INFINITESIMAL GROUP SCHEMES AS ITERATIVE DIFFERENTIAL GALOIS GROUPS

ANDREAS MAURISCHAT

ABSTRACT. This article is concerned with Galois theory for iterative differential fields (ID-fields) in positive characteristic. More precisely, we consider purely inseparable Picard-Vessiot extensions, because these are the ones having an infinitesimal group scheme as iterative differential Galois group. In this article we prove a necessary and sufficient condition to decide whether an infinitesimal group scheme occurs as Galois group scheme of a Picard-Vessiot extension over a given ID-field or not. In particular, this solves the inverse ID-Galois problem for infinitesimal group schemes.

1. Introduction

In recent days, Picard-Vessiot theory for differential equations in characteristic zero and for iterative differential equations in positive characteristic has been extended to the case of non algebraically closed fields of constants (cf. [Dyc08] resp. [Mau08]). In the classical setting the Galois group of a PV-extension is given by the points of a linear algebraic group over the constants. In characteristic zero, one then has a Galois correspondence between all intermediate differential fields and the Zariski closed subgroups of the Galois group. In positive characteristic this correspondence was restricted to intermediate iterative differential fields over which the PV-field is separable. This restriction in positive characteristic and similar problems in the case of a non algebraically closed field of constants have been removed in [Dyc08] resp. [Mau08] by regarding the Galois group as a group scheme and not as the group of rational points. Every intermediate (iterative) differential field is then obtained as the field of invariants of some closed subgroup scheme. For example an intermediate ID-field over which the PV-field is inseparable is the field of invariants of a nonreduced subgroup scheme. In general, a PV-extension $E/F$ can be inseparable itself and in this case the fixed field of $E$ under the full group of iterative differential automorphisms of $E$ over $F$ is strictly bigger than $F$. Since classically one assumes equality, the more general extensions are called pseudo Picard-Vessiot extensions (PPV-extensions) here.

In this article, we treat questions concerning purely inseparable PPV-extensions. This is done in the setting of fields with a multivariate iterative...
derivation and having a perfect field of constants. (Although some of the minor results hold without the assumption of perfectness.) We first show that a PPV-extension is purely inseparable if and only if its Galois group scheme is an infinitesimal group scheme and that the exponent of the extension and the height of the group scheme are equal (cf. Cor. 3.6). The main result is a necessary and sufficient condition to decide whether an infinitesimal group scheme occurs as Galois group of a PPV-extension over a given ID-field or not (cf. Thm. 4.5 and Cor. 4.6).

In Section 2 we introduce the reader to the basic notation of multivariate iterative differential rings and PPV-extensions. Some properties, general results on PPV-extensions and the Galois correspondence are given in Section 3 and can also be found in [Man05] (see also [Hei07]). Section 4 is dedicated to purely inseparable PPV-extensions and the corresponding infinitesimal group schemes. In the last section, we give some examples to illustrate the previous results.

Acknowledgements: I would like to thank J. Hartmann, B. H. Matzat and M. Wibmer for helpful comments and suggestions on the paper.

2. Basic notation

All rings are assumed to be commutative with unit. We use the usual notation for multiindices, namely \((i_1 + j_1) = \prod_{\mu=1}^m (i_{\mu} + j_{\mu})\) and \(T^i = T_1^{i_1} T_2^{i_2} \cdots T_m^{i_m}\) for \(i = (i_1, \ldots, i_m), j = (j_1, \ldots, j_m) \in \mathbb{N}^m\) and \(T = (T_1, \ldots, T_m)\).

An \textit{m-variate iterative derivation} on a ring \(R\) is a homomorphism of rings \(\theta : R \to R[[T_1, \ldots, T_m]]\), such that \(\theta(0) = \text{id}_R\) and for all \(i, j \in \mathbb{N}^m\), \(\theta(i) \circ \theta(j) = \sum_{t \in \mathbb{N}^m} \theta(i)(r)T^i\) (cf. [Hei07], Ch. 4). In the case \(m = 1\) this is equivalent to the usual definition of an iterative derivation given for example in [Mat01].

The pair \((R, \theta)\) is then called an ID-ring and \(C_R := \{ r \in R \mid \theta(r) = r \}\) is called the \textit{ring of constants} of \((R, \theta)\). An ideal \(I \subseteq R\) is called an \textit{ID-ideal} if \(\theta(I) \subseteq I[T]\) and \(R\) is \textit{ID-simple} if \(R\) has no nontrivial ID-ideals.

Iterative derivations are extended to localisations by \(\theta(\frac{T}{s}) := \theta(r)\theta(s)^{-1}\) and to tensor products by

\[
\theta^k(r \otimes s) = \sum_{i+j=k} \theta^i(r) \otimes \theta^j(s)
\]

for all \(k \in \mathbb{N}^m\). The \(m\)-variate iterative derivation \(\theta\) is called \textit{non-degenerate} if the \(m\) additive maps \(\theta^{(0,0,\ldots,0)}, \theta^{(0,1,0,\ldots,0)}, \ldots, \theta^{(0,\ldots,0,1)}\) (which actually are derivations on \(R\)) are \(R\)-linearly independent.

Given an ID-ring \((R, \theta_R)\) over an ID-field \((F, \theta)\), we call an element \(x \in R\) \textit{differentially finite over} \(F\) if the \(F\)-vector space spanned by all \(\theta^k(x)\) \((k \in \mathbb{N}^m)\) is finite dimensional. It is easy to see that the set of elements which are differentially finite over \(F\) form an ID-subring of \(R\) that contains \(F\).

