LOG-CONVEXITY AND THE CYCLE INDEX POLYNOMIALS WITH RELATION TO COMPOUND POISSON DISTRIBUTIONS

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Abstract. We extend the exponential formula by Bender and Canfield (1996), which relates log-concavity and the cycle index polynomials. The extension clarifies the log-convexity relation. The proof is by noticing the property of a compound Poisson distribution together with its moment generating function. We also give a combinatorial proof of extended “log-convex part” referring Bender and Canfield’s approach, where the formula by Bruijn and Erdős (1953) is additionally exploited. The combinatorial approach yields richer structural results more than log-convexity. Furthermore, we consider normal and binomial convolutions of sequences which satisfy the exponential formula. The operations generate interesting examples which are not included in well known laws about log-concavity/convexity.

1. Introduction

The focus is to study properties of non-negative sequences \((a_k)_{k \geq 0}\) and \((b_k)_{k \geq 0}\) from those of \((c_k)_{k \geq 1}\) such that they are linked by

\[
\sum_{k=0}^{\infty} a_k u^k = \sum_{k=0}^{\infty} \frac{b_k u^k}{k!} = \exp\left(\sum_{j=1}^{\infty} \frac{c_j u^j}{j}\right).
\]

(1.1)

The sequence \((a_k)\) is regarded as the cycle index polynomials of symmetric group \([10, 11]\). From the relation (1.1) Bender and Canfield have shown the log-concavity and almost log-convexity of \((a_k)\), assuming that \((c_k)_{k \geq 0}\) with \(c_0 = 1\) is log-concave \([1\text{, Theorem 1}]\).

The first main result of this paper (Section 2) is an extension of \([1\text{, Theorem 1}]\), namely, we show the log-convexity of \((a_k)\) from that of \((c_k)\). Our approach is to notice the probabilistic interpretation of (1.1) together with distributional properties of compound Poisson (CP for short) distributions. This approach leads to an alternative proof for the previous log-concavity result \([1\text{, Theorem 1}]\) which was originally derived from combinatorial study. In the same section, we specify (probably unknown) common characteristics between these combinatorial and probabilistic approaches toward \([1\text{, Theorem 1}]\). Inspired by this specification, we consider combinatorial proof for the extended “log-convex part”. This combinatorial proof yields richer results more than log-convexity, which are related to the structure of the cycle index and which were analyzed as \([1\text{, Theorem 2}]\) in the log-concave case.

In Section 3, we consider log-convex/concave properties of normal convolution of \((a_k)\) and binomial convolution of \((b_k)\) which satisfy the exponential formula (1.1). Both convolutions are obtained by multiplying the r.h.s. of (1.1). The operations generate interesting sequences which are not included in known principles for log-concavity/convexity. Our example sequences are mostly by those of probability mass functions.

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In the rest, we state related literature of both combinatorics and probability and statistics. This is worth describing since they developed the similar theories independently without having proper intersections.

In combinatorics the log-concavity/convexity of combinatorial sequences has been intensively studied (see Stanley [14] and Brenti [2] for log-concavity, and see Lin and Wang [18] for log-convexity and references therein). Especially log-concavity is closely related with unimodality and they has been studied together. Bender and Canfield’s result [1] serves as a tool to judge log-concavity. The method is to investigate the property of the original sequence by its exponential generating function (GF for abbreviation). Indeed the relation (1.1) corresponds to the GF for the cycle index of symmetric groups (see Remark 2.5 (i) bellow). The method by GFs is a powerful tool for solving combinatorial problems (see [19]).

The properties are also significant in probability and statistics. To confine the related topics, they play a crucial role in the class of infinitely divisible (ID for short) distributions, one of most important probability distributions in both theory and applications. The log-concavity is a useful tool for investigating the unimodality of ID distributions (see e.g. Sato [13] and Steutel [?]). Indeed, the class of strong unimodal probability density/mass functions is equivalent to that of log-concavity density/masss functions. The log-convexity characterizes ID distributions on $\mathbb{Z}_+$, i.e. it gives a sufficient condition for distributions to be ID [13] Theorem 51.3. A sufficient conditions for log-concavity/convexity of $\mathbb{Z}_+$ valued ID distributions is given by [5]. For ID distributions since explicit expressions of most density/masss functions are unavailable, the most useful tools are their characteristic functions ([13, p.7]) which are substantially equivalent to GFs.

Therefore quite similar problems are investigated with similar/different methods in these two fields, and connections are sometimes discovered and mentioned : e.g. nth Bell number corresponds to nth moment of the CP with mean 1. But they are sparse and not systematic. Here we point out an almost complete correspondence between the two, i.e. the cycle index of the symmetric group corresponds to the probability mass function of ID on $\mathbb{Z}_+$ (Remark 2.5 (ii)). This correspondence is justified by the uniqueness of the GF. All presented results of Section 2 stem essentially from this correspondence. We expect that the relation gives a perspective to previous miscellaneous results and promote further exchanges of the two fields.

Note that we could give results only by combinatorial methods. However, since the relation between combinatorics and probability and statistics are interesting and worth describing, we present both approaches.

2. Main results

We consider the probabilistic proof of [1, Theorem 1], which clarifies the relation between non-negative log-concave sequences and the cycle index polynomials. By the probabilistic proof, we extended the relation to that with non-negative log-convex sequences. Furthermore, we give a combinatorial proof of the extended part which yields additional new results about the cycle index polynomials.

Our focus is on the following extended theorem.

