ENUMERATION OF RESTRICTED WORDS AND LINEAR
RECURRENCE EQUATIONS

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Abstract. In previous papers, for an arithmetical function \( f_0 \), we defined functions \( f_m \) and \( c_m \) and designated numbers of restricted words over a finite alphabet counted by these functions. In this paper, we examine the reverse problem for five specific types of restricted words. Namely, we find the initial function \( f_0 \) such that \( f_m \) and \( c_m \) enumerate these words. In each case, we derive explicit formulas for \( f_m \) and \( c_m \).

Fibonacci, Merssen, Pell, Jacobsthal, Tribonacci, and Padovan numbers all appear as values of \( f_m \), so we obtain new formulas for these numbers. Also, we combinatorially derive explicit formulas for the solutions of five types of homogenous linear recurrence equations.

1. Introduction

This paper is a continuation of the investigations of the problem of restricted words enumeration from the author’s previous papers [2, 3, 4], where two functions \( f_m \) and \( c_m \) were defined as follows. For an initial arithmetic function \( f_0 \), the function \( f_m, (m \geq 1) \) is the \( m \)th invert transform of \( f_0 \). The function \( c_m(n, k) \) was defined as

\[
(1) \quad c_m(n, k) = \sum_{i_1 + i_2 + \cdots + i_k = n} f_{m-1}(i_1) \cdot f_{m-1}(i_2) \cdots f_{m-1}(i_k),
\]

where the sum is over positive \( i_1, i_2, \ldots, i_k \). For \( m \geq 1 \), the following formula holds:

\[
(2) \quad f_m(n) = \sum_{k=1}^{n} c_m(n, k).
\]

The functions \( f_m \) and \( c_m \) depend only on the initial function \( f_0 \), and are related to the enumeration of weighted compositions. Namely, if weights are \( \{ f_{m-1}, f_{m-2}, \ldots \} \), then \( f_m(n) \) is the number of all weighted compositions of \( n \), and \( c_m(n, k) \) is the number of weighted compositions of \( n \) into \( k \) parts.

In Janjić [2, 3, 4], weighted compositions were related to restricted words over a finite alphabet. For a given initial function \( f_0 \), we investigated restricted words counted by \( f_m \) and \( c_m \). In this paper, we reverse the problem. Namely, for a particular type of restricted words, we first find the initial function \( f_0 \) which count such words. We then derive formulas for \( f_m \) and \( c_m \) and give its combinatorial meanings in terms of restricted words.

We restate [4 Propositions 12], which will be used frequently in the paper.

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Proposition 1. Assume that $f_0(1) = 1$ and $m > 1$. Assume next that, for $n \geq 1$, we have $f_{m-1}(n)$ words of length $n - 1$ over a finite alphabet $\alpha$. Let $x$ be a letter which is not in $\alpha$. Then, $c_m(n, k)$ is the number of words of length $n - 1$ over the alphabet $\alpha \cup \{x\}$ in which $x$ appears exactly $k - 1$ times.

We also restate the result in [4, Proposition 6]. The following formula holds:

$$c_m(n, k) = \sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} c_1(n, i), \quad (1 \leq k \leq n).$$

We consider the following five types of restricted words:

1. Words over the alphabet $\{0, 1, \ldots, a-1, \ldots, m+a-1\}$ such that no two adjacent letters from $\{0, 1, \ldots, a-1\}$ are the same.
2. Words over the alphabet $\{0, 1, \ldots, a-1, \ldots, a + m - 1\}$ such that letters $0, 1, \ldots, a-1$ avoid a run of odd length.
3. Words over the alphabet $\{0, 1, \ldots, b-1, \ldots, m+a-1\}$ avoiding subwords of the form $0i$, $(i = 1, \ldots, b)$.
4. Words over the alphabet $\{0, 1, \ldots, m+1\}$ such that 0 and 1 appear only as subwords of the form $1i$, where $i$ is a run of zeros of length at least 1.
5. Words over the alphabet $\{0, 1, \ldots, m+1\}$ in which 0 appears only in a run of even length, and 1 appears only in a run the length of which is divisible by 3.

We note that the initial function $f_0$ is defined by a linear homogeneous recurrence in all cases.

