19) and the boundedness conditions to obtain

$$ -\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad r \frac{\partial K}{\partial u^+} + \mathcal{O} \bigg|_L = \frac{1}{M^2} O \left( \frac{\partial r}{\partial u^+} \right). \quad (7.20) $$

Provided that the line $L$ passes in the region of strong field, the solution (7.8) applies again:

$$ -\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad \frac{\partial K}{\partial u^+} \bigg|_L = -\frac{1}{M^2} O \left( \frac{\partial r}{\partial u^+} \right) + \frac{1}{M^2} O \left( \frac{\partial r}{\partial u^+} \right). \quad (7.21) $$

On the other hand, in the region of overlap one can differentiate eq. (7.15):

$$ \text{OVERLAP}: \quad \frac{\partial K}{\partial u^+} = \frac{O(1)}{M^2} \frac{\partial r}{\partial u^+} = \frac{1}{M^2} O \left( \frac{\partial r}{\partial u^+} \right). \quad (7.22) $$

Hence, choosing the line $L$ as in eq. (7.16), one obtains

$$ -\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad \frac{\partial K}{\partial u^+} = \mathcal{O}. \quad (7.23) $$

In view of condition (7.19), the same is true of any power of $K$, e.g.,

$$ -\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad \frac{1}{\partial u^+} K = \mathcal{O}. \quad (7.24) $$
Eqs. (7.9) and (7.19) enable one to integrate in the region of strong field any equation of the form

\[ r \partial_u Y = O \left( \frac{\partial r}{\partial u} \right). \]  

(7.25)

With the data for \( Y \) on an arbitrary line \( L \):

\[ u = f(v) \]  

(7.26)

passing in the region of strong field, one obtains

\[ -\frac{\partial r}{\partial u^+} = O: \quad Y = Y \bigg|_L - O \left( \frac{\partial r}{\partial u^+} \right)_L + O \left( \frac{\partial r}{\partial u^+} \right). \]  

(7.27)

In particular, the boundedness condition (3.11) solves as

\[ -\frac{\partial r}{\partial u^+} = O: \quad \partial_v B = \partial_u B \bigg|_L - \frac{1}{M} O \left( \frac{\partial r}{\partial u^+} \right)_L + \frac{1}{M} O \left( \frac{\partial r}{\partial u^+} \right). \]  

(7.28)

On the other hand, in the region of weak field one can differentiate the solution (6.41):

\[ -\frac{\partial r}{\partial u^+} > O(\sqrt{A}): \quad \partial_v B = \left( \frac{\nabla r}{\nabla u} \right)^2 \left( -\frac{M(u)}{r^2} + 2 \frac{Q^2(u)}{r^3} + O \right). \]  

(7.29)

Hence in the region of overlap

\[ \text{OVERLAP:} \quad \partial_v B = \frac{1}{M} O \left( \frac{\partial r}{\partial u^+} \right) \]  

(7.30)

where use is made of eq. (6.27) and conditions (7.5). The line \( L \) can be chosen as \( L \) in eq. (7.16). Indeed, by eq. (7.17), the equation of the line \( L \) is solvable with respect to \( u \) as well as with respect to \( v \). As a result, from eqs. (7.28) and (7.30) one obtains

\[ -\frac{\partial r}{\partial u^+} = O: \quad \partial_v B = O. \]  

(7.31)

Then, for \( B \) at the AH, one may write

\[ \frac{dB_{\text{AH}}}{dv} = (\partial_v B) \bigg|_{\text{AH}} + \frac{du_{\text{AH}}}{dv} (\partial_u B) \bigg|_{\text{AH}} \]  

(7.32)

and use eqs. (7.31), (4.15), and (3.9). In this way for the data at the AH one obtains the condition

\[ \frac{dB_{\text{AH}}}{dv} = O \]  

(7.33)

and, thereby, confirms the assumption (4.16).
The line \( L \) in eq. (7.26) can, alternatively, be chosen as the AH. Then, since eqs. (4.7)-(4.9) are of the form (7.25), they solve as

\[
- \frac{\partial r}{\partial u} = O : \quad B = B_{\text{AH}}(v) - O(\beta(v)) + O\left(\frac{\partial r}{\partial u^+}\right), \quad (7.34)
\]

\[
(\nabla r, \nabla v) = \alpha(v) \left[ 1 - O(\beta(v)) + O\left(\frac{\partial r}{\partial u^+}\right) \right], \quad (7.35)
\]

\[
r = 2E_{\text{AH}}(v) \left[ 1 - O(\beta(v)) + O\left(\frac{\partial r}{\partial u^+}\right) \right]. \quad (7.36)
\]

