BOXCITY OF GRAPHS ON SURFACES

LOUIS ESPERET AND GWENAËL JORET

Abstract. The boxicity of a graph $G = (V, E)$ is the least integer $k$ for which there exist $k$ interval graphs $G_i = (V, E_i)$, $1 \leq i \leq k$, such that $E = E_1 \cap \cdots \cap E_k$. Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two and Thomassen proved in 1986 that planar graphs have boxicity at most three. In this note we prove that the boxicity of toroidal graphs is at most 7, and that the boxicity of graphs embeddable in a surface $\Sigma$ of genus $g$ is at most $5g + 3$. This result yields improved bounds on the dimension of the adjacency poset of graphs on surfaces.

1. Introduction

Given a collection $\mathcal{C}$ of subsets of a set $\Omega$, the intersection graph of $\mathcal{C}$ is defined as the graph with vertex set $\mathcal{C}$, in which two elements of $\mathcal{C}$ are adjacent if and only if their intersection is non empty. A $d$-box is the Cartesian product $[x_1, y_1] \times \cdots \times [x_d, y_d]$ of $d$ closed intervals of the real line.

The boxicity $\text{box}(G)$ of a graph $G$ is the least integer $d \geq 1$ such that $G$ is the intersection graph of a collection of $d$-boxes. An interval graph is a graph of boxicity one.

The intersection $G_1 \cap \cdots \cap G_k$ of $k$ graphs $G_1, \ldots, G_k$ defined on the same vertex set $V$ is the graph $(V, E_1 \cap \cdots \cap E_k)$, where $E_i$ ($1 \leq i \leq k$) denotes the edge set of $G_i$. Observe that the boxicity of a graph $G$ can equivalently be defined as the least integer $k$ such that $G$ is the intersection of $k$ interval graphs.

The concept of boxicity was introduced in 1969 by Roberts [17]. It is used as a measure of the complexity of ecological [18] and social [10] networks, and has applications in fleet maintenance [16]. Graphs with boxicity one (that is, interval graphs) can be recognized in linear time. On the
other hand, Kratochvıl [13] proved that determining whether a graph has boxicity at most two is NP-complete.

Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two [19] and Thomassen proved in 1986 that planar graphs have boxicity at most three [23]. Other results on the boxicity of graphs can be found in [2, 6, 7] and the references therein.

Related to boxicity is the notion of adjacency posets of graphs, which was introduced by Felsner and Trotter [9]. The adjacency poset of a graph $G = (V, E)$ is the poset $P_G = (W, \leq)$ with $W = V \cup V'$, where $V'$ is a disjoint copy of $V$, and such that $u \leq v$ if and only if $u = v$, or $u \in V$ and $v \in V'$ and $u, v$ correspond to two distinct vertices of $G$ which are adjacent in $G$. Let us recall that the dimension $\dim(P)$ of a poset $P$ is the minimum number of linear orders whose intersection is exactly $P$.

Felsner, Li, and Trotter [8] recently showed that $\dim(P_G) \leq 5$ for every outerplanar graph $G$, and that $\dim(P_G) \leq 8$ for every planar graph $G$. They also proved that $\dim(P_G) \leq \frac{3}{2} \chi_a(G)(\chi_a(G) - 1)$ for every graph $G$ with $\chi_a(G) \geq 2$, where $\chi_a(G)$ denotes the acyclic chromatic number of $G$ (the least integer $k$ so that $G$ can be properly colored with $k$ colors, in such a way that every two color classes induce a forest). Using a result of Alon, Mohar, and Sanders [5], this implies that the dimension of $P_G$ is $O(g^{8/7})$ when $G$ is embeddable in a surface of genus $g$. At the end of their paper, the authors of [8] write that it is likely that the $O(g^{8/7})$ upper bound on $\dim(P_G)$ could be improved to $O(g)$.

