PLÜCKER EMBEDDING OF CYCLIC ORBIT CODES∗

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Abstract. Cyclic orbit codes are a family of constant dimension codes used for random network coding. We investigate the Plücker embedding of these codes and show how to efficiently compute the Grassmann coordinates of the code words.

Key words. random network coding, subspace codes, Grassmannian, finite fields, Plücker embedding

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1. Introduction. In network coding one is looking at the transmission of information through a directed graph with possibly several senders and several receivers. One can increase the throughput by linearly combining the information vectors at intermediate nodes of the network. If the underlying topology of the network is unknown we speak about random linear network coding. Since linear spaces are invariant under linear combinations, they are what is needed as codewords. It is helpful (e.g. for decoding) to constrain oneself to subspaces of a fixed dimension, in which case we talk about constant dimension codes.

The general linear group, consisting of all invertible transformations acts naturally on the set of all k-dimensional vector spaces, called the Grassmann variety. Orbits under this action are called orbit codes. Orbit codes have useful algebraic structure and can be seen as subspace analogues of linear block codes in some sense.

One can describe the balls of subspace radius 2t in the Grassmann variety in its Plücker embedding. Such an algebraic description of the balls of radius 2t is potentially important if one is interested in an algebraic decoding algorithm for constant dimension codes. For instance, a list decoding algorithm requires the computation of all code words which lie in some ball around a received message word. In this work we characterize the Plücker embedding of orbit codes generated by cyclic subgroups of the general linear group. The case of irreducible cyclic subgroups has already been studied in [7]. This work generalizes and completes those results for general cyclic orbit codes.

The paper is structured as follows: In Section 2 we give some preliminaries on random network coding and orbit codes in particular. Moreover, we define irreducible and completely reducible matrices and groups. Section 3 contains the main results of this work. We first investigate the balls around subspaces of radius 2t with respect to the Grassmann coordinates. Then we recall how to efficiently compute these coordinates of irreducible cyclic orbit codes and show how the same can be done for reducible cyclic orbit codes. For this we distinguish between completely reducible and non-completely reducible generating groups. In Section 4 we conclude this work.

2. Preliminaries. Let \( F_q \) be the finite field with \( q \) elements, where \( q \) is a prime power. We will denote the set of all \( k \)-dimensional subspaces of \( F_q^n \), called the Grassmannian, by \( G_q(k,n) \) and the general linear group over \( F_q \) by \( GL_n \). Moreover, the set of all \( k \times n \)-matrices over \( F_q \) is denoted by \( \text{Mat}_{k\times n} \).

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Let \( U \in \text{Mat}_{k \times n} \) be a matrix of rank \( k \) and

\[
\mathcal{U} = \text{rs}(U) := \text{row space}(U) \in \mathbb{G}_q(k,n).
\]

One can notice that the row space is invariant under \( \text{GL}_k \)-multiplication from the left, i.e. for any \( T \in \text{GL}_k \) it holds that \( \mathcal{U} = \text{rs}(U) = \text{rs}(TU) \). Thus, there are several matrices that represent a given subspace.

The **subspace distance** \( d_S \), given by

\[
d_S(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U}) + \dim(\mathcal{V}) - 2 \dim(\mathcal{U} \cap \mathcal{V}) \quad \forall \mathcal{U}, \mathcal{V} \in \mathbb{G}_q(k,n)
\]

is a metric on \( \mathbb{G}_q(k,n) \) and is suitable for coding over the operator channel [5]. A **constant dimension codes** is simply a subset of the Grassmannian \( \mathbb{G}_q(k,n) \). The minimum distance is defined in the usual way. Different constructions of constant dimension codes can be found in e.g. [2, 4, 5, 6, 8, 10].

Given \( U \in \text{Mat}_{k \times n} \) of rank \( k \), \( U \in \mathbb{G}_q(k,n) \) its row space and \( A \in \text{GL}_n \), we define \( U^A := \text{rs}(UA) \).

