General Solution of the Inhomogeneous Div-Curl System and Consequences

Briceyda B. Delgado
R. Michael Porter

Departamento de Matemáticas,
Cinvestav-Querétaro, Mexico
March 7, 2024

Abstract

We consider the inhomogeneous div-curl system (i.e. to find a vector field with prescribed div and curl) in a bounded star-shaped domain in $\mathbb{R}^3$. An explicit general solution is given in terms of classical integral operators, completing previously known results obtained under restrictive conditions. This solution allows us to solve questions related to the quaternionic main Vekua equation $DW = (Df/f)\overline{W}$ in $\mathbb{R}^3$, such as finding the vector part when the scalar part is known. In addition, using the general solution to the div-curl system and the known existence of the solution of the inhomogeneous conductivity equation, we prove the existence of solutions of the inhomogeneous double curl equation, and give an explicit solution for the case of static Maxwell’s equations with only variable permeability.

Keywords: div-curl system, conductivity equation, Maxwell’s equations, double curl equation, quaternionic analysis, monogenic function, hyperholomorphic function, hyperconjugate pair, Vekua equation.

Classification: 35Q60 (35Q61 30G20 30G35 32A26 35F35 35J15)

1 Introduction

We will give a complete solution to the reconstruction of a vector field from its divergence and curl, i.e., the system

$$\begin{align*}
\text{div } \vec{w} &= g_0, \\
\text{curl } \vec{w} &= \vec{g},
\end{align*}$$

for appropriate assumptions on the scalar field $g_0$ and the vector field $\vec{g}$ and their domain of definition in three-space.
This first order partial differential system governs, for example, static electromagnetic fields. In fact, Maxwell’s equations consist of two simultaneous div-curl systems which describe how electrical and magnetic fields are generated by charges and currents together with their variations. Basic references to the theory of the classical Maxwell’s equations are [6, 22]. Chapters 3 and 4 of [25] develop a quaternionic treatment for different systems of Maxwell’s equations, and [26, Chapter 2] does this for electrodynamical models.

The div-curl system has been studied from very many points of view. In [7] an existence result for a solution of the related Moisil-Teodorescu equation \( Dw = g \) was proved, and the div-curl problem consists of finding a purely vectorial solution. Explicit solutions have been found under diverse restrictive conditions, either on the data \( g_0 \) and \( \vec{g} \) (beyond the evident requirement that \( \vec{g} \) be solenoidal) or on the domain. For example, in [1, Section 4] a particular div-curl system with \( g_0 = 0 \) and \( \vec{g} = 0 \) is examined. On the other hand, for a solenoidal vector field, that is, for \( g_0 = 0 \), the Biot-Savart vector fields [12, 15] give a particular solution. In [21, Chapter 5] a numerical solution is given for the div-curl system under certain boundary conditions, based on the Least-Squares finite element method. Another important solution of the div-curl system is given in the reference book [23, p. 166] based on the Helmholtz Decomposition Theorem, representing the solution as an integral operator over all of three-space; this formula is not applicable for, say, a bounded domain. A solution for star-shaped domains, based on a radial integral operator, was recently provided by Yu. M. Grigor’ev in [16, Th. 3.2], valid when the original data \( g_0, \vec{g} \) in the system [1] are harmonic scalar and vectorial functions, respectively. Somewhat earlier, Colombo et. al. [9] produced a right inverse of curl under the condition that certain functions lie in the kernel of one of the components of the Teodorescu operator. This permits expressing the general solution for [1] under the assumption that a certain scalar field admits a hyperconjugate harmonic function.

The present work may be considered as a completion of the analysis in [9]. We will show that in fact that the required hyperconjugate harmonic function exists whenever \( \vec{g} \) is solenoidal. As in [9] we rely heavily on the classical Teodorescu operator, and for that reason we begin in Section 4 by presenting the terminology in the language of quaternionic analysis, which we will mix freely with the notation of the classical operators on vector fields. All results obtained in this paper are also valid for functions that take values in the the algebra \( \mathbb{H}(\mathbb{C}) \) of biquaternions (complex quaternions) but for simplicity we
will work with the real quaternions \( \mathbb{H} \). In Section 3 we study the components of the Teodorescu transform, which we apply in Section 4 to solve the div-curl problem in Theorem 4.4 by first constructing an explicit inverse to the curl, a result which is of independent interest. With this inverse we solve the homogeneous div-curl system (in which \( g_0 \) vanishes), and then follow [9] to show how to apply a correction to obtain the solution for the inhomogeneous system. In the remaining sections we apply this solution to several related problems, including some Dirichlet-type problems, the conductivity equation, the main Vekua equation, and the double curl-type equation, the latter of which is then used in a fundamental way for solving the static Maxwell’s equations with variable permeability (system [37] below). To make the work self-contained and to highlight the beauty of the interrelationships involved, we have included proofs of many facts which can be found elsewhere.

The authors are pleased to express their gratitude to V. V. Kravchenko for his extremely valuable suggestions and encouragement, without which this work would have been impossible.

2 Quaternionic analysis focused on \( \mathbb{R}^3 \)

The multiplicative unit of the non-commutative algebra \( \mathbb{H} \) of quaternions is denoted \( e_0 = 1 \), while the nonscalar units are \( e_1, e_2, e_3 \). We generally consider an element \( x = x_0 + \sum_{i=1}^{3} e_i x_i \in \mathbb{H} \) (\( x_i \in \mathbb{R} \)) to be decomposed as \( x = \text{Sc} x + \text{Vec} x \), where \( \text{Sc} x = x_0 \); thus we have a direct sum \( \mathbb{H} = \text{Sc} \mathbb{H} \otimes \text{Vec} \mathbb{H} \). From now on, we will freely identify \( \text{Sc} \mathbb{H}, \text{Vec} \mathbb{H} \) with the real numbers \( \mathbb{R} \) and Euclidean space \( \mathbb{R}^3 \) respectively. Thus we have function spaces such as \( C^r(\Omega, \mathbb{R}), C^r(\Omega, \mathbb{R}^3) \subseteq C^r(\Omega, \mathbb{H}) \) for a domain \( \Omega \), which will always be contained in \( \mathbb{R}^3 \). We largely follow the notation in [19].

2.1 Monogenic functions

From now on \( \vec{x} \in \mathbb{R}^3 \). The Moisil-Teodorescu differential operator \( D \) (also known as the Cauchy-Riemann or occasionally the Dirac operator) is defined by

\[
D = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.
\] (2)
As $D$ may be applied from both the left and right sides, we write out for clarity that for with a scalar function $w_0(\vec{x})$, and a vectorial function $\vec{w}(\vec{x})$,

$$Dw_0 = w_0D = \text{grad } w_0,$$
$$D\vec{w} = -\text{div } \vec{w} + \text{curl } \vec{w}, \quad \vec{w}D = -\text{div } \vec{w} - \text{curl } \vec{w},$$

expressing $D$ in terms of the gradient $\nabla$, the divergence $\nabla \cdot$, and the curl (or rotational) $\nabla \times$. Thus for $w = w_0 + \vec{w}$ the left and right operators are

$$Dw = Dw_0 + D\vec{w} = -\text{div } \vec{w} + \text{grad } w_0 + \text{curl } \vec{w},$$
$$wD = Dw_0 - D\vec{w} = -\text{div } \vec{w} + \text{grad } w_0 - \text{curl } \vec{w}.$$  \hspace{1cm} (4)

The following \cite{17,18} is a generalization of the Leibniz rule:

$$D[vw] = D[v]w + vD[w] + 2(\text{Sc}(vD))[w],$$ \hspace{1cm} (5)

where we write

$$(\text{Sc}(vD))[w] = -\sum_{i=1}^{3} v_i \partial_i w$$

and $\overline{x} = \text{Sc } x - \text{Vec } x$ denotes quaternionic conjugation. When $\text{Vec } v = 0$, this simplifies to

$$D[vw] = D[v]w + vD[w].$$

Let $\Omega \subseteq \mathbb{R}^3$ be an open subset. A function $w \in C^1(\Omega, \mathbb{H})$ is called left-monogenic (respectively right-monogenic) in $\Omega$ when $Dw = 0$ (respectively $wD = 0$) and we write $\mathcal{M}(\Omega) = \mathcal{M}(\Omega, \mathbb{H})$ and $\mathcal{M}^r(\Omega) = \mathcal{M}^r(\Omega, \mathbb{H})$ for the spaces of left-monogenic and right-monogenic functions. The unqualified term “monogenic” will refer to left-monogenic functions; the term “hyperholomorphic” is also commonly used. By (4),

$$w \in \mathcal{M}(\Omega) \iff \begin{cases} \text{div } \vec{w} = 0, \\ \text{grad } w_0 = -\text{curl } \vec{w}. \end{cases}$$ \hspace{1cm} (6)

