Quaternionic equation for electromagnetic fields in inhomogeneous media

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Abstract
We show that the Maxwell equations for arbitrary inhomogeneous media are equivalent to a single quaternionic equation which can be considered as a generalization of the Vekua equation for generalized analytic functions.

1 Introduction
Quaternionic reformulation of the Maxwell equations in a vacuum is quite well known (see, for example, [1], [6], [7], [13]). The system of Maxwell equations

\[ \text{div } D = \text{div } B = 0 \]

\[ \text{rot } E = -\partial_t B, \quad \text{rot } H = \partial_t D, \]

where \( D = \varepsilon_0 E \) and \( B = \mu_0 H \) (\( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability of free space), is equivalent to the quaternionic equation

\[ \left( \frac{1}{c_0} \partial_t + iD \right) \vec{f} = 0, \]

where \( c_0 \) is the speed of light in a vacuum, \( D \) is the Moisil-Theodoresco operator (see the definition below) and \( \vec{f} \) is a purely vectorial complex quaternion. The equivalence between (1) and (2) can be established by putting

\[ \text{Re } \vec{f} = \sqrt{\varepsilon_0} E \quad \text{and} \quad \text{Im } \vec{f} = \sqrt{\mu_0} H. \]

The quaternionic approach to Maxwell’s equations for homogeneous media was intensively used in a number of works (e.g., [8], [9], [10], [13]), but even the
question as to how to write the Maxwell equations for arbitrary inhomogeneous media in a compact quaternionic form remained open until recently (an attempt in this direction can be found in [5, Section 4.5]). In [12] such a reformulation was proposed in the case of a time-harmonic electromagnetic field and in [11] for the time-dependent case. Here we make one additional step which leads us to Maxwell’s system in the form of a single quaternionic equation, which can be considered as a generalization of the well known in complex analysis Vekua equation describing generalized analytic functions [15].

2 Preliminaries

We will consider continuously differentiable functions of four variables \((t, x_1, x_2, x_3)\) with values in the algebra of complex quaternions \(\mathbb{H}(\mathbb{C})\). By \(D\) we denote the operator \(D = \sum_{k=1}^{3} i_k \partial_k\). Here \(\partial_k = \frac{\partial}{\partial x_k}\) and \(i_k\) are the imaginary quaternionic units. Let us notice the following property of the operator \(D\). Let \(\varphi\) be a scalar complex function and \(g\) be an \(\mathbb{H}(\mathbb{C})\)-valued function. Then

\[ D[\varphi \cdot g] = D[\varphi] \cdot g + \varphi \cdot D[g]. \tag{3} \]

Taking into account that \(D[\varphi] = \text{grad} \, \varphi = i_1 \partial_1 \varphi + i_2 \partial_2 \varphi + i_3 \partial_3 \varphi\) and assuming that \(\varphi\) is different from zero we can rewrite (3) in the form

\[ (D + \frac{\text{grad} \, \varphi}{\varphi})g = \frac{1}{\varphi} D[\varphi \cdot g]. \tag{4} \]

We will use the following notations for the operators of multiplication from the left-hand side and from the right-hand side

\[ \alpha^\alpha g := \alpha \cdot g \quad \text{and} \quad M^\alpha g := g \cdot \alpha, \]

where \(\alpha \in \mathbb{H}(\mathbb{C})\). The usual complex conjugation we denote by “*”. Vectors from \(\mathbb{C}^3\) are identified with purely vectorial complex quaternions. Note that for an \(\mathbb{H}(\mathbb{C})\)-valued function \(g = g_0 + \overline{g}\) the action of the operator \(D\) can be represented as follows

\[ Dg = -\text{div} \, \overline{g} + \text{grad} \, g_0 + \text{rot} \, \overline{g}. \]

