Hydrodynamic equations for granular mixtures

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Many features of granular media can be modeled by a fluid of hard spheres with inelastic collisions. Under rapid flow conditions, the macroscopic behavior of grains can be described through hydrodynamic equations accounting for dissipation among the interacting particles. A basis for the derivation of hydrodynamic equations and explicit expressions appearing in them is provided by the Boltzmann kinetic theory conveniently modified to account for inelastic binary collisions. The goal of this review is to derive the hydrodynamic equations for a binary mixture of smooth inelastic hard spheres. A normal solution to the Boltzmann equation is obtained via the Chapman-Enskog method for states near the local homogeneous cooling state. The mass, heat, and momentum fluxes are obtained to first order in the spatial gradients of the hydrodynamic fields, and the set of transport coefficients are determined in terms of the restitution coefficients and the ratios of mass, concentration, and particle sizes. As an example of their application, the dispersion relations for the hydrodynamic equations linearized about a homogeneous state are obtained and the conditions for stability are identified as functions of the wave vector, the dissipation, and the parameters of the mixture. The analysis shows that the homogeneous reference state is unstable to long enough wavelength perturbations and consequently becomes inhomogeneous for long times. The relationship of this instability to the validity of hydrodynamics is discussed.

I. INTRODUCTION

Granular media under rapid flow conditions exhibit a great similarity to molecular fluids. This fact has stimulated the use of hydrodynamic-like type equations to describe the macroscopic behavior of such systems. The main difference from ordinary fluids is the absence of energy conservation, leading to both obvious and subtle modifications of the Navier-Stokes hydrodynamic equations. To isolate the effects of such collisional dissipation from other important properties of granular media, an idealized microscopic model system is usually considered: a system composed by smooth hard spheres with inelastic collisions. As in the elastic case, the collisions are specified in terms of the change in relative velocity at contact but with a decrease in the magnitude of the normal component measured by a positive restitution coefficient $\alpha \leq 1$. This parameter distinguishes the ideal granular fluid ($\alpha < 1$) from the ideal normal fluid ($\alpha = 1$).

Although many efforts have been devoted in the past few years in the understanding of granular fluids, the derivation of the form of the transport coefficients is still a subject of interest and controversy. In the low-density regime, this problem can be addressed by using the Boltzmann kinetic theory conveniently modified to account for inelastic collisions. The conditions for hydrodynamics are expected to be similar to those for normal fluids. For a given initial state there are two stages of evolution. First, during the kinetic stage there is rapid velocity relaxation to a “universal” velocity distribution that depends on the average local density, temperature, and flow velocity. Subsequently, the hydrodynamic stage is described through a slower evolution of the local hydrodynamic fields as they approach uniformity. The solution to the Boltzmann equation in this second stage is said to be “normal”, where all space and time dependence of the distribution function occurs through the hydrodynamic fields. The Chapman-Enskog method provides a means to construct explicitly the form of this normal solution as a perturbation expansion in the spatial gradients of the fields. This solution is then used to evaluate the fluxes in the macroscopic balance equations in terms of these gradients. To lowest order the balance equations become the granular Euler equations; to second order they are the granular Navier-Stokes equations. In carrying out this analysis, explicit forms for the transport coefficients are obtained as functions of the restitution coefficient and other parameters of the collision operator. This derivation of hydrodynamics from the Boltzmann equation has been widely covered in the case of a monocomponent gas where the particles are of the same mass and size. However, a real granular system is generally
characterized by some degrees of polidispersity in density and size, which leads to phenomena very often observed in nature and experiments, such as separation or segregation. Several attempts to apply the Boltzmann equation to derive transport coefficients for a multicomponent system began time ago [4], but the technical difficulties of the analysis entailed approximations that limited their accuracy. In addition, all these works are based on the assumption of energy equipartition so that the partial temperatures $T_i$ are made equal to the global granular temperature $T$. Nevertheless, kinetic theory studies [5, 6, 7, 8], computer simulations [9, 10, 11], and even real experiments [12, 13] have clearly shown the breakdown of energy equipartition. As a consequence, many of the previous results obtained for granular mixtures must be reexamined by using a kinetic theory which takes into account the effect of temperature differences on the transport coefficients.

The goal of this paper is to derive the hydrodynamic equations of a granular binary mixture from the Boltzmann kinetic theory. These equations are derived by applying the Chapman-Enskog expansion about a local homogeneous cooling state (LHCS) that is analogous to the local equilibrium state for a gas with elastic collisions. In the first order of the expansion, the irreversible parts of the mass, heat, and momentum fluxes are calculated and the eight transport coefficients identified. These coefficients are expressed in terms of the solutions to a set of coupled linear integral equations [1]. The analysis carried out here is more complete than previous studies [4]; it is exact to leading order in the dissipation but not limited to weak dissipation. However, for practical purposes, the above integral equations are solved approximately by using the leading terms in a Sonine polynomial expansion. Such approach compares quite well with the results obtained from numerical solutions [14, 15] of the Boltzmann equation by means of the Direct Simulation Monte Carlo (DSMC) method [16]. The explicit knowledge of the transport coefficients allows quantitative application of the nonlinear hydrodynamic equations to a number of interesting problems for mixtures, such as segregation and separation. Here, the simplest example of an application is considered, small perturbations of a spatially homogeneous state. The dispersion relations for the hydrodynamic modes are obtained from the hydrodynamic equations linearized about the homogeneous form of the LHCS. Linear stability analysis shows two shear modes and a heat mode to be unstable for long wavelength excitations. This means that small perturbations or fluctuations about the homogeneous state will grow. The implications of this for the derivation of the hydrodynamic equations are discussed here as well.

