The uniformly most powerful test of statistical significance for counting-type experiments with background.

L. Fleysher,1 R. Fleysher,1 T. J. Haines,2 A. I. Mincer,1 and P. Nemethy1

1Department of Physics, New York University, New York, New York 10003
2Los Alamos National Laboratory, Los Alamos, New Mexico 87545

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In this paper, after a discussion of general properties of statistical tests, we present the construction of the most powerful hypothesis test for determining the existence of a new phenomenon in counting-type experiments where the observed Poisson process is subject to a Poisson distributed background with unknown mean.

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I. INTRODUCTION

Typical experiments which search for new phenomena such as rare decays (see, for example [1]), new particles (see, for example [2]) and astronomical gamma-ray and X-ray sources (see, for example [3, 7]) are counting-type experiments. In such experiments the number of observed events is distributed according to a Poisson distribution with some average rate. Unfortunately, often such experiments are subject to unwanted background, i.e., even if the new phenomenon is not present, the experiment will register some number of counts with average background rate. Only in rare cases is the expected background rate known. Therefore, to overcome this difficulty, these experiments typically utilize one of several available techniques. One of the possibilities is to perform two observations — one for which some of the observed counts are believed to originate from the new phenomenon and the other for which all observed counts are known to originate due to background only; all other conditions of the observations are kept intact. Thus, the two observations will yield two observed counts \( n_1 \) and \( n_2 \) made during observation times \( t_1 \) and \( t_2 \) respectively. The number of events \( n_{1,2} \) in each observation is drawn from the corresponding parent Poisson distribution. If the new phenomenon exists, the observations will come from the Poisson distributions with different average event rates. If, on contrary, the new phenomenon does not exist, the observations will come from the Poisson distributions with identical average event rates. When it is not possible to obtain data due to background only or to otherwise determine the average expected background rate another approach is often used: the first observation is made as before, but the second one is made with the help of computer simulations of exactly the same experiment with the new phenomenon “turned off”. In other words, the observation \( n_1 \) during time \( t_1 \) is obtained as in the previous case. The second observation
$n_2$ during time $t_2$ is obtained by simulating the experiment using only the established laws of physics. Since it is believed that the computer simulation correctly describes the data collection procedure, $n_2$ can be regarded as drawn from the Poisson distribution with average background rate. In either case, a decision as to the plausibility of the existence of the phenomenon is made based on the outcomes of the two observations. Because the outcomes of the observations represent random numbers drawn from their respective parent distributions, the question of existence of the new phenomenon is here addressed by a hypothesis test.

Various statistical tests have been developed which address the question of testing the hypothesis that two independent observations $n_1$ and $n_2$ made during times $t_1$ and $t_2$ are due to common background only. The methods, an overview of which can be found in [8], mostly use Gaussian-type approximations to the Poisson distribution and are not reliable for small numbers of observed events. In this paper we present the construction of the most powerful hypothesis test for this situation. That is, we calculate the critical region to be used which, for any given probability of claiming consistency with background fluctuation (typically a number such as $10^{-3}$ or less), maximizes the probability of detecting a signal.

II. TESTING A STATISTICAL HYPOTHESIS

We begin by reviewing the general procedure of hypotheses tests outlined in [4]. A statistical hypothesis is a statement concerning the distribution of a random variable $Y$. A hypothesis test is a rule for accepting or rejecting the hypothesis based on the outcome of an experiment. The hypothesis being tested, called the null hypothesis $H_0$, is formulated in such a way that all prior knowledge strongly supports it. The hypothesis is rejected if the

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1 The variable $Y$ may be a multidimensional vector. Throughout, we will use capital letters (such as $Y$) to denote random variables and small letters (such as $y$) to denote their particular realizations.
observed value $y$ of the random variable $Y$ lies within a certain critical region $w$ of the space $W$ of all possible outcomes of $Y$ and accepted or doubted otherwise. It follows, then, that if there are two tests for the same hypothesis $H_0$, the difference between them consists in the difference in critical regions. It also follows that $H_0$ can be rejected when, in fact, it is true (error of the first kind); or it can be accepted when some other alternative hypothesis $H_1$ is true (error of the second kind). The existence of an alternative hypothesis is clear, otherwise the null hypothesis would not be questioned. The probabilities of errors of these two kinds depend on the choice of the critical region. The definitions are illustrated in figure 1. Each probability $p(y)$ of occurrence of every event $y$ is given a subscript corresponding to its progenitor hypothesis, such as $p_0(y)$ for $H_0$. The region to the right of $y_c$ is selected to be the critical one and the two types of errors of the test are marked with different hatching.

