Extended Group Analysis
of Variable Coefficient Reaction–Diffusion Equations
with Exponential Nonlinearities

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The group classification of a class of variable coefficient reaction–diffusion equations with exponential nonlinearities is carried out up to both the equivalence generated by the corresponding generalized equivalence group and the general point equivalence. The set of admissible transformations of this class is exhaustively described via finding the complete family of maximal normalized subclasses and associated conditional equivalence groups. Limit processes between variable coefficient reaction–diffusion equations with power nonlinearities and those with exponential nonlinearities are simultaneously studied with limit processes between objects related to these equations (including Lie symmetries, exact solutions and conservation laws).

1 Introduction

The problem of extended group symmetry analysis of variable coefficient reaction–diffusion equations of the general form

\[ f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u), \] (1)

where \( fgA \neq 0 \), was initiated in [23,24]. The case of \( A \) and \( B \) being power functions, i.e., the class of equations having the form

\[ f(x)u_t = (g(x)u^n u_x)_x + h(x)u^m, \] (2)

where \( fg \neq 0 \) and \((n,mh) \neq (0,0)\), was successfully investigated. For many reasons, the natural continuation of that study is to consider equations of the form \((1)\) with \( A \) and \( B \) being exponential functions,

\[ f(x)u_t = (g(x)e^{nu} u_x)_x + h(x)e^{mu}. \] (3)

Here \( f = f(x) \), \( g = g(x) \) and \( h = h(x) \) are arbitrary smooth functions of the variable \( x \), \( fg \neq 0 \) and \( n \) and \( m \) are arbitrary constants. The linear case, which is singled out by the condition \( n = m = 0 \), is excluded from consideration as it is well investigated. The semilinear equations of the form \((3)\), which correspond to the constraints \( n = 0 \) and \( m \neq 0 \), were already considered in [22,25]. Moreover, equations of the form \((3)\) with \( n \neq 0 \) are not related to linear and semilinear equations of the same form via point transformations. This is why in the present paper we study only the class of equations of the form \((3)\) with \( fgn \neq 0 \), which we briefly call \textit{class (3)}.

Note that the parameter \( n \) can be gauged to 1 by a simple scaling of variables from the very beginning but we will not use this gauge in the present paper and will deal with the general form \((3)\).
Motivated by the work in [23] as a general outline for similar studies, we carry out the
complete group classification of class (3). In order to achieve this in the easiest way, we apply
a number of modern tools of group analysis of differential equations: generalized extended
equivalence groups, conditional equivalence groups, maximal normalized subclasses, the method
of furcate split, variable gauges of arbitrary elements by equivalence transformations, additional
equivalence transformations, etc. [10, 15, 19, 20, 23, 24].

The structure of this paper is as follows. At first, in Section 2 using the direct method we
derive the determining equations for admissible point transformations in class (3). In Section 3
we find equivalence groups of different kinds for class (3) and show that the consideration of
this class can be simplified by setting the gauge $g = 1$. Moreover, two possibilities for further
simplification in the case $m = n$ via using the associated conditional equivalence group of class (3)
are discussed. In Section 4 the group classification of class (3) is carried out up to equivalence
generated by the generalized extended equivalence group of this class. The usage of the method
of furcate split for solving the group classification problem is explained in detail. Admissible
point transformations of equations from class (3) are exhaustively described in Section 5 in terms
of conditional equivalence groups and normalized subclasses. Section 6 is devoted to the study
of contractions (nontrivial limit processes) between equations from classes (2) and (3). Using the
derived contractions and the results obtained for class (3) in [23], we construct exact solutions
and conservation laws for equations from class (3). The results of the paper are summed up in
the conclusion. The thorough gauging of constant parameters in the group classification list for
class (2), which was not presented in [23], is discussed in the appendix.

2 Preliminary study of admissible transformations

Due to a special structure of equations from class (3), the following problem can be solved
completely. To describe all point transformations each of which connects a pair of equations from
class (3). Such transformations are called form-preserving [12] or admissible [18] or allowed [26]
transformations. See [18] for stronger definitions. They can be naturally interpreted in terms of
the category theory [21]. Note that there exists an infinitesimal equivalent of this notion [2].

At first we make a preliminary study of admissible transformations for class (3) using the
direct method [12]. We briefly describe the corresponding procedure. In what follows we refer
to (3) as a single equation under assuming that the arbitrary elements $f$, $g$, $h$, $n$ and $m$ are fixed
and as a class of equations if the arbitrary elements are varied. We use the same agreement for
other classes of equations. Consider a pair of equations from the class under consideration, i.e.,
equation (3) and the equation

$$
\tilde{f}(\tilde{x})\tilde{u}_t = (\tilde{g}(\tilde{x})e^{\tilde{n}\tilde{u}}\tilde{u}_x)\tilde{x} + \tilde{h}(\tilde{x})e^{\tilde{n}\tilde{u}},
$$

and assume that these equations are connected via a point transformation $T$ of the general form

$$
\tilde{t} = T(t, x, u), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u),
$$

where $|\partial(T, X, U)/\partial(t, x, u)| \neq 0$. We have to derive the determining equations for the functions $T$, $X$ and $U$ and then to solve them. Simultaneously we have to find the connection
between arbitrary elements of equations (3) and (4).

After substituting the expressions for the new variables (with tildes) into (4), we obtain
an equation in the old variables (without tildes). It should be an identity on the manifold $L$
determined by (3) in the second-order jet space $J^2$ with the independent variables $(t, x)$ and
the dependent variable $u$. To involve the constraint between variables of $J^2$ on the manifold $L$, we
substitute the expression of $u_t$ implied by equation (3). The splitting of this identity with respect to
the derivatives $u_{tx}$, $u_{tt}$, $u_{xx}$ and $u_x$ implies the determining equations for the functions $T$,
X and U. In particular, we obtain that $T_u = T_x = X_u = 0$ (and hence $T_t X_u U_u \neq 0$). This well agrees with results on more general classes of evolution equations \[12,13,19,21\]. The restrictions $T_u = T_x = 0$ are common for point transformations between $(1 + 1)$-dimensional evolution equations of order greater than one. The restriction $X_u = 0$ follows from the additional condition that the related equations are quasilinear, i.e., they have the form $u_t = F(t, x, u)u_{xx} + G(t, x, u, u_x)$ with $F \neq 0$.

We take into account the above determining equations. Then collecting the coefficients of $u_{xx}$ gives $\tilde{f} g X^2 e^{nu} = f \tilde{g} T_e e^{\tilde{n} U}$. Dividing the last equation by $e^{nu}$ and differentiating the resulting equation with respect to $u$, we obtain that $f \tilde{g} T_i e^{\tilde{n} U} (\tilde{n} U_u - n) = 0$, i.e., $\tilde{n} U_u = n$. Therefore, $U_u$ is a nonvanishing constant, which we denote by $\delta_3$, and the component $U$ can be represented in the form $U = \delta_3 u + \psi(t, x)$ with the smooth function $\psi$ of $t$ and $x$. The collecting coefficients of $u_x$ leads to the equation

$$
\left(2 \tilde{g} V_x X_x - \frac{\tilde{f}}{f} X^3 g_x + \tilde{g}_x V X_x^2 - \tilde{g} V X_{xx}\right) T_e e^{nu} + \tilde{f} X^2 X_t = 0,
$$

where by $V$ we denote $e^{\tilde{g} \psi}$, which is a nonvanishing function of $t$ and $x$. The further splitting with respect to $u$ implies, in particular, that $X_t = 0$. Taking into account the found constraints, we deduce the following expressions for components of the point transformation $T$:

$$
T = T(t), \quad X = \varphi(x), \quad U = \delta_3 u + \psi(t, x), \quad \tilde{n} = \frac{n}{\delta_3},
$$

(5)

where $T$, $\varphi$ and $\psi$ are smooth functions of their arguments, $\delta_3$ is a constant and $T_t \varphi \delta_3 \neq 0$. The remainder of determining equations for $T$ is given by the unused constraints associated with the coefficients $u_{xx}$ and $u_x$ and the collection of the term without derivatives of $u$,

$$
f \tilde{g} VT_t - \tilde{f} g \varphi^2_x = 0, \quad 2 \frac{V_x}{V} = \frac{g_x}{g} + \frac{\tilde{g}_x}{\tilde{g}} \varphi_x - \frac{\varphi_{xx}}{\varphi_x} = 0,
$$

(6)

$$
\frac{T_i}{\varphi_x} \left(\frac{\tilde{g}_x}{\varphi_x}\right)_x e^{nu} - \frac{\tilde{f} V_i}{V} - \frac{n}{\tilde{f} f} \tilde{h} e^{\tilde{nu}} + \frac{n}{\delta_3} T_i e^{\tilde{n} \psi} \tilde{h} e^{\tilde{n} \delta_3 u} = 0.
$$

(7)

(We additionally simplify the above system by substituting the expression for the ratio $\tilde{f}/f$ from the first equation into the second one.)

3 Equivalence groups

In order to solve a group classification problem it is essential to derive the equivalence transformations which preserve the differential structure of the class of differential equations under consideration and transform only its arbitrary elements. These transformations form a group \[17\], which is called the equivalence group of the class.

There exist several kinds of equivalence groups. The usual equivalence group consists of the nondegenerate point transformations of independent and dependent variables as well as arbitrary elements of the class. Moreover, transformations of independent and dependent variables do not depend on arbitrary elements. If such dependence arises then the corresponding equivalence group is called generalized \[14\]. If new arbitrary elements are expressed via old ones in some non-point, possibly nonlocal, way (e.g. new arbitrary elements are determined via integrals of old ones) then the equivalence transformations are called extended. The first examples of a generalized equivalence group and of an extended equivalence group were presented in \[14\] and \[9\], respectively. See a number of examples on different equivalence groups and their role in solving complicated group classification problems, e.g., in \[10\,23\,24\].
It appears that the generalized extended equivalence group $\hat{G}^\sim$ of class (3) is nontrivial, i.e., it is wider than the usual equivalence group of the same class. We construct the group $\hat{G}^\sim$ using the constraints derived in the course of the preliminary study of admissible transformations and varying arbitrary elements in equations (5) and (7), which form the complete system of classifying conditions for admissible transformations in class (3).

For the general values of $n$ and $m$ the exponents $e^{\delta_4 u}$, $e^{\delta_5 u}$ and $e^{\delta_6 u}$ are linearly independent. This is why due to varying of arbitrary elements we necessarily have that $\tilde{m} = m/\delta_3$. Splitting of equation (7) with respect to $u$ results in $V_t = 0$ (i.e., $\psi(t, x) = \psi(x)$) and therefore differentiating both sides of the first equation of (6) with respect to $t$ we have $T_H = 0$, i.e., $T = \delta_1 t + \delta_2$. Then solving equations (6) we obtain

$$\tilde{g} = \frac{\delta_0 \varphi_x}{V^2} g, \quad \tilde{f} = \frac{\delta_0 T_i}{V \varphi_x} f,$$

where $\delta_0$ is a nonvanishing constant. The remaining equations are

$$\left( \frac{g V_x}{\varphi_x} \right)_x = 0, \quad \tilde{h} f e^{\frac{m}{\delta_3} \psi} \delta_1 - \delta_3 \tilde{f} h = 0.$$

Solving these equations, we obtain

$$\tilde{h} = \frac{\delta_0 \delta_3}{V \varphi_x} e^{-\frac{m}{\delta_3} \psi} h, \quad V = \left( \delta_4 \int \frac{dx}{g(x)} + \delta_5 \right)^{-1},$$

where $\delta_4$ and $\delta_5$ are arbitrary constants with $(\delta_4, \delta_5) \neq (0, 0)$.

The equivalence group of the subclass singled out from class (3) by the constraint $m = n$ is wider than the usual equivalence group of the same class. In other words, the constraint $m = n$ is associated with a nontrivial conditional equivalence group of class (3). In this case we have less splitting with respect to $u$. As a result, $V$ (or, equivalently, $\psi$) is an arbitrary smooth function of $t$ and

$$\tilde{h} = \frac{\delta_0 \delta_3}{nV \varphi_x} \left[ nh \frac{V}{V} \left( \frac{gV_x}{V^2} \right)_x \right].$$

The above results are summarized in Theorems 1 and 2. In what follows we use the notation $\Psi = 1/V = e^{-\frac{m}{\delta_3} \psi}$ in order to simplify formulas.

**Theorem 1.** The generalized extended equivalence group $\hat{G}^\sim$ of class (3) is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(x),$$

$$\tilde{f} = \frac{\delta_0 \delta_1}{\varphi_x} \Psi f, \quad \tilde{g} = \delta_0 \varphi_x \Psi^2 g, \quad \tilde{h} = \frac{\delta_0 \delta_3}{\varphi_x} e^{-\frac{m}{\delta_3} \psi} \Psi h, \quad \tilde{n} = \frac{n}{\delta_3}, \quad \tilde{m} = \frac{m}{\delta_3},$$

where $\varphi$ is an arbitrary smooth function of $x$, $\varphi_x \neq 0$; $\psi$ and $\Psi$ are determined by the formulas

$$\psi(x) = -\frac{\delta_3}{n} \ln |\Psi(x)|, \quad \Psi(x) = \delta_4 \int \frac{dx}{g(x)} + \delta_5.$$

$\delta_j$, $j = 0, \ldots, 5$, are arbitrary constants, $\delta_0 \delta_1 \delta_3 \neq 0$ and $(\delta_4, \delta_5) \neq (0, 0)$.

Thus, elements of $\hat{G}^\sim$ are parameterized by six arbitrary constants and a single arbitrary smooth function of $x$. The usual equivalence group $G^\sim$ of class (3) is the subgroup of the generalized extended equivalence group $\hat{G}^\sim$, which is singled out with the condition $\delta_4 = 0$. 

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Theorem 2. The generalized extended equivalence group of the class of equations

\[ f(x)u_t = (g(x)e^{nu}u_x)_x + h(x)e^{nu} \quad \text{with} \quad nf \neq 0, \]

(9)

coincides with the usual equivalence group \( G_{m=n}^\sim \) of this class and consists of the transformations

\[ \tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(x), \]

\[ \tilde{f} = \delta_0 \phi \Psi f, \quad \tilde{g} = \delta_0 \varphi \Psi^2 g, \quad \tilde{h} = \frac{\delta_0 \delta_3}{n \varphi} [nh \Psi + (g \Psi_x)_x] \Psi, \quad \tilde{n} = \frac{n}{\delta_3}, \]

where \( \delta_j, j = 0, 1, 2, 3, \) are arbitrary constants, \( \delta_0 \delta_1 \delta_3 \neq 0, \varphi \) and \( \psi \) are arbitrary smooth functions of \( x \) with \( \varphi_x \neq 0 \), \( \Psi(x) = e^{-\frac{n \psi(x)}{\delta_3}} \).

Elements of \( G_{m=n}^\sim \) are parameterized by four arbitrary constants and two arbitrary smooth functions of the single variable \( x \). Therefore, the group \( G_{m=n}^\sim \) is really a nontrivial conditional equivalence group of class \( \mathcal{G} \).