The plan of the paper is as follows. In Sec. II we review the Boltzmann equation and associated macroscopic conservation laws. We also give a brief survey of the application of the Chapman-Enskog method around the LHCS. This state is analyzed in Section III while in Sec. IV we give the form of the Navier-Stokes hydrodynamic equations. Theoretical results are compared with simulation data for the temperature ratio and the shear viscosity coefficient. Section V deals with an analysis of the stability of the linearized hydrodynamic equations. We close the paper in Section VI with a discussion of the results presented.

II. BOLTZMANN EQUATION AND CONSERVATION LAWS

We consider a binary mixture of smooth hard spheres of masses $m_1$ and $m_2$, and diameters $\sigma_1$ and $\sigma_2$. The inelasticity of collisions among all pairs is characterized by three independent constant coefficients of normal restitution $\alpha_{11}$, $\alpha_{22}$, and $\alpha_{12} = \alpha_{21}$, where $\alpha_{ij}$ is the restitution coefficient for collisions between particles of species $i$ and $j$. In the low-density regime, the distribution functions $f_i(r, v; t)$ ($i = 1, 2$) for the two species are determined from the set of nonlinear Boltzmann equations [2]

$$ (\partial_t + v_1 \cdot \nabla) f_i = \sum_j J_{ij} [v_1 | f_i(t), f_j(t)] , \tag{1} $$

where the Boltzmann collision operator $J_{ij} [v_1 | f_i, f_j]$ describing the scattering of pairs of particles is

$$ J_{ij} [v_1 | f_i, f_j] = \sigma_{ij}^2 \int dv_2 \int d\hat{\sigma} \Theta (\hat{\sigma} \cdot g_{12}) (\hat{\sigma} \cdot g_{12}) 
\times [\alpha_{ij}^{-2} f_i(r, v'_1, t) f_j(r, v'_2, t) - f_i(r, v_1, t) f_j(r, v_2, t)] , \tag{2} $$

where $\sigma_{ij} = (\sigma_i + \sigma_j) / 2$, $\hat{\sigma}$ is a unit vector along their line of centers, $\Theta$ is the Heaviside step function, and $g_{12} = v_1 - v_2$. The primes on the velocities denote the initial values $\{v'_1, v'_2\}$ that lead to $\{v_1, v_2\}$ following a binary collision:

$$ v'_1 = v_1 - \mu_{ij} (1 + \alpha_{ij}^{-1}) (\hat{\sigma} \cdot g_{12}) \hat{\sigma}, \quad v'_2 = v_2 + \mu_{ij} (1 + \alpha_{ij}^{-1}) (\hat{\sigma} \cdot g_{12}) \hat{\sigma}, \tag{3} $$

where $\mu_{ij} = m_i / (m_i + m_j)$. 
The relevant hydrodynamic fields are the number densities \( n_i \), the flow velocity \( \mathbf{u} \), and the “granular” temperature \( T \). They are defined in terms of moments of the distributions \( f_i \) as

\[
n_i = \int d\mathbf{v} f_i(\mathbf{v}) , \quad \rho \mathbf{u} = \sum_i \int d\mathbf{v} m_i \mathbf{v} f_i(\mathbf{v}) ,
\]

\[
n_i T = p = \sum_i \int d\mathbf{v} \frac{m_i}{2} V^2 f_i(\mathbf{v}) ,
\]

where \( n = n_1 + n_2 \) is the total number density, \( \rho = m_1 n_1 + m_2 n_2 \) is the total mass density, \( p \) is the hydrostatic pressure, and \( \mathbf{V} = \mathbf{v} - \mathbf{u} \) is the peculiar velocity. At a kinetic level, it is convenient to introduce the kinetic temperatures \( T_i \) for each species defined as

\[
\frac{3}{2} n_i T_i = \int d\mathbf{v} \frac{m_i}{2} V^2 f_i ,
\]

