The $q$-Racah polynomials from scalar products of Bethe states

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Abstract

The $q$-Racah polynomials are expressed in terms of certain ratios of scalar products of Bethe states associated with Bethe equations of either homogeneous or inhomogeneous type. This result is obtained by combining the theory of Leonard pairs and the modified algebraic Bethe ansatz.

Keywords: Askey–Wilson algebra, Leonard pairs, orthogonal polynomials, Bethe ansatz

1. Introduction

At the top of the hierarchy of discrete classical $q$-hypergeometric orthogonal polynomials known as the discrete Askey scheme [AW79, KS96], the $q$-Racah polynomials$^3$ satisfy a bispectral property that is encoded into the representation theory of the Askey–Wilson algebra [T87, Zh91] using the theory of Leonard pairs [TV03, T04]. In particular, as will be reviewed in section 2, the $q$-Racah polynomials can be computed from ratios of scalar products of eigenvectors and dual eigenvectors associated with elements of the corresponding Leonard pair.

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$^3$ Note that all other families of orthogonal polynomials of the discrete Askey scheme can be reached from the $q$-Racah polynomials by various limit transitions (either in the scalar parameters entering into the definition of the polynomials, or $q \to 1$). See [KS96, K10] for details.
Recently, the spectral problem for certain combinations of elements of a Leonard pair of \(q\)-Racah type—known as the Heun–Askey–Wilson operator—has been solved using the theory of Leonard pairs combined with the framework of the modified algebraic Bethe ansatz\(^4\) \cite{BaP19}. In this approach, the eigenvectors are Bethe states and the eigenvalues are expressed in terms of Bethe roots satisfying a system of transcendental equations known as the Bethe equations. For generic parameters, the diagonalization of the Heun–Askey–Wilson operator is associated with Bethe equations of inhomogeneous type\(^5\). For special choices of parameters, preliminary results in \cite{BaP19} however suggest the existence of a rigorous correspondence between eigenbases for elements of Leonard pairs and different Bethe states associated with either homogeneous or inhomogeneous Bethe equations. Therefore, a relation between the \(q\)-Racah polynomials and scalar products of Bethe states is also expected.

The purpose of this paper is to establish the precise correspondence between eigenvectors and dual eigenvectors of elements of a Leonard pair of \(q\)-Racah type and Bethe states/dual Bethe states of homogeneous or inhomogeneous type. A straightforward consequence is various expressions of the \(q\)-Racah polynomials in terms of ratios of scalar product of Bethe states of homogeneous or inhomogeneous type. To achieve that, we have to perform two calculations in addition to those in \cite{BaP19}, namely, the construction of the dual Bethe states and the correct identification of normalization factors. Furthermore, it also leads to a series of unexpected identities relating Bethe states of homogeneous type to Bethe states of inhomogeneous type.

To the best of our knowledge, this correspondence is new and may motivate further advances, on the one hand, in the theory of quantum integrable spin chains and, on the other hand, in the theory of special functions. For instance, it is known that computing correlations for the XXZ chain with boundaries is a challenging task due to the intricacy of the reflection algebra and Bethe states, see e.g. \cite{KKMNST07, BelPS21, NT22}. By extending the approach in this paper to the \(q\)-Onsager case \cite{BaK07} (of which the Askey–Wilson algebra is the simplest quotient), we expect to gain a better understanding of the Bethe states, their scalar products and their correlations for the open XXZ chain. Meanwhile, new special functions generalizing the \(q\)-Racah polynomials may be obtained from the scalar products of the open XXZ chain with certain boundary conditions. Additional details are provided in section 4.

The text is organized as follows. In section 2, the concept of Leonard pairs, related eigenbases and the relationship between the entries of the transition matrix and the \(q\)-Racah polynomials are recalled. In section 3, eigenbases and dual eigenbases for a Leonard pair of \(q\)-Racah type are constructed in terms of Bethe states and dual Bethe states associated with Bethe equations of either homogeneous or inhomogeneous type. The interpretation of the \(q\)-Racah orthogonal polynomials in terms of ratios of scalar products of Bethe states follows, see section 3.5. Concluding remarks are given in the last section, where some perspectives are given. Part of the necessary material for the proofs can be found in \cite{BaP19} and appendix A. Some numerical solutions of the Bethe equations are given in appendix B.

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\(^4\) The modified algebraic Bethe ansatz was developed to analyze the eigenproblem of quantum spin chains with U(1) symmetry breaking boundary fields \cite{BelC13, Bel15, C15, ABGP15, BelP15}.

\(^5\) Inhomogeneous Bethe equations were originally discovered in the context of the open XXZ spin chain \cite{WYCS15}.
Notations: The parameter $q$ is assumed not to be a root of unity and $q \neq 1$. We write $[X, Y]_q = qXY - q^{-1}YX$ and $|n|_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The identity element is denoted $I$. We use the standard $q$-shifted factorials:

$$ (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q), \quad (1.1) $$

and define

$$ b(x) = x - x^{-1}. \quad (1.2) $$

2. The Askey–Wilson algebra, Leonard pairs and transition matrix

In this section, we introduce the defining relations of the Askey–Wilson algebra with generators $A, A^*$. The concept of a Leonard pair associated with $A, A^*$ is briefly reviewed. The eigenbases and dual eigenbases for the Leonard pair are defined. Then, we recall the transition matrices relating those eigenbases, and how they are expressed in terms of the $q$-Racah orthogonal polynomials and scalar products of Leonard pairs’ eigenvectors [Zh91, T03, TV03, T04].

2.1. Askey–Wilson algebra and Leonard pairs

Let $\rho, \omega, \eta, \eta^* \in C^*$ be generic. The Askey–Wilson algebra (AW) is generated by $A, A^*$ subject to the relations [T87, Zh91]

$$ [A, [A, A^*]]_q^{(1)} = \rho A^* + \omega A + \eta I, \quad (2.1) $$

$$ [A^*, [A^*, A]]_q^{(1)} = \rho A + \omega A^* + \eta^* I. \quad (2.2) $$

Let $V$ be a vector space of positive finite dimension $\dim(V) = 2s + 1$, where $s$ is an integer or half-integer. Define the algebra homomorphism $\pi : AW \to \text{End}(V)$. Assume $\pi(A), \pi(A^*)$ are diagonalizable on $V$, each multiplicity-free and $V$ is irreducible. Then, by [TV03, theorem 6.2], $\pi(A), \pi(A^*)$ is a Leonard pair [TV03, definition 1.1]. Given the eigenvalue sequence $\{\theta_M\} \ (M = 0)$ associated with $\pi(A)$ (resp. the eigenvalue sequence $\{\theta_N^*\} \ (N = 0)$ associated with $\pi(A^*)$), one associates an eigenbasis with vectors $\{\theta_M\} \ (M = 0)$ (resp. an eigenbasis with vectors $\{\theta_N^*\} \ (N = 0)$). For a Leonard pair, recall that: (i) in the eigenbasis of $\pi(A)$, then $\pi(A^*)$ acts as a tridiagonal matrix; (ii) in the eigenbasis of $\pi(A^*)$, then $\pi(A)$ acts as a tridiagonal matrix:

$$ \pi(A)(\theta_M) = \theta_M(\theta_M), \quad \pi(A^*)(\theta_M) = A_{M+1,M}^* \theta_{M+1} + A_M^* \theta_M + A_M^{*,-1} \theta_{M-1}. \quad (2.3) $$

$$ \pi(A^*)(\theta_N^*) = \theta_N^*(\theta_N^*), \quad \pi(A)(\theta_N^*) = A_{N+1,N} \theta_{N+1} + A_N \theta_N + A_N^{*,-1} \theta_{N-1}. \quad (2.4) $$

Here $A_{1,0}^* = A_{2,1,2}^* = A_{-1,0} = A_{2,1,2} = 0$, and the explicit expressions for the coefficients $\{A_{M+1,M}, A_M^*, A_M^{*,-1}\}$ and $\{A_{N+1,N}, A_N, A_N^{*,-1}\}$ in terms of the eigenvalue sequences are given in [T04].

Let $\hat{V}$ be the dual vector space of $V$, i.e. the vector space $\hat{V}$ of all linear functionals from $V$ to $C$. Define the family of covectors $\{\langle \theta_M \rangle \} \in \hat{V}$ (resp. $\{\langle \theta_N^* \rangle \} \in \hat{V}$) associated with the eigenvalue sequence $\{\theta_M\} \ (M = 0)$ (resp. $\{\theta_N^*\} \ (N = 0)$) such that:

$$ \langle \theta_M \rangle \pi(A) = \langle \theta_M \rangle \theta_M, \quad \langle \theta_M \rangle \pi(A^*) = \langle \theta_M \rangle A_{M+1,M}^* \theta_{M+1} + \langle \theta_M \rangle A_M^* \theta_M + \langle \theta_M \rangle A_M^{*,-1} \theta_{M-1}, \quad (2.5) $$

$$ \langle \theta_N^* \rangle \pi(A^*) = \langle \theta_N^* \rangle \theta_N^*, \quad \langle \theta_N^* \rangle \pi(A) = \langle \theta_N^* \rangle A_{N+1,N} \theta_{N+1} + \langle \theta_N^* \rangle A_N \theta_N + \langle \theta_N^* \rangle A_N^{*,-1} \theta_{N-1}. \quad (2.6) $$


are introduced. Now, using (case I):

\[ \langle \theta_{M'} | \theta_{M'} \rangle = \delta_{MM'} \xi_{M} \cdot \langle \theta_{N'} | \theta_{N'} \rangle = \delta_{NN'} \xi_{N} \cdot \]