This multiplication with \( \text{GL}_n \)-matrices defines a group action from the right on the Grassmannian:

\[
\mathbb{G}_q(k,n) \times \text{GL}_n \rightarrow \mathbb{G}_q(k,n) \quad (U, A) \mapsto UA
\]

For a subgroup \( G \) of \( \text{GL}_n \) the set \( C = \{UA \mid A \in G\} \) is called an **orbit code** [10]. There are different subgroups that generate the same orbit code. An orbit code is called **cyclic** if it can be defined by a cyclic subgroup \( G \leq \text{GL}_n \).

**Definition 2.1.**

1. A matrix \( G \in \text{GL}_n \) or a subgroup \( G \leq \text{GL}_n \) is called **irreducible** if \( \mathbb{F}_q^n \) contains no non-trivial \( G \)-invariant subspace, otherwise it is called **reducible**.
2. A matrix \( G \in \text{GL}_n \) or a subgroup \( G \leq \text{GL}_n \) is called **completely reducible** if \( \mathbb{F}_q^n \) is the direct sum of \( G \)-invariant subspaces which do not have any non-trivial \( G \)-invariant proper subspaces.
3. An orbit code \( C \subseteq \mathbb{G}_q(k,n) \) is called **irreducible**, respectively **completely reducible**, if \( C \) can be generated by an irreducible, respectively a completely reducible, group.

A cyclic group is irreducible (resp. completely reducible) if and only if its generator matrix is irreducible (resp. completely reducible). Moreover, an invertible matrix is irreducible if and only if its characteristic polynomial is irreducible. An invertible matrix is completely reducible if and only if the blocks of its rational canonical form are companion matrices of irreducible polynomials. For more information the reader is referred to [9], where one can also find that orbit codes arising from conjugate groups are equivalent from a coding theoretic perspective. Therefore, we can restrict our studies to cyclic subgroups generated by matrices in rational canonical form.

One can describe the multiplicative action of companion matrices of irreducible polynomials via the Galois extension field isomorphism. Let \( p(x) = \sum p_i x^i \) be a monic irreducible polynomial over \( \mathbb{F}_q \) of degree \( n \) and \( P \) its companion matrix

\[
P = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-p_0 & -p_1 & -p_2 & \ldots & -p_{n-1}
\end{pmatrix}
\]
Furthermore, let $\alpha \in \mathbb{F}_q^n$ be a root of $p(x)$. By $\phi^{(n)}$ we denote the canonical isomorphism

$$\phi^{(n)} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \cong \mathbb{F}_q[\alpha]$$

$$(v_1, \ldots, v_n) \mapsto \sum_{i=1}^{n} v_i \alpha^{i-1}.$$ 

The multiplication with $P$, respectively $\alpha$, commutes with $\phi^{(n)}$ [7], i.e. for $v \in \mathbb{F}_q^n$ it holds that

$$\phi^{(n)}(vP) = \phi^{(n)}(v)\alpha.$$ 

For a matrix $A \in \text{Mat}_{m \times n}$ denote by $A_{i_1,\ldots,i_k}$ the submatrix of $A$ consisting of the complete rows $i_1, \ldots, i_k$ and by $A[j_1,\ldots,j_l]$ the submatrix of $A$ with the complete columns $j_1, \ldots, j_l$. Consequently, $A_{i_1,\ldots,i_k}[j_1,\ldots,j_l]$ denotes the submatrix defined by rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_l$.

3. Plücker Embedding of Cyclic Orbit Codes. Let us first recall from [7] how to describe the balls of radius $2t$ (with respect to the subspace distance) around some $\mathcal{U} \in \mathcal{G}_q(k, n)$ with the help of the Grassmann coordinates.

The map $\varphi : \mathcal{G}_q(k, n) \rightarrow \mathbb{P}(\mathbb{F}_q)^{-1}(\mathbb{F}_q)$ that maps $\text{rs}(U)$ to

$$[\det(U[1,\ldots,k])] : \det(U[1,\ldots,k-1,k+1]) : \cdots : \det(U[n-k+1,\ldots,n])]$$

is an embedding of the Grassmannian in the projective space of dimension $\binom{n}{k}-1$. It is called the Plücker embedding of the Grassmannian $\mathcal{G}_q(k, n)$. The projective coordinates

$$[\det(U[1,\ldots,k]) : \cdots : \det(U[n-k+1,\ldots,n])] : \mathbb{F}_q^* \cdot [\det(U[1,\ldots,k]), \ldots, \det(U[n-k+1,\ldots,n])]$$

are often referred to as the Plücker or Grassmann coordinates of $\text{rs}(U)$. For more information on the Plücker embedding of the Grassmannian the reader is referred to [7].

