ONE-DIMENSIONAL ELEMENTARY ABELIAN EXTENSIONS
HAVE GALOIS SCAFFOLDING

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Abstract. We define a variant of normal basis, called a Galois scaffolding, that allows for an easy determination of valuation, and has implications for Galois module structure. We identify fully ramified, elementary abelian extensions of local function fields of characteristic \(p\), called one-dimensional, that, in a particular sense, are as simple as cyclic degree \(p\) extensions, and prove the statement in the title above.

1. Introduction

The Normal Basis Theorem states that in a finite, Galois extension \(L/K\) with \(G = \text{Gal}(L/K)\), there are elements \(\rho \in L\) whose conjugates \(\{\sigma \rho : \sigma \in G\}\) provide a field basis for \(L\) over \(K\). In the setting of local field extensions, the most important property of an element is its valuation and so we asked in [BEb] about the valuation of these elements: Are there are valuations (integer certificates) that guarantee that any element bearing the specified valuation be a normal basis generator? (i.e. \(v \in \mathbb{Z}\) so that \(\rho \in L\) and \(v_L(\rho) = v\) implies \(\{\sigma \rho : \sigma \in G\}\) is a basis for \(L\) over \(K\).)

In this paper, we ask for more. Let \(L/K\) be a fully ramified \(p\)-extension of local fields with finite residue field of characteristic \(p\), and let \(v_L\) denote the normalized, additive valuation. We ask, in addition to the above property, that there be an explicit basis \(\{\Theta_i\}\) of the group algebra \(K[G]\) over \(K\), which may depend upon the extension \(L/K\) but should be independent of the element \(\rho\), with the additional property that the valuations associated with this basis, \(\{v_L(\Theta_i \rho)\}\), give a complete set of residues modulo \([L : K]\). These two ingredients, an integer certificate and a basis, make up what we call a Galois scaffolding.

Prototype: Cyclic extensions of degree \(p\). Let \(L/K\) be a ramified, cyclic, degree \(p\) extension of local fields with \(\text{Gal}(L/K) = \langle \sigma \rangle\). Assume that the ramification break number for \(L/K\) is \(b\) and \(\text{gcd}(p, b) = 1\). Note that this does not restrict the extension when \(K\) has characteristic \(p\) and is only a minor restriction when \(K\) has characteristic \(0\) [EV02 III. Prop 2.3]. Let \(\rho \in L\) be any element with \(v_L(\rho) \equiv b \mod p\). Then \(v_L((\sigma - 1)^i \rho) \equiv (i + 1)b \mod p\) for \(0 \leq i \leq p - 1\). In particular, \(v_L((\sigma - 1)^i \rho)\) yields a complete set of residues modulo \(p\), and so we have a Galois scaffolding: Pick any integer \(\equiv b \mod p\) and the basis, \(\{(\sigma - 1)^i : 0 \leq i \leq p - 1\}\).

Galois scaffolding should be viewed as normal bases with the important advantage that the valuation of any element expressed in terms of the Galois scaffolding can be easily determined. In the example above, since \(L/K\) is fully ramified, every
element $\alpha \in L$ can be expressed as $\alpha = \sum_{i=0}^{p-1} a_i(\sigma - 1)^i \rho$ for certain $a_i \in K$. Then $v_L(\alpha) = \min\{v_L(a_i) + ib + v_L(\rho) : 0 \leq i \leq p - 1\}$. We repeat ourselves for emphasis. Normal bases and power bases (polynomial bases) in a prime element are two common bases. The first allows the Galois action to be easily followed. The second allows for an easy determination of valuation. These two properties are usually at tension and so Galois scaffolding are remarkable for the delicate balance that they achieve.

Galois scaffolding in ramified, cyclic, degree $p$ extensions have made Galois module structure in these extensions tractable [BF72 BV73 Al603 IST04], along with Galois module structure in fully ramified, cyclic, degree $p^2$ extensions [Eld95]. In this paper, we will restrict our attention to fully ramified elementary abelian extensions of local function fields that are, in a particular sense, as simple as a ramified cyclic extension of degree $p$, and give an explicit Galois scaffolding for these extensions. We are motivated by the fact that much about Galois module structure in wildly ramified extensions remains poorly understood despite the topic’s venerable age.

1.1. Notation. Let $p$ be a prime integer and let $\mathbb{F}_p$ be the finite field with $p$ elements. Let $K = \mathbb{F}((t))$ be a local function field with residue field $\mathbb{F}_q$, which is either $\mathbb{F}_q$, a finite field with $q$ elements where $q$ is a power of $p$, or $\mathbb{F}_p$, the algebraic closure. Let $\varphi : K \rightarrow K$ denote the $\mathbb{F}_p$-linear map $\varphi(x) = x^p - x$, and let $\phi$ denote the ring homomorphism $\phi(x) = x^p$. Use subscripts to denote field of reference. So $\pi_K$ is a prime element of $K$, and $v_K$ is the valuation normalized so that $v_K(\pi_K^n_{K}) = n$. Let $\mathcal{O}_K = \{x \in K : v_K(x) \geq 0\}$ be the valuation ring, and let $\mathfrak{p}_K = \pi_K \mathcal{O}_K$ be its maximal ideal. Let $L/K$ denote a fully ramified, Galois $p$-extension, with $G = \text{Gal}(L/K)$. Define its ramification filtration by

$$G_i = \{\sigma \in G : v_L((\sigma - 1)\pi) \geq i + 1\}.$$  

1.2. One-dimensional elementary abelian extensions. It is a basic observation in Artin-Schreier Theory that the elementary abelian extensions of $K$ lie in one-to-one correspondence with the finite subspaces of the $\mathbb{F}_p$-vector space, $K/K^p$, where $K^p$ denotes the image of $\varphi$.

Assume for the moment that the residue field of $K$ is algebraically closed, $\mathbb{F} = \mathbb{F}_p$. Define $K_{(n)} = \phi^n(K) = \mathbb{F}((x^n))$ for $n \geq 1$. Of course, $K/K_{(n)}$ is an inseparable field extension, and so, in particular, $K$ is a vector space over $K_{(n)}$. Since the residue field of $K$ is algebraically closed, $K/K^p$ is also a vector space over $K_{(n)}$. We define one-dimensional elementary abelian extensions to be those fully ramified elementary abelian extensions of degree $p^i$ with $i \leq n + 1$ that correspond to an $i$-dimensional $\mathbb{F}_p$-subspace of a one-dimensional $K_{(n)}$-subspace of $K/K^p$. Of course, we are principally interested in maximal extensions where $i = n + 1$.

