Nielsen complexity of coherent spin state operators

Kunal Pal‡ Kuntal Pal§ and Tapobrata Sarkar¶

Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India
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We calculate Nielsen’s circuit complexity of coherent spin state operators. An expression for the complexity is obtained by using the small angle approximation of the Euler angle parametrisation of a general SO(3) rotation. This is then extended to arbitrary times for systems whose time evolutions are generated by couplings to an external field, as well as non-linearly interacting Hamiltonians. In particular, we show how the Nielsen complexity relates to squeezing parameters of the one-axis twisted Hamiltonians in a transverse field, thus indicating its relation with pairwise entanglement. We further point out the difficulty with this approach for the Lipkin-Meshkov-Glick model, and resolve the problem by computing the complexity in the Tait-Bryan parametrisation.

I. INTRODUCTION

In recent years, the notion of complexity has been popular in the study of quantum systems. Broadly speaking, complexity of a system measures the difficulty in preparing a target state, starting from a given reference state. A standard approach of measuring complexity is the one pioneered by Nielsen [1–3]. In this geometric approach, the difficulty in constructing a unitary operator that relates the target state to the reference is quantified by a cost functional that encodes the information of the reference and target states, respectively. Next, we represent $U(\tau)$ as a path ordered exponential by expanding the Hamiltonian generator in a given basis,

$$U(\tau) = \mathcal{P} \exp \left( \int_{0}^{\tau} d\tau' \sum_{a} Y^{a}(\tau') J_{a} \right),$$

where, for our purposes, $J_{a}$ will denote the generators of the SO(3) group and the coefficients $Y^{a}$ are called the control functions. Now, by differentiating Eq. (1) and using the orthogonality relation between the SO(3) generators, one obtains

$$Y^{a} = \frac{1}{\text{Tr}[J^{a}J_{a}']} \text{Tr} \left( \partial_{\tau} U \cdot U^{-1}(\tau) \cdot (J^{a})' \right),$$

where from now on, repeated indices will imply summation over these, in the usual sense. Then, we will define a length functional which makes the geometric analysis clear [3, 4]

$$C(U) = \int_{0}^{1} d\tau \sqrt{G_{ab} Y^{a}(\tau) Y^{b}(\tau)} = \int_{0}^{1} ds .$$

The minimum value of the functional defined in Eq. 4 gives the required complexity. In order to proceed further, a metric on the space of unitary transformations is obtained from

$$ds^{2} = G_{ab} \left[ \frac{1}{\text{Tr}[J^{a}J_{a}']} \text{Tr} \left( dU(\tau) \cdot U^{-1}(\tau) \cdot (J^{a})' \right) \right] \times \left[ \frac{1}{\text{Tr}[J^{b}J_{b}']} \text{Tr} \left( dU(\tau) \cdot U^{-1}(\tau) \cdot (J^{b})' \right) \right],$$

where for simplicity, we choose $G_{ab} = 4 \delta_{ab}$. Now, this is equivalently written as $ds^{2} = g_{ij} dy_{i} dy_{j}$, with $y_{i}$ denoting a set of coordinates on the space of unitaries. Then, the length functional can be written as

$$C(U) = \int_{0}^{1} d\tau \sqrt{g_{ij} \frac{dy_{i}(\tau)}{d\tau} \frac{dy_{j}(\tau)}{d\tau}},$$

and minimising this amounts to finding the geodesics on the space of unitary transformations. Here, the boundary condition that encodes the information of the reference (target) state is at $\tau = 0$ ($\tau = 1$). On geodesics, the quantity $K = \sqrt{g_{ij} \frac{dy_{i}(\tau)}{d\tau} \frac{dy_{j}(\tau)}{d\tau}}$ is a constant, and $K$, computed with appropriate boundary conditions at $\tau = 0$ and $\tau = 1$ is then the measure of the Nielsen complexity $C(U)$, as follows from Eq. 5.

This approach has been immensely popular of late in the context of the holographic principle, where the gauge-gravity duality relates strongly coupled quantum field theories to weakly coupled gravity. Complexity in spin systems have also been well studied in the literature. In
particular, \cite{8,9} studied Nielsen complexity in the context of quantum phase transitions in the Kitaev model and the transverse field XY model, respectively. These works indicate that circuit complexity is a strong indicator of such phase transitions, i.e., it becomes non-analytic at critical lines (see also \cite{10}). The work of \cite{9} also explored a different definition of complexity, namely the Fubini-Study complexity \cite{11}, arising out of the definition of the quantum information metric \cite{12}, and established that this also shows special properties across a quantum phase transition.

What is relatively unexplored during the current flurry of activities in the study of complexity, is its behaviour in statistical and spin systems away from phase transitions. Indeed, such limits often show interesting behaviour and are useful in understanding many of the founding principles of quantum mechanics. These are important in their own right, and might throw light on the foundational issues of quantum mechanics. Fortunately, several methods developed in the context of the holographic principle give us a precise mathematical formalism to tackle such problems.

In this spirit, in the present work we study Nielsen complexity of operators corresponding to coherent spin states (CSS) \cite{13,14}. We note here that circuit complexity associated with coherent states have been studied recently in the literature. In \cite{8}, the authors calculated the complexity of bosonic coherent states, in \cite{15} the complexity between two coherent states was studied (see also \cite{16}), and the complexity of generalised coherent states was obtained in \cite{17} using the covariance matrix method of calculating circuit complexity developed in \cite{20}. Coherent states are one of the most fundamental objects in quantum physics, and the closest cousins to classical states since the standard harmonic oscillator coherent states as well as the CSS saturate the corresponding uncertainty bounds. These have now been studied for many decades, and have found many applications. For example, bosonic squeezed states, which are obtained by acting a non-linear operator on a bosonic coherent state is a widely studied subject in quantum optics. Similarly, squeezed spin states are also extremely well studied in the literature and are known to give rise to pairwise entanglement \cite{21} (for further details, see the review \cite{22}).

Here, we study circuit complexity of the infinitesimal form of coherent spin state operators. We first review the basic details in the next section, and show that the Euler angle representation provides a convenient way to compute the complexity. Next, in section \textbf{III} we present our main formula for the complexity of these states, followed by a generalisation to arbitrary times in section \textbf{IV}. There, we apply the formalism to models well studied, both theoretically and experimentally in the context of spin squeezing, and explicitly show how complexity is related to the squeezing parameter of the initial state. Next, in section \textbf{V} we argue that the Euler angle representation is not always useful, and compute the complexity in the Tait-Bryan parametrisation and exemplify this in the context of the Lipkin-Meshkov-Glick model.

\section{The Coherent Spin States}

The CSS (or Bloch state) can be constructed by applying the unitary operator (which we will call $D^j$) on a normalised Hilbert space \cite{13,14}. We want to calculate the circuit complexity of creating the operator $D^j$, starting from the unity operator. This is related to the circuit complexity of the CSS (here the target state) starting from a state of the Hilbert space (the reference state) in the following way. As we want to construct the state closest to the classical one, we take the reference state to be lowest or the highest weight state. Then we have to calculate the complexity associated with all the unitary operators that connect these two states and the minimum value among these is the circuit complexity of the CSS \cite{22,25}.