**Definition 3.1.** Consider the set $[\binom{n}{k}] := \{(i_1, \ldots, i_k) \mid i_l \in \{1, \ldots, n\} \forall l\}$ and define a partial order on it:

$$(i_1, \ldots, i_k) > (j_1, \ldots, j_k) \iff \exists N \in \mathbb{N}_0 : i_l = j_l \forall l < N \text{ and } i_N > j_N.$$ 

Denote by $I_k$ the identity matrix of size $k$ and by $0_{k \times m}$ the $k \times m$-matrix with only zero entries. It is easy to compute the balls of radius $2t$ around a vector space $\mathcal{U}$, denoted by $B_{2t}(\mathcal{U})$, in the following special case.

**Proposition 3.2.** [7] Proposition 3/ Define $\mathcal{U}_0 := \text{rs}[ I_k \ 0_{k \times n-k} ]$. Then

$$B_{2t}(\mathcal{U}_0) = \{ \mathcal{V} \in \mathcal{G}_q(k, n) \mid \det(\mathcal{V}[i_1,\ldots,i_k]) = 0$$

$$\forall (i_1,\ldots,i_k) \not\leq (t+1,\ldots,k,n-t+1,\ldots,n) \}.$$ 

The proposition shows that $B_{2t}(\mathcal{U}_0)$ is described in the Plücker space $\mathbb{P}(\mathbb{F}_q)^{-1}(\mathbb{F}_q)$ as a point in the Grassmannian together with linear constraints on the Grassmann coordinates.
To derive the equations for a ball $B_{2t}(\mathcal{U})$ around an arbitrary subspace $\mathcal{U} \in \mathcal{G}_q(k, n)$ note, that for any $\mathcal{U} \in \mathcal{G}_q(k, n)$ there exists an $A \in \text{GL}_n$ such that $\mathcal{U}A = \mathcal{U}$. Then a direct computation shows that

$$B_{2t}(\mathcal{U}) = B_{2t}(\mathcal{U}A) = B_{2t}(\mathcal{U}).$$

Thus, the multiplication by $A$ transforms the linear equations $\det(\mathcal{V}[i_1, \ldots, i_k]) = 0$ for all $\mathcal{U} \in \mathcal{G}_q(k, n)$ into a new set of linear equations in the Grassmannian coordinates.

Now we want to show how to explicitly compute the Grassmann coordinates of cyclic orbit codes. To do so we first recall the results for the Plücker embedding of irreducible cyclic orbit codes from [7]. Thereafter, we will extend this theory first to completely reducible and then to non-completely reducible orbit codes.

### 3.1. Irreducible Cyclic Orbit Codes

For this subsection let $\mathcal{U} \in \mathcal{G}_q(k, n)$, $p(x) = \sum_{i=0}^n p_i x^i \in \mathbb{F}_q[x]$ be a monic irreducible polynomial of degree $n$ and $\alpha$ a root of it. The companion matrix of $p(x)$ is denoted by $P$. Moreover, we define $\mathbb{F}^*_q := \mathbb{F}_q \setminus \{0\}$ and $\mathbb{F}^*_q[\alpha] := \mathbb{F}_q[\alpha] \setminus \{0\}$.

**Definition 3.3.** We define the following operation on the $k$-th outer product $\Lambda^k(\mathbb{F}_q^*[\alpha]) \cong \Lambda^k(\mathbb{F}^*_q)$:

$$*: \Lambda^k(\mathbb{F}_q^*[\alpha]) \times \mathbb{F}^*_q[\alpha] \longrightarrow \Lambda^k(\mathbb{F}_q^*[\alpha])$$

$$(v_1 \wedge \ldots \wedge v_2) \ast \beta := (v_1 \wedge \ldots \wedge v_2) \ast \beta := (v_1 \beta \wedge \ldots \wedge v_k \beta).$$

