Fitted Tension Spline Method for Singularly Perturbed Time Delay Reaction Diffusion Problems

Ermias Argago Megiso, Mesfin Mekuria Woldaregay, and Tekle Gemechu Dinka

Department of Applied Mathematics, Adama Science and Technology University, Adama, Ethiopia

Correspondence should be addressed to Mesfin Mekuria Woldaregay; msfnmkr02@gmail.com

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A uniformly convergent numerical method is presented for solving singularly perturbed time delay reaction-diffusion problems. Properties of the continuous solution are discussed. The Crank–Nicolson method is used for discretizing the temporal derivative, and an exponentially fitted tension spline method is applied for the spatial derivative. Using the comparison principle and solution bound, the stability of the method is analyzed. The proposed numerical method is second-order uniformly convergent. The theoretical analysis is supported by numerical test examples for various values of perturbation parameters and mesh size.

1. Introduction

Delay differential equations in which its highest order derivative term is multiplied by a small parameter $\varepsilon$ are known as singularly perturbed delay differential equations (SPDDEs). A good example of SPDDEs is the mathematical model for the control system of the furnace used to produce metal sheets which is given by

$$
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} = v g (u(x, t - \tau)) + c [f (u(x, t - \tau) - u(x, t))],
$$

(1)

where $u$ is the temperature distribution in a metal sheet moving at an instantaneous material strip velocity $v$ and heated by a distributed temperature source given by the function $f$; both $v$ and $f$ are dynamically changing by a controller monitoring the current temperature distribution. The controller’s finite speed causes a fixed delay of length $\tau$ [1].

When the perturbation parameter $\varepsilon$ tends to zero, the smoothness of the solution of SPDDEs descends, and it forms a boundary layer [2]. Since most of the problems have no known analytical/exact solution, formulating numerical methods becomes mandatory [3]. It is of theoretical and practical interest to consider numerical methods for solving SPDDEs. Developing a numerical method whose convergence does not depend on the perturbation parameter has great importance [4]. Owning this, different authors have developed numerical schemes for the solution of SPDDEs. Chakravarthy et al. [5–7] used an adaptive mesh method and fitted operator methods for solving singularly perturbed differential-difference equations. Kumar and Kadhalabjao [8] computed the numerical solution of SPDDEs using a B-spline collocation method on Shishkin mesh. Kanth and Kumar in [9–11] used the tension spline method to solve singularly perturbed convection-dominated differential equations. Woldaregay et al. [3, 12–16] developed an exponentially fitted numerical method for solving different singularly perturbed differential-difference equations. In [14, 17], they proposed the nonstandard finite difference techniques and the fitted mesh methods, respectively. In [18, 19], an initial value technique with an exponential fitting factor is applied to solve convection-dominated delay differential equations. The difference method on Shishkin for solving a singularly perturbed linear second-order delay differential equation is developed in [20].

Govindarao et al. [2] solved a singularly perturbed delay parabolic reaction-diffusion problem using the implicit Euler scheme for time derivative on the uniform mesh, and
they used the central difference scheme on the Shishkin mesh for spatial discretization. Kumar and Chandra Sekhara Rao [21] developed a numerical scheme using a suitable combination of fourth order compact difference scheme and central difference scheme on generalized Shishkin mesh. Gowrisankar and Natesan [22] proposed a numerical scheme using the backward-Euler method for temporal discretization and the upwind finite difference scheme for spatial derivative. Erdogan and Amiraliyev [23] treated singularly perturbed second-order delay differential equations using an exponentially fitted difference scheme on a uniform mesh.

In some cases, the cubic spline does not provide an accurate picture of the shape of the graph. In such cases, the tension spline method is preferable. A tension spline is a cubic spline that has a tension factor applied to it to stretch the graph closer to the given points. A tension factor δ is a number that is used to generate different conditions for the spline. When the tension factor is set to zero, it gives the usual cubic spline. The objective of this study is to formulate a uniform numerical method for solving singularly perturbed time delay reaction-diffusion problems. Furthermore, we need to establish the stability and uniform convergence of the method. To the best of our knowledge, the exponentially fitted tension spline method has not been used for treating SPDDEs. In this paper, we employ the Crank–Nicolson method to temporal discretization and the exponentially fitted tension spline method to spatial discretization.

