SOME GENERALIZATIONS OF DELAY INTEGRAL INEQUALITIES OF GRONWALL-BELLMAN TYPE WITH POWER AND THEIR APPLICATIONS

HAOYUE SONG AND FANWEI MENG*

School of Mathematical Sciences, Qufu Normal University
Qufu 273165, Shandong, China

(Communicated by Ting Hu)

Abstract. Noting the diverse generalizations of the Gronwall-Bellman inequality, this paper investigates some new delay integral inequalities with power, deriving explicit bound on the solution and providing an example. The inequalities given here can act as powerful tools for studying qualitative properties such as existence, uniqueness, boundedness, stability and asymptotics of solutions of differential and integral equations.

1. Introduction. In recent years, many scholars have studied the Gronwall-Bellman inequality and its applications [1-15]. In [5], Gronwall established the well-known Gronwall inequality. In [10], Pachpatte extended this to nonlinear forms. In [7-9], some scholars studied the retarded integral inequalities. In [11-15], a variety of generalizations have been researched by many scholars. In particular, a recent paper [9] established a class of delay integral inequality (see (1.1)). In [12], some retarded integral inequalities with power were investigated (see (1.2) and (1.3)).

\begin{equation}
\begin{aligned}
  u(t) &\leq a(t) + \int_{0}^{\alpha(t)} \left[ f(s) \left[ u^m(s) + \int_{0}^{s} g(\tau)u^n(\tau)d\tau \right]^p \right] ds, \\
&\quad \text{where } 0<m \leq 1, 0<n \leq 1, 0<p \leq 1.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
  u(t) &\leq a(t) + \int_{0}^{\alpha(t)} \left[ b(s) \left[ u^m(s) + \int_{0}^{s} c(\xi)u^n(\xi)d\xi \right]^p \right] ds, \\
&\quad \text{where } 0<m \leq 1, 0<n \leq 1, p>1.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
  u^q(t) &\leq a(t) + \int_{0}^{\alpha(t)} \left[ b(s) \left[ u^m(s) + \int_{0}^{s} c(\xi)u^n(\xi)d\xi \right]^p \right] ds, \\
&\quad \text{where } q \geq m > 0, q \geq n > 0, p>0.
\end{aligned}
\end{equation}

In this paper, we make a further extension of the inequalities in [9, 12], investigate some new delay integral inequalities with power, obtain upper bound estimations for the solutions of these inequalities, and give an example.

2020 Mathematics Subject Classification. Primary: 39A12; Secondary: 26A33.

Key words and phrases. Nonlinear integral inequality, delay, power, boundedness, differential equation.

* Corresponding author: Fanwei Meng.
2. Main results. In this paper, $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$, $C(D, E)$ and $C^1(D, E)$ denote the set of all continuous functions from $D$ to $E$ and the set of all continuous differentiable functions, respectively.

**Lemma 2.1.** ([7]) Let $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$. Then $a^{\frac{p}{q}} \leq \frac{2}{p}a + \frac{p-q}{p}$.

Especially, if $a \geq 0$, $0 \leq m \leq 1$, then $a^m \leq ma + (1-m)$.

**Lemma 2.2.** ([12]) Let $u, v \geq 0$ and $p \geq 0$. Then $(u + v)^p \leq K_p(u^p + v^p)$, where

$$K_p = \begin{cases} 1 & (0 \leq p \leq 1) \\ 2^{p-1} & (p > 1). \end{cases}$$

**Theorem 2.3.** Suppose that $u, a, b, c, d, e, f, g \in C(R^+, R^+)$, $g(t)$ is nondecreasing on $R^+$. Let $\alpha \in C^1(R^+, R^+)$ be nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$. Let $p, q > 0$ be constants. If $u(t)$ satisfies the inequality

$$u(t) \leq g(t) + \int_0^{\alpha(t)} b(s)[a(s)u(s) + c(s)]^p ds + \int_0^t d(s)[e(s)u(s) + f(s)]^q ds,$$

then we have the following results:

(i) When $0 < p \leq 1$, $0 < q \leq 1$, we obtain

$$u(t) \leq g(t) + G(t) + \exp \left( \int_0^t h(s) ds \right) \int_0^t k(s) \exp \left( - \int_0^s h(\tau) d\tau \right) ds. \quad (2.2)$$

(ii) When $0 < p \leq 1$, $q > 1$, and

$$B_1^{1-q}(t) + \int_0^t C_1(s) ds > 0,$$

we obtain

$$u(t) \leq A_1(t) + \exp \left( \int_0^{\alpha(t)} p b(s) a^p(s) ds \right) \left( B_1^{1-q}(t) + \int_0^t C_1(s) ds \right)^{\frac{1}{1-q}}. \quad (2.4)$$

