Neutron Production Rates by Inverse-Beta Decay in Fully Ionized Plasmas

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Abstract

Recently we showed that the nuclear transmutation rates are largely overestimated in the Widom-Larsen theory of the so called ‘Low Energy Nuclear Reactions’. Here we show that unbound plasma electrons are even less likely to initiate nuclear transmutations.

Introduction

Claims of electron-proton conversion into a neutron and a neutrino by inverse beta decay in metallic hydrides have recently been raised [1, 2], in the context of the so called Low Energy Nuclear Reactions (LENR). The condition for the reaction to occur is a considerable mass renormalization of the electrons, to overcome the negative Q-value that, otherwise, would forbid the reaction to occur. Defining a dimensionless parameter, $\beta$, in terms of the electron effective mass, $m^*$, one needs:

$$\beta = \frac{m^*}{m} \geq \frac{m_n - m_p}{m} \approx 2.8 \quad (1)$$

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a reference value, $\beta = 20$ was estimated in [1].

1To avoid confusion, we underscore that the mass renormalization in [1] has nothing to do with the velocity dependent relativistic mass. We consider extremely non-relativistic electrons. The situation is closely analogous to muon capture in muonic atoms, in that case $m^*$ being replaced by the muon mass.
It is not clear at all if such spectacularly large values of $\beta$ can be obtained in metallic hydrides and under which conditions. Nonetheless, assuming a given value of $\beta$, a calculation of the neutron rate can be obtained in a straightforward fashion from known electroweak physics. A calculation along these lines has been presented in Ref. [3] for the case of an electron bound to a proton, superseding the order-of-magnitude estimate presented in [1].

More recently, the authors of Ref. [2] have argued that nuclear transmutations should most likely be started by unbound plasma electrons. Assuming a fully ionized plasma and completely unscreened electrons, they find a rate which is enhanced, with respect to the value obtained for bound electrons, by the so-called Sommerfeld factor, $S_0 (c = 1)$:

$$S_0 = \frac{2\pi\alpha}{v}$$

(2)

where $\alpha$ is the fine structure constant and $v$ is the average thermal velocity of the electrons defined by:

$$v_{th} = \sqrt{\frac{3kT}{m^*}} = \beta^{-1/2} \sqrt{\frac{3kT}{m}} = 3.6 \times 10^{-4} \left[ \left( \frac{T}{5 \times 10^3 \text{ K}} \right) \left( \frac{20}{\beta} \right) \right]^{1/2}$$

(3)

with the numerical value in correspondence to $\beta = 20$ and to the temperature $T \approx 5 \times 10^3 \text{ K}$, estimated in [2] as the temperature that can be reached by hydride cathodes. However, the assumption of completely unscreened electrons may be unrealistic. We consider here the situation in presence of Debye screening, which, in a different context, has been recently analysed in Ref. [4]. We find that at large densities, the plasma enhancement saturates to a value determined by the Debye length, $a_D$:

$$S_0 \rightarrow S = \frac{a_D}{a_B^*}$$

(4)

with:

$$a_B^* = \frac{1}{\alpha m^*} = \beta^{-1} a_B$$

(5)

and $a_B$ the Bohr radius.

\footnote{We shall use the numerical values: $k = 8.617 \times 10^{-5} \text{ eV/K}$, $e^2/\hbar c = \alpha = 1/137.043$ and set $c = \hbar c = 1.$}
Debye Length

Static charges are screened in a plasma. The potential of the electric field of a test charge at rest in a plasma is (in Gaussian units)

\[ \phi = \frac{e}{r} e^{-r/a_D} \]  

(6)

where \( a_D \) is the Debye length defined by:

\[ \frac{1}{a_D^2} = \frac{1}{a_e^2} + \frac{1}{a_i^2} \]  

(7)

The two lengths \( a_{e,i} \) are associated to electrons and ions respectively and are given by [5]

\[ a_e = \left( \frac{kT_e}{4\pi n_e e^2} \right)^{1/2} \]  

(8)

and:

\[ a_i = \left( \frac{kT_i}{4\pi n_i (Ze)^2} \right)^{1/2} \]  

(9)

The difference in temperature between electrons and ions is expected to occur naturally because of the large difference of mass which impedes the exchange of energy in electron-ion collisions. Here we will make the approximation \( a_D = a_e \), which leads to the numerical value:

\[ a_D = 4.87 \AA \times \left[ \left( \frac{T}{5000 \text{K}} \right) \left( \frac{10^{20} \text{cm}^{-3}}{n_e} \right) \right]^{1/2} \]  

