Minimal driver sets on path and cycle graphs with arbitrary non-zero weights

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Abstract

Let $G$ be a simple, undirected graph on the vertex set $V = \{1, 2, \ldots, n\}$ and let $A$ be the adjacency matrix of $G$. A non-empty subset $\{i_1, i_2, \ldots, i_k\}$ of $V$ is called a driver set for $G$ if the system $\dot{x} = Ax + u_1 e_{i_1} + \cdots + u_k e_{i_k}$ is controllable.

In this paper we classify the minimal driver sets for the path and cycle graphs $P_n$ and $C_n$ for all values of $n$ and we determine which of those minimal driver sets render the system to be strongly structural controllable with respect to the family of all symmetric matrices $X$ satisfying $x_{ij} = 0 \iff a_{ij} = 0$.

Note that this new type of strong structural controllability requires all diagonal elements of the system matrix to be equal to zero so for example the Laplacian matrix is not included in the family.

Keywords: System, graph, (structural) controllability, driver set.

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1 Introduction

Let $G = (V, E)$ be a simple, undirected graph on the vertex set $V = \{1, 2, \ldots, n\}$ with adjacency matrix $A$. For each non-empty subset $S = \{i_1, i_2, \ldots, i_k\}$ of $V$ let $B_S$ be the $(n \times k)$-matrix with columns $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$.
A non-empty subset $S$ is called a driver set for $G$ if the system $\dot{x} = Ax + B_S u$, or the pair $(A, B_S)$, is controllable.

Let $\text{Sym}(G)$ be the set of symmetric $(n \times n)$-matrices $X$ with free diagonal elements and off-diagonal elements $x_{ij}$ unequal to zero if and only if $(i, j) \in E$. If $(A, B_S)$ is controllable then $(X, B_S)$ is controllable not just for $X = A$ but for almost all $X \in \text{Sym}(G)$, a property which is referred to as structural controllability in the literature. The subject of structural controllability of networks has been studied intensively during the last two decades by many researchers in the systems and control community, in view of applications where the weights of the edges are not fixed due to lack of information or numerical instability.

A stronger version of structural controllability is the property that $(X, B_S)$ is controllable for all $X \in \text{Sym}(G)$. This property is referred to as strong structural controllability in the literature.

Perhaps surprisingly, it turned out that this notion of strong structural controllability of a network is connected to the notion of a zero forcing set of the underlying graph. It has been proved in [6] that $(X, B_S)$ is controllable for all $X \in \text{Sym}(G)$ if and only if $S$ is a zero forcing set of $G$.

A zero forcing set is a special type of driver set but in general not every driver set is a zero forcing set. The discovery of the connection between strong structural controllability and zero forcing sets has understandably caused a surge of research in the latter. We believe there are several good reasons for studying all minimal driver sets for the system $(A, B_S)$, such as the following:

- The minimal size of a driver set could be smaller than the minimal size of a zero forcing set (see example 9 in section 3), which could be relevant in applications where using a driver set of minimum cardinality is essential.

- Additional requirements about the relative positions of the vertices in a driver set may exist which might not be satisfied by the zero forcing sets.

- Strong structural controllability with respect to $\text{Sym}(G)$ allows for $|V| + |E|$ degrees of freedom in the system matrix. For simple graphs $G$ it seems more natural to study strong structural controllability with

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The notion of a zero forcing set (briefly summarized in section 4) had been introduced several years earlier in a different context [1].
respect to the smaller family Sym\(_0(G)\) consisting of all matrices in Sym(G) with zeros on the diagonal, allowing for \(|E|\) degrees of freedom only. Driver sets \(S\) for which \((X, B_S)\) is controllable for all \(X \in \) Sym\(_0(G)\) are not necessarily a zero forcing set.

In this paper we determine all minimal driver sets for the path and cycle graphs for all values of \(n\), using a simple controllability test in terms of the eigenspaces of the adjacency matrices. We also determine for which of those minimal driver sets the system is strongly structural controllable with respect to the family Sym\(_0(G)\). It will turn out that not all such sets are zero forcing sets, so we have discovered new types of minimal driver sets that render the systems to be controllable for all non-zero weights on the edges of the path and cycle graphs. These (non-trivial) results for the path and cycle graphs could provide ideas for a similar classification of minimal driver sets for other types of simple graphs.

The organization of the paper after the introduction is as follows. In section 2 we present some relevant background information and notations. In section 3 we derive a necessary and sufficient condition for controllability of a system on a graph in terms of the eigenspaces of the adjacency matrix of the graph and give three illustrative examples. In section 4 we introduce a new type of strong structural controllability which we believe to be natural for systems on simple graphs. In sections 5 and 6 we present our results about the minimal driver sets for the path and cycle graphs.

2 Preliminaries and notations

2.1 Controllability of linear systems

Let \(A\) and \(B\) be matrices of sizes \((n \times n)\) and \((n \times k)\), respectively. A system \(\dot{x} = Ax + Bu\), or the pair \((A, B)\), is controllable if any initial state vector can be steered by the system to any other state vector in finite time. There are several equivalent ways to state the Popov-Belevich-Hautus (PBH) controllability test. Each of the following four properties is a necessary and sufficient condition for \((A, B)\) to be controllable:

1. \(\text{rank } \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n\) for all \(\lambda \in \mathbb{C}\)
2. \(\text{rank } \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n\) for all eigenvalues \(\lambda\) of \(A\)
3. no eigenvector \(v\) of \(A^T\) exists with \(B^Tv = 0\)
4. \(\text{Nul } B^T \cap E_\lambda = \{0\}\) for each eigenspace \(E_\lambda\) of \(A^T\)
We shall refer to these statements as PBH 1, . . . , PBH 4 in the sequel of this paper. The first three conditions are well-known and used very often in the literature. We add PBH 4 to the list because it will turn out to be useful in this paper (see section 3).

