A Decidable Fragment of Strategy Logic

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Abstract Strategy Logic (SL, for short) has been recently introduced by Mogavero, Murano, and Vardi as a useful formalism for reasoning explicitly about strategies, as first-order objects, in multi-agent concurrent games. This logic turns to be very powerful, subsuming all major previously studied modal logics for strategic reasoning, including ATL, ATL\textsuperscript{*}, and the like. Unfortunately, due to its expressiveness, SL has a non-elementarily decidable model-checking problem and a highly undecidable satisfiability problem, specifically, $\Sigma_{1}^{1}$-HARD. In order to obtain a decidable sublogic, we introduce and study here One-Goal Strategy Logic (SL\textsuperscript{1G}, for short). This logic is a syntactic fragment of SL, strictly subsuming ATL\textsuperscript{*}, which encompasses formulas in prenex normal form having a single temporal goal at a time, for every strategy quantification of agents. SL\textsuperscript{1G} is known to have an elementarily decidable model-checking problem. Here we prove that, unlike SL, it has the bounded tree-model property and its satisfiability problem is decidable in 2ExpTime, thus not harder than the one for ATL\textsuperscript{*}.

1 Introduction

In open-system verification \cite{5,17}, an important area of research is the study of modal logics for strategic reasoning in the setting of multi-agent games \cite{2,14,24}. An important contribution in this field has been the development of Alternating-Time Temporal Logic (ATL, for short), introduced by Alur, Henzinger, and Kupferman \cite{2}. ATL allows reasoning about strategic behavior of agents with temporal goals. Formally, it is obtained as a generalization of the branching-time temporal logic CTL\textsuperscript{*} \cite{7}, where the path quantifiers $\exists$ and $\forall$ are replaced with strategic modalities of the form $\langle \langle A \rangle \rangle$ and $\llbracket A \rrbracket$, for a set $A$ of agents. Such strategic modalities are used to express cooperation and competition among agents in order to achieve certain temporal goals. In particular, these modalities express selective quantifications over those paths that are the results of infinite games between a coalition and its complement. ATL\textsuperscript{*} formulas are interpreted over concurrent game structures (CGS, for short) \cite{2}, which model interacting processes. Given a CGS $G$ and a set $A$ of agents, the ATL\textsuperscript{*} formula $\langle \langle A \rangle \rangle \psi$ holds at a state $s$ of $G$ if there is a set of strategies for the agents in $A$ such that, no matter which strategy is executed by the agents not in $A$, the resulting outcome of the interaction in $G$ satisfies $\psi$ at $s$. Several decision problems have been investigated about ATL\textsuperscript{*}; both its model-checking and satisfiability problems are decidable in 2ExpTime \cite{27}. The complexity of the latter is just like the one for CTL\textsuperscript{*} \cite{8,9}.

Despite its powerful expressiveness, ATL\textsuperscript{*} suffers from the strong limitation that strategies are treated only implicitly through modalities that refer to games between
competing coalitions. To overcome this problem, Chatterjee, Henzinger, and Piterman introduced Strategy Logic (CHP-SL, for short) \[3\], a logic that treats strategies in two-player turn-based games as first-order objects. The explicit treatment of strategies in this logic allows the expression of many properties not expressible in ATL*.

Although the model-checking problem of CHP-SL is known to be decidable, with a non-elementary upper bound, it is not known if the satisfiability problem is decidable \[4\]. While the basic idea exploited in \[4\] of explicitly quantify over strategies is powerful and useful \[11\], CHP-SL still suffers from various limitations. In particular, it is limited to two-player turn-based games. Furthermore, CHP-SL does not allow different players to share the same strategy, suggesting that strategies have yet to become truly first-class objects in this logic. For example, it is impossible to describe the classic strategy-stealing argument of combinatorial games such as Chess, Go, Hex, and the like \[1\].

These considerations led us to introduce and investigate a new Strategy Logic, denoted SL, as a more general framework than CHP-SL, for explicit reasoning about strategies in multi-agent concurrent games \[20\]. Syntactically, SL extends the linear-time temporal-logic LTL \[26\] by means of strategy quantifiers, the existential \(\langle x \rangle\) and the universal \([x]\), as well as agent binding \((a, x)\), where \(a\) is an agent and \(x\) a variable. Intuitively, these elements can be read as “there exists a strategy \(x\)”, “for all strategies \(x\)”, and “bind agent \(a\) to the strategy associated with \(x\)”, respectively. For example, in a CGS \(G\) with agents \(\alpha\), \(\beta\), and \(\gamma\), consider the property “\(\alpha\) and \(\beta\) have a common strategy to avoid a failure”. This property can be expressed by the SL formula \(\langle x \rangle [y](\alpha, x)(\beta, x)(\gamma, y)(G \neg fail)\). The variable \(x\) is used to select a strategy for the agents \(\alpha\) and \(\beta\), while \(y\) is used to select another one for agent \(\gamma\) such that their composition, after the binding, results in a play where \(fail\) is never met. Further examples, motivations, and results can be found in a technical report \[19\].

The price that one has to pay for the expressiveness of SL w.r.t. ATL* is the lack of important model-theoretic properties and an increased complexity of related decision problems. In particular, in \[20\], it was shown that SL does not have the bounded-tree model property and the related satisfiability problem is highly undecidable, precisely, \(\Sigma_1^1\)-HARD.

The contrast between the undecidability of the satisfiability problem for SL and the elementary decidability of the same problem for ATL*, provides motivation for an investigation of decidable fragments of SL that subsume ATL*. In particular, we would like to understand why SL is computationally more difficult than ATL*.

We introduce here the syntactic fragment One-Goal Strategy Logic (SL[1G], for short), which encompasses formulas in a special prenex normal form having a single temporal goal at a time. This means that every temporal formula \(\psi\) is prefixed with a quantification-binding prefix that quantifies over a tuple of strategies and bind strategies to all agents. With SL[1G] one can express, for example, visibility constraints on strategies among agents, i.e., only some agents from a coalition have knowledge of the strategies taken by those in the opponent coalition. Also, one can describe the fact that, in the Hex game, the strategy-stealing argument does not let the player who adopts it to win. Observe that both the above properties cannot be expressed neither in ATL* nor in CHP-SL.

In a technical report \[19\], we showed that SL[1G] is strictly more expressive that
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Atl$, yet its model-checking problem is 2ExpTime-complete, just like the one for $\text{Atl}^*$, while the same problem for $\text{Sl}$ is non-elementarily decidable. Our main result here is that the satisfiability problem for $\text{Sl}[1g]$ is also 2ExpTime-complete. Thus, in spite of its expressiveness, $\text{Sl}[1g]$ has the same computational properties of $\text{Atl}^*$, which suggests that the one-goal restriction is the key to the elementary complexity of the latter logic too.

To achieve our main result, we use a fundamental property of the semantics of $\text{Sl}[1g]$ called elementariness, which allows us to simplify reasoning about strategies by reducing it to a set of reasonings about actions. This intrinsic characteristic of $\text{Sl}[1g]$ means that, to choose an existential strategy, we do not need to know the entire structure of universally-quantified strategies, as it is the case for $\text{Sl}$, but only their values on the histories of interest. Technically, to formally describe this property, we make use of the machinery of dependence maps, which is introduced to define a Skolemization procedure for $\text{Sl}$, inspired by the one in first-order logic. Using elementariness, we show that $\text{Sl}[1g]$ satisfies the bounded tree-model property. This allows us to efficiently make use of a tree automata-theoretic approach to solve the satisfiability problem. Given a formula $\phi$, we build an alternating co-Büchi tree automaton whose size is only exponential in the size of $\phi$, accepting all bounded-branching tree models of the formula. Then, together with the complexity of automata-nonnecessity checking, we get that the satisfiability procedure for $\text{Sl}[1g]$ is 2ExpTime. We believe that our proof techniques are of independent interest and applicable to other logics as well.

Related works. Several works have focused on extensions of $\text{Atl}^*$ to incorporate more powerful strategic constructs. Among them, we recall the Alternating-Time $\mu$CALCULUS (Alternating-Time $\mu$CALCULUS, for short) [2], Game Logic (GL, for short) [2], Quantified Decision Modality $\mu$CALCULUS (QDMU, for short) [25], Coordination Logic (CL, for short) [10], and some other extensions considered in [6], [21], and [31]. $\mu$CALCULUS and QDMU are intrinsically different from $\text{Sl}[1g]$ (as well as from CHP-SL and $\text{Atl}^*$) as they are obtained by extending the propositional $\mu$-calculus with strategic modalities. CL is similar to QDMU, but with LTL temporal operators instead of explicit fixpoint constructors. GL and CHP-SL are orthogonal to $\text{Sl}[1g]$. Indeed, they both use more than a temporal goal, GL has quantifier alternation fixed to one, and CHP-SL only works for two agents.

The paper is almost self contained; all proofs are reported in the appendixes. In Appendix 8 we recall standard mathematical notation and some basic definitions that are used in the paper. Additional details on $\text{Sl}[1g]$ can be found in the technical report [19].

2 Preliminaries

A concurrent game structure (CGS, for short) [2] is a tuple $\mathcal{G} \triangleq \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \lambda, \tau, s_0 \rangle$, where AP and Ag are finite non-empty sets of atomic propositions and agents, Ac and St are enumerable non-empty sets of actions and states, $s_0 \in \text{St}$ is a designated initial state, and $\lambda : \text{St} \rightarrow 2^{\text{AP}}$ is a labeling function that maps each state to the set of atomic propositions true in that state. Let $\text{Dc} \triangleq \text{Ac}^{\text{Ag}}$ be the set of decisions, i.e., functions from Ag to Ac representing the choices of an action for each agent. Then, $\tau : \text{St} \times \text{Dc} \rightarrow \text{St}$ is a transition function mapping a pair of a state and a decision to a state. If the set of actions is finite, i.e., $b = |\text{Ac}| < \omega$, we say that $\mathcal{G}$ is $b$-bounded, or
simply bounded. If both the sets of actions and states are finite, we say that $\mathcal{G}$ is finite.

A track (resp., path) in a CGS $\mathcal{G}$ is a finite (resp., an infinite) sequence of states $\rho \in \text{St}^*$ (resp., $\pi \in \text{St}^\omega$) such that, for all $i \in [0, |\rho| - 1]$ (resp., $i \in \mathbb{N}$), there exists a decision $d \in \text{Dc}$ such that $(\rho)_{i+1} = \tau((\rho)_i, d)$ (resp., $(\pi)_{i+1} = \tau((\pi)_i, d)$). A track $\rho$ is non-trivial if $|\rho| > 0$, i.e., $\rho \neq \varepsilon$. Trk $\subseteq \text{St}^+$ (resp., Pth $\subseteq \text{St}^\omega$) denotes the set of all non-trivial tracks (resp., paths). Moreover, Trk$(s) \triangleq \{ \rho \in \text{Trk} : \text{fst}(\rho) = s \}$ (resp., Pth$(s) \triangleq \{ \pi \in \text{Pth} : \text{fst}(\pi) = s \}$) indicates the subsets of tracks (resp., paths) starting at a state $s \in \text{St}$.

A strategy is a partial function $f : \text{Trk} \to \text{Ac}$ that maps each non-trivial track in its domain to an action. For a state $s \in \text{St}$, a strategy $f$ is said $s$-total if it is defined on all tracks starting in $s$, i.e., $\text{dom}(f) = \text{Trk}(s)$. $\text{Str} \triangleq \text{Trk} \to \text{Ac}$ (resp., $\text{Str}(s) \triangleq \text{Trk}(s) \to \text{Ac}$) denotes the set of all (resp., $s$-total) strategies. For all tracks $\rho \in \text{Trk}$, by $(f)_\rho \in \text{Str}$ we denote the translation of $f$ along $\rho$, i.e., the strategy with $\text{dom}((f)_\rho) \triangleq \{ l \text{st}(\rho) \cdot \rho' : \rho \cdot \rho' \in \text{dom}(f) \}$ such that $(f)_\rho(l \text{st}(\rho) \cdot \rho') \triangleq f(\rho \cdot \rho')$, for all $\rho \cdot \rho' \in \text{dom}(f)$.

Let $\text{Var}$ be a fixed set of variables. An assignment is a partial function $\chi : \text{Var} \cup \text{Ag} \to \text{Str}$ mapping variables and agents in its domain to a strategy. An assignment $\chi$ is complete if it is defined on all agents, i.e., $\text{Ag} \subseteq \text{dom}(\chi)$. For a state $s \in \text{St}$, it is said that $\chi$ is $s$-total if all strategies $\chi(l)$ are $s$-total, for $l \in \text{dom}(\chi)$. $\text{Asg}(s) \triangleq \text{Var} \cup \text{Ag} \to \text{Str}$ (resp., $\text{Asg}(s) \triangleq \text{Var} \cup \text{Ag} \to \text{Str}(s)$) denotes the set of all (resp., $s$-total) assignments. Moreover, $\text{Asg}(X) \triangleq X \to \text{Str}$ (resp., $\text{Asg}(X, s) \triangleq X \to \text{Str}(s)$) indicates the subset of $X$-defined (resp., $s$-total) assignments, i.e., (resp., $s$-total) assignments defined on the set $X \subseteq \text{Var} \cup \text{Ag}$. For all tracks $\rho \in \text{Trk}$, by $(\chi)_\rho \in \text{Asg}(\text{fst}(\rho))$ we denote the translation of $\chi$ along $\rho$, i.e., the $\text{fst}(\rho)$-total assignment with $\text{dom}((\chi)_\rho) \triangleq \text{dom}(\chi)$, such that $(\chi)_\rho(l) \triangleq (\chi(l))_\rho$, for all $l \in \text{dom}(\chi)$. For all elements $l \in \text{Var} \cup \text{Ag}$, by $\chi[l \mapsto f] \in \text{Asg}$ we denote the new assignment defined on $\text{dom}(\chi[l \mapsto f]) \triangleq \text{dom}(\chi) \cup \{ l \}$ that returns $f$ on $l$ and $\chi$ otherwise, i.e., $\chi[l \mapsto f](l) \triangleq f$ and $\chi[l \mapsto f](l') \triangleq (\chi(l'))_{\rho'}$, for all $l' \in \text{dom}(\chi) \setminus \{ l \}$.

A path $\pi \in \text{Pth}(s)$ starting at a state $s \in \text{St}$ is a play w.r.t. a complete $s$-total assignment $\chi \in \text{Asg}(s)$, $(\chi, s)$-play, for short if, for all $i \in \mathbb{N}$, it holds that $(\pi)_i = \tau((\pi)_i, d)$, where $d(a) = \chi(a)((\pi)_{\leq i})$, for each $a \in \text{Ag}$. The partial function play : $\text{Asg} \times \text{St} \to \text{Pth}$, with $\text{dom}(\text{play}) \triangleq \{ (\chi, s) : \text{Ag} \subseteq \text{dom}(\chi) \land \chi \in \text{Asg}(s) \land s \in \text{St} \}$, returns the $(\chi, s)$-play $\text{play}(\chi, s) \in \text{Pth}(s)$, for all $(\chi, s)$ in its domain.

For a state $s \in \text{St}$ and a complete $s$-total assignment $\chi \in \text{Asg}(s)$, the $i$-th global translation of $(\chi, s)$, with $i \in \mathbb{N}$, is the pair of a complete assignment and a state $(\chi, s)_i \triangleq ((\chi)_<(s) \leq i, (\pi)_i)$, where $\pi = \text{play}(\chi, s)$.

From now on, we use the name of a CGS as a subscript to extract the components from its tuple-structure. Accordingly, if $\mathcal{G} = \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \lambda, s_0 \rangle$, we have $\text{Ac}_\mathcal{G} = \text{Ac}$, $\text{Ag}_\mathcal{G} = \lambda$, $s_{0\mathcal{G}} = s_0$, and so on. Also, we use the same notational concept to make explicit to which CGS the sets $\text{Dc}$, $\text{Trk}$, $\text{Pth}$, etc. are related to. Note that, we omit the subscripts if the structure can be unambiguously individuated from the context.
3 One-Goal Strategy Logic

In this section, we introduce syntax and semantics of One-Goal Strategy Logic (SL[1G], for short), as a syntactic fragment of SL, which we also report here for technical reasons. For more about SL[1G], see [19].

St. Syntax SL syntactically extends LTL by means of two strategy quantifiers, existential {⟨x⟩} and universal [x], and agent binding (a, x), where a is an agent and x is a variable. Intuitively, these elements can be read, respectively, as "there exists a strategy x", "for all strategies x", and "bind agent a to the strategy associated with the variable x". The formal syntax of SL follows.

Definition 1 (St. Syntax). SL formulas are built inductively from the sets of atomic propositions AP, variables Var, and agents Ag, by using the following grammar, where p ∈ AP, x ∈ Var, and a ∈ Ag:

\[ \varphi ::= p \mid \neg\varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid X\varphi \mid \varphi U \varphi \mid \varphi R \varphi \mid \langle a \rangle \varphi \mid \llbracket x \rrbracket \varphi \mid (a,x)\varphi. \]

By sub(φ) we denote the set of all subformulas of the SL formula φ. For instance, with φ = ⟨⟨x⟩⟩(α, x)(F p), we have that sub(φ) = {φ, (a, x)(F p), (F p), p, t}. By free(φ) we indicate the set of free agents/variables of φ defined as the subset of Ag ∪ Var containing (i) all the agents for which there is no variable application before the occurrence of a temporal operator and (ii) all the variables for which there is an application but no quantification. For example, let φ = ⟨⟨x⟩⟩(α, x)(β, y)(F p) be the formula on agents Ag = {α, β, γ}. Then, we have free(φ) = {γ, y}, since γ is an agent without any application before F p and y has no quantification at all. A formula φ without free agents (resp., variables), i.e., with free(φ) ∩ Ag = ∅ (resp., free(φ) ∩ Var = ∅), is named agent-closed (resp., variable-closed). If φ is both agent- and variable-closed, it is named sentence. By snt(φ) we denote the set of all sentences that are subformulas of φ.

St. Semantics As for ATL*, we define the semantics of SL w.r.t. concurrent game structures. For a CGS G, a state s, and an s-total assignment χ with free(φ) ⊆ dom(χ), we write G, χ, s \models φ to indicate that the formula φ holds at s under the assignment χ. The semantics of SL formulas involving p, ¬, ∧, and ∨ is defined as usual in LTL and we omit it here (see [19], for the full definition). The semantics of the remaining part, which involves quantifications, bindings, and temporal operators follows.

Definition 2 (St. Semantics). Given a CGS G, for all SL formulas φ, states s ∈ St, and s-total assignments χ ∈ Asg(s) with free(φ) ⊆ dom(χ), the relation G, χ, s \models φ is inductively defined as follows.

1. G, χ, s \models ⟨⟨x⟩⟩φ iff there exists an s-total strategy f ∈ Str(s) such that G, χ[x \rightarrow f], s \models φ;
2. G, χ, s \models [x]φ iff for all s-total strategies f ∈ Str(s) it holds that G, χ[x \rightarrow f], s \models φ.

Moreover, if free(φ) ∪ {x} ⊆ dom(χ) ∪ {a} for an agent a ∈ Ag, it holds that:
Finally, if $\chi$ is also complete, it holds that:

4. $G, \chi, s \models X \varphi$ if $G, (\chi, s)^{1} \models \varphi$;

5. $G, \chi, s \models \varphi_{1} \cup \varphi_{2}$ if there is an index $i \in \mathbb{N}$ with $k \leq i$ such that $G, (\chi, s)^{i} \models \varphi_{2}$ and, for all indexes $j \in \mathbb{N}$ with $k \leq j < i$, it holds that $G, (\chi, s)^{j} \models \varphi_{1}$;

6. $G, \chi, s \models \varphi_{1} \cap \varphi_{2}$ if, for all indexes $i \in \mathbb{N}$ with $k \leq i$, it holds that $G, (\chi, s)^{i} \models \varphi_{2}$ or there is an index $j \in \mathbb{N}$ with $k \leq j < i$ such that $G, (\chi, s)^{j} \models \varphi_{1}$.

Intuitively, at Items 3 and 2, respectively, we evaluate the existential $\langle \{x\} \rangle$ and universal $\langle [x] \rangle$ quantifiers over strategies, by associating them to the variable $x$. Moreover, at Item 4 by means of an agent binding $(a, x)$, we commit the agent $a$ to a strategy associated with the variable $x$. It is evident that the LTL semantics is simply embedded into the SL one.

A C$\chi$G $S$ is a model of an SL sentence $\varphi$, denoted by $G \models \varphi$, iff $G, \emptyset, s_{0} \models \varphi$, where $\emptyset$ is the empty assignment. Moreover, $\varphi$ is satisfiable iff there is a model for it. Given two C$\chi$G $S$’s $G_{1}$ and $G_{2}$ and a sentence $\varphi$, we say that $\varphi$ is invariant under $G_{1}$ and $G_{2}$ iff it holds that: $G_{1} \models \varphi \iff G_{2} \models \varphi$. Finally, given two SL formulas $\varphi_{1}$ and $\varphi_{2}$ with free($\varphi_{1}$) = free($\varphi_{2}$), we say that $\varphi_{1}$ implies $\varphi_{2}$, in symbols $\varphi_{1} \Rightarrow \varphi_{2}$, if, for all C$\chi$G $S$ states $s \in \text{St}$, and free($\varphi_{1}$)-defined $s$-total assignments $\chi \in \text{Asg}(\text{free}(\varphi_{1}), s)$, it holds that if $G, \chi, s \models \varphi_{1}$ then $G, \chi, s \models \varphi_{2}$. Accordingly, we say that $\varphi_{1}$ is equivalent to $\varphi_{2}$, in symbols $\varphi_{1} \equiv \varphi_{2}$, if $\varphi_{1} \Rightarrow \varphi_{2}$ and $\varphi_{2} \Rightarrow \varphi_{1}$.

