Formula Method for Bound State Problems

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Abstract

In this work, we present a simple formula for finding bound states solution of any wave equation which can be simplified to the form of

\[ s(1 - k_3s)F''(s) + \left[ \gamma - (\alpha + \beta + 1)k_3s \right] F'(s) - \alpha\beta F(s) = 0 \]

with \( k_3 = 1, 2, 3, \ldots \), in terms of generalized hypergeometric functions \( _2F_1(\alpha, \beta; \gamma; k_3s) \). In order to show the accuracy of this proposed formula, we have re-obtained the bound state solutions of some already solved eigenvalues problems. This propose method have been demonstrated to be accurate, efficient, reliable and very easy to use.

Keywords: Schrödinger equation; asymptotic iteration; eigenvalue; eigenfunction; potential model.

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I. INTRODUCTION

In quantum mechanics, while solving Schrödinger, Dirac, Klein-Gordon, spinless Salpeter and Duffin-Kemmer-Petiau wave equations in the presence of some typical central or non-central potential model, we do often come across differential equation of the form

\[ \Psi''(s) + \frac{(k_1 - k_2 s)}{s(1 - k_3 s)} \Psi'(s) + \frac{(A s^2 + B s + C)}{s^2(1 - k_3 s)^2} \Psi(s) = 0, \]  

(1)

after which an appropriate coordinate transformation of the form \( s = s(r) \) has been used. Up till now, several approaches have been developed to exactly solve quantum systems. They include the asymptotic iteration method (AIM) [1-24], Feynman integral formalism [25-27], hypergeometric [28-30], exact quantization rule method [31-36], proper quantization rule [37-39], Nikiforov-Uvarov (NU) method [40-49], supersymmetric quantum mechanics [50-55], etc.

II. STATEMENT OF PROBLEM

Most of the above methods involves some complicated integrals while some involves great deal in algebraic manipulations. As a result, one needs a great efforts in mathematical skills before these methods could be applied to eigenvalues problems. Thus, a simple efficient methodology will have its merit in Mathematical-Physics society.

III. OBJECTIVE

- The objective of this research work is therefore to derive a very simple formula to exactly solve equation of form (1) for energy eigenvalues and wavefunctions. On contrary to other methods presented in the literature, this method is quite simple and could easily be followed, even by young scholars of Theoretical Physics solving eigenvalues problems.

- Furthermore, to test the accuracy of this new method, we investigate the solution of Schrödinger equation with several central and non-central potential models such as the Manning-Rosen, Eckart, Kratzer-type and generalized non-central Coulomb potentials for any arbitrary orbital quantum number \( l \neq 0 \). It is worth mentioning
that this method yielded results which are in excellent agreement with the existing
ones obtained by other approximate methods.

IV. FORMULA METHOD AND THEIR APPLICATIONS

In this section, we present a simple formula for finding bound state solutions of both the
relativistic and nonrelativistic wave equations. The derivation of this method can be found
in the appendix. We also proceed to demonstrate the accuracy and convienency of these
formulas by finding eigenvalues and eigenfunction of some already solved quantum system,
obtained by other methods early stated in in the introductory text.

A. Brief Overview of the Method

For a given Schrödinger-like equation including the centrifugal barrier and/or the spin-
orbit coupling term in the presence of any potential which can be written as

$$
\Psi''(s) + \frac{(k_1 - k_2 s)}{s(1 - k_3 s)} \Psi'(s) + \frac{(As^2 + Bs + C)}{s^2(1 - k_3 s)^2} \Psi(s) = 0.
$$

(2)

We propose that the energy eigenvalues and the corresponding wavefunction can be obtain
by using the following formulas

$$
\left[ k_4^2 - k_5^2 - \left[ \frac{1-2n}{2} - \frac{1}{2k_3} \left( k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right]^2 \right] \left[ \frac{1-2n}{2} - \frac{1}{2k_3} \left( k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right] - k_5^2 = 0, k_3 \neq 0,
$$

(3a)

$$
\Psi(s) = N_n s^{k_4}(1 - k_3 s)^{k_5} \, _2F_1 \left( -n, n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1; 2k_4 + k_1, k_3 s \right),
$$

