A Characterization of the Unit Ball by a Kähler–Einstein Potential

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Abstract
We will show that a universal covering of a compact Kähler manifold with ample canonical bundle is the unit ball if it admits a global potential function of the Kähler–Einstein metric whose gradient length is a minimal constant. As an application, we will extend the Wong–Rosay theorem to a complex manifold without boundary.

Keywords The Kähler–Einstein metric · Complete holomorphic vector fields · The unit ball · Automorphism groups

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1 Introduction

Most, but not all, negatively curved compact Kähler manifolds are covered by bounded symmetric domains. Thus it is natural to distinguish bounded symmetric domains from exceptional spaces, such as the universal covering of the Mostow–Siu surface [15].

Dedicated to Professor Kang-Taee Kim on the occasion of his 65th birthday.

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From this point of view, bounded symmetric domains have been characterized as a bounded domain with relevant conditions to the Cartan/Harish-Chandra realization of Hermitian symmetric space of noncompact type \([6, 17, 18]\), and as an irreducible complex manifold with nontrivial, holomorphic transformation group \([7, 16]\). An important aspect in these studies is the fact that the Bergman metric of a bounded symmetric domain is the unique, biholomorphically invariant Kähler metric and its Ricci curvature is negative; thus a canonical bundle of any compact quotient is positive so ample. For the uniformization of a compact complex manifold with ample canonical bundle, it makes more sense to regard the Bergman metric as a complete Kähler–Einstein metric with negative Ricci curvature. In this paper, we shall characterize the unit ball (the bounded symmetric domain of rank 1) by an existence of a certain potential function of the Kähler–Einstein metric.

As pointed out by Kai–Ohsawa \([11]\) (see also Theorem 2.5), a bounded homogeneous domain (so symmetric one also) admits a global potential function of its Bergman metric whose gradient length is constant, and a value of constant gradient length is indeed uniquely determined. We shall show that the unit ball has the minimal value for constant gradient length among bounded symmetric domains of the same dimension (Lemma 2.6). Moreover we can characterize the unit ball by the same minimal value condition among universal coverings of compact Kähler manifolds with ample canonical bundle.

**Theorem 1.1** Let \(X^n\) be a simply connected complex manifold of dimension \(n\) which covers a compact complex manifold and admits a complete Kähler–Einstein metric \(\omega\) with negative Ricci curvature \(-K\). Suppose that there is a global potential function \(\varphi : X \rightarrow \mathbb{R}\) of \(\omega\) satisfying

\[
\|d\varphi\|^2_\omega \equiv \frac{2(n + 1)}{K}.
\]

Then, \(X\) is biholomorphic to the unit ball \(\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}\).

By Yau \([20]\), a compact Kähler manifold with ample canonical bundle has a Kähler–Einstein metric with negative Ricci curvature, so the lifted metric structure to its covering is complete. On the other way, Yau’s–Schwarz Lemma \([19]\) implies that a compact quotient of a complete Kähler–Einstein manifold with negative Ricci curvature is a Riemannian quotient, so its canonical bundle is positive. Thus, Theorem 1.1 characterizes the unit ball in the class of universal coverings of compact Kähler manifolds with ample canonical bundle. Usually, \(-1\) or \(-(n + 1)\) have been used as a normalized condition to the Ricci curvature for the uniqueness and the biholomorphical invariance, but we shall take an arbitrary negative constant \(-K\) because the minimal value \(2(n + 1)/K\) depends on \(K\) and we would like to indicate where the Ricci curvature is involved.

Due to the Nadel–Frankel theorem \([7, 16]\), the manifold \(X\) in Theorem 1.1 is a product space of a bounded symmetric domain and a complex manifold of discrete automorphism group (a rigid factor) since it covers a compact Kähler manifold with ample canonical bundle. On the other hand, the condition to the gradient length of the potential function \(\varphi\) allows us to construct a complete holomorphic vector field.
from the gradient vector field \( \text{grad}(\varphi) \) (Theorem 3.2 in [2], see also Theorem 2.7 of this manuscript). We will mainly show in the proof (Sect. 3.1) that the specific construction of \( V \) forces the rigid factor of \( X \) to be trivial, so \( X \) is equivalent to a bounded symmetric domain. Lemma 2.6, a characterization of the unit ball among bounded symmetric domains, implies that the unit ball is the only candidate to be biholomorphic to \( X \).

Theorem 1.1 gives an intrinsic generalization of the Wong–Rosay theorem [17, 18] which says that a smoothly bounded domain in \( \mathbb{C}^n \) which admits a compact quotient is biholomorphically equivalent to the unit ball.

Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. If the holomorphic automorphism group \( \text{Aut}(\Omega) \) of \( \Omega \) admits a discrete, torsion-free, cocompact subgroup \( \Gamma \), that is, the quotient complex space \( \Gamma \backslash \Omega \) is compact, then there is a sequence \( \{ f_j \} \subset \Gamma \) whose orbit \( \{ f_j(p) \} \) for \( p \in \Omega \) accumulates at a strongly pseudoconvex boundary point \( q \in \partial \Omega \). In [17, 18], it is proved that the asymptotic value of the ratio of Eisenman–Kobayashi and Carathéodory measures at a strongly pseudoconvex boundary point is 1. Since the ratio is invariant under automorphisms, one can see the ratio at \( p \) is 1 so the domain \( \Omega \) is biholomorphic to the unit ball. From an improvement by Efimov [5], we can directly construct a biholomorphism to the unit ball using the affine rescaling method even in case of a complex manifold with boundary (see [8]).