One sometimes says that $w_0, \vec{w}$ form a hyperconjugate pair. We write $\text{Har}(\Omega, A) = \{w: \Omega \rightarrow A, \Delta w = 0\}$, where $A = \mathbb{R}, \mathbb{R}^3$ or $\mathbb{H}$, for the corresponding sets of harmonic functions defined in $A$. Since the Laplacian of a scalar function is obtained by $\Delta w_0 = -D^2 w_0$, left and right monogenic functions are harmonic. When both $Dw = 0$ and $wD = 0$, $w$ is called a monogenic constant. By (4), $w$ is a monogenic constant if and only if $w_0$
is constant and $\vec{w}$ satisfies $\text{div} \, \vec{w} = 0$ and $\text{curl} \, \vec{w} = 0$. If $w \in \mathcal{M}(\Omega)$ with $\text{Sc} \, w = 0$ or $\text{Vec} \, w = 0$, then $w$ is a monogenic constant. From this it can be seen that the space $\mathcal{M}^c(\Omega) = \mathcal{M}(\Omega, \mathbb{H}) = \mathcal{M}(\Omega) \cap \mathcal{M}'(\Omega)$ of monogenic constants in $\Omega$ can be decomposed as

$$\mathcal{M}^c(\Omega) = \mathbb{R} \oplus \mathcal{M}(\Omega),$$

where

$$\mathcal{M}(\Omega) = \text{Sol} (\Omega, \mathbb{R}^3) \cap \text{Irr} (\Omega, \mathbb{R}^3),$$

with

$$\text{Sol} (\Omega, \mathbb{R}^3) = \{ \vec{w} : \text{div} \, \vec{w} = 0 \text{ in } \Omega \} \subseteq C^1(\Omega, \mathbb{R}^3),$$

$$\text{Irr} (\Omega, \mathbb{R}^3) = \{ \vec{w} : \text{curl} \, \vec{w} = 0 \text{ in } \Omega \} \subseteq C^1(\Omega, \mathbb{R}^3).$$

Elements of $\text{Sol} (\Omega, \mathbb{R}^3)$ are called solenoidal (or incompressible, or divergence free) fields, while elements of $\text{Irr} (\Omega, \mathbb{R}^3)$ are called irrotational vector fields. Elements of $\mathcal{M}(\Omega)$ are called SI-vector fields and are studied in [14, 30, 31]. Locally they are gradients of real valued harmonic functions.

### 2.2 Standard integral operators

The operators and results in this subsection are all well known. Let $\vec{g} = g_1 \vec{e}_1 + g_2 \vec{e}_2 + g_3 \vec{e}_3$ be a vector field such that $\text{curl} \, \vec{g} = 0$. Define [24, 27]

$$\mathcal{A}[\vec{g}] (x_1, x_2, x_3) = \int_{a_1}^{x_1} g_1(t, a_2, a_3) \, dt + \int_{a_2}^{x_2} g_2(x_1, t, a_3) \, dt$$

$$+ \int_{a_3}^{x_3} g_3(x_1, x_2, t) \, dt.$$  

(8)

Then the scalar function $\psi = \mathcal{A}[\vec{g}]$ is a potential (or antigradient) for $\vec{g}$; i.e. $\text{grad} \, \psi = \vec{g}$. Since potentials are defined up to an arbitrary additive constant, this local definition can be extended to give $\mathcal{A} : \text{Irr} (\Omega, \mathbb{R}^3) \to C^2(\Omega, \mathbb{R})$ whenever $\Omega$ is simply connected.

It is also well known [32] that every real-valued harmonic function is the scalar part of a monogenic function; conversely, the condition for completing a vector part to a hyperconjugate pair is for $\vec{w}$ to be harmonic and solenoidal:
Proposition 2.1. Let \( \vec{w} \in \text{Har}(\Omega, \mathbb{R}^3) \) where \( \Omega \) is simply connected. A necessary and sufficient condition for there to exist \( w \in M(\Omega) \) such that \( \text{Vec} w = \vec{w} \) is that \( \text{div} \vec{w} = 0 \).

Proof. The necessity is given by (6). To prove the sufficiency, let \( \vec{w} \) be solenoidal. Then \( \text{curl} \text{curl} \vec{w} = \text{grad} \text{div} \vec{w} - \Delta \vec{w} = 0 \), where \( \Delta \vec{w} \) is the Laplacian applied to each component of the vector field. Thus we can define \( w_0 = -A[\text{curl} \vec{w}] \) so that \( \text{curl} \vec{w} = -\text{grad} w_0 \) as required by (6).

The radial moment operator, applicable to \( \mathbb{R}^n \)-valued functions in general, is
\[
I^\alpha[\vec{x}](\vec{x}) = \int_0^1 t^\alpha w(t\vec{x}) \, dt
\]  
(9)
in star-shaped domains, where usually \( \alpha > -1 \). Via relations such as \( \partial w_0(t\vec{x})/\partial t = \vec{x} \cdot \text{grad} w_0(t\vec{x}) \) one verifies the following

\[
I^\alpha[\vec{x} \cdot \text{grad} w] = w - (\alpha + 1)I^\alpha[w].
\]

A further property we will need is \( I^\alpha[\vec{x} \cdot \text{curl} \vec{w}] = \vec{x} \cdot \text{curl} I^\alpha[\vec{w}] \), which yields \( I^\alpha[\vec{x} \cdot \text{Vec} D\vec{u}] = \vec{x} \cdot \text{Vec} DI^\alpha[\vec{w}] \).

The monogenic completion operator \( \vec{S}_\Omega : \text{Har}(\Omega, \mathbb{R}) \to \text{Har}(\Omega, \mathbb{R}^3) \) is the composition
\[
\vec{S}_\Omega = I^0[\text{Vec} \vec{x}D]
\]  
(10)
for star-shaped open sets \( \Omega \) with respect to the origin. (Recall that \( Du \) is vectorial for scalar valued \( u \); we have written \( \vec{x}D \) for the operator \( (\vec{x}D)[u](\vec{x}) = \vec{x}Du(\vec{x}) \), which involves a quaternionic multiplication.) Explicitly this is
\[
\vec{S}_\Omega[w_0](\vec{x}) = \text{Vec} \left( \int_0^1 t\vec{x}Dw_0(t\vec{x}) \, dt \right) = \int_0^1 t\vec{x} \times \nabla w_0(t\vec{x}) \, dt, \quad \vec{x} \in \Omega.
\]

When \( \Omega \) is star-shaped with respect to some other point, the definition of \( \vec{S}_\Omega \) is adjusted by shifting the values of \( \vec{x} \) accordingly. Versions of \( \vec{S}_\Omega \) in \( \mathbb{R}^n \) can be found in greater generality in [8] and [18, Sect. 2.1.5]; we give the proof of the following here for completeness, modifying slightly the argument which was given in [32] for functions in domains in \( \mathbb{H} \).
Proposition 2.3. Let $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set. The operator $\tilde{S}_\Omega$ sends $Har(\Omega, \mathbb{R})$ to $Har(\Omega, \mathbb{R}^3)$. For every real-valued harmonic function $w_0 \in Har(\Omega, \mathbb{R})$,

$$w_0 + \tilde{S}_\Omega[w_0] \in \mathfrak{M}(\Omega).$$

Thus there is a monogenic function $w$ such that $Scw = w_0$.

Proof. Let $w(\vec{x}) = w_0(\vec{x}) + \tilde{S}_\Omega[w_0](\vec{x})$. Then since $Sc \vec{x}D[w_0] = -\vec{x} \cdot \text{grad} w_0$, by Lemma 2.2 we have $Sc I^0[-\vec{x}D[w_0]] = w_0 - I^0[w_0]$, so (10) says

$$w = -I^0[D[w_0]\vec{x}] + I^0[w_0]$$

$$= \int_0^1 -t Dw_0(t\vec{x})\vec{x} dt + \int_0^1 w_0(t\vec{x}) dt,$$

when $D[w_0]\vec{x}$ means the quaternionic multiplication $D[w_0](\vec{x})\vec{x}$. We apply $D$ and change the order of integration and derivation since $w_0$ and $Dw_0$ have continuous partial derivatives in $\Omega$:

$$(Dw)(\vec{x}) = \int_0^1 -tD_{\vec{x}}D_{\vec{x}}w_0(t\vec{x})\vec{x} dt + \int_0^1 D_{\vec{x}}[w_0(t\vec{x})] dt.$$ (11)

The subscript in $D_{\vec{x}}$ is the variable with respect to which we apply the operator. Using the Leibniz formula (5),

$$D_{\vec{x}}(D_{\vec{x}}w_0(t\vec{x})\vec{x}) = -\Delta_{\vec{x}}(w_0(t\vec{x}))\vec{x} + \frac{D_{\vec{x}}w_0(t\vec{x})}{4\pi|\vec{y} - \vec{x}|} d\vec{y}$$

$$= t\Delta_{\vec{x}}w_0(t\vec{x}) + 3D_{\vec{x}}w_0(t\vec{x}) - 2D_{\vec{x}}w_0(t\vec{x})$$

$$= D_{\vec{x}}w_0(t\vec{x})$$

since $w_0$ is harmonic. Finally, since the second integrand in (11) is $tD_{\vec{x}}w_0(t\vec{x})$, we conclude $Dw = 0$ as required. \qed

Henceforth $\Omega \subseteq \mathbb{R}^3$ will always be a bounded domain. For a bounded function $w \in C(\Omega, \mathbb{H})$, we can define the volume integral

$$L[w](\vec{x}) = -\int_{\Omega} \frac{w(\vec{y})}{4\pi|\vec{y} - \vec{x}|} d\vec{y}, \quad \vec{x} \in \Omega,$$ (12)

while for $w \in C(\partial \Omega, \mathbb{H})$ the single-layer potential [10, p. 38] is the surface integral

$$M[w](\vec{x}) = \int_{\partial \Omega} \frac{w(\vec{y})}{4\pi|\vec{y} - \vec{x}|} d\vec{s}, \quad \vec{x} \in \mathbb{R}^3 \setminus \partial \Omega.$$ (13)
The Cauchy kernel is the vector field
\[ E(\vec{x}) = \frac{\vec{x}}{4\pi|\vec{x}|^3}, \quad \vec{x} \in \mathbb{R}^3 - \{0\}, \]
which is a monogenic constant. For bounded \( w \in C(\Omega, \mathbb{H}) \), the Teodorescu transform of \( w \) is defined by
\[ T_\Omega[w](\vec{x}) = -\int_\Omega E(\vec{y} - \vec{x})w(\vec{y}) \, d\vec{y}, \quad \vec{x} \in \mathbb{R}^3. \tag{14} \]

