Zero-mass limit of a Dirac spinor with general spin orientation

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Abstract
The helicity eigenstates that describe fermions with a special spin orientation (parallel or antiparallel to the direction of momentum) provide a considerable simplification in calculations. Hence, it is generally preferred to use the helicity basis during calculations in relativistic quantum mechanics or the quantum field theory. Possibly for the above reason, Dirac spinors describing a general spin orientation have been ignored in many textbooks. Although the helicity eigenstates give an almost complete understanding of the behavior of the free Dirac solutions, the zero-mass limit is one of its exceptions. The zero-mass behavior of the free spinor with general spin orientation and its relation with chirality eigenstates has been skipped in textbooks and hence it deserves a clear, detailed investigation. In this paper we obtain the free Dirac spinors describing a general spin orientation and examine their zero-mass limit. We also briefly discuss some of the implications of this small-mass behavior of the spinors on particle physics.

Keywords: Dirac equation, ultrarelativistic limit, chirality and helicity

1. Introduction

Chirality is defined by the operator Dirac matrix $\gamma^5$. Its eigenstates with eigenvalues $+1$ and $-1$ are sometimes called right- and left-handed or positive and negative chirality eigenstates. Chirality has a fundamental importance in the Standard Model of particle physics. Weak interactions behave differently for different chirality fermion fields [1]. Although chirality is a Lorentz invariant quantity it is not conserved for massive fermions. This is obvious from the fact that the chirality operator does not commute with the mass term in the Dirac Hamiltonian. On the other hand, helicity, which is simply defined as the projection of the spin onto the
direction of momentum\(^1\), is a conserved quantity but not Lorentz invariant. Therefore, in this perspective chirality and helicity have somewhat opposite characteristics for massive fermions [2]. However, for massless fermions they become both conserved and Lorentz invariant.

It is well known that the helicity and the chirality eigenstates coincide in the zero-mass limit [3–5]. On the other hand, a general spin state does not exhibit such behavior. For instance, the spinor for a free fermion which is transversely polarized relative to the direction of its momentum is always given by the same superposition \(u^{(T)} = \frac{1}{\sqrt{2}} (u^{(+)} + u^{(-)})\) of positive and negative helicity spinors no matter how small the mass parameter is. Since the positive and negative helicity spinors converge to positive and negative chirality eigenstates for \(m \to 0\), the transversely polarized spinor \(u^{(T)}\) is always given by a mixed chirality eigenstate and hence does not converge to one of the chirality eigenstates left-handed or right-handed even for infinitesimal values of the mass. Thus we conclude that the zero-mass limit of the free Dirac spinor with arbitrary spin orientation does not necessarily result in a chirality eigenstate. Although the solutions of the free Dirac equation that describe noninteracting spin-\(1/2\) particles such as electrons have been widely studied in many textbooks, the free solutions describing a general spin orientation have been commonly omitted. Many books on this subject consider only a special type of Dirac solutions: the so-called helicity states. Only in a few of the books that study the Dirac equation do the authors implicitly demonstrate the methodology which can be used to obtain the free solutions for general spin states, but the explicit expressions are not given. We can give the references [3–5] as an example in this respect. On the other hand, the spinors describing general spin orientations are essential to understanding the ultrarelativistic/zero-mass behavior of the free Dirac solutions. As far as we know this fact has been ignored in all textbooks, and the ultrarelativistic/zero-mass limit has been analyzed using only the special solutions, namely helicity states. The main purpose of this paper is to fill this gap. Our another purpose is to deliver a better understanding of the spin phenomena in relativistic quantum mechanics. The comparison of two different methods given in sections 2 and 3 makes the concepts clearer. The second method demonstrated in section 3 provides a new simple way to obtain free spinors with general spin. The paper is essentially prepared for educational purposes and addresses graduate level students, but it is also useful for undergraduates who have attended courses on relativity and quantum mechanics.

In the following sections, we are going to obtain Dirac spinors describing a general spin orientation using two different methods and then analyze their zero-mass behavior. In section 2 the spinors describing a general spin orientation are going to be obtained by applying Lorentz transformations to a spinor at rest. In section 3 the spinors for a general spin are going to be obtained with the help of the covariant spin projection operator. In the discussion section (section 4) we are going to examine the zero-mass behavior of a spinor with general spin orientation and discuss briefly some of its implications.

