OPEN–CONSTRUCTIBLE FUNCTIONS
(CORRECTED VERSION, JANUARY 2014)

ALEXEY OSTROVSKY

Abstract. Let $f$ be a continuous function between subspaces $X, Y$ of the Cantor set $C$. We prove that:
if $f$ is one-to-one and maps open sets into resolvable, then $f$ is a piecewise homeomorphism and
if $f$ maps discrete subsets into resolvable, then $f$ is piecewise open.

1. Introduction

The present paper continues the series of publications about decomposibility of Borel functions [6], [8] - see also [2], [3] where functions of such type are the main subject.
A subset $E$ of a topological space $X$ is resolvable if for each nonempty closed in $X$ subset $F$ we have

$$\text{cl}_X(F \cap E) \cap \text{cl}_X(F \setminus E) \neq F$$

Recall that a function $f$ is open if it maps open sets into open ones. More generally, a function $f$ is said to be open–resolvable (resolvable in [1]) if $f$ maps open sets into resolvable ones.

A function $f : X \to Y$ for which $X \subset C$ admits a countable, closed and disjoint cover $C$, such that for each $C \in C$ the restriction $f|C$ is open, is called piecewise open.

**Theorem 1.** Let $f$ be a continuous, one-to-one and open–resolvable function between $X, Y \subset C$. Then $f$ is piecewise open and hence is piecewise homeomorphism.

Note, that using standard sets $H$, we obtain the proof for open-resolvable functions simpler than for open-constructible.

Analogously, using $H$-sets we can easy extend the proof for closed-constructible functions in [6] to the case of closed-resolvable.

Standard set $H$ will be used in the proof of the following theorem:

**Theorem 2.** If a continuous function $f : X \to Y$ between $X, Y \subset C$ maps discrete subsets in $X$ onto resolvable, then $f$ is piecewise open.
1.1. Standard set $H$.

The set $H$ was introduced by W. Hurewicz [4] and has a lot of applications. $H$ (called standard in [7],[5]) is a countable set without isolated points:

$$H = \{p, \ldots, p_{i_1 \ldots i_k} : k \in \mathbb{N}^+; i_1, \ldots, i_k \in \mathbb{N}^+\}$$

such that

$$p_{i_1} \to p, \text{ as } i_1 \to \infty$$

$$p_{i_1 \ldots i_k, i_{k+1}} \to p_{i_1 \ldots i_k}, \text{ as } i_{k+1} \to \infty$$

Obviously, $H$ is homeomorphic to the space of rational $\mathbb{Q}$.

Using the metric in $X \supset H$ we can suppose additionally that there are decreasing bases $U_i(p)$, and $U_i(p_{i_1 \ldots i_k})$ at points $p$ and $p_{i_1 \ldots i_k}$ satisfying conditions a), b) and c) below:

a) $U_i^{i_k}(p_{i_1 \ldots i_k}) \supset U_1(p_{i_1 \ldots i_{k+1}}), i_k \in \mathbb{N}^+$

b) for $i_k' \neq i_k''$ we have $U_1(p_{i_1 \ldots i_k'}) \cap U_1(p_{i_1 \ldots i_k''}) = \emptyset$.

c) $\text{diam}(U_i^{i_k}(p_{i_1 \ldots i_k})) < 1/(i_1 + \ldots + i_k)$

2. Construction of $Z \subset X$ for which $f|Z$ is nowhere open

Given a function $f : X \to Y$, we shall construct in the next Lemma 1 a subset $Z \subset X$ on which the restriction $f|Z$ is nowhere open on $Z$; i.e. for every clopen in $X$ subset $U$ the restriction $f|(U \cap Z)$ is not open.

Lemma 1. Let $f : X \to Y$ be a continuous function from a subspace $X$ of the Cantor set $\mathcal{C}$ onto a metrizable space $Y$. Then there is a closed subset $Z \subset X$ such that the restriction $f|Z$ is nowhere open on $Z$ and the restriction $f|(X \setminus Z)$ is piecewise open.

Proof of Lemma 1. Let us begin by proving the first part of the assertion from lemma stating that for some $Z$ the restriction $f|Z$ is nowhere open on $Z$. Indeed, if for some nonempty clopen set $V \subset X$ the restriction $f|V$ is open, then we could construct the closed set

$$X_1 = X \setminus V$$

and the corresponding restriction

$$f|X_1 : X_1 \to f(X_1).$$

Repeating this process, we could also construct a chain of closed sets ($X_\gamma = \bigcap_{\beta < \gamma} X_\beta$ for a limit $\gamma$)

$$X \supset X_1 \supset \ldots \supset X_\gamma \supset \ldots$$

which, as we know, stabilizes at some $\gamma_0 < \omega_1$. Therefore, there exists a subspace $Z$ for which holds true

$$Z = X_{\gamma_0} = X_{\gamma_0 + 1} = \ldots$$

and the restriction $f|Z$ is nowhere open on $Z$. 
pairwise disjunct clopen neighborhoods constructed.

obviously, satisfied.

