Asymptotic Expansions of Unstable and Stable Manifolds in Time-Discrete Systems

Shin-itirot Goto and Kazuhiro Nozaki

Department of Physics, Nagoya University, Nagoya 464-8602, Japan

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By means of an updated renormalization method, we construct asymptotic expansions for unstable manifolds of hyperbolic fixed points in the double-well map and the dissipative Hénon map, both of which exhibit strong homoclinic chaos. In terms of the asymptotic expansion, a simple formulation is presented to give the first homoclinic point in the double-well map. Even a truncated expansion of the unstable manifold is shown to reproduce the well-known many-leaved (fractal) structure of the strange attractor in the Hénon map.

§1. Introduction

The bifurcation of separatrices and homoclinic and heteroclinic structures are known to lead to chaos in conservative dynamical systems.1) While numerical iterations of low-dimensional mappings easily produce these complicated structures, it is extremely difficult to derive them analytically. For nearly integrable systems, this difficulty may be overcome by an asymptotic expansion beyond all orders supplemented with a Borel sum and the Stokes phenomenon.2) - 6) However, this method cannot be applied to systems that are far from integrable, such as finite-time discrete dynamical systems with double-well potentials (double-well maps) and the strange attractor in the dissipative Hénon map. Such systems are analyzed in the present paper.

Recently, a novel method based on perturbative renormalization group theory has been developed as an asymptotic singular perturbation technique.7) The renormalization group (RG) method removes secular or divergent terms from a perturbation series by renormalizing integral constants of lower order solutions. This RG method was reformulated on the basis of a naive renormalization transformation and the Lie group.8) The application of the RG method to some non-chaotic discrete systems was achieved in the framework of the envelope method,9) by which asymptotic solutions are constructed near a fixed point, except on unstable or stable manifolds.

The purposes of this paper are the following. The first purpose is to apply the reformulated RG method to chaotic discrete systems and construct asymptotic expansions of unstable and stable manifolds. Secondly, it is shown that the reformulated RG method can describe asymptotic solutions on unstable and stable manifolds near a fixed point as well as the other solutions obtained in Ref. 9) Our approach is straightforward and much simpler than the original RG method7) and the envelope method,9) and we obtain asymptotic expansions of unstable and stable manifolds constituting homoclinic tangles in the double-well map and an unstable manifold included in the closure of the strange attractor of the dissipative Hénon map.
§2. Double-well map

Let us analyze a symplectic map obtained by time-discretization of canonical equations for dynamical systems with a single degree of freedom:

\[ x_{n+1} - x_n = p_{n+1}, \quad p_{n+1} - p_n = \delta^2 f(x_n). \]  

(2.1)

For concreteness, we choose the double-well potential

\[ f(x_n) = x_n - 2x_n^3. \]

Then we obtain the second-order difference equation for \( x_n \)

\[ Lx_n \equiv x_{n+1} - (2 + \delta^2) x_n + x_{n-1} = -2\delta^2 x_n^3, \]  

(2.2)

where \( \delta^2 \) represents the time difference, which is not necessarily small. This map is area preserving and has a hyperbolic fixed point at \((x, p) = (0, 0)\). Although we concentrate on the system with a double-well potential in this section, the following analysis holds for more general systems, such as the standard map. For \( \delta^2 \to 0 \) (i.e. in the continuous limit), the system becomes integrable and the phase space is occupied by regular orbits separated by a separatrix. If \( \delta^2 \neq 0 \), splitting of the separatrix occurs, and unstable and stable manifolds of the hyperbolic fixed point cross each other at an infinite number of points (homoclinic points) that accumulate to the hyperbolic fixed point. This complex homoclinic structure (homoclinic tangles) is typical of chaos generated by the Birkhoff-Smale horseshoe mechanism. When \( \delta^2 \) is sufficiently small, the splitting of separatrix is very small, and an asymptotic expansion of unstable and stable manifolds can be obtained by means of a singular perturbation method of beyond all orders.\(^3\) Here, we are interested in the case that \( \delta^2 \sim O(1) \) or larger. Let us construct a formal series solution \( x_n^u \) near the hyperbolic fixed point \((x, p) = (0, 0)\) along the unstable manifold, that is, for \( \lim_{n \to -\infty} x_n^u = 0 \). (For brevity, \( x_n^u \) will be written \( x_n \) in the following.) We have

\[ x_n = x_n^{(1)} + x_n^{(2)} + x_n^{(3)} + x_n^{(4)} + \cdots. \]  

(2.3)

Here \( \delta^2 \) is not necessarily small, and \( |x_n^{(1)}| > |x_n^{(2)}| > \cdots \). Then, naive perturbed equations for \( x_n^{(j)} \) can be written

\[
\begin{align*}
L x_n^{(1)} &= 0, & L x_n^{(2)} &= 0, \\
L x_n^{(3)} &= -2\delta^2 x_n^{(1)} x_n^{(3)}, & L x_n^{(4)} &= -6\delta^2 x_n^{(1)} x_n^{(2)} x_n^{(2)}, \\
& \quad \cdots. 
\end{align*}
\]

