A KAM THEOREM FOR THE ELLIPTIC LOWER DIMENSIONAL TORI WITH ONE NORMAL FREQUENCY IN REVERSIBLE SYSTEMS

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Abstract. In this paper we consider the persistence of elliptic lower dimensional invariant tori with one normal frequency in reversible systems, and prove that if the frequency mapping \( \omega(y) \in \mathbb{R}^n \) and normal frequency mapping \( \lambda(y) \in \mathbb{R} \) satisfy that

\[
\text{deg}(\omega/\lambda, \mathcal{O}, \omega_0/\lambda_0) \neq 0,
\]

where \( \omega_0 = \omega(y_0) \) and \( \lambda_0 = \lambda(y_0) \) satisfy Melnikov’s non-resonance conditions for some \( y_0 \in \mathcal{O} \), then the direction of this frequency for the invariant torus persists under small perturbations. Our result is a generalization of X. Wang et al [Persistence of lower dimensional elliptic invariant tori for a class of nearly integrable reversible systems, Discrete and Continuous Dynamical Systems series B, 14 (2010), 1237-1249].

1. Introduction and main results. The study of many problems of Physics and Mechanics leads to considering a reversible system [16]. Reversible systems form a class of special conservative systems with an involution structure. During the last 50 years, many authors study the persistence of invariant tori for reversible systems and obtained many kinds of KAM theorems (see [1, 5, 6, 7, 8, 9, 11, 17, 19, 20, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 36, 38] and the references therein).
Consider the existence of $n$-dimensional invariant tori of the following dynamical system:

$$
\begin{align*}
\dot{x} &= \omega(y) + P^1(x, y, u, v), \\
\dot{y} &= P^2(x, y, u, v), \\
\dot{u} &= A(y)v + P^3(x, y, u, v), \\
\dot{v} &= -A(y)u + P^4(x, y, u, v),
\end{align*}
$$

(1.1)

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p$, $A(y) = \text{diag}(\lambda_1(y), \cdots, \lambda_p(y))$, $\omega(y) = (\omega_1(y), \cdots, \omega_n(y)) \in \mathbb{R}^n$ is called frequency vector, $A(y)$, $\omega(y)$ and the perturbation term $P^i$ $(i = 1, 2, 3, 4)$ are analytic with respect to $(x, y, u, v)$ on a complex domain

$$
M = D(\mathbb{T}^n) \times D(M) \times \left\{(u, v) \mid |u| \leq r, |v| \leq r \right\}
$$

$$
\subset \mathbb{C}^n / 2\pi\mathbb{Z}^n \times \mathbb{C}^m \times \mathbb{C}^p \times \mathbb{C}^p,
$$

where $M$ is a bounded open domain. Suppose that the system (1.1) is reversible with respect to the involution $G : (x, y, u, v) \rightarrow (-x, y, -u, v)$, that is,

$$
DG \cdot F = -F \circ G,
$$

(1.2)

where $F = (\omega + P^1, P^2, Au + P^3, -Au + P^4)^T$. By (1.2), it follows that the system (1.1) is reversible if

$$
P^j(-x, y, -u, v) = (-1)^{j+1}P^j(x, y, u, v), \quad j = 1, 2, 3, 4.
$$

A mapping $\Phi : (x, y, u, v) \rightarrow (x_+, y_+, u_+, v_+)$ is called compatible with the involution $G$ if $\Phi$ and $G$ commute. The compatible transformations transform reversible systems into systems reversible with respect to the same involution.

If $P^j = 0$ $(j = 1, 2, 3, 4)$, then reversible system (1.1) becomes

$$
\dot{x} = \omega(y), \quad \dot{y} = 0, \quad \dot{u} = A(y)v, \quad \dot{v} = -A(y)u.
$$

(1.3)

The reversible system (1.3) has an invariant subspace $\{u = 0, v = 0\}$, which foliated by a family of invariant tori $\mathbb{T}^n \times \{y_0\} \times \{0\} \times \{0\}$ with the frequency $\omega(y_0)$ for all $y_0 \in M$. Some of the invariant tori can be destroyed by an arbitrarily small perturbation. Whether some invariant tori can persist under small perturbation is an important problem in the perturbation theory of reversible system and is studied by many authors.

In the special case of $p = 0$, that is, where there is no normal frequency, the classical KAM theorem says that if $m \geq n$ and the frequency mapping $y \rightarrow \omega(y)$ is submersive in $M$, i.e.,

$$
\text{Rank} \frac{\partial \omega}{\partial y} = n, \quad \text{for all } y \in M,
$$

(1.4)

then the majority of invariant tori survive of small perturbations [1] [24].

In the case of $p > 0$, the invariant $n$-tori of reversible system (1.3) are called lower dimensional. Many authors studied the persistence of lower dimensional invariant tori for reversible systems of the form (1.1) under the non-degeneracy condition (1.4) and obtained many kinds of KAM theorems [5] [6] [7] [8] [25] [26] [27] [38]. There are also some similar results in the Hamiltonian framework [10] [11] [12] [23] and in the context of celestial mechanics [3] [4] [13].

Later, the non-degeneracy condition (1.4) has been weakened to Rüssmann’s non-degeneracy condition [5] [30]:

$$
a_1\omega_1(y) + a_2\omega_2(y) + \cdots + a_n\omega_n(y) \neq 0 \quad \text{on } \mathcal{O},
$$

(1.5)
for all $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n \setminus \{0\}$. However, under Rüssmann’s non-degeneracy condition, the range of the frequency mapping $\omega$ may be on a sub-manifold and the frequencies of persisting invariant tori may not come from unperturbed ones. Therefore, it is difficult to provide accurate information about the frequencies of KAM tori under Rüssmann’s non-degeneracy condition.

In this paper, we are mainly interested in the persistence of invariant tori with prescribed frequency. There are already some results about on the above problem for reversible systems [31, 32, 35].

Recently, some authors turn to study the persistence of invariant tori with prescribed frequency by the theory of topological degree [31, 39]. We first recall the definition for Brouwer degree. If $f(y)$ is a continuous mapping from $O \subset \mathbb{R}^m$ into $\mathbb{R}^m$ satisfying the condition $p_0 \notin f(\partial O)$ ($\partial O$ is the boundary of $O$), then one can define Brouwer degree $\deg(f, O, p_0)$ by using an approximation scheme introduced by Nagumo [21]. The idea consists in defining it first for $f$ smooth and a regular value of $p_0$ through the formula

$$
\deg(f, O, p_0) := \sum_{y \in f^{-1}(p_0)} \text{sign} \, Df (y)
$$

with $Df(y)$ the Jacobian of $f$, and then to approximate the continuous function $f$ and the point $p_0$ above by a sequence of such functions and points for which this definition holds. This is possible by the Weierstrass approximation theorem and the Sard theorem. The degrees of the approximations stabilize to a common value, denoted by $\deg(f, O, p_0)$ and being an algebraic count of the number of counter-images $p_0$ of $f$ under in $O$, which is stable for small perturbations of $p_0$ and $f$. For the details, see [2, 15, 18].

In the paper [31], the authors considered the reversible system (1.1) with $m = n + p$. Let $\lambda(y) = (\lambda_1(y), \cdots, \lambda_p(y))$, $\Omega(y) = (\omega(y), \lambda(y))$ and $\Omega_0 = (\omega_0, \lambda_0) = (\omega(y_0), \lambda(y_0))$ with a certain $y_0 \in \mathcal{M}$ satisfying the following non-resonance condition:

$$
|\langle k, \omega_0 \rangle - \langle l, \lambda_0 \rangle| \geq \frac{\alpha}{(1 + |k|)^\tau}, \quad (k, l) \in \mathbb{Z}^n \times \mathbb{Z}^p, \quad |l| \leq 2, |k| + |l| \neq 0,
$$

where $\alpha > 0$ and $\tau > n - 1$ are some constants. Let $O \subset \mathcal{M}$ be a small open neighborhood of $y_0$. The authors proved that if Brouwer’s degree of $\Omega$ at $\Omega_0$ on $O$ is not zero, i.e,

$$
\deg(\Omega, O, \Omega_0) \neq 0,
$$

then the invariant torus with the given frequency persists under small perturbations.

Noting that $m = n + p$ and $y \in \mathbb{R}^m$, the system (1.1) has enough variables and parameters to control, simultaneously, the values of the the tangential frequency and the normal frequency of the torus. A natural question is what happens when $m = n$? By observation we find that a similar result also holds under the condition $m = n$ if $p = 1$. Moreover, the non-degeneracy condition (1.6) can be weakened to

$$
\deg(\omega/\lambda, O, \omega_0/\lambda_0) \neq 0.
$$

The aim of this paper is to prove that a similar result also holds under the above non-degeneracy condition for the case of $m = n$ and $p = 1$. 

