On the relation between multifield and multidimensional integrable equations

V.E. Adler

Ufa Institute of Mathematics, 112 Chernyshevsky str.,
Ufa 450077, Russia
e-mail: adler@imat.rb.ru

Abstract

The new examples are found of the constraints which link the 1+2-dimensional and multifield integrable equations and lattices. The vector and matrix generalizations of the Nonlinear Schrödinger equation and the Ablowitz-Ladik lattice are considered among the other multifield models. It is demonstrated that using of the symmetries belonging to the hierarchies of these equations leads, in particular, to the KP equation and twodimensional analogs of the dressing chain, Toda lattice and dispersive long waves equations. In these examples the multifield equation and its symmetry have meaning of the Lax pair for the corresponding twodimensional equation under some compatible constraint between field variables and eigenfunctions.

1 Introduction. Vector NLS equation

Let us consider the vector generalization of the Nonlinear Schrödinger equation (NLS) [1, 2] and its third order symmetry:

\begin{align}
\psi_y &= \psi_{xx} + 2\langle \psi, \varphi \rangle \psi, \\
\psi_t &= \psi_{xxx} + 3\langle \psi, \varphi \rangle \psi_x + 3\langle \psi_x, \varphi \rangle \psi, \\
\varphi_t &= \varphi_{xxx} + 3\langle \psi, \varphi \rangle \varphi_x + 3\langle \psi, \varphi_x \rangle \varphi
\end{align}

where \( \psi, \varphi \in \mathbb{R}^N \) and the brackets \( \langle , \rangle \) denote scalar product. It is an easy exercise to check that the quantities

\begin{align}
u &= -2\langle \psi, \varphi \rangle, \\
q &= 2\langle \psi, \varphi_x \rangle - 2\langle \psi_x, \varphi \rangle
\end{align}
solve, in virtue of these equations and independently on the dimension $N$, the Kadomtsev-Petviashvili equation (KP)

$$4u_t = u_{xxx} - 6uu_x + 3q_y, \quad q_x = u_y.$$  \hspace{1cm} (4)

This observation was done in the papers \[3, 4\] for the first time. It reduces the problem of solving of the 1+2-dimensional equation to the simpler problem of construction of the common solution of the pair of 1+1-dimensional systems, although of the large size. Apparently, this trick is a sort of nonlinear analog of the separation of variables method. It is rather effective and even the simplest case $N = 1$ have brought to some new exact solutions of KP \[5\].

We mention also the further contributions \[6, 7, 8, 9\]. Within this approach eqs. (1), (2) are treated as the pair of the formally conjugated auxiliary linear problems for the KP equation

$$\psi_y = \psi_{xx} - u\psi, \quad -\varphi_y = \varphi_{xx} - u\varphi,$$

$$\psi_t = \psi_{xxx} - \frac{3}{2}u\psi_x - \frac{3}{4}(u_x + q)\psi, \quad \varphi_t = \varphi_{xxx} - \frac{3}{2}u\varphi_x - \frac{3}{4}(u_x - q)\varphi$$  \hspace{1cm} (5)

and the relations \[3\] are treated as the constraint between the potential and the eigenfunctions, which is consistent with the dynamics on $y$ and $t$. Bearing this interpretation in mind we always denote the multifield variables (vectors or matrices) as $\psi, \varphi$.

If the 1+2-dimensional equation is given then the finding of the corresponding 1+1-dimensional systems can be rather nontrivial task, even if the $L - A$-pair is known. Several solutions can exist. For example, it turns out that eqs. (4) \-- (6) admit also other compatible constraint:

$$u = 2\langle \psi_x, \varphi_x \rangle - 2\frac{\langle \psi, \varphi_x \rangle \langle \psi_x, \varphi \rangle}{\langle \psi, \varphi \rangle + 1},$$

$$q = 2\langle \psi_{xx}, \varphi_x \rangle - 2\langle \psi, \varphi_{xx} \rangle + 2\frac{\langle \psi, \varphi_x \rangle \langle \psi_x, \varphi \rangle - \langle \psi, \varphi_x \rangle \langle \psi_x, \varphi \rangle_x}{\langle \psi, \varphi \rangle + 1}$$

$$+ 2\frac{\langle \psi, \varphi_x \rangle \langle \psi_x, \varphi \rangle (\langle \psi, \varphi_x \rangle - \langle \psi_x, \varphi \rangle)}{(\langle \psi, \varphi \rangle + 1)^2}$$

(its origin will be explained in Section \[3\]). One more representation for the KP equation is given in the Example 3 below, but in this case the equations for $\psi$ and $\varphi$ are not conjugated.