The leading-order solution satisfying the condition \( \lim_{n \to -\infty} x_n = 0 \) is given by

\[ x_n^{(1)} = AK^n, \]  

(2.4)

where \( A \) is an integral constant, \( \delta = 2\sinh(k/2) > 0 \), and \( K = \exp k > 1 \). The series solution (2.3) is obtained as

\[ x_n = AK^n \left( 1 - c_2 A^2 K^{2n} + c_4 A^4 K^{4n} - c_6 A^6 K^{6n} + \cdots \right), \]  

(2.5)
where \( c_2, c_4 \) and \( c_6 \) are constants defined as

\[
\begin{align*}
  c_2 &= \frac{\delta^2}{D_3}, \\
  c_4 &= \frac{3\delta^4}{D_3D_5}, \\
  c_6 &= \frac{3\delta^6}{D_7D_3} \left( \frac{1}{D_3} + \frac{3}{D_5} \right), \\
  D_j &= \cosh jk - \cosh k \quad (j \in \mathbb{N}).
\end{align*}
\]

Now we introduce a renormalized variable \( \tilde{A}(n) \) so that all the higher-order terms in the series solution (2.5) are renormalized into the integral constant \( A \); that is,

\[
x_n = \tilde{A}(n) K^n, \tag{2.6}
\]

where

\[
\tilde{A}(n) = A \left( 1 - c_2 A^2 K^{2n} + c_4 A^4 K^{4n} - c_6 A^6 K^{6n} + \cdots \right), \tag{2.7}
\]

which is called a renormalization transformation: \( A \rightarrow \tilde{A} \). In order to derive a difference equation (a renormalization equation) for the renormalized variable \( \tilde{A}(n) \), we express the difference \( \tilde{A}(n+1) - \tilde{A}(n) \) in terms of \( \tilde{A} \). From (2.7), we have

\[
\tilde{A}(n+1) - \tilde{A}(n) = -c_2 (K^2 - 1) K^{2n} A^3 + c_4 (K^4 - 1) K^{4n} A^5 + \cdots. \tag{2.8}
\]

By solving (2.7) with respect to \( A \) iteratively, \( A \) can be expressed in terms of \( \tilde{A} \) and \( n \) as

\[
A = \tilde{A}(n) \{ 1 + c_2 \tilde{A}(n)^2 K^{2n} + (3c_2^2 - c_4) \tilde{A}(n)^4 K^{4n} + \cdots \}.
\]

Thus we can replace \( A \) in (2.8) by \( \tilde{A} \) and obtain a renormalization equation:

\[
\tilde{A}(n+1) - \tilde{A}(n) = -c_2 (K^2 - 1) K^{2n} \tilde{A}(n)^3 + \{ c_4 (K^4 - 1) - 3c_2^2 (K^2 - 1) \} \tilde{A}(n)^5 - \{ c_6 (K^6 - 1) - 5c_2 c_4 (K^4 - 1) + 3c_2 (K^2 - 1)(4c_2^2 - c_4) \} \tilde{A}(n)^7 + \cdots, \tag{2.9}
\]

which is a non-autonomous difference equation. In terms of the original variable \( x_n \) (see (2.6)), the renormalization equation (2.9) is transformed into an autonomous one as

\[
x_{n+1} = K x_n - K c_2 (K^2 - 1) x_n^3 + K \{ c_4 (K^4 - 1) - 3c_2^2 (K^2 - 1) \} x_n^5 - K \{ c_6 (K^6 - 1) - 5c_2 c_4 (K^4 - 1) + 3c_2 (K^2 - 1)(4c_2^2 - c_4) \} x_n^7 + \cdots \\
\equiv g_1(x_n), \tag{2.10}
\]

which gives the first branch \( g_1 \) of a return map on the unstable manifold: \( x_{n+1} = g_j(x_n) \), where \( g_j \) is a multi-valued function of \( x_n \) and \( j = (1, 2, \cdots) \) denotes a branch number. When \( K \gg 1 \) (strong chaos), it is easy to see that

\[
g_1(x) = K [f(x) + O(K^{-1})], \tag{2.11}
\]
in general. If \((K - 1) \ll 1\) (weak chaos), we have,

\[
g_1(x) = x + (K - 1)x(1 - x^2)^{1/2} + \mathcal{O}((K - 1)^2) \tag{2.12}
\]

for the double-well map. For \(K = 2.1\), an asymptotic expansion of \(g_1\) truncated at \(x_n^7\) is depicted in Fig. 1.

