Persistent current in an almost staggered Harper model

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In this paper we study the persistent current (PC) of a staggered Harper model, close to the half-filling. The Harper model, which is a quasi-periodic system, is different than other one dimensional systems with uncorrelated disorder in the fact that it can be in the metallic regime. Nevertheless, the PC for a wide range of parameters of the Harper model does not show typical metallic behavior, although the system is in the metallic regime. This is a result of the nature of the central band states, which are a hybridization of Gaussian states localized in superlattice points. When the superlattice is not commensurate with the system length, the PC behaves as in an insulator. Thus even in the metallic regime a typical finite Harper model may exhibit a PC expected from an insulator.

In this paper, we study the PC of a Harper model of spinless fermions on a ring threaded by a magnetic flux. This is a tight-binding model in which the on-site potential is spatially modulated with an irrational frequency. We would focus here on irrational frequencies which their modulus is close to half. This corresponds to a fast (two site) modulation with a slow envelope. These frequencies exhibit an increase of the compressibility when the electron-electron interactions are increased, opposite to the influence of interactions in regular disordered systems [20, 21]. Here we will show that for this range of frequencies close to half-filling, the PC shows a non-monotonous dependence on the systems size, where for most values of $L$ the PC is strongly suppressed.

The tight-binding Harper model Hamiltonian for spinless fermions on a ring threaded by a magnetic flux is:

$$H = \sum_{j=1}^{L} \lambda \cos(2\pi bj + \phi) \hat{c}_j^\dagger \hat{c}_j - \sum_{j=1}^{L-1} (e^{i\varphi_0/\phi} \hat{c}_j^\dagger \hat{c}_{j+1} + h.c.)$$

where $\hat{c}_j$ is the single particle annihilation operator on site $j$, $t$ is a real hopping amplitude. The magnetic flux is denoted by $\phi$, and $\varphi_0$ is the quantum flux quanta $\varphi_0 = hc/e$. The strength of the on-site potential is controlled by $\lambda > 0$. The on-site potential is modulated by a frequency $b$, and $\phi$ is an arbitrary phase factor. It should be clear that since we are interested in a ring, $\phi$ is irrelevant and will be ignored through the rest of this paper. We will be interested in the metallic regime of the model, i.e. $\lambda < 2t$. The irrational frequency may be written as $b = \mathbb{Z} + 1/2 + \epsilon$, where $\epsilon$ is irrational. Therefore, we can write the on-site potential as:

$$\cos(2\pi bj) = \cos(2\pi (\mathbb{Z} j + \pi j + 2\pi \epsilon j)) = (-1)^{\mathbb{Z}} \cos(2\pi \epsilon j)$$

When $\epsilon \ll 1/2$ the system is called an almost staggered Harper model for which the fast frequency of the $(-1)^\mathbb{Z}$ term is modulated by the slow frequency, $\epsilon$, of the
cos(2πεj) term. In the almost staggered case the energy spectrum of system shows unique features such as a central band that is separated from the other bands by a large gap, of order λ, as can be seen in Fig.1. Also the two bands sandwiching the central band show similar features, i.e., a rather narrow (flat) band and a large gap to the next band. Changing the length of the sample length from $L = 900$ to $L = 1000$ does not change its gross features, although some difference in the energies of the edge states in the gaps are apparent. As detailed in Ref. 20, for $ε \ll 1$, there are $L_n = 2|ε|L$ states in the central band, corresponding to the number of intersections with zero of the slow modulation envelope, which occur at $cos(2πεj) = 0$. These valleys are shown in Fig.2 for a smaller system ($L = 200$) in order not to clutter the figure. The frequency of the envelope is $ε$ and the distance between two consecutive valleys is half the period, i.e., the distance is $1/2|ε|$. For example, for the systems depicted in Fig.1 $b = \sqrt{30} = 5.477226$, and therefore $ε = -0.022774$, resulting in $L_n \approx 50$ for $L = 900$ and $L_n \approx 54.55$ for $L = 1000$. For the $L = 200$ case shown in Fig.2 $L_n \sim 9.1$. Indeed, the 9 valley states are clearly seen, as well as the fact that the valley positions are not exactly commensurate.

