AN INTEGRAL VERSION OF ZARISKI DECOMPOSITIONS ON
NORMAL SURFACES

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Abstract. We show that any pseudo-effective divisor on a normal surface decomposes
uniquely into its “integral positive” part and “integral negative” part, which is an in-
tegral analog of Zariski decompositions. As an application, we give a generalization of
the Kawamata-Viehweg vanishing, Ramanujam’s 1-connected vanishing and Miyaoka’s
vanishing theorems on surfaces. By using this vanishing result, we give a simple proof
of Reider-type theorems including the log surface case and the relative case.

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1. INTRODUCTION

In 1962, Zariski showed the following decomposition theorem [24]:

Theorem 1.1 (Zariski decomposition). Let $D$ be an effective $\mathbb{Q}$-divisor on a smooth
projective surface $X$. Then there exists the unique decomposition $D = P + N$ such that
the following hold.

(i) $P$ is a nef $\mathbb{Q}$-divisor on $X$ (called the positive part of $D$).
(ii) $N = 0$ or $N > 0$ is a negative definite $\mathbb{Q}$-divisor on $X$ (called the negative part of $D$).
(iii) $PC = 0$ for any curve $C \subset \text{Supp}(N)$.

For a more general setting including the pseudo-effective case ([2]) and the relative case,
see Theorem 3.1. This decomposition $D = P + N$ is called the Zariski decomposition of
$D$. It is a fundamental tool in algebraic geometry (for the higher dimensional analog
of Zariski decompositions, see for example [17]). Note that the positive and negative parts
$P, N$ are not necessarily $\mathbb{Z}$-divisors even if $D$ is a $\mathbb{Z}$-divisor. The main theorem in this
paper is an integral analog of the Zariski decomposition on a surface as follows (for a more
general setting including relative cases, see Theorem 3.5).

Key words and phrases. Zariski decomposition, vanishing theorem, Reider-type theorem.
Theorem 1.2 (Integral Zariski decomposition). Let $D$ be a pseudo-effective divisor on a normal complete surface $X$. Then there exists the unique decomposition $D = P_Z + N_Z$ such that the following hold.

(i) $P_Z$ is a $\mathbb{Z}$-positive divisor on $X$.

(ii) $N_Z = 0$ or $N_Z > 0$ is a negative definite divisor on $X$.

(iii) $-P_Z$ is nef over $N_Z$.

Here, a divisor $D$ on $X$ is called $\mathbb{Z}$-positive if $B - D$ is not nef over $B$ for any effective negative definite divisor $B > 0$ on $X$. Typical examples of $\mathbb{Z}$-positive divisors are the round-up of nef $\mathbb{R}$-divisors and numerically 1-connected (and not negative definite) divisors. In the usual Zariski decomposition, the positive part of $D$ measures the asymptotic behavior of the cohomology with respect to $mD, m \gg 0$. For example, the section ring of a big divisor $D$ on $X$ is finitely generated if and only if the positive part $P$ of $D$ is semiample (cf. [13] Corollary 2.3.23). Moreover, if further assume $D = P$, then the cohomology $H^i(X, \mathcal{O}_X(K_X + mD))$ vanishes for any $i > 0$ and $m > 0$ by the Kawamata-Viehweg vanishing theorem [5, 23]. On the other hand, the $\mathbb{Z}$-positive part of $D$ in the integral Zariski decomposition measures the cohomology with respect to $D$ itself. Indeed, the first cohomology $H^1(X, \mathcal{O}_X(K_X + D))$ can be computed by some cohomology on the $\mathbb{Z}$-negative part $N_Z$ for a big divisor $D$ on $X$.

Theorem 1.3. Let $D$ be a big divisor on a normal complete surface $X$, algebraic over a field of characteristic 0 or analytic. Let $D = P_Z + N_Z$ be the integral Zariski decomposition of $D$. Then we have

$$H^1(X, \mathcal{O}_X(K_X + D)) \cong H^1(N_Z, \mathcal{L}_D),$$

where $\mathcal{L}_D$ is the rank 1 sheaf on $N_Z$ defined by the cokernel of the homomorphisms $\mathcal{O}_X(K_X + P_Z) \to \mathcal{O}_X(K_X + D)$ induced by multiplying a defining section of $N_Z$.

This is a generalization of the Kawamata-Viehweg vanishing, Ramanujam’s 1-connected vanishing [18] and Miyaoka’s vanishing [15] on surfaces. The relative version for Theorem 1.3 also holds (see Theorem 4.1), even when positive characteristics, which is a generalization of the local vanishing due to Sakai [20] and Kollár-Kovács [7].

As an application of Theorem 1.3, we can prove some Reider-type theorems. For example, the following can be shown:

Theorem 1.4. Let $D$ be a big divisor on a normal complete surface $X$, algebraic over a field of characteristic 0 or analytic. Let $x \in X$ be a closed point at which $K_X + D$ is Cartier. We further assume that $P^2 > \delta_x$ (resp. $P_Z^2 > \delta_x$), where $\delta_x$ is the invariant of the germ $(X, x)$ satisfying $0 \leq \delta_x \leq 4$ (for the details, see Section 5) and $D = P + N$ (resp. $D = P_Z + N_Z$) the (resp. integral) Zariski decomposition. If $x$ is a base point of the linear system $|K_X + D|$, then there exists a curve $B$ on $X$ passing through $x$ such that $(D - B)B \leq \delta_x/4$ (resp. $D + N_Z - 2B$ is big).

It is a generalization of Reider’s theorem for base points ([19] Theorem 1 (1)). More general statements including the higher order case and the relative case, see Theorem 5.2. As an advantage of Theorem 1.3, Reider-type theorems (e.g., Theorem 1.4) can be proved easily. The relative version of Theorem 1.3 also works, even for positive characteristics, and contains the results of Shepherd-Barron [22], Laufer [12] and Sakai [21] about base points of linear systems on a resolution space of a normal surface singularity. For example, we obtain as a corollary of Theorem 5.2 the following famous results:
Corollary 1.5 (Corollaries 5.8, 5.9 and 5.10). Let \( f : X \to Y \) be one of the following (i), (ii) and (iii).

(i) \( X \) is a minimal smooth projective surface of general type over a field of characteristic 0 with \( K_X^2 \geq 5 \) and \( Y \) is a point.

(ii) \( f \) is a relatively minimal fiber space from a regular surface \( X \) to a curve \( Y \) whose general fiber has arithmetic genus greater than 1.

(iii) \( f \) is a minimal resolution of a normal surface singularity \((Y, y)\).

Then the natural homomorphism \( f^* f_* \mathcal{O}_X(mK_X) \to \mathcal{O}_X(mK_X) \) is surjective for \( m \geq 2 \).

The present paper is organized as follows. In Section 2, we fix some notations and terminology used in this paper. In Section 3, we first give another proof of Zariski decompositions in the usual sense, which is as simple as Bauer’s one in [1]. Next, we give a proof of integral Zariski decompositions. We study several properties of \( Z \)-positive divisors in the rest of the section. The \( Z \)-positivity is characterized by using the usual Zariski decomposition and connecting chains (Proposition 3.10). One important property is the \( Z \)-positivity of divisors are preserved by the round-up of the pull-back by a proper birational morphism (Proposition 3.18). In Section 4, we study vanishing theorems for adjoint divisors \( K_X + D \). Following Sakai’s argument in [20], we give a proof of the (semi-)local vanishing theorem (Theorem 4.1 (2)). We prove the absolute version of the vanishing theorem (Theorem 4.1 (1)) by using the Kawamata-Viehweg vanishing theorem as a starting point. In Section 5, we study higher order separations of adjoint (relative) linear systems of \( K_X + D \). We first define the invariant \( \delta_\zeta \) for a 0-dimensional subscheme \( \zeta \) and give a simple proof of the Reider-type theorem (Theorem 5.2). In Appendix A, we collect some basic results for Mumford’s intersection theory on a normal surface for the convenience of readers.

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2. Notations

- In this paper, we work in the category of algebraic schemes over a base field \( k \) or that of complex analytic spaces.
- A variety means an irreducible and reduced scheme which is separated and of finite type over \( k \) or an irreducible, reduced, paracompact and Hausdorff analytic space.
- A surface means a variety of dimension 2. A curve on a surface means a non-zero effective divisor on it (not necessarily irreducible and reduced).
- A divisor means a Weil divisor (not necessarily \( \mathbb{Q} \)-Cartier). We sometimes call it a \( \mathbb{Z} \)-divisor. A \( \mathbb{Q} \)-(or \( \mathbb{R} \))-divisor is a \( \mathbb{Q} \)-(or \( \mathbb{R} \))-linear combination of divisors.
- For an \( \mathbb{R} \)-divisor \( D = \sum a_i C_i \) where the \( C_i \) are distinct prime divisors, we define the round-down \( \lfloor D \rfloor := \sum a_i \lfloor a_i \rfloor C_i \), the round-up \( \lceil D \rceil := \sum \lceil a_i \rceil C_i \) and the fractional part \( \{ D \} := D - \lfloor D \rfloor \) of \( D \), where \( \lfloor a \rfloor, \lceil a \rceil \) are the greatest integer not exceeding \( a \), the least integer not less than \( a \), respectively.
- For an \( \mathbb{R} \)-divisor \( D \), we denote by \( D \geq 0 \) (resp. \( > 0 \)) that \( D \) is (resp. non-zero and) effective. \( D \geq E \) means \( D - E \) is effective for two \( \mathbb{R} \)-divisors \( D \) and \( E \).
• Throughout this paper, $X$ is a normal surface and $f: X \to Y$ is a proper surjective morphism to a variety $Y$ unless otherwise stated. In the analytic setting, we always assume that $Y$ is compact or $f$ can be extended to a proper surjective morphism $\overline{f}: \overline{X} \to \overline{Y}$ from a normal surface $\overline{X}$ to a variety $\overline{Y}$ such that $Y$ is a relatively compact open subset of $\overline{Y}$.

• An $\mathbb{R}$-divisor $D$ on $X$ is $f$-exceptional if each irreducible component of $\text{Supp}(D)$ maps to a point by $f$ (note that this is not a standard definition).

• A non-zero effective $\mathbb{R}$-divisor $D = \sum_i a_i C_i$ on $X$ is negative (semi-)definite if the intersection matrix $(C_i C_j)_{ij}$ of irreducible components of $D$ is negative (semi-)definite.

• Let $D$ be an $\mathbb{R}$-divisor and $B$ an effective $f$-exceptional divisor on $X$. Then $D$ is nef over $B$ if $DC \geq 0$ holds for each irreducible component $C$ of $B$.

3. Zariski decomposition on a normal surface

Let $f: X \to Y$ be a proper surjective morphism from a normal surface $X$ to a variety $Y$. Recall the following famous result of Zariski [24] (the pseudo-effective case is due to Fujita [2]).

**Theorem 3.1 (Zariski decomposition).** Let $D$ be an $f$-pseudo-effective $\mathbb{R}$-(resp. $\mathbb{Q}$-)divisor on $X$. Then there exists the unique decomposition $D = P + N$ such that the following hold.

(i) $P$ is an $f$-nef $\mathbb{R}$-(resp. $\mathbb{Q}$-)divisor on $X$.

(ii) $N = 0$ or $N > 0$ is a negative definite $f$-exceptional $\mathbb{R}$-(resp. $\mathbb{Q}$-)divisor on $X$.

(iii) $PC = 0$ holds for any curve $C \subset \text{Supp}(N)$.

First, we define the notion of the positivity of divisors on a surface $X$.

**Definition 3.2.** Let $R := \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. For an $\mathbb{R}$-divisor $D$ on $X$, let $N_R(D)$ be the set of negative definite $f$-exceptional $R$-divisors $B > 0$ on $X$ with $B - D$ nef over $B$. Then $D$ is called $f$-$R$-positive if $N_R(D) = \emptyset$.

**Lemma 3.3.** Let $D$ be an $f$-pseudo-effective $\mathbb{R}$-divisor on $X$. Then the following are equivalent.

(1) $D$ is $f$-nef.

(2) $D$ is $f$-$\mathbb{R}$-positive.

(3) $D$ is $f$-$\mathbb{Q}$-positive.

