Optimal decay rate of the bipolar Euler-Poisson system with damping in $\mathbb{R}^3$

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Abstract: By rewriting a bipolar Euler-Poisson equations with damping into an Euler equation with damping coupled with an Euler-Poisson equation with damping, and using a new spectral analysis, we obtain the optimal decay results of the solutions in $L^2$-norm, which improve the those in [17, 29]. More precisely, the velocities $u_1, u_2$ decay at the $L^2$-rate $(1 + t)^{-\frac{3}{4}}$, which is faster than the normal $L^2$-rate $(1 + t)^{-\frac{3}{4}}$ for the Heat equation and the Navier-Stokes equations. In addition, the disparity of two densities $\rho_1 - \rho_2$ and the disparity of two velocities $u_1 - u_2$ decay at the $L^2$-rate $(1 + t)^{-\frac{3}{4}}$.

Key Words: Bipolar damped Euler-Poisson system; decay estimates.

1. Introduction

The compressible bipolar Euler-Poisson equations with damping (BEP) takes the following form

$$
\begin{align*}
\partial_t \rho_1 + \text{div}(\rho_1 u_1) &= 0, \\
\partial_t (\rho_1 u_1) + \text{div}(\rho_1 u_1 \otimes u_1) + \nabla P(\rho_1) &= \rho_1 \nabla \phi - \rho_1 u_1, \\
\partial_t \rho_2 + \text{div}(\rho_2 u_2) &= 0, \\
\partial_t (\rho_2 u_2) + \text{div}(\rho_2 u_2 \otimes u_2) + \nabla P(\rho_2) &= -\rho_2 \nabla \phi - \rho_2 u_2, \\
\Delta \phi &= \rho_1 - \rho_2, \quad x \in \mathbb{R}^3, \ t \geq 0.
\end{align*}
$$

(1.1)

Here the unknown functions $\rho_i(x, t), u_i(x, t) (i = 1, 2), \phi(x, t)$ are the charge densities, current densities, velocities and electrostatic potential, respectively, and the pressure $P = P(\rho_i)$ is a smooth function with $P'(\rho_i) > 0$ for $\rho_i > 0$. The system (1.1) is described charged particle fluids, for example, electrons and holes in semiconductor devices, positively and negatively charged ions in a plasma. We refer to [6, 19, 21] for the physical background of the system (1.1).

In this paper, we want to study the system (1.1) with the Cauchy data

$$
\rho_i(x, 0) = \rho_{i0}(x) > 0, \ u_i(x, 0) = u_{i0}(x), \ i = 1, 2.
$$

(1.2)

A lot of important works for the system (1.1) have been done. For one-dimensional case, we refer to Zhou and Li [31], and Tsuge [23] for the unique existence of the stationary solutions, Natalini [20] and Hsiao and Zhang [1] for global entropy weak solutions on the whole real line and bounded domain respectively, Huang and Li [8] for the large-time behavior and quasi-neutral limit of $L^\infty$-solution, Natalini [20], Hsiao and Zhang [10] for the relaxation-time limit, Gasser and Marcat [4] for the combined limit, Zhu and Hattori [32] for the stability of steady-state solutions to a recombined one-dimensional bipolar hydrodynamical model, Gasser, Hsiao and Li [3] for large-time behavior of smooth solutions with small initial data.

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For the multi-dimensional case, Lattanzio [11] studied the relaxation limit, and Li [16] investigated the global smooth solutions of the Cauchy problem in the Sobolev’s space and Besov’s space, respectively. Later, Ju [7] investigated the global existence of smooth solution to the IBVP for the system (1.1).

Recently, by using the standard energy method together with the analysis of the Green’s function, Li and Yang [17] investigated the decay rate of the Cauchy’s problem of the system (1.1) of the classical solution when the initial data are small in the space $H^3 \cap L^1$. They deduced that the electric field (a nonlocal term in hyperbolic-parabolic system) slows down the decay rate of the velocity of the BEP system. More precisely, they obtained $\|\rho_i - \bar{\rho}\|_{L^2} \sim (1 + t)^{-\frac{3}{4}}$, $\|u_i\|_{L^2} \sim (1 + t)^{-\frac{3}{4}}$. We refer to the relevant works on the unipolar Navier-Stokes-Poisson equations (NSP) and unipolar Euler-Poisson equations with damping [13, 14, 30, 25, 27, 28]. The decay result in [17] was improved by us in [29] as $\|u_i\|_{L^2} \sim (1 + t)^{-\frac{3}{4}}$ by using the method introduced by Guo and Wang [5, 26], which is based on a family of energy estimates with minimum derivative counts and interpolations among them without linear decay analysis. The additional assumption on the initial perturbation in [29] is $\nabla\phi_0 \in L^2$, but it need not the smallness of the initial perturbation in $L^1$-norm.

On the other hand, recently, Tan and Wu [22] gave a new and detailed spectral analysis for the Euler equations with damping by taking Hodge decomposition on the velocity. In fact, they deduced the decay rate: $\|\rho - \bar{\rho}\|_{L^2} \sim (1 + t)^{-\frac{3}{4}}$, $\|u\|_{L^2} \sim (1 + t)^{-\frac{3}{4}}$, which improved the former decay rate in Wang and Yang [24].