Thus, to first approximation, there is a superlattice of valleys at points $j_n$, with $n = 1, \ldots, L_n$, each with a zero-energy state, $|n⟩$, centered around $j = j_n$. These states can be written as Gaussians falling off at a length scale of $ξ = \sqrt{1/|ε|}$. The central band eigenfunctions are composed of the hybridization of these localized states. For periodic boundary conditions, and when $L_n = [L_n]$, the eigenstates of the central band are plane waves composed of the valley Gaussian, $|k⟩ = L^{-1/2} \sum_{n=1}^{L} S_n e^{i k n} |n⟩$, where $S_n = \sqrt{2}\cos(nπ/2 - π/4) = \ldots, 1, 1, -1, -1, 1, 1, \ldots$. The spectrum $E_{\text{central}}(k) = -2t\cos k$ [20], where $k = 2πn/L_n$ ($m = 0, ±1, ±2, \ldots$) and $E = \exp(-ξ^2/(4ξ^2k^2))(2t\exp(-1/4ξ^2)\sinh[(4ξ^2)|r|^{-1}]-λ\exp(-π^2ξ^2))$. Thus, the central band spectrum is expected to show a degeneracy since $E_{\text{central}}(k) = E_{\text{central}}(-k)$. A closer look at this issue reveals that if the system is not exactly periodic, the degeneracy will be broken by a non perfect periodicity. One may think of the effect of the non perfect periodicity as an impurity at the region of the non-periodicity (i.e., around $n = 0$). For low-lying states in the central band, the impurity acts as a hard-wall, leading to low-lying states of the form: $|k⟩ = \sqrt{2/L} \sin(kn)$, with $k = πm/L_n$ ($m = 0, 1, 2, \ldots$), and eigenvalues $E_{\text{central}}(k) = -2t\cos k$. Thus, as can be seen in the inset of Fig. 1 for the low-lying states in the central band the degeneracy in the eigen-values is lost, both for the almost periodic case ($L = 900$), as for the non-periodic one ($L = 1000$). The low-lying wave functions are depicted in the upper panel of Fig. 3. For comparison the wave functions of a clean ring of the same length, with a single impurity at $n = 0$ (weak for $L = 900$, strong for $L = 1000$), is drawn. For the clean system the ground state wave function corresponds to a half-sine, while the first excited state to a sine. This behavior is not very sensitive to the impurity strength. A similar situation can be seen for the Harper model, where the half-sine and sine envelopes are composed of the Gaussian superlattice states at points $j_n$. As expected for the low-lying states there is no essential difference between the almost periodic case and the non-periodic one.

This changes when higher levels are considered. For higher energies, one expects that the effect of the impurity will be weaker. Indeed, as can be seen for the clean system depicted in the lower panel of Fig. 1 if the strength of the impurity is weak (the $L = 900$ case), the wave function is homogeneous. Only for the stronger impurity (the $L = 1000$ case), the wave function shows a signature of the impurity. A similar behavior is seen for the central band of the Harper model. For the almost commensurate case the states in the middle of the central band show only a slight degeneracy breaking (see the inset of Fig. 1) where the energies for $L = 900$ clearly appear in pairs). Nevertheless, when the non-commensurability is stronger, as for $L = 1000$, the levels are non-degenerate even in the middle, thus corresponding to a hard wall boundary condition even for higher energies. This distinction can be seen also for the wave functions, where for the almost commensurate Harper model the wave function is almost homogeneous, while for the non-commensurate case the wave function is non-homogeneous (see Fig. 4).