2. Case 1

To solve the problem posed in Case 1, we consider the following linear recurrence:

$$f_0(1) = 1, \quad f_0(a) = m, \quad f_0(n + 2) = (m - 1)f_0(n + 1) + mf_0(n), \quad (n \geq 1),$$

where $a > 0$. It is easy to see that

$$f_0(n) = a(a - 1)^{n-2}, \quad (n \geq 2).$$

Remark 1. This formula appears in Birmajer et al. [1] Example 17. Also, the case $a = 1$ is considered in [1] Example 18.

The function $f_0$ has the following combinatorial interpretation:

Proposition 2. The number $f_0(n)$ is the number of words of length $n - 1$ over the alphabet $\{0, 1, \ldots, a - 1\}$ such that no two adjacent letters are the same.

Proof. We have $f_0(1) = 1$, since only the empty word has length 0. Also, $f_0(2) = m$, since a word of length 1 may consist of an arbitrary letter. To obtain a word of length $n + 2$, for $n > 0$, we need to insert $a - 1$ letters in front of each word of length $n + 1$. □

As an immediate consequence of Janjić [2] Corollary 9, we obtain

Corollary 1. For $m \geq 0$, the following recurrence holds:

$$f_m(1) = 1, \quad f_m(2) = m + a, \quad f_m(n + 2) = (m + a - 1)f_m(n + 1) + mf_m(n), \quad (n \geq 1).$$

We next prove that $f_m$ counts the desired words.
Proposition 3. The number $f_m(n)$ is the number of words of length $n - 1$ over the alphabet $\{0, 1, \ldots, a - 1, a, \ldots, m + a - 1\}$, such that no two adjacent letters from $\{0, 1, \ldots, a - 1\}$ are the same.

Proof. We have $f_m(1) = 1$, since only the empty word has length 0. Also, $f_m(2) = m + a$ since a word of length 1 may consist of any letter of the alphabet. Assume that $n > 2$. Consider a word of length $n + 1$. In front of such a word, we insert a letter different from the first letter of the word. In this way, we obtain all words of length $n + 2$ beginning with two different letters. The remaining words must begin with two same letters. Since there are $m f_m(n)$ such words, the statement is true. \hfill \Box

Remark 2. The continued fraction $[f_0(1); f_0(2), f_0(3), \ldots]$ equals $\sqrt{2}$. Also, the sequence $f_1(1), f_2(2), \ldots, f_1(n)$ is the numerator of the $n$th convergent of $\sqrt{2}$.

Since $f_m(1) = 1$, we may apply Proposition 1 to obtain

Corollary 2. The number $c_m(n, k)$ is the number of words of length $n - 1$ over $\{0, 1, \ldots, a - 1, \ldots, m + a - 1\}$ in which $k - 1$ letters equal $m + a - 1$, and no two letters from $\{0, 1, \ldots, a - 1\}$ are identical.

We next derive an explicit formula for $c_1(n, k)$.

Proposition 4. We have

\begin{equation}
(4) \quad c_1(n, k) = 1, c_1(n, k) = \sum_{i=0}^{k-1} \binom{k}{i} (n - k - 1)^i (a - 1)^{n-2i}, (k < n).
\end{equation}

Proof. From (1), we firstly obtain $c_1(n, n) = 1$. Assume that $k < n$. Since at most $k - 1$ of the $i_t$ ($t = 1, 2, \ldots, k$) may equal 1, then

\begin{align*}
c_1(n, k) &= \sum_{i=0}^{k-1} \binom{k}{i} \sum_{i_1 + i_2 + \cdots + i_{k-1} = n-i} f(j_1) f(j_2) \cdots f(j_{k-1}) \\
&= \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} (a - 1)^{n-2i} \sum_{i_1 + i_2 + \cdots + i_{k-1} = n-i} 1 \\
&= \sum_{i=0}^{k-1} a^{k-i} (a - 1)^{n-2i} \binom{k}{i} (n - k - 1)^i (k - i - 1).
\end{align*}

\hfill \Box

Remark 3. Note that, in (4), terms in which $i < 2k - n$ would equal zero.

