MULTIPLE SOLUTIONS AND THEIR ASYMPTOTICS FOR LAMINAR FLOWS THROUGH A POROUS CHANNEL WITH DIFFERENT PERMEABILITIES

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Abstract. The existence and multiplicity of similarity solutions for the steady, incompressible and fully developed laminar flows in a uniformly porous channel with two permeable walls are investigated. We shall focus on the so-called asymmetric case where the upper wall is with an amount of flow injection and the lower wall with a different amount of suction. We show that there exist three solutions designated as type I, type II and type III for the asymmetric case. The numerical results suggest that a unique solution exists for the Reynolds number $0 \leq R < 14.10$ and two additional solutions appear for $R > 14.10$. The corresponding asymptotic solution for each of the multiple solutions is constructed by the method of boundary layer correction or matched asymptotic expansion for the most difficult high Reynolds number case. Asymptotic solutions are all verified by their corresponding numerical solutions.

Key words. Navier-Stokes equation, multiple solutions, laminar flow, similarity solution, asymmetric flow

AMS subject classifications. 34B15, 34E10, 76D03, 76M45

1. Introduction. Laminar flows in various geometries with porous walls are of fundamental importance to biological organisms, contaminant transports in aquifers and fractures, air circulation in the respiratory system, membrane filtration, control of boundary layer separation, automotive filters, etc. Hence, laminar flows through permeable walls have been extensively studied by researchers during the past several decades. The analysis for Navier-Stokes equation which describes the two-dimensional steady laminar flows of a viscous incompressible fluid through a porous channel with uniform injection or suction was initiated by Berman [1]. He assumed that the flow is symmetric about the centre line of the channel and is of similarity form, and reduced the problem to a fourth order nonlinear ordinary differential equation with four boundary conditions and a cross-flow Reynolds number $R$. He also gave an asymptotic solution for small Reynolds number. Then, numerous studies have been done about the laminar flows in a channel or tube with permeable walls. Yuan [30], Terrill and Shrestha [25] and Sellars [18] obtained an asymptotic solution for the large injection and large suction cases, respectively. Terrill [25, 24] and Sherstha [19] derived a series of asymptotic solutions using the method of matched asymptotic expansion for the large injection and large suction cases with a transverse magnetic field.

All these works mentioned above had produced only one solution for each value of $R$. Raithby [16] was the first to find that there is a second solution for values of $R$. This manuscript is for review purposes only.
$R > 12$ in a numerical investigation of the flow in a channel with heat transfer. Then, some studies were also devoted to the analysis of multiple solutions for the symmetric porous channel flow problem. Robinson\cite{17} conjectured that there are three types of solutions which were classified as type $I$, type $II$ and type $III$. His conclusion was based on the numerical solutions and he also derived the asymptotic solutions of types $I$ and $II$ for the large suction case. Zarturska et al \cite{31} and Cox and King \cite{6} also analyzed multiple solutions for the flow in a channel with porous walls. Lu et al \cite{12} investigated the asymptotic behaviour of the solutions. Lu and Macgillivray \cite{9, 11, 10, 13} mainly obtained the type $III$ solution. Brady and Acrivos \cite{2} presented three solutions for the flow in a channel or tube with an accelerating surface velocity.

As mentioned above, the evidence for multiple solutions was either numerical or asymptotic. Shih \cite{20} proved theoretically, applying a fixed point theorem, that there exists only one solution for injection case for the flow in a channel with porous walls. Topological and shooting methods were used by Hastings et al \cite{8} to prove the existence of all three of Robinson’s conjectured solutions. He also presented the asymptotic behaviour for the flow as $|R| \rightarrow \infty$. Terrill \cite{23} proposed a transformation to convert the two-point boundary value problem into an initial value problem to facilitate the numerical calculation of solutions for an arbitrary Reynolds number. Based on the transformation proposed by Terrill, Skalak and Wang \cite{22} described analytically the number and character of the solutions for each given Reynolds number under fairly general assumptions for the symmetrical channel and tube flow. The similar method was used by Cox \cite{4} to analyze the symmetric solutions when the two walls are accelerating equally and when one wall is accelerating and the other is stationary. The uniqueness of similarity solution was investigated theoretically by Chennellam and Liu \cite{3} and their work mainly considered the symmetric flow in a channel with slip boundary conditions.

All studies mentioned above are for symmetrical flows. The class of asymmetrical flows which may be driven by imposing different velocities on the walls turn out to be very interesting. The study of asymmetric laminar steady flow may be traced back to Proudman \cite{15} who obtained the asymptotic solutions in the core area. Then, Terrill and Shrestha \cite{27, 26, 21} extended Proudman’s work and constructed one asymptotic solution using the method of matched asymptotic expansion for the large injection, large suction and mixed cases, respectively. Here the mixed cases means that one wall is with injection while the other is with suction. Cox \cite{5} considered the practical case of an impermeable wall opposing a transpiring wall. Watson et al \cite{29} also investigated the case of asymmetrical flow in a channel which one wall is stationary and the other is accelerating.

The purpose of this paper is not to reconsider any of these previously considered problems, but instead to provide a thorough analysis for the asymmetric flow in a channel of porous walls with different permeabilities, where the upper wall is with injection and the lower wall is with suction. We will show that there exist three multiple solutions in this asymmetric case. We also mark them as type $I$, type $II$ and type $III$ solutions as people do for the symmetric case. We should remark here that type $I$, type $II$ and type $III$ solutions for the asymmetric case are much different from those for the symmetric case. We will numerically give the range of the Reynolds number where there exist three solutions. We will then construct asymptotic solutions for each solution for the most difficult case of high Reynolds number and numerically validate the constructed solutions. The paper is organized as follows. In section 2, a similarity transformation is introduced and the Navier-Stokes equation is reduced to a single fourth order nonlinear ordinary differential
equation with a Reynolds number $R$ and four boundary conditions. In section 3, we theoretically analyze that there exist three solutions of similarity transformed equation under fairly general assumptions. In section 4, we compute the multiple solutions numerically. We also sketches velocity profiles and streamlines for these asymmetric flows. In section 5, for the most difficult high Reynolds number case, the asymptotic solution for each type of multiple solutions will be constructed using the method of boundary layer correction or matched asymptotic expansion. In section 6, all the asymptotic solutions are verified by numeral solutions and meanwhile these asymptotic solutions may serve as a validation for the numerical method used in the paper.

2. Mathematical formulation. We consider the two-dimensional, viscous, incompressible asymmetric laminar flows in a porous and elongated rectangular channel. The channel exhibits a sufficiently small depth-width ratio of semi-height $h$ to length $L$. Despite the channel’s finite body length, it is reasonable to assume a semi-infinite length in order to neglect the influence of the opening at the end [28]. The flow is driven by uniform injection through the upper wall of the channel with speed $-v_2$ and suction through the lower wall with speed $-v_1$, where $v_2 > v_1 > 0$. We define an asymmetric parameter $a = \frac{v_1}{v_2}$, where $0 < a < 1$. With $\hat{x}$ representing the streamwise direction and $\hat{y}$ the transverse direction, the corresponding streamwise and transverse velocity components are defined as $u$ and $v$, respectively. The streamwise velocity is zero at the closed headend ($\hat{x} = 0$).

The equations of the continuity and momentum for the steady laminar flows of an incompressible viscous fluid through a porous channel are [26]

\begin{align}
\nabla \cdot \mathbf{V} &= 0, \\
(\mathbf{V} \cdot \nabla) \mathbf{V} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V},
\end{align}

where the symbol $\mathbf{V} = (u, v)$ represents the velocity vector, $p$ the pressure, $\rho$ the density and $\nu$ the viscosity of the fluid. The boundary conditions necessary for describing the asymmetric flow and solving the continuity and momentum equations are

\begin{align}
&u(\hat{x}, -h) = 0, \quad v(\hat{x}, -h) = -v_1, \quad u(\hat{x}, h) = 0, \quad v(\hat{x}, h) = -v_2. 
\end{align}

As we know, the study of the fluid flow equations with a high Reynolds number is most challenging. A similarity transformation provides a way to explicitly express the solutions for the channel flow. Such explicit solutions are often preferred in understanding the fluid properties especially the boundary layers. For this purpose we introduce a streamfunction and express the velocity components in terms of the streamfunction [14]:

\[ \phi = \frac{\nu \hat{x}}{h} F(y), \]

where $y = \frac{\hat{y}}{h}$ which is the non-dimensional transverse coordinate and $F(y)$ is independent to the streamwise coordinate. Then the velocity components are given by

\begin{align}
&u = \frac{\partial \phi}{\partial \hat{y}} = \frac{\nu \hat{x}}{h^2} F'(y), \quad v = -\frac{\partial \phi}{\partial \hat{x}} = -\frac{\nu}{h} F(y),
\end{align}

so that the continuity equation (2.1) is satisfied. Substituting (2.4) into equations (2.1) and (2.2) and eliminating the pressure term, a third order nonlinear ordinary
differential equation with a parameter and an integration constant is developed. It is
\[ f''' + R(f'f'' - f'^2) = K, \]
where \( K \) is an integration constant, \( R = \frac{h_2}{b} \) is the Reynolds number based on the fluid velocity \( v_2 \) through the upper wall and the semi-height \( h \) of channel and \( f = \frac{\nu}{R} \).

The boundary conditions given by (2.3) can now be updated to the normalized form
\[ f(-1) = a, \quad f'(-1) = 0, \quad f(1) = 1, \quad f'(1) = 0. \]

3. Existence of multiple solutions. Skalak [22], Cox [4] and Chenllam [3] considered symmetric flow in a channel with porous walls, accelerating walls and slip boundary conditions, respectively. In this section, we extend previous analysis [22, 4, 3] to investigate asymmetric flow in the channel with porous walls and to discuss the existence of multiple solutions.

The two-point boundary value problem, (2.5) and (2.6), can be converted into an initial value problem. Rescalling (2.5) and (2.6) by introducing \( f(y) = \frac{1}{2} bg(\xi)/R \) and \( \xi = \frac{1}{2} b(y + 1) [23] [22]:
\[ g''' + gg'' - g'^2 = k, \]
\[ g(0) = 2aR/b, \quad g'(0) = 0, \quad g''(0) = A, \quad g'''(0) = B, \]
where \( k = \frac{16RK}{b}, \ a > 0, \ b > 0 \) and \( R > 0 \). Assume all the initial conditions of (3.1) can be
\[ g(0) = 2aR/b, \quad g'(0) = 0, \quad g''(0) = A, \quad g'''(0) = B, \]
where \( A, B \neq 0 \). If we find a point \( \xi^* \) such that \( g'(<\xi) = 0 \), we obtain \( b \) by setting \( \xi^* = b \) and the value of \( g(b) \) at this point gives the Reynolds number \( R = \frac{1}{2} bg(b) \).