Let us now consider the effects of commensurability on the persistent current following through a Harper ring. The PC of the $j$-th level is defined as $I_j(σ, φ) = \partial E_j/\partial φ|_φ$, while the total persistent current up to a given level $n$ is $I_n(φ) = \sum_{j=1}^{n} I_j(φ)$. The total persistent current for any value of $n$ for length $L = 900$ and $L = 1000$ are presented in Fig. 5 with the data pertaining to the energy of the $n$-th level. Gaps between the bands are clearly seen by the jumps in the energy between adjacent levels (as in Fig. 1). First one notes that as expected, the total persistent current reaches a maximum at the middle of the bands, while it is zero at the edge of the band. For the bands far from middle, there is no essential difference in $I_n$ between the almost commensurate ($L = 900$) length and the incommensurate length ($L = 1000$).

Since we are considering here the metallic regime of the Harper model ($λ = 1$), a metallic behavior of the PC is expected. The most obvious difference between the PC of a metallic and a localized system is the amplitude of the PC. As can be seen from the energy dependence on the threading flux presented in Fig. 6 there is a huge difference in the dependence of the states between almost commensurate length ($L = 900$ and $L = 483$) and in-
FIG. 1: (Color online) The energy spectrums of two different length, one with $L = 900$ (black symbols) and the other with $L = 1000$ (red symbols), for both length $b = \sqrt{30}$ and $\lambda = 1$. The superlattice length correspond to $L_n \sim 50$ for $L = 900$ and $L_n \sim 54.55$ for $L = 1000$. The gross features of the spectrum (except for the edge states appearing in the gaps) do not essentially change between the commensurate and incommensurate length. Inset: a zoom into the central bad for both length. Please note the change in the scale of the y axis. Here there is a clear difference between the commensurate and incommensurate systems.

FIG. 2: (Color online) The on-site potential of the Harper model for $L = 200$, with $\lambda = 1$ and $b = \sqrt{30}$. The envelope corresponds to $\cos(2\pi \xi n)$. The distance between the valleys is $1/2\xi$. The number of valleys correspond to $L_n = 2|\xi|L$ which for the system depicted in this figure corresponds to $L_n \sim 9.1$, which is a slightly incommensurate case.

FIG. 3: (Color online) The ground and first excited state wave functions for $L = 900$ (upper panel) and $L = 1000$ (lower panel). For the Harper model the ground state wave function squared corresponds to the black continuous line, while the dashed curve corresponds to the first excited state. For the clean ring with an impurity the same notation is used with red curves. For the Harper model the Gaussian localized states at the superlattice points are seen. It is also evident that the Harper states have a similar envelope to the clean state with commensurate length ($L = 1000$ and $L = 405$). In this figure the flux dependence of total PC $I_{L/2}(\phi)$ is plotted. For the commensurate case a high amplitude of the PC is seen, which is in line with our expectations from a metallic system. Moreover, a unique sawtooth dependence as function of the flux is observed, similar to the typical behavior of the total PC in clean and weakly disordered systems [23]. On the other hand, for the incommensurate case $I_{L/2}(\phi)$ is almost flat and has a sine flux dependence, expected in the localized regime.

An additional indicative feature for determining whether the PC corresponds to a metallic or a local-
The total PC at a given flux $I$ is described by the first harmonic, i.e., for the metallic regime, the weight of higher harmonics is negligible. Thus, the PC falls off rather slowly [24]. This shows that both the amplitude and dependence on the flux of the almost commensurate and non-commensurate length demonstrate a completely different behavior at the middle of the central band. This is also evident from the Fourier transform of the PC (Fig. 6), where for the non-commensurate cases only the first harmonic is non-zero, while for almost commensurate cases the contribution of higher harmonics is still significant.