To be more precise, we consider the reversible system of the following form:

$$
\begin{align*}
\dot{x} &= \omega(y) + P_1(x, y, u, v), \\
\dot{y} &= P_2(x, y, u, v), \\
\dot{u} &= \lambda(y)v + P_3(x, y, u, v), \\
\dot{v} &= -\lambda(y)u + P_4(x, y, u, v),
\end{align*}
$$

(1.7)

where \((x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \omega(y) = (\omega_1(y), \ldots, \omega_n(y)) \in \mathbb{R}^n, \lambda(y) \in \mathbb{R}, \ P_1, P_2, P_3, \text{ and } P_4 \text{ are small perturbations. The corresponding involution } G \text{ is }

\begin{align*}
(x, y, u, v) &\rightarrow (-x, y, -u, v).
\end{align*}

To state the main result of this paper, we suppose that the following conditions hold:

**Assumption 1.** (Non-resonance conditions). Let \(\omega_0 = \omega(y_0)\) and \(\lambda_0 = \lambda(y_0)\) with a certain \(y_0 \in \mathcal{M} \subset \mathbb{R}^n\). Assume \(\omega_0\) and \(\lambda_0\) satisfy Melnikov’s non-resonance conditions:

$$
|\langle k, \omega_0 \rangle - \langle l, \lambda_0 \rangle| \geq \frac{\alpha}{(1 + |k|)^\tau}, \quad (k, l) \in \mathbb{Z}^n \times \mathbb{Z}, \quad |l| \leq 2, |k| + |l| \neq 0, \quad (1.8)
$$

where \(\alpha > 0\) and \(\tau > n - 1\) are some constants. Moreover, \(|\lambda(y)| \geq \sigma > 0\) holds for all \(y \in \mathcal{O}\), where \(\mathcal{O} \subset \mathcal{M}\) is a small open neighborhood of \(y_0\).

**Assumption 2.** (The non-degeneracy condition). Brouwer’s degree of \(\omega/\lambda\) at \(\omega_0/\lambda_0\) on \(\mathcal{O}\) is not zero, i.e.,

$$
\deg(\omega/\lambda, \omega_0/\lambda_0) \neq 0. \quad (1.9)
$$

**Assumption 3.** \(\omega(y), \lambda(y)\) and the perturbation terms \(P^j (1 \leq j \leq 4)\) are analytic with respect to \((x, y, u, v)\) on a complex domain

$$
M = \mathcal{D}(\mathbb{T}^n) \times \mathcal{D}(\bar{\mathcal{O}}) \times \left\{(u, v) \mid |u| \leq r, |v| \leq r\right\}
$$

$$
\subset \mathbb{C}^n/2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C},
$$

where \(\bar{\mathcal{O}}\) is the closure of \(\mathcal{O}\), \(\mathcal{D}(\mathbb{T}^n)\) is a neighborhood of \(\mathbb{T}^n\) in \(\mathbb{C}^n/2\pi\mathbb{Z}^n\) and \(\mathcal{D}(\bar{\mathcal{O}})\) is a neighborhood of \(\bar{\mathcal{O}}\) in \(\mathbb{C}^n\).

**Theorem 1.1.** Consider the reversible system (1.7). Suppose that the above assumptions (1)-(3) hold. Then there exists a sufficiently small positive constant \(\epsilon > 0\), such that if

$$
\max \left\{ \sup_M |P^1|, \frac{1}{r} \sup_M |P^2|, \frac{1}{r} \sup_M |P^3|, \frac{1}{r} \sup_M |P^4| \right\} \leq \epsilon,
$$

then the reversible system (1.7) has an invariant torus with the tangential frequency \(\omega_*\) and the normal frequency \(\lambda_*\). Here \(\omega_*\) and \(\lambda_*\) have the following form:

$$
\omega_* = \left( \frac{\lambda(y_*) + \hat{\lambda}_*(y_*)}{\lambda_0} \right) \omega_0, \quad \lambda_* = \left( \frac{\lambda(y_*) + \hat{\lambda}_*(y_*)}{\lambda_0} \right) \lambda_0, \quad (1.10)
$$

with a certain \(y_* \in \mathcal{O}\) and \(\hat{\lambda}_* = \mathcal{O}(\epsilon)\). Thus the direction of this frequency for the invariant torus \(\mathbb{T}^n \times \{ y_0 \} \times \{ 0 \} \times \{ 0 \}\) persists under small perturbations.

**Remark 1.** An example to which the above theorem can be applied is \(\omega(y) = \omega_0 + (y_1^2, y_2^2, \ldots, y_n^2)\) and \(\lambda(y) = \lambda_0 + y_1^2\) with \(\lambda_0 > 0\) and \(y_0 = 0 \in \mathcal{O} \subset \mathbb{R}^n\). Let \(\Omega(y) = (\omega(y), \lambda(y))\) and \(\Omega_0 = (\omega_0, \lambda_0)\). It is easy to see that \(\Omega(y)\) is a mapping from \(\mathcal{O} \subset \mathbb{R}^n\) into \(\mathbb{R}^{n+1}\) and \(\Omega(y)\) does not meet the condition (1.6). Hence, the
result of [31] cannot be applied. Moreover, \( \omega(y) \) does not meet the Kolmogorov’s non-degeneracy condition [4]. Thus the classic KAM theorem cannot be applied. Although \( \omega(y) \) satisfies Rüssmann’s non-degeneracy condition, the previous KAM theorems cannot provide any information on the persistence of the lower dimensional invariant tori \( \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \) for the reversible system (1.7).

In the special case that \( \lambda(y) \equiv \lambda_0 \) for all \( y \in \mathcal{O} \) for the reversible system (1.7), then the non-degeneracy condition (1.9) becomes \( \text{deg}(\omega, \mathcal{O}, \omega_0) \neq 0 \). Moreover, the equations (1.10) become \( \omega_j = (1 + \hat{\eta}_*)\omega_0 \) and \( \lambda_j = (1 + \hat{\eta}_*)\lambda_0 \) with \( \hat{\eta}_* = \frac{\hat{\lambda}}{\lambda_0} = O(\epsilon) \). Obviously, our condition (1.9) is weaker than (1.6). By the above discussions, it is easy to see that our condition (1.9) is a generalization of the above non-degeneracy conditions [5, 6, 31] for the reversible system (1.7).

Remark 2. Noting that \( y \in \mathbb{R}^{n+p} \) in the paper [31], the reversible system in [31] has enough parameters to remove the shifts of the tangential frequency and the normal frequency and make the frequency \( \omega_0 \) and \( \lambda_0 \) fixed in KAM steps. The principal difficulty of this paper is that the system (1.7) does not have enough parameters to control, simultaneously, the values of the tangential frequency and the normal frequency of the torus in the KAM steps. However, by observation we find that it is not necessary to remove the shift of the normal frequency by adjusting parameters in the KAM steps for the case of \( p = 1 \). In the proof of this theorem, we first reduce the system (1.7) to a parameterized system by the transformation \( y = \xi + y_+ \). Then we introduce an artificial external parameter \( \gamma \) and rewrite the normal frequency \( \lambda_j \) as

\[
\lambda_j(\xi, \gamma) = \left( \frac{\lambda(\xi) + \hat{\lambda}_j(\xi, \gamma)}{\lambda_0} \right) \lambda_0
\]

in the KAM steps, where \( \hat{\lambda}_j \) is the shift of the normal frequency. By adjusting the parameters and the implicit function theorem, we can also rewrite the tangential frequency \( \omega_j \) as

\[
\omega_j(\xi, \gamma) = \left( \frac{\lambda(\xi) + \hat{\lambda}_j(\xi, \gamma)}{\lambda_0} \right) \omega_0
\]

in the KAM steps. Noting that \( \omega_0 \) and \( \lambda_0 \) satisfy Melnikov’s non-resonance conditions (1.8), then \( \omega_j \) and \( \lambda_j \) also meet the Melnikov’s non-resonance conditions in the KAM steps. The above factorization method plays an important role in weakening the Brouwers degree condition (1.6).

Remark 3. In the proof of this paper, we use the topological degree theory to obtain a solution corresponding to an invariant torus of the reversible system (1.7). Actually, the reversible system (1.7) may admit many invariant tori because the solution obtained by the topological degree theory is not unique.

2. Proof of Theorem 1.1. Motivated by [23, 31, 39], we first reduce the system (1.7) to a parameterized system. By a transformation \( y = \xi + y_+ \), we introduce a parameter \( \xi \) and then the reversible system (1.7) is equivalent to a parameterized system:

\[
\begin{align*}
\dot{x} &= \omega(\xi) + f^1(x, y, u, v; \xi), \\
\dot{y} &= f^2(x, y, u, v; \xi), \\
\dot{u} &= \lambda(\xi) v + f^3(x, y, u, v; \xi), \\
\dot{v} &= -\lambda(\xi) u + f^4(x, y, u, v; \xi),
\end{align*}
\tag{2.1}
\]
where
\[ f^1 = P^1(x, y + \xi, u, v) + \omega(y + \xi) - \omega(\xi), \]
\[ f^2 = P^2(x, y + \xi, u, v), \]
\[ f^3 = P^3(x, y + \xi, u, v) + \left( \lambda(y + \xi) - \lambda(\xi) \right)v \]
\[ f^i = P^i(x, y + \xi, u, v) - \left( \lambda(y + \xi) - \lambda(\xi) \right)u \]
and \( \xi \in O \) is regarded as parameter. Note that for simplicity we have used \( y \) instead of the new variables \( y_+ \) in the transformed equations. Below we consider the persistence of invariant tori for the reversible systems (2.1) instead of the original system (1.7).

