Seiberg–Witten Floer spectra and contact structures

B. R. S. Roso

Abstract. In this article, the author defines an invariant of rational homology 3-spheres equipped with a contact structure as an element of a cohomotopy set of the Seiberg–Witten Floer spectrum as defined in Manolescu (Geometry Topol 7(2):889–932, 2003). Furthermore, in light of the equivalence established in Lidman and Manolescu (Astérisque 399:25, 2018) between the Borel equivariant homology of said spectrum and the Seiberg–Witten Floer homology of Kronheimer and Mrowka (Monopoles and three-manifolds, vol. 10, Cambridge University Press, Cambridge, 2007), the author shall show that this homotopy theoretic invariant recovers the already well known contact element in the Seiberg–Witten Floer cohomology (vid. e.g. Kronheimer et al. in Ann Math 20:457–546, 2007) in a natural fashion. Next, the behaviour of the cohomotopy invariant is considered in the presence of a finite covering. This setting naturally asks for the use of Borel cohomology equivariant with respect to the group of deck transformations. Hence, a new equivariant contact invariant is defined and its properties studied. The invariant is then computed in one concrete example, wherein the author demonstrates that it opens the possibility of considering scenarios hitherto inaccessible.

Mathematics Subject Classification. 57R58, 57K33, 57M10, 37B30.

Keywords. Seiberg–Witten Floer spectra, contact structures, regular coverings, attractor-repeller pairs.

1. Introduction

It is well known from the work of Taubes [48] that symplectic 4-manifolds with $b_2^+ > 1$ have non-trivial Seiberg–Witten invariants. This is accomplished by perturbing the Seiberg–Witten equations in a manner dictated by the symplectic form multiplied by a large positive real number. The effect of doing so is that the Seiberg–Witten equations gain an obvious canonical solution, which turns out to be unique and non-degenerate.
Perhaps a little less well known is that one can do something similar for contact 3-manifolds. This was pursued in Taubes [50] and was ultimately crucial for the proof of the Weinstein conjecture. In the 3-dimensional case, one adds, as a perturbation, the contact form multiplied, again, by a positive real number. A canonical solution to the Seiberg–Witten equations immediately becomes apparent, and, if the real number be made large enough, one finds that this solution is automatically non-degenerate. Uniqueness does not hold in the 3-dimensional case in the same form as it does in the 4-dimensional case; if it did, all contact rational homology 3-spheres would be \textit{L-spaces}. Nonetheless, another sort of uniqueness does hold but one which concerns Seiberg–Witten trajectories. The distinguished contact monopole does not admit any non trivial Seiberg–Witten trajectories coming into it in the forward time limit. This implies that this solution defines a cocycle and therefore a class in monopole Floer cohomology. As it turns out, this class is the well known contact invariant studied in Kronheimer et al. [22] and is equivalent to the contact invariants in Heegaard Floer and embedded contact homologies.

In Manolescu [35], a Seiberg–Witten Floer spectrum was defined, which was later shown to recover the monopole Floer cohomologies through its Borel \textit{U}(1)-equivariant cohomology. An important detail here is that the construction of the spectrum avoids altogether the use of any generic perturbations. Taubes’ approach to defining the contact invariant, despite requiring a generic perturbation in order to work in a Morse theoretic setting, has the property that the contact monopole is already non-degenerate before the addition of the generic perturbation. In the present article, the author applies Taubes’ approach to the contact invariant in the context of the Seiberg–Witten Floer spectrum in order to conveniently avoid the use of generic perturbations altogether.

\textbf{Theorem 1.1.} \textit{Given a contact rational homology 3-sphere} \((Y, \lambda)\) \textit{there exists a cohomotopical contact invariant,}

\[ \text{SWF}(Y, s_{\lambda}) \to T(\lambda), \]

\textit{where} \(s_{\lambda}\) \textit{is the Spin}^C \textit{structure on} \(Y\) \textit{defined by the contact form} \(\lambda\) \textit{and} \(T(\lambda)\) \textit{is a (de)suspension of a} \textit{U}(1)-\textit{equivariant Thom space of a vector bundle over the} \textit{U}(1)-\textit{orbit of the contact monopole in global Coulomb gauge}. \textit{Moreover, the classical cohomological contact invariant is recovered by pulling back via this map a class in the Borel} \textit{U}(1)-\textit{equivariant cohomology of} \(T(\lambda)\).

A similar invariant is constructed in Iida and Taniguchi [18]; however, the work in the present article is entirely independent and differs in significant ways. Firstly, the author uses a different set of analytical results to ensure the existence of his invariant and also relies more heavily on certain aspects of Conley theory to define it. Secondly, here, \textit{U}(1)-\textit{equivariance is kept manifest throughout, which means one can consider implications in Borel equivariant cohomology; indeed, the author was able to prove, via the techniques developed in Lidman and Manolescu [31], how the cohomotopical invariant recovers the well known cohomological invariant by passing to Borel equivariant cohomology. It should also be noted that the construction presented}
in the present article and the one of Iida and Taniguchi [18] are sufficiently
different that it is not clear if the two invariants are indeed equivalent or not.
Of course, one would be inclined to think that two such invariants should
really be holding the same information. However, proving their equivalence
might be a difficult task due to the different analytical foundations used, so
this goal is not pursued in the present article.

The author’s main goal, after arming himself with the cohomotopical in-
variant, was to study covering spaces. The avoidance of generic perturbations
is important in this context, as demonstrated in Lidman and Manolescu [30],
due to the impossibility of producing sufficiently generic equivariant per-
turbations. The author’s cohomotopical contact invariant can also be made
$G$-equivariant for $G$ the group of deck transformations of a finite regular
covering. This allows him to consider Borel $G$-equivariant cohomology and
deduce certain vanishing and non-vanishing results via the use of the localization theorem.

The central question one considers is whether the lift of a tight contact
structure remains tight or becomes overtwisted in the covering. The results
derived via use of the contact invariant shed light on this problem. In the work
of Lin and Lipnowski [33], the term minimal $L$-space is introduced to refer to
rational homology 3-spheres having a single solution to the Seiberg–Witten
equations for any Spin$^C$ structure. For such a manifold, the Seiberg–Witten
Floer spectrum is always the sphere spectrum. Moreover, in the case of a finite
covering, the Seiberg–Witten Floer $G$-spectrum is the sphere $G$-spectrum. In
this context, the author shall establish the following theorem.

**Theorem 1.2.** Let $\pi : Y \to Y/G$ be a regular prime order covering of mini-
mal $L$-spaces and suppose that $\lambda$ be a tight contact form on $Y/G$ with non-
vanishing cohomological contact invariant. Then, the lifted contact form, $\pi^*\lambda$,
has non-vanishing cohomological contact invariant (and, therefore, is tight)
provided that $d_3(\text{Ker} \pi^*\lambda) + 1/2 = d(Y, \pi^*s_\lambda)$, where $d_3$ denotes Gompf’s
three-dimensional invariant of hyperplane fields and $d$ denotes the Ozsváth–
Szabó invariant.

Examples of minimal $L$-spaces include all sol rational homology 3-
spheres due to the work of Lin [32]. Another example is the Hantsche–Wendt
manifold, the unique flat rational homology 3-sphere. A few more examples
exist amongst the hyperbolic manifolds as shown in Lin and Lipnowski [33].
However, the most evident class of examples of minimal $L$-spaces is that of the
elliptic manifolds. In this case, increased knowledge of the contact topology,
in particular the fact that the cohomological contact invariant never van-
ishes for tight contact structures and no two distinct contact structures have
the same Spin$^C$ structure, allows the author to prove the following stronger
theorem.

**Theorem 1.3.** Let $\pi : Y \to Y/G$ be a prime order regular covering of elliptic
manifolds and suppose that $\lambda$ be a tight contact form on $Y/G$. Then, the lifted
contact form, $\pi^*\lambda$, is isotopic to a tight contact form $\lambda'$ on $Y$ if and only if
it be homotopic to $\lambda'$.
This result leads to a scheme for determining tightness of the lift of a tight contact structure on an elliptic manifold based purely on the homotopy theoretic obstruction classes of the contact structures involved. This reduces significantly the complexity of the problem and can be used in concrete calculations. The rationale is to try to determine the obstruction theoretic invariants of hyperplane fields – that is, the $d_3$ invariant and the Spin$^C$ structure – for the lifted contact structure, $\pi^*\lambda$, from those of $\lambda$. If those be seen to match the values for a known tight contact structure on $Y$, then one shall know that $\pi^*\lambda$ is isotopic to it. The issue that arises is that it is non trivial to determine the lifting behaviour of the obstruction theoretic invariants, especially of the $d_3$ invariant.

In order to solve this problem, the author found himself having to develop techniques which seem not be discussed in the literature in a particularly well detailed manner. These shall be detailed in the present article and rely mostly on use of the Kirby calculus. As an example, the author shall study the case a certain tight contact structure on the $(-8)$-surgery on the left-handed trefoil which shall be shown to lift to a virtually overtwisted contact structure on the lens space $L(12, 7)$ via the double covering.

The main ingredient needed to perform these calculations is a form of $G$-equivariant almost-complex filling for the given covering of contact manifolds, which consists of an almost complex 4-manifold-with-boundary extending the given $G$-action on its contact boundary and potentially having a branching surface in its interior. Such a filling can often be produced by appealing to the notion of equivariant handle attachments in the context of the Kirby calculus. With such a filling at hand, one can apply the $G$-signature theorem, as was done by Khuzam [19], to deduce the lifting behaviour of $d_3$ invariants.

The other matter that one must understand carefully is the lifting behaviour of Spin$^C$ structures. This is more elementary, albeit still difficult in practice, and can be tackled in different manners. The method pursued here shall follow a similar approach to the lifting of $d_3$ invariants by using Kirby calculus to express Spin structures in terms of obstruction theory and then studying the lifting behaviour of Spin structures. The behaviour of Spin$^C$ structures follows easily thence.

This work was extracted from parts of the author’s Ph.D. dissertation, Roso [47].

2. Seiberg–Witten equations and contact structures

This section shall introduce the basic definitions and analytical results required from Seiberg–Witten theory. Throughout this article, the author shall use a version of the Seiberg–Witten equations adapted to the presence of a contact form which was first introduced in Taubes [50] and Taubes [51] and was subsequently used in Taubes’ work in the correspondence between monopole Floer homology and embedded contact homology.

Consider an oriented 3-manifold $Y$ satisfying $b_1(Y) = 0$. A contact form $\lambda$ is a 1-form on $Y$ satisfying $\lambda \wedge d\lambda > 0$. The subbundle of $TY$ given by $\text{Ker} \lambda$
is called a coorientable contact structure. In this article, all contact structures shall be assumed coorientable. As shall be seen, the version of the Seiberg–Witten equations which shall be used always admits a canonical solution, $C_\lambda$, which, provided a certain parameter $r > 0$ be made large enough, is nicely behaved in two fundamental ways. It is non-degenerate irrespective of any genericity requirements, and it is not the forward time limit of any Seiberg–Witten trajectory. These two properties shall be instrumental later in the present article. The solution $C_\lambda$ is essentially defined in a manner that make its spinor component bounded away from zero; a feature which is unique to this solution provided $r$ be made large enough.

In what follows, agree to fix a metric $g$ on $Y$ with the property that $\lambda \wedge d\lambda = \text{Vol}_g$. Use $\xi := \text{Ker} \lambda$. Fix a complex structure $J \in \text{End}(\xi)$ on the bundle $\xi$ compatible with $g$ in the sense that $g(-, -) = d\lambda|_{\xi}(-, J-)$. Use $R \in \Gamma TY$ to denote the Reeb vector field; that is, the vector field satisfying $\iota_R d\lambda = 0$, $\iota_R \lambda = 1$. Write $\xi \otimes C = \Lambda^{0,1} \xi \oplus \Lambda^{1,0} \xi$, where $\Lambda^{1,0} \xi$ and $\Lambda^{0,1} \xi$ are, respectively, the $(\pm i)$-eigenbundles of $J$. Likewise, for the dual, write $\xi^* \otimes C = \Lambda^{1,0} \xi^* \oplus \Lambda^{0,1} \xi^*$. Denote $\Lambda^{p,q} \xi^* := \Lambda^p C \Lambda^{q,0} \xi^* \otimes \Lambda^q C \Lambda^{0,1} \xi^*$.

There is canonical Spin$^C$ structure on $Y$ defined via the specification of its spinor representation bundle in the following way.

**Definition 2.1.** Define the spinor bundle $S_\lambda := \bigoplus_q \Lambda^{0,q} \xi^* = \Lambda^{0,0} \xi^* \oplus \Lambda^{0,1} \xi^*$ with Clifford multiplication $c_\ell : TY \to \text{End}_C(S_\lambda)$ defined so as to satisfy, for $\alpha \in \Lambda^{0,q} \xi^*$ and $X \in \xi$, $c_\ell(R)\alpha = (-1)^{q+1} i\alpha$, $c_\ell(X)\alpha = \sqrt{2} \left((X^{0,1})^* \wedge \alpha - \iota_{(X^{0,1})} \alpha\right)$.

**Remark 2.2.** The notation $(X \mapsto X^*) : TY \otimes C \to T^* Y \otimes C$ denotes the $C$-antilinear isomorphism induced by the metric.

**Remark 2.3.** Note that this fully determines the Clifford action due to the fact that $TY = \langle R \rangle_R \oplus \xi$. This map can be checked to indeed define an irreducible Clifford module; vid. Petit [44] for a proof and more details about this matter.

**Definition 2.4.** Denote the underlying Spin$^C$ structure, that is, the principal Spin$^C(3)$-bundle, by $s_\lambda \to Y$.

**Remark 2.5.** Notice that the determinant line bundle is simply $\text{det} s_\lambda = \Lambda^{0,1} \xi^* \cong \xi^*$ where $\xi^*$ is equipped with the complex structure induced by $J$.

**Definition 2.6.** Use $A(E)$ to denote the affine space of Hermitian connexions on a Hermitian vector bundle $E \to Y$.

**Definition 2.7.** Define the configuration space by $C(Y, s_\lambda) := A(\text{det} s_\lambda) \times \Gamma(S_\lambda)$. Use $C(Y, s_\lambda)_k$ to denote its completion in the topology induced by the Sobolev norm $L^2_k$. 
Definition 2.8. The tangent bundle $\mathcal{T}C(Y, s_\lambda) \to \mathcal{C}(Y, s_\lambda)$ is the bundle having as fibre over $C \in \mathcal{C}(Y, s_\lambda)$ the space $\mathcal{T}C(Y, s_\lambda)|_C = \Gamma(i\mathbb{T}^*Y \oplus S_\lambda)$. Use $\mathcal{T}C(Y, s_\lambda)_k$ for its Sobolev $L^2_k$ completion.

Definition 2.9. Use $\tau : S_\lambda \to i\mathbb{T}^*Y$ to denote the quadratic map defined by sending $\psi \mapsto c \ell^{-1} \left( \psi \otimes \psi^* - \frac{1}{2} |\psi|^2 \text{id} \right)$.

Definition 2.10. Denote $D_A : \Gamma S_\lambda \to \Gamma S_\lambda$ the Dirac operator defined by a connexion $A$ on $\text{det} S_\lambda$.

Definition 2.11. Define the contact configuration, $C_\lambda \equiv (A_\lambda, \psi_\lambda) \in \mathcal{A}(\text{det} S_\lambda)$, by setting its spinor component to be the constant function $\psi_\lambda := 1 \in \Gamma(\xi \ast_0 0, 0) \approx Y \times C$ and by requiring its connexion component $A_\lambda$ to solve the Dirac equation $D_A \psi_\lambda = 0$.

Remark 2.12. This canonical configuration shall play a pivotal rôle in this article.

Definition 2.13. Let $r > 0$. The canonically perturbed Seiberg–Witten vector field of $(Y, \lambda)$ is the vector field $\mathcal{X}_{\lambda,r} : \mathcal{C}(Y, s_\lambda) \to \mathcal{T}C(Y, s_\lambda)$ given by

$$\mathcal{X}_{\lambda,r}(A, \psi) = \left( \frac{1}{2} (F_A - F_{A_\lambda}) + r \tau(\psi) - \frac{ir}{2} \lambda, \mathcal{D}_A \psi \right).$$

Remark 2.14. Notice that, for any value of $r > 0$, the contact configuration solves the Seiberg–Witten equation $\mathcal{X}_{\lambda,r}(C_\lambda) = 0$.

That is, $C_\lambda$ is a fixed point of the Seiberg–Witten vector field.

Definition 2.15. Denote the gauge group by $\mathcal{G}(Y) := C^\infty(Y, U(1))$. Use $\mathcal{G}(Y)_k$ for its completion in the Sobolev $L^2_k$ norm.

Definition 2.16. For $(A, \psi) \in \mathcal{C}(Y, s_\lambda)$, define the linearized gauge action by

$$\mathcal{L}_{(A, \psi)} : C^\infty(Y, i\mathbb{R}) \to \mathcal{T}C(Y, s_\lambda)|_{(A, \psi)}, \quad u \mapsto (-du, u\psi).$$

Use $\mathcal{L}^*_{(A, \psi)}$ for its formal $L^2$-adjoint.

Definition 2.17. The local Coulomb gauge is the subspace $\mathcal{K}_C := \text{Ker} \mathcal{L}^*_C \subset \mathcal{T}C(Y, s_\lambda)|_C$.

Denote by $\mathcal{K} \to \mathcal{C}(Y, s_\lambda)$ the vector bundle with fibres the local Coulomb gauges. Use $\mathcal{K}_C, k$ and $\mathcal{K}_k$ to denote the $L^2_k$ Sobolev completions.

Definition 2.18. Denote by $\Pi^L_C : \mathcal{T}C(Y, s_\lambda)|_C \to \mathcal{K}_C$ the $L^2$-orthogonal projection.

Definition 2.19. A configuration $(A, \psi) \in \mathcal{C}(Y, s_\lambda)$ is said to be irreducible if $\psi$ is not identically zero.
Definition 2.20. An irreducible solution \( C \in \mathcal{X}_{\lambda,r}^{-1}(0) \) is said to be non-degenerate if the derivative
\[
\Pi_C^L \circ D_C \mathcal{X}_{\lambda,r} : \mathcal{K}_C \rightarrow \mathcal{K}_C
\]
be surjective.

Remark 2.21. It is customary in Seiberg–Witten theory to ensure non-degeneracy of all solutions through the addition of a generic perturbation; however, often, doing so has its downside. One of the key advantages of the Seiberg–Witten Floer spectrum construction ([35]) is that it avoids the need for such a perturbation altogether. This was crucial in the results of Lidman and Manolescu [30] concerning the Smith-type inequality of Seiberg–Witten Floer homology. Avoiding the use of a generic perturbation shall also be exploited in the present article; however, as will be seen, it is still necessary to make sure that \( C_\lambda \) be non-degenerate. This is ensured by the following theorem of Taubes.

Remark 2.22. For ease of reference, the author shall make use of the following convention. When a proposition state the existence of a certain constant which shall be of use later in the text, that constant shall be labelled with the number of the proposition.

Theorem 2.23. [50] There exists \( r_{2.23} > 0 \) such that, for \( r > r_{2.23} \), the contact configuration is non-degenerate.

Proof. Vid. Taubes [50], Lemma 3.3. □

Besides the non-degeneracy property, the configuration \( C_\lambda \) enjoys two uniqueness properties that shall prove important.

Theorem 2.24. [51] There exists \( r_{2.24} > 0 \) and \( \delta_{2.24} > 0 \) such that, for \( r > r_{2.24} \), the only configuration \( C = (a, \psi) \), up to a gauge transformation, satisfying
\[
\mathcal{X}_{\lambda,r}(C) = 0, \quad |\psi| \geq 1 - \delta_{2.24}
\]
is the contact configuration \( C = C_\lambda \).

Proof. Vid. Taubes [51], Proposition 2.8. □

Theorem 2.25. There exists \( r_{2.25} > 0 \) such that, for \( r > r_{2.25} \), any trajectory \( \gamma : \mathbb{R} \rightarrow \mathcal{C}(Y, s_\lambda) \) satisfying
\[
\frac{d}{dt} \gamma(t) = -\mathcal{X}_{\lambda,r}(\gamma(t)), \quad \lim_{t \to -\infty} \gamma(t) = C, \quad \lim_{t \to \infty} \gamma(t) = C_\lambda,
\]
where the limits are with respect to the Sobolev norm \( L_k^2 \) for any \( k \geq 5 \), must satisfy \( \gamma(t) = C_\lambda \) for all \( t \in \mathbb{R} \).

Proof. This is nearly what is stated by Taubes [51], Proposition 5.15, but not quite. Please find an adaptation of Taubes’ proof in the appendix of the present article. □
3. Review of Seiberg–Witten Floer spectra

Armed with the analytic results from the previous section, the first goal of the present article shall be to define a homotopy theoretic invariant emerging from the contact configuration. This invariant shall live in an equivariant cohomotopy set of the Seiberg–Witten Floer Spectrum \( \text{SWF}(Y, s_\lambda) \) defined by Manolescu [35]. This section shall review that construction with a slight adaptation; the Seiberg–Witten flow used shall be the one canonically perturbed by the contact form as was described in the previous section. This shall allow for the definition of the contact invariant in the next section.

Let \( Y, \lambda \) and \( g \) be as in the previous section.

**Definition 3.1.** The unperturbed Seiberg–Witten vector field is

\[
\mathcal{X} : \mathcal{C}(Y, s_\lambda) \rightarrow \mathcal{T}\mathcal{C}(Y, s_\lambda), \quad \mathcal{X}(A, \psi) = \left( \frac{1}{2} * F_A - \tau(\psi), \mathcal{D}_A \psi \right).
\]

**Remark 3.2.** The construction in Manolescu [35] uses \( \mathcal{X} \) as the Seiberg–Witten vector field. The version of the Seiberg–Witten vector field used in the present article is \( \mathcal{X}_{\lambda,r} \) and it differs from \( \mathcal{X} \) in two ways. Firstly, the spinor component of \( \mathcal{X}_{\lambda,r} \) is scaled by \( r^{1/2} \) compared to \( \mathcal{X} \); this distinction shall be evidently immaterial in the construction. Secondly, \( \mathcal{X}_{\lambda,r} \) contains the constant term \( ((ir/2)\lambda - * (1/2) F_{A_\lambda}, 0) \) added on. In the language of Lidman and Manolescu [31], this amounts to the addition of a “very tame” perturbation (vid. [31], Definition 4.4.2), which, as demonstrated there (vid. [31], Proposition 6.1.6), does not affect the construction of the spectrum. Hence, the spectra defined with \( \mathcal{X}_{\lambda,r} \) and \( \mathcal{X} \) shall be the same.

**Definition 3.3.** The normalized Gauge group is the subgroup \( G^\circ(Y) \subset \mathcal{G}(Y) \) consisting of those \( u \in \mathcal{G}(Y) \) which can be written as \( u = e^{if} \) such that \( \int_Y * f = 0 \).

**Definition 3.4.** By the global Coulomb gauge with respect to the connexion \( A_\lambda \), one means

\[
W := (A_\lambda + \text{Ker}(d^* : i\Omega^1(Y) \rightarrow i\Omega^2(Y))) \oplus \Gamma(S_\lambda) \subset \mathcal{C}(Y, s_\lambda).
\]

**Remark 3.5.** The affine space \( W \) shall be thought of as a vector space with zero being \( (A_\lambda, 0) \); that is, connexions shall be thought of as purely imaginary 1-forms by subtracting \( A_\lambda \).

**Remark 3.6.** Any \( \mathcal{G}^\circ(Y) \)-equivalence class \( [(A_\lambda + a, \psi)] \in \mathcal{C}(Y, s_\lambda)/\mathcal{G}^\circ(Y) \) has a unique representative in global Coulomb gauge; that is, there is a unique \( (A_\lambda + a', \psi') \in W \) such that \( [(A_\lambda + a', \psi')] = [(A_\lambda + a, \psi)] \in \mathcal{C}(Y, s_\lambda)/\mathcal{G}^\circ(Y) \).

**Definition 3.7.** The global Coulomb projection, \( \Pi^{GC} : \mathcal{C}(Y, s_\lambda) \rightarrow W \), is given by sending a configuration \( (A_\lambda + a, \psi) \in \mathcal{C}(Y, s_\lambda) \) to its unique \( \mathcal{G}^\circ(Y) \)-equivalent in global Coulomb gauge.

**Remark 3.8.** The map \( \Pi^{GC} \) may be computed as follows. Let \( G : L^2_m(Y) \rightarrow L^2_{m+2}(Y) \) denote the Green’s operator of the Laplacian \( \Delta : \Omega^0(Y) \rightarrow \Omega^0(Y) \). One can show that

\[
\Pi^{GC}(A_\lambda + a, \psi) = (A_\lambda + a - dGd^*a, e^{Gd^*a} \psi).
\]

As a consequence, note that \( \Pi^{GC} \) maps bounded sets to bounded sets.
Definition 3.9. The enlarged local Coulomb slice is the subspace
\[ K^E_{(A, \psi)} \subset TC(Y, s_\lambda)|_{(A, \psi)} \]
defined as the $L^2$-orthogonal complement to the orbits of $G^\circ(Y)$.

Definition 3.10. Denote by $\mathfrak{g}^\circ(Y)$ the Lie algebra of $G^\circ(Y)$.

Remark 3.11. Any equivalence class $[(b, \phi)] \in TC(Y, s_\lambda)|_{(A, \psi)}/\mathfrak{g}^\circ(Y)$ has a unique representative in enlarged local Coulomb gauge.

Definition 3.12. By the enlarged local Coulomb projection, one means
\[ \Pi^{ELC}_{(A, \psi)} : TC(Y, s_\lambda)|_{(A, \psi)} \to K^E_{(A, \psi)} \]
defined by sending a vector to the unique representative in enlarged local Coulomb gauge of its equivalence class in the quotient $TC(Y, s_\lambda)|_{(A, \psi)}/\mathfrak{g}^\circ(Y)$.

Remark 3.13. Note that $\Pi^{LC}$ and $\Pi^{ELC}$ are maps defined on the tangent bundle $TC(Y, s_\lambda)$, whereas $\Pi^{GC}$ is defined on $C(Y, s_\lambda)$. Of course, $\Pi^{GC}$ induces a map $TC(Y, s_\lambda) \to TW$ via the pushforward $\Pi^{GC}_*$.

Definition 3.14. Set $X_{\lambda,r}^{GC} := \Pi^{GC}_* X_{\lambda,r}$.

Remark 3.15. Fix some integer $k \geq 5$ and consider, henceforth, $X_{\lambda,r}^{GC}$ as a map $W_k \to W_{k-1}$ where $W_m$ denotes the completion of $W$ in the Sobolev norm $L^2_m$.

Definition 3.16. Define the Fredholm linear operator $\ell : W_k \to W_{k-1}$ by the formula
\[ \ell(A_\lambda + a, \psi) = \left( A_\lambda + \frac{1}{2} * da, D_{A_\lambda} \psi \right). \]

Definition 3.17. Define the (non-linear) operator $c : W_k \to W_{k-1}$ by $c := X_{\lambda,r}^{GC} - \ell$.

Remark 3.18. Note that $X_{\lambda,r}^{GC} = \ell + c$ where $\ell$ is linear and Fredholm and $c$ is compact as explained in Manolescu [35, Sect. 4].

Definition 3.19. For $\mu > 1$, denote by $W^\mu \subset W_k$, the subspace consisting of the span of the eigenvectors of $\ell$ with eigenvalues in the interval $(-\mu, \mu)$. Use $\tilde{p}^\mu : W_k \to W^\mu$ to denote the $L^2$-orthogonal projection.

The family of operators $\tilde{p}^\mu$ must now be smoothed out in a particular way. For that end, fix a smooth function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying $\text{supp}\beta = [0, 1]$ and $\int_{\mathbb{R}} \beta(x) dx = 1$. A preliminary version of the smoothed out family is as follows.

Definition 3.20. Define a family of operators $p^\mu_{\text{prel}} : W_k \to W^\mu$ by
\[ p^\mu_{\text{prel}} := \int_0^1 \beta(t) \tilde{p}^{\mu-t} dt \]
Remark 3.21. This preliminary version could well be used to define the Seiberg–Witten Floer spectrum and, indeed, is essentially the operator family which appears in the original definition in Manolescu [35]. However, in Lidman and Manolescu [31], the authors use a slightly modified version which turns out to be needed in proving some technical results. Some of those technical results shall be used in the present article. To define the final version of the operator family, a few more data need to be fixed. Firstly, choose an unbounded strictly increasing sequence \( \{ \mu_i \} \subset \mathbb{R} \) such that, for no \( i \), be \( \mu_i \) an eigenvalue of \( \ell \).

