Optics in nonuniform media 
and Lagrange geometry

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Abstract
In this paper the equations of motion associated with a Lagrangian inspired by relativistic optics in nonuniform moving media are considered. The model describes optical effects in the nonuniform dispersionless moving medium. When using the optical metric restricted to the Minkowski manifold, we have established the Euler-Lagrange equations for geodesics. We have specified the general model to the special case when the refractive index increases along the direction $Z$. The exact analytical solutions of the corresponding Euler-Lagrange equations have been constructed. Analysis of the solutions shows that the light beams are bending to the axes $Z$ along which the refractive index increases.

Mathematics Subject Classification (2010): 53C60, 53C80, 83C10.
Keywords and phrases: optical metric, Minkowski metric, nonuniform media, anisotropic optics, Lagrange geometry.

1 Introduction

In geometrical optics [7], a special role is played by the Beil metric (see [1]-[4], [6], [8]-[10])

$$g_{\alpha\beta}(x, y) = \varphi_{\alpha\beta}(x) + \gamma(x)y_\alpha y_\beta,$$

where $\gamma(x) \geq 0$ is a smooth function on the space-time $M^4$, and $\varphi_{\alpha\beta}(x)$ is a pseudo-Riemannian metric on $M^4$. One assumes that the manifold $M^4$ is endowed with local coordinates $(x^\alpha)_\alpha = (x^1, x^2, x^3, x^4)$, and $(y^\alpha)$ is the Liouville vector field on the total space of the tangent bundle $TM^4$; the following rule holds $y_\alpha = \varphi_{\alpha\mu}y^\mu$. Since the components of $\varphi_{\alpha\beta}(x)$ are dimensionless, the same are $\gamma y_\alpha$; so we have dimensionless combinations $[\varphi_{\alpha\beta}(x)] = 1$, $[\gamma y_\alpha] = 1$.

In this context, let us restrict our study to the Minkowski manifold $M^4 = (\mathbb{R}^4, \eta_{ij})$ which has the local coordinates $(x) := (x^i)_{i = 1}^4$. The dimension of the corresponding tangent bundle $T\mathbb{R}^4$ is equal to eight, and its local coordinates
are
\[(x, y) := (x^i, y^i)_{i=1,4} = (x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4).\]

Emerging from formula (1), we introduce the following metric on \(T\mathbb{R}^4\), which is inspired by the optics framework developed in the papers [6],[9],[11],[15]–[18] for the nonuniform moving medium:

\[g_{ij}(x, y) = \eta_{ij} + \gamma^2(x) y_i y_j, \tag{2}\]

where \(\eta = (\eta_{ij}) = \text{diag} (-1,1,1,1)\) is the Minkowski metric, and \(y_i = \eta_{ir} y^r\). Commonly, the following parametrization for \(\gamma^2(x)\) is used

\[\gamma^2(x) = \frac{1}{c^2} \left( 1 - \frac{1}{n^2(x)} \right), \tag{3}\]

where \(n = n(x)\) is interpreted as the local refractive index of the nonuniform medium (see [2], [7], [11]–[14]).

From the physical point of view, geometrical optics in moving media is an interesting object because the effects of velocity vector field are similar to the action of gravitational or magnetic fields on charged matter waves. In paper [8], the Lagrangian and the metric related to the light in moving dispersion less media have been established and the gravitation-like effects for the light deflection at a vortex has been studied (so called an optical black hole). It should be noted that in such models [8] the second term in the Gordon’s optical metric \(g_{ij} = \eta_{ij} + (1 - n^{-2}) u_i u_j\) (4)
describes medium velocity effects; here the four-velocity is defined by the formula

\[u_i(x) = \alpha \left( 1, -\frac{u}{c} \right), \quad \alpha = \left( 1 - \frac{u^2}{c^2} \right)^{-1/2}.\]

In the sequel, we will examine the special case of an anisotropic dynamical model, which is governed by the Lagrangian [13]:

\[L(x, y) = \frac{1}{2} g_{ij}(x, y) y^i y^j = \frac{1}{2} (\eta_{ij} + \gamma^2 y_i y_j) y^i y^j = \frac{1}{2} \eta_{ij} y^i y^j + \frac{\gamma^2}{2} \|y\|^4, \tag{5}\]

where the notation is used

\[\|y\|^2 = -(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = \eta_{ij} y^i y^j.\]