More generally, we can include the finite residue field case and define one-dimensional elementary abelian extensions of degree $p^{n+1}$ to be those that can be expressed as $L = K(x_0, \ldots, x_n)$ with $\varphi(x_i) = x_i^p - x_i = \phi^n(\Omega_i) \cdot \beta$ for some $\beta \in K$ with $v_K(\beta) = -b, b > 0$ and gcd$(b, p) = 1$; and some $\Omega_i \in K$ that span an $n + 1$-dimensional subspace over $\mathbb{F}_p$, with $\Omega_0 = 1$ and

$$v_K(\Omega_0) \leq \cdots \leq v_K(\Omega_1) \leq v_K(\Omega_0) = 0.$$  

\footnote{It is easy to see that Galois scaffolding are not universally available. Considering any unramified extension, where there can be no integer certificate.}

Without any loss of generality, we can assume moreover that whenever \( v_K(\Omega_i) = \cdots = v_K(\Omega_j) \) for \( i < j \), the projections of \( \Omega_i, \ldots, \Omega_j \) into \( \phi^n(\Omega_i) / \phi^n(\Omega_j) \) are linearly independent over \( \mathbb{F}_p \). It should be clear from this construction that the upper ramification numbers in one-dimensional elementary abelian extensions of degree \( p^{n+1} \) are congruent to each other modulo \( p^n \). Of course, the converse is not necessarily true.

Simple examples of a one-dimensional elementary abelian extensions are

\[
\begin{align*}
(1) & \quad \text{extensions of the form } K(y) \text{ with } y^d - y = \beta \text{ (Lemma 5.2).} \\
(2) & \quad \text{fully ramified biquadratic extensions (Lemma 5.1), and} \\
(3) & \quad \text{fully and weakly ramified } p\text{-extensions (Lemma 5.3).}
\end{align*}
\]

Evidently, our Galois scaffolding is not effected by small errors. This last observation can be rephrased in terms of twists by characters of Galois representations, along the lines of [BEa, §2.2.3].

1.3. **Galois scaffolding.** Assume the notation of the previous section and assume that \( L/K \) is near one-dimensional elementary abelian.

Relabel \( \Omega_j^{(0)} = \Omega_j \), and perform the following elementary row operations on the matrix \( [\phi(\Omega_j^{(0)})]_{0 \leq i,j \leq n} \), which in passing we note resembles the square root of a discriminant matrix. The first column is a column of 1’s. So start with the \( i = n \) row and work down to the \( i \) = 1 row, subtracting the \( i - 1 \)st row from the \( i \)th row. The \( i \) = 0 row and \( i \) = 0 column of our matrix now agree with (1) below. If we ignore them, the result is \( [\phi^{i-1}(\phi(\Omega_j^{(0)}))]_{1 \leq i,j \leq n} \). Divide each entry in a row by the first entry of the row. The result is \( [\phi^{i-1}(\phi(\Omega_j^{(0)}))/\phi(\Omega_1^{(0)})]_{1 \leq i,j \leq n} \). Define \( \Omega_j^{(1)} = \phi(\Omega_j^{(0)})/\phi(\Omega_1^{(0)}) \) for \( 1 \leq j \leq n \). Observe that \( v_K(\Omega_1^{(1)}) \leq \cdots \leq v_K(\Omega_n^{(1)}) = 0 \) and that the \( \{\Omega_j^{(1)}\}_{1 \leq j \leq n} \) span an \( n \) dimensional vector space over \( \mathbb{F}_p \). Again we have a matrix \( [\phi^{i-1}(\phi(\Omega_j^{(1)}))]_{1 \leq i,j \leq n} \) whose first column is a column of 1’s. Again, starting with the \( i = n \) row and working down to the \( i = 2 \) row, we subtract the \( i - 1 \)st row from the \( i \)th row. If we continue, following the same sequence of steps as above, and repeat as often as necessary, we get

\[
[\Omega] = \begin{bmatrix}
1 & \Omega_1^{(0)} & \Omega_2^{(0)} & \cdots & \Omega_n^{(0)} \\
0 & 1 & \Omega_2^{(1)} & \cdots & \Omega_n^{(1)} \\
& & & \ddots & \\
0 & 0 & \cdots & 1 & \Omega_n^{(n-1)} \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

where \( \Omega_j^{(0)} = \Omega_j \) and \( \Omega_j^{(j)} = 1 \) for \( 0 \leq j \leq n \); and the \( \Omega_j^{(i)} \in K \) for \( 1 \leq i \leq n \) and \( j > i \) are defined recursively by \( \Omega_j^{(i)} = \phi(\Omega_j^{(i-1)})/\phi(\Omega_1^{(i-1)}) \). Apply \( \phi^{n-i-1} \) to row...
where \( Z \), and get
\[
[\Omega^\varphi] = [\varphi^{n-i-1}(\Omega_{ij}^{(i)})]_{0 \leq i,j \leq n}.
\]

If we define the binomial coefficient \( \binom{X}{Y} \) by \( X \cdot (X-1) \cdots (X-i+1)/i! \in \mathbb{Q}[X] \),
then we can define truncated exponentiation to be the polynomial that results from
the truncation of the binomial series at the \( p \)th term:
\[
(1 + X)^{[Y]} := \sum_{i=0}^{p-1} \binom{Y}{i} X^i \in \mathbb{Z}(p)[X,Y]
\]
where \( \mathbb{Z}(p) \) denotes the integers localized at \( p \).

Choose \( \sigma_i \in G = \text{Gal}(L/K) \) based upon our choice of generators for \( L/K \) by
asking that
\[
[(\sigma_i - 1)x_j] = [\delta_{ij}] = I
\]
(i.e. \( \sigma_i x_i = x_i + 1 \) and \( \sigma_i x_j = x_j \) for \( j \neq i \)). Define
\[
[\Delta_{i,j}] = [\Omega^\varphi]^{-1}.
\]
Now for \( 0 \leq i \leq n \) define \( \Theta_{(i)} \in K[\sigma_n, \sigma_{n-1}, \ldots, \sigma_{n-i}] \) recursively by
\[
\Theta_{(i)} = \sigma_{n-i} \Theta_{(0)}^{\Delta_{n-i,n}} \Theta_{(1)}^{\Delta_{n-i,n-1}} \cdots \Theta_{(i-1)}^{\Delta_{n-i,n-(i-1)}}.
\]

Note that each \( \Theta_{(i)} \) is a 1-unit, i.e. \( \Theta_{(i)} + 1 \in (\sigma - 1 : \sigma \in G) \subseteq K[G] \) where
\( (\sigma - 1 : \sigma \in G) \) can be viewed as the augmentation ideal, the Jacobson radical, or
the nilradical. In particular, \( \alpha^p = 0 \) for all \( \alpha \in (\sigma - 1 : \sigma \in G) \). This means that
\( (\Theta_{(i)} - 1)^p = 0 \), and so \( \Theta_{(i)}^{\Delta_{i,s}} \Theta_{(i)}^{[-\Delta_{i,s}]} = 1 \). As a result, and since \( \Delta_{n-r,n-r} = 1 \),
the recursive definition for \( \Theta_{(i)} \) can be rewritten as
\[
\sigma_{n-i} = \Theta_{(0)}^{\Delta_{n-i,n}} \Theta_{(1)}^{\Delta_{n-i,n-1}} \cdots \Theta_{(i-1)}^{\Delta_{n-i,n-(i-1)}} \Theta_{(i)}^{\Delta_{n-i,n-i}}
\]
which suggests the matrix equation:
\[
\begin{bmatrix}
\Delta_{0,0} & \Delta_{0,1} & \cdots & \Delta_{0,n} \\
0 & \Delta_{1,1} & \cdots & \Delta_{1,n} \\
& & \ddots & \\
0 & \cdots & 0 & \Delta_{n,n}
\end{bmatrix}
\begin{bmatrix}
\Theta_{(n)} \\
\Theta_{(n-1)} \\
\vdots \\
\Theta_{(0)}
\end{bmatrix}
= \begin{bmatrix}
[\sigma_0] \\
[\sigma_1] \\
\vdots \\
[\sigma_n]
\end{bmatrix},
\]
where addition is replaced by multiplication and scalar multiplication is replaced
truncated exponentiation. Since truncated exponentiation does not distribute,
\( (\Theta_{(i)} \Theta_{(j)})^{[\Delta]} \neq \Theta_{(i)}^{[\Delta]} \Theta_{(j)}^{[\Delta]} \) (which is easy to check with \( p = 2 \)), we have \( [\Theta_{(n-j)}] \neq [\Omega^\varphi] \cdot [\sigma_j] \),
despite the fact that \( [\Omega^\varphi] = [\Delta_{i,j}]^{-1} \). In other words, this matrix equation
is simply a convenient way to express a recursive definition – no more, no less.

