1 Some Transversality Results

1.1 Introduction

This note is an exposition of the proof of Thom’s Conjecture, namely that algebraic curves minimise genus within their homology class in \( \mathbb{C}P^2 \), due to Kronheimer-Mrowka (see [KM]). New invariants for 4-manifolds, called Seiberg-Witten invariants, are used in the proof.

For the sake of completeness we have included an appendix at the end on Fredholm theory. As background material, the reader may wish to consult the references [D],[PP], [W].

1.2 Metrics

Let \( X \) be a connected, compact, oriented 4-manifold. Fix a reference metric \( g_0 \) on \( X \), which defines a volume form \( dV_{g_0} \) on \( X \), and hence a trivialisation of the determinant bundle \( \Lambda^4(T^*X) \) of the real cotangent bundle. This trivialisation of \( \Lambda^4 \) will be fixed throughout, and will be denoted \( dV \) without any ambiguity.

Thus the structure group for \( X \) is reduced to \( SL(4, \mathbb{R}) \), and one lets \( P_X SL(4) \) denote the principal \( SL(4, \mathbb{R}) \)-bundle on \( X \) consisting of all frames in \( TX \) on which \( dV \) yields the constant function 1 on \( X \).

The corresponding ad-bundle, denoted \( ad_{P_X} \), the vector bundle on \( X \) whose fibre over \( x \in X \) is the vector space of traceless endomorphisms \( \text{End}^0(T_xX) \) of \( T_xX \) splits into the direct sum of \( ad_{so_4} \) and \( ad_P \) (abuse of notation since \( P \) isn’t a lie algebra) corresponding to the Cartan decomposition : \( sl_4 = so_4 \oplus P \). Here, the bundle \( ad_{so_4} \) is the adjoint bundle corresponding to the principal \( SO(4, \mathbb{R}) \)-bundle \( P_X SO(4) \) consisting of \( g_0 \)-orthonormal oriented frames. It has fibre consisting of traceless \( g_0 \)-skew-symmetric endos of \( T_xX \) over \( x \). \( ad_P \) has fibre consisting of traceless \( g_0 \)-symmetric endos of \( T_xX \) over \( x \). Both are real rank-3 bundles.

Clearly, if \( g \) is another Riemannian metric on \( X \) with \( dV_g = dV \), pointwise, then \( g(u,v) = g_0((\exp h)u, (\exp h)v) \) for some section \( h \) of \( ad_P \), and thus the space of all \( C^r \) Riemannian metrics \( g \) whose volume form coincides with the prescribed one \( dV \) is precisely the space of \( C^r \)-sections \( \Gamma^r(adP) \). This space clearly contains exactly one representative from each conformal class of \( C^r \)-metric on \( X \), and may consequently be thought of as the space of equivalence classes of conformal \( C^r \)-metrics on \( X \).

The space \( C := \Gamma^r(adP) \) maybe given a Banach space structure (the reference metric \( g_0 \) is the origin, corresponds to the zero section) via the norm
which is the topology of uniform convergence of all covariant derivatives up to order $r$. We will fix $r$ to be suitably large later on.

### 1.3 $*_g$ and Self-Duality

In whatever follows, all metrics $g$ will be elements of $C$, unless indicated otherwise.

Let $\Omega^i(X)$ denote the space of smooth sections of $\Lambda^i(T^*X)$. The bundle $\Lambda^i(T^*X)$ will be denoted simply as $\Lambda^i(X)$ in future. The bundle $\Lambda^4(X)$ is identified with $\Lambda^0(X)$, the trivial bundle on $X$, via the trivialisation defined in the last section, i.e., the volume form $dV = dV_g$ goes to the constant function 1 under this identification.

The Hodge star-operator $*_g$ is the pointwise operator which makes the following diagram commute:

\[
\begin{array}{ccc}
\Lambda^2(X) & \otimes & \Lambda^2(X) \\
\uparrow & \uparrow & \uparrow \\
\Lambda^2(X) & \otimes & \Lambda^2(X) \\
\end{array}
\xrightarrow{\begin{array}{c}
\Lambda^0(X) \\
\uparrow \; *_g \\
\Lambda^0(X) \\
\end{array}}
\]

(1)

where $(\cdot, \cdot)_g$ denotes the pointwise $g$ inner product on 2-forms. One easily checks that $*_g \circ *_g = \text{id}$, and the fact that $*_g$ is an isometry with respect to the global $g$-inner product defined on $\Omega^2(X)$ by:

\[
\langle \omega, \tau \rangle_g = \int_X (\omega, \tau)_g dV = \int_X \omega \wedge *_g \tau
\]

**Remark 1.3.1** We remark here that on 2-forms the $*$-operator is a conformal invariant of the metric. This follows because if $g' = \lambda g$, where $\lambda$ is a $C^\infty$ function, then $(\omega, \tau)_{g'} = \lambda^{-2} (\omega, \tau)_g$ whereas $dV_{g'} = \lambda^2 dV_g$, so that $(\omega, \tau)_{g'} dV_{g'} = (\omega, \tau)_g dV_g$. This shows that the $*$-operator and the global inner product $\langle \cdot, \cdot \rangle_g$ are both invariant under conformal changes of the metric.

We then have the eigenbundle decomposition with respect to the involution $*_g$, namely

\[
\Lambda^2(X) = \Lambda^2_g^+ \oplus \Lambda^2_g^-
\]

where $\Lambda^2_g^\pm$ denotes the $\pm 1$-eigenspaces of $*_g$ in $\Lambda^2(X)$. The projections $\pi^\pm_g(\omega)$ are $\frac{1}{2} (\omega \pm *_g \omega)$, and $g$-orthogonal projections with respect to the pointwise $g$-inner product. The space $\Omega^2(X)$ then decomposes correspondingly to $\Omega^2_g^\pm(X)$, and this decomposition is orthogonal with respect to the global inner-product $\langle \cdot, \cdot \rangle_g$.

**Lemma 1.3.2 (Decomposition for $\Omega^2_g^+$)** There is a $\langle \cdot, \cdot \rangle_g$-orthogonal decomposition:

\[
\Omega^2_g^+ = \text{Im} d^+_g \oplus \mathcal{H}_g^{2+}
\]

where $d^+_g := \pi^+_g \circ d$ and $\mathcal{H}_g^{2+}$ denotes the $\Delta_g$-harmonic forms in $\Omega^2_g^+$. 

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Proof: By the Hodge decomposition theorem, if \( \alpha \in \Omega^2_{+} \), we may write

\[
\alpha = \alpha_1 + \alpha_2 + \alpha_3
\]

where \( \alpha_1 \in \mathcal{H}^2_{+} \), \( \alpha_2 \in \text{Im} \delta_1 \), and \( \alpha_3 \in \text{Im} \delta_2 \), where \( \delta_2 = \ast_g d \ast_g \) is the adjoint of \( d \) with respect to the global \( L^2 \)-inner product \( \langle , \rangle \), and the decomposition is orthogonal with respect thereto. Since \( \ast_g \) and \( \Delta_g \) commute, it follows that \( \ast_g \) maps \( \mathcal{H}^2_{+} \) to itself, and interchanges \( \text{Im} \delta_2 \) and \( \text{Im} d \). Thus \( \alpha = \ast_g \alpha \) implies that \( \alpha_1 = \ast_g \alpha_1 \), \( \alpha_2 = \ast_g \alpha_2 \), and \( \alpha_3 = \ast_g \alpha_3 \). Writing \( \alpha_2 = d\gamma \), we have \( \alpha = \alpha_1 + d\gamma + \ast_g d\gamma \), and since \( d\gamma + \ast_g d\gamma = 2\pi^+_g d\gamma = 2d^+_g \gamma \), the lemma follows. \( \square \).

In future, it will be our convention to identify \( H^2(X, \mathbb{R}) \) with the space of \( g_0 \)-harmonic 2-forms \( \mathcal{H}^2_{g_0} \), which will henceforth be simply written \( \mathcal{H}^2 \). Similarly, the symbols \( \ast, \pi^\pm, \Omega^\pm, \mathcal{H}^2^\pm, \) and \( d^\pm \) with subscripts will mean that the reference metric \( g_0 \) is understood.

Remark 1.3.3 If \( \alpha \in \mathcal{H}^2_{+} \), and \( \alpha \neq 0 \), then \( [\alpha] \cup [\alpha] = < \alpha, \ast \alpha >_g = ||\alpha||^2 >_g 0 \), and similarly \( \alpha \in \mathcal{H}^2_{-} \) implies \( [\alpha] \cup [\alpha] < 0 \). Thus if \( \alpha \) is self (resp. anti-selfdual) with respect to any metric \( g \), its cup product with itself is positive (resp. negative). So, for any metric \( g \), the cup product pairing is positive definite (resp. negative definite) on \( \mathcal{H}^2_{+} \) (resp. \( \mathcal{H}^2_{-} \)). Hence the numbers \( b_2^+ = \dim \mathcal{H}^2_{+} \) give the signature type of the cup product pairing \( \cup \) for any metric \( g \).

Proposition 1.3.4 Let \( \dim H^2(X, \mathbb{R}) = r, b_2^+ = 1, \) and \( b_2^- = r - 1 \). Let the positive cone of the \( \cup \)-product pairing be denoted by

\[
C := \{ \alpha \in H^2(X, \mathbb{R}) : \alpha \cup \alpha > 0 \}
\]

and let \( C_+ \) and \( C_- \) denote the two components of \( C \) determined by the sign of \( \pi^+ \alpha \). Then for any metric \( g \in C \), there exists a unique \( g \)-harmonic 2-form \( \omega_g \in \mathcal{H}^2_{g} \) satisfying (i) \( [\omega_g] \in C_+ \), (ii) \( [\omega_g] \cup [\omega_g] = < \omega_g, \ast_g \omega_g > = 1 \). Finally, (iii) \( [\omega_g] = [\omega_{g'}] \) if \( g \) and \( g' \) are conformally equivalent.

Proof: For any Riemannian metric \( g \), there is the composite map :

\[
\mathcal{H}^2 \hookrightarrow \Omega^2 \rightarrow \mathcal{H}^2_g
\]

which we call \( \psi_g \). For all \( g \), this is an isomorphism, taking a cohomology class (=\( \Delta_{g_0} \)-harmonic form) to its \( \Delta_g \)-harmonic representative. Clearly \( \psi^{-1}_g(\alpha) = [\alpha] \) for \( \alpha \in \mathcal{H}^2_g \). Via \( \psi^{-1}_g \), \( \ast_g \) maybe be regarded as an involution of \( H^2(X, \mathbb{R}) = \mathcal{H}^2 \). Choose a \( <, >_g \) unit length element of the +1-eigenspace of \( \ast_g \) (which is one dimensional by assumption) in \( H^2(X, \mathbb{R}) \). Changing the sign of this element if necessary, one can ensure that it lies in \( C_+ \). This cohomology class has unique \( g \)-harmonic representative, denoted by \( \omega_g \), and is the required element, proving (i) and (ii). Finally, (iii) follows from the Remark [1.3.3]. \( \square \)
1.4 The Map $\rho_g$

We recall the bundle isometry $\rho_g : \Lambda^{2+}_g \rightarrow su(W_+)$ of §1.4 in [PP], where $W_\pm$ are the Hermitian rank 2 bundles arising from the Spin$_c$ structure on $X$ compatible with the Riemannian metric $g$. The metrics on the bundles $h_\pm$ is to be fixed once and for all, and thus the corresponding (Hilbert-Schmidt) inner product on the bundles $su(W_+)$ are also fixed once and for all. Let us make the isometric identification

$$\rho_0 : \Lambda^{2+}_{g_0} := \Lambda^{2+} \rightarrow su(W_+)$$

where $\rho_0 := \rho_{g_0}$, with respect to the reference metric $g_0$ once and for all. Then we can view $\rho_g$ as a bundle isometry

$$\rho_g : \Lambda^{2+}_g \rightarrow \Lambda^{2+}$$

with $\rho_0 = Id$. This map can be explicitly described as follows. Write

$$g(v, w) = g_0(\exp(h)v, \exp(h)w)$$

where $h$ is a smooth section of the bundle $adP$ of §1.2 above. Then for $\omega, \tau \in \Lambda^2(T^*_x(X))$, one has

$$(\omega, \tau)_g = (\Lambda^2(\exp(-h)\omega, \Lambda^2(\exp(-h)\tau))_0$$

Thus $\rho_g = \Lambda^2(\exp(-h))$.

Note that $\Lambda^4(\exp(-h)) = \Lambda^2(\rho_g) = Id$, since $dV_g = dV_{g_0} = dV$, i.e. $h$ is a traceless endomorphism of the tangent bundle. Now one can calculate the Hodge star operator $*_g$ in terms of $* := *_{g_0}$.

**Lemma 1.4.1** The Hodge star operators $*_g$ and $*$ are related by the formula :

$$\rho_g *_g = *_g$$

Consequently, $\rho_g \pi^+_g = \pi^+_g \rho_g$, and the bundle map $\rho_g : \Lambda^2 \rightarrow \Lambda^2$ is an automorphism of the bundle $\Lambda^2$, effecting an isometry between the metrics $(\ , \ )_g$ and $(\ , \ )_0$, carrying $\Lambda^{2+}_g$ onto $\Lambda^{2+}$ and the $g$-self-dual 2-forms $\Omega^{2+}_g$ to the $g_0$-self-dual 2-forms $\Omega^{2+}$.

**Proof:**

$$\omega \wedge *_g \tau = (\omega, \tau)_g dV_g = (\rho_g \omega, \rho_g \tau)_0 dV = \rho_g \omega \wedge *_g \tau = \Lambda^2(\rho_g)(\omega \wedge \rho^{-1}_g * \rho_g \tau)$$

$$= \omega \wedge \rho^{-1}_g * \rho_g \tau$$

since $\Lambda^2(\rho_g) = Id$. Thus $*_g = \rho^{-1}_g * \rho_g$, and the result follows. \qed
1.5 Transversality

**Notation: 1.5.1** Let $b_2^+ \geq 1$. Let $C$ be as in the subsection 1.2, and, as per our convention, $\Omega^{2+}$, $\mathcal{H}^{2+}$ be the $+1$ eigenspaces with respect to $*_g$ where $g$ is the reference metric. Let $\pi_\mathcal{H}$ denote the orthogonal (with respect to $(\cdot,\cdot)_0$) projection from $\Omega^{2+}$ to $\mathcal{H}^{2+}$ from Lemma 1.3.2. Let $\delta \in \Omega^{2+}$, and let $c = c_1(L)$ denote a fixed element of $H^2(X,\mathbb{R}) = \mathcal{H}^2 \subset \Omega^2$ such that $c \cup c < 0$.