A critical region is said to be the best critical region for testing hypothesis $H_0$ with regard to $H_1$ if it is the one which minimizes the probability of the error of the second kind (to accept $H_0$ when $H_1$ is true) among all regions which give the same fixed value of the probability of the error of the first kind (to reject $H_0$ when it is true). The construction of the best critical region, resulting in the most powerful test of $H_0$ with regard to $H_1$ was considered in 4 where the problem is solved for the general case of simple hypotheses. A hypothesis is said to be simple if it completely specifies the probability of the outcome of the experiment; it is composite if the probability is given only up to some unspecified parameters. In general, if at least one of the hypotheses is composite, the best critical region may not exist 4.

Critical regions $w(\alpha)$ corresponding to different probabilities $\alpha$ of errors of the first kind are engineered before the test is performed. When experimental data $y$ is obtained, the smallest $\alpha$ is found such that $y \in w(\alpha)$. It is then said, that the observed experimental data can be characterized by the $p$-value equal to $\alpha$.

The maximum $p$-value at which the null hypothesis is rejected is called significance of the test and will be denoted as $\alpha_c$. The corresponding critical region will be denoted as $w_c = w(\alpha_c)$. The significance $\alpha_c$ is set in advance, before the test is performed and its choice is based on the penalty for making the error of the first kind. (False scientific discoveries should not happen very often, and thus the significance is often selected as $\alpha_c = 10^{-3}$.) One minus the probability of the error of the second kind is called the power of the test which we denote as $(1 - \beta)$.

If as the result of the experiment the observed data lies inside of the critical region $w_c$, it is concluded that the null hypothesis is rejected in favor of the alternative one with significance $\alpha_c$ and power $(1 - \beta)$. If, however, the observed data lies outside of the critical region $w_c$, it is concluded that the null hypothesis is not rejected in favor of the alternative one with significance $\alpha_c$ and power $(1 - \beta)$.

Special consideration must be given to the case of a composite null hypothesis $H_0$ of the form $p_0(y; \{\lambda\})$ with unknown parameters $\{\lambda\}$. Indeed, suppose a critical region $w$ is specified. Then, it is possible to perform the test: if the obtained value of the observable $Y$ is inside the critical region $w$, the null hypothesis is rejected, if $y \notin w$, it is accepted. However, the probability of the error of the first kind $\alpha(\{\lambda\}) = \int_y p_0(y; \{\lambda\})dy$ in general depends on unknown values of parameters $\{\lambda\}$ of the null hypothesis. Thus, the $p$-value $\alpha$ can not be assigned and the conclusion of the test can not be stated. It is therefore desired to construct such critical regions $w$, that the probability of the error of the first kind does not depend on the values of unknown parameters. Such regions are called similar to $W$ with regard to parameters $\{\lambda\}$. A method for construction of such regions was found in 4 under limited conditions which in the case of one parameter $\lambda$ are:
the probability distribution \( p_0(y; \lambda) \) is infinitely differentiable with respect to \( \lambda \) and

the probability distribution \( p_0(y; \lambda) \) is such that if \( \Phi(y) = \frac{d \log p_0}{d \lambda} \), then

\[
\frac{d \Phi}{d \lambda} = A + B \cdot \Phi \tag{1}
\]

where the coefficients \( A \) and \( B \) are functions of \( \lambda \), but are independent of observations \( y \).

If the above conditions are satisfied, critical regions \( w \) similar to \( W \) with regard to \( \lambda \) are built up of pieces of the hypersurfaces \( \Phi = \text{const} \) defined by the likelihood ratio \( p_1/p_0 > q \).

