Offline Reinforcement Learning: 
Fundamental Barriers for Value Function Approximation

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Abstract

We consider the offline reinforcement learning problem, where the aim is to learn a decision making policy from logged data. Offline RL—particularly when coupled with (value) function approximation to allow for generalization in large or continuous state spaces—is becoming increasingly relevant in practice, because it avoids costly and time-consuming online data collection and is well suited to safety-critical domains. Existing sample complexity guarantees for offline value function approximation methods typically require both (1) distributional assumptions (i.e., good coverage) and (2) representational assumptions (i.e., ability to represent some or all Q-value functions) stronger than what is required for supervised learning. However, the necessity of these conditions and the fundamental limits of offline RL are not well understood in spite of decades of research. This led Chen and Jiang (2019) to conjecture that concentrability (the most standard notion of coverage) and realizability (the weakest representation condition) alone are not sufficient for sample-efficient offline RL. We resolve this conjecture in the positive by proving that in general, even if both concentrability and realizability are satisfied, any algorithm requires sample complexity either polynomial in the size of the state space or exponential in other parameters to learn a non-trivial policy.

Our results show that sample-efficient offline reinforcement learning requires either restrictive coverage conditions or representation conditions that go beyond supervised learning, and highlight a phenomenon called over-coverage which serves as a fundamental barrier for offline value function approximation methods. A consequence of our results for reinforcement learning with linear function approximation is that the separation between online and offline RL can be arbitrarily large, even in constant dimension.

1 Introduction

In offline reinforcement learning, we aim to evaluate or optimize decision making policies using logged transitions and rewards from historical experiments or expert demonstrations. Offline RL has great promise for decision making applications where actively acquiring data is expensive or cumbersome (e.g., robotics (Pinto and Gupta, 2016; Levine et al., 2018; Kalashnikov et al., 2018)), or where safety is critical (e.g., autonomous driving (Sallab et al., 2017; Kendall et al., 2019) and healthcare (Gottesman et al., 2018, 2019; Wang et al., 2018; Yu et al., 2019; Nie et al., 2021)). In particular, there is substantial interest in combining offline reinforcement learning with function approximation (e.g., deep neural networks) in order to encode inductive biases and enable generalization across large, potentially continuous state spaces, with recent progress on both model-free and model-based approaches (Ross and Bagnell, 2012; Laroche et al., 2019; Fujimoto et al., 2019; Kumar et al., 2019; Agarwal et al., 2020). However, existing algorithms are extremely data-intensive, and offline RL methods—to date—have seen limited deployment in the aforementioned applications. To enable practical deployment going forward, it is paramount that we develop a strong understanding of the statistical foundations for reliable, sample-efficient offline reinforcement learning with function approximation, as well as an understanding of when and why existing methods succeed and how to effectively collect data.

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Compared to the basic supervised learning problem, offline reinforcement learning with function approximation poses substantial algorithmic challenges due to two issues: distribution shift and credit assignment. Within the literature on value function approximation (or, approximate dynamic programming), all existing methods require both (1) distributional conditions, which assert that the logged data has good coverage (addressing distribution shift), and (2) representational conditions, which assert that the function approximator is flexible enough to represent value functions induced by certain policies (addressing credit assignment). Notably, sample complexity analyses for standard offline RL methods (e.g., fitted Q-iteration) require representation conditions considerably more restrictive than what is required for supervised learning (Munos, 2003, 2007; Munos and Szepesvári, 2008; Antos et al., 2008), and these methods can diverge when these conditions do not hold (Gordon, 1995; Tsitsiklis and Van Roy, 1996, 1997; Wang et al., 2021a). Despite substantial research effort, it is not known whether these conditions constitute fundamental limits or whether the algorithms can be improved. Resolving this issue would serve as a stepping stone toward developing a theory for offline reinforcement learning that parallels our understanding of supervised (statistical) learning.

The lack of understanding of fundamental limits in offline reinforcement learning was highlighted by Chen and Jiang (2019), who observed that all existing finite-sample analyses for offline RL algorithms based on concentrability (Munos, 2003)—the most ubiquitous notion of data coverage—require representation conditions significantly stronger than realizability, a standard condition from supervised learning which asserts that the function approximator can represent optimal value functions. Chen and Jiang (2019) conjectured that realizability and concentrability alone do not suffice for sample-efficient offline RL and noted that proving such a result seemed to be out of reach for existing lower bound techniques. Subsequent progress led to positive results for sample-efficient offline RL under coverage conditions stronger than concentrability (Xie and Jiang, 2021) and impossibility results under weaker coverage conditions (Wang et al., 2020; Zanette, 2021), but the original conjecture remained open.

Contributions. We provide information-theoretic lower bounds which show that, in general, concentrability and realizability together are not sufficient for sample efficient offline reinforcement learning. Our first result concerns the standard offline RL setup, where the data collection distribution is only required to satisfy concentrability, and establishes a sample complexity lower bound scaling polynomially with the size of the state space. This result resolves the conjecture of Chen and Jiang (2019) in the positive. For our second result, we further restrict the data distribution to be induced by a policy (i.e., admissible), and show that any algorithm requires sample complexity either polynomial in the size of the state space or exponential in other problem parameters. Together, our results establish that sample-efficient offline RL in large state spaces is not possible unless more stringent conditions, either distributional or representational, hold.

Our lower bound constructions are qualitatively different from previous approaches and hold even when the number of actions is constant and the value function class has constant size. Our first lower bound highlights the role of a phenomenon we call strong over-coverage (first documented by Xie and Jiang (2021)), wherein the data collection distribution is supported over spurious states that are not reachable by any policy. Despite the irrelevance of these states for learning in the online setting, their inclusion in the offline dataset creates significant uncertainty. Our second lower bound discovers a weak variant of over-coverage, wherein the data collection distribution is induced by running an exploratory policy in particular time steps, but many of the states supported by this distribution are not reachable in other time steps, creating spurious correlations. Our work shows that both the strong and weak over-coverage phenomena serve as fundamental, information-theoretic barriers for the design of offline reinforcement learning algorithms.

1.1 Offline Reinforcement Learning Setting

Markov decision processes. We consider the infinite-horizon discounted reinforcement learning setting. Formally, a Markov decision process \( M = (S, A, P, R, \gamma, d_0) \) consists of a (potentially large/continuous) state space \( S \), action space \( A \), probability transition function \( P : S \times A \rightarrow \Delta(S) \), reward function \( R : S \times A \rightarrow [0, 1] \), discount factor \( \gamma \in [0, 1] \), and initial state distribution \( d_0 \in \Delta(S) \). Each (randomized) policy \( \pi : S \rightarrow \Delta(A) \) induces a distribution over trajectories \( (s_0, a_0, r_0), (s_1, a_1, r_1), \ldots \) via the following process. For \( h = 0, 1, \ldots \) : \( a_h \sim \pi(s_h) \), \( r_h = R(s_h, a_h) \), and \( s_{h+1} \sim P(s_h, a_h) \), with \( s_0 \sim d_0 \). We let \( \mathbb{E}^{M, \pi}[\cdot] \) and \( \mathbb{P}^{M, \pi}(\cdot) \) denote expectation and probability under this process, respectively.
The expected return for policy \( \pi \) is defined as
\[
J_M(\pi) := \mathbb{E}^{M,\pi}[\sum_{h=0}^{\infty} \gamma^h r_h],
\]
and the value function for \( \pi \) is given by
\[
V_M^\pi(s) := \mathbb{E}^{M,\pi}[\sum_{h=0}^{\infty} \gamma^h r_h | s_0 = s], \quad \text{and} \quad Q_M^\pi(s, a) := \mathbb{E}^{M,\pi}[\sum_{h=0}^{\infty} \gamma^h r_h | s_0 = s, a_0 = a].
\]

It is well-known that there exists a deterministic policy \( \pi_M^* : \mathcal{S} \to \mathcal{A} \) that maximizes \( V_M^\pi(s) \) for all \( s \in \mathcal{S} \) simultaneously and thus also maximizes \( J_M(\pi) \). Letting \( V_M^* := V_M^{\pi_M^*} \) and \( Q_M^* := Q_M^{\pi_M^*} \), we have \( \pi_M^*(s) = \text{arg max}_{a \in \mathcal{A}} Q_M^*(s, a) \) for all \( s \in \mathcal{S} \). Finally, we define the occupancy measure for policy \( \pi \) via \( d_M^\pi(s, a) := (1-\gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{E}^{M,\pi}(s_h = s, a_h = a) \). We drop the dependence on the model \( M \) when it is clear from context.

**Offline policy learning.** In the offline policy learning (or, optimization) problem, we do not have direct access to the underlying MDP and instead receive a dataset \( D_n \) of tuples \((s, a, r, s')\) with \( r = R(s, a) \), \( s' \sim P(s, a) \), and \( (s, a) \sim \mu \) i.i.d., where \( \mu \in \Delta(\mathcal{S} \times \mathcal{A}) \) is the data collection distribution. The goal of the learner is to use the dataset \( D_n \) to learn an \( \varepsilon \)-optimal policy \( \hat{\pi} \), that is:
\[
J(\pi^*) - \mathbb{E}[J(\hat{\pi})] \leq \varepsilon,
\]
where the expectation \( \mathbb{E}[\cdot] \) is over the draw of \( D_n \) and any randomness used by the algorithm.

In order to provide sample-efficient learning guarantees that do not depend on the size of the state space, value function approximation methods take advantage of the following conditions.

- **Realizability.** This condition asserts that we have access to a class of candidate value functions \( \mathcal{F} \subseteq (\mathcal{S} \times \mathcal{A} \to \mathbb{R}) \) (e.g., linear models or neural networks) such that \( Q^* \in \mathcal{F} \). Realizability (that is, a well-specified model) is the most common representation condition in supervised learning and statistical estimation (Bousquet et al., 2003; Wainwright, 2019) and is also widely used in contextual bandits (Agarwal et al., 2012; Foster et al., 2018).

- **Concentrability.** Call a distribution \( \nu \in \Delta(\mathcal{S} \times \mathcal{A}) \) admissible for the MDP \( M \) if there exists a (potentially stochastic and non-stationary\(^1\)) policy \( \pi \) and index \( h \) such that \( \nu(s, a) = \mathbb{P}^\pi[s_h = s, a_h = a] \). This condition asserts that there exists a constant \( C_{\text{conc}} < \infty \) such that for all admissible \( \nu \),
\[
\left\| \frac{\nu}{\mu} \right\|_{\infty} := \sup_{(s, a) \in \mathcal{S} \times \mathcal{A}} \left\{ \frac{\nu(s, a)}{\mu(s, a)} \right\} \leq C_{\text{conc}}.
\]

Concentrability is a simple but fairly strong notion of coverage which demands that the data distribution uniformly covers all reachable states.

Under these conditions, an offline RL algorithm is said to be sample-efficient if it learns an \( \varepsilon \)-optimal policy with \( \text{poly}(\varepsilon^{-1}, (1-\gamma)^{-1}, C_{\text{conc}}, \log|\mathcal{F}|) \) samples. Notably, such a guarantee depends only on the complexity \( \log|\mathcal{F}| \) for the value function class, not on the size of the state space.\(^2\)

**Are realizability and concentrability sufficient?** While realizability and concentrability are appealing in their simplicity, these assumptions alone are not known to suffice for sample-efficient offline RL. The most well-known line of research (Munos, 2003, 2007; Munos andSzepesvári, 2008; Antos et al., 2008; Chen andJiang, 2019) analyzes offline RL methods such as fitted Q-iteration under the stronger representation condition that \( \mathcal{F} \) is closed under Bellman updates ("completeness"),\(^3\) and obtains \( \text{poly}(\varepsilon^{-1}, (1-\gamma)^{-1}, C_{\text{conc}}, \log|\mathcal{F}|) \) sample complexity. Completeness is a widely used assumption, but it is substantially more restrictive than realizability and can be violated by adding a single function to \( \mathcal{F} \). Subsequent years have seen extensive research into algorithmic improvements and alternative representation and coverage conditions, but the question of whether realizability and concentrability alone are sufficient remains open.

\(^1\)A non-stationary policy is a sequence \( \{\pi_h\}_{h \geq 0} \), which generates a trajectory via \( a_h \sim \pi_h(s_h) \).

\(^2\)For infinite function classes \( \|\mathcal{F}\| = \infty \), one can replace \( \log|\mathcal{F}| \) with other standard measures of statistical capacity, such as Rademacher complexity or metric entropy. For example, when \( \mathcal{F} \) is a class of \( d \)-dimensional linear functions, \( \log|\mathcal{F}| \) can be replaced by the dimension \( d \), which is an upper bound on the metric entropy.

\(^3\)Precisely, \( TF \subseteq \mathcal{F} \), where \( T \) is the Bellman operator: \( [Tf](s, a) := R(s, a) + \mathbb{E}_{s' \sim P(s, a)}[\max_{a'} f(s', a')] \).
1.2 Main Results

The first of our main results is an information-theoretic lower bound which shows that realizability and concentrability are not sufficient for sample-efficient offline RL.

**Theorem 1.1 (Main theorem).** For all $S \geq 9$ and $\gamma \in (1/2, 1)$, there exists a family of MDPs $\mathcal{M}$ with $|S| \leq S$ and $|A| = 2$, a value function class $\mathcal{F}$ with $|\mathcal{F}| = 2$, and a data distribution $\mu$ such that:

1. We have $Q^\pi \in \mathcal{F}$ for all $\pi : S \to \Delta(\mathcal{A})$ (all-policy realizability) and $C_{\text{conc}} \leq 16$ (concentrability) for all models in $\mathcal{M}$.

2. Any algorithm using less than $c \cdot S^{1/3}$ samples must have $J(\pi^*) - E[J(\hat{\pi})] \geq c'(1 - \gamma)$ for some instance in $\mathcal{M}$, where $c$ and $c'$ are absolute numerical constants.

This result shows that even though realizability and concentrability are satisfied, any algorithm requires at least $\Omega(S^{1/3})$ samples to learn a near-optimal policy. Since $S$ can be arbitrarily large, this establishes that sample-efficient offline RL in large state spaces is impossible without stronger representation or coverage conditions and resolves the conjecture of Chen and Jiang (2019).

In fact, the theorem establishes hardness under a substantially stronger representation condition than realizability—all policy realizability—which requires that $Q^\pi \in \mathcal{F}$ for every policy $\pi$, rather than just for $\pi^*$. When one has the ability to interact with the MDP starting from the data collection distribution $\mu$ (e.g., via a generative model), it is known that all policy realizability and concentrability suffice for approximate policy iteration methods (Antos et al., 2008; Lattimore et al., 2020). However, the offline RL setting does not permit interaction, and so Theorem 1.1 yields a separation between offline RL and online RL with a generative model (and an exploratory distribution). The lower bound construction can also be extended to related settings, including policy evaluation and linear function approximation; see Section 2.4 for discussion.

Theorem 1.1 relies on a strong version of the over-coverage phenomenon, where the data distribution contains states not visited by any admissible policy. The issue of over-coverage was first noted by Xie and Jiang (2021), who observed that it can lead to pathological behavior in certain algorithms. Our result shows—somewhat surprisingly—that this phenomenon is a fundamental barrier that applies to any value approximation method.

In particular, we show that over-coverage causes spurious correlations across reachable and unreachable states which leads to significant uncertainty in the dynamics when the number of states is large.

Theorem 1.1 has constant suboptimality gap for $Q^*$, which rules out gap-dependent regret bounds as a path toward sample-efficient offline RL. We focus on policy optimization and infinite-horizon RL for concreteness, but the lower bound readily extends to the finite-horizon setting (in fact, with $H = 3$), and provides, to our knowledge, the first impossibility result for offline RL with constant horizon.

A lower bound for admissible data distributions. Up to this point, we have considered the most ubiquitous formulation of the offline RL problem, in which $\mu \in \Delta(S \times \mathcal{A})$ is an arbitrary distribution over state-action pairs. Theorem 1.1 exploits this formulation by placing mass on states not reachable by any policy, leading to a strong version of the over-coverage phenomenon. Our next result shows concentrability and realizability are still insufficient for sample-efficient offline RL even when the data distribution $\mu$ is admissible, in the sense that it is induced by a policy or mixture of policies. While strong over-coverage is impossible in this setting, the lower bound relies on a weak notion of over-coverage in which $\mu$ places significant mass on low-probability states.

**Theorem 1.2 (Lower bound for admissible data).** For any $S \geq 9$, $\gamma \in (1/2, 1)$, and $C \geq 64$, there exists a family of MDPs $\mathcal{M}$ with $|S| = S$ and $|A| = 2$, a value function class $\mathcal{F}$ with $|\mathcal{F}| = 2$, and a data distribution $\mu$ which is a mixture of admissible distributions, such that:

1. We have $Q^\pi \in \mathcal{F}$ for all $\pi : S \to \Delta(\mathcal{A})$ (all-policy realizability) and $C_{\text{conc}} \leq C$ (concentrability) for all models in $\mathcal{M}$.

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4Note that while the states may not be reachable for a given MDP in the family $\mathcal{M}$, in our construction, all states are reachable for some MDP in the family.
2. Any algorithm using less than \( c \cdot \min\{S^{1/3}/(\log S)^2, 2C/32, 2^{1/(1-\gamma)}\} \) samples must have \( J(\pi^*) - \mathbb{E}[J(\pi)] \geq c' \) for some instance in \( \mathcal{M} \), where \( c \) and \( c' \) are absolute numerical constants.

