Leonard pairs and the $q$-Racah polynomials

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Abstract

Let $K$ denote a field, and let $V$ denote a vector space over $K$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following two conditions:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal and the matrix representing $A^*$ is irreducible tridiagonal.

We call such a pair a Leonard pair on $V$. In the appendix to [9] we outlined a correspondence between Leonard pairs and a class of orthogonal polynomials consisting of the $q$-Racah polynomials and some related polynomials of the Askey scheme. We also outlined how, for the polynomials in this class, the 3-term recurrence, difference equation, Askey-Wilson duality, and orthogonality can be obtained in a uniform manner from the corresponding Leonard pair. The purpose of this paper is to provide proofs for the assertions which we made in that appendix.

1 Leonard pairs

We begin by recalling the notion of a Leonard pair [5], [9], [10], [11], [12], [13], [14], [15], [16]. We will use the following terms. Let $X$ denote a square matrix. Then $X$ is called tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume $X$ is tridiagonal. Then $X$ is called irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper $K$ will denote a field.

Definition 1.1 [9] Let $V$ denote a vector space over $K$ with finite positive dimension. By a Leonard pair on $V$, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both (i), (ii) below.

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

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(ii) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal and the matrix representing $A^*$ is irreducible tridiagonal.

**Note 1.2** According to a common notational convention $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$, the linear transformations $A$ and $A^*$ are arbitrary subject to (i), (ii) above.

## 2 An example

Here is an example of a Leonard pair. Set $V = \mathbb{K}^4$ (column vectors), set

$$
A = \begin{pmatrix}
0 & 3 & 0 & 0 \\
1 & 0 & 2 & 0 \\
0 & 2 & 0 & 1 \\
0 & 0 & 3 & 0 \\
\end{pmatrix}, \quad A^* = \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3 \\
\end{pmatrix},
$$

and view $A$ and $A^*$ as linear transformations from $V$ to $V$. We assume the characteristic of $\mathbb{K}$ is not 2 or 3, to ensure $A$ is irreducible. Then $A, A^*$ is a Leonard pair on $V$. Indeed, condition (i) in Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of the 4 by 4 identity matrix. To verify condition (ii), we display an invertible matrix $P$ such that $P^{-1}AP$ is diagonal and $P^{-1}A^*P$ is irreducible tridiagonal. Set

$$
P = \begin{pmatrix}
1 & 3 & 3 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
1 & -3 & 3 & -1 \\
\end{pmatrix}.
$$

By matrix multiplication $P^2 = 8I$, where $I$ denotes the identity, so $P^{-1}$ exists. Also by matrix multiplication,

$$
AP = PA^*.
$$

(1)

Apparently $P^{-1}AP$ is equal to $A^*$ and is therefore diagonal. By [11] and since $P^{-1}$ is a scalar multiple of $P$, we find $P^{-1}A^*P$ is equal to $A$ and is therefore irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of $P$.

The above example is a member of the following infinite family of Leonard pairs. For any nonnegative integer $d$ the pair

$$
A = \begin{pmatrix}
0 & d & 0 & 0 \\
1 & 0 & d - 1 & 0 \\
& 2 \\
& & \ddots \\
& & & 1 \\
0 & & & d \\
\end{pmatrix}, \quad A^* = \text{diag}(d, d - 2, d - 4, \ldots, -d)
$$

(2)
is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is zero or an odd prime greater than $d$. This can be proved by modifying the proof for $d = 3$ given above. One shows $P^2 = 2^d I$ and $AP = PA^*$, where $P$ denotes the matrix with $ij$ entry
\[
P_{ij} = \binom{d}{j} F_1 \left( \begin{array}{c} -i, -j \\ -d \\ \end{array} \right) 2 \quad (0 \leq i, j \leq d)
\]
(3)

[11] Section 16]. We follow the standard notation for hypergeometric series [4, p. 3].

3 Leonard systems

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract object called a Leonard system. In order to define this we first make an observation about Leonard pairs.

**Lemma 3.1** [9, Lemma 1.3] Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension and let $A, A^*$ denote a Leonard pair on $V$. Then the eigenvalues of $A$ are mutually distinct and contained in $\mathbb{K}$. Moreover, the eigenvalues of $A^*$ are mutually distinct and contained in $\mathbb{K}$.

To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let $d$ denote a nonnegative integer and let $\text{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d + 1$ by $d + 1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. We let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of all $d + 1$ by 1 matrices that have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\text{Mat}_{d+1}(\mathbb{K})$. We observe this module is irreducible. For the rest of this paper we let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\text{Mat}_{d+1}(\mathbb{K})$. When we refer to an $\mathcal{A}$-module we mean a left $\mathcal{A}$-module. Let $V$ denote an irreducible $\mathcal{A}$-module. We remark that $V$ is unique up to isomorphism of $\mathcal{A}$-modules, and that $V$ has dimension $d + 1$. Let $v_0, v_1, \ldots, v_d$ denote a basis for $V$. For $X \in \mathcal{A}$ and $Y \in \text{Mat}_{d+1}(\mathbb{K})$, we say $Y$ represents $X$ with respect to $v_0, v_1, \ldots, v_d$ whenever $X v_j = \sum_{i=0}^{d} Y_{ij} v_i$ for $0 \leq j \leq d$. Let $A$ denote an element of $\mathcal{A}$. We say $A$ is multiplicity-free whenever it has $d + 1$ mutually distinct eigenvalues in $\mathbb{K}$. Let $A$ denote a multiplicity-free element of $\mathcal{A}$. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of $A$, and for $0 \leq i \leq d$ put
\[
E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j},
\]
where $I$ denotes the identity of $\mathcal{A}$. We observe (i) $AE_i = \theta_i E_i$ $(0 \leq i \leq d)$; (ii) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq d)$; (iii) $\sum_{i=0}^{d} E_i = I$; (iv) $A = \sum_{i=0}^{d} \theta_i E_i$. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Using (i)--(iv) we find the sequence $E_0, E_1, \ldots, E_d$ is a basis for the $\mathbb{K}$-vector space $\mathcal{D}$. We call $E_i$ the primitive idempotent of $A$ associated with $\theta_i$. It is helpful to think of these primitive idempotents as follows. Observe
\[
V = E_0 V + E_1 V + \cdots + E_d V \quad \text{(direct sum)}.
\]
For $0 \leq i \leq d$, $E_i V$ is the (one dimensional) eigenspace of $A$ in $V$ associated with the
eigenvalue $\theta_i$, and $E_i$ acts on $V$ as the projection onto this eigenspace. We remark that
$\{A^j|0 \leq i \leq d\}$ is a basis for the $\mathbb{K}$-vector space $D$ and that $\prod_{i=0}^{d}(A-\theta_i I) = 0$. By a
Leonard pair in $\mathcal{A}$ we mean an ordered pair of elements taken from $\mathcal{A}$ that act on $V$ as a
Leonard pair in the sense of Definition 1.1. We call
$(A; A^*, \{E_i\}_{i=0}^{d}; \{E^*_i\}_{i=0}^{d})$ a Leonard pair in the sense of Definition 1.1.

We comment on how Leonard pairs and Leonard systems are related. In the following dis-

Definition 3.2 [2] By a Leonard system in $\mathcal{A}$ we mean a sequence $\Phi := (A; A^*; \{E_i\}_{i=0}^{d}; \{E^*_i\}_{i=0}^{d})$ that satisfies (i)–(v) below.

(i) Each of $A, A^*$ is a multiplicity-free element in $\mathcal{A}$.

(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of $A$.

(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of $A^*$.

(iv) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

(v) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

We refer to $d$ as the diameter of $\Phi$ and say $\Phi$ is over $\mathbb{K}$. We call $\mathcal{A}$ the ambient algebra of $\Phi$.

We comment on how Leonard pairs and Leonard systems are related. In the following dis-

Lemma 3.3 Let $A$ and $A^*$ denote elements of $\mathcal{A}$. Then the pair $A, A^*$ is a Leonard pair in $\mathcal{A}$ if and only if the following (i), (ii) hold.

(i) Each of $A, A^*$ is multiplicity-free.

(ii) There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $A$ and there exists an ordering $E_0^*, E_1^*, \ldots, E_d^*$ of the primitive idempotents of $A^*$ such that $(A; A^*; \{E_i\}_{i=0}^{d}; \{E^*_i\}_{i=0}^{d})$ is a Leonard system in $\mathcal{A}$.
We recall the notion of isomorphism for Leonard pairs and Leonard systems.

**Definition 3.4** Let $A, A^*$ and $B, B^*$ denote Leonard pairs over $\mathbb{K}$. By an *isomorphism of Leonard pairs* from $A, A^*$ to $B, B^*$ we mean an isomorphism of $\mathbb{K}$-algebras from the ambient algebra of $A, A^*$ to the ambient algebra of $B, B^*$ that sends $A$ to $B$ and $A^*$ to $B^*$. The Leonard pairs $A, A^*$ and $B, B^*$ are said to be *isomorphic* whenever there exists an isomorphism of Leonard pairs from $A, A^*$ to $B, B^*$.

Let $\Phi$ denote the Leonard system from Definition 3.2 and let $\sigma : A \to A'$ denote an isomorphism of $\mathbb{K}$-algebras. We write $\Phi^\sigma := (A^\sigma; A^\sigma^*; \{E_i^\sigma\}_{i=0}^d; \{E_i^\sigma^*\}_{i=0}^d)$ and observe $\Phi^\sigma$ is a Leonard system in $A'$.

**Definition 3.5** Let $\Phi$ and $\Phi'$ denote Leonard systems over $\mathbb{K}$. By an *isomorphism of Leonard systems* from $\Phi$ to $\Phi'$ we mean an isomorphism of $\mathbb{K}$-algebras $\sigma$ from the ambient algebra of $\Phi$ to the ambient algebra of $\Phi'$ such that $\Phi^\sigma = \Phi'$. The Leonard systems $\Phi$, $\Phi'$ are said to be *isomorphic* whenever there exists an isomorphism of Leonard systems from $\Phi$ to $\Phi'$.

We have a remark. Let $\sigma : A \to A$ denote any map. By the Skolem-Noether theorem [8, Corollary 9.122], $\sigma$ is an isomorphism of $\mathbb{K}$-algebras if and only if there exists an invertible $S \in A$ such that $X^\sigma = SXS^{-1}$ for all $X \in A$.

### 4 The $D_4$ action

A given Leonard system can be modified in several ways to get a new Leonard system. For instance, let $\Phi$ denote the Leonard system from Definition 3.2. Then each of the following three sequences is a Leonard system in $A$.

- $\Phi^* := (A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d)$,
- $\Phi^↓ := (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$,
- $\Phi^\Downarrow := (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$.

Viewing $\ast, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

- $\ast^2 = \downarrow^2 = \Downarrow^2 = 1$,
- $\Downarrow \ast = \ast \Downarrow$, $\downarrow \ast = \ast \downarrow$, $\Downarrow \downarrow = \downarrow \Downarrow$.

The group generated by symbols $\ast, \downarrow, \Downarrow$ subject to the relations (5), (6) is the dihedral group $D_4$. We recall $D_4$ is the group of symmetries of a square, and has 8 elements. Apparently $\ast, \downarrow, \Downarrow$ induce an action of $D_4$ on the set of all Leonard systems.

For the rest of this paper we will use the following notational convention.

**Definition 4.1** Let $\Phi$ denote a Leonard system. For any element $g$ in the group $D_4$ and for any object $f$ that we associate with $\Phi$, we let $f^g$ denote the corresponding object for the Leonard system $\Phi^{g^{-1}}$. We have been using this convention all along; an example is $E_i^*(\Phi) = E_i(\Phi^*)$. 

5
5 The structure of a Leonard system

In this section we establish a few basic facts concerning Leonard systems. We begin with a definition and two routine lemmas.

**Definition 5.1** Let $\Phi$ denote the Leonard system from Definition 3.2. For $0 \leq i \leq d$, we let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $E_i$ (resp. $E_i^*$). We refer to $\theta_0, \theta_1, \ldots, \theta_d$ as the eigenvalue sequence of $\Phi$. We refer to $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ as the dual eigenvalue sequence of $\Phi$. We observe $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct and contained in $\mathbb{K}$. Similarly $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct and contained in $\mathbb{K}$.

**Lemma 5.2** Let $\Phi$ denote the Leonard system from Definition 3.2 and let $L$ denote an irreducible $A$-module. For $0 \leq i \leq d$ let $v_i$ denote a nonzero vector in $E_i^*L$ and observe $v_0, v_1, \ldots, v_d$ is a basis for $V$. Then (i), (ii) hold below.

(i) For $0 \leq i \leq d$ the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents $E_i^*$ with respect to $v_0, v_1, \ldots, v_d$ has $ii$ entry 1 and all other entries 0.

(ii) The matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents $A^*$ with respect to $v_0, v_1, \ldots, v_d$ is equal to $\text{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$.

**Lemma 5.3** Let $A$ denote an irreducible tridiagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$. Pick any integers $i, j$ ($0 \leq i, j \leq d$). Then (i)–(iii) hold below.

(i) The entry $(A^*)_{ij} = 0$ if $r < |i - j|$, \quad (0 \leq r \leq d).

(ii) Suppose $i \leq j$. Then the entry $(A^*_{i-j})_{ij} = \prod_{h=i}^{j-1} A_{h,h+1}$. Moreover $(A^*_{i-j})_{ij} \neq 0$.

(iii) Suppose $i \geq j$. Then the entry $(A^*_{i-j})_{ij} = \prod_{h=j}^{i-1} A_{h+1,h}$. Moreover $(A^*_{i-j})_{ij} \neq 0$.

**Theorem 5.4** Let $\Phi$ denote the Leonard system from Definition 3.2. Then the elements

$$A^r E_0^s A^s \quad (0 \leq r, s \leq d) \quad (7)$$

form a basis for the $\mathbb{K}$-vector space $A$.

**Proof:** The number of elements in (7) is equal to $(d+1)^2$, and this number is the dimension of $A$. Therefore it suffices to show the elements in (7) are linearly independent. To do this, we represent the elements in (7) by matrices. Let $V$ denote an irreducible $A$-module. For $0 \leq i \leq d$ let $v_i$ denote a nonzero vector in $E_i^*V$, and observe $v_0, v_1, \ldots, v_d$ is a basis for $V$. For the purpose of this proof, let us identify each element of $A$ with the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents it with respect to the basis $v_0, v_1, \ldots, v_d$. Adopting this point of view we find $A$ is irreducible tridiagonal and $A^*$ is diagonal. For $0 \leq r, s \leq d$ we show the entries of $A^r E_0^s A^s$ satisfy

$$ (A^r E_0^s A^s)_{ij} = \begin{cases} 0, & \text{if } i > r \text{ or } j > s; \\ \neq 0, & \text{if } i = r \text{ and } j = s \end{cases} \quad (0 \leq i, j \leq d). \quad (8)$$
By Lemma 5.2(i) the matrix $E^*_0$ has 00 entry 1 and all other entries 0. Therefore

$$(A^r E^*_0 A^s)_{ij} = (A^r)_{i0} (A^*)_{0j} \quad (0 \leq i, j \leq d). \quad (9)$$

We mentioned $A$ is irreducible tridiagonal. Applying Lemma 5.3 we find that for $0 \leq i \leq d$ the entry $(A^r)_{i0}$ is zero if $i > r$, and nonzero if $i = r$. Similarly for $0 \leq j \leq d$ the entry $(A^*)_{0j}$ is zero if $j > s$, and nonzero if $j = s$. Combining these facts with (9) we routinely obtain (8) and it follows the elements (7) are linearly independent. Apparently the elements (7) form a basis for $\mathcal{A}$, as desired.