On the other hand, now we know a large number of separate examples and the whole classes of integrable multifield equations (see, e.g. the papers \[4, 11, 12, 13, 14, 15, 16\] and the review \[17\]) and the natural question arise, which 1+2-dimensional equations are related to them. One of the goals of this paper is to obtain some new examples of such relation.
Our second goal is to apply this approach to the differential-difference equations (lattices). For example, let us consider the lattice generated by iterations of the Bäcklund transformation (BT) for the equation (1):

\[
\psi_x = \psi_1 + \beta \psi + \langle \psi, \varphi_1 \rangle \psi, \quad -\varphi_x = \varphi_{-1} + \beta_{-1} \varphi + \langle \psi_{-1}, \varphi \rangle \varphi,
\]

(7)

where \( \beta_j \) are arbitrary scalar parameters [18, 19]. Here and further on, when working with the lattices, we omit for short the dummy integer subscript \( j \), so that \( \psi \) means \( \psi_j \), \( \psi_1 \) means \( \psi_{j+1} \) and so on.

One obtains, after introducing the quantities

\[
\begin{align*}
 f &= -\langle \psi, \varphi_1 \rangle - \beta, \\
p &= \langle \psi, \varphi_1, x \rangle - \langle \psi_x, \varphi_1 \rangle + \langle \psi, \varphi_1 \rangle^2 - \beta^2,
\end{align*}
\]

the well known Miura-type transformations [20, 21]

\[
\begin{align*}
u_1 &= u + 2 f_x, \\
u &= f^2 - f_x + p, \\
p_x &= f_y,
\end{align*}
\]

which connect eq. (4) with the modified Kadomtsev-Petviashvili equation (mKP)

\[
4 f_t = f_{xxx} - 6(f^2 + p)f_x + 3p_y, \\
p_x &= f_y.
\]

(8)

The variables \( f, p \) satisfy the lattice

\[
\begin{align*}
f_{1,x} + f_x = f_1^2 - f^2 + p_1 - p, \\
p_x &= f_y.
\end{align*}
\]

(9)

Investigation in this direction can be useful for constructing of the explicit solutions of 1+2-dimensional equations (however, it falls beyond the scope of the present paper). Let us recall that if the dependence on \( y \) is neglected then eq. (4) turns into the Korteweg-de Vries equation (KdV), and the lattice (4) turns into the dressing chain. It was proved in [22] that the dressing chain under the periodicity condition \( f_j = f_{j+K} \) is a completely integrable finite-dimensional Hamiltonian system, and all the \( k \)-gap solutions of KdV can be derived from it at \( K = 2k + 1 \). In the two-dimensional case, some large class of the KP solutions can be obtained under the periodic condition \( \psi_j = \psi_{j+K}, \varphi_j = \varphi_{j+K} \) for the lattice (7) which brings to the Hamiltonian system with \( KN \) degrees of freedom. The question if all algebraic-geometric solutions of KP can be obtained in such a way remains open. The multisoliton solutions can be constructed also by means of the nonlinear superposition principle for the lattice (7) [19].

The twodimensional examples considered in this paper include several KP- and mKP-like equations, dispersive long waves equations and, in the discrete case, the twodimensional dressing chain and Toda lattice. Moreover,
the matrix versions of these equations are considered. On the other hand we have the vector and matrix generalizations of NLS, modified NLS and derivative NLS. Probably, most beautiful example is related to the vector generalization of the Ablowitz-Ladik lattice which was introduced recently in the papers [23]. It should be mentioned that the scalar Ablowitz-Ladik lattice has been already used for constructing of the solutions of 1+2-dimensional equation [24]. As a rule, the order of nonlinear terms in multifield equations is odd, however exceptions are possible (see the last Section).

2 Jordan NLS

We start from the further generalizations of NLS which were proved to be integrable in [12]. These system are of the form

\[ \psi_y = \psi_{xx} + 2\{\psi \varphi \psi\}, \quad -\varphi_y = \varphi_{xx} + 2\{\varphi \psi \varphi\} \tag{10} \]

where the braces denote triple product which satisfies the axioms of the Jordan pair:

\[ \{abc\} = \{cba\}, \]
\[ \{ab\{cde\}\} - \{cd\{abc\}\} = \{\{abc\}de\} - \{c\{bad\}e\}. \tag{11} \]

Below we consider only the vector and matrix examples which correspond to the following, most important, rules of multiplication:

\[ 2\{abc\} = \langle a, b \rangle c + \langle c, b \rangle a, \quad a, b, c \in \mathbb{R}^N, \tag{12} \]
\[ \{abc\} = \langle a, b \rangle c + \langle c, b \rangle a - \langle a, c \rangle b, \quad a, b, c \in \mathbb{R}^N, \tag{13} \]
\[ 2\{abc\} = abc + cba, \quad a, c \in \text{Mat}_{M,N}, \quad b \in \text{Mat}_{N,M}. \tag{14} \]

Nevertheless, it is convenient to formulate some statements in general algebraic terms. In particular, the eq. (10) admits the third order symmetry

\[ \psi_t = \psi_{xxx} + 6\{\psi \varphi \psi_x\}, \quad \varphi_t = \varphi_{xxx} + 6\{\varphi \psi \varphi_x\} \tag{15} \]

and the Bäcklund transformation [13, 14]

\[ \psi_x = \psi_1 + \beta \psi + \{\psi \varphi_1 \psi\}, \quad -\varphi_x = \varphi_{-1} + \beta_{-1} \varphi + \{\varphi \psi_{-1} \varphi\}. \tag{16} \]

