MOTIVES ASSOCIATED TO SUMS OF GRAPHS

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1. Introduction

In quantum field theory, the path integral is interpreted perturbatively as a sum indexed by graphs. The coefficient (Feynman amplitude) associated to a graph $\Gamma$ is a period associated to the motive given by the complement of a certain hypersurface $X_{\Gamma}$ in projective space. Based on considerable numerical evidence, Broadhurst and Kreimer suggested [4] that the Feynman amplitudes should be sums of multizeta numbers. On the other hand, Belkale and Brosnan [2] showed that the motives of the $X_{\Gamma}$ were not in general mixed Tate.

A recent paper of Aluffi and Marcolli [1] studied the images $[X_{\Gamma}]$ of graph hypersurfaces in the Grothendieck ring $K_0(Var_k)$ of varieties over a field $k$. Let $\mathbb{Z}[\mathbb{A}^1_k] \subset K_0(Var_k)$ be the subring generated by $1 = [\text{Spec } k]$ and $[\mathbb{A}^1_k]$. It follows from [2] that $[X_{\Gamma}] \notin \mathbb{Z}[\mathbb{A}^1_k]$ for many graphs $\Gamma$.

Let $n \geq 3$ be an integer. In this note we consider a sum $S_n \in K_0(Var_k)$ of $[X_{\Gamma}]$ over all connected graphs $\Gamma$ with $n$ vertices, no multiple edges, and no tadpoles (edges with just one vertex). (There are some subtleties here. Each graph $\Gamma$ appears with multiplicity $n!/|\text{Aut}(\Gamma)|$. For a precise definition of $S_n$ see (5.1) below.) Our main result is

**Theorem 1.1.** $S_n \in \mathbb{Z}[\mathbb{A}^1_k]$.

For applications to physics, one would like a formula for sums over all graphs with a given loop order. I do not know if such a formula could be proven by these methods.

Dirk Kreimer explained to me the physical interest in considering sums of graph motives, and I learned about $K_0(Var_k)$ from correspondence with H. Esnault. Finally, the recently paper of Aluffi and Marcolli [1] provides a nice exposition of the general program.

2. Basic Definitions

Let $E$ be a finite set, and let

$$0 \rightarrow H \rightarrow \mathbb{Q}^E \rightarrow W \rightarrow 0; \quad 0 \rightarrow W^\vee \rightarrow \mathbb{Q}^E \rightarrow H^\vee \rightarrow 0$$

(2.1)
be dual exact sequences of vector spaces. For \( e \in E \), let \( e^\vee : \mathbb{Q}^E \to \mathbb{Q} \) be the dual functional, and let \((e^\vee)^2\) be the square, viewed as a quadratic function. By restriction, we can view this as a quadratic function either on \( H \) or on \( W^\vee \). Choosing bases, we get symmetric matrices \( M_e \) and \( N_e \). Let \( A_e, e \in E \) be variables, and consider the homogeneous polynomials

\[
\Psi(A) = \det(\sum A_e M_e); \quad \Psi^\vee(A) = \det(\sum A_e N_e).
\]

**Lemma 2.1.** \( \Psi(\ldots A_e, \ldots) = c \prod_{e \in E} A_e \Psi^\vee(\ldots A_e^{-1}, \ldots) \), where \( c \in k^\times \).

**Proof.** This is proposition 1.6 in [3]. \( \square \)

Let \( \Gamma \) be a graph. Write \( E, V \) for the edges and vertices of \( \Gamma \). We have an exact sequence

\[
0 \to H_1(\Gamma, \mathbb{Q}) \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^V \to H_0(\Gamma, \mathbb{Q}) \to 0.
\]

We take \( H = H_1(\Gamma) \) and \( W = \text{Image}(\partial) \) in (2.1). The resulting polynomials \( \Psi = \Psi_\Gamma, \quad \Psi^\vee = \Psi^\vee_\Gamma \) as in (2.2) are given by [3]

\[
\Psi_\Gamma = \sum_{t \in T} \prod_{e \notin t} A_e; \quad \Psi^\vee_\Gamma = \sum_{t \in T} \prod_{e \in t} A_e.
\]

Here \( T \) is the set of spanning trees in \( \Gamma \).