Proposition 2.4 ([17, Prop. 2.4.2]). Let \( w \in C(\Omega, \mathbb{H}) \) be bounded. Then \( T_\Omega(w) \in C^1(\Omega, \mathbb{H}) \). Further, the Teodorescu transform acts as the right inverse operator of \( D \):
\[ DT_\Omega[w] = w. \]

3 Components of the Teodorescu operator

As a preliminary to providing the general solution to the div-curl system in Theorem 4.4 below, we begin by analyzing the elements which form the Teodorescu operator. The following operators were introduced in [9]:
\[ T_{0, \Omega}[^w](\vec{x}) = \int_\Omega E(\vec{y} - \vec{x}) \cdot \vec{w}(\vec{y}) \, d\vec{y}, \]
\[ \overrightarrow{T}_{1, \Omega}[w_0](\vec{x}) = -\int_\Omega w_0(\vec{y})E(\vec{y} - \vec{x}) \, d\vec{y}, \]
\[ \overrightarrow{T}_{2, \Omega}[^w](\vec{x}) = -\int_\Omega E(\vec{y} - \vec{x}) \times \vec{w}(\vec{y}) \, d\vec{y}, \tag{15} \]
where \( \cdot \) denotes the scalar (or inner) product of vectors and \( \times \) denotes the cross product. Note that \( \overrightarrow{T}_{1, \Omega} \) acts on \( \mathbb{R} \)-valued functions, while \( T_{0, \Omega}, \overrightarrow{T}_{2, \Omega} \) act on \( \mathbb{R}^3 \)-valued functions, and \( T_{0, \Omega} \) produces scalar-valued functions. Furthermore,
\[ T_\Omega[w_0 + \vec{w}] = T_{0, \Omega}[\vec{w}] + \overrightarrow{T}_{1, \Omega}[w_0] + \overrightarrow{T}_{2, \Omega}[\vec{w}]. \tag{16} \]
This is an expression of the quaternionic multiplication formula \( \vec{a}\vec{b} = -\vec{a} \cdot \vec{b} + b_0\vec{a} + \vec{a} \times \vec{b} \).

The first statement of the following was noted in [9, Prop. 3.8].
Proposition 3.1. Suppose that \( \vec{w} \) is bounded. (i) \( T_{0,\Omega}[\vec{w}] \in \text{Har}(\Omega, \mathbb{R}) \) if and only if \( \vec{w} \in \text{Sol}(\Omega, \mathbb{R}^3) \); (ii) \( \vec{T}_{2,\Omega}[\vec{w}] \in \text{Har}(\Omega, \mathbb{R}^3) \) if and only if \( \vec{w} \in \text{Irr}(\Omega, \mathbb{R}^3) \).

Proof. Using \( \Delta = -D^2 \) and the property \( DT_\Omega = I \) of Proposition 2.4 together with the decomposition of the operator \( D \) given in (4) it follows that

\[
\Delta T_{0,\Omega}[\vec{w}] = -D^2 T_{0,\Omega}[\vec{w}] = -D \vec{w} = \text{div} \vec{w} - \text{curl} \vec{w}.
\]

The scalar and vector parts are \( \Delta T_{0,\Omega}[\vec{w}] = \text{div} \vec{w} \) and \( \Delta \vec{T}_{2,\Omega}[\vec{w}] = -\text{curl} \vec{w} \) by (16).

The operators \( T_{0,\Omega}, \vec{T}_{1,\Omega}, \vec{T}_{2,\Omega} \) can be expressed in terms of the operator \( L \) given in (12) acting on continuous functions and fields.

Proposition 3.2. For bounded \( w_0 \in C(\Omega, \mathbb{R}) \), \( \vec{w} \in C(\Omega, \mathbb{R}^3) \),

\[
T_{0,\Omega}[\vec{w}] = \nabla \cdot L[\vec{w}]
\]

\[
\vec{T}_{1,\Omega}[w_0] = -\nabla L[w_0],
\]

\[
\vec{T}_{2,\Omega}[\vec{w}] = -\nabla \times L[\vec{w}].
\]

Consequently, \( \vec{T}_{1,\Omega}[w_0] \in \text{Irr}(\Omega, \mathbb{R}^3) \) and \( \vec{T}_{2,\Omega}[\vec{w}] \in \text{Sol}(\Omega, \mathbb{R}^3) \).

Proof. The proof is a direct calculation, using \( \nabla \cdot (1/|\vec{x} - \vec{y}|) = -4\pi E(\vec{y} - \vec{x}) \) and the product rules of vector analysis [17] Cor. 1.3.4. The conclusion regarding the images of \( \vec{T}_{1,\Omega}, \vec{T}_{2,\Omega} \) was already noted in [9, Prop. 3.2].

Proposition 3.3. The operator \( L \) given by (12) is a right inverse of the Laplacian \( \Delta \) on the space of bounded functions in \( C^1(\Omega, \mathbb{R}) \).

Proof. Let \( w = w_0 + \vec{w} \in C^1(\Omega, \mathbb{R}) \). Using Proposition 2.4, the identity \( \text{curl} \text{curl} \vec{w} = \text{grad} \text{div} \vec{w} - \Delta \vec{w} \) and the expressions given in Proposition 3.2, we have that

\[
w = DT_\Omega w = D(T_{0,\Omega}[\vec{w}] + \vec{T}_{1,\Omega}[w_0] + \vec{T}_{2,\Omega}[\vec{w}])
\]

\[
= \text{div} (\text{grad} L[w_0]) + \text{grad} (\text{div} L[\vec{w}]) - \text{curl} (\text{curl} L[\vec{w}])
\]

\[
= \Delta L[w].
\]
Proposition 3.3 may also be proved using the fact that $1/(4\pi |\vec{x} - \vec{y}|)$ is a fundamental solution for the Laplacian. From the development given in the above proof we also have

$$\text{div} \ T_{1,\Omega} = -\text{div grad} \ L = -\Delta L = -\text{identity}, \quad (17)$$

which gives the following.

**Corollary 3.4.** [9, Prop. 3.1] The operator $-\vec{T}_{1,\Omega}$ acting on bounded functions in $C(\Omega, \mathbb{R})$ is a right inverse for the divergence $\text{div}$.

Another useful relation is the following.

**Proposition 3.5.** The relation $\text{grad} T_{0,\Omega} + \text{curl} \vec{T}_{2,\Omega} = \text{identity}$ holds on $C(\Omega, \mathbb{R}^3)$.

**Proof.** We have $T_\Omega[\vec{g}] = T_{0,\Omega}[\vec{g}] + \vec{T}_{2,\Omega}[\vec{g}]$, since $\vec{g}$ has no scalar part. The statement is obtained from the vectorial part of the relation of Proposition 2.4 according to (4). \qed

Most of what we have done will go through equally well in the context of generalized derivatives. Thus (12)–(14) make sense when $w$ is integrable, and Proposition 2.3 is shown in [17] to hold in $L^p(\Omega, \mathbb{H})$. Thus Proposition 3.1 extends to the situation in which $\text{div} \vec{w} = 0$ or $\text{curl} \vec{w} = 0$ holds in the sense of distributions, because by Weyl’s Lemma [34], [13, Th. 24.9], the weak solutions $T_{0,\Omega}[\vec{w}]$ and $\vec{T}_{2,\Omega}[\vec{w}]$ of the Laplace equation are smooth solutions.

## 4 Solution to the div-curl system

The first step in solving the div-curl system is to obtain an inverse for the curl operator, an object which is of independent interest. We will use the monogenic completion operator $\vec{S}_\Omega$ of (10) and the component Teodorescu operators of (15). Note that the vanishing divergences $\text{div} \vec{S}_\Omega[T_{0,\Omega}[\vec{w}]] = \text{div} \vec{T}_{2,\Omega}[\vec{w}] = 0$ imply the a priori fact that

$$\vec{T}_{2,\Omega} - \vec{S}_\Omega T_{0,\Omega} : \text{Sol} (\Omega, \mathbb{R}^3) \to \text{Sol} (\Omega, \mathbb{R}^3).$$
Theorem 4.1. Let \( \Omega \subseteq \mathbb{R}^3 \) be a star-shaped open set. The operator
\[
\vec{T}_{2,\Omega} - \vec{S}_\Omega T_{0,\Omega}
\]
is a right inverse for the curl acting on the class of bounded functions in \( \text{Sol}(\Omega, \mathbb{R}^3) \).

Proof. Let \( \vec{g} \in \text{Sol}(\Omega, \mathbb{R}^3) \) and let \( \vec{w} = \vec{T}_{2,\Omega}[\vec{g}] - \vec{S}_\Omega[v_0] \) where \( v_0 = T_{0,\Omega}[\vec{g}] \).