In this note, we first observe that the boxicity can also be bounded from above by a function of the acyclic chromatic number, namely $\text{box}(G) \leq \chi_a(G)(\chi_a(G) - 1)$ for every graph $G$ with $\chi_a(G) \geq 2$. Next, using a result of Adiga, Bhowmick, and Chandran [2], we relate $\dim(P_G)$ to $\text{box}(G)$ by observing that $\dim(P_G) \leq 2\text{box}(G) + \chi(G) + 4$ for every graph $G$ (here $\chi(G)$ denotes the chromatic number of $G$). Then we prove that $\text{box}(G) \leq 5g + 3$ for every graph embeddable in a surface of genus $g$. This implies that $\dim(P_G)$ is bounded by a linear function of $g$, thus confirming the suggestion of Felsner et al. [8] mentioned above. We also consider more closely the case of toroidal graphs and show that every such graph has boxicity at most 7, while there are toroidal graphs with boxicity 4. We conclude the paper with several remarks and open problems about the boxicity of graphs on surfaces.
2. Boxicity and acyclic coloring

It can be deduced from [2], or directly from [21], that the graph obtained from the complete graph $K_n$ by subdividing each edge precisely once has boxicity $\Theta(\log \log n)$. This graph is 2-degenerate, hence the boxicity of a graph cannot be bounded from above by a function of its degeneracy\(^3\) or chromatic number alone. However, the boxicity can be bounded by a function of the acyclic chromatic number, as we now show.

For a graph $G$ and a subset $X$ of vertices of $G$, we let $G[X]$ denote the subgraph of $G$ induced by $X$, and let $G \setminus X$ denote the graph obtained from $G$ by removing all vertices in $X$.

Consider a graph $G$ and a subset $X$ of vertices of $G$ together with an interval graph $I$ on the vertex set $X$, such that $I$ is a supergraph of $G[X]$. Assume without loss of generality that $I$ maps all the vertices of $X$ to subintervals of some interval $[l, r]$ of $\mathbb{R}$. We call a canonical extension of $I$ to $G$ the interval graph $I'$ defined by mapping the vertices of $X$ to their corresponding intervals in $I$, and all other vertices of $G$ to the interval $[l, r]$. Observe that a canonical extension of $I$ to $G$ is a supergraph of $G$.

**Lemma 1.** $\text{box}(G) \leq \chi_a(G)(\chi_a(G) - 1)$ for every graph $G$ with $\chi_a(G) \geq 2$.

**Proof.** Consider an acyclic coloring $c$ of $G$ with $k \geq 2$ colors. For any two distinct colors $i < j$, we consider the graph $G_{i,j}$ obtained from $G$ by adding an edge between every pair of non-adjacent vertices $u, v$ such that at most one of $u, v$ is colored $i$ or $j$.

We first show that $G = \bigcap_{i<j} G_{i,j}$. Since all $G_{i,j}$’s are supergraphs of $G$, we only need to show that for every pair $u, v$ of non-adjacent vertices in $G$, there exist $i < j$ so that $u$ and $v$ are not adjacent in $G_{i,j}$. Exchanging $u$ and $v$ if necessary, we may assume that $c(u) \leq c(v)$. If $c(u) < c(v)$ then $G_{c(u),c(v)}$ does not contain the edge $uv$. If $c(u) = c(v)$ then $G_{c(u),k}$ (if $c(u) < k$) or $G_{1,c(u)}$ (if $c(u) = k$) does not contain the edge $uv$.

We now prove that for every $i < j$, $\text{box}(G_{i,j}) \leq 2$. This implies that $\text{box}(G) \leq 2 \binom{k}{2} = k(k - 1)$. Observe that since $c$ is an acyclic coloring of $G$, the subgraph $H_{i,j}$ of $G$ induced by the vertices colored $i$ or $j$ is a forest, and thus has boxicity at most two (this follows from [19] but can also be proven independently fairly easily). Let $I_{i,j}$ and $J_{i,j}$ be two interval graphs on the vertex set

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\(^3\) A graph $G$ is $k$-degenerate if every subgraph of $G$ has a vertex with degree at most $k$. The degeneracy of $G$ is the least integer $k$ such that $G$ is $k$-degenerate.
Lemma 2. Let $G = (V, E)$ be a graph and let $X \subseteq V$ be such that $G[X]$ contains $k$ pairwise disjoint pairs of non-adjacent vertices. Then $\text{box}(G) \leq \text{box}(G \setminus X) + |X| - k$.

