HAMILTONIAN ACTIONS AND HOMOGENEOUS LAGRANGIAN SUBMANIFOLDS

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Dedicated to Professor Francesco Mercuri on his sixtieth birthday

Abstract. We consider a connected symplectic manifold $M$ acted on properly and in a Hamiltonian fashion by a connected Lie group $G$. Inspired to the recent paper [3], see also [12] and [24], we study Lagrangian orbits of Hamiltonian actions. The dimension of the moduli space of the Lagrangian orbits is given and we also describe under which condition a Lagrangian orbit is isolated. If $M$ is a compact Kähler manifold we give a necessary and sufficient condition to an isometric action admits a Lagrangian orbit. Then we investigate Lagrangian homogeneous submanifolds on the symplectic cut and on the symplectic reduction. As an application of our results, we give new examples of Lagrangian homogeneous submanifolds on the blow-up at one point of the complex projective space and on the weighted projective spaces. Finally, applying Proposition 3.7 that we may call Lagrangian slice theorem for group acting with a fixed point, we give new examples of Lagrangian homogeneous submanifolds on irreducible Hermitian symmetric spaces of compact and noncompact type.

1. Introduction

Let $(M, \omega)$ be a symplectic manifold. A Lagrangian submanifold of $M$ is a submanifold of half dimension of $M$ on which the symplectic form $\omega$ vanishes. Lagrangian submanifolds are intensively studied and they have classically played an important role in symplectic geometry (see [15], [30], [21] [22]). Recently their role has been expanded beyond that of their use in understanding symplectic diffeomorphisms.

In [21] the author asks for a group theoretical machinery producing Lagrangian submanifolds in Hermitian symmetric spaces and in a recent paper [3] the existence problem of homogeneous Lagrangian submanifolds in compact Kähler manifolds is studied, obtaining a characterization of isometric actions admitting a Lagrangian orbit for a large class of compact Kähler manifolds including irreducible Hermitian symmetric spaces.

In this paper we first study Lagrangian homogeneous submanifolds in a symplectic manifold. We give the dimension of the moduli space of Lagrangian orbits, describing under which condition a Lagrangian orbit of a reductive Lie group $G$ is isolated. The uniqueness result generalizes Theorem 2 in [3].

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Our main tool will be the moment map that can be defined whenever we consider Hamiltonian action on $M$. More precisely, let $(M, \omega)$ be a connected symplectic manifold, $G$ be a connected Lie group of symplectic diffeomorphisms acting in a Hamiltonian fashion. This means there exists a map $\mu : M \to g^*$, where $g$ is the Lie algebra of $G$, which is called *moment map*, satisfying:

1. For each $X \in g$ let
   
   \[ \mu^X : M \to \mathbb{R}, \quad \mu^X(p) = \mu(p)(X), \]
   
   the component of $\mu$ along $X$.

   Then
   
   \[ d\mu^X = i_{X\#} \phi, \]
   
   i.e. $\mu^X$ is a Hamiltonian function for the vector field $X\#$.

2. $\mu$ is $G$-equivariant, i.e. $\mu(gp) = \text{Ad}^*(g)(\mu(p))$, where $\text{Ad}^*$ is the coadjoint representation on $g^*$.

In general the matter of existence/uniqueness of $\mu$ is delicate. However whenever $g$ is semisimple the moment map exists and it is unique [13]. If $(M, \omega)$ is a compact Kähler manifold and $G$ is a connected compact Lie group of holomorphic isometries then the existence problem is resolved (see [17]): a moment map exists if and only if $G$ acts trivially on the Albanese torus $\text{Alb}(M)$.

In the sequel we always assume that for every $\alpha \in g^*$, $G\alpha$ is a locally closed coadjoint orbit of $G$. Observe that the condition of a coadjoint orbit being locally closed is automatic for reductive group and for their semidirect products with vector spaces. There exists an example of a solvable Lie group due to Mautner [29] p.512, with non-locally closed coadjoint orbits. These assumptions are needed to applying the symplectic slice, see [2], [13], [23], [28], and the symplectic stratification of the reduced space given in [2].

Before we give the statement of our first main result, we fix our notation.

Let $Gx = G/G_x$ be a $G$-orbit. Since $G_x$ is compact, we may split $g = g_x \oplus m$, as $G_x$-modules. We denote by $n(g_x)$ the Lie algebra of $N(G_x)$ the normalizer of $G_x$ in $G$. Let $v = v_x + v_m \in g_x \oplus m$ be an element of $n(g_x)$. Then $[v_m, g_x] \subseteq g_x$, i.e. $v_m \in n(g_x)$, which implies that $[v_m, g_x] = 0$, since $m$ is $G_x$-invariant. This means

\[ n(g_x) = g_x \oplus \{ v \in m : [v, g_x] = 0 \} = g_x \oplus s. \]

Let $z(g)$ be the Lie algebra of the center of $G$. Clearly $z(g) \subseteq n(g_x)$ and the projection $\pi : n(g_x) \to g_x$ maps $z(g)$ to $z(g_x)$.

**Theorem 1.1.** Let $(M, \phi)$ be a connected symplectic $G$-Hamiltonian manifold with moment map $\mu : M \to g$. Assume that $G/G_x = Gx$ is a Lagrangian orbit. Then the dimension of the moduli space of the Lagrangian orbits which contains $Gx$ is $\dim z(g) \cap s$. Therefore the dimension of the moduli space of the Lagrangian orbits is equal or less than
dim $N(G_x)/G_x$ and $Gx$ is an isolated Lagrangian orbit, i.e. there exists a $G$-invariant neighborhood on which if $Gy$ is Lagrangian then $Gy = Gx$, if and only if the projection $\pi : n(g_x) \to g_x$ maps $z(g)$ one-to-one to $z(g_x)$. Hence if $G$ is a semisimple Lie group, a Lagrangian orbit, if there exists, is isolated. Moreover, at any level set $\mu^{-1}(c)$ if there exists a Lagrangian orbit, it is isolated.

