Scalar-field theory of dark matter

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We develop a theory of dark matter based on a previously proposed picture, in which a complex vacuum scalar field makes the universe a superfluid, with the energy density of the superfluid giving rise to dark energy, and variations from vacuum density giving rise to dark matter. We formulate a nonlinear Klein-Gordon equation to describe the superfluid, treating galaxies as external sources. We study the response of the superfluid to the galaxies, in particular, the emergence of the dark-matter galactic halo, contortions during galaxy collisions, and the creation of vortices due to galactic rotation.

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1. Introduction and summary

The standard model of particle theory postulates the existence of at least one complex vacuum scalar field (the Higgs field), in order to generate certain masses in a gauge-invariant manner, mainly in the weak sector, through spontaneous symmetry breaking. Direct experimental support of this field comes from the discovery of the associated field quantum, the Higgs boson. Extension of the standard model to supersymmetric or grand unified theories will bring in still more complex vacuum fields. We may therefore suppose that the universe is permeated with complex scalar fields, whose manifestation in the large has not been the preoccupation of particle theory, but is the main focus of this paper.

As we learn from condensed matter physics, a complex scalar field on the macroscopic scale gives rise to superfluidity. The field serves as an order parameter that
expresses quantum phase coherence over macroscopic distances, and the superfluid velocity corresponds to the gradient of the phase of the complex field. From this point of view, we must view the universe as superfluid. It has been proposed that this cosmic superfluidity offers explanations to both “dark energy” and “dark matter”:

- Dark energy is the energy density of the cosmic superfluid, which drives an accelerated expansion of the universe;
- Dark matter is the manifestation of local density variations of the superfluid, which are detectable through gravitational lensing.

In this paper, we formulate and apply a phenomenological theory of dark matter based on this picture.

In Ref., the dark-energy aspect is explored in a model using a self-interacting scalar field coupled to gravity through Einstein’s equation with Robertson-Walker metric, in the era immediately following the big bang. The Robertson-Walker metric sets the only length scale in the problem, and the field potential must change with this scale through quantum field-theoretic renormalization. For mathematical consistency, one requires that the potential be asymptotically free (i.e., vanish in the small-distance limit,) and this determines it to be the Halpern-Huang potential derived from renormalization-group studies. It is a non-polynomial potential with exponential behavior for large fields, a feature leading to the generation of an effective cosmological constant that decrease in time like a power, and this gives dark energy (accelerated expansion of the universe), while avoiding the usual “fine-tuning” problem.

The same model gives a new scenario of the inflation era. After the big bang, matter (modeled as a perfect fluid coupled to the scalar field) was created in the quantum turbulence of the cosmic superfluid, in the form of a vortex tangle. It grows and decays in about $10^{-23}$s, and is described phenomenologically by Vinen’s equation originally used in liquid helium. The inflation era corresponds to the lifetime of quantum turbulence, during which the radius of the universe increases by a factor $10^{27}$, and all the matter needed for subsequently nucleosynthesis was created. At the end of the inflation era, the usefulness of this model is ended, for density variations in the universe became important. The model passes control to the standard hot big bang theory, with an important condition: the universe remains a superfluid, and all astrophysical processes take place in this superfluid.

The foregoing describes how the cosmic superfluid comes into being, and its role during the inflation era. Now we have to describe it post inflation, in which conditions have become somewhat different.

The big bang era was governed by the Planck scale of $10^{18}$GeV, which was built into Einstein’s equation through the Robertson-Walker metric. With the emergence of matter comes the nucleon mass scale of 1 GeV. This scale ultimately arises in scale-invariant QCD through spontaneous breaking of chiral symmetry — an effect referred to as “dimensional transmutation”. (The observable mass of the universe
comes mainly from protons and neutrons, and has little to do with the Higgs.) Since the matter length scale is much smaller than the Planck scale, it should prevail in the dynamics of stars and galaxies. The decoupling between the nuclear scale and the Planck scale is illustrated in Ref.\(^4\) In the present work, which is largely concerned with the present universe, we simply ignore the influence of the Planck scale, and use a simple \(\phi^4\) theory with constant phenomenological coupling constants.

In principle, one should use an updated Halpern-Huang potential, with needed renormalization to bridge the Planck era and the present. Everything that happens between then and now — the expansion of the universe, emergence of the matter scale, interactions between matter and the scalar field — will contribute to this “running” of the potential, with the length scale expanding by 18 orders of magnitude. We bypass this problem here by adopting the phenomenological \(\phi^4\) potential.

The order parameter originates in Landau\(^6\) theory of phase transitions, in which a high-temperature phase with less “order” (i.e., higher degree of symmetry) changes into a low-temperature phase with more “order” (less symmetry), through what is now called spontaneous symmetry breaking. The order parameter is something that gives a measure of “order”, and it vanishes above the critical temperature. The generic example is the magnetic moment density in a paramagnetic phase transition, which can be represented by a scalar field \(\phi\) with a symmetric field potential \(V(\phi)\), which has only one minimum with \(\phi = 0\) at high temperatures, but exhibits two minima at nonzero values \(\phi = \pm\phi_0\) below the critical temperature. This is a phenomenological way to describe spontaneous symmetry breaking. In the Ginsburg-Landau theory of superconductivity\(^7\) the order parameter is a complex scalar field coupled to the electromagnetic field, and it successfully describes the Meissner effect, in which the photon acquires mass inside a superconductor. (This is the first instance of the so-called “Higgs mechanism”.) We now know that this order parameter corresponds to the complex wave function of Cooper pairs in the microscopic BCS theory\(^8\) and the Ginsburg-Landau theory has been derived\(^9\) from BCS. However, it continues to serve as an indispensable practical tool.

The Landau approach is also important for understanding superfluidity in liquid helium and cold-trapped atoms\(^10\) The order parameter in this case is the complex wave function of the Bose-Einstein condensate, and it obeys the Gross-Pitaevskii equation, an instance of the NLSE (nonlinear Schrödinger equation) first used in nonlinear optics\(^11\).