Let
\[ D(s, r) = \{ (x, y, u, v) \mid \text{Im} x \leq s, |y| \leq r, |u| \leq r, |v| \leq r \}, \]
\[ \Pi = \{ \xi \in O \mid \text{dist}(\xi, \partial O) \geq d \}, \]
where \( \partial O \) denotes the boundary of \( O \), and \( d > r > 0 \) is a small constant. Let \( \Pi_d \) be the complex neighborhood of \( \Pi \) in \( \mathbb{C}^n \) with the radius \( d \), that is,
\[ \Pi_d = \{ \xi \in \mathbb{C}^n \mid \text{dist}(\xi, \Pi) \leq d \}. \]

Then the reversible system (2.1) is real analytic in \( (x, y, u, v, \xi) \) on \( D(s, r) \times \Pi_d \).

We first introduce some notations. If \( f(x, y, u, v, \xi) \) is analytic on \( D(s, r) \times \Pi_d \), we expand \( f \) in Taylor-Fourier series as:
\[ f(x, y, u, v, \xi) = \sum_{k \in \mathbb{Z}^n, i \in \mathbb{Z}^+, i,j \in \mathbb{Z}_+} f_{klij}^{s,r} e^{i(k,x)}. \]

Let \( M_{f_k}(y, u, v; \xi) = \sum_{i,j} |f_{klij}^{s,r}| |y|^i |u|^j \) and define
\[ \|f\|_{D(s, r) \times \Pi_d} = \sum_{k \in \mathbb{Z}^n} |M_{f_k}|_{r,d} e^{||k||}, \]
where \( |M_{f_k}|_{r,d} \) denotes the sup-norm of \( M_{f_k} \) over the domain \( D(s, r) \times \Pi_d \). Let \( f = (f^1, f^2, f^3, f^4) \) be a vector field depending on \( x, y, u, v \) and \( \xi \). Define a weighted norm by
\[ \|\|f\|\|_{D(s, r) \times \Pi_d} = \|f^1\|_{D(s, r) \times \Pi_d} + \cdots + \sum_{i=2}^4 \frac{1}{r^i} \|f^i\|_{D(s, r) \times \Pi_d}. \]

**Theorem 2.1.** Suppose the reversible system (2.1) is analytic in \( (x, y, u, v; \xi) \) on \( D(s, r) \times \Pi_d \), where \( \Pi \subset \mathbb{R}^n \) is a bounded domain. Let \( |\lambda(\xi)| \geq \sigma > 0 \) holds for all \( \xi \in \Pi \). Set \( \omega_0 = \omega(\xi_0) \) and \( \lambda_0 = \lambda(\xi_0) \) with \( \xi_0 \in \Pi \). Suppose that \( \omega_0 \) and \( \lambda_0 \) satisfy the non-resonant condition (1.8) and \( \text{deg}(\omega/\lambda, \Pi, \omega_0/\lambda_0) \neq 0 \). Then there exists a sufficiently small positive constant \( \epsilon > 0 \), such that if
\[ \|\|f\|\|_{D(s, r) \times \Pi_d} \leq \epsilon \] with \( f = (f^1, f^2, f^3, f^4) \),
then there exists \( \xi_* \in \Pi \) such that the reversible system (2.1) at \( \xi_* \) has an invariant torus with the tangential frequency \( \omega_* = \left( \frac{\lambda(\xi_*) + \lambda(\xi_0)}{\lambda_0} \right) \omega_0 \) and the normal frequency \( \lambda_* = \left( \frac{\lambda(\xi_*) + \lambda(\xi_0)}{\lambda_0} \right) \lambda_0 \), with a certain \( \xi_* \in \mathcal{O} \) and \( \lambda_* = O(\epsilon) \).

**Remark 4.** By the above discussions, Theorem 1.1 can be reduced to Theorem 2.1. Actually, the invariant torus for (2.1) at the parameter \( \xi_* \) corresponds to an invariant torus in Theorem 1.1. So below we mainly prove Theorem 2.1.
We use the Herman method to prove Theorem 2.1. The Herman method is a well-known KAM technique that introduces an artificial external parameter to make the unperturbed system highly non-degenerate. This method has been used in [5, 27, 28, 31, 39]. Now we introduce an artificial external parameter $\gamma$ and consider the following reversible system

$$
\begin{align*}
\dot{x} &= \omega(x) + \gamma + f^1(x, y, u, v; \xi), \\
\dot{y} &= f^2(x, y, u, v; \xi), \\
\dot{u} &= \lambda(x) v + f^3(x, y, u, v; \xi), \\
\dot{v} &= -\lambda(x) u + f^4(x, y, u, v; \xi),
\end{align*}
$$

where $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n$ is an external parameter. Obviously the reversible system (2.1) corresponds to the reversible system (2.2) with $\gamma = 0$. We will first give a KAM theorem for the reversible system (2.2) with parameters $(\xi, \gamma)$ and delay the proof of Theorem 2.1 later.

Let $\mu = \max_{\xi, \gamma \in \Pi_d} |\omega(\xi) - \omega(\xi)|$ and define $B(\omega, \mu) = \{\gamma \in \mathbb{C}^n \mid \text{dist}(\gamma, \omega) < \mu\}$. Let $Q = (\bigcup_{\xi \in \Pi} B(\omega(\xi), \mu)) \cap \mathbb{R}^n$. It follows that $\omega(\Pi) = \{\omega(\xi) \mid \xi \in \Pi\} \subset Q$.

Set

$$
Q_\alpha = \left\{ \Omega = (\omega, \lambda) \in Q \times \mathbb{R} \middle| \langle k, \omega \rangle - \langle l, \lambda \rangle \geq \frac{4\alpha}{(1 + |k|)^r}, \quad (k, l) \in \mathbb{Z}^n \times \mathbb{Z}, \quad |l| \leq 2, |k| + |l| \neq 0\right\}.
$$

Let $W = \Pi_d \times B(0, 2\mu + 1)$. It is easy to see that the reversible system (2.2) is analytic with respect to $(x, y, u, v, \xi, \gamma)$ on $D(s, r) \times W$.

In order to prove Theorem 2.1, we first give the following theorem.

**Theorem 2.2.** Let $f = (f^1, f^2, f^3, f^4)$. There exists a $\epsilon > 0$ such that if $\|f\|_{D(s,r) \times W} \leq \epsilon$, then we have a Cantor-like family of analytic curves in $W$:

$$
\{\Gamma^*_0 \colon \gamma = \gamma_s(\xi) \mid \Omega_0 = (\omega_0, \lambda_0) \in Q_\alpha\},
$$

which are determined by the equation

$$
\gamma + \omega(\xi) + \dot{\omega}_s(\xi, \gamma) = \left(\frac{\lambda(x) + \dot{\lambda}_s(\xi, \gamma)}{\lambda_0}\right)\omega_0,
$$

where $\dot{\omega}_s(\xi, \gamma)$ and $\dot{\lambda}_s(\xi, \gamma)$ are $C^\infty$-smooth functions on $W$. Let $\Lambda_s(\xi, \gamma) = \dot{\omega}_s(\xi, \gamma) - \dot{\lambda}_s(\xi, \gamma)$. We have

$$
|\Lambda_s(\xi, \gamma)| \leq 2\left(1 + \frac{|\omega_0|}{|\lambda_0|}\right)\epsilon \quad \text{and} \quad |\Lambda_\xi(\xi, \gamma)| + |\Lambda_\gamma(\xi, \gamma)| \leq \frac{1}{2}.
$$

Moreover, we have a parameterized family of compatible transformations

$$
\Phi_s(\xi, \gamma, \delta, \epsilon) \colon D(s, r) \rightarrow D(s, r), \quad (\xi, \gamma) \in \Gamma_s = \bigcup_{\Omega_0 \in Q_\alpha} \Gamma^*_0,
$$

where $\Phi_s(\xi, \gamma, \delta, \epsilon)$ smoothly depends on $\delta, \epsilon$. The family $\Pi_\alpha$ is determined by $\Pi_\alpha = \{\Omega_0 \in Q_\alpha \mid \omega_0(\xi) = \omega(\xi)\}$.
where $\Phi_*$ is analytic in $(x, y, u, v)$ on $D(\xi, \gamma)$ and $C^\infty$-smooth in $(\xi, \gamma)$ on $\Gamma_*$, such that for each $(\xi, \gamma) \in \Gamma_0^*$, the compatible transformation $\Phi(\cdot, \cdot, \cdot; \xi, \gamma)$ transforms the reversible system (2.2) into

$$
\begin{aligned}
\dot{x} &= \omega_*(\xi, \gamma) + f_1^*(x, y, u, v; \xi, \gamma), \\
\dot{y} &= f_2^*(x, y, u, v; \xi, \gamma), \\
\dot{u} &= \lambda_*(\xi, \gamma) v + f_3^*(x, y, u, v; \xi, \gamma), \\
\dot{v} &= -\lambda_*(\xi, \gamma) u + f_4^*(x, y, u, v; \xi, \gamma),
\end{aligned}
$$

where $f_j^*$ satisfy $f_j^*(x, 0, 0; \xi, \gamma) = 0$ ($j = 1, 2, 3, 4$). Moreover, $\omega_*$ and $\lambda_*$ have the following form:

$$
\omega_*(\xi, \gamma) = \left(\frac{\lambda(\xi) + \lambda_*(\xi, \gamma)}{\lambda_0}\right) \omega_0, \quad \lambda_*(\xi, \gamma) = \left(\frac{\lambda(\xi) + \lambda_*(\xi, \gamma)}{\lambda_0}\right) \lambda_0,
$$

where $\lambda_* = O(\epsilon)$. Hence, the reversible system (2.4) has an invariant torus $\Phi(\mathbb{T}^n, 0, 0, 0; \xi, \gamma)$ with the tangential frequency $\omega_*(\xi, \gamma)$ and the normal frequency $\lambda_*(\xi, \gamma)$.