Therefore, if one uses the total PC at half-filling, or the persistent current of a particular state in the middle of the central band to decide whether the Harper model is metallic or localized, one would get different results as function of the length of the sample. The behavior is metallic for the almost commensurate length and localized like for the non-commensurate length. Of course the system remains essentially metallic in both cases. Nevertheless, as we have seen from a direct inspection of the wave functions in the central band (Figs. 34) the non commensurate segment has an influence similar to an impurity embedded in the ring. Since this impurity acts like a barrier, we expect that the PC amplitude will be suppressed proportionally to the overlap between two superlattice states, i.e., proportionally to $\exp(C/\xi^2)$ (where $C$ is a constant). In Fig. 7 we plot the first harmonic of the PC of the middle state $\nu_{L/2}(1)$ as a function of $\xi^2 = t/\pi \lambda |\epsilon|$. Here we keep the strength of the potential $\lambda$ and system size $L$ constant, while changing $|\epsilon|$. It can be seen that $\nu_{L/2}$ is suppressed exponentially as function of $\xi^2$. This fits well with the description of the non-commensurate central band as a system with an impurity. It also confirms that PC cannot be used in these systems to determine whether the system is metallic or localized.

In conclusion, we have considered the PC in the staggered Harper model, mainly close to the half-filling. Contrary to naive expectations the PC for most systems do not show typical metallic behavior, although the system is in the metallic regime. This stems from the unique character of the central band states, which are a hybridization of Gaussian states localized in super lattice points. If the distance between the superlattice sites are (almost) commensurate with the system size, then the PC has metallic properties. On the other hand, when the superlattice sites are incommensurate the system has an effective impurity at the incommensurate section and the PC shows the signature of an insulator. Since for most combinations of the Harper frequency $b$ (close to the staggered condition $Z + 1/2$) and length $L$ are incommensurate, a typical Harper model will actually exhibit PC corresponding to an insulator.

Thus, studying the PC at half-filling for the finite almost staggered Harper model is not indicative of the phase of the system. Moreover, also other stout method for identifying the metal-insulator transition, such as the level spacing statistics [29, 30], will not show the expected

![Figure 5](https://example.com/image5.png)  
**FIG. 5:** (Color online) The total persistent current for $b = \sqrt{30}$ ($\epsilon = -0.22774$) and $\lambda = 1$ for different sample length. Upper panel: The total PC at a given flux $I_{\phi}(\phi = \pi/2)$ as function of the number of states $n$ for the $L = 900$ and $L = 1000$ length is represented by the black lines. The red curve corresponds (with an arbitrary scale) to the energy of the $n$-th level. Jumps in the red curve indicates a gap between the states. Close to the gaps (i.e., at the band edge) the total PC goes to zero. The main difference between the almost commensurate ($L = 900$) and non commensurate ($L = 1000$) length is the amplitude of the total PC in the central band (the states around $L/2$). Lower panel: The total PC at the center of the central band as function of the threading flux $\phi$. For the almost commensurate superlattice cases ($L = 900$, and $L = 483$) the PC is large and it has a saw tooth like shape. For the incommensurate cases ($L = 1000$, and $L = 405$) the PC is small and the shape is sine like.

ized behavior, is the number of harmonics composing the PC. Generally, the PC may be written as a harmonic expansion of the form $\nu = \sum f(\nu_0) \sin(2f\pi \phi/\phi_0)$ and $I = \sum f(I_0) \sin(2f\pi \phi/\phi_0)$. For the localized regime the PC is described by the first harmonic, i.e., $\nu \sim \nu(1) \sin(2\pi \phi/\phi_0)$ and $I \sim I(1) \sin(2\pi \phi/\phi_0)$ [23], while for the metallic regime, the weight of higher harmonics
FIG. 6: (Color online) Fourier transform of the total PC at the middle of the spectrum. For the incommensurate cases ($L = 1000$, and $L = 405$) the Fourier transform has only one significant component which is relatively small. For the almost commensurate superlattice cases ($L = 900$, and $L = 483$) the first component is the biggest, but the contribution from the higher components goes down only gradually.

transition in the level statistics from the Poisson distribution for the localized regime to the Wigner distribution in the metallic regime. As is evident from the inset in Fig. 4, the distribution will change even in the metallic regime, between a picket fence distribution to a bi-model one. Therefore, when studying the Harper model in order to investigate the metal-insulator transition one must be careful to choose the appropriate measures.