In our investigation here, we consider a generic complex scalar field in curved space-time, which may correspond to some vacuum field in particle theories, such as supersymmetry or grand unified theories. We assume that the field obey the nonlinear Klein-Gordon equation (NLKG) in curved space-time. We shall discuss the definition of superfluid density and superfluid velocity in a relativistic setting. Matter will be treated as a perfect fluid with given density current, interacting with the scalar field via current-current interaction. In this work we have not considered
the dynamics of matter itself.

The NLKG describes a pure superfluid at absolute zero temperature. At higher temperatures, there will appear a normal fluid, consisting of phonons, field quanta, as well as particles coupled to it, existing with the superfluid in thermal equilibrium. The so-called WIMPs (weakly interacting massive particles) searched for in particle-physics setups\textsuperscript{12} would be a component of our normal fluid, if they exist, and this would explain how they came to permeate all space. However, we have not taken the normal fluid into account in this paper.

The most persuasive evidence for dark matter is the detection of galactic halos via gravitational lensing. In our picture, the dark-matter halo is made up of extra superfluid drawn around a galaxy, due to attractions between galaxy and superfluid. The halo can follow the motion of a galaxy like a “soliton”. It can follow the rotation of a galaxy by developing quantized vortices. When two galaxies collide, their halos will merge and undergo contortions. All these phenomena are part of the superfluid hydrodynamics that follows from the NLKG. Observed dark matter formations like the so-called “bullet cluster”\textsuperscript{13} and “train wreck”\textsuperscript{14} are, in our picture, manifestations of superfluid hydrodynamics.

The main topics investigated in this paper are

- fitting model parameters to cosmological data,
- dark matter in galactic halos,
- quantized superfluid vortices,
- numerical simulations of dark-matter halos, galaxy collisions, vortices.

2. Nonlinear Klein-Gordon equation (NLKG)

We use units in which $\hbar = c = 1$, and a metric corresponding to $(-1, 1, 1, 1)$ in flat space-time. The vacuum complex scalar field is denoted by

$$\phi(x) = F(x) e^{i\sigma(x)}$$

The classical Lagrangian density in the absence of galactic matter is

$$\mathcal{L} = -g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - V$$

where $g^{\mu\nu}$ is the metric tensor, and $V(\phi^* \phi)$ is the self-interaction potential. The action is

$$S = -\int dt d^3 x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + V)$$

where $g = \det(g_{\mu\nu})$. This leads to the equation of motion

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \frac{\partial V}{\partial \phi^*} = 0$$

We use a phenomenological $\phi^4$ potential, which is the simplest way to maintain a nonzero vacuum field $F_0$:

$$V = \frac{\lambda}{2} \left( |\phi|^2 - F_0^2 \right)^2 + V_0$$
where $V_0$ is the vacuum energy density, the effective cosmological constant that gives rise to dark energy. It is easy to see that the action (6) has a global U(1) symmetry since the potential $V$ is invariant under the transformation $\phi \rightarrow \phi e^{i\alpha}$, $\phi^* \rightarrow \phi^* e^{-i\alpha}$.

In flat space-time, in the absence of galactic matter, the equation of motion is the nonlinear Klein-Gordon equation (NLKG)

$$\left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \phi - \lambda \left( |\phi|^2 - F^2_0 \right) \phi = 0$$

(6)

with a conserved current density

$$j^\mu = \frac{1}{2i} \left( \phi^\star \partial^\mu \phi - \phi \partial^\mu \phi^\star \right) = F^2 \partial^\mu \sigma$$

(7)

In the presence of a galaxy (a generic term that includes a star), we add to the Lagrangian density an interaction term $L_{\text{int}}$, which represents the non-gravitational interaction between galaxy and scalar field. (Gravity will be included in the next section). The galaxy is introduced as an external source, with a given energy current density $J^\mu$. Its interaction with the scalar field is described phenomenologically through a current-current interaction, as dictated by Lorentz covariance:

$$L_{\text{int}} = -\eta J^\mu j_\mu$$

(8)

where $\eta$ is a coupling constant, and $J^\mu$ is a four-vector:

$$J^\mu = (\rho, J)$$

(9)

where the given function $\rho(x)$ describes the density profile of the galaxy. Note that from (7) and (8) we have $L_{\text{int}} = -\eta F^2 J^\mu \partial_\mu \sigma$ which has, from particle physics point of view, the typical form of a coupling between the Noether current $J^\mu$ from the normal matter sector and the Nambu-Goldstone boson $\sigma$, if the U(1) symmetry is spontaneously broken. In fact $\eta F^2_0$ can be parametrized as $f^{-1}_\sigma$ in analogy with the pion decay constant $f_{\pi}$. In this article we shall treat it phenomenologically as an effective coupling at the macroscopic level, and fit it to galactic data. (See section 8 for the estimate of the range of $\eta$, and section 9 for the discussion on the long-range force induced by the scalar $\phi$).

As an example we consider the rotation of a galaxy submerged in the cosmic superfluid. For a galaxy rotating as a rigid body with angular velocity $\Omega$, we have

$$J = \rho \Omega \times r$$

(10)

where $r$ is the distance vector from the center of the galaxy. In the equation of motion, $\eta$ and $\rho$ occur only in the combination $\eta \rho$. We define the coupling constant $\eta$ precisely by normalizing $\rho$:

$$\int d^3 x \rho = M_{\text{galaxy}}$$

(11)

where $M_{\text{galaxy}}$ is the total mass of the galaxy.
The NLKG now reads

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) \phi - \lambda \left(|\phi|^2 - F_0^2\right) \phi - i\eta J^\mu \partial_\mu \phi = 0 \quad (12)$$

or

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) \phi - \lambda \left(|\phi|^2 - F_0^2\right) \phi - i\eta \rho \left(\frac{\partial \phi}{\partial t} + \Omega \times \mathbf{r} \cdot \nabla \phi\right) = 0 \quad (13)$$

