Percolation of finite clusters and existence of infinite shielded paths

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Abstract

In independent bond percolation on \( \mathbb{Z}^d \) with parameter \( p \), if one removes the vertices of the infinite cluster (and incident edges), for which values of \( p \) does the remaining graph contain an infinite cluster? Grimmett-Holroyd-Kozma used the triangle condition to show that for \( d \geq 19 \), the set of such \( p \) contains values strictly larger than the percolation threshold \( p_c \). With the work of Fitzner-van der Hofstad, this has been reduced to \( d \geq 11 \). We reprove this result by showing that for \( d \geq 11 \) and some \( p > p_c \), there are infinite paths consisting of “shielded” vertices — vertices all whose adjacent edges are closed — which must be in the complement of the infinite cluster. Using numerical values of \( p_c \), this bound can be reduced to \( d \geq 8 \). Our methods are elementary and do not require the triangle condition.

1 Introduction

In bond percolation, we declare each edge \( e \) in the the set \( E^d \) of nearest-neighbor edges of \( \mathbb{Z}^d \) to be open or closed with probability \( p \) or \( 1 - p \), independently of each other. The resulting product measure \( \mathbb{P}_p \) on the space \( \{0, 1\}^{E^d} \) then has \( \mathbb{P}_p(\omega(e) = 1) = p = 1 - \mathbb{P}_p(\omega(e) = 0) \) for all \( e \), and edges \( e \) with \( \omega(e) = 1 \) (respectively 0) we call open (respectively closed). The main object of study in bond percolation is the connectivity of open clusters, in particular

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whether there are infinite open clusters. Specifically, two vertices $x$ and $y$ are said to be connected by an open path (written $x \rightarrow y$) if there is a path (an alternating sequence of vertices and edges $v_0, e_0, v_1, \ldots, e_{n-1}, v_n$ such that $e_i$ has endpoints $v_i$ and $v_{i+1}$) from $x$ to $y$ whose edges are open. The (open) clusters in an outcome $\omega$ are the connected components of $\mathbb{Z}^d$ under the equivalence relation $x \rightarrow y$. If one defines
\[ p_c = p_c(d) = \inf \{ p \in [0, 1] : \mathbb{P}_p(\text{there is an infinite cluster}) > 0 \}, \]
then one can show [7, Theorem 1.10] that $p_c \in (0, 1)$ for all $d \geq 2$ and $p_c(d) \sim \frac{1}{2d}$ as $d \to \infty$ (see [5, 10]).

A natural question for $p > p_c$ is to determine the geometric properties of infinite clusters. It is known [7, Theorem 8.1] that a.s., there is a unique infinite cluster and its asymptotic density is $\theta(p) = \mathbb{P}_p(0 \text{ is in an infinite cluster}) > 0$. In this paper, following Grimmett-Holroyd-Kozma [8], we study the complement of the infinite cluster. Let $X$ be the subgraph of $\mathbb{Z}^d$ obtained after removing all vertices in the infinite cluster. The complementary critical value, $p_{fin}$, is defined as
\[ p_{fin} = p_{fin}(d) = \sup \{ p \in [0, 1] : \mathbb{P}_p(X \text{ has an infinite connected component}) > 0 \}. \]
In dimension $d = 2$, it is known that $p_c = 1/2$ [9] and that for each $p > p_c$, the infinite cluster contains infinitely many circuits (paths whose initial and final points coincide) around the origin. This implies that $p_{fin}(2) \leq p_c(2)$. Because the definition of $p_{fin}(d)$ implies
\[ p_{fin}(d) \geq p_c(d) \text{ for all } d, \]
we obtain $p_{fin}(2) = 1/2$.

Due to (1.1), one is led to ask: for which $d$ do we have $p_{fin}(d) > p_c(d)$? It is natural to believe that this is true for large $d$ because $\theta(p_c) = 0$ [7, Section 10.3] and so for $p = p_c + \epsilon$ and $\epsilon > 0$ small, one expects an infinite cluster with small asymptotic density whose removal is likely to leave much of $\mathbb{Z}^d$ intact. The inequality $p_c(d) < p_{fin}(d)$ for $d \geq 19$ was proved by Grimmett-Holroyd-Kozma in [8] using the triangle condition [1]. Later, Fitzner-van der Hofstad verified the triangle condition for $d \geq 11$ [4], so
\[ p_c(d) < p_{fin}(d) \text{ for } d \geq 11. \] (1.2)

We will develop a different approach to $p_{fin}$ involving “shielded percolation.” We call the vertex $x$ shielded if all edges incident to $x$ are closed. A path whose vertices are shielded
is called a shielded path. We define the shielded critical probability as
\[ p_{\text{shield}} := \sup\{ p \in [0, 1] : \mathbb{P}_p(\exists \text{ an infinite shielded path}) > 0 \} \]

Contrary to the situation with the critical probability \( p_c \), there a.s. exists an infinite shielded path if \( p < p_{\text{shield}} \). Furthermore, by the definition of the critical shielded probability, if \( p < p_{\text{shield}} \), then there exists an infinite connected component in \( X \). Thus, for any \( d \),
\[ p_{\text{shield}}(d) \leq p_{\text{fin}}(d). \quad (1.3) \]

By giving lower bounds on \( p_{\text{shield}} \), we therefore obtain them for \( p_{\text{fin}} \). Our goal in this paper is to reprove the Grimmett-Holroyd-Kozma result \((1.2)\) using shielded percolation. Furthermore, using numerical values of \( p_c \) from \([6, 11]\), we will also verify that \((1.2)\) should hold for all \( d \geq 8 \).

The idea for proving lower bounds for \( p_{\text{shield}} \) is adapted from Cox-Durrett \([2]\), in their study on the asymptotics of the threshold for oriented percolation. (In that paper, the idea is attributed to Kesten.) One shows that for certain values of \( p \), the number of open oriented paths from 0 to distant hyperplanes has uniformly positive mean, and suitably bounded second moment. The Paley-Zygmund inequality then implies that there are oriented infinite clusters for such \( p \). In running a version of this argument for shielded paths, we obtain the existence of infinite oriented shielded paths for certain values of \( p \). Because the oriented shielded value is smaller than \( p_{\text{shield}} \), it is conceivable that more sophisticated lower bounds for \( p_{\text{shield}} \) would allow to reduce the dimensions (11 and 8) in our results.

1.1 Main results

We begin with an explicit upper bound for \( p_{\text{shield}} \). Let \( \lambda(d) \) be the connective constant for vertex self-avoiding walks on \( \mathbb{Z}^d \). It is defined (by sub-multiplicativity) as
\[ \lambda(d) = \lim_{n \to \infty} \left( \frac{\#\{\text{vertex self-avoiding paths with } n \text{ vertices, started at } 0\}}{n} \right)^{1/n}. \]

**Theorem 1.1.** For any \( d \geq 1 \),
\[ p_{\text{shield}}(d) \leq 1 - \lambda(d)^{-\frac{1}{2d-1}}. \]

Using the elementary bound \( \lambda(d) \leq 2d - 1 \), Theorem 1.1 implies
\[ p_{\text{shield}}(d) \leq 1 - \left( \frac{1}{2d - 1} \right)^{\frac{1}{2d-1}}. \quad (1.4) \]
Therefore $p_{\text{shield}}(2) \leq 1 - \left(\frac{1}{3}\right)^{\frac{1}{3}} \sim 0.306... < \frac{1}{2}$, and we obtain
\[
p_{\text{shield}}(2) < p_{\text{fin}}(2) = p_c(2).
\]
(For $d = 3$, we obtain $p_{\text{shield}}(3) \leq 1 - \left(\frac{1}{3}\right)^{\frac{1}{3}} \sim 0.275...$, which is larger than $p_c(3) \sim 0.248...$.)

