VARIATIONAL PRINCIPLES FOR FELDMAN-KATOK METRIC MEAN DIMENSION

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ABSTRACT. We introduce the notion of Feldman-Katok metric mean dimensions in this note. We show metric mean dimensions defined by different metrics coincide under weak tame growth of covering numbers, and establish variational principles for Feldman-Katok metric mean dimensions in terms of FK Katok $\epsilon$-entropy and FK local $\epsilon$-entropy function.

1. INTRODUCTION

By a pair $(X, T)$ we mean a topological dynamical system (TDS for short), where $X$ is a compact metric space with metric $d$ and $T$ is a homeomorphism on $X$. By $M(X), M(X, T), E(X, T)$ we denote the sets of Borel probability measures on $X$, $T$-invariant Borel probability measures on $X$, $T$-invariant ergodic Borel probability measures on $X$, respectively.

For each TDS, one can assign a non-negative number to characterize the topological complexity of system. It is well-known that the classical topological entropy, defined by Bowen dynamical balls, is an important topological invariant to help us understand the dynamical systems. Besides, the Bowen dynamical balls with mistake function [PS07], dynamical balls defined by mean metric [GJ16] and Feldman-Katok metric [CL21], are also invoked and do not change the value of classical topological entropy. It turns out that these different dynamical balls become a critical role in solving the problems toward Sarnak’s conjecture, multifractal analysis, the classification problems of measure-preserving systems and the other fields.

The present note mainly involves Feldman-Katok metric. In [Orn74, Kak43, Fel76], the authors showed that edit distance is closely associated with the classification problems of measure-preserving systems. Feldman-Katok metric [KL17](FK metric for short) is the topological counterpart of edit distance. Replacing Bowen metric by FK metric, Cai and Li [CL21] proved that topological entropy defined by FK metric

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coincides classical topological entropy [Wal82]. Later, Nie and Huang [NH22] investigated the restricted sensitivity, return time and local Brin-Katok entropy in context of FK metric. Different from Bowen metric, the work [KL17, KL17, NH22] suggests that the advantage to use FK metric is that it allows time delay by ignoring the synchronization of points in orbits with only order preserving required.

A compact metric space \((X, d)\) is said to have tame growth of covering numbers if for each \(\theta > 0\),

\[
\lim_{\epsilon \to 0} \epsilon^{\theta} \log \#(X, d, \epsilon) = 0,
\]

where \((X, d, \epsilon)\) denotes the smallest cardinality of open balls \(B_d(x, \epsilon)\) covering \(X\). For example, the compact subsets of \(\mathbb{R}^n\) equipped with the Euclidean distance have tame growth of covering numbers. More generally, it is shown that [LT18, Lemma 4] every compact metrizable topological space admits a distance satisfying such condition.

For TDS with infinite topological entropies, Lindenstrauss and Weiss [LW00] introduced the notion of metric mean dimension to classify such dynamical systems and established analogous variational principles for metric mean dimensions in terms of \(L^\infty\)-rate distortion functions and \(L^p(1 \leq p < \infty)\)-rate distortion functions under the assumption of tame growth of covering numbers.

**Theorem A.** Let \((X, T)\) be a TDS with a metric \(d\). Then

\[
\overline{\text{mdim}}(T, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X, T)} R_{\mu, L^\infty}(\epsilon).
\]

Additionally, if \(d\) has tame growth of covering numbers, then for any \(1 \leq p < \infty\),

\[
\overline{\text{mdim}}(T, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X, T)} R_{\mu, L^p}(\epsilon),
\]

where \(\overline{\text{mdim}}(T, X, d)\) denotes the upper metric mean dimension of \(X\), \(R_{\mu, L^p}(\epsilon), R_{\mu, L^\infty}(\epsilon)\) denote the \(L^p, L^\infty\) rate-distortion function, respectively.

