ON INTERSECTION OF SIMPLY CONNECTED SETS IN THE PLANE

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Abstract. Several authors [2] and [7] have recently attempted to show that the intersection of three simply connected subcontinua of the plane is simply connected provided it is non-empty and the intersection of each two of the continua is path connected. In this note we give a very short complete proof of this fact. We also confirm a related conjecture of Karimov and Repovš [7].

1. Introduction

A homology (resp., singular) cell is a compact metric space whose Vietoris (resp., singular) homology groups are trivial. Helly [6] proved the following result which is now known as the Topological Helly Theorem:

Theorem 1.1. Let \( S = \{S_0, \ldots, S_m\} \), \( m \geq n \), be a finite family of homology cells in \( \mathbb{R}^n \) such that the intersection of every subfamily \( \mathcal{H} \) of \( S \) is nonempty if the cardinality \( |\mathcal{H}| \leq n + 1 \) and it is a homology cell if \( |\mathcal{H}| \leq n \). Then \( \bigcap_{i=0}^{m} S_i \) is a homology cell.

Versions of Theorem 1.1 for singular homology have been proved by Debrunner [5] and Alexandroff and Hopf [1, p. 295] for open sets in \( \mathbb{R}^n \) and simplicial complexes in \( \mathbb{R}^n \), respectively.

A topological space is said to be simply connected if it is path connected and has trivial fundamental group. It is known [4] that a compact subspace of the plane is a singular cell if and only if it is simply connected.

In section 2 of the paper [6] Helly proved that if \( S_i, i = 1, \ldots, 4 \), are singular cells in \( \mathbb{R}^2 \) such that all intersections \( S_{i_1} \cap S_{i_2} \cap S_{i_3} \) are singular cells, then \( \bigcap_{i=1}^{4} S_i \) is not empty. Hence to prove the Topological Helly Theorem for singular cells in \( \mathbb{R}^2 \), it suffices to prove the following:

Proposition 1.2. Let \( S_0, S_1 \) and \( S_2 \) be three simply connected compacta in the plane such that the intersection of any two of them is path connected and \( \bigcap_{i=0}^{2} S_i \neq \emptyset \). Then \( \bigcap_{i=0}^{2} S_i \) is simply connected.

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Bogatyi [2] has pointed out that no complete proof of this proposition can be found in the literature. He proved the proposition in the special case that \( S_i \) are Peano continua. Karimov and Repovš [7], established that, with the hypotheses of Proposition 1.2, \( \bigcap_{i=0}^{2} S_i \) is cell-like connected (i.e., every two points can be connected by a cell-like continuum). We prove Proposition 1.2 by showing that \( \bigcap_{i=0}^{2} S_i \) is path connected. We also give an affirmative answer to a conjecture of Karimov and Repovš [7] by proving the following proposition:

**Proposition 1.3.** If \( X \) and \( Y \) are compact AR’s in the plane, then so is each component of \( X \cap Y \).

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## 2. Proof of Proposition 1.2

Since the intersection of any family of simply connected sets in the plane has a trivial fundamental group with respect to each of its points, it suffices to show that \( \bigcap_{i=0}^{2} S_i \) is path connected. Suppose this is not the case and let 0 and 1 be two points in distinct arc components of \( \bigcap_{i=0}^{2} S_i \). Let \( I \subset S_0 \cap S_1 \), \( J \subset S_0 \cap S_2 \) and \( K \subset S_1 \cap S_2 \) be arcs from 0 to 1. Consider the components \( J_n, n = 1, 2, \ldots \), of \( J \setminus (I \cup K) \) which are not in \( S_1 \). Since 0 and 1 are end-points of \( J \) it follows that no \( J_i \) separates \( I \cup J \cup K \). Also, no \( J_i \) lies in a bounded component of \( \mathbb{R}^2 \setminus (I \cup K) \) because if the locally connected continuum \( I \cup K \) separates \( J_i \) from \( \infty \) in \( \mathbb{R}^2 \), then some simple closed curve in \( I \cup K \subset S_1 \) would do so as well. Since \( S_1 \) is simply connected, this would imply \( J_i \subset S_1 \).