The purpose of this paper is to improve the $L^2$-norm decay estimates in Li and Yang [17] and Wu and Wang [29]. Comparing with [17, 29], our essential idea is to change the system (1.1) into an Euler equation with damping coupled with an Euler-Poisson equation with damping. Our another observation is that when two densities $\rho_1$ and $\rho_2$ are the perturbations of $\bar{\rho}$ and $u_1$ and $u_2$ are the perturbations of zero, then $\rho_1 - \rho_2$ and $u_1 - u_2$ can be small in $H^3 \cap L^1$. Hence, by the spectral analysis in [22] for the Euler equations with damping and the spectral analysis in Wu and Wang [29] for the Euler-Poisson equations with damping, together with the global existence and energy estimates in [29], we can achieve expected decay results for the rewritten system (2.1). As a byproduct, we obtain the improved decay results of the solution for the original system (1.1).

Our main results are stated in the following theorems.

**Theorem 1.1.** ([29]) Let $P'(\rho_i) > 0 (i = 1, 2)$ for $\rho_i > 0$, and $\bar{\rho} > 0$. Assume that $(\rho_i - \bar{\rho}, u_{i0}) \in H^3(\mathbb{R}^3)$ for $i = 1, 2$, with $\epsilon_0 =: \|\rho_{i0} - \bar{\rho}, u_{i0}\|_{H^3(\mathbb{R}^3)}$ small. Then there is a unique global classical solution $(\rho_i, u_i, \nabla\phi)$ of the Cauchy problem (1.1)-(1.2) satisfying

$$\|\rho_i - \bar{\rho}, u_i, \nabla\phi\|_{H^3}^2 \leq C\epsilon_0.$$  

(1.3)

**Theorem 1.2.** Under the assumptions of Theorem 1.1. If further,

$$\|\rho_{i0} - \rho_{20}, \rho_{10} + \rho_{20} - 2\bar{\rho}, u_{10}, u_{20}\|_{L^1} \leq \epsilon_1 \ (\epsilon_1 \ll 1),$$

(1.4)

we have

$$\|\partial_t^k (\rho_1 + \rho_2 - 2\bar{\rho})\|_{L^2} \leq C(\epsilon_0 + \epsilon_1)(1 + t)^{-\frac{3}{4} - \frac{|k|}{2}}, \ k = 0, 1,$$

(1.5)

$$\|\partial_x^k (u_1 + u_2)\|_{L^2} \leq C(\epsilon_0 + \epsilon_1)(1 + t)^{-\frac{3}{4} - \frac{|k|}{2}}, \ k = 0, 1,$$

(1.6)

$$\|\rho_1 - \rho_2, u_1 - u_2\|_{L^2} \leq C(\epsilon_0 + \epsilon_1)(1 + t)^{-2},$$

(1.7)

$$\|\partial_x (\rho_1 - \rho_2, u_1 - u_2)\|_{L^2} \leq C(\epsilon_0 + \epsilon_1)(1 + t)^{-\frac{5}{2}}.$$
Corollary 1.1. Under the assumptions of Theorem 1.2, we have
\[ \|\partial_k^k (\rho_1 - \bar{\rho}, \rho_2 - \bar{\rho})\|_{L^2} \leq (\epsilon_0 + \epsilon_1)(1 + t)^{-\frac{3}{4} - \frac{\|\xi\|}{2}}, \quad k = 0, 1, \]  \hfill (1.9)
and
\[ \|\partial_k^k (u_1, u_2)\|_{L^2} \leq (\epsilon_0 + \epsilon_1)(1 + t)^{-\frac{5}{4} - \frac{\|\xi\|}{2}}, \quad k = 0, 1. \]  \hfill (1.10)

Remark 1.1. The decay rates in Corollary 1.1 are optimal. This can be understood from Theorem 1.2 in Li et al. [12], where the same decay rates as (1.9) and (1.10) from below were shown.

Remark 1.2. From Corollary 1.1, we know the density \( \rho_i - \bar{\rho} \) have the same decay rate in \( L^2 \)-norm as the solution of the Navier-Stokes equations, while the velocity \( u_i \) in [17] decays faster than the densities. That is, we have improved the decay result in [17, 29]. In addition, the decay rates of the disparity of two densities \( \rho_1 - \rho_2 \) and the disparity of two velocities \( u_1 - u_2 \) in (1.7) are surprising and satisfactory, because of the exponential decay rate for the Euler-Poisson equations with damping and the coupling of two systems in (2.4).

Remark 1.3. The authors in [2, 29] obtained the optimal decay rate based on some Lyapunov energy functionals and a family of energy estimates with minimum derivative counts, respectively. Consequently, both [2] and [29] need not the smallness of \( L^1 \)-norm of the initial data. However, the usual energy method can not be used to the rewritten system (2.4), thus, it is also interesting to get rid of the smallness of the assumption \( \| (\rho_{10} - \rho_{20}, \rho_{10} + \rho_{20} - 2\bar{\rho}, u_{10}, u_{20}) \|_{L^1} \) in Theorem 1.2.

Remark 1.4. The similar problem for the bipolar quantum hydrodynamic model will be investigated in a forthcoming paper.

