A vanishing identity on adjoint Reidemeister torsions of twist knots

SEOKBEOM YOON

For a compact oriented 3-manifold with torus boundary the adjoint Reidemeister torsion is defined as a function on the \( \text{SL}_2(\mathbb{C}) \)-character variety depending on a choice of a boundary curve. Under reasonable assumptions, it is conjectured that the adjoint torsion satisfies a certain type of vanishing identities. In this paper, we prove that the conjecture holds for all hyperbolic twist knot exteriors by using Jacobi’s residue theorem.

1 Introduction

1.1 Overview

Let \( M \) be a compact oriented 3-manifold with torus boundary and let \( \mathcal{X}^{\text{irr}}(M) \) be the character variety of irreducible representations \( \pi_1(M) \to \text{SL}_2(\mathbb{C}) \). We assume that every irreducible component of \( \mathcal{X}^{\text{irr}}(M) \) is of dimension 1. Note that there are many known examples satisfying the assumption: for instance, whenever \( M \) contains no closed incompressible surface [2, §2.4].

In [14] Porti defined the \emph{adjoint torsion}, denoted by \( \tau_\gamma \), as a function on a Zariski open subset of \( \mathcal{X}^{\text{irr}}(M) \) depending on a choice of a boundary curve \( \gamma \). Here a boundary curve means a simple closed curve in \( \partial M \) with a non-trivial class in \( \text{H}_1(\partial M; \mathbb{Z}) \). Roughly speaking, at the character \( \chi_\rho \) of an irreducible representation \( \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) the value \( \tau_\gamma(\chi_\rho) \) is the sign-refined Reidemeister torsion twisted by the adjoint action associated to \( \rho \). The definition involves the choice of a boundary curve \( \gamma \) so as to specify a basis of the twisted (co-)homology. We briefly recall the definition in Section 2.1.

Since Witten’s monumental paper [22], there have been various studies on adjoint torsion in terms of quantum field theory. Several recent studies [7, 1, 8] on the relation with the Witten index suggested the following conjecture.
Conjecture 1  Suppose that every component of $\chi_{irr}(M)$ is of dimension 1 and that the interior of $M$ admits a hyperbolic structure of finite volume. Then for any boundary curve $\gamma \subset \partial M$ we have

$$\sum_{\chi_{\rho} \in \text{tr}^{-1}(C)} \frac{1}{\tau_{\gamma}(\chi_{\rho})} = 0$$

for generic $C \in \mathbb{C}$ where $\text{tr}_{\gamma} : \chi_{irr}(M) \to \mathbb{C}$ is the trace function of $\gamma$.

A knot in $S^3$ with a diagram as in Figure 1 is called a twist knot. We denote by $K_n$ for $n \neq 0 \in \mathbb{Z}$ the twist knot having $|n|$ right-handed half twists in the box (left-handed, if $n$ is negative). We may focus on twist knots $K_{2n}$ since $K_{2n+1}$ is equivalent to the mirror image of $K_{-2n}$.

![Figure 1: A diagram of a twist knot.](image)

It is known that every knot exterior of $K_{2n}$ satisfies the assumptions in Conjecture 1 except for $n = 1$ (the trefoil knot). Namely, every twist knot $K_{2n}$ with knot exterior $M$ is hyperbolic except for $n = 1$ [13] and has the character variety $\chi_{irr}(M)$ consisting of 1-dimensional components. The latter can be derived from explicit computations [15, 12] or the fact that every twist knot exterior contains no closed incompressible surface [10]. Main aim of this paper is to prove that Conjecture 1 holds for all hyperbolic twist knots.

Theorem 1  Let $M$ be the knot exterior of the twist knot $K_{2n}$ for $n \neq 0, 1$ and $\gamma \subset \partial M$ be a boundary curve. Then we have

$$\sum_{\chi_{\rho} \in \text{tr}^{-1}(C)} \frac{1}{\tau_{\gamma}(\chi_{\rho})} = 0$$

for generic $C \in \mathbb{C}$. 
The adjoint torsion is quite hard to compute in general and its concrete computation is known only in a few examples. For twist knots the adjoint torsion $\tau_{\lambda}$ with respect to the canonical longitude $\lambda$ was first computed in [6] by using the relation with the twisted Alexander polynomial [23] and the computation were remarkably simplified in [16] (also in [18]). To prove Theorem 1, we extend the computation to an arbitrary boundary curve. We summarize our computation here for convenience of the reader. 

Let $S_k(z)$ be the Chebyshev polynomials defined by $S_{k+1}(z) = zS_k(z) - S_{k-1}(z)$ for all $k \in \mathbb{Z}$ with the initial condition $S_0(z) = 0$, $S_1(z) = 1$. Throughout the paper, the Chebyshev polynomials are always given in the variable $z$ and thus we often write $S_k(z)$ simply as $S_k$. We fix $n \neq 0 \in \mathbb{Z}$ and let $M$ be the knot exterior of the twist knot $K_{2n}$. Reformulating [15], the character variety $X_{\text{irr}}(M)$ is given by the zero set of $F(m, z) = S_n(S_n - S_{n-1})(m^2 + m^{-2}) - zS_n(S_n - S_{n-1}) + 1$ in $\mathbb{C}^* \times \mathbb{C}$ with the quotient identifying $(m, z)$ and $(m^{-1}, z)$. For a boundary curve $\gamma \subset \partial M$ we let $E_\gamma(m, z) = m^p \left( \frac{(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n-1}} m^2 + (z - 2)(S_n + S_{n-1})S_n + 1 \right)^q$ where $p/q \in \mathbb{Q} \cup \{\infty\}$ is the slope of $\gamma$.

**Theorem 2** Let $(m, z) \in \mathbb{C}^* \times \mathbb{C}$ be a solution to the Laurent polynomial $F$ and $\chi_\rho$ be the corresponding character. Then $E_\gamma(m^{\pm 1}, z)$ are eigenvalues of $\rho(\gamma)$ and

$$\tau_\gamma(\chi_\rho) = -\frac{m}{2E_\gamma} \det \left( \frac{\partial (F, E_\gamma)}{\partial (m, z)} \right)$$

if $\chi_\rho$ is $\gamma$-regular.

**Remark 1** Choosing a boundary curve $\gamma$ as a meridian $\mu$, we have $E_\gamma = m$ and the equation (1) reduces to

$$\tau_\mu(\chi_\rho) = \frac{1}{2} \frac{\partial F}{\partial z}.$$ 

It is interesting that the adjoint torsion with respect to a meridian is related to a derivative of the defining equation of the character variety. A similar observation in terms of the A-polynomial was pointed out in [4, Remark 4.5].
1.2 Global residue

We remark some relation of Theorem 1 with the global residue theorem. Recall that the global residue theorem says that any top-dimensional meromorphic form defined on a compact complex manifold has global residue zero. Here global residue means the total sum of local residues. We refer to [9, 19] for general references on residue theory.

Theorem 2 gives us that for generic \( c \in \mathbb{C}^* \) the level set \( \text{tr}_{\gamma}^{-1}(c + \frac{1}{c}) \) is the common zero set \( Z_{F,G} \) of \( F(m,z) \) and \( G(m,z) := E_{\gamma}(m,z) - c \) in \( \mathbb{C}^* \times \mathbb{C} \) and

\[
\sum_{\chi_{\rho} \in \text{tr}_{\gamma}^{-1}(c + \frac{1}{c})} \frac{1}{\tau_{\gamma}(\chi_{\rho})} = \sum_{(m,z) \in Z_{F,G}} \frac{-2c}{m \det \left( \frac{\partial (F,G)}{\partial (m,z)} \right)}.
\]

The above equation implies that Theorem 1 is equivalent to that the global residue of the meromorphic 2-form

\[
\omega = \frac{1}{F \cdot G} \frac{dm}{m} \wedge dz
\]

defined on \( \mathbb{C}^* \times \mathbb{C} \) is zero for generic \( c \in \mathbb{C}^* \).