Note that we have proved under which condition an action have infinitely many Lagrangian orbits. The next result characterizes ones having isolated Lagrangian orbit, generalizing Theorem 2 in [3].

**Theorem 1.2.** Let $G$ be a reductive Lie group acting properly and in a Hamiltonian fashion on a connected symplectic manifold $M$. A Lagrangian $G$-orbit, $Gp$, is isolated if and only if there exists a semisimple closed Lie subgroup $G'$ of $G$ such that $Gp = G'p$.

We may also characterize isometric actions on compact Kähler manifold admitting a Lagrangian orbit, applying a result of Kirwan [18] and the symplectic stratification of the reduced space given in [2].

**Theorem 1.3.** Let $G$ be a compact connected Lie group acting isometrically and in a Hamiltonian fashion on a compact Kähler manifold $M$. If we denote by $\mu$ the corresponding moment map, then $G$ admits a Lagrangian orbit if and only if there exists $c \in z(g)$ such that $\mu^{-1}(c)$ is a Lagrangian submanifold.

As an immediately corollary we have the following result.

**Corollary 1.4.** If $G$ is a compact Lie group acting isometrically in a Hamiltonian fashion on compact Kähler manifold then at any level set $\mu^{-1}(c)$ there exists at most one Lagrangian orbit. Moreover, if $G$ is a compact semisimple Lie group, then $G$ admits a Lagrangian orbit if and only if $\mu^{-1}(0)$ is a Lagrangian submanifold of $M$.

If $G$ is compact, it is standard fix an $Ad(G)$-invariant scalar product $\langle \cdot, \cdot \rangle$ and we identify $g$ with $g^*$ by means $\langle \cdot, \cdot \rangle$, we can think $\mu$ as a $g$-valued map. It is also natural study the squared moment map $\| \mu \|^2$ and its critical set. This function has been intensively studied in [18] obtaining strong information on the topology of $M$. In [5] we have proved that if a point $x$ realizes a local maximum of $\| \mu \|^2$ then the $G$-orbit through $x$ is symplectic. It is natural to study the “dual” problem: the points which realize the minimum of $\| \mu \|^2$. Note that a Lagrangian orbit could describe this set by Theorem 1.3 whenever $M$ is a compact Kähler manifold.

Next, we use Hamiltonian actions to construct non-standard homogeneous Lagrangian submanifolds on Kähler manifolds. The first “non-standard” examples on the complex projective space appeared in [12]. More recently, see [3], the classification of isometric actions of simple Lie groups admitting a Lagrangian orbit on complex projective spaces is given.
We study Lagrangian homogeneous submanifolds on the symplectic reduction ([10]) and on the symplectic cut (see [9] and [20]) making a connection between Lagrangian orbits on \( M \) and Lagrangian orbits on symplectic reduction and on symplectic cut, see Proposition 3.1 and Proposition 3.5. These results are interesting since the complex projective space is the reduced space of the standard \( S^1 \)-action on \( \mathbb{C}^n \) and the symplectic cut can be obtained by blowing-up \( M \) along a symplectic submanifold, see [9] and [20] for more details.

We use these results to construct non-standard homogeneous Lagrangian submanifolds on the blow-up at one point of the complex projective space (example 3.6 and subsection 3.2). and we also deduce, using the classification given in [3], see Corollary 3.3 and Remark 3.4, the classification of the simple compact Lie groups \( K \) acting isometrically on \( \mathbb{C}^n \) such that \( S^1 \times K \) admits a Lagrangian orbit on \( \mathbb{C}^n \).

Since our result holds also when the symplectic reduction is an orbifold, see [10] for more detail about orbifold, in subsection 3.1 we give new examples of homogeneous Lagrangian submanifolds on weighted projective spaces.

Finally, applying Proposition 3.7 which we may call Lagrangian slice for group acting with a fixed point, we give new examples of Lagrangian homogeneous submanifolds on irreducible Hermitian symmetric spaces of compact and noncompact type.

**Proposition 1.5.** Let \( G \) be a group appears in Table 1. Then \( G \) admits a Lagrangian orbit.

| \( G \) | \( M \) | \( M^* \) |
| --- | --- | --- |
| \( G_2 \subseteq SO(7) \subseteq SO(8) \) | \( SO(8)/U(4) \) | 
| \( SO(2) \times SO(n), \ n \geq 3 \) | \( SO(n + 2)/SO(2) \times SO(n) \) | \( SO_2(2, n)/SO(2) \times SO(n) \) |
| \( S(U(1) \times U(n)) \) | \( \mathbb{CP}^n \) | \( SU(1, n)/SU(U(1) \times U(n)) \) |
| \( Z(S(U(2) \times U(2n))) \times Sp(n), \ n \geq 2 \) | \( SU(2n + 2)/SU(U(2) \times U(2n)) \) | \( SU(2, 2n)/SU(U(2) \times U(2n)) \) |
| \( U(2n) \) | \( SU(4n)/U(2n) \) | \( SU^*(4n)/U(2n) \) |
| \( U(2n + 1) \subseteq U(2n + 2), \ n \geq 2 \) | \( SU(4n + 4)/U(2n + 2) \) | \( SU^*(4n + 4)/U(2n + 2) \) |
| \( U(n) \) | \( Sp(n)/U(n) \) | \( Sp(n, \mathbb{R})/U(n) \) |
| \( T^1 \cdot E_6 \) | \( E_7/T^1 \cdot E_6 \) | \( E_7^{10}/T^1 \cdot E_6 \) |
| \( T^1 \cdot \text{Spin}(9) \subseteq T^1 \cdot \text{Spin}(10) \) | \( E_6/T^1 \cdot \text{Spin}(10) \) | \( E_6^{14}/T^1 \cdot \text{Spin}(10) \) |
| \( \text{Spin}(7) \subseteq SO(8) \subseteq SU(8) \) | \( SU(8)/SU(U(2) \times U(6)) \) | 

### 2. Existence and uniqueness

Here we follows the notation as in [2] and in the introduction. The first easy remark is the following: if \( Gx \) is a Lagrangian submanifold of \( M \) then \( \mu(x) \in \mathfrak{j}(\mathfrak{g})^* \). Indeed, since \( \text{Ker} \mu = (T_xG(x))^{1, \omega} \) we have \( \mu|_{Gx} = c \in \mathfrak{g}^* \) which implies that \( c \in \mathfrak{j}(\mathfrak{g})^* \).

