MACKENZIE THEORY AND Q-MANIFOLDS

THEODORE TH. VORONOV

Abstract. We give a simple characterization of Mackenzie’s double Lie algebroids in terms of homological vector fields. Application to the ‘Drinfeld double’ of Lie bialgebroids is given and an extension to the multiple case is suggested.

Introduction

Double Lie algebroids arose in the works on double Lie groupoids [5, 6] and in connection with an analog for Lie bialgebroids of the Drinfeld double of Lie bialgebras [7, 8, 9]. They originally appeared as the tangent objects for double Lie groupoids, and later their properties were axiomatized to give the abstract notion. Their immediate application, as well as that of double Lie groupoids, is in Poisson geometry or more generally in the larger subject that may be called Bracket Geometry, which embraces topics from (multiple) groupoid and algebroid theory through homotopy algebras to geometrical structures arising in deformation and quantization theory.

The richness of the theory of double Lie algebroids, due to Kirill Mackenzie, can be seen in numerous non-obvious structures, isomorphisms and dualities arising in it. Notice, for example, a non-trivial duality theory for double and triple vector bundles, where interesting discrete symmetry groups appear [13].

For a long time, the application of double Lie algebroids was somewhat hindered by the complexity of their original definition. An obvious part of the definition is, of course, the structure of two Lie algebroids (in fact, four, on the four sides of a double vector bundle; but they can be reduced to the two ‘main’ ones). The difficulty was to state a compatibility condition for them. A system of conditions, highly non-trivial in formulation, was found by Mackenzie [7] as an abstraction of the Lie functor of double Lie groupoids, and was proved to be the correct one, for example, by showing that it is satisfied by the so-called ‘cotangent doubles’ of Lie bialgebroids.

In this paper we analyze Mackenzie’s conditions and prove that they are equivalent to a simple commutativity condition for homological vector fields on a supermanifold naturally associated with a given double vector bundle. This radically simplifies the theory and opens ways to an immediate extension to the multiple case, i.e., n-fold Lie algebroids,
as well as $n$-fold ‘bi’ Lie algebroids. (The latter will be the subject of a forthcoming paper with Kirill Mackenzie [17].)

Our main statement (Theorem 1 below) establishes equivalence between two notions: double Lie algebroids in the sense of Mackenzie and double Lie antialgebroids as defined in this paper.

In the course of a proof, we show that Mackenzie’s ‘Condition III’ (see Section 1 below), which pertains to a certain bialgebroid, actually subsumes his other conditions.

It all fits into a big picture, which is as follows. For a given double vector bundle we consider all its neighbors, that is, the double vector bundles obtained by dualization and parity reversion. (There are twelve of them, including the original bundle.) We can say that a structure such as that of a double Lie algebroid is manifested in various ways in particular structures on all of these neighbor double vector bundles. This extends the idea that, say, a Lie algebra $\mathfrak{g}$ has equivalent manifestations as a linear Poisson bracket (on the coalgebra $\mathfrak{g}^*$), as a linear Schouten bracket (on the anticoalgebra $\Pi\mathfrak{g}$) and as a quadratic homological vector field (on the antialgebra $\Pi\mathfrak{g}$). See, for example, [23]. For double Lie algebroids, out of the twelve neighbors, five allow structures with easily formulated compatibility conditions. It has turned out that four of them are reformulations of Mackenzie’s Condition III, and the remaining one is precisely our commutativity condition.

We wish to emphasize that in our work, supermanifolds provide powerful tools that we apply to ordinary (“purely even”) objects. Although we show that everything works also in a ‘superized’ context, this was not the main goal.

Some parts of the proofs are calculations in coordinates. They can no doubt be replaced by coordinate-free arguments, by extending methods used by Mackenzie. However, I wish to note that a ‘motivated’ calculation in coordinates is sometimes the quickest way to get to the crux of the matter and allows to notice facts sometimes obscured by a more ‘abstract’ presentation.

It has been of considerable interest among experts to give an alternative simpler description of double Lie algebroids since the notion first appeared. I have always believed that such a description should be in terms of supermanifolds and homological vector fields. My own earliest notes on the problem, motivated by numerous inspiring discussions with Mackenzie, date back to 2002. Unfortunately, this work was interrupted, which prevented me from giving a solution at that time. I came back to it in 2003 and again in June 2006 right before the Białowieża conference (see below), wishing to discuss the problem with Kirill Mackenzie there, and at this time solved it completely. I wrote about the solution to Kirill Mackenzie, Yvette Kosmann-Schwarzbach, Alan Weinstein and Dmitry Roytenberg. Roytenberg, after learning about the statement of Theorem 1 below, told me that it was known to
him, but he did not possess a proof. As I learned from Alan Weinstein’s
email (even before my work was completed), his former Ph.D. students
A. Gracia-Saz and R. A. Mehta were also working on the problem;
there is a reference to a work in progress in Mehta’s thesis [18], but I
am not aware of any outcome.

The paper is organized as follows.
In Section 1 we recall the definition of double Lie algebroids.
In Section 2 we recall the description of (ordinary) Lie algebroids
in the language of homological vector fields, and revise double vector
bundles. In particular, we introduce partial reversions of parity.
In Section 3 we define double Lie antialgebroids and give our main
statement (Theorem 1).
In Section 4 we analyze the three conditions appearing in the defini-
tion of double Lie algebroids and give a proof of Theorem 1.
In Section 5 we show how the equivalence of Mackenzie’s notion of
double Lie algebroids and our notion of double Lie antialgebroids is a
part of a bigger picture. Modulo some facts established in Section 4
this provides an alternative proof of Theorem 1.
In Section 6 we show the equivalence of Mackenzie’s and Royten-
berg’s doubles of Lie bialgebroids and discuss an extension of the whole
theory to the multiple case.

Terminology and notation. We use the standard language of super-
manifolds. The letter Π denotes the parity reversion functor, and
notation such as ΦΠ is used for linear maps induced on the opposite
(parity reversed) objects. Commutators and similar notions are al-
ways understood in the \( \mathbb{Z}_2 \)-graded sense. A tilde over an object is
used to denote its parity. A Q-manifold means a supermanifold en-
dowed with a homological vector field; likewise, P- and S-manifolds
mean those with a Poisson or Schouten (= odd Poisson) bracket. A
QS-manifold means one with Q- and S- structures that are compatible
(the vector field is a derivation of the bracket, cf. [2]). In general, nota-
tion and terminology are close to our paper [23]. We wish to draw the
reader’s special attention to our normally dropping the prefix ‘super-’
when this cannot cause confusion and speaking, as a rule, of ‘manifolds’
meaning supermanifolds, ‘Lie algebras’ meaning superalgebras, etc.

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zie, Yvette Kosmann-Schwarzbach and Hovhannes Khudaverdian , for
inspiring discussions, most valuable criticism, and advice. Kirill Macken-
zie has pioneered the whole subject of multiple and bi- structures in the

\[1\] For specialists we may note that we do not make a distinction in notation
between, say, \( E \) and \( E^* \), though, practically, we use \( E^\Pi \) and \( \Pi E^* \) for dual vector
bundles \( E \) and \( E^* \) to avoid extra signs. We use left coordinates on \( E \) and right
coordinates on \( E^* \).
groupoid and algebroid world. I thank most cordially Yvette Kosmann-Schwarzbach for numerous comments and remarks that helped to improve the original manuscript. My special thanks go to the organizers of the annual international Workshops on Geometric Methods in Physics in Białowieża, notably to Anatol Odzijewicz, for the highly inspiring atmosphere. Some of my notes on the subject of this paper were made in Warsaw after the XXII Białowieża Workshop, and the final result was reported at the jubilee XXV Białowieża meeting in July 2006.

1. Double Lie algebroids according to Mackenzie

Double Lie algebroids were introduced by Mackenzie in [7, 8], see also [9], as the infinitesimal counterparts of double Lie groupoids. The latter notion is a double object in the sense of Ehresmann, i.e., a groupoid object in the category of groupoids. Therefore, it has a natural categorical formulation. Compared to it, the abstract notion of a double Lie algebroid is rather complicated and non-obvious. One of the reasons for this, is that properties of brackets for Lie algebroids are not expressed diagrammatically, so one cannot approach double objects for them by methods of category theory. Mackenzie’s conditions (see below) come about as an abstraction of the properties of the double Lie algebroid of a double Lie groupoid discovered in [5, 6].

Definition 1. A double vector bundle

\[
\begin{array}{cccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
\]  

(1)

is a double Lie algebroid if all sides are (ordinary) Lie algebroids and the following conditions I, II, and III are satisfied:

**Condition I:** With respect to the vertical structures of Lie algebroids, \( D \to B \) and \( A \to M \), all maps related with the horizontal vector bundle structures are Lie algebroid morphisms (more precisely, it includes the projections, the zero sections, the fiber-wise addition and multiplication by scalars). The same holds with vertical/horizontal structures interchanged.

**Condition II:** The horizontal arrows in the diagram

\[
\begin{array}{cccc}
D & \longrightarrow & TB \\
\downarrow & & \downarrow \\
A & \longrightarrow & TM
\end{array}
\]

where at the right there is the tangent prolongation of the Lie algebroid \( B \to M \), and \( a \) stands for the anchors, define a Lie
algebroid morphism. The same holds with vertical/horizontal structures interchanged.

Condition III: The vertical arrows in the diagram

\[
\begin{array}{c}
D^* \rightarrow K^* \\
\downarrow \quad \downarrow \\
A \rightarrow M
\end{array}
\]

define a Lie algebroid morphism. Here \( K \) denotes the core. The same holds with vertical/horizontal, and \( A \) and \( B \), interchanged. The vector bundles in duality \( D^*A \rightarrow K^* \) and \( D^*B \rightarrow K^* \) define a Lie bialgebroid.