**Corollary 5.5** Let $\Phi$ denote the Leonard system from Definition 3.2. Then the elements $A, E^*_0$ together generate $\mathcal{A}$. Moreover the elements $A, A^*$ together generate $\mathcal{A}$.

**Proof:** The first assertion is immediate from Theorem 5.4. The second assertion follows from the first assertion and the observation that $E^*_0$ is a polynomial in $A^*$.

The following is immediate from Corollary 5.5.

**Corollary 5.6** Let $A, A^*$ denote a Leonard pair in $\mathcal{A}$. Then the elements $A, A^*$ together generate $\mathcal{A}$.

We mention a few implications of Theorem 5.4 that will be useful later in the paper.

**Lemma 5.7** Let $\Phi$ denote the Leonard system from Definition 3.2. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Let $X_0, X_1, \ldots, X_d$ denote a basis for the $K$-vector space $\mathcal{D}$. Then the elements

$$X_r E^*_0 X_s \quad (0 \leq r, s \leq d) \quad (10)$$

form a basis for the $K$-vector space $\mathcal{A}$.

**Proof:** The number of elements in (10) is equal to $(d+1)^2$, and this number is the dimension of $\mathcal{A}$. Therefore it suffices to show the elements (10) span $\mathcal{A}$. But this is immediate from Theorem 5.4 and since each element in (10) is contained in the span of the elements (10). □

**Corollary 5.8** Let $\Phi$ denote the Leonard system from Definition 3.2. Then the elements

$$E_r E^*_0 E_s \quad (0 \leq r, s \leq d) \quad (11)$$

form a basis for the $K$-vector space $\mathcal{A}$.

**Proof:** Immediate from Lemma 5.7 with $X_i = E_i$ for $0 \leq i \leq d$. □

**Lemma 5.9** Let $\Phi$ denote the Leonard system from Definition 3.2. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Let $X$ and $Y$ denote elements in $\mathcal{D}$ and assume $XE^*_0 Y = 0$. Then $X = 0$ or $Y = 0$. 
Proof: Let $X_0, X_1, \ldots, X_d$ denote a basis for the $K$-vector space $D$. Since $X \in D$ there exists $\alpha_i \in K$ $(0 \leq i \leq d)$ such that $X = \sum_{i=0}^{d} \alpha_i X_i$. Similarly there exists $\beta_i \in K$ $(0 \leq i \leq d)$ such that $Y = \sum_{i=0}^{d} \beta_i X_i$. Evaluating $0 = X E_0^* Y$ using these equations we get $0 = \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \beta_j X_i E_0^* X_j$. From this and Lemma 5.7 we find $\alpha_i \beta_j = 0$ for $0 \leq i, j \leq d$. We assume $X \neq 0$ and show $Y = 0$. Since $X \neq 0$ there exists an integer $i$ $(0 \leq i \leq d)$ such that $\alpha_i \neq 0$. Now for $0 \leq j \leq d$ we have $\alpha_i \beta_j = 0$ so $\beta_j = 0$. It follows $Y = 0$. \[ \square \]

We finish this section with a comment.

Lemma 5.10 Let $\Phi$ denote the Leonard system from Definition 3.2. Pick any integers $i, j$ $(0 \leq i, j \leq d)$. Then (i)–(iv) hold below.

(i) $E_i^* A^r E_j^* = 0$ if $r < |i - j|$, $(0 \leq r \leq d)$.

(ii) Suppose $i \leq j$. Then

$$E_i^* A^{j-i} E_j^* = E_i^* A E_{i+1}^* A \cdots E_{j-1}^* A E_j^*. \tag{12}$$

Moreover $E_i^* A^{j-i} E_j^* \neq 0$.

(iii) Suppose $i \geq j$. Then

$$E_i^* A^{i-j} E_j^* = E_i^* A E_{i-1}^* A \cdots E_{j+1}^* A E_j^*. \tag{13}$$

Moreover $E_i^* A^{i-j} E_j^* \neq 0$.

(iv) Abbreviate $r = |i - j|$. Then $E_i^* A^r E_j^*$ is a basis for the $K$-vector space $E_i^* A E_j^*$.

Proof: Represent the elements of $\Phi$ by matrices as in the proof of Theorem 5.4 and use Lemma 5.3. \[ \square \]

6 The antiautomorphism $\dagger$

We recall the notion of an antiautomorphism of $A$. Let $\gamma : A \to A$ denote any map. We call $\gamma$ an antiautomorphism of $A$ whenever $\gamma$ is an isomorphism of $\mathbb{K}$-vector spaces and $(XY)^\gamma = Y^\gamma X^\gamma$ for all $X, Y \in A$. For example assume $A = \text{Mat}_{d+1}(\mathbb{K})$. Then $\gamma$ is an antiautomorphism of $A$ if and only if there exists an invertible element $R$ in $A$ such that $X^\gamma = R^{-1} X^t R$ for all $X \in A$, where $t$ denotes transpose. This follows from the Skolem-Noether theorem [S Corollary 9.122].

Theorem 6.1 Let $A, A^*$ denote a Leonard pair in $A$. Then there exists a unique antiautomorphism $\dagger$ of $A$ such that $A^\dagger = A$ and $A^* \dagger = A^*$. Moreover $X^{\dagger \dagger} = X$ for all $X \in A$. 

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Proof: Concerning existence, let $V$ denote an irreducible $\mathcal{A}$-module. By Definition [1.1][i] there exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal. Let us denote this basis by $v_0, v_1, \ldots, v_d$. For $X \in \mathcal{A}$ let $X^\sigma$ denote the matrix in $\text{Mat}_{d+1}(K)$ that represents $X$ with respect to the basis $v_0, v_1, \ldots, v_d$. We observe $\sigma : \mathcal{A} \to \text{Mat}_{d+1}(K)$ is an isomorphism of $K$-algebras. We abbreviate $B = A^\sigma$ and observe $B$ is irreducible tridiagonal. We abbreviate $B^* = A^{*\sigma}$ and observe $B^*$ is diagonal. Let $D$ denote the diagonal matrix in $\text{Mat}_{d+1}(K)$ that has $ii$ entry

$$D_{ii} = \frac{B_{0i}B_{12} \cdots B_{i-1,i}}{B_{10}B_{21} \cdots B_{i,i-1}} \quad (0 \leq i \leq d).$$

It is routine to verify $D^{-1}B^tD = B$. Each of $D, B^*$ is diagonal so $DB^* = B^*D$; also $B^{*t} = B^*$ so $D^{-1}B^{*t}D = B^*$. Let $\gamma : \text{Mat}_{d+1}(K) \to \text{Mat}_{d+1}(K)$ denote the map that satisfies $X^\gamma = D^{-1}X^tD$ for all $X \in \text{Mat}_{d+1}(K)$. We observe $\gamma$ is an antiautomorphism of $\text{Mat}_{d+1}(K)$ such that $B^\gamma = B$ and $B^{*\gamma} = B^*$. We define the map $\dag : \mathcal{A} \to \mathcal{A}$ to be the composition $\dag = \sigma \gamma \sigma^{-1}$. We observe $\dag$ is an antiautomorphism of $\mathcal{A}$ such that $A^\dag = A$ and $A^{*\dag} = A^*$. We have now shown there exists an antiautomorphism $\dag$ of $\mathcal{A}$ such that $A^\dag = A$ and $A^{*\dag} = A^*$. This antiautomorphism is unique since $A, A^*$ together generate $\mathcal{A}$. The map $X \to X^{\dag}$ is an isomorphism of $K$-algebras from $\mathcal{A}$ to itself. This isomorphism is the identity since $A^{\dag} = A$, $A^{*\dag} = A^*$, and since $A, A^*$ together generate $\mathcal{A}$.

Definition 6.2 Let $A, A^*$ denote a Leonard pair in $\mathcal{A}$. By the antiautomorphism which corresponds to $A, A^*$ we mean the map $\dag : \mathcal{A} \to \mathcal{A}$ from Theorem 6.1. Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^\dag\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. By the antiautomorphism which corresponds to $\Phi$ we mean the antiautomorphism which corresponds to the Leonard pair $A, A^*$.

Lemma 6.3 Let $\Phi$ denote the Leonard system from Definition 3.2 and let $\dag$ denote the corresponding antiautomorphism. Then the following (i), (ii) hold.

(i) Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Then $X^\dag = X$ for all $X \in \mathcal{D}$; in particular $E_i^\dag = E_i$ for $0 \leq i \leq d$.

(ii) Let $\mathcal{D}^*$ denote the subalgebra of $\mathcal{A}$ generated by $A^*$. Then $X^\dag = X$ for all $X \in \mathcal{D}^*$; in particular $E_i^{*\dag} = E_i^*$ for $0 \leq i \leq d$.

Proof: (i) The sequence $A^i$ $(0 \leq i \leq d)$ is a basis for the $K$-vector space $\mathcal{D}$. Observe $\dag$ stabilizes $A^i$ for $0 \leq i \leq d$. The result follows.

(ii) Similar to the proof of (i) above.

7 The scalars $a_i, x_i$

In this section we introduce some scalars that will help us describe Leonard systems.
Definition 7.1 Let Φ denote the Leonard system from Definition 3.2. We define

\[ a_i = \text{tr}(E^*_i A) \quad (0 \leq i \leq d), \]
\[ x_i = \text{tr}(E^*_i A E^*_{i-1} A) \quad (1 \leq i \leq d), \]

where \( \text{tr} \) denotes trace. For notational convenience we define \( x_0 = 0 \).

We have a comment.

Lemma 7.2 Let Φ denote the Leonard system from Definition 3.2 and let \( V \) denote an irreducible \( A \)-module. For \( 0 \leq i \leq d \) let \( v_i \) denote a nonzero vector in \( E^*_i V \) and observe \( v_0, v_1, \ldots, v_d \) is a basis for \( V \). Let \( B \) denote the matrix in \( \text{Mat}_{d+1}(K) \) that represents \( A \) with respect to \( v_0, v_1, \ldots, v_d \). We observe \( B \) is irreducible tridiagonal. The following (i)–(iii) hold.

(i) \( B_{ii} = a_i \) \( (0 \leq i \leq d) \).

(ii) \( B_{i,i-1}B_{i-1,i} = x_i \) \( (1 \leq i \leq d) \).

(iii) \( x_i \neq 0 \) \( (1 \leq i \leq d) \).

Proof: (i), (ii) For \( 0 \leq i \leq d \) the matrix in \( \text{Mat}_{d+1}(K) \) that represents \( E^*_i \) with respect to \( v_0, v_1, \ldots, v_d \) has \( ii \) entry 1 and all other entries 0. The result follows in view of Definition 7.1.

(iii) Immediate from (ii) and since \( B \) is irreducible. \( \square \)

Theorem 7.3 Let \( \Phi \) denote the Leonard system from Definition 3.2. Let \( V \) denote an irreducible \( A \)-module and let \( v \) denote a nonzero vector in \( E^*_0 V \). Then for \( 0 \leq i \leq d \) the vector \( E^*_i A^i v \) is nonzero and hence a basis for \( E^*_i V \). Moreover the sequence

\[ E^*_i A^i v \quad (0 \leq i \leq d) \]

is a basis for \( V \).

Proof: We show \( E^*_i A^i v \neq 0 \) for \( 0 \leq i \leq d \). Let \( i \) be given. Setting \( j = 0 \) in Lemma 5.10(iii) we find \( E^*_i A^i E^*_0 v \neq 0 \). Therefore \( E^*_i A^i E^*_0 v \neq 0 \). The space \( E^*_0 v \) is spanned by \( v \) so \( E^*_i A^i v \neq 0 \) as desired. The remaining claims follow. \( \square \)

Theorem 7.4 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( a_i, x_i \) be as in Definition 7.1. Let \( V \) denote an irreducible \( A \)-module. With respect to the basis for \( V \) given in (16) the matrix that represents \( A \) is equal to

\[
\begin{pmatrix}
  a_0 & x_1 & 0 \\
  1 & a_1 & x_2 \\
  & & \ddots & \ddots \\
  & & & 1 & x_d \\
  0 & & & & 1 & a_d
\end{pmatrix}
\]
Proof: With reference to (16) abbreviate \( v_i = E_i^*A^i v \) for \( 0 \leq i \leq d \). Let \( B \) denote the matrix in Mat\(_{d+1}(\mathbb{K})\) that represents \( A \) with respect to \( v_0, v_1, \ldots, v_d \). We show \( B \) is equal to (17). In view of Lemma 7.2 it suffices to show \( B_{i,i-1} = 1 \) for \( 1 \leq i \leq d \). For \( 0 \leq i \leq d \) the matrix \( B^i \) represents \( A^i \) with respect to \( v_0, v_1, \ldots, v_d \); therefore \( A^i v_0 = \sum_{j=0}^{d} (B^i)_{j0} v_j \).

Applying \( E_i^* \) and using \( v_0 = v \) we find \( v_i = (B^i)_{d0} v_i \) so \( (B^i)_{i0} = 1 \). By Lemma 5.3 we have \( (B^i)_{d0} = B_{i,i-1} \cdots B_{21} B_{10} \) so \( B_{i,i-1} \cdots B_{21} B_{10} = 1 \). We now have \( B_{i,j-1} \cdots B_{21} B_{10} = 1 \) for \( 1 \leq i \leq d \) so \( B_{i,i-1} = 1 \) for \( 1 \leq i \leq d \). We now see \( B \) is equal to (17).

\[ \square \]

Lemma 7.5 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( a_i, x_i \) be as in Definition 7.1. Then the following (i)–(iii) hold.

(i) \( E_i^* A E_i^* = a_i E_i^* \) \( (0 \leq i \leq d) \).

(ii) \( E_i^* A E_{i-1}^* A E_i^* = x_i E_i^* \) \( (1 \leq i \leq d) \).

(iii) \( E_{i-1}^* A E_i^* A E_{i-1}^* = x_i E_{i-1}^* \) \( (1 \leq i \leq d) \).

Proof: (i) Setting \( i = j \) and \( r = 0 \) in Lemma 5.10(iv) we find \( E_i^* \) is a basis for \( E_i^* A E_i^* \). By this and since \( E_i^* A E_i^* \) is contained in \( E_i A E_i \) we find there exists \( \alpha_i \in \mathbb{K} \) such that \( E_i^* A E_i^* = \alpha_i E_i^* \). Taking the trace of both sides and using \( \text{tr}(XY) = \text{tr}(YX), \text{tr}(E_{i}^*) = 1 \) we find \( \alpha_i = \alpha_i \).

(ii) We mentioned above that \( E_i^* \) is a basis for \( E_i A E_i \). By this and since \( E_i^* A E_{i-1}^* A E_i^* \) is contained in \( E_i A E_i^* \) we find there exists \( \beta_i \in \mathbb{K} \) such that \( E_i^* A E_{i-1}^* A E_i^* = \beta_i E_i^* \). Taking the trace of both sides we find \( x_i = \beta_i \).

