Univoque numbers and an avatar of Thue-Morse

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Abstract

Univoque numbers are real numbers $\lambda > 1$ such that the number 1 admits a unique expansion in base $\lambda$, i.e., a unique expansion $1 = \sum_{j \geq 0} a_j \lambda^{-j+1}$, with $a_j \in \{0, 1, \ldots, \lceil \lambda \rceil - 1\}$ for every $j \geq 0$. A variation of this definition was studied in 2002 by Komornik and Loreti, together with sequences called admissible sequences. We show how a 1983 study of the first author gives both a result of Komornik and Loreti on the smallest admissible sequence on the set $\{0, 1, \ldots, b\}$, and a result of de Vries and Komornik (2007) on the smallest univoque number belonging to the interval $(b, b + 1)$, where $b$ is any positive integer. We also prove that this last number is transcendental. An avatar of the Thue-Morse sequence, namely the fixed point beginning in 3 of the morphism $3 \rightarrow 31, 2 \rightarrow 30, 1 \rightarrow 03, 0 \rightarrow 02$, occurs in a “universal” manner.

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1 Introduction

Komornik and Loreti determined in [18] the smallest univoque real number
a unique expansion \( 1 = \sum_{j \geq 0} a_j/\lambda^{j+1} \) with \( a_j \in \{0,1\} \) for every \( j \geq 0 \).

The word “univoque” in this context seems to have been introduced (with a slightly different meaning) by Daróczy and Katái in \([12,13]\), while characterizing unique expansions of the real number 1 was done by Erdős, Joó, and Komornik in \([15]\). The first author and Cosnard showed in \([4]\) how the result of \([18]\) parallels (and can be deduced from) their study of a certain set of binary sequences arising in the iteration of unimodal continuous functions of the unit interval that was done in \([11,2,1]\). The relevant sets of binary sequences occurring in references \([2,1]\), resp. in reference \([18]\), can be defined by

\[
\Gamma := \{ A \in \{0,1\}^\mathbb{N}, \forall k \geq 0, \overline{A} \leq \sigma^k A \leq A \}
\]

\[
\Gamma_{\text{strict}} := \{ A \in \{0,1\}^\mathbb{N}, \forall k \geq 1, \overline{A} < \sigma^k A < A \}
\]

where \( \sigma \) is the shift on sequences and the bar operation replaces 0’s by 1’s and 1’s by 0’s, i.e., if \( A = (a_n)_{n \geq 0} \), then \( \sigma A := (a_{n+1})_{n \geq 0} \), and \( \overline{A} := (1-a_n)_{n \geq 0} \); furthermore \( \leq \) denotes the lexicographical order on sequences induced by \( 0 < 1 \), the notation \( A < B \) meaning as usual that \( A \leq B \) and \( A \neq B \). The smallest univoque number in \((1,2)\) and the smallest nonperiodic sequence of the set \( \Gamma \) both involve the Thue-Morse sequence (see for example \([6]\) for more on this sequence).

It is tempting to generalize these sets to alphabets with more than 2 letters.

**Definition 1** For \( b \) a positive integer, we will say that the real number \( \lambda > 1 \) is \( \{0,1,\ldots,b\}\)-univoque if the number 1 has a unique expansion as \( 1 = \sum_{j \geq 0} a_j \lambda^{-(j+1)} \), where \( a_j \) belongs to \( \{0,\ldots,b\} \) for all \( j \geq 0 \). Furthermore, if \( \lambda > 1 \) is \( \{0,1,\ldots,[\lambda]-1\}\)-univoque, we will simply say that \( \lambda \) is univoque.

**Remark 1** If \( \lambda > 1 \) is \( \{0,1,\ldots,b\}\)-univoque for some positive integer \( b \), then \( \lambda \leq b+1 \). Also note that any integer \( q \geq 2 \) is univoque, since there is exactly one expansion of 1 as \( 1 = \sum_{j \geq 0} a_j q^{-(j+1)} \), with \( a_j \in \{0,1,\ldots,q-1\} \), namely \( 1 = \sum_{j \geq 0} (q-1)q^{-(j+1)} \).