(An explication of these conditions will be given below.)

Recall that a **double vector bundle** such as (1) is defined by the condition that all vector bundle structure maps in one direction (horizontal or vertical) are vector bundle morphisms for another direction. The core \( K \) is defined as the intersection of the kernels of the projections \( D \rightarrow A \) and \( D \rightarrow B \) considered as vector bundle morphisms (w.r.t. the other structure). \( K \) is a vector bundle over \( M \). It is a theorem due to Mackenzie that taking the two duals of \( D \) considered as a vector bundle either over \( A \) or over \( B \) leads to two double vector bundles

\[
\begin{array}{c}
D^*A \rightarrow K^* \\
\downarrow \quad \downarrow \\
A \rightarrow M
\end{array}
\quad \text{and} \quad \begin{array}{c}
D^*B \rightarrow B \\
\downarrow \quad \downarrow \\
K^* \rightarrow M
\end{array}
\]

where the vector bundles \( D^*A \rightarrow K^* \) and \( D^*B \rightarrow K^* \) over the co-core \( K^* \) are—unexpectedly—in a natural duality. All these facts, as well as the notion of the **tangent prolongation** of a Lie algebroid, can be found in [14, Ch. 9], see also [13, 15]. **Lie bialgebroids** were introduced by Mackenzie and Xu [15]. Their theory was advanced by Y. Kosmann-Schwarzbach [2], who in particular gave a very handy form of the definition, which we use. See [14].

2. **Lie Algebroids and Double Vector Bundles: Some Background**

In this section we develop tools that will be later used for an alternative description of double Lie algebroids (our main goal).

Henceforth we work in the ‘super’ setup, i.e., we consider supermanifolds and bundles of supermanifolds. However, we systematically skip the prefix ‘super-’ except when we wish to make an emphasis. All the constructions from the previous section carry over to the super case.

We use **graded manifolds** as defined in [23], i.e., supermanifolds endowed with an extra \( \mathbb{Z} \)-grading in the algebras of functions, in general not related with parity. We refer to such grading as **weight**.
Let us recall some known facts concerning Lie algebroids.

It was first shown by Vaintrob [22] that Lie algebroids can be described by homological vector fields. We shall recall this correspondence using the description given in [23] in the language of derived brackets. As mentioned, we consider the ‘superized’ version (i.e., ‘super’ Lie algebroids) by default.

Let $F \to M$ be a vector bundle. The total space $F$ is naturally a graded manifold, the (pullbacks of) functions on the base $M$ having weight 0 and linear functions on the fibers, weight 1. Using weights is very helpful for describing various geometric objects. For example, vector fields of weight $-1$ on $F$ correspond to sections of $F$ (or $\Pi F$, see below). Vector fields of weight 0 are generators of fiberwise linear transformations. Vector fields of weight 1 can be used to generate brackets of sections. More precisely: a Lie antialgebroid structure on $F \to M$, by definition, is given by a homological field $Q \in \text{Vect}(F)$ of weight 1.

There is a one-to-one correspondence between Lie antialgebroids and Lie algebroids, as follows.

Let $\Pi$ denote the parity reversion functor, and $F = \Pi E$ for a vector bundle $E \to M$. Then $F$ is a Lie antialgebroid if and only if $E$ is a Lie algebroid. The anchor and the bracket for the sections of $E$ are given by the following formulas:

\[ a(u)f := [[Q, i(u)], f] \quad (2) \]

and

\[ i([u, v]) := (-1)^{\delta}[[Q, i(u)], i(v)]. \quad (3) \]

Here $f \in C^\infty(M)$, and $u, v \in C^\infty(M, E)$ are sections. We use the natural odd injection $i: C^\infty(M, E) \to \text{Vect}(\Pi E)$, which sends a section $u \in C^\infty(M, E)$ to a vector field $i(u) \in \text{Vect}(\Pi E)$ of weight $-1$. The map $i$ is an odd isomorphism between the space of sections $C^\infty(M, E)$ and the subspace $\text{Vect}_{-1}(\Pi E) \subset \text{Vect}(\Pi E)$ of all vector fields of weight $-1$. By counting weights, one can see that the LHS’s of (2) and (3) are well-defined. The properties of the bracket and anchor are deduced from the identity $Q^2 = 0$ as is standard in the derived brackets method. Conversely, starting from a Lie algebroid structure in $E \to M$, one can reconstruct $Q$ on $\Pi E$ with the desired properties.

All these facts can be checked without coordinates; however, introducing local coordinates makes them particularly transparent. Let $x^a$ denote local coordinates on the base $M$. We shall use $u^i$ and $\xi^i$ for linear coordinates in the fibers of $E$ and $F = \Pi E$, respectively. Changes of coordinates have the following form:

\[ x^a = x^a(x'), \]
\[ u^i = u^i T_{\nu}^i(x'), \]
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and

\[ \xi^i = \xi^j T^j_i(x'). \]

The map \( i: C^\infty(M, E) \to \text{Vect}(\Pi E) \) has the following appearance in coordinates:

\[ i: u = u^i(x) e_i \mapsto i(u), \]

where

\[ i(u) = (-1)^{u_i(x)} \frac{\partial}{\partial \xi^i}. \]

Equation (4) clearly shows that the RHS is the general form of a vector field of weight \(-1\) on \( F \). A vector field \( Q \) of weight 1 on \( F \) in coordinates has the form

\[ Q = \xi^i Q^a_i(x) \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q^k_{ji}(x) \frac{\partial}{\partial \xi^k}. \]

Equations (2) and (3) produce the following formulas for the anchor:

\[ a(u) = u^i(x) Q^a_i(x) \frac{\partial}{\partial x^a}, \]

and for the brackets:

\[ [u, v] = \left( u^i Q^a_i \partial_a v^k - (-1)^{\tilde{\alpha} + 1} v^i Q^a_i \partial_a u^k - (-1)^{i + 1} \tilde{u}^i v^j Q^k_{ji} \right) e_k, \]

where we abbreviated \( \partial_a = \partial/\partial x^a \). In particular, for the elements of the local frame \( e_i \) we have

\[ [e_i, e_j] = (-1)^{i \tilde{b}} Q^k_{ij}(x) e_k. \]

Let us now proceed to **double vector bundles**. Let

\[
\begin{array}{ccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
\]

be a double vector bundle (in the category of supermanifolds). The manifold \( D \) is naturally bi-graded, by weights corresponding to the two vector bundle structures. If necessary we denote these weights by \( w_1 \) and \( w_2 \), or by \( w_A \) and \( w_B \), as convenient.

Double vector bundles allow fiberwise reversion of parity in both directions, horizontal and vertical. We denote the corresponding operations by \( \Pi_1 \) and \( \Pi_2 \) (or by \( \Pi_A \) and \( \Pi_B \) when convenient). (Such operations should be studied together with the operations of dualization in Mackenzie’s theory \[13\].) For example, for a double vector

\[ u_i(x) \] and \[ \tilde{u}_i(x) \] of a section \( u = u^i(x) e_i \) should not be confused with the coordinates \( u^i \), which are functions on the total space \( E \). In particular, each of the coordinates \( u^i \) has a certain fixed parity, and the correspondent component \( u'_i(x) \) of a section \( u \) has the same or the opposite parity as the coordinate \( u^i \) depending on whether the section \( u \) is even or odd.
bundle given by \((5)\), the vertical reversion of parity \(\Pi_1 = \Pi_A\) gives
\[
\begin{array}{ccc}
\Pi_A D & \longrightarrow & \Pi B \\
\downarrow & & \downarrow \\
A & \longrightarrow & M \\
\end{array}
\tag{6}
\]
which is a new double vector bundle. One can apply horizontal reversion of parity to it (applying the vertical reversion again takes us back), or do it the other way round.

Proposition 2.1. The operations \(\Pi_1\) and \(\Pi_2\) commute:
\[\Pi_1 \Pi_2 = \Pi_2 \Pi_1.\]

More precisely, for a double vector bundle given by \((5)\), there is a natural isomorphism of double vector bundles
\[
\begin{array}{ccc}
\Pi_B \Pi_A D & \longrightarrow & \Pi B \\
\Pi_A \Pi_B D & \longrightarrow & \Pi B \\
\Pi A & \longrightarrow & M \\
\Pi A & \longrightarrow & M \\
\end{array}
\tag{7}
\]
where we used the more suggestive notation \(\Pi_A = \Pi_1\) and \(\Pi_B = \Pi_2\).

Denote the common value of the ultimate total spaces in \((7)\) by \(\Pi^2 D := \Pi_A \Pi_B D = \Pi_B \Pi_A D\), and call it the complete reversion of parity of \(D\).

Remark 2.1. Everything extends immediately to the \(n\)-fold case, with \(\Pi^n D\) being the complete parity reversion of the ultimate total space \(D\) of an \(n\)-fold vector bundle. There are partial parity reversion operations \(\Pi_r\) such that \(\Pi_r \Pi_r = \Pi_r \Pi_r\) and \(\Pi^n D = \Pi_1 \cdot \ldots \cdot \Pi_n D\). (See Section 6.)