This equation contains a correlation length \(\xi_0\) arising from the vacuum field \(F_0\):

$$\xi_0 \equiv \frac{1}{\sqrt{\lambda F_0}} \quad (14)$$

which we identify with the correlation length in galaxy clustering\(^{15}\)

$$\xi_0 \approx 11 \text{ Mpc} = 3.3 \times 10^{25} \text{ cm} \quad (15)$$

This gives the mass of scalar-field quanta as

$$m_0 = \xi_0^{-1} \approx 1.8 \times 10^{-30} \text{ eV} \quad (16)$$

Another estimate of \(\xi_0\) can be obtained from the power spectrum of the CMB (cosmic microwave background). When analyzed in terms of Legendre polynomials \(P_\ell(\cos \theta)\), the angular distribution of thermal fluctuations in the CMB shows a peak at \(\ell \approx 200\).\(^{16}\) Now, the angular pattern \(P_\ell(\cos \theta)\) divides the circle into \(2\ell\) sectors, each subtending an angle \(\Delta \theta \approx \pi/\ell\), and stuff within this angle should be correlated. So the correlation length would be of the order of \(\xi_0 \approx \pi R/\ell\), where \(R \approx 10^{28} \text{ cm}\) is the distance of last scattering. Putting \(\ell = 200\), we get \(\xi_0 \approx 10^{26} \text{ cm}\), which is not inconsistent with \((15)\), considering the qualitative nature of the argument.

The NLKG here is a field equation, and is quite different from the relativistic N-body equations used in astrophysics. The scalar field here is the vacuum expectation value of a quantum field, which contains a high-momentum cutoff \(\Lambda\). When \(\Lambda\) changes, the coupling \(\lambda\) “renormalizes”, in order to maintain the identity of the theory. For the \(\phi^4\) theory, \(\lambda(\Lambda) \propto \ln \Lambda\). We take \(\Lambda = \Lambda_{\text{QCD}} + \Lambda_{\text{cosmo}}\), where \(\Lambda_{\text{QCD}}\) refers to the nuclear energy scale, and is taken to be time-independent, while \(\Lambda_{\text{cosmo}}\) is the energy scale of the space-time metric, and decreases with the expanding universe. The latter is ignored because \(\Lambda_{\text{cosmo}} \ll \Lambda_{\text{QCD}}\). This is why we can treat \(\lambda\) as a constant.
3. Gravitational interactions

We now include gravitational interactions in the Newtonian limit. The metric tensor is given by

\[ g_{00} = -(1 + 2U) \quad (U << 1) \]
\[ g^{00} = \frac{1}{1 + 2U} \]
\[ g_{jk} = \delta_{jk} \]
\[ \sqrt{-g} = \sqrt{1 + 2U} \]

where \( U \) is the gravitational potential. The equation of motion becomes

\[-(1 - 2U) \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi + \frac{\partial U}{\partial t} \frac{\partial \phi}{\partial t} + \nabla U \cdot \nabla \phi - \lambda \left( |\phi|^2 - F_0^2 \right) \phi - i\eta \rho \left( \frac{\partial \phi}{\partial t} + \Omega \times r \cdot \nabla \phi \right) = 0 \]

The gravitational potential has contributions from the self-gravitation of the scalar field, and from the galaxy's gravitational attraction:

\[ U(x) = -G \int d^3x' \frac{\rho_{\text{sc}}(x') + \rho_{\text{galaxy}}(x')}{|x - x'|} \]
\[ \rho_{\text{sc}} = T^{00} - V_0 = \left| \frac{\partial \phi}{\partial t} \right|^2 + \left| \nabla \phi \right|^2 + \frac{\lambda}{2} \left( |\phi|^2 - F_0^2 \right)^2 \]
\[ \rho_{\text{galaxy}} = M_{\text{galaxy}} \rho \]

where \( M_{\text{galaxy}} \) is the galaxy's total mass. With \( h = c = 1 \), the gravitational constant \( G \) is given by

\[ G = \frac{1}{4\pi} \left( \text{Planck length} \right)^2 = 3 \times 10^{-66} \text{ cm}^2 \]

4. Galactic dark-matter halo

A galaxy immersed in the cosmic superfluid will become surrounded by a halo of higher superfluid density than the vacuum. This can be observed through gravitational lensing, and interpreted by us as dark matter.

To see how the halo is formed, consider the situation in which a hypothetical galaxy had been formed in the vacuum scalar field long enough for the entire system to develop into a stationary state. We would have \( \phi \propto e^{-i\omega t} \). The eigenfrequency \( \omega \) reflects the strength of the interaction \( \eta \rho \), and may be estimated to be

\[ \omega \sim \eta \langle \rho \rangle \]

where \( \langle \rho \rangle \) denotes the spatial average of the galaxy’s density, weighted by the squared modulus of the field \( \phi^* \phi \). This formula can be derived in a variational formulation of the problem.\[12\] The NLKG would then become

\[ \nabla^2 \phi - \lambda \left( |\phi|^2 - F_0^2 - \frac{\omega^2}{\lambda} \right) \phi - \eta \rho (\omega \phi + i\Omega \times r \cdot \nabla \phi) = 0 \]
Fig. 1. Scalar-field profile showing the dark-matter halo around a galaxy. The vacuum scalar field has modulus $F_0$ outside, and $F_1 > F_0$ inside. This creates a higher energy density in the halo, which can be observed via gravitational lensing.

According to this equation, the vacuum field would be changed from $F_0$ to $F_1$, with

$$F_1^2 = F_0^2 + \frac{\omega^2}{\lambda}$$

(24)

The situation described above is of course hypothetical. What actually happens is that a dark-matter halo will initially develop around the galaxy and spread, as illustrated schematically in Fig. 1. The initial speed of the spreading is the velocity of light, the character speed in the NLKG, but it will be slowed or stopped by gravitational interactions. To estimate the overall average speed, we assume that the halos around galaxies emerged with their formation, some $10^{10}$ years ago. We take the present radius to be 10 times galactic radius, or $10^6$ ly, judging from gravitational lensing pictures. This gives an average expansion speed less than $10^{-4} c$.