**Lemma 2.2.** Let \( e \in \Gamma \) be an edge. Let \( \Gamma/e \) be the graph obtained from \( \Gamma \) by shrinking \( e \) to a point and identifying the two vertices. We do not consider \( \Gamma/e \) in the degenerate case when \( e \) is a loop, i.e. if the two vertices coincide. Let \( \Gamma - e \) be the graph obtained from \( \Gamma \) by cutting \( e \). We do not consider \( \Gamma - e \) in the degenerate case when cutting \( e \) disconnects \( \Gamma \) or leaves an isolated vertex. Then

\[
\Psi_{\Gamma/e} = \Psi_\Gamma|_{A_e=0}; \quad \Psi_{\Gamma-e} = \frac{\partial}{\partial A_e} \Psi_\Gamma.
\]

\[
\Psi^\vee_{\Gamma/e} = \frac{\partial}{\partial A_e} \Psi^\vee_\Gamma; \quad \Psi^\vee_{\Gamma-e} = \Psi^\vee_\Gamma|_{A_e=0}.
\]

(In the degenerate cases, the polynomials on the right in (2.5) and (2.6) are zero.)

**Proof.** The formulas in (2.5) are standard [3]. The formulas (2.6) follow easily using lemma 2.1. (In the case of graphs, the constant \( c \) in the lemma is 1.) \( \square \)

More generally, we can consider strings of edges \( e_1, \ldots, e_p \in \Gamma \). If at every stage we have a nondegenerate situation we can conclude inductively

\[
\Psi^\vee_{\Gamma-e_1-\cdots-e_p} = \Psi^\vee_\Gamma|_{A_{e_1}=\cdots=A_{e_p}=0}
\]
In the degenerate situation, the polynomial on the right will vanish, i.e. $X_{\Gamma}$ will contain the linear space $A_{e_1} = \cdots = A_{e_p} = 0$.

For example, let $\Gamma = e_1 \cup e_2 \cup e_3$ be a triangle, with one loop and three vertices. We get the following polynomials

$$
\Psi_{\Gamma} = A_{e_1} + A_{e_2} + A_{e_3}; \quad \Psi^\vee_{\Gamma} = A_{e_1}A_{e_2} + A_{e_2}A_{e_3} + A_{e_1}A_{e_3}
$$

$$
\Psi_{\Gamma-e_i} = 1; \quad \Psi^\vee_{\Gamma-e_i} = A_{e_j}A_{e_k} = \Psi^\vee_{\Gamma} |_{A_{e_i} = 0}
$$

(2.8) \hspace{1cm} (2.9)

The sets $\{e_i, e_j\}$ are degenerate because cutting two edges will leave an isolated vertex.

3. THE GROTHENDIECK GROUP AND DUALITY

Recall $K_0(Var_k)$ is the free abelian group on generators isomorphism classes $[X]$ of quasi-projective $k$-varieties and relations

$$
[X] = [U] + [Y]; \quad U \xrightarrow{open} X, \quad Y = X - U.
$$

(3.1)

In fact, $K_0(Var_k)$ is a commutative ring with multiplication given by cartesian product of $k$-varieties. Let $\mathbb{Z}[A_k^1] \subset K_0(Var_k)$ be the subring generated by $1 = [\text{Spec } k]$ and $[A^1_k]$. Let $\mathbb{P}_{\Gamma}$ be the projective space with homogeneous coordinates $A_e, e \in E$. We write $X_{\Gamma} : \Psi_{\Gamma} = 0, X_{\Gamma}^\vee : \Psi^\vee_{\Gamma} = 0$ for the corresponding hypersurfaces in $\mathbb{P}_{\Gamma}$. We are interested in the classes $[X_{\Gamma}], [X_{\Gamma}^\vee] \in K_0(Var_k)$.

