THE STRUCTURE OF ALGEBRAIC VARIETIES

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Abstract. The aim of this address is to give an overview of the main questions and results of the structure theory of higher dimensional algebraic varieties.

1. Early history: Euler, Abel, Jacobi, Riemann

Our story, like many others in mathematics, can be traced back at least to Euler who studied elliptic integrals of the form

\[ \int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}. \]

The study of integrals of algebraic functions was further developed by Abel and Jacobi. From our point of view the next major step was taken by Riemann. Instead of dealing with a multi-valued function like \( \sqrt{x^3 + ax^2 + bx + c} \), Riemann looks at the complex algebraic curve

\[ C := \{(x, y) : y^2 = x^3 + ax^2 + bx + c\} \subset \mathbb{C}^2. \]

Then the above integral becomes

\[ \int_{\Gamma} \frac{dx}{y} \]

for some path \( \Gamma \) on the algebraic curve \( C \). More generally, a polynomial \( g(x, y) \) implicitly defines \( y := y(x) \) as a multi-valued function of \( x \) and for any meromorphic function \( h(u, v) \), the multi-valued integral

\[ \int h(x, y(x)) \, dx \]

becomes a single valued integral

\[ \int_{\Gamma} h(x, y) \, dx \]

for some path \( \Gamma \) on the algebraic curve \( C(g) := (g(x, y) = 0) \subset \mathbb{C}^2 \). Substitutions that transform one integral associated to a polynomial \( g_1 \) into another integral associated to a \( g_2 \) can be now seen as algebraic maps between the curves \( C(g_1) \) and \( C(g_2) \).

Riemann also went further. As a simple example, consider the curve \( C \) defined by \( y^2 = x^3 + x^2 \) and notice that \( (t^3 - t)^2 \equiv (t^2 - 1)^3 + (t^2 - 1)^2 \). Thus the substitution \( x = t^2 - 1, \ y = t^3 - t \) (with inverse \( t = y/x \)) allows us to transform any integral

\[ \int h(x, \sqrt{x^3 + x^2}) \, dx \]

into

\[ \int h(t^2 - 1, t^3 - t) \cdot 2tdt. \]
To put it somewhat differently, the map 
\[ t \mapsto (x = t^2 - 1, y = t^3 - t) \] and its inverse \((x, y) \mapsto t = y/x\) establish an isomorphism
\[
\left\{ \text{meromorphic functions on the curve } (y^2 = x^3 + x^2) \right\} \leftrightarrow \left\{ \text{meromorphic functions on the complex plane } \mathbb{C} \right\}.
\]
It is best to work with meromorphic functions on \(\mathbb{C}\) that are also meromorphic at infinity; these live naturally on the Riemann sphere \(\mathbb{CP}^1\). We can now state Riemann’s fundamental theorem as follows.

**Theorem 1.1** (Riemann, 1851). *For every algebraic curve \(C \subset \mathbb{C}^2\) there is a unique, compact Riemann surface \(S\) and a meromorphic map \(\phi : S \rightarrow C\) with meromorphic inverse \(\phi^{-1} : C \rightarrow S\) such that*
\[
f_C \mapsto f_S := f_C \circ \phi \quad \text{and} \quad f_S \mapsto f_C := f_S \circ \phi^{-1}
\]
*establish an isomorphism between the meromorphic function theory on \(C\) and the meromorphic function theory on \(S\).*

## 2. Main questions, informally

We can now give an initial formulation of the two main problems that we consider; the precise versions are stated in Sections 6 and 10. The first is a direct higher-dimensional analog of the results of Riemann. (See Section 3 for basic definitions.)

**Main Question 2.1.** Given an algebraic variety \(X\), is there another algebraic variety \(X^m\) such that

1. the meromorphic function theories of \(X\) and of \(X^m\) are isomorphic and
2. the geometry of \(X^m\) is the “simplest” possible?

Riemann’s theorem says that, in dimension 1, “simplest” should mean smooth and compact, but in higher dimensions smoothness is not the right notion. One of the hardest aspects of the theory was to understand what the correct concept of “simplest” should be.

So far we have dealt with individual algebraic varieties. A salient feature of algebraic geometry is that by continuously varying the coefficients of the defining polynomials we get continuously varying families of algebraic varieties. We can thus study how to transform a family \(\{X_t : t \in T\}\) of varieties into its “simplest” form. A tempting idea is to take the “simplest” forms \(\{X_t^m : t \in T\}\) obtained previously. Unfortunately, this fails already in dimension 1. Starting with a family of curves \(\{C_t : t \in T\}\), the corresponding Riemann surfaces \(\{S_t : t \in T\}\) form a continuously varying family over a dense open subset \(T^0 \subset T\) but not everywhere.

For curves the correct answer was found by Deligne and Mumford in 1969. We use the guidance provided by this 1-dimensional case and the answer to the first Main Question to answer the second.

**Main Question 2.2.** What are the “simplest” families of algebraic varieties? How can one transform an arbitrary family into one of the “simplest” families?
3. What are algebraic varieties?

Here we quickly recall the basic concepts and definitions that we use. For general introductory texts, see [73, 66, 30].

An affine algebraic set in \( \mathbb{C}^N \) is the common zero-set of some polynomials

\[
X^{\text{aff}} = X^{\text{aff}}(f_1, \ldots, f_r) = \{(x_1, \ldots, x_N) : f_1(x_1, \ldots, x_N) = \cdots = f_r(x_1, \ldots, x_N) = 0\} \subset \mathbb{C}^N.
\]

It is especially easy to visualize hypersurfaces \( X(f) \subset \mathbb{C}^N \) defined by 1 equation. Usually we count complex dimensions, thus \( \dim \mathbb{C}^N = N \) and \( \dim X \) is one half of the usual topological dimension of \( X \). In low dimensions we talk about curves, surfaces, 3-folds. Thus, somewhat confusingly, an algebraic curve is a (possibly singular) Riemann surface.

An affine algebraic set \( X \) is called irreducible if it can not be written as a union of two algebraic sets in a nontrivial way. Such sets are called affine algebraic varieties. Every algebraic set \( X \) is a finite union of algebraic varieties \( X = \bigcup_i X_i \) such that \( X_i \nsubseteq X_j \) for \( i \neq j \). Such a decomposition is unique, up to permuting the indices. Thus from now on we are interested mainly in algebraic varieties.

For example, the irreducible components of a hypersurface \( X(f) \) correspond to the irreducible factors of \( f \), thus \( X(f) \) is irreducible iff \( f \) is a power of an irreducible polynomial.

An affine algebraic set \( X^{\text{aff}} \) is compact iff it is 0-dimensional, thus it is almost always better to work with the closure of \( X^{\text{aff}} \) in the complex projective space

\[
X := X^{\text{proj}} \subset \mathbb{CP}^N.
\]

Thus we get projective algebraic sets and projective varieties. Finally, a quasi-projective variety is an open subset \( U \) of a projective variety \( X \) whose complement \( X \setminus U \) is a projective algebraic set. Note that \( U \) is a “very large” subset of \( X \), in particular \( U \) is dense in \( X \). This is a key feature of algebraic geometry: all open subsets are “very large.”

