Abstract

For a representation of a Lie algebra, one can construct a diagram of the representation, i.e., a directed graph with edges labeled by matrix elements of the representation. This article explains how to use these diagrams to describe normal forms, orbits and invariants of the representation, especially for the case of nilpotent Lie algebras.

1 Introduction

For a representation of a Lie algebra, one can construct a diagram of the representation, i.e., a directed graph with edges labeled by matrix elements of the representation. In section 2 I give the detailed definition and a few examples. In section 3 I describe how to construct the diagrams of symmetric and exterior powers of representations, with a few examples. Section 4 is the main section of this work. I introduce there the diagram method for the description of normal forms on the orbits of a strictly triangular representation of a Lie algebra, usually nilpotent or pronilpotent. Some results in the beginning of this section (without using the diagrams) were obtained independently and earlier by Igor Brodski [1]. Then, in section 5, I show how to apply the diagram method to the description of normal forms of quadratic differentials on a line. First these normal forms were found by Alexandre Kirillov [3]. In section 6 I give the analogous description of normal forms of generalized tensor fields on a line, using the diagram method. The last two sections are closely connected to my previous work [9].
Let $f$ be a field of characteristic $p \geq 0$, $L$ a Lie $f$-algebra and $E$ an $f$-space. Denote $L^* = \text{Hom}_f(L, f)$ and $E^* = \text{Hom}_f(E, f)$, the $f$-spaces of $f$-linear forms on $L$ and $E$. Fix a basis $(e_i)_{i \in I}$ of $E$ and the dual basis $(x_i)_{i \in I}$ of $E^*$.

For a representation $T$ of a Lie algebra $L$ in $E$, define an $f$-bilinear mapping
\[ \phi_T : E^* \times E \to L^*, \quad \phi_T(x, e)l = x(T(l)e). \quad (1) \]

**Definition 1.** A diagram of a representation $T$ of a Lie algebra $L$ in $E$, corresponding to a basis $(e_i)_{i \in I}$ of $E$ is a directed graph with the set of vertices $I$, having an edge $(i, j)$ iff $\phi_T(x_j, e_i) \neq 0$, in which case this edge is labeled by $\phi_T(x_j, e_i) \in L^*$. We suppose here that this graph doesn’t have multiple edges, but can have loops. $(x_j)_{j \in I}$ denotes a basis of $E^*$, dual to $(e_i)_{i \in I}$, as usual.

**Example 1.** For the adjoint representation of a Heisenberg Lie algebra $H$ with generators $X, Y, Z$ satisfying
\[ [X, Y] = Z, \quad [X, Z] = [Y, Z] = 0, \quad (2) \]
the diagram is

\[ \begin{array}{ccc}
X & \nearrow & Y \\
& \searrow & \ \ \\
& & Z \\
\end{array} \]

where $(x, y, z)$ is the basis of $H^*$, dual to $(X, Y, Z)$.

The diagrams are very useful for the description of various operations on representations. For instance,

**Proposition 1.** The diagram of the dual representation $T^*$ corresponding to the basis $(x_j)_{j \in I}$ of $E^*$, can be obtained from the diagram of the representation $T$ corresponding to the dual basis $(e_i)_{i \in I}$ of $E$, by changing the directions of all arrows and all the signs of their labels.

**Proof.** By definition.
Example 2. The diagram of the co-adjoint representation of $H$, see example 1, is

![Diagram](image)

The underlying graph of a diagram contains important information about the representation. For the adjoint representation, it contains a lot of information about the Lie algebra. In particular,

**Proposition 2.** Let $G$ be the underlying directed graph of the adjoint representation of a Lie algebra $L$. If $G$ is a graph without edges, $L$ is a commutative Lie algebra. If $G$ is a graph without oriented cycles, $L$ is a nilpotent Lie algebra of class of nilpotency $\leq l(G) + 1$ where $l(G)$ is the length of $G$, i.e. the number of edges in the longest directed path which is a subgraph of $G$.

*Proof.* Again, everything follows directly from the definitions. \[\square\]

Example 3. For the Heisenberg Lie algebra $H$, see example 1, $l(G) = 1$. By Proposition 2, $H$ is a nilpotent Lie algebra of class $\leq 2$, i.e., a metabelian Lie algebra.

Definition 2. If the underlying graph of a diagram $C$ is a subgraph of the underlying graph of a diagram $D$ containing with each vertex all the edges starting from this vertex, with the same labels, we’ll say that the diagram $C$ is a subdiagram of the diagram $D$. If the underlying graph of a diagram $C$ is a subgraph of the underlying graph of a diagram $D$ containing with each vertex all the edges ending in this vertex, with the same labels, we’ll say that the diagram $C$ is a quotient diagram of the diagram $D$.

**Proposition 3.** If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in an $f$-space $E$ corresponding to the basis $(e_i)_{i \in I}$ and $C$ is a subdiagram of $D$, then the $f$-linear subspace $E_C$ spanned by $(e_i)_{i \in J}$ where $J$ is the set of the vertices of $C$, is $T$-invariant and $C$ is the diagram of the restriction of $T$ on $E_C$, corresponding to the basis $(e_i)_{i \in J}$. Moreover, if we denote $\overrightarrow{C}$ the diagram with the set of vertices $I \setminus J$, containing all the edges of $D$ between them, with the same labels, then $\overrightarrow{C}$ is a quotient diagram coinciding with the diagram of the quotient representation of $L$ in the quotient space $E/E_C$, corresponding to the basis $(e_i + E_C)_{i \in I \setminus J}$.  