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References

[1] P. G. Harper, Proc. Phys. Soc. London A 68, 874 (1955).
[2] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
[3] S. Aubry and G. Andre, Ann. Isr. Phys. Soc. 3, 133 (1980).
[4] For a review see: H. Hiramoto and M. Kohmoto, Int. J. Mod. Phys. B 6, 281 (1992).
[5] G. Roati, C. Derrico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno and M. Inguscio, Nature 453, 895 (2008).
[6] J. Chabe, G. Lemarie, B. Gremaud, D. Delande, P. Szriftgiser, and J. C. Garreau, Phys. Rev. Lett. 101, 255702 (2008).
[7] G. Modugno, Rep. Prog. Phys. 73, 102401 (2010).
[8] M. Modugno, New Journal of Physics 11, 033023 (2009).
[9] Y. Lahini, R. Pugatch, F. Pozzi, M. Sorel, R. Morandotti, N. Davidson, and Y. Silberberg, Phys. Rev. Lett. 103, 013901 (2009).
[10] L. Tanzi, E. Lucioni, S. Chaudhuri, L. Gori, A. Kumar, C. D’Errico, M. Inguscio,and G. Modugno, Phys. Rev. Lett. 111, 115301 (2013).
[11] C. D’Errico, E. Lucioni, L. Tanzi, L. Gori, G. Roux, I. P. McCulloch, T. Giamarchi, M. Inguscio, and G. Modugno, Phys. Rev. Lett. 113, 095301 (2014).
[12] E. Abrahams, P. Anderson, D. Licciardello, and T. Ramakrishnan, Phys. Rev. Lett., 42, 673 (1979).
[13] J. Vidal, D. Mouhanna, and T. Giamarchi, Phys. Rev. Lett. 83, 3908 (1999).
[14] J. Vidal, D. Mouhanna, and T. Giamarchi, Phys. Rev. B 65, 014201 (2001).
[15] C. Schuster, R. A. Romer, and M. Schreiber, Phys. Rev. B 65, 115114 (2002).
[16] S. Iyer, V. Oganesyan, G. Refael, and D. A. Huse, Phys. Rev. B 87, 134202 (2013).
[17] V.P. Michal, B.L. Altshuler, and G.V. Shlyapnikov, Phys. Rev. Lett. 113, 045304 (2014).
[18] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Phys. Rev. Lett. 109, 106402 (2012).
[19] M. Verbin, O. Zilberberg, Y. E. Kraus, Y. Lahini, and Y. Silberberg, Phys. Rev. Lett. 110, 076403 (2013).
[20] Y. E. Kraus, O. Zilberberg and R. Berkovits, Phys. Rev. B 89, 161106(R) (2014).
[21] B. Friedman and R. Berkovits (Preprint arXiv:1412.8382).
[22] M. Buttiker, Y. Imry and R. Landauer, Phys. Lett. A
96, 365 (1983).
[23] H.F. Cheung, Y. Gefen, E.K. Riedel and W.H. Shih, Phys. Rev. B 37, 6050 (1988); H.F Cheung, E.K. Riedel and Y. Gefen, Phys. Rev. Lett. 62, 587 (1989).
[24] H. Bouchiat and G. Montambaux, J. Phys. (Paris) 50, 2695 (1989); G. Montambaux, H. Bouchiat, D. Sigeti and R. Friesner, Phys. Rev. B 42, 7647 (1990).
[25] F. von Oppen and E.K. Riedel, Phys. Rev. Lett. 66, 80 (1991).
[26] B. L. Altshuler, Y. Gefen, and Y. Imry, Phys. Rev. Lett. 66, 88 (1991).
[27] E. Akkermans and G. Montambaux, Phys. Rev. Lett. 68, 642 (1992).
[28] R. Berkovits and Y. Avishai, Phys. Rev. Lett. 76, 291 (1996).
[29] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides and H. B. Shore, Phys. Rev. B 47, 11487 (1993).
[30] R. Berkovits and Y. Avishai, Phys. Rev. B 53, R16125 (1996).