Komornik and Loreti studied in \([19]\) the reals \( \lambda \) belonging to the interval \((1,b+1]\) that are \( \{0,1,\ldots,b\}\)-univoque. For their study, they introduced admissible sequences on the alphabet \( \{0,1,\ldots,b\} \). Denote, as above, by \( \sigma \) the shift on sequences, and by \( \overline{\text{bar}} \) the operation that replaces every \( t \in \{0,1,\ldots,b\} \) by \( b-t \), i.e., if \( A = (a_n)_{n \geq 0} \), then \( \overline{A} := (b-a_n)_{n \geq 0} \). Also denote by \( \leq \) the lexicographical order on sequences induced by the natural order on \( \{0,1,\ldots,b\} \).
order on \( \{0,1,\ldots,b\} \). Then, a sequence \( A = (a_n)_{n \geq 0} \) on \( \{0,1,\ldots,b\} \) is admissible if

\[
\forall k \geq 0 \text{ such that } a_k < b, \quad \sigma^{k+1}A < A, \\
\forall k \geq 0 \text{ such that } a_k > 0, \quad \sigma^{k+1}A > \frac{A}{A}.
\]

(Note that our notation is not exactly the notation of [19], since our sequences are indexed by \( \mathbb{N} \) and not \( \mathbb{N} \setminus \{0\} \).) These sequences have the following property: the map that associates with a real \( \lambda \in (1,b+1] \) the sequence of coefficients \( (a_j)_{j \geq 0} \in \{0,1,\ldots,b\} \) of the greedy (i.e., the lexicographically largest) expansion of 1, \( 1 = \sum_{j \geq 0} a_j \lambda^{-j+1} \), is a bijection from the set of \( \{0,1,\ldots,b\} \)-univoque \( \lambda \)'s to the set of admissible sequences on \( \{0,1,\ldots,b\} \) (see [19]).

Now there are two possible generalizations of the result of [18] about the smallest univoque number in \( (1,2) \), i.e., the smallest admissible binary sequence. One is to look at the smallest (if any) admissible sequence on the alphabet \( \{0,1,\ldots,b\} \), as did Komornik and Loreti in [19], the other is to look at the smallest (if any) univoque number in \( (b,b+1) \), as did de Vries and Komornik in [14].

It happens that the first author already studied a generalization of the set \( \Gamma \) to the case of more than 2 letters (see [1, Part 3]). Interestingly enough this study was not related to the iteration of continuous functions as was the study of \( \Gamma \), but only introduced as a tempting formal arithmetico-combinatorial generalization of the study of the set of binary sequences \( \Gamma \) to a similar set of sequences with more than two values.

The purpose of the present paper is threefold:

1) to show how the 1983 study [1, Part 3, p. 63–90] gives both the result of Komornik and Loreti in [19] on the smallest admissible sequence on \( \{0,1,\ldots,b\} \), and the result of de Vries and Komornik in [14] on the smallest number univoque number \( \lambda \) belonging to \( (b,b+1) \) where \( b \) is any positive integer;

2) to bring to light a universal morphism that governs the smallest elements in 1) above, and to show that the infinite sequence generated by this morphism is an avatar of the Thue-Morse sequence;

3) to prove that the smallest univoque number belonging to \( (b,b+1) \) (where \( b \) is any positive integer) is transcendental.

The paper consists of five sections. In Section 2 below we recall some results of [1, Part 3, p. 63–90] on the generalization of the set \( \Gamma \) to a \( (b+1) \)-
nonperiodic sequence of this set, completing results of [1, Part 3, p. 63–90]. In Section 3 we give two corollaries of the properties of this least sequence: one gives the result in [19], the other gives the result in [14]. The transcendence results are proven in the last section.

2 The generalized $\Gamma$ and $\Gamma_{\text{strict}}$ sets

Definition 2 Let $b$ be a positive integer, and $\mathcal{A}$ be a finite ordered set with $b + 1$ elements. Let $\alpha_0 < \alpha_1 < \ldots < \alpha_b$ be the elements of $\mathcal{A}$. The bar operation is defined on $\mathcal{A}$ by $\overline{\alpha_j} = \alpha_{b-j}$. We extend this operation to $\mathcal{A}^\mathbb{N}$ by $(a_n)_{n \geq 0} := (\overline{a_n})_{n \geq 0}$. Let $\sigma$ be the shift on $\mathcal{A}^\mathbb{N}$, defined by $\sigma((a_n)_{n \geq 0}) := (a_{n+1})_{n \geq 0}$.