(10)

or a Debye mass \( m_D \):

\[ m_D = \frac{\hbar}{a_D} = 404 \text{ eV} \]  

(11)

We therefore get a Debye length of about nine atoms (compared to \( a_B = 0.5 \AA \)) in correspondence to the reference temperature \( T \approx 5 \cdot 10^3 \text{ K} \) and a reference density \( n_e = 10^{20} \text{ cm}^{-3} \). When considering the \( n \) dependence, we shall restrict to the range:

\[ 10^{14} \text{ cm}^{-3} \leq n \leq 6 \times 10^{23} \text{ cm}^{-3} \]  

(12)

Values between \( 10^8 \) and \( 10^{14} \) cm\(^{-3} \) are typical of glow discharges and arcs whereas a value of about \( 10^{22} \) cm\(^{-3} \) is the free electron density in Copper [6]. Around \( 2.5 \times 10^{21} \) cm\(^{-3} \) the Debye length equals the Bohr radius [3].

\[ ^3 \text{Electron capture occurs spontaneously during the formation of neutron stars, when the} \]
Critical Velocity

The Sommerfeld factors in a plasma, Eqs. (40) and (43), can be obtained from an intuitive argument as follows (see the Appendix for a derivation from the Schrödinger equation following [4]).

We consider a critical value of the velocity, defined as:

$$\frac{2\pi \alpha}{v_{\text{crit}}} = \alpha m^* a_D$$

(13)

In this condition, the de Broglie wavelength of the particle, is equal to the Debye length:

$$\lambda = \frac{2\pi}{m^* v_{\text{crit}}} = a_D$$

(14)

For larger velocities, the wavelength is smaller and the particle probes a region of space smaller than $a_D$, where it sees an essentially unscreened Coulomb potential. In these conditions, we have to use $S_0$, Eq. (2).

For smaller velocities, as $v \to 0$, the wavelength gets larger than $a_D$. The Sommerfeld factor saturates to the value on the r.h.s. of (13), since the particle explores increasingly large portions of neutral plasma, and the screened Sommerfeld factor in Eq. (4) has to be considered.

The critical velocity defined by (13) is:

$$v_{\text{crit}} = 2.48 \cdot 10^{-4} \left(\frac{20}{\beta}\right) \left(\frac{n}{10^{20} \text{cm}^{-3}}\right) \left(\frac{5000 \text{K}}{T}\right)$$

(15)

We consider our electrons to be at $v_{\text{th}}$, Eq. (3). At the reference point, this is larger than $v_{\text{crit}}$, hence we should apply the unscreened result, $S_0$. With increasing density, however, $v_{\text{crit}}$ goes above $v_{\text{th}}$ (at $n \sim 2 \cdot 10^{20} \text{ cm}^{-3}$) and one should apply the screened result, $S$.

Transmutation Rates

To translate the previous discussion into the expected rates for transmutation from electrons in a plasma, we first recall the rate for the transmutation from Fermi energy of the electrons increases above the threshold value, due to the gravitational pressure. This occurs at electron densities $\gtrsim 10^{31} \text{ cm}^{-3}$.

*we use $h = 1$, so that $h = 2\pi$. 
bound electrons [3]:

\[ \Gamma(\bar{e}p \to n\nu_e)_{\text{bound}} = |\psi(0)|^2 \times \frac{1}{2\pi}(G_F m_e)^2 \left[ 1 + 3 \left( \frac{g_A}{g_V} \right)^2 \right] \times (\beta - \beta_0)^2; \]

\[ |\psi(0)|^2 = \frac{\beta^3}{\pi a_B^3} \]

\[ \Gamma_{\text{bound}}[\beta = 20] = 1.8 \cdot 10^{-3} \text{ Hz} \quad (16) \]

The total rate is obtained by multiplying the result \( \Gamma_{\text{bound}} \) by the volume and by the ion density, which we take equal to the electron density, \( n \), because of global neutrality:

\[ \text{Rate}_{\text{bound}} = n \cdot V \cdot \Gamma_{\text{bound}} \quad (17) \]

In the case of plasma electrons, screened and unscreened rates are obtained by the substitution:

\[ |\psi(0)|^2 \to n \cdot (S \text{ or } S_0) \quad (18) \]

and the rate is proportional to \( n^2 \):

\[ \text{Rate}_{\text{plasma}} = n \cdot V \cdot \frac{\Gamma_{\text{bound}}}{|\psi(0)|^2} \cdot n \cdot (S \text{ or } S_0) \quad (19) \]

\( S \) and \( S_0 \) corresponding respectively to the screened Debye plasma and to the unscreened Coulomb case.