We start by considering the angular momentum operator in three dimensional space, collectively denoted by the vector $\mathbf{J} = \{ J_x, J_y, J_z \}$. They satisfy the usual commutation relations $[J_i, J_j] = i \hbar \delta_{ij}$ with $i, j, k = x, y, z$. The operators $J_\pm = J_x \pm i J_y$ and $J_3$ satisfy the Lie algebra of the group $SO(3)$. Thus if we denote a basis element of the Hilbert space as $|j, m \rangle$, with $j$ being the eigenvalue of the Casimir operator $\mathbf{J}$ and $m$ the eigenvalue of $J_3$, then the result of acting the operators $J_\pm$ and $J_3$ on them is the same that of the action of a unitary irreducible representation of the three dimensional rotation group for an infinitesimal rotation. Now since a general rotation in three dimensional space can be represented by the axis angle representation, as well as in terms of the Euler angles \cite{26}, we can write the operator $D^j$ in either of the two representations. However as we shall see, Euler angles are easier to use in the calculation of circuit complexity, because in terms of Euler angles a general rotation can be written in terms of three rotations about two fixed axes, so that the matrix form of a general unitary operator $U$ associated with the rotation group is simpler. Below we shall write the matrix form of operator $D^j$ and the associated CSS in both representations.

\subsection{The axis-angle representation}

In the axis-angle representation, a general rotation in three dimensional space is specified by the unit vector $\mathbf{n}$ along the axis about which the rotation is performed, and an angle $\theta$ which gives the amount of rotation around $\mathbf{n}$. The unitary operator associated with this rotation is \( D(R(\mathbf{n}, \theta)) = \exp \left[-i\theta \mathbf{n} \cdot \mathbf{J} \right] \). To construct a CSS in this representation, let us consider a unit vector $\mathbf{n}_\phi$ in the $xy$ plane, making an angle $\phi$ ($0 \leq \phi \leq 2\pi$) with the $y$ axis, so that its components are $\mathbf{n}_\phi = \{-\sin\phi, \cos\phi, 0\}$. We are interested in the particular rotation about this axis, which brings the unit vector along the $z$ direction ($z =
(0, 0, 1)) to an arbitrary unit vector \( r(\theta, \phi) \) for our three dimensional space. The corresponding unitary operator is given by

\[
D^j(R_\tau) = \exp \left[ -i\theta(-\sin \phi J_z + \cos \phi J_y) \right] = \exp \left[ i\xi_+ - i\xi_- \right],
\]

where the complex number \( \xi \) is defined by \( \xi = -i\theta \exp \left[ -i\phi \right] \), and an overhead bar denotes its complex conjugate. Now consider an arbitrary normalised state \( |\Psi_0\rangle = \sum_{m=-j}^j c_m |j, m\rangle \) with \( \sum_{m=-j}^j |c_m|^2 = 1 \). The CSS associated with \( |\Psi_0\rangle \) is obtained by acting it with the unitary operator \( D^j(R_\tau) \) i.e. \( |r\rangle = |\theta, \phi\rangle = D^j(R_\tau)|\Psi_0\rangle \).

To construct a generalised coherent state, we can act the operator \( D^j(R_\tau) \) on any arbitrary (normalised) linear combination of the base states. However, as is well known \[13\], the CSS which has properties closest to the classical case is obtained when one chooses \( |\Psi_0\rangle \) to be the highest weight state \( |j, j\rangle \) or the lowest weight state \( |j, -j\rangle \). The standard expressions for the CSS in this case can be written in terms of the angles \( \theta, \phi \) as a linear combination of states \( |j, m\rangle \), and is given by \[14,16\]

\[
|r\rangle = \sum_{m=-j}^j \sqrt{\frac{2j}{j+m}} \left( \cos \frac{\theta}{2} \right)^{j+m} \left( \sin \frac{\theta}{2} \right)^{j-m} \exp \left[ i(j-m)\phi \right] |j, m\rangle.
\]

As mentioned above, the circuit complexity of preparing the target state \( |r\rangle \) starting from the reference state \( |j, j\rangle \) (or \( |j, -j\rangle \)) is related to the complexity of obtaining the unitary operator \( D^j(R_\tau) \) starting from the identity operator \( I \). To calculate the latter in Nielsen’s geometric approach, we need a matrix representation of the linearised version of \( D^j(R_\tau) \), written in terms of a suitable basis.

To this end, note that we can write \[14,16\] \( D^j(R_\tau) = \exp \left[ -i\eta J_+ \right] \exp \left[ \ln \left( 1 + |\eta|^2 \right) J_x \right] \exp \left[ i\eta J_- \right] \) where \( \eta = \tan \frac{\theta}{2} \exp \left[ i\phi \right] \). To down the matrix form of \( D^j(R_\tau) \), we use the standard \( 3 \times 3 \) matrix representation of the generators \( J_i \) of rotation \[20\] to get

\[
D^j(R_\eta) = \begin{pmatrix}
1 & -if(\eta) & \bar{\eta} + \eta \\
if(\eta) & 1 & i(\eta - \bar{\eta}) \\
-\bar{\eta} + \eta & i(\eta - \bar{\eta}) & 1
\end{pmatrix},
\]

where \( f(\eta) = \ln \left( 1 + |\eta|^2 \right) \).

### B. The Euler angles

In terms of the three Euler angles \( (\alpha, \beta, \gamma) \), the general rotation operator \( R(\alpha, \beta, \gamma) \) is given by the product of three rotations performed in the order \[24,27\]

\[
R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma).
\]

In the product, all the rotations are performed with respect to the fixed \( z, y \) and \( z \) axes respectively. As before, in the construction of the CSS, we are interested in a particular rotation, namely the one which brings the unit vector \( z \) along the \( z \) axis to the position of the unit vector \( r(\theta, \phi) \) in spherical polar coordinates. The required rotation is given by the operator \( R(\theta, \phi, 0) = R_z(\phi)R_y(\theta) \). Using the same matrix form of the generators as the ones used in Eq. \[8\], we now have the following simplified matrix form of this operator, up to first order in rotation angles,

\[
D^j(R_\tau(\theta, \phi)) = \begin{pmatrix}
1 & -\phi & \theta \\
\phi & 1 & 0 \\
-\theta & 0 & 1
\end{pmatrix}.
\]

Comparing with Eq. \[8\], this form is easier to use. So is the form of the general unitary operator which acts on the states of the Hilbert space and is given in a standard fashion by \( U(\alpha, \beta, \gamma) = \exp \left( -i\alpha J_z \right) \exp \left( -i\beta J_y \right) \exp \left( -i\gamma J_z \right) \). For infinitesimal rotations, the matrix form of this operator can be calculated to be

\[
U(\alpha, \beta, \gamma) = \begin{pmatrix}
1 & -\alpha + \gamma & \beta \\
\alpha + \gamma & 1 & 0 \\
-\beta & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & -\gamma & \beta \\
\gamma & 1 & 0 \\
-\beta & 0 & 1
\end{pmatrix},
\]

where, in the last line, we have redefined the angle \( \gamma \). As can be seen from the last equation, in the infinitesimal version of the general rotation, only the combination of two rotations (by angles \( \alpha \) and \( \gamma \)) around the fixed \( z \) axis appears, so that the redefinition in the last expression does not affect the calculations below.