This is a group action since $((v_1 \wedge \ldots \wedge v_2) \ast \beta) \ast \gamma = (v_1 \wedge \ldots \wedge v_2) \ast (\beta \gamma)$.\[3.4.\] Theorem 7. The following map is an embedding of the Grassmannian:

$$\varphi' : \mathcal{G}_q(k, n) \longrightarrow \mathbb{P}(\Lambda^k(\mathbb{F}_q^*[\alpha]))$$

$$\text{rs}(U) \mapsto (\varphi^{(n)}(U_1) \wedge \ldots \wedge \varphi^{(n)}(U_k)) \ast \mathbb{F}_q^*$$

It is isomorphic to $\varphi(\mathcal{G}_q(k, n))$ through the mapping

$$\varphi'(\text{rs}(U)) = \sum_{0 \leq i_1 < \cdots < i_k < n} \mu_{i_1, \ldots, i_k} (\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}) \ast \mathbb{F}_q^*$$

$$\longleftrightarrow \varphi(\text{rs}(U)) = [\mu_{0, \ldots, k-1} : \cdots : \mu_{n-k, \ldots, n-1}].$$

**Theorem 3.5.** Theorem 8. It holds that $\varphi'(\mathcal{U}P) = \varphi'(\mathcal{U}) \ast \alpha$. Hence, an irreducible cyclic orbit code $\mathcal{C} = \{\mathcal{U}P^i \mid i = 0, \ldots, \text{ord}(P) - 1\}$ has a corresponding “Plücker orbit”:

$$\varphi'(\mathcal{C}) = \{\varphi'(\mathcal{U}) \ast \alpha^i \mid i = 0, \ldots, \text{ord}(\alpha) - 1\} = \varphi'(\mathcal{U}) \ast (\alpha).$$

### 3.2. Completely Reducible Cyclic Orbit Codes

Completely reducible cyclic subgroups of $\text{GL}_n$ are exactly the ones where the blocks of the rational canonical form (RCF) of the generator matrix are companion matrices of irreducible polynomials. Because of this property one can use the theory of irreducible cyclic orbit codes block-wise to compute the minimum distances of the block component codes and hence the minimum distance of the whole code.
For simplicity, we will explain how the theory from before generalizes in the case of generator matrices whose RCF has two blocks that are companion matrices of the irreducible polynomials. The generalization to an arbitrary number of blocks is then straightforward.

Assume our generator matrix $P$ is of the type

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

where $P_1, P_2$ are companion matrices of the monic primitive polynomials $p_1(x), p_2(x) \in \mathbb{F}_q[x]$ with $\deg(p_1) = n_1, \deg(p_2) = n_2$, respectively. Let $\alpha_1, \alpha_2$ be roots of $p_1(x), p_2(x)$, respectively. Furthermore, let

$$U = [ \begin{array}{cc} U_1 & U_2 \end{array} ]$$

be a matrix representation of $U \in G_q(k, n)$ such that $U_1 \in \text{Mat}_{k \times n_1}, U_2 \in \text{Mat}_{k \times n_2}$. Then

$$UP_i = \text{rs} [ \begin{array}{cc} U_1P_{1i} & U_2P_{2i} \end{array} ].$$

By $\phi^{(n_1)} : \mathbb{F}_q^{n_1} \rightarrow \mathbb{F}_q^{n_1} \cong \mathbb{F}_q[\alpha_1]$ and $\phi^{(n_2)} : \mathbb{F}_q^{n_2} \rightarrow \mathbb{F}_q^{n_2} \cong \mathbb{F}_q[\alpha_2]$ we denote the standard vector space isomorphisms.

**Theorem 3.6.** [3] Theorem 3.6 The map

$$\phi^{(n_1, n_2)} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q[\alpha_1] \times \mathbb{F}_q[\alpha_2]$$

$$(u_1, \ldots, u_n) \mapsto (\phi^{(n_1)}(u_1, \ldots, u_{n_1}), \phi^{(n_2)}(u_{n_1+1}, \ldots, u_n))$$

is a vector space isomorphism.