We are prepared to state the main result of the paper, which is proven in \( \S 3, \S 4 \).

**Theorem 1.1.** Let \( L/K \) be a near one-dimensional elementary abelian extension,
as defined in \( \S 1.2 \). Let \( \Theta_{(i)} \in K[\text{Gal}(L/K)] \) be defined as in (3). For
\( 1 \leq i \leq n \), let \( m_i = v_K(\Omega_{i-1}) - v_K(\Omega_i) \), and choose any \( \alpha_j \in K \) with
\( v_K(\alpha_j) = p^{a_j - 1} \sum_{i=j+1}^n p^{m_i} \). Let \( b_m \) be the largest (lower) ramification break number of
\( L/K \). Given any \( \rho \in L \) with \( v_L(\rho) \equiv b_m \mod p^{a_{n+1}} \) and \( a_s \in \{0, \ldots, p-1\} \),
\[
v_L \left( \prod_{s=0}^n \alpha_{n-s}^{a_s}(\Theta_{(s)} - 1)^{a_s} \rho \right) = v_L(\rho) + \sum_{s=0}^n a_s p^s b_m.
\]
As the integers $\sum_{n=0}^{m} a_n p^n$ run through all possibilities from 0 to $p^{n+1} - 1$, the integers $(\sum_{n=0}^{m} a_n p^n)b_m$ run through all residues modulo $p^{n+1}$. Therefore

**Corollary 1.2.** $L$ has a Galois scaffolding.

**Corollary 1.3.** Any element in $L$ of valuation $b_m$ generates a normal field basis.

This last corollary provides evidence for the Conjecture in [BEo].

2. **Cyclic extensions of degree $p$**

This paper is concerned with Galois, fully and thus wildly ramified $p$-extensions that are, in a certain sense, as simple as cyclic extensions of degree $p$. And so, we should take a moment to consider the prototype: Let $L/K$ be a cyclic, ramified extension of degree $p$. So $L = K(x)$ where $x$ satisfies $\varphi(x) = x^p - x = \beta$ for some $\beta \in K$ with $v_\beta(\beta) = -b$, $b > 0$ and $\gcd(b, p) = 1$. Let $\langle \sigma \rangle = \text{Gal}(L/K)$ with $\sigma x = x + 1$. Since $v_L((\sigma - 1)x) = 0$, it is easy to see that the integer $b$ is the ramification break number for $L/K$. Since $\varphi(x) = \beta$ is really a statement about the norm of $x$, namely $N_{L/K}(x) = \beta$, we have $v_L(x) = -b$ as well.

We may rewrite $x^p - x = \beta$ as $x \cdot (x - 1)^{p-1} = -\beta$, where $(x - 1)^{p-1}$ is a binomial coefficient. Then

$$\frac{x - 1}{p - 1} \in L$$

generates $L/K$, satisfies $v_L((x - 1)^{p-1}) = -(p - 1)b \equiv b \mod p$ and, we contend, is a particularly insightful element to consider. Recall the definition of truncated exponentiation and notice the striking similarity between

$$\sigma^{[i]} \left( \frac{x - 1}{p - 1} \right) = \sigma^i \left( \frac{x - 1}{p - 1} \right) = \left( \frac{x - 1 + i}{p - 1} \right)$$

for $0 \leq i \leq p - 1$,

and the equation in

**Lemma 2.1.** Let $L = K(x)$ with $x^p - x = \beta \in K$ be a cyclic extension with $\langle \sigma \rangle = \text{Gal}(L/K)$. Given $A \in L$,

$$\sigma^{[A]} \left( \frac{x - 1}{p - 1} \right) = \left( \frac{x - 1 + A}{p - 1} \right).$$

**Proof.** Recall Pascal’s identity $\binom{x}{i-1} + \binom{x}{i} = \binom{x+1}{i} \in \mathbb{Q}[X]$, which can be rewritten as $\binom{x+1}{i} - \binom{x}{i} = \binom{x}{i-1}$. This leads to the nice observation, used in [dST07], that $\binom{\sigma - 1}{i} \binom{x}{i-1} = \binom{x-1}{i-1}$ for $0 \leq i \leq p - 1$, and therefore

$$\sigma^{[i]} \left( \frac{x - 1}{p - 1} \right) = \left( \frac{x - 1}{p - 1 - i} \right)$$

for $0 \leq i \leq p - 1$.

Under the substitution $X = \sigma - 1$ and $Y = A \in L$, we find

$$\sigma^{[A]} \left( \frac{x - 1}{p - 1} \right) = \sum_{i=0}^{p-1} \binom{A}{i} \sigma^{-1} \binom{x - 1}{p - 1 - i} = \sum_{i=0}^{p-1} \binom{A}{i} \binom{x - 1}{p - 1 - i} \in L.$$

Vandermonde’s Convolution Identity $\sum_{i=0}^{p-1} \binom{X}{i} \binom{Y}{p - 1 - i} \in \mathbb{Z} [X, Y]$ results from considering the coefficient of $Z^{p-1}$ in the identity $(1+Z)^X \cdot (1+Z)^Y = (1+Z)^{X+Y} \in \mathbb{Q}[X, Y][[Z]]$. If we replace $X = A$ and $Y = x$, we find $\sum_{i=0}^{p-1} \binom{A}{i} \binom{x - 1}{p - 1 - i} = \binom{x - 1 + A}{p - 1} \in L$. □
In [BE05] a refined ramification filtration was introduced. It grew out of the possibility that the natural \( \mathbb{F}_p \)-action on \( \sigma \) could be extended to a residue field “action;” a possibility that is certainly suggested by this striking similarity.