**III. NULL HYPOTHESIS BEING TESTED**

As was pointed out above, in a typical counting-type experiment two independent observations yielding two counts \( n_1 \) and \( n_2 \) are made during time periods \( t_1 \) and \( t_2 \) respectively with all other conditions being equal. Because it is assumed that each event carries no information about another, each of the observed counts can be regarded as being drawn from a Poisson distribution with some value of the parameter. Inasmuch as an attempt is being made to establish the existence of a new phenomenon, the null hypothesis \( H_0 \) is formulated as: \( n_1 \) and \( n_2 \) constitute an independent sample of size 2 from a single Poisson distribution (adjusted for the duration of observation) which is due to a common background with some unknown count rate \( \lambda \), or

\[
H_0 : \ p_0(n_1, n_2) = \frac{(\lambda t_1)^{n_1}}{n_1!} \frac{(\lambda t_2)^{n_2}}{n_2!} e^{-\lambda (t_1 + t_2)}
\]

The alternative hypothesis is that the two observations are due to Poisson distributions with different unknown count rates \( \lambda_1 \) and \( \lambda_2 \) respectively \( (\lambda_1 \neq \lambda_2) \):

\[
H_1 : \ p_1(n_1, n_2) = \frac{(\lambda_1 t_1)^{n_1}}{n_1!} e^{-\lambda_1 t_1} \frac{(\lambda_2 t_2)^{n_2}}{n_2!} e^{-\lambda_2 t_2}
\]

The usual physical situation is that one of the count rates is considered to have some amount due to the new process and the remainder due to the background process [e.g. \( \lambda_1 = (\lambda + \lambda_{\text{signal}}) \) and \( \lambda_2 = \lambda \)], thus the formulated hypothesis test matches the physical problem given in the introduction.

It is seen that for the case of interest both hypotheses are of composite type (\( \lambda \)'s are unspecified) which complicates the construction of the test.

**IV. THE MOST POWERFUL TEST**

The formulated probability distribution \( p_0(n_1, n_2) \) satisfies the conditions of a special case considered in [4], which facilitates the search for the best critical region in \( (n_1, n_2) \) space. The probability distribution \( p_0(n_1, n_2) \) satisfies the following conditions:

- \( p_0(n_1, n_2) \) is infinitely differentiable with respect to \( \lambda \),
the function $\Phi(n_1, n_2)$ defined as $\Phi(n_1, n_2) = \frac{d \log m}{d \lambda}$ satisfies equation (1) with $\Phi(n_1, n_2) = \frac{n_1 + n_2}{\lambda} - (t_1 + t_2)$, $A = -\frac{t_1 + t_2}{\lambda}$ and $B = -\frac{1}{\lambda}$.

Therefore, the best critical region corresponding to the error of the first kind $\alpha$, determined from $\Phi = const$, is built up of pieces of the lines $n_t = n_1 + n_2 = const$. The segments of each line are those for which the ratio of likelihoods $q$ does not exceed the preset level $\alpha$. This translates to:

$$
(n_1 - n_1) \left( \frac{\lambda_1}{\lambda} \right)^{n_1} \left( \frac{\lambda_2}{\lambda} \right)^{n_2-n_1} e^{-(\lambda_1-\lambda)t_1} e^{-(\lambda_2-\lambda)t_2} \geq q_\alpha \quad \Rightarrow \quad n_1 \ln \frac{\lambda_1}{\lambda_2} \geq q_\alpha'
$$

which can be written as:

$$n_1 \geq n_\alpha, \quad \lambda_1 > \lambda_2$$  \hspace{1cm} (2)

$$n_1 \leq n_\alpha, \quad \lambda_1 < \lambda_2$$  \hspace{1cm} (3)

where the critical value $n_\alpha$ is chosen to satisfy the desired probability of the error of the first kind $\alpha$:

$$\alpha \sum_{k=0}^{n_t} p_0(k, n_t - k) = \sum_{k=n_\alpha}^{n_t} p_0(k, n_t - k), \quad \lambda_1 > \lambda_2$$

$$\alpha \sum_{k=0}^{n_t} p_0(k, n_t - k) = \sum_{k=0}^{n_\alpha} p_0(k, n_t - k), \quad \lambda_1 < \lambda_2$$

