Bi–Hamiltonian manifolds, quasi-bi-Hamiltonian systems and separation variables

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Abstract

We discuss from a bi-Hamiltonian point of view the Hamilton–Jacobi separability of a few dynamical systems. They are shown to admit, in their natural phase space, a quasi–bi–Hamiltonian formulation of Pfaffian type. This property allows us to straightforwardly recover a set of separation variables for the corresponding Hamilton–Jacobi equation.

1 Introduction

The notion of quasi-bi-Hamiltonian (QBH) systems has been introduced quite recently [1]; it originates from the study of dynamical systems which are associated with two compatible Poisson tensors ($P_0, P_1$), but do not admit a bi–Hamiltonian formulation, in the given phase space, w.r.t. to these tensors.

At least for a class of such systems (the so-called Pfaffian QBH) it has been shown [2] how to construct a set of separation variables for the corresponding Hamilton-Jacobi equation, so that a Pfaffian QBH system can be integrated by quadratures. In our opinion, this relation between QBH formulation and separation of variables is remarkable.

On the other hand, since at present there is not, at the best of our knowledge, a satisfactory general scheme encompassing these properties, we believe that it may be preliminarily useful to study some concrete examples of such systems; as a matter of fact, it turns out that the QBH formulation is shared by a few classical separable systems (as for other cases previously considered in the literature, see [3, 4, 5, 6]). In the next section we briefly review the main definitions and results to be used in Sect. 3, where the following systems are presented: the

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Kepler problem with a homogeneous force, the Euler problem with two fixed centers and an elastic force, the motion on the ellipsoid in an elastic potential and a solvable $n$-body problem introduced in [7].

2 Bi-Hamiltonian manifolds and quasi–bi–Hamiltonian systems

A bi–Hamiltonian (BH) manifold \cite{8} is a smooth manifold $M$ endowed with two compatible Poisson tensors $P_0, P_1 : T^*M \rightarrow TM$ ($TM$ and $T^*M$ being the tangent and cotangent bundle of $M$, respectively). As it is known, if $M$ is even dimensional ($\dim M = 2n$) and at least a Poisson tensor, say $P_0$, is invertible, then $N = P_1 P_0^{-1}$ is a Nijenhuis tensor \cite{9} (a hereditary operator in the terminology of \cite{10}). In particular, if $N$ is maximal, i.e., it has $n$ functionally independent eigenvalues $(\lambda_1, \ldots, \lambda_n)$, one can introduce a set of canonical coordinates $(\lambda; \mu)$ $(\lambda := (\lambda_1, \ldots, \lambda_n); \mu := (\mu_1, \ldots, \mu_n))$ referred to as Darboux-Nijenhuis coordinates such that $P_0$, $P_1$ and $N$ take the matrix form

$$
P_0 = \begin{bmatrix} 0 & I \\
-I & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & \Lambda \\
-\Lambda & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \Lambda & 0 \\
0 & \Lambda \end{bmatrix},
$$

(1)

where $I$ denotes the $n \times n$ identity matrix and $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n)$. The first $n$ coordinates are just the eigenvalues of $N$ whereas the remaining ones can be constructed by quadratures \cite{11}. The above form will be referred to as the canonical form of the previous BH structure.

2.1 Bi–Hamiltonian systems

A vector field $X$ is said to be bi-Hamiltonian w.r.t. a pair of Poisson tensors $(P_0, P_1)$ if there exist two smooth functions $h_0$ and $h_1$ such that

$$
X = P_0 dh_1 = P_1 dh_0,
$$

(2)

d denoting the exterior derivative.

If $N$ exists and it is maximal then a BH vector field is completely integrable, a set of independent involutive integrals being just the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ \cite{12}. Conversely, a strong condition has to be satisfied by a completely integrable system in order to admit a BH formulation in a neighborhood of a Liouville torus, at least if one searches for a second Poisson tensor compatible with $P_0$ \cite{13, 14}. Hereafter, this kind of structures will be referred to as standard bi–Hamiltonian structures.

Nevertheless, the property of Liouville integrability can be related with geometric formulations which are actually different from the BH one w.r.t. a standard BH structure. This can be done in (at least) three distinct ways.

i) Searching for a BH structure $(Q_0, Q_1)$ not including the Poisson tensor $P_0$ \cite{15}; hereafter such structures will be referred to as alternative BH structures.
Admitting a degenerate BH formulation; this is the case, for instance, of the rigid body with a fixed point [14, 17, 18] and of the stationary flows of the KdV hierarchy [19, 20].