PBH 2 implies that\( \text{rank } B \geq \text{gm (}\lambda\text{)} \) for each eigenvalue \( \lambda \) of \( A \), hence
\[
\text{rank } B \geq \max_{\lambda \in \sigma(A)} \{\text{geometric multiplicity } \lambda\}. \tag{1}
\]

Two systems \((A, B)\) and \((A', B')\) are called equivalent if there exists an invertible matrix \(T\) such that
\[
\begin{align*}
A' &= TAT^{-1} \\
B' &= TB
\end{align*}
\]

If two systems are equivalent then controllability of the one is equivalent to controllability of the other.

### 2.2 Graphs

An undirected graph \(G = (V, E)\) consists of a set \(V = \{1, 2, \ldots, n\}\) and a set \(E\) of unordered pairs \(\{i, j\}\) of vertices. The elements of \(V\) and \(E\) are called the vertices and edges of \(G\), respectively. Two vertices \(i\) and \(j\) are called adjacent if \(\{i, j\} \in E\). In this paper we only consider graphs without loops, i.e., graphs without edges of the form \(\{i, i\}\); such graphs are called simple graphs. A path of length \(k\) between two vertices \(i\) and \(j\) is a sequence of vertices \(i_1 = i, i_2, \ldots, i_{k-1}, i_k = j\) such that \(i_t\) and \(i_{t+1}\) are adjacent for all \(t \in \{1, 2, \ldots, k-1\}\). The distance between two vertices \(i\) and \(j\) in a graph, denoted by \(d(i, j)\), is the shortest length of a path between \(i\) and \(j\). The adjacency matrix of a graph \(G = (V, E)\) with \(V = \{1, 2, \ldots, n\}\) is the symmetric \((n \times n)\)-matrix \(A = [a_{ij}]\) with \(a_{ij} = 1\) if \(\{i, j\} \in E\) and \(a_{ij} = 0\) otherwise.

An automorphism of a graph \(G = (V, E)\) is a permutation \(\sigma\) of \(V\) which satisfies the property \(\{\sigma(i), \sigma(j)\} \in E\) if and only if \(\{i, j\} \in E\). The set of all automorphisms of \(G\) forms a group and is denoted by \(\text{Aut}(G)\). Every element of \(\text{Aut}(G)\) can be represented uniquely by a permutation matrix \(P\) satisfying \(A = P^TAP\).

In this paper we will pay special attention to two special graphs with vertex set \(V = \{1, 2, \ldots, n\}\), viz. the path graphs denoted by \(P_n\) and the cycle graphs denoted by \(C_n\). The edge sets are \(\{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}\)
for $P_n$ and $\{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{1,n\}\}$ for $C_n$. The automorphism group $Aut(P_n)$ is generated by the reflection $\sigma$ defined by $\sigma(i) = v_{n+1-i}$ for all $i \in \{1,2,\ldots,n\}$, hence $Aut(P_n) \cong \mathbb{Z}_2$. The automorphism group of $C_n$ is generated by the rotation $\sigma = (1,2,\ldots,n)$ and the reflection $\tau$ about the axis that passes through the vertex 1 and the centre of $C_n$, hence $Aut(C_n) \cong D_{2n}$, the dihedral group of order $2n$ (the group of symmetries of a regular $n$-gon).

2.3 Plücker coordinates of subspaces

Let $W$ be an $m$-dimensional subspace of $\mathbb{R}^n$ and $\{w_1, w_2, \ldots, w_m\}$ a basis of $W$. The $\binom{n}{m}$ maximal minors of the $(n \times m)$-matrix $[w_1 \ w_2 \ \cdots \ w_m]$ are the Plücker coordinates of the subspace $W$. These coordinates are homogeneous coordinates as they are determined up to a joint non-zero factor: if we change the basis of $W$ all Plücker coordinates are multiplied by the determinant of the $(m \times m)$-matrix that represents the change of basis. The Plücker coordinates of $W$ are indexed by the $\binom{n}{m}$ sets $\{i_1, i_2, \ldots, i_m\}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$.

3 Minimal driver sets on graphs

Let $G = (V, E)$ be a simple, undirected graph on the vertex set $V = \{1,2,\ldots,n\}$ with adjacency matrix $A = A(G)$. For each non-empty subset $S = \{i_1, i_2, \ldots, i_k\}$ of $V$ let $B_S$ be the $(n \times k)$-matrix with columns $e_{i_1}$, $e_{i_2}$, $\ldots$, $e_{i_k}$. We start with some definitions.

**Definition 1** A non-empty subset $S$ of $V$ is called a driver set for $G$ if the system $(A, B_S)$ is controllable.

**Definition 2** $D(G)$ denotes the minimum cardinality of a driver set for the graph $G$.

**Definition 3** $N_D(G)$ denotes the number of minimal driver sets for $G$.

**Definition 4** $M(G)$ denotes the maximum of all geometric multiplicities of the adjacency matrix $A(G)$.

Since rank $B_S = |S|$ inequality (1) yields

$$D(G) \geq M(G).$$

(2)
Application of PBH 4 to the pair \((A, B_S)\) with \(A = A(G)\) (which is a symmetric matrix hence \(A^T\) can be replaced by \(A\)) yields the statement

\[ S \text{ is a driver set for } G \iff \text{Nul } B_S^T \cap E_\lambda = \{0\} \]

for each of the eigenspaces \(E_\lambda\) of \(A\).