Although the truncated expression agrees well with the exact numerical result for \(x_n \lesssim 0.75\), it deviates considerably from the exact result near a singular or branch point \((x_n \approx 1.1)\). In order to avoid this considerable discrepancy near such a point, we restrict the domain of \(g_1\) so that the return map \(g_1\) is reversible, that is, we consider \(x_n \lesssim 0.75\). The reversible branch thus obtained is denoted by \(\tilde{g}_1\). Then, from (2.2), we have the following functional map: \(\tilde{g}_j \rightarrow \tilde{g}_{j+1}\)

\[
\tilde{g}_{j+1}(x_n) = 2x_n - \tilde{g}_j^{-1}(x_n) + \delta^2 f(x_n), \tag{2.13}
\]

where the domain of \(x_n\) is chosen for each \(j\) so that the new branch \(\tilde{g}_{j+1}\) is reversible. Using the functional map (2.13), we can construct each reversible branch of the return map \(\tilde{g}_j\) from the first branch \(\tilde{g}_1\) or \(g_1\) step by step. As shown in Figs. 2 and 3, the result agrees well with the exact one even if a truncated expansion of \(g_1\) up to \(x_n^7\) is used for \(\tilde{g}_1\). Once the return map \(\tilde{g}_j\) on the unstable manifold is constructed, the unstable manifold \((x^u, p^u)\) can be written as

\[
p^u = x^u - \tilde{g}_j^{-1}(x^u), \tag{2.14}
\]

because \(p_n^u = x_n^u - x_{n-1}^u = x_n^u - \tilde{g}_j^{-1}(x^u)\).

Using the symmetry of the map (2.1),

\[
x' = x - p, \quad p' = -p, \tag{2.15}
\]
the stable manifold \((x^s, p^s)\) is constructed as follows. The symmetry (2.15) gives

\[
x^u = x^s - p^s, \quad p^u = -p^s.
\] (2.16)

From (2.16) and (2.14), we obtain the stable manifold in terms of \(\tilde{g}_j\) are

\[
p^s = x^s - \tilde{g}_j(x^s).
\] (2.17)

Therefore, the \(x\) coordinate of a homoclinic point \((x^h, p^h)\) is given by

\[
x^h = \tilde{g}_j(x^h).
\] (2.18)

Since the first intersection point \((x^*, p^*)\) is located at \(p^* = 0\) on the first branch \(g_1\) or \(\tilde{g}_2\) in the present case, \(x^*\) is a fixed point of the first branch of the return map:

\[
x^* = g_1(x^*) = \tilde{g}_2(x^*).
\] (2.19)
For $K \gg 1$ and $(K - 1) \ll 1$, (2.11), (2.12) and (2.19) yield $x^* \approx \pm 1/\sqrt{2}$ and $x^* \approx \pm 1$, respectively.

§3. Hénon map

Here, we construct an unstable manifold in the strange attractor of a dissipative dynamical system known as the Hénon map:

\begin{align*}
  x_{n+1} &= 1 - ax_n^2 + y_n, \quad (3.1) \\
  y_{n+1} &= bx_n, \quad (3.2)
\end{align*}

with $a = 1.4$ and $b = 0.3$. This map has the hyperbolic fixed points

\begin{align*}
  x^* &= \frac{b - 1 \pm \{(b - 1)^2 + 4a\}^{1/2}}{2a}, \\
  y^* &= bx^*.
\end{align*}

Let us consider the unstable manifold of one of the fixed points, $x^* = [b - 1 + \{(b - 1)^2 + 4a\}^{1/2}]/2a$, $y^* = bx^*$. This fixed point is known to be included in the closure of the strange attractor of the Hénon map. Setting $\hat{x}_n = x_n - x^*$ and eliminating the $y$ variable, we obtain a second-order difference equation from (3.1) and (3.2):

\begin{equation}
  \hat{x}_{n+1} + 2ax^* \hat{x}_n - b\hat{x}_{n-1} = -a\hat{x}_n^2. \quad (3.3)
\end{equation}

Near the origin of $\hat{x}$, we obtain the following formal series solution which vanishes as $n \to -\infty$:

\begin{equation}
  \hat{x}_n = K^n [A - c_2 A^2 K^n + c_3 A^3 K^{2n} - c_4 A^4 K^{3n} + \cdots].
\end{equation}

Here $A$ is an arbitrary integral constant, $K = -ax^* + \{(ax^*)^2 + b\}^{1/2} \approx -1.923739$ is one of the eigenvalues of the linearized map of (3.3), and

\begin{align*}
  c_2 &= \frac{aK^2}{D(K^2)}, \\
  c_3 &= \frac{2a^2 K^5}{D(K^3)D(K^2)}, \\
  c_4 &= \frac{a^3 K^8}{D(K^4)} \left\{ \frac{1}{D(K^2)^2} + \frac{4K}{D(K^3)D(K^2)} \right\}, \\
  D(K^j) &= K^{2j} + 2ax^* K^j - b.
\end{align*}

