Aspects of spinorial geometry

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Abstract

We review some aspects of the spinorial geometry approach to the classification of supersymmetric solutions of supergravity theories. In particular, we explain how spinorial geometry can be used to express the Killing spinor equations in terms of a linear system for the fluxes and the geometry of spacetime. The solutions of this linear system express some of the fluxes in terms of the spacetime geometry and determine the conditions on the spacetime geometry imposed by supersymmetry. We also present some of the recent applications like the classification of maximally supersymmetric G-backgrounds in IIB, this includes the most general pp-wave solution preserving 1/2 supersymmetry, and the classification of $N = 31$ backgrounds in ten and eleven dimensions.
1 Introduction

Ten- and eleven-dimensional supergravities are the low energy effective theories of string and M-theory. As such they have proved instrumental for exploring the brane solitons of string/M-theory, string dualities, string/M-theory compactifications and more recently the AdS/CFT correspondence, see e.g. [1, 2, 3, 4]. All these applications have been mediated by the use of a special class of supergravity solutions, those solutions that admit Killing spinors, and thus preserve some of the spacetime supersymmetry. Most of these solutions have been constructed using Ansätze adapted to the requirements of physical problems. However, it has become increasingly clear that it will be advantageous to classify all supersymmetric supergravity solutions.

Our knowledge of the space of all supersymmetric supergravity solutions in ten and eleven dimensions is rather limited. It was a surprise for example that IIB supergravity admits an additional maximally supersymmetric solution [5, 6], which has found applications in the AdS/CFT correspondence [7]. The maximally supersymmetric solutions of $D=10$ and $D=11$ supergravities have only recently been classified [8, 9]. The Killing spinor equations (KSE) for $N=1$ backgrounds of $D=11$ supergravity have also been solved in [10, 11] using $G$-structures. It is expected that there are many more supersymmetric solutions in ten and eleven dimensions that remain to be uncovered. In lower dimensional supergravities much more progress has been made initiated by Tod in [12], see e.g. [13].

The classification of supersymmetric solutions is an attractive geometric problem. The Riemannian analogue is to find the manifolds that admit parallel spinors. These are identified as a consequence of Berger classification of irreducible Riemannian manifolds. In supergravity, the Berger classification cannot be applied because of the presence of fluxes.

Spinorial geometry, proposed in [14], is a direct and effective method of solving the KSE of supergravity theories. These consist of a parallel transport equation for the supercovariant connection and algebraic conditions derived from the vanishing condition of the supersymmetry transformations of the fermions. The spinorial geometry method is based on the following ingredients:

- The gauge symmetry of the Killing spinor equations.
- A description of spinors in terms of forms.
- An oscillator basis in the space of spinors.

The gauge transformations are those that leave the form of the KSE invariant. The gauge group of the KSE is typically a $Spin$ group. The holonomy of the supercovariant connections [15, 16, 17, 18] on the other hand, is typically an $SL$ group, which contains the gauge group, and does not preserve the form of the KSE. Backgrounds related by gauge transformations are identified. The gauge transformations can be used to orient the Killing spinors along some directions. This has proved instrumental for solving the KSE with small and near maximal number of supersymmetries.

$N$ denotes the number of Killing spinors a background admits.
An elegant way to represent spinors is in terms of (multi)-forms proposed by Cartan. This notation gives a geometric insight into the KSE and simplifies many of the computations.

The oscillator basis in the space of spinors is used to write the KSE, or their integrability conditions, as a linear system for the fluxes and the geometry of the spacetime. The linear system can then be solved to express some of the fluxes in terms of the geometry and to determine the conditions on the spacetime geometry imposed by supersymmetry [19, 20].

Effective use of the spinorial geometry method requires all three of the above ingredients. The method has, for example, been used to solve the KSE of $N = 1$ IIB backgrounds in [21, 22], and to considerably simplify the analogous computation for $N = 1$ backgrounds in eleven-dimensional supergravity [14], originally done in [10, 11]. It has also been applied to classify, under some mild assumptions, the geometry of all supersymmetric heterotic string [23] and $N = 2$ common sector backgrounds [24]. More recently, the spinorial geometry method has been adapted to near maximally supersymmetric backgrounds and it has been used to show that $N = 31$ IIB and $N = 31$ (simply connected) eleven-dimensional backgrounds are maximally supersymmetric [25, 26]. Some other applications can be found in [27, 28]. In particular, in [28] all the supersymmetric IIB backgrounds which admit the maximal number of Killing spinors invariant under some non-trivial Lie subgroup of $\text{Spin}(9,1)$ have been classified.

In this review, we first describe how spinorial geometry can be used to write the KSE for any number of Killing spinors in terms of a linear system for the fluxes and the geometry of spacetime, i.e. we give a systematic way to solve the KSE for any background. Then we present two applications. The first application is the classification of maximally supersymmetric IIB $G$-backgrounds. Some of these can be thought of as the vacua of IIB string compactifications. These also include the most general pp-wave solution preserving (at least) 16 supersymmetries. The other application is the classification of $N = 31$ IIB and $D = 11$ supergravity backgrounds. In particular, we show that in both cases the $N = 31$ backgrounds admit an additional Killing spinor and therefore they are (locally) isometric to maximally supersymmetric ones.

This review is organized as follows: In section two, we present the systematics of spinorial geometry. In section three, we give the classification of maximally supersymmetric IIB $G$-backgrounds. In section four, we investigate the backgrounds with $31$ supersymmetries, and in section five, we give our conclusions.

## 2 The linear system of Killing spinor equations

To explain the construction of the linear system associated to the KSE, we shall first give a description of spinors in terms of forms. Then we shall use this to construct the linear system. Similarly, we shall also discuss how the integrability conditions of the KSE reduce to a linear system for the bosonic supergravity field equations.