If \( F \) and \( G \) are generic in some sense, then there exists a toric compactification of \( \mathbb{C}^* \times \mathbb{C} \) so that one can deduce that the global residue of \( \omega \) is zero, as an application of the global residue theorem. We refer to [11, 21] for precise conditions of genericness; the condition in [21] relaxes that in [11]. Unfortunately, we however do not know a direct way to obtain Theorem 1 from [21]. Checking the generic condition in [21] seems to require heavy complex analysis (see Remark 9). We thus take a practical detour to prove Theorem 1 by reducing the problem to a one-variable problem. We refer to Section 3 for details.

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2 Adjoint Reidemeister torsion of twist knots

2.1 A brief review on definitions

We briefly recall a definition of the adjoint torsion for a knot exterior. We refer to [14, 20, 5] for more details.

Let $C_s = (0 \to C_n \to \cdots \to C_0 \to 0)$ be a chain complex of vector spaces over a field $\mathbb{F}$ with a boundary map $\partial$. For a basis $c_s$ of $C_s$ and a basis $h_s$ of the homology $H_s(C_s)$, the sign-refined Reidemeister torsion $\text{Tor}(C_s, c_s, h_s)$ is defined as follows. For each $0 \leq i \leq n$ we choose a tuple $b_i$ of vectors in $C_i$ such that $\partial b_i$ is a basis of $\partial C_i$, and a representative $\tilde{h}_i$ of $h_i$ in $C_i$. Then one can check that a tuple $c'_i = (\partial b_{i+1}, \tilde{h}_i, b_i)$ is a basis of $C_i$. Letting $A_i$ be the transition matrix taking the basis $c_i$ to the other basis $c'_i$, we have

$$\text{Tor}(C_s, c_s, h_s) := (-1)^{\sum_{i=0}^{n} \alpha_i \beta_i} \prod_{i=0}^{n} \det A_i^{(-1)^{i+1}} \in \mathbb{F}^*$$

where $\alpha_i = \sum_{i=0}^{j} \dim C_i$ and $\beta_i = \sum_{i=0}^{j} \dim H_i(C_s)$. Note that the equation (2) does not depend on the choices of $b_s$ and $h_s$.

Let $M$ be the knot exterior of a knot $K \subset S^3$. We fix any triangulation of $M$ and an orientation of each cell so that the cells, say $c_1, \cdots, c_m$, form a basis of $C_s(M; \mathbb{R})$. It is well-known that $\dim H_i(M; \mathbb{R}) = 1$ for $i = 0, 1$ and $\dim H_i(M; \mathbb{R}) = 0$ otherwise. We choose a basis $h_s$ of $H_s(M; \mathbb{R})$ as $h_s = \{[pt], [\mu]\}$ where $pt$ is a point in $M$ and $\mu$ is a meridian of $K$. Let

$$\epsilon = \text{sgn}(\text{Tor}(C_s(M; \mathbb{R}), c_s, h_s)) \in \{\pm 1\}$$

where $\text{sgn}(x)$ denotes the sign of $x \in \mathbb{R}^*$.

Let $\tilde{M}$ be the universal cover of $M$ with the induced triangulation and $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ be an irreducible representation. Viewing the Lie algebra $\mathfrak{g}$ of $\text{SL}_2(\mathbb{C})$ as a $\mathbb{Z}[\pi_1(M)]$-module through the adjoint representation $\text{Ad}_\rho : \pi_1(M) \to \text{Aut}(\mathfrak{g})$ associated to $\rho$, we consider a chain complex

$$C_*(M; \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\tilde{M}; \mathbb{Z})$$

with a basis $C = \{h, e, f\} \otimes \{{\tilde{c}}_1, \cdots, {\tilde{c}}_m\}$. Here $\{h, e, f\}$ is a basis of $\mathfrak{g}$ and $\tilde{c}_i$ is any lift of $c_i$ to $\tilde{M}$. The chain complex $C_*(M; \mathfrak{g})$ usually has non-trivial homology $H_* (M; \mathfrak{g})$. We choose a boundary curve $\gamma \subset \partial M$ and fix a basis $\mathcal{H}$ of $H_* (M; \mathfrak{g})$ as follows under the assumption that $\rho$ is $\gamma$-regular [14], i.e.,
• \( \dim H_1(M; g) = 1; \)
• the inclusion \( \gamma \hookrightarrow M \) induces an epimorphism \( H_1(\gamma; g) \rightarrow H_1(M; g); \)
• if \( \text{tr}(\rho(\pi_1(\partial M))) \subset \{ \pm 2 \} \), then \( \rho(\gamma) \neq \pm \text{Id}. \)

**Remark 2** The \( \gamma \)-regularity is invariant under conjugating \( \rho \) and thus the notion of \( \gamma \)-regular character is well-defined. Most of irreducible characters are \( \gamma \)-regular. Precisely, non-\( \gamma \)-regular irreducible characters are contained in the zero set of the differential of the trace function \( \text{tr}_\gamma : \mathcal{X}^\text{irr}(M) \rightarrow \mathbb{C} \). See [14, Proposition 3.26] and [6, Remark 9].

From the Poincare duality, we have \( \dim H_i(M; g) = 1 \) for \( i = 1, 2 \) and \( \dim H_i(M; g) = 0 \) otherwise. We choose any non-zero element \( v \in g \) invariant under \( \text{Ad}_\rho(g) \) for all \( g \in \pi_1(\partial M) \) and let \( \mathcal{H} \) consist of the images of \( v \otimes \gamma \) and \( v \otimes \partial M \) under the canonical maps \( H_1(\gamma; g) \rightarrow H_1(M; g) \) and \( H_2(\partial M; g) \rightarrow H_2(M; g) \), respectively. We finally define the \textit{adjoint torsion} \( \tau_\gamma(\chi_\rho) \) as
\[
\tau_\gamma(\chi_\rho) = \epsilon \cdot \text{Tor}(C_\ast(M; g), \mathcal{C}, \mathcal{H}) \in \mathbb{C}^\ast.
\]

**Remark 3** The definition of \( \tau_\gamma(\chi_\rho) \) involves several choices: a triangulation of \( M \), an order/orientations of the cells of \( M \), lifts of the cells of \( M \) to \( \tilde{M} \), a basis of \( g \), and the vector \( v \). However, it turns out that the adjoint torsion does not depend on these choices. Also, the definition of \( \tau_\gamma(\chi_\rho) \) involves the choices of orientations of \( K, \mu, \gamma \), and \( \partial M \). If we reverse one of these, the sign of \( \tau_\gamma(\chi_\rho) \) changes. We therefore fix these orientations once and for all. Note that choices of these orientations do not matter when we consider Conjecture 1.