3
Proof. By definition.

Dually,

**Proposition 4.** If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in an $f$-space $E$ corresponding to the basis $(e_i)_{i \in I}$ and $C$ is a quotient diagram of $D$, then the diagram $\overline{C}$ defined the same as in Proposition 3 as a complete subgraph of the underlying graph of $D$ with the set of vertices $I \setminus J$ where $J$ is the set of vertices of $C$, with the same labeling of the edges, is a subdiagram of $D$ and $\overline{\overline{C}} = C$.

Proof. Again, it follows directly from the definitions.

**Proposition 5.** Let $T$ and $U$ be representations of a Lie algebra $L$ in $f$-spaces $V$ and $W$ with bases $(v_i)_{i \in I}$ and $(w_j)_{j \in J}$ respectively. Denote $D_T, D_U, D_{T\oplus U}, D_{T\otimes U}$ the diagrams of the representations $T, U, T\oplus U, T\otimes U$ corresponding to the bases $(v_i)_{i \in I}, (w_j)_{j \in J}, (v_i)_{i \in I} \Pi (w_j)_{j \in J}$ and $(v_i \otimes w_j)_{(i,j) \in I \times J}$. Then

$$D_{T\oplus U} = D_T \amalg D_U$$  \hfill (3)

$$D_{T\otimes U} = D_T \times D_U,$$  \hfill (4)

meaning that $D_T \amalg D_U$ is a disjoint union of $D_T$ and $D_U$ and $D_T \times D_U$ is a reduced Cartesian product of $D_T$ and $D_U$ (‘reduced’ means that we replace multiple loops in a vertex of the Cartesian product by one loop labeled by the sum of the labels of these multiple loops).

Proof. It is evident.

At the end of the section, note that the diagram contains all the information about the representation:

**Proposition 6.** If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $V$ corresponding to the basis $(e_i)_{i \in I}$, then

$$T(l) \sum_{i \in I} x_i e_i = \sum_{(i,j) \in E(D)} x_i w(i,j)(l)e_j$$  \hfill (5)

where $E(D)$ is the set of edges of the underlying graph of the diagram $D$, and $w(i,j) \in L^*$ denotes the label of the edge $(i,j)$.

4
Proof. It follows from Definition 1. □

In other words,

**Corollary 1.** If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $V$ corresponding to the basis $(e_i)_{i \in I}$, then for any $l \in L$, the matrix of $T(l)$ in the basis $(e_i)_{i \in I}$ is $W(D)(l)$, the transposed weight matrix of the weighted graph $D$, applied to $l$ supposing that for the matrix elements $a_{ij} \in L^*$ one has

$$
\begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \cdots & \cdots & \cdots \\
  a_{n1} & \cdots & a_{nn}
\end{pmatrix}(l) =
\begin{pmatrix}
  a_{11}(l) & \cdots & a_{1n}(l) \\
  \cdots & \cdots & \cdots \\
  a_{n1}(l) & \cdots & a_{nn}(l)
\end{pmatrix}
$$

(6)

Proof. It follows from Proposition 6. □

3 Diagrams of $\lambda$-operations on representations

Proposition 3 shows that the diagrams are useful for the description of sums and tensor products of representations. They are extremely useful for the description of other operations as well.

**Definition 3.** For a reduced weighted directed graph $G$ with linearly ordered set of vertices $V(G)$, denote $S^nG$ a reduced weighted directed graph with the set of vertices

$$V(S^nG) = S^nV(G) = \{(v_1, \ldots, v_n) \in V(G)^n \mid v_1 \leq \cdots \leq v_n\}. \quad (7)$$

For $i = 1, \ldots, n$ denote $Pr_i : V(S^nG) \to V(S^{n-1}G)$ the projection obtained by the erasing of the vertices $v_i$ standing on the $i$-th place of the sequence (7). For $A, B \in V(S^nG)$ so that $A \neq B$, the edge $(A, B) \in E(S^nG)$ exists iff for some $i, j$ one has $Pr_i(A) = Pr_j(B)$, there is an edge $(v_i(A), v_j(B)) \in E(G)$ where $v_i$ denotes the vertex of $G$ standing on the $i$-th place of the sequence (7) and

$$\text{(# of the entries of } v_i(A) \text{ in } A) \cdot w(v_i(A), v_j(B)) \neq 0$$

(8)

where $w$ denotes the weight function; in which case the weight of the edge $(A, B) \in E(S^nG)$ equals (8). For $A \in V(S^nG)$, the edge $(A, A) \in E(S^nG)$ exists iff

$$\sum_{i=1}^n w(v_i(A), v_i(A)) \neq 0,$$

(9)

in which case the weight of the edge $(A, A) \in E(S^nG)$ equals (8).
Proposition 7. If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $V$ corresponding to the basis $(e_i)_{i \in I}$, then $S^n D$ is a diagram of the representation $S^n T$ of $L$ in $S^n V = V \otimes I \mathbb{n}$ where $I \mathbb{n}$ is a subspace of $V \otimes n$ spanned by $(e_\alpha - e_\alpha')_{\alpha \in I, s \in S_n}$ where $e_{i_1 \ldots i_n} = e_{i_1} \otimes \cdots \otimes e_{i_n}$, corresponding to the basis $(e_{i_1 \ldots i_n})_{(i_1, \ldots, i_n) \in S^n I}$ where $e_{i_1 \ldots i_n}$ is the projection of $e_{i_1 \ldots i_n}$.

Proof. It follows directly from the definitions.

