ALUTHGE TRANSFORMS, TRIDIAGONAL KERNELS, AND LEFT INVERTIBLE OPERATORS

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Dedicated to Carl Pearcy on the occasion of his 85th birthday

ABSTRACT. We study Aluthge transforms of left invertible shift operators on reproducing kernel Hilbert spaces (RKHS), and in particular, on tridiagonal spaces. The Aluthge transform of a bounded linear operator $T$ on a Hilbert space is defined by $\tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2}$, where $T = U |T|$ is the polar decomposition of $T$. A RKHS $\mathcal{H}_k$ of analytic functions on the open unit disc is called tridiagonal space if there exist scalars $a_n (\neq 0)$ and $b_n$, $n \geq 0$, such that $\{(a_n + b_n z)^n\}_{n=0}^\infty$ is an orthonormal basis of $\mathcal{H}_k$. It is easy to see that $k$ is a band kernel with bandwidth 1, and

$$k(z, w) = \sum_{n=0}^\infty \left((a_n + b_n z)z^n\right)\left((\bar{a}_n + \bar{b}_n \bar{w})\bar{w}^n\right).$$

We consider bounded shift operators $M_z$ on tridiagonal spaces. In the case when $b_n = 0$, $n \geq 0$, $k$ becomes a diagonal kernel and $M_z$ becomes a weighted shift.

We prove that the shift on $\mathcal{H}_k$ is left invertible whenever the sequence $\left\{|a_n| \over |a_{n+1}|\right\}_{n \geq 0}$ is bounded away from zero. We employ two different approaches to compute Aluthge transforms of shifts on $\mathcal{H}_k$ (or, more general RKHS): the first is based on Shimorin’s analytic model; and the second approach is rather direct and based on RKHS techniques. In either case, the resulting Aluthge transforms are shifts on some RKHS.

On the other hand, unlike the case of weighted shifts, we find that Shimorin models fail to bring to the foreground the tridiagonal structure of shifts. In fact, the tridiagonal structure of a kernel $k$, as above, is preserved under Shimorin model if and only if $b_0 = 0$ or that $M_z$ on $\mathcal{H}_k$ is a weighted shift. We prove a number of concrete classification results concerning invariance of tridiagonality of kernels under the action of Aluthge transforms, Shimorin models and positive operators. We also give explicit algorithms to compute Aluthge transforms of shifts on truncated tridiagonal spaces. Curiously, in contrast to direct RKHS techniques, often (but not always) Shimorin models fail to yield tridiagonal Aluthge transforms of shifts defined on tridiagonal spaces.

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1. Introduction

This paper deals with the merge of three operator and function theoretic concepts, namely, reproducing (or tridiagonal) kernels, Aluthge transforms, and left invertible operators. There have been several motivations for our paper. However, our original motivation was to compute Aluthge transforms of shifts on “concrete analytic Hilbert spaces”, and this is where the notion of tridiagonal kernels emerged. Indeed, after the classical weighted shifts (also known as diagonal kernels), tridiagonal kernels are the “next best” concrete examples of analytic kernels.

It is “fairly easy” (/trivial) to prove that, up to unitary equivalence (/similarity), the Aluthge transform of a left invertible shift on a reproducing kernel Hilbert space is again a shift on some (explicit) reproducing kernel Hilbert space (note that a weighted shift is left invertible so long as the weight sequence is bounded away from zero). This formulation can be obtained (so far) in two different ways, either by using the reproducing kernel Hilbert space techniques (see Theorem 2.8), or by employing Shimorin’s analytic models for left invertible operators (see Theorem 2.4). This is how the idea of Shimorin models emerges in our study. After all, many researchers have put this forth as a key model for left invertible operators.

The main concern here, however, is to develop flexible and effective method of computation of Aluthge transforms. This brings us to examine and compare more closely the aforementioned methods, and subsequently we find that computing Aluthge transforms in the case of tridiagonal kernels is a quite subtle yet fruitful problem. In particular, our results, from a computational point of view, seem to indicate that Shimorin models are less effective and satisfying, in sharp contrast to its role in diagonal kernels. We also derive explicit formulas by employing the Shimorin model as well as the direct model of Aluthge transforms for some classes of tridiagonal kernels (which we call truncated tridiagonal kernels). Definite computations verifies that the direct reproducing kernel Hilbert space technique is more powerful than Shimorin models. We also prove a number of results concerning concrete representations of left inverses (see Theorem 2.6), left invertible properties of shifts on tridiagonal spaces (Theorem 3.4 and Proposition 3.5), new tridiagonal spaces from the old (see Theorem 4.2), truncated tridiagonal spaces where the Shimorin model coincides with the direct reproducing kernel Hilbert space model (see Theorem 6.3), a complete classification of quasinormal operators (see Theorem 7.2) etc. We also provide a family of instructive examples and supporting counterexamples.

We believe that some of our results may be of independent interest in respective studies and may find additional applications. To demonstrate these relationships and the main contribution of this paper, it is now necessary to disambiguate central concepts.
We begin with reproducing kernel Hilbert spaces. Reproducing kernel Hilbert space theory is an interdisciplinary subject of pure and applied mathematics. Leaving aside the prehistory, the classical reference is N. Aronszajn [6]. Briefly stated, the essential idea of this theory is to single out the role of positive definiteness of inner products, multipliers and bounded point evaluations of function Hilbert spaces (here all Hilbert spaces are assumed to be separable and over \( \mathbb{C} \)). For instance, let \( \mathcal{E} \) be a Hilbert space and \( H \) be a Hilbert space of \( \mathcal{E} \)-valued analytic functions on \( D \), where \( D \) denote the open unit disc of \( \mathbb{C} \). Suppose the evaluation operator (evaluation functional if \( \mathcal{E} = \mathbb{C} \)) \( e_w : H \to \mathcal{E} \), defined by
\[
e_w(f) = f(w) \quad (f \in H),
\]
is bounded for all \( w \in D \). Then there exists a function (we call it a kernel) \( k : D \times D \to \mathcal{B}(\mathcal{E}) \) (or \( k : D \times D \to \mathbb{C} \) if \( \mathcal{E} = \mathbb{C} \)), analytic in the first variable, such that \( k \) is positive definite, that is,
\[
\sum_{i,j=1}^{n} \langle k(z_i, z_j) \eta_j, \eta_i \rangle_{\mathcal{E}} \geq 0,
\]
for all \( \{z_i\}_{i=1}^{n} \subseteq D \), \( \{\eta_i\}_{i=1}^{n} \subseteq \mathcal{E} \) and \( n \in \mathbb{N} \). Throughout this paper, \( \mathcal{B}(\mathcal{E}) \) will denote the linear space of all bounded linear operators on \( \mathcal{E} \). Note that if \( \mathcal{E} = \mathbb{C} \), then the above positivity condition becomes \( \sum_{i,j=1}^{n} \overline{c_i} c_j k(z_i, z_j) \geq 0 \) and \( c_i \in \mathbb{C} \). It is well known that the set
\[
\{k(\cdot, w)\eta : w \in D, \eta \in \mathcal{E}\},
\]
is a total set in \( H \) and satisfies the reproducing property
\[
\langle e_w(f), \eta \rangle_{\mathcal{E}} = \langle f(w), \eta \rangle_{\mathcal{E}} = \langle f, k(\cdot, w)\eta \rangle_{H},
\]
for all \( f \in H \), \( w \in D \) and \( \eta \in \mathcal{E} \). In particular, if \( \mathcal{E} = \mathbb{C} \), then
\[
e_w(f) = f(w) = \langle f, k(\cdot, w) \rangle_{H}.
\]
Often we say that \( k \) is an analytic kernel and denote the Hilbert space \( H \) by \( H_k \) and call it analytic reproducing kernel Hilbert space (analytic Hilbert space, in short). In this case
\[
k(z, w) = \sum_{n,m=0}^{\infty} C_{mn} z^m \bar{w}^n \quad (z, w \in D),
\]
for some \( C_{mn} \in \mathcal{B}(\mathcal{E}) \), \( m, n \geq 0 \). As usual we say that \( k \) is a diagonal kernel if
\[
C_{mn} = 0 \quad (|m - n| \geq 1).
\]
Following Adams and McGuire [3] (also see the motivating paper Adams, McGuire and Paulsen [4]), we say that \( k \) is a tridiagonal kernel (or band kernel with band-width 1) if
\[
C_{mn} = 0 \quad (|m - n| \geq 2).
\]
Typical examples of diagonal kernels are the Szegö kernel, Bergman kernel, weighted Bergman kernels and Dirichlet kernel [31].
One of the basic ideas in analytic Hilbert spaces is to study the analytic properties of shift operators. The *shift operator* $M_z$ on $\mathcal{H}_k$ is defined by

$$(M_z f)(w) = w f(w) \quad (f \in \mathcal{H}_k, w \in \mathbb{D}).$$

Now let $\{a_n\}_{n \geq 0}$ be a sequence of non-zero scalars and $\{b_n\}_{n \geq 0} \subseteq \mathbb{C}$. Set

$$f_n(z) = (a_n + b_n z) z^n \quad (n \geq 0).$$

Assume that $\{f_n\}_{n \geq 0}$ is an orthonormal basis of an analytic Hilbert space $\mathcal{H}_k$. The well known fact from reproducing kernel Hilbert space theory then implies

$$(1.1) \quad k(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)} \quad (z, w \in \mathbb{D}),$$

and hence $\mathcal{H}_k$ is a tridiagonal space (rearrangement of the series follows from [3, Theorem 1]).

We now turn to Aluthge transforms. The theory of Aluthge transforms was introduced by A. Aluthge [5] in 1990 in his study of $p$-hyponormal operators. Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and let $T = U|T|$ be the polar decomposition of $T$. Here, and throughout this note, $|T| = (T^*T)^{1/2}$ and $U$ is the unique partial isometry such that $\ker U = \ker T$. The *Aluthge transform* of $T$ is the bounded linear operator

$$\tilde{T} = |T|^{1/2} U |T|^{1/2}.$$

In an appropriate sense, the Aluthge transform of an operator is closer to being normal, where, on the other hand, $T$ has a nontrivial invariant subspace if and only if $\tilde{T}$ does. Evidently, the main difficulty associated with Aluthge transform $\tilde{T}$ of $T$ is to compute or to represent the positive part $|T|$ (even when $T = M_z$ on $\mathcal{H}_k$). This is certainly not true, as we shall see later, for weighted shifts.

Finally, we turn to left invertible operators. Analytic model of a left invertible operator was introduced by Shimorin [33] to study wandering subspaces of shift invariant subspaces of a class of (mostly diagonal) reproducing kernel Hilbert spaces. His model theory says that if a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$ is a left invertible operator such that

$$\bigcap_{n=0}^{\infty} T^n \mathcal{H} = \{0\},$$

then there exists a (vector-valued and analytic) reproducing kernel Hilbert space $\mathcal{H}_k$ such that $T$ and the shift $M_z$ on $\mathcal{H}_k$ are unitarily equivalent. In this case, the kernel $k$ is explicit (see [22]) and useful in studying wandering subspace property of invariant subspaces of weighted shifts [22, 33]. We refer the reader to [29] and the extensive list of references therein for recent developments and implementations of Shimorin models.

Let us now briefly recall the concept of weighted shifts (see the classic by Shields [31]). Let $\mathcal{H}$ be a Hilbert space, $\{e_n\}_{n \geq 0}$ an orthonormal basis of $\mathcal{H}$, and let $\alpha = \{\alpha_n\}_{n \geq 0}$ a bounded sequence of positive real numbers. Then $S_\alpha e_n = \alpha_n e_{n+1}$, $n = 0, 1, 2, \ldots$, defines a *weighted shift* with weights $\{\alpha_n\}_{n \geq 0}$. In this case, $S_\alpha$ is bounded if and only if

$$\|S_\alpha\| := \sup_n \frac{\beta_{n+1}}{\beta_n} < \infty,$$
where $\beta_n = \alpha_0\alpha_1 \cdots \alpha_{n-1}$ for all $n > 1$. Moreover, in this case, $S_\alpha^* e_0 = 0$ and $S_\alpha^* e_n = \alpha_{n-1} e_{n-1}$ for all $n = 1, 2, \ldots$. In particular, if the sequence $\{\alpha_n\}_{n \geq 0}$ is bounded away from zero, then $S_\alpha$ is a left invertible but non-invertible operator. A simple computation also shows that (or see the proof of Proposition 2.10) $|S_\alpha| = \text{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots)$, and hence, $\tilde{S}_\alpha = S_{\sqrt{\alpha}}$, where 

$$\sqrt{\alpha} := \{\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \ldots\}.$$

Therefore, the Aluthge transform of the weighted shift $S_\alpha$ is also a weighted shift, namely $S_{\sqrt{\alpha}}$. Moreover, if we apply Shimorin’s model to left invertible $S_\alpha$, then the shift on the Shimorin’s reproducing kernel Hilbert space is also a weighted shift (a well-known fact, however, see Proposition 2.10). In other words, weighted shifts behaves well under Aluthge transforms and Shimorin’s analytic models.

Several natural questions emerge out of the above discussion.

- Here is perhaps the most basic: Consider the shift $M_z$ on an analytic reproducing kernel Hilbert space $\mathcal{H}_k$. Is the Aluthge transform $\tilde{M}_z$ of $M_z$ is a shift on some (explicit) reproducing kernel Hilbert space? Is there a simple/computable formula for $\tilde{M}_z$? What properties does $\tilde{M}_z$ have as an operator?

- And, here is perhaps the most concrete: What are the answers to the above questions for a tridiagonal kernel $k$?

We prove the following set of results:

In Section 2, we employ two natural approaches to prove that, up to unitary equivalence, the Aluthge transform of a left invertible shift $M_z$ on a vector-valued analytic Hilbert space $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}, \mathcal{E})$ (here $\mathcal{E}$ is a Hilbert space and $\mathcal{O}(\mathbb{D}, \mathcal{E})$ denote the space of all $\mathcal{E}$-valued analytic functions on $\mathbb{D}$) is also a left invertible shift on some vector-valued analytic Hilbert space $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}, \mathcal{W})$. The kernel $\tilde{k}$ is given by either

$$(1.2) \quad \langle \tilde{k}(z, w)\eta, \zeta \rangle_\mathcal{E} = \left\langle |M_z|^{-1}(k(\cdot, w)\eta), k(\cdot, z)\zeta \right\rangle_{\mathcal{H}_k},$$

which we call the standard Aluthge kernel of $M_z$ or, defined by

$$\tilde{k}(z, w) = P_W(I - zL)^{-1}(I - \bar{w}L^*)^{-1}|_W \quad (z, w \in \mathbb{D}),$$

which we call the Shimorin-Aluthge kernel of $M_z$ where $W = \ker M_z^*$, $P_W$ is the orthogonal projection from $\mathcal{H}_k$ onto $W$, and

$$L = |M_z|^{\frac{1}{2}}(L_{M_z^*}|M_z|^{-1}L_{M_z})|M_z|^{\frac{1}{2}},$$

and $L_{M_z}$ is the Shimorin left inverse of $M_z$. This is the content of Theorems 2.4 and 2.8. In Theorem 2.6 we prove that if $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$ and $\mathbb{C}[z] \subseteq \mathcal{H}_k$, then the left inverse $L_{M_z}$ of $M_z$ and the left inverse $L_{\tilde{M}_z}$ of $\tilde{M}_z$ (defined on the space corresponding to the Shimorin-Aluthge kernel of $M_z$) are similar up to the perturbation of an operator of rank at most one. Moreover, in this setting the Shimorin-Aluthge transforms are somewhat more explicit. This is the content of Theorem 2.7.

In Section 3, we present certain basic properties and constructions of tridiagonal spaces and study analytic tridiagonal spaces. An analytic tridiagonal kernel is a scalar tridiagonal kernel
as in (1.1) such that
\[ \mathbb{C}[z] \subseteq \mathcal{H}_k, \]
and
\[ \sup_{n \geq 0} \left| \frac{a_n}{a_{n+1}} \right| < \infty \quad \text{and} \quad \limsup_{n \geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1, \]
(which ensures that $M_z$ on $\mathcal{H}_k$ is bounded) and
\[ \left| \frac{a_n}{a_{n+1}} \right| > \epsilon \quad (n \geq 0), \]
for some $\epsilon > 0$. The latter condition means that the sequence \( \left\{ \left| \frac{a_n}{a_{n+1}} \right| \right\}_{n \geq 0} \) is bounded away from zero, and as we will see in Theorem 3.4 this also ensures that $M_z$ on $\mathcal{H}_k$ is left invertible. This assumption is natural as it is also a necessary and sufficient condition for left invertibility of weighted shifts (that is, $b_n = 0$ for all $n \geq 0$ case). In fact, the weighted shift $S_\alpha$ corresponding to the weight sequence \( \{\alpha_n\}_{n=0}^\infty \) is left invertible if and only if the sequence \( \{\alpha_n\}_{n \geq 0} \) is bounded away from zero (cf. Proposition 2.10). A scalar-valued analytic Hilbert space is called analytic tridiagonal space if the kernel function is an analytic tridiagonal kernel. We compute representations of Shimorin left inverses of shifts on analytic tridiagonal spaces (see Proposition 3.3 and Theorem 3.6). In Proposition 3.5 we prove that the shift on an analytic tridiagonal space $\mathcal{H}_k$ is analytic in the sense of Shimorin. Example 3.7 shows that Shimorin-Aluthge kernel of shifts do not necessarily preserve the structure of tridiagonal kernels.