Assuming that the refractive index \(n(x)\) is invariant with respect to Lorentz transformations, we conclude that the Lagrangian [9] is also invariant. A similar

\[\text{In this paper, the Latin letters } i, j, \ldots \text{ run from 1 to 4. The Einstein convention of summation is adopted. In order to eliminate the confusion between indices and powers, for the space-time coordinates we will use the notations: } x^i := x^{(i)}, \forall i = 1, 4.\]
3-dimensional Lagrangian was studied in [14], the corresponding non-relativistic Lagrangian being invariant with respect to the orthogonal group $O(3)$. The Lagrangian (5) produces the fundamental metric

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} = \sigma(x, y) \eta_{ij} + 2\gamma^2(x) y_i y_j,$$

where $\sigma(x, y) = (1/2) + \gamma^2(x) ||y||^2$. With notation $\tau(x, y) = (1/2) + 3\gamma^2(x) ||y||^2$, for the inverse matrix $[g^{-1}] = (g^{jk})_{j,k=1,4}$ we find

$$g^{jk}(x, y) = \frac{1}{\sigma(x, y)} \eta^{jk} - \frac{2\gamma^2(x)}{\sigma(x, y) \cdot \tau(x, y)} y^i y^k,$$

where we assume that $\sigma \cdot \tau \neq 0$. The Euler-Lagrange equations associated with the Lagrangian (5) can be written in the form (see [2])

$$\frac{d^2 x^i}{dt^2} + 2G^i(x(t), y(t)) = 0,$$

where $G^i$ is defined by the formula

$$G^i(x, y) \overset{\text{def}}{=} \frac{g^{ik}}{4} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^k} \right) = \frac{\gamma}{\sigma} ||y||^4 y^i (\gamma s y^s) - \frac{3}{2} \frac{\gamma^3}{\sigma \tau} y^i ||y||^4 (\gamma s y^s) - \frac{\gamma}{4\sigma} ||y||^4 \gamma_i \eta_{ii},$$

here $\gamma_s = \partial \gamma/\partial x^s$. The geometrical quantity $G^i(x, y)$ has the meaning of the semispray on the tangent space $T\mathbb{R}^4$ (see [10], [3]).

2 The Euler-Lagrange equations

The Euler-Lagrange equations (6) for the variables

$$(y^1, y^2, y^3, y^4) = (V^{(1)}, V^{(2)}, V^{(3)}, V^{(4)}) := V, \quad V^{(i)} = \frac{dx^{(i)}}{dt},$$

$$v^2 = |V|^2 = -(V^{(1)})^2 + (V^{(2)})^2 + (V^{(3)})^2 + (V^{(4)})^2,$$

lead to the following equations

$$\frac{dV^{(i)}}{dt} + \frac{4\gamma v^2 (1 + 3\gamma^2 v^2)}{(1 + 2\gamma^2 v^2)(1 + 6\gamma^2 v^2)} \left( \gamma_s V^{(s)} V^{(i)} \right) - \frac{\gamma v^4}{1 + 2\gamma^2 v^2} \gamma_i \eta_{ii} = 0,$$

where $\gamma_s = \partial \gamma/\partial x^s$. The geometrical quantity $G^i(x, y)$ has the meaning of the semispray on the tangent space $T\mathbb{R}^4$ (see [10], [3]).
where $t$ is some evolution parameter. In detailed form the equations (7) read as

$$
\begin{align*}
\frac{dV^{(1)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} (\gamma s V(s)) V^{(1)} + \frac{\gamma v^4}{1 + 2\gamma^2v^2} \gamma_1 &= 0, \\
\frac{dV^{(2)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} (\gamma s V(s)) V^{(2)} - \frac{\gamma v^4}{1 + 2\gamma^2v^2} \gamma_2 &= 0, \\
\frac{dV^{(3)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} (\gamma s V(s)) V^{(3)} - \frac{\gamma v^4}{1 + 2\gamma^2v^2} \gamma_3 &= 0, \\
\frac{dV^{(4)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} (\gamma s V(s)) V^{(4)} - \frac{\gamma v^4}{1 + 2\gamma^2v^2} \gamma_4 &= 0.
\end{align*}
$$

(8)

Note that, in the simplest case of the uniform medium with a constant refractive index $n(x) = n_0$, the above equations become

$$
\begin{align*}
\frac{dV^{(i)}}{dt} &= 0 \Leftrightarrow V = (V^{(1)}, V^{(2)}, V^{(3)}, V^{(4)}) = \text{constant} \Leftrightarrow \\
\frac{dx^{(i)}}{dt} &= V^{(i)} \Leftrightarrow x(t) = (V^{(1)} t + x_0^{(1)}, V^{(2)} t + x_0^{(2)}, V^{(3)} t + x_0^{(3)}, V^{(4)} t + x_0^{(4)});
\end{align*}
$$

in this case, the geodesics are the straight lines.