(2.7)

where \( \delta_{ij} \) denotes the Kronecker symbol and the normalization factors \( \xi_{M} \neq 0, \xi_{N} \neq 0 \) are introduced. Now, using (2.3), (2.5) and (2.4), together with (2.7), the equalities \( \langle \theta_{M'} | \pi(A^*) \rangle | \theta_{M} \rangle = \langle \theta_{M'} | \pi(A^*) \rangle | \theta_{M} \rangle \) and \( \langle \theta_{N'} | \pi(A) \rangle | \theta_{N} \rangle = \langle \theta_{N'} | \pi(A) \rangle | \theta_{N} \rangle \) imply the orthogonality relations

\[ \tilde{A}_{M,M'} = \tilde{A}_{M,M'} \frac{\xi_{M}}{\xi_{M'}} \text{ for } M' = M + 1, M, M - 1, \]

(2.8)

\[ \tilde{A}_{N,N'} = \tilde{A}_{N,N'} \frac{\xi_{N}}{\xi_{N'}} \text{ for } N' = N + 1, N, N - 1. \]

(2.9)

In the following, we consider the eigenvalue sequences of the form \([T99, \text{ theorem } 4.4\) (case I)]:

\[ \theta_{M} = b q^{2M} + c q^{-2M}, \quad \theta_{N} = b^* q^{2N} + c^* q^{-2N}, \]

(10.10)

where \( b, c, b^*, c^* \in \mathbb{C}^* \). For this parametrization, the structure constant \( \rho \) in (2.1), (2.2) is given by \([T99, \text{ lemma } 4.5]\):

\[ \rho = -bc(q^2 - q^{-2})^2 = -b^*c^*(q^2 - q^{-2})^2. \]

(11.11)

Without loss of generality, the other structure constants can be written in the form:

\[ \omega = (q - q^{-1})^2 \left( bc(\zeta^2 + \zeta^{-2})(q^{2r+1} + q^{-2r-1}) - (bq^{2r} + cq^{-2r})(b^*q^{2s} + c^*q^{-2s}) \right), \]

(12.12)

\[ \eta = \frac{(q^2 - q^{-2})^2}{(q + q^{-1})} bc \left( (bq^{2r} + cq^{-2r})(\zeta^2 + \zeta^{-2}) - (b^*q^{2s} + c^*q^{-2s})(q^{2r+1} + q^{-2r-1}) \right), \]

(13.13)

\[ \eta^* = -\frac{(q^2 - q^{-2})^2}{(q + q^{-1})} b^*c^* \left( (b^*q^{2s} + c^*q^{-2s})(\zeta^2 + \zeta^{-2}) - (bq^{2r} + cq^{-2r})(q^{2r+1} + q^{-2r-1}) \right), \]

(14.14)

where \( \zeta \in \mathbb{C}^* \) is generic. Adapting the notations from \([TV03, \text{ theorem } 5.3]\) with \([T04, \text{ lemma } 10.3]\) compared with (2.12)–(2.14), the coefficients \( \{\tilde{A}_{M,M'}\}, \{\tilde{A}_{N,N'}\} \) in (2.3), (2.4) read as follows:

\[ \tilde{A}_{M,M-1} = q^{4s+4} \frac{(1 - q^{2M})(c - bq^{2M+4})(b^*q^{2s-1} + cq^{2s-1})(\zeta^2 + c^*q^{2s-1})}{(c - bq^4)(c - bq^4)}, \]

(15.15)

\[ A_{M-1,M}^* = \frac{(1 - q^{2M-4s-2})(c - bq^2)(c + b^*q^{2s-2})(\zeta^2 + c^*q^{2s-2})}{(c - bq^2)(c - bq^2)}, \]

(16.16)

\[ A_{M,M}^* = \theta_{0} - \tilde{A}_{M,M+1} - \tilde{A}_{M,M-1}. \]

(17.17)

The coefficients \( \{A_{N,N'}\} \) are obtained from \( \{A_{M,M'}^*\} \) using the transformation \( (b, c, \zeta, M) \leftrightarrow (b^*, c^*, \zeta^{-1}, N) \).

\[ \{w \in V|\langle w | v \rangle = 0\} \text{ and } \{w \in V|\langle \tilde{V} | w \rangle = 0\}. \]
In addition to the Askey–Wilson relations (2.1) and (2.2), by the Cayley–Hamilton theorem one has:
\[ \prod_{M=0}^{2x} (\pi(A) - \theta_M) = 0, \quad \prod_{N=0}^{2x} (\pi(A^*) - \theta^*_N) = 0. \]  
(2.18)

Note that explicit examples of Leonard pairs associated with embeddings of the AW algebra into \( U_q(sl_2) \) or \((U_q(sl_2))^\otimes 3\) are known, see [GZ93a, GZ93b, H16].

2.2. Transition matrix and the q-Racah polynomials

Given a Leonard pair, the transition matrices relating the eigenbases \( \{|\theta_M\rangle\}_{M=0}^{2x} \) and \( \{|\theta^*_N\rangle\}_{N=0}^{2x} \) are expressed in terms of three-term recurrence relations with respect to one has:
\[ |\theta_N^*\rangle = \sum_{M=0}^{2x} P_{MN}|\theta_M\rangle \quad \text{and} \quad |\theta^*_M\rangle = \sum_{N=0}^{2x} (P^{-1})_{NM}|\theta^*_N\rangle. \]  
(2.19)

where \( P \) (resp. \( P^{-1} \)) denotes the transition matrix from the basis \( \{|\theta_M\rangle\}_{M=0}^{2x} \) to the basis \( \{|\theta^*_N\rangle\}_{N=0}^{2x} \) (resp. the inverse transition matrix from the basis \( \{|\theta^*_N\rangle\}_{N=0}^{2x} \) to the basis \( \{|\theta_M\rangle\}_{M=0}^{2x} \)). From (2.19), using (2.7), the entries of the transition matrix are given by the scalar products:
\[ P_{MN} = \langle \theta_M|\theta^*_N\rangle / \langle \theta_M|\theta_M\rangle \quad \text{and} \quad (P^{-1})_{NM} = \langle \theta^*_N|\theta_M\rangle / \langle \theta^*_N|\theta^*_N\rangle. \]  
(2.20)

Similarly, the dual eigenvectors are related as follows:
\[ |\theta^*_N\rangle = \sum_{M=0}^{2x} \sum_{\xi^*} P^{-1}_{MN}(\theta^*_M) \quad \text{and} \quad |\theta_M\rangle = \sum_{N=0}^{2x} \sum_{\xi^*} \xi_{MN}^*|\theta^*_N\rangle. \]  
(2.21)

Introduce the q-Racah polynomials:
\[ R_M(\theta_N^*) = 4\theta_3 \left[ q^{-2M} \frac{1}{\xi^*} q^{2M} \frac{1}{\xi^*} q^{-2N} \frac{1}{\xi^*} q^{2N} \frac{1}{\xi^*} - \frac{b^*}{c} q^{2s+1} \frac{1}{\xi^*} q^{-2N} \frac{1}{\xi^*} q^{2N} \frac{1}{\xi^*} q^{2}\frac{1}{\xi^*} \right]. \]  
(2.22)

Adapting the notations of [T04], the entries of the transition matrices are given by:
\[ P_{MN} = k_N R_M(\theta_N^*) \quad \text{and} \quad (P^{-1})_{NM} = \nu_0^{-1} k_M^* R_M(\theta_N^*) \]  
(2.23)

where
\[ k_N = \frac{(-\frac{b^*}{c} q^{2s+1} \frac{1}{\xi^*} q^{-2N} \frac{1}{\xi^*} q^{2N} \frac{1}{\xi^*} - \frac{b^*}{c} q^{1-2s} \frac{1}{\xi^*} q^{2s+1} \frac{1}{\xi^*} q^{2} \frac{1}{\xi^*})^N}{(\xi^*)^N (\xi^*)^N} \]  

\[ k_M^* = k_N |_{\xi^* \rightarrow M, c \rightarrow b^*, \xi^* \rightarrow c, \xi^* \rightarrow \frac{1}{\xi^*}} \]  

\[ \nu_0 = \frac{\left(\frac{b^*}{c} q^{2s+1} \frac{1}{\xi^*} q^{-2N} \frac{1}{\xi^*} q^{2N} \frac{1}{\xi^*} q^{2}\frac{1}{\xi^*}\right)^2}{2} \frac{1}{\xi^*} \frac{1}{\xi^*} \]  

From (2.19), using (2.3) and (2.4) one finds that the transition matrix coefficients satisfy three-term recurrence relations with respect to \( M, N \) [Zh91, T04]. Using (2.23), one recovers the well-known relations:
\[ \theta^*_M R_M(\theta^*_N) = A_{M,M+1}^* R_{M+1}(\theta^*_N) + A_{M,M}^* R_M(\theta^*_N) + A_{M,M-1}^* R_{M-1}(\theta^*_N), \]  
(2.24)

\[ \theta_M R_M(\theta^*_N) = A_{N,N+1} R_{N+1}(\theta^*_N) + A_{N,N} R_M(\theta^*_N) + A_{N,N-1} R_{N-1}(\theta^*_N). \]  
(2.25)
Combining the two relations in (2.19), the orthogonality relations satisfied by the $q$-Racah polynomials follow [T03, section 16]:

$$\sum_{N=0}^{2s} k_N R_M(\theta_N^+) R_M(\theta_N^-) = \nu_0(k_N)^{-1} \delta_{MM'}, \quad \sum_{M=0}^{2s} k_M R_M(\theta_N^+) R_M(\theta_N^-) = \nu_0(k_N)^{-1} \delta_{NN'}.$$  \hspace{1cm} (2.26)

