Friedmann equation and Cardy formula correspondence in brane universes

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Abstract

We study the brane with arbitrary tension $\sigma$ on the edge of various black holes with AdS asymptotics. We investigate Friedmann equations governing the motion of the brane universes and match the Friedmann equation to Cardy entropy formula.

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Motivated by the well-known example of black hole entropy, an influential holographic principle was put forward recently, which suggests that microscopic degrees of freedom that build up the gravitational dynamics do reside not in the bulk spacetime but on its boundary [1]. This principle has been viewed as a conceptual change in our thinking about gravity. Maldacena’s conjecture on AdS/CFT correspondence [2] is the first example realizing such a principle. Subsequently, Witten [3] convincingly argued that the entropy, energy and temperature of CFT at high temperatures can be identified with the entropy, mass and Hawking temperature of the AdS black hole [4], which further supports the holographic principle. In cosmological settings, testing the holographic principle is somewhat subtle. Fischler and Susskind (FS) [5] have shown that for flat and open Friedmann-Lemaître-Robertson-Walker (FLRW) universes the area of the particle horizon should bound the entropy on the backward-looking light cone. However violation of FS bound was found for closed FLRW universes. Various different modifications of the FS version of the holographic principle have been raised subsequently [6-12]. In addition to the study of holography in homogeneous cosmologies, attempts to generalize the holographic principle to a generic realistic inhomogeneous cosmological setting were carried out in [13,14].

Recently, the very interesting study of the holographic principle in FLRW universe filled with CFT with a dual AdS description has been done by Verlinde [15]. He revealed that when a universe-sized black hole can be formed, an interesting and surprising correspondence appears between entropy of CFT and Friedmann equation governing the radiation dominated closed FLRW universes. Generalizing Verlinde’s discussion to a broader class of universes including a cosmological constant [16], we found that matching of Friedmann equation to Cardy formula holds for de Sitter closed and AdS flat universes. However for the remaining de Sitter and AdS universes, the argument fails due to breaking down of the general philosophy of the holographic principle. In high dimensions, various
other aspects of Verlinde’s proposal have also been investigated in a number of works [17].

In a very recent paper [18], further light on the correspondence between Friedmann equation and Cardy formula has been shed from a Randall-Sundrum type brane-world perspective [19]. Considering the CFT dominated universe as a co-dimension one brane with fine-tuned tension in a background of an AdS black hole, Savonije and Verlinde found the correspondence between Friedmann equation and Cardy formula for the entropy of CFT when the brane crosses the black hole horizon. This result has been further confirmed by studying a brane-universe filled with radiation and stiff-matter, quantum-induced brane worlds and radially infalling brane [20]. The discovered relation between Friedmann equation and Cardy formula for the entropy shed significant light on the meaning of the holographic principle in a cosmological setting. However the general proof for this correspondence for all CFTs is still difficult at the moment. It is worthwhile to further check the validity of the correspondence in broader classes of situations than [15,18]. This is the motivation of the present paper. In addition to spherically symmetric AdS Schwarzschild black hole considered in the bulk background, we will consider various black holes with AdS asymptotics including hyperbolic AdS black holes and flat AdS black membrane. Instead of choosing a special value of the brane tension to tune the cosmological constant in the brane-universe to zero, in our following study we will choose an arbitrary value of the tension on the brane to describe the de Sitter and AdS brane universes. We will show that in de Sitter and AdS brane universes, correspondence between the Friedmann equation and Cardy formula hold for all values of curvature constants. The situation when a domain wall with matter is present in addition to the background wall tension will also be addressed.

We start by considering a four-dimensional (4D) brane in a spacetime described by a five-dimensional (5D) AdS black hole. The background metric is

\[ ds_5^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Sigma_K^2, \]  

(1)
where

\[ f = k + \frac{r^2}{L^2} - \frac{m}{r^2}. \]  

(2)

\( L \) is the curvature radius of AdS spacetime. \( k \) takes the values 0, -1, +1 corresponding to flat, open and closed geometrics, and \( d\Sigma_k^2 \) is the corresponding metric on the unit three dimensional plane, hyperboloid or sphere. In the case of \( m = 0 \), we have an exact 5D AdS space. For \( m \neq 0 \), the black hole horizon locates at

\[ r_H^2 = \frac{L^2}{2}(-k + \sqrt{k^2 + 4m/L^2}). \]  

(3)

The relation between \( m \) to the Arnowitt-Deser-Misner (ADM) mass of the hole \( M \) is [21]

\[ M = \frac{3\omega_3}{16\pi G_5}m \]  

(4)

where \( \omega_3 \) is the volume of the unit 3 sphere.