---

1The name \textit{constants} is due to the fact that all \(\theta^i (i \neq 0)\) vanish at these elements analogous to the vanishing of derivations in characteristic zero.
Remark 2.1. (see also [Hei07], Ch. 4)

Given an $m$-variate iterative derivation $\theta$ on a ring $R$, one obtains a set of $m$ (1-variate) iterative derivations $\theta_1, \ldots, \theta_m$ by defining

$$\theta^{(k)}_1 := \theta^{(k,0,\ldots,0)}, \quad \theta^{(k)}_2 := \theta^{(0,k,0,\ldots,0)}, \quad \ldots, \quad \theta^{(k)}_m := \theta^{(0,\ldots,0,k)}$$

for all $k \in \mathbb{N}$. By the iteration rule for $\theta$ these iterative derivations commute, i.e. satisfy the condition $\theta_i^{(k)} \circ \theta_j^{(l)} = \theta_j^{(l)} \circ \theta_i^{(k)}$ for all $i, j \in \{1, \ldots, m\}$, $k, l \in \mathbb{N}$. On the other hand, given $m$ commuting 1-variate iterative derivations $\theta_1, \ldots, \theta_m$ one obtains an $m$-variate iterative derivation $\theta$ by defining

$$\theta^{(k)} := \theta^{(k_1)}_1 \circ \ldots \circ \theta^{(k_m)}_m$$

for all $k = (k_1, \ldots, k_m) \in \mathbb{N}^m$.

Using the iteration rule one sees that the $m$-variate iterative derivation $\theta$ is determined by the derivations $\theta^{(1)}_1, \ldots, \theta^{(1)}_m$ if the characteristic of $R$ is zero, and by the set of maps $\{\theta^{(\ell)}_1, \ldots, \theta^{(\ell)}_m \mid \ell \in \mathbb{N}\}$ if the characteristic of $R$ is $p > 0$. Furthermore, $\theta$ is non-degenerate if and only if for all $j = 1, \ldots, m$ the derivation $\theta^{(1)}_j$ is nontrivial on $\bigcap_{i=1}^{n-1} \ker(\theta^{(1)}_i)$.

Next we consider the case that $R =: F$ is a field of positive characteristic $p$ and that $\theta$ is non-degenerate. Then the derivations $\theta^{(1)}_1, \ldots, \theta^{(1)}_m$ are nilpotent $C_F$-endomorphisms of $F$. Since they commute and $\theta$ is non-degenerate, there exist $x_1, \ldots, x_m \in F$ such that $\theta^{(1)}_i(x_j) = \delta_{ij}$ for all $i, j$, where $\delta_{ij}$ denotes the Kronecker delta. Therefore $\{x_1^{e_1} \cdots x_m^{e_m} \mid 0 \leq e_j \leq p-1\}$ is a basis of $F$ as a vector space over $F_1 := \bigcap_{i=1}^m \ker(\theta^{(1)}_i)$. Hence $F/F_1$ is a field extension of degree $p^m$. Furthermore, the maps $\theta^{(\ell)}_1, \ldots, \theta^{(\ell)}_m$ are derivations on $F_1$, they also are nilpotent and commute, and

$$\theta^{(\ell)}_i(x_j^p) = \left(\theta^{(1)}_i(x_j)\right)^p = \delta_{ij}.$$  

So by the same argument, $F_1$ is a vector space over $F_2 := F_1 \cap \bigcap_{i=1}^m \ker(\theta^{(\ell)}_i)$ and $[F_1 : F_2] = p^m$. Repeating this, one obtains a descending sequence of subfields $F_i := F_{i-1} \cap \bigcap_{i=1}^m \ker(\theta^{(\ell)}_i)$ satisfying $[F_{i-1} : F_i] = p^m$.

This sequence will be useful in Section [4].

Definition 2.2. Let $(F, \theta)$ be an ID-field, and let $A = \sum_{k \in \mathbb{N}^m} A_k T^k \in \text{GL}_n(F[[T]])$ be a matrix satisfying the properties $A_0 = 1_n$ and $(k^{l+1}) A_{k+l} = \sum_{i+j=l} \theta^{(l)}(A_k) A_j$ for all $k, l \in \mathbb{N}^m$. Then an equation

$$\theta(y) = Ay,$$

where $y$ is a vector of indeterminants, is called an iterative differential equation (IDE) over $F$.\footnote{Throughout this article, iterative derivations are applied componentwise to vectors and matrices.}
**Definition 2.3.** An ID-ring \((R, \theta_R) \geq (F, \theta)\) is called a **pseudo Picard-Vessiot ring** (PPV-ring) for \(\theta(y) = Ay\) if the following holds:

i) \(R\) is an ID-simple ring.

ii) There is a fundamental solution matrix \(Y \in \text{GL}_n(R)\), i.e. an invertible matrix satisfying \(\theta(Y) = AY\).

iii) As an \(F\)-algebra, \(R\) is generated by the coefficients of \(Y\) and \(\det(Y)^{-1}\).

iv) \(C_R = C_F\).

The quotient field \(E = \text{Quot}(R)\) (which exists, since such a PPV-ring is always an integral domain) is called a **pseudo Picard-Vessiot field** (PPV-field) for the IDE \(\theta(y) = Ay\).

**Remark 2.4.** The condition on the \(A_k\) given in the definition of the IDE is equivalent to the condition that \(\theta^{(k)}(\theta^l(Y_{ij})) = (\theta^l)^{(k+l)}(Y_{ij})\) holds for a fundamental solution matrix \(Y = (Y_{ij})_{1 \leq i,j \leq n} \in \text{GL}_n(R)\).

Furthermore, the condition \(A_0 = 1_n\) already implies that the matrix \(A\) is invertible.

**Notation** From now on, \((F, \theta)\) denotes an ID-field of positive characteristic \(p\), and \(K = C_F\) its field of constants. We assume that \(K\) is perfect, and that the \(m\)-variate iterative derivation \(\theta\) is non-degenerate.

### 3. Galois theory

In this section, we deal with the Galois group scheme corresponding to a PPV-extension. We will see various facettes of the group structure and group action, and provide the Galois correspondence for PPV-extensions.

We begin with a characterisation of the PPV-ring in a PPV-field.

**Proposition 3.1.** Let \((R, \theta_R)\) be a PPV-ring over \(F\) for an IDE \(\theta(y) = Ay\) and \(E = \text{Quot}(R)\). Then \(R\) is equal to the set of elements in \(E\) which are differentially finite over \(F\).