Proof. Let $v_{2i-1}, v_{2i}$ ($1 \leq i \leq k$) be $k$ pairwise disjoint pairs of non-adjacent vertices in $G[X]$, and let $v_{2k+1}, \ldots, v_{\ell}$ be the remaining vertices of $X$. Consider $t$ interval graphs $I_1, \ldots, I_t$ on the vertex set $V \setminus X$ such that $G \setminus X = \bigcap_{i=1}^t I_i$. We will prove that $\text{box}(G) \leq t + \ell - k$.

For every pair $v_{2i-1}, v_{2i}$ ($1 \leq i \leq k$), we consider the interval graph $J_i$ defined as follows: $v_{2i-1}$ is mapped to $\{0\}$; $v_{2i}$ is mapped to $\{2\}$; the common neighbors of $v_{2i-1}$ and $v_{2i}$ in $G$ are mapped to $[0, 2]$; the remaining neighbors of $v_{2i-1}$ are mapped to $[0, 1]$; the remaining neighbors of $v_{2i}$ are mapped to $[1, 2]$; and the remaining vertices are mapped to $\{1\}$. The graph $J_i$ is clearly a supergraph of $G$, and every non-neighbor of $v_{2i-1}$ or $v_{2i}$ in $G$ is a non-neighbor of $v_{2i-1}$ or $v_{2i}$ (respectively) in $J_i$.

Next, for every $i \in \{2k+1, 2k+2, \ldots, \ell\}$, we define an interval graph $J_i$ as follows: $v_i$ is mapped to $\{0\}$; its neighbors in $G$ are mapped to $[0, 1]$, and the remaining vertices are mapped to $\{1\}$. This is a supergraph of $G$, and every non-edge incident to $v_i$ in $G$ is a non-edge in $J_i$.

Let $I'_1, \ldots, I'_t$ denote canonical extensions of $I_1, \ldots, I_t$ to $G$. We claim that $G$ is precisely the intersection of the $I'_i$'s ($1 \leq i \leq t$), and the $J_i$'s ($i \in \{1, \ldots, k\} \cup \{2k+1, \ldots, \ell\}$). These graphs are
clearly supergraphs of $G$. Moreover, every non-edge $e$ of $G$ is a non-edge in one of these graphs, since $e$ is either a non-edge in $G \setminus X$, or is incident to some vertex $v_i$ with $i \in \{1, \ldots, \ell\}$. □

In all subsequent applications of Lemma 2, $X$ will induce a cycle in $G$. In this case we obtain $\text{box}(G) \leq \text{box}(G \setminus X) + 3$ if $|X| = 3$, and $\text{box}(G) \leq \text{box}(G \setminus X) + \lfloor |X|/2 \rfloor$ if $|X| \geq 4$.

**Lemma 3.** Let $G = (V, E)$ be a graph and let $V_1, V_2, X$ be a partition of $V$ such that no edge of $G$ has an endpoint in $V_1$ and the other in $V_2$. Let $G_1$ be a graph obtained from $G[V_1 \cup X]$ by adding a (possibly empty) set of edges between pairs of vertices from $X$. Then $\text{box}(G) \leq \text{box}(G_1) + \text{box}(G[V_2 \cup X]) + 1$. In particular $\text{box}(G) \leq \text{box}(G[V_1 \cup X]) + \text{box}(G[V_2 \cup X]) + 1$.

**Proof.** Consider $k$ interval graphs $I_1, \ldots, I_k$ on the vertex set $V_1 \cup X$ such that $G_1 = \bigcap_{i=1}^{k} I_i$, and $\ell$ interval graphs $J_1, \ldots, J_\ell$ on the vertex set $V_2 \cup X$ such that $G[V_2 \cup X] = \bigcap_{i=1}^{\ell} J_i$.