In this paper, we will develop a Galois scaffolding based on this similarity. Specifically, we suppose that \( L/K \) sits in a more general Galois extension \( M/N \), and we suppose furthermore that \( L/N \) is normal and that \( \gamma \in \text{Gal}(M/N) \). So \( \gamma^{-1}x = x + \delta \) for some \( \delta \in L \) and \( \sigma^{[\delta]}(\frac{x}{p-1}) = (\frac{x-1}{p-1})^\delta = \gamma^{-1}(\frac{x}{p-1}) \). But then
\[
\gamma\sigma^{[\delta]}(\frac{x-1}{p-1}) = \left(\frac{x-1}{p-1}\right).
\]
If \( \delta \neq 0 \) and \( \gamma \notin \langle \sigma \rangle \), then neither \( \sigma \) nor \( \gamma \) individually fix the field generator \( (\frac{x}{p-1}) \).

Yet together, using truncated exponentiation, they do. As a result, if we suppose that \( \delta \in N \) then the stabilizer of \( (\frac{x}{p-1}) \) in \( N[\text{Gal}(M/N)] \) is larger than expected.

### 3. Galois scaffolding

This section is motivated by the observation of §2 concerning the stabilizer of \( (\frac{x}{p-1}) \) and should be considered “top-down.” We begin with a generic abelian \( p \)-extension, which we “organize” using the ramification filtration. This “organization” defines a matrix \( [\Delta] \). If the coefficients of \( [\Delta] \) lie in our base field \( K \), this is easy to do. The result is a set \( G \) of the section, one question remains: Are there any elementary abelian extensions that satisfy a strong assumption, which makes it possible for us to construct a Galois scaffolding, but also makes the extension elementary abelian. At the end of the section, one question remains: Are there any elementary abelian extensions that satisfy this strong assumption? In §4 we construct extensions that do – from the “bottom-up.”

Let \( K_n/K \) be a fully ramified, abelian extension of degree \( p^{n+1} \). The case \( n = 0 \) was addressed in §1. So assume \( n \geq 1 \). Let \( G = \text{Gal}(K_n/K) \) and let \( G_i = \{ \sigma \in G : v_n((\sigma-1)\pi_n) \geq i + 1 \} \) denote the Hilbert ramification groups with break numbers \( b_1 < b_2 < \cdots < b_m \) such that \( G = G_{b_1} \), \( G_{b_i} \supseteq G_{b_{i+1}} = G_{b_{i+1}} \), and \( G_{b_{n+1}} = \langle \epsilon \rangle \). Because \( K \) is characteristic \( p \), \( \gcd(b_1, p) = 1 \), and by [Ser79] IV§2 Prop 11], \( b_i \equiv b_1 \mod p \).

Organize the extension by choosing a filtration of \( n + 1 \) subgroups that include the Hilbert ramification groups and satisfy \( G_{(i)}/G_{(i+1)} \cong C_p \).
\[
G = G_{(0)} \supseteq G_{(1)} \supseteq \cdots \supseteq G_{(n)} \supseteq G_{(n+1)} = \langle \epsilon \rangle.
\]
Indeed, since each quotient of consecutive Hilbert ramification groups is elementary abelian, this is easy to do. The result is a set \( \{\sigma_0, \sigma_1, \ldots, \sigma_n\} \) that generates \( G \) (though probably not a minimal generating set), such that \( G_{(i)} = \langle \sigma_i, \sigma_{i+1}, \ldots, \sigma_n \rangle \) and the projection of \( \sigma_i \) generates \( G_{(i)}/G_{(i+1)} \cong C_p \). For \( i \geq 0 \), let the fixed field of \( G_{(i)} \) be \( K_{i-1} \), with \( K_{-1} = K \) and define \( b_{(i)} = v_n(\sigma_i - 1)\pi_n - 1 \). This means that \( b_{(0)} \leq b_{(1)} \leq \cdots \leq b_{(n)} \) is a list of \( n + 1 \) not necessarily distinct integers and \( \{b_{(0)}, \ldots, b_{(n)}\} = \{b_1, \ldots, b_m\} \).

Since \( K_n/K \) is abelian, the Theorem of Hasse-Arf states that the upper ramification numbers are integers [Ser79] IV§3], which is equivalent to \( b_i \equiv b_m \mod [G : G_{b_{i+1}}] \) for \( 1 \leq i \leq m \), and also to
\[
b_{(i)} \equiv b_{(n)} \mod p^{i+1} \text{ for } 0 \leq i \leq n.
\]
Since \( \{b_{(0)}, \ldots, b_{(n)}\} \) is the set of ramification break numbers for \( K_n/K \), the ramification break numbers for \( K_i/K \) are \( \{b_{(0)}, \ldots, b_{(i)}\} \) [Ser79] IV §1 Prop 3 Cor].
Altogether, $\text{Gal}(K_i/K_{i-1}) = G_{b(i)}/G_{b(i+1)} = \langle \sigma_i \rangle \cong C_p$, with $K_i/K_{i-1}$ having ramification break number $b(i)$. As a result, there are $X_i \in K_i$ such that $v_i(X_i) = -b(i)$, $\varphi(X_i) = X_i^p - X_i = B_i \in K_{i-1}$ and $\sigma_i X_i = X_i + 1$. Define

$$\Delta_{i,j} = (\sigma_i - 1)X_j.$$ 

So $\Delta_{i,j} = 0$ when $i > j$, and $\Delta_{i,i} = 1$. Because $X_j \in K_j$ and $\sigma_j \sigma_j = \sigma_j \sigma_i$, we have $\Delta_{i,j} \in K_{j-1}$ when $i < j$. Furthermore, $\nu_j(\Delta_{i,j}) = \nu_j((\sigma_i - 1)X_j) = b(i) - b(j) \leq 0$. Collect these $\Delta_{i,j}$ into a matrix, whose $j$th column lies in $K_{j-1}$,

$$[\Delta] = \begin{bmatrix}
\Delta_{0,0} & \Delta_{0,1} & \cdots & \Delta_{0,n} \\
0 & \Delta_{1,1} & \cdots & \Delta_{1,n} \\
& \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \Delta_{n,n}
\end{bmatrix}.$$ 

Motivated by the final comment in §2, and the fact that we want a basis for $K[G]$ over $K$, we impose

**Assumption 1.** $\Delta_{i,j} \in K$ for all $0 \leq i, j \leq n$.

**Lemma 3.1.** Under Assumption 1, $K_n/K$ is elementary abelian.

*Proof.* Since $\Delta_{i,j} \in K$, we have $\sigma_i^p X_j = X_j + k \Delta_{i,j}$ for $0 \leq k \leq p$. This means that $\sigma_i^p X_j = X_j$ for all $0 \leq i, j \leq n$, and in particular, $\sigma_i^p X_n = X_n$ for all $0 \leq i \leq n$. Since $\nu_n(X_i) = -b(n)$, we have $\gcd(\nu_n(X_i), p) = 1$ and thus $K_n = K(X_n)$. \hfill $\square$

We will proceed in three steps towards our Galois scaffolding. First we choose a nice element $X \in K_n$ with $\nu_n(X) = b(n) = b_m$. Then we determine a basis for $K[G]$ over $K$ so that the valuations of these basis elements applied to $X$ yield a complete set of residues mod $p^{n+1}$. Finally we prove in Proposition 3.3 that this second step continues to hold when $X$ is replaced by any element of valuation $b_m$ mod $p^{n+1}$.

