COMBINATORICS OF AFFINE BIRATIONAL MAPS

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Abstract. The main object of study of the present paper is the group $\text{UAut}_n$ of unimodular automorphisms of $\mathbb{C}^n$. Taking $\text{UAut}_n$ as a working example, our intention was to develop an approach (or rather an edifice) which allows one to prove, for instance, the non-simplicity of $\text{UAut}_n$ for all $n \geq 3$. More systematic and, perhaps, general exposition will appear elsewhere.

1. Introduction

The impetus for the present paper was the article [1] in which the study of combinatorics of certain birational automorphisms of $\mathbb{k}^n$ was applied to answer a group-theoretic question. More specifically, given $f$ from the Cremona group $\text{Cr}_n$ of birational automorphisms of $\mathbb{k}^n$, the combinatorics of $f$ we have in mind is encoded (somehow) in a set of lattice points or rather a polytope, which comes for free with each $f \in \text{Cr}_n$. These discrete gadgets are constructed, as used to be common now, by fixing a (non-archimedean) valuation on the field of rational functions on $\mathbb{k}^n$ and applying this valuation to the components of various maps $f \in \text{Cr}_n$. Or, heuristically, one “brings the action of $f$ on $\mathbb{k}^n$ to infinity” (see [6], [10], [9], [8], [15], [13] for related matters).

We would like to apply the preceding point of view to study polynomial automorphisms of $\mathbb{k}^n$. Recall that the group $\text{Aut}_n$ of such automorphisms carries a structure of an infinite-dimensional algebraic group (see [20]). Then, as an algebraic group, $\text{Aut}_n$ is generated by the group of affine linear automorphisms of $\mathbb{k}^n$ and by the group of triangular automorphisms of $\mathbb{k}^n$ (see [20, Theorem 4]). The group $\text{Aut}_n$ is also non-simple because of the Jacobi map $\text{det} : \text{Aut}_n \rightarrow \mathbb{k}^\ast$. On the other hand, the kernel $\text{UAut}_n := \text{Ker}(\text{det})$ is simple as an algebraic group (see [20, Theorem 5]), but is not that as an abstract group for $n = 2$ (see [5]). The aim of the present paper is to extend the latter result to the case of arbitrary $n \geq 3$:

Theorem 1.1. The group $\text{UAut}_n$ is non-simple (as an abstract group) for all $n \geq 3$.

To prove Theorem 1.1 we introduce a subgroup $G \subset \text{Cr}_n$ which “looks like” a subgroup in $\text{SL}_n(\mathbb{Z})$ when brought to infinity, according to what we have said at the beginning (see Section 2 for the construction of $G$). Though the presence of $G$ might be interesting and important on its own (see for example Proposition 6.2), we focus on one of its subgroups, namely $G_n \subset G$, instead (see Corollary 2.9). One of the crucial features of $\mathfrak{G}_n$ is provided by Proposition 3.1. Up to this end all considerations employ only elementary algebra/combinatorics of polynomials on $\mathbb{k}^n$.

Proposition 3.1 is enough to prove Theorem 1.1 provided that $\mathfrak{G}_n$ contains “sufficiently many” normal subgroups. The latter turns out to be the case after we introduce a subset $G_e$ of generators of $\mathfrak{G}_n$ in Section 4. More precisely,
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Notations. Throughout the paper we use the following notations and conventions:

- $\mathbb{P}^n$ is the projective space with coordinates $[X_0 : \ldots : X_n]$. We denote by $S := k[X_0, \ldots, X_n]_{\text{hom}}$ the semigroup of homogeneous polynomials in the ring $k[X_0, \ldots, X_n]$.

- We fix the lattice $\mathbb{Z}^{n+1}$ with the basis dual to $\{X_0, \ldots, X_n\}$. We also fix the sublattice $\mathbb{Z}^n \subset \mathbb{Z}^{n+1}$ corresponding to $\{X_1, \ldots, X_n\}$. Both $\mathbb{Z}^{n+1}$ and $\mathbb{Z}^n$ are equipped with the standard lexicographical order for which $X_0 \geq X_1 \geq \ldots \geq X_n$.

- We set $X := (X_1, \ldots, X_n)$, $X^I := X_1^{i_1} \ldots X_n^{i_n}$ for $I \in \mathbb{Z}^n$, $I := (i_1, \ldots, i_n)$. $M_n(R)$ denotes the set of all $(n \times n)$-matrices $M$ with entries $M_{i,j} \in R$, $1 \leq i, j \leq n$, in a ring $R$.

- Given $h \in k[X_0, \ldots, X_n]$, we set $d_h$ to be the degree of $h$ in $X_0$. We denote by $\text{Supp } h$ the support of $h$ (i.e. $\text{Supp } h$ is the collection of all monomials that appear in $h$ with non-zero coefficients). We will identify $\text{Supp } h \subset \mathbb{Z}^{n+1}$ with its dual and denote by $I_h$ the maximal vector among those $I \in \mathbb{Z}^n$ with $(d_h, I) \in \text{Supp } h$. We also put $\langle h \rangle := (d_h, I_h)$ (thus $\langle h \rangle$ is the monomial $X_0^{d_h}X_1^{I_h}$).

- Every $f \in \text{Cr}_n$ (and, more generally, every rational self-map of $\mathbb{P}^n$) is represented by an $(n+1)$-tuple $[f_0 : \ldots : f_n]$ of (not necessarily coprime) polynomials $f_0, \ldots, f_n \in k[X_0, \ldots, X_n]_{\text{hom}}$. In particular, if all $f_i$ are coprime, then $f$ is uniquely determined by $[f_0 : \ldots : f_n]$.

- $f \circ g$ (or $fg$) denotes the composition $f(g)$ of two rational self-maps of $\mathbb{P}^n$.

- For a group $G$ and any $a_1, a_2, b \in G$, we put $a_1^b := ba_1b^{-1}$, $C_{a_1} := \{a_1^b\}_{b \in G}$ (the conjugacy class of $a_1$), and write $a_1 \sim a_2$ if $a_1 \in C_{a_2}$. $N \triangleleft G$ signifies that $N$ is a normal subgroup in $G$ such that $N \neq G, \{1\}$ (1 $\in G$ is the unit element).