From (2.20), it turns out that the scalar products between eigenvectors of Leonard pairs are the basic building blocks for the $q$-Racah polynomials. As $R_0(\theta_N^+) = 1$ and $R_M(\theta_N^-) = 1$, from (2.23) one gets $k_N = \langle \theta_N | \theta_N^+ \rangle / \langle \theta_0 | \theta_0 \rangle$ and $\nu_0^{-1} k_N^2 = \langle \theta_N^+ | \theta_0 \rangle / \langle \theta_0 | \theta_0 \rangle$. The following result can be viewed as a variation of [Zh91]. Up to normalization, see also [T03, theorems 14.6 and 15.6]. For any $0 \leq N, M \leq 2s$, the $q$-Racah polynomials are given by:

$$R_M(\theta_N^+) = \frac{\langle \theta_M | \theta_N^+ \rangle}{\langle \theta_0 | \theta_0 \rangle} \frac{\langle \theta_M | \theta_N^- \rangle}{\langle \theta_M | \theta_N^- \rangle} \quad (2.27)$$

$$= \frac{\langle \theta_N^+ | \theta_M \rangle}{\langle \theta_N^- | \theta_M \rangle} \frac{\langle \theta_N^+ | \theta_N^- \rangle}{\langle \theta_N^- | \theta_N^- \rangle} \quad (2.28)$$

### 3. Eigenbases for Leonard pairs from Bethe states

The AW algebra admits a presentation in the form of a reflection algebra [Za95, Ba04]. This presentation allows one to apply the technique of algebraic Bethe ansatz in order to diagonalize $\pi(A), \pi(A^*)$. The purpose of this section is to recall the eigenbases—see [BaP19] for details—and construct the dual eigenbases associated with the Leonard pair $\pi(A), \pi(A^*)$ in terms of the so-called Bethe states and dual Bethe states. The Bethe eigenstates and dual eigenstates built are essentially of two different types, called either of homogeneous type or of inhomogeneous type. The necessary material for the analysis below is found in appendix A and [BaP19].

#### 3.1. Bethe states and dual Bethe states

In the algebraic Bethe ansatz approach, the main ingredients are the ‘dynamical’ operators [CLSW03] $\{ \mathcal{D}^\alpha(u,m), \mathcal{D}^\beta(u,m), \mathcal{D}^\gamma(u,m), \mathcal{D}^\epsilon(u,m) \}$ that satisfy a set of exchange relations (A.1)–(A.7). Importantly, the dynamical operators are polynomials of maximal degree 2 in the elements $A, A^*$, depending on $\alpha, \beta \in \mathbb{C}$ and the so-called spectral parameter $u \in \mathbb{C}^*$. We refer the reader to [BaP19, appendix A] for their explicit expressions.

The starting point of the construction of Bethe states is the identification of so-called reference states. Given a Leonard pair with (2.3) and (2.4), the following lemma is derived [BaP19, propositions 3.1 and 3.2]. Let $m_0$ be an integer.

**Lemma 3.1.** If the parameter $\alpha$ is such that:

$$(q^2 - q^{-2}) \chi^{-1} \alpha C^* q^{m_0} = 1 \quad \text{(resp. } (q^2 - q^{-2}) \chi^{-1} \alpha B q^{-m_0} = -1) \quad (3.1)$$

then

$$\pi(\mathcal{E}^+(u,m_0)) \theta_0^+ = 0 \quad \text{(resp. } \pi(\mathcal{E}^-(u,m_0)) \theta_0^- = 0). \quad (3.2)$$

In the dual vector space, for (2.5), (2.6) and using [BaP19, appendix A], analog results are derived along the same line.

**Lemma 3.2.** If the parameter $\beta$ is such that:

$$(q^2 - q^{-2}) \chi^{-1} \beta B^* q^{m_0+2} = 1 \quad \text{(resp. } (q^2 - q^{-2}) \chi^{-1} \beta C q^{m_0+2} = -1) \quad (3.3)$$
The following holds:

\[ \langle \theta_0^+ | \pi(\mathcal{B}^+ (u, m_0 - 2)) = 0 \quad \text{(resp. } \langle \theta_0^- | \pi(\mathcal{B}^- (u, m_0 - 2)) = 0 \text{).} \quad (3.4) \]

Within the algebraic Bethe ansatz framework, the fundamental eigenvectors for the Leonard pair and their duals find a natural interpretation. Let \(|\Omega^\pm\rangle\) and \(|\Omega^\mp\rangle\) denote the so-called reference and dual reference states with respect to the dynamical operators \{\mathcal{B}^+(u, m), \mathcal{C}^+(u, m)\}.

**Definition 3.1.**

\[ |\theta_0\rangle = |\Omega^-\rangle, \quad |\theta_0^+\rangle = |\Omega^+\rangle, \quad \langle \theta_0 | = \langle \Omega^- |, \quad \langle \theta_0^+ | = \langle \Omega^+ |. \]

According to the choice of parameters \(\alpha, \beta\), the action of the dynamical operators \(\mathcal{A}^\pm(u, m_0)\) and \(\mathcal{G}^\pm(u, m_0)\) on the reference states \(|\Omega^\pm\rangle\) and duals \(|\Omega^\mp\rangle\) are computed. Recall the parametrization (2.12)–(2.14) and define: The vector space \(\mathcal{V}\) and its dual \(\mathcal{V}^*\) being finite dimensional, the following actions of dynamical operators are considered. Recall (2.3)–(2.6). The following result extends [BaP19, Lemma 3.4], thus we skip the proof.

**Lemma 3.3.** The following holds:

\[
\pi(\mathcal{B}^+(u, m_0 + 4s))|\theta^+_2\rangle = 0 \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\beta b^* q^{-m_0} = 1, \quad (3.5)
\]

\[
\pi(\mathcal{B}^-(u, m_0 + 4s))|\theta^-_2\rangle = 0 \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\beta c q^m = -1, \quad (3.6)
\]

\[
\langle \theta^+_2 | \pi(\mathcal{C}^+(u, m_0 + 4s)) = 0 \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\alpha c^* q^m = 1, \quad (3.7)
\]

\[
\langle \theta^-_2 | \pi(\mathcal{C}^-(u, m_0 + 4s)) = 0 \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\alpha b q^{-m_0} = -1. \quad (3.8)
\]

By straightforward calculations, it follows (see [BaP19, Lemma 3.3]) for the proof of (3.9):

**Lemma 3.4.** Let \(\alpha, \beta\) be fixed according to lemmas 3.1 and 3.2. Then, the dynamical operators act as:

\[
\pi(\mathcal{A}^\pm(u, m_0))|\Omega^\pm\rangle = \Lambda^\pm_1(u)|\Omega^\pm\rangle \quad \text{and} \quad \pi(\mathcal{G}^\pm(u, m_0))|\Omega^-\rangle = \Lambda^\pm_2(u)|\Omega^\pm\rangle, \quad (3.9)
\]

\[
|\Omega^\pm\rangle \pi(\mathcal{A}^\pm(v, m_0)) = |\Omega^\pm\rangle|\Lambda^\pm_1(v)\rangle \quad \text{and} \quad \langle \Omega^\pm | \pi(\mathcal{G}^\pm(v, m_0)) = \langle \Omega^\pm | \Lambda^\pm_2(v)\rangle, \quad (3.10)
\]

where the eigenvalues take the factorized form:

\[
\Lambda^1_1(u) = \frac{q^{-2s-1}}{u^{-1}}(q^{2s} + u^2 - 1)(q^{2s+1}u^2 - u^{-1}) \times (c^2u - q^{-2s} + u^{-1}) \left(\frac{c}{b}\right)^{1+s} + u^{-1}\left(\frac{b^*}{b}\right)^{1+s},
\]

\[
\Lambda^1_2(u) = \frac{(q^2 - u^{-2})q^{-2s-1}}{u^2(q^2 - q^{-2s})} \left(\frac{q^2 - u^{-2}q^{-1} - 1}{q^{-2s} - u^{-1}q^{-2s} - 1} \right) \left(\frac{q^2 - u^{-2}q^{-1} - 1}{q^{-2s} - u^{-1}q^{-2s} - 1} \right) \left(\frac{c}{b}\right)^{1+s} + u^{-1}\left(\frac{b^*}{b}\right)^{1+s}.
\]

Different types of Bethe states may be considered, built from successive actions of the dynamical operators \(\mathcal{B}^\pm(u, m)\) (resp. \(\mathcal{C}^\pm(u, m)\)) on each reference state \(|\Omega^\pm\rangle\) (resp. its dual \(|\Omega^\mp\rangle\)). Consider the strings of dynamical operators

\[
B^i(u, m, M) = B^i(u, m + 2(M - 1)) \cdots B^i(u, m)
\]

and

\[
C^i(v, m, N) = C^i(v, m + 2) \cdots C^i(v, m + 2N),
\]

(3.11) (3.12)
where we denote the set of variables \( \bar{u} = \{u_1, u_2, \ldots, u_M\} \), \( \bar{v} = \{v_1, v_2, \ldots, v_N\} \). Taking into account the parameters \( \alpha \) and \( \beta \) given in (3.1) and (3.3), define the following vectors and dual vectors,

\[
\begin{align*}
|\Psi_M^\pm(\bar{u}, m_0)\rangle &= \pi(B^+(\bar{u}, m_0, M))|\Omega^-\rangle \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\alpha \beta q^{-m_0} = -1 \quad \text{and} \quad \beta = 0 , \quad (3.13) \\
|\Psi_M^+(\bar{v}, m_0)\rangle &= \pi(B^+(\bar{v}, m_0, M))|\Omega^+\rangle \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\alpha \gamma q^{m_0} = 1 \quad \text{and} \quad \beta = 0 , \quad (3.14) \\
\langle \Psi_N^-(\bar{v}, m_0) | &= \langle \Omega^- | \pi(C^- (\bar{v}, m_0, N)) \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\beta \gamma q^{m_0} = -1 \quad \text{and} \quad \alpha = 0 , \quad (3.15) \\
\langle \Psi_N^+(\bar{v}, m_0) | &= \langle \Omega^+ | \pi(C^+(\bar{v}, m_0, N)) \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\beta \gamma q^{m_0} = 1 \quad \text{and} \quad \alpha = 0 . \quad (3.16)
\end{align*}
\]

As usual, if \( \bar{u} \) (or \( \bar{w}, \bar{v}, \bar{y} \)) is a set of variables satisfying certain Bethe ansatz equations, the Bethe states (3.13)–(3.16), are called ‘on-shell’. On the other hand, if the set of variables \( \bar{u} \) is arbitrary, the Bethe states are called ‘off-shell’.