The collision operators conserve the particle number of each species and the total momentum but the total energy is not conserved:

\[
\int d\mathbf{v} J_{ij}[\mathbf{v}|f_i, f_j] = 0 , \quad \sum_{i,j} \int d\mathbf{v} m_i \mathbf{V} J_{ij}[\mathbf{v}|f_i, f_j] = 0 ,
\]

\[
\sum_{i,j} \int d\mathbf{v} \frac{1}{2} m_i V^2 J_{ij}[\mathbf{v}|f_i, f_j] = - \frac{3}{2} n T \zeta ,
\]

where \( \zeta \) is identified as the cooling rate due to inelastic collisions among all species. The macroscopic balance equations follow from the Boltzmann equation (1) and Eqs. (7) and (8). They are given by

\[
D_t n_i + n_i \nabla \cdot \mathbf{u} + \nabla \cdot m_i \frac{\mathbf{j}_i}{m_i} = 0 ,
\]

\[
D_t \mathbf{u} + \rho^{-1} \nabla P = 0 ,
\]

\[
D_t T - \frac{T}{n} \sum_i \nabla \cdot \mathbf{j}_i + \frac{2}{3n} (\nabla \cdot \mathbf{q} + P : \nabla \mathbf{u}) = -\zeta T .
\]

In the above equations, \( D_t = \partial_t + \mathbf{u} \cdot \nabla \) is the material derivative,

\[
\mathbf{j}_i = m_i \int d\mathbf{v} \mathbf{V} f_i(\mathbf{v})
\]

is the mass flux for species \( i \) relative to the local flow,

\[
P = \sum_i \int d\mathbf{v} m_i \mathbf{V} \mathbf{V} f_i(\mathbf{v})
\]

is the total pressure tensor, and

\[
\mathbf{q} = \sum_i \int d\mathbf{v} \frac{1}{2} m_i V^2 \mathbf{V} f_i(\mathbf{v})
\]

is the total heat flux.

The utility of the balance equations (9)–(11) is limited without further specification of the fluxes and the cooling rate, which in general have a complex dependence on space and time. However, for sufficiently large space and time scales, one expects that the system reaches a hydrodynamic regime in which all the space and time dependence is
given entirely through a functional dependence on the hydrodynamic fields \( n_i, \mathbf{u}, \) and \( T \). The corresponding functional dependence of \( j_i, P, \) and \( q \) on these fields are called constitutive equations and define the transport coefficients of the mixture. The primary feature of a hydrodynamic description is the reduction of the description from many microscopic degrees of freedom to a set of equations involving only five local fields. At a kinetic level, the constitutive equations are obtained when one admits the existence of a normal solution to the Boltzmann equation where the velocity distribution function depends on \( \mathbf{r} \) and \( t \) only through its functional dependence on the fields, namely,

\[
f_i(\mathbf{r}, \mathbf{v}_1, t) = f_i[\mathbf{v}_1, n_1(\mathbf{r}, t), T(\mathbf{r}, t), \mathbf{u}(\mathbf{r}, t)].
\]  

(15)

The Chapman-Enskog method \(^3\) generates this normal solution explicitly by means of an expansion in gradients of the fields:

\[
f_i = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \cdots,
\]

(16)

where \( \epsilon \) is a formal parameter measuring the nonuniformity of the system. The local reference state \( f_i^{(0)} \) is chosen to give the same first moments as the exact distribution \( f_i \). The time derivatives of the fields are also expanded as \( \partial_t = \partial_t^{(0)} + \epsilon \partial_t^{(1)} + \cdots \). The coefficients of the time derivative expansion are identified from the balance equations (9)–(11) after expanding the fluxes, and the cooling rate \( \zeta \) in a similar series as (10). We close this section with some comments about the partial temperatures \( T_i \) introduced in Eq. (9). In contrast to the global temperature \( T \) these partial temperatures are not hydrodynamic fields and do not appear directly in the balance equations (9)–(11) and in the normal solution (14). However, as indicated in the next section, these partial temperatures characterize differences in the velocity distributions for the different species and therefore are important for their quantitative description. Ultimately, the transport coefficients obtained depend on the ratio \( T_1/T_2 \) which is expressed in terms of the mechanical properties of the particle collisions, through the requirement that the solution be normal.

### III. LOCAL HOMOGENEOUS COOLING STATE

To zeroth order in \( \epsilon \), the kinetic equations (11) become

\[
\partial_t^{(0)} f_i^{(0)} = \sum_j J_{ij}[f_i^{(0)}, f_j^{(0)}].
\]

(17)

The normal form of \( f_i^{(0)} \) requires that all its time dependence occurs only \( n_i, \mathbf{u}, \) and \( T \). Consequently, \( f_i^{(0)} \) must be of the form

\[
f_i^{(0)}(\mathbf{v}) = n_i v_0^{-3} \Phi_i (V/v_0),
\]

(18)

where \( v_0^2(t) = 2T(t)/(m_1 + m_2)/(m_1 m_2) \) is a thermal velocity defined in terms of the temperature \( T(t) \) of the mixture. The balance equations (9)–(11) to this order become \( \partial_t^{(0)} x_i = \partial_t^{(0)} u = 0, T^{-1} \partial_t^{(0)} T = -\zeta^{(0)} \). Here, \( x_i = n_i/n \) is the mole fraction of species \( i \) and the cooling rate \( \zeta^{(0)} \) is determined by Eq. (8) to zeroth order. Hence, all the time dependence to this order is through the temperature and so, in dimensionless form, Eq. (18) becomes the following time independent equation for \( \Phi_i \):

\[
\frac{1}{2} \zeta^* \frac{\partial}{\partial V_i^*} \cdot (V_i^* \Phi_i) = \sum_j J_{ij}^* [\Phi_i, \Phi_j],
\]