3 Maxwell equations

We assume that the relative permittivity \(\varepsilon_r\) and the relative permeability \(\mu_r\) of the material are differentiable functions of coordinates \(\varepsilon_r = \varepsilon_r(x_1, x_2, x_3)\) and \(\mu_r = \mu_r(x_1, x_2, x_3)\). The permittivity and the permeability of the medium are introduced as follows

\[ \varepsilon = \varepsilon_0 \varepsilon_r \quad \text{and} \quad \mu = \mu_0 \mu_r. \]
Then Maxwell’s equations for an inhomogeneous medium have the form
\[ \text{rot} \mathbf{H} = \varepsilon \partial_t \mathbf{E} + \mathbf{j}, \]  
(5)
\[ \text{rot} \mathbf{E} = -\mu \partial_t \mathbf{H}, \]  
(6)
\[ \text{div}(\varepsilon \mathbf{E}) = \rho, \]  
(7)
\[ \text{div}(\mu \mathbf{H}) = 0, \]  
(8)
where all the magnitudes are real. Equations (7) and (8) can be written as follows
\[ \text{div} \mathbf{E} + \frac{\text{grad} \varepsilon}{\varepsilon} \cdot \mathbf{E} = \frac{\rho}{\varepsilon}, \]  
and
\[ \text{div} \mathbf{H} + \frac{\text{grad} \mu}{\mu} \cdot \mathbf{H} = 0, \]
where \( \cdot \cdot \cdot \) denotes the usual scalar product. Combining these equations with (3) and (4) we obtain the Maxwell system in the form
\[ D \mathbf{E} = \frac{\text{grad} \varepsilon}{\varepsilon} \cdot \mathbf{E} - \mu \partial_t \mathbf{H} - \frac{\rho}{\varepsilon}, \]  
(9)
and
\[ D \mathbf{H} = \frac{\text{grad} \mu}{\mu} \cdot \mathbf{H} + \varepsilon \partial_t \mathbf{E} + \mathbf{j}. \]  
(10)
Let us make a simple observation: the scalar product of two vectors \( \mathbf{p} \) and \( \mathbf{q} \) can be represented as follows
\[ \langle \mathbf{p}, \mathbf{q} \rangle = -\frac{1}{2}(\mathbf{p} \cdot \mathbf{M} + \mathbf{M} \cdot \mathbf{p}) \mathbf{q}. \]
Using this fact, from (3) and (4) we obtain the pair of equations
\[ (D + \frac{1}{2} \frac{\text{grad} \varepsilon}{\varepsilon}) \mathbf{E} = -\frac{1}{2} \mathbf{M} \cdot \text{grad} \varepsilon \mathbf{E} - \mu \partial_t \mathbf{H} - \frac{\rho}{\varepsilon}, \]  
(11)
and
\[ (D + \frac{1}{2} \frac{\text{grad} \mu}{\mu}) \mathbf{H} = -\frac{1}{2} \mathbf{M} \cdot \text{grad} \mu \mathbf{H} + \varepsilon \partial_t \mathbf{E} + \mathbf{j}. \]  
(12)
Note that
\[ \frac{1}{2} \frac{\text{grad} \varepsilon}{\varepsilon} = \frac{\text{grad} \sqrt{\varepsilon}}{\sqrt{\varepsilon}}. \]
Then using (4), equation (11) can be rewritten in the following form

\[
\frac{1}{\sqrt{\varepsilon}} D(\sqrt{\varepsilon} \cdot \mathbf{E}) + \mathbf{E} \cdot \varepsilon = -\mu \partial_t \mathbf{H} - \frac{\rho}{\varepsilon},
\]

(13)

where

\[
\varepsilon := \frac{\text{grad} \sqrt{\varepsilon}}{\sqrt{\varepsilon}}.
\]

Analogously, (12) takes the form

\[
\frac{1}{\sqrt{\mu}} D(\sqrt{\mu} \cdot \mathbf{H}) + \mathbf{H} \cdot \mu = \varepsilon \partial_t \mathbf{E} + j,
\]