Compared to Theorem 1.1, which shows that for general data distributions any algorithm must have sample complexity polynomial in the number of states even when concentrability is constant, Theorem 1.2 shows that, for admissible data distributions,\(^5\) any algorithm must have sample complexity that is either polynomial in the number of states or exponential in concentrability (or the effective horizon \((1 - \gamma)^{-1}\)). This result is incomparable to Theorem 1.1 since, it is quantitatively slightly weaker from a sample complexity perspective, but stronger in that applies to admissible data distributions. Since admissible distributions are perhaps more natural in practice, Theorem 1.2 serves as a strong impossibility result.

While we cannot rely on strong over-coverage to prove Theorem 1.2, we are still able to create spurious correlations between a set of states that are useful for estimation and the remaining states, which are less useful. Indeed, our construction embeds a structure used in the proof of Theorem 1.1 in a nested fashion, so that even an admissible data distribution provides insufficient information to disentangle this correlation and learn a near-optimal policy.

1.3 Related Work

We close this section with a detailed discussion of some of the most relevant related work.

**Lower bounds.** While algorithm-specific counterexamples for offline reinforcement learning algorithms have a long history (Gordon, 1995; Tsitsiklis and Van Roy, 1996, 1997; Wang et al., 2021a), information-theoretic lower bounds are a more recent subject of investigation. Wang et al. (2020) (see also Amortila et al. (2020)) consider the setting where \( \mathcal{F} \) is linear (i.e., \( Q^*(s, a) = \langle \phi(s, a), \theta \rangle \), where \( \phi(s, a) \in \mathbb{R}^d \) is a known feature map). They consider a weaker coverage condition tailored to the linear setting, which asserts that \( \lambda_{\min}(\mathbb{E}_{(s,a)\sim \mu}[\phi(s,a)\phi(s,a)^\top]) \geq \frac{1}{4} \), and they show that this condition and realizability alone are not strong enough for sample-efficient offline RL. The feature coverage condition is strictly weaker than concentrability, so this does not suffice to resolve the conjecture of Chen and Jiang (2019). Instead, the conceptual takeaway is that the feature coverage condition can lead to under-coverage and may not be the right assumption for offline RL. This point is further highlighted by Amortila et al. (2020) who show that in the infinite-horizon setting, the feature coverage condition can lead to non-identifiability in MDPs with only two states, meaning one cannot learn an optimal policy even with infinitely many samples. Concentrability places stronger restrictions on the data distribution and underlying dynamics and always implies identifiability when the state and action space are finite. Establishing impossibility of sample-efficient learning under concentrability and realizability requires very new ideas (which we provide in this paper, via the notion of over-coverage).

The results of Wang et al. (2020) and Amortila et al. (2020) are extended by Zanette (2021), who provides a slightly more general lower bound for linear realizability. The results of Zanette (2021) cannot resolve the conjecture of Chen and Jiang (2019) either, because for the family of MDPs constructed therein, no data distribution can satisfy concentrability, which means that the failure of algorithms can still be attributed to the failure of concentrability rather than the hardness under concentrability. There is also a parallel line of work providing lower bounds for online reinforcement learning with linear realizability (Du et al., 2020; Weisz et al., 2021; Wang et al., 2021b), which are based on very different constructions and techniques.

Compared to the offline RL lower bounds above (Wang et al., 2020; Amortila et al., 2020; Zanette, 2021), our lower bounds have a less geometric, more information-theoretic flavor, and share more in common with lower bounds for sparsity and support testing in statistical estimation (Paninski, 2008; Verzelen and Villers, 2010; Verzelen and Gassiat, 2018; Canonne, 2020). While previous work considers a relatively small state space but large horizon and feature dimension, we grow the state space, leading to polynomial dependence on \( S \) in our lower bounds; the horizon is somewhat immaterial in our construction.

Another interesting feature is that while previous lower bounds (Wang et al., 2020; Amortila et al., 2020; Zanette, 2021) are based on deterministic MDPs, our constructions critically use stochastic dynamics, which\(^5\) The fact that the data collection distribution is a mixture, is not critical for the result. It can be weakened to a single admissible distribution with realizability (rather than all-policy realizability).
We first provide our lower bound construction, which entails specifying the MDP family. All MDPs in this family have deterministic dynamics, and the Bellman error minimization algorithm in Chen and Jiang (2019) succeeds under concentrability and realizability when the dynamics are deterministic. Therefore, any construction involving deterministic MDPs (Wang et al., 2020; Amortila et al., 2020; Zanette, 2021) cannot be used to establish impossibility of sample-efficient learning under concentrability and realizability.

Upper bounds. Classical analyses for offline reinforcement learning algorithms such as FQI (Munos, 2003, 2007; Munos and Szepesvári, 2008; Antos et al., 2008) provide sample complexity upper bounds in terms of concentrability under the strong representation condition of Bellman completeness. The path-breaking recent work of Xie and Jiang (2021) provides an algorithm which requires only realizability, but uses a stronger coverage condition (“pushforward concentrability”) which requires that $P(s' \mid s, a) / \mu(s') \leq C$ for all $(s, a, s')$. Our results imply that this condition cannot be substantially relaxed.

A complementary line of work, primarily focusing on policy evaluation (Uehara et al., 2020; Xie and Jiang, 2020; Jiang and Huang, 2020; Uehara et al., 2021), provides upper bounds that require only concentrability and realizability, but assume access to an additional weight function class that is flexible enough to represent various occupancy measures for the underlying MDP. These results scale with the complexity of the weight function class. In general, the complexity of this class may be prohibitively large without prior knowledge; this is witnessed by our lower bound construction.

1.4 Preliminaries
For any $x \in \mathbb{R}$, let $(x)_+ := \max\{x, 0\}$. For an integer $n \in \mathbb{N}$, we let $[n]$ denote the set $\{1, \ldots, n\}$. For a finite set $\mathcal{X}$, $\text{Unif}(\mathcal{X})$ denotes the uniform distribution over $\mathcal{X}$, and $\Delta(\mathcal{X})$ denotes the set of all probability distributions over $\mathcal{X}$. For probability distributions $\mathcal{P}$ and $\mathcal{Q}$ over a measurable space $(\Omega, \mathcal{F})$ with a common dominating measure, we define the total variation distance as $D_{TV}(\mathcal{P}, \mathcal{Q}) := \sup_{A \in \mathcal{F}} |\mathcal{P}(A) - \mathcal{Q}(A)| = \frac{1}{2} \int |d\mathcal{P} - d\mathcal{Q}|$ and define the $\chi^2$-divergence as $D_{\chi^2}(\mathcal{P} \parallel \mathcal{Q}) := \mathbb{E}_\mathcal{Q}[\left(\frac{d\mathcal{P}}{d\mathcal{Q}} - 1\right)^2] = \int \frac{d\mathcal{P}}{d\mathcal{Q}} - 1$ when $\mathcal{P} \ll \mathcal{Q}$ and $+\infty$ otherwise.

2 Fundamental Barriers for Offline Reinforcement Learning
In this section we present the lower bound construction for Theorem 1.1 and prove the result, then discuss consequences. The proof of Theorem 1.2—which can be viewed as a generalization of this result, but is somewhat more involved—is deferred to Appendix E, with an overview given in Section 3.

2.1 Construction: MDP Family, Value Functions, and Data Distribution
We first provide our lower bound construction, which entails specifying the MDP family $\mathcal{M}$, the value function class $\mathcal{F}$, and the data distribution $\mu$.

All MDPs in $\mathcal{M}$ belong to a parameterized MDP family with shared transition and reward structure. In what follows, we first describe the structure of the parameterized family (Section 2.1.1) and provide intuition behind why this structure leads to statistical hardness (Section 2.1.2). We then provide a specific collection of parameters that gives rise to the hard family $\mathcal{M}$ (Section 2.1.3) and complete the construction by specifying the value function class $\mathcal{F}$ and data distribution $\mu$ (Section 2.1.4).

2.1.1 MDP Parameterization
Let the discount factor $\gamma \in (0, 1)$ be fixed, and let $S \in \mathbb{N}$ be given. Assume without loss of generality that $S > 5$ and that $(S - 5)/4$ is an integer. We consider the parameterized MDP family illustrated in Figure 1.

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6Deterministic dynamics allow one to avoid the well-known double sampling problem and in particular cause the conditional variance in Eq. (3) of Chen and Jiang (2019) to vanish.

7Compared to the first version of this paper (arXiv preprint v1), the current version uses a slight modification to the Theorem 1.1 construction. The only purpose of this change is to emphasize similarity to the construction for Theorem 1.2.
Figure 1: The MDPs in $\mathcal{M}$ are parametrized by three scalars $\alpha, \beta, w$ and a subset of states $I$. The state space consists of an initial state $s$, a large number of intermediate states $S^I$, and four self-looping terminal states $\{W, X, Y, Z\}$. From the initial state $s$, action 1 (in red) transitions to state $W$, while action 2 (in blue) transitions to a subset of intermediate states $I \subset S^I$ with equal probability. In all intermediate states and terminal states, actions 1 and 2 have the same effect, with transitions denoted in black. Among the intermediate states, $I \subset S^I$ (the gray ones) are the planted states which transition with probability $\alpha$ to state $X$ and $1 - \alpha$ to state $Y$, and the remaining $S^I \setminus I$ (the striped ones) are the unplanted states which transition with probability $\beta$ to $Z$ and $(1 - \beta)$ to $Y$. There are combinatorially many choices for $I$. Only terminal states can generate non-zero rewards: the rewards of the states $W, X, Y$ and $Z$ are $w, 1, 0$ and $\alpha/\beta$, respectively.

Each MDP takes the form $M_{\alpha, \beta, w, I} = (\mathcal{S}, \mathcal{A}, P_{\alpha, \beta, I}, R_{\alpha, \beta, w, I}, \gamma, d_0)$, and is parametrized by two probability parameters $\alpha, \beta \in (0, 1)$, a reward parameter $w \in [0, 1]$, and a subset of states $I$. All MDPs in the family $\{M_{\alpha, \beta, w, I}\}$ share the same state space $\mathcal{S}$, action space $\mathcal{A}$, discount factor $\gamma$, and initial state distribution $d_0$, and differ only in terms of the transition function $P_{\alpha, \beta, I}$ and the reward function $R_{\alpha, \beta, w, I}$.

**State space.** We consider a state space $\mathcal{S} := \{s\} \cup S^I \cup \{W, X, Y, Z\}$, where $s$ is the single initial state (occurring at $h = 0$), $W, X, Y, Z$ are four self-looping terminal states, and $S^I$ is a collection of intermediate (i.e., neither initial nor terminal) states which may occur between the initial state and the terminal states $\{X, Y, Z\}$. The number of intermediate states is $S_I := |S^I| = S - 5$ which ensures $|\mathcal{S}| = S$.

**Action space.** Our action space is given by $\mathcal{A} = \{1, 2\}$. For the initial state $s$, the two actions have distinct effects, while for all other states in $\mathcal{S} \setminus \{s\}$ both actions have identical effects. As a result, the value of a given policy only depends on the action it selects in $s$. For the sake of compactness, we use the symbol $a$ as a placeholder to denote either action when taken in $s \in \mathcal{S} \setminus \{s\}$, since the choice is immaterial.$^8$

**Transition operator.** For an MDP $M_{\alpha, \beta, w, I}$, we let $I \subset S^I$ parameterize a subset of the intermediate states. We call each $s \in I$ a planted state and $s \in I^\perp := S^I \setminus I$ an unplanted state. The dynamics $P_{\alpha, \beta, I}$ for $M_{\alpha, \beta, w, I}$ are determined by $I$ and the parameters $\alpha, \beta \in (0, 1)$ as follows (cf. Figure 1):

- **Initial state $s$.** We define $P_{\alpha, \beta, I}(s, 1) = \text{Unif}(\{W\})$ and $P_{\alpha, \beta, I}(s, 2) = \text{Unif}(I)$. That is, from the initial state $s$, choosing action 1 makes the MDP transitions to state $W$ deterministically (see the red arrow in Figure 1), while choosing 2 makes the MDP transitions to each planted state in $I$ with equal probability (see the blue arrow in Figure 1); unplanted states are not reachable.

- **Intermediate states.** Transitions from states in $S^I$ are defined as follows.
  - For each planted state $s \in I$, define
    \[ P_{\alpha, \beta, I}(s, a) = \alpha\text{Unif}(\{X\}) + (1 - \alpha)\text{Unif}(\{Y\}). \]
  - For each unplanted state $s \in I^\perp$, define
    \[ P_{\alpha, \beta, I}(s, a) = \beta\text{Unif}(\{Z\}) + (1 - \beta)\text{Unif}(\{Y\}). \]

$^8$It is conceptually simpler to consider a construction where only a single action is available in $\mathcal{S} \setminus \{s\}$ and $S^I$, but this is notationally more cumbersome.
We call such a structure the \( \alpha \) which has two important features:

- **Terminal states.** All states in \( \{W, X, Y, Z\} \) self-loop indefinitely. That is \( P_{\alpha,\beta,I}(s,a) = \text{Unif}\{\{s\}\} \) for all \( s \in \{W, X, Y, Z\} \).

**Reward function.** The initial and intermediate states have no reward, i.e., \( R_{\alpha,\beta,W}(s,a) = 0 \), \( \forall s \in \{s\} \cup S^1 \), \( \forall a \in A \). Each of the self-looping terminal states \( \{W, X, Y, Z\} \) has a fixed reward determined by the parameters \( \alpha \), \( \beta \) and \( w \). In particular, we define \( R_{\alpha,\beta,w}(W,a) = w \), \( R_{\alpha,\beta,w}(X,a) = 1 \), \( R_{\alpha,\beta,w}(Y,a) = 0 \), and \( R_{\alpha,\beta,w}(Z,a) = \alpha/\beta \).

**Initial state distribution.** All MDPs in \( \mathcal{M}_{\alpha,\beta,w,I} \) start at \( s \) deterministically (that is, the initial state distribution \( d_0 \) puts all the probability mass on \( s \)). Note that since \( d_0 \) does not vary between instances, it may be thought of as known to the learning algorithm.

### 2.1.2 Intuition Behind the Construction

The family of MDPs \( \mathcal{M} \) that witnesses our lower bound is a subset of the collection \( \{\mathcal{M}_{\alpha,\beta,w,I}\} \). Before specifying the family precisely, we give intuition as to why this MDP structure leads to statistical hardness for offline reinforcement learning.

Evidently, for any MDP \( \mathcal{M}_{\alpha,\beta,w,I} \), there is only a single effective decision that the learner needs to make: to choose action 1 in \( s \) (whose value is completely determined by \( w \)) or to choose action 2 in \( s \) (whose value is completely determined by \( \alpha \)). In our construction of the MDP family (to be specified shortly), we keep \( w \) fixed over all MDPs (i.e., we make it known to the learner), so the only challenge left to the learner is to learn the value of \( \alpha \) of the underlying MDP. As we will explain, this seemingly simple task is surprisingly hard, leading to the hardness of offline reinforcement learning. The hardness arises as a result of two general principles, planted subset structure and (strong) over-coverage.

**Planted Subset Structure.** The intermediate states in \( S^1 \) are partitioned into planted and unplanted states. Each planted state in \( I \) (the planted subset) transitions to \( X \) and \( Y \) with probability \( \alpha \) and \( 1-\alpha \) respectively, while each unplanted state in \( I \) transitions to \( Z \) and \( Y \) with probability \( \beta \) and \( 1-\beta \) respectively. We call such a structure the planted subset structure, which has two important features:

- The choice of \( I \subset S^1 \) is combinatorial in nature (for example, the number of all planted subsets of size \( S_1/2 \) is \( \binom{S_1}{S_1/2} \), which is exponential in \( S_1 \)).
- Planted and unplanted states have the same value, which only depends on \( \alpha \). This holds because the rewards of \( X, Y, Z \) are 1, 0, \( \alpha/\beta \) respectively (note that \( \alpha - 1 \beta \cdot (\alpha/\beta) \)). As a result, the choice of \( I \subset S^1 \) does not affect the value function at all.

The first feature serves as the basis for statistical hardness and leads to the appearance of the state space size in the sample complexity lower bound. For intuition as to why, suppose we are given a batch dataset of independent examples in which a state in \( S^1 \) is selected uniformly at random and we observe a sample from the next state distribution. One can show that basic statistical inference tasks such as estimating the size \( |I| \) of the planted subset require \( \text{poly}(S_1) \) samples, as this entails detecting the subset based on data generated from a mixture of planted and unplanted states. For example, it is well known that testing if a distribution is uniform on a set \( I \subset [N] \) with \( |I| = \Theta(N) \) versus uniform on all of \([N]\) requires \( \text{poly}(N) \) samples (see e.g., Paninski, 2008; Ingster and Suslina, 2012; Canonne, 2020, Section 5.1).

Building on this hardness, we can show that any algorithm requires at least \( \text{poly}(S_1) \) samples to reliably estimate the transition probability parameter \( \alpha \) if \( \beta \) and \( |I| \) are unknown. Intuitively, this arises because the only way to avoid estimating \( |I| \) (which is hard) as a means to estimate \( \alpha \) is to directly look at the marginal distribution over \( \{X, Y, Z\} \). However, the marginal distribution is uninformative for estimating \( \alpha \) when there is uncertainty about \( \beta \) and \( |I| \). For example, the marginal probability of transitioning to \( X \) is \( \alpha |I| / S_1 \), from which \( \alpha \) cannot be directly recovered if \( |I| \) is unknown.
The takeaway is that while estimating \( \alpha \) would be trivial if the dataset only consisted of transitions generated from planted states, estimating this parameter when states are drawn uniformly from \( S^1 \) is very difficult because an unknown subset comes from unplanted states. This is relevant because—as we will show—in our construction, any near-optimal policy learning algorithm must have the ability to recover the value of \( \alpha \).