2. Derivation of the spinor for general spin via Lorentz boost

The orientation of spin relative to the direction of momentum is a reference frame-dependent quantity for massive fermions. For instance, we can perform a Lorentz transformation into a reference frame in which the momentum of the particle is reversed. This transformation

\(^1\) If a fermion is polarized in the direction parallel (antiparallel) to the momentum then it is described by a positive or right-handed helicity (negative or left-handed helicity) state.
obviously flips the helicity. One way to derive a Dirac spinor describing a general spin orientation is to use Lorentz transformations. In this method, the spinor for a moving fermion is obtained by applying a Lorentz boost to the spinor in the rest frame of the particle [3–6]. In the rest frame we can use the prescription of non-relativistic quantum mechanics to define a spinor with general spin. By means of Lorentz transformations we can easily get the relativistic description of a general spinor, i.e., free solutions of the Dirac equation describing a general spin orientation.

The Lorentz transformations of the Dirac spinors can be studied by the Lie group \( SL(2, C) \). The generators of \( SL(2, C) \) can be written in the following \( 4 \times 4 \) matrix form\(^2\):

\[
\begin{pmatrix}
\frac{\sigma_n}{2} & 0 \\
0 & \frac{-\sigma_n}{2}
\end{pmatrix}
\]

where \( \sigma_m, m = 1, 2, 3 \) are Pauli spin matrices. The generators \( J_m \) and \( K_m \) are responsible for spatial rotations and Lorentz boosts respectively. Without loss of generality, let us consider a pure Lorentz boost along the \( z \)-axis. The group element corresponding to this transformation is given by

\[
\exp(-i\xi K_3) = \begin{pmatrix}
\exp\left(\frac{\xi \sigma_3}{2}\right) & 0 \\
0 & \exp\left(-\frac{\xi \sigma_3}{2}\right)
\end{pmatrix}
\]

Here, \( \xi \) is the rapidity parameter which is related to the \( \beta \) parameter of the Lorentz transformation as \( \tanh \xi = \beta \). It is easy to deduce the following identities from the expansion of exponential functions and using the identities \((\sigma_3)^{2k} = \mathbb{I}, (\sigma_3)^{2k+1} = \sigma_3\):

\[
\exp\left(\frac{\xi \sigma_3}{2}\right) = \cosh\left(\frac{\xi}{2}\right)\mathbb{I} + \sinh\left(\frac{\xi}{2}\right)\sigma_3
\]

\[
\exp\left(-\frac{\xi \sigma_3}{2}\right) = \cosh\left(\frac{-\xi}{2}\right)\mathbb{I} - \sinh\left(\frac{-\xi}{2}\right)\sigma_3.
\]

Hence the Lorentz boost along the \( z \)-axis can be written as

\[
\exp(-i\xi K_3) = \begin{pmatrix}
\cosh\left(\frac{\xi}{2}\right)\mathbb{I} + \sinh\left(\frac{\xi}{2}\right)\sigma_3 & 0 \\
0 & \cosh\left(\frac{-\xi}{2}\right)\mathbb{I} - \sinh\left(\frac{-\xi}{2}\right)\sigma_3
\end{pmatrix}
\]

Assume that \( S \) is the rest frame of the spin-1/2 fermion defined by the coordinates \( x^\mu = (x, y, z, t) \). In the rest frame we can use Pauli spinors to define the spin of the fermion. Let \( \vec{n} \) be the unit vector in direction of the spin quantization axis in the rest frame \( S \). Then the non-relativistic \( 2 \times 2 \) spin matrix is \( S_\theta = \frac{1}{2}(\vec{n} \cdot \hat{\sigma}) \). Without loss of generality, choose \( \vec{n} = \sin \theta \hat{x} + \cos \theta \hat{z} \), i.e., \( \vec{n} \) is in the \( z-x \) plane which makes an angle \( \theta \) (polar angle) with respect to the \( z \)-axis. We do not lose generality if we choose the azimuthal angle zero since we are going to boost along the \( z \)-axis and the angle between the direction of the boost and the unit vector \( \vec{n} \) is the polar angle \( \theta \). Therefore, this particular choice does not change the general results. The eigenvectors of \( S_\theta = \frac{1}{2}(\sin \theta \sigma_x + \cos \theta \sigma_z) \) are