(iii) Similar to the proof of (ii) above. \[ \square \]

Lemma 7.6 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( x_i \) be as in Definition 7.1. Then the following (i), (ii) hold.

(i) \( E_i^* A_j^{-i} E_i^* A_j^{-i} E_j^* = x_{i+1} x_{i+2} \cdots x_j E_j^* \) \( (0 \leq i \leq j \leq d) \).

(ii) \( E_i^* A_j^{-i} E_j^* A_j^{-i} E_i^* = x_{i+1} x_{i+2} \cdots x_j E_i^* \) \( (0 \leq i \leq j \leq d) \).

Proof: (i) Evaluate the expression on the left using Lemma 5.10(ii), (iii) and Lemma 7.5(ii).

(ii) Evaluate the expression on the left using Lemma 5.10(ii), (iii) and Lemma 7.5(iii). \[ \square \]

8 The polynomials \( p_i \)

In this section we begin our discussion of polynomials. We will use the following notation. Let \( \lambda \) denote an indeterminate. We let \( \mathbb{K}[\lambda] \) denote the \( \mathbb{K} \)-algebra consisting of all polynomials in \( \lambda \) that have coefficients in \( \mathbb{K} \). For the rest of this paper all polynomials that we discuss are assumed to lie in \( \mathbb{K}[\lambda] \).
Definition 8.1 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $a_i, x_i$ be as in Definition 8.1. We define a sequence of polynomials $p_0, p_1, \ldots, p_{d+1}$ by

\begin{align}
p_0 &= 1, \\
p_i &= p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d),
\end{align}

where $p_{-1} = 0$. We observe $p_i$ is monic with degree exactly $i$ for $0 \leq i \leq d + 1$.

**Lemma 8.2** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Let $V$ denote an irreducible $A$-module and let $v$ denote a nonzero vector in $E_0^* V$. Then $p_i(A)v = E_i^* A_i v$ for $0 \leq i \leq d$ and $p_{d+1}(A)v = 0$.

**Proof:** We abbreviate $v_i = p_i(A)v$ for $0 \leq i \leq d + 1$. We define $v'_i = E_i^* A_i v$ for $0 \leq i \leq d$ and $v'_{d+1} = 0$. We show $v_i = v'_i$ for $0 \leq i \leq d + 1$. From the construction $v_0 = v$ and $v'_0 = v$ so $v_0 = v'_0$. From (19) we obtain

$A v_i = v_{i+1} + a_i v_i + x_i v_{i-1} \quad (0 \leq i \leq d)$

where $v_{-1} = 0$. From Theorem 7.4 we find

$A v'_i = v'_{i+1} + a_i v'_i + x_i v'_{i-1} \quad (0 \leq i \leq d)$

where $v'_{-1} = 0$. Comparing (20), (21) and using $v_0 = v'_0$ we find $v_i = v'_i$ for $0 \leq i \leq d + 1$. The result follows. \hfill \Box

We mention a few consequences of Lemma 8.2.

**Theorem 8.3** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Let $V$ denote an irreducible $A$-module. Then

$p_i(A) E_0^* V = E_i^* V \quad (0 \leq i \leq d)$.

**Proof:** Let $v$ denote a nonzero vector in $E_0^* V$. Then $p_i(A)v = E_i^* A_i v$ by Lemma 8.2. Observe $v$ is a basis for $E_0^* V$. By Theorem 7.3 we find $E_i^* A_i v$ is a basis for $E_i^* V$. Combining these facts we find $p_i(A) E_0^* V = E_i^* V$. \hfill \Box

**Theorem 8.4** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Then

$p_i(A) E_0^* E_0^* = E_i^* A_i E_0^* \quad (0 \leq i \leq d)$.

**Proof:** Let the integer $i$ be given and abbreviate $\Delta = p_i(A) - E_i^* A_i$. We show $\Delta E_0^* = 0$. In order to do this we show $\Delta E_0^* V = 0$, where $V$ denotes an irreducible $A$-module. Let $v$ denote a nonzero vector in $E_0^* V$ and recall $v$ is a basis for $E_0^* V$. By Lemma 8.2 we have $\Delta v = 0$ so $\Delta E_0^* V = 0$. Now $\Delta E_0^* = 0$ so $p_i(A) E_0^* = E_i^* A_i E_0^*$. \hfill \Box
Theorem 8.5 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomial $p_{d+1}$ be as in Definition 8.1. Then the following (i), (ii) hold.

(i) $p_{d+1}$ is both the minimal polynomial and the characteristic polynomial of $A$.

(ii) $p_{d+1} = \prod_{i=0}^{d}(\lambda - \theta_i)$.

Proof: (i) We first show $p_{d+1}$ is equal to the minimal polynomial of $A$. Recall $I, A, \ldots, A^d$ are linearly independent and that $p_{d+1}$ is monic with degree $d + 1$. We show $p_{d+1}(A) = 0$.

Let $V$ denote an irreducible $A$-module. Let $v$ denote a nonzero vector in $E^*_0V$ and recall $v$ is a basis for $E^*_0V$. From Lemma 8.2 we find $p_{d+1}(A)v = 0$. It follows $p_{d+1}(A)E^*_0 = 0$ so $p_{d+1}(A)E^*_0 = 0$. Applying Lemma 5.9 (with $X = p_{d+1}(A)$ and $Y = I$) we find $p_{d+1}(A) = 0$.

We have now shown $p_{d+1}$ is the minimal polynomial of $A$. By definition the characteristic polynomial of $A$ is equal to $\det(\lambda I - A)$. This polynomial is monic with degree $d + 1$ and has $p_{d+1}$ as a factor; therefore it is equal to $p_{d+1}$.

(ii) For $0 \leq i \leq d$ the scalar $\theta_i$ is an eigenvalue of $A$ and therefore a root of the characteristic polynomial of $A$. □

Theorem 8.6 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Let the scalars $x_i$ be as in Definition 7.1. Then

$$E^*_i = \frac{p_i(A)E^*_0p_i(A)}{x_1x_2 \cdots x_i} \quad (0 \leq i \leq d).$$

(23)

Proof: Let $\dagger : A \to A$ denote the antiautomorphism which corresponds to $\Phi$. From Theorem 8.4 we have $p_i(A)E^*_0 = E^*_iA^iE^*_0$. Applying $\dagger$ we find $E^*_0p_i(A) = E^*_0A^iE^*_i$. From these comments we find

$$p_i(A)E^*_0p_i(A) = E^*_iA^iE^*_0A^iE^*_i = x_1x_2 \cdots x_iE^*_i$$

in view of Lemma 7.6(i). The result follows. □

We finish this section with a comment.

Lemma 8.7 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Let the scalars $a_i$ be as in Definition 7.1. Then for $0 \leq i \leq d$ the coefficient of $\lambda^i$ in $p_{i+1}$ is equal to $-\sum_{j=0}^{i} a_j$.

Proof: Let $\alpha_i$ denote the coefficient of $\lambda^i$ in $p_{i+1}$. Computing the coefficient of $\lambda^i$ in (19) we find $\alpha_{i-1} = \alpha_i + a_i$ for $0 \leq i \leq d$, where $\alpha_{-1} = 0$. It follows $\alpha_i = -\sum_{j=0}^{i} a_j$ for $0 \leq i \leq d$. □
9 The scalars $\nu, m_i$

In this section we introduce some more scalars that will help us describe Leonard systems.

**Definition 9.1** Let $\Phi$ denote the Leonard system from Definition 3.2. We define

$$m_i = \text{tr}(E_i E_0^*) \quad (0 \leq i \leq d).$$

**Lemma 9.2** Let $\Phi$ denote the Leonard system from Definition 3.2. Then (i)--(v) hold below.

(i) $E_i E_0^* E_i = m_i E_i \quad (0 \leq i \leq d)$.

(ii) $E_0^* E_i E_0^* = m_i E_0^* \quad (0 \leq i \leq d)$.

(iii) $m_i \neq 0 \quad (0 \leq i \leq d)$.

(iv) $\sum_{i=0}^d m_i = 1$.

(v) $m_0 = m_0^*$.

**Proof:** (i) Observe $E_i$ is a basis for $E_i A E_i$. By this and since $E_i E_0^* E_i$ is contained in $E_i A E_i$, there exists $\alpha_i \in \mathbb{K}$ such that $E_i E_0^* E_i = \alpha_i E_i$. Taking the trace of both sides in this equation and using $\text{tr}(XY) = \text{tr}(YX)$, $\text{tr}(E_i) = 1$ we find $\alpha_i = m_i$.

(ii) Similar to the proof of (i).

(iii) Observe $m_i E_i$ is equal to $E_i E_0^* E_i$ by part (i) above and $E_i E_0^* E_i$ is nonzero by Corollary 5.8. It follows $m_i E_i \neq 0$ so $m_i \neq 0$.

(iv) Multiply each term in the equation $\sum_{i=0}^d E_i = I$ on the right by $E_0^*$, and then take the trace. Evaluate the result using Definition 9.1.

(v) The elements $E_0 E_0^*$ and $E_0^* E_0$ have the same trace. \hfill \Box

**Definition 9.3** Let $\Phi$ denote the Leonard system from Definition 3.2. Recall $m_0 = m_0^*$ by Lemma 9.2(v); we let $\nu$ denote the multiplicative inverse of this common value. We observe $\nu = \nu^*$. We emphasize

$$\text{tr}(E_0 E_0^*) = \nu^{-1}.$$ 

**Lemma 9.4** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalar $\nu$ be as in Definition 9.3. Then the following (i), (ii) hold.

(i) $\nu E_0 E_0^* E_0 = E_0$.

(ii) $\nu E_0^* E_0^* E_0 = E_0^*$.

**Proof:** (i) Set $i = 0$ in Lemma 9.2(i) and recall $m_0 = \nu^{-1}$.

(ii) Set $i = 0$ in Lemma 9.2(ii) and recall $m_0 = \nu^{-1}$. \hfill \Box
Theorem 9.5 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Let the scalars $\theta_i$ be as in Definition 5.1 and let the scalars $m_i$ be as in Definition 9.1. Then

$$p_i(\theta_j) = m_j^{-1} \text{tr}(E_j E_i^* A^i E_0^*) \quad (0 \leq i, j \leq d).$$

(26)

Proof: Using Theorem 8.4 we find

$$\text{tr}(E_j E_i^* A^i E_0^*) = \text{tr}(E_j p_i(A) E_0^*)$$

$$= p_i(\theta_j) \text{tr}(E_j E_0^*)$$

$$= p_i(\theta_j) m_j.$$

The result follows. \qed

10 The standard basis

In this section we discuss the notion of a standard basis. We begin with a comment.

Lemma 10.1 Let $\Phi$ denote the Leonard system from Definition 3.2 and let $V$ denote an irreducible $A$-module. Then

$$E_i^* V = E_i^* E_0 V \quad (0 \leq i \leq d).$$

(27)

Proof: The space $E_i^* V$ has dimension 1 and contains $E_i^* E_0 V$. We show $E_i^* E_0 V \neq 0$. Applying Corollary 5.8 to $\Phi^*$ we find $E_i^* E_0 \neq 0$. It follows $E_i^* E_0 V \neq 0$. We conclude $E_i^* V = E_i^* E_0 V$. \qed

Lemma 10.2 Let $\Phi$ denote the Leonard system from Definition 3.2 and let $V$ denote an irreducible $A$-module. Let $u$ denote a nonzero vector in $E_0 V$. Then for $0 \leq i \leq d$ the vector $E_i^* u$ is nonzero and hence a basis for $E_i^* V$. Moreover the sequence

$$E_0^* u, E_1^* u, \ldots, E_d^* u$$

(28)

is a basis for $V$.

Proof: Let the integer $i$ be given. We show $E_i^* u \neq 0$. Recall $E_0 V$ has dimension 1 and $u$ is a nonzero vector in $E_0 V$ so $u$ spans $E_0 V$. Applying $E_i^*$ we find $E_i^* u$ spans $E_i^* E_0 V$. The space $E_i^* E_0 V$ is nonzero by Lemma 10.1 so $E_i^* u$ is nonzero. The remaining assertions are clear. \qed

Definition 10.3 Let $\Phi$ denote the Leonard system from Definition 3.2 and let $V$ denote an irreducible $A$-module. By a $\Phi$-standard basis for $V$, we mean a sequence

$$E_0^* u, E_1^* u, \ldots, E_d^* u,$$

where $u$ is a nonzero vector in $E_0 V$. 

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We give a few characterizations of the standard basis.

**Lemma 10.4** Let Φ denote the Leonard system from Definition 3.2 and let V denote an irreducible A-module. Let \( v_0, v_1, \ldots, v_d \) denote a sequence of vectors in V, not all 0. Then this sequence is a Φ-standard basis for V if and only if both (i), (ii) hold below.

(i) \( v_i \in E_i^*V \) for \( 0 \leq i \leq d \).

(ii) \( \sum_{i=0}^d v_i \in E_0V \).

**Proof:** To prove the lemma in one direction, assume \( v_0, v_1, \ldots, v_d \) is a Φ-standard basis for V. By Definition 10.3 there exists a nonzero \( u \in E_0V \) such that \( v_i = E_i^*u \) for \( 0 \leq i \leq d \). Apparently \( v_i \in E_i^*V \) for \( 0 \leq i \leq d \) so (i) holds. Let I denote the identity element of A and recall \( I = \sum_{i=0}^d E_i^* \). Applying this to \( u \) we find \( u = \sum_{i=0}^d v_i \) and (ii) follows. We have now proved the lemma in one direction. To prove the lemma in the other direction, assume \( v_0, v_1, \ldots, v_d \) satisfy (i), (ii) above. We define \( u = \sum_{i=0}^d v_i \) and observe \( u \in E_0V \). Using (i) we find \( E_i^*v_j = \delta_{ij}v_j \) for \( 0 \leq i, j \leq d \); it follows \( v_i = E_i^*u \) for \( 0 \leq i \leq d \). Observe \( u \neq 0 \) since at least one of \( v_0, v_1, \ldots, v_d \) is nonzero. Now \( v_0, v_1, \ldots, v_d \) is a Φ-standard basis for V by Definition 10.3.

We recall some notation. Let \( d \) denote a nonnegative integer and let \( B \) denote a matrix in \( \text{Mat}_{d+1}(\mathbb{K}) \). Let \( \alpha \) denote a scalar in \( \mathbb{K} \). Then \( B \) is said to have constant row sum \( \alpha \) whenever \( B_{i0} + B_{i1} + \cdots + B_{id} = \alpha \) for \( 0 \leq i \leq d \).

**Lemma 10.5** Let Φ denote the Leonard system from Definition 3.2 and let the scalars \( \theta_i, \theta_i^* \) be as in Definition 3.1. Let V denote an irreducible A-module and let \( v_0, v_1, \ldots, v_d \) denote a basis for V. Let \( B \) (resp. \( B^* \)) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{K}) \) that represents \( A \) (resp. \( A^* \)) with respect to this basis. Then \( v_0, v_1, \ldots, v_d \) is a Φ-standard basis for V if and only if both (i), (ii) hold below.