Let $G$ denote the Grassmanian of $b_2^-$-dimensional subspaces of $\mathcal{H}^2$. There is a natural rank $b_2^+$ bundle $\gamma^+$ on $G$ whose fibre over $P \in G$ is $\mathcal{H}^2/P$. One may regard this as the quotient bundle of $G \times \mathcal{H}^2$ by the tautological subbundle on $G$. By definition, there is the natural quotient map $\tau : G \times \mathcal{H}^2 \to \gamma^+$ of bundles on $G$, viz. $\tau(P,\alpha) = \alpha \pmod{P}$.

If $c \in \mathcal{H}^2$ is a cohomology class, then $c$ defines the constant section of $G \times \mathcal{H}^2$, also denoted by $c$. Let its image $\tau(c)$ be denoted by $s_c$, a section of $\gamma^+$. The zero locus of this section is:

$$S_c = \{P \in G : c \in P\}$$

If $g \in C$ is a Riemannian metric, we have the $g$-self dual projector $\pi_g^+ : \mathcal{H}^2_g \to \mathcal{H}^{2+}_g$. Thus one has a natural map:

$$P : C \to G$$

$$g \mapsto \text{Ker}(\pi_g^+ \circ \psi_g : \mathcal{H}^2 \to \mathcal{H}^{2+}_g)$$

where $\psi_g$ is the isomorphism from $\mathcal{H}^2$ to $\mathcal{H}^{2+}$ introduced in 1.3.4. This map is easily seen to be $C^\nu$.

Note that if $P(g) \in S_c$, we have that $\psi_g(c) \in \mathcal{H}^{2+}_{g^2}$, which implies $[\psi_g(c)] \cup [\psi_g(c)] = c \cup c < 0$, by 1.3.3. So $\text{Im} P \cap S_c = \phi$ if $c \cup c > 0$. When $c \cup c < 0$, We have the following lemma, proved in §5.4 of [PP] (see also [DK], §4.3.14):

**Lemma:** Let $c \in \mathcal{H}$ satisfy $c \cup c < 0$. The map $P : C \to G$ defined above is transverse to $S_c$, and its inverse image (the submanifold of ‘$c$-bad metrics’) $B_c = P^{-1}(S_c)$ is therefore a $C^\nu$ submanifold of $C$ of codimension $b_2^+$. 

Now let $\delta \in \Omega^{2+}$. Then $\rho_g^{-1}(\delta) \in \Omega^{2+}_g$. Its $g$-harmonic component $\rho_g^{-1}(\delta)_{\mathcal{H}_g}$ is in $\mathcal{H}^{2+}_g$, and thus defines a cohomology class $[\rho_g^{-1}(\delta)]_{\mathcal{H}_g}$ in $H^2(X,\mathbb{R}) = \mathcal{H}^{2+}$. Consider the element $c(g,\delta) = c - \frac{1}{2\pi}(\rho_g^{-1}(\delta)_{\mathcal{H}_g}) \in \mathcal{H} = H^2(X,\mathbb{R})$. (Box brackets denote cohomology class.) Note $c(g,0) \equiv c$. We then have the section $(g,c(g,\delta))$ of the trivial bundle $C \times \mathcal{H}^2$. Let $\sigma_{c,\delta}$ denote the image of this section under the bundle map $P^*\tau : C \times \mathcal{H} \to P^*(\gamma^+)$. Note that $\sigma_{c,0}$ is just the section $P^*s_c$ where $s_c$ is defined above. Let us denote $\sigma_{c,\delta}$ by $\sigma_c$.

Let $B_{c,\delta}$ denote the zero locus of this section. Note $B_{c,0}$ is just the submanifold $B_c$ defined above.

Let $I$ be a $C^\nu$-embedded arc, parametrised by $[-1,1]$ in $C$, meeting $B_c$ transversely. We will denote a typical element of $I$ by $g_t$. We will also use the subscript $t$ wherever a subscript $g_t$ occurs, e.g. $\omega_t$ for $\omega_{g_t}$ etc., for notational convenience. We now have a couple of transversality lemmas, for the two separate cases $b_2^+ \geq 2$ and $b_2^+ = 1$. 

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Proposition 1.5.2 Let $b_2^+ \geq 2$, and $\mathcal{I}$ be transverse to $B_c$ as above. In this case of $b_2^+ \geq 2$, this means $\mathcal{I} \cap B_c = \emptyset$. Then, for all $\delta$ in an $\epsilon$-ball $U = B(0, \epsilon)$ of the origin in $\Omega^{2+}$, the intersection $\mathcal{I} \cap B_{c,\delta}$ is empty.

Proof: We are given that $\mathcal{I} \cap B_c = \emptyset$. This means that $\mathcal{I}$ does not meet the zero locus of the section $\sigma_c = \sigma_c,0$ defined above, i.e. that $\sigma_c,0(g_t) \neq 0$ for all $t \in [-1,1]$. If $\| \|$ is some bundle metric on $P^*(\gamma^+)$, let $a := \min(||\sigma_c,0(g_t)|| : t \in [-1,1])$, so that $a > 0$. Since $\sigma_{c,\delta}(g_t)$ varies continuously with $\delta$, if one chooses $\epsilon$ small enough, then one can arrange that $||\sigma_{c,0}(g_t) - \sigma_{c,\delta}(g_t)|| < \frac{a}{2}$ for all $t \in \mathcal{I}$ and all $\delta \in B(0, \epsilon)$. It will then follow that $\sigma_{c,\delta}(g_t)$ is non-zero for all $t \in [-1,1]$, i.e. that $\mathcal{I} \cap B_{c,\delta} = \emptyset$, proving the proposition. □

Proposition 1.5.3 Let $b_2^+ = 1$, and let $\mathcal{I}$ meet the submanifold $B_c$ transversally at a single point, say $\{g_a\}$. Then, for all $\delta$ in an $\epsilon$-ball $U = B(0, \epsilon)$ of the origin in $\Omega^{2+}$, the map

$$f_\delta : \mathcal{I} \to \mathbb{R}$$

$$g_t a \mapsto \left( c - \frac{1}{2\pi} (\rho^{-1} \delta)_{H_g} \right) \cup [\omega_g] = c(g, \delta) \cup [\omega_g]$$

where $[\omega_g] = \psi^{-1}_g(\omega_g)$ is the cohomology class of $\omega_g \in H^2_g$, has a unique zero, and this zero is a regular value.

Proof: Let $\sigma_c(g_t)$ denote the section $\sigma_c$ restricted to the arc $\mathcal{I}$. Let $p : P^*(\gamma^+) \to \mathcal{C}$ denote the bundle projection, and $Z$ the zero section of this bundle. Since $b_2^+ = 1$, this a real line bundle, and may be regarded as the line subbundle of $\mathcal{C} \times \Omega^2$ whose fibre over $g$ is $H^2_g$. This bundle has a trivialisation over all of $\mathcal{C}$, defined by $g \mapsto \omega_g$, where $\omega_g \in H^2_g \subset \Omega^2$ is the form defined in [3.4]. By the fact that the zero section of $\gamma^+$ and the section $s_c(G)$ meet transversely inside $\gamma^+$, and the transversality of $P$ to $S_c$, it easily follows that $\sigma_c(\mathcal{C})$ and $Z$ meet transversely in $P^*(\gamma^+)$. By assumption, the intersection:

$$\mathcal{I} \cap B_c = p \sigma_c(\mathcal{I}) \cap p(\sigma_c(\mathcal{C}) \cap Z)$$

is a transverse intersection at the single point $\{g_a\}$. Since $p : \sigma_c(\mathcal{C}) \to \mathcal{C}$ is a diffeomorphism, which carries $\sigma_c(\mathcal{I})$ and $\sigma_c(\mathcal{C}) \cap Z$ diffeomorphically to $\mathcal{I}$ and $B_c$ respectively, it follows that inside the manifold $\sigma_c(\mathcal{C})$, $\sigma_c(\mathcal{I})$ meets $\sigma_c(\mathcal{C}) \cap Z$ transversely at the single point $\sigma_c(g_a)$.

This means that the tangent space $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{I}))$, which is a one dimensional subspace of $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C}))$ is linearly independent of the subspace $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C}) \cap Z) = T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C})) \cap T_{\sigma_c(g_a)}(Z)$, which is of codimension one in $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{C}))$. This means that the tangent space $T_{\sigma_c(g_a)}(\sigma_c(\mathcal{I}))$ is not contained in $T_{\sigma_c(g_a)}(Z)$. Since this last space is of codimension one in the tangent space $T_{\sigma_c(g_a)}(P^*(\gamma^+))$, one has that the curve $\sigma_c(\mathcal{I})$ meets the zero section $Z$ transversally in the singleton $\{\sigma_c(g_a)\}$. Now let $\theta : P^*(\gamma^+) \to \mathbb{R}$ be the map $\theta(\alpha) = [\alpha] \cup [\omega_g] = \int_X \alpha \wedge \omega_g$, where $\alpha \in H^2_g$, coming from the trivialisation of the line bundle $P^*(\gamma^+)$ described above. Then $\theta$ is a submersion, and $Z = \theta^{-1}(0)$. So saying that the curve $\sigma_c(\mathcal{I})$ meets the zero section $Z$ transversally in the singleton $\{\sigma_c(g_a)\}$ is equivalent to the statement that $f_0 = \theta \circ \sigma_c : \mathcal{I} \to \mathbb{R}$ has a unique zero at $g_a$, and $f_0'(a) := f_0'(g_a)$ is non-zero. Say $f_0'(a) > 0$. Then,
(i) \( f'_0(t) \) will be strictly positive on a closed neighborhood \( V \) of \( g_a \), and also

(ii) \( \min\{|f_0(g_t)| : g_t \in I - V^o\} = b > 0 \)

Since \( \sigma_{c,\delta} \) varies smoothly with \( \delta \), (i) and (ii) above continue to hold good for \( f_\delta := \theta \circ \sigma_{c,\delta} \) for all \( \delta \in U \), where \( U \) is an \( \epsilon \)-ball around 0 for \( \epsilon \) suitably small. This proves the proposition. \( \Box \)

2 The Parametrised Seiberg-Witten Moduli Space

2.1 The Gauge Group Action

In the sequel, we assume \( b_2^+(X) \geq 1 \). Let \( L, W^+, W^- \) be a Spin\(_c\) structure on \( X \). \( c \) will always denote \( c_1(L) \in H^2(\bar{X}, \mathbb{R}) = \mathcal{H}^{2+} \). Let \( g_0 \) be the reference metric as before. Let \( k \geq 6 \).

We need to define various spaces:

\[ \mathcal{A} := \text{the completion of } C^\infty \text{ U(1)-connections on } L \text{ (an affine space modelled on } \Omega_1^1(X)) \text{, with respect to Sobolev k-norm } L_k^2. \]

By abuse of language, we shall denote the \( L_k^2 \)-completion of \( \Omega_1 \) as \( \Omega^1 \).

\[ \Gamma(W^+) := L_k^2 \text{ completion of } C^\infty \text{ complex-valued sections of } W^+. \]

\[ \Gamma(W^-) := L_{k-1}^2 \text{ completion of } C^\infty \text{ complex-valued sections of } W^- . \]

\[ \mathcal{G} := \text{Map}(X, S^1). \text{ A Hilbert manifold whose lie algebra is the } L_{k+1}^2 \text{-completion of } \Omega^0(X). \]

\[ \Omega^{2+} := L_{k-1}^2 \text{ completion of real valued } \ast_{g_0} \text{ self-dual 2-forms, again denoted by same symbol by abuse of language.} \]

\[ \mathcal{N} := \mathcal{A} \times \Gamma(W^+). \]

\[ \mathcal{N}^* := \mathcal{A} \times (\Gamma(W^+) - \{0\}). \]

The choice of \( k \geq 6 \) implies, by Sobolev’s Lemma, that elements of each of the Sobolev spaces defined above are at least twice continuously differentiable.

\( \mathcal{G} \) acts on \( \mathcal{N} \) by the action:

\[ g.(A, \Phi) = (A - \frac{1}{2\pi i}g^{-1}dg, g\Phi) = (gA, g\Phi) \]

and the choice of norms above makes this a smooth action. Note that \( g.(A, 0) = (A, 0) \) for all \( g \in S^1 \subset \mathcal{G} \), so that the gauge group action of \( \mathcal{G} \) on \( \mathcal{N} \) is free on \( \mathcal{N}^* \), and has (as can be checked easily) isotropy \( S^1 \) on \( (A, 0) \).

**Remark 2.1.1** The decomposition lemma 1.3.2 for \( \Omega^{2+} = \Omega^{2+}_{g_0} \) continues to hold for the Sobolev completions defined above because \( d : \Omega^1_{L_k^2} \rightarrow \Omega^2_{L_{k-1}^2} \) has closed range and \( \mathcal{H}^{2+}_{L_{k-1}^2} \) consists of smooth forms by elliptic regularity.
Let \( I \) denote a compact arc in \( C \), meeting \( B_c \) as in the hypotheses of Propositions 1.5.2 and 1.5.3. A typical point on \( I \) will be denoted by \( g_t \), and a subscript \( t \) anywhere will mean that the metric \( g_t \) is being used. Absence of subscript, as always, will mean that the reference metric \( g_0 \) is understood. We also recall the map \( f_\delta : I \to \mathbb{R} \) in the setting of \( b^+_2 = 1 \), that was defined in Corollary 1.5.3. For simplicity let us assume that \( f_\delta^{-1}(0) \) is the reference metric \( g_0 \), and 0 is a regular value as stated there. We recall the notation and definitions of §3, 4 and 5 of [PP]. For a metric \( g_t \) on \( X \), \((A, \Phi)\) is called a ‘monopole’ if it satisfies the following equations:

\[
D_{A,t} \Phi = 0 \\
\rho_t(\tilde{F}_A^{+,t}) - \sigma(\Phi, \Phi) = 0
\]

where \( \tilde{F}_A^{+,t} = \pi_t^+(F_A) \), and we have, once and for all, identified \( \Gamma(su(W)_+) \) with \( \Omega^2_+ = \Omega^2 \) via \( \rho_0 \), (see §1.4), and the pairing \( \sigma : W_+ \otimes \overline{W}_+ \to \Omega^2_+ \simeq \Gamma(su(W)_+) \) is the pairing defined in [PP], §1.5. \( \rho_t = \rho_{g_t} \) is the isomorphism identifying \( \Omega^2_+ \) with \( \Gamma(su(W)_+) = \Omega^2_+ \) as defined in §1.4 (see also §1.4 of [PP]).

Let \( G \) act trivially on \( \Omega^2_+ \) and \( I \), and consider the \( G \)-equivariant map :

\[
Q : A \times \Gamma(W_+) \times I \to \Gamma(W_-) \times \Omega^2_+ \\
(A, \Phi, g_t) \mapsto (D_{A,t} \Phi, \rho_t(\tilde{F}_A^{+,t}) - \sigma(\Phi, \Phi))
\]

Thus the solutions to the Seiberg-Witten equations (2) are precisely the elements of \( Q^{-1}(0,0) \). Since \((0,0)\) won’t in general be a regular value for \( Q \), we need to consider the \( \delta \)-perturbed Seiberg-Witten equations, viz. \( Q^{-1}(0, \delta) \), where \( \delta \) is a suitably small element of \( \Omega^2_+ \). Since the map \( Q \) is \( G \) equivariant, and \((0, \delta)\) is fixed by \( G \), it is natural to quotient out \( Q^{-1}(0, \delta) \) by the \( G \)-action. We shall proceed to do this in detail, in the sequel.