2. Continuous Problem

On the domain \( D = \Omega_x \times \Omega_t = (0, 1) \times (0, T) \), we consider singularly perturbed time delay reaction-diffusion problems of the form

\[
L_u(x, t) = \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)u(x, t),
\]

\[
= -b(x, t)u(x, t - \tau) + f(x, t),
\]

with the interval condition

\[
u(x, t) = \psi_b(x, t), \ (x, t) \in \overline{\Omega_x} \times [-\tau, 0],
\]

\[
u = [0, 1] \times [-\tau, 0],
\]

and the boundary conditions

\[
u(0, t) = \psi_l(t), \nu(1, t),
\]

\[
u(x, t) = \psi_r(t), t \in \overline{\Omega_t} = [0, T],
\]

where \( \varepsilon \in (0, 1) \) is the perturbation parameter, and \( \tau \) is the delay parameter. The functions \( a, b, \psi_l, \psi_r, \psi_b \), and \( f \) are assumed to be sufficiently smooth and bounded that satisfy \( a(x) \geq \alpha > 0 \), \( b(x, t) \geq \beta > 0 \). The compatibility conditions at the corner points \((0, 0), (1, 0), (0, -\tau), \) and \((1, -\tau)\) are fulfilled. That is

\[
\begin{align*}
\psi_b(0, 0) &= \psi_l(0), \\
\psi_b(1, 0) &= \psi_r(0), \\
\frac{\partial \psi_l(0)}{\partial t} - \varepsilon \frac{\partial^2 \psi_b(0, 0)}{\partial x^2} + a(0)\psi_b(0, 0), \\
&= -b(0, 0)\psi_b(0, -\tau) + f(0, 0), \\
\frac{\partial \psi_r(0)}{\partial t} - \varepsilon \frac{\partial^2 \psi_b(1, 0)}{\partial x^2} + a(1)\psi_b(1, 0), \\
&= -b(1, 0)\psi_b(1, -\tau) + f(1, 0).
\end{align*}
\]

Solution of the problem in (2)–(4) satisfies the following maximum principle.

**Lemma 1** (see [24]). (The maximum principle) Assume that \( a, b \in C^0(\overline{D}) \) and let \( u \in C^2(\overline{D}) \cap C^0(\overline{D}) \). Suppose that \( u \geq 0 \) on \( \partial D = \overline{\Omega} - D \). Then, \( L_u \geq 0 \) in \( D \) implies that \( u \geq 0 \) in \( \overline{D} \).

The stability of the solution of (2)–(4) in the maximum norm is established by the following lemma:

**Lemma 2** (see [25]). Let \( u \) be the solution of the problem in (2)–(4). Then, it satisfies the \( \varepsilon \)-uniform upper bound

\[
\|u\| \leq (1 + \alpha T)\max\left\{\|L_u\|, \|u\|_{\partial D}\right\},
\]

where the constant \( \alpha = \max_{x \in [0, 1]} \{1 - \alpha \} \leq 1 \) and the norm \( \| \cdot \| \) is denoted for the maximum norm which is defined as \( \|u\| = \max_{x \in [0, 1]} \|u\| \).

**Lemma 3** (see [26]). Derivatives of the solution of (2)–(4) satisfy the following bound:

\[
\frac{\partial^{k+l} u}{\partial x^k \partial t^l} \leq C \left[1 + \varepsilon^{-k/2} \left(\exp\left(-\sqrt{\frac{\alpha}{\varepsilon}}\right) + \exp\left(-\frac{\alpha}{\varepsilon(1 - x)}\right)\right)\right],
\]

for \( k = 0, 1, 2, 3, 4 \) and \( l = 0, 1, 2 \).

3. Numerical Scheme

3.1. Temporal Semidiscretization. Let \( M \) be the number of mesh points in the discretization of \( \Omega_t = [0, T] \), and \( m \) is the number of mesh points in the discretization of the interval \([ -\tau, 0] \). Note that \( T = kr \) for some positive integer \( k \). The time domain \([0, T]\) is discretized using a uniform mesh with time step \( \Delta t \) as \( \Omega_t^M = \{ t_j = j\Delta t, \ j = 0, 1, 2, \ldots, M \} \) and \( \Omega_t^M = \{ t_j = j\Delta t, \ j = 0, 1, 2, \ldots, M \} \). We approximated the temporal derivative term of (2)–(4) using the averaged Crank–Nicolson method, which gives a system of BVPs
$$(1 + \frac{\Delta t}{2} L)U_j^{j+1}(x) = \begin{cases} 
\left(1 - \frac{\Delta t}{2} L\right)U_j^j(x) - \Delta t b^{j+1/2}(x)\psi_b(x) + \Delta t f^{j+1/2}(x), & \text{for } j = 0, 1, \ldots, m, \\
\left(1 - \frac{\Delta t}{2} L\right)U_j^j(x) - \Delta t b^{j+1/2}(x)U_{j-m}^j(x) + \Delta t f^{j+1/2}(x), & \text{for } j = m + 1, \ldots, M - 1, 
\end{cases}$$