(i) \( B \) has constant row sum \( \theta_0 \).

(ii) \( B^* = \text{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*) \).

**Proof:** Observe \( A \sum_{j=0}^d v_j = \sum_{i=0}^d v_i(B_{i0} + B_{i1} + \cdots + B_{id}) \). Recall \( E_0V \) is the eigenspace for \( A \) and eigenvalue \( \theta_0 \). Apparently \( B \) has constant row sum \( \theta_0 \) if and only if \( \sum_{i=0}^d v_i \in E_0V \). Recall that for \( 0 \leq i \leq d \), \( E_i^*V \) is the eigenspace for \( A^* \) and eigenvalue \( \theta_i^* \). Apparently \( B^* = \text{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*) \) if and only if \( v_i \in E_i^*V \) for \( 0 \leq i \leq d \). The result follows in view of Lemma 10.4.

**Definition 10.6** Let Φ denote the Leonard system from Definition 3.2. We define a map \( \flat : A \to \text{Mat}_{d+1}(\mathbb{K}) \) as follows. Let V denote an irreducible A-module. For all \( X \in A \) we let \( X^\flat \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{K}) \) that represents \( X \) with respect to a Φ-standard basis for V. We observe \( \flat : A \to \text{Mat}_{d+1}(\mathbb{K}) \) is an isomorphism of \( \mathbb{K} \)-algebras.
Lemma 10.7 Let Φ denote the Leonard system from Definition 3.2 and let the scalars \( \theta_i, \theta^*_i \) be as in Definition 5.1. Let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. Then (i)–(iii) hold below.

(i) \( A^\flat \) has constant row sum \( \theta_0 \).

(ii) \( A^{\flat^*} = \text{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d) \).

(iii) For \( 0 \leq i \leq d \) the matrix \( E^\flat_i \) has \( ii \) entry 1 and all other entries 0.

Proof: (i), (ii) Combine Lemma 10.5 and Definition 10.6. (iii) Immediate from Lemma 5.2(i). □

Let Φ denote the Leonard system from Definition 3.2 and let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. Let \( X \) denote an element of \( A \). In Theorem 10.9 below we give the entries of \( X^\flat \) in terms of the trace function. To prepare for this we need a lemma.

Lemma 10.8 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. Then for \( X \in A \) the entries of \( X^\flat \) satisfy

\[ E^\flat_i X E^\flat_j E_0 = (X^\flat)_{ij} E^\flat_i E_0 \quad (0 \leq i, j \leq d). \]  (29)

Proof: Let the integers \( i, j \) be given and abbreviate \( \Delta = E^\flat_i X E^\flat_j E_0 - (X^\flat)_{ij} E^\flat_i E_0 \). We show \( \Delta E_0 = 0 \). In order to do this we show \( \Delta E_0 V = 0 \), where \( V \) denotes an irreducible \( A \)-module. Let \( u \) denote a nonzero vector in \( E_0 V \). By Definition 10.3 the sequence \( E_0^\flat u, E_1^\flat u, \ldots, E_d^\flat u \) is a \( \Phi \)-standard basis for \( V \). Recall \( X^\flat \) is the matrix in \( \text{Mat}_{d+1}(K) \) that represents \( X \) with respect to this basis. Applying \( X \) to \( E^\flat_j u \) we find \( X E^\flat_j u = \sum_{r=0}^d (X^\flat)_{rj} E^\flat_r u \). Applying \( E^\flat_i \) we obtain \( E^\flat_i X E^\flat_j u = (X^\flat)_{ij} E^\flat_i E_0 u \). By this and since \( u \) spans \( E_0 V \) we find \( \Delta E_0 V = 0 \). Therefore \( \Delta E_0 = 0 \) and the result follows. □

Theorem 10.9 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( m^*_i \) be as in Definition 9.1. Let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. Then for \( X \in A \) the entries of \( X^\flat \) are given as follows.

\[ (X^\flat)_{ij} = m^*_i m^*_j - (X^\flat)_{ij} \quad (0 \leq i, j \leq d). \]  (30)

Proof: In equation (29), take the trace of both sides and observe \( m^*_i = \text{tr}(E^\flat_i E_0) \) in view of Definition 9.1. □

Referring to Theorem 10.9 we consider the case \( X = E_0 \).

Lemma 10.10 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( m^*_i \) be as in Definition 9.1. Let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. Then for \( 0 \leq i, j \leq d \) the \( ij \) entry of \( E^\flat_0 \) is \( m^*_j \).

Proof: Set \( X = E_0 \) in (30). Simplify the result using \( E_0 E^\flat_j E_0 = m^*_j E_0 \) and \( m^*_i = \text{tr}(E^\flat_i E_0) \). □
Theorem 10.11 Let \( \Phi \) denote the Leonard system from Definition 3.2, and let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. For \( 0 \leq i, j \leq d \) define \( \Psi_{ij} = m_j^{-1}E^*_iE_0E^*_j \), where \( m_j \) is from Definition 9.1. Then the matrix \( \Psi_{ij} \) has \( ij \) entry 1 and all other entries 0.

Proof: Immediate from Lemma 10.7(iii), Lemma 10.10, and since \( \flat \) is an isomorphism of \( K \)-algebras. \( \square \)

11 The scalars \( b_i, c_i \)

In this section we consider some scalars that arise naturally in the context of the standard basis.

Definition 11.1 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the map \( \flat : A \to \text{Mat}_{d+1}(K) \) be as in Definition 10.6. For \( 0 \leq i \leq d - 1 \) we let \( b_i \) denote the \( i, i+1 \) entry of \( A^\flat \). For \( 1 \leq i \leq d \) we let \( c_i \) denote the \( i, i-1 \) entry of \( A^\flat \). We observe

\[
A^\flat = \begin{pmatrix}
a_0 & b_0 & 0 \\
c_1 & a_1 & b_1 \\
& \ddots & \ddots \\
0 & & \ddots & b_{d-1} \\
& & & c_d & a_d \\
\end{pmatrix},
\]

(31)

where the \( a_i \) are from Definition 7.1. For notational convenience we define \( b_d = 0 \) and \( c_0 = 0 \).

Lemma 11.2 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( b_i, c_i \) be as in Definition 11.1. Then with reference to Definition 5.1 and Definition 7.1 the following (i), (ii) hold.

(i) \( b_{i-1}c_i = x_i \) \( (1 \leq i \leq d) \).

(ii) \( c_i + a_i + b_i = \theta_0 \) \( (0 \leq i \leq d) \).

Proof: (i) Apply Lemma 7.2(ii) with \( B = A^\flat \).

(ii) Combine (31) and Lemma 10.7(i). \( \square \)

Lemma 11.3 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( b_i, c_i \) be as in Definition 11.1. Let the polynomials \( p_i \) be as in Definition 8.1 and let the scalar \( \theta_0 \) be as in Definition 5.1. Then the following (i)–(iii) hold.

(i) \( b_i \neq 0 \) \( (0 \leq i \leq d - 1) \).

(ii) \( c_i \neq 0 \) \( (1 \leq i \leq d) \).

(iii) \( b_0b_1 \cdots b_{i-1} = p_i(\theta_0) \) \( (0 \leq i \leq d + 1) \).
Proof: (i), (ii) Immediate from Lemma 11.2(i) and since each of $x_1, x_2, \ldots, x_d$ is nonzero. (iii) Assume $0 \leq i \leq d$; otherwise each side is zero. Let $\dagger : A \to A$ denote the antiautomorphism which corresponds to $\Phi$. Applying $\dagger$ to both sides of (22) we get $E_{0}^{*}p_{i}(A) = E_{i}^{*}A^{i}E_{0}^{*}$. We may now argue

\[ b_{0}b_{1}\ldots b_{i-1} = (A^{i})_{0i} \quad \text{(by 31)} \]
\[ = m_{0}^{i-1}\text{tr}(E_{0}^{*}A^{i}E_{i}^{*}E_{0}) \quad \text{(by Theorem 10.9)} \]
\[ = m_{0}^{i-1}\text{tr}(E_{0}^{*}p_{i}(A)E_{0}) \]
\[ = m_{0}^{i-1}p_{i}(\theta_{0})\text{tr}(E_{0}^{*}E_{0}) \]
\[ = p_{i}(\theta_{0}) \quad \text{(by Definition 9.1)}. \]

□

Theorem 11.4 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_{i}$ be as in Definition 8.1. Let the scalar $\theta_{0}$ be as in Definition 5.1. Then $p_{i}(\theta_{0}) \neq 0$ for $0 \leq i \leq d$. Let the scalars $b_{i}, c_{i}$ be as in Definition 11.1. Then

\[ b_{i} = \frac{p_{i+1}(\theta_{0})}{p_{i}(\theta_{0})} \quad (0 \leq i \leq d) \]

and

\[ c_{i} = \frac{x_{i}p_{i-1}(\theta_{0})}{p_{i}(\theta_{0})} \quad (1 \leq i \leq d). \]

Proof: Observe $p_{i}(\theta_{0}) \neq 0$ for $0 \leq i \leq d$ by Lemma 11.3(i), (iii). Line (32) is immediate from Lemma 11.3(iii). To get (33) combine (32) and Lemma 11.2(i). □

Lemma 11.5 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $b_{i}, c_{i}$ be as in Definition 11.1. Then the following (i), (ii) hold.

(i) $E_{i}^{*}AE_{i+1}^{*}E_{0} = b_{i}E_{i}^{*}E_{0} \quad (0 \leq i \leq d - 1)$.

(ii) $E_{i}^{*}AE_{i-1}^{*}E_{0} = c_{i}E_{i}^{*}E_{0} \quad (1 \leq i \leq d)$.

Proof: (i) This is (29) with $X = A$ and $j = i + 1$.

(ii) This is (29) with $X = A$ and $j = i - 1$. □

Theorem 11.6 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $b_{i}, c_{i}$ be as in Definition 11.1. Let the scalars $m_{i}^{*}$ be as in Definition 9.1. Then the following (i), (ii) hold.

(i) $b_{i} = m_{i}^{*}\text{tr}(E_{i}^{*}AE_{i+1}^{*}E_{0}) \quad (0 \leq i \leq d - 1)$.

(ii) $c_{i} = m_{i}^{*}\text{tr}(E_{i}^{*}AE_{i-1}^{*}E_{0}) \quad (1 \leq i \leq d)$.

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Proof: (i) This is (30) with $X = A$ and $j = i + 1$.
(ii) This is (30) with $X = A$ and $j = i - 1$.

We finish this section with a comment.

**Theorem 11.7** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $c_i$ be as in Definition 11.1. Let the scalars $\theta_i$ be as in Definition 5.1 and let the scalar $\nu$ be as in Definition 9.3. Then

$$
(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d) = \nu c_1 c_2 \cdots c_d.
$$

(34)

**Proof:** Let $\delta$ denote the expression on the left-hand side of (34). Setting $i = 0$ in (4) we find

$$
\delta E_0 = \prod_{j=1}^d (A - \theta_j I).
$$

We multiply both sides of this equation on the left by $E^*_0$ and on the right by $E^*_d$. Evaluating the resulting equation using Lemma 5.10(i) to obtain

$$
\delta E^*_d E_0 E^*_0 E_0 = E^*_d A^d E^*_0.
$$

(35)

We evaluate each side of (35). The left-hand side of (35) is equal to $\delta \nu^{-1} E^*_d E_0$ in view of Lemma 9.4(i). We now consider the right-hand side of (35). Observe $E^*_d A^d E^*_0 = E^*_d A E^*_d A^d E^*_0$ by (13). Evaluating the right-hand side of (35) using this and Theorem 11.5(ii) we find it is equal to $c_1 c_2 \cdots c_d E^*_d E_0$. From our above comments we find $\delta \nu^{-1} E^*_d E_0 = c_1 c_2 \cdots c_d E^*_d E_0$. Observe $E^*_d E_0 \neq 0$ by Lemma 10.1 so $\delta \nu^{-1} = c_1 c_2 \cdots c_d$. The result follows.

$$
\square
$$

12 **The scalars $k_i$**

In this section we consider some scalars that are closely related to the scalars from Definition 9.1.

**Definition 12.1** Let $\Phi$ denote the Leonard system from Definition 3.2. We define

$$
k_i = m^*_i \nu \quad (0 \leq i \leq d),
$$

where the $m^*_i$ are from Definition 9.1 and $\nu$ is from Definition 9.3.

**Lemma 12.2** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $k_i$ be as in Definition 12.1. Then (i) $k_0 = 1$; (ii) $k_i \neq 0$ for $0 \leq i \leq d$; (iii) $\sum_{i=0}^d k_i = \nu$.

**Proof:** (i) Set $i = 0$ in (36) and recall $m^*_0 = \nu^{-1}$.
(ii) Applying Lemma 9.2(iii) to $\Phi^*$ we find $m^*_i \neq 0$ for $0 \leq i \leq d$. We have $\nu \neq 0$ by Definition 9.3. The result follows in view of (36).
(iii) Applying Lemma 9.2(iv) to $\Phi^*$ we find $\sum_{i=0}^d m^*_i = 1$. The result follows in view of (36). $$
\square
$$
Lemma 12.3 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $k_i$ be as in Definition 12.1. Then with reference to Definition 5.1, Definition 7.1, and Definition 8.1,

$$k_i = \frac{p_i(\theta_0)^2}{x_1x_2 \cdots x_i} \quad (0 \leq i \leq d). \quad (37)$$

Proof: We show that each side of (37) is equal to $\nu(E_i^*E_0)$. Using (24) and (36) we find $\nu(E_i^*E_0)$ is equal to the left-hand side of (37). Using Theorem 8.6 we find $\nu(E_i^*E_0)$ is equal to the right-hand side of (37). □

Theorem 12.4 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the scalars $k_i$ be as in Definition 12.1. Let the scalars $b_i, c_i$ be as in Definition 11.1. Then

$$k_i = \frac{b_0b_1 \cdots b_{i-1}}{c_1c_2 \cdots c_i} \quad (0 \leq i \leq d). \quad (38)$$

Proof: Evaluate the expression on the right in (37) using Lemma 11.2(i) and Lemma 11.3(iii). □

13 The polynomials $v_i$

Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. The $p_i$ have two normalizations of interest; we call these the $u_i$ and the $v_i$. In this section we discuss the $v_i$. In the next section we will discuss the $u_i$.

Definition 13.1 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. For $0 \leq i \leq d$ we define the polynomial $v_i$ by

$$v_i = \frac{p_i}{c_1c_2 \cdots c_i}, \quad (39)$$

where the $c_j$ are from Definition 11.1. We observe $v_0 = 1$.

Lemma 13.2 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $v_i$ be as in Definition 13.1. Let the scalar $\theta_0$ be as in Definition 5.1 and let the scalars $k_i$ be as in Definition 12.1. Then

$$v_i(\theta_0) = k_i \quad (0 \leq i \leq d). \quad (40)$$

Proof: Use Lemma 11.3(iii), Theorem 12.4 and (39). □
Lemma 13.3 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the polynomials \( v_i \) be as in Definition 13.1. Let the scalars \( a_i, b_i, c_i \) be as in Definition 7.1 and Definition 11.1. Then

\[
\lambda v_i = c_{i+1}v_{i+1} + a_iv_i + b_{i-1}v_{i-1} \quad (0 \leq i \leq d-1),
\]

where \( b_{-1} = 0 \) and \( v_{-1} = 0 \). Moreover

\[
\lambda v_d - a_d v_d - b_{d-1}v_{d-1} = (c_1c_2\cdots c_d)^{-1}p_{d+1}.
\]

Proof: In (19), divide both sides by \( c_1c_2\cdots c_d \). Evaluate the result using Lemma 11.2(i) and (39).