### 2.2 The Derivative of \( Q \)

**Lemma 2.2.1** The derivative of \( Q \) is as follows :

(i) \( DQ_{(A, \Phi, g_t)}(a, \phi, 0) = \left(D_{A,t} \phi + 2\pi i a \circ \Phi, \rho_t(d^+_t a) - 2\Im \sigma(\Phi, \phi)\right) \)

where \( \Im \) denotes imaginary part, and \( \circ \) Clifford multiplication.

(ii) In the case when \( b^+_2 = 1 \), let \( \delta := Q(A, 0, g_0) = F^+_A \), and \( I \), \( f_\delta \) be as in Proposition 1.5.3. Then:

\[
DQ_{A,0,g_0}(0,0,g'(0)) = (0, d^+_t a + 2\pi f'_\delta(0)\omega_0)
\]

where \( \alpha \) is some 1-form and \( \omega_0 := \omega_{g_0} \) is as in Proposition 1.3.4, and \( g_0 \) is the unique zero of the function \( f_\delta \), as in 1.5.3.

**Proof:** Using the fact that \( D_A = \sum c(e_i) \circ \nabla_A e_i \) and that \( \nabla_A + sa, e_i = \nabla_A, e_i + 2\pi isa(e_i) \), we obtain that \( D_{A+sa} = D_A + 2\pi isa \circ (-) \). Thus:
\[
\frac{d}{ds}|_{s=0} (D_{A+sa,t}(\Phi + s\phi)) = D_{A,t}\phi + 2\pi i a \circ \Phi.
\]

Finally, since \(F_{A+sa} = F_A + sa\), we have \(F_{A+sa}^{+,t} = F_A^{+,t} + sa\), so that
\[
\frac{d}{ds}|_{s=0} \left( \rho_t(F_{A+sa}^{+,t}) \right) = \rho_t(d_t^+ a)
\]

Skew-sesquilinearity of \(\sigma\) implies that:
\[
\frac{d}{ds}|_{s=0} \sigma(\Phi + s\phi, \Phi + s\phi) = \sigma(\Phi, \phi) + \sigma(\phi, \Phi) = 2\Im(\Phi, \phi)
\]
and we have (i).

To see (ii), since we are computing the derivative at \(\Phi = 0\), in the direction of \(\phi = 0\), we have \(D_{A,t}\Phi = 0\) and \(\sigma(\Phi, \Phi) = 0\) for all \(t\), so the first coordinate of the right hand side of the equation in the statement is clearly zero. Now one just needs to compute \(\frac{d}{dt}|_{t=0} \left( \rho_t(F_A^{+,t}) \right)\).

By [1.3.2] we have
\[
\frac{d}{dt}|_{t=0} \left( \rho_t(F_A^{+,t}) \right) = \left\langle \frac{d}{dt}|_{t=0} \left( \rho_t(F_A^{+,t}) \right), \omega_0 \right\rangle_0 \omega_0 + d^+ \alpha
\]
for some 1-form \(\alpha\), where \(\langle , \rangle_0\) is the global inner product on \(\Omega^2\) with respect to \(g_0\). Recall from §1.4 the facts that \(\rho_0 = \Id\) and \(\langle \rho_t(-), \rho_t(-) \rangle_0 = (, )_t\), for the pointwise inner product, and so we have the same formula for the global inner product, viz. \(\langle \rho_t(-), \rho_t(-) \rangle_0 = (, )_t\).

We compute the first term:
\[
\frac{d}{dt}|_{t=0} \left( \rho_t(F_A^{+,t}) \right) = \left\langle \frac{d}{dt}|_{t=0} \left( \rho_t(F_A^{+,t}) \right), \rho_t(\omega_t) \right\rangle_0 - \left\langle F_A^{+,0}, \frac{d}{dt}|_{t=0} \rho_t(\omega_t) \right\rangle_0
\]
\[
= \frac{d}{dt}|_{t=0} \left( F_A^{+,t}, \omega_t \right)_t - \frac{d}{dt}|_{t=0} \left\langle \delta, \rho_t(\omega_t) \right\rangle_0 = \frac{d}{dt}|_{t=0} \left( \left\langle F_A, \omega_t \rightangle_t - \rho_t^{-1}\delta, \omega_t \right)_t
\]
\[
= \frac{d}{dt}|_{t=0} \left( 2\pi [c] \cup [\omega_t] - \left( (\rho_t^{-1}\delta)_{H_t}, \omega_t \right)_t \right) = 2\pi \frac{d}{dt}|_{t=0} \left( [c] \cup [\omega_t] - \frac{1}{2\pi} [(\rho_t^{-1}\delta)]_{H_t} \cup [\omega_t] \right)
\]
\[
= 2\pi \frac{d}{dt}|_{t=0} \left( c - \frac{1}{2\pi} [(\rho_t^{-1}\delta)]_{H_t} \right) \cup [\omega_t] = 2\pi f'_0(0)
\]
because for \(\alpha \in \mathcal{H}_t^2\), the \(\ast_t\) self duality of \(\omega_t\) implies \([\alpha] \cup [\omega_t] = \langle \alpha, \omega_t \rangle_t\). This proves the proposition.

\begin{proof}
\end{proof}

\section{2.3 Moduli Spaces}

\begin{proposition}
There is a Baire subset \(U\) of an \(\epsilon\)-ball around 0 in \(\Omega^{2+}\) for which both of the following hold:
\begin{enumerate}[(i)]
\item \((0, \delta)\) is a regular value for \(Q|_{\mathcal{N}^{*} \times \mathcal{I}}\)
\item The conclusions of Proposition [1.5.2] when \(b_2^+ \geq 2\), and Proposition [1.5.3] when \(b_2^+ = 1\) are satisfied by \(\delta\).
\end{enumerate}
\end{proposition}
Proof: Since \( Q \) is not a Fredholm map, one cannot directly apply Proposition 5.1.2 or Corollary 5.1.3 of the Appendix to it. However, consider the space:

\[
\mathcal{M}^*(\mathcal{I}) := \{(A, \Phi, g_t) : D_{A,t} \Phi = 0, \Phi \neq 0, g_t \in \mathcal{I}\}
\]

Since \( Q \) is \( G \)-equivariant, and \( G \) acts trivially on \( \mathcal{I} \) and \( \Omega^{2,+} \), it follows that the \( G \) action on \( \mathcal{M}^*(\mathcal{I}) \) is free. Also, \( \mathcal{M}^*(\mathcal{I}) \) is fibred over \( \mathcal{I} \) via projection to the last coordinate. Note that \( \mathcal{M}^*(\mathcal{I}) = Q_1^{-1}(0) \) where

\[
Q_1 = \pi_{\Gamma(W_-)} \circ Q : N^* \times \mathcal{I} \to \Gamma(W_-)
\]

By §3.4 of [PP], we know that 0 is a regular value of \( Q_{1,t} : N^* \times \{g_t\} \to \Gamma(W_-) \). Thus it is a regular value of \( Q_1 \), so that \( \mathcal{M}^*(\mathcal{I}) \) is an (infinite dimensional) submanifold of \( N^* \times \mathcal{I} \), whose fibre over \( g_t \in \mathcal{I} \) is \( \mathcal{M}^*(g_t) \) (as defined in § 3.4 of [PP]). Now, by Lemma 5.1.4 of the Appendix, regular values \((0, \delta)\) will exist for \( Q_{\mathcal{I} \times \{g\}} \) whenever regular values exist for the map:

\[
Q_2 = \pi_{\Omega^{2,+}} \circ Q : \mathcal{M}^*(\mathcal{I}) \to \Omega^{2,+}
\]

Again, \( Q_2 \) is not a Fredholm map. On the other hand, it is constant along \( G \)-orbits.

Claim 1: \( Q_2 \) descends to a map \( \mathcal{M}^*(\mathcal{I})/\mathcal{G} \to \Omega^{2,+} \) (which we also denote by \( Q_2 \)), and this map is Fredholm of index \( d(L) + 1 \) where

\[
d(L) := \frac{1}{4} \left( c_1(L)^2 - 3\sigma(X) - 2\chi(X) \right)
\]

(\( \sigma(X) \) is the signature, and \( \chi(X) \) the Euler characteristic of \( X \)).

Proof of Claim 1: Since \( \rho_t(F^+_{A,t}) - \sigma(\Phi, \Phi) = \rho_t(F^+_{gA,t}) - \sigma(g\Phi, g\Phi) \) for all \( g \in \mathcal{G} \), it is clear that \( Q_2 \) descends to the quotient space \( \mathcal{M}^*(\mathcal{I})/\mathcal{G} \).

Now, the inclusion of tangent spaces: \( T_x(\mathcal{M}^*(g_t)/\mathcal{G}) \hookrightarrow T_x(\mathcal{M}^*(\mathcal{I})/\mathcal{G}) \) has codimension one at each \( x \) (= dimension of the tangent space to \( \mathcal{I} \)), it is enough to prove that

\[
Q_{2,t} : \mathcal{M}^*(g_t)/\mathcal{G} \to \Omega^{2,+}
\]

is a Fredholm map of index \( d(L) \). The diagram:

\[
\begin{array}{ccccccc}
0 & \to & T_x(\mathcal{G}) & \to & T_{(A,\Phi,g_t)}(\mathcal{M}^*(g_t)) & \to & T_{([A,\Phi]_g)}(\mathcal{M}^*(g_t)/\mathcal{G}) \\
\downarrow d^* d = \Delta & & \downarrow \chi & & \downarrow DQ_{2,t} & & 0 \\
0 & \to & \Omega^0 & \hookrightarrow & \Omega^{2,+} \oplus \Omega^0 & \to & \Omega^{2,+} & \to & 0
\end{array}
\]

shows that \( DQ_{2,t} \) on the right is Fredholm of index \( d(L) \) iff the middle map \( \chi \) given by \( \chi(a, \phi, 0) = (DQ_{2,t}(a, \phi), d^* a) \) is Fredholm of index \( d(L) \), since the laplacian \( \Delta \) has one-dimensional kernel and cokernel, \( X \) being connected.

However, the diagram:
\[
0 \to T(A,\Phi,g_t)(\mathcal{M}^*(g_t)) \to T(A,\Phi,g_t)(\mathcal{N}^* \times \{g_t\}) \xrightarrow{DQ_{1,t}} \Gamma(W_-) \to 0
\]
\[
\downarrow \chi_1 \quad \Gamma(W_-) \oplus \Omega^{2+} \oplus \Omega^0 \to \Gamma(W_-) \oplus \Omega^{2+} \oplus \Omega^0 \to \Gamma(W_-) \to 0
\]

where \(\chi_1(a,\phi,0) = (DQ_t(a,\phi), d^*a)\), implies that \(\chi\) is a Fredholm operator iff \(\chi_1\) is a Fredholm operator, and of the same index since the vertical map on the right is an equality. Now, note that by the Proposition 2.2.1, part (i), we have:

\[
\chi_1(a,\phi,0) = (D_{A,t}\phi + 2\pi ia \circ \Phi, \rho_t(d_t^a) - 2\Im(\sigma(\Phi,\phi)), d^*a)
\]

Now, \(a \in L^2_k, \Phi \in L^2_k\) implies that \(a \circ \Phi \in L^2_k\), (Leibnitz rule for k-th derivative of a product and Schwartz inequality), and since we have \(L^2_{k-1}\) Sobolev norm on \(\Gamma(W_-)\), Rellich’s Lemma implies that the map \(a \mapsto a \circ \Phi\) is a compact operator from \(\Omega^1\) into \(\Gamma(W_-)\). Similar considerations apply to \(\Im\sigma(\Phi,\phi)\). Thus \(\chi_1\) is a compact perturbation of the map:

\[
\Gamma(W_+) \times \Omega^1 \to \Gamma(W_-) \times \Omega^{2+} \times \Omega^0
\]

whose index is clearly \(\text{index}(D_A) + \text{index}(d^+, d^*)\). The index of the second map is the negative of the Euler characteristic of the complex

\[
\Omega^0 \to \Omega^1 \to \Omega^{2+}
\]

which is \((- \dim H^2 - \dim H^0 + \dim H^1)\), which is \(-\frac{1}{2}(\sigma(X) + \chi(X))\). The Atiyah-Singer index theorem for the Dirac operator implies:

\[
\text{index}D_A = -\frac{1}{4}\left(\sigma(X) - c_1(L)^2\right)
\]

Combining these, we have the index of \(Q_2 : \mathcal{M}^*(\mathcal{I})/G \to \Omega^{2+}\) to be

\[
d(L) := \frac{1}{4}\left(c_1(L)^2 - 3\sigma(X) - 2\chi(X)\right)
\]

which proves Claim 1.

We now return to the proof of our Proposition. By 5.1.3 and 5.1.6, we have a Baire subset of a neighbourhood of 0 in \(\Omega^{2+}\) such that for \(\delta\) in this subset, \((0,\delta)\) is a regular value for \(Q_{\lambda\mathcal{N}^* \times \mathcal{I}}\). This proves (i). To get (ii), one just intersects this Baire subset with the \(\epsilon\)-ball \(U\) that is guaranteed by Propositions 1.5.2 and 1.5.3. □

**Notation : 2.3.2** We now fix a \(\delta\) so that both (i) and (ii) of the last Proposition 2.3.1 are satisfied. In the case \(b_2^+ = 1\), we assume that the function \(f_\delta : \mathcal{I} \to \mathbb{R}\) has its unique zero at \(g_0\), and 0 is a regular value for it. For notational simplicity we modify \(Q\) to \(Q_\delta\), a translate of \(Q\), by the formula:

\[
Q_\delta(A,\Phi,g_t) = Q(A,\Phi,g_t) - (0,\delta)
\]

We thus have the following consequence to the propositions and corollaries 5.1.2, 5.1.3, 5.1.6, 5.1.6 and 2.3.1.
Proposition 2.3.3 We have the following facts about $Q_\delta$:

(i) The map:

$$Q_\delta : \mathcal{N} \times \mathcal{I} \to \Gamma(W_-) \times \Omega^2$$

is a $\mathcal{G}$-equivariant map with $(0,0)$ as a regular value for $Q_\delta |_{\mathcal{N} \times \mathcal{I}}$, so that $Q_\delta^{-1}(0,0) \cap \mathcal{N} \times \mathcal{I}$ is a Banach manifold, which we will denote as $\mathcal{M}_\delta^*(\mathcal{I})$. It is fibred over $\mathcal{I}$ with fibre $\mathcal{M}_\delta^*(g_t)$ on $g_t$.