Next, fix a sequence of small real numbers \( \{ \epsilon_i \} \subset \mathbb{R} \) such that the intervals \([\mu_i - \epsilon_i, \mu_i + \epsilon_i]\) be disjoint and not contain any eigenvalue of \( \ell \). At last, pick smooth bump functions \( \{ \beta_i : \mathbb{R} \to [0, 1] \} \) such that \( \text{supp} \beta_i \subset [\mu_i - \epsilon_i, \mu_i + \epsilon_i] \).

Definition 3.22. Define the family of operators \( p^\mu : W_k \to W^\mu \) by

\[
p^\mu := \sum_i \beta_i(\mu) \tilde{p}^\mu + \left( 1 - \sum_i \beta_i(\mu) \right) p^\mu_{\text{prel}}.
\]

Remark 3.23. The family of operators \( p^\mu \) is smooth in \( \mu \) but still has the property that, for all \( i \), \( p^{\mu_i} = \tilde{p}^{\mu_i} \).

Definition 3.24. By the canonically perturbed Chern-Simons-Dirac functional, one means

\[
\text{CSD}_{\lambda,r} : C(Y, s_\lambda) \to \mathbb{R},
\]

\[
\text{CSD}_{\lambda,r}(A_\lambda + a, \psi) := \int_Y *r(\psi, D_{A_\lambda + a} \psi) - \int_Y a \wedge da - \frac{ri}{2} \int_Y \lambda \wedge da.
\]

Definition 3.25. A finite type curve \( \gamma : \mathbb{R} \to C(Y, s_\lambda) \), \( \gamma = (A, \psi) \), is a curve such that the maps \( t \mapsto \text{CSD}_{\lambda,r}(\gamma(t)) \) and \( t \mapsto \|\psi(t)\|_{C^0} \) be bounded as functions \( \mathbb{R} \to \mathbb{R} \).

Definition 3.26. A curve \( \gamma : \mathbb{R} \to W_k \) is said to be a Seiberg–Witten trajectory in global Coulomb gauge if

\[
\frac{d}{dt} \gamma = -\mathcal{X}_{\lambda,r}^{GC}(\gamma(t)).
\]

Definition 3.27. For \( R > 0 \), and a normed vector space \( V \), use \( B(V, R) \subset D(V, R) \subset V \) to denote, respectively, the open and closed balls of radius \( R \). Use \( S(V, R) = D(V, R) \setminus B(V, R) \) to denote the sphere of radius \( R \).

Theorem 3.28. (cf. [35], Proposition 1) There exists \( R > 0 \) such that all finite type trajectories of \( \mathcal{X}_{\lambda,r}^{GC} \) are contained in the ball \( B(W_k, R) \subset W_k \).

Proof. Firstly, note that the proof in Manolescu [35] can be easily adapted to the present case of the perturbed Seiberg–Witten flow. That result provides a constant \( R' > 0 \) such that, up to a gauge transformation, all Seiberg–Witten trajectories of finite type sit inside the ball \( B(W_k, R') \subset C(Y, s_\lambda) \).

Therefore, a Seiberg–Witten trajectory in global Coulomb gauge is, locally, the global Coulomb projection of a Seiberg–Witten trajectory residing in the ball \( B(W_k, R') \subset C(Y, s_\lambda) \). But the global Coulomb projection map \( \Pi^{GC} : C(Y, s_\lambda) \to W_k \) maps bounded sets to bounded sets. \( \square \)
Remark 3.29. Henceforth, assume $R > 0$ to be such that all finite type Seiberg–Witten trajectories in Coulomb gauge fit in $B(W_k, R)$.

Remark 3.30. Also, fix a family of $U(1)$-equivariant bump functions $u^\mu : W^\mu \to \mathbb{R}$ satisfying
\[ u^\mu|_{D(W^\mu, 2R)} = 1, \quad u^\mu|_{W^\mu \setminus B(W^\mu, 3R)} = 0 \]
and, $u^\mu$ constant on the sphere $S(W^\mu, t) \subset W^\mu$ of radius $t$ for all $t \in [0, \infty)$. Note that the norm on $W^\mu \subset W_k$ is defined by the Sobolev norm $L^2_k$ of $W_k$.

Definition 3.31. Define the finite dimensional approximation to the Seiberg–Witten vector field as
\[ X^\mu_{\lambda, r} = u^\mu \cdot (\ell + p^\mu c). \]
Furthermore, use $\varphi^\mu_{\lambda, r} : W^\mu \times \mathbb{R} \to W^\mu$ to denote the flow given by the O.D.E.
\[ \frac{d}{dt} \Big|_{t=0} \gamma(t) = -X^\mu_{\lambda, r}(\gamma(t)). \]
The flow lines of $\varphi^\mu_{\lambda, r}$ are called approximate Seiberg–Witten trajectories in global Coulomb gauge.

Definition 3.32. Use $S^k_{\lambda, r} \subset B(W^\mu, R)$ to denote the union of all flow lines of $\varphi^\mu_{\lambda, r}$ which remain inside of $B(W^\mu, R)$ for all time.

Theorem 3.33. ([35], Proposition 3) For $\mu > 0$ sufficiently large compared to $R$, any flow line of $\varphi^\mu_{\lambda, r}$ which be contained in the disk $D(W^\mu, 2R)$ is, in fact, contained in the open ball $B(W^\mu, R)$.

Proof. The proof in Manolescu [35] of the non-perturbed case of this theorem can be trivially adapted to the case at hand. Alternatively, this result is a special case of Lidman and Manolescu [31], Proposition 6.1.2(i) and Proposition 6.1.5, noting that the perturbation used in this article is “very tame”.

The author shall now recall the pertinent definitions from Conley theory. For a reference, the reader is directed to Conley [6], Floer [8] and Mishchaikow [38]. In what follows, suppose that $G$ be a compact Lie group, $\Gamma$ be a locally compact Hausdorff space with a continuous $G$-action and $\phi : \Gamma \times \mathbb{R} \to \Gamma$ a continuous and equivariant flow.

Definition 3.34. Let $U \subset \Gamma$ be a $G$-invariant subset. The maximal invariant set of $U$ is
\[ \text{Inv}(U) := \{ u \in U \mid (\forall t \in \mathbb{R})(\phi(u, t) \in U) \}. \]

Definition 3.35. Let $S \subset \Gamma$ be a $G$-invariant compact subset. $S$ is called an isolated invariant set if there be a compact neighbourhood $U \supset S$ such that $\text{Inv}(U) = S$.

Definition 3.36. Let $S \subset \Gamma$ be an isolated invariant set. A pair of $G$-invariant compact sets $(M, N)$ satisfying $N \subset M \subset \Gamma$ is called an index pair for $S$ when:
(i) $M \setminus N$ be an isolating neighbourhood for $S$;
(ii) for all $t \geq 0$ and $x \in N$, if $\varphi(\{x\} \times [0, t]) \subset M$, then $\varphi(\{x\} \times [0, t]) \subset N$;
(iii) for all $t \geq 0$ and $x \in M$, if $\varphi(x, t) \not\in M$, then $\varphi(\{x\} \times [0, t]) \cap N \neq \emptyset$.

**Theorem 3.37.** ([6], non-equivariant; [8] and [9], equivariant) For any isolated invariant set $S \subset \Gamma$, there exists an index pair $(M, N)$ and the $G$-equivariant pointed homotopy type $M/N$ is independent of the choice of $(M, N)$.

**Definition 3.38.** The $G$-equivariant homotopy type of $M/N$ where $(M, N)$ is an index pair for an isolated invariant set $S$ is called the Conley index of $S$ and denoted $I_G(S, \varphi)$.

Theorem 3.33 can now be reinterpreted in this language.

**Corollary 3.39.** $S_{\mu, \lambda, r}^\mu$ is a $U(1)$-invariant isolated invariant set with isolating neighbourhood $D(W_{\mu, 2R})$ for the $U(1)$-equivariant flow $\varphi_{\lambda, r}^\mu$.

The author shall now introduce the relevant definitions from equivariant stable homotopy theory. Here, the author shall deviate slightly from the route taken in Manolescu [35]; this is done in the interests of later sections that shall deal with Seiberg-Witten Floer spectra equivariant with respect to the deck transformations of a covering. In Manolescu [35], in order to perform the required desuspensions, an ad hoc version of the Spanier-Whitehead category is used. In the present article, instead, the author shall use the, by now, more standard category of spectra. This increases slightly the complexity of the definitions, but nothing new is gained as the Spanier-Whitehead category embeds into the category of spectra in a simple way. For more details, the reader is directed to May et al. [37]. In what follows, let $G$ be a compact Lie group. Whenever the author say $G$-space, he means in fact pointed $G$-space.

**Definition 3.40.** A $G$-universe $U$ is an orthogonal $G$-representation of countable dimension having the following two properties:

(i) For each finite dimensional subrepresentation $V \subset U$, the direct sum of $V$ with itself countably many times, $V^\infty$, also occurs as a subrepresentation in $U$.
(ii) The trivial representation $R$, therefore also $R^\infty$, occurs as a subrepresentation in $U$.

**Definition 3.41.** For $V$ a $G$-representation, denote by $V^+$ its one-point compactification; note that $V^+$ is a $G$-space and call it a representation sphere of $G$. For $X$ a $G$-space, the $V$th suspension of $X$ is the smash product $\Sigma^V X := V^+ \wedge X$. The $V$th loop space of $X$ is the $G$-space $\Omega^V X$ of all maps $V^+ \to X$ with $G$ acting by conjugation.

**Remark 3.42.** There is an adjunction between the suspension functor $\Sigma^V$ and the loop space functor $\Omega^V$ on $G$-spaces. That is to say that, for $G$-spaces $X_1$ and $X_2$, there is a natural bijection between the space of maps $\Sigma^V X_1 \to X_2$ and the space of maps $X_1 \to \Omega^V X_2$.

**Definition 3.43.** A $G$-prespectrum $E$ indexed on the $G$-universe $U$ consists of the following data.
(i) A set of $G$-spaces, $E_V$, one for each finite dimensional subrepresentation $V$ in $\mathcal{U}$.

(ii) A set of $G$-equivariant structure maps, $\sigma_{V,W} : \Sigma^{W-V}E_V \to E_W$, one for each pair of nested finite dimensional subrepresentations, $V \subset W \subset \mathcal{U}$, where $W - V$ denotes the orthogonal complement of $V$ in $W$.

**Definition 3.44.** A map of $G$-prespectra $f$ from a $G$-prespectrum $E$ to a $G$-prespectrum $F$, both indexed on the same universe $\mathcal{U}$, consists of a set of $G$-equivariant maps of $G$-spaces $f_V : E_V \to F_V$, one for each finite dimensional subrepresentation $V \subset \mathcal{U}$, such that the evident diagrams

$$
\begin{align*}
\Sigma^{W-V}E_V & \to E_W \\
\downarrow & \downarrow \\
\Sigma^{W-V}F_V & \to F_W
\end{align*}
$$

all commute for any pair of nested representations $V \subset W \subset \mathcal{U}$. A homotopy between two maps of prespectra $f, g : E \to F$ is a map of prespectra $h : E \wedge I_+ \to F$ where $I_+$ denotes the interval $[0, 1]$ with a disjoint base point added; here, the smash $E \wedge I_+$ between a prespectrum and a space is simply to be interpreted spacewise. A map of prespectra $f$ is called a weak equivalence if all of its constituent maps of spaces, $f_V$, be weak equivalences.

**Definition 3.45.** The suspension prespectrum functor, $\Sigma^\infty_U$, from $G$-spaces to $G$-prespectra indexed on the universe $\mathcal{U}$ is defined by assigning to a $G$-space $X$ the prespectrum $\Sigma^\infty_U X$ consisting of $(\Sigma^\infty_U X)_V := \Sigma^V X$ and structure maps the identity maps.

**Definition 3.46.** Given a $G$-representation $V$ in the universe $\mathcal{U}$ and $X$ a $G$-space, the $V$th desuspension of $X$, denoted $\Sigma^{-V}\Sigma^\infty_U X$, is a $G$-prespectrum indexed on $\mathcal{U}$ defined as follows. For $W \subset \mathcal{U}$, the $W$th space of $\Sigma^{-V}\Sigma^\infty_U X$ is either a single point, in the event that $V$ not be contained in $W$, or it is the $G$-space $\Sigma^{W-V}X$, in the event that $V$ be contained in $W$. Meanwhile, the structure maps $\Sigma^{U-W}\Sigma^{W-V}X \to \Sigma^{U-V}X$ are the evident ones.

**Definition 3.47.** A $G$-prespectrum $E$ indexed on the $G$-universe $\mathcal{U}$ is called a spectrum whenever all the adjoints, $E_V \to \Omega^{W-V}E_W$, to the structure maps be homeomorphisms. The category of $G$-spectra and maps as defined above for prespectra is denoted $G\mathcal{S}\mathcal{U}$. The homotopy category of $G$-spectra is defined as the category with the same objects as $G\mathcal{S}\mathcal{U}$ but with morphisms being the homotopy classes of maps of prespectra; this category is denoted $hG\mathcal{S}\mathcal{U}$. The stable homotopy category of $G$-spectra consists of the category $hG\mathcal{S}\mathcal{U}$ together with formal inverses for all the weak equivalences; this category is denoted $\overline{hG\mathcal{S}\mathcal{U}}$.

**Theorem 3.48.** The forgetful functor from spectra to prespectra has a right adjoint called the spectrification functor.

**Proof.** Vid. May et al. [37, Sect. XII.2]
Definition 3.49. By composing the suspension prespectrum functor, $\Sigma_{iU}^{\infty}$, with the spectrification functor, one obtains a functor from $G$-spaces to $GSU$. This functor shall be called the suspension spectrum functor and shall also be denoted by $\Sigma_{iU}^{\infty}$. Likewise, the desuspension of a $G$-space $X$, $\Sigma^{-V}\Sigma_{iU}^{\infty}X$ can be regarded as being in $GSU$.

Remark 3.50. As the notation suggests, there is a desuspension functor, $\Sigma^{-V}$, defined for all spectra, not just for suspension spectra. However, it is somewhat more subtle to define and the author shall not require it in the present article.

Remark 3.51. The stronger notion of spectra as opposed to prespectra is not so important in the present article because the main desire is simply to be able to perform desuspensions. Nonetheless, it has become standard in the literature to work with spectra because of their ability to classify homology and cohomology theories and also the superior properties that the category of spectra enjoys. Therefore, the author decided to phrase everything in terms of spectra for ease of reference.

Remark 3.52. Notice that the Coulomb gauge $W$ is a $U(1)$-universe isomorphic to $\mathbb{R}^\infty \oplus \mathbb{C}^\infty$. Such an isomorphism can be defined by picking a basis of eigenvectors of $\ell$. Note that this universe does not contain all representations of $U(1)$. Indeed, if $C$ denote the standard representation, where $U(1) \hookrightarrow \mathbb{C}$ is the unit circle, then the tensor product $C \otimes C$ is not in the universe. This choice of universe is compatible with what is done in Manolescu [35], where desuspensions are only allowed with respect to $\mathbb{R}$ and $\mathbb{C}$.

Definition 3.53. When thinking of $W$ as a universe, denote it by $W$.

Definition 3.54. Given an interval $I \subset \mathbb{R}$, use $W^I$ to denote the span of the eigenvectors of $\ell$ with eigenvalues in the interval $I$. Hence, e.g., $W^\mu = W^{(-\mu,\mu)}$.

Theorem 3.55. ([35], Sect. 7) Up to canonical isomorphism in $\bar{h}U(1)SW$, the spectrum

$$\Sigma^{-W^{(-\mu,0)}}\Sigma_{W}^{\infty}I_{U(1)}(S_{\lambda,r}^{\mu},\varphi_{\lambda,r}^{\mu})$$

only depends on $Y$, the Spin$^C$ structure $s_{\lambda}$ and the metric $g$.

A disadvantage of working with the category $\bar{h}U(1)SW$ is that the universe $W$ depends, at least superficially, on the metric $g$. This can be addressed by applying a change of universe to pass over to a standard choice of universe isomorphic to $W$. In order to demonstrate that the choices involved are immaterial to the final result, the author shall invoke one more concept from stable homotopy theory.

Definition 3.56. Given two $G$-universes $U_0$ and $U_1$ and a linear isometry $f : U_0 \to U_1$, define the associated (restrictive) change of universe functor,

$$f^* : \bar{h}GSU_1 \to \bar{h}GSU_0,$$
by defining it on prespectra as the functor that sends a prespectrum $E$ with structure maps $\sigma$ to the prespectrum $f^*E$ with structure maps $f^*\sigma$ such that

$$(f^*E)_V = E_{f(V)}, \quad (f^*\sigma)_V,W = \sigma_{f(V),f(W)}.$$  

Likewise, define the associated (inductive) change of universe functor,

$$f_* : \bar{h}GSU_0 \to \bar{h}GSU_1,$$

by defining it on a prespectrum $E$ with structure maps $\sigma$ to be the prespectrum $f_*E$ with structure maps $f_*\sigma$ satisfying the following. For finite dimensional subrepresentations $V \subset W \subset U_1$, denote $V' := V \cap f(U_0)$ and $W' := W \cap f(U_0)$. Then, set

$$(f_*E)_V := E_{f^{-1}(V')} \wedge (V - V')^+, \quad (f_*\sigma)_V,W = \sigma_{f^{-1}(V'),f^{-1}(W')}.$$  

Proposition 3.57. ([29], Proposition 1.2) $f_*$ is left adjoint to $f^*$.

Theorem 3.58. ([29], Theorem 1.7 and Corollary 1.8) The functors $f^* : \bar{h}GSU_1 \to \bar{h}GSU_0$ defined by different choices of linear isometry $f$ are canonically and coherently isomorphic. The same holds for the functors $f_*$. Moreover, if $f$ be an isomorphism, $f^*$ is an equivalence of categories with its inverse being $f_*$.

Remark 3.59. The author shall not expand on the precise meaning of canonical nor coherent in the present article; for that, the reader is directed to the proof of Lewis Jr., May and Steinberger [29, Theorem 1.7]. Suffice it to say that, by canonical, one means that the choices which appear in the proof are immaterial, and, by coherent, one means that certain diagrams that should commute do indeed commute.

Definition 3.60. Define $\mathcal{U}$ to be the $U(1)$-universe $R^\infty \oplus C^\infty$.

The essential point to be taken from Theorem 3.58 is that, since one may pass between the categories $\bar{h}U(1)SU$ and $\bar{h}U(1)SW$ in a natural fashion, one can think of the desuspension functor $\Sigma^{-W(-\mu,0)}$ as being defined on $\bar{h}U(1)SU$ by intertwining with the changes of universe.

Definition 3.61. Given a finite dimensional subrepresentation $V \subset \mathcal{W}$, define the desuspension functor

$$\Sigma^{-V} : \bar{h}U(1)SU \to \bar{h}U(1)SU$$

as the composite

$$\bar{h}U(1)SU \xrightarrow{f} \bar{h}U(1)SW \xrightarrow{\Sigma^{-V}} \bar{h}U(1)SW \xrightarrow{f^*} \bar{h}U(1)SU,$$

where $f : \mathcal{U} \to \mathcal{W}$ is any isometric isomorphism defined by choosing a basis of eigenvectors for $\ell$. 


Proposition 3.62. The endofunctor $\Sigma^{-V}$ on $\tilde{h}U(1)SU$ is well defined up to canonical isomorphism.

Proof. Immediate from Theorem 3.58. \qed

With this understood, henceforth, the author shall often drop $f^*$ and $f_*$ from the notation.

Definition 3.63. Define the metric dependent Seiberg–Witten Floer spectrum as

$$\text{SWF}(Y, s_\lambda, g) := \Sigma^{-W(-\mu, 0)} \Sigma_{U(1)}^\infty I_{U(1)}(S^\mu_\lambda, \varphi^\mu_\lambda) \approx f^* \Sigma^{-W(-\mu, 0)} \Sigma_{W}^\infty I_{U(1)}(S^\mu_\lambda, \varphi^\mu_\lambda) \in \tilde{h}U(1)SU,$$

where $f : U \to W$ denotes any isometric isomorphism defined by a basis of eigenvectors for $\ell$.

Corollary 3.64. The spectrum $\text{SWF}(Y, s_\lambda, g)$ is well defined up to canonical isomorphism in $\tilde{h}U(1)SU$.

Proof. Corollary of Theorems 3.55 and 3.58. \qed

Next, the author proceeds to explain how to (de)suspend away the metric dependence.

Definition 3.65. Let $X$ be an oriented 4-manifold-with-boundary such that $\partial X = Y$. Given a class $c \in \text{H}^2(X, \mathbb{Z})$ define its square, $c^2 \in \mathbb{Q}$, as follows. Let $c' \in \text{H}^2(X; \mathbb{Q})$ be the image of $c$ under the change of coefficients $\text{H}^2(X; \mathbb{Z}) \to \text{H}^2(X; \mathbb{Q})$. Since $b_2(Y) = 0$, there is an exact sequence

$$\text{H}^2(X; \mathbb{Q}) \to \text{H}^2(X) \to 0.$$

Pick any preimage $\tilde{c} \in \text{H}^2(X, Y; \mathbb{Q})$ for $c'$. Define

$$c^2 := (\tilde{c} \smile \tilde{c})[X, Y] \in \mathbb{Q}.$$

Definition 3.66. Define a number $n(Y, s_\lambda, g) \in \mathbb{Q}$ as follows. Choose some simply-connected 4-manifold-with-boundary $X$ with Spin$^C$ structure $t$ such that $\partial X = Y$ and such that $t$ agree with $s_\lambda$ on $Y$. Assume further that $X$ have a neighbourhood of its boundary which be isometric to $[0, 1] \times Y$. Fixing some connexion $B$ on $\text{det} t$ extending arbitrarily the connexion $A_\lambda$ on $\text{det} s_\lambda$, use $\mathcal{D}_B^+$ to denote the Dirac operators of $(X, t)$. Denote the signature of $X$ by $\sigma(X)$. With this notation in place, let

$$n(Y, s_\lambda, g) := \text{ind}_C \mathcal{D}_B^+ - \frac{1}{8} (c_1(\text{det} t)^2 - \sigma(X)).$$

Proposition 3.67. [35, Sect. 6] The number $n(Y, s_\lambda, g)$ does not depend on the choices involved in its definition. Indeed,

$$n(Y, s_\lambda, g) = \frac{1}{2} \left( \eta(\mathcal{D}_{A_\lambda}, 0) - \dim_{\mathbb{R}} \text{Ker} \mathcal{D}_{A_\lambda} - \frac{1}{4} \eta(\text{Sign}, 0) \right)$$

where $\eta(D, z)$ denotes the $\eta$ function of an operator $D$ (vid. [1]), and $\text{Sign}$ is the operator on $\Omega^1(Y) \oplus \Omega^0(Y)$ given by

$$
\begin{pmatrix}
* d & -d \\
- d^* & 0
\end{pmatrix}.$$
Proposition 3.68. [35, Sect. 7] If $N$ denote the cardinality of the finite set $H_1(Y; \mathbb{Z})$, then $8Nn(Y, s, g)$ is an integer and its residue modulo $8N$ does not depend on $g$.

Definition 3.69. Let $n(Y, s) \in \mathbb{Q}$ be such that $8Nn(Y, s) \in \{0, \ldots, 8N-1\}$ be the residue modulo $8N$ of $8Nn(Y, s, g)$.

Theorem 3.70. [35, Sect. 7] Up canonical to isomorphism in $\bar{h}U(1)SU$, the spectrum

$$\text{SWF}(Y, s, \lambda) := \Sigma C^{n(Y, s)-n(Y, s, \lambda, g)} \text{SWF}(Y, s, \lambda, g)$$

only depends on $Y$ and the $\text{Spin}^C$ structure $s, \lambda$.

Remark 3.71. The author is deviating from what is stated in Manolescu [35] slightly in using the (de)suspension by $C^{n(Y, s)-n(Y, s, \lambda, g)}$ instead of by $C^{-n(Y, s, \lambda, g)}$. The reason being his desire to operate in the standard category of spectra, $U(1)\text{SW}$, which does not allow for a desuspension by a “rational” dimensional representation in any obvious manner. In Manolescu [35], a variant of the Spanier-Whitehead category is used, which can easily be made to formally admit such (de)suspensions.

The main result of Lidman and Manolescu [31] is that the $U(1)$-equivariant Borel cohomology of the SWF spectrum corresponds to the classical monopole Floer cohomology. In order to state this theorem, the author should firstly clarify what is meant by the cohomology of a spectrum in the present context. The rightful sort of $G$-equivariant cohomology theory to consider for $G$-spectra indexed on a universe $\mathcal{V}$ is that of $\text{RO}(G; \mathcal{V})$-graded cohomology (vid. [37, Sect. XIII]). In order to avoid having to deal with such complexities, the author decided to introduce the following language which shall simplify considerably the treatment.

Definition 3.72. Let $h^*_G$ denote a $\mathbb{Z}$-graded $G$-equivariant (reduced) cohomology theory on $G$-spaces. The author shall say that $h^*_G$ satisfies the suspension axiom with respect to the universe $\mathcal{V}$ when, for any $G$-space $X$ and any finite dimensional subrepresentation $V \subset \mathcal{V}$, there be a natural isomorphism

$$h^n_G(X) \cong h^{n+\dim V}_G(\Sigma^V X).$$

Definition 3.73. Given a $G$-prespectrum $E$ indexed on a universe $\mathcal{V}$ and $G$-equivariant cohomology theory $h^*_G$ on $G$-spaces satisfying the suspension axiom with respect to the universe $\mathcal{V}$, define the cohomology of $E$ as

$$h^n_G(E) := \text{Colim}_{V \subset \mathcal{V}} h^{n+\dim V}_G(E_V).$$

Remark 3.74. It is easy to see from this definition that, for a desuspension spectrum $\Sigma^{-V}\Sigma^\infty VX$, the cohomology is simply given by the cohomology of the space $X$ but with a grading shift.

$$h^n_G(\Sigma^{-V}\Sigma^\infty VX) = h^{n+\dim V}_G(X).$$
Definition 3.75. The Borel $G$-equivariant cohomology theory for $G$-spaces with coefficients in an abelian ring $R$ is defined as follows. Denote by $EG$ a free contractible $G$-space and by $BG$ the quotient $EG/G$. Then, for a $G$-space $X$,

$$c\tilde{H}^*_G(X; R) := \tilde{H}^*((EG \times X)/G; R).$$

Remark 3.76. Following May et al. [37], the notation $c$ indicates the “geometric completion” involved in obtaining the underlying spectrum from the equivariant Eilenberg-MacLane spectrum and helps one distinguish Borel from Bredon cohomology.

Proposition 3.77. For a $G$-universe $V$ containing only finite dimensional subrepresentations $V$ such that the vector bundle $(V \times EG)/G$ over $BG$ be $R$-orientable, the Borel cohomology theory with $R$-coefficients satisfies the suspension axiom for $V$.

Proof. This follows directly from the Thom isomorphism theorem. \qed

Remark 3.78. In the case at hand of $G = U(1)$ and universe $U = \mathbb{R}^\infty \oplus \mathbb{C}^\infty$, it is clear, therefore, that the Borel cohomology theory with $\mathbb{Z}$-coefficients satisfies the suspension axiom for $U$ because a direct sum of a complex representation and a trivial real representation always define $\mathbb{Z}$-orientable vector bundles over $BG$. Hence, one can speak of the Borel cohomology of the SWF spectrum.

Theorem 3.79. [31, Theorem 1.2.1] Letting $\widehat{HM}^*(Y, s)$ denote the $\mathbb{Q}$-graded “from” monopole Floer cohomology of Kronheimer and Mrowka [21], there is an isomorphism

$$\widehat{HM}^q(Y, s) \cong c\tilde{H}^{q-n(Y, s)}_{U(1)}(\text{SWF}(Y, s); \mathbb{Z}).$$

In particular, letting the “tilde” monopole Floer cohomology, $\tilde{HM}^*(Y, s)$, be the mapping cone of the $U$-map on $\widehat{HM}^*(Y, s)$, there is an isomorphism

$$\tilde{HM}^q(Y, s) \cong \tilde{H}^{q-n(Y, s)}(\text{SWF}(Y, s); \mathbb{Z}).$$

Definition 3.80. For future reference, define the metric dependent tilde monopole Floer cohomology to be

$$\tilde{HM}^*(X, s, g) := H^*(\text{SWF}(X, s, g); \mathbb{Z}).$$

4. The cohomotopical contact invariant

This section shall fulfil the first goal of the present article. The analytic results concerning the contact monopole $C_\lambda$ shall allow the author to define a cohomotopical contact invariant in a manner that can be roughly outlined in the following way. Given a top dimensional cell in a CW-complex, if one were to quotient the complex by all other cells, one obtains a map to a sphere; this is an element of the cohomotopy of the complex. In the event that the cell not be top dimensional, but, instead, all the higher dimensional cells attach null-homotopically onto the given cell, one can still perform the same
quotient and obtain an element in the cohomotopy set of the complex. In the case of a $G$-CW-complex, a similar story can be told about a $G$-cell; here, one needs to be more careful with what is meant by cohomotopy. In any event, one obtains a map from the $G$-CW-complex to the Thom space of a vector bundle over a $G$-orbit; analogously to how, in the non-equivariant setting, the sphere is the Thom space of a bundle over the orbit of the trivial group; that is, the point.