**Theorem 2.1.** Let $(c_k)_{k \geq 1}$ be a sequence of non-negative real numbers such that $(c_k/k)_{k \geq 0}$ are infinitely summable and define the sequence $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$ by (1.1). Suppose that $(c_k)_{k \geq 1}$ is log-concave and $c_1^2 \geq c_2$, then

$$a_{k-1}a_{k+1} \leq a_k^2 \leq \frac{k+1}{k}a_{k-1}a_{k+1},$$

(2.1)
(2.2) \[ b_{k-1}b_{k+1} \geq b_k^2 \geq \frac{k}{k+1}b_{k-1}b_{k+1}. \]

Moreover, the left ineq. in (2.1) or equivalently the right ineq. of (2.2) holds only if \( c_1^2 \geq c_2. \)

Conversely, suppose that \((c_k)_{k \geq 1}\) is log-convex, then

(2.3) \[ a_{k-1}a_{k+1} \geq a_k^2, \]

(2.4) \[ b_k^2 \leq \frac{k}{k+1}b_{k-1}b_{k+1} \]

holds if and only if \( c_1^2 \leq c_2. \)

**Remark 2.2.** We mention the relation between Theorem 2.1 and [1, Theorem 1].

(i) The extended part is the convexity results of \((a_k)_{k \geq 0}\) : (2.3) or equivalently (2.4), and “only if” parts in the log-concave case.

(ii) The summability condition of \((c_k)_{k \geq 1}\) is additionally assumed in Theorem 2.1

(iii) Log-concavity assumption of (1, c_1, c_2, ...) in [1; Theorem 1] and that of \((c_k)_{k \geq 1}\) together with \( c_1^2 \geq c_2 \) are the same. Since the latter seems more clear to mention “only if” condition, we adopt this. The same argument holds in log-convex case.

(iv) We can not recover the structural result [1, Theorem 2] for \((b_k)\) from probabilistic proof, i.e. it seems difficult to prove that \((n + 1)b_m b_n - mb_{m-1}b_{n+1}\) for \( 1 \leq m \leq n \) could be expressed as a polynomial in \( \mathcal{B} = \{c_1, c_2, \ldots\} \cup \{c_j c_k - c_j 1 c_{k+1} : 0 < j \leq k\} \) with non-negative integer coefficients.

We begin to see the probabilistic proof of Theorem 2.1. For \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) let \((X_i)_{i=1,2,\ldots}\) be an independent and identically distributed sequence of \( \mathbb{Z}_+\)-valued random variables (r.v.’s for short). Let \( N \) be Poisson r.v. with parameter \( \lambda > 0 \) of which probability is \( P(N = k) = \frac{\lambda^k}{k!} e^{-\lambda}, k \in \mathbb{Z}_+ \). We consider CP r.v. \( S_N := \sum_{j=1}^{N} X_j \). Writing \( P_n = P(S_N = n), n \in \mathbb{Z}_+ \) and \( f_n := P(X_i = n) \), we have an expression of probability GF of \( S_N \) by

(2.5) \[ \sum_{k=0}^{\infty} P_{k} u^k = \exp \left( \lambda \sum_{j=0}^{\infty} f_j u^j - 1 \right) = e^{\lambda (f_0 - 1)} \exp \left( \sum_{j=1}^{n} \frac{\lambda j f_j}{j} u^j \right), \quad |u| \leq 1. \]

Here putting \( a_k = e^{-\lambda (f_0 - 1)} P_k \) and \( c_k = \lambda k f_k \), the relation (1.1) is recovered. Then, once log-concavity (resp. log-convexity) of \((P_k)\) is proven, that of \((a_k)\) follows.

In the probabilistic proof, the relations (2.5) is not directly used and our main tool is the following well-known recursion for \((P_k)\)

(2.6) \[ P_0 = e^{\lambda (f_0 - 1)}, \quad (n + 1)P_{n+1} = \sum_{k=0}^{n} \lambda (k + 1) f_{k+1} P_{n-k}, \]

which yields two key results used in the proof.

One is the following theorem, an explanation of which is given in Appendix for reader’s sake.

**Theorem 2.3** (Hansen [5, Theorem 1 and 2]). Let \((P_n)_{n \geq 0}\) and \((f_k)_{k \geq 1}\) be connected by (2.6). Assume \((k f_k)_{k \geq 1}\) is log-concave (resp. log-convex), then \((P_n)\) is log-concave (resp. log-convex) if and only if \( \lambda f_1^2 - 2 f_2 \geq 0 \) (resp. \( \lambda f_1^2 - 2 f_2 \leq 0 \)).

Another one is nearly log-convex result when \((P_n)\) is log-convex.
Remark 2.5. (i) The recursion (2.6) is equivalent to that for the cycle index polynomials in Theorem 2.3 is satisfied, then (P_k) further fulfills

\[ P_{n-1}P_{n+1} \leq P_n^2 \leq \frac{n+1}{n} P_{n-1}P_{n+1}. \]

Proof. Due to the relation (2.6), we have

\[ (n+1)P_{n-1}P_{n+1} - nP_n^2 = \lambda(n+1)f_{n+1}P_0P_{n-1} + \sum_{k=0}^{n-1} \lambda(k+1)f_{k+1}(P_{n-k}P_{n-1} - P_{n-1-k}P_n). \]

Since \((P_n)_{n \geq 0}\) is a log-concave sequence, the right-hand side is non-negative.

Proof of Theorem 2.1. We write the relation (1.1) in the form of (2.5). Noticing \(\sum_{k=1}^{\infty} (c_k/k) < \infty\), we can put \(\lambda = \frac{1}{1-f_0} \sum_{k=1}^{\infty} c_k/k\) since \(\lambda\) could be any positive constant. Let \(c_k = \lambda f_k, k \geq 1\) in (1.1) and put \(f_0 = 1 - \sum_{k=1}^{\infty} f_k\). Then multiply both sides by \(e^{\lambda(f_0-1)}\) to obtain

\[ \sum_{k=0}^{\infty} e^{\lambda(f_0-1)a_ku^k} = e^{\lambda(f_0-1)} \exp\left( \sum_{j=1}^{\infty} \frac{\lambda jf_j}{j} u^j \right). \]

Since the right-hand side is probability GF of \(P_k\) by the uniqueness \(P_k = e^{\lambda(f_0-1)a_k}\) holds.

Now we check conditions of Theorem 2.3 in terms of \((c_k)\). The log-concavity (resp. log-convexity) of \((c_k)_{k \geq 1}\) and that of \((kf_k)_{k \geq 1}\) are equivalent. Moreover \(\lambda f_1^2 - 2f_2 \geq 1/\lambda \cdot (c_1^2 - c_2 - 1) \geq 0\). Thus conditions follow from those of Theorem 2.1. Then conclusions of Theorem 2.1 are implied by \(a_k^2 - a_k a_{k+1} = e^{-2\lambda(f_0-1)}(P_k^2 - P_{k-1}P_{k+1})\) together with results of Theorem 2.3. The second inequality of (2.1) is implied by Lemma 2.4. Finally (2.2) follows from (2.1).

