Stability analysis of black holes in massive gravity: a unified treatment

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We consider the analytic solutions of massive (bi)gravity which can be written in a simple form using advanced Eddington-Finkelstein coordinates. We analyse the stability of these solutions against radial perturbations. First we recover the previously obtained result on the instability of the bidiagonal bi-Schwarzschild solutions. In the non-bidiagonal case (which contains, in particular, the Schwarzschild solution with Minkowski fiducial metric) we show that generically there are physical spherically symmetric perturbations, but no unstable modes.

General Relativity (GR) theory is very successful in the explanation of various gravity phenomena. In particular, GR passes the local gravity tests, which give the most impressive constraints. In spite of this obvious success, there are still unresolved puzzles, e.g. Dark Energy and Dark Matter, which motivate to search ways to modify GR. One way to modify gravity follows an idea of Fierz and Pauli to equip the graviton with a non-zero mass \[ m \]. However, a close examination of the massive gravity models revealed pathologies, in particular, the van Dam-Veltman-Zakharov discontinuity \[ \varepsilon \] as well as the presence of a ghost in the spectrum \[ \omega \]. The solution to the first problem has been conjectured by Vainshtein \[ \omega \] and confirmed much later in \[ \varepsilon \] (for a recent review see, e.g. \[ \varepsilon \]). On the other hand, the latter problem (related to the presence of the Ostrogradski ghost \[ \omega \]) has been recently addressed in a series of works \[ \omega \], where it was shown that for a subclass of massive potentials the Ostrogradski ghost does not appear, both in the model with one and two dynamical metrics \[ \omega \]. Other possible problems have been also discussed in the literature \[ \omega \]. For a review on massive gravity see \[ \omega \].

Black hole solutions in massive gravity are of two types, see e.g. the recent reviews \[ \omega \]. In the first, both the physical \( g \) and fiducial \( f \) metrics are diagonal (bidiagonal case). In the second \( g \) and \( f \) are not simultaneously diagonal (non-bi-diagonal case).

For the bi-diagonal case, it was shown in \[ \omega \] that regularity at the horizon requires that, for spherically symmetric solutions, the two metrics must share the same Killing horizon. This condition is straightforwardly satisfied when we take \( f = g \), in which case the massive interaction term between the two metrics vanishes and the equations of motion reduce to those of GR. In \[ \omega \] we considered the two metrics to be equal to the same (up to a conformal factor) Schwarzschild solution (bi-Schwarzschild solution) and showed that, despite the ‘triviality’ of the background, the massive interaction term shows up nontrivially in the linear perturbations featuring the Gregory-Laflamme (GL) instability \[ \omega \] of higher-dimensional black strings. This is however a mild instability, since for the physically meaningful case where the graviton mass is \( m \sim \frac{1}{H} \), where \( H \) is the Hubble scale, the timescale of the instability \( \tau_{GL} \sim \frac{1}{m} \) is of the order of the Hubble time. This result is valid for both one \( g \) or two \( g \) and \( f \) dynamical metrics and, generically, when the two Schwarzschild metrics are proportional with a constant conformal factor. The existence of such instability was confirmed numerically in \[ \omega \]. Subsequent investigations by the same authors found possible candidates (typically, in the \( m \sim \frac{1}{r_{S}} \) regime, where \( r_{S} \) is the Schwarzschild radius) for the end-point evolution \[ \omega \].

It is important to understand whether or not the presence of this instability is a generic feature of (physically relevant) static massive gravity black holes. For this reason, in this Letter we shall extend our perturbation analysis to the technically more involved case of non-bi-diagonal black hole solutions. Another motivation of this work is that, as already mentioned, in the bi-diagonal case regularity conditions \[ \omega \] do not allow the fiducial metric \( f \) (fixed or dynamical) to be flat (Minkowski) and this might seem unnatural. In the non bi-diagonal case both metrics can be nicely expressed in ingoing Eddington-Finkelstein (EF) coordinates, a choice that makes the regularity on the future horizon apparent. We shall write down solutions that contain the most interesting Schwarzschild \( g \) – Minkowski \( f \) case. Remarkably, the s-wave perturbation equations can be solved analytically. We find, generically, a nontrivial massive perturbation term but, unlike the bi-diagonal bi-Schwarzschild case, no unstable modes.