Let $I_{1}', \ldots, I_{k}'$ be canonical extensions of $I_1, \ldots, I_k$ to $G$, and let $J_{1}', \ldots, J_{\ell}'$ be canonical extensions of $J_1, \ldots, J_\ell$ to $G$. Finally, let $K$ be the interval graph defined by mapping all vertices of $V_1$ to $\{0\}$, all vertices of $V_2$ to $\{1\}$, and all vertices of $X$ to $[0, 1]$.

It is clear that all the $I_i$'s, $J_i$'s and $K$ are supergraphs of $G$, and that every non-edge of $G$ appears in one of these graphs. Hence, $\text{box}(G) \leq k + \ell + 1$. □

The next lemma is a variation of [3, Lemma 7], where the same idea is used to obtain a slightly different result.

**Lemma 4.** Let $G = (V, E)$ be a graph and let $K \subseteq V$ be a clique in $G$. Let $H$ be a graph obtained from $G$ by removing some set of edges having their two endpoints in $K$. Then $\text{box}(G) \leq 2 \text{box}(H)$.

**Proof.** Consider $k$ interval graphs $I_1, \ldots, I_k$ on the vertex set $V$ such that $H = \bigcap_{i=1}^{k} I_i$. For each $1 \leq i \leq k$, we define two interval graphs $I_i'$ and $I_i''$ in the following way. Assume that $I_i$ maps every vertex $v \in V$ to some interval $[l(v), r(v)]$ of $\mathbb{R}$. Set $a = \min\{l(v), v \in V\}$ and $b = \max\{r(v), v \in V\}$. The interval graph $I_i'$ is obtained by mapping each vertex $v$ to $[a, r(v)]$ if $v \in K$ and to $[l(v), r(v)]$ otherwise. The interval graph $I_i''$ is obtained by mapping each vertex $v$ to $[l(v), b]$ if $v \in K$ and to $[l(v), r(v)]$ otherwise. It is readily seen that $G = \bigcap_{i=1}^{k} (I_i' \cap I_i'')$, which shows that $\text{box}(G) \leq 2k$. □

Combining Lemmas 3 and 4, we immediately obtain the following corollary.

**Corollary 5.** Let $G = (V, E)$ be a graph and let $V_1, V_2, X$ be a partition of $V$ such that no edge of $G$ has an endpoint in $V_1$ and the other in $V_2$. Let $G_1$ be a graph obtained from $G[V_1 \cup X]$
by adding and/or removing any set of edges between pairs of vertices of \( X \). Then \( \text{box}(G) \leq 2\text{box}(G_1) + \text{box}(G[V_2 \cup X]) + 1 \).

We now turn to the boxicity of graphs on surfaces. In this paper, a surface is a non-null compact connected 2-manifold without boundary. We refer the reader to the book by Mohar and Thomassen [14] for background on graphs on surfaces.

Consider a graph \( G \) embedded in a surface \( \Sigma \). For simplicity, we use \( G \) both for the corresponding abstract graph and for the subset of \( \Sigma \) corresponding to the drawing of \( G \). A cycle \( C \) of \( G \) is said to be noncontractible if \( C \) is noncontractible (as a closed curve) in \( \Sigma \). Also, \( C \) is called surface separating if \( C \) separates \( \Sigma \) in two connected pieces. The facewidth \( \text{fw}(G) \) of \( G \) is the least integer \( k \) such that \( \Sigma \) contains a noncontractible simple closed curve intersecting \( G \) in \( k \) points. If \( G \) triangulates \( \Sigma \) then its facewidth is equal to the length of a shortest noncontractible cycle in \( G \).

Two cycles of \( G \) are (freely) homotopic in \( \Sigma \) if there is a continuous deformation mapping one to the other.

The following well-known fact (often called the 3-Path Property) will be used: If \( P_1, P_2, P_3 \) are three internally disjoint paths with the same endpoints in an embedded graph, and \( P_1, P_2 \) are such that \( P_1 \cup P_2 \) is a noncontractible cycle, then at least one of the two cycles \( P_1 \cup P_3, P_2 \cup P_3 \) is also noncontractible (see for instance [14, Proposition 4.3.1]). This implies the following lemma.