Then, we can obtain the original solution \( f(y) \). Since our analysis is restricted to positive \( R \), there must be \( g(b) > 0 \) (i.e. \( g(\xi^*) > 0 \)). Thus, we will discuss analytically the number of possible roots \( \xi^* \) of \( g'(\xi) \) (each with \( g(\xi^*) > 0 \)) covering the entire range of \( \xi > 0 \).

Let \( h_1(\xi) = g'(\xi), \ h_2(\xi) = g''(\xi), \ h_3(\xi) = h'''(\xi), \ h_4(\xi) = g^{(4)}(\xi) \) and \( h_5(\xi) = g^{(5)}(\xi) \) for all \( \xi \geq 0 \). Differenting equation (3.1) twice gives
\[ h_4 = h_1h_2 + h_3, \]
\[ h_5 + gh_4 = (h_2)^2. \]

Lemma 3.1. If \( B < 0 \), then \( g^{(4)}(\xi) > 0 \) for all \( \xi \geq 0 \). In particular, \( g'''(\xi) \) is a strictly increasing function of \( \xi \) on \([0, +\infty)\).

Proof. By equation (3.4), since \( g'(0) = 0 \), then \( h_4(0) = g'(0)g''(0) - g(0)h_3(0) = -g(0)B \). Since \( g(0) > 0 \) and \( B < 0 \), then \( h_4(0) > 0 \). By equation (3.5), then \( h_4 + gh_4 \geq 0 \) for all \( \xi \geq 0 \), that is, \( e^{\int_0^\xi g(t)dt}h_4(\xi) \geq 0 \) for all \( \xi \geq 0 \), which implies that \( e^{\int_0^\xi g(t)dt}h_4(\xi) \geq h_4(0) > 0 \) for all \( \xi \geq 0 \). Hence \( h_4(\xi) = g^{(4)}(\xi) > 0 \) for all \( \xi \geq 0. \]

Remark 3.2. The proof of Lemma 3.1 only uses the condition \( g(0)B < 0 \).

Proposition 3.3. Assume that \( A > 0 \) and \( B < 0 \).
(a) If there exists some \( x_0 > 0 \) such that \( g''(x_0) = 0 \), then there is no point \( \zeta > x_0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0 \).

(b) If \( g''(\xi) < 0 \) for all \( \xi \geq 0 \) and \( h_2(\alpha) = 0 \) for some \( \alpha > 0 \), then there exists some \( \zeta > \alpha \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0 \).

Proof. (a) By Lemma 3.1, then \( h_3(\xi) > 0 \) for all \( \xi \geq 0 \). Since \( h_3(x_0) = 0 \), then \( h_3(\xi) < 0 \) for all \( \xi \in [0, x_0) \) and \( h_3(\xi) > 0 \) for all \( \xi > x_0 \), which implies that \( h_2(\xi) \) is strictly decreasing on \([0, x_0)\) and strictly increasing on \((x_0, +\infty)\). Hence \( x_0 \) is the unique global minimum point of \( h_2(\xi) \) on \([0, +\infty)\). Since \( h_3(x_0) = 0 \), by equation (3.4) and Lemma 3.1, then \( h_2(x_0) = h_1(x_0)h_2(x_0) - g(x_0)h_3(x_0) = h_1(x_0)h_2(x_0) > 0 \).

Let’s assume that there is some \( \zeta > x_0 \) such that \( h_1(\zeta) = g'(\zeta) = 0 \) and \( g(\zeta) > 0 \). Since \( g'(0) = g'(\zeta) = 0 \), by Lagrange’s mean value theorem, then there exists some \( \alpha \in (0, \zeta) \) such that \( h_2(\alpha) = 0 \). Since \( x_0 \) is the unique global minimum point of \( h_2(\xi) \) on \([0, +\infty)\) and \( h_2(x_0) \neq 0 \), then \( h_2(x_0) < 0 \). Since \( h_2(\xi) > 0 \) for all \( \xi \geq 0 \) and \( h_3(\xi) > 0 \) for all \( \xi > x_0 \), then \( \lim_{\xi \to +\infty} h_2(\xi) = +\infty \). Since \( h_2(0) = A > 0 \), \( h_2(x_0) < 0 \), and \( h_4(\xi) > 0 \) for all \( \xi \geq 0 \), then there exists a unique \( \beta > x_0 \) such that \( h_2(\xi) < 0 \) for all \( \alpha < \xi < \beta \), and \( h_2(\xi) > 0 \) for all \( 0 \leq \xi < \alpha \) and all \( \xi > \beta \). Since \( h_1(x_0)h_3(x_0) > 0 \) and \( h_2(x_0) < 0 \), then \( h_1(x_0) < 0 \). Since \( h_1(\xi) = 0 \), by equation (3.4), then \( h_4(\xi) = h_1(\xi)h_2(\xi) - g(\xi)h_3(\xi) = -g(\xi)h_3(\xi) \). Since \( h_2(\xi) > 0 \) for all \( \xi > 0 \) and \( g(\xi) > 0 \), then \( h_3(\xi) < 0 \), which implies that \( \zeta < x_0 \), this leads to a contradiction.

(b) Since \( g''(\xi) < 0 \) for all \( \xi \geq 0 \) and \( h_2(\alpha) = 0 \) for some \( \alpha > 0 \), then \( h_2(\xi) > 0 \) for all \( 0 \leq \xi < \alpha \) and \( h_2(\xi) < 0 \) for all \( \xi > \alpha \), which implies that \( h_1(\xi) \) is strictly increasing on \([0, \alpha)\) and strictly decreasing on \((\alpha, +\infty)\). Hence \( \alpha \) is the unique global maximum point of \( h_1(\xi) \) on \([0, +\infty)\). Since \( h_1(0) = 0 \), then \( h_1(\alpha) > 0 \). Since \( h_2(\xi) < 0 \) for all \( \xi > \alpha \) and \( h_3(\xi) < 0 \) for all \( \xi > 0 \), then \( \lim_{\xi \to +\infty} h_1(\xi) = -\infty \). Since \( h_1(\alpha) > 0 \) and \( h_2(\xi) < 0 \) for all \( \xi > \alpha \), then there exists a unique \( \zeta > \alpha \) such that \( g'(\zeta) = h_1(\zeta) = 0 \). Since \( h_1(\xi) = 0 \), by equation (3.4), then \( h_4(\xi) = h_1(\xi)h_2(\xi) - g(\xi)h_3(\xi) = -g(\xi)h_3(\xi) \). Since \( \zeta > \alpha \), \( h_3(\xi) < 0 \) for all \( \xi \geq 0 \) and Lemma 3.1, then \( g(\zeta) > 0 \). Therefore, in this case, there exists a solution of (2.5) and (2.6). We will designate this solution as type I solution.

Proposition 3.4. Assume that \( A < 0 \) and \( B < 0 \).

(a) Then there exists some \( x_0 > 0 \) such that \( g''(x_0) = 0 \).

(b) Then there is no point \( \zeta > 0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0 \).

Proof. (a) If the statement is not right, since \( B = g'''(0) < 0 \), then \( h_3(\xi) < 0 \) for all \( \xi \geq 0 \). Since \( A = g''(0) < 0 \), then \( h_2(\xi) \leq A \) for all \( \xi \geq 0 \), which implies that \( h_1(\xi) - h_1(0) \leq A\xi \) for all \( \xi \geq 0 \). Since \( h_1(0) = 0 \), then \( h_1(\xi) < 0 \) and \( h_3(\xi) \leq A\xi \) for all \( \xi \geq 0 \), which implies that \( g(\xi) - g(0) \leq \frac{A}{2}\xi^2 \) for all \( \xi \geq 0 \). Hence \( \lim_{\xi \to +\infty} g(\xi) = -\infty \). By equation (3.5), then \( h_5 = (h_2)^2 - gh_4 \) for all \( \xi \geq 0 \). By Lemma 3.1, then \( h_4(\xi) > 0 \) for all \( \xi \geq 0 \). Since \( \lim_{\xi \to +\infty} g(\xi) = -\infty \), then there exists some \( R > 0 \) such that \( h_5(\xi) > 0 \) for all \( \xi \geq R \). Since \( h_4(\xi) > 0 \) and \( h_5(\xi) > 0 \) for all \( \xi \geq R \), then \( \lim_{\xi \to +\infty} h_3(\xi) = +\infty \), this leads to a contradiction.

(b) By the result of part (a) then there exists some \( x_0 > 0 \) such that \( g'''(x_0) = 0 \). Since \( h_4(\xi) \geq 0 \) for all \( \xi \geq 0 \), then \( h_3(\xi) < 0 \) for all \( \xi \in [0, x_0) \) and \( h_3(\xi) > 0 \) for all \( \xi > x_0 \), which implies that \( h_2(\xi) \) is strictly decreasing on \([0, x_0)\) and strictly increasing on \((x_0, +\infty)\). Hence \( x_0 \) is the unique global minimum point of \( h_2(\xi) \) on \([0, +\infty)\). Since \( g''(0) = A < 0 \) and \( h_3(\xi) < 0 \) for all \( \xi \in [0, x_0) \), then \( h_2(\xi) < A < 0 \) for all \( \xi \in (0, x_0) \). Since \( h_1(0) = 0 \), then \( h_1(\xi) \leq A\xi < 0 \) for all \( \xi \in (0, x_0) \). Since \( h_3(\xi) > 0 \) for all
\[ \xi > x_0 \text{ and } h_4(\xi) > 0 \text{ for all } \xi \geq 0, \text{ then } \lim_{\xi \to +\infty} h_2(\xi) = +\infty. \text{ Since } h_2(x_0) < 0 \text{ and } h_3(\xi) > 0 \text{ for all } \xi > x_0, \text{ then there exists a unique } x_1 > x_0 \text{ such that } h_2(\xi) < 0 \text{ for all } \xi \in (x_0, x_1), \text{ and } h_2(\xi) > 0 \text{ for all } \xi > x_1. \] Since \( h_2(\xi) < A < 0 \) for all \( \xi \in (0, x_1) \), then \( h_2(\xi) < 0 \) for all \( 0 \leq \xi < x_1 \) and \( h_2(\xi) > 0 \) for all \( \xi > x_1 \), which implies that \( h_1(\xi) \) is strictly decreasing on \([0, x_1]\) and strictly increasing on \([x_1, +\infty)\). Hence \( x_1 \) is the unique global minimum point of \( h_3(\xi) \) on \([0, +\infty)\). Since \( h_1(\xi) \leq A\xi < 0 \) for all \( \xi \in (0, x_0]\), then \( h_1(x_1) < 0 \). Since \( h_2(\xi) > 0 \) for all \( \xi > x_1 \) and \( h_3(\xi) > 0 \) for all \( \xi > x_0 \), then \( \lim_{\xi \to +\infty} h_1(\xi) = +\infty. \) Since \( h_1(\xi) < 0 \) for all \( 0 < \xi \leq x_1 \) and \( h_2(\xi) > 0 \) for all \( \xi > x_1 \), then there exists a unique \( x_2 > x_1 \) such that \( h_1(x_2) = 0 \). We know that \( x_2 \) is the unique solution of \( h_1(\xi) = 0 \) for all \( \xi > 0 \). On the other hand, since \( x_2 > x_1 > x_0 \), then \( h_4(x_2) > 0 \) and \( h_3(x_2) > 0 \). Since \( h_1(x_2) = 0 \), by equation (3.4), then \( h_4(x_2) = h_1(x_2)h_2(x_2) - g(x_2)h_3(x_2) = -g(x_2)h_3(x_2) > 0 \). Since \( h_3(x_2) > 0 \), then \( g(x_2) < 0 \). Since \( x_2 \) is the unique solution of \( h_1(\xi) = 0 \) for all \( \xi > 0 \), and \( g(x_2) < 0 \), then there is no point \( \xi > 0 \) such that \( g'(\xi) = 0 \) and \( g(\xi) > 0 \).

