Estimation of divergence measures via weighted Jensen inequality on time scales

Iqrar Ansari¹, Khuram Ali Khan¹, Ammara Nosheen², Dilda Pečarić³ and Josip Pečarić⁴

Abstract
The main purpose of the presented paper is to obtain some time scale inequalities for different divergences and distances by using weighted time scales Jensen’s inequality. These results offer new inequalities in $h$-discrete calculus and quantum calculus and extend some known results in the literature. The lower bounds of some divergence measures are also presented. Moreover, the obtained discrete results are given in the light of the Zipf–Mandelbrot law and the Zipf law.

Keywords: Time scales; Jensen’s inequality; Csiszár divergence; Zipf and Zipf–Mandelbrot law

1 Introduction
Distance or divergence measures are of key importance in statistics and information theory. Depending upon the nature of the problem, different divergence measures are suitable. A number of measures of divergence that compare two probability distributions have been proposed (see [15, 16, 23, 24, 31, 37] and the references therein). Csiszár [12] introduced the $f$-divergence functional as follows.

Definition 1.1 Suppose that $f: \mathbb{R}^+ \to (0, \infty)$ is a convex function. Let $\tilde{r} = (r_1, \ldots, r_n)$ and $\tilde{s} = (s_1, \ldots, s_n)$ be such that $\sum_{k=1}^{n} r_k = 1$ and $\sum_{k=1}^{n} s_k = 1$. Then an $f$-divergence functional is stated as

$$I_f(\tilde{r}, \tilde{s}) := \sum_{k=1}^{n} s_k f\left(\frac{r_k}{s_k}\right),$$

where $f$ bears the following requirements:

$$f(0) := \lim_{\epsilon \to 0^+} f(\epsilon); \quad 0f\left(\frac{0}{0}\right) := 0; \quad 0f\left(\frac{a}{0}\right) := \lim_{\epsilon \to 0^+} \epsilon f\left(\frac{a}{\epsilon}\right), \quad a > 0.$$

The Csiszár’s $f$-divergence is a broad class of divergences which consists of various divergence measures used in finding out the difference between two probability densities. A significant property of Csiszár’s $f$-divergence is that several well-known divergence measures

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can be deduced from this divergence measure by suitable substitutions to the convex function \( f \). In recent years, several researchers have done a considerable work providing various kinds of bounds on the divergences and distances, see e.g. [13, 14, 25, 33]. Jensen's inequality has an important role in obtaining inequalities for divergence measures. It helps to compute useful upper bounds for several entropic measures used in information theory. In [18], Jain et al. established an information inequality regarding Csiszár \( f \)-divergence by utilizing the convexity condition and Jensen's inequality. This inequality is applied in comparing some well-known divergences which play a significant role in information theory. In [19], Khan et al. obtained new results for the Shannon and Zipf–Mandelbrot entropies. They also computed different bounds for these entropies by using some refinements of the Jensen inequality. In [21], the authors established various inequalities for convex functions and applied them to Csiszár divergence. They also obtained several results for Zipf–Mandelbrot entropy. In [27], Mehmood et al. obtained a new generalized form of cyclic refinements of Jensen's inequality from convex to higher order convex functions by utilizing Taylor's formula. They also computed bounds for various notable inequalities utilized in information theory. In [11], Butt et al. used discrete and continuous cyclic refinements of Jensen's inequality and extended them from convex to higher order convex function by using new Green functions and Abel–Gontscharoff interpolating polynomial. As an application, they established a connection between new entropic bounds for relative, Shannon, and Mandelbrot entropies. In [22], Khan et al. established an elegant refinement of Jensen's inequality related to two finite sequences. The obtained inequality used to compute bounds for Csiszár divergence, variational distance, Shannon entropy, and Zipf–Mandelbrot entropy. In [29], Pečarić et al. obtained refinements of the integral version of Jensen's inequality and the Lah–Ribarič inequality and deduced estimates for the integral form of Csiszár divergence and its important particular cases. In [2], Ahmad et al. utilized some results of Jensen's inequality for convex functions and obtained various estimates for Shannon and generalized Zipf–Mandelbrot entropies. In [10], Butt et al. proved various Jensen–Grüüss type inequalities under certain conditions.

The development of the theory of time scales was initiated by Hilger in 1988. The books of Bohner and Peterson [8, 9] related to time scales are compact and resolve a lot of time scales calculus. In the past years, new developments in the theory and applications of dynamic derivatives on time scales emerged. Many results from the continuous case are carried over to the discrete one very easily, but some seem to be completely different. The study on time scales comes to reveal such discrepancies and to make us understand the difference between the two cases. The Jensen inequality has been extended to time scales by Agarwal et al. (see [1, 8]). Various classical inequalities and their converses for isotonic linear functionals on time scales are established in [5]. In [6], Anwar et al. gave the properties and applications of Jensen functionals on time scales for one variable. Further in [7], the authors obtained the Jensen inequality for several variables and deduced Jensen functionals. They also derived properties of Jensen functionals and applied them to generalized means. In recent years, the study of dynamic inequalities on time scales has been considered by several authors, see [1, 28, 30, 32, 36, 39, 40]. In [3], Ansari et al. obtained Shannon type inequalities on an arbitrary time scale. They also deduced bounds of differential entropy on time scale for various distributions. Further in [4], the authors established several inequalities for Csiszár \( f \)-divergence among two probability densities on
time scales. They also obtained new results for divergence measures in $h$-discrete calculus and quantum calculus.

Quantum calculus or $q$-calculus is usually called calculus without limits. In 1910, Jackson [17] described a $q$-analogue of derivative and integral operator along with their applications. He was the first to establish $q$-calculus in an organized form. It is important to note that quantum integral inequalities are more significant and constructive than their classical counterparts. It has been primarily for the reason that quantum integral inequalities can interpret the hereditary properties of the fact and technique under consideration. Recently, there has been a rapid development in $q$-calculus. Consequently, new generalizations of the classical approach of quantum calculus have been proposed and analyzed in various literature works. The concepts of quantum calculus on finite intervals were given by Tariboon and Ntouyas [34, 35], and they obtained certain $q$-analogues of classical mathematical objects, which motivated numerous researchers to explore the subject in detail. Subsequently, several new results related to the quantum counterpart of classical mathematical results have been established.

2 Preliminaries
An arbitrary nonempty closed subset of the real line is known as time scale $T \subset \mathbb{R}$. The subsequent results and definitions are given in [8].

**Definition 2.1** Suppose that $T$ is a time scale and $\zeta \in T$, then the forward, respectively backward, jump operators $\sigma, \rho : T \rightarrow T$ are defined as follows:

$$\sigma(\zeta) = \inf\{\nu \in T : \nu > \zeta\} \quad \text{and} \quad \rho(\zeta) = \sup\{\nu \in T : \nu < \zeta\}.$$  

**Definition 2.2** Let $T$ be a time scale and $z : T \rightarrow \mathbb{R}$ be a function, then $z$ is known as rd-continuous or right-dense continuous if its left-sided limits exist (finite) at left-dense points in $T$ and it is continuous at right-dense points in $T$. The set of rd-continuous functions $z : T \rightarrow \mathbb{R}$ is usually denoted by $C_{rd}$.

Let us introduce the set $T^k$ as follows:

$$T^k = \begin{cases} T \setminus (\rho(\sup T), \sup T) & \text{if } \sup T < \infty, \\ T & \text{if } \sup T = \infty. \end{cases}$$

**Definition 2.3** Consider a function $z : T \rightarrow \mathbb{R}$ and $\zeta \in T^k$. Then we define $z^\Delta(\zeta)$ to be the number (when it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $\zeta$ such that

$$|z(\sigma(\zeta)) - z(\nu) - z^\Delta(\zeta)(\sigma(\zeta) - \nu)| \leq \epsilon |\sigma(\zeta) - \nu| \quad \text{for all } \nu \in U.$$  

In this case, $z$ is said to be delta differentiable at $\zeta$.