We define the sets $\Gamma(\mathcal{A})$ and $\Gamma_{\text{strict}}(\mathcal{A})$ by:

$\Gamma(\mathcal{A}) := \{A = (a_n)_{n \geq 0} \in \mathcal{A}^\mathbb{N}, a_0 = \max \mathcal{A}, \forall k \geq 0, \overline{A} \leq \sigma^k A \leq A\}$,

$\Gamma_{\text{strict}}(\mathcal{A}) := \{A = (a_n)_{n \geq 0} \in \mathcal{A}^\mathbb{N}, a_0 = \max \mathcal{A}, \forall k \geq 1, \overline{A} < \sigma^k A < A\}$.

Remark 2 The set $\Gamma(\mathcal{A})$ was introduced by the first author in [1, Part 3, p. 63–90]. Note that there is a misprint in the definition given on p. 66 in [1]: $a_{\beta-i}$ should be changed into $a_{\beta-1-i}$ as confirmed by the rest of the text.

Remark 3 A sequence belongs to $\Gamma_{\text{strict}}(\mathcal{A})$ if and only if it belongs to $\Gamma(\mathcal{A})$ and is nonperiodic. Namely, $\sigma^k A = A$ if and only if $A$ is $k$-periodic; if $\sigma^k A = \overline{A}$, then $\sigma^{2k} A = A$, and the sequence $A$ is $2k$-periodic.

Remark 4 If the set $\mathcal{A}$ is given by $\mathcal{A} := \{i, i+1, \ldots, i+z\}$ where $i$ and $z$ are integers, equipped with the natural order, then for any $x \in \mathcal{A}$, we have $\overline{x} = 2i + z - x$. Namely, following Definition 2 above, we write $a_0 := i, a_1 := i + 1, \ldots, a_z := i + z$. Hence, for any $j \in [0, z]$, we have $\overline{a_j} = a_{2-j}$, which can be rewritten $\overline{i + j} = i + z - j$, i.e., for any $x$ in $\mathcal{A}$, we have $\overline{x} = i + z - (x-i) = 2i + z - x$.

A first result is that the sets $\Gamma_{\text{strict}}(\mathcal{A})$ are closely linked to the set of admissible sequences whose definition was recalled in the introduction.

Proposition 1 Let $A = (a_n)_{n \geq 0}$ be a sequence in $\{0, 1, \ldots, b\}^\mathbb{N}$, such that $a_0 = t \in [0, b]$. Suppose that the sequence $A$ is not equal to $b \ b \ b \ldots$ Then the sequence $A$ is admissible if and only if $2t > b$ and $A$ belongs to the set $\Gamma_{\text{strict}}(\{b-t, b-t+1, \ldots, t\})$. (The order on $\{b-t, b-t+1, \ldots, t\}$ is induced by the order on $\mathbb{N}$. From Remark 7 the bar operation is given by
Proof. Let $A = (a_n)_{n \geq 0}$ be a sequence belonging to $\{0, 1, \ldots, b\}^\mathbb{N}$ such that $a_0 = t \in [0, b - 1]$, and such that $A \neq b b b \ldots$

* First suppose that $2t > b$ and $A$ belongs to $\Gamma_{\text{strict}}(\{b - t, b - t + 1, \ldots, t\})$. Then, for all $k \geq 1$, $A < \sigma^k A < A$, which clearly implies that $A$ is admissible.

* Now suppose that $A$ is admissible. We thus have

$$
\forall k \geq 1 \text{ such that } a_{k-1} < b, \quad \sigma^k A < A,
\forall k \geq 1 \text{ such that } a_{k-1} > 0, \quad \sigma^k A > A.
$$

We first prove that, if the sequence $A$ is not a constant sequence, then

$$
\forall k \geq 1, \quad A < \sigma^k A < A.
$$

We only prove the inequalities $\sigma^k A < A$. The remaining inequalities are proved in a similar way. If $a_{k-1} < b$, the inequality $\sigma^k A < A$ holds. If $a_{k-1} = b$, there are two cases:

- either $a_0 = a_1 = \ldots = a_{k-1} = b$, then, if $a_k < b$ we clearly have $\sigma^k A < A$; if $a_k = b$, then the sequence $\sigma^k A$ begins with some block of $b$'s followed by a letter $< b$, thus it begins with a block of $b$'s shorter than the initial block of $b$'s of the sequence $A$ itself, hence $\sigma^k A < A$;

- or there exists an index $\ell$ with $1 < \ell < k$, such that $a_{\ell - 1} < b$, and $a_\ell = a_{\ell + 1} = \ldots = a_{k-1} = b$. As $A$ is admissible, we have $\sigma^\ell A < A$. It thus suffices to prove that $\sigma^k A \leq \sigma^\ell A$. This is clearly the case if $a_k < b$. On the other hand, if $a_k = b$, the sequence $\sigma^k A$ begins with a block of $b$'s which is shorter than the initial block of $b$'s of the sequence $\sigma^\ell A$, hence $\sigma^k A \leq \sigma^\ell A$.

