Existence of double Walsh series universal in weighted $L^1_\mu [0, 1]^2$ spaces

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Abstract

In this paper we consider a question on existence of double Walsh series universal in weighted $L^1_\mu [0, 1]^2$ spaces. We construct a weighted function $\mu(x, y)$ and a series by double Walsh system of the form

$$\sum_{n,k=1}^{\infty} c_{n,k} W_n(x) W_k(y)$$

with

$$\sum_{n,k=1}^{\infty} |c_{n,k}|^q < \infty \text{ for all } q > 2,$$

which is universal in $L^1_\mu [0, 1]^2$ concerning subseries with respect to convergence, in the sense of both spherical and rectangular partial sums.

Keywords: Walsh series, measurable function, universal series.

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1 Introduction

Let $\mu(x), 0 < \mu(x) \leq 1, x \in [0, 1]$ be a measurable on $[0, 1]$ function and let $L^1_\mu [0, 1]$ be a space of real measurable functions $f(x), x \in [0, 1]$ with

$$\int_0^1 |f(x)| \mu(x) dx < \infty.$$ 

Definition 1. A functional series

$$\sum_{k=1}^{\infty} f_k(x), \quad f_k(x) \in L^1_\mu [0, 1] \quad (1.1)$$

is called universal in weighted space $L^1_\mu [0, 1]$ with respect to rearrangements, if for any function $f(x) \in L^1_\mu [0, 1]$ the terms of (1.1) can be rearranged so that the obtained series

$$\sum_{k=1}^{\infty} f_{\sigma(k)}(x)$$

converges to $f(x)$ in the metric $L^1_\mu [0, 1]$, i.e.

$$\lim_{n \to \infty} \int_0^1 \left| \sum_{k=1}^{n} f_{\sigma(k)}(x) - f(x) \right| \cdot \mu(x) dx = 0.$$

Definition 2. The series (1.1) is called universal in $L^1_\mu [0, 1]$ concerning subseries, if for any function $f(x) \in L^1_\mu [0, 1]$ there exists a subseries $\sum_{k=1}^{\infty} f_{n_k}(x)$ of (1.1), which converges to $f(x)$ in the metric $L^1_\mu [0, 1]$. 


The above mentioned definitions are given not in the most general form but only in the form which will be applied in the present paper. Note, that for one-dimensional case there are many papers are devoted to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure (see [1], [3]-[9], [11], [12]).

The first usual universal in the sense of convergence almost everywhere trigonometric series were constructed by D.E. Menshov [8] and V.Ya. Kozlov [7]. The series of the form

\[ \frac{1}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \]  

was constructed just by them such that for any measurable on \([0, 2\pi]\) function \(f(x)\) there exists the growing sequence of natural numbers \(n_k\) such that the series \((A)\) having the sequence of partial sums with numbers \(n_k\) converges to \(f(x)\) almost everywhere on \([0, 2\pi]\).

Note that in this result, when \(f(x) \in L^1([0,2\pi])\), it is impossible to replace convergence almost everywhere by convergence in the metric \(L^1([0,2\pi])\).

This result was distributed by A.A. Talalian on arbitrary orthonormal complete systems (see [11]). He also established (see [12]), that if \(\{\phi_n(x)\}_{n=1}^{\infty}\) - the normalized basis of space \(L^p_{[0,1]}\), \(p > 1\), then there exists a series of the form

\[ \sum_{k=1}^{\infty} a_k \phi_k(x), \quad a_k \to 0. \]  

which has property: for any measurable function \(f(x)\) the members of series \((B)\) can be rearranged so that the again received series converge on a measure on \([0,1]\) to \(f(x)\).

In [2] these results are transferred to two-dimensional case.

W. Orlicz [9] observed the fact that there exist functional series that are universal with respect to rearrangements in the sense of a.e. convergence in the class of a.e. finite measurable functions. It is also useful to note that even Rieman proved that every convergent numerical series which is not absolutely convergent is universal with respect to rearrangements in the class of all real numbers. In [4] and [3] the following results are proved.

**Theorem 1.** There exists a series of the form

\[ \sum_{k=1}^{\infty} c_k W_k(x) \text{ with } \sum_{k=1}^{\infty} |c_k|^q < \infty \text{ for all } q > 2 \]  

such that for any number \(\varepsilon > 0\) a weighted function \(\mu(x)\) with

\[ 0 < \mu(x) \leq 1, \left|\{x \in [0,1] : \mu(x) \neq 1\}\right| < \varepsilon \]  

can be constructed, so that the series \((1.2)\) is universal in \(L^1_{\mu}[0,1]\) with respect to rearrangements.