Example 4. Let $T = \rho_1$ be the standard 2-dimensional representation of $sl(2)$, the Lie algebra of $2 \times 2$ matrices with trace 0. Denote

$$X_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the standard basis elements of $sl(2)$ and $(x_+, x_-, h)$ the dual basis of $sl(2)^*$. The diagram of $T$ corresponding to the standard basis $(u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 \\ 1 \end{pmatrix})$, is

$$\begin{tikzpicture}
  \draw (0,0) node[above] {h} -- (0,0) node[below] {u} -- (1,0) node[above] {x_+} -- (1,0) node[below] {v} -- (0,0) node[above] {-h} -- (0,0) node[below] {-x_-};
\end{tikzpicture}$$

It follows from Proposition 7 that the diagram for $S^2 T = \rho_2$ is

$$\begin{tikzpicture}
  \draw (0,0) node[above] {2h} -- (0,0) node[below] {u^2} -- (1,0) node[above] {2x_+} -- (1,0) node[below] {uv} -- (2,0) node[above] {x_-} -- (2,0) node[below] {v^2} -- (1,0) node[above] {-2h} -- (0,0) node[below] {-x_-} ;
\end{tikzpicture},$$

and the diagram for $S^3 T = \rho_3$ is

$$\begin{tikzpicture}
  \draw (0,0) node[above] {3h} -- (0,0) node[below] {u^3} -- (1,0) node[above] {3x_+} -- (1,0) node[below] {u^2v} -- (2,0) node[above] {2x_+} -- (2,0) node[below] {uv} -- (3,0) node[above] {x_-} -- (3,0) node[below] {v^2} -- (2,0) node[above] {-3h} -- (1,0) node[below] {-x_-} -- (0,0) node[above] {-3h} -- (0,0) node[below] {-x_-};
\end{tikzpicture}.$$  

We supposed in (12) and (13) that $p = \text{char} f \neq 2$ and $\neq 3$. For $p = 2$ or $p = 3$ one has to delete from the diagrams (12) and (13) all the edges labeled $\pm px_+$ and $\pm ph$.

Definition 4. For a reduced weighted directed graph $G$ with linearly ordered set of vertices $V(G)$ denote $\Lambda^n G$ a reduced weighted directed graph with the set of vertices

$$V(\Lambda^n G) = \Lambda^n V(G) = \{(v_1, \ldots, v_n) \in V(G)^n \mid v_1 < \cdots < v_n\}$$

(14)
For $i = 1, \ldots, n$ denote $Pr_i : V(\Lambda^n G) \to V(\Lambda^{n-1} G)$ the projection obtained by the erasing of the vertices $v_i$ standing on the $i$-th place of the sequence (14). For $A, B \in V(\Lambda^n G)$ so that $A \neq B$, the edge $(A, B) \in E(\Lambda^n G)$ exists iff for some $i, j$ one has $Pr_i(A) = Pr_j(B)$ and there is an edge $(v_i(A), v_j(B)) \in E(G)$ where $v_i$ denotes the vertex of $G$ standing on the $i$-th place of the sequence (14), in which case the weight of the edge $(A, B) \in E(\Lambda^n G)$ is

$$w(A, B) = (-1)^{j-i}w(v_i, v_j)$$

(15)

where $w$ denotes the weight. For $A \in V(\Lambda^n G)$, there is an edge $(A, A) \in E(\Lambda^n G)$ iff

$$\sum_{i \in A} w(i, i) \neq 0,$$

(16)
in which case the weight of the edge $(A, A) \in E(\Lambda^n G)$ equals (16).

**Proposition 8.** If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $V$ corresponding to the basis $(e_i)_{i \in I}$, then $\Lambda^n D$ is a diagram of the representation $\Lambda^n T$ of $L$ in $\Lambda^n V = V^{\otimes n}/J_n$ where $J_n$ is a subspace of $V^{\otimes n}$ spanned by such tensor products $e_{i_1} \otimes \cdots \otimes e_{i_n}$ that $i_k = i_l$ for some $k \neq l$; corresponding to the basis $(e_{i_1} \wedge \cdots \wedge e_{i_n})_{(i_1 < \ldots < i_n) \in \Lambda^n I}$ where $e_{i_1} \wedge \cdots \wedge e_{i_n}$ is the projection of $e_{i_1} \otimes \cdots \otimes e_{i_n}$.

*Proof.* Again, it follows directly from the definitions. \[\square\]

Sometimes one has to consider a subrepresentations $S_n T$ and $\Lambda_n T$ instead of the corresponding quotient representations described above. The diagrams of them can be easily described as well.

**Definition 5.** For a reduced weighted directed graph $G$ with a linearly ordered set of vertices $V(G)$, denote $S_n G$ a reduced weighted directed graph with the same set of vertices $V(S_n G) = V(S^n G)$ as $S^n G$, see (7). The same as for $S^n G$, for $A \in V(S_n G)$, the edge $(A, A) \in E(S_n G)$ exists iff

$$\sum_{i=1}^n w(v_i(A), v_i(A)) \neq 0,$$

(17)
in which case the weight of the edge $(A, A) \in E(S_n G)$ equals (17). The difference between Definition 3 and this definition is that for $A, B \in V(S_n G)$ so that $A \neq B$, the edge $(A, B) \in E(S_n G)$ exists iff for some $i, j$ one has $Pr_i(A) = Pr_j(B)$, there is an edge $(v_i(A), v_j(B)) \in E(G)$ and

$$\left(\# \text{ of the entries of } v_j(B) \text{ in } B\right) \cdot w(v_i(A), v_j(B)) \neq 0,$$

(18)
in which case the weight of the edge $(A, B) \in E(S_n G)$ equals (18).
Proposition 9. If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $V$ corresponding to the basis $(e_i)_{i \in I}$, then $S_nD$ is a diagram of the subrepresentation $S_nT$ of $L$ in $S_nV \subseteq V^\otimes n$ where $S_nV$ is a subspace of $V^\otimes n$ with the basis $(e^{\text{sym}}_\alpha)_{\alpha \in S^nI}$ where for $\alpha \in I^n$

$$e^{\text{sym}}_\alpha = \sum_{\beta \in S_n(\alpha)} e_\beta$$  \hspace{1cm} (19)

with $e_{i_1 \ldots i_n} = e_{i_1} \otimes \cdots \otimes e_{i_n}$; corresponding to this basis.

