On the integrability of Einstein–Maxwell–(A) dS gravity in the presence of Killing vectors

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Abstract

We study symmetry and integrability properties of four-dimensional Einstein–Maxwell gravity with a nonvanishing cosmological constant in the presence of Killing vectors. First of all, we consider stationary spacetimes, which lead, after a timelike Kaluza–Klein reduction followed by a dualization of the two vector fields, to a three-dimensional nonlinear sigma model coupled to gravity, whose target space is a noncompact version of $\mathbb{P}^2$ with an $\text{SU}(2,1)$ isometry group. It is shown that the potential for the scalars that arises from the cosmological constant in four dimensions breaks three of the eight $\text{SU}(2,1)$ symmetries, corresponding to the generalized Ehlers and the two Harrison transformations. This leaves a semidirect product of a one-dimensional Heisenberg group and a translation group $\mathbb{R}^2$ as residual symmetry. We show that, under the additional assumptions that the three-dimensional manifold is conformal to a product space $\mathbb{R} \times \Sigma$ and all fields depend only on the coordinate along $\mathbb{R}$, the equations of motion are integrable. This generalizes the results of Leigh et al in arXiv:1403.6511 to the case where electromagnetic fields are also present. In the second part of the paper we consider the purely gravitational spacetime admitting a second Killing vector that commutes with the timelike one. We write down the resulting two-dimensional action and discuss its symmetries. If the fields depend only on one of the two coordinates, the equations of motion are again integrable, and the solution turns out to be one constructed by Krasiński many years ago.

Keywords: black holes, classical theories of gravity, integrable equations in physics
1. Introduction

Exact solutions to Einstein’s field equations and to their supergravity generalizations have been playing an important role in many developments of general relativity, string theory and high energy physics [1]. For instance, they can give us a lot of insight into the theoretical aspects of general relativity such as the vacuum structure, uniqueness theorems and so on. The importance of exact solutions, though, is not limited to the classical situation, but extends also to quantum gravity. Indeed, much of our current knowledge on quantum effects in strong gravitational fields comes from the study of classical black hole solutions.

Generically, the construction of exact solutions to general relativity is a notoriously difficult problem, since the underlying field equations are a system of coupled nonlinear partial differential equations of second order. Nevertheless, one may hope that this system becomes integrable when a sufficient amount of symmetry is present. During the late 1970s, a variety of independent groups established the integrability of Einstein’s vacuum equations in stationary and axisymmetric systems. This includes the discovery of Bäcklund transformations by Harrison [2] and by Neugebauer [3], and a Lax-pair representation by Belinsky and Zakharov (BZ) [4]. In particular, the results of [4] have been generalized in many directions, e.g. to the Einstein–Maxwell system [6, 7], five-dimensional general relativity [8] and five-dimensional minimal ungauged supergravity [9] (see also [10–12]). These techniques of integrable systems allow us to construct nontrivial new solutions starting from a given seed by adding solitons in a simple algebraic manner. Perhaps one of the most exciting recent achievements for the applications of these techniques is the inverse scattering construction of black objects admitting multiple horizons with various topologies in five-dimensional pure gravity (see, e.g., [13] for a comprehensive review).

Apart from the construction of numerous exact solutions, the underlying mathematical structure behind these integrable systems has been worked out by many authors. Prior to the studies of integrability, Geroch made a pioneering analysis of solution-generating methods for Ricci flat spaces in presence of a single [14] and two mutually commuting Killing vectors [15] (see [16–19] for an electromagnetic generalization). In the presence of two commuting Killing vectors, the target space isometry does not commute with the internal SL(2, R) symmetry, giving rise to an infinite affine Lie group called the Geroch group. Hauser and Ernst were able to prove the conjecture that any stationary and axisymmetric solution can be derived in principle from a Minkowski seed by the Geroch group [20]. Cosgrove addressed the interrelationships between several solitonic systems and the Geroch group [21]. Later on, Breitenlohner and Maison (BM) unraveled the group theoretical structure of solitonic methods from the standpoint of a Riemann-Hilbert problem [22]. A very nice general analysis of the relation between the BM group structure and the inverse scattering method of the BZ approach was recently given in [23]. In addition, a close connection to nonlinear sigma models has also been widely discussed [24].

In view of possible AdS/CFT (and many other) applications, one may thus ask whether similar integrability properties still hold in the presence of a cosmological constant, for example for stationary axisymmetric Einstein spaces in four dimensions. The introduction of a negative cosmological constant has a strong impact on the spectrum of black holes. One of the most prominent is that the horizon of an asymptotically AdS black hole can be a

1 Shortly before [4] appeared, Maison [5] was able to rewrite the stationary axisymmetric vacuum Einstein equations as a ‘linear eigenvalue problem in the spirit of Lax’, and noted that this could ‘nourish some hope that a method similar to the inverse scattering method may be developed’. 
compact Riemann surface of any genus [25–28]. This is in contrast to black holes in asymptotically flat spacetimes [29, 30]. This is not the end of the story, since even more exotic possibilities exist, e.g. noncompact horizons with finite area [31, 32] (for generalizations to higher dimensions and further discussions of the physics of these solutions cf. also [33, 34]). One may thus expect a rich spectrum of black objects in the presence of a cosmological constant, with many of them perhaps still to be discovered. It is clear that the integrability of stationary axisymmetric Einstein spaces would enormously simplify the construction of such solutions. A main obstruction for this program is that the metric cannot be cast into the Weyl-Papapetrou form in the presence of a cosmological constant. It is therefore obvious that the techniques available in the absence of \( \Lambda \) cannot be straightforwardly applied.

First steps in the investigation of the integrability properties with nonvanishing \( \Lambda \) were undertaken in [35–38]. These papers developed solution-generating techniques for the stationary vacuum [36, 38] and electrovac [37] Einstein equations with a cosmological constant, in four [37, 38] and higher [36] dimensions. The cosmological constant leads to a potential in the dimensionally reduced system, breaking the symmetries of the original sigma model, and thus the usual solution-generating techniques cannot be applied anymore. Still, a restricted formalism of the solution-generating method is still applicable. In spite of this limited utility, they turn out to be indeed fruitful in generating new exact solutions [37].

Here we shall make a first step towards a systematic investigation of the integrability properties of Einstein–Maxwell gravity with nonvanishing cosmological constant in four dimensions, by extending the work of [38]. In the first part of this paper we consider stationary spacetimes which are described, after a timelike Kaluza–Klein reduction followed by a dualization of the two vector fields, by a three-dimensional nonlinear sigma model coupled to gravity, whose target space admits an \( \text{SU}(2, 1) \) isometry group. It is shown that the potential for the scalars that arises from the cosmological constant in four dimensions breaks three of the eight \( \text{SU}(2, 1) \) symmetries, namely the generalized Ehlers and the two Harrison transformations. This leaves a semidirect product of a one-dimensional Heisenberg group and a translation group \( \mathbb{R}^2 \) as residual symmetry. We show that, under the additional assumptions that the three-dimensional manifold is conformal to a product space \( \mathbb{R} \times \Sigma \) and all fields depend only on the coordinate along \( \mathbb{R} \), the equations of motion are integrable. Subsequently, we consider the purely gravitational case and assume the existence of a second Killing vector that commutes with the timelike one, i.e., we focus on stationary and axisymmetric Einstein spaces. We write down the resulting two-dimensional action and discuss its symmetries. If the fields depend only on one of the two coordinates, the equations of motion are again integrable, and the solution turns out to be one constructed by Krasiński many years ago [39–41].

The remainder of this paper is organized as follows. In the next section, we discuss the integrability of the stationary Einstein–Maxwell–\( \Lambda \) system by assuming that the base space takes a product structure \( \mathbb{R} \times \Sigma \) and the target space variables depend only on a single coordinate. We derive the Reissner-Nordström–Taub–NUT–(A)dS metric by exploiting the Hamilton-Jacobi method. This generalizes the results of [38] to the case where electromagnetic fields are also present. In section 3, we address the integrability of the Einstein–\( \Lambda \) system by assuming a second independent Killing vector and derive an interesting class of solutions. Finally, we conclude in section 4 with some final remarks. An appendix provides an attempt at a higher-dimensional generalization.
2. Einstein–Maxwell–Λ system

In this paper, we focus on 3+1-dimensional Einstein–Maxwell–(A)dS gravity, with action

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda \right), \]

and equations of motion

\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 2\left( F_{\mu\rho}F^{\rho\nu} - \frac{1}{4}g_{\mu\nu}F^{\sigma\rho}F_{\sigma\rho} \right), \quad \nabla_{\mu}F^{\mu\nu} = 0. \]

The Faraday tensor can be locally expressed in terms of a gauge potential as

\[ F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \]

We shall investigate the integrability properties of the stationary Einstein–Maxwell–Λ system, which is an extension of the work in [38].