\(^2\) In this paper we work in the Weyl representation of the Dirac matrices and use the natural \( \hbar = c = 1 \) unit system.
The above spinors are given as a function of the rapidity \( \xi \). One can obtain the spinors in (9) as a function of the energy and the momentum using the equality \( e^{\xi/2} = \left( \frac{E + |p|}{E - |p|} \right)^{1/4} \). This equality is evident from the following definitions of the rapidity parameter: \( \sinh \xi = \gamma \beta \) and \( \cosh \xi = \gamma \). In the \( c = 1 \) unit system the identities \( \gamma = E/m \) and \( \gamma \beta = |p|/m \) hold. Let us discuss some special spin orientations. One of the important special types of Dirac solutions are the helicity eigenstates. The helicity operator (\( \hat{\Lambda} \)) is defined by the following formula [5]

\[
\hat{\Lambda} = \vec{S} \cdot \frac{\vec{p}}{|p|}
\]

(10)

where \( \vec{S} \) is the \( 4 \times 4 \) spin matrix defined by

\[
\vec{S} = \frac{1}{2} \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}
\]

(11)

It is obvious from (10) that the helicity is simply the projection of the spin onto the direction of momentum. The solutions of the free Dirac equation which satisfy the eigenvalue equations \( \hat{\Lambda} u^{(+) \downarrow}(p) = u^{(+) \downarrow}(p) \) and \( \hat{\Lambda} u^{(-) \uparrow}(p) = -u^{(-) \uparrow}(p) \) are called the helicity eigenstates. We observe from these equations that \( u^{(+)}(p) \) and \( u^{(-)}(p) \) are the eigenvectors of the helicity operator with eigenvalues +1 and −1 respectively. The helicity operator has no other eigenvalues than \( \pm 1 \). The helicity eigenstates can be obtained by the choices \( \theta = 0 \) and \( \theta = \pi \) in (9). According to the notation used in this paper, spin-up (\( \uparrow \)) and spin-down (\( \downarrow \)) spinors given in (9) correspond to positive (\( + \)) and negative (\( − \)) helicity spinors respectively.
if we choose $\theta = 0$. We observe from (9) that the spinors describing a general spin can be written by the following superposition of the helicity eigenstates [7]

$$u^{(+)}(p) = \cos \left( \frac{\theta}{2} \right) u^{+\gamma}(p) + \sin \left( \frac{\theta}{2} \right) u^{-\gamma}(p)$$

$$u^{(-)}(p) = \cos \left( \frac{\theta}{2} \right) u^{\gamma-}(p) - \sin \left( \frac{\theta}{2} \right) u^{\gamma+}(p).$$

Another interesting special orientation is the transverse polarization which corresponds to the choice $\theta = \pi/2$. We see from equations (12) and (13) that the spinors for transverse polarization are mixed states composed of helicity eigenstates where each helicity eigenstate has equal probability.

The general spinors for antifermions can also be obtained in a similar way. Here one should be careful about the fact that opposite to the fermion case spin-up and spin-down states for antifermions correspond to eigenvalues $\lambda = -1$ and $\lambda = +1$ of the non-relativistic spin matrix $\mathbf{S}_n$ respectively (see the paragraph below equation (6)). Moreover there is a relative minus sign between upper and lower 2-component subspinors of the 4-component spinor.

