SELF-DUAL POLYHEDRA OF GIVEN DEGREE SEQUENCE

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Abstract

Given vertex valencies admissible for a self-dual polyhedral graph, we describe an algorithm to explicitly construct such a polyhedron. Inputting in the algorithm permutations of the degree sequence can give rise to non-isomorphic graphs.

As an application, we find as a function of \( n \geq 3 \) the minimal number of vertices for a self-dual polyhedron with at least one vertex of degree \( i \) for each \( 3 \leq i \leq n \), and construct such polyhedra. Moreover, we find a construction for non-self-dual polyhedral graphs of minimal order with at least one vertex of degree \( i \) and at least one \( i \)-gonal face for each \( 3 \leq i \leq n \).

Keywords: Algorithm, planar graph, degree sequence, polyhedron, self-dual, quadrangulation, radial graph, valency.

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1 Introduction

1.1 Results

This paper is about topological properties of polyhedra, namely the number of edges incident to a given vertex (degrees or valencies of vertices), and the number of faces adjacent to a given face (‘degrees’ or valencies of faces).

The 1-skeleton of a polyhedron is a planar, 3-connected graph – the Rademacher-Steinitz Theorem. These graphs are embeddable in a sphere in a unique way (an observation due to Whitney). We will call them polyhedral graphs, or polyhedra for short. The dual graph of a polyhedron is a polyhedron. Vertex and face valencies swap in the dual. We call a

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polyhedron self-dual if it is isomorphic to its dual. A self-dual polyhedron on $p$ vertices has $p$ faces and $2p - 2$ edges (straightforward consequence of Euler’s formula).

In [6] we considered the problem of minimising the number of vertices of a polyhedron containing at least one vertex of valency $i$, for each $3 \leq i \leq n$. We established, among other results, that the minimal order (i.e. number of vertices) for such graphs is

$$\left\lfloor \frac{n^2 - 11n + 62}{4} \right\rfloor, \quad n \geq 14.$$  

The dual problem, that has therefore the same answer, is about imposing instead that there is at least one $i$-gonal face, for each $3 \leq i \leq n$. In this paper, we assume both conditions.

**Definition 1.** We say that a polyhedron $G$ has the property $S_n$ if it comprises at least one vertex of degree $i$ for every $3 \leq i \leq n$, and at least one $i$-gonal face for every $3 \leq i \leq n$.

Our first consideration is that if we ask instead for the minimal number of faces, and assume we have such a graph $G$, then its dual $G^*$ also satisfies $S_n$, and has minimal vertices. Thereby, the answer to both questions must be the same. Moreover, it is natural to also seek self-dual solutions.

**Theorem 2.** Let $n \geq 3$ and $G$ be a polyhedral graph satisfying $S_n$. Then the minimal number of vertices of $G$ is

$$\frac{n^2 - 5n + 14}{2}, \quad \forall n \geq 3. \quad (1.1)$$

Moreover, Algorithm 3 constructs for each $n \geq 6$ a non-self-dual polyhedron $H_n$ of order (1.1) satisfying $S_n$, whereas Algorithm 9 constructs for each $n \geq 3$ a self-dual polyhedron $G_n$ of order (1.1) satisfying $S_n$. The speed of the said algorithms is quadratic in $n$, i.e., linear in the graph order.

Theorem 2 will be proven in section 2. The construction of the self-dual solutions is a special case of the following more general result, to be proven in section 3.

**Theorem 3.** Let $k \geq 0$ and

$$t_1, t_2, \ldots, t_k, 3^m \quad (1.2)$$

be given, where the integers $t_i$ are not necessarily distinct, each $t_i \geq 4$, and

$$m = 4 + \sum_{i=1}^{k} (t_i - 4). \quad (1.3)$$

Then Algorithm 9 constructs a self-dual polyhedral graph of degree sequence (1.2). Inputting in Algorithm 9 a permutation of the $t_i$ produces, in general, non-isomorphic graphs. The speed of the algorithm is linear in the graph order.