Consider the shift $M_z$ on a tridiagonal space $\mathcal{H}_k$. We are specifically interested in the tridiagonal structure of the standard Aluthge kernel $\tilde{k}$ (the scalar version of (1.2)). Thus, if we wish the kernel $\tilde{k}$ to be a tridiagonal one, then we are reduced to prove that
\[ (z, w) \mapsto \left\langle |M_z|^{-1} k(\cdot, w), k(\cdot, z) \right\rangle_{\mathcal{H}_k}, \]
defines a tridiagonal kernel on $\mathbb{D}$. Note that $|M_z|^{-1}$ is a positive operator on $\mathcal{H}_k$. From this point of view, the main result of Section 4, namely Theorem 4.2 classifies positive operators $P$ on a tridiagonal space $\mathcal{H}_k$ such that
\[ K(z, w) := \left\langle P k(\cdot, w), k(\cdot, z) \right\rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}), \]
defines a tridiagonal kernel on $\mathbb{D}$. More specifically, if
\[ P = \begin{bmatrix}
  c_{00} & c_{01} & c_{02} & c_{03} & \cdots \\
  \bar{c}_{01} & c_{11} & c_{12} & c_{13} & \ddots \\
  \bar{c}_{02} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\
  c_{03} & \bar{c}_{13} & \bar{c}_{23} & c_{33} & \ddots \\
  \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}, \]
represent the matrix representation of $P$ with respect to the basis $\{(a_n + b_n z^n)z^n\}_{n\geq 0}$ of the tridiagonal space $\mathcal{H}_k$, then the positive definite scalar kernel $K$, defined as above, is tridiagonal if and only if

$$c_{0n} = (-1)^{n-1} \frac{\bar{b}_1 \cdots \bar{b}_{n-1}}{a_2 \cdots a_n},$$

for all $n \geq 2$, and

$$c_{mn} = (-1)^{n-m+1} \frac{\bar{b}_{m+1} \cdots \bar{b}_{n-1}}{a_{m+2} \cdots a_n} c_{m,m+1},$$

for all $1 \leq m \leq n - 2$. In Section 5, we discuss tridiagonal representations of Shimorin-Aluthge kernels. It is easy to see that if $k$, as defined in (1.1), is a diagonal kernel (that is, if $b_n = 0$ for all $n \geq 0$), then the Shimorin-Aluthge kernel is also a diagonal kernel (cf. Proposition 2.10). However (as already pointed out), Shimorin models do not necessarily preserve the structure of tridiagonal kernels. We are nevertheless able to prove in Theorem 5.1 that it does for an analytic tridiagonal kernel $k$ if and only if

$$b_0 = 0,$$

or that $M_z$ on $\mathcal{H}_k$ is a weighted shift.

In Section 6 we consider truncated spaces (subclass of analytic tridiagonal spaces) in order to pinpoint more definite results, instructive examples and counterexamples on Shimorin-Aluthge kernels of shifts. A truncated space of order $r (> 2)$ is an analytic tridiagonal space $\mathcal{H}_k$ such that

$$b_n = 0 \quad (n \neq 2, 3, \ldots, r).$$

The computational advantage of a truncated space is that it annihilate a rank one operator (see (2.7)) associated with the Shimorin left inverse of the corresponding shift $M_z$. As a result, in this case we are able to prove a complete classification of tridiagonal Shimorin-Aluthge kernels of shifts. This is the content of Theorem 6.3. Curiously, the classification criterion of Theorem 6.3 is also the classification criterion of tridiagonality of standard Aluthge kernels. That is, in this case (see Corollary 6.4), tridiagonality of Shimorin-Aluthge kernels implies and implied by tridiagonality of standard Aluthge kernels of Theorem 6.3. On the other hand, if we consider a tridiagonal kernel $k$ with

$$b_0 = b_1 = 1 \text{ or } b_0 = 1,$$

and all other $b_i$’s are equal to 0, then the corresponding standard Aluthge kernel of $M_z$ is a tridiagonal kernel but the corresponding Shimorin-Aluthge kernel of $M_z$ is not a tridiagonal kernel. This is the main content of Example 6.5.

In the theory of Aluthge transforms, quasinormal operators have played an important role. In Theorem 7.2 we prove that $M_z$ on an analytic tridiagonal space $\mathcal{H}_k$ (here we do not need left invertibility of shifts) is quasinormal if and only if there exists a positive number $r$ such that

$$M_z^* M_z - M_z M_z^* = r P_{\mathbb{C}f_0},$$

where $P_{\mathbb{C}f_0}$ denote the orthogonal projection of $\mathcal{H}_k$ onto the one dimensional space $\mathbb{C}f_0$ and $\mathbb{C}f_0 = \ker M_z^*$. This is the main content of Section 7.
Finally, we come to some of the important contributions to the theory of Aluthge transforms and related areas. For a thorough analysis of Aluthge transforms we refer the reader to the series of papers by Jung, Ko and Pearcy [21, 22, 23, 24] (also see Ito, Yamazaki and Yanagida [20], Furuta [18] and Furuta and Yanagida [17]). Ando [7] exploited Aluthge transforms to study the relation between numerical ranges and the convex hull of spectrums of linear operators. We also point out another important and related notion, namely, Duggal transforms, and refer the reader to Foias, Jung, Ko and Pearcy [16] on complete contractivity of algebra homomorphisms of Aluthge and Duggal transforms in the sense of Riesz-Dunford functional calculus.

The theory of iterations of Aluthge transforms \( \{\tilde{T}^{(n)}\}_{n \geq 0} \), where \( \tilde{T}^{(0)} = T \) and \( \tilde{T}^{(n)} = \tilde{T}^{(n-1)} \), \( n \geq 1 \) is intricate. Dykema and Schultz [14] proved that the iterated Aluthge transform of an operator \( T \) converges to a normal operator whose Brown measure agrees with that of \( T \). Also see Jung, Kim and Ko [25] in the context of iteration of Aluthge transforms of composition operators, Benhida and Zerouali [10] in the context of backward Aluthge iterates of hyponormal operators. In the setting of finite dimensional Hilbert spaces, iterations of Aluthge transforms is also a challenging problem. See Ando and Yamazaki [9] on 2 \( \times \) 2 matrices, and most notably Antezana, Pujał and Stojanoff [8] on convergence of iterations of matrices. Also see Exner [15] on Aluthge transforms and Agler contractivity. For the study of Aluthge transforms in the setting of multivariable weighted shifts, we refer the reader to Curto and Yoon [13] and the references therein. We refer Abu and Kittaneh [1] and Chabbabi and Mbekhta [12] for spectral radius and numerical radius of Aluthge transforms. We also refer [11, 27, 30] and the references therein for more on Aluthge transforms.

2. ALUTHGE TRANSFORMS OF SHIFTS

In this section, we prove that the Aluthge transform of a left invertible shift on an analytic Hilbert space is again a shift on some analytic Hilbert space. We present two approaches to this problem, one based on Shimorin’s analytic models of left invertible operators and one is based on rather direct reproducing kernel Hilbert space techniques.

We begin with a brief introduction to the construction of Shimorin’s analytic models of left invertible operators. Let \( \mathcal{H} \) be a Hilbert space, and let \( T \in \mathcal{B}(\mathcal{H}) \). We say that \( T \) is left invertible if there exists \( X \in \mathcal{B}(\mathcal{H}) \) such that \( XT = I_\mathcal{H} \). It is easy to check that this equivalently means that \( T \) is bounded below, that is, there exists \( M > 0 \) such that \( \|Tf\| \geq M\|f\| \) for all \( f \in \mathcal{H} \), which is also equivalent to the invertibility of \( T^*T \). A bounded linear operator \( X \in \mathcal{B}(\mathcal{H}) \) is said to be analytic [33] if

\[
\bigcap_{n=1}^{\infty} X^n\mathcal{H} = \{0\}
\]

Note that from the viewpoint of shifts on analytic reproducing kernel Hilbert spaces, analyticity is a natural condition (cf. Proposition 3.3). Let \( T \in \mathcal{B}(\mathcal{H}) \) be a bounded below operator. Then

\[
L_T = (T^*T)^{-1}T^* ,
\]
is an left inverse of $T$. We call $L_T$ the Shimorin left inverse, to distinguish it from other left inverses of $T$. Set

$$W = \ker T^* = \mathcal{H} \ominus \mathcal{T} \mathcal{H},$$

and $\Omega = \{z \in \mathbb{C} : |z| < r(T)\}$, where $r(T)$ is the spectral radius of $T$. Then it follows from Shimorin \cite{33} Corollary 2.14] that

$$k_T(z, w) = P_W(I - zL_T)^{-1}(I - \bar{w}L_T^*)^{-1}|_W (z, w \in \Omega),$$

defines a $\mathcal{B}(W)$-valued analytic kernel $k_T : \Omega \times \Omega \to \mathcal{B}(W)$, which we call the Shimorin kernel (of $T$). We lose no generality by assuming, as we shall do, that $\Omega = \mathbb{D}$. If, in addition, $T$ is analytic, then we have the following (see Shimorin, \cite{33}):

**Theorem 2.1.** Let $T \in \mathcal{B}(\mathcal{H})$ be an analytic left invertible operator. Then $T$ on $\mathcal{H}$ and $M_z$ on $\mathcal{H}_{k_T}$ are unitarily equivalent.

It is easy to see that

$$P_W = I_\mathcal{H} - TL_T,$$

where $P_W$ denotes the orthogonal projection onto $W = \ker T^*$. This plays an important role in the proof of the above theorem.

Now we observe the following simple fact concerning Aluthge transforms of left invertible operators:

**Lemma 2.2.** Let $T$ be a left invertible operator on $\mathcal{H}$. Then

$$\tilde{T} = |T|^{\frac{1}{2}} T |T|^{-\frac{1}{2}},$$

and

$$\ker \tilde{T}^* = |T|^{-\frac{1}{2}} \ker T^*.$$

In particular, $\tilde{T}$ is similar to $T$.

**Proof.** Indeed, $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} (U |T|) |T|^{-\frac{1}{2}} = |T|^{\frac{1}{2}} T |T|^{-\frac{1}{2}}$, as $T^*T$ is invertible. The second equality follows from the first. \qed

However, from reproducing kernel Hilbert space point of view, if we assume that $T$ is a shift on an analytic Hilbert space, then $\tilde{T}$, up to unitary equivalence, is also a shift on an explicit analytic Hilbert space. We will get back to this in Theorem 2.8 and continue our discussion of Shimorin models.

**Proposition 2.3.** Let $T$ be an analytic left invertible operator on $\mathcal{H}$. Then

$$L_T L_T^* = |T|^{-2},$$

and the Shimorin left inverse $L_{\tilde{T}}$ of the Aluthge transform $\tilde{T}$ is given by

$$L_{\tilde{T}} = |T|^{\frac{1}{2}} \left( (L_T |T| T)^{-1} L_T \right) |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} \left( (T^* |T| T)^{-1} T^* \right) |T|^{\frac{1}{2}}.$$
Proof. The first equality follows from
\[ L_T L_T^* = (T^* T)^{-1} T^* T (T^* T)^{-1} = (T^* T)^{-1}. \]
To prove the second one, note that by Lemma 2.2, we have \( \tilde{T} \tilde{T} = |T|^{-\frac{1}{2}} (T^* |T| |T|)^{-\frac{1}{2}} \). Since \( T^* |T| |T| \) is invertible, it follows that
\[ (\tilde{T} \tilde{T})^{-1} = |T|^{\frac{1}{2}} (T^* |T| |T|)^{-\frac{1}{2}} |T|^{\frac{1}{2}}. \]
Then
\[ L_\tilde{T} = (\tilde{T} \tilde{T})^{-1} \tilde{T}^* = (|T|^{\frac{1}{2}} (T^* |T| |T|)^{-\frac{1}{2}} |T|^{\frac{1}{2}})^{-1} \tilde{T}^* = |T|^{\frac{1}{2}} (T^* |T| |T|)^{-\frac{1}{2}} |T|^{\frac{1}{2}}. \]
On the other hand, since \( T^* = |T|^2 L_T \), we have \( T^* |T| |T| = |T|^2 L_T |T| |T| \), and hence
\[ (T^* |T| |T|)^{-1} = (L_T |T| |T|)^{-1} |T|^{-2}. \]
Therefore
\[ (\tilde{T} \tilde{T})^{-1} = |T|^{\frac{1}{2}} (L_T |T| |T|)^{-1} |T|^{-\frac{3}{2}}, \]
which gives
\[ L_\tilde{T} = (\tilde{T} \tilde{T})^{-1} \tilde{T}^* = |T|^{\frac{1}{2}} (L_T |T| |T|)^{-1} |T|^{-2} (T^* |T| |T|)^{-\frac{1}{2}} = |T|^{\frac{1}{2}} (L_T |T| |T|)^{-1} L_T |T|^{\frac{1}{2}}, \]
and completes the proof. \( \blacksquare \)

The above equality \( L_T L_T^* = |T|^{-2} \) will be useful in what follows. As a consequence of the above and Theorem 2.1 we easily derive the following:

**Theorem 2.4.** Let \( E \) be a Hilbert space, and let \( k : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(E) \) be an analytic kernel. Assume that \( M_z \) is an analytic left invertible operator on \( H_k \). Then the Aluthge transform \( M_\tilde{z} \) of \( M_z \) on \( H_k \) is unitarily equivalent to the multiplication operator \( M_z \) on \( H_k \subseteq \mathcal{O}(\mathbb{D}, W) \), where
\[ \tilde{k}(z, w) = P_\tilde{W}(I - z L)^{-1} (I - \tilde{w} L^*)^{-1} |\tilde{W} \rangle \langle \tilde{W}| \quad (z, w \in \mathbb{D}), \]
where \( \tilde{W} = \ker M_\tilde{z}^* \), and
\[ L = |M_z|^{\frac{1}{2}} \left( (L_{M_z} |M_z|^2 M_z)^{-1} L_{M_z} \right) |M_z|^{\frac{1}{2}}. \]

**Definition 2.5.** The kernel \( \tilde{k} \) is called the Shimorin-Aluthge kernel of \( M_z \).

Under some additional assumptions on scalar-valued analytic kernels, we now prove that, up to similarity and a perturbation of an operator of rank at most one, \( L_{\tilde{M}_z} \) and \( L_{M_z} \) are the same. As far as concrete examples are concerned, and as we will see in the case of tridiagonal spaces, these assumptions are indispensable and natural (see for instance Lemma 3.2).

**Theorem 2.6.** Let \( k : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \) be an analytic kernel, \( \mathbb{C}[z] \subseteq \mathcal{H}_k \), and let \( \{ f_n \} \subseteq \mathbb{C}[z] \) be an orthonormal basis of \( \mathcal{H}_k \). Assume that \( M_z \) is an analytic left invertible operator on \( H_k \), \( \ker M_z^* = \mathbb{C} f_0 \), and
\[ f_n \in \text{span}\{z^m : m \geq 1\} \quad (n \geq 1). \]
Then \( L_{\tilde{M}_z} \) and \( L_{M_z} \) are similar up to the perturbation of an operator of rank at most one.
Proof. Since ker $M_z^* = \mathbb{C}f_0$, by the definition of Shimorin left inverse, $L_{M_z}f_0 = 0$ and
\[ L_{M_z}z^n = L_{M_z}M_z(z^{n-1}) = z^{n-1}, \]
that is
\[ L_{M_z}z^n = z^{n-1} \quad (n \geq 1). \]
In particular, $L_{M_z}f_n \in \mathbb{C}[z]$ for all $n \geq 0$. Moreover, for each $n \geq 1$, we have
\[
L_{\tilde{M}_z}(|M_z|^\frac{1}{2}z^n) = |M_z|^\frac{1}{2}(L_{M_z}|M_z|^{-1}L_{M_z})|M_z|z^n = |M_z|^\frac{1}{2}(L_{M_z}|M_z|^{-1}L_{M_z}|M_z|z^{n-1}) = |M_z|^\frac{1}{2}z^{n-1}.
\]
Therefore, we have
\[
\left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2}\right)z^n = L_{M_z}z^n = z^{n-1} \quad (n \geq 1).
\]
Then
\[
\left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2} - L_{M_z}\right)f_n = 0,
\]
that is
\[
\left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2} - L_{M_z}\right)|\text{span}\{f_n: n \geq 1\} = 0.
\]
Finally, we have clearly
\[
\left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2} - L_{M_z}\right)f_0 = \left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2}\right)f_0,
\]
and hence
\[ F := \left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2} - L_{M_z}\right)|_{\mathbb{C}f_0}, \]
is of rank at most one, and consequently $L_{\tilde{M}_z}|M_z|^\frac{1}{2} = |M_z|^\frac{1}{2}(L_{M_z} + F)$. This completes the proof of the theorem.