(3b)

respectively, where

$$
k_4 = \frac{(1 - k_1) + \sqrt{(1 - k_1)^2 - 4C}}{2}, \quad k_5 = \frac{1}{2} + \frac{k_1}{2k_3} - \frac{k_2}{2k_3} + \sqrt{\left[ \frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3} \right]^2 - \left[ \frac{A}{k_3^2} + \frac{B}{k_3} + C \right]},
$$

(4)

and $N_{nf}$ is the normalization constant. Before proceeding to the applications of these formulae, let us first discuss a special case where $k_3 \rightarrow 0$. In this regard, the eigenvalues and the corresponding wave function can be obtained as

$$
\left[ \frac{B - k_4 k_2 - nk_2}{2k_4 + k_1 + 2n} \right]^2 - k_5^2 = 0,
$$

(5)
and

\[
\Psi(s) = N_n s^{k_4} e^{x_p(-k_5 s)} \, _1F_1(-n, 2k_4 + k_1, (2k_5 + k_2)s),
\]

respectively.

B. Application of the Formulas

In this section, in order to show the accuracy of these proposed formulas, we apply them to find the bound state solution of some quantum mechanical problems studied previously in the literature.

1. Schrödinger equation with rotational \((\ell \neq 0)\) Manning-Rosen potential

The radial Schrödinger equation with the Manning-Rosen potential is given by \[56\]

\[
\frac{d^2 u_{n\ell}}{dr^2} + \left[ \frac{2\mu E_{n\ell}}{\hbar^2} - \frac{1}{b^2} \left( \frac{\alpha(\alpha - 1)}{(e^{r/b} - 1)^2} - \frac{\tilde{A}}{e^{r/b} - 1} \right) - \frac{\ell(\ell + 1)}{r^2} \right] u_{n\ell}(r) = 0.
\]

(7)

To solve the above equation for \(\ell \neq 0\) states, we apply the following approximation scheme \[56\]

\[
\frac{1}{r^2} \approx \frac{1}{b^2} \left[ D_0 + D_1 \frac{1}{e^{r/b} - 1} + D_2 \frac{1}{(e^{r/b} - 1)^2} \right],
\]

(8)

where \(D_1 = D_2 = 1\) and \(D_0 = 1/12\) \[56–58\]. Now we recast the differential equation (7) into the form given in \[2\] by introducing the appropriate change of variables \(r \rightarrow z\) through the mapping function \(z = e^{-r/b}\) and defining

\[
\xi_{n\ell} = \sqrt{-\frac{2\mu b^2 E_{n\ell}}{\hbar^2} + \ell(\ell + 1)D_0}, \quad \beta_1 = \tilde{A} - \ell(\ell + 1)D_1 \quad \text{and} \quad \beta_2 = \alpha(\alpha - 1) + \ell(\ell + 1)D_2,
\]

(9)

to obtain

\[
\frac{d^2 u_{n\ell}(z)}{dz^2} + \frac{(1 - z)}{z(1 - z)} \frac{d u_{n\ell}(z)}{dz} + \left\{ \frac{(-\xi_{n\ell}^2 - \beta_1 - \beta_2) + z(2\xi_{n\ell}^2 + \beta_1) + (-\xi_{n\ell}^2)}{z^2(1 - z)^2} \right\} u_{n\ell}(z).
\]

(10)

By comparing equation (10) with equation (2), \(k_1, k_2, k_3, A, B\) and \(C\) can be easily determined together with \(k_4\) and \(k_5\) can be obtained as:

\[
k_4 = \xi_{n\ell}, \quad \text{and} \quad k_5 = \frac{1}{2}(1 + \sqrt{1 + 4\beta_2}).
\]