**Theorem 2.** There exists a series of the form \((1.2)\) such that for any number \(\varepsilon > 0\) a weighted function \(\mu(x)\) with \((1.3)\) can be constructed, so that the series \((1.2)\) is universal in \(L^1_{\mu}[0,1]\) concerning subseries.

In this paper we prove that Theorems 1 and 2 can be transferred from one-dimensional case to two-dimensional one.
Moreover, the following statements are true.

**Theorem 3.** There exists a double series of the form

\[
\sum_{n,k=1}^{\infty} c_{n,k} W_n(x) W_k(y) \text{ with } \sum_{n,k=1}^{\infty} |c_{n,k}|^q < \infty \text{ for all } q > 2 \tag{1.4}
\]

with the following property: for any number \( \varepsilon > 0 \) a weighted function \( \mu(x, y) \) satisfying

\[
0 < \mu(x, y) \leq 1, \left| \{(x, y) \in T = [0, 1]^2 : \mu(x, y) \neq 1\} \right| < \varepsilon \tag{1.5}
\]

can be constructed so that the series (1.4) is universal in \( L_1^{\mu}(T) \) concerning subseries with respect to convergence in the sense of both spherical and rectangular partial sums.

**Theorem 4.** There exists a double series of the form (1.4) with the following property: for any number \( \varepsilon > 0 \) a weighted function \( \mu(x, y) \) with (1.5) can be constructed, so that the series (1.4) is universal in \( L_1^{\mu}(T) \) concerning rearrangements with respect to convergence in the sense of both spherical and rectangular partial sums.

**Remark.** We can also prove that Theorems 3 and 4 remain true for double trigonometric system.

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## 2 Notations

First we will give a definition of one-dimensional Walsh-Paley system (see [10]).

\[
W_0(x) = 1, \quad W_n(x) = \prod_{s=1}^{n} r_{m_s}(x), \quad n = \sum_{s=1}^{k} 2^{m_s}, \quad m_1 > m_2 > ... > m_s \tag{2.1}
\]

where \( \{r_k(x)\}_{k=0}^{\infty} \) is the system of Rademacher given as follows:

\[
r_0(x) = 1 \text{ for } x \in \left[0, \frac{1}{2}\right], \quad r_0(x) = -1 \text{ for } x \in \left(\frac{1}{2}, 1\right];
\]

\[
r_0(x + 1) = r_0(x) \quad \text{and} \quad r_k(x) = r_0(2^k x) \text{ for } k = 1, 2, ...
\]

The rectangular and spherical partial sums of the double series

\[
\sum_{k, \nu=1}^{\infty} c_{k, \nu} W_k(x) W_\nu(y)
\]

will be denoted by

\[
S_{n,m}(x, y) = \sum_{k=1}^{n} \sum_{\nu=1}^{m} c_{k, \nu} W_k(x) W_\nu(y)
\]

and

\[
S_R(x, y) = \sum_{\nu^2 + k^2 \leq R^2} c_{k, \nu} W_k(x) W_\nu(y).
\]
If \( g(x, y) \) is a continuous function on \( T = [0, 1]^2 \), then we set
\[
||g(x, y)||_C = \max_{(x,y) \in T} |g(x, y)|.
\]

For an interval \( \Delta \) in the form
\[
\Delta^{(i)}_m = \left[ \frac{i-1}{2^m}, \frac{i}{2^m} \right], \quad 1 \leq i \leq 2^m.
\]
we let \( \chi_\Delta(x) \) denote the characteristic function of \( \Delta \).

3 Some basic lemmas

In [3] we proved the following lemma.

**Lemma 1.** For any given numbers \( 0 < \varepsilon < 1, N_0 > 2 \) and a step function
\[
f(x) = \sum_{s=1}^{q} \gamma_s \cdot \chi_{\Delta_s}(x),
\]
where \( \Delta_s \) is an interval of the form \( \Delta^{(i)}_m = \left[ \frac{i-1}{2^m}, \frac{i}{2^m} \right], \quad 1 \leq i \leq 2^m \), there exist a measurable set \( E \subset [0, 1] \) and a polynomial \( P(x) \) of the form
\[
P(x) = \sum_{k=N_0}^{N} c_k W_k(x)
\]
which satisfy the conditions:

1. \( P(x) = f(x) \) on \( E \),
2. \( |E| > (1 - \varepsilon) \),
3. \( \sum_{k=N_0}^{N} |c_k|^{2+\varepsilon} < \varepsilon \),
4. \( \max_{N_0 \leq m < N} \left[ \int_{e} \left| \sum_{k=N_0}^{m} c_k W_k(x) \right| \, dx \right] < \varepsilon + \int_{e} |f(x)| \, dx \),

for every measurable subset \( e \) of \( E \).