On a complex projective space \( \mathbb{CP}^N \) the homogeneous coordinates \( (x_0: \cdots : x_N) \) are defined only up to multiplication by a scalar. Thus one can not evaluate a polynomial \( p(x_0, \ldots, x_N) \in \mathbb{C}[x_0, \ldots, x_N] \), at a point of \( \mathbb{CP}^N \). However, if \( p \) is homogeneous of degree \( d \) then

\[
p(\lambda x_0, \ldots, \lambda x_N) = \lambda^d p(x_0, \ldots, x_N).
\]

Thus the zero set of \( p \) is well-defined and a quotient of two homogeneous polynomials of the same degree

\[
f(x_0, \ldots, x_N) = \frac{p_1(x_0, \ldots, x_N)}{p_2(x_0, \ldots, x_N)}
\]

is also well-defined (except where \( p_2 \) vanishes). These are the rational functions on \( \mathbb{CP}^N \). By restriction, we get rational functions on any projective variety \( X \subset \mathbb{CP}^N \).

At first sight these seem downright antiquated definitions; a modern theory ought to be local. That is, one should consider varieties that are locally defined by analytic functions and work with meromorphic functions on them. However, we know that every meromorphic function on \( \mathbb{CP}^1 \) is rational and the same holds in all dimensions.

**Theorem 3.1** (Chow, 1949; Serre, 1956). Let \( M \subset \mathbb{CP}^N \) be a closed subset that can be locally given as the common zero set of analytic functions. Then
(1) \( M \) is algebraic, that is, it can be globally given as the common zero set of homogeneous polynomials and

(2) every meromorphic function \( f \) on \( M \) is algebraic, that is, \( f \) can be globally given as the quotient of two homogeneous polynomials of the same degree.

Now we come to a key feature of algebraic geometry. There are two competing notions of “map” and two competing notions of “isomorphism.”

**Definition 3.2 (Map and morphism).** Let \( X \subset \mathbb{CP}^N \) be an algebraic variety and \( f_0, \ldots, f_M \) nonzero rational functions on \( X \). They define a map (or rational map)

\[
f : X \dashrightarrow \mathbb{CP}^M \quad \text{given by} \quad p \mapsto (f_0(p) : \cdots : f_M(p)) \in \mathbb{CP}^M.
\]

To start with, \( f \) is only defined at a point \( p \) if none of the \( f_i \) has a pole at \( p \) and not all of the \( f_i \) vanish at \( p \). However, since the projective coordinates are defined only up to a scalar multiple, \( (g f_0, \ldots, g f_M) \) define the same map for any rational function \( g \), thus it can happen that \( f \) is everywhere defined. In this case it is called a morphism. A map is denoted by \( \dashrightarrow \) and a morphism by \( \rightarrow \).

For example, projecting \( \mathbb{CP}^2 \) from the origin \((0:0:1)\) to the line at infinity is given by

\[
\pi : (x:y:z) \mapsto (\frac{x}{z} : \frac{y}{z}) = (1 : 1).
\]

Thus \( \pi \) is defined everywhere except at \((0:0:1)\).

**Definition 3.3 (Isomorphism).** Two quasi-projective varieties \( X \subset \mathbb{CP}^N \) and \( Y \subset \mathbb{CP}^M \) are isomorphic if there are morphisms

\[
f : X \rightarrow Y \quad \text{and} \quad g : Y \rightarrow X
\]

that are inverses of each other. Isomorphism is denoted by \( X \cong Y \).

We will think of isomorphic varieties as being essentially the same. Using maps instead of morphisms in the above definition yields the notion of birational equivalence. This notion is unique to algebraic geometry; it has no known analog in topology or differential geometry.

**Definition 3.4 (Birational equivalence).** Two quasi-projective varieties \( X \subset \mathbb{CP}^N \) and \( Y \subset \mathbb{CP}^M \) are birational (in old terminology, birationally isomorphic) if there are rational maps

\[
f : X \dashrightarrow Y \quad \text{and} \quad g : Y \dashrightarrow X
\]

such that the following equivalent conditions hold.

1. \( \phi_Y \mapsto \phi_X := \phi_Y \circ f \) and \( \phi_X \mapsto \phi_Y := \phi_X \circ g \) establish an isomorphism between the meromorphic (=rational) function theory on \( X \) and the meromorphic (=rational) function theory on \( Y \).

2. There are algebraic subsets \( Z \subseteq X \) and \( W \subseteq Y \) such that \( (X \setminus Z) \cong (Y \setminus W) \).

As an example, consider the affine surface \( S := (x^2 + y^2 = z^3) \subset \mathbb{C}^3 \). It is birational to \( \mathbb{C}^2_{uv} \) as shown by the rational maps

\[
f : (x, y, z) \mapsto (\tfrac{x}{z}, \frac{y}{z}) \quad \text{and} \quad g : (u, v) \mapsto (u(u^2 + v^2), v(u^2 + v^2), u^2 + v^2).
\]

Here \( f \) is not defined if \( z = 0 \) while \( g \) is everywhere defined but it maps the pair of lines \( (u = \pm iv) \) to the origin \((0, 0, 0)\). Thus

\[
S \setminus \{z = 0\} \cong \mathbb{C}^2 \setminus \{(u^2 + v^2 = 0)\} \quad \text{but} \quad S \not\cong \mathbb{C}^2.
\]
Basic rule of thumb 3.5. Let \( X, Y \) be algebraic varieties that are birational to each other. Many questions of algebraic geometry about \( X \) can be answered by

- first studying the same question on \( Y \) and then
- studying a similar question involving the lower dimensional algebraic sets \( Z \) and \( W \) as in (3.4.2).

The aim of the Minimal Model Program is to exploit this in two steps.

- Given a question and a variety \( X \), find a variety \( Y \) that is birational to \( X \) such that the geometry of \( Y \) is “best adapted” to studying the particular question. This is a variant of the first Main Question.
- Set up the appropriate dimension induction to deal with the exceptional sets \( Z \subset X \) and \( W \subset Y \).

Important aside. More generally, if we decompose an algebraic variety into disjoint locally closed pieces, then the collection of the pieces carries a lot of information about the variety. I would like to stress that this is a rather noteworthy fact about algebraic geometry. For instance, if we decompose a simplicial complex into its simplices, then usually the only information we retain is the dimension and the Euler characteristic. By contrast, all the homology groups of a smooth, projective algebraic variety can be recovered from the pieces. This is a key consequence of Hodge Theory, as formulated by Deligne, and is a starting point of Grothendieck’s theory of motives.

4. Classical results

After the study of algebraic curves, two main avenues of investigations were pursued. One direction focused on the local study of varieties with a main aim of resolving them completely. The other direction aimed to understand the global structure of algebraic surfaces. These are both still very active research areas. We recall a few of the main results that are relevant for the general theory. For detailed treatments and for references see [6, 45].

Resolution of singularities.

Riemann’s theorem says that every singular algebraic curve \( C \) is birational to a smooth, compact curve (or Riemann surface). The first steps toward answering the Main Questions in higher dimensions focused on this problem: Is every algebraic variety birational to a smooth, projective variety?

Definition 4.1. A variety \( X \subset \mathbb{C}^N \) is smooth and has dimension \( d \) at a point \( p \in X \) iff the following equivalent conditions hold.