The second feature, that all states in \( S^1 \) share the same value, is also essential. Since the choice of \( I \subset S^1 \) does not affect the value function at all, this feature allows us to consider exponentially many choices of \( I \) while ensuring that realizability is satisfied with a value function class \( \mathcal{F} \) of constant size. Thus, the \( \text{poly}(|S|) \) factor in our lower bound cannot be attributed to any other problem parameter, such as \( \log |\mathcal{F}| \).

**(Strong) Over-coverage.** It remains to show that the hardness described above can be embedded in the offline RL setting, since (i) we must ensure concentrability is satisfied, and (ii) the learner observes rewards, not just transitions. Returning to Figure 1, we observe that the transitions from the initial state \( s \) are such that all planted states in \( I \) are reachable, but the unplanted states in \( S^1 \setminus I \) are not reachable by any policy. In particular, since all unplanted states are unreachable, any state that can only be reached from unplanted states is also unreachable, and hence we can achieve concentrability (1) without covering such states. This allows us to choose the data distribution \( \mu \) to be (roughly) uniform over all states except for the unreachable state \( Z \). This choice satisfies concentrability, but renders all reward observations uninformative (cf. Section 2.1.1). As a consequence, we show that the task described in Section 2.1.2, i.e., detecting \( \alpha \) based on transition data, is unavoidable for any algorithm with non-trivial offline RL performance.

The key principle at play here is the over-coverage phenomenon (in particular, the strong version, where \( \mu \) is supported over unreachable states). Per the discussion above, we know that if the data distribution \( \mu \) were supported only over reachable states for a given MDP, all “time step 1” examples \((s,a,r,s')\) in \( D_n \) would have \( s \in I \), which would make estimating \( \alpha \) trivial. Our construction for \( \mu \) is uniform over all states in \( S^1 \), and hence satisfies over-coverage, since it is supported over a mix of planted states and spurious (unplanted) states not reachable by any policy. This makes estimating \( \alpha \) challenging because—due to correlations between planted and unplanted states—no algorithm can accurately estimate \( \alpha \) or recover the planted states until the number of samples scales with the number of states. We emphasize, however, that while strong over-coverage makes the construction for Theorem 1.1 comparatively simple, the weak variant of over-coverage (where all states are reachable, but the offline data distribution creates a spurious correlation by favoring unplanted states) still presents a fundamental barrier and is the mechanism behind Theorem 1.2.

### 2.1.3 Specifying the MDP Family

Using the parameterized MDP family \( \{M_{\alpha,\beta,w,I}\} \), we construct the hard family \( \mathcal{M} \) for our lower bound by selecting a specific collection of values for the parameters \((\alpha,\beta,w,I)\). Define \( I_8 := \{I : |I| = \theta S_1\} \) for all \( \theta \in (0,1) \) such that \( \theta S_1 \) is an integer. We define two sub-families of MDPs,

\[
\mathcal{M}_1 := \bigcup_{I \in I_{\alpha_1}} \{M_{\alpha_1,\beta_1,w,I}\}, \quad \text{and} \quad \mathcal{M}_2 := \bigcup_{I \in I_{\alpha_2}} \{M_{\alpha_1,\beta_1,w,I}\},
\]

where \( w := \gamma(\alpha_1 + \alpha_2)/2 \) is fixed for all MDPs, \( \mathcal{M}_1 \) is specified by \((\theta_1,\alpha_1,\beta_1) = (1/2,1/4,3/4)\), and \( \mathcal{M}_2 \) is specified by \((\theta_2,\alpha_2,\beta_2) = (1/4,1/2,1/2)\).\(^9\) Finally, we define the hard family \( \mathcal{M} \) via

\[
\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2.
\]

Let us discuss some basic properties of the construction that are used to prove the lower bound.

- For all MDPs in \( \mathcal{M} \), the rewards of the terminal states \( W, X, Y \) are the same. This means there is no uncertainty in the reward function outside of state \( Z \), which has \( R_{\alpha_1,\beta_1}(Z,a) = \alpha_1/\beta_1 = 1/3 \) when \( M \in \mathcal{M}_1 \) and \( R_{\alpha_2,\beta_2}(Z,a) = \alpha_2/\beta_2 = 1 \) when \( M \in \mathcal{M}_2 \). As we mentioned, the reward of \( Z = \alpha/\beta \) is chosen to ensure that all states in \( S^1 \) have the same value \((= \gamma \alpha/(1 - \gamma))\), given the choice for \((\alpha,\beta)\).

---

\(^9\)Recall that we assume without loss of generality that \( S_1/4 \) is an integer.
• All MDPs in $M_1$ (resp. $M_2$) differ only in the choice of $I \subset S^1$. This property, along with the aforementioned fact that all states in $S^1$ have the same value $\gamma \alpha_1/(1-\gamma)$ (resp. $\gamma \alpha_2/(1-\gamma)$), ensures that $Q_M^\pi$ is the same for all $M \in M_1$ (resp. $M \in M_2$). Furthermore, our choice for $w \in (\gamma \alpha_1, \gamma \alpha_2)$ ensures that the optimal action in $s$ is action 1 (resp. action 2) for all MDPs in $M_1$ (resp. $M_2$).

• Our choice for $(\theta_1, \alpha_1, \beta_1)$ and $(\theta_2, \alpha_2, \beta_2)$ ensures that the marginal distribution of $s'$ under the process $s \sim \text{Unif}(S^1), s' \sim P(s, a)$ is the same for all $M \in M$. This property is motivated by the hard inference task described in Section 2.1.2, which requires an uninformative marginal distribution.

The exact numerical values for the MDP parameters chosen above are not essential to the result. Any tuple $(\theta_1, \alpha_1, \beta_1; \theta_2, \alpha_2, \beta_2; w)$ can be used to establish a result similar to Theorem 1.1, as long as it satisfies certain properties described in Appendix A.

2.1.4 Finishing the Construction: Value Functions and Data Distribution

We complete our construction by specifying a value function class $\mathcal{F}$ that satisfies (all-policy) realizability and a data distribution $\mu$ that satisfies concentratability (1).

**Value function class.** Define functions $f_1, f_2 : S \times A \to \mathbb{R}$ as follows; differences are highlighted in blue:

$$f_1(s, a) := \begin{cases} \frac{3}{\gamma}, & s = s, a = 1 \\ \frac{1}{\gamma}, & s = s, a = 2 \\ \frac{3}{\gamma}, & s \in S^1 \\ 1, & s = W \\ 0, & s = Y \\ 1, & s = Z \end{cases} \quad \text{and} \quad f_2(s, a) := \begin{cases} \frac{3}{\gamma}, & s = s, a = 1 \\ \frac{1}{\gamma}, & s = s, a = 2 \\ \frac{3}{\gamma}, & s \in S^1 \\ 1, & s = X \\ 0, & s = Y \\ 1, & s = Z \end{cases} \quad (2)$$

The following result is elementary; see Appendix B for a detailed calculation.

**Proposition 2.1.** For all $M \in M_1$, we have $Q^\pi_M = f_1$ for all $\pi : S \to \Delta(A)$. For all $M \in M_2$, we have $Q^\pi_M = f_2$ for all $\pi : S \to \Delta(A)$.

It follows that by choosing $\mathcal{F} := \{f_1, f_2\}$, all-policy realizability holds for all $M \in M$. Note that the all-policy realizability condition ($Q^\pi_M \in \mathcal{F}$ for all $M \in M$ and for all policies $\pi$) is substantially stronger than the standard realizability condition ($Q^\pi_M \in \mathcal{F}$ for all $M \in M$), as it requires $Q^\pi \in \mathcal{F}$ for every policy rather than just for $\pi^*$. Since the conjecture of Chen and Jiang (2019) only asks for a construction that satisfies standard realizability, by considering all-policy realizability, we are proving a stronger hardness result. This is possible because in our construction, different actions have identical effects on all states except for the initial state $s$; as a result, $Q^\pi$ does not depend on $\pi$ at all (in other words, our construction ensures that $Q^\pi$ is always the same as $Q^\pi$).

**Data distribution.** Recall that the learner is provided with an i.i.d. dataset $D_n = \{(s_i, a_i, r_i, s'_i)\}_{i=1}^n$ where $(s_i, a_i) \sim \mu$, $s'_i \sim P(\cdot | s_i, a_i)$, and $r_i = R(s_i, a_i)$ (here $P$ and $R$ are the transition and reward functions for the underlying MDP). We define the data collection distribution via:

$$\mu := \frac{1}{8} \text{Unif}(\{s\} \times \{1, 2\}) + \frac{1}{2} \text{Unif}(S^1 \times \{1, 2\}) + \frac{3}{8} \text{Unif}(\{W, X, Y\} \times \{1, 2\})$$

This choice for $\mu$ forces the learner to suffer from the hardness described in Section 2.1.2. Salient properties include: (i) both planted and unplanted states in $S^1$ are covered, and (ii) the state $Z$ is not covered. Property (i) results in strong over-coverage, which makes estimating the parameters of the underlying MDP from transitions statistically hard, while property (ii) hides the difference between the rewards of $Z$ for the two-subfamilies of MDPs and hence makes all reward observations uninformative.

We now verify the concentratability condition (1):
- For time step $h = 0$, for any $\pi : S \rightarrow \Delta(A)$, the distribution of $(s_0, a_0)$ is $d_0 \times \pi$. It follows that

$$\left\| \frac{d_0 \times \pi}{\mu} \right\|_\infty \leq \frac{1}{2} \cdot \frac{1}{2} = 16.$$ 

- For time step $h = 1$, for any $\pi : S \rightarrow \Delta(A)$, the distribution of $(s_1, a_1)$ is $\text{Unif}(I) \times \pi$. We conclude that

$$\left\| \frac{\text{Unif}(I) \times \pi}{\mu} \right\|_\infty \leq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 16,$$

where we have used that $|I| \geq S_1/4$.

- For time step $h \geq 2$, for any $\pi : S \rightarrow \Delta(A)$, the distribution of $(s_h, a_h)$ (denoted by $d_\pi^h$) is supported on $\{W, X, Y\} \times \{1, 2\}$. Therefore, we have

$$\left\| \frac{d_\pi^h}{\mu} \right\|_\infty \leq \frac{1}{8} \cdot \frac{1}{2} = 16.$$

We conclude that the construction satisfies concentrability with $C_{\text{conc}} = 16$.

### 2.2 Proof of Theorem 1.1

Having specified the lower bound construction, we proceed to prove Theorem 1.1. For any MDP $M \in \mathcal{M}$, we know from (2) that the optimal policy $\pi_M^*$ has

$$\pi_M^*(s) = \begin{cases} 1, & \text{if } M \in \mathcal{M}_1 \\ 2, & \text{if } M \in \mathcal{M}_2 \end{cases},$$

and that $Q_M^*$ has a constant gap in value between the optimal and suboptimal actions in the initial state $s$:

$$Q_M^*(s, \pi_M^*(s)) - Q_M^*(s, a) \geq \frac{1}{8} \frac{\gamma^2}{1 - \gamma}, \quad \forall a \neq \pi_M^*(s). \quad (3)$$

This implies that any policy $\pi : S \rightarrow \Delta(A)$ with high value must choose action 1 in $s$ with high probability when $M \in \mathcal{M}_1$, and must choose action 2 in $s$ with high probability when $M \in \mathcal{M}_2$. As a result, any offline RL algorithm with non-trivial performance must reliably distinguish between $M \in \mathcal{M}_1$ and $M \in \mathcal{M}_2$ using the offline dataset $D_n$. In what follows we make this intuition precise.

For each $M \in \mathcal{M}$, let $P_n^M$ denote the law of the offline dataset $D_n$ when the underlying MDP is $M$, and let $P_n^M$ be the associated expectation operator. We formalize the idea of distinguishing between $M \in \mathcal{M}_1$ and $M \in \mathcal{M}_2$ using Lemma 2.1, which reduces the task of proving a policy learning lower bound to the task of upper bounding the total variation distance between two mixture distributions $P_n^1 := \frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}_1} P_n^M$ and $P_n^2 := \frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}_2} P_n^M$.

**Lemma 2.1.** Let $\gamma \in (0, 1)$ be fixed. For any offline RL algorithm which takes $D_n = \{(s_i, a_i, r_i, s'_i)\}_{i=1}^n$ as input and returns a stochastic policy $\hat{\pi}_{D_n} : S \rightarrow \Delta(A)$, we have

$$\sup_{M \in \mathcal{M}} \left\{ J_M(\pi_M^*) - E_n^M[J_M(\hat{\pi}_{D_n})]\right\} \geq \frac{\gamma^2}{16(1 - \gamma)} \left( 1 - D_{TV}(P_n^1, P_n^2) \right).$$

Lemma 2.1 implies that if the difference between the average dataset generated by all $M \in \mathcal{M}_1$ and that generated by all $M \in \mathcal{M}_2$ is sufficiently small, no algorithm can reliably distinguish $M \in \mathcal{M}_1$ and $M \in \mathcal{M}_2$ based $D_n$, and hence must have poor performance on some instance. See Appendix C for a proof.

We conclude by bounding $D_{TV}(P_n^1, P_n^2)$. Since directly calculating the total variation distance is difficult, we proceed in two steps. We first design an auxiliary reference measure $P_0$, and then bound $D_{TV}(P_n^1, P_0)$.
and $D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2)$ separately. For the latter step, we move from total variation distance to $\chi^2$-divergence and bound $D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2)$ (resp. $D_{\chi^2}(\mathbb{P}_n^1, \mathbb{P}_n^2)$) using a mix of combinatorial arguments and concentration inequalities. This constitutes the most technical portion of the proof, and formalizes the intuition about hardness of estimation under planted subset structure described in Section 2.1.2. Our final bound on the total variation distance (proven in Appendix D) is as follows.

**Lemma 2.2.** For all $n \leq \sqrt{(S-5)/20}$, we have

$$D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2) \leq 1/2.$$  

Theorem 1.1 immediately follows by combining Lemma 2.1 and Lemma 2.2.

### 2.3 Discussion

Having proven Theorem 1.1, we briefly interpret the result and discuss some additional consequences. We refer to Section 2.4 for extensions and further discussion.

**Separation between online and offline reinforcement learning.** In the online reinforcement learning setting, the learner can execute any policy in the underlying MDP and observe the resulting trajectory. Our results show that in general, the separation between the sample complexity of online RL and offline RL can be arbitrarily large, even when concentrability is satisfied. To see this, recall that in the online RL setting, we can evaluate any fixed policy to precision $\varepsilon$ using $\mathsf{poly}((1-\gamma)^{-1}) \cdot \varepsilon^{-2}$ trajectories via Monte-Carlo rollouts. Since the class $\mathcal{M}$ we construct essentially only has two possible choices for the optimal policy and has suboptimality gap $\frac{1}{\gamma}$, we can learn the optimal policy in the online setting using $\mathsf{poly}((1-\gamma)^{-1})$ trajectories, with no dependence on the number of states. On the other hand, Theorem 1.1 shows that the sample complexity of offline RL for this family can be made arbitrarily large.

**Linear function approximation.** The observation above is particularly salient in the context of linear function approximation, where $\mathcal{F} = \{(s,a) \mapsto \langle \phi(s,a), \theta \rangle : \theta \in \mathbb{R}^d \}$ for a known feature map $\phi(s,a)$. Our lower bound construction for Theorem 1.1 can be viewed as a special case of the linear function approximation setup with $d = 2$ by choosing $\phi(s,a) = (f_1(s,a), f_2(s,a))$. Consequently, our results show that the separation between the complexity of offline RL and online RL with linearly realizable function approximation can be arbitrarily large, even when the dimension is constant. This strengthens one of the results of Zanette (2021), which provides a linearly realizable construction in which the separation between online and offline RL is exponential with respect to dimension.

**Why aren’t stronger coverage or representation conditions satisfied?** While our construction satisfies concentrability and realizability, it fails to satisfy stronger coverage and representation conditions for which sample-efficient upper bounds are known. This is to be expected, (or else we would have a contradiction!) but understanding why is instructive. Here we discuss connections to some notable conditions.

**Pushforward concentrability.** The stronger notion of concentrability that $P(s’ | s,a)/\mu(s’) \leq C$ for all $(s,a,s’)$, which is used in Xie and Jiang (2021), fails to hold because the state $Z$ is not covered by $\mu$. This presents no issue for standard concentrability because $Z$ is not reachable starting from $s$.