**Example 1.** Obviously, the example discussed in Introduction corresponds to the multiplication (12) which is the particular case of the matrix multiplication (14) at \( M = 1 \). In the general case consider the matrix NLS

\[ \psi_y = \psi_{xx} + 2\psi \varphi \psi, \quad -\varphi_y = \varphi_{xx} + 2\varphi \psi \varphi \]
where \( \psi \in \text{Mat}_{M,N}, \varphi \in \text{Mat}_{N,M} \). Let us introduce the \( M \times M \) matrices

\[
\begin{align*}
    u &= -2\psi \varphi, \\
    q &= 2\psi \varphi_x - 2\psi_x \varphi, \\
    f &= -\psi \varphi_1 - \beta I_M, \\
    p &= \psi \varphi_{1,x} - \psi_x \varphi_1 + \psi \varphi_1 \psi \varphi_1 - \beta^2 I_M
\end{align*}
\]

where \( I_M \) is the identity matrix, then using the symmetry (15) yields the matrix KP equation (here and below the square brackets denote commutator \([a,b] = ab - ba\))

\[
4u_t = u_{xxx} - 3(uu_x + u_x u - q_x + [q,u]), \quad q_x = u_y
\]

while the lattice (16) yields the Bäcklund transformation

\[
\begin{align*}
    u_1 &= u + 2f_x, \\
    u &= f^2 - f_x + p, \\
    f_{1,x} + f_x &= f_1^2 - f^2 + p_1 - p, \quad p_x = f_y + [p,f].
\end{align*}
\]

It should be stressed that, in contrast to the scalar case, the matrix \( f \) does not satisfy the matrix mKP equation.

**Example 2.** The version of the vector NLS equation related to the multiplication (13) was introduced in [25] for the first time:

\[
\begin{align*}
    \psi_y &= \psi_{xx} + 2(\psi,\varphi)\psi - \langle \psi,\psi \rangle \varphi, \\
    -\varphi_y &= \varphi_{xx} + 2(\psi,\varphi)\varphi - \langle \varphi,\varphi \rangle \psi.
\end{align*}
\]

In this example the quantities

\[
\begin{align*}
    u &= -4(\psi,\varphi), \\
    v &= 2(\psi,\psi), \\
    w &= 2(\varphi,\varphi), \\
    q &= 4(\psi,\varphi_x) - 4(\varphi_x,\varphi)
\end{align*}
\]

satisfy, in virtue of (15), the 1+2-dimensional system

\[
\begin{align*}
    4u_t &= u_{xxx} - 6uu_x + 3q_y + 6(vw)_x, \quad q_x = u_y, \\
    -2v_t &= v_{xxx} - 3v_{xy} - 3uv_x + 3vq, \\
    -2w_t &= w_{xxx} + 3w_{xy} - 3uw_x - 3wq
\end{align*}
\]

which generalize KP (reduction \( vw = 0 \)).

In conclusion we notice that actually the equation (10) admits the infinite hierarchy of the higher symmetry (it can be constructed, for example, by means of the recursion operator [12]) and using them instead of (15) brings to the higher symmetries of the corresponding KP-like equations. In some sense this solves the problem of nonlocalities which is a serious technical obstacle in the theory of twodimensional equations. However, we always consider only the simplest representatives of the integrable hierarchies in the further examples.
3 Derivative NLS

There are two different scalar systems known under the name of the derivative NLS equation:

\[
\begin{align*}
\psi_y &= \psi_{xx} + 2(\psi^2 \varphi)_x, \\
-\varphi_y &= \varphi_{xx} - 2(\psi \varphi^2)_x
\end{align*}
\]

and

\[
\begin{align*}
\psi_y &= \psi_{xx} + 2\psi \varphi \psi_x, \\
-\psi_y &= \varphi_{xx} - 2\psi \varphi \varphi_x
\end{align*}
\]

which were introduced in the papers [26] and [27] correspondingly. They are related by differential substitution. In the multifield case the difference becomes more deep: the first system admits the natural generalization in the Jordan pairs, as NLS, while the second one is generalized in associative algebras. We consider the second version first.

Theorem 1 (Olver, Sokolov[15]) Let \( \psi, \varphi \) belong to an associative algebra, then the system

\[
\begin{align*}
\psi_y &= \psi_{xx} + 2\psi \varphi \psi_x, \\
\varphi_y &= -\varphi_{xx} + 2\varphi \varphi \varphi_x
\end{align*}
\]

possesses the third order symmetry

\[
\begin{align*}
\psi_t &= \psi_{xxx} + 3\psi_{xx} \varphi \psi + 3\psi_x \varphi \psi_x + 3\psi_x \varphi \varphi \psi_x + 3\varphi \varphi \varphi \varphi \psi_x, \\
\varphi_t &= \varphi_{xxx} - 3\varphi \psi \varphi_{xx} - 3\varphi_x \psi \varphi_x + 3\varphi \varphi \varphi \psi \varphi_x.
\end{align*}
\]

We need some extension of this result. Namely, it is easy to see that actually the associativity of multiplication is sufficient, without the requirement that \( \psi, \varphi \) belong to the same space. In particular, we consider the case \( \psi \in \text{Mat}_{M,N}, \varphi \in \text{Mat}_{N,M} \). Apparently, the equations (18), (19) can be treated as an auxiliary linear problem in two ways, one leads to the matrix KP and another one to matrix mKP equation.