For $T = \mathbb{R}$, $z^\Delta$ becomes ordinary derivative $z'$, while if $T = \mathbb{Z}$, then $z^\Delta$ turns into the usual forward difference operator $\Delta z(\zeta) = z(\zeta + 1) - z(\zeta)$. If $T = q^n = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ is
the so-called $q$-difference operator, with $q > 1$, then

$$z^\Delta(\zeta) = \frac{z(q\zeta) - z(\zeta)}{(q - 1)\zeta}, \quad z^\Delta(0) = \lim_{v \to 0} \frac{z(v) - z(0)}{v}.$$ 

**Theorem 2.1** (Existence of antiderivatives) Every rd-continuous function has an antiderivative. If $x_0 \in \mathbb{T}$, then $F$ is defined by

$$F(\zeta) := \int_{x_0}^x f(\zeta) \Delta \xi \quad \text{for} \ x \in \mathbb{T}^k$$

is an antiderivative of $f$.

For $\mathbb{T} = \mathbb{R}$, we have $\int_a^b f(\zeta) \Delta \zeta = \int_a^b f(\zeta) \, d\zeta$, and if $\mathbb{T} = \mathbb{N}$, then $\int_a^b f(\zeta) \Delta \zeta = \sum_{\zeta = a}^{b-1} f(\zeta)$, where $a, b \in \mathbb{T}$ with $a \leq b$.

In [38], Wong et al. gave the weighted Jensen inequality on time scales which is stated as follows.

**Theorem 2.2** Assume that $I \subset \mathbb{R}$, and let $r \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ be a positive function with

$$\int_a^b r(\zeta) \Delta \zeta > 0,$$

where $a, b \in \mathbb{T}$. If $f \in C(I, \mathbb{R})$ is convex and $g \in C_{rd}([a, b]_{\mathbb{T}}, I)$, then

$$f\left(\frac{\int_a^b r(\zeta) g(\zeta) \Delta \zeta}{\int_a^b r(\zeta) \Delta \zeta}\right) \leq \frac{\int_a^b r(\zeta) f'(g(\zeta)) \Delta \zeta}{\int_a^b r(\zeta) \Delta \zeta}. \quad (1)$$

When $f$ is a strictly convex function, the inequality sign in (1) is strict.

### 3 Divergences on time scales

Consider the set of rd-continuous functions on time scale $\mathbb{T}$ to be

$$\Omega := \left\{ r \in C_{rd}([a, b]_{\mathbb{T}}, (0, \infty)), r(\zeta) > 0, \int_a^b r(\zeta) \Delta \zeta > 0 \right\}.$$

In the sequel, we assume that $r, s \in \Omega$ and the following integrals exist:

$$R = \int_a^b r(\zeta) \Delta \zeta \quad \text{and} \quad S = \int_a^b s(\zeta) \Delta \zeta.$$

#### 3.1 Csiszár $f$-divergence

Csiszár $f$-divergence on time scale is defined in [4] as follows:

$$D_f(s, r) := \int_a^b r(\zeta) f\left(\frac{s(\zeta)}{r(\zeta)}\right) \Delta \zeta, \quad (2)$$

where $f$ is convex on $(0, \infty)$. 

Theorem 3.1 Assume that $I \subset \mathbb{R}$, and if $f \in C(I, \mathbb{R})$ is convex, then

$$Rf\left(\frac{S}{R}\right) \leq D_f(s, r),$$

(3)

where $D_f(s, r)$ is given in (2).

Proof Put $g(\zeta) = \frac{\Omega(\zeta)}{\Omega(\zeta)}$ in (1) to get (3). \qed

Example 3.1 For $\mathbb{T} = \mathbb{R}$, Theorem 3.1 becomes [20, Theorem 5.2 on p. 10].

Example 3.2 Choose $T = \mathbb{hZ}$, $\mathbb{h} > 0$ in Theorem 3.1 to get a lower bound for Csiszár divergence in $\mathbb{h}$-discrete calculus

$$\sum_{l=\frac{a}{\mathbb{h}}}^{b-1} r(\mathbb{l}) h f\left(\frac{\sum_{l=\frac{a}{\mathbb{h}}}^{b-1} s(l) h}{\sum_{l=\frac{a}{\mathbb{h}}}^{b-1} r(l) h}\right) \leq \sum_{l=\frac{a}{\mathbb{h}}}^{b-1} r(\mathbb{l}) h f\left(\frac{s(l) h}{r(l) h}\right).$$

Remark 3.1 Choose $\mathbb{h} = 1$ in Example 3.2, and let $a = 0, b = n, r(l) = r_j, s(l) = s_j$ to get the discrete Csiszár divergence

$$\sum_{j=1}^{n} r f\left(\frac{\sum_{j=1}^{n} s_j}{\sum_{j=1}^{n} r_j}\right) \leq I_f(s, r),$$

(4)

where

$$I_f(\mathbf{s}, \mathbf{r}) = \sum_{j=1}^{n} r f\left(\frac{s_j}{r_j}\right),$$

(5)

$\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{r} = (r_1, \ldots, r_n)$.

Example 3.3 Choose $\mathbb{T} = \mathbb{qZ_0}$ ($q > 1$) in Theorem 3.1 to have a new lower bound of the Csiszár divergence in quantum calculus

$$\sum_{l=0}^{n-1} q^{l+1} r(q^l) f\left(\frac{\sum_{l=0}^{n-1} q^{l+1} s(q^l)}{\sum_{l=0}^{n-1} q^{l+1} r(q^l)}\right) \leq \sum_{l=0}^{n-1} q^{l+1} r(q^l) f\left(\frac{s(q^l)}{r(q^l)}\right).$$

3.2 Differential entropy (continuous entropy)

Consider a positive density function $r$ on time scale $\mathbb{T}$ to a continuous random variable $X$ with $\int_{\mathbb{T}} r(\zeta) \Delta \zeta = 1$, wherever the integral exists. In [3], Ansari et al. defined the so-called differential entropy on time scale by

$$h_b(X) := \int_{a}^{b} r(\zeta) \log \frac{1}{r(\zeta)} \Delta \zeta,$$

(6)

where $\tilde{b} > 1$ is the base of log. In the sequel, we assume that the base of log is greater than 1.
**Theorem 3.2** Suppose that $r, s \in C_{\text{rd}}([a, b], \mathbb{R})$ are $\Delta$-integrable functions and $r$ is a positive probability density function with $S = \int_a^b s(\zeta) \Delta \zeta > 0$. If $f \in C(I, \mathbb{R})$ is convex and $\bar{b} > 1$, then

$$h_{\bar{b}}(X) \leq \int_a^b r(\zeta) \log \frac{1}{s(\zeta)} \Delta \zeta + \log(S),$$

(7)

where $h_{\bar{b}}(\zeta)$ is defined in (6) and $a, b \in T$.

