Refined Asymptotics for Multigraded Sums of Squares

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Abstract

To prove that a polynomial is nonnegative on $\mathbb{R}^n$ one can try to show that it is a sum of squares of polynomials. The latter problem is now known to be reducible to a semidefinite programming computation much faster than classical algebraic methods, thus enabling new speed-ups in algebraic optimization. However, exactly how often nonnegative polynomials are in fact sums of squares of polynomials remains an open problem. Blekherman was recently able to show that for degree $k$ polynomials in $n$ variables — with $k \geq 4$ fixed and $n \to \infty$ — those that are SOS occupy a vanishingly small fraction of those that are nonnegative.

With an eye toward the case of small $n$, we refine Blekherman’s bounds by incorporating the underlying Newton polytope, simultaneously sharpening some of his older bounds along the way. Our refined asymptotics show that certain Newton polytopes may lead to families of polynomials where efficient SDP can still be used for most inputs.

1 Introduction

Much recent work in optimization and algorithmic real algebraic geometry (see, e.g., [Las07] and the references therein) is based on the fact that deciding whether a polynomial is a sum of squares of polynomials (is SOS) can be done efficiently via semi-definite programming (SDP) [Par03]. (See [VB96] for a nice review of SDP and its applications.) In particular, for certain $n$-variate degree $k$ polynomials, it was shown in [PS03] that one could approximate their real minima within $n^{O(1)}$ arithmetic operations via SOS techniques and SDP. This is in sharp contrast to the $k^{O(n)}$ complexity bounds coming from the best known algorithms from real algebraic geometry [BPR06]. However, for an approach via SDP to be practical, one obviously needs to know how often nonnegative polynomials are in fact SOS.

In one variable, nonnegative polynomials are actually always SOS (see, e.g., [Rez00]). However, since the classical technique of Sturm-Habicht sequences is already known to have complexity near-linear in the degree [LM01], the potential complexity savings of SDP over Sturm-Habicht are not clear. Whether SDP can provide a significant gain in speed for larger $n$, for a large fraction of inputs, is thus an important question. Similarly, many algebraic algorithms lack provable speed-ups when the input polynomials are sparse or have structured

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Newton polytopes, and thus one should also ask if SDP can provides speed gains in these settings as well. (See [Roj02] for an introduction to quantitative results on root counting involving Newton polytopes.)

In two variables, one already begins to see nonnegative polynomials that are not SOS. For instance, Motzkin’s 1967 example

\[ p(x, y) := 1 + x^2y^2(x^2 + y^2 - 3) \]

is well known for being a nonnegative polynomial not expressible as a sum of squares of polynomials (see [Rez00] for an enlightening historical discussion). However, even for bivariate sextics, it is still unknown whether such examples are rare or bountiful.

Let \( \Sigma_{k,n} \) (resp. \( P_{k,n} \)) denote those homogeneous polynomials in \( \mathbb{R}[x_1, \ldots, x_n] \) that are SOS (resp. nonnegative on \( \mathbb{R}^n \)). The first major quantitative result on the proportion of nonnegative polynomials that are SOS was derived by Blekherman [Ble06]: observing that \( \Sigma_{k,n} \) and \( P_{k,n} \) can be identified with cones in \( \mathbb{R}^{\binom{k+n-1}{n-1}} \), and using a natural convex-geometric measure, he showed that the fraction of \( P_{2k,n} \) occupied by \( \Sigma_{2k,n} \) is \( O\left(\frac{16^k(2k)!}{k! \sqrt{nk(k+1)^2}}\right) \) [Ble06].

Theorems 4.1 & 6.1]. (It is easy to show that polynomials of odd degree always have real range \( \mathbb{R} \), so our restriction to even degree is natural.) We clarify the precise underlying measures in Section 1.1 following the statement of our main theorem below.

It is worth noting that Blekherman’s result still leaves open the possibility that \( \Sigma_{2k,n} \) nevertheless occupies a large fraction of \( P_{2k,n} \) when \( n \) is small. Indeed, one can easily check via Maple that Blekherman’s bounds imply that one needs \( n \geq 36, 238, 786, 561 \) in order for the fraction of nonnegative homogeneous n-variate quartics that are SOS to be provably less than 50%. Furthermore, precious little is known about the analogues of the cones \( P_{2k,n} \) and \( \Sigma_{2k,n} \) for sparse polynomials [Rez74, BHPR10]. So let us introduce multihomogeneous polynomials.

Fix a partition \( N := (n_1, \ldots, n_\ell) \) of \( n \), i.e., any such positive sequence satisfying \( n = n_1 + \cdots + n_\ell \). Fixing a sequence of positive integers \( K := (k_1, \ldots, k_\ell) \), we say that \( f \in \mathbb{R}[x_1, \ldots, x_n] \) is multihomogeneous of type \( (n_1, k_1; \ldots; n_\ell, k_\ell) \) if we can partition the variables \( x_i \) into groups of sizes \( n_1, \ldots, n_\ell \) so that \( f \) is homogeneous of degree \( k_i \) with respect to the \( i \)-th group. We also let \( P_{K,N} \) (resp. \( \Sigma_{K,N} \)) denote the set of all \( f \in \mathbb{R}[x_1, \ldots, x_n] \) that are multihomogeneous of type \( (n_1, k_1; \ldots; n_\ell, k_\ell) \) and nonnegative (resp. SOS).

**Theorem 1.1** In the convex-geometric sense described in Section 1.1 below, the fraction of \( P_{2K,N} \) occupied by \( \Sigma_{2K,N} \) is bounded above by \( 9e^2 \times \sqrt{24 \max \{n_i \log(2k_i + 1)\}} \prod_{i=1}^\ell \frac{16^{k_i}(2k_i)!}{k_i! n_i^2} \).

Our upper bound includes (and improves) the older bound we quote above as the special case \( \ell = 1 \). In particular, since \( P_{K,N} \) is a proper subset of \( P_{k_1 + \cdots + k_\ell, 1 + n_1 + \cdots + n_\ell - \ell} \) when \( \ell \geq 2 \), our upper bound gives new information on families of polynomials with Newton polytopes more general than scaled standard simplices. Theorem 1.1 follows directly from explicit lower and upper bounds we find on the volume of certain slices of the cones \( P_{2K,N} \) and \( \Sigma_{2K,N} \). (See Theorem 2.1 in Section 2) In particular, we can also derive explicit lower bounds for our fraction of interest.