- We denote by $\mathbb{F}_2$ the free group in two generators ($\mathbb{F}_2$ always comes with the word metric w.r.t. to a fixed set of generators). We will also use standard notions and facts from the geometric group theory (see e.g. [2]). For instance, given two metric spaces $X$ and $Y$, $X \sim_{q,i.} Y$ (or $X$ is $q$-i. to $Y$) signifies that $X$ is quasi-isometric to $Y$.

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2) The results of Section 4 were motivated by (and are a group-theoretic counterpart of) the Splitting theorem for compact Lorentz manifolds with an isometric $\text{SL}_2(\mathbb{R})$-action (cf. [4] §4 and Question 6.4 below).
2. Preliminaries

2.1. The set-up. We assume for simplicity that $k = \mathbb{C}$. Consider $f \in Cr_n$ given by some $f_0, \ldots, f_n \in S$ (we assume $n \geq 3$ in what follows). Suppose that

\begin{equation}
(2.2) \quad f_0 = a_0 X_0^{d_f} X_{I_f} + \sum_{k \geq 1} X_0^{d_f - k} F_k(X),
\end{equation}

\begin{equation}
 f_j = a_{j,0} X_0^{d_f - 1} X_{I_f} + \sum_{k \geq 1} X_0^{d_f - 1 - k} F_{j,k}(X)
\end{equation}

for all $j \geq 1$ and some $d_f \in \mathbb{N}$, where $a_0, a_j \in \mathbb{C}^*$, $F_k(X), F_{j,k}(X) \in \mathbb{C}[X_1, \ldots, X_n]_{\text{hom}} = S$. Note that the condition $d_f - d_j = 1$ (for all $j \geq 1$) is satisfied by every $(n+1)$-tuple $(f_0^*, \ldots, f_n^*)$ such that $hf_i^* = f_i$ for all $i$ and any (fixed) $h \in S$ (with $f_i$ replaced by $f_i^*$ in $d_i$ for all $i$). In particular, we may take $f_i$ to be coprime, $0 \leq i \leq n$, so that (2.2) is a property of the map $f$.

Let us also assume that $f^{-1}$ satisfies (2.2) and denote by $G$ the set of all such $f$. Then, clearly, $G \neq \{1\}$:

Example 2.3. $G$ contains the following groups:

- the group $D_n := (\mathbb{C}^*)^n$ of diagonal automorphisms of $\mathbb{P}^n$;
- the subgroup in $U \text{Aut}_n$ of those $f$ which preserve the origin in $\mathbb{C}^n$ and have identity Jacobi matrix;
- for each $M \in SL_n(\mathbb{Z})$ with the $j$-th column $I_j$ contained in the hyperplane $\sum_{i=1}^n X_i = 1$ for all $j$, the birational transformation $[1 : X_1 : \ldots : X_n] \mapsto [1 : X_{I_1} : \ldots : X_{I_n}]$ (we identify $X_j$ with $X_j/X_0$) also satisfies (2.2). Note that all such $M$ form a group isomorphic to the subgroup $SL'_n(\mathbb{Z})$ of those elements in $SL_n(\mathbb{Z})$ that fix the vector $(1, \ldots, 1)$ (see 5.1 below for an explicit example of two $a_1, a_2 \in SL'_n(\mathbb{Z}) \subset G$).

Less trivial examples are provided by the groups $E_n \subset G$ and $S_n \subset \mathfrak{S}_n$ below.

Example 2.3 justifies the existence of the group $G$.

2.4. Group structure on $G$. Put $h(\langle f \rangle) := h(\langle f_0 \rangle, \ldots, \langle f_n \rangle) = h(1, \langle f_1 \rangle, \ldots, \langle f_n \rangle) \langle f_0 \rangle^\deg(h)$ for every $h \in S$ and $f \in G$ as above. Let also $M_f$ be the $(n \times n)$-matrix whose $j$-th column equals $I_{j-1} - I_{j^0}$, $1 \leq j \leq n$.

Suppose that $(d_h, I) \in \text{Supp } h$, $I \in \mathbb{Z}^n$, yields $I = I_h$. Then we get the following:

Lemma 2.5. The equality

$$\langle h(\langle f \rangle) \rangle = (\deg(h)(d_f - 1) + d_h, M_f I_h + \deg(h) I_{f_0})$$

holds.

Proof. Indeed, since $f \in G$, we get

$$\sigma(\langle f \rangle) = (\deg(h)(d_f - 1) + d_\sigma, M_f I + \deg(h) I_{f_0}) < (\deg(h)(d_f - 1) + d_h, M_f I_h + \deg(h) I_{f_0}) = \langle h(\langle f \rangle) \rangle$$

for all $\sigma := X_0^{d_\sigma} X^I \in \text{Supp } h \setminus \{\langle h \rangle\}$. \qed
Put \( h(f) := h(f_0, \ldots, f_n) = h \left( \frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0} \right)^{\deg(h)} \). Note that \( \langle h(f) \rangle \in \text{Supp} \ h(f) \). Then from Lemma 2.5 we get
\[
\langle h(f) \rangle = (\deg(h)(d_f - 1) + d_h, M_f I_h + \deg(h) I_{f_0}).
\]

This leads to the anticipated

**Proposition 2.7.** \( G \) is a subgroup in \( C r_n \).

**Proof.** Take \( f \in G \) as above. Consider also \( g \in G \) given by some \( g_0, \ldots, g_n \) in \( S \). Then from (2.6) (for \( h = g_0, \ldots, g_n \)) we obtain that \( g \circ f \) is of the form (2.2). Recall also that \( f^{-1}, g^{-1} \in G \) by definition. Thus we get \( (g \circ f)^{-1} = f^{-1} \circ g^{-1} \in G \), which proves the assertion.

2.8. Homomorphism \( \rho \). Consider the map \( \rho : G \longrightarrow \mathbb{Z}_0^\geq \) defined as follows:
\[
v : h \mapsto \langle h \rangle = (d_h, I_h) \mapsto I_h
\]
for all \( h \in S \). Then \( v \) is a \( \mathbb{Z}_0^\geq \)-valuation on \( S \). Furthermore, \( v \) (obviously) extends to a \( \mathbb{Z}^n \)-valuation on \( \mathbb{C}(X_1/X_0, \ldots, X_n/X_0) \), providing a particular case of valuations considered in [16], [17]. This determines a map
\[
\rho : G \longrightarrow M_n(\mathbb{Z}), \ f \mapsto \rho(f) := M_f
\]
for all \( f \in G \), with \( v(f_i/f_0) = I_{f_i} - I_{f_0} \), the \( i \)-th column of \( M_f \), \( 1 \leq i \leq n \).

