A uniformly spread measure criterion

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Abstract

We prove that if all shifts of a measure in the Euclidean space are close in a sense to each other, then this measure is close to the Lebesgue one.

Let \((a_n)_{n \in \mathbb{N}}\) be a discrete sequence in \(\mathbb{R}^d\), i.e., a map \(\mathbb{N} \to \mathbb{R}^d\) such that its image \(\{a_n\}_{n \in \mathbb{N}}\) has no limit point in \(\mathbb{R}^d\) and each \(x \in \{a_n\}_{n \in \mathbb{N}}\) has at most a finite multiplicity. Following Laczkovich \[1\], \[2\], we say that the sequence is uniformly spread over \(\mathbb{R}^d\), if there is \(\alpha > 0\) such that

\[
\inf \sup \left| a_n - \alpha \psi(n) \right| < \infty, \quad (1)
\]

where the infimum is taken over all bijections \(\psi : \mathbb{N} \to \mathbb{Z}^d\).

Using the idea of the mass transfer (see, for example, \[4\]), M. Sodin, B. Tsirelson extended the above definition to measures in \(\mathbb{R}^d\). In \[3\] they introduce a transportation distance between arbitrary locally finite positive measures \(\nu_1\) and \(\nu_2\)

\[
\text{Tra}(\nu_1, \nu_2) = \inf \sup \{ |x - y| : x, y \in \text{supp} \gamma \}.
\]

Here the infimum is taken over all transportation measures \(\gamma\) between measures \(\nu_1\) and \(\nu_2\), where the latter means

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x) \ d\gamma(x, y) = \int_{\mathbb{R}^d} \varphi(x) \ d\nu_1(x), \quad (2)
\]

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(y) \ d\gamma(x, y) = \int_{\mathbb{R}^d} \varphi(y) \ d\nu_2(y), \quad (3)
\]

for all continuous functions \(\varphi : \mathbb{R}^d \to \mathbb{R}\) with a compact support.
The degree of concentration of $\gamma$ near diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ shows the closeness of $\nu_1$ and $\nu_2$ to each other.

A continuous analogue of definition of uniformly spreading \[^{(1)}\] (actually belonging to M. Sodin, B. Tsirelson \[^{[3]}\]) has the form

**Definition 1.** A locally finite positive measure $\nu$ on $\mathbb{R}^d$ is uniformly spread over $\mathbb{R}^d$, if there is $\beta > 0$ such that

$$\text{Tra}(\nu, \beta \omega) < \infty,$$

where $\omega$ is the Lebesgue measure on $\mathbb{R}^d$.

In what follows we denote by $\nu^x$ the shift of the measure $\nu$ along $x \in \mathbb{R}^d$, and for any $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ put

$$Q(x, r) = \{y = (y^1, \ldots, y^d) \in \mathbb{R}^d : x^j - r/2 \leq y^j < x^j + r/2, \; j = 1, \ldots, d\}.$$

Also, we denote by $\chi^m(y), \; m \in \mathbb{Z}^d$, the indicator of the cube $Q(m, 1)$.

For a discrete sequence $(a_n)_{n \in \mathbb{N}}$ we set

$$\nu = \sum_n \delta_{a_n},$$

where $\delta$ is a unit mass sitting in the origin. Then

$$C_1 \text{Tra}(\nu, \alpha^{-d} \omega) \leq \inf_{\psi} \sup_n |a_n - \alpha \psi(n)| \leq C_2 \text{Tra}(\nu, \alpha^{-d} \omega).$$

As above, the infimum is taken over all bijections $\psi : \mathbb{N} \rightarrow \mathbb{Z}^d$, and the constants $C_1$ and $C_2$ depend only on the dimension $d$.

In fact, if the sequence satisfies \[^{(1)}\], then the measure $\gamma = \sum \delta_{a_n}(x) \otimes \alpha^{-d} \chi^{\alpha \psi(n)}(y/\alpha) \omega(y)$ is a transportation measure between $\nu$ and $\alpha^{-d} \omega$, and the first inequality in \[^{(6)}\] follows easily. The second inequality in \[^{(6)}\] is nontrivial. Its proof in \[^{[2]}\] is based on the Rado Lemma from the graph theory.

The main result of our article is the following theorem.

**Theorem 1.** A positive locally finite measure $\nu \neq 0$ is uniformly spread over $\mathbb{R}^d$ if and only if there exists a constant $C_3 < \infty$ such that

$$\text{Tra}(\nu, \nu^z) < C_3 \quad \forall \; z \in \mathbb{R}^d.$$

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Proof of the Theorem \[1\] By \[3\] Theorem 1.2, we have

$$\text{Tra}(\nu_1, \nu_3) \leq \text{Tra}(\nu_1, \nu_2) + \text{Tra}(\nu_2, \nu_3).$$

Hence, (4) and the equality \(\omega^z \equiv 1\) imply (7).