2.1. Dimensional reduction

Let us consider stationary spacetimes admitting a Killing field that is timelike at infinity. Applying the algorithm of the Kaluza–Klein reduction along the timelike direction, the metric and the gauge field can be decomposed as

\[ ds^2 = -e^{-\phi}(dt + K_{\alpha}dx^\alpha)^2 + e^{\phi}h_{\alpha\beta}dx^\alpha dx^\beta, \quad A = B\left(dt + K_{\alpha}dx^\alpha\right) + B_{\alpha}dx^\alpha, \]

where early greek indices refer to three dimensions, and the fields \( h_{\alpha\beta}, K_{\alpha}, B_{\alpha}, \phi \) and \( B \) are \( t \)-independent. Here and in what follows, the indices \( \alpha, \beta, \gamma \) are raised and lowered by \( h_{\alpha\beta} \) and its inverse. Then the effective three-dimensional Lagrangian derived from (2.1) becomes

\[ L^{(3)} = \sqrt{h} \left[ R^{(3)} - \frac{1}{2} \partial_{\alpha}\phi \partial^\alpha\phi + \frac{1}{4} e^{-2\phi} K^{\alpha\beta}K^{\gamma\delta} + 2e^{\phi}\partial_{\alpha}B\partial_{\beta}B \right. \]

\[ \left. - e^{-\phi}(G_{\alpha\beta} + K_{\alpha\beta}B)(G^{\alpha\beta} + K^{\alpha\beta}B) - 2\Lambda e^{-\phi} \right], \]

where \( G_{\alpha\beta} \equiv \partial_{\alpha}B_{\beta} - \partial_{\beta}B_{\alpha} \) and \( K_{\alpha\beta} \equiv \partial_{\alpha}K_{\beta} - \partial_{\beta}K_{\alpha} \). It is convenient to dualize the two vector fields to scalars, which can be implemented by adding to (2.4) a piece containing two Lagrange multipliers \( C \) and \( \tilde{\psi} \) that ensure the Bianchi identities,

\[ \tilde{L}^{(3)} = L^{(3)} + 2C e^{\alpha\beta}\partial_\gamma G_{\beta\gamma} + \left( \tilde{\psi} + CB \right) e^{\alpha\beta}\partial_\gamma K_{\beta\gamma}. \]

Variation of (2.5) w.r.t. \( K_{\alpha\beta} \) and \( G_{\alpha\beta} \) yields

\[ K^{\alpha\beta} = \frac{2}{\sqrt{h}} e^{2\phi} \epsilon^{\alpha\beta\gamma}\omega_\gamma, \quad \omega_\gamma \equiv \partial_\gamma \tilde{\psi} + C\partial_\gamma B - B\partial_\gamma C, \]

and

\[ G^{\alpha\beta} + K^{\alpha\beta}B = -\frac{1}{\sqrt{h}} e^{\alpha\beta}\partial_\gamma C. \]

These equations express the field strengths in terms of the twist potential \( \tilde{\psi} \) and the magnetic potential \( C \). Plugging (2.6) and (2.7) back into (2.5) leads (after dropping a tilde on \( \tilde{L}^{(3)} \)) to

\[ L^{(3)} = \sqrt{h} \left[ R^{(3)} - \left( J_{\alpha\beta} - 2\Lambda \right) e^{\alpha\beta} \right]. \]

---

\( ^2 \) We use the signature \( (-,+,+,+) \). The Ricci tensor is defined as

\[ R_{\mu\nu} = R^\rho_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\rho}. \]
where we have introduced the notation

\[ \langle J_\alpha, J_\beta \rangle \equiv \frac{1}{2} \left[ \partial_\alpha \phi \partial_\beta \phi + 4 e^{2\phi} \omega_\alpha \omega_\beta - 4 e^{\phi} \left( \partial_\alpha B \partial_\beta B + \partial_\alpha C \partial_\beta C \right) \right]. \] (2.9)

The equations of motion following from the Lagrangian (2.8) are the three-dimensional Einstein equations

\[ G_{\alpha\beta}^{(3)} + \Lambda e^{\phi} h_{\alpha\beta} = \langle J_\alpha, J_\beta \rangle - \frac{1}{2} h_{\alpha\beta} \langle J_\gamma, J^\gamma \rangle, \] (2.10)

supplemented by the divergence-type equations of motion

\[ \nabla_\alpha \left[ \partial^\alpha \phi + 2 e^\phi \left( B \partial^\alpha B + C \partial^\alpha C \right) - 4 e^{2\phi} \tilde{\omega}_\alpha \right] = 2 \Lambda e^\phi, \quad \nabla_\alpha \left( e^{2\phi} \omega^\alpha \right) = 0, \] (2.11)

\[ \nabla_\alpha \left( e^\phi \partial^\alpha B - 2 e^{2\phi} C \omega^\alpha \right) = 0, \quad \nabla_\alpha \left( e^\phi \partial^\alpha C + 2 e^{2\phi} B \omega^\alpha \right) = 0. \] (2.12)

Equation (2.8) describes a nonlinear \( \sigma \) model with pseudo-Riemannian target space coupled to Euclidean gravity in \( d = 3 \), with a potential. The latter breaks part of the target space isometries.

### 2.2. Nonlinear \( \sigma \) model and broken symmetries

The target space \( \Phi \) of the scalars in (2.8) is a Bergmann space corresponding to a noncompact version of \( \mathbb{CP}^2 \) [24, 37, 42], namely it describes a coset space \( \text{SU}(2, 1)/\text{SU}(1, 1) \times \text{U}(1) \), endowed with the metric

\[ d\sigma^2 = \mathcal{G}_{IJ}(\varphi) d\varphi^I d\varphi^J = d\phi^2 + 4 e^{2\phi} \left( d\tilde{\psi} + C dB - BdC \right)^2 - 4 e^{\phi} \left( dB^2 + dC^2 \right), \] (2.13)

where \( \varphi^I = (\phi, \tilde{\psi}, B, C) \). One can easily verify that

\[ R_{IJ} = -\frac{3}{2} \mathcal{G}_{IJ}, \quad C_{IJKL} = -\frac{1}{2} \epsilon_{IJKMN} C^{MN}_{KL}, \quad D_I R_{KLM} = 0. \] (2.14)

Here \( R_{IJ} \) and \( C_{IJKL} \) are the Riemann and Weyl tensors constructed from the target space metric \( \mathcal{G}_{IJ} \) and the covariant derivative \( D_I \). The Bergmann space is a special Kähler-Einstein manifold with negative curvature. The last equation of (2.14) is a differential characterization of a symmetric space, while the second equation implies a quaternionic structure [43].

The eight Killing vectors of \( \Phi \) generating the isometry algebra \( \mathfrak{su}(2, 1) \) are given by

\[ \xi_1 = \partial_\phi, \quad \xi_2 = C \partial_\phi + \partial_B, \quad \xi_3 = -B \partial_\phi + \partial_C, \]

\[ \xi_4 = -C \partial_B + B \partial_C, \quad \xi_5 = -2 \partial_\phi + 2 \tilde{\psi} \partial_\psi + B \partial_B + C \partial_C, \]

\[ \xi_6 = 4 \tilde{\psi} \partial_\psi + \left[ \frac{1}{2} \left( e^{\phi} - (B^2 + C^2) \right)^2 - 2 \tilde{\psi}^2 \right] \partial_\phi \]

\[ + \left[ C \left( e^{\phi} - (B^2 + C^2) \right) - 2 \tilde{\psi} B \right] \partial_B \]

\[ - \left[ B \left( e^{\phi} - (B^2 + C^2) \right) + 2 \tilde{\psi} C \right] \partial_C, \]

\[ \xi_7 = -4 B \partial_\phi + \left[ 2 \tilde{\psi} B - C \left( e^{\phi} - (B^2 + C^2) \right) \right] \partial_\psi \]

\[ + \left( e^{\phi} + B^2 - 3 C^2 \right) \partial_B + \left( 4 BC - 2 \tilde{\psi} \right) \partial_C, \]

\[ \xi_8 = -4 C \partial_\phi + \left[ 2 \tilde{\psi} C + B \left( e^{\phi} - (B^2 + C^2) \right) \right] \partial_\psi \]

\[ + \left( 4 BC + 2 \tilde{\psi} \right) \partial_B + \left( e^{\phi} + C^2 - 3 B^2 \right) \partial_C. \] (2.15)
The first five Killing vectors represent infinitesimal transformations that are linear in the scalars and comprehend a twist transformation, two electromagnetic gauge transformations, an internal $U(1)$ transformation and a scaling one. The remaining three are the most interesting, due to the nonlinearity in the fields, and they are usually called the generalized Ehlers transformation $\{\xi_7, \xi_8\}$ [44] and two Harrison transformations $\{\xi_7, \xi_8\}$ [45].