(19)

where \( V^* = V/v_0, \zeta^* = \zeta_i n \sigma_{12}^2 v_0 \), and \( J_{ij}^* = v_0^2 J_{ij}/n \sigma_{12}^2 \). Since the distribution functions are isotropic, it follows from Eqs. (12)–(13) that the zeroth order mass and heat fluxes vanish while, for the same reason, the momentum flux is diagonal with a coefficient that is just the sum of the partial pressures, i.e.,

\[
J_i^{(0)} = 0, \quad q^{(0)} = 0, \quad \rho_{ij}^{(0)} = \rho \delta_{ij}.
\]

(20)

In summary, Eq. (18) provides the normal solution to lowest order in the gradients and use of Eq. (20) in the balance equations gives the corresponding Euler level hydrodynamics. Note that \( \zeta^{(0)} \) is determined self-consistently in the solution to Eq. (18).

Just as for the one component gas case\(^{17,18}\), the exact form of \( \Phi_i \) has not yet been found, although a good approximation for thermal velocities can be obtained from an expansion in Sonine polynomials\(^5\). In the leading order, \( \Phi_i \) is given by

\[
\Phi_i(V^*) \rightarrow \left( \frac{\theta_i}{\pi} \right)^{3/2} e^{-\theta_i V^2} \left[ 1 + \frac{C_i}{4} \left( \theta_i V^4 - 5\theta_i V^2 + \frac{15}{4} \right) \right].
\]

(21)
FIG. 1: Plot of the coefficients $c_i$ versus $\alpha$ for $\sigma_1/\sigma_2 = 1$, $x_1 = \frac{1}{2}$ and $m_1/m_2 = 2$. The solid line and the circles refer to $c_1$ while the dashed line and the squares correspond to $c_2$. The dotted line and the triangles refer to the common value in the monocomponent case. The lines are the theoretical results and the symbols are the simulation results.

Here, $\theta_i = (m_i/\gamma_i) \sum_j m_j^{-1}$ and $\gamma_i = T_i/T$. The coefficients $c_i$ (which measure the deviation of $\Phi_i$ from the reference Maxwellian) are determined consistently from the Boltzmann equation. An important observation is that the partial temperatures have been introduced here since they provide the actual mean kinetic energies of the distributions through Eq. (6). This would appear to contradict the assumption of a normal solution by introducing additional variables in the distribution functions. To see that this is not the case, the time evolution for the temperature ratio $\gamma = T_1/T_2$ is calculated to be

$$\frac{\partial_t}{\partial t} \ln \gamma = \zeta_2^{(0)} - \zeta_1^{(0)}$$

(22)

where we have introduced the cooling rates $\zeta_i^{(0)}$ for the partial temperatures $T_i$ as

$$\zeta_i^{(0)} = -\frac{2}{3n_i T_i} \sum_j \int dV \frac{1}{2} m_i V^2 J_{ij}[v|f_i^{(0)}, f_j^{(0)}].$$

(23)

The total cooling rate $\zeta^{(0)}$ can be expressed in terms of the coolings $\zeta_i^{(0)}$ as

$$\zeta^{(0)} = T^{-1} \sum_i x_i T_i \zeta_i^{(0)}.$$  

(24)

The fact that $f_i$ depends on time only through $T(t)$ necessarily implies that the temperature ratio $\gamma$ must be independent of time and so, Eq. (22) gives the condition $\zeta_1^{(0)} = \zeta_2^{(0)} = \zeta^{(0)}$. In the elastic case, where $f_i^{(0)}$ is a local Maxwellian, the above condition yields $T_1(t) = T_2(t) = T(t)$ and the energy equipartition applies. However, in the inelastic case, the equality of the cooling rates leads to different values for the partial temperatures, even if one considers the Maxwellian approximation to $f_i^{(0)}$. Nevertheless, the constancy of $\gamma$ assures that the time dependence of the distribution is entirely through $T(t)$, and in fact the condition of equal cooling rates assures that the partial temperatures can be expressed in terms of $T$.