By replacing rate-distortion functions, the authors [VV17, TWL20, GS21, Shi22] verified that Lindenstrauss-Tsukamoto’s variational principles still hold for other measure-theoretic \(\epsilon\)-entropies. Inspired by the work of [LW00, LT18, CL21, GZ22], the aim of present note is to introduce the notion of Feldman-Katok metric mean dimensions and establish variational principles for Feldman-Katok metric mean dimensions. The first question that we encounter is whether the metric mean dimensions defined by different metrics have the same metric mean dimension compared with Feldman-Katok metric mean dimensions.
A compact metric space \((X, d)\) is said to have \textit{weak tame growth of covering numbers} if

\[
\lim_{\epsilon \to 0} \epsilon \log \#(X, d, \epsilon) = 0.
\]

Obviously, this condition is weaker than tame growth of covering numbers. The following theorem shows that different metric mean dimensions have the same metric mean dimension.

**Theorem 1.1.** Let \((X, T)\) be a TDS with a metric \(d\) admitting weak tame growth of covering numbers. Suppose that \(g\) is a mistake function. Then

\[
\underline{\text{mdim}}_{FK}(T, X, d) = \underline{\text{mdim}}_{M}(T, X, d) = \underline{\text{mdim}}_{\hat{M}}(g; T, X).
\]

Consequently, if \(d\) has tame growth of covering numbers, then

\[
\underline{\text{mdim}}_{FK}(T, X, d) = \underline{\text{mdim}}_{M}(T, X, d)
= \underline{\text{mdim}}_{\hat{M}}(T, X, d) = \underline{\text{mdim}}_{M}(g; T, X).
\]

where \(\underline{\text{mdim}}_{FK}(T, X, d), \underline{\text{mdim}}_{M}(T, X, d), \underline{\text{mdim}}_{\hat{M}}(T, X, d), \underline{\text{mdim}}_{M}(g; T, X)\) are metric mean dimensions defined by and FK metric, Bowen metric, mean metric and dynamical balls with mistake function \(g\).

The following variational principle for Feldman-Katok metric mean dimension in terms of Katok \(\epsilon\)-entropies allows us to link the ergodic theory and metric mean dimension theory.

**Theorem 1.2.** Let \((X, T)\) be a TDS with a metric \(d\) satisfying weak tame growth of covering numbers. Then for every \(\delta \in (0, 1)\)

\[
\underline{\text{mdim}}_{FK}(T, X, d) = \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}(X, T)} h_{\mu, FK}(T, \epsilon, \delta)
= \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X, T)} h_{\mu, FK}(T, \epsilon, \delta)
= \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X)} h_{\mu, FK}(T, \epsilon, \delta),
\]

where \(h_{\mu, FK}(T, \epsilon, \delta)\) denotes FK Katok's \(\epsilon\)-entropies of \(\mu\).

The last variational principle suggests that metric mean dimension can also be determined by FK local \(\epsilon\)-entropy function.

**Theorem 1.3.** Let \((X, T)\) be a TDS with a metric \(d\). Then

\[
\underline{\text{mdim}}_{FK}(T, X, d) = \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in X} h_{FK}(x, \epsilon),
\]

where \(h_{FK}(x, \epsilon)\) denotes the FK local \(\epsilon\)-entropy function of \(x\).
We remark that Theorem 1.1, Theorem 1.2 and Theorem 1.3 also hold for $\text{mdim}_{FK}(T, X, d)$ by changing $\limsup_{\epsilon \to 0}$ into $\liminf_{\epsilon \to 0}$.

The rest of this paper is organized as follows. In section 2, we introduce the notions of FK metric mean dimension, FK Katok $\epsilon$-entropy and FK local $\epsilon$-entropy function. In section 3, we give the proofs of Theorems 1.1, 1.2 and 1.3.

2. Preliminary

In subsection 2.1, analogous to the metric mean dimension defined by Bowen metric [LW00] we introduce the notions of Feldman-Katok metric mean dimensions. In subsection 2.2, we introduce the notions of FK Katok’s $\epsilon$-entropies for Borel probability measure and FK local $\epsilon$-entropy function on $X$ to pursue the variational principle for FK metric mean dimension.