Theorem 13.4 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the polynomials \( v_i \) be as in Definition 13.1. Let \( V \) denote an irreducible \( \mathcal{A} \)-module and let \( u \) denote a nonzero vector in \( E_0V \). Then

\[
v_i(A)E_0^*u = E_i^*u \quad (0 \leq i \leq d).
\]

Proof: For \( 0 \leq i \leq d \) we define \( w_i = v_i(A)E_0^*u \) and \( w'_i = E_i^*u \). We show \( w_i = w'_i \). Each of \( w_0, w'_0 \) is equal to \( E_0^*u \) so \( w_0 = w'_0 \). Using Lemma 13.3 we obtain

\[
Aw_i = c_{i+1}w_{i+1} + a_iw_i + b_{i-1}w_{i-1} \quad (0 \leq i \leq d-1)
\]

where \( w_{-1} = 0 \) and \( b_{-1} = 0 \). By Definition 10.6, Definition 11.1 and since \( w_0', w_1', \ldots, w'_d \) is a \( \Phi \)-standard basis,

\[
Aw'_i = c_{i+1}w'_{i+1} + a_iw'_i + b_{i-1}w'_{i-1} \quad (0 \leq i \leq d-1)
\]

where \( w'_{-1} = 0 \). Comparing (44), (45) and using \( w_0 = w'_0 \) we find \( w_i = w'_i \) for \( 0 \leq i \leq d \). The result follows.

We finish this section with a comment.

Lemma 13.5 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the polynomials \( v_i \) be as in Definition 13.1. Let the scalar \( \nu \) be as in Definition 9.3. Then the following (i), (ii) hold.

(i) \( v_i(A)E_0^*E_0 = E_i^*E_0 \quad (0 \leq i \leq d) \).

(ii) \( v_i(A)E_0^* = \nu E_i^*E_0 \quad (0 \leq i \leq d) \).

Proof: (i) Let the integer \( i \) be given and abbreviate \( \Delta = v_i(A)E_0^* - E_i^* \). We show \( \Delta E_0 = 0 \). In order to to do this we show \( \Delta E_0 V = 0 \), where \( V \) denotes an irreducible \( \mathcal{A} \)-module. Let \( u \) denote a nonzero vector in \( E_0V \) and recall \( u \) spans \( E_0V \). Observe \( \Delta u = 0 \) by Theorem 13.3 so \( \Delta E_0 V = 0 \). Now \( \Delta E_0 = 0 \) so \( v_i(A)E_0^*E_0 = E_i^*E_0 \).

(ii) In the equation of (i) above, multiply both sides on the right by \( E_0^* \) and simplify the result using Lemma 9.4(ii).
14 The polynomials \( u_i \)

Let \( \Phi \) denote the Leonard system from Definition \([3.2]\) and let the polynomials \( p_i \) be as in Definition \([8.1]\). In the previous section we gave a normalization of the \( p_i \) that we called the \( v_i \). In this section we give a second normalization for the \( p_i \) that we call the \( u_i \).

**Definition 14.1** Let \( \Phi \) denote the Leonard system from Definition \([3.2]\) and let the polynomials \( p_i \) be as in Definition \([8.1]\). For \( 0 \leq i \leq d \) we define the polynomial \( u_i \) by

\[
 u_i = \frac{p_i}{p_i(\theta_0)}, \tag{46}
\]

where \( \theta_0 \) is from Definition \([5.1]\). We observe \( u_0 = 1 \). Moreover

\[
 u_i(\theta_0) = 1 \quad (0 \leq i \leq d). \tag{47}
\]

**Lemma 14.2** Let \( \Phi \) denote the Leonard system from Definition \([3.2]\) and let the polynomials \( u_i \) be as in Definition \([14.1]\). Let the scalars \( a_i, b_i, c_i \) be as in Definition \([7.1]\) and Definition \([11.1]\). Then

\[
 \lambda u_i = b_i u_{i+1} + a_i u_i + c_i u_{i-1} \quad (0 \leq i \leq d-1), \tag{48}
\]

where \( u_{-1} = 0 \). Moreover

\[
 \lambda u_d - c_d u_{d-1} - a_d u_d = p_d(\theta_0)^{-1} p_{d+1}, \tag{49}
\]

where \( \theta_0 \) is from Definition \([5.1]\).

**Proof:** In \([19]\), divide both sides by \( p_i(\theta_0) \) and evaluate the result using Lemma \([11.2](i)\), \([32]\), and \([46]\). \( \square \)

The above 3-term recurrence is often expressed as follows.

**Corollary 14.3** Let \( \Phi \) denote the Leonard system from Definition \([3.2]\) and let the polynomials \( u_i \) be as in Definition \([14.1]\). Let the scalars \( \theta_i \) be as in Definition \([7.1]\). Then for \( 0 \leq i, j \leq d \) we have

\[
 \theta_j u_i(\theta_j) = b_i u_{i+1}(\theta_j) + a_i u_i(\theta_j) + c_i u_{i-1}(\theta_j), \tag{50}
\]

where \( u_{-1} = 0 \) and \( u_{d+1} = 0 \).

**Proof:** Apply Lemma \([14.2]\) (with \( \lambda = \theta_j \)) and observe \( p_{d+1}(\theta_j) = 0 \) by Theorem \([8.5](ii)\). \( \square \)

**Lemma 14.4** Let \( \Phi \) denote the Leonard system from Definition \([3.2]\). Let the polynomials \( u_i, v_i \) be as in Definition \([14.1]\) and Definition \([13.1]\) respectively. Then

\[
 v_i = k_i u_i \quad (0 \leq i \leq d), \tag{51}
\]

where the \( k_i \) are from Definition \([12.1]\).
Proof: Compare (39) and (46) in light of Lemma [11.3](iii) and Theorem [12.4] □

Let Φ denote the Leonard system from Definition 3.2 and let the polynomials $u_i$ be as in Definition 14.1. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote the eigenvalue sequence of Φ. Our next goal is to compute the $u_i(\theta_j)$ in terms of the trace function. To prepare for this we give a lemma.

**Lemma 14.5** Let Φ denote the Leonard system from Definition 3.2 and let the polynomials $v_i$ be as in Definition 13.1. Let the scalars $\theta_i$ be as in Definition 5.1 and let the scalars $m_i, m_i^\ast$ be as in Definition 9.1. Then

$$E_0 E_i^\ast E_j E_0^\ast = v_i(\theta_j) m_j E_0 E_0^\ast \quad (0 \leq i, j \leq d). \quad (52)$$

Proof: Let $\dagger : A \to A$ denote the antiautomorphism which corresponds to Φ. Applying $\dagger$ to the equation in Lemma 13.5(i) we find

$$E_0 E_i^\ast E_j E_0^\ast = v_i(\theta_j) E_0 E_0^\ast E_j E_0^\ast = v_i(\theta_j) m_j E_0 E_0^\ast.$$

□

**Theorem 14.6** Let Φ denote the Leonard system from Definition 3.2 and let the polynomials $u_i$ be as in Definition 14.1. Let the scalars $\theta_i$ be as in Definition 5.1 and let the scalars $m_i, m_i^\ast$ be as in Definition 9.1. Then

$$u_i(\theta_j) = m_i^\ast m_j^{-1} \text{tr}(E_0 E_i^\ast E_j E_0^\ast) \quad (0 \leq i, j \leq d). \quad (53)$$

Proof: In (52), take the trace of both sides and simplify the result using (25), (36), (51). □

**Theorem 14.7** Let Φ denote the Leonard system from Definition 3.2. Let the polynomials $u_i$ be as in Definition 14.1 and recall the $u_i^\ast$ are the corresponding polynomials for $\Phi^\ast$. Let the scalars $\theta_i, \theta_i^\ast$ be as in Definition 5.1. Then

$$u_i(\theta_j) = u_i^\ast(\theta_j^\ast) \quad (0 \leq i, j \leq d). \quad (54)$$

Proof: Applying Theorem 14.6 to $\Phi^\ast$ we find

$$u_i^\ast(\theta_j^\ast) = m_i^\ast m_j^{-1} \text{tr}(E_0 E_i^\ast E_j E_0^\ast). \quad (55)$$

Interchanging the roles of $i, j$ in (55) we obtain

$$u_j^\ast(\theta_i^\ast) = m_i^\ast m_j^{-1} \text{tr}(E_0 E_j E_i^\ast E_0^\ast). \quad (56)$$

Let $\dagger : A \to A$ denote the antiautomorphism which corresponds to Φ. Observe

$$(E_0 E_i^\ast E_j E_0^\ast)^\dagger = E_0^\ast E_j E_i^\ast E_0$$

(57)
in view of Lemma 6.3. The trace function is invariant under † so
\[
\text{tr}(E_0 E_i^* E_j^* E_0^*) = \text{tr}(E_0^* E_j E_i E_0^*).
\] (58)
Combining (53), (56), (58) we obtain (54).

In the following two theorems we show how (54) looks in terms of the polynomials \( v_i \) and \( p_i \).

**Theorem 14.8** Let \( \Phi \) denote the Leonard system from Definition 3.2. With reference to Definition 5.1, Definition 4.1 and Definition 13.1,
\[
v_i(\theta_j)/k_i = v_j^*(\theta_i^*)/k_j^* \quad (0 \leq i, j \leq d).
\] (59)
**Proof:** Evaluate (54) using Lemma 14.4.

**Theorem 14.9** Let \( \Phi \) denote the Leonard system from Definition 3.2. With reference to Definition 5.1, Definition 4.1, and Definition 8.1,
\[
p_i(\theta_j)/p_i(\theta_0) = p_j^*(\theta_i^*)/p_j^*(\theta_0^*) \quad (0 \leq i, j \leq d).
\] (60)
**Proof:** Evaluate (54) using Definition 14.1.

The equations (54), (59), (60) are often referred to as Askey-Wilson duality.

We finish this section with a few comments.

**Lemma 14.10** Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the polynomials \( u_i \) be as in Definition 14.1. Then for \( 0 \leq i, j \leq d \) we have
\[
\theta_i^* u_i(\theta_j) = b_j^* u_i(\theta_{j+1}) + a_j^* u_i(\theta_j) + c_j^* u_i(\theta_{j-1}),
\] (61)
where \( \theta_{-1}, \theta_{d+1} \) denote indeterminates.
**Proof:** Apply Corollary 14.3 to \( \Phi^* \) and evaluate the result using Theorem 14.7.

We refer to (61) as the difference equation satisfied by the \( u_i \).

**Lemma 14.11** Let \( \Phi \) denote the Leonard system from Definition 3.2 and assume \( d \geq 1 \). Let the polynomials \( u_i \) be as in Definition 14.1. Then
\[
u_i(\theta_1) = \frac{\theta_i^* - a_0^*}{\theta_i^* - a_0^*} \quad (0 \leq i \leq d).
\] (62)
**Proof:** Setting \( j = 1 \) in (54) we find \( u_i(\theta_1) = u_i^*(\theta_i^*) \). Applying (46) to \( \Phi^* \) we find \( u_i^* = p_i^*/p_i^*(\theta_0^*) \). Applying Definition 8.1 to \( \Phi^* \) we find \( p_i^* = \lambda - a_0^* \). Combining these facts we get the result.
Lemma 14.12 Let \( \Phi \) denote the Leonard system from Definition 32 and assume \( d \geq 1 \). Then
\[
b_i \theta_{i+1}^* + a_i \theta_i^* + c_i \theta_{i-1}^* = \theta_i \theta_i^* + a_0 \theta_0 - \theta_1 \quad (0 \leq i \leq d),
\]
where \( \theta_{-1}, \theta_{d+1} \) denote indeterminates.

Proof: Set \( j = 1 \) in (50). Evaluate the result using Lemma 11.2(ii) and (62). \( \square \)

15 A bilinear form

In this section we associate with each Leonard pair a certain bilinear form. To prepare for this we recall a few concepts from linear algebra.

Let \( V \) denote a finite dimensional vector space over \( \mathbb{K} \). By a bilinear form on \( V \) we mean a map \( \langle \, , \rangle : V \times V \to \mathbb{K} \) that satisfies the following four conditions for all \( u, v, w \in V \) and for all \( \alpha \in \mathbb{K} \): (i) \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \); (ii) \( \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \); (iii) \( \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \); (iv) \( \langle u, \alpha v \rangle = \alpha \langle u, v \rangle \). We observe that a scalar multiple of a bilinear form on \( V \) is a bilinear form on \( V \). Let \( \langle \, , \rangle \) denote a bilinear form on \( V \). This form is said to be symmetric whenever \( \langle u, v \rangle = \langle v, u \rangle \) for all \( u, v \in V \). Let \( \langle \, , \rangle \) denote a bilinear form on \( V \). Then the following are equivalent: (i) there exists a nonzero \( u \in V \) such that \( \langle u, v \rangle = 0 \) for all \( v \in V \); (ii) there exists a nonzero \( v \in V \) such that \( \langle u, v \rangle = 0 \) for all \( u \in V \). The form \( \langle \, , \rangle \) is said to be degenerate whenever (i), (ii) hold and nondegenerate otherwise. Let \( \gamma : \mathcal{A} \to \mathcal{A} \) denote an antiautomorphism and let \( V \) denote an irreducible \( \mathcal{A} \)-module. Then there exists a nonzero bilinear form \( \langle \, , \rangle \) on \( V \) such that \( \langle Xu, v \rangle = \langle u, X^\gamma v \rangle \) for all \( u, v \in V \) and for all \( X \in \mathcal{A} \). The form is unique up to multiplication by a nonzero scalar in \( \mathbb{K} \). The form in nondegenerate. We refer to this form as the bilinear form on \( V \) associated with \( \gamma \). This form is not symmetric in general.

We now return our attention to Leonard pairs.

Definition 15.1 Let \( \Phi = (A; A^*; \{ E_i \}_{i=0}^d; \{ E_i^* \}_{i=0}^d) \) denote a Leonard system in \( \mathcal{A} \). Let \( \dagger : \mathcal{A} \to \mathcal{A} \) denote the corresponding antiautomorphism from Definition 6.2. Let \( V \) denote an irreducible \( \mathcal{A} \)-module. For the rest of this paper we let \( \langle \, , \rangle \) denote the bilinear form on \( V \) associated with \( \dagger \). We abbreviate \( \| u \|^2 = \langle u, u \rangle \) for all \( u \in V \). By the construction, for \( X \in \mathcal{A} \) we have
\[
\langle Xu, v \rangle = \langle u, X^\dagger v \rangle \quad (\forall u \in V, \forall v \in V). \tag{63}
\]

We make an observation.