Let $\Delta : \prod_{e \in E} A_e = 0$ in $\mathbb{P}_{\Gamma}$, and let $T = T_{\Gamma} = \mathbb{P}_{\Gamma} - \Delta$ be the torus. Define

$$
X_{0\Gamma} = X_{\Gamma} \cap T_{\Gamma}; \quad X_{0\Gamma}^\vee = X_{\Gamma}^\vee \cap T_{\Gamma}.
$$

(3.2)

Lemma 2.1 translates into an isomorphism (Cremona transformation)

$$
X_{\Gamma}^0 \cong X_{0\Gamma}^\vee.
$$

(3.3)

(In fact, this is valid more generally for the setup of (2.1) and (2.2).)

We can stratify $X_{\Gamma}^\vee$ by intersecting with the toric stratification of $\mathbb{P}_{\Gamma}$ and write

$$
[X_{\Gamma}^\vee] = \sum_{\{e_1, \ldots, e_p\} \subset E} [(X_{\Gamma}^\vee \cap \{A_{e_1} = \cdots = A_{e_p} = 0\})^0] \in K_0(Var_k)
$$

(3.4)

where the sum is over all subsets of $E$, and superscript $0$ means the open torus orbit where $A_e \neq 0, e \not\in \{e_1, \ldots, e_p\}$. We call a subset $\{e_1, \ldots, e_p\} \subset E$ degenerate if $\{A_{e_1} = \cdots = A_{e_p} = 0\} \subset X_{\Gamma}^\vee$. Since $[G_m] = [A^1] - [pt] \in K_0(Var_k)$ we can rewrite (3.4)

$$
[X_{\Gamma}^\vee] = \sum_{\{e_1, \ldots, e_p\} \subset E \text{ nondegenerate}} [(X_{\Gamma}^\vee \cap \{A_{e_1} = \cdots = A_{e_p} = 0\})^0] + t
$$

(3.5)
where \( t \in \mathbb{Z}[\mathbb{A}^1] \subset K_0(Var_k) \). Now using (2.7) and (3.3) we conclude

\[
[X^\vee_{\Gamma}] = \sum_{\{e_1, \ldots, e_p\} \subset E_{\text{nondeg}}} [(X^0_{\Gamma}-e_1, \ldots, e_p)] + t.
\]

4. Complete Graphs

Let \( \Gamma_n \) be the complete graph with \( n \geq 3 \) vertices. Vertices of \( \Gamma_n \) are written \((j), 1 \leq j \leq n\), and edges \( e_{ij} \) with \( 1 \leq i < j \leq n\). We have \( \partial e_{ij} = (j) - (i) \).

**Proposition 4.1.** We have \([X^\vee_{\Gamma_n}] \in \mathbb{Z}[\mathbb{A}^1_k]\).

**Proof.** Let \( \mathbb{Q}^{n,0} \subset \mathbb{Q}^n \) be row vectors with entries which sum to 0. We have

\[
0 \to H_1(\Gamma_n) \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^{n,0} \to 0.
\]

In a natural way, \((\mathbb{Q}^{n,0})^\vee = \mathbb{Q}^n/\mathbb{Q}\). Take as basis of \( \mathbb{Q}^n/\mathbb{Q} \) the elements \((1), \ldots, (n-1)\). As usual, we interpret the \((e_{ij})^2\) as quadratic functions on \( \mathbb{Q}^n/\mathbb{Q} \). We write \( N_e \) for the corresponding symmetric matrix.