With these expressions, eq. (4.10) solves as

\[
- \frac{\partial r}{\partial u} = O : \quad - \left(1 + O\left(\frac{\partial r}{\partial u^+}\right)\right) \frac{\partial r}{\partial u^+} = \left(1 + O(\beta(u))\right) \beta(u) \times \exp\left[ \int_{\nu_{\text{AH}}(u)}^{u} dv \frac{B_{\text{AH}}(v)}{4\alpha(v)E_{\text{AH}}(v)} \left(1 + O(\beta(v))\right) \right]. \quad (7.37)
\]

Hence

\[
- \frac{\partial r}{\partial u^+} = O : \quad - \frac{\partial}{\partial u^+} \frac{\partial r}{\partial u^+} = \left(\kappa(u) + O\right) \frac{\partial r}{\partial u^+} \quad (7.38)
\]

with the function \( \kappa \) in eq. (4.13). From eqs. (7.9) and (7.38) one infers

\[
- \frac{\partial r}{\partial u^+} = O : \quad K = \kappa(u) + O, \quad (7.39)
\]

and then eqs. (7.19) and (7.24) yield for the data at the AH the conditions

\[
|O| < \kappa < \frac{1}{|O|}, \quad \frac{d}{du^+} \frac{1}{\kappa} = O. \quad (7.40)
\]

Thereby, one confirms the assumptions (4.17).

In the region of weak field, differentiate again the solution (6.41), this time with respect to \( u \):

\[
- \frac{\partial r}{\partial u^+} > O(\sqrt{A}) : \quad \partial_u B = -\frac{2}{r} \left( \frac{M(u)}{r} - \frac{2Q^2(u)}{r^2} + O \right) \partial_u r + \frac{2}{r} \left( \frac{dM(u)}{du} - \frac{1}{r} \frac{dQ^2(u)}{du} \right). \quad (7.41)
\]

By the boundedness conditions (3.9) and (7.5), this relation can be written in the form

\[
- \frac{\partial r}{\partial u^+} > O(\sqrt{A}) : \quad \frac{1}{r} \left( \frac{dM(u)}{du} - \frac{1}{r} \frac{dQ^2(u)}{du} \right) = \frac{O(1)}{M} \frac{\partial r}{\partial u}. \quad (7.42)
\]
From eq. (6.44) and the boundedness of $D$, one obtains another such relation:

$$ - \frac{\partial r}{\partial u^+} > O(\sqrt{A}) : \quad \frac{dM(u)}{du} - \frac{1}{2r} \frac{dQ^2(u)}{du} = O(1) \frac{\partial r}{\partial u}. $$  \hspace{1cm} (7.43)

The point is that the region of validity of these relations includes the region of overlap, and, there, $\frac{\partial r}{\partial u^+} = O$. Hence, for all $u^+$ for which the rays $u = \text{const.}$ reach the region of overlap, one obtains the conditions

$$ \frac{dM(u)}{du^+} = O, \quad \frac{dQ^2(u)}{du^+} = O $$ \hspace{1cm} (7.44)

limiting the data at $I^+$. Thereby, one proves eq. (6.45). This limitation, like eqs. (7.33) and (7.40), is a condition of the existence of the global solution.

Eq. (7.44) is not the only limitation on the data at $I^+$ that follows from the consistency requirements in the compact domain. The requirement that the weak-field solution be consistent in the region of overlap and that, moreover, conditions (3.12) be fulfilled:

$$ \text{OVERLAP : } rB > M |O| $$ \hspace{1cm} (7.45)

brings via eqs. (6.39), (6.41), and (7.5) to the following limitation on the data functions:

$$ \frac{Q^2(u)}{M^2(u)} < 1 - |O|. $$ \hspace{1cm} (7.46)

For sufficiently early $u$, conditions (7.44) hold by the correspondence principle, and condition (7.46) is fulfilled in consequence of the initial conditions (6.5). If it will turn out that they are valid only up to some finite value of $u^+$, then the line $u = \text{const.}$ with this value of $u^+$ bounds the region of validity of the present solution.