### 3.2. Eigenbases of homogeneous type for the Leonard pair

The eigenvectors and dual eigenvectors for the Leonard pair \( \pi(A), \pi(A^+) \), are now constructed in terms of Bethe states and dual Bethe states associated with Bethe equations of homogeneous type. Let us define the set of functions:

\[
E^M_{\pm}(u, \bar{u}) = -\frac{b(u_i^2)}{b(qu_i^4)} \prod_{j=1,j \neq i}^M f(u_i, u_j) \Lambda^\pm_i(u) + \prod_{j=1,j \neq i}^M b(u_i, u_j) \Lambda^\mp_i(u), \quad (3.17)
\]

for \( i = 1, \ldots, M \). The set of equations \( E^M_{\pm}(u, \bar{u}) = 0 \) for \( i = 1, \ldots, M \) are called the Bethe ansatz equations of homogeneous type associated with the set of Bethe roots \( \bar{u} \). Note that the set of equations \( E^M_{\pm}(u, \bar{u}) = 0 \) for \( i = 1, \ldots, M \) contain trivial solutions where \( u_i^2 = u_i^{-2} \) or \( u_i = 0 \) for \( i = 1, \ldots, M \), recall the expression of \( \Lambda_i^\pm(u) \) in lemma 3.4. These solutions must be discarded since they lead to identically null or ill defined Bethe states. To see that, we refer the reader to [BaP19, corollary 3.1] where the expansion of a Bethe state in the Poincaré–Birkhoff–Witt basis of the Askey–Wilson algebra is given. Also, after extracting the denominator in \( E^M_{\pm}(u, \bar{u}) = 0 \), we may find solutions for which \( U_i = U_j \) for \( i \neq j \) where the symmetrized Bethe root \( U_i = (qu_i^2 + q^{-1}u_i^{-2})/(q + q^{-1}) \) is introduced. The solutions with coincident symmetrized Bethe roots such that \( U_i = U_j \) for \( i \neq j \) are not admissible, since they may lead to Bethe states which are not eigenstates of \( \pi(A) \) or \( \pi(A^+) \). For additional discussion on this subject we refer the reader to [BaP19, section 3.5]. Below, the solutions \( U_i \neq U_j \) are called admissible. For a discussion of coincident Bethe roots for other integrable systems see [S22, section 2.3]. Based on these observations and supported by numerical analysis (see appendix B), we formulate the following hypothesis.

**Hypothesis 1.** For each integer \( M \) (resp. \( N \)) with \( 0 \leq M, N \leq 2s \), there exists at least one set of non trivial admissible Bethe roots \( S^M_{\pm} = \{u_1, \ldots, u_M\} \) (resp. \( S^N_{\pm} = \{w_1, \ldots, w_N\} \)) such that

\[
E^M_{\pm}(u, \bar{u}) = 0 \quad \text{for} \quad \bar{u} = S^M_{\pm}, \quad \text{(resp.} \quad E^N_{\pm}(w, \bar{w}) = 0 \quad \text{for} \quad \bar{w} = S^N_{\pm}). \quad (3.18)
\]

**Lemma 3.5.** Assume hypothesis 1. The following relations hold:

\[
\begin{align*}
|\theta_M^u\rangle &= N_M(\bar{u}) |\Psi^M_{\pm}(\bar{u}, m_0)\rangle \quad \text{for} \quad \bar{u} = S^M_{\pm}, \\
|\theta_N^w\rangle &= N_N(\bar{w}) |\Psi^N_{\pm}(\bar{w}, m_0)\rangle \quad \text{for} \quad \bar{w} = S^N_{\pm}.
\end{align*}
\]

(3.19) (3.20)
with
\[ N_M(\tilde{u}) = \prod_{k=1}^{M} (qu_k b(u_k^2)A_k^+ - \tilde{u})^{-1}, \quad N_N(\tilde{w}) = \prod_{k=1}^{N} (-q^{-1}w_k^{-1}b(w_k^2)A_k)_{-}^{-1}, \] (3.21)
and \( \tilde{N}_0(.) = \tilde{N}_0^+(.) = 1. \)

**Proof.** Consider (3.19). By [BaP19, proposition 3.1], it is known that:
\[ \pi(A)|\psi_M^M(\tilde{u}, m_0)\rangle = \theta_M^M|\psi_M^M(\tilde{u}, m_0)\rangle \quad \text{for} \quad \tilde{u} = s_M^{M(b)} . \] (3.22)

By definition of a Leonard pair, the spectrum of \( \pi(A) \) is non-degenerate with (2.3). So, if there exists a solution of the Bethe equations associated with the eigenvalue \( \theta_M \), it must be such that \( |\theta_M\rangle \) is proportional to \( |\psi_M^M(\tilde{u}, m_0)\rangle \). Let \( \tilde{N}_M(\tilde{u}) \) denote the normalization factor in the r.h.s. of (3.19). To fix it, observe that \( B^- (\tilde{u}, m_0, M) \) is a polynomial in \( A, A^+, AA^+, A^+ A \) [BaP19, appendix A]. Using (2.3), one extracts the coefficient of \( |\theta_M\rangle \) from \( |\psi_M^M(\tilde{u}, m_0)\rangle \) which, by definition, is the inverse of \( \tilde{N}_M(\tilde{u}) \). The proof of (3.20) is done along the same line, starting from [BaP19, proposition 3.2] and using (2.4).

The proof of the following lemma is analog to lemma 3.5, so we skip the details.

**Lemma 3.6.** Assume hypothesis 1. The following relations hold:
\[ \langle \theta_M \rangle = \tilde{N}_M(\tilde{v})|\psi_M^M(\tilde{v}, m_0)\rangle \quad \text{for} \quad \tilde{v} = S_M^{M(b)} , \] (3.23)
\[ \langle \tilde{\theta}_N \rangle = \tilde{N}_N^+(\tilde{y})|\psi_N^N(\tilde{y}, m_0)\rangle \quad \text{for} \quad \tilde{y} = S_N^{N(b)} \] (3.24)

with
\[ \tilde{N}_M(\tilde{v}) = \prod_{k=1}^{M} (q^{-1}v_k b(v_k^2)A_k^+ - \tilde{v})^{-1}, \quad \tilde{N}_N^+(\tilde{y}) = \prod_{k=1}^{N} (-q y_k^{-1}b(y_k^2)A_k)_{-}^{-1} \] (3.25)
and \( \tilde{N}_0(.) = \tilde{N}_0^+(.) = 1. \)

For the homogeneous case, it should be stressed that the construction of the Bethe states and dual Bethe states lead to the same set of Bethe equations (3.17).

### 3.3. Eigenbases of inhomogeneous type for the Leonard pair

The eigenvectors and dual eigenvectors for the Leonard pair \( \pi(A), \pi(A^+) \), can be alternatively constructed in terms of Bethe states and dual Bethe states associated with Bethe equations of inhomogeneous type. However, contrary to the homogeneous case, for the inhomogeneous case two different sets of Bethe equations characterizing respectively to Bethe states and dual Bethe states are obtained.

A first set of Bethe equations of inhomogeneous type is now introduced. Let us define\(^7\):
\[ E_\pm(u_i, \tilde{u}) = \frac{b(u_i^2)}{b(q u_i^2)} \prod_{j=1, j\neq i}^{2\nu} f(u_i, u_j)A_i^\pm_1(u_i) - (q^2 u_i^3)^\pm_1 \prod_{j=1, j\neq i}^{2\nu} h(u_i, u_j)A_i^\pm_1(u_i) \]
(3.26)
\[ + \nu \pm \frac{u_i^2 b(u_i^2)}{b(q)} \prod_{k=1}^{2\nu} b(q^{1/2+k-i}u_i) b(q^{1/2+k-i}u_i) \prod_{j=1, j\neq i}^{2\nu} b(u_j u_i^{-1}) b(q u_i u_j) = 0, \]

\(^7\) We observe that there is a typo in equation (3.78) of the published version of [BaP19]: the second and third terms in equation (3.78) have wrong sign. The arXiv version was corrected.
where

\[ \nu_+ = q^{-1-\gamma_1}c^*, \quad \nu_- = q^{1+\gamma_1}\],

(3.27)

for \( i = 1, \ldots, 2s \). The set of equations \( E_\pm(u, \bar{u}) = 0 \) for \( i = 1, \ldots, 2s \) are the Bethe ansatz equations of inhomogeneous type for the set of Bethe roots \( \bar{u} \) with \( M = 2s \).

Recall the structure of the spectra (2.10) for a Leonard pair. Similarly to the previous subsection, and supported by numerical analysis (see appendix B), the following hypothesis is formulated.

**Hypothesis 2.** For each integer \( M \) (or \( N \)) with \( 0 \leq M, N \leq 2s \), there exists at least one set of non trivial admissible Bethe roots \( S^M_+ = \{ u_1, \ldots, u_{2s} \} \) (resp. \( S^N_- = \{ w_1, \ldots, w_{2s} \} \)) such that

\[ E_+(u_j, \bar{u}) = 0 \text{ for } \bar{u} = S^M_+ \]  

(3.28)

and associated with the following equality

\[ \theta_M = q^{-2i} \left( c^* (\zeta^2 + \zeta^{-2}) |2s\rangle_q + q^{2s} (b^2 q^{2s} + c q^{-2s} - \sum_{j=1}^{2s} (q^{2s} + q^{-1} u_j^2)) \right) \text{ for } \bar{u} = S^M_+ \]

(3.29)

\[ \text{resp. } \theta_N = q^{2i} \left( b (\zeta^2 + \zeta^{-2}) |2s\rangle_q + q^{-2s} (b^2 q^{2s} + c q^{-2s} - \sum_{j=1}^{2s} (q^{2s} + q^{-1} w_j^2)) \right) \text{ for } \bar{w} = S^N_- \]

(3.30)

The equality (3.29) is proven in the next lemma. The equality (3.30) is proven along the same line. For numerical examples of (3.29), see [BaP19, equation (4.6) and table 1].