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1Artin’s 1927 solution to Hilbert’s 17th Problem states, in essence, that nonnegative polynomials are always expressible as sums of squares of rational functions, but efficiently reducing the latter type of sum of squares to semidefinite programming is not yet clear.
Let us state clearly that while no current bounds (including our own) adequately describe classes of multigraded polynomial where $\Sigma_{2K,N}$ occupies a provably large fraction of $P_{2K,N}$, our results are at least a first step toward incorporating Newton polytopes and sparsity in the quantitative study of $P_{n,k}$ and $\Sigma_{n,k}$. In particular, we can at least point out new families where there are significantly more nonnegative polynomials than sums of squares.

**Corollary 1.2** Suppose $\ell$ and $k_1, \ldots, k_\ell$ are fixed, with $k_1 \geq 2$. Then the fraction\(^2\) of $P_{2K,N}$ occupied by $\Sigma_{2K,N}$ tends to 0 as $n_1 \to \infty$. ■

The corollary follows immediately upon observing that its hypotheses reduce the upper bound from Theorem 1.1 to $O\left(\sqrt{\max_i \{n_i\} \prod_{i=1}^{\ell} n_i^{1/2}}\right)$.

Let us now describe our underlying measures.

### 1.1 Comparing Cones of Polynomials

As before, we let $N:=(n_1, \ldots, n_\ell)$ denote a fixed positive $\ell$-tuple with $n_1 + \cdots + n_\ell = n$. Let $S^{t-1}$ denote the unit sphere in $\mathbb{R}^t$ and let $S^N$ denote the product of spheres $S^{n_1-1} \times \cdots \times S^{n_\ell-1}$. Note that by multihomogeneity, the values of $f$ on $S^N$ determine the values of $f$ on all of $\mathbb{R}^n$. So it will suffice to study the values of $f$ on $S^N$. We will also let $d\sigma$ denote the restriction of the natural Euclidean measure to $S^N$.

The following facts follow easily from basic convexity (see, e.g., [Bar02a]):

**Proposition 1.3** The dimension of the space of all multihomogeneous polynomials of type $(n_1, k_1; \ldots; n_\ell, k_\ell)$ is $D_{N,K}:=\prod_{i=1}^{\ell} \left(\frac{n_i + k_i - 1}{k_i}\right)$. In particular, $P_{N,K}$ and $\Sigma_{N,K}$ are full-dimensional convex cones in $\mathbb{R}^{D_{N,K}}$. ■

To compare $P_{N,2K}$ and $\Sigma_{N,2K}$ in a natural way, we will need two special hyperplane sections within $P_{N,2K}$, and a distinguished polynomial:

$$L_{N,2K} := \left\{ f \in P_{N,2K} \mid \int_{S^N} f d\sigma = 1 \right\}$$

$$M_{N,2K} := \left\{ f \in P_{N,2K} \mid \int_{S^N} f d\sigma = 0 \right\}$$

$$F := \prod_{i=1}^{\ell} (x_{i,1}^2 + \cdots + x_{i,n_i}^2)^{k_i},$$

where $\{x_{i,1}, \ldots, x_{i,n_i}\}$ is simply the $i$th block of our partition of the variables $x_1, \ldots, x_n$. We then define

$$P'_{N,2K} := P_{N,2K} \cap L_{N,2K}$$

$$\tilde{P}_{N,2K} := \left\{ f \in M_{N,2K} \mid f + F \in P'_{N,2K} \right\}$$

$$\Sigma'_{N,2K} := \Sigma_{N,2K} \cap L_{N,2K}$$

\(^2\)...in the convex-geometric sense described in Section 1.1 below...
\[ \widetilde{\Sigma}_{N,2K} := \{ f \in M_{N,2K} \mid f + F \in \Sigma'_{N,2K} \} \]

In particular, our underlying theory relies heavily on results for compact convex bodies containing the origin in their interiors. Since \( P_{N,K} \) and \( \Sigma_{N,K} \) are non-compact and have the origin has a vertex, the primed bodies allow us to compactify, and the tilded bodies allow a convenient translation to the origin.

We are now ready to rigourously define what we meant by the “fraction of \( P_{N,2K} \) occupied by \( \Sigma_{N,2K} \).”

**Definition 1.4** We define the relative volume of a convex body \( K \) in \( \mathbb{R}^D \) (with respect to the unit ball \( B \) in \( \mathbb{R}^D \), and the standard Euclidean volume \( \text{Vol}(\cdot) \) as
\[
\mu(K) := \left( \frac{\text{Vol}(K)}{\text{Vol}(B)} \right)^{1/D}.
\]
Recall also that \( \text{Vol}(B) = \pi^{D/2} \frac{\Gamma(1 + D/2)}{\Gamma(1/2)} \), where \( \Gamma(t) = (t-1)! \Gamma(t-1) \) for all \( t > 0 \) and \( \Gamma(1/2) = \sqrt{\pi} \).

In particular, in the introduction, Theorem 1.1, and Corollary 1.2, we were merely discussing the fraction \( \mu(\tilde{\Sigma}_{N,2K}) / \mu(\tilde{P}_{N,2K}) \).

## 2 Lower and Upper Bounds on Relative Volumes

Theorem 1.1 follows immediately from the following lower and upper bounds.

**Theorem 2.1** We have the following inequalities:
\[
\frac{1}{9e^2 \times \sqrt{\max_{i \in \{1, \ldots, \ell\}} n_i \log(2k_i + 1)}} \leq \mu(\tilde{P}_{N,2K}) \leq 4 \prod_{i=1}^{\ell} \sqrt{\frac{2k_i^2}{4k_i^2 + n_i - 2}},
\]
\[
\frac{1}{\sqrt{24} \prod_{i=1}^{\ell} \frac{k_i!}{4k_i \sqrt{(2k_i)!(\frac{n_i}{2} + k_i)k_i/2}}} \leq \mu(\tilde{\Sigma}_{N,2K}) \leq \sqrt{24} \prod_{i=1}^{\ell} \frac{16k_i^2(2k_i)!}{k_i!n_i^{k_i/2}}.
\]

In particular, Theorem 2.1 includes Theorems 4.1 and 6.1 of [Ble06] as the special case \( \ell = 1 \). Moreover, in the special case \( \ell = 1 \), our lower bounds are sharper than those in [Ble06 Theorems 4.1 & 6.1].