$\mathbb{F}_q[\alpha_1] \times \mathbb{F}_q[\alpha_2]$ forms a multiplicative group with the multiplication

$$(\beta_1, \beta_2)(\gamma_1, \gamma_2) = (\beta_1\gamma_1, \beta_2\gamma_2).$$

**Definition 3.7.** We define the following group action:

$$* : \Lambda^k(\mathbb{F}_q[\alpha_1] \times \mathbb{F}_q[\alpha_2]) \times (\mathbb{F}_q[\alpha_1] \times \mathbb{F}_q[\alpha_2]) \rightarrow \Lambda^k(\mathbb{F}_q[\alpha_1] \times \mathbb{F}_q[\alpha_2])$$

$$((v_{11}, v_{12}) \land \cdots \land (v_{k1}, v_{k2}), (\beta_1, \beta_2)) \mapsto ((v_{11} \beta_1, v_{12} \beta_2) \land \cdots \land (v_{k1} \beta_1, v_{k2} \beta_2)).$$

**Theorem 3.8.** The following map is an embedding of the Grassmanian:

$$\varphi^{(n_1, n_2)} : G_q(k, n) \rightarrow \mathbb{P}(\Lambda^k(\mathbb{F}_q[\alpha_1] \times \mathbb{F}_q[\alpha_2]))$$

$$\text{rs}(U) \mapsto (\phi^{(n_1, n_2)}(U_1) \land \cdots \land \phi^{(n_1, n_2)}(U_k)) \ast \mathbb{F}_q^*$$

It is isomorphic to $\varphi(G_q(k, n))$ with

$$((v_{11}, v_{12}) \land \cdots \land (v_{k1}, v_{k2})) = \sum_{(i_1, \ldots, i_k) \in \{1, 2\}^k} v_{1i_1} \land \cdots \land v_{ki_k}.$$
loss of generality assume that \( n_1 \geq n_2 \). We will use the superscript \(^{(j)}\) as an additional index for the used variables and set \( \lambda^{(2)}_{hi} := 0 \) for \( h, i \in \{0, \ldots, n_1 - 1\}, i \geq n_2 \).

\[
(\phi^{(n_1,n_2)}_{hi}(U_1) \land \cdots \land \phi^{(n_1,n_2)}_{hi}(U_k)) \ast \mathbb{F}_q^*
\]

\[
= \left( \sum_{i=0}^{n_1-1} \lambda^{(1)}_{hi} \alpha^1_i \right) \land \cdots \land \left( \sum_{i=0}^{n_1-1} \lambda^{(2)}_{hi} \alpha^2_i \right) \ast \mathbb{F}_q^*
\]

\[
= \sum_{(j_1,\ldots,j_k) \in \{1,2\}^k} \left( \sum_{i=0}^{n_1-1} \lambda_{j_1i}^{(j_1)} \alpha^1_{j_1i} \right) \land \cdots \land \left( \sum_{i=0}^{n_1-1} \lambda_{j_ki}^{(j_k)} \alpha^2_{j_ki} \right) \ast \mathbb{F}_q^*
\]

where

\[
\mu^{(j_1,\ldots,j_k)}_{i_1,\ldots,i_k} := \sum_{\sigma \in S_k} (-1)^{\sigma} \lambda^{(j_1)}_{1\sigma(i_1)} \cdots \lambda^{(j_k)}_{k\sigma(i_k)} \in \mathbb{F}_q
\]

and

\[
\psi : [\nu_0,\ldots,\nu_{k-1} : \cdots : \nu_{n-k-1},\ldots,\nu_{n-1}] \to [\nu_0,\nu_1] \ast \cdots \ast [\nu_{n-k-1}] \ast \cdots \ast [\nu_{n-1}]
\]

Since \( \psi \) is an isomorphism and \( \varphi' = \psi^{-1} \circ \varphi \), it follows that \( \varphi' \) is an embedding as well.