3. Derivation of the spinor for general spin via the covariant spin projection operator

Dirac spinors for general spin can also be obtained by means of the covariant spin projection operator. The covariant spin projection operator is the covariant generalization of the non-relativistic spin projection operator. Hence it is a reference frame independent quantity and valid in any reference frame. Its explicit form is defined as [5, 8]

$$\hat{S}(s) \equiv \frac{1}{2} (1 + \gamma_5 \gamma_\mu s^\mu)$$

where $s^\mu$ is the spin-four vector which is the Lorentz transformation of the rest vector $(s^\mu)_{RF} = (0, \vec{n})$ into a reference frame where the particle has a momentum $p^\mu = (E, \vec{p})$. Therefore, if the fermion is polarized in the direction $\vec{n}$ in its rest frame then its spin-four vector can be obtained through the Lorentz transformation $s^\mu = L^\mu_\nu (s^\nu)_{RF}$. It is easy to show that the spin-four vector can be written in the following form [5, 8]

$$s^\mu = \left( \frac{\vec{p} \cdot \vec{n}}{m}, \vec{n} + \frac{\vec{p} \cdot \vec{n}}{m(E + m)} \vec{p} \right).$$

The three-vector $\vec{n}$ is sometimes called the spin three-vector and denoted by $\vec{s}$. Here the prime symbol is used to indicate that $\vec{s}$ is not the spatial component of $s^\mu$, but they are defined in different reference frames. Let $u^{(s)}(p)$ be the spinor which describes a fermion with spin-four vector $s^\mu$. Then in the rest frame of the particle its spin is quantized in the direction $\vec{n}$. According to our notation, the sign of the superscript $s$ represents the sign of the spin three-vector $\vec{n}$. Therefore $u^{(s)}(p)$ describes a fermion which is quantized in the direction $-\vec{n}$ in the rest frame of the particle. Obviously $+s (-s)$ corresponds to a spin-up (spin-down) state. The orthogonality of spin-up and spin-down states can be written in this notation as: $\bar{u}^{(s)}(p) u^{(-s)}(p) = 0$. The covariant spin projection operator $\hat{S}(s)$ projects to a polarized state which is described by the spin-four vector $s^\mu$. Hence it has the following properties

$$\hat{S}(s) u^{(s)}(p) = u^{(s)}(p), \quad \hat{S}(s) u^{(-s)}(p) = 0.$$
Here we should note that the covariant spin projection operator not only performs a projection into a polarized particle state but also performs a projection into a polarized antiparticle state. Therefore similar equations can also be written for antispinors \( \bar{v}^{(\pm)}(p) \). Let us choose \( \vec{n} = \sin \theta \hat{x} + \cos \theta \hat{z} \). With this choice \( \Sigma(s) \) performs a projection into a state with general spin orientation (again without loss of generality we assume that the azimuthal angle is zero and \( \vec{n} \) lies in the z-x plane). The desired spinor \( \bar{u}^{(\pm)}(p) \) can be obtained by applying \( \Sigma(s) \) to a subspace spanned by the set of particle states \( \{ |u^{(\pm)}(p)\rangle \} \). The projection operator onto this subspace is given by \( \sum_{s'} u^{(s')}(p) \bar{u}^{(s')}(p) = u^{(s)}(p) \bar{u}^{(s)}(p) + u^{(-s)}(p) \bar{u}^{(-s)}(p) \). This projection operator is independent from the choice of \( \vec{n} \) since for every fixed spin axis the spin-up and the spin-down spinors span the same subspace and constitute its basis. Its explicit form can be found in standard textbooks [4, 5] and given by

\[
\sum_{s'} u^{(s')}(p) \bar{u}^{(s')}(p) = \frac{p^\mu \gamma_\mu + m}{2m}.
\] (17)

Equation (17) is sometimes called the completeness relation. The completeness relation implicitly contains the set of states that we are looking for. If we apply \( \Sigma(s) \) from the left and \( \Sigma(s) \) from the right to the completeness relation (17) we obtain

\[
\bar{u}^{(s)}(p) \bar{u}^{(s)}(p) = \bar{\Sigma}(s) \left( \frac{p^\mu \gamma_\mu + m}{2m} \right) \Sigma(s).
\] (18)