**Proof of Theorem 1.1** Suppose \( Gx \) is a Lagrangian orbit. We may assume that \( \mu(x) = 0 \).
From the symplectic slice, there exists a $G$-invariant neighborhood which is symplectomorphic to $Y = G \times_{G_x} (\mathfrak{g}/\mathfrak{g}_x)^*$ and the moment map is given by

$$\mu([g, v]) = Ad(g)^*(j(v)).$$

We recall that we may split $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{m}$ as $G_x$-modules and $j$ is induced by the above decomposition, see [2]. It is well-known, shrinking the neighborhood of the zero section of $Y$ if necessary, that $\dim G[g, v] \geq \dim Gx$. Hence $G[e, v]$ is Lagrangian if and only if $\mu([e, v]) \in \mathfrak{z}(\mathfrak{g})^*$ if and only if $v \in \mathfrak{m}^* \cap \mathfrak{z}(\mathfrak{g})^*$ if and only if $v \in \mathfrak{z}(\mathfrak{g})^* \cap s^*$ by [11]. This claims that the dimension of the moduli space of Lagrangian orbits which contains $Gx$ is isolated, since the moment map of the $G$-action on $Y$ is isolated, there exists a $G$-invariant neighborhood which is symplectomorphic to $Y$ if necessary, given $q \in Y$, one may note that $\mathfrak{z}(\mathfrak{g}) = \{0\}$ if and only if the projection of $\mathfrak{g}$ into $\mathfrak{g}_x$ maps $\mathfrak{z}(\mathfrak{g})$ one-to-one to $\mathfrak{z}(\mathfrak{g}_x)$. Finally, since in a $G$-invariant neighborhood of a Lagrangian orbit, the moment map is given by $\mu([g, m]) = Ad^*(g)(j(m))$, the moment map locally separates $G$-orbits, concluding our proof.

$\square$

**Example 2.1.** We consider $G = T^1 \times SU(2)$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^2$ as follows

$$(t, A)(v, w) = (A t^{-1} v, A w)$$

This action is Hamiltonian with moment map

$$\mu((z_1, z_2), (w_1, w_2)) = \frac{i}{2} \left( \frac{||w_1||^2 - ||w_2||^2}{z_2 \bar{z}_1 + w_2 \bar{w}_1} \right) + \frac{i}{2} \frac{z_1 \bar{w}_2 + w_1 \bar{w}_2}{||w_2||^2 - ||w_1||^2}.$$ 

The orbit through $((1, 1), (1, -1))$ is Lagrangian and the $T^1$-orbit through $((1, 1), (1 - 1))$ induces a curve in the moduli space of the Lagrangian orbits.

**Proof of Theorem** In the sequel we denote by $G_{ss}$ the closed Lie semisimple subgroup of $G$ whose Lie algebra is $\mathfrak{g}_s \mathfrak{g}$.

Assume there exists a closed semisimple Lie subgroup $G'$ of $G$ such that $Gp = G'p$ is Lagrangian. It is well-known, see [23], there exists a $G$-invariant neighborhood $Y$ of $Gp$ such that $\dim Gq \geq \dim Gp$ for every $q \in Y$. Since $G' \subset G_{ss}$ we get $G_{ss}p = Gp$. If we denote by $\mu$ the moment map corresponding to the $G_{ss}$-action on $M$, one may note that $\mu(p) = 0$ if and only if $\mu(q) \in \mathfrak{z}(\mathfrak{g})$, where $\mu$ is the moment map for the $G$-action on $M$, since the moment map of the $G_{ss}$-action is the projection on $\mathfrak{g}_{ss}$ of $\mu$. Therefore, shrinking $Y$ if necessary, given $q \in Y$ we get that $Gq$ is Lagrangian if and only if $G_{ss}q$ is, proving $Gp$ is isolated, since $G_{ss}$ is semisimple.

Vice-versa, let $Gp$ be an isolated Lagrangian orbit. We may assume $\mu(p) = 0$. Since $Gp$ is isolated, there exists a $G$-invariant neighborhood $Y$ such that $\mu(Y) \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$. Hence, given $x \in Y$ we have $\mu(x) = 0$ if and only if $\mu(x) \in \mathfrak{z}(\mathfrak{g})$ if and only if $\mu(x) = 0$. This proves $Gp = \mu^{-1}(0) \cap Y = \mu^{-1}(0) \cap Y$. 

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Let $H$ be a principal isotropy to the $G_{ss}$-action on $Gp$. It is well-known, see [2] and [28], that the intersection of the stratum $M^{(H)}$ of orbit type $(H)$ with the zero level set of the moment map $\mathbf{\pi}$ is a submanifold of $M$ of constant rank and the orbit space

$$(M_\alpha)^{(H)} = (\mathbf{\pi}^{-1}(0) \cap M^{(H)})/G_{ss},$$

has a natural symplectic structure $(\omega_\alpha)^{(H)}$ whose pullback to $\mathbf{\pi}^{-1}(0) \cap M^{(H)}$ coincides with the restriction to $\mathbf{\pi}^{-1}(0) \cap M^{(H)}$ of the symplectic form on $M$. Since $\mathbf{\pi}^{-1}(0) \cap Y$ is Lagrangian, we get $(\omega_\alpha)^{(H)} = 0$ which claims $(M_\alpha)^{(H)}$ are points. This implies $G_{ss}$ acts transitively on $Gp$ concluding our proof.

\[\square\]

**Proof of the Proposition** Assume that $G$ admits a Lagrangian orbit. We may suppose that $Gp$ is Lagrangian and $\mu(p) = 0$, where $\mu$ is the corresponding moment map. It is easy to check that $G^c p$ is open in $M$. Therefore $M$ is projective algebraic and there exists a $G$-equivariant embedding in some projective space [17]. In particular, since the $G^c$-action on $M$ is algebraic, $G^c p$ is Zariski open in $M$.