Substituting explicitly the expression for $p_0(n_1, n_2)$ in to these equations, we obtain

$$\alpha = (1 + \gamma)^{-n_t} \sum_{k=n_\alpha}^{n_t} C_{n_t}^k \gamma^k = I_{\frac{\gamma}{1+\gamma}}(n_\alpha, n_t - n_\alpha + 1), \quad \lambda_1 > \lambda_2$$  \hspace{1cm} (4)

$$\alpha = (1 + \gamma)^{-n_t} \sum_{k=0}^{n_\alpha} C_{n_t}^k \gamma^k = I_{\frac{\gamma}{1+\gamma}}(n_t - n_\alpha, n_\alpha + 1), \quad \lambda_1 < \lambda_2$$  \hspace{1cm} (5)

where $\gamma = t_1/t_2 > 0$, $C_m = \frac{n!}{m!(n-m)!}$ are binomial coefficients and $I_x(a, b)$ is the normalized incomplete beta function. It must be emphasized that the critical value $n_\alpha$ depends on the parameters $\lambda_{1,2}$ of the alternative hypothesis only via the relation $\lambda_1 < \lambda_2$ or $\lambda_1 > \lambda_2$. The best critical region for testing the null hypothesis against the alternative with $\lambda_1 \neq \lambda_2$ does not exist, but it does exist for testing against $\lambda_1 > \lambda_2$ or $\lambda_1 < \lambda_2$ separately, that is when the signal is a source or a sink respectively. The equations (4) with (2) or (5) with (3) define the best critical region $w(\alpha)$ in the space $(n_1, n_2)$ for testing $H_0$ with regard to $H_1$ (defined for $\lambda_1 < \lambda_2$ and $\lambda_1 > \lambda_2$ separately) corresponding to the probability of the error of the fist kind $\alpha$. The boundary of this critical region is found by solving the equation (4) or (5) with respect to $n_\alpha$ for all possible values of $n_t$. Owing to the discrete nature of the observed number of events these equations might not have solutions for the specified level of significance $\alpha_c$. Nevertheless, it is possible to construct a conservative critical region such that the probability to observe data within the region does not exceed the preset level $\alpha_c$ if the null hypothesis is true. This is done by requiring:
\[
\begin{cases}
\alpha_c \geq I_{\frac{1}{1+\gamma}}(n_\alpha, n_t - n_\alpha + 1) \\
\alpha_c < I_{\frac{1}{1+\gamma}}(n_\alpha - 1, n_t - n_\alpha + 2)
\end{cases}
\]

\[
\begin{cases}
\alpha_c \geq I_{\frac{1}{1+\gamma}}(n_t - n_\alpha, n_\alpha + 1) \\
\alpha_c < I_{\frac{1}{1+\gamma}}(n_t - n_\alpha - 1, n_\alpha + 2)
\end{cases}
\]

The power \((1 - \beta)\) of the test will, of course, depend on the values of the parameter \(s\) of the alternative hypothesis:

\[
(1 - \beta) = \sum_{n_t=0}^{\infty} \sum_{k=n_\alpha}^{n_t} p_1(k, n_t - k), \quad \lambda_1 > \lambda_2
\]

\[
(1 - \beta) = \sum_{n_t=0}^{n_\alpha} \sum_{k=0}^{n_t} p_1(k, n_t - k), \quad \lambda_1 < \lambda_2
\]

After explicit substitution of \(p_1(n_1, n_2)\) into these equations, we obtain:

\[
(1 - \beta) = \sum_{n_t=0}^{\infty} \frac{(\lambda_1 t_1 + \lambda_2 t_2)^{n_t}}{n_t!} e^{-(\lambda_1 t_1 + \lambda_2 t_2)} I_{\frac{\lambda_1 t_1}{\lambda_1 t_1 + \lambda_2 t_2}} (n_\alpha, n_t - n_\alpha + 1), \quad \lambda_1 > \lambda_2
\]

\[
(1 - \beta) = \sum_{n_t=0}^{\infty} \frac{(\lambda_1 t_1 + \lambda_2 t_2)^{n_t}}{n_t!} e^{-(\lambda_1 t_1 + \lambda_2 t_2)} I_{\frac{\lambda_2 t_2}{\lambda_1 t_1 + \lambda_2 t_2}} (n_t - n_\alpha, n_\alpha + 1), \quad \lambda_1 < \lambda_2
\]

For the purposes of the hypothesis test itself, equations (4,5) provide the method for the \(p\)-value calculation without the need for solving them. To do this, \(n_t\) must be set to \((n_1 + n_2)\) and \(n_\alpha\) to \(n_1\) then \(\alpha\) is computed from the equation (4) or (5). If the obtained \(p\)-value \(\alpha\) is not greater than \(\alpha_c\) the null hypothesis is rejected. This is the uniformly most powerful test of \(H_0\) with regard to \(H_1\).