**Proof.** If $N_{\mathbb{R}}(D)$ has an element $B$, then $DB = -(B - D)B + B^2 < 0$ holds since $B - D$ is nef over $B$ and $B > 0$ is negative definite. Hence (1) implies (2). It is clear that $f$-$\mathbb{R}$-positive implies $f$-$\mathbb{Q}$-positive. Suppose that $D$ is $f$-$\mathbb{Q}$-positive. Let $C$ be an arbitrary $f$-exceptional irreducible curve on $X$. If $C^2 \geq 0$, then $DC \geq 0$ since $C$ is nef and $D$ is $f$-pseudo-effective. Then we may assume that $C^2 < 0$, that is, $C$ is negative definite. Let $\varepsilon > 0$ be a positive rational number and put $B := \varepsilon C$. Since $D$ is $f$-$\mathbb{Q}$-positive, we have $(B - D)C < 0$. Thus $DC > \varepsilon C^2$ holds. Since $\varepsilon > 0$ is arbitrary, we have $DC \geq 0$. Hence $D$ is $f$-nef. \hfill $\square$

**Remark 3.4.** For not necessarily $f$-pseudo-effective $\mathbb{R}$-divisors on $X$, we have

$$f\text{-nef} \implies f\text{-}\mathbb{R}\text{-positive} \implies f\text{-}\mathbb{Q}\text{-positive} \implies f\text{-}\mathbb{Z}\text{-positive}.$$
The main theorem in this section is the following, which is a $\mathbb{Z}$-version of Theorem 3.1.

**Theorem 3.5** (Integral Zariski decomposition). Let $D$ be an $f$-pseudo-effective $\mathbb{R}$-divisor on $X$. Then there exists the unique decomposition $D = P + N$ such that the following hold.

(i) $P$ is an $f$-Z-positve $\mathbb{R}$-divisor on $X$.

(ii) $N = 0$ or $N > 0$ is a negative definite $f$-exceptional $\mathbb{Z}$-divisor on $X$.

(iii) $-P$ is nef over $N$.

3.1. Proof of Zariski decompositions. We start with an easy lemma.

**Lemma 3.6.** Let $D$ be an $\mathbb{R}$-divisor on $X$ and $D = F + E = A + B$ two decompositions of $D$ as $\mathbb{R}$-divisors which satisfies the following conditions.

(i) $E$ is effective.

(ii) $B$ is non-zero effective, negative definite and $f$-exceptional.

(iii) $F$ and $-A$ are nef over $B$.

Then $B \leq E$ holds.

**Proof.** Let $B = \sum b_i C_i$ and $E = \sum e_i C_i$ be irreducible decompositions and put $G := \sum \min \{ b_i, e_i \} C_i$. Clearly, we have $G \leq B$ and $G \leq E$. In order to prove $B \leq E$, it suffices to show that $G = B$. Suppose that $G < B$. Then $(B - G)^2 < 0$ holds since $B - G$ is non-zero and negative definite. Moreover, we have $(E - G)(B - G) \geq 0$ and $A(B - G) \leq 0$ since $E - G$ and $B - G$ has no common components and $-A$ is nef over $B$. Thus we have

$$F(B - G) = (A + B - E)(B - G) = A(B - G) + (B - G)^2 - (E - G)(B - G) < 0,$$

which contradicts the condition that $F$ is nef over $B$. Hence we obtain $B \leq E$. □

**Lemma 3.7.** Let $D$ be an $\mathbb{R}$-divisor on $X$. If $\mathcal{N}_D(D) \neq \emptyset$, then it has a maximal element.

**Proof.** By Zorn’s lemma, it is enough to show that $\mathcal{N}_D(D)$ is inductive, that is, any non-empty totally ordered subset $T \subset \mathcal{N}_D(D)$ has an upper bound, where we consider the inclusion order on $\mathcal{N}_D(D)$. For such a subset $T \subset \mathcal{N}_D(D)$, we put $B_T := \sum b_i C_i$, where the sum is taken over all $f$-exceptional irreducible reduced curves $C_i$ and $b_i := \sup \{ \mathrm{mult}_{C_i}(B) \mid B \in T \}$. We will show that this is well-defined and gives an upper bound of $T$. We first assume that there is an infinite sequence $\{ C_n \}_n$ with $b_n > 0$ for each $n$. Then there exist elements $B_n$ of $T$ with $\mathrm{mult}_{C_n}(B_n) > 0$. Since $T$ is totally order, we have $B_1 \leq B_2$ or $B_2 \leq B_1$. Thus we may assume that $\mathrm{mult}_{C_0}(B_2) > 0$. Similarly, we can take an element $B_n$ of $T$ for each $n$ such that the number of irreducible component of $B_n$ is not less than $n$. Taking $n > \dim N^1(X/Y)$, it is a contradiction because $B_n$ is negative definite and then all irreducible components of $B_n$ are linearly independent in $N^1(X/Y)$. Hence we have $b_i = 0$ except for a finite number of $C_i$’s. Next, we assume that $b_i = \infty$ holds for some $i$. Then we can take an infinite sequence $\{ B_n \}_n$ in $T$ such that $\lim_{n \to \infty} \mathrm{mult}_{C_i}(B_n) = \infty$. In particular, we have $\lim_{n \to \infty} B_n = -\infty$ since $B_n$ is negative definite and any irreducible component of $B_n$ is contained in $\mathrm{Supp}(B_T)$. On the other hand, it follows that $(B_n - D)B_n \geq 0$ since $B_n - D$ is nef over $B_n$. Thus we have $DB_n \leq B_n$, or equivalently, $D(B_n/\sqrt{-B_n^2}) \leq -\sqrt{-B_n^2}$.

Since the multiplication map $D - : N_1(X/Y) \to \mathbb{R}$ has the minimum on the compact subset $K := \{ B \in \bigoplus_{C_i \subset \mathrm{Supp}(B_T)} \mathbb{R}C_i \mid B^2 = -1 \}$, we have a contradiction by taking $n \to \infty$. 


Hence we have \( b_i < \infty \) for each \( i \) and then \( B_T \) is well-defined as an \( \mathbb{R} \)-divisor. Moreover, \( B_T \) is negative definite since we can take an element \( B \in \mathcal{T} \) with \( \text{Supp}(B) = \text{Supp}(B_T) \).

For any irreducible component \( C_i \) of \( B_T \) and \( \varepsilon > 0 \), we can take \( B_\varepsilon \in \mathcal{T} \) satisfying \( \text{mult}_{C_i}(B_\varepsilon) > 0 \) and \( 0 \leq b_i - \text{mult}_{C_i}(B_\varepsilon) < \varepsilon \). Since \( B_\varepsilon - D \) is nef over \( B_\varepsilon \), we have

\[
(B_T - D)C_i = (B_\varepsilon - D)C_i + (B_T - B_\varepsilon)C_i \geq (b_i - \text{mult}_{C_i}(B_\varepsilon))C_i^2 > \varepsilon C_i^2.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have \( (B_T - D)C_i \geq 0 \), whence \( B_T - D \) is nef over \( B_T \). Thus \( B_T \) is an element of \( \mathcal{N}_\mathbb{R}(D) \), which is an upper bound of \( \mathcal{T} \) by definition. \( \square \)

**Lemma 3.8.** Let \( D \) be an \( f \)-pseudo-effective \( \mathbb{R} \)-divisor on \( X \). Then \( D - B \) is \( f \)-pseudo-effective for any \( B \in \mathcal{N}_\mathbb{R}(D) \).

**Proof.** We take effective \( \mathbb{R} \)-divisors \( D_n \) on \( X \) such that \( D_n \rightarrow D \) in \( N^1(X/Y) \) \((n \rightarrow \infty)\).

Let us write \( D_n - B = G^+_n - G^-_n \), where \( G^+_n \), \( G^-_n \) are effective \( \mathbb{R} \)-divisors having no common components. In order to prove \( D - B \) is \( f \)-pseudo-effective, it is enough to show that \( G^-_n \rightarrow 0 \) in \( N^1(X/Y) \) \((n \rightarrow \infty)\). Note that \( G^-_n \leq B \) and then \( G^-_n \) is negative definite and \( f \)-exceptional. Thus we have

\[
(B - D_n)G^-_n = (G^-_n - G^+_n)G^-_n \leq (G^-_n)^2 \leq 0.
\]

Since the sequence of multiplication maps \( \{(B - D_n) - : N_1(X/Y) \rightarrow \mathbb{R}\}_n \) converges uniformly to the non-negative function \((B - D) - \) on the compact subset \( K := \{E \mid 0 \leq E \leq B\} \) of \( N_1(X/Y) \), we have \( (B - D_n)G^-_n \rightarrow 0 \) \((n \rightarrow \infty)\), whence \( (G^-_n)^2 \rightarrow 0 \) \((n \rightarrow \infty)\).

Since \( G^-_n \) is negative definite, we have \( G^-_n \rightarrow 0 \) \((n \rightarrow \infty)\). \( \square \)

For an \( f \)-pseudo-effective \( \mathbb{R} \)-divisor \( D \) on \( X \), we take a maximal element \( N_\mathbb{R} \) in \( \mathcal{N}_\mathbb{R}(D) \).

If \( \mathcal{N}_\mathbb{R}(D) = \emptyset \), we define \( N_\mathbb{R} := 0 \). Put \( P_\mathbb{R} := D - N_\mathbb{R} \).

**Lemma 3.9.** \( P_\mathbb{R} \) is \( f \)-nef.

**Proof.** If \( P_\mathbb{R} = D \), the claim holds from Lemma 3.3. Thus we may assume that \( N_\mathbb{R} > 0 \).

By Lemma 3.8, \( P_\mathbb{R} \) is \( f \)-pseudo-effective. Thus it suffices to show that \( P_\mathbb{R} \) is \( f \)-nef over \( C \) on \( X \) such that \( P_\mathbb{R}C \leq 0 \). Take a small number \( \varepsilon > 0 \) such that \( (P_\mathbb{R} - \varepsilon C)C < 0 \). Since \( -P_\mathbb{R} = N_\mathbb{R} - D \) is nef over \( N_\mathbb{R} \), it follows that \( \varepsilon C + N_\mathbb{R} - D \) is nef over \( \varepsilon C + N_\mathbb{R} \). Hence, it suffices to show that \( \varepsilon C + N_\mathbb{R} \) is negative definite since it contradicts the maximality of \( N_\mathbb{R} \). If \( \varepsilon C + N_\mathbb{R} \) is not negative definite, then there exists an effective \( f \)-exceptional nef divisor \( Z \) on \( X \) with \( \text{Supp}(Z) = \text{Supp}(\varepsilon C + N_\mathbb{R}) \) by Lemma A.3. Since \( P_\mathbb{R} \) is \( f \)-pseudo-effective, it follows that \( (D - N_\mathbb{R})Z \geq 0 \). On the other hand, since \( N_\mathbb{R} - D \) is nef over \( N_\mathbb{R} \) and \( (N_\mathbb{R} - D)C > 0 \), we have \( (N_\mathbb{R} - D)Z \leq 0 \). Hence we have \( (N_\mathbb{R} - D)Z = 0 \) and \( \text{Supp}(Z) \subset \text{Supp}(N_\mathbb{R}) \), which is a contradiction. \( \square \)

**Corollary 3.10.** If \( \mathcal{N}_\mathbb{R}(D) \neq \emptyset \), then \( N_\mathbb{R} \) is the maximum element of \( \mathcal{N}_\mathbb{R}(D) \).

**Proof.** We take two maximal elements \( N_\mathbb{R} \) and \( N'_\mathbb{R} \) of \( \mathcal{N}_\mathbb{R}(D) \) and write

\[
D = P_\mathbb{R} + N_\mathbb{R} = P'_\mathbb{R} + N'_\mathbb{R}.
\]

Since \( P_\mathbb{R} \) and \( -P'_\mathbb{R} \) are nef over \( N'_\mathbb{R} \), we have \( N'_\mathbb{R} \leq N_\mathbb{R} \) by Lemma 3.6. By maximality, we have \( N'_\mathbb{R} = N_\mathbb{R} \). \( \square \)

**Remark 3.11.** We can take a maximal element of \( \mathcal{N}_\mathbb{R}(D) \) from Lemma 3.7 even if \( D \) is not \( f \)-pseudo-effective. But it is not necessarily unique in general.
Proof of Theorem 3.1. The decomposition \( D = P_R + N_R \) satisfies the condition (i), (ii) and (iii) in Theorem 3.5 by Lemma 3.9 and the definition of \( P_R \). The uniqueness of the decomposition satisfying (i), (ii) and (iii) follows from Lemma 3.6. If \( D \) is a \( \mathbb{Q} \)-divisor, then the negative part \( N_R = \sum a_i C_i \) is also a \( \mathbb{Q} \)-divisor since \( DC_j = N_R C_j = \sum_i a_i (C_i C_j) \) is rational for each \( j \) and the matrix \( (C_i C_j)_{ij} \) is negative definite over \( \mathbb{Q} \). \( \square \)

Let \( D \) be an \( f \)-pseudo-effective \( \mathbb{R} \)-divisor on \( X \) and \( D = P_R + N_R \) the Zariski decomposition in Theorem 3.5.