**Proposition 3.5.** Assume that \( B > 0 \).

(a) If \( h_4(\xi) \neq 0 \) for all \( \xi > 0 \), then \( h_4(\xi) < 0 \) for all \( \xi \leq 0 \).

(b) If \( h_4(\xi_0) = 0 \) for some \( \xi_0 > 0 \), then \( h_4(\xi) < 0 \) for all \( 0 \leq \xi < \xi_0 \) and \( h_4(\xi) > 0 \) for all \( \xi > \xi_0 \).

**Proof.** (a) By equation (3.4), since \( g'(0) = 0 \), then \( h_4(0) = h_1(0)h_2(0) - g(0)h_3(0) = -g(0)B \). Since \( g(0) > 0 \) and \( B > 0 \), then \( h_4(0) < 0 \). By the assumption that \( h_4(\xi) \neq 0 \) for all \( \xi > 0 \), then \( h_4(\xi) < 0 \) for all \( \xi \leq 0 \). (b) By the proof of part (a), we know that \( h_4(0) < 0 \). Let \( \Sigma = \{ \xi > 0 : h_4(\xi) = 0 \} \), by the assumption, then \( \xi_0 \in \Sigma \). Since \( h_4(0) < 0 \), by the continuity, we know that \( \eta := \inf_{\xi \in \Sigma} \xi \in (0, \xi_0] \), which implies that \( \xi_0 \geq \eta \), \( h_4(\eta) = 0 \) and \( h_4(\xi) < 0 \) for all \( 0 \leq \xi < \eta \). By equation (3.5), then \( h_4' + gh_4 \geq 0 \) for all \( \xi \geq 0 \), that is, \( \left( e^{\int_0^\xi g(t)dt}h_4(\xi) \right)' \geq 0 \) for all \( \xi \geq 0 \), which implies that \( e^{\int_0^\xi g(t)dt}h_4(\xi) \geq h_4(\eta) = 0 \) for all \( \xi \geq \eta \). Hence \( h_4(\xi) = g^{(4)}(\xi) \geq 0 \) for all \( \xi \geq \eta \).

**Claim 3.6.** \( h_4(\xi) > 0 \) for all \( \xi > \eta \).

**Proof.** If not, since \( h_4(\xi) \geq 0 \) for all \( \xi \geq \eta \), then there exists some \( \alpha > \eta \) such that \( h_4(\alpha) = 0 \). If \( h_4(\xi) \equiv 0 \) for all \( \eta \leq \xi \leq \alpha \), by equation (3.5), then \( h_2(\xi) \equiv 0 \) for all \( \eta \leq \xi \leq \alpha \). Rewrite equation (3.4) as \( h''_2 + gh' - h_1h_2 = 0 \) for all \( \xi \geq 0 \), since \( h_2(\eta) = h_2(\xi) = 0 \), by the uniqueness theorem of solutions to the ODE, then \( h_2(\xi) = 0 \) for all \( \xi \geq 0 \), which implies that \( B = h_2(0) = h_2(0) = 0 \), this leads to a contradiction. Since \( h_4(\xi) \geq 0 \) for all \( \xi \geq \eta \), then there exists some \( \beta \in (\eta, \alpha) \) such that \( h_4(\beta) > 0 \). Since \( \left( e^{\int_0^\xi g(t)dt}h_4(\xi) \right)' \geq 0 \) for all \( \xi \geq 0 \) and \( h_4(\beta) > 0 \), then \( h_4(\xi) > 0 \) for all \( \xi > \beta \). Since \( \alpha > \beta \), then \( h_4(\beta) < h_4(\alpha) = 0 \), this leads to a contradiction. Therefore, we know that \( h_4(\xi) > 0 \) for all \( \xi > \eta \).

Since \( h_4(\xi_0) = h_4(\eta) = 0, \xi_0 \geq \eta \), by Claim 3.6, then \( \xi_0 = \eta \). By the definition of \( \eta \) and Claim 3.6, then \( h_4(\xi) < 0 \) for all \( 0 \leq \xi < \xi_0 \) and \( h_4(\xi) > 0 \) for all \( \xi > \xi_0 \).

**Proposition 3.7.** Assume that \( A > 0 \) and \( B > 0 \), if \( g^{(4)}(\xi) < 0 \) for all \( \xi \geq 0 \), then \( g''(\xi) > 0 \) for all \( \xi \geq 0 \). In particular, since \( g''(0) = A > 0 \), then \( g''(\xi) > A > 0 \) for all \( \xi > 0 \). Moreover, since \( g'(0) = 0 \), then \( g'(\xi) > 0 \) for all \( \xi > 0 \).

**Proof.** If not, since \( h_3(0) = B > 0 \) and \( h_4(\xi) < 0 \) for all \( \xi \geq 0 \), then there exists a unique \( x_0 > 0 \) such that \( h_3(\xi) > 0 \) for all \( 0 \leq \xi < x_0 \), \( h_3(x_0) = 0 \), and \( h_3(\xi) < 0 \) for all \( \xi > x_0 \). Since \( h_2(0) = A > 0 \) and \( h_3(\xi) > 0 \) for all \( 0 \leq \xi < x_0 \), then \( h_2(\xi) > 0 \).
for all $0 \leq \xi \leq x_0$. Since $h_1(0) = 0$, then $h_1(\xi) > 0$ for all $0 \leq \xi \leq x_0$, which implies that $h_1(x_0)h_2(x_0) > 0$. Since $h_3(x_0) = 0$ and $h_4(\xi) < 0$ for all $\xi \geq 0$, by equation (3.4), we know that $h_1(x_0)h_2(x_0) = h_4(x_0) - g(x_0)h_3(x_0) = h_4(x_0) < 0$, this leads to a contradiction.

**Proposition 3.8.** Assume that $A > 0$ and $B > 0$, and there exists some $\xi_0 > 0$ such that $g^{(4)}(\xi) < 0$ for all $\xi \in (0, \xi_0)$ and $g^{(4)}(\xi) > 0$ for all $\xi > \xi_0$. Then $\xi_0$ is the unique global minimum point of $g''''(\xi)$ on $[0, \infty)$ and $g''''(\xi_0) > 0$. In particular, since $g''''(0) = A > 0$, then $g''''(\xi) > A > 0$ for all $\xi > 0$. Moreover, since $g'(0) = 0$, then $g''''(\xi) > 0$ for all $\xi > 0$.

**Proof.** Since $g^{(4)}(\xi) < 0$ for all $\xi \in (0, \xi_0)$ and $g^{(4)}(\xi) > 0$ for all $\xi > \xi_0$, then $g''''(\xi)$ is strictly decreasing on $[0, \xi_0]$ and strictly increasing on $[\xi_0, +\infty)$, which implies that $\xi_0$ is the unique global minimum point of $g''''(\xi)$ on $[0, \infty)$. Now let’s decide the sign of $h_3(\xi_0) = g''''(\xi_0)$. If $h_3(\xi_0) \leq 0$, since $h_3(0) = B > 0$ and $g^{(3)}(\xi) < 0$ for all $\xi \in (0, \xi_0)$, then there exists a unique $\eta \in (0, \xi_0]$ such that $h_3(\xi) > 0$ for all $0 \leq \xi < \eta$, $h_3(\eta) = 0$, and $h_3(\xi) < 0$ for all $\xi > \eta$. Since $h_2(0) = A > 0$, then $h_2(\xi) > A > 0$ for all $0 \leq \xi < \eta$. Since $h_1(0) = 0$, then $h_1(\xi) > 0$ for all $0 \leq \xi < \eta$, which implies that $h_1(\eta)h_2(\eta) > 0$. Since $h_2(\eta) = 0$, $h_4(\xi) < 0$ for all $0 \leq \xi < \et$ and $0 \leq \eta \leq \xi_0$, by equation (3.4), we know that $h_1(\eta)h_2(\eta) = h_4(\eta) - g(\eta)h_3(\eta) = h_4(\eta) < 0$, this leads to a contradiction.

**Proposition 3.9.** Assume that $A < 0$ and $B > 0$, and $g^{(4)}(\xi) < 0$ for all $\xi \geq 0$.

(a) If $g''''(x_0) = 0$ for some $x_0 > 0$, then there is no point $\zeta > 0$ such that $g'(\zeta) = 0$ and $g(\zeta) > 0$. In particular, there is no point $\zeta > x_0$ such that $g''''(\zeta) = 0$ and $g(\zeta) > 0$.

(b) If $g''''(\xi) > 0$ for all $\xi \geq 0$, then there is no point $\zeta > 0$ such that $g''''(\zeta) = 0$ and $g(\zeta) > 0$.