(11)
By using equations (3a) and (3b), we can easily calculate the energy eigenvalues and the corresponding wave function as

$$E_{n\ell} = \frac{\hbar^2}{24\mu b^2} \ell(\ell+1) - \frac{\hbar^2}{2\mu b^2} \left[ A + \alpha(\alpha - 1) - \left[ n + \frac{1}{2} + \frac{1}{2}\sqrt{(1-2\alpha)^2 + 4\ell(\ell+1)} \right]^2 \right], \quad (12)$$

$$u_{n\ell}(z) = N_{n\ell} z^{\xi_{n\ell}} (1-z)^{\frac{1}{2}[1+\sqrt{1+4\beta^2}]} _2F_1 \left( -n, n + 2(\xi_{n\ell} + \frac{1}{2}[1 + \sqrt{1 + 4\beta^2}]); 2\xi_{n\ell} + 1, z \right), \quad (13)$$

respectively, where $N_{n\ell}$ is the normalization constant. The above results are identical to the ones obtained in the literature [56, 59]. It should be noted that the solution of Hulthén potential can be easily obtained by setting $\alpha$ to 0 or 1 and $(A\hbar^2/2\mu b^2)$ to $Ze^2/\delta$ in equations (12) and (13).

2. Schrödinger equation with rotational Eckart potential

The radial Schrödinger equation with the Eckart potential is given by [3, 60] (in units $\hbar = c = \mu = 1$)

$$\frac{d^2R_{n\ell}(r)}{dr^2} + \left[ 2E_{n\ell} - \frac{2\beta e^{-r/a}}{(1 - e^{-r/a})^2} + \frac{2\alpha e^{-r/a}}{(1 - e^{-r/a})} - \frac{\ell(\ell+1)}{r^2} \right] R_{n\ell}(r) = 0. \quad (14)$$

By using an appropriate approximation given in [3, 60] to deal with the centrifugal term and introducing a new variable of the form $z = e^{-r/a}$, we have

$$\frac{d^2R_{n\ell}(z)}{dz^2} + \left[ \frac{(1-z)}{z(1-z)} \frac{dR_{n\ell}}{dz} + \frac{z^2(2a^2[E_{n\ell} - \alpha]) + z(2a^2[\beta - \alpha - 2E_{n\ell}] - \ell(\ell+1)) + 2E_{n\ell}a^2}{z^2(1-z)^2} \right] R_{n\ell}(z) = 0. \quad (15)$$

By comparing equation (15) with equation (2), $k_1, k_2, k_3, A, B$ and $C$ can be easily determined and $k_4$ and $k_5$ can be obtained as:

$$k_4 = \sqrt{-2E_{n\ell}a^2}, \quad k_5 = \frac{1}{2} + \frac{1}{2}\sqrt{(2\ell+1)^2 + 8a^2\beta}. \quad (16)$$

Thus, the energy eigenvalues can be easily obtained by using formula (3a) as

$$E_{n\ell} = \alpha - \frac{1}{8a^2} \left[ 2\alpha a^2 + \left( n + \frac{1}{2} + \frac{1}{2}\sqrt{(2\ell+1)^2 + 8\beta a^2} \right)^2 \right]^2, \quad (17)$$
and the corresponding wave functions can be obtained from (3b) as:

\[ R_{n\ell}(z) = N_{n\ell} z^{\sqrt{-2E_{n\ell}a^2}}(1 - z)^{1/2 + \sqrt{(2\ell + 1)^2 + 8a^2\beta}} \times _2F_1\left(-n, n + 2, \frac{1}{2} + \frac{1}{2} \sqrt{(2\ell + 1)^2 + 8a^2\beta}; 2\sqrt{-2E_{n\ell}a^2} + 1, z\right), \]  

(18)

where \( N_{n\ell} \) is the normalization constant. These results are in excellent agreement with the ones obtained previously [3, 60].

3. Schrödinger equation with the rotational Kratzer potential

The radial Schrödinger equation with the Kratzer potential is given by [7]

\[ \frac{d^2R_{n\ell}(s)}{ds^2} + \left[ s^2 \left( \frac{2\mu E_{n\ell}}{h^2} + s \left( \frac{4\mu D_ea^2}{h^2} \right) + \left( -\frac{2\mu D_ea^2}{h^2} - \ell(\ell + 1) \right) \right) s \right] R_{n\ell}(s) = 0, \]  