Lemma 3.12. If \( N_Z(D) \neq \emptyset \), it has the maximum element.

Proof. Note that each element \( B \) of \( N_Z(D) \) is a subdivisor of \( N_R \) since \( P_R \) and \( B - D \) are nef over \( B \) and Lemma 3.6. In particular, \( N_Z(D) \) is a finite set. Let \( B = \sum_i b_i C_i \) and \( B' = \sum_i b'_i C_i \) be two divisors in \( N_Z(D) \) and put \( B'' := \sum_i \max\{b_i, b'_i\} C_i \). For each irreducible curve \( C_i \leq B'' \), we have
\[
(B'' - D)C_i = (B - D)C_i + (B'' - B)C_i \geq 0
\]
when \( b'_i \leq b_i \). Similarly, we also have \( (B'' - D)C_i \geq 0 \) in the case of \( b_i \leq b'_i \). Thus \( B'' \) belongs to \( N_Z(D) \), whence the claim holds. \( \square \)

Let \( N_Z \) be the maximum element of \( N_Z(D) \) if \( N_Z(D) \neq \emptyset \), or \( N_Z := 0 \) if \( N_Z(D) = \emptyset \). Put \( P_Z := D - N_Z \). Note that the decomposition \( P_Z = P_R + (N_R - N_Z) \) is the Zariski decomposition of \( P_Z \) since \( N_Z \leq N_R \).

Lemma 3.13. \( P_Z \) is \( f \)-Z-positive.

Proof. We may assume that \( N_Z > 0 \). Suppose that there exists an element \( B \) of \( N_Z(P_Z) \). Then we obtain two decompositions \( P_R + (N_R - N_Z) = (P_Z - B) + B \) of \( P_Z \). Since \( P_R \) and \( B - P_Z \) are nef over \( B \), we have \( B \leq N_R - N_Z \) by Lemma 3.6. In particular, \( B + N_Z \) is negative definite. Thus it suffices to show that \( B + N_Z - D \) is nef over \( B + N_Z \) since it contradicts the maximality of \( N_Z \). For any subcurve \( C \leq B \), we have \( (B + N_Z - D)C \geq 0 \) since \( B - P_Z = B + N_Z - D \) is nef over \( B \). For any irreducible curve \( C \) in \( N_Z \) not contained in \( B \), we have
\[
(B + N_Z - D)C = BC + (N_Z - D)C \geq 0,
\]
since \( N_Z - D \) is nef over \( N_Z \). Hence the claim holds. \( \square \)

Proof of Theorem 3.1. The decomposition \( D = P_Z + N_Z \) satisfies the condition (i), (ii) and (iii) in Theorem 3.5 by Lemma 3.13. Let \( D = P'_Z + N'_Z \) be another decomposition satisfying (i), (ii) and (iii) in Theorem 3.5. If \( N'_Z = 0 \), then \( D = P'_Z \) is \( f \)-Z-positive and so \( N_Z = 0 \). If \( N'_Z > 0 \), then \( N'_Z \) belongs to \( N_Z(D) \) by the condition (iii). Thus \( N'_Z \leq N_Z \) holds. Suppose \( N'_Z < N_Z \). Then \( N_Z - N'_Z \) is an element of \( N_Z(P'_Z) \) since it is negative definite and \( (N_Z - N'_Z) - P'_Z = -P_Z \) is nef over \( N_Z - N'_Z \), which contradicts the condition (i) on \( P'_Z \). Hence we have \( P_Z = P'_Z \) and \( N_Z = N'_Z \). \( \square \)

Remark 3.14. (1) Taking (integral) Zariski decompositions defines self-maps \( P_R, N_R \) on \( N^1(X/Y) \) with \( P_R^2 = P_R, N_R^2 = N_R \) and \( P_R + N_R = \text{id}_{N^1(X/Y)} \), where \( R = \mathbb{R} \) or \( \mathbb{Z} \). The maps \( P_R \) and \( N_R \) are continuous, but \( P_Z \) and \( N_Z \) are not continuous in general.

(2) Given a property \( \mathcal{P} \) of negative definite \( f \)-exceptional irreducible curves on \( X \), we can consider the (resp. integral) Zariski decomposition \( D = P + N \) (resp. \( D = P_Z + N_Z \)) with the additional condition that each irreducible component of \( N \) (resp. \( N_Z \)) has the property \( \mathcal{P} \). Indeed, all the argument in this subsection works by replacing \( N_R(D) \) (\( R = \mathbb{Z} \) or \( \mathbb{R} \)) by the subset consisting of \( B \in N_R(D) \) any component of which has the property \( \mathcal{P} \).
3.2. \(Z\)-positive divisors.

**Definition 3.15.** Let \(A\) and \(B\) be \(\mathbb{R}\)-divisors on \(X\) such that \(A - B\) is an effective \(f\)-exceptional \(\mathbb{Z}\)-divisor. Then the sequence of divisors \(B = D_0 < D_1 < \cdots < D_m = A\) is called a connecting chain from \(B\) to \(A\) if \(C_i := D_i - D_{i-1}\) is a non-zero effective \(\mathbb{Q}\)-divisor and \(D_i - C_i > 0\) for any \(i = 1, \ldots, m\). Note that \(B = A\) is regarded as a connecting chain from \(B\) to \(A\) (\(m = 0\) case).

The following is a characterization of \(f\)-\(Z\)-positive divisors.

**Proposition 3.16.** Let \(D\) be an \(f\)-pseudo-effective \(\mathbb{R}\)-divisor on \(X\) with the Zariski decomposition \(D = P + N\). Then the following are equivalent.

1. \(D\) is \(f\)-\(Z\)-positive.
2. For any \(D - \mathcal{N} \leq D_0 \leq D\) with \(D - D_0\) integral, there exists a connecting chain from \(D_0\) to \(D\).
3. There exists a connecting chain from \(D - \mathcal{N} \) to \(D\).

**Proof.** Assume that \(D\) is \(f\)-\(Z\)-positive. Let \(D_0\) be an \(\mathbb{R}\)-divisor on \(X\) as in (2) and we may assume that \(D - D_0 > 0\). In order to prove (2), it suffices to show by induction on the number of irreducible components of \(D - D_0\) that there exists an irreducible subcurve \(C \leq D - D_0\) such that \(D_0C > 0\) holds. Since \(-D_0 = (D - D_0) - D\) is not nef over \(D - D_0\) by the \(f\)-\(Z\) positivity of \(D\), the claim follows. The condition (2) trivially implies (3). We assume that \(D\) satisfies (3) and fix a connecting chain \(D - \mathcal{N} \leq D_0 < D_1 < \cdots < D_m = D\). We will show that \(D\) is \(f\)-\(Z\)-positive by induction on \(m\). We first assume \(m = 0\), that is, \(\mathcal{N} = 0\). Suppose that there is a non-zero effective \(f\)-exceptional \(\mathbb{Z}\)-divisor \(B\) on \(X\) such that \(B - D\) is nef over \(B\). From Lemma 3.6 we have \(B \leq N\), which contradicts \(\mathcal{N} = 0\). Hence \(D\) is \(f\)-\(Z\)-positive. Assume that \(m > 0\) and the claim holds when the length of the connecting chain is less than \(m\). In particular, \(D_{m-1}\) is \(f\)-\(Z\)-positive. Assume that there exists a non-zero effective divisor \(B\) on \(X\) such that \(B - D\) is nef over \(B\). Then \(B\) is contained in \(N\) by Lemma 3.6. It is easy to see that \(C_m := D - D_{m-1}\) is not contained in \(B\) since \(D_{m-1}\) is \(f\)-\(Z\)-positive and \(D_{m-1}C_m > 0\). Thus we have \(C_mB \geq 0\). Then \(B - D_{m-1} = B - D + C_m\) is nef over \(B\), which contradicts that \(D_{m-1}\) is \(f\)-\(Z\)-positive. Hence \(D\) is \(f\)-\(Z\)-positive, whence (1) holds.

**Corollary 3.17.** Let \(D = M + Z\) be an \(\mathbb{R}\)-divisor on \(X\) such that \(M\) is an \(f\)-\(\text{ nef}\) \(\mathbb{R}\)-divisor and \(Z\) is an \(f\)-\(\text{exceptional}\) \(\mathbb{R}\)-divisor with \(\mathcal{N} = 0\). Then \(D\) is \(f\)-\(Z\)-positive. In particular, \(D = \mathcal{P} M\) is \(f\)-\(Z\)-positive for any \(f\)-\(\text{ nef}\) \(\mathbb{R}\)-divisor \(M\) on \(X\).

**Proof.** Note that \(D\) is \(f\)-\(\text{pseudo-effective}\) by Lemma 3.2 (2). Let \(D = M + Z = P + N\) be the Zariski decomposition of \(D\) in Theorem 3.1. Since \(M\) and \(-P\) is \(f\)-\(\text{nef}\) over \(N\), we have \(N \leq Z\) by Lemma 3.6. In particular, we have \(\mathcal{N} = 0\). Hence \(D\) is \(f\)-\(Z\)-positive by Proposition 3.16.

The following property is important.

**Proposition 3.18.** Let \(\pi : X' \rightarrow X\) be a proper birational morphism between normal surfaces. Let \(D\) be an \(f\)-\(Z\)-positive \(\mathbb{R}\)-divisor on \(X\) and \(Z\) a \(\pi\)-exceptional \(\mathbb{R}\)-divisor on \(X\) with \(\mathcal{N} = 0\). Then \(\pi^* D + Z\) is \((f \circ \pi)\)-\(Z\)-positive. In particular, \(\mathcal{P} \pi^* D\) is \((f \circ \pi)\)-\(Z\)-positive for any \(f\)-\(Z\)-positive \(\mathbb{Z}\)-divisor \(D\) on \(X\).
Proof. We put $D' := \pi^*D + Z$. Suppose that there is a negative definite $(f \circ \pi)$-exceptional divisor $B' > 0$ on $X'$ such that $B' - D'$ is nef over $B'$. Let us denote $B' = \pi^*B + B_{\pi}$ for some $\pi$-exceptional $\mathbb{Q}$-divisor $B_{\pi}$ and $B := \pi_*B' \geq 0$. Let $D = P + N$ be the Zariski decomposition. Then $D' = \pi^*P + (\pi^*N + Z)$ gives the Zariski decomposition of $D'$ since $Z$ is effective and $\pi$-exceptional. Since $\pi^*P$ and $B' - D'$ is nef over $B'$, we have $B' \leq \pi^*N + Z$ by Lemma 3.6. Taking $\pi_*$, we obtain $B \leq N$. In particular, $B = 0$ or $B > 0$ is negative definite $f$-exceptional. We write $B_{\pi} - Z = G^+ - G^-$, where $G^+$ and $G^-$ are effective $\pi$-exceptional $\mathbb{R}$-divisors having no common components. Note that $\text{Supp}(G^+) \subset \text{Supp}(B')$. If $G^+ > 0$, then we can take an irreducible curve $C$ in the support of $G^+$ such that $G^+C < 0$ since $G^+$ is negative definite. Hence we have $C \leq B'$ and

\[(B' - D')C = (\pi^*(B - D) + B_{\pi} - Z)C = (G^+ - G^-)C < 0,\]

which contradicts the nefness of $B' - D'$ over $B'$. Thus $G^+ = 0$ holds. Then we have $B > 0$ because $\mathcal{L}Z \subset 0$. Since $D$ is $f$-$\mathcal{L}$-positive, there exists an irreducible curve $C$ in $B$ such that $(B - D)C < 0$. Let $\hat{C}$ be the proper transform of $C$ on $X'$. Then $\hat{C}$ is contained in $B'$ and

\[(B' - D')\hat{C} = (B - D)C - G^-\hat{C} < 0,\]

which contradicts the nefness of $B' - D'$ over $B'$. Hence $D'$ is $(f \circ \pi)$-$\mathcal{L}$-positive. \qed

Definition 3.19 (cf. [9, 15]). Let $D$ be an effective divisor on a regular complete surface $X$ and $m$ an integer. Then $D$ is called \textit{numerically $m$-connected} or $m$-connected for short (resp. \textit{chain-connected} or series-connected) if $D_1D_2 \geq m$ (resp. $-D_1$ is not nef over $D_2$) holds for any decomposition $D = D_1 + D_2$ with $D_1, D_2 > 0$. Clearly, 1-connected implies chain-connected.