We are going to construct for every \( n \geq 1 \) an arc \( J^n \subset S_0 \cap S_1 \) in \( I \cup J \cup K \) from 0 to 1 such that \( J^n \cap (J_1 \cup \ldots \cup J_n) = \emptyset \). Let \( D_1 \) be the bounded component of \( \mathbb{R}^2 \setminus (I \cup J \cup K) \) whose boundary contains \( J_1 \) and \( J_1 \cup I \cup K \) separates \( D_1 \) from infinity in \( \mathbb{R}^2 \). Then \( D_1 \subset D_I \cap D_K \), where \( D_I \) (resp., \( D_K \)) is the component of \( \mathbb{R}^2 \setminus (I \cup J) \) (resp., \( \mathbb{R}^2 \setminus (J \cup K) \)) containing \( D_1 \). Then \( D_I \) and \( D_K \) are bounded because \( J_1 \) does not separate \( I \cup J \cup K \). Note that \( D_I \subset S_0 \) and \( D_K \subset S_2 \). Let \( D \) be the component of \( D_I \cap D_K \) containing \( D_1 \). Then \( D \subset S_0 \cap S_2 \) and \( J_1 \subset \partial D \subset S_0 \cap S_2 \). Moreover, \( Fr(D) \subset I \cup J \cup K \). It is well known [9] that each continuum contained in the union of finitely many arcs is rim-finite and, hence, locally connected. So \( Fr(D) \) is locally connected. Let \( C \subset Fr(D) \) be the simple closed curve that separates \( D \) from \( \infty \) in \( \mathbb{R}^2 \). Then there is an arc \( J^1 \subset (J \cup C) \setminus J_1 \subset S_0 \cap S_2 \) from 0 to 1. Obviously, \( J^1 \subset \mathbb{R}^2 \setminus J_1 \) since \( J_1 \subset D \). Suppose we already constructed an arc \( J^n \subset S_0 \cap S_1 \) in \( I \cup J \cup K \).
from 0 to 1 such that \( J^n \cap (J_1 \cup \ldots \cup J_n) = \emptyset \). If \( J^n \cap J_{n+1} = \emptyset \), let \( J^{n+1} = J^n \). If \( J^n \cap J_{n+1} \neq \emptyset \), we repeat the above arguments with \( J^n \) in place of \( J \) and \( J_{n+1} \) in place of \( J_1 \) to obtain an arc \( J^{n+1} \subset S_0 \cap S_2 \cap \left( J^n \cup I \cup \bigcup_{j=1}^{n} J_j \right) \) from 0 to 1. By induction, we construct a sequence of arcs \( \{ J^n \}_{n=1}^{\infty} \) from 0 to 1 with \( J^{n+1} \subset S_0 \cap S_1 \cap \left( I \cup J \cup K \setminus \bigcup_{i=1}^{n} J_i \right) \). Let \( J^* = \lim \sup J^n \). Then \( J^* \subset (S_0 \cap S_2) \cap \left( I \cup J \cup K \setminus \bigcup_{i=1}^{\infty} J_i \right) \subset S_1 \) is a continuum from 0 to 1. As above, \( J^* \) is locally connected. So, there is an arc in \( J^* \) from 0 to 1 which contradicts the fact that 0 and 1 are in distinct arc components of \( \bigcap_{n=0}^{\infty} S_i \).

3. PROOF OF PROPOSITION 1.3

Let \( C \) be a component of \( X \cap Y \). If \( K \) is the topological hull of \( C \), then \( K \subset X \) and \( K \subset Y \) since neither \( X \) nor \( Y \) separates \( \mathbb{R}^2 \). So, \( K = C \). By unicoherence of \( \mathbb{R}^2 \) it follows that \( Fr(C) \), the boundary of \( C \) in \( \mathbb{R}^2 \), is connected.

By the well-known result of Borsuk [3] (that every locally connected plane continuum not separating the plane is an \( AR \)), it remains to prove that \( C \) is locally connected. Since \( C \) is a continuum in the plane, it suffices to prove that \( Fr(C) \) is locally connected. To prove this it suffices to show that every pair of points of \( Fr(C) \) is separated by a finite set (see [10, p. 99]).

Since \( X \) is simply connected, locally connected subcontinuum in the plane, by [10, ch. IV], all true cyclic elements of \( X \) are topological disks \( D_i \) such that the cardinality of \( D_i \cap D_j \) is at most 1 for \( i \neq j \) and, if the sequence \( \{ D_i \} \) is infinite, then \( \lim \text{diam} D_i = 0 \). Hence, each \( Fr(D_i) \) is a simple closed curve and \( Fr(X) = X \setminus \bigcup \text{int}(D_i) \) is a locally connected continuum with a particularly simple structure. Let \( x \) and \( y \) be distinct points in \( Fr(C) \subset Fr(X) \cup Fr(Y) \). If \( x \) and \( y \) do not both lie in any one cyclic element of \( X \), then an one point set separates \( x \) and \( y \) in \( X \) and, hence, in \( C \). Thus, we may suppose that there are cyclic elements \( D \) in \( X \) and \( E \) in \( Y \) with \( x, y \in D \cap E \). Now \( x \) in \( \text{int}(D) \) implies there is a neighborhood \( W \) of \( x \) in \( Fr(X) \cup Fr(Y) \) with \( \overline{W} \subset \text{int}(D) \). Then a finite set \( P \) separates \( Fr(Y) \setminus W \) from \( x \) in \( Fr(Y) \) since \( Fr(Y) \) is rim-finite. Hence, \( P \) separates \( x \) from \( Fr(X) \cup Fr(Y) \setminus W \). So we may suppose \( x, y \in Fr(D) \cap Fr(E) \) (see [8, 49.V, Theorem 3, p. 244]).

Let \( F \) be a two-point set in \( Fr(E) \) which separates \( x \) and \( y \) in \( Fr(E) \). Then \( F \) separates \( x \) and \( y \) in \( Fr(Y) \) [10, IV.3.1, p. 67]. Since \( D \) is hereditary normal, there is a closed set \( A \subset D \) which separates \( x \) and \( y \) in \( D \) and such that \( A \cap Y \subset F \). Since \( D \) is unicoherent, a component \( A' \) of \( A \) separates \( x \) and \( y \) in \( D \). It is now a routine exercise to construct an arc \( A'' \subset D \) such that \( A'' \) separates \( x \) and \( y \) in \( D \) and \( A'' \cap Y \subset F \). If we also take \( A'' \) to be irreducible
with respect to separating \( x \) and \( y \) in \( D \) (see [8, V.49, Theorem 3, p.244]), then \( A'' \cap Fr(D) \) will contain just two points \( c \) and \( d \). As above, \( A'' \) separates \( x \) and \( y \) in \( X \) because \( D \) is a cyclic element of \( X \). So \( A'' \cap (Fr(X) \cup Fr(Y)) \subset F \cup \{c, d\} \) separates \( x \) and \( y \) in \( Fr(C) \subset (Fr(X) \cup Fr(Y)) \subset X \). So, \( Fr(C) \) is rim-finite, hence, locally connected.

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