(19)

where we have introduced a new transformation of the form \( r = s \). Now by comparing equation (19) with equation (2), we can easily find the values of parameters \( A, B \) and \( C \). It is also clear that \( k_1 = k_2 = k_3 = 0 \) and the other parameters are obtained as

\[ k_4 = \frac{1}{2} + \sqrt{\left( \frac{1}{2} + \ell \right)^2 + \frac{2\mu D_ea^2}{h^2}} \quad \text{and} \quad k_5 = -\frac{2\mu E_{n\ell}}{h^2}. \]  

(20)

The energy eigenvalues can now be obtained by means of equation (3a) as

\[ E_{n\ell} = -\frac{2\mu D_e^2a^2}{h^2} \left[ n + \frac{1}{2} + \sqrt{\left( \frac{1}{2} + \ell \right)^2 + \frac{2\mu D_ea^2}{h^2}} \right]^{-2}, \]  

(21)

and the corresponding wave function can be found through equation (3b) as:

\[ R_{n\ell}(r) = r^{\frac{1}{2}} e^{-\sqrt{\frac{1}{2} + \ell} + \frac{2\mu D_ea^2}{h^2} r} \exp \left(-\sqrt{-\frac{2\mu E_{n\ell}}{h^2}} r \right) \times _1F_1\left(-n, 1 + 2, \frac{1}{2} + \sqrt{\left( \frac{1}{2} + \ell \right)^2 + \frac{2\mu D_ea^2}{h^2}} + 2r \sqrt{-\frac{2\mu E_{n\ell}}{h^2}} \right). \]  

(22)

These results are identical with the ones obtained in refs [7, 61].

4. Schrödinger equation with generalized non central Coulomb potential model

The ring-shaped coulombic (Hartmann) potential in spherical coordinates is given by [62–64]

\[ V(r, \theta) = -\frac{Ze^2}{r} + \frac{\beta}{r^2 \sin^2 \theta} + \frac{\gamma \cos \theta}{r^2 \sin^2 \theta}, \]  

(23)
The solution of the Schrödinger equation with the combined Coulomb plus Aharonov-Bohm potential and the Hartmann ring-shaped potential, which was originally proposed as a model for benzene molecules, can be obtained as a special case of potential (23). Now, the Schrödinger equation in the presence of potential \( V(r, \theta) \) can be reduced to the two ordinary differential equations [62–64]

\[
\frac{d^2 R_{n\ell}(r)}{dr^2} + \frac{2}{r} \frac{d R_{n\ell}(r)}{dr} + \left[ \frac{2\mu}{\hbar^2} \left( \frac{E_{n\ell}}{r} + \frac{Ze^2}{r^2} \right) - \frac{\ell(\ell + 1)}{r^2} \right] R_{n\ell}(r) = 0, \quad (24a)
\]

\[
\frac{d^2 \Theta(\theta)}{d\theta^2} + \text{coth} \theta \frac{d \Theta(\theta)}{d\theta} + \left[ \ell(\ell + 1) - \left( \frac{m^2 + \beta + \gamma \cos \theta}{\sin^2 \theta} \right) \right] \Theta(\theta) = 0, \quad (24b)
\]

where the corresponding total wave function is taken as \( \Psi(r, \theta, \phi) = R_n(r) \Theta(\theta) e^{im\phi} \). Now by introducing a new transformation of the form \( s = r \) and \( s = \frac{\cos \theta - 1}{2} \) in equations (24a) and (24b) respectively, we obtain

\[
\frac{d^2 R_{n\ell}(s)}{ds^2} + \frac{2}{s} \frac{d R_{n\ell}(s)}{ds} + \left[ \left( \frac{2\mu E}{\hbar^2} \right) s^2 + \left( \frac{2\mu Z e^2}{\hbar^2} \right) s + (-\ell(\ell + 1)) \right] R_{n\ell}(s) = 0, \quad (25a)
\]

\[
\frac{d^2 \Theta(s)}{ds^2} + \frac{2s + 1}{s(1 + s)} \frac{d \Theta(s)}{ds} + \left[ \left( -\ell(\ell + 1) \right) s^2 + (-\ell(\ell + 1) - \gamma/2)s + (-[m^2 + \beta + \gamma]/4) \right] \Theta(s) = 0. \quad (25b)
\]