1. \( X \subset \mathbb{C}^N \cong \mathbb{R}^{2N} \) is a \( C^\infty \)-submanifold of (real) dimension \( 2d \) near \( p \).
2. One can choose coordinates \( z_1, \ldots, z_N \) and equations \( f_1, \ldots, f_{N-d} \) of \( X \) such that \( (f_1 = \cdots = f_{N-d} = 0) \) coincides with \( X \) near \( p \) and the Jacobian matrix \( (\partial f_i/\partial z_j : 1 \leq i,j \leq N-d) \) is invertible at \( p \).
3. There are holomorphic functions \( \phi_i = \phi_i(w_1, \ldots, w_d) \) defined near the origin and constants \( c_i \) such that
   \[
   (w_1, \ldots, w_d) \mapsto (\phi_1(w), \ldots, \phi_{N-d}(w), w_1 + c_1, \ldots, w_d + c_d)
   \]
   maps a small ball \( 0 \in \mathbb{B}^d(\epsilon) \subset \mathbb{C}^d \) onto a neighborhood of \( p \in X \).
In the latter case we view \((w_1, \ldots, w_d)\) as local analytic coordinates on \(X\) near \(p\). (It is an ever present technical problem that there is no good notion of local algebraic coordinates. Open algebraic neighborhoods are too large to admit a single-valued coordinate system.)

On an algebraic variety \(X\) the set of singular points turns out to be an algebraic subset, denoted by \(\text{Sing } X \subset X\). For every variety \(X\), a generalization of Riemann’s method (1.1) produces a new variety \(X' \to X\), called the normalization of \(X\), such that \(\text{Sing } (X')\) has codimension \(\geq 2\) in \(X'\). Thus, in higher dimensions, one usually works with normal varieties whose singular set has codimension \(\geq 2\).

To make the singular set even smaller, or to get rid of it completely, turned out to be very difficult. The final result was established by Hironaka in 1964.

**Theorem 4.2** (Resolution of singularities). For every algebraic variety \(X\), there are (very many) smooth, projective varieties \(X' \to X\) such that \(\text{Sing } (X')\) has codimension \(\geq 2\) in \(X'\). Thus, in higher dimensions, one usually works with normal varieties whose singular set has codimension \(\geq 2\).

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(2) $S$ is birational to $C \times \mathbb{CP}^1$ for a unique, smooth, projective curve $C$.

4.6 (Du Val singularities). It was also gradually understood that instead of working with the minimal model $S^m$, it is sometimes better to use a slightly singular canonical model $S^{\text{can}}$. The resulting singularities were first classified by Du Val in 1934; the list is quite short, ranging from the simplest $(x^2 + y^2 + z^2 = 0)$ to the most complicated $(x^2 + y^3 + z^5 = 0)$. They are also called rational double points.

Their importance was not generally recognized until the 1960’s when they were rediscovered from many different points of view; see [18] for a survey.

5. The first Chern class and the Ricci curvature

The first Chern class, which is closely related to the Ricci curvature, carries much of the important information about the structure of a variety. We follow the differential geometry sign conventions; algebraic geometers usually work with the canonical class, which is (a slight refinement of) the negative of the first Chern class.

5.1 (Complex volume forms). A measure on $\mathbb{R}^n$ can be identified with an $n$-form

$$s(x_1, \ldots, x_n) \cdot dx_1 \wedge \cdots \wedge dx_n.$$ Thus a measure on a real manifold $M$ is an $n$-form that in local coordinates can be written as

$$h(z_1, \ldots, z_n) \cdot dz_1 \wedge \cdots \wedge dz_n.$$ Thus a complex volume form $\omega$ gives a real volume form $(\sqrt{-1})^n \omega \wedge \bar{\omega}$ where the constant comes from the formula

$$dz \wedge d\bar{z} = (dx + \sqrt{-1}dy) \wedge (dx - \sqrt{-1}dy) = -2\sqrt{-1} \, dx \wedge dy.$$ (There is usually an additional $\pm$, depending on one’s orientation conventions.)

From the point of view of differential geometry, one would like to use $C^\infty$ complex volume forms, that is, the $h(z_1, \ldots, z_n)$ should be nowhere zero $C^\infty$-functions. Algebraic geometry, however, prefers meromorphic volume forms where the $h(z_1, \ldots, z_n)$ are meromorphic functions. (See [9.2.1] for some explicit examples.) Thus the ideal situation is when a complex volume form is given by nowhere zero holomorphic functions $h(z_1, \ldots, z_n)$. This is possible only for Calabi–Yau varieties; they form a very special but important subclass [6.2].

Thus in general we try to understand how to connect $C^\infty$ and meromorphic volume forms.

On the differential geometry side the key notion is the curvature which defines the Chern form.

**Definition 5.2** (Chern form and Chern class). Let $\omega$ be a $C^\infty$ complex volume form. The first Chern form or Ricci curvature form of $(X, \omega)$ is the 2-form

$$\hat{c}_1(X, \omega) := \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |h(z_1, \ldots, z_n)| = \frac{\sqrt{-1}}{\pi} \sum_{ij} \frac{\partial^2 \log |h(z)|}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$
As a 2–form, this depends on the choice of the volume form $\omega$, but it gives a well defined De Rham cohomology class $c_1^R(X) \in H^2_{DR}(X, \mathbb{R})$ which actually lifts to an integral cohomology class

$$c_1(X) \in H^2(X, \mathbb{Z}),$$

called the first Chern class of $X$.

**Definition 5.3 (Algebraic degree).** Let $X$ be a smooth, projective variety and $C \subset X$ an algebraic curve. It is not hard to see that there is always a meromorphic volume form $\omega_m$ that is defined and nonzero at all but finitely many points of $C$. We define the degree of $\omega_m$ on $C$ as

$$\text{deg}_C \omega_m := \#(\text{zeros of } \omega_m \text{ on } C) - \#(\text{poles of } \omega_m \text{ on } C),$$

where both zeros and poles are counted with multiplicities.

The Chern form and the algebraic degree are connected by the Gauss–Bonnet theorem.

**Theorem 5.4.** Let $X$ be a smooth, projective variety. Let $\omega_r$ be a $C^\infty$ complex volume form and $\omega_m$ a meromorphic volume form. Then, for every algebraic curve $C \subset X$

$$\int_C c_1(X) = \int_C \tilde{c}_1(X, \omega_r) = - \text{deg}_C \omega_m. \quad (5.4.1)$$

(The minus sign comes from the happenstance that differential geometers prefer to work with the tangent bundle while the volume forms use the (determinant of the) cotangent bundle.)

**Positivity/negativity and complex differential geometry.**

In differential geometry it is especially nice to work with metrics whose curvature is everywhere positive (or everywhere zero or everywhere negative) but these rarely exist. A usual weakening is to work with Kähler metrics that satisfy the *Einstein condition*: the Ricci curvature should be a constant multiple of the metric; see [64, Chap.19] for definitions and an introduction.

If this *Einstein constant* is positive, then in (5.4.1) we integrate an everywhere positive form. Thus $\int_C c_1(X)$ is positive for every curve $C$. We hope that in this case there are meromorphic volume forms with poles (but no zeros).

Similarly, if the Einstein constant is negative, then in (5.4.1) we integrate an everywhere negative form. Thus $\int_C c_1(X)$ is negative for every curve $C$. We hope that in this case there are holomorphic volume forms (usually with zeros).

Algebraic geometry can be used to understand the numbers $\text{deg}_C \omega_m$, hence the values of the integrals $\int_C c_1(X)$. It is a very difficult task to use the positivity/negativity of the integrals $\int_C c_1(X)$ to obtain a Kähler metric with positive/negative Einstein constant.