In order to see that these Killing vectors indeed generate the $SU(2,1)$ symmetry, let us define

$$E_2^1 = -\frac{1}{4}\left[\xi_7 + i\xi_8 + i(\xi_3 - i\xi_2)\right], \quad E_2^3 = -\frac{1}{4}\left[-(\xi_7 + i\xi_8) + i(\xi_3 - i\xi_2)\right].$$

$$E_1^3 = \frac{1}{4}(2\xi_5 + i\xi_4 + 2i\xi_6), \quad E_1^1 = H_1 + E_3^3, \quad E_2^2 = H_2 + E_3^3,$$

$$E_3^3 = -\frac{1}{3}(H_1 + H_2), \quad E_3^2 = -(E_2^3)^*, \quad E_3^1 = (E_1^3)^*, \quad E_3^2 = (E_2^3)^*,$$

where $H_1, H_2$ are Cartan generators defined by

$$H_1 = \frac{1}{2}\xi_1 - i\xi_6, \quad H_2 = \frac{1}{4}(\xi_1 - 6\xi_4 - 2\xi_6), \quad [H_1, H_2] = 0.$$

One can easily verify that these vectors $E_i^j$ ($i, j = 1, 2, 3$) satisfy the $su(2,1)$ algebra

$$[E_i^j, E_i^k] = \delta_k^j E_i^l - \delta_l^j E_i^k, \quad E_i^j = 0.$$

Note that the dependence of the scalar potential

$$V(\phi) = -2\Lambda e^\phi$$

on the dilaton $\phi$ breaks the invariance under nonlinear isometries and scalings. It is easy to see that the latter is recovered if we admit a rescaling of $\Lambda$. The theory described by (2.8) is thus invariant only under $SU(2,1)/H_0$, where $H_0 \subset SU(2,1)$ is a subgroup generated by $\xi_6, \xi_7, \xi_8$ corresponding to the Heisenberg algebra

$$[\xi_7, \xi_8] = 4\xi_6, \quad [\xi_7, \xi_6] = [\xi_8, \xi_6] = 0.$$

The five unbroken generators close themselves to form another one-dimensional Heisenberg subalgebra in semidirect sum with $\mathbb{R}^2$,

$$[\xi_2, \xi_3] = -2\xi_1, \quad [\xi_2, \xi_1] = [\xi_3, \xi_1] = 0,$$

$$[\xi_i, \xi_4] = (\alpha_i)^j/\xi_j, \quad [\xi_i, \xi_5] = (\sigma_i)^j/\xi_j,$$

where $i, j = 1, 2, 3$ and

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Heisenberg algebra (2.20) realizes the fact that the constant $\phi$ space constitutes a Nil manifold, viz, one can view the four-dimensional metric (2.13) as a Wick-rotated Bianchi-II Universe.

The well-known solution-generating techniques [23, 46, 47] based on group theory thus cannot be applied in the presence of a cosmological constant. Moreover, the broken symmetries are also a first sign of the loss of complete integrability, valid for $\Lambda = 0$ after another
dimensional reduction [22, 48, 49]. This also implies the inapplicability of the inverse scattering method [4, 23]. In what follows, we shall perform an analysis of some remaining integrability properties, extending the results of [38].

2.3. Hamiltonian formalism and first integrals

In the spacetime admitting a single Killing field, the sigma model still couples to the base space \( h_{\alpha\beta} \) represented by three-dimensional Einstein gravity according to (2.10). Because of the intricacy of this system, we usually simplify the problem by assuming further symmetries. In the absence of \( \Lambda \), the base space is decoupled from the sigma model by assuming an axial Killing field. More precisely, the metric without \( \Lambda \) can be cast into the Weyl-Papapetrou form, and the base space part can be obtained by quadrature once the sigma model on \( \mathbb{R}^2 \) is solved. Unfortunately, this decoupling does not occur in the presence of \( \Lambda \) as we will see in section 3.

In this section, we follow a different path to arrive at an integrable system. Along the lines of the argument in [38], we consider the case in which the base space admits only a single degree of freedom. Now we suppose that \( h_{\alpha\beta} \) describes a warped product space \( \mathbb{R} \times \Sigma \), with \( \Sigma \) a two-dimensional manifold. Moreover we assume that all the scalar fields depend only on the coordinate representing \( \mathbb{R} \). To capture this more conveniently, let us introduce another scalar field \( k \) that describes a rescaling of the three-dimensional metric \( h_{\alpha\beta} \),

\[
h_{\alpha\beta} = k \hat{h}_{\alpha\beta}.
\]

Absorbing the warp factor into \( k \), \( \hat{h}_{\alpha\beta} \) can be taken to be an unwarped product,

\[
\hat{h}_{\alpha\beta} dx^\alpha dx^\beta = d\sigma^2 + d\Omega^2,
\]

where \( d\Omega^2 \) is the line element on \( \Sigma \). Under these settings, every quantity depends only on a single variable \( \sigma \). In this case the trace and the \( \sigma\sigma \) component of the Einstein equations (2.10) become respectively

\[
\hat{R}^{(3)} = \frac{1}{2k^2} \left( \frac{dk}{d\sigma} \right)^2 - \left( J_\sigma, J_\sigma \right) + 2\Lambda k e^\sigma,
\]

\[
\frac{1}{k} \left( \frac{d^2k}{d\sigma^2} \right) = \frac{1}{k^2} \left( \frac{dk}{d\sigma} \right)^2 - \left( J_\sigma, J_\sigma \right) - 2\Lambda k e^\sigma.
\]

It is clear that the scalar curvature \( \hat{R}^{(3)} \) must be constant as a consequence of the fact that the r.h.s. of (2.23) depends only on \( \sigma \) and the l.h.s. is independent of \( \sigma \). Without further restrictions we can thus take \( \hat{R}^{(3)} = 2l \) with \( l = 0, \pm 1 \), so that \( \Sigma \) must be a maximally symmetric space, \( d\Omega^2_l = d\theta^2 + f_l^2(\theta) d\phi^2 \), where

\[
f_l(\theta) = \frac{1}{\sqrt{l}} \sin \left( \sqrt{l} \theta \right) = \begin{cases} \sin \theta, & l = 1, \\ \theta, & l = 0, \\ \sinh \theta, & l = -1. \end{cases}
\]

One obtains then a classical dynamical system with five degrees of freedom, with action

\[
S = \int d\sigma k^2 \left[ \frac{1}{2k^2} \left( \frac{dk}{d\sigma} \right)^2 - \left( J_\sigma, J_\sigma \right) + 2l - 2\Lambda k e^\sigma \right].
\]

For future convenience we introduce a new evolution parameter \( \tau \) defined by

\[
k^3 e^\sigma d\sigma = d\tau.
\]
With the new potential \( \dot{V} = 2\Lambda - \frac{2i}{k}e^{-\phi} \) and \( \omega \equiv \omega_r \), the action (2.26) can be expressed as
\[
S = \int L d\tau \quad \text{with a Lagrangian}
\]
\[
L = \frac{1}{2} \left[ e^{\phi}k^2 - k^2e^\phi \phi'^2 - 4k^2e^{3\phi} \omega^2 + 4e^{2\phi}k^2 \left( B'^2 + C'^2 \right) \right] - \dot{V},
\]
where a prime denotes a derivative w.r.t. \( \tau \). It is easy to see that (2.23) is the constraint \( H \equiv L + 2\dot{V} = 0 \). It then turns out to be more convenient to pass to a Hamiltonian formulation rather than working in a Lagrangian description. After a Legendre transformation one gets
\[
H = \frac{1}{2} \left[ e^{-\phi}p_k^2 - \frac{e^{-\phi}}{k^2}p_C^2 - \frac{e^{-3\phi}}{4k^2}p_{\psi}^2 \right. \\
\left. + \frac{e^{-2\phi}}{4k^2} \left( p_B^2 + p_C^2 - 2Cp_Bp_{\psi} + 2Bp_Cp_{\psi} + \left( B^2 + C^2 \right)p_{\psi}^2 \right) \right] + \dot{V}.
\]

The solution of this dynamical system is highly linked to the existence of commuting constants of motion. The Killing vector fields of \( su(2, 1) \) can be promoted to functions in phase space, realizing a Lie algebra isomorphism, by means of the substitutions\( ^3 \)
\[
\partial_{\varphi_i} \mapsto p_{\varphi_i}, \quad \{ \cdot, \cdot \} \mapsto \{ \cdot, \cdot \}_{PB}, \quad \xi_i \mapsto -C_i,
\]
where \( \{ \varphi_i \} = \{ \phi, \psi, B, C \} \) and \( i = 1, \ldots, 8 \). The minus sign in front of \( C_i \) reflects the fact that the infinitesimal generators and the corresponding charges obey the same algebra up to the sign of the structure constants\( ^4 \). The only nonvanishing Poisson brackets between the \( C_i \) and the Hamiltonian are given by
\[
\text{the only nonvanishing Poisson brackets between the constants of motion read}
\]
\[
\{ C_2, C_3 \} = -2C_1, \quad \{ C_2, C_4 \} = C_3, \quad \{ C_3, C_4 \} = -C_2.
\]

\( ^3 \) Our convention for the Poisson bracket is \( \{ A, B \} \equiv \Omega_{MN} \partial_M A \partial_N B = \sum \left( \frac{\partial A}{\partial q^M} \frac{\partial B}{\partial p^N} - \frac{\partial A}{\partial p^N} \frac{\partial B}{\partial q^M} \right) \), where \( \Omega = i\pi_2 \) is the symplectic form.