Notations. In this paper, \( D^l = \partial_x^l \) with an integer \( l \geq 0 \) stands for the usual any spatial derivatives of order \( l \). For \( 1 \leq p \leq \infty \) and an integer \( m \geq 0 \), we use \( L^p \) and \( W^{m,p} \) denote the usual Lebesgue space \( L^p(\mathbb{R}^3) \) and Sobolev spaces \( W^{m,p}(\mathbb{R}^3) \) with norms \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{W^{m,p}} \), respectively, and set \( H^m = W^{m,2} \) with norm \( \| \cdot \|_{H^m} \) when \( p = 2 \). In addition, we define the homogeneous Sobolev’s space \( H^s \) of all \( g \) for which \( \| g \|_{H^s} \) is finite, where
\[ \| g \|_{H^s} := \| \Lambda^s g \|_{L^2} = \| |\xi|^s \hat{g} \|_{L^2}. \]
Here, for \( s \in \mathbb{R} \), a pseudo-differential operator \( \Lambda^s \) by
\[ \Lambda^s g(x) = \int_{\mathbb{R}^n} \hat{g}(\xi)e^{2\pi i \langle x, \xi \rangle} d\xi, \]
where \( \hat{g} \) denotes the Fourier transform of \( g \).

Throughout this paper, for simplicity, we write \( \| \cdot \|_{L^\infty} = \| \cdot \|_{\infty}, \| \cdot \|_{L^2} = \| \cdot \|. \) And we will use \( C \) or \( C_i \) denotes a positive generic (generally large) constant that may vary at different places. And the sign “\( \sim \)” means
\[ f \sim g \iff \text{there exists two positive constants } C_1, C_2 \text{ such that } C_1 |f| \leq |g| \leq C_2 |f|. \]

The rest of the paper is arranged as follows. In Section 2, we reformulate the original system, then give detailed spectral analysis for the Euler equations with damping and Euler-Poisson equations with damping. The proof of temporal decay results of the solution will be derived in Section 3.
2. Spectral analysis and linear $L^2$ estimates

2.1 Reformulation of original problem

Assume $\bar{\rho} = 1$ and $P'(\bar{\rho}) = 1$ without loss of generality. Then the Cauchy problem (1.1)-(1.2) is reformulated as

\[
\begin{align*}
\partial_t \rho_1 + \text{div} u_1 &= -u_1 \cdot \nabla \rho_1 - \rho_1 \text{div} u_1, \\
\partial_t u_1 + u_1 + \nabla \rho_1 - \nabla \phi &= -u_1 \cdot \nabla u_1 - h(\rho_1) \nabla \rho_1, \\
\partial_t \rho_2 + \text{div} u_2 &= -u_2 \cdot \nabla \rho_2 - \rho_2 \text{div} u_2, \\
\partial_t u_2 + u_2 + \nabla \rho_2 + \nabla \phi &= -u_2 \cdot \nabla u_2 - h(\rho_2) \nabla \rho_2, \\
\Delta \phi &= \rho_1 - \rho_2, \\
(\rho_1, u_1, \rho_2, u_2)(x, 0) &= (\rho_{10}, u_{10}, \rho_{20}, u_{20})(x),
\end{align*}
\] (2.1)

where $h(\rho_i) = \frac{P'(\rho_i)}{\rho_i} - 1$.

Next, set

\[
n_1 = \rho_1 + \rho_2 - 2, \ n_2 = \rho_1 - \rho_2, \ w_1 = u_1 + u_2, \ w_2 = u_1 - u_2,
\] (2.2)

which give equivalently

\[
\rho_1 = \frac{n_1 + n_2}{2} + 1, \ \rho_2 = \frac{n_1 - n_2}{2} + 1, \ u_1 = \frac{w_1 + w_2}{2}, \ u_2 = \frac{w_1 - w_2}{2}.
\] (2.3)

From (2.2) and (2.3), it follows that the Cauchy problem (2.1) can be reformulated into the following Cauchy problem for the unknown $(n_1, w_1, n_2, w_2, \phi)$

\[
\begin{align*}
\partial_t n_1 + \text{div} w_1 &= f_1(n_1, w_1, n_2, w_2), \\
\partial_t w_1 + w_1 + \nabla n_1 &= f_2(n_1, w_1, n_2, w_2), \\
\partial_t n_2 + \text{div} w_2 &= f_3(n_1, w_1, n_2, w_2), \\
\partial_t w_2 + w_2 + \nabla n_2 - 2\nabla \phi &= f_4(n_1, w_1, n_2, w_2), \\
\Delta \phi &= n_2,
\end{align*}
\] (2.4)

where $(n_{10}, w_{10}, n_{20}, w_{20}) := (\rho_{10} + \rho_{20} - 2, u_{10} + u_{20}, \rho_{10} - \rho_{20}, u_{10} - u_{20})$, and

\[
\begin{align*}
f_1 &= -\frac{1}{2}[w_2 \nabla n_1 + w_1 \nabla n_2 + n_1 \text{div} w_1 + n_2 \text{div} w_2]; \\
f_2 &= -\frac{1}{2}[w_1 \nabla w_1 + w_2 \nabla w_2 + (h(\frac{n_1+n_2}{2} + 1) + h(\frac{n_1-n_2}{2} + 1))\nabla n_1 \\
&\quad + (h(\frac{n_1+n_2}{2} + 1) - h(\frac{n_1-n_2}{2} + 1))\nabla n_2]; \\
f_3 &= -\frac{1}{2}[w_1 \nabla n_2 + w_2 \nabla n_1 + n_1 \text{div} w_2 + n_2 \text{div} w_1]; \\
f_4 &= -\frac{1}{2}[w_1 \nabla w_2 + w_2 \nabla w_1 + (h(\frac{n_1+n_2}{2} + 1) - h(\frac{n_1-n_2}{2} + 1))\nabla n_1 \\
&\quad + (h(\frac{n_1+n_2}{2} + 1) + h(\frac{n_1-n_2}{2} + 1))\nabla n_2].
\end{align*}
\] (2.5)