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2 It would be of interest to re-derive these results using $G$-structures.
2.1 Expansion in terms of basis of forms

A description of spinors in terms of forms \([29, 30, 31, 14]\) is advantageous because it gives a geometric insight into the nature of spinors. In turn this makes many computations necessary for the classification of supersymmetric backgrounds straightforward, e.g. the construction of the linear systems that we explain below and the computation of spinor isotropy groups.

As a paradigm, let us construct the Majorana representation of \(Spin(10, 1)\). For this we begin with the spinor representation of \(Spin(10)\). Let \(U = \mathbb{C} < e_1, \ldots, e_5 >\) be a complex vector space spanned by the orthonormal vectors \(e_1, \ldots, e_5\) and equipped with the Hermitian inner product

\[
< z^i e_i, w^j e_j > = \sum_{i=1}^{10} \bar{z}^i w^i , \tag{2.1}
\]

where \(\bar{z}^i\) is the standard complex conjugate of \(z^i\) in \(U\). The space of Dirac \(Spin(10)\) spinors is \(\Delta_c = \Lambda^*(U)\). The above inner product can be easily extended to \(\Delta_c\) and it is called the Dirac inner product on the space of spinors. The gamma matrices act on \(\Delta_c\) as

\[
\Gamma_i \eta = e_i \wedge \eta + e_i \eta , \quad i \leq 5 , \quad \Gamma_{5+i} \eta = i e_i \wedge \eta - i e_i \eta , \quad i \leq 5 , \tag{2.2}
\]

where \(e_i \wedge\) is the adjoint of \(e_i \wedge\) with respect to \(\langle,\rangle\). The linear maps \(\Gamma_i\) are Hermitian with respect to the inner product \(\langle,\rangle\), \(\langle \Gamma_i \eta, \theta \rangle = \langle \eta, \Gamma_i \theta \rangle\), and satisfy the Clifford algebra relations \(\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2 \delta_{ij}\). It will be convenient to define a Hermitian basis for the \(\Gamma_i\) as

\[
\Gamma_\alpha = \frac{1}{\sqrt{2}} (\Gamma_\alpha + i \Gamma_{\alpha+5}) , \quad \alpha = 1, \ldots, 5 , \tag{2.3}
\]

and \(\Gamma^\alpha = g^{\alpha\beta} \Gamma_\beta\), where \(g^{\alpha\beta} = \delta_{\alpha\beta}\). The Dirac \(Spin(10, 1)\) representation, \(\Delta_c = \Lambda^*(U)\), is constructed by identifying \(\Gamma_0 = \Gamma_1 \ldots \Gamma_5\), where \(\Gamma_5 = \Gamma_{10}\).

The Majorana spinor inner product on \(\Delta_c\) is

\[
B(\eta, \theta) = < B(\bar{\eta}), \theta > , \tag{2.4}
\]

where \(\bar{\eta}\) is the standard complex conjugate of \(\eta\) in \(\Lambda^*(U)\) and \(B = \Gamma_0 \ldots \Gamma_5\). It is easy to verify that \(B(\eta, \theta) = - B(\theta, \eta), \ i.e.\ B\) is skew-symmetric. The Majorana condition can be easily imposed by setting

\[
\bar{\eta} = \Gamma_0 B(\eta) , \quad \eta \in \Delta_c . \tag{2.5}
\]

The Majorana spinors \(\Delta_{32}\) of eleven-dimensional supergravity are those spinors in \(\Delta_c\) which obey the Majorana condition (2.5). The \(Pin(10)\)-invariant inner product \(B\) induces a \(Spin(10, 1)\) invariant inner product on \(\Delta_{32}\) which is the usual skew-symmetric inner product on the space of spinors of eleven-dimensional supergravity.
A consequence of the construction above is that any Majorana spinor of \( \text{Spin}(10,1) \) can be written as
\[
\epsilon = f(1 + e_{12345}) + ig(1 - e_{12345}) + \sqrt{2}u^i(e_i + \frac{1}{4!}\epsilon_{ijklm}e_{jklm}) \\
+ i\sqrt{2}v^i(e_i - \frac{1}{4!}\epsilon_{ijklm}e_{jklm}) + \frac{1}{2}w^{ij}(e_{ij} - \frac{1}{3!}\epsilon_{ijklm}e_{jklm}) \\
+ \frac{1}{2}z^{ij}(e_{ij} + \frac{1}{3!}\epsilon_{ijklm}e_{jklm}),
\]
where \( i, j, \ldots = 1, \ldots, 5 \) and \( f, g, u^i, v^i, w^{ij} \) and \( z^{ij} \) are real spacetime functions. Clearly, \( \epsilon \) can also be expressed in the Hermitian basis
\[
\epsilon = f^I\sigma_I, \quad \sigma_I = (1, e_i, e_{ij}, e_{ijk}, e_{ijkl}, e_{12345}),
\]
where \( f^I \) are complex functions. The six types of spinors \( \sigma_I \) correspond to the irreducible representations of \( U(5) \) on \( \Lambda^*(\mathbb{C}^5) \). This basis of spinors, for reasons that we shall not explain here, is referred to as a timelike basis [19].

Backgrounds that are related by a gauge transformation of the KSE are identified. Because of this, if two sets of Killing spinors are related by such a gauge transformation, then they give rise to the same supersymmetric background. So to classify the different supersymmetric backgrounds, one has to identify the inequivalent classes of Killing spinors up to gauge transformations. In turn, this leads to choosing representatives of the orbits of the gauge group in the space of spinors. Considerable simplification can be made in the various computations, if one chooses carefully such representatives. For example, there are two types of orbits of \( \text{Spin}(10,1) \) in \( \Delta_{32} \), one has stability subgroup \( \text{SU}(5) \) while the other has stability subgroup \( (\text{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R} \) [32, 33]. So there are two different types of geometries that can occur in \( N = 1 \) eleven-dimensional backgrounds. A representative of the \( \text{SU}(5) \) orbit is
\[
\epsilon = f(1 + e_{12345}).
\]
Compared to the general spinor in (2.6), this representative is much simpler. In turn, the linear system associated with (2.8) is rather simple and can be straightforwardly solved [14]. The use of the gauge group is essential for the analysis of solutions preserving a large number of supersymmetries as well.