The action for the dRGT (bi-gravity) model can be
written as follows

\[ S = M_P^2 \int d^4x \sqrt{-g} \left( \frac{R}{2} + m^2 U(g, f) + m^2 \Lambda_f \right) + S_m[g] \]
\[ + \frac{\kappa M_P^2}{2} \int d^4x \sqrt{-f} \left( R[f] - m^2 \Lambda_f \right). \quad (1) \]

The interaction potential \( U(g, f) \) is expressed in terms of the matrix \( K_{\mu\nu} = \delta_{\mu\nu} - \gamma_{\mu\nu} \), where \( \gamma_{\mu\nu} = \sqrt{g} f_{\alpha\beta} \). The potential \( U \) consists of three pieces, \( U \equiv U_2 + \alpha_2 U_3 + \alpha_4 U_4 \), with \( \alpha_3 \) and \( \alpha_4 \) being parameters of the theory, and each of them reads,

\[ U_2 = \frac{1}{2t} (|K|^2 - |K^2|), \]
\[ U_3 = \frac{1}{3t} (|K|^3 - 3|K||K^2| + 2|K|^3) \], \( U_4 = \det(K) \), \quad (2) \]

where \( |K| = K^\rho_\rho \) and \( |K^n| = (K^n)^\rho_\rho \).

The variation of the action with respect to \( g \) and \( f \) in the vacuum gives,

\[ G^\mu_\nu = m^2 \left( T^\mu_\nu + \Lambda_f \delta^\mu_\nu \right), \]
\[ G^\mu_\nu = m^2 \left( \sqrt{-g} \frac{\partial T^\mu_\nu}{\partial f} - \Lambda_f \delta^\mu_\nu \right), \quad (3) \]

where \( G^\mu_\nu \) and \( G^\mu_\nu \) are the corresponding Einstein tensors for the two metrics \( g \) and \( f \), \( T^\mu_\nu \equiv U \delta^\mu_\nu - 2g^\mu_\alpha \frac{\partial U}{\partial g_\alpha} \) and \( T^\mu_\nu = -\tilde{T}^\mu_\nu + U \delta^\mu_\nu \). When \( f \) is not dynamical then Eq. \( (1) \) is absent.

We shall write down the metric solutions in the bi-advanced Eddington-Finkelstein (biEF) form

\[ ds^2 = -\left(1 - \frac{r_g}{r} \right) dt^2 + 2dvdr + r^2 d\Omega^2, \]
\[ ds_f^2 = C^2 \left[ -\left(1 - \frac{rf}{r} \right) dt^2 + 2dvdr + r^2 d\Omega^2 \right], \quad (4) \]

where \( C \) is a constant (conformal factor) and \( r_g \) and \( r_f \) are the two (in general different) Schwarzschild radii for the two metrics. The only non-diagonal terms of \( T^\mu_\nu \) and \( \tilde{T}^\mu_\nu \) read,

\[ T^\mu_\nu = -T^\nu_\mu = \frac{C (\beta(C-1)^2 - 2\alpha(C-1) + 1) (r_f - r_g)}{2r}, \quad (5) \]

where we defined \( \alpha \equiv 1 + \alpha_3 \), \( \beta \equiv \alpha_3 + \alpha_4 \). These off-diagonal terms must vanish due to the choice of the metrics \( (3) \) and \( (4) \). This implies either \( r_s = r_f \), which is equivalent to the (bi-diagonal) bi-Schwarzschild solution analysed in [14], or

\[ \beta(C-1)^2 - 2\alpha(C-1) + 1 = 0, \quad (6) \]

that we will consider in detail below. The case with a flat (Minkowski) fiducial metric \( (f) \) falls within this class of solutions. For all of them we need to fine tune the two cosmological constants to \( \Lambda_g = \frac{1}{2}(C-1)(3-\beta(C-1)^2) \), \( \Lambda_f = \frac{1}{2\alpha}(C-1)C^{-2}(\beta(C-1)^2 - 2C-1) \) to balance the corresponding \( (\sim \delta^\mu_\nu \) contributions coming from \( T^\mu_\nu \) and \( \tilde{T}^\mu_\nu \).

At this point we would like to stress that the non-biagonal solutions require a specific choice of the conformal factor \( C \), given via Eq. \( (6) \), for each set of the parameters \( \alpha \) and \( \beta \). On the other hand, the bi-diagonal solutions exist for any value of \( C \). Therefore the bi-diagonal solutions seem to be more likely to form, e.g. in the process of gravitational collapse from (almost) bi-flat initial conditions with \( C \) different from \( (6) \).