To achieve this in the present context, the author shall make use of a fundamental construction in the Conley theory; namely, the notion of attractor-repeller pairs. What has already been said about the contact configuration shall be summarised as saying that the orbit of the contact configuration defines a repeller in the isolated invariant set defining the Seiberg–Witten Floer spectrum. Well known results on Conley theory then provide a cofibration involving Conley indices. Due to the non-degeneracy of the contact configuration, the cofibre map of this cofibration can be interpreted, in the presence of a choice of $U(1)$-CW-structure on the Seiberg–Witten Floer spectrum, exactly as the quotient of all but one special $U(1)$-cell defined by the orbit of the contact monopole. This cofibre map shall be declared the contact invariant.

Lastly, one must take care to stabilize everything so as to make the cofibre map really an invariant with respect to the spectral cut-off parameter $\mu$ and the metric $g$. As it turns out, this does not provide any difficulty beyond what was already encountered in Manolescu [35], and the proofs shall follow closely what is said there only with a few extra Conley theoretic inputs concerning the naturality of attractor-repeller pair cofibrations.

Let $Y$, $\lambda$, and $g$ be as in the preceding sections.

**Remark 4.1.** Notice that $C_\lambda$ is, by definition, in the global Coulomb slice with respect to $A_\lambda$; that is, $C_\lambda \in W$. Moreover, since $\mathfrak{D}_{A_\lambda} \psi_\lambda = 0$, it is also true that $C_\lambda \in \text{Ker } \ell$. Therefore, for $R > 0$ sufficiently large, $C_\lambda \in B(W^\mu, R)$ for all $\mu > 0$. Henceforth, agree to set $R > 0$ large enough so that this last inclusion hold.

**Remark 4.2.** It is worth emphasising that the global Coulomb gauge does not fix a gauge with respect to the entire gauge group $G(Y)$ but, rather, with respect to the normed gauge group $G^0(Y)$. Hence, there is a circle’s worth of fixed points of $X_{\lambda,r}^{GC}$ in $W_k$ which are gauge equivalent to $C_\lambda$.

**Definition 4.3.** Denote by $U_\lambda \subset W_k$ the circle of configurations gauge equivalent to $C_\lambda$ and call it the contact circle.

**Definition 4.4.** Use $\mathcal{J}_C$ to denote the tangent space to the $U(1)$-orbit at $C \in W_k$.

**Definition 4.5.** Let $\tilde{g}$ denote the metric on $W_k$ defined by assigning to tangent vectors $(a, \psi), (b, \phi) \in T_C W_k \cong W_k$ the value $\Re \langle \Pi_{C}^{\text{ELC}}(a, \psi), \Pi_{C}^{\text{ELC}}(b, \phi) \rangle$, where $\Pi_{C}^{\text{ELC}}$ is the projection to the enlarged local Coulomb gauge defined in Definition 3.12.
Remark 4.6. The metric $\tilde{g}$ is the one used in many of the technical results of Lidman and Manolescu [31]. It is notable because it turns the Seiberg–Witten vector field $X^{GC}_{\lambda,r}$ in global Coulomb gauge into the $\tilde{g}$-gradient of the CSD functional restricted to $W_k$. In the present situation, it shall be necessary to invoke some of those results of Lidman and Manolescu [31] which make reference to this metric. The $\tilde{g}$ metric leads to the following definition.

Definition 4.7. Define the local anticircular slice in the global Coulomb gauge at $C \in W_k$, denoted $K_{AGC}^{C}$, as the $\tilde{g}$-orthogonal complement to $J^{C}$ in $W_k$. Use $K_{AGC}^{J,C}$ for its Sobolev completion of regularity $j \in \mathbb{Z}$ and use $\Pi^{AGC}_{C} : W_k \to K_{AGC}^{C}$ to denote the $\tilde{g}$-orthogonal projection.

Proposition 4.8. For sufficiently large $r > 0$, at any $C \in U_{\lambda}$, the derivative

$$\Pi^{AGC}_{C} \circ D_{C}X^{\lambda,r}_{\lambda,r} \circ \Pi^{ELC}_{C} : K_{k,C}^{AGC} \to K_{k-1,C}^{AGC}$$

is surjective.

Proof. By gauge equivariance, it suffices to prove the result for $C = C_{\lambda}$. Theorem 2.23 has established that, for sufficiently large $r > 0$, the map

$$\Pi^{LC}_{C_{\lambda}} \circ D_{C_{\lambda}}X_{C_{\lambda}} : K_{C_{\lambda}} \to K_{X_{C_{\lambda}}(C_{\lambda})}$$

is surjective. Hence, the result follows directly from Lidman and Manolescu [31], Lemma 5.6.1, "(ii) $\Rightarrow$ (iv)"; cf. Lidman and Manolescu [31], Formulae (97) and (94).

Remark 4.9. Henceforth, fix $r > 0$ so as to make Proposition 4.8 hold.

Remark 4.10. Bear in mind that the vector space $W^{\mu}$ is the direct sum of a real vector space and a complex vector space, whence comes its $U(1)$-action. Note that $U_{\lambda} \hookrightarrow W^{\mu}$ is the $U(1)$-equivariant embedding of a $U(1)$-manifold.

Definition 4.11. Let $E^{\mu}_{\lambda} \to U_{\lambda}$ denote the $U(1)$-equivariant normal bundle of $U_{\lambda}$ as a submanifold of $W^{\mu}$.

Proposition 4.12. For sufficiently large $\mu > 0$, $U_{\lambda}$ is a hyperbolic fixed set of the flow $\varphi^{\mu}_{\lambda,r}$ in $W^{\mu}$. In other words, for any $C \in U_{\lambda}$, the derivative

$$D_{C}X^{\mu}_{\lambda,r} : E^{\mu}_{\lambda}|_{C} \to E^{\mu}_{\lambda}|_{C}$$

has no eigenvalue with vanishing real part. In particular, $U_{\lambda}$ is a non-degenerate fixed set.

Proof. By Proposition 4.8, $U_{\lambda}$ consists of non-degenerate irreducible fixed points of the flow $X^{GC}_{\lambda,r}$ on $W_k$. Hence, apply the same argument used in the proof of Lidman and Manolescu [31, Proposition 7.3.1], to find that the same remains true when passing to a finite dimensional approximation provided one choose a sufficiently large $\mu$.

Remark 4.13. Fix $\mu > 0$ large enough so as to make Proposition 4.12 hold.
Remark 4.14. Identify $E^\mu_\lambda$ with a sufficiently small tubular neighbourhood of $U_\lambda$ so as to not contain any other fixed points of the flow $\varphi^\mu_{\lambda,r}$. This is possible due to the non-degeneracy ensured by Proposition 4.12. Now, as a vector bundle, one can split $E^\mu_\lambda$ into stable and unstable subbundles as $E^{s,\mu}_\lambda \oplus E^{u,\mu}_\lambda$, where

$$\forall v \in E^{s,\mu}_\lambda, \quad \langle v, D_C X^\mu_{\lambda,r}(v) \rangle \geq m|v|^2,$$

$$\forall v \in E^{u,\mu}_\lambda, \quad \langle v, D_C X^\mu_{\lambda,r}(v) \rangle \leq -m|v|^2$$

for some constant $m > 0$. Moreover, this splitting is preserved by $D_C X^\mu_{\lambda,r}$ and, hence, $U_\lambda$ is an isolated invariant set with index pair $(D(E^\mu_\lambda), S(E^{u,\mu}_\lambda))$ where $D$ and $S$ denote the unit disk and unit sphere bundles.

Definition 4.15. For $E$ a vector $G$-bundle over a compact Hausdorff space, denote its $G$-equivariant Thom space by $\Theta^G_E := D_E/S_E$.

Corollary 4.16. The Conley index $I_{U(1)}(U_\lambda, \varphi^\mu_{\lambda,r})$ is $U(1)$-equivariantly homotopy equivalent to the $U(1)$-equivariant Thom space $\Theta_{U(1)}(E^{u,\mu}_\lambda)$ of the bundle $E^{u,\mu}_\lambda \to U_\lambda$.

As a direct consequence, by using a Morse decomposition, one finds that the contact circle $U_\lambda$ defines a $U(1)$-cell in some $U(1)$-CW-complex decomposition of the space $I_{U(1)}(S^\mu_{\lambda,r}, \varphi^\mu_{\lambda,r})$. If one were to add a generic perturbation on top of the canonical perturbation in use, one would find that a possible choice of $U(1)$-CW-decomposition would have a one to one correspondence between its $U(1)$-cells and the set of fixed points and fixed circles of the Seiberg–Witten flow. It is preferable, of course, to avoid using such a generic perturbation. In order to derive a cohomotopical invariant from a given cell, it is necessary to establish something about the attaching maps of the higher dimensional cells. In particular, if the given cell be itself top dimensional or, more generally, if one find that all higher dimensional cells attach null-homotopically onto the given cell, then it follows that one can quotient all but the given cell in the $U(1)$-CW-complex and obtain a map to the quotient of the contact cell by its boundary. This is morally the strategy which shall be pursued next.

Theorem 4.17. For sufficiently large $r > 0$ and $\mu > 0$, there are no non-constant approximate Seiberg–Witten trajectories in the set $S^\mu_{\lambda,r}$ which have plus infinity limit in the contact circle $U_\lambda$.

Proof. The author starts with the argument in Step 1 of the proof of Proposition 3 in Manolescu [35]. Suppose the result not hold. Then, there exists an increasing sequence $\mu_n \to \infty$ and a sequence of non-constant trajectories $\gamma_n : \mathbb{R} \to D(W^{\mu_n}, 2R)$ satisfying

$$\frac{\partial}{\partial t} \gamma_n(t) = -X^\mu_{\lambda,r}(\gamma_n(t)), \quad \lim_{t \to \infty} \gamma_n(t) = C_\lambda.$$

Notice that there is a bound

$$\left\| \frac{\partial}{\partial t} \gamma_n(t) \right\|_{L^2_{k-1}} \leq \| \ell \gamma_n(t) \|_{L^2_{k-1}} + \| p^\mu c \gamma_n(t) \|_{L^2_{k-1}} \leq KR.$$
where $K > 0$ is a constant independent of both $n$ and $t$. This implies that the set of functions
\[ \{ \gamma_n : \mathbb{R} \to D(W_{k-1}, 2R) \} \]
is equicontinuous. By use of the Arzelà-Ascoli theorem, one can replace this sequence by a subsequence which converge to some $\gamma : \mathbb{R} \to D(W_{k-1}, 2R)$ in the compact-open topology. Now, due to compactness of $c$, the sequence of operators $(1 - p^\mu_n)c : W_k \to W_{k-1}$ converges to zero weakly. Given this, observe that
\[ \frac{\partial}{\partial t} \gamma_n = -(\ell + c)\gamma_n + (1 - p^\mu_n)c\gamma_n \xrightarrow{n \to \infty} -(\ell + c)\gamma. \]
uniformly as functions from compact subsets of $\mathbb{R}$ to $W_{k-1}$. Hence,
\[ \gamma_n(t) - \gamma_n(0) = \int_0^t \frac{\partial}{\partial s} \gamma_n(s)ds \xrightarrow{n \to \infty} -\int_0^t (\ell + c)\gamma(s)ds. \]
Together, these two last assertions imply that
\[ \frac{\partial}{\partial t} \gamma(t) = -(\ell + c)\gamma(t). \]
Moreover, observe that $\gamma(t) \xrightarrow{t \to \infty} U_\lambda$. Therefore, $\gamma$ is the Coulomb projection of a Seiberg–Witten trajectory with positive infinity limit gauge equivalent to the contact configuration. By Theorem 2.25, such a trajectory must be constant. Let $C_n := \lim_{t \to -\infty} \gamma_n(t)$. By assumption, these cannot be equal to $C_\lambda$. Note also that $\lim_{n \to \infty} C_n = \lim_{t \to -\infty} \gamma(t) = C_\lambda$. However, Proposition 4.8 combined with Lidman and Manolescu [31], Proposition 7.2.2, guarantee that, for sufficiently large $n$ and some small neighbourhood $N \supset W_k$ of $C_\lambda$, there cannot be any solution to $X^\mu_{\lambda,r} = 0$ inside of $N$ other than $C_\lambda$ itself, thereby contradicting convergence of the sequence $\{C_n\}$ to $C_\lambda$. \qed

A few more concepts from Conley theory shall be needed next. In what follows, suppose that $G$ be a compact Lie group, $\Gamma$ be a locally compact Hausdorff space with a continuous $G$-action and $\phi : \Gamma \times \mathbb{R} \to \Gamma$ a continuous and equivariant flow.

**Definition 4.18.** For a $G$-invariant subset $T \subset \Gamma$, define its $\omega$-limit, $\omega(T)$, as the maximal invariant set of the closure of $\varphi(T \times [0, \infty))$. Likewise, define its $\omega^*$-limit, $\omega^*(T)$, as the maximal invariant set of the closure of $\varphi(T \times (-\infty, 0])$.

**Definition 4.19.** A $G$-invariant subset $A \subset S$ is an **attractor** when there is a neighbourhood $U \subset S$ of $A$ as a subspace of $S$ such that $A = \omega(U)$. Likewise, a $G$-invariant subset $R \subset S$ of an isolated invariant set $S \subset \Gamma$ is called a **repeller** when there is a neighbourhood $U \subset S$ of $R$ as a subspace of $S$ such that $R = \omega^*(U)$.

**Definition 4.20.** Given an attractor $A \subset S$ in the isolated invariant set $S \subset \Gamma$, one defines its complementary repeller, $A^* \subset S$, as the set $\{x \in S \mid \omega(\{x\}) \cap A = \emptyset\}$. Given a repeller $R \subset S$, one defines its complementary attractor, $R^* \subset S$, similarly.
Definition 4.21. An attractor-repeller pair \((A, R)\) of an isolated invariant set \(S \subset \Gamma\) consists of an attractor \(A\) and a repeller \(R\) in \(S\) such that \(A = R^*\), or, equivalently, \(R = A^*\).

Definition 4.22. Let \((A, R)\) be an attractor-repeller pair for of an isolated invariant set \(S \subset \Gamma\). A triple of \(G\)-invariant compact sets \((L, M, N)\), \(N \subset M \subset L \subset \Gamma\), is called an index triple for \((A, R)\) when \((L, M)\) is an index pair for \(R\), \((L, N)\) is an index pair for \(S\) and \((M, N)\) is an index pair for \(A\).

Theorem 4.23. ([6], Sect. I.7, albeit non-equivariantly) For any attractor-repeller pair \((A, R)\) of an isolated invariant set \(S \subset \Gamma\), there exists an index triple \((L, M, N)\) for it and the induced cofibration,

\[ I_G(A, \phi) \to I_G(S, \phi) \to I_G(R, \phi), \]

is independent of the choice of \((L, M, N)\) up to \(G\)-equivariant homotopy.

Now, one can reinterpret the analytic results concerning the contact circle in the Conley theoretic language.

Theorem 4.24. The contact circle \(U_\lambda \subset S_{\lambda, r}^\mu\) is a repeller in \(S_{\lambda, r}^\mu\). Hence, there exists a cofibration

\[ I_{U(1)}(U^\mu_\lambda, \varphi^\mu_{\lambda, r}) \to I_{U(1)}(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r}) \to I_{U(1)}(U_\lambda, \varphi^\mu_{\lambda, r}). \]

Proof. Follows from Theorems 4.17 and 4.23.

Definition 4.25. Define the spectral cut-off and metric dependent cohomotopical contact invariant as the cofibre map

\[ \Psi(\lambda, g, \mu) : I_{U(1)}(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r}) \to I_{U(1)}(U_\lambda, \varphi^\mu_{\lambda, r}) \]

of Theorem 4.24.

Proposition 4.26. \(\Psi(\lambda, g, \mu)\) does not depend on the choice of \(r > 0\).

Proof. Suppose one chose two sufficiently large values for \(r\), call them \(r_0 < r_1\). Without loss of generality, assume \(|r_0 - r_1|\) to be small. Pick the value of \(R > 0\) large enough so that it satisfy Theorem 3.28 for both \(r = r_0\) and \(r = r_1\). Then, \(D(W^\mu, 2R)\) serves as an isolating neighbourhood for all the isolated invariant sets \(S^\mu_{\lambda, r}\) under the parametrised family of flows \(\varphi^\mu_{\lambda, r}\) as \(r\) varies in \([r_0, r_1]\). The continuation properties of Conley theory then provide a diagram of the form

\[ I_{U(1)}(S^\mu_{\lambda, r_0}, \varphi^\mu_{\lambda, r_0}) \to I_{U(1)}(U_\lambda, \varphi^\mu_{\lambda, r_0}) \]

\[ \downarrow \]

\[ I_{U(1)}(S^\mu_{\lambda, r_1}, \varphi^\mu_{\lambda, r_1}) \to I_{U(1)}(U_\lambda, \varphi^\mu_{\lambda, r_1}) \]

which commutes up to homotopy and where the vertical arrows are homotopy equivalences (cf. [6, Sect. III.3.1], and [23] for the non-equivariant case).

Definition 4.27. The metric dependent contact Thom space is the desuspension

\[ T(\lambda, g) := \Sigma^\infty_{U} \Sigma^\infty_{U^0} I_{U(1)}(U_\lambda, \varphi^\mu_{\lambda, r}) \in \hU(1)SU. \]
Definition 4.28. The *metric dependent cohomotopical contact invariant*,
\[ \Psi(\lambda, g) : \text{SWF}(Y, s_\lambda, g) \to T(\lambda, g), \]
is the desuspension \( \Sigma^{-W(-\mu, 0)} \Psi(\lambda, g, \mu) \) as a morphism in \( \bar{h}\text{U}(1)S\text{U} \).

Proposition 4.29. \( \Psi(\lambda, g) \) does not depend on the choice of \( \mu > 0 \).

Proof. This proof shall start by recalling the setup of the first half of the proof of Theorem 1 of Manolescu [35]. Assume two values for \( \mu \) be given; call them \( 0 < \mu_0 < \mu_1 \) and assume both be large enough so as to satisfy Theorem 3.33. Consider \( \mu \in [\mu_0, \mu_1] \). Denote \( \bar{\varphi}_{\lambda, r}^\mu \), the flow of the vector field 
\[ -(\ell + p^\mu \cdot c p^\mu) \] 
on \( W^{\mu_1} \). It is easy to check that, for any \( \mu \in [\mu_0, \mu_1] \),
all finite type trajectories of \( -(\ell + p^\mu \cdot c p^\mu) \) on \( W^{\mu_1} \) are, in fact, contained in \( W^\mu \subset W^{\mu_1} \). As a consequence, notice that the set \( S^{\mu}_{\lambda, r} \subset W^{\mu} \) can be identified with the union of the finite type trajectories of \( \bar{\varphi}_{\lambda, r}^\mu \) contained in \( B(W^{\mu_1}, R) \). Hence, abuse notation and write \( S^{\mu}_{\lambda, r} \subset W^{\mu_1} \) for all \( \mu \in [\mu_0, \mu_1] \).
Moreover, in this description, \( D(W^{\mu_1}, 2R) \) is an isolating neighbourhood for \( S^{\mu}_{\lambda, r} \) for all \( \mu \in [\mu_0, \mu_1] \). By the continuation properties of the Conley index,
\[ I_{U(1)}(S^{\mu_1}_{\lambda, r}, \varphi_{\lambda, r}^{\mu_1}) \simeq I_{U(1)}(S^{\mu_0}_{\lambda, r}, \varphi_{\lambda, r}^{\mu_0}) \]
Write \( W^{\mu_1} = W^{\mu_0} \oplus W' \), where \( W' \) is \( L^2 \)-orthogonal to \( W^{\mu_0} \). Of course, \( W' \)
is simply the span of the eigenvectors of \( \ell \) with eigenvalues in \( (-\mu_1, -\mu_0) \cup [\mu_0, \mu_1] \). Use \( D \subset W' \) to denote a small disk around the origin. Then, if one

care to check, one finds that
\[ D(W^{\mu_0}, 3R/2) \times D \subset W^{\mu_1} \]
is also an isolating neighbourhood for \( S^{\mu_0}_{\lambda, r} \). Furthermore, with respect to this product, one can show that the flow \( \bar{\varphi}_{\lambda, r}^\mu \) is homotopic to a product flow \( \varphi^{\mu_0} \times \bar{r} \), where \( \bar{r} \) is the flow on \( D \) induced by \( -\ell \).
Notice that
\[ I_{U(1)}(\{0\}, \bar{r}) = \left(W(-\mu_1, -\mu_0)\right)^+. \]
By the behaviour of the Conley index under product flows, it follows that
\[ I_{U(1)}(S^{\mu_1}_{\lambda, r}, \varphi_{\lambda, r}^{\mu_1}) \simeq I_{U(1)}(S^{\mu_0}_{\lambda, r}, \varphi_{\lambda, r}^{\mu_0}) \wedge \left(W(-\mu_1, -\mu_0)\right)^+. \]
Now, focus is turned to the contact circle. Recall the assumption from Remark 4.1 that \( \mu_0, \mu_1 \) be large enough so that \( U_{\lambda} \subset B(W^{\mu_0}, R) \subset B(W^{\mu_1}, R) \).
As above, observe that
\[ I_{U(1)}(U_{\lambda}, \varphi_{\lambda, r}^{\mu_1}) \simeq I_{U(1)}(U_{\lambda}, \varphi_{\lambda, r}^{\mu_0}). \]
On the other hand, considering again the product flow \( \varphi_{\lambda, r}^{\mu_0} \times \bar{r} \), one finds
\[ I_{U(1)}(U_{\lambda}, \varphi_{\lambda, r}^{\mu_1}) \simeq I_{U(1)}(U_{\lambda}, \varphi_{\lambda, r}^{\mu_0}) \wedge \left(W(-\mu_1, -\mu_0)\right)^+. \]
This leads to a diagram of the form
\[ I_{U(1)}(S^{\mu_0}_{\lambda, r}, \varphi_{\lambda, r}^{\mu_0}) \wedge \left(W(-\mu_1, -\mu_0)\right)^+ \longrightarrow I_{U(1)}(U_{\lambda}, \varphi_{\lambda, r}^{\mu_0}) \wedge \left(W(-\mu_1, -\mu_0)\right)^+ \]
\[ \downarrow \]
\[ I_{U(1)}(S^{\mu_1}_{\lambda, r}, \varphi_{\lambda, r}^{\mu_1}) \longrightarrow I_{U(1)}(U_{\lambda}, \varphi_{\lambda, r}^{\mu_1}) \]
which, due to the naturality of the attractor-repeller cofibration under continuation, commutes up to homotopy. Desuspending everything as needed provides the required invariance. □

Definition 4.30. Define the contact Thom space as the desuspension
\[ \mathcal{T}(\lambda) := \Sigma C^{n(Y, s_\lambda)} - n(Y, s_\lambda, g) \mathcal{T}(\lambda, g) \in \tilde{h}U(1)SU. \]

Definition 4.31. Define the cohomotopical contact invariant of the contact rational homology sphere \((Y, \lambda)\),
\[ \Psi(\lambda) : \text{SWF}(Y, s_\lambda) \to \mathcal{T}(\lambda), \]
to be the (de)suspension \( \Sigma C^{n(Y, s_\lambda)} - n(Y, s_\lambda, g) \Psi(\lambda, g) \) as a morphism of \( \tilde{h}U(1)SU. \)

Proposition 4.32. \( \Psi(\lambda) \) does not depend on the choice of metric \( g \).

Proof. Again, the proof shall start by recalling what is said by Manolescu in Manolescu [35] and then extending the argument to deal with the contact circle. Since the space of compatible metrics is connected, it suffices to prove the result for nearby metrics, so consider two such metrics \( g_0, g_1 \) and a smooth path \( t \mapsto g_t \) interpolating them. One can choose \( \mu > 0 \) and \( R > 0 \) large enough so as to satisfy the usual requirements for all metrics along the path \( g_t \). The author shall use a subscripted \( t \) to denote the versions of objects constructed with the metric \( g_t \); for example, \( W^\mu_t \) denotes the version of \( W^\mu \) constructed with \( g_t \). Notice that, perhaps after increasing \( \mu \) slightly, one can assume that \( \mu \) not be an eigenvalue of \( \ell_t \) for any \( t \in [0, 1] \). As a consequence, the spaces \( W^\mu_t \) have constant dimension as \( t \) varies. This means the spaces \( W^\mu_t \) form a vector bundle over \([0, 1]\) and, therefore, the vector spaces \( W^\mu_t \) for different values of \( t \) may be identified via a trivialisation of this bundle; hence, use \( W^\mu \) to denote any of the spaces \( W^\mu_t \). For different values of \( t \), consider the balls \( B(W^\mu, R)_t \) all as subsets of this same \( W^\mu \). By assuming the metrics \( g_0, g_1 \) sufficiently close to one another, one also finds that, for any \( t_1, t_2 \in [0, 1] \), \( B(W^\mu, R)_{t_1} \subset B(W^\mu, 2R)_{t_2} \). From this, it follows that
\[ \bigcap_{t \in [0, 1]} D(W^\mu, 2R)_t \]
is an isolating neighbourhood for \((S^\mu_{\lambda, r})_t \) with respect to the flow \( (\varphi^\mu_{\lambda, r})_t \) for all \( t \in [0, 1] \). The flow \( (\varphi^\mu_{\lambda, r})_t \) varies continuously with \( t \in [0, 1] \); hence, by Conley theory,
\[ I_{U(1)} \left( (S^\mu_{\lambda, r})_0, (\varphi^\mu_{\lambda, r})_0 \right) \simeq I_{U(1)} \left( (S^\mu_{\lambda, r})_1, (\varphi^\mu_{\lambda, r})_1 \right). \quad (*) \]
Consider now the contact circle. One easily checks that, under the change of metric, the contact circle, \((U_{\lambda})_t \), moves smoothly in \( W^\mu \) (it is not fixed under changes of metric because it depends on the global Coulomb projection). By assuming the metrics to be sufficiently close, one also sees that
\[ \bigcap_{t \in [0, 1]} D((E^\mu_\lambda)_t) \]
is an isolating neighbourhood for all \((U_\lambda)_t\), where \((E^u_\lambda)_t \to (U_\lambda)_t\) is the \(g_t\) version of the tubular neighbourhood introduced in Remark 4.14. Therefore,

\[
I_{U(1)}\left((U_\lambda)_0, (\varphi^\mu_{\lambda,r})_0\right) \simeq I_{U(1)}\left((U_\lambda)_1, (\varphi^\mu_{\lambda,r})_1\right),
\]

which is not much to say due to the characterisation of these as Thom spaces in Corollary 4.16; however, the fact that both this homotopy equivalence and the one of \((\ast)\) come from the same deformation of the flow allows one to use the naturality of the attractor-repeller cofibration sequence so as to have the following diagram commute up to homotopy.

\[
\begin{array}{ccc}
I_{U(1)}\left((S^\mu_{\lambda,r})_0, (\varphi^\mu_{\lambda,r})_0\right) & \longrightarrow & I_{U(1)}\left((U_\lambda)_0, (\varphi^\mu_{\lambda,r})_0\right) \\
\downarrow & & \downarrow \\
I_{U(1)}\left((S^\mu_{\lambda,r})_1, (\varphi^\mu_{\lambda,r})_1\right) & \longrightarrow & I_{U(1)}\left((U_\lambda)_1, (\varphi^\mu_{\lambda,r})_1\right).
\end{array}
\]

Now, the effect of the desuspensions shall be addressed. Consider the subspace \((W(-\mu,0))_t \subset W^\mu\). Note that, despite all the \((W^\mu)_t \cong W^\mu\) being identified, the subspaces \((W(-\mu,0))_t\) shall still vary with \(t \in [0,1]\). Recall that \(W\) is defined as the direct sum of a real and a complex space; use \(n_{\mu,t}\) to denote the complex dimension of this complex summand appearing in \((W(-\mu,0))_t\). Notice that \(n_{\mu,1} - n_{\mu,0}\) is the spectral flow of the family of Dirac operators \((D_A^\lambda)_t\) defined as the metric \(g_t\) varies. One can then check with Proposition 3.67 that

\[
n(Y, s_\lambda, g_0) - n(Y, s_\lambda, g_1) = n_{\lambda,1} - n_{\lambda,0}.
\]

Without loss of generality, suppose \(n(Y, s_\lambda, g_0) \leq n(Y, s_\lambda, g_1)\). Together with the fact that the operator family \(*_{\ast,1} : \Omega^1(Y) \to \Omega^1(Y)\) has zero spectral flow due to \(H_1(Y; \mathbb{R}) = 0\), the above implies that

\[
(W(-\mu,0))_0 \cong (W(-\mu,0))_1 \oplus \mathbb{C}^{n(Y, s_\lambda, g_1) - n(Y, s_\lambda, g_0)}.
\]

The result follows by combining this with the commuting diagram above. \(\square\)

**Remark 4.33.** In the same vein as in Manolescu [35], the metric invariance can be strengthened to invariance up to canonical isomorphism, which is to say, in this context, that the isomorphism does not depend on the path of metrics interpolating the given two metrics, but the details shall be left out.