**Lemma 6.** Suppose that \( C \) is a noncontractible cycle of a graph \( G \) embedded in a surface. Then there exists a noncontractible induced cycle \( C' \) of \( G \) with \( V(C') \subseteq V(C) \).

The next lemma is a standard fact about noncontractible cycles in embedded graphs, see [14, Chapter 4.2].

**Lemma 7.** Suppose that \( C \) is a noncontractible cycle of a graph \( G \) embedded in a surface of genus \( g \geq 1 \). Then each component of \( G \setminus V(C) \) is embeddable in a surface of genus \( g - 1 \).

Recall that Thomassen [23] proved that \( \text{box}(G) \leq 3 \) for every planar graph \( G \). We are now ready to state and prove the main result of this note, extending Thomassen’s bound to general surfaces.

**Theorem 8.** Let \( G \) be a graph embedded in a surface \( \Sigma \) of genus \( g \). Then \( \text{box}(G) \leq 5g + 3 \).
Proof. We prove the result by induction on $g$. If $g = 0$ the bound follows from [23], so we can assume that $g \geq 1$. We can also assume that $G$ triangulates $\Sigma$, since $G$ is an induced subgraph of a triangulation\(^4\) of $\Sigma$ and the boxicity is monotone by taking induced subgraphs.

First suppose that $fw(G) \leq 5$. Since $G$ is a triangulation, there exists a noncontractible cycle $C$ of length at most 5. Using Lemma 6, we can further assume that $C$ is an induced cycle of $G$. The boxicity of a graph is clearly the maximum boxicity of its components. Thus by Lemma 7 we obtain by induction that $\text{box}(G \setminus V(C)) \leq 5(g-1)+3$, and by Lemma 2 that $\text{box}(G) \leq 5(g-1)+3+3 \leq 5g+3$.

From now on we assume that $fw(G) \geq 6$, and we consider a shortest noncontractible cycle $C$ in $G$. It follows from Lemma 6 that $C$ is an induced cycle (otherwise, we could shorten it). Let $V'$ be the set of vertices from $V(G) \setminus V(C)$ having at least one neighbor in $C$. We will construct a graph $H$ with $\text{box}(H) \leq 2$ such that $H$ can be obtained from $G[V' \cup V(C)]$ by only adding and removing edges having both endpoints in $V'$. By Corollary 5, we then have $\text{box}(G) \leq 2 \text{box}(H) + \text{box}(G[V(C)]) + 1 \leq \text{box}(G[V(C)]) + 5$. This in turn implies the theorem since Lemma 7 and the induction hypothesis give that $\text{box}(G[V(C)]) \leq 5(g-1)+3$, implying $\text{box}(G) \leq 5g+3$.

We remark that every vertex from $V'$ has at most three neighbors in $C$. More precisely, if some vertex of $V'$ does not belong to one of these four disjoint sets:

- $S_1$: the set of vertices of $V'$ with exactly one neighbor in $C$;
- $S_2$: the set of vertices of $V'$ with exactly two neighbors in $C$ and such that these vertices are consecutive in $C$;
- $S_3$: the set of vertices of $V'$ with exactly two neighbors in $C$ and such that these vertices are at distance two in $C$;
- $S_4$: the set of vertices of $V'$ with exactly three neighbors in $C$ and such that these vertices are consecutive in $C$;

then, since $C$ has length at least 6, the 3-Path Property implies that $G$ contains a noncontractible cycle that is shorter than $C$, which is a contradiction.