We give a remark about further relations between CP distribution and the cycle index.

**Remark 2.5.** (i) The recursion (2.6) is equivalent to that for the cycle index polynomials \(A(\Sigma_n)\) of symmetric group \(\Sigma_n\) with variables \((x_1, x_2, \ldots, x_n)\):

\[ A(\Sigma_0) = 1 \quad \text{and} \quad A(\Sigma_n) = \frac{1}{n} \sum_{\ell=1}^{n} x_\ell A(\Sigma_{n-\ell}) \]

(see e.g. Harary and Palmer [4] p.120]). Namely, scale-adjusted \(A(\Sigma_{n+1})\) has a correspondence with \(P_n\). This is rationalized by comparing their GFs. The GF of \(A(\Sigma_n)\) is given by (1.1), i.e.

\[ \sum_{n=0}^{\infty} A(\Sigma_n)t^n = \exp\left( \sum_{j=1}^{\infty} \frac{x_j}{j} t^j \right) \]

(see e.g. [19] Theorem 4.7.2 for elementary proof). Assume that \(\sum_{j=1}^{\infty} x_j/j < \infty\). Let \(\lambda = \frac{1}{1-f_0} \sum_{k=1}^{\infty} x_k/k, f_k = x_k/(\lambda k), k \geq 1\) and \(f_0 = 1 - \sum_{k=1}^{\infty} f_k\) as before and then (2.8) coincides with (2.5). Hence by the uniqueness of GFs we conclude that for any \(A(\Sigma_{n+1})\) there exists probability mass function \(P_n\), and stated properties of \(P_n\) (Theorem 2.3 and Lemma 2.4) hold true for \(A(\Sigma_n)\).

(ii) Since CP distribution coincides with ID distributions on \(\mathbb{Z}_k\) (see [15] Theorem 3.2, III), \(A(\Sigma_n)\)
is relevant to ID distributions. Indeed, it is known that a distribution $P_k$, $k \in \mathbb{Z}_+$ with $P_0 > 0$ is ID if and only if the quantity $r_k$ with $k \in \mathbb{Z}_+$ determined by

$$P_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} r_k P_{n-k}$$

are non-negative (see [15 Theorem 4.4, II] or [13 Corollary 51.2]). In view of (2.9) and (2.7), one see the correspondence between the cycle index of symmetric group and ID distributions. In each topic there are established properties. Thus further investigation of the relation would be our next interest.

Next we consider a combinatorial proof of the extended part based on the cycle index polynomials as done in [1]. Let $\Sigma_m$ denote the symmetric group of degree $m$ and let $N_j(\sigma)$ be the number of $j$-cycle in the permutation $\sigma$. Then the cycle index polynomials $(a_m)$ and related polynomials $(b_m)$ are defined as

$$a_m(c_1, c_2, \ldots, c_m) = \frac{1}{m!} b_m(c_1, c_2, \ldots, c_m) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \text{wt}(\sigma),$$

where $\text{wt}(\sigma) = c_1^{N_1(\sigma)} \cdots c_m^{N_m(\sigma)}$. In what follows, we give several properties as derived in [1] which are used. For $\sigma_1 \in \Sigma_{m+1}$ let $\sigma_1'$ be $\sigma_1$ with $m + 1$th element deleted from the cycle containing it. The summation of $\text{wt}(\sigma_1')$ over all $\sigma_1 \in \Sigma_{m+1}$ yields

$$\sum_{\sigma_1 \in \Sigma_{m+1}} \text{wt}(\sigma_1') = (m + 1)b_m.$$ 

If $m + 1$ element belongs to a $j$-cycle of $\sigma_1$, then

$$c_{j-1} \text{wt}(\sigma_1) = c_j \text{wt}(\sigma_1')$$

holds. We have two formulas for $b_{m+1}$ and $(m + 1)b_m$: let $(m)_k$ denote the falling factorial $m(m - 1) \cdots (m - k + 1)$, then

$$b_{m+1} = \sum_{j=1}^{m+1} (m)_{j-1} c_j b_{m+1-j},$$

$$b_m = \sum_{j=1}^{m} (m)_{j-1} c_j b_{m-j},$$

both of which have combinatorial interpretation. For (2.12), $j$th term in the sum, $(m)_{j-1} c_j b_{m-j}$ implies the sum of $\text{wt}(\sigma_1')$ over all combinations in $\sigma_1$ such that $j$-cycle contains $m + 1$th element. There are $(m)_{j-1}$ ways to construct $j$-cycle which contains $m + 1$th element and $b_{m-j}$ is the sum of weights over all permutations for remaining $m - j$ elements. Here $c_j$ is the weight of $j$-cycle. For (2.13), $j$th term implies the sum of $\text{wt}(\sigma_1')$ over all permutations in $\sigma_1$ such that $m + 1$th element is removed from $j$-cycle of $\sigma_1$, so that $c_j (m)_{j-1} c_j b_{m-j}$ in (2.12) is replaced by $c_{j-1}$. Here we additionally use (2.10).

Now we give a combinatorial proof of the extended part. We get the idea from the formula (5) by [5], though an analogue of the formula has already been used in Bruijn and Erdős [3].