### 2.2 Adjoint torsions of twist knots

We fix \( n \neq 0 \in \mathbb{Z} \) and let \( M \) be the knot exterior of the twist knot \( K_{2n} \). Let \( \mathcal{X}^\text{irr}(M) \) be the character variety of irreducible representations \( \pi_1(M) \rightarrow \text{SL}_2(\mathbb{C}) \). Recall that \( \mathcal{X}^\text{irr}(M) \), as a set, is the set of conjugacy classes of irreducible representations [3]:
\[
\mathcal{X}^\text{irr}(M) = \{ \rho : \pi_1(M) \rightarrow \text{SL}_2(\mathbb{C}) : \text{irreducible} \} / \text{Conjugation}.
\]

It is well-known that the fundamental group of \( M \) has a presentation
\[
\pi_1(M) = \langle a, b \mid w^a = bw^a \rangle \quad \text{where} \quad w = ba^{-1}b^{-1}a
\]
and that up to conjugation an irreducible representation \( \rho : \pi_1(M) \rightarrow \text{SL}_2(\mathbb{C}) \) is given by
\[
(4) \quad \rho(a) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} m & 0 \\ -u & m^{-1} \end{pmatrix}
\]
Lemma 1  The variable $z$ determines the variable $u$ uniquely as

$$u = \frac{(z - 2)S_n}{S_n - S_{n-1}}. \tag{7}$$

Proof  From the equation (6) we have $m^2 + m^{-2} = u + 2 + (2 - z)u^{-1}$. Plugging it into the equation (5), we obtain

$$R(m, u) = S_{n+1} - (u^2 - (u + 1)(m^2 + m^{-2} - 3))S_n = S_{n+1} - (u^2 - (u + 1)(u - 1 + (2 - z)u^{-1}))S_n = S_{n+1} - (z - 1 + (z - 2)u^{-1})S_n = S_n - S_{n-1} - (z - 2)S_nu^{-1}.$$

On the other hand, on the zero set of $R(m, u)$, both $S_n - S_{n-1}$ and $(z - 2)S_n$ cannot be zero, as they have no common zero. Therefore, $u$ is determined as in the equation (7).

Using Lemma 1, one can use the variable $z$ instead of the variable $u$. In particular, we may describe $\mathcal{X}^{\text{irr}}(M)$ in terms of variables $m$ and $z$ as follow.
Lemma 2  One has

\[ X^{irr}(M) = \{(m, z) \in \mathbb{C}^* \times \mathbb{C} : F(m, z) = 0\} / (m, z) \sim (m^{-1}, z) \]

where

\[ F(m, z) = S_n(S_n - S_{n-1})(m^2 + m^{-2}) - zS_n(S_n - S_{n-1}) + 1 \in \mathbb{C}[m^\pm 1, z]. \]

Proof  From the equation (7) we have

\[ u + 1 = \frac{(z - 1)S_n - S_{n-1}}{S_n - S_{n-1}} = \frac{S_{n+1} - S_n}{S_n - S_{n-1}} \quad \text{and} \quad u + 2 = \frac{zS_n - 2S_{n-1}}{S_n - S_{n-1}} = \frac{S_{n+1} - S_{n-1}}{S_n - S_{n-1}}. \]

We then have

\[
R(m, u) = S_{n+1} - (u^2 - (u + 1)(m^2 + m^{-2} - 3))S_n = S_n(u + 1)(m^2 + m^{-2}) + S_{n+1} - S_n - (u + 1)(u + 2)S_n = S_{n+1} - S_n \frac{S_n(m^2 + m^{-2}) + S_n - S_{n-1} - \frac{(zS_n - 2S_{n-1})S_n}{S_n - S_{n-1}}}{S_n - S_{n-1}} = S_{n+1} - S_n \frac{S_n(m^2 + m^{-2}) - \frac{(z - 1)S_n^2 - S_{n-1}^2}{S_n - S_{n-1}}}{S_n - S_{n-1}} = \frac{S_{n+1} - S_n}{(S_n - S_{n-1})^2} F(m, z). \]

Note that we used the fact \( S_n^2 - zS_nS_{n-1} + S_{n-1}^2 = 1 \) in the fifth equation.

On the other hand, if \( S_{n+1} - S_n = 0 \), then we have \( u = -1 \) from the equation (9) and \( z = m^2 + m^{-2} + 1 \) from the equation (6). It then follows that

\[
F(m, z) = S_n(S_n - S_{n-1})(m^2 + m^{-2}) - zS_n(S_n - S_{n-1}) + 1 = S_n(S_n - S_{n-1})(z - 1) - zS_n(S_n - S_{n-1}) + 1 = -S_n(S_n - S_{n-1}) + 1 = -(z - 1)S_nS_{n+1} + S_{n+1}^2 = -S_{n+1}(S_{n+1} - S_n) = 0. \]

Note that we used the fact \( S_n^2 - zS_nS_{n-1} + S_{n-1}^2 = 1 \) in the forth equation. This completes the proof of \( R(m, u) = 0 \Leftrightarrow F(m, z) = 0. \) \( \square \)
Theorem 3  The adjoint torsion to $\lambda$

We fix a point $(m, \mu)$ with respect to $\mu$ and let $\rho$ be the irreducible representation given as in the equation (4). We denote by $F_g$ the function $g \in \mathbb{C}[m^{\pm 2}, z]$ into the form $A(z)m^2 + B(z)$. For instance, we have

$$F_g = (B - Af)m^2 + C - A.$$ 

This would be useful when we try to verify $g_1(m, z) = g_2(m, z) \in \mathbb{C}[m^{\pm 2}, z]$; we first simplify them into $g_1(m, z) = A_1(z)m^2 + B_1(z)$ and $g_2 = A_2(z)m^2 + B_2(z)$ by using $F(m, z) = 0$ and then check one-variable problems $A_1(z) = A_2(z)$ and $B_1(z) = B_2(z)$.

We fix a point $(m, z) \in \mathbb{C}^* \times \mathbb{C}$ satisfying $F(m, z) = 0$ and let $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ be the irreducible representation given as in the equation (4). We denote by $\chi_\rho$ the character of $\rho$. We choose a meridian $\mu = a$ and let $\lambda$ be the canonical longitude with respect to $\mu$. A simple formula for computing the adjoint torsion $\tau_\lambda(\chi_\rho)$ with respect to $\lambda$ is given in [16]. We express the formula in terms of the variable $z$ as follows.

Theorem 3  The adjoint torsion $\tau_\lambda(\chi_\rho)$ is given by

$$\tau_\lambda(\chi_\rho) = -(2n + 1)S_n^2 + 2nS_nS_{n-1} - \frac{2(n - 1)S_n + nS_{n-1}}{(z + 2)(S_n - S_{n-1})}$$

if $\chi_\rho$ is $\lambda$-regular.

Proof  It is computed in [16, Corollary 2.6] that

(10)  

$$\tau_\lambda(\chi_\rho) = -\frac{1}{(y + 2 - x^2)(y^2 - xy + x^3)} \left( \frac{(2n - 1)y^2 + xy^2 - 2nx^2(x^2 - 2)}{y^2 - xy + x^3} + 2n \right)$$

where $x = m + m^{-1}$ and $y = u + 2$. A straightforward computation gives that (recall the equation (9))

$$y + 2 - x^2 = u + 2 - (m^2 + m^{-2}) = \frac{zS_n - 2S_{n-1}}{S_n - S_{n-1}} - \frac{zS_n(S_n - S_{n-1}) - 1}{S_n(S_n - S_{n-1})} = 1 - \frac{S_{n-1}}{S_n} = 1 - \alpha$$

Note that we used $F(m, z) = 0$ and $S_n^2 - zS_nS_{n-1} + S_{n-1}^2 = 1$ in the second and third
equations, respectively. It follows from $x^2 = y + 1 - \alpha$ that

\[
(2n - 1)y^2 + yx^2 - 2nx^2(x^2 - 2) = (2n - 1)y^2 + y(y + 1 + \alpha) - 2n(y + 1 + \alpha)(y - 1 + \alpha) = ((1 - 4n)\alpha + 1)y + 2n(1 - \alpha^2)
\]

\[
= ((1 - 4n)\alpha + 1)\frac{zS_n - 2S_{n-1}}{S_n - S_{n-1}} + 2n(1 - \alpha^2)
\]

\[
= ((1 - 4n)\alpha + 1)\frac{z - 2\alpha}{1 - \alpha} + 2n(1 - \alpha^2).
\]

Similar computations give

\[
y^2 - yx^2 + x^2 = \frac{1}{S_n^2(1 - \alpha)} \quad \text{and} \quad y^2 - yx^2 + 2x^2 = z + 2.
\]