In contrast, the next result implies that $p_{\text{shield}}(d) > p_c(d)$ for large $d$.

Write $e_i$ for the $i$-th standard basis vector, and let $(X_n), (X'_n)$ be i.i.d. with $\mathbb{P}(X_n = e_i) = \mathbb{P}(X'_n = e_i) = \frac{1}{d}$ for $1 \leq i \leq d$. $S_n, S'_n$ are defined as the sum of the first $n$ terms respectively with $S_0 = S'_0 = 0$. Define the return probability
\[
p_2 = \mathbb{P}(\|S_n - S'_n\|_1 = 2 \text{ for some } n \geq 2 \mid \|S_1 - S'_1\|_1 = 2).
\]

**Theorem 1.2.** Suppose that $d \geq 4$ and that $p$ satisfies the conditions

1. $p < 1 - \left(\frac{1}{d}\right)^{\frac{1}{d-1}}$ and
2. $\frac{1}{(1-p)^2} \left( p_2 - \frac{1}{d^2} + \frac{1}{d} \left(1 - \frac{1}{d}\right)(d(1-p)^{d-1} - 1)^{-1}\right) < 1.$

Then $p_{\text{shield}}(d) \geq p$.

The previous result states that $p_{\text{shield}}$ can be bounded in terms of the return probability $p_2$. It is difficult to find the exact value of $p_2$, but at least we can calculate bounds for it. As a result of above theorems, we get the following corollaries. Write $a_n \sim b_n$ for real sequences $(a_n)$ and $(b_n)$ if $a_n/b_n \to 1$ as $n \to \infty$.

**Corollary 1.3.**
\[
p_{\text{shield}}(d) \sim \frac{\log d}{2d} \text{ as } d \to \infty. \tag{1.5}
\]

**Remark 1.4.** Corollary 1.3 implies that
\[
\liminf_{d \to \infty} \frac{p_{\text{fin}}(d)}{\log d} \geq 1.
\]

It would be interesting to have asymptotic upper bounds for $p_{\text{fin}}(d)$. Is $\frac{\log d}{2d}$ the correct order of $p_{\text{fin}}$, as it is \([8]\) on the $2d$-regular tree?

**Corollary 1.5.** For $d \geq 11$,
\[
p_c(d) < p_{\text{shield}}(d) \leq p_{\text{fin}}(d). \tag{1.6}
\]

If numerical values of $p_c(d)$ from \([6]\) or \([11]\) are used, we can improve the dimension in Corollary 1.5 to $d \geq 8$. This is shown in Table 2 in the appendix.
1.2 Outline of the paper

In the next section, we give a short proof that $p_{\text{shield}}(d) > p_c(d)$ for $d$ large enough. The proof we give would be difficult to make quantitative, since it uses (far from optimal) estimates from 1-dependent percolation. We present it because it gives a simple explanation for the inequality in high dimension. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we prove Corollaries 1.3 and 1.5. Last, in the appendix, we explain how we show numerically that for $d \geq 8$, one has $p_{\text{shield}}(d) > p_c(d)$.

2 Short proof of $p_{\text{shield}}(d) > p_c(d)$ for large $d$

In this section we give a short proof that $p_{\text{shield}}(d)$ (and therefore $p_{\text{fin}}(d)$) is larger than $p_c(d)$ if $d$ is large. Let $a > 1$ and fix $d_*$ so that

$$p_{\text{site}}(d_*) < e^{-a}.$$  \hspace{1cm} (2.1)

(This is possible since $p_{\text{site}}(d) \to 0$ as $d \to \infty$.) For $d \geq d_*$, set

$$\mathbb{Z}_d^* = \{x \in \mathbb{Z}^d : x \cdot e_i = 0 \text{ for } i = d_* + 1, \ldots, d\}.$$  

We say that a vertex $x \in \mathbb{Z}_d^*$ is partially shielded if all edges of the form $\{x, x \pm e_i\}$ are closed for $i = 1, \ldots, d_*$. Note that the partially shielded vertices form an independent site percolation process on $\mathbb{Z}_d^*$ with parameter $(1 - p)^{2(d-d_*)}$. Set $p = p(d) = \frac{a}{2d}$ so that, because $p_c(d) \sim \frac{1}{2d}$, we have $p > p_c$ for large $d$. Furthermore, for any $x \in \mathbb{Z}_d^*$,

$$\mathbb{P}_p(x \text{ is partially shielded}) = \left(1 - \frac{a}{2d}\right)^{2(d-d_*)} \to e^{-a} \text{ as } d \to \infty.$$  

For $x \in \mathbb{Z}_d^*$, we define $Y_x$ to be the indicator of the event that all edges of the form $\{x, x \pm e_i\}$ are closed for $i = 1, \ldots, d_*$. Then the $Y_x$’s form a 1-dependent site percolation process on $\mathbb{Z}_d^*$ (independent of the process of partially shielded vertices) such that for any $x \in \mathbb{Z}_d^*$,

$$\mathbb{P}_p(Y_x = 1) = \left(1 - \frac{a}{2d}\right)^{2d_*} \to 1 \text{ as } d \to \infty.$$  

Therefore the result of Liggett-Schonmann-Stacey [7, Theorem 7.65] implies that $(Y_x)$ is stochastically bounded below by an independent site percolation process $(Z_x)$ with $\mathbb{P}(Z_x = 1) \to 1$ as $d \to \infty$. We will assume that the variables $Z_x$ are coupled with the original
percolation process so that if $Z_x = 1$, then $Y_x = 1$ and that the $Z_x$’s are independent of the process of partially shielded vertices.

Call $x \in \mathbb{Z}^d$ green if $x$ is partially shielded and $Z_x = 1$. Then the set of shielded vertices in $\mathbb{Z}^d_*$ contains the set of green vertices. Since

$$\mathbb{P}_p(x \text{ is green}) = \left(1 - \frac{a}{2d}\right)^{2(d-d_*)} \mathbb{P}(Z_x = 1) \to e^{-a} \text{ as } d \to \infty,$$

inequality 2.1 implies that for $d$ large, this probability is $> p_{\text{site}}^{\text{site}}(d_*)$. Because the green sites form an independent site percolation process on $\mathbb{Z}^d_*$, one has

$$\mathbb{P}_p(\text{there is an infinite component of green vertices}) > 0 \text{ for large } d.$$

This implies that for large $d$, one has $p_{\text{shield}}(d) \geq p = \frac{a}{2d} > p_c(d)$.

### 3 Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** Suppose $p < p_{\text{shield}}$; that is, there is a.s. an infinite shielded path $\Gamma$. This is a path, which we will take to be vertex self-avoiding, whose vertices are all shielded. By translation invariance, the probability that the origin is contained in such a path is positive. We will use a Peierls-type argument to show that $p$ cannot be too large.

To do this, we enumerate the vertices of $\Gamma$ as $0 = x_0, x_1, \ldots$, and use a type of loop-erasure to produce from them another vertex self-avoiding path with shielded vertices $0 = y_0, y_1, \ldots$.