The explicit form of the approximation (21) provides detailed predictions for the temperature ratio $T_1/T_2$ (through calculation of the cooling rates) and for the cumulants $c_i$ [from Eq. (19)] as functions of the mass ratio, size ratio, composition and restitution coefficients[5]. This completely fixes the approximate distribution function. In Fig. 1 we show the dependence of $c_1$ and $c_2$ on the (common) restitution coefficient $\alpha_{ij} \equiv \alpha$ for $\sigma_1/\sigma_2 = 1$, $x_1 = \frac{1}{2}$, and $m_1/m_2 = 2$. We also present the corresponding results obtained from a numerical solution[9] of the Boltzmann equation by means of the DSMC method[16]. The agreement between the simulation data and the theoretical results is excellent. Further, the small values of the coefficients $c_i$ supports the assumption of a low-order truncation in the polynomial expansion of the distribution functions. One of the main findings of our theory is that, except for mechanically equivalent particles, the partial temperatures are in general different ($\gamma \equiv T_1/T_2 \neq 1$). This conclusion
FIG. 2: Plot of the temperature ratio $\gamma \equiv T_1/T_2$ versus $\alpha$ for $\sigma_1/\sigma_2 = 1$, $x_1 = \frac{2}{3}$ and three different values of the mass ratio: $m_1/m_2 = 0.1$ (dotted line and triangles), $m_1/m_2 = 2$ (dashed line and squares), and $m_1/m_2 = 10$ (solid line and circles). The lines are the theoretical results and the symbols are the simulation results.

contrasts with all previous results\[4]\ derived for granular mixtures, where it was implicitly assumed the equipartition of granular energy between both species (i.e., $\gamma = 1$). The breakdown of the energy equipartition has been confirmed by computer simulations\[6, 7, 10, 11]\ and by real experiments of vibrated mixtures in two\[12]\ and three\[13]\ dimensions. For the sake of illustration, in Fig. 2 we plot the temperature ratio $\gamma$ versus $\alpha$ for $\sigma_1/\sigma_2 = 1$, $x_1 = \frac{2}{3}$, and several values of the mass ratio as given by the first Sonine approximation and by Monte Carlo simulations. The comparison between theory and simulation data shows an excellent agreement, even for strong dissipation. We also observe that for large differences in the mass ratio, the temperature differences are quite important.

IV. NAVIER-STOKES HYDRODYNAMIC EQUATIONS

Implementation of the Chapman-Enskog method to the first order in $\epsilon$ has been carried out and without approximation recently\[14]. Technical details of these calculations will not be repeated here and only the main results are offered. The solution of the Boltzmann equation to first order in the spatial gradients can be written as

\[ f^{(1)}_1 = A_i \cdot \nabla x_1 + B_i \cdot \nabla p + C_i \cdot \nabla T + D_{i,k\ell} \nabla u_{k\ell}. \]  

(25)

The coefficients $A_i$, $B_i$, $C_i$, and $D_{i,k\ell}$ are functions of the peculiar velocity and the hydrodynamic fields. These coefficients obey inhomogeneous integral equations involving the linearized Boltzmann operators. Solubility conditions for the existence of solutions has been also proven. The corresponding constitutive equations found to this order are

\[ j^{(1)}_1 = -\frac{m_1 m_2 n}{\rho} D \nabla x_1 - \frac{\rho}{p} D_p \nabla p - \frac{\rho}{T} D' \nabla T, \quad j^{(1)}_2 = -j^{(1)}_1, \]  

(26)

\[ q^{(1)} = -T^2 D'' \nabla x_1 - L \nabla p - \Lambda \nabla T, \]  

(27)

\[ P^{(1)}_{k\ell} = -\eta \left( \nabla_{\ell} u_k + \nabla_{k} u_\ell - \frac{2}{3} \delta_{k\ell} \nabla \cdot u \right). \]  

(28)

Furthermore, there is not contribution to the cooling rate at this order, i.e., $\zeta^{(1)} = 0$. This is a consequence of the symmetry of the distribution function $f^{(1)}_1$ since the latter does not contain any contribution proportional to the divergence of the flow velocity field. This property is special to the low density Boltzmann kinetic theory and such terms occur at higher densities\[13]. The transport coefficients appearing in Eqs. (26–28) are the diffusion coefficient
we plot the ratio be measured from simulations and compared with the predictions given by the Chapman-Enskog method. In Fig. 3, that the shearing work still heats the system. In the long time limit, the Navier-Stokes shear viscosity coefficient can state of uniform shear flow. An external thermostat is introduced to compensate for the energy lost in collisions so contributions were not taken into account in previous work on granular mixtures. The accuracy of the approximate results obtained from the Boltzmann equation in the first Sonine approximation. We see that in general the deviation

\begin{equation}
D_{x} x_1 + \frac{\rho}{n^2 m_{1} m_{2}} \nabla \cdot j_1 = 0,
\end{equation}

\begin{equation}
D_{x} p + p \nabla \cdot u + \frac{2}{3} (\nabla \cdot q + P : \nabla u) = -\zeta p.
\end{equation}