Now, since $a_0 = t$ and $\sigma^k A < A$ for all $k \geq 1$, we have $a_k \leq t$ for all $k \geq 0$. Similarly, since $\sigma^k A > \overline{A}$ for all $k \geq 1$, we have $a_k \geq b - t$ for all $k \geq 1$. Finally $A > \overline{A}$ implies that $t = a_0 \geq b - t$. We thus have that $2t \geq b$ and $A$ belongs to $\Gamma(\{b - t, b - t + 1, \ldots, t\})$. Now, if $b = 2t$, then $\{b - t, b - t + 1, \ldots, t\} = \{t\}$ and $\bar{t} = t$. This implies that $A = t t t \ldots$, which is not an admissible sequence. □

**Remark 5** For $b = 1$, this (easy) result is noted without proof in [15] and proved in [4].

We need another definition from [1].

**Definition 3** Let $b$ be a positive integer, and $A$ be a finite ordered set with
that \( \mathcal{A} \) is equipped with a bar operation as in Definition 2. Let \( A = (a_n)_{n \geq 0} \) be a periodic sequence of smallest period \( T \), and such \( a_{T-1} < \max \mathcal{A} \). Let \( a_{T-1} = \alpha_j \) (thus \( j < b \)). Then the sequence \( \Phi(A) \) is defined as the 2\( T \)-periodic sequence beginning with \( a_0 \ a_1 \ldots \ a_{T-2} \ \alpha_{j+1} \ \bar{a}_0 \ \bar{a}_1 \ldots \ \bar{a}_{T-2} \ \alpha_{b-j-1} \), i.e.,

\[
\Phi((a_0 \ a_1 \ldots \ a_{T-2} \ \alpha_j)^\infty) := (a_0 \ a_1 \ldots \ a_{T-2} \ \alpha_{j+1} \ \bar{a}_0 \ \bar{a}_1 \ldots \ \bar{a}_{T-2} \ \alpha_{b-j-1})^\infty.
\]

We first prove the following easy claim.

**Proposition 2** The smallest element of \( \Gamma\{\{b-t, b-t+1, \ldots, t\}\} \) (where \( 2t > b \)) is the 2-periodic sequence \( (t (b-t))^\infty = (t (b-t) \ t (b-t) \ t \ldots) \).

**Proof.** Since any sequence \( A = (a_n)_{n \geq 0} \) belonging to \( \Gamma\{\{b-t, b-t+1, \ldots, t\}\} \) begins in \( t \), and satisfies \( \sigma A \geq T \), then it must satisfy \( a_0 = t \) and \( a_1 \geq b-t \).

Now if a sequence \( A \) belonging to \( \Gamma\{\{b-t, b-t+1, \ldots, t\}\} \) is such that \( a_0 = t \) and \( a_1 = (b-t) \), then it must be equal to the 2-periodic sequence \( (t (b-t))^\infty \) (Lemma 2, b, p. 73]). Since this periodic sequence trivially belongs to \( \Gamma\{\{b-t, b-t+1, \ldots, t\}\} \), it is its smallest element. \[\square\]

Denoting as usual by \( \Phi^s \) the \( s \)-th iterate of \( \Phi \), we state the following theorem which is a particular case of the theorem on pages 72–73 of [1] about the smallest elements in certain subintervals of \( \Gamma\{\{0,1, \ldots, b\}\} \), and of the definition of \( q \)-mirror sequences given in [1] Section II, p. 67] (here \( q = 2 \)).

**Theorem 1 ([1])** Define \( P := (t (b-t))^\infty = (t (b-t) \ t (b-t) \ t \ldots) \).

The smallest nonperiodic sequence in the set \( \Gamma\{\{b-t, b-t+1, \ldots, t\}\} \) (i.e., the smallest element of \( \Gamma_{\text{stric}}(\{b-t, b-t+1, \ldots, t\}) \)) is the sequence \( M \) defined by

\[
M := \lim_{s \to \infty} \Phi^s(P),
\]

that actually takes the (not necessarily distinct) values \( b-t, b-t+1, t-1, t \). Furthermore, this sequence \( M = (m_n)_{n \geq 0} = t \ b-t+1 \ b-t \ t \ b-t \ t \ldots \) can be recursively defined by

\[
\forall k \geq 0, \ m_{2k+1} = t, \\
\forall k \geq 0, \ m_{2k+1+1} = b + 1 - t, \\
\forall k \geq 0, \forall j \in [0, 2^{k+1} - 2], \ m_{2k+1+j} = m_j.
\]