In view of Theorem 1, the family of equivalence transformations

\[ \tilde{t} = t, \quad \tilde{x} = \int_{x_0}^x \frac{dy}{g(y)}, \quad \tilde{u} = u, \]

(10)

parameterized by the arbitrary element \( g \) maps class \( \mathcal{G} \) onto its subclass consisting of equations of the form

\[ \tilde{f} \tilde{u}_t = (e^{\tilde{n} \tilde{u}} \tilde{u}_x)_{\tilde{x}} + \tilde{h} e^{\tilde{n} \tilde{u}}, \]

with \( \tilde{g} = 1 \). The new arbitrary elements are expressed via the old ones in the following way:

\[ \tilde{f} = f g, \quad \tilde{h} = gh, \quad \tilde{m} = m, \quad \tilde{n} = n. \]

Hence, within the framework of symmetry analysis it suffices, without loss of generality, to investigate the equations of the general form

\[ f(x)u_t = (e^{nu}u_x)_x + h(x)e^{nu} \quad \text{with} \quad nf \neq 0 \]

(11)

instead of class \( \mathcal{G} \) because all results on symmetries, solutions and conservation laws of equations from subclass \( \mathcal{G}_1 \) can be extended to the entire class \( \mathcal{G} \) with transformations \( \mathcal{G}_1 \). In other words, up to \( G^\sim \)-equivalence we can assign the gauge \( g = 1 \) for the arbitrary element \( g \).

The description of the generalized equivalence group of class \( \mathcal{G}_1 \) and the generalized conditional equivalence group of the same class which is associated with the condition \( m = n \) is deduced from Theorems 1 and 2 by setting \( \tilde{g} = g = 1 \).

Theorem 3. The generalized equivalence group \( \hat{G}_1^\sim \) of class \( \mathcal{G}_1 \) is formed by the transformations

\[ \tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \frac{\delta_6 x + \delta_7}{\delta_4 x + \delta_5}, \quad \tilde{u} = \delta_3 u - \frac{\delta_3}{n} \ln |\delta_4 x + \delta_5|, \]

\[ \tilde{f} = \frac{\delta_1}{\delta_4^2} [\delta_1 x + \delta_5]^3 f, \quad \tilde{h} = \frac{\delta_3}{\delta_4^2} [\delta_4 x + \delta_5] m + h, \quad \tilde{n} = \frac{n}{\delta_3}, \quad \tilde{m} = \frac{m}{\delta_3}, \]

where \( \delta_j, j = 1, \ldots, 7, \) are arbitrary constants such that \( \delta_1 \delta_3 \neq 0, \Delta = \delta_5 \delta_6 - \delta_4 \delta_7 \neq 0 \) and the tuple \( (\delta_1, \delta_5, \delta_6, \delta_7) \) is defined up to a nonzero multiplier; e.g., we can set \( \Delta = \pm 1 \).

\( ^1 \)Different simple gauges of arbitrary elements by equivalence transformations are possible. For example, instead of \( g = 1 \) we can set \( f = 1 \) but the gauge \( g = 1 \) is more convenient because it results in simpler group classification of class \( \mathcal{G} \). Simultaneously, we can assign the value 1 to the constant arbitrary element \( n \). It has been noted in the introduction that this gauge is not essential in the course of group classification and hence it will be used only in the presentation of the final classification list.
Theorem 4. The class of equations

\[ f(x)u_t = (e^{nu} u_x)_x + h(x)e^{nu} \quad \text{with} \quad nf \neq 0 \]  \hspace{1cm} (12)

admits the generalized equivalence group \( \hat{G}_{1,m=n} \) consisting of the transformations

\[
\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \frac{\delta_3}{2n} \ln |\delta_0 \varphi_x|,
\]

\[
\tilde{f} = \delta_0 \delta_1 |\varphi_x|^{-\frac{2}{3}} f, \quad \tilde{h} = \delta_3 \varphi_x^{-2} h + \frac{\delta_3}{n} |\varphi_x|^{-\frac{2}{3}} \left(|\varphi_x|^{-\frac{2}{3}}\right)_{xx}, \quad \tilde{n} = \frac{n}{\delta_3},
\]

where \( \delta_j, j = 0, \ldots, 3, \) are arbitrary constants with \( \delta_0 \delta_1 \delta_3 \neq 0 \) and \( \varphi = \varphi(x) \) is an arbitrary smooth function with \( \varphi_x \neq 0. \)

Class (12) can be mapped onto a proper subclass with only one arbitrary element depending on \( x \) using an appropriate family of point transformations from the group \( \hat{G}_{1,m=n}. \) The most convenient gauges for arbitrary elements are the gauges \( f = 1 \) and \( h = 0. \)

The first gauge can be realized by the transformation

\[
\tilde{t} = \text{sign}(f)t, \quad \tilde{x} = \int_{x_0}^x f(y)\frac{2}{3} dy, \quad \tilde{u} = u + \frac{1}{3n} \ln |f|,
\]

which maps an equation of the form (12) to the equation \( \tilde{u}_\tilde{t} = (e^{\tilde{n} \tilde{u}_\tilde{x}})_\tilde{x} + \tilde{h} e^{\tilde{n} \tilde{u}}, \) i.e., \( \tilde{f} = 1. \) The other new arbitrary elements are expressed via the old ones in the following way:

\[
\tilde{h} = \frac{1}{f} \left(f^{-\frac{2}{3}} h + \frac{1}{n}(f^{-\frac{2}{3}})_{xx}\right), \quad \tilde{n} = n.
\]

Theorem 5. The generalized equivalence group \( \hat{G}_{f=g=1,m=n} \) of the class of equations

\[ u_t = (e^{nu} u_x)_x + h(x)e^{nu} \quad \text{with} \quad n \neq 0, \]  \hspace{1cm} (15)

consists of the transformations

\[
\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_4 x + \delta_5, \quad \tilde{u} = \delta_3 u + \frac{\delta_3}{n} \ln \frac{\delta_4}{\delta_1}, \quad \tilde{h} = \frac{\delta_3}{\delta_4} h, \quad \tilde{n} = \frac{n}{\delta_3},
\]

where \( \delta_j, j = 1, \ldots, 5, \) are arbitrary constants, \( \delta_1 \delta_3 \delta_4 \neq 0, \delta_1 > 0. \)

The gauge \( h = 0 \) for class (12) can be realized by the family of transformations (13), where \( \delta_1 = \delta_3 = 1, \delta_2 = 0 \) and the function \( \varphi \) satisfies the ODE \( (|\varphi_x|^{-\frac{2}{3}})_{xx} + n h |\varphi_x|^{-\frac{2}{3}} = 0. \)

Theorem 6. The generalized equivalence group \( \hat{G}_{1,h=0} \) of the class of equations

\[ f(x)u_t = (e^{nu} u_x)_x \quad \text{with} \quad nf \neq 0, \]  \hspace{1cm} (16)

is projection of the group \( \hat{G}_1 \) on the space \((t, x, u, f, n).\)

It is possible to carry out the complete group classification of equations from class (12) using either the gauge \( f = 1 \) or the gauge \( h = 0. \) The choice of the first gauge is justified in Section 4.
4 Lie symmetries

In general, the group classification of class (3) is carried out within the framework of the classical Lie approach [16,17]. At the same time, we additionally apply a number of modern tools of symmetry analysis, which essentially simplify both related calculations and the presentation of final results. Thus, gauging the arbitrary element $g$ to 1 by equivalence transformations, we in fact classify subclass (11) instead of the entire class. The main equivalence relation involved in the consideration is generated by the generalized equivalence group $\hat{G}_1^*$ of this subclass, which contains the usual equivalence group of the same subclass as a proper subgroup. In other words, we use $\hat{G}_1^*$-equivalence, which is stronger than $G_1^*$-equivalence prescribed by the classical Lie approach. Moreover, for group classification of equations from subclass (11) with $m = n$ we involve the equivalence relation which is generated by the conditional generalized equivalence group $\hat{G}_{1,m=n}$ and is even stronger than $\hat{G}_1^*$-equivalence. In total, this leads to the reduction of classification cases and lowering the number of additional equivalence transformations to be constructed. Lie symmetry extensions are separated using the method of furcate split [10,15].

The reduction of group classification of class (3) to that of its subclass (11) is possible due to combining two facts. Namely, 1) each equation from class (3) is mapped by a transformation from $\hat{G}_1^*$ to 1 by equivalence transformations, we in fact classify subclass (11) instead of the entire class. The main equivalence relation involved in the consideration is generated by the generalized equivalence group $\hat{G}_1^*$ of this subclass, which contains the usual equivalence group of the same subclass as a proper subgroup. In other words, we use $\hat{G}_1^*$-equivalence, which is stronger than $G_1^*$-equivalence prescribed by the classical Lie approach. Moreover, for group classification of equations from subclass (11) with $m = n$ we involve the equivalence relation which is generated by the conditional generalized equivalence group $\hat{G}_{1,m=n}$ and is even stronger than $\hat{G}_1^*$-equivalence. In total, this leads to the reduction of classification cases and lowering the number of additional equivalence transformations to be constructed. Lie symmetry extensions are separated using the method of furcate split [10,15].

We look for vector fields of the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

which generate one-parameter groups of point symmetry transformations of an equation $\mathcal{L}$ from class (11). These vector fields form the maximal Lie invariance algebra $A^{\text{max}} = A^{\text{max}}(\mathcal{L})$ of the equation $\mathcal{L}$. Any such vector field $Q$ satisfies the criterion of infinitesimal invariance, i.e., the action of the second prolongation $Q^{(2)}$ of $Q$ on equation (11) results in the conditions being an identity for all solutions of this equation. Namely, we require that

$$Q^{(2)}\{f(x)u_t - e^{nu}u_{xx} - ne^{nu}u_x^2 - h(x)e^{mu}\} = 0$$

(17)

identically, modulo equation (11).

After the elimination of $u_t$ due to (11), equation (17) becomes an identity in six variables, $t$, $x$, $u$, $u_x$, $u_{xx}$ and $u_{tx}$. In fact, equation (17) is a multivariable polynomial in the variables $u_x$, $u_{xx}$ and $u_{tx}$. The coefficients of different powers of these variables must be zero, giving the determining equations on the coefficients $\tau$, $\xi$ and $\eta$. Solving these equations, we immediately find that $\tau = \tau(t)$, $\xi = \xi(t,x)$. This completely agrees with the general results on point transformations between evolution equations [12]. Then the remaining determining equations take the form

$$\xi_t f = (\xi_{xx} - 2n\eta_x)e^{nu}, \quad \xi_{tx} f = \tau_t - 2\xi_x + n\eta,$$

$$\eta_t f = \eta_{xx}e^{nu} + (\xi h_x + (2\xi_x + (m - n)\eta) h) e^{mu}.$$  

Splitting of the first equation with respect to $u$ and the subsequent integration imply that

$$\xi = \xi(x), \quad \eta = \frac{1}{2\eta} \xi_x + \eta^0(t),$$

where $\eta^0 = \eta^0(t)$ is a smooth function of $t$. 


Finally we obtain the classifying equations which essentially include both the residuary uncertainties in the coefficients of the vector field $Q$ and the arbitrary elements of the class under consideration:

$$\xi f_x = \tau_t - \frac{3}{2} \xi_x + mn^0, \quad (18)$$

$$n^0_x f = \frac{1}{2n} \xi_x e^{nu} + \left( \xi h_x + \left( \frac{3n + m}{2n} \xi_x + (m - n) n^0 \right) h \right) e^{mu}. \quad (19)$$

In order to find the common part of Lie symmetries for all equations from class (11) (resp. class (3)), we should vary the arbitrary elements and split with respect to them in the determining equations. This results in $\tau_t = \xi = \eta = 0$.

**Proposition 1.** The kernel algebra, i.e., the intersection of the maximal Lie invariance algebras of equations from class (11) (resp. class (3)) is the one-dimensional algebra $A^{\cap} = \langle \partial_t \rangle$.

The classification of possible extensions of $A^{\cap}$ is reduced to the simultaneous solution of equations (18) and (19) with respect to both the residuary uncertainties in the coefficients of the vector field $Q$ and the arbitrary elements up to $\sim_1$-equivalence. Further splitting with respect to $u$ in the last equation depends on the value of $m$. Namely, we should separately consider three exclusive cases

1) $m \neq 0, n$; 2) $m = 0$; 3) $m = n$.

The case $h = 0$ is special as the value of $m$ is undefined in this case but it can be included to the case $m = n$ (see Section 2). This is why in the first two cases we assume that $h \neq 0$.

### 4.1 General case of $m$

In the first (general) case splitting of (19) gives the equations $n^0_x = 0$ and $\xi_x = 0$. Differentiation of (18) with respect to $t$ leads to the equation $\tau_{tt} = 0$. Therefore,

$$\tau = c_4 t + c_5, \quad \xi = nc_2x^2 + c_1x + c_0, \quad \eta = c_2x + c_3. \quad (20)$$

The classifying equations take the form

$$(nc_2x^2 + c_1x + c_0) f_x f = -3nc_2x + nc_3 + c_4 - 2c_1, \quad (21)$$

$$(nc_2x^2 + c_1x + c_0) h_x h = -(3n + m)c_2x + (n - m)c_3 - 2c_1. \quad (22)$$

Further investigation will be carried out using the method of furcate split [10, 15]. For any operator $Q$ from $A^{\max}$ the substitution of its coefficients into equations (21) and (22) gives some equations on $f$ and $h$ of the general form

$$(nax^2 + bx + c) f_x f = -3nax + d, \quad (23)$$

$$(nax^2 + bx + c) h_x h = -(3n + m)ax + (n - m)p - 2b, \quad (24)$$

where $a, b, c, d$ and $p$ are constants which are defined up to nonzero multiplier. The set $\mathcal{V}$ of values of the coefficient tuple $(a, b, c, d, p)$ obtained by varying of an operator from $A^{\max}$ is a linear space. Note that the first three components of any nonzero element from $\mathcal{V}$ form a nonzero triple (otherwise the corresponding system with respect to $f$ and $h$ would be inconsistent). The dimension $k = k(A^{\max})$ of the space $\mathcal{V}$ is not greater than 2 otherwise the corresponding
equations form an incompatible system with respect to \( f \) and \( h \). The value of \( k \) is an invariant of the transformations from \( \tilde{G}_1 \). Therefore, there exist three \( \tilde{G}_1 \)-inequivalent cases for the value of \( k \): \( k = 0 \), \( k = 1 \) and \( k = 2 \). We consider these possibilities separately.

I. The condition \( k = 0 \) means that (21)-(22) are not equations with respect to \( f \) and \( h \) but identities. Therefore, \( f \) and \( h \) are not constrained and \( c_i = 0 \), \( i = 0, \ldots, 4 \). We obtain nothing but the kernel algebra \( A^\gamma \) presented by Case 1 of Table 1.