Let us now review some background needed to prove Theorem 2.1.

## 3 Background on Metrics and Convex Bodies

The key new ingredient in our bounds is to extend earlier metric results on convex cones by using of a natural action of a product of orthogonal groups.

Let \( SO(t) \) denote the group of orthogonal \( t \times t \) matrices with determinant 1, and let \( SO(N) := \prod_{i=1}^{\ell} SO(n_i) \). Noting that \( \langle f, g \rangle := \int_{S^{n-1}} fg \text{d}\sigma \) defines an \( O(t) \)-invariant inner product on the space of homogeneous \( n \)-variate polynomials, we can extend this inner product to multihomogeneous polynomials as follows. First, observe that the space of multihomogeneous polynomials of type \((n_1, k_1; \ldots; n_\ell, k_\ell)\) can be identified with \( \mathbb{R}^{D_{n,K}} \), which in turn can be identified with \( \mathbb{R}^{D_{n_1,k_1}} \otimes \cdots \otimes \mathbb{R}^{D_{n_\ell,k_\ell}} \) (following the notation of Proposition 1.3). One
then simply defines \( \langle f_1 \otimes \cdots \otimes f_\ell, g_1 \otimes \cdots \otimes g_\ell \rangle := \prod_{i=1}^\ell \int_{S^{n_i-1}} f_i g_i \, d\sigma_i \) (with \( d\sigma_i \) denoting the natural Euclidean measure restricted to \( S^{n_i-1} \)), and extends by multi-linearity.

We then need to recall some geometric constructions.

**Definition 3.1** For a convex body \( K \) (in a vector space \( V \)) containing the origin in its interior, we define the gauge function \( G_K \) as follows:

\[
G_K(x) := \sup\{ \lambda \geq 0 : \lambda x \in K \}
\]

**Lemma 3.2** [Bal97] Let \( K \) be a convex body in \( V \) with origin in its interior and let \( \langle \cdot, \cdot \rangle \) be an inner product on \( V \). Let \( S \) be the unit sphere in \( V \) and \( d\mu \) the \( SO(V) \) invariant measure on \( S \). Then we have the following formula for the volume of \( K \):

\[
\frac{\text{Vol}(K)}{\text{Vol}B} = \int_S G_K^D(x) \, d\mu
\]

where \( D \) is the dimension of the vector space \( V \) and \( B \) is the unit ball in \( V \). ■

**Lemma 3.3** The gauge \( G_{\tilde{P}} \) of \( \tilde{P}_{N,2K} \) at any polynomial \( f \in M_{N,2K} \) is given by,

\[
G_{\tilde{P}_{N,2K}}(f) = \left| \inf_{v \in S^N} f(v) \right|^{-1}
\]

**Proof:** From Definition 3.1 we have,

\[
G_{\tilde{P}_{N,2K}}(f) = \sup\{ \lambda > 0 : \lambda f \in \tilde{P}_{N,2K} \}
\]

We know from the definition of \( \tilde{P}_{N,2K} \) that if \( g \in M_{N,2K} \), then \( g \) is in \( \tilde{P}_{N,2K} \) if \( \inf_{v \in S^N} g(v) \geq -1 \). Let \( m_g = \inf_{v \in S^N} g(v) \). Clearly if \( g \in M_{N,2K} \), then \( m_g < 0 \). Now \( \inf_{v \in S^N} g/|m_g| \geq -1 \) and \( \inf_{v \in S^N} g/|m_g + \varepsilon| \leq -1 \) for every \( \varepsilon > 0 \). ■

A multihomogeneous polynomial \( f \) can be viewed as a function on the sphere product \( S^N \), and then identified with the restriction of a linear functional (defined by the coefficients of \( f \)) to an \( SO(N) \)-orbit. We will make use of certain norms of \( f \) to estimate certain relative volumes, so the following result will be extremely useful.

**Barvinok’s Theorem** [BB06] Let \( G \) be a compact group acting on a finite dimensional vector space \( V \). Let \( v \in V \) be a point and let \( l : V \to \mathbb{R} \) be a linear functional. Let us define \( f : G \to \mathbb{R} \) by \( f(g) := l(gv) \) for all \( g \in G \). For \( k > 0 \), let \( d_k \) be the dimension of the subspace spanned by the orbit \( \{ gv^{\otimes k}, g \in G \} \) in \( V^{\otimes k} \). In particular \( d_k \leq \left( \frac{\dim V + k - 1}{k} \right) \). Then,

\[
\| f \|_{2k} \leq \max_{g \in G} | f(g) | \leq \sqrt[2k]{d_k} \left\| f \right\|_{2k}
\]  (1)

This theorem enables us to bound the sup-norm of a function \( f \) via its \( 2k \)-norms. We shall also use the following result to prove our lower bound results.

**Barvinok’s Lemma** [BB06] Let \( G \) be a compact group acting on a finite \( d \)-dimensional real vector space \( V \) endowed with a \( G \)-invariant scalar product \( \langle \cdot, \cdot \rangle \) and let \( v \in V \) be a point. Let \( S^{d-1} \subseteq V \) be the unit sphere endowed with the Haar probability measure \( dc \). Then, for every positive integer \( k \), we have,

\[
\int_{S^{d-1}} \left( \int_G \langle c, gv \rangle^{2k} \, dg \right)^{1/2k} \, dc \leq \sqrt{\frac{2k \langle v, v \rangle}{d}}
\]
3.1 The Blaschke-Santaló Inequality

Let us first recall the following classical construction.

**Definition 3.4** Let $K$ be a convex subset of a vector space $V$. The polar of $K$ is then $K^\circ := \{ l \in V^* : l(v) \leq 1, \forall v \in K \}$.

In case the vector space $V$ is endowed with an inner product $\langle \cdot , \cdot \rangle$, then we can identify $K^\circ$ with a subset of $V$ itself.

$$K^\circ = \{ w \in V : \langle v, w \rangle \leq 1, \forall v \in K \}$$

We shall first describe the Blaschke-Santaló inequality [San49], [Bla15], [Sch93], [Gru07]. This will prove quite useful for transferring lower bound results into upper bound results.