A coordinate language, particularly handy for visualizing double (and multiple) vector bundles, is as follows. Consider a double vector bundle given by \((5)\). As above, denote local coordinates on \(M\) by \(x^a\). Let \(u^i\) and \(w^\alpha\) be linear coordinates on the fibers of \(A \to M\) and \(B \to M\), respectively. On \(D\) we have coordinates \(x^a, u^i, w^\alpha, z^\mu\) so that \(u^i, z^\mu\) are linear fiber coordinates for \(D \to B\) and \(w^\alpha, z^\mu\) for \(D \to A\). Coordinate changes have the form:
\[
x^a = x^a(x'), \tag{8}
\]
\[
u^i = u'^i T_{i'}^i(x'), \tag{9}
\]
\[
w^\alpha = w^\alpha T^{\alpha \beta}(x'), \tag{10}
\]
\[
z^\mu = z^\mu T_{\mu \nu}(x') + u'^i w^\alpha T_{i' \nu}(x'). \tag{11}
\]
This is a convenient description of a double vector bundle structure. (The reader who lacks a taste for coordinate calculations may translate it into a language of local trivializations.) In particular, for two weights we have \(w_1 = \#u + \#z\) and \(w_2 = \#w + \#z\), where \(\#\) denotes
the degree in the respective variable. Everything extends directly to multiple vector bundles.

Partial parity reversion is described as follows. For (6) we have \( x^a, u^i, \eta^\alpha, \theta^\mu \) as coordinates on \( \Pi_A D \), so that \( \tilde{\eta}^\alpha = \tilde{w}^\alpha + 1 = \tilde{\alpha} + 1 \), \( \tilde{\theta}^\mu = \tilde{z}^\mu + 1 = \tilde{\mu} + 1 \), and changes of coordinates have the form

\[
\begin{align*}
  u^i &= u'^i T_{\nu^i}^i, \\
  \eta^\alpha &= \eta'^\alpha T_{\alpha'}^\alpha, \\
  \theta^\mu &= \theta'^\mu T_{\mu'}^\mu + (-1)^\nu \tilde{u}^\nu \eta'^\alpha T_{\alpha'}^\alpha \nu^\mu,
\end{align*}
\]

where we suppress coordinates on \( M \). Here \( \eta^\alpha, \theta^\mu \) are fiber coordinates for \( \Pi_A D \rightarrow A \) and \( \eta^\alpha \) are fiber coordinates for \( \Pi B \rightarrow M \). Similarly, for

\[
\begin{array}{ccc}
  \Pi_B D & \longrightarrow & B \\
  \downarrow & & \downarrow \\
  \Pi A & \longrightarrow & M 
\end{array}
\]

we have \( x^a, \xi^i, w^\alpha, \theta^\mu \) as coordinates on \( \Pi_B D \), where \( \tilde{\xi}^i = \tilde{u}^i + 1 = \tilde{i} + 1 \) and \( \tilde{\theta}^\mu = \tilde{z}^\mu + 1 = \tilde{\mu} + 1 \), with changes of coordinates of the form

\[
\begin{align*}
  \xi^i &= \xi'^i T_{\nu^i}^i, \\
  w^\alpha &= w'^\alpha T_{\alpha'}^\alpha, \\
  \theta^\mu &= \theta'^\mu T_{\mu'}^\mu + \xi'^i u'^\nu \eta'^\alpha T_{\alpha'}^\alpha \nu^\mu.
\end{align*}
\]

(Note different meanings of \( \theta^\mu \) for \( \Pi_A D \) and \( \Pi_B D \).) Applying parity reversion once again we obtain

\[
\begin{array}{ccc}
  \Pi_B \Pi_A D & \longrightarrow & \Pi B \\
  \downarrow & & \downarrow \\
  \Pi A & \longrightarrow & M 
\end{array}
\]

with coordinates on \( x^a, \xi^i, \eta^\alpha, z^\mu \) on \( \Pi_B \Pi_A D \) with the transformation law

\[
\begin{align*}
  \xi^i &= \xi'^i T_{\nu^i}^i, \\
  \eta^\alpha &= \eta'^\alpha T_{\alpha'}^\alpha, \\
  z^\mu &= z'^\mu T_{\mu'}^\mu + (-1)^\nu \tilde{x}^\nu \eta'^\alpha T_{\alpha'}^\alpha \nu^\mu,
\end{align*}
\]

and

\[
\begin{array}{ccc}
  \Pi_A \Pi_B D & \longrightarrow & \Pi B \\
  \downarrow & & \downarrow \\
  \Pi A & \longrightarrow & M 
\end{array}
\]
with coordinates $x^a, \xi^i, \eta^\alpha, z^\mu$ on $\Pi_A \Pi_B D$ with the transformation law

$$
\begin{align*}
\xi^i &= \xi'^i T^i, \\
\eta^\alpha &= \eta'^\alpha T^\alpha, \\
z^\mu &= z'^\mu T^\mu + (-1)^{i+1} \xi'^i \eta'^\alpha T^\alpha T^\mu.
\end{align*}
$$

Note that these transformation laws are the same up to a change of sign of the $z$-coordinate. In particular, this gives a proof for Proposition 2.1.

3. Main statement

In this section we give our main statement, which is a characterization of Mackenzie’s double Lie algebroids in terms of graded $Q$-manifolds.

**Definition 2.** A double vector bundle

$$
\begin{array}{ccc}
H & \longrightarrow & G \\
\downarrow & & \downarrow \\
F & \longrightarrow & M
\end{array}
$$

is a double Lie antialgebroid if it is endowed with two homological vector fields $Q_1$ and $Q_2$ on the manifold $H$ of weights $(1,0)$ and $(0,1)$, respectively, such that

$$[Q_1, Q_2] = 0.$$  

**Remark 3.1.** Equivalently, a double Lie antialgebroid can be defined as a double vector bundle such as (15) with a (single) homological vector field $Q$ on $H$ of total weight 1. Then $Q$ can be decomposed into the sum $Q_1 + Q_2$ of fields of weights $(1,0)$ and $(0,1)$, and $Q^2 = 0$ is equivalent to $Q_1^2 = Q_2^2 = [Q_1, Q_2] = 0$.

**Remark 3.2.** An extension to multiple Lie antialgebroids is immediate. An $n$-fold Lie antialgebroid is an $n$-fold vector bundle, which therefore gives rise to an $n$-graded structure on its (ultimate) total space, endowed with a homological vector field $Q$ of total weight 1. In greater detail, we obtain $n$ commuting homological fields $Q_r$, $r = 1, \ldots, n$, on the ultimate total space of weights $(0, \ldots, 1, \ldots, 0)$, respectively. (Here 1 stands at the $r$-th place, all other weights being zero.) Then

$$Q = Q_1 + \ldots + Q_n.$$  

We shall come back to this in Section 6.

**Example 3.1.** Let $(E, E^*)$ be a Lie bialgebroid with base $M$. Then

$$
\begin{array}{ccc}
T^* \Pi E & \longrightarrow & \Pi E^* \\
\downarrow & & \downarrow \\
\Pi E & \longrightarrow & M
\end{array}
$$

(17)
is a double vector bundle, as one can check. This is a superization of Mackenzie [7, 8]. The natural diffeomorphism in the upper-left corner of (17) is a superized version of a theorem of Mackenzie [7, 8] extending a statement of Tulczyjew [21]. The Lie algebroid structures in $E$ and $E^*$ give rise to homological fields on $\Pi E$ and $\Pi E^*$, respectively. These two vector fields correspond to two functions (‘linear Hamiltonians’) on $T^*\Pi E = T^*\Pi E^*$, of weights $(1, 0)$ and $(0, 1)$. That $(E, E^*)$ is a bialgebroid is equivalent to the commutativity of these Hamiltonians (due to Roytenberg [20], see also [23]). Therefore the corresponding Hamiltonian vector fields make the double vector bundle (17) a double Lie antialgebroid. We shall come back to this example in Section 6.

**Theorem 1** (A Characterization of Double Lie Algebroids). A double Lie algebroid structure in a double vector bundle such as (5) is equivalent to a double Lie antialgebroid structure in the complete parity reversion double vector bundle

$$
\begin{array}{c}
\Pi^2 D \\
\downarrow \\
\Pi A
\end{array}
\begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow \\
\Pi B \\
\Pi A \longrightarrow M
\end{array}
(18)
$$

i.e., to a homological field $Q = Q_1 + Q_2$ on $\Pi^2 D$ of total weight 1.

To appreciate the statement one may compare the three conditions of Definition 1 with the commutativity equation (16) of Definition 2.

Let us show how a homological vector field $Q = Q_1 + Q_2$ of total weight 1 on $\Pi^2 D$ generates Lie algebroid structures on all sides of (5). As in our discussion of a (single) Lie algebroid above, everything can be formulated in a coordinate-free setting. However, using coordinates sheds some extra light.