From the foregoing, the Cauchy problem (2.4) can be formally divided into the Cauchy problem for the Euler equations with damping (2.4)$_{1,2}$ and the Euler-Poisson equations with damping (2.4)$_{3,4,5}$, which interact each other through the nonlinear inhomogeneous terms on the right-hand side.
2.2 Spectral analysis of linearized Euler equations with damping

In this subsection, we shall give a spectral analysis for the linearized Euler equations with damping. A similar analysis can be founded in [22]. For the convenience of the readers, the proof will be also sketched here.

We take Hodge decomposition to analyze the following linearized equations corresponding to system (2.4)_{1,2}

\[
\begin{aligned}
\partial_t n_1 + \text{div} w_1 &= 0, \\
\partial_t w_1 + w_1 + \nabla n_1 &= 0, \\
(n_1, w_1)(t = 0) &= (n_{10}, w_{10}).
\end{aligned}
\]

Let \( v_1 = \Lambda^{-1} \text{div} w_1 \) be the “compressible part” of the velocity and \( d_1 = \Lambda^{-1} \text{curl} w_1 \) be the “incompressible part”, then the system (2.6) can be rewritten as

\[
\begin{aligned}
\partial_t n_1 + \Lambda v_1 &= 0, \\
\partial_t v_1 + w_1 + \Lambda n_1 &= 0, \\
\partial_t d_1 + d_1 &= 0.
\end{aligned}
\]

Obviously, by the definition of \( v_1 \) and \( d_1 \), and the relation

\[
w_1 = -\Lambda^{-1} \nabla v_1 - \Lambda^{-1} \text{div} d_1,
\]

the estimates in space \( H^l(\mathbb{R}^3) \) for the original function \( w_1 \) is the same as for the functions \((v_1, d_1)\).

By the semigroup theory for evolutionary equation, the solution \((n_1, v_1)\) of the first two equations of the system (2.7) can be expressed via the Cauchy problem for \(X = (n_1, v_1)^T\) as

\[
X_t = AX, \quad X(0) = X_0, \quad t \geq 0,
\]

which leads to

\[
X(t) = S_1(t)X_0 = \exp^{tA}X_0, \quad t \geq 0.
\]

Taking the Fourier transform with respect to the space variable yields

\[
\frac{d}{dt} \hat{X} = A(\xi)\hat{X} \text{ with } A(\xi) = \begin{pmatrix} 0 & -|\xi| \\ |\xi| & -1 \end{pmatrix}.
\]

The characteristic polynomial of \(A(\xi)\) is \(\lambda^2 + \lambda + |\xi|^2\) and has two distinct roots:

\[
\lambda_{\pm}(\xi) = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - |\xi|^2}.
\]

By a direct computation, we can verify the exact expression the Fourier transform \(\hat{G}_1(\xi, t)\) of Green function \(G_1(x, t) = \exp^{tA}\) as

\[
\hat{G}_1(\xi, t) = \begin{pmatrix}
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} & \frac{|\xi|}{\lambda_+ - \lambda_-} e^{\lambda_+ t} - \frac{|\xi|}{\lambda_+ - \lambda_-} e^{\lambda_- t} \\
-\frac{|\xi|}{\lambda_+ - \lambda_-} e^{\lambda_+ t} + \frac{|\xi|}{\lambda_+ - \lambda_-} e^{\lambda_- t} & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}
\end{pmatrix}.
\]

We use the standard higher-lower frequency decomposition to derive the long-time decay rate of solutions in \(L^2\) framework. For \(|\xi| \ll 1\), it holds that

\[
\begin{aligned}
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} &\sim e^{-|\xi|^2 t} - |\xi|^2 e^{-t}, \\
\frac{|\xi|}{\lambda_+ - \lambda_-} e^{\lambda_+ t} - \frac{|\xi|}{\lambda_+ - \lambda_-} e^{\lambda_- t} &\sim -|\xi| e^{-t} + |\xi| e^{-|\xi|^2 t}, \\
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} &\sim (1 - |\xi|^2) e^{-t}.
\end{aligned}
\]
On the other hand, as we know, the higher frequency part of $\hat{G}_1(\xi, t)$ in $L^2$-norm has the exponential decay rate.