**Completeness.** Bellman completeness requires that the value function class $\mathcal{F}$ has $T_M \mathcal{F} \subseteq \mathcal{F}$ for all $M \in \mathcal{M}$, where $T_M$ is the Bellman operator for $M$. We show in (2) that the set of optimal Q-value functions $\{Q_M^*(s)\}_{M \in \mathcal{M}}$ is small, but completeness requires that the class remains closed even when we mix and match value functions and Bellman operators from $\mathcal{M}_1$ and $\mathcal{M}_2$, which results in an exponentially large class in our construction. To see why, first note that by Bellman optimality, we must have $\{Q_M^*(s)\}_{M \in \mathcal{M}_1} \subseteq \mathcal{F}$ if $\mathcal{F}$ is complete. We therefore also require $T_{M’}Q_M^* \in \mathcal{F}$ for $M’ \in \mathcal{M}_1$ and $M’ \in \mathcal{M}_2$. Unlike the optimal Q-functions, which are constant across $S^1$, the value of $[T_{M’}Q_M^*](s,a)$ for $s \in S^1$ depends on whether $s \in I$ or $s \in S^1 \setminus I$, where $I$ is the collection of planted states for $M’$.\footnote{Recall that $f_1$ is the optimal Q-function for any $M \in \mathcal{M}_1$ and consider $T_{M’}f_1$ where $M’ \in \mathcal{M}_2$ has planted set $I$. For $s \in I$, we have $[T_{M’}f_1](s,a) = (1/2 \cdot 1 + 1/2 \cdot 0) \gamma = \gamma/2$ while for $s \in S^1 \setminus I$, we have $[T_{M’}f_1](s,a) = (1/2 \cdot 1/3 + 1/2 \cdot 0) \gamma = \gamma/6$.} As a result, there are $|S^1|$ possible values for the Bellman backup,
Theorem 1.1 presents the simplest variant of our lower bound for clarity of exposition. In what follows we which means that the cardinality of \( F \) must be exponential in \( S \).

### 2.4 Extensions

Theorem 1.1 presents the simplest variant of our lower bound for clarity of exposition. In what follows we sketch some straightforward extensions.

- **Policy evaluation.** Our lower bound immediately extends from policy optimization to policy evaluation. Indeed, letting \( \pi_1^* \) and \( \pi_2^* \) denote the optimal policies for \( M_1 \) and \( M_2 \) respectively, we have \( |J_M(\pi_1^*) - J_M(\pi_2^*)| \propto \frac{\gamma^2}{1 - \gamma} \) for all \( M \in \mathcal{M} \), and we know that \( J_M(\pi_1^*) \) is constant across all \( M \in \mathcal{M} \). It follows that any algorithm which evaluates policy \( \pi_2^* \) to precision \( \varepsilon \cdot \frac{\gamma^2}{1 - \gamma} \) with probability at least \( 1 - \delta \) for sufficiently small numerical constants \( \varepsilon, \delta > 0 \) can be used to select the optimal policy with probability \( (1 - \delta) \), and thus guarantee \( J(\pi^*) - \mathbb{E}[J(\hat{\pi})] \lesssim \delta \cdot \frac{\gamma^2}{1 - \gamma} \). Hence, such an algorithm must use \( n = \Omega(|S|^{1/3}) \) samples by our policy optimization lower bound.

To formally cast this setup in the policy evaluation setting, we take \( \Pi = \{ \pi_2^* \} \) as the class of policies to be evaluated, and we require a value function class \( \mathcal{F} \) such that \( Q_M^\pi \in \mathcal{M} \) for all \( \pi \in \Pi, M \in \mathcal{M} \). By Proposition 2.1, it suffices to select \( \mathcal{F} = \{ f_1, f_2 \} \).

- **Learning an \( \varepsilon \)-suboptimal policy.** Theorem 1.1 shows that for any \( \gamma \in (1/2, 1), n \gtrsim S^{1/3} \) samples are required to learn a \((1 - \gamma)^{-1}\)-optimal policy. We can extend the construction to show that more generally, for any \( \varepsilon \in (0, 1), n \gtrsim \frac{S^{1/3}}{\varepsilon} \) samples are required to learn an \( \varepsilon \cdot (1 - \gamma)^{-1}\)-optimal policy. We modify the MDP family \( M_{a,b,w,I} \) by adding a single dummy state \( t \) with a self-loop and zero reward. The initial state distribution is changed so that \( d_0(t) = 1 - \varepsilon \) and \( d_1(s) = \varepsilon \). That is, with probability \( 1 - \varepsilon \), the agent begins in \( t \) and stays there forever, collecting no reward, and otherwise the agent begins at \( s \) and proceeds as in the original construction. Analogously, we replace the original data distribution \( \mu \) with \( \mu' := (1 - \varepsilon)\delta_t + \varepsilon\mu \), where \( \delta_t \) is a point mass on \( t \). This preserves the concentrability bound \( C_{\text{conc}} \leq 16 \). This modification rescales the optimal value functions, and the conclusion of Lemma 2.1 is replaced by

\[
\sup_{M \in \mathcal{M}} \{ J_M(\pi_M^*) - \mathbb{E}_n^M[\hat{J}_M(\pi_{D_n})] \} \geq \varepsilon \cdot \frac{\gamma^2}{16(1 - \gamma)} \left( 1 - D_{TV}(\frac{1}{3}_n, \frac{2}{3}_n) \right).
\]

On the other hand, since samples from the state \( t \) provide no information about the underlying instance, the effective number of samples is reduced to \( \varepsilon n \). One can make this intuition precise and prove that \( D_{TV}(\frac{1}{3}_n, \frac{2}{3}_n) \leq 3/4 \) whenever \( \varepsilon n \leq c \cdot S^{1/3} \) for a numerical constant \( c \). Combining this with the previous bound yields the result.

- **Linear function approximation.** As discussed above, Theorem 1.1 can be viewed as a special case of linear function approximation with \( d = 2 \) and \( \phi(s, a) = (f_1(s, a), f_2(s, a)) \). Compared with recent lower bounds in the linear setting (Wang et al., 2020; Zanette, 2021), this result is significantly stronger in that (a) it considers a stronger coverage condition, (b) holds with constant dimension and constant effective horizon, and (c) scales with the number of states, which can be arbitrarily large.

Lastly, it should be clear at this point that our lower bound construction extends to the finite-horizon setting with \( H = 3 \) by simply removing the self-loops from the terminal states. The only difference is that the optimal Q-value functions require a new calculation since rewards are no longer discounted.

### 3 Proof Overview for Theorem 1.2

In this section we present a high-level overview of the construction and proof for Theorem 1.2. We defer the complete proof, as well as additional discussion, to Appendix E. The proof is based on an extension of the construction used in Theorem 1.1. We still use the concept of planted and unplanted states, but since the data collection distribution must be admissible, we cannot rely on strong over-coverage to create spurious correlations. In particular, a naive adaptation would require that the unplanted states are reachable with sufficient probability, which would necessitate that state \( Z \) is supported by \( \mu \) with sufficient probability. The
resulting construction would not lead to a meaningful lower bound, as the reward information from $Z$ can be used to learn the optimal policy.

To avoid this issue, we modify the construction in Theorem 1.1 to replace $Z$ with another “layer” of states with planted subset structure (see Appendix E.1 and Footnote 13 for the precise definition of a layer). By repeating this several times, we obtain a family of MDPs with $L > 3$ layers of planted subset structure, connected in the manner displayed in Figure 2 (see Appendix E.1 for the details). Specifically, taking action 2 (in blue) from the initial state $s$, the $l$th layer is selected with probability $\propto 1/2^l$, and we transit uniformly to the states in the $l$th layer. In each layer, the planted states behave similarly to the construction for Theorem 1.1, transitioning to terminal states $X$ and $Y$ with specific probabilities that are chosen such that the marginal distribution provides no information. However, except for at the last layer, the unplanted states do not transition directly to the terminal state $Z$, but rather to the planted states of the next layer. Overall, $Z$ can be reached with only $O(1/2^L)$ probability.

Similar to our previous construction, the new multi-layer construction ensures that every MDP in the family differs only in terms of the reward of $Z$ and the transition probabilities for planted and unplanted states. Moreover, while the state $Z$ is no longer unreachable, we know that since all policies only reach $Z$ with exponentially small (in $L$) probability, we can satisfy concentrability with a data collection distribution that places exponentially small mass on $Z$ (which—if it appeared in the dataset—would reveal the optimal policy).

As a result, we have that with high probability, all reward observations in the dataset provide no information. Intuitively, this allows us to apply an inductive argument to show that one cannot learn the value function. As the base case, when the reward for $Z$ is unobserved (which happens with high probability), the $L$th layer resembles an instance of the construction used to prove Theorem 1.1. Then, going backwards, if one cannot estimate the $(l + 1)$st layer, we can view the $l$th layer as an instance of the previous construction to show that one cannot estimate the value of this layer as well. This induction relies on the delicate design of the data collection distribution $\mu$, which is supported on both planted and unplanted states, but nevertheless exhibits a weak notion of over-coverage resulting in spurious correlations. The argument also requires gradually decreasing the difference in the value function (between the two MDP families) from the $L$th layer to the first layer; however, we can ensure that the rate of decrease is very slow, which leads to statistical hardness.

On a technical level, after constructing the MDP family, many of the calculations are similar to those used to prove Theorem 1.1. Analogously to Lemma 2.1, we lower bound the suboptimality of any algorithm by the total variation distance between two mixture distributions. Then we bound this TV distance by constructing auxiliary reference measures and passing to the $\chi^2$-divergence, analogously to Lemma 2.2. Finally, since we rely on a similar planted subset structure, the $\chi^2$-divergence calculation shares many technical elements with the proof of Lemma 2.2.

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Footnote 11: The inductive argument is discussed here mainly for providing intuition. The proof in Appendix E is more direct and does not involve a formal inductive argument.
4 Conclusion

We have proven that concentrability and realizability alone are not sufficient for sample-efficient offline reinforcement learning, resolving the conjecture of Chen and Jiang (2019). Our results establish that sample-efficient offline RL requires coverage or representation conditions beyond what is required for supervised learning and show that over-coverage is a fundamental barrier for offline RL.

For future research, an immediate question is whether it is possible to circumvent our lower bound by considering trajectory-based data rather than \((s,a,r,s')\) tuples. More broadly, while our results elucidate the role of concentrability and realizability, it remains to obtain a sharp, distribution-dependent characterization for the sample complexity of offline RL with general function approximation. Such a characterization would need to recover our result and previous results—both positive and negative—as special cases.

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Part I

Proofs for Theorem 1.1

A General Scheme to Construct Hard Families of Instances

Recall that Section 2.1.3 gives specific numerical values for the parameters that define the model class \( \mathcal{M} \) used in our lower bound construction. The precise values are not critical for our proof, and in this section we give general conditions on the parameters under which one can derive a similar lower bound. In doing so, we also provide some intuition behind the specific choice of parameters used for Theorem 1.1.

In more detail, for any tuple of parameters \((\theta_1, \alpha_1, \beta_1; \theta_2, \alpha_2, \beta_2; w)\), we consider the family of MDPs \( \mathcal{M} \) given by

\[
\mathcal{M}_1 := \bigcup_{I \in \mathcal{I}_1} M_{\alpha_1, \beta_1, w, I}, \quad \mathcal{M}_2 := \bigcup_{I \in \mathcal{I}_2} M_{\alpha_2, \beta_2, w, I}, \quad \mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2.
\]

There are 7 independent scalars in the tuple, all of which lie in \([0, 1]\): \(\theta_1, \alpha_1, \beta_1, \theta_2, \alpha_2, \beta_2, w\); note that the parameter \(w\) is shared between \(\mathcal{M}_1\) and \(\mathcal{M}_2\). The family \(\mathcal{M}\) above can be used to derive a hardness result similar to Theorem 1.1 as long as the following three general equality and inequality constraints are satisfied:

- All \(M \in \mathcal{M}\) have the same marginal distribution for \(s'\) under the process \(s \sim \text{Unif}(S^1), s' \sim P(s, a)\):
  \[
  \theta_1 \alpha_1 = \theta_2 \alpha_2, \quad \text{and} \quad (1 - \theta_1) \beta_1 = (1 - \theta_2) \beta_2.
  \]  
  (4)

This ensures that the learner cannot trivially test whether \(M \in \mathcal{M}_1\) or \(\mathcal{M}_2\) using marginals, which is tacitly used in the proof of Lemma 2.2.

- The parameters \(\theta_1, \alpha_1, \beta_1, \theta_2, \alpha_2, \beta_2\) are bounded away from 0 and 1:
  \[
  \theta_1, \alpha_1, \beta_1, \theta_2, \alpha_2, \beta_2 \in (0, 1).
  \]  
  (5)

In particular, the distance from the boundary should be a constant independent of \(\frac{1}{|S|}\) and \(\gamma\).

- In state \(s\) (i.e., the only state where the two actions have distinct effects), action 1 is strictly better (resp. worse) than action 2 if \(M \in \mathcal{M}_1\) (resp. \(M \in \mathcal{M}_2\)), which means \(w/(1 - \gamma) = Q^\pi_M(s, 1) > Q^\pi_M(s, 2) = \gamma \alpha_1 (1 - \gamma)\) (resp. \(w/(1 - \gamma) = Q^\pi_M(s, 1) < Q^\pi_M(s, 2) = \gamma \alpha_2 (1 - \gamma)\)). This means
  \[
  \gamma \alpha_1 < w < \gamma \alpha_2.
  \]  
  (6)

The final lower bound depends on this separation quantitatively.

Any tuple simultaneously satisfying Eqs. (4) to (6) is sufficient for our proof (modulo numerical differences). Naturally, the numerical values for the function class \(\mathcal{F}\) defined in (2) must be changed accordingly so that the class contains \(Q^\pi\) for both \(\mathcal{M}_1\) and \(\mathcal{M}_2\).

B Computation of Value Functions (Proposition 2.1)

In this section, we verify Proposition 2.1, which asserts that for all \(\pi\), \(Q^\pi_M = f_1\) for all \(M \in \mathcal{M}_1\) and \(Q^\pi_M = f_2\) for all \(M \in \mathcal{M}_2\), where \(f_1\) and \(f_2\) are defined in (2). Note that the calculation we present here is based on the precise values for the parameters \((\theta_1, \alpha_1, \beta_1; \theta_2, \alpha_2, \beta_2; w)\) given in Section 2.1.3, not the general scheme given in Appendix A.

Proof of Proposition 2.1. Suppose \(M \in \mathcal{M}_1\). Let \(I_M\) denote the planted subset associated with \(M\). First, for any self-looping terminal state \(s \in \{W, X, Y, Z\}\), since all actions in \(A\) have identical effects, we have

\[
V^\pi_M(s) = Q^\pi_M(s, a) = \sum_{h=0}^\infty \gamma^h R_{\alpha_1, \beta_1, w}(s, a) = \begin{cases} 
\frac{3}{8} \gamma, & s = W \\
1, & s = X \\
0, & s = Y \\
\frac{1}{3}, & s = Z
\end{cases}
\]
for all $\pi : S \rightarrow \Delta(A)$, where we utilize the fact that $R_{\alpha_1, \beta_1, w}(W, a) = w = \gamma(\alpha_1 + \alpha_2)/2 = 3\gamma/8$ and $R_{\alpha_1, \beta_1, w}(Z, a) = \alpha_1 / \beta_1 = 1/3$.

Next, for any intermediate state $s \in S^I$, since all actions in $A$ have identical effects, we have

\[
V^\pi_M(s) = Q^\pi_M(s, a) = R_{\alpha_1, \beta_1, w}(s, a) + \gamma E_{s' \sim P_{\alpha_1, \beta_1, w}}(s, a)[V^\pi_M(s')] = 0 + \gamma V^\pi_M(W) = \frac{\gamma^2}{1 - \gamma} \frac{3}{8},
\]

for all $\pi : S \rightarrow \Delta(A)$.

Thus, for the initial state $s$, we have

\[
Q^\pi_M(s, 1) = R_{\alpha_1, \beta_1, w}(s, 1) + \gamma E_{s' \sim P_{\alpha_1, \beta_1, w}}(s, 1)[V^\pi_M(s')] = 0 + \gamma V^\pi_M(W) = \frac{\gamma^2}{1 - \gamma} \frac{3}{8},
\]

Now suppose $M \in M_2$. Let $I_M$ denote the planted subset associated with $M$. For any self-looping terminal state $s \in \{W, X, Y, Z\}$, since all actions in $A$ have identical effects, we have

\[
V^\pi_M(s) = Q^\pi_M(s, a) = \sum_{h=0}^{\infty} \gamma^h R_{\alpha_2, \beta_2, w}(s, a) = \frac{1}{1 - \gamma},
\]

for all $\pi : S \rightarrow \Delta(A)$, where we utilize the fact that $R_{\alpha_2, \beta_2, w}(W, a) = w = \gamma(\alpha_1 + \alpha_2)/2 = 3\gamma/8$ and $R_{\alpha_2, \beta_2, w}(Z, a) = \alpha_2 / \beta_2 = 1$. For any intermediate state $s \in S^I$, since all actions in $A$ have identical effects, we have

\[
V^\pi_M(s) = Q^\pi_M(s, a) = R_{\alpha_2, \beta_2, w}(s, a) + \gamma E_{s' \sim P_{\alpha_2, \beta_2, w}}(s, a)[V^\pi_M(s')] = 0 + \gamma V^\pi_M(W) = \frac{\gamma^2}{1 - \gamma} \frac{3}{8},
\]

for all $\pi : S \rightarrow \Delta(A)$. Thus, for the initial state $s$, we have

\[
Q^\pi_M(s, 1) = R_{\alpha_2, \beta_2, w}(s, 1) + \gamma E_{s' \sim P_{\alpha_2, \beta_2, w}}(s, 1)[V^\pi_M(s')] = 0 + \gamma V^\pi_M(W) = \frac{\gamma^2}{1 - \gamma} \frac{3}{8},
\]

for all $\pi : S \rightarrow \Delta(A)$. It follows that $Q^\pi_M(s, a) = f_2(s, a)$ for all $(s, a) \in S \times A$, for all $\pi : S \rightarrow \Delta(A)$.\]
C Proof of Lemma 2.1

We now prove Lemma 2.1. Before proceeding, let us note that this lemma is proven only for the precise values for the parameters $\theta_1, \alpha_1, \beta_1; \theta_2, \alpha_2, \beta_2; w$ given in Section 2.1.3. One could establish a more general lemma using the generic parameters introduced in Appendix A, but this would require changing the numerical constants appearing in the statement.