with the boundary conditions

$$U_j^{j+1}(0) = \psi_l(t_{j+1}), U_j^{j+1}(1) = \psi_r(t_{j+1}),$$

where $L^j U_j^{j+1}(x) = -e U_j^{j+1}(x) + a(x)U_j^{j+1}(x)$. Here, $U_j^{j+1}(x)$ is denoted for the approximation of $u(x, t)$ at the $(j+1)^{th}$ time level and $b^{j+1/2}(x) = b(x, t_{j+1}) + b(x, t_j)/2$, similarly for $f^{j+1/2}(x)$ also.

**Lemma 4.** For each $j = 0, 1, 2, \ldots M - 1$, the local truncation error in the temporal direction is given by

$$\|LTE_j^{j+1}\| \leq C_1 (\Delta t)^3,$$

where $C_1$ is a constant independent of $\varepsilon$ and mesh size $\Delta t$.

**Proof.** Using Taylor’s series approximation for $u(x, t_j)$ and $u(x, t_{j+1})$ centering at $t_{j+1/2}$, we obtain

$$\begin{align*}
\frac{u(x, t_j) - u(x, t_{j+1/2})}{\Delta t} &= \frac{\partial u(x, t_j)}{\partial t} + O(\Delta t), \\
\frac{u(x, t_{j+1}) - u(x, t_{j+1/2})}{\Delta t} &= \frac{\partial u(x, t_{j+1})}{\partial t} + O(\Delta t).
\end{align*}$$

From (11), we obtain

$$\frac{u(x, t_j) - u(x, t_{j+1/2})}{\Delta t} = \frac{\partial u(x, t_{j+1/2})}{\partial t} + O(\Delta t^2).$$

Substituting (12) into (2), we obtain

$$\frac{u(x, t_{j+1/2}) - u(x, t_j)}{\Delta t} = \frac{\partial u(x, t_{j+1/2})}{\partial t} + \frac{(\Delta t)^2}{8} u_{tt}(x, t_{j+1/2}) + O((\Delta t)^3).$$

Since the error $LTE_j^{j+1}(x)$: $= u(x, t_{j+1}) - U_j^{j+1}(x)$ satisfies the semidiscrete difference scheme

$$\left(1 + \frac{\Delta t}{2} L\right)LTE_j^{j+1}(x) = O((\Delta t)^3),$$

where $LTE_j^{j+1}(0) = 0$

$$\|LTE_j^{j+1}\| \leq C_1 (\Delta t)^3.$$

Next, we need to show the bound for the global error of temporal discretization. Let us denote $GTE_j^{j+1}$ as the global error up to the $(j+1)^{th}$ time step.

**Lemma 5.** The global error in temporal discretization at $t_{j+1}$ time step is given by

$$\|GTE_j^{j+1}\| \leq C_1 (\Delta t)^3, j = 1, 2, \ldots, M - 1.$$
Proof. Using the local truncation error up to the \((j + 1)\)th time step given in the above lemma, we obtain the global error up to the \((j + 1)\)th time step as

\[
\|GTE^{j+1}\| = \sum_{i=1}^{M} \|LTE^i\| + \|LTE^2\| + \ldots + \|LTE^{j+1}\| \\
\leq C, T(\Delta t)^2 \text{ since } (j + 1)\Delta t \leq T = C(\Delta t)^2, C, T = C,
\]

where \(C\) is a constant independent of \(\epsilon\) and \(\Delta t\). □

Lemma 6. Let \(U_{j+1}(x)\) be a solution of the semidiscrete problem in (7), (8) at the \(j + 1\) time level. Then, its derivatives satisfy the bound

\[
\|\frac{d^3 U_{j+1}(x)}{dx^3}\| \leq C \left[ 1 + e^{-k2} \left( \exp\left(\frac{\alpha}{\epsilon (1-x)}\right) + \exp\left(\frac{\alpha}{\epsilon (1-x)}\right)\right) \right],
\]

where \(a(x) \geq \alpha > 0\) for \(k = 0, 1, \ldots, 4\).