Let \(W\) be an \(m\)-dimensional subspace of \(\mathbb{R}^n\) and \(S\) a subset of \(V\) with \(|S| = k \geq 1\). Then \(\text{Nul } B_S^T \cap W = \{0\}\) if and only if \(\{B_S^T w_1, B_S^T w_2, \ldots, B_S^T w_m\}\) is linearly independent for each basis \(\{w_1, w_2, \ldots, w_m\}\) of \(W\). This condition can be rephrased as

\[ \text{rank } B_S^T M = m \]

for each \((n \times m)\)-matrix \(M\) which satisfies \(\text{Col } M = W\) (i.e., the columns of \(M\) form a basis of \(W\)). Note that \(B_S^T M\) is a \((k \times m)\)-matrix and that the condition \(\text{rank } B_S^T M = m\) implies \(k \geq m\), i.e.,

\[ |S| \geq \dim W. \]

If \(|S| = \dim W = k\) then \(B_S^T M\) is a \((k \times k)\)-matrix in which case the condition \(\text{rank } B_S^T M = k\) is equivalent to the condition \(\det B_S^T M \neq 0\).

The determinant of \(B_S^T M\) is the homogeneous Plücker coordinate indexed by \(S\) of the subspace \(W\). Hence we have the following lemma, which is a useful tool for constructing minimal driver sets for graphs \(G\) with \(D(G) = M(G)\).

**Lemma 5** Let \(G\) be a graph with \(D(G) = M(G) = k\). If \(S\) is a minimal driver set for \(G\) then for each \(k\)-dimensional eigenspace \(E_\lambda\) of \(A(G)\) the Plücker coordinate of \(E_\lambda\) indexed by \(S\) is unequal to zero.

**Example 6** \(G = P_5\). The eigenvalues of \(A\) are \(-1, 0, 1, -\sqrt{3}, \sqrt{3}\). Basis vectors for the corresponding eigenspaces are the columns of the matrix \(M\) given by

\[
M = \begin{bmatrix}
-1 & 1 & -1 & 1 & 1 \\
1 & 0 & -1 & -\sqrt{3} & \sqrt{3} \\
0 & -1 & 0 & 2 & 2 \\
-1 & 0 & 1 & -\sqrt{3} & \sqrt{3} \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

\(D(P_5) = M(P_5) = 1\) and \(N_D(P_5) = 2\). The two minimal driver sets are \(\{1\}\) and \(\{5\}\) because rows 1 and 5 of the matrix \(M\) do not contain a zero. The two minimal driver sets lie in a single orbit under the action of the automorphism group \(\text{Aut}(P_5) = \langle (1, 5)(2, 4) \rangle\). The path graphs \(P_n\) for general \(n\) will be discussed in section 5.
Example 7 $G = C_6$. The eigenvalues of $A$ are $-2, 2, -1^2, 1^2$ hence $M(C_6) = 2$, which implies $D(C_6) \geq 2$. Basis vectors for the corresponding eigenspaces are collected in the following block matrix

$$M = [M_1|M_2|M_3|M_4] = \begin{bmatrix}
  -1 & 1 & -1 & -1 & 1 & -1 \\
  1 & 1 & 0 & 1 & 0 & -1 \\
  -1 & 1 & 1 & 0 & -1 & 0 \\
  1 & 1 & -1 & -1 & -1 & 1 \\
  -1 & 1 & 0 & 1 & 0 & 1 \\
  1 & 1 & 1 & 0 & 1 & 0 
\end{bmatrix}.$$ 

By looking at these bases of the eigenspaces we can immediately observe that $D(C_6) = 2$. It turns out that the nonzero Plücker coordinates of $E_{-1}$ and $E_1$ are precisely the ones indexed by the 12 elements $\{i, j\}$ with $d(i, j) \in \{1, 2\}$. Since the basis vectors of the remaining eigenspaces $E_{-2}$ and $E_2$ don’t have two zeros in any of these pairs of positions we can conclude $D(C_6) = 2$ with $N_D(C_6) = 12$. The sets $\{i, j\}$ with $d(i, j) = 3$ are the sets of cardinality 2 that are not a driver set. For example $\{1, 4\}$ is not a driver set because $\det B_{\{1,4\}}^T M_3 = 0$ or $\det B_{\{1,4\}}^T M_4 = 0$ (in this example both are true):

$$B_{\{1,4\}}^T M_3 = \begin{bmatrix}
  -1 & -1 \\
  -1 & -1 
\end{bmatrix}$$ and $$B_{\{1,4\}}^T M_4 = \begin{bmatrix}
  1 & -1 \\
  -1 & 1 
\end{bmatrix}.$$ 

The minimal driver sets fall into the two orbits $\{\{i, j\} \mid d(i, j) = 1\}$ and $\{\{i, j\} \mid d(i, j) = 2\}$ under the group $\text{Aut}(C_6) = \langle (1, 2, 3, 4, 5, 6), (1, 2)(3, 6)(4, 5) \rangle$. The cycle graphs $C_n$ for general $n$ will be discussed in section 6.