We begin the proof by lower bounding the regret for any MDP in the family $\mathcal{M}$. For any $i \in \{1, 2\}$, any MDP $M \in \mathcal{M}_i$, and any policy $\pi : \mathcal{S} \rightarrow \Delta(\mathcal{A})$, we have

$$J_M(\pi_M^*) - J_M(\pi) = Q_M^*(s, \pi_M^*(s)) - Q_M^*(s, \pi(s))$$

where the second equality follows because $s$ is the only state where different actions have distinct effects, and the last inequality follows from (3).

Now, consider any fixed offline reinforcement learning algorithm which takes the offline dataset $D_n$ as an input and returns a stochastic policy $\hat{\pi}_D_n : \mathcal{S} \rightarrow \Delta(\mathcal{A})$. For each $i \in \{1, 2\}$, we apply (7) to all MDPs in $\mathcal{M}_i$ and average to obtain

$$\frac{1}{|\mathcal{M}_i|} \sum_{M \in \mathcal{M}_i} \mathbb{E}_n^M [J_M(\pi_M^*) - J_M(\hat{\pi}_D_n)] \geq \frac{\gamma^2}{8(1 - \gamma)} \frac{1}{|\mathcal{M}_i|} \sum_{M \in \mathcal{M}_i} \mathbb{P}_n^M (\hat{\pi}_D_n(s) \neq i).$$

Applying the inequality above for $i = 1$ and $i = 2$ and combining the results, we have

$$\max_{M \in \mathcal{M}} \mathbb{E}_n^M [J_M(\pi_M^*) - J_M(\hat{\pi}_D_n)]$$

$$\geq \frac{1}{2|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{E}_n^M [J_M(\pi_M^*) - J_M(\hat{\pi}_D_n)] + \frac{1}{2|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{E}_n^M [J_M(\pi_M^*) - J_M(\hat{\pi}_D_n)]$$

$$\geq \frac{\gamma^2}{16(1 - \gamma)} \left\{ \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{P}_n^M (\hat{\pi}_D_n(s) \neq 1) + \frac{1}{|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{P}_n^M (\hat{\pi}_D_n(s) \neq 2) \right\}$$

$$\geq \frac{\gamma^2}{16(1 - \gamma)} \left( 1 - D_{\text{TV}} \left( \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{P}_n^M : \frac{1}{|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{P}_n^M \right) \right),$$

where the last inequality follows because $\mathbb{P}(E) + \mathbb{Q}(E^c) \geq 1 - D_{\text{TV}}(\mathbb{P}, \mathbb{Q})$ for any event $E$. \qed

D Proof of Lemma 2.2

This proof is organized as follows. In Appendix D.1, we introduce a reference measure and move from the total variation distance to the $\chi^2$-divergence. This allows us to reduce the task of upper bounding $D_{\text{TV}}(\mathbb{P}_n^1, \mathbb{P}_n^2)$ to the task of upper bounding two manageable density ratios (Eqs. (9) and (10) in the sequel). We develop several intermediate technical lemmas related to the density ratios in Appendix D.2, and in Appendix D.3 we put everything together to bound the density ratios, thus completing the proof of Lemma 2.2.

For the statement of Lemma 2.2 and the main subsections of this appendix (Appendices D.1 and D.3), we only consider the specific values for the parameters $\theta_1, \alpha_1, \beta_1; \theta_2, \alpha_2, \beta_2; w$ given in Section 2.1.3. However, in Appendix D.2, which contains intermediate technical lemmas, results are presented under a slightly more general setup, as explained at the beginning of the subsection.
D.1 Introducing a Reference Measure and Moving to $\chi^2$-Divergence

Directly calculating the total variation distance $D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2)$ is challenging, so we design an auxiliary reference measure $\mathbb{P}_n^0$ which serves as an intermediate quantity to help with the upper bound. The reference measure $\mathbb{P}_n^0$ lies in the same measurable space as $\mathbb{P}_n^1$ and $\mathbb{P}_n^2$, and is defined as follows:

$$\mathbb{P}_n^0\left(\{(s_i, a_i, r_i, s'_i)\}_{i=1}^{n}\right) := \prod_{i=1}^{n} \mu(s_i, a_i) \mathbb{I}(r_i = R_0(s_i, a_i)) P_0(s'_i \mid s_i, a_i), \quad \forall \{(s_i, a_i, r_i, s'_i)\}_{i=1}^{n},$$

where

$$R_0(s, a) := \begin{cases} 
0, & s \in \{s\} \cup S^1, \\
w = 3\gamma/8, & s = W, \\
1, & s = X, \\
0, & s = Y, \\
0, & s = Z,
\end{cases}$$

and

$$P_0(\cdot \mid s, 1) := W, \text{ w.p. 1},$$
$$P_0(\cdot \mid s, 2) := \text{Unif}(S^1),$$
$$\forall s \in S^1: \quad P_0(\cdot \mid s, a) := \begin{cases} 
X, & \text{w.p. } \theta_1\alpha_1, \\
Y, & \text{w.p. } 1 - \theta_1\alpha_1 - (1 - \theta_1)\beta_1, \\
Z, & \text{w.p. } (1 - \theta_1)\beta_1,
\end{cases}$$
$$\forall s \in \{W, X, Y, Z\}: \quad P_0(\cdot \mid s, a) := s, \text{ w.p. 1}.$$

The reference measure $\mathbb{P}_n^0$ can be understood as the law of $D_n$ when the data collection distribution is $\mu$ and the underlying MDP is $M_0 := (S, A, P_0, R_0, \gamma, d_0)$. Note that although we define the transition operator $P_0$ above based on the tuple $(\theta_1, \alpha_1, \beta_1)$, substituting in $(\theta_2, \alpha_2, \beta_2)$ leads to the same operator — this is guaranteed by an important feature of our construction: the families $M_1$ and $M_2$ in our construction satisfy the constraint (4), so that $\theta_1\alpha_1 = \theta_2\alpha_2$ and $(1 - \theta_1)\beta_1 = (1 - \theta_2)\beta_2$.

In what follows, we provide more explanations on the design of the transition operator $P_0$ and the reward function $R_0$.

Properties and Intuition of $P_0$. There are two ways to understand $P_0$. Operationally, $P_0$ is simply the pointwise average transition operator of the MDPs in $M_1$ or $M_2$, in the sense that

$$\forall s \in S, a \in A : \quad P_0(\cdot \mid s, a) = \frac{1}{|M_1|} \sum_{M \in M_1} P_M(\cdot \mid s, a) = \frac{1}{|M_2|} \sum_{M \in M_2} P_M(\cdot \mid s, a),$$

where $P_M$ is the transition operator associated with each MDP $M$. More conceptually, $P_0$ is the transition operator obtained by performing state aggregation using the value function class $\mathcal{F} = \{f_1, f_2\}$, where states with the same values for both $f_1$ and $f_2$ are viewed as identical and constrained to share dynamics (which is induced by averaging over the data collection distribution).

Properties and Intuition of $R_0$. Outside of state $Z$, the reward function $R_0$ is the same as the reward function of any MDP in $\mathcal{M}$, i.e.,

$$\forall s \neq Z, a \in A : \quad R_0(s, a) = R_M(s, a), \quad \forall M \in \mathcal{M},$$

where $R_M$ is the transition operator associated with each MDP $M$. The value of $R_0(Z, a)$ is immaterial, as the data collection distribution $\mu$ is not supported on $(Z, a)$ (in other words, different values of $R_0(Z, a)$ lead to essentially the same reference measure $\mathbb{P}_n^0$); we choose $R_0(Z, a) = 0$ for concreteness.
Moving to $\chi^2$-Divergence. Equipped with the definition of the reference measure $\mathbb{P}_n^0$, we proceed to bound $D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2)$. By the triangle inequality for the total variation distance, we have

$$D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2) \leq D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^0) + D_{TV}(\mathbb{P}_n^2, \mathbb{P}_n^0) \leq \frac{1}{2} \sqrt{D_{\chi^2}(P_n^1 || P_n^0)} + \frac{1}{2} \sqrt{D_{\chi^2}(P_n^2 || P_n^0)},$$

(8)

where the last inequality follows from the fact that $D_{TV}(P, Q) \leq \frac{1}{2} \sqrt{D_{\chi^2}(P \parallel Q)}$ for any $P, Q$ (see Proposition 7.2 or Section 7.6 of Polyanskiy (2020)).

In what follows, we derive simplified expressions for $D_{\chi^2}(P_n^1 || P_n^0)$ and $D_{\chi^2}(P_n^2 || P_n^0)$. We first expand and simplify $D_{\chi^2}(P_n^1 || P_n^0)$, then obtain a similar expression for $D_{\chi^2}(P_n^2 || P_n^0)$.

For each MDP $M \in \mathcal{M}$, let $P_M$ and $R_M$ denote the associated transition and reward functions. Observe that our construction for $P_M, R_M,$ and $\mu$ (see Section 2.1) ensures that for any $(s, a, r, s') \in S \times A \times [0, 1] \times S$ with $\mu(s, a) \mathbb{1}_{\{r = R_M(s, a)\}} P_M(s' \mid s, a) = 0$, we have $\mu(s, a) \mathbb{1}_{\{r = R_M(s, a)\}} P_M(s' \mid s, a) = 0$. As a result, we have $P_M^M \leq P_n^0$ for any $M \in \mathcal{M}$, which implies that $P_n^1, P_n^2 \leq P_n^0$. Hence, we can expand the $\chi^2$-divergence as

$$D_{\chi^2}(P_n^1 || P_n^0) = \frac{1}{|\mathcal{M}_1|^2} \sum_{M, M' \in \mathcal{M}_1} \mathbb{E}_{\{(s, a, r, s')\}_{i=1}^n \sim P_n^0}[\frac{1}{\prod_{i=1}^n \mu(s_i, a_i) \mathbb{1}_{\{r_i = R_M(s_i, a_i)\}} P_M(s_i' \mid s_i, a_i)}] - 1,$$

(9)

where the third equality follows because (i) $R_M(s, a) = R_0(s, a), \forall M \in \mathcal{M}, \forall a \in A, \forall s \neq Z$, and (ii) state $Z$ is not covered by $\mu$. Indeed, since the reward function for every MDP in $\mathcal{M}$ is the same as $R_0$ for all $(s, a)$ covered by $\mu$, the rewards $r_1, \ldots, r_n$ in $D_n$ are completely uninformative in our construction—they have the same distribution regardless of the underlying MDP. This is why the final expression for $D_{\chi^2}(P_n^1 || P_n^0)$ in (9) is completely independent of the reward distribution for both measures.

Using an identical calculation, we also have

$$D_{\chi^2}(P_n^2 || P_n^0) = \frac{1}{|\mathcal{M}_2|^2} \sum_{M, M' \in \mathcal{M}_2} \mathbb{E}_{\{(s, a, r, s')\}_{i=1}^n \sim P_n^0}[\frac{1}{\prod_{i=1}^n \mu(s_i, a_i) \mathbb{1}_{\{r_i = R_M(s_i, a_i)\}} P_M(s_i' \mid s_i, a_i)}] - 1.$$

(10)

Equipped with these expressions for the $\chi^2$-divergence, the next step in the proof of Lemma 2.2 is to upper bound the right-hand side for Eqs. (9) and (10). This is done in Appendix D.3, but before proceeding we require several intermediate technical lemmas.

### D.2 Technical Lemmas for Density Ratios

In this subsection, we state a number of technical lemmas which can be used to bound the density ratio appearing inside the square in Eqs. (9) and (10) for generic MDPs $M_{\alpha, \beta, w, I}$ with $I \in I_{\mathcal{M}}$. The lemmas hold
for any choice of \( (\theta, \alpha, \beta) \), and are independent of the reward parameter \( w \). For this general setup, we work with a variant of the reference operator \( P_0 \) defined based on the values \( (\theta, \alpha, \beta) \) via

\[
P_0(:, |s, 1) := W, \text{ w.p. } 1, \]

\[
P_0(:, |s, 2) := \text{Unif}(S^1),
\]

\[
\forall s \in S^1 : P_0(:, |s, a) := \begin{cases} X, \text{ w.p. } \theta \alpha, \\ Y, \text{ w.p. } 1 - \theta \alpha - (1 - \theta) \beta, \\ Z, \text{ w.p. } (1 - \theta) \beta, \end{cases}
\]

\[
\forall s \in \{W, X, Y, Z\} : P_0(:, |s, a) := s, \text{ w.p. } 1.
\]

In Appendix D.3, we instantiate the results from this subsection with \( (\theta_i, \alpha_i, \beta_i) \) for \( i \in \{1, 2\} \). Recall that per the discussion in Appendix D.1, our specific parameter choices for the families \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) induce the same reference operator \( P_0 \).

**Lemma D.1.** For all \( I, I' \in \mathcal{I}_\theta \), \( (2\theta - 1) + S_1 \leq |I \cap I'| \leq \theta S_1 \).

**Proof.** Since \( |I| = |I'| = \theta S_1 \), we have \( |I \cap I'| \leq |I| = \theta S_1 \) and

\[
|I \cap I'| = |I| + |I'| - |I \cup I'| \geq |I| + |I'| - S_1 = (2\theta - 1)S_1.
\]

Since \( |I \cap I'| \geq 0 \) trivially, the result follows. \( \square \)

The next lemma controls the density ratio for states in \( S^1 \). To state the result compactly, we define

\[
\phi_{\theta, \alpha, \beta} := \theta^2 \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} \right).
\]  

(11)

Since Lemma D.2 is stated for any given \( \theta, \alpha, \beta \), we use \( P_I \) to denote \( P_{\theta, \alpha, \beta} \) to keep notation compact.

**Lemma D.2.** For all \( I, I' \in \mathcal{I}_\theta \), we have

\[
\mathbb{E}_{s \sim \text{Unif}(S^1), s' \sim P_0(:, |s, a)} \left[ \frac{P_I(s' | s, a)P_I(s' | s, a)}{P_0^2(s' | s, a)} \right] = 1 + \phi_{\theta, \alpha, \beta} \cdot \left( \frac{|I \cap I'|}{\theta^2 S_1} - 1 \right).
\]

**Proof.** For any \( I, I' \in \mathcal{I}_\theta \), we observe that

\[
\mathbb{E}_{s \sim \text{Unif}(S^1), s' \sim P_0(:, |s, a)} \left[ \frac{P_I(s' | s, a)P_I(s' | s, a)}{P_0^2(s' | s, a)} \right] = \mathbb{E}_{s \sim \text{Unif}(S^1)} \left[ \sum_{s' \in \{X, Y, Z\}} \frac{P_I(s' | s, a)P_I(s' | s, a)}{P_0(s' | s, a)} \right].
\]

To proceed, we calculate the value of the ratio \( \frac{P_I(s' | s, a)P_I(s' | s, a)}{P_0(s' | s, a)} \) for each possible choice for \( s \in S^1 \) and \( s' \in \{X, Y, Z\} \) in Table 1 below.

| \( s \in I \cap I' \) | \( s' = X \) | \( s' = Y \) | \( s' = Z \) |
| --- | --- | --- | --- |
| \( \alpha / \theta \) | \( (1 - \alpha) \beta / (\theta(1 - \alpha) + (1 - \theta)(1 - \beta)) \) | 0 | 0 |
| \( s \in (I \cup I') \setminus (I \cap I') \) | 0 | \( (1 - \alpha)(1 - \beta) / (\theta(1 - \alpha) + (1 - \theta)(1 - \beta)) \) | 0 |
| \( s \notin (I \cup I') \) | 0 | \( (1 - \beta)^2 / (\theta(1 - \alpha) + (1 - \theta)(1 - \beta)) \) | \( \beta / (1 - \theta) \) |

Table 1: Value of \( \frac{P_I(s' | s, a)P_I(s' | s, a)}{P_0(s' | s, a)} \) for all possible pairs \( (s, s') \).