By Proposition 3.1, \( v_0 \in \text{Har}(\Omega, \mathbb{R}) \), so by Proposition 2.3, \( v_0 + \vec{S}_\Omega[v_0] \) is a monogenic function whose equivalent system (6) is
\[
\begin{align*}
\text{div } \vec{S}_\Omega[v_0] &= 0, \\
\text{curl } \vec{S}_\Omega[v_0] &= -\nabla v_0.
\end{align*}
\]
(19)
Combining these equations equations with Proposition 3.5, we have that
\[
\text{curl } \vec{w} = -\nabla v_0 + \vec{g} + \nabla v_0 = \vec{g}.
\]

In [9] it was shown that \( \vec{T}_{2,\Omega} \) acts as a right inverse for curl for elements of the kernel of \( T_{0,\Omega} \). Indeed, let \( T_{0,\Omega}[\vec{g}] = 0 \). By Proposition 3.1, the field \( \vec{g} \) is indeed solenoidal, and since by (19) \( \text{div } \vec{S}_\Omega[T_{0,\Omega}[\vec{g}]] = 0 \), Theorem 4.1 says that \( \text{curl } \vec{T}_{2,\Omega}[\vec{g}] = \vec{g} \). It was recognized in [9] that to require \( T_{0,\Omega}[\vec{g}] \) to vanish would be too strong a condition; now we see that the precise condition is for it to be harmonic.

Corollary 4.2. Let \( \Omega \subseteq \mathbb{R}^3 \) be a star-shaped open set. Let \( \vec{g} \in \text{Sol}(\Omega, \mathbb{R}^3) \) be a bounded divergence free vector field. Then the general solution of the homogeneous system
\[
\text{div } \vec{w} = 0, \quad \text{curl } \vec{w} = \vec{g}
\]
in \( \Omega \) has the form
\[
\vec{w} = \vec{T}_{2,\Omega}[\vec{g}] - \vec{S}_\Omega[T_{0,\Omega}[\vec{g}]] + \nabla h,
\]
(20)
where \( h \in \text{Har}(\Omega, \mathbb{R}) \) is an arbitrary real-valued harmonic function.

Proof. First let \( \vec{w} \) be given by (20). Then \( \text{div } \vec{w} = 0 \) by the observations preceding the statement of the Corollary. A difference \( \vec{v} \) of two solutions of the div-curl system satisfies \( \text{div } \vec{v} = 0, \text{curl } \vec{v} = 0 \), i.e., \( \vec{v} \) is a monogenic constant. Since \( \Omega \) is star-shaped and therefore simply connected, \( \vec{v} \) is the gradient of a harmonic function \( h \).
Corollary 4.3. The operator

\[-L - I^{-1} \left[ \frac{|\vec{x}|^2}{2} \text{grad} T_{0,\Omega} \right] \]  

(21)

is a right inverse for the double curl operator \( \text{curl} \text{curl} \) acting on the class of bounded functions \( \vec{w} \in \text{Sol}(\Omega, \mathbb{R}^3) \).

(Note that \( I^\alpha \) for the exponent \( \alpha = -1 \) can be applied because of the factor \( |\vec{x}|^2 \) in the operand.)

Proof. It is enough to show that the curl applied after the operator (21) produces the right inverse of curl given in (18). But Proposition 3.2 with (10) gives

\[
\text{curl} I^{-1} \left[ \frac{|\vec{x}|^2}{2} \text{grad} T_{0,\Omega} \right] = I^0 \left[ \text{curl} \left( \frac{|\vec{x}|^2}{2} \text{grad} T_{0,\Omega} \right) \right] = I^0 \left[ \text{grad} \frac{|\vec{x}|^2}{2} \times \text{grad} T_{0,\Omega} \right] = I^0 \left[ \vec{x} \times \text{grad} T_{0,\Omega} \right] = \vec{S}_{\Omega} T_{0,\Omega},
\]

while by Lemma 2.2,

\[
\text{curl} L = -\vec{T}_{2,\Omega}.
\]

Adding these equalities we have the result. \( \square \)

Now we can proceed to solve the inhomogeneous div-curl system with data \( g_0, \vec{g} \). Assuming as always that \( \vec{g} \) is solenoidal, the function \( v = T_{\Omega}[-g_0 + \vec{g}] \) satisfies \( Dv = -g_0 + \vec{g} \) and therefore is a quaternionic solution to (11). We seek to construct a vector solution \( \vec{w} \) by subtracting a monogenic function whose scalar part is precisely the scalar part of \( v \). Thus the key consists in taking the \( T_{0,\Omega} \) component of \( T_{\Omega}(\vec{g}) \), and using Proposition 2.3 to construct the monogenic conjugate of the \( \mathbb{R} \)-valued function \( T_{0,\Omega}[\vec{g}] \). This is accomplished in the following result.

Theorem 4.4. Let \( \Omega \) be a star-shaped open set. Let \( g_0 \in C(\Omega, \mathbb{R}) \) and \( \vec{g} \in \text{Sol}(\Omega, \mathbb{R}^3) \) be bounded. The general solution of the inhomogeneous div-curl system (1) is given by

\[
\vec{w} = -\vec{T}_{1,\Omega}[g_0] + \vec{T}_{2,\Omega}[\vec{g}] - \vec{S}_{\Omega}[T_{0,\Omega}[\vec{g}]] + \nabla h,
\]

where \( h \in \text{Har}(\Omega, \mathbb{R}) \) is arbitrary.
Proof. Since \( \text{div} \vec{g} = 0 \), Proposition 3.1 says that \( T_{0,\Omega}[\vec{g}] \) is an \( \mathbb{R} \)-valued harmonic function, so Proposition 2.3 permits us to complete it to the monogenic function \( T_{0,\Omega}[\vec{g}] + \vec{S}_{\Omega}[T_{0,\Omega}[\vec{g}]] \). By (16), the difference
\[
T_{\Omega}[-g_0 + \vec{g}] - (T_{0,\Omega}[\vec{g}] + \vec{S}_{\Omega}[T_{0,\Omega}[\vec{g}]]) = -\overrightarrow{T}_{1,\Omega}[g_0] + \overrightarrow{T}_{2,\Omega}[\vec{g}] - \vec{S}_{\Omega}[T_{0,\Omega}[\vec{g}]]
\]
is purely vectorial. By Proposition 2.4 and the fact that \( D\nabla h = 0 \),
\[
D\vec{w} = D(T_{0}[-g_0 + \vec{g}]) = -g_0 + \vec{g}.
\]
so (1) is satisfied because of (3). As in the proof of Corollary 4.2, we obtain the general solution adding to \( \vec{w} \) the gradient of an arbitrary harmonic function.

Consider the following Sobolev spaces,
\[
H^1(\Omega, A) = \left\{ u \in L^2(\Omega, A) : \text{grad} u \in L^2(\Omega, A) \right\},
H^1_0(\Omega, A) = C^\infty_0(\Omega, A) \subseteq H^1(\Omega, A),
H^{1/2}(\partial\Omega, A) = \left\{ \varphi \in L^2(\partial\Omega, A) : \exists u \in H^1(\Omega, A) \text{ with } u|_{\partial\Omega} = \varphi \right\}.
\]
Again \( A = \mathbb{R}, \mathbb{R}^3, \mathbb{H} \) while \( C^\infty_0 \) denotes smooth functions of compact support; the space \( H^{1/2}(\partial\Omega, A) \) consists of the boundary values of functions in \( H^1(\Omega, A) \). (Whenever \( \partial\Omega \) is mentioned we will assume it is sufficiently smooth to apply the basic facts about Sobolev spaces.) We write \( [\varphi] = u + H^1_0(\Omega, A) \subseteq H^1(\Omega, A) \) for the equivalence class of functions with common boundary values \( \varphi \).

Proposition 3.1 and the results of this section are valid even in the distributional sense, because the properties of the Teodorescu transform \( T_{\Omega} \) (see [17, Prop. 2.4.2]) as well as Proposition 2.3 are also valid in Sobolev spaces. Thus in Theorem 4.4, we may take \( g_0 \in L^2(\Omega, \mathbb{R}), \vec{g} \in L^2(\Omega, \mathbb{R}^3) \) and \( \text{div} \vec{g} = 0 \) in the weak sense,
\[
\int_{\Omega} \vec{g} \cdot \nabla v \, dx = 0,
\]
for all test functions \( v \in H^1_0(\Omega) \). Then (22) is a weak solution of the div-curl system (1).

Example 4.5. Let \( \Omega \) be the unit ball in \( \mathbb{R}^3 \) minus any ray emanating from the origin. Take \( g_0 = 0 \) and \( \vec{g} = \vec{x}/|\vec{x}|^3 \) in the div-curl system (1). Since \( \Omega \) is
star-shaped with respect to any of its points \( \vec{a} \) opposite to the ray, we shift the origin to \( \vec{a} \) in the formula (10) for \( \vec{S}_\Omega \) (cf. [32]) as follows:

\[
\vec{S}_\Omega[w_0](\vec{x}) = \operatorname{Vec} \int_0^1 -t Dw_0((1-t)\vec{a} + t\vec{x})(\vec{x} - \vec{a})dt.
\]

(24)

Since the removed ray is of zero measure, we may use the explicit formula for the Teodorescu transform of \( \vec{g} \) for the unit ball given in [18, p. 324, formula 28], namely

\[
T_{0,\Omega}[\vec{g}] = 1 - \frac{1}{|\vec{x}|} \quad \text{while} \quad -\vec{T}_{1,\Omega}[\vec{g}] = 0.
\]

Since \( \vec{g} \) is solenoidal, the div-curl solution of Theorem 4.4 is

\[
\vec{w}(\vec{x}) = -\vec{S}_\Omega \left[ \frac{1}{|\vec{x}|} + 1 \right] \frac{\vec{a} \times \vec{x}}{|\vec{x}|(\vec{a} \cdot \vec{x} - |\vec{a}||\vec{x}|)}.
\]

(25)

Since \( \text{div} \vec{w} = 0 \) and \( \text{curl} \vec{w} = \vec{x}/|\vec{x}|^3 \) are independent of \( \vec{a} \), the difference of two such solutions is an SI field, as would be expected.