For the sake of concreteness, we consider $\Pi^2 D = \Pi_B \Pi_A D$ with natural coordinates $x^a, \xi^i, \eta^\alpha, z^\mu$ thereon. A vector field $Q_1 \in \text{Vect}(\Pi^2 D)$ of weight $(1, 0)$ has the form

$$
Q_1 = \xi^i Q_1^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_1^k \frac{\partial}{\partial \xi^k} + \left( \xi^i \eta^\alpha Q_1^j \eta^\beta + z^\mu Q_1^j \right) \frac{\partial}{\partial \eta^\beta} + \left( \frac{1}{2} \xi^i \eta^\alpha Q_1^j + \xi^i Q_1^j \right) \frac{\partial}{\partial z^\lambda},
(19)
$$

while $Q_2 \in \text{Vect}(\Pi^2 D)$ of weight $(0, 1)$, the form

$$
Q_2 = \eta^\alpha Q_2^a \frac{\partial}{\partial x^a} + \left( \eta^\alpha \xi^i Q_2^j \xi^\alpha + z^\mu Q_2^j \right) \frac{\partial}{\partial \xi^j} + \left( \frac{1}{2} \eta^\alpha \eta^\beta \xi^i Q_2^\gamma \eta^\gamma + \eta^\alpha z^\mu Q_2^\gamma \right) \frac{\partial}{\partial z^\lambda},
(20)
$$

All coefficients here are functions of $x^a$. Now, *due to the fact that $Q_1$ has weight 0 w.r.t. the horizontal fiber coordinates, it admits partial
parity reversion in this direction, giving a vector field on \( \Pi_B D \):

\[
Q_1^\Pi := \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_j^k \frac{\partial}{\partial \xi^k} + \left( \xi^i w^\alpha Q_{\alpha j}^\beta + \theta^\mu Q_{\mu j}^\lambda \right) \frac{\partial}{\partial w^\beta} + \left( \frac{1}{2} \xi^i \xi^j w^\alpha Q_{\alpha j i}^\lambda + \xi^i \theta^\mu Q_{\mu j}^\lambda \right) \frac{\partial}{\partial \theta^\lambda} .
\]

Similarly, \( Q_2 \) allows vertical parity reversion, which gives

\[
Q_2^\Pi := \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \left( \eta^\alpha u^i Q_{i \alpha}^j + \theta^\mu Q_{\mu j}^\lambda \right) \frac{\partial}{\partial u^j} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta \alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} + \left( \frac{1}{2} \eta^\alpha \eta^\beta u^i Q_{i \beta \alpha}^\lambda + \eta^\alpha \theta^\mu Q_{\mu j}^\lambda \right) \frac{\partial}{\partial \theta^\lambda} .
\]

on \( \Pi_A D \).

Both \( Q_1^\Pi \) and \( Q_2^\Pi \) are homological fields. They define Lie antialgebroid structures on the vector bundles \( \Pi_B D \to B \) and \( \Pi_A D \to A \), which correspond to Lie algebroid structures on \( D \to B \) and \( D \to A \).

The restrictions of \( Q_1^\Pi \) and \( Q_2^\Pi \) on \( \Pi_A \) and \( \Pi_B \), respectively, treated as submanifolds (zero sections) in \( \Pi_B D \) and \( \Pi_A D \) are tangent to these submanifolds and define homological vector fields

\[
Q_1^{(0)} = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_j^k \frac{\partial}{\partial \xi^k} \quad (23)
\]

on \( \Pi_A \) and

\[
Q_2^{(0)} = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta \alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} \quad (24)
\]

on \( \Pi_B \). This leads to Lie algebroid structures on \( A \to M \) and \( B \to M \).

4. Analysis of Mackenzie’s conditions

To prove Theorem 1, we can do the following. Consider a double vector bundle given by (1). Assume that all four sides are Lie algebroids and describe them by homological vector fields. Then we can study conditions I, II and III of Definition 1 and see what they mean in terms of these fields.

Recall the notion of a Lie algebroid morphism. It is non-obvious for the case of different bases. See [14, §4.3] for the definition. Instead of it, we shall use the following statement:

**Proposition 4.1.** Suppose \( E_1 \to M_1 \) and \( E_2 \to M_2 \) are Lie algebroids defined by homological vector fields \( Q_1 \in \text{Vect} \, \Pi E_1 \) and \( Q_2 \in \text{Vect} \, \Pi E_2 \). A vector bundle map given by the horizontal arrows of

\[
\begin{align*}
E_1 & \xrightarrow{\Phi} E_2 \\
\downarrow & \downarrow \\
M_1 & \xrightarrow{\varphi} M_2
\end{align*}
\]

...
is a morphism of Lie algebroids if and only if the vector fields $Q_1$ and $Q_2$ are $\Phi^\Pi$-related, where

$$\Phi^\Pi : \Pi E_1 \to \Pi E_2$$

is the induced map of the opposite vector bundles.

This statement first appeared, without a proof, in Vaintrob [22]. In our language, it is equivalent to saying that the map $\Phi^\Pi : \Pi E_1 \to \Pi E_2$ is a morphism of Lie antialgebroids. It is much easier to handle than the original definition of morphisms of Lie algebroids.

Recall that vector fields on (super)manifolds are related by a smooth map $F$ if the pull-back of functions $F^*$ commutes with them. In terms of the local flows $g_t$ and $h_t$ generated by these fields, this means that $F$ commutes with the flows: $g_t \circ F = F \circ h_t$.

Let us introduce a necessary notation. We have ‘horizontal’ Lie algebroid structures, i.e., in the vector bundles $D \to B$ and $A \to M$, and ‘vertical’, i.e., in $D \to A$ and $B \to M$. Hence we have homological vector fields $Q_{DB} \in \text{Vect}(\Pi B D)$, $Q_{AM} \in \text{Vect}(\Pi A)$, $Q_{DA} \in \text{Vect}(\Pi A D)$ and $Q_{BM} \in \text{Vect}(\Pi B)$. If we use the notation for coordinates from Section 2 then $Q_{DB}$ has weight 1 in variables $\xi^i, \theta^\mu$ and $Q_{AM}$, in variables $\xi^i$. Similarly, $Q_{DA}$ has weight 1 in variables $\eta^\alpha, \theta^\mu$ and $Q_{BM}$, in variables $\eta^\alpha$.

We shall study conditions I, II and III one by one.

4.1. Condition I. Condition I is the easiest for analysis.

Consider for concreteness the horizontal algebroid structures. Condition I requires that all vertical structure maps: bundle projections, zero sections, fiberwise addition and multiplication by scalars, give morphisms of Lie algebroids. We have the following diagrams to analyze:

\[
\begin{align*}
D \to B & \quad & D \to B \\
p \downarrow & \quad & p \downarrow \\
A \to M & \quad & i \uparrow \\
\end{align*}
\]

for morphisms

\[
\begin{align*}
D \to B & \quad & D \to B \\
\downarrow & \quad & \uparrow \\
A \to M & \quad & A \to M
\end{align*}
\]

and

\[
\begin{align*}
D \times_A D \to B \times_M B & \quad & D \to B \\
+ \downarrow & \quad & + \downarrow \\
D \to B & \quad & t_A \downarrow \\
\end{align*}
\]
for morphisms

\[
\begin{array}{ccc}
D \times_A D & \longrightarrow & B \times_M B \\
\downarrow & & \downarrow \\
D & \longrightarrow & B
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
D & \longrightarrow & B
\end{array}
\]

In the language of homological vector fields, we see that the flows generated by the vector fields \( Q_{DB} \) and \( Q_{AM} \) should commute with all the vertical structure maps above, more precisely, with the maps induced on the total spaces of the parity reversed horizontal vector bundles. Commuting with the projection means that (the flow of) the vector field \( Q_{DB} \) acts fiberwise on the total space of \( \Pi_B D \to \Pi_A \) and induces on \( \Pi_A \) (the flow of) the vector field \( Q_{AM} \). Hence \( Q_{AM} \) is completely determined by \( Q_{DB} \). Consider the action of the flow of \( Q_{DB} \) on the fibers of \( \Pi_B D \to \Pi_A \). Commutativity with fiberwise multiplication by scalars, \( t_A : \Pi_B D \to \Pi_B D \), and addition, \( +_A : \Pi_B D \times \Pi_A \Pi_B D \to \Pi_B D \), means that the flow of \( Q_{DB} \) is fiberwise linear (over \( \Pi_A \)). This is equivalent to the vector field \( Q_{DB} \) having weight 0 w.r.t. fiber coordinates on \( \Pi_B D \to \Pi_A \). Commutativity with the zero section \( \Pi A \to \Pi_B D \) then comes about automatically.

We may summarize: if the horizontal Lie algebroid structures are described by homological vector fields \( Q_{DB} \) and \( Q_{AM} \), then Condition I of Definition \( \Pi \) is equivalent to \( Q_{DB} \) having vertical weight 0 (its horizontal weight is 1) and \( Q_{AM} \) being the restriction of \( Q_{DB} \) to the base \( \Pi A \subset \Pi_B D \).

In coordinates we obtain the following expressions for \( Q_{DB} \) and \( Q_{AM} \):

\[
Q_{DB} = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{jk} \frac{\partial}{\partial \xi^k} + \left( \xi^i w^\alpha Q_{\alpha i} + \theta^\mu Q_{i}^\alpha \right) \frac{\partial}{\partial w^\beta} + \left( \frac{1}{2} \xi^i \xi^j w^\alpha Q_{j \alpha i} + \xi^i \theta^\mu Q_{i}^\lambda \right) \frac{\partial}{\partial \theta^\lambda}, \quad (25)
\]
a vector field of weight \((1,0)\) on \( \Pi_B D \), and

\[
Q_{AM} = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{jk} \frac{\partial}{\partial \xi^k}, \quad (26)
\]
a vector field on \( \Pi A \) of weight 1, the restriction of \( Q_{DB} \) on \( \Pi A \).

In the same way we deduce, for the vector fields \( Q_{DA} \in \text{Vect}(\Pi A D) \) and \( Q_{BM} \in \text{Vect}(\Pi B) \) describing vertical Lie algebroid structures, that \( Q_{DA} \) should have weight \((0,1)\) on \( \Pi A D \) and \( Q_{BM} \) be its restriction to \( \Pi B \). Hence the coordinate expressions:

\[
Q_{DA} = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \left( \eta^\alpha u^\gamma Q_\alpha^\gamma + \theta^\mu Q_{i}^\gamma \right) \frac{\partial}{\partial w^\beta} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta \alpha} \frac{\partial}{\partial \eta^\gamma} + \left( \frac{1}{2} \eta^\alpha \eta^\beta u^\gamma Q_{i \beta \alpha} + \eta^\alpha \theta^\mu Q_{i}^\lambda \right) \frac{\partial}{\partial \theta^\lambda}, \quad (27)
\]
and
\[ Q_{BM} = \eta^\alpha Q^\alpha_a \frac{\partial}{\partial x^a} + \frac{1}{2} \eta^\alpha \eta^\beta Q^\gamma_{\beta\alpha} \frac{\partial}{\partial \eta^\gamma}. \] (28)

We have recovered formulas (21)–(24).