Let us now consider the linear perturbations around the non bi-diagonal black hole solutions. The metric perturbations \( h^{(g)}_{\mu\nu} \) and \( h^{(f)}_{\mu\nu} \) satisfy the linearised equations

\[ \delta G^\mu_\nu = m^2 \delta T^\mu_\nu, \quad \delta G^\mu_\nu = \frac{m^2}{\kappa} \delta \left( \frac{\sqrt{-g}}{\sqrt{-f}} \tilde{T}^\mu_\nu \right). \quad (7) \]

We shall look for unstable \((\Omega > 0)\) spherically-symmetric modes of the form

\[ h^{(g)}_{\mu\nu} = e^{\Omega v} \left( \begin{array}{ccc} h^{gg}_{\mu\nu}(r) & h^{gr}_{\mu\nu}(r) & 0 \\ 0 & h^{rr}_{\mu\nu}(r) & 0 \\ 0 & 0 & h^{\theta\theta}_{\mu\nu}(r) \end{array} \right), \quad (8) \]

along with a similar expression for \( h^{(f)}_{\mu\nu} \), but with an overall \( 1/C^2 \) term, which are regular at the horizon and vanish asymptotically. As the advanced time \( v \) is regular at the future horizon, we require the metric perturbations in our biEF frame to be regular at \( r = r_g \) and \( r = r_f \). At infinity, instead, it is more suitable to use the Schwarzschild time \( t = v - r - r_0 \ln(r/r_0 - 1) \sim v - r \) to separate between temporal and spatial components. Therefore, in the asymptotic region the \( r \)-dependent part of our perturbations \( h^{(g)}_{\mu\nu}(r) \) must behave as \( o(e^{-\Omega r}) \) to be physically acceptable.

It turns out that the massive terms in the perturbations equations take the remarkably simple form

\[ \delta T^\mu_\nu = \frac{A(r_s - r_f)}{4r} e^{\Omega v} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & h^{gg}_{\mu\nu} & 0 \\ 0 & 0 & 0 \end{array} \right), \quad (9) \]

and \( \delta \left( \frac{\sqrt{-g}}{\sqrt{-f}} \tilde{T}^\mu_\nu \right) = -\delta T^\mu_\nu \), where we defined \( A \equiv C^2 (\beta(C-1)^2 - 2\alpha(C-1) + 1) \), \( h^{(g)}_{\mu\nu} = h^{(f)}_{\mu\nu} - C^2 h^{(f)}_{\mu\nu} \). So that, e.g. \( h^{(g)}_{\mu\nu}(r) = h^{(f)}_{\mu\nu}(r) - h^{(f)}_{\mu\nu}(r) \), taking into account the factor \( 1/C^2 \) in the definition of \( h^{(f)}_{\mu\nu} \). This is to be compared with the Pauli-Fierz-like form in the bi-diagonal (bi-Schwarzschild) case

\[ \delta T^\mu_\nu = \frac{C}{2} (\beta(C-1)^2 - 2\alpha(C-1) + 1) \times e^{\Omega v} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{h^{gg}_{\mu\nu}}{2} & 0 \\ 0 & 0 & \frac{h^{\theta\theta}_{\mu\nu}}{2} \end{array} \right), \quad (10) \]
together with, again, \( \delta \left( \sqrt{-g} T^\mu_\nu \right) = -\delta T^\mu_\nu \). It is interesting to note that at the intersection of the two branches of solutions, \( r_S = r_f \) in (11) and \( C \) fixed by (8) in (12), we get \( \delta T^\mu_\nu = 0 \), i.e. the perturbation equations are those of GR. This is not true for \( r_S \neq r_f \), and in particular when the fiducial metric \( f \) is flat (\( r_f = 0 \)), unless \( \beta = (C - 1)^{-2} \) implying, from (8), \( \beta = \alpha^2 \). This choice, that was also made in [18], is not generic, since we see that in general the perturbation equations contain a nontrivial massive term (11) when \( r_g \) (or \( r_f \)) is nonzero, i.e. when one or both metrics are curved.

Note the presence of (some components of) \( h^{\mu\nu}_{(-)} \) in (11) and (12). Under an infinitesimal coordinate transformation \( x^\alpha \rightarrow x^\alpha + \xi^\alpha \) the perturbed metric components transform as \( h^{\mu\nu}_{(g)} \rightarrow h^{\mu\nu}_{(g)} + \nabla^\mu (g) \xi^\nu - \nabla^\nu (g) \xi^\mu \) (where the index \( g \) in the covariant derivative indicates that the derivative is taken with respect to metric \( g \)) along with a similar expression for \( h^{\mu\nu}_{(f)} \). All the components of \( h^{\mu\nu}_{(-)} \) are gauge-invariant in the bi-diagonal case, whereas for a similar expression for \( h^{\mu\nu}_{(g)} \) we obtain the condition (14) made in [18], is not generic, since we see that in general the perturbation equations contain a nontrivial massive term (11) when \( r_g \) (or \( r_f \)) is nonzero, i.e. when one or both metrics are curved.