**Proof.** (Compare [Mat01], Thm. 4.9, for the case when \(K\) is algebraically closed and \(\theta\) is univariate.)

Let \(Y \in \text{GL}_n(R)\) be a fundamental solution matrix for the IDE. Then by definition \(\theta^{(k)}(Y) = A_k Y\) and hence for all \(i,j\) and all \(k \in \mathbb{N}^m\) the derivatives \(\theta^{(k)}(Y_{ij})\) are in the \(F\)-vector space spanned by all \(Y_{ij}\), i.e. all \(Y_{ij}\) are differentially finite. Furthermore, one has \(\theta(\det(Y)^{-1}) = \det(\theta(Y))^{-1} = \det(AY)^{-1} = \det(A)^{-1} \det(Y)^{-1}\), i.e. \(\det(Y)^{-1}\) is differentially finite. Therefore, \(R\) is generated by differentially finite elements, and since the differentially finite elements form a ring, all elements of \(R\) are differentially finite.

On the other hand, let \(x \in E\) be differentially finite over \(F\) and let \(W_F(x)\) be the \(F\)-vector space spanned by all \(\theta^{(k)}(x)\) \((k \in \mathbb{N}^m)\). Then the set \(I_x := \{ r \in R \mid r \cdot W_F(x) \subseteq R \}\) is an ID-ideal of \(R\). Since \(W_F(x)\) is finite dimensional and \(E\) is the quotient field of \(R\), one has \(I_x \neq 0\). Since \(R\) is ID-simple, this implies \(I_x = R\). Hence \(1 \cdot W_F(x) \subseteq R\), and in particular \(1 \cdot x = x \in R\). \(\square\)
From this characterisation of the PPV-ring as the ring of differentially finite elements, we immediately get the following.

\textbf{Corollary 3.2.} Let $E$ be a PPV-field over $F$ for several IDEs. Then the PPV-ring inside $E$ is unique and independent of the particular IDE.

\section{The Galois group scheme.} For a PPV-ring $R/F$ we define the functor
\[
\text{Aut}^{\text{ID}}(R/F) : (\text{Algebras}/K) \to (\text{Groups}), L \mapsto \text{Aut}^{\text{ID}}(R \otimes_K L/F \otimes_K L)
\]
where $L$ is provided with the trivial iterative derivation.

In \cite{Mau08}, Sect. 10, it is shown that the functor $\mathcal{G} := \text{Aut}^{\text{ID}}(R/F)$ is representable by a $K$-algebra of finite type and hence is an affine group scheme of finite type over $K$, which is called the (iterative differential) \textbf{Galois group scheme} of the extension $R$ over $F$ – denoted by $\text{Gal}^{\text{ID}}(R/F)$ –, or also the Galois group scheme of the extension $E$ over $F$, $\text{Gal}^{\text{ID}}(E/F)$, where $E = \text{Quot}(R)$ is the corresponding PPV-field.\footnote{This is justified by the fact given in Corollary 3.2 that the PPV-ring can be recovered from the PPV-field without regarding an IDE. Also take care that the functor $\text{Aut}^{\text{ID}}(E/F)$ is not isomorphic to $\text{Aut}^{\text{ID}}(R/F)$. Hence the Galois group scheme of $E/F$ has to be defined using the PPV-ring.}

Furthermore $\text{Spec}(R)$ is a $(\mathcal{G} \times_K F)$-torsor and the corresponding isomorphism of rings
\[
\gamma : R \otimes_F R \to R \otimes_K K[\mathcal{G}]
\]
is an $R$-linear ID-isomorphism.

By restricting $\gamma$ to the constants, one obtains that $K[\mathcal{G}]$ is isomorphic to $C_{R \otimes_F R}$. One checks by calculation (see also \cite{Tak89}) that the comultiplication on $K[\mathcal{G}]$ is induced via this isomorphism by the map
\[
R \otimes_F R \to (R \otimes_F R) \otimes_R (R \otimes_F R), a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b),
\]
and the counit map $\text{ev} : K[\mathcal{G}] \to K$ is induced by the multiplication
\[
R \otimes_F R \to R, a \otimes b \mapsto ab.
\]

Let $\mathcal{H} \leq \mathcal{G}$ be a subgroup functor, i.e. for every $K$-algebra $L$, the set $\mathcal{H}(L)$ is a group acting on $R_L := R \otimes_K L$ and this action is functorial in $L$. An element $r \in R$ is then called \textbf{invariant} under $\mathcal{H}$ if for all $L$, the element $r \otimes 1 \in R_L$ is invariant under $\mathcal{H}(L)$. The ring of invariants is denoted by $R^\mathcal{H}$. (In \cite{Jan03}, I.2.10 the invariant elements are called “fixed points”.)

Let $E = \text{Quot}(R)$ be the quotient field and for all $L$ let $\text{Quot}(R \otimes_K L)$ be the localisation by all nonzero divisors. Since every automorphism of $R \otimes_K L$ extends uniquely to an automorphism of $\text{Quot}(R \otimes_K L)$, the functor $\text{Aut}(R/F)$ is a subgroup functor of the group functor
\[
(\text{Algebras}/K) \to (\text{Groups}), L \mapsto \text{Aut}(\text{Quot}(R \otimes_K L)/ \text{Quot}(F \otimes_K L)).
\]
In this sense, we call an element $e = \frac{r}{s} \in E$ invariant under $H$, if for all $K$-algebras $L$ and all $h \in H(L)$,

$$\frac{h.(r \otimes 1)}{h.(s \otimes 1)} = \frac{r \otimes 1}{s \otimes 1} = e \otimes 1.$$ 

The ring of invariants of $E$ is denoted by $E^H$.

**Remark 3.3.** The action of $G := \text{Gal}(R/F)$ on $R$ is fully described by the ID-homomorphism $\rho := \gamma|_{\otimes R} : \hat{R} \to R \otimes_K K[G]$. Namely, for a $K$-algebra $L$ and $g \in G(L) \cong \text{Hom}(K[G], L)$, one has $g.(r \otimes 1) = (1 \otimes g)(\rho(r)) \in R \otimes_K L$ for all $r \in R$.

**Proposition 3.4.** Let $E/F$ be a PPV-extension with PPV-ring $R$ and Galois group scheme $G$. An ID-field $\tilde{F}$, with $F \leq \tilde{F} \leq E$, is a PPV-field over $F$, if and only if it is stable under the action of $G$, i.e. if $\rho(R \cap \tilde{F}) \subseteq (R \cap \tilde{F}) \otimes K[G]$.