(ii) When $b_2^+ \geq 2$, $Q_\delta^{-1}(0,0) \subset \mathcal{N}^* \times \mathcal{I}$. In this case the group of gauge transformations $\mathcal{G}$ acts freely on all of $Q_\delta^{-1}(0,0)$, and consequently the quotient space $Q_\delta^{-1}(0,0)/\mathcal{G} = \mathcal{M}_\delta^*(\mathcal{I})/\mathcal{G} := M_{c,\delta}(\mathcal{I})$ is a manifold of dimension $d(L) + 1$. This manifold may also be regarded as $Q_{2\delta}^{-1}(0)$, where $Q_{2\delta} = \pi_1\Omega^2 \circ Q_\delta$ is as in Claim 1 in the proof of Proposition 2.3.1. Its fibre over $g_t$, $M_{c,\delta}(g_t)$ is a compact manifold of dimension $d(L)$ for all $g_t \in \mathcal{I}$. Thus $M_{c,\delta}(\mathcal{I})$ is also compact.

(iii) If $b_2^+ = 1$, and $H^1(X, \mathbb{R}) = 0$, the space

$$\left( Q_\delta^{-1}(0,0) \cap (\mathcal{A} \times \{0\} \times \mathcal{I}) \right)/\mathcal{G}$$

is just a single point (called a reducible solution), say $([A_0,0], g_0)$. A neighbourhood of this point in $Q_\delta^{-1}(0,0)/\mathcal{G}$ is homeomorphic to $\phi^{-1}(0,0)/S^1$ where:

$$\phi : \text{Ker} D_{A_0,0} \to \text{Coker} D_{A_0,0}$$

is a smooth map which is (a) $S^1$-equivariant, and (b) has 0 as a regular value in a small deleted neighbourhood $U - \{0\}$ of 0 in Ker $D_{A_0,0}$. For metrics $g_t$ such that $t \neq 0$ the moduli space $M_{c,\delta}(g_t)$ is a finite set of points if $d(L) = 0$.

Proof:

(i) follows immediately from Proposition 2.3.1. For (ii), note that $Q_\delta(A, \Phi, g_t) = (0,0)$ implies that $D_{A,t}\Phi = 0$, and $\rho_t(F_{A,t}^-) = \sigma(\Phi, \Phi) + \delta$. This implies

$$\frac{1}{2\pi} \left( \rho_t^{-1}(\Phi, \Phi) \right)_{\mathcal{H}_t} = \frac{1}{2\pi} \left( F_{A,t}^- - \rho_t^{-1}(\delta) \right)_{\mathcal{H}_t} = \pi_t^+ \circ \psi_t(c(g_t, \delta)) = \sigma_{c,\delta}(g_t)$$

in the notation of [1.5.1]. But $\mathcal{I} \cap B_{c,\delta} = \phi$, i.e. the section $\sigma_{c,\delta}$ is nonvanishing on $\mathcal{I}$ by our choice of $\mathcal{I}$ and $\delta$, so $\sigma(\Phi, \Phi) \neq 0$, and so $\Phi \neq 0$. Thus there are no reducible solutions, and $Q_\delta^{-1}(0,0) \subset \mathcal{N}^* \times \mathcal{I}$. Hence $Q_\delta^{-1}(0,0)/\mathcal{G}$ is a smooth manifold, since the $\mathcal{G}$ action is free on $\mathcal{N}^* \times \mathcal{I}$, and since each fibre $M_{c,\delta}(g_t)$ (notation of §3.5 of [PP]) is compact by §5.2 of [PP]. Since $\mathcal{I}$ is compact, so is $M_{c,\delta}(\mathcal{I})$. In the case when $d(L) = 0$, $M_{c,\delta}(g_t)$ is then a finite set of points, and $M_{c,\delta}(g_a)$ and $M_{c,\delta}(g_b)$ are cobordant, and so have the same cardinality (modulo 2). In particular, $M_{c,\delta}(g_1)$ and $M_{c,\delta}(g_{-1})$ have the same cardinality modulo 2. Thus we have (ii), since the rest of it follows from Proposition 2.3.1 and Lemma 2.1.6 of the Appendix.
To get (iii), note that when \( b^+_2 = 1 \), \( Q_{2, \delta}(A, 0, g_t) = (0, 0) \) implies that \( \sigma_{e, \delta}(g_t) = 0 \). This happens (see Proposition \[1.5.3\]) iff \( f_\delta(g_t) = 0 \). By our choice of \( \mathcal{I} \) and \( \delta \), this happens only at \( g_t = g_0 \). Now choose a point \((A_0, 0, g_0) \in Q^{-1}_\delta(0, 0) \cap (A \times \{0\} \times \mathcal{I}) \). Then \( Q_\delta(A_0 + a, 0, g_0) = (0, 0) \) implies \( \rho_0(d^+_a a) = d^+ a = 0 \). Let \( g \in G \). The gauge action takes \( A_0 \) to \( A_0 + g^* \omega \), where \( \omega \in H^1(S^1, \mathbb{R}) \) is the generating “angle” 1-form of \( S^1 \). But \( g^* \omega \) is thus a closed 1-form on \( X \). Since \( H^1(X, \mathbb{R}) = 0 \) by assumption, \( g^* \omega = da \) for some function \( \alpha \in \Omega^0 \). Conversely, given an \( \alpha \in \Omega^0 \), the map \( g : X \to S^1 \) defined by \( g(x) = e^{2\pi i \alpha(x)} \) satisfies \( g^* \omega = da \). Thus \( \left( Q^{-1}_\delta(0, 0) \cap (A \times \{0\} \times \mathcal{I}) \right) / G \simeq \frac{\text{Ker} d^+}{\text{Im} d} = H^1(X, \mathbb{R}) = 0 \). So it consists of a single point \([(A_0, 0, g_0)]\).

Now we need to get a model for a neighbourhood of this point. It is well known that for a smooth action, the neighborhood of a point in the orbit space corresponding to an orbit with isotropy \( G \) is homeomorphic to a neighborhood in the orthogonal slice of that orbit divided by the isotropy \( G \). A slice in \( \Omega^1 \times \Gamma(W_+) \times \mathcal{I} \) orthogonal to the \( \mathcal{G} \)-orbit of \((A_0, 0, g_0) \), which we will take as \((0, 0, g_0) \) by setting the origin at \( A_0 \), is clearly \((\text{Im} d)^+ \times \Gamma(W_+) \times \mathbb{R} \) because the \( \mathcal{G} \)-orbit has tangent space \( \text{Im} d \times 0 \times 0 \) at \((0, 0, g_0) \). Also \( DQ_\delta;A_0,0,g_0 \) is an \( S^1 \)-equivariant map, (because \( Q_\delta \) is \( G \) equivariant), where \( S^1 \) is the isotropy group of the point \((A_0, 0, g_0) \), and of course, this \( S^1 \)-action is orthogonal and linear (it is the derivative of the \( S^1 \)-action on the space \( A \times \Gamma(W_+) \times \mathcal{C} \)). By Lemma \[2.2.1\] we have:

\[
\begin{align*}
DQ_\delta;A_0,0,g_0(a, \phi, 0) &= (D_{A_0} \phi, d^+ a) \\
DQ_\delta;A_0,0,g_0(0, 0, \lambda g'(0)) &= (0, \lambda(2\pi f_\delta'(0)\omega_{g_0} + d^+ a))
\end{align*}
\]

We now claim that:

\[
DQ_\delta;A_0,0,g_0 : (\text{Im} d)^+ \times \Gamma(W_+) \times \mathbb{R} \to \Gamma(W_-) \times \Omega^{2+}
\]

is a Fredholm operator, whose kernel is \( \text{Ker} D_{A_0} \subset \Gamma(W_+) \), and cokernel is \( \text{Coker} D_{A_0} \subset \Gamma(W_-) \).

From the above formulae it follows that

\[
\text{Ker} (DQ_\delta;A_0,0,g_0) = \{(a, \phi, \lambda g'(0)) : a \perp \text{Im} d, D_{A_0} \phi = 0, d^+ a + \lambda(d^+ a + 2\pi f_\delta'(0)\omega_{g_0}) = 0\}
\]

Now, \( \omega_{g_0} \perp \text{Im} d^+ \), so on the right hand side we must have \( \lambda 2\pi f_\delta'(0)\omega_{g_0} = 0 \). But \( f_\delta'(0) \neq 0 \), by our choice of \( \mathcal{I} \), \( \delta \) and Proposition \[1.5.3\], so \( \lambda = 0 \). But this implies that \( D_{A_0} \phi = 0 \) and \( d^+ a = 0 \). Thus

\[
\text{Ker} (DQ_\delta;A_0,0,g_0) = \left( (\text{Im} d)^+ \cap \text{Ker} d^+ \right) \times \text{Ker} D_{A_0}
\]

Since \( (\text{Im} d)^+ \cap \text{Ker} d^+ \simeq \text{Ker} d^+/\text{Im} d \subset H^1(X, \mathbb{R}) = 0 \), this is just \( \text{Ker} D_{A_0} \). It is finite dimensional by ellipticity of the Dirac operator.

To show that

\[
DQ_\delta;A_0,0,g_0((\text{Im} d)^+ \times \Gamma(W_+) \times \mathbb{R})
\]

is closed, it is enough to show that \( DQ_\delta;A_0,0,g_0((\text{Im} d)^+ \times \Gamma(W_+) \times 0) \) is closed. But this is just \( \text{Im} D_{A_0} \times d^+ (\text{Im} d)^+ \). Now \( (\text{Im} d)^+ = \text{Ker} d^* = \mathcal{H}^1 \oplus d^* \Omega^2 \), so \( d^+ (\text{Im} d)^+ = d^+ d^* \Omega^2 \). Since \( \Omega^1 = \mathcal{H}^1 \oplus d^* \Omega^2 \oplus d\Omega^0 \), \( d^+ \Omega^1 = d^+ d^* \Omega^2 \) as well. Thus \( d^+ (\text{Im} d)^+ = d^+ \Omega^1 \). This is clearly closed by the decomposition of Lemma \[1.3.3\]. Since \( D_{A_0} \), the Dirac operator, is elliptic, its range is closed too, so the range of

\[
DQ_\delta;A_0,0,g_0 : (\text{Im} d)^+ \times \Gamma(W_+) \times \mathbb{R} \to \Gamma(W_-) \times \Omega^{2+}
\]

is closed.
is closed. Its cokernel is:

\[ \{(\psi, \tau) : \psi \perp \operatorname{Im} D_{A_0}; \; \tau \perp d^+a + \lambda(d^+\alpha + 2\pi f')_0(0)\omega_0 \; \forall \lambda \in \mathbb{R}, \; a \in (\operatorname{Im} d)^\perp \} \]

This implies \( \psi \in \operatorname{Coker} D_{A_0} \). Also, setting \( \lambda = 0 \), one finds that \( \tau \perp d^+((\operatorname{Im} d)^\perp) \), i.e. \( \tau \perp d^+\Omega^1 \). Since \( \tau \in \Omega^2 \), the Lemma \([1.3.2]\) implies that \( \tau \in \mathcal{H}^2 \). However, by setting \( a = 0 \) and \( \lambda = 1 \), \( \tau \) is also orthogonal to \( 2\pi f'_0(0)\omega_0 + d^+\alpha \), and since we already have \( \tau \perp d^+\Omega^1 \), it follows that \( \tau \perp \omega_0 \), since \( f'_0(0) \neq 0 \) by our choice of \( \mathcal{I} \) (from \([1.5.3]\)). We now note that \( \omega_0 \) was chosen as the basis element of \( \mathcal{H}^2 \), so \( \tau = 0 \). Thus the cokernel of our map is just \( \operatorname{Coker} D_{A_0} \). This proves our Fredholm-ness assertion, and the identifications of kernel and cokernel.

Thus, Proposition \([5.1.3]\) applies, and a neighbourhood of \((A, 0, g_0)\) in \( Q^{-1}_\delta(0, 0) / \mathcal{G} \) is homeomorphic to a neighbourhood of \( 0 \) in \( \phi^{-1}(0) / S^1 \), where the \( S^1 \)-equivariant map:

\[ \phi : \operatorname{Ker} D_{A_0} \to \operatorname{Coker} D_{A_0} \]

has \( 0 \) as a regular value when restricted to \( \phi^{-1}(0) \setminus \{0\} \). The last statement follows from the fact that if \( t \neq 0 \) then \( f'_0(g_t) \neq 0 \), and \( M_\delta^0(g_t) = M_\delta(g_t) \) is a Banach manifold (by part (i)), since there are no reducible solutions, and its quotient by \( \mathcal{G} \) has dimension \( d(L) = 0 \) by part (ii) above. This proves (iii), and the proposition. \( \square \)

### 3 Computations

#### 3.1 The case of \( \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2} \)

Let \( X = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2} \), which is just the blow-up of \( \mathbb{C}P^2 \) at \( n \) points. The case \( n = 0 \) is just \( \mathbb{C}P^2 \), which is a special case of the ensuing discussion. Since \( X \) is a complex manifold, we have a canonical spin\(_c \) structure on \( X \) (as in § 6.1 of [PP]), with \( L = K_X^{-1}, \; W_+ = K_X^{-1} \oplus 1_{\mathbb{C}P^2}, \; W_- = T_{\text{hol}}(X) \). Also

(i) \( H^1(X, \mathbb{R}) = 0 \), and

(ii) \( H^2(X, \mathbb{R}) = RH \oplus \sum_{i=1}^n R E_i \), where \( H \) is the pullback of the hyperplane class in \( \mathbb{C}P^2 \) via the blow up map \( \pi \) and will, from here on, be called the hyperplane class by abuse of language. \( E_i \) is the generating \( \overline{\mathbb{C}P^2} \) in the \( i \)-th copy of \( \mathbb{C}P^2 \). The cup products between these classes are:

\[ H \cup H = 1, \; E_i \cup E_j = -\delta_{ij}, \; H \cup E_i = 0 \; \forall \; 1 \leq i, j \leq n \]

Thus \( b^+_2 = 1, \; b^-_2 = n \) and the cup pairing is of type \((1, n)\).

(iii) From the formula \( K_X = \pi^*(K_{\mathbb{C}P^2}) \otimes [E] \), where \( E := \sum_{i=1}^n E_i \) is the exceptional divisor, and that \( c_1(K_X^{-1}) = c_1(\mathbb{C}P^2) = 3H_{\mathbb{C}P^2} \), it follows that \( c_1(L) = c_1(K_X^{-1}) = 3H - E \). Thus \( c_1(L)^2 = 9 - n \), and this is \( < 0 \) whenever \( n > 9 \). Since \( \chi(X) = n + 3 \), and \( \sigma(X) = 1 - n \), we have \( d(L) = \frac{1}{4}(c_1(L)^2 - 3\sigma - 2\chi)(X) = 0 \) (see Proposition \([2.3.1]\) for the definition of \( d(L) \)).
**Proposition 3.1.1** For \( X = \mathbb{C}P^2 \# n \mathbb{C}P^2 \), the Seiberg-Witten Moduli Space \( M_{c, \delta}(g_t) \) consists of finitely many points whenever \( f_\delta(g_t) \neq 0 \), (i.e. when \( t \neq 0 \)) where \( f_\delta \) is the function defined in \([1.5.3]\).