For smooth varieties Aubin and Yau proved existence in 1977 when $\int_C c_1(X)$ is always negative or when $\int_C c_1(X)$ is always zero. The singular case is treated in [20, 7]. The positive curvature case is more subtle; a complete answer is not yet known.

While our approach to the structure of varieties is guided by these curvature considerations, in algebraic geometry we can understand only the algebraic degree of the first Chern class. Thus we look at the functional

$$C \mapsto \int_C c_1(X)$$
and focus on those varieties where this is everywhere negative (or everywhere zero or everywhere positive).

The Main Conjecture then asserts that every variety can be built up from these special varieties in a rather clear process.

6. The Main Conjecture

On a typical variety $X$, the Chern class $c_1(X)$ is positive on some curves and negative on others, in a rather unpredictable way. Using the first Chern class and the theory of algebraic surfaces as our guide, we focus on three basic “especially simple” types of smooth, projective varieties. These are the “building blocks” of all algebraic varieties.

6.1 (Negatively curved). These are the varieties where $\int_C c_1(X)$ is negative for every curve $C \subset X$. This is the largest class of the three.

6.2 (Flat or Calabi–Yau). Here $\int_C c_1(X)$ is zero for every curve $C \subset X$. They play an especially important role in string theory and mirror symmetry; see [77, 32] for introductions.

6.3 (Positively curved or Fano). Here $\int_C c_1(X)$ is positive for every curve. There are few of these varieties in each dimension, but they occur especially frequently in applications.

A simple set of examples to keep in mind is the following. A smooth hypersurface $X_d \subset \mathbb{CP}^n$ of degree $d$ is negatively curved if $d > n+1$, flat if $d = n+1$ and positively curved if $d < n+1$.

A variety in any of these 3 classes is considered “simplest,” but we do not yet have enough “simplest” varieties for answering the first Main Question. For example, taking products of these we get examples where $c_1(X)$ has different signs on different curves. Two of these possible “mixed types” are relevant for us.

Consider a product $X := N \times F$ of a negatively curved and of a flat variety. It is clear that $\int_C c_1(X) \leq 0$ for every curve $C \subset X$ and $\int_C c_1(X) = 0$ only if $C$ lies in a fiber of the first projection $N \times F \to N$. This observation leads to the 4th class.

6.4 (Semi-negatively curved or Kodaira–Iitaka type). Here $\int_C c_1(X) \leq 0$ for every curve $C \subset X$ and there is a unique morphism $I_X : X \to I(X)$ such that $\int_C c_1(X) = 0$ iff $C$ is contained in a fiber of $I_X$.

This includes the classes 6.1, 6.2. $I_X$ is an isomorphism for negatively curved varieties and a constant map in the flat case.

In the intermediate cases, when $0 < \dim I(X) < \dim X$, almost all fibers of $I_X$ are Calabi–Yau varieties. Thus one can view these as families of (lower dimensional) Calabi–Yau varieties parametrized by the (lower dimensional) variety $I(X)$. If we understand families of (lower dimensional) varieties well enough, we understand $X$. (This is one of the reasons we are interested in the second Main Question.) Furthermore, in these cases $I(X)$ is negatively curved in a “suitable sense,” though we do not yet have a final agreed-upon definition of what this means.

Next consider a product $X := N \times P$ of a negatively curved and of a positively curved variety. If a curve $C$ lies in a fiber of the first projection then $\int_C c_1(X) > 0$, but there are many other such curves. Nonetheless, the first projection is uniquely determined by $X$ and this leads to the definition of the 5th class.
6.5 (Positive fiber type). I really would like to say that in these cases there is a unique morphism $m_X : X \to M(X)$ such that $M(X)$ is semi-negatively curved and $c_1(X)$ is positive on all the fibers. (To avoid trivial cases, we also assume that $\dim M(X) < \dim X$.) This, unfortunately, still does not give enough “simplest” varieties for the first Main Question. It took quite some time to arrive at the correct definition, to be discussed in Section 7.

We can now state a precise version of the first Main Question.

**Main Conjecture 6.6.** Every algebraic variety $X$ is birational to a variety $X^m$ that is either of type (6.4) or of type (6.5).

**Complement.** $X^m$ – especially in case (6.4) – is called a minimal model of $X$.

In the semi-negatively curved case $I(X^m)$ is unique but $X^m$ itself is not. However, it is quite well understood how the different $X^m$ are related to each other. (This is the story of flops, see [37, 27].) By contrast, in case (6.5) it is very hard to determine when two such varieties $X_1^m$ and $X_2^m$ are birational.

**Caveat.** While the Main Conjecture is expected to be true, in general one has to allow terminal singularities – to be defined in (9.3) – on $X^m$.

This was a rather difficult point historically since over a century of experience suggested that singularities should be avoided. For surfaces terminal = smooth, thus the issue of singularities did not come up in Theorem 4.5.

By now the correct classes of singularities have been established and, for many questions we consider, they do not seem to cause any problems. We describe these singularities in Section 9.

6.7 (Traditional names). A variety $X$ is said to be of general type if $\dim I(X^m) = \dim X$. In this case $X \dashrightarrow I(X^m)$ is birational and $I(X) := I(X^m)$ is called the canonical model of $X$; it has canonical singularities (9.4). We see in Section 10 that the second Main Question has a good answer for families of canonical models.

The Kodaira dimension of a variety $X$ is the dimension of $I(X^m)$.

The Kodaira dimension is defined to be $-\infty$ for the class (6.5).

The Main Conjecture is usually broken down into two parts that are, in principle, independent of each other. The first part separates the classes 6.4 and 6.5 from each other and the second part provides the structural description in case 6.4. These forms first appear in Reid’s paper [71, Sec.4].

6.7.1 Minimal Model Conjecture. Every algebraic variety $X$ is birational to a variety $X^m$ such that either $c_1(X^m)$ is semi-negative or there is a morphism to a lower dimensional variety $\pi : X^m \to S$ such that $\int_C c_1(X^m) > 0$ if $C$ is contained in a fiber of $\pi$. (In the second case the map $\pi$ need not be unique and it does not give the best structural description.)

6.7.2 Abundance Conjecture. If $c_1(Y)$ is semi-negative then there is a unique morphism $I_Y : Y \to I(Y)$ such that $\int_C c_1(Y) = 0$ iff $C$ is contained in a fiber of $I_Y$.

7. Rationally connected varieties

Before we consider minimal models, we describe the structure we expect for varieties in the 5th class (6.5). An introduction aimed at non-specialists is given in [33]. More detailed accounts are in [5, 39].
Clebsch and Max Noether noticed around 1860–1870 that, when the numerical invariants suggest that a surface could be birational to $\mathbb{CP}^2$, then it is. The final result along these lines was established by Castelnuovo in 1896.

Analogous questions in higher dimension turned out to be much harder. Fano classified smooth positively curved 3–folds around 1930. (He missed some cases though, so did subsequent “complete” lists produced in the 1970’s and then in the 1980’s. The (hopefully) final list was not established until 2003.) This is, however, one area where the singularities do matter; we still do not know all positively curved 3–folds with terminal singularities.

It appears that instead of global descriptions we should focus on rational curves in a variety; these are the images of morphisms $\phi : \mathbb{CP}^1 \to X$. For a projective variety $X$, the following dichotomy is quite easy to establish.

i) either the rational curves cover a subset of $X$ which is meager (that is, a countable union of nowhere dense closed subsets)

ii) or the rational curves cover all of $X$.