If $x \in \mu^{-1}(0)$ lies in different $G$-orbit, then there exist, due a results of Kirwan [18], two $G^c$-invariant disjoint neighborhood $U_p$ and $U_x$ of $p$ and $x$ respectively. Since $U_p$ contains $Gp$, the neighborhood $U_p$ must meet $U_x$, which is an absurd. We point out that, this claim that there exists at most one Lagrangian orbit at any level set $\mu^{-1}(c)$.

Vice-versa, assume that $\mu^{-1}(c)$ is Lagrangian. Let $H$ be a isotropy subgroup for the $G$-action on $M$. Since $M^{(H)} \cap \mu^{-1}(c)/G$ has a natural symplectic structure $(\phi)^{(H)}$ whose pullback to $M^{(H)} \cap \mu^{-1}(c)$ coincides with the restriction to $M^{(H)} \cap \mu^{-1}(c)$ of the symplectic form on $M$, we get that $M^{(H)} \cap \mu^{-1}(c)/G$ are points. This claim that given $p \in \mu^{-1}(c)$ we have $\mu^{-1}(c) = Gp$, since $\mu^{-1}(c)$ is connected, concluding our proof.

\[\square\]

**Remark 2.2.** In the sequel we always assume that $G$ acts properly and coisotropically on $M$. This means that there exists an open dense subset $U$ such that for every $x \in U$, $Gx$ is a coisotropic submanifold of $M$ with respect to $\phi$, i.e. $(T_x Gx)^{\perp_\phi} \subset T_x Gx$. Coisotropic actions are intensively studied in [17] and [14] and in the following paper [26], [6], [7] the complete classification of compact connected Lie groups acting isometrically and in a Hamiltonian fashion on irreducible compact Hermitian symmetric spaces are given. In a forthcoming paper [8], we shall study coisotropic actions of Lie groups acting properly and in a Hamiltonian fashion on a symplectic manifold $M$. An equivalent condition for a connected Lie group $G$ acts coisotropically on $M$ is that for every $\alpha \in \mathfrak{g}^*$ the set $G\mu^{-1}(\alpha)/G$ are points [8]. In particular if the fibers of the moment map $\mu : M \rightarrow \mathfrak{g}^*$ are connected, this holds if $M$ and $G$ are compact, then $G$ admits a Lagrangian orbit if and only if $\mu^{-1}(z)$ is a Lagrangian submanifold for some $z \in \mathfrak{z}(\mathfrak{g})^*$. 
Now, assume that a principal orbit is Lagrangian. Then there exists an abelian closed Lie group $T$ which have the same orbits of the $G$-action on $M$. Indeed, let $H$ be a principal isotropy. Since $\mu(M(H)) \subset \mathfrak{z}(\mathfrak{g})$, $M$ is mapped by $\mu$ to $\mathfrak{z}(\mathfrak{g})$. Therefore, given $x \in M^H$, from symplectic slice, we have that the $H$-submodule $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is abelian. Let $T$ be the closure of the torus whose Lie algebra is $\mathfrak{m}$. Note that $Tx = Gx$. Therefore, the set of regular points of $T$ is a subset of the set of $G$-regular points which implies that $T$ and $G$ have the same orbits, since both the $T$-action and the $G$-action are proper actions.

3. **Symplectic cut, symplectic reduction and homogeneous Lagrangian submanifolds**

In this section we shall investigate how the existence of homogeneous Lagrangian submanifolds on $M$ implies the existence of homogeneous Lagrangian submanifold on the symplectic cut and on the symplectic reduction. For sake of completeness we describe briefly these constructions and we invite the reader to see [10], [20] and [9] for a good exposition of these subjects.

Let $M$ be a connected symplectic manifold and let $G$ be a compact connected Lie group acting in a Hamiltonian fashion on $M$. Let $K$ be a semisimple compact Lie subgroup of $G$ and let $T^k$ be a $k$-dimensional torus which centralizes $K$ in $G$, i.e. $T^k \subset C_G(K)$. In the sequel we denote by

$$\phi : M \longrightarrow \mathfrak{k} \oplus \mathfrak{t}_k,$$

where $\mathfrak{t}_k = \text{Lie}(T^k)$, the moment map of the $T^k \cdot K$-action on $M$ and with $\mu$, respectively with $\psi$, the moment map of the $K$-action on $M$, respectively a moment map of the $T^k$-action on $M$.

Let $\lambda \in \mathfrak{t}_k$ be such that $T^k$ acts freely on $\psi^{-1}(\lambda)$. Then the symplectic reduction

$$M_\lambda = \psi^{-1}(\lambda)/T^k,$$

is a symplectic manifold on which $K$ acts, since it commutes with $T^k$, in a Hamiltonian fashion with moment map

$$\mathfrak{T} : M_\lambda \longrightarrow \mathfrak{k}, \quad \mathfrak{T}(x) = \mu(x).$$

This proves that $\mathfrak{T}(x) = 0$ if and only if $\mu(x) = 0$ if and only if $\phi(x) \in \mathfrak{t}_k$.