2.1. Feldman-Katok metric mean dimensions. Fix $x, y \in X$, $n \in \mathbb{N}$, and $\delta > 0$, we define an $(n, \delta)$-match of $x$ and $y$ to be an order preserving (i.e. $\pi(i) < \pi(j)$ whenever $i < j$) bijection $\pi : D(\pi) \to R(\pi)$ so that $D(\pi), R(\pi) \subset \{0, 1, \cdots, n-1\}$ and $d(T^i x, T^{\pi(i)} y) < \delta$ for every $i \in D(\pi)$. Set

$$\bar{f}_{n, \delta}(x, y) = 1 - \frac{1}{n} \max \{|\pi| : \pi \text{ is an } (n, \delta)\text{-match of } x \text{ and } y\},$$

where $|\pi|$ denotes the cardinality of the set $D(\pi)$.

The Feldman-Katok metric (or FK metric for short) on $X$ is given by

$$d_{FK}(x, y) = \inf \{\delta > 0 : \bar{f}_{n, \delta}(x, y) < \delta\}.$$

Let $Z$ be a non-empty subset of $X$. Given $n \in \mathbb{N}$ and $\epsilon > 0$, a set $E \subset X$ is said to be a FK-$n(\epsilon)$ spanning set of $Z$ if for any $x \in Z$, there exists $y \in E$ such that $d_{FK}(x, y) < \epsilon$. Denote by $r_{FK}(T, Z, n, d, \epsilon)$ the smallest cardinality of FK-$n(\epsilon)$ spanning sets of $Z$. Put

$$r_{FK}(T, Z, d, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_{FK}(T, Z, n, d, \epsilon).$$

We define Feldman-Katok upper and lower metric mean dimensions of $X$ as

$$\underline{\text{mdim}}_{FK}(T, X, d) = \limsup_{\epsilon \to 0} \frac{r_{FK}(T, X, d, \epsilon)}{\log \frac{1}{\epsilon}},$$

$$\overline{\text{mdim}}_{FK}(T, X, d) = \liminf_{\epsilon \to 0} \frac{r_{FK}(T, X, d, \epsilon)}{\log \frac{1}{\epsilon}}.$$

In [CL21], Cai and Li defined the Feldman-Katok topological entropy of $X$ as $h_{FK}(T, X) = \lim_{\epsilon \to 0} r_{FK}(T, Z, d, \epsilon)$. Then for sufficiently $\epsilon > 0$, one may think of

$$r_{FK}(T, Z, d, \epsilon) \approx \text{mdim}_{FK}(T, X, d) \cdot \log \frac{1}{\epsilon}.$$
Hence, FK metric mean dimensions can be interpreted as how fast the term \( r_{FK}(T, X, d, \epsilon) \) approximates the infinite Feldman-Katok topological entropy as \( \epsilon \to 0 \).

The authors [Wal82, PS07, GJ16, CL21] showed that different dynamical balls defined by Bowen metrics, mean metrics, FK metrics, Bowen balls with mistake function leads to the same topological entropy. We briefly recall their definitions and then define metric mean dimension for these metrics.

- Mean metric: the \( n \)-th mean metric on \( X \) is given by

\[
\bar{d}_n(x, y) = \frac{1}{n} \sum_{j=0}^{n-1} d(T^j x, T^j y).
\]

The upper metric mean dimension of \( X \) defined by mean metric is given by

\[
\text{mdim}_M(T, X, d) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log \hat{r}_n(T, X, d, \epsilon)}{n \log \frac{1}{\epsilon}},
\]

where \( \hat{r}_n(T, X, d, \epsilon) \) denotes the smallest cardinality of \((n, \epsilon)\) spanning sets of \( X \) in mean metric.

- Bowen metric: the \( n \)-th mean metric on \( X \) is given by

\[
d_n(x, y) = \max_{0 \leq j \leq n-1} d(T^j x, T^j y).
\]

The upper metric mean dimension of \( X \) defined by Bowen metric is given by

\[
\text{mdim}_M(T, X, d) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log r_n(T, X, d, \epsilon)}{n \log \frac{1}{\epsilon}},
\]

where \( r_n(T, X, d, \epsilon) \) denotes the smallest cardinality of \((n, \epsilon)\) spanning sets of \( X \) in Bowen metric.

