ESCAPE DYNAMICS FOR INTERVAL MAPS

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ABSTRACT. We study the structure of the escape orbits for a certain class of
interval maps. This structure is encoded in the escape transition matrix \( \hat{A}_f \)
of an interval map \( f \), extending the traditional matrix \( A_f \) which considers the
transition among the Markov subintervals. We show that the escape transition
matrix is a topological conjugacy invariant. We then characterize the 0–1
matrices that can be fabricated as escape transition matrices of Markov interval
maps \( f \) with escape sets. This shows the richness of this class of interval maps.

1. Introduction. In this paper we further investigate the class of open dynamical
systems, see [1, 3, 4, 13], that arise from Markov interval maps with non trivial
escape set, see [10, 11].

For a Markov interval map \( f : I \to I \), with non trivial escape set, we introduce a
0–1 matrix \( \hat{A}_f \) that encapsulates not only the transition among the Markov intervals
of \( f \) (which is the usual transition matrix) but also the transitions from the Markov
intervals to the escape intervals. This leads us to naturally study the influence of \( \hat{A}_f \)
in the dynamics of \( (I, f) \). In particular, to study and classify different interval maps,
with nontrivial escape sets, whose restrictions to the respective maximal invariant
Cantor sets are topological conjugated. The traditional transition matrix \( A_f \) that
encodes the transitions of the Markov subintervals in the partition of the domain
of \( f \) was called Stefan matrix in [8] and later used as a (Markov) transition matrix,
see [12, 10, 11].

Besides the intrinsic interest in the open dynamical systems, the other motivation
to further the study of this escape matrix \( \hat{A}_f \) arises from the undesirable non-
faithfulness of the representations \( \nu_x \) of the Toeplitz algebra \( \mathcal{T}_{A_f} \) on the orbits of
points \( x \) in the underlying open dynamical systems (provided by interval maps with

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trivial escape sets) that we constructed in [6]. Two interval maps \( f \) and \( g \) can have the same Markov transition matrix \( A_f = A_g \) but different escape matrices such that the representation \( \nu_x \) of \( T_{A_f} \) is faithful for the open dynamical system \( (I, f) \) but non-faithful for \( (I, g) \). In this paper we characterize the 0–1 matrices \( X \) that can be realised by interval maps \( f \) (which is non-unique), thus \( \hat{A}_f = X \), and then use this in [7] as one of the key ingredients to successfully find a graph algebra for which \( \nu_x \) is indeed a faithful representation. The construction of these representations shows one clear advantage of considering \( \hat{A}_f \) instead of \( A_f \).

We leave this representation theory viewpoint as it is and concentrate on the framework of the problems we tackle in this paper. More precisely, in [6] we considered the interval maps \( f \) where the orbit of a point \( x \) can be in the escape set of \( f \) – see [9] – namely, \( f^k(x) \in I \) does not belong to the domain of \( f \) for a certain \( k \in \mathbb{N} \). Besides this, we defined a 0–1 transition matrix \( \hat{A}_f \) that captures not only the transition among the \( n \) Markov subintervals (the traditional transition matrix \( A_f \)) but also the transitions from the Markov subintervals to the \( m \) escape subintervals (giving rise to \( B_f \)). If we gather the Markov subintervals first and then the escape subintervals, then there exists a permutation matrix \( P \) such that

\[
P \hat{A}_f P^T = \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}
\]

(1)

where the lower blocks are matrices with all entries equal to zero.

With this transition matrix \( \hat{A}_f \) defined, we tackle two natural problems.

The first one concerns topological conjugacy where we indeed prove that two Markov interval maps \( f, g \) are topological conjugated whenever \( \hat{A}_f = P \hat{A}_g P^T \) for a certain permutation matrix \( P \), which is in principle unrelated to the permutation that appears in Eq. (1).

The second problem we tackle is to characterize the 0–1 matrices in the block shape \( \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix} \) that can actually be realised as an escape transition matrix \( \hat{A}_f \) for some Markov interval map \( f \) (up to permutation as explained in Eq. (1)). If such realisation exists, then it is highly not unique. The rows of \( \hat{A}_f \) which are entirely null detect the positions of the escape sets of \( f \). This is crucial for the study of the dynamics. The related problem, with no escape sets, was studied in [5], and the answer was that for every row, the entries of ones must be filled in consecutive entries, which we called interval type matrix, see Definition 2.5. Our analysis also shows that such realisations \( f \) are restrictions of Markov interval maps \( g \) without escape sets.

The plan for the rest of the paper is as follows. In Sect. 2 we first review the dynamical systems background for the interval maps with escape sets, emphasizing the notion of escape transition matrix as in Definition 2.3. Then in Proposition 2.8 we prove that the escape transition matrix is a topological conjugacy invariant. With the help of the notion of \( m \)-configuration, in Subsect. 3 we characterize the block matrices \( \begin{bmatrix} A & B \\ 0_{n \times n} & 0_{n \times m} \end{bmatrix} \) that can be realised as escape transition matrices of interval maps (up to row and column permutations), see the main Theorem 3.7 as well as Propositions 3.4 and 3.5. We also conclude that any such interval map \( f \) (with escape points) is a restriction of a certain interval map without escape points (see Corollary 3.10).
2. Dynamics of interval maps with escape sets. Given $n \in \mathbb{N}$, let $\Gamma$ be an ordered set $\{c_0, c_1, c_1^+, \ldots, c_n, c_n^+, c_n\}$ of (at most) $2n$ real numbers such that

$$c_0 < c_1 < c_1^+ \leq c_2 < \ldots < c_{n-1} \leq c_n < c_n^+.$$  \hfill (2)

Given $\Gamma$ as above, we define the collection of closed intervals $C = \{I_1, \ldots, I_n\}$, with

$$I_1 = [c_0, c_1], \ldots, I_j = [c_{j-1}, c_j], \ldots, I_n = [c_{n-1}, c_n].$$  \hfill (3)

We also consider the collection of open intervals $\{E_1, \ldots, E_{n-1}\}$, with

$$E_1 = \left]c_1^-, c_1^+\right[ , \ldots, E_{n-1} = \left]c_{n-1}^-, c_n^+\right[ ,$$  \hfill (4)

in such a way that $I := [c_0, c_n] = \bigcup_{j=1}^n I_j \bigcup \left(\bigcup_{j=1}^{n-1} E_j\right)$.

We now consider the interval maps for which we can construct partitions of the interval $I$ as in (2), (3) and (4).

**Definition 2.1.** Let $I \subset \mathbb{R}$ be an interval. A map $f$ is in the class $\mathcal{M}(I)$ if it satisfies the properties (P1), (P2), (P3'), (P4), presented below, and is in the class $\mathcal{PL}(I)$ if it satisfies the properties (P1), (P2), (P3), (P4):

(P1) [Existence of a finite partition in the domain of $f$] There is a partition $C_f = \{I_1, \ldots, I_n\}$ of closed intervals with $\#(I_i \cap I_j) \leq 1$ for $i \neq j$, $\text{dom}(f) = \bigcup_{i=1}^n I_i \subset I$ and $\text{im}(f) = I$.

(P2) [Markov property] For every $i = 1, \ldots, n$ the set $f(I_i) \cap \left(\bigcup_{j=1}^n I_j\right)$ is a non-empty union of intervals from $C_f$.

(P3) [Piecewise linear and expansive map] $f_{|I_j} \in \mathcal{C}(I_j)$, $|f'_{|I_j}(x)| = d_j > 1$, for every $x \in I_j$, $j = 1, \ldots, n$.

(P3') [Expansive map] $f_{|I_j} \in \mathcal{C}(I_j)$, monotone and $|f'_{|I_j}(x)| > b > 1$, for every $x \in I_j$, $j = 1, \ldots, n$, and some $b$.

(P4) [Aperiodicity] For every interval $I_j$ with $j = 1, \ldots, n$ there is a natural number $q$ such that $\text{dom}(f) \subset f^q(I_j)$.

See Figures 1–4 for examples of maps satisfying the conditions of Definition 2.1. Clearly $\mathcal{PL}(I) \subset \mathcal{M}(I)$. The minimal partition $C$ satisfying the Definition 2.1 is denoted by $C_f$. We remark that the Markov property (P2) allows us to encode the transitions between the intervals in the so-called (Markov) transition $n \times n$ matrix $A_f = (a_{ij})$, defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } f(\bar{I}_i) \supset \bar{I}_j, \\ 0 & \text{otherwise} \end{cases}$$ \hfill (5)

where $\bar{J}$ denotes the interior of a set $J$. A map $f \in \mathcal{M}(I)$ uniquely determines (together with the minimal partition $C_f = \{I_1, \ldots, I_n\}$):

(i) The $f$-invariant set $\Omega_f := \{x \in I : f^k(x) \in \text{dom}(f) \text{ for all } k = 0, 1, \ldots\}$. Note that $\Omega_f$ is colloquially called the survivor set of $f$.