Example 3. Let us consider \( M \times M \) matrices

\[
u = -2\psi_x \varphi, \quad q = 2\psi_x \varphi - 2\psi_{xx} \varphi - 4\psi_x \varphi \psi \varphi
\]

then \( \psi \) satisfy equations of the form

\[
\begin{align*}
\psi_y &= \psi_{xx} - \nu \psi, \\
\psi_t &= \psi_{xxx} - \frac{3}{2} \psi_x \psi_x - \frac{3}{4} (\nu_x + q) \psi
\end{align*}
\]

and, in virtue of their consistency, \( \nu \) and \( q \) satisfy eq. (17). In particular, a new representation of the scalar KP arise at \( M = 1 \).
**Example 4.** Now let us consider the $N \times N$ matrices 

$$
f = \varphi \psi, \quad p = \varphi x - \varphi x \psi + f^2.
$$

They satisfy the matrix mKP equation

$$
4f_t = f_{xxx} + 3([f_{xx}, f] - 2f_x f + p_y + [p, f^2] + f_x p + p f_x), \quad f_y = p_x + [p, f]^2
$$

in virtue of (18), (19)

Next, let us discuss the multifield analogs of the Kaup-Newell system. They were studied, e.g. in [10, 28].

**Theorem 2** Let triple product \{ \} satisfy axioms (11) then the system

$$
\psi_y = \psi_{xx} + 2\{\psi \varphi \psi\}_x, \quad \varphi_y = -\varphi_{xx} + 2\{\varphi \psi \varphi\}_x
$$

admits the symmetry of third order

$$
\psi_t = \langle \psi_{xx} + 6\{\psi \varphi \psi\}_x + 6\{\varphi \psi \varphi\}_x \rangle_x,
\varphi_t = \langle \varphi_{xx} - 6\{\varphi \psi \varphi\}_x + 6\{\varphi \psi \varphi\}_x \rangle_x.
$$

This statement was proved in [28] in the case of the Jordan triple systems, but, as before, it remains valid also for the Jordan pairs, that is when $\psi$ and $\varphi$ belong to the different spaces. We just write down the resulting two-dimensional equations.

**Example 5.** In the case of multiplication (14) the matrices

$$
f = \varphi \psi, \quad p = \varphi x - \varphi x \psi + 3f^2
$$

solve, in virtue of (21), (22) the same equation (20). This coincidence, although unexpected at a first glance, is easily explained if one consider the corresponding auxiliary linear problems, which are formally conjugated.

**Example 6.** In the case of multiplication (13) the quantities

$$
f = -2\langle \psi, \varphi \rangle, \quad g = \langle \psi, \psi \rangle, \quad h = \langle \varphi, \varphi \rangle, \quad p = 2\langle \psi, \varphi x \rangle - 2\langle \psi x, \varphi \rangle - 3f^2 + 6gh
$$

satisfy the system of equations

$$
4f_t = f_{xxx} + 3(p - 2gh)_y + 6(g_{xx}h - gh_{xx})
- 2(6fgh + f^3)_x - 6pf_x, \quad p_x = f_y
-2g_t = g_{xxx} - 3g_{xy} + 6f(g_y - g_{xx}) + 3(g(p - f_x))_x
+ 6g^2h_x + 6gf f_x + 9g_x f^2,
-2h_t = h_{xxx} + h_{xy} + 6f(h_y + h_{xx}) + 3(h(p + f_x))_x
+ 6h^2g_x + 6hf f_x + 9h_x f^2
$$

which is reduced to mKP (8) at $gh = 0$. 

7
4 Modified Jordan NLS

Let us consider the lattice (16), assuming \( \beta_j = 0 \) for sake of simplicity, and make renaming \( \psi = \tilde{\psi}, \varphi_1 = \tilde{\varphi} \). This give rise to the differential substitution

\[
\psi = \tilde{\psi}, \quad \varphi = -\tilde{\varphi}_x - \{\tilde{\varphi}\tilde{\psi}\tilde{\varphi}\}
\]

and one can easily see that eq. (10) can be rewritten in new variables. The result is the modified Jordan NLS (the tilde is omitted) \[29\]

\[
\psi_y = \psi_{xx} - 2\{\psi\varphi_x\psi\} - 2\{\psi\varphi\psi\}\psi, \\
-\varphi_y = \varphi_{xx} + 2\{\varphi\psi_x\varphi\} - 2\{\varphi\psi\varphi\}\varphi
\]

(23)

and the chain of its Bäcklund transformations

\[
\psi_x = \psi_1 + \{\psi\varphi\psi\}, \quad -\varphi_x = \varphi_{-1} + \{\varphi\psi\varphi\}. \quad (24)
\]

We will not rewrite the symmetry (13) since it is clear without calculations that the resulting twodimensional equations are exactly the same as in the Section 2 and the difference is only in the formulæ which define the constraints between potential and eigenfunctions.