**Proof.** (a) If there exists some $\zeta > 0$ such that $g'(\zeta) = 0$ and $g(\zeta) > 0$, since $h_4(\zeta) < 0$ for all $\zeta \geq 0$, by equation (3.4), then $h_4(\zeta) = h_1(\zeta)h_2(\zeta) - g(\zeta)h_3(\zeta) = -g(\zeta)h_3(\zeta) < 0$. Since $g(\zeta) > 0$, then $h_3(\zeta) > 0$. Since $h_4(\xi) < 0$ for all $\xi \geq 0$ and $h_3(x_0) = 0$, then $0 < \xi < x_0$. Since $g(0) = g'(\xi) = 0$, by Lagrange’s mean value theorem, then $h_2(\eta) = 0$ for some $\eta \in (0, \zeta)$. Since $h_4(\xi) < 0$ for all $\xi \geq 0$ and $h_3(x_0) = 0$, then $h_3(\xi) > 0$ for all $\xi \in [0, x_0)$ and $h_3(\xi) < 0$ for all $\xi > x_0$, which implies that $h_2(\xi)$ is strictly decreasing on $[0, x_0]$ and strictly decreasing on $[x_0, +\infty)$. Hence $x_0$ is the unique global maximum point of $h_2(\xi)$ on $[0, +\infty)$. Since $h_2(\eta) = 0$, then $h_2(x_0) > 0$. Since $h_4(\xi) < 0$ for all $\xi \geq 0$ and $h_3(x_0) = 0$, by equation (3.4), then $h_4(x_0) = h_1(x_0)h_2(x_0) - g(x_0)h_3(x_0) = h_1(x_0)h_2(x_0) < 0$. Since $h_2(x_0) > 0$, then $h_1(x_0) < 0$. Since $h_3(\xi) > 0$ for all $\xi \in [0, x_0)$, then $h_2(\xi) < 0$ for all $\xi \in [0, \eta]$ and $h_2(\xi) > 0$ for all $\xi \in (\eta, x_0)$, which implies that $h_1(\xi)$ is strictly decreasing on $[0, \eta]$ and strictly increasing on $(\eta, x_0)$. Since $h_1(0) = 0$, then $h_1(\xi) < 0$ for all $\xi \in (0, x_0)$. Since $\xi \in (0, x_0)$, then $g'(\xi) = h_1(\xi) < 0$, this leads to a contradiction.

(b) If there exists some $\zeta > 0$ such that $g'(\zeta) = 0$ and $g(\zeta) > 0$, since $g'(0) = 0$, by Lagrange’s mean value theorem, then $h_2(\eta) = 0$ for some $\eta \in (0, \zeta)$. Since $h_3(\xi) > 0$ for all $\xi \geq 0$, then $h_2(\xi) < 0$ for all $\xi \in [0, \eta]$ and $h_2(\xi) > 0$ for all $\xi > \eta$, which implies that $h_1(\xi)$ is strictly decreasing on $[0, \eta]$ and strictly increasing on $[\eta, +\infty)$. Since $\eta < \zeta$ and $h_1(0) = h_1(\zeta) = 0$, then $h_1(\xi) < 0$ for all $0 < \xi < \zeta$ and $h_1(\xi) > 0$ for all $\xi > \zeta$, which implies that $\zeta$ is the unique global minimum point of $g(\xi)$ on $[0, +\infty)$. Since $\Delta := g(\zeta) > 0$, then $g(\xi) \geq \Delta > 0$ for all $\xi \geq 0$. Since $h_4(\xi) < 0$ for all $\xi \geq 0$, by equation (3.5), then $h_3(\xi) = (h_2(\xi))^2 - g(\xi)h_4(\xi) > 0$ for all $\xi \geq 0$. Since $h_3(\xi) > 0$ and $h_4(\xi) < 0$ for all $\xi \geq 0$, then there exists some $L \geq 0$ such that
\[
\lim_{\xi \to +\infty} h_3(\xi) = L \text{ and } h_3(\xi) > L \geq 0 \text{ for all } \xi \geq 0. \]

By Lagrange’s mean value theorem, there exists some sequence \( \{y_n\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} y_n = +\infty \) and \( \lim_{n \to \infty} h_4(y_n) = 0. \) Since \( h_3(\xi) > 0 \) for all \( \xi \geq 0, \) then \( h_4(\xi) \) is strictly increasing on \([0, +\infty), \) which implies that \( \lim_{\xi \to +\infty} h_4(\xi) = 0. \) By Lagrange’s mean value theorem, there exists some sequence \( \{x_n\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} x_n = +\infty \) and \( h_3(x_n) = 0. \) Since \( g(\xi) > 0 \) and \( h_4(\xi) < 0 \) for all \( \xi \geq 0, \) then \( \limsup_{n \to \infty} [h_3(x_n) + g(x_n) h_4(x_n)] = 0. \) On the other hand, since \( h_2(\xi) > 0 \) for all \( \xi > \eta \) and \( h_3(\xi) > 0 \) for all \( \xi \geq 0, \) then \( \limsup_{n \to \infty} (h_2(x_n))^2 > 0, \) which contradicts with equation (3.5).

\[\square\]

**Proposition 3.10.** Assume that \( A < 0 \) and \( B > 0, \) and there exists some \( \xi_0 > 0 \) such that \( g^{(4)}(\xi) < 0 \) for all \( \xi \in (0, \xi_0) \) and \( g^{(4)}(\xi) > 0 \) for all \( \xi > \xi_0. \)

(a) If \( g'''(x_0) \geq 0 \) for some \( x_0 \geq \xi_0, \) then there is no point \( \zeta > x_0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0. \) In particular, if \( g'''(\xi_0) \geq 0, \) then there is no point \( \zeta > \xi_0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0. \)

(b) If \( g'''(\xi) < 0 \) for all \( \xi \in [\xi_0, \infty) \) and \( g'''(\xi) \) has only one zero on \([0, \infty), \) then there is no point \( \zeta > \xi_0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0. \)

(c) If \( g'''(\xi) < 0 \) for all \( \xi \in [\xi_0, \infty) \) and \( g'(\alpha) = 0 \) and \( g'(\gamma) > 0 \) for some point \( \alpha > \gamma, \) then \( g'(\xi) \) has a unique \( \zeta > \alpha \) such that \( g'(\zeta) = 0 \) and \( g(\xi) > 0. \)

In particular, there exists a unique \( \gamma \in (0, \xi_0) \) such that \( g'(\gamma) = 0. \) If \( \gamma < \xi_0, \) then \( g(\gamma) < 0. \) If \( \gamma > \xi_0, \) then \( g(\gamma) > 0. \)

**Proof.** (a) Assume that there is some point \( \zeta > x_0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0, \) since \( h_4(\xi) = g^{(4)}(\xi) > 0 \) for all \( \xi \geq \xi_0, \) and \( g'''(x_0) \geq 0, \) then \( h_3(\xi) > 0. \)

Since \( h_4(\xi) > 0 \) for all \( \xi > \xi_0, \) by equation (3.4), then \( h_4(\zeta) = h_1(\zeta) h_2(\zeta) - g(\zeta) h_3(\zeta) = -g(\zeta) h(\zeta) > 0. \) Since \( g(\zeta) > 0, \) then \( h_3(\zeta) < 0, \) this leads to a contradiction.

(b) Assume that there is some point \( \zeta > \xi_0 \) such that \( g'(\zeta) = 0 \) and \( g(\zeta) > 0, \) since \( g'(0) = 0, \) by Lagrange’s mean value theorem, then there exists some \( \alpha \in (0, \zeta) \) such that \( h_2(\alpha) = g'(\alpha) = 0. \) Since \( g''(0) = A < 0 \) and \( g''(\xi) \) has only one zero on \([0, \infty), \) then \( h_2(\xi) \) is strictly increasing on all \( \frac{\xi}{\xi_0} = \alpha, \) which implies that \( \alpha \) is the unique global maximum point of \( g''(\xi). \) Hence \( h_3(0) = 0 \) and \( h_4(\alpha) \leq 0. \) Since \( g^{(4)}(\xi) < 0 \) for all \( \xi \in (0, \xi_0) \) and \( g^{(4)}(\xi) > 0 \) for all \( \xi > \xi_0, \) then \( 0 < \alpha \leq \xi_0. \) Since \( h_3(\alpha) = 0 \) and \( h_3(\xi) = g'''(\xi) < 0 \) for all \( \xi > \xi_0, \) then \( 0 < \alpha < \xi_0, \) which implies that \( h_4(\alpha) < 0. \) On the other hand, since \( h_2(\alpha) = h_3(\alpha) = 0, \) by equation (3.4), then \( h_4(\alpha) = h_1(\alpha) h_2(\alpha) - g(\alpha) h_3(\alpha) = 0, \) which leads to a contradiction.

(c) Since \( g^{(4)}(\xi) < 0 \) for all \( \xi \in (0, \xi_0) \) and \( g^{(4)}(\xi) > 0 \) for all \( \xi \geq \xi_0, \) then \( h_3(\xi) \) strictly decreasing on \([0, \xi_0], \) and strictly increasing on \([\xi_0, +\infty), \) which implies that \( \xi_0 \) is the unique global minimum point of \( h_3(\xi) \) on \([0, +\infty), \) since \( h_3(\xi_0) = 0, \) \( h_3(0) = B > 0 \) and \( h_4(\xi) < 0 \) for all \( 0 \leq \xi \leq \xi_0, \) then there exists a unique \( x_1 \in (0, \xi_0) \) such that \( h_3(\xi) > 0 \) for all \( 0 \leq \xi < x_1 \) and \( h_3(\xi) < 0 \) for all \( x_1 < \xi < \xi_0. \) Since \( h_3(\xi) < 0 \) for all \( \xi \geq \xi_0, \) then \( h_3(\xi) > 0 \) for all \( 0 \leq \xi < x_1 \) and \( h_3(\xi) < 0 \) for all \( \xi > x_1, \) which implies that \( h_3(\xi) \) strictly increasing on \([0, x_1], \) and strictly decreasing on \([x_1, +\infty), \) then \( h_2(0) = A < 0 \) and \( h_2(\alpha) = 0, \) then \( h_2(\xi) \) has either 1 or 2 zeros on \([0, +\infty). \) If \( h_2(\xi) \) has only one zero on \([0, +\infty), \) since \( h_2(0) = A < 0 \) and \( h_2(\alpha) = 0, \) then \( \alpha \) is the only global maximum point of \( h_2(\xi) \) on \([0, +\infty). \) Since \( h_2(\xi) \) is strictly increasing on \([0, x_1], \) and strictly decreasing on \([x_1, +\infty), \) then \( \alpha = x_1, \) which implies that \( \alpha = x_1 < \xi_0, \) this contradicts \( \alpha > \xi_0. \) If \( h_2(\xi) \) has two zeros on \([0, +\infty), \) then there exists a unique \( \beta \in (0, x_1) \) such that \( h_2(\xi) > 0 \) for all \( \beta \leq \xi < \alpha \) and \( h_2(\xi) < 0 \) for all \( \xi \in (0, \beta) \cup (\alpha, +\infty). \) Since \( h_2(\xi) < 0 \) for all \( \xi > \alpha \) and \( h_3(\xi) < 0 \) for all \( \xi > x_1, \) then
As can be seen from Figure 2b, the flows form a thin boundary layer structure near the lower wall of the channel for the relatively high Reynolds number. The increasing Reynolds number has little influence on the flow character, but the
boundary layer is thinner and thinner with the increasing $R$.