The same analysis can be done for the integrability conditions \( \mathcal{I}\epsilon = 0 \) of a Killing spinor \( \epsilon \). Since these conditions are linear, we have
\[
\mathcal{I}\epsilon = f^I\mathcal{I}\sigma_I.
\]
Therefore to find which field equations are determined by the KSE, it suffices to compute \( \mathcal{I}\sigma_I \). In this way the integrability conditions give rise to a linear system in terms of the field equations. The solution to this linear system shows which field equations are already satisfied or related to others, and which field equations are independent and still have to be imposed. See [19] for the explicit expressions for each \( e_{i_1 \ldots i_4} \) and applications of this linear system in some \( N = 1, 2 \) and 4 examples.

The Majorana representation of \( \text{Spin}(10,1) \) can also be constructed from the spinor representations of \( \text{Spin}(9,1) \). This leads to another (null) basis in the space of \( \text{Spin}(10,1) \)
spinors from the timelike basis constructed above. One advantage of the null basis is that one can easily investigate the cases where spinors are in the \((\text{Spin}(7) \ltimes \mathbb{R}^5) \times \mathbb{R}\) orbit. The null basis is also the preferred basis to investigate the KSE of IIB and type I supergravities. Details of the construction of this basis and some applications can be found in \([21, 20]\).

2.2 Systematics of Killing spinor equations

To explain the construction of the linear system associated with the KSE, we observe, using (2.7), that

\[ \mathcal{D}_A f = \partial_A f^I \sigma_I + f^I \mathcal{D}_A \sigma_I, \quad (2.10) \]

where \(\mathcal{D}\) is the supercovariant connection of eleven-dimensional supergravity. Thus the KSE reduce to the evaluation of \(\mathcal{D}\) on the basis spinors \(\sigma_I\). So it remains to compute these and express the result in the basis (2.7). To do this, first write

\[ \sigma_{i_1 \ldots i_L} = e_{i_1 \ldots i_L} \frac{1}{2!} \Gamma_i \ldots \Gamma_L \sigma, \quad (2.11) \]

where the indices \(i_1, \ldots, i_L\) pick out \(I\) holomorphic indices (with \(0 \leq I \leq 5\)) from the range \(\alpha = 1, \ldots, 5\). It will be convenient to distinguish between the indices that do appear in the basis element (2.11) and those that do not: we split the holomorphic indices \(\alpha\) into the indices \(a = (i_1, \ldots, i_L)\) and the remaining \(5 - I\) indices \(p\), and similarly for the anti-holomorphic indices \(\bar{\alpha}\). Note that \(\Gamma^a\) and \(\Gamma^p\) annihilate the spinor \(e_{i_1 \ldots i_L}\) while \(\Gamma^a\) and \(\Gamma^\bar{p}\) act as creation operators. For this reason it is useful to define the new indices \(\rho, \sigma, \tau\) consisting of the combination

\[ \rho = (\bar{a}_1, \ldots, \bar{a}_I, p_1, \ldots, p_{5-I}) \), \quad \bar{\rho} = (a_1, \ldots, a_I, \bar{p}_1, \ldots, \bar{p}_{5-I}), \quad (2.12) \]

where \(\Gamma^p\) and \(\Gamma^\bar{p}\) are the annihilation and creation operators, respectively, for the spinor \(e_{i_1 \ldots i_L}\). Note that the indices \(\alpha, \rho\) are identical for \(I = 0\), i.e. for the spinor \(1\). For \(I > 0\), i.e. for any other basis element, these indices differ.

In terms of the basis \(\{e_{i_1 \ldots i_L}, \Gamma^{a_1} e_{i_1 \ldots i_L}, \ldots, \Gamma^{a_5} e_{i_1 \ldots i_L}\}\), the supercovariant derivative with \(A = 0\) can be expanded in the following contributions:

\[ \mathcal{D}_0 e_{i_1 \ldots i_L} = \left[ \frac{1}{2} \Omega_0, \tau \right] + (\mathbf{1})^I + \frac{\mathcal{F}_{\tau_1, \tau_2}}{24} e_{i_1 \ldots i_L} + \left[ (\mathbf{1})^I + \frac{\mathcal{F}_{\sigma_1, \sigma_2}}{24} \right] \Gamma^{a_1} e_{i_1 \ldots i_L} + \frac{1}{6} G_{\sigma_1, \sigma_2, \sigma_3} \Gamma^{a_1} e_{i_1 \ldots i_L} + \frac{1}{35} G_{\sigma_1, \sigma_2, \sigma_3} \Gamma^{a_1} e_{i_1 \ldots i_L} + \frac{1}{288} F_{\sigma_1 \ldots \sigma_4} \Gamma^{a_1} e_{i_1 \ldots i_L}. \quad (2.14) \]

The \(i_1, \ldots, i_L\) should not be thought of as indices in this context, but rather as fixed labels for a particular spinor.