**Proof.** Our goal is to prove (2.4) by the induction, and then (2.3) follows by the equivalence. Since $b_0 = 1, b_1 = c_1$ and $b_2 = c_1^2 + c_2$, we have $b_0 b_2 - 2b_1^2 = c_2 - c_1^2 \geq 0$. Assume that (2.4) holds with
From the property (2.10), the right hand-side is
\[
\begin{align*}
\sum_{\sigma \in S_{m+1}} (c_{m} \cdot wt(\sigma_{1}) - c_{m+1} \cdot wt(\sigma_{1}')) + (m + 1) \sum_{\sigma_{2} \in S_{m}} (c_{m} \cdot wt(\sigma_{2}) - c_{m+1} \cdot wt(\sigma_{2}')), \\
\end{align*}
\]
where \(\sigma_{i}, i = 1, 2\) are permutations of the symmetric group but with different degrees. We further obtain from (2.12) and (2.13) that
\[
\begin{align*}
mb_{m-1} \left( \sum_{j=1}^{m+1} c_{m}(m-j)c_{j}b_{m+1-j} - c_{m+1}(m)j_{1}c_{j_1}b_{m+1-j} \right) \\
- (m + 1)b_{m} \left( \sum_{j=1}^{m} c_{m}(m-1)_j b_{m-j} - c_{m+1}(m-1)_j b_{m-j} \right) \\
= mb_{m-1} \sum_{j=1}^{m}(m-j)b_{m+1-j}(c_{m}c_{j} - c_{m+1}c_{j+1}) \\
- mb_{m} \sum_{j=1}^{m}(m-1)_j b_{m-j}(c_{m+1}c_{j+1} - c_{m}c_{j}) \\
= m \sum_{j=1}^{m}(c_{m+1}c_{j-1} - c_{m}c_{j})(m-1)_j (m+1-j)b_{m-j} - mb_{m-1}b_{m+1-j} \\
+ b_{m} \sum_{j=1}^{m}(m-1)_j b_{m-j}(c_{m+1}c_{j+1} - c_{m}c_{j}).
\end{align*}
\]
Now since (2.4) holds for \(k = 1, \ldots, m - 1\) by the induction hypothesis, one could see from the last expression that (2.4) is satisfied also with \(m\).

By combinatorial approach, we can prove further results more than (2.1)–(2.4). The following is an extension of Theorem 2 in [1] which is related with the log-convexity result in Theorem 2.1.

**Theorem 2.6.** Let \(c_0 = 1\) and let \(c_1, c_2, \ldots\) be indeterminates. Further let
\[
\begin{align*}
\mathcal{X} &= \{c_1, c_2, \ldots\}, \\
\mathcal{Y} &= \mathcal{X} \cup \{c_j c_k - c_{j-1} c_{k+1} : 0 < j \leq k\}, \\
\mathcal{Z} &= \mathcal{X} \cup \{c_j c_{k+1} - c_j c_k : 0 < j \leq k\}
\end{align*}
\]
and define the sequence \((b_k)_{k \geq 0}\) by (1.1). Then
\[
\begin{align*}
(n + 1)b_{m}b_{n} - mb_{m-1}b_{n+1} &\in \mathbb{N}[\mathcal{Y}], \\
mb_{m-1}b_{n+1} - (n + 1)b_{m}b_{n} &\in \mathbb{N}[\mathcal{Z}]/\mathbb{N}[\mathcal{X}],
\end{align*}
\]
for \(1 \leq m \leq n\), namely, \((n + 1)b_{m}b_{n} - mb_{m-1}b_{n+1}\) can be expressed as a polynomial in \(\mathcal{Y}\) with non-negative integer coefficients and \(mb_{m-1}b_{n+1} - (n + 1)b_{m}b_{n}\) can be expressed as a ratio of polynomials in \(\mathcal{X}\) and those in \(\mathcal{Z}\) with respectively non-negative integer coefficients.

Note that proof for (2.15) is given in [1] and further extended by [12], where \(b_{m}b_{n} - b_{m-1}b_{n+1} \in \mathbb{N}[\mathcal{Y}]\) is shown. Therefore we give a proof only for (2.16).
Proof. We show (2.16) by the induction. It is immediate to see
\[ b_0b_2 - 2b_1^2 = c_2 - c_1^2 \in \mathbb{N}[X]. \]
Assume that
\[ (2.17) \quad kb_{k-1}b_{k+1} - (k + 1)b_k^2 \in \mathbb{N}[X]/\mathbb{N}[X] \]
holds for \( k \leq m - 1 \), then
\[ (2.18) \quad kb_{k-1}b_{k+1} - (\ell + 1)b_{\ell}b_{\ell} \in \mathbb{N}[X]/\mathbb{N}[X] \]
holds for all \( 1 \leq k \leq \ell \leq m - 1 \). Indeed, we observe that
\[ (2.19) \quad \frac{b_{\ell+1}}{(\ell + 1)b_{\ell} - kb_{k-1}} = \left( \frac{b_{\ell+1}}{(\ell + 1)b_{\ell} - \ell b_{\ell-1}} \right) + \cdots + \left( \frac{b_{k+1}}{(k + 1)b_k - b_k} \right) \]
is included in \( \mathbb{N}[X]/\mathbb{N}[X] \) by the induction hypothesis. Here we also use \( b_\ell \in \mathbb{N}[X], k \leq m - 1 \) which follows from the recursion (2.12). Then we multiply (2.19) by \( (\ell + 1)b_{\ell} \cdot kb_{k-1} \) to conclude (2.18). Now we recall the previous equality in the proof of Theorem 2.1:
\[ c_m(mb_{m-1}b_{m+1} - (m + 1)b_m^2) \]
\[ = m \sum_{j=1}^{m} (m - 1)j - (c_{m+1}c_j - c_mc_j)(m + 1 - j)b_{m-j}b_m - mb_{m-1}b_{m+1-j} \]
\[ + b_m \sum_{j=1}^{m} (m - 1)j - (c_{m+1}c_{j-1} - c_mc_j). \]
Then since (2.18) holds for \( 1 \leq k \leq \ell \leq m - 1 \) by the assumption we conclude (2.17) with \( k = m \). \qed

Remark 2.7. We do not know whether \( mb_{m-1}b_{n+1} - (n + 1)b_nb_n \in \mathbb{N}[X] \) holds or not. Our conjecture is negative but we do not prove it.

3. Convolution and binomial convolution of cycle index polynomials

We first review a generalization of the Bender and Canfield exponential formula (Theorem 2 in [1]) to convoluted sequences which is done by Schirmacher [12]. At there we give resulting consequences of the extension together with its alternative proof. Then we investigate the “log-convex counter part”. These generalizations to normal and binomial convolutions may provide interesting examples of log-concave/convex sequences which are not included in known laws.