Now we first use Theorem 2.2 to prove Theorem 2.1 and delay the proof of Theorem 2.2 until later. Let $\Omega_0 = (\omega_0, \lambda_0) \in \mathcal{Q}_\alpha$, then we have an analytic curve

$$
\Gamma_{\Omega_0} : \gamma = \gamma_*(\xi), \quad \xi \in \Pi,
$$
determined by the following equation

$$
\gamma + \omega(\xi) + \hat{\omega}_*(\xi, \gamma) = \left(\frac{\lambda(\xi) + \lambda_*(\xi, \gamma)}{\lambda_0}\right) \omega_0.
$$

It follows easily that

$$
\gamma = \lambda(\xi) \left(\omega_0/\lambda_0 - \omega(\xi)/\lambda(\xi)\right) - \Lambda_*(\xi, \gamma).
$$

By the implicit function theorem we have

$$
\gamma_*(\xi) = \lambda(\xi) \left(\omega_0/\lambda_0 - \omega(\xi)/\lambda(\xi) - \Lambda_*(\xi, \gamma_*(\xi))/\lambda(\xi)\right), \quad \xi \in \Pi.
$$

Noting that $|\Lambda_*(\xi, \gamma)/\lambda(\xi)| \leq 2 \left(1 + \frac{\omega_0}{\lambda(\xi)}\right) \frac{\pi}{\lambda(\xi)}$ and deg$(\omega_0/\lambda_0 - \omega/\lambda, \Pi, 0) \neq 0$, if $\epsilon$ is sufficiently small, we have $\xi_* \in \Pi$ such that

$$
\omega_0/\lambda_0 - \omega(\xi_*)/\lambda(\xi_*) - \Lambda_*(\xi_*, \gamma_*(\xi_*))/\lambda(\xi_*) = 0,
$$

which implies $\gamma_*(\xi_*) = 0$. Let

$$
\omega_* = \left(\frac{\lambda(\xi_*) + \lambda_*(\xi_*, \gamma_*(\xi_*))}{\lambda_0}\right) \omega_0 = \left(\frac{\lambda(\xi_*) + \lambda_*(\xi_*, 0)}{\lambda_0}\right) \omega_0,
$$

$$
\lambda_* = \left(\frac{\lambda(\xi_*) + \lambda_*(\xi_*, \gamma_*(\xi_*))}{\lambda_0}\right) \lambda_0 = \left(\frac{\lambda(\xi_*) + \lambda_*(\xi_*, 0)}{\lambda_0}\right) \lambda_0.
$$

Therefore, the reversible system (2.1) has an invariant torus $\Phi(\mathbb{T}^n, 0, 0, 0; \xi_*, \gamma_*(\xi_*)) = \Phi(\mathbb{T}^n, 0, 0, 0; \xi_*, \gamma_*(\xi_*))$ with the tangential frequency $\omega_*$ and the normal frequency $\lambda_*$. This completes the proof of Theorem 2.1. Noting that the invariant torus for (2.1) at the parameter $\xi_*$ corresponds to an invariant torus in Theorem 1.1. Theorem 1.1 is also proved. Now it remains to prove Theorem 2.2.
3. **Proof of Theorem 2.2.** Below we mainly use the KAM technique developed in [31] to prove Theorem 2.2. Our method is the standard KAM iteration. In the proof of this theorem, it is not necessary to remove the shift of the normal frequency by adjusting parameters as in [31]. We rewrite the normal frequency \( \omega_j \) as \( \omega_j = \frac{\lambda_j + \hat{\lambda}}{\lambda_0} \omega_0 \) in the KAM steps. By the implicit function theorem, we can also rewrite the tangential frequency \( \omega_j \) as \( \omega_j = \frac{\lambda_j + \hat{\lambda}}{\lambda_0} \omega_0 \) in the KAM steps. Then the condition (1.6) can be reduced to (1.9).

**KAM-Step.** In this section, we outline the formal process of one cycle of the KAM iteration. To simplify notations, in what follows, the quantities without subscripts refer to those at the \( j \)-th step, while the quantities with subscripts “+” denote the corresponding ones at the \((j+1)\)-th step. We will use the same notation \( c \) to indicate different constants, which are independent of the iteration process.

Suppose at the \( j \)-th step, we have arrived at the following reversible system:

\[
\dot{x} = \omega(\xi, \gamma) + f^1(x, z; \xi, \gamma), \quad \dot{z} = A(\xi, \gamma)z + g(x, z; \xi, \gamma),
\]

where \( \omega(\xi, \gamma) = \gamma + \omega^0(\xi) + \hat{\omega}(\xi, \gamma) \),

\[
z = \begin{pmatrix} y \\ u \\ v \end{pmatrix}, \quad g = \begin{pmatrix} f^2 \\ f^3 \\ f^4 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda(\xi, \gamma) \\ 0 & -\lambda(\xi, \gamma) & 0 \end{pmatrix},
\]

with \( \lambda(\xi, \gamma) = \lambda^0(\xi) + \hat{\lambda}(\xi, \gamma) \). We describe one step of KAM iteration in more details in the following lemma.

**Lemma 3.1.** (Iteration Lemma) Consider the reversible system (3.1) with \( |\lambda^0(\xi)| \geq \sigma \) for all \( \xi \in \Omega \) and

\[
|||f|||_{D(s, r) \times W} \leq \epsilon = \alpha \rho^{r+1} E.
\]

Assume that \( \Lambda(\xi, \gamma) = \hat{\omega}(\xi, \gamma) - \frac{\sigma}{\lambda_0} \hat{\lambda}(\xi, \gamma) \) satisfies

\[
|\hat{\lambda}(\xi, \gamma)| \leq \frac{\sigma}{2}, \quad |\Lambda_\epsilon(\xi, \gamma)| + |\Lambda_\gamma(\xi, \gamma)| \leq \frac{1}{2}, \quad \forall (\xi, \gamma) \in W,
\]

where \( \Omega_0 = (\omega_0, \lambda_0) \in \mathcal{Q}_\alpha \). Then the equation

\[
\gamma + \omega^0(\xi) + \hat{\omega}(\xi, \gamma) = \left( \frac{\lambda^0(\xi) + \hat{\lambda}(\xi, \gamma)}{\lambda_0} \right) \omega_0,
\]

defines implicitly an analytic mapping

\[
\gamma : \xi \in \Pi_d \rightarrow \gamma(\xi) \in B(0, 2\mu + 1),
\]

such that

\[
\Gamma_{\Omega_0} := \{(\xi, \gamma(\xi)) \mid \xi \in \Pi_d \} \subset W.
\]

Let \( 0 < \rho < \frac{\xi}{4} \) and \( K > 0 \) such that \( e^{-K\rho} = E \). Let \( \delta = \frac{\sigma |\omega_0|}{\lambda_0 + |\omega_0| (1+K)^{r+1}} \) and we have

\[
U(\Gamma_{\Omega_0}, \delta) = \left\{ (\xi, \gamma') \in \Pi_d \times \mathbb{C}^n \mid |\gamma' - \gamma(\xi)| \leq \delta \right\} \subset W.
\]

Set

\[
s_+ = s - 3\rho, \quad \rho_+ = \frac{1}{2} \rho, \quad \eta = E^{\frac{1}{2}}, \quad r_+ = \eta r, \quad E_+ = cE^{\frac{1}{2}}, \quad d_+ = d - \frac{\delta}{2}.
\]
Then, there exists
\[ W_+ = \left\{ (\xi, \gamma') \in \Pi_{d^+} \times \mathbb{C}^n \mid \xi \in \Pi_{d^+}, (\xi, \gamma) \in \Gamma = \bigcup_{\Omega_0 \in \mathcal{O}_a} \Gamma_{\Omega_0} \right\} \]
\[ |\gamma' - \gamma(\xi)| \leq \frac{\delta}{2} \subset W, \] (3.4)
such that for any \((\xi, \gamma) \in W_+\), there exists a compatible transformation \(\Phi(\cdot, \cdot; \xi, \gamma) : D(s, r) \to D(s, r)\) which changes the reversible system (3.1) to
\[ \dot{x} = \omega(\xi, \gamma) + f_+^0(x, z; \xi, \gamma), \quad \dot{z} = A_+(\xi, \gamma)z + g_+(x, z; \xi, \gamma), \] (3.5)
where \(\omega(\xi, \gamma) = \omega(\xi, \gamma) + \Delta \tilde{\omega}(\xi, \gamma), \)
\[ A_+ = \begin{pmatrix} 0 & 0 & \lambda_+(\xi, \gamma) \\ 0 & 0 & 0 \\ -\lambda_+(\xi, \gamma) & 0 \end{pmatrix}, \quad g_+ = \begin{pmatrix} f_2^0 \\ f_3^0 \\ f_4^0 \end{pmatrix}, \]
with \(\lambda_+(\xi, \gamma) = \lambda^0(\xi, \gamma) + \Delta \lambda^0(\xi, \gamma)\) and \(\Delta \lambda^0(\xi, \gamma) = \lambda^0(\xi, \gamma) + \Delta \lambda(\xi, \gamma)\). Moreover, we have the following conclusions:

(i) The compatible transformation \(\Phi\) satisfies
\[ \|\Xi(\Phi - id)\|_{D(s+, r+) \times W_+} \leq cE, \] (3.6)
\[ \|\Xi(D\Phi - I_{2n+2})\Xi^{-1}\|_{D(s+, r+) \times W_+} \leq cE, \] (3.7)
where \(\Xi = \text{diag}(I_n, \frac{1}{2} I_n, \frac{1}{2}, \frac{1}{2})\).