Note that in this basis \(e_{i_1 \ldots i_L}\) is the Clifford algebra vacuum.
Observe that the component $\Gamma^a_{\bar{a}a} e_{i_1 \cdots i_f}$ vanishes. Similarly, the expression for $A = \rho$ read

$$
\begin{align*}
\mathcal{D}_\rho e_{i_1 \cdots i_f} &= \left[ \frac{1}{2} \Omega^\rho_{\rho, \sigma} + (-1)^f \frac{i}{2} G_{\rho, \sigma} \right] e_{i_1 \cdots i_f} \\
&+ \left[ (-1)^f \frac{i}{2} \Omega^\rho_{\rho, 0} + \frac{1}{4} F_{\rho \tau} \right] e_{i_1 \cdots i_f} \\
&+ \left[ \frac{1}{2} \Omega^\rho_{\rho, \sigma_1} \right] e_{i_1 \cdots i_f} + \left[ \frac{1}{4} \Omega^\rho_{\rho, \sigma_2} + \left[ (-1)^f \frac{i}{12} g_{\rho \sigma_1} G_{\sigma_2} \right] \right] e_{i_1 \cdots i_f} \\
&+ \left[ \frac{1}{24} F_{\rho \sigma_1 \sigma_2 \sigma_3} - \frac{1}{24} g_{\rho \sigma_1} F_{\sigma_2 \sigma_3} \right] \Gamma^\rho_1 \bar{\sigma}_2 \bar{\sigma}_3 e_{i_1 \cdots i_f} \\
&+ \left[ (-1)^f \frac{i}{8} g_{\rho \sigma_1} G_{\sigma_2} + \left[ (-1)^f \frac{i}{9} g_{\rho \sigma_1} G_{\sigma_2} \right] \right] e_{i_1 \cdots i_f} \\
&+ \left[ -\frac{1}{288} g_{\rho \sigma_1} F_{\sigma_2 \sigma_3} \right] \Gamma^\rho_1 \bar{\sigma}_2 \bar{\sigma}_3 e_{i_1 \cdots i_f}.
\end{align*}
$$

Finally, for $A = \bar{\rho}$ we find

$$
\begin{align*}
\mathcal{D}_{\bar{\rho}} e_{i_1 \cdots i_f} &= \left[ \frac{1}{2} \Omega^\rho_{\rho, \sigma} + (-1)^f \frac{i}{2} G_{\rho, \sigma} \right] e_{i_1 \cdots i_f} + \left[ (-1)^f \frac{i}{2} \Omega^\rho_{\rho, 0} + \frac{1}{4} F_{\rho \tau} \right] e_{i_1 \cdots i_f} \\
&+ \left[ \frac{1}{4} \Omega^\rho_{\rho, \sigma_1} \right] e_{i_1 \cdots i_f} + \left[ \frac{1}{4} \Omega^\rho_{\rho, \sigma_2} + \left[ (-1)^f \frac{i}{24} G_{\rho \sigma_1 \sigma_2} \right] \right] e_{i_1 \cdots i_f} \\
&+ \left[ \frac{1}{24} F_{\rho \sigma_1 \sigma_2 \sigma_3} \right] \Gamma^\rho_1 \bar{\sigma}_2 \bar{\sigma}_3 e_{i_1 \cdots i_f}.
\end{align*}
$$

Observe that the components along $\Gamma^a_{\bar{a}a} e_{i_1 \cdots i_f}$ and $\Gamma^a_{\bar{a}a} e_{i_1 \cdots i_f}$ vanish. It is convenient to convert the above expressions from basis (2.13) to the”canonical” basis (2.7). For this, we expand the products of $\Gamma^\rho$ matrices, which are creation operators on $e_{i_1 \cdots i_f}$, into a sum of products of $\Gamma^\rho$ and $\Gamma^\rho$ matrices, which are annihilation and creation operators, respectively, on 1. Then we act on $e_{i_1 \cdots i_f}$ with the annihilation operators. In particular, we have

$$
\begin{align*}
\mathcal{D}_{\rho} e_{i_1 \cdots i_f} &= \sum_k [\mathcal{D}_{\rho} e_{i_1 \cdots i_f}] \bar{\rho}_1 \cdots \bar{\rho}_k \Gamma^{\rho_1 \cdots \rho_k} e_{i_1 \cdots i_f} \\
&= \sum_k \sum_{m+n=k} \frac{k!}{m! n!} [\mathcal{D}_{\rho} e_{i_1 \cdots i_f}]_{a_1 \cdots a_m \bar{\rho}_1 \cdots \bar{\rho}_n} \Gamma^{a_1 \cdots a_m} \bar{\rho}_1 \cdots \bar{\rho}_n e_{i_1 \cdots i_f} \\
&= \sum_k \sum_{m+n=k} \frac{k!}{m! n!} (-1)^{m/2 + n I} e^{a_1 \cdots a_m} \bar{a}_{m+1} \cdots \bar{a}_I \\
&\quad \left[ \mathcal{D}_{\rho} e_{i_1 \cdots i_f} \right]_{a_1 \cdots a_m \bar{\rho}_1 \cdots \bar{\rho}_n} \Gamma^{a_{m+1} \cdots a_I} \bar{\rho}_1 \cdots \bar{\rho}_n 1,
\end{align*}
$$

with the obvious restrictions $m \leq I$ and $n \leq 5 - I$ and the convention that $e_{i_1 \cdots i_f} = 1$. Using the expressions (2.13), (2.15) and (2.16) for the components of $\mathcal{D}_{\rho} e_{i_1 \cdots i_f}$ in the basis (2.13) which appear in square brackets in (2.17), one can easily compute the components of $\mathcal{D}_{\rho} e_{i_1 \cdots i_f}$ in the canonical basis (2.7). The explicit expressions for the different basis elements are given in (19).

Substituting (2.17) into (2.10) and setting each component in the basis (2.7) equal to zero, one derives a linear system with variables the spin connection $\Omega$ of the geometry and the fluxes that appear in the supercovariant derivative, in this case the four-form $F$. In addition there are terms consisting of the differentials of the functions $f^I$ that define the Killing spinors. Solving this linear system is equivalent to solving the KSE for any number of spinors. In particular, one can derive all the conditions on a background imposed by supersymmetry.
3 Maximally supersymmetric $G$-backgrounds

Amongst the various supersymmetric IIB backgrounds are those for which the Killing spinors are invariant under the action of some proper Lie subgroup $G$ of $Spin(9,1)$. Let $\Delta^G$ be the space of $G$-invariant spinors. If $\{\eta_p : p = 1, \ldots, m\}$ is a maximal set of linearly independent Majorana-Weyl spinors in $\Delta^G$, a basis for $\Delta^G$ is given by $\{\eta_i : i = 1, \ldots, 2m\} = \{\eta_p, i\eta_p : p = 1, \ldots, m\}$, so $\Delta^G$ is $2m$-dimensional.