Theorem 3.1 (Theorem 3 in [12]). For \( i = 1, \ldots, w, w \in \mathbb{N} \) let \( c_{i,0} = 1 \) and let \( c_{i,1}, c_{i,2}, \ldots \) be indeterminates. Let
\[ Y_i = \{ c_{i,1}, c_{i,2}, \ldots \} \cup \{ c_{i,j}c_{i,k} - c_{i,j-1}c_{i,k+1} : 0 < j \leq k \} \]
and let
\[ (3.1) \quad \sum_{k=0}^{\infty} A_k u^k = \sum_{k=0}^{\infty} B_k u^k = \exp \left( \sum_{j=0}^{\infty} \frac{\sum_{i=1}^{w} c_{i,j}}{j} u^j \right). \]
Then for \( 1 \leq m \leq n \),
\[ (3.2) \quad A_mA_n - A_{m-1}A_{n+1} \in \mathbb{R}_+ \cup \bigcup_{j=1}^{w} \mathbb{Y}_j, \]
\[ (3.3) \quad (n + 1)A_{m-1}A_{n+1} - mA_mA_n \in \mathbb{R}_+ \cup \bigcup_{j=1}^{w} \mathbb{Y}_j. \]
Namely \( A_m A_n - A_{m-1} A_{n+1} \) are polynomials in \( \bigcup_{i=1}^w \mathcal{Y}_i \) with non-negative rational coefficients and so are \((n + 1)A_m A_{n+1} - mA_m A_n \) The differences (3.2) and (3.3) are equivalent to

\[
(n + 1)B_n B_n - m B_{m-1} B_{n+1} \in \mathbb{R}_+[\bigcup_{i=1}^w \mathcal{Y}_i],
\]

(3.4)

\[
B_{m-1} B_{n+1} - B_m B_n \in \mathbb{R}_+[\bigcup_{i=1}^w \mathcal{Y}_i].
\]

The formula (3.1) implies that \((A_k)_{k \geq 0} \) and \((B_k)_{k \geq 0} \) are normal and binomial convolutions respectively ([19 Chapters 2.2 and 2.3]).

In [12] a general proof based on Cauchy-Binet formula was just suggested (see also [14]) and explicit expressions were skipped. In order to see clearly the reason why log-concavity is preserved and log-convexity is not in the normal convolution we give a direct proof. The following lemma is crucial of which proof is given in Appendix.

**Lemma 3.2.** Let \((x_k)_{k \geq 0} \) and \((y_k)_{k \geq 0} \) are indeterminates. Let \( D_n = \sum_{k=0}^n x_k y_{n-k} \), then for \( 1 \leq m \leq n \),

\[
D_m D_n - D_{m-1} D_{n+1}
\]

\[
= x_0 y_0 x_m y_n + y_0 \sum_{k=0}^{n-m-1} x_m x_{k+1} y_{n-1-k} + y_0 \sum_{k=0}^{m-1} y_k (x_m x_{n-k} - x_{n+1} x_{m-k-1})
\]

(3.6)

\[
+ \sum_{k=0}^{m-1} \sum_{\ell=0}^{n-m} x_k x_\ell (y_{m-k} y_{n-1-\ell} - y_{m-1-k} y_{n+1-\ell})
\]

\[
+ \sum_{k=0}^{m-1} \sum_{\ell=0}^{n+1} (x_k x_{m+1+\ell} - x_\ell x_{m+1+k}) (y_{m-k} y_{n+1-\ell} - y_{m-1-k} y_{n-\ell}).
\]

Moreover, putting

\[
\mathcal{G} = \{x_0, x_1, \ldots, x_n\} \cup \{x_m x_n - x_{m-1} x_{n+1} : 1 \leq m \leq n\},
\]

\[
\mathcal{F} = \{y_0, y_1, \ldots, y_n\} \cup \{y_m y_n - y_{m-1} y_{n+1} : 1 \leq m \leq n\},
\]

we obtain

\[
D_m D_n - D_{m-1} D_{n+1} \in \mathbb{N}[\mathcal{G} \cup \mathcal{F}], \quad 1 \leq m \leq n.
\]

**Proof of Theorem 3.1** Observe that \( A_k \) is \( w \)th convolution of sequence \( (a_{i,k})_i \), \( i = 1, 2, \ldots, w \) such that

\[
\sum_{k=0}^\infty a_{i,k} u^k = \exp \left( \sum_{j=0}^\infty c_{i,j} u^j \right)
\]

holds. By the induction, it suffices to prove the result for the bivariate convolution. Let \( x_i \in \mathbb{R}_+[\mathcal{Y}_1] \) and \( y_j \in \mathbb{R}_+[\mathcal{Y}_2] \) and suppose that for \( 1 \leq m \leq n \)

\[
x_m x_n - x_{m-1} x_{n+1} \in \mathbb{R}_+[\mathcal{Y}_1] \quad \text{and} \quad y_m y_n - y_{m-1} y_{n+1} \in \mathbb{R}_+[\mathcal{Y}_2].
\]

Then due to Lemma 3.2, the bivariate convolution \( D_n = \sum_{k=0}^n x_k y_{n-k} \), \( n \geq 0 \) satisfies

\[
D_m D_n - D_{m-1} D_{n+1} \in \mathbb{R}_+[\mathcal{Y}_1 \cup \mathcal{Y}_2], \quad 1 \leq m \leq n.
\]

Therefore (3.2) holds. Equation (3.3) follows from Lemma 2 of [12].

**Remark 3.3** (Remark for Lemma 3.2). (i) Equation (3.6) would be calculated from the Cauchy-Binet formula. In the proof of [12] only the formula is mentioned and the detail is omitted. We confirm the derivation by applying the formula to

\[
A = \begin{pmatrix}
0 & y_1 & \cdots & y_n & y_{n+1} \\
0 & 0 & \cdots & y_{n-1} & y_n
\end{pmatrix}
\]
\[ B = \left( \begin{array}{ccccccc} x_m & x_{m-1} & \cdots & x_0 & 0 & \cdots & 0 & 0 \\ x_{n+1} & x_n & \cdots & x_{n-m+1} & x_{n-m} & \cdots & x_1 & x_0 \end{array} \right)'. \]

See also [14].