Here, we will try to extend the Wong–Rosay theorem to a complex manifold without boundary. Let us go back to the domain \( \Omega \subset \mathbb{C}^n \) and its cocompact subgroup \( \Gamma \subset \text{Aut}(\Omega) \) as above. Admitting a compact quotient \( \Gamma \backslash \Omega \) implies that \( \Omega \) is pseudoconvex so has a complete Kähler–Einstein metric with Ricci curvature \(-K\) by Cheng–Yau [1] and Mok–Yau [14]. Then, the gradient length \( \|d\varphi\|_{\omega}^2 \) of the canonical potential function \( \varphi \) of \( \omega \) is uniformly bounded near a strongly pseudoconvex boundary point \( q \) and the asymptotic value at \( q \) is \( 2(n+1)/K \):

\[
\lim_{z \to q} \|d\varphi\|_{\omega}^2(z) = \frac{2(n+1)}{K}
\]

(see Proposition 2.2). Using the method of potential rescaling as in [13], we can construct a global potential function \( \tilde{\varphi} \) of \( \omega \) whose gradient length is the minimal constant:

\[
\|d\tilde{\varphi}\|_{\omega}^2 \equiv \frac{2(n+1)}{K}.
\]

Since \( \Omega \) is simply connected [18], Theorem 1.1 implies that \( \Omega \) is the unit ball up to biholomorphic equivalence. More generally, we have

**Theorem 1.2.** Let \( X^n \) be a complex manifold equipped with the complete Kähler–Einstein metric \( \omega \) with negative Ricci curvature \(-K\) and suppose that there is a discrete, torsion-free, cocompact subgroup \( \Gamma \) of \( \text{Aut}(X) \). If

1. there is a sequence \( \{ f_j \} \subset \Gamma \) with a localizing neighborhood \( U \);
(2) there is a local potential function $\varphi : U \to \mathbb{R}$ of $\omega$ which has uniformly bounded gradient length and satisfies
\[
\lim_{j \to \infty} \|d\varphi\|^2_{\omega}(f_j(x)) = \frac{2(n + 1)}{K} \text{ for any } x \in X,
\]
then $X$ is a quotient space of the unit ball $\mathbb{B}^n$. If $U$ is simply connected in addition, then $X$ is biholomorphic to the unit ball.

Here, a localizing neighborhood $U$ of $\{f_j\}$ means that $f_j(K) \subset U$ for any compact subset $K \subset X$ with sufficiently large $j$. Thus the value $\|d\varphi\|^2_{\omega}(f_j(x))$ in (1.1) is defined well. In case of a domain $\Omega \subset \mathbb{C}^n$ with a cocompact subgroup $\Gamma$, an intersection $\Omega \cap U$ for any open neighborhood $U$ of a strongly pseudoconvex boundary point $q \in \partial \Omega$ should be a localizing neighborhood of a sequence $\{f_j\} \subset \text{Aut}(\Omega)$ whose orbit accumulates at $q$. As we mentioned, Condition (1.1) also holds for $\{f_j\}$.

In the proof of Theorem 1.2 (Sect. 3.2), we will apply the method of potential rescaling in [2, 13] to construct a global potential function with constant gradient length $2(n + 1)/K$. Then, Theorem 1.1 allows us to conclude that the universal covering space is equivalent to the unit ball.

The organization of this paper is as follows. We will introduce the gradient length of potentials, some relevant identities and basic materials for the main results in Sect. 2; especially the characterization of the unit ball among bounded symmetric domains by the minimal length constant $2(n + 1)/K$ (Sect. 2.3) and the existence of a complete holomorphic vector field in case of the minimal length constant (Sect. 2.4, see also [2]). Then we will prove Theorems 1.1 and 1.2 in Sect. 3.

2 A Kähler–Einstein Potential with Constant Gradient Length and the Existence of a Complete Holomorphic Vector Field

We will discuss now a negatively curved, complete Kähler–Einstein manifold admitting a global potential function with constant gradient length. In Lemma 2.6, we will characterize the unit ball among bounded symmetric domains by a minimal constant of gradient length. Then, the existence theorem of a complete holomorphic vector field in case of the minimal constant will be introduced.

Throughout this section, $X^n$ is an $n$-dimensional complex manifold and $\omega$ is its complete Kähler–Einstein metric with negative Ricci curvature $-K$, that is,
\[
\text{Ric}(\omega) = -K\omega,
\]
where $\omega$ also stands for its Kähler form. We will employ $\z = (\z^1, \ldots, \z^n)$ as a local coordinate system for local expressions of quantities. In this local coordinates, $(g_{\alpha\bar{\beta}})$ stands for the metric tensor of $\omega$:
\[
\omega = \sqrt{-1}g_{\alpha\bar{\beta}}d\z^\alpha \wedge d\z^{\bar{\beta}},
\]
where the indices $\alpha, \beta, \ldots$ run from 1 to $n$ and the summation convention for duplicated indices is always assumed. We denote the complex conjugate of a tensor by taking the bar on the indices: $\overline{z^\alpha} = z^\alpha$, $\overline{g_{\alpha\beta}} = g_{\overline{\alpha}\overline{\beta}}$. We will also use the matrix $(g_{\alpha\overline{\beta}})$ and its inverse matrix $(g^{\overline{\beta}\alpha})$ to raise and lower indices:

$$\varphi^\alpha = g^{\alpha\overline{\beta}} \varphi_{\overline{\beta}}, \quad R_{\overline{\beta}}^{\overline{\alpha}}{}_{\overline{\mu}\overline{\nu}} = g^{\overline{\gamma}\overline{\alpha}} R_{\overline{\beta}}^{\overline{\gamma}}{}_{\overline{\mu}\overline{\nu}}.$$

### 2.1 Gradient Length of a Potential Function

A potential function $\varphi$ of $\omega$ is a local smooth function satisfying

$$\dd c \varphi = \omega.$$

Here $\dd c = \frac{1}{2} (\overline{\partial} - \partial)$ so $\dd c = \frac{1}{2} i \partial \overline{\partial}$. Locally, a potential function is naturally given by

$$\varphi = \frac{1}{K} \log \det(g_{\alpha\overline{\beta}}) \quad (2.1)$$

since $\omega$ is Kähler–Einstein and the Ricci form is written by

$$\text{Ric}(w) = -\dd c \log \det(g_{\alpha\overline{\beta}}).$$

For a bounded pseudoconvex domain $\Omega$ in $\mathbb{C}^n$ and its Kähler–Einstein metric $\omega$, the function given by (2.1) with respect to the standard coordinates of $\mathbb{C}^n$ is a global potential function of $\omega$, so called a canonical potential of $(\Omega, \omega)$.