Let $[p] \in M_\lambda$. It is easy to see that $k[p] = [p]$ if and only if there exists $r(k) \in T^k$, which is unique since $T^k$ acts freely on $\psi^{-1}(\lambda)$, such that $kp = r(k)p$. This means that the following homomorphism of Lie groups

$$K[p] \longrightarrow (T^k \cdot K)p, \quad F(k) = kr(k)^{-1},$$

is a covering map, due the fact that $K$ is semisimple. Hence

$$\dim K[p] = \dim(T^k \cdot K)p - \dim T^k.$$
Since \( \dim M = \dim M_\lambda - 2 \dim T^k \), we have proved the following result.

**Proposition 3.1.** Let \( M \) be a connected symplectic manifold on which a compact connected Lie group \( G \) acts in a Hamiltonian fashion on \( M \). Let \( K \) be a compact semisimple Lie subgroup of \( G \) and let \( T^k \) be a \( k \)-dimensional torus which centralizes \( K \) in \( G \). Let \( \lambda \in \mathfrak{t}_k \) be such that \( T^k \) acts freely on \( \psi^{-1}(\lambda) \), where \( \psi \) is a moment map of the \( T^k \)-action on \( M \). Then \( (T^k \cdot K)p, p \in \psi^{-1}(\lambda) \), is Lagrangian if and only if \( K[p] \) in \( M_\lambda \) is.

**Remark 3.2.** We would like to point out that Proposition 3.1 holds if we assume only that \( \lambda \in \text{Lie}(T^k) \) is a regular value. In this case the symplectic reduction could be an orbifold, see [10], and \( T^k \) could act almost freely on the level set \( \psi^{-1}(\lambda) \). Hence the following map

\[
F : K[p] \longrightarrow (T^k \cdot K)p/T^k_p,
\]

is a covering map which implies, since \( T^k_p \) is finite, \( \dim K[p] = \dim (T^k \cdot K)_p \).

It is well-known that the complex projective space is the Kähler reduction for the standard \( S^1 \)-action on \( \mathbb{C}^{n+1} \), see [10]. Therefore, from Proposition 3.1, we deduce the following result.

**Corollary 3.3.** Let \( K \) be a compact semisimple Lie group acting in a Hamiltonian fashion on \( \mathbb{C}P^n \). Then \( K \) admits a Lagrangian orbit if and only if the \( S^1 \times K \)-action on \( \mathbb{C}^{n+1} \) admits.

**Remark 3.4.** From the classification given in [3], we deduce which compact simple Lie groups \( K \) satisfy \( K \times S^1 \) act with a Lagrangian orbit on \( \mathbb{C}^n \).

Now, we consider a one-dimensional torus \( T^1 \) which centralizes a connected compact Lie group \( K \). We may consider the symplectic cut which is briefly described in the following (see also [20] and [9]). In the sequel we think the moment maps for the \( T^1 \)-actions as \( \mathbb{R} \)-valued maps.

We consider the symplectic manifold \( M \times \mathbb{C} \) with symplectic form \( \omega = \frac{i}{2}dz \wedge d\overline{z} \). \( T^1 \) acts diagonally on \( M \times \mathbb{C} \) as \( t(m,z) = (tm,tz) \) with moment map \( \overline{\psi} = \psi + \|z\|^2 \).

Let \( \lambda \in \mathbb{R} = \text{Lie}(T^1) \) be such that \( T^1 \) acts freely on \( \psi^{-1}(\lambda) \). Then \( T^1 \) acts freely on \( \overline{\psi}^{-1}(\lambda) \) so we may consider the symplectic reduction,

\[
M^\lambda = \overline{\psi}^{-1}(\lambda)/T^1,
\]

which is called symplectic cut. Note that \( \dim M = \dim M^\lambda \). \( K \) acts on \( M^\lambda \) as \( k[m,z] = [km,z] \) with moment map

\[
\overline{\mu}([x,z]) = \mu(x),
\]

where \( \mu \) is the moment map of the \( K \)-action on \( M \). Since if \( z \neq 0 \), \( K_{[m,z]} = K_m \) we deduce that \( Km \) is Lagrangian if and only if \( K[m,z] \) is.
Proposition 3.5. Let \((M, \omega)\) be a symplectic manifold on which a compact connected Lie group \(G\) acts in a Hamiltonian fashion. Assume that \(K\) is a compact subgroup of \(G\) whose centralizer in \(G\) contains a one-dimensional torus \(T^1\). Then \(K\) admits a Lagrangian orbit in the open subset \(\{x \in M : \psi(x) < \lambda\}\), where \(\psi\) is the corresponding moment map of the \(T^1\)-action on \(M\), if and only if the \(K\)-action on the symplectic cut, obtained from the \(T^1\)-action on \(M\), admits.

Example 3.6. Let \(T^1\) acting on \(\mathbb{C}P^n\) as follows
\[
(t, [z_o, \ldots, z_n]) \mapsto [t^{-1}z_o, z_1, \ldots, z_n].
\]
This is a Hamiltonian action with moment map
\[
\phi([z_o, \ldots, z_n]) = \frac{1}{2} \| z_o \|^2 + \cdots + \| z_n \|^2.
\]
One can prove that \(\phi([1, \ldots, 0])\) is the global maximum of the moment map \(\phi\) and \(\phi^{-1}(\frac{1}{2}) = [1, \ldots, 0]\). Hence, as in [9] page 5, if \(\lambda = \frac{1}{2} - \epsilon\), \(\epsilon \geq 0\), then the Kähler cut \((\mathbb{C}P^n)^\lambda\) is the blow-up of \(\mathbb{C}P^n\) at \([1, \ldots, 0]\) that we shall indicate with \(\mathbb{C}P^n_{[1, \ldots, 0]}\). The torus action \(T^n\) on \(\mathbb{C}P^n\) given by
\[
(t_1, \ldots, t_n)([z_o, \ldots, z_n]) = [z_o, t_1^{-1}z_1, \ldots, t_n^{-1}z_n],
\]
is Hamiltonian and the principal orbits are Lagrangian. Note that \(T^n\) acts in a Hamiltonian fashion on \(\mathbb{C}P^n_{[1, \ldots, 0]}\) and for the above discussion we conclude that the \(T^n\)-action on \(\mathbb{C}P^n_{[1, \ldots, 0]}\) admits a Lagrangian orbit. Moreover, it acts coisotropically and its principal orbits are Lagrangian.