**Proof.** If $\tilde{F}$ is a PPV-field, its PPV-ring $\tilde{R}$ is the set of elements in $\tilde{F}$ which are differentially finite over $F$ (cf. Prop 3.1), in particular we have $\tilde{R} = \tilde{F} \cap R$. Hence we obtain a commutative diagram:

$$\begin{array}{ccc}
\tilde{R} \otimes_F \tilde{R} & \cong & \tilde{R} \otimes_K K[\text{Gal}(\tilde{R}/F)] = \tilde{R} \otimes_K C_{\tilde{R} \otimes_F \tilde{R}} \\
\downarrow & & \downarrow \\
R \otimes_F R & \cong & R \otimes_K K[G] = R \otimes_K C_{R \otimes_F R}
\end{array}$$

But this implies $\rho(\tilde{R}) \subseteq \tilde{R} \otimes_K C_{\tilde{R} \otimes_F \tilde{R}} \subseteq \tilde{R} \otimes_K K[G]$, i.e. $\tilde{F}$ is stable under the action of $G$.

The converse is given in Theorem 3.5,iii). □

**Theorem 3.5.** (Galois correspondence) Let $E/F$ be a PPV-extension with PPV-ring $R$ and Galois group scheme $G$.

i) There is an antiisomorphism of the lattices

$$\mathcal{H} := \{ H \mid H \leq G \text{ closed subgroup scheme of } G \}$$

and

$$\mathfrak{M} := \{ M \mid F \leq M \leq E \text{ intermediate ID-field} \}$$

given by $\Psi : \mathcal{H} \to \mathfrak{M}, H \mapsto E^H$ and $\Phi : \mathfrak{M} \to \mathcal{H}, M \mapsto \text{Gal}(E/M)$.

ii) If $H \leq G$ is normal, then $E^H = \text{Quot}(R^H)$ and $R^H$ is a PPV-ring over $F$ with Galois group scheme $\text{Gal}(R^H/F) \cong G/H$.

iii) If $M \in \mathfrak{M}$ is stable under the action of $G$, then $H := \Phi(M)$ is a normal subgroup scheme of $G$, $M$ is a PPV-extension of $F$ and $\text{Gal}(M/F) \cong G/H$.

iv) For $H \in \mathcal{H}$, the extension $E/E^H$ is separable if and only if $H$ is reduced.

**Proof.** See [Mau08], Thm. 11.5. □
For a purely inseparable field extension $E/F$ one denotes by $e(E/F)$ the **exponent** of the extension, i.e. the minimal number $e \in \mathbb{N}$ such that $E^{p^e} \subseteq F$. For an infinitesimal group scheme $G$ over $K$, the **height** of $G$, denoted by $h(G)$, is the minimal number $h \in \mathbb{N}$ such that $x^{ph} = 0$ for all $x \in K[G]^+$. (Here $K[G]^+$ is the kernel of the counit map $ev : K[G] \to K$ and is a nilpotent ideal by the definition of an infinitesimal group scheme.)

**Corollary 3.6.** Let $E/F$ be a PPV-extension with Galois group scheme $G$. Then $E/F$ is a purely inseparable extension if and only if $G$ is an infinitesimal group scheme. In this case, the exponent $e(E/F)$ and the height $h(G)$ are equal.

**Proof.** Let $G$ be infinitesimal of height $h$ and let $ev : K[G] \to K$ denote the evaluation map corresponding to the neutral element of the group. Then for any $\frac{r}{s} \in E$, we have $(id \otimes ev)(\gamma(r \otimes s - s \otimes r)) = 0$, i.e. $\gamma(r \otimes s - s \otimes r) \in R \otimes_K K[G]^+$. Since $G$ is of height $h$, we obtain $(r \otimes s - s \otimes r)^{ph} = 0$. Therefore $r^{pe} \otimes s^{pe} = s^{pe} \otimes r^{pe} \in R \otimes_F R$ which means that $\frac{r^{pe}}{s^{pe}} \in F$.

So $E/F$ is purely inseparable of exponent $\leq h$. On the other hand, let $E/F$ be purely inseparable of exponent $e$. For arbitrary $x \in K[G]^+$, let $\gamma^{-1}(1 \otimes x) =: \sum_j r_j \otimes s_j$. Then

$$1 \otimes x^{pe} = \gamma \left( \sum_j r_j^{pe} \otimes s_j^{pe} \right) = \gamma \left( \sum_j r_j^{pe} s_j^{pe} \otimes 1 \right) = \sum_j r_j^{pe} s_j^{pe} \otimes 1.$$ 

Hence (e.g. by applying id $\otimes ev$), one obtains $\sum_j r_j^{pe} s_j^{pe} = 0$ and $x^{pe} = 0$. Therefore $G$ is infinitesimal of height $\leq e$. \hfill $\square$

**4. Purely inseparable extensions**

As in the previous section, $F$ denotes a field of positive characteristic $p$ with a non-degenerate $m$-variate iterative derivation $\theta$ and a perfect field of constants $K = CF$.

**Notation** For all $\ell \in \mathbb{N}$, let $J_\ell := \{(j_1, \ldots, j_m) \in \mathbb{N}^m \setminus \{0\} \mid \forall i : j_i < p^\ell\}$ and let

$$F_\ell := \bigcap_{j \in J_\ell} \ker(\theta_{F_{j}}^{\ell j}).$$

Actually, the subfields $F_\ell$ are the same as the ones defined in Remark 2.7. Since $\theta_F(F_\ell) \subseteq F_\ell[[T_{1}^{p^\ell}, \ldots, T_{m}^{p^\ell}]]$, one obtains an iterative derivation on $F_{[\ell]} := (F_\ell)^{p^{-\ell}}$ by $\theta_{F_{[\ell]}}(x) := \left(\theta_F(x^{p^\ell})\right)^{p^{-\ell}}$. Obviously, it is the unique iterative derivation which turns $F_{[\ell]}$ into an ID-extension of $F$.