- Mistake function: A non-decreasing unbounded map \( g: \mathbb{N} \to \mathbb{N} \) is called a mistake function if \( g(n) < n \) and

\[
\lim_{n \to \infty} \frac{g(n)}{n} = 0.
\]

The mistake Bowen ball \( B_n(g; x, \epsilon) \) centered at \( x \) with radius \( \epsilon \) and length \( n \) w.r.t \( g \) is given by

\[
B_n(g; x, \epsilon) = \{ y \in X : \max_{j \in \Lambda} d(T^j x, T^j y) < \epsilon \text{ for some } \Lambda \in I(g; n) \},
\]

where \( I(g; n) \) is the set of all subsets \( \Lambda \) of \( \lambda_n \) satisfying \( |\Lambda| \geq n - g(n) \) and \( \Lambda_n = \{0, 1, \ldots, n-1\} \). Then \( g(n) \) is the number how many mistakes that we are allowed to shadow an orbit of length \( n \).

A set \( E \subset X \) is a \((g; n, \epsilon)\) spanning set of \( X \) if for any \( x \in X \), there exists \( y \in E \) and \( \Lambda \in I(g; n) \) such that \( \max_{j \in \Lambda} d(T^j x, T^j y) < \epsilon \).
The smallest cardinality of \((g; n, \epsilon)\) spanning sets of \(X\) is denoted by \(r_n(g; T, X, \epsilon)\). Put
\[
  r(g; T, X, \epsilon) = \limsup_{n \to \infty} \frac{\log r_n(g; T, X, \epsilon)}{n}.
\]
We define the upper metric mean dimension of \(X\) with mistake function \(g\) as
\[
  \underbar{mdim}_M(g; T, X) = \limsup_{\epsilon \to 0} \frac{r(g; T, X, \epsilon)}{\log \frac{1}{\epsilon}}.
\]

2.2. FK Katok \(\epsilon\)-entropy and local \(\epsilon\)-entropy function. Let \(\mu \in M(X)\), \(\epsilon > 0\), \(n \in \mathbb{N}\) and \(\delta \in (0, 1)\). Put
\[
  R_{\mu, FK}(T, n, \delta, \epsilon) = \min \{ \#E : E \subset X and \mu(\bigcup_{x \in E} B_{FK_n}(x, \epsilon)) > 1 - \delta \},
\]
where \(B_{FK_n}(x, \epsilon)\) denotes the ball with center \(x\) and radius \(\epsilon\) defined by FK metric \(d_{FK_n}\).

Following the idea of [BK83], we define FK Katok’s \(\epsilon\)-entropies of \(\mu\) as
\[
  h_{\mu, FK}(T, \epsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log R_{\mu, FK}(T, n, \delta, \epsilon).
\]
The entropy function \(h(x)\) (in terms of Bowen metric) was introduced by Ye and Zhang [YZ07] to study uniform entropy points. Besides, they showed that topological entropy of \(X\) is equal to the supremum of \(h(x)\) over all points of \(X\). Next, we introduce the notion of local \(\epsilon\)-entropy function in terms of FK metric to establish a variational principle for FK metric mean dimensions.

Given \(\epsilon > 0, x \in X\), we define the FK local \(\epsilon\)-entropy function of \(x\) as
\[
  h_{FK}(x, \epsilon) = \inf \{ r_{FK}(T, K, d, \epsilon) : K is a closed neighborhood of x \}.
\]

3. Proofs of main results

In this section, we prove Theorems 1.1, 1.2 and 1.3.

We first give the proof of Theorem 1.1.

Let \(U\) be a finite open cover of \(X\). By \(\text{diam}(U) = \max_{U \in \mathcal{U}} \text{diam}U\) we denote the diameter of \(U\). By \(\text{Leb}(U)\), the Lebesgue number of \(U\), we denote the maximal positive number \(\epsilon > 0\) such that every open ball \(B_d(x, \epsilon) = \{ y \in X : d(x, y) < \epsilon \}\) is contained in some element of \(U\).