Proposition 3.20. Let $D$ be a chain-connected divisor on a regular complete surface $X$, which is not negative definite. Then $D$ is $\mathcal{L}$-positive.

Proof. By Proposition 1.2 in [9], $D$ is chain-connected if and only if for any subdivider $0 < D_0 \leq D$, there exists a connecting chain from $D_0$ to $D$. Let $D = P + N$ be the Zariski decomposition. Then we have $P > 0$ since $D$ is not negative definite. Thus $D - \mathcal{L}N \subset 0$ and so the claim follows from Proposition 3.16. \qed

Remark 3.21. (1) Ramanujam’s connectedness lemma implies that any nef and big effective divisor is 1-connected (cf. [4] p.242). In summary, the following implication holds for an effective and not negative definite divisor:

\[
\text{nef and big} \implies \text{1-connected} \implies \text{chain-connected} \implies \text{Z-positive}.
\]

(2) Any chain-connected divisor $D$ is connected, i.e., $H^0(D, \mathcal{O}_D)$ is a field ([15] Corollary 3.6). On the other hand, Z-positive effective divisors are not necessarily connected. For example, a finite sum of fibers of a fiber space over a curve is Z-positive. However, if further assume that $D$ is big, it is connected since $H^1(X, \mathcal{O}_X(-D)) = 0$ holds by Theorem 4.1 proved in the next section.

4. Vanishing theorem on a normal surface

In this section, we prove the following vanishing theorem.
Theorem 4.1. Let $D$ be an $f$-big divisor on $X$ and $D = P_Z + N_Z$ the integral Zariski decomposition in Theorem 3.5. Let $\mathcal{L}_D$ and $\mathcal{L}'_D$ respectively be the rank $1$ sheaves on $N_Z$ defined by the cokernel of the homomorphisms $\mathcal{O}_X(K_X + P_Z) \to \mathcal{O}_X(K_X + D)$ and $\mathcal{O}_X(-D) \to \mathcal{O}_X(-P_Z)$ induced by multiplying a defining section of $N_Z$.

1) Assume that $\dim(Y) = 0$ and the base field $k$ is of characteristic $0$ if we consider the algebraic setting. Then we have

$$H^1(X, \mathcal{O}_X(K_X + D)) \cong H^1(N_Z, \mathcal{L}_D),$$

and

$$H^1(X, \mathcal{O}_X(-D)) \cong H^0(N_Z, \mathcal{L}'_D).$$

2) Assume that $\dim(Y) \geq 1$. Then we have

$$R^1f_*\mathcal{O}_X(K_X + D) \cong R^1f_*\mathcal{L}_D.$$ 

In particular, we have

$$\text{length} R^1f_*\mathcal{O}_X(K_X + D) = \dim H^1(N_Z, \mathcal{L}_D).$$

Remark 4.2. Theorem 4.1 (1) is a generalization of the Kawamata-Viehweg vanishing [8, 23], Ramanujam’s 1-connected vanishing [18], Miyaoka’s vanishing [15] on a surface. Langer’s vanishing ([11] Theorem 3.2 and its remarks) is essentially the same as Theorem 4.1 (1) in the case that $X$ is projective and $D$ is $\mathbb{Z}$-positive, which was proved by using a log version of Reider’s method [19]. Our proof is more elementary. Theorem 4.1 (2) is a generalization of the relative vanishing theorem due to Sakai and Kollár-Kovács (cf. [20], [7]).

4.1. Proof of Theorem 4.1 (2). First, we assume that $\dim(Y) \geq 1$. We follow Sakai’s argument in [20] in this subsection.

Lemma 4.3 ((Semi-)local vanishing on a regular surface. cf. [20], [7]). Assume that $X$ is regular. Let $D$ be an $f$-big $f$-$\mathbb{Z}$-positive divisor on $X$. Then $R^1f_*\mathcal{O}_X(K_X + D) = 0$ holds.

Proof. In order to prove the claim, it suffices to show the completion $R^1f_*\mathcal{O}_X(K_X + D)_y$ is $0$ for any closed point $y \in Y$. By the formal function theorem, it suffices to show that for any $f$-exceptional divisor $B$ on $X$, the vanishing $H^1(B, \mathcal{O}_B(K_X + D)) = 0$ holds, which is equivalent to $H^0(B, \mathcal{O}_B(B - D)) = 0$ by the Serre duality. We will show this by the induction on the number of irreducible components of $B$. Note that $B$ is negative definite when $\dim(Y) = 2$, or negative semi-definite when $\dim(Y) = 1$ by Zariski’s lemma. If $B$ is irreducible, then it follows that $(B - D)B < 0$ since $D$ is big and $f$-$\mathbb{Z}$-positive (note that when $\dim(Y) = 1$, the bigness of $D$ implies that $B - D$ is not nef over $B$ if $\text{Supp}(B)$ contains the support of a fiber $F$ of $f$ since $DF > 0$, see Corollary A.6). Hence the claim follows. We assume that $B$ is not irreducible. Then there is an irreducible component $C$ of $B$ such that $(B - D)C < 0$ since $D$ is big and $f$-$\mathbb{Z}$-positive. Taking $H^0$ of the exact sequence

$$0 \to \mathcal{O}_{B-C}(B - C - D) \to \mathcal{O}_B(B - D) \to \mathcal{O}_C(B - D) \to 0,$$

we have $H^0(B, \mathcal{O}_B(B - D)) = 0$ by the inductive assumption. Hence the claim holds. \Box

Proposition 4.4 (Projection formula. cf. [20]). Let $\pi: X' \to X$ be a proper birational morphism between normal surfaces. Let $D$ be an $\mathbb{R}$-divisor on $X$ and $Z$ a $\pi$-exceptional effective $\mathbb{R}$-divisor on $X'$. Then $\pi_*\mathcal{O}_{X'}(\mathcal{L}^\ast D + Z, \mathcal{D}) \cong \mathcal{O}_X(\mathcal{L}D, \mathcal{D})$ holds.
Proof. We prove this in the algebraic setting. In the analytic case, the proof is similar and proved in [20]. First, we show the claim when $E := X$. Hence it is sufficient to show that $H^0(X', \mathcal{O}_{X'}(\langle \pi^*D + Z \rangle)) \rightarrow H^0(X' \setminus E, \mathcal{O}_{X'}(\langle \pi^*D + Z \rangle)) \rightarrow H^0(\mathcal{O}_{X'}(\langle \pi^*D + Z \rangle))$.

Because $X$ is normal, we have natural isomorphisms

$$H^0(X' \setminus E, \mathcal{O}_{X'}(\langle \pi^*D + Z \rangle)) \cong H^0(X \setminus \{x\}, \mathcal{O}_X(\langle \pi D \rangle)) \cong H^0(X, \mathcal{O}_X(\langle \pi D \rangle)).$$

Hence it is sufficient to show that $H^1(E, \mathcal{O}_{X'}(\langle \pi^*D \rangle)) = 0$. By the local duality (cf. [3] Corollary 3.5.15), it suffices to show that $R^1\pi_*\mathcal{O}_{X'}(K_{X'} - \langle \pi^*D \rangle) = 0$. Since $-\langle \pi^*D \rangle = \pi^{-1}\mathbb{Z}$ is $\pi$-Z-positive by Corollary [3.17], the claim follows from Lemma [4.3].

Next, we consider the case that $X'$ is not necessarily regular. We take a resolution $\pi': X' \rightarrow X$. Then we have

$$\pi_*\mathcal{O}_{X'}(\langle \pi^*D + Z \rangle) \cong \pi'_*\mathcal{O}_{X'}(\langle \pi'^*D + Z \rangle) = (\pi \circ \pi')_*\mathcal{O}_{X'}(\langle \pi \circ \pi' \rangle^*D + \pi'^*Z),$$

where the first and the last isomorphisms are due to the projection formula in the regular case.

\[\square\]

**Proposition 4.5** (Semi-)local vanishing on a normal surface. cf. [20]. Let $D$ be an $f$-big $\pi$-$\mathbb{Z}$-positive divisor on the normal surface $X$. Then $R^1\pi_*\mathcal{O}_X(K_X + D) = 0$ holds.

Proof. Let $\pi: X' \rightarrow X$ be a resolution. By using the Leray spectral sequence

$$E_2^{pq} = R^p\pi_*R^q\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D \rangle) \Rightarrow E^m = R^m(f \circ \pi)_*\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D \rangle)$$

and $E^1 = 0$ by Proposition [3.18] and Lemma [4.3] we have $R^1\pi_*\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D \rangle) = 0$. Let $Z$ be an effective $\pi$-exceptional divisor on $X'$. There is an exact sequence

$$0 \rightarrow \pi_*\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D \rangle) \rightarrow \pi_*\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D + Z \rangle) \rightarrow T \rightarrow 0,$$

where $T$ is a torsion sheaf on $X$ whose support is of dimension 0. Taking $R^1\pi_*$, we have $R^1\pi_*\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D + Z \rangle) = 0$. Let us write $K_{X'} = \pi^*K_X - \Delta$, where $\Delta$ is a $\pi$-exceptional $\mathbb{Q}$-divisor on $X'$. By taking $Z$ sufficiently effective, we have

$$\pi_*\mathcal{O}_{X'}(K_{X'} + \langle \pi^*D + Z \rangle) = \pi_*\mathcal{O}_{X'}(\langle \pi^*K_X + \Delta \rangle) \cong \mathcal{O}_X(K_X + D)$$

by Proposition [4.4], where $Z' := Z - \Delta + \langle \pi^*D - \pi^*D + \{\pi^*(K_X + D)\}$ is an effective $\pi$-exceptional divisor. Hence $R^1\pi_*\mathcal{O}_X(K_X + D) = 0$ follows.

\[\square\]

**Proof of Theorem 4.1** (2). Let $D = P_Z + N_Z$ be the integral Zariski decomposition as in Theorem [3.5]. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(K_X + P_Z) \rightarrow \mathcal{O}_X(K_X + D) \rightarrow \mathcal{L}_D \rightarrow 0.$$ 

Since $R^1\pi_*\mathcal{O}_X(K_X + P_Z) = 0$ for $i \geq 1$ by Proposition [4.5] we get $R^1\pi_*\mathcal{O}_X(K_X + D) \cong R^1\pi_*\mathcal{L}_D$.

\[\square\]
4.2. Proof of Theorem 4.1 (1). Next, we assume that \( \dim(Y) = 0 \) and \( X \) is a normal complete surface, algebraic over a field of characteristic 0 or in the analytic setting. The following lemma can be proved similarly to Proposition 4.5 by using Proposition 4.4.

**Lemma 4.6.** Let \( \pi: X' \to X \) be a proper birational morphism between normal complete surfaces. Let \( M \) be an \( \mathbb{R} \)-divisor on \( X \). If \( H^i(X', \mathcal{O}_{X'}(K_{X'} + [\pi^* M'])) = 0 \) for some \( i \geq 1 \), then \( H^i(X, \mathcal{O}_X(K_X + [M'])) = 0 \) holds.

Sakai proved in [20] the following vanishing theorem by using the usual Kawamata-Viehweg vanishing theorem and Lemma 4.6.

**Proposition 4.7** ([20] Theorem 5.1). Let \( M \) be a nef and big \( \mathbb{R} \)-divisor on \( X \). Then we have \( H^i(X, \mathcal{O}_X(K_X + [M'])) = 0 \) for \( i \geq 1 \).

**Proof.** Taking a resolution and using Lemma 4.6, we may assume that \( X \) is smooth. Shift-ing the coefficients of prime divisors in \( M \), we may assume that \( M \) is ample. Moreover, taking the base change to the algebraic closure, we may assume that the base field is algebraically closed. By taking a log resolution \( \pi: X' \to X \) of \( (X, \{ M \}) \) and using Lemma 4.6 again, the assertion follows from the Kawamata-Viehweg vanishing theorem. \( \square \)

The following corollary in the smooth surface case is a special version of the Miyaoka vanishing theorem in [15].

**Corollary 4.8** (Miyaoka vanishing theorem on a normal surface, cf. [15]). Let \( D \) be a big divisor on \( X \) with the Zariski decomposition \( D = P + N \). If \( \lceil N \rceil = 0 \), then \( H^i(X, \mathcal{O}_X(K_X + D)) = 0 \) for any \( i \geq 1 \).

**Proof.** Since \( P = D - N \) is a nef and big \( \mathbb{Q} \)-divisor on \( X \) and \( \lceil P \rceil = D - \lceil N \rceil = D \), the claim holds from Proposition 4.7. \( \square \)

By using Corollary 4.8, we will prove the big \( \mathbb{Z} \)-positive vanishing, the smooth case of which is essentially the same as Theorem 2.7 in [15].