**Remark 4.1.** Note that homological vector fields \( Q_{DB} \) and \( Q_{DA} \) determining the horizontal and vertical Lie algebroid structures are defined on different supermanifolds, \( \Pi_B D \) and \( \Pi_A D \). The crucial fact is that both have weight zero “in the other direction”. In particular, this allows to additionally reverse parity and obtain vector fields defined on a common domain \( \Pi^2 D \) (so that the commutativity condition makes sense).

4.2. **Condition II.** Consider Condition II of Definition 1. For diagram
\[
\begin{array}{ccc}
D & \xrightarrow{a} & TB \\
\downarrow & & \downarrow \\
A & \xrightarrow{a} & TM
\end{array}
\] (29)

which is supposed to give a Lie algebroid morphism
\[
\begin{array}{ccc}
D & \longrightarrow & TB \\
\downarrow & & \downarrow \\
A & \longrightarrow & TM
\end{array}
\] (30)

we first need to explicate the tangent prolongation Lie algebroid \( TB \to TM \). The definition is in [14, §9.7]. However, we shall use the following proposition instead.

**Proposition 4.2.** The tangent prolongation Lie algebroid \( TE \to TM \) of a Lie algebroid \( E \to M \) is given by the tangent prolongation of the corresponding homological vector field \( Q \in \text{Vect}(\Pi E) \), which is an (automatically homological) vector field on \( T(\Pi E) = \Pi_{TM} TE \).

Note that for any vector bundle \( E \to M \) taking tangents leads to a double vector bundle
\[
\begin{array}{ccc}
TE & \longrightarrow & E \\
\downarrow & & \downarrow \\
TM & \longrightarrow & M
\end{array}
\]
(see, e.g., [14, §3.4]), so partial parity reversions make sense.

By differentiating the field \( Q_{BM} \) given by (28), we immediately obtain a vector field \( \hat{Q}_{BM} \) on \( T(\Pi B) = \Pi_{TM} TB \),
\[
\hat{Q}_{BM} = \eta^\alpha Q^\alpha_a \frac{\partial}{\partial x^a} + \frac{1}{2} \eta^\alpha \eta^\beta Q^\gamma_{\beta\alpha} \frac{\partial}{\partial \eta^\gamma}
\]
\[
+ \left( \eta^\alpha Q^\alpha_a + \eta^\alpha \hat{Q}^\alpha_a \right) \frac{\partial}{\partial x^a} + \left( \eta^\alpha \eta^\beta Q^\gamma_{\beta\alpha} + \frac{1}{2} \eta^\alpha \eta^\beta \hat{Q}^\gamma_{\beta\alpha} \right) \frac{\partial}{\partial \eta^\gamma}, \] (31)
which is the homological vector field defining the tangent prolongation Lie algebroid in (33). Here, expressions such as $\dot{Q}_\alpha^a$ stand for $\dot{x}^k \partial_k Q_\alpha^a$, etc.

In order to simplify notation, below we shall write formulas for the case when $D, A, B, M$ are ordinary manifolds, not supermanifolds. This will allow us to avoid some signs. Obviously, everything carries over to the general super case.

Recall the formula for the anchor map $a: D \to TB$. From (25) we can extract

$$\dot{x}^a = u^i Q_i^a, \quad \dot{\eta}^\beta = u^i \eta^a Q_\alpha^\beta + \theta^\mu Q_\mu^\beta,$$

for the corresponding map $\Pi_A D \to T(\Pi B)$.

We see that the condition that (30) is a morphism of Lie algebroids translates into the condition that the map given by (32) relates the vector fields given by (27) and (31). After a simplification, this gives the following four equations:

$$Q_\beta^\beta Q_\alpha^a = Q_\mu^\beta Q_j^a,$$

$$Q_\alpha^\beta Q_\beta^a + Q_\beta^a \partial_\beta Q_\alpha^a = Q_\alpha^\beta \partial_\alpha Q_i^a + Q_i^a Q_\alpha^a,$$  

$$Q_\beta^a Q_{\beta \gamma} + \frac{1}{2} Q_\beta^a \partial_\beta Q_{\gamma \alpha} = Q_\alpha^a \partial_\alpha Q_{\beta \gamma} + Q_j^a Q_{\beta \gamma} + \frac{1}{2} Q_\beta^\alpha Q_\delta^\gamma + \frac{1}{2} Q_\gamma^\beta Q_\delta^\gamma,$$  

$$Q_{\alpha \beta}^a = -Q_{\beta \alpha}^a Q_\gamma^\alpha + Q_j^a Q_{\beta \gamma} - Q_\beta^a Q_\gamma^\alpha,$$

(square brackets denote alternation).

To complete the analysis of Condition II, we have to consider diagrams similar to (29) and (30):

$$\begin{array}{ccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
TA & \longrightarrow & TM
\end{array}$$

and

$$\begin{array}{ccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
TA & \longrightarrow & TM
\end{array}$$

where $A$ and $B$ have exchanged places. This adds two equations to the system (33) – (35):

$$Q_{\beta \alpha}^a Q_{\gamma \beta} + \frac{1}{2} Q_\beta^a \partial_\beta Q_{\gamma \alpha} = Q_\alpha^a \partial_\alpha Q_{\beta \gamma} + Q_j^a Q_{\beta \gamma} + \frac{1}{2} Q_\beta^\alpha Q_\delta^\gamma + \frac{1}{2} Q_\gamma^\beta Q_\delta^\gamma,$$

and

$$Q_{\alpha \beta}^a = -Q_{\beta \alpha}^a Q_\gamma^\alpha + Q_j^a Q_{\beta \gamma} - Q_\beta^a Q_\gamma^\alpha.$$  

(Note that equations (33), (34) are symmetric w.r.t. $A$ and $B$.)

The system of equations (33) – (38) is equivalent to Condition II of Definition 7. Notice that it is bilinear in vector fields $Q_{DA}$ and $Q_{DB}$. 
4.3. Condition III, and conclusion of the proof. Condition III requires “deciphering” more than the previous ones. First of all, we have to consider two diagrams:

\[ D^*A \longrightarrow K^* \]
\[ \downarrow \quad \downarrow \]
\[ A \longrightarrow M \]

(39)

and

\[ D^*B \longrightarrow B \]
\[ \downarrow \quad \downarrow \]
\[ K^* \longrightarrow M \]

(40)

and understand why the top horizontal arrow in (39) and the left vertical arrow in (40) are Lie algebroids.

Recall that the core \( K \) is a vector bundle over the base \( M \). In coordinates, \( K \) is described by \( x^a, z^\mu \) with the transformation law

\[ z^\mu = z^\nu T_{\nu\mu}^\mu(x^i) \]

obtained from (11) by setting \( u^i \) and \( w^\alpha \) to zero. When we dualize the vertical bundle \( D \rightarrow A \) in (11) we obtain the bundle \( D^*A \rightarrow A \) with fiber coordinates \( w^\alpha, z^\mu \) (with lower indices) so that the form \( w^\alpha w_\alpha + z^\mu z_\mu \) giving the pairing in coordinates, is invariant. We arrive at the following transformation laws

\[ w^\alpha' = T^\alpha_{\alpha'} w_\alpha + u^i T^\mu_{\alpha'} T_{\mu\nu}^\nu z_\mu \]

(41)

\[ z^\mu' = T^\mu_{\mu'} z_\mu \]

(42)

where \( (T^\mu_{\alpha'}) \) and \( (T^\mu_{\mu'}) \) are reciprocal matrices, and the transformation of \( u^i \) remains as in (9). This explains the double vector bundle structure of (39), in particular the vector bundle \( D^*A \rightarrow K^* \) (note that \( z_\mu \) can be considered as fiber coordinates for \( K^* \rightarrow M \)).

The same holds when we dualize over \( B \). The total space of the vector bundle \( D^*B \rightarrow B \) has coordinates \( x^a, u_i, w^\alpha, z_\mu \), the coordinates \( (u_i, z_\mu) \) being dual to \( (u^i, z^\mu) \) on \( D \). Hence the transformation law

\[ u'_i = T^i_{\alpha'} u_\alpha + w^\alpha T^\mu_{\alpha'} T_{\mu\nu}^\nu z_\mu \]

(43)

\[ z'_\mu = T^\mu_{\mu'} z_\mu \]

(44)

from which we immediately obtain the double vector bundle structure of (40). Treating \( D^*A \) and \( D^*B \) as bundles over \( K^* \), with fiber coordinates \( (u^i, w^\alpha) \) and \( (u_i, w^\alpha) \), respectively, we arrive at a surprising natural duality between them (see also [16] and [14, §9.2]), with the pairing given by the form

\[ u^i u_i - w^\alpha w_\alpha \]

(45)

(where the minus sign is absolutely essential, in order to cancel terms with \( z_\mu \) appearing in changes of coordinates).
The Lie algebroid structure in both $D^*A \to K^*$ and $D^*B \to K^*$ is a consequence of two facts: the above duality between $D^*A \to K^*$ and $D^*B \to K^*$ and the linearity over the base $K^*$ of the Poisson brackets induced on each $D^*A$ and $D^*B$ by the Lie algebroid structures in $D \to A$ and $D \to B$, respectively. After dualizing over $K^*$, these linear Poisson structures give Lie algebroid structures.