**Proof** The function $f(\zeta) = -\log \zeta$ is convex. Use $f(\zeta) = -\log \zeta$ with $\int_a^b r(\zeta) \Delta \zeta = 1$ in (3) to get

$$-\log(S) \leq \int_a^b -r(\zeta) \log \left( \frac{s(\zeta)}{r(\zeta)} \right) \Delta \zeta,$$

$$= \int_a^b (r(\zeta) \log r(\zeta) - r(\zeta) \log s(\zeta)) \Delta \zeta,$$

$$= \int_a^b r(\zeta) \log r(\zeta) \Delta \zeta - \int_a^b r(\zeta) \log s(\zeta) \Delta \zeta,$$

$$= - \int_a^b r(\zeta) \log \frac{1}{r(\zeta)} \Delta \zeta + \int_a^b r(\zeta) \log \frac{1}{s(\zeta)} \Delta \zeta,$$

$$= -h_{\bar{b}}(X) + \int_a^b r(\zeta) \log \frac{1}{s(\zeta)} \Delta \zeta,$$

the stated result. □

**Remark 3.2** The inequality in (7) holds in the opposite direction for the base of log less than 1.

**Example 3.4** For $T = \mathbb{R}$, Theorem 3.2 becomes [26, Theorem 21(a)].

**Example 3.5** Choose $T = h\mathbb{Z}$, $h > 0$ in Theorem 3.2 to get an upper bound for entropy in $h$-discrete calculus

$$\sum_{l=-\frac{b}{h}}^{\frac{b-1}{h}} r(lh)h \log \left( \frac{1}{r(lh)h} \right) \leq \sum_{l=-\frac{b}{h}}^{\frac{b-1}{h}} r(lh)h \log \left( \frac{1}{s(lh)h} \right) + \log \left( \sum_{l=-\frac{b}{h}}^{\frac{b-1}{h}} s(lh)h \right).$$

(8)

**Remark 3.3** Put $h = 1$ in (8) to get [26, Theorem 8 (i)].

**Example 3.6** Choose $T = q^{\mathbb{N}}$ ($q > 1$) in Theorem 3.2 to have

$$\sum_{l=0}^{n-1} q^{sl} r(q^l) \log \left( \frac{1}{r(q^l)} \right) \leq \sum_{l=0}^{n-1} q^{sl} r(q^l) \log \left( \frac{1}{s(q^l)} \right) + \log \left( \sum_{l=0}^{n-1} q^{sl+s(q^l)} \right).$$

(9)

**Remark 3.4** (9) contains Shannon entropy which is new in quantum calculus up to the knowledge of authors.
3.3 Karl Pearson $\chi^2$-divergence

The $\chi^2$-divergence on time scale is defined in [4] as follows:

$$D_{\chi^2}(s, r) := \int_a^b r(\zeta) \left[ \left( \frac{s(\zeta)}{r(\zeta)} \right)^2 - 1 \right] \Delta \zeta. \quad (10)$$

**Theorem 3.3** Assume the conditions of Theorem 3.1 to get

$$\frac{1}{R} [S^2 - R^2] \leq D_{\chi^2}(s, r), \quad (11)$$

where $D_{\chi^2}(s, r)$ is defined in (10).

**Proof** Consider $f(\zeta) = \chi^2 - 1$ in (3) to obtain

$$\left( \frac{S}{R} \right)^2 - 1 \leq \frac{1}{R} \int_a^b r(\zeta) \left[ \left( \frac{s(\zeta)}{r(\zeta)} \right)^2 - 1 \right] \Delta \zeta,$$

after simplification we get

$$S^2 - R^2 \leq R \int_a^b r(\zeta) \left[ \left( \frac{s(\zeta)}{r(\zeta)} \right)^2 - 1 \right] \Delta \zeta,$$

the desired result. □

**Example 3.7** If $T = \mathbb{R}$, then (11) takes the form

$$\frac{1}{\int_a^b r(\zeta) d\zeta} \left[ \left( \int_a^b s(\zeta) d\zeta \right)^2 - \left( \int_a^b r(\zeta) d\zeta \right)^2 \right] \leq \int_a^b r(\zeta) \left[ \left( \frac{s(\zeta)}{r(\zeta)} \right)^2 - 1 \right] d\zeta.$$

**Example 3.8** Choose $T = h\mathbb{Z}$, $h > 0$ in Theorem 3.3 to get a new lower bound for $\chi^2$-divergence in $h$-discrete calculus

$$\frac{1}{\sum_{l=\frac{a}{h}}^{\frac{b-1}{h}} r(lh)h} \left[ \left( \sum_{l=\frac{a}{h}}^{\frac{b-1}{h}} s(lh)h \right)^2 - \left( \sum_{l=\frac{a}{h}}^{\frac{b-1}{h}} r(lh)h \right)^2 \right] \leq \sum_{l=\frac{a}{h}}^{\frac{b-1}{h}} r(lh)h \left[ \left( \frac{s(lh)}{r(lh)} \right)^2 - 1 \right]. \quad (12)$$

**Remark 3.5** Choose $h = 1$ in (12), let $a = 0$, $b = n$, $r(l) = r_j$, and $s(l) = s_j$ to get $\chi^2$-divergence

$$\frac{1}{\sum_{j=1}^{n} r_j} \left[ \left( \sum_{j=1}^{n} s_j \right)^2 - \left( \sum_{j=1}^{n} r_j \right)^2 \right] \leq \chi^2(\bar{s}, \bar{r}),$$

where

$$\chi^2(\bar{s}, \bar{r}) = \sum_{j=1}^{n} r_j \left[ \left( \frac{s_j}{r_j} \right)^2 - 1 \right]. \quad (13)$$
Example 3.9 Choose $T = q^{n_0} (q > 1)$ in Theorem 3.3 to have a new lower bound for $\chi^2$-divergence in quantum calculus

\[
\sum_{l=0}^{n-1} q^{l+1} r(q^l) \left( \left( \sum_{l=0}^{n-1} q^{l+1} s(q^l) \right)^2 - \left( \sum_{l=0}^{n-1} q^{l+1} r(q^l) \right)^2 \right) \\
\leq \sum_{l=0}^{n-1} q^{l+1} r(q^l) \left( \left( \frac{s(q^l)}{r(q^l)} \right)^2 - 1 \right). \tag{14}
\]

3.4 Kullback–Leibler divergence

Kullback–Leibler divergence on time scale is defined in [4] as follows:

\[
D(s, r) = \int_a^b s(\zeta) \ln \left[ \frac{s(\zeta)}{r(\zeta)} \right] d\zeta. \tag{15}
\]

Theorem 3.4 Assume the conditions of Theorem 3.1, then we have

\[
S \ln \left( \frac{S}{R} \right) \leq D(s, r), \tag{16}
\]

where $D(s, r)$ is defined in (15).

Proof Consider $f(\zeta) = \zeta \ln \zeta$ in (3) to get

\[
\frac{S}{R} \ln \left( \frac{S}{R} \right) \leq \frac{1}{R} \int_a^b s(\zeta) \ln \left( \frac{s(\zeta)}{r(\zeta)} \right) d\zeta,
\]

or we have

\[
S \ln \left( \frac{S}{R} \right) \leq \int_a^b s(\zeta) \ln \left( \frac{s(\zeta)}{r(\zeta)} \right) d\zeta,
\]

the desired result. \qed

Example 3.10 For $T = \mathbb{R}$, (16) becomes

\[
\int_a^b s(\zeta) d\zeta \ln \left( \frac{\int_a^b s(\zeta) d\zeta}{\int_a^b r(\zeta) d\zeta} \right) \leq \int_a^b s(\zeta) \ln \left( \frac{s(\zeta)}{r(\zeta)} \right) d\zeta.
\]

Example 3.11 Choose $T = h\mathbb{Z}$, $h > 0$ in Theorem 3.4 to get a new lower bound in $h$-discrete calculus

\[
\sum_{l=a}^{b-1} s(lh) h \ln \left( \frac{\sum_{l=a}^{b-1} s(lh) h}{\sum_{l=a}^{b-1} r(lh) h} \right) \leq \sum_{l=a}^{b-1} s(lh) h \ln \left( \frac{s(lh)}{r(lh)} \right). \tag{17}
\]

Remark 3.6 Choose $h = 1$ in (17), let $a = 0$, $b = n$, $r(l) = r_j$, and $s(l) = s_j$ to get the discrete Kullback–Leibler divergence

\[
\sum_{j=1}^{n} s_j \ln \left( \frac{\sum_{l=0}^{n-1} s_j l}{\sum_{l=0}^{n-1} r_j l} \right) \leq \text{KL} (\mathbf{s}, \mathbf{r}),
\]
where
\[ \text{KL}(\tilde{s}, \tilde{r}) = \sum_{j=1}^{n} s_j \ln \left( \frac{s_j}{r_j} \right) \] (18)