Here the projection ensures that the sum over \( s' \) yields just one term with \( s' = s \). Equation (18) gives a set of algebraic equations for the components of \( \bar{u}^{(s)}(p) \). The right-hand side of equation (18) can be calculated using the explicit expressions for Dirac matrices. For simplicity we assume that the fermion is moving along the z-axis. Then the right-hand side of equation (18) gives a real \( 4 \times 4 \) matrix. Hence we can choose the components of the \( \bar{u}^{(s)}(p) \) spinor all real, \( \bar{u}^{(s)}(p)_i = \alpha_i, \alpha_i \in \Re, \quad i = 1, 2, 3, 4 \). After some algebra we obtain the following matrix equation

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{4} \cos^2 \left( \frac{E - m}{2m} \right) & \frac{1}{4} \sin^2 \left( \frac{E - m}{2m} \right) & \frac{1}{4} \sin \theta & \frac{1}{4} \sin \theta \\
\frac{1}{4} \sin^2 \left( \frac{E - m}{2m} \right) & \frac{1}{4} \cos^2 \left( \frac{E - m}{2m} \right) & \frac{1}{4} \sin \theta & \frac{1}{4} \sin \theta \\
\frac{1}{4} \sin \theta & \frac{1}{4} \sin \theta & \frac{1}{4} \sin^2 \left( \frac{E - m}{2m} \right) & \frac{1}{4} \sin \theta \\
\frac{1}{4} \sin \theta & \frac{1}{4} \sin \theta & \frac{1}{4} \sin \theta & \frac{1}{4} \sin^2 \left( \frac{E - m}{2m} \right)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}
\] (19)

The solution of the matrix equation (19) gives the components of the spin-up spinor \( \bar{u}^{(s)}(p) \) for general spin orientation. Similarly the components of spin-down spinor \( \bar{u}^{(s)}(p) \) can be solved from (19) with the replacement \( \theta \rightarrow \theta + \pi \). This is obvious because spin-down state corresponds to the orientation \( \vec{n} = -\sin \theta \hat{x} - \cos \theta \hat{z} \) in the rest frame of the particle. We obtain the following spinors from the solution of equation (19)

\[
\bar{u}^{(s)}(p) = \bar{u}^{(s)}(p) \frac{1}{2} \left( 1 + \gamma_\mu \bar{v}^\mu \right)
\]

where \( \bar{a} = \bar{u}^{(s)} \). Thus the bar `-' operation does not change the form of the spin projection operator.
The above spinors exactly coincide with the spinors (9) obtained from Lorentz transformation.

The same method can be applied in the case of antifermions. However, one should use the completeness relation \(\sum_n \psi^{(\uparrow)}_n(p)\psi^{(\downarrow)}_n(p) = \frac{\gamma_m - m}{2\pi}\) for antifermions instead of (17). Furthermore the definition of spin-up and spin-down orientations should be reversed compared to the fermion case.

4. Discussion

Let us examine the zero-mass limit of the spinors given in (9) or (20). The zero-mass limit corresponds to ultrarelativistic limit \(\xi \to \infty\) or equivalently \(|\vec{p}| \to E\). We observe from equation (9) that the terms proportional to \(e^{\xi/2}\) diverge in the zero-mass limit. Hence the spinors given in (9) or (20) become undefined in this limit. Therefore we should renormalize these spinors to a finite value. If we take the zero-mass limit of the spinors describing a general spin orientation (spinors in (9) or (20)) and normalize them according to the formula \(\bar{u}^{(\alpha)}u^{(\alpha)} = \delta^{\alpha\beta}\), we get the following spinors

\[
\bar{u}^{(\uparrow)}(p) = \lim_{\xi \to \infty} Nu^{(\uparrow)}(p) = \begin{pmatrix}
\cos \theta \\
0 \\
\sin \theta
\end{pmatrix}
\]

\[
u^{(\uparrow)}(p) = \lim_{\xi \to \infty} Nu^{(\downarrow)}(p) = \begin{pmatrix}
0 \\
0 \\
\cos \theta
\end{pmatrix}.
\]