To obtain an explicit formula for $c_m(n, k)$, we use (3). We first extract the term for $i = n$ to obtain

\begin{equation}
c_m(n, k) = m^{n-k} \binom{n-1}{k-1} + \sum_{i=k}^{n-1} \binom{i-1}{k-1} c_1(n, i).
\end{equation}

It follows that

\begin{equation}
c_m(n, k) = m^{n-k} \binom{n-1}{k-1} + \sum_{i=k}^{n-1} \sum_{j=0}^{i-1} (m-1)^i-1 a^{i-j} (a - 1)^{n-2i-j} \binom{i-1}{k-1} \binom{n-i-1}{i-j} \binom{i}{j}.
\end{equation}
Using (2), we obtain the following formula for \( f_m(n) \):

\[
f_m(n) = m^{n-1} + \sum_{k=1}^{n-1} \sum_{i=k}^{n-1} \sum_{j=0}^{k-1} (m-1)^{i-k} a^{i-j} (a-1)^{n-2i+j} \left( \frac{i}{k-1} \right)^{i} \left( \frac{n-i-1}{i-1} \right)^{i-j-1}.
\]

3. Case 2

Let \( a \) be a positive integer. Define \( f_0 \) as follows:

\[
f_0(1) = 1, f_0(2) = 0, f_0(n+2) = af_0(n), (n \geq 1).
\]

We firstly describe the restricted words counted by \( f_0 \).

**Proposition 5.** For \( a > 0 \), the number \( f_0(n) \) is the number of words of length \( n-1 \) over the alphabet \( \{0, 1, \ldots, a-1\} \) in which there are no runs of odd length.

**Proof.** Let \( d(n) \) denote the number of words of length \( n \), which we wish to count. Firstly, \( d(0) = 1 \) since only the empty word has length 0. Next, \( d(1) = 0 \) as there are no runs of length 1. Assume that \( n > 2 \). A word of length \( n \) must begin with two identical letters. Hence, there are \( ad(n-2) \) such words. We conclude that the following recurrence holds:

\[
d(0) = 1, d(1) = 0, d(n) = ad(n-2), (n \geq 2),
\]

which yields \( d(n-1) = f_0(n), (n \geq 1) \). \( \square \)

From (5), we easily obtain the following explicit formula for \( f_0 \):

\[
f_0(n) = \begin{cases} 
0, & \text{if } n = 2t; \\
a^t, & \text{if } n = 2t + 1.
\end{cases}
\]

**Corollary 3.** For \( m \geq 0 \), the following recurrence holds:

\[
f_m(1) = 1, f_m(2) = m,
\]

\[
f_m(n+2) = mf_m(n+1) + af_m(n), (n \geq 1).
\]

**Proof.** The proof is a consequence of (2 Corollary 9). \( \square \)

**Proposition 6.** The number \( f_m(n) \) is the number of words of length \( n-1 \) over the alphabet \( \{0, 1, \ldots, a-1, \ldots, a+m-1\} \), such that letters \( 0, 1, \ldots, a-1 \) avoid runs of odd length.

**Proof.** We let \( d(n) \) denote the number of desired words of length \( n-1 \). It is clear that \( d(0) = 1 \) and \( d(1) = m \). A word of length \( n+1 \) may begin with a letter from \( \{a, a+1, \ldots, a+m-1\} \). There are \( nd(n) \) such word. If a word begins with a letter from \( \{0, 1, \ldots, a-1\} \), it must be followed by the same letter. Hence, there are \( ad(n-1) \) such words. We conclude that \( d(n) = f_m(n+1) \). \( \square \)

Some well-known classes of numbers satisfy the recurrence from Corollary 3. We give the appropriate combinatorial meaning for some of them.

1. The case \( a = 1, m = 1 \) concerns the Fibonacci numbers. The number of binary words of length \( n-1 \) in which 0 avoids a run of odd length is \( F_n \).
2. The case \( a = 1, m = 2 \) concerns the Pell numbers \( P_n \). The number of ternary words of length \( n-1 \) in which 0 avoids runs of odd length is \( P_n \).
(3) The case \( a = 2, m = 1 \) concerns the Jacobsthal numbers \( J_n \). The number of ternary words of length \( n - 1 \) in which 0 and 1 avoid runs of odd length is \( J_n \).

From the combinatorial interpretation, we easily derive an explicit formula for \( f_m(n) \).