**Remark 4.6.** Suppose now that two given scalar and vectorial functions \( g_0 \) and \( \vec{g} \) are harmonic. Under this additional hypothesis (and of course \( \text{div} \vec{g} = 0 \)), a solution

\[
\vec{w} = -\vec{x} \times \int^1 I_1[\vec{g}] + \text{grad} \left( \frac{|\vec{x}|^2}{4} I^{1/2}[g_0 - \vec{x} \cdot I^2[\text{curl} \vec{g}]] \right)
\]

(26)

for the div-curl system (1) was given by Yu. M. Grigor’ev in [16, Th. 3.2], where the integrals \( I^\alpha \) were defined in (9). We relate this to our solution (22).

By additivity we may consider \( g_0 \) and \( \vec{g} \) independently. Suppose \( g_0 = 0 \), and let \( \vec{w} \) be given by (26). Substitute \( \vec{g} = \text{curl} \overrightarrow{T_{2,\Omega}[\vec{g}]} + \text{grad} T_{0,\Omega}[\vec{g}] \) (Proposition 3.5) to obtain

\[
\vec{w} = \vec{f} - \vec{S}_\Omega T_{0,\Omega}[\vec{g}]
\]

where

\[
\vec{f}(\vec{x}) = -\vec{x} \times \int^1 [\text{curl} \overrightarrow{T_{2,\Omega}[\vec{g}]} + \text{grad} \left( \frac{|\vec{x}|^2}{4} I^{1/2}[-\vec{x} \cdot I^2[\text{curl} \vec{\overrightarrow{T_{2,\Omega}[\vec{g}]}]]] \right)].
\]

Here we have used the fact that \( \text{curl curl} \overrightarrow{T_{2,\Omega}[\vec{g}]} = \text{curl} [\vec{g} - \text{grad} T_{0,\Omega}[\vec{g}]] = \text{curl} \vec{g} \).
The function $\vec{v} = \vec{T}_{2,\Omega}[\vec{g}]$ trivially satisfies
\[
\text{div } \vec{v} = 0, \quad \text{curl } \vec{v} = \text{curl } \vec{T}_{2,\Omega}[\vec{g}],
\]
while by (26), $\vec{f}$ satisfies the same system. Thus $\vec{f}$ differs from $\vec{T}_{2,\Omega}[\vec{g}]$ by an SI field. Since $\vec{T}_{1,\Omega}[g_0] = 0$, we have $\vec{w} = \vec{T}_{2,\Omega}[\vec{g}] - \vec{S}_\Omega T_{0,\Omega}[\vec{g}] + \nabla h$ for some harmonic $h$, which agrees with our solution (22).

For $\vec{g} = 0$, we simply note that both (22) and (26) are left inverses of div applied to $g_0$.

It is easily seen that $\vec{g}$ in Example 4.5 is harmonic, and that when one shifts appropriately the base point of integration in (26), the same solution (25) is obtained.

### 4.1 Div-curl system with boundary data

We will rewrite the right inverse for the curl (18) in terms of boundary value integral operators. The following result tells us that $T_{0,\Omega}$ is in some sense a boundary integral operator, as it can be expressed in terms of the single-layer operator $M$ of (13) when the boundary values of $\vec{w}$ are known. Here $\eta$ denotes the unit normal vector to the boundary $\partial \Omega$, which from here on will be assumed to be smooth, and $\Omega$ will be bounded.

**Proposition 4.7.** For every $\vec{w} \in \text{Sol}(\overline{\Omega}, \mathbb{R}^3)$,
\[
T_{0,\Omega}[\vec{w}] = M[\vec{w}]_{\partial \Omega} \cdot \eta.
\]

**Proof.** Using that
\[
\nabla_{\vec{x}} \left( \frac{1}{4\pi |\vec{x} - \vec{y}|} \right) = -\frac{\vec{x} - \vec{y}}{4\pi (\vec{x} - \vec{y})^3} = -\nabla_{\vec{y}} \left( \frac{1}{4\pi |\vec{x} - \vec{y}|} \right),
\]
we find that
\[
T_{0,\Omega}[\vec{w}](\vec{x}) = \int_{\Omega} \frac{\vec{x} - \vec{y}}{4\pi |\vec{x} - \vec{y}|^3} \cdot \vec{w}(\vec{y}) \, d\vec{y}
\]
\[
= \int_{\Omega} \nabla_{\vec{y}} \left( \frac{1}{4\pi |\vec{x} - \vec{y}|} \right) \cdot \vec{w}(\vec{y}) \, d\vec{y}
\]
\[
= \int_{\Omega} \nabla_{\vec{y}} \cdot \left( \frac{\vec{w}(\vec{y})}{4\pi |\vec{x} - \vec{y}|} \right) \, d\vec{y}. \tag{27}
\]
since \( \vec{w} \in \text{Sol}(\Omega, \mathbb{R}^3) \). By the Divergence Theorem,

\[
T_{0,\Omega}[\vec{w}](\vec{x}) = \int_{\partial \Omega} \frac{\vec{w}(\vec{y})}{4\pi |\vec{x} - \vec{y}|} \cdot \eta(\vec{y}) \, ds_y
\]

as desired. \( \square \)

Analogously to Proposition 4.7, the operator \( \overrightarrow{T}_2, \Omega \) can be recovered when we know the boundary values of functions in \( \text{Irr}(\overline{\Omega}, \mathbb{R}^3) \):

**Proposition 4.8.** For every \( \vec{w} \in \text{Irr}(\overline{\Omega}, \mathbb{R}^3) \),

\( \overrightarrow{T}_2, \Omega[\vec{w}] = -M[\vec{w}|_{\partial \Omega} \times \eta]. \) (28)

**Proof.**

\[
\overrightarrow{T}_2, \Omega[\vec{w}](\vec{x}) = \int_{\Omega} -\frac{\vec{x} - \vec{y}}{4\pi |\vec{x} - \vec{y}|^3} \times \vec{w}(\vec{y}) \, dy
\]

\[
= -\int_{\Omega} \nabla_y \left( \frac{1}{4\pi |\vec{x} - \vec{y}|} \right) \times \vec{w}(\vec{y}) \, dy
\]

\[
= -\int_{\Omega} \nabla_y \times \left( \frac{\vec{w}(\vec{y})}{4\pi |\vec{x} - \vec{y}|} \right) \, dy + \int_{\Omega} \nabla_y \times \vec{w}(\vec{y}) \, dy.
\]

Applying \( \vec{w} \in \text{Irr}(\overline{\Omega}, \mathbb{R}^3) \) and Stokes’ theorem to obtain the desired result. \( \square \)

We will write \( \overrightarrow{M}(\partial \Omega) \) for the space of boundary values of SI vector fields in \( \overline{\Omega} \), which we recall from (7) are the purely vectorial monogenic constants. Since SI vector fields are harmonic, the extension of \( \varphi \in \overrightarrow{M}(\partial \Omega) \) to the interior is unique. We rewrite the right inverse of curl given in Theorem 4.1 as a boundary integral operator, under the condition that boundary data \( \vec{\varphi} \) has an irrotational and solenoidal extension:

**Proposition 4.9.** Let \( \vec{\varphi} \in \overrightarrow{M}(\partial \Omega) \) be the boundary values of the vector field \( \vec{g} \) defined in \( \Omega \). Define

\[
\vec{w} = -M[\vec{\varphi} \times \eta] - \vec{S}_\Omega[M[\vec{\varphi} \cdot \eta]]
\] (29)

where again \( \eta \) is the outward normal. Then

\[
\text{div} \vec{w} = 0, \quad \text{curl} \vec{w} = \vec{g}.
\]
Proof. By Propositions 4.7 and 4.8, $T_{0,\Omega}[\vec{w}] = M[\vec{\varphi} \cdot \eta]$ and $\vec{T}_{2,\Omega}[\vec{g}] = -M[\vec{\varphi} \times \eta]$, respectively. The statement now follows from the Corollary 4.2. 

Let $\vec{\varphi} \in \mathbb{M}(\partial \Omega)$. Since $\Delta \vec{w} = -\text{curl } \vec{g}$, the solution (29) solves the following Dirichlet-type problem

\[
\begin{align*}
\Delta \vec{w} &= 0, \\
\text{curl } \vec{w}|_{\partial \Omega} &= \vec{\varphi}.
\end{align*}
\]

5 Application to the main Vekua equation and conductivity equation on $\mathbb{R}^3$

The Vekua equation, whose theory was introduced in [5, 33] for functions in $\mathbb{R}^2$, plays an important role in the theory of pseudo-analytic functions (sometimes called generalized analytic functions). We will study a special Vekua equation, which in [24] is called the main Vekua equation. We are interested in the natural generalization of this equation to the quaternionic case [24, Ch. 16], which possesses properties similar to those of the complex Vekua equation, including an intimate relation with the conductivity equation. The conductivity equation appears in many aspects of physics, and gives rise to inverse problems with applications to fields such as tomography. Here we apply the results obtained on the div-curl system to study solutions of these equations.

5.1 The main Vekua equation and equivalent formulations

The main Vekua equation is

\[
DW = \frac{Df}{f} \vec{W},
\]

with $D$ the Moisil-Teodorescu operator given in (2), and $f$ a nonvanishing smooth function. We are interested in solutions $W = W_0 + \vec{W} \in C^1(\Omega, \mathbb{H})$ for a domain $\Omega \subseteq \mathbb{R}^3$. 