Fig. 2: Velocity profiles of type I at $a = 0.8$ with $R = 40$, $R = 70$, $R = 100$ and $R = 130$.

Typical velocity profiles for type II solution are presented in Figure 3. All of these flows occur as $R > 14.10$. As $R$ is increased, the minimum of transverse velocity in the reverse region is decreasing and the turning points which are the points such that $f(y) = 0$ are moving towards the walls of the channel. The maximum of streamwise velocity is increasing and the minimum is decreasing with the increasing $R$.

Fig. 3: Velocity profiles of type II at $a = 0.8$ with $R = 40$, $R = 70$, $R = 100$ and $R = 130$.

The type III solutions, shown in Figure 4, have an unusual shape. The rapid decay occurs not only for the streamwise velocity but also for the transverse velocity near the lower wall. With the increasing $R$, the region between the lower wall and the minimum velocity become thinner. There is a region of reverse flow near the lower wall for the streamwise velocity.

All numerical results indicate that the solutions consist of inviscid solution and boundary layer solution which is confined to the viscous layer near the lower wall of the channel. It is obvious that the flow direction of streamwise velocity inside the boundary layers for type II and type III is opposite to the type I. The reversal flow occurs for both type II and type III.
In an effort to develop a better understanding of the flow character, we show in Figure 5 sketches of the streamlines to describe the flow behaviour corresponding to the different branches of solutions. These graphs depict all three types of solutions and enable us to deduce their fundamental characteristics.

![Streamline patterns](image)

Fig. 5: Streamline patterns of types I, II and III solutions from top to bottom at $a = 0.8$ with $R = 40$, $R = 70$, $R = 100$ and $R = 130$.

5. Asymptotic multiple solutions for high Reynolds number $R$. We have shown the existence of multiple solutions and from the numerical solutions we know that when $R$ is relatively large, there exists three solutions. Since the upper wall is
with injection while the lower wall is with suction which indicates that the flow may exhibit a boundary layer structure near the lower wall for high Reynolds number, it is of considerable theoretical interest to construct asymptotic solution for the three types solutions which can help us to develop a better understanding of the characteristics of boundary layer.

By treating $\epsilon = \frac{1}{R}$ as a small perturbation parameter, equation (2.5) can be written as

$$\epsilon f'' + (ff'' - f'^2) = k,$$

where $k = K/R$.

**5.1. Asymptotic solution of type I.** From the numerical solution of type I in Figure 2, we can see that the streamwise velocity rapidly decays near the lower wall ($y = -1$). Hence, by the method of boundary layer correction, $f(y)$ and $k$ can be expanded as follows

$$f(y) = f_0(y) + \epsilon(f_1(y) + h_1(\eta)) + \epsilon^2(f_2(y) + h_2(\eta)) + \cdots,$$

$$k = k_0 + \epsilon k_1 + \epsilon^2 k_2 + \cdots,$$

where $\eta = \frac{1+yu}{\epsilon}$ is a stretching transformation near $y = -1$ and $h_i(\eta), i = 1, 2 \cdots$ are boundary layer functions. By substituting (5.2) into (2.6) and collecting the equal power of $\epsilon$, the boundary conditions become

$$f_0|_{y=1} = 1, \quad f_0'|_{y=1} = 0, \quad f_0|_{y=-1} = a,$$

$$f_i'|_{y=1} = 0, \quad f_i'|_{y=-1} + h_i|_{\eta=0} = 0, \quad i = 1, 2 \cdots,$$

$$f_i|_{y=1} = 0, \quad f_i|_{y=-1} + h_i|_{\eta=0} = 0, \quad i = 1, 2 \cdots,$$

where $h_i$ denotes the derivative of $h_i$ with respect to $\eta$. We note here that $f_0(\eta)$ is the solution of the reduced problem

$$f_0f''_0 - f_0'^2 = k_0$$

satisfying boundary conditions (5.4).

The construction is similar to that of subsection 4.1 in [7], where additional factors such as a magnetic force and a boundary expansion rate are considered. So we omit the details here and only provide the asymptotic solution of (2.5) and (2.6) for type I solution

$$f(y) = \cos z + \epsilon \{(Q(z) + b)\sin z + \frac{\lambda}{2b}(z\sin z + \cos z) + \frac{\lambda}{2b} + \frac{b}{2}(z\tan^2 z - \tan z)$$

$$+ \frac{b}{2}(\ln(1 - \sin z) - \ln \cos z)(z\sin z + \cos z) + \frac{b}{a}\sin 2b \cdot e^{-\alpha n} + O(\epsilon^2),$$

where $\eta = \frac{1+yu}{\epsilon}, \quad z = by - \frac{\cos^{-1} a}{2}, \quad Q(z) = b \int_0^z \phi \sec \phi(1 - \sec^2 \phi) d\phi$ and

$$\lambda = \frac{1}{2a(b \sin(2b) + \cos(2b))}(2(b - ab - aQ(-2b)) \sin(2b) + ab \tan(2b)$$

$$- 2ab^2 \tan^2(2b) + ab(\cos(2b) + 2b \sin(2b))(\ln(1 + \sin(2b)) - \ln \cos(2b))).$$

Terrill [26] considered a similar case where the lower wall is with injection and the upper wall is with suction and constructed an asymptotic solution of the type I solution with the method of matched asymptotic expansion. of the construction of type I solution, the details are similar to those of Subsection 4.1 in [7].
5.2. Asymptotic solution of type II. Constructing an asymptotic expansion as \( R \to \infty \) for the solution of type II is a more complicated process than that presented in the previous subsection. From the numerical solution of type II in Figure 3, we know that \( f(y) \) vanishes at exactly two points \( y_1 \) in \((-1,0)\) and \( y_2 \) in \((0,1)\) (called turning points \([13]\)). The technique used in this section follows the symmetric flow case in \([9, 10, 13]\) where there exists only one turning point.

Define that the distance between \( y = -1 \) and \( y = y_1 \) is \( \Delta_1 \) and the distance between \( y = 1 \) and \( y = y_2 \) is \( \Delta_2 \), hence, it follows that \( y_1 = -1 + \Delta_1 \) and \( y_2 = 1 - \Delta_2 \) which are unknown a priori. By differentiating \((5.1)\), we obtain

\[
\epsilon f^{iv} + (ff^{iv} - ff''') = 0.
\]

1) Asymptotic solution between the turning points \( y_1 \) and \( y_2 \)
Letting \( \epsilon = 0 \), equation \((5.9)\) become

\[
ff''' - ff'' = 0.
\]

We observe three types of solutions for the equation: \( cy, c \sinh(dy+e) \) and \( c \sin(dy+e) \). But, to have the solution be valid uniformly in \([y_1, y_2]\) and satisfy the conditions \( f(y_1) = 0 \) and \( f(y_2) = 0 \), the following has to hold:

\[
(5.11a) \quad f(y) \sim \Lambda \sin \frac{\pi}{2 - \Delta_1 - \Delta_2} (y - (-1 + \Delta_1))
\]

\[
(5.11b) \quad = -\Lambda \sin \frac{\pi}{2 - \Delta_1 - \Delta_2} (y - (1 - \Delta_2)),
\]

where \( \Lambda < 0 \) is a constant. Figure 3a shows that the turning points \( y_1 \) and \( y_2 \) are moving towards the left-end point and the right-end point of the interval \([-1,1]\), respectively, with increasing \( R \). The quantities \( \Delta_1, \Delta_2 \) and \( \Lambda \) which are related to \( \epsilon \), will be determined by matching as \( \epsilon \to 0 \).

2) Asymptotic solution near \( y = y_1 \) and inner solution near \( y = -1 \)
We introduce a variable transformation

\[
\tau = \frac{-1 + \Delta_1 - y}{\Delta_1}, \quad y \in [-1, -1 + \Delta_1].
\]

Letting \( f(y) = f(-1 + \Delta_1 - \tau \Delta_1) = \overline{f}(\tau) \), then, \((5.9)\) becomes

\[
\epsilon \overline{f}^{iv} + (\overline{f}\overline{f}^{iv} - \overline{f}\overline{f}'') = 0,
\]

where \( \overline{\tau} = \frac{\tau}{\Delta_1} \). The boundary conditions to be satisfied by \((5.13)\) are

\[
(5.14) \quad \overline{f}(0) = 0, \quad \overline{f}(1) = a, \quad \overline{f}'(1) = 0.
\]

Since \( \tau \to 0 \) as \( \epsilon \to 0 \), \((5.13)\) subject to \((5.14)\) is still a singular perturbation problem.

(1) Outer solution
Setting \( \tau = 0 \), the reduced equation is

\[
\overline{ff}''' - \overline{f}\overline{f}'' = 0,
\]

satisfying the boundary condition \( \overline{f}(0) = 0, \overline{f}(1) = a \) and \( \overline{f}'(\tau) > 0 \) for all \( \tau \). Equation \((5.15)\) may have three possible solution: \( \sigma \tau, a \sin \overline{\tau} \) and \( a \sinh(\ln(\frac{1 + \overline{\tau}}{1 - \overline{\tau}})) \). By the proof of Proposition 3.10(c) for the type II solution, we know that \( \frac{1}{2}b(y_1 + 1) < \gamma < \xi_0 \),

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then \( g^v(\xi) < 0 \) in \( (0, \frac{1}{2} b(y_1 + 1)) \), thus \( F^v(\tau) < 0 \) in \( (0, 1) \). Hence, trigonometric functions and hyperbolic functions can be excluded. The outer solution is

\[
(5.16) \quad \tilde{f}(\tau) = \sigma \tau + \cdots,
\]

where \( \sigma \) can be determined by matching.