Next, suppose that the measure \(\nu\) satisfies (7). We decompose \(\mathbb{R}^d\) into the cubes \(Q(m, 1), m \in \mathbb{Z}^d\). Let \(\gamma_m\) be a transportation measure between \(\nu\) and \(\nu^m\). For fixed \(k \in \mathbb{Z}^d\) set

$$\lambda_{k,n} = \frac{1}{(2n + 1)^d} \sum_{\|m-k\|_{\infty} \leq n} \gamma_m.$$

It follows from definition of a transportation measure (2) and (3) that \(\lambda_{k,n}\) is a transportation measure between \(\nu\) and

$$\mu_{k,n} = \frac{1}{(2n + 1)^d} \sum_{\|m-k\|_{\infty} \leq n} \nu^m.$$

Note that we can replace in (2) and (3) the function \(\varphi\) by the indicator function of any bounded Borel subset of \(\mathbb{R}^d\). Therefore, by (7), we get for any \(m, k \in \mathbb{Z}^d\)

$$\nu^m(Q(k, 1)) = \nu^{m+k}(Q(0, 1)) = \gamma_{m+k}(Q(0, 1) \times \mathbb{R}^d) \leq \gamma_{m+k}(\mathbb{R}^d \times Q(0, C_3 + 1)) = \nu(Q(0, C_3 + 1)).$$

Hence the measures \(\mu_{k,n}\) are uniformly bounded on every compact subset of \(\mathbb{R}^d\), and the measures \(\lambda_{k,n}\) are uniformly bounded on every compact subset of \(\mathbb{R}^d \times \mathbb{R}^d\). By (7),

$$\text{Tra}(\nu, \mu_{k,n}) \leq \sup_{m \in \mathbb{Z}^d} \text{Tra}(\nu, \nu^m) \leq C_3. \quad (8)$$

Take a subsequence \(n' \to \infty\) such that for each \(k \in \mathbb{Z}^d\) the measures \(\lambda_{k,n}\) weakly converge to some measure \(\lambda_k\), and the measures \(\mu_{k,n}\) weakly converge to some measure \(\mu_k\). Note that for any \(k, k' \in \mathbb{Z}^d\)

$$\mu_{k,n} - \mu_{k',n} = \frac{1}{(2n + 1)^d} \left[ \sum_{\|m-k\|_{\infty} \leq n, \|m-k'\|_{\infty} > n} \nu^m - \sum_{\|m-k\|_{\infty} \leq n, \|m-k\|_{\infty} > n} \nu^m \right],$$

and

$$\text{card}\{m : \|m - k\|_{\infty} \leq n, \|m - k'\|_{\infty} > n\} = O(n^{d-1}),$$

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\[
\text{card}\{\|m - k'\|_{\infty} \leq n, \|m - k\|_{\infty} > n\} = O(n^{d-1}),
\]
as \(n \to \infty\). Hence, for each \(k, k' \in \mathbb{Z}^d\) the variations of measures \(\mu_{k,n} - \mu_{k',n}\) on every compact subset tend to zero as \(n \to \infty\). The same assertion is valid for the differences \(\lambda_{k,n} - \lambda_{k',n}\). Therefore, for all \(k \in \mathbb{Z}^d\) we have \(\mu_k \equiv \mu\) and \(\lambda_k \equiv \lambda\) for some measures \(\lambda\) and \(\mu\). Moreover,

\[
\mu(Q(k, 1)) = \mu(Q(k', 1)) \quad \forall \ k, k' \in \mathbb{Z}^d.
\]

It can easily be checked that the measure \(\lambda\) is a transportation measure between \(\nu\) and \(\mu\). By (8), we get

\[
\text{Tra}(\nu, \mu) \leq C_3.
\]

Hence, \(\mu \not\equiv 0\). Let \(\rho_m\) be the restriction of the measure \(\mu\) to \(Q(m, 1)\). We obtain that the measure

\[
\sum_{m \in \mathbb{Z}^d} \rho_m(x) \otimes \chi^m(y) \omega(y)
\]
is a transportation measure between \(\mu\) and \(\beta \omega\) with \(\beta = \mu(Q(0, 1))\). Finally,

\[
\text{Tra}(\nu, \beta \omega) \leq \text{Tra}(\nu, \mu) + \text{Tra}(\mu, \beta \omega) \leq C_3 + 1.
\]

Theorem 1 is proved.

For discrete sequences in \(\mathbb{R}^d\) we obtain the following result.

**Theorem 2.** A discrete sequence \((a_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^d\) satisfies (7) if and only if for any \(z \in \mathbb{R}^d\) there is a bijection \(\sigma : \mathbb{N} \to \mathbb{N}\) such that

\[
\sup_n |a_n + z - a_{\sigma(n)}| \leq C_7 < \infty. \quad (9)
\]

**Proof.** Clearly, (1) yields (9). On the other hand, if (9) holds and \(\nu\) is the measure defined in (5), then the measure

\[
\sum_{n \in \mathbb{N}} \delta^{a_n + z}(x) \otimes \delta^{a_{\sigma(n)}}(y)
\]
is a transportation measure between \(\nu^z\) and \(\nu\). Condition (9) implies \(\text{Tra}(\nu^z, \nu) < \infty\). By Theorem 1 for some \(\alpha > 0\) we have \(\text{Tra}(\nu, \alpha^{-d} \omega) < \infty\). Using (4), we obtain (1).

\[\square\]
References

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