II. If \( k = 1 \) then we have, up to a nonzero multiplier, exactly one system of the form (23)-(24) with respect to the functions \( f \) and \( h \). Integrating this system up to \( \tilde{G}_1 \)-equivalence gives

\[
f = \exp \left( \int \frac{-3na x + d}{nax^2 + bx + c} \, dx \right), \quad h = \varepsilon \exp \left( - \int \frac{(3n + m)ax + 2b + (m - n)p}{nax^2 + bx + c} \, dx \right),
\]

where \( \varepsilon = \pm 1 \mod \tilde{G}_1 \), and the coefficients of \( Q \) take the form

\[
\tau = (d + 2b - np)\lambda t + c_5, \quad \xi = (nax^2 + bx + c)\lambda, \quad \eta = (ax + p)\lambda.
\]

Here \( \lambda \) and \( c_5 \) are arbitrary constants. Hence \( A^{\text{max}} \) is the two-dimensional algebra given by Case 2 of Table 1.

It is necessary to investigate equivalence of cases with the arbitrary elements \( f \) and \( h \) of the above form, corresponding to different values of the tuple \((a, b, c, d, p)\).

**Lemma 1.** Up to \( \tilde{G}_1 \)-equivalence the parameter tuple \((a, b, c, d, p)\) can be assumed to belong to the set \( \{(0, 1, 0, \hat{d}, \hat{p}), (0, 0, 1, 1, \hat{p}), (0, 0, 1, 0, 1), (0, 0, 1, 0, 0), (1/n, 0, 1, \hat{d}, \hat{p})\} \), where \( \hat{d} \geq -3/2 \) and, if \( \hat{d} = -3/2 \), \( \hat{n}p \geq 1/2 \); \( \hat{d} \geq 0 \) and, if \( \hat{d} = 0 \), \( \hat{p} \geq 0 \).

**Proof.** Combined with the multiplication by a nonzero constant, each transformation from the equivalence group \( \tilde{G}_1 \) is extended to the coefficient tuple of system (23)-(24):

\[
\tilde{n}a = \nu(\delta_2^5 na - \delta_4 \delta_5 b + \delta_4^2 c), \quad \tilde{b} = \nu(-2\delta_5 \delta_7 na + (\delta_4 \delta_7 + \delta_5 \delta_6)b - 2\delta_4 \delta_6 c),
\]

\[
\tilde{c} = \nu(\delta_2^5 na - \delta_6 \delta_7 b + \delta_6^2 c), \quad \tilde{d} = \nu \Delta d + 3\nu(\delta_5 \delta_7 na - \delta_4 \delta_7 b + \delta_4 \delta_6 c),
\]

\[
\tilde{n}p = \nu \Delta np - \nu(\delta_5 \delta_7 na - \delta_4 \delta_7 b + \delta_4 \delta_6 c),
\]

where \( \Delta = \delta_5 \delta_6 - \delta_4 \delta_7 \neq 0 \), \( \nu \) is an arbitrary nonzero constant.

There are only three \( \tilde{G}_1 \)-inequivalent values of the triple \((a, b, c)\) depending on the sign of \( D = b^2 - 4nac \),

\[
(0, 1, 0) \quad \text{if} \quad D > 0, \quad (0, 0, 1) \quad \text{if} \quad D = 0, \quad (1/n, 0, 1) \quad \text{if} \quad D < 0.
\]

Indeed, if \( D > 0 \) then there exist two linearly independent pairs \((\delta_4, \delta_5)\) and \((\delta_6, \delta_7)\) such that \( \delta_2^5 na - \delta_4 \delta_5 b + \delta_4^2 c = 0 \) and \( \delta_2^5 na - \delta_6 \delta_7 b + \delta_6^2 c = 0 \). For these values of \( \delta \)'s we have \( \tilde{a} = \tilde{c} = 0 \).

In the case \( D = 0 \) we choose values of \( \delta_4, \delta_5, \delta_6 \) and \( \delta_7 \) for which \( \delta_2^5 na - \delta_4 \delta_5 b + \delta_4^2 c = 0 \) and the pair \((\delta_6, \delta_7)\) is not proportional to the pair \((\delta_4, \delta_5)\). Then we obtain that \( \tilde{a} = 0 \) and \( \tilde{b} = \nu \delta_7(\delta_4 b - 2\delta_5 na) + \nu \delta_6(\delta_5 b - 2\delta_4 c) = 0 \). The residual coefficient \( \tilde{b} \) if \( D > 0 \) and \( \tilde{c} \) if \( D = 0 \) is necessarily nonzero and hence can be scaled to 1 by choosing the appropriate value of \( \nu \). If \( D < 0 \), we have \( ac \neq 0 \) and can set \( a > 0 \). The matrix

\[
\begin{pmatrix}
na & -b/2 \\
-b/2 & c
\end{pmatrix}
\]

is symmetric and positive. Hence the corresponding bilinear form is a well-defined scalar product. We choose \( \nu = 1 \) and pairs \((\delta_4, \delta_5)\) and \((\delta_6, \delta_7)\) which are orthonormal with respect to this product. Then \( \tilde{n}a = \tilde{c} = 1 \) and \( \tilde{b} = 0 \).

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Certain freedom in varying group parameters is preserved even after fixing one of the above inequivalent forms for both the tuples \((a, b, c)\) and \((\tilde{a}, \tilde{b}, \tilde{c})\). This allows us to set constraints for the coefficients \(d\) and \(p\).

Thus, it follows from the equality \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (0, 1, 0)\) that \(\delta_4\delta_6 = \delta_6\delta_7 = 0\) and \(\delta_4\delta_5 + \delta_5\delta_6 = \nu^{-1}\). There are two cases for the solution of the above system in \(\delta\)'s: either \(\delta_4 = \delta_7 = 0\) and \(\Delta = \delta_5\delta_6 = \nu^{-1}\) or \(\delta_4 = \delta_5 = 0\) and \(\Delta = -\delta_4\delta_7 = -\nu^{-1}\). In the first case the coefficients \(d\) and \(np\) are differently transformed. In the second case the transformation takes the form \(\tilde{d} = -d - 3\), \(\tilde{p} = -np + 1\). This is why up to \(G_1^-\)-equivalence we can assume that \(\tilde{d} \geq -3/2\) and, if \(\tilde{d} = -3/2\), \(\tilde{p} \geq 1/2\).

The equality \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (0, 0, 1)\) implies \(\delta_4 = 0\), \(\nu\delta_6^2 = 1\) and \(\Delta = \delta_5\delta_6 \neq 0\). The transformation of \(d\) and \(np\) is reduced to simultaneous scaling with the same multipliers, \(\tilde{d} = \nu\Delta d\) and \(\tilde{p} = \nu\Delta np\). This allows us either to set \(\tilde{d} = 1\) if \(d \neq 0\) or to scale \(\tilde{p}\) if \(d = 0\) and hence \(\tilde{d} = 0\). Therefore, we obtain the following inequivalent tuples \((0, 0, 1, 1, \tilde{p})\), if \(d \neq 0\), \((0, 0, 1, 0, 1)\) if \(d = 0\), \(\tilde{p} \neq 0\), and \((0, 0, 1, 0, 0)\) in the case \(d = p = 0\). The last tuple corresponds to the constant-coefficient equations (11) (Case 3 of Table 1).

Setting \((na, b, c) = (n\tilde{a}, \tilde{b}, \tilde{c}) = (1, 0, 1)\) results in \(\delta_4^2 + \delta_5^2 = \delta_6^2 + \delta_7^2 = \nu^{-1}\) and \(\delta_4\delta_6 + \delta_5\delta_7 = 0\). Hence \(\delta_6 = \varepsilon\delta_5\) and \(\delta_7 = -\varepsilon\delta_4\), where \(\varepsilon = \pm 1\). The transformation of the coefficients \(d\) and \(np\) is reduced to the multiplication by \(\varepsilon\), \(\tilde{d} = \varepsilon d\) and \(\tilde{p} = \varepsilon np\). This is why we can only set \(\varepsilon = 0\) and, if \(\tilde{d} = 0\), \(\tilde{p} \geq 0\).

Lemma 1 implies that up to \(G_1^-\)-equivalence the case \(k = 1\) is partitioned into the three inequivalent subcases:

1. \((f, h) = (|x|^d, \varepsilon|x|^q), q = (n - m)p - 2; \quad \langle \partial_t, (d + 2 - pm)t \partial_t + x \partial_x + p \partial_u \rangle;\)
2. \((f, h) = (e^{dx}, \varepsilon e^{dx}), q = (n - m)p; \quad \langle \partial_t, (d - pm)t \partial_t + x \partial_x + p \partial_u \rangle;\)
3. \((f, h) = \left((x^2 + 1)^{-\frac{3}{2}} e^{\varepsilon \arctan x}, \varepsilon (x^2 + 1)^{-\frac{3}{2}} - \frac{m}{2n} e^{\varepsilon \arctan x}\right), q = (n - m)p: \quad \langle \partial_t, (d - pm)t \partial_t + (x^2 + 1) \partial_x + (x/n + p) \partial_u \rangle.\)

Here \((d, q) \neq (0, 0), (-3, -3 - m/n)\) and \((d, q) \neq (0, 0)\) for the first and second subcases, respectively. (Otherwise \(k = 2\), see below.) It additionally follows from Lemma 1 that up to \(G_1^-\)-equivalence we can set certain constraints for the parameters \(d\) and \(q\). (It is convenient to use \(q\) instead of \(p\) as a parameter.) For the first subcase an exhaustive gauge implied by \(G_1^-\)-equivalence consists of the inequalities \(d \geq -3/2\) and, if \(d = -3/2\), \(q \geq -3/2 - m/(2n)\).

They can be set using the equivalence transformation

\[
\tilde{t} = t, \quad \tilde{x} = \frac{1}{x}, \quad \tilde{u} = u - \frac{1}{n} \ln |x|,
\]

whose extension to the parameters \(d\) and \(q\) is given by \(\tilde{d} = -d - 3\) and \(\tilde{q} = -q - 3 - m/n\). In the second subcase the parameters \(d\) and \(q\) can be gauged using a scaling of \(x\). More precisely, \(d = 1 \mod G_1^-\) if \(d \neq 0\) and \(q = 1 \mod G_1^-\) if \(d = 0\). In the last subcase we can just simultaneously alternate the signs of \(d\) and \(q\). Hence the exhaustive gauge is presented by \(\tilde{d} \geq 0\) and, if \(d = 0\), \(q \geq 0\).

III. Let \(k = 2\). We choose a basis \(\{(a_i, b_i, c_i, d_i, p_i), i = 1, 2\}\) of the space \(V\) of tuples \((a, b, c, d, p)\) associated with \(A^{\text{max}}\) and introduce the notation

\[
P_i = na_ix^2 + b_ix + c_i, \quad Q_i = -3na_ix + d_i, \quad R_i = -(3n + m)a_ix + (n - m)p_i - 2b_i.
\]

Then \(P_i \neq 0, i = 1, 2\), and the system for the functions \(f\) and \(h\) is written in the form

\[
\frac{f_x}{f} = \frac{Q_1}{P_1} = \frac{Q_2}{P_2}, \quad \frac{h_x}{h} = \frac{R_1}{P_1} = \frac{R_2}{P_2}.
\]
Consider two different cases depending on whether the first components of all elements from the space $V$ are zero.

If this is true, we can choose a basis of $V$ with $b_1 = c_2 = 1$ and $b_2 = c_1 = 0$. Then system (26) obviously implies that $f_x = h_x = 0$. As $fh \neq 0$, up to $\hat{G}_1^\sim$-equivalence we obtain Case 3 of Table 1.

If the space $V$ contains tuples with nonzero first components, we can assume without loss of generality that $a_1 = 1$ and $a_2 = 0$. Then we have $b_2 \neq 0$ as otherwise system (26) is not compatible. We set $b_2 = 1$ and $b_1 = 0$ by changing the basis of $V$ and then set $c_2 = 0$ using a shift of $x$ allowed by $\hat{G}_1^\sim$-equivalence. Equations (25) are consistent if and only if $c_1 = 0$. They imply that $(f, h) = (x^{-3}, \varepsilon |x|^{-3-\frac{4\alpha}{3}}) \bmod \hat{G}_1^\sim$, where $\varepsilon = \pm 1$. The corresponding maximal Lie invariance algebra

$$\langle \partial_t, nx^2 \partial_x + x \partial_u, (m + n)t \partial_t + (m - n)x \partial_x - 2 \partial_u \rangle$$

is a two-dimensional extension of the kernel algebra $A^0 = \langle \partial_t \rangle$. This Lie symmetry extension is not included in Table 1 because it is reduced to Case 3 by the transformation (25) from the equivalence group $\hat{G}_1^\sim$.

4.2 Case $m = 0$

If $m = 0$ then $\xi = nc_2 x^2 + c_1 x + c_0$ and the classifying equations have the form

$$\frac{f}{x} = \eta_i^0 f = (nc_2 x^2 + c_1 x + c_0) h_x + \left(3nc_2 x + \frac{3}{2} c_1 - n\eta^0 \right) h. \tag{27}$$

Differentiating the latter equation with respect to $t$, we obtain the equation $f \eta_i^0 t = -nh \eta^0$. If $(h/f)x \neq 0$ then $\eta^0$ is a constant and hence the corresponding equation possesses the same maximal Lie invariance algebra as in the case of general value of $m$ for the same values of $f$ and $h$. In what follows we assume that $(h/f)x = 0$, i.e., $\varepsilon = h/f$ is a constant which equals $\pm 1$ modulo $\hat{G}_1^\sim$. Substituting the condition $h = \varepsilon f$ into (27), we obtain that

$$\tau = c_4 e^{-\varepsilon nt} + c_5, \quad \xi = nc_2 x^2 + c_1 x + c_0, \quad \eta = c_2 x + \varepsilon c_4 e^{-\varepsilon nt} + c_3,$$

and the classifying equation on the function $f$ is the following one

$$\frac{f}{x} = -3nc_2 x - 2c_1 + nc_3.$$

It is obvious that this equation has the form (23). Applying the method of furcate split in the same way as in Section 4.1, we obtain only three $\hat{G}_1^\sim$-inequivalent Lie symmetry extensions, namely, Cases 4–6 of Table 1. Case 5 is partitioned into the three $\hat{G}_1^\sim$-inequivalent subcases:

1. $f = |x|^d$: $\langle \partial_t, e^{-\varepsilon nt} (\partial_t + \varepsilon \partial_u), nx \partial_x + (d + 2) \partial_u \rangle$;

2. $f = e^x$: $\langle \partial_t, e^{-\varepsilon nt} (\partial_t + \varepsilon \partial_u), n \partial_x + \partial_u \rangle$;

3. $f = (x^2 + 1)^{-\frac{d}{2}} e^{d \arctan x}$: $\langle \partial_t, e^{-\varepsilon nt} (\partial_t + \varepsilon \partial_u), n(x^2 + 1) \partial_x + (x + d) \partial_u \rangle$.

Here always $h = \varepsilon f$. In the first subcase we should assume that $d \neq -3, 0$ as otherwise this subcase is reduced to Case 6 with a wider Lie invariance algebra, and $d \geq -3/2 \bmod \hat{G}_1^\sim$, cf. Lemma 1 and the transformation (25). In the last subcase we can just alternate the sign of $d$. Hence the exhaustive gauge up to $\hat{G}_1^\sim$-equivalence is presented by $d \geq 0$. 