Let $K$ be a convex set in an $n$ dimensional vector space $V$, endowed with an inner product $\langle \cdot , \cdot \rangle$. We introduced the concept of a polar body above. This can be generalized to what is called the polar of $K$ with respect to an arbitrary point $z \in V$:

$$K_z^\circ := \{ y + z | \langle y, x + z \rangle \leq 1 \text{ for all } x \in K \}$$

Let

$$p(K) := \inf \{ (\text{Vol} K_z^\circ)(\text{Vol} K) | z \in \text{int}(K) \}$$

This infimum is reached for a unique point in $V$ called the Santaló point of $K$, $s(K)$.

**The Blaschke-Santaló Inequality** Let $B_M$ be the unit ball in $V$. Then

$$\text{Vol}(K)(\text{Vol}(K^{s(K)}) \leq \text{Vol}(B_M)^2.$$ 

This was proved by fairly technical arguments in [San49]. Saint Raymond in 1981 [StR81] gave a simple proof of this for the special case of centrally symmetric convex bodies. A simple proof of a generalization of the Blaschke-Santaló inequality was provided by Meyer and Pajor [MP90] using Steiner symmetrization.

**Definition 3.5** The sup norm of $f \in P_{N,K}$ is defined as,

$$\| f \|_\infty := \sup \{ f(x) | x \in S^N \}$$

**Definition 3.6** The unit ball in $M_{N,2K}$ under the sup norm is as follows:

$$B_\infty := \{ f \in M_{N,2K} | \| f \|_\infty \leq 1 \}$$

In what follows, we let Conv($S$) denote the convex hull of a set $S$.

**Lemma 3.7** For any convex subsets $A$ and $B$ of a vector space $V$ we have $(A \cap B)^\circ = \text{Conv}(A^\circ \cup B^\circ)$.

**Proof:** $f \in \text{Conv}(A^\circ \cup B^\circ)$, implies that there exist $f_1 \in A^\circ$ and $f_2 \in B^\circ$, such that, $f = \lambda f_1 + (1 - \lambda)f_2$. Now $\langle f, g \rangle = \lambda \langle f_1, g \rangle + (1 - \lambda) \langle f_2, g \rangle$. For $g \in A \cap B$, we have $\langle f_1, g \rangle \leq 1$ and $\langle f_2, g \rangle \leq 1$. This clearly means, $\langle f, g \rangle \leq 1$. ■
3.2 Exotic Metrics

We shall now briefly discuss a couple of important metrics we will need in our proofs later. Let us begin by defining a multigradient on $P_{N,2K}$ along the lines of our earlier inner product on $P_{N,2K}$. The idea is that given gradients $\nabla_i$ on $P_{n_i,k_i}$, we can tensor them together to obtain a gradient on $P_{N,2K}$:

$$\nabla := \otimes_{i=1}^l \nabla_i,$$

where for every $f_i \in P_{n_i,k_i}$ we have $\nabla_i(f_i) = \left(\frac{\partial f_i}{\partial x_{i,1}}, \ldots, \frac{\partial f_i}{\partial x_{i,n_i}}\right)$ and $\nabla(f)$ is then $\otimes_{i=1}^l \nabla_i(f_i)$, when $f = \otimes_{i=1}^l f_i$ is a decomposable tensor. Using multi-linearity like before, we can extend this to the whole of $P_{N,2K}$.

**Definition 3.8** For $f, g \in P_{N,2K}$, we define the gradient metric as follows:

$$\langle f, g \rangle_G := \frac{1}{4^l \prod_{i=1}^l k_i^2} \int_{S(N)} \langle \nabla(f), \nabla(g) \rangle \, d\sigma$$

We have the following useful result.

**Lemma 3.9** [Kel28] For any $f \in P_{N,2K}$ we have, $\|f\|_{\infty} \geq \|f\|_G$.

We now extend the differential metric of Blekherman's paper to the multihomogeneous case. Let $x_1 = (x_{11}, \ldots, x_{1n_1})$, $x_2 = (x_{21}, \ldots, x_{2n_2})$, $\ldots$, $x_l = (x_{l1}, \ldots, x_{ln_l})$ denote the $l$ sets of variables. Let $x = (x_1, \ldots, x_l) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_l}$. We shall use the following compact notation to name the associated differential operators of monomials. Let

$$\alpha = (\alpha_1, \ldots, \alpha_{l}, \ldots, \alpha_l) = (\alpha_{l1}, \ldots, \alpha_{ln_l}).$$

Then,

$$x^\alpha = \prod_{i=1}^l x_{i,1}^{\alpha_{i,1}} \cdots x_{i,n_i}^{\alpha_{i,n_i}}$$

and,

$$D_x^\alpha := \otimes_{i=1}^l \frac{\partial^{\alpha_{i,1}} \cdots \partial^{\alpha_{i,n_i}}}{\partial x_{i,1}^{\alpha_{i,1}} \cdots \partial x_{i,n_i}^{\alpha_{i,n_i}}}$$

Now for a form $f \in P_{N,2K}$,

$$f = \sum_{|\alpha|=2K} c_\alpha x^\alpha$$

we can define an associated linear operator as follows:

$$D_f := \sum_{|\alpha|=2K} c_\alpha D_x^\alpha$$

Now we finally get to defining the differential metric using a positive definite bilinear form using $D_f$.

$$\langle f, g \rangle_D := D_f(g)$$
3.3 Representation theory

We will need some simple concepts from representation theory for our proof of the upper bound for nonnegative polynomials. The basic reference [FH91] is more than adequate for our purpose. In particular we shall need the following lemma. If $G_1$ and $G_2$ are two groups and $V_1$ and $V_2$ are representations of $G_1$ and $G_2$, then the tensor product, $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$, by $(g_1 \times g_2).(v_1 \otimes v_2) = g_1 v_1 \otimes g_2 v_2$. To distinguish this “external” tensor product from the “internal” tensor product when $G_1 = G_2$, we denote this by, $V_1 \odot V_2$.

Lemma 3.10 If $V_1$ and $V_2$ are irreducible then $V_1 \odot V_2$ is also irreducible and every irreducible representation of $G_1 \times G_2$ arises this way.

4 The Proof of Theorem 2.1

There are of course 4 inequalities to be proved. For the sake of brevity, we will omit the proof of the lower bound on $\tilde{\Sigma}$, particularly since it is very similar in flavor to the other three proofs.