## 5. Recovery of the cohomological invariant

In light of the equivalence, proved by Lidman and Manolescu [31], between the Borel U(1)-equivariant cohomology of the Seiberg–Witten Floer spectrum and the monopole Floer “from” cohomology, this section shall discuss the relation between the cohomotopical contact invariant defined in the previous section and the well known contact invariants in Floer cohomology.

In Kronheimer et al. [22, Sect. 6.3], a distinguished element of the monopole Floer cohomology group \(\widehat{HM}^*(Y, s_\lambda)\), therein denoted \(\tilde{\psi}\), is defined (up to sign) from the datum of a contact structure \(\text{Ker} \lambda\). Here, this class shall
be denoted $\psi(\lambda)$. In fact, most of the groundwork for the definition of this invariant was done a decade earlier in Kronheimer and Mrowka [20], except that the machinery of monopole Floer homology had not yet been developed. This same invariant was studied in Taubes [51, Sect. 4], therein denoted $t_r$, and was shown [51, Proposition 4.3] to be generated by a single generator of the monopole Floer cochain complex. This generator is essentially the contact configuration, herein denoted $C_\lambda$, with the caveat that a generic perturbation must be used in that context, else the monopole Floer cohomology groups cannot be defined.

By work of Taubes [52–56], it was established that there is a natural equivalence between the monopole Floer cohomology $\widehat{HM}^*(Y, s_\lambda)$ and the embedded contact homology $ECH_*(Y, \lambda; 0)$ of Hutchings (vid. [17]). Furthermore, in ECH, there is a very simply defined contact invariant, which is the class generated by the empty set of Reeb orbits. In Taubes [56], Taubes established that, under his isomorphism, the ECH contact invariant corresponds to $\psi(\lambda)$.

There is yet another guise under which $\psi(\lambda)$ appears. In Kutluhan et al. [24–28], the authors construct isomorphisms between a variant of the monopole Floer homologies, called the balanced monopole Floer homologies, and the Heegaard Floer homologies of Ozsváth and Szabó [42]; in case $b_1 = 0$, these balanced monopole Floer homologies agree with the usual monopole Floer homologies. In [43], an invariant of contact structures is defined (up to sign) which lives in the group $HF^+(-Y, s_\lambda)$; this is usually denoted $c^+(Y, \lambda)$. This invariant was identified with the ECH version of the contact invariant in Colin et al. [4,5].

The manner via which the cohomotopical contact invariant $\Psi(\lambda)$ recovers the cohomological invariant $\psi(\lambda)$ is fundamentally rather simple. Returning to the motif of the collapse of all but a single cell in a CW-complex, one can obtain a class in the cohomology of that complex by pulling back the generator of the cohomology of the sphere. In the case of a U(1)-CW-complex, as has been said in the preceding section, the same construction leads to a map from the complex to the Thom space of a vector bundle over an orbit space. One then proceeds by understanding the Borel cohomology of the Thom space as being generated by an equivariant Thom class and the pullback of this Thom class provides a cohomological invariant. Since the classical cohomological contact invariant is the class in monopole Floer cohomology defined by the cochain consisting only of the generator associated to the contact monopole, it is not at all surprising that this approach shall work.

A quick review of the pertinent definitions is in line. In what follows, let $G$ denote a compact Lie group.

**Definition 5.1.** Suppose a $G$-equivariant (reduced) cohomology theory $h_G^*$ be given. By a $G$-equivariant Thom class, one means, for a $G$-vector bundle $V \to X$, a class $\theta_G(V) \in h_G^*(\Theta_G(V))$, where $\Theta_G(V)$ denotes the Thom space, subject to the requirement that, given the inclusion of any $G$-orbit
\[ i : \mathcal{O} \rightarrow X \], the restriction \( i^* \theta_G(V) \) generate \( h^*_G(O_+(i^*V)) \) as a free \( h^*_G(O_+) \)-module, where \( O_+ \) denotes the orbit space \( O \) with a disjoint base point added.

**Remark 5.2.** Specialising this definition to when \( G \) act trivially on the base space \( X \), the only type of orbit is \( i : G/G \rightarrow X \), so the requirement is that \( i^* \theta_G(V) \) generate \( h^*_G(\Theta_G(V|_p)) \) as a free \( h^*_G(S^0) \)-module.

**Remark 5.3.** In the other extreme, specialising to when \( G \) act freely on \( X \), the only type of orbit is \( i : G/1 \rightarrow X \), so the requirement is that \( \theta_G(V) \) generate \( h^*_G(\Theta_G(V)) \) as a free \( h^*_G(X_+) \)-module.

**Remark 5.4.** It is not at all certain if, given a bundle, such an equivariant Thom class exists, or even if the prerequisite that \( h^*_G(\Theta_G(V|_p)) \) be a free \( h^*_G((G/H)_+) \)-module with a single generator is satisfied. In the event that it exist, one shall say that \( V \) is \( h^*_G \)-orientable.

**Remark 5.5.** Note that, for \( G = U(1) \) acting on \( X \cong U(1) \) freely, there always exists a Thom class for any \( U(1) \)-bundle \( E \rightarrow X \).

**Definition 5.6.** Use \( \theta_\lambda \in c\tilde{H}_U^*(T(\lambda)) \) to denote the desuspension of the Thom class \( \theta_U(E^\mu_{\lambda,u}) \) suitably suspended or desuspended according to Definition 4.31, where \( E^\mu_{\lambda,u} \rightarrow U_\lambda \) is the unstable bundle of \( U_\lambda \) in \( W^\mu \).

**Remark 5.7.** Of course, this is only defined up to sign which shall be consistent with the familiar sign ambiguity of the contact invariant in monopole Floer cohomology (cf. [34]).

**Remark 5.8.** In order to draw the comparison between the cohomological and cohomotopical invariants, one must work in the presence of an appropriate generic perturbation as that is how the cohomological invariant is defined in Kronheimer et al. [22] and Taubes [51]. It is not difficult to see that the construction of the invariant \( \Psi(\lambda) \) is not affected by the addition of a small generic perturbation. Indeed, in the case of a small generic perturbation \( p \), there still is a distinguished solution nearby \( C_\lambda \) and unique in that it satisfies the lower bound in its spinor component as in Theorem 2.24. Call this solution \( C_\lambda(p) \). Furthermore, \( C_\lambda(p) \) still satisfies the conclusion of Theorem 2.25; that is, there is no perturbed Seiberg–Witten trajectory whose forward limit be \( C_\lambda(p) \). This follows from the compactness results in Kronheimer and Mrowka [21], Chapter 16. The precise details shall be left out, but the idea is as follows. If this were not true for generic perturbations of arbitrarily small norm, one could form a sequence of such perturbations converging to zero, together with respective perturbed Seiberg–Witten trajectories; however, the compactness properties of the Seiberg–Witten map lead to a convergent subsequence; the limit of this subsequence provides a non-trivial Seiberg–Witten trajectory into the contact configuration thereby contradicting Theorem 2.25. For a detailed discussion of the generically perturbed case, vid. Roso [47, Sect. 5].

**Theorem 5.9.** The cohomological contact invariant is recovered by the cohomotopical invariant via

\[ \pm \psi(\lambda) = \Psi(\lambda)^* (\theta_\lambda). \]
Proof. The author shall begin by recalling the setup used in Lidman and Manolescu [31] to equate the cohomology of the SWF spectrum and the monopole Floer cohomology. To avoid potential confusion, the reader is cautioned that, in Lidman and Manolescu [31], the letter $\lambda$ is used in the place where $\mu$ is in the present article. According to Lidman and Manolescu [31], Chapter 14, one can find a perturbation $p$ with the properties that it be “very tame” (vid. [31, Definition 4.4.2]), “admissible” (vid. [31, Definition 4.5.8] or [21, Definition 22.1.1]) and that it satisfy the conclusions of Lidman and Manolescu [31, Proposition 7.4.1, Proposition 8.0.1, and Proposition 10.0.2].

Denote by $\text{SWF}_p(Y, s)$ the version of the Seiberg–Witten Floer spectrum constructed with the finite dimensional approximations to the generically perturbed Seiberg–Witten vector field $X_{GC, \lambda, r}^p := X_{GC, \lambda, r} + \Pi_{\ast}^{GC} p$.

Then, according to Lidman and Manolescu [31, Proposition 6.1.6], the spectra $\text{SWF}_p(Y, s_\lambda)$ and $\text{SWF}(Y, s_\lambda)$ are $U(1)$-equivariantly stably homotopy equivalent. The point of using $p$ is that the cohomology of $\text{SWF}_p(Y, s_\lambda)$ may be computed via a Morse complex technique which allows one to equate it to $\hat{\text{HM}}^\ast(Y, s_\lambda)$.

To be more precise, define the anti-circular global Coulomb projection of the Seiberg–Witten vector field as $X_{AGC, \lambda, r}^p := \Pi_{K_{AGC}} k X_{GC, \lambda, r}^p$, where $\Pi_{K_{AGC}}$ denotes the pointwise $\tilde{g}$-orthogonal projection to the local slices $K_{k, C}^{AGC} \subset W_k$ (vid. Definitions 4.5, 4.7). One can consider finite dimensional approximations of this vector field by setting $X_{AGC, \lambda, r; p}^\mu := \mu^\nu X_{AGC, \lambda, r; p}$.

Then, one can extend this to a vector field on $(W^\mu)^\sigma$; that is, the blow-up of $W^\mu$ at the reducibles. Since $U(1)$-acts freely on the blow-up, one can quotient it and consider $(W^\mu)^\sigma / U(1)$. The vector field $X_{AGC, \lambda, r; p}^\mu$ then uniquely defines a vector field on $(W^\mu)^\sigma / U(1)$ which shall be denoted here by $[X_{AGC, \lambda, r; p}^\mu]^{\sigma}$. One can form a Morse complex $(\hat{C}_\mu, \hat{\partial}_\mu)$, where the abelian group $\hat{C}_\mu$ is generated by the stationary points of $[X_{AGC, \lambda, r; p}^\mu]^{\sigma}$ inside $B(W^\mu, 2R)^\sigma / U(1)$, and the boundary map $\hat{\partial}_\mu$ is defined by a signed count of trajectories of $[X_{AGC, \lambda, r; p}^\mu]^{\sigma}$ having index 1.

According to Lidman and Manolescu [31, Equation (272)], the cohomology $\hat{\text{HM}}^i(U(1))$ ((SWF$p(Y, s_\lambda)$) is identified with the cohomology of the Morse complex $(\hat{C}_\mu, \hat{\partial}_\mu)$ within a grading range $i \in \{-M_\mu, \ldots, M_\mu\}$, where $M_\mu \in \mathbb{Q}_{>0}$ is a certain number which is shown to go to infinity as the spectral cut-off parameter $\mu$ goes to infinity. On the other hand, Lidman and Manolescu [31, Equation (277)], asserts that the monopole Floer cohomology $\hat{\text{HM}}^i(Y, s_\lambda)$ is also computed from the same complex $(\hat{C}_\mu, \hat{\partial}_\mu)$ within some other grading range $i \in \{-N + 1, \ldots, N - 1\}$. This is done by directly identifying the monopoles in the grading range $\{-N + 1, \ldots, N - 1\}$ with the fixed points.
of the vector field $[X^\Lambda_{\lambda, r, p}^\mu]^\sigma$ (vid. [31, Proposition 9.3.1]) and by identifying the trajectories of spectral flow 1 counted in the differential of the monopole Floer cochain complex with those counted by $\partial_\mu$ (vid. [31, Chapter 13]). One then makes sure to pick $\mu$ so that $M_\mu \geq N - 1$ and so that the grading range $\{-N + 1, \ldots, N - 1\}$ cover at least all the irreducible monopoles.

Consider now how the contact invariant $\psi(\lambda)$ manifests itself in this picture. Firstly, note that the gauge equivalence class $[C_\lambda(p)]$ is an irreducible monopole; therefore, it is covered by the chain level identification above. It is established in Taubes [51, Proposition 4.3], that, perhaps after yet another increase to the parameter $r$, the cohomological invariant may be described as the class in $\hat{HM}^*(Y, s_\lambda)$ given by the cochain consisting only of the generator $[C_\lambda(p)]$. Hence, in the cohomology of the Morse complex $(\hat{C}_\mu, \hat{\partial}_\mu)$, the element $\psi(\lambda)$ is represented by the gauge equivalence class $[C_\lambda(p)]$. On the other hand, the cohomology class $\Psi(\lambda)^*(\theta_\lambda)$ is precisely defined so that it be generated by the class $[C_\lambda(p)]$ as well; this is because $\Psi(\lambda)$ is the quotient map of the quotient of $\text{SWF}_p(Y, s_\lambda)$ by every cell other than the distinguished $U(1)$-cell associated to $[C_\lambda(p)]$ (vid. [47, Sect 5]). □

Given Theorem 5.9, the author intends to use the newly constructed invariant to deduce results about $\psi(\lambda)$ in contexts in which it has proven difficult to do so while relying solely on the machinery of monopole Floer homology, Heegaard Floer homology and embedded contact homology. Note that it is not clear, at the moment, if there is any case in which $\Psi(\lambda)$ may hold any more information than $\psi(\lambda)$ does. The key advantage of $\Psi(\lambda)$ that the author wishes to emphasise is that it does not require a generic perturbation for its definition. This shall be exploited in the next section.

6. Finite coverings

In Lidman and Manolescu [30], the authors studied the Seiberg–Witten Floer spectrum in the presence of a finite regular covering. The key to their results was the observation that the spectrum of the manifold upstairs in the covering acquires an action of the deck transformation group $G$ and, upon taking appropriate fixed points of this action, one obtains the downstairs spectrum. A Smith-type inequality is then derived through actual application of Smith theory.

In this section, the author’s goal is to formulate the contact invariant in this same scenario of a finite regular covering. This shall involve studying the attractor-repeller pair cofibration used to define the contact invariant in the $G$-equivariant setting. In doing so, one encounters no real difficulty and it is straightforward to obtain a $G$-equivariant cohomotopical contact invariant.

Consider a finite group $G$ and a rational homology sphere $Y$ equipped with a free $G$-action. Use $\pi : Y \to Y/G$ to denote the quotient map. Agree to fix a metric $g$ on $Y/G$ and use, on $Y$, the induced $G$-invariant metric $\pi^*g$. Suppose $\lambda$ be a contact form on $Y/G$ so that $\pi^*\lambda$ be a $G$-equivariant contact form on $Y$. Notice that the canonical Spin$^C$ structure defined by $\lambda$ on $Y/G$ naturally lifts to $Y$ as the canonical Spin$^C$ structure defined by $\pi^*\lambda$; that is,
\[ \pi^*s_\lambda = s_{\pi^*\lambda}. \] Likewise, the connexion \( A_{\pi^*\lambda} \) on \( \det s_{\pi^*\lambda} \) is the lift \( \pi^*A_\lambda \) of the connexion \( A_\lambda \) on \( \det s_\lambda \). Denote by \( W \) the global Coulomb slice with respect to \( A_{\pi^*\lambda} \) on \( Y \) and by \( W' \) the global Coulomb slice with respect to \( A_\lambda \) on \( Y/G \).

It is difficult to study this scenario in the classical setting of monopole Floer homology due to the need for generic perturbations in order to achieve the required Morse-Smale condition of Morse theory. Indeed, there is no guarantee that a sufficiently generic perturbation chosen for \( Y \) in order to satisfy the conditions for the construction of the group \( \widehat{HM}^*(Y, \pi^*s) \) can be made \( G \)-equivariant so that it define a valid perturbation for the construction of \( \widehat{HM}^*(Y/G, s) \). As a consequence, the behaviour of the monopole Floer homology groups under coverings has proven elusive to study via the classical Morse theoretic approach. As the construction of the SWF spectrum avoids the addition of a generic perturbation, one can say something significant using this machinery. The author starts by recalling the main observations of Lidman and Manolescu [30].

**Remark 6.1.** Note that \( G \) acts linearly on the Coulomb slice \( W \). Moreover, the quotient map \( \pi : Y \to Y/G \) induces an inclusion \( W'/\to W \) which identifies \( W' \) with the fixed point space \( W^G \). This inclusion is *not*, however, an isometry in the \( L^2 \)-norm. Nonetheless, the \( L^2 \) ball \( B(W', R') \) is identified with the \( L^2 \) ball \( B(W, |G| R') \subset W \), which allows one to construct the Sobolev norm on \( W \) so as to have it be \( G \)-invariant. As a consequence, in the Sobolev completions, one still has \( W'_k = W^G_k \).

**Remark 6.2.** The Fredholm operator \( \ell \) on \( Y \) is \( G \)-equivariant. Hence, its restriction to \( W^G \) agrees with the analogous operator defined on \( Y/G \). Therefore, use \( \ell \) to denote both these operators. Furthermore, note that \( (W')^\mu = (W^\mu)^G \). The map \( c \) is also \( G \)-equivariant, so a finite-type Seiberg–Witten trajectory in \( W'_k \) is the same thing as a \( G \)-fixed finite-type Seiberg–Witten trajectory in \( W^G_k \). The same identification can be made between the Seiberg–Witten trajectories in the finite dimensional approximations.

Use \( R > 0 \) to denote the constant provided by Theorem 3.28 for the case of the manifold \( Y \) and \( R' > 0 \) to denote this constant for the quotient manifold \( Y/G \). By perhaps increasing \( R \) or \( R' \), one can ensure that \( R' = R/|G| \). This means that \( R \) will also satisfy the conclusions of Theorem 3.28 for \( Y/G \).

Next, recall from Remark 3.30 that the choice of bump functions \( u^\mu \) for \( Y \) was made so as to have it constant on the spheres centred at zero in the Sobolev norm; therefore, the \( u^\mu \) are automatically \( G \)-invariant. Therefore, their restrictions to \( (W^\mu)^G \) can be used to define the bump functions required for \( Y/G \). With these conditions, the finite dimensional Seiberg–Witten flow \( \varphi_{\mu, \lambda}^r \) of \( Y \) is \( G \)-equivariant and restricts to \( (W^\mu)^G \) as the finite dimensional Seiberg–Witten flow of \( Y/G \). Now, if the reader agree to fix \( \mu > 0 \) large enough so as to have Theorem 3.33 hold for both \( Y \) and \( Y/G \), then, it follows that the isolated invariant set \( S^\mu_{\pi^*\lambda, r} \) is \( G \)-invariant and its \( G \)-fixed subset, \( (S^\mu_{\pi^*\lambda, r})^G \), is precisely the isolated invariant set used to define the SWF spectrum of \( Y/G \). Moreover, since the \( G \)-action is linear on
$W^\mu$, the isolating neighbourhood $D(W^\mu, 2R)$ of $S^\mu_{\pi^\ast \lambda, r}$ is $G$-invariant and $D(W^\mu, R)^G = D((W^\mu)^G, R)$ serves as an isolating neighbourhood for the construction of the SWF spectrum on $Y/G$.

A few more notions from equivariant stable homotopy theory shall be required. In what follows, use $H$ to denote an arbitrary compact Lie group. As before, the reader is directed to May et al. [37] for further details.

**Theorem 6.3.** Given a closed normal subgroup $K \subset H$, there exists a functor, called the geometric fixed points functor, from $H$-spectra indexed over a universe $\mathcal{V}$ to $(H/K)$-spectra indexed over the universe $\mathcal{V}^K$,

$$\Phi^K : H\mathcal{V} \to (H/K)\mathcal{V}^K,$$

satisfying the properties that

(i) $\Phi^K \Sigma^\infty X \cong \Sigma^\infty_{\mathcal{V}^K} X^K$,

(ii) $\Phi^K E \wedge \Phi^K F \cong \Phi^K(E \wedge F)$.

**Proof.** Vid. May et al. [37, Sect. XVI.3] \hfill \square

**Remark 6.4.** Since the author is simply dealing with suspension spectra, these two properties suffice in understanding the geometric fixed points of the spectra at hand. In particular, note that, for an $H$-space $X$ and an $H$-representation $\mathcal{V}$, it follows that

$$\Phi^K \Sigma^{-\mathcal{V}} \Sigma^\infty X = \Sigma^{-\mathcal{V}^K} \Sigma^\infty_{\mathcal{V}^K} X^K.$$

**Definition 6.5.** An $H$-universe is called complete if one can find, for any finite dimensional $H$-representation $\mathcal{V}$, a sub-representation in $\mathcal{V}$ isomorphic to $\mathcal{V}$.

**Remark 6.6.** For $G$ the group of deck transformations of the covering $\pi : Y \to Y/G$, notice that the universe $\mathcal{W}$ defined by the Coulomb gauge of $Y$ is naturally a $G \times U(1)$-universe and its $G$-fixed point space, $\mathcal{W}^G$, is the $U(1)$-universe defined by the Coulomb gauge of $Y/G$. Let $\mathcal{U}''$ denote a complete $G$-universe and $\mathcal{U}' := \mathcal{U} \oplus \mathcal{U}'$ denote a $G \times U(1)$-universe. By intertwining with change of universe functors defined by an isometry $\mathcal{W} \to \mathcal{U}'$, as was done in Definition 3.61, one can consider the functor $\Sigma^{-W^{(-\mu, 0)}}$ as an endofunctor of the category $\tilde{h}(U(1) \times G)\mathcal{SU}'$.

**Definition 6.7.** Define the metric dependent $G$-equivariant Seiberg–Witten Floer spectrum as

$$\text{SWF}_G(Y, \pi^\ast s, \pi^\ast g) := \Sigma^{-W^{(-\mu, 0)}} \Sigma^\infty_{U(1) \times G} S^\mu_{\pi^\ast \lambda, r} \in \tilde{h}(U(1) \times G)\mathcal{SU}'$$

**Remark 6.8.** The author believes it to be possible to (de)suspend away the metric dependence in an analogous fashion to the non-equivariant case; however, it seems the details are somewhat subtle, so he chose not to pursue that goal in this article. This would involve considering a localization of the representation ring $RO(U(1) \times G)$, equivariant spectral flow and equivariant eta invariants.

**Theorem 6.9.** [30] The Seiberg–Witten Floer spectra for $Y$ and $Y/G$ are related by

$$\Phi^G \text{SWF}_G(Y, \pi^\ast s, \pi^\ast g) = \text{SWF}(Y/G, s, g).$$
Proof. Although the statement in Lidman and Manolescu [30] is made in terms of $U(1) \times G$-spaces instead of spectra, the properties of the geometric fixed points functor mentioned above make clear that this is the correct statement for spectra. □

Now, the contact invariant construction shall be considered in this setting. If necessary, increase $R$, $\mu$ and the parameter $r$ used in the Seiberg–Witten equations so that Remark 4.1, Proposition 4.8 and Theorem 4.17 be satisfied for both $Y$ and $Y/G$.

Notice that the upstairs contact circle, $U_{\pi^* \lambda} \subset (W^\mu)^G \subset W^\mu$, is naturally identified with the downstairs contact circle, $U_\lambda$. Beware, however, that the dual attractors to $U_{\pi^* \lambda}$ in $S_{\pi^* \lambda, r}^\mu$ and in $(S_{\pi^* \lambda, r}^\mu)^G$ are, of course, not the same.

Definition 6.10. Let

$$T_G(\pi^* \lambda, g) := \Sigma W^{(-\mu,0)} \Sigma_{U'} I_{U(1) \times G}(U_{\pi^* \lambda}, \varphi_{\pi^* \lambda, r}^\mu) \in \bar{h}(U(1) \times G)SU'$$

denote the Thom space appearing in the codomain of the cohomotopical contact invariant but now seen as a $(U(1) \times G)$-spectrum (cf. Definition 4.27).

The attractor-repeller cofibration

$$I_{U(1) \times G}((U_{\pi^* \lambda})^*, \varphi_{\pi^* \lambda, r}^\mu) \to I_{U(1) \times G}(S_{\pi^* \lambda, r}^\mu, \varphi_{\pi^* \lambda, r}^\mu) \to I_{U(1) \times G}(U_{\pi^* \lambda}, \varphi_{\pi^* \lambda, r}^\mu)$$

can be formed $G$-equivariantly as well. This is done by constructing a $(U(1) \times G)$-invariant index triple $(L, M, N)$ according to Theorem 4.23. The consequence is that the contact invariant gains a $G$-equivariant version.

Definition 6.11. Define the $G$-equivariant metric dependent contact invariant as the map

$$\Psi_G(\pi^* \lambda, \pi^* g) : SWF_G(Y, \pi^* \lambda, \pi^* g) \to T_G(\pi^* \lambda, \pi^* g)$$

given by desuspending the above cofibre map by the $(U(1) \times G)$-representation $W^{(-\mu,0)}$.

Theorem 6.12. The cohomotopical contact invariants of $(Y, \pi^* \lambda)$ and $(Y/G, \lambda)$ are related by

$$\Phi^G \Psi_G(\pi^* \lambda, \pi^* g) = \Psi(\lambda, g).$$

Proof. If $(L, M, N)$ be a $U(1) \times G$-invariant index triple for $S_{\pi^* \lambda, r}^\mu$, then the triple of fixed point sets $(L^G, M^G, N^G)$ is a $U(1)$-invariant index triple for $(S_{\pi^* \lambda, r}^\mu)^G = S_{\lambda, r}^\mu$. The result follows at once. □

The extra data available due to the $G$-action allows one to define a $G$-equivariant (or $(U(1) \times G)$-equivariant) version of the monopole Floer cohomologies via the $G$-equivariant cohomology of the SWF spectrum since the spectrum has naturally acquired a $G$-action. There are three choices of equivariant cohomology theory which spring to mind in this context: Borel cohomology, equivariant K-theory and Bredon cohomology. In the present article, the author shall pursue the use of Borel cohomology as that turned out to be the most readily applicable.
7. Borel cohomology

This section shall deal with the Borel $G$-equivariant cohomology of the spectrum $\text{SWF}_G$ and consider the resulting equivariant contact class. Despite the spectrum $\text{SWF}_G$ being a $U(1) \times G$-spectrum, in the interests of simplicity, the author shall ignore the $U(1)$-action and only consider cohomologies equivariant with respect to the $G$-action. Application of Borel cohomology to the cohomotopical contact invariant leads to a $G$-equivariant cohomological contact invariant $\psi_G$, which, as shall be seen, holds information about both the upstairs and downstairs contact structures of the covering. Verily, if one know the value of $\psi_G$, one can recover the value of both cohomological contact invariants of the covering.

Ideally, what one really wishes is to infer something about the upstairs contact invariant from knowledge about the downstairs contact invariant; the new equivariant contact invariant, therefore, may seem not to be leading in that direction. However, Borel cohomology, under appropriate circumstances, enjoys the very powerful property that the cohomology of the fixed points of a $G$-space conditions significantly the Borel cohomology of the whole space; this is the content of the localization theorem. As a consequence of localization, one sometimes can determine the equivariant contact invariant from knowledge of the downstairs contact invariant. And, as the equivariant contact invariant recovers the non-equivariant upstairs contact invariant, this allows one to infer the upstairs contact invariant starting only from knowledge of the downstairs contact invariant.

The applicability of localization, however, is limited to scenarios where one know precisely what the SWF$_G$ spectrum is. This happens, for instance, in the event that there be a unique solution to the Seiberg–Witten equations, which implies that SWF$_G$ is an equivariant sphere spectrum. Nonetheless, such manifolds are sufficiently abundant that the results derived say something quite non-trivial. A special case of interest shall be that of elliptic manifolds, where a very general theorem may be stated which aids greatly in determining whether the lift of a tight contact structure remains tight or becomes overtwisted.

For further simplicity, the author shall avoid the language of $G$-spectra and work instead at the level of $G$-spaces as much as possible in this section. This is similar to what is done in Lidman and Manolescu [30]. The main reason being that the appropriate form of the localization theorem is more difficult to describe for spectra as questions about the various forms of fixed points functors come into play.

Let $Y$, $G$, $\pi$, $g$ and $\lambda$ be as in the previous section.

**Definition 7.1.** Define the $G$-equivariant Borel metric dependent monopole Floer cohomology as

$$c\tilde{H}_G^*(Y, \pi^*s, \pi^*g; R) := c\tilde{H}_G^*(\text{SWF}_G(Y, \pi^*s, \pi^*g); R)$$

where $R$ is some commutative ring, which will be left implicit in the notation henceforth.
Remark 7.2. In order to make the best out of Borel cohomology, it is best to focus on the case $G = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$. In this case, the coefficient ring $R$ shall be chosen to be $\mathbb{Z}/p\mathbb{Z}$. For the remainder of this section, these choices shall be implicit lest the notation become overloaded.