**Proposition 7.** We have

\[
f_m(n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} m^{n-2j-1} \binom{n-1-j}{j}.
\]

**Proof.** According to Proposition 6 in a word counted by \( f_m \), the letters from \( \{0, 1, \ldots, a-1\} \) may appear only in pairs. There are \( a \) such pairs. We may choose \( j \), \((0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor)\) pairs in a word of length \( n - 1 \). These \( j \) pairs may be chosen in \( \binom{n-1-j}{j} \) ways. When we have chosen \( j \) pairs from \( \{0, 1, \ldots, a-1\} \), the remaining \( n - 1 - 2j \) letters are from \( \{a, a+1, \ldots, a+m-1\} \), which are \( m \) in number. \( \square \)

As a consequence, we obtain the following similar explicit formulas for the Fibonacci, Pell and Jacobsthal numbers:

\[
F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j},
\]

\[
P_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2j-1} \binom{n-j-1}{j},
\]

\[
J_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^j \binom{n-j-1}{j}.
\]

**Corollary 4.** The number \( c_m(n, k) \) is the number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, a-1, \ldots, a+m-1\} \) in which the letter \( a+m-1 \) appears \( k - 1 \) times, and letters from \( \{0, 1, \ldots, a-1\} \) avoid runs of odd length.

**Proof.** The proof follows from Proposition 1. \( \square \)

We now derive an explicit formula for \( c_1(n, k) \).

**Proposition 8.** The following equation holds:

\[
c_1(n, k) = \begin{cases} 
\frac{n-k}{k-1} \left( \binom{n+k}{k-1} \right), & \text{if } n-k \text{ is even;} \\
0, & \text{if } n-k \text{ is odd.}
\end{cases}
\]

**Proof.** Each term in \( \{1\} \) in which \( i_t \) is even equals zero. Hence, \( \{1\} \) becomes

\[
c_1(n, k) = \sum_{2j_1+1+2j_2+1+\cdots+2j_k+1=n} a^{2j_1} \cdot a^{2j_2} \cdots a^{2j_k}
\]

\[
= a^{\frac{n-k}{k-1}} \sum_{j_1+j_2+\cdots+j_k=\frac{n+k}{2}} 1 = a^{\frac{n-k}{k-1}} \binom{n+k}{k-1}.
\]

\( \square \)
As a consequence of (2), we obtain the following explicit formulas for the Fibonacci and Jacobsthal numbers:

\[
F_{2n} = \sum_{k=1}^{n} \binom{n+k-1}{n-k}, \quad F_{2n-1} = \sum_{k=1}^{n} \binom{n+k-2}{n-k},
\]

\[
J_{2n} = \sum_{k=1}^{n} 2^{n-k} \binom{n+k-1}{n-k}, \quad J_{2n-1} = \sum_{k=1}^{n} 2^{n-k} \binom{n+k-2}{n-k}.
\]

Furthermore, we derive an explicit formula for \(c_2(n, k)\). Using (3), for even \(n\), we obtain

\[
c_2(2n, k) = \sum_{i=k}^{2n} \binom{i-1}{k-1} c_1(2n, i) = \sum_{j=\left\lceil \frac{k}{2} \right\rceil}^{n} \binom{2j-1}{k-1} c_1(2n, 2j) = \sum_{j=\left\lceil \frac{k}{2} \right\rceil}^{n} a^{n-j} \binom{2j-1}{k-1} \binom{n+j-1}{n-j}.
\]

For odd \(n\), we have

\[
c_2(2n-1, k) = \sum_{i=k}^{2n-1} \binom{i-1}{k-1} c_1(2n, i) = \sum_{j=\left\lceil \frac{k+1}{2} \right\rceil}^{n} \binom{2j-2}{k-1} c_1(2n-1, 2j-1) = \sum_{j=\left\lceil \frac{k+1}{2} \right\rceil}^{n} a^{n-j} \binom{2j-2}{k-1} \binom{n+j-2}{n-j}.
\]

In particular, for \(a = 1\), we obtain the following formulas for Pell numbers:

\[
P_{2n} = \sum_{k=1}^{2n} \sum_{j=\left\lceil \frac{k}{2} \right\rceil}^{n} \binom{2j-1}{k-1} \binom{n+j-1}{n-j},
\]

\[
P_{2n-1} = \sum_{k=1}^{2n-1} \sum_{j=\left\lceil \frac{k+1}{2} \right\rceil}^{n} \binom{2j-2}{k-1} \binom{n+j-2}{n-j}.
\]

**Remark 4.** Using (3), we may obtain an explicit formula for \(c_m(n, k)\).

4. **Case 3**

Let \(a > b > 0\) be integers. We define \(f_0\) by the following recurrence:

\[
f_0(1) = 1, \quad f_0(2) = a, \quad f_0(n+2) = af_0(n+1) - bf_0(n), (n \geq 1).
\]

**Proposition 9.** The number \(f_0(n)\) is the number of words of length \(n-1\) over the alphabet \(\{0, 1, \ldots, a-1\}\), avoiding subwords \(0i, (i = 1, \ldots, b)\).