The gradient length of a potential function $\varphi$ is the length of the 1-form $d\varphi$ measured by $\omega$ which is locally expressed by

$$\|d\varphi\|^2_\omega = \left\| \varphi_\alpha dz^\alpha + \varphi_{\overline{\beta}} d\overline{z}^{\overline{\beta}} \right\|^2_\omega = 2 \varphi_\alpha \varphi_{\overline{\beta}} g_{\alpha\overline{\beta}} = 2 \varphi_\alpha \varphi^\alpha,$$

where $\varphi_\alpha = \partial \varphi / \partial z^\alpha$, $\varphi_{\overline{\beta}} = \partial \varphi / \partial \overline{z}^{\overline{\beta}}$. For the sake of simplicity, we will often employ

$$\|\partial \varphi\|_\omega^2 = \varphi_\alpha \varphi^\alpha = \frac{1}{2} \|d\varphi\|_\omega^2$$

as a gradient length.

**Remark 2.1** If $\omega'$ is another complete Kähler–Einstein metric with Ricci curvature $-K'$, then by the uniqueness of Kähler–Einstein metric [19], we have

$$K_\omega = K'_{\omega'}$$
since the Ricci curvature tensor is invariant under the scalar multiplication to the metric so

\[-K\omega = \text{Ric}(\omega) = \text{Ric}(\omega') = -K'\omega'.\]

Given potential \(\varphi'\) of \(\omega'\), a corresponding potential of \(\omega\) is of the form \(\varphi = K'\varphi'/K\), so we have a relation of their gradient length:

\[
\|\partial\varphi\|^2_\omega = \left(\frac{K'}{K}\right)^2 \|\partial\varphi'\|^2_\omega = \frac{K'}{K} \|\partial\varphi'\|^2_{\omega'}. \tag{2.2}
\]

### 2.2 Boundary Behavior of Gradient Length

Let us consider the Kähler–Einstein metric of the unit ball \(B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}\). The defining function \(\rho(z) = \|z\|^2 - 1\) gives a potential function

\[
\varphi_\rho = -\log(-\rho) = -\log(1 - \|z\|^2) = \frac{1}{n+1} \log \frac{1}{(1 - \|z\|^2)^{n+1}}, \tag{2.3}
\]

of the complete Kähler–Einstein metric \(\omega_{B^n}\) with Ricci curvature \(-(n + 1)\) (i.e. \(K = n + 1\)), whose metric tensor is given by

\[
g_{\alpha\bar{\beta}}^{B^n} = \frac{n+1}{(1 - \|z\|^2)^2} \left(\delta_{\alpha\bar{\beta}}(1 - \|z\|^2) + z^\alpha \bar{z}^\beta\right).
\]

One can easily see that \(\varphi_\rho\) is the canonical potential function of \(\omega_{B^n}\). From

\[
g_{\alpha\bar{\beta}}^{\mathbb{B}^n} = \frac{(1 - \|z\|^2)}{n+1} \left(\delta_{\alpha\bar{\beta}} - z^\alpha \bar{z}^\beta\right),
\]

we have

\[
\|\partial\varphi_\rho\|^2_\omega = \|z\|^2.
\]

This implies that the boundary value of \(\|\partial\varphi_\rho\|^2_\omega\) is 1. For the Kähler–Einstein metric \(\omega = \frac{n+1}{K}\omega_{B^n}\) with Ricci curvature \(-K\) and its potential function \(\varphi = \frac{n+1}{K}\varphi_\rho\), one can see that

\[
\|\partial\varphi\|^2_\omega(z) \to \frac{n+1}{K}
\]
as \(z\) tends to \(\partial B^n\) from (2.2).

In Proposition 4.3 in [2], we proved that such boundary behavior of gradient length still holds for any strongly pseudoconvex bounded domains. From [9], we can also have the same estimate near a strongly pseudoconvex boundary point.
Proposition 2.2 Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with $C^k$ boundary where $k \geq \max \{2n + 9, 3n + 6\}$ and let $\varphi$ be the canonical potential of the complete Kähler–Einstein metric $\omega$ with Ricci curvature $-K$. If $q \in \partial \Omega$ is a strongly pseudoconvex boundary point, then

$$\|\partial \varphi\|_{\omega}^2(z) \to \frac{n+1}{K}$$

as $z \to q$.

Proof This is proved in [2] when $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$ is a bounded strongly pseudoconvex domain with a smooth boundary by using the boundary behavior of the solution of the following complex Monge–Ampere equation [1]:

$$(\omega + dd^c u)^n = e^{\kappa u + F} \omega^n,$$

$$\omega + dd^c u > 0,$$  \hspace{1cm} (2.4)

where $\omega = -\log(-r)$ and $F = \log \det (r_{\alpha \bar{\beta}})(-r + |dr|^2)$. More precisely, if $r$ is a good defining function, which is called an approximate solution of the complex Monge–Ampere equation, satisfying

$$F = \log \det (r_{\alpha \bar{\beta}})(-r + |\partial r|^2) = O(|r|^{n+1}),$$

then $u$ satisfies that

$$|D^p u| (x) = O(|r|^{n+1/2-p-\varepsilon}) \quad \text{for} \quad \varepsilon > 0.$$  \hspace{1cm} (2.5)

Since $\varphi = -\log(-r) + u$, one can easily compute the boundary behavior of $\|\partial \varphi\|^2$.