Let \(T^1 = Z(SU(1) \times U(4))\) and \(K = T^1 \times U(2)\) acting on \(\mathbb{C}P^4\) as follows
\[
(t, A)(Z) = (t, A)[z_o, z_1, z_2, w_1, w_2] = (t, A)[z_o, z, w] = [tz_o, At^{-4}z, At^{-4}w].
\]
This action is Hamiltonian with moment map
\[
\mu(Z) = \frac{i}{2} \| Z \|^2 \left( \frac{\| z \|^2 + \| w \|^2}{z_2 \overline{z}_1 + w_2 \overline{w}_1} - \frac{z_1 \overline{w}_2 + w_1 \overline{z}_2}{z_2 \overline{z}_1 + w_2 \overline{w}_1} \right) + \frac{i}{2} \| Z \|^2 \left( -\| z_o \|^2 + \frac{1}{2}(\| z \|^2 + \| w \|^2) \right)
\]
Let \(p = [z_o, 1, 1, 1, -1] \in \mathbb{C}P^4\). Note that \(\mu(p) \in \text{Lie}(T^1) \oplus 3(u(2))\), and if \(z_o \neq 0\) then \(\dim Kp = 4\), which implies \(Kp\) is Lagrangian.

Since \(K\) commutes with the above \(T^1\)-action, we obtain that the \(K\)-action on \(\mathbb{C}P^4_{[1, \ldots, 0]}\) admits a Lagrangian orbit, which is \(K[p]\).

Next, we claim the Lagrangian slice theorem for \(G\)-action with a fixed point.

Proposition 3.7. Let \(G\) be a compact Lie group acting in a Hamiltonian fashion with a fixed point on a symplectic manifold \(M\). If the slice representation at the fixed point has a Lagrangian orbit then so has the \(G\)-action on \(M\).
Proof. It follows immediately from symplectic slice. Indeed, if \( Gp = p \) then \( \mu(p) = \beta \in \mathfrak{z}(\mathfrak{g}) \) and from symplectic slice, the moment map is locally given by
\[
\mu(gp, m) = \beta + \text{Ad}^*(g)(\mu_V(m))
\]
If \( Gm \) is a Lagrangian orbit of the \( G \)-action on the slice then \( \dim Gm = \frac{1}{2} \dim M \) and \( \mu_V(m) \in \mathfrak{z}(\mathfrak{g}) \). In particular \( \dim G[p, m] = \frac{1}{2} \dim M \) and \( \mu([p, m]) \in \mathfrak{z}(\mathfrak{g}) \) which implies \( G[p, m] \) is Lagrangian. \( \square \)

Proof of the Proposition 1.5. Except for the first and the last cases, the prove follows from the classification given in [3], Corollary 3.3 and finally from Proposition 3.7. In the sequel we always refer to Table 1 p. 16 in [3], which has been enclosed at the end of this section, for compact simple Lie groups acting with a Lagrangian orbit in \( \mathbb{C}P^n \). We briefly explain our method.

The semisimple part of \( G \) admits a Lagrangian orbit on the projective space of the slice, by Table 1. Hence, by Corollary 3.3 \( G \) has a Lagrangian orbit on the slice which implies, from Proposition 3.7, \( G \) admits a Lagrangian orbit on \( M \). We consider only the Hermitian symmetric spaces of compact type since for the non compact case we have only to change \( M \) with \( M^* \).

1. \( G_2 \) acting on \( M = \text{SO}(8)/\text{U}(4) \). We use the same argument as in [7]. Since \( G_2 \cap \text{U}(4) = \text{SU}(3) \), the orbit through \([\text{U}(4)]\) is Lagrangian. Indeed, let
\[
\phi : M \rightarrow \mathfrak{g}_2^*
\]
be the moment map of the \( G_2 \)-action on \( M \). One may check that
\[
\dim G_2([\text{U}(4)]) = \frac{1}{2} \dim \text{SO}(8)/\text{U}(4),
\]
\( G_2\phi([\text{U}(4)]) = G_2/P \) is a flag manifold and \( \text{SU}(3) \subset P \). Since \( \text{SU}(3) \) is a maximal subgroup of \( G_2 \) which does not centralize a torus, we deduce that \( P = G_2 \) proving \( \phi([\text{U}(4)]) = 0 \). Hence \( G_2[\text{U}(4)] \) is Lagrangian.

2. \( G = \text{SO}(2) \times \text{SO}(n) \) acting on \( M = \text{SO}(n+2)/\text{SO}(2) \times \text{SO}(n) \). The slice is \( \mathbb{C}^n \) on which \( \text{SO}(n) \) acts with \( \Lambda_1 \). Since \( \text{SO}(n) \) admits a Lagrangian orbit on \( \mathbb{C}P^{n-1} \), from Corollary 3.3 \( G \) admits a Lagrangian orbit on the slice which implies, from Proposition 3.7, \( G \) admits on \( M \);

3. \( G = S(\text{U}(1) \times \text{U}(n)) \) acting on \( M = \mathbb{C}P^n \). Since \( \text{SU}(n) \) has a Lagrangian orbits on \( \mathbb{C}P^{n-1} \) then \( G \) admits a Lagrangian orbit on the slice which implies that \( G \) has a Lagrangian orbit on \( M \).

4. \( G = Z(\text{S}(\text{U}(2) \times \text{U}(2n)))) \times \text{Sp}(n) \) acting on \( M = \text{SU}(2n+2)/\text{S}(\text{U}(2) \times \text{U}(2n)) \). The slice is given by \( \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \) on which \( \text{Sp}(n) \) acts diagonally while the one dimensional torus acts as
\[
t(v, w) = (t^{\frac{1-n}{n}} v, t^{\frac{1-n}{n}} w)
\]
Since $\text{Sp}(n)$ admits a Lagrangian orbit on $\mathbb{CP}^{4n-1}$, we get that $G$ admits on $M$.