Remark 7.3. Since $\text{SWF}_G(Y, \pi^*s, \pi^*g)$ depends on $g$ only up to suspension by $G$-representations, it follows that $\text{cHM}_G^*(Y, \pi^*s, \pi^*g)$ depends on $g$ only up to shifts in grading, which, although cosmetically unpleasant, is not a major issue in practice.

Definition 7.4. Given a $G$-representation $V$ and a ring $R$, one says that $V$ is $G$-equivariantly $R$-orientable if the vector bundle $V_G := (V \times EG)/G$ over $BG$ be $R$-orientable. In which case, one writes $e_G(V) \in \widetilde{cH}_G^*(S^0; R)$ for its Euler class.

Remark 7.5. Since the author is using coefficients $R = \mathbb{Z}/p\mathbb{Z}$ for the group $G = \mathbb{Z}/p\mathbb{Z}$ and $p$ is a prime, it follows that all $G$-representations are $G$-equivariantly $R$-orientable. In the case $p = 2$, all vector bundles are $R$-orientable anyway. Otherwise, if $p$ be an odd prime, then any non-trivial representation of $G$ is complex and, therefore, the vector bundle defined over $BG$ shall also be complex and therefore orientable over any field. So, in any event, an equivariant Euler class always exists for the purposes being pursued here.

Remark 7.6. Note that, due to the Thom isomorphism, for an orientable $G$-representation $V$, the ring $\widetilde{cH}_G^*(V^+)$ is isomorphic to $\widetilde{cH}_G^*(S^0)$ with its grading shifted by $\dim V$. Moreover, under this isomorphism, notice that the Thom class $\theta_G(V_G)$ of the bundle $V_G \to BG$ gets sent to $1 \in \widetilde{H}_G^0(S^0)$. In other words, one can think of $\theta_G(V_G)$ as a $G$-equivariant fundamental class of the $G$-manifold $V^+$.

Theorem 7.7. (Localization theorem) Let $\Gamma$ be a finite $G$-CW-complex and $S \subset \widetilde{H}^* (BG; \mathbb{Z}/p\mathbb{Z})$ consist of those elements which be Euler classes of $G$-representations having no trivial summand. The inclusion of fixed points $\Gamma^G \hookrightarrow \Gamma$ induces an isomorphism between cohomologies localized with respect to $S$,

$$S^{-1}\widetilde{cH}_G^*(\Gamma; \mathbb{Z}/p\mathbb{Z}) \sim S^{-1}\widetilde{cH}_G^*(\Gamma^G; \mathbb{Z}/p\mathbb{Z}).$$

Proof. Vid. [57, Theorem 3.13]  □

Example 7.8. Consider the inclusion of fixed points

$$(V^+)^G \hookrightarrow V^+$$

of a $G$-representation sphere coming from a $G$-representation $V$ potentially having $V^G \neq 0$. In this example, assume $p > 2$. Firstly, recall that, in this case, the cohomology of $BG$ is the ring

$$\widetilde{cH}_G^*(S^0) = (\mathbb{Z}/p\mathbb{Z})[u, v]/(v^2)$$

where $\deg u = 2$, $\deg v = 1$. It is worth paying close attention to this ring. In the following diagram, the top row indicates the abelian subgroups at each
degree, the middle row indicates the generator with the corresponding degree and the bottom row indicates the numerical value of the degree.

\[ \cdots 0 0 \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots \]

\[ \langle 1 \rangle \langle v \rangle \langle u \rangle \langle uv \rangle \langle u^2 \rangle \]

\[ (−2) (−1) (0) (1) (2) (3) (4). \]

Now, because \( p > 2 \) is an odd prime, all representations of \( G \) are complex and therefore define orientable bundles over \( BG \). It follows from the Thom isomorphism that

\[ c\tilde{H}^*_G(V^+) \cong c\tilde{H}^{*−\dim V}(S^0) \]

and likewise for \((V^+)^G\). Hence, before localization, the inclusion of fixed points \((V^+)^G \hookrightarrow V^+\) induces a map

\[ c\tilde{H}^{*−\dim V}(S^0) \rightarrow c\tilde{H}^{*−\dim V^G}(S^0) \]

which, in the notation above, is depicted as

\[ \cdots 0 0 \cdots 0 \mathbb{Z}/p\mathbb{Z} \cdots \]

\[ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \]

\[ \cdots 0 \mathbb{Z}/p\mathbb{Z} \cdots \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots , \]

where all the \( \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) maps are isomorphisms. Now, consider the effect of localization. The set \( S \) with respect to which one must perform localization is the set \( \{u^n | n > 0\} \); this can be seen from looking at the representation theory of \( G \). The localized Borel cohomology therefore is

\[ S^{−1}c\tilde{H}^*_G(S^0) = (\mathbb{Z}/p\mathbb{Z})[u, u^{-1}, v]/(v^2). \]

Schematically,

\[ \cdots \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots \]

\[ \langle u^{-1} \rangle \langle u^{-1} v \rangle \langle 1 \rangle \langle v \rangle \langle u \rangle \langle uv \rangle \langle u^2 \rangle \]

\[ (−2) (−1) (0) (1) (2) (3) (4). \]

After localization, the map induced by the inclusion of fixed points takes the form

\[ \cdots \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots \]

\[ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \]

\[ \cdots \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots , \]

where all maps are isomorphisms.

**Example 7.9.** Now, consider the same scenario but with \( p = 2 \). In this case, recall that the cohomology of \( BG \) is the ring

\[ c\tilde{H}^*_G(S^0) = (\mathbb{Z}/2\mathbb{Z})[u] \]

where \( \deg u = 1 \). In particular, as an abelian group, this is the same as in the case \( p > 2 \); only the ring structures differ. Again, draw the same sort of diagram as before.

\[ \cdots 0 0 \mathbb{Z}/2\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \cdots \]

\[ \langle 1 \rangle \langle u \rangle \langle u^2 \rangle \langle u^3 \rangle \langle u^4 \rangle \]

\[ (−2) (−1) (0) (1) (2) (3) (4). \]
Unlike in the $p > 2$ case, here, there are non-trivial real representations of $G$ which define non-orientable bundles over $BG$. No matter; in this case, the coefficient ring being used for cohomology theories is $\mathbb{Z}/2\mathbb{Z}$, and all vector bundles are $\mathbb{Z}/2\mathbb{Z}$-orientable. Hence, again by the Thom isomorphism,
\[ c\tilde{H}_G^*(V^+) \cong c\tilde{H}_G^{* - \dim V}(S^0) \]
and likewise for $(V^+)^G$. As before, the inclusion of fixed points $(V^+)^G \hookrightarrow V^+$ induces a map
\[ c\tilde{H}_G^{* - \dim V}(S^0) \rightarrow c\tilde{H}_G^{* - \dim V^G}(S^0) \]
which one depicts as
\[
\begin{array}{cccccccc}
\cdots & 0 & 0 & \cdots & 0 & \mathbb{Z}/2\mathbb{Z} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & 0 & \mathbb{Z}/2\mathbb{Z} & \cdots & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \cdots ,
\end{array}
\]
where all the $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ maps are $1 \mapsto 1$. Localization, in this case, is with respect to the set $S = \{ u^n | n > 0 \}$. Therefore,
\[ S^{-1}c\tilde{H}_G(S^0) = (\mathbb{Z}/2\mathbb{Z})[u, u^{-1}]. \]
Schematically,
\[
\begin{array}{cccccccc}
\cdots & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \cdots \\
\langle u^{-2} \rangle & \langle u^{-1} \rangle & \langle 1 \rangle & \langle u \rangle & \langle u^2 \rangle & \langle u^3 \rangle & \langle u^4 \rangle & \\
(-2) & (-1) & (0) & (1) & (2) & (3) & (4). & \\
\end{array}
\]
The end result is effectively the same as in the $p > 2$ case; the map induced by the inclusion of fixed points is
\[
\begin{array}{cccccccc}
\cdots & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \cdots & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \cdots ,
\end{array}
\]
where all maps are isomorphisms.

Remark 7.10. Recall the unstable normal bundle $E_{u,\mu}^{\pi,\lambda} \rightarrow U_{\pi,\lambda}$ from Remark 4.14. As was seen earlier, the $G$-equivariant Conley index of $U_{\pi,\lambda}$ is the Thom space of this bundle,
\[ I_G(U_{\pi,\lambda}, \varphi_{\pi,\lambda}^\mu, r) = \Theta_G(E_{u,\mu}^{\pi,\lambda}). \]
Let $\mathbb{M} \hookrightarrow E_{\lambda}^\mu$ denote a fibre of the bundle. Note that $\mathbb{M}$ is a $G$-representation and decompose it as $\mathbb{M}^G \oplus \mathbb{F}$ where $\mathbb{F}$ is a $G$-representation with trivial fixed points, $\mathbb{F}^G = 0$. Next, consider the unstable normal bundle $E_{u,\mu}^{\pi,\lambda} \hookrightarrow U_{\lambda}$. The Conley index is again computed as
\[ I(U_{\lambda}, \varphi_{\lambda, r}^\mu) = \Theta(E_{\lambda}^{u,\mu}). \]
One can ensure that the unstable normal bundles be related as
\[ E_{\lambda}^{u,\mu} = (E_{\pi,\lambda}^{u,\mu})^G. \]
Hence, the Conley indices of $U_{\lambda}$ in $(W_{\mu}^\pi)^G$ and of $U_{\pi,\lambda}$ in $W_\mu$ are related by
\[ I_G(U_{\pi,\lambda}, \varphi_{\pi,\lambda}^\mu, r) = I(U_{\lambda}, \varphi_{\lambda, r}^\mu) \wedge \mathbb{F}^+. \]
Since $I(U_\lambda, \varphi^\mu_{\lambda, r})$ is $G$-trivial, one has that
\[
c{\tilde{H}}_G^*(I(U_\lambda, \varphi^\mu_{\lambda, r})) \cong \tilde{H}^*(I(U_\lambda, \varphi^\mu_{\lambda, r})) \otimes c{\tilde{H}}_G^*(S^0)
\]
and, similarly,
\[
c{\tilde{H}}_G^*(I(U_\lambda, \varphi^\mu_{\lambda, r}) \wedge \bar{f}^+) \cong \tilde{H}^*(I(U_\lambda, \varphi^\mu_{\lambda, r})) \otimes c{\tilde{H}}_G^*(\bar{f}^+).
\]

**Remark 7.11.** Next, consider a non-equivariant Thom class
\[
\theta(E^\mu_\chi) \in \tilde{H}^{\dim{G}}(I(U_\lambda, \varphi^\mu_{\lambda, r}))
\]
for the bundle $E^\mu_{\pi^r \chi} \to U_\chi$. Bearing in mind the isomorphisms outlined in the previous remark, define a class
\[
\theta_G(E^\mu_{\pi^r \chi}) := \theta(E^\mu_\chi) \otimes \theta_G(\bar{f}_G) \in \tilde{H}^*(\Theta(E^\mu_{\pi^r \chi})) \otimes \mathbb{Z}/p \mathbb{Z} \ c{\tilde{H}}_G^*(\bar{f}^+) \cong c{\tilde{H}}_G^*(\Theta_G(E^\mu_{\pi^r \chi})).
\]
It is easy to verify that the class $\theta_G(E^\mu_{\pi^r \chi})$ serves as a $G$-equivariant Thom class for the $G$-bundle $E^\mu_{\pi^r \chi} \to U_{\pi^r \chi}$.

**Remark 7.12.** Due to the difficulties of working with localization in the context of spectra, the author decided to proceed with certain arguments applied prior to desuspension. For that end, it becomes useful to define contact invariants dependent on the sufficiently large spectral cut-off parameter. In what follows, let $\mu > 0$ again be large enough to satisfy what is said in Theorem 3.33 and Proposition 4.12.

**Definition 7.13.** Let the cohomological contact invariant in finite dimensional approximation be the class
\[
\psi(\lambda, g, \mu) \in \tilde{H}^*(I(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r}))
\]
given as the pullback of $\theta(E^\mu_\chi)$ via the cofibre map
\[
I(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r}) \to \Theta(E^\mu_\chi).
\]

**Definition 7.14.** Let the equivariant cohomological contact invariant in finite dimensional approximation be the class
\[
\psi_G(\pi^*\lambda, \pi^*g, \mu) \in c{\tilde{H}}_G^*(I_G(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\pi^*\lambda, r}))
\]
given as the pullback of $\theta_G(E^\mu_{\pi^r \lambda})$ via the cofibre map
\[
I_G(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\pi^*\lambda, r}) \to \Theta_G(E^\mu_{\pi^r \lambda}).
\]

**Proposition 7.15.** Under the map induced by inclusion of fixed points
\[
c{\tilde{H}}_G^*(I_G(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\pi^*\lambda, r})) \to c{\tilde{H}}_G^*(I(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r})) \cong \tilde{H}^*(I(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r})) \otimes c{\tilde{H}}_G^*(S^0)
\]
the class $\psi_G(\pi^*\lambda, \pi^*g, \mu)$ gets sent to $\psi(\lambda, g, \mu) \otimes e_G(\bar{f})$.

**Proof.** Consider the commuting diagram
\[
\begin{array}{ccc}
c{\tilde{H}}_G^*(I_G(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\pi^*\lambda, r})) & \to & c{\tilde{H}}_G^*(\Theta_G(E^\mu_{\pi^r \lambda})) \\
\downarrow & & \downarrow \\
c{\tilde{H}}_G^*(I(S^\mu_{\lambda, r}, \varphi^\mu_{\lambda, r})) & \to & c{\tilde{H}}_G^*(\Theta(E^\mu_\chi)),
\end{array}
\]
where the horizontal arrows come from the cofibre maps and the vertical from the inclusion of fixed points. By the isomorphisms discussed in Remark 7.10, one can rewrite the right vertical map as

\[ \tilde{H}^*(\Theta(E^u_{\lambda, \mu})) \otimes c\tilde{H}_G^*(f^+ \rightarrow \tilde{H}^*(\Theta(E^u_{\lambda, \mu})) \otimes c\tilde{H}_G^*(S^0). \]

Moreover, with respect to these tensor products, this map is simply \( \mathcal{id} \otimes i^* \) where \( i \) is the inclusion of fixed points \( i : S^0 \hookrightarrow f^+ \). But note that \( i^*\theta_G(f_G) \) is, by definition, the Euler class \( e_G(f_G) \). The result then follows by commutativity of the diagram. \( \square \)

**Proposition 7.16.** If \( \psi_G(\pi^*\lambda, \pi^* g, \mu) = 0 \), then \( \psi(\lambda, g, \mu) = 0 \).

**Proof.** Since the representation \( f \) has no trivial summand, the Euler class \( e_G(f) \) is never zero. The result then follows immediately from Proposition 7.15. \( \square \)

One can now desuspend appropriately to obtain a result that is not dependent on the spectral cut-off \( \mu \).

**Definition 7.17.** Define the metric dependent equivariant cohomological contact invariant

\[ \psi_G(\pi^*\lambda, \pi^* g) \in c\tilde{HM}_G^*(Y, \pi^* s_\lambda, \pi^* g) \]

as the desuspension of the class \( \psi_G(\pi^*\lambda, \pi^* g, \mu) \) by the \( G \)-representation \( W(\mu, 0) \).

**Theorem 7.18.** If \( \psi_G(\pi^*\lambda, \pi^* g) = 0 \), then \( \psi(\lambda, g) = 0 \).

**Proof.** Follows directly from Proposition 7.16 after desuspension. \( \square \)

**Remark 7.19.** Recall that there is a natural transformation \( c\tilde{H}^*_G \rightarrow \tilde{H}^* \) induced by the inclusion of the fibre of the Borel construction. That is, if \( \Gamma \) be a \( G \)-space, \( \{p\} \times \Gamma \hookrightarrow \Gamma_G := (E_G \times \Gamma)/G \). In the present context, this translates to a forgetful map

\[ c\tilde{HM}_G^*(Y, \pi^* s_\lambda, \pi^* g) \rightarrow \tilde{HM}^*(Y, \pi^* s_\lambda, \pi^* g). \]

**Theorem 7.20.** Under the map

\[ c\tilde{HM}_G^*(Y, \pi^* s_\lambda, \pi^* g) \rightarrow \tilde{HM}^*(Y, \pi^* s_\lambda, \pi^* g), \]

the class \( \psi_G(\pi^*\lambda, \pi^* g) \) gets sent to \( \psi(\pi^*\lambda, \pi^* g) \).

**Proof.** This follows simply from the fact that the equivariant Thom class

\[ \theta_G(E^u_{\pi^*\lambda}) \in c\tilde{H}_G^*(I_G(U_{\pi^*\lambda})) \]

is sent to a non-equivariant Thom class for the bundle \( E^u_{\pi^*\lambda} \rightarrow U_{\pi^*\lambda} \) via the map

\[ c\tilde{H}_G^*(I_G(U_{\pi^*\lambda})) \rightarrow \tilde{H}^*(I(U_{\pi^*\lambda})). \] \( \square \)

**Corollary 7.21.** If the equivariant contact invariant \( \psi_G(\pi^*\lambda, \pi^* g) \) vanish, then the non-equivariant contact invariants of both \( Y \) and \( Y/G \) shall also vanish,
\[ \psi(\pi^*\lambda, \pi^*g) = 0, \quad \psi(\lambda, g) = 0. \]

**Proof.** Corollary of Theorems 7.18 and 7.20. \hfill \square

In order to seek computable examples, the author shall make use of the following definition first used in the work of Lin and Lipnowski \[33\].

**Definition 7.22.** By saying that \( Y \) is a *minimal L-space*, one means that the perturbed Seiberg–Witten equations admit no irreducible solutions for some perturbation of arbitrarily small norm.

**Example 7.23.** Perhaps the best known examples of minimal L-spaces are the *elliptic manifolds* due to their metrics of positive scalar curvature (cf. Kronheimer and Mrowka \[21, Sect. 22.7\]).

**Example 7.24.** Another well known example of a minimal L-space is the *Hantzsche–Wendt* manifold, that is, the unique flat rational homology 3-sphere (cf. Kronheimer and Mrowka \[21, Sect. 37.4\]).

**Example 7.25.** By recent work of Lin \[32\], it is known that all rational homology 3-spheres which have *sol geometry* are minimal L-spaces.

**Example 7.26.** According to the surprising work of Lin and Lipnowski \[33\], it is known that certain well known *hyperbolic* rational homology 3-spheres are also minimal L-spaces.

**Remark 7.27.** Assume that \( X \) be a minimal L-space. In this case, it is easy to see that

\[ \text{SWF}(X, \mathfrak{s}, g) = \Sigma W(-\mu,0) S \]

where \( S \in \mathcal{SR}^{\infty} \) is the sphere spectrum (cf. \[35, Sect. 10\], example about the Poincaré homology sphere). Likewise, supposing \( Y \) to be a minimal L-space upon which \( G \) freely act, it is just as easy to see that

\[ \text{SWF}_G(Y, \pi^*\mathfrak{s}, \pi^*g) = \Sigma W(-\mu,0) S_G, \]

where \( S_G \in G\mathcal{SU}' \) is the \( G \)-equivariant sphere spectrum for the \( G \)-universe \( \mathcal{U}' \); this is because, there being a unique solution to the Seiberg–Witten equations, \( G \) can only act trivially on it and, therefore, the action on the Conley index will come entirely from the linear action on \( W^\mu \).

**Remark 7.28.** In order to apply the localization theorem, it is easiest to work at the level of spaces instead of spectra and consider the inclusion of fixed points (cf. Theorem 6.9)

\[ I(\mathcal{U}_\lambda^\mu, \varphi_{\lambda,r}^\mu) \rightarrow I_G(\mathcal{U}_\lambda^\mu, \varphi_{\lambda,r}^\mu). \]

This is simply the inclusion of fixed points of a \( G \)-representation sphere, exactly as was studied in Examples 7.8 and 7.9. The crux of the matter is to deduce what this says about the contact invariants at play. The first observation is the following.
Proposition 7.29. Let $S \subset H^*(BG; \mathbb{Z}/p\mathbb{Z})$ be the set of Euler classes of $G$-representations having no trivial summands. The map

$$S^{-1}c\tilde{H}_G^*(I(G(S_{\pi^*,\lambda,r}^\mu, \varphi_{\pi^*,\lambda,r}^\mu))) \to S^{-1}c\tilde{H}_G^*(I(S_{\lambda,r}^\mu, \varphi_{\lambda,r}^\mu))$$

on localized Borel cohomology induced by the inclusion of fixed points maps $\psi_G(\pi^*\lambda, \pi^* g, \mu)$ to $\psi(\lambda, g, \mu) \otimes e_G(f)$, where these classes are being seen now as classes in localized Borel cohomology.

Proof. Follows from Proposition 7.15 after localizing. □

Theorem 7.30. Suppose $\pi : Y \to Y/G$ be a regular $p$-fold covering of minimal $L$-spaces where $p$ be prime and let $\lambda$ be a contact form on $Y/G$. It follows that $\psi_G(\pi^*\lambda, \pi^* g) = 0$ if and only if $\psi(\lambda, g) = 0$.

Proof. It suffices to work with the invariants $\psi_G(\pi^*\lambda, \pi^* g, \mu)$ and $\psi(\lambda, g, \mu)$ dependent on the spectral cut-off parameter as $\psi_G(\pi^*\lambda, \pi^* g)$ and $\psi(\lambda, g)$ are just grading-shifted versions of those. The localization theorem asserts that

$$S^{-1}c\tilde{H}_G^*(I(G(S_{\pi^*,\lambda,r}^\mu, \varphi_{\pi^*,\lambda,r}^\mu))) \to S^{-1}c\tilde{H}_G^*(I(S_{\lambda,r}^\mu, \varphi_{\lambda,r}^\mu))$$

is an isomorphism. Since $I(G(S_{\pi^*,\lambda,r}^\mu, \varphi_{\pi^*,\lambda,r}^\mu))$ is a $G$-representation sphere, its Borel cohomology is a free one-dimensional $c\tilde{H}_G^*(S^0)$-module; hence, $\psi_G(\pi^*\lambda, \pi^* g, \mu)$ cannot be a torsion element (with respect to the $c\tilde{H}_G^*(S^0)$-module structure) of $c\tilde{H}_G^*(I(G(S_{\pi^*,\lambda,r}^\mu, \varphi_{\pi^*,\lambda,r}^\mu)))$. The result then follows from Proposition 7.29 and, again, the observation that $e_G(f) \neq 0$ since $f^G = 0$. □

Theorem 7.31. Suppose $\pi : Y \to Y/G$ be a regular $p$-fold covering of minimal $L$-spaces where $p$ be prime and let $\lambda$ be a contact form on $Y/G$. The contact invariant $\psi(\lambda)$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients on $Y/G$ vanishes only if the contact invariant $\psi(\pi^*\lambda)$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients on $Y$ vanishes.

Proof. Noting, again, that it suffices to work with the analogous invariants dependent on the metric and the spectral cut-off parameter, combine Theorems 7.30 and 7.20. □

Definition 7.32. For a rational homology 3-sphere $Y$ with $\text{Spin}^C$ structure $\mathfrak{s}$, the Frøyshov invariant, $h(Y, \mathfrak{s})$, is defined as follows. Recall firstly that, as a $\mathbb{Z}[U]$-module, $\tilde{HM}^*(Y, \mathfrak{s})$ splits as a rank one free summand and a torsion summand. Let $h(Y, \mathfrak{s})$ be minus one half of the minimal degree, with respect to the absolute $\mathbb{Q}$-grading, in which the free summand be non-zero.

Remark 7.33. There is another well known invariant extracted from the $\mathbb{Q}$-grading of a Floer theory; in this case, from the Heegaard Floer theory. This one is denoted $d(Y, \mathfrak{s})$ and was first defined by Ozsváth and Szabó [41] in a similar fashion but referencing $HF^+(Y, \mathfrak{s})$ instead of $\tilde{HM}^*(Y, \mathfrak{s})$. According to the corpus that has equated Heegaard Floer homology, monopole Floer cohomology and embedded contact homology, the invariants $h$ and $d$, in fact, hold the same information but, for historical reasons, they are related by...
\[ -2d(Y, s) = h(Y, s). \]

This is achieved via the isomorphisms that preserve the absolute grading of the Floer theories by hyperplane fields (vid. [7, 45]) and the observation that the \( \mathbb{Q} \)-grading, in each case, is recovered through Gompf’s \( d_3 \) invariant of the hyperplane fields, perhaps plus one half depending on one’s conventions. The advantage of working with the Ozsváth-Szabó invariant is that it is readily computable for many important classes of manifolds.

**Theorem 7.34.** Suppose \( \pi : Y \to Y/G \) be a regular \( p \)-fold covering of minimal L-spaces where \( p \) be prime and let \( \lambda \) be a contact form on \( Y/G \). If \( \psi(\lambda) \neq 0 \) and \( \deg \psi(\pi^*\lambda) = -2h(Y, \pi^*s_\lambda) \), it follows that \( \psi(\pi^*\lambda) \neq 0 \).

**Proof.** As before, work with the metric and spectral cut-off dependent versions. Then, \( \psi(\lambda, g, \mu) \neq 0 \) implies \( \psi_G(\pi^*\lambda, \pi^*g, \mu) \neq 0 \) by localization. Meanwhile, the map
\[
\tilde{c}_{H^*}(I_G(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\lambda, r})) \to \tilde{H}^*(I(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\lambda, r})),
\]
which occurs in Theorem 7.20, can be written in terms of each degree as
\[
\cdots 0 \mathbb{Z}/p\mathbb{Z} \mathbb{Z}/p\mathbb{Z} \cdots
\]
\[
\downarrow \downarrow \downarrow \cdots
\]
\[
\cdots 0 \mathbb{Z}/p\mathbb{Z} 0 \cdots,
\]
where the central map \( \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) is an isomorphism. Hence, the invariant \( \psi(\pi^*\lambda, \pi^*g, \mu) \) is non-zero precisely when
\[
\tilde{H}^\deg \psi(\pi^*\lambda, \pi^*g, \mu)(I(S^\mu_{\pi^*\lambda, r}, \varphi^\mu_{\lambda, r})) = \mathbb{Z}/p\mathbb{Z}.
\]
Or, equivalently, after desuspensions,
\[
\tilde{HM}^\deg \psi(\pi^*\lambda)(Y, \pi^*s) = \mathbb{Z}/p\mathbb{Z}.
\]

Now, to relate this to the Froyslash invariant, recall that, for an L-space, the value of \( -2h(Y, \pi^*s_\lambda) \) is the grading of the only degree in which \( HM^*(Y, \pi^*s_\lambda) \) is non-trivial. \( \square \)

**Remark 7.35.** Note that \( \deg \psi(\pi^*\lambda) = -2h(Y, \pi^*s_\lambda) \) is also an obvious necessary condition for \( \psi(\pi^*\lambda) \neq 0 \). Therefore, the above theorem can be seen as strengthening this to a necessary and sufficient condition.

**Theorem 7.36.** Suppose \( \pi : Y \to Y/G \) be a regular \( p \)-fold covering of minimal L-spaces where \( p \) be prime and let \( \lambda \) be a contact form on \( Y/G \). If \( \deg \psi(\pi^*\lambda) < -2h(Y, \pi^*s_\lambda) \), then \( \psi(\lambda) = 0 \).

**Proof.** Note \( \deg \psi_G(\pi^*\lambda, \pi^*g) = \deg \psi(\pi^*\lambda, \pi^*g) \), so, if \( \deg \psi(\pi^*\lambda) < -2h(Y, \pi^*s_\lambda) \), then \( \psi_G(\pi^*\lambda, \pi^*g) = 0 \) because \( \tilde{HM}_G^\deg \psi_G(\pi^*\lambda, \pi^*g)(Y, \pi^*s_\lambda, \pi^*g) = 0 \). The result now follows from Theorem 7.18. \( \square \)

**Remark 7.37.** Recall that the grading of the contact invariant is given simply by the 3-dimensional obstruction theoretic invariant of hyperplane fields as
\[
\deg \psi(\lambda) = d_3(\text{Ker } \lambda) + \frac{1}{2}.
\]
The reader can find a precise definition of $d_3$, originally due to Gompf [14], in the following section, which shall be dedicated to understanding its behaviour under coverings.

**Theorem 7.38.** [11,12,36,58] The tight contact structures on a small Seifert fibred L-space all have non-vanishing contact invariant and no pair of non-isotopic tight contact structures share the same Spin$^C$ structure.

**Remark 7.39.** Elliptic manifolds are small Seifert fibred L-spaces, and this theorem allows one to state a stronger version of the above results.