(2) Inner solution

The lower wall of the channel is with suction, hence, we introduce a stretching variable \( x^* = \frac{1 - \tau}{\epsilon} \). Letting \( \tilde{f}(\tau) = f(x^*) \), then, (5.13) becomes

\[
(5.17) \quad \tilde{f}'''' + \tilde{f}'''' = 0.
\]

The conditions at point \( x^* = 0 \) are \( \tilde{f}(0) = a \) and \( \tilde{f}'(0) = 0 \).

The inner solution can be expanded as:

\[
(5.18) \quad \tilde{f}(x^*) = a + \epsilon \tilde{f}_1(x^*) + \cdots.
\]

Substituting (5.18) into (5.17) and collecting the terms of \( O(\epsilon) \), we can obtain the equation of \( \tilde{f}_1(x^*) \)

\[
(5.19) \quad \tilde{f}_1'''' + a \tilde{f}_1'''' = 0,
\]

satisfying \( \tilde{f}_1(0) = \tilde{f}_1'(0) = 0 \). Then, the expression of \( \tilde{f}_1(x^*) \) is

\[
(5.20) \quad \tilde{f}_1(x^*) = b_1(e^{-ax^*} + ax^* - 1) + b_2 x^2,
\]

where \( b_1 \) and \( b_2 \) will be determined by matching. Hence, the inner solution becomes

\[
(5.21) \quad \tilde{f}(x^*) = a + \epsilon(b_1(e^{-ax^*} + ax^* - 1) + b_2 x^2) + \cdots.
\]

Meanwhile, assume that the expression of outer solution can be written as

\[
(5.22) \quad F(\tau) = \sigma \tau + \epsilon F_1(\tau) + \epsilon^2 F_2(\tau) + \cdots.
\]

Substituting (5.22) into (5.13) yields that \( F_1(\tau) \) satisfies

\[
(5.23) \quad \tau F_1'''' - F_1'' = 0.
\]

The corresponding condition is \( F_1(0) = 0 \). Then, the expression of \( F_1(\tau) \) is

\[
(5.24) \quad F_1(\tau) = \frac{1}{6} c_1 \tau^3 + d_1 \tau,
\]

where \( c_1 \) and \( d_1 \) are constants. The outer solution (5.22) expressing in terms of inner variable \( x^* \) is

\[
(5.25) \quad \tilde{f}(\tau) = \sigma(1 - \tilde{\tau}x^*) + \epsilon(c_1(1 - \tilde{\tau}x^*)^3 + d_1(1 - \tilde{\tau}x^*)) + \cdots
\]

\[
= \sigma + \epsilon(-\sigma x^* + c_1 + d_1) + \cdots.
\]

Matching the inner solution (5.21) with the outer solution (5.25) gives \( \sigma = a, b_1 = -1, b_2 = 0 \) and \( c_1 + d_1 = 0 \). Following the analysis in [10], we know that \( F_1(\tau) \) are all linear, \( i = 1, 2, \cdots \), where \( c_1 = 0 \) and \( d_1 = 1 \). Hence, \( \tilde{f}(\tau) \) can be written as

\[
(5.26) \quad \tilde{f}(\tau) \sim \theta(\tau) \tau,
\]
where $\theta(\tau) = a + d_1\tau + d_2\tau^2 + \cdots$ and $\theta(\tau) \to a$ as $\tau \to 0$. The inner solution has exponentially small terms and outer solution has to be more precise, we assume that $\overline{f}(\tau)$ is as follow:

$$(5.27) \quad \overline{f}(\tau) = \theta(\tau)\tau + \sum_{i=1}^{\infty} \delta_i h_i(\tau),$$

where $\delta_i = \delta_i(\tau) = o(\tau^n)$ and $\delta_{i+1} \ll \delta_i$ for all positive integers $n$ and $i$. $(5.27)$ is valid in the small neighborhood of the turning point $y_1 = -1 + \Delta_1$. Substituting $(5.27)$ into $(5.13)$ and collecting the terms of $O(\delta_1)$ yield

$$(5.28) \quad \overline{e} h_1^{iv} - \theta \tau h_1''' + \theta h_1'' = 0,$$

satisfying the condition $h_1(0) = 0$. Following the similar analysis [13], one solution of $h_1$ is

$$(5.29) \quad h_1(\tau) = \frac{a}{6} \tau^3 + r_1 \tau,$$

where $r_1$ is a constant. Setting $\delta_2 = \delta_2(\tau) = o(\tau^n)$ and $\delta_{i+1} \ll \delta_i$ for all positive integers $n$ and $i$. $(5.27)$ is valid in the small neighborhood of the turning point $y_1 = -1 + \Delta_1$. Substituting $(5.27)$ into $(5.13)$ and collecting the terms of $O(\delta_1)$ yield

$$(5.28) \quad \overline{e} h_1^{iv} - \theta \tau h_1''' + \theta h_1'' = 0,$$

$$(5.29) \quad \overline{e} h_2^{iv} - \theta \tau h_2''' + \theta h_2'' + \tau^3 = 0.$$

Differentiate $(5.29)$ and multiply by the integrating factor $e^{-\theta \tau^2}$, then, we can obtain

$$(5.30) \quad h_2^{iv} = -\frac{1}{e} \frac{\theta}{\tau^2} \tau^2 \left\{ \left( \frac{2\tau}{\theta} \right)^{3/2} \left[ \frac{\sqrt{\pi}}{4} - \frac{1}{2} \sqrt{\frac{\theta}{2\tau}} \tau \right] e^{-\theta \tau^2} + \cdots \right\} - Ce^{\theta \tau^2},$$

where $C$ is a constant. If we choose $\tau < 0$ which is away from zero, $h_2^{iv}$ will have exponentially large term. Then, we can choose $C$ to eliminate the exponentially large term. Evaluating $(5.30)$ leads to

$$(5.31) \quad h_2^{iv} = \frac{1}{e} \frac{\theta}{\tau^2} \tau^2 \left\{ \left( \frac{2\tau}{\theta} \right)^{3/2} \left[ \frac{\sqrt{\pi}}{4} - \frac{1}{2} \sqrt{\frac{\theta}{2\tau}} \tau \right] e^{-\theta \tau^2} + \cdots \right\} - Ce^{\theta \tau^2}.$$

Hence, we choose $C = \frac{\sqrt{\pi}}{2\theta \sqrt{\theta}}$. Evaluating $(5.31)$, we obtain asymptotic expression

$$(5.32) \quad h_2^{iv} \sim \theta^{-1} \tau.$$

Hence, the expression for $\tau < 0$ is

$$(5.33) \quad \overline{f}(\tau) = \theta \tau + \delta_1 \left( \frac{\tau^3}{6} + r_1 \tau \right) + \delta_1 \theta^{-1} \left( \frac{\tau^5}{90} + \cdots \right) + \cdots.$$

Then, expanding $(5.11a)$ at the turning point $y_1 = -1 + \Delta_1$ yields

$$(5.34) \quad f(y) \sim \Lambda \sin \left( \frac{\pi}{2 - \Delta_1 - \Delta_2} \left( y - (-1 + \Delta_1) \right) \right)$$

$$= -\Lambda \left( \frac{\pi \Delta_1 \tau}{2 - \Delta_1 - \Delta_2} \right) + \Lambda \left( \frac{\pi \Delta_1 \tau}{2 - \Delta_1 - \Delta_2} \right)^3 - \Lambda \left( \frac{\pi \Delta_1 \tau}{2 - \Delta_1 - \Delta_2} \right)^5 + \cdots.$$

Comparing the linear term in $(5.33)$ and $(5.34)$, we can obtain

$$(5.35) \quad \Lambda \sim -\frac{a(2 - \Delta_1 - \Delta_2)}{\pi \Delta_1}.$$
where $\theta \sim a$ is used.
Then, comparing the cubic term, we get

\begin{equation}
\delta_1 \sim -a(\frac{\pi \Delta_1}{2 - \Delta_1 - \Delta_2})^2.
\end{equation}

Hence, the asymptotic expansion of $\overline{f}(\tau)$ is

\begin{equation}
\overline{f}(\tau) = \theta \tau - a(\frac{\pi \Delta_1}{2 - \Delta_1 - \Delta_2})^2 \frac{\tau^3}{6} + a^2(\frac{\pi \Delta_1}{2 - \Delta_1 - \Delta_2})^4 \theta^{-1} h_2 + \cdots.
\end{equation}

3) The determination of $\Delta_1$ and $\Delta_2$

In this section, we will find the asymptotic relationship between $\Delta_1$, $\Delta_2$ and $\epsilon$ by matching near $\tau = 1$. From (5.30) with $C = \sqrt{\frac{2}{29} \theta}$, we can know

\begin{equation}
h_2^iv = -\frac{1}{\pi} e^{\frac{\epsilon}{2} \theta x^2} \int_0^r s^2 e^{-\frac{\epsilon}{2} \theta x^2} ds - \frac{\sqrt{2\pi}}{2\theta \sqrt{\theta}} e^{\frac{\epsilon}{2} \theta x^2} + \frac{1}{\theta} + \cdots.
\end{equation}

Then, from (5.37) and (5.38), we can have

\begin{equation}
\begin{aligned}
\frac{d^4 \overline{f}}{d\tau^4} &= \delta_1\delta^iv(\tau) \\
&= \frac{1}{\pi} \Delta_1^{7/2} \frac{(2 - \Delta_1 - \Delta_2)^3}{\theta} e^{\frac{\epsilon}{2} \theta x^2} + a^2(\frac{\pi \Delta_1}{2 - \Delta_1 - \Delta_2})^4 \frac{1}{\theta} + \cdots.
\end{aligned}
\end{equation}

The outer solution (5.39) expressing in the terms of inner variable $x^*$ is

\begin{equation}
\frac{1}{\pi^4} \frac{d^4 \overline{f}}{dx^4} = -a^{1/2} \pi^{9/2} \frac{\Delta_1^{7/2}}{(2 - \Delta_1 - \Delta_2)^3} \sqrt{2\epsilon} e^{\frac{\epsilon}{2} \theta x^*} e^{\frac{\epsilon}{2} \theta x^*} e^{-\theta x^*} \\
+ a^2(\frac{\pi \Delta_1}{2 - \Delta_1 - \Delta_2})^4 \frac{1}{\theta} (1 - \tau x^*) + \cdots.
\end{equation}

Differentiate (5.21) four times gives

\begin{equation}
\frac{1}{\pi^4} \frac{d^4 \overline{f}}{dx^4} = -a^4 \frac{1}{\pi^3} e^{-ax^*} + \cdots.
\end{equation}