3 Nonuniform nondispersive medium

We will consider a nonuniform dispersionless medium, whose refractive index $n(x)$ depends only on space coordinates $(x^{(2)}, x^{(3)}, x^{(4)}) = (X, Y, Z)$.

Let $\gamma = \gamma_4 x^{(4)} = \gamma_4 Z$, this means that the function $\gamma(x)$ linearly increases on the $Z$-axis. From the relation (3), one can find the explicit dependence of the refractive index on $Z$:

$$
n^2 = \frac{1}{1 - c^2 \gamma_4^2 Z^2}.
$$

The shapes of this dependence at different $\gamma_4$ are shown in Figure 11.

Then the system (8) takes the form

$$
\begin{align*}
\frac{dV^{(1)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} \gamma_4 V^{(1)} V^{(4)} &= 0, \\
\frac{dV^{(2)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} \gamma_4 V^{(2)} V^{(4)} &= 0, \\
\frac{dV^{(3)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} \gamma_4 V^{(3)} V^{(4)} &= 0, \\
\frac{dV^{(4)}}{dt} + \frac{4\gamma v^2(1 + 3\gamma^2v^2)}{(1 + 2\gamma^2v^2)(1 + 6\gamma^2v^2)} \gamma_4 V^{(4)} V^{(4)} - \frac{\gamma v^4}{1 + 2\gamma^2v^2} \gamma_4 &= 0.
\end{align*}
$$

(9)

3 By the symmetry of the system (8), we can treat by analogy the following similar cases: $\gamma = \gamma_3 Y$ or $\gamma = \gamma_2 X$. 

4
For shortness, in the following we use the notation $v^2 \equiv W$. From (9) we can derive the following equations for variables $W$ and $V^{(4)}$:

\[
\begin{align*}
\frac{dW}{dt} + \frac{8\gamma (1 + 3\gamma^2 W)}{(1 + 2\gamma^2 W)(1 + 6\gamma^2 W)} \gamma_4 V^{(4)} W^2 - \frac{2\gamma}{1 + 2\gamma^2 W} \gamma_4 V^{(4)} W^2 &= 0, \\
\frac{dV^{(4)}}{dt} + \frac{4\gamma (1 + 3\gamma^2 W)}{(1 + 2\gamma^2 W)(1 + 6\gamma^2 W)} \gamma_4 (V^{(4)})^2 W - \frac{\gamma}{1 + 2\gamma^2 W} \gamma_4 W^2 &= 0,
\end{align*}
\]

or differently

\[
\begin{align*}
\frac{dW}{dt} + \frac{6\gamma \gamma_4}{1 + 6\gamma^4 W} V^{(4)} W^2 &= 0, \\
\frac{dV^{(4)}}{dt} + \frac{4\gamma (1 + 3\gamma^2 W)}{(1 + 2\gamma^2 W)(1 + 6\gamma^2 W)} \gamma_4 (V^{(4)})^2 W - \frac{\gamma}{1 + 2\gamma^2 W} \gamma_4 W^2 &= 0.
\end{align*}
\]

Taking into account the identities

\[
\gamma = \gamma_4 x^{(4)}, \quad V^{(4)} = \frac{dx^{(4)}}{dt}, \quad x^{(4)} = Z,
\]

we rewrite the equations (11) as follows

\[
\begin{align*}
\frac{dW}{dt} + \frac{6Z \gamma^2}{1 + 6\gamma^4 Z^2 W} W^2 \frac{dZ}{dt} &= 0, \\
\frac{d^2 Z}{dt^2} + \frac{4\gamma_4 Z (1 + 3\gamma^2 Z^2 W)}{(1 + 2\gamma^2 Z^2 W)(1 + 6\gamma^2 Z^2 W)} \gamma_4 \left(\frac{dZ}{dt}\right)^2 W - \frac{\gamma_4 Z}{1 + 2\gamma^2 Z^2 W} \gamma_4 W^2 &= 0.
\end{align*}
\]

