Strong edge coloring of planar graphs

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Abstract

A strong edge coloring of a graph is a proper edge coloring where the edges at distance at most two receive distinct colors. It is known that every planar graph with maximum degree $\Delta$ has a strong edge coloring with at most $4 \Delta + 4$ colors. We show that $3 \Delta + 6$ colors suffice if the graph has girth 6, and $3 \Delta$ colors suffice if the girth is at least 7. Moreover, we show that cubic planar graphs with girth at least 6 can be strongly edge colored with at most 9 colors.

Keywords: Strong edge coloring, strong chromatic index, planar graph, discharging method

1 Introduction

A strong edge coloring of a graph $G$ is a proper edge coloring where every color class induces a matching, i.e., every two edges at distance at most two receive distinct colors. The smallest number of colors for which a strong edge coloring of a graph $G$ exists is called the strong chromatic index, $\chi'_s(G)$. In 1985, Erdős and Nešetřil posed the following conjecture during a seminar in Prague.

Conjecture 1 (Erdős, Nešetřil). Let $G$ be a graph with maximum degree $\Delta$. Then,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4} \Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4} (5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

They also presented the construction, which shows that Conjecture 1, if true, is tight. In 1997, Molloy and Reed [7] established currently the best known upper bound for the strong chromatic index of graphs with sufficiently large maximum degree.

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Theorem 2 (Molloy, Reed). For every graph $G$ with sufficiently large maximum degree $\Delta$ it holds that
\[
\chi'_s(G) \leq 1.998 \Delta^2.
\]

In 1990, Faudree et al. [2] proposed several problems regarding subcubic graphs.

Problem 1 (Faudree et al.). Let $G$ be a subcubic graph. Then,
1. $\chi'_s(G) \leq 10$;
2. $\chi'_s(G) \leq 9$ if $G$ is bipartite;
3. $\chi'_s(G) \leq 9$ if $G$ is planar;
4. $\chi'_s(G) \leq 6$ if $G$ is bipartite and the weight of each edge is at most 5;
5. $\chi'_s(G) \leq 7$ if $G$ is bipartite of girth 6;
6. $\chi'_s(G) \leq 5$ if $G$ is bipartite and has girth large enough.

Andersen [1] confirmed that Conjecture 1 holds for subcubic graphs, i.e., that the strong chromatic index of any subcubic graph is at most 10, which solves also the first item of Problem 1. The second item of Problem 1 was confirmed by Steger and Yu [8].

In this paper, we consider planar graphs with bounded girth. In 1990, Faudree et al. [2] found a construction of planar graphs showing that for every integer $k \geq 2$ there exists a planar graph $G$ with maximum degree $k$ and $\chi'_s(G) = 4k - 4$. Moreover, they proved the following theorem.

Theorem 3 (Faudree et al.). Let $G$ be a planar graph with maximum degree $\Delta$. Then,
\[
\chi'_s(G) \leq 4 \Delta + 4.
\]

The proof of Theorem 3 is short and simple, so we present it here also.

Proof. By Vizing’s theorem [9] every graph is $(\Delta + 1)$-edge colorable. Moreover, if $\Delta \geq 7$, $\Delta$ colors suffice [10]. Let $M_i$ be the set of the edges colored by the same color. Let $G(M_i)$ be the graph obtained from $G$ where every edge from $M_i$ is contracted. Note that the vertices corresponding to the edges of $M_i$ that are incident to a common edge are adjacent in $G(M_i)$. Since $G(M_i)$ is planar, we can color the vertices with 4 colors by the Four Color Theorem, and therefore all the edges of $M_i$ with a common edge receive distinct colors in $G$. After coloring each of the $k$ graphs $G(M_i)$, for $i \in \{1, 2, \ldots, k\}$ and $k$ being the chromatic index of $G$, we obtain a strong edge coloring of $G$. \hfill \Box

Notice that if a planar graph has girth at least 7, the bound is decreased to $3\chi'(G)$, due to Grötzsch’s theorem [3]. Moreover, Kronk, Radlowski, and Franen [6] showed that if a planar graph has maximum degree $\Delta$ at least 4 and girth at least 5, its chromatic index equals $\Delta$. This fact, combined with our result in Theorem 8 gives us the following.