(ii) Menon [8] proved that log-concavity is preserved under convolution by a direct calculation, i.e. proved the case \( n = m \) in Lemma 3.2. Our results with \( n = m \) coincides the corresponding calculation in [8].

(iii) From (3.6) one could observe that if one of sequences \( (x_k) \) and \( (y_k) \) is log-convex, then both positive and negative terms appear in (3.6). Accordingly we could not judge the sign of \( D_mD_n - D_{m-1}D_{n+1} \) without assuming additional conditions.

Remark 3.4 (Remark for Theorem 3.1). Assume that \( (c_{i,j})_{j \geq 1}, i = 1, \ldots, w \) are log-concave in \( j \) with \( c_{i,0} = 1 \).

(i) It is immediate from equations (3.2) and (3.4) that

\[
(n + 1)B_mB_n - mB_{m-1}B_{n+1} \geq 0 \iff A_mA_n - A_{m-1}A_{n+1} \geq 0.
\]

Notice that the result of Theorem 3.1 is not covered by the original version (Theorem 2.1) since the sum \( C_j = \sum_{i=1}^w c_{i,j} \) in the relation

\[
\sum_{k=0}^\infty A_ku^k = \sum_{k=0}^\infty \frac{B_ku^k}{k!} = \exp\left( \sum_{j=0}^\infty \frac{C_j}{j!}u^j \right)
\]

does not always log-concave (see example 3.27). Namely, operations of (normal or binomial) convolution to sequences which satisfy (1.1) widens the class of sequence \( (c_j) \) from which log-concavity of \( (a_k) \) follows.

(ii) The log-concavity of \( (A_k) \) (3.2) and the log-convexity of \( (B_k) \) (3.5) are concluded from general theory : The log-concavity is preserved by both ordinary and binomial convolution while log-convexity is preserved only by binomial convolution (see p.455 in [18]).

(iii) We do not know whether \( (n + 1)B_mB_n - mB_{m-1}B_{n+1} \in \mathbb{N}[\bigcup_{i=1}^w \mathcal{Y}_i] \) follows as in (2.15) or not. Our conjecture is positive, but it remains to be seen.

The following is log-convex counterpart.

Corollary 3.5. For \( w \in \mathbb{N} \) let

\[
\sum_{k=0}^\infty A_ku^k = \sum_{k=0}^\infty \frac{B_ku^k}{k!} = \exp\left( \sum_{j=1}^\infty \sum_{i=1}^w c_{i,j}u^j \right).
\]

Suppose that \( (c_{i,k})_{k \geq 1}, i = 1, \ldots, w \) are respectively log-convex in \( k \). Then for \( 1 \leq m \leq n \)

\[
mB_{m-1}B_{n+1} - (n + 1)B_mB_n \geq 0
\]

if and only if

\[
\sum_{i=1}^w c_{i,2} - \left( \sum_{i=1}^w c_{i,1} \right)^2 \geq 0.
\]

Equation (3.7) is equivalent to

\[
A_{m-1}A_{n+1} - A_mA_n \geq 0.
\]
Example 3.8

Example such that log-convexity is preserved under normal convolution.

Due to Theorem 2.1 together with \( c \) and further define \( \mathcal{P} = \mathcal{W} \cup \{ C_{j-1}C_{k+1} - C_jC_k : 0 < j \leq k \} \).

Then the relation (1.1) concludes via Theorem 2.6 that

\[
mB_{m-1}B_{n+1} - (n + 1)B_mB_n \in \mathbb{N}[\mathcal{W}] \text{ for } 1 \leq m \leq n.
\]

Since log-convexity is preserved under the summation, \( (C_k)_{k \geq 1} \) is again log-convex. Moreover, since

\[
C_0C_2 - C_1^2 = \sum_{i=1}^{w} c_{i,2} - \left( \sum_{i=1}^{w} c_{i,1} \right)^2 \geq 0
\]

implies \( C_0C_{k+1} - C_1C_k \geq 0, \ k \geq 1 \), we conclude

\[
mB_{m-1}B_{n+1} - (n + 1)B_mB_n \geq 0 \iff A_{m-1}A_{n+1} - A_mA_n \geq 0
\]

follows. Notice that \( B_0 = 1, B_1 = C_1 \) and \( B_2 = C_2 + C_1^2 \) imply

\[
B_0B_2 - B_1^2 = C_0C_2 - C_1^2,
\]

which proves the ’only if’ part.

\[\square\]

Remark 3.6. (i) Log-convexity is generally not preserved by convolution. However, under the condition (3.8) of Corollary 3.5 it is preserved.

(ii) Log-convexity is preserved under binomial convolution. The result (3.7) in Corollary 3.5 is slightly stronger but with the condition (3.8).

In what follows, we see examples of convoluted sequences which reflect obtained theories in this section. We use statistical distributions for construction of sequences.

Example 3.7 (Geometric distribution sequence). Let \( c_k = p(1 - p)^{k-1}, k = 1, 2, \ldots, p \in (0, 1) \) and \( c_0 = 1 \) in (1.1). We observe that \( c_k^2 - c_{k-1}c_{k+1} = 0 \), so that \( (c_k) \) is both log-concave and log-convex. Due to Theorem 2.7 together with \( c_k^2 - c_0c_2 = p(2p - 1) \), the sequence \( (a_k) \) is log-concave (resp. log-convex) for \( p \geq 1/2 \) (resp. \( p \leq 1/2 \)). Now in (3.1) put \( c_{1,k} = p(1 - p)^{k-1} \) with \( c_{1,0} = 1 \) and prepare another Geometric sequence by \( c_{2,k} = p'(1 - p')^{k-1}, p' \in (0, 1), c_{2,0} = 1 \) and further define \( C_k = c_{1,k} + c_{2,k} \). Note that \( (C_k)_{k \geq 1} \) is log-convex since

\[
C_k^2 - C_{k-1}C_{k+1} = -pp'(1 - p)^{k-1}(1 - p')^{k-1}(2 - p - p')^2 \leq 0.
\]

However, from Theorem 3.7 if \( p, p' \geq 1/2 \), \( (A_k) \) of (3.1) is log-concave, the result is not included in the original Theorem 2.7. On the other hand even if \( p, p' \leq 1/2 \), \( (A_k) \) is not always log-convex since

\[
C_1^2 - C_0C_2 = (p + p')(2(p + p') - 1) - 2pp' > 0
\]

for \( p = p' = 1/2 \). Furthermore if \( p + p' \leq 1/2 \), then \( (A_k) \) is log-convex, so that this provides an example such that log-convexity is preserved under normal convolution.