We begin with $y_0 = 0$. We then define $k_1$ as the last index such that $x_{k_1}$ is adjacent to $y_0$. We “remove the loop” between $x_0$ and $x_{k_1}$ by setting $y_1 = x_{k_1}$. Continuing, assuming we have defined $y_0, \ldots, y_j$ and $k_1, \ldots, k_j$ for $j \geq 1$, we let $k_{j+1}$ be the last index such that $x_{k_{j+1}}$ is adjacent to $y_j$, and set $y_{j+1} = x_{k_{j+1}}$. Note that there is always at least one such vertex because $x_{k_{j+1}}$ is adjacent to $y_j$. Therefore the sequence $(k_j)$ is strictly increasing, and the $y_j$’s are all distinct.

Clearly each of the $y_k$’s is shielded. We note that the sequence $(y_j)$ has the following properties:

$$y_j \text{ is adjacent to } y_{j-1} \text{ for } j \geq 1 \quad \text{ (3.1)}$$

and

$$y_j \text{ is not adjacent to any of } y_0, \ldots, y_{j-2} \text{ for } j \geq 2. \quad \text{ (3.2)}$$

Indeed, (3.1) holds by the definition of $y_j$ (it is $x_{k_j}$, which is adjacent to $x_{k_{j-1}} = y_{j-1}$). To see (3.2), note that if $i = 0, \ldots, j - 2$, then $k_{i+1}$ is the last index such that $x_{k_{i+1}}$ is adjacent
to \(y_i\), and since \(i + 1 \leq j - 1\), the number \(k_j\) must be strictly larger than \(k_{i+1}\). Therefore \(y_j\) cannot be adjacent to \(y_i\).

Let \(\Xi_n\) be the set of sequences \(0 = y_0, \ldots, y_n\) of distinct vertices with properties (3.1) (for \(j = 1, \ldots, n\)) and (3.2) (for \(j = 2, \ldots, n\)). Then the probability that any \(\gamma \in \Xi_n\) is shielded is \(q^{2d}(q^{2d-1})^n\), where \(q = 1 - p\). Because \(p < p_{\text{shield}}\), for each \(n\),

\[
0 < \inf_{m} P_p (\text{some } \gamma \in \Xi_m \text{ is shielded}) \leq \sum_{\gamma \in \Xi_n} q^{2d}(q^{2d-1})^n = q^{2d}(q^n(2d-1)\#\Xi_n).
\]

Because of property (3.1), each \(\gamma \in \Xi_n\) is a vertex self-avoiding path with \(n + 1\) vertices, started at 0. The number of such paths equals \(\lambda(d) + o(1)\) as \(n \to \infty\), so

if \(p < p_{\text{shield}}(d)\), then \(q^{2d}(q^n(2d-1)(\lambda(d) + o(1))^{n+1})\) is bounded away from 0 as \(n \to \infty\).

This implies \(q^{2d-1}\lambda(d) \geq 1\), and so we find \(p \leq 1 - \lambda(d)^{\frac{1}{2d-1}}\) for any \(p\) satisfying \(p < p_{\text{shield}}(d)\). This completes the proof. \(\square\)

Next, we move to lower bounds for \(p_{\text{shield}}\).

**Proof of Theorem 1.2.** We use a version of the second moment method from oriented percolation in [2]. Let \(R_n\) be the set of oriented paths from the origin to

\[
H_n := \\{ y \in \mathbb{Z}^d : \sum_{i=1}^{d} (y \cdot e_i) = n \}\. \tag{3.3}
\]

Let \(N_n\) be the (random) number of shielded paths in \(R_n\). Then,

\[
\mathbb{E}_p N_n = \sum_{\gamma \in R_n} \mathbb{P}_p (\text{all sites in } \gamma \text{ are shielded}) = q^{2d}(dq^{2d-1})^n, \quad \text{and} \quad \tag{3.4}
\]

\[
\mathbb{E}_p N_n^2 = \sum_{\gamma, \gamma' \in R_n} \mathbb{P}_p (\text{all sites in } \gamma \cup \gamma' \text{ are shielded}),
\]

where \(q = 1 - p\). The object is now to find values of \(p\) for which \(\frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2}\) is bounded away from infinity. If we do this, then \(\mathbb{P}_p(N_n \geq 1) = \mathbb{P}_p(N_n > 0) \geq \frac{(\mathbb{E}_p N_n)^2}{\mathbb{E}_p N_n^2}\) will be bounded away from zero, and there will be an infinite (oriented) shielded path with positive probability. For such values of \(p\), then, we will have \(p \leq p_{\text{shield}}\), and this produces a lower bound on \(p_{\text{shield}}\). In other words,

\[
\text{if } \sup_{n} \frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2} < \infty, \text{ then } p \leq p_{\text{shield}}. \tag{3.5}
\]
We now write the probability in the sum for $\mathbb{E}_p N_n^2$ as a product of many factors. First, for the path $\gamma$, we get a factor $q^{2d}q^{(2d-1)n}$. For the other path, write the vertices of $\gamma$ (in order) as $x_0, \ldots, x_n$ and the vertices of $\gamma'$ as $x_0', \ldots, x_n'$. Let $k$ satisfy $0 \leq k \leq n$. If $x_k = x_k'$, then all edges incident to $x_k'$ have already been counted in the factor for $\gamma$. If instead $||x_k - x_k'|| = 1$, then $x_k'$ is adjacent to at most two vertices of $\gamma$, so we get a factor of at most $q^{2d-3}$. Last, $x_k'$ is not adjacent to any vertices of $\gamma$ if $||x_k - x_k'|| > 2$, so we get a factor of $q^{2d-1}$. Let $Z_n(\gamma, \gamma') = \#\{k = 1, \ldots, n : x_k = x_k'\}$, and $O_n(\gamma, \gamma') = \#\{k = 1, \ldots, n : ||x_k - x_k'|| = 2\}$. Then we get the upper bound

$$\mathbb{E}_p N_n^2 \leq \sum_{\gamma, \gamma' \in R_n} q^{2d} q^{(2d-1)n} q^{(2d-3)O_n(\gamma, \gamma')} q^{(2d-1)(n-O_n(\gamma, \gamma')-Z_n(\gamma, \gamma'))} \sum_{\gamma, \gamma' \in R_n} q^{-2O_n(\gamma, \gamma')} q^{-(2d-1)Z_n(\gamma, \gamma')} = (3.6)$$

We now represent $O_n(\gamma, \gamma')$ and $Z_n(\gamma, \gamma')$ using random walks. Let $(X_k), (X'_k)$ be i.i.d. sequences with $P(X_k = e_i) = P(X'_k = e_i) = \frac{1}{d}$ for $1 \leq i \leq d$. $S_n$ and $S'_n$ are defined as the sum of the first $n$ terms respectively with $S_0 = S'_0 = 0$. Let $Z_n = \#\{k = 1, \ldots, n : S_k = S'_k\}$ and $O_n = \#\{k = 1, \ldots, n : ||S_k - S'_k||_1 = 2\}$. Using these variables and (3.6), we have the representation

$$\frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2} \leq \frac{1}{q^{2d}} \mathbb{E} q^{-2O_n} q^{-(2d-1)Z_n},$$

so by the monotone convergence theorem,

$$\frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2} \leq \frac{1}{q^{2d}} \mathbb{E} q^{-2O-(2d-1)Z} \text{ for all } n,$$

where $Z = \lim_{n \to \infty} Z_n$ and $O = \lim_{n \to \infty} O_n$. Putting this in (3.5), we obtain

$$\text{if } \mathbb{E} q^{-2O-(2d-1)Z} < \infty, \text{ then } p \leq p_{\text{shield}}. \quad (3.7)$$