Define $$\rho = \prod_{j=0}^n \left( \frac{X_j}{p-1} \right) \in K_n.$$ 

Because of (4), we may choose $\alpha_j \in K$ such that $v_j(\alpha_j) = b(n) - b(j)$. Therefore $v_j(\alpha_j^{(p-1)}(\frac{X_j}{p-1})) = (p-1)b(n)$ for $0 \leq j \leq n$. Choose $\alpha \in K$ with $\nu_K(\alpha) = b(n)$ Define $A = \alpha \prod_{j=0}^n \alpha_j^{(p-1)} \in K$. So $\nu_n(A) \equiv 0 \mod p^{n+1}$ and 

$$X = A \rho = \alpha \prod_{j=0}^n \alpha_j^{(p-1)}\left( \frac{X_j}{p-1} \right)$$

has valuation $\nu_n(X) = p^{n+1} \cdot b(n) - (p-1) \cdot \sum_{j=0}^n p^{n-j} \cdot b(n) = b(n) = b_m$.

Recall (3), namely the recursive definition for $\Theta_{(i)} \in K[G]$ for $0 \leq i \leq n$.

**Lemma 3.2.** For $0 \leq i, j \leq n$, 

$$\Theta_{(i)} \left( \frac{X_j}{p-1} \right) = \begin{cases} 
\left( \frac{X_j}{p-1} \right) & \text{if } j \neq n - i, \\
\left( \frac{X_j}{p-1} \right)^{p^{n-j}} & \text{if } j = n - i.
\end{cases}$$

*Proof.* We proceed by induction. For $i = 0$, $\Theta_{(i)} = \Theta_{(0)} = \sigma_n$ and since $\sigma_n$ fixes $K_{n-1}$ while $(\frac{X_j}{p-1}) \in K_j$, the result is clear. Now assume the result for $0 \leq i < k$ and consider $\Theta_{(k)} \left( \frac{X_j}{p-1} \right)$. Because $\Theta_{(k)}$ is a product (3), we need to examine the
effect of each factor $\Theta_{(i)}^{[-\Delta_{n-k,n-1}]}$ in that product, namely $\Theta_{(i)}^{[-\Delta_{n-k,n-1}]}(X_i)$ for $0 \leq i < k$. By induction $(\Theta_{(i)} - 1)^r \binom{X_{n-i}}{p-1} = \binom{X_{n-i}}{p-1-r}$ for $0 \leq r \leq p-1$, and $(\Theta_{(i)} - 1)^r \binom{X_{p-1}}{p-1} = 0$ for $j \neq n - i$. Therefore using Lemma 2.1, we have

$$\Theta_{(i)}^{[-\Delta_{n-k,n-1}]}(X_j) = \begin{cases} \binom{X_j}{p-1} & \text{for } j \neq n - i, \\ \binom{X_j - \Delta_{n-k,j}}{p-1} & \text{for } j = n - i. \end{cases}$$

If $j < n - k$, then every factor of $\Theta_{(k)}$ and thus $\Theta_{(k)}$ acts trivially on $\binom{X_i}{p-1}$. If $j = n - k$ then the only factor of $\Theta_{(k)}$ to act non-trivially is $\sigma_{n-k} = \sigma_j$. As a result, $\Theta_{(k)} \binom{X_j}{p-1} = \sigma_j \binom{X_j}{p-1} = \binom{X_j+1}{p-1}$. If $j > n - k$, then exactly two factors of $\Theta_{(k)}$ to act non-trivially, namely $\sigma_{n-k}$ and $\Theta_{(n-j)}^{[-\Delta_{n-k,j}]}$. So

$$\Theta_{(k)}(X_j) = \sigma_{n-k} \Theta_{(n-j)}^{[-\Delta_{n-k,j}]}(X_j) = \sigma_{n-k} \binom{X_j - \Delta_{n-k,j}}{p-1} = \binom{X_j}{p-1}.$$

Now notice that for $0 \leq r \leq p-1$, we have

$$(\Theta_{(i)} - 1)^r \mathbb{X} = (\Theta_{(i)} - 1)^r A \prod_{j=0}^n \binom{X_j}{p-1} = A \prod_{j \neq i} \binom{X_j}{p-1} \cdot (\Theta_{(i)} - 1)^r \binom{X_{n-i}}{p-1} = A \prod_{j \neq i} \binom{X_j}{p-1} \binom{X_{n-i}}{p-1-r}.$$  

Therefore $(\Theta_{(i)} - 1)^r \mathbb{X} = \mathbb{X} \binom{X_{n-i}}{p-1-r} \binom{X_{n-i}}{p-1}^{-1}$ and so $v_n((\Theta_{(i)} - 1)^r \mathbb{X}) = b_n + rp^i b_{n-i}$. Moreover given $c_i \in \{0, 1, \ldots, p-1\}$, we have

$$\prod_{i=0}^n (\Theta_{(i)} - 1)^{c_i} \mathbb{X} = A \prod_{j=0}^n \binom{X_j}{p-1 - c_{n-j}},$$

and using the $\alpha_j \in K$ with $v_j(\alpha_j) = b_{(n)} - b_{(j)}$,

$$(5) \quad v_n \left( \prod_{i=0}^n \alpha_{n-i}^{c_i} (\Theta_{(i)} - 1)^{c_i} \mathbb{X} \right) = \left( 1 + \sum_{i=0}^n c_i p^i \right) b_{(n)}.$$

Therefore

$$\left\{ \prod_{i=0}^n \alpha_{n-i}^{c_i} (\Theta_{(i)} - 1)^{c_i} : 0 \leq c_i \leq p-1 \right\}$$

is the desired basis.

**Proposition 3.3.** Under Assumption 1, we have a Galois scaffolding. Let $\mathbb{X} \in K_n$ be any element with $v_n(\mathbb{X}) \equiv b_{(n)} \mod p^{n+1}$. Let $\Theta_{(i)} \in K[G]$ be as defined in (3), and let $\alpha_j \in K$ with $v_K(\alpha_j) = (b_{(n)} - b_{(j)})/p^{i+1} \in \mathbb{Z}$, then

$$v_n \left( \prod_{i=0}^n \alpha_{n-i}^{c_i} (\Theta_{(i)} - 1)^{c_i} \mathbb{X} \right) = v_n(\mathbb{X}) + \sum_{i=0}^n c_i p^i b_m.$$
Proof. Using (5), we can express $X$ as a linear combination of $\prod_{i=0}^{n} \alpha_{n-i}^{c_i}(\Theta(i) - 1)^{c_i}X$ with coefficients in $K$. It is enough therefore to show that when we apply $\prod_{i=0}^{n} \alpha_{n-i}^{d_i}(\Theta(i) - 1)^{d_i}$ with $0 \leq d_i \leq p - 1$ to any term in this linear combination, we increase valuation by at least $\sum_{i=0}^{n} d_i p^ib_m$, namely

$$v_n\left(\prod_{i=0}^{n} \alpha_{n-i}^{c_i+d_i}(\Theta(i) - 1)^{c_i+d_i}X\right) \geq v_n\left(\prod_{i=0}^{n} \alpha_{n-i}^{c_i}(\Theta(i) - 1)^{c_i}X\right) + \sum_{i=0}^{n} d_i p^ib_m.$$ 