Note that in the examples above the property of being a minimal driver set is invariant under the action of the automorphism group $\text{Aut}(G)$. This is true in general:

**Proposition 8** Let $\pi \in \text{Aut}(G)$. Then $S$ is a driver set for $G$ if and only if $\pi(S)$ is a driver set for $G$.

**Proof.** Let $P$ denote the permutation matrix that corresponds to $\pi \in \text{Aut}(G)$. Then $B_{\pi(S)} = PB_S$ and $A = PAP^T$, hence the systems $(A, B_S)$ and $(A, B_{\pi(S)})$ are equivalent. ■

Suppose we know that $D(G) = k$ and we also know the different orbits of $k$-sets under the group $\text{Aut}(G)$. Then the set all of minimal driver sets can be simply determined by investigating one representative of each orbit. The following example illustrates this method.
Example 9 Let $Q_n, n \geq 1$, denote the hypercube graph with $2^n$ vertices, i.e., the graph with vertex set $V = \{0,1\}^n$ and the following definition of adjacency: $x$ and $y$ are adjacent (form an edge) if and only if $x$ and $y$ differ in one coordinate position only. The adjacency matrices of $Q_n$ can be defined recursively as follows:

$$A(Q_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A(Q_{n+1}) = \begin{bmatrix} A(Q_n) & I_n \\ I_n & A(Q_n) \end{bmatrix}, n \geq 1.$$ 

In this example we consider $Q_3$. The eigenvalues of $A(Q_3)$ are $3^1, (-3)^1, 1^3$ and $(−1)^3$ hence $M(G) = 3$, which implies $D(Q_3) \geq 3$. Basis vectors for the corresponding eigenspaces are collected in the following block matrix

$$M = [M_1|M_2|M_3|M_4] = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$ 

By looking at these bases of the eigenspaces we can immediately observe that $D(Q_3) = 3$: the maximal minors of $M_3$ and $M_4$ from the first three rows (for example) are both unequal to zero while the first three elements of $M_1$ and of $M_2$ are not equal to zero. The minimal size of a zero forcing set for $Q_3$ is equal to 4 hence none of the minimal driver sets for $Q_3$ is a zero forcing set. Now let’s look at the total picture of minimal driver sets for $Q_3$. There are three orbits of subsets of vertices of cardinality 3 under the group $\text{Aut}(G) \cong S_3 \times S_3^2$ (with representatives $\{1,2,3\}, \{1,2,4\}$ and $\{1,2,7\}$). It is readily seen that both $\{1,2,3\}$ and $\{1,2,4\}$ are a minimal driver set and $\{1,2,7\}$ is not. The orbits of $\{1,2,3\}$ and $\{1,2,4\}$ have sizes 24 and 8, respectively, hence $N_D(Q_3) = 32$.

4 Strong structural controllability

Let $S$ be a driver set for a graph $G = (V,E)$ with $|V| = n$. Let $\text{Sym}(G)$ be the set of all symmetric $(n \times n)$-matrices $X = [x_{ij}]$ satisfying $x_{ij} \neq 0 \iff (i,j) \in E$
for all pairs \((i, j)\) with \(i \neq j\). Hence \(\text{Sym}(G)\) is the largest set of symmetric matrices that have their non-zero off-diagonal entries in precisely the same positions as the adjacency matrix \(A\). The following type of strong structural controllability is well-known:

**Definition 10** \((G, S)\) is strongly \(\text{Sym}(G)\)-controllable if \((X, B_S)\) is controllable for all \(X \in \text{Sym}(G)\).

Note that this formulation is a succinct alternative to the more elaborate version ‘\((G, S)\) is strongly structurally controllable with respect to \(\text{Sym}(G)\) if \((X, B_S)\) is controllable for all \(X \in \text{Sym}(G)\)’. More generally we replace ‘strongly structurally controllable with respect to \(F\)’ by ‘strongly \(F\)-controllable’ (where \(F\) is a set of matrices having the same zero/non-zero pattern in the off-diagonal entries as \(A\)).

It has been proved in [6] that \((G, S)\) is strongly \(\text{Sym}(G)\)-controllable if and only if \(S\) is a zero forcing set of \(G\).

The process of zero forcing, which was introduced in [1] and independently in [2], can be briefly summarized in the following way.

Let \(S\) be a non-empty subset of vertices of \(G\) and suppose all vertices from \(S\) are colored black and all vertices from \(V \setminus S\) are colored white. If there exists a black vertex with exactly one white neighbour \(j\) then change the color of \(j\) to black and extend the set \(S\) to \(S \cup \{j\}\) and repeat this process until no color change is possible anymore.

**Definition 11** The set \(S\) is called a zero forcing set if the coloring process described above results in all vertices being colored black.

**Definition 12** The zero forcing number of \(G\), denoted by \(Z(G)\), is the minimum cardinality of a zero forcing set.

The zero forcing number and minimal zero forcing sets for the path and cycle graphs are well-known:

| \(G\) | \(Z(G)\) | Zero forcing sets \(S\) with \(|S| = Z(G)\) |
|-------|--------|----------------------------------|
| \(P_n\) | 1 | \{1\} and \{\(n\)\} |
| \(C_n\) | 2 | \{i, j\} with \(d(i, j) = 1\) |

Each zero forcing set is a driver set hence for each graph \(G\) we have

\[
D(G) \leq Z(G). \tag{3}
\]
Note that for the path and cycle graphs all minimal zero forcing sets lie in the same orbit under the action of the automorphism groups of the graphs. In general the minimal zero forcing sets of $G$ could lie in different orbits but the property of being a zero forcing set is indeed invariant under the action of $Aut(G)$. This follows immediately from the definition of a zero forcing set, which is based on the adjacency structure of $G$ only. Equivalently we have the following property:

**Proposition 13** Let $\pi \in Aut(G)$. Then $(G, S)$ is strongly $Sym(G)$-controllable if and only if $(G, \pi(S))$ is strongly $Sym(G)$-controllable.