Example 3.8 (Log-series distribution sequence). Prepare \( n \) sequences \( c_{i,k} = p_k^i/(k \log(1 - p_i)), p_i \in (0, 1), i = 1, 2, \ldots, n, k = 1, 2, \ldots, \) each of which is known to be a log-series distribution. For each \( i \),

\[
c_{i,k}^2 - c_{i,k-1}c_{i,k+1} \leq 0,
\]
Now we consider the \( n \)th convolution and put \( C_k = \sum_{i=1}^n c_{i,k} \) in the relation (1.1). Then Theorem 2.1 concludes log-convexity of \( (a_k) \) if \( p_1 \geq 1 - e^{-2} \) since

\[
c_2^n c_2 = \frac{p_1^2}{2 \log^2 (1 - p_1)} (2 + \log(1 - p_1)).
\]

Now we consider the \( n \)th convolution and put \( C_k = \sum_{i=1}^n c_{i,k} \) in (3.1). Here \( (C_k)_{k \geq 0} \) with \( C_0 = 1 \) is log-convex and \( (A_k) \) is \( n \)th convolution of the corresponding \( (a_i) \)'s, \( i = 1, 2, \ldots, n \). Since

\[
C_1^n C_2 = \left( \sum_{i=1}^n c_{i,1} \right)^2 - \sum_{i=1}^n c_{i,2}
\]

\[
\leq n \sum_{i=1}^n c_{i,1}^2 - \sum_{i=1}^n c_{i,2}
\]

\[
\leq \sum_{i=1}^n \frac{p_i^2}{2 \log^2 (1 - p_i)} (2n + \log(1 - p_i)) \leq 0
\]

if \( p_{i,k} \geq 1 - e^{-2n} \) respectively, \( (A_k) \) is shown to be log-convex. This is an example such that convolution of log-convex sequences yields again log-convex sequences.

Appendix A. Proof of Lemma [12]

By adjusting number of components in sums and changing subscripts, we have

\[
D_m D_n - D_{m-1} D_{n+1} = x_0 (y_n D_m - y_{n+1} D_{m-1}) := I
\]

\[
+ y_0 \left( x_m \sum_{k=0}^{n-1} x_{k+1} y_{n-1-k} - x_{n+1} \sum_{k=0}^{m-1} x_k y_{n-1-k} \right) := II
\]

\[
+ \sum_{k=0}^{m-1} \sum_{t=0}^{n-1} x_k x_{t+1} (y_{m-k} y_{n-1-t} - y_{m-1-k} y_{n-1-t}) := III.
\]

Equation (I) corresponds to the first term and the sum with \( \ell = 0 \) in the 4th term of (3.6). Moreover, we observe that

II = \( y_0 x_m \sum_{k=0}^{n-m-1} x_{k+1} y_{n-1-k} + y_0 \sum_{k=0}^{m-1} y_{m-1-k} (x_m x_{n-m+1+k} - x_{n+1} x_k) \).

Then by reversing indices of \( k \) in the latter sum, the expression is shown to coincide with sum of the second and the third terms of (3.6) respectively. Again by shifting indices, we obtain

III = \( \sum_{k=0}^{m-1} \sum_{t=1}^{n-m} x_k x_{t} (y_{m-k} y_{n-\ell} - y_{m-1-k} y_{n-\ell+1}) \)

\[
+ \sum_{k=0}^{m-1} \sum_{t=0}^{n-1} x_k x_{n-m+1+t} (y_{m-k} y_{n-1-t} - y_{n-1-k} y_{m-\ell}).
\]

The first sum is equivalent to the sum with \( \ell \geq 1 \) in 4th sum of (3.6), and due to the symmetry of indices, the second sum is written as the 5th sum of (3.6). Now the proof is over. \( \square \)
APPENDIX B. RESULTS IN HANSEN [5]

Our results are heavily depends on Hansen [5] Lemmas 1,2 and Theorems 1,2]. In [5], proofs are sometimes omitted. For readers’ sake we restate results and review proofs recovering omitted parts. For convenience write \( r_k = \lambda(k + 1)f_{k+1} \) so that the recursion (2.6) has

\[(n + 1)P_{n+1} = \sum_{k=0}^{n} r_k P_{n-k}. \]

We are starting with two Lemmas.

**Lemma B.1** (Lemma 2 of [5]). Assume (B.1) and let \( P_{-1} = 0. \) Then

\[ m(m + 2)(P_{m+1}^2 - P_mP_{m+2}) \]

(B.2)

\[ = P_{m+1}(r_0 P_m - P_{m+2}) + \sum_{\ell=0}^{m} \sum_{k=0}^{\ell} (P_{m-\ell}P_{m-k-1} - P_{m-k}P_{m-\ell-1})(r_{k+1}r_{\ell} - r_{\ell+1}r_k), \]

\[ r_{m+1}(m + 2)(P_{m+1}P_{m+3} - P_{m+2}^2) \]

(B.3)

\[ = P_{m+1}(r_{m+2}P_{m+2} - r_{m+1}P_{m+3}) + \sum_{k=0}^{m} (P_{m-k}P_{m+2} - P_{m+1}P_{m-k+1})(r_{m+2}r_k - r_{k+1}r_{m+1}). \]