These two cases correspond to the alternatives in the Main Conjecture. That is, if $X$ is birational to a semi-negatively curved variety then rational curves cover a meager subset and, conjecturally, the converse also holds.

The correct approach to the best structural description of the 5th class $6.5$ was not discovered until 1992 (Kollar–Miyaoka–Mori [50]). The key observation is that we should even change the class $6.3$. Instead of a curvature description, we should focus on rational curves contained in a variety.

Definition 7.1. A projective variety $X$ is called rationally connected if, for any number of points $x_1, \ldots, x_r \in X$, there is a morphism $\phi : \mathbb{CP}^1 \to X$ whose image passes through $x_1, \ldots, x_r$.

I claim that rationally connected varieties constitute the “correct” birational version of being positively curved. This is not a precise mathematical assertion since not every rationally connected variety is birational to a positively curved variety, not even when singularities are allowed. Rather, the assertion is that any answer to the first Main Question needs to work with rational connectedness instead of positivity of curvature.

7.2 (Supporting evidence).

It is easy to see that $\mathbb{CP}^n$ is rationally connected. More generally, every positively curved variety is rationally connected (Nadel [57], Campana [10], Kollár–Miyaoka–Mori [49], Zhang [80]).

Being rationally connected is invariant under smooth deformations and birational maps [49].

Rationally connected varieties share key arithmetic properties of rational varieties over $p$-adic fields (Kollár [42]), finite fields (Kollár–Szabó [54], Esnault [19]) and function fields of curves (Graber–Harris–Starr [22], de Jong–Starr [16]).

The loop space of a rationally connected variety is also rationally connected (Lempert–Szabó [59]).

The notion of rational connectedness allows us to give the correct description of the class $6.5$. A weaker variant is proved in [50]; the form below combines this with [22].

Theorem 7.3. Let $X$ be a variety that is covered by rational curves. Then there is a unique (up to birational equivalence) map $m_X : X \dasharrow M(X)$ such that
almost all fibers of $m_X$ are rationally connected and
rational curves cover only a meager subset of $M(X)$.

There are two main open geometric problems about rationally connected varieties. The first concerns a topological characterization. In its naive form the question asks: What can we tell about a variety from its underlying topological space? It seems that the answer is: not much. However, the underlying topological space of a smooth variety carries a natural symplectic structure and this seems to incorporate much more information.

**Conjecture 7.4.** [41, Conj.4.2.7] Being rationally connected is a property of the underlying symplectic structure.

For partial results see [41,78].

The other problem asks if we could strengthen the definition of rationally connected varieties. Note that $\mathbb{CP}^n$ contains not just many rational curves but also many higher dimensional rational subvarieties (hyperplanes, hyperquadrics, ...). Maybe this is also a general property of rationally connected varieties? As far as I know, 3-dimensional rationally connected varieties always contain rational surfaces. I believe, however, that this is not the case in higher dimension.

**Conjecture 7.5.** [43, Prob.56] Many rationally connected varieties do not contain any rational surface.

8. Minimal Models

This is a short history of Mori’s program, also called the Minimal Model Program and frequently abbreviated as MMP. For general introductions see [12,52], or the technically more detailed [33,13,25].

8.1 (Iitaka’s program, 1970–85). This approach predates the Main Conjecture. At the beginning it was not even suspected that the Main Conjecture could be true, in fact, lacking the right class of singularities, it was assumed that the Main Conjecture would fail for most varieties. Thus the aim of Iitaka’s program was to sort varieties into 5 broad types that (as we now know) exactly correspond to the ones in (6.1–6.5). The main contributors were, in rough historical order, Iitaka, Ueno, Fujita, Kawamata, Viehweg and Kollár; see [74,62] for surveys.

8.2 (Canonical and terminal singularities, Reid 1980–83). Reid was studying higher dimensional analogs of Du Val singularities of surfaces [1,6]; obtaining rather complete descriptions in dimension 3. It was quite important that when Mori’s program lead to singularities, the relevant classes were already there and were known to be well behaved. An especially readable account is [72].

8.3 (The birth of Mori’s program, 1981–88). Mori’s groundbreaking paper [61] introduces 3 new ideas.

If $c_1(X)$ is not semi-negative then, by definition, $c_1(X)$ is positive on some curve $C \subset X$. Mori first proves that there is such a rational curve; that is, there is a morphism $\phi: \mathbb{CP}^1 \to X$ such that $c_1(X)$ is positive on its image. It is quite remarkable that the proof goes through algebraic geometry over finite fields. To this day there is no proof known that avoids this; in particular this step is not yet known for complex manifolds that are not algebraic.
Second, he identifies the “most positive” such maps $\phi : \mathbb{CP}^1 \to X$; this is called extremal ray theory.

Third, in dimension 3 he gives a complete description of all extremal rays and the resulting map $X \to X_1$ that removes the “most positive” part of $X$.

The program now seems clear (at least in dimension 3). Repeat the procedure for $X_1$ and prove that after finitely many steps we end up with $X \to X_1 \to \cdots \to X_r$ such that $c_1(X_r)$ is semi-negative. This is called Mori’s program or Minimal Model program.

There are two, rather formidable, problems. In many cases the new variety $X_1$ is smooth but sometimes it is singular. Luckily, these singularities have been studied by Reid, at least in dimension 3. Still, it is necessary to establish the above 3 steps for singular varieties. This was accomplished rather rapidly by Kawamata, Reid, Shokurov and Kollár. The program was first written down in [71 Sec.4].

The more serious problem is that in some cases taking the contraction $X_i \to X_{i+1}$ is clearly not the right step. Instead we have to take a step back and construct a new variety $X_i^+$ that sits in a flip diagram

\[
\begin{array}{ccc}
X_i & \xrightarrow{\phi_i} & X_i^+ \\
\downarrow p_i & & \downarrow p_i^+ \\
X_{i+1} & \xleftarrow{} & X_{i+1}^+
\end{array}
\]

Geometrically, we start with $X_i$, find an especially badly behaving $\mathbb{CP}^1 \cong C_i \subset X_i$ and remove it. Then we compactify the resulting $X_i \setminus C_i$ by attaching another curve $C_i^+ \cong \mathbb{CP}^1$ but differently. The key difference is a sign change:

$$\int_{C_i} c_1(X_i) > 0 \quad \text{but} \quad \int_{C_i^+} c_1(X_i^+) < 0.$$  

This operation is called a flip. For more about flips, see [37, 27].

Flips are reminiscent of Dehn surgery in 3–manifold topology where we remove a circle and put it back differently.

In dimension 3 the existence of flips is proved in a very difficult paper by Mori [63], which completes the program in this case. A detailed description of 3–dimensional flips is given in [51]. The list is rather lengthy; this makes it unlikely that a similarly complete answer will ever be worked out in higher dimensions.

8.4 (Log variants: Kawamata, Shokurov, 1984–1992). The Iitaka program established that for many results one can work with cohomology classes in $H^2(X, \mathbb{R})$ that are close enough to the first Chern class. This turned out to be a very powerful tool. By choosing the perturbations appropriately, we can focus our attention on one or another part of a variety. These are somewhat technical questions but by now we understand how to work with them and most applications of the Minimal Model Program use a perturbed case.