Define \( t := |I \cap I'| \). From Lemma D.1, we must have \( t \in [(2\theta - 1)S_1, \theta S_1] \). We also have \( |I \cup I'| = |I| + |I'| - |I \cap I'| = 2\theta S_1 - t \). Hence, the event in the first row of Table 1 occurs with probability \( |I \cap I'| / S_1 = t / S_1 \), the event in the second row occurs with probability \( |(I \cup I') \setminus (I \cap I')| / S_1 = (2\theta S_1 - 2t) / S_1 \), and the event in the third row occurs with probability \( (2\theta S_1 - 2t) / S_1 \).
and the event in the third row occurs with probability $|S_1 \setminus (I \cup I')|/S_1 = ((1 - 2\theta)S_1 + t)/S_1$. Using these values, we obtain
\[
\mathbb{E}_{s \sim \text{Unif}(S^1)} \left[ \sum_{s' \in \{X, Y, Z\}} \frac{P_I(s' \mid s, a) P_{I'}(s' \mid s, a)}{P_0(s' \mid s, a)} \right] 
\]
\[
= \frac{t}{S_1} \cdot \left( \frac{\alpha}{\theta} + \frac{(1 - \alpha)^2}{\theta(1 - \alpha) + (1 - \theta)(1 - \beta)} \right) + \left( \frac{2\theta - 2t}{S_1} \right) \cdot \frac{(1 - \alpha)(1 - \beta)}{\theta(1 - \alpha) + (1 - \theta)(1 - \beta)} 
+ \left( 1 - 2\theta + \frac{t}{S_1} \right) \cdot \left( \frac{(1 - \beta)^2}{\theta(1 - \alpha) + (1 - \theta)(1 - \beta)} + \frac{\beta}{1 - \theta} \right) 
= \frac{t}{S_1} \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\alpha}{\theta} + \frac{\beta}{1 - \theta} \right) 
+ \left( \frac{\theta(\beta - \alpha) + 1 - \beta}{1 - \theta} \right) \cdot \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\alpha}{\theta} + \frac{\beta}{1 - \theta} \right) 
+ \frac{2\theta(\beta - \alpha)(1 - \beta) + (1 - \beta)^2}{\theta(\beta - \alpha) + 1 - \beta} + \theta - \frac{\theta(\beta - \alpha) + 1 - \beta}{1 - \theta} 
= \left( \frac{t}{S_1} - \theta^2 \right) \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\alpha}{\theta} + \frac{\beta}{1 - \theta} \right) 
+ \theta^2 \cdot \frac{\theta}{\theta(\beta - \alpha) + 1 - \beta} + \theta^2 \cdot \frac{\alpha}{\theta} + \frac{\beta}{1 - \theta} + \theta - \frac{\theta(\beta - \alpha) + 1 - \beta}{1 - \theta} 
= \frac{2\theta(\beta - \alpha)(1 - \beta) + (1 - \beta)^2}{\theta(\beta - \alpha) + 1 - \beta} + \theta - \frac{\theta(\beta - \alpha) + 1 - \beta}{1 - \theta} + 1.
\]
Grouping the terms in the second line together, we find that (i) $= \theta(\beta - \alpha) + 1 - \beta$, (ii) $= \theta\alpha + \beta$, and (iii) $= -\theta\beta$, and by summing, \[(i) + (ii) + (iii) = 1.\]
Hence, the above expression is equal to
\[
\left( \frac{t}{S_1} - \theta^2 \right) \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\alpha}{\theta} + \frac{\beta}{1 - \theta} \right) + 1 
= \left( \frac{t}{S_1} - \theta^2 \right) \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} \right) + 1.
\]
Recalling the definition of $\phi_{\theta, \alpha, \beta}$, this completes the proof. \[
\]
The next lemma bounds the magnitude of $\phi_{\theta, \alpha, \beta}$ in terms of the parameter $\theta$.

**Lemma D.3.** For any $\alpha, \beta, \theta \in (0, 1)$, we have
\[
\theta^2 |\alpha - \beta| \leq \phi_{\theta, \alpha, \beta} \leq \frac{\theta}{1 - \theta} \max \{\alpha, \beta\} \leq \frac{\theta}{1 - \theta}.
\]
**Proof.** Recall that $\phi_{\theta, \alpha, \beta} = \theta^2 \left( \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} \right)$. We consider two cases.

**Case 1:** $\alpha \leq \beta$. Assume $\alpha < \beta$, as the result is immediate if $\alpha = \beta$. We have
\[
\frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} \geq 0 + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} \geq \frac{\theta(\beta - \alpha)}{\theta(1 - \theta)} = \frac{\beta - \alpha}{1 - \theta} > |\alpha - \beta|
\]
and
\[
\frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} = \frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\beta - \alpha}{1 - \theta} + \frac{\alpha}{\theta(1 - \theta)} 
\leq \frac{\beta - \alpha}{1 - \theta} + \frac{\beta - \alpha}{1 - \theta} + \frac{\alpha}{\theta(1 - \theta)} 
= \frac{\beta}{\theta(1 - \theta)},
\]

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where the inequality above follows since \( \theta(\beta - \alpha) + 1 - \beta > \theta(\beta - \alpha) > 0 \).

**Case 2: \( \alpha > \beta \).** We have

\[
\frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} \geq 0 + \frac{\theta(\beta - \alpha) + \alpha - \beta}{\theta(1 - \theta)} = \frac{\alpha - \beta}{\theta} > |\alpha - \beta|
\]

and

\[
\frac{(\beta - \alpha)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\theta(\beta - \alpha) + \alpha}{\theta(1 - \theta)} = \frac{(\alpha - \beta)^2}{1 - \theta(\alpha - \beta)} + \frac{\beta - \alpha}{1 - \theta} + \frac{\alpha}{\theta(1 - \theta)} \\
\leq \frac{(\alpha - \beta)^2}{\theta(\beta - \alpha) + 1 - \beta} + \frac{\beta - \alpha}{1 - \theta} + \frac{\alpha}{\theta(1 - \theta)} \\
= \frac{\alpha}{\theta(1 - \theta)},
\]

where the inequality uses that \( 1 - \beta > \alpha - \beta > \theta(\alpha - \beta) \).

The lemma immediately follows. \( \square \)

The final lemma in this section controls the density ratio for the initial state \( s \) when \( \alpha = 0 \). Again, we use \( P_t \) to denote \( P_{\alpha, \beta, t} \) to keep notation compact.

**Lemma D.4.** For any \( I, I' \in \mathcal{I}_\theta \), we have

\[
\mathbb{E}_{s' \sim P_0(\cdot \mid s, 2)} \left[ \frac{P_t(s' \mid s, 2)P_{t'}(s' \mid s, 2)}{P_0^2(s' \mid s, 2)} \right] = \frac{|I \cap I'|}{\theta^2 S_1}.
\]

**Proof.** Let \( I, I' \in \mathcal{I}_\theta \) be given and observe that

\[
\mathbb{E}_{s' \sim P_0(\cdot \mid s, 2)} \left[ \frac{P_t(s' \mid s, 2)P_{t'}(s' \mid s, 2)}{P_0^2(s' \mid s, 2)} \right] = \mathbb{E}_{s' \sim \text{Unif}(S')} \left[ \frac{1_{\{s' \in I \cap I'\}}}{\theta^2} \right] = \frac{|I \cap I'|}{\theta^2 S_1}.
\]

\( \square \)

### D.3 Completing the Proof

To keep notation compact, define

\[
g_{\theta, \alpha, \beta}(t; n) := \left( \left( \frac{t}{\theta^2 S_1} - 1 \right) \frac{8\phi_{\theta, \alpha, \beta} + 1}{16} + 1 \right)^n.
\]

For all \( M \in \mathcal{M} \), \( P_M(\cdot \mid s, a) \) and \( P_0(\cdot \mid s, a) \) differ only when \( (s, a) = (s, 2) \) or \( (s, a) \in S_1 \times A \), so—recalling the value of \( \mu \)—we have

\[
\frac{1}{|\mathcal{M}|^2} \sum_{M, M' \in \mathcal{M}} \left( \mathbb{E}_{(s,a) \sim \mu, s' \sim P_0(\cdot \mid s, a)} \left( \frac{P_M(s' \mid s, a)P_{M'}(s' \mid s, a)}{P_0^2(s' \mid s, a)} \right) \right)^n
\]

\[
= \frac{1}{|\mathcal{M}|^2} \sum_{M, M' \in \mathcal{M}} \left( \frac{1}{2} \mathbb{E}_{s' \sim \text{Unif}(S')} \left[ \frac{P_M(s' \mid s, a)P_{M'}(s' \mid s, a)}{P_0^2(s' \mid s, a)} \right] + \frac{1}{16} \mathbb{E}_{s' \sim P_0(\cdot \mid s, 2)} \left[ \frac{P_M(s' \mid s, 2)P_{M'}(s' \mid s, 2)}{P_0^2(s' \mid s, 2)} \right] + \frac{7}{16} \right)^n
\]

\[
= \frac{1}{|S_1|} \sum_{t} \sum_{I, I' \in \mathcal{I}_\theta \mid |I \cap I'| = t} \left( \frac{t}{\theta^2 S_1} - 1 \right) \phi_{\theta_1, \alpha_1, \beta_1} + 1 \right)^n + \frac{1}{16} \frac{t}{\theta^2 S_1} + \frac{7}{16} \right)^n,
\]

26
where we have used the expressions for the density ratio from Lemmas D.2 and D.4. We further simplify to

\[
\frac{1}{(\theta_1 S_1)^2} \sum_{t} t \left( \sum_{I, I' \in I_{s,1}: |I \cap I'| = t} \left( \frac{t}{\theta_1^2 S_1} - 1 \right) \frac{8 \phi_{\theta_1, \alpha_1, \beta_1} + 1}{16} + 1 \right)^n
\]

as

\[
\frac{\theta_2 S_1}{t} \left( \frac{\theta_1}{\theta_1 S_1 - t} \right) \left( \frac{S_1 - \theta_2 S_1}{\theta_2 S_1} \right) g_{\theta_2, \alpha_2, \beta_2}(t; n),
\]

where the second equality uses Lemma D.1. Applying the same calculation for \(M_2\), we also have that

\[
\frac{1}{|M_2|^2} \sum_{M, M' \in M_2} \left( \mathbb{E}_{(s, a) \sim \mu, s' \sim \nu} \left[ \frac{P_M(s' | s, a) P_{M'}(s' | s, a)}{P_0^2(s' | s, a)} \right] \right)^n
\]

Therefore, to upper bound the right-hand sides of Eqs. (9) and (10), we only need to upper bound the quantity

\[
\sum_{t=(2\theta-1)+S_1}^{\theta S_1} \frac{g_{\theta_2} S_1}{t} \left( \frac{g_{\theta_1} S_1 - t S_1}{\theta S_1} \right) g_{\theta, \alpha, \beta}(t; n),
\]

for both \((\theta, \alpha, \beta) = (\theta_1, \alpha_1, \beta_1)\) and \((\theta, \alpha, \beta) = (\theta_2, \alpha_2, \beta_2)\). To upper bound this quantity, we use the following two lemmas.

**Lemma D.5** (Monotonicity of \(g_{\theta, \alpha, \beta}\)). For any \(\theta, \alpha, \beta \in (0, 1)\) and any \(n \in \mathbb{N}\), the function \(t \mapsto g_{\theta, \alpha, \beta}(t; n)\) is non-decreasing for \(t \in [2(\theta - 1) + S_1, \theta S_1]\).

**Proof.** By Lemma D.2, we have \(\left( \frac{t}{\theta^2 S_1} - 1 \right) \phi_{\theta, \alpha, \beta} + 1 \geq 0\) for all \(t \in [(2\theta - 1) + S_1, \theta S_1]\), and hence

\[
\left( \frac{t}{\theta^2 S_1} - 1 \right) \frac{8 \phi_{\theta, \alpha, \beta} + 1}{16} + 1 = \frac{1}{2} \left( \frac{t}{\theta^2 S_1} - 1 \right) \phi_{\theta, \alpha, \beta} + 1 + \frac{1}{16} \frac{t}{\theta^2 S_1} + \frac{7}{16} \geq 0
\]

for all \(t \in [(2\theta - 1) + S_1, \theta S_1]\). This ensures that we are in the domain where \(x \mapsto x^n\) is non-decreasing. Next, by Lemma D.3, we know that \(\phi_{\theta, \alpha, \beta} \geq 0\), so the coefficient on \(t\) is non-negative. It follows that \(g_{\theta, \alpha, \beta}(t; n)\) is non-decreasing in \(t \in [(2\theta - 1) + S_1, \theta S_1]\).

**Lemma D.6** (Hypergeometric tail bound). For any \(\theta \in \{\theta_1, \theta_2\}\) and \(\epsilon \in (0, \theta^2 S_1)\), we have

\[
\sum_{t \geq (\theta + \epsilon) S_1} \frac{g_{\theta} S_1}{t} \left( \frac{S_1 - t S_1}{\theta S_1} \right) \leq \exp(-2\epsilon^2 \theta S_1).
\]

**Proof.** Let \(\text{Hyper}(t; K, N, N') := \binom{K}{t} \frac{C_{N-K}}{C_{N'}}\) denote the hypergeometric probability mass function, which corresponds to the probability that exactly \(t\) balls are blue when \(N'\) balls are sampled without replacement from a jar containing \(N\) total balls, \(K\) of which are blue (see, e.g., Chapter 2.1.4 of Rice (2006) for background). We observe that the term \(\frac{g_{\theta_1} S_1}{t} \left( \frac{S_1 - t S_1}{\theta S_1} \right) \) arising in Eqs. (12) and (13) is precisely \(\text{Hyper}(t; \theta S_1, S_1, \theta S_1)\), which corresponds to the process in which we sample \(\theta S_1\) balls without replacement from a jar with \(S_1\) balls, \(\theta S_1\) of which are blue.

We now apply a classical tail bound for hypergeometric random variables.
Lemma D.7 (Hoeffding (1963)). Let $X \sim \text{Hyper}(K, N, N')$ and define $p = K/N$. Then for any $0 < \epsilon < pN'$, we have

$$
\Pr[X \geq (p + \epsilon)N'] \leq \exp(-2\epsilon^2 N').
$$

Instantiating this bound with $\text{Hyper}(\theta S_1, S_1, \theta S_1)$ (since $\theta S_1$ is an integer), we have $p = \theta$ and

$$
\sum_{t \geq (\theta + \epsilon) \cdot \theta S_1} \frac{\binom{\theta S_1}{t} \binom{S_1 - \theta S_1}{\theta S_1 - t}}{\binom{S_1}{\theta S_1}} g_{\theta, \alpha, \beta}(t; n) = \Pr[X \geq (\theta + \epsilon) \cdot \theta S_1] \leq \exp(-2\epsilon^2 \theta S_1).
$$

Returning to the quantity in (12), for any $(\theta, \alpha, \beta) \in \{(\theta_1, \alpha_1, \beta_1), (\theta_2, \alpha_2, \beta_2)\}$ and any $\epsilon \in (0, \theta^2 S_1)$ we can split the sum and upper bound as follows:

$$
\leq \sum_{t \geq (\theta + \epsilon) \cdot \theta S_1} \frac{\binom{\theta S_1}{t} \binom{S_1 - \theta S_1}{\theta S_1 - t}}{\binom{S_1}{\theta S_1}} g_{\theta, \alpha, \beta}(t; n) + \exp(-2\epsilon^2 \theta S_1) \cdot g_{\theta, \alpha, \beta}(\theta S_1; n)
$$

$$
\leq \sum_{t \geq (\theta + \epsilon) \cdot \theta S_1} \frac{\binom{\theta S_1}{t} \binom{S_1 - \theta S_1}{\theta S_1 - t}}{\binom{S_1}{\theta S_1}} g_{\theta, \alpha, \beta}(\theta \cdot \theta S_1; n) + \exp(-2\epsilon^2 \theta S_1) \cdot g_{\theta, \alpha, \beta}(\theta S_1; n)
$$

$$
\leq g_{\theta, \alpha, \beta}(\theta \cdot \theta S_1; n) + \exp(-2\epsilon^2 \theta S_1) \cdot g_{\theta, \alpha, \beta}(\theta S_1; n), \quad (14)
$$

where the first two inequalities follow from Lemmas D.5 and D.6 and the last uses that the sum in the penultimate line is at most 1. We further calculate

$$
g_{\theta, \alpha, \beta}(\theta \cdot \theta S_1; n) = \left(\left(\frac{(\theta + \epsilon) \cdot \theta S_1}{\theta^2 S_1} - 1\right) \frac{8\phi_{\theta, \alpha, \beta} + 1}{16} + 1\right)^n
$$

$$
= \left(\frac{\epsilon \cdot 8\phi_{\theta, \alpha, \beta} + 1}{\theta} + 1\right)^n
$$

$$
\leq \left(\frac{\epsilon}{2(1 - \theta)} + 1\right)^n, \quad (15)
$$

where the inequality follows from Lemma D.3. Similarly, we have

$$
\exp(-2\epsilon^2 \theta S_1) \cdot g_{\theta, \alpha, \beta}(\theta S_1; n) = \exp(-2\epsilon^2 \theta S_1) \cdot \left(\left(\frac{\theta S_1}{\theta^2 S_1} - 1\right) \frac{8\phi_{\theta, \alpha, \beta} + 1}{16} + 1\right)^n
$$

$$
\leq \exp(-2\epsilon^2 \theta S_1) \cdot \left(\left(\frac{1}{\theta} - 1\right) \frac{8\theta/(1 - \theta) + 1}{16} + 1\right)^n
$$

$$
\leq \exp(-2\epsilon^2 \theta S_1) \cdot \left(\frac{1}{2\theta} + 1\right)^n
$$

$$
= \exp(n \ln(1 + 1/(2\theta)) - 2\epsilon^2 \theta S_1)
$$

$$
\leq \exp(n/(2\theta) - 2\epsilon^2 \theta S_1), \quad (16)
$$

where the first inequality follows from Lemma D.3 and the last inequality uses that $\log(1 + x) \leq x$.