Theorem 5 (Hocquard, Valicov). Let $G$ be a planar subcubic graph with girth $g$. Then,
(i) if $g \geq 30$, then $\chi'_s(G) \leq 6$;
(ii) if \( g \geq 11 \), then \( \chi'_s(G) \leq 7 \);
(iii) if \( g \geq 9 \), then \( \chi'_s(G) \leq 8 \);
(iv) if \( g \geq 8 \), then \( \chi'_s(G) \leq 9 \).

Later, these bounds were improved in [4] to the following:

**Theorem 6** (Hocquard et al.). Let \( G \) be a planar subcubic graph with girth \( g \). Then,

(i) if \( g \geq 14 \), then \( \chi'_s(G) \leq 6 \);
(ii) if \( g \geq 10 \), then \( \chi'_s(G) \leq 7 \);
(iii) if \( g \geq 8 \), then \( \chi'_s(G) \leq 8 \);
(iv) if \( g \geq 7 \), then \( \chi'_s(G) \leq 9 \).

In this paper we consider planar graphs with girth 6 and introduce the following results.

**Theorem 7.** Let \( G \) be a planar graph with girth at least 6 and maximum degree \( \Delta \geq 4 \). Then,

\[ \chi'_s(G) \leq 3 \Delta + 6. \]

**Theorem 8.** Let \( G \) be a subcubic planar graph with girth at least 6. Then,

\[ \chi'_s(G) \leq 9. \]

Theorem 8 partially solves the third item of Problem 1. The proposed bound, if true, is realized by the complement of \( C_6 \) (see Fig. 1). Here, let us remark that very recently Hocquard et al. [4] obtained an improved result of Theorem 8, proving that every subcubic planar graph without cycles of length 4 and 5 admits a strong edge coloring with at most 9 colors.

![Figure 1: Subcubic planar graph with the strong chromatic index equal to 9.](image)

All the graphs considered in the paper are simple. We say that a vertex of degree \( k \), at least \( k \), and at most \( k \) is a \( k \)-vertex, a \( k^+ \)-vertex, and a \( k^- \)-vertex, respectively. Similarly, a \( k \)-neighbor, a \( k^+ \)-neighbor, and a \( k^- \)-neighbor of a vertex \( v \) is a neighbor of \( v \) of degree \( k \), at least \( k^+ \), and at most \( k \), respectively. A 2-neighborhood of an edge \( e \) is comprised of the edges at distance at most two from \( e \). Here, the distance between the edges \( e \) and \( e' \) in a graph \( G \) is defined as the distance between the vertices corresponding to \( e \) and \( e' \) in the line graph \( L(G) \).
2 Proof of Theorem 7

We prove the theorem using the discharging method. First, we list some structural properties of a minimal counterexample $G$ to the theorem and then, we show that a planar graph with such properties cannot exist.

2.1 Structure of minimal counterexample

Since $G$ is a minimal counterexample, a graph obtained from $G$ by removing any edge or vertex has some strong $(3\Delta + 6)$-edge coloring $\sigma$. In every proof we show that $\sigma$ can be extended to $G$, establishing a contradiction. We note here that in the proofs by removing an edge from a graph and adding it back after coloring, we do not decrease the distance between the edges of the same color such that the distance between them would be less than 3.

A 4-vertex is a $4_2$-vertex if it has at most two 2-neighbors and a $4_3$-vertex if it has three 2-neighbors. We call a 2-vertex weak if it has a $3^-$-neighbor, semiweak if it has a $4_3$-neighbor, and strong otherwise.

First, we show that vertices of degree 1 have neighbors of degree at least 5.

Lemma 9. Every 1-vertex in $G$ is adjacent to a $5^+$-vertex. Moreover, if it is adjacent to a 5-vertex $u$, all the other neighbors of $u$ have degree at least 3.