**Proof.** Let $P$ denote the permutation matrix that corresponds to $\pi \in Aut(G)$. Then $B_{\pi(S)} = PB_S$. The systems $(X, B_S)$ and $(PX^TP, PB_S)$ are equivalent hence controllability of the one is equivalent to controllability of the other. On the other hand, $Sym(G)$ is invariant under the transformation $X \mapsto PX^TP$, which permutes the free parameters on the diagonal and the free parameters on the off-diagonal positions $(i, j) \in E$.

Strong $Sym(G)$-controllability allows for $|V| + |E|$ degrees of freedom in the system matrix. In applications with simple graphs it seems more natural to require strong structural controllability with respect to the smaller family $Sym_0(G)$ consisting of all matrices in $Sym(G)$ with zeros on the diagonal, allowing for $|E|$ degrees of freedom only. Driver sets $S$ for which $(G, S)$ is $Sym_0(G)$-controllable are not necessarily a zero forcing set. Note that Proposition 13 holds for the smaller family $Sym_0(G)$ as well, because the transformation $X \mapsto PX^TP$ doesn’t change the zeros on the diagonal.

The chain $Sym_0(G) \subset Sym(G)$ gives rise to the following two types of driver sets $S$:

**Definition 14** A driver set $S$ is

- **of type I** if $(G, S)$ is strongly $Sym(G)$-controllable

- **of type II** if $\begin{cases} (G, S) \text{ is strongly } Sym_0(G)\text{-controllable}, \\
\text{ but not strongly } Sym(G)\text{-controllable} \end{cases}$

Driver sets of type I are zero forcing sets, driver sets of type II are not zero forcing sets but could still be useful for certain applications. Since each of the two types defined above is $Aut(G)$-invariant we could also speak of *orbits* of type I, II.
To prove that \((G, S)\) is strongly \(F\)-controllable we can proceed as follows. Due to PBH 1 \((X, B_S)\) is controllable for each \(X \in F\) if and only if

\[
\text{rank } \left[ \begin{array}{cc} X - \lambda I & B_S \end{array} \right] = n
\]

for all \(\lambda \in \mathbb{C}\) and \(X \in F\). The rows of \(\left[ \begin{array}{cc} X - \lambda I & B_S \end{array} \right]\) are linearly independent if and only if the rows of \((X - \lambda I)_{V \setminus S}\) are linearly independent, where \((X - \lambda I)_{V \setminus S}\) is the submatrix of \(X - \lambda I\) which is obtained by deleting all rows \(i\) with \(i \in S\). Hence \((X, B_S)\) is controllable for each \(X \in F\) if and only if

\[
\text{rank } (X - \lambda I)_{V \setminus S} = n - |S|
\]

for all \(\lambda \in \mathbb{C}\) and \(X \in F\). We shall use this method in the next two sections where we determine all the orbits of type II minimal driver sets for the path and cycle graphs.

5 Path graphs

Since \(Z(P_n) = 1\) and \(D(P_n) \leq Z(P_n)\) it follows that \(D(P_n) = 1\) as well. In the following theorem \(\phi\) denotes the Euler totient function.

**Theorem 15** \(\{i\}\) is a driver set for the graph \(P_n\) if and only if

\[
\gcd(i, n + 1) = 1,
\]

hence \(N_D(P_n) = \phi(n + 1)\).

**Proof.** The eigenvalues of \(A = A(P_n)\) are given by \(\lambda_k = 2 \cos \left( \frac{k\pi}{n+1} \right)\) with \(k = 1, 2, \ldots, n\) and all eigenvalues have multiplicity equal to 1. The vector

\[
\begin{bmatrix}
\sin \left( \frac{k\pi}{n+1} \right) \\
\sin \left( \frac{2k\pi}{n+1} \right) \\
\vdots \\
\sin \left( \frac{nk\pi}{n+1} \right)
\end{bmatrix}^T
\]

is an eigenvector of \(A\) belonging to the eigenvalue \(\lambda_k\). Due to PBH 3 (with \(A^T = A\) \(\{i\}\) is not a driver set if and only if there exists an eigenvector of \(A\) whose \(i\)-th entry is equal to 0 hence if and only if \(\sin \left( \frac{ik\pi}{n+1} \right) = 0\) for at least one \(k \in \{1, 2, \ldots, n\}\). The latter is true if and only if \(ik \equiv 0 \mod n + 1\) for at least one \(k \in \{1, 2, \ldots, n\}\), which is equivalent to \(\gcd (i, n + 1) \neq 1\). ■

The orbits of minimal driver sets under the group \(\text{Aut}(P_n) \cong S_2\) are simply the pairs \(\{\{i\}, \{n + 1 - i\}\}\) with \(\gcd (i, n + 1) = 1\) hence the number of orbits is equal to \(\frac{1}{2} \phi(n + 1)\).
Driver sets of type I have to be zero forcing sets \(^4\). It is obvious that \(\{1\}\) and \(\{n\}\) are the only zero forcing sets for \(P_n\) and that this is true for all \(n \geq 2\). It is easy to see that the orbit \(\{\{1\}, \{n\}\}\) is of type I without resorting to the notion of zero forcing sets. We only need to show this for one representative of the orbit. For each \(X = [x_{ij}] \in \text{Sym}(P_n)\) the matrix \((X - \lambda I)_{\{2,\ldots,n\}}\) is an echelon matrix with \(n-1\) pivots \(x_{12}, x_{23}, \ldots, x_{n-1,n}\) hence \(\text{rank} \,(X - \lambda I)_{\{2,\ldots,n\}} = n-1\) for all \(X \in \text{Sym}(P_n)\) and \(\lambda \in \mathbb{C}\). Before examining the other orbits of minimal driver sets we present some useful lemmas.