**Lemma 3.7.** Assume hypothesis 2. The following relations hold:

\[ \theta_M = N^{(i)}_M(\bar{u}^\prime) \Psi^2_+(\bar{u}^\prime, m_0) \text{ for } \bar{u}^\prime = S^M_+ \]

(3.31)

\[ \theta_N = N^{(i)}_N(\bar{w}^\prime) \Psi^2_-(\bar{w}^\prime, m_0) \text{ for } \bar{w}^\prime = S^N_- \]

(3.32)

with

\[ N^{(i)}_M(\bar{u}^\prime) = N^{(i)}_M(\bar{u}^\prime)(P^{-1})_{2s,M} , \quad N^{(i)}_N(\bar{w}^\prime) = N^{(i)}_N(\bar{w}^\prime)P_{2s,N} . \]

(3.33)

**Proof.** Consider (3.31). Specializing [BaP19, proposition 3.3] for \( (\kappa, \kappa^*) = (1, 0) \), it follows:

\[ \pi(A) |\Psi^2_+(\bar{u}^\prime, m_0)\rangle = \theta(\bar{u}^\prime) |\Psi^2_+(\bar{u}^\prime, m_0)\rangle \text{ for } E_+(\bar{u}^\prime, \bar{u}^\prime) = 0 , \]

(3.34)

where \( \theta(\bar{u}^\prime) \) denotes the r.h.s. of (3.29) with \( u_j \to u_j^\prime \). Now, by definition of a Leonard pair and our choice of parameterization of the structure constants (2.12)-(2.14), the eigenvalues of \( \pi(A) \) are of the form (2.10). So, if there exists a set of solutions \( \{ u_1^\prime, \ldots, u_{2s}^\prime \} \) of \( E_+(u_j, \bar{u}) = 0 \) for \( j = 1, \ldots, 2s \), it always exists an integer \( M \) such that \( \theta_M = \theta(\bar{u}^\prime) \). Let us denote the corresponding set by \( \bar{u}^\prime = S^M_+ \). The absence of degeneracies in the spectrum of \( \pi(A) \) implies that \( |\theta_M\rangle \) is proportional to \( |\Psi^2_+(\bar{u}^\prime, m_0)\rangle \). To determine the normalization coefficient \( N^{(i)}_M(\bar{u}^\prime) \) in (3.31), one compares

\[ N^{(i)}_M(\bar{u}^\prime) |\Psi^2_+(\bar{u}^\prime, m_0)\rangle = N^{(i)}_M(\bar{u}^\prime) (N^{(i)}_M(\bar{u}^\prime))^{-1} |\theta_M\rangle + \cdots \]

(3.35)
with the second equation in (2.19):

$$|\theta_M\rangle = (P^{-1})_{2sM} \theta_{2s}^* + \cdots$$  (3.36)

The proof of (3.32) is done along the same line, starting from the specialization of [BaP19, proposition 3.3] for $$(k, \kappa^*) = (0, 1)$$.  

We now turn to the construction of the dual eigenstates for $${\hat{\pi}}(A), \pi(A^*)$$. To this end, a second set of Bethe equations of inhomogeneous type is introduced. Let us define:

$$\tilde{\mathbb{E}}_{\pm}(y_i, \bar{y}_i) = \frac{b(y_i^2)}{b(q^2)} \prod_{j=1, j \neq i}^{2s} f(y_j) \Lambda_{\pm 1}^{\pm 1}(y_j) - (q^2 y_j^3)^{\pm 1} \prod_{j=1, j \neq i}^{2s} h(y_j) \Lambda_{\pm 1}^{\pm 1}(y_j)$$  (3.37)

and associated with the following equality

$$\tilde{\mathbb{E}}_{\pm}(y_i, \bar{y}_i) = 0 \quad \text{for} \quad \bar{y} = dS^M(\bar{y}) \quad \text{(resp.} \quad \bar{y} = dS^N(\bar{y}))$$  (3.39)



\[ \begin{align*}
\theta_M &= q^{4s} \left( b^*(\zeta^2 + \zeta^{-2})[2s]_q + q^{-2s}(b q^{2s} + c q^{-2s}) - q^{-1} b^* \sum_{j=1}^{2s} (q y_j^2 + q^{-1} y_j^{-2}) \right) \\
\quad \text{for} \quad \bar{y} = dS^M(\bar{y})
\end{align*} \]

\[ \begin{align*}
\theta_N &= q^{-4s} \left( c(\zeta^2 + \zeta^{-2})[2s]_q + q^{2s}(b q^{2s} + c q^{-2s}) - q c \sum_{j=1}^{2s} (q y_j^2 + q^{-1} y_j^{-2}) \right) \\
\quad \text{for} \quad \bar{y} = dS^N(\bar{y})
\end{align*} \]

**Lemma 3.8.** Assume hypothesis 3. The following relations hold:

$$\langle \theta_M \rangle = \tilde{N}_M^{(i)}(\bar{y}')(\bar{y'}^2(\bar{y'}, m_0)) \quad \text{for} \quad \bar{y}' = dS^M(\bar{y})$$  (3.43)

$$\langle \theta_N \rangle = \tilde{N}_N^{(i)}(\bar{y}')(\bar{y'}^2(\bar{y'}, m_0)) \quad \text{for} \quad \bar{y}' = dS^N(\bar{y})$$

with

$$\tilde{N}_M^{(i)}(\bar{y}') = \tilde{N}_M^{(i)}(\bar{y}') P_{M,2s} \frac{\xi_M}{\xi_{2s}} \quad \text{and} \quad \tilde{N}_N^{(i)}(\bar{y}') = \tilde{N}_N^{(i)}(\bar{y}') (P^{-1})_{N,2s} \frac{\xi_N}{\xi_{2s}}.$$  (3.44)
Proof. Consider (3.42) and assume hypothesis 3. Let us show:

\[
\langle \Psi^2_+ (v', m_0) | \pi (A) = \langle \Psi^2_+ (v', m_0) \rangle \theta (v') \quad \text{for} \quad v' = dS_+^{(i)},
\]

where \( \theta (v') \) denotes the r.h.s. of (3.40) with \( v_j \to v'_j \). The proof of (3.45) follows standard computations within the algebraic Bethe ansatz approach, using the material given in appendix A together with the following relation that holds for all \( v, v_i, i = 1, \ldots, 2s \):

\[
\langle \Psi^2_+ (v, m_0) | \# (\Psi^2_+ (v, m_0 + 4x)) = \delta \frac{b(v)}{v^2} \prod_{v_j \in v} b(q^{1/2 + k - s v_j} \zeta) h(v, v_j) \frac{h(v, v_j)}{b(v, v_j) b(q v v_j)} \langle \Psi^2_+ (v, m_0) | \rangle

- \delta \frac{b(v)}{v^2} \prod_{v_j \in v} b(q^{1/2 + k - s v_j} \zeta) h(v, v_j) \frac{h(v, v_j)}{b(v, v_j) b(q v v_j)} \langle \Psi^2_+ (\{v, v_i\}, m_0) | \rangle
\]

(3.46)

with \( \delta = b^* q^{2i} \). Note that the proof of this latter relation being analog to [BaP19, appendix C], we skip it. Firstly, one obtains:

\[
\langle \Psi^2_+ (v, m_0) | \pi (A) = \langle \Psi^2_+ (v, m_0) \rangle \lambda_2^2 (v, v) - \frac{v b(q)}{b(v^2)} \sum_{v_j \in v} \frac{E_+(v, v_j)}{b(v, v_j) b(q v v_j)} \langle \Psi^2_+ (\{v, v_j\}, m_0) | \rangle
\]

with

\[
\lambda_2^2 (v, v) = \frac{v^{-1}}{b(v^2) b(q v^2)} \prod_{v_j \in v} f(v, v_j) \Lambda_2^2 (v) + \frac{q^2 v^3}{b(v^2) b(q v^2)} \prod_{v_j \in v} h(v, v_j) \Lambda_2^2 (v)

- q \delta \prod_{k=0}^{2s} b(q^{1/2 + k - s v_j} \zeta) h(q^{1/2 + k - s v_j} \zeta) \frac{h(q^{1/2 + k - s v_j} \zeta)}{b(q^{1/2 + k - s v_j} \zeta)} \frac{h(v, v_j)}{b(v, v_j) b(q v v_j)} + \frac{(q v v_j(v) + q^{-1} - 1) \bar{v}(v)^{-1}}{b(v^2) b(q v^2)}
\]

and

\[
E_+(v, v_j) = \frac{b(v_j^2)}{v_j b(q v_j^2)} \prod_{v_j \in v} f(v_j, v_j) \Lambda_2^2 (v_j) - q^2 v_j^3 \prod_{v_j \in v} h(v_j, v_j) \Lambda_2^2 (v_j)