Proof. Let $v$ be a 1-vertex with a $4^-$-neighbor $u$. Let $\sigma$ be a strong edge coloring of $G - v$. The number of colored edges in the 2-neighborhood of the edge $uv$ is at most $3\Delta$, hence there are at least 6 colors available for $uv$, and so $\sigma$ can be extended to $G$, a contradiction.

If $v$ is adjacent to a 5-vertex $u$ with another $2^-$-neighbor, the number of colored edges in the 2-neighborhood of the edge $uv$ is at most $3\Delta + 2$, and so $\sigma$ can be extended again, since $uv$ has at least 4 available colors.

Next, in the minimal counterexample $G$ every 2-vertex is adjacent to some vertex of degree at least 5.

Lemma 10. Every 2-vertex has at least one $5^+$-neighbor.

Proof. Suppose that $v$ is a 2-vertex with two $4^-$-neighbors $u$ and $w$. Let $\sigma$ be the strong $(3\Delta + 6)$-edge coloring of $G - v$. Each of the two noncolored edges $uv, vw$ in $G$ has at most $3\Delta + 3$ colored edges in the 2-neighborhood, hence there are at least 3 available colors for each, which means that both can be easily colored.

The following lemma shows that every vertex of $G$ has at least one neighbor of degree at least 3.

Lemma 11. Every vertex in $G$ has at least one $3^+$-neighbor.

Proof. Suppose, to the contrary, that $v$ is a $k$-vertex of $G$ having only $2^-$-neighbors. Then, by the minimality of $G$, we have a strong $(3\Delta + 6)$-coloring $\sigma$ of $G - v$. By applying $\sigma$ to $G$, only the edges incident to $v$ remain noncolored. As each of these edges has at least $2\Delta - k + 7$ available colors, we can color them greedily one by one, and so extend $\sigma$ to $G$.

Now, we prove that if a $k$-vertex of $G$ has $k - 1$ $2^-$-neighbors, all of them are strong.

Lemma 12. Every $k$-vertex, $k \geq 5$, with $k - 1$ $2^-$-neighbors has only strong $2^-$-neighbors.
Proof. Suppose that \( v \) is a \( k \)-vertex, \( k \geq 5 \), with one \( 3^+ \)-neighbor and at least one \( 2^- \)-neighbor \( u \) which is not strong and has another neighbor \( w \). Since \( u \) is not strong, \( w \) is either a \( 4_3 \)-vertex or a \( 3^- \)-vertex. Let \( \sigma \) be a strong \((3\Delta + 6)\)-edge coloring of \( G - u \). The edge \( uv \) has at most \( \Delta + 2(k-2) + 3 \leq 3\Delta - 1 \) colored edges in the \( 2^- \)-neighborhood, while the number of edges in the \( 2^- \)-neighborhood of \( uw \) is at most \( 2\Delta + k \leq 3\Delta \). Hence, both edges can be colored.

Finally, we show that every \( k \)-vertex of \( G \) with \( k-2 \) \( 2^- \)-neighbors has at least three \( 2^- \)-neighbors that are not weak.

Lemma 13. Every \( k \)-vertex, \( k \geq 5 \), with \( k-2 \) \( 2^- \)-neighbors has at least three nonweak \( 2^- \)-neighbors.

Proof. Let \( v \) be a \( k \)-vertex, \( k \geq 5 \), with two \( 3^+ \)-neighbors. Moreover, suppose that \( v \) is adjacent to exactly two \( 2^- \)-neighbors that are not weak. Let \( u_1, u_2, \ldots, u_{k-4} \) be the weak neighbors of \( v \), and \( w_1, w_2, \ldots, w_{k-4} \) their neighbors distinct from \( v \). Let \( \sigma \) be the strong \((3\Delta + 6)\)-edge coloring of \( G - \{u_1, u_2, \ldots, u_{k-4}\} \). We extend \( \sigma \) to \( G \) in the following way. First, we color the edges \( vu_1, vu_2, \ldots, vu_{k-4} \) one by one. Such a coloring is possible since the number of colored edges in the \( 2^- \)-neighborhood of such an edge never exceeds \( 3\Delta - 1 \). Then, we color the edges \( u_iw_i, i \in \{1,2,\ldots,k-4\} \). Again, every edge has at most \( 3\Delta \) colored edges in the \( 2^- \)-neighborhood, hence there is a free color by which we color it. This establishes the lemma.