Before ending this section, we note that here we have used the infinitesimal form of the rotation operator, and this will be used to find the metric on the space of unitaries. Since the metric is a local quantity, it indeed suffices to work with this infinitesimal form. We will keep this in mind in our analysis below.

### III. CIRCUIT COMPLEXITY IN THE EULER ANGLE REPRESENTATION

To calculate the circuit complexity of forming the operator \( D^j(R_\tau(\theta, \phi)) \) given in Eq. \[4\] starting from the identity operator \( I \), we need to find out the geodesic connecting these two unitary operators in the space of unitary operators \( U \). The minimum length of such a geodesic is the required circuit complexity. As mentioned in the introduction, this procedure of calculating the circuit complexity using Nielsen’s geometric approach is by now standard, and more details can be found in \[14,16,28\]. We shall only outline the procedure briefly. To describe a geodesic trajectory on the space of unitary operators, we use the parameter \( \tau \) along the trajectory, with the parameter is chosen in such a way that at the starting point of the geodesic we have \( \tau = 0 \) and at the ending \( \tau = 1 \). Hence the boundary conditions at the two points are respectively

\[
U(\tau = 0) = I; \quad U(\tau = 1) = D^j(R_\tau(\theta, \phi)).
\]
Comparing these boundary conditions with the infinitesimal form of unitary operator of the rotation group in Eq. (10), we can express them in terms of $\beta, \gamma$ at the two points to be

$$
\beta(\tau = 0) = 0 , \beta(\tau = 1) = \theta , \gamma(\tau = 1) = \phi . \quad (12)
$$

After setting the boundary conditions, the next step is to find out the metric on the space of unitaries, given by Eq. (4). Writing $y_i = \{ \beta, \gamma \}$, and taking $G_{ab} = 4\delta_{ab}$ for convenience, the line element $ds^2 = g_{ij} dy^i dy^j$ is given by

$$
ds^2 = \frac{1}{(1 + \beta^2 + \gamma^2)^2} \left[ (\gamma^4 + \gamma^2 (\beta^2 + 5) + 4) d\beta^2 + (\beta^4 + \beta^2 (\gamma^2 + 5) + 4) d\gamma^2 - 2\beta \gamma (\beta^2 + \gamma^2 + 5) d\beta d\gamma \right].
$$

This form of the metric is non-diagonal, and therefore cumbersome to deal with. However, there is a hidden symmetry here. To see this, we introduce two new coordinates $\{ \rho, \Theta \}$ related to $\{ \beta, \gamma \}$ by

$$
\beta(\rho, \Theta) = \rho \sin \Theta , \quad \gamma(\rho, \Theta) = \rho \cos \Theta . \quad (14)
$$

Substituting $d\beta(\rho, \Theta)$ and $d\gamma(\rho, \Theta)$ in Eq. (13), we have the following simplified diagonal form of the metric,

$$
ds^2 = \frac{4}{(1 + \rho^2)^2} d\rho^2 + \frac{\rho^2 (4 + \rho^2)}{(1 + \rho^2)} d\Theta^2 . \quad (15)
$$

In terms of our new coordinates, the metric has not only become diagonal, but $\Theta$ has also become cyclic, indicating the hidden symmetry. This is indeed a big advantage, as we will see. Before proceeding, we need to write the boundary conditions in Eq. (12) in the $\{ \rho, \Theta \}$ coordinates. These are given by

$$
\rho(\tau = 0) = 0 , \quad \rho(\tau = 1) = \sqrt{\theta^2 + \phi^2} , \\
\Theta(\tau = 1) = \arctan \left[ \frac{\theta}{\phi} \right] , \quad (16)
$$

with the value of $\Theta(\tau = 0)$ being indeterminate.

To calculate the circuit complexity, we minimise the complexity functional in Eq. (4) with the coordinates $y_i = \{ \rho, \Theta \}$. This is done by solving the geodesic equations subject to boundary conditions in Eq. (10), in the space of unitaries equipped with the metric of Eq. (13). The geodesic equations are equivalent to the ones obtained from the Lagrangian $\mathcal{L} = g_{ij} \dot{y}^i \dot{y}^j$, where an overdot indicates a derivative with respect to $\tau$. We find that these are given by

$$
\ddot{\rho} - \frac{2 \rho \rho'^2}{\rho^2 + 1} - \frac{1}{4} \rho (\theta^4 + 2 \rho^2 + 4) \dot{\Theta}^2 = 0 , \\
\ddot{\Theta} + \frac{2 (\rho^4 + 2 \rho^2 + 4)}{\rho (\rho^2 + 1) (\rho^2 + 4)} \dot{\rho} \dot{\Theta} = 0 . \quad (17)
$$

From Eq. (17), we see that $\Theta = \text{constant}$ are geodesics. Then, from the first equation in Eq. (17), we obtain with $g_{ij} \dot{y}^i \dot{y}^j = K^2$,

$$
\rho = \frac{K}{2} (1 + \rho^2) , \quad (18)
$$
a conclusion that is also reached from the definition of $K^2$, along with the fact that $\Theta$ is a constant. We then have the solutions,

$$
\Theta = \Theta_0 , \quad \rho(\tau) = \tan \left[ \frac{K \tau}{2} - C_1 \right] , \quad (19)
$$

where $\Theta_0$ and $C_1$ are two constants that have to be fixed from boundary conditions. Note that we must have $\Theta_0 \neq 0$, as otherwise Eq. (13) would imply that $\beta = 0$ for all values of $\tau$, a condition that is clearly incompatible with that at $\tau = 1$ in Eq. (12).

Now, translating back to the $\{ \beta, \gamma \}$ coordinates, we have

$$
\beta = \tan \left[ \frac{K \tau}{2} - C_1 \right] \sin \Theta_0 , \quad \gamma = \tan \left[ \frac{K \tau}{2} - C_1 \right] \cos \Theta_0 . \quad (20)
$$

Eq. (12) then determines that the constants $C_1 = n\pi$ and $\Theta_0 = \arctan(\theta/\phi)$. Setting $\tau = 1$, either of the two relations in Eq. (20) then gives the complexity

$$
C \left( D^i \left( R_{\tau}(\theta, \phi) \right) \right) = K \left[ 2 \left( \arctan \sqrt{\theta^2 + \phi^2} + n\pi \right) \right] . \quad (21)
$$

The metric in Eq. (13) and the above expression for the circuit complexity is valid for small values of rotation angles $\theta, \phi$ such that we can neglect the $O(\theta \phi)$ and higher order terms in the operator $D_i \left( R_{\tau}(\theta, \phi) \right)$ and the matrix form of the operators given in Eqs. (9) and (10) provide good approximation to the exact expressions.