**Proof.** For (B.2), we have

\[ m(m + 2)P_{m+1}^2 - mP_{m}(m + 2)P_{m+2} \]

\[ = (m + 1)^2 P_{m+1}^2 - mP_{m}(m + 2)P_{m+2} - P_{m+1}^2 \]

\[ = \left( \sum_{k=0}^{m} r_k P_{m-k} \right)^2 - \left( \sum_{k=0}^{m-1} r_k P_{m-1-k} \right) \left( \sum_{k=0}^{m+1} r_k P_{m+1-k} \right) - P_{m+1}^2 \]

\[ = \sum_{k=0}^{m} r_k P_{m-k} \sum_{\ell=1}^{m} r_{\ell} P_{m-\ell} - \sum_{k=0}^{m-1} r_k P_{m-1-k} \sum_{\ell=1}^{m+1} r_{\ell} P_{m+1-\ell} + r_0 P_{m}P_{m+1} - P_{m+1}^2 \]

\[ = P_{m+1}(r_0 P_m - P_{m+1}) + \sum_{k=0}^{m} \sum_{\ell=0}^{m} (r_k r_{\ell+1} - r_{k+1}r_{\ell}) P_{m-k}P_{m-1-\ell} = (B.2), \]

where we notice that terms of \( k = \ell \) are zero in the sum. Next we see (B.3) and write

\[ r_{m+1}(m + 2)(P_{m+1}P_{m+3} - P_{m+2}^2) \]

\[ = r_{m+1}(m + 3)P_{m+1}P_{m+3} - r_{m+1}(m + 2)P_{m+2}^2 - r_{m+1}P_{m+1}P_{m+3} \]

\[ + (m + 2)r_{m+2}P_{m+1}P_{m+2} - (m + 2)r_{m+2}P_{m+1}P_{m+2} \]

\[ = r_{m+1}(m + 3)P_{m+1}P_{m+3} - r_{m+1}(m + 2)P_{m+2}^2 + (m + 1)r_{m+2}P_{m+1}P_{m+2} - (m + 2)r_{m+2}P_{m+1}P_{m+2} \]

\[ + r_{m+2}P_{m+1}P_{m+2} - r_{m+1}P_{m+1}P_{m+3} \]

\[ = r_{m+1}P_{m+1} \sum_{k=0}^{m+2} r_k P_{m+2-k} - r_{m+1}P_{m+2} \sum_{k=0}^{m+1} r_k P_{m+1-k} + r_{m+2}P_{m+2} \sum_{k=0}^{m} r_k P_{m-k} - r_{m+2}P_{m+1} \sum_{k=0}^{m+1} r_k P_{m+1-k} \]

\[- r_{m+1}P_{m+1}P_{m+3} + r_{m+2}P_{m+1}P_{m+2} \]

\[ = r_{m+1}P_{m+1}r_0 P_{m+2} + r_{m+1}P_{m+1}r_{m+2}P_0 + r_{m+1}P_{m+1} \sum_{k=1}^{m+1} r_k P_{m+2-k} \]
when we were working in a joint work.

Acknowledgment:
I am grateful to Tomasz Rolski for indicating the paper by Bender and Canfield (B.3) and Lemma B.1.

Log-convex part : The proof is similar to "Log-concave part", except for applying the induction to sequences, the proof is completed.

\[ P_{m+1}(r_{m+2}P_{m+2} - r_{m+1}P_{m+3}) = r_{m+1}P_{m+1} \]

\[ + r_{m+1}P_{m+1} \sum_{k=0}^{m} r_{k+1}P_{m+1-k} - r_{m+2}P_{m+1} \sum_{k=0}^{m} r_{k},P_{m+1-k} \]

\[ - \sum_{k=0}^{m} r_{m+2}\sum_{k=0}^{m} P_{m-k} + r_{m+2}P_{m+2} \sum_{k=0}^{m} r_{k},P_{m-k} \]

\[ = P_{m+1}(r_{m+2}P_{m+2} - r_{m+1}P_{m+3}) \]

\[ + \sum_{k=0}^{m} P_{m+1}P_{m+1-k}(r_{m+1}r_{k+1} - r_{m+2}r_{k}) + \sum_{k=0}^{m} P_{m+2}P_{m-k}(r_{m+2}r_{k} - r_{m+1}r_{k+1}) \]

\[ = (B.3). \]

\[ \square \]

**Lemma B.2** (Lemma 2 of [5]), Assume (B.1) and \( P_{0} > 0 \), then

(i) if \( (P_{n}) \) is strictly log-concave for \( n = 1, 2, \ldots, m \), then \( r_{0}P_{m} - P_{m+1} > 0 \),

(ii) if \( (r_{n}) \) is strictly log-convex and \( r_{0} > r_{1} < 0 \), then \( r_{m+2}P_{m+2} - r_{m+1}P_{m+3} > 0 \).

**Proof.** (i) Since \( (P_{n+1}/P_{n}) \) is decreasing, we have \( r_{0} = P_{1}/P_{0} > P_{m+1}/P_{m} \).

(ii) Since \( (r_{m+1}/r_{m}) \) is increasing, the recursion (B.1) yields

\[ (m + 3)P_{m+3} = r_{0}P_{m+2} + \sum_{k=0}^{m+1} r_{k+1}P_{m+1-k} \frac{r_{k+1}}{r_{k}} \]

\[ \leq P_{m+2}r_{m+2} + (m + 2)P_{m+2} \max_{1 \leq k \leq m+3} \left\{ \frac{r_{k}}{r_{k-1}} \right\} \]

\[ = (m + 3)P_{m+2}r_{m+2}. \]

\[ \square \]

**Proof of Theorem 2.3** Log-concave part : Suppose \( (r_{m}) \) (or equivalently \( (k_{f}) \)) is strictly log-concave and \( r_{0} > r_{1} \geq 0 \) \( \Leftrightarrow \lambda f_{1}^{2} - 2f_{2} \). Then

\[ 2(P_{1}^{2} - P_{0}P_{2}) = P_{0}^{2}(r_{0}^{2} - r_{1}) > 0. \]

Now with this and Lemma (i), we apply induction to (B.2) to see that \( (P_{n}) \) is strictly log-concave. Since any log-concave sequence can be written as a limit of strictly log-concave sequences, the proof is completed.

Log-convex part : The proof is similar to "Log-concave part", except for applying the induction to (B.3) and Lemma (ii).

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