8.5 (Abundance: Kawamata, Miyaoka, 1987–1992). Even for surfaces, the Abundance Conjecture 6.7.2 is a rather subtle result. It is even harder for 3–folds. The proofs use many special properties of surfaces; this is why the higher dimensional cases are still not well understood. A rather complete account of the 3–dimensional methods is given in [38].
8.6 (Inductive approach in low dimensions: Shokurov, 1992–2003). In retrospect, the key development of the decade was an inductive approach to flips. A detailed treatment of the 3-dimensional case is given in [38]. For the rest of the nineties progress was slow, culminating in a treatment of 4-dimensional flips. There were many technical difficulties to overcome and the importance of these methods was not fully appreciated at first since the dimension reduction leads to a much more complicated problem that seems to fail in higher dimensions.

8.7 (The Corti seminar, 2003–2005). Over the course of several years a group led by Corti developed the previous ideas further and integrated them with the rest of the program [13]. This provided the bridge to the general case.

8.8 (The general type case: Hacon and M\textsuperscript{c}Kernan, 2005–2010). The real breakthrough was achieved in [26] where the existence of flips in dimension $n$ was reduced to an instance of the MMP in dimension $n-1$. This left a series of global questions to resolve. The paper [9] settled everything for varieties of general type. A good introduction is in [14].

At about the same time Siu started to develop an analytic approach which aims to get $I(X^w)$, without going through the individual steps; see [70] for an overview. An algebraic variant of this is in [11].

8.9 (Abundance: Hacon and Xu, 2012–). Although the Abundance conjecture is known in very few cases, there has been significant progress when $\dim I(X)$ is expected to be close to $\dim X$. The log version of the special case when $\dim I(X) = \dim X$ is especially important for applications in moduli theory. These have been settled in [28, 68].

8.10 (Positive characteristic, Hacon and Xu, Birkar, Patakfalvi, 2012–). Mori’s original works are very geometric and these ideas quickly lead to a simple proof of the 2-dimensional case of the Main Conjecture in positive characteristic. However, subsequent developments rely very heavily on Kodaira-type vanishing theorems that are known to fail in positive characteristic, although no actual failure is known in the cases used by the program. The 3-dimensional case was recently settled in [29, 8]. Substantial parts of the Iitaka program are proved in positive characteristic in [69].

8.11 (Open problems). From our point of view, the main open problem is to complete the missing parts of the Main Conjecture.

It is known that the MMP always runs, that is, the sequence of contractions and flips $X = X_1 \rightarrow X_2 \rightarrow \cdots$ exists. The problem is that it is not clear how to prove that the process eventually stops. In the 3-dimensional case, Mori’s approach provides a rather complete description of the steps of the MMP. This gives many ways to show that each step improves various invariants and that eventually the process stops. By contrast, the method of Hacon–McKernan produces the steps of the MMP in a rather indirect way. We have very little information about the steps beyond their existence.

9. SINGULARITIES OF THE MINIMAL MODEL PROGRAM

So far we have been sweeping the singularities of the minimal models under a rug, but it is time for a look at them. Understanding the correct class of singularities is crucial in the development of the structure theory of algebraic varieties. This is a
somewhat technical subject with many difficult questions and methods but by now we understand these singularities well enough that in many questions they do not cause any problems. A rather complete treatment is given in [47]. Here I focus on the main ideas behind the definitions.

Given a variety $Y$, one frequently looks at a resolution of singularities $f : X \to Y$ as in Theorem 4.2 and translates problems on $Y$ to questions on $X$. Then the hard part is to interpret the answer obtained on $X$ in terms of $Y$. Here the key seems to be the inverse function theorem.

**9.1 (The inverse function theorem).** The classical inverse function theorem says that if $f := (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map then $f$ has a local inverse at a point $p \in \mathbb{R}^n$ iff the Jacobian determinant

$$\text{Jac}(f) := \det \left( \frac{\partial f_i}{\partial x_j} \right)$$

does not vanish at $p$. We can also think about it in terms of the “standard” volume forms $\omega_x := dx_1 \wedge \cdots \wedge dx_n$ and $\omega_y := dy_1 \wedge \cdots \wedge dy_n$. Then

$$f^* \omega_y = \text{Jac}(f) \cdot \omega_x,$$

thus the vanishing/non-vanishing of the Jacobian tells us how the pull-back of the “standard” volume form of the target compares to the “standard” volume form of the source.

Note that the Jacobian itself depends on the choice of the coordinates, but its vanishing or non-vanishing depends only on $f$.

In the complex analytic setting one can use the “standard” complex volume forms $\omega_z := dz_1 \wedge \cdots \wedge dz_n$ and $\omega_w := dw_1 \wedge \cdots \wedge dw_n$ on the unit balls $B^n_z \subset \mathbb{C}^n_z$ and $B^n_w \subset \mathbb{C}^n_w$. Given a holomorphic map $f := (f_1, \ldots, f_n) : B^n_z \to B^n_w$ we get that

$$f^* \omega_w = \det \left( \frac{\partial f_i}{\partial z_j} \right) \cdot \omega_z =: \text{Jac}(f) \cdot \omega_z,$$

and $f$ has a local inverse iff $\text{Jac}(f)$ does not vanish at $p$.

**9.2 (The Jacobian in the singular case).** Let $X$ be a normal algebraic variety and $p \in X$ a singular point. It is quite easy to see that if $\omega_1, \omega_2$ are two holomorphic volume forms on $X \setminus \text{Sing} X$ in a neighborhood of a singular point $p \in \text{Sing} X$ then there is a unique holomorphic function $\phi$ such that $\omega_1 = \phi \cdot \omega_2$ and $\phi(p) \neq 0$. Thus all holomorphic volume forms on $X \setminus \text{Sing} X$ have the same asymptotic behavior near $\text{Sing} X$. The local existence of such forms is a slightly technical question, so let us just focus on an example. If $Y := \{f(w_1, \ldots, w_{n+1}) = 0\} \subset \mathbb{C}^{n+1}$ is a hypersurface then the “standard” volume form is given by

$$\omega_Y = (-1)^i \frac{dw_1 \wedge \cdots \wedge dw_{i-1} \wedge df_{i+1} \wedge \cdots \wedge dw_{n+1}}{\partial f_i/\partial w_i}. \tag{9.2.1}$$

(It is easy to check that this is independent of $i$. Note also that $\omega_Y$ is not defined when all of the $\partial f_i/\partial w_i$ vanish; which happens exactly on $\text{Sing} Y$.) Thus if $f : B^n_z \to Y$ is holomorphic then we can define the Jacobian of $f$ by the formula

$$\text{Jac}(f) := \frac{f^* \omega_Y}{\omega_z}.$$

Note that due to the denominators in (9.2.1), in general $\text{Jac}(f)$ can have poles.
For example, consider the singularity $Y_{n,d} := (w_1^d + \cdots + w_n^d = w_{n+1}^d) \subset \mathbb{C}^{n+1}$ and a holomorphic map $f : \mathbb{B}_n^d \to Y$ given by

$$f : (z_1, \ldots, z_n) \to (z_1, z_1 z_2, \ldots, z_1 z_n, z_1^d \sqrt{1 + z_2^d + \cdots + z_n^d}).$$

Then $\omega_{Y_{n,d}} = -d^{-1}w_1^{-d}dw_2 \wedge \cdots \wedge dw_{n+1}$ and we easily compute that the Jacobian of $f$ has a zero/pole of order $n - d$ along the hyperplane ($z_1 = 0$).