Then by using the explicit expression of $\hat{n}_1$ and $\hat{v}_1$:

$$\hat{n}_1 = \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \hat{n}_{10} + |\xi| e^{\lambda_+ t} \hat{\nu}_{10},$$

$$\hat{v}_1 = -|\xi| e^{\lambda_+ t} - e^{\lambda_- t} \hat{n}_{10} + \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \hat{\nu}_{10},$$

we have

$$\|\hat{n}_1(t)\|^2 = \int_{|\xi| \leq \eta} |\hat{n}_1(\xi, t)|^2 d\xi + \int_{|\xi| \geq \eta} |\hat{n}_1(\xi, t)|^2 d\xi$$

$$\leq C \int_{|\xi| \leq \eta} e^{-|\xi|^2 t} (|\hat{n}_{10}|^2 + |\hat{v}_{10}|^2) d\xi + C e^{-bt} \int_{|\xi| \geq \eta} (|\hat{n}_{10}|^2 + |\hat{v}_{10}|^2) d\xi$$

$$\leq C \|(n_{10}, v_{10})\|_{L^2}^2 \int_{|\xi| \leq \eta} e^{-|\xi|^2 t} d\xi + C e^{-bt} \|(n_{10}, v_{10})\|_{L^2}^2$$

$$\leq C(1 + t)^{-\frac{3}{2}} \|(n_{10}, v_{10})\|_{L^2 \cap L^1}^2, \quad \text{where } b > 0,$$

(2.13)

and

$$\|\hat{v}_1(t)\|^2 = \int_{|\xi| \leq \eta} |\hat{v}_1(\xi, t)|^2 d\xi + \int_{|\xi| \geq \eta} |\hat{v}_1(\xi, t)|^2 d\xi$$

$$\leq C \int_{|\xi| \leq \eta} |\xi|^2 e^{-|\xi|^2 t} (|\hat{n}_{10}|^2 + |\hat{v}_{10}|^2) d\xi + C e^{-bt} \int_{|\xi| \geq \eta} (|\hat{n}_{10}|^2 + |\hat{v}_{10}|^2) d\xi$$

$$\leq C \|(n_{10}, v_{10})\|_{L^2}^2 \int_{|\xi| \leq \eta} |\xi|^2 e^{-|\xi|^2 t} d\xi + C e^{-bt} \|(n_{10}, v_{10})\|_{L^2}^2$$

$$\leq C(1 + t)^{-\frac{3}{2}} \|(n_{10}, v_{10})\|_{L^2 \cap L^1}^2, \quad \text{where } b > 0.$$

(2.14)

Similarly, the $L^2$-decay rate for the derivatives of $\hat{n}_1$ and $\hat{v}_1$ are

$$\|\partial_x^k n_1\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{\delta}{2}} \|(n_{10}, v_{10})\|_{H^k \cap L^1},$$

(2.15)

and

$$\|\partial_x^k v_1\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{\delta}{2}} \|(n_{10}, v_{10})\|_{H^k \cap L^1}.$$

(2.16)

From (2.15) and (2.16), and using the relation $w_1 = -\Lambda^{-1} \nabla v_1 - \Lambda^{-1} \text{div} d_1$ and the fact $\|d_1\|_{L^2} \sim e^{-t}$, one can easily obtain the following $L^2$-decay result for the linearized Euler equation with damping.

**Lemma 2.1.** Let $(n_{10}, w_{10}) \in H^l \cap L^1$ and $n_1, w_1$ satisfy the system (2.6). Then there exists a constant $C$ such that for $0 \leq k \leq l$

$$\|\partial_x^k n_1\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{\delta}{2}} \|(n_{10}, w_{10})\|_{H^k \cap L^1},$$

(2.17)

and

$$\|\partial_x^k w_1\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{\delta}{2}} \|(n_{10}, w_{10})\|_{H^k \cap L^1}.$$

(2.18)
2.3 Spectral analysis of linearized Euler-Poisson equations with damping

In this subsection, we shall give a spectral analysis for the linearized Euler-Poisson equations with damping.

We take Hodge decomposition to analyze the following linearized equations corresponding to system (2.4)\ref{euler-poisson}:

\[
\begin{aligned}
\partial_t n_2 + \text{div} w_2 &= 0, \\
\partial_t w_2 + w_2 + \nabla n_2 - 2\nabla \phi &= 0, \\
\Delta \phi &= n_2, \\
(n_2, w_2)(t = 0) &= (n_{20}, w_{20}).
\end{aligned}
\]  

(2.19)

Let \( v_2 = \Lambda^{-1} \text{div} w_2 \) be the “compressible part” of the velocity and \( d_2 = \Lambda^{-1} \text{curl} w_2 \) be the “incompressible part”, then the system (2.19) can be rewritten as:

\[
\begin{aligned}
\partial_t n_2 + \Lambda v_2 &= 0, \\
\partial_t v_2 - \Lambda n_2 + 2\Lambda^{-1}n_2 &= 0, \\
\partial_t d_2 + d_2 &= 0.
\end{aligned}
\]  

(2.20)

By the semigroup theory for evolutionary equations, the solution \((n_2, v_2)\) of the first two equations of the system (2.20) can be expressed via the Cauchy problem for \( Y = (n_2, v_2)^T \) as

\[ Y_t = BY, \ Y(0) = Y_0, \ t \geq 0, \]

which leads to

\[ Y(t) = S_1(t)Y_0 = e^{tB}Y_0, \ t \geq 0. \]

Taking the Fourier transform with respect to the space variable yields

\[
\frac{d}{dt} \hat{Y} = B(\xi)\hat{Y} \quad \text{with} \quad B(\xi) = \begin{pmatrix} 0 & -|\xi| \\ |\xi| - 2|\xi|^{-1} & -1 \end{pmatrix}.
\]