All this is readily expressed in our language. For example, the Lie algebroid structure in $D^*B \to K^*$ is given by a homological vector field on $\Pi K^*D^*B$, which is essentially the ‘transpose’ of the vector field $Q_{DA}$ on $\Pi A^D$. Indeed, $Q_{DA}$ generates linear transformations of the fibers of $\Pi B^D \to \Pi A^D$. In coordinates $x^a, \xi_i, \eta^\alpha, z^\mu$ on $\Pi K^*D^*A$ we get the field

$$Q = \eta^\alpha Q^a_\alpha \frac{\partial}{\partial x^a} - \left( \eta^a Q^i_{ai} \xi_j + \frac{1}{2} \eta^a \eta^\beta u^i Q^\lambda_{\beta|a} z_\lambda \right) \frac{\partial}{\partial \xi_i} + \frac{1}{2} \eta^a \eta^\beta Q^i_{\beta|o} \frac{\partial}{\partial \eta^o} - \left( Q^i_\mu \xi_j + \eta^a \theta^\mu Q^\lambda_{\alpha|a} z_\lambda \right) \frac{\partial}{\partial z^\mu}. \tag{46}$$

The Schouten bracket on $\Pi K^*D^*B$ corresponding to the Lie algebroid structure of $D^*A \to K^*$ (but initially arising from the Lie algebroid structure in $D \to B$) is defined by

$$\{x^a, \xi_i\} = Q^a_i \tag{47}$$

$$\{\xi_i, \xi_j\} = Q^i_{ij} \xi_l + \eta^a Q^\lambda_{aij} z_\lambda \tag{46}$$

$$\{\xi_i, \eta^a\} = \eta^\beta Q^a_{\beta i} \tag{47}$$

$$\{\xi_i, z_\mu\} = -Q^i_\mu z_\lambda \tag{47}$$

$$\{\eta^a, z_\mu\} = -Q^a_\mu, \tag{47}$$

the other brackets between coordinates being zero. (Formulas (46) and (47) are given for the setup where the initial vector bundles are purely even; the general case is similar but contains extra signs.)

Similarly, we can write down the homological vector field and Schouten bracket on $\Pi K^*D^*A$. The expressions will be ‘dual’ to (46) and (47).

One can immediately see that the field given by (46) is related with $Q_{BM}$ by the projection $\Pi K^*D^*B \to \Pi B$, which is the functor $\Pi$ applied to the projection $D^*B \to B$. Hence the horizontal projections in (46) give a Lie algebroid morphism. The same is true for the vertical projections in (46). We see that this part of Condition III holds automatically!

Let us examine the other part of the condition, that is, that the dual bundles $D^*A \to K^*$ and $D^*B \to K^*$ form a Lie bialgebroid. It turns out to be the main one. By the self-duality of the notion of a Lie bialgebroid, it suffices to check the two structures on one of the bundles. Taking $D^*B \to K^*$, we see that the bialgebroid condition is
equivalent to the vector field (46) being a derivation of the Schouten bracket given by (47). (We have to check not only (47), but also all the zero brackets such as \( \{ x^a, x^b \} = 0 \), etc.)

A direct calculation leads to the following set of nine equations:

\[
Q^\alpha_\mu Q^\mu_\alpha - Q^\mu_\alpha Q^\alpha_\mu = 0 \quad (48)
\]

\[
Q^\lambda_\mu Q^\mu_\nu + Q^\alpha_\mu Q^\mu_\nu + Q^\mu_\nu Q^\lambda_\mu + Q^\nu_\mu Q^\lambda_\nu = 0 \quad (49)
\]

\[
Q^\beta_\alpha Q^\alpha_\beta + Q^\nu_\alpha \partial_\beta Q^\alpha_\nu - Q^\gamma_\alpha Q^\alpha_\gamma = -Q^\alpha_\beta \partial_\alpha Q^\gamma_\beta \quad (50)
\]

\[
Q^\gamma_\alpha \partial_\beta Q^\gamma_\mu + Q^\nu_\alpha Q^\gamma_\nu - Q^\mu_\nu Q^\gamma_\mu = Q^\lambda_\mu Q^\gamma_\lambda \quad (51)
\]

\[
- Q^\alpha_\mu Q^\beta_\lambda - Q^\alpha_\nu Q^\gamma_\nu + Q^\beta_\lambda Q^\nu_\gamma + Q^\alpha_\nu Q^\gamma_\nu - Q^\beta_\alpha Q^\lambda_\nu - Q^\beta_\gamma Q^\lambda_\nu - Q^\alpha_\gamma Q^\lambda_\gamma - Q^\alpha_\beta Q^\gamma_\beta - Q^\beta_\gamma Q^\gamma_\beta = 0 \quad (52)
\]

\[
Q^\gamma_\alpha Q^\gamma_\mu - Q^\gamma_\lambda Q^\mu_\lambda - Q^\beta_\lambda Q^\gamma_\nu - Q^\mu_\nu Q^\gamma_\mu = Q^\alpha_\mu Q^\gamma_\alpha \quad (53)
\]

\[
- Q^\gamma_\alpha Q^\beta_\lambda - Q^\gamma_\nu Q^\gamma_\nu + Q^\beta_\lambda Q^\nu_\gamma + Q^\gamma_\nu Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\gamma - Q^\gamma_\nu Q^\gamma_\nu = Q^\alpha_\nu Q^\gamma_\nu + Q^\beta_\nu Q^\gamma_\nu + Q^\gamma_\nu Q^\gamma_\nu \quad (54)
\]

\[
Q^\gamma_\alpha Q^\gamma_\mu + Q^\gamma_\nu Q^\gamma_\nu + Q^\beta_\gamma Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\nu = Q^\alpha_\gamma Q^\gamma_\gamma + Q^\beta_\gamma Q^\gamma_\gamma + Q^\gamma_\gamma Q^\gamma_\gamma \quad (55)
\]

\[
Q^\gamma_\alpha Q^\gamma_\mu + Q^\gamma_\nu Q^\gamma_\nu + Q^\beta_\gamma Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\nu - Q^\beta_\gamma Q^\gamma_\nu = Q^\alpha_\gamma Q^\gamma_\gamma + Q^\beta_\gamma Q^\gamma_\gamma + Q^\gamma_\gamma Q^\gamma_\gamma \quad (56)
\]

**Condition III of Definition 4** is equivalent to equations (18)–(56). Notice that they contain equations (33)–(38), which are equivalent to Condition II. Hence, **Condition III contains Condition II**.

To conclude the proof of Theorem 1 it remains to consider the vector fields \( Q_1 \) and \( Q_2 \) on \( TN_D \), given by (21) and (22), and find their commutator. A direct calculation shows that the commutativity condition

\[
[Q_1, Q_2] = 0
\]

produces a system of equations that coincides with equations (18)–(56) of Condition III (and containing, as we showed, Condition II).

Hence we see that Mackenzie’s Definition 1 is equivalent to the commutativity of the homological fields \( Q_1 \) and \( Q_2 \), *quod erat demonstrandum*. 

5. The big picture

After presenting a ‘computational’ proof of our main statement, we shall now explain why it all works. The argument in this section gives an alternative proof of Theorem 1, almost without calculations.

Let us again consider a double vector bundle

\[ D \longrightarrow B \]
\[ \downarrow \quad \downarrow \]
\[ A \longrightarrow M \]

We shall assume that all sides of it have Lie algebroid structures and that each of these structures is compatible with the linear structure in the other direction. Thus we assume the obvious part of the definition of a double Lie algebroid (i.e., Condition I). As we have seen, this is equivalent to saying that the Lie algebroid structures on \( D \rightarrow A \) and \( D \rightarrow B \) are defined by homological vector fields of weights \((0, 1)\) and \((1, 0)\) on the ultimate total spaces of

\[ \Pi_A D \longrightarrow \Pi B \] \[ \Pi_B D \longrightarrow B \] \[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ A \longrightarrow M \] \[ \Pi A \longrightarrow M \]

respectively.

We want to formulate a ‘compatibility condition’.

Let us return for a moment to ordinary Lie algebroids (or just Lie algebras). Suppose \( E \rightarrow M \) is a vector bundle. It has three neighbors: the dual bundle \( E^* \), the opposite bundle \( \Pi E \) and the antidual \( \Pi E^* \). A Lie algebroid structure in \( E \) (which is a structure on the module of sections) is equivalently expressed by each of the following structures on its neighbors: a homological vector field of weight 1 on \( \Pi E \), a linear Poisson bracket on \( E^* \), and a linear Schouten bracket on \( \Pi E^* \). All axioms of a Lie algebroid are contained in the equation \( Q^2 = 0 \) or in the Jacobi identities for the Poisson or Schouten bracket.

Acting in a similar way, let us consider all the neighbors of our double vector bundle \((57)\). There are four operations that can be applied: vertical dual, horizontal dual, vertical reversion of parity, and horizontal reversion of parity. Besides \((58)\) one can obtain the following double vector bundles that are neighbors of \((57)\).