If any sum $c_i + d_i \geq p$ then $(\Theta(i) - 1)^{c_i+d_i} = 0$ and the valuation of the left-hand-side is infinite. So we are left with the case where all sums $c_i + d_i < p$. But in this case, we can use (5) to determine that we have equality. \qed

4. Near One-dimensional Elementary Abelian Extensions

In contrast with §3, this section is “bottom-up”. Motivated by the idea of maximal refined ramification in [BEa], we follow §1.2 and define the class of near one-dimensional elementary abelian extensions, by describing how the generators of each extension are related. We organize these generators by size (by valuation) as in §1.2, and then define the matrix $[\Omega^\phi]$, over $K$ as in §1.3. Our organization of the generators, “organizes” the matrix $[\Omega^\phi]$. The main result of the section is that this also “organizes” the extension in essentially the same fashion as in §3. In particular, $[\Omega^\phi] \cdot [\Delta] = I$, which means that near one-dimensional elementary abelian extension satisfy Assumption 1 and thus possess Galois scaffolding.

Recall the notation of §1.2: Let $L = K(x_0, \ldots, x_n)$ with $\varphi(x_i) = \phi^n(\Omega_i) \cdot \beta + \epsilon_i$ for some $\beta \in K$ with $v_K(\beta) = -b$, $b > 0$ and gcd$(b, p) = 1$; some $\Omega_i \in K$ that span an $n + 1$-dimensional subspace over $\mathbb{F}_p$; and some “error terms” $\epsilon_i \in K$, whose size will be controlled by (6) below. Initially, we merely assume $v_K(\epsilon_i) > v_K(\phi^n(\Omega_i))$, so the ramification break number of $K(x_1)/K$ is $-v_K(\phi^n(\Omega_i)\beta)$.

Furthermore recall $\Omega_0 = 1$ and that the other $\Omega_i$ are “organized” (relabelled) so that $v_K(\Omega_n) \leq \cdots \leq v_K(\Omega_1) \leq v_K(\Omega_0) = 0$, and if $v_K(\Omega_i) = \cdots = v_K(\Omega_j)$ for $i < j$, the projections of $\Omega_i, \ldots, \Omega_j$ into $\phi^n(\Omega_i)\beta\Omega_i/\phi^n(\Omega_i)\beta\mathbb{Q}_K$ are linearly independent over $\mathbb{F}_p$. This means that $K(x_i, \ldots, x_j)$ has one break in its ramification filtration at $-v_K(\phi^n(\Omega_i)\beta)$.

For $1 \leq i \leq n$, define $m_i = v_K(\Omega_{i-1}) - v_K(\Omega_i) \geq 0$. We control the size of the error terms with: For $1 \leq i \leq n$,

$$v_K(\epsilon_i) < -\frac{b}{p^n} - \sum_{j=1}^{i} p^j m_j + \sum_{j=i+1}^{n} (p^n - p^j)m_j = v_K(\phi^n(\Omega_i)\beta) + \frac{(p^n-1)b}{p^n} - (p-1)\sum_{j=1}^{n-i} p^j v_K(\Omega_j),$$

which since $v_K(\Omega_j) \leq 0$ is clearly stronger than our initial assumption, $v_K(\epsilon_i) > v_K(\phi^n(\Omega_i)\beta)$. Notice further that if, for a particular $i$, the right-hand-side of (6) is zero, then (6) is equivalent to “no error” (i.e. $\epsilon_i = 0$), since the inequality $v_K(\epsilon_i) > 0$ implies $\epsilon_i \in K^p$.

Choose $\sigma_i \in G = \text{Gal}(L/K)$ based upon our generators so that $[(\sigma_i - 1)x_j] = [\delta_{ij}] = I$. Define $H(i) = \langle \sigma_i, \ldots, \sigma_n \rangle$, and let $K_{i-1} = K(x_0, \ldots, x_{i-1})$ be the fixed
field of $H_{(i)}$. So $K_{-1} = K$ and $K_n = L$. As noted earlier,

$$u_{(i)} = b + p^n \sum_{j=1}^{i} m_j$$

is the ramification number of $K(x_i)/K$, and is therefore an upper ramification number of $L/K$. By considering our assumptions on the $\Omega_i$, one sees that the set of upper ramification numbers is $\{u_{(0)}, \ldots, u_{(n)}\}$. We may pass to the lower ramification numbers using the Herbrand function $\psi(x)$ [Ser73, IV §3]. Again considering our assumptions on the $\Omega_i$, one sees that $\{b_{(0)}, \ldots, b_{(n)}\}$ is the set of lower ramification numbers where

$$b_{(i)} = b + p^n \sum_{j=1}^{i} p^j m_j.$$ 

Moreover, $b_{(i)}$ is the ramification number of $K_{i}/K_{i-1}$, and it is clear that the groups $H_{(i)}$ are the groups $G_{(i)}$ defined in §3. We can express the restriction on the error terms in (6) in terms of ramification numbers: $v_{K}(e_i) > -b_{(n)}/p^n + u_{(n)} - u_{(i)}$.

Our next step is to construct the $X_i \in K_i$ of §3. Recall the $\Omega_{(i)}$ defined in §1.3. Define $X_{(0)} = x_j$. And for $j \geq i$, recursively define

$$X_{(i)} = X_{(i-1)} - \phi^{n-1}(\Omega_{(i-1)}) X_{(i-1)}.$$ 

If we use this definition to replace $X_{(i-1)}$ in (7) with $X_{(i-2)} - \phi^{n-1}(\Omega_{(i-2)}) X_{(i-2)}$, we find that $X_{(i)} = X_{(i-2)} - \phi^{n-1}(\Omega_{(i-2)}) X_{(i-2)} - \phi^{n-1}(\Omega_{(i-1)}) X_{(i-1)}$. If we continue in this way, we eventually find $X_{(i)} = X_{(0)} - \sum_{k=0}^{i-1} \phi^{n-1}(\Omega_{(k)}) X_{(k)}$. Consider the case $i = j$. Since $x_j = X_{(0)}$ and $\Omega_{(j)} = 1$, this can be rewritten as

$$x_j = \sum_{k=0}^{j} \phi^{n-k-1}(\Omega_{(k)}) X_{(k)}.$$ 

Recall that $[\Omega^0] = [\phi^{n-1}(\Omega_{(i)})]_{0 \leq i, j \leq n}$. Therefore

$$[X_{(0)}, X_{(1)}, \ldots, X_{(n)}] : [\Omega^0] = [x_0, x_1, x_2, \ldots, x_n].$$