Therefore, when the expressions of the fluxes and the cooling rate $\zeta \rightarrow \zeta^{(0)}$ are substituted into the balance equations and one gets a closed set of hydrodynamic equations for $x_1$, $u$, $T$, and $p$. These are the Navier-Stokes equations for a mixture, which are given by

\begin{equation}
(D_t + \zeta) p + \frac{5}{3} p \nabla \cdot u = \frac{2}{3} \nabla \cdot (T^2 D'' \nabla x_1 + L \nabla p + \lambda \nabla T) + \frac{2}{3} \eta (\nabla \epsilon u_k + \nabla k u_\ell - \frac{2}{3} \delta_{k\ell} \nabla \cdot u) \nabla \epsilon u_k,
\end{equation}

\begin{equation}
(D_t + \zeta) T + \frac{5}{3} p \nabla \cdot u = -\frac{T}{n} \left(1 - \frac{1}{m_1} \right) \nabla \cdot \left(\frac{m_1 m_2 n}{\rho} D \nabla x_1 + \frac{D_p}{p} \nabla p + \frac{D'}{T} D' \nabla T \right) + \frac{2}{3n} \nabla \cdot (T^2 D'' \nabla x_1 + L \nabla p + \lambda \nabla T) + \frac{2}{3n} \eta (\nabla \epsilon u_k + \nabla k u_\ell - \frac{2}{3} \delta_{k\ell} \nabla \cdot u) \nabla \epsilon u_k,
\end{equation}

\begin{equation}
D_{t} u_\ell + \nabla \epsilon u_p = \rho^{-1} \eta (\nabla \epsilon u_k + \nabla k u_\ell - \frac{2}{3} \delta_{k\ell} \nabla \cdot u).
\end{equation}

In general, the influence of dissipation on the transport coefficients is quite important even for moderate values of the restitution coefficient (say $\alpha \approx 0.9$). Further, the effect of the additional contributions to the transport coefficients coming from the partial temperature differences can be quite significant, even for weak dissipation. These new contributions were not taken into account in previous work on granular mixtures. The accuracy of the approximate Sonine solution to the resulting integral equations has been recently tested at the level of the shear viscosity coefficient. Specifically, the DSMC method has been used to solve the Boltzmann equation in the special hydrodynamic state of uniform shear flow. An external thermostat is introduced to compensate for the energy lost in collisions so that the shearing work still heats the system. In the long time limit, the Navier-Stokes shear viscosity coefficient can be measured from simulations and compared with the predictions given by the Chapman-Enskog method. In Fig. 3, we plot the ratio $\eta^*(\alpha)/\eta^*(1)$ versus the mass ratio $m_1/m_2$ for $\sigma_1/\sigma_2 = 1$, $x_1 = 1/2$, and three different values of the (common) restitution coefficient $\alpha$. Here, $\eta^* = \eta nT$, with $\nu = \sqrt{\pi n \sigma_1^2 v_0}$ and $\eta^*(1)$ refers to the elastic value of the shear viscosity coefficient. Again, the symbols represent the simulation data while the lines refer to the theoretical results obtained from the Boltzmann equation in the first Sonine approximation. We see that in general the deviation
of $\eta^*(\alpha)$ from its functional form for elastic collisions is quite important. This tendency becomes more significant as the mass disparity increases. The agreement between the first Sonine approximation and simulation is seen to be in general excellent. This agreement is similar to the one previously found in the monocomponent case\cite{20, 21}. At a quantitative level, the discrepancies between theory and simulation tend to increase as the restitution coefficient decreases, although these differences are quite small (say, for instance, around 2% at $\alpha = 0.7$ in the disparate mass case $m_1/m_2 = 10$). The influence of the size ratio on the shear viscosity is shown in Fig. 4 for $m_1/m_2 = 4$ and $x_1 = \frac{1}{2}$. We observe again a strong dependence of the shear viscosity on dissipation. However, for a given value of $\alpha$, the influence of $\sigma_1/\sigma_2$ on $\eta^*$ is weaker than the one found before in Fig. 3 for the mass ratio. The agreement for
both $\alpha = 0.9$ and $\alpha = 0.8$ is quite good, except for the largest size ratio at $\alpha = 0.8$. These discrepancies become more significant as the dissipation increases (say $\alpha = 0.7$), especially for mixtures of particles of very different sizes. In summary, according to the comparison carried out in Figs. 3 and 4, one can conclude that the agreement between theory and simulation extends over a wide range values of the restitution coefficient, indicating the reliability of the first Sonine approximation for describing granular flows beyond the quasielastic limit.

V. LINEARIZED HYDRODYNAMIC EQUATIONS AND STABILITY

The simplest application of the hydrodynamic equations is to small initial perturbations about the strictly homogeneous state for a large system. Then the nonlinear equations can be expanded to linear order in the deviations of $x_1, u, T,$ and $p$ about their homogeneous values. This leads to partial differential equations with coefficients that are independent of space but they depend on time since the reference state is cooling. As in the monocomponent case, this time dependence can be eliminated through a change in the time and space variables, and a scaling of the hydrodynamic fields. Let $\delta y_0(r,t) = y_0(r,t) - y_H(t)$ denote the deviation of $\{x_1, u, T, p\}$ from their values in the homogeneous state. We introduce the following dimensionless space and time variables:

$$\tau = \int_0^t dt' \nu_H(t'), \quad \xi = \frac{\nu_H(t)}{v_0H(t)} r,$$

where $v_0H(t)$ and $\nu_H(t)$ has been introduced before. The superscript $H$ indicates that the quantity is evaluated in the homogeneous cooling state. This means that

$$\partial_\tau x_{1H} = 0, \quad u_H = 0, \quad \partial_\tau \ln T_H = \partial_\tau \ln p_H = -\zeta_H$$