From Proposition 2.7 we get the following:

**Corollary 2.9.** \( \rho(G) \subset GL_n(\mathbb{Z}) \) and \( \rho \) is a group homomorphism. In particular, for \( \mathcal{E}_n := \text{Ker}(\rho) \) the group \( \text{Out}(\mathcal{E}_n) \) of outer automorphisms of \( \mathcal{E}_n \) contains \( \mathbb{F}_2 \).

**Proof.** Let us use the notations from the proof of Proposition 2.7. Recall that \( d_{g_0(f)} - d_{g_1(f)} = 1 \) for all \( j \geq 1 \) and \( I_{g_i(f)} = M_f I_{g_i} + \deg(g_i) I_{f_0} \) for all \( i \geq 0 \) (see (2.6)). This implies that \( \rho(g \circ f) = M_{g \circ f} = M_f M_g = \rho(g) \rho(f) \). Note also that \( \rho \) splits over \( SL'_n(\mathbb{Z}) \subset G \) and \( \rho(G) = SL'_n(\mathbb{Z}) \) by construction (see (2.2) and Example 2.3). Thus we get \( G = \mathcal{E}_n \times SL'_n(\mathbb{Z}) \) and a homomorphism \( \mathbb{F}_2 \longrightarrow \text{Out}(\mathcal{E}_n) \) (cf. Lemma 2.5 below). Let us show that the latter is injective.

Consider an arbitrary \( f \in D_n \subset G \) (see Example 2.3). We may assume that \( f \) coincides with the map
\[
[X_0 : X_1 : \ldots : X_n] \mapsto [X_0 : \lambda_1 X_1 : \ldots : \lambda_n X_n]
\]
for some fixed \( \lambda_i \in \mathbb{C}^* \). Now take any \( a \in \mathbb{F}_2 \subseteq SL'_n(\mathbb{Z}) \). Then \( a(f) := af a^{-1} \) in \( G \) also belongs to \( D_n \) and is obtained from \( f \) by replacing every \( \lambda_i, 1 \leq i \leq n \), by the products \( \prod_{j=1}^{n} \lambda_{j,i}^{k_{j,i}} \) for some \( k_{j,i} \in \mathbb{Z} \) such that \( \sum_j k_{j,i} = 1 \).

In particular, one may always choose \( f \) (for \( a \neq 1 \)) to be such that \( \prod_{j=1}^{n} \lambda_{j,i}^{k_{j,i}} \neq \lambda_i \) for at least one \( i \), so that \( a(f) \neq f \) in this case.

On the other hand, if \( a(f) = f^g \) for some \( g \in \mathcal{E}_n \), then it follows from Lemma 2.5 that \( f^g = f \) (cf. 3.2 below for the “typical” shape of \( g \)). This together with \( a(f) \neq f \) shows that \( \mathbb{F}_2 \) injects into \( \text{Out}(\mathcal{E}_n) \).
3. The group $\mathcal{E}_n$

We retain the notations of Section 2

**Proposition 3.1.** There exists a subgroup $\mathcal{E}_n \subset \mathfrak{S}_n$ and a surjective homomorphism $\xi : \mathcal{E}_n \twoheadrightarrow \mathbf{UAut}_{n-1}$.

**Proof.** Consider a rational map $\Lambda : \mathbb{P}^n \to \mathbb{P}^n$ defined as follows:

\begin{align*}
X_0 &\mapsto \alpha_0 X_0^d + \alpha X_0 X_1^{d-1} =: \Lambda_0, \\
X_1 &\mapsto \alpha_1 X_0^{d-1} X_1 + \beta X_1^d =: \Lambda_1, \\
X_j &\mapsto \alpha_j X_0^{d-1} X_j + \Lambda_j(X) =: \Lambda_j,
\end{align*}

for all $j \geq 2$ and some $d \in \mathbb{N}$, $\alpha_1, \alpha, \beta \in \mathbb{C}$, $\Lambda_j^*(X) \in \mathbb{C}[X_1, \ldots, X_n]_{\text{hom}}$. Let us additionally suppose that the map $\Lambda^* : [X_1 : \ldots : X_n] \mapsto [X_1^d : \Lambda_1^*(X) : \ldots : \Lambda_n^*(X)]$ is a birational automorphism of $\mathbb{P}^{n-1}$ that coincides with a polynomial automorphism on $\mathbb{C}^{n-1} = \mathbb{P}^{n-1} \cap (X_1 \neq 0)$. Denote by $\mathcal{E}_n$ the set of all such $\Lambda$ contained in $\text{Cr}_n$.

**Lemma 3.3.** We have $\mathcal{E}_n \neq \{1\}$. More precisely, to each element in $\mathbf{UAut}_{n-1}$ there corresponds an element in $\mathcal{E}_n$, similarly as $\Lambda \in \mathcal{E}_n$ above corresponds to $\Lambda^* \in \mathbf{UAut}_{n-1}$.

**Proof.** We may take $\alpha_0, \alpha_1, \alpha, \beta$ in $\mathfrak{S}_2$ to be such that the corresponding polynomials $\Lambda_0, \Lambda_1$ have exactly $d - 1$ common roots. Then we identify $\Lambda$ with the map

\begin{align*}
(X_0, X_2, \ldots, X_n) &\mapsto \left(\Lambda_0/\Lambda_1 = X_0, \ldots, \Lambda_n/\Lambda_1\right)
\end{align*}

on the affine subset $\mathbb{C}^n = \mathbb{P}^n \cap (X_1 \neq 0)$. Now, if $\Lambda_j (= \Lambda_j/\Lambda_1$ on $\mathbb{C}^n$) are linear for all $j \geq 2$, the assertion is obvious. Otherwise, we may assume the linear part of $\Lambda_j$ coincides with $X_j$, all $j \geq 2$. Then it is easy to see that $\Lambda^{-1}(O)$ equals the (non-multiple) point $O := [0 : 1 : 0 : \ldots : 0]$. Indeed, any point from $\Lambda^{-1}(O)$ has $X_0 = 0$ (see (3.2)), i.e. is contained in $\mathbb{C}^{n-1}$ (see definition of $\Lambda^*$ above). But then $\Lambda^{-1}(O) = \Lambda^*-1(O) = \text{non-multiple } O$. This shows that the degree of the map $\Lambda$ equals 1.