**Lemma 16** For each \(X = [x_{ij}] \in \text{Sym}_0(P_n)\) we have

\[
\det X = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
(-1)^{\frac{n}{2}}x_{12}^2x_{34}^2\cdots x_{n-1,n}^2 & \text{if } n \text{ is even}
\end{cases}
\]

**Proof.** Let \(d_n = \det X\) with \(X = [x_{ij}] \in \text{Sym}_0(P_n)\). Then \(d_1 = 0\) and \(d_2 = -x_{12}^2\) and expansion along the last column and then along the last row yields the recurrence relation

\[
d_n = -x_{n-1,n}^2d_{n-2}
\]

for all \(n \geq 3\). \(\blacksquare\)

For each \(X \in \text{Sym}_0(P_n)\) with \(n \geq 3\) and \(i \in \{2,\ldots,n-1\}\) the matrix \(X_{V\setminus\{i\}}\) has the block structure

\[
\begin{bmatrix}
Y & 0 & \vdots \\
0 & \ddots & 0 \\
x_{i-1,i} & \ddots & \ddots \\
0 & \ddots & \ddots \\
0 & \ddots & \ddots & Z
\end{bmatrix}
\]

with \(Y \in \text{Sym}_0(P_{i-1})\) and \(Z \in \text{Sym}_0(P_{n-i})\).

**Lemma 17** \(^3\) Let \(X \in \text{Sym}_0(P_n)\) with \(n \geq 3\) and \(i \in \{2,\ldots,n-1\}\) and \(Y\) and \(Z\) as in (4). Then \(\text{rank} \,(X - \lambda I)_{V\setminus\{i\}} < n-1\) if and only if \(Y\) and \(Z\) have a common eigenvalue \(\lambda\).

\(^3\)A similar result has been proved in \(^2\) with respect to the system \((L_n, B_{\{i\}})\), where \(L_n\) is the Laplacian matrix of \(P_n\).
Proof. The linear system $(X - \lambda I)^T_{V \setminus \{i\}} v = 0$ breaks down into

(1) $(Y - \lambda I)v_{\{1,2,\ldots,i-1\}} = 0$
(2) $x_{i-1,i}v_{i-1} + x_{i,i+1}v_i = 0$
(3) $(Z - \lambda I)v_{\{i+1,i+2,\ldots,n-1\}} = 0$

Equation (2) implies that either $v_{i-1} = v_i = 0$ or $v_{i-1}v_i \neq 0$. If $v_{i-1} = v_i = 0$ then it follows from (1) and (3) that $v = 0$. Suppose rank $(X - \lambda I)^T_{V \setminus \{i\}} < n-1$, i.e., suppose the system above does have a non-trivial solution $v$. Then $v_{i-1} \neq 0$ and $v_i \neq 0$ hence $v_{\{1,2,\ldots,i-1\}} \neq 0$ and $v_{\{i+1,i+2,\ldots,n-1\}} \neq 0$, so (1) and (3) show that $\lambda$ is an eigenvalue of $Y$ and $Z$. Conversely, suppose $Y$ and $Z$ have a common eigenvalue $\lambda$, i.e., suppose (1) and (3) have non-trivial solutions. These solutions can be scaled in such a way that $v_{i-1}$ and $v_i$ satisfy equation (2), hence a non-trivial solution of the linear system $(X - \lambda I)^T_{V \setminus \{i\}} v = 0$ exists. 

Now let us examine the orbit $\{\{2\}, \{n-1\}\}$. Due to Theorem 15, $\{2\}$ is a driver set if and only if gcd$(2, n+1) = 1$, i.e., if and only if $n$ is even. It is a zero forcing set for $n = 2$, so we consider $n \geq 4$.

Theorem 18 For all even $n \geq 4$ the minimal driver sets $\{2\}$ and $\{n-1\}$ for the graph $P_n$ are of type II.

Proof. We only need to show this for one representative of the orbit. Due to Lemma 17, $(P_n, \{2\})$ is not strongly Sym$_0(P_n)$-controllable if and only if there exists an $X \in$ Sym$_0(P_n)$ such that $Y \in$ Sym$_0(P_1)$ and $Z \in$ Sym$_0(P_{n-2})$ (as defined in (4)) have a common eigenvalue. In this case $Y = [0]$ so $(P_n, \{2\})$ is not strongly Sym$_0(P_n)$-controllable if and only if $Z$ is singular. It follows from Lemma 16 that det $Z = 0$ if and only if $n-2$ is odd. 

Finally we show that the remaining orbits are not of type II.

Theorem 19 Let $\{i\}$ be a minimal driver set for $P_n$ with $3 \leq i \leq n-2$. $(P_n, \{i\})$ is not strongly Sym$_0(P_n)$-controllable.