+ \delta \frac{q b(v_j^2)}{b(q)} \prod_{k=0}^{2s} b(q^{1/2 + k - s v_j} \zeta) h(q^{1/2 + k - s v_j} \zeta) \frac{h(q^{1/2 + k - s v_j} \zeta)}{b(q^{1/2 + k - s v_j} \zeta)} \frac{h(v_j, v_j)}{b(v_j, v_j) b(q v_j v_j)}
\]

for \( i = 1, \ldots, 2s \). Secondly, by hypothesis 3, equation (3.47) reduces to (3.45) for \( v = v' = dS_+^{(i)} \), where \( \theta (v') = \lambda_2^2 (v, v') \). By studying the singular part of \( \lambda_2^2 (v, v') \), one finds \( \theta (v') \) reduces to the r.h.s. of (3.40). This concludes the proof of (3.45). The absence of degeneracies in the spectrum of \( \pi (A) \) in (2.5) implies that \( \langle \theta_M \rangle \) is proportional to \( \langle \Psi^2_+ (v', m_0) \rangle \). The normalization coefficient \( \tilde{N}_M^{(i)} (v') \) in (3.42) is determined through the comparison between

\[
\tilde{N}_M^{(i)} (v') \langle \Psi^2_+ (v', m_0) \rangle = \tilde{N}_M^{(i)} (v') \left( \tilde{N}_2^2 (v')^{-1} \langle \theta_M \rangle + \cdots \right)
\]

and the second equation in (2.21):

\[
\langle \theta_M \rangle = p_{M, 2^i} \frac{\xi_M}{\xi_{2a}} \langle \theta_{2a} \rangle + \cdots
\]

(3.50)

The proof of (3.43) is done along the same line.
3.4. Relating homogeneous and inhomogeneous Bethe states

In the algebraic Bethe ansatz framework, in general relating solutions of eigenproblems of homogeneous and inhomogeneous types might appear as a complicated problem. In the case studied in the present letter, the correspondence between the eigenbases of Leonard pairs and Bethe eigenstates implies the following identities, straightforward consequences of previous results. From lemmas 3.5–3.8, it follows:

\begin{align}
\mathcal{N}_M(\bar{u})|\Psi^M(\bar{u},m_0)\rangle &= \mathcal{N}^M_0(\bar{u}')|\Psi^2_+(\bar{u}',m_0)\rangle \quad \text{for} \quad \bar{u} = S^{M(h)}_-, \bar{u}' = S^{M(i)}_+, \tag{3.51} \\
\mathcal{N}^N_0(\bar{w})|\Psi^N(\bar{w},m_0)\rangle &= \mathcal{N}^{N(i)}_N(\bar{w}')|\Psi^2_-(\bar{w}',m_0)\rangle \quad \text{for} \quad \bar{w} = S^{N(h)}_+, \bar{w}' = S^{N(i)}_-. \tag{3.52}
\end{align}

and

\begin{align}
\tilde{\mathcal{N}}_M(\bar{v})|\Psi^M(\bar{v},m_0)\rangle &= \tilde{\mathcal{N}}^M_0(\bar{v}')|\Psi^2_+(\bar{v}',m_0)\rangle \quad \text{for} \quad \bar{v} = S^{M(h)}_-, \bar{v}' = dS^{M(i)}_+, \tag{3.53} \\
\tilde{\mathcal{N}}^N_0(\bar{y})|\Psi^N(\bar{y},m_0)\rangle &= \tilde{\mathcal{N}}^{N(i)}_N(\bar{y}')|\Psi^2_-(\bar{y}',m_0)\rangle \quad \text{for} \quad \bar{y} = S^{N(h)}_+, \bar{y}' = dS^{N(i)}_. \tag{3.54}
\end{align}

Specializing above identities, in particular the reference states of definition 3.1 can be written as inhomogeneous Bethe states:

\begin{align}
|\Omega^-\rangle &= \mathcal{N}^N_0(\bar{u}')|\Psi^2_+(\bar{u}',m_0)\rangle \quad \text{for} \quad \bar{u}' = S^{0(i)}_+, \tag{3.55} \\
|\Omega^+\rangle &= \mathcal{N}^{N(i)}_N(\bar{w}')|\Psi^2_-(\bar{w}',m_0)\rangle \quad \text{for} \quad \bar{w}' = S^{N(i)}_. \tag{3.56}
\end{align}

3.5. The $q$-Racah polynomials

From the results of the previous sections, the $q$-Racah polynomials can be expressed in terms of ratios of certain scalar products of Bethe states, either of homogeneous or inhomogenous type. Three examples are now displayed.

**Proposition 3.1.** The $q$-Racah polynomials are given by:

\begin{equation}
R_M(\theta^*_N) = \mathcal{N}^*_N(\bar{v})^{-1} \frac{|\Psi^N(\bar{v},m_0)\rangle \langle \Psi^M(\bar{u},m_0)\rangle}{|\Omega^+\rangle \langle \Psi^M(\bar{u},m_0)\rangle} \frac{\langle \Omega^+|\Omega^+\rangle}{\langle \Psi^N(\bar{v},m_0)|\Psi^M(\bar{v},m_0)\rangle} \tag{3.57}
\end{equation}

for $\bar{u} = S^{M(h)}_-, \bar{v} = S^{N(h)}_+$. 

**Proof.** Recall hypothesis 1. Consider the expression of the $q$-Racah polynomials given by (2.28). Insert (3.19), (3.20), (3.24).

Variations of above expression can be derived using the correspondence between eigenvectors of elements of the Leonard pair and inhomogeneous Bethe states. For instance, recall hypothesis 1 and 2. Insert (3.23), (3.31), (3.32) in (2.27). The $q$-Racah polynomials are now given by:

\begin{equation}
R_M(\theta^*_N) = \mathcal{N}^*_N(\bar{u})^{-1} \frac{|\Psi^M(\bar{v},m_0)\rangle \langle \Psi^2_+(\bar{w},m_0)\rangle}{|\Omega^-\rangle \langle \Psi^2_+(\bar{w},m_0)\rangle} \frac{\langle \Omega^-|\Omega^-\rangle}{\langle \Psi^M(\bar{v},m_0)|\Psi^2_+(\bar{w},m_0)\rangle} \tag{3.58}
\end{equation}

for $\bar{v} = S^{M(h)}_-, \bar{w} = S^{N(i)}_+, \bar{u} = S^{M(i)}_+$. Another example is obtained as follows. Insert (3.31), (3.32), (3.43) in (2.28). One gets:
\[
R_M(\theta_N^*) = \tilde{N}_N^*(\tilde{y}')^{-1} \frac{\langle \Psi_+^{2s}(\tilde{y}', m_0) | \Psi_+^{2s}(\tilde{u}, m_0) \rangle}{\langle \Omega^+ | \Psi_+^{2s}(\tilde{u}, m_0) \rangle} \cdot \frac{\langle \Omega^+ | \Omega^+ \rangle}{\langle \Psi_+^{2s}(\tilde{y}', m_0) | \Psi_+^{2s}(\tilde{y}, m_0) \rangle} \tag{3.59}
\]
for \(\tilde{u} = S^N(i), \tilde{y} = S^{sN(N)}(i), \tilde{y}' = dS^{sN(N)}(i)\).

4. Concluding remarks

In this letter, a correspondence between eigenbases/dual eigenbases of Leonard pairs and Bethe states/dual Bethe states of homogeneous and inhomogeneous type has been obtained. It follows that certain ratios of scalar products of corresponding Bethe states produce \(q\)-Racah polynomials. Clearly, eigenbases for other examples of Leonard pairs and related orthogonal polynomials of the discrete Askey-scheme may be studied along the same line as limiting cases. For instance, a correspondence for the Racah type \(q = 1\) may be studied starting from [BerCSV20, BerCCV22a].

Some perspectives are now presented. From the point of view of applications in physics, as mentioned in Introduction, the following problems may be considered.

(i) The open XXZ spin chain with integrable boundary conditions has been extensively studied in the literature, see e.g. the Introduction of [Bel15] for an account. For left generic and right special boundary conditions, the framework of the \(q\)-Onsager approach shows that the diagonalization of the transfer matrix is reduced to the diagonalization of the so-called alternating generators \(\{W_{-k} | k \in \mathbb{N}\}\) [BaK07]. These mutually commuting generators are the natural generalizations of the single element \(A\) of the Leonard pair \(\mathbb{A}, \mathbb{A}^*\). Extending the analysis presented here based on the modified algebraic Bethe ansatz, a correspondence between homogeneous and inhomogenous Bethe eigenstates is expected, as well as a relation between ratios of scalar products of Bethe states and certain special functions generalizing the \(q\)-Racah polynomials.

(ii) Scalar products of Bethe states are one of the main object of interest in the analysis of correlation functions for integrable spin chains. It is well-known that scalar products of Bethe states may be written in terms of certain compact determinant formulas involving the Bethe roots. It is certainly worth trying to obtain analogous expressions for the various types of scalar products entering the \(q\)-Racah formulae, or more generally between on-shell and off-shell Bethe states. The best route to that is to use the modern method introduced in [BelS19]. Indeed, this method was successfully applied to compute scalar products similar to those that enter the \(q\)-Racah formulae [BelPS21, BelS21]. This will be considered in a forthcoming paper.

(iii) Furthermore, the results in this paper may also find applications in the study of the entanglement entropy and other correlations of inhomogeneous free fermionic models, since the associated reduced correlation matrix commutes with the Heun–Askey–Wilson operator, see [CNV19, BerCCVV22b].

From the point of view of the theory of special functions and related integrable systems, we see at least two directions of research. Firstly, although not discussed here, let us mention that the modified algebraic Bethe ansatz formalism applied to the diagonalization of \(\pi(\mathbb{A}^*)\) (or
equivalently $\pi(A)$) generates three different Baxter TQ-relations. The first TQ-relation is of homogeneous type, and coincides with the second-order $q$-difference equation for the Askey–Wilson polynomials. As a well-known fact, the zeroes of the Askey–Wilson polynomials are characterized by Bethe equations of homogeneous type. For details, see [BaP19, section 4.3]. The second and third TQ-relations are of inhomogeneous type, and also admit polynomial solutions. In those two cases, the zeroes of the new polynomials satisfy Bethe equations of inhomogeneous type. Due to the fact that each TQ-relation inherits from the theory of Leonard pairs of $q$-Racah type, it suggests that a classification of the corresponding inhomogeneous Bethe equations according to the Askey-scheme—as well as related new polynomials—should be investigated further.