Condition (3.12) for $B$ used in eqs. (6.40) and (6.41) implies also that in the region of weak field

$$ - \frac{\partial r}{\partial u^+} > O(\sqrt{A}) : \quad E > \frac{1}{2} M(u) + |O|. $$ \hspace{1cm} (7.47)

Then consider the weak-field solution in the region of overlap. In this region one has $r = 2E(1 + O)$, and, therefore, eq. (6.40) becomes the following equation for $E$:

$$ \text{OVERLAP : } E = M(u) - \frac{Q^2(u)}{4E} + O. $$ \hspace{1cm} (7.48)
Condition (7.47) singles out the solution

\[ E = \frac{1}{2} \left( \mathcal{M}(u) + \sqrt{\mathcal{M}^2(u) - Q^2(u) + \mathcal{O}} \right). \]  

(7.49)

Hence for \( B \) one obtains

\[ B = 2 - \frac{2\mathcal{M}(u)}{\mathcal{M}(u) + \sqrt{\mathcal{M}^2(u) - Q^2(u) + \mathcal{O}}} + \mathcal{O}. \]  

(7.50)

Finally, in the region of weak field one has eqs. (6.56) and (6.26) whence

\[ (\nabla r, \nabla v) = 1 + \mathcal{O}, \quad (\nabla r, \nabla u^+) = -1 + \mathcal{O}. \]  

(7.51)

Consider now the strong-field solution. Any path connecting the AH with the asymptotically flat infinity crosses the region of overlap. The paths \( v = \text{const.} \) cross it when

\[ O \left( \frac{1}{\beta(v)} \right) > e^{F_2} > O \left( \frac{1}{\sqrt{\beta(v)}} \right). \]  

(7.52)

This is seen from eqs. (7.4), (4.29), and (4.20). Therefore,

\[ \Gamma_2 = \frac{1}{|\mathcal{O}|}. \]  

(7.53)

Using eq. (7.53) and eqs. (4.46)-(4.47), one obtains from the strong-field solution (4.20)-(4.29)

\[ (\nabla r, \nabla v) = \alpha(v)(1 + \mathcal{O}), \]  

(7.54)

\[ (\nabla r, \nabla u) = -\frac{B_{AH}(u)}{4E_{AH}(u)\kappa(u)}(1 + \mathcal{O}), \]  

(7.55)

\[ E = E_{AH}(u)(1 + \mathcal{O}), \]  

(7.56)

\[ B = B_{AH}(u) + \mathcal{O}. \]  

(7.57)

Equating the functions in eqs. (7.54)-(7.57) and (7.49)-(7.51) relates the data at the AH to the data at infinity:

\[ \alpha = 1 + \mathcal{O}, \]  

(7.58)

\[ \kappa = \frac{B_{AH}}{4E_{AH}}(1 + \mathcal{O}), \]  

(7.59)

\[ E_{AH}(u) = \frac{1}{2} \left( \mathcal{M}(u) + \sqrt{\mathcal{M}^2(u) - Q^2(u) + \mathcal{O}} \right) + \mathcal{O}, \]  

(7.60)
\[ B_{AH}(u) = 2 - \frac{2\mathcal{M}(u)}{\mathcal{M}(u) + \sqrt{\mathcal{M}^2(u) - Q^2(u)} + \mathcal{O}} + \mathcal{O}. \] (7.61)

Relations (7.58) and (7.59) result from fixing the normalizations of \( v \) and \( u \). In particular, the choice \( u = u^+ \) results in the determination of \( \kappa \) as in eq. (7.59). Insertion of expression (4.13) in eq. (7.59) yields the differential constraint

\[
\frac{d \ln \beta}{du^+} = \frac{B_{AH}}{4E_{AH}} \left( \frac{1}{\alpha} \frac{dv_{AH}}{du^+} - 1 + \mathcal{O} \right), \quad \beta = -2 \frac{dE_{AH}}{du^+} \]

(7.62)

which, combined with eq. (7.58), leaves two independent data functions at the AH: \( E_{AH} \) and \( B_{AH} \). These are related to the data at \( I^+ \) by eqs. (7.60) and (7.61). In consequence of relation (7.61), condition (4.79) imposes a new limitation on the data at \( I^+ \):

\[
-|\mathcal{O}| \leq \frac{Q^2(u)}{M^2(u)}. \] (7.63)

The inequalities (7.63) and (7.46) clutch the ratio \( Q^2/M^2 \).