Proof. It follows from Lemma 3. □

3.2. Spatial Discretization. A function \(S(x, \delta) = S(x)\) is a class of \(C^2(\overline{I}_x)\), which interpolates \(u(x)\) at the mesh points \(x_i\), depending on the parameter \(\delta\), reduces to cubic spline in \(\overline{I}_x\) as \(\delta \rightarrow 0\), and it is termed as a parametric cubic spline function [27, 28]. The spatial domain is discretized into \(N\) equal number of subintervals, each of length \(h = 1/N\). Let \(0 = x_0, x_N = 1\) and \(x_i = ih, i = 0, 1, 2, \ldots, N\) be the mesh points. For spatial discretization, we apply an exponentially fitted tension spline method which helps us control the influence of the singular perturbation parameter. In \([x_{i-1}, x_i]\), the spline function \(S(x)\) satisfies the differential equation

\[
S''(x) - \delta S(x) = [S''(x) - S(x)] \frac{x_{i+1} - x}{h} + [S''(x) - S(x)] \frac{x - x_i}{h}
\]

where \(S(x_i) = u(x_i)\) and \(\delta > 0\) is termed as a cubic spline in compression. Solving the linear second-order differential equation in (19) and determining the arbitrary constants from the interpolation conditions \(S(x_{i+1}) = u(x_{i+1}), S(x_i) = u(x_i)\), we get

\[
-\varepsilon M_k + a_k U^{j+1}(x_k) = \left\{ \begin{array}{ll}
\varepsilon M_{k-1} - a_k U^j(x_k) + \Delta t \left[ -b_{i+1/2}^j \psi_k(x_k) + f^{j+1/2}_k \right], & \text{for } j = 0, 1, 2, \ldots, m, \\
\varepsilon M_k - a_k U^j(x_k) + \Delta t \left[ -b_{i+1/2}^{j+1} U^{j-m}(x_k) + f^{j+1/2}_k \right], & \text{for } j = m + 1, \ldots, M - 1,
\end{array} \right.
\]

for \(k = i, i \pm 1\), where \(M_k = u''(x_k), a_k = a(x_k)\) and \(b_{i+1/2}^j = b_{i+1/2}^{j+1/2}(x_k)\).

To handle the effect of the perturbation parameter \(\epsilon\), we use the exponential fitting factor

\[
S(x) = \frac{h^2}{\lambda^2 \sinh\lambda} \left[ M_{i+1} \sinh\left(\frac{\lambda (x - x_i)}{h}\right) + M_i \sinh\left(\frac{\lambda (x_{i+1} - x)}{h}\right) \right].
\]

\[
\frac{h^2}{\lambda} \left[ M_{i+1} - \frac{\lambda^2}{h^2} u(x_{i+1}) \right] \left( \frac{\lambda (x - x_i)}{h} \right) + M_i \frac{\lambda^2}{h^2} u(x_i) \left( \frac{\lambda (x_{i+1} - x)}{h} \right).
\]

where \(\lambda = h\delta^{1/2}\) and \(M_k = u''(x_k)\) for \(k = i, i \pm 1\). Now, differentiating (19) and letting \(x \to x_i\), on the interval \([x_{i-1}, x_i]\), we obtain

\[
S'(x_i^+) = \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{h}{\lambda^2} \left[ M_{i+1} \left( 1 - \frac{\lambda}{\sinh\lambda} \right) + M_i \left( \lambda \coth\lambda - 1 \right) \right].
\]

Similarly, on the interval \([x_{i-1}, x_i]\), we obtain

\[
S'(x_i^-) = \frac{u(x_i) - u(x_{i-1})}{h} + \frac{h}{\lambda^2} \left[ M_i \left( \lambda \coth\lambda - 1 \right) + M_{i-1} \left( 1 - \frac{\lambda}{\sinh\lambda} \right) \right].
\]

Equating the left and right hand derivatives at \(x_i\) gives

\[
\frac{u(x_i) - u(x_{i-1})}{h} + \frac{h}{\lambda^2} \left[ M_i \left( \lambda \coth\lambda - 1 \right) + M_{i-1} \left( 1 - \frac{\lambda}{\sinh\lambda} \right) \right],
\]

\[
\frac{u(x_{i+1}) - u(x_i)}{h} - \frac{h}{\lambda^2} \left[ M_{i+1} \left( 1 - \frac{\lambda}{\sinh\lambda} \right) + M_i \left( \lambda \coth\lambda - 1 \right) \right].
\]

Rearranging, we obtain the tridiagonal system

\[
\lambda_1 M_{i+1} + 2\lambda_2 M_i + \lambda_1 M_{i-1} = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}
\]

for \(i = 1, 2, 3, \ldots, N - 1\) where \(\lambda_1 = 1/\delta^2 \left( \lambda/\sinh\lambda - 1 \right)\) and \(\lambda_2 = 1/\delta \left( 1 - \lambda \coth\lambda \right)\).