It was proven in [1] that the sequence \( \lim_{s \to \infty} \Phi^s((t (b-t))^\infty) \) is 2-automatic (for more about automatic sequences, see [7]). The second author noted that this sequence is actually a fixed point of a uniform morphism of level \( 2 \) and proved the uniform law of large numbers ([8, theorems 3.1, 3.2]).
least equal to 4, i.e., $2t \geq b + 3$. (Recall that we always have $t \geq b - t$, i.e., $2t \geq b$.) More precisely we have Theorem 2 below, where the Thue-Morse sequence pops up, as in [1] and in [19], but also as in [2] and [18]. Before stating this theorem we give a definition.

**Definition 4** The “universal” morphism $\Theta$ is defined on $\{e_0, e_1, e_2, e_3\}$ by

$$
\Theta(e_3) := e_3 e_1, \quad \Theta(e_2) := e_3 e_0, \quad \Theta(e_1) := e_0 e_3, \quad \Theta(e_0) := e_0 e_2.
$$

Note that this morphism has an infinite fixed point beginning in $e_3$

$$
\Theta^\infty(e_3) = \lim_{k \to \infty} \Theta^k(e_3) = e_3 e_1 e_0 e_3 e_0 e_2 e_3 e_1 e_0 e_2 \ldots.
$$

**Theorem 2** Let $(\varepsilon_n)_{n \geq 0}$ be the Thue-Morse sequence, defined by $\varepsilon_0 = 0$ and for all $k \geq 0$, $\varepsilon_{2k} = \varepsilon_k$ and $\varepsilon_{2k+1} = 1 - \varepsilon_k$. Then the smallest nonperiodic sequence $M = (m_n)_{n \geq 0}$ belonging to $\Gamma(\{b - t, b - t + 1, \ldots, t\})$ satisfies

$$
\forall n \geq 0, \quad m_n = \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1.
$$

Using the morphism $\Theta$ introduced in Definition 4 above we thus have

- if $2t \geq b + 3$, then the sequence $M$ is the fixed point beginning in $t$ of the morphism deduced from $\Theta$ by renaming $e_0, e_1, e_2, e_3$ respectively $b - t, b - t + 1, t - 1, t$ (note that the condition $2t \geq b + 3$ implies that these four numbers are distinct);

- if $2t = b + 2$ (thus $b - t + 1 = t - 1$), then the sequence $M$ is the pointwise image of the fixed point beginning in $e_3$ of the morphism $\Theta$ by the map $g$ defined by $g(e_3) := t$, $g(e_2) = g(e_1) := t - 1$, $g(e_0) := b - t$;

- if $2t = b + 1$ (thus $b - t = t - 1$ and $b - t + 1 = t$), then the sequence $M$ is the pointwise image of the fixed point beginning in $e_3$ of the morphism $\Theta$ by the map $h$ defined by $h(e_3) = h(e_1) := t$, $h(e_2) = h(e_0) := t - 1$.

**Proof.** Let us first prove that the sequence $M = (m_n)_{n \geq 0}$ is equal to the sequence $(u_n)_{n \geq 0}$, where $u_n := \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1$. It suffices to prove that the sequence $(u_n)_{n \geq 0}$ satisfies the recursive relations defining $(m_n)_{n \geq 0}$ that are given in Theorem 1. Recall that the sequence $(\varepsilon_n)_{n \geq 0}$ has the property that $\varepsilon_n$ is equal to the parity of the sum of the binary digits of the integer $n$ (see [6] for example). Hence, for all $k \geq 0$, $\varepsilon_{2k+1} = 1$, and $\varepsilon_{2k+1} = \varepsilon_{2k+1} = 1$. This implies that for all $k \geq 0$, $u_{2k-1} = t$ and $u_{2k+1-1} = b - t + 1$. Furthermore, for all $k \geq 0$, and for all $j \in [0, 2^{k+1} - 2]$, we have $\varepsilon_{2k+1+j} = 1 - \varepsilon_j$ and $\varepsilon_{2k+1+j+1} = 1 - \varepsilon_{j+1}$. Hence
To show how the “universal” morphism $\Theta$ enters the picture, we study the sequence $(v_n)_{n \geq 0}$ with values in $\{0,1\}^2$ defined by: for all $n \geq 0$, $v_n := (\varepsilon_n, \varepsilon_{n+1})$. Since we have, for all $n \geq 0$, $v_{2n} = (\varepsilon_n, 1-\varepsilon_n)$ and $v_{2n+1} = (1-\varepsilon_n, \varepsilon_{n+1})$, we clearly have