**Lemma 3.1.** Let \((X, d)\) be a compact metric space. Then for every \(\epsilon > 0\), there exists a finite open cover \(\mathcal{U}\) of \(X\) with \(|\mathcal{U}| = \#(X, d, \frac{\epsilon}{4})\), such that \(\text{diam}(\mathcal{U}) \leq \epsilon\) and \(\text{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}\).

**Proof.** Let \(Z\) be a subset of \(X\) so that \(X = \bigcup_{x \in Z} B_d(x, \frac{\epsilon}{4})\) with smallest cardinality \(\#(X, d, \frac{\epsilon}{4})\). Then \(\mathcal{U} = \{ B(x, \frac{\epsilon}{2}) : x \in Z \}\) is the open cover that we need. \(\Box\)
Proof of theorem 1.1. We divide the proof into two steps.

Step 1. we show

$$\liminf_{n \to \infty} \mathcal{M}(T, X, d) = \mathcal{M}(T, X, d).$$

The inequality $$\liminf_{n \to \infty} \mathcal{M}(T, X, d) \leq \mathcal{M}(T, X, d)$$ holds by using the fact $$d_{\mathcal{M}} \leq d_n.$$ By Lemma 3.1, there exists a finite open cover $$\mathcal{U}$$ of $$X$$ with $$|\mathcal{U}| = \#(X, d, \frac{\epsilon}{4})$$ such that $$\text{diam}(\mathcal{U}) \leq \epsilon$$ and $$\text{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}.$$ Let $$E_1$$ be a $$FK(n, \frac{\epsilon}{4})$$ spanning set of $$X$$ with the smallest cardinality $$r_{FK}(T, X, n, d, \frac{\epsilon}{4}).$$ Then

$$X = \bigcup_{x \in E_1} \bigcup_{k \in \{1, \ldots, n\}} \bigcup_{|\pi|=k, \pi \text{ is order preserving}} \bigcap_{i \in D(\pi)} T^{-i}B_d(T^{\pi(i)}x, \frac{\epsilon}{4}).$$

Since each open ball $$B_d(T^{\pi(i)}x, \frac{\epsilon}{4})$$ is contained in some element of $$\mathcal{U},$$ then $$\bigcap_{i \in D(\pi)} T^{-i}B_d(T^{\pi(i)}x, \frac{\epsilon}{4})$$ is contained in some element of $$\bigvee_{i \in D(\pi)} T^{-i}\mathcal{U}.$$ Note that $$\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U} = (\bigvee_{i \in D(\pi)} T^{-i}\mathcal{U}) \cup (\bigvee_{i \in D(\pi)} T^{-i}\mathcal{U})$$ and $$|\bigvee_{i \in D(\pi)} T^{-i}\mathcal{U}| \leq |\mathcal{U}|^{n-|\pi|}.$$ Then $$\bigcap_{i \in D(\pi)} T^{-i}B_d(T^{\pi(i)}x, \frac{\epsilon}{4})$$ can be at most covered by $$|\mathcal{U}|^{n-|\pi|}$$ elements of $$\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}.$$ Since the number of order preserving bijection $$\pi$$ with $$|\pi|= k$$ is not more than $$(C_n^k)^2,$$ then $$X$$ can be covered by $$|E_1| \sum_{k \in \{1, \ldots, n\}} (C_n^k)^2|\mathcal{U}|^{n-k}$$ elements of $$\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}.$$ By $$N(\mathcal{U})$$ we denote the smallest cardinality of subcover of $$\mathcal{U}$$ covering $$X.$$ Recall that the topological entropy of $$\mathcal{U}$$ [Wal82] is given by $$h_{top}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}).$$ By (3.1), this yields that

$$N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) \leq |E_1| \sum_{k \in \{1, \ldots, n\}} (C_n^k)^2|\mathcal{U}|^{n-k}$$

$$\leq |E_1| \sum_{k \in \{1, \ldots, n\}} 4^n |\mathcal{U}|^{n-k}$$

$$\leq r_{FK}(T, X, n, d, \frac{\epsilon}{4}) \cdot 4^n \cdot |\mathcal{U}|^{n+1} \cdot \left(\frac{n\epsilon}{4} + 1\right).$$