It can be seen that the application of the method of best critical region construction to the problem of testing whether two observations came from the Poisson distributions with the same parameter or not have led us to the criterion suggested on intuitive grounds in [6]. The presented discussion, however, shows that it is not possible to construct a better test for the hypotheses under consideration. The practical use of equations (4,5,6,7) should not present any difficulty using modern computers.

V. COMPOUNDING RESULTS OF INDEPENDENT TESTS

It is often the case that the complete data set consists of several runs of the experiment, each of which belongs to the counting type subjected to Poisson distributed background with unknown means. The data set is then a set of pairs \((n_{1,r}, n_{2,r})\) with corresponding durations of observations \((t_{1,r}, t_{2,r})\), where subscript \(r\) enumerates all the runs of the experiment. Here we distinguish two cases: first where the parameters of both hypotheses do not depend on the run number \(r\) and second where such independence can not be asserted because of
modifications made to the apparatus between the runs. In the first case it can be seen that the critical region must be constructed out of pieces of surfaces \( n_t = \sum_r (n_{1,r} + n_{2,r}) = \text{const} \), on which \( \sum_r n_{1,r} \geq n_\alpha \) for the case of \( \lambda_1 > \lambda_2 \) or \( \sum_r n_{1,r} \leq n_\alpha \) for the case of \( \lambda_1 < \lambda_2 \). It is thus seen that equations (11,12) provide a method for the \( p \)-value calculation: \( n_t = \sum_r (n_{1,r} + n_{2,r}), \ n_\alpha = \sum_r n_{1,r} \) and \( \gamma = \sum_r \frac{n_{1,r}}{n_{2,r}} \). In other words, if the parameters of both hypotheses do not depend on the run number \( r \), the corresponding observations can simply be added.

In the second case, the derivation proceeds in the fashion similar to the presented derivation of equations (11,15), the critical region is built up of surfaces \( n_{t,r} = n_{1,r} + n_{2,r} = \text{const} \) such that

\[
\sum_r n_{1,r} \log(\frac{\lambda_{1,r}}{\lambda_{2,r}}) \geq q'_\alpha
\]

The critical value \( q'_\alpha \) is chosen to satisfy the desired probability of the error of the first kind \( \alpha \):

\[
\alpha = \prod_r (1 + \gamma_r)^{-n_{t,r}} \sum_{k_r \in [0, n_{t,r}]} C_{n_{t,r}}^{k_r} \gamma_r^{k_r} \left\{ \sum_r k_r \log(\frac{\lambda_{1,r}}{\lambda_{2,r}}) \geq q'_\alpha \right\}
\]

The \( p \)-value is obtained by setting \( q'_\alpha \) equal to \( \sum_r n_{1,r} \log(\frac{\lambda_{1,r}}{\lambda_{2,r}}) \). In this case, the critical region depends on the parameters of the alternative hypothesis, but the test becomes uniformly most powerful if it is known that the ratios \( \lambda_{1,r}/\lambda_{2,r} \) do not depend on the run number \( r \). The latter situation is common in practice because it reflects change of the level of pre-scaling of events in the data acquisition system or degradation of efficiency of sensors occurred between the experimental runs.

VI. CONCLUSION

In this paper we have reviewed the basic concepts of statistical hypothesis tests and underlined the relevant aspects often employed. The difficulty arises because frequently both the null and the alternative hypotheses are of composite type.

We have considered typical counting experiments and constructed the most powerful statistical test. In doing so, we have insisted on the ability to quantify the error of the first kind although the parameter of the composite null hypothesis is unknown. The test also happens to be the uniformly most powerful with regard to the composite alternative hypothesis with \( \lambda_1 < \lambda_2 \) or \( \lambda_1 > \lambda_2 \) separately. The constructed test is especially important for the case of small number of events where previously used methods are inadequate because the usual Gaussian-type approximations break down. Fortuitously, this is the case for which the proposed test can most easily be performed. The existence of the most powerful statistical test allows comparisons with other computationally less demanding methods to be made which may be important for some applications.

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