Secondly, it is known that irreducible finite dimensional representations of the $q$-Onsager algebra [T99, Ba04] with generators $W_0, W_1$ are classified according to the theory of tridiagonal pairs of $q$-Racah type [IT09] that generalizes the theory of Leonard pairs. For a tridiagonal pair of $q$-Racah type associated with $W_0, W_1$, a characterization of the entries of the transition matrices relating the eigenbases of $W_0$ to the eigenbases of $W_1$ (and more generally the eigenbases of each family of so-called alternating generators [Ba04, BaS09, BaB17, T21a, T21b]) in terms of special functions is an open problem. Using the modified algebraic Bethe ansatz, we expect the special functions of interest can be computed in terms of ratios of scalar products of Bethe states associated with the open XXZ spin chain for special boundary conditions as mentioned in (i) above.

Some of these problems will be discussed elsewhere.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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**Appendix A. Askey–Wilson generators and the dynamical operators**

In this section, we recall the precise relationship between the generators of the Askey–Wilson algebra $A, A^*$ and the so-called dynamical operators $\{\mathcal{A}(u,m), \mathcal{B}(u,m), \mathcal{C}(u,m), \mathcal{D}(u,m)\}$ that arise within the inverse scattering framework. The exchange relations satisfied by the dynamical operators are first recalled. We refer to [BaP19] for details. Note that for the analysis in this paper, the left and right actions of the dynamical operators $\mathcal{A}(u,m), \mathcal{B}(u,m)$ on certain products of $\mathcal{B}(u,m)$ and $\mathcal{C}(u,m)$ are also needed.
A.1. The dynamical operators and exchange relations

Let \( m \) be a positive integer and \( \epsilon = \pm 1 \). The dynamical operators \( \mathcal{A}_\epsilon(u,m), \mathcal{B}_\epsilon(v,m), \mathcal{C}_\epsilon(u,m), \mathcal{D}_\epsilon(u,m) \) are subject to the exchange relations:

\[
\mathcal{B}_\epsilon(u,m+2)\mathcal{B}_\epsilon(v,m) = \mathcal{B}_\epsilon(v,m+2)\mathcal{B}_\epsilon(u,m),
\]
\[
\mathcal{A}_\epsilon(u,m+2)\mathcal{B}_\epsilon(v,m) = f(u,v)\mathcal{A}_\epsilon(v,m)\mathcal{A}_\epsilon(u,m) + g(u,v,m)\mathcal{B}_\epsilon(u,m)\mathcal{A}_\epsilon(v,m) + w(u,v,m)\mathcal{B}_\epsilon(u,m)\mathcal{D}_\epsilon(v,m),
\]
\[
\mathcal{D}_\epsilon(u,m+2)\mathcal{B}_\epsilon(v,m) = h(u,v)\mathcal{D}_\epsilon(v,m)\mathcal{B}_\epsilon(u,m),
\]
\[
+ k(u,v,m)\mathcal{B}_\epsilon(u,m)\mathcal{D}_\epsilon(v,m) + n(u,v,m)\mathcal{B}_\epsilon(u,m)\mathcal{A}_\epsilon(v,m),
\]
\[
\mathcal{C}_\epsilon(u,m+2)\mathcal{B}_\epsilon(v,m) = R(u,v,m-2)\mathcal{C}_\epsilon(u,m)
\]
\[
+ q(u,v,m)\mathcal{A}_\epsilon(v,m)\mathcal{D}_\epsilon(u,m) + r(u,v,m)\mathcal{A}_\epsilon(u,m)\mathcal{B}_\epsilon(v,m)
\]
\[
+ s(u,v,m)\mathcal{A}_\epsilon(u,m)\mathcal{A}_\epsilon(v,m) + x(u,v,m)\mathcal{C}_\epsilon(v,m)\mathcal{A}_\epsilon(u,m)
\]
\[
+ y(u,v,m)\mathcal{D}_\epsilon(u,m)\mathcal{A}_\epsilon(v,m) + z(u,v,m)\mathcal{D}_\epsilon(u,m)\mathcal{D}_\epsilon(v,m)
\]

and

\[
\mathcal{C}_\epsilon(u,m+2)\mathcal{C}_\epsilon(v,m+2) = f(u,v)\mathcal{C}_\epsilon(v,m)\mathcal{C}_\epsilon(u,m+2) + g(u,v,m)\mathcal{C}_\epsilon(v,m)\mathcal{A}_\epsilon(u,m+2)
\]
\[
+ w(u,v,m)\mathcal{B}_\epsilon(u,m)\mathcal{D}_\epsilon(v,m) + n(u,v,m)\mathcal{A}_\epsilon(v,m)\mathcal{C}_\epsilon(u,m+2)
\]
\[
+ q(u,v,m)\mathcal{A}_\epsilon(v,m)\mathcal{D}_\epsilon(u,m) + r(u,v,m)\mathcal{A}_\epsilon(u,m)\mathcal{B}_\epsilon(v,m)
\]
\[
+ s(u,v,m)\mathcal{A}_\epsilon(u,m)\mathcal{A}_\epsilon(v,m) + x(u,v,m)\mathcal{C}_\epsilon(v,m)\mathcal{A}_\epsilon(u,m)
\]
\[
+ y(u,v,m)\mathcal{D}_\epsilon(u,m)\mathcal{A}_\epsilon(v,m) + z(u,v,m)\mathcal{D}_\epsilon(u,m)\mathcal{D}_\epsilon(v,m)
\]

where the coefficients are given by

\[
f(u,v) = \frac{b(qv/u)b(uv)}{b(qv)b(uv)}, \quad h(u,v) = \frac{b(q^2uv)b(qu/v)}{b(qv)b(u/v)},
\]
\[
g(u,v,m) = \frac{\gamma(u/v,m+1)}{\gamma(1,m+1)} b(q^2b(\frac{v}{u})) b(\frac{q}{b}), \quad w(u,v,m) = -\frac{\gamma(u,v,m)}{\gamma(1,m+1)} b(qv), \quad n(u,v,m) = \frac{\gamma(1/uv,m+2)}{\gamma(1,m+1)} b(qv)b(q^2v)b(qv),
\]
\[
k(u,v,m) = \frac{\frac{\gamma(u/v,m+1)}{\gamma(1,m+1)} b(q)b(\frac{q^2u^2}{b})}{b(qv)b(b(\frac{q}{v}))},
\]
\[
q(u,v,m) = \frac{\frac{\gamma(u/v,m)}{\gamma(1,m+1)} b(q)b(\frac{uv}{b})}{\frac{\gamma(1,m+1)}{b(\frac{q}{u})}}, \quad r(u,v,m) = \frac{\frac{\gamma(1,m+1)}{b(\frac{q}{u})}}{\frac{\gamma(1,m+1)}{b(\frac{q}{v})}}, \quad s(u,v,m) = \frac{b(q)b(a_2^2)(uv)}{\gamma(1,m+1)}, \quad x(u,v,m) = \frac{b(q)b(u^2v)}{\gamma(1,m+1)}, \quad y(u,v,m) = \frac{b(q)b(\frac{uv}{v})}{\gamma(1,m+1)}, \quad z(u,v,m) = \frac{b(q)b(uv)}{\gamma(1,m+1)}.
\]

where

\[
\gamma^\epsilon(u,m) = \alpha^{\frac{\epsilon+1}{\beta}} \beta^{\frac{1}{\beta}} q^{-m} u - \alpha^{\frac{\epsilon+1}{\beta}} \beta^{\frac{1}{\beta}} q^{m} u^{-1}.
\]
A.2. Products of dynamical operators and exchange relations

Using the exchange relations above, the action of the entries on products of $R^e(u, m)$, $C^e(u, m)$ can be derived. Recall the strings of length $M$ of operators $R^e(u, m)$ defined in (3.11). Define

$$B^e(\{u, u_i\}, m, M) = R^e(u, m + 2(M - 1)) \cdots R^e(u, m + 2(M - i)) \cdots R^e(u, m).$$  \hspace{1cm} (A.10)

Using the relations (A.1)–(A.3), one shows that the action of the dynamical operators $\{a^e(u, m), D^e(u, m)\}$ on the string (3.11) is given by:

$$a^e(u, m + 2M)B^e(\bar{u}, m, M) = \prod_{i=1}^{M} f(u, u_i)B^e(\bar{u}, m, M)a^e(u, m)$$  \hspace{1cm} (A.11)

$$+ \sum_{i=1}^{M} g(u, u_i, m + 2(M - 1)) \prod_{j=1, j \neq i}^{M} f(u, u_j)B^e(\{u, \bar{u}_i\}, m, M)a^e(u, m)$$

$$+ \sum_{i=1}^{M} w(u, u_i, m + 2(M - 1)) \prod_{j=1, j \neq i}^{M} h(u, u_j)B^e(\{u, \bar{u}_i\}, m, M)a^e(u, m).$$

and

$$D^e(u, m + 2M)B^e(\bar{u}, m, M)$$

$$= \prod_{i=1}^{M} h(u, u_i)B^e(\bar{u}, m, M)D^e(u, m)$$

$$+ \sum_{i=1}^{M} k(u, u_i, m + 2(M - 1)) \prod_{j=1, j \neq i}^{M} h(u, u_j)B^e(\{u, \bar{u}_i\}, m, M)D^e(u, m)$$

$$+ \sum_{i=1}^{M} n(u, u_i, m + 2(M - 1)) \prod_{j=1, j \neq i}^{M} f(u, u_j)B^e(\{u, \bar{u}_i\}, m, M)a^e(u, m).$$  \hspace{1cm} (A.12)