\( ^4 \) This can be shown as follows. Let \( Q_i = Q_i(q^I, p_I) \) be first integrals obeying the Lie algebra \( \{ Q_1, Q_2 \} = f_1 ^I \partial_I Q_1 \) and let us denote the corresponding Hamiltonian vector fields by \( V^{Q_i} = \Omega^{MN} \partial_N Q_i \). For any function \( F = F(q^I, p_I) \) in phase space, we have a formula \( V^{Q_i} \partial_i F = -\{ Q_i, F \} \). It follows that for the vector field \( V^{\psi} = \Omega^{MN} \partial_N \dot{Q}_\psi \psi = f_\psi ^I \partial_I \dot{Q}_\psi \), we obtain \( V^{\psi} \partial_i F = -\{ Q_\psi, F \} = -\{ \dot{Q}_\psi, F \}, \quad \{ \dot{Q}_\psi, F \} = -\{ \dot{V}_\psi, F \} \), where at the second equality we used the Jacobi identity. This establishes \( \{ V_i, V_j \} = -f_i ^k V_k \), as desired.
\[ \{ \hat{C}_5, C_1 \} = -2C_1, \quad \{ \hat{C}_5, C_2 \} = -C_2, \quad \{ \hat{C}_5, C_3 \} = -C_3. \] (2.33)

Among \( C_i, C_2, C_3, C_4, \hat{C}_5 \) and the operators composed of them, the maximal set of commuting first integrals is given by \( H, C_i, C_4, C_2^2 + C_3^2 \), and we fix the values of these first integrals with four constants \( E, v, K_i, K_5 \),

\[ H = E, \quad p_\psi = 4v, \quad \dot{p}_\psi - C p_B = K_i, \quad \left( p_B + C p_\psi \right)^2 + \left( p_\psi - B p_\psi \right)^2 = K_2. \] (2.34)

We want to use these equations to solve the system, so we shall set \( E = 0 \) only at the end of the integration procedure.

### 2.4. Integrability: RN-TN-(A)dS solution

Using (2.34), the Hamiltonian can be rewritten as

\[ H = \frac{e^{-\phi}}{2} p_\psi^2 - \frac{e^{-\phi}}{2k^2} p_\psi^2 - \frac{e^{-3\phi}}{8k^2}(4v)^2 + \frac{e^{-2\phi}}{8k^2}(K_2 + 16vK_i) + \hat{V}, \] (2.35)

and thus the electromagnetic and twist part has decoupled from the other fields. In order to solve the Hamilton-Jacobi equation

\[ H \left( k, \phi, \frac{\partial S}{\partial k}, \frac{\partial S}{\partial \phi} \right) + \frac{\partial S}{\partial \tau} = 0, \] (2.36)

we use the separation ansatz

\[ S = W(k, \phi) = E\tau, \] (2.37)

which leads to

\[ \frac{e^{-\phi}}{2} \left( \frac{\partial W}{\partial k} \right)^2 - \frac{e^{-\phi}}{2k^2} \left( \frac{\partial W}{\partial \phi} \right)^2 - \frac{e^{-3\phi}}{8k^2}(4v)^2 + \frac{e^{-2\phi}}{8k^2}(K_2 + 16vK_i) + \hat{V} = E. \] (2.38)

Equation (2.38) can be solved by defining the new variables \( x = ke^\tau, y = e^{-\phi} \) and applying the Charpit-Lagrange method. The result is

\[ W(x, y) = \frac{1}{6a^2} \sqrt{2ax - y^2} \left( \hat{E}(v^2 + ax) - 6ad - 12a^2y \right) + \frac{1}{8v}(K_2 + 16vK_i) \arccot \left( \frac{v}{\sqrt{2ax - y^2}} \right). \] (2.39)

where \( \hat{E} \equiv 2\Lambda - E \) and \( a \) is an integration constant. Following the Hamilton-Jacobi technique we can introduce two other constants \( \beta_1, \beta_2 \) according to

\[ \beta_1 = \frac{\partial S}{\partial E}, \quad \beta_2 = \frac{\partial S}{\partial a}. \] (2.40)

Using the dynamical constraint \( H = 0 \), they are given by

\[ \beta_1 = \frac{1}{6a^2} \sqrt{2ax - y^2} \left( v^2 + ax \right) + \tau, \] (2.41)

\[ \beta_2 = \frac{\Lambda \left( 2v^4 - 2av^2x - a^2x^2 \right)}{3a^3 \sqrt{2ax - y^2}} + \frac{K_2 + 16vK_i}{16a \sqrt{2ax - y^2}} - \frac{lt^2 - lax + 2a^2xy}{a^2 \sqrt{2ax - y^2}}. \] (2.42)
To simplify the solution, it is convenient to define a new evolution parameter \( r \) by
\[
\tau = \frac{1}{\sqrt{2a}} \left( \frac{r^3}{3} + r \frac{v^2}{2a} \right)
\] (2.43)

To solve the two algebraic equations (2.41) and (2.42), we note that it is possible to set \( \beta_1 = 0 \) without loss of generality by shifting \( \tau \). Then (2.41) gives
\[
x = r^2 + \frac{v^2}{2a}.
\] (2.44)

Plugging this into (2.42) yields
\[
y = \frac{1}{2a} \left( \frac{K_2 + 16 \nu K_i}{16} - \sqrt{2} \beta_2 a^{3/2} r + \frac{\hbar^2}{2a} - \frac{3r_{\nu}^2 v^2 - 3v^4}{a} \right).
\]

Using the original expression for \( H \), (2.28), the Hamilton equations for the electromagnetic part become
\[
\frac{dp_B}{dr} = -\frac{v}{\sqrt{2a} \left( r^2 + \frac{v^2}{2a} \right)} \left( p_C + 4vB \right),
\]
\[
\frac{dp_C}{dr} = -\frac{v}{\sqrt{2a} \left( r^2 + \frac{v^2}{2a} \right)} \left( -p_B + 4vC \right).
\] (2.45)

\[
\frac{dB}{dr} = \frac{1}{4\sqrt{2a} \left( r^2 + \frac{v^2}{2a} \right)} \left( p_B - 4vC \right),
\]
\[
\frac{dC}{dr} = \frac{1}{4\sqrt{2a} \left( r^2 + \frac{v^2}{2a} \right)} \left( p_C + 4vB \right).
\] (2.46)

Using the gauge freedom generated by \( \xi_1 \) and \( \xi_2 \), we can implement a boundary condition in such a way that \( B \) and \( C \) vanish at infinity [42]. This eliminates two integration constants and the solutions are given by
\[
B = \frac{\beta_3 + r_{\beta_4}}{r^2 + \frac{v^2}{2a}}, \quad C = \frac{\sqrt{2a} r_{\beta_3} - \frac{v^2 \beta_4}{2a}}{v \left( r^2 + \frac{v^2}{2a} \right)}.
\] (2.47)

Finally, the twist potential \( \tilde{\psi} \) can be found by inverting the equation \( p_{\tilde{\psi}} = 4v \), which leads to
\[
\tilde{\psi} = \int \frac{d\tau}{\sqrt{2a} \left( r^2 + \frac{v^2}{2a} \right)} + c \frac{dB}{dr} + B \frac{dC}{dr}.
\] (2.48)

The integration procedure is now complete. Since the constants defining the solution are not very illuminating, we define the new constants.
\[ m = \frac{\beta_3 a^{3/2}}{\sqrt{2}}, \quad n = \frac{v}{\sqrt{2a}}, \quad Q = \sqrt{2a} \beta_3, \quad P = \frac{-\sqrt{2a} \beta_3}{n}, \quad 2a = m^2 + \ell^2 n^2, \]

(2.49)

which give \( K_1 = 0 \) and \( K_2 = 16(P^2 + Q^2) \). It turns out that the four-dimensional metric and \( U(1) \) gauge field take the form of the RN-TN-(A)dS solution [50],

\[
\begin{aligned}
\text{d}s^2 &= -e^{-\varphi}(dr + K_\varphi d\varphi)^2 + k e^\varphi \left( \frac{dr^2}{\Delta} + d\theta^2 + f_1^2(\theta)d\varphi^2 \right), \\
A_\nu d\chi^\nu &= B dt + A_\varphi d\varphi,
\end{aligned}
\]