It follows that $$h_{top}(T, \mathcal{U}) \leq r_{FK}(T, X, d, \frac{\epsilon}{4}) + \frac{\epsilon}{4} \log |\mathcal{U}| + \log 4.$$ Since $$\text{diam}(\mathcal{U}) \leq \epsilon,$$ one has

$$r(T, X, d, 2\epsilon) := \limsup_{n \to \infty} \frac{\log r_n(T, X, d, 2\epsilon)}{n} \leq h_{top}(T, \mathcal{U})$$

and hence

$$r(T, X, d, 2\epsilon) \leq r_{FK}(T, X, d, \frac{\epsilon}{4}) + \frac{\epsilon}{4} \log \#(X, d, \frac{\epsilon}{4}) + \log 4.$$
Since \( d \) has weak tame growth, one has \( \overline{\dim}_M(T, X, d) \leq \overline{\dim}_{FK}(T, X, d) \).

Step 2. We continue to show
\[
\overline{\dim}_M(T, X, d) = \overline{\dim}_M(g; T, X).
\]

The inequality \( \overline{\dim}_M(g; T, X) \leq \overline{\dim}_M(T, X, d) \) follows by the fact that \( B_n(x, \epsilon) \subset B_n(g; x, \epsilon) \). Fix \( \epsilon > 0 \). Let \( E_2 \) be a \( (g; n, \frac{\delta}{4}) \) spanning set of \( X \) with \( |E_2| = r_n(g; T, X, \frac{\epsilon}{4}) \). So we have
\[
X = \bigcup_{x \in E_2} \bigcup_{k=|n-g(n)|}^{n} \bigcap_{i=1}^{k} T^{-i}B(T^i x, \frac{\epsilon}{4}).
\]

Similar to Step 1, one can get that
\[
N(\bigvee_{i=0}^{n-1} T^{-i}U) \leq |E_2| \left[ \sum_{k=|n-g(n)|}^{n} C^n_k |U|^{n-k} \right]
\leq r_n(g; T, X, \frac{\epsilon}{4}) \cdot (g(n) + 1) \cdot |U|^{g(n)+1} \cdot 2^n.
\]

Therefore,
\[
\frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}U) \leq \frac{\log r_n(g; T, X, \frac{\epsilon}{4})}{n} + \frac{\log(g(n) + 1)}{n} + \frac{(g(n) + 1) \log r_1(T, X, d, \frac{\epsilon}{4})}{n}.
\]

So \( r(T, X, d, 2\epsilon) \leq h_{\log}(T, U) \leq r(g; T, X, \frac{\epsilon}{4}) + \log 2 \). This shows that \( \overline{\dim}_M(T, X, d) \leq \overline{\dim}_M(g; T, X) \).

If \( d \) has tame growth of covering numbers, we have \( \overline{\dim}_M(T, X, d) = \overline{\dim}_M(T, X, d) \) [LT18]. Together with Steps 1 and 2, this finishes the proof. \( \square \)

In fact, by the proof of Step 2, we have \( \overline{\dim}_M(T, X, d) = \overline{\dim}_M(g; T, X) \) without the assumption of weak tame growth for \( d \). Next, we proceed to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Fix \( \epsilon > 0, \delta \in (0, 1) \) and let \( \mu \in M(X) \). Define
\[
h_\mu(T, \epsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log R_\mu(T, n, \delta, \epsilon),
\]
where 
\[
R_\mu(T, n, \delta, \epsilon) := \min \{ \#E : E \subset X \text{ and } \mu(\cup_{x \in E} B_{dn}(x, \epsilon)) > 1 - \delta \}.
\]