We shall consider solutions for which the Killing spinors span $\Delta^G$; such solutions are called maximally supersymmetric $G$-backgrounds. These types of solutions were first investigated in [22], where the KSE corresponding to $G = Spin(7) \ltimes \mathbb{R}^8$ ($\dim \Delta^G = 2$), $G = SU(4) \ltimes \mathbb{R}^8$ ($\dim \Delta^G = 4$) and $G = G_2$ ($\dim \Delta^G = 4$) were examined. The integrability conditions of these examples were then analyzed in [20]. The remaining examples of maximally-supersymmetric $G$-backgrounds with $G = Sp(2) \ltimes \mathbb{R}^8$ ($\dim \Delta^G = 6$), $G = (SU(2) \times SU(2)) \ltimes \mathbb{R}^8$ ($\dim \Delta^G = 8$), $G = \mathbb{R}^8$ ($\dim \Delta^G = 16$), $G = SU(3)$ ($\dim \Delta^G = 8$), $G = SU(2)$ ($\dim \Delta^G = 16$) and $G = \{1\}$ ($\dim \Delta^G = 32$) were later constructed in [28].

For maximally-supersymmetric $G$-backgrounds, the Killing spinors $\epsilon_i$ are given by

$$\epsilon_i = \sum_{r=1}^{2m} f_{ir} \eta_r, \quad i = 1, \ldots, 2m,$$

where $f_{ir}$ is a $2m \times 2m$ invertible real matrix, whose components $f_{ij}$ in general are not constant, but depend on the spacetime co-ordinates. Using these properties of $f$, it follows that the IIB KSE can be written as

$$\sum_{j=1}^{2m} (f^{-1} \partial_M f)_{ij} \eta_j + D_M \eta_i = 0,$$

$$P_A \Gamma^A \eta_i + \frac{i}{24} G_{ABC} \Gamma^{ABC} \eta_i = 0,$$

for $i = 1, \ldots, 2m$, where $D_M$ is the supercovariant derivative. First consider the algebraic constraint given in (3.2). Evaluating this constraint for $i = \ell$ and $i = \ell + m$ for $\ell = 1, \ldots, m$ we obtain

$$P_A \Gamma^A \eta_\ell = 0,$$

$$G_{ABC} \Gamma^{ABC} \eta_\ell = 0.$$

(3.3)

Next, by using the constraint on $G$ given in (3.3), note that the supercovariant derivative acting on $\eta_i, i = 1, \ldots, 2m$, simplifies to give

$$D_M \eta_\ell = \nabla_M \eta_\ell + \frac{i}{48} F_{MN1N2N3N4} \Gamma^{N1N2N3N4} \eta_\ell + \frac{1}{8} G_{MAB} \Gamma^{AB} \eta_\ell,$$

$$D_M \eta_{\ell+m} = i \nabla_M \eta_\ell - \frac{i}{48} F_{MN1N2N3N4} \Gamma^{N1N2N3N4} \eta_\ell - \frac{1}{8} G_{MAB} \Gamma^{AB} \eta_\ell,$$

(3.4)

for $\ell = 1, \ldots, m$, where $\nabla_M = \partial_M - \frac{i}{4} Q_M + \frac{i}{4} \Omega_{MAB} \Gamma^{AB}$. Substituting these expressions back into the gravitino Killing spinor equation in (3.2) and evaluating for $i = \ell$ and
\[ i = \ell + m \text{ we find the conditions} \]
\[
\frac{1}{2} \sum_{j=1}^{2m} (f^{-1} \partial_M f)_{\ell j} \eta_j - i \sum_{j=1}^{2m} (f^{-1} \partial_M f)_{(\ell+m)j} \eta_j \\
+ \nabla_M \eta_\ell + \frac{i}{48} \Gamma^{N_1 \ldots N_4} F_{N_1 \ldots N_4} \eta_\ell = 0 ,
\]
\[
\sum_{j=1}^{2m} (f^{-1} \partial_M f)_{\ell j} \eta_j + i \sum_{j=1}^{2m} (f^{-1} \partial_M f)_{(\ell+m)j} \eta_j + \frac{1}{4} G_{MBC} \Gamma^{BC} \eta_\ell = 0 . \quad (3.5)
\]

Hence we observe that the KSE can be split into equations involving only \( P \), or \( G \), or \( \Omega \) and \( F \). The fact that the equations factorize in this fashion means that the geometry and fluxes of the solutions are significantly constrained. It has been shown that the integrability conditions of the KSE of maximally supersymmetric \( G \)-backgrounds also factorize \[20\].

On evaluating the constraints \((3.5)\) together with \((3.3)\), one obtains a parallel transport equation for the matrix \( f \) of the form
\[
(f^{-1} \partial_M f)_{ij} + (C_M)_{ij} = 0 , \quad (3.6)
\]
where \( C \) is a connection whose components are obtained from the spacetime Levi-Civita connection and the fluxes of the supergravity theory. This connection is the restriction of the supergravity supercovariant connection to the subbundle of the Killing spinors. A necessary condition for the condition \((3.6)\) to admit a solution is that the curvature \( F(C) \) associated with \( C \) vanish
\[
F(C) \equiv dC - C \wedge C = 0 . \quad (3.7)
\]
In general, it is not possible to choose \( f \) to be the identity matrix, although it is possible to pre-multiply \( f \) by an arbitrary invertible constant matrix, as \((3.6)\) is invariant under the transformation \( f \rightarrow gf \), where \( g \) is an invertible constant \( 2m \times 2m \) matrix.