**Proof.** We let \(d(n)\) denote the number of the words of length \(n-1\). Firstly, \(d(0) = 1\), since only the empty word has length 0. Next, \(d(1) = a\), since there are no restrictions on words of length 1. Assume that \(n > 1\). A word of length \(n\) may begin with any letter. We have \(a \cdot d(n-1)\) such words. From this number, we must subtract words which begin with subwords \(0i, (i = 1, 2, \ldots, b)\). Hence, \(d(n)\) satisfies the same recurrence as \(f_0(n)\), and the proposition is proved. \(\square\)
Example 1. (1) If \(a = 2, b = 1\), we have
\[
 f_0(1) = 1, f_0(2) = 2, f_0(n + 2) = 2f_0(n + 1) - f_0(n), (n \geq 1),
\]
which yields that \(f_0(n) = n\). Hence, \(n\) is the number of binary words of length \(n - 1\) avoiding subword 01.

(2) If \(a = 3, b = 1\), we have
\[
 f_0(1) = 1, f_0(2) = 3, f_0(n + 2) = 3f_0(n + 1) - f_0(n), (n \geq 1),
\]
which is a well-known recurrence for the Fibonacci numbers \(F_{2n}\). Hence, \(F_{2n}\) is the number of ternary words of length \(n - 1\) avoiding subword 01.

We now consider the particular case \(a = b + 1\).

Corollary 5. If \(b > 1\) and \(a = b + 1\), then
\[
 f_0(n) = \frac{b^n - 1}{b - 1}.
\]

Proof. We have \(f_0(1) = 1, f_0(2) = 1 + b = a\). Further,
\[
 f_0(n + 2) = \frac{b^{n+2} - 1}{b - 1}.
\]

On the other hand, we have
\[
 (b + 1)f_0(n + 1) - bf_0(n) = (b + 1) \cdot \frac{b^{n+1} - 1}{b - 1} - b \cdot \frac{b^n - 1}{b - 1} = \frac{b^{n+2} - 1}{b - 1}.
\]

\[\square\]

In particular, for \(a = 3, b = 2\), we have \(f_0(n) = 2^n - 1\), which yields

Corollary 6. The Mersenne number \(2^n - 1\) is the number of ternary words of length \(n - 1\) avoiding 01 and 02.

Using \[2\] Corollary 9, we obtain
\[
 f_m(1) = 1, f_m(2) = m + a; f_m(n + 2) = (a + m)f_m(n + 1) - bf_m(n), (n \geq 1).
\]

This means that \(f_m\) counts the same sort of words as \(f_0\), with \(m + a\) instead of \(a\).

Using Proposition \[1\] we obtain the following combinatorial interpretation of \(c_m(n, k)\).

Corollary 7. The number \(c_m(n, k)\) is the number of words of length \(n - 1\) over the alphabet \(\{0, 1, \ldots, b - 1, b, \ldots, m + a - 1\}\) having exactly \(k\) letters equal \(m + a - 1\) and avoiding subwords \(0j, (j = 1, 2, \ldots, b)\).

We next derive an explicit formula for \(c_1(n, k)\). A generating function for the sequence \(f_0(1), f_0(2), \ldots\) is \(\frac{1}{bx^2 - ax + 1}\). According to \[4\] Equation (1), we have
\[
 \frac{x^k}{(bx^2 - ax + 1)^k} = \sum_{n=k}^{\infty} c_1(n, k)x^k.
\]

The numbers \(\alpha = \frac{a + \sqrt{a^2 - 4b}}{2b}\) and \(\beta = \frac{a - \sqrt{a^2 - 4b}}{2b}\) are the solutions of the equation \(bx^2 - ax + 1 = 0\).