(2.21)

The characteristic polynomial of \( B(\xi) \) is \( \lambda^2 + \lambda + |\xi|^2 + 2 \) and has two distinct roots:

\[ \lambda_{\pm}(\xi) = -\frac{1}{2} \pm \frac{1}{2} \sqrt{-7 - 4|\xi|^2}. \]  

(2.22)

By a direct computation, we can verify the exact expression the Fourier transform \( \hat{G}_2(\xi, t) \) of Green function \( G_2(x, t) = e^{tB} \) as

\[
\hat{G}_2(\xi, t) = \begin{pmatrix} \lambda_+ e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} |\xi| e^{\lambda_+ t} - e^{\lambda_- t} \\ -|\xi| e^{\lambda_+ t} - e^{\lambda_- t} \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \lambda_+ - \lambda_- - e^{\lambda_+ t} - e^{\lambda_- t} \end{pmatrix}.
\]

(2.23)

We use the standard higher-lower frequency decomposition to derive the long-time decay rate of solutions in \( L^2 \) framework. For \( |\xi| \ll 1 \), it holds that

\[
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} \sim -(1 + |\xi|^2) e^{-t},} {\lambda_+ - \lambda_- - e^{\lambda_+ t} - e^{\lambda_- t} \sim -|\xi| e^{-t}.}
\]

(2.24)

On the other hand, it is obvious that the higher frequency part of \( \hat{G}(\xi, t) \) in \( L^2 \)-norm has also the exponential decay rate.

Then by using the explicit expression of \( \hat{n}_2 \) and \( \hat{v}_2 \), we have the following \( L^2 \)-decay rate for \( \hat{n}_2 \) and \( \hat{v}_2 \):

\[
\| \partial_x^k(n_2, w_2) \|_{L^2} \leq Ce^{-bt} \| (n_{10}, v_{10}) \|_{\tilde{H}^{k\cap L^2}}, \quad \text{for some constant } b > 0.
\]

(2.25)

From (2.25) and the relation \( w_2 = -\Lambda^{-1} \nabla v_2 - \Lambda^{-1} \text{div} d_2 \) and the fact \( \| d_2 \|_{L^2} \sim e^{-t} \), one can easily obtain the following \( L^2 \)-decay result for the linearized Euler equation with damping.
Lemma 2.2. Let \((n_{20}, w_{20}) \in H^l \cap L^1\) and \(n_2, w_2\) satisfy the system (2.19). Then there exists a positive constant \(C\) such that for \(0 \leq k \leq l\)
\[
\|\partial^k_x(n, w)\|_{L^2} \leq C e^{-bt}(n_{20}, w_{20})\|_{H^k \cap L^1}, \text{ for some constant } b > 0. \tag{2.26}
\]

3. Proof of the main results

In this section, we will mainly use Lemma 2.1 and Lemma 2.2 to prove the optimal decay rate of the nonlinear system (2.4).

Recall that
\[
n_1 = \rho_1 + \rho_2 - 2 \bar{\rho}, \quad n_2 = \rho_1 - \rho_2, \quad w_1 = u_1 + u_2, \quad w_2 = u_1 - u_2. \tag{3.1}
\]

Then the assumption (1.4) implies that
\[
\|(n_{10}, n_{20}, w_{10}, w_{20})\|_{L^1} \leq \epsilon_1, \quad \epsilon_1 \ll 1. \tag{3.2}
\]

On the other hand, the global existence in Theorem 1.1 implies that
\[
\frac{1}{2} \leq n_1, n_2 \leq 2, \quad |n|^{1/2} + |n| \leq C|n|, \quad |n| \leq C, \quad i = 1, 2. \tag{3.3}
\]

Now, we are in a position to prove Theorem 1.2. Define
\[
M(t) = \sup_{0 \leq s \leq t} \left\{ \sum_{k=0}^{1} [(1+s)^{1/2} \|D^k n_1\|_{L^2} + (1+s)^{1/2} \|D^k n_2\|_{L^2} + (1+s)^2 \|n_2\|_{L^2}] \right. \nonumber
\]
\[
\left. + (1+s)^{5/4} \|D^2 n_2\|_{L^2} + (1+s)^{3/2} \|D^2 n_1\|_{L^2} + \|D^2 n_2\|_{L^2} + \|D^2 n_1\|_{L^2} \right\}. \tag{3.4}
\]

We claim that it holds for any \(t \in [0, T]\),
\[
M(t) \leq C(\epsilon_1 + \epsilon_2). \tag{3.5}
\]

First, by Duhamel principle and Lemma 2.1, Lemma 2.2, it is easy to verify that \(L^2\)-norm of the solution \(n_1, n_2, w_2\) of the problem (2.1) can be expressed as
\[
\|D^k n_1\|_{L^2} \leq C \|(n_{10}, w_{10})\|_{L^1}(1+t)^{-1/2} + \int_0^t (1+t-s)^{-1/2} \|D^k f_1, f_2\|_{L^1} + \|D^k f_1, f_2\|_{L^1} \|n\| ds, \nonumber
\]
\[
\|D^k n_2\|_{L^2} \leq C \|(n_{10}, w_{10})\|_{L^1}(1+t)^{-1/2} + \int_0^t (1+t-s)^{-1/2} \|D^k f_1, f_2\|_{L^1} + \|D^k f_1, f_2\|_{L^1} \|n\| ds, \nonumber
\]
\[
\|D^k w_2\|_{L^2} \leq C e^{-bt} \|(n_{20}, w_{20})\|_{L^1} + \int_0^t e^{-b(t-s)} \|f_3, f_4\|_{L^1} + \|D^k f_3, f_4\|_{L^1} \|n\| ds, \nonumber
\]
\[
\|D^k w_2\|_{L^2} \leq C e^{-bt} \|(n_{20}, w_{20})\|_{L^1} + \int_0^t e^{-b(t-s)} \|f_3, f_4\|_{L^1} + \|D^k f_3, f_4\|_{L^1} \|n\| ds, \quad b > 0. \tag{3.6}
\]