In addition to (3.12), the following strings of dynamical operators $C^e(v, m)$ with length $N$ is defined:

$$C^e(\{v, \bar{v}_i\}, m, N) = C^e(v_1, m + 2) \cdots C^e(v, m + 2) \cdots C^e(v_N, m + 2N).$$  \hspace{1cm} (A.13)

Using the relations (A.5)–(A.7), we can similarly show that the left action of the ‘diagonal’ dynamical operators $\{a^e(v, m), D^e(v, m)\}$ on the string (3.12) is given by,

$$C^e(\bar{v}, m, N)a^e(v, m + 2N)$$

$$= f(\bar{v}, \bar{v})a^e(v, m)C^e(\bar{v}, m, N) + \sum_{i=1}^{N} g(v, v_i, m + 2(N - 1))f(v, \bar{v}_i)a^e(v, m)C^e(\{v, \bar{v}_i\}, m, N)$$

$$+ \sum_{i=1}^{N} w(v, v_i, m + 2(N - 1))h(v, \bar{v}_i)D^e(v, m)C^e(\{v, \bar{v}_i\}, m, N)$$  \hspace{1cm} (A.14)
and
\[ C'(v,m,N) \mathcal{D}'(v,m+2N) \]
\[ = h(v) \mathcal{D}'(v,m)C'(v,m,N) + \sum_{i=1}^{N} k(v,v_i,m+2(N-1))h(v,v_i)\mathcal{D}'(v_i,m)C'(v,v_i,m,N) \]
\[ + \sum_{i=1}^{N} n(v,v_i,m+2(N-1))f(v_i)v_i\mathcal{D}'(v_i,m)C'(v,v_i,m,N). \]  \quad (A.15)

### A.3. Expressions of $A,A^*$ in terms of the dynamical operators

The expression of the Askey–Wilson algebra generators in terms of the dynamical operators previously introduced is now recalled. As shown in [BaP19], according to the choice $\epsilon = \pm 1$ the element $A$ can be expressed in two different ways. For instance, in terms of the dynamical operators for $\epsilon = -1$ one has:
\[ A = \tilde{A}^-(u,m) + \frac{(qu\eta(u) + q^{-1}u^{-1}\tilde{\eta}(u^{-1}))}{(u^2 - u^{-2})(q^2u^2 - q^{-2}u^2)} \quad \text{with} \quad \tilde{\eta}(u) = (q + q^{-1})\rho^{-1}(\eta u + \eta^* u^{-1}), \]  \quad (A.16)

where, for further convenience, we have introduced:
\[ \tilde{A}^-(u,m) = \frac{u^{-1}}{(u^2 - u^{-2})} \left( \frac{1}{(qu^2 - q^{-1}u^{-2})}\mathcal{A}^-(u,m) + \frac{1}{(q^2u^2 - q^{-2}u^2)}\mathcal{D}^-(u,m) \right). \]  \quad (A.17)

Alternatively, for $\epsilon = +1$ a slightly more complicated expression in terms of the dynamical operators is obtained. Namely:
\[ A = \tilde{A}^+(u,m) + \frac{(qu\tilde{\eta}(u) + q^{-1}u^{-1}\tilde{\eta}(u^{-1}))}{(u^2 - u^{-2})(q^2u^2 - q^{-2}u^2)} \quad \text{with} \quad \tilde{\eta}(u) = (q + q^{-1})\rho^{-1}(\eta u + \eta^* u^{-1}), \]  \quad (A.18)

where (recall the definition (A.9))
\[ \tilde{A}^+(u,m) = \frac{u}{(u^2 - u^{-2})} \left( \frac{\gamma^+ (q^{-1}u^{-2},m)}{(qu^2 - q^{-1}u^{-2})\gamma^+(1,m+1)}\mathcal{A}^+(u,m) + \frac{\gamma^+ (qu^2,m)}{(q^2u^2 - q^{-2}u^2)\gamma^+(1,m+1)}\mathcal{D}^+(u,m) \right) \]
\[ + \frac{\beta q_{m}}{\gamma^+(1,m)}\mathcal{D}^+(u,m) - \frac{\beta q_{m}}{\gamma^+(1,m)}\mathcal{A}^+(u,m) \right). \]

In the main text, we also need the expression of $A^*$ for the cases $\epsilon = \pm 1$. For $\epsilon = +1$, it reads:
\[ A^* = \tilde{A}^+(u,m) + \frac{(qu\tilde{\eta}(u^{-1}) + q^{-1}u^{-1}\tilde{\eta}(u))}{(u^2 - u^{-2})(q^2u^2 - q^{-2}u^2)} \quad \text{with} \quad \tilde{\eta}(u) = (q + q^{-1})\rho^{-1}(\eta u + \eta^* u^{-1}), \]  \quad (A.19)

where we have introduced:
\[ \tilde{A}^+(u,m) = \frac{u}{(u^2 - u^{-2})} \left( \frac{1}{(qu^2 - q^{-1}u^{-2})}\mathcal{A}^+(u,m) + \frac{1}{(q^2u^2 - q^{-2}u^2)}\mathcal{D}^+(u,m) \right). \]  \quad (A.20)
and metrized form, that is, a particular set of numerical parameters. Here we solve the homogeneous Bethe equations. In this appendix we give examples of numerical solutions to the homogeneous Bethe equations.

\[ \tilde{\alpha}^{-1}(u,m) = \begin{cases} \frac{u^{-1}}{(u^2 - u^2)} \left( \frac{\gamma^{-1}(q^{-1}u^{-2}m)}{(qu^2 - q^{-1}u^{-2})\gamma^{-1}(1,m+1)} \tilde{\alpha}^{-1}(u,m) + \frac{\gamma^{-1}(qu^2,m)}{(q^2u^2 - q^{-2}u^{-2})\gamma^{-1}(1,m+1)} \tilde{\beta}^{-1}(u,m) \right) + \frac{\alpha_q^{-1}}{\gamma^{-1}(1,m)} \tilde{\beta}^{-1}(u,m) - \frac{\beta_q^{-1}}{\gamma^{-1}(1,m)} \tilde{\alpha}^{-1}(u,m) \right) \end{cases} \]

The expressions of \( A, A^* \) in terms of the dynamical operators together with the exchange relations above are used to compute the action of \( A, A^* \) on the so-called Bethe states.

**Table 1.** Examples of symmetrized Bethe roots for homogeneous and inhomogeneous Bethe equations.

| \( s \)  | \( N \) | \( \theta_0^s \) | Homogeneous \( \{W_1, \ldots, W_n\} \) | Inhomogeneous \( \{\tilde{W}_1, \ldots, \tilde{W}_n\} \) | Inhomogeneous \( \{V_1, \ldots, V_n\} \) |
|------|------|-------|-----------------|-----------------|-----------------|
| 1 | 0 | 2.5 | \{1.53952 + 0.500417i, 0.45684 + 0.505017i\} | \{2.2728 - 0.966683i, 0.0745008 - 1.00293i\} |
| 1 | 1.91511 - 0.96418i | \{1.46846 - 1.4873i\} | \{2.07086 + 0.294415i, 0.525409 + 0.698016i\} | \{1.60724 - 0.632627i, 0.388364 - 0.810955i\} |
| 1 | 2 | 0.43412 - 1.47721i | \{0.794456 - 0.710359i, 1.16871 + 0.954638i\} | \{2.05797 - 0.508163i, 0.722852 - 0.484161i\} |
| 3/2 | 0 | 2.5 | \{0.880037 + 0.753915i\} | \{0.0633231 + 0.39849i, 1.88459 + 0.122195i\} | \{0.10921 + 1.557i\} |
| 1 | 1.91511 - 0.96418i | \{1.25794 - 1.74953i\} | \{1.1076 + 0.856377i\} | \{1.0032 - 1.03993i\} |
| 2 | 0.43412 - 1.47721i | \{0.613402 - 0.900527i, 1.87298 - 0.201354i\} | \{1.60039 + 0.436105i\} | \{0.951734 - 0.702797i\} |
| 3 | -1.25 - 1.29904i | \{0.388149 - 0.390414i, 1.109403 - 0.362482i\} | \{1.55272 - 0.317027i\} | \{0.900883 - 0.307861i\} |

For \( \epsilon = -1 \), one has:

\[ A^* = \tilde{\alpha}^{-1}(u,m) + \left( \frac{q(u)\tilde{\alpha}^{-1}(u^{-1}) + q^{-1}\tilde{\alpha}^{-1}(u)}{(u^2 - u^2)(q^2u^2 - q^{-2}u^{-2})} \right) \]  

\[(A.21)\]

The expressions of \( A, A^* \) in terms of the dynamical operators together with the exchange relations above are used to compute the action of \( A, A^* \) on the so-called Bethe states.

**Appendix B. Numerical solution of Bethe equations**

In this appendix we give examples of numerical solutions to the homogeneous (3.17) and inhomogeneous (3.26), (3.37) Bethe equations. Therefore, we verify hypothesis 1–3 for a particular set of numerical parameters. Here we solve \( E^0_s \) \( \langle w_j, \tilde{w}_j \rangle = 0 \) in (3.17), \( E_- (w_j, \tilde{w}_j) = 0 \) in (3.26) and \( E_- (v_j, \tilde{v}_j) = 0 \) in (3.37). We consider \( s = 1, 3/2 \) and the numerical parameters \( b = c = \zeta = 1, b^* = 1/2, c^* = 2 \) and \( q = e^{-\pi} \). The roots are displayed in Table 1 in the symmetrized form, that is, \( W_i = (q\tilde{w}_i^2 + q^{-1}\tilde{w}_i)^{-2}/(q + q^{-1}) \), \( W_i' = (q\tilde{w}_i^{2'} + q^{-1}\tilde{w}_i^{2'})^{-2}/(q + q^{-1}) \) and \( V_i = (qv_i^2 + q^{-1}v_i^{-2})/(q + q^{-1}) \).
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