The following is true.
Lemma 2. For any numbers \( \gamma \neq 0, 0 < \delta < 1, N > 1 \) and for any square \( \Delta = \Delta_1 \times \Delta_2 \subset T = [0, 1]^2 \) there exists a measurable set \( E \subset T \) and a polynomial \( P(x, y) \) of the form

\[
P(x, y) = \sum_{k,s=N}^{M} c_{k,s}W_k(x) \cdot W_s(y),
\]

with the following properties:

1. \(|E| > 1 - \delta,\)

2. \(\sum_{k,s=N}^{M} |c_{k,s}|^{2+\delta} < \delta,\)

3. \(P(x, y) = \gamma \cdot \chi_\Delta(x, y) \) for \((x, y) \in E,\)

4. \[\begin{align*}
&\max_{N \leq m, n \leq M} \left[ \int \int_E \left| \sum_{k,s=N}^{m} c_{k,s}W_k(x) \cdot W_s(y) \right| \, dx \, dy \right] \\
&+ \max_{\sqrt{2}N \leq R \leq \sqrt{2}M} \left[ \int \int_E \left| \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s}W_k(x) \cdot W_s(y) \right| \, dx \, dy \right] \leq 16 \cdot |\gamma| \cdot |\Delta|,
\end{align*}\]

for every measurable subset \( e \) of \( E. \)

Proof of Lemma 2. We apply Lemma 1, setting

\[f(x) = \gamma \cdot \chi_\Delta(x), \quad N_0 = N, \quad \varepsilon = \frac{\delta}{2} .\]

Then we can define a measurable set \( E_1 \subset [0, 1] \) and a polynomial \( P_1(x) \) of the form

\[P_1(x) = \sum_{k=N}^{N_1} a_kW_k(x)\]

which satisfy the conditions:

1. \(P_1(x) = \gamma \cdot \chi_{\Delta_1}(x) \) for \( x \in E_1,\)

2. \(|E_1| > 1 - \frac{\delta}{2},\)

3. \(\sum_{k=N}^{N_1} |a_k|^{2+\delta} \leq \delta,\)
\[(4^0) \quad \max_{N \leq k \leq N_1} \left[ \int_{e_1} \left| \sum_{k=N}^{N_1} a_k W_k(x) \right| \, dx \right] \leq 2 \cdot |\gamma| \cdot |\Delta_1|,\]

for every measurable subset \(e_1\) of \(E_1\).

Set
\[M_0 = 2 \cdot \left( N_2^2 + 1 \right) \quad (3.1)\]

and apply Lemma 1 again, setting
\[f(y) = \chi_{\Delta_2}(y), \quad N_0 = M_0, \quad \varepsilon = \frac{\delta}{2}.\]

Then we can define a measurable set \(E_2 \subset [0, 1]\) and a polynomial \(P_2(y)\) of the form
\[P_2(y) = \sum_{s=M_0}^{M} b_s W_s(y),\]

which satisfy the conditions:
\[(1^{00}) \quad P_2(y) = \chi_{\Delta_2}(y) \text{ for } y \in E_2,\]
\[(2^{00}) \quad |E_2| > 1 - \frac{\delta}{2},\]
\[(3^{00}) \quad \sum_{s=M_0}^{M} |b_s|^{2+\delta} < \delta,\]
\[(4^{00}) \quad \max_{M_0 \leq m \leq M} \left[ \int_{e_2} \left| \sum_{s=M_0}^{M} b_s W_s(y) \right| \, dy \right] \leq 2 \cdot |\Delta_2|,\]

for every measurable subset \(e_2\) of \(E_2\).

Set
\[E = E_1 \times E_2, \quad (3.2)\]
\[P(x, y) = P_1(x) \cdot P_2(x) = \sum_{k,s=N}^{M} c_{k,s} W_k(x) \cdot W_s(y), \quad (3.3)\]

where
\[c_{k,s} = a_k \cdot b_s, \quad if \ N \leq k \leq N_1, \ M_0 \leq s \leq M \quad (3.4)\]

and
\[c_{k,s} = 0, \text{ for other } k \text{ and } s.\]

By \((1^0) - (3^0)\), \((1^{00}) - (3^{00})\) and \((3.2) - (3.4)\) we obtain
\[|E| > 1 - \delta,\]

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\[
\sum_{k,s=N}^{M} |c_{k,s}|^{2+\delta} = \sum_{k=N}^{N_1} |a_k|^{2+\delta} \cdot \sum_{s=M_0}^{M} |b_s|^{2+\delta} < \delta,
\]

Thus, the statements 1) - 3) of Lemma 2 are satisfied. Now we will check the fulfillment of statement 4) of Lemma 2.