Proposition 10. We have
\[
 c_1(n, k) = \frac{1}{bk} \sum_{j=0}^{n-k} \frac{1}{\alpha^{j+k} \beta^{n-j}} \binom{n-j-1}{k-1} \binom{k+j-1}{k-1}.
\]
Proof. We expand $\frac{x^k}{(\alpha - x)(\beta - x)}$ into powers of $x$. Since
$$\frac{1}{(\gamma - x)^k} = \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} x^i \gamma^i,$$
we easily obtain
$$\frac{x^k}{b^k(\alpha - x)^k(\beta - x)^k} = \sum_{i=0}^{\infty} \left[ \sum_{j=0}^{i} \frac{1}{b^k \alpha^j \beta^{i-j}} \binom{k+j-1}{k-1} \binom{k+i-j-1}{k-1} \right] x^{i+k},$$
and the statement follows by replacing $i$ by $n-k$. □

In the case $a = b + 1$, we have $\alpha = 1$ and $\beta = \frac{1}{b}$. Therefore, the following formula holds:

$$c_1(n, k) = \sum_{i=0}^{n-k} b^{n-k-i} \binom{n-i-1}{k-1} \binom{k+i-1}{k-1},$$

Using (11), we obtain

Identity 1.

$$\sum_{i_1+i_2+\ldots+i_k=n} \left[ \prod_{t=1}^{k} (b^{i_t} - 1) \right] = \sum_{i=0}^{n-k} b^{n-k-i} (b-1)^k \binom{n-i-1}{k-1} \binom{k+i-1}{k-1},$$

where $i_t > 0, (t = 1, 2, \ldots, k)$.

Remark 5. Using (11) and (22), we obtain explicit formulas for $c_m(n, k)$ and $f_m(n)$.

5. Case 4

We let $R$ denote the condition given in this case. We first solve the problem for binary words.

Proposition 11. Let $f_0(n)$ be the number of binary words satisfying $R$. Then,

1. $f_0(1) = 1, f_0(2) = 0, f_0(n+2) = f_0(n+1) + f_0(n), (n > 1)$.
2. For $n > 1$, we have $f_0(n) = F_{n-2}$.

Proof. (1) We have $f_0(1) = 1$, since the empty word has length 0. Next, $f_0(2) = 0$, since no words of length 1 satisfy $R$. Also, $f_0(3) = 1$, since 10 is the only word of length 2 satisfying $R$. Next, $f_0(4) = 1$, since 100 is the only word of length 3 satisfying $R$. Assume that $n > 1$. Then,
$$f_0(n+4) = f_0(n+2) + f_0(n+1) + \cdots,$$

since a word of length greater than 3 must begin with a subword of the form 1000. Analogously, we obtain
$$f_0(n+5) = f_0(n+3) + f_0(n+2) + \cdots.$$

Comparing these two equations, we get
$$f_0(n+5) = f_0(n+4) + f_0(n+3).$$

(2) The formula follows from the preceding recurrence. □

Since $f_0(1) = 1$, and so $f_m(1) = 1$, using Proposition 11 and (22), we obtain the following combinatorial interpretations of $f_m$ and $c_m(n, k)$.
Corollary 8. (1) The number \( c_m(n, k) \) is the number of words over the alphabet \( \{0, 1, \ldots, m + 1\} \) of length \( n - 1 \) having \( k - 1 \) letters equal \( m + 1 \) and satisfying \( R \).

(2) The number \( f_m(n) \) is the number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, m\} \) satisfying \( R \).

We next derive an explicit formula for \( c_1(n, k) \). It is known that \( c_1(n, k) \) is the coefficient of \( x^n \) in the expansion of \( (\sum_{i=1}^{\infty} F_{i-2}x^i)^k \) into powers of \( x \).

We consider the following auxiliary initial function:

\[
\tau_0(1) = 0, \quad \tau_0(n) = 1 \quad (n > 1).
\]

From [2, Proposition 23], we obtain \( \tau_1(n) = F_{n-1} \). It is proved in [3, Proposition 13] that \( \tau_2(n, k) = \binom{n-k-1}{k-1} \), \( (k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor) \), and \( \tau_2(n, k) = 0 \) otherwise.

Using [4, Proposition 6] yields \( \tau_3(n, k) = \sum_{i=k}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{i-1}{k-1} \binom{n-i-1}{i-1} \).