We compute the expectation in the following lemma. Along with (3.7), it immediately implies the main result, Theorem 1.2. (The condition $p_2 < 1$ for $d \geq 4$ will be verified in Lemma 4.1.)
Lemma 3.1. Assume that $p_2 < 1$ and that $q = 1 - p$ satisfies 

$$dq^{2d-1} > 1$$

and $f(q) < q^2$, 

where

$$f(q) = \left(\frac{1}{d} - \frac{1}{d^2}\right)(dq^{2d-1} - 1)^{-1} + p_2 - \frac{1}{d^2}.$$ 

Then

$$E q^{-2O - (2d-1)Z} = (1 - p_2) \left(1 - \frac{1}{d}\right) \frac{dq^{2d-1}}{dq^{2d-1} - 1} (q^2 - f(q))^{-1} < \infty.$$ 

Proof. Let $h_n = ||S_n - S'_n||_1$ for $n \geq 0$, and note that $(S_n - S'_n)_{n \geq 0}$ is a Markov chain on $\mathbb{Z}^d$ started at the origin. The sequence $(h_n)$ takes values in $\{0, 2, \ldots\}$, but is not a Markov chain. However, computations give the following probabilities for it:

$$\mathbb{P}(h_k = 0 \mid h_{k-1} = 0) = \frac{1}{d}, \quad \mathbb{P}(h_k = 2 \mid h_{k-1} = 0) = 1 - \frac{1}{d}, \text{ for } k \geq 1 \text{ and}$$

$$\mathbb{P}(h_k = 2 \mid h_{k-1} = 2) = \frac{3d - 4}{d^2}, \quad \mathbb{P}(h_k = 0 \mid h_{k-1} = 2) = \frac{1}{d^2} \text{ for } k \geq 2. \quad (3.8)$$

Furthermore, since $p_2 < 1$, the strong Markov property implies $O < \infty$ a.s.

Let $(\mathcal{F}_n)$ be the filtration generated by $(X_k, X'_k : k = 1, \ldots, n)$, and define the stopping times

$$\tau_0 = 0, \quad \tau_1 = \inf\{n \geq 1 : h_n = 2\}, \text{ and generally}$$

$$\tau_k = \inf\{n \geq \tau_{k-1} + 1 : h_n = 2\} \text{ for } k \geq 1.$$ 

We then decompose the value of $Z$ according to “excursions” from the set $\{h_n = 2\}$. In other words, on the event $\{O = k\}$ for $k \geq 1$, we can write $Z = Z_1 + \cdots + Z_k$, where

$$Z_i = \#\{n \in [\tau_{i-1} + 1, \tau_i] : h_n = 0\}.$$ 

(For this decomposition to hold, we need that $\#\{n \geq \tau_k + 1 : h_n = 0\} = 0$. This holds a.s. on $\{O = k\}$, since after time $\tau_k$, the chain must move from $\{h_n = 2\}$ to $\{h_n = 4\}$, and never come back — if it moves to $\{h_n = 0\}$, it will a.s. move back to $\{h_n = 2\}$ eventually by (3.8).)

Now we compute the expectation in the lemma iteratively, conditioning on each $\mathcal{F}_{\tau_k}$:

$$E q^{-2O - (2d-1)Z} = \sum_{k=1}^{\infty} E \left[q^{-2k - (2d-1)(Z_1 + \cdots + Z_k)} 1_{\{O=k\}}\right].$$

9
\[ 
\sum_{k=1}^{\infty} q^{-2k} \mathbb{E} \left[ q^{-(2d-1)(Z_1+\ldots+Z_k)} 1_{\{\tau_k<\infty, \tau_{k+1}=\infty\}} \mid \mathcal{F}_{\tau_k} \right] 
\]

\[ 
= \sum_{k=1}^{\infty} \left( q^{-2k} \mathbb{E} \left[ q^{-(2d-1)(Z_1+\ldots+Z_k)} 1_{\{\tau_k<\infty\}} \right] P(\tau_{k+1}=\infty \mid \mathcal{F}_{\tau_k}) \right). 
\]

By the strong Markov property, \( P(\tau_{k+1}=\infty \mid \mathcal{F}_{\tau_k}) = P(h_n \neq 2 \text{ for all } n \geq 2 \mid S_1 - S_1' = x) \) for some (random) \( x = S_\tau - S_\tau' \) in the set \( \{ z \in \mathbb{Z}^d : \|z\|_1 = 2 \} \). These \( x \) are all of the form \( e_i - e_j \) with \( i \neq j \). By symmetry, these probabilities are the same for all \( x \), and can be written as \( P(h_n \neq 2 \text{ for all } n \geq 2 \mid h_1 = 2) = 1-p_2 \). (This argument is similar to the one that gives that \( p_2 < 1 \) implies \( O < \infty \text{ a.s.} \), stated below (3.8).) Therefore

\[ 
\mathbb{E}q^{-2O-(2d-1)}Z = (1-p_2) \sum_{k=1}^{\infty} q^{-2k} \mathbb{E} \left[ q^{-(2d-1)(Z_1+Z_2+\ldots+Z_k)} 1_{\{\tau_k<\infty\}} \right]. 
\]

Now conditioning on \( \mathcal{F}_{\tau_{k-1}} \), this equals

\[ 
(1-p_2) \sum_{k=1}^{\infty} \left( q^{-2k} \mathbb{E} \left[ q^{-(2d-1)(Z_1+\ldots+Z_{k-1})} 1_{\{\tau_{k-1}<\infty\}} \right] \mathbb{E} \left[ q^{-(2d-1)Z_k} 1_{\{\tau_k<\infty\}} \mid \mathcal{F}_{\tau_{k-1}} \right] \right). \tag{3.9}
\]

As before, by the strong Markov property, the term \( \mathbb{E} \left[ q^{-(2d-1)Z_k} 1_{\{\tau_k<\infty\}} \mid \mathcal{F}_{\tau_{k-1}} \right] \) for \( k \geq 2 \) is equal to \( \mathbb{E} \left[ q^{-(2d-1)Z} 1_{\{\tau_2<\infty\}} \mid S_1 - S_1' = x \right] \) for some random \( x = S_{\tau_{k-1}} - S_{\tau_{k-1}'} \) of the form \( e_i - e_j \) for some \( i \neq j \). These expectations are all the same by symmetry, so if we set

\[ 
f(q) = \mathbb{E} \left[ q^{-(2d-1)Z} 1_{\{\tau_2<\infty\}} \mid h_1 = 2 \right],
\]

then (3.9) gives us

\[ 
\mathbb{E}q^{-2O-(2d-1)}Z = (1-p_2) \left[ q^{-2} \mathbb{E} \left[ q^{-(2d-1)Z_1} 1_{\{\tau_1<\infty\}} \right] + \sum_{k=2}^{\infty} \left( q^{-2k} f(q) \mathbb{E} \left[ q^{-(2d-1)(Z_1+\ldots+Z_{k-1})} 1_{\{\tau_{k-1}<\infty\}} \right] \right) \right].
\]