**Proof:** Follows immediately from (iii) of Proposition 2.3.3 \( \square \)

Of course, this proposition does not tell us what the cardinality (mod 2) of the moduli space \( M_{c, \delta}(g_t) \) might be. To show that there exist metrics for which this cardinality is non-zero will be our next goal.

**Proposition 3.1.2** Let \( X = \mathbb{C}P^2 \# n \mathbb{C}P^2 \), as above, with \( n > 9 \). Let \( I \) be an arc in \( \mathcal{C} \) chosen in accordance with the Proposition 2.3.3 (i.e. \((0,0)\) is a regular value of \( Q_\delta|N^* \times I \), and the function \( f_\delta : I \rightarrow \mathbb{R} \) satisfies the conclusion of Proposition 1.5.3. Then :

\[
\# M_{c, \delta}(g_{-1}) - \# M_{c, \delta}(g_1) = 1 \pmod{2}
\]

**Proof:** By (iii) of Proposition 2.3.3, a neighbourhood of \((A_0,0,g_0)\), the unique reducible point in \( M_{c, \delta}(I) \) is homeomorphic to a neighbourhood of 0 in \( \phi^0/S^1 \), where \( \phi : \text{Ker } D_{A_0} \rightarrow \text{Coker } D_{A_0} \) is a smooth \( S^1 \)-equivariant map, and \( \phi \) has 0 as a regular value in a deleted neighbourhood of 0 in \( \text{Ker } D_{A_0} \). We need to show that an odd number of arcs emerge from 0 in this neighbourhood.

Now

\[
\text{index}_{\mathbb{R}}(D_{A_0}) = \frac{1}{2} \text{index}_{\mathbb{R}}(D_{A_0}) = \frac{1}{8}(c_1(L)^2 - \sigma(X)) = \frac{1}{8}(9 - n - 1 + n) = 1
\]

Thus if \( \text{dim}_{\mathbb{R}}(\text{Coker } D_{A_0}) = r \), then \( \text{dim}_{\mathbb{R}}(\text{Ker } D_{A_0}) = r + 1 \). So our \( \phi \) is an \( S^1 \)-equivariant smooth map (with \( S^1 \) acting as scalar multiplication on both sides) from \( \mathbb{C}^{r+1} \) to \( \mathbb{C}^r \), with 0 a regular value for \( \phi|_{\mathbb{C}^{r+1}-0} \). Let \( \mathcal{O}(-1) \) denote the tautological bundle on \( \mathbb{C}P^r \), and let

\[
\pi : (\mathbb{C}^{r+1} - 0)/S^1 \simeq \mathbb{C}P^r \times \mathbb{R}^+ \rightarrow \mathbb{C}P^r
\]
denote the projection into the first factor. If we then denote by \( <z_0, ... z_r> \) the \( S^1 \)-equivalence class of \((z_0, ... z_r)\) in the orbit space \( (\mathbb{C}^{r+1} - 0)/S^1 \), we get a natural section \( s \) of the bundle \( \pi^* \text{hom}(\mathcal{O}(-1), \mathbb{C}^r) \simeq \pi^*(\mathbb{C}^r \otimes \mathcal{O}(1)) \) by setting :

\[
\pi(<z_0, ... , z_r>)(u) = \phi(z_0, ..., z_r)
\]

where \( u \) is the unit vector \( \frac{(z_0, ... , z_r)}{\| (z_0, ... , z_r) \|} \). The \( S^1 \)-equivariance of \( \phi \) implies that this map \( s \) is well defined, and that 0 is a regular value for \( \phi \) on a deleted neighbourhood means that \( s \)
is transverse to the zero-section. Now, $s^{-1}(0)$ is clearly $φ^{-1}(0,0) - \{0\}/S^1$. Thus, modulo 2, the number of arcs going to 0 is precisely the number of points in $s^{-1}(0) \cap (\mathbb{CP}^r \times \{ε\})$ modulo 2, for generic $ε$. But this is just the Euler number of the bundle $\mathcal{C}^r \otimes O(1)$, which is 1. Thus

$$\#M_{c,δ}(g_{-1}) - \#M_{c,δ}(g_1) = 1 \pmod{2}$$

and we are done. 

**Proposition 3.1.3** (Hitchin) There exist metrics $g$ on $X = \mathbb{CP}^2 \# n\mathbb{CP}^2$, such that:

(i) $g$ is Kahler.

(ii) $[ω_g] \cup c_1(L)$ has the same sign as the scalar curvature $s_g$, which is positive.

(iii) For any metric $g'$ conformally equivalent to a Kahler metric $g$ satisfying (i) and (ii) above, $[ω_{g'}] \cup c_1(L) > 0$.

**Proof:** See reference number [4] in [KM] for the proof of (i) and (ii). For (iii), note that $[ω_{g'}] = [ω_g]$ when $g$ and $g'$ are conformally equivalent, by (iii) of 1.3.4. 

Clearly, for a metric as in Proposition 3.1.3 above, the moduli space $M_{c,δ}(g)$ is empty, by § 5.2 of [PP]. Consequently, we have the following corollary to Proposition 3.1.2.

**Corollary 3.1.4** If $g$ is a metric on $X = \mathbb{CP}^2 \# n\mathbb{CP}^2$ such that $c_1(L) \cup [ω_g] < 0$, then the moduli space $M_{c,δ}(g) \neq \emptyset$.

**Proof:** Assuming there is such a metric, join it by an arc $I$ in $C$ meeting $B_c$ transversely, at one point, and apply Propositions 1.5.3, 3.1.2. 

### 3.2 The Tubing Construction

Let $X$ be a compact, connected oriented Riemannian 4-manifold, and let $Y$ be a compact 3-manifold, also oriented (so that its normal bundle in $X$ is trivial) such that:

$$X - Y = X_+ \cup X_-$$

as two disjoint components. Assume further that there exists a Riemannian metric $g$ on $X$ such that $g_{|\nu(ε)} = dl^2 \times g_Y$, where $ν(ε)$ is an $ε$-tubular neighbourhood of $Y$ in $X$, and $g_Y$ is a smooth Riemannian metric on $Y$. Let us denote $g_{|X_\pm} := g_{\pm}$. 

Definition 3.2.1 Define the \( R \)-tubing of \( X \), denoted \((X_R, g_R)\) to be the manifold.

\[
X_R = (X_- \cup \{[-R, 0] \times Y\}) \cup \{(0, R] \times Y\} \cup X_+
\]

with the metric \( g_R \) being defined by \( g_R|_{X_\pm} = g_\pm \) and \( g_{[-R, R]\times Y} = dt^2 \times g_Y \) on the piece \([-R, R] \times Y\). Note the ends \( \{\pm R\} \times Y \) are identified with \( \partial X_\pm \).

Clearly, since \( X_R \) is diffeomorphic to \( X \), one may view \( g_R \) as a new metric on \( X \). Of course, its volume form \( dV_{g_R} \) will change, but by a global conformal change one may restore the old volume form. Note that this conformal change does not affect \( \omega \) clearly enough to show that \( c \), its volume form, which contains the class \( H \), and \( [\omega] \) lie in the bounded region \( C_+ \cap \{\beta : \beta \cup H = 1\} \), so have a (with respect to the global inner product \( \langle \cdot, \cdot \rangle_{g_0} \) \( L^2 \)-convergent subsequence, which we continue to call \( \omega_i \).

Lemma 3.2.2 \( \tilde{\Sigma} \) has trivial normal bundle in \( X \).

Proof: Let \( \nu = \nu(\tilde{\Sigma}) \) be the normal bundle of \( \tilde{\Sigma} \) in \( X \). Then we know that the Euler number of this bundle is precisely the self-intersection number of \( \tilde{\Sigma} \) in \( X \), i.e. \( (dH - E_i).(dH - E_i) = d^2H^2 + E^2 = d^2 + \sum E_i^2 = d^2 - d^2 = 0 \). Thus \( \nu(\tilde{\Sigma}) \) has a nowhere vanishing section, i.e. has a trivial line bundle as a summand. Since it is orientable, it follows that it is trivial. \( \square \)

Thus if \( \nu_\epsilon(\tilde{\Sigma}) \) is an \( \epsilon \)-tubular neighbourhood of \( \tilde{\Sigma} \), we have \( \nu_\epsilon(\tilde{\Sigma}) \simeq D_\epsilon \times \tilde{\Sigma} \), and its boundary \( \partial\nu_\epsilon(\tilde{\Sigma}) \simeq S^1 \times \tilde{\Sigma} \). Call this last space \( Y \). Now apply the tubing construction of Lemma 3.2.1 to \( X, Y \), with \( X_- = X - \nu_\epsilon(\tilde{\Sigma}) \), \( X_+ = \nu_\epsilon(\tilde{\Sigma}) \), \( Y = \partial\nu_\epsilon(\tilde{\Sigma}) \). Note that an \( \epsilon \)-neighborhood of \( Y \) is diffeomorphic to \( (-\epsilon, \epsilon) \times Y \).

Proposition 3.2.3 Assume there exists a metric \( g_0 \) on \( X \) such that \( g_0|_{\nu_\epsilon(Y)} \) is a product metric, and let \((X(R), g_R)\) denote the \( R \)-tubing of \( g \) as defined above. Then, for \( R \) sufficiently large, \( c_1(L) \cup [\omega_{g_R}] < 0 \).

Proof: For notational ease we shall denote \( \omega_{g_R} \) by \( \omega_R \). Let \( R_i \) be a sequence of positive numbers tending to \( \infty \). We know by Proposition 3.3.1 that in \( H^2(X, R) = H, [\omega_{R_i}] \cup [\omega_{R_i}] = 1 \), and \([\omega_{R_i}] \in C_+ \) where \( C_+ \) is the preferred component of the positive cone of the cup-product form, which contains the class \( H \) defined above. So \( [\omega_{R_i}] \cup H > 0 \) for all \( i \). Let us normalise and define \( \omega_i \) to be the unique \( \Delta_0 \) harmonic 2-form representing \(([\omega_{R_i}] \cup H)^{-1}[\omega_{R_i}] \). It is clearly enough to show that \( c_1(L) \cup \omega_i < 0 \) for \( i \) large enough. Since \( \omega_i \cup H = 1 \) for all \( i \), and hence \( \omega_i \) lie in the bounded region \( C_+ \cap \{\beta : \beta \cup H = 1\} \), so have a (with respect to the global inner product \( \langle \cdot, \cdot \rangle_{g_0} \) \( L^2 \)-convergent subsequence, which we continue to call \( \omega_i \).
Since \( \omega_i \) are \( \Delta_{g_0} \)-harmonic, the Garding inequality says that they converge in all Sobolev \( L^2_k \)-norms (with respect to the \( g_0 \)-metric on \( X \)), and hence uniformly on compact subsets \( K \subset X_+ \cup (0, \varepsilon] \times Y \subset X \), in particular. Now, there is a diffeomorphism \( \phi_i : X(R_i) \to X \), which takes the piece \( X_+ \cup (0, R_i) \times Y \) to the piece \( X_+ \cup (0, \varepsilon) \times Y \), taking the metric \( g_{R_i} := g_i \) on \( X(R_i) \) to the metric \( g_i \) on \( X \). Note that \( \phi_i \) are identity on \( X_+ \), and just a scaling of \( R_i \) of the \( t \) variable on \( (0, R_i] \times Y \) and identity on the \( Y \) variable. So, on the piece \( (0, \varepsilon) \times Y \), the 1-form \( ds \) has length \( \frac{1}{R_i} \), with respect to \( g_i \), but length 1 with respect to \( g_0 \). The 1-forms \( ds \) on \( Y \) have the same length with respect to \( g_i \) and \( g_0 \). It is now more convenient to change the variable \( t \) to \( R - t \), which replaces \( X_+ \cup (0, R] \times Y \) with the isometric manifold \( X_+ \cup [0, R] \times Y \), where \( \partial X_+ \) is glued to \( \{0\} \times Y \), and similarly \( s \) to \( \varepsilon - s \) taking \( (0, \varepsilon) \) to \( [0, \varepsilon) \). Let us define \( X_{i+} = X_+ \cup [0, R_i] \times Y \), and \( \phi_i \) is the composite diffeomorphism \( \phi_i : X_{i+} \to X_+ \cup [0, \varepsilon) \times Y \). Also, since \( \phi_i \) and \( \phi_j \) are both identity on \( Y \), and \( \phi_i^* ds = \frac{1}{R_i} dt \), \( \phi_j^* ds = \frac{1}{R_j} dt \) for \( j \geq i \), where \( s \in [0, \varepsilon) \), it follows that for \( j \geq i \) we have, for any \( i \)-form \( \omega \) on \( (0, \varepsilon) \times Y \), the inequalities:

\[
\left\| \phi_i^* \omega \right\|_{g_i, X_{i+}} \leq \left\| \omega \right\|_{g_0, X_+ \cup (0, \varepsilon) \times Y} \\
\left\| \phi_i^* \omega - \phi_j^* \omega \right\|_{g_j, X_{j+}} \leq \varepsilon \left( \frac{1}{R_i} - \frac{1}{R_j} \right) \left\| \omega \right\|_{g_0, X_+ \cup (0, \varepsilon) \times Y} 
\]