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4.3 Case \( m = n \)

If \( m = n \) then \( \eta^0 \) is a constant and the classifying equations take the form

\[
\frac{\xi_f}{f} = \tau_t - \frac{3}{2} \xi_x + m \eta^0, \quad \frac{1}{2n} \xi_{xxx} + \xi h_x + 2\xi_x h = 0. \tag{28}
\]

To carry out the group classification in this case, it is necessary to use an additional gauge of the arbitrary elements of class \([12]\). Consider both the gauges proposed in Section 2, namely, \( f = 1 \) and \( h = 0 \).

Choosing the gauge \( f = 1 \) results in subclass \([15]\). The first equation of \((28)\) implies that \( \xi_{xx} = 0 \), i.e., \( \xi = c_1 x + c_2 \), where \( c_1 \) and \( c_2 \) are constants and \( \eta = \frac{1}{2n} \xi_x + \eta^0 \) is a constant, which is denoted by \( c_3 \). Then \( \tau = (2c_1 - nc_3) t + c_2 \). The classifying equation on the function \( h \) can be written in the form

\[
(c_1 x + c_2) h_x + 2c_1 h = 0.
\]

It is easy to derive Cases 7, 8, 9 and 10 of Table 1.

Introducing the gauge \( h = 0 \), we replace the study of class \([12]\) by the study of class \([16]\). Under this gauge, determining equations gives \( \xi_{xxx} = 0 \) and hence \( \tau, \xi \) and \( \eta \) have the form \((20)\). The classifying equation on the function \( f \) coincides with equation \((21)\). The solution of \((21)\) results in the following cases of Lie symmetries extensions:

- arbitrary \( f \): \( \langle \partial_t, nt \partial_t - \partial_u \rangle \);
- \( f = \exp \left( \int \frac{-3nx^2 + d}{nax^2 + bx + c} \, dx \right) \): \( \langle \partial_t, nt \partial_t - \partial_u, (d + 2b)t \partial_t + (nax^2 + bx + c) \partial_x + ax \partial_u \rangle; \)
- \( f = 1 \): \( \langle \partial_t, nt \partial_t - \partial_u, \partial_x, 2t \partial_t + x \partial_x \rangle \).

The first and third cases obviously correspond to Cases 7 and 10 of Table 1, respectively.

Similarly to the general case of \( m \) and the case \( m = 0 \), the second case of Lie symmetry extensions with \( m = n \) under the gauge \( h = 0 \) is partitioned into the following three \( \hat{G}_1 \)-inequivalent subcases:

1. \( f = |x|^d \): \( \langle \partial_t, nt \partial_t - \partial_u, (d + 2)t \partial_t + x \partial_x \rangle \);  
2. \( f = e^x \): \( \langle \partial_t, nt \partial_t - \partial_u, t \partial_t + \partial_x \rangle \);  
3. \( f = (x^2 + 1)^{-\frac{3}{2}} e^{d \arctan x} \): \( \langle \partial_t, nt \partial_t - \partial_u, ndt \partial_t + n(x^2 + 1) \partial_x + x \partial_u \rangle \).

In the first subcase we should again assume that \( d \neq -3, 0 \) as the value \( d = 0 \) corresponds to the case of \( f = 1 \) and \( h = 0 \) with a wider Lie invariance algebra (Case 10 of Table 1) and the value \( d = -3 \) is \( \hat{G}_1 \)-equivalent to the value \( d = 0 \), cf. the consideration for \( k = 2 \) in Section 4.1. Additionally, Lemma \([1]\) implies that using the transformation \((25)\) we can set the gauge and \( d \geq -3/2 \) mod \( \hat{G}_1 \). In the last subcase the equivalence transformation alternating the sign of \( x \) leads to the gauge \( d \geq 0 \) mod \( \hat{G}_1 \).

**Remark 1.** The above subcases are related to cases of Table 1 via point transformations of the form \([14]\) which belong to the equivalence group \( \hat{G}_{1,m=n} \). Thus, the first subcase with \( d \in \{-3, -3/2, 0\} \), the second subcase and the third subcase with \( d \neq 0 \) are mapped to Case 8 with

\[
\alpha = \frac{d(d + 3)}{4n(d + 3/2)^2}, \quad \alpha = \frac{1}{4n} \quad \text{and} \quad \alpha = \frac{d^2 + 9}{4nd^2},
\]

respectively. The first subcase with \( d = -3/2 \) and the third subcase with \( d = 0 \) are mapped to Case 9 with \( \varepsilon = -1/(4n) \) and \( \varepsilon = 1/n \), respectively.
Analyzing the group classifications of the class $[12]$ under both the gauges, we conclude that each of the gauges $f = 1$ and $h = 0$ has certain advantages and disadvantages. More precisely, under the gauge $h = 0$ equations have a less number of summands and the group classification list in the case $m = n$ is similar to the group classification lists in the cases of general $m$ and $m = 0$. The gauge $h = 0$ is also convenient in order to look for additional equivalence transformations between the cases $m = 0$ and $m = n$. At the same time, the gauging transformation, the corresponding equivalence group and the inequivalent values of the residiary arbitrary element, $h$, arising in the course of group classification are much simpler for the gauge $f = 1$. In particular, it is very easy to partition into inequivalent cases under this gauge. Moreover, the group classification list presented in Table 1, which involves the gauge $f = 1$ in the case $m = n$, is well consistent with the group classification list found for class $[2]$ in [23]. This is important for the study of contractions between cases of Lie symmetry extensions.

### 4.4 Classification list and additional equivalence transformations

Results of group classification for subclasses of class $[3]$ are collected in Table 1.

| no. | $f(x)$ | $h(x)$ | Basis of $A^\text{max}$ |
|-----|--------|--------|-------------------------|
| 1   | $\forall$ | $\forall$ | $\partial_t$ | General case of $m$ |
| 2   | $f_1(x)$ | $h_1(x)$ | $\partial_t, (d + 2b - pm)\partial_t + (nax^2 + bx + c)\partial_x + (ax + p)\partial_u$ |
| 3   | 1 | $\varepsilon$ | $\partial_t, \partial_x, 2nt\partial_t + (m - n)x\partial_x - 2\partial_u$ |
|     |       | $m = 0$, $h \neq 0$, $(h/f)_t = 0$ |
| 4   | $\forall$ | $\varepsilon f(x)$ | $\partial_t, e^{-\varepsilon n t}(\partial_h + \varepsilon \partial_u)$ |
| 5   | $f_1(x)$ | $\varepsilon f_1(x)$ | $\partial_t, e^{-\varepsilon n t}(\partial_h + \varepsilon \partial_u), n(nax^2 + bx + c)\partial_x + (nax + 2b + d)\partial_u$ |
| 6   | 1 | $\varepsilon$ | $\partial_t, e^{-\varepsilon n t}(\partial_h + \varepsilon \partial_u), \partial_x, nx\partial_x + 2\partial_u$ |
|     |       | $m = n$ or $h = 0$ |
| 7   | $\forall$ | $\forall$ | $\partial_t, nt\partial_t - \partial_u$ |
| 8   | 1 | $\alpha x^{-2}$ | $\partial_t, nt\partial_t - \partial_u, nx\partial_x + 2\partial_u$ |
| 9   | 1 | $\varepsilon$ | $\partial_t, nt\partial_t - \partial_u, \partial_x$ |
| 10  | 1 | 0 | $\partial_t, nt\partial_t - \partial_u, \partial_x, nx\partial_x + 2\partial_u$ |

Here $n, \alpha$ and $\varepsilon$ are nonzero constants, $n = 1 \pmod{\tilde{G}_1}$, $\varepsilon = \pm 1 \pmod{\tilde{G}_1}$,

$$f_1(x) = \exp \left( \int -\frac{3nax + d}{nax^2 + bx + c} \, dx \right), \quad h_1(x) = \varepsilon \exp \left( - \int \frac{(3n + m)ax + 2b + (m - n)p}{nax^2 + bx + c} \, dx \right),$$

and up to $\tilde{G}_1$-equivalence the parameter tuple $(a, b, c, d, p)$ can be assumed to belong to the set

$$\{(0, 0, 0, \tilde{d}, (\tilde{q} + 2)/(n - m)), (0, 0, 1, \tilde{p}), (0, 0, 1, 0), (1/n, 0, 1, \tilde{d}, \tilde{p}),\}$$

where $(\tilde{d}, \tilde{q}) \neq (0, 0), (-3, -3 - m/n)$ and modulo $\tilde{G}_1$ we can also set $\tilde{d} \geq -3/2$ and, if $\tilde{d} = -3/2$, $\tilde{q} \geq -3/2 - m/(2n); \tilde{d} \geq 0$ and, if $\tilde{d} = 0$, $\tilde{p} \geq 0$. In Case 5 the parameter $p$ should be neglected. In Case 7 the arbitrary element $f$ (resp. $h$) can be additionally gauged by transformations from $\tilde{G}_{1, m=n}$. For example, we can set $f = 1$.

There exist additional equivalence transformations between classification cases presented in Table 1. Namely the point transformation

$$t' = \frac{1}{\varepsilon nt} e^{\varepsilon n t}, \quad x' = x, \quad u' = u - \varepsilon t$$

(29)
links the equations \( f(x)u_t = (g(x)e^{nu}u_x)_x + \varepsilon f(x) \) and \( f(x')u'_{t'} = (g(x')e^{nu'}u'_{x'})_x' \). This transformation belongs to no equivalence group found in Section 2 and reduces Cases 4–6 of Table 1 to the set of cases ‘\( m = n \) or \( h = 0 \)’. For the reduction to be precise to Cases 7–10 of Table 1, for Case 5 transformation (29) should be composed with an appropriate transformation of the form (14). Such compositions map subcases of Case 5 to subcases of Case 8 and 9, cf. Remark [1].

The transformations described exhaust additional equivalence transformations within the classification list from Table 1. It is proved in the next section within the framework of admissible transformations. Thus, transformation (29) can also be included in the framework of conditional equivalence but the corresponding conditional equivalence group is too complicated.

As a result, we obtain the following assertion.

**Theorem 7.** Up to point transformations, a complete list of Lie symmetry extensions for equations from class (3) is exhausted by Cases 1–3 and 7–10 of Table 1.

Therefore, the maximal dimension of Lie symmetry algebras for equations from class (3) equals four, and for equations with \( m \neq 0, n \) this dimension equals three.

In the Table 1 we do not present the partitions of Cases 2 and 5 into inequivalent subclasses in detail. The complete description of these partitions is given in Sections 4.1 and 4.2.

**Corollary 1.** If an equation from class (3) is invariant with respect to a four-dimensional Lie algebra then it is reduced using point transformations to the equation \( u_t = (e^{u_x}u_x)_x \).

**Corollary 2.** If an equation from class (3) with \( m \neq 0, n \) possesses a three-dimensional Lie invariance algebra then it is mapped by a point transformation to the equation \( u_t = (e^{u_x}u_x)_x \pm e^{nu} \).

## 5 Classification of admissible (form-preserving) transformations

In contrast to the construction of equivalence groups, in the course of the complete study of admissible transformations we cannot vary arbitrary elements. The description of the set of admissible transformations will be presented in terms of conditional equivalence groups and normalized subclasses.

The consideration of Section 2 implies that any admissible transformation in class (3) is of the general form (5), where the parameters satisfy the inequality \( T_t \varphi_x \delta_3 \neq 0 \) and equations (6) and (7). Differentiating the first equation of (6) with respect to \( t \), we have that \( V = \theta(x)/T_t \). Recall that \( V \) denotes the expression \( e^{nu} \varphi \). Then the determining equations become

\[
\begin{align*}
2 f \tilde{f} \theta - \tilde{f} g \varphi_x^2 & = 0, \\
2 \theta_x - \frac{g_x}{g} \tilde{g} \varphi_x - \varphi_{xx} & = 0, \\
\frac{1}{\varphi_x} \left( \frac{\theta_x}{\varphi_x} \right) e^{nu} - \tilde{f} T_t \left( \frac{1}{T_t} \right)_t - n \tilde{f} h e^{nu} + \frac{n}{\delta_3} T_t e^{\tilde{m} \varphi} h e^{\tilde{m} \delta_1 u} & = 0.
\end{align*}
\]

Solving equations (30), we express the new arbitrary elements \( \tilde{f} \) and \( \tilde{g} \) via the old ones:

\[
\tilde{f} = \frac{\delta_0}{\theta \varphi_x} f, \quad \tilde{g} = \frac{\delta_0 \varphi_x}{\theta^2} g,
\]

where \( \delta_0 \) is an arbitrary nonzero constant.

The splitting of equation (31) with respect to \( u \) and the subsequent integration of the determining equations appreciably depend on values of \( m \) and \( \tilde{m} \). Consider possible cases for these values.
If \( \tilde{m} = m/\delta_3 \neq 0 \), equation (31) implies \( T_{tt} = 0 \), i.e., \( T = \delta_1 t + \delta_2 \) and therefore \( V = \theta(x)/\delta_1 \). The other conditions obtained from (31) result in
\[
\tilde{m} = \frac{m}{\delta_3} \neq 0, \quad \tilde{n}: \quad \tilde{h} = \frac{\delta_0 \delta_3}{\delta_1 \varphi_x} e^{-\tilde{m} \psi} h, \quad \text{where} \quad \theta = \left( \delta_4 \int \frac{dx}{g(x)} + \delta_5 \right)^{-1};
\]
\[
\tilde{m} = \frac{m}{\delta_3} = \frac{n}{\delta_3}: \quad \tilde{h} = \frac{\delta_0 \delta_3}{n \varphi_x} \left[ \frac{nh}{\theta} - \left( \frac{\theta}{\delta_3^2} \right)_x \right], \quad \text{where} \quad \theta \text{ is an arbitrary function.}
\]
Therefore, in the case \( m \neq 0, n \) (resp. the case \( m = n \)) any admissible transformation is induced by a transformation from the generalized extended equivalence group \( \hat{G}^0 \) (resp. the conditional equivalence group \( G^{m=n} \)).

If \( \tilde{m} = m = 0 \) then equation (31) implies an expression for \( \theta \) and an equation connecting the ratios \( \tilde{h}/\tilde{f} \) and \( h/f \),
\[
\theta = \left( \delta_4 \int \frac{dx}{g(x)} + \delta_5 \right)^{-1}, \quad \frac{\tilde{h}}{\tilde{f}} = \frac{\delta_3 h}{T t} + \frac{\delta_3 n}{T t} \left( \frac{1}{T t} \right)_t.
\]
In view of the simplest differential consequence
\[
\left( \frac{\tilde{h}}{\tilde{f}} \right)_x = \frac{\delta_3}{T t} \left( \frac{h}{f} \right)_x
\]
of the last equation we have \( \left( \frac{\tilde{h}}{\tilde{f}} \right)_x \neq 0 \) and \( T_{tt} = 0 \) if \( (h/f)_x \neq 0 \). Therefore, any admissible transformation for equations with \( m = 0 \) and \( (h/f)_x \neq 0 \) is induced by a transformation from the equivalence group \( G^0 \).