As a simple application of Eq. (21), we can compute the complexity of the Dicke model coherent states, which can be written as the tensor product of the coherent state of the harmonic oscillator and the coherent state of the annihilation operator $a \alpha = a \alpha$. The state $\alpha$ can be obtained from the unit mass harmonic oscillator ground state by applying the displacement operator $D(\alpha)$ i.e.

$$
|\alpha, r \rangle = |\alpha \rangle \otimes | r \rangle . \quad (22)
$$

Here $\alpha$ is a complex number which is the eigen value of the annihilation operator $a |\alpha \rangle = a |\alpha \rangle$. The state $\alpha$ can be obtained from the unit mass harmonic oscillator ground state by applying the displacement operator $D(\alpha)$ i.e.

$$
|\alpha \rangle = D(\alpha) |0 \rangle , \quad D(\alpha) = \exp \left[ iv\sqrt{2} \left( \sqrt{\omega} a Q - \frac{\alpha r P}{\sqrt{\omega}} \right) \right] , \quad (23)
$$

where $Q$ and $P$ are the operator form of the position and momenta respectively, and $\alpha_r, \alpha_i$ are the real and imaginary parts of $\alpha$. The complexity of creating the Dicke model CS $(|\alpha, r \rangle)$ from the product state $(|0 \rangle \otimes | j, -j \rangle)$ can be viewed as the sum of complexities of creating the displacement operator $D(\alpha)$ and the rotation operator $D^i \left( R_{\tau} \right)$ from the identity (see [23, 24]). It follows
straightforwardly that
\[
C_{\text{Dicke}} = C(D(\alpha)) + C(D_j(R_\tau)) = \sqrt{2 \left( \frac{\alpha^2}{\omega} + \omega \alpha^2 \right) + 2 \left( \arctan \left[ \sqrt{\theta^2 + \phi^2} \right] + n\pi \right)},
\]
where \( f(\theta, \phi, t) \) and \( g(\theta, \phi, t) \) are two functions whose forms depend on the Hamiltonian of the system. The modified boundary conditions at the starting and the end point of the geodesic now read
\[
\begin{align*}
\beta, \gamma (\tau = 0) &= 0, \quad \beta (\tau = 1) = g(\theta, \phi, t), \\
\gamma (\tau = 1) &= f(\theta, \phi, t).
\end{align*}
\]

The calculation of the circuit complexity is the same as before, and we will write down the final expression only,
\[
C(\theta, \phi, t) = 2 \arctan \left[ \sqrt{f(\theta, \phi, t)^2 + g(\theta, \phi, t)^2} \right] + 2n\pi.
\]

We shall now consider three different systems having Hamiltonian with increasing intricacy, and we will see the time evolved operator \( D^j(\theta, \phi, t) \) for each of these systems will fall into a distinct category depending upon the spin components present. Namely, in the standard Euler angle representation of Eq. (10), rotations about only two fixed axes appear (here \( x \) and \( z \) axes). In the first two examples below, at time \( t \), only rotations about these two axes appear with the difference between them is that in the first case, rotation about the third axis (here \( x \)) is absent in any order of rotation angle while in the second case the third axis rotation is absent only in the lowest order of the rotation angles. In the third case however, in \( D^j(\theta, \phi, t) \) a rotation about the \( x \) axis is present even in the lowest order in \( \theta, \phi \), and hence to obtain an analytic expression for the complexity, we have to switch to the alternative Tait-Bryan parametrisation for the rotation matrix.

\section*{IV. TIME EVOLUTION OF CIRCUIT COMPLEXITY}

So far we have computed the circuit complexity associated with CSS at a fixed time. It is more interesting to compute this in the case where the system evolves in time, with the time evolution being governed by the Hamiltonian operator \( H \). The procedure of calculating the complexity remains the same as in previous section, the only thing that is different is the expressions of the matrix \( D^j(R_\tau(\theta, \phi)) \) (see Eq. (9)), which now becomes a function of time as well, and subsequently the boundary conditions of Eq. (12) are also changed.

At an arbitrary time \( t \), we have
\[
D^j(\theta, \phi, t) = \begin{pmatrix}
1 & -f(\theta, \phi, t) & g(\theta, \phi, t) \\
f(\theta, \phi, t) & 1 & 0 \\
-g(\theta, \phi, t) & 0 & 1
\end{pmatrix},
\]
where \( f(\theta, \phi, t), g(\theta, \phi, t) \) are two functions whose forms depend on the Hamiltonian of the system. The modified boundary conditions at the starting and the end point of the geodesic now read
\[
\begin{align*}
\beta, \gamma (\tau = 0) &= 0, \quad \beta (\tau = 1) = g(\theta, \phi, t), \\
\gamma (\tau = 1) &= f(\theta, \phi, t).
\end{align*}
\]

The calculation of the circuit complexity is the same as before, and we will write down the final expression only,
\[
C(\theta, \phi, t) = 2 \arctan \left[ \sqrt{f(\theta, \phi, t)^2 + g(\theta, \phi, t)^2} \right] + 2n\pi.
\]

This result follows from standard textbook material with the only non trivial point being the transformation between the Euler angle and the axis angle representations. The necessary details are summarised in Appendix A. For given non-zero values of \( \theta \) and \( \phi \), the complexity thus monotonically increases and saturates to a maximum value of \( \pi \) at large times.

\section*{A. Class-1 : Spin magnet interaction}

First we consider the simple case of a spin \( S \) interacting with a constant external magnetic field \( B \) along the \( z \) direction via an interaction Hamiltonian given by \( -S \cdot B = -S_z B \). In this case, it is straightforwardly shown that at any time \( t \) of the evolution, the CSS operator remains coherent (which is not true for two other examples we consider below) and hence the rotation operator at \( t \) can be obtained by replacing \( \phi \) by \( \phi + Bt \), in the final rotation about the \( z \) axis. Now taking the linear form of this operator for small values of rotation angles, we find that the complexity at time \( t \) is
\[
C(\theta, \phi, t) = 2 \arctan \left[ \sqrt{\theta^2 + (\phi + Bt)^2} \right] + 2n\pi.
\]

\section*{B. Class-2 : Non-linear interactions}

Another interesting application of the results derived so far can be envisaged via non-linear interactions. We consider a collection of \( N \) spins and assume that there is a nonlinear interaction present between the individual spins. We also assume that this nonlinear interaction term is of the form \( J_i^2 \) where \( i = x, y, z \). The resulting model is known in the literature as the one axis twisting model. Specifically, we are interested in is the one axis counter twisting model in the presence of a transverse field, and the Hamiltonian reads
\[
H_{\text{non}} = 2\delta J_z^2 + \Omega J_x,
\]
with the first term being the nonlinear interaction between the spins, and the second represents an interaction

\footnote{If the nonlinear interaction term between the spins is of the form \( J_i J_j + J_j J_i \) with \( i \neq j \) and \( J_i \) denoting \( i \)th component of collective angular momentum operator, then the model is known as two axis counter twisting model. Here we shall consider only the one axis twisting model.}
with a transverse external field of frequency $\Omega$ \[^{31}\] (see also \[^{32}\]). As before, our aim is to find out the operator $D^{J}(\theta, \phi, t)$ at a time $t$. For this, we need the expressions of the operators $J_z(t)$ and $J_y(t)$.