(2.50)

(2.51)

where

\[
\Delta = l(r^2 - n^2) - 2mr - \frac{\Lambda}{3}(r^4 + 6r^2n^2 - 3n^4) + P^2 + Q^2,
\]

\[
k = \frac{\Delta}{m^2 + \ell^2 n^2}, \quad e^{-\varphi} = \frac{k}{r^2 + n^2}, \quad K_\varphi = -4n\sqrt{m^2 + \ell^2 n^2}f_1^2(\theta/2),
\]

\[
B = \frac{Qr - nP}{\sqrt{m^2 + \ell^2 n^2}(r^2 + n^2)}, \quad A_\varphi = \frac{2f_1^2(\theta/2)\left(P\left(n^2 - r^2\right) - 2nQr\right)}{n^2 + r^2}.
\]

(2.52)

Note that the fields \( K_\varphi \) and \( A_\varphi \) are obtained from the dualization of (2.6) and (2.7), which involves

\[
\tilde{\psi} = \frac{n}{3(m^2 + \ell^2 n^2)}(\Lambda r + \frac{3l(r - 3m - 4\Lambda n^2r)}{r^2 + n^2}), \quad C = -\frac{nQ + rP}{\sqrt{m^2 + \ell^2 n^2}(r^2 + n^2)}.
\]

For \( P = Q = 0 \) we recover the results of [38], and thus the integrability properties described in [38] are still valid in the case of nonvanishing electromagnetic charges. We saw that, even if the cosmological constant reduces the internal symmetry group from \( \text{SU}(2, 1) \) to \( \text{SU}(2, 1)/\mathbb{H}_1 \), it has not spoiled the integrability once we restrict to the subspace (2.22). This condition reduces the infinite number of degrees of freedom to effectively five. Only the three nonlinear generators of \( \text{su}(2, 1) \) are broken and the remaining commuting first integrals are enough to decouple the electromagnetic and twist potentials and to integrate the system in three steps. The general case remains unsolved and is highly linked to the broken affine Kac–Moody algebra arising after another dimensional reduction [49, 51, 52]. The action of \( \text{SU}(2, 1)/\mathbb{H}_1 \) on the fields generates a transformation on the parameter space, and in particular a scale transformation also requires a rescaling of \( \Lambda \). Unfortunately these surviving symmetries alone are useless to produce new interesting solutions.

### 3. Einstein-\( \Lambda \) system

In this section, we shall consider the action (2.1) with vanishing electromagnetic field \( F_{\mu\nu} = 0 \), and assume the existence of an additional Killing vector that commutes with \( \partial_t \). For \( \Lambda = 0 \), this system is described by the Weyl-Papapetrou formalism which allows us to utilize certain integrability techniques. We see that the \( \Lambda \) term destroys the reduction to the Weyl-Papapetrou system. In spite of this, a further reduction to \( d = 1 \) with a suitable choice of variables enables us to solve the Einstein-\( \Lambda \) system in full generality.
3.1. Effective field theory in two dimensions

In general, a solution with an \( \text{SO}(2) \) isometry group cannot be written in the Lewis form \[ (3.1) \], but this becomes true if the line element admits a two-dimensional foliation orthogonal to the one in which the action of \( \text{SO}(2) \) is transitive. With this additional hypothesis we take

\[
\,\text{dx}^2 = -e^{-\phi}(dt + Kd\varphi)^2 + e^{\phi}\left(e^{2\alpha}h_{mn}dx^m dx^n + e^{2\lambda}d\varphi^2\right),
\]

where \( t \) and \( \varphi \) are Killing coordinates, and the metric depends only on \( x^m \) \((m = 1, 2)\). Plugging this into \((2.1)\) (with \( F_{\mu\nu} = 0 \)) yields the two-dimensional Lagrangian

\[
\mathcal{L}^{(2)} = \sqrt{g}e^\chi\left[R^{(2)} + 2\partial_{\mu}\chi\partial^\mu \psi - \frac{1}{2}\left(\partial_{\mu}\phi\partial^\mu \phi - e^{-2(\phi+\chi)}\partial_{\mu}K \partial^\mu K\right) - 2\Lambda e^{\phi+2\psi}\right].
\]

\( R^{(2)} \) is the two-dimensional Ricci scalar associated with \( h_{mn} \). In order to emphasize the nonlinear \( \text{SL}(2, \mathbb{R}) \) symmetry present for \( \Lambda = 0 \), we define new fields \( \tilde{\psi}, \tilde{\phi} \) by \( 2\tilde{\psi} = -\phi + \alpha \chi + 2\tilde{\psi} \) \((\alpha \) is an arbitrary constant to be specified later), \( \phi = \tilde{\phi} - \chi \), which leads to

\[
\mathcal{L}^{(2)} = \sqrt{g}e^\chi\left[R^{(2)} + 2\partial_{\mu}\chi\partial^\mu \tilde{\psi} + \left(\frac{1}{2} + \alpha\right)\partial_{\mu}\chi\partial^\mu \chi - \langle \mathcal{J}_\phi, J^\phi \rangle - 2\Lambda e^{\alpha\chi+2\tilde{\psi}}\right].
\]

where the \( \sigma \) model target space is \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \), and thus

\[
\langle \mathcal{J}_\phi, J^\phi \rangle = \frac{1}{2}\left(\partial_{\mu}\phi\partial^\mu \phi - e^{-2\tilde{\phi}}\partial_{\mu}K \partial^\mu K\right).
\]

When \( \Lambda \) is turned off, we can recover another \( \text{SL}(2, \mathbb{R}) \) via dualization in \( d = 3 \) \([22, 24, 42]\). Since these two nonlinear representations of \( \text{SL}(2, \mathbb{R}) \) are linked by a nonlinear and nonlocal relation, they do not commute. This non-commutativity can be used to generate new solutions \([46, 47]\) and to study the symmetries of the effective field theory \((3.3)\) \((\Lambda = 0)\) \([22, 48]\).

Varying \((3.3)\) and using the fact that in two dimensions every metric is conformally flat\(^6\), one obtains the equations of motion

\[
\Delta e^\chi + 2\Lambda e^{(1+\alpha)\chi+2\tilde{\psi}} = 0,
\]

\[
\partial_{(\mu}\left(e^{\phi}\partial_{\nu)\phi} - e^{-2\tilde{\phi}+\chi}K \partial_{\nu)\phi}\right) = 0,
\]

\[
\partial^2 e^\chi = e^\chi\left[\frac{1}{2}\left(\langle \mathcal{J}_\phi, \mathcal{J}^\phi \rangle - \langle \mathcal{J}_\psi, \mathcal{J}^\psi \rangle\right) - \partial_{\nu}\chi \partial^\nu \tilde{\psi} + \partial_{\nu}\chi \partial^\nu \tilde{\psi}\right.

\[\left. - \frac{1}{2}\left(\frac{1}{2} + \alpha\right)\left(\partial_{\nu}\chi \partial^\nu \chi - \partial_{\nu}\phi \partial^\nu \phi\right) - \Lambda e^{\alpha\chi+2\tilde{\psi}}\right],
\]

\[
\partial_{(\mu}\partial_{\nu)\phi} \equiv e^\chi\left[\partial_{\nu}\chi \partial_{\nu} \tilde{\phi} + \partial_{\nu}\chi \partial_{\nu} \tilde{\phi} + \left(\frac{1}{2} + \alpha\right)\partial_{\nu}\chi \partial_{\nu} \chi - \langle \mathcal{J}_\phi, \mathcal{J}^\phi \rangle\right],
\]

\[
\Delta \tilde{\psi} - \frac{1}{2}\left(\frac{1}{2} + \alpha\right)\partial_{\mu}\chi \partial^\mu \chi + \frac{1}{2}\langle \mathcal{J}_\phi, J^\phi \rangle - \Lambda \alpha e^{\alpha\chi+2\tilde{\psi}} = 0,
\]

where \( \Delta = \partial^2_{\nu}\partial_{\nu} + \partial^2_{\chi} \). It is interesting to note that for the choice \( \alpha = 0 \), \((3.5)\) becomes the massive Klein–Gordon equation on a curved space with metric \( e^{2\tilde{\psi}}(d\rho^2 + dz^2) \) and mass

\(^5\) The first \( \text{SL}(2, \mathbb{R}) \) is called the Matzner-Misner \([54]\) transformation and the second one the Ehlers transformation \([44]\).