Plugging above equations into the equation (10), we have

\[
\tau_\lambda(\chi_\rho) = -S_n^2 \left( \frac{(1 - 4n)\alpha + 1)(z - 2\alpha)}{(1 - \alpha)(z + 2)} + \frac{2n(1 - \alpha^2)}{z + 2} + 2n \right).
\]

\[
= -(2n + 1)S_n^2 + 2nS_nS_n - 2((n - 1)S_n + nS_{n-1}) \frac{2((n - 1)S_n + nS_{n-1})}{(z + 2)(S_{n} - S_{n-1})}.
\]

Note that the last equation is obtained from the fact $S_n^2 - zS_nS_{n-1} = 1$.

\[\square\]

**Remark 5** One can easily check that $S_k$ has value $(-1)^{k+1}k$ at $z = -2$ for all $k \in \mathbb{Z}$. It follows that $(n - 1)S_n + nS_{n-1}$ has a zero at $z = -2$ and thus $(n - 1)S_n + nS_{n-1}$ is divided by $z + 2$. Precisely, it is known that $(k - 1)S_k + kS_{k-1} = (z + 2)(S'_k - S'_{k-1})$ for all $k \in \mathbb{Z}$ where $S'_k$ is the derivative of $S_k$.

Let $\gamma \subset \partial M$ be a boundary curve of slope $p/q \in \mathbb{Q} \cup \{\infty\}$. Namely, $\gamma = \mu^p\lambda^q$ for coprime integers $p$ and $q$. We need eigenvalues of $\rho(\gamma)$ in order to compute the adjoint torsion $\tau_\gamma(\chi_\rho)$.

**Lemma 3** The matrix $\rho(\gamma)$ is of the form

\[
\rho(\gamma) = \rho(\mu^p\lambda^q) = \begin{pmatrix} m^p & m^q \\ 0 & m^{-p}l^{-q} \end{pmatrix}
\]

where

\[
l = -(z - 2)(S_{n+1} - S_{n-1})S_{n}^2m^2 + (z - 2)(S_n + S_{n-1})S_n + 1.
\]
Proof It is enough to show that the \((1, 1)\)-entry of \(\rho(\lambda)\) coincides with the given \(l\). It is known that \(\lambda = w^*w^n\) where \(w^*\) is the word obtained by writing \(w\) in the reversed order (see e.g. [15]). From [17, §3.2] we have
\[
\rho(w^n) = \left( \frac{S_{n+1} - (1 + (2 - m^{-2})u + u^2)S_n}{((m - m^{-1})u + mu^2)S_n} \right) \left( \frac{(m^{-1} - m - mu)S_n}{S_{n+1} - (1 - u^2)S_n} \right),
\]
\[
\rho(w^*) = \left( \frac{S_{n+1} - (1 - m^{-2}u)S_n}{((m^{-1} - m - m^3)u + m^{-3}u^2)S_n} \right) \left( \frac{(m^{-1} + m - m^3)S_n}{S_{n+1} - (1 + (2 - m^2)u + u^2)S_n} \right).
\]
It follows that the \((1, 1)\)-entry of \(\rho(\lambda) = \rho(w^n)\rho(w^*)\) is
\[
(S_{n+1} - (1 - m^{-2}u)S_n) \cdot (S_{n+1} - (1 + (2 - m^{-2})u + u^2)S_n)
+ (m - m^{-1} - m^{-1}u)S_n \cdot ((m - m^{-1})u + mu^2)S_n
= S_{n+1}^2 - (z + (m^2 - m^{-2})u)S_nS_{n+1}
+ (u(u + 1)m^2 - (u^3 + u^2 - 1) - u(u^2 + u + 1)m^{-2} + u^2m^{-4})S_n^2
= 1 - (m^2 - m^{-2})uS_nS_{n+1}
+ (u(u + 1)m^2 - (u^3 + u^2 - 1) - u(u^2 + u + 1)m^{-2} + u^2m^{-4} - 1)S_n^2
= 1 - (m + m^{-1})((u + 1)m - m^{-1})(z - 2)S_n^2.
\]
Note that we used the fact \(S_{n+1}^2 - zS_{n+1}S_n + S_n^2 = 1\) and the equation \((7) \iff uS_n = ((z - 1)u + z - 2)S_{n-1}\) for the third and fourth equations, respectively. Then the desired expression \((11)\) is obtained by plugging \(u + 1 = \frac{S_{n+1} - S_{n-1}}{S_n - S_{n-1}}\) (recall the equation \((9)\)) and then simplifying it as in Remark 4. \(\square\)

We define a function \(E_{\gamma}\) on the zero set of \(F\) in \(\mathbb{C}^* \times \mathbb{C}\) as
\[
E_{\gamma}(m, z) = mp\left( \frac{-(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n-1}} m^2 + (z - 2)(S_n + S_{n-1})S_n + 1 \right)^q.
\]

It is clear from Lemma 3 that \(E_{\gamma}(m, z)\) is an eigenvalue of \(\rho(\gamma)\).

Remark 6 A similar computation given as in the proof of Lemma 3 shows that
\[
l^{-1} = - \frac{(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n-1}} m^2 + (z - 2)(S_n + S_{n-1})S_n + 1
\]
and thus \(E_{\gamma}(m, z)\) is an eigenvalue of \(\rho(\gamma)\).

Remark 7 Using the equation \(F(m, z) = 0\), one can re-express
\[
l = -(z - 2)S_n^2m^2 - (z - 2)(S_n - S_{n-1})m^2 + (S_n - S_{n-1})^2
\]
\[
l^{-1} = -(z - 2)S_n^2m^{-4} - (z - 2)(S_n - S_{n-1})m^{-2} + (S_n - S_{n-1})^2
\]
so that \(l^\pm 1\) become Laurent polynomials. In particular, using these expressions, \(E_{\gamma}\) becomes a Laurent polynomial.
Theorem 4 The adjoint torsion $\tau_\gamma(\chi_\rho)$ is given by

\begin{equation}
\tau_\gamma(\chi_\rho) = -\frac{m}{2E_\gamma} \det \left( \frac{\partial(F, E_\gamma)}{\partial(m, z)} \right)
\end{equation}

if $\chi_\rho$ is $\gamma$-regular.

Note that as a consequence, the right-hand side of the equation (13) has the same value at $(m, z)$ and $(m^{-1}, z)$.

Proof For simplicity let $F(m, z) = f_1(z)(m^2 + m^{-2}) + f_2(z)$ and $E_\gamma(m, z) = m^p(g_1(z)m^2 + g_2(z))q$. See the equations (8) and (12). Also, we let $f_3 = f_2 / f_1$ so that $m^2 + m^{-2} + f_3(z) = 0$ and $dz/dm = -2(m - m^{-3})/f'_3$. Straightforward computations give that

\[ \frac{dl}{dm} = \frac{d}{dm}(g_1 m^2 + g_2) \]
\[ = 2g_1 m + (g'_1 m^2 + g'_2) \frac{dz}{dm} \]
\[ = 2g_1 m - \frac{2(m - m^{-3})(g'_1 m^2 + g'_2)}{f'_3} \]

and that

\[ \det \left( \frac{\partial(f_1, E_\gamma)}{\partial(m, z)} \right) \]
\[ = \det \left( \frac{\partial(m^2 + m^{-2} + f_3, m^p(g_1 m^2 + g_2)q)}{\partial(m, z)} \right) \]
\[ = \det \left( \frac{2(m - m^{-3})}{f'_3} m^{p-1}(g_1 m^2 + g_2)q + qm^p(g_1 m^2 + g_2)^{q-1} \cdot 2g_1 m}{g_1 m^2 + g_2} \right) \]
\[ = \det \left( \frac{2(m - m^{-3})}{f'_3} \frac{pE_\gamma}{m} + \frac{qE_\gamma}{l} \cdot 2g_1 m}{g_1 m^2 + g_2} \right) \]
\[ = -\frac{pE_\gamma}{m} f'_3 + \frac{qE_\gamma}{l} (2(m - m^{-3})(g'_1 m^2 + g'_2) - 2g_1 f'_3 m) \]
\[ = \frac{2E_\gamma}{l} \left( (m - m^{-3})(g'_1 m^2 + g'_2) - g_1 f'_3 m \right) \left( \frac{1}{m} \frac{dm}{dl} + q \right) \]
\[ = \frac{2E_\gamma}{l} \left( (-g'_1 f_3 + 2g'_2 - g_1 f'_3) m + (-2g'_1 + g'_2 f_3) m^{-1} \right) \left( \frac{1}{m} \frac{dm}{dl} + q \right) . \]