**Corollary 7.40.** Suppose $\pi : Y \to Y/G$ be a regular $p$-fold covering of elliptic manifolds where $p$ be prime and let $\lambda$ be a tight contact form on $Y/G$. It follows that $\pi^* \lambda$ is tight if and only if $d_3(\pi^* \lambda) + \frac{1}{2} = -2h(Y, \pi^* s_\lambda)$.

**Remark 7.41.** One can avoid computing the Frøyshov invariant here if one know precisely the list of tight contact structures on $Y$ together with their corresponding Spin$^C$ structures and $d_3$ invariants. In that event, it suffices for one to compute $\pi^* s_\lambda$ and $d_3(\pi^* \text{Ker} \lambda)$ and compare with the entries of that list. If there be a match, $\pi^* \lambda$ must be tight, if not, $\pi^* \lambda$ must be overtwisted. This may be useful in some cases where computing Frøyshov invariants is difficult and the classification of tight contact structures is known by other methods, but it should be noted that, in many cases of interest, the calculations required to classify tight contact structures on small Seifert fibred L-spaces require comprehensive knowledge of the Frøyshov invariants (cf. [36]). In summary, this can be phrased as follows.

**Corollary 7.42.** Suppose $\pi : Y \to Y/G$ be a regular $p$-fold covering of elliptic manifolds where $p$ be prime and let $\lambda$ be a tight contact form on $Y/G$. It follows that $\pi^* \lambda$ is isotopic to a given tight contact structure—and therefore is also tight—if and only if it is homotopic to said tight contact structure.

Use $Y_{HW}$ to denote the Hantzsche–Wendt manifold, that is, the unique rational homology 3-sphere with Euclidean geometry. A curious fact about $Y_{HW}$ is that there are cyclic self-coverings $Y_{HW} \to Y_{HW}$ of any odd degree (vid. [3]). As $Y_{HW}$ is also known to be a minimal L-space, one could hope to apply the above methods here. Unfortunately, this fails because these self-coverings are never regular due to the first homology of $Y_{HW}$ having even order whilst the coverings odd order.

The case of the rational homology spheres admitting Sol geometry may be one where the methods developed here can work. The problem, however, lies in the limited knowledge available about the contact topology of such manifolds; there is no known classification of the isotopy classes of tight contact structures. Hence, before being able to apply Theorem 7.34 in this case, one would have to produce tight contact structures and develop a way to compute the respective Frøyshov invariants. The author intends to pursue that route in later work.

A similar story can be told of the few hyperbolic manifolds known to be minimal L-spaces; although the methods above make non-trivial statements about lifting tight contact structures, one hardly has the ability to apply them...
due to lack of information about the contact topology of those manifolds. One case where the methods developed here can be applied is the double covering of the hyperbolic manifold $m007(3,1)$ by $m036(-3,2)$, where the notation being used is that of the Hodgson-Weeks census of small volume closed hyperbolic 3-manifolds from the well known SnapPy software of Culler, Dunfield, Goerner & Weeks. That both these manifolds are minimal L-spaces is the consequence of the work of Lin and Lipnowski [33, Theorem 1].

This leaves the elliptic manifolds currently as the best place in which to seek to perform concrete computations. In the next few sections, the author shall proceed to develop certain topological techniques which shall be required for these calculations. The goal shall be to apply Corollary 7.42 to find certain tight contact structures which lift to tight contact structures by showing simply that the lift have the same homotopy theoretic invariants as a known tight contact structure. In order to do that, one must address the problem of computing the obstruction theoretic invariants of the lifted contact structure. There are two obstruction theoretic invariants: the Spin$^C$ structure and the $d_3$ invariant of Gompf. Lifting either of them provides a challenge of its own that must be overcome. The next two sections shall deal with these two problems.

8. The $d_3$ invariant and finite coverings

Consider the following problem. Say $\lambda$ be a contact form on a rational homology 3-sphere $Y/G$ regularly and finitely covered by $\pi : Y \to Y/G$. Given the value of $d_3(\text{Ker } \lambda)$ is it possible to say something about the value of $d_3(\pi^* \text{Ker } \lambda)$? This problem was studied by Khuzam [19] via use of the $G$-signature theorem. In summary, one can solve the problem by seeking an almost complex 4-manifold-with-boundary having the contact manifold $(Y, \pi^* \lambda)$ as its almost complex boundary and extending the $G$-action into its interior. The action need not remain free inside the 4-manifold-with-boundary; it need only have a properly embedded closed surface as its $G$-fixed points. Having such an equivariant almost complex filling, one can infer the lifting behaviour of the $d_3$ invariant of the contact form $\lambda$.

Producing such a 4-manifold-with-boundary is typically a difficult task. The goal of this section is to describe a method that the author devised by use of Kirby calculus to construct such a filling. The method consists, in essence, of the following. One starts with a Kirby diagram of $Y/G$, which also defines a 4-dimensional 2-handlebody having $Y/G$ as its boundary. Then, one removes a set of 2-handles judiciously so that the resulting handlebody evidently have a branched covering with branching locus a properly embedded surface with boundary at the curves along which the removed 2-handles were originally attached. Next, one computes this branched covering, and proceeds to equivariantly attach 2-handles in the hope that, in the quotient, these 2-handles be precisely the 2-handles that were removed initially. This procedure is not always easy to carry out, but, in simple cases such as when $Y/G$ be a surgery on a knot, it can be readily done. One must also be careful with
the almost complex structure on the filling. To achieve that, one considers Legendrian Kirby diagrams for the contact manifold \((Y/G, \lambda)\).

As an application of the method, the author shall calculate the \(d_3\) invariant of the lift of a tight contact structure of the \((-8)\)-surgery on the left-handed trefoil via a double covering. This is an example of a prism manifold, therefore elliptic, so one can apply the results of the preceding section to try and determine if the lifted contact structure remains tight. Recall that, for the lift to be tight, one needs the lift to have an appropriate pair of \(d_3\) invariant and Spin\(^C\) structure. It shall be shown that it does indeed have the appropriate \(d_3\) invariant, whereas, understanding the behaviour of the Spin\(^C\) structures under coverings is the content of the next section.

To begin, the author shall recall the relevant definitions and the standard results that shall be needed. Let \(Y, G, \pi, g\) and \(\lambda\) be as in the previous section.

**Definition 8.1.** A contact 3-manifold \((Y, \lambda)\) is called the almost complex boundary of an almost complex 4-manifold-with-boundary \((M, J)\) if \(\partial M = Y\) and \(\text{Ker } \lambda = TY \cap JTY\).

**Definition 8.2.** [14] Given a contact structure \(\text{Ker } \lambda\) on a rational homology 3-sphere \(Y\), the \(d_3\) invariant is defined as follows. Choose some almost complex 4-manifold-with-boundary \((M, J)\) such that \(Y\) be its almost complex boundary. Then, define

\[
d_3(\text{Ker } \lambda) := \frac{1}{4} \left( c_1(M, J)^2 - 2\chi(M) - 3\sigma(M) \right)
\]

where the notation \(c^2 \in Q\), for a class \(c \in H^2(X; \mathbb{Z})\), is defined in Definition 3.65.

**Remark 8.3.** The author shall sometimes write \(d_3(\lambda)\) instead of \(d_3(\text{Ker } \lambda)\) as that is more convenient in the present context.

**Remark 8.4.** Gompf originally called this invariant \(\theta\) and the factor of \(1/4\) was not present in his definition. There is another convention sometimes found in the literature, particularly in the context of Heegaard Floer theory, wherein the value of \(d_3\) has \(1/2\) added to what was defined above. The appeal of doing so is that the grading of the contact invariant becomes \(d_3\) instead of \(d_3 + 1/2\).

Now, consider again the case of finite coverings. The following result of Khuzam summarizes what can be said by means of the \(G\)-signature theorem about the lifting behaviour of \(d_3\).

**Theorem 8.5.** [19, Theorem 2] Suppose \(\pi : Y \to Y/G\) be an \(m\)-fold regular cyclic covering. Let \(\lambda\) be a contact form on \(Y/G\) and \(\pi^* \lambda\) its lift to \(Y\). Suppose further that one have \(\Pi : M \to M/G\) an \(m\)-fold cyclic branched covering of almost complex 4-manifolds-with-boundary where the branching locus be a closed surface \(S \subset M\) satisfying \(S \cap \partial M = \emptyset\) and where \((Y, \pi^* \lambda)\) be the almost complex boundary of \(M\). Then, the \(d_3\) invariants of \(\lambda\) and \(\pi^* \lambda\) are related by

\[
d_3(\pi^* \lambda) = md_3(\lambda) + \frac{3}{4}m\sigma(M/G) - \frac{3}{4}\sigma(M) - \frac{3}{4} \sum_{k=1}^{m-1} (S \cdot S) \csc^2(\gamma_k/2),
\]

where the notation \(\gamma_k\) is defined in Definition 3.65.
where \( \gamma_k \) denotes the angle of rotation given by the action of the element \( k \in \mathbb{Z}/m\mathbb{Z} \) on the normal planes to the surface \( S \) when \( X \) be equipped with a \( \mathbb{Z}/m\mathbb{Z} \)-invariant metric. In fact, this metric can always be chosen so that \( \gamma_k = 2\pi k/m \).

**Remark 8.6.** In the present article, the author follows the convention that the lens space \( L(p,1) \) is the one given by \((-p)\)-surgery on the unknot in \( S^3 \).

**Example 8.7.** In Khuzam [19], the only example studied is that of the universal covering \( S^3 \to L(p,1) \) of the lens space \( L(p,1) \). As a first example here, the author shall extend that to the lens covering \( L(p,1) \to L(mp,1) \).

One can construct the branched covering \( \Pi : M \to M/G \) as follows. Let \( M \) be the disk bundle over \( S^2 \) of Euler class \(-p\) so that \( \partial M = L(p,1) \). Note, \( G := \mathbb{Z}/m\mathbb{Z} \) acts on \( M \) by rotating the disk fibres. The action is free away from the zero section, which is fixed; therefore, \( S := M/G \simeq S^2 \). The manifold \( M/G \) is no other than the disk bundle of Euler class \(-mp\) over \( S^2 \), so its boundary is \( L(mp,1) \). Both \( M \) and \( M/G \) have almost complex structures coming from their disk bundle structures and \( \Pi \) is pseudoholomorphic with respect to these. Clearly, this is precisely the scenario of Theorem 8.5 and it becomes possible to compute \( d_3(\pi^*\lambda) \) from the value of \( d_3(\lambda) \). The lens space \( L(mp,1) \) admits precisely \( mp - 1 \) tight contact forms \( \lambda_1, \ldots, \lambda_{mp-1} \) (vid. [16] or [13]) satisfying

\[
d_3(\lambda_k) = \frac{1}{4} \left( -mp - 1 + 4k - \frac{4k^2}{mp} \right).
\]

Noting that \( S \cdot S = -p \), one computes

\[
d_3(\pi^*\lambda_k) = \frac{m}{4} \left( -mp - 1 + 4k - \frac{4k^2}{mp} \right) + \frac{3}{4} \cdot m(-1) - \frac{3}{4} \cdot (-1) - \frac{3}{4} \sum_{j=1}^{m-1} (-p) \csc^2(j\pi/m) \]
\[
= -m + km - \frac{k^2}{p} + \frac{3}{4} - \frac{p}{4} \]
\[
= \frac{1}{4} \left( -p + 3 + 4m(k-1) - \frac{4k^2}{p} \right),
\]

where the author used the well known identity \( \sum_{j=1}^{m-1} \csc^2(j\pi/m) = (m^2 - 1)/3 \), cf. Cauchy [2], Note VIII. Comparing this with the possible values of the \( d_3 \) invariants for the tight contact structures on \( L(p,1) \), one easily sees that the only values of \( k \) for which \( d_3(\pi^*\lambda) \) matches the value of the \( d_3 \) invariant of some tight contact structure on \( L(p,1) \) are \( k = 1 \) and \( k = mp - 1 \). These are the **universally** tight contact structures of \( L(mp,1) \) and, in any event, one knows that they lift to the universally tight contact structures on \( L(p,1) \), so there is nothing interesting happening in this family of examples; that is, all virtually overtwisted contact structures immediately lift to overtwisted contact structures.
Remark 8.8. As the reader may have foreseen, finding the equivariant almost complex filling of the contact $G$-manifold $(Y, \pi^* \lambda)$ in order to apply Theorem 8.5, is not typically an easy task. The remainder of this section, shall introduce a method for producing examples of such fillings by use of Kirby calculus. The rationale is the following. Firstly, one constructs a branched covering of 4-manifolds-with-boundary where the branching locus is allowed to intersect the boundary; secondly, a set of 2-handles is equivariantly glued in order to cap off the branching locus thereby making the boundary freely acted by the deck transformations.

Remark 8.9. Such matters shall require working with knot diagrams. The author prefers to draw what are called grid diagrams. These are knot diagrams where only vertical and horizontal lines occur, and apparent corners should be understood as being arcs with of a very small radius. This style of diagram has the advantage of bringing isotopies and Reidemeister moves more evidently into the realm of combinatorics. Figure 1 shows the example of the left-handed trefoil as a grid diagram.

Remark 8.10. Now, follows a series of standard definitions and propositions which shall be needed in pursuing the goal of the remainder of this section.

Definition 8.11. By a Legendrian knot or link in a contact 3-manifold $(Y, \lambda)$, one means a knot or link everywhere tangent to the contact structure $\text{Ker} \lambda$.

Remark 8.12. It is standard to represent Legendrian knots in $S^3 = (\mathbb{R}^3)^+$, with its standard contact form $\lambda = dz + xdy$, via their front projection diagrams. Those are the knot diagrams one obtains when projecting a knot to the $yz$-plane. This leads to a diagram that possesses no tangencies parallel to the $y$-axis but may have cusps pointing in directions parallel to the $z$-axis and whose transverse crossings must always have the strand passing underneath be the one for which the slope $dz/dy$ be highest. Conversely, any such diagram where all crossings be transverse is the front projection of a Legendrian knot. The grid knot diagrams, which the author favours, can readily be used to represent front projection diagrams if the following convention be agreed: the $y$-axis is understood to point in the southeast direction and the $z$-axis in the northeast direction. Here, the corners of the curve pointing in the
**Figure 2.** Another Legendrian left-handed trefoil

*southwest* and *northeast* directions are understood to be *smoothed* as before whereas those pointing in the *northwest* and *southeast* are understood to be *cusps*. Note that the diagram in Fig. 1 can be reinterpreted as a front projection diagram as it satisfies the required conditions. Figure 2 shows another inequivalent example of a Legendrian left-handed trefoil. When the author desire for the reader to interpret a particular diagram as Legendrian, he shall make it clear from the context; otherwise, the diagrams are to be understood simply as representing smooth links.

**Definition 8.13.** The *contact framing* of a Legendrian knot $K$ in $(Y, \lambda)$ is the framing induced by the contact structure $\text{Ker} \lambda$.

**Definition 8.14.** The *Thurston-Bennequin invariant*, $tb(K) \in \mathbb{Z}$, of a Legendrian knot $K \subset S^3$ is the value of the contact framing relative to the 0-framing; that is, relative to the framing induced by a Seifert surface.

**Proposition 8.15.** For a Legendrian knot $K \subset S^3$, if $w(K)$ denote its writhe and $c(K)$ denote the number of cusps in one of its front projection diagrams, then

$$tb(K) = w(K) - \frac{1}{2}c(K).$$

**Proof.** Vid. e.g. Geiges [10], Proposition 3.5.9. □

**Example 8.16.** The Legendrian left-handed trefoil $K$ in Fig. 2 has $tb(K) = -7$.

**Definition 8.17.** The *rotation number*, $r(K)$, of an oriented Legendrian knot in $S^3$ is the degree of the map $f : S^1 \to S^1$ defined as follows. Take some parametrization $\gamma : S^1 \to S^3$ of $K$. Let $\Sigma \subset S^3$ denote a Seifert surface for $K$. Trivialize the standard contact structure of $S^3$ over $\Sigma$ yielding a bundle isomorphic to $\Sigma \times \mathbb{R}^2$. Since $K$ is Legendrian, the derivative $\gamma'$ together with this trivialization defines a map $S^1 \to \mathbb{R}^2 \setminus \{0\}$. Given a deformation retraction $\mathbb{R}^2 \setminus \{0\} \to S^1$, this defines the desired map $f : S^1 \to S^1$.

**Definition 8.18.** For an oriented Legendrian knot $K \subset S^3$, denote, respectively, by $c_+(K)$ and $c_-(K)$ the number of upwardly and downwardly oriented cusps of a front projection diagram of $K$. In a Legendrian grid diagram
of the sort being used by the author, $c_+(K)$ is the number of corners which start by pointing east and finish by pointing north as one traverses it along the orientation of $K$, while $c_-(K)$ is the number of corners which start by pointing south and finish by pointing west.

**Proposition 8.19.** For an oriented Legendrian knot $K \subset S^3$, the rotation number can be computed by

$$r(K) = \frac{1}{2} (c_-(K) - c_+(K)).$$

**Proof.** Vid. e.g. Geiges [10], Proposition 3.5.19. □

**Example 8.20.** The Legendrian left-handed trefoil $K$ in Fig. 2 has $r(K) = 0$.

**Definition 8.21.** A 3-manifold $Y$ given as surgery on a Legendrian link in $S^3$ where the framing of a link component $K$ is precisely the contact framing minus 1 is called a Legendrian surgery. A Legendrian surgery naturally inherits a tight contact structure from $S^3$. Moreover, the 4-manifold-with-boundary obtained from the corresponding Kirby diagram has a natural complex structure; indeed, it is a Stein surface. For a reference, vid. Gompf [14]. This means that $d_3$ invariants can be read off this Kirby diagram as follows.

**Proposition 8.22.** [14] Let $(Y, \lambda)$ be a contact rational homology 3-sphere given as Legendrian surgery on a Legendrian link $\bigsqcup_{i=1}^n K_i$ in $S^3$. Use $L$ to denote the linking matrix of the link and $\sigma(L)$ its signature as a symmetric bilinear form. Then,

$$d_3(\lambda) = \frac{1}{4} \left( \sum_{i,j=1}^n r(K_i)(L^{-1})_{ij}r(K_j) - 2(n + 1) - 3\sigma(L) \right).$$

**Remark 8.23.** Let $(X, \lambda)$ be a contact rational homology 3-sphere. Suppose one have a Kirby diagram for an almost complex 4-manifold-with-boundary $(N, J)$ consisting entirely of 2-handles with its almost complex boundary being $\partial(N, J) = (X, \lambda)$. Label the 2-handles of $N$ as $\{h_i \mid i \in I\}$ and their respective attaching knots in $S^3$ as $\{K_i \mid i \in I\}$. Now, consider a subset of these 2-handles, denote it $\{h_i \mid i \in I'\}$, and remove them from the Kirby diagram. This yields another almost complex 4-manifold-with-boundary, call it $(N', J')$, having as its 2-handles the set $\{h_i \mid i \in I \setminus I'\}$; denote its almost complex boundary by $(X', \lambda') := \partial(N', J')$. Consider the link $L := \bigsqcup_{i \in I'} K_i$ as a link in $X'$. Fix some pseudoholomorphic curve $S' \subset N'$ with $\partial S' = L$ and $\text{int} S' \subset \text{int} N'$. Now, the $p$-fold branched covering of $N'$ with branching locus $S'$ exists precisely when the homology class $[S'] \in H_2(N', \partial N'; \mathbb{Z})$ be divisible by $p$. Assume that this be indeed the case. Then, in fact, the branched cover, call it $M'$, can be made to come with an almost complex structure and the covering map $\Pi' : M' \rightarrow N'$ can be made to respect the almost complex structures. Write $Y' := \partial M'$; this is a contact manifold. The restriction to the boundaries gives a branched covering of 3-manifolds $\pi' : Y' \rightarrow X'$ having branching locus the link $L \subset X'$. Required now is the concept of equivariant handle attachment.
Definition 8.24. Consider a 2-handle $h$ as a copy of $D^2 \times D^2$ to be attached along $S^1 \times D^2$. Regard $D^2 \times D^2$ as having the action of $G := \mathbb{Z}/p\mathbb{Z}$ defined by $e^{2\pi i/p} \cdot (z_1, z_2) := (z_1, e^{2\pi i/p} z_2)$. This action defines a $g$-fold branched covering $D^2 \times D^2 \to D^2 \times D^2$ with branching locus the core disk $D^2 \times \{0\}$. Think of this as the $G$-2-handle. When one be in possession of a $G$-4-manifold-with-boundary $M$ with a $G$-fixed surface $S$ having $\text{int} \ S \subset \text{int} \ M$ and $\partial S \subset \partial M$, define the equivariant handle attachment of $h$ along a component $K$ of $\partial S$ to be the $G$-4-manifold-with-boundary $M \cup h$ given by gluing $h$ to $M$ via an equivariant framing $\varphi$ of $K$; that is, an equivariant diffeomorphism onto its image $\varphi : S^1 \times D^2 \to M$ such that its image be a tubular neighbourhood of $K$.

Remark 8.25. Notice that the resulting manifold $M \cup h$ has one of the boundary components of the $G$-fixed surface $S$ capped by the $G$-fixed core disk of $h$. Hence, $M \cup h$ has one less $G$-fixed circle on its boundary compared to $M$.

Remark 8.26. Returning now to the context of Remark 8.23, one desires to equivariantly attach 2-handles to the manifold $M'$ in order to convert the branched covering of 3-manifolds $\pi : Y' \to X'$ to a genuine covering $\pi : Y \to X$, where $X$ be the 3-manifold with which one has started. Whether this is possible or not is a matter about the framings of the handles $\{h_i \mid i \in I'\}$ which were initially removed from $N$ resulting in the manifold $N'$. Consider a fixed $i \in I'$. Now, equivariantly attach a 2-handle $\tilde{h}_i$ along the knot $(\Pi')^{-1}(K_i)$ with framing determined by an integer $m_i \in \mathbb{Z}$. One hopes to be able to choose $m_i$ so that the corresponding quotient handle $\tilde{h}_i/G$ have the same framing as the handle $h_i$ of $N$; thereby allowing one to identify them. The main issue is that determining the behaviour of framings under the quotient turns out to be a subtle matter.

Instead of considering this general case, simplify the scenario as follows. Consider instead a manifold $N$ as above but consisting of a single 2-handle $h$ attached along a knot $K \subset S^3$ with framing $n$. Then, one forms $N'$ by removing $h$ to obtain, of course, the disk $D^4$. The manifold $M'$, therefore, is a familiar sort of manifold; it is the branched covering of $D^4$ branched over the Seifert surface of $K$ with its interior pushed into the interior of $D^4$. In this case, it is an easy matter to determine the behaviour of framings under the quotient. If one equivariantly attach a 2-handle $\tilde{h}$ to $M'$ along $(\Pi')^{-1}(K)$ with framing $m$, the framing of the quotient handle $\tilde{h}/G$ shall be $mp$. Hence, for the procedure to work, one needed $n$ to be a multiple of $p$.

The process of computing branched coverings in Kirby calculus is outlined in Gompf and Stipsicz [15, Sect. 6.3]. In general, it can be a difficult procedure where one obtains a Kirby diagram for the complement of a tubular neighbourhood of the branching surface, computes the genuine covering of the resulting manifold and then glues handles to fill in the hole left by the removal of the branching surface. There is a simpler approach, discussed in Rolfsen [46], which has the disadvantage of requiring blow-ups to be performed. By a blow-up the author means the formation of the connected sum.
with a copy of $\overline{\mathbb{CP}^2}$ or $\mathbb{CP}^2$. In the case of $\overline{\mathbb{CP}^2}$, almost complex complex structures can be naturally carried to the blow-up; in the case of $\mathbb{CP}^2$, this is not the case. Hence, for the purposes being pursued in this section, one may only perform blow-ups of the type where one takes the connected sum with a copy $\mathbb{CP}^2$. The procedure is as follows. Starting with the Kirby diagram of $N$ consisting of a single 2-handle $h$ attached along a knot $K$, perform a sequence of blow-ups in order to untie $K$ without changing the framing $n$ of the handle $h$. Now, the branched covering that one needs to compute is simply branched over a disk since $K$ has become the unknot. This is a significantly easier task than the general case.

Consider now a concrete example in which to apply this procedure. Start with the manifold $X$ given as $(-8)$-surgery on the left handed trefoil knot $K$. This manifold admits a double covering, which is the one that shall be studied here. One can perform a single blow-up to untie $K$ as depicted in Fig. 3. Hence, the manifold $N$ shall be the 2-handlebody defined by the right hand side Kirby diagram of Fig. 3. After a series of isotopies, one can achieve the form on the left hand side of Fig. 4.
After another isotopy, one obtains the diagram in right hand side of Fig. 4. What one desires to do now is to remove the \((-8)\)-framed handle in order to form the manifold \(N'\) as depicted in the left hand side of the Fig. 5. Once that be done, one can read off the double branched covering branched over the obvious Seifert disk for the knot \(K\). The resulting manifold, \(M'\), is depicted on the right hand side of Fig. 5. The more lightly stroked curve indicates the handle which has been removed, which is, therefore, also the boundary of the branching locus.

**Remark 8.27.** Notice that the framings behave under the branched covering in such a manner that the *blackboard framing* be preserved irrespective of the change in the writhe. This is why the \((-1)\)-framed handle lifts to a pair of \((+1)\)-framed handles.

Next, one equivariantly attaches a 2-handle along \((\Pi')^{-1}(K)\) in order to cap off the branching surface \(S\) leading to the branched covering of 4-manifolds-with-boundary seen in Fig. 6 where the branching locus no longer intersects the boundary.
**Proposition 8.28.** The manifold $X$ defined by $(-8)$-surgery on the left handed trefoil admits a contact form $\lambda$ which lifts via a double covering $Y \to X$ to a contact form $\pi^* \lambda$ having $d_3(\pi^* \lambda) = 1/4$.

*Proof.* Define $N$ and $M$ to be the 4-manifolds-with-boundary on the left and right hand side of Fig. 6 respectively. Here, the branching locus no longer intersects the boundaries $X = \partial N$ and $Y = \partial M$; therefore, one can apply Theorem 8.5 but only to the subset – and this subset may be proper – of the contact structures on $X$ which occur as almost complex boundaries of $N$. The author shall focus on a particular contact structure. Let $\lambda$ be the contact structure on $X$ defined as the Legendrian surgery according to the Legendrian representative of $K$ seen in Fig. 2. Recall, from Example 8.16, that the Thurston-Bennequin invariant of this Legendrian knot is $-7$; hence, the Legendrian surgery is indeed the topological $(-8)$-surgery. Since, in defining the 4-manifold-with-boundary $N$, only a $(-1)$-blow-up was performed, this contact structure is still an almost complex boundary of $N$. From Example 8.20, one knows that the rotation number is 0. With these data, one applies Proposition 8.22 to find that

$$d_3(\pi^* \lambda) = \frac{1}{4} (0 - 4 + 3) = -\frac{1}{4}.$$

Now, using Theorem 8.5, one computes

$$d_3(\pi^* \lambda) = -\frac{1}{2} + \frac{3}{4} \left( + 2 \sigma \begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix} - \sigma \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix} - \sum_{k=1}^{2} (-4) \csc^2 (\pi k/m) \right)$$

$$= \frac{1}{4}, \quad (2)$$

where the matrices are read off from Fig. 6. □

**Proposition 8.29.** The manifold $X$ defined by $(-8)$-surgery on the left handed trefoil is the prism manifold having Seifert fibration $(S^2; (1, -1), (3, 2), (2, 1), (2, 1))$.

*Proof.* The left-handed trefoil is a torus knot; hence, one can establish the Seifert invariants of $X$ from a well known theorem of Moser [39]. □

**Corollary 8.30.** The double cover $Y$ of $X$ is the lens space $L(12, 7)$.

*Proof.* One can compute coverings directly from the Seifert invariants. The Seifert manifold $(S^2; (1, -1), (3, 2), (2, 1), (2, 1))$ has a horizontal double covering (meaning it lifts Seifert fibres to Seifert fibres 1-to-1) with invariants

$$(S^2; (1, -1), (1, -1), (3, 2), (3, 2), (1, 1), (1, 1))$$

and this is one of the Seifert structures on the lens space $L(12, 7)$. □

**Remark 8.31.** The lens space $L(12, 7)$ can be obtained as surgery on a chain of three unknots with framings respectively $-2$, $-4$ and $-2$. According to the classification of tight contact structures on lens spaces due, independently, to...
Honda [16] and Giroux [13], the manifold $Y$ has precisely 3 isotopy classes of tight contact structures. Each of these contact structures comes from Legendrian surgery on the three possible Legendrian stabilizations of this chain of unknots having the appropriate Thurston-Bennequin invariants. The three Legendrian surgery diagrams are depicted in Fig. 7.

**Proposition 8.32.** The three tight contact structures on $Y = L(12, 7)$ depicted in Fig. 7 have $d_3$ invariants, respectively, equal to $-1/12$, $1/4$ and $-1/12$.