\(^6\) The conformal factor can be absorbed into \( \tilde{\psi} \), see \((3.1)\).
$m^2 = -2\Lambda$. In the case $\Lambda = 0$ it boils down to $\Delta e^\chi = 0$, which allows us to take $e^\chi = \rho$ without loss of generality, but for $\Lambda \neq 0$ $\chi$ does not decouple anymore from the other fields. This explains the failure of the metric to fall into the Weyl-Papapetrou class. Note also that (3.9) follows from the other equations. To see this, consider $\partial_t(\mathbf{3.7}) - \partial_t(\mathbf{3.8})$ and use (3.5) and (3.6). This leads to

$$\frac{1}{2} \partial^2 t e^\chi (\partial_t^2 e^\chi - \partial_t^2 \chi) + \frac{1}{2} \partial_t (\partial_t e^\chi)^2 - \partial_t \tilde{\psi} \partial_t e^\chi \partial_t^2 \chi = \alpha \partial_k \partial^k \chi \partial_t \chi - \left( J_e, J^k_e \right) \partial_k \chi - \partial_t \chi \left( \Delta \tilde{\psi} - \Lambda \alpha e^\alpha + 2\tilde{\psi} \right).$$

(3.10)

Moreover one can rewrite $\mathbf{3.7} \partial_t \chi - \mathbf{3.8} \partial_t \chi$ in the form

$$\frac{1}{2} \partial_t \chi (\partial_t^2 e^\chi - \partial_t^2 \chi) + \frac{1}{2} \partial_t (\partial_t e^\chi)^2 - \partial_t \tilde{\psi} \partial_t e^\chi \partial_t^2 \chi = \left( \alpha - \frac{1}{4} \right) \partial_k \partial^k \chi \partial_t \chi - \left( J_e, J^k_e \right) \partial_k \chi + \frac{1}{2} \partial_t \chi \left( \langle J_e, J_e \rangle - \langle J_e, J_e \rangle \right).$$

(3.11)

Finally, the difference between (3.10) and (3.11) implies (3.9).

Note that the $\sigma$ model isometry group $\text{SL}(2, \mathbb{R})$ acts only on $\tilde{\phi}$ and $K$. Since the potential in (3.3) is independent of $\tilde{\phi}$ and $K$, this $\text{SL}(2, \mathbb{R})$ is a symmetry of the complete Lagrangian (3.3). However, one easily checks that from a four-dimensional point of view, a transformation with this $\text{SL}(2, \mathbb{R})$ corresponds merely to a diffeomorphism, and thus cannot be used to generate new solutions.

### 3.2. Further reduction to $d = 1$

If we make the additional assumption that all the fields depend only on $\rho$, the two-dimensional effective field theory (3.3) boils down to a dynamical system with four degrees of freedom described by

$$S = \int d\rho e^\chi \left[ -\frac{1}{2} \partial^2 \rho \partial_\rho e^\chi + \left( \frac{1}{2} + \alpha \right) \left( \partial_\rho e^\chi \right)^2 - \frac{1}{2} \left( \partial_\rho \tilde{\phi} \right)^2 - \frac{e^{-2\tilde{\phi}}}{2} \left( \partial_\rho K \right)^2 - 2\Lambda e^\alpha + 2\tilde{\psi} \right].$$

(3.12)

which turns out to be exactly solvable. Introducing the new coordinate $r$ by $e^{-\chi} d\rho = dr$ and defining $V = 2\Lambda e^{2\tilde{\phi}}$, the system (3.12) is described by the Lagrangian $S = \int Ldr$ as

$$L = 2\chi' \tilde{\psi}' + \left( \frac{1}{2} + \alpha \right) \chi'^2 - \frac{1}{2} \left( \tilde{\phi}'^2 - e^{-2\tilde{\phi}} K'^2 - e^{(2+\alpha)\chi} V \right),$$

(3.13)

where a prime denotes a derivative w.r.t. $r$. In what follows, we shall make the choice $\alpha = -2$, for which $\chi$ becomes cyclic. Then the equations of motion following from (3.13) are given by

$$\tilde{\phi}'' - e^{2\tilde{\phi}} A^2 = 0, \quad K' = e^{2\tilde{\phi}} A,$$

(3.14)

$$\tilde{\psi}'' + 3\Lambda e^{2\tilde{\phi}} = 0, \quad \chi'' + 2\Lambda e^{2\tilde{\phi}} = 0,$$

(3.15)

where $A$ is an integration constant, together with the constraint $H \equiv L + 2V = 0$, that emerges from (3.7). Equations (3.14) and (3.15) are easily solved, and one finds that the

7 The case $A = 0$ leads, after a change of coordinates, to (22.8) of [1], and it cannot be recovered smoothly after the integration.
metric in four dimensions is given by

\[ ds^2 = -e^{-\bar{\phi}+\chi}(dt + K d\varphi)^2 + e^{2\bar{\phi}} dr^2 + e^{2(\bar{\phi}-\chi)} dz^2 + e^{\bar{\phi}+\chi} d\varphi^2, \]  

where

\[ e^{-\bar{\phi}} = \frac{A}{\sqrt{C_1}} \cos \left[ \sqrt{C_1} (r + C_2) \right], \quad e^{2\bar{\phi}} = \frac{C_3}{3\Lambda \cosh^2 \left( \sqrt{C_3} r \right)}, \quad e^\chi = C_4 \frac{e^{\frac{4(\sqrt{C_1} + \sqrt{C_3} r)}{3}}}{\cosh \left( \sqrt{C_3} r \right)} \, \tan \left[ \frac{\sqrt{C_1}}{A} \tan \left( \sqrt{C_1} (r + C_2) \right) \right], \]  

and \( C_1, C_2, C_3, C_4 \) are integration constants. For generic values of these constants, \( \partial_t, \partial_\varphi \) and \( \partial_z \) constitute an exhaustive list of Killing vectors, as one can verify by checking the integrability of the Killing equation. This solution falls into Petrov type I and the Kretschmann invariant is

\[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{8\Lambda^2}{3C_1^{3/2}} \left[ 3C_3^{3/2} \cosh^4 \left( \sqrt{C_3} r \right) + C_3^{3/2} \sinh^4 \left( \sqrt{C_3} r \right) - 2(C_3 + 3C_1) \sqrt{4C_3 + 3C_1} \cosh \left( \sqrt{C_3} r \right) \sinh \left( \sqrt{C_3} r \right) \right], \]

while the Chern-Pontryagin one vanishes. Hence \( r \to \infty \) corresponds to a curvature singularity. This class of solutions has already been found by Santos and MacCallum in [55, 56] and previously by Krasiński in [39–41], in both cases in a different coordinate system. The connection between the metric (3.16) and those of [55, 56] is easy to find after the identification \( e^{(\mu)} = \Psi(r) \). Other explicit forms of the fields defining (3.16) can be recovered through the combination of analytic continuations, diffeomorphisms and nontrivial limits on the constants, requiring the final metric to be real.

An example of a well-known static solution inside the class of (3.16) is found if we define \( A = \sqrt{C_1} \alpha \) and make the choice

\[ C_1 = 0, \quad \alpha = 1, \quad C_3 = 9M^2, \quad C_4 = \left( \ell^2 M \right)^{2/3}, \]  

where the parameter \( \ell \) is related to the cosmological constant by \( \Lambda = -3\ell^{-2} \). Then the line element boils down to

\[ ds^2 = -e^{\chi} dr^2 + e^{2\chi} d\varphi^2 + e^{2\bar{\phi}} dr^2 + e^{2(\bar{\phi}-\chi)} dz^2, \]  

with

\[ e^{2\bar{\phi}} = -\frac{M^2 \ell^2}{\cosh^2(3Mr)}, \quad e^\chi = \left( \ell^2 M \right)^{2/3} \frac{e^{2M r}}{[\cosh(3Mr)]^{2/3}}. \]  

Introducing the new radial coordinate \( R \) by

\[ R = \left( \ell^2 M \right)^{1/3} \frac{e^{M r}}{[\cosh(3Mr)]^{1/3}}, \]

the solution becomes

\[ ds^2 = R^2 \left( -dt^2 + d\varphi^2 \right) + \left( -\frac{2M}{R} + \frac{R^2}{\ell^2} \right)^{-1} dR^2 + \left( -\frac{2M}{R} + \frac{R^2}{\ell^2} \right) dz^2. \]  

This is the well-known planar AdS soliton which turns into a planar black hole after the analytic continuation \( t \mapsto iz, \ z \mapsto it \).
Another interesting limit of (3.16) is obtained if we choose $C_3 = 3 \Lambda$, $C_4 = \beta^2$, and, after the limit $\Lambda \to 0$ and the rescaling $(r, z) \to \left( \sqrt{C_4} r, \sqrt{C_4} z \right)$, we fix the other parameters as

$$A = \beta = \frac{3}{\sqrt{C_4}}, \quad C_2 = 0, \quad C_4 = \frac{1}{k^2}. \quad (3.23)$$