As an example, consider the SL sentence $\varphi = \langle [x] \rangle \langle [y] \rangle \langle [z] \rangle (((\alpha, x)(\beta, y)(X p) \land (\alpha, y)(\beta, z)(X q))$. Note that both agents $\alpha$ and $\beta$ use the strategy associated with $y$ to achieve simultaneously the LTL goals $X p$ and $X q$, respectively. A model for $\varphi$ is the C$\chi$G $G \triangleq \langle \{p, q\}, \langle \alpha, \beta \rangle, \{0, 1\}, \{s_{0}, s_{1}, s_{2}, s_{3}\}, \lambda, \tau, s_{0}\rangle$, where $\lambda(s_{0}) \triangleq 0$, $\lambda(s_{1}) \triangleq \{p\}$, $\lambda(s_{2}) \triangleq \{p, q\}$, $\lambda(s_{3}) \triangleq \{q\}$, $\tau(s_{0}, (0, 0)) \triangleq s_{1}$, $\tau(s_{0}, (1, 0)) \triangleq s_{2}$, $\tau(s_{0}, (1, 0)) \triangleq s_{3}$, and all the remaining transitions go to $s_{0}$. See the representation of $G$ depicted in Figure 1 in which vertexes are states of the game and labels on edges represent decisions of agents or sets of them, where the symbol $*$ is used in place of every possible action. Clearly, $G \models \varphi$ by letting, on $s_{0}$, the variables $x$ to chose action 0 (the formula $(\alpha, x)(\beta, y)(X p)$ is satisfied for any choice of $y$, since we can move from $s_{0}$ to either $s_{1}$ or $s_{2}$, both labeled with $p$) and $z$ to choose action 1 when $y$ has action 0 and, vice versa, 0 when $y$ has 1 (in both cases, the formula $(\alpha, y)(\beta, z)(X q)$ is satisfied, since one can move from $s_{0}$ to either $s_{2}$ or $s_{3}$, both labeled with $q$).

### SL[1G] Syntax

To formalize the syntactic fragment SL[1G] of SL, we need first to define the concepts of quantification and binding prefixes.

**Definition 3 (Prefixes).** A quantification prefix over a set $V \subseteq \text{Var}$ of variables is a finite word $\varphi \in \{\langle \{x\} \rangle, \langle [x] \rangle : x \in V\}^{|V|}$ of length $|V|$ such that each variable $x \in V$ occurs just once in $\varphi$. A binding prefix over a set $V \subseteq \text{Var}$ of variables is a finite word
We denote the set of \( \psi \) where \( G \) is a C\(_G\) model of length \(|Ag|\) of the form \( L[1]G \) such that each agent \( a \in Ag \) occurs just once in \( b \). Finally, \( \text{Qnt}(V) \subseteq \{ [[x]], [[x]] : x \in V \}^{|V|} \) and \( \text{Bnd}(V) \subseteq \{ (a, x) : a \in Ag \land x \in V \}^{|Ag|} \) denote, respectively, the sets of all quantification and binding prefixes over variables in \( V \).

We can now define the syntactic fragment we want to analyze. The idea is to force each group of agent bindings, represented by a binding prefix, to be coupled with a quantification prefix.

**Definition 4 (\( SL[1]G \) Syntax).** \( SL[1]G \) formulas are built inductively from the sets of atomic propositions \( AP \), quantification prefixes \( \text{Qnt}(V) \), for \( V \subseteq \text{Var} \), and binding prefixes \( \text{Bnd}(\text{Var}) \), by using the following grammar, with \( p \in AP \), \( \varphi \in \cup_{V \subseteq \text{Var}} \text{Qnt}(V) \), and \( b \in \text{Bnd}(\text{Var}) \):

\[
\varphi ::= p | \neg \varphi | \varphi \land \varphi | \varphi \lor \varphi | X \varphi | \varphi U \varphi | \varphi R \varphi | \varphi b \varphi,
\]

with \( \varphi \in \text{Qnt}(\text{free}(b \varphi)) \), in the formation rule \( \varphi b \varphi \).

In the following, for a *goal* we mean an \( SL \) agent-closed formula of the kind \( b \psi \); where \( \psi \) is variable-closed and \( b \in \text{Bnd(}\text{free} (\psi)\)\). Note that, since \( b \psi \) is a goal, it is agent-closed, so, \( \text{free}(b \psi) \subseteq \text{Var} \). Moreover, an \( SL[1]G \) sentence \( \varphi \) is *principal* if it is of the form \( \varphi = \psi b \psi \), where \( \psi \) is a goal and \( \varphi \in \text{Qnt}(\text{free}(b \psi)) \). By \( \text{psnt}(\varphi) \subseteq \text{snt}(\varphi) \) we denote the set of principal sub-sentences of the \( SL[1]G \) formula \( \varphi \).

As an example, let \( \varphi_1 = \psi \psi_1 \) and \( \varphi_2 = \psi(\psi_1 \psi_2 \psi_2) \), where \( \psi = [[x]]([[y]]([[z]]) \rightend{array} \leftend{array}

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\( b_1 = (\alpha, x)(\beta, y)(\gamma, z), b_2 = (\alpha, y)(\beta, z)(\gamma, y), \psi_1 = X p, \psi_2 = X q. \) Then, it is evident that \( \varphi_1 \in \text{SL}[1G] \) but \( \varphi_2 \notin \text{SL}[1G] \), since the quantification prefix \( \varphi \) of the latter does not have in its scope a unique goal.

It is fundamental to observe that the formula \( \varphi_1 \) of the above example cannot be expressed in \( ATL^* \), as proved in \([19]\) and reported in the following theorem, since its 2-quantifier alternation cannot be encompassed in the 1-alternation \( ATL^* \) modalities. On the contrary, each \( ATL^* \) formula of the type \( \langle A \rangle \psi \), where \( A = \{ \alpha_1, \ldots, \alpha_n \} \subseteq Ag = \{ \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \} \) can be expressed in \( SL[1G] \) as follows: \( \langle x_1 \rangle \ldots \langle x_n \rangle \langle y_1 \rangle \ldots \langle y_m \rangle \langle \alpha_1, x_1 \rangle \cdots \langle \alpha_n, x_n \rangle \langle \beta_1, y_1 \rangle \cdots \langle \beta_m, y_m \rangle \psi \).

**Theorem 1.** \( SL[1G] \) is strictly more expressive than \( ATL^* \).

We now give two examples in which we show the importance of the ability to write specifications with alternation of quantifiers greater than 1 along with strategy sharing.

**Example 1 (Escape from Alcatraz).** Consider the situation in which an Alcatraz prisoner tries to escape from jail with the help of an external accomplice of him, by helicopter. Due to his panoramic point of view, assume that the accomplice has the full visibility on the behaviors of guards, while the prisoner does not have the same ability. Therefore, the latter has to put in practice an escape strategy that, independently from guards moves, can be supported by his accomplice to escape. We can formalize such an intricate situation by means of an \( SL[1G] \) sentence as follows. First, let \( G_A \) be a CGS modeling the possible situations in which the agents “\( p \)” prisoner, “\( g \)”

\(^1\) We thank Luigi Sauro for having pointed out this example.
We now define the concept of a reasoning about strategies by reducing it to a set of local reasonings.

The goal of each player is to be the first to form a path connecting the opposing sides of the board marked by his color. It is easy to prove that the stealing-strategy argument does not lead to a winning strategy in Hex, i.e., if the player that moves second copies the moves of the opponent, he surely loses the play. It is possible to formalize this fact in SL[1G] as follows. First model Hex with a CGS \( G_H \) whose states represent a possible possible configurations reached during a play between “r” red and “b” blue. Then, express the negation of the stealing-strategy argument by asserting that \( G_H \models \langle \langle x \rangle \rangle (r, x)(b, x)(F \text{ cnc} r) \). Intuitively, this sentence says that agent \( r \) has a strategy that, once it is copied (binded) by \( b \) it allows the former to win, i.e., to be the first to connect the related red edges (\( F \text{ cnc} r \)).

4 Strategy Quantifications

We now define the concept of a dependence map. The key idea is that every quantification prefix occurring in an SL formula can be represented by a suitable choice of a dependence map over strategies. Such a result is at the base of the definition of the elementariness property and allows us to prove that SL[1G] is elementarily satisfiable, i.e., we can simplify a reasoning about strategies by reducing it to a set of local reasonings about actions [19].

**Dependence map** First, we introduce some notation regarding quantification prefixes. Let \( \varphi \in \text{Qnt}(V) \) be a quantification prefix over a set \( V(\varphi) \triangleq V \subseteq \text{Var} \) of variables. By \( \langle \langle \varphi \rangle \rangle \triangleq \{ x \in V : \exists i \in [0, |\varphi|] . (\varphi)_i = \langle \langle x \rangle \rangle \} \) and \( \llbracket \varphi \rrbracket \triangleq \langle \langle \varphi \rangle \rangle \setminus \langle \langle \varphi \rangle \rangle \) we denote, respectively, the sets of existential and universal variables quantified in \( \varphi \). For two variables \( x, y \in V \), we say that \( x \) precedes \( y \) in \( \varphi \), in symbols \( x \prec \varphi y \), if \( x \) occurs before \( y \) in \( \varphi \). Moreover, by \( \text{Dep}(\varphi) \triangleq \{ (x, y) \in V \times V : x \in \llbracket \varphi \rrbracket, y \in \llbracket \varphi \rrbracket \wedge x \prec \varphi y \} \) we denote the set of dependence pairs, i.e., a dependence relation, on which we derive the parameterized version \( \text{Dep}(\varphi, y) \triangleq \{ x \in V : (x, y) \in \text{Dep}(\varphi) \} \) containing all variables from which \( y \) depends. Also, we use \( \overline{\varphi} \in \text{Qnt}(V) \) to indicate the quantification derived from \( \varphi \) by dualizing each quantifier contained in it, i.e., for all \( i \in [0, |\varphi|] \), it holds that \( (\overline{\varphi})_i = \langle (\varphi)_i \rangle \text{ iff } (\varphi)_i = \llbracket x \rrbracket \), with \( x \in V \). Clearly, \( \langle \langle \overline{\varphi} \rangle \rangle = \langle \langle \varphi \rangle \rangle \) and \( \llbracket \overline{\varphi} \rrbracket = \llbracket \varphi \rrbracket \). Finally, we define the notion of valuation of variables over a generic set \( D \) as a partial function \( v : \text{Var} \to D \) mapping every variable in its domain to an element in \( D \). By \( \text{Val}_D(V) \triangleq V \to D \) we denote the set of all valuation functions over \( D \) defined on \( V \subseteq \text{Var} \).

We now give the semantics for quantification prefixes via the following definition of dependence map.
Definition 5 (Dependence Maps). Let $\varphi \in \text{Qnt}(V)$ be a quantification prefix over a set of variables $V \subseteq \text{Var}$, and $D$ a set. Then, a dependence map for $\varphi$ over $D$ is a function $\theta : \text{Val}_D([\varphi]) \to \text{Val}_D(V)$ satisfying the following properties:

1. $\theta(\varphi)_{[[\varphi]]} = v$, for all $v \in \text{Val}_D([\varphi])$;
2. $\theta(v_1)(x) = \theta(v_2)(x)$, for all $v_1, v_2 \in \text{Val}_D([\varphi])$ and $x \in [\varphi]$ such that $v_1|_{\text{Dep}(\varphi, x)} = v_2|_{\text{Dep}(\varphi, x)}$.

$\text{DM}_D(\varphi)$ denotes the set of all dependence maps for $\varphi$ over $D$.

Intuitively, Item 1 asserts that $\theta$ takes the same values of its argument w.r.t. the universal variables in $\varphi$ and Item 2 ensures that the value of $\theta$ w.r.t. an existential variable $x$ in $\varphi$ does not depend on variables not in $\text{Dep}(\varphi, x)$. To get better insight into this definition, a dependence map $\theta$ for $\varphi$ can be considered as a set of Skolem functions that, given a value for each variable in $V$ that is universally quantified in $\varphi$, returns a possible value for all the existential variables in $\varphi$, in a way that is consistent w.r.t. the order of quantifications.

We now state a fundamental theorem that describes how to eliminate strategy quantifications of an St. formula via a choice of a dependence map over strategies. This procedure, easily proved to be correct by induction on the structure of the formula in [19], can be seen as the equivalent of the Skolemization in first order logic [13].

Theorem 2 (St. Strategy Quantification). Let $\mathcal{G}$ be a CGS and $\varphi = \varphi\psi$ an St. sentence, where $\psi$ is agent-closed and $\varphi \in \text{Qnt}(\text{free}(\psi))$. Then, $\mathcal{G} \models \varphi$ if there exists a dependence map $\theta \in \text{DM}_{\text{Str}(s_0)}(\varphi)$ such that $\mathcal{G}, \theta(\chi), s_0 \models \psi$, for all $\chi \in \text{Asg}([\varphi], s_0)$.

The above theorem substantially characterizes the St. semantics by means of the concept of dependence map. In particular, it shows that if a formula is satisfiable then it is always possible to find a suitable dependence map returning the existential strategies in response to the universal ones. Such a characterization lends itself to define alternative semantics of St., based on the choice of a subset of dependence maps that meet a certain given property. We do this on the aim of identifying semantic fragments of ST having better model properties and easier decision problems. With more details, given a CGS $\mathcal{G}$, one of its states $s$, and a property $P$, we say that a sentence $\varphi\psi$ is $P$-satisfiable, in symbols $\mathcal{G} \models P \varphi\psi$, if there exists a dependence map $\theta$ meeting $P$ such that, for all assignment $\chi \in \text{Asg}([\varphi], s)$, it holds that $\mathcal{G}, \theta(\chi), s \models \psi$. Alternative semantics identified by a property $P$ are even more interesting if there exists a syntactic fragment corresponding to it, i.e., each satisfiable sentence of such a fragment is $P$-satisfiable and vice versa. In the following, we put in practice this idea in order to show that $\text{St}[1G]$ has the same complexity of $\text{ATL}^*$ w.r.t. the satisfiability problem.

Elementary quantifications According to the above description, we now introduce a suitable property of dependence maps, called elementariness, together with the related alternative semantics. Then, in Theorem 5 we state that $\text{St}[1G]$ has the elementariness property, i.e., each $\text{St}[1G]$ sentence is satisfiable iff it is elementary satisfiable.

Intuitively, a dependence map $\theta \in \text{DM}_{T \rightarrow D}(\varphi)$ over functions from a set $T$ to a set $D$ is elementary if it can be split into a set of dependence maps over $D$, one for each element of $T$, represented by a function $\theta : T \to \text{DM}_D(\varphi)$. This idea allows us to
enormously simplify the reasoning about strategy quantifications, since we can reduce them to a set of quantifications over actions, one for each track in their domains.

Note that sets D and T, as well as U and V used in the following, are generic and in our framework they may refer to actions and strategies (D), tracks (T), and variables (U and V). In particular, observe that functions from T to D represent strategies. We prefer to use abstract names, as the properties we describe hold generally.

To formally develop the above idea, we have first to introduce the important variant of S

Definition 6 (Adjoint Functions). Let D, T, U, and V be four sets, and \( m : (T \to D)^U \to (T \to D)^V \) and \( \tilde{m} : T \to (D^U \to D^V) \) two functions. Then, \( \tilde{m} \) is the adjoint of \( m \) if \( \tilde{m}(t)(g)(x) = m(g)(x)(t) \), for all \( g \in (T \to D)^U \), \( x \in V \), and \( t \in T \).

Intuitively, a function \( m \) transforming a map of kind \( (T \to D)^U \) into a new map of kind \( (T \to D)^V \) has an adjoint \( \tilde{m} \) if such a transformation can be done pointwisely w.r.t. the set \( T \), i.e., we can put out as a common domain the set \( T \) and then transform a map of kind \( D^U \) in a map of kind \( D^V \). Observe that, if a function has an adjoint, this is unique. Similarly, from an adjoint function it is possible to determine the original function unambiguously. Thus, it is established a one-to-one correspondence between functions admitting an adjoint and the adjoint itself.

The formal meaning of the elementariness of a dependence map over generic functions follows.

Definition 7 (Elementary Dependence Maps). Let \( \phi \in \text{Qut}(V) \) be a quantification prefix over a set \( V \subseteq \text{Var} \) of variables, D and T two sets, and \( \theta \in \text{EDM}_{T \to D}(\phi) \) a dependence map for \( \phi \) over \( T \to D \). Then, \( \theta \) is elementary if it admits an adjoint function. \( \text{EDM}_{T \to D}(\phi) \) denotes the set of all elementary dependence maps for \( \phi \) over \( T \to D \).

As mentioned above, we now introduce the important variant of SL[1G] semantics based on the property of elementariness of dependence maps over strategies. We refer to the related satisfiability concept as \textit{elementary satisfiability}, in symbols \( \models_E \).

The new semantics of SL[1G] formulas involving atomic propositions, Boolean connectives, temporal operators, and agent bindings is defined as for the classic one, where the modeling relation \( \models \) is substituted with \( \models_E \), and we omit to report it here. In the following definition, we only describe the part concerning the quantification prefixes. Observe that by \( \zeta_\phi : \text{Ag} \to \text{Var}, \) for \( \phi \in \text{Bnd}(\text{Var}) \), we denote the function associating to each agent the variable of its binding in \( \phi \).

Definition 8 (SL[1G] Elementary Semantics). Let \( \mathcal{G} \) be a CGS, \( s \in \text{St} \) one of its states, and \( \phi \psi \) an SL[1G] principal sentence. Then \( \mathcal{G}, \varnothing, s \models_E \phi \psi \) iff there is an elementary dependence map \( \theta \in \text{EDM}_{\text{Str}(s)}(\phi) \) for \( \phi \) over \( \text{Str}(s) \) such that \( \mathcal{G}, \theta(\chi) \circ \zeta_\phi, s \models_E \psi, \) for all \( \chi \in \text{Asg}(\llbracket \phi \rrbracket), s \).
It is immediate to see a strong similarity between the statement of Theorem 2 of SL strategy quantification and the previous definition. The only crucial difference resides in the choice of the kind of dependence map. Moreover, observe that, differently from the classic semantics, the quantifications in a prefix are not treated individually but as an atomic block. This is due to the necessity of having a strict correlation between the point-wise structure of the quantified strategies.

Finally, we state the following fundamental theorem which is a key step in the proof of the bounded model property and decidability of the satisfiability for SL$^1_G$, whose correctness has been proved in [19]. The idea behind the proof of the elementariness property resides in the strong similarity between the statement of Theorem 2 of SL strategy quantification and the definition of the winning condition in a classic turn-based two-player game. Indeed, on one hand, we say that a sentence is satisfiable iff “there exists a dependence map such that, for all all assignments, it holds that ...”. On the other hand, we say that the first player wins a game iff “there exists a strategy for him such that, for all strategies of the other player, it holds that ...”. The gap between these two formulations is solved in SL$^1_G$ by using the concept of elementary quantification. So, we build a two-player turn-based game in which the two players are viewed one as a dependence map and the other as a valuation over universal quantified variables, both over actions, such that the formula is satisfied iff the first player wins the game. This construction is a deep technical evolution of the proof method used for the dualization of alternating automata on infinite objects [22]. Precisely, it uses Martin’s Determinacy Theorem [18] on the auxiliary turn-based game to prove that, if there is no dependence map of a given prefix that satisfies the given property, there is a dependence map of the dual prefix satisfying its negation.

**Theorem 3 (SL$^1_G$ Elementariness).** Let $\mathcal{G}$ be a CGS and $\varphi$ an SL$^1_G$ sentence. Then, $\mathcal{G} \models \varphi$ iff $\mathcal{G} \models E \varphi$.

In order to understand what elementariness means from a syntactic point of view, note that in SL$^1_G$ it holds that $\varphi \Box X \psi \equiv \varphi \Box X \varphi \psi$, i.e., we can requantify the strategies to satisfy the inner subformula $\psi$. This equivalence is a generalization of what is well known to hold for CTL*: $EX \psi \equiv EX E \psi$. Moreover, note that, as reported in [19], elementariness does not hold for more expressive fragments of SL, such as SL$^1_B G$.

## 5 Bounded Dependence Maps

Here we prove a boundedness property for dependence maps crucial to get, in Section 6, the bounded tree-model property for SL$^1_G$, which is a preliminary step towards our decidability proof for the logic.

As already mentioned, on reasoning about the satisfiability of an SL$^1_G$ sentence, one can simplify the process, via elementariness, by splitting a dependence map over strategies in a set of dependence maps over actions. Thus, to gain the bounded model property, it is worth understanding how to build dependence maps over a predetermined finite set of actions, while preserving the satisfiability of the sentence of interest.

The main difficulty here is that, the verification process of a sentence $\varphi$ over an (unbounded) CGT $\mathcal{T}$ may require some of its subsentences, perhaps in contradiction among them, to be checked on disjoint subtrees of $\mathcal{T}$. For example, consider the formula
\[ \varphi = \phi_1 \land \phi_2, \text{ where } \phi_1 = \wp_1 \forall X p \text{ and } \phi_2 = \wp_2 \forall X \neg p \text{ with } \wp = (\alpha, x)(\beta, y)(\gamma, z). \]

It is evident that, if \( T \models \varphi \), the two strategy quantifications made via the prefixes \( \wp_1 \) and \( \wp_2 \) have to select two disjoint subtrees of \( T \) on which verify the temporal properties \( X p \) and \( X \neg p \), respectively. So, a correct pruning of \( T \) in a bounded tree-model has to keep the satisfiability of the subsentences \( \phi_1 \) and \( \phi_2 \) separated, by avoiding the collapse of the relative subtrees, which can be ensured via the use of an appropriate number of actions.

By means of characterizing properties named overlapping (see Definitions 12 and 13) on quantification-binding prefixes and sets of dependence maps, called signatures (see Definition 9) and signature dependences (see Definition 13), respectively, we ensure that the set of required actions is finite. Practically, we prove that sentences with overlapping signatures necessarily share a common subtree, independently from the number of actions in the model (see Corollary 1). Conversely, sentences with non-overlapping signatures may need different subtrees. So, a model must have a sufficient big set of actions, which we prove to be finite anyway (see Theorem 5). Note that, in the previous example, \( \varphi \) to be satisfiable needs to have non-overlapping signatures, since otherwise there is at least a shared outcome on which verify the incompatible temporal properties \( X p \) and \( X \neg p \).