Proof. It follows directly from the definitions. \hfill \square

Example 5.

$$e^{\text{sym}}_{111} = e_1 \otimes e_1 \otimes e_1,$$  \hspace{1cm} (20)

$$e^{\text{sym}}_{112} = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1.$$  \hspace{1cm} (21)

Example 6. Note that the underlying directed graphs (unweighted) of $S^nD$ and $S_nD$ are the same for $p = \text{char } f = 0$, but can be different for $p > 0$. The diagram of $S_3T$ where $T$ is the standard 2-dimensional representation of $sl(2)$ considered in Example 4 is

$$3h \quad x_+ \quad h \quad 2x_+ \quad -h \quad 3x_+ \quad -3h \quad \begin{array}{c}
  u^3 \\
  \longrightarrow x_-
\end{array} \quad \begin{array}{c}
  u^2v \\
  \longrightarrow 2x_-
\end{array} \quad \begin{array}{c}
  uv^2 \\
  \longrightarrow 3x_-
\end{array} \quad \begin{array}{c}
  v^3
\end{array}.$$  \hspace{1cm} (22)

The edges labeled by $\pm 3x_\pm$ that we have to delete for $p = 3$ from the diagrams (13) and (22), have opposite directions.

Proposition 10. If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $V$ corresponding to the basis $(e_i)_{i \in I}$, then $\Lambda^nD$ is a diagram of the subrepresentation $\Lambda_nT$ of $L$ in $\Lambda_nV \subseteq V^\otimes n$ where $\Lambda_nV$ is a subspace of $V^\otimes n$ with the basis $(e_{i_1} \wedge \cdots \wedge e_{i_n})_{(i_1 < \cdots < i_n) \in \Lambda^nI}$ where

$$e_{i_1} \wedge \cdots \wedge e_{i_n} = \sum_{s \in S_n} (-1)^{\text{inv}(s)} e_{s(i_1)} \otimes \cdots \otimes e_{s(i_n)}$$  \hspace{1cm} (23)

where $\text{inv}(s)$ is the number of inversions in $s \in S_n$; corresponding to this basis.

Proof. Again, it follows directly from the definitions. \hfill \square
4 Diagram method

I describe here the general method of finding normal forms for the orbits of strictly triangular representations of Lie algebras, utilizing the diagrams of representations. For simplicity, let us suppose that $f$ is a field of characteristic 0.

**Definition 6.** We’ll call the representation $T$ of a Lie algebra $L$ in the space $E$ with linearly ordered basis $(e_i)_i \in I$, (strictly) triangular, if linear transformations $T(l)$ for all $l \in L$ have (strictly) lower triangular matrices in the given basis.

**Proposition 11.** The representation $T$ is triangular iff for any edge $(i, j)$ of its diagram, $i \leq j$. The representation $T$ is strictly triangular iff for any edge $(i, j)$ of its diagram, $i < j$.

**Proof.** It follows directly from Definition 6.

**Definition 7.** For a strictly triangular representation $T$, denote $\Gamma(T)$ the group of automorphisms of $E$ of the form

$$\exp T(l) = \sum_{i=0}^{\infty} \frac{1}{i!} T(l)^i$$

with $l \in L$.

**Lemma 1.** If $D$ is a diagram of a representation $T$ of a Lie algebra $L$ in $E$ corresponding to the basis $(e_i)_i \in I$, then for any $l \in L$, the matrix of $\exp T(tl)$ in the basis $(e_i)_i \in I$ is $\Gamma(T)(l)$, the transposed walk matrix of the weighted graph $D$, applied to $l$, where walk matrix $C(D)$ is an $I \times I$ matrix such that

$$C(D)_{ij} = \sum_{n=0}^{\infty} \frac{c_n(D, i \rightarrow j)}{n!} t^n$$

where

$$c_n(D, i \rightarrow j) = \sum w(i_1, j_1) \ldots w(i_n, j_n)$$

with summation over all the walks $(i_1, j_1), \ldots, (i_n, j_n)$ of length $n$ between $i$ and $j$, and $w$ denotes the weight function.

**Proof.** It follows from (24) and the definitions of exp and matrix multiplication.
Lemma 2. If for a strictly triangular representation $T$, $l \in L$, $k \in I$ and $x \in E$, one has $((\exp (tl))x)_j = x_j$ for all $j < k$, $t \in f$ and $((\exp (l))x)_k = x_k + c$ with $c \neq 0$, then

$$((\exp (tl))x)_k = x_k + ct$$

for any $t \in f$. In particular,

$$((\exp (-\frac{x_k}{c}l))x)_k = 0.$$  \hspace{1cm} (28)