- if $v_n = (0,0)$, then $v_{2n} = (0,1)$ and $v_{2n+1} = (1,0)$,
- if $v_n = (0,1)$, then $v_{2n} = (0,1)$ and $v_{2n+1} = (1,1)$,
- if $v_n = (1,0)$, then $v_{2n} = (1,0)$ and $v_{2n+1} = (0,0)$,
- if $v_n = (1,1)$, then $v_{2n} = (1,0)$ and $v_{2n+1} = (0,1)$.

This exactly means that the sequence $(v_n)_{n \geq 0}$ is the fixed point beginning in $(0,1)$ of the 2-morphism

$$
(0,0) \rightarrow (0,1)(1,0), \\
(0,1) \rightarrow (0,1)(1,1), \\
(1,0) \rightarrow (1,0)(0,0), \\
(1,1) \rightarrow (1,0)(0,1).
$$

We may define $e_0 := (1,0)$, $e_1 := (1,1)$, $e_2 := (0,0)$, $e_3 := (0,1)$. Then the above morphism can be written

$$
e_3 \rightarrow e_3 e_1, \ e_2 \rightarrow e_3 e_0, \ e_1 \rightarrow e_0 e_3, \ e_0 \rightarrow e_0 e_2$$

which is the morphism $\Theta$. The above construction shows that the sequence $(v_n)_{n \geq 0}$ is a fixed point of $\Theta$.

Now, define the map $\omega$ on $\{0,1\}^2$ by

$$
\omega((x,y)) := y - (2t - b - 1)x + t - 1.
$$

We have $\omega(v_n) = m_n$ for all $n \geq 0$. Thus

- if $2t \geq b + 3$, the sequence $(m_n)_{n \geq 0}$ takes exactly four distinct values, namely $b-t, b-t+1, t-1, t$. This implies that $(m_n)_{n \geq 0}$ is the fixed point beginning in $t$ of the morphism obtained from $\Theta$ by renaming the letters, i.e., $e_3 \rightarrow t$, $e_2 \rightarrow (t-1)$, $e_1 \rightarrow (b-t+1)$, $e_0 \rightarrow (b-t)$. The morphism can thus be written $t \rightarrow t (b-t+1), (t-1) \rightarrow t (b-t), (b-t+1) \rightarrow (b-t) t, (b-t) \rightarrow (b-t) (t-1)$;

- if $2t = b + 2$ (resp. $2t = b + 1$) the sequence $(m_n)_{n \geq 0}$ takes exactly three (resp. two) values, namely $b-t, t-1, t$ (resp. $t-1, t$). It is still the pointwise image by $\Theta$ of the sequence $(v_n)_{n \geq 0}$. Renaming $\Theta$ as $g$ (resp. $h$) as in the statement of Theorem 2 only takes into account that the integers $b-t, b-t+1, t-1, t$ are not distinct. $\square$
Remark 6 The reason for the choice of indexes for $e_3, e_2, e_1, e_0$ is that the order of indexes is the same as the natural order on the integers $t, t - 1, b - t + 1, b - t$ to which they correspond when $2t \geq b + 3$. In particular if $b = t = 3$, the morphism reads: $3 \rightarrow 31, 2 \rightarrow 30, 1 \rightarrow 03, 0 \rightarrow 02$. Interestingly enough, though not surprisingly, this morphism also occurs (up to renaming once more the letters) in the study of infinite square-free sequences on a 3-letter alphabet. Namely, in the paper [9], Berstel proves that the square-free Istrail sequence [16], originally defined (with no mention of the Thue-Morse sequence) as the fixed point of the (non-uniform) morphism $0 \rightarrow 12, 1 \rightarrow 102, 2 \rightarrow 0$, is actually the pointwise image of the fixed point beginning in 1 of a 2-morphism $\Theta'$ on the 4-letter alphabet $\{0, 1, 2, 3\}$ by the map $0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 0$. The morphism $\Theta'$ is given by

$$
\Theta'(0) = 12, \quad \Theta'(1) = 13, \quad \Theta'(2) = 20, \quad \Theta'(3) = 21.
$$