Last, we iterate the above procedure, conditioning successively on \( \mathcal{F}_{\tau_{k-1}}, \mathcal{F}_{\tau_{k-2}}, \ldots, \mathcal{F}_{\tau_1} \), to obtain

\[ 
\mathbb{E}q^{-2O-(2d-1)}Z = (1-p_2) \mathbb{E} \left[ q^{-(2d-1)Z_1} 1_{\{\tau_1<\infty\}} \right] \sum_{k=1}^{\infty} \left( q^{-2k} f(q)^{k-1} \right),
\]

or, because \( \tau_1 < \infty \text{ a.s.} \) (see (3.8)),

\[ 
\mathbb{E}q^{-2O-(2d-1)}Z = (1-p_2) \mathbb{E} q^{-(2d-1)Z_1} \sum_{k=1}^{\infty} \left( q^{-2k} f(q)^{k-1} \right).
\]
\[
(1 - p_2) \mathbb{E}q^{-(2d-1)Z_1}(q^2 - f(q))^{-1} \text{ if } f(q) < q^2. \tag{3.10}
\]

We now set out to compute the terms in (3.10). Beginning with \( f(q) \), because \( h_2 = 0 \) almost surely implies \( \tau_2 < \infty \), we obtain

\[
f(q) = \mathbb{E} \left[ q^{-(2d-1)Z_2} 1_{\{\tau_2 < \infty, h_2 = 0\}} \mid h_1 = 2 \right] + \mathbb{P} (\tau_2 < \infty \text{ and } h_2 \neq 0 \mid h_1 = 2)
\]

\[
= \mathbb{E} \left[ q^{-(2d-1)Z_2} 1_{\{h_2 = 0\}} \mid h_1 = 2 \right] + p_2 - \frac{1}{d^2}. \tag{3.11}
\]

Furthermore, using (3.8), the first term of (3.11) equals

\[
\frac{1}{d^2} \mathbb{E} \left[ q^{-(2d-1)Z_2} \mid h_1 = 2, h_2 = 0 \right]
\]

\[
= \frac{1}{d^2} \sum_{j=1}^{\infty} q^{-(2d-1)j} \mathbb{P} (h_2 = \cdots = h_{j+1} = 0, h_{j+2} = 2 \mid h_1 = 2, h_2 = 0)
\]

\[
= \frac{1}{d^2} \sum_{j=1}^{\infty} q^{-(2d-1)j} \left( \frac{1}{d} \right)^{j-1} \left( 1 - \frac{1}{d} \right)
\]

\[
= \frac{1}{d} \left( 1 - \frac{1}{d} \right) (dq^{2d-1} - 1)^{-1} \text{ if } dq^{2d-1} > 1.
\]

Putting this in (3.11), we obtain

\[
f(q) = \left( \frac{1}{d} - \frac{1}{d^2} \right) (dq^{2d-1} - 1)^{-1} + p_2 - \frac{1}{d^2} \text{ when } dq^{2d-1} > 1. \tag{3.12}
\]

For the other term in (3.10), we similarly compute when \( dq^{2d-1} > 1 \)

\[
\mathbb{E}q^{-(2d-1)Z_1} = \sum_{j=0}^{\infty} q^{-(2d-1)j} \mathbb{P} (h_1 = \cdots = h_j = 0, h_{j+1} = 2)
\]

\[
= \left( 1 - \frac{1}{d} \right) \sum_{j=0}^{\infty} q^{-(2d-1)j} \left( \frac{1}{d} \right)^j
\]

\[
= \left( 1 - \frac{1}{d} \right) \frac{dq^{2d-1}}{dq^{2d-1} - 1}
\]

We place this and (3.12) into (3.10) to complete the proof.

\[
\square
\]

4 Proofs of Corollaries 1.3 and 1.5

We will use the following result in the proofs of both corollaries.

**Lemma 4.1.** For \( d \geq 4 \), one has \( p_2 < 1 \). Furthermore, if we define

\[
p_d = \mathbb{P}(S_n = S'_n \text{ for some } n \geq 1),
\]

then
1. \( p_2 = \frac{(d^2+1)p_d - d - 1}{dp_d - d} \), and

2.

\[
p_d \leq \frac{1}{d} + \left(1 - \frac{1}{d}\right) \frac{1}{d^2} + \frac{1}{d^2} \left(1 - \frac{1}{d}\right) \left(\frac{3d - 4}{d^2}\right) + \frac{1}{d^2} \left(\frac{\left(\frac{3d - 4}{d^2}\right)^2}{\left(\frac{d^2 - 3d + 3}{d^2}\right)} \left(\frac{4}{d^2}\right)\right) + \sum_{k=5}^{d} \frac{k!}{d^k} + \sum_{j=1}^{\infty} \left(\frac{j!}{d}\right)^j.
\]

**Proof.** We begin with item 1. We continue with the sequence \((h_n)\) from the previous section, where \( h_n = \|S_n - S'_n\|_1 \). As before, let

\[
Z_n = \#\{k = 1, \ldots, n : h_k = 0\} \quad \text{and} \quad O_n = \#\{k = 1, \ldots, n : h_k = 2\}.
\]

Then, recalling the probabilities in (3.8), we compute

\[
\mathbb{E}(1 + Z_n) = 1 + \sum_{k=1}^{n} \mathbb{P}(h_k = 0)
\]

\[
= 1 + \sum_{k=1}^{n} \left(\mathbb{P}(h_k = 0, h_{k-1} = 0) + \mathbb{P}(h_k = 0, h_{k-1} = 2)\right)
\]

\[
= 1 + \sum_{k=1}^{n} \left(\frac{1}{d} \mathbb{P}(h_{k-1} = 0) + \frac{1}{d^2} \mathbb{P}(h_{k-1} = 2)\right)
\]

\[
= 1 + \frac{1}{d} \mathbb{E}(1 + Z_{n-1}) + \frac{1}{d^2} \mathbb{E}O_{n-1}.
\]

By the monotone convergence theorem, for \( Z = \lim_{n \to \infty} Z_n \) and \( O = \lim_{n \to \infty} O_n \), we have

\[
\mathbb{E}(1 + Z) = 1 + \frac{1}{d} \mathbb{E}(1 + Z) + \frac{1}{d^2} \mathbb{E}O. \tag{4.1}
\]

To write (4.1) in terms of \( p_2 \) and \( p_d \), we note that by the strong Markov property,

\[
\mathbb{P}(Z = k) = p_d^k (1 - p_d) \quad \text{for} \quad k \geq 0, \quad \text{and} \quad \mathbb{P}(O = k) = (1 - p_2) p_2^{k-1} \mathbb{P}(h_k = 2 \text{ for some } k \geq 1)
\]

\[
= (1 - p_2) p_2^{k-1} \quad \text{for} \quad k \geq 1. \tag{4.3}
\]
Therefore
\[ EZ = \frac{p_d}{1 - p_d} \quad \text{and} \quad EO = \frac{1}{1 - p_2}, \]
and (4.1) becomes
\[ \frac{1}{1 - p_d} = 1 + \frac{1}{d(1 - p_d)} + \frac{1}{d^2(1 - p_2)}. \]
This implies the first item of the lemma.