We now give few more details on the idea behind the properties described above. Suppose to have a set of quantification prefixes \( Q \subseteq \text{Qut}(V) \) over a set of variables \( V \). We ask whether there is a relation among the elements of \( Q \) that forces a set of related dependence maps to intersect their ranges in at least one valuation of variables. For instance, consider in the previous example the prefixes to be set as follows: \( \wp_1 = [x] [y] [z] \) and \( \wp_2 = [x] [y] [z] \). Then, we want to know whether an arbitrary pair of dependence maps \( \theta_1 \in \text{DM}_D(\wp_1) \) and \( \theta_2 \in \text{DM}_D(\wp_2) \) has intersecting ranges, for a set \( D \). In this case, since \( y \) is existentially quantified in both prefixes, we can build \( \theta_1 \) and \( \theta_2 \) in such a way they choose different elements of \( D \) on \( y \), when they do the same choices on the other variables, supposed that \( |D| > 1 \). Thus, if the prefixes share at least an existential variable, it is possible to find related dependence maps that are non-overlapping. Indeed, in this case, the formula \( \varphi \) is satisfied on the C\( G \)S \( G_{SA} \) of Figure 2 since we can allow \( y \) on \( s_0 \) to chose 0 for \( \wp_1 \) and 1 for \( \wp_2 \).

Now, let consider the following prefixes: \( \wp_1 = [x] [z] [y] \) and \( \wp_2 = [z] [y] [x] \). Although, in this case, each variable is existentially quantified at most once, we have that \( x \) and \( z \) mutually depend in the different prefixes. So, there is a cyclic dependence that can make two related non-overlapping dependence maps. Indeed, suppose to have \( D = \{0, 1\} \). Then, we can choose \( \theta_1 \in \text{DM}_D(\wp_1) \) and \( \theta_2 \in \text{DM}_D(\wp_2) \) in the way that, for all valuations \( v_1 \in \text{dom}(\theta_1) \) and \( v_2 \in \text{dom}(\theta_2) \), it holds that \( \theta_1(v_1)(z) \equiv v_1(x) \) and \( \theta_2(v_2)(x) \equiv 1 - v_2(z) \). Thus, \( \theta_1 \) and \( \theta_2 \) do not intersect their ranges. In-
deed, with the considered prefixes, the formula \( \varphi \) is satisfied on the CGS \( G_{CD} \) of Figure [3] by using the dependence maps described above.

Finally, consider a set of prefixes in which there is neither a shared existential quantified variable nor a cyclic dependence, such as the following: \( \varphi_1 \triangleq \{ x \} \{ y \} \{ z \} \), \( \varphi_2 \triangleq \{ y \} \{ x \} \{ z \} \), and \( \varphi_3 \triangleq \{ y \} \{ x \} \{ z \} \). We now show that an arbitrary choice of dependence maps \( \theta_1 \in DM_D(\varphi_1) \), \( \theta_2 \in DM_D(\varphi_2) \), and \( \theta_3 \in DM_D(\varphi_3) \) must have intersecting ranges, for every set \( D \). Indeed, since \( y \) in \( \varphi_2 \) does not depend from any other variable, there is a value \( d_y \in D \) such that, for all \( v_2 \in \text{dom}(\theta_2) \), it holds that \( \theta_2(v_2)(y) = d_y \). Now, since \( x \) in \( \varphi_3 \) depends only on \( y \), there is a value \( d_x \in D \) such that, for all \( v_3 \in \text{dom}(\theta_3) \) with \( v_3(y) = d_y \), it holds that \( \theta_3(v_3)(x) = d_x \). Finally, we can determine the value \( d_\varphi \in D \) of \( z \) in \( \varphi_1 \) since \( x \) and \( y \) are fixed. So, for all \( v_1 \in \text{dom}(\theta_1) \) with \( v_1(x) = d_x \) and \( v_1(y) = d_y \), it holds that \( \theta_1(v_1)(z) = d_\varphi \). Finally, we run this procedure since we can find at each step an existential variable that depends only on universal variables previously determined.

In order to formally define the above procedure, we need to introduce some preliminary definitions. As first thing, we generalize the described construction by taking into account not only quantification prefixes but binding prefixes too. This is due to the fact that different principal subsentences of the specification can share the same quantification prefix by having different binding prefixes. Moreover, we need to introduce a tool that gives us a way to differentiate the check of the satisfiability of a given sentence in different parts of the model, since it can use different actions when starts the check from different states. For this reason, we introduce the concepts of signature and labeled signature. The first is used to arrange opportunistically prefixes with bindings, represented in a more general form through the use of a generic support set \( E \), while the second allows us to label signatures, by means of a set \( L \), to maintain an information on different instances of the same sentence.

**Definition 9 (Signatures).** A signature on a set \( E \) is a pair \( \sigma \triangleq (\varphi, b) \in \text{Qnt}(V) \times V^E \) of a quantification prefix \( \varphi \) over \( V \) and a surjective function \( b \) from \( E \) to \( V \), for a given set of variables \( V \subseteq \text{Var} \). A labeled signature on \( E \) w.r.t. a set \( L \) is a pair \( (\sigma, l) \in (\text{Qnt}(V) \times V^E) \times L \) of a signature \( \sigma \) on \( E \) and a labeling \( l \) in \( L \). The sets \( \text{Sig}(E) \triangleq \bigcup_{V \subseteq \text{Var}} \text{Qnt}(V) \times V^E \) and \( \text{LSig}(E, L) \triangleq \text{Sig}(E) \times L \) contain, respectively, all signatures on \( E \) and labeled signatures on \( E \) w.r.t. \( L \).

We now extend the concepts of existential quantification and functional dependence from prefixes to signatures. By \( \ll Br(\sigma) \rr \triangleq \{ e \in E : b(e) \in \ll \varphi \rr \} \), \( \text{Dep}(\sigma) \triangleq \{ (e', e'') \in E \times E : b(e') \neq b(e'') \} \), and \( \text{Col}(\sigma) \triangleq \{ (e', e'') \in E \times E : b(e') = b(e'') \in \ll \varphi \rr \} \), with \( \sigma = (\varphi, b) \in \text{Sig}(E) \), we denote the set of existential elements, and the relation sets of functional dependent and collapsing elements, respectively. Moreover, for a set \( S \subseteq \text{Sig}(E) \) of signatures, we define \( \text{Col}(S) \triangleq (\bigcup_{\sigma \in S} \text{Col}(\sigma))^+ \) as the transitive relation set of collapsing elements and \( \ll S \rr \triangleq \bigcup_{\sigma \in S} \ll S, \sigma \rr \), with \( \ll S, \sigma \rr \triangleq \{ e \in \ll \sigma \rr : \exists \sigma' \in S, e' = (\varphi', b') \in \ll \sigma' \rr, (\sigma \neq \sigma' \lor b(e) \neq b'(e')) \land (e, e') \in \text{Col}(S) \} \), as the set of elements that are existential in two signatures, either directly or via a collapsing chain. Finally, by \( \text{Dep}'(\sigma) \triangleq \{ (e', e'') \in E \times E : \exists e''' \in E : (e', e''') \in \text{Col}(S) \land (e', e''') \in \text{Dep}(\sigma) \} \) we indicate the relation set of functional dependent elements connected via a collapsing chain.
As described above, if a set of prefixes has a cyclic dependence between variables, we are sure to find a set of dependence maps, bijectively related to such prefixes, that do not share any total assignment in their codomains. Here, we formalize this concept of dependence by considering bindings too. In particular, the check of dependences is not done directly on variables, but by means of the associated elements of the support set $E$. Note that, in the case of labeled signatures, we do not take into account the labeling component, since two instances of the same signature with different labeling cannot have a mutual dependent variable.

To give the formal definition of cyclic dependence, we first provide the definition of S-chain.

**Definition 10 (S-Chain).** An S-chain for a set of signatures $S \subseteq \text{Sig}(E)$ on $E$ is a pair $(\vec{e}, \vec{\sigma}) \in E^k \times S^k$, with $k \in [1, \omega]$, for which the following hold:

1. $\text{lst}(\vec{e}) \in \llbracket \text{lst}(\vec{\sigma}) \rrbracket$;
2. $((\vec{e})_i, (\vec{e})_{i+1}) \in \text{Dep}^l((\vec{\sigma})_i)$, for all $i \in [0, k - 1]$;
3. $(\vec{\sigma})_i \neq (\vec{\sigma})_j$, for all $i, j \in [0, k]$ with $i < j$.

It is important to observe that, due to Item 3, each S-chain cannot have length greater than $|S|$.

Now we can give the definition of cyclic dependence.

**Definition 11 (Cyclic Dependences).** A cyclic dependence for a set of signatures $S \subseteq \text{Sig}(E)$ on $E$ is an S-chain $(\vec{e}, \vec{\sigma})$ such that $(\text{lst}(\vec{e}), \text{fst}(\vec{e})) \in \text{Dep}^l(\text{lst}(\vec{\sigma}))$. Moreover, it is a cyclic dependence for a set of labeled signatures $P \subseteq \text{LSign}(E, L)$ on $E$ w.r.t. $L$ if it is a cyclic dependence for the set of signatures $\{\sigma \in \text{Sig}(E) : \exists l \in L. (\sigma, l) \in P\}$. The sets $C(S), C(P) \subseteq E^+ \times S^+$ contain, respectively, all cyclic dependences for signatures in $S$ and labeled signatures in $P$.

Observe that $|C(S)| \leq |E|^{|S|}.|S|!$, so, $|C(P)| \leq |E|^{|P|}.|P|!$.

At this point, we can formally the property of overlapping for signatures. According to the above description, this implies that dependence maps related to prefixes share at least one total variable valuation in their codomains. Thus, we say that a set of signatures is overlapping if they do not have common existential variables and there is no cyclic dependence. Observe that, if there are two different instances of the same signature having an existential variable, we can still construct a set of dependence maps that do not share any valuation, so we have to avoid this possibility too.

**Definition 12 (Overlapping Signatures).** A set $S \subseteq \text{Sig}(E)$ of signatures on $E$ is overlapping if $\llbracket S \rrbracket = \emptyset$ and $C(S) = \emptyset$. A set $P \subseteq \text{LSign}(E, L)$ of labeled signatures on $E$ w.r.t. $L$ is overlapping if $\{\sigma \in \text{Sig}(E) : \exists l \in L. (\sigma, l) \in P\}$ is overlapping and, for all $(\sigma, l'), (\sigma, l'') \in P$, if $\llbracket \sigma \rrbracket \neq \emptyset$ then $l' = l''$.

Finally, to manage the one-to-one connection between signatures and related dependence maps, it is useful to introduce the simple concept of signature dependence, which associates to every signature a related dependence map. We also define, as expected, the concept of overlapping for these functions, which intuitively states that the contained dependence maps have identical valuatuons of variables in their codomains, once they are composed with the related functions on the support set.
Definition 13 (Signature Dependences). A signature dependence for a set of signatures \( S \subseteq \text{Sig}(E) \) on \( E \) over \( D \) is a function \( \nu : S \rightarrow \bigcup_{(p, b) \in S} \text{DM}_D(p) \) such that, for all \( (p, b) \in S \), it holds that \( \nu((p, b)) \in \text{DM}_D(p) \). A signature dependence for a set of labeled signatures \( P \subseteq \text{LSig}(E, L) \) on \( E \) w.r.t. \( L \) over \( D \) is a function \( \nu : P \rightarrow \bigcup_{(p, b, l) \in P} \text{DM}_D(p) \) such that, for all \( ((p, b), l) \in P \), it holds that \( \nu(((p, b), l)) \in \text{DM}_D(p) \). The sets \( \text{SigDep}_D(S) \) and \( \text{LSigDep}_D(P) \) contain, respectively, all signature dependences for \( S \) and labeled signature dependences for \( P \) over \( D \). A signature dependence \( \nu \in \text{SigDep}_D(S) \) is overlapping if \( \cap_{(p, b) \in S} \{ v \circ b : v \in \text{rng}(\nu((p, b))) \} \neq \emptyset \). A labeled signature dependence \( \nu \in \text{LSigDep}_D(P) \) is overlapping if \( \cap_{((p, b), l) \in P} \{ v \circ b : v \in \text{rng}(\nu(((p, b), l))) \} \neq \emptyset \).

As explained above, signatures and signature dependences have a strict correlation w.r.t. the concept of overlapping. Indeed, the following result holds. The idea here is to find, at each step of the construction of the common valuation, a variable, called pivot, that does not depend on other variables whose value is not already set. This is possible if there are no cyclic dependences and each variable is existential in at most one signature.

Theorem 4 (Overlapping Dependence Maps). Let \( S \subseteq \text{Sig}(E) \) be a finite set of overlapping signatures on \( E \). Then, for all signature dependences \( \nu \in \text{SigDep}_D(S) \) for \( S \) over a set \( D \), it holds that \( \nu \) is overlapping.

This theorem can be easily lifted to labeled signatures, as stated in the following corollary.

Corollary 1 (Overlapping Dependence Maps). Let \( P \subseteq \text{LSig}(E, L) \) be a finite set of overlapping labeled signatures on \( E \) w.r.t. \( L \). Then, for all labeled signature dependences \( \nu \in \text{LSigDep}_D(P) \) for \( P \) over a set \( D \), it holds that \( \nu \) is overlapping.

Finally, if the set \( D \) is sufficiently large, in the case of non-overlapping labeled signatures, we can find a signature dependence that is non-overlapping too, as reported in following theorem. The high-level combinatorial idea behind the proof is to assign to each existential variable, related to a given element of the support set in a signature, a value containing a univocal flag in \( P \times V(P) \), where \( V(P) \triangleq \bigcup_{(p, b) \in P} V(p) \), representing the signature itself. Thus, signatures sharing an existential element surely have related dependence maps that cannot share a common valuation. Moreover, for each cyclic dependence, we choose a particular element whose value is the inversion of that assigned to the element from which it depends, while all other elements preserve the related values. In this way, in a set of signature having cyclic dependences, there is one of them whose associated dependence maps have valuations that differ from those in the dependence maps of the other signatures, since it is the unique that has an inversion of the values.

Theorem 5 (Non-Overlapping Dependence Maps). Let \( P \subseteq \text{LSig}(E, L) \) be a set of labeled signatures on \( E \) w.r.t. \( L \). Then, there exists a labeled signature dependence \( \nu \in \text{LSigDep}_D(P) \) for \( P \) over \( D \triangleq P \times V(P) \times \{0, 1\}^{C(P)} \) such that, for all \( P' \subseteq P \), it holds that \( \nu_{|P'} \in \text{LSigDep}_D(P') \) is non-overlapping, if \( P' \) is non-overlapping.
6 Model Properties

We now investigate basic model properties of $S_{L[1]}$ that turn out to be important on their own and useful to prove the decidability of the satisfiability problem.

First, recall that the satisfiability problem for branching-time logics can be solved via tree automata, once a kind of bounded tree-model property holds. Indeed, by using it, one can build an automaton accepting all models of formulas, or their encoding. So, we first introduce the concepts of concurrent game tree, decision tree, and decision-unwinding and then show that $S_{L[1]}$ is invariant under decision-unwinding, which directly implies that it satisfies a unbounded tree-model property. Finally, by using the techniques previously introduced, we further prove that the above property is actually a bounded tree-model property.

Tree-model property  We now introduce two particular kinds of $C_{GS}$ whose structure is a directed tree. As already explained, we do this since the decidability procedure we give in the last section of the paper is based on alternating tree automata.

Definition 14 (Concurrent Game Trees). A concurrent game tree (CGT, for short) is a $C_{GS}$ $T \equiv \langle AP, Ag, Ac, St, \lambda, \tau, \varepsilon \rangle$, where (i) $St \subseteq \Delta^*$ is a $\Delta$-tree for a given set $\Delta$ of directions and (ii) if $t \cdot e \in St$ then there is a decision $d \in Dc$ such that $\tau(t, d) = t \cdot e$, for all $t \in St$ and $e \in \Delta$. Furthermore, $T$ is a decision tree (DT, for short) if (i) $St = Dc^*$ and (ii) if $t \cdot d \in St$ then $\tau(t, d) = t \cdot d$, for all $t \in St$ and $d \in Dc$.

Intuitively, CGTs are $C_{GS}$s with a tree-shaped transition relation and DTs have, in addition, states uniquely determining the history of computation leading to them.

At this point, we can define a generalization for $C_{GS}$s of the classic concept of unwinding of labeled transition systems, namely decision-unwinding. Note that, in general and differently from $A_{TL^*}$, $St$ is not invariant under decision-unwinding, as we show later. On the contrary, $S_{L[1]}$ satisfies such an invariance property. This fact allows us to show that this logic has the unbounded tree-model property.

Definition 15 (Decision-Unwinding). Let $G$ be a $C_{GS}$. Then, the decision-unwinding of $G$ is the DT $G_{DU} \equiv \langle AP, Ag, Ac_G, Dc_G^*, \lambda_G, \tau_G, \varepsilon \rangle$ for which there is a surjective function $\text{unw} : Dc_G^* \to St_G$ such that (i) $\text{unw}(\varepsilon) = s_{0G}$, (ii) $\text{unw}(\tau(t, d)) = \tau_G(\text{unw}(t), d)$, and (iii) $\lambda(t) = \lambda_G(\text{unw}(t))$, for all $t \in Dc_G^*$ and $d \in Dc_G$.

Note that each $C_{GS}$ $G$ has a unique associated decision-unwinding $G_{DU}$.

We say that a sentence $\varphi$ has the decision-tree model property if, for each $C_{GS}$ $G$, it holds that $G \models \varphi$ iff $G_{DU} \models \varphi$. By using a standard proof by induction on the structure of $S_{L[1]}$ formulas, we can show that this logic is invariant under decision-unwinding, i.e., each $S_{L[1]}$ sentence has decision-tree model property, and, consequently, that it satisfies the unbounded tree-model property. For the case of the combined quantification and binding prefixes $\varphi \psi$, we can use a technique that allows to build, given an elementary dependence map $\theta$ satisfying the formula on a $C_{GS}$ $G$, an elementary dependence map $\theta'$ satisfying the same formula over the DT $G_{DU}$, and vice versa. This construction is based on a step-by-step transformation of the adjoint of a dependence maps into another, which is done for each track of the original model. This means that
we do not actually transform the strategy quantifications but the equivalent infinite set of action quantifications.

**Theorem 6 (SL[1G] Positive Model Properties).**

1. SL[1G] is invariant under decision-unwinding;
2. SL[1G] has the decision-tree model property.

Although this result is a generalization of that proved to hold for ATL*, it actually represents an important demarcation line between SL[1G] and SL. Indeed, as we show in the following theorem, SL does not satisfy neither the tree-model property nor, consequently, the invariance under decision-unwinding.

**Theorem 7 (SL Negative Model Properties).**

1. SL does not have the decision-tree model property;
2. SL is not invariant under decision-unwinding.

**Bounded tree-model property** We now have all tools we need to prove the bounded tree-model property for SL[1G], which we recall SL does not satisfy [20]. Actually, we prove here a stronger property, which we name **bounded disjoint satisfiability**.

To this aim, we first introduce the new concept regarding the satisfiability of different instances of the same subsentence of the original specification, which intuitively states that these instances can be checked on disjoint subtrees of the tree model. With more detail, this property asserts that, if two instances use part of the same subtree, they are forced to use the same dependence map as well. This intrinsic characteristic of SL[1G] is fundamental to build a unique automaton that checks the truth of all subsentences, by simply merging their respective automata, without using a projection operation that eliminates their proper alphabets, which otherwise can be in conflict. In this way, we are able to avoid an exponential blow-up. A clearer discussion on this point is reported later in the paper.

**Definition 16 (SL[1G] Disjoint Satisfiability).** Let $\mathcal{T}$ be a CGT, $\varphi \triangleq \varphi \psi$ an SL[1G] principal sentence, and $S \triangleq \{ s \in St : T, \emptyset, s \models \varphi \}$. Then, $T$ satisfies $\varphi$ disjointly over $S$ if there exist two functions $\text{head} : S \to \text{DM}_{\mathcal{Ac}}(\varphi)$ and $\text{body} : \text{Trk}(\varepsilon) \to \text{DM}_{\mathcal{Ac}}(\psi)$ such that, for all $s \in S$ and $\chi \in \text{Asg}(\models \varphi, s)$, it holds that $T, \emptyset(\chi), s \models \psi$, where the elementary dependence maps $\emptyset \in \text{EDM}_{\text{Str}(s)}(\varphi)$ is defined as follows: (i) $\emptyset(s) \triangleq \text{head}(s)$; (ii) $\emptyset(\rho) \triangleq \text{body}(\rho' \cdot \rho)$, for all $\rho \in \text{Trk}(s)$ with $|\rho| > 1$, where $\rho' \in \text{Trk}(\varepsilon)$ is the unique track such that $\rho' \cdot \rho \in \text{Trk}(\varepsilon)$.

In the following theorem, we finally describe the crucial step behind our automata-theoretic decidability procedure for SL[1G]. At an high-level, the proof proceeds as follows. We start from the satisfiability of the specification $\varphi$ over a DT $\mathcal{T}$, whose existence is ensured by Item 2 of Theorem 6 of SL[1G] positive model properties. Then, we construct an intermediate DT $\mathcal{T}_2$, called flagged model, which is used to check the satisfiability of all subsentences of $\varphi$ in a disjoint way. Intuitively, the flagged model adds a controller agent, named sharp that decides on which subtree a given subsentence has to be verified. Now, by means of Theorem 3 on the SL[1G] elementariness, we construct the adjoint functions of the dependence maps used to verify the satisfiability of
the sentences on $T$. Then, by applying Corollary 1 and Theorem 5 of overlapping and non-overlapping dependence maps, respectively, we transform the dependence maps over actions, contained in the ranges of the adjoint functions, in a bounded version, which preserves the satisfiability of the sentences on a bounded pruning $T'$ of $T$. Finally, we remove the additional agent $\hat{\xi}$ obtaining the required bounded DT $T'$. Observe that, due to the particular construction of the bounded dependence maps, the disjoint satisfiability is preserved after the elimination of $\hat{\xi}$.

**Theorem 8 (SL[1G] Bounded Tree-Model Property).** Let $\varphi$ be an SL[1G] satisfiable sentence and $P \triangleq \{(\varphi, b), (\psi, i)\} \in L\text{Sig}(Ag, SL \times \{0, 1\}) : \varphi \psi \in psnt(\varphi) \land i \in \{0, 1\}$ the set of all labeled signatures on Ag w.r.t. $SL \times \{0, 1\}$ for $\varphi$. Then, there exists a $b$-bounded DT $T$, with $b = |P| \cdot |V(P)| \cdot 2^{|L(P)|}$, such that $T \models \varphi$. Moreover, for all $\phi \in psnt(\varphi)$, it holds that $T$ satisfies $\phi$ disjointly over the set $\{s \in St : T, s \models \varphi\}$.