Step 1. Basic estimates From (3.3), the nonlinear terms \(f_1, f_2, f_3, f_4\) can be rewritten into
\[
\begin{align*}
 f_1 & \sim O(1)(w_2 D_1 + w_1 D_2 + n_1 D_1 + n_2 D_2), \\
 f_2 & \sim O(1)(w_1 D_1 + w_2 D_2 + n_1 D_1 + n_2 D_2), \\
 f_3 & \sim O(1)(w_2 D_1 + w_1 D_2 + n_1 D_1 + n_2 D_2), \\
 f_4 & \sim O(1)(w_2 D_2 + w_1 D_1 + n_1 D_1 + n_2 D_2 + n_1 D_2 + n_2 D_2). \tag{3.7}
\end{align*}
\]

Thus, we have
\[
\begin{align*}
\| & (f_1, f_2)\|_{L^1}(s) \\
& \leq C \|w_2 D_1 + w_1 D_2 + n_1 D_1 + n_2 D_2\|_{L^1} + \|n_2\|_{L^1}\|D_1\| + \|n_1\|_{L^1}\|D_2\| + \|n_2\|_{L^1}\|n_1\| \\
& \leq CM^2(t)(1+s)^{-2}. \tag{3.8}
\end{align*}
\]
In fact, we just need to consider the key term $\|n_1Dn_1\|_{L^1}(s)$ with the “worst” decay rate (see ansatz (3.4)), which can be estimated as

$$\|n_1Dn_1\|_{L^1}(s) \leq C\|n_1\|\|Dn_1\| \leq CM^2(t)(1+s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}} = CM^2(t)(1+s)^{-2}. \quad (3.9)$$

Then, for $\|(f_1, f_2)\|(s)$, we get

$$\|(f_1, f_2)\|(s) \leq C\|(w_2Dn_1, w_1Dn_2, n_1Dw_1, n_2Dw_2, w_1Dw_1, w_2Dw_2, n_1Dn_1, n_2Dn_2, n_1Dn_2, n_2Dn_2)\|$$

$$\leq C\{\|w_2\|\|Dn_1\| + \|w_1\|\|Dn_2\| + \|n_1\|\|Dw_1\| + \|n_2\|\|Dw_2\| + \|n_1\|\|Dw_2\| + \|n_2\|\|Dw_1\| + \|n_1\|\|Dn_2\| + \|n_2\|\|Dn_1\| \}.$$ \quad (3.10)

The key term $\|n_1Dn_1\|$ in (3.10) with the “worst” decay rate (see ansatz (3.4)) can be estimated as

$$\|n_1Dn_1\|(s) \leq C\|n_1\|\|Dn_1\| \leq C\|Dn_1\|^{\frac{1}{2}}\|D^2n_1\|$$

$$\leq CM^2(t)(1+s)^{-\left(\frac{3}{2} + \frac{3}{2} + \frac{3}{2} + \frac{3}{2}\right)} = CM^2(t)(1+s)^{-\frac{3}{2}}, \quad (3.11)$$

where we have used the fact (the Nirenberg inequality)

$$\|u\|_{\infty} \leq C\|Du\|^{\frac{1}{2}}\|D^2u\|^{\frac{1}{2}}. \quad (3.12)$$

Thus from (3.6) to (3.11), we get

$$\|n_1\|(t) \leq \epsilon_1(1+t)^{-\frac{3}{4}} + CM^2(t)(1+t)^{-\frac{3}{4}}, \quad (3.13)$$

and

$$\|w_1\|(t) \leq \epsilon_1(1+t)^{-\frac{3}{4}} + CM^2(t)(1+t)^{-\frac{3}{4}}. \quad (3.14)$$

Similarly, we can easily obtain

$$\|(f_3, f_4)\|_{L^1}(s) \leq CM^2(t)(1+s)^{-2}, \quad (3.15)$$

and

$$\|(f_3, f_4)\|(s) \leq CM^2(t)(1+s)^{-\frac{3}{2}}, \quad (3.16)$$

which together with (3.6) yield that

$$\|(n_2, w_2)\|(t) \leq \epsilon_1e^{-bt} + CM^2(t)(1+t)^{-2}, \quad b > 0. \quad (3.17)$$