Proof. Since $((\exp (tl))x)_j - x_j$ for $j < k$ are polynomials of $t$ equal to 0 for all $t \in f$ and we suppose that $f$ is of characteristic 0, hence infinite, all the coefficients of these polynomials must be 0. It means that $c_n(D, i \rightarrow j) = 0$ for all $j < k$ and $n \geq 1$, where $D$ is the diagram of $T$, as usual. Now, writing each walk of length $n \geq 2$ from $i$ to $k$ as a composition of a walk of length $n - 1$ from $i$ to $j$ and the last edge of the original walk, from $j$ to $k$, we obtain from (26)

$$c_n(D, i \rightarrow k) = \sum_{j \in I} c_{n-1}(D, i \rightarrow j) w(j, k) = 0$$ \hspace{1cm} (29)

for $n \geq 2$. Thus,

$$((\exp (tl))x)_k = x_k + \sum_{j \in I} x_j c_1(D, j \rightarrow k)(tl) = x_k + \sum_{j < k} x_j w(j, k)(l)t.$$ \hspace{1cm} (30)

Substituting $t = 1$, we find

$$c = \sum_{j < k} x_j w(j, k)(l).$$ \hspace{1cm} (31)

Equations (30) and (31) give us (27) and (28) follows from (27). \hfill \Box

Definition 8. For a linearly ordered basis $(e_i)_{i \in I}$ of $E$, introduce the following quasi-ordering on $E$:

$$\sum_{i \in I} x_i e_i \preceq \sum_{i \in I} y_i e_i \iff (x_j \neq 0, y_j = 0) \Rightarrow \exists i \leq j_i, (x_i = 0, y_i \neq 0).$$ \hspace{1cm} (32)

For elements $x, y \in E$, we’ll write $x \prec y$ iff $x \preceq y$ and $y \npreceq x$.

In other words, $x \prec y$ means that in the first place where $x_i y_i = 0$ and coefficients $x_i, y_i$ are not both 0, one finds $x_i = 0, y_i \neq 0$. If $x_i y_i = 0$ implies $x_i = y_i = 0$, then $x \sim y$, i.e., $x \preceq y$ and $y \preceq x$. 

10
Theorem 1. For a strictly triangular representation $T$ in a finite dimensional $f$-space $E$, every orbit of $\Gamma(T)$ in $E$ has the unique lowest point with respect to $\prec$; i.e., the point $x$ such that $x \prec y$ for any $y \in \Gamma(T)x$, $y \neq x$.

Proof. Choose an arbitrary $\Gamma(T)$-orbit and a point $x^{(0)}$ on it. Look at the coordinates of this point according to the order of $I$, trying to turn non-zero coordinates into 0 by applying elements of $\Gamma(T)$ to $x^{(0)}$. Since $T$ is strictly triangular, the action of $\Gamma(T)$ on each coordinate $x^{(0)}_j$ depends only on the preceding coordinates $x^{(0)}_i$, with $i < j$. In particular, we can’t change the first non-zero coordinate. Let $i_0$ be the smallest index of a non-zero coordinate of $x^{(0)}$ that we can change, leaving all the preceding 0’s as 0’s. Then, applying Lemma 2, we can make $(\alpha_1x^{(0)})_{i_0} = 0$ by action of some element $\alpha_1 \in \Gamma(T)$, leaving all the preceding 0’s as 0’s. Denote $x^{(1)} = \alpha_1x^{(0)}$. The element $x^{(1)}$ has the same coordinates $x^{(1)}_i = x^{(0)}_i$ as the element $x^{(0)}$ for $i < i_0$, by construction, and $x^{(1)}_{i_0} = 0$ while $x^{(0)}_{i_0} \neq 0$. Thus, $x^{(1)} \prec x^{(0)}$. Analogously, denoting $i_1$ the smallest index of a non-zero coordinate of $x^{(1)}$ that we can change, leaving all the preceding 0’s as 0’s, applying Lemma 2, we can make $(\alpha_2x^{(1)})_{i_1} = 0$ by action of some element $\alpha_1 \in \Gamma(T)$, leaving all the preceding 0’s as 0’s. Denote $x^{(2)} = \alpha_2x^{(1)}$. The same as above, we have $x^{(2)} \prec x^{(1)} \prec x^{(0)}$. Continuing this construction for a sequence of indices $i_0 < i_1 < \cdots < i_k$, we can construct a sequence of points $x^{(k+1)} \prec \cdots \prec x^{(0)}$ of the $\Gamma(T)$-orbit of $x^{(0)}$. Because of the finiteness of $I$, earlier or later, we must stop on the point of $x^{(k+1)}$ of this sequence. By our construction, we can’t change any non-zero coordinates of $x^{(k+1)}$ leaving all the preceding 0’s as 0’s, by elements of $\Gamma(T)$. It means that $x^{(k+1)}$ is $\prec$ than any other point on the orbit. We constructed the lowest point on the orbit. □

Definition 9. We’ll call the lowest point on an orbit with respect to $\prec$, by the normal form of the orbit.

Lemma 3. If for a strictly triangular representation $T$, $l \in L$, $k \in I$ and $x \in E$, one has $(T(l)x)_j = 0$ for all $j < k$ and $(T(l)x)_k = c \neq 0$, then $((\exp T(tl))x)_j = x_j$ for all $j < k$, $t \in f$ and

\[ ((\exp T(tl))x)_k = x_k + ct \tag{33} \]

for any $t \in f$. In particular,

\[ ((\exp T(-\frac{x_k}{c}))x)_k = 0. \tag{34} \]
Proof. Note that \((T(l)x)_j = 0\) implies \((T(tl)x)_j = 0\). Now, because \(T\) is strictly triangular and \((T(tl)x)_j = 0\) for all \(j < k\), we have \((T(tl)x)^2)_j = 0\) for all \(j \leq k\). Thus

\[(\exp T(tl)x)_j = ((1 + T(tl)x))_j = x_j \tag{35}\]

for \(j < k\) and

\[(\exp T(tl)x)_k = ((1 + T(tl)x))_k = x_k + ct, \tag{36}\]

q.e.d. Formula \((34)\) follows from \((33)\).