It has been shown in \[28\] that the form of the maximally supersymmetric \( G \)-background solutions depends on whether \( G \) is compact or non-compact. In the non-compact cases with \( G = K \ltimes \mathbb{R}^8 \) for \( K = Spin(7), SU(4), Sp(2), SU(2) \times SU(2) \) or \( K = \{1\} \), the space-time geometry always admits a null vector field \( X = e^- \) which is parallel with respect to the Levi-Civita connection, \( \nabla X = 0 \); and the holonomy of the Levi-Civita connection is contained in \( K \ltimes \mathbb{R}^8 \), \( \text{hol}(\nabla) \subseteq K \ltimes \mathbb{R}^8 \). Co-ordinates \( u, v, y^I \) for \( I = 1, \ldots, 8 \) can be chosen such that \( X = \frac{\partial}{\partial u} \), and the spacetime metric can be written as
\[
\text{ds}^2 = 2dv(du + Vdv + n_Idy^I) + \gamma_{IJ}dy^Idy^J , \quad (3.8)
\]
where \( V = V(v, y), n_I = n_I(v, y), \gamma_{IJ} = \gamma_{IJ}(v, y) \), i.e. the spacetime is a pp-wave. The above holonomy condition implies that the holonomy of the Levi-Civita connection of the transverse 8-manifold, \( Y_8 \), defined by \( u, v = \text{const} \), is contained in \( K \). In addition, the fluxes take the form
\[
P = P_e^-, \quad G = e^- \wedge L, \quad F = e^- \wedge M , \quad e^- = dv , \quad (3.9)
\]
where $L$ is a 2-form on $Y^8$, and $M$ is a self-dual 4-form on $Y^8$ and may depend on both $y^I, v$.

For example, the fluxes for solutions with $G = \text{Spin}(7) \ltimes \mathbb{R}^8$ are

$$G = e^- \wedge L^{\text{spin}(7)}, \quad F = e^- \wedge (\frac{1}{14} Q^- \psi + M^{27}) ,$$

where $L^{\text{spin}(7)} \in \text{spin}(7)$, $M^{27}$ lies in the 27 irreducible representation in the decomposition of 4-forms with respect to $\text{Spin}(7)$ and $\psi$ is the $\text{Spin}(7)$-invariant four-form.

The fluxes for the case $G = SU(4) \ltimes \mathbb{R}^8$ are given by

$$G = e^- \wedge (L^{\text{su}(4)} + \ell \omega) , \quad F = e^- \wedge (-\frac{1}{12} Q^- \omega \wedge \omega + \text{Re}(m \chi) + \tilde{M}) ,$$

where $L^{\text{su}(4)} \in \text{su}(4)$, $\omega$ is the Hermitian $(1,1)$ form, $\chi$ is the $SU(4)$-invariant $(4,0)$ form and $\tilde{M}$ is a traceless $(2,2)$ form. The remaining fluxes for non-compact $G$ are presented in [28].

In addition a straightforward consequence of the results of [28] is that the most general solution in the $\mathbb{R}^8$ case, assuming that the transverse metric does not depend on $v$, is

$$ds^2 = 2dv(du + Vdv) + \delta_{IJ}dy^I dy^J , \quad P = P_-(v)e^-, \quad G = e^- \wedge L , \quad F = e^- \wedge M ,$$

where

$$L = \frac{1}{2} L_{IJ}(v)dy^I \wedge dy^J , \quad M = \frac{1}{4} M_{I_1...I_4}(v)dy^{I_1} \wedge \cdots \wedge dy^{I_4} ,$$

$$V = \frac{1}{2} A_{IJ}(v)y^I y^J + B_I(v)y^I + H(y, v) , \quad \partial_I^2 H(y, v) = 0 ,$$

$$\text{tr} A = -\frac{1}{6} M_{I_1...I_4} M^{I_1...I_4} - \frac{1}{4} L_{IJ}^* L^{IJ} - 2P_- P^* ,$$

i.e. $A_{IJ}, B_I, P, G$ and $F$ depend only on $v$ but otherwise unrestricted, $M$ is self-dual in $\mathbb{R}^8$ and $H$ is a harmonic function in $\mathbb{R}^8$, e.g. $H = b(v) + \sum_k \frac{a_k(v)}{|y - y_k(v)|^8}$. This is the most general pp-wave solution of IIB supergravity, under the assumption mentioned above, which preserves at least 16 supersymmetries.

Next let us turn to backgrounds with Killing spinors invariant under compact $G$ groups. There are several cases for each choice of group $G$. All such backgrounds can be found in [28]. The simplest case is for $G = G_2$ where

$$ds^2 = ds^2(\mathbb{R}^{2,1}) + ds^2(Y^7) , \quad G = P = F = 0 ,$$

where $Y^7$ is a $G_2$-holonomy manifold.

As another example consider $G = SU(3)$. The spacetime geometry is the product of a four-dimensional Lorentzian symmetric space with a Calabi-Yau manifold $Y^6$. The

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5 The recently announced solution in [34] is included.
6 We can without loss of generality set $n = 0$ in the corresponding solution of [28].
supersymmetry conditions on the fluxes imply that $P = G = 0$. There are then three sub-cases to consider. First $M = \text{AdS}_2 \times S^2 \times Y^6$, and the metric and fluxes are

\[
\begin{align*}
  ds^2 &= ds^2(\text{AdS}_2) + ds^2(S^2) + ds^2(Y^6), \\
  ds^2(\text{AdS}_2) &= -(e^0)^2 + (e^1)^2, \\
  ds^2(S^2) &= (e^5)^2 + (e^6)^2, \\
  F &= \frac{1}{2\sqrt{2}}[H^1 \wedge \text{Re}\chi - H^2 \wedge \text{Im}\chi], \\
  H^1 &= \lambda_1 e^0 \wedge e^1 + \lambda_2 e^5 \wedge e^6, \\
  H^2 &= -\lambda_1 e^5 \wedge e^6 + \lambda_2 e^0 \wedge e^1, \\
  (3.15)
\end{align*}
\]

for constants $\lambda_1, \lambda_2$, and the scalar curvature of $\text{AdS}_2$ and $S^2$ are $R_{\text{AdS}_2} = -R_{S^2} = -4(\lambda_1^2 + \lambda_2^2)$.