(ii) The collection of open intervals $\{E_1, \ldots, E_{n-1}\}$, such that $I \setminus \bigcup_{j=1}^{n-1} I_j = \bigcup_{j=1}^{n-1} E_j$.

(iii) The transition matrix $A_f = (a_{ij})_{i,j=1,\ldots,n}$.

The subintervals $I_1, \ldots, I_n$ are sometimes called the Markov subintervals of $f$ whereas $E_1, \ldots, E_{n-1}$ the escape subintervals (possibly empty) of $f$.

Matrices $A$ for which there exists a positive integer $m$ such that all the entries of $A^m$ are non-zero are called primitive. We note that the matrix $A_f$ is primitive (thus irreducible) whenever $f \in \mathcal{M}(I)$, see Definition 2.1.
Definition 2.2. The address map \( ad : \bigcup_{j=1}^{n} I_j \rightarrow \{1, 2, ..., n\} \) is defined as follows: \( ad(x) = i \) if \( x \in \hat{I}_i \), where \( \hat{I}_i \) denotes the interior of \( I_i \). The itinerary map \( it_f : \bigcup_{j=1}^{n} I_j \rightarrow \{1, 2, ..., n\}^{\aleph_0} \) is defined as \( it_f(x) = (ad(f^n(x)))_{n=0,1,...} \).

Note that \( \Omega_f \) is the set of points that remain in \( \text{dom}(f) \) under iteration of \( f \), and is usually called a cookie-cutter set, see [9]. The open set

\[
E_f := I \setminus \Omega_f = \bigcup_{k=0}^{\infty} f^{-k} \left( \bigcup_{j=1}^{n-1} E_j \right)
\]

is usually called the escape set. Every point in \( E_f \) will eventually fall, under iteration of \( f \), into the interior of some interval \( E_j \) (where \( f \) is not defined) and the iteration process ends. We may say that \( x \) is in the escape set \( E_f \) of \( f \) if and only if there is \( k \in \mathbb{N} \) such that \( f^k(x) \notin \text{dom}(f) \). If \( c_j^- = c_j^+ \), for some \( j \), then \( E_j = \emptyset \) and \( c_j \) is a singular point, either a critical point or a discontinuity point of \( f \).

We will consider the equivalence relation \( R_f \) defined by

\[
R_f := \{(x,y) : f^n(x) = f^n(y) \text{ for some } n, m \in \mathbb{N}_0 \}.
\]

The relation \( R_f \) is a countable equivalence relation in the sense that the equivalence class \( R_f(x) \) of \( x \in I \), is a countable set. We denote \( x \sim y \) whenever \((x,y) \in R_f\).

Let \( f \in \mathcal{M}(I) \) be such that \( E_f \neq \emptyset \). This means that there is at least one nonempty open interval \( E_j = \]c_j^-,c_j^+[ \), with \( c_j^- \neq c_j^+ \), with \( j \in \{1,...,n-1\} \). The non-empty open subinterval \( E_j \) is called an escape interval.

If \( x \in \Omega_f \) then \( R_f(x) \) has a graph structure without a preferable vertex. If \( x \in E_f \) then \( R_f(x) \) has a natural structure of a rooted tree. The root of \( R_f(x) \) is the point \( e(x) \) with no outgoing edge, so \( f^{-1}(e(x)) \in \text{dom}(f) \) but \( e(x) \notin \text{dom}(f) \).

For every \( y \in E_f \) there is a least natural number \( \tau(y) \) such that \( f^{\tau(y)}(y) \notin \text{dom}(f) \), which means that, \( f^{\tau(y)}(y) \in E_j \), for some \( j \) such that \( E_j \neq \emptyset \). The final escape point, for the orbit of \( y \), is then denoted by \( e(y) := f^{\tau(y)}(y) \) and the final escape interval index is denoted by \( \iota(y) \), that is, if \( f^{\tau(y)}(y) \in E_j \) then \( \iota(y) = j \).

In order to describe symbolically the escape orbits, we extend the symbol space adding a symbol for each escape interval \( E_j \), which will represent an end for the symbolic sequence. For each escape interval \( E_j \) we associate a symbol \( \hat{j} \) to distinguish the symbol associated with the interval partition \( I_j \). That is, we consider the symbols ordered by:

\[
1 < \hat{1} < 2 < \hat{2} < ... < n - 1 < \hat{n-1} < n.
\]

If \( E_j \) is not an interval, that is \( E_j = \emptyset \), then there is no symbol \( \hat{j} \). Moreover, we define

\[
\Sigma_{E_f} = \{ \hat{j} : E_j \neq \emptyset, j \in \{1,...,n-1\} \}.
\]

The address map \( ad \) (see Definition 2.2) is extended to the escape set \( E_f \) with \( ad(x) = \hat{j} \in \Sigma_{E_f} \) if \( x \in E_j \). Therefore, the address map is defined for all points of the interval \( I \) except for the points of the boundary of the subintervals \( I_1,...,I_n \), see (2) and (3), that is

\[
ad: I \setminus \{c_0,c_1^+,c_n : i = 1,...,n-1\} \rightarrow \{1,...,n\} \cup \Sigma_{E_f}.
\]

The itinerary map (see Definition 2.2) is also extended such that

\[
\text{it}_f(x) = ad(x) ad(f(x)) \ldots ad\left(f^{\tau(x)-1}(x)\right) ad(e(x)), \quad x \in E_f.
\]
The itinerary of a point \( x \in E_f \) is always a finite word terminating in a symbol \( \hat{j} \in \Sigma_{E_f} \).

An admissible escape word is a word occurring as the itinerary of an escape point \( x \in E_f \). These words are formed by

\[
\xi = \xi_1 \xi_2 \ldots \xi_k \hat{j}
\]

such that \( a_{\xi_i, \xi_{i+1}} = 1 \) for \( i = 1, 2, \ldots, k - 1 \), and terminating on an escape symbol \( \hat{j} \).

2.1. The escape transition matrix and topological conjugacy. Let \( f \in \mathcal{M}(I) \). As in the last section, we thus have an index set \( \{1, \ldots, n\} \cup \Sigma_{E_f} \) which is ordered as in (8) and (9). In order to deal with the possible transitions from Markov transition intervals to escape intervals we define the escape transition matrix \( \hat{A}_f \) as follows.

**Definition 2.3** (see Def. 3.2 in [6]). Given the transition matrix \( A_f \) as in (5), we define a matrix \( \hat{A}_f = (\hat{a}_{ij}) \) indexed by \( \{1, \ldots, n\} \cup \Sigma_{E_f} \) such that

\[
\hat{a}_{ij} = \begin{cases} 
  a_{ij} & \text{if } i, j \in \{1, \ldots, n\}, \\
  1 & \text{if } i \in \{1, \ldots, n\}, j \in \Sigma_{E_f}, \text{ and } \hat{I}_i \cap f^{-1}(E_j) \neq \emptyset, \\
  0 & \text{otherwise}.
\end{cases}
\]

For row and column labeling, the matrix \( \hat{A}_f \) respects the order given in (8).

**Example 2.4.** Let \( f \) be the map, see Figure 1,

\[
f(x) = \begin{cases} 
  5x + 2/5 & \text{if } x \leq 3/25 \\
  5x - 2 & \text{if } 2/5 \leq x \leq 3/5 \\
  5x - 4 & \text{if } 4/5 \leq x \leq 1
\end{cases}
\]

The domain of \( f \) is given by \( \text{dom}(f) = I_1 \cup I_2 \cup I_3 \), with \( I_1 = [0, 3/25], I_2 = [2/5, 3/5], I_3 = [4/5, 1] \). The escape intervals are \( E_1 = ]3/25, 2/5[ \), \( E_2 = ]3/5, 4/5[ \). The escape symbols set is \( \Sigma_{E_f} = \{\hat{1}, \hat{2}\} \) and the transition matrix associated with \( f \) is

\[
A_f = \begin{bmatrix}
  0 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1 \\
\end{bmatrix}
\]

whereas the matrix \( \hat{A}_f \) is as follows (see the order in (8)):

\[
\hat{A}_f = \begin{bmatrix}
  0 & 0 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

because \( f(\hat{I}_1) = \hat{I}_2 \cup \hat{E}_2 \cup \hat{I}_3 \) and \( f(\hat{I}_2) = f(\hat{I}_3) = \hat{I}_1 \cup \hat{E}_1 \cup \hat{I}_2 \cup \hat{E}_2 \cup \hat{I}_3 \).