**Proposition 4.9.** Let \( D \) be a big \( \mathbb{Z} \)-positive divisor on \( X \). Then we have \( H^1(X, \mathcal{O}_X(K_X + D)) = 0 \).

**Proof.** First, we assume that \( X \) is smooth. Let \( D = P + N \) be the Zariski decomposition. Since \( D \) is \( \mathbb{Z} \)-positive, there exists a connecting chain \( D - \lceil N \rceil =: D_0 < D_1 < \cdots < D_m := D \) by Proposition 3.16. Putting \( C_i := D_i - D_{i-1} \), we have \( D_{i-1} C_i > 0 \). From Corollary 4.8 we have \( H^1(X, \mathcal{O}_X(K_X + D_0)) = 0 \). By taking the cohomologies of the exact sequences

\[
0 \to \mathcal{O}_X(K_X + D_{i-1}) \to \mathcal{O}_X(K_X + D_i) \to \mathcal{O}_{C_i}(K_X + D_i) \to 0
\]

and using the vanishing \( H^1(C_i, \mathcal{O}_{C_i}(K_X + D_i)) \cong H^0(C_i, \mathcal{O}_{C_i}(-D_{i-1}))^* = 0 \), the claim holds by the induction on \( m \).

Next, we will prove the claim in general. Take a resolution \( \pi: X' \to X \). Then \( \lceil \pi^* D \rceil \) is big and \( \mathbb{Z} \)-positive by proposition 3.18. Thus we have \( H^1(X', \mathcal{O}_{X'}(K_{X'} + [\pi^* D'])) = 0 \). By using Lemma 4.6, we get \( H^1(X, \mathcal{O}_X(K_X + D)) = 0 \). \( \square \)

**Proof of Theorem 4.1 (1).** Let \( D = P_2 + N_2 \) be the integral Zariski decomposition. Then \( H^1(X, \mathcal{O}_X(K_X + D)) \cong H^1(N_2, \mathcal{L}_D) \) holds by using the exact sequence

\[
0 \to \mathcal{O}_X(K_X + P_2) \to \mathcal{O}_X(K_X + D) \to \mathcal{L}_D \to 0
\]
and \(H^1(X, O_X(K_X + P_Z)) = 0\) by Proposition 4.9. The dual case \(H^1(X, O_X(-D)) \cong H^0(N_Z, \mathcal{L}_D')\) can be proved similarly.

5. Reider-type theorems

In this section, we apply Theorem 4.11 to the criterion of higher order separations of adjoint (relative) linear systems. Let \(f: X \to Y\) be a proper surjective morphism from a normal surface. If \(\dim(Y) = 0\), we assume that the base field is of characteristic 0 if we consider the algebraic setting.

5.1. The main theorem. Let \(\zeta \subset X\) be a cluster, that is, a subscheme (or an analytic subset) of dimension 0. First, we define an invariant \(\delta_\zeta\) of the germ \((X, \zeta)\).

**Definition 5.1.** We consider resolutions \(\pi: X' \to X\) of the singularities contained in \(\zeta\) and \(\pi\)-exceptional divisors \(Z > 0\) satisfying \(\pi_*O_{X'}(-Z) \subset I_\zeta\), where \(I_\zeta\) is the sheaf of ideals corresponding to \(\zeta\). For such \(\pi\) and \(Z\), we define \(\delta_\zeta(\pi, Z)\) as the number \(-|\Delta - Z|^2\) if \(\Delta - Z\) is not effective, or 0 otherwise, where \(\Delta := \pi^*K_X - K_{X'}\) is the anti-canonical cycle of \(\pi\). The invariant \(\delta_\zeta\) of the germ \((X, \zeta)\) is defined as

\[
\delta_\zeta := \inf \{\delta_\zeta(\pi, Z) \mid \pi \text{ and } Z \text{ are as above}\}.
\]

Note that there exist \(\pi\) and \(Z\) such that \(\delta_\zeta = \delta_\zeta(\pi, Z)\) since \(Z\) is integral and there is a sufficiently divisible integer \(s\) such that \(s\Delta\) is integral for any resolution \(\pi\). By definition, we can write \(\delta_\zeta = \sum_x \delta_{\zeta_x}\), where \(\zeta_x\) is the subcluster of \(\zeta\) supported at the point \(x\).

The main theorem in this section is the following.

**Theorem 5.2.** Let \(D\) be an \(f\)-big divisor on \(X\) and \(D = P + N\) (resp. \(D = P_Z + N_Z\)) the Zariski decomposition (resp. the integral Zariski decomposition as in Theorem 3.5). Let \(\zeta\) be a cluster on which \(K_X + D\) is Cartier. When \(\dim(Y) = 0\), we further assume that \(P^2 > \delta_\zeta\) (resp. \(P^2_Z > \delta_\zeta\)). Then the natural map \(f_*O_X(K_X + D) \to f_*O_X(K_X + D)|_\zeta\) is surjective, or there exists an \(f\)-exceptional divisor \(B > 0\) on \(X\) intersecting \(\zeta\) such that \((D - B)B \leq \delta_\zeta/4\) (resp. and \(D + N_Z - 2B\) is big).

**Proof.** Assume that \(f_*O_X(K_X + D) \to f_*O_X(K_X + D)|_\zeta\) is not surjective. Then we have \(R^1 f_*I_\zeta O_{X'}(K_{X'} + D) \neq 0\).

First, we consider the case that \(D\) is \(f\)-Z-positive. We take a resolution \(\pi: X' \to X\) of the singularities contained in \(\zeta\) and a \(\pi\)-exceptional divisor \(Z > 0\) on \(X'\) such that \(\pi_*O_{X'}(-Z) \subset I_\zeta\) and \(\delta_\zeta = \delta_\zeta(\pi, Z)\). Then \(\pi_*O_{X'}(\pi^*(K_X + D) - Z)\) is a subsheaf of \(I_\zeta O_{X'}(K_{X'} + D)\) whose cokernel is supported on \(\zeta\). Hence we have \(R^1 f_*\pi_*O_{X'}(\pi^*(K_X + D) - Z) \neq 0\). By using the Leray spectral sequence

\[
E_2^{p,q} = R^p f_* R^q \pi_* O_{X'}(\pi^*(K_X + D) - Z) \Rightarrow E_\infty = R^m f_* O_{X'}(\pi^*(K_X + D) - Z),
\]

we have \(R^1 f_* O_{X'}(K_{X'} + D') \neq 0\), where we put \(D' := \pi^* D + \Delta - Z\) and \(f' := f \circ \pi\). Note that \(\Delta - Z\) is not effective in this case. Indeed, if \(\Delta \leq \Delta\), then we may assume that \(D' = \pi^* D + \Delta\), which is \(f'\)-big and \(f'\)-Z-positive by Proposition 3.17. By Theorem 4.11, it is a contradiction to the non-vanishing of \(R^1 f_* O_{X'}(K_{X'} + D')\). When \(\dim(Y) = 0\), \(D'\) is \(f'\)-big since \(\pi^* P + \Delta - Z\) is \(f'\)-big from the assumption \(P^2 > \delta_\zeta\) and Lemma A.7. In the case of \(\dim(Y) \geq 1\), \(D'\) is always \(f'\)-big from the bigness of \(D\). Hence the non-vanishing of \(R^1 f_* O_{X'}(K_{X'} + D')\) implies that \(D'\) is not \(f'\)-Z-positive by Theorem 4.11. Let \(B' > 0\) be the Z-negative part of \(D'\) as in Theorem 3.5 and put \(B := \pi_* B'\). Then \(B\) is non-zero. Indeed, if \(B'\) is \(\pi\)-exceptional, then we have \(R^1 f_* O_{X'}(K_{X'} + D') \neq 0\).
\[ D' - B' \neq 0 \] by the same argument as above after replacing \( Z \) with \( Z + B' \), which is a contradiction to Theorem 3.3. Let us write \( B' = \pi^* B + B_{\pi} \) for some \( \pi \)-exceptional \( \mathbb{Q} \)-divisor \( B_{\pi} \) on \( X' \). Then we have \( 0 \leq (B' - D')B' = (B - D)B + (B_{\pi} - \Delta + Z)B_{\pi} \).

\[
(D - B)B \leq (B_{\pi} - \Delta + Z)B_{\pi} = \left( B_{\pi} - \frac{1}{2}(\Delta - Z) \right)^2 - \frac{1}{4}(\Delta - Z)^2 \leq \frac{1}{4} \delta_\zeta.
\]

We show that \( B \) intersects \( \zeta \). Suppose that \( B \cap \zeta = \emptyset \). Then for any subcurve \( C \leq B \), the proper transform \( \hat{C} \leq B' \) of \( C \) equals the total transform \( \pi^* C \). Thus \( B \) is negative definite and we have

\[
0 \leq (B' - D')\hat{C} = (\pi^*(B - D) + B_{\pi} - \Delta + Z)\pi^* C = (B - D)C.
\]

Hence \( B - D \) is nef over \( B \), which contradicts the \( f \)-Z-positivity of \( D \). In the case of \( \dim(Y) = 0 \) and \( D^2 > \delta_\zeta \), we will show that \( D - 2B \) is \( f \)-big. For this, it is enough to show that \( D' - 2B' \) is big since \( D - 2B = \pi_*(D' - 2B') \) and Lemma A.1 (1). This follows from Corollary A.3.

For any \( f \)-big divisor \( D \) which is not \( f \)-Z-positive, we consider the integral Zariski decomposition \( D = P_Z + N_Z \) as in Theorem 3.3. If \( N_Z \) intersects \( \zeta \), then \( B := N_Z \) satisfies the properties in Theorem 5.2. Then we may assume that \( N_Z \) and \( \zeta \) are disjoint. Hence \( f_*O_X(K_X + P_Z) \to f_*O_X(K_X + P_Z)|_{\zeta} \) is also not surjective. As shown in the first half, we can take an \( f \)-exceptional curve \( B \) on \( X \) intersecting \( \zeta \) such that \( (P_Z - B)B \leq \delta_\zeta/4 \). If \( N_Z B \leq 0 \), then we have \( (D - B)B \leq (P_Z - B)B \leq \delta_\zeta/4 \). Assume that \( N_Z B > 0 \). Putting \( B := B + N_Z \), we have

\[
(D - B)B = (P_Z - B)(B + N_Z) = (P_Z - B)B + P_Z N_Z - B N_Z < (P_Z - B)B \leq \delta_\zeta/4.
\]

Replacing \( B \) to \( B \), we have the desired inequality. The bigness of \( D + N_Z - 2B \) follows from the fact that \( P_Z - 2B \) is big.

\[ \square \]

\textbf{Remark 5.3.} Reider’s original proof [19] uses the vector bundle technique, especially, Serre’s construction and the Bogomolov inequality. Many authors generalize Reider’s theorem to singular surfaces (e.g., [21], [4], [11]). There are mainly two ways of the proof. The first one is to use the vector bundle technique along the original one and the second one is to use the Kawamata-Viehweg vanishing theorem. Our proof of Theorem 5.2 belongs to the latter one. Moreover, Theorem 5.2 contains log versions \( D = ^{\operatorname{rel}} M \), a nef and big \( \mathbb{R} \)-divisor \( M \) and relative versions of Reider-type theorems. For the case of normal projective surfaces, Theorem 5.2 is slightly weaker than Theorem 3.2 in [11] because the invariant \( \delta_\zeta \) in this paper may be greater than the \( \delta_\zeta \) defined in [11].

\textbf{5.2. Upper bound of \( \delta_\zeta \).} Before stating corollaries of Theorem 5.2 we collect some upper bounds of the invariants \( \delta_\zeta \) with length(\( \zeta \)) = 1 or 2 for the readers’ convenience (more details, see [4], [10]).

First, we consider the case that \( \zeta = x \) is a closed point of a normal surface \( X \). The following is a non-standard definition.

\textbf{Definition 5.4.} The germ \((X, x)\) is called \( \textit{log terminal} \) if there exists a resolution \( \pi : X' \to X \) of \((X, x)\) such that \( c_1(\Delta) \leq 0 \), where \( \Delta = \pi^* K_X - K_{X'} \) is the anti-canonical cycle of \( \pi \).
The following is well-known to experts. For the convenience of readers, we will prove this.

**Lemma 5.5** ([1] Corollary 2.2.11). Any log terminal germ \((X, x)\) is rational, that is, \(R^1\pi_*O_{X'} = 0\) and \(R^2\pi_*O_{X'}(K_{X'}) = 0\) holds for any resolution \(\pi: X' \to X\) of \((X, x)\).