In the second case $M = CW_4(-2\mu^21) \times Y^6$, and the metric and fluxes are

\[
\begin{align*}
  ds^2 &= ds^2(CW_4) + ds^2(Y^6), \\
  F &= \frac{1}{2\sqrt{2}}[H^1 \wedge \text{Re}\chi - H^2 \wedge \text{Im}\chi], \\
  H^1 &= \mu e^- \wedge e^1, \\
  H^2 &= \mu e^- \wedge e^6, \\
  (3.16)
\end{align*}
\]

where $CW_4$ is the four-dimensional Cahen-Wallach space.

In the third case, $M = \mathbb{R}^{3,1} \times Y^6$, and the metric and fluxes are

\[
\begin{align*}
  ds^2(M) &= ds^2(\mathbb{R}^{3,1}) + ds^2(Y^6), \\
  F &= 0. \\
  (3.17)
\end{align*}
\]

Further details concerning these solutions as well as the solutions for $G = SU(2)$ can be found in [28]. We remark that a key part of the computation is the understanding of the flatness condition of the $C$ connection. It is not always possible to set $f = 1$. This can only happen whenever $C$ takes values in a subalgebra of $\text{Spin}(9,1)$ which preserves $\Delta^G$. This is the case for $G = G_2$.

## 4 Near maximal supersymmetry

The method of spinorial geometry has recently been adapted to backgrounds with near maximal number of supersymmetries [25, 26]. The basic idea is that instead of specifying $N$ Killing spinors we can specify $N_{\text{max}} - N$ “normal” spinors, where $N_{\text{max}}$ is the maximal number of supersymmetries for the theory under study, e.g. $N_{\text{max}} = 32$ for IIB and $D = 11$ supergravity. The gauge symmetry can then be used to simplify the form of the normal spinor(s), which in turn also simplifies the expressions for the Killing spinors. For $N = 31$ we need to study 3 cases, corresponding to the number of orbits of the normal spinors under the action of the gauge group, in type IIB and 2 cases in $D = 11$ supergravity. Essential for the definition of normal spinors is the existence of a non-degenerate pairing between spinors and normal spinors, see [25] for details.
4.1 \( N = 31 \) in IIB supergravity

This case simplifies as it turns out to be enough to study the algebraic KSE as we will see below. The Killing spinors can be written as

\[
\epsilon_r = \sum_{i=1}^{32} f^i_r \eta_i , \quad r = 1, \ldots, N ,
\]

(4.1)

where \( \eta_p, \quad p = 1, \ldots, 16 \), is a basis in the space of positive chirality Majorana-Weyl spinors, \( \eta_{6+p} = \eta_p \) and \( f \) are real spacetime functions. Consider first a normal spinor in the orbit with stability subgroup \( \text{Spin}(7) \ltimes \mathbb{R}^8 \). A representative for this orbit is

\[
\nu = (n + im)(e_5 + e_{1234}) .
\]

(4.2)

Rewriting (4.1) as

\[
\epsilon_r = f^i_r (1 + e_{1234}) + f^{17}_r i (1 + e_{1234}) + f^k_r \eta_k ,
\]

(4.3)

where \( \eta_k \) are the remaining basis elements, and using the orthogonality condition between \( \nu \) and \( \epsilon_r \), we get

\[
\epsilon_r = \frac{f^{17}_r}{\eta} (m + in)(1 + e_{1234}) + f^k_r \eta_k .
\]

(4.4)

Note that the orthogonality relation has allowed us to eliminate one function from each Killing spinor. Substituting the form of \( \epsilon_r \) into the algebraic KSE and using that the rank of the matrix \( (f^i_r) \) is 31, one finds

\[
P_M \Gamma^M C \ast [(m + in)(1 + e_{1234})] + \frac{1}{24} G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} (m + in)(1 + e_{1234}) = 0 ,
\]

\[
P_M \Gamma^M \eta_p = 0 , \quad G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} \eta_p = 0 , \quad p = 2, \ldots, 16 .
\]

(4.5)

The factorization of the constraints on \( P \) and \( G \) is similar to that occurring for the maximal \( G \)-backgrounds as explained in the previous section. Noting that \( \eta_p \) is only annihilated by either \( \Gamma^+ \) or \( \Gamma^- \) implies that the only non-vanishing component of \( P \) is either \( P_+ \) or \( P_- \), respectively. Since both types of spinors occur \( P = 0 \). Using that \( P = 0 \), we find that \( G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} \eta_i = 0 \) for all the basis spinors \( \eta_i \), which implies that \( G = 0 \). If both \( P \) and \( G \) vanish the gravitino KSE is linear over the complex numbers implying that there is always an even number of Killing spinors, hence \( N = 31 \) implies \( N = 32 \). The analysis for the other two orbits is analogous [25]. To our knowledge this is the first example which demonstrates that there are restrictions on the number of supersymmetries of backgrounds in a maximal supergravity theory. A similar result was found afterwards for IIA supergravity [35].