(ii) The new perturbation term \(f_+ = (f_1^+, f_2^0, f_3^0, f_4^0)\) satisfies
\[ \|f_+\|_{D(s+, r+) \times W_+} \leq \epsilon_+ = \alpha_{\rho^+}^{1 + 1}E_+. \] (3.8)

(iii) \(\Delta \tilde{\omega}(\xi, \gamma), \Delta \lambda^0(\xi, \gamma)\) and \(\Delta \lambda(\xi, \gamma) = \Delta \lambda^0(\xi, \gamma) - \frac{\lambda^0}{\lambda_0} \Delta \lambda(\xi, \gamma)\) satisfy
\[ |\Delta \tilde{\omega}(\xi, \gamma)| \leq \epsilon, \quad |\Delta \lambda^0(\xi, \gamma)| \leq \epsilon, \quad \forall (\xi, \gamma) \in W, \] (3.9)
\[ |\Delta \lambda^0(\xi, \gamma)| + |\Delta \lambda(\xi, \gamma)| \leq 2 \left(1 + \frac{|\omega_0|}{|\lambda_0|}\right) \frac{\epsilon}{\delta}, \quad \forall (\xi, \gamma) \in W_+. \] (3.10)

Thus, if
\[ \left(1 + \frac{|\omega_0|}{|\lambda_0|}\right)\epsilon \leq \frac{\delta}{8}, \] (3.11)
the equation
\[ \gamma + \omega^0(\xi) + \omega_+(\xi, \gamma) = \gamma + \omega^0(\xi) + \omega(\xi, \gamma) + \Delta \tilde{\omega}(\xi, \gamma) = \left(\frac{\lambda^0(\xi) + \lambda^0_+(\xi, \gamma)}{\lambda_0}\right)\omega_0, \]
defines implicitly an analytic mapping
\[ \gamma_+ : \quad \xi \in \Pi_{d^+} \rightarrow \gamma_+(\xi) \in B(0, 2\mu + 1), \]
satisfying
\[ |\gamma_+(\xi) - \gamma(\xi)| \leq 2 \left(1 + \frac{|\omega_0|}{|\lambda_0|}\right) \epsilon \leq \frac{\delta}{4} \] (3.12)
and
\[ \Gamma^+_{\Omega_0} := \left\{ (\xi, \gamma_+(\xi)) \mid \xi \in \Pi_{d^+} \right\} \subset W_. \] (3.13)
Let
\[ \delta_+ = \frac{\sigma|\omega_0|}{\lambda_0^0|\lambda_0|0} \frac{\alpha}{(1 + K_+)^{1 + 2}} \text{ with } K_+ \text{ satisfying } e^{-K_+\rho^+} = E_+. \] (3.14)
then for all \(\Omega_0 \in \mathcal{Q}_\alpha\), we have \(U(\Gamma^+_{\Omega_0}, \delta_+) \subset W_+\).
We divide the proof of Lemma 3.1 into the following several parts.

A. **Truncation.** If \( f(x; \xi, \gamma) \) is analytic in \( x \) on \( \mathbb{T}^n \), expanding \( f \) as Fourier series with respect to \( x \), we have
\[
f(x; \xi, \gamma) = \sum_{k \in \mathbb{Z}^n} f_k(\xi, \gamma)e^{\sqrt{-1}(k,x)}.
\]
Define the truncation of the Fourier series by
\[
T_k f(x; \xi, \gamma) = \sum_{|k| \leq K} f_k(\xi, \gamma)e^{\sqrt{-1}(k,x)}.
\]
Denote by
\[
\begin{align*}
T_K f^1(x, 0; \xi, \gamma) &= T_k(f^1|_{z=0}), \\
T_K g(x, 0; \xi, \gamma) &= T_k(g|_{z=0}), \\
T_K g_z(x, 0; \xi, \gamma) &= T_k\left(\frac{\partial g}{\partial z}|_{z=0}\right).
\end{align*}
\]

B. **Constructing compatible transformation.** Define a transformation
\[
\Phi: (x_+, z_+) \rightarrow (x, z)
\]
by
\[
x = x_+ + h(x_+), \quad z = z_+ + a(x_+) + b(x_+)z_+,
\]
where \( h, a \) are vector functions and \( b \) is an \((n+2) \times (n+2)\)-matrix function. By definition, it is easy to see that \( \Phi \) is compatible with respect to the involution \( G \) if only if
\[
h(-x) = -h(x), \quad Sa(-x) = a(x), \quad Sb(-x)S = b(x).
\]
where \( S = \text{diag}(I_n, -1, 1) \). Under the transformation \( \Phi \) the system (3.1) is changed to
\[
\begin{cases}
\dot{x} = \omega^+ + \left(I_n + \partial_x h(x)\right)^{-1}\left(-\partial_x h(x) + T_K f^1(x, 0) - \Delta \hat{\omega}\right) + f^1_+(x, z), \\
\dot{z} = A_+z + \left(I_{n+2} + b(x)\right)^{-1}\left(-\partial_x a(x) + Aa(x) + T_K g(x, 0)\right) + \\
\phantom{\dot{z} = A_+z + \left(I_{n+2} + b(x)\right)^{-1}\left(-\partial_x a(x) + Aa(x) + T_K g(x, 0)\right) + } \\
\phantom{\dot{z} = A_+z + \left(I_{n+2} + b(x)\right)^{-1}\left(-\partial_x a(x) + Aa(x) + T_K g(x, 0)\right) + } + g_+(x, z),
\end{cases}
\]
where \( \omega^+ = \omega + \Delta \hat{\omega} \) and \( A_+ = A + \Delta \hat{A} \) with \( \Delta \hat{\omega} \) and \( \Delta \hat{A} \) being decided later,
\[
\partial_\omega h = \sum_{k \in \mathbb{Z}^n} \sqrt{-1}(k, \omega)h_k e^{\sqrt{-1}(k,x)},
\]
\( \partial_\omega a \) and \( \partial_\omega b \) are defined similarly. Note that the parameters \((\xi, \gamma)\) are implied in the above equations. Moreover, we have
\[
\begin{align*}
f_+^1(x, z) &= \left(I_n + \partial_x h(x)\right)^{-1}\left(f^1 \circ \Phi(x, z) - T_K f^1(x, 0) - \partial_{\Delta \hat{\omega}} h(x)\right), \\
g_+(x, z) &= \left(I_{n+2} + b(x)\right)^{-1}\left(g \circ \Phi(x, z) - T_K g(x, 0) - T_K g_z(x, 0)\right) \\
&\quad - b(x) \Delta \hat{A} z - (\partial_{\Delta \hat{\omega} f^1_+(x, z)}(a(x) + b(x)z))
\end{align*}
\]
Here we have used \((x, z)\) instead of the new variables \((x_+, z_+)\) in the transformed equations for simplicity.
We want to find \( h(x), a(x) \) and \( b(x) \) such that
\[
\begin{align*}
\partial_\omega h(x) &= T_K f^1(x, 0) - \Delta \dot{\omega}, \\
\partial_\omega a(x) &= T_K g(x, 0), \\
\partial_\omega b(x) &= -A \partial_x h(x) + b(x) \mathcal{A} = T_K g_z(x, 0) - \Delta \dot{\mathcal{A}}.
\end{align*}
\] (3.20) (3.21) (3.22)

Then the system (3.17) becomes
\[
\dot{x} = \omega^+(\xi) + f^1_+(x, z, \xi), \quad \dot{z} = \mathcal{A}_+(\xi)z + g_+(x, z, \xi)
\] (3.23)

with \( f^1_+ \) and \( g_+ \) being much smaller perturbations than before.

C. Extension of small divisors. By the definition of \( \Gamma_{\Omega_0} \) in (3.3) and noting that
\[
\lambda(\xi, \gamma) = \left( \begin{array}{c}
\lambda^0(\xi) + \lambda(\xi, \gamma) \\
\lambda_0
\end{array} \right)
\]
with \( \lambda^0(\xi) \) and \( \lambda_0 \) being much smaller perturbations than before.