**Example 3.12** Choose \( T = q^{N_0} (q > 1) \) in Theorem 3.4 to have a new lower bound in quantum calculus
\[ \sum_{l=0}^{n-1} q^{l+1} s(q^l) \ln \left( \frac{\sum_{l=0}^{n-1} q^{l+1} s(q^l)}{\sum_{l=0}^{n-1} q^{l+1} r(q^l)} \right) \leq \sum_{l=0}^{n-1} q^{l+1} s(q^l) \ln \left( \frac{s(q^l)}{r(q^l)} \right). \]

### 3.5 Hellinger discrimination

Hellinger discrimination on time scale is defined in [4] as follows:
\[ h^2(s,r) = \frac{1}{2} \int_{a}^{b} \left[ \sqrt{s(\zeta)} - \sqrt{r(\zeta)} \right]^2 d\zeta. \] (19)

**Theorem 3.5** Assume the conditions of Theorem 3.1 to obtain
\[ \frac{1}{2}(\sqrt{S} - \sqrt{R})^2 \leq h^2(s,r), \] (20)

where \( h^2(s,r) \) is defined in (19).

**Proof** Consider \( f(\zeta) = \frac{1}{2}(\sqrt{\zeta} - 1)^2 \) in (3) to get
\[ \frac{1}{2} \left( \frac{\sqrt{S}}{R} - 1 \right)^2 \leq \frac{1}{2R} \int_{a}^{b} r(\zeta) \left( \sqrt{\frac{s(\zeta)}{r(\zeta)}} - 1 \right)^2 d\zeta, \] (21)

after simplification we obtain
\[ \frac{1}{2}(\sqrt{S} - \sqrt{R})^2 \leq \frac{1}{2} \int_{a}^{b} \left( \sqrt{s(\zeta)} - \sqrt{r(\zeta)} \right)^2 d\zeta, \]

the desired result. \( \square \)

**Example 3.13** For \( T = \mathbb{R}, (20) \) becomes
\[ \frac{1}{2} \left( \left[ \int_{a}^{b} s(\zeta) d\zeta \right]^2 - \left[ \int_{a}^{b} r(\zeta) d\zeta \right]^2 \right) \leq \frac{1}{2} \int_{a}^{b} (\sqrt{s(\zeta)} - \sqrt{r(\zeta)})^2 d\zeta. \]

**Example 3.14** Choose \( T = h\mathbb{Z}, h > 0 \) in Theorem 3.5 to get a new lower for Hellinger discrimination in \( h \)-discrete calculus
\[ \frac{1}{2} \left( \sum_{l=-\frac{h}{2}}^{b-h} s( lh ) \right)^2 - \left( \sum_{l=-\frac{h}{2}}^{b-h} r( lh ) \right)^2 \leq \frac{1}{2} \sum_{l=-\frac{h}{2}}^{b-h} (\sqrt{s( lh )} - \sqrt{r( lh )})^2. \] (22)
Remark 3.7 Choose $h = 1$ in (22), let $a = 0$, $b = n$, $r(l) = r_j$, and $s(l) = s_j$ to get the Hellinger distance

$$\frac{1}{2} \left[ \left( \sum_{j=1}^{n} s_j \right)^{1/2} - \left( \sum_{j=1}^{n} r_j \right)^{1/2} \right]^2 \leq h^2(\bar{s}, \bar{r}),$$

where

$$h^2(\bar{s}, \bar{r}) = \frac{1}{2} \sum_{j=1}^{n} (\sqrt{s_j} - \sqrt{r_j})^2.$$  \hspace{1cm} (23)

Example 3.15 Choose $T = q^{N_0}$ $(q > 1)$ in Theorem 3.5 to have a new lower for Hellinger discrimination in quantum calculus

$$\frac{1}{2} \left[ \left( \sum_{l=0}^{n-1} q^{l+1} s(q^l) \right)^{1/2} - \left( \sum_{l=0}^{n-1} q^{l+1} r(q^l) \right)^{1/2} \right]^2 \leq \frac{1}{2} \sum_{l=0}^{n-1} q^{l+1} [\sqrt{s(q^l)} - \sqrt{r(q^l)}]^2.$$  \hspace{1cm} (24)

3.6 Bhattacharyya coefficient

The Bhattacharyya coefficient on time scale is defined in [4] as follows:

$$D_B(s, r) = \int_{a}^{b} \sqrt{r(\zeta)s(\zeta)} \Delta \zeta.$$  \hspace{1cm} (25)

Theorem 3.6 Assume the conditions of Theorem 3.1 to get

$$D_B(s, r) \leq \sqrt{RS},$$  \hspace{1cm} (26)

where $D_B(s, r)$ is defined in (25).

Proof Consider $f(\zeta) = -\sqrt{\zeta}$ in (3) to get

$$- \sqrt{\frac{S}{R}} \leq - \frac{1}{R} \int_{a}^{b} \sqrt{r(\zeta)s(\zeta)} \Delta \zeta,$$

after simplification we obtain

$$\int_{a}^{b} \sqrt{r(\zeta)s(\zeta)} \Delta \zeta \leq \sqrt{RS},$$

the desired result. \hspace{1cm} \Box

Example 3.16 If $T = \mathbb{R}$, then (26) takes the form

$$\int_{a}^{b} (r(\zeta)s(\zeta))^{1/2} d\zeta \leq \left( \int_{a}^{b} r(\zeta) d\zeta \int_{a}^{b} s(\zeta) d\zeta \right)^{1/2}.$$
Example 3.17 Choose $T = h\mathbb{Z}, \ h > 0$ in Theorem 3.6 to get a new upper bound for the Bhattacharyya coefficient in $h$-discrete calculus

$$\frac{1}{2} \left( \frac{1}{h} \sum_{l=-\frac{h}{2}}^{\frac{h}{2}-1} r(l)hs(l)h \right)^{\frac{1}{2}} \leq \left( \frac{1}{h} \sum_{l=-\frac{h}{2}}^{\frac{h}{2}-1} r(l)h \sum_{l=-\frac{h}{2}}^{\frac{h}{2}-1} s(l)h \right)^{\frac{1}{2}}. \tag{27}$$

Remark 3.8 Choose $h = 1$ in (27), let $a = 0$, $b = n$, $r(l) = r_j$, and $s(l) = s_j$ to get the Bhattacharyya coefficient

$$B(\tilde{s}, \tilde{r}) \leq \left( \sum_{j=1}^{n} r_j \sum_{j=1}^{n} s_j \right)^{\frac{1}{2}},$$

where

$$B(\tilde{s}, \tilde{r}) = \sum_{j=1}^{n} \sqrt{r_j s_j}. \tag{28}$$

Example 3.18 Choose $T = q^{\mathbb{N}_0} (q > 1)$ in Theorem 3.6 to have a new upper bound for the Bhattacharyya coefficient in quantum calculus

$$\sum_{l=0}^{n-1} q^{-l} \left[ r(q^l)s(q^l) \right]^2 \leq \left( \sum_{l=0}^{n-1} q^{-l} r(q^l) \sum_{l=0}^{n-1} q^{-l} s(q^l) \right)^{\frac{1}{2}}. \tag{29}$$

3.7 Jeffreys distance

Jeffreys distance on time scale is defined in [4] as follows:

$$D_J(s, r) = \int_{a}^{b} (s(\xi) - r(\xi)) \ln \left[ \frac{s(\xi)}{r(\xi)} \right] \Delta \xi. \tag{30}$$

Theorem 3.7 Assume the conditions of Theorem 3.1 to get

$$(S - R) \ln \left( \frac{S}{R} \right) \leq D_J(s, r),$$

where $D_J(s, r)$ is defined in (29).