**Proof.** \(R^1\pi_*O_{X'}(K_{X'}) = 0\) follows from Theorem 4.1. If \((X, x)\) is regular and \(\pi: X' \to X\) is the blow-up at \(x\), then \(\Delta\) is \(\pi\)-nef. If \((X, x)\) is a log terminal singularity and \(\pi: X' \to X\) is a minimal resolution of \((X, x)\), then \(\Delta = \{\Delta\}\) is \(\pi\)-Z-positive by Proposition 3.16. Then we have

\[ R^1\pi_*O_{X'} = R^1\pi_*O_{X'}(K_{X'} - \pi^*K_X + \Delta) = 0 \]

for such \((X, x)\) and \(\pi\) by Theorem 4.1. For any resolution \(\pi: X' \to X\), we also obtain \(R^1\pi_*O_{X'} = 0\) by using the Leray spectral sequence \(R^p\pi_*R^q\pi'_*O_X \to R^m\pi_*O_{X'}\) for a composition \(\pi = \pi' \circ \pi''\) inductively. \(\square\)

**Lemma 5.6** (cf. [1] Theorem 2). We have \(\delta_x \leq 4\) if \((X, x)\) is regular, \(\delta_x \leq 2\) if \((X, x)\) is a log terminal singularity and \(\delta_x = 0\) otherwise.

**Proof.** Let \(\pi: X' \to X\) be the blow-up at \(x\) if \((X, x)\) is regular, or the minimal resolution of \((X, x)\) otherwise and \(Z\) the fundamental cycle of \(\pi\). When \((X, x)\) is regular, then \(\Delta - Z = -2E, E\) is the exceptional \((-1)\)-curve on \(X'\) and so \(\delta_x \leq -4E^2 = 4\) holds. When \((X, x)\) is a log terminal singularity, then we have

\[ \delta_x \leq -((\Delta - Z)^2 = K_X(\Delta - Z) - Z(X_X + Z) = -K_X(Z - \Delta) - (2p_a(Z) - 2) = -K_X(Z - \Delta) + 2 \leq 2, \]

where \(p_a(Z) = 0\) since \(Z\) is the fundamental cycle on a resolution of a rational singularity and the last inequality follows from the fact that \(K_X\) is \(\pi\)-nef and \(Z - \Delta\) is effective. The assertion for the not log terminal case is clear. \(\square\)

Next, we consider a cluster \(\zeta\) of length 2. For simplicity, we assume that \(X\) is regular.

**Lemma 5.7.** \(\delta_\zeta \leq 8\) holds.

**Proof.** If the support of \(\zeta\) consists of two points \(x, y\), we can write \(\delta_\zeta = \delta_x + \delta_y\). Then the claim follows from Lemma 5.6. We assume that the support of \(\zeta\) is one point \(x\). The defining ideal sheaf \(I_\zeta\) (or its completion) is of the form \((z, w^2)\) for a local coordinate \((z, w)\) on \(X\) around \(x\). Hence we can take a proper birational morphism \(\pi: X' \to X\) which is a composite of two blow-ups \(\pi_x\) and \(\pi_y\) at \(x\) and a point \(y\) infinitely near to \(x\), and a \(\pi\)-exceptional divisor \(Z = E_y + \pi_x^*E_x\) such that \(\pi_*O_{X'}(-Z) = I_\zeta\). Then we have \(\delta_\zeta \leq 8\). \(\square\)

5.3. **Corollaries of Theorem 5.2.** In this subsection, we give some corollaries of Theorem 5.2 for freeness and very ampleness of adjoint linear systems.

**Corollary 5.8.** Let \(X\) be a normal complete surface, algebraic over a field of characteristic 0 or analytic. Let \(D\) be a nef divisor on \(X\) and \(x \in X\) a point with \(K_X + D\) Cartier at \(x\). Assume that
(i) $D^2 > \delta_x$, and

(ii) $DB \geq \frac{1}{2} \delta_x$ for any curve $B$ on $X$ passing through $x$.

Then $x$ is not a base point of $|K_X + D|$.

Proof. If $x$ is a base point of $|K_X + D|$, then there is a curve $B$ passing through $x$ on $X$ such that $D - 2B$ is big from Theorem 5.2. Since $D$ is nef, $(D - 2B)D \geq 0$ holds. Hence $DB \leq D^2/2 < \delta_x/2$, which contradicts the assumption. $\square$

**Corollary 5.9** (cf. [8]). Let $f : X \to Y$ be a fiber space from a regular surface $X$ to a curve $Y$. Let $D$ be an $f$-nef divisor on $X$ with $DF > 0$ for a fiber $F$ of $f$. Then for any base point $x$ of $f_*\mathcal{O}_X(K_X + D)$, there exists an $f$-exceptional curve $B$ on $X$ passing through $x$ such that one of the following holds.

(i) $DB = 0$ and $B^2 = -1$.

(ii) $DB = 1$ and $B^2 = 0$.

In particular, if one of the following conditions (a) and (b) holds, then the natural map $f^*f_*\mathcal{O}_X(K_X + D) \to \mathcal{O}_X(K_X + D)$ is surjective.

(a) $D = mH$, $H$ is $f$-ample and $m \geq 2$.

(b) $f$ is a relatively minimal fibration of genus greater than 1 and $D - K_X$ is $f$-nef.

Proof. Let $x$ be a base point of $f_*\mathcal{O}_X(K_X + D)$. Then Theorem 5.2 implies that there is an $f$-exceptional curve $B$ passing through $x$ on $X$ such that $(D - B)B \leq 1$. Since $B^2 \leq 0$ by Zariski’s lemma, we have $DB \leq 1 + B^2 \leq 1$. Hence $B^2 = 0$ or $-1$. When $B^2 = 0$, then we can write $B = aF$ for some $a \in \mathbb{Q}_{\geq 0}$. Then $DB > 0$ by assumption. Hence the first half of the claim follows. The rest of the claim follows easily by using the fact that $K_XB + B^2$ is even. $\square$

Note that the last assertion of Corollary 5.9 also follows from Lemmas 1.3.2 and 4.2.1 in [8].

Similarly to Corollary 5.9 the following can be proved.

**Corollary 5.10** (cf. [21] Theorem 7, [12] Theorem 3.1). Let $f : X \to Y$ be a resolution of a normal surface singularity $(Y, y)$. Let $D$ be an $f$-nef divisor on $X$. Then for any base point $x$ of $f_*\mathcal{O}_X(K_X + D)$, there exists an $f$-exceptional curve $B$ on $X$ passing through $x$ such that $DB = 0$ and $B^2 = -1$ holds. In particular, if $D$ is $f$-ample, or $f$ is a minimal resolution and $D - K_X$ is $f$-nef, then the natural map $f^*f_*\mathcal{O}_X(K_X + D) \to \mathcal{O}_X(K_X + D)$ is surjective.

We can show the very ample cases similarly. In the rest of the section, we assume that the base field is algebraically closed if we consider the algebraic setting. Note that for a proper morphism $f : X \to Y$ between varieties, a Cartier divisor $L$ on $X$ is $f$-very ample if and only if the natural map $f_*\mathcal{O}_X(L) \to f_*\mathcal{O}_X(L)|_{\zeta}$ is surjective for any cluster $\zeta$ of length 2 in a fiber of $f$.

**Corollary 5.11.** Let $X$ be a normal complete surface, algebraic over a field of characteristic 0 or analytic. Let $D$ be a nef divisor on $X$ and $\zeta$ a cluster on which $K_X + D$ is Cartier. Assume that

(i) $D^2 > \delta_\zeta$, and

(ii) $DB \geq \frac{1}{2} \delta_\zeta$ for any curve $B$ on $X$ intersecting $\zeta$. 

Then $|K_X + D|$ separates $\zeta$. In particular, if $X$ is smooth, $D^2 > 8$ and $DB \geq 4$ for any curve $B$ on $X$, then $K_X + D$ is very ample.

**Corollary 5.12.** Let $f : X \to Y$ be a fiber space from a smooth surface $X$ to a curve $Y$. Let $D$ be an $f$-nef divisor on $X$ with $DF > 0$ for a fiber $F$ of $f$. If $f_*$ $\mathcal{O}_X(K_X + D)$ does not separate a cluster $\zeta$ of length 2, then there exists an $f$-exceptional curve $B$ on $X$ intersecting $\zeta$ such that one of the following holds.

(i) $DB = 0$ and $B^2 = -2$ or $-1$.
(ii) $DB = 1$ and $B^2 = -1$ or $0$.
(iii) $DB = 2$ and $B^2 = 0$.

In particular, if one of the following conditions (a) and (b) holds, then $K_X + D$ is $f$-very ample.

(a) $D = mH$, $H$ is $f$-ample and $m \geq 3$.
(b) $f$ is a relatively minimal fibration of genus greater than 1, $D - K_X$ is $f$-nef and there are no curves $B$ with $p_a(B) = i$ and $B^2 = i - 2$ ($i = 0, 1, 2$) contained in fibers.

**Corollary 5.13.** Let $f : X \to Y$ be a resolution of a normal surface singularity $(Y, y)$. Let $D$ be an $f$-nef divisor on $X$. If $f_* \mathcal{O}_X(K_X + D)$ does not separate a cluster $\zeta$ of length 2, then there exists an $f$-exceptional curve $B$ on $X$ intersecting $\zeta$ such that one of the following hold.

(i) $DB = 0$ and $B^2 = -2$ or $-1$.
(ii) $DB = 1$ and $B^2 = -1$.

In particular, if one of the following conditions (a) and (b) holds, then $K_X + D$ is $f$-very ample.

(a) $D = mH$, $H$ is $f$-ample and $m \geq 2$.
(b) $f$ is a minimal resolution, $D - K_X$ is $f$-nef and there are no $f$-exceptional $(-2)$-curves and $f$-exceptional curves $E$ with $p_a(E) = 1$ and $E^2 = -1$.

**Appendix A. Intersection theory on a normal surface**

In this appendix, we recall some fundamental results for Mumford’s intersection theory on a normal surface [10].

Let $X$ be a normal surface and $f : X \to Y$ a proper surjective morphism to a variety $Y$. Let $\text{WDiv}(X)$ be the group of Weil divisors on $X$. Let $\text{WDiv}(X/Y)$ be the subgroup of $\text{WDiv}(X)$ consisting of $f$-exceptional Weil divisors on $X$. Let $\text{CDiv}(X)$ be the group of Cartier divisors on $X$, that is, $\text{CDiv}(X) = H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, where $\mathcal{K}_X^*$ is the sheaf of invertible rational (or meromorphic) functions on $X$. For a Cartier divisor $D$ on $X$ and an $f$-exceptional curve $C$ on $X$, the intersection number of $D$ and $C$ is defined as $DC := \deg(\nu^* \mathcal{O}_X(D))$, where $\nu : \tilde{C} \to C$ is the normalization of $C$. By this intersection pairing, we can define the intersection form $\text{CDiv}(X) \times \text{WDiv}(X/Y) \to \mathbb{Z}$. On the other hand, since $X$ is normal, the natural cycle map

$$\text{CDiv}(X) \to \text{WDiv}(X); \quad D \mapsto \sum_{C: \text{prime divisor}} \text{ord}_C(D)C$$

is injective. According to [10], we can define the extended intersection form

$$\text{WDiv}(X) \times \text{WDiv}(X/Y) \to \mathbb{Q}$$
as follows. Let $\pi: X' \to X$ be a resolution of singularities (for existence, see [14]). For a Weil divisor $D$ on $X$, we define the Mumford pull-back $\pi^* D$ as a $\mathbb{Q}$-divisor $\hat{D} + \sum_i a_i E_i$, where $\hat{D}$ is the proper transform of $D$ and the rational coefficients $a_i$ of the $\pi$-exceptional prime divisors $E_i$ are uniquely determined by the equations $\hat{DE}_j + \sum_i a_i E_i E_j = 0$ for each $j$ since $\pi$-exceptional divisors $E_i$ form a negative definite matrix (cf. [16, 3]). For a Weil divisor $D$ and an $f$-exceptional Weil divisor $E$, we define the intersection number $DE := \pi^* D \cdot \pi^* E$. In general, the Mumford pull-back is defined for any proper birational morphism $\pi: X' \to X$ from a normal surface $X'$ as a group homomorphism $\pi^*: \text{WDiv}(X) \to \text{WDiv}(X') \otimes \mathbb{Q}$ by the same way. Let

$$N^1(X/Y)_\mathbb{Z} := \text{WDiv}(X)/\{D \in \text{WDiv}(X) \mid DE = 0 \text{ for } E \in \text{WDiv}(X/Y)\}$$

and

$$N_i(X/Y)_\mathbb{Z} := \text{WDiv}(X/Y)/\{E \in \text{WDiv}(X/Y) \mid DE = 0 \text{ for } D \in \text{WDiv}(X)\}$$

be the groups consisting of the numerical equivalence classes of Weil divisors and of $f$-exceptional Weil divisors, respectively. Let us denote

$$N^1(X/Y)_\mathbb{Q} := N^1(X/Y)_\mathbb{Z} \otimes \mathbb{Q}, \quad N^1(X/Y) := N^1(X/Y)_\mathbb{Z} \otimes \mathbb{R}$$

and

$$N_i(X/Y)_\mathbb{Q} := N_i(X/Y)_\mathbb{Z} \otimes \mathbb{Q}, \quad N_i(X/Y) := N_i(X/Y)_\mathbb{Z} \otimes \mathbb{R}.$$ 