2.2 Discharging

Now, we show that a minimal counterexample \( G \) with the described properties does not exist. In order to prove this, we set the charges to all vertices and faces in such a way that the sum of all charges is negative. Then, we redistribute charges among the vertices and faces so that every single object has non-negative charge, clearly obtaining a contradiction on the existence of \( G \).

The initial charge of vertices and faces is set as follows:

\[
\begin{align*}
\text{ch}_0(v) &= 2d(v) - 6, \quad v \in V(G); \\
\text{ch}_0(f) &= l(f) - 6, \quad f \in F(G).
\end{align*}
\]

By Euler’s formula, it is easy to compute that the sum of all charges is \(-12\). We redistribute the charge among the vertices and faces by the following discharging rules:

(R1) Every face sends 2 to every incident 1-vertex.
(R2) Every \( 5^+ \)-vertex sends 2 to every adjacent 1-vertex.
(R3) Every \( 5^+ \)-vertex sends 2 to every adjacent weak 2-vertex.
(R4) Every \( 5^+ \)-vertex sends \( \frac{4}{3} \) to every adjacent semiweak 2-vertex.
(R5) Every \( 5^+ \)-vertex sends 1 to every adjacent strong 2-vertex.
(R6) Every \( 4_2 \)-vertex sends 1 to each of the adjacent 2-vertices.
(R7) Every \( 4_3 \)-vertex sends \( \frac{2}{3} \) to each of the three adjacent 2-vertices.

Now, we are ready to prove Theorem 7.
Proof. Suppose, to the contrary, that $G$ is a minimal counterexample to the theorem. We use the structural properties of $G$ to show that after applying the discharging rules the charge of all vertices and faces is nonnegative.

First, consider the faces of $G$. It is easy to see that since $G$ has girth at least 6 the initial charge of every face is nonnegative. Faces only send charge by the rule (R1), i.e., to every incident 1-vertex they send 2 of charge. Let the base of $f$ be a face with all incident 1-vertices removed. Notice that, by the girth condition, the bases of all faces have length at least 6 and that every incident 1-vertex increases the length of the base by 2. So, the number of 1-vertices incident to a face $f$ is at most $2(l(f) - 6)$ and the final charge of $f$ is at least $l(f) - 6 - 2 \cdot \frac{1}{2}(l(f) - 6) = 0$.

Now, we consider the final charge of a vertex $v$ regarding its degree:

- **$v$ is a 1-vertex.** By Lemma 9, the unique neighbor $u$ of $v$ is of degree at least 5. By (R1), $v$ receives 2 of charge from its incident face and 2 of charge from $u$ by (R2). Hence it receives 4 in total and its final charge is 0.

- **$v$ is a 2-vertex.** By Lemma 10, $v$ has at least one $5^+$-neighbor $u$. In order to have nonnegative charge, $v$ needs to receive at least 2 of charge. If $v$ is weak, it receives 2 from $u$ by (R3). In case when $v$ is semiweak, it receives $\frac{3}{2}$ from $u$ by (R4) and $\frac{2}{3}$ from the 4$\gamma$-neighbor by (R7), again 2 in total. Otherwise $v$ is strong and it receives 1 from each of the two neighbors by (R5) and (R6).

- **$v$ is a 3-vertex.** The initial charge of $v$ is 0 and it neither sends nor receives any charge, hence its final charge is also 0.

- **$v$ is a 4-vertex.** By Lemma 9, $v$ has no 1-neighbor and by Lemma 11, $v$ has at most three 2-neighbors. In case when $v$ has precisely three 2-neighbors it sends $\frac{2}{3}$ to each of them by (R7), which is 2 in total, otherwise it may send 1 to each 2-neighbor by (R6). The final charge of $v$ is thus at least $8 - 6 - 2 = 0$.