Note that if we use the row and column labeling order \( I_1 \ldots I_n E_1 \ldots E_m \) with \( \#\Sigma_{E_f} = m \), then \( \hat{A}_f \) becomes

\[
\hat{A}_f = \begin{bmatrix}
  A_f & B_f \\
  0_{m \times n} & 0_{m \times m}
\end{bmatrix}
\]

where \( A_f \) is the Markov transition matrix of \( f \) and \( B_f \) is the transition matrix from Markov subintervals to the escape subintervals. We write \( 0_{p \times q} \) for the \( p \times q \) matrix with zeros everywhere, whereas \( 1_{p \times q} \) is the \( p \times q \) matrix with ones everywhere.
In order to understand the relation between the escape matrices $\hat{A}_f$ and $\hat{A}_g$ of two interval maps $f, g$, we put forward the following definition in the context of generic 0–1 matrices.

**Definition 2.5.** Let $X$ be an $r \times r$ matrix with entries in $\{0, 1\}$ with index set $\alpha_X = \{1, 2, ..., r\}$.

1. The matrix $X$ is said to be of interval type if in every row, the entries equal to 1 are all consecutive (cf. Def. 2 in [5]).
2. If $\pi$ is a permutation of the rows of an interval type matrix $X$ and $P_\pi$ its permutation matrix, then we say that $\pi$ preserves interval type if the matrix $P_\pi X P_\pi^T$ is an interval type matrix.

We can write the above definition as follows. Let $X = (x_{ij})$. The follower set of the $i \in \alpha_X$ is

$$F_X (i) := \{ j \in \alpha_X : x_{ij} = 1 \}.$$

(13)

If no confusion arises, we denote $F_X(i)$ by $F(i)$. Then the matrix $X$ is of interval type if and only if the follower set $F_X(i)$ is a set of consecutive numbers (for every $i \in \alpha_X$), i.e. $F_X(i) = \{p, p + 1, ..., p + t\}$ for some $t \in \mathbb{N}$. We remark that if a row $i$ of $X$ is zero, then is $F_X(i) = \emptyset$.

**Lemma 2.6.** Let $X$ be an interval type matrix. A permutation $\pi$ preserves the interval type of $X$ if and only if

$$\pi (F_X (i))$$

is a set of consecutive numbers for every $i \in \alpha_X$.

**Proof.** Given $i \in \alpha_X$ the set $F_X (i)$ corresponds to the positions of entries equal to 1 in the row $i$ of $X$. The set $\pi (F_X (i))$ corresponds to the row $\pi (i) \in \alpha_{P_\pi X P_\pi^T}$ of the matrix $P_\pi X P_\pi^T$. The matrix $P_\pi X P_\pi^T$ is of interval type if and only if for every
row \( s \in \alpha_{P_nX^T} \) of \( P_nX^T \) the follower set \( F_{P_nX^T}(s) \), is a set of consecutive numbers. Since \( F_{P_nX^T}(s) = \pi\left(F_X(\pi^{-1}(s))\right) \), setting \( i = \pi^{-1}(s) \) concludes the proof. \( \square \)

Now the following result is straightforward.

**Corollary 2.7.** Let \( X \) be an interval type matrix. If \( \pi \) leaves the followers sets of \( X \) invariant then it preserves interval type.

We now derive the relation between the escape transition matrices for topologically conjugated interval maps, by considering an interval type matrix \( X \) as an escape matrix. Let \( f^o \) and \( g^o \) denote the restrictions of each map to the finite union of the interior of the intervals in the domain of \( f \) and \( g \), respectively, see (P1) in Definition 2.1. Let \( f \) and \( g \in \mathcal{M}(I) \). Let \( D_1, \ldots, D_{n_f}, \ldots, D_{n_g} \) be the intervals in the partition of the domains of \( f \) and \( g \), respectively, as in Definition 2.1.

We recall that \( f^o \) and \( g^o \) are topologically conjugated if there exists a continuous bijective map

\[
\phi : \bigcup_{i=1}^{n_f} \hat{D}_i \to \bigcup_{j=1}^{n_g} \hat{G}_j
\]

with \( \phi^{-1} \) continuous such that \( \hat{g}^o = \phi \circ f^o \circ \phi^{-1} \).

Note that escape intervals for \( f \) are escape intervals for \( f^o \) and there is a transition between \( I_i \) and \( I_j \) through \( f \) if and only if there is a transition between \( \hat{I}_i \) and \( \hat{I}_j \) through \( f^o \).

Note that \( \hat{A}_f \) and \( \hat{A}_g \) are necessarily of interval type.

**Proposition 2.8.** Let \( f \in \mathcal{M}(I_f) \) and \( g \in \mathcal{M}(I_g) \), for certain intervals \( I_f \) and \( I_g \). The maps \( f^o \), \( g^o \) are topologically conjugated if and only if there is a permutation \( \pi \) such that \( \hat{A}_f = P_{\pi} \hat{A}_g P_{\pi}^T \).

**Proof.** Let \( D_1, D_2, \ldots, D_r, \ldots, D_{n_f+m_f} \) be the intervals partitioning \( I_f \), including escape intervals, with the order of real line, and \( G_1, G_2, \ldots, G_r, \ldots, G_{n_g+m_g} \) be the intervals partitioning \( I_g \). We assume that \( n_f \) are the Markov subintervals (say \( D_{m_1}, \ldots, D_{m_{n_f}} \)) and \( m_f \) are the escape intervals for \( f \), and similarly for \( g \).

Suppose that \( f^o \) and \( g^o \) are topologically conjugated. So there is a bijective and bicontinuous map \( \phi : \bigcup_{i=1}^{n_f} \hat{D}_{m_i} \to \bigcup_{j=1}^{n_g} \hat{G}_{m_j} \) satisfying

\[
g^o = \phi \circ f^o \circ \phi^{-1}.
\]

If \( D_i \) is an escape interval for \( f \) then \( \phi(D_i) \) is an escape interval for \( g \) since

\[
\phi \circ f(D_i) \text{ is not defined}
\]

and therefore

\[
g \circ \phi(D_i) \text{ is not defined}.
\]

Using the escape intervals \( G_i \) and \( \phi^{-1} \) instead of \( \phi \) leads us to \( m_f = m_g = ; m \). If \( D_i \) is not an escape interval for \( f \) then \( \phi(D_i) \) is not an escape interval for \( g \). In fact, the interior of an interval \( D_i \) in \( C_f \) is sent to the interior of an interval in \( C_g \), since

\[
\phi \circ f(D_i) = g \circ \phi(D_i),
\]

consequently \( n_f = n_g = n \) and there is a permutation \( \pi \in S_{n+m} \) so that

\[
\phi(D_i) = G_{\pi(i)}.
\]
Let \( \hat{A}_f = (\hat{a}(f)_{ij}) \) and \( \hat{A}_g = (\hat{a}(g)_{ij}) \). The existence of \( x \in \hat{D}_i \) with \( f(x) \in \hat{D}_j \) is equivalent to \( \hat{a}(f)_{ij} = 1 \). On the other hand \( x \in \hat{D}_i \) if and only if \( \phi(x) \in \hat{G}_{\pi(i)} \) and \( f(x) \in \hat{D}_j \) if and only if \( \phi(f(x)) = g(\phi(x)) \in \hat{G}_{\pi(j)} \), therefore \( \hat{a}(g)_{\pi(i)\pi(j)} = 1 \). This means that \( \hat{a}(f)_{ij} = 1 \) implies \( \hat{a}(g)_{\pi(i)\pi(j)} = 1 \). The same argument shows that \( \hat{a}(g)_{\pi(i)\pi(j)} = 1 \) then \( \hat{a}(f)_{ij} = 1 \) because if \( y \in \hat{G}_{\pi(i)} \) with \( g(y) \in \hat{G}_{\pi(j)} \), then \( x := \phi^{-1}(y) \in \hat{D}_i \) and \( f(x) \in \hat{D}_j \). Therefore \( \hat{A}_f(\pi, j) = (\hat{A}_g(\pi(i), \pi(j))) \), so that \( \hat{A}_f = P_\pi \hat{A}_g P_\pi^T \).