The new examples arise if one consider the time with a “negative number” from the hierarchy of eq. (23) symmetries.

**Theorem 3** For any Jordan pair equations (23) define the symmetry of the following hyperbolic system

\[
\psi_{xz} = 2\{\psi\varphi\psi\}_z - \psi, \quad \varphi_{xz} = -2\{\varphi\psi\varphi\}_z - \varphi. \quad (25)
\]

The proof is straightforward and use only identities (11).

**Example 7.** If multiplication is defined by formula (12) then eqs. (23), (26) take the form

\[
\psi_y = \psi_{xx} - 2(\langle\psi, \varphi_x\rangle + (\psi, \varphi)^2)\psi, \\
-\varphi_y = \varphi_{xx} + 2(\langle\psi_x, \varphi\rangle - (\psi, \varphi)^2)\varphi, \\
\psi_{xz} = \langle\psi, \varphi\rangle\psi_z + (\langle\psi_z, \varphi\rangle - 1)\psi, \quad \varphi_{xz} = -\langle\psi, \varphi\rangle\varphi_z - (\langle\psi, \varphi_x\rangle + 1)\varphi. \quad (27)
\]

In virtue of them the quantities

\[
u = \langle\psi, \varphi\rangle, \quad v = \langle\psi_z, \varphi\rangle - 1, \quad q = \langle\psi, \varphi_x\rangle + u^2
\]
satisfy the well known two-dimensional generalization of the dispersive long waves equations (DLW) [30, 31, 32]

\[ u_y = (u_x + u^2 - 2q)_x, \quad -v_t = (v_x - 2uv)_x, \quad q_z = v_x. \]  

(28)

The lattice \((24)\) takes form

\[ \psi_x = \psi_1 + \langle \psi, \varphi \rangle \psi, \quad -\varphi_x = \varphi_{-1} + \langle \psi, \varphi \rangle \varphi \]

and is the member of the hierarchy which was studied in details in the paper [33]. In particular, it was proved there that this lattice admits the symmetry of the form

\[ \psi_z = \frac{\psi_{-1}}{\langle \psi_{-1}, \varphi \rangle - 1}, \quad -\varphi_z = \frac{\varphi_{-1}}{\langle \psi, \varphi_{-1} \rangle - 1}. \]  

(29)

One can easily check that this pair of lattices define the BT for the system (27). Moreover, the variables \(u, v\) satisfy the two-dimensional Toda lattice

\[ u_z = v - v_1, \quad v_x = v(u - u_{-1}), \]  

(30)

which define the BT for the system (28) [35, 34].

**Example 8.** Analogs of the lattice (29) exist for other Jordan pairs as well. Let \(\psi \in \text{Mat}_{M,N}, \varphi \in \text{Mat}_{N,M}\), then the following differentiations commute in the matrix case (14):

\[ \psi_x = \psi_1 + \psi \varphi \psi, \quad -\varphi_x = \varphi_{-1} + \varphi \psi \varphi, \]

\[ \psi_z = (\psi_{-1} \varphi - I_M)^{-1} \psi_{-1}, \quad -\varphi_z = \varphi_{1}(\psi \varphi_{-1} - I_M)^{-1}, \]

\[ \psi_y = \psi_{xx} - 2\psi \varphi_x \psi - 2\psi \psi \varphi \psi \varphi, \quad -\varphi_y = \varphi_{xx} + 2\varphi \psi_x \varphi - 2\varphi \psi \varphi \psi \varphi \]

and the matrices of the \(M \times M\) size

\[ u = \psi \varphi, \quad v = \psi_z \varphi - I_M = (\psi_{-1} \varphi - I_M)^{-1}, \quad q = \psi \varphi_x + u^2 = -\psi \varphi_{-1} \]

satisfy the nonabelian twodimensional DLW equations and Toda lattice [35]

\[ u_y = u_{xx} + 2(u_x u - q_x + [u, q]), \quad -v_y = v_{xx} - 2(v u)_x - 2[v, q], \quad q_z = v_x, \]

\[ u_z = v - v_1, \quad v_x = uv - vu_{-1}. \]
5 Vector Ablowitz-Ladik lattice

Probably one of the most beautiful examples is related to the multifield Ablowitz-Ladik lattices. It yields the same twodimensional equations as in the previous section, but is more symmetric.