Let \(N^2 + M_0^2 < R^2 < N_1^2 + M^2\), then for some \(m_0 > M_0\) we have \(m_0 < R^2 < m_0 + 1\) and from (3.1) it follows, that \(R^2 - N_1^2 > (m_0 - 1)^2\).

Consequently taking relations (4.0), (4.00) and (3.2) - (3.4) for any measurable set \(e \subset E\) \((e = e_1 \times e_2, \ e_1 \subset E_1, \ e_2 \subset E_2)\) we obtain

\[
\int \int_{e} \left| \sum_{k,N^2 + M^2 \leq k^2 + s^2 \leq R^2} c_{k,s} W_k(x) \cdot W_s(y) \right| \, dx \, dy
\]

\[
\leq \int \int_{e} \left| \sum_{k=N}^{N_1} \sum_{s=M_0}^{m_0-1} c_{k,s} W_k(x) \cdot W_s(y) \right| \, dx \, dy
\]

\[
+ \max_{N < n \leq N_1} \left[ \int \int_{e} \left| \sum_{k=N}^{n} c_{k,m_0} W_k(x) \cdot W_{m_0}(y) \right| \, dx \, dy \right]
\]

\[
\leq \left[ \int_{e_1} \left| \sum_{k=N}^{N_1} a_k W_k(x) \right| \, dx \right] \cdot \left[ \int_{e_2} \left| \sum_{s=M_0}^{m_0-1} b_s W_s(y) \right| \, dy \right]
\]

\[
+ |b_{m_0}| \cdot \left[ \int_{e_2} \left| W_{m_0}(y) \right| \, dy \right] \cdot \max_{N < n \leq N_1} \left[ \int_{e_1} \left| \sum_{k=N}^{n} a_k W_k(x) \right| \, dx \right]
\]

\[
\leq 12 \cdot |\gamma| \cdot |\Delta|.
\]

Similarly, for \(N \leq \overline{m} \leq N_1, \ M_0 \leq \overline{m} \leq M\), we get

\[
\int \int_{e} \left| \sum_{k,s=N}^{\overline{m},\overline{m}} c_{k,s} W_k(x) \cdot W_s(y) \right| \, dx \, dy \leq 4 \cdot |\gamma| \cdot |\Delta|.
\]

Lemma 2 is proved.

**Lemma 3.** For any numbers \(\varepsilon > 0, \ N > 1\) and a step function

\[
f(x, y) = \sum_{\nu=1}^{\nu_0} \gamma_{\nu} \cdot \chi_{\Delta_{\nu}}(x, y),
\]

there exists a measurable set \(E \subset T\) and a polynomial \(P(x, y)\) of the form

\[
P(x, y) = \sum_{k,s=N}^{M} c_{k,s} W_k(x) \cdot W_s(y),
\]

\[
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\]
which satisfy the following conditions:

(I) \( P(x, y) = f(x, y) \) for \((x, y) \in E\),

(II) \( |E| > 1 - \varepsilon \),

(III) \[ \sum_{k, s = N}^{M} |c_{k,s}|^{2+\varepsilon} < \varepsilon, \]

(IV) \[ \max_{N \leq \nu < M} \left[ \int \int_{\Delta_{\nu}} \left| \sum_{k, s = N}^{M} c_{k,s} W_k(x) \cdot W_s(y) \right| dx dy \right] \]
\[ + \max_{\sqrt{2}N \leq R \leq \sqrt{2}M} \left[ \int \int_{2N^2 \leq k^2 + s^2 \leq R^2} \left| \sum_{k, s = N}^{M} c_{k,s} W_k(x) \cdot W_s(y) \right| dx dy \right] \]
\[ \leq 2 \cdot \int_{E} |f(x, y)| dx dy + \varepsilon, \]

for every measurable subset \( E \) of \( E \).

**Proof of Lemma 3.** Without any loss of generality, we assume that

\[ \max_{1 \leq \nu \leq \nu_0} \left( |\gamma_{\nu}| \cdot |\Delta_{\nu}| \right) < \frac{\varepsilon}{32}, \quad (3.5) \]

(\( \Delta_{\nu}, \; 1 \leq \nu \leq \nu_0 \) are the constancy rectangular domain of \( f(x, y) \), i.e., where the function \( f(x, y) \) is constant).