It can be deduced from equation (25a) that \( k_1 = 2, k_2 = k_3 = 0 \) and the values of \( A, B, C \) can be clearly seen. The parameters \( k_4 \) and \( k_5 \) can therefore be calculated as

\[
k_4 = \ell, \quad k_5 = \sqrt{-\frac{2\mu E}{\hbar^2}}, \quad (26)
\]

and hence the eigenvalues equation can be found by using formula (3a) as

\[
E_{N\ell} = -\frac{\mu Z^2 e^4}{2\hbar^2 [N + \ell + 1]^2}. \quad (27)
\]

Again, by comparing equation (25b) with equation (2), \( k_1, k_2, k_3, A, B \) and \( C \) can be easily determined then \( k_4 \) and \( k_5 \) can be obtained as:

\[
k_4 = \frac{1}{2} \sqrt{m^2 + \beta + \gamma}, \quad k_5 = \frac{1}{2} \sqrt{m^2 + \beta - \gamma}, \quad (28)
\]

and by using equation (3a) the following results can be obtained:

\[
(\ell - n)^2 = \left[ \frac{1}{2} \sqrt{m^2 + \beta + \gamma} + \frac{1}{2} \sqrt{m^2 + \beta - \gamma} \right]^2
\]

\[
\ell = n + \left[ \frac{(m^2 + \beta) + \sqrt{(m^2 + \beta)^2 - \gamma^2}}{2} \right]^{1/2}. \quad (29)
\]
Making use of equations (29) and (27), the final energy levels for a real bound charged particle in a Coulombic field plus a combination of non central potentials given by equation (25b) are

$$E_{n,N,m} = -\frac{\mu Z^2 e^4}{2\hbar^2 \left[ N + 1 + n + \left( \frac{m^2 + \beta}{2} \right)^{1/2} \right]^2}$$

(30)

and the complete eigenfunctions (radial×angular) can be obtained from formula (3b) as

$$\Psi(r, \theta, \phi) = r^\ell \exp \left( -\sqrt{\frac{-2\mu E}{\hbar^2}} \right) \left( \frac{\cos \theta - 1}{2} \right)^{1/2} \sqrt{m^2 + \beta + \gamma} \left( \frac{\cos \theta + 1}{2} \right)^{1/2} \sqrt{m^2 + \beta - \gamma + 1} e^{\pm im\phi}$$

$$\times _2F_1 \left( -n, n + \sqrt{m^2 + \beta + \gamma} + \sqrt{m^2 + \beta - \gamma + 1} + 1 + \sqrt{m^2 + \beta + \gamma}, \left( \frac{1 - \cos \theta}{2} \right) \right)$$

$$\times _1F_1 \left( -n, 2(\ell + 1), 2r \sqrt{\frac{-2\mu E}{\hbar^2}} \right).$$

(31)

These results are in agreement with the ones obtained previously [62, 63, 65]

V. CONCLUDING REMARKS

In this work, we proposed a simple formula for finding eigensolutions (energy eigenvalues and wave function) of any Schrödinger-like differential equation. The proposed formula is derived by using the asymptotic iteration method (AIM) and functional analysis approach (FAA). This approach presents a new alternative and accurate method to obtain single particle bound and resonant states with different potential fields. To show the accuracy and effectiveness of this method, we obtained the solution of Schrödinger equation with central and non central potential models like Manning-Rosen potential, Hulthén potential, Eckart potential, Kratzer-type potential and generalized non central Coulomb potential. We considered the cases of $k_3 = 0$ and $k_3 \neq 0$ for exact and approximate solutions. It is worth being paid attention that all of our results are in excellent agreement with the ones obtained previously by other methods.

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Appendix A: Derivation of the Formulas

Firstly, let us analyze the asymptotic behavior at the origin and at infinity for the finiteness of our solution. It can be tested when \( s \to 0 \) by taking the solution of equation (7) as

\[
\Psi(s) = s^{k_4}.
\]

Again, it can also be proved that when \( s \to \frac{1}{k_3} \), the solution of equation (7) is \( \Psi(s) = (1 - k_3 s)^{k_5} \), where

\[
k_5 = \frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3} + \sqrt{\left[ \frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3} \right]^2 - \left[ \frac{A}{k_3^2} + \frac{B}{k_3} + C \right]}.
\]