Hence,

\[
\left( \sum_{i=1}^{\infty} F_{i-2}x^i \right)^k = \sum_{n=k}^{\infty} \tau_3(n, k)x^n.
\]

We let \( X \) denote \( \sum_{i=1}^{\infty} F_{i-2}x^i \). We have to expand the expression \( (\sum_{i=1}^{\infty} F_{i-2}x^i)^k \), which we denote by \( Y \). It follows that \( Y = x^k(1 + X)^k \). Hence,

\[
Y = x^k \left( 1 + \sum_{i=1}^{k} \binom{k}{i} X^i \right)^k = \sum_{n=k}^{\infty} c_1(n, k)x^n.
\]

Using \( Y \) yields

\[
Y = x^k + \sum_{i=1}^{k} \sum_{j=i}^{\infty} \binom{k}{i} \tau_3(j, i)x^{j+k}.
\]

It is easy to see that, in the case \( j + k = n \), the coefficient of \( x^n \) on the right-hand side of this equation equals \( \sum_{i=1}^{k} \binom{k}{i} \tau_3(n-k, i) \). We thus obtain

Proposition 12. The following equations hold:

\[
c_1(k, k) = 1,
\]

\[
c_1(n, k) = \sum_{i=1}^{k} \sum_{j=i}^{\infty} \binom{k}{i} \binom{n-k-j-1}{j-1}, \quad (n > k).
\]

Using [2 Corollary 9], we easily obtain the following recurrence for \( f_m \):

\[
f_m(1) = 1, \quad f_m(2) = m, \quad f_m(n + 2) = (m + 1)f_m(n + 1) - (m - 1)f_m(n).
\]

Some particular cases are of note. In the case \( m = 1 \), we obtain

\[
f_1(1) = 1, \quad f_1(2) = 1, \quad f_1(n + 2) = 2f_1(n + 1), \quad (n > 1),
\]
which implies
\[ f_1(1) = f_1(2) = 1, \quad f_1(n) = 2^{n-2}, \quad (n > 2). \]
We thus obtain the following property of powers of 2.

**Corollary 9.** For \( n \geq 2 \), the number \( 2^{n-2} \) is the number of ternary words of length \( n - 1 \) satisfying \( \mathcal{R} \).

It yields that the following Euler-type identity holds:

**Identity 2.** For \( n > 2 \), the number of binary words of length \( n - 2 \) is the number of ternary words of length \( n - 1 \), in which 0 and 1 appear only in a run of the form \( 1i \), where \( i \) is the run of zeros of length \( i \geq 1 \).

From Propositions 12 and 2, we obtain the following identity for the Merse

**Identity 3.**
\[
2^{n-2} - 1 = \sum_{k=1}^{n} \sum_{i=1}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{n-k-j-1}{j-1}, \quad (n > 2).
\]

We now consider the second particular case \( m = 2 \). We have
\[ f_2(1) = 1, \quad f_2(2) = 2, \quad f_2(n+2) = 3f_2(n+1) - f_2(n), \]
which is the recurrence for Fibonacci numbers \( F_{2n-1} \). We thus have

**Corollary 10.** The Fibonacci number \( F_{2n-1} \) is the number of quaternary words of length \( n - 1 \) satisfying \( \mathcal{R} \).

Calculating values for \( c_2(n, k) \), we obtain

**Identity 4.**
\[
F_{2n-1} = \sum_{k=1}^{n} \sum_{i=1}^{k} \sum_{j=i}^{n-k} \binom{i-1}{k-1} \binom{j-1}{t-1} \binom{n-i-j-1}{j-1}.
\]

**Remark 6.** Using 3 and 2, we obtain the explicit formulas for \( c_m(n, k) \) and \( f_m(n) \).

6. Case 5

We let \( \mathcal{R} \) denote the given condition. Again, we first consider the binary words.

**Proposition 13.**

1. The following formula holds:
\[ f_0(1) = 1, \quad f_0(2) = 0, \quad f_0(3) = 1; \]
\[ f_0(n+3) = f_0(n+1) + f_0(n), \quad (n \geq 1). \]

2. We have \( f_0(n) = p_{n+2} \), where \( p_n \) is the \( n \)th Padovan number.

**Proof.** The first statement is easy to prove. Since (1) is essentially the recurrence for the Padovan numbers, the statement (2) is true. \qed

This means that the Padovan number \( p_{n+2} \) is the number of binary words of length \( n - 1 \) in which 0 appears in runs of even length, while 1 appears in runs, the lengths of which are divisible by 3. This means that the Padovan numbers count the compositions into parts 2 and 3, which is a well-known.
Corollary 11. (1) The function \( f_m \) satisfies the following recurrence:

\[
    f_m(1) = 1, f_m(2) = m, f_m(3) = m^2 + 1,
    f_m(n + 3) = mf_m(n + 2) + f_m(n + 1) + f_m(n), (n > 1).
\]

(2) Then, \( f_m(n) \) is the number of words of length \( n - 1 \) over \( \{0, 1, \ldots, m + 1\} \) satisfying \( R \).