Lemma 15.2 With reference to Definition 15.1 let \( \mathcal{D} \) (resp. \( \mathcal{D}^* \)) denote the subalgebra of \( \mathcal{A} \) generated by \( A \) (resp. \( A^* \)). Then for \( X \in \mathcal{D} \cup \mathcal{D}^* \) we have
\[
\langle Xu, v \rangle = \langle u, Xv \rangle \quad (\forall u \in V, \forall v \in V). \tag{64}
\]
Proof: Combine (63) and Lemma 6.3

With reference to Definition 15.1, our next goal is to show \(\langle , \rangle\) is symmetric. We will use the following lemma.

**Theorem 15.3** With reference to Definition 15.1 let \(u\) denote a nonzero vector in \(E_0 V\) and recall \(E_0^* u, E_1^* u, \ldots, E_d^* u\) is a \(\Phi\)-standard basis for \(V\). We have
\[
\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i \nu^{-1} \|u\|^2 \quad (0 \leq i, j \leq d),
\]
where the \(k_i\) are from Definition 12.1 and \(\nu\) is from Definition 9.3.

**Proof:** By (64) and since \(E_0 u = u\) we find \(\langle E_i^* u, E_j^* u \rangle = \langle u, E_0 E_i^* E_j^* E_0 u \rangle\). Using Lemma 9.2(ii) and (36) we find \(\langle u, E_0 E_i^* E_j^* E_0 u \rangle = \delta_{ij} k_i \nu^{-1} \|u\|^2\). \(\square\)

**Corollary 15.4** With reference to Definition 15.1 the bilinear form \(\langle , \rangle\) is symmetric.

**Proof:** Let \(u\) denote a nonzero vector in \(E_0 V\) and abbreviate \(v_i = E_i^* u\) for \(0 \leq i \leq d\). From Theorem 15.3 we find \(\langle v_i, v_j \rangle = \langle v_j, v_i \rangle\) for \(0 \leq i, j \leq d\). The result follows since \(v_0, v_1, \ldots, v_d\) is a basis for \(V\). \(\square\)

We have a comment.

**Lemma 15.5** With reference to Definition 15.1 let \(u\) denote a nonzero vector in \(E_0 V\) and let \(v\) denote a nonzero vector in \(E_0^* V\). Then the following (i)–(iv) hold.

(i) Each of \(\|u\|^2, \|v\|^2, \langle u, v \rangle\) is nonzero.

(ii) \(E_0^* u = \langle u, v \rangle \|v\|^{-2} v\).

(iii) \(E_0 v = \langle u, v \rangle \|u\|^{-2} u\).

(iv) \(\nu \langle u, v \rangle^2 = \|u\|^2 \|v\|^2\).

**Proof:** (i) Observe \(\|u\|^2 \neq 0\) by Theorem 15.3 and since \(\langle , \rangle\) is not 0. Similarly \(\|v\|^2 \neq 0\). To see that \(\langle u, v \rangle \neq 0\), observe that \(v\) is a basis for \(E_0^* V\) so there exists \(\alpha \in \mathbb{K}\) such that \(E_0^* u = \alpha v\). Recall \(E_0^* u \neq 0\) by Lemma 10.2 so \(\alpha \neq 0\). Using (64) and \(E_0^* v = v\) we routinely find \(\langle u, v \rangle = \alpha \|v\|^2\) and it follows \(\langle u, v \rangle \neq 0\).

(ii) In the proof of part (i) we found \(E_0^* u = \alpha v\) where \(\langle u, v \rangle = \alpha \|v\|^2\). The result follows.

(iii) Similar to the proof of (ii) above.

(iv) Using \(u = E_0^* u\) and \(v E_0^* E_0 = E_0\) we find \(\nu^{-1} u = E_0 E_0^* u\). To finish the proof, evaluate \(E_0 E_0^* u\) using (ii) above and then (iii) above. \(\square\)

**Theorem 15.6** With reference to Definition 15.1 let \(u\) denote a nonzero vector in \(E_0 V\) and let \(v\) denote a nonzero vector in \(E_0^* V\). Then
\[
\langle E_i^* u, E_j v \rangle = \nu^{-1} k_i k_j^* u_i(\theta_j) \langle u, v \rangle \quad (0 \leq i, j \leq d).
\]
Proof: Using Theorem 13.4 we find
\[ \langle E^*_i u, E_j v \rangle = \langle v_i(A)E^*_0u, E_j v \rangle \]
\[ = \langle E^*_0u, v_i(A)E_j v \rangle \]
\[ = v_i(\theta_j)\langle E^*_0u, E_j v \rangle \]
\[ = v_i(\theta_j)\langle E^*_0u, v^*_j(A^*)E^*_0v \rangle \]
\[ = v_i(\theta_j)\langle v^*_j(A^*)E^*_0u, E^*_0v \rangle \]
\[ = v_i(\theta_j)E^*_j v \langle E^*_0u, E^*_0v \rangle. \quad (67) \]

Using Lemma 15.5(ii)–(iv) we find \( \langle E^*_0u, E^*_0v \rangle = \nu^{-1}\langle u, v \rangle \). Observe \( v_i(\theta_j) = u_i(\theta_j)k_i \) by (51). Applying Lemma 13.2 to \( \Phi^* \) we find \( v^*_j(\theta^*_i) = k^*_j \). Evaluating (67) using these comments we obtain (66). \( \square \)

Remark 15.7 Using Theorem 15.6 and the symmetry of \( \langle \, , \rangle \) we get an alternate proof of Theorem 14.7.

Theorem 15.8 With reference to Definition 15.1, let \( u \) denote a nonzero vector in \( E^*_0V \) and let \( v \) denote a nonzero vector in \( E^*_0V \). Then for \( 0 \leq i \leq d \), both
\[ E^*_i u = \frac{\langle u, v \rangle}{\|v\|^2} \sum_{j=0}^{d} v_i(\theta_j)E_j v, \quad (68) \]
\[ E_i v = \frac{\langle u, v \rangle}{\|u\|^2} \sum_{j=0}^{d} v^*_i(\theta^*_j)E^*_j u. \quad (69) \]

Proof: We first show (68). To do this we show each side of (68) is equal to \( v_i(A)E^*_0u \). By Theorem 13.4 we find \( v_i(A)E^*_0u \) is equal to the left-hand side of (68). To see that \( v_i(A)E^*_0u \) is equal to the right-hand side of (68), multiply \( v_i(A)E^*_0u \) on the left by the identity \( I \), expand using \( I = \sum_{j=0}^{d} E_j \), and simplify the result using \( E_j A = \theta_j E_j \) \( (0 \leq j \leq d) \) and Lemma 15.5(ii). We have now proved (68). Applying (68) to \( \Phi^* \) we obtain (69). \( \square \)

Definition 15.9 Let \( \Phi \) denote the Leonard system from Definition 3.2. We define a matrix 
\( P \in \text{Mat}_{d+1}(\mathbb{K}) \) as follows. For \( 0 \leq i, j \leq d \) the entry \( P_{ij} = v_j(\theta_i) \), where \( \theta_i \) is from Definition 5.1 and \( v_j \) is from Definition 13.1.

Theorem 15.10 Let \( \Phi \) denote the Leonard system from Definition 3.2. Let the matrix \( P \) be as in Definition 15.9 and recall \( P^* \) is the corresponding matrix for \( \Phi^* \). Then \( P^*P = \nu I \), where \( \nu \) is from Definition 9.3.

Proof: Compare (68), (69) and use Lemma 15.5(iv). \( \square \)
Theorem 15.11 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the matrix $P$ be as in Definition 15.9. Let the map $\flat : A \to \text{Mat}_{d+1}(\mathbb{K})$ be as in Definition 10.6 and let $\sharp : A \to \text{Mat}_{d+1}(\mathbb{K})$ denote the corresponding map for $\Phi^*$. Then for all $X \in A$ we have

$$X^\flat P = PX^\flat.$$  

(70)

Proof: Let $V$ denote an irreducible $A$-module. Let $u$ denote a nonzero vector in $E_0^*V$ and recall $E_{0}^*u, E_{1}^*u, \ldots, E_{d}^*u$ is a $\Phi$-standard basis for $V$. By Definition 10.6, $X^\flat$ is the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents $X$ with respect to $E_{0}^*u, E_{1}^*u, \ldots, E_{d}^*u$. Similarly for a nonzero $v \in E_0^*V$, $X^\sharp$ is the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents $X$ with respect to $E_0^*v, E_1^*v, \ldots, E_d^*v$. In view of (68), the transition matrix from $E_0^*v, E_1^*v, \ldots, E_d^*v$ to $E_{0}^*u, E_{1}^*u, \ldots, E_{d}^*u$ is a scalar multiple of $P$. The result follows from these comments and elementary linear algebra. □

16 The orthogonality relations

In this section we show that each of the polynomial sequences $p_i, u_i, v_i$ satisfy an orthogonality relation. We begin with the $v_i$.

Theorem 16.1 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $v_i$ be as in Definition 13.1. Then both

$$\sum_{r=0}^{d} v_i(\theta_r)v_j(\theta_r)k_r^* = \delta_{ij}\nu k_i \quad (0 \leq i, j \leq d),$$

(71)

$$\sum_{i=0}^{d} v_i(\theta_r)v_i(\theta_s)k_i^{-1} = \delta_{rs}\nu k_i^{-1} \quad (0 \leq r, s \leq d).$$

(72)

Proof: We refer to Theorem 15.10. To obtain (71) compute the $ij$ entry in $P^*P = \nu I$ using matrix multiplication and evaluate the result using Theorem 14.8. To obtain (72) compute the $ij$ entry of $PP^* = \nu I$ using matrix multiplication and evaluate the result using Theorem 14.8. □

We now turn to the polynomials $u_i$.

Theorem 16.2 Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $u_i$ be as in Definition 14.1. Then both

$$\sum_{r=0}^{d} u_i(\theta_r)u_j(\theta_r)k_r^* = \delta_{ij}\nu k_i^{-1} \quad (0 \leq i, j \leq d),$$

$$\sum_{i=0}^{d} u_i(\theta_r)u_i(\theta_s)k_i = \delta_{rs}\nu k_i \quad (0 \leq r, s \leq d).$$
Proof: Evaluate each of (71), (72) using Lemma 14.4.

We now turn to the polynomials $p_i$.

**Theorem 16.3** Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $p_i$ be as in Definition 8.1. Then both

$$\sum_{r=0}^{d} p_i(\theta_r)p_j(\theta_r)m_r = \delta_{ij}x_1x_2\cdots x_i \quad (0 \leq i, j \leq d),$$

$$\sum_{r=0}^{d} \frac{p_i(\theta_r)p_i(\theta_s)}{x_1x_2\cdots x_i} = \delta_{rs}m_r^{-1} \quad (0 \leq r, s \leq d).$$

Proof: Applying Definition 12.1 to $\Phi^*$ we find $k_r^* = m_r\nu$ for $0 \leq r \leq d$. Evaluate each of (71), (72) using this and Definition 13.1, Lemma 11.2(i), (38).

\[\square\]

17  Everything in terms of the parameter array

In this section we express all the polynomials and scalars that came up so far in the paper, in terms of a short list of parameters called the parameter array. The parameter array of a Leonard system consists of its eigenvalue sequence, its dual eigenvalue sequence, and two additional sequences called the first split sequence and the second split sequence. The first split sequence is defined as follows. Let $\Phi$ denote the Leonard system from Definition 3.2. We showed in [9, Theorem 3.2] that there exists nonzero scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathbb{K}$ and there exists an isomorphism of $\mathbb{K}$-algebras $\mathcal{A} \to \text{Mat}_{d+1}(\mathbb{K})$ such that

$$\mathcal{A}^\sharp = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & 0 \\ 1 & \theta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & \theta_d \end{pmatrix}, \quad A^{\sharp^*} = \begin{pmatrix} \theta_0^* & \varphi_1^* & \varphi_2^* & \cdots & 0 \\ \varphi_1^* & \theta_1^* & \varphi_2^* & \cdots & 0 \\ \varphi_2^* & \theta_2^* & \cdots & \varphi_d^* \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_d^* \end{pmatrix}, \quad (73)$$

where the $\theta_i, \theta_i^*$ are from Definition 5.1. The sequence $\sharp, \varphi_1, \varphi_2, \ldots, \varphi_d$ is uniquely determined by $\Phi$. We call the sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ the first split sequence of $\Phi$. We let $\phi_1, \phi_2, \ldots, \phi_d$ denote the first split sequence of $\Phi^{\sharp}$ and call this the second split sequence of $\Phi$. For notational convenience we define $\varphi_0 = 0$, $\varphi_{d+1} = 0$, $\phi_0 = 0$, $\phi_{d+1} = 0$.

**Definition 17.1** Let $\Phi$ denote the Leonard system from Definition 3.2. By the parameter array of $\Phi$ we mean the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$, where $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$ and $\varphi_1, \varphi_2, \ldots, \varphi_d$ (resp. $\phi_1, \phi_2, \ldots, \phi_d$) is the first split sequence (resp. second split sequence) of $\Phi$.

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We remark that two Leonard systems over $\mathbb{K}$ are isomorphic if and only if they have the same parameter array \cite[Theorem 1.9]{9}.

The following result shows that the parameter array behaves nicely with respect to the $D_4$ action given in Section 4.

**Theorem 17.2** \cite[Theorem 1.11]{9} Let $\Phi$ denote a Leonard system with parameter array $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$. Then (i)–(iii) hold below.

(i) The parameter array of $\Phi^*$ is $(\theta_i^*, \theta_i, i = 0..d; \varphi_j, \phi_{d-j+1}, j = 1..d)$.

(ii) The parameter array of $\Phi^\downarrow$ is $(\theta_i, \theta_{d-i}^*, i = 0..d; \varphi_{d-j+1}, \phi_{d-j+1}, j = 1..d)$.

(iii) The parameter array of $\Phi^\downarrow$ is $(\theta_{d-i}, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$.

For the rest of this paper we will use the following notation.

**Definition 17.3** Suppose we are given an integer $d \geq 0$ and two sequences of scalars $\theta_0, \theta_1, \ldots, \theta_d; \theta_0^*, \theta_1^*, \ldots, \theta_d^*$ taken from $\mathbb{K}$. Then for $0 \leq i \leq d+1$ we let $\tau_i, \tau_i^*, \eta_i, \eta_i^*$ denote the following polynomials in $\mathbb{K}[\lambda]$.

\[
\tau_i = \prod_{h=0}^{i-1} (\lambda - \theta_h), \quad \tau_i^* = \prod_{h=0}^{i-1} (\lambda - \theta_h^*),
\]

\[
\eta_i = \prod_{h=0}^{i-1} (\lambda - \theta_{d-h}), \quad \eta_i^* = \prod_{h=0}^{i-1} (\lambda - \theta_{d-h}^*).
\]

We observe that each of $\tau_i, \tau_i^*, \eta_i, \eta_i^*$ is monic with degree $i$.

**Theorem 17.4** Let $\Phi$ denote the Leonard system from Definition 3.2 and let $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ denote the corresponding parameter array. Let the polynomials $u_i$ be as in Definition 14.1. Then

\[
u^i = \sum_{h=0}^{i} \frac{\tau_i^*(\theta_h^*)}{\varphi_1 \varphi_2 \cdots \varphi_h} \tau_h (0 \leq i \leq d).
\]

We are using the notation \cite[14]{4}.