The particular (gauge-invariant) solution is given by a single nonzero component for each metric perturbation

\[
h^{\mu\nu}_{(g)} = h^{\mu\nu}_{GR} + h^{\mu\nu}_{(m)}.
\]

leaving though the non-zero \( h_{GR}^{\mu\nu} \),

\[
h_{GR}^{\mu\nu} = \epsilon^{\mu\nu} \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ \Omega_1 & c_0 (\Omega - \frac{r_m}{r}) & c_0 r^{-3} & 0 \\ 0 & c_0 r^{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

This completes the derivation of the full solution Eqs. (9). We immediately see that (18), (19) and (21) are regular at the horizon, but not at infinity. Therefore the bi-diagonal black holes (8), (9), (8) do not have unstable modes. The physical perturbations (18), (19) are regular when \( \Omega = i w \), in which case they describe "ingoing" waves. The existence of physical static perturbations (\( \Omega = 0 \)) seems to be excluded due to the term \( \Omega^{-1} \) in (18), (19) unless we take \( c_0 \sim \Omega \), in which case \( h_{GR}^{\mu\nu} \) vanishes and the only nonvanishing contribution left in \( h^{\mu\nu} \) is \( \sim 1/r \), which describes the same solutions

which is known to exhibit GL unstable modes for \( 0 < m' < O(1/r_S) \) [13] [22]. Taking the sum of the two Eqs. (9), the massive terms cancel out and we obtain perturbations equations as in GR for the massless component \( h^{\mu\nu}_{GR} = h^{\mu\nu}_{(g)} + k h^{\mu\nu}_{(f)} \), which is a pure gauge for the s-wave (10).

In non bi-diagonal case we cannot single out the gauge-invariant (massive) metric component in the same way by considering linear combinations of Eqs. (9). Nevertheless, the simple form of the massive matrix (11), together with the constraints (14) lead to an analytical resolution of the perturbation equations. General solutions for \( h^{\mu\nu}_{(g)} \) and \( h^{\mu\nu}_{(f)} \) contain a part that is gauge-dependent (same as in GR) plus a particular solution to the full equations, i.e.

\[
h^{\mu\nu}_{(g)} = h^{\mu\nu}_{GR} + h^{\mu\nu}_{(m)}.
\]

The particular (gauge-invariant) solution is given by a single nonzero component for each metric perturbation

\[
h_{(m)}^{\mu\nu} = \frac{A(r_g - r_f) \epsilon^{\mu\nu}}{4 \Omega} m^2 h_{(-)}^{\mu\nu},
\]

\[
h_{(m)}^{\mu\nu} = \frac{\Omega_1 c_0 (\Omega - \frac{r_m}{r})}{\Omega_1 c_0 r^{-3}} 0 0.
\]

(21)
seen directly from our analysis: going to spatial infinity, the case of bi-flat spacetime, i.e. the helicity-0 mode is turbulence $h_{\mu\nu}^{(0,0)}$. This can be seen from (18), (19), and (13). The presence of curvature restores the scalar degree of freedom, giving the non-trivial (again, regular $\Omega = 0$) solution (15) and (19), which cannot be gauged away.

To summarize, we presented a stability analysis of both bidiagonal and non-bidiagonal black hole solutions against spherically symmetric perturbations. We considered solutions of massive gravity which can be written in the simple form (5) and (6) using advanced Eddington-Finkelstein coordinates. We then studied the perturbations of the metric(s) around these solutions. We confirmed our previous result on the instability of bi-diagonal solutions [14]. On the other hand, we found that non-bidiagonal solutions generically possess physical s-wave perturbations, Eq. (21), however, there are no unstable spherically symmetric modes. For a particular choice of the parameters of the model (namely, $\beta = \alpha^2$) the non-bidiagonal solutions do not contain propagating s-wave modes, since the mass term in the equations of motion for the metric perturbations is identically zero. There are several open questions, which go beyond the scope of this Letter. In particular, we focussed on spherically symmetric perturbations, while it would be important to study the stability of non-spherical modes as well. Also we have not investigate in detail the ghost issue (e.g. for the non-bidiagonal case), namely whether the perturbations contain ghosts or not. These questions are left for future work.

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[21] The absence of this ghost degree of freedom does not necessarily mean that though all other degree of freedom are healthy.  
[22] Note that there is a region in parameter space for which $m^2_{\gamma f}$ becomes negative, and hence Eq. (10) does not correspond to the case studied in [13]. The flip of sign for $m^2_{\gamma f}$, however, signals that the spin-0 part of the graviton becomes a ghost, therefore the vacuum is unstable at the quantum level. This can also be seen from the Higuchi bound [19] when the de-Sitter curvature goes to zero.