17
The operator $D - (Df/f)C_{\mathbb{H}}$ corresponding to (30), and similar expressions, appear in various factorizations. For example, when $u$ is scalar,

$$\nabla \cdot f^2 \nabla u = -f \left( D + M^\partial f \right) \left( D - \frac{Df}{f} C_{\mathbb{H}} \right) f u,$$

$$\left( \Delta - \frac{\Delta f}{f} \right) u = \left( D + M^\partial f \right) \left( D - M^\partial f \right) u,$$

where $C_{\mathbb{H}}$ is the quaternionic conjugate operator and $M^a$ denotes the operator of multiplication on the right by the function $a$.

The set

$$\mathcal{M}_f(\Omega) = \left\{ W : DW = \frac{Df}{f} W \right\} \subseteq C^1(\Omega, \mathbb{H})$$

(31)

of solutions of the main Vekua equation (30) is a nontrivial linear subspace over $\mathbb{R}$. In [24, Chapter 16] we find results that relate solutions of the main Vekua equation to solutions of other differential equations. In particular, (30) is related to the $\mathbb{R}$-linear Beltrami equation, a fact which was essential in the solution of the Calderón problem in the complex case [2]. A similar fact is the following.

**Lemma 5.1.** [24, Th. 161] $W \in \mathcal{M}_f(\Omega)$ if and only if the scalar part $W_0$ and the vector part $\vec{W}$ satisfy the homogeneous div-curl system

$$\text{div}(f\vec{W}) = 0,$$

$$\text{curl}(f\vec{W}) = -f^2 \nabla \left( \frac{W_0}{f} \right).$$

(32)

This system reduces to (6) when $f$ is constant. Thus it is natural to wish to generalize results concerning monogenic functions to solutions of the main Vekua equation. This is one of our main goals in this section.

Suppose that $W \in C^2(\Omega, \mathbb{H})$. From the second equation of (32) we obtain by applying div, curl that

$$\nabla \cdot f^2 \nabla \left( \frac{W_0}{f} \right) = 0,$$

(33)

$$\text{curl} \left( f^{-2} \text{curl} \left( f\vec{W} \right) \right) = 0.$$  

(34)
The first equation is the so-called *conductivity equation* and the second one is called the *double curl-type equation* for the conductivity $f^2$. These equations are satisfied separately by the scalar and vector parts of $W$ in analogously to the way that two harmonic conjugates satisfy separately the Laplace equation; together they are not sufficient for $W_0 + \vec{W}$ to satisfy (32). The conductivity equation is equivalent to the Schrödinger equation

$$\Delta W_0 - \frac{\Delta f}{f} W_0 = 0.$$ 

Using (4) and the fact that $f\vec{W}$ is vectorial, we have the equivalence

$$DW = \frac{Df}{f} \vec{W} \iff D(f\vec{W}) = -f^2 \nabla \left( \frac{W_0}{f} \right).$$ \hspace{1cm} (35)

For brevity we will say that $f^2$ is a *conductivity* when $f$ is a non-vanishing $\mathbb{R}$-valued function in the domain under consideration. The conductivity will be called *proper* when $f$ and $1/f$ are bounded.

### 5.2 Completion of Vekua solutions from partial data

It is important to know what type of functions can be solutions to some main Vekua equation (i.e., for some $f$). Another question is how to complete an $f^2$-hyperconjugate pair, i.e. to recover the vector part $\vec{W}$ such that $W = W_0 + \vec{W} \in \mathcal{M}_f(\Omega)$ when the scalar part $W_0: \Omega \rightarrow \mathbb{R}$ is known, or vice versa. We apply the results of Section 4 to these questions. First we treat the generalization of Proposition 2.3 for nonconstant conductivity.

**Theorem 5.2.** Let $f^2$ be a conductivity of class $C^2$ in an open star-shaped set $\Omega \subseteq \mathbb{R}^3$. Suppose that $W_0 \in C^2(\Omega, \mathbb{R})$ satisfies the conductivity equation (33) in $\Omega$. Then there exists a function $\vec{W}$ such that $W_0 + \vec{W} \in \mathcal{M}_f(\Omega)$. The function $f\vec{W}$ is unique up to a purely vectorial additive monogenic constant, i.e., the gradient of a real harmonic function.

**Proof.** Observe that (32) is a homogeneous div-curl system (11) in the unknown $\vec{w} = f\vec{W}$, with $g_0 = 0$ and $\vec{g} = -f^2 \nabla (W_0/f) \in \text{Sol}(\Omega, \mathbb{R}^3)$, since by hypothesis $W_0$ satisfies (33). By Corollary 4.2, the general solution $\vec{W}$ is given by

$$f\vec{W} = \vec{T}_{2,\Omega} \left[ -f^2 \nabla \left( \frac{W_0}{f} \right) \right] + \vec{S}_\Omega \left[ T_{0,\Omega} \left[ f^2 \nabla \left( \frac{W_0}{f} \right) \right] \right] + \nabla h,$$

(36)
where $h$ is an arbitrary harmonic function. By Lemma 5.1, $W_0 + \vec{W} \in \mathfrak{M}_f(\Omega)$.

Of course, the assumptions of Theorem 5.2 can be relaxed to say that $f, W_0 \in H^1(\Omega, \mathbb{R})$ and (33) is satisfied weakly. Then $\vec{W} \in H^1(\Omega, \mathbb{R}^3)$ given by (36) produces a weak solution $W_0 + \vec{W}$ of the main Vekua equation.

Analogously to the well-known $\partial$-problem in the complex case, the completion of the vector part of solutions of the main Vekua equation is given in terms of the integral operator $T_\Omega$. In [29] there is a generalization for the quaternionic case; however, the solution given there is not purely vectorial.

### 5.3 A result on Vekua-type operators

As was noted at the beginning of this section, there are other operators quite similar to the Vekua operator which appear in factorizations of second order operators. We write $D_r[f] = f D$ for the right-sided operator of [41]. In [27], certain relations were established among the operators

$$
\mathbf{V} = D - \frac{Df}{f} C_{\mathbb{H}}, \quad \mathbf{V}_r = D_r - M \frac{Df}{f} C_{\mathbb{H}},
$$

$$
\mathbf{V}_1 = D_r + \frac{Df}{f}, \quad \mathbf{V}_1 = D + M \frac{Df}{f}.
$$

where $M \frac{Df}{f}$ denotes right multiplication. The following is a a right inverse of the operator $\mathbf{V}$ on a subspace analogous to the condition that $\vec{g} \in \text{Sol}(\Omega, \mathbb{R}^3)$ for [41].

**Theorem 5.3.** Let $f^2$ be a conductivity in the star-shaped open set $\Omega \subseteq \mathbb{R}^3$. Let $\vec{G} \in C^1(\Omega, \mathbb{R}^3)$ be a purely vectorial solution of the equation $\mathbf{V}_1 \vec{G} = 0$. Then the general solution of the system $\mathbf{V} W = 0$ and $\mathbf{V} W = \vec{G}$ in $\Omega$ is given by

$$
W = \frac{1}{2} \left( f A \left[ \frac{\vec{G}}{f} \right] - \frac{1}{f} \vec{T}_{2,\Omega}[f \vec{G}] + \frac{1}{f} \vec{S}_{\Omega}[T_{0,\Omega}[f \vec{G}]] + \nabla h \right),
$$

where $h \in \text{Har}(\Omega, \mathbb{R})$ is arbitrary.

A similar formula was given in [27, Theorem 9] but with an incorrect expression in place of $\vec{T}_{2,\Omega} - \vec{S}_{\Omega} T_{0,\Omega}$ for the inverse of curl. Otherwise the proof is essentially the same.
5.4 Vekua boundary value problems

The following fact is essential to the solution of the Calderón problem in the plane [2]; see [20, Th. 4.1] and the references therein, and a sketch of a proof in $\mathbb{R}^n$ in [3, p. 407]. The following conductivity problem reduces to the Dirichlet problem in the case where $f$ is constant.

**Proposition 5.4.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with connected complement and $f^2$ a measurable proper conductivity in $\Omega$. Given prescribed boundary values $\varphi \in H^{1/2}(\partial \Omega, \mathbb{R})$, there is a unique solution $u \in H^1(\Omega, \mathbb{R})$ to the conductivity boundary value problem

$$
\nabla \cdot f^2 \nabla u = 0,
$$

$$
u|_{\partial \Omega} = \varphi.
$$

The methods of variational calculus applied in Section 6.2 can be used to obtain the existence of solutions of second-order elliptic equations such as this one.

**Theorem 5.5.** Let $f^2$ be a proper conductivity in the bounded, star-shaped open set $\Omega \subseteq \mathbb{R}^3$, $f \in H^1(\Omega, \mathbb{R})$ and suppose that $\varphi \in H^{1/2}(\partial \Omega, \mathbb{R})$. Then there exists a function $W : \Omega \to \mathbb{H}$ that satisfies the main Vekua equation (30) weakly and has boundary values $ScW|_{\partial \Omega} = \varphi$.

**Proof.** Proposition 5.4 gives a solution $u \in H^1(\Omega, \mathbb{R})$ of $\nabla \cdot f^2 \nabla u = 0$ with boundary values $u|_{\partial \Omega} = \varphi/f \in H^{1/2}(\partial \Omega, \mathbb{R})$. The function $W_0 = fu$ satisfies the conditions of Theorem 5.2 and therefore has a completion $W_0 + \bar{W}$ satisfying the Vekua equation weakly.

**Remark 5.6.** Theorem 5.5 provides a way to define a “Hilbert transform”

$$
\mathcal{H}_f : H^{1/2}(\partial \Omega, \mathbb{R}) \to H^{1/2}(\partial \Omega, \mathbb{R}^3)
$$

associated to the main Vekua equation (30), by

$$
\mathcal{H}_f[\varphi] = \bar{W}|_{\partial \Omega},
$$

where $\bar{W}$ is given by (36).