Proof. Due to Lemma 17, $(P_n, \{i\})$ is not strongly Sym$_0(P_n)$-controllable if and only if there exists an $X \in$ Sym$_0(P_n)$ such that $Y$ and $Z$ have a common eigenvalue. For each $i \in \{3, \ldots, n-2\}$ such a pair $Y, Z$ is easily constructed in the following way. Choose any $Y \in$ Sym$_0(P_{i-1})$ and $Z \in$ Sym$_0(P_{n-i})$ and a pair $\lambda_0, \mu_0$ of non-zero eigenvalues of $Y$ and $Z$ respectively. Then $\mu_0 Y \in$ Sym$_0(P_{i-1})$ and $\lambda_0 Z \in$ Sym$_0(P_{n-i})$ share the eigenvalue $\lambda_0 \mu_0$. 

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6 Cycle graphs

Let \( \omega = \exp(i \frac{2 \pi}{n}) \). The eigenvalues of the adjacency matrix \( A = A(C_n) \) are given by \( \lambda_k = \omega^k + \omega^{n-k} = 2 \cos \left( \frac{2k\pi}{n} \right) \) with \( k = 0, 2, \ldots, n-1 \). The algebraic multiplicities (equal to the geometric ones because \( A \) is symmetric hence diagonalizable) are all equal to 2 with the exceptions of \( \lambda_0 = 2 \) for all \( n \) and \( \lambda_{\frac{n}{2}} = -2 \) for all even \( n \). Hence \( M(C_n) = 2 \) which implies \( D(C_n) \geq 2 \). On the other hand \( Z(C_n) = 2 \) hence \( D(C_n) = 2 \) as well. The following Theorem specifies which pairs of vertices do in fact form a minimal driver set.

**Theorem 20** \( \{i, j\} \) is a driver set for the graph \( C_n \) if and only if

\[
\gcd(2d, n) \in \{1, 2\},
\]

where \( d = d(i, j) \) denotes the distance between the vertices \( i \) and \( j \).

**Proof.** The 1-dimensional eigenspaces of \( A \) is/are given by

\[
\text{Span} \left\{ \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \right\} \text{ for all } n \text{ and }
\text{Span} \left\{ \begin{bmatrix} 1 & -1 & \cdots & 1 & -1 \end{bmatrix}^T \right\} \text{ for all even } n,
\]

hence all entries of the eigenvectors from these eigenspaces are unequal to 0. Therefore it follows from Lemma\[ that \( \{i, j\} \) is not a driver set if and only the Plücker coordinate \( p_{ij} \) of at least one 2-dimensional eigenspace of \( A \) is equal to zero. Let \( \lambda_k \) be an eigenvalue of \( A \) of multiplicity 2 of \( A \), i.e., let

\[
\lambda_k = \omega^k + \omega^{n-k},
\]

with \( \omega = \exp(i \frac{2 \pi}{n}) \) for \( k \in \{1, \ldots, n\} \) except \( k = \frac{n}{2} \) if \( n \) is even. The corresponding 2-dimensional eigenspace \( E_{\lambda_k} \) is given by

\[
E_{\lambda_k} = \text{Span} \left\{ v_k, \bar{v}_k \right\},
\]

where \( v_k = \begin{bmatrix} 1 & \omega^k & \cdots & \omega^{(n-1)k} \end{bmatrix}^T \). The Plücker coordinate \( p_{ij} \) of \( E_{\lambda} \) is given by

\[
p_{ij} = \det \left[ \begin{array}{cc} (v_k)_i & (\bar{v}_k)_i \\ (v_k)_j & (\bar{v}_k)_j \end{array} \right],
\]

hence

\[
p_{ij} = \det \begin{bmatrix} \omega^{(i-1)k} & \omega^{n-(i-1)k} \\ \omega^{(j-1)k} & \omega^{n-(j-1)k} \end{bmatrix} = \omega^{n+k(i-j)} - \omega^{n+k(j-i)}.
\]
Now \( \omega^{n+k(i-j)} - \omega^{n+k(j-i)} = 0 \) if and only if \( k(i-j) \equiv k(j-i) \mod n \), i.e., if and only if \( 2k(j-i) \equiv 0 \mod n \). The trivial solutions \( k = 0 \) and \( k = \frac{n}{2} \) for even \( n \) do not correspond to a 2-dimensional eigenspace hence \( \{i,j\} \) is not a driver set if and only if \( \gcd(2(j-i),n) \notin \{1,2\} \). This condition can be replaced by \( \gcd(2d,n) \notin \{1,2\} \) because \( d = \min\{j-i, n+i-j\} \) and \( 2k(j-i) \equiv 0 \mod n \) if and only if \( 2k(n+i-j) \equiv 0 \mod n \).

The orbits of minimal driver sets under the group \( \text{Aut}(C_n) \cong D_n \) are the sets \( \Omega_d \) defined by

\[
\Omega_d = \{\{i,j\} \in \binom{V}{2} | d(i,j) = d\}
\]

for fixed values of \( d \in \{1,\ldots,\left\lfloor \frac{n}{2}\right\rfloor\} \) satisfying \( \gcd(2d,n) \in \{1,2\} \). Since the size of each orbit is equal to \( n \) the number of orbits is equal to \( \frac{1}{n}N_D(C_n) \).