A set of Fourier transformed dimensionless variables are then defined by

$$\delta \bar{y}_{k\alpha}(\tau) = \int d\xi e^{-ik\xi} \delta y_0(\xi, \tau),$$

where $\delta y_{k\alpha}(\tau)$ denotes the four variables $(\rho_k, \theta_k, \Pi_k)$. These variables separate into a degenerate pair of equations for the transverse velocity components $w_{k\perp}$ (orthogonal to $k$)

$$(\partial_{\tau^*} - \frac{\zeta^*}{2} + \eta^* k^2) w_{k\perp} = 0,$$

and a coupled set of equations for $\rho_k, \theta_k, \Pi_k$, and the longitudinal velocity component $w_{k\parallel}$ (parallel to $k$)

$$\frac{\partial \delta y_{k\alpha}(\tau)}{\partial \tau} = \left( M_{\alpha\beta}^{(0)} + i k M_{\alpha\beta}^{(1)} + k^2 M_{\alpha\beta}^{(2)} \right) \delta y_{k\beta}(\tau),$$

where now $\delta y_{k\alpha}(\tau)$ denotes the four variables $(\rho_k, \theta_k, \Pi_k, w_{k\parallel})$. The matrices in this equation are

$$M^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_1 (\frac{\partial^*}{\partial \tau^*})_{T,p} & \frac{\zeta^*}{2} & -\zeta^* & 0 \\ -x_1 (\frac{\partial^*}{\partial \tau^*})_{T,p} & \frac{\zeta^*}{2} & -\zeta^* & 0 \\ 0 & 0 & 0 & \zeta^* \end{pmatrix},$$

$$M^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \frac{\mu^4}{\mu_{x1} + \mu_{x2}} & 0 & 0 \end{pmatrix},$$

$$M^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
where positive for $k < k^*$ and they are not restricted to specific values of the parameters of the mixture. The explicit knowledge of the transport coefficients is obtainable by taking the leading terms in a Sonine polynomial expansion. In contrast to previous studies [4], our results apply for arbitrary degree of dissipation.

Equation (40) is decoupled from the rest and can be integrated directly leading to

$$M^{(2)} = \begin{pmatrix}
-x_1 \left( \frac{2}{3} D''_s - \frac{1-\mu}{x_1 \mu + x_2} D_s \right) & -x_1^{-1} D'_s - \frac{2}{3} \lambda^* & -x_1^{-1} D'_p \left( \mu \right) & 0 \\
-x_1^{-1} D'_s - \frac{2}{3} \lambda^* & -\frac{2}{3} \lambda^* + \frac{1-\mu}{x_1 \mu + x_2} D'_p \left( \mu \right) & 0 & 0 \\
-\frac{2}{3} x_1 D''_s & 0 & 0 & -\frac{4}{3} \eta^* \\
0 & 0 & 0 & -\frac{4}{3} \eta^*
\end{pmatrix}.$$  

(44)

In these equations, $\mu = m_1 / m_2$, $x_i = n_i H / n_H$, $\zeta^* = \zeta^*_H / \nu_H$. Moreover, we have introduced the reduced Navier-Stokes transport coefficients

$$D^* = \frac{\nu_H D}{n_H v_0^2}, \quad D'_p = \frac{\rho_H^2 \nu_H D_p}{m_1 n_2 v_H^2 v_0^2}, \quad D''_s = \frac{\rho_H^2 \nu_H D''}{m_1 n_2 v_H^2 v_0^2},$$  

(45)

$$\eta^* = \frac{\nu_H \eta}{\rho_H v_0^2},$$  

(46)

$$D''_s = \frac{\nu_H T_H D''}{n_H v_0^2}, \quad L^* = \frac{\nu_H L}{v_0^2}, \quad \lambda^* = \frac{\nu_H \lambda}{n_H v_0^2}.$$  

(47)

Equation (40) is decoupled from the rest and can be integrated directly leading to

$$w_{k,\perp}(\tau) = w_{k,\perp}(0) \exp(s_{\perp} \tau),$$  

(48)

where

$$s_{\perp} = \frac{1}{2} \zeta^* - \eta^* k^2.$$  

(49)

This identifies two shear modes. The remaining modes have the form $\exp(s_n \tau)$ for $n = 1, 2, 3, 4$, where $s_n$ are the solutions of the corresponding quartic equation. A detailed study on the dependence of the hydrodynamic modes on the inelasticity and the parameters of the mixture will be given elsewhere. The linear hydrodynamic equations characterize the stability of the homogeneous cooling state. We see from Eq. (40) that $s_{\perp}$ for the two shear modes is positive for $k < k^*_{\perp}$, where

$$k^*_{\perp} = \left( \frac{\zeta^*}{2 \eta^*} \right)^{1/2}.$$  

(50)

Thus, initial long wavelength perturbations of the HCS that for instance, excite the shear mode will grow exponentially, representing an instability of the reference state.