For convenience, we normalize the rates in plasma to the rate in Eq. (17), computed for \( \beta = 20 \), already a considerably large rate, although a factor of \( \sim 300 \) smaller than claimed in [1], and see if we can get anywhere close to unity or higher.

The formulae are

\[ \eta_{\text{Debye}}(n, \beta) = \frac{\text{Rate}_{\text{Debye}}}{\text{Rate}_{\text{bound}}[\beta = 20]} = n \frac{\pi a_B^3}{\beta^3} \frac{(\beta - \beta_0)^2}{(20 - \beta_0)^2} S = \]

\[ \pi (na_B^3) \frac{a_D}{a_B} \frac{(\beta - \beta_0)^2}{\beta^2} \frac{(20 - \beta_0)^2}{(20 - \beta_0)^2} \quad (20) \]

and

\[ \eta_{\text{Coul}}(n, \beta) = \frac{\text{Rate}_{\text{Coul}}}{\text{Rate}_{\text{bound}}[\beta = 20]} = n \frac{2\pi \alpha}{v} \frac{\pi a_B^3}{\beta^3} \frac{(\beta - \beta_0)^2}{(20 - \beta_0)^2} \quad (21) \]

for the two cases.
Figure 1: Ratios corresponding to the screened plasma (Sommerfeld factor $S$) and to the unscreened one (Sommerfeld factor $S_0$), for the case $\beta = 20$. The previous discussion indicates that we must use $S_0$ for $v_{\text{crit}} \leq v_{\text{th}}$ and $S$ for $v_{\text{crit}} \geq v_{\text{th}}$. The result is represented by the thick line.

In Fig. 1 we display the ratios corresponding to the screened plasma (Sommerfeld factor $S$) and to the unscreened one (Sommerfeld factor $S_0$), for the case $\beta = 20$. The previous discussion indicates that we must use $S_0$ for $v_{\text{crit}} \leq v_{\text{th}}$ and $S$ for $v_{\text{crit}} \geq v_{\text{th}}$. The result is represented by the thick line.

The rate for electron capture from plasma never goes anywhere close to the capture rate for bound electrons derived in [3] for the same value of $\beta$, let alone to the larger rate quoted in [1]. Our results are in line with the lack of observation of neutrons in plasma discharge experiments recently reported in [8].

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APPENDIX: Sommerfeld factor for electrons in screened and unscreened plasma

Let us consider an attractive screened potential in the plasma in the form:

\[ V(r) = -\frac{\alpha}{r} e^{m_D r} \]  (22)

The radial Schrödinger equation for the two body (e--ion) wave-function, \( \chi(r) \), reads:

\[ \frac{d^2 \chi(r)}{d^2 r} + 2m^* \left( \frac{m^* v^2}{2} - V(r) \right) \chi(r) = 0 \]  (23)

Changing \( r \) into the adimensional variable \( x \):

\[ r = a_B^* x = \frac{1}{\alpha m^*} x \]  (24)

we get:

\[ \chi''(x) + \left( \frac{v^2}{\alpha^2} + \frac{2}{x} e^{-\epsilon x} \right) \chi(x) = 0 \]  (25)

In the limit of small or vanishing \( v \) we write the equation as:

\[ \chi''(x) + k^2(x) \chi(x) = 0 \]  (26)

in terms of an effective momentum:

\[ k^2(x) = \frac{2}{x} e^{-\epsilon x} \]  (27)

and solve it by the WKB method, which gives:

\[ \chi(x) = A \frac{1}{\sqrt{k(x)}} e^{\pm i \int k(x') dx'} \]  (28)

We can use the WKB approximation as long as

\[ \left| \frac{k'(x)}{k^2(x)} \right| \ll 1 \]  (29)

that is:

\[ \frac{e^{\epsilon x/2}}{2\sqrt{2x}(1 + \epsilon x)} \ll 1 \]  (30)
At the value where the exponential bends, namely $\epsilon x = 1$, we have:

$$\left| \frac{k'(x)}{k^2(x)} \right|_{x=1/\epsilon} = \sqrt{\frac{\epsilon}{2}} e^{1/2} = \frac{\epsilon}{k(x=1/\epsilon)} \equiv \frac{\epsilon}{k_{\text{eff}}}$$

(31)

and the condition that this region is within the range of validity of WKB is then:

$$\frac{v_{\text{eff}}}{\alpha} = k_{\text{eff}} \gg \epsilon = \frac{a_B}{a_D} = \frac{a_B}{\beta a_D}$$

(32)

with $\beta$ defined as in Eq. (1).