Setting \( u = u^+ \) and using relations (7.58), (7.59) simplify the equations of section 4, and it will be added that these equations enable one to calculate the red-shift factor. Inserting the solution (4.22) in eq. (3.17), one obtains

\[
\frac{du^-}{du^+} = 2\beta_0 \exp \left( - \int_{u^-}^{u^+} du^+ \kappa(u) \right), \quad (7.64)
\]

\[
\frac{d}{du^+} \ln \frac{du^+}{du^-} = \frac{B_{AH}(u)}{4E_{AH}(u)} \left( 1 + \mathcal{O} \right). \quad (7.65)
\]

For the curvature \( D \) in the region of strong field, integration of eq. (3.8) yields the result similar to (4.27)

\[
-\frac{\partial r}{\partial u^+} = \mathcal{O}: \quad D = D_{AH}(v) + \mathcal{O} \]

(7.66)

but, within the strong-field solution, one is unable to express \( D_{AH} \) through the independent data. Adjoining the weak-field solution (6.46) through the region of overlap, one obtains \( D_{AH} \) and infers that the equation

\[
D = \mathcal{O} \]

(7.67)

holds globally. The equations (5.1) and (5.2) also hold globally. Thus, in the basis (1.8)-(1.11), only one projection of \( T_{\nu\mu}^{\text{vac}} \) can become macroscopic in the semiclassical region: \( T_1 \).
Global solution (continued)

Missing now are analytic expressions for the solution that could be used in both the weak-field and strong-field regions. Outside the AH, such expressions can be obtained. Of the main interest is the global solution for an outgoing light ray. It can be obtained as follows.

In the region of weak field one can integrate the identity (4.37) by making use of eq. (6.39). However, for making use also of eq. (6.56), \( r < M/|O| \) because, for large \( r \), the correction in eq. (6.58) cannot be discarded. The initial condition can be taken at the line \( L \) in the region of overlap. Then, for

\[
(\nabla r)^2 \bigg|_L \leq (\nabla r)^2 < 1 - |O|,
\]

one obtains

\[
\left(v - v_{AH}(u)\right) = 2r + 4\mathcal{M}(u)\ln\frac{r}{2E_{AH}(u)} + \frac{4E_{AH}(u)}{B_{AH}(u)}\ln\frac{(\nabla r)^2}{(\nabla r)^2 \bigg|_L} - 4E_{AH}(u)(1 + \mathcal{O})
\]

\[
- \frac{4E_{AH}(u)}{B_{AH}(u)} \left[1 + \left(1 - B_{AH}(u)\right)^2\right] \left\{\ln \left[1 - \frac{2E_{AH}(u)}{r} (1 - B_{AH}(u))\right] - \ln B_{AH}(u)\right\}
\]

where the identifications (7.60), (7.61) are used.

In the region of strong field one has eqs. (4.24) and (4.18) in which one may insert the expression (7.59) for \( \kappa \). Outside the AH, one may use also eqs. (4.46)-(4.47) to obtain

\[
(\nabla r)^2 = |O|: \quad (\nabla r)^2 + 2\beta(v)(1 + \mathcal{O}) = 2\beta(u) e^{\Gamma_1}(1 + \mathcal{O}) .
\]

It follows from the derivation of eqs. (4.46)-(4.47) that, for a point \((u, v)\) outside the AH, the integrand of \( \Gamma_1 \) in eq. (4.19) is constant up to \( \mathcal{O} \):

\[
(\nabla r)^2 = |O|: \quad \Gamma_1 = \frac{B_{AH}(u)}{4E_{AH}(u)} (v - v_{AH}(u))(1 + \mathcal{O}) .
\]

Hence one obtains

\[
(\nabla r)^2 = |O|: \quad (1 + \mathcal{O}) (v - v_{AH}(u)) = \frac{4E_{AH}(u)}{B_{AH}(u)} \ln \frac{(\nabla r)^2 + 2\beta(v)}{2\beta(u)} + \mathcal{O} .
\]
When the point \((u, v)\) is at the AH, \(\beta(v) = \beta(u)\). In this way eq. (8.5) verifies at the AH.