### 7 Satisfiability Procedure

We finally solve the satisfiability problem for SL[1G] and show that it is $2\text{ExpTime}$-complete, as for ATL*. The algorithmic procedures is based on an automata-theoretic approach, which reduces the decision problem for the logic to the emptiness problem of a universal Co-Büchi tree automaton (UCT, for short) [12]. From an high-level point of view, the automaton construction seems similar to what was proposed in literature for CTL* [16] and ATL* [27]. However, our technique is completely new, since it is based on the novel notions of elementariness and disjoint satisfiability.

**Principal sentences** To proceed with the satisfiability procedure, we have to introduce a concept of encoding for an assignment and the labeling of a DT.

**Definition 17 (Assignment-Labeling Encoding).** Let $T$ be a DT, $t \in St_T$ one of its states, and $\chi \in Asg_T(V, t)$ an assignment defined on the set $V \subseteq \text{Var}$. A $(\text{Val}_{AcT}(V) \times 2^{\text{AP}})$-labeled $DcT$-tree $T' \triangleq (St_T, u)$ is an assignment-labeling encoding for $\chi$ on $T$ if $uivist(p) = (\hat{\chi}(\rho), \lambda_T(ivist(p)))$, for all $\rho \in Trk_T(t)$.

Observe that there is a unique assignment-labeling encoding for each assignment over a given DT.

Now, we prove the existence of a UCT $U^{AcC}_v$ for each SL[1G] goal $\psi$ having no principal subsentences. $U^{AcC}_v$ recognizes all the assignment-labeling encodings $T'$ of an a priori given assignment $\chi$ over a generic DT $T$, once the goal is satisfied on $T$ under $\chi$. Intuitively, we start with a UCW, recognizing all infinite words on the alphabet $2^{\text{AP}}$ that satisfy the LTL formula $\psi$, obtained by a simple variation of the Vardi-Wolper construction [29]. Then, we run it on the encoding tree $T'$ by following the directions imposed by the assignment in its labeling.

**Lemma 1 (SL[1G] Goal Automaton).** Let $\psi$ be an SL[1G] goal without principal subsentences and $Ac$ a finite set of actions. Then, there exists an UCT $U^{AcC}_v \triangleq (\text{Val}_{AcC}(\text{free}(\psi))) \times 2^{\text{AP}}, \text{Dec}, Q_v, \delta_v, q_{0v}, N_v$ such that, for all DTs $T$ with $AcT = Ac$, states $t \in St_T$, and assignments $\chi \in Asg_T(\text{free}(\psi), t)$, it holds that $T, \chi, t \models \psi$ iff $T' \in L(U^{AcC}_v)$, where $T'$ is the assignment-labeling encoding for $\chi$ on $T$. 


We now introduce a new concept of encoding regarding the elementary dependence maps over strategies.

**Definition 18 (Elementary Dependence-Labeling Encoding).** Let $T$ be a DT, $t \in \text{St}_{\text{T}}$ one of its states, and $\theta \in \text{EDM}_{\text{St}_{\text{T}}(t)}(\psi)$ an elementary dependence map over strategies for a quantification prefix $\psi \in \text{Qnt}(V)$ over the set $V \subseteq \text{Var}$. $A (\text{DM}_{\text{Ac}}(\phi) \times 2^\mathbb{N})$-labeled $\Delta$-tree $T' \triangleq (\text{St}_T, u)$ is an elementary dependence-labeling encoding for $\theta$ on $T$ if $u(\text{lst}((\rho)_{\geq 1})) = (\bar{\theta}(\rho), \lambda_T(\text{lst}(\rho)))$, for all $\rho \in \text{Trk}_T(t)$.

Observe that also in this case there exists a unique elementary dependence-model encoding for each elementary dependence map over strategies.

Finally, in the next lemma, we show how to handle locally the strategy quantifications on each state of the model, by simply using a quantification over actions modeled by the choice of an action dependence map. Intuitively, we guess in the labeling what is the right part of the dependence map over strategies for each node of the tree and then verify that, for all assignments of universal variables, the corresponding complete assignment satisfies the inner formula.

**Lemma 2 (SL[1G] Sentence Automaton).** Let $\phi \psi$ be an SL[1G] principal sentence without principal subsentences and $A\phi$ a finite set of actions. Then, there exists an UCT $U^{A\phi}_{\phi \psi} \triangleq \langle \text{DM}_{A\phi}(\psi) \times 2^\mathbb{N}, \text{Dc}, \text{Q}_{\phi \psi}, \delta_{\phi \psi}, q_0, q_1, N_{\phi \psi} \rangle$ such that, for all DTs $T$ with $A\phi = A\phi$, states $t \in \text{St}_T$, and elementary dependence maps over strategies $\theta \in \text{EDM}_{\text{St}_{\text{T}}(t)}(\psi)$, it holds that $T, \theta(\chi), t \models_E \forall \psi$, for all $\chi \in \text{Asg}_T([\psi], t)$, iff $T' \in L(U^{A\phi}_{\phi \psi})$, where $T'$ is the elementary dependence-labeling encoding for $\theta$ on $T$.

**Full sentences** By summing up all previous results, we are now able to solve the satisfiability problem for the full SL[1G] fragment.

To construct the automaton for a given SL[1G] sentence $\phi$, we first consider all UCT $U^{A\phi}_{\phi}$, for an assigned bounded set $A\phi$, previously described for the principal sentences $\phi \in \text{pnt}(\phi)$, in which the inner subsentences are considered as atomic propositions. Then, thanks to the disjoint satisfiability property of Definition[15] we can merge them into a unique UCT $U_{\phi}$ that supplies the dependence map labeling of internal components $U^{A\phi}_{\phi}$, by using the two functions head and body contained into its labeling. Moreover, observe that the final automaton runs on a $b$-bounded decision tree, where $b$ is obtained from Theorem[8] on the bounded-tree model property.

**Theorem 9 (SL[1G] Automaton).** Let $\phi$ be an SL[1G] sentence. Then, there exists an UCT $U_{\phi}$ such that $\phi$ is satisfiable iff $L(U_{\phi}) \neq \emptyset$.

Finally, by a simple calculation of the size of $U_{\phi}$ and the complexity of the related emptiness problem, we state in the next theorem the precise computational complexity of the satisfiability problem for SL[1G].

**Theorem 10 (SL[1G] Satisfiability).** The satisfiability problem for SL[1G] is 2EXPTIME-complete.
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8 Mathematical Notation

In this short reference appendix, we report the classical mathematical notation and some common definitions that are used along the whole work.

Classic objects We consider \( \mathbb{N} \) as the set of natural numbers and \( [m, n] \triangleq \{ k \in \mathbb{N} : m \leq k \leq n \} \), \( [m, n) \triangleq \{ k \in \mathbb{N} : m \leq k < n \} \), \( (m, n] \triangleq \{ k \in \mathbb{N} : m < k \leq n \} \), and \( [m, n) \triangleq \{ k \in \mathbb{N} : m < k < n \} \) as its interval subsets, with \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \triangleq \mathbb{N} \cup \{ \omega \} \), where \( \omega \) is the numerable infinity, i.e., the least infinite ordinal.

Given a set \( X \) of objects, we denote by \( |X| \in \mathbb{N} \cup \{ \infty \} \) the cardinality of \( X \), i.e., the number of its elements, where \( \infty \) represents a more than countable cardinality, and by \( 2^X \triangleq \{ Y : Y \subseteq X \} \) the powerset of \( X \), i.e., the set of all its subsets.

Relations By \( R \subseteq X \times Y \) we denote a relation between the domain \( \text{dom}(R) \triangleq X \) and codomain \( \text{codom}(R) \triangleq Y \), whose range is indicated by \( \text{rng}(R) \triangleq \{ y \in Y : \exists x \in X. (x, y) \in R \} \) to represent the inverse of \( R \) itself. Moreover, by \( S \circ R \), with \( R \subseteq X \times Y \) and \( S \subseteq Y \times Z \), we denote the composition of \( R \) with \( S \), i.e., the relation \( S \circ R \triangleq \{ (x, z) \in X \times Z : \exists y \in Y. (x, y) \in R \land (y, z) \in S \} \). We also use \( R^n \triangleq R^{n-1} \circ R \), with \( n \in [1, \omega] \), to indicate the \( n \)-iteration of \( R \subseteq X \times Y \), where \( Y \subseteq X \) and \( R^0 \triangleq \{(y, y) : y \in Y\} \) is the identity on \( Y \). With \( R^+ \triangleq \bigcup_{n=1}^{\omega} R^n \) and \( R^* \triangleq R^+ \cup R^0 \) we denote, respectively, the transitive and reflexive-transitive closure of \( R \). Finally, for an equivalence relation \( R \subseteq X \times X \) on \( X \), we represent with \( (X/R) \triangleq \{[x]_R : x \in X\} \), where \([x]_R \triangleq \{ x' \in X : (x, x') \in R \} \), the quotient set of \( X \) w.r.t. \( R \), i.e., the set of all related equivalence classes \( \{[\cdot]_R\} \).

Functions We use the symbol \( Y^X \subseteq 2^{X \times Y} \) to denote the set of total functions \( f \) from \( X \) to \( Y \), i.e., the relations \( f \subseteq X \times Y \) such that for all \( x \in \text{dom}(f) \) there is exactly one element \( y \in \text{cod}(f) \) such that \( (x, y) \in f \). Often, we write \( f : X \to Y \) and \( f : X \to Y \) to indicate, respectively, \( f \in Y^X \) and \( f \in \bigcup_{X \subseteq Y} Y^X \). Regarding the latter, note that we consider \( f \) as a partial function from \( X \) to \( Y \), where \( \text{dom}(f) \subseteq X \) contains all and only the elements for which \( f \) is defined. Given a set \( Z \), by \( f|Z \triangleq f \cap (Z \times Y) \) we denote the restriction of \( f \) to the set \( X \cap Z \), i.e., the function \( f|Z : X \cap Z \to Y \) such that, for all \( x \in \text{dom}(f) \cap Z \), it holds that \( f|Z(x) = f(x) \). Moreover, with \( \emptyset \) we indicate a generic empty function, i.e., a function with empty domain. Note that \( X \cap Z = \emptyset \) implies \( f|Z = \emptyset \). Finally, for two partial functions \( f, g : X \to Y \), we use \( f \cup g \) and \( f \cap g \) to represent, respectively, the union and intersection of these functions defined as follows: \( \text{dom}(f \cup g) \triangleq \text{dom}(f) \cup \text{dom}(g) \setminus \{ x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x) \} \), \( \text{dom}(f \cap g) \triangleq \{ x \in \text{dom}(f) \cap \text{dom}(g) : f(x) = g(x) \} \), \( (f \cup g)(x) = f(x) \) for \( x \in \text{dom}(f \cup g) \cap \text{dom}(f) \), \( (f \cup g)(x) = g(x) \) for \( x \in \text{dom}(f \cup g) \cap \text{dom}(g) \), and \( (f \cap g)(x) = f(x) \) for \( x \in \text{dom}(f \cap g) \).

Words By \( X^n \), with \( n \in \mathbb{N} \), we denote the set of all \( n \)-tuples of elements from \( X \), by \( X^* \triangleq \bigcup_{n=0}^{\infty} X^n \) the set of finite words on the alphabet \( X \), by \( X^+ \triangleq X^* \setminus \{ \epsilon \} \) the set of non-empty words, and by \( X^\omega \) the set of infinite words, where, as usual, \( \epsilon \in X^* \).
is the empty word. The length of a word \( w \in X^\infty \triangleq X^* \cup X^\omega \) is represented with \( |w| \in \mathbb{N} \). By \( (w)_i \) we indicate the \( i \)-th letter of the finite word \( w \in X^* \), with \( i \in [0, |w|) \). Furthermore, by \( \text{fst}(w) \triangleq (w)_0 \) (resp., \( \text{lst}(w) \triangleq (w)_{|w|-1} \)), we denote the first (resp., last) letter of \( w \). In addition, by \( (w)_{<i} \) (resp., \( (w)_{>i} \)), we indicate the prefix up to (resp., suffix after) the letter of index \( i \) of \( w \), i.e., the finite word built by the first \( i + 1 \) (resp., last \( |w| - i - 1 \) letters \( (w)_0, \ldots, (w)_i \)) (resp., \( (w)_{i+1}, \ldots, (w)_{|w|-1} \)). We also set \( (w)_{<0} \triangleq \varepsilon \), \( (w)_{<i} \triangleq (w)_{i-1} \), \( (w)_{\geq 0} \triangleq w \), and \( (w)_{\geq i} \triangleq (w)_{i-1} \) for \( i \in [1, |w|) \).

Mutatis mutandis, the notations of \( i \)-th, letter, first, prefix, and suffix apply to infinite words too. Finally, by \( \text{pf}(w_1, w_2) \in X^\infty \) we denote the maximal common prefix of two different words \( w_1, w_2 \in X^\infty \), i.e., the finite word \( w \in X^* \) for which there are two words \( w_1', w_2' \in X^\infty \) such that \( w_1 = w \cdot w_1', w_2 = w \cdot w_2' \), and \( \text{fst}(w_1') \neq \text{fst}(w_2') \). By convention, we set \( \text{pf}(w, w) \triangleq w \).

Trees
For a set \( \Delta \) of objects, called directions, a \( \Delta \)-tree is a set \( T \subseteq \Delta^* \) closed under prefix, i.e., if \( t \cdot d \in T \), with \( d \in \Delta \), then also \( t \in T \). We say that it is complete if it holds that \( t \cdot d' \in T \) whenever \( t \cdot d \in T \), for all \( d' < d \), where \( \subseteq \subseteq \Delta \times \Delta \) is an a priori fixed strict total order on the set of directions that is clear from the context. Moreover, it is full if \( T = \Delta^* \). The elements of \( T \) are called nodes and the empty word \( \varepsilon \) is the root of \( T \). For every \( t \in T \) and \( d \in \Delta \), the node \( t \cdot d \in T \) is a successor of \( t \) in \( T \). The tree is \( b \)-bounded if the maximal number \( b \) of its successor nodes is finite, i.e., \( b = \max_{t \in T} |\{ t \cdot d \in T : d \in \Delta \}| < \omega \). A branch of the tree is an infinite word \( w \in \Delta^\omega \) such that \( (w)_{<i} \in T \), for all \( i \in \mathbb{N} \). For a finite set \( \Sigma \) of objects, called symbols, a \( \Sigma \)-labeled \( \Delta \)-tree is a quadruple \( (\Sigma, \Delta, T, \nu) \), where \( T \) is a \( \Delta \)-tree and \( \nu : T \to \Sigma \) is a labeling function. When \( \Delta \) and \( \Sigma \) are clear from the context, we call \( (T, \nu) \) simply a (labeled) tree.

9 Proofs of Section 5

In this appendix, we give the proofs of Theorem 4 and Corollary 1 of overlapping dependence maps and Theorem 5 of non-overlapping dependence maps. In particular, to prove the first two results, we need to introduce the concept of pivot for a given set of signatures and then show some useful related properties. Moreover, for the latter result, we define an apposite ad-hoc signature dependence, based on a sharp combinatorial construction, in order to maintain separated the dependence maps associated to the components of a non-overlapping set of signatures.

Pivot
To proceed with the definitions, we have first to introduce some additional notation. Let \( E \) be a set and \( \sigma \in \text{Sig}(E) \) a signature. Then, \( \llbracket \sigma \rrbracket \triangleq E \setminus \llbracket \langle \sigma \rangle \rrbracket \) indicates the set of elements of \( E \) associated to universal quantified variables. Moreover, for an element \( e \in E \), we denote by \( \text{Dep}(\sigma, e) \triangleq \{ e' \in E : (e', e) \in \text{Dep}(\sigma) \} \) the set of elements from which \( e \) is functional dependent. Given another element \( e' \in E \), we say that \( e \) precedes \( e' \) in \( \sigma \), in symbols \( e <_{\sigma} e' \), if \( b(e) <_{\psi} b(e') \), where \( \sigma = (\psi, b) \). Observe that this kind of order is, in general, not total, due to the fact that \( b \) is not necessarily injective. Consequently, by \( \min_{<_{\sigma}} F \), with \( F \subseteq E \), we denote the set of minimal elements of \( F \) w.r.t. \( <_{\sigma} \).

Finally, for a given set of signatures \( S \subseteq \text{Sig}(E) \), we indicate by \( \llbracket S \rrbracket \triangleq \bigcap_{\sigma \in S} \llbracket \sigma \rrbracket \) the
set of elements that are universal in all signatures of \( S \), by \( \text{Col}(S, e) \triangleq \{ e' \in E \setminus [S] \} \) the set of existential elements that form a collapsing chain with \( e \), and by \( \text{Col}(S, \sigma) \triangleq \{ e \in E : \exists e' \in \langle \sigma \rangle . (e', e) \in \text{Col}(S) \} \) the set of elements that form a collapsing chain with at least one existential element in \( \sigma \).

Intuitively, a pivot is an element on which we can extend a partial assignment that is shared by a set of dependence maps related to signatures via a signature dependence, in order to find a total assignment by an iterative procedure. Let \( F \) the domain of a partial function \( d : E \rightarrow D \) and \( e \) an element not yet defined, i.e., \( e \in E \setminus F \). If, on one hand, \( e \) is existential quantified over a signature \( \sigma = \langle \nu, b \rangle \) and all the elements from which \( e \) depends on that signature are in the domain \( F \), then the value of \( e \) is uniquely determined by the related dependence map. So, \( e \) is a pivot. If, on the other hand, \( e \) is universal quantified over all signatures \( \sigma \in S \) and all elements that form a collapsing chain with \( e \) are in the domain \( F \), then, also in this case we can define the value of \( e \) being sure to leave the possibility to build a total assignment. So, also in this case \( e \) is a pivot. For this reason, pivot plays a fundamental role in the construction of such shared assignments. The existence of a pivot for a given finite set of signatures \( S \subseteq \text{Sig}(E) \) w.r.t. a fixed domain \( F \) of a partial assignment is ensured under the hypothesis that there are no cyclic dependences in \( S \). The existence proof passes through the development of three lemmas describing a simple seeking procedure.

With the previous description and the examples of Section 5 in mind, we now formally describe the properties that an element of the support set has to satisfy in order to be a pivot for a set of signatures w.r.t. an a priori given subset of elements.

**Definition 19 (Pivots).** Let \( S \subseteq \text{Sig}(E) \) be a set of signatures on \( E \) and \( F \subseteq E \) a subset of elements. Then, an element \( e \in E \) is a pivot for \( S \) w.r.t. \( F \) if \( e \not\in F \) and either one of the following items holds:

1. \( e \in [S] \) and \( \text{Col}(S, e) \subseteq F \);
2. there is a signature \( \sigma \in S \) such that \( e \in \langle \sigma \rangle \) and \( \text{Dep}(\sigma, e) \subseteq F \).

Intuitively, Item 1 asserts that the pivot is universal quantified over all signatures and all existential elements that form a collapsing chain starting in the pivot itself are already defined. On the contrary, Item 2 asserts that the pivot is existential quantified and, on the relative signature, it depends only on already defined elements.

Before continuing, we provide the auxiliary definition of minimal \( S \)-chain.

**Definition 20 (Minimal \( S \)-Chain).** Let \( S \subseteq \text{Sig}(E) \) be a set of signatures on \( E \) and \( F \subseteq E \) a subset of elements. A pair \((\bar{e}, \bar{\sigma}) \in E^k \times S^k\), with \( k \in [1, \omega]\), is a minimal \( S \)-chain w.r.t. \( F \) if it is an \( S \)-chain such that \((\bar{e})_i \in \text{min}(\sigma)_i . (E \setminus F)\), for all \( i \in [0, k[ \).

In addition to the definition of pivot, we also give the formal concept of pivot seeker that is used, in an iterative procedure, to find a pivot if this exists.

**Definition 21 (Pivot Seekers).** Let \( S \subseteq \text{Sig}(E) \) be a set of signatures on \( E \) and \( F \subseteq E \) a subset of elements. Then, a pair \((e \cdot \bar{e}, \sigma \cdot \bar{\sigma}) \in E^k \times S^k\) of sequences of elements and signatures of length \( k \in [1, \omega]\) is a pivot seeker for \( S \) w.r.t. \( F \) if the following hold:

1. \( e \in \text{min}_\omega(E \setminus F) \);
2. \( \text{fst}(\bar{e}) \in (\langle \sigma \rangle \cup \text{Col}(S, \sigma)) \setminus F \), if \( k > 1 \);
Lemma 4 (Pivot Seeker Extension). Let $e$ be a pivot seeker of just one further element. Item 1 ensures that the element $e$ we are going to consider as a candidate for pivot depends only on the elements defined in $F$. Item 2 builds a link between the signature $\sigma$ of the present candidate and the head element $\text{fst}(\vec{e})$ of the previous step, in order to maintain information about the dependences that are not yet satisfied. Finally, Item 3 is used to ensure that the procedure avoids loops by checking pivots on signature already considered.

As shown through the above mentioned examples, in the case of overlapping signatures, we can always find a pivot w.r.t. a given set of elements already defined, by means of a pivot seeker.

The following lemma ensures that we can always start the iterative procedure over pivot seekers to find a pivot.

**Lemma 3 (Pivot Seeker Existence).** Let $S \subseteq \text{Sig}(E)$ be a set of signatures on $E$ and $F \subseteq E$ a subset of elements. Then, there exists a pivot seeker for $S$ w.r.t. $F$ of length 1.

**Proof.** Let $\sigma \in S$ be a generic signature and $e \in E$ an element such that $e \in \min_{\sigma}(E \setminus F)$. Then, it is immediate to see that the pair $(e, \sigma) \in E^1 \times S^1$ is a pivot seeker for $S$ w.r.t. $F$ of length 1, since Item 1 of Definition 21 of pivot seekers is verified by construction and Items 2 and 3 are vacuously satisfied.