Here, \(N\) is the normalization factor and the upper tilde symbol is used to indicate that these are normalized spinors in the \(\xi \to \infty\) limit. We see from equation (21) that the helicity eigenstates which correspond to special spin orientation \(\theta = 0\), converge to the chirality eigenstates in the \(\xi \to \infty\) limit. Indeed if we apply the right and left chirality projection operators \(\hat{R} = \frac{1}{2}(1 + \gamma_5)\) and \(\hat{L} = \frac{1}{2}(1 - \gamma_5)\) to the helicity eigenstates we obtain

\[
\hat{R} \bar{u}^{(+)}(p) = \bar{u}^{(+)}(p), \quad \hat{R} \bar{u}^{(-)}(p) = 0
\]

\[
\hat{L} \bar{u}^{(-)}(p) = \bar{u}^{(-)}(p), \quad \hat{L} \bar{u}^{(+)}(p) = 0.
\]

Here we should recall that for special spin orientation \(\theta = 0\), spin-up (spin-down) spinor corresponds to positive (negative) helicity and instead of the notation \(\uparrow\) and \(\downarrow\) we use \(\uparrow\) and \(\uparrow\downarrow\). On the other hand this zero-mass behavior is not valid for spinors with general spin orientation. If we apply the chirality projection operators \(\hat{R}\) and \(\hat{L}\) to the spinors (21) describing a general spin orientation we get the following equalities which hold in the zero-mass limit.
\[
\hat{R} \hat{a}^{(1)}(p) = \cos \left( \frac{\theta}{2} \right) \hat{a}^{(+)}(p), \quad \hat{R} \hat{a}^{(1)}(p) = -\sin \left( \frac{\theta}{2} \right) \hat{a}^{(+)}(p)
\]
\[
\hat{L} \hat{a}^{(1)}(p) = \sin \left( \frac{\theta}{2} \right) \hat{a}^{(-)}(p), \quad \hat{L} \hat{a}^{(1)}(p) = \cos \left( \frac{\theta}{2} \right) \hat{a}^{(-)}(p).
\]

We see from equation (23) that the spinors describing a general spin orientation do not converge to one of the chirality eigenstates left-handed or right-handed in the zero-mass limit. They are always given by a mixed chirality. This fact can also be understood intuitively from (8) and (9). We see from (8) and (9) that the operator \( \exp(-i\xi K) \) tends to the projection onto the first and last component of the spinor as the rapidity parameter \( \xi \) increases. In the limit \( \xi \rightarrow \infty \) the second and third components of \( \exp(-i\xi K) \hat{u}^{(1)}_{\chi_{\pm}} \) disappear whereas the first and last component diverge. We can regulate these divergent components through the normalization procedure discussed in the previous page (see equation (21) and the paragraph above). If the result of the limit operator applied to a spinor of the form (7) is to be an eigenvector of \( \gamma_{5} \), then one component of \( \chi_{\pm} \) must be equal to 0. This is the case if and only if \( \theta = 0 \) or \( \theta = \pi \).

The zero-mass behavior observed from (23) is also evident from equations (12) and (13) where the spinors are written explicitly as a mixed state composed of helicity eigenstates. Since the helicity and chirality eigenstates coincide in the zero-mass limit, this mixed helicity state represents a mixed chirality state for \( \xi \rightarrow \infty \). The amount of mixture is determined by the angle \( \theta \). For special orientation \( \theta = \pi/2 \) (transverse polarization) we get the maximum mixing. Here, we would like to draw the reader's attention to the fact that \( \theta \) is not a dynamical variable. It does not depend on the rapidity parameter \( \xi \). \( \theta \) is the angle measured in the frame in which the particle is at rest. Hence, one may call it 'proper angle' in analogy with the term proper time. Strictly speaking the spinors given in (9) or (20) describe a fermion which in its rest frame has a spin orientation \( \vec{n} = \sin \theta \hat{x} + \cos \theta \hat{\xi} \). Therefore the angle \( \theta \) is not affected by relativistic aberration.

It is well known that the Dirac equation decomposes into two Weyl equations when mass is zero. Hence one may expect that the free solutions of the Dirac equation converge to the solutions of one of the Weyl equations in the \( m \rightarrow 0 \) limit. Moreover the behavior of the helicity eigenstates in the zero-mass limit is compatible with this expectation. According to our opinion, the above arguments may mislead some students or non-experts to think that the expectation is valid in a general case. On the contrary, as we have shown, the free spinor with general spin orientation cannot be described by one of the Weyl equations in the zero-mass limit. We have deduced that the spinor describing a general spin orientation is always given by a mixed chirality state and thus both of the Weyl equations are necessary simultaneously to describe the spinor with general spin orientation even in the zero-mass limit.