Let \( Z = \{ z_1, \ldots, z_l \} \) be a subset of \( X \) so that \( X = \cup_{1 \leq j \leq l} B_d(z_j, \frac{\delta}{4}) \) with smallest cardinality \( l = \#(X, d, \frac{\delta}{4}) \). Let \( E \) be a subset of \( X \) with the smallest cardinality \( R_{\mu, FK}(T, n, \delta, \frac{\epsilon}{4}) \) so that \( \mu(\cup_{x \in E} B_{FK_n}(x, \frac{\epsilon}{4})) > 1 - \delta \). Note that for each \( x \in E \),
\[
B_{FK_n}(x, \frac{\epsilon}{4}) = \bigcup_{k=|1-\frac{\epsilon}{4}n|}^{n} \bigcup_{\pi : |\pi| = k, \pi \text{ is order preserving} \atop \pi \in D(\pi)} \bigcap_{i \in D(\pi)} T^{-i}B_d(T^\pi(i) x, \frac{\epsilon}{4}).
\]
Choose \( x_y \in \bigcap_{i \in D(x)} T^{-i} B_d(T^\pi(i)x, \frac{\epsilon}{4}) \) with \( |\pi| = k \). Then

\[
(3.5) \quad A := \bigcap_{i \in D(x)} T^{-i} B_d(T^\pi(i)x, \frac{\epsilon}{4}) \subset \bigcap_{i \in D(x)} T^{-i} B_d(T^i x_y, \frac{\epsilon}{2}).
\]

Let \( \{0, \ldots, n-1\} \backslash D(\pi) = \{j_1, j_2, \ldots, j_a\} \) with \( a = n - k \). Then

\[
(3.6) \quad A \subset \bigcup_{1 \leq m_1, \ldots, m_a \leq l} \left( \bigcap_{i=1}^a T^{-j_i} B_d(z_{m_i}, \frac{\epsilon}{2}) \right) \cap A
\]

for some \( x, z_{m_1}, \ldots, z_{m_a} \in \bigcap_{i=1}^n T^{-j_i} B_d(z_{m_i}, \frac{\epsilon}{2}) \cap A \neq \emptyset \). Therefore, by (3.4), (3.5) and (3.6) one has

\[
R_\mu(T, n, \delta, \epsilon) \leq R_{\mu, FK}(T, n, \delta, \frac{\epsilon}{4}) \sum_{k=(1-\frac{\epsilon}{4})n}^n (C^k_n \frac{\#(X, d, \frac{\epsilon}{2})}{4})^{n-k}.
\]

Similar to (3.2), we have

\[
(3.7) \quad h_\mu(T, \epsilon, \delta) \leq h_{\mu, FK}(T, \frac{\epsilon}{4}, \delta) + \frac{\epsilon}{4} \log \#(X, d, \frac{\epsilon}{2}) + \log 4.
\]

Hence,

\[
\text{mdim}_{FK}(T, X, d) = \text{mdim}_{M}(T, X, d) \text{ by Theorem 1.1}
\]

\[
= \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in E(X,T)} h_\mu(T, \epsilon, \delta) \text{ by [Shi22, Theorem 4.2]}
\]

\[
\leq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in E(X,T)} h_{\mu, FK}(T, \epsilon, \delta) \text{ by (3.7)}.
\]

One the other hand, \( h_{\mu, FK}(T, \epsilon, \delta) \leq r_{FK}(T, X, d, \epsilon) \) holds for every \( \mu \in M(X) \). We complete the proof. \( \square \)

Finally, we give the proof of Theorem 1.3.

**Proposition 3.2.** Let \((X, T)\) be a TDS with a metric \(d\). Suppose that \(Z_1, Z_2, \ldots, Z_m\) are closed subsets of \(X\). Then for every \(\epsilon > 0\),

\[
r_{FK}(T, \cup_{j=1}^m Z_j, d, \epsilon) = \max_{1 \leq j \leq m} r_{FK}(T, Z_j, d, \epsilon).
\]

**Proof.** Fix \(\epsilon > 0\). It suffices to show

\[
r_{FK}(T, \cup_{j=1}^m Z_j, d, \epsilon) \leq \max_{1 \leq j \leq m} r_{FK}(T, Z_j, d, \epsilon).
\]