(5) $G = U(2n)$ acting on $M = SU(4n)/U(2n)$. The slice is given by $\Lambda^2(\mathbb{C}^{2n})$ and $SU(2n)$ acts with a Lagrangian orbit on its complex projective space. Hence $G$ admits a Lagrangian orbit on $M$.

(6) $G = U(2n + 1)$ acting on $M = SU(4n + 4)/U(2n + 2)$. The slice is given by

$$\Lambda^2(\mathbb{C}^{2n+1}) \oplus \mathbb{C} \otimes \mathbb{C}^{2n+1}.$$ 

Note that the center acts as

$$t(X, v) = (t^2 X, tv).$$

As in [3], one may prove that $Gp$ is Lagrangian, where

$$p = \left( \begin{array}{cc} 0 & 0 \\ 0 & J_n \end{array} \right), e_1,$$

and

$$J_n = \left( \begin{array}{cc} 0 & -I_n \\ I_n & 0 \end{array} \right),$$

which implies that $G$ admits a Lagrangian orbit on $M$;

(7) $G = U(n)$ acting on $\text{Sp}(n)/U(n)$. The slice is given by $S^2(\mathbb{C}^n)$ and $SU(n)$ has a Lagrangian orbit on the projective space of $S^2(\mathbb{C}^n)$. Then $G$ admits a Lagrangian orbit on $M$.

(8) $G = T^1 \cdot E_6$ acting on $M = E_7/T^1 \cdot E_6$. The slice is given by $\mathbb{C}^{27}$ on which $G$ acts with the $\Lambda_1$ representation. Since $E_6$ admits a Lagrangian orbit on $\mathbb{CP}^{26}$, $G$ admits on $M$.

(9) $G = T^1 \cdot \text{Spin}(9) \subseteq T^1 \cdot \text{Spin}(10)$ acting on $E_6/T^1 \cdot \text{Spin}(10)$. The slice is given by $\mathbb{C}^{16}$ on which $G$ acts with the spin-representation. Since $\text{Spin}(9)$ has a Lagrangian orbit on $\mathbb{CP}^{15}$, $G$ admits a Lagrangian orbit on $M$ from Corollary 3.3 and Proposition 3.7.

(10) This case is proved in [6] p. 1736.

3.1. Homogeneous Lagrangian submanifolds on weighted projective Spaces. In [3] it was proved that the following action

| $G$        | $\rho$ | $\dim_{\mathbb{C}} \mathbb{P}(V)$ | $G^\circ_p$          |
|------------|--------|----------------------------------|----------------------|
| $SU(n)$    | $\Lambda_1 \oplus \Lambda_1^*$ | $2n - 1$            | $\text{SO}(n)$      |
| $SU(2n + 1)$| $\Lambda_2 \oplus \Lambda_1$ | $2n^2 + 3n + 1$ | $\text{Sp}(n)$      |
| $\text{Sp}(n)$ | $\Lambda_1 \oplus \Lambda_1$ | $4n - 1$            | $\text{Sp}(n - 1)$ |
| $\text{Spin}(10)$ | $\Lambda_e \oplus \Lambda_e$ | $31$                | $\text{SU}(5)$      |
admit Lagrangian orbit. We briefly explain the notation, which follows one given in [8]. $\rho$ denotes the representation and $G^o$ denotes the connected component of the identity of the isotropy of the Lagrangian orbit. Moreover we have identified the fundamental weights $\Lambda_i$ with the corresponding irreducible representations.

We shall prove that if $G$ appears in the above table, then it induces an action on some weighted projective space with a Lagrangian orbit.

Let $T^1$ be a one-dimensional torus acting on $V = V_1 \oplus V_2$ as

$$t(v, w) = (t^{-k}v, t^{-s}),$$

where $k, s$ are distinct natural numbers. This actions is Hamiltonian with moment map

$$\psi((v, w)) = \frac{i}{2} (k \parallel v \parallel^2 + s \parallel w \parallel^2).$$

Let $\lambda = \frac{i}{2}$. The reduced space $M_\lambda$ is the weighted projective space, which we denote by $\mathbb{P}(V)[k, s]$, on which $G$ acts in a Hamiltonian fashion since it commutes with the $T^1$-action. The map $\mu : V \rightarrow g^*$ defined for every $(v, w) \in V$ and for every $X \in g$ by

$$\mu((v, w)) = -i\langle X(v, w), (v, w) \rangle$$

is the moment map for the $G$-action on $V$, where $\langle \cdot, \cdot \rangle$ denotes the natural hermitian scalar product on $V$.

We claim that $L = G \times T^1$ admits a Lagrangian orbit which lies in $\psi^{-1}(\frac{\lambda}{2})$. Note that the corresponding moment map for the $L$-action is $\xi = \mu + \psi$. Our approach is the following.

Let $p = (v, w) \in V$ be such that $G[p]$ is Lagrangian in $\mathbb{P}(V)$, which has been calculated in [8]. Then we may prove that $L\overline{p}$, where $\overline{p} = \frac{1}{\sqrt{k\parallel v \parallel^2 + s\parallel w \parallel^2}}p$, is Lagrangian, which implies that $G[\overline{p}]$ is Lagrangian in $\mathbb{P}(V)[k, s]$.