Hence, the wave function in the intermediate region, for this problem, can be taken as

\[
\Psi(s) = s^{k_4} (1 - k_3 s)^{k_5} F(s).
\]

The substitution of the above equation (A3) into equation (A1) leads to the following second-order differential equations

\[
F''(s) + F'(s) \left[ \frac{(2k_4 + k_1) - sk_3(2k_4 + 2k_3 + k_2)}{s(1 - k_3 s)} \right] - \left[ \frac{2k_3 k_4 (k_4 - 1) + k_5 k_3 (2k_4 + k_1) + k_4 (k_1 k_3 + k_2) - B}{s(1 - k_3 s)} \right] F(s) = 0,
\]

(A4a)
\[ F''(s) + F'(s) \left[ \frac{(2k_4 + k_1) - sk_3(2k_4 + 2k_5 + \frac{k_2}{k_1})}{s(1-k_3s)} \right] - \left[ \frac{k_3(k_4 + k_5)^2 + (k_4 + k_5)(k_2 - k_3) + \frac{4}{k_3^2}}{s(1-k_3s)} \right] F(s) = 0. \]

(A4b)

It should be noted that equation (A4a) is equivalent to equation (A4b), but equation (A4b) becomes more complicated during the course of our calculations. Therefore, we shall continue our derivation with equation (A4b). At this stage, we shall obtain the solution of equation (A4b) by using the asymptotic iteration method and also in terms of hypergeometric functions.

**Derivation from AIM**

Because of unfamiliar readers, firstly, we give a brief review of AIM with all necessary formulas for our derivation. For a given potential the idea is to convert the Schrödinger-like equation to the homogenous linear second-order differential equation of the form:

\[ y''_n(x) = \lambda_0(x)y'_n(x) + s_0(x)y_n(x), \quad (A5) \]

where \( \lambda_0(x) \neq 0 \) and the prime denote the derivative with respect to \( x \), the extral parameter \( n \) denotes the radial quantum number. The variables, \( s_0(x) \) and \( \lambda_0(x) \) are sufficiently differentiable. To find energy spectrum equation of any Schrödinger-like equation, the equation is first transformed to form of (A5). Then, one need to obtain \( \lambda_k(x) \) and \( s_k(x) \) with \( k > 0 \), i.e.,

\[ \lambda_k(x) = \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \quad s_k(x) = s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x). \quad (A2) \]

With \( \lambda_k(x) \) and \( s_k(x) \) values, one can obtain the quantization condition

\[ \delta_k(x) = \begin{vmatrix} \lambda_k(x) & s_k(x) \\ \lambda_{k-1}(x) & s_{k-1}(x) \end{vmatrix} = 0, \quad k = 1, 2, 3, \ldots \quad (A7) \]

The energy eigenvalues are then obtained by the condition given by equation (A7) if the problem is exactly solvable. For nontrivial potentials that have no exact solutions, for a specific \( n \) principal quantum number, we choose a suitable \( x_0 \) point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential and the approximate energy eigenvalues are obtained from the roots of equation (A7) for sufficiently great values of \( k \) with iteration for which \( k \) is always greater than \( n \) in the
numerical solutions. Furthermore, the eigenfunction can be obtained by transforming the Schrödinger-like equation to the form of

\[ y''(x) = 2 \left( \frac{a x^{N+1}}{1 - b x^{N+2}} - \frac{m + 1}{x} \right) y'(x) - \frac{W x^N}{1 - b x^{N+2}} y(x). \]  

(A8)

The exact solution \( y_n(x) \) can be expressed as

\[ y_n(x) = (-1)^n C_2 (N + 2)^n (\sigma)_n 2F_1(-n, \rho + n; \sigma; bx^{N+2}), \]  

(A9)

where the following notations has been used

\[ (\sigma)_n = \frac{\Gamma(\sigma + n)}{\Gamma(\sigma)}, \quad \sigma = \frac{2m + N + 3}{N + 2} \quad \text{and} \quad \rho = \frac{(2m + 1)b + 2a}{(N + 2)b}. \]  

(A10)