Searching for a QBH formulation of the vector field \( X \) w.r.t. a standard BH structure [1, 2].

### 2.2 Quasi–bi–Hamiltonian systems

In connection with the third approach, we recall that the vector field \( X \) is said to be quasi-bi-Hamiltonian w.r.t. a pair of Poisson tensors \((P_0, P_1)\) [1] if there exist three smooth functions \( H, \ K, \ \rho \) such that

\[
X = P_0 \, dH = \frac{1}{\rho} P_1 \, dK .
\]

(3)

One could say that the given dynamical system is Hamiltonian also w.r.t. the Poisson tensor \( P_1 \), provided that a nontrivial change in time: \( dt \to d\tau = \left(\frac{1}{\rho}\right) dt \) is introduced. In particular, a QBH vector field \( X \) is said to be Pfaffian if \( N \) exists and \( \rho = \prod_{i=1}^{n} \lambda_i \), i.e., it is just the product of the eigenvalues of \( N \). The relevance of QBH vector fields is based upon the following two results.

- Any completely integrable system with two degrees of freedom admits a QBH formulation in a neighborhood of a Liouville torus [1].
- Any Pfaffian QBH vector field with \( n \) degrees of freedom is separable (in the sense of Hamilton–Jacobi) in the Darboux–Nijenhuis coordinates [3]. Indeed, the general solution of Eq. (3), written in these coordinates, is

\[
H = \sum_{i=1}^{n} \frac{f_i(\lambda_i; \mu_i)}{\prod_{j=1 \atop j \neq i}^{n} (\lambda_i - \lambda_j)} , \quad K = \sum_{i=1}^{n} \rho_i \frac{f_i(\lambda_i; \mu_i)}{\prod_{j=1 \atop j \neq i}^{n} (\lambda_i - \lambda_j)} , \quad (\rho_i := \prod_{j=1 \atop j \neq i}^{n} \lambda_j) , \tag{4}
\]

where each function \( f_i \) is an arbitrary smooth function, depending at most on one pair of variables \( (\lambda_i; \mu_i) \). (Obviously enough, if \( n = 1 \) it is: \( \prod_{j=1}^{n} (\lambda_i - \lambda_j) := 1, \rho_i = 1, P_1 = \lambda P_0, H = K = f(\lambda, \mu) \). A remarkable feature of \( H \) and \( K \) is that they are separable as they verify the Levi–Civita condition [21], so that the corresponding Hamilton equations are integrable by quadratures. We stress the fact that, owing to the arbitrariness of \( f_i \), the functions \( H \) and \( K \) (4) provide a class of separable functions generally different from the known Stäckel class, quadratic in the momenta (e.g., see [22, p. 101]).

The results of [2] have been completed in [4], where it has been shown that a QBH vector field \( X \) admits \( n \) integrals of motion in involution \( F_k(k = 1, \ldots, n) \)

\[
F_k = \sum_{i=1}^{n} \frac{\partial c_k}{\partial \lambda_i} \frac{f_i(\lambda_i; \mu_i)}{\prod_{j=1 \atop j \neq i}^{n} (\lambda_i - \lambda_j)} , \tag{5}
\]
$c_1, \ldots, c_n$ being the coefficients of the minimal polynomial of $N$

$$
\lambda^n + \sum_{i=1}^{n} c_i \lambda^{n-i} = \prod_{i=1}^{n} (\lambda - \lambda_i) ;
$$

(6)

in particular, $F_1 = -H, F_n = (-1)^n K$. Furthermore, each function $F_k$ turns out to be separable.

**Remark.** Let $H, K, \rho$ be of the form (4), i.e., the general solutions of the QBH Eq. (3) in the Pfaffian case. If $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are $C^1$ functions with nonvanishing derivatives $\varphi', \psi'$, let us define on the phase space the functions $\Phi, \Psi$ given by

$$
\Phi(\lambda; \mu) := \varphi(H(\lambda; \mu)) , \quad \Psi(\lambda; \mu) := \psi(K(\lambda; \mu)) .
$$

(7)

Then $\Phi, \Psi$ are solutions of the QBH Eq. (3) with a function $\tilde{\rho}$ given by

$$
\tilde{\rho} = \rho \frac{\psi'(K)}{\varphi'(H)} .
$$

(8)

Hence, for each QBH system of Pfaffian type there is a class of QBH systems of non Pfaffian type, with $\tilde{\rho}$ of the form (3). Moreover, it is straightforward to show that $\Phi$ and $\Psi$ are separable, since they satisfy the Levi–Civita condition, provided that this condition be fulfilled by $H$ and $K$ (as in the Pfaffian case), whatever the particular form of the functions $\varphi$ and $\psi$.