(12)

To resolve the first equation in (12), let us make two substitutions:

\[
3\gamma^2 Z^2 = F_1, \quad \frac{1}{W} = F_2 \implies F'_1 = 6\gamma^2 ZZ', \quad W' = -\frac{1}{F_2^2} F'_2,
\]

where the derivative over $t$ is denoted by a prime. The first equation in (12) takes the form

\[
-F'_2 + F'_1 \frac{1}{1 + 2F_1/F_2} = 0.
\]

Making an additional substitution $F_1/F_2 = G$, we arrive at a differential equation with separable variables:

\[
\frac{F'_1}{F_2} = \frac{G'}{1 + G} \implies G = c_1 F_2 - 1.
\]

Further turning to the initial variables $Z$ and $W$, we find the following relation between the variables $Z$ and $W$:

\[
Z^2 = \frac{1}{3\gamma^2} \frac{c_1 - W}{W^2}.
\]

(13)
where $c_1$ is an arbitrary constant. From the physical point of view, the variable $Z$ should be real one, so the difference $c_1 - W$ has to be positive. At a chosen metric signature $v^2 = W = -(V^{(1)})^2 + (V^{(2)})^2 + (V^{(3)})^2 + (V^{(4)})^2$ and assuming $V^{(1)} \equiv c^2$, where $c$ is the velocity of the light in the vacuum, one should conclude that $W < 0$. From the other side, $Z^2 > 0$, so the difference $c_1 - W$ should satisfy the requirement of positiveness: $c_1 > W$.

![Figure 1: (a) Dependence of the refractive index $n$ on coordinate $Z$ at different values of $\gamma_4$; (b) and (c) Trajectory of a ray. We use the following parameters: $c = 1$; $\gamma_4 = 0.06$ (dotted line), 0.03 (dashed line), 0.01 (solid line). In (b) $c_1 = -0.1$; $c_2 = 1$ (dotted line), 0.2 (dashed line), 0.02 (solid line). In (c) $c_1 = 0$; $Z_0 = 15$; $X_0 = 0$; $V_{0(4)} = -0.9$; $V_{0(2)} = 0.05$](image)

From the relation (13) it follows the expression for $W$:

$$W = \frac{-1 \pm \sqrt{12c_1\gamma_4^2Z^2 + 1}}{6\gamma_4^2Z^2}. \quad (14)$$

Let us introduce a new variable $f$ defined as

$$f^2 = 12c_1\gamma_4^2Z^2 + 1. \quad (15)$$

Then the formula (14) takes the form:

$$W = \frac{2c_1(f - 1)}{f^2 - 1}; \quad (16)$$

here the negative values of $f$ correspond to a sign "-" in (14), while positive values of $f$ are in the range of $W$ defined by a sign "+". Taking into account $W < 0$, one finds that $f < 1$. As $Z^2$ should be positive, the following restriction on $f$ follows from the formula (15): at $c_1 > 0$, $f^2 > 1$; at $c_1 < 0$, $f^2 < 1$.

Substituting $W$ from (14) in the second equation of (12), we get the nonlinear differential equation of the second order for the variable $Z$:

$$Z'' + \frac{12c_1\gamma_4^2Z}{12c_1\gamma_4^2Z^2 + 2\sqrt{12c_1\gamma_4^2Z^2 + 1} + 1} Z'^2 - \left(\frac{\sqrt{12c_1\gamma_4^2Z^2 + 1} - 1}{\sqrt{12c_1\gamma_4^2Z^2 + 1} + 2}\right)^2 = 0; \quad (17)$$
in terms of new variable \( f \) we have

\[
f''f + \frac{(f^3 - 2f - 2)}{(f + 2)(f^2 - 1)} (f')^2 - \frac{12c_1^2 \gamma_2^2 (f - 1)}{(f + 1)(f + 2)} = 0. \tag{18}
\]

Taking in mind the identity

\[
\gamma_4^2 \gamma^2 = \gamma^2 = \frac{1}{c^2}(1 - n^{-2}),
\]

from (15) one finds

\[
12(1 - n^{-2}) = f^2 - 1, \quad f^2 = 13 - 12n^{-1}.
\]

At \( n = 1 \) we have \( f = 1 \). The increasing of the refractive index \( n \) leads to the rising of the variable \( f \), however we should remember that the value \( f \) is restricted by the inequality \( f^2 < 13 \), or \(-13 < f < 13\). Taking in mind the previously determined restrictions, one get \(-13 < f < -1\) at \( c_1 > 0 \) and \(-1 < f < 1\) at \( c_1 < 0 \).