(14)

where

\[
\mu := \frac{\text{grad} \sqrt{\mu}}{\sqrt{\mu}}.
\]

Introducing the notations

\[
\mathbf{E} := \sqrt{\varepsilon} \mathbf{E}, \quad \mathbf{H} := \sqrt{\mu} \mathbf{H}
\]

and multiplying (13) by \(\sqrt{\varepsilon}\) and (14) by \(\sqrt{\mu}\) we arrive at the equations

\[
(D + M \varepsilon) \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{H} - \frac{\rho}{\sqrt{\varepsilon}},
\]

(15)

and

\[
(D + M \mu) \mathbf{H} = \frac{1}{c} \partial_t \mathbf{E} + \sqrt{\mu} j,
\]

(16)

where \(c = 1/\sqrt{\varepsilon\mu}\) is the speed of propagation of electromagnetic waves in the medium.

Equations (13) and (14) can be rewritten even in a more elegant form. Consider the function

\[
\mathbf{f} := \mathbf{E} + i \mathbf{H}
\]

Let us apply to it the quaternionic Maxwell operator

\[
\frac{1}{c} \partial_t + iD.
\]

We obtain

\[
\left(\frac{1}{c} \partial_t + iD\right) \mathbf{f} = \frac{1}{c} \partial_t \mathbf{E} - D \mathbf{H} + i\left(\frac{1}{c} \partial_t \mathbf{H} + D \mathbf{E}\right).
\]

For the real part of this expression we use equation (15) and for the imaginary part equation (16). Then we have
\[ (\frac{1}{c} \partial_t + iD) \vec{f} = -i(M^{\perp} \vec{E} + iM^{\parallel} \vec{H}) - \sqrt{\mu} j - \frac{i\rho}{\sqrt{\varepsilon}}. \]  \hspace{1cm} (17)

Note that
\[ \vec{E} = \frac{1}{2}(\vec{f} + \vec{f}^*) \quad \text{and} \quad \vec{H} = \frac{1}{2i}(\vec{f} - \vec{f}^*). \]

Hence
\[ M^{\perp} \vec{E} + iM^{\parallel} \vec{H} = \frac{1}{2}(M^{\perp+\parallel} \vec{f} + M^{\perp-\parallel} \vec{f}^*). \]

Let us notice that
\[ \vec{e}^\parallel + \vec{\mu} = -\text{grad} \frac{c}{c} \quad \text{and} \quad \vec{e}^\perp - \vec{\mu} = -\frac{\text{grad} W}{W}, \]

where \( W = \sqrt{\mu/\varepsilon} \) is the intrinsic wave impedance of the medium. Denote
\[ \vec{e}^\parallel := \frac{\text{grad} \sqrt{c}}{\sqrt{c}} \quad \text{and} \quad \vec{W} := \frac{\text{grad} \sqrt{W}}{\sqrt{W}}. \]

Then
\[ M^{\perp} \vec{E} + iM^{\parallel} \vec{H} = -(M^{\perp} \vec{f} + M^{\parallel} \vec{f}^*). \]

From (17) we obtain the Maxwell equations for an inhomogeneous medium in the following form
\[ (\frac{1}{c} \partial_t + iD) \vec{f} - M^{\perp} \vec{E} - M^{i\parallel} \vec{f}^* = -\left( \sqrt{\mu} j + \frac{i\rho}{\sqrt{\varepsilon}} \right) \]  \hspace{1cm} (18)

(compare with (8)). This equation is completely equivalent to the Maxwell system (5)-(8) and represents Maxwell’s equation for inhomogeneous media in a quaternionic form.

**Remark 1** Equation (18) can be considered as a generalization of the well known in complex analysis Vekua equation describing generalized analytic functions [13]. Recently in [14] using the L. Bers approach, another quaternionic generalization of the Vekua equation was considered. Probably some of the interesting results discussed in [14] can be obtained for (18) also. Then their physical meaning would be of a great interest.

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