Viceversa, let us search for the general solution of Eq. (3) w.r.t. $\tilde{H}, \tilde{K}$ and $\tilde{\rho} = \rho f'(H)/g'(K)$ ($\rho = \prod_{i=1}^{n} \lambda_i$), with $f$ and $g$ arbitrarily chosen functions; it is easy to prove that $\tilde{H}$ and $\tilde{K}$ are given by

$$
\tilde{H} = f^{-1}(H) , \quad \tilde{K} = g^{-1}(K)
$$

(9)

with $H$ and $K$ of the form (4).

This kind of generalization of the Pfaffian case has been recently considered in [23] only for the case $n = 2$. As it is evident, the above results hold for QBH dynamical systems with an arbitrary number of degrees of freedom.

### 2.3 The “origin” of some QBH vector fields

Let us describe a possible situation in which some interesting QBH vector fields arise. Let $(M, \mathcal{P}_0, \mathcal{P}_1)$ be a BH manifold and $X$ the vector field of a given dynamical system on $M$, Hamiltonian w.r.t. $\mathcal{P}_0$. If neither $\mathcal{P}_0$ nor $\mathcal{P}_1$ is invertible, a possible way to analyse the integrability of $X$ is to eliminate the Casimir functions of one Poisson tensor, say $\mathcal{P}_0$, by fixing their values; of course, both $\mathcal{P}_0$ and $X$ can be restricted to a symplectic leaf $S_0$ of $\mathcal{P}_0$, so that $X$ is still a Hamiltonian vector field on $S_0$. However, if $\mathcal{P}_1$ cannot be restricted to $S_0$, the BH formulation (3) is lost on $S_0$, even if $X$ is BH on $M$. As a matter of fact, in a few cases the following situation occurs:
i) there is a fibration \( \pi : M \to M' = M/\pi \) such that both \( P_0 \) and \( P_1 \) are projectable along \( \pi \); since \( \pi \) turns out to be transversal to \( S_0 \) and \( S_1 \) (a symplectic leaf of \( P_1 \)), the quotient space \( M' \) and the symplectic manifold \( S_0 \) are diffeomorphic and \( S_0 \) itself is a BH manifold, with \( P_0 \) and \( P_1 \) invertible, \( P_0 \) and \( P_1 \) denoting the reduced tensors of \( P_0 \) and \( P_1 \) respectively. If this is the case, and if the eigenvalues of the Nijenhuis tensor \( N = P_1 P_0^{-1} \) are independent, then one can introduce the Darboux-Nijenhuis coordinates on \( S_0 \).

ii) There is a function \( \rho \) such that the restricted field \( X \) admits the QBH formulation (3), with \( H \) and \( K \) given by the restriction to \( S_0 \) of integrals of motion of \( X \). The interest of this result is that if \( X \) is Pfaffian, then in the Darboux-Nijenhuis chart it is separable [2].

We remark that the situation described in i) and ii) is, so to say, “experimental”, a sound theoretical foundation of these results being lacking. The peculiarity of the above reduction is given by the fact that two different geometric processes are used simultaneously: the restriction for the vector field and the projection for the BH structure. Due to this fact, one maintains the BH structure but loses the BH formulation for the vector field, recovering in some cases the QBH formulation. This happens, for instance, for:

- the integrable Hénon–Heiles system and its multidimensional generalizations obtained by reduction from the stationary flows of the KdV hierarchy [20, 24];
- a class of permutationally symmetric potentials recovered from the restricted flows of the coupled KdV systems, the most representative member being the Garnier system [4].

Both classes of dynamical systems live on a BH manifold \( M \) of maximal rank (\( \dim M = 2n + 1 \)) and their QBH formulation is obtained by the reduction to a symplectic \( 2n \) dimensional manifold according to the above scheme i) , ii). The study of examples with BH structures of non–maximal rank such as the stationary flows of the Boussinesq hierarchy will appear elsewhere [25]. In this regard, we recall that the geometry of \( (2n + 1) \)-dimensional BH manifolds of maximal rank has been completely described by Gelfand and Zakharevich [26], whereas there is not a similar analysis for non–maximal BH manifolds.