**Proposition 4.1.**

i) For all $\ell \in \mathbb{N}$, $F_{[\ell]}$ is the unique maximal purely inseparable ID-extension of $F$ of exponent $\leq \ell$.

ii) For all $\ell_1, \ell_2 \in \mathbb{N}$, $(F_{[\ell_1]})(\ell_2) = F_{[\ell_1 + \ell_2]}$.

iii) If $F_{[1]} = F$ then $F_{[\ell]} = F$ for all $\ell \in \mathbb{N}$.
iv) If \( F_{1|} \neq F \) and \( \theta \) is non-degenerate, then for all \( \ell \in \mathbb{N} \), the exponent of \( F_{\ell}|/F \) is exactly \( \ell \).

**Proof.** For the proof of part [i], we have already seen that \( F_{\ell}|/F \) is an ID-extension, and by definition it is purely inseparable of exponent \( \leq \ell \). If \( E \) is a purely inseparable ID-extension of \( F \) of exponent \( \leq \ell \), then \( E^{p^\ell} \subseteq F \cap E_{\ell} \subseteq F_{\ell} \) and therefore \( E \subseteq F_{\ell} \). Hence \( F_{\ell}| \) is the unique maximal ID-extension of this kind.

By definition \((F_{[\ell_1]}|_{[\ell_2]}\) is an ID-extension of \( F \) of exponent \( \leq \ell_1 + \ell_2 \). Hence by part [i], we have \((F_{[\ell_1]}|_{[\ell_2]} \subseteq F_{[\ell_1+\ell_2]} \). On the other hand \((F_{[\ell_1+\ell_2]}|^{p^{\ell_1+\ell_2}} \subseteq F \) and so \((F_{[\ell_1+\ell_2]}|^{p^{\ell_2}} \subseteq F_{[\ell_1]} \). Hence \( F_{[\ell_1+\ell_2]} \) is an ID-extension of \( F_{[\ell_1]} \) of exponent \( \leq \ell_2 \) and therefore contained in \((F_{[\ell_1]}|_{[\ell_2]} \). This proves part [ii].

Part [iii] is a direct consequence of part [ii]. So it remains to prove [iv]. For this it suffices to show that \( F_{[\ell_1]} \neq F_{[\ell]} \) for all \( \ell \), because this implies that \( e(F_{[\ell]}|F) \geq e(F_{[\ell_1-1]}|F) + 1 \geq \cdots \geq e(F_{[1]}|F) + \ell - 1 = \ell \).

By Remark [2.1], one has \( \dim_{F_{[\ell_1+1]}(F_{\ell})} = p^m \), since \( \theta \) is non-degenerate. Assume that \( F_{[\ell_1+1]} = F_{[\ell]} \). Then \( F_{\ell+1} = (F_{[\ell_1+1]}|^{p^{\ell_1+1}} = (F_{[\ell]}|^{p^{\ell_1+1}} = (F_{\ell})^{p^2} \) and therefore \( F \) is a finite extension of \( F_{\ell} \) of degree \( [F : (F_{\ell})^{p^2}] = [F : F_{\ell+1}] = p^{(\ell+1)m} \). On the other hand,

\[
[F : (F_{\ell})^{p^2}] = [F : F^{p^2}] \cdot [F^{p^2} : (F_{\ell})^{p^2}] = [F : F] \cdot [F : F_{\ell}] = p^m[F : F^{p^2}].
\]

So \([F : F_{\ell}] = p^m = [F : F_{1}] \), and hence \( F_{1} = F_{\ell} \), in contradiction to \( F_{[1]} \neq F \).

**Theorem 4.2.** Let \( E/F \) be a PPV-extension and let \( \ell \in \mathbb{N} \). Then \( E_{[\ell]}|/F_{[\ell]} \) is a PPV-extension, and its Galois group scheme is related to \( \text{Gal}(E/F) \) by \((\text{Frob}^\ell)^* (\text{Gal}(E_{[\ell]}|F_{[\ell]})) \cong \text{Gal}(E/F) \), where \( \text{Frob} \) denotes the Frobenius morphism on \( \text{Spec}(K) \).

**Proof.** Let \( R \subseteq E \) be the corresponding PPV-ring and \( Y \in \text{GL}_n(R) \) a fundamental solution matrix for a corresponding IDE \( \theta(y) = Ay \). Since the \( m \)-variate iterative derivation is non-degenerate on \( F \), on has \( [F : F_{\ell}] = p^{m\ell} = [E : E_{\ell}] \). Hence, there is a matrix \( D \in \text{GL}_n(F) \) such that \( \tilde{Y} := D^{-1}Y \in \text{GL}_n(RE_{\ell}) \). The matrix \( \tilde{Y} \) satisfies

\[
\theta(\tilde{Y}) = \theta(D^{-1}Y) = \theta(D)^{-1}AD\tilde{Y},
\]

i.e. it is a fundamental solution matrix for the IDE \( \theta(y) = \tilde{A}y \), where \( \tilde{A} = \theta(D)^{-1}AD \in \text{GL}_n(F[[T]]) \).

We first show that \( \tilde{A} \in \text{GL}_n(F_{[\ell]}[[T_1^{p^\ell}, \ldots, T_m^{p^\ell}]]) \). Clearly \( \tilde{A} \in \text{GL}_n(F[[T^{p^\ell}]]) \), since \( \theta^{(k)}(\tilde{Y}) = 0 \) for all \( k \in J_\ell \) and since \( \theta \) is iterative. Then for all \( j \in \mathbb{N}^m \) and all \( k \in J_\ell \) we have

\[
\theta^{(k)} \left( \theta^{(j)}(\tilde{Y}) \right) = \theta^{(j)} \left( \theta^{(k)}(\tilde{Y}) \right) = 0,
\]

and

\[
\theta^{(k)} \left( \theta^{(j)}(\tilde{Y}) \right) = \theta^{(k)} \left( \tilde{A}_j \cdot \tilde{Y} \right) = \theta^{(k)}(\tilde{A}_j)\tilde{Y}.
\]
Hence, $\theta^{(k)}(\tilde{A}_j) = 0$. Therefore $\tilde{A}_j$ has coefficients in $F_\ell$.

Since $\tilde{A} \in \text{GL}_n(F_\ell[[T^{p^\ell}]])$, $R_\ell$ is actually a PPV-ring over $F_\ell$ with fundamental solution matrix $Y$.