The complete parity reversion of \((57)\):

\[ \Pi^2 D \longrightarrow \Pi B \]
\[ \downarrow \quad \downarrow \]
\[ \Pi A \longrightarrow M \]
The two duals of (57):

\[ D^*A \longrightarrow K^* \quad \text{and} \quad D^*B \longrightarrow B \]

\[ A \longrightarrow M \quad K^* \longrightarrow M \]

(60)

The parity reversions of each of the duals:

\[ \Pi_A D^*A \longrightarrow \Pi K^* \]

\[ A \longrightarrow M \]

(61)

\[ \Pi_K \cdot D^*A \longrightarrow K^* \]

\[ \Pi A \longrightarrow M \]

(62)

\[ \Pi^2 D^*A \longrightarrow \Pi K^* \]

\[ \Pi A \longrightarrow M \]

(63)

and

\[ \Pi_K \cdot D^*B \longrightarrow \Pi B \]

\[ K^* \longrightarrow M \]

(64)

\[ \Pi_B D^*B \longrightarrow B \]

\[ \Pi K^* \longrightarrow M \]

(65)

\[ \Pi^2 D^*B \longrightarrow \Pi B \]

\[ \Pi K^* \longrightarrow M \]

(66)

This is the full list up to natural isomorphisms. These twelve objects, including the original double vector bundle (57), can be arranged into a four-valent graph. More precisely, from each vertex emanate two edges corresponding to taking duals and two edges corresponding to parity reversions. We can think of the edges being colored by two colors.

**Remark 5.1.** In the multiple case, for an n-fold vector bundle, the number of edges emanating from each vertex will be \( n + n = 2n \). Question: what is the total number of neighbors (the number of vertices in the graph)?
Lie algebroid structures on the sides of (57) satisfying the linearity conditions, which were expressed above in terms of weights, generate various combinations of structures on each of the neighbors (58)–(66), a pair for each. More symmetrically, we can say that each pair of structures for a particular double vector bundle from (57)–(66) is just a manifestation of one ‘double structure’. (All pairs contain the same information.) One can make a list of such structures. The next step will be to look for suitable compatibility conditions for each pair. The philosophy is that one should look for pairs where a compatibility condition is formulated naturally, and take it as the definition of compatibility for an equivalent pair where such a condition does not come about in an obvious way. Suppose we do not know what a double Lie algebroid is (a compatibility of Lie algebroids on the sides of (57)): for the right notion, examine its neighbors.

For each of the double bundles (58), the ultimate total space is a Lie algebroid (over the base $K^*$) and simultaneously possesses a linear Poisson bracket (linear over both bases). That means that the dual bundle over $K^*$ is also a Lie algebroid, and one may ask whether they form a Lie bialgebroid. As the analysis of the previous section shows, this may be considered as the Mackenzie definition of a double Lie algebroid for (57), since it subsumes all other conditions from Definition 1.

As the example of Lie algebras (or algebroids) shows, to define a Lie algebra or algebroid in terms of a structure on the manifold itself\footnote{We mean a structure in the algebra of functions as opposed to, say, a structure on sections of a vector bundle.}, one has to replace $E$ by one of the neighbors: $E^*$, $\Pi E$, or $\Pi E^*$, with the corresponding structure. For the “bi-” case (Lie bialgebras or bialgebroids), remarkably, it all boils down to one type of structure, should we take $E^*$, $\Pi E$, or $\Pi E^*$ as a model: namely, a QS-structure, i.e., a homological vector field and a Schouten bracket, with the derivation condition (see [23]).

Among the twelve double vector bundles (57)–(66) there are precisely five with a structure induced on the total space, thus allowing a verifiable compatibility condition. They are: (59), (62), (63), (64), (66).

On the total space of (59) there are two homological vector fields of weights $(0,1)$ and $(1,0)$. A compatibility condition for them is commutativity. This is precisely our Definition 2.

On the total space of (62) or (64) there is a Schouten bracket of weight $(-1,-1)$ and a homological vector field of weight $(0,1)$ or $(1,0)$, respectively. On the total space of (63) or (66) there is a Poisson bracket of weight $(-1,-1)$ and a homological vector field of weight $(0,1)$ or $(1,0)$. A compatibility condition in each case is the derivation property of the vector field w.r.t. the bracket.
Proposition 5.1. The compatibility conditions for (62), (64), (63) and (66) are equivalent, and are different ways of saying that \((D^A, D^B)\) is a Lie bialgebroid over \(K^*\).

Proof. Indeed, for any Lie bialgebroid \((E, E^*)\) the compatibility can be stated in terms of either \(E\) or \(E^*\) (as a QS-structure on either \(\Pi E\) or \(\Pi E^*\), respectively). This corresponds to (62) or (64) in our case. In our special case there is also an extra option of changing parity in the second direction, which adds (63) and (66) to the picture. □

We see now that there are essentially two conditions to compare: the Mackenzie bialgebroid condition, which lives on one of (62), (64), (63) and (66), and our commutativity condition for (59).

Proposition 5.2. The Mackenzie bialgebroid condition and the commutativity of the two homological vector fields for (59) are equivalent.

Proof. Consider one of the manifestations of the bialgebroid condition, say, for concreteness, (62). The derivation property means that the flow of the vector field preserves the bracket. On the other hand, the commutativity condition for (59) means that the flow of one field preserves the other. Now the claim follows from functoriality: notice that a linear transformation preserves a Lie bracket if and only if the adjoint map preserves the corresponding linear Poisson bracket and if and only if the ‘\(\Pi\)-symmetric’ map preserves the corresponding homological vector field. □

Propositions 5.1 and 5.2 together imply Theorem 4.

6. Applications and generalizations

In this section we consider some examples and applications, as well as discuss an extension to multiple Lie algebroids.

6.1. Doubles of Lie bialgebroids. Recall that Drinfeld’s classical double of a Lie bialgebra is again a Lie bialgebra with “good” properties. An analog of this construction for Lie algebroids turned out to be a puzzle. Three constructions of a ‘double’ of a Lie bialgebroid have been suggested. Suppose \((E, E^*)\) is a Lie bialgebroid over a base \(M\). Liu, Weinstein and Xu [4] suggested to consider as its double a structure of a Courant algebroid on the direct sum \(E \oplus E^*\). Mackenzie in [7, 8, 9, 11] and Roytenberg in [20] suggested two different constructions based on cotangent bundles. Though they look very different (in particular, Roytenberg’s double is a supermanifold, and Mackenzie stays in the classical world), we shall show now that they are essentially the same.

Roytenberg previously showed [20] that the Liu–Weinstein–Xu double is recovered from his own construction as a derived bracket, generalizing the results of C. Roger [19] and Y. Kosmann-Schwarzbach [11, 8] for MACKENZIE THEORY AND Q-MANIFOLDS
Lie bialgebras. Therefore, proving that the Mackenzie and Roytenberg pictures are equivalent or, actually, the same, if understood properly, shows conclusively that this ‘cotangent double’ is fundamental, and should be regarded as the correct extension of Drinfeld’s double of Lie bialgebras to Lie bialgebroids.

Both Roytenberg’s and Mackenzie’s construction use the statement that the cotangent bundles of dual vector bundles are isomorphic ([15], an extension of [21]; see also [12], [20], [23]). Hence there is a double vector bundle

\[ T^*E = T^*E^* \longrightarrow E^* \]

\[ \downarrow \quad \downarrow \]

\[ E \longrightarrow M \]

(67)

Mackenzie shows that it is a double Lie algebroid. He calls it the \textit{cotangent double} of a Lie bialgebroid \((E, E^*)\). Note that the canonical symplectic structure on \(T^*E\) corresponds to the invariant scalar product on Drinfeld’s double \(d(b) = b \oplus b^*\) of a Lie bialgebra \(b\).

On the other hand, Roytenberg uses the description of Lie algebroids via homological vector fields. He considers the double vector bundle

\[ T^*\Pi E = T^*\Pi E^* \longrightarrow \Pi E^* \]

\[ \downarrow \quad \downarrow \]

\[ \Pi E \longrightarrow M \]

(68)

and homological vector fields \(Q_E \in \text{Vect}(\Pi E)\) and \(Q_{E^*} \in \text{Vect}(\Pi E^*)\) defining Lie algebroid structures on \(E \to M\) and \(E^* \to M\), respectively. Recall that vector fields on a manifold correspond to fiberwise linear functions (Hamiltonians) on the cotangent bundle so that the commutator maps to the Poisson bracket. Denote the functions corresponding to \(Q_E\) and \(Q_{E^*}\) by \(H_E\) and \(H_{E^*}\), respectively. Roytenberg shows that under the natural symplectomorphism \(T^*\Pi E = T^*\Pi E^*\) the linear function \(H_{E^*}\) on \(T^*\Pi E^*\) corresponding to the vector field \(Q_E\) transforms precisely into the fiberwise quadratic function \(S_E\) on \(T^*\Pi E\) specifying the Schouten bracket on \(\Pi E\) induced by the Lie structure on \(E^*\). The derivation property of \(Q_E\) w.r.t. the Schouten bracket on \(\Pi E\) is one of the equivalent definitions of a Lie bialgebroid [2], and the most convenient. Hence, Roytenberg’s statement means that it is also equivalent to the commutativity of the Hamiltonians \(H_E\) and \(H_{E^*}\) under the canonical Poisson bracket. They generate commuting homological vector fields \(X_{H_E}\) and \(X_{H_{E^*}}\) on the cotangent bundle \(T^*\Pi E\). In our language, \(X_{H_E}\) and \(X_{H_{E^*}}\) make a Lie antialgebroid. (One can see that the conditions for weights are satisfied.) Roytenberg [20] calls the supermanifold \(T^*\Pi E = T^*\Pi E^*\) together with the homological vector field \(Q = X_{H_E} + X_{H_{E^*}}\) on it, the \textit{Drinfeld double} of \((E, E^*)\).

If we slightly refine Roytenberg’s picture, considering the double Lie antialgebroid given by \(X_{H_E}\) and \(X_{H_{E^*}}\) rather than a single \(Q\)-manifold,
we can immediately see that by Theorem 1 his picture becomes identical to that of Mackenzie.