**Lemma 4.2.** The \( N_{e_{ij}} \) form a basis for the space of all \((n-1) \times (n-1)\) symmetric matrices.

**Proof of lemma.** The dual map \( \mathbb{Q}^n/\mathbb{Q} \to \mathbb{Q}^E \) carries

\[
(k) \mapsto \sum_{\mu>k} -e_{k\mu} + \sum_{\nu<k} e_{\nu k}; \quad k \leq n-1.
\]

We have

\[
(e_{ij})^2(\sum_{k=1}^{n-1} a_k \cdot (k)) = \begin{cases} a_i^2 - 2a_i a_j + a_j^2 & i < j < n \\ a_i^2 & j = n. \end{cases}
\]

It follows that if \( j < n \), \( N_{e_{ij}} \) has \(-1\) in positions \((ij)\) and \((ji)\) and \(+1\) in positions \((ii), (jj)\) (resp. \( N_{e_{in}} \) has \(1\) in position \((ii)\) and zeroes elsewhere). These form a basis for the symmetric \((n-1) \times (n-1)\) matrices. \( \square \)

It follows from the lemma that \( X^\vee_{\Gamma_n} \) is identified with the projectivized space of \((n-1) \times (n-1)\) matrices of rank \( \leq n-2 \). In order to compute the class in the Grothendieck group we detour momentarily into classical algebraic geometry. For a finite dimensional \( k \)-vector space \( U \), let \( \mathbb{P}(U) \) be the variety whose \( k \)-points are the lines in \( U \). For a \( k \)-algebra \( R \), the \( R \)-points \( \text{Spec} \ R \to \mathbb{P}(U) \) are given by pairs \((L, \phi)\) where \( L \) on \( \text{Spec} \ R \) is a line bundle and \( \phi : L \to U \otimes_k R \) is a locally split embedding.
Suppose now \( U = \text{Hom}(V, W) \). We can stratify \( \mathbb{P}(\text{Hom}(V, W)) = \bigsqcup_{p>0} \mathbb{P}(\text{Hom}(V, W))^p \) according to the rank of the homomorphism. Looking at determinants of minors makes it clear that \( \mathbb{P}(\text{Hom}(V, W))^p \) is closed. Let \( R \) be a local ring which is a localization of a \( k \)-algebra of finite type, and let \( a \) be an \( R \)-point of \( \mathbb{P}(\text{Hom}(V, W))^p \). Choosing a lifting \( b \) of the projective point \( a \), we have

\[
0 \to \ker(b) \to V \otimes R \xrightarrow{b} W \otimes R \to \text{coker}(b) \to 0,
\]

and \( \text{coker}(b) \) is a finitely generated \( R \)-module of constant rank \( \dim W - p \) which is therefore necessarily free.

Let \( Gr(\dim V - p, V) \) and \( Gr(p, W) \) denote the Grassmann varieties of subspaces of the indicated dimension in \( V \) (resp. \( W \)). On \( Gr(\dim V - p, V) \times Gr(p, W) \) we have rank \( p \) bundles \( E, F \) given respectively by the pullbacks of the universal quotient on \( Gr(\dim V - p, V) \) and the universal subbundle on \( Gr(p, W) \). It follows from the above discussion that

\[
\mathbb{P}(\text{Hom}(V, W))^p \cong \mathbb{P}(\text{Isom}(E, F)) \subset \mathbb{P}(\text{Hom}(E, F)).
\]

Suppose now that \( W = V^\vee \). Write \( \langle \ , \ \rangle : V \otimes V^\vee \to k \) for the canonical bilinear form. We can identify \( \text{Hom}(V, V^\vee) \) with bilinear forms on \( V \)

\[
\rho : V \to V^\vee \leftrightarrow (v_1, v_2) \mapsto \langle v_1, \rho(v_2) \rangle.
\]

Let \( \text{SHom}(V, V^\vee) \subset \text{Hom}(V, V^\vee) \) be the subspace of \( \rho \) such that the corresponding bilinear form on \( V \) is symmetric. Equivalently, \( \text{Hom}(V, V^\vee) = V^{\vee \otimes 2} \) and \( \text{SHom}(V, V^\vee) = \text{Sym}^2(V^\vee) \subset V^{\vee \otimes 2} \).