This is not the case if \( (h/f)_x = 0 \). Then \( \left( \frac{\tilde{h}}{\tilde{f}} \right)_x = 0 \), i.e., the condition \( (h/f)_x = 0 \) is invariant with respect to \( G^0 \). We denote the constants \( h/f \) and \( \tilde{h}/\tilde{f} \) by \( \alpha \) and \( \tilde{\alpha} \), respectively. Then we get the following equation for the function \( T(t) \):
\[
\left( \frac{1}{T t} \right)_t = -n \alpha \frac{1}{T t} + \tilde{n} \tilde{\alpha},
\]
where \( \tilde{n} = n/\delta_3 \). We integrate this equation and present the general solution in such a form that continuous dependence of it on the parameters \( \alpha \) and \( \tilde{\alpha} \) is obvious:
\[
\begin{align*}
\alpha \tilde{\alpha} \neq 0: & \quad \frac{e^{\tilde{\alpha} T} - 1}{\tilde{n} \tilde{\alpha}} = \delta_1 \frac{e^{n \alpha t} - 1}{n \alpha} + \delta_2, \quad \alpha = 0, \quad \tilde{\alpha} \neq 0: \quad \frac{e^{\tilde{\alpha} T} - 1}{\tilde{n} \tilde{\alpha}} = \delta_1 t + \delta_2, \\
\alpha \neq 0, \quad \tilde{\alpha} = 0: & \quad T = \delta_1 \frac{e^{n \alpha t} - 1}{n \alpha} + \delta_2, \quad \alpha = \tilde{\alpha} = 0: \quad T = \delta_1 t + \delta_2.
\end{align*}
\] (32)
Here \( \delta_1 \neq 0 \) as \( T_t \neq 0 \).

**Theorem 8.** The generalized extended equivalence group \( \hat{G}^0_{m=0,(h/f)_x=0} \) of the subclass of class (33), which is singled out by the conditions \( m = 0 \) and \( (h/f)_x = 0 \), consists of the transformations
\[
\begin{align*}
\tilde{t} = T(t), & \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u + \psi(t, x), \\
\tilde{\varphi} = \frac{\delta_0}{\varphi_x} \Psi f, & \quad \tilde{g} = \delta_0 \varphi_x \Psi^2 g, \quad \tilde{n} = \frac{n}{\delta_3},
\end{align*}
\]
where the smooth function \( T = T(t) \) with \( T_t \neq 0 \) is defined by (32), \( \alpha = h/f \) and \( \tilde{\alpha} = \tilde{h}/\tilde{f} \) are constants, \( \varphi \) is an arbitrary smooth function of \( x \) with \( \varphi_x \neq 0 \), the functions \( \psi \) and \( \Psi \) are defined by the formulas
\[
\psi(t, x) = -\frac{\delta_3}{n} \ln |T(t)\Psi(x)|, \quad \Psi(x) = \delta_4 \int \frac{dx}{g(x)} + \delta_5,
\]
\( \delta_j, j = 0, \ldots, 5, \) are arbitrary constants, \( \delta_0 \delta_1 \delta_3 \neq 0 \) and \( (\delta_4, \delta_5) \neq (0, 0) \).
In contrast to $G_{m=n}^\sim$, we do not use $\hat{G}_{m=0,(h/f)_x=0}$ in the course of group classification of class (\textbf{3}) because the application of this conditional equivalence group does not have a crucial influence on classification, and the corresponding system $m = 0$, $(h/f)_x = 0$ for arbitrary elements is less obvious. At the same time, transformations from $\hat{G}_{m=0,(h/f)_x=0}$ play the role of additional equivalence transformations after completing the classification (see the previous section).

If $h\tilde{h} \neq 0$, $m = 0$ and $\tilde{m} = \tilde{n}$ then the determining equations yield

$$\frac{h}{f} = \frac{1}{n} T_{tt} \quad \text{i.e.,} \quad \left( \frac{h}{f} \right)_x = 0, \quad \tilde{h} = -\delta (\tilde{m}/\varphi x^y) \left( \theta_x g \right)_x.$$

Hence both equations (\textbf{3}) and (\textbf{4}) are mapped to the subclass '$h = 0$' of the class under consideration. These mappings are realized by the transformations from the corresponding conditional equivalence groups. We can assume that their images coincide. Therefore, the admissible transformation is composition of a transformation $T_1$ of (\textbf{3}) to the equation $f(x)u_t = (g(x)e^{\tilde{m}u_x})_x$ and a transformation $T_2$ of the equation $f(x)u_t = (g(x)e^{\tilde{m}u_x})_x$ to (\textbf{4}). The transformation $T_1$ belongs to $\hat{G}_{m=0,(h/f)_x=0}$ and the transformation $T_2$ belongs to $G_{m=n}^\sim$, and we can choose $\varphi = x$ in $T_1$.

The case $\tilde{h}h \neq 0$, $m = n$ and $\tilde{m} = 0$ is considered in similar way.

All the possible cases are exhausted. We summarize the investigation of admissible transformations in class (\textbf{3}) in the following assertion.

**Theorem 9.** Let the equations

$$f(x)u_t = (g(x)e^{\tilde{m}u_x})_x + h(x)e^{\tilde{m}u} \quad \text{and} \quad \tilde{f}(\tilde{x})u_t = (\tilde{g}(\tilde{x})e^{\tilde{m}\tilde{u}_x})_x + \tilde{h}(\tilde{x})e^{\tilde{m}\tilde{u}}$$

be connected via a point transformation $T$ in the variables $t$, $x$ and $u$. Then

- either $\frac{\tilde{m}}{\tilde{n}} = \frac{m}{n}$ or $(m, \tilde{m}) = (0, \tilde{n})$ or $(m, \tilde{m}) = (n, 0)$.

The transformation $T$ is induced by a transformation from

- a) $\hat{G}$ if either $m \neq 0, n$ or $m = 0$, $(h/f)_x \neq 0$;
- b) $G_{m=n}^\sim$ if $m = n$ and $\tilde{m} \neq 0$, then also $\tilde{m} = \tilde{n}$;
- c) $\hat{G}_{m=0,(h/f)_x=0}$ if $m = \tilde{m} = 0$, $(h/f)_x = 0$, then also $(\hat{h}/\hat{f})_x = 0$.

If $m = 0$ and $\tilde{m} = \tilde{n}$ then $(h/f)_x = 0$ and the transformation $T$ is the composition of two transformations, from $\hat{G}_{m=0,(h/f)_x=0}$ and $G_{m=n}^\sim$, with the intermediate equation having $h = 0$.

The case with $m = n$ and $\tilde{m} = 0$ is similar to the previous one.

**Corollary 3.** Class (\textbf{3}) is represented as the union of its three maximal normalized subclasses separated by the conditions

\[ (h \neq 0, m \neq 0, n) \quad \text{or} \quad (m = 0, (h/f)_x \neq 0) \quad \text{or} \quad (m = 0, (h/f)_x = 0, m = n). \]

Only the latter two subclasses have a non-empty intersection, and the intersection being the normalized subclass '$h = 0$'.

Recall that the class of differential equations is called normalized if any admissible transformation in this class belongs to its equivalence group. See \textbf{18} for strong definitions.

The subsets of equations appearing in Corollary 3 are really subclasses of class (\textbf{3}) since they are singled out from class (\textbf{3}) by usual systems of differential equations and/or inequalities with respect to arbitrary elements. This is not obvious only for the first subclass. The condition corresponding to it is in fact equivalent to a single inequality, $(m - n)h(m^2 + ((h/f)_x)^2) \neq 0$. 

\[ \]
Contractions

Examples of nontrivial limits between equations admitting Lie symmetry extensions are known for a long time. For instance, in [1] equations with exponential nonlinearities were excluded from the group classification list of nonlinear diffusion equations as a separate case and were just considered as a limiting case of equations with power nonlinearities. At the same time, it looks more convenient to include such cases to classification lists and then indicate connections between different classification cases via limiting processes. Using the analogy with theory of Lie algebras such connections are called contractions. A theoretical background on contractions of differential equations, their Lie symmetry algebras and solutions was first discussed in [11].

In this section we relate, via contractions, the group classification lists obtained for classes (3) and (2) in the present paper and in [23], respectively. For convenience of the presentation, in-equivalent cases of Lie symmetry extension in class (2) are collected in Table 2. Necessary explanations on the equivalence involved and thorough gauges of involved parameters by equivalence transformations are given in the appendix. Then contractions are used to construct exact solutions of equations from class (3) using known solutions of equations from class (2). In Section 6.3 we demonstrate three different ways of a similar consideration for conservation laws: in terms of contractions of associated characteristic or conserved vectors or divergence expressions themselves.

6.1 Contractions of equations and of Lie invariance algebras

At first we apply the equivalence transformation

\[
\begin{align*}
\tilde{t} &= \delta t, \\
\tilde{x} &= \sqrt{\delta} x, \\
\tilde{u} &= \delta(u - 1), \\
\tilde{n} &= \frac{n}{\delta}, \\
\tilde{m} &= \frac{m}{\delta}
\end{align*}
\]  

(33)

parameterized by a positive constant parameter \(\delta\) to the equation from class (2) with the values arbitrary elements \(g = 1\) and \(f\) and \(h\) presented in Case 2 of Table 2. The constant parameters \(a, b, c, d\) and \(p\) are transformed in the following way

\[
\begin{align*}
\tilde{a} &= a, \\
\tilde{b} &= \frac{b}{\sqrt{\delta}}, \\
\tilde{c} &= c, \\
\tilde{d} &= \frac{d}{\sqrt{\delta}}, \\
\tilde{p} &= \sqrt{\delta} p, \\
\tilde{\alpha} &= \delta \alpha
\end{align*}
\]  

(34)

wherever this is relevant, i.e., we change parameters if and only if they appear in the values of arbitrary elements of the initial equation. Then, we take the imaged equation and proceed to the limit \(\delta \to +\infty\). This results in the equation from class (11) with the values of the arbitrary elements \(f\) and \(h\) presented in Case 2 of Table 1. The same procedure establishes a contraction between the associated Lie algebras of vector fields. The corresponding notation will be 2.2 \(\to\) 1.2, where the first numbers indicate the numbers of the tables and the second numbers indicate the numbers of cases within the tables. We present the complete list of contractions which replace power nonlinearities by exponential ones and, therefore, connect cases of Lie symmetry extensions for classes (3) and (2):

\[
\begin{align*}
2.1 &\to 1.1, \quad 2.2 \to 1.2, \quad 2.3 \to 1.3, \quad 2.4 \to 1.4, \quad 2.5 \to 1.5, \\
2.6 &\to 1.6, \quad 2.8 \to 1.7, \quad 2.9 \to 1.8, \quad 2.10 \to 1.9, \quad 2.11 \to 1.10.
\end{align*}
\]

6.2 Contractions of Lie reductions and exact solutions

In [23] we carried out Lie reductions and constructed Lie exact solution for equations from class (2) with the values of arbitrary elements presented in Cases 9 and 12 of Table 2, which admit three-dimensional Lie symmetry algebras. It is shown in the previous subsection that
Parameter-functions $f$ from classes (3) and (2) are whose maximal Lie invariance algebras $\mathfrak{g}$ and $\mathfrak{g}$ are generated by the vector fields $\tilde{X}_1 = \partial_t$, $\tilde{X}_2 = n\partial_t - u\partial_x$, $\tilde{X}_3 = n\partial_x + 2u\partial_u$, and $X_1 = \partial_t$, $X_2 = n\partial_t - u\partial_u$, $X_3 = nx\partial_x + 2u\partial_u$.

### Table 2. Results of group classification of class (2) under the gauge $g = 1$

| no. | $n$ | $f(x)$ | $h(x)$ | Basis of $A^{max}$ |
|-----|-----|--------|--------|---------------------|
| 1   | $\forall$ | $\forall$ | $\forall$ | $\partial_t$ |
| 2   | $\forall$ | $f_1(x)$ | $h_1(x)$ | $\partial_t, (d + 2b - nm)t\partial_t + ((n + 1)ax^2 + bx + c)\partial_x + (ax + p)u\partial_u$ |
| 3   | $\forall$ | 1 | $\varepsilon$ | $\partial_t, \partial_x, 2(1 - m)t\partial_t + (1 + n - m)x\partial_x + 2u\partial_u$ |
|     |     |     |     | $m = 1, \ h \neq 0, \ (h/f)_x = 0$ |
| 4   | $\forall$ | $\varepsilon f$ | $\varepsilon f$ | $\partial_t, e^{-\varepsilon t}(\partial_t + \varepsilon u\partial_u)$ |
| 5   | $\forall$ | $f_1(x)$ | $\varepsilon f$ | $\partial_t, e^{-\varepsilon t}(\partial_t + \varepsilon u\partial_u), n((n + 1)ax^2 + bx + c)\partial_x + (nax + 2b + d)u\partial_u$ |
| 6   | $\neq -\frac{4}{3}$ | 1 | $\varepsilon$ | $\partial_t, \partial_x, e^{-\varepsilon t}(\partial_t + \varepsilon u\partial_u), nx\partial_x + 2u\partial_u$ |
| 7   | $\neq -\frac{4}{3}$ | 1 | $\varepsilon$ | $\partial_t, \partial_x, e^{\varepsilon t}(\partial_t + \varepsilon u\partial_u), -\frac{1}{4}x\partial_x + 2u\partial_u, -\frac{1}{4}x^2\partial_x + xu\partial_u$ |
|     |     |     |     | $m = n + 1$ or $h = 0$ |
| 8   | $\forall$ | $\forall$ | $\partial_t, nt\partial_t - u\partial_u$ |
| 9   | $\neq -\frac{4}{3}$ | 1 | $ax^{-2}$ | $\partial_t, nt\partial_t - u\partial_u, 2t\partial_t + x\partial_x$ |
| 10  | $\neq -\frac{4}{3}$ | 1 | $\varepsilon$ | $\partial_t, nt\partial_t - u\partial_u, \partial_x$ |
| 11  | $\neq -\frac{4}{3}$ | 1 | 0 | $\partial_t, \partial_x, nt\partial_t - u\partial_u, 2t\partial_t + x\partial_x$ |
| 12  | $\neq -\frac{4}{3}$ | $e^\alpha$ | $\alpha$ | $\partial_t, t\partial_t + \frac{4}{4}u\partial_u, \partial_x - \frac{4}{4}u\partial_u$ |
| 13  | $\neq -\frac{4}{3}$ | 1 | 0 | $\partial_t, \partial_x, \frac{1}{2}t\partial_t + u\partial_u, 2t\partial_t + x\partial_x, -\frac{1}{4}x^2\partial_x + xu\partial_u$ |

Here $\alpha$ is arbitrary constant, $\alpha \neq 0$ in Case 9, $\varepsilon = \pm 1$, 

$$f_1(x) = \exp \left[ \int \frac{-(3n + 4)ax + d}{(n + 1)ax^2 + bx + c} \, dx \right], \quad h_1(x) = \exp \left[ \int \frac{-(3(n + 1) + m)ax + (n - m + 1)p - 2b}{(n + 1)ax^2 + bx + c} \, dx \right],$$

and it can be assumed up to $G_{n-1}^m$-equivalence (see Theorem 11) that, if $n \neq -1$, the parameter tuple $(a, b, c, d, p)$ takes only the following inequivalent values:

$\{(0, 1, 0, \tilde{d}, (\bar{q} + 2)/(n + m - 1)), (0, 0, 1, 1, \tilde{p}), (0, 0, 1, 0, 1), (1/(n + 1), 0, 1, \tilde{d}, \tilde{p})\}$,

where $\tilde{p}$ is an arbitrary constant; $\tilde{d} \geq 0$ and, if $\tilde{d} = 0$, $\tilde{p} \geq 0$;

$$(\tilde{d}, \bar{q}) \neq (0, 0), \left( -\frac{3n + 4}{n + 1}, -3 - \frac{m}{n + 1} \right); \tilde{d} \geq -\frac{3n + 4}{2(n + 1)} \text{ and, if } \tilde{d} = -\frac{3n + 4}{2(n + 1)} \bar{q} \geq \frac{3}{2} - \frac{m}{2(n + 1)}.$$
respectively. The contraction $2.9 \to 1.8$ can be realized using the simpler transformation
\[
\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = \delta(u - 1), \quad \tilde{n} = \frac{n}{\delta}, \quad \tilde{\alpha} = \delta \alpha
\] 
(37)

than transformation (33). In the course of this contraction the algebra $\mathfrak{g}$ is contracted to the algebra $\tilde{\mathfrak{g}}$ as a Lie algebra of vector fields in the space of $(t, x, u)$. Namely, $X_1 \to \tilde{X}_1$, $X_2 \to \tilde{X}_2$ and $X_3 \to \tilde{X}_3$.