The reader will note immediately that $\Theta'$ is another avatar of $\Theta$ obtained by renaming letters as follows: $0 \rightarrow 2, 1 \rightarrow 3, 2 \rightarrow 0, 3 \rightarrow 1$. This, in particular, shows that the sequence $(m_n)_{n \geq 0}$, in the case where $2t = b + 2$, is the fixed point of the non-uniform morphism $t \rightarrow t (t - 1) (b - t), (t - 1) \rightarrow t (b - t), (b - t) \rightarrow (t - 1)$, i.e., an avatar of Istrail’s square-free sequence. Furthermore it results from [9] that this sequence on three letters cannot be the fixed point of a uniform morphism. A last remark is that the square-free Braunholtz sequence on three letters given in [10] (see also [9, p. 18-07]) is exactly our sequence $(m_n)_{n \geq 0}$ when $t = b = 2$, i.e., the sequence $2 1 0 2 0 1 2 0 1 2 0 \ldots$

3 Small admissible sequences and small univoque numbers with given integer part

3.1 Small admissible sequences with values in the set $\{0, 1, \ldots, b\}$

In [19] the authors are interested in the smallest admissible sequence with values in the set $\{0, 1, \ldots, b\}$, where $b$ is an integer $\geq 1$. They prove in particular the following result, which is an immediate corollary of our Theorem [2].

Corollary 1 (Theorems 4.3 and 5.1 of [19]) Let $b$ be an integer $\geq 1$. The smallest admissible sequence with values in $\{0, 1, \ldots, b\}$ is the sequence $(z + \varepsilon_{n+1})_{n \geq 0}$ if $b = 2z + 1$, and $(z + \varepsilon_{n+1} - \varepsilon_{n})_{n \geq 0}$ if $b = 2z$. 
Proof. Let \( A = (a_n)_{n \geq 0} \) be the smallest (non-constant) admissible sequence with values in \( \{0, 1, \ldots, b\} \). Since \( A > \overline{A} \), we must have \( a_0 \geq \overline{a_0} = b - a_0 \).

Thus, if \( b = 2z + 1 \) we have \( a_0 \geq z + 1 \). We also have, for all \( i \geq 0 \), \( \overline{a_0} \leq a_i \leq a_0 \). Now the smallest element of the set \( \Gamma(\{b - z - 1, b - z, \ldots, z - 1, z + 1\}) \) is the smallest admissible sequence on \( \{0, 1, \ldots, b\} \) that begins in \( z + 1 \). Hence this is the smallest admissible sequence with values in \( \{0, 1, \ldots, b\} \). Theorem 2 gives that this sequence is \((m_n)_{n \geq 0}\) with, for all \( n \geq 0 \), \( m_n = \varepsilon_{n+1} + z \).

If \( b = 2z \), we have \( a_0 \geq z \). But if \( a_0 = z \), then \( \overline{a_0} = z \), and the conditions of admissibility implies that \( a_n = z \) for all \( n \geq 0 \) and \((a_n)_{n \geq 0}\) would be the constant sequence \( (z z z \ldots) \). Hence we must have \( a_0 \geq z + 1 \). Now the smallest element of the set \( \Gamma(\{b - z - 1, b - z, \ldots, z - 1, z + 1\}) \) is the smallest admissible sequence on \( \{0, 1, \ldots, b\} \) that begins in \( z + 1 \). Hence this is the smallest admissible sequence with values in \( \{0, 1, \ldots, b\} \). Theorem 2 gives that this sequence is \((m_n)_{n \geq 0}\) with, for all \( n \geq 0 \), \( m_n = \varepsilon_{n+1} + z \). \( \square \)

3.2 Small univoque numbers with given integer part

We are interested here in the univoque numbers \( \lambda \) in an interval \((b, b+1]\) with \( b \) a positive integer. This set was studied in [17], where it was proven of Lebesgue measure 0. Since \( 1 = \sum_{j \geq 0} a_j \lambda^{-(j+1)}, \lambda \in (b, b+1] \) and \( a_0 \leq b \), the fact that the expansion of 1 is unique, hence equal to the greedy expansion, implies that \( a_0 = b \). In other words, we study the admissible sequences with values in \( \{0, 1, \ldots, b\} \) that begin in \( b \), i.e., the set \( \Gamma_{\text{strict}}(\{0, 1, \ldots, b\}) \).