We start with the Heisenberg equation of motion for the spin operators, given respectively by $\[^{31}\]$:

$$
\frac{dJ_z(t)}{dt} = -4\delta J_y(t)J_z(t),
$$

$$
\frac{dJ_y(t)}{dt} = -\Omega J_z(t) + 4\delta J_x(t)J_z(t),
$$

$$
\frac{dJ_z(t)}{dt} = \Omega J_y(t).
$$

Here $J_{j}(J_{j}) = \frac{1}{2}[J_{j}, J_{j} + J_{j}, J_{j}]$ indicates symmetrization of the indices. The general solutions of these coupled first order equations are difficult to obtain, and they are usually solved by employing an approximation scheme. Here we shall use the so called frozen spin approximation, which is justified when the condition $\Omega >> \delta$ is satisfied.

In that case, the force of the external field is much greater than the nonlinear interaction strength, and hence $J_z(t)$ remains fixed at its value at $t = 0$ during the entire evolution. This value of conveniently fixed to be $-J$ \[^{31}\]. With the frozen spin approximation, the solution to Eq. \[^{30}\] is given as (with the notation $J_{i} = J_{i}(t = 0)$):

$$
J_z(t) \approx J_z \cos[\omega_0 t] + \frac{\Omega J_x}{\omega_0} \sin[\omega_0 t],
$$

$$
J_y(t) \approx -\frac{\omega_0 J_z}{\Omega} \sin[\omega_0 t] + J_y \cos[\omega_0 t],
$$

where $\omega_0 = \sqrt{\Omega^2 + 4\delta} \Omega$ is known as frozen spin frequency. Hence at time $t$, we have

$$
D^{J}(\theta, \phi, t) = e^{iH_{ss}t}D^{J}(\theta, \phi, t = 0)e^{-iH_{ss}t},
$$

$$
= \exp\left(-i\phi J_z(t)\right) \exp\left(-i\theta J_y(t)\right).
$$

Now we substitute the solutions of Eq. \[^{31}\] to get

$$
D^{J}(\theta, \phi, t) = 1 - iJ_z \left(\phi \cos[\omega_0 t] - \frac{\theta \omega_0}{\Omega} \sin[\omega_0 t]\right) - iJ_y \left(\frac{\phi \Omega}{\omega_0} \sin[\omega_0 t] + \theta \cos[\omega_0 t]\right) + O(\theta \phi).
$$

Comparing with the previous example of the class-1 model, here no term proportional to $J_z$ appears in Eq. \[^{33}\]. The reason for this is that in the previous case, the external field was applied along $z$ direction, but in contrast here it is along the $x$ direction. Since we are working in the frozen spin approximation, the time variation of $J_x$ is fixed by the external field. Had we applied the field in $z$ or $y$ direction, such component would arise, as we shall soon see in the context of Lipkin-Meshkov-Glick model.

This expression for the operator $D^{J}(\theta, \phi, t)$ in Eq. \[^{33}\] is a periodic function of time, and hence the complexity of creating such an operator from the identity also varies periodically with time. This is in contrast to the spin-magnetic interaction that we studied before. As mentioned above, the reason for such behaviour has also to do with the direction of the applied field. As before, substituting the matrix form of the operator, and comparing with the general matrix operator of Eq. \[^{25}\], we get the unknown functions in this case to be

$$
f(\theta, \phi, t) = \left(\phi \cos[\omega_0 t] - \frac{\theta \omega_0}{\Omega} \sin[\omega_0 t]\right),
$$

$$
g(\theta, \phi, t) = \left(\frac{\phi \Omega}{\omega_0} \sin[\omega_0 t] + \theta \cos[\omega_0 t]\right).
$$

Thus the circuit complexity of the CSS is given by the expression

$$
C(\theta, \phi, t) = 2 \arctan\left[\sqrt{\left(\phi \cos[\omega_0 t] - \frac{\theta \omega_0}{\Omega} \sin[\omega_0 t]\right)^2 + \left(\frac{\phi \Omega}{\omega_0} \sin[\omega_0 t] + \theta \cos[\omega_0 t]\right)^2}\right] + 2n\pi
$$

Since we are working with the frozen spin approximation, if we take the limit $\delta \to 0$, i.e., $\omega_0 \to \Omega$ in this equation, we get back the time independent expression for the complexity of Eq. \[^{24}\], as expected.

Having derived the expression for the circuit complexity, it is interesting to note that this can be written in terms of the squeezing parameters of the single axis twisting model. We first consider the unit vector $n$ having component $n_i$ with respect to a set of mutually orthogonal unit vectors. Then, for a many particle system which has the collective spin components $J_n = n \cdot J$, we will, following \[^{32}\] \[^{22}\], define the squeezing parameter along a direction $n_1$ as

$$
\xi_{n_1}^2 = \frac{N \langle (\Delta J_{n_1})^2 \rangle}{\langle J_{n_2} \rangle^2 + \langle J_{n_3} \rangle^2},
$$

and a state is said to be spin squeezed along the direction $n$, if $\xi_{n_i}^2 < 1$ for that state. It is well known that the applied external field increases squeezing compared to the zero external field case, and that this squeezing can be maintained for a longer period of time as well \[^{51}\].

Now we assume that at $t = 0$ the single axis twisting model is prepared in an eigenstate of the operator $J_z$. Then we can calculate the variance of the spin components in the frozen spin approximation

$$
\langle (\Delta J_y(t))^2 \rangle = \frac{J}{2} \left(\cos^2[\omega_0 t] + \frac{\omega_0^2}{\Omega^2} \sin^2[\omega_0 t]\right),
$$

$$
\langle (\Delta J_z(t))^2 \rangle = \frac{J}{2} \left(\cos^2[\omega_0 t] + \frac{\Omega^2}{\omega_0^2} \sin^2[\omega_0 t]\right).
$$

We also calculate the correlation between the operators
\( J_z(t) \) and \( J_y(t) \), which is given by

\[
\langle J_z(t)J_y(t) + J_y(t)J_z(t) \rangle = J \cos \left[ \omega_0 t \right] \sin \left[ \omega_0 t \right] \left( \frac{\Omega}{\omega_0} - \omega_0 \right) \Omega. \tag{38}
\]

When the nonlinear interaction between the spins is zero, i.e., \( \delta = 0 \) we have \( \omega_0 = \Omega \) and hence this correlation function vanishes as expected. Using Eqs. \((37)\) and \((38)\), we can write the final expression for the complexity obtained in Eq. \((35)\) as

\[
C(\theta, \phi, t) = 2 \arctan \left[ \frac{2}{J} \left( \phi^2 \langle (\Delta J_y(t))^2 \rangle + \phi^2 \langle (\Delta J_z(t))^2 \rangle + \theta \phi \langle J_z(t)J_y(t) + J_y(t)J_z(t) \rangle \right) \right] + 2n\pi. \tag{39}
\]