**Theorem 3.9.** In analogy to the irreducible case the following holds:

\[
\varphi^{(n_1,n_2)}(UP) = \varphi^{(n_1,n_2)}(U) \ast (\alpha_1, \alpha_2).
\]

Hence, the code \( C = \{UP^i \mid i = 0, \ldots, \text{ord}(P) - 1\} \) has a corresponding “Plücker orbit”

\[
\varphi^{(n_1,n_2)}(C) = \{\varphi^{(n_1,n_2)}(U) \ast (\alpha_1, \alpha_2)^i \mid i = 0, \ldots, \text{ord}(P) - 1\} = \varphi^{(n_1,n_2)}(U) \ast (\alpha_1, \alpha_2)^\ast.
\]

**Example 3.10.** Over \( \mathbb{F}_2 \) let \( p_1(x) = p_2(x) = (x^2 + x + 1) \),

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

and \( U = rs \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix} \). Then \( \text{ord}(P) = 3 \) and \( \varphi^{(2)}(U) = ((1_1, 0_2) \land (\alpha_1, 1_2)) \) where we use the subscripts to distinguish between the elements of \( \mathbb{F}_q[\alpha_1] \) and \( \mathbb{F}_q[\alpha_2] \). The elements of the Plücker orbit \( \varphi^{(2)}(U) \ast (\alpha_1, \alpha_2)^\ast \) are

\[
(1_1, 0_2) \land (\alpha_1, 1_2),
\]

\[
(\alpha_1, 0_2) \land (\alpha_1 + 1_1, 0_2),
\]

\[
((\alpha_1, 0_2) \land (\alpha_1 + 1_1, 0_2)) = (\alpha_1 \land \alpha_1 + 1_1) + (\alpha_1, \alpha_2) = (\alpha_1 \land 1_1) + (\alpha_1 \land \alpha_2),
\]

\[
((\alpha_1 + 1_1, 0_2) \land (1_1, \alpha_2 + 1_2)) = (\alpha_1 + 1_1 \land 1_1) + (\alpha_1 + 1_1, \alpha_2 + 1_2)
\]

\[
= (\alpha_1 \land 1_1) + (\alpha_1 \land \alpha_2) + (\alpha_1 \land 1_2) + (1_1 \land \alpha_2) + (1_1 \land 1_2).
\]
The corresponding Grassmann coordinates are
\[ [1:1:0:0:0:0], [1:0:0:1:0], [1:1:1:1:0]. \]

One can easily verify that these are the Grassmann coordinates of the corresponding orbit code \( \mathcal{U}(P) \).

### 3.3. Non-Completely Reducible Cyclic Orbit Codes

But what happens if one of the blocks of the rational canonical form is not irreducible, i.e. it is the companion matrix of a reducible polynomial? For simplicity we restrict our investigation to the case that \( P \) is the companion matrix of \( p(x) = p_1(x)^2 \) where \( p_1(x) \in \mathbb{F}_q[x] \) is irreducible over \( \mathbb{F}_q \) and of degree \( n/2 \). Together with the results of the previous subsection one can then easily generalize the results of this subsection to any other non-completely reducible generator matrix.

Consider \( \mathbb{F}_q[x]/p(x) \) as the isomorphic representation of \( \mathbb{F}_q^n \) via
\[
\bar{\phi}^{(n)} : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q[x]/p(x)
\]
\[(v_1, \ldots, v_n) \longmapsto \sum_{i=1}^{n} v_i x^{i-1}.
\]

In this case the multiplication with \( P \) still corresponds to multiplication with \( x \mod p(x) \), the difference is that \( \mathbb{F}_q[x]/p(x) \) is not a field anymore. This is why we cannot use the \( \mathbb{F}_q \)-algebra of some \( \alpha \) anymore but have to work with \( \mathbb{F}_q[x]/p(x) \).

Since we did not use the field structure, the definitions and results of Section 3.1 are straightforwardly carried over to this setting. Therefore the proofs of the results are omitted in this subsection.

**Definition 3.11.** We define the following operation on \( \Lambda^k(\mathbb{F}_q[x]/p(x)) \equiv \Lambda^k(\mathbb{F}_q^n) \):

\[
* : \Lambda^k(\mathbb{F}_q[x]/p(x)) \times (\mathbb{F}_q[x]/p(x)) \setminus \{0\} \longrightarrow \Lambda^k(\mathbb{F}_q[x]/p(x))
\]
\[
((v_1 \wedge \cdots \wedge v_k), \beta) \longmapsto (v_1 \wedge \cdots \wedge v_k) * \beta := (v_1 \wedge \cdots \wedge v_k \beta).
\]

This is a group action since \((v_1 \wedge \cdots \wedge v_k) * \beta * \gamma = (v_1 \wedge \cdots \wedge v_k) * (\beta \gamma)\).