Indeed, apply the complete reversion of parity to (67). Notice that \( \Pi^2 T^* E = \Pi^2 T^* E^* \) coincides with \( T^* \Pi E = T^* \Pi E^* \) (easily checked in coordinates). By Theorem 1 the double vector bundle (67) is a double Lie algebroid if and only if the corresponding double vector bundle

\[
\begin{array}{c}
\Pi^2 T^* E = \Pi^2 T^* E^* \quad \longrightarrow \quad \Pi E^* \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Example 6.1. Consider again the double vector bundle given by (67). Notice that the core of it is the cotangent bundle $T^*M \to M$. Take the two duals of (67). We obtain

$$
\begin{array}{c}
TE \longrightarrow TM \\
\downarrow \\
E \longrightarrow M
\end{array}
$$

for the vertical dual and

$$
\begin{array}{c}
TE^* \longrightarrow E^* \\
\downarrow \\
TM \longrightarrow M
\end{array}
$$

for the horizontal dual. Both double vector bundles (70) and (71) are known to be double Lie algebroids [11]. Three double Lie algebroids (67), (70), (71), which are double vector bundles in duality

$$
\begin{array}{c}
TE^* \longrightarrow TM \\
\downarrow \\
T^*E \longrightarrow E^* \\
\downarrow \\
E \longrightarrow M
\end{array}
$$

make a double bialgebroid (or ‘bi double Lie algebroid’), for example, in the sense that the triple vector bundle

$$
\begin{array}{c}
T^*T^*E \longrightarrow TE^* \\
\downarrow \\
T^*E \longrightarrow E^* \\
\downarrow \\
E \longrightarrow M
\end{array}
$$

where they make a corner, is a triple Lie algebroid. (It is the second cotangent double of $(E, E^*)$.)

This example should be considered as a preliminary announcement of results to follow in [17].

6.3. Higher Lie algebroids. We have already mentioned that the methods of this paper naturally allow us to consider multiple Lie algebroids. (A part of the motivation for doing this comes from the theory of doubles discussed above.) A detailed theory will be presented elsewhere. Here we wish to give a sketch.

First we need a language for describing multiple vector bundles. Fix a natural number $n$. To define $n$-fold vector bundles, consider vector spaces $V_r$, $V_{r_1r_2}$, $V_{r_1r_2r_3}$, $\ldots$, of arbitrary dimensions $d_r, d_{r_1r_2}, d_{r_1r_2r_3},$
etc., numbered by increasing sequences \( r_1 < \ldots < r_k \), where \( 0 < k \leq n \) and all \( r_i \) run from 1 to \( n \).

**Example 6.2.** When \( n = 1 \), we have just one vector space \( V = V_1 \). When \( n = 2 \), we have \( V_1, V_2 \) and \( V_{12} \). For \( n = 3 \), we have 7 spaces: \( V_1, V_2, V_3, V_{12}, V_{13}, V_{23}, \) and \( V_{123} \). In general the number of spaces is \( 2^n - 1 \).

For convenience of notation let us fix linear coordinates on each of the spaces, denoting them \( v^i_{(r)}, v^i_{(r_1 r_2)}, \) etc. (Each index such as \( i_r \) runs over its own set of values, of cardinality equal to the dimension of the respective space.)

**Definition 3.** An \( n \)-fold vector bundle over a base \( M \) is a fiber bundle over \( M \) with the standard fiber

\[
\prod_r V_r \times \prod_{r_1 < r_2} V_{r_1 r_2} \times \ldots \times V_{12 \ldots n}
\]

where the transition functions have the form:

\[
v^i_{(r)} = v^i_{(r)} T^i_{i'_{r}},
\]

\[
v^i_{(r_1 r_2)} = v^i_{(r_1 r_2)} T^i_{i'_{r_1 r_2}} + v^i_{(r_1)} v^i_{(r_2)} T^i_{i'_{r_2} i'_{r_1}},
\]

\[
\ldots \ldots \ldots 
\]

\[
v^i_{(12 \ldots n)} = v^i_{(12 \ldots n)} T^i_{i'_{12 \ldots n}} + \ldots + v^i_{(1)} \ldots v^i_{(n)} T^i_{i'_{n} \ldots i'_{1}}.
\]

In other words, the transformation for each of \( V_r \) is linear; for \( V_{r_1 r_2} \) it is linear plus an extra term bilinear in \( V_{r_1} \) and \( V_{r_2} \), etc.

**Example 6.3.** For a triple vector bundle \( (n = 3) \), we have fiber coordinates: \( v^i_{(1)}, v^i_{(2)}, v^i_{(3)}, v^i_{(12)}, v^i_{(13)}, v^i_{(23)}, \) and \( v^i_{(123)} \). The transformation law is as follows:

\[
v^i_{(1)} = v^i_{(1)} T^i_{i'_{1}},
\]

\[
v^i_{(2)} = v^i_{(2)} T^i_{i'_{2}},
\]

\[
v^i_{(3)} = v^i_{(3)} T^i_{i'_{3}},
\]

\[
v^i_{(12)} = v^i_{(12)} T^i_{i'_{12}} + v^i_{(1)} v^i_{(2)} T^i_{i'_{2} i'_{1}},
\]

\[
v^i_{(13)} = v^i_{(13)} T^i_{i'_{13}} + v^i_{(1)} v^i_{(3)} T^i_{i'_{3} i'_{1}},
\]

\[
v^i_{(23)} = v^i_{(23)} T^i_{i'_{23}} + v^i_{(2)} v^i_{(3)} T^i_{i'_{3} i'_{2}},
\]

\[
v^i_{(123)} = v^i_{(123)} T^i_{i'_{123}} + v^i_{(1)} v^i_{(23)} T^i_{i'_{23} i'_{1}} + v^i_{(2)} v^i_{(13)} T^i_{i'_{13} i'_{2}} + v^i_{(3)} v^i_{(12)} T^i_{i'_{12} i'_{3}}
\]

\[+ v^i_{(1)} v^i_{(2)} v^i_{(3)} T^i_{i'_{1} i'_{2} i'_{3}}.\]
Remark 6.1. Triple vector bundles — with the quaternary case briefly mentioned — were introduced and studied in [13] from a different viewpoint (not using local trivializations and transition functions). Paper [13] also contains conjectured ‘likely principles’ of duality for general multiple case.

A multiple vector bundle has faces, which are also multiple vector bundles. A face is obtained by choosing indices \( r_1 < \ldots < r_k \); fiber coordinates for it will be the coordinates \( v^{i_1 \ldots i_k}_{(r_1 \ldots r_k)} \) and all other coordinates labelled by subsets of \( r_1, \ldots, r_k \). For example, for a triple vector bundle there are faces that are (ordinary) vector bundles and double vector bundles, corresponding to the edges and 2-faces of a 3-cube. In a natural way various partial projections and zero sections are defined.

The total space of a multiple vector bundle is a multi-graded manifold. More precisely, there are weights \( w_{r}, r = 1, \ldots, n \), each of them being a degree in all coordinates containing a given label \( r \). For example, \( w_2 \) is the total degree in \( v^{(2)}, v^{(12)}, v^{(23)}, \ldots, v^{(12 \ldots n)} \). We define total weight as \( w = w_1 + \ldots + w_n \).

Due to the multilinearity of transition functions, for a multiple vector bundle the operations of partial parity reversion \( \Pi_r \) and partial dual \( D_r \) in the \( r \)-th direction, make sense for each \( r = 1, \ldots, n \).

Definition 4. An \( n \)-fold Lie antialgebroid is an \( n \)-fold vector bundle endowed with a homological vector field \( Q \) of total weight 1 on the total space \( E \).

Clearly, this is the same as having \( n \) odd vector fields \( Q_r \) of weights \((0, \ldots, 1, \ldots, 0)\) such that

\[ [Q_r, Q_s] = 0 \]

for all \( r, s \).

Definition 5. An \( n \)-fold Lie algebroid is an \( n \)-fold vector bundle such that the \( n \)-fold vector bundle obtained by the complete parity reversion \( \Pi^n = \Pi_1 \ldots \Pi_n \) is an \( n \)-fold Lie antialgebroid.

In other words, we take the statement of Theorem 1 as a definition for the multiple case.

Each face of a multiple Lie (anti)algebroid is also a multiple Lie (anti)algebroid.

We expect the following to be true: the possibility to define multiple Lie algebroids à la Mackenzie, via duals and bialgebroids, and the equivalence of such a definition with Definition 4 (i.e., the analog of Theorem 1). This requires an analysis of the structures induced on all neighbors of a multiple Lie algebroid.

An \( n \)-fold Lie bialgebroid (or a “bi-” \( n \)-fold Lie algebroid) can be defined, in supergeometry terms, as an \( n \)-fold Lie algebroid such that all
its duals are also \( n \)-fold Lie algebroids with the compatibility condition that reads as follows: on the total space with completely reversed parity there is a homological vector field \( Q \) of total weight 1, which defines the algebroid structure, and an odd or even (depending on the parity of the number \( n \)) bracket of an appropriate weight, which corresponds to algebroid structures on all the duals; the field \( Q \) should be a derivation of the bracket. Shortly, it can be described as a \( QS \)- or \( QP \)-structure on the total space with particular conditions on weights.

It should be possible to prove that this is equivalent to the cotangent double being an \((n + 1)\)-fold Lie algebroid. (Which we used as a definition in Example 6.1.) The main statement then should be that the \textbf{general principle} stated above holds: that the cotangent double is, moreover, a “bi-” \((n + 1)\)-fold Lie algebroid. See [17] for all this.

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School of Mathematics, University of Manchester, Sackville Street, Manchester, M60 1QD, United Kingdom
E-mail address: theodore.voronov@manchester.ac.uk