**Proof:** Let the integer $i$ be given. The polynomial $u_i$ has degree $i$ so there exists scalars $\alpha_0, \alpha_1, \ldots, \alpha_i$ in $\mathbb{K}$ such that

\[
u^i = \sum_{h=0}^{i} \alpha_h \tau_h.
\]
We show
\[ \alpha_h = \frac{\tau_h^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_h} \]  \hspace{1cm} (0 \leq h \leq i). \tag{78}

In order to do this we show \( \alpha_0 = 1 \) and \( \alpha_{h+1} \varphi_{h+1} = \alpha_h (\theta_i^* - \theta_h^*) \) for \( 0 \leq h \leq i - 1 \). We now show \( \alpha_0 = 1 \). We evaluate \( \tau_h(\theta_0) \) at \( \lambda = \theta_0 \) and find \( u_i(\theta_0) = \sum_{h=0}^{i} \alpha_h \tau_h(\theta_0) \). Recall \( u_i(\theta_0) = 1 \) by \( \{17\} \). Using \( \{74\} \) we find \( \tau_h(\theta_0) = 1 \) for \( h = 0 \) and \( \tau_h(\theta_0) = 0 \) for \( 1 \leq h \leq i \). From these comments we find \( \alpha_0 = 1 \). We now show \( \alpha_{h+1} \varphi_{h+1} = \alpha_h (\theta_i^* - \theta_h^*) \) for \( 0 \leq h \leq i - 1 \). Let \( V \) denote an irreducible \( \mathcal{A} \)-module. From \( \{73\} \) there exists a basis \( e_0, e_1, \ldots, e_d \) for \( V \) that satisfies \( (A - \theta_j I)e_j = e_{j+1} \) \( (0 \leq j \leq d - 1) \), \( (A - \theta_d I)e_d = 0 \) and \( (A^* - \theta_d^* I)e_j = \varphi_j e_{j-1} \) \( (1 \leq j \leq d) \), \( (A^* - \theta_0^* I)e_0 = 0 \). From the action of \( A \) on \( e_0, e_1, \ldots, e_d \) we find \( e_j = \tau_j(A)e_0 \) for \( 0 \leq j \leq d \). Observe \( A^* e_0 = \theta_0^* e_0 \) so \( e_0 \in E_0^* V \). Combining Theorem \( 8.3 \) and \( \{46\} \) we find \( u_i(A)E_0^* V = E_i^* V \). By this and since \( e_0 \in E_0^* V \) we find \( u_i(A)e_0 \in E_i^* V \). Apparently \( u_i(A)e_0 \) is an eigenvector for \( A^* \) with eigenvalue \( \theta_i^* \). We may now argue

\[
\begin{align*}
0 &= (A^* - \theta_i^* I)u_i(A)e_0 \\
&= (A^* - \theta_i^* I) \sum_{h=0}^{i} \alpha_h \tau_h(A)e_0 \\
&= (A^* - \theta_i^* I) \sum_{h=0}^{i} \alpha_h e_h \\
&= \sum_{h=0}^{h=i-1} e_h (\alpha_{h+1} \varphi_{h+1} - \alpha_h (\theta_i^* - \theta_h^*)).
\end{align*}
\]

By this and since \( e_0, e_1, \ldots, e_d \) are linearly independent we find \( \alpha_{h+1} \varphi_{h+1} = \alpha_h (\theta_i^* - \theta_h^*) \) for \( 0 \leq h \leq i - 1 \). Line \( \{78\} \) follows and the theorem is proved.

\[ \square \]

**Lemma 17.5** Let \( \Phi \) denote the Leonard system from Definition \( \{3.2\} \) and let \( (\theta_i, \theta_i^*, i = 0.d; \varphi_j, \phi_j, j = 1.d) \) denote the corresponding parameter array. Let the polynomials \( p_i \) be as in Definition \( \{8.7\} \). With reference to Definition \( \{17.3\} \) we have
\[
p_i(\theta_0) = \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\tau_i^*(\theta_i^*)} \]  \hspace{1cm} (0 \leq i \leq d). \tag{79}

**Proof:** In equation \( \{76\} \), each side is a polynomial of degree \( i \) in \( \lambda \). For the polynomial on the left in \( \{76\} \) the coefficient of \( \lambda^i \) is \( p_i(\theta_0) \) by \( \{46\} \) and since \( p_i \) is monic. For the polynomial on the right in \( \{76\} \) the coefficient of \( \lambda^i \) is \( \tau_i^*(\theta_i^*) (\varphi_1 \varphi_2 \cdots \varphi_i)^{-1} \). Comparing these coefficients we obtain the result.

\[ \square \]

**Theorem 17.6** Let \( \Phi \) denote the Leonard system from Definition \( \{3.3\} \) and let \( (\theta_i, \theta_i^*, i = 0.d; \varphi_j, \phi_j, j = 1.d) \) denote the corresponding parameter array. Let the polynomials \( p_i \) be as in Definition \( \{8.7\} \). Then with reference to Definition \( \{17.3\} \)
\[
p_i = \sum_{h=0}^{i} \frac{\varphi_1 \varphi_2 \cdots \varphi_i \tau_h^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_h \tau_i^*(\theta_i^*)} \tau_h \]  \hspace{1cm} (0 \leq i \leq d).
Proof: Observe \( p_i = p_i(\theta_0)u_i \) by (46). In this equation we evaluate \( p_i(\theta_0) \) using (79) and we evaluate \( u_i \) using (76). The result follows. □

Theorem 17.7 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let \((\theta_i, \theta^*_i; i = 0..d; \varphi_j, \phi_j; j = 1..d)\) denote the corresponding parameter array. Let the scalars \( b_i, c_i \) be as in Definition 11.1. Then with reference to Definition 17.3 the following (i), (ii) hold.

(i) \[ b_i = \frac{\tau^*_i(\theta^*_i)}{\tau^*_{i+1}(\theta^*_{i+1})} \] \[ (0 \leq i \leq d-1). \]

(ii) \[ c_i = \frac{\eta^*_{i-1}(\theta^*_i)}{\eta^*_{d-i+1}(\theta^*_{i-1})} \] \[ (1 \leq i \leq d). \]

Proof: (i) Evaluate (32) using Lemma 17.5.

(ii) Comparing the formulae for \( b_i, c_i \) given in Theorem 11.6 we find, with reference to Definition 4.1, that \( c_i = b^*_{d-i} \). Applying part (i) above to \( \Phi \downarrow \) and using Theorem 17.2(ii) we routinely obtain the result. □

Let \( \Phi \) denote the Leonard system from Definition 3.2 and let the scalars \( a_i \) be as in Definition 7.1. We mention two formulae that give \( a_i \) in terms of the parameter array of \( \Phi \). The first formula is obtained using Lemma 11.2(ii) and Theorem 17.7. The second formula is given in the following theorem. This theorem was proven in [9, Lemma 5.1]; however we give an alternate proof that we find illuminating.

Theorem 17.8 [9, Lemma 5.1] Let \( \Phi \) denote the Leonard system from Definition 3.2 and let \((\theta_i, \theta^*_i; i = 0..d; \varphi_j, \phi_j; j = 1..d)\) denote the corresponding parameter array. Let the scalars \( a_i \) be as in Definition 7.1. Then

\[ a_i = \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta^*_{i+1} - \theta^*_{i+1}} \] \[ (0 \leq i \leq d), \] \hspace{1cm} (80)

where we recall \( \varphi_0 = 0, \varphi_{d+1} = 0, \) and where \( \theta^*_{-1}, \theta^*_{d+1} \) denote indeterminates.

Proof: Let the polynomials \( p_0, p_1, \ldots, p_{d+1} \) be as in Definition 8.1 and recall these polynomials are monic. Let \( i \) be given and consider the polynomial

\[ \lambda p_i - p_{i+1}. \] \hspace{1cm} (81)

From (19) we find the polynomial (81) is equal to \( a_ip_i + x_ip_{i-1} \). Therefore the polynomial (81) has degree \( i \) and leading coefficient \( a_i \). In order to compute this leading coefficient, in (81) we evaluate each of \( p_i, p_{i+1} \) using Theorem 8.5(ii) and Theorem 17.6. By this method we routinely obtain (80). □

Theorem 17.9 Let \( \Phi \) denote the Leonard system from Definition 3.2 and let \((\theta_i, \theta^*_i; i = 0..d; \varphi_j, \phi_j; j = 1..d)\) denote the corresponding parameter array. Let the scalars \( x_i \) be as in Definition 7.1. Then with reference to Definition 17.3

\[ x_i = \frac{\varphi_i \eta_{i-1}^*(\theta_{i-1}^*) \eta_{d-i}^*(\theta_i^*)}{\tau^*_i(\theta^*_i) \eta_{d-i+1}^*(\theta_{i-1}^*)} \] \[ (1 \leq i \leq d). \] \hspace{1cm} (82)
Proof: Use $x_i = b_{i-1}c_i$ and Theorem 17.7.

**Theorem 17.10** Let $\Phi$ denote the Leonard system from Definition 3.2 and let $(\theta_i, \theta^*_i, i = 0..d; \phi_j, \phi_j, j = 1..d)$ denote the corresponding parameter array. Let the scalar $\nu$ be as in Definition 9.3. Then with reference to Definition 17.3,

$$\nu = \frac{\eta_d(\theta_0)\eta_d^*(\theta^*_0)}{\phi_1\phi_2\cdots\phi_d}. \quad (83)$$

Proof: Evaluate (34) using Theorem 17.7(ii).

**Theorem 17.11** Let $\Phi$ denote the Leonard system from Definition 3.2 and let $(\theta_i, \theta^*_i, i = 0..d; \phi_j, \phi_j, j = 1..d)$ denote the corresponding parameter array. Let the scalars $k_i$ be as in Definition 12.1. Then with reference to Definition 17.3,

$$k_i = \frac{\varphi_1\varphi_2\cdots\varphi_i\eta_d^*(\theta^*_0)}{\phi_1\phi_2\cdots\phi_i \tau_i^*(\theta^*_i)\eta_{d-i}^*(\theta^*_i)} \quad (0 \leq i \leq d). \quad (84)$$

Proof: Evaluate (88) using Theorem 17.7.

**Theorem 17.12** Let $\Phi$ denote the Leonard system from Definition 3.2 and let $(\theta_i, \theta^*_i, i = 0..d; \phi_j, \phi_j, j = 1..d)$ denote the corresponding parameter array. Let the scalars $m_i$ be as in Definition 9.1. Then with reference to Definition 17.3,

$$m_i = \frac{\varphi_1\varphi_2\cdots\varphi_i\phi_1\phi_2\cdots\phi_{d-i}}{\eta_d^*(\theta^*_0)\tau_i(\theta_i)\eta_{d-i}(\theta_i)} \quad (0 \leq i \leq d). \quad (85)$$

Proof: Applying Definition 12.1 to $\Phi^*$ we find $m_i = k_i^*\nu^{-1}$. We compute $k_i^*$ using Theorem 17.11 and Theorem 17.2(i). We compute $\nu$ using Theorem 17.10. The result follows.

18 Some polynomials from the Askey scheme

Let $\Phi$ denote the Leonard system from Definition 3.2 and let the polynomials $u_i$ be as in Definition 14.1. In this section we discuss how the $u_i$ fit into the Askey scheme [6], [3, p260]. Our argument is summarized as follows. In [16] we displayed 13 families of parameter arrays. By [16, Theorem 5.16] every parameter array is contained in at least one of these families. In (76) the $u_i$ are expressed as a sum involving the parameter array of $\Phi$. In [16, Examples 5.3-5.15] we evaluated this sum for the 13 families of parameter arrays. We found the corresponding $u_i$ form a class consisting of the $q$-Racah, $q$-Hahn, dual $q$-Hahn, $q$-Krawtchouk, dual $q$-Krawtchouk, quantum $q$-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito, and orphan polynomials. This class coincides with the terminating branch of the Askey scheme. We remark the Bannai/Ito polynomials...
can be obtained from the q-Racah polynomials by letting \( q \) tend to \(-1\) \cite[p.260]{3}. The orphan polynomials exist for diameter \( d = 3 \) and \( \text{Char}(\mathbb{K}) = 2 \) only \cite[Example 5.15]{16}. We will not reproduce all the details of our calculations here; instead we illustrate what is going on with some examples. We will consider two families of parameter arrays. For the first family the corresponding \( u_i \) will turn out to be some Krawtchouk polynomials. For the second family the corresponding \( u_i \) will turn out to be the q-Racah polynomials.

Our first example is associated with the Leonard pair \( (2) \). Let \( d \) denote a nonnegative integer and consider the following elements of \( \mathbb{K} \).

\[
\begin{align*}
\theta_i &= d - 2i, & \theta^*_i &= d - 2i & (0 \leq i \leq d), \\
\varphi_i &= -2i(d - i + 1), & \phi_i &= 2i(d - i + 1) & (1 \leq i \leq d).
\end{align*}
\]

(86) \hspace{1cm} (87)

In order to avoid degenerate situations we assume the characteristic of \( \mathbb{K} \) is zero or an odd prime greater than \( d \). By \cite[Theorem 1.9]{9} we find there exists a Leonard system \( \Phi \) over \( \mathbb{K} \) that has parameter array \( (\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d) \). Let the scalars \( a_i \) for \( \Phi \) be as in (14). Applying Theorem 17.8 to \( \Phi \) we find

\[
a_i = 0 \quad (0 \leq i \leq d).
\]

(88)

Let the scalars \( b_i, c_i \) for \( \Phi \) be as in Definition 11.1. Applying Theorem 17.7 to \( \Phi \) we find

\[
b_i = d - i, \quad c_i = i \quad (0 \leq i \leq d).
\]

(89)

Pick any integers \( i, j \) \((0 \leq i, j \leq d)\). Applying Theorem 17.4 to \( \Phi \) we find

\[
u_i(\theta_j) = \sum_{n=0}^{d} \frac{(-i)_n(-j)_n2^n}{(-d)_n n!},
\]

(90)

where

\[
(a)_n := a(a + 1)(a + 2)\cdots(a + n - 1) \quad n = 0, 1, 2, \ldots
\]

Hypergeometric series are defined in \cite[p. 3]{4}. From this definition we find the sum on the right in (90) is the hypergeometric series

\[
_{2}F_{1}\left( \begin{array}{c}
-i, -j \\
-d
\end{array} \right| 2 \right).
\]

(91)

A definition of the Krawtchouk polynomials can be found in \cite{1} or \cite{6}. Comparing this definition with (90), (91) we find the \( u_i \) are Krawtchouk polynomials but not the most general ones. Let the scalar \( \nu \) for \( \Phi \) be as in Definition 9.3. Applying Theorem 17.10 to \( \Phi \) we find \( \nu = 2^d \). Let the scalars \( k_i \) for \( \Phi \) be as in Definition 12.1. Applying Theorem 17.11 to \( \Phi \) we obtain a binomial coefficient

\[
k_i = \binom{d}{i} \quad (0 \leq i \leq d).
\]

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Let the scalars \( m_i \) for \( \Phi \) be as in Definition 9.1. Applying Theorem 17.2 to \( \Phi \) we find
\[
m_i = \left( \frac{d}{i} \right) 2^{-d} \quad (0 \leq i \leq d).
\]