**Lemma 3.5.** For every $\Lambda \in \mathcal{E}_n$, we have $\Lambda^{-1} \in \mathcal{E}_n$.

**Proof.** Consider the hyperplane $\Pi := \langle X_0 = 0 \rangle \subset \mathbb{P}^n$. Then, by construction, $\Lambda|_{\Pi \cap (X_1 \neq 0)}$ is a polynomial automorphism of $\Pi \cap (X_1 \neq 0) = \mathbb{C}^{n-1}$. We also have $\Lambda(O) = O$ for the point $O := [1 : 0 : \ldots : 0]$. Then $\Lambda^{-1}$ also preserves $O$ and induces a polynomial automorphism on $\Pi \cap (X_1 \neq 0)$. In particular, this implies that both $\Lambda$ and $\Lambda^{-1}$ are biholomorphic maps near $O \in \mathbb{C}^n = \mathbb{P}^n \cap (X_0 \neq 0)$, with (diagonal) Jacobi matrices inverse to one another. We leave it to the reader to write down the defining functions for $\Lambda^{-1}$ and make sure of they look like (3.2). Hence $\Lambda^{-1} \in \mathcal{E}_n$.

Similar argument as in the proof of Lemma 3.3 shows that $\Lambda_1 \circ \Lambda_2 \in \mathcal{E}_n$ for any $\Lambda_1, \Lambda_2 \in \mathcal{E}_n$. Thus $\mathcal{E}_n$ is a (non-trivial) subgroup in $\text{Cr}_n$. Moreover, we have $\mathcal{E}_n \subset \mathfrak{S}_n$ by construction, and restricting to $\Pi$ we get a homomorphism $\xi : \mathcal{E}_n \to \mathbf{UAut}_{n-1}$. The latter is also onto (see Lemma 3.3 and Proposition 3.1) is proved.

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\footnote{Note that $\Lambda^{-1}(\Pi \cap (X_1 \neq 0)) = \Pi \cap (X_1 \neq 0)$, where $\Lambda^{-1}$ is for the proper birational transform, and $\Lambda$ is smooth and bijective (hence biregular) near $\Pi \cap (X_1 \neq 0)$ (cf. 3.2 with $\Lambda_0, \Lambda_1$ (resp. $X_0, X_1$) interchanged).}
Remark 3.6. Let \( L \subseteq \mathbb{C}(X_1, \ldots, X_n) \) be the linear subspace spanned by the rational functions \( 1, \Lambda_0/\Lambda_1, \Lambda_2/\Lambda_1, \ldots, \Lambda_n/\Lambda_1 \). Then, as \( \dim L = n + 1 \), the (self) intersection index \([L, \ldots, L]\) (see e.g. [12]) is equal to the degree of \( \Lambda \), which is 1 (cf. the proof of Lemma 3.3). Let also \( v \) be the valuation as in [28]. It follows from [32] that \( \{v(\Lambda_1/\Lambda_0), \ldots, v(\Lambda_n/\Lambda_0)\} \) is the standard basis in \( \mathbb{Z}^n \). Denote by \( \Delta \) (resp. by \( \Delta(S(\Lambda)) \)) the corresponding simplex (resp. Newton convex body) in \( \mathbb{R}^n \) (note that \( \Delta \subseteq \Delta(S(\Lambda)) \)). Then from [12, Theorem 11.2] we obtain that \( 1 = \text{Vol}(\Delta(S(\Lambda))) \geq \text{Vol}(\Delta) = 1 \). So the Newton convex body of the rational map \( \Lambda \) is the standard simplex in \( \mathbb{R}^n \). It would be interesting to study the class of algebraic varieties \( X \) for which the latter property is satisfied for any birational map \( X \rightarrow X \).

4. Intermedia: One Group-Geometric Argument

4.1. Two sets of generators in \( \mathfrak{G}_n \). Let \( G_{ne} \) be the set of all \( f \in \mathfrak{G}_n \) such that \( a(f) \neq f \) for every \( a \in \mathbb{F}_2 \setminus \{1\} \). Similarly, let \( G_e \) be the set of all \( f \in \mathfrak{G}_n \) such that \( a(f) \sim f \) for all \( a \in \mathbb{F}_2 \). In general, for any \( g \in \mathfrak{G}_n \), let \( E_g \subseteq \mathbb{F}_2 \) be the group of those \( a \) for which \( a(g) \sim g \) (i.e. \( E_f = \mathbb{F}_2 \) for all \( f \in G_e \)).

Example 4.2. It is easy to see that both sets \( \mathfrak{D}_n \cap G_{ne} \) and \( \mathfrak{D}_n \cap G_e \) are infinite (cf. the proof of Corollary 2.9). Note also that \( G_{ne} \) and \( G_e \) are stable under the conjugation and inversion in \( \mathfrak{G}_n \).

4.3. The tree \( T \). Recall that \( \mathbb{F}_2 \) acts freely, transitively and isometrically on a (four-valent) tree \( T \) (see Figure 1 below). Furthermore, if \( \mathcal{X} \) is a Riemann surface of genus 2, the group \( \mathbb{F}_2 \) appears in the Schottky uniformization of \( \mathcal{X} \) (see e.g. [14]). Namely, since \( \mathbb{F}_2 \subset \mathbf{PGL}_2(\mathbb{C}) \), one obtains a natural \( \mathbb{F}_2 \)-action on \( \mathbb{P}^1(\mathbb{C}) \). Let \( S \subset \mathbb{P}^1(\mathbb{C}) \) be the closure of the set of attractive and repulsive fixed points for all \( \gamma \in \mathbb{F}_2 \). The complement \( \Omega := \mathbb{P}^1(\mathbb{C}) \setminus S \) is connected, \( \Omega = \bigcup_{\gamma \in \mathbb{F}_2} \gamma \cdot D \) for \( D \) being the exterior domain of four non-intersecting circles on the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \), and \( \mathcal{X} = \mathbb{F}_2 \setminus \Omega \) for the proper discontinuous action \( \mathbb{F}_2 \curvearrowright \Omega \).

This amounts to the next

Lemma 4.4. \( T \sim_{q, 1} \Omega \) (the latter being a domain in \( \mathbb{P}^1(\mathbb{C}) \)).