Given an integer \( 1 \leq \nu \leq \nu_0 \), by applying Lemma 2 with \( \delta = \frac{\varepsilon}{1000} \), we find that there exists a measurable set \( E_{\nu} \subset T \) and a polynomial \( P_{\nu}(x, y) \) of the form

\[ P_{\nu}(x, y) = \sum_{k, s = N_{\nu}}^{M_{\nu}} c_{k,s}^{(\nu)} W_k(x) \cdot W_s(y) \quad (3.6) \]

with the following properties:

\[ |E_{\nu}| > 1 - \frac{\varepsilon}{2\nu}, \quad (3.7) \]

\[ \sum_{k, s = N_{\nu}}^{M_{\nu}} |c_{k,s}^{(\nu)}|^{2+\varepsilon} < \frac{\varepsilon}{\nu_0}, \quad (3.8) \]

\[ P_{\nu}(x, y) = \gamma_{\nu} \cdot \chi_{\Delta_{\nu}}(x, y) \text{ for } (x, y) \in E_{\nu}, \quad (3.9) \]

\[ \max_{N_{\nu} \leq \nu < M_{\nu}} \left[ \int \int_{E_{\nu}} \left| \sum_{k, s = N_{\nu}}^{M_{\nu}} c_{k,s}^{(\nu)} W_k(x) \cdot W_s(y) \right| dx dy \right] \]
\[ + \max_{\sqrt{2}N, \leq R \leq \sqrt{2}M} \left[ \int \int e \left( \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s}^{(\nu)} W_k(x) \cdot W_s(y) \right) dxdy \right] \]
\[ \leq 16 \cdot |\gamma_\nu| \cdot |\Delta_\nu| < \frac{\varepsilon}{2}. \] (3.10)

for every measurable subset \( e \) of \( E_\nu \) (see (3.5)).

Thus, for any fixed number \( \nu, 1 \leq \nu \leq \nu_0 \) we can define a measurable set \( E_\nu \subset T \) and a polynomial \( P_\nu(x, y) \) of the form (3.6), which satisfy the conditions (3.7) - (3.9). Then we can take
\[ N_1 = N, \ n_\nu = M_{\nu-1} + 1, \ 1 \leq \nu \leq \nu_0. \]

Set
\[ E = \bigcap_{\nu=1}^{\nu_0} E_\nu, \] (3.11)
\[ P(x, y) = \sum_{\nu=1}^{\nu_0} P_\nu(x, y) = \sum_{k,s=N}^{M} c_{k,s} W_k(x) \cdot W_s(y), \ M = M_{\nu_0}, \] (3.12)

where
\[ c_{k,s} = c_{k,s}^{(\nu)}, \text{ for } N_\nu \leq k, s \leq M_\nu, \ 1 \leq \nu \leq \nu_0 \] (3.13)

and
\[ c_{k,s} = 0, \text{ for other } k \text{ and } s. \]

From (3.7) - (3.9), (3.11) - (3.13) we obtain:
\[ P(x, y) = f(x, y) \text{ for } (x, y) \in E, \]
\[ |E| > 1 - \varepsilon, \]
\[ \sum_{k,s=N}^{M} |c_{k,s}|^{2+\varepsilon} < \sum_{\nu=1}^{\nu_0} \left[ \sum_{k,s=N_\nu}^{M_\nu} |c_{k,s}^{(\nu)}|^{2+\varepsilon} \right] < \varepsilon, \]

i.e the statements I) - III) of Lemma 3 are satisfied. Now we will check the fulfillment of statement IV) of Lemma 3.

Let \( R \in [\sqrt{2}N, \sqrt{2}M] \), then for some \( \nu', 1 \leq \nu' \leq \nu_0 \) we have \( \sqrt{2}N_{\nu'} \leq R \leq \sqrt{2}N_{\nu'+1} \), consequently from (3.12) and (3.13) we have
\[ \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s} W_k(x) \cdot W_s(y) = \sum_{\nu=1}^{\nu'-1} P_\nu(x, y) \]
\[ + \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s}^{(\nu')} W_k(x) \cdot W_s(y). \]

In view of the conditions (3.9) - (3.12) and the equality \( P(x, y) = f(x, y) \) on \( E \), for any measurable set \( e \subset E \) we obtain
\[ \int \int e \left( \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s} W_k(x) \cdot W_s(y) \right) dxdy \]
\[ \leq \int \int \left| \sum_{\nu=1}^{\nu'-1} P_{\nu}(x, y) \right| dxdy \]

\[ + \int \int \left| \sum_{2N_{s}^{2} \leq k^{2} \leq s^{2}} c_{k,s}^{(\nu')} W_{k}(x) \cdot W_{s}(y) \right| dxdy \]

\[ \leq \int \int |f(x, y)| dxdy + \epsilon. \]

Similarly, for any \( e \subset E \) we have

\[ \max_{N \leq M} \left[ \int \int \left| \sum_{k, s = N}^{\infty} c_{k,s} W_{k}(x) \cdot W_{s}(y) \right| dxdy \right] \]

\[ \leq \int \int |f(x, y)| dxdy + \epsilon. \]

Lemma 3 is proved.