(3) Also, \( c_m(n, k) \) is the number of words of length \( n - 1 \) over \( \{0, 1, \ldots, m + 1\} \) having \( k - 1 \) letters equal to \( m + 1 \), and satisfying \( R \).

Proof. The claim (1) easily follows from \([2, \text{Theorem 6}]\). The claims (2) and (3) follow from Proposition 1.

We add a short combinatorial proof for (2). Equation \( f_m(1) = 1 \) means that the empty word satisfies \( R \). Further, \( f_m(2) = m \) means that a word of length 1 may consist of any letter except 0 and 1. Next, \( f_m(3) = m^2 + 1 \) means that a word of length 2 may consist of pairs from \( \{2, 3, \ldots, m + 1\} \), which are \( m^2 \) in number, plus the word 00. Finally, a word of length \( n > 2 \) may begin with any letter from \( \{2, 3, \ldots, m + 1\} \), or from 00, or from 111. \( \square \)

The case \( m = 1 \) in Corollary 11 is the recurrence for Tribonacci numbers. Hence,

Corollary 12. The sequence \( 1, 1, 2, 4, 7, \ldots \) of the Tribonacci numbers is the invert transform of the sequence \( 1, 0, 1, 1, 1, 2, \ldots \) of the Padovan numbers.

Also, Tribonacci numbers count ternary words satisfying \( R \).

Finally, we calculate \( c_1(n, k) \). We define the arithmetic function \( \overline{f}_0 \) such that \( \overline{f}_0(2) = \overline{f}_0(3) = 1 \), and \( \overline{f}_0(n) = 0 \) otherwise. It follows from \([3, \text{Proposition 5}]\) that \( \tau_1(n, k) = \binom{n - 2k}{k} \). Also, using \([2, \text{Theorem 6}]\), we obtain

\[
    \overline{f}_1(1) = 0, \overline{f}_1(2) = 1, \overline{f}_1(3) = 1,
    \overline{f}_1(n + 3) = \overline{f}_0(n + 1) + \overline{f}_0(n).
\]

This implies that \( \overline{f}_1(n) = f_0(n - 1), (n > 1) \). The sequence \( f_0(1), f_0(2), \ldots \) is thus obtained by inserting 1 at the beginning of the sequence \( \overline{f}_1(1), \overline{f}_1(2), \ldots \).

Using \([4, \text{Equation (10)}]\), we obtain

\[
    \tau_2(n, k) = \sum_{i=k}^{n-k} \binom{i-1}{k-1} \binom{i}{n-2-i}.
\]

On the other hand, \([4, \text{Proposition 2}]\) yields

\[
    \left( \sum_{i=1}^{\infty} \overline{f}_1(i)x^i \right)^k = \sum_{n=k}^{\infty} \tau_2(n, k)x^n.
\]

To obtain an explicit formula for \( c_1(n, k) \), we need to expand the expression \( X \) given by \( X = (\sum_{i=1}^{\infty} f_0(i)x^i)^k \) into powers of \( x \). We have

\[
    X = \left( x + \sum_{i=2}^{\infty} f_0(i)x^i \right)^k = (x + Y)^k,
\]

where \( Y = \sum_{i=1}^{\infty} \overline{f}_1(i)x^i \). Hence,

\[
    X = x^k \sum_{i=0}^{k} \binom{k}{i} Y^i.
\]
Applying Equation (10) yields
\[ X = \sum_{i=0}^{k} \binom{k}{i} \sum_{j=i}^{\infty} c_2(j, i)x^{j+k}. \]

Taking \( n = j + k \), we get

**Proposition 14.** The following formula holds:
\[ c_1(n, k) = \sum_{i=0}^{k} \binom{n-k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}. \]

We thus obtain the following identity for the Tribonacci numbers \( T_n \):

**Identity 5.**
\[ T_n = \sum_{k=1}^{n} \sum_{i=0}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}. \]

**Remark 7.** Using (3) and (2), we obtain explicit formulas for \( c_m(n, k) \) and \( f_m(n) \).

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