- **$v$ is a 5-vertex.** Suppose first that $v$ is adjacent to a 1-vertex $u$. By Lemma 9, $u$ is the only 2$^-$-neighbor of $v$, hence $v$ sends 2 by (R2) and its final charge is 2. Therefore, we may assume that $v$ has no 1-neighbor. If $v$ has at most two 2-neighbors it sends at most 4 of charge in total by the rules (R3)–(R5), and it retains nonnegative charge. If $v$ has three 2-neighbors none of them is weak by Lemma 13, and so it sends at most 4 by (R4) or (R5). Finally, if $v$ has four 2-neighbors, all of them are strong by Lemma 12, so $v$ sends 4 by (R5). It follows that $v$ has nonnegative final charge.

- **$v$ is a $k$-vertex, $k \geq 6$.** Let $n_1$ and $n_2$ be the numbers of 1-neighbors and 2-neighbors of $v$. By Lemma 11, we have that $n_1 + n_2 \leq k - 1$. If $n_1 + n_2 = k - 1$, by Lemma 12, it follows that $v$ has only strong 2-neighbors, and so $n_1 = 0$. Hence the final charge of $v$ is $2k - 6 - (k - 1) = k - 5 \geq 0$. In case when $n_1 + n_2 = k - 2$, by Lemma 13, we have that there are at least three nonweak 2-neighbors of $v$, so its final charge is at least $2k - 6 - 2(k - 2 - 3) - 3 \cdot \frac{4}{3} = 0$. Finally, if $n_1 + n_2 \leq k - 3$, the final charge of $v$ is at least $2k - 6 - 2(k - 3) = 0$.

We have shown that the final charge of every vertex and face in $G$ is nonnegative, and so is the sum of all charges. Hence, a minimal counterexample to Theorem 7 does not exist. \qed
3 Proof of Theorem 8

To prove the theorem we follow the same procedure as in Section 2. First, we list some properties of a minimal counterexample to the theorem. Recall that $G$ is subcubic.

3.1 Structure of minimal counterexample

The first lemma considers the minimum degree and the neighborhood of 2-vertices.

**Lemma 14.** For a minimal counterexample $G$ the following claims hold:

(a) the minimum degree of $G$ is at least 2;
(b) every 2-vertex has two 3-neighbors;
(c) every 3-vertex has at least two 3-neighbors.

**Proof.** (a) Suppose $v$ is a 1-vertex in $G$ and let $u$ be its unique neighbor. By the minimality, there is a strong 9-edge coloring $\sigma$ of $G - v$. It is easy to see that there are at most 6 colored edges in the 2-neighborhood of $uv$, hence we easily extend $\sigma$ to $G$.

(b) Suppose, to the contrary, that $u$ and $v$ are adjacent 2-vertices in $G$. Let $w$ be the second neighbor of $v$. Let $G' = G - v$ and $\sigma$ a strong edge coloring of $G'$ which, by the minimality of $G$, uses at most 9 colors. An easy calculation shows that $vw$ has at least two, and $uv$ has at least three available colors. Hence, $\sigma$ can be extended to $G$.

(c) Let $v$ be a 3-vertex with 2-neighbors $u$ and $w$, and let $z$ be the second neighbor of $w$. By the minimality, the graph $G' = G - w$ has a strong edge coloring $\sigma$ with at most 9 colors. Notice that $wz$ has at most eight used colors in the 2-neighborhood, so we can color it. Finally, we color $vw$ which also has at most eight colors used in the 2-neighborhood, and so establish the claim.

In the next two lemmas we consider 6 and 7-faces incident to 2-vertices.

**Lemma 15.** There is no 6-face incident to a 2-vertex in $G$.

**Proof.** Let $f$ be a 6-face with an incident 2-vertex and let the vertices of $f$ be labeled as in Fig. 2. By the minimality of $G$, there is a strong 9-edge coloring $\sigma$ of $G' = G - v_0$. Now, we only need to color the edges $v_0v_1$ and $v_0v_5$. Since there are only eight colored edges in the 2-neighborhood of $v_0v_5$, we color it with a free color.