Comparing (5.40) and (5.41) suggests that the overlap domain must satisfy the conditions: $\frac{d^3}{dx^3} \overline{f} << 1$ and $x^2 > 1$. It is obvious that

\begin{equation}
- a^{1/2} \pi^{9/2} \frac{\Delta_1^{7/2}}{(2 - \Delta_1 - \Delta_2)^3} \sqrt{2\epsilon} e^{\frac{\epsilon}{2} \theta x^*} e^{-\theta x^*} \sim -a^4 \frac{1}{\pi^3}.
\end{equation}

Finally, setting $\theta \sim a + \tau$, we obtain the asymptotic relationship:

\begin{equation}
\frac{\Delta_1}{\epsilon} e^{a \frac{\epsilon}{2} \theta} = a^7(2 - \Delta_1 - \Delta_2)^8 e^{2\pi^3 \epsilon^8}.
\end{equation}

4) Asymptotic solution near $y = y_2$

In order to analyze the asymptotic behaviour near $y = y_2$, we also introduce a variable transformation

\begin{equation}
\eta = \frac{y - 1 + \Delta_2}{\Delta_2}, \quad y \in [1 - \Delta_2, 1].
\end{equation}
Letting \( f(y) = f(1 - \Delta_2 + \eta \Delta_2) = \tilde{f}(\eta) \), then, (5.9) becomes

\[
(5.45) \quad \tilde{\epsilon} \tilde{f}^{iv} + (\tilde{f}^{iv} - \tilde{f}^{iv}) = 0,
\]

where \( \tilde{\epsilon} = \frac{\epsilon}{\Delta_2} \). The boundary conditions to be satisfied by (5.45) are

\[
(5.46) \quad \tilde{f}(0) = 0, \quad \tilde{f}(1) = 1, \quad \tilde{f}'(1) = 0,
\]

\( \tilde{\epsilon} \to 0 \) as \( \epsilon \to 0 \), but there is no boundary layer near \( \eta = 1 \) (or \( y = 1 \)), hence, (5.13) and (5.14) form a regular perturbation problem. Setting \( \tilde{\epsilon} = 0 \), the reduced equation is

\[
(5.47) \quad \tilde{f}^{iv} - \tilde{f}^{iv} = 0
\]

satisfying the boundary condition (5.46). The corresponding solution is

\[
(5.48) \quad \tilde{f}(\eta) = \sin \frac{\pi}{2} \eta.
\]

Since there is no boundary layer near the upper wall of the channel, we expand \( \tilde{f}(\eta) \) at the point \( \eta = 0 \)

\[
(5.49) \quad \tilde{f}(\eta) = \frac{\pi}{2} - \frac{1}{3!} \left( \frac{\pi}{2} \eta \right)^3 + \frac{1}{5!} \left( \frac{\pi}{2} \eta \right)^5 + O(\tilde{\epsilon}).
\]

Then, expand (5.11b) at the turning point \( y_2 = 1 - \Delta_2 \)

\[
(5.50) \quad f(y) \sim -\Lambda \sin \frac{\pi}{2 - \Delta_1 - \Delta_2} (y - (1 - \Delta_2))
\]

\[
= -\Lambda \frac{\pi \Delta_2 \eta}{2 - \Delta_1 - \Delta_2} + \frac{\Lambda}{3!} \left( \frac{\pi \Delta_2 \eta}{2 - \Delta_1 - \Delta_2} \right)^3 - \frac{\Lambda}{5!} \left( \frac{\pi \Delta_2 \eta}{2 - \Delta_1 - \Delta_2} \right)^5 + \cdots.
\]

Comparing the linear term in (5.49) and (5.50): \( \frac{\pi}{2} \sim -\Lambda \frac{\pi \Delta_2 \eta}{2 - \Delta_1 - \Delta_2} \), then we can obtain

\[
(5.51) \quad \Lambda \sim -\frac{2 - \Delta_1 - \Delta_2}{2 \Delta_2}.
\]

From (5.35) and (5.51), the relationship between \( \Delta_1 \) and \( \Delta_2 \) is obvious:

\[
(5.52) \quad \frac{\Delta_2}{\Delta_1} = \frac{\pi}{2a}.
\]

5.3. Asymptotic solution of type III. The numerical solution for type III in Figure 4 shows that, as \( R \to \infty \), the flow should consist of an inviscid core and a thin boundary layer near the lower wall. Both transverse and streamwise velocities rapidly decay and then the streamwise velocity rapidly increases near the lower wall for type III solution while only streamwise velocity rapidly decays for type I and type II solutions. Therefore, it is reasonable to expect that the high Reynolds number structure of the flow can be determined by boundary layer theory near the lower wall. Further, in this case we expect from numerical results that only two boundary conditions at the upper wall \( (y = 1) \) are satisfied by the reduced problem. This makes the construction much harder than that of type I solution. We expand \( k \) as (5.3) and \( f \) as follow

\[
(5.53) \quad f(y) = f_0(y) + h_0(\eta) + \epsilon(f_1(y) + h_1(\eta)) + \epsilon^2(f_2(y) + h_2(\eta)) + \cdots,
\]
where \( \eta = \frac{1 + \beta}{y} \) is a stretching transformation near the lower wall dimensionless height \( y = -1 \) and \( h_i(\eta), \ i = 0, 1, 2 \cdots \) are boundary layer functions. By substituting (5.53) into (2.6), the boundary conditions become

\[
\begin{align*}
(5.54) & \quad f_0|_{y=1} = 1, \quad f_0'|_{y=1} = 0, \\
(5.55) & \quad h_0|_{\eta=0} = a - f_0|_{y=-1}, \quad h_0'|_{\eta=0} = 0, \\
(5.56) & \quad f_i|_{y=1} = 0, \quad f_i'|_{y=1} = 0, \quad i = 1, 2, \cdots, \\
(5.57) & \quad h_i|_{\eta=0} = -f_i|_{y=-1}, \quad h_i'|_{\eta=0} = -f_i'|_{y=-1}, \quad i = 1, 2, \cdots,
\end{align*}
\]

where \( \dot{h}_0 \) denotes the derivative of \( h_0 \) with respect to \( \eta \). Substituting (5.53) and (5.3) into (5.1) and collecting the terms of \( O(1) \), we can obtain the equation of \( f_0 \) (same as (5.7)):

\[
(5.58) \quad f_0 f_0'' - f_0'^2 = k_0,
\]

satisfying boundary conditions (5.54) (different from (5.4)). Similarly, collecting the terms of \( O(\epsilon^{-2}) \), we can obtain the equation of \( h_0 \):

\[
(5.59) \quad \ddot{h}_0 + (h_0 + f_0(-1))\dot{h}_0 - h_0^2 = 0,
\]

satisfying boundary conditions (5.55).

One expression of \( f_0 \) with the boundary conditions (5.54) is

\[
(5.60) \quad f_0 = \cos(by - b),
\]

where \( b \) is an undetermined parameter and we denote \( f_0(-1) = \cos 2b \) as \( \beta \). We shall determine \( \beta \) such that equation (5.59) subject to boundary conditions (5.55) has a boundary layer solution. Usually we request a boundary layer function to tend to zero as \( \eta \to \infty \). However for problem (5.59) with (5.55) such a solution may not exist. A rigorous proof is highly nontrivial, we will report it in a forthcoming paper. For the purpose of the construction of the first order asymptotic solution here in the paper, it is enough to request a boundary layer function \( h_0(2/\epsilon) \to 0 \) (or much smaller than \( O(\epsilon) \)) when \( \epsilon \) is sufficiently small. It is obvious that \( h_0(\eta) = a - \beta \) and \( (a - \beta) e^{-\beta \eta} \) are two solutions of (5.59), but the former is not a boundary layer function and the latter doesn’t satisfy (5.55). It is hardly possible, however, to obtain any other explicit solution for the nonlinear equation (5.59) with (5.55). We thus make use of both analytic and numerical tools to predict \( \beta \).

Next, we shall show that \( \beta < 0 \) is impossible.

**Proposition 5.1.** Let \( h_0(\eta) \) be a boundary layer function solution of (5.59) and (5.55) in \([0, 2/\epsilon]\), then we can have:

(a) If \( h_0''(\eta_1) > 0 \) for some \( \eta_1 \geq 0 \), then \( h_0''(\eta) > 0 \) for all \( \eta \geq \eta_1 \).

(b) There holds that \( h_0'(\eta) \leq 0 \) for all \( \eta \geq 0 \).

(c) There holds that \( h_0'(\eta) < 0 \) for all \( \eta > 0 \).

(d) There exists some \( \eta_2 > 0 \) such that \( h_0''(\eta) > 0 \) for all \( \eta \geq \eta_2 \).

(e) \( \beta < 0 \) is impossible.

**Proof.** (a) Let \( h_2(\eta) = h_0''(\eta) \) for all \( \eta \geq 0 \), by equation (5.59), then

\[
(5.61) \quad h_2 + (h_0 + \beta)h_2 = (h_0')^2 \geq 0 \quad \text{for all} \quad \eta \geq 0,
\]

which implies that \( e^{\int_{\eta_1}^\eta (h_0(t)+\beta)dt} h_2(\eta) \geq 0 \) for all \( \eta \geq \eta_1 \). So we get \( e^{\int_{\eta_1}^\eta (h_0(t)+\beta)dt} h_2(\eta) \geq h_2(\eta_1) = h_0''(\eta_1) > 0 \) for all \( \eta \geq \eta_1 \), which implies that \( h_0''(\eta) = h_2(\eta) > 0 \) for all \( \eta \geq \eta_1 \).
(b) If not, that is, there exists some $\lambda_0 > 0$ such that $h''_0(\lambda_0) > 0$. Since $h''_0(0) = 0$, then there exists some $b \in [0, \lambda_0)$ such that $h''_0(b) = 0$ and $h''_0(\eta) > 0$ for all $\eta \in (b, \lambda_0]$. Then $h''_0(b) > 0$. By the result of part (a), then $h''_0(\eta) > 0$ for all $\eta \geq b$, which implies that $h''_0(\eta)$ is increasing in $[b, 2/\epsilon)$. So $h''_0(\eta) \geq h''_0(\lambda_0) := \sigma > 0$ for all $\eta \geq \lambda_0$, which implies that $h''_0(\eta) - h''_0(\lambda_0) \geq \sigma(\eta - \lambda_0)$ for all $\eta \geq \lambda_0$. Since $\sigma > 0$, by taking $\eta \to 2/\epsilon$ sufficiently large (or $\epsilon$ sufficiently small), then $h_0(2/\epsilon)$ is large, and then is in contradiction to a boundary layer function.