\(^4\)We can first assume that $G$ is connected and has no embedding on a surface with lower genus, and then using [14, Proposition 3.4.1] that $G$ has an embedding in $\Sigma$ in which every face is homeomorphic to an open disk. In this embedding, for every face $f$ consider a boundary walk $e_1, \ldots, e_k$ around $f$, add a triangulated polygon on $k$ vertices $v_1, \ldots, v_k$ inside $f$, and join each $v_i$ with the endpoints of $e_i$. 
Enumerate the vertices of $C$ as $v_1, \ldots, v_k$, in order. Let $H$ be the intersection graph of the 2-dimensional boxes depicted in Figure 1. Vertices $v_1$, $v_{k-1}$, and $v_k$ are mapped to boxes with corners $(-1,0)$ and $(0,3)$, $(k-3,0)$ and $(k-2,3)$, $(0,3)$ and $(k-3,4)$, respectively. Every other vertex $v_i$ is mapped to the box with corners $(i-2, i \mod 2)$ and $(i-1, 1 + i \mod 2)$. Consider now the vertices of $V'$: Each vertex $v \in S_1$ is mapped to an arbitrary point in the interior of the box of $v_i$, where $v_i$ is the unique neighbor of $v$ in $C$. Each vertex $v \in S_2$ is mapped to the point $(0,3)$ if $v$ is adjacent to $v_1$ and $v_k$ in $C$; to the point $(k-3,3)$ if $v$ is adjacent to $v_{k-1}$ and $v_k$ in $C$, and to the point $(i-1,1)$ if $v$ is adjacent to $v_i$ and $v_{i+1}$ in $C$ for some $i \in \{1, \ldots, k-2\}$.

Consider now a vertex $v \in S_3$ with neighbors $v_i$ and $v_{i+2}$ in $C$, where $i \in \{1, \ldots, k-3\}$. Then $v$ is mapped to the horizontal segment with endpoints $(i-1, \frac{1}{2} + i \mod 2)$ and $(i, \frac{1}{2} + i \mod 2)$. Vertices of $S_3$ that are adjacent to $v_2$ and $v_k$ in $C$ are mapped to the vertical segment with endpoints $(\frac{1}{2}, 1)$ and $(\frac{1}{2}, 3)$. Vertices of $S_3$ that are adjacent to $v_{k-2}$ and $v_k$ in $C$ are similarly mapped to the vertical segment with endpoints $(k-\frac{7}{2}, 1 + k \mod 2)$ and $(k-\frac{7}{2}, 3)$. Vertices of $S_3$ that are adjacent to $v_1$ and $v_{k-1}$ in $C$ are mapped to the horizontal segment with endpoints $(0, \frac{5}{2})$ and $(k-3, \frac{5}{2})$.

Each vertex $v \in S_4$ that is adjacent to $v_i, v_{i+1}, v_{i+2}$ in $C$ where $i \in \{1, \ldots, k-3\}$ is mapped to the horizontal segment with endpoints $(i-1,1)$ and $(i,1)$. Each vertex $v \in S_4$ that is adjacent to $v_1, v_2, v_k$ in $C$ is mapped to the vertical segment with endpoints $(0,1)$ and $(0,3)$. Each vertex $v \in S_4$ that is adjacent to $v_{k-2}, v_{k-1}, v_k$ in $C$ is mapped to the vertical segment with endpoints $(k-3,1)$ and $(k-3,3)$. Finally, each vertex $v \in S_4$ that is adjacent to $v_1, v_{k-1}, v_k$ in $C$ is mapped to the horizontal segment with endpoints $(0,3)$ and $(k-3,3)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The graph $H$ as intersection of 2-dimensional boxes.}
\end{figure}
It follows from the definition of $H$ that the vertices of $V(C)$ induce the cycle $C$ in $H$ and that
the neighbors on $V(C)$ of any vertex $v \in V'$ are the same in $G$ and in $H$. This shows that $H$
can be obtained from $G[V' \cup V(C)]$ by only adding and removing edges having both endpoints in $V'$,
which concludes the proof.

Theorem 8 implies that toroidal graphs have boxicity at most 8. We improve on this bound by
using the following remarkable result of Schrijver [20]: Every graph embedded in the torus with
facewidth $k$ contains $\lfloor 3k/4 \rfloor$ vertex-disjoint noncontractible cycles. (Note that on the torus, such
cycles are necessarily homotopic.)

**Theorem 9.** $\text{box}(G) \leq 7$ for every toroidal graph $G$.

*Proof.* Again, we may assume that $G$ triangulates the torus.