For the sake of clarity, we recall that we are considering from the very beginning a well specified vector field \( X \) which is associated to the dynamical system under investigation. So, the situation described above does not contradict the following known results: if one has been given a BH structure with \( P_0 \) invertible, one can construct a BH vector field whose Hamiltonian functions are the traces (or the eigenvalues) of the Nijenhuis tensor \( N \) [12]; moreover, a QBH vector field can always be constructed for any maximal Nijenhuis tensor [25]. However, it turns out that these BH and QBH fields do not in general coincide with the vector field \( X \) of the given dynamical system.

3 Examples of dynamical systems with QBH formulation

We present four examples of integrable systems which can be given a QBH formulation of Pfaffian type w.r.t. a standard BH structure.
The first three systems (see, e.g., [27, p.126–129]), are defined on the cotangent bundles of Riemannian manifolds; therefore they are of St¨ackel type and the Nijenhuis tensor of their QBH formulation can be constructed by lifting, in a suitable way, a conformal Killing tensor [28] from the configuration space to the corresponding cotangent bundle. (Such a construction will be the subject of a further publication). The dynamical system considered in the last example is naturally Pfaffian QBH, the physical coordinates being just Darboux-Nijenhuis coordinates w.r.t. a standard BH structure written in the canonical form (1).

3.1 The Kepler problem with a homogeneous force (Lagrange 1766)

Let us consider the classical problem of a particle in the plane under the influence of the Kepler potential and of a homogeneous field force. The Hamiltonian function is

\[ H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{a}{\sqrt{q_1^2 + q_2^2}} - bq_2, \quad (10) \]

where \((q_1, q_2; p_1, p_2)\) are the cartesian coordinates of the particle and the conjugate momenta, respectively; \(a\) and \(b\) are real constants. The vector field is \(X = P_0 \, dH\), \(P_0\) being the canonical Poisson tensor. There is a second independent integral of motion

\[ K = \frac{p_1}{2}(q_2p_1 - q_1p_2) - \frac{aq_2}{2\sqrt{q_1^2 + q_2^2}} + \frac{bq_1^2}{4}, \quad (11) \]

which allows us to give \(X\) a QBH formulation. Indeed, we can write \(X = \frac{1}{\rho} P_1 \, dK\) with \(\rho = -\frac{q_2}{4}\) and

\[ P_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & q_1 \\ 0 & 0 & q_1 & 2q_2 \\ 0 & -q_1 & 0 & -p_1 \\ -q_1 & -2q_2 & p_1 & 0 \end{bmatrix} \quad (12) \]

Since the minimal polynomial of the Nijenhuis tensor \(N = P_1 P_0^{-1}\) is

\[ m(\lambda) = \lambda^2 - q_2 \lambda - \frac{q_1^4}{4}, \quad (13) \]

one easily checks that \(X\) is a Pfaffian QBH vector field. The Darboux–Nijenhuis coordinates can be constructed following [11]; they are

\[ \lambda_{1,2} = \frac{1}{2}(q_2 \pm \sqrt{q_1^2 + q_2^2}) \quad \mu_{1,2} = p_2 - \frac{m}{q_1}(q_2 \pm \sqrt{q_1^2 + q_2^2}). \quad (14) \]

One can easily verify that \((\lambda_1, \lambda_2)\) are just parabolic coordinates in the plane with focus at the point \(q_1 = q_2 = 0\) and axis the \(q_2\)-axis. In the above coordinates the two integrals of motion read
On account of the general result proved in [2], one recovers that \( H \) and \( K \) are separable.

### 3.2 The Euler problem with the two fixed centers and an elastic force (Euler 1760–Lagrange 1766)

The Hamiltonian function for this problem can be written as follows

\[
H = \frac{\lambda_1 \mu_1^2 - 2b \lambda_1^2 + a}{2(\lambda_1 - \lambda_2)} + \frac{\lambda_2 \mu_2^2 - 2b \lambda_2^2 + a}{2(\lambda_2 - \lambda_1)} \tag{15}
\]

\[
K = \frac{\lambda_1 \mu_1^2 - 2b \lambda_1^2 + a}{2(\lambda_1 - \lambda_2)} + \frac{\lambda_2 \mu_2^2 - 2b \lambda_2^2 + a}{2(\lambda_2 - \lambda_1)} \tag{16}
\]

On account of the general result proved in [3], one recovers that \( H \) and \( K \) are separable.