4 Proofs of the theorems

Proof of Theorem 3.

Let

\[ \{ f_{s}(x, y) \}_{s=1}^{\infty}, \quad (x, y) \in T \]  

(4.1)

be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers. Applying Lemma 3 consecutively, we can find a sequence \( \{ E_{s} \}_{s=1}^{\infty} \) of sets and a sequence of polynomials

\[ P_{s}(x, y) = \sum_{k, \nu = N_{s} - 1}^{N_{s} - 1} c_{k,s}^{(\nu')} W_{k}(x)W_{\nu}(y), \]

(4.2)

\[ 1 = N_{0} < N_{1} < ... < N_{s} < ... , \quad s = 1, 2, .... \]

which satisfy the conditions:

\[ P_{s}(x, y) = f_{s}(x, y), \quad (x, y) \in E_{s}, \]  

(4.3)

\[ |E_{s}| > 1 - 2^{-2(s+1)}, \quad E_{s} \subset T, \]  

(4.4)

\[ \sum_{k, \nu = N_{s} - 1}^{N_{s} - 1} \left| c_{k,s}^{(\nu')} \right|^{2+2^{-2s}} < 2^{-2s}, \]  

(4.5)
where

\[ \text{for every measurable subset } e \text{ of } E_s. \]

Denote

\[ \sum_{k, \nu = 1}^{\infty} c_{k, \nu} W_k(x) W_\nu(y) = \sum_{s=1}^{\infty} \left[ \sum_{k, \nu = N_s}^{N_s-1} c_{k, \nu}^{(s)} W_k(x) W_\nu(y) \right], \]  

(4.7)

where

\[ c_{k, \nu} = c_{k, \nu}^{(s)}, \text{ for } N_{s-1} \leq k, \nu < N_s, \ s = 1, 2, \ldots. \]

For an arbitrary number \( \varepsilon > 0 \) we set

\[ \Omega_n = \bigcap_{s=n}^{\infty} E_s, \ n = 1, 2, \ldots; \]

\[ E = \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, \ n_0 = \lfloor \log_{1/2} \varepsilon \rfloor + 1; \]  

(4.8)

\[ B = \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \bigcup \left( \bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1} \right) \]

It is obvious (see (4.4), (4.8)) that \( |B| = 1 \) and \( |E| > 1 - \varepsilon. \)

We define a function \( \mu(x, y) \) in the following way:

\[ \mu(x, y) = 1, \ for \ (x, y) \in E \cup (T \setminus B); \]

\[ \mu(x, y) = \mu_n, \ for \ (x, y) \in \Omega_n \setminus \Omega_{n-1}, \ n \geq n_0 + 1, \]  

(4.9)

where

\[ \mu_n = \left[ 2^{2n} \cdot \prod_{s=1}^{n} h_s \right]^{-1}; \]

(4.10)

\[ h_s = \| f_s \|_C + \max_{N_{s-1} \leq M < N_s} \left[ \sum_{k, \nu = N_{s-1}}^{N_s} c_{k, \nu}^{(s)} W_k(x) W_\nu(y) \right]_C \]

\[ + \max_{\sqrt{2}N_{s-1} \leq R < \sqrt{2}N_s} \left[ \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} W_k(x) W_\nu(y) \right]_C + 1. \]
From (4.5), (4.7) - (4.10) we obtain

(A) \(-1 < \mu(x, y) \leq 1, \mu(x, y)\) is a measurable function and

\[ |\{(x, y) \in T : \mu(x, y) \neq 1\}| < \varepsilon. \]

(B) \(-\sum_{k, \nu=1}^{\infty} |c_{k, \nu}|^q < \infty\) for all \(q > 2\).

Hence, obviously we have (see (4.5) and (4.7))

\[ \lim_{k, \nu \to \infty} c_{k, \nu} = 0. \]

It follows from (4.8) - (4.10) that for all \(s \geq n_0\) and \(N_{s-1} \leq \overline{n}, \underline{m} < N_s\)

\[ \int \int_{T \setminus \Omega_s} \left| \sum_{k, \nu=1}^{\infty} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy \]

\[ = \sum_{n=s+1}^{\infty} \left[ \int \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k, \nu=1}^{\infty} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu_n dx dy \right] \]

\[ \leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[ \int \int_{T} \left| \sum_{k, \nu=1}^{\infty} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| h^{-1}_s dx dy \right] < \frac{1}{3} 2^{-2s}. \] (4.11)