Proof Consider $f(\xi) = (\xi - 1) \ln \xi$ in (3) to get

$$R \left( \frac{S}{R} - 1 \right) \ln \left( \frac{S}{R} \right) \leq \int_{a}^{b} r(\xi) \left( \frac{s(\xi)}{r(\xi)} - 1 \right) \ln \left( \frac{s(\xi)}{r(\xi)} \right) \Delta \xi,$$

or we have

$$(S - R) \ln \left( \frac{S}{R} \right) \leq \int_{a}^{b} (s(\xi) - r(\xi)) \ln \left( \frac{s(\xi)}{r(\xi)} \right) \Delta \xi,$$

the desired result. \qed
Example 3.19 For \( T = \mathbb{R} \), (30) takes the form

\[
\left( \int_a^b s(\zeta) \, d\zeta - \int_a^b r(\zeta) \, d\zeta \right) \ln \left( \frac{\int_a^b s(\zeta) \, d\zeta}{\int_a^b r(\zeta) \, d\zeta} \right) \leq \int_a^b \left[ s(\zeta) - r(\zeta) \right] \ln \left( \frac{s(\zeta)}{r(\zeta)} \right) \, d\zeta.
\]

Example 3.20 Choose \( T = h \mathbb{Z} \), \( h > 0 \) in Theorem 3.7 to get a new lower bound for Jeffreys distance in \( h \)-discrete calculus

\[
\left( \sum_{l=0}^{b-h} s(\zeta)h - \sum_{l=0}^{b-h} r(\zeta)h \right) \ln \left( \frac{\sum_{l=0}^{b-h} s(\zeta)h}{\sum_{l=0}^{b-h} r(\zeta)h} \right)
\leq \sum_{l=0}^{b-h} \left( s(\zeta)h - r(\zeta)h \right) \ln \left( \frac{s(\zeta)h}{r(\zeta)h} \right),
\]

(31)

Remark 3.9 Choose \( h = 1 \) in (31), let \( a = 0 \), \( b = n \), \( r(l) = r_j \), and \( s(l) = s_j \) to get Jeffreys distance

\[
\left( \sum_{j=1}^{n} s_j - \sum_{j=1}^{n} r_j \right) \ln \left( \frac{\sum_{j=1}^{n} s_j}{\sum_{j=1}^{n} r_j} \right) \leq D_J(\tilde{s}, \tilde{r}),
\]

where

\[
D_J(\tilde{s}, \tilde{r}) = \sum_{j=1}^{n} (s_j - r_j) \ln \left( \frac{s_j}{r_j} \right).
\]

(32)

Example 3.21 Choose \( T = q^n \mathbb{N} \) (\( q > 1 \)) in Theorem 3.7 to have a new lower bound for the Jeffreys distance in quantum calculus

\[
\left( \sum_{l=0}^{n-1} q^{l+1}s(q^l) - \sum_{l=0}^{n-1} q^{l+1}r(q^l) \right) \ln \left( \frac{\sum_{l=0}^{n-1} q^{l+1}s(q^l)}{\sum_{l=0}^{n-1} q^{l+1}r(q^l)} \right)
\leq \sum_{l=0}^{n-1} q^{l+1} \left( s(q^l) - r(q^l) \right) \ln \left( \frac{s(q^l)}{r(q^l)} \right),
\]

3.8 Triangular discrimination

Triangular discrimination on time scale is defined in [4] as follows:

\[
D_\Delta(r,s) = \int_a^b \frac{[s(\zeta) - r(\zeta)]^2}{s(\zeta) + r(\zeta)} \, d\zeta.
\]

(33)

Theorem 3.8 Assume the conditions of Theorem 3.1 to obtain

\[
\frac{[S - R]^2}{S + R} \leq D_\Delta(r,s),
\]

(34)

where \( D_\Delta(r,s) \) is defined in (33).
Proof Consider \( f(\zeta) = \frac{(\zeta - 1)^2}{\zeta + 1} \) in (3) to get

\[
R \left( \frac{S - R}{S + R} \right)^2 \leq \int_a^b r(\zeta) \left( \frac{d(\zeta)}{r(\zeta)} - 1 \right)^2 \Delta \zeta
\]
or

\[
\frac{[S - R]^2}{S + R} \leq \int_a^b \frac{[s(\zeta) - r(\zeta)]^2}{s(\zeta) + r(\zeta)} \Delta \zeta.
\]

□

Example 3.22 For \( T = \mathbb{R} \), (34) becomes

\[
\frac{\int_a^b s(\zeta) d\zeta - \int_a^b r(\zeta) d\zeta}{\int_a^b s(\zeta) d\zeta + \int_a^b r(\zeta) d\zeta} \leq \int_a^b \frac{[s(\zeta) - r(\zeta)]^2}{s(\zeta) + r(\zeta)} d\zeta.
\]

Example 3.23 Choose \( T = \mathbb{h} \mathbb{Z}, h > 0 \) in Theorem 3.8 to get a new lower bound for the triangular discrimination in \( h \)-discrete calculus

\[
\frac{\left( \sum_{l=0}^{b-1} h(l) - \sum_{l=0}^{b-1} r(l) \right) h}{\sum_{l=0}^{b-1} s(l) + \sum_{l=0}^{b-1} r(l) h} \leq \sum_{l=0}^{b-1} \frac{h(s(l) - r(l))}{h(l) + r(l)}. \tag{35}
\]

Remark 3.10 Choose \( h = 1 \) in (35), let \( a = 0, b = n, r(l) = r_j, \) and \( s(l) = s_j \) to get the triangular discrimination

\[
\frac{\left( \sum_{j=1}^{n} s_j - \sum_{j=1}^{n} r_j \right)^2}{\sum_{j=1}^{n} s_j + \sum_{j=1}^{n} r_j} \leq \Delta(\tilde{s}, \tilde{r}),
\]

where

\[
\Delta(\tilde{s}, \tilde{r}) = \sum_{j=1}^{n} \frac{(s_j - r_j)^2}{s_j + r_j}. \tag{36}
\]

Example 3.24 Choose \( T = q^{\mathbb{N}_0} (q > 1) \) in Theorem 3.8 to have a new lower bound for the triangular discrimination in quantum calculus

\[
\frac{\left( \sum_{l=0}^{n-1} q^{l+1} s(q^l) - \sum_{l=0}^{n-1} q^{l+1} r(q^l) \right)^2}{\sum_{l=0}^{n-1} q^{l+1} s(q^l) + \sum_{l=0}^{n-1} q^{l+1} r(q^l)} \leq \sum_{l=0}^{n-1} q^{l+1} \frac{[s(q^l) - r(q^l)]^2}{s(q^l) + r(q^l)}. \tag{37}
\]

4 Zipf–Mandelbrot law

The Zipf–Mandelbrot law is a discrete probability distribution and is defined via a probability mass function which is given as follows:

\[
f(j; N, a, b) = \frac{1}{(j + b)^a} H_{N, a, b}, \quad j = 1, \ldots, N, \tag{37}
\]

where

\[
H_{N, a, b} = \sum_{l=1}^{N} \frac{1}{(l + b)^a}. \tag{38}
\]
is a generalization of a harmonic number and $N \in \{1, 2, \ldots\}$, $a > 0$ and $b \in [0, \infty)$ are parameters.

If $b = 0$ and $N$ is finite, then the Zipf–Mandelbrot law is commonly known as the Zipf law. By expression (37), the probability mass function in connection with the Zipf law is

$$f(j; N, a) = \frac{1}{(j)^a H_{N,a}}, \quad j = 1, \ldots, N,$$

(39)

where

$$H_{N,a} = \sum_{i=1}^{N} \frac{1}{(i)^a}.$$

(40)

Using $r_j = f(j; N, a, b)$ in (37) as a probability mass function, we observe the obtained results via the Zipf–Mandelbrot law.