(14) \[ \]

Note that the last equation is obtained from the equation $m^2 + m^{-2} + f_3 = 0$ as in Remark 4.
We now claim that
\begin{align*}
(15) \quad -g_1'f_3 + 2g_2' - g_1f_3' + g_1\tau_\lambda(\chi_\rho)/f_1 &= 0 \quad \text{and} \quad -2g_1' + g_2f_3' + g_2\tau_\lambda(\chi_\rho)/f_1 = 0.
\end{align*}
Note these equations are only in the variable $z$ due to Theorem 3. From the identity $S_n^2 - zS_nS_{n-1} + S_{n-1}^2 = 1$, we have
\[2S_nS_n' - S_nS_{n-1} - zS_nS_{n-1} - zS_nS_n' + 2S_{n-1}S_{n-1}' = 0.\]
Together with the identity $nS_{n-1} + (n-1)S_n = (z+2)(S_n' - S_{n-1}')$, we obtain
\begin{equation}
(16) \quad S_n' = \frac{(n-1)zS_n - 2nS_{n-1}}{z^2 - 4} \quad \text{and} \quad S_{n-1}' = \frac{2(n-1)S_n - znS_{n-1}}{z^2 - 4}.
\end{equation}
Recall that $g_1, g_2, f_1, f_2$, and $f_3$ are given in terms of $S_n, S_{n-1}$, and $S_{n+1}$. Plugging $S_{n+1} = zS_n - S_{n-1}$, they are given in terms of $S_n$ and $S_{n-1}$ and thus the equation (15) is given in terms of $S_n, S_{n-1}, S_n', S_{n-1}'$, and $S_n'$. Then plugging the equation (16) into the equation (15), we obtain two equations, each of which consists of terms in $S_n$ and $S_{n-1}$. With the aid of Mathematica, one can check that both equations have a factor $S_n^2 - zS_nS_{n-1} + S_{n-1}^2 - 1$ which is identically zero. This proves the equation (15).

Combining the equations (14) and (15), we have
\[
\det \left( \frac{\partial (F, E_\gamma)}{\partial (m, z)} \right) = -\frac{2E_\gamma f_1}{l} \left( \frac{g_1\tau_\lambda(\chi_\rho)}{f_1} m + \frac{g_2\tau_\lambda(\chi_\rho)}{f_1} m^{-1} \right) \left( \frac{l}{m} \frac{dm}{dl} + q \right)
\]
\[
= -\frac{2E_\gamma f_1}{l} \left( \frac{l}{mf_1} \tau_\lambda(\chi_\rho) \right) \left( \frac{l}{m} \frac{dm}{dl} + q \right)
\]
\[
= -\frac{2E_\gamma f_1}{mf_1} \tau_\lambda(\chi_\rho).
\]
Recall that the last equation follows from [14, Theorem 4.1]:
\[
\tau_\gamma(\chi_\rho) = \tau_\lambda(\chi_\rho) \frac{d\log(m^p |\rho|)}{d\log l} = \tau_\lambda(\chi_\rho) \left( \frac{l}{m} \frac{dm}{dl} + q \right).
\]
This completes the proof, since we have
\[
\det \left( \frac{\partial (F, E_\gamma)}{\partial (m, z)} \right) = \frac{1}{f_1} \det \left( \frac{\partial (F, E_\gamma)}{\partial (m, z)} \right)
\]
on the zero set of $F(m, z)$.

\[\square\]

**Remark 8** In this paper, we use two variables for computational simplicity. However, using three variables with the variable $l$ seems also natural. Precisely, if we let $E_\gamma = m^p |\rho|$ and
\[
H(m, z, l) = l - \left( \frac{(z-2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n-1}} m^2 + (z-2)(S_n + S_{n-1})S_n + 1 \right),
\]
then we have
\[ \tau_\gamma(\chi_\rho) = -\frac{m}{2E_\gamma} \det \left( \frac{\partial(F, E_\gamma, H)}{\partial(m, z, l)} \right) \]
for \((m, z, l) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*\) satisfying \(F = H = 0\).

3 Proof of Theorem 1

We fix \(n \in \mathbb{Z} \setminus \{0, 1\}\) and let \(M\) be the knot exterior of the twist knot \(K_{2n}\). We also fix a boundary curve \(\gamma \subset \partial M\) of slope \(p/q \in \mathbb{Q} \cup \{\infty\}\). We may assume that \(q \geq 0\). In Section 2, we computed that for generic \(c \in \mathbb{C}^*\)
\[ \sum_{\chi_\rho \in \text{tr}_{\gamma}^{-1}(c + \frac{1}{c})} \frac{1}{\tau_\gamma(\chi_\rho)} = \sum_{(m, z) \in Z_{F,G}} \frac{-2c}{m \det \left( \frac{\partial(F, G)}{\partial(m, z)} \right)} \]
where
\[ F(m, z) = S_n(S_n - S_{n-1})(m^2 + m^{-2}) - zS_n(S_n - S_{n-1}) + 1 \]
\[ G(m, z) = m^p \left( \frac{(z - 2)(S_n + 1 - S_{n-1})S_n^2 m^2 + (z - 2)(S_n + S_{n-1})S_n + 1}{S_n - S_{n-1}} \right)^q - c \]
and \(Z_{F,G}\) is the common zero set of \(F\) and \(G\) in \(\mathbb{C}^* \times \mathbb{C}\). Note that genericity of \(c\) guarantees that \(\text{tr}_{\gamma}^{-1}(c + \frac{1}{c})\) consists of \(\gamma\)-regular characters (see Remark 2).

Remark 9 Recall Remark 7 that we can take \(G\) as a Laurent polynomial. If the system \((F, G)\) of Laurent polynomials is \((\Delta_F, 0)\)-proper (see [21] for the definition), where \(\Delta_F\) is the Newton polygon of \(F\), then Theorem 1 is directly obtained from [21, Theorem 1.2].

3.1 For even \(p\)

For simplicity let \(F(m, z) = f_1(z)(m^2 + m^{-2}) + f_2(z), G(m, z) = m^p(g_1(z)m^2 + g_2(z))^{q} - c, \) and \(f_3(z) = f_2(z)/f_1(z)\). Note that by definition if \(f_1(z_0) = 0\) for some \(z_0 \in \mathbb{C}\), then \(f_2(z_0) = 1\). In particular, \(f_1\) and \(f_2\) has no common zero.

From the equation \(F/f_1 = m^2 + m^{-2} + f_3 = 0\), we recursively obtain (see Remark 4)
\[ m^{2k} = h_k(z)m^2 - h_{k-1}(z) \]
for all \( k \in \mathbb{Z} \) where \( h_0(z) = 0 \), \( h_1(z) = 1 \), and \( h_{k+1}(z) = -f_3(z)h_k(z) - h_{k-1}(z) \). It follows that
\[
G(m, z) = m^q (g_1 m^2 + g_2)^q - c
= \sum_{k=0}^{q} \binom{q}{k} g_1^k g_2^{q-k} m^{2k+p} - c
= \left( \sum_{k=0}^{q} \binom{q}{k} g_1^k g_2^{q-k} h_{k+\frac{p}{2}} \right) m^2 - \left( \sum_{k=0}^{q} \binom{q}{k} g_1^k g_2^{q-k} h_{k+\frac{p}{2} - 1} + c \right)
= \alpha(z)m^2 - \beta(z)
\]
on the zero set of \( F(m, z) \).

