Critical behavior near the many-body localization transition in driven open systems

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In a many-body localized (MBL) system, the coupling to an external bath typically breaks local integrals of motion. Thus the system relaxes to a unique thermal steady state. When the bath is non-thermal or when the system is weakly driven out of equilibrium, local conservation laws can be excited far from any thermal equilibrium value. We show how this property can be used to study the many-body localization phase transition in weakly open systems. Here, the strength of the coupling to the non-thermal bath plays a similar role as a finite temperature in a \( T = 0 \) quantum phase transition. By tuning this parameter, we can detect key features of the MBL transition: the divergence of dynamical exponent due to Griffiths effects in one dimension and the critical disorder strength. On the ergodic side of the MBL phase transition, fluctuations in the local temperature grow with a fractional exponent related to the inverse dynamical exponent. On the MBL side they decrease monotonically. We apply these ideas to study the MBL critical point numerically using a truncated matrix-product operator solution to the steady state of a Lindblad equation.

Many-body localization is a state of interacting quantum systems, which fail to thermalize subject to their intrinsic dynamics due to the effect of strong disorder \( \epsilon \). A pertinent question, currently under intense theoretical study and debate, concerns the nature of the phase transition between the ergodic and localized phases. This transition represents a new class of dynamical quantum phase transitions, which involves a fundamental change of the entanglement structure in all, or at least many, of the eigenstates.

Unlike equilibrium phase transitions, the many-body localization transition is sharp only if the system is completely isolated, which imposes severe limitations on the ability to study it using standard theoretical, numerical, and experimental approaches. In particular, the requirement of a closed system appears to preclude experiments with solid state materials, which, due to coupling to a thermal phonon bath cannot be many-body localized. Even in experiments with small systems of ultra-cold atoms and ion traps, which are usually considered to be exquisitely isolated, signatures of many-body localization are visibly polluted by extrinsic decay processes that may mask the critical point \( \epsilon = 0 \). At the same time, numerical experiments are also severely limited. Because of the need to address closed system dynamics, these have been mostly restricted to exact diagonalization of very small systems \( S \leq 14 \). There is increasing evidence that such simulations are overwhelmed by transient finite-size effects that supersede the critical scaling behavior \( \epsilon \approx 0 \).

In this paper, we propose to bypass the limitations posed by closed systems by studying sharp signatures of the MBL transition in weakly open driven systems. In a previous work \( \epsilon = 0 \), some of us showed that in the limit of vanishing coupling to a bath \( \epsilon \) and concomitantly weak drive strength \( \epsilon \theta \), the MBL transition shows up as a singular change in the temperature variations across the sample. On the thermalizing side of the critical point (i.e. for sufficiently weak disorder) the temperature fluctuations vanish in the limit \( \epsilon \to 0 \), while they remain finite on the MBL side. At small, but non-vanishing, coupling to the external baths one expects this transition to broaden into a universal crossover governed by the critical point located at \( \epsilon = 0 \). Thus, a finite external coupling \( \epsilon \) has a role similar to turning on a non zero temperature above a \( T = 0 \) quantum phase transition. Studying the leading dependence of the spatial temperature fluctuations on \( \epsilon \) in the vicinity of the critical point is analogous to studying the leading dependence of the order parameter on the temperature in a conventional quantum phase transition. Similarly such a measurement could allow to determine critical exponents as well as the critical disorder strength. Furthermore, the MBL transition in one dimensional systems is thought to be preceded by a thermal Griffiths regime, a scenario that explains observed subdiffusive transport \( \epsilon > 0 \). We argue, using an effective model of the Griffiths phase, that measuring the leading \( \epsilon \) dependence of the temperature variations can reveal the continuously varying dynamical exponent in this regime.