**Theorem 3.12.** The following map is an embedding of the Grassmannian:

\[
\bar{\varphi} : \mathcal{G}_q(k, n) \longrightarrow \mathbb{P}(\Lambda^k(\mathbb{F}_q[x]/p(x)))
\]
\[
\text{rs}(U) \longmapsto (\bar{\phi}^{(n)}(U_1) \wedge \cdots \wedge \bar{\phi}^{(n)}(U_k)) * \mathbb{F}_q^*
\]

It is isomorphic to \( \varphi(\mathcal{G}_q(k, n)) \).

**Theorem 3.13.** In analogy to the irreducible case the following holds:

\[
\bar{\varphi}(\text{UP}) = \bar{\varphi}(U) * x \mod p(x)
\]

Hence, the code \( \mathcal{C} = \{\text{UP}^i \mid i = 0, \ldots, \text{ord}(P) - 1\} \) has a corresponding “Plücker orbit”:

\[
\bar{\varphi}(\mathcal{C}) = \{\bar{\varphi}(U) * x^i \mid i = 0, \ldots, q^n - 1\} = \bar{\varphi}(U) * \langle x \rangle
\]

**Example 3.14.** Over \( \mathbb{F}_2 \) let \( P \) be the companion matrix of \( p(x) = (x^2 + x + 1)^2 = x^4 + x^2 + 1 \) and \( U \in \mathcal{G}_2(2, 4) \) such that \( \bar{\phi}(U) = \{0, 1, x + x^2, 1 + x + x^2\} \), i.e.

\[
U = \text{rs} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.
\]
Then $\bar{\varphi}(U) = (1 \wedge x + x^2)$. The elements of the Plücker orbit $\bar{\varphi}(U) * \langle x \rangle$ are

$$
\begin{align*}
(1 \wedge x + x^2) &= (1 \wedge x) + (1 \wedge x^2) \\
(x \wedge x^2 + x^3) &= (x \wedge x^2) + (x \wedge x^3) \\
(x^2 \wedge x^3 + x^4) &= (x^2 \wedge x^3 + x^2 + 1) = (x^2 \wedge x^3) + (x^2 \wedge 1) \\
(x^3 \wedge x^4 + x^3 + x) &= (x^3 \wedge x^3 + x^2 + x + 1) = (x^3 \wedge x^2) + (x^3 \wedge x) + (x^3 \wedge 1) \\
(x^4 \wedge x^4 + x^3 + x^2 + x) &= (x^2 + 1 \wedge x^3 + x + 1) \\
&= (x^2 \wedge x^3) + (x^2 \wedge x) + (x^2 \wedge 1) + (1 \wedge x^3) + (1 \wedge x) \\
(x^3 + x \wedge x^4 + x^2 + x) &= (x^3 + x \wedge x + 1) = (x^3 \wedge x) + (x^3 \wedge 1) + (x \wedge 1)
\end{align*}
$$

The corresponding Grassmann coordinates are

$$
[1 : 1 : 0 : 0 : 0 : 0], [0 : 0 : 0 : 1 : 1 : 0], [0 : 1 : 0 : 0 : 0 : 1], \\
[0 : 0 : 1 : 0 : 1 : 1], [1 : 1 : 1 : 0 : 1 : 0], [1 : 0 : 1 : 0 : 1 : 0].
$$

One can easily verify that these are the Grassmann coordinates of the corresponding orbit code.

4. Conclusion. We described cyclic orbit codes within the Plücker space and showed that the orbit structure is preserved. For this we distinguished between irreducible, completely reducible and non-completely reducible generator matrices. Theorems 3.4 and 3.8 show how this structure can be used for efficiently computing the Grassmann coordinates of the elements of a given cyclic orbit code. These coordinates can then be used for describing the balls of subspace radius $2t$ around the code words which is potentially important for new algebraic decoding algorithms.

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