Now, suppose to have a pivot seeker of length not greater than the size of the support set $F$ and that no pivot is already found. Then, in the case of signatures without cyclic dependences, we can always continue the iterative procedure, by extending the previous pivot seeker of just one further element.

**Lemma 4 (Pivot Seeker Extension).** Let $S \subseteq \text{Sig}(E)$ be a set of signatures on $E$ with $C(S) = \emptyset$ and $F \subseteq E$ a subset of elements. Moreover, let $(e, \vec{e}, \sigma, \vec{\sigma}) \in E^k \times S^k$ be a pivot seeker for $S$ w.r.t. $F$ of length $k \in [1, \omega]$. Then, if $e$ is not a pivot for $S$ w.r.t. $F$, there exists a pivot seeker for $S$ w.r.t. $F$ of length $k + 1$.

**Proof.** By Item 1 of Definition 21 of pivot seekers, we deduce that $e \notin F$ and $\text{Dep}(\sigma, e) \subseteq F$. Thus, if $e$ is not a pivot for $S$ w.r.t. $F$, by Definition 19 of pivot, we have that $e \notin [\sigma]$ or $\text{Col}(S, e) \not\subseteq F$ and, in both cases, $e \in \llbracket e \rrbracket$. We now distinguish the two cases.

- $e \notin \llbracket S \rrbracket$.

  There exists a signature $\sigma' \in S$ such that $e \in \llbracket \sigma' \rrbracket$. So, consider an element $e' \in \min_{\sigma'}(E \setminus F)$. We now show that the pair of sequences $(e' \cdot e, \sigma' \cdot \vec{\sigma}) \in E^{k+1} \times S^{k+1}$ of length $k + 1$ satisfies Items 1 and 2 of Definition 21. The first item is trivially verified by construction. Moreover, $\text{fst}(e' \cdot e) = e \in \llbracket \sigma' \rrbracket \setminus F$. Hence, the second item holds as well.

- $e \in \llbracket S \rrbracket$.

  We necessarily have that $\text{Col}(S, e) \not\subseteq F$. Thus, there is an element $e' \in E \setminus ([S] \cup F)$ such that $(e', e) \in \text{Col}(S)$. Consequently, there exists also a signature $\sigma' \in S$ such that $e' \in \llbracket \sigma' \rrbracket \setminus F$. So, consider an element $e'' \in \min_{\sigma'}(E \setminus F)$. We now show that the pair of sequences $(e'' \cdot e, \sigma' \cdot \vec{\sigma}) \in E^{k+1} \times S^{k+1}$ of length $k + 1$ satisfies Items 1 and 2 of Definition 21. The first item is trivially verified by construction. Moreover, since $(e', e) \in \text{Col}(S)$, by the definition of $\text{Col}(S, \sigma')$, we have that $\text{fst}(e' \cdot e) = e \in \text{Col}(S, \sigma') \setminus F$. Hence, the second item holds as well.
At this point, it only remains to show that Item 3 of Definition 21 holds, i.e., that 
\((e \cdot \bar{e}, \sigma \cdot \bar{\sigma})\) is a minimal S-chain w.r.t. F. For \(k = 1\), we have that Items 2 and 3 of Definition 10 of S-chain are vacuously verified. Moreover, since \(e \in \| \sigma \|\), also Item 1 of the previous definition holds. Finally, the S-chain is minimal w.r.t. F, due to the fact that \(e \in \min_\sigma (E \setminus F)\). Now, suppose that \(k > 1\). Since \((\bar{e}, \bar{\sigma})\) is already an S-chain, to prove Items 2 and 3 of Definition 10 of S-chain, we have only to show that 
\((e, \fst(\bar{e})) \in \Dep(\sigma)\) and \(\sigma \neq (\bar{\sigma})_i\), for all \(i \in [0, k - 1]\), respectively.

By Items 1 and 2 of Definition 21 we have that \(e \in \min_\sigma (E \setminus F)\) and \(\fst(\bar{e}) \in (\| \sigma \| \cup \Col(S, \sigma)) \setminus F\). So, two cases arise.

- \(\fst(\bar{e}) \in \| \sigma \| \setminus F\).
  
  Since \(e \in \| \sigma \| \cap \min_\sigma (E \setminus F)\), we can deduce that 
  \((e, \fst(\bar{e})) \in \Dep(\sigma) \subseteq \Dep'(\sigma)\).

- \(\fst(\bar{e}) \in \Col(S, \sigma) \setminus F\).

  By the definition of \(\Col(S, \sigma)\), there exists \(e' \in \| \sigma \| \setminus F\) such that 
  \((e', \fst(\bar{e})) \in \Col(S)\). Now, since \(e \in \| \sigma \| \cap \min_\sigma (E \setminus F)\), we can deduce that 
  \((e, e') \in \Dep(\sigma)\).

  Thus, by definition of \(\Dep'(\sigma)\), it holds that 
  \((e, \fst(\bar{e})) \in \Dep'(\sigma)\).

Finally, suppose by contradiction that there exists \(i \in [0, k - 1]\) such that 
\(\sigma = (\bar{\sigma})_i\). Two cases can arise.

- \(i = k - 2\).

  Then, by Item 1 of Definition 10 we have that 
  \((\bar{e})_i = \lst(\bar{e}) \in \([\lst(\bar{\sigma})] = \[(\bar{\sigma})_i]\];

- \(i < k - 2\).

  Then, by Item 2 of Definition 10 we have that 
  \((((\bar{e})_i, (\bar{e})_{i+1}) \in \Dep'(((\bar{\sigma})_i))\). Consequently, 
  \((\bar{e})_i \in \[(\bar{\sigma})_i]\).

By Definition 20 of minimal S-chain, since \((\bar{e}, \bar{\sigma})\) is minimal w.r.t. F, it holds that 
\((\bar{e})_i \in \min_\sigma (E \setminus F)\). So, \((\bar{e})_i \in \[(\bar{\sigma})_i] \cap \min_\sigma (E \setminus F)\). Moreover, by Item 2 of Definition 21 we have that 
\((\bar{e})_0 \in (\| \sigma \| \cup \Col(S, \sigma)) \setminus F = (\| (\bar{\sigma})_i \| \cup \Col(S, (\bar{\sigma})_i)) \setminus F\). Thus, by applying a reasoning similar to the one used above to prove that 
\((e, \fst(\bar{e})) \in \Dep(\sigma)\), we obtain that 
\((\bar{e})_i, (\bar{e})_0) \in \Dep'(((\bar{\sigma})_i))\). Hence, 
\((((\bar{e})_i, (\bar{e})_{i+1}) \in \Dep'(((\bar{\sigma})_i))\).

Finally, if we have run the procedure until all elements in E are visited, the first one of 
the last pivot seeker is necessarily a pivot.

Lemma 5 (Seeking Termination). Let \(S \subseteq \Sig(E)\) be a finite set of signatures on \(E\) with \(C(S) = \emptyset\) and \(F \subset E\) a subset of elements. Moreover, let 
\((e \cdot \bar{e}, \sigma \cdot \bar{\sigma}) \in E^k \times S^k\) be a pivot seeker for \(S\) w.r.t. \(F\) of length \(k \doteq |S| + 1\). Then, \(e\) is a pivot for \(S\) w.r.t. \(F\).

Proof. Suppose by contradiction that \(e\) is not a pivot for \(S\) w.r.t. \(F\). Then, by Lemma 4 of pivot seeker extension, there exists a pivot seeker for \(S\) w.r.t. \(F\) of length \(k + 1\), which is impossible due to Item 3 of Definition 21 of pivot seekers, since an S-chain of length \(k\) does not exist.

By appropriately combining the above lemmas, we can prove the existence of a 
pivot for a given set of signatures having no cyclic dependences.

Lemma 6 (Pivot Existence). Let \(S \subseteq \Sig(E)\) be a finite set of signatures on \(E\) with \(C(S) = \emptyset\) and \(F \subset E\) a subset of elements. Then, there exists a pivot for \(S\) w.r.t. \(F\).
Proof. By Lemma 3 of pivot seeker existence, there is a pivot seeker of length 1 for S w.r.t. F, which can be extended, by using Lemma 4 of pivot seeker extension, at most |S| < ω times, due to Lemma 5 of seeking termination, before the reach of a pivot e for S w.r.t. F.

Big signature dependences. In order to prove Theorem 5 we first introduce big signature map w.

Definition 22 (Big Signature Dependences). Let $P \subseteq L\text{Sig}(E)$ be a set of labeled signatures over a set $E$, and $D = P \times V(P) \times \{0, 1\}^{C(P)}$. Then, the big signature dependence $w \in \text{SigDep}_D(P)$ for $P$ over $D$ is defined as follow. For all $(\sigma, l) = (\langle \varphi, b \rangle, l) \in P$, and $\nu \in \text{Val}_D(\langle \varphi \rangle)$, we have that:

1. $w((\sigma, l))(\langle \varphi \rangle)(x) \triangleq \nu(x)$, for all $x \in \langle \varphi \rangle$;
2. $w((\sigma, l))(\langle \varphi \rangle)(x) \triangleq (\sigma, l, x, h)$, for all $x \in \langle \varphi \rangle$, where $h \in \{0, 1\}^{C(P)}$ is such that, for all $(\vec{e}, \vec{\sigma})$, the following hold:
   (a) if $\sigma = \text{fst}(\vec{\sigma})$ and $x = b(\text{fst}(\vec{e}))$ then $h((\vec{e}, \vec{\sigma})) \triangleq 1 - h((\vec{e}, \vec{\sigma}))$, where $h' \in \{0, 1\}^{C(P)}$ is such that $\nu(b(\text{fst}(\vec{e}))) = ((\sigma', l'), x', h')$, for some $(\sigma', l') \in P$ and $x' \in V(P)$;
   (b) if there exists $i \in [1, |\vec{\sigma}|]$ such that $\sigma = (\vec{\sigma})_i$ and $x = b((\vec{e})_i)$, then $h((\vec{e}, \vec{\sigma})) \triangleq h((\vec{e}, \vec{\sigma}))$, where $h' \in \{0, 1\}^{C(P)}$ is such that $\nu(b((\vec{e})_i)) = ((\sigma', l'), x', h')$, for some $(\sigma', l') \in P$ and $x' \in V(P)$;
   (c) if none of the above cases apply, set $h((\vec{e}, \vec{\sigma})) \triangleq 0$.

Note that Items 2a and 2b are mutually exclusive since, by definition of cyclic dependence, each signature $(\vec{\sigma})_i$ occurs only once in $\vec{\sigma}$.

It is easy to see that the previous definition is well formed, i.e., that $w$ is actually a labeled signature dependence. Indeed the following lemma holds.

Lemma 7. Let $P \subseteq L\text{Sig}(E)$ be a set of labeled signatures over a set $E$ and $D = P \times V(P) \times \{0, 1\}^{C(P)}$. Then the big signature dependence $w$ for $P$ over $D$ is a labeled signature dependence for $P$ over $D$.

Proof. We have to show that $w((\langle \varphi, b \rangle, l))$ is a dependence map for $\varphi$ over $D$, for all $(\sigma, l) \in P$.

1. By Item 1 of Definition 22 it holds that $w((\sigma, l))(\langle \varphi \rangle)(x) = \nu(x)$, for all $x \in \langle \varphi \rangle$ and $\nu \in \text{Val}_D(\langle \varphi \rangle)$, which means $w((\sigma, l))(\langle \varphi \rangle)|_{\nu} = \nu$, that means that Item 1 of Definition 5 holds.
2. For the Item 2 of Definition 5 let $\nu_1, \nu_2 \in \text{Val}_D(\langle \varphi \rangle)$ and $x \in \langle \varphi \rangle$ such that $(\nu_1)_{\text{Dep}(\varphi, x)} = (\nu_2)_{\text{Dep}(\varphi, x)}$. We have to prove that $w((\sigma, l))(\nu_1)(x) = w((\sigma, l))(\nu_2)(x)$.
   By definition, we have that $w((\sigma, l))(\nu_1)(x) = ((\sigma, l), x, h_1)$ and $w((\sigma, l))(\nu_2)(x) = ((\sigma, l), x, h_2)$. So, we have only to show that $h_1 = h_2$. To do this, consider a cyclic dependence $(\vec{e}, \vec{\sigma}) \in C(P)$ for which there exists $i \in [0, |\vec{\sigma}|]$ such that $\sigma = (\vec{\sigma})_i$ and $x = b((\vec{e})_i)$. Then, we have that $\nu_1(y) = \nu_2(y) = ((\sigma', l'), y', h')$ for $y = b((\vec{e})_{i-1} \mod |\vec{\sigma}|)$. Then, we have the following:
   - by Item 2a of Definition 22 if $i = 1$ then $h_1((\vec{e}, \vec{\sigma})) = 1 - h_1((\vec{e}, \vec{\sigma})) = h_2((\vec{e}, \vec{\sigma}))$;

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We finally able to show the proofs of the above mentioned results.

**Theorem 4 (Overlapping Dependence Maps).** Let \( S \subseteq \text{Sig}(E) \) be a finite set of overlapping signatures on \( E \). Then, for all signature dependences \( w \in \text{SigDep}_D(S) \) for \( S \) over a set \( D \), it holds that \( w \) is overlapping.

**Proof.** By Definition 13 of signature dependence, to prove the statement, i.e., that \( \bigcap_{(\varphi,b) \in S} \varphi \circ b : v \in \text{rng}(w(\varphi,b))) \neq \emptyset \), we show the existence of a function \( d \in D^E \) such that, for all signatures \( \sigma = (\varphi, b) \in S \), there is a valuation \( v_\sigma \in \text{rng}(w(\sigma)) \) for which \( d = v_\sigma \circ b \).

We build \( d \) iteratively by means of a succession of partial functions \( d_j : E \rightarrow D \), with \( j \in [0, |E|] \), satisfying the following invariants:

1. \( d_j(e') = d_j(e'') \), for all \( (e', e'') \in \text{Col}(S) \cap (\text{dom}(d_j) \times \text{dom}(d_j)) \);
2. for all \( e \in \text{dom}(d_j) \), there is \( i \in [0, j] \) such that \( e \) is a pivot for \( S \) w.r.t. \( \text{dom}(d_i) \);
3. \( \text{dom}(d_j) \subset \text{dom}(d_{j+1}) \), where \( j < |E| \);
4. \( d_j = d_{j+1} |_{\text{dom}(d_j)} \), where \( j < |E| \).

Before continuing, observe that, since \( \langle S \rangle = \emptyset \), for each element \( e \in E \setminus \langle S \rangle \), there exists exactly one signature \( \sigma_e = (\varphi_e, b_e) \in S \) such that \( e \in \langle \sigma_e \rangle \).

As base case, we simply set \( d_0 \triangleq \emptyset \). It is immediate to see that the invariants are vacuously satisfied.

Now, consider the iterative case \( j \in [0, |E|] \). By Lemma 6 of pivot existence, there is a pivot \( e_j \in E \) for \( S \) w.r.t. \( \text{dom}(d_j) \). Remind that \( e_j \notin \text{dom}(d_j) \). At this point, two cases can arise.

| Case | Description |
|------|-------------|
| 1.   | \( e_j \in \langle S \rangle \). If there is an element \( e \in \text{dom}(d_j) \) such that \( (e, e_j) \in \text{Col}(S) \) then set \( d_{j+1} \triangleq d_j \cup \{e_j \mapsto d_j(e)\} \). By Invariant 1 at step \( j \), the choice of such an element is irrelevant. Otherwise, choose a value \( c \in D \), and set \( d_{j+1} \triangleq d_j \cup \{e_j \mapsto c\} \). In both cases, all invariants at step \( j + 1 \) are trivially satisfied by construction. |
| 2.   | \( e_j \notin \langle S \rangle \). Consider a valuation \( v_j \in \text{Val}_D(\langle \sigma_{e_j} \rangle) \) such that \( v_j(b_{e_j}(e)) = d_j(e) \), for all \( e \in \text{dom}(d_j) \cap \langle \sigma_{e_j} \rangle \). The existence of such a valuation is ensured by Invariant 1 at step \( j \), since \( d_j(e') = d_j(e'') \), for all \( e', e'' \in \text{dom}(d_j) \) with \( b_{e_j}(e') = b_{e_j}(e'') \). Now, set \( d_{j+1} \triangleq d_j \cup \{e_j \mapsto \nu(\sigma_{e_j})(v_j)(b_{e_j}(e_j))\} \). It remains to verify the invariants at step \( j + 1 \). Invariants 2, 3, and 4 are trivially satisfied by construction. For Invariant 1 instead, suppose that there exists \( (e_j, e) \in \text{Col}(S) \cap (\text{dom}(d_{j+1}) \times \text{dom}(d_{j+1})) \) with \( e_j \neq e \). By Invariant 2 at step \( j \), there is \( i \in [0, j] \) such that \( e \) is a pivot for \( S \) w.r.t. \( \text{dom}(d_i) \), i.e., \( e = e_i \). At this point, two subcases can arise, the first of which results to be impossible. |
• $e_i \in [S]$. By Item 1 of Definition 19 of pivot, it holds that $\text{Col}(S, e_i) \subseteq \text{dom}(d_i)$. Moreover, since $e_j \notin [S]$ and $(e_j, e_i) \in \text{Col}(S)$, it holds that $e_j \notin \text{Col}(S, e_i)$. Thus, by a repeated application of Invariant 3 from step $i$ to step $j$, we have that $e_j \in \text{dom}(d_j) \subseteq \text{dom}(d_i) \not\ni e_j$, which is a contradiction.

• $e_i \notin [S]$. Since $e_j, e_i \notin [S]$ and $(e_j, e_i) \in \text{Col}(S)$, it is easy to see that $\sigma_e_j = \sigma_e$ and $b_{e_j} = b_{e_i}(e_i)$. Otherwise, we have that $e_j \in \langle S \rangle = \emptyset$, which is impossible. Hence, it follows that $d_{i+1}(e_j) = w(\sigma_{e_j})(v_j)(b_{e_i}(e_j)) = w(\sigma_e)(v_j)(b_{e_i}(e_i))$. Moreover, $d_{i+1}(e_i) = w(\sigma_e)(v_i)(b_{e_i}(e_i))$. Now, it is easy to observe that $\text{Dep}(v_j, b_{e_j}(e_j)) = v_j|_{\text{Dep}(v_j, b_{e_j}(e_j))}$, from which we derive that $v_j|_{\text{Dep}(v_j, b_{e_j}(e_j))} = v_j|_{\text{Dep}(v_i, b_{e_i}(e_i))}$. At this point, by Item 2 of Definition 5 of dependence maps, it holds that $w(\sigma_{e_i})(v_j)(b_{e_i}(e_i)) = w(\sigma_{e_i})(v_i)(b_{e_i}(e_i))$, so $d_{j+1}(e_j) = d_{j+1}(e_i)$. Finally, by a repeated application of Invariant 4 from step $i+1$ to step $j$, we obtain that $d_{i+1}(e_i) = d_{j+1}(e_i)$. Hence, $d_{j+1}(e_j) = d_{j+1}(e_i)$.

By a repeated application of Invariant 3 from step 0 to step $|E| - 1$, we have that $d_{|E|}$ is a total function. So, we can now prove that $d \triangleq d_{|E|}$ satisfies the statement, i.e., $d \in \Gamma_{(b, b) \in S}\{v \circ b : v \in \text{ran}(w(\sigma, b))\}$.

For each signature $\sigma = (\emptyset, b) \in S$, consider the universal valuation $v_\sigma \in \text{Val}_D(\langle \sigma \rangle)$ such that $v_\sigma(b(e)) = d(e)$, for all $e \in \langle \sigma \rangle$. The existence of such a valuation is ensured by Invariant 1 at step $|E|$. Moreover, let $v_\sigma \triangleq w(\sigma)(v_\sigma)$. It remains to prove that $d = v_\sigma \circ b$, by showing separately that $d_{\langle \sigma \rangle} = (v_\sigma \circ b)_{\langle \langle \sigma \rangle \rangle}$ and $d_{\langle \langle \sigma \rangle \rangle} = (v_\sigma \circ b)_{\langle \langle \sigma \rangle \rangle}$ hold.

On one hand, by Item 1 of Definition 5 for each $x \in \langle \langle \sigma \rangle \rangle$, it holds that $v_\sigma(x) = w(\sigma)(v_\sigma(x))$. Thus, for each $e \in \langle \sigma \rangle$, we have that $v_\sigma(b(e)) = w(\sigma)(v_\sigma(b(e))) = w(\sigma)(v_\sigma)(b(e)) = v_\sigma(b(e)) = (v_\sigma \circ b)(e)$. So, $d_{\langle \sigma \rangle} = (v_\sigma \circ b)_{\langle \langle \sigma \rangle \rangle}$.

On the other hand, consider an element $e \in \langle \langle \sigma \rangle \rangle$. By Invariant 2 at step $|E|$, there is $i \in [0, |E|]$ such that $e$ is a pivot for $S$ w.r.t. $\text{dom}(d_i)$. This means that $e_i = e$ and so $\sigma_{e_i} = \sigma$. So, by construction, we have that $d_{i+1}(e_i) = w(\sigma)(v_i)(b_{e_i}(e_i))$. Moreover, $w(\sigma)(v_{e_i})(b(e)) = v_{e_i}(b(e)) = (v_\sigma \circ b)(e)$. Thus, to prove the required statement, we have only to show that $d(e) = d_{i+1}(e)$ and $w(\sigma)(v_i)(b(e)) = w(\sigma)(v_{e_i})(b(e))$. By a repeated application of Invariants 3 and 4 from step $i$ to step $|E| - 1$, we obtain that $\text{dom}(d_i) \subseteq \text{dom}(d)$, $d_i = d_{|\text{dom}(d_i)|}$, and $d_{i+1}(e) = d(e)$. Thus, by definition of $v_i$ and $v_{e_i}$, it follows that $v_i(b(e')) = d_i(e') = d'(e') = v_{e_i}(b(e'))$, for all $e' \in \text{dom}(d_i)$.

At this point, by Item 2 of Definition 19 it holds that $\text{Dep}(\sigma, e) \subseteq \text{dom}(d_i)$, which implies that $v_i|_{\text{Dep}(v_i, b(e))} = v_{e_i}|_{\text{Dep}(v_{e_i}, b(e))}$. Hence, by Item 2 of Definition 5 we have that $w(\sigma)(v_i)(b(e)) = w(\sigma)(v_{e_i})(b(e))$. So, $d_{\langle \langle \sigma \rangle \rangle} = (v_\sigma \circ b)_{\langle \langle \sigma \rangle \rangle}$.