(iii) When $0 < q \leq 1$, $p > 1$, and

$$D_1^{1-p}(t) + \int_0^t E_1(s) ds > 0,$$

we obtain

$$u(t) \leq A_2(t) + \exp \left( \int_0^t q d(s) e^q(s) ds \right) \left( D_1^{1-p}(t) + \int_0^t E_1(s) ds \right)^{\frac{1}{1-p}}, \quad (2.6)$$

where

$$A_1(t) = g(t) + \int_0^{\alpha(t)} b(s)c^p(s)ds + \int_0^t 2^{q-1}d(s)f^q(s)ds,$$

$$A_2(t) = g(t) + \int_0^{\alpha(t)} 2^{p-1}b(s)e^p(s)ds + \int_0^t d(s)f^q(s)ds,$$

$$B_1(t) = \int_0^{\alpha(t)} [pb(s)a^p(s)A_1(s) + (1-p)b(s)a^p(s)]ds + \int_0^t 2^{2(q-1)}d(s)e^q(s)A_1^q(s)ds,$$

$$C_1(t) = (1-q)2^{2(q-1)}d(t)e^q(t)\exp \left( \int_0^{\alpha(t)} (q-1)pb(s)a^p(s)ds \right).$$
\[ D_1(t) = \int_0^{\alpha(t)} 2^{(p-1)}b(s)a^p(s)A_2^2(s)ds + \int_0^t [qd(s)e^q(s)A_2(s) + (1 - q)d(s)e^q(s)]ds, \]
\[ E_1(t) = (1 - p)2^{(p-1)}\alpha'(t)b(\alpha(t))a^p(\alpha(t)) \exp \left( \int_0^t (p - 1)qd(s)e^q(s)ds \right). \]
\[ G(t) = \int_0^{\alpha(t)} [pb(s)(a(s)g(s) + c(s)) + (1 - p)b(s)]ds \]
\[ + \int_0^t [qd(s)(e(s)g(s) + f(s)) + (1 - q)d(s)]ds, \]
\[ h(t) = p\alpha'(t)b(\alpha(t))a(\alpha(t)) + qd(t)e(t), \]
\[ k(t) = p\alpha'(t)b(\alpha(t))a(\alpha(t))G(\alpha(t)) + qd(t)e(t)G(t). \]

(2.7)

**Proof.** (i) When \(0 < p \leq 1, 0 < q \leq 1\), denoting
\[ w(t) = \int_0^{\alpha(t)} b(s)[a(s)u(s) + c(s)]^p + \int_0^t d(s)[e(s)u(s) + f(s)]^q ds, \]
then \(w(t)\) is a nondecreasing function on \(R^+\), and
\[ u(t) \leq g(t) + w(t). \]

(2.9)

Applying Lemma 2.1 to (2.8) and substituting (2.9) into it, we get
\[ w(t) \leq \int_0^{\alpha(t)} [pb(s)(a(s)u(s) + c(s)) + (1 - p)b(s)]ds \]
\[ + \int_0^t [qd(s)(e(s)u(s) + f(s)) + (1 - q)d(s)]ds \]
\[ \leq \int_0^{\alpha(t)} [pb(s)(a(s)(g(s) + w(s)) + c(s)) + (1 - p)b(s)]ds \]
\[ + \int_0^t [qd(s)(e(s)(g(s) + w(s)) + f(s)) + (1 - q)d(s)]ds \]
\[ \leq G(t) + \int_0^{\alpha(t)} pb(s)a(s)w(s)ds + \int_0^t qd(s)e(s)w(s)ds, \]

(2.10)

where the definition of \(G(t)\) is given by (2.7).

Let
\[ v(t) = \int_0^{\alpha(t)} pb(s)a(s)w(s)ds + \int_0^t qd(s)e(s)w(s)ds, \]

(2.11)

then \(v(t)\) is a nondecreasing function on \(R^+\), and
\[ v(0) = 0, w(t) \leq G(t) + v(t), v(\alpha(t)) \leq v(t). \]

(2.12)

Differentiating (2.11), from (2.12), we have
\[ v'(t) = p\alpha'(t)b(\alpha(t))a(\alpha(t))w(\alpha(t)) + qd(t)e(t)w(t) \]
\[ \leq p\alpha'(t)b(\alpha(t))a(\alpha(t))(G(\alpha(t)) + v(\alpha(t))) + qd(t)e(t)(G(t) + v(t)) \]
\[ \leq p\alpha'(t)b(\alpha(t))a(\alpha(t))(G(\alpha(t)) + v(t)) + qd(t)e(t)(G(t) + v(t)) \]
\[ \leq h(t)v(t) + k(t), \]

(2.13)

where the definitions of \(h(t)\) and \(k(t)\) are given by (2.7).
From (2.9), (2.12) and (2.14), we obtain

\[ v(t) \leq \exp \left( \int_0^t h(s) ds \right) \int_0^t k(s) \exp \left( - \int_0^s h(\tau) d\tau \right) ds. \]  