Now, comparing equation (A11) with equation (A9), we can determine the following parameters

\[ m = k_4 + \frac{k_1}{2} - 1, \quad a = \left( k_5 - \frac{k_1}{2} + \frac{k_2}{2 k_3} \right) k_3, \quad \sigma = 2k_4 + k_1 \quad \text{and} \quad \rho = 2(k_4 + k_5) + \frac{k_2}{k_3} - 1, k_3 \neq 0. \]  

(A11)

Having determined these parameters, we can easily find the eigenfunction \( \Psi(s) \) expressed in terms of the hypergeometric function as

\[ \Psi(s) = (-1)^n C_2 \frac{\Gamma(2k_4 + k_1 + n)}{\Gamma(2k_4 + k_1)} s^{k_4}(1 - k_3 s)^{k_5} 2F_1 \left( -n, n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1; 2k_4 + k_1, k_3 s \right) 
\]

\[ = N_n s^{k_4}(1 - k_3 s)^{k_5} 2F_1 \left( -n, n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1; 2k_4 + k_1, k_3 s \right), k_3 \neq 0, \]  

(A12)

where \( N_n \) is the normalization constant. Now, let us turn to the derivation of the energy eigenvalues. By using the equation (A5) we can rewrite the \( \lambda_0(s) \) and \( s_0(s) \) and consequently we can calculate \( \lambda_k(s) \) and \( s_k(s) \). Thus, it gives

\[
\lambda_0 = \frac{sk_3(2k_4 + 2k_5 + \frac{k_2}{k_3}) - (2k_4 + k_1)}{s(1 - k_3 s)}
\]

\[
s_0 = \frac{k_3(k_4 + k_5)^2 + (k_4 + k_5)(k_2 - k_3) + \frac{A}{k_3}}{s(1 - k_3 s)}
\]

\[
\lambda_1 = \frac{2k_3(k_4 + k_5) + k_2}{s(1 - k_3 s)} - \frac{kk_1(1 - k_3 s + kk_1)}{s^2(1 - k_3 s)^2} + \frac{k_3 k_1}{s(1 - k_3 s)^2} + \frac{k_3(k_4 + k_5)^2 + (k_2 - k_3)(k_4 + k_5) + \frac{A}{k_3}}{s(1 - k_3 s)}
\]

\[
s_1 = \frac{2k k_2 k_3 s(1 + k_4 + k_5) - (1 + 2k_4 + k_1 - sk_2)}{s^2 k_3(1 - k_3 s)^2}
\]

...etc
where \( kk_1 = -2k_4 - k_1 + 2s(k_4k_3 + k_5k_3 + k_2/2) \) and \( kk_2 = k_3(k_4 + k_5)^2 + k_2k_3(k_4 + k_5) - k_3^2(k_5 + k_4) + A \). Combining these results with the termination condition given by equation (A7) gives

\[
\begin{align*}
\lambda_0 s_1 &= \Rightarrow k_4 + k_5 = -\frac{1}{2k_3} \left( k_2 - k_3 - \sqrt{(k_3 - k_2)^2 - 4A} \right), \\
\lambda_1 s_2 &= \Rightarrow k_4 + k_5 = -\frac{1}{2k_3} \left( k_2 + k_3 - \sqrt{(k_3 - k_2)^2 - 4A} \right), \\
\lambda_2 s_3 &= \Rightarrow k_4 + k_5 = -\frac{1}{2k_3} \left( k_2 + 3k_3 - \sqrt{(k_3 - k_2)^2 - 4A} \right), \\
\lambda_3 s_4 &= \Rightarrow k_4 + k_5 = -\frac{1}{2k_3} \left( k_2 + 5k_3 - \sqrt{(k_3 - k_2)^2 - 4A} \right),
\end{align*}
\] (A14)

etc.