As discussed in [18], we regard the brane as a boundary of background AdS geometry. The location and the metric on the boundary become time dependent. Choosing the gauge where

\[ \dot{r}^2 = f^2 \dot{t}^2 - f, \]  

(5)

the metric on the brane is given by

\[ ds_4^2 = -d\tau^2 + r^2(\tau)d\Sigma_3^2. \]  

(6)

Following [3,18], we learn that CFT lives on the brane, which is the boundary of the AdS hole. The energy for a CFT on a sphere with volume \( V = r^3\omega_3 \) is given by \( E = \frac{L}{r}M \). The density of the CFT energy can be expressed as

\[ \rho_{CFT} = \frac{E}{V} = \frac{3mL}{16\pi G_5 r^4}. \]  

(7)

According to [3,22], the entropy of the CFT on the brane is equal to the Bekenstein-Hawking entropy of the AdS black hole

\[ S_{CFT}(4D) = S_{BH}(5D) = \frac{V_H}{4G_5}, \quad V_H = r_H^3\omega_3. \]  

(8)
The area of AdS equals to the volume of the corresponding spatial section for an observer on the brane. The entropy density in the brane is

\[ s = S/V = \frac{r^3_H}{4G_5r^3} = \frac{r^3_H}{2G_4Lr^3} \]  

(9)

where \( G_5 = \frac{G_4L}{2} \).

The induced CFT temperature on the brane is

\[ T = \frac{L}{r}T_H, \]  

(10)

where \( T_H \) is the Hawking temperature of the bulk black hole

\[ T_H = \frac{1}{4\pi} f'(r_H) = \frac{1}{2\pi} \left[ \frac{r_H}{L^2} + \frac{m}{r^3_H} \right]. \]  

(11)

The first law of thermodynamics expressed in terms of densities has the form on the brane

\[ T ds = d\rho_{\text{CFT}} + 3(\rho_{\text{CFT}} + P_{\text{CFT}} - Ts)\frac{dr}{r}. \]  

(12)

Considering the equation of state \( P_{\text{CFT}} = \rho_{\text{CFT}}/3 \) and taking \( m = k r_H^2 + r_H^4/L^2 \), the term representing Casimir energy density derived from eqs(7,10,11) is

\[ \frac{3}{2}(\rho_{\text{CFT}} + P_{\text{CFT}} - Ts) = \frac{\gamma}{r^2} \]  

(13)

where \( \gamma = \frac{3kr^2_H}{8\pi G_4r^2} \).

With the given expressions for the entropy density \( s \), CFT energy density \( \rho_{\text{CFT}} \) and \( \gamma \), the entropy density can be expressed in the form of Cardy’s formula

\[ s^2 = \left( \frac{4\pi}{3\sqrt{k}} \right)^2 \gamma(\rho_{\text{CFT}} - \frac{\gamma}{r^2}). \]  

(14)

It is easy to check that this formula agrees to Eq.(30) in [18] regardless of different values of \( k \).
Now we start to study the motion of brane-universe and examine the relation between the Friedmann equation and Cardy formula in the brane-cosmology.

The equation of motion for the scalar factor $r(\tau)$ can be obtained from Israel’s matching conditions under the assumption of $z_2$ symmetry:

$$K_{\mu\nu} = -8\pi G_5 [T_{\mu\nu} - \frac{1}{3} T^c_{\gamma\mu\nu}],$$  \hspace{1cm} (15)

where $T_{\mu\nu}$ is the energy momentum tensor on the brane and $K_{\mu\nu}$ is the extrinsic curvature.

Introduce a stress-energy tensor on the brane [23]

$$T_{ab} = -\sigma \gamma_{ab}$$  \hspace{1cm} (16)

where $\sigma$ is an arbitrary brane tension. From (15), the space component of the junction condition gives

$$H^2 = -\frac{k}{r^2} + \frac{m}{r^4} - \frac{1 - (\sigma/\sigma_c)^2}{L^2}$$  \hspace{1cm} (17)

where $\sigma_c = \frac{3}{8\pi G_5 L}$ is the critical brane tension. Taking $\sigma = \sigma_c$, (17) reduces to the Friedmann equation of CFT radiation dominated brane universe without cosmological constant discussed in [18]. If $\sigma > \sigma_c$ or $\sigma < \sigma_c$, the brane-world is a de Sitter universe or AdS universe, respectively.

When the brane crosses the horizon $r_H$, eq(17) becomes

$$H^2 = \frac{1}{L^2} (\frac{\sigma}{\sigma_c})^2$$  \hspace{1cm} (18)

by taking $r = r_H$ and $m = kr_H^2 + r_H^4/L^2$. The entropy density (9) can be expressed at $r = r_H$ as

$$s = \frac{H}{2G_4} (\sigma_c/\sigma).$$  \hspace{1cm} (19)

Putting (19) in (14) and taking $r = r_H$ and $m = kr_H^2 + r_H^4/L^2$, (14) exactly reproduces the first Friedmann equation.
Using the fact that $\dot{\rho}_{CFT} = -3H(P_{CFT} + \rho_{CFT})$ and $P_{CFT} = \rho_{CFT}/3$, the second Friedmann equation can be obtained
\[
\dot{H} = \frac{k}{r^2} - \frac{2m}{r^4}
\] (20)
where (7) has been used. When the brane crosses the black hole horizon, (20) can be expressed as
\[
\dot{H} = -\frac{k}{r_H^2} - \frac{2}{L^2}
\] (21)
Employing (21) and (18), the CFT temperature at the black hole horizon is
\[
T = -\frac{\dot{H}}{2\pi H} \left( \frac{\sigma}{\sigma_c} \right).
\] (22)
From (13), we know
\[
T = \left[ \rho_{CFT} + P_{CFT} - \frac{2\gamma}{3r^2} \right]/s.
\] (23)
Substituting (22) and the values of $s, \gamma, \rho_{CFT}$ at the horizon, the second Friedmann equation can be reproduced.