We now characterize the elements of the space $\text{Vec} \mathfrak{M}_f(\Omega)$ of vector parts of solutions to the main Vekua equation, that is the $f^2$-hyperconjugate pair.
Proposition 5.7. [27, Th. 10] Let $\vec{W} \in C^2(\Omega, \mathbb{R}^3)$ where $\Omega$ is a simply connected domain in $\mathbb{R}^3$. For the existence of $W \in \mathfrak{M}_f(\Omega)$ such that $\text{Vec} W = \vec{W}$ it is necessary and sufficient that $\text{div} (f\vec{W}) = 0$ together with the double curl-type equation (34).

Proof. The necessity is given by (32). For the sufficiency, the second condition implies that $f^{-2} \text{curl} (f\vec{W})$ admits a potential $W_0$ obtained by applying $\mathcal{A}$ of (8). The function $W = W_0 + \vec{W}$ then satisfies (32) and hence also (30).

6 Equation of double curl type

The following system of equations corresponds to the static Maxwell system, in a medium when just the permeability $f^2$ is variable ([25, Ch. 4] or [6, Ch. 2]):

\[
\begin{align*}
\text{div} (f^2 \vec{H}) &= 0, \\
\text{div} \vec{E} &= 0, \\
\text{curl} \vec{H} &= \vec{g}, \\
\text{curl} \vec{E} &= f^2 \vec{H}.
\end{align*}
\] (37)

Here $\vec{E}$ and $\vec{H}$ represent electric and magnetic fields, respectively. We will apply our results to this system and to the double curl-type equation

\[
\text{curl} (f^{-2} \text{curl} \vec{E}) = \vec{g},
\] (38)

which is immediate from the last two equations of (37).

6.1 Generalized solutions of the Maxwell system

To obtain a general solution of (37) we will use the existence of solutions of the inhomogeneous conductivity problem

\[
\begin{align*}
\text{div} (f^2 \nabla W_0) &= g_0, \\
W_0|_{\partial \Omega} &= \varphi.
\end{align*}
\] (39)
Theorem 6.1 ([28, p. 197, Th. 10]). Suppose that $f^2$ is a continuous proper conductivity in $\Omega$ and $g_0 \in L^2(\Omega, \mathbb{R})$. Let $\varphi \in H^{1/2}(\partial \Omega, \mathbb{R})$. Then there exists a unique generalized solution $W_0 \in H^1(\Omega, \mathbb{R})$ to the boundary value problem (39). Furthermore, $W_0$ satisfies

$$
\|W_0\|_{H^1(\Omega)} \leq c(\|g_0\|_{L^2(\Omega)} + \inf \{\|v\|_{H^1(\Omega)} : v \in H^1(\Omega, \mathbb{R}) \text{ and } v|_{\partial \Omega} = \varphi\})
$$

for some constant $c$ which does not depend on $g_0$ or $\varphi$.

Corollary 6.2 ([28, p. 173, Th. 1]). Suppose that $f^2$ is a continuous proper conductivity in $\Omega$ and $g_0 \in L^2(\Omega, \mathbb{R})$. Then there exists a unique generalized solution $W_0 \in H^1_0(\Omega, \mathbb{R})$ to the boundary value problem (39). Furthermore,

$$
\|W_0\|_{H^1_0(\Omega)} \leq c\|g_0\|_{L^2(\Omega)},
$$

for some constant $c$ which does not depend on $g_0$.

The right inverse of the curl given by Theorem 4.1 permits us to invert the composed operator $\text{curl} f^{-2} \text{curl}$, providing of course that this right inverse is applied to weakly solenoidal fields. The pair of fields $(\vec{E}, \vec{H})$ in the following result is constructed explicitly in terms of the operators defined in this paper.

Theorem 6.3. Let the domain $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set, and assume that $f^2$ is a continuous proper conductivity in $\Omega$. Let $\vec{g} \in L^2(\Omega, \mathbb{R}^3)$ satisfy $\text{div}\vec{g} = 0$. Then there exists a generalized solution $W_0 \in H^1_0(\Omega, \mathbb{R})$ to the boundary value problem (39) and its general form is given by

$$
\vec{E} = \vec{T}_{2,\Omega}[f^2(\vec{B} + \nabla h)] - \vec{S}_{1,\Omega}[T_{0,\Omega}[f^2(\vec{B} + \nabla h)]] + \nabla h_1,
\vec{H} = \vec{B} + \nabla h,
$$

where $h_1$ is an arbitrary real valued harmonic function.

Proof. Since $\text{div}\vec{g} = 0$, by Corollary 4.2 the vector field

$$
\vec{B} = \vec{T}_{2,\Omega}[\vec{g}] - \vec{S}_{1,\Omega}[T_{0,\Omega}[\vec{g}]]
$$

satisfies $\text{curl}\vec{B} = \vec{g}$ and $\text{div}\vec{B} = 0$ weakly. To solve

$$
\text{curl}\vec{E} = f^2(\vec{B} + \nabla h),
$$

23
we must find an $\mathbb{R}$-valued function $h$ such that

$$\text{div} \left( f^2 (\vec{B} + \nabla h) \right) = 0.$$  

Since $\text{div} \left( f^2 \vec{B} \right) = \nabla f^2 \cdot \vec{B}$, we need to solve the inhomogeneous conductivity equation

$$\text{div} \left( f^2 \nabla h \right) = -\nabla f^2 \cdot \vec{B}, \tag{42}$$

It is no loss of generality to take the boundary condition $h|_{\partial \Omega} = 0$ in (33). By Corollary 6.2, this determines a unique generalized solution of (42) provided that $\nabla f^2 \cdot \vec{B} \in L^2(\Omega, \mathbb{R})$. But since $T_\Omega : L^2(\Omega, \mathbb{H}) \to H^1(\Omega, \mathbb{H})$ is bounded \cite[Theorem 8.4]{19}, in fact $T_{0, \Omega}[\vec{g}] \in H^1(\Omega, \mathbb{R})$ and $\vec{T}_{2, \Omega}[\vec{g}] \in H^1(\Omega, \mathbb{R}^3)$. Combining with the fact that $\vec{S}_{\Omega}[T_{0, \Omega}[\vec{g}]]$ is harmonic by Proposition 2.3, we have $\vec{B} \in L^2(\Omega, \mathbb{R}^3)$. Thus, the hypothesis is fulfilled, and the desired $h$ exists. Applying the right inverse of curl to (41) we have the solution (40) where $h_1$ is an arbitrary harmonic function. Then div $\vec{E} = 0$ by Corollary 4.2 and the remaining equations of (37) are easily verified.

\subsection{6.2 Variational methods for double curl boundary value problems}

In this section we will prove that given $\vec{\varphi} \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$ there exists an extension to the interior of $\Omega$ satisfying the double curl-type equation (34). Let $f \in H^{1/2}(\Omega, \mathbb{R})$ be a measurable proper conductivity. Let us define the nonlinear functional $\epsilon = \epsilon_f : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R}$ by

$$\epsilon[\vec{W}] = \int_{\Omega} f^{-2} \text{curl} \vec{W} \cdot \text{curl} \vec{W} \, dx. \tag{43}$$

We are interested in proving that for fixed $\vec{\varphi} \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$, there exists at least one element that minimizes $\epsilon$; we will use the results of the variational calculus which can be found, for example, in \cite[Ch. 3]{11}. Let $X$ be a reflexive Banach space and let $I : X \to \mathbb{R}$. We say that $I$ is weakly lower semicontinuous (w.l.s.) if $\liminf_{k \to \infty} I(u_k) \geq I(u)$ whenever $u_k \to u$ weakly in $X$. A functional $I$ is called coercive when there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $I(u) \geq \alpha \|u\|_X + \beta$ for all $u \in X$. 

24
Proposition 6.4. (Ch. 3, Th. 1.1) Let $X$ be a reflexive Banach space and let $I : X \to \mathbb{R}$ be a w.l.s. and coercive functional. Then there exists at least one element $u_0 \in X$ such that

$$I(u_0) = \inf \{ I(u) : u \in X \}.$$  

Corollary 6.5. Under the hypotheses of Proposition 6.4, if $Y \subseteq X$ is a closed (in the norm of $X$) and convex subset, then exists $u_1 \in Y$ such that

$$I(u_1) = \inf \{ I(u) : u \in Y \}.$$  

We apply these facts to the reflexive Banach space $X = H^1(\Omega, \mathbb{R}^3)$, and the functional $I = \varepsilon$ of (43), with $Y \subseteq X$ defined as follows:

$$Y = \{ \vec{W} \in H^1(\Omega, \mathbb{R}^3) : \vec{W}|_{\partial \Omega} = \vec{\varphi} \}.$$  

Proposition 6.6. $Y \subseteq X$ and $\varepsilon$ satisfy the hypothesis of Corollary 6.5: (a) $Y$ is convex; (b) $Y$ is closed; (c) $\varepsilon$ is coercive; (d) $\varepsilon$ is w.l.s.