For any \((\xi, \gamma) \) in \( \Gamma_{\Omega_0} \), there exists \( (\xi, \gamma) \) in \( \Gamma \) such that \( |\gamma' - \gamma| < \delta \). So it follows that
\[
\begin{align*}
|\langle k, \omega(\xi, \gamma') - \omega(\xi, \gamma) \rangle + \langle l, \lambda(\xi, \gamma') - \lambda(\xi, \gamma) \rangle| \\
&\leq |\langle k, \lambda(\xi, \gamma') - \lambda(\xi, \gamma) \rangle| + |\lambda(\xi, \gamma') - \lambda(\xi, \gamma)\rangle| \\
&\leq \frac{1}{2}(K + 2) \left( 1 + \frac{\lambda_0}{|\omega_0|} \right) \delta \leq \frac{\alpha \sigma}{\lambda_0(1 + |k|)^2} \leq \frac{\alpha \sigma}{\lambda_0(1 + |k|)^2},
\end{align*}
\] (3.24)

for \( |k| \leq K, |l| \leq 2 \) and \( |k| + |l| \neq 0 \). By (3.24) and (3.25), we can extend the small divisor conditions to the neighborhood \( \hat{U}(\Gamma_{\Omega_0}, \delta) = \bigcup_{0 \in \Omega_0} U(\Gamma_{\Omega_0}, \delta) \) of \( \Gamma \). That is, for \((\xi, \gamma) \) in \( U(\Gamma_{\Omega_0}, \delta), (k, l) \in \mathbb{Z}^n \times \mathbb{Z}, |k| \leq K, |l| \leq 2, |k| + |l| \neq 0 \), we have
\[
\begin{align*}
\left| \langle k, \omega(\xi, \gamma') \rangle + \langle l, \lambda(\xi, \gamma') \rangle \right| \geq \frac{\alpha \sigma}{|\lambda_0(1 + |k|)^2|},
\end{align*}
\] (3.25)

D. Solving linear homological equations. In the following, we solve the linear homological equations (3.20)-(3.22). We first solve the equation (3.20). Let
\[
\begin{align*}
h(x) &= \sum_{0 < |k| \leq K} h_k e^{\sqrt{-1}(k,x)}, \quad f^1(x, 0) = \sum_k f^1_k e^{\sqrt{-1}(k,x)}.
\end{align*}
\]

Set \( \Delta \dot{\omega} = f^1_0 \). Comparing the Fourier coefficients in (3.20), we have \( h_k = \frac{f^1_k}{\sqrt{-1}(k, \omega)} \) for all \( 0 \neq |k| \leq K \). By (3.26) it follows that
\[
\|h\|_{D(s-\rho, r) \times U(\Gamma, \delta)} \leq \frac{\epsilon \rho}{\alpha \rho^2}.
\] (3.27)

Next we solve the equation (3.21). Set
\[
P = \begin{pmatrix}
I_n & 0 & 0 \\
0 & 1 & 1 \\
0 & \sqrt{-1} & -\sqrt{-1}
\end{pmatrix}.
\] (3.28)
It is not difficult to verify that
\[ M = P^{-1}AP = \sqrt{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda(\xi, \gamma) & 0 \\ 0 & 0 & -\lambda(\xi, \gamma) \end{pmatrix}. \] (3.29)

Set \( \tilde{a} = P^{-1}a \) and \( \tilde{c}(x) = T_K P^{-1}g(x, 0) \). Then the linear homological equation \[ (\sqrt{-1}(\omega, k)I_{n+2} + M)\tilde{a}_k = \tilde{c}_k, \] (3.31)

where \( \{\tilde{a}_k\} \) and \( \{\tilde{c}_k\} \) are the Fourier coefficients of \( \tilde{a} \) and \( \tilde{c} \), respectively. If \( |k| \neq 0 \), by the small divisor conditions \[ |\tilde{a}_k| \leq \frac{(1 + |k|)^r}{\alpha} |\tilde{c}_k|. \]

If \( |k| = 0 \), the coefficient matrix of the linear equation \[ M \] is singular. However, by the reversible structure we can choose a special solution for our problem. Let \( [g(\cdot, 0)] \) be the average of \( g(x, 0) \) with respect to \( x \) on \( \mathbb{T}^n \). Since the system \[ (3.1) \] is reversible, we have
\[ [g(\cdot, 0)] = \begin{pmatrix} 0_{n \times 1} \\ d_{1 \times 1} \\ 0_{1 \times 1} \end{pmatrix}. \]

Noting that
\[ A\left(0, \ldots, 0, -\frac{d}{\lambda(\xi, \gamma)}\right)^T = [g(\cdot, 0)], \]
we have \( \tilde{a}_0 = P^{-1}\left(0, \ldots, 0, -\frac{d}{\lambda(\xi, \gamma)}\right)^T \) and \( |\tilde{a}_0| \leq c\varepsilon r/\alpha \). Then it is easy to see that
\[ ||a||_{D(s-p, r) \times U(\Gamma, \delta)} \leq \frac{c\varepsilon r}{\alpha \rho^s}. \] (3.32)

Now we consider the last equation \[ (3.22) \]. Let \( \tilde{b} = P^{-1}bP \), \( \Delta \hat{A} = P^{-1}\Delta \hat{A}P \) and \( \tilde{d}(x) = T_K P^{-1}g_\omega(x, 0)P \). Then the linear homological equation \[ \partial_x \tilde{b}(x) - M\tilde{b}(x) + \tilde{b}(x)M = \tilde{d}(x) - \Delta \hat{A}. \] (3.33)

Similarly, the above equation is equivalent to
\[ -M\tilde{b}_0 + \tilde{b}_0M = \tilde{a}_0 - \Delta \hat{A}, \quad |k| = 0, \]
\[ \sqrt{-1}(\omega, k)\tilde{b}_k - M\tilde{b}_k + \tilde{b}_kM = \tilde{d}_k, \quad 0 \neq |k| \leq K, \]
where \( \{\tilde{b}_k\} \) and \( \{\tilde{d}_k\} \) are the Fourier coefficients of \( \tilde{b} \) and \( d \), respectively. Let \( \sqrt{-1}\lambda_1, \sqrt{-1}\lambda_2, \ldots, \sqrt{-1}\lambda_{n+2} \) be the eigenvalues of \( M \) with \( \lambda_i = 0 \) for \( 1 \leq i \leq n, \lambda_{n+1} = \lambda(\xi, \gamma) \) and \( \lambda_{n+2} = -\lambda(\xi, \gamma) \). Write the matrices \( \tilde{b}_k = (\tilde{b}_{ij}) \) and \( \tilde{d}_k = (\tilde{d}_{ij}) \). If \( |k| \neq 0 \), then we have
\[ \sqrt{-1}(\omega, k)\tilde{b}_{ij} = \tilde{d}_{ij}. \]

In the same way as the above we can solve \( \tilde{b}_{ij} \) and have
\[ |\tilde{b}_{ij}| \leq \frac{(1 + |k|)^r}{\alpha} |\tilde{d}_{ij}|. \]
For $|k| = 0$ solving the equation is more complicated. Let $[g_z(\cdot, 0)]$ be the average of $g_z(x, 0)$ with respect to $x$ on $\mathbb{T}^n$. Since the system (3.1) is reversible, we have

$$[g_z(\cdot, 0)] = \begin{pmatrix} 0 & C_{n \times 1} & 0 \\ D_{1 \times n} & 0 & E_{1 \times 1} \\ 0 & F_{1 \times 1} & 0 \end{pmatrix}.$$ 

Let $\Delta \hat{\lambda} = \frac{1}{2}(E - F)$. Set 

$$\Delta \hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta \hat{\lambda} \\ 0 & -\Delta \hat{\lambda} & 0 \end{pmatrix}.$$ 

Then $\tilde{d}_0 - \Delta \hat{A} = P^{-1}([g_z(\cdot, 0)] - \Delta \hat{A})P$ has the following form

$$\tilde{d}_0 - \Delta \hat{A} = \begin{pmatrix} 0 & C & C \\ \frac{1}{2}D & 0 & -\sqrt{\frac{1}{2}}(E + F) \\ \frac{1}{2}D & \sqrt{\frac{1}{2}}(E + F) & 0 \end{pmatrix}.$$ 

Thus we have a solution by letting

$$\tilde{b}_0^{ij} = \begin{cases} \frac{d_0^{ij}}{\sqrt{-1}(-\lambda_i + \lambda_j)}, & i \geq n + 1, i \neq j \quad \text{or} \quad j \geq n + 1, i \neq j, \\ 0, & \text{else.} \end{cases}$$

Note that $\|d\|_{D(s, r) \times W} \leq c\|g_z(x, 0)\|_{D(s, r) \times W} \leq c\epsilon$. By (3.26) it follows that

$$\|\tilde{b}\|_{D(s-r, r) \times U(r, \delta)} \leq \frac{c\epsilon}{\alpha \rho^2},$$

which implies that

$$\|b\|_{D(s-r, r) \times U(r, \delta)} \leq \frac{c\epsilon}{\alpha \rho^2}. \quad (3.34)$$

Now the transformation $\Phi$ is defined well by (3.15). The symmetry of (3.16) can be proved in the same way as in [17, 30]. Hence, $\Phi$ is a compatible transformation. Set $s_+ = s - 3\rho, r_+ = \eta r$. Then it is easy to see that

$$\Phi : (x_+, z_+) \in D(s_+, r_+) \to (x, z) \in D(s - 2\rho, 3\eta r) \subset D(s, r).$$

Moreover, we have

$$\|\Xi(\Phi - id)\|_{D(s_+, r_+) \times U(r, \delta)} \leq cE, \quad \|\Xi(D\Phi - I_{2n+2})\Xi^{-1}\|_{D(s_+, r_+) \times U(r, \delta)} \leq cE, \quad (3.35)$$

where $\Xi = \text{diag}(I_{n+1}, 1, I_n, 1, 1, \tau)$.