The Hamiltonian function for this problem can be written as follows

\[
H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{b_1}{\sqrt{q_1^2 + (q_2 + c)^2}} + \frac{b_2}{\sqrt{q_1^2 + (q_2 - c)^2}} + \frac{k}{2}(q_1^2 + q_2^2) \tag{17}
\]

where: \((q_1, q_2; p_1, p_2)\) are as in the previous example; the two centers are at the points \( F_1 = (0, -c) \) and \( F_2 = (0, c) \); \( b_1, b_2, k \) are real constants. The corresponding vector field is \( X = P_0 dH \).

As above, also this system does not admit a BH formulation w.r.t. a second Poisson tensor \( P_1 \), but it can be given a QBH formulation. Indeed, one can write Eq. (3) with integral of motion \((K, -cF_1)\) and the following Poisson tensor \( P_1 \)

\[
P_1 = \begin{pmatrix}
0 & 0 & a_1 - q_1^2 & -q_1 q_2 \\
0 & 0 & -q_1 q_2 & a_2 - q_2^2 \\
-(a_1 - q_1^2) & q_1 q_2 & 0 & -q_1 p_2 + q_2 p_1 \\
q_1 q_2 & -(a_2 - q_2^2) & q_1 p_2 - q_2 p_1 & 0
\end{pmatrix} \tag{18}
\]

The minimal polynomial of the Nijenhuis tensor \( N = P_1 P_0^{-1} \) is

\[
m(\lambda) = \lambda^2 + (q_1^2 + q_2^2 - a_1 - a_2)\lambda + a_1 a_2 - a_2 q_1^2 - a_1 q_2^2 \tag{19}
\]

so one easily recognizes the Pfaffian property of \( X \). The Darboux–Nijenhuis coordinates are given by

\[
\lambda_{1,2} = \frac{(a_1 + a_2 - q_1^2 - q_2^2)}{2} \mp \frac{\sqrt{(a_1 + a_2 - q_1^2 - a_1 - a_2)^2 - 4(a_1 a_2 - a_2 q_1^2 - a_1 q_2^2)}}{2} \tag{20}
\]

\[
\mu_{1,2} = \frac{p_1 q_2 + p_2 q_1}{4 q_1 q_2} + \frac{(p_1 q_2 - p_2 q_1)(q_1^2 + q_2^2)}{4(a_2 - a_1) q_1 q_2} \mp \frac{(p_1 q_2 - p_2 q_1) \sqrt{(a_2 - a_1)(a_2 - a_1 + 2 q_1^2 - 2 q_2^2) + (q_1^2 + q_2^2)^2}}{4(a_2 - a_1) q_1 q_2} \tag{21}
\]
One can verify that \((\lambda_1, \lambda_2)\) are just the Jacobi elliptic coordinates in the plane (already known to Euler) with foci at the points \(F_1, F_2\) and axis the \(q_2\)-axis. In the above coordinates the two integrals of motion read

\[
H = \frac{-2(a_1 - \lambda_1)(a_2 - \lambda_1)\mu_1^2 - (b_1 + b_2)\sqrt{a_2 - \lambda_1} - \frac{k}{2}\lambda_1^2}{\lambda_1 - \lambda_2} + \frac{-2(a_1 - \lambda_2)(a_2 - \lambda_2)\mu_2^2 + (b_2 - b_1)\sqrt{a_2 - \lambda_2} - \frac{k}{2}\lambda_2^2}{\lambda_2 - \lambda_1} \tag{23}
\]

\[
K = \frac{\lambda_2}{\lambda_1} \frac{-2(a_1 - \lambda_1)(a_2 - \lambda_1)\mu_1^2 - (b_1 + b_2)\sqrt{a_2 - \lambda_1} - \frac{k}{2}\lambda_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_1}{\lambda_2} \frac{-2(a_1 - \lambda_2)(a_2 - \lambda_2)\mu_2^2 + (b_2 - b_1)\sqrt{a_2 - \lambda_2} - \frac{k}{2}\lambda_2^2}{\lambda_2 - \lambda_1} \tag{24}
\]

Also in this case they are separable, since they have the general form (4).