Theorem 2. For a strictly triangular representation \(T\) of a Lie algebra \(L\) in a finite dimensional \(f\)-space \(E\), the element \(x \in E\) is a normal form on its \((\exp T)\)-orbit, iff

\[x \preceq x + T(l)x \tag{37}\]

for all \(l \in L\).

Proof. Proof by contradiction. Suppose that an \(x \in E\) is a normal form on its orbit, and the set of \(l \in L\) such that

\[x + T(l)x < x, \tag{38}\]

is not empty, i.e. there exist such \(k \in I\), \(l \in L\) that \(x_k \neq 0\) but \((x + T(l)x) = 0\) and \((T(l)x)_j = 0\) if \(j < k\) and \(x_j = 0\). In the last case \(x_k \neq 0\), but

\[(T(l)x)_k = (x + T(l)x)_k - x_k = -x_k \neq 0. \tag{39}\]

Let \(k \in I\) be the smallest, for all \(l \in L\), index such that \(x_k \neq 0\) and \((T(l)x)_k \neq 0\) and \((T(l)x)_j = 0\) if \(j < k\) and \(x_j = 0\). Then \((T(l)x)_j = 0\) for all \(j < k\). Applying Lemma \([3]\) we get

\[(\exp T(-\frac{x_k}{c}l))x_j = \begin{cases} x_j & \text{for } j < k, \\ 0 & \text{for } j = k, \end{cases} \tag{40}\]

that means

\[(\exp T(-\frac{x_k}{c}l))x \preceq x; \tag{41}\]

a contradiction. Thus, for every normal form we have \((37)\) for all \(l \in L\). Conversely, if for some \(x \in E\) we have \((37)\) for all \(l \in L\), then let \(k \in I\) be the smallest, for all \(l \in L\), index such that \(x_k \neq 0\)
and \((\exp T(l)x)_k = x_k + c\) with \(c \neq 0\) and \((\exp T(l)x)_j = 0\) if \(j < k\) and \(x_j = 0\). Applying Lemma 2, we get

\[ x + T(-\frac{x_k}{c}l)x < x, \]

that is a contradiction to (37). Thus if for some \(x \in E\) we have (37) for all \(l \in L\), then \(x\) is a normal form on its \(\exp(T)\)-orbit.

**Corollary 2.** For a strictly triangular representation \(T\) of a Lie algebra \(L\) in a finite dimensional \(f\)-space \(E\), the element \(x \in E\) is a normal form on its \(\exp(T)\)-orbit, iff for every \(k \in I\) with \(x_k \neq 0\),

\[ (T(l)x)_k \subseteq \text{span}\{(T(l)x)_i : i < k\} \subseteq L^*. \]  

**Proof.** The condition (43) means that we can’t change non-zero coordinates of \(x\) by adding \(T(l)x\) without changing the previous coordinates, which is the same as the stament of Theorem 2.

The diagram \(D\) of a representation \(L\) gives us a convenient way of writing down the elements

\[ (T(l)x)_j = \sum_{i < j} x_i w(i,j)(l) \]  

for a strictly triangular representation \(T\), see (5) and Proposition 11.

Using diagrams and Corollary 2, we can construct all the normal forms of strictly triangular representations. Let \(T\) be a strictly triangular representation of a Lie algebra \(L\) in the \(f\)-space \(E\) with linearly ordered basis \((e_i)_{1 \leq i \leq n}\). Denote \(D\) the diagram of \(T\) and \(w\) its weight function. First we construct the normal form ‘in a general position’. Put \(x_1 = c_1 \in f^*\) with an arbitrary \(c_1 \in f^*\). If \(w(1,2) \neq 0\), put \(x_2 = 0\) and remember the element \(y_2 = c_1 w(1,2) \in L^*\), if \(w(1,3) \notin y_2 f \subseteq L^*\), put \(x_3 = 0\) and remember \(y_3 = c_1 w(1,3) \in L^*\), \(\ldots\), until \(w(1,k) \in \text{span}\{y_2, \ldots, y_{k-1}\} \subseteq L^*\), in which case put \(x_k = c_k \in f^*\); now if \(c_1 w_{1,k+1} + c_k w_{k,k+1} \notin \text{span}\{y_2, \ldots, y_{k-1}\} \subseteq L^*\), put \(x_{k+1} = 0\) and remember \(y_{k+1} = c_1 w_{1,k+1} + c_k w_{k,k+1} \in L^*\), otherwise put \(x_{k+1} = c_{k+1} \in f^*\) and don’t remember the value of \(y_{k+1}\) and so on, assigning \(x_j = 0\) and memorizing the corresponding value of \((44)\), denoting it \(y_j\) if it does not belong to the \(\text{span}\{y_i\}_{i<j}\) and assigning \(x_j = c_j \in f^*\) otherwise, without remembering \(y_j\) in that case, until we define the value of the last coefficient, \(x_n\).

**Theorem 3.** The element

\[ x = \sum_{i \in I} x_i e_i \in E \]  

13
with coefficients $x_i$ described above, is a normal form.

Proof. It follows from Corollary 2, formula (44) and our construction. \qed

**Definition 10.** Let us call the element $x \in E$ of form (45) with $x_i$ given by the construction above, a normal form in a general position.