**Step 2. Estimates for higher order derivatives** Firstly, we consider the first order derivatives. From (3.7), we get

$$\|D(f_1, f_2)\|(s) \leq C\|D(w_2Dn_1, w_1Dn_2, n_1Dw_1, n_2Dw_2, w_1Dw_1, w_2Dw_2, n_1Dn_1, n_2Dn_1, n_1Dn_2, n_2Dn_2)\|$$

$$\leq C\{\|D(w_2Dn_1, w_2D^2n_1, w_1Dn_2, n_1D^2w_1, w_1Dw_1, w_2Dw_2, n_1Dn_1, n_1D^2n_1, n_2Dn_2, n_1Dw_1, n_2Dw_2, n_1D^2w_2, n_2D^2w_2, n_1Dn_2, n_2D^2n_2, n_1D^2n_2, n_2Dn_2, n_2D^2n_2)\| \}.$$ \quad (3.18)
In the same way, from ansatz (3.4), we only consider the terms containing \( n_1 \). In fact, from (3.12) we have
\[
\|n_1 D^2 n_1\| (s) \leq C\|n_1\| \|D^2 n_1\| \leq C\|Dn_1\| \|D^2 n_1\|^{\frac{3}{2}} \\
\leq CM^2(t)(1 + s)^{-\left(\frac{1}{2} + \frac{3}{4} + \frac{3}{2}\right)} = CM^2(t)(1 + s)^{-\frac{5}{2}},
\]
(3.19)
\[
\|Dn_1 Dn_1\| (s) \leq C\|Dn_1\| \|D^2 n_1\| \leq C\|D^2 n_1\| \|D^3 n_1\|^{\frac{1}{2}} \|Dn_1\| \\
\leq CM^2(t)(1 + s)^{-\left(\frac{1}{2} + \frac{3}{4} + \frac{3}{2}\right)} = CM^2(t)(1 + s)^{-\frac{5}{2}}.
\]
(3.20)
Consequently, from (3.6) and (3.8), we have
\[
\|Dn_1\| (t) \leq \epsilon_1 (1 + t)^{-\frac{7}{4}} + CM^2(t)(1 + t)^{-\frac{13}{8}},
\]
(3.21)
and
\[
\|Dw_1\| (t) \leq \epsilon_1 (1 + t)^{-\frac{7}{4}} + CM^2(t)(1 + t)^{-\frac{13}{8}}.
\]
(3.22)
Similarly, (3.19) and (3.20) together with (3.6) and (3.8) yield that
\[
\|D(n_2, w_2)\| (t) \leq \epsilon_1 e^{-bt} + CM^2(t)(1 + t)^{-\frac{15}{8}}, b > 0.
\]
(3.23)
Next, we can use the same method as above to derive the estimates of second order derivatives. From (3.7), we have
\[
\|D^2 (f_1, f_2)\| (s) = \|(D^2 w_2 Dn_1, D^2 w_3 D^2 n_1, w_2 D^3 n_1, D^2 n_1 Dw_1, Dn_1 D^2 w_1, \\
D^2 n_1 Dn_1, n_1 D^3 n_1, D^2 n_1 Dn_2, Dn_1 D^2 n_2, n_1 D^3 n_2, \ldots)\| (s),
\]
(3.24)
where “…” denotes the other terms containing no variable \( n_1 \). Indeed, because of the “worst” decay rate of \( n_1 \) in the ansatz (3.4), we just need to consider the terms listed above, especially two terms \( \|Dn_1 D^2 n_1\| \) and \( \|n_1 D^3 n_1\| \). In fact, we have
\[
\|Dn_1 D^2 n_1\| (s) \leq C\|Dn_1\| \|D^2 n_1\| \leq C\|D^2 n_1\|^{\frac{3}{2}} \|D^3 n_1\|^{\frac{1}{2}} \leq CM^2(t)(1 + s)^{-\frac{15}{8}},
\]
(3.25)
and
\[
\|n_1 D^3 n_1\| (s) \leq C\|n_1\| \|D^3 n_1\| \leq C\|Dn_1\| \|D^2 n_1\| \|D^3 n_1\| \\
\leq CM^2(t)(1 + s)^{-\left(\frac{1}{2} + \frac{3}{4} + \frac{3}{2}\right)} = CM^2(t)(1 + s)^{-\frac{5}{2}}.
\]
(3.26)
Then, (3.25), (3.26), (3.6) and (3.8) imply that
\[
\|D^2 (n_1, w_1)\| (t) \leq \epsilon_1 (1 + t)^{-\frac{7}{4}} + CM^2(t)(1 + t)^{-\frac{5}{4}}.
\]
(3.27)
Similarly, we have
\[
\|D^2 (n_2, w_2)\| (t) \leq \epsilon_1 (1 + t)^{-\frac{7}{4}} + CM^2(t)(1 + t)^{-\frac{5}{4}}.
\]
(3.28)
Lastly, Theorem 1.1 and the relation (2.2) lead to that
\[
\|D^3 (n_1, n_2, w_1, w_2)\| (t) \leq C\epsilon_0.
\]
(3.29)
In summary, from (3.13), (3.14), (3.17), (3.21)-(3.23) and (3.27)-(3.29), we deduce that
\[
M(t) \leq C(\epsilon_0 + \epsilon_1) + CM^2(t).
\]
(3.30)
By the standard continuous argument, we know that there exists a constant \( C > 0 \) such that
\[
M(t) \leq C(\epsilon_0 + \epsilon_1), t \in [0, T].
\]
(3.31)
This proves Theorem 1.2. Immediately, Corollary 1.1 can be derived from Theorem 1.2 and (2.2).

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