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1The notation $3^m$ indicates that the number 3 is repeated $m$ times.
Remark 4. For fixed $t_1, \ldots, t_k$, we need equality (1.3) to hold in order for (1.2) to be the degree sequence of a self-dual polyhedron. Indeed, we have

$$
\sum_{i=1}^{k} t_i + 3m = 2q = 2(2p - 2) = 4(k + m) - 4
$$

by the handshaking lemma and self-duality. Algorithm 3 thereby constructs a self-dual polyhedral graph for any given admissible degree sequence.

1.2 Discussion and related work

Theorem 2 solves a natural modification of the questions investigated in [6], as mentioned in section 1.1. The method is to establish a lower bound on the minimal order of graphs satisfying the property $S_n$, and then to explicitly construct, for each $n$, solutions of such order via an algorithm. Here the expression for the minimal order (1.1) is cleaner, and the constructions more straightforward than in [6]. Theorem 2 will be proven in section 2. The self-dual construction of Theorem 2 is an application of Theorem 3.

Theorem 3 is about constructing self-dual polyhedra for any admissible degree sequence. The notions of duality and self-duality have been investigated since antiquity, with the Platonic solids. However, it was only relatively recently that the cornerstone achievement of generating all self-dual polyhedra was carried out [1]. This was done by constructing all their radial graphs, to be defined in section 3. Indeed, there is a one-to-one correspondence between self-dual polyhedra and their radial graphs.

Their radial graphs are certain 3-connected quadrangulations of the sphere (i.e. polyhedra where all faces are cycles of length 4), namely, those with no separating 4-cycles (i.e. all 4-cycles are faces). Self-duals and these quadrangulations are thereby intimately related (there is a caveat, a 3-connected quadrangulation of this type is not necessarily the radial of a self-dual polyhedron). Now, the generation of all quadrangulations of the sphere is another cornerstone result in graph theory [3, 2]. Equipped with this knowledge, we will prove Theorem 3 (section 3.2).

Notation. We will usually denote vertex and edge sets of a graph $G$ by $V(G)$ and $E(G)$, and their cardinality by $p = |V(G)|$ (order) and $q = |E(G)|$ (size). We will work with simple graphs (no loops or multiple edges). For $p \geq 4$, we call $W_p$ the $p - 1$-gonal pyramid (or wheel graph), of $p$ vertices.

Let $\mathcal{P}$ be an operation on a graph $G$, that modifies a given subgraph $H$ of $G$. The notation $\mathcal{P}(G)$ is not well-defined as $G$ may contain no subgraph isomorphic to $H$, or may be ambiguous when the choice of $H$ is not unique. Given the graphs $G, G'$, we will write $\mathcal{P}[G] \cong G'$ when there exists a subgraph $H$ of $G$ such that the graph obtained from $G$ on applying $\mathcal{P}$ to $H$ is isomorphic to $G'$.
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2 Proof of Theorem 2

Lower bound. Let the graph $G$ satisfy property $S_n$. In particular, $G$ has at least one vertex of valency $i$ for every $3 \leq i \leq n$. As shown in [6, proof of Lemma 7], we then have a lower bound on the edges $q = |E(G)|$,

$$2q \geq \frac{(n-2)(n-3)}{2} + 3p.$$

On the other hand, imposing that $G$ has at least one $i$-gonal face for all $4 \leq i \leq n$ yields

$$2q \leq 6p - 12 - 2 \sum_{i=4}^{n} (i - 3) = 6p - 12 - (n - 3)(n - 2).$$

Combining the two inequalities yields the lower bound in Theorem 2

$$p \geq \frac{n^2 - 5n + 14}{2}, \quad \forall n \geq 3.$$

Construction. We now turn to actually constructing 3-polytopes of such order. Consulting [5, Table I], we find that entries 1 (tetrahedron), 2 (square pyramid) and 34 (Figure 1a) are the unique polyhedra of minimal order satisfying $S_3, S_4$, and $S_5$ respectively. These are all self-dual. Next, we construct for each $n \geq 6$ a non-self-dual polyhedron of minimal order satisfying $S_n$.