Let us study the related contractions of Lie reductions of equation (36) to ones of equation (35). Inequivalent Lie reductions of these equations with respect to one-dimensional subalgebras of the corresponding maximal Lie invariance algebras are exhausted by those presented in Tables 3 and 4. For convenience we omit tildes in Table 4. The transformations of the invariant independent and dependent variables, which are induced by transformation (37), take the form $\tilde{\phi} = \delta(\varphi - 1)$ and $\tilde{\omega} = \omega$ in all cases of Table 3.

| N | $X$ | $\omega$ | $u = \varphi(\omega)$ | Reduced ODE |
|---|---|---|---|---|
| 1 | $X_2 - \mu X_3$ | $x|t|^\mu$ | $|t|^{\frac{1 + 2n}{n}} \varphi(\omega)$ | $(\varphi^n \varphi_\omega)_{\omega} - \mu \omega e \varphi_\omega + (1 + 2\mu)\varepsilon + \alpha n \omega^{-2} e^{n\varphi} = 0$, $\varepsilon = \text{sign } t$ |
| 2 | $X_3$ | $t$ | $|t|^\frac{2}{n} \varphi(\omega)$ | $n^2 \varphi_\omega - (an^2 + 2n + 4)\varphi^{n+1} = 0$ |
| 3 | $X_3 \pm X_1$ | $xe^{\mp t}$ | $e^{\frac{2\alpha}{n}} \varphi(\omega)$ | $(\varphi^n \varphi_\omega)_{\omega} \pm \omega \varphi_\omega \alpha + \frac{2}{n} \varphi + \alpha n \omega^{-2} \varphi^{n+1} = 0$ |
| 4 | $X_1$ | $x$ | $\varphi(\omega)$ | $(\varphi^n \varphi_\omega)_{\omega} + \omega n \varphi_\omega e^{n\varphi} = 0$ |

Table 3. Lie reductions for Case 9 of Table 2.

| N | $X$ | $\omega$ | $u = \varphi(\omega)$ | Reduced ODE |
|---|---|---|---|---|
| 1 | $X_2 - \mu X_3$ | $x|t|^\mu$ | $\varphi(\omega) - \frac{1 + 2n}{n} \ln |t|$ | $(e^{n\varphi})_{\omega} - \mu e \varphi_\omega + (1 + 2\mu)\varepsilon + \alpha n \omega^{-2} e^{n\varphi} = 0$, $\varepsilon = \text{sign } t$ |
| 2 | $X_3$ | $t$ | $\varphi(\omega) + \frac{2}{n} \ln |x|$ | $n \varphi_\omega - (an + 2) e^{n\varphi} = 0$ |
| 3 | $X_3 \pm X_1$ | $xe^{\mp t}$ | $\varphi(\omega) \pm \frac{2}{n} t$ | $(e^{n\varphi})_{\omega} \pm \omega \varphi_\omega \alpha + \frac{2}{n} \varphi + \omega n \omega^{-2} e^{n\varphi} = 0$ |
| 4 | $X_1$ | $x$ | $\varphi(\omega)$ | $(e^{n\varphi})_{\omega} + \omega n \varphi_\omega e^{n\varphi} = 0$ |

Table 4. Lie reductions for Case 8 of Table 1.

Consider Case 1 of Table 3 in detail. The transformed version and the corresponding limit of the ansatz $u = |t|^{-\frac{1 + 2n}{n}} \varphi(\omega)$ are
\[
\left(1 + \frac{\tilde{u}}{\delta}\right)^{\frac{\delta n}{\varphi}} = |\tilde{t}|^{-\left(1 + 2\mu\right)} \left(1 + \frac{\varphi}{\delta}\right)^{\frac{\delta n}{\varphi}} \rightarrow e^{\tilde{\varphi} \frac{\delta n}{\varphi}} = |\tilde{t}|^{-\left(1 + 2\mu\right)} e^{\tilde{\varphi} \frac{\delta n}{\varphi}} \quad \text{at} \quad \delta \to +\infty.
\]

Therefore, the contracted ansatz is $\tilde{u} = \tilde{\varphi}(\tilde{\omega}) \cdot |\tilde{t}|^{-\frac{1 + 2n}{n}} \ln |\tilde{t}|$. The reduced equation from Case 1 of Table 3 is mapped by transformation (37) to the equation
\[
\frac{\delta \tilde{n}}{\delta \varphi} + 1 \left(1 + \frac{\varphi}{\delta}\right)^{\frac{\delta n}{\varphi} + 1} \tilde{\omega} \tilde{\omega} = -\mu \tilde{\varphi} \tilde{\omega} \tilde{\varphi}_\omega + (1 + 2\mu) \varepsilon \left(1 + \frac{\varphi}{\delta}\right) + \frac{\tilde{\varphi} \tilde{n}}{\tilde{\omega}^2} \left(1 + \frac{\varphi}{\delta}\right)^{\frac{\delta n}{\varphi} + 1} = 0.
\]

Then the limit process at $\delta \to +\infty$ leads to the equation
\[
(e^{\tilde{\varphi} \frac{\delta n}{\varphi}})_{\tilde{\omega} \tilde{\omega}} - \mu \tilde{\varphi} \tilde{\omega} \tilde{\varphi}_\omega + (1 + 2\mu) \varepsilon + \frac{\tilde{\varphi} \tilde{n}}{\tilde{\omega}^2} e^{\tilde{\varphi} \frac{\delta n}{\varphi}} = 0
\]
which is also obtained from equation (35) by the reduction with respect to the contracted ansatz and presented by Case 1 of Table 4.
Analogously we obtain contractions of reductions 3.2 → 4.2, 3.3 → 4.3 and 3.4 → 4.4.

For Cases 2 and 4 of Table 3 exact solutions of reduced equations were found in [23]. The substitution of these solutions to the respective ansatzes results in the following exact solutions of equation (36):

\[ u = \left| \frac{x^2}{C - (\alpha n + 2 + 4n^{-1})t} \right|^{\frac{1}{n}}, \quad \text{for equation (35)} \]

\[ u = \begin{cases} 
  C_1 \sqrt{x} \ln x + C_2 \sqrt{x} \bigg|^{\frac{1}{n+1}}, & \text{if } \alpha' = 0, \\
  C_1 x^{\kappa_1} + C_2 x^{\kappa_2} \bigg|^{\frac{1}{n+1}}, & \text{if } \alpha' > 0, \\
  C_1 \sqrt{x} \sin(\sigma \ln x) + C_2 \sqrt{x} \cos(\sigma \ln x) \bigg|^{\frac{1}{n+1}}, & \text{if } \alpha' < 0,
\end{cases} \]

where

\[ \alpha' = 1 - 4\alpha(n + 1), \quad \kappa_{1,2} = \frac{1 \pm \sqrt{\alpha'}}{2}, \quad \sigma = \frac{\sqrt{-\alpha'}}{2}. \]

Here and in what follows C, C_1 and C_2 are arbitrary constants. Applying transformation (37) to solution (38) and proceeding with the limit \( \delta \to +\infty \), we obtain

\[ (1 + \frac{\ddot{u}}{\delta}) = \ddot{x}^2 \left( C - (\ddot{\alpha} n + 2 + \frac{4}{n\delta}) t \right)^{-1} \to e^{\ddot{\alpha} n} = \ddot{x}^2 \left( C - (\ddot{\alpha} n + 2) \ddot{t} \right)^{-1}. \]

As a result, we construct the exact solution

\[ \ddot{u} = \frac{1}{\ddot{n}} \ln \left| \frac{x^2}{C - (\ddot{\alpha} n + 2) \ddot{t}} \right|. \]

for equation (35). Applying the same technique to solutions (39) leads to the steady-state exact solutions of (35):

\[ \ddot{u} = \begin{cases} 
  \frac{1}{\ddot{n}} \ln |C_1 \sqrt{x} \ln x + C_2 \sqrt{x}|, & \text{if } \ddot{\alpha}' = 0, \\
  \frac{1}{\ddot{n}} \ln |C_1 \ddot{x}^{\kappa_1} + C_2 \ddot{x}^{\kappa_2}|, & \text{if } \ddot{\alpha}' > 0, \\
  \frac{1}{\ddot{n}} \ln |C_1 \sqrt{x} \sin(\sigma \ln \ddot{x}) + C_2 \sqrt{x} \cos(\sigma \ln \ddot{x})|, & \text{if } \ddot{\alpha}' < 0,
\end{cases} \]

where

\[ \ddot{\alpha}' = 1 - 4\ddot{\alpha} n, \quad \kappa_{1,2} = \frac{1 \pm \sqrt{\ddot{\alpha}'}}{2}, \quad \sigma = \frac{\sqrt{-\ddot{\alpha}'}}{2}. \]

Another way for finding this solution is to integrate the reduced equation of Case 4 from Table 4. By the obvious transformation \( \dot{\varphi} = e^{\alpha \varphi} \) the reduced equation is mapped to the Euler equation \( \omega^2 \ddot{\varphi} + \alpha n \dot{\varphi} = 0 \).

### 6.3 Contractions of conservation laws

We use contractions in order to construct conservation laws of equations from class (3) with \( g = 1 \) using results obtained in [23] for equations from class (2) with the same gauge of \( g \). Note that the consideration can be easily extended to the entire classes (3) and (2) using transformations from the corresponding equivalence groups.

Roughly speaking, a conservation law of a system \( \mathcal{L} \) of differential equations is a divergence expression that vanishes on solutions of this system. Thus, in the case of two independent variables \( t \) and \( x \) and one unknown function \( u \) the general form of conservation laws is
\[ D_t F(t, x, u_{(r)}) + D_x G(t, x, u_{(r)}) = 0 \] whenever \( u \) is a solution of \( \mathcal{L} \). Here \( D_t \) and \( D_x \) are the operators of total differentiation with respect to \( t \) and \( x \), respectively, and \( u_{(r)} \) denotes the set of all the derivatives of the functions \( u \) with respect to \( t \) and \( x \) of order not greater than \( r \), including \( u \) as the derivative of the zero order. The components \( F \) and \( G \) of the conserved vector \((F, G)\) are called the density and the flux of the conservation law. Two conserved vectors \((F, G)\) and \((F', G')\) are equivalent if there exist such functions \( \tilde{F} \), \( \tilde{G} \) and \( H \) of \( t \), \( x \) and derivatives of \( u \) that \( \tilde{F} \) and \( \tilde{G} \) vanish for all solutions of \( \mathcal{L} \) and \( F' = F + \tilde{F} + D_x \mathcal{H} \), \( G' = G + \tilde{G} - D_t \mathcal{H} \). A conserved vector is called trivial if it is equivalent to the zero conserved vector.

It is found in \([23]\) that there are only two subclasses of equations of form \((2)\) which admit nontrivial conserved vectors. Thus, assuming the gauge \( g = 1 \), each equation from class \((2)\) with \( m = n + 1 \), i.e., an equation of the form \( f(x)u_t = (u^n u_x)_x + h(x)u^{n+1}_x \), admits two linearly independent conservation laws with the following conserved vectors \((F^i, G^i)\) and the characteristics \( \lambda^i, i = 1, 2 \):

\[
\begin{align*}
n \neq -1: & \quad \left( \phi^i f u, -\phi^i u^n u_x + \phi^i u^{n+1} \frac{x}{n+1} \right), \quad \lambda^1 = \phi^i, \ i = 1, 2; \\
n = -1: & \quad (x f u, -x u^{-1} u_x + \ln u), \quad \lambda^1 = x; \quad (f u, -u^{-1} u_x), \quad \lambda^2 = 1.
\end{align*}
\]

Here \( \beta_1 \) and \( \beta_2 \) are arbitrary constants. The functions \( \phi^i = \phi^i(x) \), \( i = 1, 2 \), form a fundamental set of solutions of the second-order linear ordinary differential equation \( \phi_{xx} + (n + 1)h \phi = 0 \).

The other subclass of equations admitting nontrivial conserved vectors is singled out from \((2)\) under the gauge \( g = 1 \) by the conditions \( m = 1 \) and \( h = \alpha f \), where \( \alpha \) is an arbitrary constant, i.e., it consists of equations of the form

\[ f(x)u_t = (u^n u_x)_x + \alpha f(x)u. \] (40)

The corresponding conserved vectors and characteristics are

\[
\begin{align*}
n \neq -1: & \quad (xe^{-\alpha t} f u, e^{-\alpha t} (-xu^n u_x + u^{n+1} \frac{x}{n+1})), \quad \lambda^1 = xe^{-\alpha t}, \ \
& \quad (e^{-\alpha t} f u, -e^{-\alpha t} u^n u_x), \quad \lambda^2 = e^{-\alpha t}; \\
n = -1: & \quad (xe^{-\alpha t} f u, e^{-\alpha t} (-xu^{-1} u_x + \ln u)), \quad \lambda^1 = xe^{-\alpha t}, \\
& \quad (e^{-\alpha t} f u, -e^{-\alpha t} u^{-1} u_x), \quad \lambda^2 = e^{-\alpha t}.
\end{align*}
\] (41)

In order to contract equations from class \((2)\) to equations from class \((3)\), we should vary the arbitrary element \( n \). This is why only the case of general \( n \) is appropriate for contractions. There are three different ways in order to realize contractions of conservation laws. Namely, we can contract characteristics of conservation laws, their conserved vectors or conservation laws as divergent expressions themselves.