**Theorem 4** Let $\psi \in \text{Mat}_{M,N}$, $\varphi \in \text{Mat}_{N,M}$ then the lattices

$$
\left\{ \begin{array}{ll}
\psi_x = \psi_{-1} + \psi_{-1} \varphi \\
-\varphi_x = \varphi_1 + \varphi \varphi_1,
\end{array} \right. \quad \left\{ \begin{array}{ll}
\psi_z = \psi_1 + \psi \varphi_1 \\
-\varphi_z = \varphi_{-1} + \varphi_{-1} \varphi
\end{array} \right. \quad (31)
$$

commute and the variables $\psi, \varphi$ satisfy the following system in virtue of these lattices:

$$
\psi_{xz} = \psi_x \varphi (\psi \varphi + I_M)^{-1} \psi_z + \psi \varphi \psi + \psi,
\varphi_{xz} = \varphi_z (\psi \varphi + I_M)^{-1} \psi_x + \psi \varphi \varphi + \varphi. \quad (32)
$$

In the scalar case the sum of the flows (31) and the dilation symmetry $\psi_\tau = -2\psi$, $\varphi_\tau = 2\varphi$ is the Ablowitz-Ladik lattice [36] and (32) is Pohlmeyer-Lund-Regge system (this observation due to [37]). The second order symmetries of the eq. (31) hierarchy is nothing but derivative NLS equations (cf (18)):

$$
\psi_y = \psi_{xx} + 2\psi_x \varphi_1 \psi, \quad \varphi_y = -\varphi_{xx} + 2\varphi \varphi_{-1} \varphi_x, \quad (33)
\psi_\eta = \psi_{zz} + 2\psi \varphi_{-1} \psi_z, \quad \varphi_\eta = -\varphi_{zz} + 2\varphi_z \varphi_1 \varphi. \quad (34)
$$

Of course, one should eliminate the variables $\psi_{\pm 1}, \varphi_{\pm 1}$ in virtue of the lattices in order two treat these equations as higher symmetries of the system (32).

I will consider only the vector case $M = 1$ :

$$
\left\{ \begin{array}{ll}
\psi_x = \psi_{-1} + \langle \psi_{-1}, \varphi \rangle \psi \\
-\varphi_x = \varphi_1 + \langle \psi, \varphi_1 \rangle \varphi,
\end{array} \right. \quad \left\{ \begin{array}{ll}
\psi_z = \psi_1 + \langle \psi, \varphi \rangle \psi_1 \\
-\varphi_z = \varphi_{-1} + \langle \psi, \varphi \rangle \varphi_{-1},
\end{array} \right. \quad (35)
$$

$$
\left\{ \begin{array}{ll}
\psi_{xz} = \frac{\langle \psi_x, \varphi \rangle}{\langle \psi, \varphi \rangle + 1} \psi_z + \langle \psi, \varphi \rangle \psi + \psi, \\
\varphi_{xz} = \frac{\langle \psi, \varphi_x \rangle}{\langle \psi, \varphi \rangle + 1} \varphi_z + \langle \psi, \varphi \rangle \varphi + \varphi.
\end{array} \right. \quad (36)
$$

Now the sum of the lattices (35) (plus dilation symmetry) defines the so-called “asymmetric discretization of vector NLS” introduced in [23]. It should be mentioned that another discretization was considered in these papers as well, which can be obtained by summing the second lattice (35) and the lattice symmetric with respect to reflection $j \to -j$. Integrability of this version was
established in [38], however the question about the structure of the higher symmetries remains open.

The quantities

\[ v = \langle \psi, \varphi \rangle + 1, \quad u = -\langle \psi, \varphi_1 \rangle = \langle \psi, \varphi_x \rangle/v \]

satisfy the Toda lattice (30) in virtue of (35). In order to reproduce the DLW equations, it is sufficient to consider the symmetries (33), (34). First one takes the form

\[ \psi_y = \psi_{xx} - 2q\psi, \quad -\varphi_y = \varphi_{xx} - 2q\varphi, \quad q = \langle \psi_x, \varphi_x \rangle - \frac{\langle \psi, \varphi_x \rangle \langle \psi_x, \varphi \rangle}{\langle \psi, \varphi \rangle + 1} \]

after elimination of \( \psi_{-1}, \varphi_1 \) and brings to eq. (28), up to the change of the sign of \( y \). Notice, that this formula coincide, up to the renaming of the potential, with the auxiliary linear problem (5) for KP. It can be easily proved that the form of the second auxiliary problem (6) is uniquely derived from the compatibility condition and yields the third order symmetry of the system (36). This gives the new example of the constraint for KP, which was displayed in Introduction.

The symmetry (34) is rewritten as follows

\[ \psi_\eta = \psi_{zz} - 2\frac{\langle \psi, \varphi_z \rangle}{\langle \psi, \varphi \rangle + 1} \psi_z, \quad \varphi_\eta = -\varphi_{zz} + 2\frac{\langle \psi_z, \varphi \rangle}{\langle \psi, \varphi \rangle + 1} \varphi_z \]

and denoting \( p = \langle \psi_z, \varphi \rangle/v \) one obtains another equation from the DLW hierarchy:

\[ u_\eta = u_{zz} - 2v_z + 2pu_z, \quad v_\eta = -(v_z - 2pv)_z, \quad p_x = u_z. \]

6 Concluding examples

Notice, that the order of nonlinear terms was odd for all examples of multifield equations considered above, while the constraints were defined by expressions with even order products of \( \psi \)-functions. Here we present two examples with the different structure of nonlinearities.