In terms of the squeezing parameters introduced earlier, we have

\[
C(\theta, \phi, t) = 2 \arctan \left[ \sqrt{\frac{\theta^2 \xi_n^2 + \phi^2 \xi_n^2 + \frac{2\theta \phi}{J} \langle J_z(t)J_y(t) + J_y(t)J_z(t) \rangle} \right] + 2n\pi. \tag{40}
\]

To glean insight into Eq. \((40)\), note that if we take \( \theta = 0 \) or \( \phi = 0 \) as an example, the tangent of the complexity is proportional to the squeezing parameter. With say \( \theta = 0 \), \( n = 0 \) and for small values of \( \phi \), we have \( C \sim 2\phi \xi_n \). Now, \( \xi_n^2 \) also is a measure of pairwise entanglement, with \( \xi_n^2 < 1 \) implying that the many-body density matrix is not separable \(^{32}\). Here we see that the operator complexity is proportional to the entanglement measure of the system, for some specific choices of the rotation angles. Before closing this section, we point out that we can also write down the expression for the complexity in terms of the pairwise correlation function \(^{22}\)

\[
G_{1n,2n} = \frac{1}{N-1} \left[ \frac{4}{N^2} \left( N \langle (\Delta J_n(t))^2 \rangle + \langle J_n^2 \rangle \right) - 1 \right]. \tag{41}
\]

Assuming \( \theta = 0 \) we then have

\[
C(\phi, t) = 2 \arctan \left[ \phi \sqrt{(N-1)G_{12,22} + 1} \right]. \tag{42}
\]

V. COMPLEXITY IN THE TAIT-BRYAN PARAMETRISATION

Computation of the Nielsen complexity in the Euler angle parametrisation might not always be useful. For example, in the Lipkin-Meshkov-Glick model (to be elaborated upon shortly), it can be checked that during the time evolution, \( J_y(t) \) not only has a component along \( J_y \) but also along \( J_x \). This causes a problem in writing the rotation operator at an arbitrary time \( t \), because the general unitary rotation matrix of Eq. \((20)\) does not have any \( J_x \) component in the lowest order of rotation angles.

Whereas this problem can still be dealt with by using the Euler angles parametrisation, but the price we pay is that the angles themselves become function of time and satisfy coupled differential equations \(^{33,34}\). The solutions for these equations are difficult to obtain analytically even with simple functional forms of the time dependent frequency.

Here we establish that this problem can be circumvented by using a different parametrisation of the general rotation operator, known in the literature as the Tait-Bryan (TB) angles \(^{35}\). In that case, instead of rotations only along fixed \( y \) and \( z \) axis, a general rotation is written in terms of a single rotation about all of the three fixed axis, i.e., the rotation operator now is given by

\[
U(\alpha, \beta, \gamma) = \exp \left( -i\alpha J_x \right) \exp \left( -i\beta J_y \right) \exp \left( -i\gamma J_z \right). \tag{43}
\]

Note that the last rotation is around the fixed \( x \) axis instead of \( z \). Also, the TB rotations angles are denoted in this section by \( \alpha, \beta, \gamma \) for convenience, and should not be confused with the Euler angles used earlier.

As before the matrix form of the operator \( U(\alpha, \beta, \gamma) \) in a preferred basis of the generators is given by

\[
U(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{pmatrix}. \tag{44}
\]

The general unitary operator now has three independent angles and thus the unitary geometry is three dimensional. The line element of this geometry obtained by methods similar to the ones that we have used before is lengthy and we omit it for brevity, while noting that on the \( \alpha = 0 \) hypersurface, it reduces to the one in Eq. \((13)\). As before, we find that there is a hidden symmetry here, and that the metric can be put into a diagonal form by the writing it in terms of the new coordinates \( (\rho, \Theta, \Phi) \) given in in terms of TB angles by the relations

\[
\alpha(\rho, \Theta, \Phi) = \rho \sin \Phi, \quad \beta(\rho, \Theta, \Phi) = \rho \sin \Theta \cos \Phi, \quad \gamma(\rho, \Theta, \Phi) = \rho \cos \Theta \cos \Phi. \tag{45}
\]

In this new coordinate system, the metric acquires a simple diagonal form

\[
ds^2 = \frac{1}{(1 + \rho^2)^2} \left[ 4d\rho^2 + \rho^2 (\rho^4 + 5\rho^2 + 4) \left( d\Phi^2 + \cos^2 \Phi d\Theta^2 \right) \right]. \tag{46}
\]
Once again the coordinate $\Theta$ is cyclic, making the computations simple.

We want to calculate the circuit complexity of creating the operator of $D^i(\theta, \phi, t)$ from the identity operator. However, the matrix form of this operator, given by

$$D^i(\theta, \phi, t) = \begin{pmatrix} 1 & f_3(\theta, \phi, t) & f_2(\theta, \phi, t) \\ f_3(\theta, \phi, t) & 1 & -f_1(\theta, \phi, t) \\ -f_2(\theta, \phi, t) & f_1(\theta, \phi, t) & 1 \end{pmatrix}$$

now depends on three unknown functions $f_1$, $f_2$, $f_3$, whose form is to be determined by the Hamiltonian. To calculate this complexity we have to solve the geodesic whose form is to be determined by the Hamiltonian. To see, e.g., the authoritative account in [38], we refer the reader to the following separable form

$$\partial \tau = \frac{L(1 + \rho^2)}{2\rho^2(\rho^2 + 4) \cos^2 \Phi}.$$ (49)

and the equations for $\Phi(\tau)$ and $\rho(\tau)$ are now given by

$$\frac{d}{d\tau} \left[ \frac{2\rho^2(\rho^2 + 4) \Phi}{(1 + \rho^2)} \right] - \frac{2\rho^2(\rho^2 + 4) \cos \Phi \sin \Theta \Omega^2}{(1 + \rho^2)} = 0,$$

$$\rho^2 = \frac{K^2(1 + \rho^2)^2}{4} - \frac{L^2(1 + \rho^2)^3}{16\rho^2(\rho^2 + 4) \cos^2 \Phi} - \frac{\rho^2(\rho^2 + 4)(1 + \rho^2)}{4} \Phi^2,$$ (50)

where $K^2 = g_{ij} y^i y^j$. It is difficult to eliminate $\Phi$ from Eq. (50) to obtain a first order equation for $\rho(\tau)$ only. Instead of directly solving these equations, we will use the well known Hamilton-Jacobi method to separate these. For a geodesic $y^i(\tau)$ of the metric $g_{ij}$ the Hamilton-Jacobi (HJ) equation for the Hamilton principal function $S$ is defined by (the method is standard and is routinely used in general relativity, and can be found in many textbooks, see, e.g., the authoritative account in [38])

$$2 \frac{\partial S}{\partial \tau} = g^{ab} \frac{\partial S}{\partial y^a} \frac{\partial S}{\partial y^b}.$$ (51)