Further, given \( f \in G_e \) let us suppose for a moment that \( a(f^c) \neq f^c \) for all \( a \in \mathbb{F}_2, c \in \mathfrak{G}_n \). Identify \( T \) with its set of vertices \( \{a(f)\}_{a \in \mathbb{F}_2} \) and similarly introduce the tree \( T^f_c := \{a(f^c)\}_{a \in \mathbb{F}_2} \) (thus \( T^f_c \) is another copy of \( T = T^1_f \)). Then \( T^f_c \) carries a metric \( \text{dist}(\ast, \ast) \), coming from the word metric on \( \mathbb{F}_2 \), so that \( \text{dist}(a(f^c), b(f^c)) := \text{dist}(ab^{-1}, 1) \) for all \( a, b \in \mathbb{F}_2 \).[4] Now, gluing \( T^f_c \) with \( T^f_{(a)(c)} \) (isometrically) via \( b(f^c) \mapsto b^a(f^{a(c)a}) \) for all \( a, b \in \mathbb{F}_2 \) (we regard \( a, b, c \) as elements in \( \mathfrak{G}_n \) acting on \( f \) by conjugation), we may identify the metric space

\[
\mathcal{C}_f := \bigsqcup_{c \in \mathfrak{G}_n} T^f_c / \sim
\]

with \( T \) (as sets). Here \( \sim \) is the equivalence relation such that \( b(f^c) \sim b^a(f^{a(c)a}) \) for all \( a, b, c \).[5] (Note also that the assertion of Lemma 4.4 obviously holds for \( \mathcal{C}_f \) in place of \( T \).)

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[4] Note that \( f^c \in G_e \) and \( E_{fc} = E_f \) for all \( c \) (cf. Example 22).

[5] Indeed, we have \( a' a(f) = a'(f^a) = f^{a'(a)a'} \) for all \( a', a \in \mathbb{F}_2 \), which implies that \( \sim \) is symmetric and transitive.
Lemma 4.5. \( C_f \) is defined for any \( f \in G_e \). More precisely, this \( C_f \) is q.-i. to \( \Omega \) and coincides with \( T \) set-theoretically, similarly as above.

Proof. We use notations from Section 2. Put \( x := [X_0, \ldots : X_n] \) and fix an arbitrary \( a \in \mathbb{F}_2 \setminus \{1\} \). Then, since there is no \( c \in \mathfrak{S}_n \) such that \( c^{-1}(g(c(x))) = a^{-1}(g(a(x))) \) for all \( g \in \mathfrak{S}_n \) (because \( a \in \text{Out}(\mathfrak{S}_n) \)), we can associate with \( a(f^c) \) an ordered pair \( \{a;c\} \). Now, since \( \{a;c\} \) are all distinct for different \( a, c \), we repeat the previous construction of the trees \( T_f \) (with \( a(f^c) \) replaced by \( \{a;c\} \)). Finally, we use the fact that \( f \in G_e \) to glue the trees \( T_f \) and \( T_{a(c)a} \) via \( \sim \) as earlier, which gives \( C_f \) (\( \sim \)-q.-i. \( \Omega \)) as wanted. \( \square \)

Remark 4.6. To say it in words, every \( T_f \) in the definition of \( C_f \) corresponds to a “coloring” of \( T \) (one for each \( c \in \mathfrak{S}_n \)), compatible with the \( \mathbb{F}_2 \)-action (cf. Figure 1). In turn, the pairs \( \{a,c\} \) from the proof of Lemma 4.5 can be considered as “local coordinates” (with \( T_e \) being “local charts”) on \( T \), where the \( \mathfrak{S}_n \)-part corresponds to “coordinate bases”, while the \( \mathbb{F}_2 \)-part is the “coordinate values”.

In view of Lemma 4.5 we will not distinguish between \( T \) and \( C_f \) in what follows, so that the tree \( T \) comes enhanced with additional structure (cf. Remark 4.6).

The next result may be considered as the glimpse of a certain “Anosov property” enjoyed by the elements from \( G_e \), for one may observe an analogy between the (hyperbolic) \( \mathbb{Z} \)-action on \( \text{Diff} \) (see the discussion in [11] §2 or in [11] §5 for instance) and the \( \mathbb{F}_2 \)-action on \( G_e \) in our case, with assertions “two elements \( f, g \in \text{Diff} \) are homotopic, \( C^r \)-close, etc.” being replaced by “\( f \sim g, f, g \in G_e \)”.

Proposition 4.7. For every \( f, g \in G_e \), the group \( E_{fg} \) is non-cyclic.

Proof. Suppose that \( E_{fg} = \langle b \rangle \) for some \( b \in \mathbb{F}_2 \). Let us glue \( C_f \) with \( C_g \) as follows:

\[ a(f^c) \sim a(g^c) \text{ for all } a \in \mathbb{F}_2, c \in \mathfrak{S}_n. \]

(Obviously, the latter \( \sim \) is compatible with the equivalence relation used to construct \( C_f \) and \( C_g \) above, and so we keep the same notation for both.) Again, since \( \mathbb{F}_2 \subseteq \text{Out}(\mathfrak{S}_n) \) and \( f, g \in G_e \), this construction is compatible with the \( \mathbb{F}_2 \)-action. In particular (to simplify the notations), we will assume that \( a(f^c) \neq f^c, a(g^c) \neq g^c \) for all \( a \in \mathbb{F}_2, c \in \mathfrak{S}_n \), as in Figure 1 below.

Further, in the preceding definition of the trees \( T_f \) we can formally replace \( a(f^c) \) by \( C_a(f^c g^c) \), with arbitrary \( a \in \mathbb{F}_2, c, c' \in \mathfrak{S}_n \), where again \( C_a(f^c g^c) \) is regarded as a “\( \{\text{value};\ \text{coordinate}\} \)” triple \( \{a; c, c'\} \), analogous to that in the proof of Lemma 4.5. Then we repeat the isometric gluings of \( T_f \) with \( T_{a(c)a} \) (resp. of \( T_{c} \) with \( T_{a(c'):a} \)) to get the tree \( T' \sim q.-i. T \) (the former being (formally) identified with \( \{C_a(fg)\}_{a \in \mathbb{F}_2} \), carrying a free, transitive and isometric \( \mathbb{F}_2 \)-action. Thus the construction/enhancement of \( T' \) is essentially the repetition verbatim of that for \( T \).