Finally let us see that the permutation \( \pi \) preserves interval type. The image of any \( \hat{D}_i \), through \( f \), must be the union of intervals whose labeling consist of consecutive natural numbers, that is, there are two natural numbers, \( r_i \) and \( t_i \) so that

\[
f(\hat{D}_i) = \bigcup_{j: \hat{a}(f)_{ij} = 1}^\circ D_j = \bigcup_{j=0}^{t_i-1} D_{r_i+j}.
\]

The number \( r_i \) is the position the block of 1’s in the row \( i \) and \( t_i \) is the length of the block. On the other hand, the fact they are consecutive means that \( \bigcup_{j=0}^{t_i-1} D_{r_i+j} \) is an interval. Therefore, \( \phi \circ f(\hat{D}_i) = g \circ \phi(\hat{D}_i) \) is also an interval, and \( g \circ \phi(\hat{D}_i) = g(\hat{G}_{\pi(i)}) \) is an interval. Consequently, the permutation preserves interval type.

Conversely, let us assume that we are given Markov matrices \( f \) and \( g \) such that their escape matrices \( \hat{A}_f, \hat{A}_g \) are conjugated by a permutation matrix \( P_\pi \) with permutation \( \pi \) which preserves interval type and the escape state set. Since \( P_\pi \) preserves escape states also preserve regular states. The matrices have the same dimension and its dimension is partitioned into \( m + n \) (\( m \) the number of escape states and \( n \) the number of regular states). Next, we show that the permutation matrix determine directly an invertible map \( \phi \) preserving the topological structure. Each nonzero entry in the permutation matrix is in the entry \((i, \pi(i))\). The map \( \phi \) is defined as the linear map sending the interior of each interval \( D_i \) to the interior of the interval \( G_{\pi(i)} \). If \( D_i \) is a escape interval then the sign of \( \phi \) on \( D_i \) is arbitrary.

Now, \( g(\phi(\hat{D}_i)) = g(\hat{G}_{\pi(i)}) = \left( \bigcup_{j: \hat{a}(g)_{\pi(i)j} = 1}^\circ G_j \right) \).

On the other hand,

\[
\phi(f(\hat{D}_i)) = \phi(\bigcup_{j: \hat{a}(f)_{ij} = 1}^\circ D_j) = \left( \bigcup_{j: \hat{a}(f)_{ij} = 1}^\circ G_{\pi(j)} \right).
\]

Let \( t = \pi(j) \) and recall that \( \hat{a}(g)_{\pi(i)t} = \hat{a}(f)_{ij} \). Since the matrices are conjugated precisely by the permutation \( \pi \), we obtain \( g(\phi(\hat{D}_i)) = \phi(f(\hat{D}_i)) \), for every \( D_i \) and therefore for every point in the interior of the domain of \( f \). Therefore, by construction, \( \phi \) is invertible and preserving continuity on the interior of the domain of \( f \). Therefore, if we start the above construction using \( \hat{A}_g \) and \( g \), then we obtained the topological conjugacy. \( \square \)
Example 2.9. Let

\[
\hat{A}_f = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

for a certain \( f \in \mathcal{M}(I) \) (see the graph \( f \) in Fig. 2). The index set of \( \hat{A}_f \) is \( \alpha_{\hat{A}_f} = \{1, 2, 3, 4, 5, 6, 7\} \), with the domain intervals \( D_1 = I_1, D_2 = I_2, D_3 = I_3, D_5 = I_4, D_7 = I_5 \), and the escape intervals are \( D_3 = E_2, D_6 = E_3 \). The follower set for each state is \( F(1) = \{2, 3, 4\}, F(2) = \{5, 6, 7\}, F(3) = \emptyset, F(4) = \{1, 2\}, F(5) = \{3, 4\}, F(6) = \emptyset, F(7) = \{5, 6, 7\} \). Any map topologically conjugated to \( f \) will have associated a permutation so that \( \pi(F(i)) \) is a set of consecutive numbers, for every \( i \in \alpha_{\hat{A}_f} \). Consider the permutation

\[
\pi_1 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
4 & 5 & 6 & 7 & 3 & 2 & 1 \\
\end{pmatrix}
\]

This permutation satisfies the referred property (of preserving interval type) since

\[
\begin{align*}
\pi_1(F_{A_f}(1)) &= \pi_1\{2, 3, 4\} = \{5, 6, 7\}, \\
\pi_1(F_{A_f}(2)) &= \pi_1\{5, 6, 7\} = \{1, 2, 3\}, \\
\pi_1(F_{A_f}(3)) &= \pi_1(\emptyset) = \emptyset, \\
\pi_1(F_{A_f}(4)) &= \pi_1\{1, 2\} = \{4, 5\}, \\
\pi_1(F_{A_f}(5)) &= \pi_1\{3, 4\} = \{6, 7\}, \\
\pi_1(F_{A_f}(6)) &= \pi_1(\emptyset) = \emptyset,
\end{align*}
\]
Figure 3. Graph of the function $g_1$ in Example 2.9

$$\pi_1 \left( F_{A_f} (7) \right) = \pi_1 \{5, 6, 7\} = \{1, 2, 3\}.$$ 

The permutated transition matrix

$$\hat{A}_{g_1} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}$$

is of interval type and can be associated with a certain interval map $g_1$ (see the graph $g_1$ in Fig. 3) which is conjugated to $f$ through a certain $\phi_1$ with $g_1^* = (\phi_1)^{-1} \circ f \circ \phi_1$. The graph of $\phi_1$ can be determined by the permutation matrix $P_{\pi_1}$ where

$$P_{\pi_1} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} ;$$

see the graph of $\phi_1$ in Fig. 4.

**Example 2.10.** Consider now the permutation

$$\pi_2 = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7 & 2 & 3 & 4 & 5 & 6 & 1 \\
\end{bmatrix}$$

This permutation does not preserve interval type:

$$\pi_2 \left( F_{A_f} (1) \right) = \pi_2 \{2, 3, 4\} = \{2, 3, 4\},$$
Example 2.9. Let

\[ \hat{A}_{f} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

This gives us the transition matrix

\[ \pi_2 (F_{A_f} (2)) = \pi_2 \{5, 6, 7\} = \{1, 5, 6\} \text{ is not consecutive,} \]
\[ \pi_2 (F_{A_f} (3)) = \pi_2 (\emptyset) = \emptyset, \]
\[ \pi_2 (F_{A_f} (4)) = \pi_2 \{1, 2\} = \{2, 7\} \text{ is not consecutive,} \]
\[ \pi_2 (F_{A_f} (5)) = \pi_2 \{3, 4\} = \{3, 4\}, \]
\[ \pi_2 (F_{A_f} (6)) = \pi_2 (\emptyset) = \emptyset, \]
\[ \pi_2 (F_{A_f} (7)) = \pi_2 \{5, 6, 7\} = \{1, 5, 6\} \text{ is not consecutive.} \]

Consider the permutation (different from identity)

\[ \pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 6 & 7 & 5 & 4 \end{pmatrix} \]
This permutation preserves the followers sets (invariant sets for \(\pi_3\)): 
\[
\begin{align*}
\pi_3(F_{A_f}(1)) &= \pi_3\{1, 2, 3\} = \{1, 2, 3\}, \\
\pi_3(F_{A_f}(2)) &= \pi_3\{4, 5, 6, 7\} = \{4, 5, 6, 7\}, \\
\pi_3(F_{A_f}(3)) &= \pi_3(\emptyset) = \emptyset, \\
\pi_3(F_{A_f}(4)) &= \pi_3\{4, 5\} = \{4, 5\}, \\
\pi_3(F_{A_f}(5)) &= \pi_3\{6, 7\} = \{6, 7\}, \\
\pi_3(F_{A_f}(6)) &= \pi_3(\emptyset) = \emptyset, \\
\pi_3(F_{A_f}(7)) &= \pi_3\{1, 2, 3\} = \{1, 2, 3\}.
\end{align*}
\]

The conjugated matrix is, naturally, of interval type

\[
\hat{A}_{g_3} = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

3. Realising matrices as Markov interval maps with escape sets. If we are given a pair \((A, \Gamma)\) of a primitive and interval type matrix \(A\) and an ordered set of points \(\Gamma\) of \(I = [0, 1]\) as in (2), then [5, Proposition 4] provides (a piecewise linear) interval map \(f(\lambda, \Gamma) \in PL(I)\) such that the slopes of \(f(\lambda, \Gamma)\) are positive, the transition matrix of \(f(\lambda, \Gamma)\) is the original matrix \(A_{f(\lambda, \Gamma)} = A\) and the minimal Markov partition \(C_{f(\lambda, \Gamma)}\) of \(f(\lambda, \Gamma)\) is \(\{I_1, \ldots, I_n\}\) given by (3).