Figure 1

Algorithm 5.

Input. A natural number $N \geq 6$.  

(a) The only polyhedron $H_5 = G_5$ of minimal order satisfying $S_5$.

(b) The polyhedron $H_7$ constructed in Algorithm 5.
Output. For each $6 \leq n \leq N$, a non-self-dual polyhedron $H_n$ of minimal order satisfying $S_n$.

Description. We start by considering the graph $H_5$ in Figure 1a with its attached vertex labelling, by setting the integer $n := 6$, and the set of $n-3$ triples

$$S := \{(v_1, v_4, v_6), (v_5, v_1, v_7), (v_6, v_1, v_5)\}.$$ 

At each step, given $H_{n-1}$, we perform the operation depicted in Figure 2, 'edge splitting', to each vertex triple of $S$ in turn, taking for $u_1, u_2, u_3$ the entries of the triple in order. We label successively $v_8, v_9, \ldots$ the newly inserted vertices via the edge splitting. This yields the graph $H_n$. The graph $H_7$ is illustrated in Figure 1b. At the same time, we modify $S$ in the following way. Upon applying edge splitting to $(a, b, c)$, say, we replace it by the new triple $(a, b, v)$, where $v$ is the new vertex introduced by the splitting. Lastly, calling $a'$ the first vertex of the last triple in $S$, we insert the further triple $(v_{|V(H)_{n-1}|+1}, v_1, a')$, and increase $n$ by 1. The algorithm stops as soon as $n = N + 1$.

![Figure 2: Edge splitting operation on the vertices $(u_1, u_2, u_3)$, consecutive on the boundary of a face.](image)

Remark 6. Edge splitting has the effect of raising by one the valencies of the vertex $u_1$ and of the face containing $u_2, u_3$ but not $u_1$. It also introduces the new vertex $u_4$ of degree 3, and the new triangular face $u_1, u_2, u_4$.

It is straightforward to check by induction, with base case $n = 6$, that the $H_n$ of Algorithm 5 indeed satisfy the sought properties of Theorem 2. First, edge splitting is well-defined, as it is always performed on a triple of vertices forming a triangular face. Indeed, once we replace $(u_1, u_2, u_3)$ of Figure 2 with $(u_1, u_2, u_4)$ as in the algorithm, the latter triple forms a face. As for the last triple inserted at each step, note that it is simply

$$(v_{|V(H)_{n-1}|+1}, v_1, v_{|V(H)_{n-2}|+1}).$$

The vertices $u_1 = v_{|V(H)|+1} = v_6$, $u_2 = v_1$, $u_3 = v_{|V(H)|+1} = v_5$ form a triangle in $H_5$. Therefore, after edge splitting, $u_1, u_2$, and $u_4 = v_{|V(H)|+1} = v_8$ are the vertices of a triangle in $H_6$, and so forth in this fashion.

Second, for the graph order (1.1), each step adds $n-3$ vertices, and we have by induction

$$\frac{(n - 1)^2 - 5(n - 1) + 14}{2} + n - 3 = \frac{n^2 - 5n + 14}{2}.$$
Third, to obtain $H_n$ from $H_{n-1}$, we perform $n - 3$ edge splittings. These transform a vertex of degree $i$ into one of degree $i + 1$, for $3 \leq i \leq n - 1$ respectively. Moreover, at the same time an $i$-gon gets replaced by an $i + 1$-gon: indeed, in Figure 2 the face different containing $u_2, u_3$ but not $u_1$ loses the edge $u_2u_3$ and acquires $u_2u_4, u_4u_3$. We conclude that $H_n$ satisfies $S_n$.

Fourth, we show that $H_n$ is not self-dual for any $n \geq 6$. On one hand, $\deg_{H_n}(v_1) = n$, $\deg_{H_n}(v_5) = n - 1$, and $v_1v_5 \notin E(H_n)$. On the other hand, in $H_n$ the $n$-gon and the $n - 1$-gon share the edge $v_2v_3$.