The open systems approach suggests the use of new computational techniques to investigate the MBL transition without the limitations of exact diagonalization. In this paper we demonstrate a calculation of the spatial temperature fluctuations at steady state as a function of a coupling strength \( \epsilon \) to a set of Lindblad operators leading to a non equilibrium steady state. The relative coupling to the different Lindblad terms can be arranged such that the steady state density matrix is well described by a matrix product operator with low bond dimension, allowing efficient computation. We find a sharp signature of the the Griffith regime with a continuously varying dynamical exponent that diverges at the critical point. Before proceeding we note the connection to Ref. \( \epsilon \approx 0 \), where Griffiths exponents have been computed numerically for a similar spin-chain coupled to Lindblad operators placed at the two ends of the chain to drive a
steady state current. Because in our case the coupling to the drive and the bath are in the bulk the calculation can converge faster and we are therefore able to access parameter regimes much closer to the MBL transition.

Hydrodynamic description—The simplest case for obtaining the dependence of the temperature fluctuations on the drive strength $\epsilon$ is the diffusive thermal regime. Assuming for simplicity that energy is the only conserved quantity in the limit $\epsilon \to 0$, we derive the long wavelength temperature variations from an energy diffusion equation, supplemented with a source and sink terms representing the bath and drive respectively,

$$\partial_t \epsilon - \nabla (\kappa(r) \nabla T(r)) = -\epsilon g_1(r)(T(r)-T_0) + \epsilon \theta g_2(r)$$

(1)

The disorder is represented in this model by a weak modulation of the conductivity $\kappa(r)$ and couplings to the external sources $g_{1(2)}(r)$: $\kappa(r) = \bar{\kappa} + \delta \kappa(r)$, $g_{1(2)}(r) = \bar{g} + \delta g_{1(2)}(r)$. The temperature profile varies around the mean value, $T(r) = \bar{T} + \delta T(r)$, where the average effective temperature is $\bar{T} = T_0 + \theta$ is determined by the relative strength of the drive compared to the coupling to the bath.

Linearizing Eq. (1) in the disorder strength gives

$$(-\bar{\kappa} \nabla^2 + \epsilon \bar{g}) \delta T(r) = \epsilon \theta \delta g(r)$$

(2)

where $\delta g(r) = \delta g_{1(2)}(r) - \delta g_1(r)$. This equation is solved for the local temperature variations using the Green’s function, $\delta T(r) = \epsilon \int d\tau \delta G(r-r') \delta g(r')$, which in momentum space is given by $G(k) = (\bar{\kappa} k^2 + \bar{g} \epsilon)^{-1}$. We can generalize to the sub-diffusive regime heuristically by modifying the Green’s function to $G(k) = \langle \tilde{g} | k |^z + \bar{g} \epsilon \rangle^{-1}$ in order to impose dynamical scaling with an exponent $z \geq 2$. Now, assuming Gaussian disorder with short range correlations, $(\delta g(r) \delta g(r')) \equiv \langle \delta g^2 \rangle \delta(r-r')$, we find in d dimensions,

$$\delta T \sim \theta |\delta g/\bar{g}| \langle \delta g/\bar{g} \rangle^{d/2z} \epsilon^{d/2z}$$

(3)

We note however, that using a disorder averaged Green’s function to get this result may not be properly accounting for the effect of rare regions that dominate the transport in the Griffiths regime. Below we compute the temperature variations in a minimal model that takes this physics into account.

Thermal resistor network – As a minimal model for the Griffiths regime we consider a chain of conducting islands, each characterized by its own temperature $T_i$, coupled by links representing insulating regions of size $\ell$. The link lengths are drawn from the probability distribution $P(\ell) = \frac{1}{\ell} e^{-\ell/\xi}$, where $\xi$ is identified with the correlation length that diverges toward the MBL critical point. Each insulating region gives rise to a conductance exponentially small in its length $\Gamma_{\text{ins}}(\ell) = \Gamma_0 e^{-\ell/a}$, where $a$ is a microscopic length scale. Without the coupling to the drive and the bath, the distribution of insulating links leads to a power-law distribution of inter-island conductances $P(\Gamma) \sim (\Gamma/\Gamma_0)^{\alpha-1}$ with $\alpha = \frac{d}{2}$. The range $0 < \alpha < 1$, for which the average resistance $\langle \Gamma^{-1} \rangle$ diverges corresponds to the subdiffusive regime $[20] [30].$

Coupling this system to a bath and to an energy source ultimately destroys the insulating behavior of the links, adding a channel of conductance through the link with conductivity $\epsilon \Gamma_0$. Thus we take the heat conductance through a link in presence of this coupling to be $\Gamma = \epsilon^2\Gamma_0 + \Gamma_0 e^{-\ell/a}$, with an implicit cutoff $\ell \geq a$.