In case that $\Omega$ is a bounded pseudoconvex domain with the hypothesis, Gontard proved that there exists a local approximate solution $r$ in a neighborhood $U \subset \mathbb{C}^n$ of $q \in \partial \Omega$ [9]. He also proved that the local solution $u$ of (2.4) in $U \cap \Omega$ with a local approximate solution $r$ satisfies the boundary behavior (2.5). Then the same computation as in the previous case gives the conclusion. (For the detailed proof, see Sect. 4 in [2].) \qed

2.3 The Minimal Value of Constant Gradient Length

For a bounded symmetric domain $\Omega$ in $\mathbb{C}^n$, the Bergman metric $\omega_{\Omega} = dd^c \log K_{\Omega}$ given by the Bergman kernel function $K_{\Omega}$ is the complete Kähler–Einstein metric. As an extended study of [4] for the $L^2$-cohomology vanishing, Donnelly [3] proved that the potential function $\log K_{\Omega}$ has bounded gradient length in order to get a kind of Kähler-hyperbolicity of the Bergman metric [10]. In the paper of Kai–Ohsawa [11], they considered the Cayley transform to a Siegel domain of the second kind. Since a homogeneous Siegel domain of the second kind is affine-homogeneous, its Bergman kernel function generates a potential function with constant gradient length. Moreover a value of constant gradient length does not depend on any choice of potentials (see
also Theorem 2.5). Under the normalized condition $-K$ to the Ricci curvature, we will prove that the gradient length constant of the unit ball is minimal among bounded symmetric domains.

Let us denote by $\Delta_\omega$ the Laplace–Beltrami operator of $\omega$ with non-positive eigenvalues which is locally written by

$$\Delta_\omega = g^{\alpha\bar{\beta}} \nabla_\alpha \nabla_{\bar{\beta}}$$

where $\nabla$ is the covariant derivative of $\omega$ and $\nabla_\alpha = \nabla_{\partial/\partial z^\alpha}$, $\nabla_{\bar{\beta}} = \nabla_{\partial/\partial \bar{z}^\beta}$.

**Proposition 2.3** (Proposition 3.1 in [2]) Let $\phi$ be a local potential function of a complete Kähler–Einstein manifold $(X^n, \omega)$ with negative Ricci curvature $-K$. Then

$$\Delta_\omega \left\| \partial \phi \right\|^2_\omega = \left\| \nabla^2 \phi \right\|^2_\omega + n - K \left\| \partial \phi \right\|^2_\omega .$$

Here, $\nabla'$ is the $(1, 0)$-part of $\nabla$. The length $\left\| \nabla^2 \phi \right\|^2_\omega$ can be locally written by

$$\left\| \nabla^2 \phi \right\|^2_\omega = \left\| (\nabla_\beta \nabla_\alpha \phi) dz^\alpha \otimes dz^\beta \right\|^2_\omega = \left\| \varphi_{\alpha;\beta} dz^\alpha \otimes dz^\beta \right\|^2_\omega$$

and coincides with the trace of the semi-positive symmetric operator

$$\varphi^{\alpha;\bar{\beta}} \frac{\partial}{\partial z^\alpha} \otimes dz^\gamma$$

(2.6)

of $T^{(1,0)}X$.

Suppose that $\left\| \partial \phi \right\|^2_\omega$ is locally constant. Then, we have

$$0 = \partial \left( \left\| \partial \phi \right\|^2_\omega \right) = (\varphi_\alpha \varphi^{\alpha})_\beta dz^\beta = (\varphi_{\alpha;\beta} \varphi^\alpha + \varphi_\alpha \delta^\alpha_{\beta}) dz^\beta$$

$$= (\varphi_{\alpha;\beta} \varphi^\alpha + \varphi_\alpha \delta^\alpha_{\beta}) dz^\beta = (\varphi_{\alpha;\beta} \varphi^\alpha + \varphi_\beta) dz^\beta .$$

It follows that

$$\varphi_{\alpha;\beta} \varphi^\alpha = \varphi_{\beta;\alpha} \varphi^\alpha = -\varphi_\beta .$$

(2.7)

This means that at each point of $X$ where $\phi$ is defined, the gradient vector

$$\text{grad}(\phi) = \varphi^\alpha \frac{\partial}{\partial z^\alpha} = g^{\alpha\bar{\beta}} \varphi_{\bar{\beta}} \frac{\partial}{\partial z^\alpha}$$
is an eigenvector of the semi-positive symmetric operator in (2.6) with the eigenvalue 1. Therefore $\|\nabla^2 \varphi\|_\omega^2 \geq 1$. As a conclusion, we have

$$0 = \Delta_\omega \| \partial \varphi \|_\omega^2 = \| \nabla^2 \varphi \|_\omega^2 + n - K \| \partial \varphi \|_\omega^2 \geq 1 + n - K \| \partial \varphi \|_\omega^2 .$$

This implies the following.

**Proposition 2.4** Let $\varphi$ be a local potential function of the Kähler–Einstein metric $\omega$ with negative Ricci curvature $-K$. If the length $\| \partial \varphi \|_\omega$ is constant, then

$$\| \partial \varphi \|_\omega^2 \geq \frac{n + 1}{K} .$$

Now we will see that only the unit ball has a global Kähler–Einstein potential function whose gradient length attains the optimal (so minimal) constant $(n + 1)/K$.

**2.3.1 A Potential of the Unit Ball $\mathbb{B}^n$**

Let $\omega_{\mathbb{B}^n}$ be the Kähler–Einstein metric of $\mathbb{B}^n$ with Ricci curvature $-(n + 1)$ and $\varphi_\rho$ be its canonical potential as in Sect. 2.2. One can construct a potential function

$$\tilde{\varphi} = \varphi_\rho + 2 \log \left| 1 + z^1 \right| = \frac{1}{n + 1} \log \frac{\left| 1 + z^1 \right|^{2(n + 1)}}{(1 - \| z \|^2)^{n + 1}},$$

applying the method of potential rescaling in [13] to a sequence of hyperbolic automorphisms whose orbit accumulates at $(1, 0, \ldots, 0) \in \partial \mathbb{B}^n$. By Proposition 2.2 in [13], this $\tilde{\varphi}$ has a constant gradient length

$$\| \tilde{\varphi} \|_{\omega_{\mathbb{B}^n}}^2 \equiv 1 .$$

Therefore

$$\varphi = \frac{n + 1}{K} \tilde{\varphi} = \frac{1}{K} \log \frac{\left| 1 + z^1 \right|^{2(n + 1)}}{(1 - \| z \|^2)^{n + 1}} ,$$

is a potential function of the complete Kähler–Einstein metric $\omega$ with Ricci curvature $-K$ satisfying

$$\| \partial \varphi \|_\omega^2 \equiv \frac{n + 1}{K} .$$
2.3.2 A Potential for the Bounded Symmetric Domain