Assume first that $\text{fw}(G) \leq 5$. Since $G$ is a triangulation, there exists a noncontractible cycle $C$
of length at most 5 such that $G \setminus V(C)$ is planar. Using Lemma 6, we can further assume that $C$
is an induced cycle of $G$. Then, using Lemma 2 and the result of Thomassen about the boxicity of
planar graphs, we deduce that $\text{box}(G) \leq 3 + 3 = 6$.

Assume now that $\text{fw}(G) \geq 6$. The aforementioned result of Schrijver implies that $G$ contains 4
pairwise vertex-disjoint noncontractible cycles, say $C_1, C_2, C_3, C_4$ in this order. Because of $C_2$ and
$C_4$ there are no edges between $C_1$ and $C_3$ in $G$. We may further assume by Lemma 6 that $C_1$ and
$C_3$ are induced cycles in $G$. (Observe that every noncontractible cycle in $G[V(C_1)]$ or in $G[V(C_3)]$
is again homotopic to the four cycles $C_1, C_2, C_3, C_4$, because such a cycle is vertex-disjoint from
$C_2$ and $C_4$.) The removal of $C_1$ and $C_3$ cuts the torus into two connected pieces $\Sigma_1$ and $\Sigma_2$. Let
$V_i$ ($i = 1, 2$) be the set of vertices lying on $\Sigma_i$, and set $X = V(C_1) \cup V(C_3)$. Since $G[V_1 \cup X]$ and
$G[V_2 \cup X]$ are planar, it follows from Lemma 3 that $\text{box}(G) \leq 3 + 3 + 1 = 7$.  

It was proved in [17] that for every $n \geq 1$, the graph $G_{2n}$ obtained from $K_{2n}$ by removing a
perfect matching has boxicity exactly $n$. Since $G_8$ can be embedded on the torus (see Figure 2),
there exist toroidal graphs with boxicity four.

Recall that, for a graph $G = (V, E)$, the adjacency poset $\mathcal{P}_G$ of $G$ is defined as the poset
$\mathcal{P}_G = (W, \leq)$ with $W = V \cup V'$, where $V'$ is a disjoint copy of $V$, and $u \leq v$ if and only if $u = v$, or
$u \in V$ and $v \in V'$ and $u, v$ correspond to two distinct vertices of $G$ which are adjacent in $G$. Let
$\mathcal{P}_G^*$ denote the poset obtained from $\mathcal{P}_G$ by adding that $u \leq v$ for every $(u, v) \in V \times V'$ such that
$u$ and $v$ correspond to the same vertex of $G$. Adiga, Bhowmick, and Chandran [2] recently proved
that \( \dim(P_G^*)/2 - 2 \leq \text{box}(G) \leq 2 \dim(P_G^*) \) for every graph \( G \). Using this result, we may bound the dimension of \( P_G \) as follows.

**Theorem 10.** \( \dim(P_G) \leq 2 \text{box}(G) + \chi(G) + 4 \) for every graph \( G = (V,E) \).

**Proof.** We have that \( \dim(P_G^*) \leq 2 \text{box}(G) + 4 \) by the aforementioned result of Adiga et al. [2], thus it is enough to show that \( \dim(P_G) \leq \dim(P_G^*) + \chi(G) \). Consider a (proper) coloring \( V_1, V_2, \ldots, V_k \) of \( G \) with \( k = \chi(G) \) colors, and let \( V_1', V_2', \ldots, V_k' \) denote the corresponding partition of \( V' \). For \( i \in \{1, \ldots, k\} \), let \( L_i = (W, \leq_i) \) be an arbitrary linear order satisfying that

\[
V_1 \cup \cdots \cup V_{i-1} \cup V_i \cup V_{i+1} \cup \cdots \cup V_k \leq_i V_1' \cup \cdots \cup V_i' \cup V_{i+1}' \cup \cdots \cup V_k'.
\]