When we have \(n + m\) (closed) subintervals \(J_1, \ldots, J_{m+n}\) of \(I\) and want to declare that the (interior) of \(m\) of them are escape subintervals for some Markov map \(f\), then the following notion is helpful.

**Definition 3.1.** Let \(n \in \mathbb{N}\) and \(0 \leq m < n - 1\) be an integer. An \(m\)-configuration \(C_{m,n}\) of \(n + m\) determines numbers \(\{m(j)\}_{j=1,\ldots,n}\) such that

\(1 < \epsilon(1) < \epsilon(2) < \ldots < \epsilon(i) < \ldots < \epsilon(m) < n + m.\)

An \(m\)-configuration \(C_{m,n}\) of \(n + m\) determines numbers \(\{m(j)\}_{j=1,\ldots,n}\) such that

\(1 = m(1) \leq m(2) \leq \ldots \leq m(n - 1) \leq m(n) = n + m\) and

\(\{\epsilon(i) : i = 1, \ldots, m\} \cup \{m(j) : j = 1, \ldots, n\} = \{1, \ldots, n + m\}.
\)

Note that \(m(i) \neq m(j)\) for \(i \neq j\), as a consequence of Definition 3.1. In view of (3) and (4), an \(m\)-configuration provides us a way to order the Markov and escape subintervals of an interval map \(f\) as follows:

\(1, \ldots, \epsilon(1) - 1, \epsilon(1), \epsilon(1) + 1, \ldots, \epsilon(i) - 1, \epsilon(i), \epsilon(i) + 1, \ldots, \epsilon(m) - 1, \epsilon(m), \epsilon(m) + 1, \ldots, n + m.\) (14)

We note that for every \(i = 1, \ldots, m\), \(\epsilon(i) \pm 1 \in \{m(j)\}_{j=1,\ldots,n}\) so that there exists \(j_i = 1, \ldots, n\) such that \(\epsilon(i) - 1 = m(j_i)\) (thus \(\epsilon(i) + 1 = m(j_i + 1)\)). From the viewpoint of symbolic dynamics, this is the natural order for the rows and columns of the escape transition matrix \(\hat{A}_f\) in Definition 2.3, see also (8). When \(m = n - 1\), there is only one \(m\) configuration of \(n + m\), with \(\epsilon(i) = 2i\) for \(i = 1, \ldots, m\), \(m(j) = 2j - 1\) for \(j = 1, \ldots, n\) and \(n + m = 2n - 1\). If \(m = 0\), there is no configuration (there is no \(\epsilon(i)\) and we set \(m(j) = j\) for all \(j = 1, \ldots, n\)).
Every $m$-configuration $C_{m,n}$ of $n+m$ gives rise to a choice of the relative locations of the $m$ escape subintervals in $n$ Markov subintervals of $I$, as in (14), but it does not give the points in (2) neither the interval map $f$. However, if such map exists, then the escape symbols are precisely

$$\Sigma_{E_f} = \{ \hat{e}(i) - i : i = 1, ..., m \},$$

see (9) and, the interval map $f$ is such that

$$P \hat{A}_f P^T = \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}$$

with $P$ the permutation matrix determined by the following permutation

$$\pi_{C_{n,m}} := \begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & n+m \\ m(1) & m(2) & \cdots & m(n) & \epsilon(1) & \epsilon(2) & \cdots & \epsilon(m) \end{pmatrix}.$$  

Of course, given an interval map $f$, then the ordering (8) naturally gives rise to an $m$-configuration of $n+m$ and therefore the above permutation matrix $P$ can be denoted by $P_f$.

**Example 3.2.** Recall the interval map $f$ from Example 2.9. In this case we have a 2-configuration $\{3, 6\}$, with $n = 5$, $m = 2$ and the total number of intervals $n+m = 7$. Therefore, $m(1) = 1$, $m(2) = 2$, $m(3) = 4$, $m(4) = 5$, $m(5) = 7$, corresponding to the Markov intervals, and $\epsilon(1) = 3$, $\epsilon(2) = 6$, corresponding to the escape intervals. Note that the correspondence with the symbolic representation is obtained through $\hat{e}(i) - i : \hat{e}(1) - 1 = 2$ and $\hat{e}(2) - 2 = 6 - 2 = 4$.

The permutation $\pi_{C_{5,2}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 5 & 7 & 3 & 6 \end{pmatrix}$ transforms $\hat{A}_f$ into its normal form $N(\hat{A}_f)$.

Note that for every reducible non-negative matrix $X$ there is a permutation matrix $P$ so that

$$PX^TP^T = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ 0 & X_{22} & \cdots & X_{2n} \\ 0 & \cdots & X_{nn} \end{bmatrix}$$

where each $X_{ii}$, $i = 1, ..., n$ is either an irreducible block matrix or an $1 \times 1$ matrix equal to 0, see [2]. If $X$ is itself irreducible we have $X = X_{11}$ and in this case $P = I$. We then say that $PX^TP^T$ is a normal form of $X$ and we denote it by $N_P(X)$ or simply by $N(X)$. This form is not necessarily unique. For example, if the dimension of the irreducible block $X_{11}$ is equal to $n_1$ greater than 1 we have $n_1!$ different ways of representing $X$ in irreducible blocks, considering the permutations which fix all the lines and columns except the lines and columns of the block $X_{11}$. We will then have the following equivalent representation

$$\tilde{P}X^T\tilde{P} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \cdots & \tilde{X}_{1n} \\ 0 & \tilde{X}_{22} & \cdots & \tilde{X}_{2n} \\ 0 & \cdots & \tilde{X}_{nn} \end{bmatrix}.$$  

Different interval maps $f$ and $g$ may give rise to different permutation matrices $P_f$ and $P_g$, but with the same normal form $N_{P_f}(\hat{A}_f) = N_{P_g}(\hat{A}_g)$. In this case, $A_f = A_g$ and $B_f = B_g$.

Now we summarize in the following result what we discussed so far.
Proposition 3.3. Let \( \hat{A}_f \) be the \( n \times n \) escape transition matrix of a Markov interval map \( f \), \( A_f \) the Markov transition matrix of \( f \) and \( B_f \) the \( n \times m \) transition matrix from the Markov subintervals to the escape subintervals. Then \( N_{P_f}(\hat{A}_f) = \begin{bmatrix} A_f & B_f \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix} \).

Proof. Given such an interval map \( f \), (8) gives rise to an \( m \)-configuration \( C_{n,m} \) of \( n + m \). The row and columns labeling of \( \hat{A}_f \) is (14) whereas that of \( \begin{bmatrix} A_f & B_f \\ 0 & 0 \end{bmatrix} \) is given by \( m(1), \ldots, m(n), e(1), \ldots, e(m) \). Now if we consider the permutation matrix \( P \) associated to \( C_{n,m} \) as in (16), then we have \( P\hat{A}_fP^T = \begin{bmatrix} A_f & B_f \\ 0 & 0 \end{bmatrix} \). If we fix the labeling of the matrix \( A_f \), then the normal form of \( \hat{A}_f \) is unique (note that \( A_f \) is primitive because we assume that \( f \) is a Markov map, see Definition 2.1).

Given \( m, n \in \mathbb{N} \) such that \( 0 \leq m < n \), we now study the conditions over pairs of matrices \( A \) and \( B \) with sizes \( n \times n \) and \( n \times m \) (respectively) such that there exists an interval map \( f \) for which:

\[
N_{P_f}(\hat{A}_f) = \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}.
\]

We will simply write \( N(\hat{A}_f) \) instead of \( N_{P_f}(\hat{A}_f) \).

We note that if we fix an \( m \)-configuration \( C_{m,n} \), and thus a permutation matrix \( P_{C_{m,n}} \) as in (16), then we may find a matrix \( \hat{A} \) such that \( N_{P_{C_{m,n}}}(\hat{A}) = \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix} \), but this does not guarantee the existence of a solution for (17) as we will see in the sequel.