Irreducible bounded symmetric domains consist of the following four classical type domains,

\[
\Omega_{I}^{p, q} = \left\{ Z \in \mathbb{M}^{C}(p, q) : I_{p} - ZZ^{*} > 0 \right\},
\]

\[
\Omega_{II}^{m} = \left\{ Z \in \mathbb{M}^{C}(m, m) : I_{m} - ZZ^{*} > 0, \ Z' = -Z \right\},
\]

\[
\Omega_{III}^{m} = \left\{ Z \in \mathbb{M}^{C}(m, m) : I_{m} - ZZ^{*} > 0, \ Z' = Z \right\},
\]

\[
\Omega_{IV}^{m} = \left\{ Z = (z_{1}, \ldots, z_{m}) \in \mathbb{C}^{m} : ZZ^{*} < 1, \ 0 < 1 - 2ZZ^{*} + |ZZ^{t}|^{2} \right\},
\]

and two exceptional type domains,

\[
\Omega_{V}^{16}, \ \Omega_{VI}^{27}.
\]

Here, \(\mathbb{M}^{C}(p, q)\) denotes the set of \(p \times q\) complex matrices and \(Z^{*}\) the complex conjugate transpose of the matrix \(Z \in \mathbb{M}^{C}(p, q)\).

Let \(\Omega\) be an irreducible bounded symmetric domain in \(\mathbb{C}^{n}\). The Bergman kernel \(K_{\Omega}\) is of the form

\[
K_{\Omega}(z, z) = cN_{\Omega}(z, z)^{-c_{\Omega}}
\]

for the generic norm \(N_{\Omega}\) of \(\Omega\) and some positive constants \(c, c_{\Omega}\). The constant \(c_{\Omega}\), the dimension \(n\) and the rank \(r\) are given by as follows.

By abuse of notation, we will denote the Bergman kernel function by \(K_{\Omega}\), that is, \(K_{\Omega}(z) = K_{\Omega}(z, z)\). Let us consider the Bergman metric \(\omega_{\Omega}\) of \(\Omega\),

\[
\omega_{\Omega} = dd^{c} \log K_{\Omega},
\]

which is also a complete Kähler–Einstein metric with Ricci curvature \(-1\) (i.e. \(K = 1\)). We remark that the potential function log \(K_{\Omega}\) does not have a constant gradient length.

Let us consider a Cayley transform \(\sigma : \Omega \to S\) where \(S\) is a Siegel domain of the second kind. Then the Bergman kernel function \(K_{S}\) of \(S\) gives a potential function \(\sigma^{*} \log K_{S}\) of \((\Omega, \omega_{\Omega})\) with constant gradient length.

**Theorem 2.5** (Kai–Ohsawa [11]) *The potential function \(\sigma^{*} \log K_{S}\) of \(\omega_{\Omega}\) has a constant gradient length \(L_{\Omega}\):

\[
\left\| \partial (\sigma^{*} \log K_{S}) \right\|_{\omega_{\Omega}}^{2} \equiv L_{\Omega}.
\]

If there is another potential function \(\varphi\) with constant \(\left\| \partial \varphi \right\|_{\omega_{\Omega}}^{2}\), then \(\left\| \partial \varphi \right\|_{\omega_{\Omega}}^{2} \equiv L_{\Omega}\).
For a maximal totally geodesic polydisc $\Delta^r$ in $\Omega$ where $r$ is the rank of $\Omega$, we may assume that

$$\Delta^r = \left\{ (z^1, \ldots, z^r, 0, \ldots, 0) : \left| z^\alpha \right| < 1 \text{ for } \alpha = 1, \ldots, r \right\}$$

by a change of coordinates of $\mathbb{C}^n$. Then we can take a Cayley transform $\sigma : \Omega \rightarrow S$ such that

1. $S$ is a Siegel domain of the second kind in $\mathbb{C}^n$,
2. the restriction $\sigma|_{\Delta^r} : \Delta^r \rightarrow \mathbb{H}^r \subset S$ is a Cayley transformation of the polydisc to the $r$-product of the right half plane $\mathbb{H} = \{ \zeta \in \mathbb{C} : \text{Re } \zeta < 0 \}$,

$$\mathbb{H}^r = \{ (w^1, \ldots, w^r, 0, \ldots, 0) : \text{Re } w^\alpha < 0 \text{ for } \alpha = 1, \ldots, r \},$$

more precisely

$$\sigma (z^1, \ldots, z^r, 0, \ldots, 0) = \left( \frac{z^1 - 1}{z^1 + 1}, \ldots, \frac{z^r - 1}{z^r + 1}, 0, \ldots, 0 \right). \tag{2.8}$$

Note that at $w = (w^1, \ldots, w^r, 0, \ldots, 0) \in \mathbb{H}^r$, the Bergman kernel $K_S$ of $S$ is written by

$$K_S(w, w) = C \left( K_{\mathbb{H}}(w^1, w^1) \cdots K_{\mathbb{H}}(w^r, w^r) \right)^{c_\Omega/2} \tag{2.9}$$

for some positive constant $C$ where $K_{\mathbb{H}}$ is the Bergman kernel of $\mathbb{H}$.