By taking $p^\ell$-th roots, we obtain that $R_{[\ell]}$ is a PPV-ring over $F_{[\ell]}$ with fundamental solution matrix $\left((\tilde{Y}_{i,j})^{p^\ell}\right)_{i,j}$.

For obtaining the relation between the Galois groups, we first observe that $F$ and $R_\ell$ are linearly disjoint over $F_\ell$ and hence $F \otimes_{F_\ell} R_\ell \cong R$, which induces a natural isomorphism of the Galois groups $\text{Gal}(R/F) \cong \text{Gal}(R_\ell/F_\ell)$.

Furthermore the $p^\ell$-th power Frobenius endomorphism induces an isomorphism

$$R_{[\ell]} \otimes_{F_{[\ell]}} R_{[\ell]} \xrightarrow{(p^\ell)^\ell} R_\ell \otimes_{F_\ell} R_\ell.$$

Since $\text{Gal}(R_\ell/F_\ell)$ (resp. $\text{Gal}(R_{[\ell]}/F_{[\ell]})$) is isomorphic to a $K$-group scheme, using the previous theorem, $E_{[\ell]}/F$ is a PPV-extension and therefore $E_{[\ell]}/E$ is a PPV-extension.

From this theorem we obtain a criterion for $E_{[\ell]}/E$ being a PPV-extension.

**Corollary 4.3.** Let $E/F$ be a PPV-extension and suppose that $F_1 = F^p$. Then the extension $E_{[\ell]}/E$ is a PPV-extension, for all $\ell \in \mathbb{N}$.

**Proof.** From $F_1 = F^p$, it follows that $F_{[\ell]} = F$ for all $\ell$. Hence by the previous theorem, $E_{[\ell]}/F$ is a PPV-extension and therefore $E_{[\ell]}/E$ is a PPV-extension. \hfill \Box

**Proposition 4.4.** Let $E$ be a finite ID-extension of some ID-field $F$ with $C_E = K$. Then there is a finite field extension $L$ over $K$ such that $E$ is contained in a PPV-extension of $FL = F \otimes_K L$.

**Proof.** Let $e_1, \ldots, e_n \in E$ be an $F$-basis of $E$. Then there are unique $A_k \in F^{n \times n}$, such that $\theta_E^{(k)}(e_i) = \sum_{j=1}^n (A_k)_{ij} e_j$ for all $k \in \mathbb{N}^m$ and $i = 1, \ldots, n$. Since the $A_k$ are unique, the property of $\theta_E$ being an iterative derivation implies that $\theta(y) = Ay$ is an iterative differential equation, where $A = \sum_{k \in \mathbb{N}^m} A_k T_k \in \text{GL}_n(F[[T]])$. Let $U := E[X_{ij}, \text{det}(X)^{-1}]$ be the universal solution ring for this IDE over $E$ (i.e., $\theta_U(X) = AX$). Then the ideal $(x_{11} - e_1, x_{21} - e_2, \ldots, x_{n1} - e_n) \subseteq U$ is an ID-ideal and there is a maximal ID-ideal $P$ containing $(x_{11} - e_1, \ldots, x_{n1} - e_n)$. Then the field of constants $L := C_U/P$ of $U/P$ is a finite field extension of $K$ and by construction $U/P$ is a PPV-extension of $FL$ which contains $E$. \hfill \Box

**Theorem 4.5.** Let $F$ be an ID-field with $C_F = K$ perfect.

Let $\bar{C}_\ell$ denote the maximal subalgebra of $C_{F_{[\ell]}}$ which is a Hopf algebra with respect to the comultiplication induced by $F_{[\ell]} \otimes_F F_{[\ell]} \rightarrow (F_{[\ell]} \otimes_F F_{[\ell]}) \otimes F_{[\ell]} (F_{[\ell]} \otimes_F F_{[\ell]}), a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b)$.
Then an infinitesimal group scheme of height $\leq \ell$ is realisable as ID-Galois group scheme over $F$, if and only if it is a factor group of $\text{Spec}(\tilde{C}_\ell)$.

**Proof.** Let $\tilde{G}$ be an infinitesimal group scheme of height $\leq \ell$ which is realisable as Galois group scheme over $F$ and let $F'/F$ be an extension with Galois group scheme $\tilde{G}$. By Cor. 3.6 and Prop. 4.1 $F'$ is an ID-subfield of $F[\ell]$. Therefore, $K[\tilde{G}] \cong C_{F' \otimes_F F'}$ is a subalgebra of $C_{F[\ell] \otimes_F F[\ell]}$ and is a Hopf algebra with comultiplication as given in the statement. Hence it is a sub-Hopf algebra of $\tilde{C}_\ell$ and so $\tilde{G}$ is a factor group of $\text{Spec}(\tilde{C}_\ell)$.

For the converse, we first assume that there is a PPV-extension $E/F$ such that $E \supseteq F[\ell]$. Let $R$ denote the corresponding PPV-ring and $\mathcal{G} := \text{Gal}(E/F)$ the Galois group scheme. Since $F[\ell]$ is an intermediate ID-field, there is a subgroup $\mathcal{H} \leq \mathcal{G}$ such that $F[\ell] = E^{\mathcal{H}}$. Since all elements in $F[\ell]$ are differentially finite over $F$ we even have $F[\ell] = R^{\mathcal{H}}$. Then $\tilde{C}_\ell \subseteq C_{F[\ell] \otimes_F F[\ell]} \subseteq C_{R \otimes_F R} \cong K[\mathcal{G}]$ is a sub-Hopf algebra, i.e. $\text{Spec}(\tilde{C}_\ell)$ is a factor group of $\mathcal{G}$. If $\mathcal{G}$ is a factor group of $\text{Spec}(\tilde{C}_\ell)$ then it is a factor group of $\mathcal{G}$ and therefore there is a normal subgroup $\mathcal{G}' \leq \mathcal{G}$ such that $\tilde{G} \cong \mathcal{G} / \mathcal{G}'$. Then by the Galois correspondence, $\tilde{F} := E\mathcal{G}'$ is a PPV-extension of $F$ with Galois group scheme $\tilde{G}$.