Corollary 1 (Overlapping Dependence Maps). Let $P \subseteq \text{LSig}(E, L)$ be a finite set of overlapping labeled signatures on $E$ w.r.t. $L$. Then, for all labeled signature dependences $w \in \text{LSigDep}_D(P)$ for $P$ over a set $D$, it holds that $w$ is overlapping.

Proof. Consider the set $P' \triangleq \{(\sigma, l) \in P : \langle \sigma \rangle \neq \emptyset\}$ of all labeled signatures in $P$ having at least one existential element. Since $P$ is overlapping, by Definition 12 of overlapping signatures, we have that, for all $(\sigma, l_1), (\sigma, l_2) \in P'$, it holds that $l_1 = l_2$. 
So, let $S \triangleq \{ \sigma \in \text{Sig}(E) : \exists l \in L . (\sigma, l) \in P' \}$ be the set of first components of labeled signatures in $P'$ and $h : S \rightarrow P'$ the bijective function such that $h(\sigma) \triangleq (\sigma, l)$, for all $\sigma \in S$, where $l \in L$ is the unique label for which $(\sigma, l) \in P'$ holds. Now, since $S$ is overlapping, by Theorem 4 of overlapping dependence maps, we have that the signature dependence $w \circ h \in \text{SigDep}_D(S)$ is overlapping as well. Thus, it is immediate to see that $w|_{P'}$ is also overlapping, i.e., by Definition 13 of signature dependencies, there exists $d \in D$ such that $d \in \cap_{((\varphi, b), l) \in P'} \{ v \circ b : v \in \text{rng}(w((\varphi, b), l)) \} \neq \emptyset$. At this point, consider the labeled signatures $(\sigma, l) = ((\varphi, b), l) \in P \setminus P'$. Since $\langle \sigma \rangle = \emptyset$, i.e., $\langle \varphi \rangle = \emptyset$, we derive that $w((\sigma, l)) \in \text{DM}_D(\varphi)$ is the identity dependence map, i.e., it is the identity function on $\text{Val}_D(V(\varphi))$. Thus, $\text{rng}(w((\sigma, l))) = \text{Val}_D(V(\varphi))$. So, we have that $d \in \cap_{((\varphi, b), l) \in P'} \{ v \circ b : v \in \text{rng}(w((\varphi, b), l)) \} \neq \emptyset$. Hence, again by Definition 13 it holds that $w$ is overlapping.

**Theorem 5 (Non-Overlapping Dependence Maps).** Let $P \subseteq \text{LSig}(E, L)$ be a set of labeled signatures on $E$ w.r.t. $L$. Then, there exists a labeled signature dependence $w \in \text{LSigDep}_D(P)$ for $P$ over $D \triangleq P \times \{ 0, 1 \}^{C(P)}$ such that, for all $P' \subseteq P$, it holds that $w|_{P'} \in \text{LSigDep}_D(P')$ is non-overlapping, if $P'$ is non-overlapping.

**Proof.** Let $S' \triangleq \{ \sigma \in \text{Sig}(E) : \exists l \in L . (\sigma, l) \in P' \}$ be the set of signatures that occur in some labeled signature in $P'$.

If $P'$ is non-overlapping, we distinguish the following three cases.

1. There exist $(\sigma, l_1), (\sigma, l_2) \in P'$, with $\sigma = (\varphi, b)$, such that $\langle \sigma \rangle \neq \emptyset$ and $l_1 \neq l_2$.

   Then, for all valuations $v \in \text{Val}_D(\langle \varphi \rangle)$ and variables $x \in \langle \varphi \rangle$, we have that $w((\sigma, l_1))(v)(x) = ((\sigma, l_1), x, h_1) \neq ((\sigma, l_2), x, h_2) = w((\sigma, l_1))(v)(x)$. Thus, $w((\sigma, l_1))(v)(x) \circ b \neq w((\sigma, l_2))(v)(x) \circ b$, for all $v \in \text{Val}_D(\langle \varphi \rangle)$. Hence, $w$ is non-overlapping.

2. $\langle S' \rangle \neq \emptyset$.

   Then, there exist $\sigma' = (\varphi', b')$, $\sigma'' = (\varphi'', b'') \in S'$, $e' \in \langle \sigma' \rangle$, and $e'' \in \langle \sigma'' \rangle$ such that $\sigma' \neq \sigma''$ or $b'(e') \neq b''(e'')$ and, in both cases, $(e', e'') \in \text{Col}(S')$. By contradiction, let $d \in \cap_{((\varphi, b), l) \in P'} \{ v \circ b : v \in \text{rng}(w((\varphi, b), l)) \}$. Observe that $d(e') = d(e'')$, for all $(e', e'') \in \text{Col}(S')$. So, there exist $v' \in \text{Val}_D(\langle \varphi' \rangle)$ and $v'' \in \text{Val}_D(\langle \varphi'' \rangle)$ such that $v'(b'(e)) = d(e)$, for all $e \in \langle \sigma' \rangle$, and $v''(b''(e')) = d(e)$, for all $e \in \langle \sigma'' \rangle$. Observe that there are $l', l'' \in L$ such that $(\sigma', l')$, $(\sigma'', l'') \in P'$. So, by the hypothesis of the existence of $d$, we have that $w((\sigma', l'))(v')(b'(e')) = d(e') = d(e'') = w((\sigma'', l''))(v'')(b''(e''))$.

   Now, the following cases arise.

   - $\sigma' \neq \sigma''$.

     By Definition 22 of big signature dependence, it holds that $w((\sigma', l'))(v')(b'(e')) = ((\sigma', l'), b'(e'), h') \neq ((\sigma'', l''), b''(e''), h'') = w((\sigma'', l''))(v'')(b''(e''))$, which is a contradiction.

   - $\sigma' = \sigma''$.

     Then, we have that $b'(e') \neq b''(e'')$. By Definition 22 it holds that $w((\sigma', l'))(v')(b'(e')) = ((\sigma', l'), b'(e'), h') \neq ((\sigma'', l''), b''(e''), h'') = w((\sigma'', l''))(v'')(b''(e''))$, which is a contradiction.

3. $\text{Col}(S') \neq \emptyset$.

   Then, there exists $(e, \vartheta) \in \text{Col}(S')$. Let $n \triangleq |\vartheta| - 1$. Assume, by contradiction, that there exists $d \in \cap_{((\varphi, b), l) \in P'} \{ v \circ b : v \in \text{rng}(w((\varphi, b), l)) \}$. Observe again that
we can satisfy both the goals \( \pi \) where there exists \( l_i \in L \) such that \( ((\sigma_i), l_i) \in P' \). Moreover, let \( v_i \in Val_D([\sigma_i]) \) such that \( v_i(b_i(e)) = d(e) \) for all \( e \in [\sigma_i] \). Then, there exist \( n+1 \) functions \( h_0, \ldots, h_n \in \{0,1\}^{C(p)} \) such that, for all \( i \in [0,n] \), we have that \( d((\sigma_i), e) = w(((\sigma_i), l_i))(v_i)(b_i((\epsilon_i), l_i)) = (((\sigma_i), l_i), b_i((\epsilon_i), l_i)) \). Observe that, by Item 24 of Definition 22 for all \( i \in [0,n] \), it holds that \( h_{i+1}(\langle \epsilon, \sigma \rangle) = h_i(\langle \epsilon, \sigma \rangle) \) and, in particular, \( h_0(\langle \epsilon, \sigma \rangle) = h_n(\langle \epsilon, \sigma \rangle) \). However, by Item 24 of Definition 22 it holds that \( h_0(\langle \epsilon, \sigma \rangle) = 1 - h_n(\langle \epsilon, \sigma \rangle) \). So, \( h_0(\langle \epsilon, \sigma \rangle) \neq h_n(\langle \epsilon, \sigma \rangle) \), which is a contradiction.

10 Proofs of Section 6

In this appendix, we prove Theorem 7 on the negative properties for Stl. Successively, we introduce the concept of flagged model and flagged formulas. Finally, we prove Theorem 8.

Theorem 7 (Stl. Negative Model Properties). For Stl, it holds that:

1. it is not invariant under decision-unwinding;
2. it does not have the decision-tree model property.

Proof. [Item (1)]. Assume by contradiction that Stl is invariant under decision-unwinding and consider the two CGSS \( G_1 \triangleq (AP, Ag, Ac, St, \lambda, \tau_{G_1}, s_0) \) and \( G_2 \triangleq (AP, Ag, Ac, St, \lambda, \tau_{G_2}, s_0) \), with \( AP = \{p\} \), \( Ag = \{\alpha, \beta\} \), \( Ac = \{0,1\} \), \( St = \{s_0, s'_1, s''_1, s'_2, s''_2, s'_3, s''_3\} \), \( \lambda(s''_2) = \lambda(s''_3) = \{p\} \) and \( \lambda(s) = \emptyset \), for all \( s \in St \setminus \{s'_2, s''_2, s'_3, s''_3\} \), and \( \tau_{G_1} \) and \( \tau_{G_2} \) given as follow. If by ab we indicate the decision in which agent \( \alpha \) takes the action \( a \) and agent \( \beta \) the action \( b \), then we set \( \tau_{G_1} \) and \( \tau_{G_2} \) as follow: \( \tau_{G_1}(s_0,0^*) = \tau_{G_2}(s_0,0^*) = s'_1 \), \( \tau_{G_1}(s_0,1^*) = \tau_{G_2}(s_0,1^*) = s''_1 \), \( \tau_{G_1}(s'_1,0^*) = \tau_{G_2}(s'_1,0^*) = s'_2 \), \( \tau_{G_1}(s'_1,1^*) = \tau_{G_2}(s'_1,1^*) = s''_2 \), \( \tau_{G_1}(s''_2,0^*) = \tau_{G_2}(s''_2,0^*) = s'_3 \), \( \tau_{G_1}(s''_2,1^*) = \tau_{G_2}(s''_2,1^*) = s''_3 \), and \( \tau_{G_1}(s,**') = \tau_{G_2}(s,**') = s \), for all \( s \in \{s'_2, s''_2, s'_3, s''_3\} \). Observe that \( G_{1DU} = G_{2DU} \).

Then, it is evident that \( G_1 \models \varphi \) iff \( G_{1DU} \models \varphi \) iff \( G_{2DU} \models \varphi \) iff \( G_2 \models \varphi \). In particular, the property does have to hold for the Stl sentence \( \varphi = \langle \langle x \rangle \rangle (\langle y_0 \rangle \langle y_1 \rangle (\langle \alpha, x \rangle (\langle \beta, y_0 \rangle (X X p) \wedge ((\alpha, y_0) (\beta, y_1) (X X \neg p)) \wedge ((\alpha, x) (\beta, y_0) (X X \neg p)) \wedge ((\alpha, y_0) (\beta, y_1) (X X \neg p)))) \rangle \). It is easy to see that \( G_1 \not\models \varphi \), while \( G_2 \models \varphi \). Thus, Stl cannot be invariant under decision-unwinding.

Indeed, each strategy \( f_x \) of the agent \( \alpha \) in \( G_1 \) forces to reach only one state at a time among \( s'_2, s''_2, s'_3, \) and \( s''_3 \). Formally, for each strategy \( f_x \in \text{Str}_{G_1}(s_0) \), there is a state \( s \in \{s'_2, s''_2, s'_3, s''_3\} \) such that, for all strategies \( f_y \in \text{Str}_{G_1}(s_0) \), it holds that \( (\pi_2) = s \), where \( \pi \triangleq \text{play}(\alpha \mapsto f_x)(\beta \mapsto f_y), s_0 \). Thus, it is impossible to satisfy both the goals \( X X p \) and \( X X \neg p \) with the same strategy of \( \alpha \).

On the contrary, since \( s_0 \in G_2 \) is owned by the agent \( \beta \), we may reach both \( s'_1 \) and \( s''_1 \) with the same strategy \( f_x \) of \( \alpha \). Thus, if \( f_x(s_0 \cdot s'_1) \neq f_x(s_0 \cdot s''_1) \), we reach, at the same time, either the pair of states \( s'_2 \) and \( s''_2 \) or \( s'_3 \) and \( s''_3 \). Formally, there are a strategy \( f_x \in \text{Str}_{G_2}(s_0) \), with \( f_x(s_0 \cdot s'_1) \neq f_x(s_0 \cdot s''_1) \), a pair of states \( (s_p, s_{\neg p}) \in \{s'_2, s''_2, s'_3, s''_3\} \), and two strategies \( f_{\beta_1}, f_{\beta_2} \in \text{Str}_{G_2}(s_0) \) such that \( (\pi_{\beta_2}) = s_p \) and \( (\pi_{\beta_2}) = s_{\neg p} \), where \( \pi_{\beta_1} \triangleq \text{play}(\alpha \mapsto f_x)(\beta \mapsto f_{\beta_1}), s_0 \) and \( \pi_{\beta_2} \triangleq \text{play}(\alpha \mapsto f_x)(\beta \mapsto f_{\beta_2}), s_0 \). Hence, we can satisfy both the goals \( X X p \) and \( X X \neg p \) with the same strategy of \( \alpha \).
[Item (2)]. To prove the statement we have to show that there exists a satisfiable sentence that does not have a DT model. Consider the St. sentence $\varphi \equiv \forall \varphi_1 \land \varphi_2$, where $\varphi_1$ is the negation of the sentence $\varphi$ used in Item (1) and $\varphi_2 \equiv [x][y][\alpha(x)(\beta,y)X ((\langle x \rangle \langle y \rangle)(\alpha,x)(\beta,y)X \land \langle \langle x \rangle \langle y \rangle \rangle(\alpha,x)(\beta,y)X \land \neg p)]$. Moreover, note that the sentence $\varphi_2$ is equivalent to the CTL formula $AX ((\exists X p) \land (\exists X \neg p))$. Then, consider the Cgs $G \equiv \langle AP, Ag, Ac, St, \lambda, \tau, s_0 \rangle$ with $AP = \{p\}$, $Ag = \{\alpha, \beta\}$, $Ac = \{0, 1\}$, $St = \{s_0, s_1, s_2, s_3\}$, $\lambda(s_0)\lambda(s_1) = \lambda(s_3) = 0$ and $\lambda(s_2) = \{p\}$, and $\tau(s_0, **) = s_1$, $\tau(s_1, *) = s_1$, $\tau(s_1, 1*) = s_3$, and $\tau(s, **) = s$, for all $s \in \{s_2, s_3\}$.

It is easy to see that $G$ satisfies $\varphi$. At this point, let $T$ be a DT model of $\varphi_2$. Then, such a tree has necessarily at least two actions and, consequently, two different successors $t_1, t_2 \in Dc^*$ of the root $\varepsilon$, where $t_1, t_2 \in Dc$ and $t_1(\alpha) = t_2(\alpha)$. Moreover, there are two decisions $d_1, d_2 \in Dc$ such that $p \in \lambda(t_1 \cdot d_1)$ and $p \notin \lambda(t_2 \cdot d_2)$. Now, let $f_{\varepsilon}, f_{\varepsilon}^{\alpha}, f_{\varepsilon}^{\beta} \in \text{Str}(\varepsilon)$ be three strategies for which the following holds: $f_{\varepsilon}(\varepsilon) = t_1(\alpha)$, $f_{\varepsilon}^{\alpha}(\varepsilon) = t_1(\alpha)$, $f_{\varepsilon}^{\beta}(\varepsilon) = t_2(\beta)$, $f_{\varepsilon}(t_1) = d_1(\alpha)$, $f_{\varepsilon}^{\alpha}(t_1) = d_1(\beta)$, $f_{\varepsilon}^{\beta}(t_2) = d_2(\alpha)$, and $f_{\varepsilon}^{\beta}(t_2) = d_2(\beta)$. Then, it is immediate to see that $T, \varphi \rightarrow f_{\varepsilon}[y_\varepsilon \rightarrow f_{\varepsilon}^{\alpha}[y_\varepsilon^{\alpha} \rightarrow f_{\varepsilon}^{\beta}[y_\varepsilon^{\beta} \rightarrow f_{\varepsilon}^{\beta}]] \varepsilon = ((\alpha,x)(\beta,y)X X p) \land ((\alpha,x)(\beta,y)X X \neg p))$. Thus, we obtain that $T \models \neg \varphi_1$. Hence, $\varphi$ does not have a DT model.

**Flagged features** A flagged model of a given Cgs $G$ is obtained adding a so-called $z$-agent to the set of agents and flagging every state with two flags. Intuitively, the $z$-agent takes control of the flag to use in order to establish which part of a given formula is checked in the Cgs. We start giving first the definition of plan and then the concepts of flagged model and flagged formulas.

**Definition 23 (Plans).** A track (resp., path) plan in a Cgs $G$ is a finite (resp., an infinite) sequence of decisions $\kappa \in Dc^*$ (resp., $\kappa \in Dc^\omega$). $\text{TPln} \triangleq Dc^*$ (resp., $\text{PPln} \triangleq Dc^\omega$) denotes the set of all track (resp., path) plans. Moreover, with each non-trivial track $\rho \in \text{Trk}$ (resp., path $\tau \in \text{Pth}$) it is associated the set $\text{TPln}(\rho) \triangleq \{\kappa \in Dc^{[\rho]} - 1: \forall i \in [0, [\kappa[.\rho)] \cdot i + 1 = \tau((\rho, i, \kappa))\} \subseteq \text{TPln}$ (resp., $\text{PPln}(\tau) \triangleq \{\kappa \in Dc^\omega: \forall i \in \mathbb{N}.(\tau(i + 1) = \tau((\tau, i, \kappa))\} \subseteq \text{PPln}$) of track (resp., path) plans that are consistent with $\rho$ (resp., $\tau$).

**Definition 24 (Flagged model).** Let $G = \langle AP, Ag, Ac, St, \lambda, \tau, s_0 \rangle$ be a Cgs with $|Ac| \geq 2$. Let $z \notin Ag$ and $c_2 \in Ac$. Then, the flagged Cgs is defined as follows:

$$G_z = \langle AP, Ag \cup \{z\}, Ac, St \times \{0, 1\}, \lambda_z, \tau_z, (s_0, 0) \rangle$$

where $\lambda_z(s, t) \equiv \lambda(s)$, for all $s \in St$ and $t \in \{0, 1\}$, and $\tau_z(s, t, d) \equiv (\tau(s, t; Ag), \iota')$, with $\iota' = 0$ if $d(z) = c_2$.

Since $G$ and $G_z$ have a different set of agents, an agent-closed formula $\varphi$ w.r.t. $Ag_G$ is clearly not agent-closed w.r.t. $Ag_{G_z}$. For this reason, we introduce the concept of flagged formulas, that represent, in some sense, the agent-closure of formulas.

**Definition 25 (Flagged formulas).** Let $\varphi \in \text{Sl}[16]$. The universal flagged formula of $\varphi$, in symbol $\varphi_{Ag}$, is obtained by replacing every principal subformula $\phi \in \text{psnt}(\varphi)$ with the formula $\varphi_{Ag} \equiv [x_2][z, x_2]$. The existential flagged formula of $\varphi$, in symbol $\varphi_{Ex}$, is obtained by replacing every principal subformula $\phi \in \text{psnt}(\varphi)$ with the formula $\varphi_{Ex} \equiv [x_2][z, x_2]$. 


Substantially, these definitions help us to check satisfiability of principal sub-sentences in a separate way. The special agent \( \sharp \) takes control, over the flagged model, of which branch to walk on the satisfiability of some \( \phi \in \text{psnt}(\varphi) \). Obviously, there is a strict connection between satisfiability of flagged formulas over \( G_{\sharp} \) and \( \varphi \) over \( G \).

Indeed, the following lemma holds.

**Lemma 8 (Flagged model satisfiability).** Let \( \varphi \in \text{SL}1\alpha \) and let \( \varphi_{A\sharp} \) and \( \varphi_{E\sharp} \) the flagged formulas. Moreover, let \( G \) be a CGS and \( G_{\sharp} \) his relative flagged CGS. Then, for all \( s \in \text{St}, \) it holds that:

1. if \( G, \emptyset, s \models \varphi \) then \( G_{\sharp}, \emptyset(s, i) \models \varphi_{A\sharp} \), for all \( i \in \{0, 1\} \);
2. if, for all \( i \in \{0, 1\} \) it holds that \( G_{\sharp}, \emptyset(s, i) \models \varphi_{E\sharp} \), then \( G, \emptyset, s \models \varphi \).

**Proof.** On the first case, let \( \theta \in \text{DM}_{\text{Str}G}(\varphi) \), we consider \( \theta_{A\sharp} \in \text{DM}_{\text{Str}G_{\sharp}}(\llbracket \varphi \rrbracket) \) such that if \( x \neq x_{\sharp} \) then \( \theta_{A\sharp}(\chi(x)) = \theta(\chi)(x) \), otherwise \( \theta_{A\sharp}(\chi(x)) = \chi(x_{\sharp}) \). On the second case, let \( \theta_{E\sharp} \in \text{DM}_{\text{Str}G_{\sharp}}(\llbracket \varphi \rrbracket) \), we consider \( \theta \in \text{DM}_{\text{Str}G}(\varphi) \) such that \( \theta(\chi)(x) = \theta_{E\sharp}(\chi(x)) \) (note that \( \text{dom}(\theta(\chi)) \) is strictly included in \( \text{dom}(\theta_{E\sharp}(\chi)) \)).