By a straightforward calculation, the Bergman metric is $c_\Omega \delta_{\alpha \bar{\beta}}$ at $0 \in \Omega$, so we get

$$L_\Omega = \| (\sigma^* \log K_S) \|_{\omega_\Omega}^2 (0) = \| \partial \log (K_S \circ \sigma) \|_{\omega_\Omega}^2 (0)$$

$$= \left\| \sum_{\alpha=1}^n \frac{\partial}{\partial z^\alpha} \log (K_S \circ \sigma) d\bar{z}^\alpha \right\|_{\omega_\Omega}^2 (0) = \frac{1}{c_\Omega} \left[ \sum_{\alpha=1}^n \left| \frac{\partial}{\partial z^\alpha} \log (K_S \circ \sigma) \right|_{z=0}^2 \right].$$

Since $\sigma (0) = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{H}^r$ and $K_{\mathbb{H}}(\zeta) = 2 / (\zeta + \bar{\zeta})^2$, Eq. (2.8) and the identity in (2.9) implies that

$$\frac{\partial}{\partial z^\alpha} \log (K_S \circ \sigma) = c_\Omega \left( \frac{\partial}{\partial \zeta} \log K_{\mathbb{H}} \right)_{\zeta=-1} = c_\Omega.$$

Therefore we have

$$L_\Omega \geq r c_\Omega.$$
Table 1  Invariants

| $\Omega$ | $\Omega^1_{\cdot,\cdot}$ | $\Omega^2_{m,m}$ | $\Omega^3_{m,m}$ | $\Omega^4_{m}$ | $\Omega^5_{16}$ | $\Omega^6_{27}$ |
|----------|----------------|-----------------|-----------------|---------------|--------------|--------------|
| $c_{\Omega}$ | $p + q$ | $2(m - 1)$ | $m + 1$ | $m$ | $12$ | $18$ |
| $n$ | $pq$ | $\frac{m(m-1)}{2}$ | $\frac{m(m+1)}{2}$ | $m$ | $16$ | $27$ |
| $r$ | $p$ | $\frac{m}{2}$ | $m$ | $2$ | $2$ | $3$ |

This gives a characterization of the unit ball among bounded symmetric domains.

**Lemma 2.6**  Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^n$ with the complete Kähler–Einstein metric $\omega$ with Ricci curvature $-K$ and let $\varphi$ be a potential function of $\omega$ with constant $\|\partial \varphi\|_\omega^2$. Then if $\Omega$ is not the unit ball up to biholomorphic equivalence, then

$$\|\partial \varphi\|_\omega^2 > \frac{n + 1}{K}.$$  

**Proof**  Since the Ricci curvature of the Bergman metric $\omega_\Omega$ is $-1$, so $\omega = (1/K)\omega_\Omega$ is the complete Kähler–Einstein metric with Ricci curvature $-K$ and $\varphi = (1/K)\sigma^* \log K_S$ is a potential function of $\omega$ satisfying

$$\|\partial \varphi\|_\omega^2 \geq \frac{L_\Omega}{K} \geq \frac{rc_{\Omega}}{K}.$$  

Table 1 shows that $rc_{\Omega} > n + 1$ if $\Omega$ is irreducible and $\Omega \neq \Omega^4_{1,n} = \mathbb{B}^n$. Thus the assertion follows for an irreducible $\Omega$.

Suppose that $\Omega$ is a product of bounded symmetric domains $\Omega_1$ and $\Omega_2$, that is, $\Omega = \Omega_1 \times \Omega_2$. Let $\omega_k$ ($k = 1, 2$) be the complete Kähler–Einstein metric of $\Omega_k$ with Ricci curvature $-K$. By the uniqueness of the Kähler–Einstein metric, we have

$$\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$$  

as a Kähler form where $\pi_k : \Omega_1 \times \Omega_2 \to \Omega_k$ is the projection. Taking a potential function $\varphi_k$ of $\omega_k$ with constant gradient length, we have a potential function $\varphi = \pi_1^* \varphi_1 + \pi_2^* \varphi_2$ of $\omega$ satisfying

$$\|\partial \varphi\|_\omega^2 \equiv \|\partial \varphi_1\|_{\omega_1}^2 + \|\partial \varphi_2\|_{\omega_2}^2.$$  

Proposition 2.4 implies that $\|\partial \varphi\|_\omega^2$ is greater than $(n + 1)/K$. This completes the proof.  

\[\square\]

### 2.4 Existence of a Complete Holomorphic Vector Field

In [2, 13], it was proved that there is a complete holomorphic vector field on a negatively curved complete Kähler–Einstein manifold admitting a global potential function with minimal constant gradient length.
Theorem 2.7 (Theorem 3.2 in [2]) Let $\omega$ be a complete Kähler–Einstein metric of $X^n$ with Ricci curvature $-K$. If there is a global potential $\varphi$ of $\omega$ satisfying

$$\|\partial \varphi\|_\omega^2 = \frac{n + 1}{K},$$

then the $(1, 0)$-vector field

$$V = \sqrt{-1} e^{\frac{\varphi}{n+1}} \text{grad}(\varphi)$$

is a complete holomorphic vector field.

The completeness of $V$: Since $\omega$ is a complete metric, the $(1, 0)$-vector field $W = \sqrt{-1} \text{grad}(\varphi)$ with constant length $(n + 1)/K$ is complete. Moreover the corresponding real vector field $\text{Re} W$ is tangent to each level set of $\varphi$ since

$$(\text{Re} W) \varphi = \sqrt{-1} \varphi^\alpha \partial_\alpha - \sqrt{-1} \varphi^\bar{\alpha} \partial_{\bar{\alpha}} = 0.$$ 

This means that an integral curve $\gamma : \mathbb{R} \to X$ of $\text{Re} W$ lies on a level subset $\{ \varphi = c \}$. Since $V = e^{\frac{\varphi}{n+1}} W = c' W$ on $\{ \varphi = c \}$ for $c' = Kc/(n + 1)$, the curve $\tilde{\gamma} : \mathbb{R} \to X$ given by $\tilde{\gamma}(t) = \gamma(c't)$ is an integral curve of $V$. This implies that $V$ is complete.