Then the metric (3.16) becomes

$$k^2 ds^2 = -e^\Phi \left[ \cos \left( \sqrt{3} r \right) \left( dr^2 - d\varphi^2 \right) + 2 \sin \left( \sqrt{3} r \right) d\varphi dr \right] + e^{-2r} dz^2 + dr^2. \quad (3.24)$$

This is the Petrov solution [53], eq. (12.14), also found as the metric induced on a constant $r$ hypersurface of the $d = 5$ Einstein–Maxwell–$\Lambda$ solution discussed in [36], eq. (5.37). It is interesting to note that (3.24) is the only vacuum solution of Einstein’s equations admitting a simply transitive four-dimensional maximal group of motions generated by the Killing vectors [53]

$$T = \partial_t, \quad Z = \partial_z, \quad \Phi = \partial_\varphi, \quad R = \partial_r + z \partial_\varphi + \frac{1}{2} \left( \sqrt{3} t - \varphi \right) \partial_\varphi - \frac{1}{2} \left( t + \sqrt{3} \varphi \right) \partial_r,$$

satisfying

$$[R, T] = \frac{1}{2} T - \frac{\sqrt{3}}{2} \Phi, \quad [R, \Phi] = \frac{1}{2} \Phi + \frac{\sqrt{3}}{2} T, \quad [R, Z] = -Z. \quad (3.25)$$

Furthermore the determinant of (3.24) is always $-1$, so $(t, \varphi, r, z)$ can be promoted to global coordinates.

It is worthwhile to note that in our approach, which stresses the symmetries of the underlying theory, a simple system of integrable equations (3.14), (3.15) emerges in a natural way. However finding the most general solution to (3.5)–(3.8) remains a difficult problem due to the breaking of the $\text{SL}(2, \mathbb{R})$ Ehlers symmetry, similar to the Einstein–Maxwell–$\Lambda$ case. As we tried to argue in the introduction, this system of equations may nevertheless admit a Lax-pair representation, which we expect (if it exists at all) to be highly nontrivial to find. We hope to come back to this point in a future publication.

4. Final remarks

The integrable nature of Einstein’s gravity in certain contexts is a useful ingredient for our understanding of the nonlinear nature of gravitational physics. In this paper, we developed some novel techniques of integrability that allowed us to obtain solutions of the Einstein–Maxwell–$\Lambda$ system. In the first part we derived the complete solution by assuming that every quantity depends only on a single variable and that the specific form of the base space is $\mathbb{R} \times \Sigma$, by extending the work of [38]. This restriction is strong enough to give a tractable system, yet turns out to be rich enough to incorporate a class of gravitational solutions of physical interest. In the second part, we found that under the codimension one assumption, an appropriate choice of variables allows us to obtain the decoupled equations (3.14), (3.15). We expect that these convenient variables would continue to be powerful for further investigations of the case with higher codimension.

This work can be generalized in various directions. An interesting plausible route is to see if other gravitational theories display integrability properties similar to the Einstein–Maxwell–$\Lambda$ system in four dimensions. As we shall discuss in the appendix, it seems that the higher-dimensional generalization is not straightforward even for the pure Einstein–$\Lambda$ case.
Nevertheless, this failed attempt also gives further insight into the integrability properties of Einstein’s equations.

Another possible future work is to extend our formalism to the case where the base space admits the dependence on more than one variable. To this aim, hints that these systems could actually be integrable come from the study of the Plebański-Demiański solution \([57]\), which is the most general known Petrov type D solution of the four-dimensional Einstein–Maxwell equations with \(\Lambda\). It possesses a very high degree of symmetry and contains a lot of subcases and parameters of physical interest, so one may think of being able to generate it from a given seed solution. Also, its relation to supersymmetry \([58]\) may be a tractable way to understand the integrable nature of Einstein’s gravity in the presence of a cosmological constant. We hope to come back to these points in future work.

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**Appendix. On the integrability of the higher-dimensional Einstein–\(\Lambda\) system**

In this appendix, we discuss if the integrability properties in four dimensions can be extended to higher dimensions. Let us consider the \(D\)-dimensional spacetime \((M_D, g_D)\) represented by Einstein’s gravity with a cosmological constant,

\[
S_D = \frac{1}{2\kappa_D^2} \int d^Dx \sqrt{-g^{(D)}} \left( R^{(D)} - 2\Lambda \right).
\]  

(A.1)

Suppose that \((M_D, g_D)\) admits \(n\) Killing vectors which mutually commute. The metric can be put into the form

\[
d\mathbf{s}_D^2 = \tilde{g}_{\mu\nu}(x) \, dx^\mu dx^\nu + \tilde{M}_{mn}(x) \left( dy^m + 2K_{\mu}^{(m)}(x) \, dx^\mu \right) \left( dy^n + 2K_{\nu}^{(n)}(x) \, dx^\nu \right).
\]  

(A.2)

Here \(\tilde{g}_{\mu\nu}\) is the metric of the external \(d \equiv D - n\) dimensional space, \(\tilde{M}_{mn}\) is the internal metric and \(K_{\mu}^{(m)}\) are the Kaluza–Klein gauge fields. Due to the isometries, we can reduce the \(D\)-dimensional gravity system down to \(d\) dimensions. Denoting the Kaluza–Klein field strength as \(K_{\mu\nu}^{(mn)} = \partial_\mu K_{\nu}^{(mn)} - \partial_\nu K_{\mu}^{(mn)}\), it is straightforward to show that the Ricci scalar is decomposed as

\[
R^{(D)} = \hat{R} - \hat{M}_{mn} K_{\mu\nu}^{(mn)} K^{(n)\mu\nu} - \hat{M}_{mn} \hat{\nabla}_\mu \hat{\nabla}_\nu \hat{M}_{mn}
\]  

\[
- \frac{1}{4} \left( \hat{M}_{mn} \hat{\nabla}_\mu \hat{M}_{mn} \right)^2 - \frac{3}{4} \hat{\nabla}_\mu \hat{M}_{mn} \hat{\nabla}_\nu \hat{M}_{mn} - \hat{m} \hat{\nabla}_\mu \hat{M}_{mn},
\]  

(A.3)

where \(\hat{\nabla}_\mu\) is the linear connection compatible with \(\tilde{g}_{\mu\nu}\). Let us perform the conformal transformation

\[
\hat{g}_{\mu\nu} = k e^{\phi} \tilde{g}_{\mu\nu}, \quad \hat{M}_{mn} = e^{\phi} \tilde{M}_{mn}, \quad \det(\hat{M}) = -s,
\]  

(A.4)

where \(s = \pm 1\) and the additional scalar field \(k\) has been introduced as in the body of the text. To achieve the Einstein frame for \(k = 1\), we choose
\[ \alpha = \sqrt{\frac{2n}{(d + n - 2)(d - 2)}}, \quad \beta = -\frac{(d - 2)\alpha}{n} \]  \hfill (A.5)

One can then verify that the \( d \)-dimensional equations of motion can be derived from the action

\[
S_d = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{|g|} k^{(d-2)/2} \left[ R - \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{4} \text{Tr} \left( \nabla M \nabla M^{-1} \right) \right] + \frac{1}{4} (d - 1)(d - 2) \frac{(\nabla k)^2}{k^2} - k^{-1} \epsilon^{(\beta - \alpha)\nu} M_{mn} K^{(m)\nu} K^{(n)\mu} - 2 \Lambda \kappa e^{\alpha \varphi} \right].
\]  \hfill (A.6)

Hence \( M_{mn} \) is the matrix corresponding to an \( \text{SL}(n, \mathbb{R})/\text{SO}(n) \) nonlinear sigma model.

In the following, we shall focus on the \( d = 3 \) case. This is a generalization of the vacuum [59] and electrovac [60] analyses. The Maxwell equations can be solved to give local twist potentials \( \psi_{(m)} \) such that

\[
k^{-1/2} \epsilon^{(\beta - \alpha)\nu} M_{mn} K^{(n)\mu} = \frac{1}{2} \epsilon^{\mu\nu\rho} \nabla_{(\rho)} \psi_{(m)},
\]  \hfill (A.7)

where \( \epsilon_{\mu\nu\rho} \) is the volume element compatible with the external metric \( g_{\mu\nu} \). It therefore turns out that the 3-dimensional Einstein and matter field equations can be derived from the action

\[
S_3 = \frac{1}{2\kappa_3^3} \int d^3 x \sqrt{|g|} k^{1/2} \left[ R + \frac{1}{4} \text{Tr} \left( \nabla M \nabla M^{-1} \right) + \frac{1}{2k^2} (\nabla k)^2 - 2 \Lambda \kappa e^{\alpha \varphi} \right],
\]  \hfill (A.8)

where

\[
\bar{M} = \begin{pmatrix} \left( \det \bar{M} \right)^{-1} & -\left( \det \bar{M} \right)^{-1} \psi_{(m)} \\ -\left( \det \bar{M} \right)^{-1} \psi_{(n)} & \bar{M}_{mn} + \left( \det \bar{M} \right)^{-1} \psi_{(m)} \psi_{(n)} \end{pmatrix}, \quad \det \bar{M} = 1, \quad \bar{M} = \bar{M}^T,
\]  \hfill (A.9)

with inverse

\[
\bar{M}^{-1} = \begin{pmatrix} \det \bar{M} + \bar{M}^{pq} \psi_{(p)} \psi_{(q)} & \bar{M}^{mp} \psi_{(p)} \\ \bar{M}^{mn} \psi_{(n)} & \bar{M}_{mn} \end{pmatrix}
\]  \hfill (A.10)

It turns out that the coset symmetry is now enhanced to \( \text{SL}(D - 2)/\text{SO}(D - 2) \).