For \( \rho \) symmetric as above, one sees easily that \( \rho(V) = \ker(V)^\perp \) so there is a factorization

\[
V \to V/\ker(\rho) \xrightarrow{\mathbb{P}} (V/\ker(\rho))^\vee = \ker(\rho)^\perp \hookrightarrow V^\vee.
\]

The isomorphism in (4.7) is also symmetric.

Fix an identification \( V = k^n \) and hence \( V = V^\vee \). A symmetric map is then given by a symmetric \( n \times n \) matrix. On \( Gr(n-p, n) \) we have the universal rank \( p \) quotient \( Q = k^n \otimes \mathcal{O}_{Gr}/K \), and also the rank \( p \) perpendicular space \( K^{\perp} \) to the universal subbundle \( K \). Note \( K^{\perp} \cong Q^\vee \).

It follows that

\[
\mathbb{P}(\text{SHom}(k^n, k^n))^p \cong \mathbb{P}(\text{SHom}(Q, Q^\vee))^p \subset \mathbb{P}(\text{SHom}(Q, Q^\vee)).
\]

This is a fibre bundle over \( Gr(n-p, n) \) with fibre \( \mathbb{P}(\text{Hom}(k^p, k^p))^p \), the projectivized space of symmetric \( p \times p \) invertible matrices.
We can now compute \([X^\vee_{\Gamma_n}]\) as follows. Write \(c(n, p) = [\mathbb{P}(\mathsf{SHom}(k^n, k^n))^p]\).

We have the following relations:

\[
(4.9) \quad c(n, 1) = [\mathbb{P}^{n-1}]; \quad \sum_{p=1}^{n} c(n, p) = [\mathbb{P}^{(n+1)/2}];
\]

\[
(4.10) \quad c(n, p) = [\mathsf{Gr}(n - p, n)] \cdot c(p, p)
\]

\[
(4.11) \quad [X^\vee_{\Gamma_n}] = \sum_{p=1}^{n-2} c(n - 1, p)
\]

Here (4.10) follows from (4.8). It is easy to see that these formulas lead to an expression for \([X^\vee_{\Gamma_n}]\) as a polynomial in the \([\mathbb{P}^N]\) and \([\mathsf{Gr}(n - p, n - 1, n - 1)]\) (though the precise form of the polynomial seems complicated). To finish the proof of the proposition, we have to show that \([\mathsf{Gr}(a, b)] \in \mathbb{Z}[\mathbb{A}_1^1]\). Fix a splitting \(k^b = k^{b-a} \oplus k^a\). Stratify \(\mathsf{Gr}(a, b) = \coprod_{p=0}^{d} \mathsf{Gr}(a, b)^p\) where

\[
(4.12) \quad \mathsf{Gr}(a, b)^p = \{ V \subset k^{b-a} \oplus k^a \mid \dim(V) = a, \ \text{Image}(V \to k^a) \ \text{has rank} \ p \} = \\
\{ \langle X, Y, f \rangle \mid X \subset k^{b-a}, \ Y \subset k^a, \ f : Y \to X \}
\]

where \(\dim X = a - p, \ \dim(Y) = p\). This is a fibration over \(\mathsf{Gr}(b - a - p, b - a) \times \mathsf{Gr}(p, a)\) with fibre \(\mathbb{A}_1^{p(b-a-p)}\). By induction, we may assume \([\mathsf{Gr}(b - a - p, b - a) \times \mathsf{Gr}(p, a)] \in \mathbb{Z}[\mathbb{A}_1^1]\). Since the class in the Grothendieck group of a Zariski locally trivial fibration is the class of the base times the class of the fibre, we conclude \([\mathsf{Gr}(a, b)^p] \in \mathbb{Z}[\mathbb{A}_1^1]\), completing the proof.

□

In fact, we will need somewhat more.

\textbf{Lemma 4.3.} Let \(\Gamma\) be a graph.