The condition of continuity in (24) ensures the continuity of the first derivatives of the spline \(S(x)\) at interior nodes. Using into (24), (8) gives

\[
\sigma = \frac{h^2}{\sinh\left( h/2\sqrt{a_i}\epsilon \right)}.
\]

that stabilizes the discrete scheme. Substituting the fitting factor into (26), (25) we obtain the difference equation as
\[
(1 + \frac{\Delta t}{2} L_{z,t}^{\Delta t,h}) U_i^{j+1} = v_i^U U_{i-1}^j + v_i^e U_i^j + v_i^l U_i^{j+1},
\]
where
\[
= z_i^U U_{i-1}^j + z_i^e U_i^j + z_i^l U_i^{j+1} + K^i,
\]

\[
v_i^e = \frac{-\Delta t}{2} \left( \frac{\alpha}{h^2} - \lambda_1 a_{i-1} \right) + \lambda_1, \\
v_i^l = \frac{-\Delta t}{2} \left( \frac{\alpha}{h^2} + \lambda_1 a_{i+1} \right) + \lambda_1,
\]

\[
K^j = \left\{ \begin{array}{l}
\lambda_1 \Delta t \left[ -b_i^{1/2} \psi_b(x_{i-1}) + f_{i}^{1/2} \right] + 2\lambda_2 \Delta t \left[ -b_i^{1/2} \psi_b(x_{i}) + f_i^{1/2} \right], \\
+\lambda_1 \Delta t \left[ -b_i^{1/2} \psi_b(x_{i+1}) + f_i^{1/2} \right], \text{for } j = 0, 1, 2, \ldots, m,
\end{array} \right.
\]

\[
\quad + \lambda_1 \Delta t \left[ -b_i^{1/2} U_{i-1}^{j-m} + f_i^{1/2} \right] + 2\lambda_2 \Delta t \left[ -b_i^{1/2} U_i^{j-m} + f_i^{1/2} \right], \text{for } j = m + 1, \ldots, M - 1.
\]

Lemma 7. (Discrete comparison principle) Let there exist a comparison function \( Q_i^{j+1} \) such that if \( U_i^{j+1} \leq Q_i^{j+1} \) and \( U_N^{j+1} \leq Q_N^{j+1} \), then \( (1 + \Delta t/2) L_{z,t}^{\Delta t,h} U_i^{j+1} \leq (1 + \Delta t/2) L_{z,t}^{\Delta t,h} Q_i^{j+1} \), \( i = 1, 2, \ldots, N - 1 \), implies that \( U_i^{j+1} \leq Q_i^{j+1} \), \( i = 0, 1, 2, \ldots, N \).

Proof. The matrix \( (1 + \Delta t/2) L_{z,t}^{\Delta t,h} U_i^{j+1} \) is a size of \((N + 1) \times (N + 1)\) with its entries for \( i = 1, 2, \ldots, N - 1 \) which are \( v_i^e < 0, v_i^l > 0 \), and \( v_i^e < 0 \). As a result, the coefficient matrix satisfies the \( M \) matrix property. So, there is an inverse matrix that is non-negative. This proves that the discrete solution exists and is unique.

Lemma 8. (Uniform stability estimate) The solution \( U_i^{j+1} \) of the discrete scheme (21) satisfies the bound
\[
|U_i^{j+1}| \leq \frac{(1 + (\Delta t/2) L_{z,t}^{\Delta t,h}) |U_i^{j+1}|}{1 + \Delta t/2} + \max \left\{ |\psi_i(t_{j+1})|, |\psi_i(t_{j+1})| \right\},
\]
where \( \alpha > 0 \) is lower bound of \( a_i = a(x_i) \).