- The case \( m = 0 \) (so \( \Sigma_{E_f} = \emptyset \)) was answered positively in [5, Proposition 6] for any primitive interval type matrix \( A \). We briefly explain the construction of an interval map \( f \) such that \( A_f = A \) and \( \Sigma_{E_f} = \emptyset \). Let \( \lambda_A \) be the Perron eigenvalue of \( A \) and \( \mu_A = (v_1, \ldots, v_n) \) the corresponding Perron eigenvector normalised so that \( \sum_{i=1}^n v_i = 1 \). Then \( \Gamma = \{ c_0, c_1^+, c_1^-, \ldots, c_n^+, c_n^- \} \) with \( c_0 = 0 \), \( c_j^+ = \sum_{i=1}^{j+1} v_i \) and \( c_n = 1 \) provide a partition of the interval into \( n \) subintervals. Besides this, any piecewise linear map \( f := f_{(A, \Gamma)} \) so that \( f'_{(A, \Gamma)}(x) = \lambda_A \) is expansive, \( \Sigma_{E_f} = \emptyset \) and its transition matrix \( A_f \) coincides with \( A \).
- For the case \( n = 2 \) and \( m = 1 \) we have the following 1-configuration of 3: \( m(1) = 1, e(1) = 2 \) and \( m(2) = 3 \). This configuration was realized in [6, Example 3.5] by the interval map \( f(x) = 3x \mod 1 \) with domain given by \( \text{dom}(f) = I_1 \cup I_2 \), where \( I_1 = [0, 1/3] \) and \( I_2 = [2/3, 1] \) (thus \( \Sigma_{E_f} = \{1\} \)). In this case we have \( E_1 = [1/3, 2/3] \). The transition matrix \( A_f \) and the escape transition matrix \( \hat{A}_f \) are as follows: is

\[
A_f = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \hat{A}_f = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.
\]

- Another example is if we consider the 2-configuration of 5 given by \( m(1) = 1, e(1) = 2, m(2) = 3, e(2) = 4 \) and \( m(3) = 5 \). In this case [6, Example 3.6] the
We consider the interval map

\[ f(x) = \begin{cases} 
5x + 4/5 & \text{if } x \leq 1/25 \\
5x - 2 & \text{if } 2/5 \leq x \leq 3/5 \\
5x - 4 & \text{if } 4/5 \leq x \leq 1 
\end{cases} \]

provided a realization of the configuration, where the domain of \( f \) is given by \( I_1 \cup I_2 \cup I_3 \), with \( I_1 = [0, 1/25], I_2 = [2/5, 3/5], I_3 = [4/5, 1] \) and the escape intervals are \( E_1 = [1/25, 2/5], E_2 = [3/5, 4/5] \). The escape symbols set is \( \Sigma_E = \{ \tilde{1}, \tilde{2} \} \). The transition matrix \( A_f \) and the escape transition matrix \( \hat{A}_f \) are as follows:

\[
A_f = \begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \quad \text{and} \quad \hat{A}_f = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

- We now recall here [6, Example 3.7] that realizes the above 2-configuration of 5 (that was considered in [6, Example 3.6]). The underlying interval map is

\[ g(x) = \begin{cases} 
5x + 2/5 & \text{if } x \leq 3/25 \\
5x - 2 & \text{if } 2/5 \leq x \leq 3/5 \\
5x - 4 & \text{if } 4/5 \leq x \leq 1 
\end{cases} \]

with domain given by \( I_1 \cup I_2 \cup I_3 \), with \( I_1 = [0, 3/25], I_2 = [2/5, 3/5], I_3 = [4/5, 1] \). The escape intervals are \( E_1 = [3/25, 2/5], E_2 = [3/5, 4/5] \). The escape symbols set is \( \Sigma_E = \{ \tilde{1}, \tilde{2} \} \). The transition matrix \( A_g \) and the escape transition matrix \( \hat{A}_g \) are as follows:

\[
A_g = \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \quad \text{and} \quad \hat{A}_g = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

As we can see, the last two examples have the same transition matrices \( A_f = A_g \) but different escape transition matrices.

The case where \( A_f \) and \( B_f \) are the full matrices (of ones) is already interesting and we can actually realise the escape matrix by a very explicit dynamical system as follows.

**Proposition 3.4.** Fix \( m, n \in \mathbb{N} \) and an \( m \)-configuration of \( n + m \). Let \( 1_{n \times n} \) and \( 1_{n \times m} \) be the \( n \times n \) and \( n \times m \) full matrices, respectively. Then there is an interval map \( f \) such that

\[
N(\hat{A}_f) = \begin{bmatrix}
1_{n \times n} & 1_{n \times m} \\
0_{m \times n} & 0_{m \times m}
\end{bmatrix}.
\]

**Proof.** We consider the interval map \( g : I \to I \) such that \( g(x) = (n + m)x \mod 1 \). By Definition 2.1 \( g \in \mathcal{M}(I) \), with \( J_1, \ldots, J_{n+m} \) its Markov partition, where

\[
J_1 = \left[ 0, \frac{1}{n + m} \right], \ldots, J_i = \left[ \frac{i - 1}{n + m}, \frac{i}{n + m} \right], \ldots, J_{n+m} = \left[ \frac{n + m - 1}{n + m}, 1 \right],
\]

\( \hat{E}_g = \emptyset \), and \( A_g \) is the full \((n + m) \times (n + m)\) matrix.
Using the $m$-configuration $\{e_i\}_{i=1,...,m}$ of $n + m$ we define $E_1 = \hat{J}_e(1),...,E_m = \hat{J}_e(m)$ and $I_1 = J_1, I_2 = J_{m(2)},...,I_j = J_{m(j)},...,I_{n-1} = J_{m(n-1)}, I_n = J_n$, and $f : \text{dom}(f) \rightarrow I$ such that $f = g|_{\text{dom}(f)}$, where $\text{dom}(f) = \bigcup_{j=1}^{n} I_j$. Then we clearly have that $A_f$ is the full $n \times n$ matrix, $\hat{A}_f$ is the full $(n + m) \times (n + m)$ except in the $m$ rows $e(1),...,e(m)$ which are all null and $E_f = \bigcup_{i=1}^{m} E_i$. \hfill \Box

The cases of realisability of $1$-configurations of $n + 1$ from interval maps is of particular interest in the sequel as well. Given $(u_1,...,u_n) \in \{0,1\}^n$, we look for the existence of interval maps $f$ such that

$$N(\hat{A}_f) = \begin{bmatrix} A & u_1 \\ \vdots & \vdots \\ 0 & u_n \end{bmatrix}$$

for some $A$ (which is the Markov transition matrix $A_f$ of $f$). To keep it simpler, we work with $e(1) = 2$ (and so $m(1) = 1, m(2) = 3,...,m(n) = n + 1$).

**Proposition 3.5.** Let $(u_1,...,u_n) \in \{0,1\}^n$. Then there is an interval map $f$ such that

$$\hat{A}_f = \begin{bmatrix} a_{11} & u_1 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ a_{21} & u_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & u_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where $A_f = (a_{ij})$ is the Markov transition matrix of $f$.

**Proof.** The case $(u_1,...,u_n) = (0,...,0)$ means that such realisation $f$ has no escape points. Therefore this can be done using any realisation $f$ of an interval type and a primitive matrix $A$ of size $n \times n$, as in [5].

Next we assume that $(u_1,...,u_n) \neq (0,...,0)$. Let us construct an interval type and primitive 0-1 matrix $K = (k_{ij})$ of size $(n + 1) \times (n + 1)$ whose 2nd column is $(u_1,1,u_2,...,u_n)$ and the 2nd row is $(1,1,...,1)$.

The rest of the matrix $K = (k_{ij})$ is constructed as follows:

If $i$ such that $k_{i2} = 1$ ($k_{i2} = u_{i-1}$ for $i \geq 3$ or $k_{i2} = u_1$) with $i = 1,3,4,...,n + 1$ then $k_{ij} = 1$ for every $j$ with $1 \leq j \leq n + 1$.

Let now $s$ be the least integer so that $u_s = 1$ and $u_i = 0$ for $i < s$. If $k_{i2} = 0$ with $i = 1,3,4,...,n + 1$ then $k_{ij} = 0$ for every $j$, $1 \leq j \leq s$ and $k_{ij} = 1$ for every $j$, $s + 1 \leq j \leq n + 1$.

This implies that $K^2 > 0$, that is, every entry of $K^2$ is positive and therefore $K$ is a primitive matrix. Indeed, if the row $i$ of $K$ is a row of $1$’s then the clearly $(K^2)_{ij} > 1$ since every column of $K$ is non-zero. If the row $i$ of $K$ is in the form $0...01s...1$ ($s$ is the position of the last 0) then the entry $(K^2)_{ij}$ has the term $K_{i,s+1} + K_{s+1,j} = 1$. Therefore $(K^2)_{ij} \geq 1$ for all $i,j$.