For every \(n \in \mathbb{N}\), one can choose \(1 \leq j_{(n, \epsilon)} \leq m\) such that

\[
\max_{1 \leq j \leq m} r_{FK}(T, Z_j, n, d, \epsilon) = r_{FK}(T, Z_{j_{(n, \epsilon)}}, n, d, \epsilon).
\]

and hence \(r_{FK}(T, \cup_{j=1}^m Z_j, n, d, \epsilon) \leq m \cdot r_{FK}(T, Z_{j_{(n, \epsilon)}}, n, d, \epsilon)\). This implies that

\[
\log r_{FK}(T, \cup_{j=1}^m Z_j, n, d, \epsilon) \leq \log m + \log r_{FK}(T, Z_{j_{(n, \epsilon)}}, n, d, \epsilon).
\]
By Pigeon principle and the definition of \( r_{FK}(T, \cup_{j=1}^{m} Z_j, d, \epsilon) \), we can choose a subsequence \( n_k \to \infty \) such that
\[
\frac{1}{n_k} \log sp_{FK}(T, \cup_{j=1}^{m} Z_j, n_k, d, \epsilon) \to r_{FK}(T, \cup_{j=1}^{m} Z_j, d, \epsilon)
\]
and \( Z_{j(n_k, \epsilon)} = Z_j \) for all \( k \), where \( 1 \leq j \leq m \) is a constant independent of the choice of \( n_k \). It follows that
\[
r_{FK}(T, \cup_{j=1}^{m} Z_j, d, \epsilon) \leq r_{FK}(T, Z_{j_\epsilon}, d, \epsilon) \leq \max_{1 \leq i \leq m} r_{FK}(T, Z_i, d, \epsilon).
\]
\( \square \)

**Proof of Theorem 1.3.** Fix \( \epsilon > 0 \). It is clear that \( \sup_{x \in X} h_{FK}(x, \epsilon) \leq r_{FK}(T, X, d, \epsilon) \).

Let \( \{B^1_{i_1}, \ldots, B^1_{m_1}\} \) be a finite closed balls family of \( X \) with radius at most 1. By Proposition 3.2, there exists \( 1 \leq j_1 \leq m_1 \) such that
\[
r_{FK}(T, X, d, \epsilon) = r_{FK}(T, B^1_{j_1}, d, \epsilon).
\]

For the closed ball \( B^1_{j_1} \), let \( \{B^2_{i_1}, \ldots, B^2_{m_2}\} \) be a finite closed balls family of \( B^1_{j_1} \) with radius at most \( \frac{1}{2} \) covering \( B^1_{j_1} \) and \( B^2_i \subset B^1_{j_1} \) for every \( 1 \leq i \leq m_2 \). Then by Proposition 3.2 again there exists \( 1 \leq j_2 \leq m_2 \) such that
\[
r_{FK}(T, B^1_{j_1}, d, \epsilon) = r_{FK}(T, B^2_{j_2}, d, \epsilon).
\]
Repeating this procedure, for every \( n \geq 2 \), there exists a closed ball \( B^1_{j_n} \subset B^1_{j_{n-1}} \) with radius at most \( \frac{1}{n} \) such that
\[
r_{FK}(T, X, d, \epsilon) = r_{FK}(T, \cap_{i=1}^{n} B^1_{j_i}, d, \epsilon).
\]
Let \( \{x_0\} = \cap_{n \geq 1} B^1_{j_n} \). For any closed neighborhood \( K \) of \( x_0 \), we can choose \( n_0 \) so that \( \cap_{i=1}^{n_0} B^1_{j_i} \subset K \). Therefore,
\[
r_{FK}(T, X, d, \epsilon) = r_{FK}(T, \cap_{i=1}^{n_0} B^1_{j_i}, d, \epsilon) \leq r_{FK}(T, K, d, \epsilon),
\]
which implies that \( r_{FK}(T, X, d, \epsilon) \leq h_{FK}(x_0, \epsilon) \leq \sup_{x \in X} h_{FK}(x, \epsilon) \).
\( \square \)

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No potential conflict of interest was reported by the authors.

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