With the link conductances in hand, we can write the rate equation for the energy transport on the chain

$$\partial_t q_i - \Gamma_{i,i+1}(T_{i+1} - T_i) + \Gamma_{i-1,i}(T_i - T_{i-1}) = -\epsilon q_{i+1} + \epsilon \theta q_i.$$

(4)

This is a discrete version of Eq. (1) with a physically motivated distribution of link conductances. We take $g_{1(2),i} = \bar{g} + \delta g_{1(2),i}$ with $\delta g_{1(2),i}$ drawn from a uniform distribution in the range $[-\delta g_{1(2),i}, \delta g_{1(2),i}]$. We solve for the steady state $\partial_t q_i = 0$ of these rate equations numerically to obtain temperature profiles and extract the normalized variation of local temperatures

$$\frac{\delta T}{T} = \sqrt{\frac{\langle \text{Var}(T_i) \rangle}{\langle T_i \rangle^2}}$$

(5)

Here $\langle \cdot \rangle$ and Var are the sample mean and variance, while $\langle \cdot \rangle$ denotes averaging over disorder realizations. Here we assumed, for simplicity, that the conducting islands are all of similar size.

The results of the numerical solution for $\delta T/T$ as a function of $\epsilon$ are shown in Fig. 1 for different values of $\alpha = a/\xi$. We get $\delta T \sim \epsilon^{1/2z}$ as anticipated in Eq. (3). Furthermore we see that for $0 < \alpha < 1$ the

![FIG. 1. Fluctuations of the local temperatures, $\delta T/T$, are computed from a resistor network model as function of the coupling strength $\epsilon$ to a thermal bath and driving. For small $\epsilon$, temperature fluctuations are described by $\delta T/T \sim \epsilon^{1/2z}$ with $z = \alpha^{-1} = \xi/a$ for $0 < \alpha < 1$, and $z = 2$ in the diffusive regime, $\alpha > 1$. Parameters: $\kappa_0 = 1, \alpha \Gamma_0 = 5.0, T_0 = 1, \delta g_1 = 0.05, \delta g_2 = 0.05, \theta = T_0, N = 1000$, averaged over $M = 500$ configurations.](image-url)
dynamical exponent grows with the correlation length as \( z \sim 1/\alpha = \xi / a \), whereas for \( \alpha > 1 \) it saturates to \( z = 2 \), as expected for a diffusive system. Thus we establish a direct relation between the leading dependence of the temperature fluctuations \( \delta T \) on \( \epsilon \) and the dynamical exponent \( z \), which governs the sub-diffusive behavior in a closed system [37, 38]. Crucially we find that the dynamical exponent diverges together with the correlation length \( \xi \) on approaching the MBL transition [37, 38].

Before proceeding we comment on the behavior of the thermal resistor network in two dimensions. It is shown in the supplementary material that in this case we have \( \delta T \sim \epsilon^{1/4} \), implying \( z = 2 \) for all \( \alpha \). This is also the expected dynamical behavior in a two dimensional closed system because any rare region with large resistance can be short circuited by surrounding smaller resistors [39].

**Charge transport** — In solid state systems it is usually much easier to measure charge transport than the local temperature profile. It is therefore natural to seek signatures of MBL or the Griffiths regimes in the resistance of a weakly open system. In order to compute how the resistance scales with the external bath or drive coupling \( \epsilon \) we consider a charge resistor network described by the rate equations

\[
\partial_t n_i - (\bar{\Gamma}_{i,i+1}(\mu_{i+1} - \mu_i) - \bar{\Gamma}_{i-1,i}(\mu_i - \mu_{i-1})) = 0 \quad (6)
\]