(2.14)

Multiplying (2.13) by \( \exp \left( \int_0^t h(s) ds \right) \), and integrating it from 0 to \( t \), we obtain

\[ u(t) \leq g(t) + G(t) + \exp \left( \int_0^t h(s) ds \right) \int_0^t k(s) \exp \left( - \int_0^s h(\tau) d\tau \right) ds. \]

From (2.9), (2.12) and (2.14), we obtain

\[ u(t) \leq g(t) + G(t) + \exp \left( \int_0^t h(s) ds \right) \int_0^t k(s) \exp \left( - \int_0^s h(\tau) d\tau \right) ds. \]

(ii) When \( 0 < p \leq 1, q > 1 \), applying Lemma 2.2 to (2.1), we get

\[ u(t) \leq g(t) + \int_0^\alpha b(s) [a^p(s)u^p(s) + c^q(s)] ds + \int_0^t 2^{q-1} d(s) [e^q(s)u^q(s) + f^q(s)] ds \]

\[ = A_1(t) + \int_0^\alpha b(s) a^p(s) u^p(s) ds + \int_0^t 2^{q-1} d(s) e^q(s) u^q(s) ds, \]

(2.15)

where the definition of \( A_1(t) \) is given by (2.7).

Define a function \( w(t) \) on \( R^+ \) by

\[ w(t) = \int_0^\alpha b(s) a^p(s) u^p(s) ds + \int_0^t 2^{q-1} d(s) e^q(s) u^q(s) ds, \]

then \( w(t) \) is a nondecreasing function, and

\[ u(t) \leq A_1(t) + w(t). \]

(2.17)

Applying Lemma 2.1 and Lemma 2.2 to (2.17), from (2.16), we have

\[ w(t) \leq \int_0^\alpha b(s) a^p(s) \left( A_1(s) + w(s) \right)^p ds + \int_0^t 2^{q-1} d(s) e^q(s) \left( A_1(s) + w(s) \right)^q ds \]

\[ \leq \int_0^\alpha b(s) a^p(s) \left[ p \left( A_1(s) + w(s) \right) + (1 - p) \right] ds \]

\[ + \int_0^t 2^{q-1} d(s) e^q(s) \left[ 2^{q-1} \left( A_1^q(s) + w^q(s) \right) \right] ds \]

\[ = B_1(t) + \int_0^\alpha b(s) a^p(s) w(s) ds + \int_0^t 2^{2(q-1)} d(s) e^q(s) w^q(s) ds, \]

(2.18)

where the definition of \( B_1(t) \) is given by (2.7).

Let \( T > 0 \) be any fixed constant on \( R^+ \). Since \( B_1(t) \) is a nondecreasing function, for all \( t \in [0, T] \), from (2.18), we get

\[ w(t) \leq B_1(T) + \int_0^\alpha b(s) a^p(s) w(s) ds + \int_0^t 2^{2(q-1)} d(s) e^q(s) w^q(s) ds. \]

(2.19)

Set \( v(t) \) as the right-hand side of the above inequality, we get

\[ v(t) = B_1(T) + \int_0^\alpha b(s) a^p(s) w(s) ds + \int_0^t 2^{2(q-1)} d(s) e^q(s) w^q(s) ds, \]

(2.20)

then \( v(t) \) is a nondecreasing function, and

\[ v(0) = B_1(T), w(\alpha(t)) \leq w(t) \leq v(t). \]

(2.21)
Differentiating \(v(t)\) with respect to \(t\), and using (2.21), we obtain
\[
v'(t) = p\alpha'(t)b(\alpha(t))a^p(\alpha(t))w(\alpha(t)) + 2^{(q-1)}d(t)c^q(t)w^q(t) \\
\leq p\alpha'(t)b(\alpha(t))a^p(\alpha(t))v(t) + 2^{(q-1)}d(t)c^q(t)v^q(t), t \in [0, T].
\] (2.22)

Consider the following initial value problem for Bernoulli differential equation
\[
\frac{dy(t)}{dt} = p\alpha'(t)b(\alpha(t))a^p(\alpha(t))y(t) + 2^{(q-1)}d(t)c^q(t)y^q(t), y(0) = B_1(T),
\] (2.23)
then we obtain a unique solution to this Bernoulli equation:
\[
y(t) = \exp\left(\int_0^{\alpha(t)} pb(s)a^p(s)ds\right) \left(B_1^{1-q}(T) + \int_0^t C_1(s)ds\right)^{\frac{1}{1-q}}, t \in [0, T],
\] (2.24)
where the definition of \(C_1(t)\) is given by (2.7).