Note that these notations are not standard because we use Mumford’s intersection form instead of the usual intersection form. In the case of $\dim(Y) = 0$, we denote by $N(X) = N^1(X) = N_1(X)$ instead of $N^1(X/Y)$ and $N_1(X/Y)$. The real vector spaces $N^1(X/Y)$ and $N_i(X/Y)$ are finite dimensional ([14], Ch. IV, Section 1, Proposition 4) and the intersection form induces the non-degenerate bilinear form $N^1(X/Y) \times N_1(X/Y) \to \mathbb{R}$. For a proper birational morphism $\pi: X' \to X$, the Mumford pull-back induces injective homomorphisms $\pi^*: N^1(X/Y) \to N^1(X')$ and $\pi^*: N_i(X/Y) \to N_i(X')$ and we can write

$$(A.1) \quad N^1(X'/Y)_\mathbb{Q} = \pi^* N^1(X/Y)_\mathbb{Q} \oplus (\oplus_i Q E_i), \quad N_i(X'/Y)_\mathbb{Q} = \pi^* N_i(X/Y)_\mathbb{Q} \oplus (\oplus_i Q E_i),$$

where $E_i$ are $\pi$-exceptional prime divisors.

Let $\text{Eff}^1(X/Y)$ (resp. $\text{Eff}_1(X/Y)$) be the subgroup of $N^1(X/Y)$ (resp. $N_1(X/Y)$) generated by numerical equivalence classes of effective (resp. $f$-exceptional effective) divisors on $X$. We define $\text{PE}^1(X/Y)$ (resp. $\text{PE}_1(X/Y)$) as the closure of $\text{Eff}^1(X/Y)$ in $N^1(X/Y)$ (resp. $\text{Eff}_1(X/Y)$ in $N_1(X/Y)$). An $\mathbb{R}$-divisor (resp. $f$-exceptional $\mathbb{R}$-divisor) $D$ on $X$ is said to be $f$-pseudo-effective (resp. $f$-exceptional pseudo-effective) if the numerical equivalence class of $D$ belongs to $\text{PE}^1(X/Y)$ (resp. $\text{PE}_1(X/Y)$).

An $\mathbb{R}$-divisor (resp. $f$-exceptional $\mathbb{R}$-divisor) $D$ on $X$ is called $f$-nef (resp. nef) if $DC \geq 0$ (resp. $CD \geq 0$) holds for any $f$-exceptional curve (resp. curve) $C$ on $X$. Let $\text{Nef}^1(X/Y)$ (resp. $\text{Nef}_1(X/Y)$) be the subgroup of $N^1(X/Y)$ (resp. $N_1(X/Y)$) generated by numerical equivalence classes of nef divisors on $X$ (resp. nef divisors on $X$). Then $\text{PE}^1(X/Y)$ and $\text{Nef}^1(X/Y)$ (resp. $\text{PE}_1(X/Y)$ and $\text{Nef}_1(X/Y)$) are closed cones in $N^1(X/Y)$ (resp. $N_1(X/Y)$) and we have the duality of cones (cf. [13] Proposition 1.4.28)

$$\text{PE}^1(X/Y) = \text{Nef}_1(X/Y)^*, \quad \text{PE}_1(X/Y) = \text{Nef}_1(X/Y)^*.$$ 

Recall that for an $n$-dimensional normal complete variety $X$, a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $X$ is big if there is a positive number $\alpha$ such that $h^0(X, \mathcal{O}_X(mD)) \geq \alpha m^n$ for all
sufficiently large and divisible integer \(m\) (cf. [13] Definition 2.2.1). For a proper morphism \(f: X \to Y\) between varieties, a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(D\) on \(X\) is called \(f\)-big if the restriction of \(D\) to the normalization of the generic fiber of the Stein factorization of \(f: X \to f(X)\) is big. For a proper surjective morphism \(f: X \to Y\) from a normal surface \(X\) to a variety \(Y\), a Weil divisor \(D\) on \(X\) is called \(f\)-big if the pull-back \(\pi^*D\) is \((f \circ \pi)\)-big for some (or equivalently, any) resolution \(\pi: X' \to X\). Let \(\text{Big}(X/Y)\) be the cone in \(N^1(X/Y)\) generated by \(f\)-big divisors on \(X\). An \(\mathbb{R}\)-divisor \(D\) on \(X\) is called \(f\)-big if the numerical class of \(D\) is contained in \(\text{Big}(X/Y)\).

An \(\mathbb{R}\)-divisor \(D\) is called numerically \(f\)-ample if \(DC > 0\) for any \(C \in \text{PE}_1(X/Y) \setminus \{0\}\). Let \(\text{NA}(X/Y)\) be the cone in \(N^1(X/Y)\) consisting of numerical equivalence classes of numerically \(f\)-ample divisors.

**Lemma A.1.** Let \(\pi: X' \to X\) be a proper birational morphism between normal surfaces and \(f':= f \circ \pi: X' \to Y\).

1. The push-forward of any \(f'\)-pseudo-effective (resp. \(f'\)-nef, \(f'\)-big, numerically \(f'\)-ample, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective) \(\mathbb{R}\)-divisor on \(X'\) by \(\pi\) is \(f\)-pseudo-effective (resp. \(f\)-nef, \(f\)-big, numerically \(f\)-ample, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective).

2. The pull-back of any \(f\)-pseudo-effective (resp. \(f\)-nef, \(f\)-big, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective, \(f\)-exceptional pseudo-effective) \(\mathbb{R}\)-divisor on \(X\) by \(\pi\) is \(f'\)-pseudo-effective (resp. \(f'\)-nef, \(f'\)-big, \(f'\)-exceptional pseudo-effective, \(f'\)-exceptional pseudo-effective).

**Proof.** The claims about nefness in (1) and (2) and numerically ampleness in (1) are easy to check. The claims about pseudo-effectiveness in (1) and (2) follows from the fact that the effectivity preserves under the push-forward and the pull-back, which are continuous maps. The claim about bigness in (2) follows by definition. The claim about bigness in (1) with \(\dim(Y) \geq 1\) is trivial since the generic fibers of \(f\) and \(f'\) are isomorphic. We assume that \(\dim(Y) = 0\). Let \(D\) be a big divisor on \(X\). Then, by Proposition [14] (cf. [20] Theorem 6.2), we have

\[
\pi_*\mathcal{O}_{X'}(\pi^*\pi_*mD_\pi + E) \cong \mathcal{O}_X(m\pi_*D)
\]

for any \(\pi\)-exceptional effective divisor \(E\) on \(X'\) and any positive integer \(m\). Since \(D\) is big and \(mD \leq \pi^*\pi_*mD_\pi + E\) for sufficiently effective \(E\), it follows that \(\pi_*D\) is big. \(\square\)

**Lemma A.2.** The following hold.

1. \(\text{NA}(X/Y) \subset \text{Big}(X/Y) \subset \text{Eff}^1(X/Y) \subset \text{PE}^1(X/Y)\).

2. \(\text{NA}(X/Y) \subset \text{Nef}^1(X/Y) \subset \text{PE}^1(X/Y)\).

**Proof.** Note that \(\text{Big}(X/Y) = \text{Eff}^1(X/Y) = \text{PE}^1(X/Y) = N^1(X/Y)\) for \(\dim(Y) = 2\). For \(\dim(Y) = 0\), the claim (1) can be shown straightforward. For \(\dim(Y) = 1\), the claim (1) follows from \(f_\*\mathcal{O}_X(mD) \otimes \mathcal{O}_Y(a)\) is globally generated for any \(f\)-big divisor \(D\) on \(X\), \(m \gg 0\) and a divisor \(a\) on \(Y\) of sufficiently high degree. The claim (2) in \(\dim(Y) = 0\) follows from the fact that \(DD' \geq 0\) holds for any nef divisors \(D, D'\) on \(X\) (cf. [6]). We assume \(\dim(Y) = 1\). Let \(E\) be an \(f\)-exceptional nef \(\mathbb{R}\)-divisor on \(X\). Since \(FE = 0\) for a fiber \(F\) of \(f\), we can write \(E = aF\) in \(N_1(X/Y)\) for some \(a \in \mathbb{R}\). If there exists an \(f\)-horizontal curve on \(X\), we have \(a \geq 0\) by the nefness of \(E\). Then any \(f\)-nef divisor \(D\) is \(f\)-pseudo-effective since \(DD' \geq 0\). If there exist no \(f\)-horizontal curves on \(X\), then \(N^1(X/Y)\) is generated by \(f\)-exceptional curves. Thus we have \(F = 0\) in \(N_1(X/Y)\) and then \(\text{Nef}_1(X/Y) = 0\). This implies \(\text{PE}^1(X/Y) = N^1(X/Y)\), whence (2) follows. \(\square\)
Lemma A.3. The following hold.

1. If \( \dim(Y) = 0 \), an \( \mathbb{R} \)-divisor \( D \) on \( X \) is numerically ample if and only if \( D^2 > 0 \) and \( DC > 0 \) for any curve \( C \) on \( X \), that is, \( D \) satisfies Nakai-Moishezon’s condition.

2. A \( \mathbb{Q} \)-divisor \( D \) on \( X \) is \( f \)-ample if and only if \( D \) is numerically \( f \)-ample and \( \mathbb{Q} \)-Cartier.

3. Assume \( \dim(Y) = 0 \). Then \( \text{NA}(X) \neq \emptyset \) if and only if \( X \) is algebraic or Moishezon. In this case, the signature of the intersection form on \( N(X) \) is \((1, \rho - 1)\), where \( \rho := \dim N(X) \). Otherwise, the intersection form on \( N(X) \) is negative definite.

4. \( \text{NA}(X/Y) = \emptyset \) if and only if \( \text{Big}(X/Y) = \emptyset \).

Proof. If \( \dim(Y) = 0 \) and \( D \) is numerically ample, then \( D^2 > 0 \) holds since \( D \) is pseudo-effective and not numerically trivial. Thus \( D \) satisfies Nakai-Moishezon’s condition. Conversely, assume that \( D \) satisfies Nakai-Moishezon’s condition and \( DC = 0 \) for some pseudo-effective divisor \( C \) on \( X \). Then the Hodge index theorem implies that \( C \) is numerically trivial. Hence \( D \) is numerically ample, which completes the proof of (1).

The claim (2) with \( \dim(Y) > 0 \) is easy to prove. The claim (2) with \( \dim(Y) = 0 \) follows from (1) and the Nakai-Moishezon criterion for ampleness (cf. [3] Chapter 3).

We assume \( \dim(Y) = 0 \). Note that a resolution of \( X \) is projective if and only if \( X \) is algebraic or Moishezon. In this case, Lemma A.1(1) implies \( \text{NA}(X) \neq \emptyset \). Conversely, let \( D \) be a numerically ample divisor on \( X \) and take a resolution \( \pi: X' \to X \). Then \( m\pi^*D - Z \) is numerically ample for sufficiently large \( m \gg 0 \), where \( Z \) is the fundamental cycle of \( \pi \) on \( X' \). Since \( X' \) is regular, it is \( \mathbb{Q} \)-Cartier. Then it is ample by (2), whence \( X' \) is projective. The rest of claim (3) follows from (A.1).

The claim (4) with \( \dim(Y) = 0 \) follows from (3). Indeed, if there exists a big divisor \( D \) on \( X \), then the mobile part \( M \) of \( mD \), \( m \gg 0 \) has \( M^2 > 0 \). Assume that \( \dim(Y) = 1 \). Let \( D \) be an \( f \)-big divisor on \( X \). Then \( DF > 0 \) holds, where \( F \) is a fiber of \( f \). For any reducible fiber \( F_i \) of \( f \), let \( Z_i \) be the fundamental cycle the support on which is the union of irreducible components \( C \) of \( F_i \) with \( DC \leq 0 \). Then \( mD - \sum Z_i \) is numerically \( f \)-ample for \( m \gg 0 \), which implies (4). The claim (4) with \( \dim(Y) = 2 \) follows from \( \text{NA}(X/Y) \) contains the fundamental cycle of \( f \) and \( \text{Big}(X/Y) = N^1(X/Y) \).