Take the points and clopen sets

Proof of Theorems 1 and 2.

2.1. Proof of Theorems 1 and 2.

On the step 1 take a point \( x \in Z \) and a base of clopen in \( X \) neighborhoods \( U^k(x) \subset X \) with diametr less than \( 1/k \).

Since \( f \) is nowhere open on \( Z \), there are

\[
U^l(x_k) \subset Z \cap (U^k(x) \setminus (U^{k+1}(x)))
\]

such that \( x_k \to x \), \( f(x_k) = p_k \to p = f(x) \) and

\[
f^{-1}(p_{k,l}) \cap U^1(x_k) = \emptyset
\]

Take \( x_{k,l} \in U(x_{k,l}) \subset \cap (U^l(x_k) \setminus (U^{l+1}(x_k))).

For the basic induction step \( m \) we suppose that the points \( p = f(x), ..., p_{i_1,...,i_k} = f(x_{i_1,...,i_k}) \) satisfying the conditions a), b),c) of definition the set \( H \) are constructed.

Analogously pick in some clopen sets \( U^1(x_{i_1,..,i_k}) \) the points \( x_k \to x \) with pairwise disjunct clopen neighborhoods \( U^1(x_k) \) such that for some sequence \( p_{k,l} \to p_k = f(x_k) \) we have

\[
f^{-1}(p_{k,l}) \cap U^1(x_k) = \emptyset
\]

Take the points

\( x_{k,l} \in f^{-1}(p_{k,l}) \setminus U^1(x_k) \)

and clopen sets

\( U^1(x_{k,l}) \subset X \setminus U^1(x_k). \)

Since \( f \) is continuous we can suppose that \( U^1(x_{k,l}) \) are disjunct with all \( U^1(x_k) \).

Since \( f \) is nowhere open on \( Z \cap U^1(x_{k,l}) \) we can repeat the construction for \( U^1(x_{k,l}) \) etc. Analogously we obtain

\[
p_{i_1,...,i_m,k,l} \to p_{i_1,...,i_m,k} \text{ as } l \to \infty
\]

\[
x_{i_1,...,i_m,k,l} \in f^{-1}(p_{i_1,...,i_m,k,l}) \setminus U^1(x_{i_1,...,i_m,k}).
\]

\[
U^1(x_{i_1,...,i_m,k}) \cap f^{-1}(p_{i_1,...,i_m,k,l}) = \emptyset, l \in N^+
\]

We can suppose that \( U^1(p_{i_1,...,i_m,k}) \) are clopen and there is a clopen neighborhood \( U^1(x_{i_1,...,i_m,k,l}) \) of point \( x_{i_1,...,i_m,k,l} \) disjunct with all \( U^1(x_{i_1,...,i_m,k}). \)

Denote

\[
D = \{ x_{i_1,...,i_2,k+1} : k \in N^+; i_1, ..., i_{2k+1} \in N^+ \}
\]

By our construction \( D \) is discrete and \( f(D) \) is dense and codense in

\[
H = \{ p_{i_1,...,i_k} : k \in N^+; i_1, ..., i_k \in N^+ \}
\]

that proves Theorem 2.
To prove Theorem 1 we note, that in case of one-to-one functions the open set $O(D) = \{U^1(x_{i_1}), \ldots, U^1(x_{i_1}, \ldots, x_{i_{2k+1}}) : k \in N^+, i_1, \ldots, i_{2k+1} \in N^+\}$ has the same image as $D$.

\[\square\]

**Question 1.** Are the continuous open–$\Delta^0_2$ functions (even for Polish or analytic spaces $X \subset C$) piecewise or countably open?

**References**

[1] Gao, S., Kieftenbeld,V.: Resolvable Maps Preserve Complete Metrizability, Proc. Am. Math. Soc. 138 (2010), 2245–2252.

[2] Ghoussoub, N., Maurey, B.: $G_\delta$–Embeddings in Hilbert Space, Journal of Functional Analysis 61 (1985), 72–97.

[3] Holicky, P.: Preservation of completeness by some continuous maps, Topology Appl. 157 (2010), 1926-1930.

[4] Hurewicz W.: *Relativ perfekte Teile von Punktmengen und Mengen (A)*, Fund. Math. 12 (1928), 78–109.

[5] Ostrovskii, A. V.: Product $F_{\alpha\beta}$-spaces and $A$-sets, Moscow University Mathematics Bulletin, 30, (1975) 95–99.

[6] Ostrovsky, A.: Closed–constructible functions are piecewise closed, Topology Appl. 160 (2013), 1675-1680.

[7] Ostrovskii, A.V.: On non separable $\tau$-analytic sets and their mappings, Soviet Math. Dokl. 17(1972) 99–102.

[8] Ostrovsky, A.: Preservation of the Borel class under open-$LC$ functions, Fund. Math., 213:2 (2011), 191–195.

E-mail address: alexei.ostrovski@gmx.de

Follow me on ResearchGate