For our case the HJ equation is given by

$$2 \frac{\partial S}{\partial \tau} = \frac{(1 + \rho^2)^2}{4} \left( \frac{\partial S}{\partial \rho} \right)^2 + \frac{(1 + \rho^2)}{\rho^2(\rho^2 + 4)} \left( \frac{\partial S}{\partial \Phi} \right)^2 + \frac{(1 + \rho^2)}{\rho^2(\rho^2 + 4) \cos^2 \Phi} \left( \frac{\partial S}{\partial \Theta} \right)^2.$$ (52)

Since we already know that the momentum associated with $\Theta$ and $g_{ab} y^a y^b$ are two constants of motion, we assume that a solution $S$ of the HJ equation can be written in the following separable form

$$S = \frac{1}{2} K^2 \tau + L \Theta + S_\Phi(\Phi) + S_\rho(\rho).$$ (53)

As indicated, $S_\rho(\rho)$ and $S_\Phi(\Phi)$ are two functions of their single arguments. Substituting $S$ in the HJ equation we have

$$K^2 = \frac{(1 + \rho^2)^2}{4} \left( \frac{dS_\rho}{d\rho} \right)^2 + \frac{(1 + \rho^2)}{\rho^2(\rho^2 + 4)} \left( \frac{dS_\Phi}{d\Phi} \right)^2 + \frac{L^2(1 + \rho^2)}{\rho^2(\rho^2 + 4) \cos^2 \Phi}.$$ (54)

After a bit of algebraic manipulation, we can rewrite this as

$$K^2 \rho^2(\rho^2 + 4) - \frac{1}{4} \rho^2(\rho^2 + 4)(\rho^2 + 1) \left( \frac{dS_\rho}{d\rho} \right)^2 = L^2 \sec^2 \Phi + \left( \frac{dS_\Phi}{d\Phi} \right)^2.$$ (55)

Now the left side of this equation is solely a function of $\rho$, and the right side is a function only of $\Phi$. Thus to satisfy this equation, both side must be equal to a constant which we call $\mathcal{M}$.

Now that we have the separated the HJ equation, we can get the first order equations in the coordinates by using the formula

$$p_a = g_{ab} \frac{dy^b}{d\tau} = \frac{\partial S}{\partial y^a},$$ (56)

and we find that these are given by

$$\frac{d\rho}{d\tau} = \frac{1}{2} \sqrt{\frac{(1 + \rho^2)^3}{\rho^2(\rho^2 + 4)} \sqrt{\frac{K^2 \rho^2(\rho^2 + 4)}{(1 + \rho^2)} - \mathcal{M}}},$$ (57)

$$\frac{d\Phi}{d\tau} = \frac{(1 + \rho^2)}{\rho^2(\rho^2 + 4)} \sqrt{\mathcal{M} - L^2 \sec^2 \Phi}.$$ (58)

We need to solve these two equations along with the first order equation for $\Theta$ (Eq. (49)), with the boundary conditions mentioned in Eq. (48). For this we first notice in order to avoid divergences at $\rho = 0$ and hence to be compatible with the boundary conditions, we need to put $L = \mathcal{M} = 0$. Then, we obtain

$$\Theta = \Theta_0, \quad \Phi = \Phi_0, \quad \rho(\tau) = \tan \left[ \frac{K \tau}{2} - C_3 \right],$$ (58)

where $\Theta_0$ and $\Phi_0$ are constants. Note that the solution for $\rho$ is the same as obtained before for the case of ordinary Euler angle parametrisation. However, the expression for $K$ is different from the previous case due to the
presence of the extra coordinate. To see this, we first note that the boundary conditions here imply that

$$\Theta_0 = \arctan \left[ \frac{f_2}{f_3} \right], \quad \Phi_0 = \arctan \left[ \frac{f_1}{\sqrt{f_2^2 + f_3^2}} \right].$$

These then give our final expression for the complexity,

$$K = 2 \arctan \left[ \sqrt{f_1^2 + f_2^2 + f_3^2} \right] + 2n\pi .$$

A. Class-3 : The Lipkin- Meshkov-Glick model with TB parameters

As a concrete example, we consider the Lipkin-Meshkov-Glick (LMG) model which describes the interaction of $N$ spin 1/2 particles in presence of an external magnetic field $\mathbf{H}_{\text{ext}}$. When the magnetic field is applied along the $z$ direction, the Hamiltonian of the LMG model can be written as

$$\mathbf{H}_{\text{LMG}} = -\frac{\lambda}{N} (J^2 \cos \theta + \kappa J_y^2) - BJ_z .$$

Here, as before, $\mathbf{J}_i$ is the total spin operator in the $i$th direction. The parameter $\kappa > 0$ characterises the strength of the interaction between the spins, and $0 \leq \kappa \leq 1$ characterises the anisotropy of this interaction.

The Heisenberg equations of motion are given by

$$\frac{d\mathbf{J}_x(t)}{dt} = -\frac{2\kappa}{N} \mathbf{J}_y(t)\mathbf{J}_z(t) + B\mathbf{J}_y(t) ,$$
$$\frac{d\mathbf{J}_y(t)}{dt} = \frac{2}{N} \mathbf{J}_x(t)\mathbf{J}_z(t) - B\mathbf{J}_x(t) ,$$
$$\frac{d\mathbf{J}_z(t)}{dt} = -\frac{2\lambda(1-\kappa)}{N} \mathbf{J}_y(t)\mathbf{J}_y(t) .$$

Since the general solution of these equations is difficult to obtain, we shall focus here on some simple cases. First consider the case of isotropic LMG model, characterised by $\kappa = 1$. In that case, the Hamiltonian commutes with $\mathbf{J}_z$ and hence is diagonal in the $|J,m\rangle$ representation. The energy eigenvalues are then given by

$$E = -\frac{\lambda}{N} (J(J+1) - m^2) - Bm ,$$

where $J$ now denotes the eigen value of the collective spin operator. This also means that $\mathbf{J}_x(t)$ is time independent, which can be seen from the last relation of Eq. (62).

Since $\mathbf{J}_x(t)$ is time independent, the other two equations of motion are easy to solve, and are given by the periodic functions

$$\mathbf{J}_x(t) = \mathbf{J}_x \cos [\omega_1 t] + \mathbf{J}_y \sin [\omega_1 t] ,$$
$$\mathbf{J}_y(t) = \mathbf{J}_y \cos [\omega_1 t] - \mathbf{J}_x \sin [\omega_1 t] ,$$

with $\omega_1 = B + 1$, where for convenience we have set $\mathbf{J}_z = -N/2$ and as before when the argument of the angular momentum operator is not explicitly written it indicates value at $t = 0$. Now the time evolution of the operator $\mathcal{D}^j (\theta, \phi)$ is obtained to be

$$\mathcal{D}^j (\theta, \phi, t) = \exp \left( -i\phi \mathbf{J}_z \exp \left( -i\theta \{ \mathbf{J}_y \cos [\omega_1 t] - \mathbf{J}_x \sin [\omega_1 t] \} \right) \right) \exp \left( -i\phi \mathbf{J}_y \cos [\omega_1 t] \right) \exp \left( -i\phi \mathbf{J}_y \cos [\omega_1 t] \right) .$$

Here, in the last step, we have employed the transformation from Euler angle to the axis angle representation described in Appendix A. As can be seen, the operator complexity in this case is the same as the spin magnet interaction.