Lastly we note that Algorithm 5 may be implemented in linear time in the graph order (quadratic in $n$).

Remark 7. There are several other constructions, similar to Algorithm 5, yielding polyhedra of minimal order satisfying $S_n$, e.g. the duals $H_n^*$. The idea is to apply $n - 3$ edge splittings at each step, where each simultaneously increases by 1 the valency of a vertex and of a face.

The self-dual case, assuming Theorem 3. For the last part of Theorem 2 we require the further condition of self-duality. However, constructions with edge splitting in general do not preserve the self-duality of $H_5$ in the new graphs obtained from it. In the next section we will present Algorithm 9 that produces a self-dual polyhedron for any given admissible degree sequence, as stated in Theorem 3. The self-dual polyhedra of Theorem 2 may be constructed independently of the arguments of section 3, although possibly in a less intuitive fashion. Here we complete the proof of Theorem 2 assuming Theorem 3. To obtain $G_n$ we simply input the tuple $(4, 5, \ldots, n)$, i.e. the sequence

$$n, n - 1, \ldots, 4, 3^{(n^2 - 7n + 20)/2},$$

into Algorithm 9.

3 Generating self-dual polyhedra

3.1 Radial graphs and quadrangulations

The radial, or vertex-face graph $R_G$ of a plane graph $G$ is obtained by taking $V(R_G)$ to be the set of vertices and regions of $G$. We have an edge between two vertices $u, v$ of $R_G$ whenever $u$ is a vertex of $G$, and $v$ a region of $G$, such that $u$ lies on the boundary of $v$ in $G$ [7] section 2.8].

If the plane graph $G$ is 2-connected, the newly constructed $R_G$ is a quadrangulation of the sphere, i.e. each region is delimited by a 4-cycle [7] section 2.8]. If $G$ is a polyhedron then
so is $RG$ [1] Lemma 2.1]. Moreover, $G$ is a polyhedron if and only if $RG$ has no separating 4-cycles (i.e. 4-cycles that are not faces, so that removing the cycle disconnects the graph) [7 Lemma 2.8.2].

The radial graph of the tetrahedron is the cube, and more generally the radial graph of the pyramid (or wheel) $W_p$, $p \geq 4$ is the so-called ‘pseudo double wheel’ $PDW_{2p}$ (of $2p$ vertices), i.e. the dual graph of the $p - 1$-gonal antiprism. As established in [3 Theorem 3], and initially stated in [2], all polyhedral quadrangulations of the sphere are obtained from the cube by applying three transformations $P_1, P_2, P_3$, sketched in [3 Figure 3] and [2 Figure 3]. We introduce the notation $C(\mathcal{G}, \mathcal{P})$ for the set of all graphs that may be obtained from an initial set of graphs $\mathcal{G}$ by applying the set of transformations $\mathcal{P}$. Under this notation, the previous statement may be rephrased as,

$$C(\{PDW_8\}, \{P_1, P_2, P_3\})$$

is the set of 3-connected quadrangulations of the sphere.

Moreover,

$$C(\{PDW_2p : p \geq 4\}, \{P_1\})$$

is the set of all polyhedral quadrangulations without separating 4-cycles [3 Theorem 4]. It follows that

$$C(\{PDW_2p : p \geq 4\}, \{P_1\}) \text{ is the set of radial graphs of polyhedra.}$$

We note that the transformation $P_2$ replaces a subgraph of $G$ that is isomorphic to $PDW_8 - v$ with a copy of $PDW_{10} - v$. In particular, $P_2[PDW_{2p}] \cong PDW_{2p+2}$. Therefore, we have

$$C(\{PDW_2p : p \geq 4\}, \{P_1\}) \subseteq C(\{PDW_8\}, \{P_1, P_2\}).$$

For $G$ a self-dual polyhedron, we have in particular $|V(R_G)| = 2|V(G)|$ and $|E(R_G)| = 2|E(G)| = 2|V(R_G^*)|$. Furthermore, we can recover $G$ from $RG$ by noting that the latter is always bipartite, and taking for $G$ all of the vertices in either part of $RG$, together with edges for $G$ between pairs of vertices belonging to the same face in $RG$. The above considerations have the following consequence.