Analogously for all \(s \geq n_0\) and \(\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s\) we have

\[ \int \int_{T \setminus \Omega_s} \left| \sum_{2N_{s-1} \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy < \frac{1}{3} 2^{-2s}. \] (4.12)

By (4.2), (4.8) - (4.10) for all \(s \geq n_0\) we have

\[ \int \int_{T} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \]

\[ = \int \int_{\Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \]

\[ + \int \int_{T \setminus \Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \]

\[ = \sum_{n=s+1}^{\infty} \left[ \int \int_{\Omega_n \setminus \Omega_{n-1}} |P_s(x, y) - f_s(x, y)| \mu_n dx dy \right] \]

\[ \leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[ \int \int_{T} \left| f_s(x, y) \right| + \sum_{k, \nu=1}^{N_{s-1}} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right] h^{-1}_s dx dy \]

\[ < \frac{1}{3} 2^{-2s} < 2^{-2s}. \] (4.13)
By (4.6) and (4.8) - (4.11) for all \( N_{s-1} \leq \pi, \mu \leq N_s \) and \( s \geq n_0 + 1 \) we obtain

\[
\int \int_T \left| \sum_{k, \nu = N_{s-1}}^{\pi, \mu} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy
\]

\[
\int \int_{\Omega_s} \left| \sum_{k, \nu = N_{s-1}}^{\pi, \mu} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy
\]

\[
\int \int_{T \setminus \Omega_s} \left| \sum_{k, \nu = N_{s-1}}^{\pi, \mu} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy
\]

\[
< \sum_{n = n_0 + 1}^{n} \left[ \int \int_{\Omega_n \setminus \Omega_{n-1}} \sum_{k, \nu = N_{s-1}}^{\pi, \mu} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy + \frac{1}{3} 2^{-2s}
\]

\[
< \sum_{n = n_0 + 1}^{n} \left[ \int \int_{\Omega_n \setminus \Omega_{n-1}} \sum_{k, \nu = N_{s-1}}^{\pi, \mu} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy + \frac{1}{3} 2^{-2s}
\]

\[
= 2^{-2(s+1)} \cdot \sum_{n = n_0 + 1}^{\infty} \mu_n + \int \int_{\Omega_s} |f_s(x, y)| \mu(x, y) dx dy + \frac{1}{3} 2^{-2s}
\]

\[
< 2 \cdot \int \int_{T} |f_s(x, y)| \mu(x, y) dx dy + 2^{-2s}. \tag{4.14}
\]

Analogously for all \( s \geq n_0 \) and \( \sqrt{2} N_{s-1} \leq R \leq \sqrt{2} N_s \) we have (4.12)

\[
\int \int_{T} \left| \sum_{2N_{s-1} \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x, y) dx dy
\]

\[
< 2 \cdot \int \int_{T} |f_s(x, y)| \mu(x, y) dx dy + 2^{-2s}. \tag{4.15}
\]

Now we’ll show that the series (4.7) is universal in \( L^1_{\mu}(T) \) concerning subseries with respect to convergence by both spherical and rectangular partial sums.

Let \( f(x, y) \in L^1_{\mu}(T) \), i.e.

\[
\int \int_{T} |f(x, y)| \mu(x, y) dx dy < \infty.
\]

It is easy to see that we can choose a function \( f_{n_1}(x, y) \) from the sequence (4.1) such that

\[
\int \int_{T} |f(x, y) - f_{n_1}(x, y)| \mu(x, y) dx dy < 2^{-2}, \quad n_1 > n_0 + 1. \tag{4.16}
\]