Lemma A.4 (Kodaira’s lemma. cf. [13] Corollary 2.2.7). Let \( D \) be an \( \mathbb{R} \)-divisor on \( X \) and assume that \( \text{NA}(X/Y) \neq \emptyset \). Then the following are equivalent.

1. \( D \) is \( f \)-big.

2. \( D = A + E \) for some numerically \( f \)-ample \( \mathbb{Q} \)-divisor \( A \) and effective \( \mathbb{R} \)-divisor \( E \) on \( X \).

3. There exists a positive number \( \alpha > 0 \) such that \( \text{rank} f_*O_X(\lfloor mD \rfloor) \geq \alpha m^{\dim X - \dim Y} \) for sufficiently large and divisible \( m \gg 0 \).

Proof. If \( \dim(Y) = 2 \), all the conditions (1), (2) and (3) are satisfied by any \( \mathbb{R} \)-divisor \( D \) on \( X \). Then we may assume \( \dim(Y) \leq 1 \). By Lemma A.2(1), numerically \( f \)-ample divisors are \( f \)-big. Thus (2) implies (1) and (3). Conversely, we will show (2) under the assumption of (1). We may assume \( D \) is integral since a numerically \( f \)-ample \( \mathbb{R} \)-divisor can be decomposed into the sum of a numerically \( f \)-ample \( \mathbb{Q} \)-divisor and an effective \( \mathbb{R} \)-divisor. By taking a resolution of \( X \), we may assume that \( X \) is regular. Take an \( f \)-ample effective divisor \( A \) on \( X \) and consider the exact sequence

\[ (A.2) \quad 0 \to O_X(mD - A) \to O_X(mD) \to O_A(mD) \to 0. \]
Note that $A$ maps onto $Y$. Since $D$ is $f$-big and $A$ is 1-dimensional, we have $f_*\mathcal{O}_X(mD - A) \neq 0$ for $m \gg 0$. Thus we can write $mD = A + E$ by taking $E \in |mD - A|$ if $\dim(Y) = 0$, or taking $E \in |mD - A + f^*a|$ for a divisor $a$ on $Y$ of sufficiently high degree and replacing $A - f^*a$ by $A$ if $\dim(Y) = 1$, whence (2) follows. Replacing $mD$ with $\lfloor mD \rfloor$ in (A.2) and tracing the same proof as above, we can prove that (3) implies (2).

\textbf{Lemma A.5.} The following hold.

(1) For any divisor $B$ and any numerically $f$-ample (resp. $f$-big) divisor $D$ on $X$, the divisor $B + nD$ is numerically $f$-ample (resp. $f$-big) for sufficiently large $n \gg 0$. In particular, $\text{NA}(X/Y)$ (resp. $\text{Big}(X/Y)$) is an open subset of $N^1(X/Y)$.

(2) If $\text{NA}(X/Y) \neq \emptyset$, then an $\mathbb{R}$-divisor $D$ is $f$-nef (resp. $f$-pseudo-effective) if and only if $D + A$ is numerically $f$-ample (resp. $f$-big) for any numerically $f$-ample $\mathbb{R}$-divisor $A$ on $X$.

(3) If $\text{NA}(X/Y) \neq \emptyset$, it follows that

$$\overline{\text{NA}(X/Y)} = \text{Nef}^1(X/Y), \quad \text{NA}(X/Y) = \text{Int}(\text{Nef}^1(X/Y)),$$

and

$$\overline{\text{Big}(X/Y)} = \text{PE}^1(X/Y), \quad \text{Big}(X/Y) = \text{Int}(\text{PE}^1(X/Y)).$$

\textbf{Proof.} The claim (1) for numerically ampleness is easy to prove. The claim (1) for bigness follows from Lemma A.4.

The claim (2) for the nef case is easy to show. We will prove (2) for the pseudo-effective case. Let $D$ be an $\mathbb{R}$-divisor on $X$ such that $D + A$ is $f$-big for any numerically $f$-ample $\mathbb{R}$-divisor $A$ on $X$. Take numerically $f$-ample $\mathbb{Q}$-divisors $A_n$ on $X$ such that $A_n \to 0$ in $N^1(X/Y)$ ($n \to \infty$). Then $D + A_n$ is $f$-big and in particular effective. Hence $D$ is $f$-pseudo-effective.

Let $D$ be an effective divisor on $X$. Then $D + A$ is $f$-big for any numerically $f$-ample divisor $A$ on $X$. Hence, it suffices to show that the property of $D$ that $D + A$ is $f$-big for any numerically $f$-ample $\mathbb{R}$-divisor $A$ is a closed condition. Let $D$ be an $\mathbb{R}$-divisor on $X$ and assume that there exists a numerically $f$-ample $\mathbb{R}$-divisor $A$ such that $D + A$ is not $f$-big. In particular, $D + (1/2)A$ is not $f$-big. We consider $D + \varepsilon B + (1/2)A$ for any divisor $B$ and $\varepsilon > 0$. Since $A$ is numerically $f$-ample, $(1/2)A - \varepsilon B$ is numerically $f$-ample for sufficiently small $\varepsilon$ from (1), which implies that $D + \varepsilon B + (1/2)A$ is not $f$-big. Hence (2) follows.

The claim (3) follows from (1), (2) and the standard argument of the topology. Note that $\text{Big}(X/Y) \neq \emptyset$ from Lemma A.4.

\textbf{Corollary A.6.} If $\dim(Y) = 1$, then an $\mathbb{R}$-divisor $D$ on $X$ is $f$-pseudo-effective (resp. $f$-big) if and only if $DF \geq 0$ (resp. $DF > 0$) for a fiber $F$ of $f$.

\textbf{Proof.} This follows from Lemmas A.4 and A.5.

\textbf{Lemma A.7.} Let $D$ be an $\mathbb{R}$-divisor on $X$ with $D^2 > 0$ (resp. $D^2 \geq 0$). The following are equivalent.

(1) $D$ is big (resp. pseudo-effective).
(2) There exists a nef and big divisor $F$ such that $DF > 0$ (resp. $DF \geq 0$).
(3) There exists a nef divisor $F$ on $X$ such that $DF > 0$ (resp. or $D \equiv 0$).
In particular, $\text{Big}(X) \cap \{D \in N(X) \mid D^2 > 0\} = C_{++}(X)$ holds.

Proof. We may assume that $D$ is integral. First, we show the equivalence for bigness. Assume that $D$ is big. Then $DF > 0$ holds for a numerically ample divisor $F$ on $X$ (note that $NA(X) \neq \emptyset$). Then $D$ satisfies (2). The implication from (2) to (3) is clear. We assume that $D$ satisfies the condition (3). By taking a resolution of $X$, we may assume that $X$ is regular by Lemma A.1 (1). By the Riemann-Roch theorem and the condition (3), we have $\dim |mD| \to \infty$ ($m \to \infty$). In particular, $D$ is pseudo-effective. That is, $C_{++}(X)$ is contained in $PE^+(X)$. Since $C_{++}(X)$ is open in $N(X)$, we have $C_{++}(X) \subset \text{Big}(X)$ by Lemma A.5 (3). Hence $D$ satisfies (1).

Next, we show the equivalence for pseudo-effectiveness. If $D \equiv 0$, then the claim is trivial. Thus we may assume that $D$ is numerically non-trivial. The proof that (1) implies (2) (or (3)) is similar to the big case. The implication from (2) (or (3)) to (1) follows from the equivalence for bigness by replacing $D$ with $D + A_n$ for numerically ample $\mathbb{Q}$-divisors $A_n$ with $A_n \to 0$ in $N(X)$ ($n \to \infty$). \hfill \Box

Corollary A.8. Let $D$ be a pseudo-effective $\mathbb{R}$-divisor on $X$ with $D = P + N$ the Zariski decomposition in Theorem [27]. Then the following hold.

1. $D$ is nef and big if and only if $D$ is nef and $D^2 > 0$. In particular, $D$ is big if and only if $P^2 > 0$.
2. If further assume that $D^2 > 0$. Then $D' := P - N$ is big and $D'^2 > 0$. In particular, this operation gives a map $C_{++}(X) \to C_{++}(X); \; D \mapsto D'$.

Proof. The claim (1) follows from Lemmas A.4 and A.7. Suppose $D^2 > 0$. Note that $P$ is nef and big since $D$ is big. Hence the first half of the claim (2) follows from Lemma A.7. $D'P = (P - N)P = P^2 > 0$ and $D'^2 = P^2 + N^2 = D^2 > 0$. Since the negative part $N$ of $D$ is determined by its numerical equivalence class, this operation defines a self-map on $C_{++}(X)$. \hfill \Box

The following lemma is purely linear-algebraic and easy to show.

Lemma A.9. Let $A = \bigcup_{i=1}^n A_i$ be the irreducible decomposition of a connected complete curve $A$ on $X$. Then the following are equivalent.

1. $\langle 1 \rangle_0$ (resp. $\langle 1 \rangle_0$, $\langle 1 \rangle_>$) There is an effective divisor $Z = \sum_{i=1}^n a_i A_i$ with $a_i \in \mathbb{Q}_{>0}$ such that $A_j Z < 0$ (resp. $A_j Z = 0$, $A_j Z > 0$) for any $j$.
2. $\langle 2 \rangle_0$ (resp. $\langle 2 \rangle_0$, $\langle 2 \rangle_>$) The matrix $(A_i A_j)_{ij}$ is negative definite (resp. negative semi-definite and not negative definite, not negative semi-definite).

Proof. It is clear that (1)$_>$ implies (2)$_>$ because $Z^2 > 0$. We assume (1)$_<$ (resp. (1)$_0$). In order to show (2)$_<$ (resp. (2)$_0$), it suffices to prove that for any non-zero $\mathbb{Q}$-divisor $B = \sum_{i=1}^n b_i A_i$, it holds $B^2 < 0$ (resp. $B^2 \leq 0$). One can check

$$B^2 \leq \sum_i \frac{b_i^2}{a_i} A_i Z < 0 \quad \text{(resp.} \leq 0)$$

by a direct computation and $A_i A_j \geq 0$ for $i \neq j$.

Next, we assume (2)$_<$ (resp. (2)$_0$, (2)$_>$). For an $\mathbb{R}$-divisor $B = \sum_{i=1}^n b_i A_i$, put $B_+ := \sum_{i=1}^n |b_i| A_i$. Then one can check $B^2 \leq B^2_+$ since $A_i A_j \geq 0$ for $i \neq j$. Let $\alpha$ be
the maximal eigenvalue of \((A_i A_j)_{ij}\) and \(V_\alpha \subset \mathbb{R}^n\) the corresponding eigenspace. Note that \(\alpha\) is not necessarily rational. For a non-zero vector \(b = (b_1, \ldots, b_n) \in V_\alpha\), we put \(B := \sum_{i=1}^{n} b_i A_i\). Since \(A_i B = \alpha b_j\) for any \(j\), we have \(B^2 = \alpha \sum_{j=1}^{n} |b_j|^2\). On the other hand, it follows that \(B^2_+ = \sum_{i,j} (A_i A_j) |b_i||b_j| \leq \alpha \sum_{i=1}^{n} |b_i|^2 = B^2\), whence \(B^2 = B^2_+\) and \(|b| = (|b_1|, \ldots, |b_n|) \in V_\alpha\) hold. Since \(A\) is connected, we can check that \(b_i \neq 0\) holds for any \(i\). Then we may assume that each element of \(b\) is positive. Thus we have \(A_i B = \alpha b_j < 0\) (resp. \(= 0, > 0\)) by the condition \((2)_{<0}\) (resp. \((2)_{=0}\), \((2)_{>0}\)). In the case of \((2)_{=0}\) (that is, \(\alpha = 0\)), we can take all the coefficients \(b_i\) are rational because \(\alpha\) is rational. Then \(Z := B\) satisfies \((1)_{=0}\). In the case of \((2)_{<0}\) (resp. \((2)_{>0}\)), then we can shift the coefficients \(b_i\) to rational numbers preserving the condition \(A_i B = A_j (\sum_{i=1}^{n} b_i A_i) < 0\) (resp. \(> 0\)) for each \(j\). Hence we may assume all the \(b_i\) are positive rational. Then \(Z := B\) satisfies \((1)_{<0}\) (resp. \((1)_{>0}\)).

\[\Box\]

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