The following analysis of the finite rank operator $F$, defined as in (2.5), will be useful. Note that
\[ L_{\tilde{M}_z}|M_z|^\frac{1}{2} = |M_z|^\frac{1}{2}(L_{M_z} + F). \]
Let $g \in \mathcal{H}_k$. Clearly, since $L_{M_z}f_0 = 0$, we have
\[
Fg = \left\langle g, f_0 \right\rangle_{\mathcal{H}_k} \left(|M_z|^\frac{1}{2}L_{\tilde{M}_z}|M_z|^\frac{1}{2}f_0\right)
\]
Then Proposition 2.3 implies that
\[ Fg = \left\langle g, f_0 \right\rangle_{\mathcal{H}_k} \left((M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0\right) \quad (g \in \mathcal{H}_k). \]

The appearance of the finite rank operator $F$ causes severe computational difficulties for Aluthge transforms in the setting of Shimorin-Aluthge kernels of shifts (see the examples and the main theorem in Section 6). On the other hand, combining Theorem 2.1, Proposition 2.3 and (2.6), we have:
**Theorem 2.7.** In the setting of Theorem 2.6, the Aluthge transform $\tilde{M}_z$ of $M_z$ on $\mathcal{H}_k$ is unitarily equivalent to the shift $M_z$ on $\mathcal{H}_k$, where

$$\tilde{k}(z, w) = P_W(I - zL)^{-1}(I - \bar{w}L^*)^{-1}|_W,$$

$W = |M_z|^{-\frac{1}{2}} \ker M_z^* = \mathbb{C}(|M_z|^{-\frac{1}{2}}f_0)$, and

$$L = |M_z|^\frac{1}{2}(LM_z + F)|M_z|^{-\frac{1}{2}},$$

and

$$Fg = \langle g, f_0 \rangle_{\mathcal{H}_k}\left((M_z^*|M_z)^{-1}M_z^*|M_z|f_0\right) \quad (g \in \mathcal{H}_k).$$

We now revisit Theorem 2.4 from a direct reproducing kernel Hilbert space standpoint. Indeed, there is a rather simpler and more direct proof of Theorem 2.4 which avoids using the analytic model of left invertible operators. In this case, also, the reproducing kernel of the corresponding Aluthge transform is explicit.

**Theorem 2.8.** Let $\mathcal{E}$ be a Hilbert space, $k : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$ be an analytic kernel. Assume that $M_z$ is an analytic left invertible operator on $\mathcal{H}_k$. Then

$$\left\langle \tilde{k}(z, w)\eta, \zeta \right\rangle_{\mathcal{E}} = \left\langle |M_z|^{-1}(k(\cdot, w)\eta), k(\cdot, z)\zeta \right\rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}, \eta, \zeta \in \mathcal{E}),$$

defines a kernel $\tilde{k} : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$, $M_z$ on $\mathcal{H}_k$ defines a bounded linear operator, and there exists a unitary $U : \mathcal{H}_k \to \mathcal{H}_{\tilde{k}}$ such that $UM_z = M_zU$.

**Proof.** First note that $\tilde{k}$, as defined in the statement, is a $\mathcal{B}(\mathcal{E})$-valued analytic kernel. Indeed

$$\left\langle |M_z|^{-1}(k(\cdot, w)\eta), k(\cdot, z)\zeta \right\rangle_{\mathcal{H}_k} = \left\langle |M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta), |M_z|^{-\frac{1}{2}}(k(\cdot, z)\zeta) \right\rangle_{\mathcal{H}_k},$$

for all $z, w \in \mathbb{D}$ and $\eta, \zeta \in \mathcal{E}$, and $k$ is a $\mathcal{B}(\mathcal{E})$-valued analytic kernel. Then

$$\mathcal{H}_{\tilde{k}} = |M_z|^{-\frac{1}{2}}\mathcal{H}_k = \mathcal{H}_k,$$

and

$$\{|M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta) : w \in \mathbb{D}, \eta \in \mathcal{E}\},$$

is a total set in $\mathcal{H}_{\tilde{k}}$. Moreover

$$\left\langle |M_z|^{-\frac{1}{2}}f, |M_z|^{-\frac{1}{2}}g \right\rangle_{\mathcal{H}_{\tilde{k}}} = \left\langle f, g \right\rangle_{\mathcal{H}_k} \quad (f, g \in \mathcal{H}_k).$$

Clearly, $h \in \mathcal{H}_k \mapsto |M_z|^{-\frac{1}{2}}h$ defines a unitary map $U : \mathcal{H}_k \to \mathcal{H}_{\tilde{k}}$. Now by Lemma 2.2, we have

$$\tilde{M}_z^* = |M_z|^{-\frac{1}{2}}M_z^*|M_z|^{\frac{1}{2}},$$

which implies that

$$\tilde{M}_z^*\left(|M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta)\right) = |M_z|^{-\frac{1}{2}}M_z^*(k(\cdot, w)\eta) = \bar{w}\left(|M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta)\right),$$

for all $z, w, \eta, \zeta \in \mathbb{D}$, and $\eta, \zeta \in \mathcal{E}$. Therefore, $U : \mathcal{H}_k \to \mathcal{H}_{\tilde{k}}$ is unitary.
for all \( w \in \mathbb{D} \) and \( \eta \in \mathcal{E} \). Then
\[
\tilde{M}_z^* U(k(\cdot, w)\eta) = \tilde{M}_z^* \left( |M_z|^{-\frac{1}{2}} (k(\cdot, w)\eta) \right)
= |M_z|^{-\frac{1}{2}} (\bar{w} k(\cdot, w)\eta)
= |M_z|^{-\frac{1}{2}} M_z^*(k(\cdot, w)\eta)
= U M_z^*(k(\cdot, w)\eta),
\]
that is \( U \tilde{M}_z = M_z U \).

**Definition 2.9.** The kernel \( \tilde{k} \) is called the standard Aluthge kernel of \( M_z \).

In particular, if \( k \) is a scalar-valued kernel, then
\[
\tilde{k}(\cdot, w) = |M_z|^{-\frac{1}{2}} k(\cdot, w),
\]
and
\[
\tilde{k}(z, w) = \left\langle |M_z|^{-1} k(\cdot, w), k(\cdot, z) \right\rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}).
\]

Therefore, if \( \mathcal{H}_k \) is a tridiagonal space, then there are two ways to compute the Aluthge kernel \( \tilde{k} \): use Theorem 2.4 or use the one above. However, it is curious to note that, from a general computational point of view, neither approach is completely satisfactory and definite. On the other hand, often the standard Aluthge kernel approach (and sometimes both standard Aluthge kernel and Shimorin-Aluthge kernel approaches) lead to satisfactory results. We will discuss this in the computational part of this paper.

In the setting of left invertible weighted shifts, it is well known that the shift on the analytic Hilbert space \( \mathcal{H}_k \) corresponding to the Shimorin kernel \( k_S \) (see the definition in (2.2)) is also a weighted shift. Nonetheless, we sketch the proof here for the sake of completeness and the reader’s convenience.

**Proposition 2.10.** Let \( S_\alpha \) be the weighted shift with weight sequence \( \{\alpha_n\}_{n \geq 0} \). If the sequence \( \{\alpha_n\}_{n \geq 0} \) is bounded away from zero, then \( S_\alpha \) is left invertible, and the Shimorin kernel \( k_{S_\alpha} \) is a diagonal kernel.

**Proof.** Let \( \{e_n\}_{n \geq 0} \) be an orthonormal basis of a Hilbert space \( \mathcal{H} \), and let \( S_\alpha e_n = \alpha_n e_{n+1} \) for all \( n \geq 0 \). Observe that \( \mathcal{W} = \ker S_\alpha^* = \mathbb{C} e_0 \) and
\[
S_\alpha^* e_n = \begin{cases} 0 & \text{if } n = 0 \\ \alpha_{n-1} e_{n-1} & \text{if } n \geq 1. \end{cases}
\]
Then
\[
S_\alpha^* S_\alpha e_n = \alpha_n^2 e_n \quad (n \geq 0).
\]
Since \( \{\alpha_n\}_{n \geq 0} \) is bounded away from zero, it follows that \( S_\alpha^* S_\alpha \) is invertible, and hence \( S_\alpha \) is left invertible. Then the Shimorin left inverse \( L_{S_\alpha} := (S_\alpha^* S_\alpha)^{-1} S_\alpha^* \) is given by
\[
L_{S_\alpha} e_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{\alpha_{n-1}} e_{n-1} & \text{if } n \geq 1. \end{cases}
\]
Therefore, $L_{S_a}$ is the backward shift, and

$$\begin{align*}
L_{S_a}^m e_n &= \begin{cases} 
0 & \text{if } m > n \\
\frac{1}{\alpha_0 \cdots \alpha_{n-1}} e_0 & \text{if } m = n \\
\frac{1}{\alpha_{n-1} \cdots \alpha_{n-m}} e_{n-m} & \text{if } m < n.
\end{cases}
\end{align*}$$

Moreover $L_{S_a}^m e_n = \frac{1}{\alpha_0 \alpha_{n+1} \cdots \alpha_{n+m-1}} e_{n+m}$ for all $n, m \geq 0$. In particular

$$L_{S_a}^m e_0 = \frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{m-1}} e_m \quad (m \geq 1).$$

Therefore we have

$$P_W L_{S_a}^m L_{S_a}^n e_0 = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{1}{(\alpha_0 \cdots \alpha_{n-1})^2} e_0 & \text{if } m = n,
\end{cases}$$

and thus we have clearly

$$k_{S_a}(z, w) = \sum_{n=0}^{\infty} \frac{1}{(\alpha_0 \cdots \alpha_{n-1})^2} (z \bar{w})^n \quad (z, w \in \mathbb{D}).$$

Notice in the above

$$L_{S_a} = \begin{bmatrix} 
0 & \frac{1}{\alpha_0} & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{\alpha_1} & 0 & \ddots \\
0 & 0 & 0 & \frac{1}{\alpha_2} & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix},$$

that is, $L_{S_a}$ is the backward shift corresponding to the weight sequence $\{\frac{1}{\alpha_i}\}_{n \geq 0}$. As pointed out earlier, this is a well known fact. However, we will see in Example 3.7 that this property does not hold for tridiagonal kernels.

### 3. Tridiagonal Kernels and Left Invertibility

We now turn to tridiagonal reproducing kernel Hilbert spaces. The main contribution of this section is the left invertibility and representations of Shimorin left inverses of shifts on tridiagonal reproducing kernel Hilbert spaces, and a counterexample relating tridiagonal kernels and Shimorin models.

The notion of a tridiagonal reproducing kernel Hilbert space was introduced by Adams and McGuire [3] (also see Adams, McGuire and Paulsen [4] for motivation). Here we avoid finer technicalities and introduce only the necessary features of tridiagonal reproducing kernel Hilbert spaces. Let $\mathcal{E}$ be a Hilbert space, $k$ be a $\mathcal{B}(\mathcal{E})$-valued analytic kernel on $\mathbb{D}$, and let
$\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}, \mathcal{E})$ be the corresponding reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. Then there exists a sequence $\{C_{mn}\}_{m,n \geq 0} \subseteq \mathcal{B}(\mathcal{E})$ such that

$$k(z, w) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n \quad (z, w \in \mathbb{D}).$$

We say that $k$ is a tridiagonal kernel if

$$C_{mn} = 0 \quad (|m - n| \geq 2).$$

In this case, we say that $\mathcal{H}_k$ is a tridiagonal space corresponding to the tridiagonal kernel $k$.

We now introduce analytic tridiagonal spaces. A tridiagonal space $\mathcal{H}_k$ is said to be analytic tridiagonal space if $\mathcal{E} = \mathbb{C}$, $\mathbb{C}[z] \subseteq \mathcal{H}_k$ and there exist a pair of sequences of complex numbers $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, $a_n \neq 0$ for all $n \geq 0$, such that $\{f_n\}_{n \geq 0}$ is an orthonormal basis of $\mathcal{H}_k$, where

$$f_n(z) = (a_n + b_n z) z^n \quad (n \geq 0),$$

and

$$\sup_{n \geq 0} \left| \frac{a_n}{a_{n+1}} \right| < \infty \quad \text{and} \quad \limsup_{n \geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1,$$

and the sequence $\left\{ \left| \frac{a_n}{a_{n+1}} \right| \right\}_{n \geq 0}$ is bounded away from zero. The latter condition means that there is a number $\epsilon > 0$ such that

$$\left| \frac{a_n}{a_{n+1}} \right| > \epsilon \quad (n \geq 0).$$

A positive definite kernel $k : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ is called analytic tridiagonal kernel if the corresponding reproducing kernel Hilbert space $\mathcal{H}_k$ is an analytic tridiagonal space.

Note that the conditions in (3.1) ensure that the shift $M_z$ is a bounded linear operator on $\mathcal{H}_k$ \cite[Theorem 5]{3}. On the other hand, as we will see in Theorem 3.4, condition (3.2) ensures that $M_z$ is left invertible. We refer the reader to \cite[Theorem 2]{3} for the containment of polynomials in tridiagonal reproducing kernel Hilbert spaces. Also, it is worthwhile recalling that a weighted shift $S_{\alpha}$ is bounded if and only if

$$\sup_{n \geq 0} \alpha_n < \infty.$$

In this case, $S_{\alpha}$ is left invertible if and only if $\{\alpha_n\}_{n \geq 0}$ is bounded away from zero (cf. Proposition 2.10). By translating this into the setting of reproducing kernel Hilbert spaces \cite[Proposition 7]{31}, it is clear that the conditions of analytic tridiagonal kernels are natural. In particular, if $b_n = 0$, $n \geq 0$, then (3.2) is a necessary and sufficient condition for left invertibility of shifts on diagonal kernels.