The wave vector dependence of the remaining four modes is more complex. However, the long wavelength stability can be obtained from the eigenvalues of $M^{(0)}$ which are easily found to be

$$s_n = \left( -\frac{1}{2} \zeta^*, 0, 0, \frac{1}{2} \zeta^* \right).$$  

(51)

Hence, there is another unstable mode. In summary, at asymptotically long wavelengths the spectrum of the linearized hydrodynamic equations is comprised of a decaying mode at $-\zeta^*/2$, a two-fold degenerate mode at 0, and a three-fold degenerate unstable mode at $\zeta^*/2$.

VI. DISCUSSION

The primary objective of this review has been to obtain the hydrodynamic description of a binary mixture of granular gases from the Boltzmann kinetic theory. The Chapman-Enskog method is used to solve the Boltzmann equation up to the Navier-Stokes order and the associated transport coefficients are given in terms of the solutions of a set of linear integral equations. A practical evaluation of these coefficients is possible by taking the leading terms in a Sonine polynomial expansion. In contrast to previous studies [3], our results apply for arbitrary degree of dissipation and they are not restricted to specific values of the parameters of the mixture. The explicit knowledge of the transport
coefficients and the cooling rate allows one to perform a study of the linearized hydrodynamic equations around the homogenous cooling state. The stability analysis shows that the homogeneous cooling state is unstable to long enough wavelength perturbations.

The reference state in our Chapman-Enskog expansion has been taken to be an exact solution of the uniform Boltzmann equation. An interesting and important result of this solution is that the partial temperatures of each species must be different. This does not mean that there are additional hydrodynamic degrees of freedom since their cooling rates are equal and consequently, the partial temperatures can be expressed in terms of the global temperature. However, the relationships between these partial temperatures are functions of the mole fractions (composition) and lead to new contributions (not considered in previous works) to the transport coefficients. The consequences of this new effect on transport are quite significant.

The dependence of the transport coefficients on the full parameter space has been explored in the case of the shear viscosity $\eta$. In addition, these theoretical predictions have been compared with the results obtained from a numerical solution of the Boltzmann equation by means of the DSMC method. The theory and simulation clearly show that in general, the influence of dissipation on $\eta$ is quite important since there is a relevant dependence of the shear viscosity on the restitution coefficient. With respect to the accuracy of the Chapman-Enskog results we see that the first Sonine approximation exhibits an excellent agreement with the simulation data. This supports the idea that the Sonine polynomial approximation for granular fluids has an accuracy comparable to that for elastic collisions. Exceptions to this agreement are extreme mass ratios and strong dissipation. These discrepancies are due basically to the approximations introduced in applying the Chapman-Enskog, and more specifically in using the first Sonine approximation.

Although the utility of a hydrodynamic description for granular media under rapid flow conditions has been recognized for many years, there are some doubts about the possibility of going from a kinetic theory to a hydrodynamic level of description by using a Chapman-Enskog expansion around the homogeneous cooling state. Given that the search for exact solutions of the Boltzmann equation is far beyond the present perspectives, an alternative to get some insight into the above question is to numerically solve the kinetic equation by using for instance the DSMC method and compare these results with the corresponding solution obtained by assuming the validity of a hydrodynamic description. The good agreement obtained here for the temperature ratio and the shear viscosity coefficient shows clearly the direct verification of hydrodynamics and the quantitative predictions for transport coefficients from kinetic theory for states with small spatial gradients but including strong dissipation.

One of the main limitations of the results obtained here from the Boltzmann equation is its restriction to the low-density regime. In this regime, the collisional transfer contributions to the fluxes are negligible and only their kinetic contributions are taken into account. Possible extension in both aspects, theory and simulation, of the present hydrodynamic description to higher densities can be carried out in the context of the revised Enskog theory. In this case, many of the phenomena appearing in dense granular fluids (such as spontaneous formation of dense clusters surrounded by regions of low-density) could be studied.

The instability of the HCS found in the previous section may also seem to represent a limitation on the validity of hydrodynamics. However, it should be noted that the Chapman-Enskog expansion is not an expansion about the HCS but rather its local form, parametrized by the exact hydrodynamic fields. These fields obey the nonlinear hydrodynamic equations which are not unstable, but rather they describe the correct subsequent evolution of the unstable linear equations. The linearized hydrodynamics is indeed limited to time scales short compared to the growth of the linear perturbations. Nevertheless, these equations describe correctly the nature of the onset of the instability. Since it is a long wavelength phenomenon, the instability can be suppressed completely for sufficiently small system sizes.

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