For the second item, we define the stopping time
\[ \tau = \inf \{ k \geq 1 : h_k = 0 \}, \]
so that \( p_d = \mathbb{P}(\tau < \infty) \). By a straightforward calculation,
\[ \mathbb{P}(\tau = 1) = \frac{1}{d}, \quad \text{and} \]
\[ \mathbb{P}(\tau = 2) = \mathbb{P}(h_2 = 0 \mid h_1 = 2)\mathbb{P}(h_1 = 2) = \left(1 - \frac{1}{d}\right)\frac{1}{d^2}. \]
We will need to compute both \( \mathbb{P}(\tau = 3) \) and \( \mathbb{P}(\tau = 4) \), and these are a little more complicated.
We first claim that
\[ \mathbb{P}(\tau = 3) = \frac{1}{d^2} \left(\frac{3d - 4}{d^2}\right) \left(1 - \frac{1}{d}\right). \]
To show this use (3.8) to write
\[ \mathbb{P}(\tau = 3) = \mathbb{P}(h_1 = 2)\mathbb{P}(h_2 = 2 \mid h_1 = 2)\mathbb{P}(h_3 = 0 \mid h_1 = 2, h_2 = 2) = \left(1 - \frac{1}{d}\right)\frac{3d - 4}{d^2}\mathbb{P}(h_3 = 0 \mid h_1 = 2, h_2 = 2). \]
The last probability is written using the Markov property at time 2 as
\[
\mathbb{E} \left[ \mathbb{P}(h_3 = 0 \mid \mathcal{F}_2)1_{\{h_1=2,h_2=2\}} \right] = \frac{\mathbb{E} \left[ \mathbb{P}(h_2 = 0 \mid S_1 - S'_1 = x)1_{\{h_1=2,h_2=2\}} \right]}{\mathbb{P}(h_1 = 2, h_2 = 2)},
\]
where \( x \) is the (random) value of \( S_2 - S'_2 \), which must be of the form \( e_i - e_j \) for some \( i \neq j \). These probabilities are constant as \( x \) varies, and are equal to \( \frac{1}{d^2} \). Therefore we obtain
\[ \mathbb{P}(h_3 = 0 \mid h_1 = 2, h_2 = 2) = \frac{1}{d^2}, \]
and this shows (4.5).

The situation with \( \{\tau = 4\} \) is somewhat worse than that for \( \{\tau = 3\} \), and the form is
\[ \mathbb{P}(\tau = 4) \leq \frac{1}{d^2} \left(\left(\frac{3d - 4}{d^2}\right)^2 + \left(\frac{d^2 - 3d + 3}{d^2}\right)\left(\frac{4}{d^2}\right)\right) \left(1 - \frac{1}{d}\right). \]
The analysis splits into 2 cases.
1. \((h_0, \ldots, h_4) = (0, 2, 2, 2, 0)\),
2. \((h_0, \ldots, h_4) = (0, 2, 4, 2, 0)\).

The first case is computed exactly as we did for \(\{\tau = 3\}\): we obtain the form

\[
\left(1 - \frac{1}{d}\right) \left(\frac{3d - 4}{d^2}\right)^2 \frac{1}{d^2}.
\] (4.7)

For the second, we get

\[
\left(1 - \frac{1}{d}\right) \mathbb{P}(h_2 = 4 \mid h_1 = 2)\mathbb{P}(h_3 = 2 \mid h_1 = 2, h_2 = 4) \frac{1}{d^2}.
\]

By (3.8),

\[
\mathbb{P}(h_2 = 4 \mid h_1 = 2) = 1 - \frac{1}{d^2} - \frac{3d - 4}{d^2} = \frac{d^2 - 3d + 3}{d^2},
\]

so we obtain

\[
\left(1 - \frac{1}{d}\right) \frac{(d^2 - 3d + 3)\mathbb{P}(h_3 = 2 \mid h_1 = 2, h_2 = 4)}{d^2} \frac{1}{d^2}. \tag{4.8}
\]

For the other term, we again use the Markov property to write it as

\[
\mathbb{E} \left[ \frac{\mathbb{P}(h_3 = 2 \mid S_2 - S'_2 = x) \mathbf{1}_{\{h_1 = 2, h_2 = 4\}}}{\mathbb{P}(h_1 = 2, h_2 = 4)} \right],
\]

where \(x\) is the (random) value of \(S_2 - S'_2\). Up to symmetry, there are 3 different values of \(x\):

(A) \(2e_i - 2e_j\) for some \(i \neq j\),

(B) \(2e_i - e_j - e_\ell\) for some distinct \(i, j, \ell\), and

(C) \(e_i + e_j - e_\ell - e_m\) for some distinct \(i, j, \ell, m\).

In case (A), \(X_3\) must be \(e_j\) and \(X'_3\) must be \(e_i\) to make \(h_3 = 2\). This gives a probability of \(\frac{1}{d^2}\). In case (B), \(X_3\) must be \(e_j\) or \(e_\ell\) and \(X'_3\) must be \(e_i\), giving a probability of \(\frac{2}{d^2}\). In case (C), \(X_3\) must be \(e_\ell\) or \(e_m\) and \(X'_3\) must be \(e_i\) or \(e_j\), giving a probability of \(\frac{4}{d^2}\). In all cases, the probability is bounded above by \(\frac{4}{d^2}\). Plugging this into (4.8) gives an upper bound of

\[
\left(1 - \frac{1}{d}\right) \frac{d^2 - 3d + 3}{d^2} \cdot \frac{4}{d^2} \cdot \frac{1}{d^2}.
\]

If we add this to (4.7), we obtain the claimed bound in (4.6).

For \(\mathbb{P}(\tau = k)\) with \(k \geq 5\), we use

\[
\mathbb{P}(\tau = k) \leq \mathbb{P}(S_k = S'_k) \leq \max_{x \in H_k} \mathbb{P}(S_k = x),
\]

14
where we recall that $H_k$ was defined in (3.3). Following [2, p. 155], for $1 \leq k \leq d$, the maximum above is attained when $x = e_1 + \cdots + e_k$, so

$$\mathbb{P}(\tau = k) \leq \max_{x \in H_k} \mathbb{P}(S_k = x) \leq \frac{k!}{d^k} \text{ for } 1 \leq k \leq d. \quad (4.9)$$

To bound $\mathbb{P}(\tau = k)$ for $k > d$, we first claim that $\max_{x \in H_j} \mathbb{P}(S_j = x)$ is nonincreasing in $j$. Indeed, if this were not true, then we could find $j$ such that $\max_{x \in H_j} \mathbb{P}(S_j = x) > \max_{y \in H_{j-1}} \mathbb{P}(S_{j-1} = y)$. Choosing $x$ corresponding to the maximum in $H_j$, we could compute

$$\mathbb{P}(S_j = x) = \sum_{y \in H_{j-1}} \mathbb{P}(S_{j-1} = y) \mathbb{P}(S_j = x | S_{j-1} = y) < \mathbb{P}(S_j = x) \sum_{y \in H_{j-1}} \mathbb{P}(X_j = x - y) = \mathbb{P}(S_j = x),$$

a contradiction. So, using the claim, if $k \geq jd$ for $j \geq 1$, we estimate, writing $y = (y_1, \ldots, y_d)$,

$$\mathbb{P}(\tau = k) \leq \mathbb{P}(S_k = S_k') \leq \max_{x \in H_k} \mathbb{P}(S_k = x) \leq \max_{y \in H_{jd}} \mathbb{P}(S_{jd} = y) = \max_{y \in H_{jd}} \left[ \frac{1}{d^d} \cdot \frac{(jd)!}{y_1! \cdots y_d!} \right] \leq \left( \frac{1}{d} \right)^j \left( \frac{(jd)!}{(j!)^d} \right). \quad (4.10)$$

If we write

$$p_d = \mathbb{P}(\tau = 1) + \mathbb{P}(\tau = 2) + \mathbb{P}(\tau = 3) + \mathbb{P}(\tau = 4) + \sum_{k=5}^d \mathbb{P}(\tau = k) + \sum_{j=1}^\infty \sum_{k=jd+1}^{jd} \mathbb{P}(\tau = k),$$

and use (4.4) for the first and second terms, (4.5) for the third, (4.6) for the fourth, (4.9) for the first sum, and (4.10) for the last, we obtain the claimed inequality in item 2.