We prove here, as a corollary of Theorem 2, that, for any positive integer \( b \), there exists a smallest univoque number belonging to \((b, b+1]\). This result was obtained in [14] (see the penultimate remark in that paper); it generalizes the result obtained for \( b = 1 \) in [18].

Corollary 2 For any positive integer \( b \), there exists a smallest univoque number in the interval \((b, b+1]\). This number is the solution of the equation \( 1 = \sum_{n \geq 0} d_n \lambda^{-n-1}, \) where the sequence \((d_n)_{n \geq 0}\) is given by, for all \( n \geq 0 \), \( d_n := \varepsilon_{n+1} - (b - 1)\varepsilon_n + b - 1 \).

Proof. It suffices to apply Theorem 2 with \( t = b \). \( \square \)

4 Transcendence results

We prove here, mimicking the proof given in [3], that numbers such that the expansion of 1 is given by the sequence \((m_n)_{n \geq 0}\) are transcendental. This
Theorem 3 Let $b$ be an integer $\geq 1$ and $t \in [0, b]$ be an integer such that
$2t \geq b + 1$. Define the sequence $(m_n)_{n \geq 0}$ as in Theorem 2 by, for all $n \geq 0$, $m_n := \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1$, thus the sequence $(m_n)_{n \geq 0}$ begins with
t $b - t + 1$ $b - t$ $t$ $b - t$ $t - 1$ ... Then the number $\lambda$ belonging to
$(1, b + 1)$ defined by
$1 = \sum_{n \geq 0} m_n \lambda^{-n-1}$ is transcendental.

Proof. Define the $\pm 1$ Thue-Morse sequence $(r_n)$ by $r_n := (-1)^{\varepsilon_n}$. We
clearly have $r_n = 1 - 2\varepsilon_n$ (recall that $\varepsilon_n$ is 0 or 1). It is also immediate that
the function $F$ defined for the complex numbers $X$ such that $|X| < 1$ by
$F(X) = \sum_{n \geq 0} r_n X^n$ satisfies $F(X) = \prod_{k \geq 0} (1 - X^{2k})$ (see, e.g., [6]). Since
$2m_n = 2\varepsilon_{n+1} - 2(2t - b - 1)\varepsilon_n + 2t - 2 = b - r_{n+1} + (2t - b - 1)r_n$
we have, for $|X| < 1$,

$$2X \sum_{n \geq 0} m_n X^n = ((2t - b - 1)X - 1)F(X) + 1 + \frac{bX}{1 - X}.$$  

Taking $X = 1/\lambda$ where $1 = \sum_{n \geq 0} m_n \lambda^{-n-1}$, we get the equation

$$2 = ((2t - b - 1)\lambda^{-1} - 1)F(1/\lambda) + 1 + \frac{b}{\lambda - 1}.$$  

Now, if $\lambda$ were algebraic, then this equation shows that $F(1/\lambda)$ would be an
algebraic number. But, since $1/\lambda$ would be an algebraic number in $(0, 1)$,
the quantity $F(1/\lambda)$ would be transcendental from a result of Mahler [20],
giving a contradiction.  $\square$

Remark 7 In particular the $\{0, 1, \ldots, b\}$-univoque number corresponding
to the smallest admissible sequence with values in $\{0, 1, \ldots, b\}$ is transcendental,
as proved in [19] (Theorems 4.3 and 5.9). Also the smallest univoque
number belonging to $(b, b + 1)$ is transcendental.

5 Conclusion

There are many papers dealing with univoque numbers. We will just men-
tion here the study of univoque Pisot numbers. The authors together with
K. G. Hare determined in [5] the smallest univoque Pisot number, which
happens to have algebraic degree 14. Note that the number corresponding
to the sequence of Proposition 2 is the larger real root of the polynomial
$X^2 - tX - (b - t + 1)$, hence a Pisot number (which is unitary if $t = b$).
Also note that for any $b \geq 2$, the real number $\beta$ such that the $\beta$-expansion
of 1 begins with $b$ $b - 1$ $b - 2$ $\ldots$ $1$ is transcendental.
It would be interesting to determine the smallest univoque Pisot number belonging to \((b, b + 1)\): the case \(b = 1\) was addressed in [3], but the proof uses heavily the fine structure of Pisot numbers in the interval \((1, 2)\) (see [8, 21, 22]). A similar study of Pisot numbers in \((b, b + 1)\) would certainly help.

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