The following table lists the values of \( N_D(C_n) \) for \( n \leq 12 \).

| \( n \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|----|----|----|
| \( N_D(C_n) \) | 3 | 4 | 10 | 12 | 21 | 16 | 27 | 40 | 55 | 24 |

It is easy to see that the orbit \( \Omega_1 \) is of type I without resorting to the notion of zero forcing sets. We only need to show this for one representative of the orbit. For each \( X = [x_{ij}] \in \text{Sym}(C_n) \) the matrix \( (X - \lambda I)_{\{3,\ldots,n\}} \) is row equivalent to an echelon form with pivots \( x_{1n}, x_{23}, x_{34}, \ldots, x_{n-2,n-1} \) hence \( \text{rank} (X - \lambda I)_{\{3,\ldots,n\}} = n-2 \) for all \( X \in \text{Sym}(C_n) \) and \( \lambda \in \mathbb{C} \). Before examining the other orbits we discuss the analogue of Lemma 17 for the cycle graphs. For each \( X \in \text{Sym}_0(C_n) \) with \( n \geq 6 \) and \( j \in \{3,\ldots,\left\lfloor \frac{n}{2}\right\rfloor\} \) the matrix \( X_{V\setminus\{1,j\}} \) has the block structure

\[
\begin{bmatrix}
  x_{12} & 0 & \cdots & 0 \\
  0 & \vdots & \ddots & 0 \\
  \vdots & & \ddots & x_{i-1,i} \\
  0 & \cdots & & \vdots \\
  x_{1n} & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix}
    0 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 0 \\
  \end{bmatrix} \\
  \begin{bmatrix}
    Y \\
    Z \\
  \end{bmatrix}
\end{bmatrix}
\]
with $Y \in \text{Sym}_0(P_{j-2})$ and $Z \in \text{Sym}_0(P_{n-j})$. The following Lemma can be proved in the same way as Lemma 17.

**Lemma 21** Let $X \in \text{Sym}_0(C_n)$ with $n \geq 6$, $j \in \{3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}$ and $Y$ and $Z$ as in (5). Then rank $(X - \lambda I)_{V \setminus \{1,j\}} < n - 2$ if and only if $Y$ and $Z$ have a common eigenvalue $\lambda$.

**Proof.** The linear system $(X - \lambda I)_{V \setminus \{1,j\}}^T v = 0$ breaks down into

\begin{align*}
(1) & \quad x_{12}v_1 + x_{1n}v_{n-2} = 0 \\
(2) & \quad (Y - \lambda I)v_{\{2,\ldots,j-1\}} = 0 \\
(3) & \quad x_{j-1,j}v_{j-1} + x_{j,j+1}v_j = 0 \\
(4) & \quad (Z - \lambda I)v_{\{j+1,\ldots,n-2\}} = 0
\end{align*}

The existence of a non-trivial solution $v$ forces $v_1, v_{j-1}, v_j$ and $v_{n-2}$ to be non-zero and $\lambda$ to be an eigenvalue of $Y$ and $Z$. Conversely, the existence of non-trivial solutions of (2) and (3) gives rise to a non-trivial solution $v$. 

Now let us examine the orbit $\Omega_2$. Due to the theorem above, $\{i, j\}$ with $d(i, j) = 2$ is a driver set if and only if $\gcd(4, n) \in \{1, 2\}$.

**Theorem 22** $\Omega_2$ is a type II orbit of minimal driver sets for $C_n$ if and only if $n$ is odd ($> 3$).

**Proof.** We only need to show this for one representative of the orbit. We consider $S = \{i, j\} = \{1, 3\}$. Due to Lemma 21 ($C_n, \{1, 3\}$) is not strongly $\text{Sym}_0(C_n)$-controllable if and only if there exists an $X \in \text{Sym}_0(C_n)$ such that $Y \in \text{Sym}_0(P_{1})$ and $Z \in \text{Sym}_0(P_{n-3})$ (as defined in (5)) have a common eigenvalue. In this case $Y = [0]$ so $(C_n, \{1, 3\})$ is not strongly $\text{Sym}_0(C_n)$-controllable if and only if $Z$ is singular. It follows from Lemma 16 that $\det Z \neq 0$ for all odd $n$. Obviously the case $n = 3$ is not included because $\{1, 3\} \in \Omega_1$ for the graph $C_3$. 

Finally we show that the remaining orbits are not of type II.

**Theorem 23** Let $\Omega_d$ be an orbit of minimal driver sets for $C_n$ with $d \geq 3$. $\Omega_d$ is not strongly $\text{Sym}_0(C_n)$-controllable.

\[\text{A similar result has been proved in [8] with respect to the system } (L_n, B_{\{i,j\}}), \text{ where } L_n \text{ is the Laplacian matrix of } C_n.\]
Proof. We only need to show this for one representative of the orbit. We consider \( \{1, j\} \) with \( j \in \{4, \ldots, \lfloor \frac{n}{2} \rfloor \}. \) Due to Lemma 21 \( (C_n, \{1, j\}) \) is not strongly \( \text{Sym}_0(C_n) \)-controllable if and only if there exists an \( X \in \text{Sym}_0(C_n) \) such that \( Y \) and \( Z \) have a common eigenvalue. Choose any \( Y \in \text{Sym}_0(P_{j-2}) \) and \( Z \in \text{Sym}_0(P_{n-j}) \) and a pair \( \lambda_0, \mu_0 \) of non-zero eigenvalues of \( Y \) and \( Z \) respectively. Then \( \mu_0 Y \in \text{Sym}_0(P_{j-2}) \) and \( \lambda_0 Z \in \text{Sym}_0(P_{n-j}) \) share the eigenvalue \( \lambda_0 \mu_0. \)

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