Now, given a binding \( b \) and its relative function \( \zeta_{t} \), consider \( b_{\sharp} \triangleq (\sharp, x_{\sharp})b \) and its relative function \( \zeta_{t, \sharp} \). We show that in both cases considered above there is some useful relation between \( \pi_{0} \triangleq \text{play}(\theta(\chi) \circ \zeta_{t}, s) \) and \( \pi_{0, \sharp} \triangleq \text{play}(\theta_{t}(\chi) \circ \zeta_{t, \sharp}(s, i)) \). Indeed, let \( \kappa_{t} \) the plan such that, for all \( i \in \mathbb{N} \), we have that \( (\pi_{0})_{i+1} = \tau((\pi_{0})_{i}, (\kappa_{t})_{i}) \) and let \( \kappa_{t, \sharp} \) the plan such that, for all \( i \in \mathbb{N} \), we have that \( (\pi_{0, \sharp})_{i+1} = \tau((\pi_{0, \sharp})_{i}, (\kappa_{t, \sharp})_{i}) \). By the definition of play, for each \( i \in \mathbb{N} \) and \( a \in \text{Ag} \), we have that \( (\kappa_{t})_{i}(a) = (\theta(\chi) \circ \zeta_{t})(a)((\pi_{0})_{i}) \) and \( (\kappa_{t, \sharp})_{i}(a) = (\theta_{t}(\chi) \circ \zeta_{t, \sharp})(a)((\pi_{0, \sharp})_{i}) \). Clearly, for all \( i \in \mathbb{N} \), we have that \( (\kappa_{t})_{i} = (\kappa_{t, \sharp})_{i} \).

Due to these facts, we can prove by induction that for each \( i \in \mathbb{N} \) there exists \( i \in \{0, 1\} \) such that \( (\pi_{0, \sharp})_{i} = (\pi_{0})_{i} \). The base case is trivial and we omit it here. As inductive case, suppose that \( (\pi_{0, \sharp})_{i} = (\pi_{0})_{i} \), for some \( i \). Then, by definition we have that \( (\pi_{0, \sharp})_{i+1} = \tau_{\sharp}((\pi_{0, \sharp})_{i}, (\kappa_{t, \sharp})_{i}) \).

Moreover, by definition of \( \tau_{\sharp} \), we have that \( (\pi_{0, \sharp})_{i+1} = \tau((\pi_{0})_{i}, (\kappa_{t})_{i}) \), for some \( i' \in \{0, 1\} \).

Since \( (\kappa_{t})_{i} = (\kappa_{t, \sharp})_{i} \), we have that \( (\pi_{0, \sharp})_{i+1} = \tau((\pi_{0})_{i}, (\kappa_{t})_{i}) \), which is the assert. It follows, by definition of \( \lambda_{t} \), that \( \lambda((\pi_{0})_{i}) = \lambda((\pi_{0, \sharp})_{i}) \), for each \( i \in \mathbb{N} \). So, every sentence satisfied on \( \pi_{0} \) is satisfied also on \( \pi_{0, \sharp} \). Now we proceed to prove Items 1 and 2 separately. Item 1 First, consider the case that \( \phi \) is of the form \( \exists x \psi \), where \( \psi \) is a quantification prefix and \( \psi \) is a boolean composition of goals. Since \( G, \emptyset, s \models \phi \), there exists \( \theta \in \text{DM}_{\text{Str}G}(\varphi) \) such that we have \( G, \theta(\chi), s \models \psi \), for all assignment \( \chi \in \text{Asg}_{G}(s) \). Now, consider \( \phi_{A\sharp} \triangleq \llbracket \exists x \psi \rrbracket \). Clearly, \( \theta_{A\sharp} \in \text{DM}_{\text{Str}G_{\sharp}}(\llbracket \exists x \psi \rrbracket) \), where \( \llbracket \exists x \psi \rrbracket \) is equivalent to \( \llbracket \exists x \psi \rrbracket \).

Now, if we have a formula \( \varphi \) embedding some proper principal sub-sentence, then by the induction hypothesis every \( \phi \in \text{psnt}(\varphi) \) is satisfied by \( G \) if and only if \( \phi_{A\sharp} \) is satisfied by \( G_{\sharp} \). By working on the structure of the formula it follows that the result holds for \( \varphi \) and \( \phi_{A\sharp} \), too, so the proof for this Item is done.

Item 2 First, consider the case of \( \phi \) is of the form \( \exists x \psi \), where \( \psi \) is a quantification prefix and \( \psi \) is a boolean composition of goals. Let \( G_{\sharp}, \emptyset(s, i) \models \phi_{E\sharp} \). Note that \( \phi_{E\sharp} \triangleq \llbracket \exists x \psi \rrbracket \), where \( \llbracket \exists x \psi \rrbracket \) is equivalent to \( \llbracket \exists x \psi \rrbracket \), so there exists \( \theta_{E\sharp} \in \text{DM}_{\text{Str}G_{\sharp}}(\llbracket \exists x \psi \rrbracket) \)
such that, for all assignment $\chi \in \text{Asg}_E((\langle x_2 \rangle)\psi)$, we have that $\mathcal{G}_\psi, \varphi_{E_2}(\chi), (s, i) \models (s, x_2)\. \psi$. Then, consider $\theta \in \text{DM}_{\text{Str}}$ given by $\theta(\chi)(x)) = \theta_\psi(\chi)(x))$. Clearly, $\theta$ is build starting from $\varphi_{E_2}$ as described above. Then, from $\mathcal{G}_\psi, \varphi_{E_2}(\chi), (s, i) \models (s, x_2)\. \psi$ it follows that $\mathcal{G}_\psi, (s, i) \models \varphi$. Now, if we have a formula $\varphi$ embedding some proper principal subsequence, then by the induction hypothesis every $\phi \in \text{psnt}(\varphi)$ is satisfied by $\mathcal{G}$ if and only if $\phi_{A_1}$ is satisfied by $\mathcal{G}_\psi$. By working on the structure of the formula it follows that the result holds for $\varphi$ and $\varphi_{E_2}$ too, so the proof for this Item is done.

**Proof of Theorem 8** From now on, by using Item 2 of Theorem 6 we can assume to work exclusively on CGTs. Let $S_\phi \triangleq \{ s \in \text{St}_T : T, \emptyset, s \models \phi \}$ and $T_\phi \triangleq S_\phi \times \{ 0, 1 \}$. By Item 1 of Lemma 3 we have that $T_\emptyset, \emptyset, t \models \phi_{A_1}$, for all $t \in T_\emptyset$. Moreover, for all $t \in T_\phi$, consider a strategy $t^T_\phi \in \text{St}_{\text{Tr}_T}(t)$ given by $t^T_\phi(\rho) = c_t$ iff $\rho = t$. Moreover, for all $\phi \in \text{psnt}(\varphi)$, consider the function $A_\phi : \text{Tr}_{T}(\varepsilon) \to 2^{(\text{St}_{\text{Tr}}(\varepsilon) \times \text{Tr}_{T}(\varepsilon))}$ given by $A_\phi(\rho) \triangleq \{ (\rho_i, \rho') : i \in \{ 0, |\rho| \} \land \rho' \in \text{Tr}_{T}(\varepsilon_0 \to f_{\rho}^\ast), \rho_i \} \land \text{lst}(\rho) = \text{lst}(\rho') \}$. Note that $(\text{lst}(\rho), \text{lst}(\rho')) \in A_\phi(\rho)$. Indeed: (i) $\text{lst}(\rho) = \rho_{|\rho|}$; (ii) $\text{lst}(\rho) \in \text{Tr}_{\text{Tr}}(\emptyset \to f_{\rho}^{\ast}(\rho), \rho_i)$; and (iii) $\text{lst}(\rho) = \text{lst}(\text{lst}(\rho))$. Observe that if $(\rho_i, \rho') \in A_\phi(\rho)$ then $\rho' = \rho_{|\rho|}$. Hence, except for $\text{lst}(\rho), \text{lst}(\rho')$, there exists at most one pair in $A_\phi(\rho)$. Indeed, by contradiction let $(\rho_i, \rho_{|\rho|})$ and $(\rho_j, \rho_{|\rho|} + 1)$ both in $A_\phi(\rho)$ with $i \leq j$ and $j \neq |\rho|$. Then, by the definition of compatible tracks $\text{Tr}_{\text{Tr}}(\emptyset \to f_{\rho}^{\ast}(\rho), \rho_i)$, there exists a plan $\kappa \in \text{Pln}(\rho_{|\rho|} + 1)$ such that for all $h \in \{ 0, |\rho| - i \}$ we have $\kappa_h(\varepsilon) = f_{\rho}^{\ast}(\rho_{|\rho|} + 1), h)$. Then, by the definition of $f_{\rho}^{\ast}$, $\kappa_h(\varepsilon) \neq c_t$. So, by the definition of plan and $T_\emptyset$, we have that $\rho_{j + 1} = (s, 1)$. On the other hand, since $(\rho_j, \rho_{|\rho|}) \in A_\phi(\rho)$, then there exists a plan $\kappa' \in \text{pln}(\rho_{|\rho|})$ such that $(\kappa')_0(\varepsilon) = f_{\rho}^{\ast}(\rho_{|\rho|}) = c_t$. Which implies, by the definition of $f_{\rho}^{\ast}$, we have that $\rho_{j + 1} = (s', 1)$, where $s' \neq s$, which is in contradiction with the fact that the second coordinate of $\rho_{j + 1}$ is 1, as shown above.

This reasoning allows us to build the functions $\text{head}_\phi$ and $\text{body}_\phi$ for the disjoint satisfiability of $\phi$ over $T_\emptyset$ on the set $T_\phi$. Indeed, the unique element $(\rho_i, \rho') \in A_\phi(\rho) \setminus \{(\text{lst}(\rho), \text{lst}(\rho'))\}$ can be used to define opportune the elementary dependence map used for such disjoint satisfiability.

**Theorem 8 (SL[1G] Bounded Tree-Model Property).** Let $\varphi$ be an SL[1G] satisfiable sentence and $P \triangleq \{ (\langle \varphi, b \rangle, (\psi, i)) \in \text{LSig}(\text{Ag}, \text{St} \times \{ 0, 1 \}) : \forall \psi \in \text{psnt}(\varphi) \land i \in \{ 0, 1 \} \}$ the set of all labeled signatures on Ag w.r.t. $\text{St} \times \{ 0, 1 \}$ for $\varphi$. Then, there exists a $b$-bounded DT $T$, with $b = |P| \cdot |V(P)| \cdot 2^{|C(P)|}$, such that $T \models \varphi$. Moreover, for all $\phi \in \text{psnt}(\varphi)$, it holds that $T$ satisfies $\phi$ disjointly over the set $\{ s \in \text{St} : T, \emptyset, s \models \phi \}$.

**Proof.** Since $\varphi$ is satisfiable, then, by Item 2 of Theorem 6 we have that there exists a DT $T$, such that $T \models \varphi$. We now prove that there exists a bounded DT $T' \triangleq (\text{AP}, \text{Ag}, \text{Ac}_T, \text{St}_T, |\lambda_T|, \text{tr}_T, \varepsilon)$ with $\text{Ac}_T \triangleq \{ 0, n \}$ and $n = |P| \cdot |V(P)| \cdot 2^{|C(P)|}$. Since $T'$ is a DT, we have to define only the labeling function $\lambda_{T'}$. To do this, we need two auxiliary functions $h : \text{St}_T \times \text{Dc}_T \to \text{Dc}_T$ and $g : \text{St}_T \to \text{St}_T$ that lift correctly the labeling function $\lambda_T$ to $\lambda_{T'}$. Function $g$ is defined recursively as follows: (i) $g(\varepsilon) \triangleq \varepsilon$, (ii) $g(t', d') \triangleq g(t') \cdot h(g(t'), d')$. Then, for all $t' \in \text{St}_T$, we define $\lambda_T(t') \triangleq \lambda_T(g(t'))$. It remains to define the function $h$. By Item 1 of Lemma 8 we have that $T_\emptyset \models \varphi_{A_1}$ and consequently that $T_\emptyset \models \varphi_{E_2}$. Moreover, applying the reasoning explained above, $T_\emptyset$ satisfies disjointly $\phi$ over $S_\phi$, for all $\phi \in \text{psnt}(\varphi)$. Then, for
all \( \phi \in \text{psnt}(\varphi) \), we have that there exist a function \( \text{head}_\phi : S_\phi \to \text{DM}_{A_{\text{CT}}} (\varphi) \) and a function \( \text{body}_\phi : \text{Trk}_T (\varepsilon) \to \text{DM}_{A_{\text{CT}}} (\varphi) \) that allow \( T \) to satisfy \( \varphi \) in a disjoint way over \( S_\phi \). Now, by Theorem 5 there exists a signature dependence \( w \in \text{LSigDep}_{A_{\text{CT}}, (P)} \) such that, for all \( P' \subseteq P \), we have that \( w_{|P'} \in \text{LSigDep}_{P_{\text{CT}}, (P)} \) is non-overlapping, if \( P \) is non-overlapping. Moreover, by Corollary 1 for all \( P' \subseteq P \), we have that \( w_{|P'} \in \text{LSigDep}_{A_{\text{CT}}, (P)} \) is overlapping, if \( P \) is overlapping. At this point, consider the function \( D : \text{DC}_{T'} \to 2^P \) that, for all \( d' \in \text{DC}_{T'} \), is given by \( D(d') \triangleq \{ ((\varphi, b), (\psi, i)) \in P : \exists e' \in \text{AC}_{T'}^{(\|\psi\|)}, d' = w(\sigma)(e') \circ \zeta \} \). Note that, for all \( d' \in \text{DC}_{T'} \), we have that \( D(d) \subseteq P \) is overlapping. Now, consider the functions \( W : \text{St}_{T_\alpha} \to \text{LSigDep}_{A_{\text{CT}}, (P)} \) such that, for all \( t \in \text{St}_{T_\alpha} \) and \( \sigma = ((\varphi, b), (\psi, i)) \in P \), is such that

\[
W(t)(\sigma) = \begin{cases} \text{head}_\phi(t) , & t \in T_\phi \\ \text{body}_\phi(\rho') , & \text{otherwise} \end{cases}
\]

where \( \phi = \varphi \psi \rho \) and \( \rho' \in \text{Trk}_{T_\alpha} (\varepsilon) \) is the unique track such that \( \text{lst}(\rho') = t \). Moreover, consider the function \( T : \text{St}_{T_\alpha} \times \text{DC}_{T'} \to 2^P \) such that, for all \( t \in \text{St}_{T_\alpha} \) and \( d \in \text{DC}_{T'} \), it is given by \( T(t, d) \triangleq \{ \sigma = ((\varphi, b), (\psi, i)) \in P : \exists e \in \text{AC}_{T'}^{(\|\psi\|)} , d = W(t)(\sigma) \circ \zeta \} \). It is easy to see that, for all \( d' \in \text{DC}_{T'} \) and \( t \in \text{St}_{T_\alpha} \), there exists \( d \in \text{DC}_{T'} \) such that \( D(d') \subseteq T(t, d) \). By Corollary 1 for all \( t \in \text{St}_{T_\alpha} \), we have that \( W(t)|_{D(d')} \) is overlapping. So, by Definition 13 for all \( t \in \text{St}_{T_\alpha} \) and \( d' \in \text{DC}_{T'} \), there exists \( d \in \text{AC}_{T_\alpha}^{\phi} \) such that \( d \in \cap_{\sigma=(\varphi,\zeta), (\psi, i) \in D(d')} \{ z \circ b : z \in \text{rg}(W(t)(\sigma)) \} \), which implies \( T(t, d) \supseteq D(d') \). Finally, by applying the previous reasoning we obtain the function \( h \) such that, for all \( (t, d') \in \text{St}_{T} \times \text{DC}_{T'} \), it associates a decision \( h(t, d') \triangleq d \in \text{DC}_{T} \). The proof that \( T' \models \varphi \) proceeds naturally by induction and it is omitted here.

### 11 Proofs of Section 7

In this appendix, we give the proofs of Lemmas 1 and 2 of SL[1g] goal and sentence automaton and Theorems 9 and 10 of SL[1g] automaton and satisfiability.

**Alternating tree automata** Nondeterministic tree automata are a generalization to infinite trees of the classical nondeterministic word automata on infinite words. Alternating tree automata are a further generalization of nondeterministic tree automata [22]. Intuitively, on visiting a node of the input tree, while the latter sends exactly one copy of itself to each of the successors of the node, the former can send several own copies to the same successor. Here we use, in particular, alternating parity tree automata, which are alternating tree automata along with a parity acceptance condition (see [12], for a survey).

We now give the formal definition of alternating tree automata.

**Definition 26 (Alternating Tree Automata).** An alternating tree automaton (ATA, for short) is a tuple \( A \triangleq (\Sigma, \Delta, Q, \delta, q_0, \mathcal{H}) \), where \( \Sigma, \Delta, \) and \( Q \) are, respectively, non-empty finite sets of input symbols, directions, and states, \( q_0 \in Q \) is an initial state, \( \mathcal{H} \) is an acceptance condition to be defined later, and \( \delta : Q \times \Sigma \to B^+ (\Delta \times Q) \) is an alternating transition function that maps each pair of states and input symbols to a positive Boolean combination on the set of propositions of the form \( (d, q) \in \Delta \times Q \), a.k.a. moves.
On one side, a non-deterministic tree automaton (NTA, for short) is a special case of 
ATA in which each conjunction in the transition function \( \delta \) has exactly one move \((d,q)\) 
associated with each direction \(d\). This means that, for all states \(q \in Q\) and symbols \(\sigma \in \Sigma\), we have that \(\delta(q, \sigma)\) is equivalent to a Boolean formula of the form \(\bigvee_i \bigwedge_{d \in \Delta} (d, q_i, d)\). 
On the other side, a universal tree automaton (UTA, for short) is a special case of ATA 
in which all the Boolean combinations that appear in \(\delta\) are conjunctions of moves. Thus, 
we have that \(\delta(q, \sigma) = \bigwedge_i (d, q_i)\), for all states \(q \in Q\) and symbols \(\sigma \in \Sigma\).

The semantics of the ATAs is given through the following concept of run.

**Definition 27 (ATA Run).** A run of an ATA \(A = \langle \Sigma, \Delta, Q, \delta, q_0, R \rangle\) on a \(\Sigma\)-labeled 
\(\Delta\)-tree \(T = \langle T, v \rangle\) is a \((\Delta \times Q)\)-tree \(R\) such that, for all nodes \(x \in R\), where 
\(x = \prod_{i=1}^{n}(d_i, q_i)\) and \(y = \prod_{i=1}^{n} d_i\) with \(n \in [0, |w|]\), it holds that (i) \(y \in T\) and (ii), there 
is a set of moves \(S \subseteq \Delta \times Q\) with \(S \models \delta(q_n, v(y))\) such that \(x \cdot (d, q) \in R\), for all 
\((d, q) \in S\).

In the following, we consider ATAs along with the parity acceptance condition 
(APT, for short) \(R_0 \triangleq (F_1, \ldots, F_k) \subseteq (2^Q)^+\) with \(F_1 \subseteq \ldots \subseteq F_k = Q\) (see \([16]\), 
for more). The number \(k\) of sets in the tuple \(R\) is called the index of the automaton. We also 
consider ATAs with the co-Büchi acceptance condition (ACT, for short) that is the 
special parity condition with index 2.

Let \(R\) be a run of an ATA \(A\) on a tree \(T\) and \(w\) one of its branches. Then, by 
\(\inf(w) \triangleq \{q \in Q : \{i \in \mathbb{N} : \exists d \in \Delta : (w)_i = (d, q)\} = \omega\}\) we denote the set of states 
that occur infinitely often as the second component of the letters along the branch \(w\). Moreover, we say that \(w\) satisfies the parity acceptance condition \(R_0 \triangleq (F_1, \ldots, F_k)\) if 
the least index \(i \in [1, k]\) for which \(\inf(w) \cap F_i \neq \emptyset\) is even.

Finally, we can define the concept of language accepted by an ATA.

**Definition 28 (ATA Acceptance).** An ATA \(A = \langle \Sigma, \Delta, Q, \delta, q_0, R \rangle\) accepts a \(\Sigma\)-labeled 
\(\Delta\)-tree \(T\) iff there exists a run \(R\) of \(A\) on \(T\) such that all its infinite branches satisfy 
the acceptance condition \(R_0\).

By \(L(A)\) we denote the language accepted by the ATA \(A\), i.e., the set of trees \(T\) accepted 
by \(A\). Moreover, \(A\) is said to be empty if \(L(A) = \emptyset\). The emptiness problem for \(A\) is to 
decide whether \(L(A) = \emptyset\).

**Proofs of theorems** We are finally able to show the proofs of the above mentioned 
results.

**Lemma I** (SL[1G] Goal Automaton). Let \(\psi\) an SL[1G] goal without principal sub-sentences and \(A_c\) a finite set of actions. Then, there exists an UCT \(U_{\psi}^{A_c} \triangleq \langle \Val_{A_c}(\free(\psi)) \times 2^{AP}, D_c, Q_{\psi}, \delta_{\psi}, q_{0(\psi)}, N_{\psi} \rangle\) such that, for all DTS \(T\) with \(A_{CT} = A_c\), states \(t \in St_T\), and assignments \(\chi \in \Asg_T(\free(\psi), t)\), it holds that \(T, \chi, t \models \psi\) iff \(T' \in L(U_{\psi}^{A_c})\), 
where \(T'\) is the assignment-labeling encoding for \(\chi\) on \(T\).

**Proof.** A first step in the construction of the UCT \(U_{\psi}^{A_c}\), is to consider the UCW \(U_{\psi} \triangleq \langle 2^{AP}, Q_{\psi}, \delta_{\psi}, q_{0(\psi)}, N_{\psi} \rangle\) obtained by dualizing the NBW resulting from the application 
of the classic Vardi-Wolper construction to the LTL formula \(\neg \psi\) \([29]\). Observe that 
\(L(U_{\psi}) = L(\psi)\), i.e., this automaton recognizes all infinite words on the alphabet \(2^{AP}\).
that satisfy the LTL formula \( \psi \). Then, define the components of \( \mathcal{U}^{\mathcal{A} C}_{\psi} \triangleq (\text{Val}_{\mathcal{A} C}(\text{free}(\psi))) \times 2^{2\mathcal{A} P}, \text{DC}, \mathcal{Q}_\psi, \delta_\psi, q_{0\psi}, \mathcal{N}_\psi \), as follows:

- \( \mathcal{Q}_\psi \triangleq \{q_{0\psi}\} \cup Q_\psi \), with \( q_{0\psi} \notin Q_\psi \);
- \( \delta_\psi(q_{0\psi}, (v, \sigma)) \triangleq \bigwedge_{q \in Q_\psi} \delta_q(q, (v, \sigma)) \), for all \((v, \sigma) \in \text{Val}_{\mathcal{A} C}(\text{free}(\psi)) \times 2^{2\mathcal{A} P} \);
- \( \delta_\psi(q, (v, \sigma)) \triangleq \bigwedge_{q' \in \mathcal{E}_{\psi}(q, \sigma)} (v \circ_\delta q', q') \), for all \( q \in Q_\psi \) and \((v, \sigma) \in \text{Val}_{\mathcal{A} C}(\text{free}(\psi)) \times 2^{2\mathcal{A} P} \);
- \( \mathcal{N}_\psi \triangleq \mathcal{N}_\psi \).