**Example 7.** The adjoint representation of the Lie algebra of strictly upper triangular $4 \times 4$ matrices, has a diagram

$$
\begin{array}{cccc}
x_4 & -a_{24} & x_5 & -a_{34} \quad x_6 \\
a_{23} & & a_{12} & \\
x_2 & -a_{34} & x_3 & a_{13} \\
& a_{23} & & \\
x_1 & & & \\
\end{array}
$$

(46)

Applying the diagram method, we get a normal form in a general position

$$
\begin{pmatrix}
c_4 & 0 & 0 \\
c_2 & 0 & 0 \\
c_1 & 0 & 0 \\
\end{pmatrix}
$$

(47)

with non-zero $c_1, c_2, c_4$. The memorized linear forms are $y_3 = c_1 a_{23} - c_2 a_{34}$, $y_5 = c_2 a_{12} - c_4 a_{23}$ and $y_6 = c_1 a_{13} - c_4 a_{24}$. Also, by Proposition 1, the co-adjoint representation of this Lie algebra has a diagram

$$
\begin{array}{cccc}
x_3 & a_{23} & x_2 & -a_{12} \quad x_5 \\
& a_{34} & & a_{14} \\
x_1 & -a_{12} & x_4 & -a_{23} \\
& & a_{13} & \\
\end{array}
$$

(48)

Applying the diagram method, we get a normal form in a general position

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & c_5 & 0 \\
c_1 & 0 & 0 \\
\end{pmatrix}
$$

(49)

with non-zero $c_1, c_5$. 

14
After describing the normal forms in a general position, we can continue the description of normal forms as follows. Take a normal form \( x \), put \( x_k = 0 \) instead of the last non-zero coefficient of it and continue the procedure as above, for \( x_{k+1} \) and so on, the same as we did calculating the normal forms in a general position, saving all the memorized linear forms \( y_i \) for \( i < k \) and changing them appropriately for \( i > k \), until we define the value of the last coefficient, \( x_n \).

**Theorem 4.** If we apply the procedure described in the previous paragraph to a normal form, we’ll get a normal form as well.

*Proof.* Indeed, all the conditions of Corollary 2\(^2\) are true for \( i < k \) because we started from a normal form and they are also true for \( i \geq k \), by construction. Thus, by Corollary 2\(^2\) we get a normal form.

Then, applying the same procedure to a new normal form, we can construct another one and so on until we get 0, where we stop. This method, I called a diagram method because of the intensive use of diagrams in it, is not an algorithm for an infinite field in a rigorous sence of this word because it sometimes requires an infinite number of steps. Nevertheless, it allows us to describe the normal forms in many cases rather easily. A few years ago I wrote down all the normal forms for the co-adjoint representations of the Lie algebras of strictly upper triangular \( n \times n \) matrices with \( n \leq 8 \) without using a computer. For a finite field, it is a real algorithm. We can’t use the exponential formula in that case in general, but we can use another correspondence between algebraic Lie groups and algebras. For the case of the Lie algebra of strictly triangular matrices over a finite field, one can use the correspondence \( T \to 1 + T \), see the details in \( \text{[4]} \).

**Example 8.** Applying the diagram method to the normal formal in a general position, \( \langle 49 \rangle \), we
obtain the following complete list of normal forms for that case:

\[
\begin{pmatrix}
0 & c_5 \\
0 & 0 & 0 \\
c_1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 \\
0 & 0 & 0 \\
c_1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & c_2 & 0 \\
0 & c_4 & c_6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & c_2 & 0 \\
0 & c_4 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & c_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & c_5 \\
0 & 0 & 0 \\
0 & 0 & c_6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & c_5 \\
0 & 0 & c_6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & c_5 \\
0 & 0 & c_6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & c_5 \\
0 & 0 & c_6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

with non-zero coefficients \(c_i\).

Notice that for the second matrix in this example, the set of memorized linear forms \((y_k)\) is exactly the same as for the first matrix which is a normal form in a general position. In such cases, when the first normal forms obtained from a normal form in a general position have the same set of the memorized \(y_k\), it is convenient to refer to these new normal forms as forms in a general position as well; and we’ll do that in the following two sections.

Normal forms in the last example form a few families parametrized as \((f^*)^k\), without any additional conditions. The same is true for the co-adjoint representations of Lie algebras of strictly upper triangular \(n \times n\) matrices with \(n \leq 8\). However, the following example shows that for \(n \geq 9\) there are some additional polynomial conditions.

**Proposition 12.** The matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{28} & c_{32} & x_{35} & 0 & 0 \\
0 & 0 & 0 & c_{22} & c_{27} & c_{31} & 0 & 0 & 0
\end{pmatrix}
\]

is
with \( c_i \in f^* \) and at least one of the coefficients \( x_{33} \) and \( x_{35} \) being non-zero, is a normal form for a co-adjoint representation of a Lie algebra of strictly upper-triangular \( 9 \times 9 \) matrices, iff

\[
\begin{vmatrix}
  c_{28} & c_{32} \\
  c_{27} & c_{31}
\end{vmatrix} = c_{28}c_{31} - c_{27}c_{32} = 0.
\] (52)

**Proof.** Only if the determinant (52) is 0, the corresponding linear forms for \( x_{33} \) or \( x_{35} \) are linearly dependent on the memorized earlier linear forms, corresponding to the diagram method. \( \square \)

## 5 Normal forms of quadratic differentials on a line

Let \( f \) be a field of characteristic 0. Denote \( L_n \) the Lie algebra generated by \((l_i)_{i \geq n}\) with \( i \in \mathbb{Z}, \, n > 0 \) and

\[
[l_i, l_j] = (j - i)l_{i+j},
\] (53)

see \[3\]. Denote \( G_n \) the group of automorphisms \( g \) of the algebra \( f[[t]] \) of formal power series, such that \( g(t) = t + o(t^n) \). \( L_n \) can be considered as a Lie algebra of \( G_n \), with

\[
l_i = t^{i+1} \frac{d}{dt}.
\] (54)

Denote also \( L_{mn} = L_m/L_n \) and \( G_{mn} = G_m/G_n \) for \( m \leq n \).