Since $I = [(\sigma_i - 1)x_j]$, we find that $[(\sigma_i - 1)X_{(j)}] : [\Omega^0] = I$. Therefore

$$[(\sigma_i - 1)X_{(j)}] = [\Omega^0]^{-1}.$$ 

Clearly $K_j = K(x_0, \ldots, x_j) = K(X_{(0)}, \ldots, X_{(j)})$. If we could determine that $v_j(X_{(j)}) = -b_{(j)}$, then we could choose the $X_{(j)} = X_{(j)}$ and find that $[\Delta_{i,j}] = [\Omega^0]^{-1}$. As a result, our extension would satisfy Assumption 1. The remainder of this section is therefore concerned with the valuation $v_j(X_{(j)})$. Since the $\Omega_{(i)}$ are an important ingredient in the definition of the $X_{(j)}$, given in (7), we need

**Lemma 4.1.** For $0 \leq i < j \leq n$

$$v_K(\Omega_{(i)}) = -p^i \sum_{k=i+1}^{j} m_k$$

**Proof.** We induct on $i$. Since $m_k = v_K(\Omega^{(0)}_{k-1}) - v_K(\Omega^{(0)}_k)$ for $1 \leq k \leq n$, the result holds for $i = 0$. For $i > 1$, we assume the result. So in particular, $v_K(\Omega^{(i-1)}_{i+1}) \leq \cdots \leq v_K(\Omega^{(i-1)}_i) \leq v_K(\Omega^{(i-1)}_i) = 0$. Then $v_K(\varphi(\Omega^{(i-1)}_i)) = pv_K(\Omega^{(i-1)}_i)$ and
thus using the definition for $\Omega_j^{(i)}$ in §1.3, we find that $v_K(\Omega_j^{(i)}) = pv_K(\Omega_j^{(i-1)}) - pv_K(\Omega_i^{(i-1)})$ and result follows.

To assist in our analysis of $v_j(X_j^{(j)})$, define $B_0 = \beta$, $E_j^{(0)} = \epsilon_j$ for $j > 0$. Then for $i > 0$ recursively define

$$B_i = -\phi^{n-i}(\phi(\Omega_i^{(i-1)}))X_{i-1}^{(i-1)} + E_i^{(i-1)}$$

and $E_j^{(i)} = E_j^{(i-1)} - \phi^{n-i}(\Omega_j^{(i)})E_i^{(i-1)}$ for $j > i$. And $E_i^{(i)} = 0$. The significance of these $B_i$ and $E_j^{(i)}$ results from

**Lemma 4.2.** For $j \geq i$

$$\varphi(X_j^{(i)}) = \phi^{n-i}(\Omega_j^{(i)})B_i + E_j^{(i)}$$

**Proof.** The statement is clear for $i = 0$. Assume that it holds for $i - 1$. Therefore $\varphi(X_j^{(i-1)}) = \phi^{n-i+1}(\Omega_j^{(i-1)})B_{i-1} + E_j^{(i-1)}$ and in particular, $\varphi(X_j^{(i-1)}) = B_{i-1}$. Consider $\varphi(X_j^{(i)})$. It is easy to see that $\varphi(aX) = \phi(a)\varphi(X) + \varphi(a)X$. Therefore using (7) we find that

$$\varphi(X_j^{(i)}) = \varphi(X_j^{(i-1)}) - \phi^{n-i+1}(\Omega_j^{(i-1)})\varphi(X_{i-1}^{(i-1)}) - \phi^{n-i}(\phi(\Omega_j^{(i-1)}))X_{i-1}^{(i-1)}$$

$$= \phi^{n-i+1}(\Omega_j^{(i-1)})B_{i-1} + E_j^{(i-1)} - \phi^{n-i+1}(\Omega_j^{(i-1)})B_{i-1} - \phi^{n-i}(\phi(\Omega_j^{(i-1)}))X_{i-1}^{(i-1)}$$

$$= E_j^{(i-1)} - \phi^{n-i}(\phi(\Omega_j^{(i-1)}))X_{i-1}^{(i-1)},$$

which, using (9), can be seen to agree with the statement for $i$.

**Lemma 4.3.** Assume the bounds given in (6). Then for $1 \leq i \leq n$, we have

$$v_K(E_i^{(i-1)}) > -b_{(i)}/p^i.$$  

**Proof.** Use Lemma 4.1 to determine that (6) is equivalent to

$$v_K(\phi^{-i}(\Omega_i^{(n)})\epsilon_i) > -b_{(n)}/p^n.$$  

We are interested in $v_K(E_i^{(i-1)})$. So recall that $E_j^{(i)} = E_j^{(i-1)} - \phi^{n-i}(\Omega_j^{(i)})E_i^{(i-1)}$ for $j > i$, which means that $E_j^{(i)} = E_j^{(0)} - \sum_{k=1}^{i} \phi^{n-k}(\Omega_j^{(k)})E_k^{(k-1)}$, and in particular,

$$(10) \quad E_i^{(i-1)} = \epsilon_i - \sum_{k=1}^{i-1} \phi^{n-k}(\Omega_i^{(k)})E_k^{(k-1)}.$$  

In order that $v_K(E_i^{(i-1)}) > -b_{(i)}/p^i$ for $1 \leq i \leq n$, it is sufficient to prove

$$(11) \quad v_K(\epsilon_i) > -b_{(i)}/p^i \text{ for } 1 \leq i \leq n, \text{ and}$$  

$$(12) \quad v_K(\phi^{n-k}(\Omega_i^{(k)})E_k^{(k-1)}) > -b_{(i)}/p^i \text{ for } 1 \leq k \leq i - 1 \leq n - 1.$$  

Let $A_i = -b_{(i)}/p^i + v_K(\phi^{-i}(\Omega_i^{(n)}))$. Using Lemma 4.1, we find that $-b_{(i)}/p^i + v_K(\phi^{-i}(\Omega_i^{(n)})) = -b_{(i-1)}/p^i + v_K(\phi^{-i+1}(\Omega_n^{(i-1)}))$. As a result, $A_i > A_{i-1}$, since $-b_{(i-1)}/p^i > -b_{(i-1)}/p^{i-1}$. We are given by (6) that $v_K(\phi^{-i}(\Omega_i^{(n)})\epsilon_i) > -b_{(n)}/p^n = A_n$. So $v_K(\phi^{-i}(\Omega_i^{(n)})\epsilon_i) > A_j$ for all $j$, including $j = i$. Therefore (11) follows from (6).
Focus on (12), which is equivalent to \( v_K(E_k^{(k-1)}) > B_k^i \) where \( B_k^i = -b_{(i)}/p^j - v_K(\phi^{n-k}(\Omega_i^{(k)})) \). Since \(-b_{(i)}/p^j - v_K(\phi^{n-k}(\Omega_i^{(k)})) = -b_{(i-1)}/p^j - v_K(\phi^{n-k}(\Omega_{i-1}^{(k)}))\), we have \( B_k^i > B_{i-1}^k \). And thus (12) is equivalent to

\[
(13) \quad v_K(\phi^{n-k}(\Omega_i^{(k)})E_k^{(k-1)}) > -b_{(i)}/p^n \text{ for } 1 \leq k \leq n-1.
\]