Now, we will solve the equation (15). Because it does not contain the variable \( t \) explicitly, we can reduce the order of equation by means of the substitution \( f' \to p, \quad f'' \to pp' \), where \( p' = dp/df \). In this way, we obtain

\[
- \frac{12c_1^2 (f - 1) \gamma_2^2}{(f + 1)(f + 2)} + \frac{(f^3 - 2f - 2) p^2}{(f + 2)(f^2 - 1)} + fpp' = 0. \tag{19}
\]

The last equation transforms to a nonhomogeneous differential equation by means of the substitution \( p^2 \to K \):

\[
K' + \frac{2(f^3 - 2f - 2)}{f(f + 2)(f^2 - 1)} K - \frac{24c_1^2 \gamma_2^2 (f - 1)}{f(f + 1)(f + 2)} = 0, \tag{20}
\]

where \( K' = dK/df \). Whence it follows the solution

\[
K = f'^2 = \frac{1}{f^2(f + 2)^2} \left[ 24c_1^2 \gamma_2^2 (f^3 - 1) - c_2 (f^2 - 1) \right]. \tag{21}
\]

The last equation can be resolved implicitly in terms of the elliptic integrals \( E[\phi|m] \) and \( F[\phi|m] \) [5]:

\[
t = -\frac{\sqrt{-c_2 f^2 + c_2 + \Gamma^2 (f^3 - 1)}}{3 \Gamma^2} \times \left\{ -\frac{4c_2 - 2\Gamma^2(f + 5)}{f - 1} - \frac{iB (c_2 + 3\Gamma^2) \sqrt{(\Gamma^2(2f + 1) - c_2)^2 - A}}{\sqrt{f - 1} (c_2 (-f - 1) + \Gamma^2 (f^2 + f + 1))} \right\}
\]
where $E[\phi|m]$ stands for the elliptic integral of the second kind, $F[\phi|m]$ is the elliptic integral of the first kind. In the above formula the following parameters are used:

$$
\Gamma^2 = 24c_1^2c_2^2, \quad A = (c_2 - \Gamma^2)(c_2 + 3\Gamma^2), \quad B = \sqrt{\frac{c_2 - 3\Gamma^2}{\Gamma^2}}.
$$

The constants $c_2$ and $\Gamma(c_1)$ are defined by the initial conditions for $(Z, Z')$.

Now, we can find the corresponding expressions for the velocities $V^{(1)}$, $V^{(2)}$, $V^{(3)}$ and the coordinates $x^{(1)}$ and $X = x^{(2)}$, $Y = x^{(3)}$ (see equations (9)).

Because the first three equations in the system (9) have the same form, it is sufficient to solve one of them, let it be the equation for $V^{(2)} = \frac{dX}{dt}$. To this end, we transform the second equation in (9) to the variable $f$. Substituting the expressions (15), (14) in the second equation in (9) and taking into account that the derivative of $V^{(2)}$ over $t$ can be represented as

$$
\frac{dV^{(2)}}{dt} = \frac{dV^{(2)}}{df} \frac{df}{dt} = \frac{dV^{(2)}}{df} f',
$$

one get the equation

$$
\frac{dV^{(2)}}{df} + \frac{1}{f + 2} V^{(2)} = 0,
$$

whose solution is

$$
V^{(2)} = \frac{V_0^{(2)}}{f + 2}.
$$

Finally, we can find the coordinate $X$ from the equation

$$
\frac{dX}{dt} = V^{(2)},
$$

having used the identity

$$
\frac{dX}{dt} = \frac{dX}{df} \frac{df}{dt} = \frac{dX}{df} f',
$$

and the expression (21). In this way we obtain the equation

$$
\frac{dX}{df} = \frac{V_0^{(2)} f}{\sqrt{\Gamma^2(f^3 - 1) - c_2(f^2 - 1)}}.
$$
The solution of this equation reads