The edge $v_0v_1$ has now nine colored edges in the 2-neighborhood. In case that these nine edges only use at most eight colors, we can color $v_0v_1$ with a free color, thus we may assume that all nine edges have different colors assigned as it is shown in Fig. 2. Moreover, suppose that the color 1 appears on $v_3v_4$, $v_4u_4$, or one of the edges incident to $u_5$. Then, there are at most seven distinct colors in the 2-neighborhood of $v_0u_5$, so we may color it with a color different from 9 and assign 9 to $v_0v_1$. Similarly, the edge $v_4u_4$ and the edges incident to $u_5$ are not colored by 4. Hence, we may assume that $\sigma(v_3u_4)$ is 2 or 3, say 2, and the edge $v_4u_4$ and one of the edges incident to $u_5$ have the colors 3 and 5. The third edge incident to $u_5$ has color 6 by the same argumentation.

Next, one of the edges incident to $u_2$ has color 7, otherwise we recolor $v_1v_2$ with 7 and color $v_0v_1$ with 4. The same reasoning shows that the edge $v_3u_3$ or one of the edges incident to $u_2$ has color 8. Equivalently, one of the edges incident to $u_4$ has color 4, for
otherwise we recolor \( v_4v_5 \) with 4 and color \( v_0v_1 \) with 7. Similarly, one of the edges \( v_3u_3 \) or an edge incident to \( u_4 \) has color 1.

It remains to consider three cases regarding the assignment of colors to the edges mentioned above.

(a) Suppose that \( \sigma(v_3u_3) = 8 \). Then, the third edge incident to \( u_4 \) has color 1. Moreover, the third edge incident to \( u_2 \) has color 2, otherwise we color \( v_2v_3 \) with 2 and \( v_3v_4 \) with 6, which enables us to color \( v_0v_1 \) with 6. One of the edges incident to \( u_3 \) has color 4, otherwise we color \( v_2v_3 \) with 4, \( v_1v_2 \) with 6, \( v_0v_5 \) with 4, and finally \( v_0v_1 \) with 9. Similarly, one of the edges incident to \( u_3 \) has color 7, otherwise we set \( \sigma(v_3v_4) = 7 \) and \( \sigma(v_4v_5) = 2 \) and color \( v_0v_1 \) with 7. But now, we can color \( v_3v_4 \) with 9 and \( v_0v_5 \) with 2 and set \( \sigma(v_0v_1) = 9 \), and so obtain the coloring of all the edges of \( G \) (see Fig. 3).

(b) Suppose that \( \sigma(v_3u_3) = 1 \). Then, one of the edges incident to \( u_2 \) has color 8. Therefore one of the edges incident to \( u_4 \) has color 6, otherwise we set \( \sigma(v_2v_3) = 2 \), \( \sigma(v_3v_4) = 6 \), and \( \sigma(v_0v_1) = 6 \). Similarly, the edges incident to \( u_3 \) must have colors 4 and 7. Otherwise, if there is no edge of color 4, we swap the colors of \( v_1v_2 \) and \( v_2v_3 \), color \( v_0v_1 \) with 9 and \( v_0v_5 \) with 4. In case when there is no edge of color 7 incident to \( u_3 \), we swap the colors of \( v_3v_4 \) and \( v_4v_5 \), and color \( v_0v_1 \) with 7. Finally, having such a coloring, we can swap the colors of \( v_3v_4 \) and \( v_0v_5 \) and color \( v_0v_1 \) with 9 (see Fig. 4).
(c) Suppose that one of the edges incident to $u_2$ has color 8 and one of the edges incident to $u_4$ has color 1. We can swap the colors of $v_2v_3$ and $v_3v_4$, which enables us to color $v_0v_1$ with 6 and hence extend the coloring $\sigma$ to $G$.