Suppose $k$ is an analytic tridiagonal kernel. It is well known and a general fact \cite{6} that

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)},$$

(3.3)
and hence
\[ k(z, w) = |a_0|^2 + \sum_{n \geq 1} (|a_n|^2 + |b_{n-1}|^2)z^n \bar{w}^n + \sum_{n \geq 0} a_n \bar{b}_n z^n \bar{w}^{n+1} + \sum_{n \geq 0} \bar{a}_n b_n z^{n+1} \bar{w}^n, \]
for all \( z, w \in \mathbb{D} \) (see [3, Theorem 1] on rearrangement of the above series). Now, for each \( n \geq 0 \), we write [3, Section 3]
\[ z^n = \sum_{m=0}^{\infty} \alpha_m f_m, \]
for some \( \alpha_m \in \mathbb{C}, m \geq 0 \). Then
\[ z^n = a_0 a_0 + \sum_{m=1}^{\infty} (\alpha_{m-1}b_{m-1} + \alpha_m a_m)z^m. \]
Thus comparing coefficients, we have
\[ \alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0 \quad \text{and} \quad \alpha_n = \frac{1}{a_n}, \]
as \( a_m \)'s are non-zero scalars. Moreover, \( \alpha_{n+j-1}b_{n+j-1} + \alpha_{n+j}a_{n+j} = 0 \), and hence
\[ \alpha_{n+j} = -\frac{\alpha_{n+j-1}b_{n+j-1}}{a_{n+j}}, \]
and thus
\[ \alpha_{n+j} = \frac{(-1)^j b_n b_{n+1} \cdots b_{n+j-1}}{a_n a_{n+1} \cdots a_{n+j}}, \]
for all \( j \geq 1 \). This implies
\[ (3.4) \quad z^n = \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \left( \prod_{j=0}^{m-1} b_{n+j} \prod_{j=0}^{m-1} a_{n+j+1} \right) f_{n+m} \quad (n \geq 0), \]
where \( \prod_{j=0}^{-1} x_{n+j} := 1 \). With this, we now proceed to compute \( M_z \). Let \( n \geq 0 \). Then
\[
M_z f_n = a_n z^{n+1} + b_n z^{n+2} \\
= \frac{a_n}{a_{n+1}} f_{n+1} + (b_n - \frac{a_n b_{n+1}}{a_{n+1}}) z^{n+2} \\
= \frac{a_n}{a_{n+1}} f_{n+1} + a_{n+2} \left( \frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1} a_{n+2}} \right) z^{n+2},
\]
that is
(3.5) \[ M_z f_n = \frac{a_n}{a_{n+1}} f_{n+1} + a_{n+2} c_n z^{n+2}, \]
where
\[ (3.6) \quad c_n = \frac{a_n}{a_{n+2}} \left( \frac{b_n}{a_{n+1}} - \frac{b_{n+1}}{a_{n+1}} \right) \quad (n \geq 0). \]
Then (3.4) implies that

\[
M_z f_n = \left( \frac{a_n}{a_{n+1}} \right) f_{n+1} + c_n \sum_{m=0}^{\infty} (-1)^m \left( \prod_{j=0}^{m-1} \frac{b_{n+j+2}}{a_{n+j+3}} \right) f_{n+2+m} \quad (n \geq 0).
\]

Hence, the matrix representation \([M_z]\) of \(M_z\) on the analytic tridiagonal space \(H_k\) with respect to the orthonormal basis \(\{f_n\}_{n \geq 0}\) is given by (also see [3, Page 729])

\[
[M_z] = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots \\
\frac{a_0}{a_1} & 0 & 0 & 0 & \ldots \\
\frac{a_1}{a_2} & \frac{a_2}{a_3} & 0 & 0 & \ldots \\
\frac{c_0}{a_3} & \frac{c_1}{a_4} & \frac{c_2}{a_5} & \frac{c_3}{a_6} & \ldots \\
\frac{c_0b_2}{a_3a_4} & \frac{c_1b_3}{a_4a_5} & \frac{c_2b_4}{a_5a_6} & \frac{c_3b_5}{a_6a_7} & \ldots \\
\frac{c_0b_2b_3}{a_3a_4a_5} & \frac{c_1b_3b_4}{a_4a_5a_6} & \frac{c_2b_4b_5}{a_5a_6a_7} & \frac{c_3b_5b_6}{a_6a_7a_8} & \ldots \\
\frac{c_0b_2b_3b_4}{a_3a_4a_5a_6} & \frac{c_1b_3b_4b_5}{a_4a_5a_6a_7} & \frac{c_2b_4b_5b_6}{a_5a_6a_7a_8} & \frac{c_3b_5b_6b_7}{a_6a_7a_8a_9} & \ldots \\
\end{bmatrix}
\]

In particular, \(M_z\) is a weighted shift if and only if \(c_n = 0\) for all \(n \geq 0\). Moreover, by (3.6), we have

\[
c_n = 0 \text{ if and only if } \frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n} \quad (n \geq 0).
\]

Therefore, we have the following useful observation:

**Lemma 3.1.** The shift \(M_z\) on an analytic tridiagonal space \(H_k\) is a weighted shift if and only if \(c_n = 0\) for all \(n \geq 0\), or, equivalently, \(\left\{ \frac{b_n}{a_n} \right\}_{n \geq 0}\) is a constant sequence.

Next, we note that the matrix representation of the conjugate of \(M_z\) is given by

\[
[M_z^*] = \begin{bmatrix}
0 & \frac{a_0}{a_1} & \frac{-c_0b_2}{a_3} & \frac{-c_0b_2b_3}{a_3a_4} & \ldots \\
0 & \frac{a_1}{a_2} & \frac{-c_1}{a_3} & \frac{-c_1b_3}{a_4a_5} & \ldots \\
0 & 0 & \frac{a_2}{a_3} & \frac{-c_2}{a_4} & \ldots \\
0 & 0 & 0 & \frac{a_3}{a_4} & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
\end{bmatrix}
\]

It will be useful to record here the following fact. The proof uses the assumption that \(\mathbb{C}[z] \subseteq H_k\).

**Lemma 3.2.** If \(H_k\) is an analytic tridiagonal space, then

\[
\ker M_z^* = \mathbb{C} f_0.
\]

**Proof.** Clearly, (3.9) implies that \(f_0 \in \ker M_z^*\). On the other hand, from \(\mathbb{C}[z] \subseteq H_k\) we deduce that

\[
f_n = M_z(a_n z^{n-1} + b_n z^n) \in \text{ran} M_z,
\]
for all $n \geq 1$, and hence
\[ \text{span}\{f_n : n \geq 1\} \subseteq \text{ran} M_z. \]
The result now follows from the fact that
\[ C f_0 = \left( \text{span}\{f_n : n \geq 1\} \right)^\perp \supseteq \text{ker} M^*_z. \]

Recall that the conditions in (3.1) on $\{f_n\}_{n \geq 0}$ (or on the tridiagonal kernel $k$) ensures that the shift $M_z$ is a bounded linear operator on $\mathcal{H}_k$. We now use condition (3.2) to prove that $M_z$ is left invertible.

Before we state and prove the result, we need to construct a specific bounded linear operator. The choice of this operator is not accidental, as we will see in Theorem 3.6 that it is nothing but the Shimorin left inverse of $M_z$. For each $n \geq 1$, set

\[ d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}. \]

**Proposition 3.3.** Let $k$ be an analytic tridiagonal kernel corresponding to the orthonormal basis $\{f_n\}_{n \geq 0}$, where $f_n(z) = (a_n + b_n z) z^n$, $n \geq 0$. Then the linear operator $L$ defined by

\[ [L] = \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & \ldots \\ 0 & d_1 & a_2 & 0 & 0 & \ldots \\ 0 & \frac{-d_1 b_2}{a_2} & d_2 & a_3 & 0 & \ldots \\ 0 & \frac{d_1 b_2 b_3}{a_2 a_3} & -\frac{d_2 b_3}{a_3} & d_3 & a_4 & \ldots \\ 0 & \frac{-d_1 b_2 b_3 b_4}{a_2 a_3 a_4} & \frac{d_2 b_3 b_4}{a_3 a_4} & -\frac{d_3 b_4}{a_4} & d_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \]

with respect to the orthonormal basis $\{f_n\}_{n \geq 0}$ defines a bounded linear operator on $\mathcal{H}_k$.

**Proof.** For each $n \geq 1$, we have clearly $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} = a_n \frac{b_n}{a_n} - a_{n-1} \frac{b_{n-1}}{a_{n-1}}$, and hence

\[ |d_n| \leq \left| \frac{a_{n+1}}{a_n} \right| \left| \frac{b_n}{a_n} \right| + \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{b_{n-1}}{a_{n-1}} \right|. \]

Since the sequence $\left\{ \left| \frac{a_{n+1}}{a_n} \right| \right\}_{n \geq 0}$ is bounded away from zero (see (3.2)), we have that

\[ \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| < \infty. \]

This and the second assumption then imply that $\{d_n\}$ is a bounded sequence. Let $S$ denote the matrix obtained from $[L]$ by deleting all but the superdiagonal elements of $[L]$. Similarly, $L_0$ denote the matrix obtained from $[L]$ by deleting all but the diagonal

\[ S = \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & \ldots \\ 0 & d_1 & a_2 & 0 & 0 & \ldots \\ 0 & \frac{-d_1 b_2}{a_2} & d_2 & a_3 & 0 & \ldots \\ 0 & \frac{d_1 b_2 b_3}{a_2 a_3} & -\frac{d_2 b_3}{a_3} & d_3 & a_4 & \ldots \\ 0 & \frac{-d_1 b_2 b_3 b_4}{a_2 a_3 a_4} & \frac{d_2 b_3 b_4}{a_3 a_4} & -\frac{d_3 b_4}{a_4} & d_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \]

with respect to the orthonormal basis $\{f_n\}_{n \geq 0}$ defines a bounded linear operator on $\mathcal{H}_k$.
elements of \([L]\), and in general, assume that \(L_i\) denote the matrix obtained from \([L]\) by deleting all but the \(i\)-th subdiagonal of \([L]\), \(i = 0, 1, 2\ldots\). Since

\[
L = S + \sum_{i \geq 0} L_i,
\]

it clearly suffices to prove that \(S\) and \(\{L_i\}_{i \geq 0}\) are bounded linear operators, and \(S + \sum_{i \geq 0} L_i\) is absolutely convergent. Note that

\[
\|S\| = \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| < \infty.
\]

Moreover, our assumption \(\lim \sup_{n \geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1\) implies that

\[
\left| \frac{b_n}{a_{n+1}} \right| < r \quad (n \geq n_0),
\]

for some \(r < 1\) and \(n_0 \in \mathbb{N}\). Set

\[
M = \sup_{n \geq 1} \left\{ \left| \frac{b_n}{a_{n+1}} \right|, \left| d_n \right| \right\}.
\]

Then \(\|L_i\| \leq M_{i+1}\) for all \(i = 0, \ldots, n_0\), and

\[
\|L_i\| \leq M_{i+1} r^{i-n_0} \quad (i > n_0),
\]

from which it follows that

\[
\|S\| + \sum_{i \geq 0} \|L_i\| = \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| + \sum_{0 \leq i \leq n_0} \|L_i\| + \sum_{i \geq n_0+1} \|L_i\| \\
\leq \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| + \sum_{0 \leq i \leq n_0} \|L_i\| + M_{n_0+1} \left( \sum_{i \geq n_0+1} r^{i-n_0} \right) \\
\leq \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| + \sum_{0 \leq i \leq n_0} \|L_i\| + M_{n_0+1} \frac{r}{1 - r},
\]

and completes the proof of the theorem.

We are now ready to prove that \(M_z\) is left invertible.

**Theorem 3.4.** In the setting of Proposition 3.3, we have

\[
LM_z = I_{\mathcal{H}_k}.
\]

In particular, \(M_z\) is left invertible.

**Proof.** We consider the matrix representations of \(M_z\) and \(L\) as in (3.8) and Proposition 3.3 respectively. Let \([L][M_z] = (\alpha_{mn})_{m,n \geq 0}\). Clearly it suffices to prove that \(\alpha_{mn} = \delta_{mn}\). It is easy to see that \(\alpha_{n,n+k} = 0\) for all \(k \geq 1\). Now by (3.6), we have

\[
(3.11) \quad c_n = -\frac{a_n}{a_{n+2}} d_{n+1} \quad (n \geq 0).
\]
Note that the \( r \)-th column, \( r \geq 0 \), of \([M_z]\) is the transpose of

\[
\begin{pmatrix}
0, \ldots, 0, \frac{a_r}{a_{r+1}}, c_r, \frac{c_r b_{r+2}}{a_{r+3}}, \ldots, (-1)^{n-r-2} \frac{c_r b_{r+2} \cdots b_{n-1}}{a_{r+3} \cdots a_n}, (-1)^{n-r-1} \frac{c_r b_{r+2} \cdots b_n}{a_{r+3} \cdots a_{n+1}}, \ldots
\end{pmatrix},
\]

and the \( n \)-th row, \( n \geq 0 \), of \([L]\) is given by

\[
\begin{pmatrix}
0, (-1)^{n-1} \frac{d_1 b_1 \cdots b_{n-1}}{a_2 \cdots a_n}, (-1)^{n-2} \frac{d_2 b_2 \cdots b_{n-1}}{a_3 \cdots a_n}, (-1)^{n-3} \frac{d_3 b_3 \cdots b_{n-1}}{a_4 \cdots a_n}, \ldots
\end{pmatrix},
\]

\[
\ldots, \frac{-d_{n-1} b_{n-1}}{a_n}, d_n, \frac{a_{n+1}}{a_n}, 0, 0, \ldots
\]

Now, if \( r \leq (n-2) \), then the \( \alpha_{nr} \) (the \((n,r)\)-th entry of \([L][M_z]\)) is given by

\[
\alpha_{nr} = (-1)^{n-r-1} \frac{d_{r+1} b_{r+1} \cdots b_{n-1}}{a_r} + (-1)^{n-r-2} \frac{d_{r+2} b_{r+2} \cdots b_{n-1}}{c_r} + \ldots
\]

and hence, using (3.11), we obtain

\[
\alpha_{nr} = (-1)^{n-r-1} \frac{a_r b_{r+1} \cdots b_{n-1}}{a_{r+1} a_{r+2} \cdots a_n} + (-1)^{n-r-2} \frac{a_r d_{r+1}}{a_{r+2}} \frac{d_{r+2} b_{r+2} \cdots b_{n-1}}{a_{r+3} \cdots a_n} + \ldots
\]

Using \( d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} \) (see (3.10)), we have

\[
\alpha_{nr} = (-1)^{n-r-1} \frac{a_r b_{r+2} \cdots b_{n-1}}{a_{r+2} a_{r+3} \cdots a_n} \left( \frac{b_{r+1} + 1}{a_{r+1}} - \frac{b_n}{a_n} + \left( \frac{b_n}{a_n} - \frac{b_{r+1}}{a_{r+1}} \right) \right) = 0.
\]
For the case \( r = n - 1 \), we have
\[
\alpha_{n,n-1} = d_n \left( \frac{a_{n-1}}{a_n} \right) + \frac{a_{n+1}}{a_n} (c_{n-1})
= \left( \frac{a_{n-1}}{a_n} \right) d_n + \frac{a_{n+1}}{a_n} \left( - \frac{a_{n-1}}{a_{n+1}} d_n \right)
= 0,
\]
and finally
\[
\alpha_{nn} = \left( \frac{a_{n+1}}{a_n} \right) \left( \frac{a_n}{a_{n+1}} \right) = 1,
\]
completes the proof. □

We now consider Shimorin’s analyticity property of shifts (see (2.1)) on analytic tridiagonal spaces.

**Proposition 3.5.** *If \( k \) is an analytic tridiagonal kernel, then \( M_z \) is an analytic left invertible operator on \( H_k \).*

**Proof.** The fact that \( M_z \) is left invertible follows from Theorem 3.4. It remains to show that \( M_z \) is analytic. For any subset \( X \subseteq H_k \), we denote by \( \bigvee X \) the norm closed linear span of \( X \) in \( H_k \). By (3.7), for all \( m, n \geq 0 \), we have
\[
M_z^m f_n \in \bigvee_{j=m+n} ^\infty f_j,
\]
which implies
\[
M_z^m H_k \subseteq \bigvee_{j=m} ^\infty f_j,
\]
and hence
\[
\bigcap_{m=0} ^\infty M_z^m H_k \subseteq \bigcap_{m=0} ^\infty \left[ \bigvee_{j=m} ^\infty f_j \right] = \{0\}.
\]
This completes the proof of the proposition. □

Now let \( H_k \) be an analytic tridiagonal space. Our aim is to compute the Shimorin left inverse \( L_{M_z} = (M_z^* M_z)^{-1} M_z^* \) of \( M_z \) on \( H_k \). What we prove in fact is that \( L \) in Proposition 3.3 is the Shimorin left inverse of \( M_z \). First note, for each \( n \geq 1 \), that
\[
L_{M_z} z^n = (M_z^* M_z)^{-1} M_z^* M_z z^{n-1} = (M_z^* M_z)^{-1} (M_z^* M_z) z^{n-1},
\]
and hence (or see (2.4))
\[
L_{M_z} z^n = z^{n-1}, \quad (3.12)
\]
for all \( n \geq 1 \), that is, \( L_{M_z} \) is the backward shift on \( H_k \) (a well known fact about Shimorin left inverses). On the other hand, by Lemma 3.2 we have \( L_{M_z} f_0 = (M_z^* M_z)^{-1} M_z^* f_0 = 0 \), and hence \( L_{M_z} f_0 = 0 \), which in particular yields
\[
L_{M_z} 1 = - \frac{b_0}{a_0}. \quad (3.13)
\]
Using (3.10), we have
\[
L_{M_z} f_n = L_{M_z} (a_n z^n + b_n z^{n+1})
= a_n z^{n-1} + b_n z^n
= \frac{a_n}{a_{n-1}} (a_{n-1} z^{n-1} + b_{n-1} z^n) + (b_n - \frac{a_n b_{n-1}}{a_{n-1}}) z^n
= \frac{a_n}{a_{n-1}} f_{n-1} + d_n a_n z^n
= \frac{a_n}{a_{n-1}} f_{n-1} + d_n (a_n z^n + b_n z^{n+1}) - d_n b_n z^{n+1},
\]
and hence by (3.4)
\[
L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n f_n - d_n \left( \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=0}^{m} b_{n+j}}{\prod_{j=0}^{m} a_{n+1+j}} f_{n+1+m} \right).
\]
This is the left inverse $L$ of $M_z$ as in Proposition 3.3. Whence the next statement:

**Theorem 3.6.** Let $\mathcal{H}_k$ be an analytic tridiagonal space. If $L$ is as in Proposition 3.3, then the Shimorin left inverse $L_{M_z}$ of $M_z$ is given by
\[
L_{M_z} = L.
\]
In particular
\[
L_{M_z} f_0 = 0,
\]
and
\[
L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n f_n - d_n \left( \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=0}^{m} b_{n+j}}{\prod_{j=0}^{m} a_{n+1+j}} f_{n+1+m} \right) \quad (n \geq 1),
\]
and the matrix representation of $L_{M_z}$ with respect to the orthonormal basis $\{f_n\}_{n \geq 0}$ is given by
\[
[L_{M_z}] = \\
\begin{bmatrix}
0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \ldots \\
0 & d_1 & \frac{a_2}{a_1} & 0 & 0 & \ldots \\
0 & -\frac{d_1 b_1}{a_2} & d_2 & \frac{a_3}{a_2} & 0 & \ldots \\
0 & \frac{d_1 b_1 b_2}{a_2 a_3} & -\frac{d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ldots \\
0 & -\frac{d_1 b_1 b_2 b_3}{a_2 a_3 a_4} & \frac{d_2 b_2 b_3}{a_3 a_4} & -\frac{d_3 b_3}{a_4} & \quad & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]

Recall from Theorem 2.1 that, if $M_z$ is left invertible on an $\mathcal{E}$-valued analytic reproducing kernel Hilbert space $\mathcal{H}_k$, then $M_z$ on $\mathcal{H}_k$ is unitarily equivalent to the shift $M_z$ on $\mathcal{H}_{kM_z}$, where
\[
k_{M_z}(z, w) = \sum_{m, n=0}^{\infty} \left( P_W L_{M_z}^m L_{M_z}^n |W \right) z^m w^n \quad (z, w \in \mathbb{D}),
\]
(see (2.2)) and $L_{M_z}$ is the Shimorin left inverse of $M_z$ on $\mathcal{H}_k$. As emphasized already in Proposition 2.10 that if $k$ is a diagonal kernel, then $k_{M_z}$ is also a diagonal kernel, that is, if $M_z$ on $\mathcal{H}_k$ is a weighted shift, then $M_z$ on $\mathcal{H}_{k_{M_z}}$ is also a weighted shift. However, the following example says that Shimorin models are not compatible with tridiagonal kernels.