Here \( \mu_i \) is the electro-chemical potential on island \( i \). \( \bar{\Gamma}_{i,j} \) are charge conductances on links, which are distributed exactly as the thermal conductances in Eq. (1). The main difference from the thermal case is that there are no source or sink terms because the external coupling to the bath and the drive are assumed to conserve charge. In fact for these measurements we can consider a system with just a drive or just tunable coupling to phonons. Both terms give rise to a parallel channel of ohmic conductivity proportional to \( \epsilon \) on the insulating links, so that

\[
\bar{\Gamma}(\ell) = \bar{\Gamma}_0 e^{-\ell/a} + \epsilon \sigma_0 / \ell \quad (7)
\]

Comparing the first term to the second we see that the insulating behavior dominates for \( \ell < \ell_* \approx a \ln \epsilon^{-1} + a \ln \ln \epsilon^{-1} \), while the bath or drive induced conductance dominates in longer links. To gain analytic insights we calculate the average resistivity of the chain

\[
\bar{\rho} = \bar{\ell}^{-1} \int d\ell P(\ell) \bar{\Gamma}(\ell)^{-1} \approx \frac{1}{\ell_0} \int_{a}^{\ell_*} d\xi^{-1} e^{\ell(a^{-1}-\xi^{-1})} \approx \frac{\alpha}{\ell_0} \left( \frac{\epsilon}{\ln \epsilon^{-1}} \right)^{\alpha-1} \quad (8)
\]

where \( \alpha = 1/z = a/\xi \) and \( \bar{\ell} = \int d\ell P(\ell) \ell \sim \xi \). A numerical solution, presented in the supplementary information, of the current in a one dimensional chain with a bias voltage confirms this result. Thus it should be possible to measure the dynamical exponent \( z \) by varying the coupling \( \epsilon \) via controlled cooling of the phonon bath or by varying the strength of an external drive.

**Numerical solution of a spin model** — Besides opening a new experimental route, consideration of weakly open driven systems, suggests a new approach for accessing the many-body localization transition numerically. Here we calculate how the local temperature variations in a spin-chain model change with the coupling to a weak drive that brings the system to a non thermal steady state.

The coherent part of the dynamics is governed by the Hamiltonian

\[
H = \sum_i S_i \cdot S_{i+1} + h(\zeta_i^z S_i^x + \zeta_i^x S_i^z) \quad (9)
\]

with open boundary conditions and disorder fields drawn uniformly from the range \( \zeta_i^\pm \in [-1, 1] \). For simplicity we have chosen a model in which energy is the only conserved quantity. The MBL transition in the Hamiltonian (9) has been studied in Ref. [30] using exact diagonalization.

We compute the properties of this system when it is weakly coupled to non-thermal baths described through the Lindblad formalism. The precise choice of Lindblad operators is not important, as long as the steady-state is non-trivial, \( \rho_\infty \neq 1 \). Therefore, at least one of the Lindblad operators has to be non-Hermitian. We choose a symmetric combination of several dissipators

\[
\begin{align*}
L_i^{(1a)} &= \frac{1}{2} S_i^z \left( \frac{1}{2} S_{i+1}^z - S_{i+1}^x \right), \\
L_i^{(1b)} &= \frac{1}{2} S_i^z - S_i^x \ , \\
L_i^{(2a)} &= \frac{1}{2} S_i^- \left( \frac{1}{2} S_{i+1}^x + S_{i+1}^z \right), \\
L_i^{(2b)} &= \frac{1}{2} S_i^+ S_i^z, \\
L_i^{(3)} &= S_i^z.
\end{align*}
\]

(10)