4.1 The Lower Bound on $\mu(\tilde{P}_{N,2K})$

The proof of the lower bound for $\mu(\tilde{P}_{N,2K})$ can be broken down into the following steps, analogous to the approach adopted in [Ble06]. For simplicity, we will restrict to the special case $\ell = 2$.

1. The volume taking into account the effect of higher dimensions is defined using the integral of gauge function of $\tilde{P}_{N,2K}$ over the unit sphere $S_M$ in $M_{N,2K}$.

2. This is then manipulated to an integral involving the sup-norm of bihomogeneous polynomials over $S_M$.

3. Using Barvinok’s Theorem we bound $\|f\|_\infty$ by the $k$-norm of $f$, $\|f\|_k$, for some suitably chosen $k \in \mathbb{N}$.

4. Using Barvinok’s Lemma we then bound $\|f\|_k$ to obtain our result.

We now cover the details: Since $\tilde{P}_{N,2K}$ is a convex body with origin in its interior, we can use Lemma 3.3 to represent $\mu(\tilde{P}_{N,2K})$. Integrating in polar coordinates we obtain

$$\left(\frac{\text{Vol} \tilde{P}_{N,2K}}{\text{Vol} B_M}\right)^{1/D_M} = \left(\int_{S_M} G_{\tilde{P}_{N,2K}}^{D_M} d\mu\right)^{1/D_M}$$
Since the right hand side is the $D_M$-norm of $G_P$, we can apply Hölder’s theorem and obtain:

$$\left( \frac{\text{Vol}\tilde{P}_{N,2K}}{\text{Vol}B_M} \right) \geq \left( \int_{S_M} G_P(f) d\mu \right) \quad \text{(by Hölder’s Inequality)} \quad (2)$$

$$\geq \left( \int_{S_M} \left| \inf_{v \in \mathbb{S}^N} f(v) \right|^{-1} d\mu \right)^{-1} \quad \text{(by Lemma 3.3)}$$

Finally, it is easy to observe that $\|f\|_{\infty} \geq |\inf_{v \in \mathbb{S}^N} f(v)|$. Hence, to bound the volume of nonnegative multihomogeneous polynomials from below, we need only estimate the integral of the sup norm over the unit sphere.

$$\left( \frac{\text{Vol}\tilde{P}_{N,2K}}{\text{Vol}B_M} \right) \geq \left( \int_{S_M} \|f\|_{\infty} d\mu \right)^{-1}$$

We proceed by bounding the $\|f\|_{\infty}$ norm by $\|f\|_{2k}$ using Barvinok’s results. To apply Barvinok’s theorem, we view an $f \in P_{N,2K}$ as the restriction of a linear functional on $(\mathbb{R}^{n_1})^{\otimes 2k_1} \otimes \cdots \otimes (\mathbb{R}^{n_\ell})^{\otimes 2k_\ell}$ to an $SO(n_1) \times \cdots \times SO(n_\ell)$ orbit in $(\mathbb{R}^{n_1})^{\otimes 2k_1} \otimes \cdots \otimes (\mathbb{R}^{n_\ell})^{\otimes 2k_\ell}$.

**Lemma 4.1** Given a vector space $V = V_1 \times \cdots \times V_\ell$ and a group action of $G = G_1 \times \cdots G_\ell$ on $V$, we have a natural $G$ action on $V^{\otimes k} := V_1^{\otimes k_1} \otimes \cdots \otimes V_\ell^{\otimes k_\ell}$. ■

Applying our lemma above, one can then derive from Barvinok’s Theorem that

$$\|f\|_{2k} \leq \|f\|_{\infty} \leq \sqrt[k]{\left( \frac{n_1 + 2k_1 k - 1}{2k_1 k} \right) \cdots \left( \frac{n_\ell + 2k_\ell k - 1}{2k_\ell k} \right)} \|f\|_{2k}.$$ 

By taking $k = \max\{n_1 \log(2k_1 + 1), \ldots, n_\ell \log(2k_\ell + 1)\}$, and some intermediate combinatorial inequalities, we obtain $\|f\|_{\infty} \leq 3e^2 \|f\|_{2k}$. Integrating these norms, and combining with Barvinok’s Lemma, we are done. ■

### 4.2 The Upper Bound on $\mu\left( \tilde{P}_{N,2K} \right)$

The proof of the upper bound for $\mu\left( \tilde{P}_{N,2K} \right)$ can then be broken down into the following steps:

1. Relate the volume of $\tilde{P}_{N,2K}$ to that of its polar, $\tilde{P}_{N,2K}^\circ$, using the Blaschke-Santaló Inequality.

2. Obtain a relation between the polar of the unit ball in the sup norm, $B_\infty^\circ$ and $\tilde{P}_{N,2K}^\circ$.

3. Introduce a multigraded extension of the gradient metric to upper bound $B_\infty^\circ$ by the unit ball in the gradient metric, $B_G$.
4. Finally bound the ratio of $B_G$ to $B_M$ using arguments from representation theory.

We now proceed with the details, focussing on the special case $\ell = 2$ for simplicity. (The general case is merely notationally more difficult.) Consider first the polar $\tilde{P}_{N,2K}$ via Definition 3.4:

$$\tilde{P}_{N,2K} = \{ f \in M_{N,2K} \mid \langle f, g \rangle \leq 1, \forall g \in \tilde{P}_{N,2K} \}$$

Since $\tilde{P}_{N,2K}$ is fixed by $SO(n_1) \times SO(n_2)$ and origin is the only point in $M_{N,2K}$ fixed by $SO(n_1) \times SO(n_2)$. From our earlier discussion of the Blaschke-Santalo Inequality, we have that the Santaló point of a convex body is unique. Hence the origin is the Santaló point of $\tilde{P}_{N,2K}$ and we thus obtain the following:

$$\left( \text{Vol} \tilde{P}_{N,2K} \right) \left( \text{Vol} \tilde{P}_{N,2K}^o \right) \leq \left( \text{Vol} B_M \right)^2$$

Therefore it suffices to show that

$$\left( \frac{\text{Vol} \tilde{P}_{N,2K}^o}{\text{Vol} B_M} \right) \geq \frac{1}{4} \left( \frac{2k_1^2}{4k_1^2 + n_1 - 2} \right)^{1/2} \left( \frac{2k_2^2}{4k_2^2 + n_2 - 2} \right)^{1/2}$$