Lemma 16. Every 7-face in $G$ is incident to at most one 2-vertex.

Proof. By Lemma 14, we infer that a 7-face with three incident 2-vertices cannot appear in a minimal counterexample. Moreover, the arrangement of the possible two 2-vertices incident to a 7-face is unique. Let $f$ be such a face and let its vertices be labeled as in Fig. 5.

By the minimality of $G$, there exists a strong 9-edge coloring $\sigma$ of $G - \{v_2, v_3, v_4, v_5\}$. It is easy to see that after assigning the colors of $\sigma$ to the edges of $G$, there are seven noncolored edges. A simple count shows that the edge $v_1v_2$ has at most 6 colored edges in the 2-neighborhood and hence at least 3 free colors. Similarly, the edges $v_2v_3$, $v_3v_4$, and $v_4v_5$ have at least 5 free colors, and $v_3v_5$, $v_4v_4$, and $v_5v_6$ have at least 3 free colors.

Now, color $v_4u_4$ with one of its free colors and $v_3v_4$ with the a color such that $v_1v_2$ retains at least three available colors. Finally, color $v_3u_3$, $v_5v_6$, $v_4v_5$, $v_2v_3$, and $v_1v_2$ in the given order. It is easy to see that each of the edges that are being colored always has at least one free color. Thus, the coloring $\sigma$ can be extended to $G$, a contradiction.
From the proof of Lemma 16, it also follows that the distance between the vertices of degree 2 in $G$ is at least 4.

### 3.2 Discharging

We use the same initial charge for vertices and faces as in the previous section, i.e.,

$$
ch_0(v) = 2d(v) - 6, \quad v \in V(G);
$$

$$
ch_0(f) = l(f) - 6, \quad f \in F(G).
$$

We redistribute the charge using just one discharging rule:

**(R)** Every face sends 1 to every incident 2-vertex.

Using the structure properties of a minimal counterexample $G$ and the discharging rule, we prove Theorem 8.

**Proof.** We show that after applying the discharging rule every vertex and face in $G$ has nonnegative charge which contradicts the fact that the total sum of initial charges is $-12$.

By Lemma 14, there are only vertices of degree 2 and 3 in $G$. Every 2-vertex is incident to two faces hence it receives 2 of charge, so its final charge is 0. On the other hand, 3-vertices have initial charge 0 and they send no charge, therefore their charge remains 0.

It remains to consider the faces. By Lemma 15, 6-faces send no charge, so their charge remains 0. By Lemma 16, 7-face sends at most 1, hence it retains nonnegative charge. Finally, every $k$-face $f$, $k \geq 8$, has at most $\left\lfloor \frac{1}{2}l(f) \right\rfloor$ incident 2-vertices by Lemma 14, therefore the final charge of $f$ is at least $l(f) - 6 - \left\lfloor \frac{1}{2}l(f) \right\rfloor \geq \left\lceil \frac{2}{3}l(f) \right\rceil - 6 \geq 0$. \qed

### 4 Discussion

In the introduction we mentioned that Faudree et al. [2] introduced a construction of planar graphs of girth 4 with the strong chromatic index equal to $4\Delta - 4$. We have shown that if the girth of a planar graph is at least 6, $3\Delta + 6$ colors suffice. This bound is not tight, however, there exist planar graphs with high girth and the strong chromatic index considerably close to the bound given by Theorem 7. Consider an odd cycle of length $k$ and append $d-2$, $d \geq 3$, leaves to each of the $k$ initial vertices (see Fig. 6 for an example). A planar graph with girth $k$ is obtained, we denote it by $C_k^d$.

**Proposition 17.** For every odd integer $k \geq 3$ and every integer $d \geq 3$, it holds

$$
\left\lceil \frac{2k(d - 1)}{k - 1} \right\rceil \leq \chi'_s(C_k^d) \leq \left\lceil \frac{2k(d - 2)}{k - 1} \right\rceil + 5.
$$

**Proof.** First, consider the lower bound. There are $(d-1)k$ edges in $C_k^d$ and each color can be assigned to at most $(k-1)/2$ of them in order to satisfy the conditions of the strong edge coloring. Hence, the strong chromatic index of $C_k^d$ is at least $\lceil 2k \cdot (d-1)/(k-1) \rceil$.