The (unique) steady state \( \rho_\infty \) is obtained by solving the Lindblad time evolution

\[
\dot{\rho} = -i[H, \rho] + \epsilon \sum_i \left( L_i^\nu \rho L_i^{\nu \dagger} - \frac{1}{2} \{ L_i^{\nu \dagger} L_i^\nu, \rho \} \right) \quad (11)
\]

using the time-evolving block decimation (TEBD) technique for a vectorized density matrix [41, 42]. The dephasing term \( L_i^{(3)} \), Eq. (10), is used to ensure that the steady state is sufficiently close to the identity, so that a bond dimension of \( \chi = 100 \) is adequate to describe a system of \( N = 20 \) sites for \( \epsilon \geq 0.01 \). Note that larger bond dimensions and longer propagation times are needed for smaller \( \epsilon \), making computation in these cases more expensive, see Suppl. Mat. At fixed \( h \), the same set of disorder configurations is used for different values of \( \epsilon \), while independent configurations are used at different values of \( h \). We find that this procedure helps to determine the exponent \( z \) as the \( \epsilon \) dependence becomes less affected by the statistical ensemble. We average over 100 \( (h = 1, 2) \) or 300-500 \( (h \geq 2) \) disorder configurations.

The goal of the calculation is to obtain the spatial variation of the local temperature for varying values of the dissipative coupling \( \epsilon \). To determine the local temperatures \( T_i \) we compare the reduced density matrix of 2 sites.
the expected non analytic behavior $\delta\beta/\bar{\beta} \sim \epsilon^{1/2z}$ at small values of $\epsilon$ (see Fig. 2(b)). The fitted dynamical exponent $z$, shown in Fig. 3 changes continuously with disorder strength, growing rapidly as the MBL transition is approached. Error bars in Fig. 2 and Fig. 3 were obtained using a jackknife resampling. The usage of a re-sampling method for error estimates is necessary because statistical errors for different $\epsilon$ at fixed $h$ are strongly correlated in our setup. As discussed above, the dynamical exponent is expected to diverge together with the correlation length $\xi$ at the MBL critical point. The apparent saturation of $z$ is an artifact of the fit procedure; it is impossible to fit a small exponent $\alpha$ to the function $\epsilon^\alpha$ for realistic values of $\epsilon \gtrsim 0.01$.

We obtain an estimate of the critical disorder strength $h_c$ by recording the fraction $P$ of disorder realizations showing $\delta\beta/\bar{\beta}$ increasing with $\epsilon$ near $\epsilon = 0.01$ (see Fig. 2(c)). From the condition $P = 0.5$ we estimate $h_c \approx 8.75 \pm 0.5$ for $N = 20$. Allowing for smaller $\epsilon$ might lead to a slightly larger estimate of $h_c$ for this system size. A previous exact-diagonalization study of the same model on 20 sites 141 estimated a range of possible values for the critical disorder strength depending on the quantity being examined, from as low as $h_c \sim 2 - 3$ to $h_c \sim 7 - 8$.

With $h_c$ at hand we extract a correlation length exponent $\nu$ from the divergence of $z \sim \xi \sim (h_c - h)^{-\nu}$. As shown in Fig. 3 fitting can be reliably performed for $h \in [1, 4]$ when the error bars on $z$ are taken into account. The result of this procedure gives the estimate $\nu \approx 4.0 \pm 0.9$ (with large errorbars due to uncertainty in $h_c$), which is consistent with the Harris-Chayes bound, $\nu > 2/d$ 114,115 and also in agreement with single param-
This approach gives rise to universal broadening of the critical point itself $h = h_c$ in the limit $\epsilon \to 0$. Our results are consistent with a jump across the transition, but they also leave open the possibility of a slow logarithmic behavior as $-1/\log \epsilon$, which would allow a continuous change of $\delta \beta/\beta$ across the transition.

Discussion – We have demonstrated the advantages of investigating the MBL transition as a function of the coupling strength $\epsilon$ to external non-equilibrium baths. In this approach the coupling gives rise to universal broadening of the critical point, governed by the coupling $\epsilon$ in the same way that temperature broadens a conventional quantum critical point. In numerical computation, the finite coupling to a bath limits the operator entanglement entropy allowing to use powerful matrix-product operator methods on both sides of the phase transition. Thus we were able to obtain quantitative information on the dynamical critical point, the correlation length exponent $\nu$, the critical disorder strength. Moreover, having a weak coupling to the baths seems to regulate the calculation by broadening the many-body energy levels facilitating faster convergence to the thermodynamic limit. Even though our calculations were performed on rather small system sizes, the approach is in principle scalable (the computational effort scales linearly with system size at fixed bond dimension) and thus offers an appealing alternative to non-scalable exact-diagonalization calculations.