(i) Let \(e_0 \in \Gamma\) be an edge. Define \(\Gamma' = \Gamma \cup \varepsilon\), the graph obtained from \(\Gamma\) by adding an edge \(\varepsilon\) with \(\partial\varepsilon = \partial e_0\). Then \(X^\vee_{\Gamma'}\) is a cone over \(X^\vee_{\Gamma}\).

(ii) Define \(\Gamma' = \Gamma \cup \varepsilon\) where \(\varepsilon\) is a tadpole, i.e. \(\partial\varepsilon = 0\). Then \(X^\vee_{\Gamma'}\) is a cone over \(X^\vee_{\Gamma}\).

\textbf{Proof.} We prove (i). The proof of (ii) is similar and is left for the reader.

Let \(E, V\) be the edges and vertices of \(\Gamma\). We have a diagram

\[
\begin{array}{ccc}
\mathbb{Q}^E & \xrightarrow{\partial} & \mathbb{Q}^V \\
\downarrow & & \downarrow \\
\mathbb{Q}^E \oplus \mathbb{Q} \cdot \varepsilon & \xrightarrow{\partial} & \mathbb{Q}^V
\end{array}
\]

(4.13)
Dualizing and playing our usual game of interpreting edges as functionals on $\text{Image}(\partial)^{\vee} \cong \mathbb{Q}^{V}/\mathbb{Q}$, we see that $\varepsilon^{\vee} = e^{\vee}_{0}$. Fix a basis for $\mathbb{Q}^{V}/\mathbb{Q}$ so the $(e^{\vee})^{2}$ correspond to symmetric matrices $M_{e}$. We have

$$ (4.14) \quad X^{\vee}_{\Gamma} : \det\left(\sum_{E} A_{e}M_{e}\right) = 0; \quad X^{\vee}_{\Gamma} : \det(A_{e}M_{e} + \sum_{E} A_{e}M_{e}) = 0. $$

The second polynomial is obtained from the first by the substitution $A_{e_{0}} \mapsto \epsilon^{\vee}A_{e_{0}} + A_{\epsilon}$. Geometrically, this is a cone as claimed. □

Let $\Gamma_{N}$ be the complete graph on $N \geq 3$ vertices. Let $\Gamma \supset \Gamma_{N}$ be obtained by adding $r$ new edges (but no new vertices) to $\Gamma_{N}$.

**Proposition 4.4.** $[X^{\vee}_{\Gamma}] \in \mathbb{Z}[\mathbb{A}^{1}] \subset K_{0}(\text{Var}_{k})$.

**Proof.** Note that every pair of distinct vertices in $\Gamma_{N}$ are connected by an edge, so the $r$ new edges $e$ either duplicate existing edges or are tadpoles ($\partial e = 0$). It follows from lemma 4.3 that $X^{\vee}_{\Gamma}$ is an iterated cone over $X^{\vee}_{\Gamma_{N}}$. In the Grothendieck ring, the class of a cone is the sum of the vertex point with a product of the base times an affine space, so we conclude from proposition 4.1. □

5. The Main Theorem

Fix $n \geq 3$. Let $\Gamma_{n}$ be the complete graph on $n$ vertices. It has $\binom{n}{2}$ edges. Recall (lemma 2.2) a set $\{e_{1}, \ldots, e_{p}\} \subset \text{edge}(\Gamma_{n})$ is nondegenerate if cutting these edges (but leaving all vertices) does not disconnect $\Gamma_{n}$. (For the case $n = 3$ see (2.8) and (2.9).) Define

$$ (5.1) \quad S_{n} := \sum_{\{e_{1}, \ldots, e_{p}\} \text{ nondegenerate}} [X_{\Gamma_{n}-\{e_{1}, \ldots, e_{p}\}}] \in K_{0}(\text{Var}_{k}). $$

Let $\Gamma$ be a connected graph with $n$ vertices and no multiple edges or tadpoles. Let $\tilde{G} \subset \text{Sym}^{\text{vert}}(\Gamma)$ be the subgroup of the symmetric group on the vertices which acts on the set of edges. Then $[X_{\Gamma}]$ appears in $S_{n}$ with multiplicity $n!/|\tilde{G}|$.