We illustrate these possibilities in detail using equations \((40)\) with \( n \neq -1 \) and their conservation laws associated the same characteristic \( \lambda^1 = xe^{-\alpha t} \). The corresponding conserved vectors are presented in \((41)\). At first we apply equivalence transformation \((37)\) to equation \((40)\) and proceed to the limit \( \delta \to +\infty \). As a result, we obtain the class of equations (tildes are omitted)

\[ f(x)u_t = (e^{nu} u_x)_x + \alpha f(x), \] (42)
i.e., equations from class \((11)\) with \( m = 0 \) and \( h = \alpha f \).

For the image \( \tilde{\lambda}^1 \) of the characteristic \( \lambda^1 = xe^{-\alpha t} \) with respect to transformation \((37)\) we have that \( \tilde{\lambda}^1 \to x \) if \( \delta \to +\infty \). Now we are able to construct the corresponding conservation law of \((42)\) using the characteristic obtained as an integrating factor. After the multiplication by \( x \) the equation \((42)\) can be written in divergent form as

\[ D_t (xf u - \alpha x f t) + D_x \left( -xe^{nu} u_x + \frac{1}{n} e^{nu} \right) = 0. \] (43)
Therefore, we construct conservation law (43) of equation (42) via carrying out a limiting process of characteristics. Another way is to directly deal with divergence expressions. Thus the conservation law

\[ D_t \left( xe^{-at} fu - e^{-at} xu^n u_x + e^{-at} u^{n+1} \right) = 0 \]

of (40) with \( n \neq -1 \) is transformed by (37) to

\[ \tilde{D}_t \left( \tilde{x} e^{-\frac{\tilde{\alpha}}{\delta} t} \left( \tilde{u} + 1 \right) \right) + \tilde{D}_\tilde{x} \left( -e^{-\frac{\tilde{\alpha}}{\delta} \tilde{x}} \left( \frac{\tilde{u}}{\delta} + 1 \right) \frac{\tilde{n}}{\delta} \tilde{u} + e^{-\frac{\tilde{\alpha}}{\delta}} \frac{1}{\tilde{n} + 1} \left( \frac{\tilde{u}}{\delta} + 1 \right) \tilde{n} + 1 \right) = 0. \]

Multiplying the obtained expression by \( \delta \) and adding the term \( -\tilde{x} f \delta \) under \( D_\tilde{t} \), we proceed to the limit \( \delta \to +\infty \). (As the term \( -\tilde{x} f \delta \) depends only on \( \tilde{x} \), it is negligible in view of the differentiation with respect to \( \tilde{t} \).) Omitting tildes, we exactly obtain the conservation law (43).

Conserved vectors can be contracted in the same way using the fact that we can add expression which depends on \( \tilde{x} \) only to the density component up to equivalence of conserved vectors.

Contracting the conservation laws obtained in [23] jointly with the corresponding equations, we obtain the following assertion.

**Theorem 10.** A complete list of equations from class (11) possessing nontrivial conservation laws is exhausted by the following ones.

1. \( m = n \):
   \[
   (\varphi^i u, -\varphi^i e^{nu} u_x + \frac{1}{n} \varphi^i e^{nu}), \quad \varphi^i, \quad i = 1, 2.
   \]

2. \( m = 0, \quad h = \alpha f \):
   \[
   (xf(u - \alpha t), -xe^{nu} u_x + \frac{1}{n} e^{nu}), \quad (f(u - \alpha t), -e^{nu} u_x), \quad 1.
   \]

Here the functions \( \varphi^i = \varphi^i(x) \), \( i = 1, 2 \), form a fundamental set of solutions of the second-order linear ordinary differential equation \( \varphi_{xx} + nh \varphi = 0 \) and \( \alpha \) is an arbitrary constant.

Simultaneously with constraints on the arbitrary elements we also present conserved vectors and characteristics of the basis elements of the corresponding space of conservation laws.

7 Conclusion

In this paper we carry out the extended Lie symmetry analysis of equations from class (1), where \( A \) and \( B \) are exponential functions of \( u \). In this sense the present paper continues the series of papers [22–25], where equations (1) with power nonlinearities and semilinear equations of this form with exponential source were studied.

Lie symmetries, admissible transformations and conservation laws of equations from class (3) are exhaustively classified. This difficult task is achieved due to the usage of generalized equivalence groups instead of usual ones, nontrivial gauging of arbitrary elements and the representation of class (3) as a union of normalized subclasses. Moreover, in the course of group classification of equations with \( m = n \) the associated conditional equivalence group is used, which essentially simplified calculations and the final result.

We also study limit processes, called contractions, between equations from classes (2) and (3) jointly with limit processes between the corresponding Lie invariance algebras. This allows us to predict and then to check the results of group classification for class (3). Moreover, we construct conservation laws and exact solutions for equations from class (3) using contractions and the related results obtained for equations of the form (2) in [23].
The feature of this paper in comparison with the preceding ones is that the method of furcate split applied to classify Lie symmetries classification as well as the usage of the direct method for finding equivalence and admissible transformations are described in detail.

Existing examples on group classification of variable–coefficient diffusion–convection equations \[8, 10, 19\] show that the subclasses with power or exponential nonlinearities \(A\) and \(B\) are usually the most complicated to be classified. In a forthcoming paper we intend to complete Lie symmetry analysis of equations from class (1), where the nonlinearities \(A\) or \(B\) are not power or exponential functions.

A Gauging of parameters in classification list with power nonlinearities

We briefly describe results on the group classification of class (2) obtained in \[23\], which are needed for the proper understanding of the final list of inequivalent Lie symmetry extensions collected in Table 2. In order to attain to the complete similarity to the presentation of the group classification of class (3) in the present paper, we carry out certain modifications and enhancements of results from \[23\]. In particular, we thoroughly gauge parameters appearing in Cases 2 and 5 of Table 2 and partition these cases into simpler inequivalent subclasses in the same way as this is done for class (3) in Section 4.

Via gauging of the arbitrary element \(g\) by transformations from the corresponding equivalence group, the group classification of class (2) is reduced to that of its subclass singled out by the constraint \(g = 1\).

**Theorem 11.** The generalized equivalence group \(G_{g=1}^{-}\) of the subclass of equations having the form (2) with \(g = 1\) consists of the transformations

\[
\begin{align*}
\tilde{t} &= \delta_1 t + \delta_2, \quad \tilde{x} = \frac{\delta_6 x + \delta_7}{\delta_4 x + \delta_5}, \quad \tilde{u} = \delta_3 \frac{\delta_4 x + \delta_5}{\delta_7 + n} u, \\
\tilde{f} &= \frac{\delta_1 \delta_2}{\Delta^2} \frac{\delta_4 x + \delta_5}{\delta_7 + n} f, \quad \tilde{h} = \frac{\delta_3 \delta_4 x + \delta_5}{\Delta^2} \frac{\delta_4 x + \delta_5}{\delta_7 + n} h, \quad \tilde{n} = n, \quad \tilde{m} = m,
\end{align*}
\]

where \(\delta_j, j = 1, \ldots, 7\), are arbitrary constants such that \(\delta_1 \delta_3 \neq 0\), \(\Delta = \delta_3 \delta_6 - \delta_4 \delta_7 \neq 0\) and the tuple \((\delta_4, \delta_5, \delta_6, \delta_7)\) is defined up to a nonzero multiplier, e.g., we can set \(\Delta = \pm 1\). The arbitrary element \(n\) is assumed to be unequal to \(-1\). For \(n = -1\) transformations from the group \(G_{g=1}^{-}\) take the form

\[
\begin{align*}
\tilde{t} &= \delta_1 t + \delta_2, \quad \tilde{x} = \frac{\delta_4 x + \delta_5}{\delta_7 + n} u, \\
\tilde{f} &= \frac{\delta_1}{\delta_4^2 \delta_3 e^\delta_6 x} f, \quad \tilde{h} = \frac{1}{\delta_4^2 \delta_3 e^\delta_6 x} h, \quad \tilde{n} = n, \quad \tilde{m} = m,
\end{align*}
\]

where \(\delta_j, j = 1, \ldots, 6\), are arbitrary constants, \(\delta_1 \delta_3 \delta_4 \neq 0\).

**Theorem 12.** The conditional equivalence group \(G_{g=1, m=1}^{-}\) of class (2) associated with the constraints \(g = 1\) and \(m = n + 1\) is formed by the transformations

\[
\begin{align*}
\tilde{t} &= \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \psi(x) u, \\
\tilde{f} &= \frac{\delta_3 \delta_1}{\psi^{3n+3}} f, \quad \tilde{h} = \frac{\delta_0 \psi^{n+1} |\psi|^{-(n+2)} \varphi_x}{\psi^{4n+4}} h, \quad \tilde{n} = n.
\end{align*}
\]

Here \(\varphi\) and \(\psi\) are arbitrary smooth functions of \(x\) and \(\delta_j, j = 0, 1, 2\), are arbitrary constants satisfying the conditions \(\delta_0 \varphi_x = \psi^{2n+2}\) and \(\delta_0 \delta_1 \psi \neq 0\).
In the course of group classification in the general case \( m \neq 1, n + 1 \) we derive that the maximal Lie invariance algebra of an equation from class (2) with \( g = 1 \) is a proper extension of the kernel algebra \( \langle \partial_t \rangle \) only if the corresponding value of the arbitrary elements \( f \) and \( h \) satisfy a system of the form

\[
\begin{align*}
((n+1)ax^2 + bx + c) \frac{f_x}{f} &= -(3n+4)ax + d, \\
((n+1)ax^2 + bx + c) \frac{h_x}{h} &= -(3(n+1)+m)ax + (1+n-m)p - 2b,
\end{align*}
\]

where \( a, b, c, d \) and \( p \) are constants which are not simultaneously equal to zero.

**Lemma 2.** If \( n \neq -1 \), up to \( G_{g=1} \)-equivalence the parameter tuple \((a,b,c,d,p)\) can be assumed to belong to the set

\[
\{(0,1,0,\hat{a},\hat{p}), (0,0,1,1,\hat{p}), (0,0,1,0,1), (0,0,1,0,0), (1/(n+1),0,1,\hat{a},\hat{p})\},
\]

where \( \hat{a} \geq 0 \) and, if \( \hat{a} = 0, \hat{p} \geq 0; \hat{p} \) is an arbitrary constant;

\[
\bar{a} \geq -\frac{3n+4}{2(n+1)} \quad \text{and, if} \quad \bar{a} = -\frac{3n+4}{2(n+1)}, \quad \bar{p} \geq \frac{1}{2(n+1)}.
\]

If \( n = -1 \), \( G_{g=1} \)-inequivalent values of the parameter tuple \((a,b,c,d,p)\) are exhausted by elements of the set

\[
\{(0,1,0,d',p'), (0,0,1,0,1), (0,0,1,0,0), (\epsilon'',0,1,0,p'')\},
\]

where \( d' \) and \( p' \) are arbitrary constants, \( \epsilon'' = \pm 1 \) and \( p'' > 0 \).

**Proof.** Combined with the multiplication by a nonzero constant, each transformation from the equivalence group \( G_{g=1} \) is extended to the coefficient tuple of the system (44). The extension takes the form

\[
\begin{align*}
(n+1)\bar{a} &= \nu (\delta^2_5(n+1)a - \delta_4 \delta_5 b + \delta^2_5 c), \\
\bar{b} &= \nu (-2\delta_5 \delta_7 (n+1)a + (\delta_4 \delta_7 + \delta_5 \delta_6)b - 2\delta_4 \delta_6 c), \\
\bar{c} &= \nu (\delta^2_7(n+1)a - \delta_6 \delta_7 b + \delta^2_6 c), \\
\bar{d} &= \nu \Delta d + \frac{3n+4}{n+1} \nu (\delta_5 \delta_7 (n+1)a - \delta_4 \delta_7 b + \delta_4 \delta_6 c), \\
\bar{p} &= \nu \Delta p - \frac{1}{n+1} \nu (\delta_5 \delta_7 (n+1)a - \delta_4 \delta_7 b + \delta_4 \delta_6 c)
\end{align*}
\]

if \( n \neq -1 \) and

\[
\begin{align*}
\bar{a} &= \frac{\nu}{\delta_4} (a + \delta_6 b), \\
\bar{b} &= \nu b, \\
\bar{c} &= \nu (\delta_4 c - \delta_5 b), \\
\bar{d} &= \nu d + \frac{\nu}{\delta_4} (\delta_5 a + \delta_6 \delta_6 b - \delta_6 \delta_4 c), \\
\bar{p} &= \nu p - \frac{\nu}{\delta_4} (\delta_5 a + \delta_5 \delta_6 b - \delta_6 \delta_4 c)
\end{align*}
\]

if \( n = -1 \). Here \( \Delta = \delta_5 \delta_6 - \delta_4 \delta_7 \neq 0 \) and \( \nu \) is an arbitrary nonzero constant.

If \( n \neq -1 \), the proof is similar to the proof of Lemma 1. In this case there are only three \( G_{g=1} \)-inequivalent values of the triple \((a,b,c)\) depending on the sign of \( D = b^2 - 4nac \); the possibilities are

\[
(0,1,0) \quad \text{if} \quad D > 0, \quad (0,0,1) \quad \text{if} \quad D = 0, \quad (1/(n+1),0,1) \quad \text{if} \quad D < 0.
\]

Indeed, if \( D > 0 \) then there exist two linearly independent pairs \((\delta_4, \delta_5)\) and \((\delta_6, \delta_7)\) such that \( \delta^2_5(n+1)a - \delta_4 \delta_5 b + \delta^2_5 c = 0 \) and \( \delta^2_7(n+1)a - \delta_6 \delta_7 b + \delta^2_6 c = 0 \). For these values of \( \delta^2 \) we have \( \bar{a} = \bar{c} = 0 \). In the case \( D = 0 \) we choose values of \( \delta_4, \delta_5, \delta_6 \) and \( \delta_7 \) for which \( \delta^2_5(n+1)a - \delta_4 \delta_5 b + \delta^2_5 c = 0 \)
and the pair \((\delta_6, \delta_7)\) is not proportional to the pair \((\delta_4, \delta_5)\). Then we obtain that \(\tilde{a} = 0\) and 
\[
\tilde{b} = \nu \delta_7(\delta_4 b - 2\delta_5 (n+1)a) + \nu \delta_6(\delta_4 b - 2\delta_5 c) = 0.
\]
The residual coefficient \(\tilde{b}\) if \(D > 0\) and \(\tilde{c}\) if \(D = 0\) is necessarily nonzero and hence it can be scaled to 1 using the multiplication by the appropriate value of \(\nu\). If \(D < 0\), we have \(ac \neq 0\) and can set \(a > 0\). As the matrix
\[
\begin{pmatrix}
(n+1)a & -b/2 \\
-b/2 & c
\end{pmatrix}
\]
is symmetric and positive, the corresponding bilinear form is a well-defined scalar product.

Choosing \(\nu = 1\) and pairs \((\delta_4, \delta_5)\) and \((\delta_6, \delta_7)\) which are orthonormal with respect to this product, we obtain \((n+1)a = \tilde{c} = 1\) and \(\tilde{b} = 0\).

Now for each of the above inequivalent form for both the tuples \((a, b, c)\) and \((\tilde{a}, \tilde{b}, \tilde{c})\) we look for possible gauges of the coefficients \(d\) and \(p\).