E. Estimates of perturbation terms. We first estimate the new perturbation term $f_1^1$. In view of $\|\partial x h\|_{D(s - 2\rho, r) \times U(r, \delta)} \leq \frac{c\epsilon}{\alpha \rho^2 \tau} \leq cE \ll 1$, it is easy to see that $\|(I_n + \partial z h)^{-1}\|_{D(s - 2\rho, r) \times U(r, \delta)} \leq 2$ if $E$ is sufficiently small. Note that $|\Delta \hat{\omega}| \leq c$. By (3.18) and (3.27) it follows that

$$\|f_1^1\|_{D(s_+, r_+) \times U(r, \delta)} \leq 2\|f_1^1 \circ \Phi(x, z) - Tk f_1^1(x, 0)\|_{D(s_+, r_+) \times U(r, \delta)} + cE \epsilon. \quad (3.36)$$

Let

$$f_1^1 \circ \Phi(x, z) - Tk f_1^1(x, 0) = F^1 + F^2 + F^3, \quad (3.37)$$
where
\[ F^1 = f^1 \circ \Phi(x, z) - f^1(x, z), \]
\[ F^2 = f^1(x, z) - f^1(x, 0), \]
\[ F^3 = f^1(x, 0) - T_K f^1(x, 0). \]

By (3.27), (3.32), (3.34) and the Cauchy estimates, it follows that
\[ \| F^1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\left( \frac{\epsilon}{\rho} \frac{\epsilon}{\alpha \rho^2} + \frac{\epsilon \rho}{r} \frac{\epsilon \rho}{\alpha \rho^2} \right) \leq cE, \]
\[ \| F^2 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\frac{\epsilon}{r} \eta r \leq c\eta e, \]
\[ \| F^3 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq \sum_{|k| > K} \| \mathcal{M}_k \|_{r_+} e^{|k|}(s-3\rho) \leq e^{-K\rho} e. \]

Noting that \( \eta = E \frac{1}{2} \) and \( e^{-K\rho} = E \). we have
\[ \| f^1 \circ \Phi - T_K f^1(x, 0) \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\eta e. \]

By (3.36) we have \( \| f^1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\eta e. \)

Now we estimate the new perturbation term \( g_+ \). Note that \( |\Delta \hat{\omega}| \leq \epsilon, |\Delta \hat{A}| \leq \epsilon \) and
\[ \| (I_{n+2} + b)^{-1} \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq 2 \]
if \( E \) is sufficiently small. Combining (3.19), (3.27), (3.32) and (3.34) it follows that
\[ \| g_+ \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq 2\| g \circ \Phi(x, z) - T_K g(x, 0) - T_K g_+(x, 0)z \|_{D(s_+, r_+ \times U(\Gamma, \delta)} + cEer + cEr \| f^1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \]

In the same way as (3.37), we have
\[ g \circ \Phi(x, z) - T_K g(x, 0) - T_K g_+(x, 0)z = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3, \]

where
\[ \mathcal{F}_1 = g \circ \Phi(x, z) - g(x, z), \]
\[ \mathcal{F}_2 = g(x, z) - g(x, 0) - g_+(x, 0)z, \]
\[ \mathcal{F}_3 = g(x, 0) + g_+(x, 0)z - T_K \left( g(x, 0) + g_+(x, 0)z \right). \]

Then, it follows that
\[ \| \mathcal{F}_1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\left( \frac{\epsilon}{\rho} \frac{\epsilon}{\alpha \rho^2} + \frac{\epsilon \rho}{r} \frac{\epsilon \rho}{\alpha \rho^2} \right) \leq cEer, \]
\[ \| \mathcal{F}_2 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\frac{\epsilon}{r} \eta r^2 \leq c\eta^2 er, \]
\[ \| \mathcal{F}_3 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq e^{-K\rho} er. \]

Noting that \( \eta = E \frac{1}{2}, e^{-K\rho} = E \) and \( \| f^1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\eta e \), it follows that
\[ \| g_+ \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \leq c\eta^2 er \leq c\eta e. \]

Thus we have \( \| f^1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} = \| f^1 \|_{D(s_+, r_+ \times U(\Gamma, \delta)} + \frac{1}{r_+} \| g_+ \|_{D(s_+, r_+ \times U(\Gamma, \delta)} \]

\( \leq c\eta e. \)

Let \( W_+ \) be defined by (3.4). Since the set \( \mathcal{Q}_0 \) is closed, it follows easily that \( W_+ \) is also closed. Obviously, we have
\[ W_+ \subset U(\Gamma, \delta) \subset W \] and \( \text{dist}(W_+, \partial W) \geq \frac{1}{2}\delta. \)
where $\partial W$ is the boundary of $W$. Note that $\Delta \Lambda (\xi, \gamma) = \Delta \hat{\omega} (\xi, \gamma) - \frac{\omega_0}{\lambda_0} \Delta \hat{\lambda} (\xi, \gamma)$, with $|\Delta \omega| \leq \epsilon$ and $|\Delta \hat{\lambda}| \leq \epsilon$. It is easy to see that (3.9) and (3.10) hold.

Set

$$
\hat{\omega}_+ (\xi, \gamma) = \hat{\omega} (\xi, \gamma) + \Delta \hat{\omega} (\xi, \gamma),
\hat{\lambda}_+ (\xi, \gamma) = \hat{\lambda} (\xi, \gamma) + \Delta \hat{\lambda} (\xi, \gamma),
\Lambda_+ (\xi, \gamma) = \hat{\omega}_+ (\xi, \gamma) - \frac{\omega_0}{\lambda_0} \hat{\lambda}_+ (\xi, \gamma).
$$

By the implicit function theorem, if

$$
\left| \frac{d \Lambda_+ (\xi, \gamma)}{d \gamma} \right| \leq \frac{1}{2}, \quad \forall (\xi, \gamma) \in M_+,
$$

the equation

$$
\gamma + \omega^0 (\xi) + \hat{\omega}_+ (\xi, \gamma) = \left( \frac{\lambda^0 (\xi) + \hat{\lambda}_+ (\xi, \gamma)}{\lambda_0} \right) \omega_0,
$$

defines implicitly an analytic mapping

$$
\gamma_+ : \xi \in \Pi_{d_+} \to \gamma (\xi) \in B(0, 2\mu + 1).
$$

Note that $\gamma$ and $\gamma_+$ satisfy that

$$
\gamma (\xi) + \omega^0 (\xi) + \hat{\omega}_+ (\xi, \gamma (\xi)) = \left( \frac{\lambda^0 (\xi) + \hat{\lambda}_+ (\xi, \gamma (\xi))}{\lambda_0} \right) \omega_0,
\gamma_+ (\xi) + \omega^0 (\xi) + \hat{\omega}_+ (\xi, \gamma_+ (\xi)) = \left( \frac{\lambda^0 (\xi) + \hat{\lambda}_+ (\xi, \gamma_+ (\xi))}{\lambda_0} \right) \omega_0.
$$

Then it is easy to see that

$$
|\gamma_+ (\xi) - \gamma (\xi)| = |\Lambda_+ (\xi, \gamma_+ (\xi)) - \Lambda_+ (\xi, \gamma (\xi))| \leq |\Lambda (\xi, \gamma_+ (\xi)) - \Lambda (\xi, \gamma (\xi))| + \frac{1}{2} |\gamma_+ (\xi) - \gamma (\xi)| + \left( 1 + \frac{|\omega_0|}{|\lambda_0|} \right) \epsilon.
$$

Hence, the conclusions (3.12) and (3.13) hold. By (3.14) we have $U (\Gamma^+_{t_0}, \delta_+) \subset W_+$. Thus, the proof of Lemma 3.1 is complete.

**Iteration.** Now we choose some suitable parameters so that the KAM step can iterate infinitely.

At the initial step, let $\omega^0 (\xi) = \omega (\xi)$, $\lambda^0 (\xi) = \lambda (\xi)$, $s_0 = s$, $r_0 = r$, $\rho_0 = \frac{s}{12}, \epsilon_0 = \epsilon$, $E_0 = \frac{\epsilon_0}{\alpha \rho_0^{\alpha}}, \eta_0 = E_0^\frac{1}{2}, d_0 = d$ and $\delta_0 = \frac{\sigma |\omega_0|}{\lambda_0^2 + |\lambda_0^{\omega_0}| (1 + K_0)^{-\tau}}$. Let $K_0$ satisfy $e^{-K_0 \rho_0} = E_0$. Assume that $s_j, r_j, \rho_j, E_j, \epsilon_j, \eta_j, K_j, d_j, \delta_j$ are all well defined for $j$-th step. Then we define

$$
\begin{align*}
& s_{j+1} = s_j - 3 \rho_j, \quad r_{j+1} = \eta_j r_j, \quad \rho_{j+1} = \frac{1}{2} \rho_j, \quad E_{j+1} = c E_j^{\frac{3}{2}}, \\
& \epsilon_{j+1} = \alpha \rho_j^{\tau + 1} E_{j+1}, \quad \eta_{j+1} = E_{j+1}^\frac{1}{2}, \quad d_{j+1} = d_j - \frac{1}{2} \delta_j, \\
& K_{j+1} = - \ln E_{j+1} / \rho_{j+1}, \quad \delta_{j+1} = \frac{\sigma |\omega_0|}{\lambda_0^2 + |\lambda_0^{\omega_0}| (1 + K_{j+1})^{-\tau}}.
\end{align*}
$$

In the following we are going to check all assumptions in Lemma 3.1 to ensure KAM steps valid for all $j \geq 0$.