**Lemma 4** For generic \( c \in \mathbb{C}^* \) one has
\[
Z_{F,G} = \left\{ (m, z) \in \mathbb{C}^* \times \mathbb{C} : H(z) = 0, f_1(z) \neq 0, m = \pm \sqrt[3]{\frac{\beta(z)}{\alpha(z)}} \right\}
\]
where \( H = f_1(b \pm \frac{\beta}{\alpha}) + f_2 \).

**Proof** For \((m_0, z_0) \in Z_{F,G} \) we have \( f_1(z_0) \neq 0 \); otherwise, we have \( f_1(z_0) = f_2(z_0) = 0 \) which contradicts the fact that \( f_1 \) and \( f_2 \) have no common zero. Since denominators of \( f_3, g_1 \), and \( g_2 \) are \( f_1 = S_n(S_n - S_{n-1}), S_n - S_{n-1}, \) and \( 1 \), respectively, denominators of \( \alpha \) and \( \beta \) divide some power of \( f_1 \). Thus the fact \( f_1(z_0) \neq 0 \) implies that both \( \alpha \) and \( \beta \) have no pole at \( z_0 \). If either \( \alpha(z_0) = 0 \) or \( \beta(z_0) = 0 \), then from \( \alpha(z_0)m_0^2 - \beta(z_0) = 0 \) we have \( \alpha(z_0) = \beta(z_0) = 0 \) which fails for generic \( c \). Therefore, both \( \alpha(z_0) \) and \( \beta(z_0) \) are non-zero complex numbers. It follows that \( m_0 = \pm \sqrt[3]{\beta(z_0)/\alpha(z_0)} \) and \( H(z_0) = 0 \).

Conversely, let \( z_0 \) be a zero of \( H \) with \( f_1(z_0) \neq 0 \). Recall that \( f_1(z_0) \neq 0 \) implies that both \( \alpha \) and \( \beta \) have no pole at \( z_0 \). For generic \( c \), we may assume that \( \alpha \) and \( \beta \) have no common zero. Then from \( H(z_0) = 0 \) we have \( \alpha(z_0) \neq 0 \) and \( \beta(z_0) \neq 0 \); otherwise, \( H \) has a pole at \( z_0 \). Letting \( m_0 = \pm \sqrt[3]{\beta(z_0)/\alpha(z_0)} \), it is clear that \( m_0 \) is a non-zero complex number satisfying \( F(m_0, z_0) = 0 \) and \( G(m_0, z_0) = 0 \).

**Lemma 5** For each \( (m, z) \in Z_{F,G} \) one has
\[
\det \left( \frac{\partial(F, G)}{\partial(m, z)} \right) = -2m\alpha H'.
\]
Proof. Recall that we have $m^{2k} = h_k m^2 - h_{k-1}$ from the equation $F(m, z) = 0$. It follows that
\[
(18) \quad \det \left( \frac{\partial (F, m^{2k} - h_k m^2 + h_{k-1})}{\partial (m, z)} \right) = 0
\]
for all $k \in \mathbb{Z}$. We then have
\[
\det \left( \frac{\partial (F, G)}{\partial (m, z)} \right) = \det \left( \frac{\partial (F, \alpha m^2 - \beta)}{\partial (m, z)} \right) = \det \left( \frac{\partial (H, \alpha m^2 - \beta)}{\partial (m, z)} \right) = -2m \alpha H'.
\]
Here the first equation follows from the equation (18) together with the fact that $\alpha m^2 - \beta$ differs from $G$ by some linear combination of $m^{2k} - h_k m^2 + h_{k-1}$. Similarly, the second equation follows from the fact that $H$ differs from $F$ by some multiple of $\alpha m^2 - \beta$.

Rewriting the equation (17) by using Lemmas 4 and 5, we have
\[
1 = \sum_{\chi \rho \in Tr^{-1}(c + \frac{1}{c})} \frac{1}{\tau_{\gamma}(X, \rho)} = \sum_{(m, z) \in Z_{r, c}} \frac{-2c}{m \det \left( \frac{\partial (F, G)}{\partial (m, z)} \right)} - c \alpha \beta H'
\]
where $m = \pm \sqrt{\frac{\beta(z)}{\alpha(z)}}$
\[
= \sum_{H(\zeta) = 0, f_1(\zeta) \neq 0} \frac{2c}{\beta H'}
\]
and the last equation follows from Lemma 6 below.

Lemma 6. \(\beta H'\) has a pole at each common zero of $H$ and $f_1$ for generic $c \in \mathbb{C}^*$.

Proof. Let $z_0$ be a common zero of $H$ and $f_1$ and let $\nu$ be the discrete valuation counting the order of zero/pole at $z_0$. Clearly, we have $\nu(H) > 0$ and thus $\nu(H') < \nu(H)$. If $\nu(\beta H) \leq 0$, then we have $\nu(\beta H') < \nu(\beta H) \leq 0$ and we are done. In what follows, we prove that the other case $\nu(\beta H) > 0$ gives us contradiction.

Let $\delta = \beta - c$ so that $\delta$ does not depend on $c$. From the fact that $E_{\gamma}(m, z) = \alpha m^2 - \delta$ on the zero set of $F$ and $E_{\gamma}(m, z)E_{\gamma}(m^{-1}, z) = 1$, we have
\[
1 = (\alpha m^2 - \delta)(\alpha m^{-2} - \delta)
= \alpha^2 + \delta^2 - \alpha \delta (m^2 + m^{-2})
= \alpha^2 + \delta^2 + f_3 \alpha \delta
\]
on the zero set of $F$. Since $\alpha^2 + \delta^2 + f_2\alpha\delta$ is a one-variable rational function, it should be identically 1.

Recall that by definition we have $\alpha\beta H = f_1(\alpha^2 + \beta^2) + f_2\alpha\beta$. From the equation (20) $\Leftrightarrow f_1 = f_1(\alpha^2 + \delta^2) + f_2\alpha\delta$, we have

$$\alpha\beta H - f_1 = f_1(\beta^2 - \delta^2) + f_2\alpha(\beta - \delta)$$
$$\Leftrightarrow \alpha\beta H - f_1 = f_1(2\delta c + c^2) + f_2\alpha c$$
$$\Leftrightarrow c^{-1}\beta H = 2\alpha^{-1}f_1\delta + \alpha^{-1}f_1(c + c^{-1}) + f_2.$$

Recall that $f_1(z_0) = 0$ implies $f_2(z_0) = 1$. From the equation (21) we conclude that for generic $c$ the only possible case of $\nu(\beta H) > 0$ is that $\nu(\delta) < 0$ and $\nu(f_1) + \nu(\delta) = \nu(\alpha)$ with $\alpha^{-1}f_1(\delta)(z_0) = -\frac{1}{2}$. On the other hand, from the equation (20) we have

$$f_1 = f_1(\alpha^2 + \delta^2) + f_2\alpha\delta$$
$$\Leftrightarrow \alpha^{-1}f_1 = f_1\alpha + \delta(\alpha^{-1}f_1\delta + f_2).$$

Since $(\alpha^{-1}f_1\delta + f_2)(z_0) = -\frac{1}{2} + 1 = \frac{1}{2}$ and $\nu(f_1\alpha) = 2\nu(f_1) + \nu(\delta) > \nu(\delta)$, we have $\nu(f_1\alpha + \delta(\alpha^{-1}f_1\delta + f_2)) = \nu(\delta)$. It follows that $\nu(\alpha^{-1}f_1) = \nu(\delta)$ and $-\nu(\delta) = \nu(\delta)$ ($\because \nu(f_1) + \nu(\delta) = \nu(\alpha)$). This contradicts to $\nu(\delta) < 0$.