Combining Eqs. (9), (10), (12) and (14) to (16) and instantiating the bounds for $(\theta_1, \alpha_1, \beta_1)$ and $(\theta_2, \alpha_2, \beta_2)$, we have

$$
D_{\chi^2}(\mathbb{P}_n^1 \parallel \mathbb{P}_n^0) \leq \inf_{\epsilon \in (0, \theta_1^2 S_1)} \left\{ \left(\frac{\epsilon}{2(1 - \theta_1)} + 1\right)^n + \exp(n/(2\theta_1) - 2\epsilon^2 \theta_1 S_1) \right\} - 1.
$$
Let \( c \in (0, 1/2) \) be an arbitrary constant. For each \( i \in \{1, 2\} \), we set \( \epsilon = 2c \cdot \frac{(1 - \theta_i \theta)}{n} \) (which belongs to \((0, \theta^2 S_1)\) because \( \epsilon < \theta_i \) since \( n \geq 1 \) and \( \theta S_1 \geq 1 \) by assumption). Then we have
\[
\left(\frac{\epsilon}{2(1 - \theta_i \theta)} + 1\right)^n \leq \left(1 + \frac{c}{n}\right)^n \leq c^n \leq 1 + 2c, \quad \forall i \in \{1, 2\},
\]
and
\[
D_{\chi^2}(\mathbb{P}_n^i \| \mathbb{P}_n^0) \leq 2c + \exp\left(\frac{n}{2\theta_i} - 8c^2 \theta_i \frac{(1 - \theta_i)^2 \theta_i^2}{n^2} S_1\right), \quad \forall i \in \{1, 2\}.
\]
In particular, whenever \( S_1 \geq \max_{i \in \{1, 2\}} \frac{n^3}{8c^2 \theta_i^2 (1 - \theta_i)^2} \), we have
\[
D_{\chi^2}(\mathbb{P}_n^i \| \mathbb{P}_n^0) \leq 2c + \exp(-n/(2\theta_i)), \quad \forall i \in \{1, 2\}.
\]
Plugging in the values \( \theta_1 = 1/2, \theta_2 = 1/4 \) and setting \( c = 1/10 \), we have that whenever \( n \geq 5 \) and \( S_1 > 6400n^3 \),
\[
D_{\chi^2}(\mathbb{P}_n^i \| \mathbb{P}_n^0) \leq \frac{1}{5} + \exp(-n) \leq \frac{1}{4}, \quad \forall i \in \{1, 2\}.
\]
Combining this with (8), we have that \( D_{\text{TV}}(\mathbb{P}_n^1, \mathbb{P}_n^2) \leq \sqrt{1/4} = 1/2 \), which proves the lemma. \( \square \)

## Part II

### Proofs for Theorem 1.2

#### E  Theorem 1.2: Lower Bound Construction and Proof

We restate Theorem 1.2 below for convenience.

**Theorem 1.2** (Lower bound for admissible data). For any \( S \geq 9, \gamma \in (1/2, 1) \), and \( C \geq 64 \), there exists a family of MDPs \( \mathcal{M} \) with \( |S| = S \) and \( |A| = 2 \), a value function class \( \mathcal{F} \) with \( |\mathcal{F}| = 2 \), and a data distribution \( \mu \) which is a mixture of admissible distributions, such that:

1. We have \( Q^\pi \in \mathcal{F} \) for all \( \pi : S \to \Delta(A) \) (all-policy realizability) and \( C_{\text{conc}} \leq C \) (concentrability) for all models in \( \mathcal{M} \).

2. Any algorithm using less than \( c \cdot \min\{S^{1/3}/(\log S)^2, 2\gamma, 2/(1 - \gamma)\} \) samples must have \( J(\pi^*) - \mathbb{E}[J(\hat{\pi})] \geq c' \) for some instance in \( \mathcal{M} \), where \( c \) and \( c' \) are absolute numerical constants.

#### E.1 Lower Bound Construction

We begin by specifying the structure of the MDPs in the family \( \mathcal{M} \) used to prove Theorem 1.2. Let \( \gamma \in (0, 1) \) be fixed, and let \( S \in \mathbb{N} \) be given. Let \( L \in \mathbb{N} \) be an integer parameter whose value will be chosen at the end of the proof (Appendix E.4). Define \( L_{\text{div}} := \sum_{i=1}^{L} (2L + 1 - i)(L + 2 - i) \leq 4L^3 \), and assume without loss of generality that \( S > 5 \) and that \((S - 5)/L_{\text{div}}\) is an integer.\(^{12}\) We consider a parameterized class of

\[\text{states and } \mathcal{F} \ni \{(S - 5)/L_{\text{div}}\} L_{\text{div}} + 5 \text{ states and then add } S - 5 \text{ arbitrary states that are not reachable by any policy. Since we are considering the case where } \mu \text{ is admissible, those non-reachable states do not affect the sample complexity of any algorithm (as they do not affect } D_n \text{ at all). It is easy to show that the conclusion of Theorem 1.2 still holds.} \]
MDPs illustrated in Figure 2. Each MDP takes the form $M_{L,\alpha,w,I} = \{S, A, P_{L,\alpha,I}, R_{L,\alpha,w}, \gamma, d_0\}$, and is parametrized by the integer $L \in \mathbb{N}$, a vector of subsets $I = (I^1, \ldots, I^L)$ where $I^l \subseteq S$, and scalars $\alpha \in (0, 1/L)$ and $w \in [0, 1]$. All MDPs in the family $\{M_{L,\alpha,w,I}\}$ share the same state space $S$, action space $A$, discount factor $\gamma$, and initial state distribution $d_0$, and differ only in terms of the transition function $P_{L,\alpha,I}$ and the reward function $R_{L,\alpha,w}$.

**State space.** We consider a layered\(^1\) state space $S = \{s\} \cup S^1 \cup \cdots \cup S^L \cup \{W, X, Y, Z\}$, where $s$ is the initial state, $S^1, \ldots, S^L$ are $L$ layers of intermediate (i.e., neither initial nor terminal) states, and $\{W, X, Y, Z\}$ are self-looping terminal states. The number of intermediate states in layer $l \in [L]$ is $S_l := \frac{\alpha^l - 1}{\alpha - 1} (2L+1-l)(L+2-l)$, which ensures that $|S| = \sum_{l=1}^L S_l + 5 = S$.\(^2\)

**Action space.** Our action space is given by $A = \{1, 2\}$. For the initial state $s$, the two actions have distinct effects, while for all other states in $S \setminus \{s\}$ both actions have identical effects. As a result, the value of a given policy only depends on the action it selects in $s$. As in the proof of Theorem 1.1, we use the symbol $a$ as a placeholder to denote either action when taken in $s \in S \setminus \{s\}$, since the choice is immaterial.

**Transition operator.** For each MDP $M_{L,\alpha,w,I}$, recalling $I = (I^1, \ldots, I^L)$, we let $I^l \subseteq S^l$ parameterize a subset of the $l$th-layer intermediate states. We call each $s \in I^l$ an $l$th-layer planted state and call $s \in I^l \setminus \{s\}$ an $l$th-layer unplanted state. The dynamics $P_{L,\alpha,I}$ for $M_{L,\alpha,w,I}$ are determined by $L$, $\alpha \in (0, 1/L)$, and $I$ as follows (cf. Figure 2):

- **Initial state $s$.** For the dynamics from the initial state $s$, we define $$P_{L,\alpha,I}(s, 1) = \text{Unif}([W]),$$ and $$P_{L,\alpha,I}(s, 2) = \frac{1}{2} \left( \sum_{l=1}^L \left( \frac{1}{2^l} \text{Unif}(S^l) \right) + \frac{1}{2^L} \text{Unif}([Z]) \right) + \frac{1}{2} \cdot \text{Unif}([X, Y]).$$

That is, from the initial state $s$, choosing action 1 always leads to state $W$ in the next time step (see the red arrow in Figure 2), while choosing 2 leads to all states in $S^1 \cup \cdots \cup S^L \cup \{X, Y, Z\}$ (i.e., $S \setminus \{s, W\}$) with certain probability (see the blue arrow in Figure 2, but note that transitions from $s$ to $\{X, Y\}$ are not displayed).

- **Intermediate states.** Transitions from states in $S^1, \ldots, S^L$ are defined as follows.

  - For each $l$th-layer planted state $s \in I^l \subseteq S^l$, define $$P_{L,\alpha,I}(s, a) = \frac{\gamma^{L-l} \alpha}{1 - (l-1)\alpha} \text{Unif}([X]) + \left( 1 - \frac{\gamma^{L-l} \alpha}{1 - (l-1)\alpha} \right) \text{Unif}([Y]).$$

  - For each $l$th-layer unplanted states $s \in I^l \subseteq S^l$, define $$P_{L,\alpha,I}(s, a) = \frac{1 - l \cdot \alpha}{1 - (l-1)\alpha} \text{Unif}(I^{l+1}) + \frac{\alpha}{1 - (l-1)\alpha} \text{Unif}([Y]),$$

with the convention that $I^{L+1} := \{Z\}$.

Since we restrict to $\alpha \leq 1/L$, one can verify that these are valid probability distributions.

- **Terminal states.** All states in $\{W, X, Y, Z\}$ self-loop indefinitely. That is $P_{L,\alpha,I}(s, a) = \text{Unif}([s])$ for all $s \in \{W, X, Y, Z\}$.

\(^1\)Importantly, one should distinguish the concept of “layer” (which we use to simply refer to a group of states) and the concept of “time step” (which indexes the sequential evolution of the MDP). A state in layer $I \in [L]$ may be reached in any time step. For example, in Figure 2, states in $I^3$ (which belongs to layer 3) can be reached in both time step 1 (through the blue arrow) and time step 2 (from $I^2$), but cannot be reached in time step 3.

\(^2\)The precise value of $S_l$ given here is not essential to our proof. Its primarily serves to avoid a rounding issue that arises in Appendix E.2, which can also be addressed through other methods.
**Reward function.** The initial and intermediate states have no reward, i.e., $R_{L,\alpha,w}(s,a) = 0, \forall s \in \{s\} \cup S^1 \ldots \cup S^L, \forall a \in A$. Each of the self-looping terminal states in $\{W, X, Y, Z\}$ has a fixed reward determined by the parameters $L$, $\alpha$ and $w$. In particular, we define $R_{L,\alpha,w}(W,a) = w, R_{L,\alpha,w}(X,a) = 1, R_{L,\alpha,w}(Y,a) = 0,$ and $R_{L,\alpha,w}(Z,a) = \alpha/(1 - L\alpha)$.

**Initial state distribution.** All MDPs in $\{M_{L,\alpha,w,I}\}$ start at $s$ deterministically (that is, the initial state distribution $d_0$ places all its probability mass on $s$). Since $d_0$ does not vary between instances, it should be thought of as known to the learning algorithm.

### E.2 Specifying the MDP Family $\mathcal{M}$

We leave $L \in \mathbb{N}$ (we interpret $\mathbb{N}$ to not include 0) as a free parameter until the end of Appendix E, where we will give a concrete $L$ that leads to Theorem 1.2. Given $L \in \mathbb{N}$, let $\alpha_1 := \frac{1}{2L}$ and $\alpha_2 := \frac{1}{L+1}$. For $\alpha \in (0, 1)$, define

$$V_\alpha := \sum_{l=1}^{L} \frac{\alpha^{l-1} \alpha}{2^{l+1} 1 - (l-1)\alpha} + \frac{1}{2L+1} \frac{\alpha}{1 - L\alpha} + \frac{1}{2},$$

which has $0 < V_{\alpha_1} < V_{\alpha_2} < 1$, and let $w := \frac{V_{\alpha_1} + V_{\alpha_2}}{2}$. Define $I_\theta := \{I : |I| = \theta_l S_l\}$ for any $\theta = (\theta_1, \ldots, \theta_L) \in (0, 1)^L$ such that $\theta_l S_l$ is an integer for all $l \in [L]$. We define two sub-families of MDPs via

$$\mathcal{M}_1 := \bigcup_{I \in I_{\theta(1)}} \{M_{L,\alpha_1,w,I}\}, \quad \text{and} \quad \mathcal{M}_2 := \bigcup_{I \in I_{\theta(2)}} \{M_{L,\alpha_2,w,I}\},$$

where $\mathcal{M}_1$ is specified by $\alpha_1$ and $\theta^{(1)} = (\theta_1^{(1)}, \ldots, \theta_L^{(1)})$ with

$$\theta_l^{(1)} := \frac{\alpha_2}{1 - (l-1)\alpha_2}, \quad \forall l \in [L],$$

and $\mathcal{M}_2$ is specified by $\alpha_2$ and $\theta^{(2)} = (\theta_1^{(2)}, \ldots, \theta_L^{(2)})$ with

$$\theta_l^{(2)} := \frac{\alpha_1}{1 - (l-1)\alpha_1}, \quad \forall l \in [L].$$

Finally, we define the hard family $\mathcal{M}$ via

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2.$$
E.3 Finishing the Construction: Value Functions and Data Distribution

Value function class. Define functions $f_1, f_2 : S \times A \to \mathbb{R}$ as follows, recalling that $w := \frac{V_{\alpha_1} + V_{\alpha_2}}{2}$ (differences are highlighted in blue):

$$f_1(s, a) := \frac{1}{1 - \gamma} \begin{cases} 
\gamma w, & s = s, a = 1 \\
\gamma V_{\alpha_1}, & s = s, a = 2 \\
\gamma \frac{L_{-(l-1)\alpha_1}}{1-(l-1)\alpha_1}, & s \in S^l, l \in [L] \\
w, & s = W \\
1, & s = X \\
0, & s = Y \\
\frac{\alpha_{l-1}}{1-(l-1)\alpha_1}, & s = Z 
\end{cases}$$

$$f_2(s, a) := \frac{1}{1 - \gamma} \begin{cases} 
\gamma w, & s = s, a = 1 \\
\gamma V_{\alpha_2}, & s = s, a = 2 \\
\gamma \frac{L_{-(l-1)\alpha_2}}{1-(l-1)\alpha_2}, & s \in S^l, l \in [L] \\
w, & s = W \\
1, & s = X \\
0, & s = Y \\
\frac{\alpha_{l-1}}{1-(l-1)\alpha_2}, & s = Z 
\end{cases}$$

The following result is an elementary calculation. See Appendix H for a detailed calculation.

**Proposition E.1.** For all $\pi : S \to \Delta(A)$, we have $Q_M^{\pi} = f_1$ for all $M \in M_1$ and $Q_M^{\pi} = f_2$ for all $M \in M_2$.

It follows that by choosing $F = \{f_1, f_2\}$, all-policy realizability holds for all $M \in M$.

Data distribution. Recall that in the offline RL setting, the learner is provided with an i.i.d. dataset $D_n = \{(s_i, a_i, r_i, s'_i)\}_{i=1}^n$ where $(s_i, a_i) \sim \mu$, $s'_i \sim P(\cdot \mid s_i, a_i)$, and $r_i = R(s_i, a_i)$. To ensure admissibility of $\mu$, we consider the exploratory policy $\pi_0$ given by

$$\pi_0(s) = \text{Unif}(A), \ \forall s \in S.$$ 

We define the data collection distribution $\mu$ via:

$$\mu(s, a) := \frac{1}{2} d_0^n + \frac{1}{2} d_1^n = \frac{1}{2} d_0^n + \frac{1}{2} d_1^n + \frac{1}{2} d_2^n(s, a),$$

which, by construction, is a mixture of admissible distributions as desired. As a reminder, we use the notation $d_{\pi}^n \in \Delta(S \times A)$ to denote the occupancy measure of $\pi$ at time step $h$, that is $d_{\pi}^n(s, a) := \mathbb{P}_{\pi}(s_h = s, a_h = a)$, where the dependence on the MDP $M$ is suppressed.

In general, this choice of $\mu$ will depend on the underlying MDP $M \in M$ through $d_1^n$. However, for our specific construction, we calculate that

$$\mu(\cdot, a) = \frac{1}{2} d_0 + \frac{1}{2} \left( \frac{1}{2} P_{L, \alpha_1}(s, 1) + \frac{1}{2} P_{L, \alpha_1}(s, 2) \right) = \frac{1}{2} d_0 + \frac{1}{4} \text{Unif}(\{W\}) + \frac{1}{4} \left( \sum_{l=1}^L \left( \frac{1}{2} \text{Unif}(S^l) + \frac{1}{2L} \text{Unif}(\{Z\}) \right) + \frac{1}{2} \cdot \text{Unif}(\{X, Y\}) \right) = \frac{1}{8} \left( \sum_{l=1}^L \left( \frac{1}{2} \text{Unif}(S^l) + \frac{1}{2L} \text{Unif}(\{Z\}) \right) + \frac{1}{2} \text{Unif}(\{Z\}) + \frac{1}{4L} \text{Unif}(\{W\}) + \frac{1}{8} \text{Unif}(\{X, Y\}) \right),$$

which is in fact independent of the choice of $M \in M$.

In addition, by a straightforward calculation, we see that this choice of $\mu$ leads to the following bound on the concentrability coefficient. See Appendix H for a detailed calculation.
Proposition E.2. We have $C_{\text{conc}} \leq 32L$ for all models in $\mathcal{M}$.

E.4 Proof of Theorem 1.2

Recall that for each $M \in \mathcal{M}$, we let $\mathbb{P}^M_n$ denote the law of the offline dataset $D_n$ when the underlying MDP is $M$, and we let $\mathbb{E}^M_n$ be the associated expectation operator. Lemma E.1, stated below, reduces the task of proving a policy learning lower bound to the task of upper bounding the total variation distance between the mixture distributions $\mathbb{P}^1_n := \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{P}^M_n$ and $\mathbb{P}^2_n := \frac{1}{|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{P}^M_n$.