Switch the roles of \( i \) and \( k \) in (10) and then apply \( \phi^{n-k}(\Omega_i^{(k)}) \) to both sides:

\[
\phi^{n-k}(\Omega_i^{(k)})E_k^{(k-1)} = \phi^{n-k}(\Omega_i^{(k)})e_k - \sum_{i=1}^{k-1} \phi^{n-k}(\Omega_i^{(k)})\phi^{n-i}(\Omega_{i}^{(i)})E_i^{(i-1)}.
\]

By Lemma 4.1, \( v_K(\phi^{n-k}(\Omega_i^{(k)})\phi^{n-i}(\Omega_{i}^{(i)})) = v_K(\phi^{n-i}(\Omega_{i}^{(i)})) \). Therefore (13) follows from (6) by induction on \( k \).

\[\square\]

**Lemma 4.4.** Assume the bounds in (6). Then for \( 0 \leq j \leq n \), \( v_j(X_j^{(0)}) = -b_{(j)} \).

**Proof.** It is clear that \( v_0(X_0^{(0)}) = -b_{(0)} \). So for \( i > 0 \), assume that \( v_{i-1}(X_{i-1}^{(i-1)}) = -b_{(i-1)} = -b - p^n \sum_{j=1}^{i-1} p^jm_j \). Using Lemma 4.1, we see that \( v_K(\phi(\Omega_{i-1}^{(i-1)})) = -p^im_i \). So \( v_K(\phi^{-i}(\phi(\Omega_{i-1}^{(i-1)}))) = -p^im_i \) and therefore \( v_{i-1}(\phi^{-i}(\phi(\Omega_{i-1}^{(i-1)}))) = -p^n(p^im_i) \). So \( v_{i-1}(\phi^{-i}(\phi(\Omega_{i-1}^{(i-1)}))) = -b_{(i)} \). By Lemma 4.3, \( v_{i-1}(E_{i-1}^{(i-1)}) > -b_{(i)} \). Therefore \( v_{i-1}(B_i) = -b_{(i)} \). Lemma 4.2 implies that in particular the norm \( N_{K_i/K_{i-1}}(X_{i}^{(i)}) = \phi(X_{i}^{(i)}) = B_i \), which means that \( v_i(X_{i}^{(i)}) = -b_{(i)} \).

As a result, we can put all this together and find

**Proposition 4.5.** Near one-dimensional elementary abelian extensions satisfy Assumption 1.

5. EXAMPLES OF NEAR ONE-DIMENSIONAL ELEMENTARY ABELIAN EXTENSIONS

**Lemma 5.1.** Fully ramified biquadratic extensions are near one-dimensional elementary abelian extensions.

**Proof.** Biquadratic extensions are special in that there is only one nontrivial residue modulo 2. Let \( L/K \) be a fully ramified biquadratic extension. We may assume that \( L = K(x_0, x_1) \) with \( x_0^2 - x_0 = \beta \), \( x_1^2 - x_1 = \beta_1 \), \( v_K(\beta_1) \leq v_K(\beta) < 0 \) and both of \( v_K(\beta_1) \) and \( v_K(\beta) \) odd. Because the difference of two odd numbers is even, there is a \( \mu_0 \in K \) such that \( \lambda_0^2\beta_1 \equiv \beta \mod \mathfrak{P}_K \). Let \( \beta = \mu_0^2\beta_1 + \tau_0 \) for some \( v_K(\tau_0) > v_K(\beta) \). Since we can replace \( \beta \) by any element in its coset \( \beta + K^\circ \), we may assume \( v_K(\tau_0) = 0 \), or \( v_K(\tau_0) < 0 \) with \( v_K(\tau_0) \) odd. If \( v_K(\tau_0) \) odd, then there is a \( \mu_1 \in K \) such that \( \mu_1^2\beta_1 \equiv \tau_0 \mod \tau_0\mathfrak{P}_K \), and thus \( \beta = (\mu_0 + \mu_1)^2\beta_1 + \tau_1 \) for \( v_K(\tau_1) > v_K(\tau_0) \). Continue in this way until \( \beta = \mu^2\beta_1 + \tau \) for some \( \mu \in K \) and either \( \tau = 0 \) or \( v_K(\tau) = 0 \).

If \( \tau = 0 \), then \( \beta_1 = \mu^{-2}\beta \) and the extension is one-dimensional. If \( v_K(\tau) = 0 \), then \( \beta_1 = \Omega_1^2\beta + \epsilon_1 \) where \( \epsilon_1 = -\tau\mu^{-2} \) and \( \Omega_1 = \mu^{-1} \). Continuing to translate into the notation of §4, we note that \( b = -v_K(\beta) \) and \( m_1 = -v_K(\Omega_1) = v_K(\mu) \). So \( v_K(\epsilon_1) = -2m_1 > -b/2 - 2m_1 \), which is the inequality given by (6). So the extension is near one-dimensional.

\[\square\]

**Lemma 5.2.** Let \( K = \mathbb{F}(t) \) with \( \mathbb{F}_q \subset \mathbb{F} \), and let \( \beta \in K \) with \( v_K(\beta) < 0 \) and \( \gcd(v_K(\beta), p) = 1 \). Then \( L = K(y) \) with \( y^q - y = \beta \) is a one-dimensional elementary abelian extension of \( K \).
Proof. Let \( q = p^f \) and let \( \{ 1 = \omega_0, \omega_1, \ldots, \omega_{f-1} \} \) be a basis for \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Then
\[
x_i = \sum_{r=0}^{f-1} \phi^r(\omega_i y) \quad \text{where} \quad y^q - y = \beta \quad \text{satisfies} \quad x_i^p - x_i = \omega_i \beta.
\]
of course \( \phi \) is an automorphism of \( \mathbb{F}_q \). So we may set \( \Omega_i = \phi^{-f+1}(\omega_i) \).

The following class of fully and weakly ramified \( p \)-extensions (i.e. with \( G = G_1 \) and \( G_2 = \{ e \} \)) is notable for being wildly ramified while possessing a normal integral basis (for the maximal ideal) [Ull70].

**Lemma 5.3.** Let \( L/K \) be a noncyclic, fully and weakly ramified \( p \)-extension, then \( L/K \) is a near one-dimensional elementary abelian extension.

Proof. The extension is elementary abelian [Ser79, IV §2], with one break in its ramification filtration at \( b = 1 \). As a result there is only one upper ramification break number, also at \( u = 1 \). Thus \( L = K(x_0, x_1, \ldots x_n) \) with \( v_K(\phi(x_i)) = -1 \).

Let \( \beta = \phi(x_0) \). Then there are units \( \omega_i \in F \) such that \( \phi(x_i) = \omega_i \beta \mod \mathfrak{O}_K \). Since \( \phi \) is an automorphism of \( F \), we may let \( \Omega_i = \phi^{-n}(\omega_i) \) and find \( \epsilon_i \in \mathfrak{O}_K \) such that \( \phi(x_i) = \phi^n(\Omega_i) \beta + \epsilon_i \) with either \( \epsilon_i = 0 \) or \( v_K(\epsilon_i) = 0 \). Using the notation of §4, we find that \( m_i = 0 \) and in all cases \( v_K(\epsilon_i) \geq 0 > \frac{-1}{p^n} = \frac{-b}{p^n} \), which is (6). \( \square \)

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