At first glance, this odd zero-mass behavior of the general spinors in (9) or (20) seems to contradict the results of the little group analysis of Wigner [9]. It is well known that massless particles are described solely by pure helicity states and that the spin orientations other than parallel or antiparallel to the direction of momentum are not allowed for massless particles [9]. Therefore it is natural to expect that a spinor with a general spin orientation gradually becomes more and more longitudinal and its transverse component gradually vanishes as the mass goes to zero. This behavior obviously contradicts that observed from equation (23). The crucial point which is generally skipped is that the transition from general spin states to the helicity eigenstates does not necessarily take place in a continuous manner. If the mass is strictly zero then the particle moves at the speed of light and consequently we cannot define its rest frame. Since it is impossible to make a Lorentz transformation to a frame moving at the speed of light, our calculations that lead to equations (9) and (20) are invalid.
for strictly massless particles. On the other hand, if the fermion has a nonzero mass we can make a Lorentz transformation to the rest frame of the fermion. Hence, the general spinors given in (9) or (20) describe the fermions with mass greater than zero, i.e., \( m > 0 \).

Here we should stress the fact that the general spinors describe massive fermions no matter how small their masses are. In the point \( m = 0 \) our calculations and their results equations (9) and (20) become invalid. Since the limit \( m \to 0 \) of a free spinor \( u^{(1, -)}(p) \) with general spin orientation exists but not equal to one of the helicity eigenstates \( u^{(+,-)}(p) \), i.e., \( \lim_{m \to 0} u^{(1, -)}(p) \neq u^{(+,-)}(p) \), we conclude that the free solutions of the Dirac equation describing a general spin orientation have a discontinuity at the point \( m = 0 \).

Therefore the zero-mass behavior observed from equation (23) does not contradict the results of [9]. Strictly massless fermions are indeed described solely by helicity eigenstates. However if the fermion has a non-zero mass then it is allowed to have an arbitrary spin orientation which is different from the momentum or opposite momentum direction. The transverse polarization does not disappear in the \( m \to 0 \) limit but vanishes instantly at the point \( m = 0 \).

In closing, let us discuss some of the implications of this zero-mass behavior of the Dirac spinors on particle physics. Experimental results obtained in the Super-Kamiokande and Sudbury Neutrino Observatory verified the existence of neutrino oscillations, which shows that neutrinos have tiny but non-zero masses [10, 11]. Since the neutrino masses are very tiny they are ultrarelativistic fermions at the energy scales of current experiments. Therefore the results obtained in the zero-mass limit can be applicable to neutrinos. In the Standard Model of particle physics, neutrinos couple minimally to other standard model particles only through the vertex \( q \gamma^5 \) and hence the interaction project out the left chiral component of the field and the right chiral component decouples completely. Consequently, the cross section for a neutrino with positive helicity is so tiny and can be neglected for energies much greater than \( m \). On the other hand, as we have deduced, the spinor with general spin orientation is always given by a mixed chirality state determined by the non-dynamical variable \( \theta \). Hence, the weak interaction does not annihilate the fermion field describing a general spin orientation in the \( m \to 0 \) limit. Eventually, although the cross section for a neutrino with positive helicity goes to zero as \( m \to 0 \), the cross section remains finite for a neutrino with general spin orientation [7]. Specifically if the neutrino is transversely polarized \((\theta = \pi/2)\) then the polarized cross section is almost half of the cross section for negative helicity neutrino\(^4\). The above results imply that the production or the absorption probability of the neutrinos with arbitrary spin orientation through Standard Model reactions cannot be neglected in general.

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\(^4\) Here we should note that the neutrinos interact in flavor eigenstates which are given by a superposition of the mass eigenstates. Hence, in the reality the calculation of the cross section is more complicated than we have discussed in this paper.
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