The holomorphicity of $V$: Let us written the vector field $V$ by

$$V = \sqrt{-1} V^\alpha \frac{\partial}{\partial z^\alpha} \text{ where } V^\alpha = e^{\frac{\varphi}{n+1}} \varphi^\alpha.$$ 

In order to prove that $V$ is holomorphic, we will see that $\nabla'' V$ is vanishing where $\nabla''$ is the $(0, 1)$-part of the Kähler connection $\nabla$, so coincides with $\partial_\beta$ on $T^{(1, 0)}X$. The tensor field $\nabla'' V$ is given by

$$\nabla'' V = \sqrt{-1} V^\alpha \frac{\partial}{\partial z^\alpha} \otimes dz^\beta,$$

where

$$V^\alpha_\beta = \frac{K}{n + 1} e^{\frac{\varphi}{n+1}} \varphi_\beta \varphi^\alpha + e^{\frac{\varphi}{n+1}} \varphi_\alpha^\beta = e^{\frac{\varphi}{n+1}} \left( \frac{K}{n + 1} \varphi_\beta \varphi^\alpha + \varphi_\alpha^\beta \right).$$

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A straightforward computation gives that
\[
\|\nabla''V\|_{\omega}^2
= e^{2K}\left(\frac{K}{n+1}\right)^2 \left(\varphi_\alpha\varphi^\alpha\right)^2 + \frac{K}{n+1}\varphi_\beta\varphi^\alpha\varphi_\alpha^{,\beta} + \frac{K}{n+1}\varphi_\alpha^{,\beta}\varphi_\beta^{,\alpha} + \varphi_\alpha^{,\beta}\varphi_\alpha^{,\beta}
\]
\[
= e^{2K}\left(\frac{K}{n+1}\right)^2 \left(\|\partial\varphi\|_{\omega}^4 - 2\frac{K}{n+1}\|\partial\varphi\|_{\omega}^2 + 1\right)
\]
\[
= e^{2K}\left(\frac{K}{n+1}\|\partial\varphi\|_{\omega}^2 - 1\right)^2 \equiv 0.
\]

Here, we used the identity in (2.7). This implies that \( V \) is holomorphic.

3 Proofs of Main Results

In this section, we will prove Theorems 1.1 and 1.2. Throughout this section, \( X^n \) is a complex manifold with a complete Kähler–Einstein metric \( \omega \) of Ricci curvature \(-K\) and \( \Gamma \) is a discrete, torsion-free, cocompact subgroup of \( \text{Aut}(X) \).

3.1 Proof of Theorem 1.1

Suppose that \( X \) is simply connected and there is a global potential function \( \varphi \) of \( \omega \) with
\[
\|\partial\varphi\|_{\omega}^2 \equiv \frac{n+1}{K}.
\]

By Lemma 2.6, it is sufficient to show that \( X \) is biholomorphic to a bounded symmetric domain, i.e. a Hermitian symmetric space of noncompact type.

Since \((X, \omega)\) is Ricci negative and \( \Gamma \) acts on \( X \) as an isometric transformation group, the quotient \( \Gamma\setminus X \) has a negative anti-canonical class so \( c_1(\Gamma\setminus X) < 0 \). By the Nadel–Frankel theorem (Theorem 0.1 in [7]), there is a finite covering \( X' \rightarrow \Gamma\setminus X \) such that \( X' \) is holomorphically factorized by
\[
X' = X'_1 \times X'_2,
\]
where \( X'_1 \) is locally symmetric and \( X'_2 \) is locally rigid (the universal covering of \( X'_2 \) has a discrete automorphism group). Then, we have the factorization
\[
X = X_1 \times X_2,
\]
where \( X_k \) is the universal covering of \( X'_k \); therefore

(1) \( X_1 \) is a Hermitian symmetric space of noncompact type;
(2) \( \text{Aut}(X_2) \) is discrete.

We will show that \( X_2 \) is a trivial factor so that \( X = X_1 \).

Suppose that \( X_2 \) is not trivial. Since \( X_1 \) is a Hermitian symmetric space of noncompact type, so it admits a complete Kähler–Einstein metric \( \omega_1 \) with \( \text{Ric}(\omega_1) = -K\omega_1 \). Moreover since \( c_1(X_2^\prime) < 0 \) by Corollary 4.5 in [7], the covering \( X_2 \) also admits a complete Kähler–Einstein metric \( \omega_2 \) with \( \text{Ric}(\omega_2) = -K\omega_2 \) by Yau [20]. Therefore \( \omega \) should be the product metric of \( \omega_1 \) and \( \omega_2 \) because of the uniqueness of negatively curved complete Kähler–Einstein metric.

By the assumption as in (3.1) and Theorem 2.7, the \((1, 0)\)-vector field

\[ V = \sqrt{-1} e^{\pi i \varphi} \nabla \varphi \]

is a complete holomorphic vector field of \( X \). Let \( \mathcal{V} = V + \overline{V} \) be the corresponding real tangent vector field and let \( \{ \mathcal{V}_t : t \in \mathbb{R} \} \) be its flow so that each \( \mathcal{V}_t \) belongs to the identity component of \( \text{Aut}(X) \) and \( \text{Isom}(X, \omega) \). By the de Rham decomposition on a product space of simply connected Riemannian manifolds (Theorem 3.5 in Chap. VI of [12]), \( \mathcal{V}_t \) can be split to isometries of \( X_1 \) and \( X_2 \); thus there is \( \mathcal{V}_{k, t} \in \text{Isom}(X_k, \omega_k) \) such that

\[ \mathcal{V}_t(x_1, x_2) = (\mathcal{V}_{1, t}(x_1), \mathcal{V}_{2, t}(x_2)) \]

for any \((x_1, x_2) \in X_1 \times X_2\). That means that we can regard \( \mathcal{V}_{2, t} \) as an isometry of \((X_2, \omega)\). Since \( \mathcal{V}_t : X \to X \) is holomorphic, the restriction \( \mathcal{V}_{2, t} : X_2 \to X_2 \) is also holomorphic, so constitutes a holomorphic transformation group of \( X_2 \). Therefore \( \mathcal{V}_{2, t} \) is just the identity mapping of \( X_2 \) because \( \text{Aut}(X_2) \) is discrete. Now we can conclude that the infinitesimal generator \( \mathcal{V} \) should be tangent to each fiber \( X_1 \) of \( X \) so orthogonal to each fiber \( X_2 \). This also holds for \( \nabla \varphi \). The identity \( \langle \nabla \varphi, \cdot \rangle_{\omega} = \overline{\partial} \varphi(\cdot) \) says that \( d\varphi(v) = d^c \varphi(v) = 0 \) for any vector \( v \) in the complexified tangent bundle \( \mathbb{C}T\mathbb{C}X_2 \) of \( X_2 \).