(Here \( A \leq_i B \) means that \( u \leq_i v \) for every \( u \in A \) and \( v \in B \).) Then it is easily checked that each \( L_i \) is a linear extension of \( P_G \), and that the intersection of these \( k \) linear orders with \( P_G^* \) is exactly \( P_G \). It follows that \( \dim(P_G) \leq \dim(P_G^*) + k \), as desired. \( \square \)

**Corollary 11.** Let \( G \) be a graph embeddable in a surface \( \Sigma \) of genus \( g \geq 1 \). Then \( \dim(P_G) \leq 10g + \frac{1}{2} (27 + \sqrt{1+48g}) \) if \( \Sigma \) is orientable, and \( \dim(P_G) \leq 10g + \frac{1}{2} (27 + \sqrt{1+24g}) \) otherwise.

**Proof.** This follows from Theorems 8 and 10, and Heawood’s upper bound on the chromatic number of \( G \), namely \( \chi(G) \leq \frac{1}{2} (7+\sqrt{1+48g}) \) if \( \Sigma \) is orientable, and \( \chi(G) \leq \frac{1}{2} (7+\sqrt{1+24g}) \) otherwise. \( \square \)

This confirms what Felsner, Li, and Trotter [8] suggested as an improvement of their result.

### 4. Open Questions

The first question is whether the bounds obtained in Section 3 are best possible. We believe that the boxicity of graphs embeddable in a surface of genus \( g \) should rather be \( O(\sqrt{g}) \). Since the complete graph \( K_{2n} \) with a perfect matching removed has boxicity \( n \), this would be optimal. This example also shows that the boxicity of graphs with no \( K_t \)-minor can be linear in \( t \), while we only
know a $O(t^4 \log^2 t)$ upper bound (see the remark after Lemma 1).

The edgewidth of a graph $G$ embedded in a surface $\Sigma$ is the length (number of edges) of a shortest noncontractible cycle in $G$. Kawarabayashi and Mohar [11] proved that for every fixed surface $\Sigma$, graphs embeddable in $\Sigma$ with sufficiently large edgewidth are acyclically 7-colorable. It then follows from Lemma 1 that these graphs have boxicity at most 42. We believe that the following stronger statement is true:

**Conjecture 12.** For every fixed surface $\Sigma$ there exists an integer $e_\Sigma$ so that every graph $G$ embeddable on $\Sigma$ with edgewidth at least $e_\Sigma$ has boxicity at most three.

It follows from a theorem of Thomassen [23] that triangle-free planar graphs have boxicity at most two. Since there exist trees that are not interval graphs, a natural question is whether, for every surface $\Sigma$, graphs embeddable in $\Sigma$ and having sufficiently large girth (length of a shortest cycle) have boxicity at most two. We prove that the following slightly weaker statement holds:

**Theorem 13.** For every fixed surface $\Sigma$ there exists some integer $g_\Sigma$ such that every graph with girth at least $g_\Sigma$ embeddable in $\Sigma$ has boxicity at most 4.

**Proof.** It is well-known (see [4]) that there exists an integer $g_\Sigma$ such that the vertex set of every graph $G$ embeddable on $\Sigma$ and having girth at least $g_\Sigma$ can be partitioned into a forest $F$ and a stable set $S$, in such way that every two vertices of $S$ are at distance at least three in $G$.

Consider the graph $G_1$ obtained from $G$ by adding an edge between every pair of non-adjacent vertices $u, v$, such that at least one of $u, v$ is in $S$. As remarked in the proof of Lemma 1, $\text{box}(G_1) \leq 2$. Observe now that every vertex of $F$ has at most one neighbor in the stable set $S$. Using this property, it can be deduced from [7, Proof of Theorem 1] that the graph $G_2$ obtained from $G$ by adding all possible edges between pairs of vertices of $F$ has boxicity at most two. Since $G = G_1 \cap G_2$, it follows that $\text{box}(G) \leq 4$. \hfill $\square$

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Laboratoire G-SCOP (CNRS, GRENOBLE-INP), GRENOBLE, FRANCE

_E-mail address: louis.esperet@g-scop.fr_

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_E-mail address: gjoret@ulb.ac.be_
Author/s:
Esperet, L; JORET, G

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