The parameters $F_1, \omega$ give rise to new correlation lengths:

$$\xi_1 = \frac{1}{\sqrt{\lambda} F_1}$$

$$\xi_\omega = \frac{1}{\omega}$$

(25)

and they are related to the vacuum correlation length through

$$\frac{1}{\xi_1^2} = \frac{1}{\xi_0^2} + \frac{1}{\xi_\omega^2}$$

(26)

The region inside the dark-matter halo is governed by $\xi_1$, the outside region is governed by $\xi_0$, and there is a transition region between the two. We may assume that the stationary equation (24) approximately holds within the halo, but of course it becomes invalid in the transition region and beyond. When the frequency $\omega$ is sufficiently large, the halo region will become non-relativistic.
Fig. 2. Formation of the dark-matter halo around a disk galaxy (side view). The halo initially grows at the speed of light normal to the galactic disk, and is eventually stabilized by gravity. For a disk-like galaxy, the halo would emerge as a surface layer on the disk, and grow in thickness under a force normal to the galactic disk, so as to lower the energy. Initially the layer would grow at the speed of light, the characteristic speed of the NLKG, but eventually it would be slowed or stopped by self gravitation of the layer, and the gravitational attraction from the galaxy. This could produce the “squashed beach ball” shape of the halo around the Milky Way. The process is schematically illustrated in Fig.2.

5. Relation between dark energy and dark matter

The universe roughly consists of $\frac{3}{4}$ dark energy and $\frac{1}{4}$ dark matter. Since dark energy and dark matter are manifestations of the same scalar field, that fact imposes relations that can be tested. In the following we regard dark matter and halo as the same. Let

$$\gamma \equiv \frac{\text{Volume of all dark-matter halos}}{\text{Volume of universe}}$$

$$f \equiv \frac{\rho_{\text{dark matter}}}{\rho_{\text{galaxy}}}$$

where $\rho$ denotes mass density. The following relation should hold between dark energy and dark matter, with neglect of observable matter:

$$\gamma \rho_{\text{dark matter}} + (1 - \gamma) \rho_{\text{dark energy}} = \rho_0$$

where $\rho_0$ is the energy density of the universe. The two terms above represent roughly $\frac{3}{4}$ and $\frac{1}{4}$ of the total energy density of the universe, respectively.
Thus
\[
\frac{(1 - \gamma) \rho_{\text{dark energy}}}{\gamma \rho_{\text{dark matter}}} = 3
\]
\[
\rho_{\text{dark energy}} = \frac{3\gamma}{1 - \gamma} \rho_{\text{dark matter}} \approx 3\gamma \rho_{\text{dark matter}}
\]
which leads to the relation
\[
\gamma = \frac{1}{3} \rho_{\text{dark energy}} = \frac{1}{3f} \rho_{\text{galaxy}}
\]
Using the values
\[
\rho_{\text{dark energy}} = V_0 = 5.4 \times 10^8 \text{ cm}^{-4}
\]
\[
\rho_{\text{galaxy}} = 1.8 \times 10^{14} \text{ cm}^{-4}
\]
from\cite{19} and\cite{20} respectively, we get
\[
\gamma = \frac{10^{-6}}{f}
\]
Since each galaxy occupies a volume the size of its halo, we should have
\[
\gamma = N_{\text{galaxy}} \left( \frac{R_{\text{halo}}}{R_{\text{universe}}} \right)^3
\]
where \( R_{\text{universe}} \) is the radius of the universe, and \( N_{\text{galaxy}} \) is the number of galaxies. Equating the above to \( f^{-1}10^{-6} \) and then solving for \( R_{\text{halo}} \), we get
\[
R_{\text{halo}} = \frac{R_{\text{universe}}}{N_{\text{galaxy}}^{1/3}} \frac{10^{-2}}{f^{1/3}} \sim \frac{10^5 \text{ ly}}{f^{1/3}}
\]
where we have used the experimental values \( R_{\text{universe}} = 5 \times 10^{28} \text{ cm}, N_{\text{galaxy}} = 2 \times 10^{11} \). The average radius of a galaxy happens to be \( 10^5 \text{ ly} \), and the above can be rewritten \( R_{\text{halo}} \sim f^{-1/3} R_{\text{galaxy}}, \) or
\[
f \sim \left( \frac{R_{\text{galaxy}}}{R_{\text{halo}}} \right)^3
\]
The definition \cite{27} is equivalent to \( f = (M_{\text{halo}}/M_{\text{galaxy}}) \left( R_{\text{galaxy}}/R_{\text{halo}} \right)^3 \), where \( M \) denotes total mass. The above thus implies
\[
\frac{M_{\text{halo}}}{M_{\text{galaxy}}} \sim 1
\]
From images of gravitational lensing, e.g. the “bullet cluster”\cite{13} we estimate \( R_{\text{halo}}/R_{\text{galaxy}} \sim 10, \) (somewhat subjectively,) and obtain
\[
f \sim 10^{-3}
\]
\[
\gamma \sim 10^{-3}
\]
This indicated that the halo is a dilute medium with mass density a thousand times smaller than that of the galaxy.
6. Superfluid velocity and quantized vortices

A superfluid can flow pass walls without friction, as long as its velocity is below a critical velocity, according to the Landau criterion of superfluidity. Similarly, a macroscopic object can move through a superfluid without friction, as long as its relative velocity is below the critical velocity, which is usually the phonon velocity associated with long-wavelength excitations, or Goldstone modes. For our cosmic superfluid, this is the velocity of light, and thus stars in a galaxy should be able to move frictionlessly through the dark-matter halo, since the latter is part of the superfluid. On the other hand, a superfluid can be induced to rotation about an axis, by creating quantized vortices in the superfluid (See below).

In non-relativistic theories, the superfluid velocity is defined by

\[ v_s = \frac{\hbar}{M} \nabla \sigma, \]

where \( \sigma \) is the phase of the complex order parameter, and \( M \) a mass. However, this is not bounded by the velocity of light, and clearly needs modification in the relativistic domain. In flat space-time, we generalize it as follows. We start with the conserved current \( j^\mu \) given by (7), which satisfies the continuity equation \( \partial_{\mu} j^\mu = 0 \), or

\[
\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0
\]

\[
\mathbf{j} = F^2 \nabla \sigma
\]

\[
n = -F^2 \dot{\sigma}
\]

(38)

where the field is represent in the form \( \phi = F e^{i\sigma} \), and a dot denotes time derivative. We interpret \( n \) as the superfluid density, and define the superfluid velocity \( \mathbf{v}_s \) through

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_s) = 0
\]

(39)

This leads to the definition

\[
\mathbf{v}_s = \xi_s \nabla \sigma
\]

\[
\xi_s = \frac{1}{\dot{\sigma}}
\]

(40)

where \( \xi_s \) is a correlation length that is generally space-time dependent. Now the magnitude of \( \mathbf{v}_s \) is bounded by the velocity of light, as long as \( \partial^\mu \sigma \) is a time-like four vector. We recover the nonrelativistic case when \( \sigma (\mathbf{r}, t) = -Mt + \nu (\mathbf{r}, t) \), with \( M \to \infty \).