To show that the upper bound holds, we first construct a strong edge coloring of the pendent edges with $2(d-2) + \ell$ colors, where $\ell = \lceil 2(d-2)/(k-1) \rceil$. Let $v_1, v_2, \ldots, v_k$ be the vertices of the cycle in $C_k^d$. The pendent edges incident to two consecutive vertices
Figure 6: The strong chromatic index of $C_5^d$ is 10.

must receive distinct colors, so $2(d-2)$ distinct colors must be used on the pendent edges incident to any pair of adjacent vertices of the cycle. We color the pendent edges incident to the vertex $v_i$ by the colors in $C_i = \{x^i_1, x^i_2, \ldots, x^i_{d-2}\}$, for $i \in \{1, 2, \ldots, 2t + 1\}$. Here, we define $t = \lceil (d-2)/\ell \rceil$, and $x^i_j = (j + (i-1) \cdot (d-2)) \mod (2(d-2) + \ell)$ for $j \in \{1, 2, \ldots, d-2\}$. Observe that $2t + 1 \leq k$.

Now, we color the pendent edges incident to the remaining vertices $v_{2t+2}, v_{2t+3}, \ldots, v_k$ with the colors from the sets $C_{2t}$ and $C_{2t+1}$. The pendent edges incident to the vertices with even indices receive the colors from $C_{2t}$, and the pendent edges incident to the vertices with odd indices receive the colors from $C_{2t+1}$. Obviously, the pendent edges incident to two adjacent vertices $v_i$ and $v_{i+1}$, for $i \leq k-1$, receive distinct colors.

Consider now the colors of the pendent edges incident to the vertex $v_k$. They receive the colors from $C_{2t+1}$, so we need to show that $C_1 \cap C_{2t+1} = \emptyset$, i.e. $(d-2) \leq 2t(d-2) \mod (2(d-2) + \ell) \leq (d-2) + \ell$. Consider the following reduction:

$$
\begin{align*}
d - 2 & \leq 2t (d-2) \mod (2(d-2) + \ell) \leq d - 2 + \ell \\
2(d-2) & \leq d - 2 + \ell \\
(d-2) + \ell - t \ell & \leq d - 2 + \ell \\
0 & \leq (d-2) + \ell - t \ell \mod (2(d-2) + \ell) \leq \ell
\end{align*}
$$

Obviously, $d - 2 \leq t \ell \leq d - 2 + \ell$, hence, $d - 2 < x^k_j \leq 2(d-2) + \ell$, for every color $x^k_j \in C_k$, so our coloring of the pendent edges of $C_k^d$ is strong. Finally, we use at most 5 additional colors to color the edges of the cycle. This establishes the upper bound.

Let us mention that 5 additional colors are used only when $k = 5$, otherwise 4 or 3 colors suffice. Moreover, a longer argument shows that instead of using only new colors on the cycle, some colors of the pendent edges could be used. Notice also that $\lceil 2k(d-2)/(k-1) \rceil + 5 \leq \lceil 2k(d-1)/(k-1) \rceil + 3$.

The graphs $C_k^d$ do not achieve the highest strong chromatic index among the planar graphs of girth $k$ and maximum degree $d$ (see Fig. 7 for an example). However, we believe that they achieve it up to the constant, thus we propose the following conjecture.

**Conjecture 18.** There exists a constant $C$ such that for every planar graph $G$ of girth
Figure 7: A planar graph with girth 5, maximum degree 3, the strong chromatic index 6, and a strong edge coloring. The graph $C_5^3$ has the strong chromatic index 5.

$k \geq 5$ and maximum degree $\Delta$ it holds

$$\chi'_s(G) \leq \left\lceil \frac{2k(\Delta - 1)}{k - 1} \right\rceil + C.$$ 

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