Lemma 4.8. There is an \( \mathbb{F}_2 \)-equivariant (continuous) map \( \varphi : T \to T' \) of metric spaces which coincides with the quotient map \( \mathbb{F}_2 \to \mathbb{F}_2 / \langle b \rangle \) on the sets of vertices. In particular, \( \varphi \) is surjective.

6) Recall that \( a, b \in G \) act on \( \mathbb{P}^n \). Then \( c^{-1}(g(c(x))) = a^{-1}(g(a(x))) = e^{-1}(g(a(x))) \) is understood as an identity between the elements in \( G \). In particular, this does not depend on the choice of \( x \), so that the further \( \{a;c\} \) is correctly defined.

7) The arguments below work for the product of any \( f_1, \ldots, f_m \in G_e \) and \( m \geq 2 \).
Proof. Identify $\mathcal{T}$ with its chart $\mathcal{T}_f^f \cong \mathcal{T}_g^g$ and set $\varphi$ to be as follows:

\begin{equation}
\label{eq:4.9}
a(f) \sim a(g) \mapsto C_a(fg), \ a \in \mathbb{F}_2.
\end{equation}

(Here we use the chart \{\(C_a(fg)\)\}_{a \in \mathbb{F}_2} for $\mathcal{T}'$ as well.) Clearly, the (set-theoretic) map $\varphi$ is surjective and coincides with $\mathbb{F}_2 \to \mathbb{F}_2 / \langle b \rangle$ on the sets of vertices, since $E_{fg} = \langle b \rangle$.

In order to extend $\varphi$ to a metric morphism, it suffices to show the definition of $\varphi$ does not depend, up to isometry, on the ($\mathbb{F}_2$-equivariant) identification of $\mathcal{T}$ with $\mathcal{T}_f^f$ (a.k.a. $\mathcal{T}_g^g$). This will follow if we check the definition of $\varphi$ does not depend on replacing $a$ by $a'$ for an arbitrary fixed $a' \in \mathbb{F}_2$ and all $a$. But that is why we need enhanced $\mathcal{T}$ and $\mathcal{T}'$. Namely, regarding (as usual) $a'$ as an element in $G_n$ acting on $f, g$ by conjugation, we simply go to the chart $\mathcal{T}_f^{a'} \cong \mathcal{T}_g^{a'}$ of $\mathcal{T}'$ so that $\varphi$ now acts like this:

$$a' aa'^{-1}(f') \sim a' aa'^{-1}(g') \mapsto C_{a'}(fg).$$

(Here $\mathcal{T}'$ is also considered in its other chart \{\(C_a(fc)\)\}_{a \in \mathbb{F}_2}.) Thus we have replaced $\mathcal{T}, \mathcal{T}'$ by their isometric copies and defined $\varphi$ for these also (compatibly with (4.9)). This is correct because $\mathcal{T}_f^f \cong \mathcal{T}_g^g, \mathcal{T}_f^{a'} \cong \mathcal{T}_g^{a'}$, etc. are (formally) distinct by construction. \hfill \Box

Lemmas 4.4 and 4.8 yield a 1-cycle fibration $\Omega \longrightarrow \Omega$ which is q.-i. to $\varphi$:

\begin{center}
\begin{tikzpicture}
\node (T_f) at (0,0) {$\mathcal{T}_f^f$};
\node (T_c) at (2,0) {$\mathcal{T}_c^c$};
\node (T) at (0,-1.5) {$\mathcal{T} \cong_{q.-i.} \Omega$};
\node (T') at (2,-1.5) {$\mathcal{T}' \cong_{q.-i.} \Omega$};
\draw[->] (T_f) -- (T_c);
\draw[->] (T) -- (T');
\end{tikzpicture}
\end{center}

Figure 1.

But the latter is impossible for the domains in $\mathbb{P}^1(\mathbb{C})$. Proposition 4.7 is proved. \hfill \Box

The set $G_e$ generates a normal subgroup $N$ in $\mathfrak{G}_n$ with $N \cap \mathcal{E}_n \supseteq G_e \cap \mathfrak{D}_n \neq \{1\}$ (see Example 4.2). We will see in Section 5 that the complement $\mathfrak{G}_n \setminus G_{ne} \cup G_e$ contains an element $\Lambda$ such that the group $E_{\Lambda}$ is cyclic. This together with Proposition 4.7 implies that $N \vartriangleleft \mathfrak{G}_n$. We will show that in fact $\mathfrak{G}_n \setminus N \ni \Lambda$ for some $\Lambda \in \mathcal{E}_n \setminus \ker(\xi)$ (cf. Proposition 4.11), which easily yields $\xi(N \cap \mathcal{E}_n) \not\subseteq \mathbf{U} \mathbf{A} \mathbf{u}t_{n-1}$ (see the discussion after Corollary 5.6), hence Theorem 1.1 (Note that $N \cap \mathcal{E}_n \not\subseteq \ker(\xi)$ because $G_e \cap \mathfrak{D}_n \not\subseteq \ker(\xi)$.)

Remark 4.10. It would be interesting to test non-simplicity of a group $G$ satisfying $\mathbb{F}_2 \subseteq \text{Out}(G)$ and $G_e \cdot G_e \subseteq G_{ne} \cup G_e$ (cf. Question 6.4 below).
5. Proof of Theorem 5.1

5.1. We keep up with the previous notations. Let us assume in addition that \( n \geq 4 \). Consider \( \Lambda \in E_n \) defined as follows (cf. 3.2):

\[
X_i \mapsto (X_0^{d-1} + X_1^{d-1})X_i,
\]

\[
X_4 \mapsto X_0^{d-1}X_4 + \Lambda_d(X_1, X_2, X_3, \ldots, X_n)
\]

for all \( i \neq 4 \). Let us also consider \( a_1, a_2 \in SL'_n(\mathbb{Z}) \) defined as follows (cf. Example 2.3):

\[
a_1 : [1 : X_1 : \ldots : X_n] \mapsto [1 : X_1 : X_2 : X_2^i : X_4 : \ldots : X_n],
\]

\[
a_2 : [1 : X_1 : \ldots : X_n] \mapsto [1 : X_1 : X_1^i : X_3 : \ldots : X_n]
\]

for \( I_1 := (1, -1, 1, 0, \ldots, 0) \) and \( I_2 := (-1, 1, 1, 0, \ldots, 0) \).