It should be noted that \( \mathbf{v}_s \) has physical meaning as a velocity field only when \( \xi_s \) is constant, or at least slowing varying.

For a manifestly Lorentz-covariant formulation, we introduce a time-like unit four vector

\[
W^\mu = \kappa \partial^\mu \sigma
\]

(41)
Fig. 3. Core size of vortex created by a rotating galaxy is governed by the local correlation length. Inside the galactic it is some 60 times smaller than that in the vacuum, according to (50). Consequently, the size of the vortex core vastly increases towards the edge of the halo.

with $W^\mu W_\mu = -1$. Thus $\kappa$ is given by

$$\kappa = \frac{1}{\sqrt{\dot{\sigma}^2 - |\nabla \sigma|^2}}$$

and

$$W = \frac{v_s}{\sqrt{1 - v_s^2}}$$

$$W^0 = \frac{1}{\sqrt{1 - v_s^2}}$$

To discuss vorticity, it is more convenient to work with $\partial^\mu \sigma$ without the non-constant factor $\kappa$, because it is $\partial^\mu \sigma$ that satisfies a quantization condition. Denote the spatial gradient by

$$u = \nabla \sigma = v_s/\xi_s$$

The line integral of $u$ over a closed loop $C$ must be a multiple of $2\pi$, by continuity of the scalar field:

$$\oint_C ds \cdot u = \oint_C ds \cdot \nabla \sigma = 2\pi N$$

When $N \neq 0$, there is a vortex line encircled by $C$. We refer the reader to Ref.4 for a discussion of vortex dynamics.

For disk-like galaxies rotating about an axis normal to the disk, vortex lines will be generated normal to the disk, and there is cylindrical symmetry at least near the surface of the galactic disk. The core size of the vortex line is proportional to the correlation length, which is $\xi_1$ inside the galactic halo, and increases as we go
away from the center of the halo, and approach the larger vacuum value $\xi_0$ outside the halo. The transition layer between the halo and the vacuum may be regarded as a fuzzy boundary where a vortex line can “terminate”. This is illustrated in Fig.3. The halo of a galaxy with one vortex would look like a fat donut with a tiny hole. The picture might be different for a black hole, for which the strong space-time curvature may have important effects on vorticity.

We can roughly estimate the critical angular velocity for the creation of one vortex, using a cylindrical geometry. Consider a circular loop $C$ of radius $R$, with an infinite vortex line normal to the loop through its center. The velocity $u$ is tangential to the circle with constant magnitude $u_0$. We find by explicit evaluation

$$\oint_C ds \cdot u = 2\pi Ru_0$$

Evaluating this to $2\pi N$ yields $u_0 = N/R$, or

$$\nu_s/c = \xi_s N/R^2$$

The corresponding angular velocity is $\Omega = \nu_s/R$. Thus, to have one vortex inside radius $R$, the angular velocity must exceed the critical value

$$\Omega_c = c\xi_s R^2$$

In Table 1, we give estimates of $\Omega_c$ for various stellar objects, putting $\xi_s \sim R$, i.e., assuming that the correlation length of the superfluid is greater than the size of the object. The actual angular velocities $\Omega$ are also cited for reference. As we can see, a black hole could generate a million vortices, the Milky Way and neutron star are marginally critical cases, while the sun and earth fall short of criticality.

When a black hole “drags” the surrounding superfluid into rotation, there would appear the order of $10^6$ vortex lines, which would form a cage around the black hole. This may explain the so-called “non-thermal filaments” observed in the neighborhood of Sagittarius near the center of the Milky Way\(^\text{[21]}\)

### 7. Determination of model parameters

Assume that the correlation length $\xi_1$ in a typical galactic halo is roughly the radius of the halo, which we take (somewhat subjectively) to be 10 times that of the disk
of a typical galaxy:

\[ \xi_1 \sim 5 \times 10^{23} \text{ cm} \quad (49) \]

This makes

\[ \frac{F_1}{F_0} = \frac{\xi_0}{\xi_1} \sim 66 \quad (50) \]

Since \( F_1 >> F_0 \), we have \( F_1^2 = F_0^2 + \omega^2/\lambda^2 \approx \omega^2/\lambda \), or

\[ \omega \approx 1/\xi_1 \quad (51) \]

Now we calculate \( \omega \) using (22) in the halo, taking \( F \) to be constant, and get

\[ \omega \sim \eta \langle \rho \rangle = \frac{\eta \int_{\text{halo}} d^3x \rho}{V_{\text{halo}}} = \frac{\eta M_{\text{galaxy}}}{V_{\text{halo}}} \quad (52) \]

\[ \sim \frac{\eta M_{\text{halo}}}{V_{\text{halo}}} = \eta \rho_{\text{dark matter}} \quad (53) \]

where \( V_{\text{halo}} \) is the volume of the halo, and we have identified the halo with dark matter. Using \( \omega \approx 1/\xi_1 \) and solving for \( \eta \), we obtain

\[ \eta \sim \frac{1}{\xi_1 \rho_{\text{dark matter}}} = 10^{-35} \text{ cm}^3 = 10^4 \text{ fermi}^3 \quad (54) \]

where fermi = 10\(^{-13}\) cm is the characteristic length of nuclear interactions. (54) is an interesting and profound result, although it is obtained through a crude estimation. Given the dark matter density that we used in Section V, we see a subatomic scale, \( \eta^{1/3} \approx 20 \) fermi, emerging from cosmological scales. The value of \( \eta^{1/3} \) may vary in different models. For example, if WIMPs are taken into consideration as a normal-fluid component, one may have weakly interacting particles with masses from \( 10^2 \) to \( 10^3 \) GeV which correspond to \( \eta^{1/3} \approx 10^{-2} \sim 10^{-3} \) fermi. In practice the value of \( \eta^{1/3} \) should be constrained by experimental observations (see the next section for discussions on the long-ranged force induced by the scalar field compared with gravity).