Let \(t = T\), we get
\[
y(T) = \exp\left(\int_0^{\alpha(T)} pb(s)a^p(s)ds\right) \left(B_1^{1-q}(T) + \int_0^T C_1(s)ds\right)^{\frac{1}{1-q}}.
\] (2.25)

According to the comparison theorem for ordinary differential equations and condition (2.3), we have
\[
v(T) \leq y(T) = \exp\left(\int_0^{\alpha(T)} pb(s)a^p(s)ds\right) \left(B_1^{1-q}(T) + \int_0^T C_1(s)ds\right)^{\frac{1}{1-q}}.
\] (2.26)

From (2.17), (2.21) and (2.26), we get
\[
u(t) \leq A_1(t) + \exp\left(\int_0^{\alpha(t)} pb(s)a^p(s)ds\right) \left(B_1^{1-q}(t) + \int_0^t C_1(s)ds\right)^{\frac{1}{1-q}}, t \in R^+.
\] (2.27)

By the arbitrariness of \(T\), we obtain
\[
u(t) \leq A_1(t) + \exp\left(\int_0^{\alpha(t)} pb(s)a^p(s)ds\right) \left(B_1^{1-q}(t) + \int_0^t C_1(s)ds\right)^{\frac{1}{1-q}}, t \in R^+.
\]

(iii) When \(0 < q \leq 1, p>1\), the proof is similar to the second case.

The proof of Theorem 2.3 is completed. \(\square\)

**Theorem 2.4.** Let \(u, a, b, c, d, e \in \mathcal{C}(R^+, R^+)\), and let \(a \in \mathcal{C}^1(R^+, R^+)\) be nondecreasing function with \(\alpha(t) \leq t\), \(\alpha(0) = 0\). Suppose that \(m, n, l, r \in (0, 1]\), \(p, q>0\) are constants. If \(u(t)\) satisfies the inequality
\[
u(t) \leq a(t) + \int_0^{\alpha(t)} b(s)\left[u^m(s) + \int_0^s c(\tau)u^n(\tau)d\tau\right]^p ds \\
+ \int_0^t d(s)\left[u^l(s) + \int_0^s e(\tau)u^r(\tau)d\tau\right]^q ds,
\] (2.28)
then we have the following results:
(i) When \(0 < p \leq 1, 0 < q \leq 1\), we obtain
\[
u(t) \leq a(t) + G_1(t) + \exp\left(\int_0^t h_1(s)ds\right) \int_0^t k_1(s) \exp\left(-\int_0^s h_1(\tau)d\tau\right)ds.
\] (2.29)
(ii) When \(0 < p \leq 1, q > 1\), and
\[
B_2^{1-q}(t) + \int_0^t C_2(s)ds > 0,
\] (2.30)
we obtain
\[ u(t) \leq a(t) + A_3(t) + \exp \left( \int_0^t pb(s) f^p_1(s) ds \right) \left( B^1_2(t) + \int_0^t C_2(s) ds \right)^{1/p}. \] (2.31)

(iii) When \( 0 < q \leq 1, \ p > 1, \) and
\[ D^1_2(t) + \int_0^t E_2(s) ds > 0, \] we obtain
\[ u(t) \leq a(t) + A_4(t) + \exp \left( \int_0^t q d(s) g^q_1(s) ds \right) \left( D^1_2(t) + \int_0^t E_2(s) ds \right)^{1/q}. \] (2.32)

where
\[ A_3(t) = \int_0^t b(s) \left[ ma(s) + 1 - m + \int_0^s c(\tau) (na(\tau) + 1 - n) d\tau \right]^p ds \]
\[ + \int_0^t 2^{q-1} d(s) \left[ la(s) + 1 - l + \int_0^s c(\tau) (ra(\tau) + 1 - r) d\tau \right]^q ds, \]
\[ A_4(t) = \int_0^t 2^{p-1} b(s) \left[ ma(s) + 1 - m + \int_0^s c(\tau) (na(\tau) + 1 - n) d\tau \right]^p ds \]
\[ + \int_0^t d(s) \left[ la(s) + 1 - l + \int_0^s c(\tau) (ra(\tau) + 1 - r) d\tau \right]^q ds, \]
\[ B_2(t) = \int_0^t \left[ pb(s) f^p_1(s) A_3(t) + (1 - p)b(s) f^p_1(s) \right] ds + \int_0^t 2^{2(q-1)} d(s) g^q_1(s) A^q_3(s) ds, \]
\[ C_2(t) = (1 - q) 2^{2(q-1)} d(t) g^q_1(t) \exp \left( \int_0^t (q - 1) pb(s) f^p_1(s) ds \right), \]
\[ D_2(t) = \int_0^t 2^{2(p-1)} b(s) f^p_1(s) A^p_3(s) ds + \int_0^t [qd(s) g^q_1(s) A_4(s) + (1 - q) d(s) g^q_1(s)] ds, \]
\[ E_2(t) = (1 - p) 2^{2(p-1)} q \alpha(t) b(\alpha(t)) f^p_1(\alpha(t)) \exp \left( \int_0^t ((p - 1) q d(s) g^q_1(s) ds \right), \]
\[ G_1(t) = \int_0^t \left[ pb(s) \left( ma(s) + 1 - m + \int_0^s c(\tau) (na(\tau) + 1 - n) d\tau \right) + (1 - p)b(s) \right] ds \]
\[ + \int_0^t \left[ q d(s) \left( la(s) + 1 - l + \int_0^s c(\tau) (ra(\tau) + 1 - r) d\tau \right) + (1 - q) d(s) \right] ds, \]
\[ f_1(t) = m + n \int_0^t c(s) ds, g_1(t) = l + r \int_0^t e(s) ds, \]
\[ h_1(t) = \rho(\alpha(t)) f_1(\alpha(t)) + q d(t) g_1(t), \]
\[ k_1(t) = \rho(\alpha(t)) f_1(\alpha(t)) G_1(\alpha(t)) + q d(t) g_1(t) G_1(t). \] (2.34)
Proof. Denoting