**Example 3.7.** Let $a_n = 1$ for all $n \geq 0$, $b_0 = \frac{1}{2}$, and let $b_n = 0$ for all $n \geq 1$. Let $\mathcal{H}_k$ denote the tridiagonal reproducing kernel Hilbert space corresponding to the orthonormal basis $\{f_n\}_{n \geq 0}$, where $f_n = (a_n + b_nz)z^n$ for all $n \geq 0$. Since $f_0 = 1 + \frac{1}{2}z$ and $f_n = z^n$ for all $n \geq 1$, by (3.8), we have

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2} & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$  

By Theorem 3.6, the Shimorin left inverse $L_{M_z} = (M_z^*M_z)^{-1}M_z^*$ is given by

$$L_{M_z} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$  

Recall, in this case, that $W = \mathbb{C}f_0$. It is easy to check that

$$L_{M_z}f_1 = f_0 - \frac{1}{2}f_1,$$

and

$$L_{M_z}^*f_0 = f_1, \quad L_{M_z}^*f_1 = -\frac{1}{2}f_1 + f_2, \quad \text{and} \quad L_{M_z}^*f_2 = f_3.$$  

Then

$$L_{M_z}^3f_0 = -\frac{1}{2}L_{M_z}^*f_1 + L_{M_z}^*f_2 = \frac{1}{4}f_1 - \frac{1}{2}f_2 + f_3,$$

and hence

$$P_WL_{M_z}L_{M_z}^3f_0 = \frac{1}{4}P_W(L_{M_z}f_1),$$  

as $P_WL_{M_z}f_j = 0$ for all $j \neq 1$. Consequently

$$P_WL_{M_z}L_{M_z}^3f_0 = \frac{1}{4}f_0 \neq 0,$$

and hence, the Shimorin kernel $\tilde{k}$, as defined in (2.2), is not a tridiagonal kernel.
This example motivates one to ask: How to determine whether or not the Shimorin kernel \( k_{M_z} \) of a tridiagonal kernel \( k \) is also tridiagonal? We have a completely satisfactory answer to this question: \( k_{M_z} \) is tridiagonal if and only if \( b_0 = 0 \) or that \( M_z \) is a weighted shift on \( \mathcal{H}_k \). However, its detailed proof is somewhat lengthy and involved. We postpone this discussion till Section 5. Along the way, we will consider a similar (and practical) question.

4. Positive operators and tridiagonal kernels

In the present section, our aim is to classify positive operators \( P \) on tridiagonal spaces \( \mathcal{H}_k \) such that

\[
\mathbb{D} \times \mathbb{D} \ni (z, w) \mapsto \langle P k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k},
\]

is also a tridiagonal kernel. While this problem is of independent interest, the motivation for our interest in this question comes from Theorem 2.8 (see the paragraph preceding Corollary 4.3). We start with a simple example.

**Example 4.1.** We consider the same example as in Example 3.7. Note that \( M_z \) is left invertible and not a weighted shift with respect to the orthonormal basis \( \{ f_n \}_{n \geq 0} \) of \( \mathcal{H}_k \). Then by Proposition 2.3, we have

\[
|M_z|^{-2} = L_{M_z} L_{M_z}^* = \begin{bmatrix}
1 & -\frac{1}{2} & 0 & 0 & \ldots \\
-\frac{1}{2} & \frac{5}{4} & 0 & 0 & \ddots \\
0 & 0 & 1 & 0 & \ddots \\
0 & 0 & 0 & 1 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Let

\[
|M_z|^{-1} = \begin{bmatrix}
\alpha & \beta & 0 & 0 & \ldots \\
\beta & \gamma & 0 & 0 & \ddots \\
0 & 0 & 1 & 0 & \ddots \\
0 & 0 & 0 & 1 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

where \( B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \) is the positive square root of \( \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{bmatrix} \). Define \( K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \) by

\[
K(z, w) = \langle |M_z|^{-1} k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}).
\]

A simple calculation then shows that

\[
\frac{\alpha}{2} + \beta \neq 0,
\]

and

\[
K(z, w) = \alpha + \frac{\alpha}{2} \bar{w} + (\alpha + \beta) z + \frac{\alpha}{4} + \beta + \gamma \bar{z} w + \sum_{n \geq 2} z^n \bar{w}^n,
\]
that is, $K$ is also a tridiagonal kernel.

Now we prove the main result of this section.

**Theorem 4.2.** Let $H_k$ be a tridiagonal space corresponding to the orthonormal basis $f_n(z) = (a_n + b_n z)z^n, n \geq 0$, and let $P$ be a positive operator on $H_k$. Suppose

$$P = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & \cdots \\ \bar{c}_{01} & c_{11} & c_{12} & c_{13} & \cdots \\ \bar{c}_{02} & \bar{c}_{12} & c_{22} & c_{23} & \cdots \\ \bar{c}_{03} & \bar{c}_{13} & \bar{c}_{23} & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

represent the matrix representation of $P$ with respect to the basis $\{f_n\}_{n \geq 0}$. Then the positive definite scalar kernel $K$, defined by

$$K(z, w) = \left\langle P k(\cdot, w), k(\cdot, z) \right\rangle_{H_k} \quad (z, w \in \mathbb{D}),$$

is tridiagonal if and only if

$$c_{0n} = (-1)^{n-1} \frac{\bar{b}_1 \cdots \bar{b}_{n-1}}{a_2 \cdots a_n} \quad (n \geq 2),$$

and

$$c_{mn} = \frac{(-1)^{n-m+1} \bar{b}_{m+1} \cdots \bar{b}_{n-1}}{a_{m+2} \cdots a_n} c_{m,m+1},$$

for all $1 \leq m \leq n - 2$. Equivalently, $K$ is tridiagonal if and only if

$$P = \begin{bmatrix} c_{00} & c_{01} & -\frac{b_1}{a_2}c_{01} & \frac{b_1 b_2}{a_2 a_3}c_{01} & \cdots \\ \bar{c}_{01} & c_{11} & c_{12} & -\frac{b_2}{a_3}c_{12} & \cdots \\ -\frac{b_1}{a_2} \bar{c}_{01} & \bar{c}_{12} & c_{22} & c_{23} & \cdots \\ \frac{b_1 b_2}{a_2 a_3} \bar{c}_{01} & -\frac{b_2}{a_3} \bar{c}_{12} & \bar{c}_{23} & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**Proof.** Note, for each $w \in \mathbb{D}$, by (3.3), we have

$$k(\cdot, w) = \sum_{m=0}^{\infty} \frac{f_m(w)}{m} f_m,$$

and thus

$$Pk(\cdot, w) = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m-1} \bar{c}_{nm} f_n(w) + \sum_{n=m}^{\infty} c_{mn} f_n(w) \right) f_m.$$
where $\sum_{n=0}^{-1} x_n := 0$. Then

$$\left\langle Pk(\cdot, w), k(\cdot, z) \right\rangle_{H_k} = \sum_{m=0}^{\infty} f_m(z) \left( \sum_{n=0}^{m-1} \bar{c}_{nm} f_n(w) + \sum_{n=m}^{\infty} c_{mn} f_n(w) \right)$$

$$= \sum_{m=0}^{\infty} (a_m z^m + b_m z^{m+1}) \left( \sum_{n=0}^{m-1} \bar{c}_{nm} (\bar{a}_n \bar{w}^n + \bar{b}_n \bar{w}^{n+1}) \right)$$

$$+ \sum_{n=m}^{\infty} c_{mn} (\bar{a}_n \bar{w}^n + \bar{b}_n \bar{w}^{n+1})$$

$$= \sum_{m,n \geq 0} \alpha_{mn} z^m \bar{w}^n,$$

where $\alpha_{mn}$ denotes the coefficient of $z^m \bar{w}^n$, $m, n \geq 0$. Our interest here is to compute $\alpha_{mn}$, $|m - n| \geq 2$. Clearly, $\alpha_{mn} = \bar{\alpha}_{nm}$ for all $m, n \geq 0$, and

$$\alpha_{0n} = a_0 (\bar{a}_n c_{0n} + \bar{b}_{n-1} c_{0,n-1}) \quad (n \geq 2), \tag{4.1}$$

and

$$\alpha_{mn} = a_m (\bar{a}_n c_{mn} + \bar{b}_{n-1} c_{m,n-1}) + b_{m-1} (\bar{a}_n c_{m-1,n} + \bar{b}_{n-1} c_{m-1,n-1}) \quad (1 \leq m < n). \tag{4.2}$$

Suppose $n \geq 2$. By (4.1), $\alpha_{0n} = 0$ if and only if

$$c_{0n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{0,n-1}.$$

In particular, if $n = 2$, then

$$c_{02} = -\frac{\bar{b}_1}{\bar{a}_2} c_{01},$$

and hence, by (4.1) again, we have

$$c_{0n} = (-1)^{n-1} \prod_{i=1}^{n-1} \frac{\bar{b}_i}{\bar{a}_i} c_{01} \quad (n \geq 2).$$

Therefore, $\alpha_{0n} = 0$ for all $n \geq 2$ if and only if the above identity hold for all $n \geq 2$.

Next we want to consider the case $m, n \neq 0$ and $|m - n| \geq 2$. Assume that $n \geq 3$. Then (4.2) along with (4.1) implies

$$\alpha_{1n} = a_1 (\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1}) + b_{m-1} (\bar{a}_n c_{0n} + \bar{b}_{n-1} c_{0,n-1})$$

$$= a_1 (\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1}) + \frac{b_{m-1}}{a_0} \alpha_{0n}.$$ 

Therefore, if $\alpha_{0n} = 0$ for all $n \geq 3$, then $\alpha_{1n} = a_1 (\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1})$. Hence $\alpha_{1n} = 0$ if and only if

$$\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1} = 0,$$
which is equivalent to
\[ c_{1n} = -\frac{\bar{b}_{n-1}}{a_n}c_{1,n-1}. \]

Therefore, under the assumption that \( \alpha_{1n} = 0 \) and \( n \geq 4 \), (4.2) along with (4.1) implies
\[
\alpha_{2n} = a_2(\bar{a}_nc_{2n} + \bar{b}_{n-1}c_{2,n-1}) + b_{m-1}(\bar{a}_nc_{1n} + \bar{b}_{n-1}c_{1,n-1})
= a_2(\bar{a}_nc_{2n} + \bar{b}_{n-1}c_{2,n-1}).
\]

Then \( \alpha_{2n} = 0 \), \( n \geq 4 \), if and only if
\[ c_{2n} = -\frac{\bar{b}_{n-1}}{a_n}c_{2,n-1}. \]

Consequently, by induction, for all \( m, n \neq 0 \) and \( |m - n| \geq 2 \), we have that \( \alpha_{mn} = 0 \) if and only if
\[ \bar{a}_nc_{mn} + \bar{b}_{n-1}c_{m,n-1} = 0, \]
or equivalently
\[ c_{mn} = -\frac{\bar{b}_{n-1}}{a_n}c_{m,n-1}. \]

Finally, observe that
\[ c_{mn} = (-1)^{n-m+1}\frac{\bar{b}_{n-1}\cdots\bar{b}_{m+1}}{a_n\cdots a_{m+2}}c_{m,m+1}, \]
for all \( 1 \leq m \leq n - 2 \). This completes the proof of the theorem.

Let \( \mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}) \) be a reproducing kernel Hilbert space. Suppose \( M_z \) on \( \mathcal{H}_k \) is left invertible and analytic. Then Theorem 2.8 says that \( \tilde{M}_z \) and \( M_z \) on \( \mathcal{H}_k(\subseteq \mathcal{O}(\mathbb{D})) \) are unitarily equivalent, where
\[ \tilde{k}(z, w) := \left\langle |M_z|^{-1}k(\cdot, w), k(\cdot, z) \right\rangle_{\mathcal{H}_k} = \left\langle |M_z|^{-1}k(\cdot, w) \right\rangle(z), \]
for all \( z, w \in \mathbb{D} \). In the following, as a direct application of Theorem 4.2, we address the issue of tridiagonal representation of the shift \( M_z \) on \( \mathcal{H}_k \).

Corollary 4.3. In the setting of Theorem 2.8, assume in addition that \( \mathcal{E} = \mathbb{C} \) and \( \mathcal{H}_k \) is a tridiagonal space with respect to the orthonormal basis \( \{f_n\}_{n \geq 0} \), where \( f_n(z) = (a_n + b_nz)z^n \), \( n \geq 0 \). Then \( \mathcal{H}_k \) is a tridiagonal space if and only if
\[
U|M_z|U^* = \begin{bmatrix}
    c_{00} & c_{01} & -\frac{\bar{b}_1}{a_2}c_{01} & \frac{\bar{b}_1\bar{b}_{2}}{a_2a_3}c_{01} & \cdots \\
    \bar{c}_{01} & c_{11} & c_{12} & -\frac{\bar{b}_2}{a_3}c_{12} & \cdots \\
    -\frac{\bar{b}_1}{a_2}c_{01} & \bar{c}_{12} & c_{22} & c_{23} & \cdots \\
    \frac{\bar{b}_1\bar{b}_{2}}{a_2a_3}c_{01} & -\frac{\bar{b}_2}{a_3}c_{12} & \bar{c}_{23} & c_{33} & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix},
\]
with respect to the basis \( \{f_n\}_{n \geq 0} \).
Proof. Recall from Theorem 2.8 that $\mathcal{H}_k = |M_z|^{-\frac{1}{2}}\mathcal{H}_k$ and $U h = |M_z|^{-\frac{1}{2}}h$, $h \in \mathcal{H}_k$, defines the intertwining unitary. Set $P := U|M_z|U^*$. Then $P \in \mathcal{B}(\mathcal{H}_k)$ is a positive operator, and for any $z, w \in \mathbb{D}$, we have

$$\langle P\tilde{k}(\cdot, w), \tilde{k}(\cdot, z) \rangle_{\mathcal{H}_k} = \langle |M_z|U^*\tilde{k}(\cdot, w), U^*\tilde{k}(\cdot, z) \rangle_{\mathcal{H}_k} = \langle |M_z|^{-\frac{1}{2}}k(\cdot, w), |M_z|^{-\frac{1}{2}}k(\cdot, z) \rangle_{\mathcal{H}_k},$$

as $Uk(\cdot, w) = |M_z|^{-\frac{1}{2}}k(\cdot, w)$. Hence

$$k(z, w) = \langle P\tilde{k}(\cdot, w), \tilde{k}(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}).$$

The result now follows from Theorem 4.2.

In particular, if $\tilde{k}$ is a tridiagonal kernel, then for $k$ to be a tridiagonal kernel, it is necessary (as well as sufficient) that $U|M_z|U^*$ is of the form as in the statement of Corollary 4.3.