Finally, we show that for $d \geq 4$, one has $p_2 < 1$. It suffices, in fact, to show that $p_d < 1$ since $p_2 = 1 - \frac{1 - p_d}{d^2 p_d - 1}$, and if $p_d < 1$ then the numerator is positive (the denominator is always positive since $p_d = \mathbb{P}(h_1 = 0) = 1/d$). To show $p_d < 1$, it is enough by (4.2) to show that $\mathbb{E}Z < \infty$ and so we estimate as above, using Stirling’s approximation with $1 \leq \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} \leq e^{\frac{1}{12n}}$ for all $n \geq 1$ from [3, Eq. (9.15)] to obtain

$$\left( \frac{1}{d} \right)^j \frac{(jd)!}{(j!)^d} \leq \sqrt{2\pi d} \cdot \frac{e^{\frac{1}{12d}}}{(2\pi)^{\frac{d}{2}}} \cdot \frac{1 - d}{2^d}.$$

and

$$\mathbb{E}Z = \sum_{k=1}^\infty \mathbb{P}(h_k = 0) \leq \sum_{k=1}^\infty \max_{x \in H_k} \mathbb{P}(S_k = x) \leq \sum_{k=1}^d \frac{k!}{d^k} + \sum_{j=1}^\infty \sum_{k=jd+1}^{jd} \frac{(jd)!}{(j!)^d}.$$
\[ \leq \sum_{k=1}^{d} \frac{k!}{d^k} + d\sqrt{2\pi d} \frac{e^{\frac{-1}{2d}}}{(2\pi)^{d/2}} \sum_{j=1}^{\infty} j^{-d}. \]

This is finite for \( d \geq 4 \).

Proof of Corollary 1.3. From the consequence (1.4) of Theorem 1.1,
\[
\limsup_{d \to \infty} \frac{p_{\text{shield}}(d)}{\log d} \leq 1.
\]

For the lower bound, we put \( p = p(d) = \frac{a \log d}{2d} \) for \( a \in (0,1) \), and check that \( p \) satisfies the conditions of Theorem 1.2. First, \( p_2 < 1 \) for large \( d \) by Lemma 4.1. Next, one has
\[
(1 - p)^{2d-1} = \exp \left( (2d - 1) \log \left( 1 - \frac{a \log d}{2d} \right) \right) = \exp (-a(1 + o(1)) \log d) = d^{-a + o(1)} \text{ as } d \to \infty.
\]

This is > \( \frac{1}{d} \) for large \( d \), so the first assumption of Theorem 1.2 holds. For the second, the calculation is similar: its left side equals
\[
\frac{1}{(1 - \frac{a \log d}{2d})^2} \left( p_2 - \frac{1}{d^2} + \frac{1}{d} \left( 1 - \frac{1}{d} \right) (d^{1-a + o(1)} - 1)^{-1} \right),
\]
which is \( p_2(1 + o(1)) \) as \( d \to \infty \). Since \( p_2 < 1 \), we see this the left side is < 1 for large \( d \), and this verifies item 2. In conclusion, we find that \( p_{\text{shield}}(d) \geq \frac{a \log d}{2d} \) for large \( d \), whenever \( a < 1 \), and so
\[
\liminf_{d \to \infty} \frac{p_{\text{shield}}(d)}{\log d} \geq 1.
\]

Proof of Corollary 1.5. To find values of \( d \) for which \( p_{\text{shield}}(d) > p_c(d) \), we will need a useful upper bound for \( p_c \). Unfortunately, we only have explicit upper bounds for the threshold of oriented percolation. We define the return probability
\[
\rho_d = \mathbb{P}(S_k = S'_k, S_{k+1} = S'_{k+1} \text{ for some } k \geq 0),
\]
and use [2, Eq. (1.1)], which states that the oriented threshold satisfies \( \overline{p}_c(d) \leq \rho_d \). Since \( p_c(d) \leq \overline{p}_c(d) \), we obtain \( p_c(d) \leq \rho_d \).

Define the stopping time \( \hat{\tau} = \inf \{k \geq 0 : S_k = S'_k, S_{k+1} = S'_{k+1} \} \), so that \( \rho_d = \sum_{k=0}^{\infty} \mathbb{P}(\hat{\tau} = k) \). By similar calculations to those in the proof of Corollary 1.3 (the following are listed in [2, p. 155-156]),
\[
\mathbb{P}(\hat{\tau} = 0) = \frac{1}{d}, \quad \mathbb{P}(\hat{\tau} = 1) = 0, \quad \mathbb{P}(\hat{\tau} = 2) = \frac{1}{d^2} - \frac{1}{d^4},
\]
for \( l = 1, \ldots, d \),

\[
\mathbb{P}(l \leq \hat{\tau} \leq d) \leq \sum_{k=l}^{d} \frac{1}{d} \cdot \frac{k!}{d^k},
\]

and

\[
\mathbb{P}(d < \hat{\tau} < \infty) \leq \sum_{j=1}^{\infty} \left( \frac{1}{d} \right)^j \frac{(jd)!}{(j!)^d}.
\]

We will again want to separate out the cases \( \hat{\tau} = k \) for \( k = 3, 4 \). Doing calculations similar to those in the proof of Lemma 4.1, we obtain

\[
\mathbb{P}(\hat{\tau} = 3) = \left( 1 - \frac{1}{d} \right) \cdot \frac{3d - 4}{d^2} \cdot \frac{1}{d^3}
\]

\[
\mathbb{P}(\hat{\tau} = 4) \leq \frac{1}{d^3} \left[ \left( \frac{3d - 4}{d^2} \right)^2 + \left( \frac{d^2 - 3d + 3}{d^2} \right) \left( \frac{4}{d^2} \right) + \left( \frac{1}{d^2} \right) \left( 1 - \frac{1}{d} \right) \right] \left( 1 - \frac{1}{d} \right).
\]

(In the first case, the relevant \((h_n)\) vector is \((h_0, \ldots, h_4) = (0, 2, 2, 0, 0)\) and for the second case, they are \((h_0, \ldots, h_5) = (0, 2, 2, 0, 0, 0)\), \((0, 2, 4, 0, 0, 0)\), and \((0, 2, 0, 2, 0, 0)\).)