The solution (8.5) is valid on the line \(L\) as well:

\[
(1 + \mathcal{O}) \left( v_L - v_{AH}(u) \right) = \frac{4E_{AH}(u)}{B_{AH}(u)} \ln \frac{(\nabla r)^2}{2\beta(u)} + \mathcal{O}.
\]  

(8.6)

Here use is made of the fact that 

\[
(\nabla r)^2 \big|_L > O \left( \beta^{1/2}(v) \right) .
\]  

(8.7)

The bounds (8.1) and (8.7) can be used to replace in equation (8.2) \((\nabla r)^2\) with \((\nabla r)^2 + 2\beta(v)\) within the accuracy of this equation. Then, combining eqs. (8.2) and (8.6), one obtains

\[
(1 + \mathcal{O}) \left( v - v_{AH}(u) \right) = 2r + 4M(u) \ln \frac{r}{2E_{AH}(u)} + \frac{4E_{AH}(u)}{B_{AH}(u)} \ln \frac{(\nabla r)^2 + 2\beta(v)}{2\beta(u)} - 4E_{AH}(u) - \frac{4E_{AH}(u)}{B_{AH}(u)} \left[ 1 + (1 - B_{AH}(u))^2 \right] \left\{ \ln \left[ 1 - \frac{2E_{AH}(u)}{r} \left( 1 - B_{AH}(u) \right) \right] - \ln B_{AH}(u) \right\} (8.8)
\]

and concludes that this expression is valid in the whole of the range

\[
0 \leq (\nabla r)^2 < 1 - |\mathcal{O}| .
\]  

(8.9)

Indeed, in the range (8.1) it is valid by derivation, and in the range \((\nabla r)^2 = |\mathcal{O}|\) it coincides with expression (8.5). Outside the AH, only in the asymptotic region \((\nabla r)^2 = 1 - |\mathcal{O}|\) does the expression (8.8) need a correction. In this region, one may use the asymptotic behaviours (6.3) and (6.58) to obtain the solution of eq. (4.37):

\[
\left( v - v_{AH}(u) \right) \big|_{I^+} = 2r + 4M \ln \frac{r}{2E_{AH}(u)} + \cdots .
\]  

(8.10)

The correction is thus in the fact that, when \(\ln r\) is large, its coefficient is constant and equal to \(4M\) as distinct from \(4M(u)\) in eq. (8.8). With this correction, eq. (8.8) is the soughtafter solution for an outgoing light ray at \((\nabla r)^2 \geq 0\).

Outside the AH, also the following relations are valid globally:

\[
(\nabla r)^2 \geq 0 : \quad -2(1 + \mathcal{O}) \frac{\partial r}{\partial u^+} = (\nabla r)^2 + 2\beta(v) ,
\]  

(8.11)

\[
(\nabla r)^2 \geq 0 : \quad (1 + \mathcal{O})(\nabla r, \nabla u^+) = -\frac{(\nabla r)^2}{(\nabla r)^2 + 2\beta(v)} ,
\]  

(8.12)
\((\nabla r)^2 \geq 0: \quad (1 + \mathcal{O})(\nabla v, \nabla u^+) = -\frac{2}{(\nabla r)^2 + 2\beta(v)}. \quad (8.13)\)

Eqs. (8.8)-(8.13) generalize the respective classical formulae.

The result (8.8) is a specific case of the following result for the integral along an outgoing light ray:

\[
(\nabla r)^2 \geq 0: \quad \int \frac{dr}{(\nabla r)^2} \left. f \right|_{u=\text{const.}} = f_{AH}(u) \frac{r_{AH}(u)}{B_{AH}(u)} \ln \frac{1}{2\beta(u)} + f \frac{r}{B} \ln \left( (\nabla r)^2 + 2\beta(v) \right) \nonumber \]

\[
- \int_0^{(\nabla r)^2} d(\nabla r)^2 \left( \ln(\nabla r)^2 \right) \frac{d}{d(\nabla r)^2} \left( f \frac{r}{B} \bigg|_{u=\text{const.}} \right). \quad (8.14)\]

In the integral remaining on the right-hand side, one may insert the explicit expressions (6.39) and (6.41) for \((\nabla r)^2\) and \(B\). The result (8.14) is valid up to \(\mathcal{O}\) for any function \(f\) that possesses the properties

\[
f = O(1), \quad \frac{d}{d(\nabla r)^2} f \bigg|_{u=\text{const.}} = O(1) \quad (8.15)\]

including at the AH. The property of derivative is possessed, in particular, by any function that depends on \(v\) only through the arguments

\[
f = f(r, E) \quad (8.16)\]

and has bounded derivatives with respect to these arguments. Indeed, by eqs. (4.37), (4.68), and (1.13),

\[
\left. \frac{d}{d(\nabla r)^2} \right|_{u=\text{const.}} = r \frac{(\nabla r)^2}{B(\nabla r)^2 - 4A} \cdot (8.17)\]

\[
\left. \frac{d}{d(\nabla r)^2} E \right|_{u=\text{const.}} = r \frac{2A}{B(\nabla r)^2 - 4A} - \frac{1}{2} r \frac{(\nabla r)^2}{B(\nabla r)^2 - 4A} \left( B - 1 + (\nabla r)^2 \right) . \quad (8.18)\]

For \(0 \leq (\nabla r)^2 < 1 - |\mathcal{O}|\), these derivatives are bounded owing to the negativity of \(A\).