Following the same procedure as in the previous section, we renormalize $A$ as

\begin{align*}
  \bar{A}(n) &= A - c_2 A^2 K^n + c_3 A^3 K^{2n} - c_4 A^4 K^{3n} + \cdots, \\
  \hat{x}_n &= \bar{A}(n) K^n, \quad (3.4)
\end{align*}

and obtain a similar non-autonomous renormalization equation of $\bar{A}$ as (2.9). Replacing $\bar{A}(n)$ by $\hat{x}_n$ in the renormalization equation of $\bar{A}$, we have
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Fig. 4. The first branch of the return map on an unstable manifold for the Hénon map. The dots (·) denote an exact solution calculated using numerical iterations, and the crosses (×) denote \( \tilde{g}_1 \) obtained by the renormalization method.

Fig. 5. The exact return map of the Hénon map’s unstable manifold is obtained from numerical iteration.

\[
\begin{align*}
\hat{x}_{n+1} &= K\hat{x}_n - c_2\hat{x}_n^2 K(K - 1) \\
&\quad + \hat{x}_n^3 K(K - 1)[c_3(K + 1) - 2c_2^2] \\
&\quad + \hat{x}_n^4 K(K - 1)[-c_4(K^2 + K + 1) + 3c_3c_2(K + 1) \\
&\quad - c_2(5c_2^2 - c_3^2)] \\
&\quad + \hat{x}_n^5 K(K - 1)[c_5(K^3 + K^2 + K + 1) - 4c_2c_4(K^2 + K + 1) \\
&\quad + 3c_3(3c_2^2 - c_3)(K + 1) - 2c_2(7c_2^3 - 6c_2c_3 + c_4)] + \cdots \\
&\equiv g_1(\hat{x}_n),
\end{align*}
\]

which gives the first branch \( g_1 \) of a return map on the unstable manifold: \( \hat{x}_{n+1} = \)
$g_j(\hat{x}_n)$, where $g_j$ is a multi-valued function of $\hat{x}_n$ and $j (=1,2,\ldots)$ denotes a branch number. An asymptotic expansion of $g_1$ truncated at $\hat{x}_n^5$ is depicted in Fig. 4. We see that the truncated expression deviates considerably from the exact numerical result near $\hat{x}_n \approx 0.7$ and for $\hat{x}_n \lesssim -1$. In order to avoid this considerable discrepancy, we restrict the domain of $g_1(\hat{x})$ to $-0.7 > \hat{x} > 0$, where the return map $g_1$ is reversible. The reversible branch obtained in this way is denoted by $\tilde{g}_1$. Then, from (3.3), we obtain the functional map $\tilde{g}_j \rightarrow \tilde{g}_{j+1}$ for neighboring reversible branches as

$$\tilde{g}_{j+1}(\hat{x}_n) = -a\hat{x}^2 - 2ax^*\hat{x}_n + b\tilde{g}_j^{-1}(\hat{x}_n), \quad (3.6)$$

where the domain of $\hat{x}_n$ is appropriately chosen for each $j$. Using the functional
map (3.6), we construct each reversible branch of the return map \( \tilde{g}_j \) from the first branch \( \tilde{g}_1 \) [given by (3.5)] step by step. Even when the asymptotic expansion of \( \tilde{g}_1 \) is truncated at \( \hat{x}_n^5 \), the result agrees well with the exact one, as shown in Figs. 5 and 6. In Fig. 7, we see how well each branch of the return map constructed from an initial truncated map \( \tilde{g}_1 \) produces the well-known many-leaved (fractal) structure of the strange attractor near the fixed point.\(^{10}\)

\section{Conclusion}

We constructed asymptotic expansions for unstable manifolds of hyperbolic fixed points of the double-well map and the dissipative Hénon map, both of which exhibit homoclinic chaos. Since the dimension of these manifolds is 1, the dynamics on them are described by a return map with an infinite number of branches. An asymptotic expansion of the first branch of the return map was obtained using an updated renormalization method, which is an extension of the RG method. In this way, the extended RG method makes it possible to construct asymptotic solutions both on and off unstable (stable) manifolds near a fixed point. We explicitly constructed a functional map, through which the other branches were obtained consecutively from the first branch. Thereby a global asymptotic form of the unstable manifold was constructed. This form yields homoclinic chaos in the double-well map and the strange attractor in the dissipative Hénon map. Using a truncated expansion of the first branch of the return map, we calculated an approximate unstable manifold. The manifold we obtained agrees well with that obtained by an exact numerical calculation. A fixed point of the first branch of the return map was found to be the first homoclinic point in the double-well map, and the approximate unstable manifold reproduces the well-known many-leaved (fractal) structure of the strange attractor in the Hénon map.

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