Proof. (a) is immediate. To prove (b), let $\{ \vec{W}_k \} \subseteq Y$ with $\vec{W}_k \to \vec{W}$; that is, $\| \vec{W}_k - \vec{W} \|_{H^1(\Omega)} \to 0$ as $k \to \infty$. By the Trace Theorem in Sobolev spaces we have $C > 0$ such that

$$\| \vec{W}_k|_{\partial \Omega} - \vec{W}|_{\partial \Omega} \|_{H^{1/2}(\partial \Omega)} \leq C \| \vec{W}_k - \vec{W} \|_{H^1(\Omega)}$$

for all $k$. And since $\vec{W}_k|_{\partial \Omega} = \vec{\varphi}$, then $\vec{W}|_{\partial \Omega} = \vec{\varphi}$ almost everywhere in $\partial \Omega$. By definition,

$$\varepsilon[\vec{W}] = \| f^{-1} \text{curl} \vec{W} \|_{L^2(\Omega)}^2,$$

so we have (c). For (d), since the norm in any Banach space is is w.l.s., we need to prove that if $\vec{W}_k \to \vec{W}$ weakly in $H^1(\Omega, \mathbb{R}^3)$, then $\text{curl} \vec{W}_k \to \text{curl} \vec{W}$ weakly in $L^2(\Omega, \mathbb{R}^3)$. But this holds because $\partial \vec{W}_k/\partial x_i \to \partial \vec{W}/\partial x_i$ weakly in $L^2(\Omega, \mathbb{R}^3) (i = 1, 2, 3)$, and because the curl is a combination of elements of $\partial/\partial x_i$.

Theorem 6.7. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary, and let $f^2$ be a measurable proper conductivity. Then given the boundary values $\vec{\varphi} \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$ there exists an extension $\vec{W} \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\text{curl} \left( f^{-2} \text{curl} \vec{W} \right) = 0,$$

$$\vec{W}|_{\partial \Omega} = \vec{\varphi}.$$
**Proof.** By Corollary 6.5 and Proposition 6.6 the nonlinear functional (43) has a minimum $\vec{W}$ over $[\vec{\varphi}] + H^1_0(\Omega, \mathbb{R}^3)$. By definition, the second equation of the system (44) holds. To prove the first one, from the integration by parts formula for Sobolev spaces we have that

$$\langle \text{curl } f^{-2} \text{curl } \vec{W}, \vec{v} \rangle = \int_\Omega f^{-2} \text{curl } \vec{W} \cdot \text{curl } \vec{v} dx$$

(45)

when $\vec{v} \in H^1_0(\Omega, \mathbb{R}^3)$. The Gâteaux derivative of $\varepsilon$ at $\vec{w}$ in the direction $\vec{v}$ is

$$\varepsilon'[\vec{W}] = \lim_{t \to 0} \frac{\varepsilon[\vec{W} + t\vec{v}] - \varepsilon[\vec{W}]}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \int_\Omega 2tf^{-2} \text{curl } \vec{v} \cdot \text{curl } \vec{W} + t^2f^{-2} \text{curl } \vec{v} \cdot \text{curl } \vec{v} dx$$

$$= 2 \int_\Omega f^{-2} \text{curl } \vec{v} \cdot \text{curl } \vec{W} dx.$$

Since $\vec{W}$ is an extreme point for $\varepsilon$, the integral vanishes, by (45) the first equation of (44) holds in the distributional sense.

The minimum is not unique, because $\varepsilon[\vec{W}] = \varepsilon[\vec{W} + \text{grad } h]$ when $h \in H^1_0(\Omega, \mathbb{R})$.

**Further applications.** The solution is analogous for the inhomogeneous counterpart of the double curl-type equation, that is, with $\text{curl } (f^{-2} \text{curl } \vec{W}) = \vec{g}$ in place of the first equation of (44). In this case the functional to minimize is

$$\varepsilon[\vec{W}] = \int_\Omega f^{-2} \text{curl } \vec{W} \cdot \text{curl } \vec{W} dx - 2 \int_\Omega \vec{g} \cdot \vec{W} dx,$$

where $\vec{g} \in L^2(\Omega, \mathbb{R}^3)$ and $\text{div } \vec{g} = 0$ weakly.

Similarly, we can find weak solutions for the inhomogeneous conductivity equation $\text{div } f^2 \nabla W_0 = g_0$ (cf. Proposition 5.4 and Theorem 6.1). Now the functional to minimize is

$$\varepsilon[W_0] = \int_\Omega f^2 \nabla W_0 \cdot \nabla W_0 dx + 2 \int_\Omega g_0 W_0 dx,$$

26
given \( g_0 \in L^2(\Omega, \mathbb{R}) \).

Consequently, we can relax the hypotheses of Theorem 6.3 to assume that the proper conductivity \( f^2 \) is measurable. As a result, in this generality there exists a pair \((\vec{E}, \vec{H})\) satisfying the differential system (37) in the distributional sense, and satisfying additionally the boundary condition
\[
\vec{E}|_{\partial \Omega} = \vec{\phi} \in H^{1/2}(\partial \Omega, \mathbb{R}^3).
\]

References

[1] R. A. Adams, J. J. F. Fournier. Sobolev spaces. Academic Press, New York (1978).

[2] K. Astala, L. Päävärinta. “Calderón’s inverse conductivity problem in the plane.” Annals of Mathematics, 163 (2006).

[3] K. Astala, T. Iwaniec, G. Martin. Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton mathematical series. Princeton University Press, Princeton, Oxford (2009).

[4] S. Bergman. Integral operators in the theory of linear partial differential equations. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 23, Springer-Verlag, New York (1969).

[5] L. Bers. Theory of pseudo-analytic functions. New York University (1953).

[6] A. Bossavit. Computational Electromagnetism. Academic Press, Boston (1998).

[7] F. Brackx, R. Delanghe, F. Sommen. Clifford analysis. Pitman Advanced Publishing Program (1982)

[8] F. Brackx, H. de Schepper. “Conjugate harmonic functions in Euclidean space: a spherical approach.” Comput. Methods Funct. Theory 6:1 (2006) 165–182.

[9] F. Colombo, M. E. Luna-Elizarrarás, I. Sabadini, M. Shapiro, D. C. Struppa, “A quaternionic treatment of the inhomogeneous div-rot system.” Moscow Math. J. 12:1 (2012) 37–48.
[10] D. Colton, R. Kress. Inverse Acoustic and Electromagnetic Scattering Theory. Springer-Verlag (1992).

[11] B. Dacorogna. Direct methods in the Calculus of Variations. Springer-Verlag (1989).

[12] R. Feynman, The Feynman Lectures on Physics (2nd ed.). Addison-Wesley (2005).

[13] O. Forster. Lectures on Riemann Surfaces. Grad. Texts in Math. vol. 81, Springer (1981).

[14] J. O. González-Cervantes, M. E. Luna-Elizarrarás, M. Shapiro. “On the Bergman theory for solenoidal and irrotational vector fields, I: General theory.” Operator Theory: Advances and Applications 210 (2010) 79–106.

[15] D. J Griffiths, Introduction to Electrodynamics (3rd ed.). Prentice Hall (1998).

[16] Yu. M. Grigor’ev. “Three-dimensional Quaternionic Analogue of the Kolosov-Muskhelishvili Formulae.” Hypercomplex Analysis: New Perspectives and Applications, Trends in Mathematics, Birkhäuser, Basel (2014) 145–166.

[17] K. Gürlebeck, W. Sprößig. Quaternionic Analysis and Elliptic Boundary Value Problems. Birkhäuser Verlag, Berlin (1990).

[18] K. Gürlebeck, W. Sprößig. Quaternionic and Clifford Calculus for Physicists and Engineers. Chichester: John Wiley & Sons (1997).

[19] K. Gürlebeck, K. Habetha, W. Sprößig. Holomorphic Functions in the Plane and n-dimensional Space. Birkhäuser (2008).

[20] V. Isakov. Inverse problems for partial differential equations. Springer-Verlag (1998).

[21] B. Jiang. The Least-Squares Finite Element Method. Springer-Verlag Berlin Heidelberg (1998).

[22] J. D. Jackson. Classical electrodynamics. John Wiley & Sons, Third edition (1999).
[23] G. A. Korn, T. M. Korn, *Mathematical Handbook for Scientists and Engineers*. Dover Publications, Inc (1968).

[24] V. V. Kravchenko. *Applied pseudoanalytic function theory*. Frontiers in mathematics. Birkhäuser, Basel (2009).

[25] V. V. Kravchenko. *Applied Quaternionic Analysis*. Heldermann Verlag: Lemgo (2003).

[26] V. V. Kravchenko, M. V. Shapiro. *Integral Representations for Spatial Models of Mathematical Physics*. Addison Wesley Longman Ltd: Harlow (1996).

[27] V. V. Kravchenko, S. Tremblay. Spatial pseudoanalytic functions arising from the factorization of liner second order elliptic operators. *Mathematical Methods in the Applied Sciences*, **34** (2011) 1999–2010.

[28] V. P. Mikhailov. *Partial differential equations*. Mir Publishers (1978).

[29] R. M. Porter, M.V. Shapiro, and N. L. Vasilevski. “On the analogue of the $\bar{\partial}$-problem in quaternionic analysis.” *Kluwer Academic Publishers Group*, Fundamental Theories of Physics **55** (1993) 167–173.

[30] M. V. Shapiro. “On the conjugate harmonic function of M. Riesz - E. Stein - G. Weiss.” *Topics in Complex Analysis, Differential Geometry and Mathematical Physics*, World Scientific (1997) 8–32.

[31] E. M. Stein, G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Univ. Press, Princeton, N.J. (1971).

[32] A. Sudbery. “Quaternionic analysis.” *Math. Proc. Cambridge Phil. Soc.* **85** (1979) 99–225.

[33] I. N. Vekua. *Generalized analytic functions*. Moscow: Nauka (in Russian) (1959); English translation Oxford: Pergamon Press (1962).

[34] H. Weyl. “The method of orthogonal projection in potential theory.” *Duke Mathematical Journal* **7** (1940) 411–444.