**Proof.** Compute using Proposition 8.22. \(\square\)

**Remark 8.33.** In light of Proposition 8.28, the contact structure depicted in the middle of Fig. 7 is the one that shall receive special attention as its $d_3$ invariant matches that of the lift of the tight contact structure on $X$ via the double cover. In order to apply Corollary 7.42 and see that these two contact structure are one and the same, what remains to be shown is whether the Spin$^C$ structures also agree. This problem shall be studied in the next section.

### 9. The spin-C structure and finite coverings

This section shall deal with a fairly elementary problem but one that turns out to be quite difficult to solve in practice. The problem is that of lifting Spin$^C$ structures across finite coverings. There are various methods which one might try to use in approaching this problem. For instance, in the case of Seifert manifolds, there is a certain canonical Spin$^C$ structure defined by the Seifert fibration. This canonical Spin$^C$ structure is pulled back naturally via coverings and, therefore, it suffices to have a good description of it upstairs and downstairs in the covering in order to understand how every other Spin$^C$ structure lifts. This is so because the set of Spin$^C$ structures on a manifold $Y$ is a $H^2(Y; \mathbb{Z})$-torsor, and, under the map induced by a covering, this torsor structure is preserved; hence, it suffices to understand the induced map in the second cohomology together with how one single Spin$^C$ structure lifts in order to deduce the action on every other. Another approach is to use contact topology. Suppose one know of a certain universally tight contact structure $\lambda$ on $Y$, then, its lift must always be universally tight via any covering. If there not be many of these upstairs, this reveals information about what the lift of

\[\text{Figure 7. Legendrian links whose Legendrian surgeries define the three contact structures of } L(12, 7)\]
the Spin\(^C\) structure associated with \(\lambda\) is. This strategy is particularly useful in the case of lens spaces. Another method is to turn to Spin structures. To any Spin structure is associated a Spin\(^C\) structure in a natural way. Hence, it is suffices to know how a Spin structure lifts via a covering to deduce how the Spin\(^C\) structures lift as well.

This last approach shall be the one pursued in the present section. Spin structures can be studied with the aid of the Kirby calculus and this fits well into the picture of lifting \(d_3\) invariants of the preceding section. Indeed, Gompf and Stipsicz [15] describe not only how to express Spin structures on a 3-manifold given by a Kirby diagram by ascribing extra decorations to it, but also how this description changes under Kirby moves. The main feat of the current section is to describe the manner in which this description of Spin structures behaves under finite coverings.

The double covering of the \((-8)\)-surgery on the left-handed trefoil considered in the previous section shall continue to furnish examples in this section. By the end, it shall be established that the tight contact structure whose lift was postulated to be tight indeed has its Spin\(^C\) structure lift to the correct one, so that the theorems concerning the equivariant contact invariant assert that the lift be tight.

**Definition 9.1.** Let \(M\) be an \(n\)-manifold with \(n \geq 3\) and use \(M_k\) to denote the \(k\)-skeleton of \(M\) in some cellular decomposition of \(M\). A Spin structure on \(M\) one means a trivialization of \(TM|_{M_1}\) which extend to \(TM|_{M_2}\).

According to Gompf and Stipsicz [15], Spin structures can be understood in terms of Kirby calculus in a rather convenient fashion which shall be recalled next. Consider a 3-manifold \(Y\) given as the boundary of a 2-handlebody \(M\). Denote by \(\{K_i \subset S^3 \mid i \in I\}\) the set of knots onto which the 2-handles of \(M\) are attached and by \(\{h_i \mid i \in I\}\) the corresponding 2-handles.

**Definition 9.2.** Given a Spin structure \(S\) on \(Y\), define the class \(w_2(M, S) \in H^2(M, Y; \mathbb{Z}/2\mathbb{Z})\) as the obstruction to extending \(S\) from \(Y\) to \(M\).

**Remark 9.3.** By standard obstruction theory, this class can be characterized in terms of the its evaluations on each of the classes \([D_i, \partial D_i] \in H_2(M, Y; \mathbb{Z}/2\mathbb{Z})\) associated to the cocores \(\{D_i\}\) of the 2-handles \(\{h_i\}\). One can ask whether \(S\) extends across \(h_i\) to a Spin structure on \(h_i \cong D^2 \times D^2\). It follows that \(w_2(M, S)\) evaluates on the class \([D_i, \partial D_i]\) as 0 when \(S\) extends across \(h_i\) and as 1 when it does not.

**Definition 9.4.** An element \(w\) of \(H^2(M, Y; \mathbb{Z}/2\mathbb{Z})\) is called characteristic when it gets mapped to the second Stiefel-Whitney class \(w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})\) via the map \(H^2(M, Y; \mathbb{Z}/2\mathbb{Z}) \to H^2(M; \mathbb{Z}/2\mathbb{Z})\).

For each \(i\), suppose \(F_i \subset M\) to be a closed surface constructed by taking a Seifert surface for the knot in \(S^3\) along which the 2-handle \(h_i\) was attached, pushing the interior of this Seifert surface into the interior of \(D^4\) and capping it by gluing it to the core disk of the handle \(h_i\). Orient the closed surface \(F_i\) in a manner compatible to the orientation of the knot along which \(h_i\) was attached. Notice that the classes \([F_i]\) span \(H_2(M; \mathbb{Z})\) as a \(\mathbb{Z}\)-module.
Proposition 9.5. A class \( w \in H^2(M, Y; \mathbb{Z}/2\mathbb{Z}) \) is characteristic if and only if, for all \( i \in I \),
\[
w([D_i, \partial D_i]) = [F_i] \cdot [F_i] \mod 2.
\]

Proof. Follows from Wu’s formula. Vid. Gompf and Stipsicz [15], Exercise 5.7.3. \( \square \)

Proposition 9.6. [15] The class \( w_2(M, S) \in H^2(M, Y; \mathbb{Z}/2\mathbb{Z}) \) completely characterizes the Spin structure \( S \) on \( Y \). Conversely, given any characteristic element \( w \in H^2(M, Y; \mathbb{Z}/2\mathbb{Z}) \) there is some Spin structure \( S \) on \( Y \) such that \( w_2(M, S) = w \).

Remark 9.7. In a Kirby diagram, the author shall denote a Spin structure \( S \) by writing a 0 or a 1 following the framing coefficient separated from it by a comma next to each 2-handle. This signifies the value of the evaluation of \( w_2(Y, S) \) on the class in \( H^2(M, Y; \mathbb{Z}/2\mathbb{Z}) \) corresponding to the respective 2-handle. Figure 8 shows this notation in the main example from the preceding section.

Remark 9.8. It is then possible to keep track of the Spin structure during the performance of Kirby moves. The rule when performing a handle slide of \( h_i \) over \( h_j \), where the respective components of \( w_2(M, S) \) be \( n_i \) and \( n_j \), is that \( n_j \) changes precisely when \( n_i = 1 \), whereas \( n_i \) always stays unchanged. For a blow-up, the component of \( w_2(M, S) \) associated to the newly attached \((\pm 1)\)-framed 2-handle is always 1. Figure 9 illustrates the result of both these operations.

The matter is now to consider how a Spin structure lifts via a finite covering in the Kirby calculus setting. As in the previous section, the procedure shall be carried out in two parts. Firstly, one performs a branched covering over a 4-manifold-with-boundary obtained by judiciously removing 2-handles and then one equivariantly glues 2-handles in the branched covering to cap the branching locus. In the first part, one needs to take care of how a Spin structure behaves near the free 2-handles; in the second part, one needs to take care of how it behaves near the non-free 2-handles.

Consider a 3-manifold \( X \), the boundary of a 4-manifold-with-boundary \( N \) having only a 0-handle and a set of 2-handles \( \{ h_i \mid i \in I \} \). Let \( S \) be a
Spin structure on $X$. Denote by $\{K_i \subset S^3\}$ the set of attaching knots of the 2-handles and by $\{D_i\}$ the cocores. Given a subset of 2-handles $\{h_i \mid i \in I'\}$, denote by $N'$ the 4-manifold-with-boundary given by removing this set of 2-handles from $N$; denote by $X'$ its boundary. Now, assume that there exist the $p$-fold branched covering $\Pi' : M' \to N'$ branched at a Seifert surface $S'$ for the link $L := \bigsqcup_{i \in I'} K_i$ with its interior pushed into the interior of $D^4$. Let $Y' := \partial M'$ and $\pi' : Y' \to X'$ be the restriction of $\Pi'$. For each 2-handle $h_i$ of $N'$, the lift $(\Pi')^{-1}(h_i)$ consists of $p$ 2-handles of $M'$ freely permuted by the deck transformations.

**Proposition 9.9.** For each handle $h$ in $(\Pi')^{-1}(h_i)$ with cocore $D$,

$$w_2(Y', (\pi')^*S)([D, \partial D]) = w_2(X', S)([D_i, \partial D_i]).$$

**Proof.** Recall that $w_2(X', S)([D_i, \partial D_i])$ is characterized by whether the Spin structure $S$ extends to the whole interior of the 2-handle $h_i$. That extension existing, the lifted Spin structure $(\pi')^*S$ shall also extend over each of the preimage 2-handles $h$ of $h_i$. Conversely, should the extension not exist downstairs, it cannot extend upstairs over any $h$ either. □

**Example 9.10.** Figure 10 exemplifies Proposition 9.9 in the already familiar setting of the branched double cover over the left-handed trefoil $K$ from the preceding section (cf. Fig. 5).

Now, one needs to understand what happens to the non-free 2-handles. Proceeding as in Remark 8.26, one equivariantly attaches a 2-handle $\tilde{h}_i$ along $(\Pi')^{-1}(K_i)$ for each $i \in I'$ in order to cap the branching locus thereby defining the manifolds $M := M' \cup \bigsqcup_{i \in I'} \tilde{h}_i$ and $N := M/G$. Let $m_i$ denote the framing coefficient of the equivariant handle $\tilde{h}_i$. As before, one hopes to be able to choose $m_i$ so that the original manifold $Y$ be $\partial N$. Assume that this be the case. The question then becomes: given the value of $w_2(X, S)([D_i, \partial D_i])$ for an $i \in I'$, what is the value of $w_2(Y, \pi^*S)([\tilde{D}_i, \partial \tilde{D}_i])$ where $\tilde{D}_i$ denotes the cocore of $\tilde{h}_i$?

**Proposition 9.11.** For each $i \in I'$, if $w_2(X, S)([D_i, \partial D_i]) = 1$, then $w_2(Y, \pi^*S)([\tilde{D}_i, \partial \tilde{D}_i])$ is always 1 as well. Conversely, if $w_2(X, S)([D_i, \partial D_i]) = 0$, then...
then \(w_2(Y, \pi^*S)([\tilde{D}_i, \partial\tilde{D}_i])\) is 1 if the covering multiplicity \(p\) be even and 0 otherwise.

**Proof.** Firstly, consider the 2-handle \(D^2 \times D^2\) having the Spin structure \(S_0\) defined along \(D^2 \times S^1\) which does not extend across \(D^2 \times D^2\). Consider the \(p\)-fold branched cover \(D^2 \times D^2 \rightarrow D^2 \times D^2\) given by rotating the second \(D^2\) factor and keeping the first one fixed. One can trivialize the bundle \(T(D^2 \times D^2)|_{D^2 \times S^1}\) by splitting it as \(\mathbb{R}^3 \oplus TS^1\). It is clear now that this trivialization is pulled back to itself along the restriction of the covering to \(D^2 \times S^1\). This means that the Spin structure \(S_0\) is pulled back to itself and the lift also does not extend across \(D^2 \times D^2\). Secondly, consider instead the Spin structure \(S\) which extend across \(D^2 \times D^2\). Then, to understand the lifting behaviour, make use of the other Spin structure already considered, that is \(S_0\), which does not extend across the covering. The obstruction class to a homotopy between \(S\) and \(S_0\) can be seen as a non-trivial class in the cohomology \(H^1(S^1; \pi_1(SO(4))) \cong \pi_1(SO(4))\). One now easily sees that, under the \(p\)-fold covering, this class in \(\pi_1(SO(4))\) gets traversed \(p\) times. Since \(\pi_1(SO(4))\) is the cyclic group in two elements, the result follows. \(\square\)

**Example 9.12.** Figure 11 continues the line of examples coming from the \((-8)\)-surgery on the left-handed trefoil and shows the perhaps slightly unexpected behaviour in this case: the Spin structure does extend across the \((-8)\)-framed handle downstairs but, upstairs, its lift does not extend across the lifted 2-handle because the cover is a double cover.

**Proposition 9.13.** Let \(X\) be the \((-8)\)-surgery on the left-handed trefoil. Let \(S\) be the Spin structure on \(X\) as defined by the Kirby diagram in Fig. 8. Let \(Y\) be the lens space \(L(12,7)\). Denote by \(\pi: Y \rightarrow X\) the double covering. Let \(M\) be the standard linearly plumbed 4-manifold-with-boundary whose boundary is \(L(12,7)\). Then, the Spin structure \(\pi^*S\) on \(Y\) is the one for which \(w_2(M, \pi^*S) = 0\).

**Proof.** Using the techniques developed above, this becomes a Kirby calculus affair. Starting from what is seen on the right hand side of Fig. 11, one computes, as shown in the following sequence of diagrams, (i) by blowing down of
one of the +1-framed unknots, (ii) blowing up twice by +1-framed unknots and (iii) blowing down the −1-framed unknot.

Now, if one pay careful attention at the signs of the crossings on the last diagram, one sees that one can perform a handle slide of any one of the 2-framed handles over the (−2)-framed handle in order to obtain a linear chain of unknots. Thence, it is a straightforward matter to perform blow-ups and blow-downs to reach the standard plumbing diagram of the
lens space $L(12,7)$. The procedure is outlined in the following diagrams.

**Proposition 9.14.** A Spin$^C$ structure on a 3-manifold $Y$ consists of precisely the same data as a complex trivialization of $TY \oplus \mathbb{R}$ over the 2-skeleton of $Y$ which extend across its 3-skeleton.

**Proof.** Follows from obstruction theory. Cf. Gompf and Stipsicz [15], Remark 5.6.9(a), for the case of Spin structures. Also vid. Gompf [14].

**Remark 9.15.** From this characterization of Spin$^C$ structures on $Y$, it is easy to see that a Spin structure $\mathcal{S}$ defines a Spin$^C$ structure by taking the trivialization of $TY|_{Y_2}$ defined by $\mathcal{S}$ and picking the trivial complex structure on $TY|_{Y_2} \oplus \mathbb{R}$. One can then check that this trivialization extends to $TY|_{Y_3}$.

**Definition 9.16.** Given a Spin structure $\mathcal{S}$, denote its induced Spin$^C$ structure by $\mathcal{S}^C$.

**Proposition 9.17.** [14, Theorem 4.12] Let $(Y, \lambda)$ be a contact 3-manifold given as Legendrian surgery on a Legendrian link $L := \bigsqcup_{i \in I} K_i$ in $S^3$. Let $M$ be the 4-manifold-with-boundary defined by the same Kirby diagram. Suppose $\mathcal{S}$ be a Spin structure on $Y$ and use $L' = \bigsqcup_{i \in I'} K_i \subset L$ to denote the sublink of $L$ consisting of those components $K_i$ for which $w_2(Y, \mathcal{S})([D_i, \partial D_i]) \neq 0$, where $D_i$ denotes the cocore of the handle attached along $K_i$. Recall that the author uses $s_\lambda$ to denote the Spin$^C$ structure defined by the contact form $\lambda$. Then, the difference class

$$s_\lambda - S^C \in H^2(Y; \mathbb{Z})$$
of Spin\(^C\) structures is determined by the restriction to \(Y\) of a class \(\rho \in H^2(M; \mathbb{Z})\) defined by its evaluations on the classes \([F_i] \in H^2(M; \mathbb{Z})\), according to the formula
\[
(s_\lambda - S^C)([F_i]) = \frac{1}{2} \left( r(K_i) + \sum_{j \in I'} [F_i] \cdot [F_j] \right).
\]

Example 9.18. In the case of the Spin 3-manifold \(X\) defined by Fig. 8 and studied in Proposition 9.13, consider the Spin\(^C\) structure \(s_\lambda\) defined by the contact structure \(\text{Ker} \lambda\) produced by Legendrian surgery on the Legendrian left-trefoil depicted in Fig. 2, that is, having Thurston-Bennequin invariant \(-7\) and rotation number zero. Computing using Proposition 9.17, one readily finds that \(s_\lambda - S^C = 0\).

Remark 9.19. Using Proposition 9.17 and the Kirby calculus techniques developed above, one can compute the lift of a Spin\(^C\)-structure \(s_\lambda\) via a finite covering by computing the lift of a Spin structure \(S\lambda\) and then computing the lift of the second cohomology class \(s_\lambda - S^C\).

Theorem 9.20. Consider the manifold \(X\) given by \((-8)\)-surgery on the left-handed trefoil with tight contact structure \(\text{Ker} \lambda\) given by Legendrian surgery according to Fig. 2. The lift of \(\text{Ker} \lambda\) via the double cover \(\pi : Y \to X\) is the tight contact structure \(\tilde{\lambda}\) on \(Y \cong L(12, 7)\) given by Legendrian surgery on the middle diagram of Fig. 7.

Proof. By Remark 8.33, \(d_3(\pi^* \lambda) = d_3(\tilde{\lambda})\). Meanwhile, Proposition 9.13 and Example 9.18 combine to assert that the Spin\(^C\) structures also match; that is, \(\pi^* s_\lambda \cong s_{\tilde{\lambda}}\). Since the \(d_3\) invariant and the Spin\(^C\) structure completely characterize the homotopy class of a hyperplane field, \(\pi^* \lambda\) and \(\tilde{\lambda}\) are homotopic. Now, according to Corollary 7.42, this is a sufficient condition for \(\pi^* \lambda\) and \(\tilde{\lambda}\) to be isotopic. \(\Box\)

Remark 9.21. To the best of the author’s knowledge, there is no other way to obtain this result short of working in coordinates, which would probably prove itself to be untenable.

Acknowledgements

The author is greatly indebted to his advisor S. Sivek for the invaluable guidance throughout his doctoral studies and the sage commentary during the writing of the present article. The author would like to thank F. Lin, L. Nicolaescu and C.H. Taubes for insightful email correspondences. He also thanks the referee for his suggestions and corrections. This work was supported by the Engineering and Physical Sciences Research Council.
Author contributions Author “Roso” is responsible for everything in the manuscript.

Declarations

Conflict of interest The authors declare no conflict of interests.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

10. Appendix

This appendix shall deal with the proof of Theorem 2.25, where it was claimed that there are no non-trivial Seiberg–Witten trajectories with positive infinity limit the contact configuration. A very similar assertion is made by Taubes [51], Proposition 5.15. The key difference is that Taubes deals with the case of a non-torsion $\det s_{\lambda}$, which is not the case here. The fact that $\det s_{\lambda}$ is non-torsion in Taubes [51], Proposition 5.15, also causes its conclusion to be somewhat weaker than what is proved here. Another minor difference is that, in the present article, the author must not assume that the negative infinity limit of the trajectory be non-degenerate. These differences require only significant modifications to the final part of the proof; therefore, the proof presented here shall be very similar to what one can find in Taubes [51] and other works of Taubes that have drawn significantly from it. However, Taubes’ proof is difficult to follow for someone who has not read Taubes [50] and Taubes [51] in their entirety; therefore, the author felt that it was necessary to include some of the details here for the reader’s convenience. The reader can also find a complete account with self-contained proofs for all lemmata in Roso [47, Sect. 12].

The overall strategy shall consist of the following. On the one hand, it shall be shown that only the contact configuration and its gauge equivalent configurations have the property that the spinor component be bounded away from zero in a certain way. On the other, it shall be shown that the trajectory necessarily satisfies the same sort of bound on its spinor component for all time. As a consequence, both endpoints shall have to be gauge equivalent.
Henceforth, the author shall deal with Seiberg–Witten trajectories. Suppose \( \gamma : \mathbb{R} \to \mathcal{C} \) to be a finite type trajectory satisfying
\[
\frac{d}{dt} \gamma(t) = -\mathcal{X}_{\lambda, r} \left( \gamma(t) \right).
\]
Assume that both limits \( \lim_{t \to \pm \infty} \gamma(t) \) be well defined in the \( L^2 \) topology. Note that, by Sobolev embedding, the limits are also well defined in \( C^2 \).
The author shall not yet be assuming that the positive time limit be the contact configuration \( C_\lambda \); that assumption shall be added later. Write \( \gamma(t) = (A(t), \psi(t)) \in \mathcal{C}(Y, s_\lambda) \) for its connexion and spinor components respectively.
Decompose the spinor as \( \psi(t) = (\alpha(t), \beta(t)) \) according to the direct sum \( S_\lambda = \Lambda^{0,0} \xi^* \oplus \Lambda^{0,1} \xi^* \).

**Lemma 10.1.** The spinor \( \psi \) satisfies the second order equation
\[
-\frac{\partial^2}{\partial t^2} \psi + \nabla_A^* \nabla_A \psi - c\ell \left( \frac{1}{2} *F_A - r\tau(\psi) - \frac{ir\lambda}{2} \right) \psi + \frac{s}{4} \psi = 0,
\]
where \( s : Y \to \mathbb{R} \) denotes the scalar curvature of \( Y \).

**Proof.** The Dirac equation
\[
\frac{\partial}{\partial t} \psi = -\mathcal{D}_A \psi
\]
implies the second order equation
\[
\frac{\partial^2}{\partial t^2} \psi = -\frac{\partial}{\partial t} \mathcal{D}_A \psi = -\mathcal{D}_A \frac{\partial}{\partial t} \psi - c\ell \left( \frac{\partial}{\partial t} A \right) \psi = \mathcal{D}^2_A \psi - c\ell \left( \frac{\partial}{\partial t} A \right) \psi.
\]
Now, apply the well known Weitzenböck formula (cf. [40], (1.3.11)),
\[
\mathcal{D}_A^* \mathcal{D}_A \psi = \nabla_A^* \nabla_A \psi - \frac{1}{2} c\ell (\ast F_A) \psi + \frac{s}{4} \psi,
\]
to find that
\[
-\frac{\partial^2}{\partial t^2} \psi + \nabla_A^* \nabla_A \psi - c\ell \left( \frac{\partial}{\partial t} A + \frac{1}{2} \ast F_A \right) \psi + \frac{s}{4} \psi = 0.
\]
The result then follows by applying the curvature component of the Seiberg–Witten trajectory equations. \( \square \)

**Remark 10.2.** A distinction worth mentioning between what is done here and what is done in Taubes [51] is that some (but not all) of the results in Taubes [51] concern a different version of the Seiberg–Witten trajectory equations. To be precise, this different version says
\[
\frac{\partial}{\partial t} A = -\frac{1}{2} (\ast (F_A - F_{A_\lambda}) - r e^{T(t)} (\tau(\psi) - \frac{ir\lambda}{2}) \), \quad \frac{\partial}{\partial t} \psi = -\mathcal{D}_A \psi,
\]
for a fixed function \( T : \mathbb{R} \to [0, \infty) \) satisfying \( T(t) = 0 \) for all \( t \leq 1 \) and \( T(t) = t \) for all \( t \geq 2 \). The equations at use in the present article are equivalent to setting \( T = 0 \) everywhere. The use of such a \( T \) is certainly important in Taubes [51] and brings about some complications which do not occur in the present context. Some of the results proved in Taubes [51] for this version of the Seiberg–Witten equations are easily adapted to the present context as is
remarked in the proof of Taubes [51, Proposition 5.15]. In particular, for the
next two lemmata, the author shall refer the reader to Taubes [51].

Remark 10.3. In what follows, a sequence of bounds on various quantities
shall be claimed. These shall involve constants $K_n$ labelled by $n$ the number
of the lemma in which they appear. These quantities may depend on the
metric of $Y$, and the contact form $\lambda$, but they shall not depend on the par-
ticular solution $(A, \psi)$ to the Seiberg–Witten equations nor shall they shall
not depend on the value of the parameter $r > 0$.

Lemma 10.4. There is a constant $K_{10.4} > 0$ such that
$$|\alpha|^2 + |\beta|^2 \leq 1 + K_{10.4}r^{-1}.$$  
Proof. Analogous to the first statement of Taubes [51], Lemma 5.1 with the
function $T$ from Remark 10.2 set to zero. Alternatively, vid. Roso [47], Lemma
12.5. □

Lemma 10.5. There are constants $K_{10.5} > 0$ and $r_{10.5} > 0$ such that for any
$r > r_{10.5}$ the following inequality hold.
$$|\beta|^2 \leq K_{10.5} (r^{-1}(1 - |\alpha|^2) + r^{-2}).$$  
Proof. Analogous to the second statement of Taubes [51], Lemma 5.1 with the
function $T$ from Remark 10.2 set to zero. Alternatively, vid. Roso [47,  
Lemma 12.10]. □

Lemma 10.6. There is a constant $K_{10.6} > 0$ such that
$$\left| \frac{\partial}{\partial t} A \right| + |F_A| \leq K_{10.6}r.$$  
Proof. Analogous to Taubes [51, Proposition 5.2] with the function $T$ from
Remark 10.2 set to zero. Alternatively, vid. Roso [47, Lemma 12.25]. □

Henceforth, add the assumption that
$$\lim_{t \to +\infty} \gamma = C_\lambda.$$  
The goal shall be bound $|\alpha|^2$ away from zero in the same manner as is the
case for $C_\lambda$ but for the entirety of the trajectory.

For that end, start by introducing the symplectic form $\omega := e^{2t}(dt \wedge
\lambda + *\lambda)$, on $\mathbb{R} \times Y$ and fixing some almost complex structure $J$ compatible
with $\omega$. Also fix a non-decreasing $C^\infty$ function $\sigma : [0, \infty) \to [0, 1]$ satisfying
$\sigma|_{[0,1/2]} = 0$ and $\sigma|_{[1,\infty)} = 1$. Use $\sigma'$ to denote its derivative. Assume further that $\sigma' : \mathbb{R} \to [0, 3]$.

Definition 10.7. Given $\delta > 0$, denote $\sigma_\delta : \mathbb{R} \times Y \to [0, 1]$ the function
$$(t, y) \mapsto \sigma (\delta^{-1} (1 - |\alpha|^2)).$$  

Definition 10.8. Given $\delta > 0$, denote $\sigma'_\delta : \mathbb{R} \times Y \to [0, 1]$ the function
$$(t, y) \mapsto \sigma' (\delta^{-1} (1 - |\alpha|^2)).$$
Now, introduce $\nabla_A$ to denote the Spin$^C$ covariant derivative on $S_\lambda$ induced by the connexion $A$ on $\det s_\lambda$. This is to say that Clifford multiplication is covariantly constant for $\nabla_A$. It shall also be necessary to work with covariant derivatives defined on each of the summands $S_\lambda \cong \Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^*$. Both these derivatives shall be denoted by the same symbol as
\[
\nabla_A' : \Gamma(\Lambda^{0,k}\xi^*) \to \Gamma(\Lambda^{0,k}\xi^* \otimes T^*Y)
\]
and shall be defined by projecting $\nabla_A$ to the respective summand of $S_\lambda$. This can be conveniently expressed in terms of Clifford multiplication by the formula
\[
\nabla_A' = \frac{1}{2} \left( 1 + i(-1)^k c\ell(\lambda) \right) \nabla_A.
\]

It shall prove advantageous to work with objects over $\mathbb{R} \times Y$ rather than time dependent objects over $Y$. For that end, introduce $\hat{d}$ and $\hat{*}$ to denote, respectively, the exterior derivative and the Hodge operator over $\mathbb{R} \times Y$. The Spin$^C$ structure $s_\lambda$ over $Y$ also defines a Spin$^C$ structure over $\mathbb{R} \times Y$, which shall be denoted $\hat{s}_\lambda$. The time dependent connexion $A$ over $\det s_\lambda$ also defines a connexion on the bundle $\det \hat{s}_\lambda$; denote this connexion by $\hat{A}$ and note that it is characterized by requiring its induced covariant derivative to be
\[
\nabla_{\hat{A}} := dt \otimes \frac{\partial}{\partial t} + \nabla_A.
\]
Hence, one sees that its curvature is given by
\[
F_{\hat{A}} = dt \wedge \frac{\partial}{\partial t} A + F_A.
\]
The covariant derivative $\nabla_{\hat{A}}'$ induces a covariant derivative on the trivial complex line bundle over $\mathbb{R} \times Y$ defined by
\[
\nabla_{\hat{A}}' := dt \otimes \frac{\partial}{\partial t} + \nabla_A'.
\]
This covariant derivative can then be extended to a covariant exterior derivative on complex differential forms, which shall be denoted
\[
\hat{d}_{\hat{A}} : \Omega^k(\mathbb{R} \times Y) \otimes \mathbb{C} \to \Omega^{k+1}(\mathbb{R} \times Y) \otimes \mathbb{C}.
\]

\textbf{Definition 10.9.} Given $\delta > 0$, define a 2-form $\varphi_\delta \in \Omega^2(\mathbb{R} \times Y)$ by the formula
\[
\varphi_\delta := \frac{1}{\delta} \sigma' \cdot \hat{d}_{\hat{A}} \alpha \wedge \hat{d}_{\hat{A}} \overline{\alpha} + \sigma_\delta \cdot F_{\hat{A}}.
\]

\textbf{Remark 10.10.} Notice that the form $\varphi_\delta$ is closed.