Example 9. Let us consider the generalization of the KdV equation which corresponds to the Jordan algebra \( D_N \) [13]:

\[ u_y = u_{xxx} + 3(u^2 - \langle \psi, \psi \rangle)_x, \quad \psi_y = \psi_{xxx} + 6(u\psi)_x, \quad u \in \mathbb{R}, \quad \psi \in \mathbb{R}^N. \]
It possesses the fifth order symmetry
\[
\begin{align*}
  u_t &= u_{xxxxx} + 5(2uu_{xx} + u_x^2 + 2u^3 - 6u\langle \psi, \psi \rangle - 2\langle \psi, \psi_{xx} \rangle - \langle \psi_x, \psi_x \rangle, \\
  \psi_t &= \psi_{xxxxx} + 10(u\psi_{xx} + u_x\psi_x + u_{xx}\psi + 3u^2\psi - \langle \psi, \psi \rangle)_{xx},
\end{align*}
\]
and the quantities
\[
v = 6u, \quad q = 6u_{xx} + 18u^2 - 18\langle \psi, \psi \rangle
\]
satisfy, in virtue of these equations, the twodimensional generalization of Sawada-Kotera equation \[39\]
\[
-9v_t = v_{xxxxx} + 5(vv_{xxx} + v_xv_{xx} + v^2v_x - v_{xxy} - vv_y - qv_x - q_y), \quad q_x = v_y.
\]

**Example 10.** The equation
\[
\psi_y = \psi_{xx} + 2\psi\psi_x + [\psi, \psi^2]
\]
on arbitrary left-symmetric algebra was proved to be integrable in \[11\]. It admits the infinite hierarchy of higher symmetries, simplest one is of the form
\[
\psi_t = \psi_{xxx} + 3(\psi\psi_{xx} + \psi_x^3 + [\psi, \psi\psi_x] + \psi(\psi_x\psi)) + [\psi, \psi\psi^2].
\]
This equation is a multifield generalization of Burgers equation and is linearizable via Cole-Hopf type substitution.

An example of left-symmetric multiplication is given by formula
\[
\psi\varphi = \langle \psi, c \rangle \varphi + \langle \psi, \varphi \rangle c, \quad \psi, \varphi, c \in \mathbb{R}^n
\]
where vector \(c\) is fixed. One can check that in this case the quantities
\[
u = \langle \psi, c \rangle, \quad q = |c|^2|\psi|^2 + u_x + u^2
\]
solve equation
\[
-2u_t = u_{xxx} + 6(uu_{xx} + u_x^2 + 2u^2u_x) - 3u_{xy} - 6uu_y - 6qu_x, \quad q_x = u_y.
\]
Of course this equation is linearizable as well: the Cole-Hopf transformation
\[
2u = v_x/v, \quad 2q = v_y/v
\]
links it with the linear equation
\[
2v_t = 3v_{xy} - v_{xxx}.
\]

**Acknowledgements.** I am grateful to S.V. Manakov who drew my attention to the paper \[3\] and B.G. Konopelchenko, A.B. Shabat and R.I. Yamilov for useful discussions and remarks. This work was supported by grants RFBR-99-01-00431 and INTAS-99-01782.
References

[1] Manakov S.V. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. Soviet Physics JETP, 38, No. 2, 248–253 (1974).

[2] Fordy A.P., Kulish P.P. Nonlinear Schrödinger equations and simple Lie algebras. Commun. Math. Phys., 89, No. 3, 427–443 (1983).

[3] Konopelchenko B.G., Strampp W. The AKNS hierarchy as symmetry constraint of the KP hierarchy, Inverse Problems, 7, L17–L24 (1991).

[4] Konopelchenko B., Sidorenko J., Strampp W. (1+1)-dimensional integrable systems as symmetry constraints of (2+1)-dimensional systems. Phys. Lett. A 157, No. 1, 17–21 (1991).

[5] Cheng Y., Li Y. The constraint of the Kadomtsev-Petviashvili equation and its special solutions. Phys. Lett. A, 157, No. 1, 22–26 (1991).

[6] Cheng Y., Li Y. Constraints of the 2+1 dimensional integrable soliton systems. J. Phys. A, 25, No. 2, 419-431 (1992).

[7] Xu B., Li Y. (1+1)-dimensional Hamiltonian systems as symmetry constraints of the Kadomtsev-Petviashvili equation. J. Phys. A, 25, No. 10, 2957–2968 (1992).

[8] Konopelchenko B.G., Strampp W. Reductions of (2+1)-dimensional integrable systems via mixed potential-eigenfunction constraints. J. Phys. A, 25, 4399–4411 (1992).

[9] Konopelchenko B.G., Strampp W. New reductions of the Kadomtsev-Petviashvili and two-dimensional Toda lattice hierarchies via symmetry constraints, J. Math. Phys., 33, 3676–3686 (1992).

[10] Fordy A.P. Derivative nonlinear Schrödinger equations and Hermitian symmetric spaces. J. Phys. A, 17, 1235–1245 (1984).

[11] Svinolupov S.I. On the analogues of the Burgers equation. Phys. Lett. A, 135, No. 1, 32 (1989).

[12] Svinolupov S.I. Generalized Schrödinger equations and Jordan pairs. Commun. Math. Phys., 143, 559–575 (1992).