We thus obtain an interval type matrix $K$ such that $K^2 > 0$. Let $\lambda_K$ be the Perron-Frobenius eigenvalue of $K$ and $\mu = (\mu_1,...,\mu_{n+1})$ the corresponding Perron-Frobenius eigenvector normalised so that $\sum_{i=1}^{n+1}\mu_i = 1$. Then if we define $c_{ij} = \sum_{i=1}^{j}\mu_i$, [5, Proposition 6] shows that there is a piecewise linear map $g \in P L(I)$ with slope $\lambda_K$ (which is greater than 1 because $K$ is primitive) such that $A_g = K$. 


and \( J_1 = [0, c_1], ..., J_i = [c_{i-1}, c_i], ..., J_{n+1} = [c_n, 1] \) is the Markov partition of \( g \) (so \( g|_{J_i} \) is a piecewise linear map such that \( g(J_i) = \bigcup_j b_{ij} J_j \).

Now, we define \( E_1 := J_2, I_1 := J_1, I_2 := J_3, ..., I_n := J_{n+1}, \) and \( f := g|_{\bigcup_{i=1}^{n+1} I_i} \). Then \( E_1 \) is the only escape subinterval of \( f \) so that \( E_f = E_1 \), and \( f \) does fulfill the requested property that there is a transition from \( I_1 \) to \( E_1 \) if \( u_1 = 1 \), and for \( i > 1 \), there is a transition from \( I_i \) to \( E_1 \) if and only if \( u_{i+1} = 1 \).

Note that if all the \( u_i \)'s are one in the last proof, then we get the matrix \( K \) as a particular case (with \( m = 1 \)) of Proposition 3.4.

We illustrate the above constructive proof.

**Example 3.6.** If \( n = 2 \), we have two nontrivial cases: \((u_1, u_2) = (1, 0)\) or \((u'_1, u'_2) = (0, 1)\) leading to

\[
K = \begin{bmatrix}
1 & u_1 & 1 \\
1 & 1 & 1 \\
1 & u_2 & 0
\end{bmatrix}
\quad \text{or} \quad
K = \begin{bmatrix}
0 & u'_1 & 1 \\
1 & 1 & 1 \\
1 & u'_2 & 1
\end{bmatrix},
\]

and two interval maps \( f_1 \) and \( f_2 \) such that \( A_{f_1} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix} \quad \text{and} \quad
A_{f_2} = \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix} \), respectively. Then \( N(\tilde{A}_{f_1}) = N \left( \begin{bmatrix}
1 & u_1 & 1 \\
0 & 0 & 0 \\
1 & u_2 & 0
\end{bmatrix} \right) = \begin{bmatrix}
A_{f_1} & u_1 \\
0 & u_2
\end{bmatrix} \) and \( N(\tilde{A}_{f_2}) = N \left( \begin{bmatrix}
0 & u'_1 & 1 \\
0 & 0 & 0 \\
1 & u'_2 & 1
\end{bmatrix} \right) = \begin{bmatrix}
A_{f_2} & u'_1 \\
0 & u'_2
\end{bmatrix} \).

If \( n = 3 \) and we consider \( u = (0, 0, 1) \), then

\[
K = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix},
A_f = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\quad \text{and} \quad
\hat{A}_f = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

We now study the general case of problem in (17). By [5], we know that \( A \) has to be an interval type matrix.

For any 0–1 matrix \( K = (k_{ij}) \), let \( \mathcal{K}_i = \{ j : k_{ij} = 1 \} \) (using the associated calligraphic letter). If we have a 0–1 matrix \( \begin{bmatrix}
A & B \\
0_{m \times n} & 0_{m \times m}
\end{bmatrix} \) of type \((n + m) \times (n + m)\) and an \( m \)-configuration \( C_{m,n} \) of \( m + n \), then for every \( i = 1, ..., n \) we set

\[
\mathcal{A}_i^{C_{m,n}} = \{ m(j) : j \in \mathcal{A}_i \} \quad \text{and} \quad \mathcal{B}_i^{C_{m,n}} = \{ e(j) : j \in \mathcal{B}_i \}
\]

where \( \mathcal{A}_i \) is the follower set as defined in Eq. (13).

**Theorem 3.7.** Let \( A = (a_{ij}) \) be a 0–1 primitive matrix of size \( n \times n \) and fix \( 1 \leq m \leq n - 1 \). Let \( B = (b_{ij}) \) be a 0–1 matrix of size \( n \times m \) without zero columns and fix an \( m \)-configuration \( C_{m,n} \) of \( n + m \).

Then there is a Markov interval map \( f \) such that its configuration is \( C_{m,n} \) and

\[
N(\hat{A}_f) = \begin{bmatrix}
A & B \\
0_{m \times n} & 0_{m \times m}
\end{bmatrix}
\]

if and only if \( \mathcal{A}_i^{C_{m,n}} \cup \mathcal{B}_i^{C_{m,n}} \) is a set of consecutive integer numbers, for every \( i = 1, ..., n \).

In that case, \( A \) and \( B \) must then be interval type matrices.
Proof. Let $C_{m,n}$ be an $m$ configuration of $n + m$, so that we have $\{\epsilon(i) : i = 1, \ldots, m\}$ and $\{m(j) : j = 1, \ldots, n\}$ such that $\{\epsilon(i) : i = 1, \ldots, m\} \cup \{m(j) : j = 1, \ldots, n\} = \{1, \ldots, n + m\}$ (see Definition 3.1).

Now assume that there is such a Markov map $f$. Then we have a partition $J_1, \ldots, J_{n+m}$ of the interval $I$. The $m$-configuration is such that $I_j = J_{m(j)}$ for $j = 1, \ldots, n$ are the Markov subintervals of $f$ and $E_{\epsilon(i) - i} = J_{\epsilon(i)}$ for $i = 1, \ldots, m$ (see eq. (14)) are the escape subintervals of $f$. Then the row and column labels of $A_f$ are as follows:

$$1, \ldots, \epsilon(1) - 1, \epsilon(1), \epsilon(1) + 1, \ldots, \epsilon(i) - 1, \epsilon(i), \epsilon(i) + 1, \ldots, \epsilon(m) - 1, \epsilon(m), \epsilon(m) + 1, \ldots, n + m.$$  

In particular, $(A_f)_{m(i) m(j)} = a_{ij}$ and $(A_f)_{m(i) \epsilon(j)} = b_{ij}$.

Now, since $f$ is a Markov map, for every $i$, $f(J_{m(i)})$ must be a union of subintervals from $(\bigcup_{i=1}^m J_{\epsilon(i)}) \cup (\bigcup_{j=1}^n J_{m(j)})$. Using the definition of $A_f$, we have

$$f(J_{m(i)}) = \bigcup_{j} \left( \bigcup_{a_{ij}J_{m(j)}} \bigcup \left( \bigcup_{b_{ij}J_{e(j)}} \right) \right).$$

Since $f$ is a continuous on $I_i = J_{m(i)}$, $\{J_{m(j)} : j \in A_i\} \cup \{J_{\epsilon(i)} : i \in B_i\}$ is a subinterval of $I$.

Conversely, let us assume that $A_f C_{m,n} \cup B_f C_{m,n}$ is a set of consecutive integer numbers (for all $i = 1, \ldots, n$). Then for the configuration $C_{m,n}$ and its associated permutation matrix $P$ (see Eq. (16)) the columns and rows of $K_0 := P^T \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix} P$ are ordered as follows

$$1, \ldots, \epsilon(1) - 1, \epsilon(1), \epsilon(1) + 1, \ldots, \epsilon(i) - 1, \epsilon(i), \epsilon(i) + 1, \ldots, \epsilon(m) - 1, \epsilon(m), \epsilon(m) + 1, \ldots, n + m.$$  

Let $K = (k_{ij})$ be the $(n + m) \times (n + m)$ matrix such that $k_{m(i) m(j)} = a_{ij}$ for $i, j = 1, \ldots, n$, $k_{m(i) \epsilon(j)} = b_{ij}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and $k_{ij} = 1$ in the other cases (i.e. the rows labeled with $\epsilon(1), \ldots, \epsilon(m)$ and all $j = 1, \ldots, n + m$).