Lemma 5.3. The elements \( a_1^2, a_2^2 \) generate a free subgroup \( \mathbb{F}_2 \subseteq SL'_n(\mathbb{Z}) \).

Proof. Take a matrix \( \star := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & b \end{pmatrix} \) with some \( a, b \in \mathbb{C} \). Then the matrix \( X' := \star a_1 \) has entries \( X'_{2,3} = a, X'_{3,3} = -a + b \). Similarly, \( X'' := \star a_2 \) has entries \( X''_{2,3} = a + b, X''_{3,3} = b \). Letting \( a := 1, b := 0 \) and \( a := 0, b := 1 \), we obtain a homomorphism from the group generated by \( a_1^2, a_2^2 \) onto the subgroup \( \Gamma \subseteq SL_2(\mathbb{Z}) \) generated by the matrices \( \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \). Hence it suffices to show that \( \Gamma \cong \mathbb{F}_2 \). But the latter follows from the Ping-Pong Lemma. \( \square \)

Note that \( a_1 \Lambda a_1^{-1} = \Lambda \) (i.e. \( a_1 \in E_\Lambda \) in the notations of 4.1). On the other hand, we have the following:

Proposition 5.4. There exists \( \Lambda \) of the form (5.2) such that \( a \Lambda a^{-1} \not\cong \Lambda \) for every \( a \in \mathbb{F}_2 \setminus \{a_1^k \}_k \in \mathbb{Z} \).

Proof. Suppose that \( a \Lambda a^{-1} = f \Lambda f^{-1} \) for some \( a \in \mathbb{F}_2 \setminus \{a_1^k \}_k \in \mathbb{Z} \) and \( f \in \mathfrak{E}_n \). We will exclude only the case \( a := a_2 \) (the general case is treated similarly). The map \( a_2 \Lambda a_2^{-1} \) acts as follows:

\[
X_1 \mapsto X_3^{d}(X_0^{d-1} + X_1^{d-1})X_1,
\]

\[
X_4 \mapsto X_0^{d-1}X_3 X_4 + \Lambda_d(X_1X_3, X_1X_2, X_4X_3, \ldots, X_nX_3)
\]

for all \( i \neq 4 \). At this stage we assume that \( \Lambda_d \neq 0 \mod (X_1, X_4, \ldots, X_n) \). Then, in particular, \( a_2 \Lambda a_2^{-1} \) contracts the hyperplane \( H := (X_3 = 0) \) to a point. On the other hand, we have

Lemma 5.5. The map \( f \Lambda f^{-1} \) does not contract \( H \).

Proof. We may assume that \( d \gg 1 \). Then, since \( f \in \mathfrak{E}_n \) and \( \Lambda(O) = 0 \) (see the proof of Lemma 5.3), one can easily see that \( f \Lambda f^{-1} \) asymptotically equals \( \Lambda \). Namely, it suffices to put \( f_i^{-1} := X_i / \varepsilon \) for the components of \( f^{-1} \), \( 1 \leq i \leq n \), and some (varying) \( 0 \leq \varepsilon \ll 1 \). Then, similarly as in the proof of Lemma 2.5, one finds that \( \Lambda f^{-1} \) acts as \( \Lambda_i + \varepsilon \) for the components of \( \Lambda f^{-1} \) and \( \Lambda \) (cf. 2.2, 5.2). Furthermore, since \( \varepsilon \) can be expressed as an analytic function in \( \Lambda_i \) locally near the point \( O \), by the same argument (with \( X_i \) replaced by \( \Lambda_i \) for all \( i \)) we get \( (f \Lambda f^{-1})_i = \Lambda_i + \varepsilon \) for the components of \( f \Lambda f^{-1} \). In particular, \( f \Lambda f^{-1} \) cannot contract \( H \). \( \square \)
Lemma 5.4 gives \( a_2Aa_2^{-1} \neq fAf^{-1} \), a contradiction.

\[ \square \]

Let \( \mathfrak{G}_n' \subseteq \mathfrak{G}_n \) be the maximal subgroup preserving the hyperplane \( \Pi = (X_0 = 0) \). Take \( \Lambda \) as in the proof of Proposition 5.4 and let \( \Lambda_0 \) be the restriction of \( \Lambda \) to \( \Pi \). Then from (the proof of) Proposition 5.4 we get the following (w.r.t. the induced \( \rho(G) \)-action on \( \Pi \)):

**Corollary 5.6.** \( a_1\Lambda_0a_1^{-1} = \Lambda_0 \) and \( a\Lambda_0a^{-1} \neq \Lambda_0 \) in \( \mathfrak{G}_n'|_{\Pi} \) for every \( a \in \mathbb{F}_2 \setminus \{a_1^{k}\}_{k \in \mathbb{Z}} \).

We have \( \mathcal{E}_n \subseteq \mathfrak{G}_n' \) and \( \mathbb{F}_2 \subseteq \text{Out}(\mathfrak{G}_n') \) via the induced \( \rho(G) \)-action on \( \Pi \) (cf. Lemma 5.3 and the proof of Corollary 2.3). Then the arguments of Section 4 with the extra condition “modulo \( \mathcal{X}_0 \)” added, apply literally to show that \( (N \cap \mathfrak{G}_n')|_{\Pi} \) is a proper normal subgroup of \( \mathfrak{G}_n'|_{\Pi} \) such that \( (N \cap \mathcal{E}_n)|_{\Pi} \neq \{1\} \) and \( \Lambda_0 \not\in N|_{\Pi} \) (for the latter we have also used Corollary 5.6).

**Lemma 5.7.** \( \xi(N \cap \mathcal{E}_n) \not\subset \mathbb{U}_{\text{Aut}}(\mathcal{E}_n) \).

**Proof.** Indeed, we have
\[
\begin{align*}
\mathcal{E}_n|_{\Pi} := \xi(\mathcal{E}_n) &= \mathbb{U}_{\text{Aut}}(\mathcal{E}_n), \\
\Lambda \in \mathcal{E}_n \text{ and } \Lambda_0 = \xi(\Lambda) \neq \xi(N \cap \mathcal{E}_n) = (N \cap \mathcal{E}_n)|_{\Pi} \text{ because } \Lambda_0 \not\in N|_{\Pi}, \\
N \cap \mathcal{E}_n \not\subset \text{Ker}(\xi).
\end{align*}
\]

This shows that \( \xi(N \cap \mathcal{E}_n) \neq \{1\}, \mathbb{U}_{\text{Aut}}(\mathcal{E}_n), \) i.e. \( \xi(N \cap \mathcal{E}_n) \not\subset \mathbb{U}_{\text{Aut}}(\mathcal{E}_n) \).