\[ w(t) = \int_0^{\alpha(t)} b(s) \left[ u^m(s) + \int_0^s c(\tau)u^n(\tau)d\tau \right]^p ds \]

then \( w(t) \) is a nondecreasing function on \( R^+ \), and

\[ u(t) \leq a(t) + w(t). \]

Applying Lemma 2.1 to (2.36), we obtain

\[ u^m(t) \leq (a(t) + w(t))^m \leq m(a(t) + w(t)) + 1 - m, \]

\[ u^n(t) \leq (a(t) + w(t))^n \leq n(a(t) + w(t)) + 1 - n, \]

\[ u'(t) \leq (a(t) + w(t))' \leq l(a(t) + w(t)) + 1 - l, \]

\[ u''(t) \leq (a(t) + w(t))'' \leq r(a(t) + w(t)) + 1 - r. \]  

Substituting (2.37) into (2.35), we get

\[ \begin{align*}
    w(t) & \leq \int_0^{\alpha(t)} b(s) \left[ (m(a(s) + w(s)) + 1 - m) + \int_0^s c(\tau)(n(a(\tau) + w(\tau)) + 1 - n)d\tau \right]^p ds \\
    & \quad + \int_0^t d(s) \left[ (l(a(s) + w(s)) + 1 - l) + \int_0^s c(\tau)(r(a(\tau) + w(\tau)) + 1 - r)d\tau \right]^q ds,
\end{align*} \]

that is,

\[ \begin{align*}
    w(t) & \leq \int_0^{\alpha(t)} b(s) \left[ f_1(s)w(s) + ma(s) + 1 - m + \int_0^s c(\tau)na(\tau) + 1 - n)d\tau \right]^p ds \\
    & \quad + \int_0^t d(s) \left[ g_1(s)w(s) + la(s) + 1 - l + \int_0^s c(\tau)ra(\tau) + 1 - r)d\tau \right]^q ds,
\end{align*} \]

where the definitions of \( f_1(s) \) and \( g_1(s) \) are given by (2.34).

Inequality (2.38) has the same form as inequality (2.1) of Theorem 2.3. By Theorem 2.3, condition (2.30) and condition (2.32), we can obtain the results.

The proof of Theorem 2.4 is completed. \( \square \)

Remark 1. When \( d(t) \equiv 0 \) and \( 0 < p \leq 1 \), Theorem 2.4 is converted into Theorem 2.1 in [9]. When \( d(t) \equiv 0 \) and \( p > 1 \), Theorem 2.4 is converted into Theorem 2.1 in [12].

Theorem 2.5. Let \( u, a, b, c, d, e \in C(R^+, R^+) \), and let \( \alpha \in C^1(R^+, R^+) \) be nondecreasing function with \( \alpha(t) \leq t, \alpha(0) = 0 \). Suppose that \( k \geq m > 0, k \geq n > 0, k \geq l > 0, k \geq r > 0, p, q > 0 \) are constants. If \( u(t) \) satisfies the inequality

\[ \begin{align*}
    u^k(t) & \leq a(t) + \int_0^{\alpha(t)} b(s) \left[ u^m(s) + \int_0^s c(\tau)u^n(\tau)d\tau \right]^p ds \\
    & \quad + \int_0^t d(s) \left[ u'(s) + \int_0^s e(\tau)u''(\tau)d\tau \right]^q ds,
\end{align*} \]

we obtain...
we have the following results:

(i) When $0 < p, 1, 0 < q \leq 1$, we have

$$u(t) \leq \left[ a(t) + G_2(t) + \exp \left( \int_0^t h_2(s)ds \right) \int_0^t k_2(s) \exp \left( -\int_0^t h_2(\tau)d\tau \right) ds \right]^{\frac{1}{p}}. \quad (2.40)$$