Proof. Let us consider a barrier function \( \mu_i^{j+1} = \| (1 + (\Delta t/2) L_{z,t}^{\Delta t,h}) U_i^{j+1} \| / (1 + \Delta t/2) + \max \left\{ |\psi_i(t_{j+1})|, |\psi_i(t_{j+1})| \right\} \). We can easily show that \( \mu_i^{j+1} = 0 \) and \( \mu_i^{j+1} = 0 \) respectively.
Using the discrete comparison principle, we obtain 
\[ \mu_{i,j+1}^+ \geq 0, \quad i = 0, 1, 2, \ldots, N \]

**Lemma 9** (see [29, 30]). For a fixed number of mesh \( N \) and for positive integer \( k \) as \( \epsilon \to 0 \)

\[
\lim_{\epsilon \to 0} \max_{1 \leq k \leq N-1} \frac{\exp(-Cx_i/\sqrt{\epsilon})}{\epsilon^{k/2}} = 0, \quad \lim_{\epsilon \to 0} \max_{1 \leq k \leq N-1} \frac{\exp(-C(1-x_i)/\sqrt{\epsilon})}{\epsilon^{k/2}} = 0.
\]

(31)

**Lemma 10.** If \( Q_{i+1}^j \) be any mesh function such that \( Q_0^1 = Q_N^1 = 0 \). Then,

\[ |Q_i^{j+1}| \leq \frac{1}{\alpha} \max_{1 \leq k \leq N-1} \left| L_{c,h}^{\Delta t,h} Q_k^{j+1} \right|. \]

(32)

**Proof.** Consider two barrier functions of the form 
\[ \mu_{i,j+1}^+ = S \pm Q_k^{j+1}, \quad \text{where} \quad S = 1/\alpha \max_{1 \leq k \leq N-1} |L_{c,h}^{\Delta t,h} Q_k^{j+1}|. \]

It is easily shown that \( \mu_{i,j+1}^+ \geq 0, \quad 0 \leq \mu_{i,j+1}^- \leq 1 \). Next, we have that

\[ L_{c,h}^{\Delta t,h} \mu_{i,j+1} = L_{c,h}^{\Delta t,h} (S \pm Q_k^{j+1}), \]

\[ = \pm L_{c,h}^{\Delta t,h} Q_k^{j+1} + \frac{\alpha}{\epsilon} \max_{1 \leq k \leq N-1} \left| L_{c,h}^{\Delta t,h} Q_k^{j+1} \right| \geq 0. \]

Hence, using the discrete comparison principles gives 
\[ |Q_i^{j+1}| \leq 1/\alpha \max_{1 \leq k \leq N-1} |L_{c,h}^{\Delta t,h} Q_k^{j+1}|. \]

Theorem 1. The discrete solution satisfies the following error bound:

\[ \|U^{j+1}(x) - U_i^{j+1}\| \leq C N^{-2}. \]

(34)

**Proof.** The truncation error of the scheme (27) is given by

\[ L_{c}^{\Delta t,U^{j+1}}(x) - L_{c}^{\Delta t,h U^{j+1}} \]

\[ \left( \frac{\epsilon}{12} (U_{xx}^{j+1}(x_i) \theta_i + U_{xxx}^{j+1}(x_i)) + \theta U_{xx}^{j+1}(x_i) \right) \frac{h^2}{2}, \]

\[ \left( \frac{\epsilon}{240} U_{xxx}^{j+1}(x_i) - \frac{\epsilon}{144} U_{jj}^{j+1}(x_i) + \theta U_{xxx}^{j+1}(x_i) \right) \frac{h^4}{2}, \]

\[ + \left( \frac{\epsilon}{2880} U_{xxxx}^{j+1}(x_i) \right) \frac{h^6}{2}. \]

Using Lemmas 6 together with Lemma 9 and the relation 
\( N^{-2} > N^{-4} > N^{-6} \ldots \), the discrete scheme satisfies the bound

\[ \|U^{j+1}(x) - U_i^{j+1}\| \leq C N^{-2}. \]

(40)

Using the bound in Lemma 10, we obtain 
\[ \|U^{j+1}(x) - U_i^{j+1}\| \leq C N^{-2}. \]

(41)

**Theorem 2.** Let \( u \) and \( U \) be the solution of (2)–(4) and (27), respectively. Then, the following uniform error bound holds

\[ \|u - U\| \leq C \left(N^{-2} + (\Delta t)^2 \right). \]

(42)