Now the unit ball with respect to the sup-norm, $B_\infty$, is the intersection of $\tilde{P}_{N,2K}$ with $-\tilde{P}_{N,2K}$: $B_\infty = \tilde{P}_{N,2K} \cap -\tilde{P}_{N,2K}$. From Lemma 3.7 we then see that,

$$B_\infty^o = \text{Conv}(\tilde{P}_{N,2K}^o \cup -\tilde{P}_{N,2K}^o) \subset \tilde{P}_{N,2K}^o \oplus -\tilde{P}_{N,2K}^o$$

We now apply the Rogers and Shephard theorem [Sch93] from convex geometry to get a bound on the polar of the sup-norm unit ball.

$$\text{Vol} B_\infty^o \leq \left( \frac{2D_M}{D_M} \right) \text{Vol} \tilde{P}_{N,2K}^o$$

**Lemma 4.2** For all $n > 0$ we have $\binom{2n}{n} \leq 4^n$.

**Proof:** The left hand side is the coefficient of the $x^n$ term in the expansion of $(1 + x)^{2n}$. Taking $x = 1$, we clearly have, $\binom{2n}{n} \leq (1 + 1)^{2n} = 4^n$. \[\blacksquare\]

From Lemma 4.2 it follows that $\left( \frac{\text{Vol} B_\infty^o}{\text{Vol} \tilde{P}_{N,2K}^o} \right)^{1/D_M} \geq \frac{1}{4}$. This reduces the proof of the upper bound to proving the following inequality:

$$\left( \frac{\text{Vol} B_\infty^o}{\text{Vol} B_M} \right)^{1/D_M} \geq \left( \frac{4k_1^2 + n_1 - 2}{2k_1^2} \right)^{1/2} \left( \frac{4k_2^2 + n_2 - 2}{2k_2^2} \right)^{1/2}$$

We now bound the sup-norm ball using the gradient metric introduced in Lemma 3.8. For $f \in M_{N,2K}$, which is decomposable, say $f = f_1 \otimes f_2$, we have,

$$\nabla f = \left( \frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_{n_1}} \right) \otimes \left( \frac{\partial f_2}{\partial y_1}, \ldots, \frac{\partial f_2}{\partial y_{n_2}} \right)$$

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From Lemma 4.3, the gradient metric of $f$ as above becomes
\[
\langle f, f \rangle_G = \frac{1}{16k_1^2k_2^2} \int_{S^{n_1-1} \times S^{n_2-1}} \left( \left( \frac{\partial f_1}{\partial x_1} \right)^2 + \ldots + \left( \frac{\partial f_1}{\partial x_{n_1}} \right)^2 \right)
\times \left( \left( \frac{\partial f_2}{\partial y_1} \right)^2 + \ldots + \left( \frac{\partial f_2}{\partial y_{n_2}} \right)^2 \right) \, d\sigma
\]

Let $B_G$ be the unit ball in the gradient metric and the corresponding norm $\|f\|_G$. From Lemma 3.9, $B_\infty \subset B_G$. Polarity reverses inclusion and thus $B_G^\circ \subset B_\infty^\circ$ and $\text{Vol} B_G^\circ = \left( \frac{\text{Vol} B_M}{\text{Vol} B_G} \right)^2$, using the Blaschke-Santaló Inequality. Consequently, we have, $\text{Vol} B_\infty^\circ \geq \left( \frac{\text{Vol} B_M}{\text{Vol} B_G} \right)^2$ and hence $\frac{\text{Vol} B_\infty^\circ}{\text{Vol} B_M} \geq \frac{\text{Vol} B_M}{\text{Vol} B_G}$. Thus, we are left with proving the following:

**Lemma 4.3**
\[
\left( \frac{\text{Vol} B_M}{\text{Vol} B_G} \right)^{1/D_M} \geq \left( \frac{4k_1^2 + n_1 - 2}{2k_1^2} \right)^{1/2} \left( \frac{4k_2^2 + n_2 - 2}{2k_2^2} \right)^{1/2}
\]

**Proof:** It is enough that we show the following is true for all $f \in M_{N, 2K}$.
\[
\langle f, f \rangle \geq \left( \frac{4k_1^2 + n_1 - 2}{2k_1^2} \right)^{1/2} \left( \frac{4k_2^2 + n_2 - 2}{2k_2^2} \right)^{1/2} \langle f, f \rangle_G
\]

By the invariance of both inner products under the action of $SO(n_1) \times SO(n_2)$, it is enough to prove the lemma in the irreducible components of the representation. We know that (Lemma 3.10), the irreducible components are $H_{n_1, 2l_1} \otimes H_{n_2, 2l_2}$ for $0 \leq l_1 \leq k_1$ and $0 \leq l_2 \leq k_2$. And,
\[
H_{n, 2l} = \{ f \in P_{n, 2k} \mid f = (x_1^2 + \ldots + x_n^2)^{k-l}h, h \in P_{n, 2l} \}
\]
If $f$ is a harmonic form of degree $2d$ in $n$ variables, Stokes’ formula gives us,
\[
\langle f, f \rangle_G = \frac{2d}{4d + n - 2} \langle f, f \rangle
\]
Also, when $f = (x_1^2 + \ldots + x_n^2)^{k-d}h$, where $h$ is a harmonic form of degree $2d \leq 2k$, it is easy to check that,
\[
\langle f, f \rangle_G = \frac{d^2}{k^2} \langle h, h \rangle_G + \frac{k^2 - d^2}{k^2} \langle h, h \rangle
\]
We now obtain the following similar results when $f_1 = (x_1^2 + \ldots + x_{n_1}^2)^{k_1-d_1}h_1$ and $f_2 = (y_1^2 + \ldots + y_{n_2}^2)^{k_2-d_2}h_2$. Observe that
\[
\langle f_1f_2, f_1f_2 \rangle_G = \langle f_1, f_1 \rangle_G \langle f_2, f_2 \rangle_G
\]
\[
= \left( \frac{d_1^2}{k_1^2} \langle h_1, h_1 \rangle_G + \frac{k_1^2 - d_1^2}{k_1^2} \langle h_1, h_1 \rangle \right)
\times \left( \frac{d_2^2}{k_2^2} \langle h_2, h_2 \rangle_G + \frac{k_2^2 - d_2^2}{k_2^2} \langle h_2, h_2 \rangle \right)
\]
\[
= \left( \frac{2d_1^2 + d_1(n_1 - 2) + 2k_1^2}{2k_1^2} \right) \left( \frac{2d_2^2 + d_2(n_2 - 2) + 2k_2^2}{2k_2^2} \right) \langle f_1f_2, f_1f_2 \rangle
\]
\[
\leq \left( \frac{4k_1^2 + n_1 - 2}{2k_1^2} \right) \left( \frac{4k_2^2 + n_2 - 2}{2k_2^2} \right) \langle f_1f_2, f_1f_2 \rangle
\]
The last step follows since the minimum clearly occurs when \( d_1 = d_2 = 1 \). This proves the lemma.