We finally claim that the equation (19) is zero due to Jacobi’s residue theorem. Recall that Jacobi’s residue theorem says that any non-constant polynomial $f$ with $f(0) \neq 0$ and no multiple zero satisfies

$$\sum_{f(z)=0} g(z) f'(z) = 0$$

for any polynomial $g$ with $\deg g \leq \deg f - 2$.

**Lemma 7** For generic $c \in \mathbb{C}^*$ one has

$$\sum_{H(z)=0} \frac{1}{\beta H'} = 0.$$  

**Proof** Recall that $\alpha$ and $\beta$ are rational functions in the variable $z$. We let $\alpha = \alpha_1/\alpha_2$ and $\beta = \beta_1/\beta_2$ for some polynomials $\alpha_1, \alpha_2, \beta_1,$ and $\beta_2$ with $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$. Here we take the greatest common divisor in $\mathbb{C}[z]$. Plugging these into $H$, we have

$$H = \frac{f_1(\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2) + f_2\alpha_1\alpha_2\beta_1\beta_2}{\alpha_1\alpha_2\beta_1\beta_2} = \frac{H_1d}{H_2d}$$

$$\Leftrightarrow \frac{1}{\beta H'} = 0.$$
where $\gcd(H_1, H_2) = 1$ and $d$ is the greatest common divisor of the numerator and denominator. Clearly, we have

$$\sum_{H_1(z) = 0} \frac{1}{\beta_1 H_1'} = \sum_{H_2(z) = 0} \frac{1}{\beta_2 (H_2/H_1)'} = \sum_{H_1(z) = 0} \frac{\beta_2 H_2}{\beta_1 H_1'}$$

Note that $\gcd(\beta_1, H_1 d) = \gcd(\beta_1, f_1 \alpha_1^2 \beta_2^2) = 1$ for generic $c$ (\because $f_1 \alpha_1^2 \beta_2$ does not depend on $c$). In particular, $\beta_1$ and $d$ have no common zero. It follows from $H_2 d = \alpha_1 \alpha_2 \beta_1 \beta_2$ that $\beta_1$ divides $H_2$ and thus $H_2/\beta_1$ is a polynomial.

For generic $c$ we may assume that $H_1$ has no multiple zero and is non-zero at $z = 0$. Lemma 8 below shows that $\deg \beta_2 H_2 - \deg \beta_1 \leq \deg H_1 - 2$. Then we have

$$\sum_{H_1(z) = 0} \frac{\beta_2 H_2/\beta_1}{H_1'} = 0$$

from Jacobi’s residue theorem.

**Lemma 8**  For generic $c \in \mathbb{C}^*$ one has $\deg H + \deg \beta \geq 2$.

Here the degree of a rational function means the degree difference of the numerator and denominator: for instance, $\deg H = \deg H_1 - \deg H_2$ and $\deg \beta = \deg \beta_1 - \deg \beta_2$.

**Proof**  Recall the proof of Lemma 6 that $\alpha^2 + \delta^2 + f_3 \alpha \delta = 1$ where $\delta = \beta - c$. Since $\deg f_3 = \deg f_2 - \deg f_1 = 1$, we have $\deg \alpha \neq \deg \delta$. On the other hand, one can simplify $H$ as

$$H = \frac{f_1(\alpha^2 + \beta^2) + f_2 \alpha \beta}{\alpha \beta} = \frac{f_1(\alpha^2 + \delta^2 + 2c \delta + c^2) + f_2 \alpha (\delta + c)}{\alpha \beta} = \frac{f_1(1 + 2c \delta + c^2) + f_2 \alpha c}{\alpha \beta} = \frac{cf_1(c^{-1} + c + 2 \delta + f_3 \alpha)}{\alpha \beta} = \frac{cf_1((c^{-1} + c) \delta + 1 - \alpha^2 + \delta^2)}{\alpha \beta \delta}.$$  

It follows that for generic $c$

$$\deg H + \deg \beta = \deg f_1 + \text{Max}(\deg \delta, \deg(1 - \alpha^2 + \delta^2)) - \deg \alpha - \deg \delta.$$
If $\deg \alpha < 0$, then
$$\deg H + \deg \beta \geq \deg f_1 + \deg \delta - \deg \alpha - \deg \delta = \deg f_1 - \deg \alpha \geq \deg f_1 + 1.$$  
If $\deg \alpha \geq 0$, then
$$\deg H + \deg \beta \geq \deg f_1 + \deg(1 - \alpha^2 + \delta^2) - \deg \alpha - \deg \delta \geq \deg f_1 + 1.$$  
Note that the last inequality follows from the fact $\deg \alpha \neq \deg \delta$. This completes the proof, since $\deg f_1 \geq 1$ for $n \neq 0, 1$.

\[\square\]

3.2 For odd $p$

As in Section 3.1, we have
$$G(m, z) = m^{-1}m^{p+1}(g_1m^2 + g_2)^q - c$$
$$= \sum_{k=0}^q \binom{q}{k} g_1^k g_2^{q-k} h_{k+\frac{m+1}{2}} m - \sum_{k=0}^q \binom{q}{k} g_1^k g_2^{q-k} h_{k+\frac{m-1}{2}} m^{-1} - c$$
$$=: \alpha(z)m - \beta(z)m^{-1} - c.$$

on the zero set of $F$. From the fact that $E_\gamma(m, z)E_\gamma(m^{-1}, z) = 1$, we have
$$1 = (\alpha m - \beta m^{-1})(\alpha m^{-1} - \beta m)$$
$$= \alpha^2 + \beta^2 - \alpha \beta (m^2 + m^{-2})$$
$$= \alpha^2 + \beta^2 + f_3 \alpha \beta$$
on the zero set of $F$ and therefore $\alpha^2 + \beta^2 + f_3 \alpha \beta$ should be identically 1.

It is not hard to solve the equations $F(m, z) = 0$ and $\alpha m - \beta m^{-1} - c = 0$ for generic $c \in \mathbb{C}^*$ (see the proof of Lemma 9 below for details):
$$Z_{F, G} = \left\{ (m, z) \in \mathbb{C}^* \times \mathbb{C} : H(z) = 0, m = \frac{c \alpha + \beta}{\alpha^2 - \beta^2} \right\}$$

where $H = (f_3 + 2)(\alpha + \beta)^2 + (c - 1/c)^2$.