For the proof of eq. (8.14), first calculate using expressions (6.39) and (6.41)

\[
(\nabla r)^2 \geq (\nabla r)^2 \bigg|_L: \quad \int \frac{dr}{(\nabla r)^2} \left. f \right|_{u=\text{const.}} = f \frac{r}{B} \ln(\nabla r)^2 - \left( f \frac{r}{B} \ln(\nabla r)^2 \right) \bigg|_L \nonumber \]

\[
- \int (\nabla r)^2 (\ln(\nabla r)^2) \frac{d}{d(\nabla r)^2} \left( f \frac{r}{B} \bigg|_{u=\text{const.}} \right). \quad (8.19)\]
By conditions (8.15) and the boundedness of the derivative (8.17), one has

\[ \int_0^t d(\nabla r)^2 \left( \ln(\nabla r)^2 \right) \frac{d}{d(\nabla r)^2} \left( f \frac{r}{B} \bigg|_{u=\text{const.}} \right) = O \, . \]  

(8.20)

Therefore, the lower limit of the integral on the right-hand side of eq. (8.19) can be shifted to zero. By conditions (8.15) one has also

\[ (\nabla r)^2 = |O| : \int_{r_{\text{AH}}}^r \frac{dr}{(\nabla r)^2} f \bigg|_{u=\text{const.}} = f_{\text{AH}}(u) \int_{r_{\text{AH}}}^r \frac{dr}{(\nabla r)^2} \bigg|_{u=\text{const.}} + O \, . \]  

(8.21)

The latter integral is the solution of eq. (4.37), and it has already been calculated in eq. (8.5):

\[ (\nabla r)^2 = |O| : \int_{r_{\text{AH}}}^r \frac{dr}{(\nabla r)^2} f \bigg|_{u=\text{const.}} = f_{\text{AH}}(u) \frac{r_{\text{AH}}(u)}{B_{\text{AH}}(u)} \ln \left( \frac{(\nabla r)^2}{2\beta(u)} \right) + O \, . \]  

(8.22)

Hence

\[ \int_{r_{\text{AH}}}^r \frac{dr}{(\nabla r)^2} f \bigg|_{u=\text{const.}} = f_{\text{AH}}(u) \frac{r_{\text{AH}}(u)}{B_{\text{AH}}(u)} \ln \left( \frac{(\nabla r)^2}{2\beta(u)} \right) + O \, . \]  

(8.23)

Combining eqs. (8.19) and (8.23) yields the result

\[ (\nabla r)^2 \geq (\nabla r)^2 \bigg|_{L} : \int_{r_{\text{AH}}}^r \frac{dr}{(\nabla r)^2} f \bigg|_{u=\text{const.}} = f_{\text{AH}}(u) \frac{r_{\text{AH}}(u)}{B_{\text{AH}}(u)} \ln \left( \frac{1}{2\beta(u)} \right) + f \frac{r}{B} \ln(\nabla r)^2 \]

\[ - \int_0^t d(\nabla r)^2 \left( \ln(\nabla r)^2 \right) \frac{d}{d(\nabla r)^2} \left( f \frac{r}{B} \bigg|_{u=\text{const.}} \right) + O \, . \]  

(8.24)

Equations (8.22) and (8.24) calculate the integral in the (overlapping) regions of strong field and weak field respectively. Using these equations, one can check that, up to \( O \), eq. (8.14) is valid globally.

The virtue of the global solutions (8.8) and (8.14) is in the fact that they calculate the respective quantities outside the region of strong field. In the range

\[ |O| < (\nabla r)^2 < 1 - |O| \, , \]  

(8.25)

all terms of expression (8.14) except the first one are \( O(1) \) and are negligible:

\[ \int_{r_{\text{AH}}}^r \frac{dr}{(\nabla r)^2} f \bigg|_{u=\text{const.}} = f_{\text{AH}}(u) \frac{r_{\text{AH}}(u)}{B_{\text{AH}}(u)} \ln \left( \frac{1}{2\beta(u)} \right) + MO(1) \, . \]  

(8.26)
Similarly, in this range expression (8.8) is

$$v - v_{AH}(u) = \frac{4E_{AH}(u)}{B_{AH}(u)} \ln \frac{1}{2\beta(u)} + MO(1) \ .$$  \hspace{1cm} (8.27)

The large term proportional to \( \ln \beta(u) \) emerges as a contribution of the strong-field region and dominates in these expressions.