Since $A$ is primitive and every column of $B$ is non-zero, the matrix $K$ is also primitive because (using the definition of $K$)

$$(K^p)_{m(i) m(i')} \geq (A^p)_{ii'},$$  

$$(K^p)_{m(i) \epsilon(j)} \geq (A^{p-1} B)_{ij}, (K^p)_{\epsilon(i) m(j')} \geq 1 \text{ and } (K^p)_{\epsilon(i) \epsilon(j')} \geq 1$$

for every $p \in \mathbb{N}$, $i, i' = 1, \ldots, n$ and $j, j' = 1, \ldots, m$. Note that if $s$ such that $A^s > 0$, then the above shows that $K^{s+1} > 0$, therefore $K$ is a primitive matrix.

Therefore we can apply [5, Proposition 6] and find a piecewise linear map $g \in \mathcal{M}(I)$ with slope the Perron-Frobenius eigenvalue $\lambda_K$ of $K$, the escape set $\Sigma_{E_g} = \emptyset$, and the Markov partition $C_g = \{J_1, \ldots, J_{m+n}\}$ such that $A_g = K$. Now we define $E_{\epsilon(i) - i} = J_{\epsilon(i)}$ and $I_j = J_{m(j)}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ and $f = g |_{\bigcup_{j=1}^n I_j}$. Then it is clear that $f$ does what we want, namely $A_f = A$, $A_f = K_0$ and $N(A_f) = \begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}$ as required.

If $A$ and $B$ are interval type matrices, then $\begin{bmatrix} A & B \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}$ might not be realisable by an escape transition matrix. This depends on the chosen configuration, as the following example shows.
Example 3.8. Let us consider two cases with $n = 3$, $m = 2$, where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} A & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} A & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. $$

There is only one 2-configuration: the first is with $e(1) = 2$ and $e(2) = 4$ (consequently $m(1) = 1$, $m(2) = 3$, $m(3) = 5$). Then $K_1$ is realisable as an escape transition matrix $A_f$ for some interval map $f \in \mathcal{M}(I)$, whereas $K_2$ is not.

It is easy to justify (without using the Theorem 3.7 that $K_2$ is not realised as an escape transition matrix (if $K_2$ were realised, then $f(\hat{I}_1) = E_1 \cup \hat{I}_3$, but then $f$ could not be continuous).

It is straightforward to verify that $K_1$ is an escape transition matrix (up to permutations) for some interval map $f \in \mathcal{M}(I)$ if we use Theorem 3.7: $A_1 = \{3\}$ and $B_1 = \{2\}$. Thus $\mathcal{A}_1^{C_{2,3}} \cup \mathcal{B}_1^{C_{2,3}} = \{m(3), e(2)\} = \{5, 4\}$ (consecutive numbers) and

$$\mathcal{A}_2^{C_{2,3}} \cup \mathcal{B}_2^{C_{2,3}} = \mathcal{A}_3^{C_{2,3}} \cup \mathcal{B}_3^{C_{2,3}} = \{m(1), m(2), m(3), e(1), e(2)\} = \{1, 2, 3, 4, 5\}$$

which are also consecutive numbers.

Note that for $K_2$, we have $\mathcal{A}_1^{C_{2,3}} \cup \mathcal{B}_1^{C_{2,3}} = \{m(3), e(1)\} = \{5, 2\}$ (non-consecutive numbers), since $B_1 = \{1\}$.

The role of the $m$-configuration in the Theorem 3.7 is essential.

Example 3.9. Let us use the same interval type matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ from Example 3.8 and consider $K = \begin{bmatrix} A & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In this case $n = 3$, $m = 1$ and there are two 1-configurations: the first is with $e(1) = 2$ (thus $m(1) = 1, m(2) = 3, m(3) = 4$) and the second is with $e(1) = 3$ (hence $m(1) = 1, m(2) = 2, m(3) = 4$).

In the first configuration, $A_1 = \{3\}$ and $B_1 = \{1\}$ thus

$$\mathcal{A}_1^{C_{1,3}} \cup \mathcal{B}_1^{C_{1,3}} = \{m(3), e(1)\} = \{4, 2\}$$

(non-consecutive numbers), consequently there is no interval map $f$ such that $\hat{A}_f$ equals $K_1$ (up to permutations).

For the second configuration, $\mathcal{A}_1^{C_{1,3}} \cup \mathcal{B}_1^{C_{1,3}} = \{m(3), e(1)\} = \{3, 4\}$ and

$$\mathcal{A}_2^{C_{1,3}} \cup \mathcal{B}_2^{C_{1,3}} = \mathcal{A}_3^{C_{1,3}} \cup \mathcal{B}_3^{C_{1,3}} = \{m(1), m(2), m(3), e(1)\} = \{1, 2, 3, 4\}$$

(consecutive numbers) implying, by Theorem 3.7, that there exists an interval map $f \in \mathcal{M}(I)$ such that $\hat{A}_f = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. So both configurations give rise to the same matrix $K$ (up to permutations), one of them is realised by interval maps but the other is not.

\[^1\text{If such } f \text{ existed then } \hat{A}_f \text{ would be given by } \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ but the first row of the latter matrix has more than one blocks of ones, i.e. the matrix is not of interval type.}\]
We also infer from the proof of Theorem 3.7 that if we use
\[ 1, \ldots, \epsilon(1) - 1, \epsilon(1), \epsilon(1) + 1, \ldots, \epsilon(i) - 1, \epsilon(i), \epsilon(i) + 1, \ldots, \epsilon(m) - 1, \epsilon(m), \epsilon(m) + 1, \ldots, n + m \]
as the order of the row and column labels of the escape transition matrix \( \hat{A}_f \), then \( \hat{A}_f \) becomes an interval type matrix for rows labeled by \( m(1), m(2), \ldots, m(n) \).

**Corollary 3.10.** Let \( \hat{A}_f \) be an escape transition matrix of an interval map \( f \). Then \( f \) is the restriction of an interval map \( g \) with escape set \( E_g = \emptyset \).

**Proof.** The map \( f \) determines an interval type matrix \( A_f \) of size \( n \times n \), \( m \) escape subintervals of \( f \) and an \( m \)-configuration \( C_{m,n} \) of \( n + m \) – giving rise to a partition \( \{1, \ldots, m\} = \{\epsilon(i) : i = 1, \ldots, m\} \cup \{m(j) : j = 1, \ldots, n\} \).

Then we consider the matrix \( K \) as in the proof of Theorem 3.7, and thus [5] guarantees the existence of an interval map \( g \in \mathcal{M}(I) \) such that \( A_g = K \) and \( f \) is indeed the restriction of \( g \) to the union of all the subintervals labelled by \( m(1), \ldots, m(n) \).

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**REFERENCES**

[1] R. Alcaraz Barrera, Topological and ergodic properties of symmetric sub-shifts, *Discrete Contin. Dyn. Syst.*, 34 (2014), 4459–4486.

[2] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, 9. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[3] S. Bundfuss, T. Krüger and S. Troubetzkoy, Topological and symbolic dynamics for hyperbolic systems with holes, *Ergodic Theory Dynam. Systems*, 31 (2011), 1305–1323.

[4] L. Clark, The \( \beta \)-transformation with a hole, *Discrete Contin. Dyn. Syst.*, 36 (2016), 1249–1269.

[5] C. Correia Ramos, N. Martins and P. R. Pinto, Interval maps from Cuntz-Krieger algebras, *J. Math. Anal. Appl.*, 374 (2011), 347–354.

[6] C. Correia Ramos, N. Martins and P. R. Pinto, Toeplitz algebras arising from escape points of interval maps, *Banach J. Math. Anal.*, 11 (2017), 536–553.

[7] C. Correia Ramos, N. Martins and P. R. Pinto, On graph algebras from interval maps, *Ann. Funct. Anal.*, 10 (2019), 203–217.

[8] B. Derrida, A. Gervois and Y. Pomeau, Iteration of endomorphisms on the real axis and representations of numbers, *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 29 (1978), 305–356.

[9] K. Falconer, *Techniques in Fractal Geometry*, John Wiley and Sons, Ltd., Chichester, 1997.

[10] J. P. Lampreia, A. Rica da Silva and J. Sousa Ramos, Construction of 0–1 matrices associated to period-doubling processes, *Stochastica*, 9 (1985), 165–178.

[11] J. P. Lampreia and J. Sousa Ramos, Symbolic dynamics of bimodal maps, *Portugal. Math.*, 54 (1997), 1–18.

[12] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995.

[13] N. Sidorov, Arithmetic dynamics, *Topics in Dynamics and Ergodic Theory*, London Math. Soc. Lecture Note Ser., *Cambridge Univ. Press, Cambridge*, 310 (2003), 145–189.

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