For this reason, we give results concerning the Csiszár functional $\tilde{I}_f(s, \tilde{r})$ for the Zipf–Mandelbrot law.

Case-1 Define $\tilde{r}$ by (37) as a Zipf–Mandelbrot law $N$-tuple, Csiszár functional (5) becomes

$$\tilde{I}_f(j; N, a_1, a_2, b_2, \tilde{s}) = \sum_{j=1}^{N} \frac{1}{(j + b_2)^a H_{N,a_2 b_2}} f\left((j + b_2)^a H_{N,a_2 b_2}\right),$$

(41)

where $f : I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$, and $N \in \mathbb{N}$, $a_2 > 0$, $b_2 > 0$ are such that $s(j + b_2)^a H_{N,a_2 b_2} \in I$, $j = 1, \ldots, N$.

Case-2 When $\tilde{s}$ and $\tilde{r}$ both are defined via the Zipf–Mandelbrot law for $N$-tuples:

$$\tilde{I}_f(j; N, a_1, a_2, b_1, b_2) = \sum_{j=1}^{N} \frac{1}{(j + b_2)^a H_{N,a_2 b_2}} f\left((j + b_2)^a H_{N,a_2 b_2}\right),$$

(42)

where $f : I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$, and $N \in \mathbb{N}$, $a_1, a_2 > 0$, $b_1, b_2 > 0$ are such that $(j + b_2)^a H_{N,a_2 b_2} \in I$, $j = 1, \ldots, N$.

Case-3 If $\tilde{s}$ and $\tilde{r}$ both are defined as the Zipf law for $N$-tuples, then Csiszár functional (5) becomes

$$\tilde{I}_f(j; N, a_1, a_2) = \sum_{j=1}^{N} \frac{1}{(j)^a H_{N,a_2}} f\left((j)^a H_{N,a_2}\right).$$

(43)

Start from case-1 which is for the single Zipf–Mandelbrot law $r_j, j = 1, \ldots, N$.

**Corollary 4.1** Assume that $I \subset \mathbb{R}$, and let $N \in \mathbb{N}$, $a_2 > 0$, $b_2 > 0$ be such that $\sum_{j=1}^{N} s(j + b_2)^a H_{N,a_2 b_2} \in I$ for $j = 1, \ldots, N$. If $f$ is a convex function, then

$$\sum_{j=1}^{n} \frac{1}{(j + b_2)^a H_{N,a_2 b_2}} f\left((\sum_{j=1}^{n} s_j)/\sum_{j=1}^{n} (j + b_2)^a H_{N,a_2 b_2}\right) \leq \tilde{I}_f(j; N, a_2, b_2, \tilde{s}).$$

(44)

**Proof** Put $r_j = \frac{1}{(j + b_2)^a H_{N,a_2 b_2}}$ for $j = 1, \ldots, N$ in (4) to get (44), where $\tilde{I}_f(j; N, a_2, b_2, \tilde{s})$ is defined in (41).
Remark 4.1 The inequality sign in (44) holds in reverse direction when \( f \) is a concave function.

The next result is for case- 2 as both \( s_j \) and \( r_j \) are defined by the Zipf–Mandelbrot law.

**Corollary 4.2** Assume that \( I \subset \mathbb{R}, \) and let \( N \in \mathbb{N}, a_1, a_2 > 0, b_1, b_2 > 0 \) be such that \( \frac{(j + b_2)^{a_2} H_{N,a_2,b_2}}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \in I \) for \( j = 1, \ldots , N. \) If \( f \) is a convex function, then

\[
\sum_{j=1}^{n} \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} f \left( \sum_{j=1}^{n} \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \right) \leq \tilde{I}_f(j, N, a_1, a_2, b_1, b_2). \tag{45}
\]

**Proof** Using \( r_j = \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} \) and \( s_j = \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \) for \( j = 1, \ldots , N, \) in (4), we get (45), where \( \tilde{I}_f(j, N, a_1, a_2, b_1, b_2) \) is defined in (42).

Remark 4.2 The inequality sign in (45) holds in reverse direction when \( f \) is a concave function.

The next result is for case- 3 as both \( s_j \) and \( r_j \) are defined by the Zipf law.

**Corollary 4.3** Assume that \( I \subset \mathbb{R}, \) and let \( N \in \mathbb{N}, a_1, a_2 > 0 \) be such that \( \frac{(j + b_2)^{a_2} H_{N,a_2,b_2}}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \in I \) for \( j = 1, \ldots , N. \) If \( f \) is a convex function, then

\[
\sum_{j=1}^{n} \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} f \left( \sum_{j=1}^{n} \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \right) \leq \tilde{I}_f(j, N, a_1, a_2). \tag{46}
\]

**Proof** Using \( r_j = \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} \) and \( s_j = \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \) for \( j = 1, \ldots , N, \) in (4), we get (46), where \( \tilde{I}_f(j, N, a_1, a_2) \) is defined in (43).

Remark 4.3 The inequality sign in (46) holds in reverse direction when \( f \) is a concave function.

To give certain results related to the particular cases of \( f \)-divergences, we begin with the well-known Kullback–Leibler divergence (18).

**Corollary 4.4** Let \( N \in \mathbb{N} \) and \( a_2 > 0, b_2 > 0. \) Then

\[
\sum_{j=1}^{n} s_j \ln \left( \frac{\sum_{j=1}^{n} s_j}{\sum_{j=1}^{n} \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}}} \right) \leq KL(j, N, a_2, b_2, \tilde{s}). \tag{47}
\]

**Proof** The function \( f(\zeta) = \zeta \ln(\zeta) \) is convex. Use \( f(\zeta) = \zeta \ln(\zeta) \) in (44) to obtain (47), where

\[
KL(j, N, a_2, b_2, \tilde{s}) = \sum_{j=1}^{N} s_j \ln(s_j(j + b_2)^{a_2} H_{N,a_2,b_2}). \tag{48}
\]

If \( s_j \) and \( r_j \) are defined by the Zipf–Mandelbrot law.
Corollary 4.5 Suppose that $N \in \mathbb{N}$ and $a_1, a_2 > 0$, $b_1, b_2 > 0$. Then

$$\sum_{j=1}^{n} \frac{1}{(j+b_1)^{a_1} H_{N,a_1,b_1}} \ln \left( \frac{\sum_{j=1}^{n} \frac{1}{(j+b_1)^{a_1} H_{N,a_1,b_1}}} {\sum_{j=1}^{n} \frac{1}{(j+b_2)^{a_2} H_{N,a_2,b_2}}} \right) \leq \tilde{\mathcal{KL}}(j, N, a_1, a_2, b_1, b_2). \quad (48)$$

Proof The function $f(\xi) = \xi \ln(\xi)$ is convex. Use $f(\xi) = \xi \ln(\xi)$ in (45) to get (48), where

$$\tilde{\mathcal{KL}}(j, N, a_1, a_2, b_1, b_2) = \sum_{j=1}^{N} \frac{1}{(j+b_1)^{a_1} H_{N,a_1,b_1}} \ln \left( \frac{(j+b_2)^{a_2} H_{N,a_2,b_2}}{(j+b_1)^{a_1} H_{N,a_1,b_1}} \right).$$

The following result holds as both $s_j$ and $r_j$ are defined by the Zipf law.

Corollary 4.6 Let $N \in \mathbb{N}$, $a_1, a_2 > 0$. Then

$$\sum_{j=1}^{n} \frac{1}{(j)^{a_1} H_{N,a_1}} \ln \left( \frac{\sum_{j=1}^{n} \frac{1}{(j)^{a_1} H_{N,a_1}}} {\sum_{j=1}^{n} \frac{1}{(j)^{a_2} H_{N,a_2}}} \right) \leq \tilde{\mathcal{KL}}(j, N, a_1, a_2). \quad (49)$$

Proof The function $f(\xi) = \xi \ln(\xi)$ is convex. Use $f(\xi) = \xi \ln(\xi)$ in (46) to have (49), where

$$\tilde{\mathcal{KL}}(j, N, a_1, a_2) = \sum_{j=1}^{N} \frac{1}{j^{a_1} H_{N,a_1}} \ln \left( \frac{j^{a_2-a_1} H_{N,a_2}}{H_{N,a_1}} \right).$$

Analogous results for the Hellinger distance (23) are given as follows.