Hence, we have

\[
\int \int_{T} |f_{n_1}(x, y)| \mu(x, y) dx dy < 2^{-2} + \int \int_{T} |f(x, y)| \mu(x, y) dx dy. \tag{4.17}
\]
From (4.13) and (4.16) we get
\[
\int \int_T |f(x, y) - P_{n_1}(x, y)| \mu(x, y) dx dy \\
\leq \int \int_T |f(x, y) - f_{n_1}(x, y)| \mu(x, y) dx dy \\
+ \int \int_T |f_{n_1}(x, y) - P_{n_1}(x, y)| \mu(x, y) dx dy < 2 \cdot 2^{-2}.
\]
Assume that numbers \(n_1 < n_2 < \ldots < n_{q-1}\) are chosen in such a way that the following condition is satisfied:
\[
\int \int_T \left| f(x, y) - \sum_{s=1}^{j} P_{n_s}(x, y) \right| \mu(x, y) dx dy < 2 \cdot 2^{-2j}, \quad 1 \leq j \leq q - 1. \tag{4.18}
\]
Now we choose a function \(f_{n_q}(x, y)\) from the sequence (4.1) such that
\[
\int \int_T \left| f(x, y) - \sum_{s=1}^{q-1} P_{n_s}(x, y) - f_{n_q}(x, y) \right| \mu(x, y) dx dy \\
< 2 \cdot 2^{-2q}, \quad n_q > n_{q-1}. \tag{4.19}
\]
This with (4.18) imply
\[
\int \int_T |f_{n_q}(x, y)| \mu(x, y) dx dy < 2^{-2q} + 2 \cdot 2^{-2(q-1)} = 9 \cdot 2^{-2}. \tag{4.20}
\]
From (4.2), (4.13) - (4.15) and (4.20) we obtain
\[
\int \int_T |f_{n_q}(x, y) - P_{n_q}(x, y)| \mu(x, y) dx dy < 2^{-2n_q}, \tag{4.21}
\]
where
\[
P_{n_q}(x, y) = \sum_{k, \nu = N_{n_q-1}}^{N_{n_q}-1} c^{(n_q)}_{k, \nu} W_k(x) W_\nu(y),
\]
\[
\max_{N_{n_q-1} \leq n, m < N_{n_q}} \left[ \int \int_T \left| \sum_{k, \nu = N_{n_q-1}}^{n, m} c^{(n_q)}_{k, \nu} W_k(x) W_\nu(y) \right| \mu(x, y) dx dy \right] \\
< 19 \cdot 2^{-2q}. \tag{4.22}
\]
Analogously we have
\[
\max_{\sqrt{N_{n_q-1} \leq R \leq \sqrt{N_{n_q}}}} \left[ \int \int_T \left| \sum_{2N_{n_q-1} \leq k^2 + \nu^2 \leq R^2} c^{(n_q)}_{k, \nu} W_k(x) W_\nu(y) \right| \mu(x, y) dx dy \right]
\]
In quality subseries of Theorem we shall take
\[
\sum_{q=1}^{\infty} P_{nq}(x,y) = \sum_{q=1}^{\infty} \left[ \sum_{k, \nu = N_{nq}-1}^{N_{nq}-1} c_{k,\nu}^{(nq)} W_k(x) W_\nu(y) \right].
\] (4.23)

From (4.19) and (4.21) we have
\[
\int \int_T |f(x,y) - \sum_{s=1}^{q-1} P_{ns}(x,y)| \mu(x,y) dx dy
\leq \int \int_T \left| \left( f(x,y) - \sum_{s=1}^{q-1} P_{ns}(x,y) \right) - f_{nq}(x,y) \right| \mu(x,y) dx dy
+ \int \int_T |f_{nq}(x,y) - P_{nq}(x,y)| \mu(x,y) dx dy < 2 \cdot 2^{-2q}.
\] (4.24)

Let \( \bar{\pi} \) and \( \bar{m} \) be arbitrary natural numbers. Then for some natural number \( q \) we have
\[
N_{nq-1} \leq \min\{\bar{\pi}, \bar{m}\} < N_{nq}.
\]
Taking into account (4.22) and (4.24) for rectangular partial sums \( S_{\bar{\pi}\bar{m}}(x,y) \) of (4.23) we get
\[
\int \int_T |S_{\bar{\pi}\bar{m}}(x,y) - f(x,y)| \mu(x,y) dx dy
\leq \int \int_T \left| f(x,y) - \sum_{s=1}^{q} P_{ns}(x,y) \right| \mu(x,y) dx dy
+ \int \int_T |f_{nq}(x,y) - P_{nq}(x,y)| \mu(x,y) dx dy
+ \max_{N_{nq-1} \leq \bar{\pi}, \bar{m} < N_{nq}} \left[ \int \int_T \left| \sum_{k, \nu = N_{nq}-1}^{\bar{\pi},\bar{m}} c_{k,\nu}^{(nq)} W_k(x) \cdot W_\nu(y) \right| \mu(x,y) dx dy \right]
< 21 \cdot 2^{-2q}.
\] (4.25)

Analogously for \( \sqrt{2}N_{nq-1} \leq R \leq \sqrt{2}N_{nq} \) we have
\[
\int \int_T |S_R(x,y) - f(x,y)| \mu(x,y) dx dy < 21 \cdot 2^{-2q},
\] (4.26)

where \( S_R(x,y) \) the spherical partial sums of (4.23).

From (4.25) and (4.26) we conclude that the series (4.7) is universal in \( L_1^\mu(T) \) concerning subseries with respect to convergence by both spherical and rectangular partial sums (see Definition 2).

Theorem 3 is proved.

Remark. We can show Theorem 4 by the method in the proof of Theorem 3.
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