5. Shimorin models and tridiagonal kernels

Throughout this section, $\mathcal{H}_k$ will be an analytic tridiagonal space corresponding to the orthonormal basis $\{f_n\}_{n \geq 0}$, where

$$f_n(z) = (a_n + b_n z)z^n \quad (n \geq 0).$$

Recall that the Shimorin kernel $k_{M_z}$ of $M_z$ on $\mathcal{H}_k$ is the operator-valued kernel function (see (2.2) and also Theorem 2.1) $k_{M_z} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$k_{M_z}(z, w) = P_{\mathcal{W}}(I - z L_{M_z})^{-1}(I - \bar{w} L_{M_z}^*)^{-1}|_{\mathcal{W}} \quad (z, w \in \mathbb{D}).$$

Here, of course, $\mathcal{W} = \mathbb{C} f_0$, the one-dimensional space generated by the vector $f_0$. So one may regard $k_{M_z}$ as a scalar kernel. The purpose of this section is to prove the following characterization:

**Theorem 5.1.** The Shimorin kernel $k_{M_z}$ of $M_z$ is tridiagonal if and only if $M_z$ on $\mathcal{H}_k$ is a weighted shift or

$$b_0 = 0.$$  

**Proof.** We split the proof into several steps.

**Step 1:** We first denote $L_{M_z} = L$ and

$$X_{mn} = P_{\mathcal{W}}L^nL^*|_{\mathcal{W}} \quad (m, n \geq 0),$$

for simplicity. First observe that Theorem 3.6 implies that

$$L^m f_0 = 0 \quad (m \geq 1),$$
and hence, $X_{m0} = 0 = X^*_{m0} = X_0m$ for all $m \geq 1$. Then the formal matrix representation of the Shimorin kernel $k_{Mz}$ is given by

$$\begin{bmatrix}
I_{Mz} & 0 & 0 & 0 & \ldots \\
0 & X_{11} & X_{12} & X_{13} & \ldots \\
0 & X_{12} & X_{22} & X_{23} & \ldots \\
0 & X_{13} & X_{23} & X_{33} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$  

Clearly, in view of the above, $k_{Mz}$ is tridiagonal if and only if

$$X_{mn}f = 0,$$

for all $m, n \neq 0$ and $|m - n| \geq 2$.

**Step 2:** In this step we aim to compute matrix representations of $L^p$ and $L^{*p}$, $p \geq 1$, with respect to the orthonormal basis $\{f_n\}_{n \geq 0}$. The matrix representation of $[L]$ in Theorem 3.6 is instructive. It also follows that

$$\begin{bmatrix}
\alpha_1 & 0 & 0 & 0 & \ldots \\
\alpha_2 & \frac{\tilde{d}_1}{d_1} & \frac{-\tilde{d}_1 b_1}{a_2} & \frac{\tilde{d}_1 b_1 b_2}{a_2 a_4} & \ldots \\
0 & \frac{\tilde{d}_2}{a_1} & \frac{-\tilde{d}_2 b_2}{a_3} & \frac{-\tilde{d}_2 b_2 b_3}{a_3 a_4} & \ldots \\
0 & 0 & \frac{\tilde{d}_3}{a_2} & \frac{-\tilde{d}_3 b_3}{a_4} & \ldots \\
0 & 0 & 0 & \frac{\tilde{d}_4}{a_3} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$  

Here we redo the construction taking into account the general $p \geq 1$, and proceed as in the proof of Theorem 3.6. However, the proofs are by no means the same and the general case is quite involved. Assume that $n \geq 1$. We need to consider two cases: $n \geq p$ and $n \leq p - 1$. Suppose $n \geq p$. By (3.12) and (3.13), we have

$$L^p f_n = a_n L^p z^n + b_n L^{*p} z^{n+1}$$

$$= a_n z^{n-p} + b_n z^{n-p+1}$$

$$= \frac{a_n}{a_{n-p}} (a_{n-p} z^{n-p} + b_{n-p} z^{n-p+1}) + \left( b_n - \frac{a_n}{a_{n-p}} b_{n-p} \right) z^{n-p+1}$$

$$= \frac{a_n}{a_{n-p}} f_{n-p} + d_n^{(p)} z^{n-p+1},$$

where

$$d_n^{(p)} = b_n - \frac{a_n}{a_{n-p}} b_{n-p} \quad (n \geq p).$$  

Hence by (3.4)

$$L^p f_n = \frac{a_n}{a_{n-p}} f_{n-p} + \frac{d_n^{(p)}}{a_{n-p+1}} \left( f_{n-p+1} - \frac{b_{n-p+1}}{a_{n-p+2}} f_{n-p+2} + \frac{b_{n-p+1} b_{n-p+2}}{a_{n-p+2} a_{n-p+3}} f_{n-p+3} + \cdots \right),$$
that is
\[ L^p f_n = \frac{a_n}{a_{n-p}} f_{n-p} + \frac{d^{(p)}_n}{a_{n-p+1}} \sum_{m=0}^{\infty} (-1)^m \left( \frac{\prod_{j=0}^{m-1} b_{n-p+j+1}}{\prod_{j=0}^{m-1} a_{n-p+j+2}} \right) f_{n-p+m+1}, \]
for all \( n \geq p \).

We now let \( p = 1 \) and \( n = 1 \). Then by Theorem 3.6, we have
\[ (5.4) \quad L f_1 = \frac{a_1}{a_0} f_0 + d_1 f_1 + (-1) \frac{d_1 b_1}{a_2} f_2 + \left( \frac{d_1 b_1 b_2}{a_2 a_3} \right) f_3 + \cdots. \]

Finally, let \( 1 \leq n \leq p - 1 \). Then \( p > 1 \), and again by (3.12) and (3.13), we have
\[ L^p f_n = L^p (a_n z^n + b_n z^{n+1}) = a_n L^{p-n} + b_n L^{p-n-1} = a_n \left( -\frac{b_0}{a_0} \right)^{p-n} + b_n \left( -\frac{b_0}{a_0} \right)^{p-n-1} = a_n \left( -\frac{b_0}{a_0} \right)^{p-n-1} \frac{b_n}{a_n - b_0}. \]

We set
\[ (5.5) \quad \beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0} \quad (n \geq 1), \]
and
\[ (5.6) \quad \beta^{(p)}_n = a_n \left( -\frac{b_0}{a_0} \right)^{p-n-1} \beta_n \quad (1 \leq n \leq p - 1). \]

Then
\[ L^p f_n = \beta_n^{(p)} \quad (1 \leq n \leq p - 1). \]

This and (3.4) implies that
\[ L^p (f_n) = \frac{\beta_n^{(p)}}{a_0} \sum_{m=0}^{\infty} (-1)^m \left( \frac{\prod_{j=0}^{m-1} b_j}{\prod_{j=0}^{m-1} a_{j+1}} \right) f_m, \]
for all \( 1 \leq n \leq p - 1 \). Then
\[ (5.7) \quad [L^2] = \begin{bmatrix} 0 & \beta^{(2)}_0 & \frac{a_2}{a_0} & 0 & 0 & \cdots \\ \frac{\beta^{(2)}_0 b_0}{a_0} & 0 & \frac{a_2}{a_0} & \frac{a_3}{a_1} & 0 & \cdots \\ \frac{\beta^{(2)}_0 a_1}{a_0} & -\frac{a_2}{a_0} & 0 & \frac{a_3}{a_1} & \frac{a_4}{a_2} & \cdots \\ \frac{\beta^{(2)}_0 b_0 b_1}{a_0 a_1} & -\frac{a_2}{a_0} & -\frac{a_3}{a_1} & 0 & \cdots & \cdots \\ \frac{\beta^{(2)}_0 a_1 a_2}{a_0 a_1} & \frac{a_2}{a_0} & \frac{a_3}{a_1} & -\frac{a_4}{a_2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \]
and in general, for each \( p \geq 2 \), we have

\[
[L^p] = \begin{bmatrix}
0 & \frac{\beta^{(p)}_1}{a_0} & \frac{\beta^{(p)}_2}{a_0} & \cdots & \frac{\beta^{(p)}_{p-1}}{a_0} & \frac{\beta^{(p)}_p}{a_0} & 0 & 0 & \cdots \\
0 & -\frac{\beta^{(p)}_2}{a_0} & -\frac{\beta^{(p)}_3}{a_0} & \cdots & -\frac{\beta^{(p)}_p}{a_0} & -\frac{\beta^{(p)}_1}{a_0} & 0 & 0 & \cdots \\
0 & \frac{\beta^{(p)}_1}{a_0}a_1 & \frac{\beta^{(p)}_2}{a_0}a_1 & \cdots & \frac{\beta^{(p)}_p}{a_0}a_1 & \frac{\beta^{(p)}_1}{a_0}a_2 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Hence, for each \( p \geq 2 \), we have

\[
[L^{*p}] = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
\beta^{(p)}_1 & -\beta^{(p)}_2 & \beta^{(p)}_3 & \cdots & \beta^{(p)}_p & -\beta^{(p)}_1 & \beta^{(p)}_2 & \cdots \\
\frac{\beta^{(p)}_1}{a_0} & -\frac{\beta^{(p)}_2}{a_0} & \frac{\beta^{(p)}_3}{a_0} & \cdots & \frac{\beta^{(p)}_p}{a_0} & -\frac{\beta^{(p)}_1}{a_0} & \frac{\beta^{(p)}_2}{a_0} & \cdots \\
\frac{\beta^{(p)}_1}{a_0}a_1 & -\frac{\beta^{(p)}_2}{a_0}a_1 & \frac{\beta^{(p)}_3}{a_0}a_1 & \cdots & \frac{\beta^{(p)}_p}{a_0}a_1 & -\frac{\beta^{(p)}_1}{a_0}a_2 & \frac{\beta^{(p)}_2}{a_0}a_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

**Step 3:** We now identify condition on the sequence \( \{\beta^{(n+2)}_n\}_{n \geq 1} \) implied by the requirement that

\[
X_{m,m+2} = 0 \quad (m \geq 1).
\]

Before proceeding further, we record here the following crucial observation: Suppose \( \beta^{(p)}_n = 0 \) for some \( p \) and \( n \) such that \( 1 \leq n \leq p - 1 \). Then by (5.6), we have

\[
(5.10) \quad \beta^{(q)}_n = 0 \quad (q \geq p).
\]

Now assume \( m \geq 1 \). The matrix representation in (5.9) implies

\[
(5.11) \quad L^{*m+2}f_0 = \frac{1}{\beta^{(m+2)}_1} \left( \beta^{(m+2)}_1f_1 + \beta^{(m+2)}_2f_2 + \cdots + \beta^{(m+2)}_{m+1}f_{m+1} + \bar{a}_{m+2}f_{m+2} \right).
\]

Observe that, by Theorem 3.7, we have

\[
P_{\mathcal{W}}L(f_i) = \begin{cases} \frac{a_i}{a_0}f_0 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}
\]
Let us now assume \( m \geq 2 \). Then (5.8) implies

\[
P_W L^m (f_i) = \begin{cases} \frac{\bar{r}^{(m)}}{a_0} f_0 & \text{if } 1 \leq i \leq m - 1 \\ \frac{\bar{a}^{(m)}}{a_0} f_0 & \text{if } i = m \\ 0 & \text{if } i \geq m + 1. \end{cases}
\]

Since \( X_{m,m+2} = P_W L^m L^{m+2} |_{\mathcal{W}} \), this yields

\[
X_{m,m+2} f_0 = \frac{1}{|a_0|^2} \left( \bar{\beta}_1^{(m+2)} \beta_1^{(m)} + \bar{\beta}_2^{(m+2)} \beta_2^{(m)} + \cdots + \bar{\beta}_{m-1}^{(m+2)} \beta_{m-1}^{(m)} + \bar{\beta}_m^{(m+2)} a_m \right) f_0.
\]

In particular, if \( m = 1 \), then we have

\[
X_{13} f_0 = \frac{1}{\bar{a}_0} \left( \beta_1^{(3)} a_1 \right) f_0,
\]

and hence, from (5.6) we have

\[
X_{13} = 0 \text{ if and only if } \beta_1^{(3)} = 0.
\]

By (5.13), applied with \( m = 2 \) we have

\[
X_{24} f_0 = \frac{1}{|a_0|^2} \left( \bar{\beta}_1^{(4)} \beta_1^{(2)} + \bar{\beta}_2^{(4)} a_2 \right) f_0.
\]

Assume that \( \beta_1^{(3)} = 0 \). By (5.10)

\[
\beta_1^{(4)} = 0,
\]

and, consequently

\[
X_{24} f_0 = \bar{\beta}_2^{(4)} a_2 f_0.
\]

Hence

\[
X_{24} = 0 \text{ if and only if } \beta_2^{(4)} = 0.
\]

Therefore, if \( X_{m,m+2} = 0 \) for all \( m \geq 1 \), then by induction, it follows that \( \beta_m^{(m+2)} = 0 \) for all \( m \geq 1 \). The converse also follows from the above computation.

Thus we have proved: \( X_{m,m+2} = 0 \) for all \( m \geq 1 \) if and only if \( \beta_m^{(m+2)} = 0 \) for all \( m \geq 1 \).

**Step 4:** Our aim is to prove the following claim: Suppose \( X_{i,i+2} = 0 \) for all \( i = 1, \ldots, m \). Then \( X_{m,n} = 0 \) for all \( n = m + 3, m + 4, \ldots \).

To this end, let \( n = m + j \) and \( j \geq 3 \). Then the matrix representation in (5.9) (or the equality (5.11) implies

\[
L^* f_0 = \frac{1}{\bar{a}_0} \left( \bar{\beta}_1^{(n)} f_1 + \bar{\beta}_2^{(n)} f_2 + \cdots + \bar{\beta}_{n-1}^{(n)} f_{n-1} + \bar{a}_n f_n \right),
\]

and then

\[
P_W L^m L^* f_0 = \left( \frac{1}{\bar{a}_0} \sum_{i=1}^{n-1} \bar{\beta}_i^{(n)} P_W L^m (f_i) \right) + \frac{\bar{a}_n}{\bar{a}_0} P_W L^m f_n
\]

\[
= \frac{1}{\bar{a}_0} \sum_{i=1}^{m} \bar{\beta}_i^{(n)} P_W L^m (f_i),
\]
since
\[ P_W L^m f_i = 0 \quad (i > m), \]
by the matrix representation of \( L^m \) in (5.8). Hence by (5.12) (or directly from (5.8)), we have
\[ P_W L^m L^* f_0 = \frac{1}{|a_0|^2} \left( \beta_1^{(n)} \beta_1^{(m)} + \beta_2^{(n)} \beta_2^{(m)} + \cdots + \beta_{m-1}^{(n)} \beta_{m-1}^{(m)} + a_m \bar{\beta}_m^{(n)} \right). \]
Now note that \( X_{i,i+2} = 0 \), that is
\[ \beta_i^{(i+2)} = 0 \quad (i = 1, \ldots, m), \]
by assumption. Since \( i + 2 \leq m + j \) for all \( i = 1, \ldots, m \), by (5.10), we have
\[ \beta_i^{(n)} = \beta_i^{(m+j)} = 0 \quad (i = 1, \ldots, m). \]
Hence \( P_W L^m L^* f_0 = 0 \), that is
\[ X_{m,m+i} = 0 \quad (i = 3, 4, \ldots), \]
which proves the claim.

Step 5: So far all we have proved is that \( X_{mn} = 0 \) for all \( |m-n| \geq 2 \) if and only if \( \beta_{m+2}^{(m+2)} = 0 \) for all \( m \geq 1 \). Now, by (5.6) and (5.5), we have
\[ \beta_n^{(n+2)} = a_n \left( -\frac{b_0}{a_0} \right) \beta_n, \]
where
\[ \beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0}, \]
for all \( n \geq 1 \). Thus \( \beta_n^{(n+2)} = 0 \) for all \( n \geq 1 \) if and only if \( b_0 = 0 \) or \( \beta_n = 0 \) for all \( n \geq 1 \). On the other hand, Lemma 3.1 implies that \( \beta_n = 0 \) for all \( n \geq 1 \) if and only if \( M_z \) is a weighted shift.

Finally, by Proposition 2.10, we know that if \( M_z \) is a left invertible weighted shift, then the Shimorin kernel is also a diagonal kernel. This completes the proof of Theorem 5.1. \( \blacksquare \)

6. Truncated tridiagonal kernels

In this section, we introduce a (perhaps both deliberate and accidental) class of analytic tridiagonal kernels from a computational point of view. Let \( \mathcal{H}_k \) be an analytic tridiagonal space corresponding to the kernel
\[ k(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)} \quad (z, w \in \mathbb{D}), \]
where \( f_n = (a_n + b_n z)z^n, \ n \geq 0 \). Suppose \( r > 2 \) is a natural number. We say that \( k \) is a truncated tridiagonal kernel of order \( r \) (in short, truncated kernel of order \( r \)) if
\[ b_n = 0 \quad (n \neq 2, 3, \ldots, r). \]
We say that an analytic tridiagonal space \( \mathcal{H}_k \) is truncated space of order \( r \) if \( k \) is a truncated kernel of order \( r \). Note that there are no restrictions imposed on the scalars \( b_2, \ldots, b_r \).
Let $\mathcal{H}_k$ be a truncated space of order $r$. Then $\tilde{M}_z$ is unitarily equivalent to $M_z$ on $\mathcal{H}_k$, where $\tilde{k}$ is either the Shimorin-Aluthge kernel or the standard Aluthge kernel of $M_z$ as in Theorem 2.4 and Theorem 2.8, respectively. Here our aim is to compute the Shimorin-Aluthge kernel of $M_z$. More specifically, we classify all truncated kernels $\tilde{k}$ such that the Shimorin-Aluthge kernel $\tilde{k}$ of $M_z$ is tridiagonal. We begin by computing $|M_z|^{-1}$.