(ii) When $0 < p \leq 1, q > 1$, and

$$B_3^{1-q}(t) + \int_0^t C_3(s)ds > 0, \quad (2.41)$$

we have

$$u(t) \leq \left[ a(t) + A_5(t) + \exp \left( \int_0^t pb(s) f_2^p(s) ds \right) \left( B_3^{1-q}(t) + \int_0^t C_3(s)ds \right) \right]^{\frac{1}{q}}. \quad (2.42)$$

(iii) When $0 < q \leq 1, p > 1$, and

$$D_3^{1-p}(t) + \int_0^t E_3(s)ds > 0, \quad (2.43)$$

we have

$$u(t) \leq \left[ a(t) + A_6(t) + \exp \left( \int_0^t qd(s) g_2^q(s) ds \right) \left( D_3^{1-p}(t) + \int_0^t E_3(s)ds \right) \right]^{\frac{1}{p}}. \quad (2.44)$$

where

$$A_5(t) = \int_0^{\alpha(t)} b(s) \left[ \frac{m}{k} a(s) + \frac{k - m}{k} + \int_0^s c(\tau) \left( \frac{n}{k} a(\tau) + \frac{k - n}{k} \right) d\tau \right] p \, ds + \int_0^t 2^{q-1} d(s) \left[ \frac{l}{k} a(s) + \frac{k - l}{k} + \int_0^s e(\tau) \left( \frac{r}{k} a(\tau) + \frac{k - r}{k} \right) d\tau \right] q \, ds,$$

$$A_6(t) = \int_0^{\alpha(t)} 2^{p-1} b(s) \left[ \frac{m}{k} a(s) + \frac{k - m}{k} + \int_0^s c(\tau) \left( \frac{n}{k} a(\tau) + \frac{k - n}{k} \right) d\tau \right] p \, ds + \int_0^t d(s) \left[ \frac{l}{k} a(s) + \frac{k - l}{k} + \int_0^s e(\tau) \left( \frac{r}{k} a(\tau) + \frac{k - r}{k} \right) d\tau \right] q \, ds,$$

$$B_3(t) = \int_0^{\alpha(t)} \left[ pb(s) f_2^p(s) A_5(s) + (1 - p)b(s) f_2^p(s) \right] ds + \int_0^t 2^{2(q-1)} d(s) g_2^q(s) A_5^q(s)ds,$$

$$C_3(t) = (1 - q) 2^{2(q-1)} d(t) g_2^q(t) \exp \left( \int_0^{\alpha(t)} (q - 1)b(s) f_2^p(s) ds \right),$$

$$D_3(t) = \int_0^{\alpha(t)} 2^{2(p-1)} b(s) f_2^p(s) A_6(s)ds + \int_0^t \left[ qd(s) g_2^q(s) A_6(s) + (1 - q)d(s) g_2^q(s) \right] ds,$$

$$E_3(t) = (1 - p) 2^{2(p-1)} a'(t) b(a(t)) f_2^p(a(t)) \exp \left( \int_0^t (p - 1)d(s) g_2^q(s) ds \right),$$
Applying Lemma 2.1 into (2.47), we obtain

Theorem 2.3, condition (2.41) and condition (2.43), we can obtain the results.

Substituting (2.48) into (2.46), we get

Proof. Set

Then

\[
G_2(t) = \int_0^{\alpha(t)} \left[ pb(s) \left( \frac{m}{k} a(s) + \frac{k-m}{k} + \int_s^\alpha c(\tau) \left( \frac{n}{k} a(\tau) + \frac{k-n}{k} \right) d\tau \right) + (1-p)b(s) \right] ds
\]

\[
+ \int_0^t \left[ qd(s) \left( \frac{l}{k} a(s) + \frac{k-l}{k} + \int_s^\alpha e(\tau) \left( \frac{r}{k} a(\tau) + \frac{k-r}{k} \right) d\tau \right) + (1-q)d(s) \right] ds,
\]

\[
f_2(t) = \frac{m}{k} + \frac{n}{k} \int_0^t c(s)ds, \quad t \in \mathbb{R}^+, \quad G_2(t) = \frac{l}{k} + \frac{r}{k} \int_0^t e(s)ds,
\]

\[
b_2(t) = p a(t)b(\alpha(t)) f_2(\alpha(t)) + qd(t)g_2(t),
\]

\[
k_2(t) = p a(t)b(\alpha(t)) G_2(\alpha(t)) + qd(t)g_2(t)G_2(t).
\]

(2.45)

\[
\text{Proof. Set}
\]

\[
v(t) = \int_0^{\alpha(t)} b(s) \left[ u^m(s) + \int_0^s c(\tau) u^n(\tau) d\tau \right]^p ds + \int_0^t d(s) \left[ u^l(s) + \int_0^s e(\tau) u^r(\tau) d\tau \right]^q ds,
\]

then \(v(t)\) is a nondecreasing function on \(R^+\), and

\[
u(t) \leq (a(t) + v(t))^\frac{1}{k}.
\]