We now estimate the vacuum field \( F_0 \). In our model, dark energy comes from \( V_0 \), and dark matter comes from deviations from the vacuum field:

\[ \frac{\rho_{\text{dark energy}}}{\rho_{\text{dark matter}}} = \frac{V_0}{\frac{3}{2} (F_1^4 - F_0^4)} = 3\gamma \quad (55) \]

where we have used (30). Since \( F_1 >> F_0 \), we have

\[ \frac{\lambda F_1^4}{2V_0} = \frac{1}{3\gamma} \quad (56) \]

With (31) and (30), we get

\[ \lambda F_1^4 = \frac{2V_0}{3\gamma} = 3.6 \times 10^{11} \text{ cm}^{-4} \quad (57) \]
We can find $F_1$ and $\lambda$ separately by noting

$$F_1^2 = \frac{\lambda F_1^4}{F_1} = (\lambda F_1^4) \xi_1^2$$

(58)

The results are

$$F_1 \sim 10^{29} \text{ cm}^{-1}$$

$$F_0 \sim 1.5 \times 10^{27} \text{ cm}^{-1}$$

$$\lambda \sim 3.6 \times 10^{-105}$$

(59)

8. Long-ranged force induced by the scalar field

The small mass (16) of the field particle means that the scalar field would induce a long-ranged force in matter. Consider two static test sources

$$\rho_i (r) = m_i \delta^3 (r - r_i) \quad (i = 1, 2)$$

(60)

The field due to one test source, in the absence of the other, is of the form $\phi (r, t) = e^{-i\omega_0 t} F (r)$, where $\omega_0 \sim \xi_0^{-1}$. The equation for $F$ reads

$$\nabla^2 F + \left[ \omega_0^2 - \lambda \left( F^2 - F_0^2 \right) \right] F = \eta \omega_0 F \rho$$

(61)

We shall neglect the term in brackets, and put $F = F_0$ on the right side. The field produced by source 1 is then given by

$$\nabla^2 F_1 (r) = \eta \omega_0 F_0 m_1 \delta (r - r_1)$$

$$F_1 (r) = - \frac{\eta \omega_0 F_0 m_1}{4\pi} \frac{1}{|r - r_1|}$$

(62)

The energy of source 2 in this field is given by

$$U_{21} = -\eta \int d^3 x \rho_2 \text{Im} \left( \phi_1^* \frac{d}{dt} \phi_1 \right) = \eta \omega_0 m_2 F_1^2 (r_2)$$

$$= \frac{\eta^3 \omega_0^3 F_0^2 m_2^2}{(4\pi)^2} \frac{1}{|r_1 - r_2|^2}$$

(63)

The total potential energy is $U_{\text{scalar}} = U_{12} + U_{21}$:

$$U_{\text{scalar}} = \frac{K m_2 m_1}{|r_1 - r_2|^2}$$

$$K = (4\pi)^{-2} \eta^3 \omega_0^3 F_0^2 (m_1 + m_2)$$

(64)

This is to be compared with the gravitational potential energy

$$U_{\text{grav}} = -\frac{G m_1 m_2}{|r_1 - r_2|}$$

(65)

which dominates at large distances. For $m_1 = m_2 = M_{\text{galaxy}}$ and $\eta \sim (20 \text{ fermi})^3$ (see eqt. (54)), the two potentials have equal magnitude at distance

$$|r_1 - r_2|_{eq} = \frac{|K|}{G} \sim 10^{19} \text{ cm}$$

(66)
which is 1% the thickness of the Milky Way disk. This distance, however, is very sensitive to the value of $\eta$ and can vary enormously in different models. As we mentioned in the previous section, if WIMPs are taken into account, one may have $\eta^{1/3} \sim 10^{-3}$ fermi, which reduces the above distance $|r_1 - r_2|_{eq}$ to $10^{-16}$ cm or below. This seems to be consistent with the short-distance character of the WIMPs’ non-gravitational interactions. Therefore at cosmological scales we may ignore the induced potential in the present work. However, it may have an effect in the distribution of galaxies, and may contribute to the formation of galaxies and galaxy clusters at the early universe.

9. Goldstone modes and vacuum stability

Being a gravitating medium, the cosmic superfluid could be unstable against gravitational clustering. This would be manifested as the spontaneous emergence of patches of dark matter, analogous to the Jeans instability of a gravitating fluid.$^{22}$ To investigate the stability of the vacuum, we perturbed it by exciting the Goldstone modes. The NLKG in the absence of matter, but with gravity in the Newtonian limit, is given by

$$\nabla^2 \phi - (1 - 2U) \frac{\partial^2 \phi}{\partial t^2} + \nabla U \cdot \nabla \phi - \lambda \left( |\phi|^2 - F_0^2 \right) \phi = 0 \quad (67)$$

where $U$ is the gravitational potential satisfying the Poisson equation

$$\nabla^2 U = 4\pi G \left[ \left| \phi \right|^2 + |\nabla \phi|^2 + \frac{\lambda}{2} \left( |\phi|^2 - F_0^2 \right)^2 \right] \quad (68)$$

The ground state corresponds to $\phi_0 = F_0$. To investigate the Goldstone modes, we put

$$\phi = F_0 + f \quad (69)$$

where $f$ is a small perturbation. When gravity is ignored, one can solve the linearized NLKG via the Bogoliubov transformation

$$f = \alpha e^{-i(\omega t - k \cdot x)} + \beta^* e^{i(\omega t - k \cdot x)} \quad (70)$$

In this case, one obtains the dispersion relation $\omega = k$, as dictated by Lorentz covariance.