Lemma 6.1. If $\mathcal{H}_k$ is a truncated space of order $r$, then

$$
|M_z|^{-1} = \begin{bmatrix}
\left| \frac{a_1}{a_0} \right| & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & c_{11} & c_{12} & \cdots & c_{1,r+1} & 0 & 0 & \cdots \\
0 & \bar{c}_{12} & c_{22} & \cdots & c_{2,r+1} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & \bar{c}_{1,r+1} & \bar{c}_{2,r+1} & \cdots & c_{r+1,r+1} & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & \left| \frac{a_{r+1}}{a_{r+2}} \right| & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \left| \frac{a_{r+1}}{a_{r+3}} \right| & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
$$

with respect to the orthonormal basis $\{f_n\}_{n \geq 0}$.

Proof. For each $n \geq 1$, by the definition of $d_n$ from (3.10), we have $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$, and hence

$$
d_1 = d_{r+i} = 0 \quad (i = 2, 3, \ldots).
$$

Then Theorem 3.6 tells us that

$$
[L_{M_z}] = \begin{bmatrix}
0 & \frac{a_1}{a_0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{a_2}{a_1} & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & d_2 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \frac{(-1)^{r-2} d_2 b_2 \cdots b_{r-1}}{a_3 \cdots a_r} & \cdots & \frac{d_r}{a_r} & \frac{a_{r+1}}{a_r} & 0 & 0 & \cdots \\
0 & 0 & \frac{(-1)^{r-1} d_2 b_2 \cdots b_r}{a_3 \cdots a_r a_{r+1}} & \cdots & -\frac{d_r b_r}{a_{r+1}} & d_{r+1} & \frac{a_{r+2}}{a_{r+1}} & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{a_{r+1}}{a_{r+2}} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

Now, by Proposition 2.3, $|M_z|^{-2} = L_{M_z} L_{M_z}^*$, which implies

$$
\begin{bmatrix}
|a_1| \left| \frac{a_1}{a_0} \right|^2 & 0 & 0 & 0 \\
0 & A_{r+1}^2 & 0 & 0 \\
0 & 0 & D^2 & 0
\end{bmatrix},
$$

where

$$
L_{M_z} = \begin{bmatrix}
0 & \frac{a_1}{a_0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{a_2}{a_1} & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & d_2 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \frac{(-1)^{r-2} d_2 b_2 \cdots b_{r-1}}{a_3 \cdots a_r} & \cdots & \frac{d_r}{a_r} & \frac{a_{r+1}}{a_r} & 0 & 0 & \cdots \\
0 & 0 & \frac{(-1)^{r-1} d_2 b_2 \cdots b_r}{a_3 \cdots a_r a_{r+1}} & \cdots & -\frac{d_r b_r}{a_{r+1}} & d_{r+1} & \frac{a_{r+2}}{a_{r+1}} & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{a_{r+1}}{a_{r+2}} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$
where $A_{r+1}^2$ is a positive definite matrix of order $r + 1$ and

$$D^2 = \text{diag}\left(\frac{|a_{r+3}|^2}{a_{r+2}}, \frac{|a_{r+3}|^2}{a_{r+2}}, \ldots\right).$$

Using this, one easily completes the proof.

Recall (Theorem 2.7) that the Shimorin-Aluthge kernel of $M_z$ is given by

$$\tilde{k}(z, w) = P_{\tilde{W}}(I - zL_{\tilde{M}_z})^{-1}(I - \tilde{w}L_{\tilde{M}_z})^{-1}|_{\tilde{W}} \quad (z, w \in D),$$

where $\tilde{W} = |M_z|^{-\frac{1}{2}} \ker M^*_z$, and

$$L_{\tilde{M}_z} = |M_z|^\frac{1}{2} (L_{M_z} + F)|M_z|^{-\frac{1}{2}},$$

and

$$Fg = \left\langle g, f_0 \right\rangle_{\mathcal{H}_k} \left( (M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0 \right) \quad (g \in \mathcal{H}_k).$$

We now come to the key point.

**Lemma 6.2.** If $k$ is a truncated kernel, then $F = 0$ and

$$L_{\tilde{M}_z}|M_z|^\frac{1}{2} = |M_z|^\frac{1}{2} L_{M_z}.$$

**Proof.** The matrix representation of $|M_z|^{-1}$ in Lemma 6.1 implies that

$$|M_z|f_0 = \frac{a_0}{a_1} f_0,$$

and hence

$$M_z^*|M_z|f_0 = \frac{a_0}{a_1} M_z^* f_0 = 0,$$

by Lemma 3.2. Therefore, the proof follows from the definition of $F$ and (6.1).

From the computational point of view, it is useful to observe that $A_{r+1}^2 = L_{r+1}L_{r+1}^*$, where

$$L_{r+1} = \begin{bmatrix}
\frac{a_2}{a_1} & 0 & 0 & 0 & 0 \\
\frac{a_1}{d_2} & \frac{a_2}{a_1} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(-1)^r \frac{a_3 b_2 \cdots b_{r-1}}{a_3 \cdots a_r a_{r+1}} & (-1)^{r-1} \frac{a_3 b_2 \cdots b_{r-1}}{a_3 \cdots a_r a_{r+1}} & \cdots & \frac{a_{r+1}}{a_r} & 0 \\
(-1)^{r-1} \frac{a_3 b_2 \cdots b_{r-1}}{a_3 \cdots a_r a_{r+1}} & (-1)^{r-2} \frac{a_3 b_2 \cdots b_{r-1}}{a_3 \cdots a_r a_{r+1}} & \cdots & \frac{a_{r+1}}{a_r} & \frac{a_{r+2}}{a_{r+1}}
\end{bmatrix}.$$ 

In other words, $A_{r+1}^2$ admits a lower-upper triangular factorization. This is closely related to the Cholesky factorizations/decompositions of positive-definite matrices in the setting of infinite dimensional Hilbert spaces (see [4] and [28]).

**Theorem 6.3.** Let $\mathcal{H}_k$ be a truncated space of order $r$. Then the Shimorin-Aluthge kernel is tridiagonal if and only if

$$c_{mn} = (-1)^{n-m+1} \frac{\tilde{b}_m + 1 \cdots \tilde{b}_{n-1}}{a_{m+2} \cdots a_{n}} c_{m,m+1},$$
for all $1 \leq m \leq n - 2$ and $3 \leq n \leq r + 1$, where $c_{mn}$ are the entries of the middle block submatrix of order $r + 1$ of $[|M_z|^{-1}]$ in Lemma 6.1.

**Proof.** We split the proof into several steps.

**Step 1:** We first observe that

$$\tilde{k}(z, w) = \sum_{m,n=0}^{\infty} \tilde{X}_{mn} z^m \bar{w}^n \quad (z, w \in \mathbb{D}),$$

where $\tilde{X}_{mn} = P_{\tilde{W}} L_m^{\tilde{L}_{M_z}} |\tilde{W}|$ for all $m, n \geq 0$. Now Lemma 6.2 implies that

$$\tilde{L}_{M_z}^{m} \tilde{L}_{M_z}^{*n} = |M_z|^{\frac{1}{2}} L_{M_z}^{m} |M_z|^{-1} L_{M_z}^{*n} |M_z|^{\frac{1}{2}}.$$

Observe that $P_{\tilde{W}} = I - \tilde{M}_z \tilde{L}_{M_z}$ (see (2.3)), and hence

$$P_{\tilde{W}} = I - (|M_z|^{\frac{1}{2}} L_{M_z} |M_z|^{-\frac{1}{2}}) (|M_z|^{\frac{1}{2}} L_{M_z} |M_z|^{-\frac{1}{2}}) = |M_z|^{\frac{1}{2}} (I - M_z L_{M_z}) |M_z|^{-\frac{1}{2}} = |M_z|^{\frac{1}{2}} P_W |M_z|^{-\frac{1}{2}},$$

that is

$$P_{\tilde{W}} |M_z|^{\frac{1}{2}} = |M_z|^{\frac{1}{2}} P_W,$$

which implies

$$(6.2) \quad \tilde{X}_{mn} = |M_z|^{\frac{1}{2}} P_W L_m^{m} |M_z|^{-1} L_{M_z}^{*n} |W| (m, n \geq 0).$$

As a passing remark, we note that the above equality holds so long as the finite rank operator $F = 0$ (this observation also will be used in Example 6.4).

**Step 2:** Now we compute the matrix representation of $L^p_{M_z}, p \geq 2$. So let $p \geq 2$. Recall from (5.5) the definition

$$\beta_n^{(p)} = a_n \left(\frac{-b_0}{a_0}\right)^{p-n} \beta_n \quad (1 \leq n \leq p - 1),$$

where $\beta_n = \frac{b_n}{a_n} - \frac{b_n}{a_0}$. Since $b_0 = 0$, we have

$$\beta_n^{(p)} = 0 \quad (1 \leq n < p - 1),$$

and

$$\beta_{p-1}^{(p)} = a_{p-1} \beta_{p-1} = a_{p-1} \left(\frac{b_{p-1}}{a_{p-1}} - \frac{b_0}{a_0}\right),$$

that is

$$\beta_{p-1}^{(p)} = b_{p-1} \quad (p \geq 2).$$

In particular, since $b_1 = 0$, we have

$$\beta_1^{(2)} = b_1 = 0.$$

Also recall from (5.3) the definition

$$d_n^{(p)} = b_n - \frac{a_n}{a_{n-p}} b_{n-p} \quad (n \geq p).$$
Therefore, by (5.7), the associated matrix of $L_{Mz}^2$ is given by

$$
[L_{Mz}^2] = \begin{bmatrix}
0 & \frac{a_2}{a_0} & 0 & 0 & \cdots \\
0 & \frac{a_3}{a_1} & \frac{a_4}{a_2} & 0 & \cdots \\
0 & 0 & \frac{a_5}{a_3} & \frac{a_6}{a_4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
$$

and in general, by (5.8), we have

$$
[L_{Mz}^p] = \begin{bmatrix}
0 & \cdots & 0 & \frac{a_{p-1}}{a_0} & \frac{a_p}{a_0} & 0 & 0 & \cdots \\
0 & \cdots & 0 & \frac{a_{p-1}}{a_1} & \frac{a_p}{a_1} & \frac{a_{p+1}}{a_2} & 0 & \cdots \\
0 & \cdots & 0 & 0 & \frac{a_{p+1}}{a_2} & \frac{a_{p+2}}{a_3} & \cdots \\
0 & \cdots & 0 & 0 & -\frac{a_{p+1}}{a_2} \frac{a_{p+2}}{a_3} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \quad (p \geq 2).
$$

Then

$$
[L_{Mz}^{*p}] = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \cdots \\
\frac{b_{p-1}}{a_0} & 0 & 0 & 0 & \cdots \\
\frac{a_p}{a_0} & \frac{d_p}{a_1} & 0 & 0 & \cdots \\
0 & \frac{a_{p+1}}{a_1} & \frac{d_{p+1}}{a_2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \quad (p \geq 2).
$$

Step 3: We prove that $\tilde{X}_{0n} = |Mz|^{\frac{3}{2}} P_W |Mz|^{-1} L_{Mz}^{*n}|W = 0$ for all $n \geq 1$. In what follows, the above matrix representations and the one of $|Mz|^{-1}$ in Lemma 6.1 will be used repeatedly. By (5.2), we have $L_{Mz}^{*} f_0 = \frac{\bar{a}_1}{a_0} f_1$, and hence

$$
\tilde{X}_{01} f_0 = |Mz|^{\frac{3}{2}} P_W |Mz|^{-1} L_{Mz}^{*} f_0 = |Mz|^{\frac{3}{2}} P_W \left( \frac{\bar{a}_1}{a_0} [c_1 f_1 + \bar{c}_1 f_2 + \cdots] \right) = 0.
$$

On the other hand, if $n \geq 2$, then

$$
L_{Mz}^{*n} f_0 = \frac{\bar{b}_{n-1}}{a_0} f_{n-1} + \frac{\bar{a}_n}{a_0} f_n,
$$

and hence $|Mz|^{-1} f_0 \perp L_{Mz}^{*n} f_0$. This implies that $\tilde{X}_{0n} = 0$ for all $n \geq 2$. Therefore, all entries in the first row (and hence, also in the first column) of the formal matrix representation of
\( \tilde{k}(z, w) \) are zero except the \((0, 0)\)-th entry (which is \( I_W \)). Hence (see also (5.1))

\[
\begin{bmatrix}
I_W & 0 & 0 & 0 \\
0 & \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\
0 & \tilde{X}_{12}^* & \tilde{X}_{22} & \tilde{X}_{23} \\
0 & \tilde{X}_{13}^* & \tilde{X}_{23}^* & \tilde{X}_{33} \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Step 4: Our only interest here is to analyze the rank one operator \( \tilde{X}_{m,m+k}, m \geq 1, k \geq 2 \). The matrix representation in (6.4) implies

\[
L_{f_0}^{m+k} = \frac{1}{\bar{a}_0} (\tilde{b}_{m+k} f_{m+k} + \tilde{a}_{m+k} f_{m+k}),
\]

and hence

\[
|M_z|^{-1} L_{f_0}^{m+k} = \frac{1}{\bar{a}_0} \left( \tilde{b}_{m+k-1} |M_z|^{-1} f_{m+k-1} + \tilde{a}_{m+k} |M_z|^{-1} f_{m+k} \right).
\]

There are three cases to be considered.

Case I \((m + k = r + 2)\): Note that \( b_{r+1} = 0 \). Then (6.6) implies

\[
|M_z|^{-1} L_{f_0}^{m+k} = \frac{1}{\bar{a}_0} \left( \tilde{a}_{r+2} |M_z|^{-1} f_{r+2} \right),
\]

and then

\[
L_{f_0}^{m} |M_z|^{-1} L_{f_0}^{m+k} = \frac{a_{r+2}}{a_0} L_{f_0}^{m} |M_z|^{-1} f_{r+2} = \frac{a_{r+3}}{a_{r+2}} L_{f_0}^{m} |M_z|^{-1} f_{r+2}.
\]

By (6.3), we have \( P_W L_{f_0}^{m} |M_z|^{-1} L_{f_0}^{m+k} f_{r+2} = 0 \) (note that \( k \geq 2 \)), and hence

\[
P_W L_{f_0}^{m} |M_z|^{-1} L_{f_0}^{m+k} f_{r+2} = 0,
\]

that is, \( \tilde{X}_{m,m+k} = 0 \).

Case II \((m + k < r + 2)\): In this case

\[
|M_z|^{-1} f_{m+k-1} = \frac{a_{m+k}}{a_{m+k-1}} f_{m+k-1}, \quad \text{and} \quad |M_z|^{-1} f_{m+k} = \frac{a_{m+k+1}}{a_{m+k}} f_{m+k}.
\]

Again, by (6.3), we have

\[
P_W L_{f_0}^{m} f_{m+k-1} = P_W L_{f_0}^{m} f_{m+k} = 0,
\]

and hence in this case also \( \tilde{X}_{m,m+k} = 0 \).