Combining these estimates and again using Stirling’s approximation, we obtain

\[
p_c \leq \rho_d \leq \frac{1}{d} + \frac{1}{d^2} - \frac{1}{d^4} + \frac{1}{d^5} \left( \frac{3d - 4}{d^2} \right) \left( 1 - \frac{1}{d} \right)
\]

\[
+ \frac{1}{d^3} \left[ \left( \frac{3d - 4}{d^2} \right)^2 + \left( \frac{d^2 - 3d + 3}{d^2} \right) \left( \frac{4}{d^2} \right) + \left( \frac{1}{d^2} \right) \left( 1 - \frac{1}{d} \right) \right] \left( 1 - \frac{1}{d} \right) \quad (4.11)
\]

\[
+ \sum_{k=5}^{d} \frac{k!}{d^k+1} + \sqrt{2\pi d} e^{\frac{1}{2d}} \frac{1}{d} \sum_{j=1}^{\infty} j^{\frac{1}{d}} =: g(d)
\]

To give an explicit lower bound on \( p_{\text{shield}}(d) \), we will show that for \( p = g(d) \), the two conditions of Theorem 1.2 hold. That is, we will show that

\[
g(d) < 1 - \left( \frac{1}{d} \right)^{\frac{1}{2d-1}} \quad (4.12)
\]

and

\[
\frac{1}{(1 - g(d))^2} \left( p_2 - \frac{1}{d^2} + \frac{1}{d} \left( 1 - \frac{1}{d} \right) (d(1 - g(d))^{2d-1} - 1)^{-1} \right) < 1. \quad (4.13)
\]

For any \( d \) such that these inequalities hold, we must have \( p_{\text{shield}}(d) > p_c(d) \). Indeed, since the left side of either inequality is a continuous function of \( g(d) \), they will also hold for some number \( \hat{p} > g(d) \) sufficiently close to \( g(d) \), and we will have \( p_c \leq g(d) < \hat{p} \leq p_{\text{shield}}(d) \).

To show the two inequalities above, we recall Lemma 4.1 and the bounds contained therein. From there, we define

\[
B(d) = \frac{1}{d} + \left( 1 - \frac{1}{d} \right) \frac{1}{d^2} + \frac{1}{d^2} \left( \frac{3d - 4}{d^2} \right) \left( 1 - \frac{1}{d} \right)
\]

17
\[ + \frac{1}{d^2} \left[ \left( \frac{3d - 4}{d^2} \right)^2 + \left( \frac{d^2 - 3d + 3}{d^2} \right) \left( \frac{4}{d^2} \right) \right] \left( 1 - \frac{1}{d} \right) + \sum_{k=5}^{d} \frac{k!}{d^k} + d^{2} \sqrt{2 \pi d} \frac{e^{\frac{1}{2d}}}{(2\pi)^{\frac{d}{2}}} \sum_{j=1}^{\infty} j^{\frac{1-d}{2}} \]

and

\[ t(x) = \frac{(d^2 + 1)x - d - 1}{d^2x - d}. \]

(The function \( t \) is defined so that \( t(p_d) = p_2 \).) Because \( t(x) = 1 - \frac{1-x}{d^2x-d} \), it is monotone nondecreasing for \( x > 1/d \). Therefore, since \( 1/d < p_d \leq B(d) \), one has \( p_2 \leq t(B(d)) \), and we see that it will suffice to show that

\[ g(d) \left( 1 - \left( \frac{1}{d} \right)^{\frac{1}{2d-1}} \right)^{-1} < 1 \]

and

\[ \frac{1}{(1-g(d))^2} \left( t(B(d)) - \frac{1}{d^2} + \frac{1}{d} \left( 1 - \frac{1}{d} \right) (d(1-g(d))^{2d-1} - 1)^{-1} \right) < 1. \]

Table 1 shows computed values of the left sides of these inequalities. Their values drop below 1 between dimensions 10 and 11.

Table 1: The values of the left sides of (4.12) and (4.13). The maximum of the two values drops below 1 between \( d = 10 \) and 11. Because both inequalities hold for \( 11 \leq d \leq 18 \), one has \( p_c(d) < p_{\text{shield}}(d) \) for these \( d \). (Values computed using Mathematica.)
A Numerical results

If we use numerical values of $p_c$, the result can be reduced to $d = 8$. In other words, we can show that $p_{\text{shield}}(d) > p_c(d)$ for $d \geq 8$. The second column of Table 2 shows numerical values of $p_c = p_c^{\text{bond}}$ for dimensions 5-10. The third column gives lower bounds for $p_{\text{shield}}(d)$ for these dimensions. The fourth gives the maximum of the left sides of (4.12) and (4.13) when setting $p$ equal to the value in the third column. Because this maximum is $< 1$, it shows that the value in the second column is indeed a lower bound for $p_{\text{shield}}$. One can see that the lower bound for $p_{\text{shield}}$ is larger than the value of $p_c$ for dimensions 8-10. In Table 2, we used $\hat{B}(d)$ as the upper bound of $p_d$ using (4.9), (4.10) (without using Stirling’s approximation):

$$p_d \leq \hat{B}(d) := \frac{1}{d} + \left(1 - \frac{1}{d}\right) \frac{1}{d^2} + \frac{1}{d^2} \left(\frac{3d - 4}{d^2}\right) \left(1 - \frac{1}{d}\right)$$

$$+ \frac{1}{d^2} \left[\left(\frac{3d - 4}{d^2}\right)^2 + \left(\frac{d^2 - 3d + 3}{d^2}\right) \left(\frac{4}{d^2}\right) \right] \left(1 - \frac{1}{d}\right) + \sum_{k=5}^{d} k! + \sum_{j=1}^{\infty} \left(\frac{1}{d}\right)^{jd-1} \frac{(j!)^d}{(j!)^d},$$

That is, the lower bound of $p_{\text{shield}}(d)$ in Table 2 is the maximum value of $p(d)$ which satisfies both of the following:

$$p(d) \left(1 - \left(\frac{1}{d}\right)^{\frac{1}{2d-1}}\right)^{-1} < 1$$

and

$$\frac{1}{(1 - p(d))^2} \left(t(\hat{B}(d)) - \frac{1}{d^2} + \frac{1}{d} \left(1 - \frac{1}{d}\right) (d(1 - p(d))^{2d-1} - 1)^{-1}\right) < 1.$$ 

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Table 2: Numerical values of $p_c = p_{c,\text{bond}}$ and lower bounds for $p_{\text{shield}}$. The top numerical value of $p_c$ comes from [6] and the bottom value comes from [11]. The fourth column is the maximum of the left sides of the first and second conditions in Theorem 1.2 when $p$ is set equal to the lower bound for $p_{\text{shield}}$, which is the value in the third column. The value in the second column increases above that in the third between $d = 7$ and 8. (Data computed using Mathematica.)

| $d$ | $p_{c,\text{bond}}$ | lower bound of $p_{\text{shield}}$ | maximum of first and condition |
|-----|----------------------|--------------------------------------|---------------------------------|
| 5   | 0.118 171 8          | 0.012726                             | 0.999998                        |
|     | 0.118 171 5          |                                      |                                 |
| 6   | 0.094 201 9          | 0.034893                             | 0.999997                        |
|     | 0.094 201 6          |                                      |                                 |
| 7   | 0.078 675 2          | 0.059902                             | 0.999998                        |
|     | 0.078 675 2          |                                      |                                 |
| 8   | 0.067 708 3          | 0.083526                             | 0.999994                        |
|     | 0.067 708 4          |                                      |                                 |
| 9   | 0.059 496 0          | 0.097006                             | 0.999993                        |
|     | 0.059 496 0          |                                      |                                 |
| 10  | 0.053 092 5          | 0.100445                             | 0.999985                        |
|     | 0.053 092 5          |                                      |                                 |

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