Corollary 4.7 Let $N \in \mathbb{N}$, $a_2 > 0$, $b_2 > 0$. Then

$$\frac{1}{2} \left[ \left( \sum_{j=1}^{n} s_j \right)^{\frac{1}{2}} - \left( \sum_{j=1}^{n} \frac{1}{(j+b_2)^{a_2} H_{N,a_2,b_2}} \right)^{\frac{1}{2}} \right]^2 \leq \tilde{h}^2(j, N, a_2, b_2, \tilde{s}). \quad (50)$$

Proof Since $f(\xi) = \frac{1}{2}(\sqrt{\xi} - 1)^2$ is a convex function, therefore we use $f(\xi) = \frac{1}{2}(\sqrt{\xi} - 1)^2$ in (44) to get (50), where

$$\tilde{h}^2(j, N, a_2, b_2, \tilde{s}) = \sum_{j=1}^{N} \frac{1}{2} \left[ (s_j)^{\frac{1}{2}} - \left( \frac{1}{(j+b_2)^{a_2} H_{N,a_2,b_2}} \right)^{\frac{1}{2}} \right]^2.$$

The following result holds as both $s_j$ and $r_j$ are defined by the Zipf–Mandelbrot law.

Corollary 4.8 Let $N \in \mathbb{N}$, $a_1, a_2 > 0$, $b_1, b_2 > 0$. Then

$$\frac{1}{2} \left[ \left( \sum_{j=1}^{n} \frac{1}{(j+b_1)^{a_1} H_{N,a_1,b_1}} \right)^{\frac{1}{2}} - \left( \sum_{j=1}^{n} \frac{1}{(j+b_2)^{a_2} H_{N,a_2,b_2}} \right)^{\frac{1}{2}} \right]^2 \leq \tilde{h}^2(j, N, a_1, a_2, b_1, b_2). \quad (51)$$
Proof Since \( f(\zeta) = \frac{1}{2}(\sqrt{\zeta} - 1)^2 \) is a convex function, therefore we use \( f(\zeta) = \frac{1}{2}(\sqrt{\zeta} - 1)^2 \) in (45) to get (51), where

\[
\tilde{h}^2(j, N, a_1, a_2, b_1, b_2) = \frac{1}{2} \sum_{j=1}^{N} \left[ \left( \frac{1}{(j + b_1)^{a_1} H_{N, a_1, b_1}} \right)^{\frac{1}{2}} - \left( \frac{1}{(j + b_2)^{a_2} H_{N, a_2, b_2}} \right)^{\frac{1}{2}} \right]^2. \tag{52}
\]

The following result holds as both \( s_j \) and \( r_j \) are defined by the Zipf law.

**Corollary 4.9** Let \( N \in \mathbb{N}, a_1, a_2 > 0 \). Then

\[
\frac{1}{2} \left[ \left( \sum_{j=1}^{n} \frac{1}{(j)^{a_1} H_{N, a_1}} \right)^{\frac{1}{2}} - \left( \sum_{j=1}^{n} \frac{1}{(j)^{a_2} H_{N, a_2}} \right)^{\frac{1}{2}} \right]^2 \leq \tilde{h}^2(j, N, a_1, a_2). \tag{53}
\]

**Proof** Since \( f(\zeta) = \frac{1}{2}(\sqrt{\zeta} - 1)^2 \) is a convex function, therefore we use \( f(\zeta) = \frac{1}{2}(\sqrt{\zeta} - 1)^2 \) in (46) to get (53), where

\[
\tilde{h}^2(j, N, a_1, a_2, b_1, b_2) = \sum_{j=1}^{N} \frac{1}{2} \left[ \left( \frac{1}{(j + b_1)^{a_1} H_{N, a_1}} \right)^{\frac{1}{2}} - \left( \frac{1}{(j + b_2)^{a_2} H_{N, a_2}} \right)^{\frac{1}{2}} \right]^2. \tag{54}
\]

Similarly, corresponding results for the Karl Pearson divergence (13) and the Jeffrey distance (32) are given below.

**Corollary 4.10** Let \( N \in \mathbb{N} \) and \( a_2 > 0, b_2 > 0 \). Then

\[
\sum_{j=1}^{n} \frac{1}{(j + b_2)^{a_2} H_{N, a_2, b_2}} \left[ \left( \sum_{j=1}^{n} s_j \right)^{\frac{1}{2}} - \left( \sum_{j=1}^{n} \frac{1}{(j + b_2)^{a_2} H_{N, a_2, b_2}} \right)^{\frac{1}{2}} \right]^2 \leq \tilde{\chi}^2(j, N, a_2, b_2, \tilde{s}). \tag{55}
\]

**Proof** Since \( f(\zeta) = \zeta^2 - 1 \) is a convex function, therefore we use \( f(\zeta) = \zeta^2 - 1 \) in (44) to obtain (55), where

\[
\tilde{\chi}^2(j, N, a_2, b_2, \tilde{s}) = \sum_{j=1}^{N} \frac{1}{(j + b_2)^{a_2} H_{N, a_2, b_2}} \left[ \left( s_j (j + b_2)^{a_2} H_{N, a_2, b_2} \right)^{\frac{1}{2}} - 1 \right]. \tag{56}
\]

If \( s_j \) and \( r_j \) are defined via the Zipf–Mandelbrot law.

**Corollary 4.11** Suppose that \( N \in \mathbb{N} \) and \( a_1, a_2 > 0, b_1, b_2 > 0 \). Then

\[
\sum_{j=1}^{n} \frac{1}{(j + b_1)^{a_1} H_{N, a_1, b_1}} \left[ \left( \sum_{j=1}^{n} s_j (j + b_1)^{a_1} H_{N, a_1, b_1} \right)^{\frac{1}{2}} - \left( \sum_{j=1}^{n} \frac{1}{(j + b_2)^{a_2} H_{N, a_2, b_2}} \right)^{\frac{1}{2}} \right]^2 \leq \tilde{\chi}^2(j, N, a_1, a_2, b_1, b_2). \tag{57}
\]
Proof Since $f(\xi) = \xi^2 - 1$ is a convex function, therefore we use $f(\xi) = \xi^2 - 1$ in (45) to get (54), where

$$\tilde{\chi}^2(j, N, a_1, a_2, b_1, b_2) = \sum_{j=1}^{N} \frac{1}{(j + b_2)^{a_2} H_{N,a_1,b_2}} \left[ (j + b_2)^{a_2} H_{N,a_1,b_2} \right]^2 - 1 \right]. \tag{56} \quad \square$$

The following result holds as both $s_j$ and $r_j$ are defined by the Zipf law.