Potentially even more exciting than the numerical advances are the experimental breakthroughs that the open system approach can facilitate in the study of MBL. In particular this approach may allow to study MBL and the MBL transition in solids in spite of the inevitable coupling to a phonon bath. The non-equilibrium conditions we considered here can be achieved by driving the system externally either with light, or with a bias voltage, to a steady state set by the ratio of couplings to the drive and the phonon bath. Both couplings can be controlled, the former through the strength of the driving field and the latter by changing the phonon temperature. We have shown that under these conditions the critical properties can be inferred from the dependence of local temperature variations on the coupling $\epsilon$. In principle local temperatures can be measured by comparing the Stokes and Anti-Stokes response in a local (tip-enhanced) Raman spectroscopy experiment [48, 49]. However, a more natural measurement in solids would be a simple resistivity measurement. We have argued that dependence of the resistivity on $\epsilon$ at small values of the coupling can also provide crucial information on the MBL transition.

We note that the sub-diffusive scaling in one dimensional systems $\epsilon^{1/2z}$ may be hard to distinguish from true localization close to the transition, where the dynamical exponent becomes very large $z \gg 2$. In two and three dimensional systems, however, we do not expect to observe sub diffusive scaling even close to a localization transition (see supplementary information for results of a random resistor model). Thus the open system approach is uniquely suitable for determining the fate of many body localization in two and three dimensional systems in view of recent arguments [50] and some numerical evidence [51] for absence of a sharp MBL transition in this case.

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diffusive behavior in two dimensions

In this appendix we generalize the random resistor model, used to describe the sub-diffusive behavior in the Griffiths regime of one dimensional systems, to the case of a two dimensional system. We will show how diffusive behavior emerges in this model throughout the thermal phase as long as the correlation length associated with the transition remains finite.

Our two dimensional toy model consists of a square lattice of conducting islands connected by insulating links of linear length \( l \) taken from a probability distribution \( P(l) \sim e^{-(l/\xi)^2} \) (and generally in \( d \) dimensions \( P(l) \sim e^{-(l/\xi)^d} \)). This takes into account that in higher dimensions insulating regions are harder to construct than in one dimension because they can be short circuited by conducting paths. Conductances across the links are given by

$$\Gamma = \epsilon \frac{\kappa_0}{l^{2-d}} + \Gamma_0 e^{-l/\alpha} \quad \alpha = \frac{a}{\xi} \quad (14)$$

This relation between the conductance and the link size implies a probability distribution of conductances that decays faster than any power law:

$$P(\Gamma) \sim \frac{1}{\Gamma} e^{-\Gamma a^d (\ln(\Gamma \kappa_0))^{d/\delta}} \quad \text{for} \ \Gamma \gg \epsilon \frac{\kappa_0}{a^{2-d}}. \quad (15)$$

This contrasts with the power-law distribution obtained in one dimension \( P(\Gamma) \sim (\Gamma_0/\Gamma)^{1-a} \).

Fig. 4 shows temperature fluctuations as a function of coupling strength to the environment \( \epsilon \) for \( d = 2 \). For all values of \( \xi \) we see a dependence \( \delta\beta/\beta \sim \epsilon^{1/2} \) consistent with diffusive behavior \( z = 2 \). This is in contrast to one dimension where we observed a continuously varying dynamical exponent \( z \sim \xi/a \). This observation confirms the expectation \[39\] that Griffiths effect are absent in \( d > 1 \) where insulating regions cannot serve as bottlenecks, but are rather short-circuited by surrounding smaller resistivities.

Current dependence on the dynamical exponent

As an alternative to temperature fluctuations, we proposed in the main text, a setup that is driven via a small...
bias at the edges. In the bulk, the system is still coupled to phonons but not necessarily driven. The information about the dynamical exponent is in this case contained in the dependence of the current on the strength of coupling to the thermal phonon bath. The average resistivity, obtained by taking an ensemble average of the local conductances, was $\bar{\rho} \sim (\epsilon / \ln \epsilon)^{-1}$, where $\alpha = 1/z = a/\xi$.