Note that all the \( X_{i+} \) are isometrically embedded in the non-compact manifold with infinite end \( X_\infty := X_+ \cup [0, \infty) \times Y \), where \( \{0\} \times Y \) is glued to \( \partial X_+ \), with the complete metric defined by \( g_+ \) on \( X_+ \) and \( dt^2 \times g_Y \) on the infinite tube \( [0, \infty) \times Y \) (call this metric \( g_\infty \)), so that \( g_\infty|X_i = g_i \). Extending \( \phi_i^* \omega_i \) on \( X_{i+} \) by 0 to all of \( X_\infty \) defines an \( L^2(g_\infty) \) form on \( X_\infty \), which we continue to denote by the same symbol. Now let \( \tilde{\omega}_i \) be the \( \Delta_{g_\infty} \)-harmonic part of \( \phi_i^* \omega_i \). This is possible in view of the Kodaira decomposition:

\[ L^2 = \mathcal{H}^2_\infty \oplus \overline{\delta \Lambda_c} \oplus \overline{\Lambda_c} \]

which is always true for a \textit{complete} Riemannian metric (see [Ko]). (For such a complete metric \( g \), \( \text{Ker} \Delta_g \) is the same as the space of closed and co-closed forms.) Now let \( K \) be a compact subset of \( X_{i+} \), and hence \( X_\infty \). Since for \( j \geq i \), we have \( X_{i+} \subset X_{j+} \subset X_\infty \) and \( g_j = g_i \) on \( X_{i+} \), we have the inequalities of sup \( (C^0) \) norms:

\[
\left\| \tilde{\omega}_i - \tilde{\omega}_j \right\|_{\infty, K, g_\infty} \leq \left\| \phi_i^* \omega_i - \phi_j^* \omega_j \right\|_{\infty, K, g_\infty} \\
= \left\| \phi_i^* \omega_i - \phi_j^* \omega_j \right\|_{\infty, K, g_j} \\
\leq \left\| \phi_i^* \omega_i - \phi_j^* \omega_i \right\|_{\infty, K, g_j} + \left\| \phi_j^* \omega_i - \phi_j^* \omega_j \right\|_{\infty, K, g_j} \\
\leq \varepsilon \left( \frac{1}{R_i} - \frac{1}{R_j} \right) \left\| \omega_i \right\|_{\infty, \phi_j(K), g_0} + \left\| \omega_i - \omega_j \right\|_{\infty, \phi_j(K), g_j} \\
\leq \varepsilon \left( \frac{1}{R_i} - \frac{1}{R_j} \right) \left\| \omega_i \right\|_{\infty, \phi_j(K), g_0} + \left\| \omega_i - \omega_j \right\|_{\infty, \phi_j(K), g_0}
\]

by using the two inequalities \([3]\) above. This shows that the \( \Delta_{g_\infty} \)-harmonic forms \( \tilde{\omega}_i \) are uniformly Cauchy on compact subsets of \( X_\infty \), and hence converge uniformly on compact sets.
to some $\Delta_{g_\infty}$-harmonic form $\tilde{\omega}$. Also,

$$
\|\tilde{\omega}\|_{g_\infty,X_\infty} = \lim_{i \to \infty} \|\tilde{\omega}_i\|_{g_i,X_i} \\
\leq \lim_{i \to \infty} \|\phi^*_i \omega_i\|_{g_i,X_i} \\
\leq \lim_{i \to \infty} \|\omega_i\|_{g_0, X_i \cup [0, \epsilon) \times Y} < \infty
$$

shows that $\tilde{\omega}$ is an $L^2$ 2-form with respect to $g_\infty$ on $X_\infty$, so it is in $\mathcal{H}^2_{g_\infty}$. However, the Kodaira decomposition above shows that this space is precisely the image of $H^2_{c}(X_\infty)$ in $H^2(X_\infty)$. However, from the definition of $X_+\times Y$, this is the image of $H^2_{c}(X_\infty)\times Y$, which is the kernel of the restriction map $H^2(\Sigma \times D^2) \to H^2(\Sigma \times S^1)$, which is zero (e.g. by Kunneth formula). Thus $\tilde{\omega} = 0$.

Now, since $\nu_\epsilon(\tilde{\Sigma})$ is a trivial bundle, we may find a copy of $\tilde{\Sigma}_1$ which lies in $\partial X_+ = \partial \nu_\epsilon(\tilde{\Sigma})$, and is homologous to $\tilde{\Sigma}$ in $X$. Note that $\tilde{\Sigma}_1$ therefore lies in $X_+$ for all $i$, and in $X_\infty$, satisfying $\phi^*_i (\tilde{\Sigma}_1) = \tilde{\Sigma}_1$ for all $i$. Now,

$$
\lim_{i \to \infty} (\omega_i \cup (dH - E)) = \lim_{i \to \infty} \int_{\tilde{\Sigma}_1} \omega_i = \lim_{i \to \infty} \int_{\tilde{\Sigma}_1} \tilde{\omega}_i = \int_{\tilde{\Sigma}_1} \tilde{\omega} = 0
$$

Consequently, since $c_1(L) = 3H - E$, we have

$$
\lim_{i \to \infty} (\omega_i \cup c_1(L)) = \lim_{i \to \infty} \omega_i \cup (dH - E) - (d - 3)(\omega_i \cup H) = 0 - (d - 3) < 0
$$

since $d > 3$ by assumption. This proves the proposition.

**Corollary 3.2.4** There exists a metric $g$ on $X$, which is a product in a tubular neighbourhood of $Y$, such that $\#M_{c,\delta}(g_R) \neq 0$ for $R$ large enough.

**Proof:** Since $Y$ has a product neighbourhood, one can put a product metric on an $\epsilon$-neighbourhood of $Y$, and extend it to all of $X$ by using a partition of unity. The rest follows from the Corollary 3.1.4 and the proposition above. \qed

### 3.3 Temporal Gauge Solutions

We go back to the setting of the tubing construction of Def. 3.2.1, and do some analysis on the tube portion $[-R,R] \times Y$. Look at the restrictions of the $U(2)$ bundles $W_\pm$ coming from the chosen Spin$_c$ structure on $X$ to $[-R,R] \times Y$, viz. $W_\pm|[-R,R] \times Y$. These are both isomorphic via $c(dt)$, Clifford multiplication by the (unit length) 1-form $dt$, and hence may both be regarded as pullbacks via the projection map $[-R,R] \times Y \to Y$ of a $U(2)$-bundle $W_3$ on $Y$. If $A$ is a connection on $[-R,R] \times Y$, then $A_{|t \times Y} := A(t)$ is a connection for
L_{|Y}$ for each $t$, and maybe regarded as a path in $A_Y$, the affine space of $U(1)$ connections on $L_{|Y}$. Similarly, if $\Phi \in \Gamma(W_3)$, one may regard the restriction to the $t$-slice $\Phi(t)$ as a path of sections of $\Gamma(W_3)$. Note that the metric on $[-R, R] \times Y$ is a product, and hence induces the same metric on each slice $\{t\} \times Y$.

**Definition 3.3.1** We say a connection $A$ on $R \times Y$ is in *temporal gauge* if it has no $dt$ component. (We are of course fixing a reference connection $A_0$, as always). Say it is *translation invariant in a temporal gauge* if it is the pullback of a connection on $L_{|Y}$. Similar definitions make sense for $[0, R) \times Y$, and $[-R, R) \times Y$. Finally, a solution $(A, \Phi)$ to the Seiberg-Witten equations on $(X_R = X, g_R)$ will be called a *temporal gauge solution* if $A$ is in temporal gauge on the tube $T = [-R, R] \times Y$, and similarly for translation invariant temporal gauge solutions.

Clearly, if a connection $A$ on $L_{|R \times Y}$ is in temporal gauge, it can be recovered from the path $A(t)$, since it has no $dt$ component.

**Remark 3.3.2** If $A = A_R$ is any connection on $X_R$ (see [3.2.1]), there exists a gauge transformation $g(R)$ in the connected component of $id_X$ in the gauge group $\mathcal{G}$ such that $g(R)A$ is in temporal gauge on $[-R, R] \times Y$. Further, if $A_{R|X_+}$ is a fixed connection $A_+$ on $L_{|X_+}$ independent of $R$, we may choose $g(R)$ to be the identity map $id_{X_+}$ on the piece $X_+$ for all $R$. Similarly for $X_-.$

**Proof of remark:**

Let $A_0(y, t)dt$ be the $dt$ component of $A$ on $[-R, R] \times Y$. There is a $C^\infty$ function $h_R(y, t)$ on $[-R, R] \times Y$ which satisfies : $A_0(t, y) = \frac{\partial h_R(t, y)}{\partial t}$. Choose a $C^\infty$ function $f$ extending $h_R$ to $X = X_R$. If $A_R(R, y) = A_+(R, y)$ is independent of $R$, we may choose $h_R(t, y)$ to be such that $h(R, y) = 0$ for all $R$, and choose $f$ to be identically 0 on $X_+$ for all $R$. Now take the gauge transformation $g(R) = \exp(2\pi i f(t, y))$. Now, since

$$g(R)^{-1}dg(R) = 2\pi i \left( \frac{\partial h_R(t, y)}{\partial t} \right) dt = 2\pi i A_0(y, t)dt$$

on $[-R, R] \times Y$, it follows that $gA = A - \frac{1}{2\pi i}g^{-1}dg$ has no $dt$ component on $[-R, R] \times Y$. $\square$

Let $\partial_{A(t)} : \Gamma(W_3) \rightarrow \Gamma(W_3)$ be the induced Dirac operator on $W_3$, with respect to the connection $A(t)$. In view of Remark 3.3.2 above, any solution $(A, \Phi)$ to the Seiberg-Witten equations on $(X_R, g_R)$ can be assumed to be a temporal gauge representative in its gauge equivalence class. Our first goal is to consider the restriction of such a temporal gauge solution to the tube $T = [-R, R] \times Y$, and view it as a time-dependent solution $(A(t), \Phi(t))$ of some equations on $Y$ involving $\partial_{A(t)}$ etc.

**Lemma 3.3.3** Let us denote by $*_Y$ the star operator on $Y$ defined by the metric $g_Y$ (induced from $g_R$ on $X$, as in [3.2.1]), and let a 2-form $\omega$ on $X$ be expressed as $\omega = dt \wedge \phi + \psi$ on the tube $T = [-R, R] \times Y$ where $\phi, \psi$ are devoid of $dt$. Then

$$*_Y \omega = *_Y \phi + dt \wedge *_Y \psi$$

where $*$ is the star operator of $g_R$ on $X$. 

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Proof: Is a straightforward exercise, since \( g_R = dt^2 \times g_Y \) and so \( dV_{g_R} = dt dV_Y \), and the star operator is characterised by the diagram in the subsection 1.3 (Self-Duality) of §1.

Corollary 3.3.4 Let \( A \) be a connection on \( X = X_R \), and let \( F_A \) be its curvature. Let \( T \) denote the tube \([-R, R] \times Y\), with the product metric \( g_R = dt^2 \times g_Y \). Then, if \( A \) is translation invariant in a temporal gauge, we have the equality of pointwise norms:

\[
(F_A^+(y), F_A^+(y))_{y,g_R} = (F_A^-(y), F_A^-(y))_{y,g_R}
\]

Proof: Write \( F_A = dt \wedge \frac{dA}{dt} + F_{A,Y} \). By the Lemma above,

\[
\begin{align*}
F_A^+ &= \frac{1}{2} (F_A + *F_A) = \frac{1}{2} \left[ dt \wedge \left( \frac{dA}{dt} + *_{Y}F_{A,Y} \right) \right] + \left( F_{A,Y} + *_{Y} \frac{dA}{dt} \right) \\
F_A^- &= \frac{1}{2} (F_A + *F_A) = \frac{1}{2} \left[ dt \wedge \left( \frac{dA}{dt} - *_{Y}F_{A,Y} \right) \right] + \left( F_{A,Y} - *_{Y} \frac{dA}{dt} \right)
\end{align*}
\]

Thus, since \( \frac{dA}{dt} = 0 \) for \( A \) translation-invariant and in temporal gauge, we have the result. \( \square \)

The proof above also shows that the spaces of \(*\)-self dual and antiself dual 2-forms on \( T \) which are \( t \)-translation invariant are both isomorphic to \( \Omega^1(Y) \). The self-dual one is given as \( \frac{1}{2}(dt \wedge *_{Y}\omega(y) + \omega(y)) \) and the anti-selfdual one as \( \frac{1}{2}(-dt \wedge *_{Y}\omega + \omega) \).

For notational convenience, we denote the isomorphism \( \omega \mapsto \frac{1}{2}(-dt \wedge *_{Y}\omega + \omega) \) by \( \theta : \Omega^1(Y) \rightarrow \Omega^2_{\text{inv}}(T) \), where the right hand side is the space of translation invariant self-dual forms on \( T \).

The Clifford structure map \( \gamma : \Lambda^1(T) \otimes \mathbb{C} \rightarrow \text{Hom}(W_+, W_-) \) restricts to the Clifford isometry \( \tilde{\gamma} : \Lambda^1(Y) \otimes \mathbb{C} \rightarrow \text{End}^0(W_3) \). It is easily checked that \( \rho \circ \theta = \tilde{\gamma} \). Finally, we recall the pairing \( \sigma \) defined by \( \sigma(\Phi, \Psi) = i \left( h_+(-, \Psi)\Phi - \frac{1}{2} h_+(\Phi, \Psi) \mathbb{I} \right) \) over the tube \( T \):

\[
\sigma : W_+ \otimes W_+ \rightarrow \text{End}^0(W_+) \simeq \Lambda^2 \otimes \mathbb{C}
\]

Since \( W_+ = \pi^*(W_3) \), we have the pairing:

\[
\tau : W_3 \times W_3 \rightarrow \Lambda^1(Y) \otimes \mathbb{C}
\]

where \( \tilde{\gamma} \circ \tau = \sigma \). So, if we regard a section \( \Phi \in \Gamma(W_+/T) \) as a section in \( \Gamma(\pi^*W_3) \), which is the same as a path of sections \( \Phi(t) \) in \( \Gamma(W_3) \), then by definition,

\[
\sigma(\Phi(t,y), \Phi(t,y)) = \tilde{\gamma}(\tau(\Phi(t)(y), \Phi(t)(y)))
\]

Now we look at the Dirac operator on \( T \). The covariant derivative with respect to \( A \) and compatible with the Levi-Civita connection of \( g_R \) is given by:

\[
\nabla_{A,dt} = i (A \downarrow dt) \otimes (-) + \frac{\partial}{\partial t}
\]

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where \( \vdash \) denotes contraction. Since \( A \) is in temporal gauge, \( A \vdash dt = 0 \), and so \( \nabla_A dt = \frac{\partial}{\partial t} \).

Of course, viewing \( \Phi(t, -) \) as a path of sections \( \Phi(t) \) of \( W_3 \), we have

\[
\frac{\partial \Phi(t, -)}{\partial t} = \frac{d \Phi(t)}{dt}
\]

Also, the Clifford multiplication \( c(dt) \) is what makes \( W_+ \) isomorphic to \( W_- \simeq \pi^*(W_3) \). So the Dirac operator for the bundle \( W \) on the tube \( T \) with induced Spin\(_c\) structure on \( T \) reads as:

\[
D_A = \frac{d A}{dt} + \partial_A(Y)
\]

where \( \partial_A(Y) \) is the Dirac operator for the induced Spin\(_c\) structure on \( Y \). It follows that time-dependent Seiberg-Witten equations read on \( Y \) as (assuming, as usual that \( A \) is in temporal gauge):

\[
\left( \frac{dA}{dt} + \ast_Y F_{A,Y} \right) - \tau(\Phi, \Phi) - \ast_Y \delta = 0
\]

\[
\frac{d\Phi(t)}{dt} = -\partial_{A(t),Y} \Phi(t)
\]

where \( \delta \in \Omega^2(Y) \). Hence we have the:

**Proposition 3.3.5** If \((A, \Phi)\) is a temporal gauge solution to the Seiberg-Witten equations on the tube \( T = \mathbb{R} \times Y \) or \([-R, R] \times Y \), then \((A(t), \Phi(t))\) is a path in \( \mathcal{A}_Y \times \Gamma(W_3) \) which is a trajectory of the equations (4).