Let us focus on the \( D = 5 \) case in what follows. Assuming two commuting Killing vectors \( \xi_1 = \partial / \partial r \) and \( \xi_2 = \partial / \partial \psi \), we parametrize

\[
\bar{M}_{mn} \text{dy}^m \text{dy}^n = -f (dr + \omega d\psi)^2 + (fh)^{-1} d\varphi^2, \quad h = e^{2\varphi / \sqrt{\gamma}}.
\]  \hfill (A.11)

It follows that the matrix \( \bar{M} \) in (A.9) can be expressed by

\[
\bar{M} = \begin{pmatrix} -h & h \psi_{(1)} & h \psi_{(2)} \\ h \psi_{(1)} & -f - h \psi_{(1)}^2 & -f \omega - h \psi_{(1)} \psi_{(2)} \\ h \psi_{(2)} & -f \omega - h \psi_{(1)} \psi_{(2)} & -h \psi_{(2)}^2 - f \omega^2 - (fh)^{-1} \end{pmatrix}.
\]  \hfill (A.12)

This coincides with the parametrization \( (q_1, q_2, q_3, p_1, p_2, p_3) \) of \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \) given in [61] by the identification

\[
q_1 \mapsto -h, \quad q_2 \mapsto -f, \quad p_1 \mapsto -\psi_{(1)}, \quad p_2 \mapsto \omega, \quad p_3 \mapsto -\psi_{(2)}.
\]  \hfill (A.13)
The Killing vectors \( \tilde{X}_i \) \((i = 1, \ldots, 8)\) of the target space \( \mathrm{d}s^2 = -(1/4)\mathrm{Tr}(dM\hat{d}M^{-1})\) are given by (5.24) of [61].

For \( k = 1 \), the potential depends only on \( \phi \). The Killing vectors that keep the potential invariant are given by

\[
\tilde{X}_- \equiv \tilde{X}_1 = 2\tilde{X}_2, \quad \tilde{X}_3, \quad \tilde{X}_4, \quad \tilde{X}_5, \quad \tilde{X}_6.
\]  

(A.14)

The remaining three generators \((\tilde{X}_+ \equiv \tilde{X}_1 - \tilde{X}_2, \tilde{X}_7, \tilde{X}_8)\) do not leave the potential invariant and form a Sim(1) \(\times\mathbb{R}\) algebra corresponding to the Bianchi III Universe,

\[
\left[ \tilde{X}_+, \tilde{X}_+ \right] = \tilde{X}_7, \quad \left[ \tilde{X}_+, \tilde{X}_8 \right] = \left[ \tilde{X}_7, \tilde{X}_8 \right] = 0.
\]  

(A.15)

We make a codimension one ansatz and require that the base space metric has an SO(3) symmetry. Then, the base space reads

\[
\mathrm{d}s_B^2 = \mathrm{d}x^2 + \sigma_1^2 + \sigma_2^2,
\]  

(A.16)

where \(\sigma_i\) are SU(2) invariant forms,

\[
\sigma_1 = -\sin \psi \mathrm{d}\theta + \cos \psi \sin \theta \mathrm{d}\varphi, \\
\sigma_2 = \cos \psi \mathrm{d}\theta + \sin \psi \sin \theta \mathrm{d}\varphi, \\
\sigma_3 = \mathrm{d}\psi + \cos \theta \mathrm{d}\varphi.
\]

Every quantity is dependent only on a single variable \(x\). This class of metrics includes the five-dimensional Myers-Perry black hole [62] with equal angular momenta. Defining

\[
k^{3/2} h \mathrm{d}x = \mathrm{d}\tau, \quad \hat{V} = 2\Lambda - 2k^{-1}h^{-1},
\]  

(A.17)

the one-dimensional Lagrangian \( S = \int L \mathrm{d}\tau \) reads

\[
L = \frac{1}{2} h k^{1/2} + \frac{1}{4} k^2 \mathrm{Tr} \left[ \left( M^\prime \right)^\dagger \left( M^{-1}\right)^\dagger \right] - \hat{V},
\]  

(A.18)

where the prime denotes differentiation w.r.t. \(\tau\). The Hamiltonian boils down to

\[
H = \frac{1}{2} \left[ h^{-1}p_k^2 + f^{-2}h^{-2}k^{-2}p_l^2 + f^{-1}h^{-1}k^{-2}p_l^2 + fh^{-2}k^{-2} \left( p_{\psi_1} + \omega p_{\psi_2} \right)^2 \right]
\]

\[
- \frac{4}{3k^2} \left( f^2h^{-1}p_l^2 - f\gamma h + hp_h^2 \right) \right] + \hat{V}.
\]  

(A.19)

The trace of Einstein’s equations gives the constraint \( H = 0 \). For \( \varphi' = \{f, \omega, h, \psi_1, \psi_2, \} \), one can verify that \( C_6, C_7, C_8, C_9 \) are obvious first integrals and other nonvanishing Poisson brackets with the Hamiltonian are

\[
\{ C_1, H \} = -\frac{4}{3}H + \frac{8}{3}\Lambda, \quad \{ C_2, H \} = -\frac{2}{3}H + \frac{4}{3}\Lambda,
\]  

(A.20)

\[
\{ C_7, H \} = -2\psi_1H + 4\Lambda\psi_1, \quad \{ C_8, H \} = -2\psi_2H + 4\Lambda\psi_2.
\]  

(A.21)

Hence \( C_1 \equiv C_1 - 2C_2 \) is another first integral. The nonvanishing Poisson bracket among these first integrals reads

\[
\{ \tilde{X}_6, \} = 2\epsilon_3 p_2 \partial_{\epsilon_3} + p_1 \partial_{p_2} + \left( \epsilon_1^{-1} q_{12} - p_2^2 \right) \partial_{p_2}.
\]

8 \( \tilde{X}_6 \) seems to have a typo in [61] and it should be modified to \( \tilde{X}_6 = 2\epsilon_3 p_2 \partial_{\epsilon_3} + p_2 \partial_{\epsilon_3} + \left( \epsilon_1^{-1} q_{12} - p_2^2 \right) \partial_{p_2} \).
\begin{align}
\{C_3, C_4\} &= -C_5, \quad \{C_4, C_6\} = C_7, \quad \{C_5, C_6\} = C_3, \quad \{C_3, C_4\} = C_3, \\
\{C_3, C_4\} &= -2C_4, \quad \{C_3, C_5\} = -C_5, \quad \{C_3, C_6\} = 2C_6.
\end{align}
(A.22)

Thus $C_3$, $C_6$, $C_7$ generate a SL(2, $\mathbb{R}$) subalgebra and its quadratic Casimir $C_1^2 - 4C_6 C_4$ is another obvious first integral. Clearly, this is not true for the SL(3, $\mathbb{R}$) quadratic Casimir $\frac{1}{2}(C_1^2 - C_2 C_1 + C_2^2 - C_4 C_6 - C_5 C_7 - C_3 C_8)$ which involves also generators of the broken symmetries.

We found that three is the maximal number of commuting first integrals in involution with $H$ and there are many sets of this type that can be built from $C_3$, $C_5$, $C_4$, $C_5$, $C_6$ and their compositions. In order to decouple some of the fields in the Hamilton-Jacobi equation associated with (A.19), the most intriguing set seems to be composed of $H$, $C_1$, $C_3$ and the cubic invariant $C(3) = C_1^2 C_4 + C_2^2 C_6 - C_3 C_5 C_7$. Unfortunately, we were unable to find a sufficient number of commuting first integrals to carry out the integration procedure as in the four-dimensional Einstein–Maxwell system. One plausible reason for this is that the target space SL(3, $\mathbb{R}$)/SO(3) is five-dimensional, admitting eight Killing vectors, while in the $D = 4$ Einstein–Maxwell case the target space (2.13) is four-dimensional with eight Killing vectors. Obviously, the former is less symmetric. It would be interesting to see if this system displays chaotic behavior. We shall leave this point for future investigation.

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