As it was explained in \[3\], we can realize \( L^*_n \) as a space of quadratic differentials with the basis \( \langle y_i = t^{-i-2}(dt)^2 \rangle_{i \geq n} \) dual to \((l_i)_{i \geq n}\) and \( L^*_{mn} \) as a subspace of \( L^*_m \) with the basis \( \langle y_i \rangle_{m \leq i < n} \).

**Theorem 5.** Let \( m \geq 0 \). If \( n \leq 2m + 2 \), then all the elements of \( L^*_{m+1,n+1} \) are normal forms in a general position. If \( n > 2m + 2 \), normal forms in a general position for the coadjoint representation of \( L^*_{m+1,n+1} \), are \( c_0y_n + c_1y_{n-1} + \cdots + c_my_{n-m} \) with any \( c_0 \in f^*, c_1, \ldots, c_m \in f \), for odd \( n \), or \( c_0y_n + c_1y_{n-1} + \cdots + c_my_{n-m} + c_{n/2}y_{n/2} \) with any \( c_0 \in f^*, c_1, \ldots, c_m, c_{n/2} \in f \), for even \( n \). Each normal form in a general position in \( L^*_{m+1,n+1} \) is a normal form in \( L^*_{m+1,i+1} \) for each \( i \geq n \) and in \( L^*_{m+1} \). Each normal form in \( L^*_{m+1,i+1} \) is a normal form in a general position in an \( L^*_{m+1,n+1} \) with \( \min\{2m+2, i\} \leq n \leq i \). Each normal form in \( L^*_{m+1} \) is a normal form in a general position in an \( L^*_{m+1,n+1} \) with \( n \geq 2m + 2 \).

**Proof.** The diagrams of the co-adjoint representations of \( L_{m+1,n+1} \) and of \( L_{m+1} \) have an edge \((k,j)\)
iff $k > j + m$ and $k \neq 2j$ and

$$w(k, j) = (k - 2j)y_{k-j}$$

in this case. Applying the diagram method, we immediately get the result of Theorem 5.

**Corollary 3.** For odd $n$, the ring of polynomial invariants of the co-adjoint representation of $L_{m+1,n+1}$ is $f[l_n, \ldots, l_{n-m}]$. For even $n$, this ring is $f[l_n, \ldots, l_{n-m}, P]$ where $P$ is a polynomial.

**Proof.** Rational invariants of a representation give us the equations parametrizing the orbits in a general position. Computing the orbits of normal forms in a general position, given by Theorem 5, we can see that they are affine subspaces $l_n = c_0, \ldots, l_{n-m} = c_m$ for odd $n$, or, for even $n$, intersections of the affine subspaces given by the same equations and an affine hypersurface, the equation of which must be given by a polynomial, by Luroth theorem.

Theorem 5 and Corollary 3 belong to Alexandre Kirillov [3]. The explicit formulas for the polynomials $P$ were found in my work [9].

### 6 Normal forms of generalized formal tensor fields on a line

Let $f$ be a field of characteristic 0, Lie algebras $L_{m+1}$ and groups $G_{m+1}$ defined in the previous section. Denote $F_{\lambda\mu} = f[[t]]t^{\mu}(dt)^{-\lambda}$. Lie algebras $L_{m+1}$ and groups $G_{m+1}$ naturally act on $F_{\lambda\mu}$. We'll use the topological basis $(e_n = t^{n+\mu}(dt)^{-\lambda})_{n \in \mathbb{Z}_{\geq 0}}$ of $F_{\lambda\mu}$.

**Theorem 6.** Let $m \geq 0$. If $\mu \neq (m + k + 1)\lambda$ with a positive integer $k$, then the normal forms in a general position in $F_{\lambda\mu}$, are $c_0e_0 + c_1e_1 + \cdots + c_me_m$ with any $c_0 \in f^*, c_1, \ldots, c_m \in f$. If $\lambda \neq 0$ and $\mu = (m + k + 1)\lambda$ for a positive integer $k$, the normal forms in general position in $F_{\lambda\mu}$, are $c_0e_0 + c_1e_1 + \cdots + c_me_m + c_{m+k}e_{m+k}$ with any $c_0 \in f^*, c_1, \ldots, c_m, c_{m+k} \in f$. Each normal form in a general position in $F_{\lambda\mu}$, is a normal form in $F_{\lambda,\mu-n}$ for every nonnegative integer $n$. Each normal form in $F_{\lambda\mu}$, is a normal form in a general position in $F_{\lambda,\mu+n}$ for a nonnegative integer $n$; except for the case $\lambda = 0$ and $\mu$ is a non-positive integer, in which case there are additional normal forms, $c + C$, for any constant $c \in f^*$ and $C$, a normal form in a general position in $F_{0n}$ with a positive integer $n$. 

18
Proof. The diagram of the representation of $L_{m+1}$ in $F_{\lambda \mu}$ has an edge $(j, k)$ iff $k > j + m$ and $j + \mu \neq (k - j + 1)\lambda$, in which case

$$w(j, k) = (j + \mu - (k - j + 1)\lambda)y_{k-j}. \quad (56)$$

Applying the diagram method, we immediately get the result of Theorem 3.

Analogously Corollary 3, we can describe the invariants of the representation of $L_{m+1}$ in $F_{\lambda \mu}$, see the explicit formulas in [9].

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