We now note that the equations (3.3.3) above are the gradient-flow equations of a functional defined on \( \mathcal{A}_Y \times \Gamma(W_3) \).

**Proposition 3.3.6** The equations (4) are the gradient flow equations for the functional defined on \( \mathcal{A}_Y \times \Gamma(W_3) \) by

\[
C_\delta(A, \Phi) = -\frac{1}{2} \left( \int_Y (A - B) \wedge (F_A - \delta) + \int_Y (\Phi, \partial_A \Phi)_{W_3,Y} dV_Y \right)
\]

where \( B \) is a reference connection on \( L_{|Y} \) and the integrand on the extreme right is the inner product on the fibre \( W_{3,Y} \), and \( \delta \) is a 2-form on \( Y \). Hence, \((A(t), \Phi(t))\) satisfying (4) is the gradient flow for this functional, and \( C_\delta \) is monotonically increasing along this trajectory.

**Proof:** For simplicity, denote \( A - B \) as \( A \), where \( B \) is the reference connection. Let \( \{e_i\}_{i=1}^2 \) be a \((-,-)_{W_3}\) unitary frame for \( W_3 \). We recall that \( \sigma(\Phi, \Psi) = i \left[ h_+(-, \Psi) \Phi - \frac{1}{2} h_+(\Phi, \Psi) \text{Id} \right] \).

The foregoing definitions lead to the following identity (since \( \tilde{\gamma} \) maps into traceless endos):

\[
< \tau(\Phi, \Phi), \omega > = < \sigma(\Phi, \Phi), \tilde{\gamma}(\omega) > = \frac{1}{2} \text{Tr}(\Phi \circ \tilde{\gamma}(\omega)^t)
\]

\[
= \frac{i}{2} \sum_{j=1}^2 \langle (\tilde{\gamma}(\omega)^t e_j, \Phi) \Phi, e_j \rangle = \frac{i}{2} \langle \Phi, \tilde{\gamma}(\omega) \Phi \rangle
\]
where the round brackets denote the hermitian inner product \((-,-)_{W_3,y}\) on the fibre \(W_3,y\) and the angular brackets denote the Riemannian inner product on \(Y\). In view of the calculation above, and that \(\partial_A = \sum_{j=1}^3 \tilde{\gamma}(\omega_j) \nabla_{A,\omega_j}\) the inner product \(\langle \Phi, \partial_A \Phi \rangle_{W_3}\) satisfies:

\[
(\Phi, \partial_A \Phi)_{W_3} - (\langle \Phi, \partial_B \Phi \rangle_{W_3} = -i \sum_{j=1}^3 \langle \Phi, A, \omega_j > \tilde{\gamma}(\omega_j) \Phi \rangle_{W_3} = -2 \langle \tau(\Phi, \Phi), A >
\]

where \(\{\omega_j\}_{j=1}^3\) is a local orthonormal frame for \(\Lambda^1(Y)\).

Thus the integrand of \(C_\delta\) becomes:

\[
Q(A, \Phi) = - \left[ \frac{1}{2} < A, *_Y (F_A - \delta) > + \frac{1}{2} (\Phi, \partial_B \Phi)_{W_3,y} - \langle \tau(\Phi, \Phi), A > \right]
\]

Thus,

\[
\frac{\partial Q}{\partial A} = - \left( < -, *_Y (F_A - \frac{1}{2} \delta) > - < -, \tau(\Phi, \Phi) > \right)
\]

\[
\frac{\partial Q}{\partial \Phi} = - \frac{1}{2} ((-, \partial_A \Phi)_{W_3.y} + (\Phi, \partial_A -)_{W_3,y})
\]

by using Stokes formula, \(F_A = dA\) and the self-adjointness of \(\partial_A\) on \(Y\). Thus the gradient flow equations for the functional, viz.,

\[
\frac{dA}{dt} = \frac{\partial Q}{\partial A}
\]

\[
\frac{d\Phi}{dt} = \frac{\partial Q}{\partial \Phi}
\]

lead, respectively, to the required equations (4), and the proposition is proved. \(\square\)

We next investigate what happens to \(C_\delta\) under the Gauge group action. We shall drop the subscript \(\delta\) from \(C_\delta\) for notational convenience.

**Proposition 3.3.7** Under a gauge transformation \(g \in \text{Map}(Y, S^1)\), we have the transformation formula:

\[
C(gA, g\Phi) = C(A, \Phi) + 2\pi^2 [g] \cup \left( c_1(L) - \frac{1}{2\pi} [\delta_H] \right)
\]

where \([g]\) denotes the cohomology class of \(g\) in \(H^1(Y, \mathbb{R})\). Since \([g]\) is an integral cohomology class, this shows that on a gauge orbit, \(C\) is well defined in \(\mathbb{R}/2\pi^2 \mathbb{Z} \cup \left( c_1(L) - \frac{1}{2\pi} [\delta_H] \right)\).

**Proof:** We apply the formulas \(gA = A - g^{-1}dg, \partial_y A(g\Phi) = g\partial_A \Phi\), and \(F_y A = F_A\) to compute:

\[
C(gA, g\Phi) = C(A, \Phi) + \frac{1}{2} \int_Y g^{-1}dg \wedge (F_A - \delta)
\]

\[
= C(A, \Phi) + \pi [g] \cup 2\pi (c_1(L) - \frac{1}{2\pi} [\delta_H])
\]

\[
= C(A, \Phi) + 2\pi^2 [g] \cup (c_1(L) - \frac{1}{2\pi} [\delta_H])
\]

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proving the proposition.

Now we are ready for the main proposition of this section.

**Proposition 3.3.8** In the notation of Definition 3.2.1, assume that \( M_{c, \delta}(g_R) \) is non-empty for \( R \) large enough. Then there exists a solution \((A, \Phi)\) to the Seiberg-Witten equations on \( R \times Y \) which is translation invariant in a temporal gauge.

**Proof:** Let \((A_R, \Phi_R)\) be a solution to the Seiberg-Witten equations on \((X_R, g_R)\) for sufficiently large \( R \). By gauge transforming if necessary (see Remark 3.3.2), let us assume that all the \( A_R \) are in temporal gauge. Take reference connections \( B_R \) on the bundle \( L \to X_R \) such that the restrictions \( B_R|_{X_\pm} := B_\pm \) are fixed, independent of \( R \).

Let us denote the change in the functional \( C \) along the tube \( T = [-R, R] \times Y \subset X_R \) by:

\[
l_{A, \Phi}(R) = C(A_R(R), \Phi_R(R)) - C(A_R(-R), \Phi_R(-R))
\]

By the Proposition 3.3.6, we have that \( C(A_R(t), \Phi_R(t)) \) are monotonic functions of \( t \). Now, \((A_R, \Phi_R)\) will restrict to solutions on \( X_\pm \). By construction, the scalar curvature of \( g_R \) has the same infimum on \( X_R \) for all \( R \). Hence, by §5.2 of [PP], there is a uniform \( C^0 \)-bound for all \( \Phi_R \) on \( X_R \) independent of \( R \). The (compactness) argument of §5.2 of [PP] shows that there exist gauge transformations \( h_R^\pm \) of \( X_\pm \) such that:

\[
h_R^\pm A_R - B_R^\pm = h_R^\pm A_R - B^\pm
\]

are both bounded in Sobolev \( L_k^2 \)-norm (for \( k \) suitably large) uniformly for all \( R \). By Sobolev’s Lemma, this implies uniform \( C^0 \) bounds on zeroth and first derivatives of \( A_R, \Phi_R \) for all \( R \). Since \( C(h_R^\pm A_R(\pm R), \Phi_R(\pm R)) \) are the evaluations of \( C \) (which only involves derivatives upto first order) to the ends \( \{\pm R\} \times Y \) of the tube \([-R, R] \times Y \), we have uniform bounds for \( C(h_R^\pm A_R(\pm R), \Phi_R(\pm R)) \) independent of \( R \). Now let \( \gamma \) be a 1-cycle Poincare-dual to \( c_1(L) - \frac{1}{2\pi} [\delta_H] \) in \( Y \). It is the intersection with \( Y \) of the 2-cycle \( \Gamma \) which is Poincare-dual to \( c_1(L) - \frac{1}{2\pi} [\delta_H] \) in \( X \).

Let \( i_\pm : Y \hookrightarrow X_\pm \) denote the inclusions (as ends), and \([h_R^\pm]\) denote the cohomology classes of \( h_R^\pm \) in \( H^1(Y)\), or \( H^1(X_\pm)\). Then,

\[
\langle c_1(L) - \frac{1}{2\pi} [\delta_H], [Y] \rangle = \langle i_\pm^* c_1(L) - \frac{1}{2\pi} [\delta_H], [h_R^\pm] \rangle
\]

\[
= \langle i_\pm^* \left( c_1(L) - \frac{1}{2\pi} [\delta_H] \right) \cup [h_R^\pm], [X_\pm] \rangle = \langle \delta_\pm i_\pm^* c_1(L) - \frac{1}{2\pi} [\delta_H], [h_R^\pm], [X_\pm] \rangle = 0
\]

where \([Y], [X_\pm]\) denote orientation classes, and \( \partial_\pm : H_4(X_\pm, Y) \to H_3(Y) \) and \( \delta_\pm : H^3(Y) \to H^4(X_\pm, Y) \) denote respectively the connecting homomorphisms in the long exact homology and cohomology sequences of the pair \((X_\pm, Y)\). Thus, by Proposition 3.3.7 we have

\[
C(h_R^\pm A_R(\pm R), h_R^\pm \Phi_R(\pm R)) = C(A_R(\pm R), \Phi_R(\pm R))
\]

Thus we have a uniform bound \( M \) on \( l_{A_R, \Phi_R} \) independent of \( R \).
Now let $R$ be a positive integer, say $R = N$, and denote by $\Delta_i$ the change in $C$ across $[i-1,i] \times Y$, viz.

$$\Delta_i = C(A_N(i), \Phi_N(i)) - C(A_N(i-1), \Phi_N(i-1))$$

We saw in Proposition 3.3.3 that $(A_N, \Phi_N)$, being solutions to Seiberg-Witten equations and $A_N$ being in temporal gauge implied that they were (time-dependent) solutions to the equations, and Proposition 3.3.6 then implied that $C$ was monotonic increasing in time for these solutions. Thus all the $\Delta_i$ are non-negative. Let $\Delta_{\min,N} = \min_i \Delta_i$. Hence we have:

$$2N \Delta_{\min,N} \leq \sum_{i=-N}^{+N} \Delta_i = C(A_N(N), \Phi_N(N)) - C(A_N(-N), \Phi_N(-N)) = l_{A_N, \Phi_N} \leq M$$

for all $N$.

Hence $\lim_{N \to \infty} \Delta_{\min,N} = 0$. Denote by $(A_{(N)}, \Phi_{(N)})$ the restriction of $(A_N, \Phi_N)$ to the interval $[i-1,i] \times Y$ on which $\Delta_{\min,N} = |\Delta_i|$. This may be viewed as a solution on $[0,1] \times Y$, denoted by the same symbol $(A_{(N)}, \Phi_{(N)})$. As we saw above, we have

$$C(A_{(N)}(1), \Phi_{(N)}(1)) - C(A_{(N)}(0), \Phi_{(N)}(0)) \leq \frac{M}{N}$$

which goes to 0 as $N \to \infty$. The uniform $C^0$ bound on $(A_N, \Phi_N)$, and hence $(A_{(N)}, \Phi_{(N)})$ gives a solution (on passing to a subsequence) $(A, \Phi)$ on $[0,1] \times Y$ for which

$$(C(A(1), \Phi(1)) - C(A(0), \Phi(0)) = \lim \left( C(A_{(N)}(1), \Phi_{(N)}(1)) - C(A_{(N)}(0), \Phi_{(N)}(0)) \right) = 0$$

The monotonicity of $C$ across $[0,1] \times Y$ implies that $(A, \Phi)$ is constant along $[0,1] \times Y$. This solution is clearly therefore a translation invariant solution on $[0,1] \times Y$, which extends to all of $\mathbb{R} \times Y$ by time-translating for all times. \qed

# 4 Proof of Thom’s Conjecture

We first need a lemma:

**Lemma 4.0.9** Let $Y = S^1 \times \tilde{\Sigma}$, where $\tilde{\Sigma}$ is a Riemannian 2-manifold of constant scalar curvature $s$ and genus $g \geq 1$. Assume the metric on $\tilde{\Sigma}$ is normalised so that its volume is 1 (and thus $s = 2\pi \chi(\tilde{\Sigma}) = 2\pi(2 - 2g)$). Let $Y$ have a metric $g_Y$ extending this metric on $\tilde{\Sigma}$, and let the infinite tube $T = \mathbb{R} \times Y$ have a product metric $dt^2 \times g_Y$, and $L$ be the line bundle associated with a compatible Spin$_c$ structure. Suppose there is a solution to the Seiberg-Witten equations on $(T, g_Y)$ which is translation invariant in a temporal gauge. Then:

$$\left| \frac{1}{2\pi} \int_{\Sigma} F_A \right| \leq 2g - 2$$
Proof: From the $C^0$ bound (see §5.2, [PP]), the sup norm satisfies $|\Phi|^2_\infty \leq 2\pi(2g-2) + \|\delta\|_\infty$. Since $|\sigma(\Phi,\Phi)|^2 = \frac{1}{2} |\Phi|^4$, we have

$$|\sigma(\Phi,\Phi)|_\infty \leq \frac{1}{\sqrt{2}} (2\pi(2g-2) + \|\delta\|_\infty)$$

Thus, from the Seiberg-Witten equations:

$$\|F_A^+\|_{\infty,Y} \leq \frac{1}{\sqrt{2}} (2\pi(2g-2) + 2\|\delta\|_\infty)$$

However, by Corollary 3.3.4, we have the pointwise norm equality $\|F_A^+\| = \|F_A^-\|$ because our solution is translation invariant in a temporal gauge. Therefore,

$$\|F_A\| \leq \sqrt{2} \|F_A^+\| \leq 2\pi(2g-2) + O(\|\delta\|_\infty)$$

Thus

$$\left|\frac{1}{2\pi} \int_{\tilde{\Sigma}} F_A\right| \leq \frac{1}{2\pi} \left[2\pi(2g-2) + O\left(\int_{\tilde{\Sigma}} \|\delta\|_\infty\right)\right]$$

This proves the lemma, since $\delta$ is arbitrarily small. \qed

Now we can prove the main theorem.