**Lemma 5.7** finishes the proof of Theorem 1.1.

### 6. Final Comments

The way how we used the groups \( G, \mathfrak{G}_n \) and \( \mathcal{E}_n \) to prove Theorem 1.1 makes it reasonable to develop the preceding arguments more systematically and study other subgroups in \( \mathbb{C} \mathfrak{r}_n \) which “behave expectedly at infinity”. Let us advocate this thesis by proving the following:

**Proposition 6.1 (cf. [4, 5.1]).** For any, not necessarily algebraically closed field \( k \subset \mathbb{C} \), the group \( \mathbb{C} \mathfrak{r}_n \) is not embedable into \( GL_m(\mathbb{C}) \) for all \( m \in \mathbb{N} \cup \{\infty\} \) and \( n \geq 2 \).

**Proof.** Take \( g_1 \in \mathfrak{s}_n \) and \( g_2 \in SL_n(\mathbb{Z}) \) any unipotent element. Consider the group \( E := \langle g_1, g_2 \rangle \subset G \). Let us also suppose that \( g_2^2 = 1 \). We can always choose \( g_1, g_2 \) in such a way that \( E = \langle g_1, g_2 \rangle \) for some \( 2 \leq k \leq n \).

**Lemma 6.2.** \( E \) is not embedable into \( GL_m(\mathbb{C}) \) for any \( m \in \mathbb{N} \cup \{\infty\} \).

**Proof.** Here we follow the paper [4]. Suppose that \( E \subset GL_m(\mathbb{C}) \) for some \( m \).

Consider a word metric \( \text{dist}_E \) on \( E \) and the corresponding metric space \( (E, \text{dist}_E) \). Then, since the extension \( \langle g_1 \rangle^{\oplus k} \times \langle g_2 \rangle \subset GL_m(\mathbb{C}) \) is non-trivial, we may assume that \( 2 \leq m < \infty \), which gives a natural isometric embedding \( (E, \text{dist}_E) \hookrightarrow (\mathbb{R}^{\oplus k} \times S^1) \times \mathbb{R}, \text{dist} \), for \( \log(\text{dist}_E) = \text{dist}_E \), such that \( (\langle g_1 \rangle^{\oplus k} \times S^1) \times \mathbb{R}, \text{dist} \) is a hyperbolic space and \( s \times \mathbb{R} \) is a horocycle for all \( s \in (\langle g_1 \rangle^{\oplus k} \times S^1) \) (see [4] 2.2, (d), (e), (f)). In particular, since \( g_1^2 = 1 \) and

\[ 8 \] After the text has been written, I was informed by S. Cantat about [http://perso.univ-rennes1.fr/serge.cantat/Articles/cnl-5.jpg](http://perso.univ-rennes1.fr/serge.cantat/Articles/cnl-5.jpg) where a similar statement had been proved (via a group-theoretic argument) for every finite \( m \in \mathbb{N} \).
\((g_2) = \mathbb{Z} \subset \mathbb{R}\), this implies that \(\text{Con}_\infty(E)\), the *asymptotic cone* of \(E\) (with induced metric), is totally disconnected (loc. cit.).

On the other hand, since \(g_2^a \circ g \circ g_2^b = g' \circ g_2^c\) for all \(a, b \in \mathbb{Z}\), \(g \in (g_1)^{\oplus k}\) and some \(c := c(a, b) \in \mathbb{Z}\), \(g' := g'(g) \in (g_1)^{\oplus k}\), the group \(E \subset \mathfrak{S}_n\) (obviously) acts as \(\mathbb{Z} = \langle g_2 \rangle\) on the Berkovich spectrum of \(\mathbb{C}^n\) (cf. \([13] \text{ Section 5}\)). In particular, we obtain that \((E, \text{dist}_E)\) is q.-i. to \(\mathbb{Z}\) with the corresponding word metric (see \([9, 0.2.C]\)), which implies that \(\text{Con}_\infty(E) = \mathbb{R}\) with the usual metric (see \([9, 2.B, (a)]\)). This contradicts the previous paragraph. \(\Box\)

Lemma 6.2 proves Proposition 6.2. \(\Box\)

**Remark 6.3.** It would be interesting to construct examples of algebraic varieties \(X\) over a number field \(F\) for which the above non-embeddability result for \(\text{Cr}_n\) provides a non-trivial obstruction to rationality of \(X\) over \(F\).

Finally, Corollary 2.9 relates \(\mathfrak{S}_n\) to hyperbolic groups and groups with small cancellation (cf. \([3]\)), which together with results of Section 3 makes one ask the next

**Question 6.4.** Let \(G\) be a group such that \(\mathbb{F}_2 \subseteq \text{Out}(G)\). Is \(G\) non-simple?

Unfortunately, the answer to Question 6.4 is negative in general, as the case of the group \(G := \text{PGL}_{n+1}(\mathbb{C})\) (with \(\mathbb{F}_2 \subset \text{Gal}(\mathbb{C}/\mathbb{Q}) \subseteq \text{Out}(G)\) shows. However, the latter indicates an interesting difference between the groups \(\mathcal{E}_n\) and \(\text{PGL}_{n+1}(\mathbb{C})\), which together with the proof of Theorem 1.1 suggests a way to attack the (non-)simplicity of \(\text{Cr}_n\) for all \(n \geq 2\) (basically, one constructs a normal subgroup \(N \subseteq \text{Cr}_n\) exactly as in Section 1 above (cf. Remark 1.11, and tries to show that \(N \neq \text{Cr}_n\), arguing as in Section 1 for instance).

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9) Pointed out by M. Gromov (though the case of discrete or, better, finitely presented \(G\) might still be of interest).

10) Provided that \(\mathbb{F}_2 \subseteq \text{Out}(\text{Cr}_n)\). For example, one may attain this for \(\text{Gal}(\mathbb{C}/\mathbb{Q}) \subseteq \text{Out}(\text{Cr}_n)\) (cf. [2]), but then the construction of \(N\) may be no longer valid.
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