Intuitively, the UCT \( \mathcal{U}^{\mathcal{A} C}_{\psi} \) simply runs the UCW \( \mathcal{U}_\psi \) on the branch of the encoding individuated by the assignment in input. Thus, it is easy to see that, for all states \( t \in \text{St}_T \) and assignments \( \chi \in \text{Asg}_T(\text{free}(\psi), t) \), it holds that \( T, \chi, t \models b \psi \) iff \( T' \in L(\mathcal{U}^{\mathcal{A} C}_{\psi}) \), where \( T' \) is the assignment-labeling encoding for \( \chi \) on \( T \).

**Lemma 2** (SL[1G] Sentence Automaton). Let \( \psi \) be an SL[1G] principal sentence without principal subsentences and \( \mathcal{A} C \) a finite set of actions. Then, there exists an UCT \( \mathcal{U}^{\mathcal{A} C}_{\psi} \triangleq (\text{DM}_{\mathcal{A} C}(\psi)) \times 2^{2\mathcal{A} P}, \text{DC}, \mathcal{Q}_{\psi\psi}, \delta_{\psi\psi}, q_{0\psi\psi}, \mathcal{N}_{\psi\psi} \) such that, for all DTS \( T \) with \( \text{Ac}_T = \mathcal{A} C \), states \( t \in \text{St}_T \), and elementary dependence maps over strategies \( \theta \in \text{EDM}_{\text{St}_T(\psi)}(\psi) \), it holds that \( T, \chi_T(\psi), t \models E b \psi \), for all \( \chi_T(\psi) \in \text{Asg}_T(\psi, t) \) iff \( T' \in L(\mathcal{U}^{\mathcal{A} C}_{\psi}) \), where \( T' \) is the elementary dependence-labeling encoding for \( \theta \) on \( T \).

**Proof.** By Lemma 1 of SL[1G] goal automaton, there is an UCT \( \mathcal{U}^{\mathcal{A} C}_{g(\psi)} \triangleq (\text{Val}_{\mathcal{A} C}(\text{free}(\psi))) \times 2^{2\mathcal{A} P}, \text{DC}, \mathcal{Q}_{g(\psi)}, \delta_{g(\psi)}, q_{0g(\psi)}, \mathcal{N}_{g(\psi)} \) such that, for all DTS \( T \) with \( \text{Ac}_T = \mathcal{A} C \), states \( t \in \text{St}_T \), and assignments \( \chi \in \text{Asg}_T(\text{free}(\psi), t) \), it holds that \( T, \chi, t \models b \psi \) iff \( T' \in L(\mathcal{U}^{\mathcal{A} C}_{g(\psi)}) \), where \( T' \) is the assignment-labeling encoding for \( \chi \) on \( T \).

Now, transform \( \mathcal{U}^{\mathcal{A} C}_{g(\psi)} \) into the new UCT \( \mathcal{U}^{\mathcal{A} C}_{\psi\psi} \triangleq (\text{DM}_{\mathcal{A} C}(\psi)) \times 2^{2\mathcal{A} P}, \text{DC}, \mathcal{Q}_{\psi\psi}, \delta_{\psi\psi}, q_{0\psi\psi}, \mathcal{N}_{\psi\psi} \), with \( \mathcal{Q}_{\psi\psi} \triangleq \mathcal{Q}_{g(\psi)}, \mathcal{Q}_{\psi\psi} \triangleq q_{0\psi\psi}, \) and \( \mathcal{N}_{\psi\psi} \triangleq \mathcal{N}_{g(\psi)} \), which is used to handle the quantification prefix \( \psi \) atomically, where the transition function is defined as follows: \( \delta_{\psi\psi}(q, (t, \sigma)) \triangleq \bigwedge_{v \in \text{Val}_{\mathcal{A} C}(\psi)} \delta_q(v(\psi), (t, \sigma)) \), for all \( q \in \mathcal{Q}_{\psi\psi} \) and \((t, \sigma) \in \text{DM}_{\mathcal{A} C}(\psi) \times 2^{2\mathcal{A} P} \). Intuitively, \( \mathcal{U}^{\mathcal{A} C}_{\psi\psi} \) reads an action dependence map \( \theta \) on each node of the input tree \( T' \) labeled with a set of atomic propositions \( \sigma \) and simulates the execution of the transition function \( \delta_{\psi\psi}(q, (v, \sigma)) \) of \( \mathcal{U}^{\mathcal{A} C}_{\psi} \), for each possible valuation \( v = \theta(v') \) on \( \text{free}(\psi) \) obtained from \( \theta \) by a universal valuation \( v' \in \text{Val}_{\mathcal{A} C}(\psi) \).

It is worth observing that we cannot move the component set \( \text{DM}_{\mathcal{A} C}(\psi) \) from the input alphabet to the states of \( \mathcal{U}^{\mathcal{A} C}_{\psi\psi} \) by making a related guessing of the dependence map \( \theta \) in the transition function, since we have to ensure that all states in a given node of the tree \( T' \), i.e., in each track of the original model \( T \), make the same choice for \( \theta \).

Finally, it remains to prove that, for all states \( t \in \text{St}_T \) and elementary dependence maps over strategies \( \theta \in \text{EDM}_{\text{St}_T(\psi)}(\psi) \), it holds that \( T, \theta(\chi), t \models E b \psi \), for all \( \chi \in \text{Asg}_T(\psi, t) \), iff \( T' \in L(\mathcal{U}^{\mathcal{A} C}_{\psi\psi}) \), where \( T' \) is the elementary dependence-labeling encoding for \( \theta \) on \( T \).

**[Only if]**. Suppose that \( T, \theta(\chi), t \models E b \psi \), for all \( \chi \in \text{Asg}_T(\psi, t) \). Since \( \psi \) does not contain principal subsentences, we have that \( T, \theta(\chi), t \models b \psi \). So, due to the property of \( \mathcal{U}^{\mathcal{A} C}_{\psi\psi} \), it follows that there exists an assignment-labeling encoding \( T'_\chi \in \mathcal{U}^{\mathcal{A} C}_{\psi\psi} \), which implies the existence of a \( (\text{DC} \times \mathcal{Q}_{\psi\psi}) \)-tree \( R_\chi \) that is an accepting run for \( \mathcal{U}^{\mathcal{A} C}_{\psi\psi} \).
on $\mathcal{T}'$. At this point, let $R \triangleq \bigcup_{\chi \in \text{Asg}_{T}([\varnothing], t)} R_{\chi}$ be the union of all runs. Then, due to the particular definition of the transition function of $\mathcal{U}_{\varphi T}^{\mathcal{L}}$, it is not hard to see that $R$ is an accepting run for $\mathcal{U}_{\varphi T}^{\mathcal{L}}$ on $\mathcal{T}'$. Hence, $\mathcal{T}' \in L(\mathcal{U}_{\varphi T}^{\mathcal{L}})$.

\[ \{\text{If}\} \text{ Suppose that } \mathcal{T}' \in L(\mathcal{U}_{\varphi T}^{\mathcal{L}}). \text{ Then, there exists a } (\mathcal{Dc} \times Q_{\varphi T})\text{-tree } R \text{ that is an accepting run for } \mathcal{U}_{\varphi T}^{\mathcal{L}} \text{ on } \mathcal{T}'. \text{ Now, for each } \chi \in \text{Asg}_{T}([\varnothing], t), \text{ let } R_{\chi} \text{ be the run for } \mathcal{U}_{\varphi T}^{\mathcal{L}} \text{ on the assignment-state encoding } \mathcal{T}'_{\chi} \text{ for } \theta(\chi) \text{ on } \mathcal{T}. \text{ Due to the particular definition of the transition function of } \mathcal{U}_{\varphi T}^{\mathcal{L}}, \text{ it is not hard to see that } R_{\chi} \subseteq R. \text{ Thus, since } R \text{ is accepting, we have that } R_{\chi} \text{ is accepting as well. So, } \mathcal{T}'_{\chi} \in L(\mathcal{U}_{\varphi T}^{\mathcal{L}}). \text{ At this point, due to the property of } \mathcal{U}_{\varphi T}^{\mathcal{L}}, \text{ it follows that } \mathcal{T}, \theta(\chi), t \models \forall \psi. \text{ Since } \psi \text{ does not contain principal subsentences, we have that } \mathcal{T}, \theta(\chi), t \models E \forall \psi, \text{ for all } \chi \in \text{Asg}_{T}([\varnothing], t). \]

**Theorem**[9] (SL[1G] Automaton). Let $\varphi$ be an SL[1G] sentence. Then, there exists an UCT $\mathcal{U}_{\varphi}$ such that $\varphi$ is satisfiable iff $L(\mathcal{U}_{\varphi}) \neq \emptyset$.

**Proof.** By Theorem[8] of SL[1G] bounded tree-model property, if an SL[1G] sentence $\varphi$ is satisfiable, it is satisfiable in a disjoint way on a $b$-bounded DT with $b \triangleq |P| \cdot |V(P)|$. 2$\mathbb{C}(V)$, where $P \triangleq \{(\varphi, \delta), (\psi, i) \in \text{LSig}(\mathcal{A}_{\varphi}, \mathcal{L} \times \{0, 1\}) : \varphi \psi \in \text{psnt}(\varphi) \land i \in \{0, 1\}\}$ is the set of all labeled signatures on $\mathcal{A}_{\varphi}$ w.r.t. $\mathcal{L} \times \{0, 1\}$. Thus, we can build an automaton that accepts only $b$-bounded tree encodings. To do this, in the following, we assume $\mathcal{A}_{\varphi} \triangleq \{0, 1\}$.

Consider each principal subsentence $\phi \in \text{psnt}(\varphi)$ of $\varphi$ as a sentence with atomic propositions in $\mathcal{A}_{\varphi}$ \ $\text{psnt}(\varphi)$ having no inner principal subsentence. This means that these subsentences are considered as fresh atomic propositions. Now, let $\mathcal{U}_{\varphi T}^{\mathcal{L}} \triangleq \langle \text{DM}_{\mathcal{A}_{\varphi}}(\varphi) \times 2^{|\mathcal{A}_{\varphi} \cup \text{psnt}(\varphi)|}, \text{DC}, Q_{\varphi}, \delta_{\varphi}, q_{0, \varphi}, \text{St}_{\mathcal{A}_{\varphi}} \rangle$ be the UCTs built in Lemma[2]. Moreover, set $M \triangleq \{m \in \text{psnt}(\varphi) \to \bigcup_{\phi \in \text{psnt}(\varphi)} \text{DM}_{\mathcal{A}_{\varphi}}(\varphi) : \forall \phi = \varphi \psi \in \text{psnt}(\varphi), m(\phi) \in \text{DM}_{\mathcal{A}_{\varphi}}(\varphi)\}$. Then, we define the components of the UCT $\mathcal{U}_{\varphi} \triangleq \langle M \times M \times 2^{|\mathcal{A}_{\varphi} \cup \text{psnt}(\varphi)|}, \text{DC}, Q_{\varphi}, \delta_{\varphi}, q_{0, \varphi}, \text{St}_{\mathcal{A}_{\varphi}} \rangle$, as follows:

- $Q \triangleq \{q_{0, \varphi}, \varphi\} \cup \bigcup_{\phi \in \text{psnt}(\varphi)} \{\varphi\} \times Q_{\varphi}$;
- $\delta(q_{0}, (m_{h}, m_{b}, \sigma)) \triangleq \delta(q_{c}, (m_{h}, m_{b}, \sigma))$, if $\sigma \models \varphi$, and $\delta(q_{0}, (m_{h}, m_{b}, \sigma)) \triangleq f$, otherwise, where $\varphi$ is considered here as a Boolean formula on $\mathcal{A}_{\varphi} \cup \text{psnt}(\varphi)$;
- $\delta(q_{c}, (m_{h}, m_{b}, \sigma)) \triangleq \bigwedge_{d \in \text{DC}} (d, q_{c}) \land \bigwedge_{\phi \in \text{psnt}(\varphi)} \delta_{\varphi}(q_{\phi}, (m_{h}, \sigma))(d, \varphi)/(d, (\sigma, q_{\varphi}));$
- $\delta((\varphi, \varphi), (m_{h}, m_{b}, \sigma)) \triangleq \delta_{\varphi}(q_{\varphi}, (m_{h}, \sigma))(d, \varphi)/(d, (\sigma, q_{\varphi}));$
- $\delta_{\varphi}(q_{\varphi}, (m_{h}, m_{b}, \sigma)) \triangleq \delta_{\varphi}(q_{\varphi}, (m_{h}, \sigma))$.

Intuitively, $\mathcal{U}_{\varphi}$ checks whether there are principal subsentences $\phi$ of $\varphi$ contained into the labeling, for all nodes of the input tree, by means of the checking state $q_{c}$. In the affirmative case, it runs the related automata $\mathcal{U}_{\varphi_{\phi}}$ by supplying them, as dependence maps on actions, the head part $m_{h}$, when it starts, and the body part $m_{b}$, otherwise. In this way, it checks that the disjoint satisfiability is verified.

We now prove that the above construction is correct.

[Only if] Suppose that $\varphi$ is satisfiable. Then, by Theorem[8], there exists a $b$-bounded DT $\mathcal{T}$ such that $\mathcal{T} \models \varphi$. In particular, w.l.o.g., assume that $\mathcal{A}_{\mathcal{T}} = \mathcal{A}_{\varphi}$. Moreover, for all $\phi = \varphi \psi \in \text{psnt}(\varphi)$, it holds that $\mathcal{T}$ satisfies $\phi$ disjointly over the set $S_{\phi} \triangleq \{t \in \text{St}_{\mathcal{T}} : \mathcal{T}, \emptyset, t \models \phi\}$. This means that, by Definition[10] of SL[1G] disjoint satisfiability, there
exist two functions $\text{head}_\phi : S_\phi \to DM_{Ac}(\phi)$ and $\text{body}_\phi : \text{Trk}_T(\varepsilon) \to DM_{Ac}(\phi)$ such that, for all $t \in S_\phi$ and $\chi \in \text{Asg}_T([\phi]), t$, it holds that $T, \theta_{\phi,t}(\chi), t \models b\psi$, where the elementary dependence map $\theta_{\phi,t} : EDM_{Str_T(t)}(\phi)$ is defined as follows: (i) $\overrightarrow{\theta_{\phi,t}}(t) \triangleq \text{head}_\phi(t)$; (ii) $\overrightarrow{\theta_{\phi,t}}(\rho) \triangleq \text{body}_\phi(\rho')$, for all $\rho \in \text{Trk}_T(t)$ with $|\rho| > 1$, where $\rho' \in \text{Trk}_T(\varepsilon)$ is the unique track such that $\rho' \cdot \rho \in \text{Trk}_T(\varepsilon)$.

Now, let $T_{\varepsilon}$ be the DT over $AP \cup \text{psnt}(\phi)$ with $Ac_{T_{\varepsilon}} = Ac$ such that (i) $\lambda_{T_{\varepsilon}}(t)$ and (ii) $\phi \in \lambda_{T_{\varepsilon}}(t)$ iff $t \in S_\phi$, for all $t \in St_{T_{\varepsilon}} = St_T$ and $\phi \in \text{psnt}(\phi)$.

By Lemma\textsuperscript{2} we have that $T'_{\phi,t} \in L(\ell_{\phi}^{\text{Ac}})$, where $T'_{\phi,t}$ is the elementary dependence-labeling encoding for $\theta_{\phi,t}$ on $T_{\varepsilon}$. Thus, there is a $(Dc \times Q)\text{-tree}$ $R_{\phi,t}$ that is an accepting run for $\mathcal{U}_{\phi}^{\text{Ac}}$ on $T'_{\phi,t}$. So, let $R'_{\phi,t}$ be the $(Dc \times Q)\text{-tree}$ defined as follows: $R'_{\phi,t} \triangleq \{(t, t', (q, \phi)) : (t', q) \in R_{\phi,t}\}$.

At this point, let $R \triangleq R_{\phi,t} \cup \{(\varepsilon) \cup \{(t, q) : t \in St_T \wedge t \neq \varepsilon\}$, and $\mathcal{T}' \triangleq (St_T, R')$ one of the $(M \times M \times 2^{AP \cup \text{psnt}(\phi)})$-labeled Dc-tree satisfying the following property: for all $t \in St_T$ and $\phi \in \text{psnt}(\phi)$, it holds that $u(t) = (m_h, m_b, \sigma)$, where (i) $\sigma \in AP = \lambda_T(t)$, (ii) $\phi \in \sigma$ iff $t \in S_\phi$, (iii) $m_h(\phi) = \text{head}_\phi(t)$, if $t \in S_\phi$, and (iv) $m_b(\phi) = \text{body}_\phi(\rho_t)$ with $\rho_t \in \text{Trk}_T(\varepsilon)$ the unique track such that $\text{lst}(\rho_t) = t$. Moreover, since $T \models \varphi$, we have that $\lambda_{T_{\varepsilon}}(\varepsilon) \models \varphi$, where, in the last expression, $\varphi$ is considered as a Boolean formula on $AP \cup \text{psnt}(\phi)$. Then, it is easy to prove that $R$ is an accepting run for $\mathcal{U}_\phi$ on $T'$, i.e., $T' \in L(\mathcal{U}_\phi)$. Hence, $L(\mathcal{U}_\phi) \neq \emptyset$.

If $\exists i$, suppose that there is an $(M \times M \times 2^{AP \cup \text{psnt}(\phi)})$-labeled Dc-tree $\mathcal{T}' \triangleq (Dc^*, u)$ such that $\mathcal{T}' \in L(\mathcal{U}_\phi)$ and let the $(Dc \times Q)$-tree $R$ be the accepting run for $\mathcal{U}_\phi$ on $\mathcal{T}'$. Moreover, let $T$ be the DT over $AP \cup \text{psnt}(\phi)$ with $Ac_T = Ac$ such that, for all $t \in St_T$, it holds that $u(t) = (m_h, m_b, \lambda_T(t))$, for some $m_h, m_b \in M$.

Now, for all $\phi \neq \psi \psi$, we make the following further assumptions:

- $S_\phi \triangleq \{t \in S_T : \exists m_h, m_b \in M, \sigma \in 2^{AP \cup \text{psnt}(\phi)}, u(t) = (m_h, m_b, \sigma) \land \phi \in \sigma\}$;
- let $R_{\phi,t}$ be the $(Dc \times Q)\text{-tree}$ such that $R_{\phi,t} \triangleq \{\varepsilon\} \cup \{(t, q) : (t', q) \in R\}$, for all $t \in S_\phi$;
- let $T'_{\phi,t}$ be the elementary dependence-labeling encoding for $\theta_{\phi,t} : EDM_{Str_T(t)}(\phi)$ on $T$, for all $t \in S_\phi$, where $\overrightarrow{\theta_{\phi,t}}(t) \triangleq m_h(\phi)$, with $u(t) = (m_h, m_b, \sigma)$ for some $m_h, m_b \in M$ and $\sigma \in 2^{AP \cup \text{psnt}(\phi)}$, and $\overrightarrow{\theta_{\phi,t}}(\rho) \triangleq m_b(\phi)$, with $u(\text{lst}(\rho)) = (m_h, m_b, \sigma)$ for some $m_h, m_b \in M$ and $\sigma \in 2^{AP \cup \text{psnt}(\phi)}$, for all $\rho \in \text{Trk}_T(t)$ with $|\rho| > 1$.

Since $R$ is an accepting run, it is easy to prove that $R_{\phi,t}$ is an accepting run for $\mathcal{U}_\phi$ on $T'_{\phi,t}$. Thus, $T'_{\phi,t} \in L(\mathcal{U}_\phi^{\text{Ac}})$. So, by Lemma\textsuperscript{2} it holds that $T, \theta_{\phi,t}(\chi), t \models b\psi$, for all $t \in S_\phi$ and $\chi \in \text{Asg}_T([\phi]), t$, which means that $S_\phi = \{t \in St_T : T, \emptyset, t \models \phi\}$.

Finally, since $\lambda_{T_{\varepsilon}}(\varepsilon) \models \varphi$, we have that $T \models \varphi$, where, in the first expression, $\varphi$ is considered as a Boolean formula on $AP \cup \text{psnt}(\phi)$.

**Theorem\textsuperscript{10}** (SL\textsuperscript{1G} Satisfiability). The satisfiability problem for SL\textsuperscript{1G} is 2EXPTIME-complete.

**Proof.** By Theorem\textsuperscript{9} of SL\textsuperscript{1G} automaton, to verify whether an SL\textsuperscript{1G} sentence $\varphi$ is satisfiable we can calculate the emptiness of the UPT $\mathcal{U}_\varphi$. This automaton is obtained
by merging all UCTs $\mathcal{U}^\omega_{\phi}$, with $\phi = \varphi \psi \in \text{psnt}(\varphi)$, which in turn are based on the UCTs $\mathcal{U}^\omega_{\psi}$ that embed the UCWs $\mathcal{U}_\varphi$. By a simple calculation, it is easy to see that $\mathcal{U}_\varphi$ has $2^{\mathcal{O}(|\varphi|)}$ states.

Now, by using a well-known nondeterminization procedure for APTs [23], we obtain an equivalent NPT $\mathcal{N}_\varphi$ with $2^{2^{\mathcal{O}(|\varphi|)}}$ states and index $2^{\mathcal{O}(|\varphi|)}$.

The emptiness problem for such a kind of automaton with $n$ states and index $h$ is solvable in time $O(n^h)$. Thus, we get that the time complexity of checking whether $\varphi$ is satisfiable is $2^{2^{\mathcal{O}(|\varphi|)}}$. Hence, the membership of the satisfiability problem for $\text{SL}[1\text{G}]$ in $2\text{ExPTIME}$ directly follows. Finally the thesis is proved, by getting the relative lower bound from the same problem for $\text{CTL}^+$.