It is interesting to note that when the brane crosses the black hole horizon, Friedmann equation coincides with Cardy formula. This result holds independently of the value of the brane tension $\sigma$ and curvature constant $k$. The correspondence between Friedmann equations and Cardy formula is valid in all universes.

Now we consider the case in which motion of the brane universe results from matter on the domain wall. We take the energy-momentum tensor $T_{ab} = -\sigma\gamma_{ab} + \rho u_a u_b + P(\gamma_{ab} + u_a u_b)$, which corresponds to matter with energy density $\rho$ and pressure $P$, in addition to the background wall tension $\sigma$ [23]. For $\rho \ll \sigma_c$, the angular components of (15) give
\[
H^2 = -\frac{k}{r^2} + \frac{m}{r^4} - [1 - (\frac{\sigma}{\sigma_c})^2 - 2(\frac{\sigma}{\sigma_c})^2 \frac{\rho}{\sigma_c}] / L^2
\] (24)
when $\sigma = \sigma_c$, (24) reduces to the description in [23] on the motion of brane universe. For $\sigma_c^2 > \sigma^2 + 2\sigma\rho$, (24) is the Friedmann equation for AdS universe; while for $\sigma_c^2 < \sigma^2 + 2\sigma\rho$,
(24) describes the de Sitter universe. The Friedmann equation discussed in [18] is a specific case of (24) where $\sigma = \sigma_c$ and $\rho = 0$.

It has been argued that there is a possible duality relation between classical gravity in 5D AdS bulk and a conformal field theory residing on its boundary brane-world [3,22]. In this context, the second term in (24) may be interpreted from the boundary theory point of view as the contribution of a true radiation bath of the conformal fields whose energy density satisfies (7).

Considering when the brane pass the black hole horizon, the Friedmann equation (24) can be expressed as

$$H^2 = \left[\frac{(\sigma/\sigma_c)^2 + 2(\sigma/\sigma_c)(\rho/\sigma_c)}{L^2}\right] = C/L^2. \quad (25)$$

The CFT entropy density (9) can thus be written as

$$s = H/(2G_4\sqrt{C}). \quad (26)$$

Substituting (26) into (14) and choosing values of $\gamma, \rho_{CFT}, \tau$ at the black hole horizon, it is easy to reproduce the first Friedmann equation from the Cardy formula (14).

The second Friedmann equation can be obtained by differentiating (24) with respect to $\tau$. For the radiation dominated universe with $P = \rho/3$,

$$\dot{H}_1 = \frac{k}{r^2} - \frac{2m}{r^4} - \frac{4\rho\sigma}{L^2\sigma_c^2}. \quad (27)$$

For matter dominated universe with $P = 0$, the second Friedmann equation reads

$$\dot{H}_2 = \frac{k}{r^2} - \frac{2m}{r^4} - \frac{3\rho\sigma}{L^2\sigma_c^2}. \quad (28)$$

The CFT temperature (10) at the horizon may be expressed in the Hubble constant and its time derivative as

$$T_1 = -\frac{\dot{H}_1 + \frac{4\rho\sigma}{L^2\sigma_c^2}}{2\pi H\sqrt{C}} \quad \text{for} \quad P = \rho/3, \quad (29)$$
\[ T_2 = -\frac{\dot{H}_2 + \frac{3\rho\sigma}{L^2\sigma^2}}{2\pi H}\sqrt{C} \quad \text{for} \quad P = 0. \] 

From (23), the second Friedmann equation can be reproduced at the black hole horizon.

In summary, we have investigated the relationship between Friedmann equation and Cardy formula in general cases of brane-universes. We found that when the brane crosses the bulk AdS black hole horizon, the correspondence between Friedmann equations and Cardy formula for CFT entropy holds for all values of curvature constant $k$ on de Sitter and AdS universes. At first sight, the result does not look consistent with that in [16], where the match between Friedmann equation and Cardy formula is obtained only for de Sitter closed and AdS flat universes. But we claim that these two cases are different. Recall that in deriving the result in [16], we need a universe-sized black hole to be formed. However here in the moving domain-wall, we cannot define a universe-sized black hole embedding into the moving brane-world [20]. The generalized Cardy formula derived in [16] is the unification between Bekenstein entropy bound and Hubble entropy bound. While the Cardy formula obtained here exactly describes the entropy of radiation bath of conformal fields. Our results generalized the discussion in [18] and shed further light on the meaning of the holographic principle in cosmological setting.

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