**Proof.** The combination of temporal and spatial error bounds gives the required result.

4. Numerical Results and Discussions

We considered three examples for validating the theoretical analysis. We computed the maximum absolute error and the rate of convergence of the scheme for different mesh lengths and the values of \( \epsilon \). For the test examples, the exact solution
is not given. The double mesh technique is applied to compute the maximum absolute error. Let \( U_{j,M} \) be the computed solution on \( N \) and \( M \) number of mesh points in space and time discretization, respectively, and \( U_{j,2M} \) be the computed solution on the double number of mesh points \( 2N \) and \( 2M \) by including the midpoints \( x_{i+1/2} = x_{i+1} + x_{i}/2 \) and \( t_{j+1/2} = t_{j+1} + t_{j}/2 \) into the mesh points. The maximum absolute error is calculated by using the formula

\[
E_{N,M}^e = \max_{i,j} \left| U_{j,M}^{i,N} - U_{j,2M}^{i,2N} \right|, 
\]

and the uniform error is calculated by using the formula

\[
E_{N,M} = \max_{\epsilon \in [0,1]} E_{N,M}^e. 
\]

The rate of convergence is calculated as

\[
r_{N,M}^e = \log_{2} \frac{E_{N,M}^e}{E_{2N,2M}^e},
\]

and the uniform rate of convergence is calculated as

\[
r_{N,M} = \log_{2} \frac{E_{N,M}}{E_{2N,2M}}.
\]

Example 1. [24] Consider the following problem with the retarded argument

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - \epsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{(1 + x^2)}{2} u(x,t) + u(x,t - \tau) &= t^3, \\
u(x,t) &= 0, (x,t) \in [0,1] \times [-1,0], \\
u(0,t) &= u(1,t) = 0, t \in [0,2].
\end{align*}
\]

Example 2. [25] Consider the following problem with the retarded argument

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - \epsilon \frac{\partial^2 u(x,t)}{\partial x^2} + x^2 u(x,t) &= t^3 - u(x,t - \tau), \\
u(x,t) &= 0, (x,t) \in [0,1] \times [-1,0], \\
u(0,t) &= u(1,t) = 0, t \in [0,2].
\end{align*}
\]

Example 3. [31] Consider the following problem with the retarded argument

Table 1: Example 1, the maximum absolute error and the rate of convergence of the scheme.

| \( \epsilon \) | \( M = 64 \) | \( M = 128 \) | \( M = 256 \) | \( M = 512 \) | \( M = 1024 \) |
|----|-----|-----|-----|-----|-----|
| 2^{-6} | 4.2439e-03 | 8.8457e-04 | 3.5017e-04 | 1.8423e-04 | 9.4623e-05 |
| 2^{-8} | 2.2925e-02 | 4.6993e-03 | 1.0750e-03 | 2.2823e-04 | 9.6683e-05 |
| 2^{-10} | 8.9288e-02 | 2.4218e-02 | 4.9511e-03 | 1.946e-03 | 2.7467e-04 |
| 2^{-12} | 8.9288e-02 | 2.4218e-02 | 4.9511e-03 | 1.946e-03 | 2.7467e-04 |
| \( E_{N,M} \) | 8.9288e-02 | 2.4218e-02 | 4.9511e-03 | 1.946e-03 | 2.7467e-04 |
| \( r_{N,M} \) | 1.8824 | 2.2903 | 2.0512 | 2.1208 | 2.2335 |

Table 2: Example 2, the maximum absolute error and the rate of convergence of the scheme.

| \( \epsilon \) | \( M = 64 \) | \( M = 128 \) | \( M = 256 \) | \( M = 512 \) | \( M = 1024 \) |
|----|-----|-----|-----|-----|-----|
| 2^{-6} | 3.9903e-03 | 8.1168e-04 | 4.1671e-04 | 2.2002e-04 | 1.1291e-04 |
| 2^{-8} | 2.2316e-02 | 4.5663e-03 | 1.0299e-03 | 2.2688e-04 | 1.1671e-04 |
| 2^{-10} | 8.8696e-02 | 2.3903e-02 | 4.8842e-03 | 1.1725e-03 | 2.6934e-04 |
| 2^{-12} | 2.1312e-01 | 9.2178e-02 | 2.4753e-02 | 5.0491e-03 | 1.2449e-03 |
| \( E_{N,M} \) | 2.1312e-01 | 9.2178e-02 | 2.4753e-02 | 5.0491e-03 | 1.2449e-03 |
| \( r_{N,M} \) | 1.2092 | 1.8968 | 2.2935 | 2.0200 | 2.0749 |
\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + (1.1 + x^2)u(x,t) &= t^3 - u(x,t - \tau), \\
u(x,t) &= 0, \quad (x,t) \in [0,1] \times [-1,0], \\
u(0,t) &= u(1,t) = 0, \quad t \in [0,2].
\end{align*}
\] (47)