### 3.3 The motion on the ellipsoid in an elastic potential (Jacobi 1843)

Let us consider a harmonic oscillator on the \(n\)-dimensional ellipsoid

\[
\sum_{i=0}^{n} \frac{x_i^2}{a_i} = 1 , \tag{25}
\]

where \(x_i (i = 0, 1, \ldots, n)\) are Cartesian coordinates in \(R^{n+1}\) and \(a_0 < a_1 < a_2 < \ldots < a_n\) are positive real constants. If \(y_i\) are the conjugate momenta, the Hamiltonian function is

\[
H = \frac{1}{2} \sum_{i=0}^{n} (y_i^2 + x_i^2) .
\]

Let us consider the canonical transformation \((x_i, y_i) \mapsto (\lambda_i, \mu_i)\) associated to the point transformation \((x_i) \mapsto (\lambda_i)\), where \(\lambda_i\) are the generalized elliptic coordinates in \(R^{n+1}\) defined as the \((n + 1)\) roots of the equation

\[
\sum_{i=0}^{n} \frac{x_i^2}{a_i} - \lambda = 1 . \tag{26}
\]

The ellipsoid is given by the submanifold \(\lambda_0 = 0\), and the Hamiltonian function of the harmonic oscillator restricted to the cotangent bundle \(\lambda_0 = 0, \mu_0 = 0\) reads (up to an inessential constant term)

\[
H = \frac{1}{2} \sum_{i=1}^{n} \frac{4 \prod_{j=0}^{n} (a_j - \lambda_i) \lambda_i^{-1} \mu_i^2 + k \lambda_i^n}{\prod_{j=1}^{n} (\lambda_i - \lambda_j)} , \tag{27}
\]

where \(\prod_{j=1}^{n} (\lambda_i - \lambda_j) := 1\) for \(n = 1\), (recall that \(\sum_{i=1}^{n} \lambda_i^n / \prod_{j=1}^{n} (\lambda_i - \lambda_j)\) can be written as \(\sum_{i=1}^{n} \lambda_i\)). Since \(H\) has the form (4), the general result proved in \(\ref{2}\) allows us to infer immediately that
i) the elliptic coordinates and the corresponding momenta $\mu_i$ are Darboux–Nijenhuis coordinates for a standard BH structure $(P_0, P_1)$;

ii) in these coordinates, $P_1$ takes the canonical form (1);

iii) the vector field $X = P_0 dH$ is a Pfaffian QBH vector field. The function $K$ takes the form

$$K = \frac{1}{2} \sum_{i=1}^{n} \rho_i \frac{4 \prod_{j=0}^{n}(a_j - \lambda_i) \lambda_i^{-1} \mu_i^2 + \lambda_i^n}{\prod_{j=1}^{n}(\lambda_i - \lambda_j)} ;$$

(28)

iv) a complete set of rational integrals of motion in involution is given by (3).

### 3.4 A solvable $n$–body problem

This system belongs to a large class of integrable $n$–body problems in the plane, recently introduced by F. Calogero [7]. Let $M = C^{2n}$ (with coordinates $\lambda := (\lambda_1, \ldots, \lambda_n)$ and the conjugate momenta $\mu := (\mu_1, \ldots, \mu_n)$) be the phase space of the dynamical system with Hamiltonian

$$H = \sum_{i=1}^{n} g_i(\lambda_i) e^{\alpha \mu_i} + b \lambda_i^n ,$$

(29)

where $g_i$ are arbitrary smooth functions, each one depending only on the corresponding coordinate $\lambda_i$, and $\alpha, b$ are arbitrary constants. The related Newton equations of motion take the form

$$\ddot{\lambda}_k = 2 \sum_{i \neq k} \frac{\dot{\lambda}_i \dot{\lambda}_k}{\lambda_k - \lambda_i} - b \dot{\lambda}_k .$$

(30)

In the case $b = 0$, a QBH formulation and an alternative BH formulation (according to the item i) of Subsec. 2.1) has been discussed in [3], while in the case $b = \sqrt{-1} \omega$, ($\omega \in \mathbb{R}$) the motion has been proved to be completely periodic in [7].

Just as in the previous example of this section, by comparing (29) with (1) one immediately concludes that:

i) the coordinates $\lambda_i$ and the corresponding momenta $\mu_i$ are Darboux–Nijenhuis coordinates for a standard BH structure $(P_0, P_1)$;

ii) in these coordinates, $P_1$ takes the canonical form (1);

iii) the vector field $X = P_0 dH$ is a Pfaffian QBH vector field. The function $K$ takes the form

$$K = \sum_{i=1}^{n} \rho_i \frac{g_i(\lambda_i) e^{\alpha \mu_i} + b \lambda_i^n}{\prod_{j=1}^{n}(\lambda_i - \lambda_j)} .$$

(31)
iv) a complete set of integrals of motion in involution is given by \( (\text{5}) \).