Eq. (8.27) gives the life-time of the "instantaneous" black hole, i.e., of what appears as a black hole at each instant of evaporation. Suppose that some falling observer hits the AH at a given value of \( u \). Then how much later should another observer fall to discover that, at this value of \( u \), there is no more black hole? The answer is in eq. (8.27). In particular, taken at the tangent ray \( u = u_0 \), eq. (8.27) gives the life-time of the "classical" black hole, i.e., the one that forms initially as a result of the collapse but then destroys itself by evaporation. In the classical geometry, this life-time is infinite. The earliest observer to discover that, at \( u = u_0 \), there is no more black hole is also the first to discover that the geometry is no more classical. Hence

$$v_{\text{crit}} - v_0 = 4M \ln \frac{1}{2\beta_0} + O(M) \ , \quad \beta_0 = - \frac{dr_{AH}}{du^+}\bigg|_{u=u_0} \ .$$  \hspace{1cm} (8.28)

This is the critical value of \( v \) provided for by the correspondence principle, eq. (2.4).
9 Summary

The above is all that kinematics can say. It expresses the metric in the semiclassical region through two Bondi charges $\mathcal{M}(u)$ and $Q^2(u)$, and this expression is valid as long as the Bondi charges satisfy the conditions

$$\mathcal{M} > |\mathcal{O}|, \quad (9.1)$$

$$\frac{d\mathcal{M}}{du^+} = \mathcal{O}, \quad \frac{dQ^2}{du^+} = \mathcal{O}, \quad \quad -|\mathcal{O}| \leq \frac{Q^2}{\mathcal{M}^2} < 1 - |\mathcal{O}|. \quad (9.2)$$

Their fulfillment is a verifiable assumption. On the basis of this assumption, the Bondi charges can be calculated. Thereby the bound in $u^+$ to the validity of these conditions will be established. For sufficiently early $u^+$, they hold deliberately but it may well be the case that there is a finite value of $u^+$ beyond which they are no longer valid. If condition (9.1) ceases being valid, then the limit of validity of semiclassical theory is reached. However, it may also be the case that, at some value of $u^+$, conditions (9.2) cease being valid while condition (9.1) still holds. It is only then and beyond this value of $u^+$ that the present approach will fail. The failure will possibly signify that there are some other semiclassical effects in the problem, different from and additional to the Hawking effect.

The Planck constant makes no appearance in the present study. It will appear at the next stage of the calculation. Given the metric in the semiclassical region, one can use any of the semiclassical techniques to calculate $T_{\mu\nu}^{\text{vac}}$ at $\mathcal{I}^+$ and, thereby, express the Bondi charges through themselves. The result will be closed equations for the data functions. In particular, the constant $\beta_0$ in eq. (8.28) will be related to the Hawking luminosity of the black hole. Kinematics reduces the problem in functions of two variables to a problem in functions of one variable. The latter problem is a subject of the quantum dynamics.
Appendix. Bound on the first derivative of the curvature $A$

Denote by prime the derivative $M(d/dr)$ along $v = \text{const}$. Then eq. (3.10) can be written as

$$|A''| < d, \quad d = O(1). \quad (A.1)$$

Eq. (4.29) implies that, in the region of strong field, $A$ satisfies the condition

$$|A| < A, \quad A = A(v) = O. \quad (A.2)$$

By eqs. (4.23), (4.20), and (4.1) this condition holds on the line $v = \text{const}$ in the interval of $r$ having the length

$$\Delta r = MO, \quad \forall O. \quad (A.3)$$

For the derivation below, it suffices that the interval of validity of condition (A.2) have the length

$$\Delta r > 2M\sqrt{A}, \quad (A.4)$$

which is, of course, secured by eq. (A.3). It will then be proved that

$$|A'| < c\sqrt{A}, \quad c = O(1). \quad (A.5)$$

The assertion to be proved is a corollary of a general lemma. Given bounds on a function and its second derivative, the lemma establishes a bound on the first derivative provided that the given bounds hold in a sufficiently large interval of the argument. For given $d$ in eq. (A.1) and $A$ in eq. (A.2), this interval of the argument should exceed $2\sqrt{A}$, and then