For $\beta = 20$ and $a_D$ from Eq. (10), we find:

$$v_{\text{eff}} > \frac{\alpha a_B}{\beta a_D} \equiv v_{\text{WKB}} \approx 3.9 \cdot 10^{-5}$$

(33)

On the other hand, the smallest velocity we consider is the thermal velocity, Eq. (3), which is safely within the region of validity of the WKB approximation. Note that $v_{\text{WKB}}$ is simply proportional to the critical velocity $v_{\text{crit}}$ defined in (13):

$$v_{\text{WKB}} = \frac{v_{\text{crit}}}{2\pi}$$

(34)

We are interested in the square modulus of the wavefunction at the origin relative to its unperturbed value (transmutation is taking place at the origin), the ratio being the Sommerfeld enhancement:

$$S_k \sim |\psi_k(0)|^2 = \left| \frac{R_{k,\ell=0}(x=0)}{A_k} \right|^2 = \left| \frac{\chi_k(0)}{A_kx_k} \right|^2$$

(35)

where we have used the fact that $R_{k,\ell}(x) \sim x^\ell$ as $x \to 0$. The constant $A$ depends on the normalization of the radial function at large distances. Since $R_{k,\ell=0}$ goes to a constant as $x \to 0$, we need that $\chi_k(x) \to 0$ as $x \to 0$ or

$$\chi_k(x) \to x\chi_k'(0) \text{ as } x \to 0$$

(36)

thus giving

$$S_k \sim \left| \frac{\chi_k'(0)}{A_k} \right|^2$$

(37)

\footnote{In the conventions of [7], $A = 2$}
Within the region of validity of the WKB approximation, \( k \gtrsim \varepsilon \), we have
\[
\chi(x) = A \frac{1}{\sqrt{k(x)}} e^{\pm i \int x' k(x')} \tag{38}
\]
where \( A \) is chosen to be the same constant which appears in (35). Therefore
\[
S_k \sim \left| \frac{1}{\sqrt{k(x)} e^{\pm i \int x' k(x')}} \left( \pm i - \frac{1}{2} k'(x) \right) \right|^2 \bigg|_{x=0} \tag{39}
\]
the last term in parenthesis being much smaller than one. The maximum value attainable by \( S_k \) is at the border of the WKB approximation limit, i.e. for \( k \sim \varepsilon \), Eq. (32)
\[
S \sim \frac{1}{\varepsilon} = \frac{a}{a_B^*} = \frac{a}{a_B} \beta \tag{40}
\]
In the limit \( \varepsilon \to 0 \), the Schrödinger equation (23) is solved analytically. The ‘in’ wavefunction in the continuous spectrum of the attractive Coulomb field is given by:
\[
\psi_k^{(+)} = e^{\pi k/2} \Gamma(1 - i/k) e^{i k \cdot r} F(i/k, 1, ik \cdot r - ikr) \tag{41}
\]
where \( F = _1 F_1 \) is the Kummer function (hypergeometric confluent). Here \( k \cdot r \) corresponds to \( mv \times r \), measured in units \( 1/m \). Thus it is the adimensional quantity \( v/\alpha \). The same would hold writing \( kr = (k/\alpha m)(\alpha m \cdot r) \).

In these respects \( k/\alpha m \to k \) is dimensionless, \( k = v/\alpha \), and we understand the factor \( e^{\pi k/2} \), or the term \( \Gamma = (1 - i/k) \). The \( k = v/\alpha \) appears in the Schrödinger equation (23).

The action of the attractive Coulomb field on the motion of the particle near the origin can be characterized by the ratio of the square modulus of \( \psi_k^{(+)}(0) \) to the square modulus of the wave function for free motion \( \psi_k(r) = e^{i k \cdot r} \). Using that \( \Gamma^*(z) = \Gamma(z^*) \), \( F(i/k, 1, 0) = 1 \) and:
\[
\Gamma(1 + i/k) \Gamma(1 - i/k) = \frac{\pi}{k \sinh(\pi/k)} \tag{42}
\]
we get the result:
\[
S = S_0 = |\psi_k^{(+)}(0)|^2 = \frac{2}{k(1 - e^{-2\pi/k})} \approx \frac{2\pi}{k} = \frac{2\pi \alpha}{v} \tag{43}
\]
for small velocities \([4, 2]\).
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