With inclusion of gravity, linearization in $f$ leads to

$$\nabla^2 f - (1 - 2U) \frac{\partial^2 f}{\partial t^2} + \nabla U \cdot \nabla f - \xi_0^{-2} (f^* + f) = 0 \quad (71)$$

This is a nonlinear equation, because $U$ depends on $f$. To solve it, we resort to an approximation through the replacements

$$\frac{\partial U}{\partial t} \frac{\partial f}{\partial t} \rightarrow -i\gamma \frac{\partial f}{\partial t}$$

$$\nabla U \cdot \nabla f \rightarrow -i\mathbf{q} \nabla f \quad (72)$$
Fig. 4. The Goldstone modes with $\omega = k$ are split into “optical” and “acoustical” branches by gravity. In the acoustical branch, the sound velocity is pure imaginary in the region $0 < k < k_0$, and this signifies vacuum instability, with formation of patches of dark matter of size $L_0 = 2\pi/k_0$. In the present universe $L_0$ is estimated to be much larger than the radius of the universe, and $k_0 \to 0$. Thus, the present vacuum should be stable against gravitational clustering.

where $f$ on the right side is given by (70), and $\gamma$, $q$ are real constants. We assume $U \ll 1$ and neglect it, and obtain a set of homogeneous algebraic equations:

\[
\begin{align*}
[k^2 - \omega^2 + \xi_0^{-2} - (q \cdot k - \gamma \omega)] \alpha + \xi_0^{-2} \beta &= 0 \\
\xi_0^{-2} \alpha + [k^2 - \omega^2 + \xi_0^{-2} + (q \cdot k - \gamma \omega)] \beta &= 0
\end{align*}
\]

(73)

We average over the directions of $q$, and take $\langle q \cdot k \rangle = 0$, $\langle (q \cdot k)^2 \rangle = \frac{1}{2} q^2 k^2$. Setting the determinant to zero, we obtain the dispersion relations

\[
\omega^2 = k^2 + \frac{1}{2} \left( A \pm \sqrt{2Bk^2 + A^2} \right)
\]

\[
A = 2\xi_0^{-2} + \gamma^2
\]

\[
B = q^2 + 2\gamma^2
\]

(74)

The Goldstone modes split into two branches ($\pm$) that are analogous to the “optical” (+) and acoustical (−) modes in solids. As depicted in Fig.4, both revert to the relation $\omega = k$ when $k \to \infty$. At the low $k$ end of the spectrum, $\omega$ approaches a finite constant for the optical mode, while it goes to zero at $k = k_0$ for the
acoustical mode, with the behavior
\[ \omega \approx \sqrt{2k_0 (k - k_0)} \]
\[ k_0 = \sqrt{\frac{1}{2} (B - 2A)} \]
(75)

If \( k_0 > 0 \), this signifies instability, in which the vacuum breaks up into patches of dark matter with characteristic size
\[ L_0 = \frac{2\pi}{k_0} \]
(76)

To estimate the instability length \( L_0 \), we use (75) in the form
\[ L_0 = 2\pi \left( \frac{B}{2} - A \right)^{-1/2} = 2\pi \left( \frac{q^2}{2} - \frac{2}{\xi_0} \right)^{-1/2} \]
(77)

This is an implicit relation, because \( q \) depends on \( L_0 \). We can estimate \( q \) as follows. Suppose, in spontaneous fluctuation, the vacuum field develops a value \( F_1 \) in a region of size \( L_0 \). The energy density in this patch would be \( \lambda (F_1^4 - F_0^4) \), and the mass of the patch would be
\[ M = \lambda (F_1^4 - F_0^4) L_0^3 \]
(78)

By dimensional analysis, we estimate that
\[ U \sim \frac{GM}{L_0} \]
\[ q \sim \frac{U}{L_0} \]
(79)

Substituting this into (77) yields a quartic equation for \( L_0 \). The solution in the limit of weak gravity gives
\[ \frac{L_0}{\xi_0} \approx \frac{2}{GF_0^2 (F_1^4/F_0^4 - 1)} \]
(80)

Using \( G = 3 \times 10^{-66} \text{ cm}^2 \), \( F_0 \sim 1.5 \times 10^{27} \text{ cm}^{-1} \), we obtain
\[ \frac{L_0}{\xi_0} \approx \frac{3 \times 10^{12}}{(F_1/F_0)^4 - 1} \]
(81)

We had obtained \( F_1/F_0 \approx 66 \) in the galactic halo, but that was due to interaction between galaxy and vacuum field. Here, \( F_1 \) is a property of the vacuum field, and is determined by balancing the weak gravity against the strong field potential. We expect that \( F_1/F_0 \approx 1 \), which means \( L_0 > 3 \times 10^{12} \xi_0 \). Since this length is overwhelmingly larger than the radius of the universe, we conclude that the present vacuum is stable against gravitational clustering. This of course does not preclude vacuum clustering during the early universe, when conditions were different.
10. Numerical exploration

We solve the NLKG with external-source galaxies whose densities have Gaussian distributions. The galaxies move in prescribed ways as external sources, and our focus is on the response of the superfluid. Computations on moving galaxies are carried out in a $512 \times 512$ two-dimensional spatial grid, with periodic boundary conditions. Those on vortex lattice creation are done in a $1024 \times 1024$ grid. For illustration purposes, model parameters are chosen arbitrarily.

Fig. 5 shows a “film strip” of the superfluid in the presence of a moving galaxy, consisting of sequential plots of the scalar-field modulus. Frame 1 is the initial state, with velocity directed upwards. Frame 2 shows the development of the dark-matter halo (white area) around the galaxy, and an outgoing transient cylindrical wave front in the superfluid. In frame 3, the transient wave continues to expand about the original center, while the halo moves with the galaxy like a soliton. In frame 4, the transient wave hits the boundaries.

Fig. 6 shows the collision between two galaxies, initially made to move towards each other head-on. They continue on such a course, go through each other, and finally recede from each other. The frames show the evolution of the field modulus in contour plots, with the white areas corresponding to the dark-matter halos. The pictures show the merging of the two halos, the contortions of the combined halo, and finally the recession of the galaxies from each other, with their proper halos restored.