**Corollary 4.12** Let $N \in \mathbb{N}$, $a_1, a_2 > 0$. Then

$$\sum_{j=1}^{N} \frac{1}{(j)^{a_1} H_{N,a_2}} \left[ \left( \frac{1}{(j)^{a_1} H_{N,a_1}} \right)^2 - \left( \sum_{j=1}^{N} \frac{1}{(j)^{a_2} H_{N,a_2}} \right)^2 \right] \leq \tilde{\chi}^2(j, N, a_1, a_2). \tag{55}$$

**Proof** Since $f(\xi) = \xi^2 - 1$ is a convex function, therefore we use $f(\xi) = \xi^2 - 1$ in (46) to have (55), where

$$\tilde{\chi}^2(j, N, a_1, a_2) = \sum_{j=1}^{N} \frac{1}{(j)^{a_2} H_{N,a_2}} \left[ (j)^{a_2} H_{N,a_2} \right]^2 - 1 \right]. \quad \square$$

**Corollary 4.13** Let $N \in \mathbb{N}$ and $a_2 > 0$, $b_2 > 0$. Then

$$\left( \sum_{j=1}^{N} s_j - \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} \right) \ln \left( \frac{\sum_{j=1}^{N} s_j}{\sum_{j=1}^{N} \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}}} \right) \leq \tilde{D}_j(j, N, a_2, b_2). \tag{56}$$

**Proof** The function $f(\xi) = (\xi - 1) \ln(\xi)$ is convex. Use $f(\xi) = (\xi - 1) \ln(\xi)$ in (44) to obtain (56), where

$$\tilde{D}_j(j, N, a_2, b_2) = \sum_{j=1}^{N} \left( s_j - \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} \right) \ln \left( \frac{(j + b_2)^{a_2} H_{N,a_2,b_2}}{s_j} \right). \quad \square$$

If $s_j$ and $r_j$ are defined via the Zipf–Mandelbrot law.

**Corollary 4.14** Suppose that $N \in \mathbb{N}$ and $a_1, a_2 > 0$, $b_1, b_2 > 0$. Then

$$\left( \sum_{j=1}^{N} \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}} - \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} \right) \ln \left( \frac{\sum_{j=1}^{N} \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}}}{\sum_{j=1}^{N} \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}}} \right) \leq \tilde{D}_j(j, N, a_1, a_2, b_1, b_2). \tag{57}$$

**Proof** The function $f(\xi) = (\xi - 1) \ln(\xi)$ is convex. Use $f(\xi) = (\xi - 1) \ln(\xi)$ in (45) to get (57), where

$$\tilde{D}_j(j, N, a_1, a_2, b_1, b_2) = \sum_{j=1}^{N} \left( \frac{1}{(j + b_1)^{a_1} H_{N,a_1,b_1}} - \frac{1}{(j + b_2)^{a_2} H_{N,a_2,b_2}} \right) \ln \left( \frac{(j + b_2)^{a_2} H_{N,a_2,b_2}}{(j + b_1)^{a_1} H_{N,a_1,b_1}} \right). \quad \square$$

The following result holds as both $s_j$ and $r_j$ are defined by the Zipf law.
Suppose that $N \in \mathbb{N}$, $a_1, a_2 > 0$. Then

\[
\left( \sum_{j=1}^{n} \frac{1}{(j)^{a_1}H_{N,a_1}} - \sum_{j=1}^{n} \frac{1}{(j)^{a_2}H_{N,a_2}} \right) \ln \left( \frac{\sum_{j=1}^{n} \frac{1}{(j)^{a_2}H_{N,a_2}}}{\sum_{j=1}^{n} \frac{1}{(j)^{a_1}H_{N,a_1}}} \right) \leq \tilde{D}_f(j,N,a_1,a_2).
\]  

(58)

**Proof** The function $f(\zeta) = (\zeta - 1) \ln(\zeta)$ is convex. Use $f(\zeta) = (\zeta - 1) \ln(\zeta)$ in (46) to have (58), where

\[
\tilde{D}_f(j,N,a_1,a_2) = \sum_{j=1}^{N} \left( \frac{1}{j^{a_1}H_{N,a_1}} - \frac{1}{j^{a_2}H_{N,a_2}} \right) \ln \left( \frac{j^{a_2-a_1}H_{N,a_2}}{H_{N,a_1}} \right).
\]

In addition to all, similar findings for triangular discrimination are given as follows.

**Corollary 4.16** Let $N \in \mathbb{N}$, $a_2 > 0$, $b_2 > 0$. Then

\[
\frac{\left( \sum_{j=1}^{n} s_j - \sum_{j=1}^{n} \frac{1}{(j)^{b_2}H_{N,a_2}} \right)^2}{\sum_{j=1}^{n} s_j + \sum_{j=1}^{n} \frac{1}{(j)^{b_2}H_{N,a_2}}} \leq \tilde{\Delta}(j,N,a_2,b_2,\tilde{s}).
\]

(59)

**Proof** Since $f(\zeta) = \frac{(\zeta - 1)^2}{\zeta + 1}$ is a convex function, therefore we use $f(\zeta) = \frac{(\zeta - 1)^2}{\zeta + 1}$ in (44) to obtain (59), where

\[
\tilde{\Delta}(j,N,a_2,b_2,\tilde{s}) = \sum_{j=1}^{N} \frac{1}{(j + b_2)^{a_2}H_{N,a_2,b_2}} \left[ s_j(j + b_2)^{a_2}H_{N,a_2,b_2} - 1 \right]^2 \frac{1}{s_j(j + b_2)^{a_2}H_{N,a_2,b_2} + 1}.
\]

If $s_j$ and $r_j$ are defined via the Zipf–Mandelbrot law.

**Corollary 4.17** Suppose that $N \in \mathbb{N}$ and $a_1, a_2 > 0$, $b_1, b_2 > 0$. Then

\[
\frac{\left( \sum_{j=1}^{n} \frac{1}{(j)^{b_1}H_{N,a_1}} - \sum_{j=1}^{n} \frac{1}{(j)^{b_2}H_{N,a_2}} \right)^2}{\sum_{j=1}^{n} \frac{1}{(j)^{b_1}H_{N,a_1}} + \sum_{j=1}^{n} \frac{1}{(j)^{b_2}H_{N,a_2}}} \leq \tilde{\Delta}(j,N,a_1,a_2,b_1,b_2).
\]

(60)

**Proof** Since $f(\zeta) = \frac{(\zeta - 1)^2}{\zeta + 1}$ is a convex function, therefore we use $f(\zeta) = \frac{(\zeta - 1)^2}{\zeta + 1}$ in (45) to get (60), where

\[
\tilde{\Delta}(j,N,a_1,a_2,b_1,b_2) = \sum_{j=1}^{N} \frac{[j + b_2)^{a_2}H_{N,a_2,b_2} - (j + b_1)^{a_1}H_{N,a_1,b_1}]^2}{(j + b_2)^{a_2}H_{N,a_2,b_2} + (j + b_1)^{a_1}H_{N,a_1,b_1}}.
\]

The following result holds when both $s_j$ and $r_j$ are defined via the Zipf law.

**Corollary 4.18** Let $N \in \mathbb{N}$, $a_1, a_2 > 0$. Then

\[
\frac{\left( \sum_{j=1}^{n} \frac{1}{(j)^{a_1}H_{N,a_1}} - \sum_{j=1}^{n} \frac{1}{(j)^{a_2}H_{N,a_2}} \right)^2}{\sum_{j=1}^{n} \frac{1}{(j)^{a_1}H_{N,a_1}} + \sum_{j=1}^{n} \frac{1}{(j)^{a_2}H_{N,a_2}}} \leq \tilde{\Delta}(j,N,a_1,a_2).
\]

(61)
Proof Since \( f(\xi) = \frac{\xi-1}{\xi+1}^2 \) is a convex function, we use \( f(\xi) = \frac{\xi-1}{\xi+1}^2 \) in (46) to have (61), where

\[
\tilde{\Delta}(j, N, a_1, a_2) = \sum_{j=1}^{N} \left( \frac{p_{2j} - a_1 H_{N, a_2} - H_{a_2}}{p_{2j} - a_2 H_{N, a_1} + H_{a_1}} \right)^2.
\]

□

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Author details
1Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan. 2Department of Mathematics, University of Lahore (Sargodha Campus), Sargodha 40100, Pakistan. 3Department of Media and Communication, University North, Trg dr. Žarka Dolinar 1, Koprivnica, Croatia. 4RUDN University, Miklukho-Maklaya str. 6, 117198 Moscow, Russia.

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