**Proposition 8.** The radial graph of any polyhedron $G$ may be obtained from the cube via the transformations $P_1, P_2$ of [3 Figure 3]. Moreover, the number of applications of $P_1$ to generate self-duals is even.

**Proof.** By the arguments of the present section, it suffices to prove that when $G$ is self-dual, the number of applications of $P_1$ on the cube to obtain $RG$ is indeed even. From [3 Figure 3], we observe that $P_1$ has the effect of adding an edge to $G$, and a vertex and an edge to $G^*$. As opposed to this, $P_2$ adds one vertex and one edge to both $G, G^*$. We have thus obtained our parity argument. 

In the next section we prove Theorem [3 putting it in the context of the above literature.
3.2 The proof of Theorem 3

As it turns out, for any \( n \geq 3 \), generating a self-dual polyhedron \( G \) of minimal order satisfying \( S_n \) may be done by applying only a transformation \( \mathcal{P} \) (to be defined below, and similar to \( \mathcal{P}_2 \) of [3, 2]) to the cube in order to construct \( R_G \), and then passing to \( G \). This generalises readily to Theorem 3 as we will now prove.

We begin by defining a function \( f \), that maps a tuple \( T = (t_1, t_2, \ldots, t_k), k \geq 0 \), of integers \( \geq 4 \) to the degree sequence (1.2)

\[
f(T) = t_1, t_2, \ldots, t_k, 3^m,
\]

where \( m \) is given by (1.3).

Algorithm 9.

Input. A \( k \)-tuple of integers \( T = (t_1, t_2, \ldots, t_k) \), with \( t_i \geq 4 \) for each \( i \).

Output. A self-dual polyhedron \( G(T) \) of degree sequence \( f(T) \).

Description. We will construct the radial graph \( R_{G(T)} \), and then pass to \( G(T) \) as explained in section 3.1. We begin by setting \( R_{G(T)} \) to be the cube \( PDW_8 \), radial graph of the tetrahedron. We also consider a subgraph \( H \) of \( R_{G(T)} \) with the vertex labelling of Figure 3a. We define the transformation \( \mathcal{P} \) that modifies a subgraph \( H \) of a graph \( G \) as shown in Figure 3b. We stop when \( T \) is empty. Each step entails \( t_i - 3 \) successive applications of \( \mathcal{P} \) to \( R_{G(T)} \). Before each subsequent application, we apply to \( H \) a graph isomorphism \( \varphi \) such that

\[
\varphi(a) = a, \quad \varphi(b) = c, \quad \varphi(c) = d, \\
\varphi(A) = A, \quad \varphi(B) = C, \quad \varphi(C) = D.
\]

as labelled in Figure 3c. Following all the \( t_i - 3 \) operations, we instead apply to \( H \) the graph isomorphism \( \psi \) satisfying

\[
\psi(a) = c, \quad \psi(b) = a, \quad \psi(c) = d, \\
\psi(A) = C, \quad \psi(B) = A, \quad \psi(C) = D.
\]

then we delete \( t_i \) from \( T \), and proceed to the next step.

Remark 10. There are in general several polyhedra for a given degree sequence (1.2). Algorithm 9 does not construct them all. On the other hand, in many cases permutations of the \( t_i \)'s give rise to non-isomorphic solutions, as may be observed via direct computation.

Remark 11. It follows from Theorem 3 that the set

\[
\mathcal{C} \{ \{PDW_8\}, \{\mathcal{P}\} \}
\]
Table 1: For given graph size $q$, the number of radial graphs $R_G$ of self-dual polyhedra $G$ with size $q$ that belong to $C(\{PDW_8\}, \{P\})$, compared to the total. For values in the last row, refer e.g. to [4].

Remark 12. The transformation $\mathcal{P}$ is similar to $\mathcal{P}_2$ of [3, 2]. More precisely, $\mathcal{P}_2$ is applicable if and only if, $\mathcal{P}$ may be applied and moreover either $a, b$ or $A, B$ belong to the same face in $R_G$ (referring to the labelling of Figure 3).