Let \( W \) be a \((1, 0)\)-vector field tangent to \( X_2 \). The Lie bracket \([W, \overline{W}]\) is also tangent to \( X_2 \), so we have \( d^c \varphi(W) = d^c \varphi(\overline{W}) = d^c \varphi([W, \overline{W}]) = 0 \). This means that the Kähler form \( \omega \) annihilates the nontrivial subbundle \( \mathbb{C}T\mathbb{C}X_2 \) since

\[ \omega(W, \overline{W}) = d\overline{d}^c \varphi(W, \overline{W}) = W(d^c \varphi(\overline{W}) - \overline{W}(d^c \varphi(W))) - d^c \varphi([W, \overline{W}]) = 0. \]

This is a contradiction, so \( X_2 \) is trivial.

\[ \square \]

### 3.2 Proof of Theorem 1.2

Let \( U \) be a localizing neighborhood \( U \) of a sequence \( \{ f_j \} \subset \Gamma \), that is, if \( K \subset X \) is compact, then

\[ f_j(K) \subset U \]
for sufficiently large $j$. Suppose that there is a local potential function $\varphi : U \to \mathbb{R}$ of $\omega$ satisfying

$$\|\partial \varphi\|_\omega < C \text{ on } U$$

(3.2)

for some constant $C$, and

$$\lim_{j \to \infty} \|\partial \varphi\|^2_\omega(f_j(x)) = \frac{n + 1}{K}$$

(3.3)

for any $x \in X$.

Let us fix a point $x_0 \in X$ and consider a $\omega$-distance ball $B_R$ centered at $x_0$ with radius $R > 0$. Then $B_R$ is relatively compact in $X$ so $f_j(B_R) \subset U$ eventually for $j$. Therefore we can consider a sequence $\{\varphi_j\}$ of functions on $B_R$ defined by

$$\varphi_j = \varphi \circ f_j - (\varphi \circ f_j)(x_0).$$

This is indeed a sequence of potentials of $\omega$ on $B_R$ since $f_j \in \text{Isom}(X, \omega)$ implies

$$\dd c \varphi_j = \dd c (f_j^* \varphi) = f_j^* \dd c \varphi = f_j^* \omega = \omega.$$

We will show that

$$\{\varphi_j\} \text{ admits a subsequence converging on } B_R \text{ in the local } C^\infty\text{-topology.}$$

If it holds, using the compact exhaustion $X = \bigcup_{j \to \infty} B_{R_j}$ and the diagonal processing, we have a global potential function $\varphi_\infty$ of $\omega$ as a subsequential limit of $\{\varphi_j\}$ in the local $C^\infty$-topology of $X$. When we assume $\varphi_j \to \varphi_\infty$ passing to a subsequence, it follows

$$\|\partial \varphi_\infty\|_\omega(x) = \lim_{j \to \infty} \|\partial \varphi_j\|_\omega(x) = \lim_{j \to \infty} \|\partial \varphi\|_\omega(f_j(x)) = \frac{n + 1}{K}$$

for any $x \in X$ from $f_j \in \text{Isom}(X, \omega)$ and the assumption of (3.3). Lifting $\varphi_\infty$ to the universal covering $\tilde{X}$ of $X$ and applying Theorem 1.1, we can see that $\tilde{X}$ is biholomorphic to the unit ball. In case of simply connected $U$, $X$ is also simply connected from the completeness of $\omega$ (see Lemma in p. 256 in [18]). Therefore it remains to show the assertion.

Let us take $R' > R$ and consider $\varphi_j$ as a potential function on $B_{R'}$ for sufficiently large $j$ so that $f_j(B_{R'}) \subset U$. Then we have

$$\frac{1}{2} \|\dd c \varphi_j\|_\omega = \|\partial \varphi_j\|_\omega = \|\partial (f_j^* \varphi)\|_\omega = \|\partial \varphi\|_\omega \circ f < C$$

uniformly on $B_{R'}$. Since $\varphi_j(x_0) = 0$ for any $j$, we can conclude that $\{\varphi_j\}$ is uniformly bounded on $B_{R'}$. 

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In order to show a subsequential convergence of \( \{\varphi_j\} \) on \( B_R \subset B_{R'} \) in the local \( C^\infty \)-topology of \( B_R \), it suffices to prove that for each point \( x \in B_R \), there is a neighborhood \( U_x \) of \( x \) such that \( \{\varphi_j\} \) converges subsequentially in the local \( C^\infty \)-topology of \( U_x \) since \( B_R \) is relatively compact.

Take a sufficiently small, local coordinate neighborhood \( U_x \subset B_{R'} \) of a given \( x \in B_R \) so that there is a local potential function \( \varphi_x \) of \( \omega \) on \( U_x \) whose gradient length \( \|d\varphi_x\|_\omega \) is bounded. Then we have a sequence \( \{\varphi_j - \varphi_x\} \) of uniformly bounded plurisubharmonic functions on \( U_x \). Now we can find \( \psi_j : U_x \to \mathbb{R} \) such that

\[
\eta_j = \varphi_j - \varphi_x + \sqrt{-1}\psi_j
\]

is holomorphic solving \( d\psi_j = 2d^c(\varphi_j - \varphi_x) \) on \( U_x \). When we normalize \( \psi_j \) by \( \psi_j(x) = 0 \), the sequence \( \{\psi_j\} \) is also uniformly bounded on \( U_x \) since

\[
\|d\psi_j\|_\omega = \|2d^c(\varphi_j - \varphi_x)\|_\omega = \|d(\varphi_j - \varphi_x)\|_\omega \leq \|d\varphi_j\|_\omega + \|d\varphi_x\|_\omega.
\]

The sequence \( \{\eta_j\} \) of holomorphic functions on \( U_x \) is uniformly bounded now; thus it admits a uniformly convergent subsequence on any compact subset of \( U_x \). Simultaneously, \( \{\varphi_j\} \) converges subsequentially in the local \( C^\infty \)-topology of \( U_x \). This proves the assertion. \( \square \)

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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