In this appendix we compute the resistance of the network in one and two dimensions obtained through numerical solution of the steady state rate equations. In the one dimensional case, for example, Kirchoff’s law is

$$(\tilde{\Gamma}_{i,j}(\mu_{i+1} - \mu_i) - \tilde{\Gamma}_{i-1,j}(\mu_i - \mu_{i-1})) = 0.$$  \hspace{1cm} (16)

In this case we set up a voltage bias across the chain $V = \mu_N - \mu_0$, then solve for the island chemical potentials $\mu_i$ to get the current. As in the thermal case, we draw the link sizes along the chain from the distribution $P(\ell) \sim \xi^{-1} e^{-(\ell/\xi)^d}$, while the link conductances are:

$$\tilde{\Gamma} = \epsilon \frac{\sigma_0}{\ell^{2-d}} + \tilde{\Gamma}_0 e^{-l/\alpha} \hspace{1cm} (17)$$

In the two dimensional case we set up a constant chemical potential $\mu = V$ on the right edge and a constant $\mu = 0$ on the left edge.

The result of the calculation of the current in a one dimensional chain for the case $\alpha < 1$ is shown in Fig. 4. The result agrees well with the analytic prediction for the resistance scaling as $\epsilon^{\alpha-1}$, which was obtained in the main text up to logarithmic corrections. In the diffusive regime ($\alpha > 1$), the current has a non-zero $\epsilon \to 0$ limit, as we expect for a system with finite intrinsic resistivity.

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Also in two dimension we always find a finite resistivity in the $\epsilon \to 0$ as well as linear corrections, as shown in Fig. 5.

Remember that the dynamical exponent is related to $\alpha$ as $z \sim \alpha^{-1}$. Measuring the dependence $j(\epsilon, \alpha)$ in a disordered system would therefore provide information on $z$ and its divergence upon approaching the MBL transition even in disordered materials that are weakly coupling to phonons. The strength of coupling to phonons $\epsilon$ could be monitored through variation of phonon temperature; see our previous work [17], where we showed that phonon temperature determines the effective coupling.

**Comparison to standard measures of subdiffusivity**

Our approach to extract dynamical exponent $\alpha$ from the temperature variation, $\delta T \sim \epsilon^{d/2z}$, is novel and applies to weakly open and driven setups. Here we compare...
FIG. 7. Comparison of the dynamical exponent $z$ calculated from the fluctuations of the local temperatures (open) or from the anomalous heat diffusion in a closed system (closed). Parameters: $\kappa_0 = 1$, $a\Gamma_0 = 5.0$, $T_0 = 1$, $\delta g_1 = 0.05$, $\delta g_2 = 0.05$, $\theta = T_0$, $N = 1000$, averaged over $M = 500$ configurations.

we determine $z$ by measuring the spreading of energy packet $\sqrt{\sum (i-x_0)^2 \tilde{e}_i(t)} \sim t^{1/z}$, where $e_i(t)$ is obtained from Eq. 4 at $\epsilon = 0$

$$\partial_t e_i - \Gamma_{i,i+1}(T_{i+1} - T_i) + \Gamma_{i-1,i}(T_i - T_{i-1}) = 0 \quad (18)$$

and $\tilde{e}_i(t) = \langle e_i(t) \rangle$ is a disorder averaged energy distribution. Note that here $\Gamma = \Gamma_0 e^{-1/\alpha}$. Fig. 7 shows a qualitative agreement between the value of $z$ obtained via these two different approaches.

**Convergence rate of $\delta\beta/\bar{\beta}$**

In Fig. 8 we show the dynamics of the order parameter $\delta\beta/\bar{\beta}$ during the TEBD evolution computed for one disorder realization. It can be seen that the convergence to the steady-state value is exponential with a rate which scales as $\epsilon^{-1}$, making computation at small $\epsilon$ increasingly more difficult.

**FIG. 8.** Evolution of $\delta\beta/\bar{\beta}$ during the TEBD computation, for one disorder realization at $h = 4$. The inset shows the same plot with the time axis rescaled by $\epsilon$. 