Furthermore, the corresponding Hamilton–Jacobi equation is separable; a complete integral is

\[
S(\lambda; b_1, \ldots, b_n) = -b_1 t + W(\lambda; b_1, \ldots, b_n)
\]

with

\[
W = \frac{1}{a} \sum_{i=1}^{n} \int_{\lambda_i}^{\lambda_i} \log \left( \frac{1}{g_i(\xi)} \sum_{j=0}^{n} b_j \xi^{n-j} \right) d\xi,
\]

(32)

with \( b_0 = b \) and \( b_1 = H \).

We wish to stress that, unlike the previous three examples, this system is not of Stäckel type, nevertheless it is separable as well.

4 Concluding remarks

The Kepler and the Euler problems considered in Subsections 3.1 and 3.2 correspond to Hamiltonian systems with two degrees of freedom. We recall that for such kind of systems there is a general result stating that a QBH formulation always exists \([1]\): however, it is essentially different from those presented above. Hence, the two systems are explicit examples of the non uniqueness of the QBH formulation.

Indeed, in \([1]\) one assumes to have a vector field \( X \), Hamiltonian w.r.t. an invertible Poisson tensor and Liouville–integrable; passing to the action–angle variables, one can conclude that, for \( n = 2 \), Eq. (3) holds for a suitable Poisson tensor \( P'_1 \), a function \( K' \) and an integrating factor \( \rho' \). In particular, \( \rho' \) depends only on the action variables, so it is a constant of motion for \( X \); this general property allows us to infer, by simple inspection, that our QBH formulations are different, since in both cases the functions \( \rho \) depend only on the configuration variables and therefore they are not integrals of motion for the corresponding dynamical systems.

Finally, let us make a few comments about the relevance of the QBH formulation (we thank an anonymous referee for arising this question).

Let us consider two different situations (as well as those presented in Subsect. 2.3 and Sect. 3, respectively), reflecting two ways in which the QBH systems arise; these situations can be classified according to the fact that the phase space \( M \) of the given dynamical system \( X \) is:

i) a BH manifold \((M, P_0, P_1)\), with both \( P_0 \) and \( P_1 \) non invertible;

ii) a symplectic manifold \((M, P_0 = \omega_0^{-1})\), \( \omega_0 \) being a symplectic tensor.

In the first case, if one wants to solve the equations of motion through a complete integral of the Hamilton–Jacobi equation, then one has to pass to a symplectic manifold (as it happens for the stationary \([2]\) and the restricted flows of the KdV hierarchy, whose phase space is odd–dimensional). As a matter of fact, the symplectic manifold is the proper geometrical setting where the Hamilton–Jacobi method must be set up \([3]\) (after having been usually considered on cotangent bundles). To the best of our knowledge, a further generalization to Poisson (not symplectic) manifolds is still lacking. A possible way to achieve this goal is to perform a “reduction procedure” to a symplectic leaf \( S_0 \) of one of the Poisson tensor, getting (if any) a
QBH formulation for $X$ and a Nijenhuis tensor $N := P_1 P_0^{-1}$ (see Subsect. 2.3 for details and notations). It is just such a tensor, living on $S_0$ and not existing on $M$, that allows us to define in an intrinsic way, by means of its spectral data, the Darboux–Nijenhuis coordinates. We recall that such coordinates are separation variables for the Hamilton–Jacobi equation corresponding to any Pfaffian QBH system.

In the second case, $M$ is a symplectic manifold. If $X$ is Liouville–integrable, it always admits (infinitely many) alternative BH formulation, not including $P_0$ (as recalled in item i) of Subsect. 2.1). However, if one wants still to exploit $P_0$ for getting a standard BH structure, in many cases a QBH formulation for $X$ can be constructed in a quite natural way, as it has been shown in the examples discussed above. Of course, in this case one could also try to embed the given dynamical system $X$ in a larger phase space and to lift the QBH formulation in order to get a BH formulation for (the lifting of) $X$, somehow reverting the above mentioned reduction procedure. However, this lifting is neither natural nor unique, and would oblige to work anew in a Poisson (not symplectic) manifold.

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