As in the classical case, the Jacobian of $f$ depends on the choice of the “standard” volume forms but the vanishing/non-vanishing or the order of vanishing of the Jacobian depends only on $f$.

We can now define terminal singularities; these form the smallest possible class needed for the Main Conjecture.

**Definition 9.3.** A normal variety $Y$ has terminal singularities iff the inverse function theorem holds for $Y$. That is, if $f : \mathbb{B}_n^d \to Y$ does not have a local inverse at $p \in \mathbb{B}_n^d$ then $\text{Jac}(f)$ vanishes at $p$. (There is a small problem when the exceptional set of $f$ is too small, we can ignore it for now.)

For canonical models and for moduli questions, two more types of singularities are needed.

**Definition 9.4.** A normal variety $Y$ has canonical singularities iff $\text{Jac}(f)$ is holomorphic for every $f : \mathbb{B}_n^d \to Y$ and log-canonical singularities iff $\text{Jac}(f)$ has at most simple poles for every $f$.

The above computations suggest (and it is indeed true) that $Y_{n,d}$ (as in 9.2.2) is terminal iff $d < n$, canonical iff $d \leq n$ and log canonical iff $d \leq n + 1$.

**9.5 (Local volume of $Y$ near Sing $Y$).** A good way to think about these singularities is as follows. Pick a point $p \in \text{Sing} Y$ and let $\omega_Y$ be a “standard” local complex volume form. Then $(\sqrt{-1}/2)^n \omega_Y \wedge \bar{\omega}_Y$ is a real volume form and we can ask about the local volume of $X$, that is, $\int_U (\sqrt{-1}/2)^n \omega_Y \wedge \bar{\omega}_Y$ for a suitably small neighborhood $p \in U \subset X$.

If $Y$ has a canonical singularity near $p$ then the local volume is finite. In the log-canonical case the local volume is infinite but barely. If $g$ is any holomorphic function vanishing on Sing $Y$ then $\int_U |g|^\epsilon (\sqrt{-1}/2)^n \omega_Y \wedge \bar{\omega}_Y$ is finite for every $\epsilon > 0$.  

**9.6 (Intermediate differential forms).** On an $n$-dimensional variety we have so far considered holomorphic $n$-forms only but for several questions one also needs to understand the pull-back $f^* \eta$ of lower degree differential forms as well. This proved to be surprisingly difficult but almost all local questions were settled by Greb–Kebekus–Kovács–Peternell [23].

10. MODULI OF VARIETIES OF GENERAL TYPE

Let $X$ be a class of projective varieties, for instance curves or surfaces of a certain type. The theory of moduli aims to find “optimal” ways to write down all varieties in the class $X$.

This is a large theory with many aspects. The 3 volumes of [21] contain surveys of most of the active areas. Here my aim is to focus on just one of them: the moduli of varieties of general type. Introductions are given in [57, 25, 46] while a detailed treatment should be in [48].

We start with the historically first example.
Example 10.1 (Elliptic curves). They can all be given by an affine equation
\[ E(a, b, c) := (y^2 = x^3 + ax^2 + bx + c) \subset \mathbb{C}^2; \]
the corresponding projective curve has a unique point \([p]\) at infinity. Here \(c_1(E) = 0\), so it is best to think of this as elliptic curves with a marked point \([p]\). The curve \(E(a, b, c)\) is smooth iff the discriminant of the cubic
\[ \Delta(a, b, c) := 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^2 \]
is not zero.

Two such curves are isomorphic iff there is an affine-linear transformation \((x, y) \mapsto (\alpha^2x + \beta, \alpha^3y)\) that transforms one equation into the other. All these transformations form a (2-dimensional) group \(G\). Thus we get the following.

Version 1. The isomorphism classes of all elliptic curves are in one-to-one correspondence with the orbits of \(G\) on \(\mathbb{C}^3 \setminus (\Delta(a, b, c) = 0)\).

Next we need to identify the \(G\)-orbits. The key is the \(j\)-invariant \(j(E(a, b, c)) := 2^8(a^2 - 3b)^3/\Delta(a, b, c)\). (The factor \(2^8\) is not important for us, it is there for number-theoretic reasons.) It is not very hard to work out the following.

Version 2. Two elliptic curves are isomorphic iff they have the same \(j\)-invariant.

We can restate this as follows:

Version 3. The moduli space of elliptic curves is the complex line \(\mathcal{M}_1 \cong \mathbb{C}\) and the value \(j(E)\) of the \(j\)-invariant gives the point in \(\mathcal{M}_1\) that corresponds to \(E\).

The only sensible compactification of \(\mathbb{C}\) is \(\mathbb{C}P^1\), so what corresponds to the point at infinity? This should be a curve where the discriminant of the cubic \(x^3 + ax^2 + bx + c\) vanishes. That is, when \(x^3 + ax^2 + bx + c\) has a multiple root. There are 2 types of such cubics. If there is a triple root we get \(y^2 = x^3\), a cuspidal curve. If there is a double root we get \(y^2 = x^3 + x^2\), a nodal curve. In this case the correct choice is to go with the nodal curve.

10.2 (The main steps of a moduli theory). We hope to do something similar with more general algebraic varieties. We proceed in several steps.

Step 1. Identify a class of projective varieties \(X\) that should have a “good” moduli theory. We aim to prove that such a theory exists for negatively curved varieties as in (6.1). We allow canonical singularities, thus this includes canonical models of varieties of general type. (It seems that in most other cases there is no “good” compactified moduli theory, unless some additional structure is added on, for instance an ample divisor as in [2].)

Step 2. Add some extra data (also called rigidification) first. A typical extra datum is an embedding \(j : X \hookrightarrow \mathbb{P}^N\) for some \(N\). Use the additional data to get a moduli space with a universal family
\[ U_{X,j} \subset \mathbb{P}^N \times \mathcal{M}_{X,j} \]
with projection \(\pi_{X,j} : U_{X,j} \to \mathcal{M}_{X,j}\) such that every pair \((X, j)\) occurs exactly once among the fibers of \(\pi_{X,j}\). (It is not easy to show that one can choose a fixed \(N\) that works for all varieties in a given class. For smooth varieties this was proved by Matsusaka in 1972; the general case was settled recently by Hacon and Xu.)

Step 3. Next we get rid of the extra data. Usually we have to take a quotient by a Lie group like \(GL(N + 1, \mathbb{C})\). This can be hard but, if everything works out, at the end we have
\[ U_X := U_{X,j}/GL(N + 1, \mathbb{C}), \quad M_X := \mathcal{M}_{X,j}/GL(N + 1, \mathbb{C}) \]
and a morphism \(\pi_X : U_X \to M_X\). (See Step 6 for the possible dependence on \(N\).)
Step 4. In almost all cases, the resulting spaces are not compact and compactifying them in a “good” way is difficult. The key step is to identify the limits of families of varieties in $X$ that should give a “good” compact moduli theory. There is no a priori reason to believe that such a choice exists or that it is unique. Finding the right choice in higher dimension was the last conceptual step in the program. For canonical models of varieties of general type we have the “right” answer, see (10.5) and (10.8).

Step 5. We have to go back and redo Steps 1–3 for this more general class of objects to get a compactified moduli theory $\overline{\pi}_X : \overline{U}_X \to \overline{M}_X$.