Now we shall consider a more general case, namely the anisotropic LMG model in the frozen spin approximation. As usual in the frozen spin approximation, we assume that under the influence of the external magnetic field, the spin component along that direction i.e., $\mathbf{J}_z$ remains fixed. Once again, we take $\kappa = -N/2$. This approximation is valid when the strength of the external field is much greater than the spin-spin interaction. Then the solutions of the equation of motion are given by (see, e.g., [41])

$$\mathbf{J}_y(t) = \mathbf{J}_y \cos [\omega_B t] - \sqrt{\frac{B+1}{B+\kappa}} \mathbf{J}_x \sin [\omega_B t] ,$$
$$\mathbf{J}_x(t) = \mathbf{J}_x \cos [\omega_B t] + \sqrt{\frac{B+1}{B+\kappa}} \mathbf{J}_y \sin [\omega_B t] ,$$

with $\omega_B = \sqrt{(B+1)(B+\kappa)}$. Notice that these solutions reduce to that of the isotropic case when $\kappa = 1$, because in both the cases, $\mathbf{J}_x(t)$ remains constant. In the isotropic case the solutions become exact.

As before, the operator $\mathcal{D}^j (\theta, \phi, t)$ is given by

$$\mathcal{D}^j (\theta, \phi, t) = \exp \left( -i\phi \mathbf{J}_z \exp \left( -i\theta \{ \mathbf{J}_y \cos [\omega_B t] - \sqrt{\frac{B+1}{B+\kappa}} \mathbf{J}_x \sin [\omega_B t] \} \right) \right) \exp \left( -i\phi \mathbf{J}_y \cos [\omega_B t] \right) .$$

Now a comparison with previous cases (for example Eq. (63)) show that since here $\mathbf{J}_y(t)$ has a nonzero component along $\mathbf{J}_x$ it gives a contribution in the first order of $\theta$. If we work with the unitary operator in the Euler angle representation of previous section then it does not have any rotation about $x$ axis. The same situation arose previously in the case of spin-magnet interaction as well as in the isotropic LMG model discussed previously. However in both the cases, the problem was solved by using the equivalence between the axis-angle and Euler angle representation described in Appendix A. Since the coefficient of $\mathbf{J}_x \sin [\omega_B t]$ is not unity, a little inspection shows that the same trick can not be used here. The advantage of the TB parametrisation used in this section over the standard Euler angles is apparent in this case - a
rotation along x, y or z axis can be incorporated with TB angles not with the Euler ones (at least when the angles themselves are time independent). To calculate the complexity of LMG model at time t we simply write down the matrix form of Eq. (65), and compare it with that of Eq. (17) of the TB parametrisation. We find,

\[ f_1 = -\theta \sqrt{\frac{B+1}{B+\kappa}} \sin [\omega_B t], \quad f_2 = \theta \cos [\omega_B t], \quad f_3 = \phi. \]  

(68)

These can be readily used in the general expression Eq. (60) to obtain the complexity.

VI. CONCLUSIONS

In this paper, we have studied Nielsen’s complexity in the context of coherent spin state operators. We have constructed a simple formula for the complexity for linearised operators that correspond to preparing a target state from a given reference state. This was achieved by constructing the metric on the space of unitaries of the rotation operator by employing the Euler angle representation, and computing analytic solutions of the resulting geodesics with appropriate boundary conditions. We considered three models, with increasing levels of complication. Our first example was the ubiquitous spin-magnetic field coupling. This was followed by the one-axis twisting model where we showed an interesting possibility of relating the complexity with pairwise entanglement. Finally we considered the LMG model, where the usual Euler angle parametrisation was found to be less useful and we resorted to a Tait-Bryan parametrisation.

It is useful to contrast the results presented here with the Fubini-Study complexity (see, e.g. [3]). This is an alternative characterisation of complexity where one works directly with the metric on the parameter manifold of quantum states. In the time-independent case, the parameter manifold of the CSS has been worked out in [12] and corresponds to the two-sphere. The geodesics then are simply great circles on this sphere and are different from the ones on the space of unitaries that we have computed for the CSS operators.

In this paper we gave indication that the Nielsen complexity of the CSS operators might be related to pairwise entanglement. It would be very interesting to quantify this further.

Appendix A

Here we detail some results of the discussion presented in subsection IV A. At time t the expression for SCS of the Hamiltonian \( H_{sm} = -S_zB \) is given by (here \( j \) in Eq. (7) are s, the eigen values of total spin operator \( S \))

\[ |\theta, \phi, t\rangle = \sum_{m=-s}^{s} \frac{1}{2} s + m \left[ \sin \frac{\theta}{2} \right]^{s-m} \times \]

\[ e^{i(s-m)\phi} e^{imBt} |J, m\rangle = e^{-iBst} |\theta, (\phi + Bt)\rangle. \]

(A1)

We can calculate \( D^j(\theta, \phi, t) \) directly in the Heisenberg picture to be

\[ D^j(\theta, \phi, t) = \exp \left( -i\phi S_z \right) \exp \left( -i\theta S_y(t) \right) \]

\[ = \exp \left( -i\phi S_z \right) \exp \left( -i\theta \left( S_y \cos [Bt] - S_x \sin [Bt] \right) \right) \].

(A2)

Note that this form of the operator \( D^j(\theta, \phi, t) \) is the same as the one at \( t = 0 \) with \( \phi \) replaced by \( (\phi + Bt) \). To see this, we compare the second exponential operator in the expression above with the rotation operator used to construct a CSS in the axis-angle representation of eq. (9). Thus the second exponential operator in Eq. (A2) is nothing but the operator which rotates the unit vector \( z \) along the \( z \) direction by an angle \( \theta \) with respect to the unit vector \( n_\phi = (-\sin [Bt], \cos [Bt], 0) \), albeit written in the axis-angle representation. But we know how to represent the same rotation in terms of Euler angles, which is given by the rotation operator \( R(\phi + Bt, \theta, 0) = R_z(\phi + Bt)R_y(\theta) \). Thus we finally have the operator at an arbitrary time to be

\[ D^j(\theta, \phi, t) = \exp \left( -i\phi S_z \right) \exp \left( -iBtS_z \right) \exp \left( -i\theta S_y \right) \]

\[ = \exp \left( -i[\phi + Bt]S_z \right) \exp \left( -i\theta S_y \right) \].

(A3)

Now taking linear form of this operator for small values of the rotation angle, the complexity at an time \( t \) is straightforwardly obtained as Eq. (28).

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