4.2 \( N = 31 \) in 11D supergravity

In \( D = 11 \) there is no algebraic KSE which makes the problem much harder to analyze. Instead we have to solve the parallel transport equation

\[
\mathcal{D}_A \epsilon_r = 0 , \quad r = 1, \ldots, 31 .
\]

(4.6)
It is convenient to study the integrability condition

\[ \mathcal{R}_{AB} \epsilon_r = [\mathcal{D}_A, \mathcal{D}_B] \epsilon_r = 0, \quad (4.7) \]

since the constraints from satisfying the field equations and Bianchi identities can easily be incorporated. In particular, \( \Gamma^N \mathcal{R}_{MN} \) is a linear combination of field equations and Bianchi identities and therefore necessarily vanishes. If one can show that \( \mathcal{R}_{AB} = 0 \), then the backgrounds will be (locally) maximally supersymmetric [8].

There are two ways of solving the integrability conditions. The first is to expand the supercovariant curvature in (a basis of) gamma matrices

\[ \mathcal{R}_{MN,ab} = \sum_{k=1}^{5} \frac{1}{k!} (T_{MN}^k)_{A_1A_2...A_k} (\Gamma^{A_1A_2...A_k})_{ab}, \quad (4.8) \]

and let the gamma matrices act on \( \epsilon_r \) and read off the conditions on \( T \). The second, more economical, way is to make use of a spinorial basis and write the supercovariant curvature as

\[ \mathcal{R}_{MN,ab} = u_{MN}^{\nu} \eta_{r,a} \nu_b, \quad (4.9) \]

where \( \nu \) is the normal spinor, \( \epsilon_r = f^r_s \eta_s \) and \( \eta_r \) is a basis in the space of Killing spinors. \( \mathcal{R}_{MN} \) can thus be written in terms of 31 two-forms \( u_{MN}^{\nu} \) which is consistent with the fact that the stability subgroup of 31 spinors in \( SL(32, \mathbb{R}) \) is \( \mathbb{R}^{31} \). \( T \) and \( u \) can easily be related by contracting the expressions above with gamma matrices.

After solving the integrability conditions \( \mathcal{R}_{MN} \) is expressed in terms of the 31 two-forms \( u_{MN}^{\nu} \), which are further constrained by the field equations and Bianchi identities. In particular the requirement that \( \Gamma^N \mathcal{R}_{MN} \) has to vanish, as linear combination of field equations and Bianchi identities, implies

\[ (T^1_{MN})^N = 0, \quad (T^2_{MN})^P = 0, \quad (T^1_{MP_1})_{P_2} + \frac{1}{2} (T^3_{MN})_{P_1P_2}^N = 0, \]
\[ (T^2_{MP_1})_{P_2P_3} - \frac{1}{3} (T^4_{MN})_{P_1P_2P_3}^N = 0, \quad (T^3_{MP_1})_{P_2P_3P_4} + \frac{1}{4} (T^5_{MN})_{P_1...P_4}^N = 0, \]
\[ (T^4_{MP_1})_{P_2...P_5} - \frac{1}{5!} \epsilon_{P_1...P_5} Q_1...Q_6 (T^5_{MQ_1})_{Q_2...Q_6} = 0. \quad (4.10) \]

From the explicit expressions for \( T \) in terms of the physical fields it follows that

\[ (T^1_{MN})_P = (T^1_{MN})_P, \quad (T^2_{MN})_{PQ} = (T^2_{PQ})_{MN}, \quad (T^3_{MN})_{PQR} = 0. \quad (4.11) \]

After showing that \( T^1 = 0 \), which implies that \( F \wedge F = 0 \), we have that

\[ (T^2_{MN})_{PQR} = \frac{1}{6} (\nabla_M F_{NPQR} - \nabla_N F_{MPQR}), \quad (4.12) \]

which finally implies that \( \mathcal{R}_{AB} = 0 \) and thus maximal supersymmetry [8].

## 5 Concluding remarks

In this review, we outlined some aspects of the spinorial geometry approach to solving the KSE of supergravity theories. In addition, we presented some of the applications,
like the classification of maximally supersymmetric IIB $G$-backgrounds and $N = 31$ supersymmetric IIB and $D = 11$ supergravity backgrounds. We have also emphasized that we have constructed the most general pp-wave solution of IIB supergravity which preserves at least 16 supersymmetries.

The question that remains is whether the supersymmetric backgrounds of IIB and $D = 11$ supergravities can be classified\footnote{IIA backgrounds are special cases of the $D = 11$ ones.}. There are two classes of supersymmetric backgrounds. One class consists of those backgrounds which admit Killing spinors that are invariant under some proper Lie subgroup of the appropriate $Spin$ gauge group. Examples of such backgrounds are those in IIB and $D = 11$ supergravities with $N = 1$ supersymmetry, and the maximally supersymmetric IIB $G$-backgrounds that we have mentioned above. The other class is those backgrounds for which the stability subgroup of their Killing spinors in the $Spin$ group is $\{1\}$. Examples of such backgrounds are those in IIB and $D = 11$ supergravities that preserve $N = 31$ and $N = 32$ supersymmetries. It can be shown that two spinors in IIB or $D = 11$ supergravities can have a trivial subgroup in the $Spin$ gauge groups. So there may exist such backgrounds for any $N \geq 2$. For the former class, it seems likely that the KSE of backgrounds whose Killing spinors are invariant under a proper Lie subgroup of $Spin$ can be solved. The invariance condition imposes a strong restriction on the form of Killing spinors and so the linear systems that arise from the spinorial geometry can be tractable. For the latter class, it is encouraging that the $N = 31$ backgrounds we have investigated are maximally supersymmetric. This indicates that if the Killing spinors are not invariant under some proper Lie subgroup of $Spin$, then the KSE together with the field equations and the Bianchi identities impose strong restrictions on the geometry and fluxes of the background. If this persists for backgrounds with fewer supersymmetries, then the classification of supersymmetric backgrounds is simplified and so the programme can be carried out in full.

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