Fig. 7 shows two galaxies colliding at a nonzero impact parameter. The superfluid between them is sheared into rotational flow, through the creation of vortices. In frame 2, two black dots appear, indicating the formation of vortices, and they continue to develop in the subsequent frames.

Fig. 8 shows one moment during the development of a vortex lattice around a rotating galaxy. Such vortex lattices have been observed in cold-trapped atoms and simulated in the NLSE. To our knowledge, this is the first simulation in the NLKG. The left panel is a contour plot of the field modulus, and the right frame is one of the phase of the field. The vortices form four rings, with the outermost ring located just outside the dark-matter halo (in white). The vortex counts of the rings are: 19, 12, 11, 9, a total of 51, with the inner most ring being somewhat irregular because it is still being formed. In general, vortices are nucleated inside the galaxy, and migrate outwards. The phase contours show a radial pattern, in which the dark radial spokes represent “strings” across which the phase jumps by $2\pi$, and these strings must be terminated by vortices. For example, the tips of the “chrysanthemum petals” coincide with the locations of the vortices on the outermost layer. Note that vortices in the outermost layer has much larger core size than those within the halo, because the correlation length is much larger. [See Fig.3 and Eq. 50.]

Fig. 9 shows the number of vortices as function the angular velocity of the galaxy, for a range of the interaction parameter between galaxy and scalar field.
Fig. 5. Contour plots of the scalar-field modulus in 2D space, showing time evolution of the superfluid in response to the presence of a moving galaxy. The white areas correspond to the dark-matter halo. A transient cylindrical wave front can be seen propagating from the initial position of the galaxy, hitting the boundaries in the last frame.

Fig. 6. Collision between two galaxies approaching each other along the vertical direction. The dark-matter halos merged, underwent deformations, and were finally restored when the galaxies recede from each other.

11. Discussions and outlook

The basic idea of this model is that the universe is filled with a superfluid, which is described by a vacuum complex scalar field, and dark matter is a manifestation of density fluctuations of the superfluid. This idea has similarities with previous theories that describe dark matter as a Bose-Einstein condensate of some kind. The difference is that our superfluid is relativistic and fills the entire universe, and, as described in, had its origin in the big bang, and played a critical role in the creation of matter through vortex activities. The big-bang theory gives a Hubble
Fig. 7. Two galaxies passing each other at nonzero impact parameter. The superfluid between them is sheared into rotation, with creation of vortices (the black dots).

Fig. 8. A vortex lattice surrounding a rotating galaxy. The left panel is a contour plot of the field modulus, showing four rings of vortices, the outermost of which lies beyond the galactic halo, and has a much larger core size. The right panel is a contour plot of the phase of the field. The radial spokes are “strings” across which the phase jumps by $2\pi$. See text for a fuller description.

parameter that decays in time like a power law. Such a behavior has been postulated phenomenologically under the name of “intermediate inflation”, and compared with
During the big-bang era, it was essential to use a scalar potential that is asymptotically free, because the length scale was inflating rapidly. In the present universe, however, we can describe the superfluid by a phenomenological NLKG, with any convenient potential. The preliminary investigation here aims mainly to show that it passes first encounters with observations, and establish some parameters in the NLKG. Important limitations of our model are the following: it treats the superfluid at absolute zero, ignoring the normal fluid component; and it treats matter as external source, ignoring its dynamics.

The normal fluid will consist of phonons of the Goldstone modes described earlier, the quanta of the scalar field, with mass given by (16), and elementary particles coupled to the scalar field, for example the WIMPs that particle physicists are searching for. If they exist, their excitation from the superfluid would explain why they are expected to gather around galaxies, but they are only secondary components of dark matter.

The present model cannot explain the velocity curve of Andromeda, the earliest indication of dark matter, which shows that “dust” surrounding the galaxy has a velocity too large to be accounted for by the visible galactic mass. To do that, we need an equation of motion for matter.

In a more general sense, a extension of the present model should look towards the inclusion of cosmic superfluidity into a hot big bang theory, and analysis of the CMB. One would add the NLKG to any quantitative treatment. Clearly, this would be a major project yet to be undertaken.

We can only speculate on the effects of the cosmic superfluidity in the hot big bang scenarios, and CMB fluctuations. They would have little influence on small-
scale phenomena, such as the nuclear and electromagnetic processes in nucleosynthesis and barogenesis\({\textsuperscript{35}}\) and the BAO (baryon acoustic oscillations) in the CMB.\({\textsuperscript{36}}\) The main impact would be associated with quantized vorticity, which could leave imprints on the CMB, and play a role in structure formation in the universe.

Appendix A. Dimensionless form of NLKG

For numerical computations, it is convenient to use the normalization convention
\(\int d^3x \rho = 1\), and absorb the mass of the source into \(\eta\). To put the equation of motion into dimensionless form, we scale all lengths with the vacuum correlation length \(\xi_0 = \left(\sqrt{\lambda F_0}\right)^{-1}\), and introduce dimensionless quantities indicated with an overhead tilde:

\[
\phi = F_0 \tilde{\phi} \\
t = \xi_0 \tilde{t} \\
x = \xi_0 \tilde{x} \\
\Omega = \xi_0^{-1} \tilde{\Omega} \\
\rho = \xi_0^{-3} \tilde{\rho} \\
\eta = \xi_0^2 \tilde{\eta}
\]

The only adjustable parameter is the mass of the galaxy \(M_{\text{galaxy}}\). It is convenient to use the Milky Way as standard:

\[
GM_{\text{Milky Way}} = 1.2 \times 10^{-15}\ \text{cm}
\]
Let
\[ \beta = \frac{M_{\text{galaxy}}}{M_{\text{Milky Way}}} \] (A.6)

then the Poisson equation takes the form
\[ \tilde{\nabla}^2 U = \beta c_1 \tilde{\rho} + c_2 \left[ \left| \frac{\partial \tilde{\phi}}{\partial t} \right|^2 + \left| \tilde{\nabla} \tilde{\phi} \right|^2 + \frac{1}{2} \left( \left| \tilde{\phi} \right|^2 - 1 \right)^2 \right] \]
\[ c_1 = 4.56 \times 10^{-40} \]
\[ c_2 = 10^{-10} \] (A.7)

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