4.3 The Upper Bound on \( \mu\left(\tilde{\Sigma}_{N,2K}\right) \)

We can outline the steps involved as follows:

1. Bound the volume of SOS polynomials by an average width using results from convexity theory \[\text{Sch93}\].

2. Express the average width in terms of an integral involving the support function of SOS polynomials.

3. Bound the support function by a max norm.

4. Use Barvinok’s method to bound the integral of the max norm by an \( L^2 \) norm for some large \( p \).

We now proceed with the details, focussing on the special case \( \ell = 2 \) for simplicity. (The general case is merely notationally more difficult.) First, we have the following bound for the volume of multihomogeneous SOS polynomials from Uryshon’s Inequality \[\text{Sch93}\]:

\[
\left( \frac{\text{Vol}\tilde{\Sigma}_{N,2K}}{\text{Vol}B_M} \right)^{1/D_M} \leq \frac{W_{\tilde{\Sigma}}}{2}
\]

Here \( W_{\tilde{\Sigma}} \) is the average width of \( \tilde{\Sigma} \) and is given by,

\[
W_{\tilde{\Sigma}} = 2 \int_{S_M} L_{\tilde{\Sigma}} d\mu
\]

where \( L_{\tilde{\Sigma}} \) is the support function of \( \tilde{\Sigma}_{N,2K} \) which can be computed by the following formula:

\[
L_{\tilde{\Sigma}}(f) = \max_{g \in \Pi} \langle f, g \rangle
\]

Thus we can obtain an upper bound for the volume of multihomogeneous SOS polynomials by bounding their average width \( W_{\tilde{\Sigma}} \).

The extreme points in \( \Sigma_{N,2K} \) are clearly perfect squares. Hence, since \( \tilde{\Sigma}_{N,2K} \) is a translation of \( \Sigma_{N,2K} \) by \((x_1^2 + \ldots + x_{n_1}^2)^{k_1}(y_1^2 + \ldots + y_{n_2}^2)^{k_2}\), the extreme points in \( \tilde{\Sigma} \) are given as follows: \( g^2 - (x_1^2 + \ldots + x_{n_1}^2)^{k_1}(y_1^2 + \ldots + y_{n_2}^2)^{k_2} \) where \( g \in P_{N,K} \) and \( \int_{S_{n_1-1} \times S_{n_2-1}} g^2 d\sigma = 1 \).

For \( f \in M_{N,2K} \),

\[
\langle f, (x_1^2 + \ldots + x_{n_1}^2)^{k_1}(y_1^2 + \ldots + y_{n_2}^2)^{k_2} \rangle = \int_{S_{n_1-1} \times S_{n_2-1}} f d\sigma = 0
\]

Hence the expression for the support function simplifies to,

\[
L_{\tilde{\Sigma}}(f) = \max_{g \in \Pi} \langle f, g^2 \rangle
\]
Now we introduce a quadratic form on $P_{N,K}$ whose norm bounds $L_\tilde{\Sigma}$:

$$H_f(g) = \langle f, g^2 \rangle$$ for $g \in P_{N,K}$

Now $L_\tilde{\Sigma}(f) \leq \|H_f\|_\infty$, so we can use Barvinok's theorem to bound $\|H_f\|_\infty$ by an $L^2$ norm of $H_f$ for some large $p$. Since $H_f$ is a form of degree 2 on the vector space $P_{N,K}$ of dimension $D_{N,K}$ we get

$$\|H_f\|_\infty \leq 2\sqrt{3}\|H_f\|_{2D_{N,K}}$$

We now proceed as in the case of non-negative multihomogeneous polynomials, using Hölder's inequality to estimate the integral of $\|H_f\|_\infty$.

$$\int_{S_M} L_\tilde{\Sigma} d\mu \leq \left( \int_{S_M} \int_{S_{PN,K}} \langle f, g^2 \rangle^{2D_{N,K}} d\sigma(g) d\mu(f) \right)^{1/2D_{N,K}}$$

Since the inner integral depends only on the projection of $g^2$ into $M_{N,2K}$, we have,

$$\int_{S_M} \langle f, g^2 \rangle^{2D_{N,K}} d\mu(f) \leq \|g^2\|_{2D_{N,K}}^{2D_{N,K}} \int_{S_M} \langle f, p \rangle^{2D_{N,K}} d\mu(f)$$

for any $p \in S_M$.

We can compute the second integral easily due to the invariance of the inner product under $SO(n_1) \times SO(n_2)$ (see for example [Bar02B] and [BB06]):

$$\int_{S_M} \langle f, p \rangle^{2D_{N,K}} d\mu(f) = \frac{\Gamma(D_{N,K} + \frac{1}{2})\Gamma(\frac{1}{2}D_{M})}{\sqrt{\Pi}\Gamma(D_{N,K} + \frac{1}{2}D_{M})}$$

Furthermore, Duoandikoetxea [Duo83] (see also [HLP88, Ch. 1]) has shown that for $g \in S_{P_{n_1,k_1}}$, $\|g^2\|_2 \leq 4^{2k_1}$. This implies that for $g \in P_{N,K}$, $\|g^2\|_2 \leq 4^{2k_1}4^{2k_2}$

Combining these two results, we have,

$$\int_{S_M} L_\tilde{\Sigma}(f) d\mu \leq 4^{2k_1}4^{2k_2} \left( \frac{\Gamma(D_{N,K} + \frac{1}{2})\Gamma(\frac{1}{2}D_{M})}{\sqrt{\Pi}\Gamma(D_{N,K} + \frac{1}{2}D_{M})} \right)^{\frac{1}{2D_{N,K}}}