(2.47)

Applying Lemma 2.1 into (2.47), we obtain

\[
u^m(t) \leq \left( a(t) + v(t) \right)^\frac{m}{k} \leq \frac{m}{k} \left( a(t) + v(t) \right) + \frac{k-m}{k},
\]

\[
u^n(t) \leq \left( a(t) + v(t) \right)^\frac{n}{k} \leq \frac{n}{k} \left( a(t) + v(t) \right) + \frac{k-n}{k},
\]

(2.48)

\[
u^l(t) \leq \left( a(t) + v(t) \right)^\frac{l}{k} \leq \frac{l}{k} \left( a(t) + v(t) \right) + \frac{k-l}{k},
\]

\[
u^r(t) \leq \left( a(t) + v(t) \right)^\frac{r}{k} \leq \frac{r}{k} \left( a(t) + v(t) \right) + \frac{k-r}{k}.
\]

Substituting (2.48) into (2.46), we get

\[
v(t) \leq \int_0^{\alpha(t)} b(s) \left[ \frac{m}{k} a(s) + v(s) \right] + \frac{k-m}{k} + \int_0^s c(\tau) \left( \frac{n}{k} a(\tau) + v(\tau) \right) + \frac{k-n}{k} d\tau \right]^p ds
\]

\[
+ \int_0^t d(s) \left[ \frac{l}{k} a(s) + v(s) \right] + \frac{k-l}{k} + \int_0^s e(\tau) \left( \frac{r}{k} a(\tau) + v(\tau) \right) + \frac{k-r}{k} d\tau \right]^q ds,
\]

that is,

\[
v(t) \leq \int_0^{\alpha(t)} b(s) \left[ f_2(s)v(s) + \frac{m}{k} a(s) + \frac{k-m}{k} + \int_0^s c(\tau) \left( \frac{n}{k} a(\tau) + \frac{k-n}{k} \right) d\tau \right]^p ds
\]

\[
+ \int_0^t d(s) \left[ g_2(s)v(s) + \frac{l}{k} a(s) + \frac{k-l}{k} + \int_0^s e(\tau) \left( \frac{r}{k} a(\tau) + \frac{k-r}{k} \right) d\tau \right]^q ds,
\]

(2.49)

where the definitions of \(f_2(t)\) and \(g_2(t)\) are given by (2.45).

Inequality (2.49) has the same form as inequality (2.1) of Theorem 2.3. By Theorem 2.3, condition (2.41) and condition (2.43), we can obtain the results.

The proof of Theorem 2.5 is completed. □
Remark 2. When $d(t) \equiv 0$, Theorem 2.5 is converted into Theorem 2.2 in [12].

3. Application. Consider the following integral equation:

$$
\dot{u}(t) = a(t) + \int_0^\alpha F(s, u(s), \int_0^s G(\tau, u(\tau))d\tau)ds
+ \int_0^t H(s, u(s), \int_0^s K(\tau, u(\tau))d\tau)ds,
$$

(3.1)

where $t \in R^+$, $u \in C(R^+, R^+)$, $a \in C(R^+, R^+)$. $\alpha \in C^1(R^+, R)$. $\alpha \in C^1(R^+, R)$ is a nondecreasing function with $\alpha(t) \leq t$, and $\alpha(t) = \varphi(t), t \in [d, 0]$ with $-\infty < d = \inf \{\alpha(t), t \in R^+\} \leq 0$, $\varphi(t) \in C([d, 0], R^+)$. $F, H \in C(R^+ \times R \times R, R), G, K \in C(R^+ \times R, R)$. $k > 0$ is constant.

Theorem 3.1. Suppose that $F, G, H, K$ satisfy the following conditions:

$$
|F(t, U, V)| \leq b(t)(U^m + V)^p, |H(t, U, V)| \leq d(t)(U^q + V)^q,

|G(t, U)| \leq c(t)U^n, |K(t, U)| \leq e(t)U^r,

(3.2)

$$

\[B_3^{1-q}(t) + \int_0^t C_3(s)ds > 0,\]

where $t \in R^+$, $U \in R$, $V \in R$, $b, c, d, e \in C(R^+, R^+)$, $k \geq m > 0$, $k \geq n > 0$, $k \geq l > 0$, $k \geq r > 0$, $0 < p \leq 1$, $q > 1$ are constants. $B_3(t)$ and $C_3(t)$ are defined in Theorem 2.5.