Let $x_j = K_j \rho_j = - \ln E_j$. We have

$$
\frac{\epsilon_j}{\delta_j} = c E_j \rho_j^{\tau + 1} (1 + K_j)^{\tau + 1} \leq c E_j (2K_j \rho_j)^{\tau + 1} \leq c (2x_j)^{\tau + 1} e^{-x_j} \to 0 \quad \text{as} \quad j \to \infty.
$$
Note that we can choose sufficiently small $E_0$ such that $x_0$ is sufficiently large. Thus, the assumption $\left(1 + \frac{|\omega_l|}{|\lambda_0|}\right)\varepsilon_j < \frac{A_6}{8}$ holds in KAM steps.

Obviously,

$$\lambda_j = \sum_{i=0}^{j-1} |\Delta \lambda_j| \leq \sum_{i=0}^{j-1} \varepsilon_i \leq \sum_{i=0}^{\infty} \varepsilon_i \leq 2E_0,$$

$$\frac{\delta_{i+1}}{\delta_j} = \left(1 + \frac{K_j}{1 + K_{j+1}}\right)^{\tau+1} = \left(\frac{1}{2}\right)^{\tau+1} \left(\rho_j + x_j\right)^{\tau+1}.$$  

Hence, for sufficiently small $E_0$, the assumptions $|\lambda_j| \leq \frac{E_0}{2}$ and $\delta_+ \leq \frac{1}{4} \delta$ hold in KAM steps.

Let $F_j = 2\left(1 + \frac{|\omega_l|}{|\lambda_0|}\right)\varepsilon_j$. It is not difficult to prove that $\frac{F_{j+1}}{F_j} \leq \frac{1}{2}$ for all $j \geq 0$ if $E_0$ is sufficiently small. Set $\Lambda_0 = 0$. By iteration we have $\Lambda_j = \sum_{i=0}^{j-1} \Delta \Lambda_i$. Combining with estimates for $\Delta \Lambda_j$, we have

$$|\Lambda_j| \leq \sum_{i=0}^{j-1} F_i \leq \sum_{i=0}^{\infty} F_i \leq 2F_0 \leq 4\left(1 + \frac{|\omega_l|}{|\lambda_0|}\right)E_0(-2\ln E_0)^{\tau+1}.$$  

Hence, if $E_0$ is sufficiently small, the assumption $|\Lambda_j| \leq \frac{E_0}{2}$ is also convergent on $W_0$. Let $F_j = \rho_j + x_j$. In the same way as in [30] we have the convergence of $\Phi_j$ to $\Phi_0 = \lim_{j \to \infty} \Phi_j$ on $D(\frac{\pi}{2}, \frac{\tau}{2})$.

Now we consider the convergence of $\hat{\omega}_j$. By the KAM step we have

$$|\hat{\omega}_{j+1} - \hat{\omega}_j| \leq |\Delta \hat{\omega}_j| \leq \varepsilon_j \to 0 \text{ as } j \to \infty \text{ for all } (\xi, \gamma) \in W_0 = \bigcap_{j \geq 0} W_j.$$  

So we have $\hat{\omega}_* = \lim_{j \to \infty} \hat{\omega}_j$ on $W_0$. Similarly, we have $\hat{\lambda}_* = \lim_{j \to \infty} \hat{\lambda}_j$ on $W_0$. Let $\hat{\eta}_0 = \frac{\lambda_0}{\omega_0}$ and $\Lambda_* = \hat{\omega}_* - \frac{\lambda_0}{\omega_0} \hat{\lambda}_*$ we have

$$|\hat{\eta}_0(\xi, \gamma)| \leq 2\varepsilon_0, \quad |\Lambda_0(\xi, \gamma)| \leq 2\left(1 + \frac{|\omega_l|}{|\lambda_0|}\right)\varepsilon_0 \text{ and } |\Lambda_0(\xi, \gamma)| + |\Lambda_0(\xi, \gamma)| \leq \frac{1}{2},$$  

for all $(\xi, \gamma) \in W_0 = \bigcap_{j \geq 0} W_j$.

Let $d_0 = d_0 - \frac{1}{2} \sum_{j=0}^{\infty} \delta_j$. It follows that $d_* = d_0 - \frac{3}{2} \delta_0$. If $E_0$ is sufficiently small such that $\delta_0 < d_0$, then we have $d_* > \frac{1}{2} d_0$. Thus $\Pi_{d_*} \subset \bigcap_{j \geq 0} \Pi_{d_j}$. By (3.12) and $\varepsilon_j \to 0$ as $j \to 0$, it is easy to see that $\{\gamma_j(\xi)\}$ is also convergent on $\Pi_{d_*}$. In fact, by the iteration lemma, for $i > j$

$$|\gamma_i(\xi) - \gamma_j(\xi)| \leq \sum_{l=j}^{i-1} \frac{\delta_l}{4} \leq \frac{\delta_j}{2}.$$  

Let $\gamma_j(\xi) \to \gamma_*(\xi)$, $\xi \in \Pi_{d_*}$. Since $\Gamma^j_{d_0} = \{(\xi, \gamma_j(\xi) | \xi \in \Pi_{d_*}) \subset W_j$ and $\gamma_j$ are all analytic on $\Pi_{d_*}$, so is the limit $\gamma_*(\xi)$. Let $i \to \infty$ and we have

$$|\gamma_0(\xi) - \gamma_j(\xi)| \leq \frac{\delta_j}{2}.$$
This implies that $\Gamma_{t_0} = \{(\xi, \gamma, \eta) \mid \xi \in \Pi_{t_0}\} \subset W_j$ and so $\Gamma = \bigcup_{t_0 \in \Omega_{t_0}} \Gamma_{t_0} \subset W_j$. So $\Gamma \subset W = \bigcap_{j \geq 0} W_j$. Obviously, for all $(\xi, \gamma) \in \Gamma_{t_0}$ we have
\[
\gamma + \omega(\xi, \gamma) = \left(\frac{\lambda(\xi) + \hat{\lambda}_s(\xi, \gamma)}{\lambda_0}\right)\omega_0.
\] (3.38)

In the same way as in [30] we can prove that $A_*$ and $\Phi_*$ are $C^\infty$ with respect to $(\xi, \gamma)$ on $W_*$ in the Whitney sense. By Whitney extension theorem [37], we can extend $A_*$ and $\Phi_*$ to be $C^\infty$-smooth on $W = \Pi_d \times B(0, 2\mu + 1)$, but it only makes sense on $W_*$ for our problem.

Moreover, by (3.38) it is easy to see that for each $(\xi, \gamma) \in \Gamma_{t_0}$ with $\Omega_0 = (\omega_0, \lambda_0) \in \Omega_\alpha$, the compatible transformation $\Phi(\cdot, \cdot, \cdot; \xi, \gamma)$ transforms the reversible system (2.4) into
\[
\begin{aligned}
\dot{x} &= \omega_{*}(\xi, \gamma) + f^1_{*}(x, y, u, v; \xi, \gamma), \\
\dot{y} &= f^2_{*}(x, y, u, v; \xi, \gamma), \\
\dot{u} &= \lambda_{*}(\xi, \gamma) v + f^3_{*}(x, y, u, v; \xi, \gamma), \\
\dot{v} &= -\lambda_{*}(\xi, \gamma) u + f^4_{*}(x, y, u, v; \xi, \gamma),
\end{aligned}
\]
where $f^j_{*}$ satisfy $f^j_{*}(x, 0, 0, 0; \xi, \gamma) = 0 \ (j = 1, 2, 3, 4)$. Moreover, $\omega_{*}$ and $\lambda_{*}$ have the following form:
\[
\omega_{*}(\xi, \gamma) = \left(\frac{\lambda(\xi) + \hat{\lambda}_s(\xi, \gamma)}{\lambda_0}\right)\omega_0, \quad \lambda_{*}(\xi, \gamma) = \left(\frac{\lambda(\xi) + \hat{\lambda}_s(\xi, \gamma)}{\lambda_0}\right)\lambda_0.
\] (3.39)

where $\hat{\lambda}_* = O(\epsilon)$. Hence, the reversible system (2.4) has an invariant torus $\Phi(T^n, 0, 0, 0; \xi, \gamma)$ with the tangential frequency $\omega_{*}(\xi, \gamma)$ and the normal frequency $\lambda_{*}(\xi, \gamma)$. This completes the proof of Theorem 2.2.

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