1. $G = SU(n)$ and $\rho = \Lambda_1 \oplus \Lambda_1^*$. $p = (e_1, e_1^*)$ and one may prove that the $L$-orbit through $\overline{p}$ has dimension $2n$. Since $\xi(\overline{p}) = \psi(\overline{p})$, $L\overline{p}$ is Lagrangian;
2. $G = SU(2n + 1)$ and $\rho = \Lambda_2 \oplus \Lambda_1$. $p = (J_n, e_1)$, where $J_n$ is the same as in case (6) of the proof of Proposition 3.7. Since $dim L\overline{p} = 2n^2 + 3n + 2$, $L\overline{p}$ is Lagrangian;
3. $Sp(n)$ and $\rho = \Lambda_1 \oplus \Lambda_1$. $p = (e_1, e_2)$ and it is easy to check that $dim L\overline{p} = 4n$, which implies that $L\overline{p}$ is Lagrangian;
4. $G = Spin(10)$ and $\rho = \Lambda_v \oplus \Lambda_v$. $p = (1 + e_{1234}, e_{15} + e_{2345})$ (see [27] for the notation).

One may prove that $dim L\overline{p} = 32$ which means that $L$ admits a Lagrangian orbit.

3.2. Homogeneous Lagrangian submanifolds on the blow-up at one point of the complex projective space. Let $K$ be compact Lie group acting linearly on $V$ with at two sub-modules, i.e. $V = V_1 \oplus V_2$ and $K$ preserves $V_i, i = 1, 2$. Suppose that $K$ admits a Lagrangian orbit on the complex projective space $P(V)$. If we consider a one-dimensional torus $T^1$ acting on $V_1$ or on $V_2$ which commutes with $K$ we may induces an Hamiltonian
action of $K$ on the Kähler cut $\mathbb{P}(V)^{\lambda}$ and this action admits a Lagrangian orbit. In [3] it was proved that the following action admits a Lagrangian orbit.

| $G$          | $\rho$ | $\dim_{\mathbb{C}} \mathbb{P}(V)$ | $G_p^\circ$ |
|--------------|--------|------------------------------------|-------------|
| $\text{SU}(n)$ | $\Lambda_1 \oplus \Lambda_1^*$ | $2n - 1$       | $\text{SO}(n)$ |
| $\text{SU}(2n + 1)$ | $\Lambda_2 \oplus \Lambda_1$ | $2n^2 + 3n + 1$ | $\text{Sp}(n)$ |
| $\text{Sp}(n)$ | $\Lambda_1 \oplus \Lambda_1$ | $4n - 1$       | $\text{Sp}(n - 1)$ |
| $\text{Spin}(10)$ | $\Lambda_e \oplus \Lambda_e$ | $31$           | $\text{SU}(5)$ |

We shall prove that these group admits a Lagrangian orbit on the blow-up at one point of $\mathbb{P}(V)$. We analyze in detail only the first case: the other ones are similar.

Let $T^1$ be a torus acting on $\mathbb{P}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ as $t[(v, w)] = [(t^{-1}v, w)]$. This action is Hamiltonian with moment map

$$\mu([ (v, w) ]) = \frac{i \| v \|^2}{2 \| [v, w] \|^2}.$$  

Hence $p = [(1, \ldots, 1), (0, \ldots, 0)]$ is the global maximum and as usual, see [9], given $\lambda = \frac{1}{2} - \epsilon$, $\epsilon \cong 0$, the Kähler-cut is the blow-up of the complex projective space $\mathbb{P}(V)$ at $p$, which we denote by $\mathbb{P}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)_{[p]}$. Since the $\text{SU}(n)$-action on $\Lambda_1 \oplus \Lambda_1^*$ commutes with the $T^1$-action, from Proposition 3.3 $\text{SU}(n)$ admits a Lagrangian orbit on $\mathbb{P}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)_{[p]}$. Summing up we have the following Homogeneous Lagrangian submanifolds.
**Table:** Lagrangian orbits of simple Lie groups in projective spaces

| $G$          | $\rho$          | $\dim_{\mathbb{C}} \mathbb{P}(V)$ | cond. |
|--------------|-----------------|------------------------------------|-------|
| $\text{SU}(n)$ | $2\Lambda_1$     | $\frac{n(n+1)}{2} - 1$             |       |
| $\text{SU}(n)$ | $\Lambda_1 \oplus \Lambda_1^*$ | $2n - 1$                           |       |
| $\text{SU}(n)$ | $\underbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}_n$ | $n^2 - 1$                           |       |
| $\text{SU}(2n)$ | $\Lambda_2$      | $n(2n - 1) - 1$                    | $n \geq 3$ |
| $\text{SU}(2n + 1)$ | $\Lambda_2 \oplus \Lambda_1$ | $2n^2 + 3n + 1$                    | $n \geq 2$ |
| $\text{SU}(2)$ | $3\Lambda_1$     |                                    | 3     |
| $\text{SU}(6)$ | $\Lambda_3$      |                                    | 19    |
| $\text{SU}(7)$ | $\Lambda_3$      |                                    | 34    |
| $\text{SU}(8)$ | $\Lambda_3$      |                                    | 55    |
| $\text{Sp}(n)$ | $\Lambda_1 \oplus \Lambda_1$ | $4n - 1$                           |       |
| $\text{Sp}(3)$ | $\Lambda_3$      |                                    | 13    |
| $\text{SO}(n)$ | $\Lambda_1$      |                                    | $n - 1$ | $n \geq 3$ |
| $\text{Spin}(7)$ | spin.rep.       |                                    | 7     |
| $\text{Spin}(9)$ | spin.rep.       |                                    | 15    |
| $\text{Spin}(10)$ | $\Lambda_e \oplus \Lambda_e$ | $31$                              |       |
| $\text{Spin}(11)$ | spin.rep.       |                                    | 31    |
| $\text{Spin}(12)$ | $\Lambda_e$      |                                    | 31    |
| $\text{Spin}(14)$ | $\Lambda_e$      |                                    | 63    |
| $\text{E}_6$   | $\Lambda_1$      |                                    | 26    |
| $\text{E}_7$   | $\Lambda_1$      |                                    | 55    |
| $\text{G}_2$   | $\Lambda_2$      |                                    | 6     |
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