Remark 13. Applying $\mathcal{P}$ to $R_G$ has the same effect on $G$ and $G^*$ as applying the edge splitting of Figure 2 to them, where $u_1 = a$, $u_2 = b$, $u_3 = c$, and $u_4 = d$ (and analogously for vertices $A, B, C, D$ of $G^*$).

Let us now complete the proof of Theorem 3. We start by justifying applicability of the transformation $\mathcal{P}$. The initial cube clearly has a subgraph isomorphic to $H$ in Figure 3a. Furthermore, the graph in Figure 3b also has a subgraph isomorphic to $H$, where the isomorphism is $\varphi \ (3.1)$. The same statement remains true for $\psi \ (3.2)$.

Starting with the cube $R_G((3,3,3,3))$, each operation $\mathcal{P}$ clearly yields another 3-connected quadrangulation of the sphere. We now check that self-duality of $G$ is preserved by the algorithm. Each operation $\mathcal{P}$ on $R_{G(T)}$ transforms $G(T)$ and $G^*(T)$ in the same way (Remark 13). As the initial $G((3,3,3,3))$ (tetrahedron) is self-dual, then so will all the successive $G(T)$’s be. Further, the following considerations for lower-case labels $a, b, c, d$ apply verbatim to the upper-case ones by duality.
We now analyse how each step affects the degrees of the vertices in $G$. First, the degree of a vertex in $G$ is the number of faces that the corresponding vertex lies on in $R_G$, i.e., $\deg_G(v) = \deg_{R_G}(v)$ for each $v$ by the definition of radial graph. Now, each application of $\mathcal{P}$ adds 1 to the degree of $a$ (and $A$ of $G^*$), introduces the new vertex $d$ (and $D$ of $G^*$), of degree 3, and leaves other valencies unchanged. When we update $H$ via $\varphi$ (3.1), $a$ is mapped to itself. Therefore, step $i = 1, \ldots, k$ has the effect of increasing by $t_i - 3$ the degree of $a$ (and $A$).

Second, we claim that the algorithm step $i$ increases by $t_i - 4$ the number of vertices of valency 3 in $G$. By the considerations above, the first application of $\mathcal{P}$ increases one valency of $G$ from 3 to 4, and adds a new vertex of degree 3. Hence the first application of each step leaves the number of vertices of valency 3 in $G(T)$ unchanged. Each subsequent application of $\mathcal{P}$ increases their total by 1. Now step $i$ entails $t_i - 3$ operations of type $\mathcal{P}$, hence the number of vertices of degree 3 increases by $(t_i - 3) - 1$ as claimed.

Third, we claim that, at the beginning of each algorithm step, in $G(T)$ with its attached labelling one has $$\deg(a) = \deg(A) = 3.$$ We show this claim by induction. In the initial cube all vertices are of valency 3. When we apply $\psi$ (3.2) to $H$, $a$ is mapped to $c$, and $\deg(c) = 3$ since $\mathcal{P}$ does not modify its degree.

Putting everything together, after $k$ algorithm steps the degree sequence of $G(T)$ will be $$t_1, t_2, \ldots, t_k, 3^4 + \sum_{i=1}^{k} (t_k-4)$$ i.e., at the end of the algorithm the resulting sequence will be $[1.2]$ as desired.

As for algorithm speed, the total number of operations to obtain $G(T)$ is proportional to the sum of the $t_i$’s, i.e. to the graph size $q$, that is to say, to its order $p$ since $q = 2p - 2$. The proof of Theorem 3 is complete.

Future work. Our investigation generates a portion of the self-dual polyhedra (recall Table 1), starting from the tetrahedron, by applying $\mathcal{P}$ to its radial graph (Figure 3). This portion includes at least one such graph for every admissible degree sequence. It would be of interest to further analyse the set $\mathcal{C}(\{PDW_8\}, \{\mathcal{P}\})$ and its properties.
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