Step 6. An extra issue that arises is that the compactifications could also depend on the dimension of the $\mathbb{P}^N$ chosen in Step 2. This does not seem to happen for $\overline{M}_X$ itself (at least for $N$ large enough) but it does happen for $\overline{M}_X$ for some of the proposed variants.

Step 7. Finally, if everything works out, we would like to study the properties of $\overline{M}_X$, $\overline{M}_X$ and to use these to prove further theorems.

Next we review the historical development of the higher dimensional theory.

10.3 (Geometric Invariant Theory: Mumford, 1965). Riemann probably knew that all smooth, compact Riemann surfaces of a given genus $g$ form a nice family, but the moduli spaces $M_g$ were first rigorously constructed by Teichmüller in 1940 as an analytic space and by Mumford in 1965 as an algebraic variety. Mumford’s book [65] presents a program to construct moduli spaces under rather general conditions and uses it to obtain $M_g$. Using these methods, moduli spaces were constructed for surfaces (Gieseker, 1977) and for higher dimensions (Viehweg, 1990).

The correct compactification of these moduli spaces was much less clear. In principle, GIT provides an answer, but the resulting compactification might depend on the embedding dimension chosen in $M_g$ (Step 2). Recently Wang–Xu [79] prove that, for surfaces and in higher dimensions, the GIT compactification does depend on the embedding dimension. (The current examples, however, do not exclude the possibility that some variant of the GIT approach does provide an answer that is independent of the embedding dimension.)

10.4 (Compact moduli of curves: Deligne and Mumford, 1969). The optimal compactification of $M_g$ is constructed in [17]. In the boundary $\overline{M}_g \setminus M_g$ we should allow reducible curves $C = \cup_i C_i$ that satisfy two restrictions.

(Local property) $C$ has only nodes as singularities. In suitable local analytic coordinates these are given by an equation $(xy = 0) \subset \mathbb{C}^2$. As in (9.2) the “standard” volume form on a node is given by $\frac{dx}{x}$ (on the line $(y = 0)$) and by $-\frac{dy}{y}$ (on the line $(x = 0)$). These forms have a simple pole at the singularity, corresponding to the restriction on log canonical singularities in (9.4).

(Global property) Instead of each $c_1(C_i)$ being negative, we assume that each $c_1(C_i) - D_i$ is negative where $D_i$ is the sum of the nodes that lie on $C_i$. (Thus we allow $C_i \cong \mathbb{P}^1$, as long as at least 3 nodes also lie on $C_i$.)

10.5 (Compact moduli of surfaces: Kollár and Shepherd–Barron, 1988). It was clear from the Mumford–Gieseker approach that one should work with the
canonical models of surfaces of general type (as in 4.6) in order to get a good moduli theory, but the correct class of singular limits was not known.

An approach using minimal models was proposed in [53]: given a family of canonical models over a punctured disc $S^* \to \Delta^*$, first construct any compactification whose central fiber is a reduced simple normal crossing divisor and then take the (relative) canonical model. It is not hard to see that this gives a unique limit. This says that at the boundary of the moduli space we should allow stable surfaces: reducible surfaces $S = \bigcup_i S_i$ that satisfy two restrictions.

(Local property) $S$ has so-called semi-log canonical singularities. What are these? First of all, aside from finitely many points $S$ is either smooth or has two local branches meeting transversally, like $(xy = 0) \subset \mathbb{C}^3$. These are the natural generalizations of nodes. Then we can have log canonical singularities (9.4). Finally, it can happen that several $S_i$ come together at a point and each of them has a log-canonical singularity there. An explicit list is given in [53].

(Global property) Instead of each $c_1(S_i)$ being negative, we assume that each $c_1(S_i) - D_i$ is negative where $D_i$ is the sum of the double curves that lie on $S_i$.

Another interesting issue that arises is that not every deformation of such singular surfaces is allowed. It turns out that even basic numerical invariants of a surface can jump if we allow arbitrary deformations. To avoid this, [53] identifies a restricted deformation theory (called QG-condition) that produces the correct boundary.

This answers our second Main Question: First, the “simplest” families of surfaces of general type are families $f : S_M \to M$ whose fibers are stable surfaces (and satisfy the QG-condition). Second, every family of surfaces of general type is birational to such a “simplest” family, at least after a generically finite-to-one change of the base $M$.

The projectivity of the resulting moduli spaces was proved in [36].

10.6 (Moduli of pairs: Alexeev, Kontsevich, 1994). Frequently we are interested in understanding all subvarieties $X$ of a given variety $Y$. All is well if $X$ is smooth, but it is less clear how to handle singular subvarieties. Various methods have been proposed, going back to Cayley in 1860.

Alexeev proposed in [1] that instead of working with very singular subvarieties, one should look at morphisms $X \to Y$ that mimic (10.5); see also [3]. Independently, Kontsevich developed this approach for curves [55]. The latter since became a standard tool in quantum cohomology theory.

10.7 (Quotient theorems: Keel, Kollár, Mori, 1997). Step 3 of (10.2) leads to the general problem of taking the quotient of a variety by a group. In our cases we have the extra information that every point has a finite stabilizer. In the sixties Artin and Seshadri proved several quotient theorems, especially when all stabilizers are trivial. The general results needed for the moduli theory were established in [10, 33]. This is a quite subtle subject since the resulting quotients are so called algebraic spaces, a concept somewhat more general than varieties (or even schemes). Using the ideas of [36] one can then show that, in the cases of interest to us, the quotients are in fact projective (Fujino, Kovács, McKernan).

10.8 (Moduli in higher dimensions). The general theory follows the outlines of (10.6) with some key differences.
First, when [53] was written, minimal models were known to exist only in dimension 3. The higher dimensional theory needs several results that were established only recently [28].

Second, it turned out to be quite difficult to understand how the irreducible components of a reducible variety \( X = \bigcup_i X_i \) glue together. For curves, as in [10.4] the well-defined residue of the 1-form \( dx / x \) is a key ingredient. The current approach in higher dimension relies on a new Poincaré–type residue theory for log canonical singularities; see [17, Chap.4]. The full theory should be written up in [48].

10.9 (Explicit examples: Alexeev, 2002–). While the above methods provide a complete answer in principle, it has been very difficult to work out a full description in concrete cases. The first such examples were Abelian varieties [2] and plane curves (Hacking, [24]). Recent surface examples are in [4].

10.10 (Hyperbolicity: Kovács, Viehweg, Zuo, 2000–2010). So far, very little is known about the moduli spaces of surfaces and higher dimensional varieties in general. The local structure of these spaces can be arbitrarily complicated [75]. Hyperbolicity properties of the moduli of smooth curves were conjectured by Shafarevich in 1962 and later extended to higher dimensions in [56, 76, 34, 58].

10.11 (Degeneration of Fano varieties: Xu, 2007–). We know much less about the moduli of Fano (=positively curved) varieties. Most of the geometric works deal with extending families \( g^* : X^* \to \Delta^* \) over a punctured disc across the puncture. Two questions turned out to be especially interesting: understanding the combinatorial structure of the central fiber \( X_0 \) for arbitrary limits and finding limits where \( X_0 \) is especially simple.

A series of papers [44, 31, 15] shows that the combinatorial structure of \( X_0 \) is contractible; this answers an old conjecture of J. Ax. Recently, [60] shows that there are limits where \( X_0 \) itself is a (singular) Fano variety, as conjectured by Tian.

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