In Tables 1, 2, and 3, respectively, we show the maximum point-wise absolute error and the rate of convergence of Examples 1, 2, and 3. In each column of the tables of 1–3 (or for each \( N \) and \( M \) as \( \varepsilon \to 0 \), the maximum absolute error becomes uniform; this shows that the convergence of the method is independent of the perturbation parameter. One can observe in these tables that the method converges with an order of convergence two for each values of the perturbation parameter. The comparison in Table 4 shows that the proposed method is more accurate than the scheme in

### Table 3: Example 3, the maximum absolute error and the rate of convergence of the scheme.

| \( \varepsilon \) | \( M = 64 \) | \( N = 20 \) | \( M = 128 \) | \( N = 40 \) | \( M = 256 \) | \( N = 80 \) | \( M = 512 \) | \( N = 160 \) | \( M = 1024 \) | \( N = 320 \) |
|-----------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \( 2^{-6} \)   | 7.2293e-03 | 1.6486e-03 | 3.5961e-04 | 1.4320e-04 | 7.3755e-05 |
|                 | 2.1326   | 2.1967   | 1.3284   | 0.95722   | 0.97366   |
| \( 2^{-8} \)   | 3.3115e-02 | 7.7596e-03 | 1.8118e-03 | 4.2553e-04 | 9.3382e-05 |
|                 | 2.0934   | 2.0986   | 2.090    | 2.1880    | 1.2673    |
| \( 2^{-10} \)  | 1.0114e-01 | 3.4597e-02 | 8.0418e-03 | 1.8959e-03 | 4.6360e-04 |
|                 | 1.5476   | 2.1051   | 2.0846   | 2.0319    | 2.0927e   |
| \( 2^{-12} \)  | 2.0856e-01 | 1.0410e-01 | 3.5372e-02 | 8.1868e-03 | 1.9433e-03 |
|                 | 1.0025   | 1.5573   | 2.1112   | 2.0748    | 2.0097    |

### Table 4: Example 1 comparison of \( \varepsilon \)-uniform error \( (E^{N,M}) \) and \( \varepsilon \)-uniform rate of convergence \( (r^{N,M}) \).

| Methods | \( M = 64 \) | \( N = 20 \) | \( M = 128 \) | \( N = 40 \) | \( M = 256 \) | \( N = 80 \) | \( M = 512 \) | \( N = 160 \) | \( M = 1024 \) | \( N = 320 \) |
|---------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| Proposed method | \( E^{N,M} \) | 8.9288e-02 | 2.4218e-02 | 4.9511e-03 | 1.1946e-03 | 2.7467e-04 |
|           | \( r^{N,M} \) | 1.8824   | 2.2903   | 2.0512   | 2.1208   | 2.2335   |
| Result in [24] | \( E^{N,M} \) | 5.3429e-02 | 2.7108e-02 | 1.3653e-02 | 6.8512e-03 | 3.4318e-03 |
|           | \( r^{N,M} \) | 0.9789   | 0.9895   | 0.9947   | 0.9974   | 0.9987   |

### Figure 1: Solution of Example 2, with boundary layer formation, (a) \( \varepsilon = 2^{-6} \) (b) \( \varepsilon = 2^{-12} \).
In Figures 1 and 2, the boundary layer formation on the solution is given as $\varepsilon$ goes small.

5. Conclusion

In this paper, the exponentially fitted tension spline method is developed for solving the singularly perturbed time delay reaction-diffusion problems. The solution of considered problems exhibits exponential boundary layers on the left and right side of the spatial domain. We employed the Crank–Nicolson method in temporal discretization and an exponentially fitted tension spline method in spatial discretization to set up the numerical method. The bounds and properties of the analytical solution are explored. An exponential fitting parameter is induced to stabilize the influence of the perturbation parameter on the discrete solution. The stability and convergence analysis of the method are discussed and proved. The performance of the method is illustrated through the numerical examples. It is proved that the method gives second-order convergence in space and in time.

Data Availability

No additional data are used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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