We now give our second example. For this example the polynomials \( u_i \) will turn out to be the \( q \)-Racah polynomials. To begin, let \( d \) denote a nonnegative integer and consider the following elements in \( \mathbb{K} \).
\[
\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})/q^i, \quad (92)
\]
\[
\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})/q^i \quad (93)
\]
for \( 0 \leq i \leq d \), and
\[
\varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-1})(1 - r_1q^i)(1 - r_2q^i), \quad (94)
\]
\[
\phi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* \quad (95)
\]
for \( 1 \leq i \leq d \). We assume \( q, h, h^*, s, s^*, r_1, r_2 \) are nonzero scalars in the algebraic closure of \( \mathbb{K} \), and that \( r_1r_2 = ss^*q^{d+1} \). To avoid degenerate situations we assume none of \( q^i, r_1q^i, r_2q^i, s^*q^i/r^1, s^*q^i/r^2 \) is equal to 1 for \( 1 \leq i \leq d \) and neither of \( sq^i, s^*q^i \) is equal to 1 for \( 2 \leq i \leq 2d \). By [9, Theorem 1.9] there exists a Leonard system \( \Phi \) over \( \mathbb{K} \) that has parameter array \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\). Let the scalars \( b_i, c_i \) for \( \Phi \) be as in Definition 11.1. Applying Theorem 17.7 to \( \Phi \) we find
\[
b_0 = \frac{h(1 - q^{-d})(1 - r_1q)(1 - r_2q)}{1 - s^*q^2},
\]
\[
b_i = \frac{h(1 - q^{-d})(1 - s^*q^{i+1})(1 - r_1q^{i+1})(1 - r_2q^{i+1})}{(1 - s^*q^{2i+1})(1 - s^*q^{2i+2})} \quad (1 \leq i \leq d - 1),
\]
\[
c_i = \frac{h(1 - q^i)(1 - s^*q^{i+d+1})(r_1 - s^*q^i)(r_2 - s^*q^i)}{s^*q^{d}(1 - s^*q^{2i+1})} \quad (1 \leq i \leq d - 1),
\]
\[
c_d = \frac{h(1 - q^d)(r_1 - s^*q^d)(r_2 - s^*q^d)}{s^*q^d(1 - s^*q^{2d})}.
\]
Pick integers \( i, j \) (\( 0 \leq i, j \leq d \)). Applying Theorem 17.4 to \( \Phi \) we find
\[
u_i(\theta_j) = \sum_{n=0}^{d} \frac{(q^{-i}; q)_n(s^*q^{i+1}; q)_n(q^{-j}; q)_n(q^{-i+j}; q)_nq^n}{(r_1q; q)_n(r_2q; q)_n(1 - aq^n)} \quad (96)
\]
where
\[
(a; q)_n := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}) \quad n = 0, 1, 2 \ldots
\]
Basic hypergeometric series are defined in [4, p. 4]. From that definition we find the sum on the right in (96) is the basic hypergeometric series
\[
_{4} \phi_3 \left( q^{-i}, s^*q^{i+1}, q^{-j}, s^*q^{j+1} \left| \begin{array}{c} 
q^{-i}, q^{-j}, q, q^{-i+j} \\
q, q^{-i-b}, q, q^{-j-c}
\end{array} \right. \right) \quad (97)
\]
A definition of the $q$-Racah polynomials can be found in [2] or [6]. Comparing this definition with (93), (97) and recalling $r_1r_2 = sq^d$, we find the $q$-Racah polynomials. Let the scalar $\nu$ for $\Phi$ be as in Definition 8.3. Applying Theorem 17.10 to $\Phi$ we find

$$\nu = \frac{(sq^2;q)_d(s^2q^2;q)_d}{r_1^d q^d (sq/r_1;q)_d (s^4q^2/r_1;q)_d}.$$  

Let the scalars $k_i$ for $\Phi$ be as in Definition 12.1. Applying Theorem 17.11 to $\Phi$ we obtain

$$k_i = \frac{(r_1q;q)_i(r_2q;q)_i(q^{-d};q)_i(s^2q;q)_i(1-s^2q^{2i+1})}{s^i q^i(q;q)_i(s^2q/r_1;q)_i(q^2q^{d+2};q)_i(1-sq^{2i})} \quad (0 \leq i \leq d).$$

Let the scalars $m_i$ for $\Phi$ be as in Definition 9.1. Applying Theorem 17.12 to $\Phi$ we find

$$m_i = \frac{(r_1q;q)_i(r_2q;q)_i(q^{-d};q)_i(s^2q;q)_i(1-s^2q^{2i+1})}{s^{i+1} q^{i+1}(q;q)_i(s^2q/r_1;q)_i(q^2q^{d+2};q)_i(1-sq^{2i})\nu} \quad (0 \leq i \leq d).$$

19 A characterization of Leonard systems

In [3] Appendix A) we mentioned that the concept of a Leonard system can be viewed as a “linear algebraic version” of the polynomial system which D. Leonard considered in [7]. In that appendix we outlined a correspondence that supports this view but we gave no proof. In this section we provide the proof.

We recall some results from earlier in the paper. Let $\Phi$ denote the Leonard system from Definition 3.2. Let the polynomials $p_0, p_1, \ldots, p_{d+1}$ be as in Definition 8.1 and recall $p_0^*, p_1^*, \ldots, p_{d+1}^*$ are the corresponding polynomials for $\Phi^*$. For the purpose of this section, we call $p_0, p_1, \ldots, p_{d+1}$ the monic polynomial sequence (or MPS) of $\Phi$. We call $p_0^*, p_1^*, \ldots, p_{d+1}^*$ the dual MPS of $\Phi$. By Definition 8.1 we have

$$p_0 = 1, \quad p_0^* = 1, \quad (98)$$

$$\lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d), \quad (99)$$

$$\lambda p_i^* = p_{i+1}^* + a_i^* p_i^* + x_i^* p_{i-1}^* \quad (0 \leq i \leq d), \quad (100)$$

where $x_0, x_0^*, p_{-1}, p_{-1}^*$ are all zero, and where

$$a_i = \text{tr}(E_i^* A), \quad a_i^* = \text{tr}(E_i A^*) \quad (0 \leq i \leq d),$$

$$x_i = \text{tr}(E_i^* A E_{i-1} A), \quad x_i^* = \text{tr}(E_i A^* E_{i-1} A^*) \quad (1 \leq i \leq d).$$

By Lemma 7.2(iii) we have

$$x_i \neq 0, \quad x_i^* \neq 0 \quad (1 \leq i \leq d). \quad (101)$$

Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$, and recall

$$\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if} \quad i \neq j, \quad (0 \leq i, j \leq d). \quad (102)$$
By Theorem 8.5(ii) we have
\[ p_{d+1}(\theta_i) = 0, \quad p_i^*(\theta_i^*) = 0 \quad (0 \leq i \leq d). \] (103)

By Theorem 11.4 we have
\[ p_i(\theta_0) \neq 0, \quad p_i^*(\theta_0^*) \neq 0 \quad (0 \leq i \leq d). \] (104)

By Theorem 14.9 we have
\[ \frac{p_i(\theta_j)}{p_i(\theta_0)} = \frac{p_j^*(\theta_i^*)}{p_j^*(\theta_0^*)} \quad (0 \leq i, j \leq d). \] (105)

In the following theorem we show the equations (98)–(105) characterize the Leonard systems.

**Theorem 19.1** Let \( d \) denote a nonnegative integer. Given polynomials
\[ p_0, p_1, \ldots, p_{d+1}, \quad (106) \]
\[ p_0^*, p_1^*, \ldots, p_{d+1}^* \quad (107) \]
in \( \mathbb{K}[\lambda] \) satisfying (98)–(101) and given scalars
\[ \theta_0, \theta_1, \ldots, \theta_d, \quad (108) \]
\[ \theta_0^*, \theta_1^*, \ldots, \theta_d^* \quad (109) \]
in \( \mathbb{K} \) satisfying (103)–(105), there exists a Leonard system \( \Phi \) over \( \mathbb{K} \) that has MPS (106), dual MPS (107), eigenvalue sequence (108) and dual eigenvalue sequence (109). The system \( \Phi \) is unique up to isomorphism of Leonard systems.

**Proof:** We abbreviate \( V = \mathbb{K}^{d+1} \). Let \( A \) and \( A^* \) denote the following matrices in \( \text{Mat}_{d+1}(\mathbb{K}) \):
\[
A := \begin{pmatrix}
a_0 & x_1 & & & 0 \\
1 & a_1 & x_2 & & \\
& 1 & & \ddots & \\
& & \ddots & \ddots & x_d \\
0 & & & 1 & a_d
\end{pmatrix}, \quad A^* := \text{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*).
\]

We show the pair \( A, A^* \) is a Leonard pair on \( V \). To do this we apply Definition 1.1. Observe that \( A \) is irreducible tridiagonal and \( A^* \) is diagonal. Therefore condition (i) of Definition 1.1 is satisfied by the basis for \( V \) consisting of the columns of \( I \), where \( I \) denotes the identity matrix in \( \text{Mat}_{d+1}(\mathbb{K}) \). To verify condition (ii) of Definition 1.1 we display an invertible matrix \( X \) such that \( X^{-1}AX \) is diagonal and \( X^{-1}A^*X \) is irreducible tridiagonal. Let \( X \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{K}) \) that has entries
\[
X_{ij} = \frac{p_i(\theta_j)p_j^*(\theta_0^*)}{x_1x_2 \cdots x_i} \quad (110)
\]
\[
= \frac{p_j^*(\theta_i^*)p_i(\theta_0)}{x_1x_2 \cdots x_i} \quad (111)
\]

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0 \leq i, j \leq d$. The matrix $X$ is invertible since it is essentially Vandermonde. Using (99) and (110) we find $AX = XH$ where $H = \text{diag}(\theta_0, \theta_1, \ldots, \theta_d)$. Apparently $X^{-1}AX$ is equal to $H$ and is therefore diagonal. Using (100) and (111) we find $A^*X = XH^*$ where

$$
H^* := \begin{pmatrix}
  a_0^* & x_1^* & 0 \\
  1 & a_1^* & x_2^* \\
  & 1 & \ddots \\
  & & \ddots & \ddots \\
  0 & & & 1 & a_d^*
\end{pmatrix}.
$$

Apparently $X^{-1}A^*X$ is equal to $H^*$ and is therefore irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of $X$. We have now shown the pair $A, A^*$ is a Leonard pair on $V$. Pick an integer $j$ $(0 \leq j \leq d)$. Using $X^{-1}AX = H$ we find $\theta_j$ is the eigenvalue of $A$ associated with column $j$ of $X$. From the definition of $A^*$ we find $\theta_j^*$ is the eigenvalue of $A^*$ associated with column $j$ of $I$. Let $E_j$ (resp. $E_j^*$) denote the primitive idempotent of $A$ (resp. $A^*$) for $\theta_j$ (resp. $\theta_j^*$). From our above comments the sequence $\Phi := (A; A^*; \{E_j\}_{j=0}^d; \{E_j^*\}_{j=0}^d)$ is a Leonard system. From the construction $\Phi$ is over $\mathbb{K}$. We show (106) is the $MPS$ of $\Phi$. To do this is suffices to show $a_i = \text{tr}(E_i^*A)$ for $0 \leq i \leq d$ and $x_i = \text{tr}(E_i^*AE_{i-1}^*)$ for $1 \leq i \leq d$. Applying Lemma 7.2(i),(ii) to $\Phi$ (with $v_i =$ column $i$ of $I$, $B = A$) we find $a_i = \text{tr}(E_i^*A)$ for $0 \leq i \leq d$ and $x_i = \text{tr}(E_i^*AE_{i-1}^*)$ for $1 \leq i \leq d$. Therefore (106) is the $MPS$ of $\Phi$. We show (107) is the dual $MPS$ of $\Phi$. Applying Lemma 7.2(i),(ii) to $\Phi^*$ (with $v_i =$ column $i$ of $X$, $B = H^*$) we find $a_i^* = \text{tr}(E_iA^*)$ for $0 \leq i \leq d$ and $x_i^* = \text{tr}(E_iAE_{i-1}A^*)$ for $1 \leq i \leq d$. Therefore (107) is the dual $MPS$ of $\Phi$. From the construction we find (108) (resp. (109)) is the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. We show $\Phi$ is uniquely determined by (106)–(109) up to isomorphism of Leonard systems. Recall that $\Phi$ is determined up to isomorphism of Leonard systems by its own parameter array. We show the parameter array of $\Phi$ is determined by (106)–(109). Recall the parameter array consists of the eigenvalue sequence, the dual eigenvalue sequence, the first split sequence and the second split sequence. We mentioned earlier that the eigenvalue sequence of $\Phi$ is (108) and the dual eigenvalue sequence of $\Phi$ is (109). By Lemma 17.5 the first split sequence of $\Phi$ is determined by (106)–(109). By this and Theorem 17.9 we find the second split sequence of $\Phi$ is determined by (106)–(109). We have now shown the parameter array of $\Phi$ is determined by (106)–(109). We now see that $\Phi$ is uniquely determined by (106)–(109) up to isomorphism of Leonard systems.

\[\Box\]

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References

[1] G.E. Andrews, R. Askey, R. Roy. Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.

[2] R. Askey and J.A. Wilson. A set of orthogonal polynomials that generalize the Racah coefficients or 6 − j symbols. SIAM J. Math. Anal., 10:1008–1016, 1979.

[3] E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin/Cummings, London, 1984.

[4] G. Gasper and M. Rahman. Basic hypergeometric series. Encyclopedia of Mathematics and its Applications, 35. Cambridge University Press, Cambridge, 1990.

[5] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes. Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 56:167–192, 2001.

[6] R. Koekoek and R. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analog, volume 98-17 of Reports of the faculty of Technical Mathematics and Informatics. Delft, The Netherlands, 1998.

[7] D. Leonard. Orthogonal polynomials, duality, and association schemes. SIAM J. Math. Anal., 13(4):656–663, 1982.

[8] J. J. Rotman. Advanced modern algebra. Prentice Hall, Saddle River NJ 2002.

[9] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. Linear Algebra Appl., 330:149–203, 2001.

[10] P. Terwilliger. Two relations that generalize the q-Serre relations and the Dolan-Grady relations. Physics and combinatorics 1999 (Nagoya), World Scientific Publishing, River Edge, NJ, 2001

[11] P. Terwilliger. Leonard pairs from 24 points of view. Rocky mountain Journal of Mathematics., 32(2):827–888, 2002.

[12] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other: the $TD-D$ and the $LB-UB$ canonical form. Preprint.

[13] P. Terwilliger. Introduction to Leonard pairs. OPSFA Rome 2001. J. Comput. Appl. Math., 153(2):463–475, 2003.

[14] P. Terwilliger. Introduction to Leonard pairs and Leonard systems. Sūrikaisekikenkyūsho Kōkyūroku, (1109):67–79, 1999. Algebraic combinatorics (Kyoto, 1999).

[15] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other: comments on the split decomposition. Preprint.
[16] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other: comments on the parameter array. *Geometric and Algebraic Combinatorics* 2, Oisterwijk, The Netherlands 2002. Submitted.

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