[13] Svinolupov S.I. Jordan algebras and integrable systems. Funct. Anal. Appl., 27, No. 4, 40–53 (1993).

[14] Svinolupov S.I., Sokolov V.V. Vector and matrix generalizations of classical integrable equations. Theor. Math. Phys. 100, No. 2, 959–962 (1994).
[15] *Olver P.J., Sokolov V.V.* Integrable evolution equations on associative algebras. Commun. Math. Phys., 193, 245–268 (1998).

[16] *Olver P.J., Sokolov V.V.* Non-abelian integrable systems of the derivative nonlinear Schrödinger type. Inverse Problems, 14, No. 6, L5–L8 (1998).

[17] *Habibullin I.T., Sokolov V.V., Yamilov R.I.* Multi-component integrable systems and nonassociative structures. Proc. of Workshop on Nonlinear Physics, World Scientific Publ., 1996, pp. 139–168.

[18] *Svinolupov S.I., Yamilov R.I.* The multi-field Schrödinger lattices. Phys. Lett. A, 160, 548–552 (1991).

[19] *Adler V.E.* Nonlinear superposition formula for Jordan NLS equations. Phys. Lett. A, 190, 53–58 (1994).

[20] *Konopelchenko B.G.* On the gauge-invariant description of the evolution equations integrable by Gelfand-Dikij spectral problems. Phys. Lett. A, 92, 323–327 (1982).

[21] *Jimbo M., Miwa T.* Publ. RIMS, 19, No. 3, 943 (1983).

[22] *Veselov A.P., Shabat A.B.* Dressing chain and spectral theory of Schrödinger operators. Funct. Anal. Appl., 27, No. 2, 81–96 (1993).

[23] *Ablowitz M.J., Ohta Y., Trubatch A.D.* On Discretizations of the Vector Nonlinear Schrödinger Equation, [solv-int/9810014](http://arxiv.org/abs/solv-int/9810014). On Integrability and Chaos in Discrete Systems. Chaos, Solitons & Fractals, 11, No. 1-3, 159–169 (2000) [solv-int/9810020](http://arxiv.org/abs/solv-int/9810020).

[24] *Vekslerchik V.E.* The Davey-Stewartson equation and the Ablowitz-Ladik hierarchy. Inverse Problems, 12, No. 6, 1057 (1996).

[25] *Kulish P.P., Sklyanin E.K.* O(N)-invariant nonlinear Schrödinger equation — a new completely integrable system. Phys. Lett. A, 84, No. 7, 349–352 (1981).

[26] *Kaup D.J., Newell A.C.* An exact solution for a derivative nonlinear Schrödinger equation. J. Math. Phys., 19, 798–801 (1978).

[27] *Chen H.H., Lee Y.C., Liu C.S.* Integrability of nonlinear Hamiltonian systems by inverse scattering method. Physica Scripta, 20, 490–492 (1979).

[28] *Adler V.E., Svinolupov S.I., Yamilov R.I.* Multi-component Volterra and Toda type integrable equations. Phys. Lett. A, 254, 24–36 (1999).

[29] *Ablowitz M.J., Segur H.* Solitons and the Inverse Scattering Transform. Philadelphia, PA: SIAM, 1981.
[30] Boiti M., Leon J.J.-P., Pempinelli F. Integrable two-dimensional generalisation of the sine- and sinh-Gordon equations. Inverse Problems, 3, No. 1, 37–49 (1987).

[31] Boiti M., Leon J.J.-P., Pempinelli F. Spectral transform for a two spatial dimension extension of the dispersive long wave equation. Inverse Problems, 3, No. 3, 371–387 (1987).

[32] Konopelchenko B.G. The two-dimensional second-order differential spectral problem: compatibility conditions, general BTs and integrable equations. Inverse Problems, 4, No. 1, 151–163 (1988).

[33] Merola I., Ragnisco O., Zhang T.G. A novel hierarchy of integrable lattices. Inverse Problems, 10, No. 6, 1315-1334 (1994). [solv-int/9401007]

[34] Shabat A.B., Yamilov R.I. To transformation theory of two-dimensional integrable systems. Phys. Lett. A, 227, 15–23 (1997).

[35] Konopelchenko B.G. The nonabelian 1+1-dimensional Toda lattice as the periodic fixed point of the Laplace transform for the 2+1-dimensional integrable system. Phys. Lett. A, 156, 221–222 (1991).

[36] Ablowitz M.J., Ladik J.F. Nonlinear differential-difference equations and Fourier analysis. J. Math. Phys. 17, No. 6, 1011–1018 (1976).

[37] Shabat A.B., Yamilov R.I. Symmetries of nonlinear chains. Leningrad Math. J. 2, No. 2, 377–400 (1991).

[38] Tsuchida T., Ujino H., Wadati M. Integrable semi-discretization of the coupled nonlinear Schrodinger equations. J. Phys. A, 32, No. 11, 2239–2262 (1999).

[39] Sawada K., Kotera T. A method for finding n-soliton solutions of the KdV equation and KdV-like equations. Prog. Theor. Phys., 51, 1355–1367 (1974).