If there is no PPV-extension $E/F$ containing $F[\ell]$, then by Prop. 4.4 there is a finite Galois extension $K'$ of $K$ such that there is a PPV-extension $E'/F K'$ containing $F[\ell]K'$. By the previous arguments there is a PPV-field $F'$ over $FK'$ with Galois group $\tilde{G} \times_K K'$. Since $F'$ is a purely inseparable extension of $FK'$, it is defined over $F$, i.e. there is an ID-field $\tilde{F}/F$ such that $F' = \tilde{F} \otimes_K K'$. Since $\text{Gal}(K'/K)$ acts on $F' = \tilde{F}K'$ by ID-automorphisms, the constants of $\tilde{F} \otimes_F \tilde{F} \cong (F' \otimes_F \tilde{F})^{\text{Gal}(K'/K)} \cong (F' \otimes_F \tilde{F})^{\text{Gal}(K'/K)} \cong K'[\mathcal{G}]$ inside $C_{F[\ell] \otimes_F F[\ell]}$ are equal to the $\text{Gal}(K'/K)$-invariants of $C_{F' \otimes_F K'}$. Hence $\mathcal{G}$ is an ID-subfield $\subseteq \mathcal{G}'$ inside $C_{F[\ell] \otimes_F F[\ell]}$, where $\subseteq$ means that $\mathcal{G}$ is a sub-Hopf algebra of $\tilde{C}_\ell$.

**Corollary 4.6.** Let $E$ be an ID-field and suppose that $E$ is a PPV-extension of some ID-field $F$ satisfying $F_1 = F^p$. An infinitesimal group scheme of height $\leq \ell$ is realisable as ID-Galois group scheme over $E$, if and only if it is a factor group of $\text{Gal}(E[\ell]/E)$.

**Proof.** This follows directly from Theorem 4.5 and the fact that in this case $E[\ell]/E$ is a PPV-extension by Corollary 4.3. $\square$

5. Examples

In this section we consider some examples. Throughout this section $K$ denotes a perfect field of characteristic $p > 0$ and $K((t))$ is equipped with the univariate iterative derivation $\theta$ given by $\theta(t) = t + T$. 


Example 1. We start with the easiest case. If \( F = K(t) \) or \( F \) is a finite ID-extension of \( K(t) \) inside \( K((t)) \), then \( F_1 = F^p \), i.e. \( F_{[t]} = F \), and therefore by Prop. 4.1, there exist no purely inseparable ID-extensions of \( F \). For \( F = K(t) \), the property \( F_1 = F^p \) is obvious, and for \( F \) being a finite extension of \( K(t) \), it is obtained by a simple dimension argument.

Example 2. We present an example for an ID-field \( F \) with \( F_{[t]} \supseteq F \) which nevertheless has no purely inseparable PPV-extensions. More precisely, we show that the constants of \( F_{[t]} \otimes_F F_{[t]} \) are equal to \( K = C_F \) for all \( t \in \mathbb{N} \).

Let \( \alpha \in \mathbb{Z}_p \setminus \mathbb{Q} \) be a \( p \)-adic integer, and for all \( k \in \mathbb{N} \), let \( \alpha_k \in \{0, \ldots, p^k - 1\} \) be chosen such that \( \alpha \equiv \alpha_k \mod p^k \). Then we define \( r := \sum_{k=1}^{\infty} t^{\alpha_k} \in K[[t]] \).

The field \( F := K(t, r) \) is then an ID-subfield of \( K((t)) \), since for all \( j \in \mathbb{N} \),

\[
\theta^{(p^j)}(r) = \sum_{k=1}^{\infty} \theta^{(p^j)}(t^{\alpha_k}) = \sum_{k=1}^{\infty} \left( \frac{\alpha_k}{p^j} \right) t^{\alpha_k - p^j} = \left( \frac{\alpha_{j+1}}{p^j} \right) t^{-p^j} \sum_{k=j+1}^{\infty} t^{\alpha_k} = \left( \frac{\alpha_{j+1}}{p^j} \right) t^{-p^j} \left( r - \sum_{k=1}^{j} t^{\alpha_k} \right) \in K(t, r).
\]

Here we used that \( \left( \frac{a}{p^j} \right) = 0 \) if \( a < p^j \) and \( \left( \frac{a}{p^j} \right) \equiv \left( \frac{b}{p^j} \right) \mod p \) if \( a \equiv b \mod p^{j+1} \).

We will show now that \( r \) is transcendental over \( K(t) \):

Let \( s \) be a solution for the 1-dimensional IDE \( \theta^{(p^j)}(y) = \left( \frac{\alpha_{j+1}}{p^j} \right) t^{-p^j} y \ (j \in \mathbb{N}) \) in some extension field of \( F \). Since \( \alpha \not\in \mathbb{Q} \), the element \( s \) is transcendental over \( K(t) \) by [Mat01], Thm. 3.13. One then easily verifies

\[
\theta^{(p^j)} \begin{pmatrix} s & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \left( \frac{\alpha_{j+1}}{p^j} \right) t^{-p^j} & -\left( \frac{\alpha_{j+1}}{p^j} \right) \sum_{k=1}^{j} t^{\alpha_k - p^j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & r \\ 0 & 1 \end{pmatrix},
\]

which shows that \( K(t, r, s) \) is a PPV-field over \( K(t) \) with Galois group inside \( \mathbb{G}_m \ltimes \mathbb{G}_a \cong \{ (x, 1) \in \text{GL}_2 \} \).

Since \( s \) is transcendental over \( K(t) \), the full subgroup \( \mathbb{G}_m \) is contained in the Galois group. The only subgroups of \( \mathbb{G}_a \) which are stable under the \( \mathbb{G}_m \)-action are the Frobenius kernels \( \alpha_{p^m} \). But all Galois groups over \( K(t) \) are reduced (cf. [Mau08], Cor. 11.7), and hence we have \( \text{Gal}(K(t, r, s)/K(t)) = \mathbb{G}_m \ltimes \mathbb{G}_a \) or \( \mathbb{G}_m \). In both cases \( K(t, r, s) \) contains no elements that are algebraic over \( K(t) \). Since the power series of \( r \) does not become eventually periodic, \( r \not\in K(t) \) and so \( r \) has to be transcendental over \( K(t) \).