Case III \((m + k < r + 2)\): We again stress that \( m \geq 1 \) and \( k \geq 2 \). It is useful to observe, by virtue of (6.3), that

\[
P_W L_{f_0}^{m} f_j = \begin{cases} 
\frac{b_{m-1}}{a_0} f_0 & \text{if } j = m-1 \\
\frac{a_{m}}{a_0} f_0 & \text{if } j = m \\
0 & \text{otherwise}
\end{cases}
\]
Now set $s = m + k - 1$. The matrix representation of $|M_z|^{-1}$ in Lemma 6.1 implies that

$$|M_z|^{-1} f_s = c_1 f_1 + c_2 f_2 + \cdots + c_{s,s+1} f_{s+1} + \cdots + \bar{c}_{s,r+1} f_{r+1}.$$  

By (6.3) and the above equality, we have

$$P_W L_M^m |M_z|^{-1} f_s = \left( c_{m-1,s} \frac{b_{m-1}}{a_0} + c_{m,s} \frac{a_m}{a_0} \right) f_0.$$  

Next, set $t = m + k$. Again, the matrix representation of $|M_z|^{-1}$ in Lemma 6.1 implies that

$$|M_z|^{-1} f_t = c_{t,t} f_1 + c_{t,t+1} f_{t+1} + \cdots + \bar{c}_{t,r+1} f_{r+1},$$  

and, again, by (6.3) and the above equality, we have

$$P_W L_M^m |M_z|^{-1} f_t = \left( c_{m-1,t} \frac{b_{m-1}}{a_0} + c_{m,t} \frac{a_m}{a_0} \right) f_0.$$  

The equality in (6.5) becomes

$$|M_z|^{-1} L^{s+m+k} f_0 = \frac{1}{a_0} \left( \bar{b}_s |M_z|^{-1} f_s + \bar{a}_t |M_z|^{-1} f_t \right),$$  

and hence, the one in (6.6) implies

$$P_W L_M^m |M_z|^{-1} L^{s+m+k} f_0 = \frac{1}{|a_0|^2} \left[ \bar{b}_s \left( c_{m-1,s} b_{m-1} + c_{m,s} a_m \right) + \bar{a}_t \left( c_{m-1,t} b_{m-1} + c_{m,t} a_m \right) \right] f_0.$$  

This shows that $P_W L_M^m |M_z|^{-1} L^{s+m+k} f_0 = 0$ if and only if

$$\bar{b}_s \left( c_{m-1,s} b_{m-1} + c_{m,s} a_m \right) + \bar{a}_t \left( c_{m-1,t} b_{m-1} + c_{m,t} a_m \right) = 0.$$  

Step 5: So far all we have proved is that $\tilde{k}$ is tridiagonal if and only if

(6.7) $b_{m-1} \left( \bar{b}_{m+k-1} c_{m-1,m+k-1} + \bar{a}_{m+k} c_{m,m+k} \right) + a_m \left( \bar{b}_{m+k-1} c_{m,m+k-1} + \bar{a}_{m+k} c_{m,m+k} \right) = 0,$

for all $m \geq 1$, $k \geq 2$ and $m + k < r + 2$.  

If $m = 1$, then using the fact that $b_0 = 0$, we have

$$c_{1,k+1} = -\frac{\bar{b}_k}{\bar{a}_{1+k}} c_{1,k} \quad (2 \leq k < r + 1),$$  

and hence

$$c_{1n} = (-1)^n \frac{\prod_{i=2}^{n-1} \bar{b}_i}{\prod_{i=3}^{n} \bar{a}_i} c_{12} \quad (3 \leq n \leq r + 1).$$  

Similarly, if $m = 2$, then (6.7) together with the assumption that $b_1 = 0$ implies that

(6.8) $c_{2n} = (-1)^{n-1} \frac{\prod_{i=3}^{n-1} \bar{b}_i}{\prod_{i=4}^{n} \bar{a}_i} c_{23} \quad (4 \leq n \leq r + 1).$

Next, if $m = 3$, then (6.7) again implies

$$b_2 \left( \bar{b}_{k+2} c_{2,k+2} + \bar{a}_{k+3} c_{2,k+3} \right) + a_3 \left( \bar{b}_{k+2} c_{3,k+2} + \bar{a}_{k+3} c_{3,k+3} \right) = 0 \quad (k < r - 1).$$
On the other hand, by (6.8), we have \( c_{2,k+3} = -\frac{b_{k+2}}{a_{k+3}} c_{2,k+2} \), and hence
\[
\bar{b}_{k+2} c_{3,k+2} + \bar{a}_{k+3} c_{3,k+3} = 0,
\]
that is
\[
c_{3,k+3} = -\frac{\bar{b}_{k+2}}{\bar{a}_{k+3}} c_{3,k+2} \quad (k < r - 1).
\]
Now, evidently the recursive situation is exactly the same as that of the proof of Theorem 4.2 (more specifically, see (4.2)). This completes the proof of the theorem.

As is clear by now, by virtue of Theorem 4.2, the classification criterion of the above theorem is also a classification criterion of tridiagonality of standard Aluthge kernels. Therefore, we have the following:

**Corollary 6.4.** If \( \mathcal{H}_k \) is a truncated space, then the Shimorin-Aluthge kernel of \( M_z \) is tridiagonal if and only if the standard Aluthge kernel of \( M_z \) is tridiagonal.

Now we comment on the assumptions in the definition of truncated kernels. The main advantage of the truncated space corresponding to an analytic tridiagonal kernel is that \( F = 0 \), where \( F \) is the finite rank operator as in (2.7). In this case, as already pointed out, we have \( L_{M_z} = |M_z|^{\frac{1}{2}} L_{M_z} |M_z|^{-\frac{1}{2}} \). This brings a big cut down in computation. On the other hand, quite curiously, if
\[
b_0 = b_1 = 1 \text{ or } b_0 = 1,
\]
and all other \( b_i \)'s are equal to 0, then the corresponding standard Aluthge kernel of \( M_z \) is tridiagonal kernel but the corresponding Shimorin-Aluthge kernel of \( M_z \) is not a tridiagonal kernel. Since computations are rather complicated, we only present the result for the following (convincing) case:

**Example 6.5.** Let \( a_n = b_0 = b_1 = 1 \) and \( b_m = 0 \) for all \( n \geq 0 \) and \( m \geq 2 \). Let \( \mathcal{H}_k \) denote the tridiagonal space corresponding to the basis \( \{(a_n + b_n z^n)z^m\}_{n \geq 0} \). By (3.8) and Theorem 3.7, we have
\[
[M_z] = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
and
\[
[L_{M_z}] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]
respectively. Hence, applying $L_{M_2} L_{M_2}^* = |M_2|^{-2}$ (see Proposition 2.3) to this, we obtain

$$|M_2|^{-2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$ 

Set $\alpha = \frac{3+\sqrt{5}}{2}$, $\beta = \frac{3-\sqrt{5}}{2}$ (note that $1 + (1-\alpha)(1-\beta) = 0$) and

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{\frac{1}{2}},$$

where

$$a = \frac{1}{\sqrt{5}}[\sqrt{\alpha}(1-\beta) - \sqrt{\beta}(1-\alpha)], \quad b = \frac{1}{\sqrt{5}}[-\sqrt{\alpha} + \sqrt{\beta}] \quad \text{and} \quad c = \frac{1}{\sqrt{5}}[-\sqrt{\alpha}(1-\alpha) + \sqrt{\beta}(1-\beta)].$$

Then

$$|M_2|^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$ 

From this it follows that

$$|M_2| f_0 = f_0,$$

and hence the finite rank operator $F$, as in (2.7), is given by

$$F g = \left\langle g, f_0 \right\rangle_{\mathcal{H}_k} \left( (M_2^* |M_2| M_2)^{-1} M_2^* |M_2| f_0 \right) = 0 \quad (g \in \mathcal{H}_k).$$

Then $F = 0$, and hence (2.6) implies that

$$L_{\tilde{M}_2} = |M_2|^{\frac{1}{2}} L_{M_2} |M_2|^{-\frac{1}{2}}.$$ 

By (6.2) (and also see Step 1 in the proof of Theorem 6.3), the coefficient of $z^m \bar{w}^n$ of the Shimorin-Aluthge kernel $\tilde{k}$ is given by

$$\tilde{X}_{mn} = |M_2|^\frac{1}{2} P_{W} L_{M_2}^m |M_2|^{-1} L_{M_2}^n |W| \quad (m, n \geq 0).$$

We compute the coefficient of $z \bar{w}^3$ of the Shimorin-Aluthge kernel function as

$$P_{W} L_{M_2} |M_2|^{-1} L_{M_2}^3 f_0 = P_{W} L_{M_2} |M_2|^{-1} L_{M_2}^2 f_1 = P_{W} L_{M_2} |M_2|^{-1} L_{M_2} f_2 = P_{W} L_{M_2} |M_2|^{-1} (-f_2 + f_3) = P_{W} L_{M_2} (-bf_1 + cf_2 + f_3) = P_{W} L_{M_2} (-bf_1) = -bf_0.$$
But
\[ b = \frac{1}{\sqrt{3}}[-\sqrt{\alpha} + \sqrt{\beta}] \neq 0, \]
and hence
\[ P_W L_{M_z} |M_z|^{-1} L_{M_z}^3 \neq 0. \]
This implies that the Shimorin-Aluthge kernel is not tridiagonal. On the other hand, the matrix representation of $|M_z^{-1}|$ implies right away that the standard Aluthge transform is tridiagonal (see Theorem 4.2).

7. Quasinormal operators and tridiagonal spaces

A bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is said to be quasinormal if $T^* T$ and $T$ commutes, that is
\[ [T^*, T]T = 0, \]
where $[T^*, T] = T^* T - TT^*$ is the commutator of $T$. Quasinormal operators plays a crucial role in the theory of Aluthge transforms. In [23], Bong, Ko and Pearcy proved that fixed points of Aluthge transforms are quasinormal operators. On the other hand, the class of normal operators are related with the strong operator topology limit of iterations of Aluthge transforms. See [22] for the issue of convergency of iterated Aluthge transforms, and also see Antezana, Pujals and Stojanoff [8] and Yamazaki [34] in the context of convergence of iterated Aluthge transforms of matrices.

In this section, we present a complete classification of quasinormality of $M_z$ on analytic tridiagonal spaces. Here, however, we do not need to assume that $M_z$ is left invertible. So, our main classification result, Theorem 7.2, is valid without the assumption (3.2).

To motivate our result on quasinormality, we first consider the known case of weighted shifts. Recall that the weighted shift $S_\alpha$ corresponding to the weight sequence (of positive real numbers) $\{\alpha_n\}_{n \geq 0}$ is given by $S_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 0$. Then (see the proof of Proposition 2.10)
\[ S_\alpha S_\alpha^* e_{n+1} = \alpha_n^2 e_{n+1}, \]
and hence $(S_\alpha^* S_\alpha - S_\alpha S_\alpha^*) S_\alpha = 0$ if and only if
\[ (S_\alpha^* S_\alpha - S_\alpha S_\alpha^*) S_\alpha e_n = 0, \]
for all $n \geq 0$, which is equivalent to
\[ \alpha_n (\alpha_{n+1}^2 - \alpha_n^2) = 0, \]
for all $n$. Thus, we have proved [19] Problem 139:

**Lemma 7.1.** The weighted shift $S_\alpha$ is quasinormal if and only if the weight sequence $\{\alpha_n\}_{n \geq 0}$ is a constant sequence.

Now we turn to $M_z$ on an analytic tridiagonal space $\mathcal{H}_k$. We first assume that
\[ [M_z^*, M_z] = rP_{f_0}, \]
where \( r \) is a non-negative real number and \( P_{f_0} \) denote the orthogonal projection of \( \mathcal{H}_k \) onto the one dimensional space \( \mathbb{C}f_0 \). Then \([M_z^*, M_z]M_z = rP_{f_0}M_z\) implies that

\[
([M_z^*, M_z]M_z)f_n = rP_{f_0}(zf_n).
\]

Now by (3.7) we have

\[
zf_n = \sum_{i=n+1}^{\infty} \beta_i f_i,
\]

for some scalar \( \beta_i \in \mathbb{C}, i \geq n + 1 \). Note that

\[
\beta_{n+1} = \frac{a_n}{a_{n+1}} \neq 0.
\]

This shows that \( P_{f_0}(zf_n) = 0 \), and hence

\[
([M_z^*, M_z]M_z)f_n = 0 \quad (n \geq 0),
\]

that is, \( M_z \) is quasinormal. Conversely, assume that \( M_z \) is quasinormal. Then \([M_z^*, M_z]M_z = 0\) implies that

\[
\text{ran} M_z \subseteq \ker [M_z^*, M_z],
\]

and therefore, by Lemma 3.2, we have

\[
\mathbb{C}f_0 = \ker M_z^* \supseteq \overline{\text{ran}[M_z^*, M_z]}.
\]

Clearly this implies

\[
[M_z^*, M_z] = rP_{f_0},
\]

for some scalar \( r \in \mathbb{C} \). Then

\[
r\|f_0\|^2 = \langle rP_{f_0}f_0, f_0 \rangle_{\mathcal{H}_k} = \langle [M_z^*, M_z]f_0, f_0 \rangle_{\mathcal{H}_k} = \|M_zf_0\|^2 - \|M_z^*f_0\|^2 = \|M_zf_0\|^2,
\]

as \( M_z^*f_0 = 0 \), which implies

\[
r = \frac{\|M_zf_0\|^2}{\|f_0\|^2} > 0.
\]

Thus, we have proved:

**Theorem 7.2.** Let \( \mathcal{H}_k \) be an analytic tridiagonal space. Then \( M_z \) on \( \mathcal{H}_k \) is quasinormal if and only if there exists a non-negative real number \( r \) such that

\[
M_z^*M_z - M_zM_z^* = rP_{f_0},
\]

where \( P_{f_0} \) denote the orthogonal projection of \( \mathcal{H}_k \) onto the one dimensional space \( \mathbb{C}f_0 \).
In more algebraic terms this result can be formulated as follows: First we recall the matrix representation of \( M_z \) (see (3.8))

\[
[M_z] = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots \\
\frac{a_2}{a_1} & 0 & 0 & 0 & \ldots \\
c_0 & \frac{a_2}{a_3} & 0 & 0 & \ldots \\
-\frac{c_0 b_2}{a_3 a_4} & c_1 & \frac{a_2}{a_3} & 0 & \ldots \\
\frac{c_0 b_2 b_3}{a_3 a_4 a_5} & -\frac{c_0 b_3}{a_4} & c_2 & \frac{a_3}{a_4} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

For each \( n \geq 0 \), we denote by \( R_n \) and \( C_n \) the \( n \)-th column and \( n \)-th row, respectively, of \([M_z]\). We then identify each of these column and row vectors with elements in \( \mathcal{H}_k \). Then \( R_n, C_n \in \mathcal{H}_k, n \geq 0 \). Using the matrix representation \([M_z^*]\) (see (3.9)) and \([M_z]\), we get

\[
\langle R_0, R_n \rangle_{\mathcal{H}_k} = 0,
\]

for all \( n \geq 0 \), and, consequently

\[
[M_z^*, M_z] = \begin{bmatrix}
\langle C_0, C_0 \rangle_{\mathcal{H}_k} & \langle C_0, C_1 \rangle_{\mathcal{H}_k} & \langle C_0, C_2 \rangle_{\mathcal{H}_k} & \ldots \\
\langle C_1, C_0 \rangle_{\mathcal{H}_k} & \langle C_1, C_1 \rangle_{\mathcal{H}_k} - \langle R_1, R_1 \rangle_{\mathcal{H}_k} & \langle C_1, C_2 \rangle_{\mathcal{H}_k} - \langle R_1, R_2 \rangle_{\mathcal{H}_k} & \ldots \\
\langle C_2, C_0 \rangle_{\mathcal{H}_k} & \langle C_2, C_1 \rangle_{\mathcal{H}_k} - \langle R_2, R_1 \rangle_{\mathcal{H}_k} & \langle C_2, C_2 \rangle_{\mathcal{H}_k} - \langle R_2, R_2 \rangle_{\mathcal{H}_k} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Therefore \([M_z^*, M_z] = \alpha P_f \) if and only if

\[
\langle C_0, C_0 \rangle_{\mathcal{H}_k} = r, \quad \langle C_0, C_1 \rangle_{\mathcal{H}_k} = 0,
\]

for all \( i \geq 1 \), and

\[
\langle C_m, C_n \rangle_{\mathcal{H}_k} - \langle R_m, R_n \rangle_{\mathcal{H}_k} = 0,
\]

for all \( 1 \leq m \leq n \).

It is easy to see that a quasinormal operator is always subnormal [19]. However, a complete classification of subnormality of \( M_z \) on tridiagonal spaces is rather more subtle and not quite as clear-cut as in the quasinormal situation. In fact the general classification of subnormality of \( M_z \) on tridiagonal spaces is not known (however, see [2] and the related work [26]).

**Concluding remarks**: Evidently, the main inconvenience of Aluthge transforms results from the representations of the positive part \([T]\) of the polar decompositions of a bounded linear operator \( T \). Broadly speaking, this paper discusses some definite problems and constraints in Aluthge transforms, Shimorin’s analytic models of left invertible operators and (tridiagonal) analytic reproducing kernel Hilbert spaces. This also discusses the use of techniques drawn from the aforementioned concepts in the setting of “simple” shifts on function Hilbert spaces. Part of the main contributions of this paper also lies perhaps in detecting more concrete problems in operator theory and function theory. We are interested, for instance, in the
following further questions: What is the limit of iterations of Aluthge transforms of shifts on analytic tridiagonal spaces? What is the invariant subspace lattice of shifts on analytic tridiagonal spaces. What is the several variable analogue of Aluthge transforms in the setting of tridiagonal spaces (cf. [13] on weighted shifts). We intend to return to some of these issues in a future paper.

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