(c) If not, by the result of part (b), then there exists some $\lambda_1 > 0$ such that $0 = h''_0(\lambda_1) = \sup_{\eta \geq 0} h''_0(\eta)$, which implies that $h''_0(\lambda_1) = 0$. By the uniqueness theorem of solution to ODE, then $h_0 = h_0(\lambda_1)$ in $[0, 2/\epsilon)$, is in contradiction to a boundary layer function.

(d) If not, that is, $h''_0(\eta) \leq 0$ for all $\eta \geq 0$, then $h''_0(\eta)$ is non-increasing on $[0, 2/\epsilon)$. By the result of part (c), then $h''_0(\eta) \leq h''_0(1) < 0$ for all $\eta \geq 1$, which implies that $h''_0(\eta) - h''_0(1) \leq h''_0(1)(\eta - 1)$ for all $\eta \geq 1$. Since $h''_0(1) < 0$, by taking $\eta \to 2/\epsilon$ sufficiently large, then $h_0(2/\epsilon)$ is negatively large, is in contradiction to a boundary layer function.

(e) If not, that is $\beta < 0$, since $h_0(2/\epsilon)$ is sufficiently small as $\epsilon$ is sufficiently small, then when $\eta < 2/\epsilon$ is sufficiently large (say $\eta > \eta_3$), we have $\beta + h_0(\eta) < 0$ for all $\eta > \eta_3$. From (5.59), we have

$$
(5.61) \quad h''_0(\eta) = h''_0(\eta_3)e^{- \int_{\eta_3}^{\eta} (h_0(t)+\beta) \, dt} + \int_{\eta_3}^{\eta} h^2_0(s)e^{- \int_{\eta_3}^{s} (h_0(t)+\beta) \, dt} \, ds.
$$

We mark the right most term as $B(\eta)$. Integrating (5.61) from $\eta_3$ to $\eta$, we can obtain

$$
(5.62) \quad h'_0(\eta) = h'_0(\eta_3) + h''_0(\eta_3) \int_{\eta_3}^{\eta} e^{- \int_{\eta_3}^{s} (h_0(t)+\beta) \, dt} \, dx + \int_{\eta_3}^{\eta} B(x) \, dx.
$$

Fixed $\eta_3$, it is obvious that the first term at the right hand of (5.62) is a negative constant and the third term is always positive. Since $\beta + h_0(\eta) < 0$, by the results of parts (a) and (d), $h''_0(\eta_3) > 0$, then we have $h''_0(\eta_3) \int_{\eta_3}^{\eta} e^{- \int_{\eta_3}^{s} (h_0(t)+\beta) \, dt} \, dx \geq h''_0(\eta_3)(\eta - \eta_3)$. Hence, $h'_0(\eta)$ is sufficiently large as $\eta$ is close to $2/\epsilon$, then $h_0(2/\epsilon)$ can not be close to $0$, in contradiction to a boundary layer function.

Although we can prove $\beta \geq 0$, it is still difficult to determine $\beta$ analytically. We thus determine $\beta$ numerically. Gradually increasing $R$ and comparing the type III numerical solution of (5.1) and (2.6) for a given boundary value $a$ with the solution of the reduced problem as in expression (5.60), we can numerically estimate $\beta$. The results are summarised in Table 1. Then, it is obvious that $b = \cos^{-1} \frac{\beta}{2}$. Then, we can solve the boundary layer equation (5.59) subject to (5.55) numerically. The numerical results for $h_0(\eta)$ show that $h_0(\eta) \to 0$ as $\eta \to 2/\epsilon$. Finally, the asymptotic solution up to $O(\epsilon)$ is $f(y) = f_0(y) + h_0(\eta) + O(\epsilon)$. This will be compared with the numerical solution in next section.

Table 1: The numerical results of $\beta$ at different given boundary values $a$ for $R = 1500.$

| $a$ | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\beta$ | 0.0889 | 0.0783 | 0.0672 | 0.0551 | 0.0417 | 0.0264 | 0.0079 |
6. Comparison of numerical and asymptotic solutions. Numerical solutions for (2.5) and (2.6) can be readily obtained by MATLAB boundary value problem solver bvp4c. Comparison of the asymptotic solution and numerical solution will be shown in the following Tables.

For the type I solution, we will make comparison between numerical and asymptotic solution for $f'(y)$ so as to see the accuracy of the type I asymptotic solution constructed in (5.8). From Table 2, it can be seen that the asymptotic solution is matched well with the numerical solution.

Table 2: Comparison between numerical and asymptotic results for $f'(y)$ at $a = 0.8$ with $R = 100$, $R = 200$ and $R = 300$.

| $f'(y)$ | $R = 100$ | $R = 200$ | $R = 300$ |
|---------|-----------|-----------|-----------|
| $y$     | Numeric   | Asymptotic| Numeric   | Asymptotic| Numeric   | Asymptotic|
| -1.0    | 0.0000    | 0.0000    | 0.0000    | 0.0000    | 0.0000    | 0.0000    |
| -0.8    | 0.1780    | 0.1781    | 0.1771    | 0.1771    | 0.1768    | 0.1768    |
| -0.6    | 0.1603    | 0.1603    | 0.1593    | 0.1593    | 0.1590    | 0.1590    |
| -0.4    | 0.1417    | 0.1417    | 0.1409    | 0.1409    | 0.1406    | 0.1406    |
| -0.2    | 0.1226    | 0.1226    | 0.1219    | 0.1219    | 0.1217    | 0.1217    |
| -0.0    | 0.0829    | 0.0829    | 0.0824    | 0.0824    | 0.0823    | 0.0823    |
| 0.2     | 0.0625    | 0.0625    | 0.0621    | 0.0621    | 0.0620    | 0.0620    |
| 0.4     | 0.0418    | 0.0418    | 0.0416    | 0.0416    | 0.0415    | 0.0415    |
| 0.8     | 0.0209    | 0.0209    | 0.0208    | 0.0208    | 0.0208    | 0.0208    |
| 1.0     | 0.0000    | 0.0000    | 0.0000    | 0.0000    | 0.0000    | 0.0000    |

For the type II solution, since the turning points $y_1 = -1 + \Delta_1$ and $y_2 = 1 - \Delta_2$ are unknown a priori, getting the values of them is very important and difficult. We will contrast numerical and asymptotic results for the turning points. The asymptotic results of $\Delta_1$ and $\Delta_2$ are from (5.43) and (5.52). From Table 3, it can be seen that the error between the numerical and asymptotic results of the turning points is decreasing with the increasing $R$ and that $|\Delta_1|$ and $|\Delta_2|$ get smaller and smaller as $R$ increases. These verify our constructing process of the type II asymptotic solution in previous section.

Table 3: Comparison between numerical and asymptotic results for the turning points $y_1 = -1 + \Delta_1$ and $y_2 = 1 - \Delta_2$ at $a = 0.8$.

| $R$ | $y_1 = -1 + \Delta_1$ | $y_2 = 1 - \Delta_2$ |
|-----|-----------------------|-----------------------|
|     | Numeric   | Asymptotic | Numeric   | Asymptotic |
| 100 | -0.7449   | -0.7345    | 0.5457    | 0.4728    |
| 200 | -0.8263   | -0.8227    | 0.6753    | 0.6619    |
| 300 | -0.8914   | -0.8921    | 0.7808    | 0.7959    |
| 400 | -0.9203   | -0.9201    | 0.8483    | 0.8434    |
| 500 | -0.9363   | -0.9363    | 0.8783    | 0.8750    |

For the type III solution, we will compare the numerical solution with the type III asymptotic solution. Because of the complexity of the boundary layer problem (5.59) and (5.55), we compute the asymptotic solution $f(y) = f_0(y) + h_0(y) + O(\epsilon)$ in the following way: $f_0(y)$ is obtained from (5.60) and $\beta$ or $b$ is estimated from numerical solution of (5.1) and (2.6), and $h_0$ is obtained numerically based on solving (5.59) and (5.55). The Table 4 shows that the asymptotic solution matches well with

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the numerical solution for large Reynolds numbers.

Table 4: Comparison between numerical and asymptotic solutions for \( f(y) \) at \( R = 800 \) with \( a = 0.652 \), \( a = 0.748 \) and \( a = 0.876 \).

| \( y \) | \( a = 0.652 \) | \( a = 0.748 \) | \( a = 0.876 \) |
|------|----------------|----------------|----------------|
| Numeric | Asymptotic | Numeric | Asymptotic | Numeric | Asymptotic |
| -1.0 | 0.6520 | 0.6520 | 0.7480 | 0.7480 | 0.8760 | 0.8760 |
| -0.6 | 0.3489 | 0.3462 | 0.3590 | 0.3516 | 0.3811 | 0.3866 |
| -0.2 | 0.6131 | 0.6103 | 0.6194 | 0.6133 | 0.6271 | 0.6169 |
| 0.0 | 0.7256 | 0.7228 | 0.7301 | 0.7246 | 0.7356 | 0.7267 |
| 0.2 | 0.8212 | 0.8187 | 0.8242 | 0.8196 | 0.8279 | 0.8204 |
| 0.4 | 0.8981 | 0.8959 | 0.8998 | 0.8960 | 0.9019 | 0.8960 |
| 0.6 | 0.9543 | 0.9526 | 0.9550 | 0.9523 | 0.9560 | 0.9518 |
| 0.8 | 0.9885 | 0.9875 | 0.9887 | 0.9872 | 0.9889 | 0.9867 |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

7. Conclusion. In this article, we have considered the multiplicity and asymptotics of similarity solutions for laminar flows in a porous channel with different permeabilities, in particular, flows permeating from upper wall of the porous channel and exiting from the lower wall. We rigorously prove that there exist three similarity solutions designated as type \( I \), type \( II \) and type \( III \) solutions, and then numerically show that these three solutions exist for \( R > 14.10 \). Meanwhile, the asymptotic solution for each of the three types of similarity solutions is constructed for the most interesting and challenging high Reynolds number case and is also verified numerically. For the type \( I \) solution, its streamwise velocity has an exponentially rapid decay. For the type \( II \) solution, there are two turning points and its streamwise velocity also has an exponentially rapid decay. For the type \( III \) solution, there exists an exponentially rapid change not only for its streamwise velocity (decay and then increase) but also for its transverse velocity (decay). The reversal flow occurs for both type \( II \) and type \( III \) solutions.

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