Then for all $t \in R^+$, any solution of the equation (3.1) satisfies

$$
|u(t)| \leq \left[ a(t) + A_5(t) + \exp \left( \int_0^{\alpha(t)} pb(s)f_2^p(s)ds \right) \left( B_3^{1-q}(t) + \int_0^t C_3(s)ds \right)^{\frac{1}{1-q}} \right]^\frac{1}{k},
$$

(3.3)

where the definitions of $A_5(t)$ and $f_2(t)$ are defined in Theorem 2.5.

Proof. Let $u(t)$ be any solution of equation (3.1), from (3.1) and (3.2), for all $t \in R^+$, we have

$$
|u(t)|^k \leq a(t) + \int_0^{\alpha(t)} \left| F(s, u(s), \int_0^s G(\tau, u(\tau))d\tau) \right| ds
+ \int_0^t \left| H(s, u(s), \int_0^s K(\tau, u(\tau))d\tau) \right| ds

\leq a(t) + \int_0^{\alpha(t)} b(s) \left| u(s) \right|^m + \int_0^s \left| G(\tau, u(\tau)) \right| d\tau \right|^p ds
+ \int_0^t d(s) \left| u(s) \right|^l + \int_0^s \left| K(\tau, u(\tau)) \right| d\tau \right|^q ds

\leq a(t) + \int_0^{\alpha(t)} b(s) \left| u(s) \right|^m + \int_0^s c(\tau) \left| u(\tau) \right|^n d\tau \right|^p ds
+ \int_0^t d(s) \left| u(s) \right|^l + \int_0^s e(\tau) \left| u(\tau) \right|^r d\tau \right|^q ds.
$$

(3.4)

Applying Theorem 2.5 to (3.4), by condition (3.2), for all $t \in R^+$, we obtain

$$
|u(t)| \leq \left[ a(t) + A_5(t) + \exp \left( \int_0^{\alpha(t)} pb(s)f_2^p(s)ds \right) \left( B_3^{1-q}(t) + \int_0^t C_3(s)ds \right)^{\frac{1}{1-q}} \right]^\frac{1}{k},
$$

where the definitions of $A_5(t)$ and $f_2(t)$ are given by Theorem 2.5.

This completes the proof. \[\Box\]
Acknowledgments. The authors gratefully acknowledge the effective and beneficial suggestions of the anonymous referees, which significantly improved the paper.

REFERENCES

[1] A. Abdeldaim, Nonlinear retarded integral inequalities of Gronwall-Bellman type and applications, J. Math. Inequal., 10 (2016), 285–299.
[2] A. Abdeldaim and M. Yakout, On some new integral inequalities of Gronwall-Bellman-Pachpatte type, Appl. Math. Comput., 217 (2011), 7887–7899.
[3] J.-C. Chang and D. Luor, On some generalized retarded integral inequalities and the qualitative analysis of integral equations, Appl. Math. Comput., 244 (2014), 324–334.
[4] Q. Feng, F. Meng and B. Zheng, Gronwall-Bellman type nonlinear delay integral inequalities on time scales, J. Math. Anal. Appl., 382 (2011), 772–784.
[5] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Annals of Mathematics, 20 (1919), 292–296.
[6] J. Gu and F. Meng, Some new nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales, Appl. Math. Comput., 245 (2014), 235–242.
[7] F. Jiang and F. Meng, Explicit bounds on some new nonlinear integral inequality with delay, J. Comput. Appl. Math., 205 (2007), 479–486.
[8] Z. Li and W.-S. Wang, Some new nonlinear powered Gronwall-Bellman type retarded integral inequalities and their applications, J. Math. Inequal., 13 (2019), 553–564.
[9] Z. Li and W.-S. Wang, Some nonlinear Gronwall-Bellman type retarded integral inequalities with power and their applications, Appl. Math. Comput., 347 (2019), 839–852.
[10] B. G. Pachpatte, Inequalities for Differential and Integral Equations, Academic Press, London, 1998.
[11] A. Shakoor, I. Ali, S. Wall and A. Rehman, Some generalizations of retarded nonlinear integral inequalities and its applications, J. Math. Inequal., 14 (2020), 1223–1235.
[12] Y. Tian and M. Fan, Nonlinear integral inequality with power and its application in delay integro-differential equations, Advances in Difference Equations, 2020 (2020), 142.
[13] Y. Tian, M. Fan and F. Meng, A generalization of retarded integral inequalities in two independent variables and their applications, Appl. Math. Comput., 221 (2013), 239–248.
[14] W.-S. Wang, X. Zhou and Z. Guo, Some new retarded nonlinear integral inequalities and their applications in differential-integral equations, Appl. Math. Comput., 218 (2012), 10726–10736.
[15] R. Xu and F. Meng, Some new weakly singular integral inequalities and their applications to fractional differential equations, J. Inequal. Appl., 2016 (2016), 78, 16 pp.

Received March 2021; revised August 2021; early access October 2021.

E-mail address: haoyue_song@163.com
E-mail address: fwmeng@qfnu.edu.cn