\textbf{Lemma 10.11.} For all $\delta \in (0, \delta_{2.24})$, there exists $s_\delta \in \mathbb{R}$, such that, the 2-form $\varphi_\delta$ vanishes identically on $[s_\delta, \infty) \times Y \subset \mathbb{R} \times Y$.

\textit{Proof.} By Theorem 2.24 and the fact that $\lim_{t \to \infty} \gamma = C_\lambda$ in the $C^0$ topology, it follows that there exists $s_\delta \in \mathbb{R}$ such that, for $t > s_\delta$, $|\psi(t)| \geq 1 - \delta_{2.24}$. Since the function $\sigma$ is supported on $[1/2, \infty)$, the claimed result follows. \qed

\textbf{Lemma 10.12.} For $r > r_{2.24}$ and $\delta \in (0, \delta_{2.24})$, $\omega \wedge \varphi_\delta$ is integrable over all of $\mathbb{R} \times Y$. 
Proof. Notice that there is some $K > 0$, which may well depend on $r$, $A$ or $\psi$ here, such that, for sufficiently negative $s < 0$, $|\omega \wedge \varphi_\delta|$ restricted to $(-\infty, s)$ is no greater than $Ke^s$. This is a consequence of the fact that the limit $\lim_{t \to -\infty} \gamma(t)$ is well defined in the Sobolev norm $L^2_t$ and, therefore, also in $C^2$ by Sobolev embedding. Together with Lemma 10.11, this implies integrability. □

Lemma 10.13. Provided $r > r_{2.24}$ and $\delta \in (0, \delta_{2.24})$, it follows $\int_{\mathbb{R} \times Y} \omega \wedge \varphi_\delta = 0$.

Proof. Note that $\omega$ is exact and $\varphi_\delta$ is closed and integrate by parts. □

Introduce the notation $\bar{\partial}_A : \Omega^{i,j}(\mathbb{R} \times Y) \to \Omega^{i+1,j}(\mathbb{R} \times Y)$, $\bar{\partial}_A : \Omega^{i,j}(\mathbb{R} \times Y) \to \Omega^{i,j+1}(\mathbb{R} \times Y)$, for the covariant Cauchy–Riemann operators associated to the connexion $\hat{A}$ and the almost complex structure $J$. It can be verified that, for $\eta \in \Omega^{0,0}(\mathbb{R} \times Y)$, they take the form

$$\partial_A \eta := \frac{1}{2} \left( \partial_A \eta + i e^{-2t} \hat{\gamma}(\omega \wedge \partial_A \eta) \right), \quad \bar{\partial}_A \eta := \frac{1}{2} \left( \bar{\partial}_A \eta - i e^{-2t} \hat{\gamma}(\omega \wedge \bar{\partial}_A \eta) \right).$$

Lemma 10.14. Vanishing of the integral in Lemma 10.13 amounts to saying

$$\int_{\mathbb{R} \times Y} \hat{\gamma}^2 \left( \delta^{-1} \sigma_\delta \left( |\partial_A \alpha|^2 - |\bar{\partial}_A \alpha|^2 \right) + r \sigma_\delta \left( 1 - |\alpha|^2 + |\beta|^2 \right) \right) = 0.$$

Proof. Can be checked directly by expanding with the expressions given above for $\partial_A$ and $\bar{\partial}_A$ acting on 0-forms and by using the four-dimensional Seiberg–Witten curvature equation. □

Lemma 10.15. (cf. [51, Lemma 5.13]) For a given $\delta \in (0, \delta_{2.24})$, there exist $r_{10.15} > 0$ and $K_{10.15} > 0$ such that, for all $r > r_{10.15}$,

$$|\bar{\partial}_A \alpha| \leq K_{10.15}.$$

Proof. It is well known (cf. [40, Sect. 1.4.3]), that, in terms of the notation above, the Dirac equation,

$$\frac{\partial}{\partial t} \psi = -\mathcal{D}_A \psi$$

takes the more familiar form

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = f_1(\psi),$$

where $f_1$ is some bundle homomorphism independent of $r$. Hence, a bound on $\bar{\partial}_A \alpha$ of the variety required follows from a bound on $\frac{\partial}{\partial t} \beta$, $\nabla_A \beta$ and $|\psi|$. The bound on $|\psi|$ was already provided by Lemma 10.4. The rest of the proof shall concern itself with the other two bounds.

Let $r_{10.15}$ be large enough so that the injectivity radius of $Y$ be strictly larger than $(r_{10.15})^{-1/2}$ and so that the bundle $\Lambda^{0,2}*(\mathbb{R} \times Y)$ be trivial when restricted to any ball of radius $(r_{10.15})^{-1/2}$ in $\mathbb{R} \times Y$. Then, let $r > r_{10.15}$, and fix a point $p \in \mathbb{R} \times Y$. Introduce $\phi_{p,r}$ to denote the Gaussian chart centred
at $p$ rescaled so that the ball of radius 1 of $\mathbb{R}^4$ be mapped to the geodesic ball of radius $r^{-1/2}$. Now, let $\beta : B(\mathbb{R}^4, 1) \to \mathbb{C}$ denote the pullback of $\beta$ via this chart seen as a complex valued function by trivialising the bundle $\Lambda^{0,2}\mathbb{T}^*(\mathbb{R} \times Y)$ on this chart via the parallel transport map of the connexion $\hat{A}$. One can check that Lemma 10.1 implies a certain second order equation for $\tilde{\beta}$ of the form

$$\Delta \tilde{\beta} + \sum_{j=1}^{4} f_{j+1} \frac{\partial}{\partial x_j} \tilde{\beta} + f_6 \tilde{\beta} + r^{-1} f_7 = 0,$$

where $f_2, \ldots, f_7$ are complex valued functions with norms bounded above by some constant $K_1 > 0$ independent of $r$ and the point $p$. Recall Lemma 10.5; it implies that, for some constant $K_2 \geq K_1$ also independent of $r$ and $p$, that $|\beta| \leq K r^{-1/2}$. Standard elliptic theory then provides a bound of the form

$$\left| \frac{\partial}{\partial x_j} \tilde{\beta}(0) \right| \leq K_3 r^{-1/2},$$

where $K_3 \geq K_2$ is some potentially larger constant independent of $r$ and $p$. Since the chart at hand was scaled so that the ball of radius 1 be mapped to the geodesic ball of radius $r^{-1/2}$, and $\Lambda^{0,2}\mathbb{T}^*(\mathbb{R} \times Y)$ was trivialized by the parallel transport of the connexion $\hat{A}$, whose curvature satisfies the bound of Lemma 10.6, if the reader care to check, it follows that, for some constant $K_4 \geq K_3$ independent of $r$ and $p$,

$$\left| \frac{\partial}{\partial t} \tilde{\beta} \right| + |\nabla_A \tilde{\beta}| \leq K_4.$$

□

The next few results needed shall require mention of a variant of the vortex equations on $\mathbb{R}^4 = \mathbb{C}^2$. These are defined next. In what follows, use $\omega_0$ to denote the standard symplectic form on $\mathbb{R}^4$.

**Definition 10.16.** Consider a pair $(a_0, \alpha_0)$, where $a_0 \in i\Omega^1(\mathbb{R}^4)$ and $\alpha_0 \in \Omega^{0,0}(\mathbb{R}^4)$. Then, $(a_0, \alpha_0)$ satisfy the vortex equations with bound $K > 0$ when

$$\bar{\partial} \alpha_0 = 0, \quad |\alpha_0| \leq 1, \quad d^+ a = \frac{1}{2} (1 - |\alpha_0|^2) \omega_0, \quad |d^- a| \leq K.$$

**Lemma 10.17.** [51, Lemma 5.14] For any $K > 0$ and $\delta \in (0, 1)$, there exist $R_{10.17} > 2$ and $K_{10.17} > 1$ such that, for any vortex $(a_0, \alpha_0)$ with bound $K$, if one use $V$ to denote the volume of the set

$$\{ x \in B(\mathbb{R}^4, R_{10.17}) \mid (1 - |\alpha_0(x)|^2) > \delta \},$$

and $V'$ denote the volume of the set

$$\{ x \in B(\mathbb{R}^4, \frac{1}{2} R_{10.17}) \mid \delta > (1 - |\alpha_0|^2) \geq \frac{1}{2} \delta \},$$

then it follows that $V' \leq K_{10.17} V$.

**Remark 10.18.** The ability to apply this lemma in the present context comes from the following.
**Lemma 10.19.** There exists $K_{10.19} > 0$ such that, for any $R \geq 1$ and $\epsilon > 0$, there exists $r_{10.19} > 0$ such that, for all $p \in \mathbb{R} \times Y$ and $r > r_{10.19}$, it follows that there exists a vortex $(a_0, \alpha_0)$ bounded by $K_{10.19}$ and a gauge transformation $u$ such that
\[
\| (a_0, (a_0, 0)) - (\phi_{p,r,R})^* u \cdot (a, (\alpha, \beta)) \|_{C^0(D(\mathbb{R}^4, R))} < \epsilon,
\]
where $\phi_{p,r,R}$ denotes the rescaled Gaussian chart centred at $p$ so that the geodesic ball of radius $Rr^{-1/2}$ be mapped to the ball of radius $R$ of $\mathbb{R}^4$.

**Proof.** According to Taubes [51], the proof is an adaptation of a similar statement made in Taubes [49]. The idea is as follows. Assume the contrary. Then, for any $K_{10.19} > 0$, there exists $R \geq 1$, $\epsilon > 0$, an unbounded sequence $\{r_n\}$, a sequence of points $p_n \in \mathbb{R} \times Y$ and a sequence of Seiberg–Witten trajectories $(A_\lambda + a_n, (\alpha_n, \beta_n))$ with parameter $r = r_n$ such that, if one define $(\tilde{a}_n, (\tilde{\alpha}_n, \tilde{\beta}_n))$ to mean the pullback of $(a_n, (\alpha_n, \beta_n))$ via the chart $\phi_{p_n,r_n,R}$ seen as functions of the ball of radius $R$ of $\mathbb{R}^4$, then $(\tilde{a}_n, (\tilde{\alpha}_n, \tilde{\beta}_n))$ do not lie in a ball of radius $\epsilon$ around any vortex bounded by $K_{10.19}$ in the $C^0(D(\mathbb{R}^4, R))$ norm. Now, the Seiberg–Witten equations imply that, after redefining $(\tilde{a}_n, (\tilde{\alpha}_n, \tilde{\beta}_n))$ by applying some gauge transformation $u_n$, these functions obey elliptic equations of the form
\[
d^+ \tilde{a}_n = f_1(\psi_n, \bar{\psi}_n) + f_2, \quad d^* \tilde{a}_n = 0, \quad \bar{\partial} \tilde{a}_n + \bar{\partial}^* \tilde{\beta}_n = f_3(\tilde{a}_n, \psi_n),
\]
for certain polynomial functions $f_1$, $f_2$ and $f_3$. On the other hand, by using Lemmas 10.4, 10.5, 10.6 and standard elliptic theory arguments, one can conclude that there must be a subsequence uniformly convergent on the disk of radius $R$. Let the limit of this convergent subsequence be denoted $(\tilde{a}_\infty, (\tilde{\alpha}_\infty, \tilde{\beta}_\infty))$. The point now is that, by Lemma 10.5, one can see further that $\beta_\infty = 0$. Hence, if one care to check, it follows, by examining carefully the terms $f_1$, $f_2$ and $f_3$, that the equations above reduce to the vortex equations in the limit $n \to \infty$. Consequently, $(a_\infty, \alpha_\infty)$ is a vortex bounded by $K_{10.19}$, which is a contradiction.

Set the following notation
\[
\Omega_{\delta} := \int_{\mathbb{R} \times Y} \hat{s}e^{2t} \sigma_{\delta}, \quad \Omega'_{\delta} := \int_{\mathbb{R} \times Y} \hat{s}e^{2t} \sigma_{\delta}'.
\]

**Lemma 10.20.** (cf. [51], Lemma 5.17) Given $\delta \in (0, \delta_{2.24})$, there exist $r_{10.20} > 0$ and $K_{10.20} > 0$ such that, for all $r > r_{10.20}$,
\[
\Omega_{\delta}' \leq K_{10.20} \Omega_{\delta}.
\]

**Proof.** The analogous statement in Taubes [51] lacks a proof as it is claimed to be similar to a previous lemma; therefore, the author shall furnish the details here. Firstly, notice that the function $\sigma_{\delta}$ is non-zero only at points where $(1 - |\alpha|^2) \geq \delta$; meanwhile, $\sigma_{\delta}'$ is non-zero only at points where $\delta \geq (1 - |\alpha|^2) \geq \frac{1}{2}\delta$. Now, If $g$ be the product metric of $\mathbb{R} \times Y$ consider instead the metric $e^{2t}g$. 

Since the set of points where \( (1 - |\alpha|^2) \geq \delta \) has \( t \) coordinate bounded above, the volume of this set in the metric \( e^{2t}g \) is, in fact, finite. Hence, note that, in order to prove the claimed result, it suffices to bound the volume of points where \( \delta \geq (1 - |\alpha|^2) \geq \frac{1}{2}\delta \) by some constant times the volume of the set of points where \( (1 - |\alpha|^2) \geq \delta \); both volumes being with respect to the metric \( e^{2t}g \). For the rest of this proof, the metric at use shall be \( e^{2t}g \) whenever the author talk of volumes or geodesy. For brevity, use \( R := R_{10.17} \) to denote the constant from Lemma 10.17. Consider a set \( \Lambda \) of disjoint geodesic balls in \( Y \) centred at points \( p \) where \( \delta \geq (1 - |\alpha(p)|^2) \geq \frac{1}{2}\delta \) and all having radius \( \frac{1}{4}Rr^{-1/2} \). Due to compactness of \( Y \), one can also assume \( \Lambda \) to be maximal with respect to inclusion. For each ball \( B \in \Lambda \), let \( B'' \supset B' \supset B \) denote the concentric geodesic balls having radii \( Rr^{-1/2} \) and \( \frac{1}{2}Rr^{-1/2} \) respectively. Now, suppose that some point \( p \in \mathbb{R} \times Y \) at which \( \delta \geq (1 - |\alpha(p)|^2) \geq \frac{1}{2}\delta \) not be in the set \( \bigcup_{B \in \Lambda} B' \). Then, if \( r \) be sufficiently large compared to \( R \), say larger than some \( r_{10.20} > 0 \), the ball of radius \( \frac{1}{4}Rr^{-1/2} \) centred at \( p \) would not intersect any of the balls in the set \( \Lambda \); so, if this were the case, \( \Lambda \) could not be maximal. Therefore, \( \bigcup_{B \in \Lambda} B' \) covers the set of points where \( \delta \geq (1 - |\alpha|^2) \geq \frac{1}{2}\delta \). Moreover, perhaps after an increase to \( r_{10.20} > 0 \), Riemannian geometry provides an upper bound for the maximal number \( n \) such that there be a set of balls \( \{B_i\} \subset \Lambda \) satisfying \( B_1'' \cap \cdots \cap B_n'' \neq \emptyset \); this upper bound is independent of \( \delta \) and \( r \) except that one must ensure \( r > r_{10.20} \).

The consequence of all of this is that it suffices to provide the desired sort of bound for each of the balls \( B \). For that end, given a ball \( B \in \Lambda \), denote by \( V_B \) the volume of the subset of \( B'' \) where \( \delta \geq (1 - |\alpha(p)|^2) \geq \frac{1}{2}\delta \); likewise, denote by \( V'_B \) the volume of the subset of \( B' \) where \( (1 - |\alpha|^2) \geq \delta \). Now, apply Lemma 10.17 in combination with Lemma 10.19 in order to obtain a constant \( K_{10.20} > 0 \) independent of \( B \), satisfying \( V'_B \leq K_{10.20}V_B \).  

**Lemma 10.21.** Given \( \delta \in (0, \delta_{2.24}) \), there exists \( r_{10.21} \) such that for all \( r > r_{10.21} \), \( \Omega_{\delta} = 0 \).

**Proof.** To start, let \( r_{10.21} = \max\{r_{10.20}, r_{10.15}\} \). Recall that Lemma 10.14 asserted that

\[
\int_{\mathbb{R} \times Y} \hat{*}e^{2t} \left( \delta^{-1} \sigma'_\delta (|\partial_A \alpha|^2 - |\bar{\partial}_A \alpha|^2) + r\sigma_\delta (1 - |\alpha|^2 + |\beta|^2) \right) = 0.
\]

Neglecting some positive terms yields the inequality

\[
\int_{\mathbb{R} \times Y} \hat{*}e^{2t} \left( -\delta^{-1} \sigma'_\delta |\bar{\partial}_A \alpha|^2 + r\sigma_\delta (1 - |\alpha|^2) \right) \leq 0.
\]

Focusing on the second term of the integrand, note that, at a point where \( \sigma_\delta \neq 0 \), it is necessarily the case that \( (1 - |\alpha|^2) \geq \delta \); hence,

\[
\int_{\mathbb{R} \times Y} \hat{*}e^{2t} \left( -\delta^{-1} \sigma'_\delta |\bar{\partial}_A \alpha|^2 + r\sigma_\delta \delta \right) \leq 0.
\]

By applying Lemmas 10.20 and 10.15, one finds

\[
( -K_{10.20}K^2_{10.15}\delta^{-1} + r\delta) \Omega_{\delta} \leq 0.
\]
But $\Omega_{\delta} \geq 0$. Hence, perhaps after increasing $r_{10,21}$, it follows that, for $r > r_{10,21}$, $\Omega_{\delta} = 0$. □

**Theorem 10.22.** Given $\delta \in (0, \delta_{2,24})$, for all $r > r_{10,21}$, it follows that, point-wise on all of $\mathbb{R} \times Y$,

$$1 - |\alpha|^2 \leq \delta.$$

**Proof.** $\Omega_{\delta}$ is the integral of $e^{2t} \sigma_{\delta}$, which is non-negative and strictly positive wherever $(1 - |\alpha|^2) > \delta$. □

**Corollary 10.23.** $C = \lim_{t \to -\infty} \gamma$ is gauge equivalent to $C_\lambda$.

**Proof.** $C$ is a solution to the Seiberg–Witten equations on $Y$. Its $\Lambda^{0,0} \xi^*$ component is $\lim_{t \to -\infty} \alpha$. Therefore, it also satisfies the bound in Theorem 10.22. Theorem 2.24 then guarantees that $C$ must be gauge equivalent to $C_\lambda$. □

**Corollary 10.24.** $\gamma$ is the constant trajectory at $C_\lambda$.

**Proof.** Assume the contrary. Because the Seiberg–Witten vector field $\mathcal{X}_{\lambda,r}$ is minus the gradient of the functional $\text{CSD}_{\lambda,r}$, the value of $\text{CSD}_{\lambda,r}$ never increases along the trajectory $\gamma$. Moreover, the contact configuration $C_\lambda$ is a non-degenerate fixed point of the downward gradient flow of $\text{CSD}_{\lambda,r}$ by Theorem 2.23, which means that the value of $\text{CSD}_{\lambda,r}$ certainly decreases along $\gamma$ for sufficiently large time as one approaches $C_\lambda$. But since both endpoints of $\gamma$ are gauge equivalent and $b_1 = 0$, the values of $\text{CSD}_{\lambda,r}$ are the same for gauge equivalent configurations. This cannot be. □

**Remark 10.25.** Hereby, the author concludes the proof of Theorem 2.25.

### References

[1] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry I. Math. Proc. Camb. Philos. Soc. 77(1), 43–69 (1975)

[2] Cauchy, A.L.: Cours d’Analyse de l’École Royale Polytechnique. de Bure, Paris (1821)

[3] Chelnokov, G., Mednykh, A.: On the coverings of Hantzsche–Wendt manifold. arXiv:2009.06691 (arXiv preprint) (2020)

[4] Colin, V., Ghiggini, P., Honda, K.: The equivalence of Heegaard Floer and embedded contact homology via open book decompositions I. arXiv:1208.1074 (arXiv preprint) (2012a)

[5] Colin, V., Ghiggini, P., Honda, K.: The equivalence of Heegaard Floer and embedded contact homology via open book decompositions II. arXiv:1208.1077 (arXiv preprint) (2012b)

[6] Conley, C.: Isolated Invariant Sets and the Morse Index, 38th edn. American Mathematical Society, Providence (1978)

[7] Cristofaro-Gardiner, D.: The absolute gradings on embedded contact homology and Seiberg–Witten Floer cohomology. Algebr. Geometric Topol. 13(4), 2239–2260 (2013)
[8] Floer, A.: A refinement of the Conley index and an application to the stability of hyperbolic invariant sets. Ergodic Theory Dyn. Syst. 7, 1 (1987)

[9] Floer, A., Zehnder, E.: The equivariant Conley index and bifurcations of periodic solutions of Hamiltonian systems. Ergodic Theory Dyn. Syst 8, 87–97 (1988)

[10] Geiges, H.: An Introduction to Contact Topology. Cambridge University Press, Cambridge (2008)

[11] Ghiggini, P.: On tight contact structures with negative maximal twisting number on small Seifert manifolds. Algebr. Geometric Topol. 8(1), 381–396 (2008)

[12] Ghiggini, P., Lisca, P., Stipsicz, A.I.: Classification of tight contact structures on small seifert 3-manifolds with $e_0 \geq 0$. Proc. Am. Math. Soc 20, 909–916 (2006)

[13] Giroux, E.: Structures de contact en dimension trois et bifurcations des feuilletages de surfaces. Invent. Math. 141(3), 615–689 (2000)

[14] Gompf, R.E.: Handlebody construction of Stein surfaces. Ann. Math. 20, 619–693 (1998)

[15] Gompf, R.E., Stipsicz, A.I.: 4-Manifolds and Kirby Calculus, 20th edn. American Mathematical Society, Providence (1999)

[16] Honda, K.: On the classification of tight contact structures I. Geometry Topol. 4, 309–368 (1999)

[17] Hutchings, M., Taubes, C.H.: Gluing pseudoholomorphic curves along branched covered cylinders I. J. Symplectic Geometry 5(1), 43–137 (2007)

[18] Iida, N., Taniguchi, M.: Seiberg–Witten Floer homotopy contact invariant. Stud. Sci. Math. Hung. 58(4), 505–558 (2021)

[19] Khuzam, M.B.: Lifting the 3-dimensional invariant of 2-plane fields on 3-manifolds. Topol. Appl. 159(3), 704–710 (2012)

[20] Kronheimer, P., Mrowka, T.: Monopoles and contact structures. Invent. Math. 130(2), 209–255 (1997)

[21] Kronheimer, P., Mrowka, T.: Monopoles and Three-Manifolds, vol. 10. Cambridge University Press, Cambridge (2007)

[22] Kronheimer, P., Mrowka, T., Ozsváth, P., Szabó, Z.: Monopoles and lens space surgeries. Ann. Math. 20, 457–546 (2007)

[23] Kurland, H.L.: Homotopy invariants of repeller–attractor pairs. I. The Püppé sequence of an RA pair. J. Differ. Equ. 46(1), 1–31 (1982)

[24] Kutluhan, Ç., Lee, Y.J., Taubes, C.H.: HF=HM, I: Heegaard Floer homology and Seiberg–Witten Floer homology. Geometry Topol. 24(6), 2829–2854 (2020)

[25] Kutluhan, Ç., Lee, Y.J., Taubes, C.H.: HF=HM, II: Reeb orbits and holomorphic curves for the ech/Heegaard Floer correspondence. Geometry Topol. 24(6), 2855–3012 (2020)

[26] Kutluhan, Ç., Lee, Y.J., Taubes, C.H.: HF=HM, III: holomorphic curves and the differential for the ech/Heegaard Floer correspondence. Geometry Topol. 24(6), 3013–3218 (2020)

[27] Kutluhan, Ç., Lee, Y.J., Taubes, C.H.: HF=HM, IV: The Seiberg–Witten Floer homology and ech correspondence. Geometry Topol. 24(7), 3219–3469 (2021)

[28] Kutluhan, Ç., Lee, Y.J., Taubes, C.H.: HF=HM, V: Seiberg–Witten Floer homology and handle additions. Geometry Topol. 24(7), 3471–3748 (2021)
[29] Lewis, L.G., Jr., May, J.P., Steinberger, M.: Equivariant stable homotopy theory. Lect. Notes Math. 20, 1213 (1986)

[30] Lidman, T., Manolescu, C.: Floer homology and covering spaces. Geometry Topol. 22(5), 2817–2838 (2018)

[31] Lidman, T., Manolescu, C.: The equivalence of two Seiberg–Witten Floer homologies. Astérisque 399, 25 (2018)

[32] Lin, F.: Monopole Floer homology and SOLV geometry. Ann. Henri Lebesgue 3, 1117–1131 (2020)

[33] Lin, F., Lipnowski, M.: The Seiberg–Witten equations and the length spectrum of hyperbolic three-manifolds. J. Am. Math. Soc. 35(1), 233–293 (2022)

[34] Lin, J., Ruberman, D., Saveliev, N.: On the Frøyshov invariant and monopole Lefschetz number. arXiv:1802.07704 (arXiv preprint) (2018)

[35] Manolescu, C.: Seiberg–Witten–Floer stable homotopy type of three-manifolds with $b_1 = 0$. Geometry Topol. 7(2), 889–932 (2003)

[36] Matković, I.: Classification of tight contact structures on small Seifert fibered L-spaces. Algebr. Geometric Topol. 18(1), 111–152 (2018)

[37] May, J.P., et al.: Equivariant Homotopy and Cohomology Theory, 91st edn. American Mathematical Society, Providence (1996)

[38] Mischaikow, K.: Conley index theory. Dyn. Syst. 20, 119–207 (1995)

[39] Moser, L.: Elementary surgery along a torus knot. Pac. J. Math. 38(3), 737–745 (1971)

[40] Nicolaescu, L.I.: Notes on Seiberg–Witten Theory, vol. 28. American Mathematical Society, Providence (2000)

[41] Ozsváth, P., Szabó, Z.: Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. Adv. Math. 173(2), 179–261 (2003)

[42] Ozsváth, P., Szabó, Z.: Holomorphic disks and topological invariants for closed three-manifolds. Ann. Math. 20, 1027–1158 (2004)

[43] Ozsváth, P., Szabó, Z.: Heegaard Floer homology and contact structures. Duke Math. J. 129, 1 (2005)

[44] Petit, R.: Spinc-structures and Dirac operators on contact manifolds. Differ. Geom. Appl. 22(2), 229–252 (2005)

[45] Ramos, V.G.B.: Absolute gradings on ECH and Heegaard Floer homology. Quantum Topol. 9(2), 207–228 (2018)

[46] Rolfsen, D.: Knots and Links, vol. 346. American Mathematical Society, Providence (2003)

[47] Roso, B.R.S.: Seiberg–Witten Floer spectra and contact structures. Ph.D. Dissertation, University College London (2023)

[48] Taubes, C.H.: The Seiberg–Witten invariants and symplectic forms. Math. Res. Lett. 1(6), 809–822 (1994)

[49] Taubes, C.H.: SW $\Rightarrow$ Gr: from the Seiberg–Witten equations to pseudoholomorphic curves. J. Am. Math. Soc. 10, 845–918 (1996)

[50] Taubes, C.H.: The Seiberg–Witten equations and the Weinstein conjecture. Geometry Topol. 11(4), 2117–2202 (2007)

[51] Taubes, C.H.: The Seiberg–Witten equations and the Weinstein conjecture II: More closed integral curves of the Reeb vector field. Geometry Topol. 13(3), 1337–1417 (2009)
[52] Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer cohomology I. Geometry Topol. 14(5), 2497–2581 (2010)
[53] Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer cohomology II. Geometry Topol. 14(5), 2583–2720 (2010)
[54] Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer cohomology III. Geometry Topol. 14(5), 2721–2817 (2010)
[55] Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer cohomology IV. Geometry Topol. 14(5), 2819–2960 (2010)
[56] Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer cohomology V. Geometry Topol. 14(5), 2961–3000 (2010)
[57] tom Dieck, T.: Transformation Groups. W. de Gruyter, Berlin (1987)
[58] Wu, H.: Legendrian vertical circles in small Seifert spaces. Commun. Contemp. Math. 8(2), 219–246 (2006)

B. R. S. Roso
University College London
London
UK
e-mail: bruno.roso.14@ucl.ac.uk

Accepted: February 19, 2023.