Remark 10 \textit{H and $f_1$ have no common zero for generic $c$. It follows that for every zero $z_0$ of $H$, $f_3(z_0) + 2$ is a certain complex number and thus $\alpha(z_0) + \beta(z_0) \neq 0$. On the other hand, using the equation $\alpha^2 + \beta^2 + f_3 \alpha \beta = 1$, we can rewrite $H$ as}

\begin{equation}
H = (f_3 - 2)(\alpha - \beta)^2 + (c + 1/c)^2.
\end{equation}

We then have $\alpha(z_0) - \beta(z_0) \neq 0$ similarly. It follows that $m = \frac{c \alpha(z_0) + \beta(z_0)}{\alpha^2(z_0) - \beta^2(z_0)}$ is a non-zero complex number.
Lemma 9  For each \((m, z) \in Z_{F,G}\) one has
\[
\det \left( \frac{\partial (F, G)}{\partial (m, z)} \right) = \frac{cf_1}{(\beta^2 - \alpha^2)m} H'.
\]

Proof  Using the Euclidean algorithm, one can find an equation, which is linear in the variable \(m\), among linear combinations of \(F\) and \(B := \alpha m - \beta m^{-1} - c\). Precisely, we have
\[
D := (\alpha \beta f_1^{-1})F + (-\beta^2 m + \alpha \beta m^{-1} - c\alpha)B \\
= c(\beta^2 - \alpha^2)m + c^2 \alpha + \beta(\alpha^2 + \beta^2 + f_3 \alpha \beta) \\
= c(\beta^2 - \alpha^2)m + c^2 \alpha + \beta.
\]
In particular, we have
\[
\det \left( \frac{\partial (F, G)}{\partial (m, z)} \right) = \det \left( \frac{\partial (F, B)}{\partial (m, z)} \right) \\
= \det \left( \begin{array}{cc}
\alpha \beta f_1^{-1} & 0 \\
1 & 1
\end{array} \right)^{-1} \det \left( \frac{\partial (D, B)}{\partial (m, z)} \right)
\]
where the first equation follows from the equation (18) (as in the proof of Lemma 5). Plugging the equation \(D\) (which is linear in \(m\)) into \(B\) to eliminate the variable \(m\), we obtain
\[
E := \frac{\alpha \beta^2 ((f_3 + 2)(\alpha + \beta)^2 + (c - 1/c)^2)}{(\alpha^2 - \beta^2)(\alpha \beta + \beta/c)}.
\]
Note that we used the equation \(\alpha^2 + \beta^2 + f_3 \alpha \beta = 1\) in the simplification. It follows that
\[
\det \left( \frac{\partial (D, B)}{\partial (m, z)} \right) = \det \left( \frac{\partial (D, E)}{\partial (m, z)} \right) = c(\beta^2 - \alpha^2)E'.
\]
Combining the above computations, we have
\[
\det \left( \frac{\partial (F, G)}{\partial (m, z)} \right) = \frac{f_1}{\alpha \beta^2 \alpha^2} \det \left( \frac{\partial (D, B)}{\partial (m, z)} \right) \\
= \frac{cf_1}{\alpha \beta^2} E' \\
= -\frac{cf_1}{c \alpha + \beta/c} H' \\
= \frac{cf_1}{(\beta^2 - \alpha^2)m} H'.
\]
Note that the third and fourth equations follow from the equations \(H = 0\) and \(D = 0\), respectively. \(\square\)
Rewriting the equation (17) by using Lemma 9, we have
\[
\sum_{\chi \in \mathcal{G}} \frac{1}{\tau(z)} = \sum_{(m,d) \in \mathcal{M}} m \det \left( \frac{\partial(F,G)}{\partial(m,z)} \right) = \sum_{H(\gamma) = 0} \frac{2(\alpha^2 - \beta^2)}{f_1 H'}.
\]
As in Section 3.1, we claim that the above equation is zero due to Jacobi’s residue theorem.

**Lemma 10** For generic \( c \in \mathbb{C}^* \) one has
\[
\sum_{H(\gamma) = 0} \frac{\alpha^2 - \beta^2}{f_1 H'} = 0.
\]

**Proof** Let \( \alpha = \alpha_1/\alpha_2 \) and \( \beta = \beta_1/\beta_2 \) for some polynomials \( \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) with \( \gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1 \). Plugging these into \( H \), we have
\[
H = \frac{(f_2 + 2f_1)(\alpha_1 \beta_2 + \alpha_2 \beta_1)^2 + (c - 1/c)^2 f_1 \alpha_2^2 \beta_2^2}{f_1 \alpha_2^2 \beta_2^2} = : \frac{H_1 \alpha_1^2}{H_2 d}
\]
with \( \gcd(H_1, H_2) = 1 \). Then we have
\[
(23) \quad \sum_{H(\gamma) = 0} \frac{\alpha^2 - \beta^2}{f_1 H'} = \sum_{H(\gamma) = 0} \frac{(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2) H_2}{f_1 \alpha_2^2 \beta_2^2 H_1} = \sum_{H(\gamma) = 0} \frac{(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2)/d}{H_1}.
\]
We claim that \( d \) divides \( \alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2 \) and thus \( (\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2)/d \) is a polynomial. It follows from the definitions of \( \alpha \) and \( \beta \) that the denominators \( \alpha_2 \) and \( \beta_2 \) divide some power of \( f_1 \). In particular, every zero \( z_0 \) of \( d \) is also a zero of \( f_1 \). Let \( \nu \) be the discrete valuation counting the order of zero/pole at \( z_0 \). From the fact that \( d \) divides \( (f_2 + 2f_1)(\alpha_1 \beta_2 + \alpha_2 \beta_1)^2 = (H_1 - (c - 1/c)^2 H_2) d \) and that \( \gcd(f_1, f_2) = 1 \), we have
\[
\nu(d) \leq 2 \nu(\alpha_1 \beta_2 + \alpha_2 \beta_1).
\]
On the other hand, using the equality \( \alpha^2 + \beta^2 + f_3 \alpha \beta = 1 \), one can rewrite \( H \) as
\[
H = \frac{(f_2 - 2f_1)(\alpha_1 \beta_2 - \alpha_2 \beta_1)^2 + (c + 1/c)^2 f_1 \alpha_2^2 \beta_2^2}{f_1 \alpha_2^2 \beta_2^2} = \frac{H_1 \alpha_1^2}{H_2 d}
\]
and then we have \( \nu(d) \leq 2 \nu(\alpha_1 \beta_2 - \alpha_2 \beta_1) \) similarly. Therefore,
\[
\nu(d) \leq \nu(\alpha_1 \beta_2 + \alpha_2 \beta_1) + \nu(\alpha_1 \beta_2 - \alpha_2 \beta_1) = \nu(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2).
\]
This proves that \( d \) divides \( \alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2 \).

Lemma 11 below shows that \( \deg(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2) - \deg d \leq \deg H_1 - 2 \), which is equivalent to \( \deg(\alpha^2 - \beta^2) \leq \deg H + \deg f_1 - 2 \) (see the equation (23)). Then the lemma follows from Jacobi’s residue theorem. \( \square \)
Lemma 11  For generic \( c \in \mathbb{C}^* \) one has \( \deg H + \deg f_1 - 2 \geq \deg(\alpha^2 - \beta^2) \).

**Proof**  Recall that \( f_1H = (f_2 + 2f_1)(\alpha + \beta)^2 + (c - 1/c)^2 f_1. \) It follows that for generic \( c \in \mathbb{C}^* \)
\[
\deg f_1 + \deg H \geq \text{Max}(\deg(f_2 + 2f_1)(\alpha + \beta)^2, \deg f_1).
\]
In particular, \( \deg f_1 + \deg H \geq \deg(f_2 + 2f_1) + 2 \deg(\alpha + \beta) \). One can easily check that \( \deg(f_2 + 2f_1) \geq 2 \) for \( n \neq 0, 1 \). Also, we have \( \deg \alpha \neq \deg \beta \) from the equation \( \alpha^2 + \beta^2 + f_3 \alpha \beta = 1 \) with \( \deg f_3 = 1 \). It follows that \( \deg(\alpha + \beta) = \deg(\alpha - \beta) \). This completes the proof, as we have
\[
\deg f_1 + \deg H \geq \deg(f_2 + 2f_1) + 2 \deg(\alpha + \beta) \\
\geq 2 + \deg(\alpha + \beta) + \deg(\alpha - \beta) \\
= 2 + \deg(\alpha^2 - \beta^2).
\]

\[\square\]

Remark 11  Lemmas 8 and 11 do not hold for \( n = 1 \) (the trefoil knot). Furthermore, the equation (17) is non-zero for the trefoil knot. This shows that the hyperbolicity assumption in Conjecture 1 is essential.

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sbyoon15@kias.re.kr