Theorem 4.0.10 (Kronheimer-Mrowka) If $\Sigma$ is an oriented 2-manifold smoothly embedded in $\mathbb{CP}^2$ so as to represent an algebraic curve of degree $d$, then the genus $g(\Sigma)$ of $\Sigma$ satisfies:

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}$$

Proof: The cases of $d = 1, 2$ are trivial, and $d = 3$ is due to Kervaire-Milnor (see reference [6] in [KM]). So we will assume $d > 3$ in the sequel. By Proposition 3.2.3, there exists a metric $g_R$ on $X = \mathbb{CP}^2 \# d^2 \mathbb{CP}^2$ such that $c_1(L) \cup [\omega_{g_R}] < 0$. By Corollary 3.2.4, the moduli space $M_{c,\delta}(g_R) \neq \emptyset$. (We just need to ensure that the reference metric we started with on $X$ is a product metric in a tubular neighbourhood of $Y = S^1 \times \tilde{\Sigma}$.) By Proposition 3.3.8, there is a solution on $\mathbb{R} \times Y$ which is translation invariant in a temporal gauge. By the Lemma 4.0.9 above, $\left|c_1(L)\cdot \tilde{\Sigma}\right| \leq 2g - 2$, which implies $c_1(L)\cdot \tilde{\Sigma} \geq 2 - 2g$. By the opening discussion of §3.1, we have $c_1(L) = 3H - E$, and by construction $[\tilde{\Sigma}] = dH - E$. Thus $(3H - E) \cdot (dH - E) \geq 2 - 2g$. Applying $H.E = 0, H.H = 1, E.E = -d^2$, we get $3d - d^2 \geq 2 - 2g$, i.e. $g \geq \frac{(d-1)(d-2)}{2}$, proving the theorem. \qed

5 Appendix :Fredholm Theory

5.1 Preliminaries

All Hilbert manifolds in the sequel are assumed to be second countable and paracompact. We recall that a bounded operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is said to be Fredholm if Ker $T$ and Coker $T$ are finite dimensional and the range Im $T$ is closed.
Definition 5.1.1 We call a smooth map:

\[ f: \mathcal{M} \to \mathcal{N} \]

a Fredholm map if the derivative \( Df(x) : T_x\mathcal{M} \to T_x\mathcal{N} \) is a Fredholm operator for each \( x \in \mathcal{M} \).

It is necessary to extend results like the implicit function theorem in the finite dimensional case to the infinite dimensional case, so as to make manifolds out of inverse images of regular values etc. The key to doing it is the following proposition, which enables one to construct a “standard local model” of a smooth map whose derivative is given to be Fredholm at a point \( p \), in a neighbourhood of \( p \). Its main utility is to decompose a (non-linear) smooth map with infinite-dimensional range into a linear map (with infinite-dimensional range) and a non-linear map with finite dimensional range, in a small neighbourhood of \( p \).

Proposition 5.1.2 Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two Hilbert spaces, and let \( f: \mathcal{H}_1 \to \mathcal{H}_2 \) be a smooth map between them, such that \( f(0) = 0 \). Assume that the derivative \( T = Df(0) \) is a Fredholm operator from \( T_0\mathcal{H}_1 = \mathcal{H}_1 \) to \( T_0\mathcal{H}_2 = \mathcal{H}_2 \). Note that we are only requiring the derivative to be Fredholm at a point, not that \( f \) necessarily be a Fredholm map. Then, with the orthogonal decomposition \( \mathcal{H}_1 = \text{Ker} T \oplus V_1 \), \( \mathcal{H}_2 = \text{Im} T \oplus V_2 \), there exists a (non-linear) map \( \phi: \mathcal{H}_1 \to V_2 = \text{Coker} T \) and a diffeomorphism \( h: U \to h(U) \) for \( U \) a neighbourhood of 0 such that:

(i) \( f \circ h(n, v_1) = (Tv_1, \phi(n, v_1)) \) for \( n \in \text{Ker} T \), \( v_1 \in V_1 \), \( (n, v_1) \in U \). So \( \phi \) is smooth with finite dimensional range \( V_2 = \text{Coker} T \). In particular,

(ii) \( f \) is a Fredholm map in the neighbourhood \( U \) of 0.

(iii) \( \phi(0) = 0, \ D\phi(0) = 0. \)

(iii) If \( G \) is a group acting via an orthogonal linear action on \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), and \( f \) is \( G \)-equivariant, then \( V_1 \), \( V_2 \) are \( G \)-invariant, and \( h \) and \( \phi \) are also \( G \)-equivariant.

Proof:

Clearly if \( f \) is \( G \)-equivariant, \( T = Df(0) \) is also \( G \)-equivariant, and the orthogonality of the action implies that both \( \text{Ker} T \) and \( \text{Im} T \) being \( G \)-invariant, their orthogonal complements \( V_1 \) and \( V_2 = \text{Coker} T \) are \( G \)-invariant.

Note that \( T_{\mid V_1}: V_1 \to \text{Im} T \) is a bounded bijective linear operator, so has a bounded inverse \( T^{-1}: \text{Im} T \to V_1 \) by the open mapping theorem. Let \( \pi: \mathcal{H}_2 \to \text{Im} T \) be the orthogonal projection onto \( \text{Im} T \). Let \( \bar{T} \) denote the composite \( T^{-1} \circ \pi: \mathcal{H}_2 \to V_1 \), and \( \theta: \mathcal{H}_1 \to \text{Ker} T \) be the orthogonal projection onto \( \text{Ker} T \). Consider the map:

\[
\chi: \mathcal{H}_1 \to \mathcal{H}_1 \\
x \mapsto (\theta(x), \bar{T}(f(x)))
\]
Then, since \( \theta(n) = n \) for \( n \in \ker T \), we compute:

\[
D\chi(0)(n, v) = (n, \tilde{T} \circ Df(0)(n, v)) = (n, \tilde{T} \circ T(n, v)) \\
= (n, T^{-1} \circ \pi \circ T(n, v)) = (n, T^{-1} \circ T(n, v)) = (n, v)
\]

Note also that in the \( G \)-setting, \( \chi \) is \( G \)-equivariant. Now \( D\chi(0) = \text{id} \) implies by the (infinite-dimensional) inverse function theorem that there exists a ball \( V = B(0, \delta) \) around the origin in \( \mathcal{H}_1 \) on which \( \chi \) is a diffeomorphism onto its image. Let \( U_1 = B(0, \varepsilon) \subset \chi(V) \), and for \( y \in U_1 \), define \( \phi_1 : U_1 \rightarrow V_2 \) by \( \phi_1 = \pi \circ f \circ \chi^{-1}(y) \). Note that again, in the \( G \)-setting, \( \phi_1 \) is \( G \)-equivariant since it is a composite of \( G \)-equivariant maps. Now for \( y \in U_1 = B(0, \varepsilon) \), we have, using \( T(\theta(u), v) = T(v) \) and the definitions above:

\[
f \circ \chi^{-1}(y) = (\pi \circ f \circ \chi^{-1}(y), \pi \circ 
\circ \chi^{-1}(y), \phi_1(y)) = (T \circ T^{-1} \circ \pi \circ f \circ \chi^{-1}(y), \phi_1(y)) \\
= (T \circ \tilde{T} \circ f(\chi^{-1}(y), \phi_1(y)) = (T(\theta(\chi^{-1}(y)), \tilde{T} \circ f(\chi^{-1}(y)), \phi_1(y)) \\
= (T \circ \chi \circ \chi^{-1}(y), \phi_1(y)) = (T(y), \phi_1(y))
\]

All we need to do now is extend \( \phi_1 : U_1 \rightarrow V_2 \) to \( \phi : \mathcal{H}_1 \rightarrow V_2 \) in a \( G \)-equivariant manner, and this is easily done by using the map:

\[
\rho : \mathcal{H}_1 \rightarrow \mathcal{H}_1, \quad \begin{array}{rcl}
x & \mapsto & \frac{\varepsilon x}{\varepsilon + \psi(\|x\|) \|x\|}
\end{array}
\]

where \( \psi : \mathbb{R} \rightarrow [0, 1] \) is a \( C^\infty \)-map which is identically zero for \( |t| < \frac{\varepsilon}{2} \) and identically 1 for \( |t| > \varepsilon \). Then \( \rho \) is \( G \)-equivariant since \( G \) preserves \( \| \| \); and maps \( \mathcal{H}_1 \rightarrow B(0, \varepsilon) \) and is equal to the identity map on \( B(0, \frac{\varepsilon}{2}) \). Thus the map \( \phi = \phi_1 \circ \rho \) is a \( G \)-equivariant map agreeing with \( \phi_1 \) on \( B(0, \frac{\varepsilon}{2}) \). Now take \( U = B(0, \frac{\varepsilon}{2}) \) and \( h = \chi^{-1} \). Clearly, \( \phi(0) = 0 \) by construction, and since \( Df(0) = T \), we have \( D\phi(0) = 0 \), and hence (i), (iii), and (iv) follow. To see (ii) note that on \( U \), we have \( f \) equivalent to the map \( (T, \phi) \) via the local diffeomorphism \( h \) applied on the domain. Since \( T \) is a linear isomorphism from \( V_1 \) to \( \text{Im} T \), and \( \phi \) is smooth with finite dimensional range, it is easy to check that \( (T, \phi) \) is Fredholm on \( U \). Thus \( f \) is also Fredholm on \( U \), proving (ii).

\[ \square \]

**Corollary 5.1.3** Let \( f, \phi, \mathcal{H}_1, \) and \( \mathcal{H}_2 \) be as in the last proposition. \( \tilde{U} := h(U) \). Then, for \( (a, \delta) \in \mathcal{H}_2 \),

(i) the germ of the inverse image of \((a, \delta)\), i.e. \( f^{-1}(a, \delta) \cap \tilde{U} \) is equivalent (i.e. via an ambient diffeomorphism \( h : U \rightarrow \tilde{U} \)) to \( \phi_b^{-1}(\delta) \), where

\[
\phi_b : U_b \rightarrow V_2, \quad n \mapsto \phi(n, b)
\]

\( b := T^{-1}(a) \) (uniquely defined as an element of \( V_1 \)), and \( U_b = \theta(U \cap \pi^{-1}_V(b)) \) is the \( b \)-slice of \( U \). Thus, with the hypotheses on \( f \) of the last proposition, a local model for \( f^{-1}(a, \delta) \) near 0 is given by the fibre of the finite dimensional smooth map \( \phi_b \). In particular \( f^{-1}(0) \) is locally homeomorphic to \( \phi_0^{-1}(0) \).
(ii) For a fixed $a$ in $\text{Im} T \subset H_2$, $(a, \delta)$ is a regular value for $f|_{\tilde{U}}$ if and only if $\delta$ is a regular value for the smooth map $\phi_{f|U_0}$, whose domain and range are finite dimensional. By the usual (finite-dimensional) Morse-Sard theorem applied to $\phi_{f|U_0}$, it follows that for a fixed $a$, there is a Baire subset of $\delta \in V_2$ such that $(a, \delta)$ is a regular value for $f|_{\tilde{U}}$. In case $\delta$ happens to be in this Baire subset, the inverse image $f^{-1}(a, \delta) \cap \tilde{U}$ is a smooth manifold of dimension $= \dim U_0 - \dim V_2 = \dim (\text{Ker} T) - \dim (\text{Coker} T) = \text{index} T$. (Note: A Baire set is a countable intersection of open dense sets.)

(iii) If $f : \mathcal{M} \to \mathcal{N}$ is a smooth Fredholm map between Hilbert manifolds, the regular values of $f$ constitute a Baire subset of $\mathcal{N}$.

**Proof:** $f|_{\tilde{U}}$ has exactly the same properties as $f \circ h|_U$ as far as regular values, inverse images of regular values etc. are concerned. Thus 5.1.2 implies (i) and (ii). (iii) follows from second countability and the fact that countable intersections of Baire sets are Baire. □

**Corollary 5.1.4** Let everything be as in Proposition 5.1.2. If $f$ is a $G$-equivariant map, (with $G$ acting orthogonally and linearly on both domain and range as before), then a local (homeomorphic) model for $(\tilde{U} \cap f^{-1}(0))/G$ is $(U_0 \cap \phi_0^{-1}(0))/G$ where $\phi_0 = \phi(-, 0)$.

**Proof:** By (i) of the corollary 5.1.3 above, $\tilde{U} \cap f^{-1}(0)$ is homeomorphic to $U_0 \cap \phi_0^{-1}(0)$ via a $G$-equivariant local diffeomorphism $h : U \to \tilde{U}$ in the ambient space. Since (iv) of 5.1.2 implies that $\phi$ is $G$-equivariant, so is $\phi_0 = \phi(-, 0)$, and the corollary follows. □

Thus we have proved the following :

**Proposition 5.1.5** (*Local finite-dimensional model for a neighbourhood of a singular point in the orbit space*) If $f : \mathcal{H}_1 \to \mathcal{H}_2$ is a $G$-equivariant map between Hilbert spaces on which $G$ acts linearly and orthogonally, such that $T = Df(0)$ is a Fredholm operator, then a local model for the germ of $f^{-1}(0)/G$ at 0 is the germ at 0 of the finite dimensional object $\phi_0^{-1}(0)/G$, where $\phi_0 : \text{Ker} T \to \text{Coker} T$ is a smooth $G$-equivariant map between finite dimensional spaces (with the restricted action of $G$).

Another lemma which will be useful in the sequel is the following:

**Lemma 5.1.6** Let $f : \mathcal{M} \to \mathcal{N}_1 \times \mathcal{N}_2$ be a smooth map between Hilbert manifolds, and let $f_i := \pi_i \circ f$ where $\pi_i$ are the projection maps to $\mathcal{N}_i$ for $i = 1, 2$. Then

(i) $(a, b) \in \mathcal{N}_1 \times \mathcal{N}_2$ is a regular value of $f$ if $a$ is a regular value of $f_1$ and $b$ is a regular value of $f_2|_{f_1^{-1}(a)}$.

(ii) If $f$ is a Fredholm map, then $f_2|_{f_1^{-1}(a)}$ is a Fredholm map for all regular values $a$ of $f_1$.

For a regular value $a$ of $f_1$, there is a Baire subset $U \subset \mathcal{N}_2$ such that $(a, b)$ is a regular value of $f$ for $b \in U$. 29
Proof: Let \( M_a := f^1_1(a) \). We have the commuting diagram:

\[
\begin{array}{ccc}
M_a & \hookrightarrow & M \\
\downarrow & & \downarrow f \\
\{a\} \times N_2 & \hookrightarrow & N_1 \times N_2 \rightarrow N_1
\end{array}
\]

which leads to the diagram of derivatives:

\[
\begin{array}{ccc}
0 & \rightarrow & T_x(M_a) \\
\downarrow Df_2|_{M_a} & & \downarrow Df(x) \\
0 & \rightarrow & T_{f_2(x)}N_2 \rightarrow T_{aN_1} \oplus T_{f_2(x)}N_2 \rightarrow T_aN_1 \rightarrow 0
\end{array}
\]

for all \( x \in M_a \). Clearly \( Df(x) \) is surjective iff \( Df_2(x) \) is surjective, for all \( x \in M_a \). Thus \((a, f_2(x))\) is a regular value of \( f \) iff \( f_2(x) \) is a regular value for \( f_2|_{M_a} \). Similarly, the Fredholm statement follows because of the snake lemma, for all \( x \in M_a \). \( \square \)

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