Thus, from \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (0, 1, 0)\) we derive \(\delta_4 \delta_5 = \delta_6 \delta_7 = 0\) and \(\delta_4 \delta_7 + \delta_5 \delta_6 = \nu^{-1}\). This system in \(\delta\)'s has two solutions, \(\delta_4 = \delta_7 = 0\) with \(\Delta = \delta_5 \delta_6 = \nu^{-1}\) and \(\delta_5 = \delta_6 = 0\) with \(\Delta = -\delta_4 \delta_7 = -\nu^{-1}\). The first solution leads to the identical transformation of the coefficients \(d\) and \(p\). For the second solution the transformation takes the form \(\tilde{d} = -d - (3n + 4)/(n+1), \tilde{p} = -p+1/(n+1)\). This is why up to \(G_\infty\)-equivalence we can assume that \(\tilde{d}\) and \(\tilde{p}\) satisfy (15).

Setting \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (0, 0, 1)\) results in \(\delta_4 = 0, \nu \delta_6^2 = 1\) and \(\Delta = \delta_5 \delta_6 \neq 0\). The transformation of \(d\) and \(p\) is reduced to simultaneous scaling with the same multipliers, \(\tilde{d} = \nu \Delta d\) and \(\tilde{p} = \nu \Delta p\). This allows us either to set \(\tilde{d} = 1\) if \(d \neq 0\) or to scale \(\tilde{p}\) if \(d = 0\) and hence \(\tilde{d} = 0\). As a result, we obtain the tuples \((0, 0, 1, 1, \tilde{p})\) if \(d \neq 0\), \((0, 0, 1, 0, 1)\) if \(d = 0\) and \(p \neq 0\), and \((0, 0, 1, 0, 0)\) in the case \(d = p = 0\). The last tuple corresponds to equations of the form (2) with constant coefficients (Case 3 of Table 2).

The equality \((n+1)a, b, c) = ((n+1)\tilde{a}, \tilde{b}, \tilde{c}) = (1, 0, 1)\) implies \(\delta_4^2 + \delta_6^2 = \delta_5^2 + \delta_7^2 = \nu^{-1}\) and \(\delta_4 \delta_6 + \delta_5 \delta_7 = 0\). Hence \(\delta_6 = \tilde{\epsilon} \delta_5\) and \(\delta_7 = -\tilde{\epsilon} \delta_4\), where \(\tilde{\epsilon} = \pm 1\). The transformation of the coefficients \(d\) and \(p\) is reduced to the multiplication by \(\tilde{\epsilon}\), \(\tilde{d} = \tilde{\epsilon} d\) and \(\tilde{p} = \tilde{\epsilon} p\). This is why we can only set \(\tilde{d} \geq 0\) and, if \(\tilde{d} = 0\), \(\tilde{\epsilon} \geq 0\).

Consider the case \(n = -1\). As the tuple \((a, c, b, d, p)\) is nonzero, system (46) implies that \((b, c) \neq (0, 0)\). If the tuple \((a, b, c)\) satisfies the condition \(b \neq 0\) (resp. \(b = a = 0\), resp. \(b = 0\) and \(a \neq 0\)) then it is \(G_\infty\)-equivalent to the tuple \((0, 1, 0)\) (resp. \((0, 0, 1)\), resp. \((\tilde{\epsilon}'', 0, 1)\)). Similarly to the case \(n \neq -1\), now we look for possibilities of gauging the parameters \(d\) and \(p\) after fixing one of the above inequivalent forms for both the tuples \((a, b, c)\) and \((\tilde{a}, \tilde{b}, \tilde{c})\).

The equality \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (0, 1, 0)\) implies \(\nu = 1\) and \(\delta_5 = \delta_6 = 0\). Therefore, the parameters \(d\) and \(p\) are identically transformed.

Setting \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (0, 0, 1)\) in (46) results in \(\delta_4 = \nu^{-1}, \tilde{d} = \nu (d - \delta_6)\) and \(\tilde{p} = \nu (p + \delta_6)\). Therefore, we can set \((\tilde{d}, \tilde{p})\) to be equal to \((0, 0)\) or \((0, 1)\).

It follows from (46) with \((a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) = (\epsilon'', 0, 1)\), where \(\epsilon'' = \pm 1\), that \(\nu = \delta_4 = \pm 1\), \(\tilde{d} = \nu d + \epsilon'' \delta_5 - \nu \delta_6\) and \(\tilde{p} = \nu p - \epsilon'' \delta_6 + \nu \delta_6\). This allows for setting one of the parameters \(d\) and \(p\) to zero. After fixing the zero value of \(d\), we can additionally alternate the sign of \(\tilde{p}\). Hence we assume \((\tilde{d}, \tilde{p}) = (0, p'')\), where \(p'' \geq 0\).

Lemma 2 implies that up to \(G_\infty\)-equivalence Case 2 of Table 2 is partitioned into three inequivalent subcases:

1. \((f, h) = (|x|^d, \epsilon |x|^q), q = (1 + n - m)p - 2, (\partial_t, (d + 2 - pn)t \partial_t + x \partial_x + pu \partial_u);\)
2. \((f, h) = (e^{dx}, \epsilon e^{x^2}), q = (1 + n - m)p, (\partial_t, (d - pn) t \partial_t + \partial_x + pu \partial_u);\)
3. \(n \neq -1, (f, h) = \left((x^2 + 1)^{-\frac{3m+1}{m+1}} e^{d \arctan x}, \epsilon (x^2 + 1)^{-\frac{3m+1}{m+1}} e^{d \arctan x}\right), q = (1 + n - m)p, (\partial_t, (d - pn) t \partial_t + (x^2 + 1) \partial_x + (x/(n + 1) + p) u \partial_u);\)
4. \(n = -1, (f, h) = (e^{\frac{1}{2} \epsilon'' x^2}, \epsilon e^{\frac{m}{2} e^{x^2 + qx}}, q = -mp, (\partial_t, qt \partial_t - m \partial_x + (q + m \epsilon'' x) u \partial_u).\)
Here $\varepsilon, \varepsilon'' = \pm 1$, $(d, q) \neq (0, 0)$ for the first and second subcases and additionally $(d, q) \neq \left(-\frac{3n + 4}{(n + 1)}, -3 - m/(n + 1)\right)$ in the first subcase with $n \neq -1$ and $(d, q) \neq (0, 0)$ in the third subcase with $n = -4/3$; otherwise we have cases of Lie symmetry extensions of greater dimensions. It follows from Lemma\(^\text{2}\) that up to $G_{g=1}^\sim$-equivalence we can set certain constraints for the parameters $d$ and $q$. (It is convenient to use $q$ instead of $p$ as a parameter.) These constraints are different for the cases $n \neq -1$ and $n = -1$.

If $n \neq -1$, for the first subcase an exhaustive gauge implied by $G_{g=1}^\sim$-equivalence consists of the inequalities
\[
d \geq -\frac{3n + 4}{2(n + 1)} \quad \text{and, if} \quad d = -\frac{3n + 4}{2(n + 1)}, \quad q \geq -\frac{3n + 3 + m}{2(n + 1)}.
\]

They can be set using the equivalence transformation
\[
\tilde{t} = t, \quad \tilde{x} = \frac{1}{x}, \quad \tilde{u} = |x|^{-\frac{1}{n+1}} u,
\]
whose extension to the parameters $d$ and $q$ is given by $\tilde{d} = -d - (3n + 4)/(n + 1)$ and $\tilde{q} = -q - 3 - m/(n + 1)$. In the second subcase the parameters $d$ and $q$ can be gauged using a scaling of $x$. More precisely, $d = 1$ mod $G_{g=1}^\sim$ if $d \neq 0$ and $q = 1$ mod $G_{g=1}^\sim$ if $d = 0$. In the last subcase we can just simultaneously alter the signs of $d$ and $q$. Hence the exhaustive gauge is presented by $d \geq 0$ and, if $d = 0$, $q \geq 0$.

Consider $n = -1$. Then in the first subcase the parameters $d$ and $q$ are not changed by transformations from $G_{g=1}^\sim$. In the second subcase the parameters $d$ and $q$ can be gauged $0$ and $1$, respectively. In the last subcase we can alternate the sign of $q$ and assume $q \geq 0$ mod $G_{g=1}^\sim$.

A similar partition can be also carried out for Case 5 of Table 2. Namely, we have the following inequivalent subcases:

1. $f = |x|^d$: \(\partial_t, e^{-\varepsilon nt}(\partial_t + \varepsilon u \partial_u), n x \partial_x + (d + 2) u \partial_u\);
2. $f = e^z$: \(\partial_t, e^{-\varepsilon nt}(\partial_t + \varepsilon u \partial_u), n x \partial_x + u \partial_u\);
3. $n \neq -1, f = (x^2 + 1)^{-\frac{3n+4}{2(n+1)}} e^{d \arctan x}$:
\[
\langle \partial_t, e^{-\varepsilon nt}(\partial_t + \varepsilon u \partial_u), n(x^2 + 1) \partial_x + (nx/(n + 1) + d) u \partial_u \rangle;
\]

\[
n = -1, f = e^{\frac{1}{2} x^2} e^{\varepsilon'' x^2}:
\langle \partial_t, e^{\varepsilon'' t}(\partial_t + \varepsilon'' u \partial_u), \partial_x + (\varepsilon'' x - d) u \partial_u \rangle.
\]

In the first subcase $d \neq 0$ and, if $n \neq -1, d \neq -(3n + 4)/(n + 1)$. In the third subcase $d \neq 0$ if $n = -4/3$ and $e'' = \pm 1$ if $n = -1$. The gauges for the parameter $d$ modulo $G_{g=1}^\sim$-equivalence coincide with the gauges of $d$ in the respective subcases of Case 2 from Table 2.

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**References**

[1] G.W. Bluman, G.J. Reid and S. Kumei, New classes of symmetries for partial differential equations, *J. Math. Phys.* 29 (1988), 806–811.

[2] A.V. Borovskikh, Group classification of the eikonal equation for a 3-dimensional inhomogeneous medium, *Mat. Sb.* 195, no. 4 (2004), 23–64 (Russian); translation in *Sb. Math.* 195 (2004), 479–520.

[3] P. Clarkson and E. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Physica D* 70 (1993), 250–288.
[4] V.A. Dorodnitsyn, Group properties and invariant solutions of a nonlinear heat equation with a source or a sink, Preprint N 57, Moscow, Keldysh Institute of Applied Mathematics of Academy of Sciences USSR, 1979.

[5] V.A. Dorodnitsyn, On invariant solutions of non-linear heat equation with a source, Zhurn. Vych. Matemat. Matemat. Fiziki 22 (1982), 1393–1400 (in Russian).

[6] H.-D. Frey, W.G. Glöckle and T.F. Nonnenmacher, Symmetries and integrability of generalized diffusion reaction equations, J. Phys. A: Math. Gen. 25 (1993), 665–679.

[7] W.I. Fushchich, Conditional symmetry of equations of nonlinear mathematical physics, Ukr. Mat. Zh. 43 (1991), 1456–1470.

[8] N.M. Ivanova, C. Sophocleous, On the group classification of variable coefficient nonlinear diffusion–convection equations, J. Comp. Appl. Math. 197 (2006), 322–344.

[9] N.M. Ivanova, R.O. Popovych and C. Sophocleous, Conservation laws of variable coefficient diffusion–convection equations, Proc. of Tenth International Conference in Modern Group Analysis (October 24-31, 2004, Larnaca, Cyprus) (2005), 107–113; arXiv:math-ph/0505015.

[10] N.M. Ivanova, R.O. Popovych and C. Sophocleous, Group analysis of variable coefficient diffusion–convection equations. I. Enhanced group classification, Lobachevskii J. Math. 31 (2010), 100–122; arXiv:0710.2731.

[11] N.M. Ivanova, R.O. Popovych and C. Sophocleous, Group analysis of variable coefficient diffusion–convection equations. II. Contractions and exact solutions, arXiv:0710.3049 19 pp.

[12] J.G. Kingston and C. Sophocleous, On form-preserving point transformations of partial differential equations, J. Phys. A: Math. Gen. 31 (1998), 1597–1619.

[13] B.A. Magadeev, Group classification of nonlinear evolution equations, Algebra i Analiz 5 (1993), 141–156 (in Russian); translation in St. Petersburg Math. J. 5 (1994), 345–359.

[14] S.V. Meleshko, Group classification of equations of two-dimensional gas motions, Prikl. Mat. Mekh. 58 (1994), 56–62 (in Russian); translation in ZJ. Appl. Math. Mech. 58 (1994), 629–635.

[15] A.G. Nikitin and R.O. Popovych, Group classification of nonlinear Schrödinger equations, Ukr. Mat. Zh. 53 (2001), 1053–1060 (Ukrainian); translation in Ukr. Math. J. 53 (2001), 1255–1265; arXiv:math-ph/0301009.

[16] P. Olver, Applications of Lie groups to differential equations, New-York, Springer-Verlag, 1986.

[17] L.V. Ovsiannikov, Group analysis of differential equations, New York, Academic Press, 1982.

[18] R.O. Popovych and H. Eshraghi, Admissible point transformations of nonlinear Schrodinger equations, Proc. of 10th International Conference in Modern Group Analysis (MOGRAN X) (October 24-31, 2004, Larnaca, Cyprus) (2005), 168–176.

[19] R.O. Popovych and N.M. Ivanova, New results on group classification of nonlinear diffusion–convection equations, J. Phys. A: Math. Gen. 37 (2004), 7547–7565; arXiv:math-ph/0306035.

[20] R.O. Popovych, M. Kunzinger and H. Eshraghi, Admissible transformations and normalized classes of nonlinear Schrödinger equations, Acta Appl. Math., 109 (2010), 315–359; arXiv:math-ph/0611061.

[21] M. Prokhorova, The structure of the category of parabolic equations, arXiv:math.AP/0512094 24 pp.

[22] O.O. Vaneeva, Group classification via mapping between classes: an example of semilinear reaction-diffusion equations with exponential nonlinearity, Proc. of the 5th. Math. Phys. Meeting: Summer School and Conf. on Modern Mathematical Physics (July 6–17, 2008, Belgrade, Serbia) (2009), 463–471; arXiv:0811.2587.

[23] O.O. Vaneeva, A.G. Johnpillai, R.O. Popovych and C. Sophocleous, Enhanced group analysis and conservation laws of variable coefficient reaction–diffusion equations with power nonlinearities, J. Math. Anal. Appl. 330 (2007), 1363–1386; arXiv:math-ph/0605081.

[24] O.O. Vaneeva, R.O. Popovych and C. Sophocleous, Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, Acta Appl. Math. 106 (2009), 1–46; arXiv:0708.3457.

[25] O.O. Vaneeva, R.O. Popovych and C. Sophocleous, Reduction operators of variable coefficient semilinear diffusion equations with an exponential source, Proceedings of 5th Workshop “Group Analysis of Differential Equations and Integrable Systems” (June 6-10, 2010, Protaras, Cyprus), 2011, 207-219, arXiv:1010.2046.

[26] P. Winternitz and J.P. Gazeau, Allowed transformations and symmetry classes of variable coefficient Korteweg-de Vries equations, Phys. Lett. A 167 (1992), 246–250.