$$|A'| < (1 + 2d)\sqrt{A}. \quad (A.6)$$

For the proof of the lemma, consider any point in the interval of validity of conditions (A.1) and (A.2). Call it point 1. Suppose that

$$|A'_1| > \sqrt{A}, \quad (A.7)$$
and let, for definiteness, $A'_1$ be positive. Consider any subinterval containing point 1:

$$\bar{r} \leq r_1 \leq \bar{r}$$  \hspace{1cm} (A.8)

and having the length

$$\bar{r} - \bar{r} = 2M\sqrt{A}.$$  \hspace{1cm} (A.9)

It will be proved that, in this subinterval, there is a point 2 at which

$$A'_2 = \sqrt{A}, \quad \bar{r} \leq r_2 \leq \bar{r}.$$  \hspace{1cm} (A.10)

Indeed, if there is no such point, then

$$A' > \sqrt{A}, \quad \bar{r} \leq r \leq \bar{r},$$  \hspace{1cm} (A.11)

and one obtains using eq. (A.2):

$$2A > |\bar{A}| + |\bar{A}| \geq |\bar{A} - \bar{A}| = \bar{A} - \bar{A} = \frac{1}{M} \int_{r}^{\bar{r}} A' dr > \sqrt{A} \frac{\bar{r} - \bar{r}}{M}.$$  \hspace{1cm} (A.12)

Hence

$$\bar{r} - \bar{r} < 2M\sqrt{A}$$  \hspace{1cm} (A.13)

which is at variance with eq. (A.9).

Thus the point 2 exists, and

$$|r_1 - r_2| \leq 2M\sqrt{A}.$$  \hspace{1cm} (A.14)

Then one obtains using eq. (A.1):

$$A'_1 - \sqrt{A} = |A'_1 - A'_2| = \frac{1}{M} \left| \int_{r_2}^{r_1} A'' dr \right| < d \frac{|r_1 - r_2|}{M} \leq 2d\sqrt{A}.$$  \hspace{1cm} (A.15)

Having conducted the same consideration for negative $A'_1$, one concludes that, if the assumption (A.7) is true, then

$$|A'_1| < (1 + 2d)\sqrt{A}.$$  \hspace{1cm} (A.16)

But, if the assumption (A.7) is not true, eq. (A.16) is true all the more. This proves the result (A.6) for any point in the region of validity of conditions (A.1) and (A.2).
Acknowledgments

The present work was supported by the Italian Ministry for Foreign Affairs, and the Ministry of Education of Japan. Essential parts of it have been done during the author’s stays at the Yukawa Institute for Theoretical Physics, and at the University of Naples by the invitation of Centro Volta. The author is especially grateful for the hospitality and care to Luigi Cappiello and Roberto Pettorino in Naples, and Masao Ninomiya and Mihoko Nojiri in Kyoto.
References

1. S.W. Hawking, Commun. math. Phys. 43 (1975) 199.

2. See A.G. Mirzabekian and G.A. Vilkovisky, Ann. Phys. 270 (1998) 391, and references therein.

3. V.P. Frolov and I.D. Novikov, Black Hole Physics, Kluwer, Dordrecht, 1998.

4. V.P. Frolov and G.A. Vilkovisky, in Proc. 2nd Marcel Grossmann Meeting on General Relativity, Trieste, 1979 (R. Ruffini, Ed.), p. 455, North-Holland, Amsterdam, 1982.

5. V.P. Frolov and G.A. Vilkovisky, Phys. Lett. B 106 (1981) 307; in Proc. 2nd Seminar on Quantum Gravity, Moscow, 1981 (M.A. Markov and P.C. West, Eds.), p. 267, Plenum, London, 1983.
Figure captions

Fig.1. Penrose diagram for the semiclassical region. The bold curve is the apparent horizon. The light lines are level lines of the advanced time $v$ and retarded time $u$. The points 0 to 6 are referred to in the text.

Fig.2. Completed diagram of Fig. 1. The bold line $u = \text{const.}$ is the event horizon. The point I on the apparent horizon is the maximum of $-(dE_{AH}/du)$. The point II is the maximum of $-\alpha(dE_{AH}/dv)$. The broken line is the line of extrema of $(\nabla r)^2$. The light solid curve is $\partial_v E = 0$. 
This figure "figone.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/0511182v1
This figure "figtwo.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/0511182v1