By finding the nth term of the above arithmetic progression, the formula for the eigenvalues can be obtained as

\[
\lambda = \frac{1 - 2n}{2} - \frac{1}{2k_3} \left( k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right)
\] (A15)

or more explicitly as

\[
\frac{2 \left( \frac{1 - 2n}{2} - \frac{1}{2k_3} \left( k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right)^2}{\left( \frac{1 - 2n}{2} - \frac{1}{2k_3} \left( k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right)^2} - k_5^2 = 0.
\] (A16)

**Derivation in terms of Generalized Hypergeometric Function**

The solutions of equation (A4b) can also be obtained in terms of the hypergeometric function as

\[
F(s) = \frac{[(\alpha + n)(\alpha + n - 1)(\alpha + n - 2)\ldots][n(\beta + n)(\beta + n - 1)(\beta + n - 2)\ldots]}{\Gamma(n)} \left( k_3s \right)^n}
\]

\[
= _2F_1(\alpha, \beta; \gamma; k_3s) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n}{n!(\gamma)_n} s^n,
\] (A17)

where \( \alpha, \beta \) and \( \gamma \) are given by

\[
\alpha = k_4 + k_5 - \frac{1}{2} + \frac{1}{2k_3} \left[ k_2 - \sqrt{(k_2 - k_3)^2 - 4A} \right], \quad (A18a)
\]

\[
\beta = k_4 + k_5 - \frac{1}{2} + \frac{1}{2k_3} \left[ k_2 + \sqrt{(k_2 - k_3)^2 - 4A} \right], \quad (A18b)
\]

\[
\gamma = 2k_4 + k_1. \quad (A18c)
\]
Considering the finiteness of the solutions, it is shown from equation (A17) that $F(s)$ approaches infinity unless $\alpha$ is a negative integer. Nonetheless, the wave function $\Psi(s)$ will not be finite everywhere unless we take
\[ \alpha = k_4 + k_5 - \frac{1}{2} + \frac{1}{2k_3} \left[ k_2 - \sqrt{(k_2 - k_3)^2 - 4A} \right] = -n, \quad n = 0, 1, 2, 3, \ldots \quad (A19) \]

Hence, with equation (A19), the expression for $\beta$ given by equation (A18b) can be rewritten as
\[ \beta = 2(k_4 + k_5) + \frac{k_2}{k_3} - 1 + n. \quad (A20) \]

Using equations (A19) and (A20), the solution of equation (A4b) can now be expressed as
\[ F(s) = \frac{1}{2} F_1 \left( -n, n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1, 2k_4 + k_1, k_3 s \right). \quad (A21) \]

With equation (A21), we can finally rewrite the wave function $\Psi(s)$ in equation (A3) as
\[ \Psi(s) = N_n s^{k_4}(1 - k_3 s)^{k_5} \frac{1}{2} F_1 \left( -n, n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1, 2k_4 + k_1, k_3 s \right), \quad (A22) \]

where $N_n$ is the normalization constant. In addition, by using equation (A19), we can find the following formula for energy eigenvalues:
\[ \left[ k_4^2 - k_5^2 - \frac{1 - 2n}{2k_3} \left( k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right]^2 - k_5^2 = 0, k_3 \neq 0. \quad (A23) \]

It should be noted that the two approaches for obtaining this formula yield the same results. Furthermore, let us discuss case where $k_3 \to 0$. Therefore, equation (A11) reduces to
\[ \Psi''(s) + \frac{(k_1 - k_2 s)}{s} \Psi'(s) + \frac{(As^2 + Bs + C)}{s^2} \Psi(s) = 0. \quad (A24) \]

Also, the proposed wave function (A3) becomes
\[ \lim_{k_3 \to 0} \Psi(s) = s^{k_4} e^{-k_5 s} F(s) \quad (A25) \]

with
\[ \lim_{k_3 \to 0} k_4 = \frac{(1 - k_1) + \sqrt{(1 - k_1)^2 - 4C}}{2}, \quad \lim_{k_3 \to 0} k_5 = \frac{k_2}{2} + \sqrt{\left( \frac{k_2}{2} \right)^2 - A}. \quad (A26) \]

By following the same procedure, the eigenvalues and the corresponding wave function can be obtained as
\[ \left[ \frac{B - k_4 k_2 - nk_2}{2k_4 + k_1 + 2n} \right]^2 - k_5^2 = 0, \quad (A27) \]
and

\[ \Psi(s) = N_n s^{k_4} \exp(-k_5 s) \, _1F_1 \left( -n, 2k_4 + k_1, (2k_5 + k_2)s \right), \quad (A28) \]

respectively. \( N_n \) is the normalization constant.