**Theorem 5.1.** $S_{n} \in \mathbb{Z}[\mathbb{A}^{1}] \subset K_{0}(\text{Var}_{k})$.

**Proof.** It follows from (3.6) and proposition 4.1 that

$$ (5.2) \quad \sum_{\{e_{1}, \ldots, e_{p}\} \text{ nondegenerate}} [X_{\Gamma_{n}-\{e_{1}, \ldots, e_{p}\}}^{0}] \in \mathbb{Z}[\mathbb{A}^{1}_{k}]. $$

Write $\bar{e} = \{e_{1}, \ldots, e_{p}\}$ and let $\bar{f} = \{f_{1}, \ldots, f_{q}\}$ be another subset of edges. We will say the pair $\{\bar{e}, \bar{f}\}$ is nondegenerate if $\bar{e}$ is nondegenerate in the above sense, and if further $\bar{e} \cap \bar{f} = \emptyset$ and the edges of $\bar{f}$ do not
support a loop. For \( \{\vec{e}, \vec{f}\} \) nondegenerate, write \((\Gamma_n - \vec{e})/\vec{f}\) for the graph obtained from \(\Gamma_n\) by removing the edges in \(\vec{e}\) and then contracting the edges in \(\vec{f}\). If we fix a nondegenerate \(\vec{e}\), we have

\[
\sum_{\vec{f}} \left[ X^0_{(\Gamma_n - \vec{e})/\vec{f}} \right] + t = [X_{\Gamma_n - \vec{e}}].
\]

Here \(t \in \mathbb{Z}[A^1]\) accounts for the \(\vec{f}\) which support a loop. These give rise to degenerate edges in \(X_{\Gamma_n - \vec{e}}\) which are linear spaces and hence have classes in \(\mathbb{Z}[A^1]\). Summing now over both \(\vec{e}\) and \(\vec{f}\), we conclude

\[
S_n \equiv \sum_{\{\vec{e}, \vec{f}\} \text{ nondeg.}} \left[ X^0_{(\Gamma_n - \vec{e})/\vec{f}} \right] \mod \mathbb{Z}[A^1].
\]

Note that if \(\vec{e}, \vec{f}\) are disjoint and \(\vec{f}\) does not support a loop, then \(\vec{e}\) is nondegenerate in \(\Gamma_n\) if and only if it is nondegenerate in \(\Gamma_n/\vec{f}\). This means we can rewrite (5.4)

\[
S_n \equiv \sum_{\vec{f}} \sum_{\vec{e} \subset \Gamma_n/\vec{f} \text{ nondegen.}} [X^0_{(\Gamma_n - \vec{e})/\vec{f}}].
\]

Let \(\vec{f} = \{f_1, \ldots, f_q\}\) and assume it does not support a loop. Then \(\Gamma_n/\vec{f}\) has \(n - q\) vertices, and every pair of distinct vertices is connected by at least one edge. This means we may embed \(\Gamma_n - q \subset \Gamma_n/\vec{f}\) and think of \(\Gamma_n/\vec{f}\) as obtained from \(\Gamma_n - q\) by adding duplicate edges and tadpoles. We then apply proposition 4.4 to conclude that \(X^0_{\Gamma_n/\vec{f}} \in \mathbb{Z}[A^1_k]\). Now arguing as in (3.6) we conclude

\[
\sum_{\vec{e} \subset \Gamma_n/\vec{f} \text{ nondegen.}} [X^0_{(\Gamma_n/\vec{f}) - \vec{e}}] \in \mathbb{Z}[A^1_k]
\]

Finally, plugging into (5.5) we get \(S_n \in \mathbb{Z}[A^1]\) as claimed. \(\square\)
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