Next we are going to calculate the constants of \( F_{[t]} \otimes_F F_{[t]} \):

It is easily seen that \( F_{[t]} = K(t, r_{[t]}) \), where

\[
r_{[t]} := \left( t^{-\alpha_{1}} (r - \sum_{k=1}^{\ell} t^{\alpha_k}) \right)^{p^{-\ell}} = \sum_{k=1}^{\infty} t^{(\alpha_{k+\ell} - \alpha_1)p^{-\ell}} \in K[[t]],
\]
and the derivatives of $r_{[\ell]}$ are given by:
\[
\theta^{(p^\ell)}(r_{[\ell]}) = \left(\frac{(\alpha j + 1 + \ell - \alpha \ell)p^{-\ell}}{p^j}\right) t^{-p^j} \left(r_{[\ell]} - \sum_{k=1}^{j} \ell(\alpha k + \ell - \alpha \ell)p^{-\ell}\right).
\]
Hence, one obtains for all $n \in \mathbb{N}$:
\[
\theta^{(n)}(r_{[\ell]}) \in \left(\frac{(\alpha - \alpha \ell)p^{-\ell}}{n}\right) t^{-n} r_{[\ell]} + K(t).
\]
For calculating the constants in $F_{[\ell]} \otimes F F_{[\ell]}$, we remark that \{\( r_{[\ell]}^i \otimes r_{[\ell]}^j \mid 0 \leq i, j \leq p^\ell - 1 \}\) is a basis of $F_{[\ell]} \otimes F F_{[\ell]}$ as an $F$-vector space. A further calculation shows that for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$
\[
\theta^{(n)}(t^{k} r_{[\ell]}^i \otimes r_{[\ell]}^j) \equiv \left(\frac{k + (i + j)(\alpha - \alpha \ell)p^{-\ell}}{n}\right) t^{-n} (t^{k} r_{[\ell]}^i \otimes r_{[\ell]}^j)
\]
modulo terms in $r_{[\ell]}^\mu \otimes r_{[\ell]}^\nu$ with $\mu + \nu < i + j$. So an element $x := \sum_{i,j} c_{i,j} r_{[\ell]}^i \otimes r_{[\ell]}^j \in F_{[\ell]} \otimes F F_{[\ell]}$ can only be constant, if for the terms of maximal degree these binomial coefficients vanish for all $n$. Since $\alpha$ is not rational, this is only possible if $i = j = 0$ is the maximal degree and if $k = 0$, i.e. $x \in K$. So we have shown that $C_{F_{[\ell]} \otimes F F_{[\ell]}} = K$ for all $\ell \in \mathbb{N}$, which implies by Theorem 4.3 that there are no purely inseparable PPV-extensions over $F = K(t, r)$.

**Example 3.** The following example is quite contrary to the previous one. In this example all purely inseparable ID-extensions are PPV-extensions. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_p$ be $p$-adic integers such that the set $\{1, \alpha_1, \ldots, \alpha_n\}$ is $\mathbb{Z}$-linear independent, and let $\alpha_i =: \sum_{k=0}^{\infty} a_{i,k} p^k$ ($i = 1, \ldots, n$) be their normal series, i.e. $a_{i,k} \in \{0, \ldots, p - 1\}$. For $i = 1, \ldots, n$, we then define
\[
s_i := \sum_{k=0}^{\infty} a_{i,k} p^k \in K((t))
\]
and consider the field $F := K(t, s_1, \ldots, s_n)$ which obviously is an ID-subfield of $K((t))$. Since $\theta^{(p^\ell)}(s_i) = a_{i,\ell}$ for all $\ell \in \mathbb{N}$ and $i = 1, \ldots, n$, the extension $F/K(t)$ is a PPV-extension and its Galois group scheme is a subgroup scheme of $\mathbb{G}_a^n$. Actually, the condition on the $\alpha_i$ implies that the $s_i$ are algebraically independent over $K(t)$ and hence the Galois group scheme is the full group $\mathbb{G}_a^n$. Therefore by Corollary 4.3 for all $\ell \in \mathbb{N}$ the extension $F_{[\ell]}/F$ is a PPV-extension and $\text{Gal}(F_{[\ell]}/F) \cong (\alpha_{p^\ell})^n$, where $\alpha_{p^\ell}$ denotes the kernel of the $p^\ell$-th power Frobenius map on $\mathbb{G}_a^n$. Furthermore, $(\alpha_{p^\ell})^n$ is a commutative group scheme and so all its subgroup schemes are normal subgroup schemes. By Theorem 3.5 this implies that every intermediate ID-field $F \leq E \leq F_{[\ell]}$ is a PPV-extension of $F$. So all purely inseparable ID-extensions of $F$ are PPV-extensions over $F$. Furthermore, by Cor. 4.6 an infinitesimal group scheme is realisable over $F$ if and only if it is a closed subgroup scheme of $(\alpha_{p^\ell})^n$ for some $\ell$, i.e. if and only if it is a closed infinitesimal subgroup scheme of $\mathbb{G}_a^n$. 
REFERENCES

[Dyc08] Dyckerhoff, T.: The Inverse Problem of Differential Galois Theory over the Field \( \mathbb{R}(t) \). eprint arXiv:0802.2897v1 (2008)

[Hei07] Heiderich, F.: Picard-Vessiot-Theorie für lineare partielle Differentialgleichungen. Heidelberg University Library, Diplom thesis (2007)

[Jan03] Jantzen, J.C.: Representations of algebraic groups. Am. Math. Soc. (2003)

[Mat01] Matzat, B.H.: Differential Galois Theory in Positive Characteristic, notes written by J. Hartmann. IWR-Preprint 2001-35 (2001)

[Mau08] Maurischat, A.: Galois theory for iterative connections and nonreduced Galois groups, to appear in TAMS. also available as preprint at http://arxiv.org/abs/0712.3748.

[Tak89] Takeuchi, M.: A Hopf Algebraic Approach to the Picard-Vessiot Theory. Journal of Algebra, 122:481-509 (1989)

Andreas Maurischat (né Rösheisen), Interdisciplinary Center for Scientific Computing, Heidelberg University, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany
E-mail address: andreas.maurischat@iwr.uni-heidelberg.de