Abramowitz and Stegun [AS72] gives the following inequality for the Gamma function:

$$\frac{\Gamma(n + a)}{\Gamma(n + b)} \leq \frac{1}{n^{b-a}}$$, for $b - a \geq 0, a \geq 0, n \in \mathbb{N}$

Using this we obtain

$$\left( \frac{\Gamma(\frac{1}{2}D_{M})}{\Gamma(D_{N,K} + \frac{1}{2}D_{M})} \right)^{\frac{1}{2D_{N,K}}} \leq \sqrt{\frac{2}{D_{M}}}

\left( \frac{\Gamma(\frac{1}{2} + D_{N,K})}{\sqrt{\Pi}} \right)^{\frac{1}{2D_{N,K}}} \leq \sqrt{\frac{D_{N,K}}{D_{M}}}

This implies,

$$\int_{S_M} L_\tilde{\Sigma}(f) d\mu \leq 4^{2k_1}4^{2k_2}2\sqrt{3}\sqrt{\frac{2D_{N,K}}{D_{M}}}

From here, we at last obtain our desired upper bound. ■
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References

[AS72] Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1972.

[Bal97] Ball, Keith, “An elementary introduction to modern convex geometry”, in *Math. Sci. Res. Inst. Publ.*, Vol. 31, Cambridge, UK: Cambridge Univ. Press, 1997, pp. 1–58.

[Bar02a] Barvinok, Alexander, *A Course in Convexity*, Graduate Studies in Mathematics, Vol. 54, AMS Press, 2002.

[Bar02b] Barvinok, Alexander, “Estimating $L_{\infty}$ norms by $L^{2k}$ norms for functions on orbits,” *Foundations of Computational Mathematics*, vol. 2, pp. 393–412, Jan. 2002.

[BB06] Barvinok, Alexander and Blekherman, Grigoriy, “Convex geometry of orbits,” in *Proceedings of the MSRI Workshop on Discrete and Computational Geometry*, AMS Publications, Boston, MA, 2006, pp. 51–77.

[BHP10] Bastani, Osbert; Hillar, Chris; Popov, Dimitar; and Rojas, J. Maurice, “Sums of Squares, Randomization, and Sparse Polynomials,” in preparation, 2010.

[BPR06] Basu, Saugata; Pollack, Ricky; and Roy, Marie-Francoise, *Algorithms in Real Algebraic Geometry*, Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, 2006.

[Bla15] Blaschke, W., “Eine erweiterung des satzes von Vitali ber eolgen analytischer funktionen berichte,” *Math. Phys. Kl., Sächs. Gesell. der Wiss. Leipzig*, vol. 67, pp. 194–200, 1915.

[Ble06] Blekherman, Grigoriy, “There are Significantly More Nonnegative Polynomials than Sums of Squares,” *Israel J. of Math.*, vol. 183 (2006), pp. 355–380.

[Duo83] Duoandikoetxea, J. “Reverse Holder inequalities for spherical harmonics,” *Proc. Amer. Math. Soc.*, vol. 101, no. 3, pp. 487–491, Mar. 1983.

[FH91] Fulton, William and Harris, Joseph, *Representation Theory - A First Course*, Springer-Verlag, New York, New York, 1991.

[Gru07] Gruber, P. M., *Convex and Discrete Geometry*, Springer-Verlag, New York, New York, 2007.

[HLP88] Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities*, Cambridge, Cambridge University Press, UK, 1988.
[Kel28] Kellogg, O., “On bounded polynomials in several variables,” Math. Zeitschrift, vol. 27, pp. 55–64, Aug. 1928.

[Las07] Lassere, Jean B., “A Sum of Squares Approximation of Nonnegative Polynomials,” SIAM Review, Vol. 49, No. 4, pp. 651–669.

[LM01] Lickteig, Thomas and Roy, Marie-Francoise, “Sylvester-Habicht Sequences and Fast Cauchy Index Computation,” J. Symbolic Computation (2001) 31, pp. 315–341.

[MP90] Meyer, M. and Pajor, A., “On the Blaschke-Santal inequality,” Arch. Math. (Basel), vol. 55, pp. 82–93, 1990.

[Par03] Parrilo, Pablo A., “Semidefinite programming relaxations for semialgebraic problems,” Algebraic and geometric methods in discrete optimization, Math. Program. 96 (2003), no. 2, Ser. B, pp. 293–320.

[PS03] Parrilo, Pablo A. and Sturmfels, Bernd, “Minimizing polynomial functions,” in Algorithmic and Quantitative Real Algebraic Geometry (S. Basu and L. Gonzalez-Vega, eds), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 60, AMS Press, 2003.

[Rez74] Reznick, Bruce, “Extremal psd forms with few terms,” Duke Math J. 45 (1978), pp. 363-374.

[Rez00] Reznick, Bruce, “Some concrete aspects of Hilbert’s 17th problem,” Real Algebraic Geometry and Ordered Structures, (C. N. Delzell, J.J. Madden eds.) Cont. Math., 253 (2000), pp. 251–272.

[Roj02] Rojas, J. Maurice, “Why Polyhedra Matter in Non-Linear Equation Solving,” paper corresponding to an invited talk delivered at a conference on Algebraic Geometry and Geometric Modelling (Vilnius, Lithuania, July 29 – August 2, 2002), Contemporary Mathematics, vol. 334, pp. 293–320, AMS Press, 2003.

[San49] Santaló, L. A., “An affine invariant for convex bodies of n-dimensional space,” Portugaliae Math., vol. 8, pp. 155–161, Aug. 1949.

[StR81] Saint-Raymond, J., “Sur le volume des corps convexes symetriques,” in Initiation Seminar on Analysis, vol. 11 (G. Choquet, M. Rogalski, and J. Saint-Raymond Eds.), Publ. Math. Univ. Pierre et Marie Curie, 1981, pp. 25–221.

[Sch93] Schneider, Rolf, Convex Bodies: the Brunn-Minkowski Theory, Cambridge, UK: Cambridge University Press, 1993.

[VB96] Vandenberghe, L. and Boyd, S., “Semidefinite programming,” SIAM Rev., 38 (1996), pp. 49–95.