\[
X(f) = X_0 + V_0^{(2)} \left[ \frac{2\sqrt{\Gamma^2 (f^2 - 1) - c_2^2 (f^2 - 1)}}{\Gamma^2 (f - 1)} \right. \\
+ \frac{iB\sqrt{f - 1}}{2\Gamma^2 \sqrt{\Gamma^2 (f^2 - 1) - c_2^2 (f^2 - 1)}} \sqrt{(c_2 - \Gamma^2 (2f + 1))^2 - A} \\
\times \left( \sqrt{\frac{\sqrt{A} - 3c_2 + 3\Gamma^2}{c_2 - \frac{3\Gamma^2}{2}}} \right) F \left( i \sinh^{-1} \left( \frac{2B}{\sqrt{f - 1}} \right) |1 - \frac{2\sqrt{A}}{3\Gamma^2 - c_2 + \sqrt{A}}\right) \\
\left. + 4E \left( i \sinh^{-1} \left( \frac{2B}{\sqrt{f - 1}} \right) |1 - \frac{2\sqrt{A}}{3\Gamma^2 - c_2 + \sqrt{A}}\right) \right].
\]  

(26)

The trajectory \(Z(X)\) has the clear physical sense that it coincides with the trajectory of ray. Expressions (26) and (15) define the \(XZ\)-projection of the trajectory of the ray implicitly. Its behavior depends on the parameters \(\Gamma\) and \(c_2\) and is illustrated in Figure 1b. The ray deflects onto the direction of higher values of the refractive index and in some point total internal reflection occurs. By a symmetry reason, this behavior remains true for any axes-direction which influences (by increasing) the refractive index.

It should be emphasized that the obtained solution does not fulfilled for a particular case at \(c_1 = 0\) (\(\Gamma = 0\)). In this case from the formula (13) we obtain

\[
Z = \frac{1}{\sqrt{3\gamma_4}} \sqrt{-\frac{1}{W}} = \frac{1}{\sqrt{3\gamma_4 c\sqrt{1 - V^2/c^2}}},
\]

(27)

where \(V^{(1)} = c, V^2 = (V^{(2)})^2 + (V^{(3)})^2 + (V^{(4)})^2\).

From the relation (13) at \(c_1 = 0\) it follows the expression for \(W\) in the form:

\[
W = -\frac{1}{3\gamma_4 Z^2}.
\]

(28)

At substitution \(W\) from (28) in the second equation of (12), the nonlinear differential equation of the second order for the variable \(Z\) reduces to the follows:

\[
\frac{d^2 Z}{dt^2} - \frac{1}{3\gamma_4 Z^3} = 0.
\]

(29)

The solution of this equation is

\[
Z(t) = \sqrt{\frac{t^2}{3\gamma_4 Z_0^2} + \left( Z_0 + V_0^{(4)} t \right)^2},
\]

(30)

where \(Z_0\) and \(V_0^{(4)}\) denote coordinate \(Z\) and velocity along the \(Z\)-axis at initial moment \(t = 0\). Now, we can find the corresponding expressions for the velocities \(V^{(1)}, V^{(2)}, V^{(3)}\) and the coordinates \(x^{(1)}\) and \(X = x^{(2)}, Y = x^{(3)}\). To do this,
we substitute the expressions (28) and (30) in the equations (9) and take into account that \( \gamma = \gamma_4 x^{(4)} = \gamma_4 Z \). Then one obtains

\[
\frac{dV^{(1)}}{dt} = 0, \quad \frac{dV^{(2)}}{dt} = 0, \quad \frac{dV^{(3)}}{dt} = 0.
\]

So, we have

\[
x^{(1)} = V_0^{(1)} t + x_0^{(1)}, \quad X = V_0^{(2)} t + X_0, \quad Y = V_0^{(3)} t + Y_0.
\]

The obtained trajectories at different \( \gamma_4 \) are shown in Fig. 1c. As one can see, the type of the trajectories is similarly to the obtained in the previously case.

4 Conclusion

The model which describes optical effects in the nonuniform dispersionless moving medium has been studied. When using the optical metric restricted to the Minkowski manifold, we have established the Euler-Lagrange equation system for geodesics. We have specified the general model to the special case when the refractive index increases along the direction \( Z \). The exact analytical solutions of the corresponding Euler-Lagrange equations have been constructed. Analysis of the solutions shows that the light beams are bending to the axes \( Z \) along which the refractive index increase.

Acknowledgements. The present work was developed under the auspices of the Project BRFFR No. F20RA-007, within the cooperation framework between Romanian Academy and Belarusian Republican Foundation for Fundamental Research. Many thanks go to Professor Y.N. Obukhov, whose useful advice helped us to improve this paper.

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