Lemma E.1. Consider any fixed $\gamma \in (0, 1)$ and $L \in \mathbb{N}$. For any offline RL algorithm which takes $D_n = \{(s_i, a_i, r_i, s'_i)\}_{i=1}^n$ as input and returns a stochastic policy $\hat{\pi}_{D_n}: S \rightarrow \Delta(A)$, we have

$$\sup_{M \in \mathcal{M}} \{ J_M(\pi^*_M) - \mathbb{E}^M_n[J_M(\hat{\pi}_{D_n})] \} \geq \frac{\gamma L}{16L(1-\gamma)}(1 - D_{TV}(\mathbb{P}^1_n, \mathbb{P}^2_n)).$$

See Appendix F for the proof of Lemma E.1. We conclude the proof of Theorem 1.2 by bounding the total variation distance $D_{TV}(\mathbb{P}^1_n, \mathbb{P}^2_n)$. Because directly calculating the total variation distance is difficult, we proceed in two steps. We first design two auxiliary reference measures $Q^1_n$ and $Q^2_n$, and then bound $D_{TV}(\mathbb{P}^1_n, Q^1_n)$, $D_{TV}(\mathbb{P}^2_n, Q^2_n)$, and $D_{TV}(Q^1_n, Q^2_n)$ separately. For the latter step, as in the proof of Theorem 1.1, we move from total variation distance to $\chi^2$-divergence, which we bound using similar arguments. Our final bound on the total variation distance, which is proven in Appendix G, is as follows.

Lemma E.2. Consider any fixed $\gamma \in (0, 1)$ and $L \in \mathbb{N}$. For all $n \leq \sqrt{(S -\bar{s})/(20L^2)}$, we have

$$D_{TV}(\mathbb{P}^1_n, \mathbb{P}^2_n) \leq 1/2 + n/(8 \cdot 2^L).$$

Theorem 1.2 immediately follows by choosing

$$L := \left\lceil \min \left\{ \frac{C}{32 \cdot (1 - \gamma)}, \log_2(S) \right\} \right\rceil$$

and combining Lemma E.1 and Lemma E.2. With this choice of $L$, Lemma E.2 implies that $D_{TV}(\mathbb{P}^1_n, \mathbb{P}^2_n) \leq 5/8$ whenever $n \leq c \cdot \min\{S^{1/3}/(\log S)^2, 2^{C/32}, 2^{1/(1-\gamma)}\}$ for a sufficiently small numerical constant $c$, and we have $C_{\text{conc}} \leq C$ as desired by Proposition E.2. Finally, whenever $\gamma \geq 1/2$, using our choice for $L$ within Lemma E.1 gives

$$\sup_{M \in \mathcal{M}} \{ J_M(\pi^*_M) - \mathbb{E}^M_n[J_M(\hat{\pi}_{D_n})] \} \geq \Omega(1) \cdot \frac{\gamma L}{L(1-\gamma)} = \Omega(1)$$

where we use the fact that

$$\frac{\gamma L}{L(1-\gamma)} \geq \frac{\gamma^{1/(1-\gamma)}(1/(1-\gamma))(1-\gamma)}{1-\gamma} = \gamma^{1/(1-\gamma)} \geq (1/2)^2$$

when $\gamma \in [1/2, 1)$.

F Proof of Lemma E.1

We begin the proof by lower bounding the regret for any MDP in the family $\mathcal{M}$. For any $i \in \{1, 2\}$, any MDP $M \in \mathcal{M}_i$, and any policy $\pi: S \rightarrow \Delta(A)$, we have

$$J_M(\pi^*_M) - J_M(\pi) = Q^*_M(s, \pi^*_M(s)) - Q^*_M(s, \pi(s))$$

$$= Q^*_M(s, \pi^*_M(s)) - Q^*_M(s, \pi(s))$$

$$= Q^*_M(s, i) - Q^*_M(s, \pi(s))$$

$$= \frac{\gamma}{1-\gamma} \frac{|V_{\alpha_1} - V_{\alpha_2}|}{2} \mathbb{P}(\pi(s) \neq i)$$

$$\geq \frac{\gamma L}{24L(1-\gamma)} \mathbb{P}(\pi(s) \neq i),$$

(20)
where the inequality follows because
\[
|V_{a_1} - V_{a_2}| = \sum_{i=1}^{L} \frac{1}{2^{i-1}} \left( \frac{\gamma^{L-(i-1)} \alpha_1}{1 - (1-\gamma) \alpha_1} - \frac{\gamma^{L-(i-1)} \alpha_2}{1 - (1-\gamma) \alpha_2} \right) + \frac{1}{2^{L+1}} \left( \frac{\alpha_1}{1 - L \alpha_1} - \frac{\alpha_2}{1 - L \alpha_2} \right)
\] 
\[
\geq \frac{1}{2} \gamma^L |\alpha_1 - \alpha_2| = \frac{\gamma^L}{2} \left( \frac{1}{L+1} - \frac{1}{2L} \right) \geq \frac{\gamma^L}{12L}
\]
when \( L \geq 2 \).

Now, consider any fixed offline reinforcement learning algorithm which takes the offline dataset \( D_n \) as an input and returns a stochastic policy \( \hat{\pi}_{D_n} : \mathcal{S} \to \Delta(\mathcal{A}) \). For each \( i \in \{1, 2\} \), we apply (20) to all MDPs in \( \mathcal{M}_i \) and average to obtain
\[
\frac{1}{|\mathcal{M}_i|} \sum_{M \in \mathcal{M}_i} \mathbb{E}_n^M[J_M(\pi_M^*) - J_M(\hat{\pi}_{D_n})] \geq \frac{\gamma^L}{24L} \frac{1}{(1-\gamma)} \frac{\gamma}{|\mathcal{M}_i|} \sum_{M \in \mathcal{M}_i} \mathbb{P}_n^M(\hat{\pi}_{D_n}(s) \neq i).
\]

Applying the inequality above for \( i = 1 \) and \( i = 2 \) and combining the results, we have
\[
\max_{M \in \mathcal{M}} \mathbb{E}_n^M[J_M(\pi_M^*) - J_M(\hat{\pi}_{D_n})] \geq \frac{1}{2|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{E}_n^M[J_M(\pi_M^*) - J_M(\hat{\pi}_{D_n})] + \frac{1}{2|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{E}_n^M[J_M(\pi_M^*) - J_M(\hat{\pi}_{D_n})]
\]
\[
\geq \frac{\gamma^L}{48L} \frac{1}{(1-\gamma)} \left( \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{P}_n^M(\hat{\pi}_{D_n}(s) \neq 1) + \frac{1}{|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{P}_n^M(\hat{\pi}_{D_n}(s) \neq 2) \right)
\]
\[
\geq \frac{\gamma^L}{48L} \frac{1}{(1-\gamma)} \left( 1 - D_{TV} \left( \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} \mathbb{P}_n^M, \frac{1}{|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} \mathbb{P}_n^M \right) \right),
\]
where the last inequality follows because \( \mathbb{P}(E) + \mathbb{Q}(E^c) \geq 1 - D_{TV}(\mathbb{P}, \mathbb{Q}) \) for any event \( E \).

\[\square\]

### G Proof of Lemma E.2

This proof is organized as follows. In Appendix G.1, we introduce two reference measures and move from the total variation distance to the \( \chi^2 \)-divergence. This allows us to reduce the task of upper bounding \( D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2) \) to the task of upper bounding two manageable density ratios (Eqs. (23) and (24) in the sequel). We develop several intermediate technical lemmas related to the density ratios in Appendix G.2, and in Appendix G.3 we put everything together to bound the density ratios, thus completing the proof of Lemma E.2.

#### G.1 Introducing Reference Measures and Moving to \( \chi^2 \)-Divergence

Directly calculating the total variation distance \( D_{TV}(\mathbb{P}_n^1, \mathbb{P}_n^2) \) is challenging, so we design two auxiliary reference measures \( \mathbb{Q}_n^1 \) and \( \mathbb{Q}_n^2 \) which serves as intermediate quantities to help with the upper bound. The reference measures \( \mathbb{Q}_n^1, \mathbb{Q}_n^2 \) lies in the same measurable space as \( \mathbb{P}_n^1 \) and \( \mathbb{P}_n^2 \), and are defined as follows:

\[
\mathbb{Q}_n^1(\{(s, a, r, s')\}_{i=1}^n) := \prod_{i=1}^n \mu(s_i, a_i) P_0(s'_i | s_i, a_i), \quad \forall \{(s, a, r, s')\}_{i=1}^n,
\]

\[
\mathbb{Q}_n^2(\{(s, a, r, s')\}_{i=1}^n) := \prod_{i=1}^n \mu(s_i, a_i) P_0(s'_i | s_i, a_i), \quad \forall \{(s, a, r, s')\}_{i=1}^n,
\]
where

\[
R_1(s, a) := \begin{cases} 
0, & s \in \{s\} \cup S^0 \cup \cdots \cup S^L, \\
w, & s = W, \\
1, & s = X, \\
0, & s = Y, \\
\frac{\alpha_1}{1-L\alpha_1}, & s = Z,
\end{cases}
\quad R_2(s, a) := \begin{cases} 
0, & s \in \{s\} \cup S^0 \cup \cdots \cup S^L, \\
w, & s = W, \\
1, & s = X, \\
0, & s = Y, \\
\frac{\alpha_2}{1-L\alpha_2}, & s = Z,
\end{cases}
\]

and

\[
P_0(s, 1) = \text{Unif}\{\{W\}\},
\]

\[
P_0(s, 2) = \frac{1}{2} \cdot \left( \sum_{l=1}^{L} \frac{1}{2^l} \text{Unif}(S^l) \right) + \frac{1}{2^L} \text{Unif}\{\{Z\}\} + \frac{1}{2} \cdot \text{Unif}\{\{X, Y\}\},
\]

\[
\forall s \in S^l, \forall l \in [L] : P_0(s, a) = \frac{(1-\alpha_1)(1-\alpha_2)}{(1-(l-1)\alpha_1)(1-(l-1)\alpha_2)} \text{Unif}(S^{l+1})
\]

\[
+ \frac{\gamma^{L-l}\alpha_1\alpha_2}{(1-(l-1)\alpha_1)(1-(l-1)\alpha_2)} \text{Unif}\{\{X\}\}
\]

\[
+ \left(1 - \frac{(1-\alpha_1)(1-\alpha_2)}{(1-(l-1)\alpha_1)(1-(l-1)\alpha_2)} - \frac{\gamma^{L-l}\alpha_1\alpha_2}{(1-(l-1)\alpha_1)(1-(l-1)\alpha_2)} \right) \text{Unif}\{\{Y\}\},
\]

\[
\forall s \in \{W, X, Y, Z\} : P_0(s, a) = \text{Unif}\{\{s\}\}.
\]

The reference measure \(\mathbb{Q}_1^n\) is the law of \(D_n\) when the data collection distribution is \(\mu\) and the underlying MDP is \(\overline{M}_1 := (\mathcal{S}, \mathcal{A}, \overline{P}_0, \overline{R}_1, \gamma, d_0)\). Notably, \(\overline{M}_1\) shares the same reward function with all MDPs in \(\mathcal{M}_1\), and differs from the MDPs in \(\mathcal{M}_1\) only in terms of the transition operator \(\overline{P}_0\).

There are two ways to understand \(\overline{P}_0\). Operationally, \(\overline{P}_0\) is simply the pointwise average transition operator of the MDPs in \(\mathcal{M}_1\), in the sense that

\[
\forall s \in \mathcal{S}, a \in \mathcal{A} : P_0(\cdot \mid s, a) = \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} P_M(\cdot \mid s, a),
\]

where \(P_M\) is the transition operator. For this reason, we call \(\overline{M}_1\) the average MDP associated with \(\mathcal{M}_1\). More conceptually, \(\overline{P}_0\) is the transition operator obtained by performing state aggregation using the value function class \(\mathcal{F} = \{f_1, f_2\}\), where states with the same values for both \(f_1\) and \(f_2\) are viewed as identical and constrained to share dynamics (which is induced by averaging over the data collection distribution).

Similarly, the reference measure \(\mathbb{Q}_2^n\) can be understood as the law of \(D_n\) when the data collection distribution is \(\mu\) and the underlying MDP is \(\overline{M}_2 := (\mathcal{S}, \mathcal{A}, \overline{P}_0, \overline{R}_2, \gamma, d_0)\), where \(\overline{M}_2\) is the average MDP associated with \(\mathcal{M}_2\). An important property is that \(\overline{M}_1\) and \(\overline{M}_2\) share the same transition operator \(\overline{P}_0\) and differs only in terms of the reward on state \(Z\). This is a consequence of our construction, as when we construct \(\mathcal{M}_1\) and \(\mathcal{M}_2\) we strive to ensure that

\[
\forall s \in \mathcal{S}, a \in \mathcal{A} : \frac{1}{|\mathcal{M}_1|} \sum_{M \in \mathcal{M}_1} P_M(\cdot \mid s, a) = \overline{P}_0(\cdot \mid s, a) = \frac{1}{|\mathcal{M}_2|} \sum_{M \in \mathcal{M}_2} P_M(\cdot \mid s, a),
\]

and there is no uncertainty in the reward function outside of state \(Z\).

Figure 3 illustrates the average MDPs \(\overline{M}_1\) and \(\overline{M}_2\) (the only difference between \(\overline{M}_1\) and \(\overline{M}_2\) is the reward on state \(Z\), which is not displayed). Note that for each \(l \in [L]\), all intermediate states in \(S^l\) have the same dynamics, so the planted subset structure is erased by averaging/aggregating.
where the first equality follows from the well-known identity between the total variation distance and the

The next lemma shows that the total variation distance between

Using Lemma G.1, we have

Starting with the triangle inequality for the total variation distance, we have

where the second inequality follows from the fact that \( D_{TV}(P, Q) \leq \frac{1}{2} \sqrt{D_{\chi^2}(P \parallel Q)} \) for any \( P, Q \) (see Proposition 7.2 or Section 7.6 of Polyanskiy (2020)).

The next lemma shows that the total variation distance between \( Q_n^1 \) and \( Q_n^2 \) is small. Intuitively, this is because the average MDPs \( \bar{M}_1 \) and \( \bar{M}_2 \) only differ in the reward on state \( Z \), but the data distribution \( \mu \)'s coverage of on \( Z \) is very small.

**Lemma G.1.** For all \( n < \infty \), we have \( D_{TV}(Q_n^1, Q_n^2) \leq n \mu(Z, a) = n/(8 \times 2^L) \).

**Proof.** Let \( R := \{1, 0, \alpha_1/(1 - L\alpha_1), \alpha_2/(1 - L\alpha_2), R(W, a)\} \), then \( R(s, a) \in R \) for all \((s, a) \in S \times A\). Since \(|S|, |A|, |R| < \infty\), the realization of the offline dataset \( D_n = \{(s_i, a_i, r_i, s_i')\}_{i=1}^n \) only has finitely many possible outcomes, and we have

\[
D_{TV}(Q_n^1, Q_n^2) = \frac{1}{2} \sum_{(s_i, a_i, r_i, s_i') \in S \times A \times R \times S, \forall i \in [n]} \left| Q_n^1((s_i, a_i, r_i, s_i')) - Q_n^2((s_i, a_i, r_i, s_i')) \right|
\]

\[
= \frac{1}{2} \sum_{(s_i, a_i, r_i, s_i') \in S \times A \times R \times S, \forall i \in [n]} \prod_{i=1}^{n} \mu(s_i, a_i) \prod_{i=1}^{n} \mathbb{1}_{\{r_i = R_1(s_i, a_i)\}} - \prod_{i=1}^{n} \mathbb{1}_{\{r_i = R_2(s_i, a_i)\}}
\]

\[
= \frac{1}{2} \sum_{(s_i, a_i) \in S \times A, \forall i \in [n]} \prod_{i=1}^{n} \mu(s_i, a_i) \prod_{i=1}^{n} \mathbb{1}_{\{r_i = R_2(s_i, a_i)\}} - \prod_{i=1}^{n} \mathbb{1}_{\{r_i = R_2(s_i, a_i)\}}
\]

\[
= \sum_{(s_i, a_i) \in S \times A, \forall i \in [n]} \mathbb{1}_{\exists i \in [n] \text{ s.t. } s_i = Z} \prod_{i=1}^{n} \mu(s_i, a_i)
\]

\[
= \mathbb{P}_{s_1, \ldots, s_n \sim \mu}(\exists i \in [n] \text{ s.t. } s_i = Z) = \mathbb{P}_{s_1, \ldots, s_n \sim \mu}(\{s_1 = Z\} \cup \cdots \cup \{s_n = Z\}) \leq n \mu(\{Z\}),
\]

where the first equality follows from the well-known identity between the total variation distance and the \( L_1 \) norm and the last inequality follows from a union bound.

Using Lemma G.1, we have

\[
D_{TV}(P_n^1, P_n^2) \leq \frac{1}{2} \sqrt{D_{\chi^2}(P_n^1 \parallel Q_n^1)} + \frac{1}{2} \sqrt{D_{\chi^2}(P_n^2 \parallel Q_n^2)} + n \mu(Z, a).
\]

Note that \( \mu(Z, a) = 1/8 \cdot 1/2^L \) which produces the final term in the bound in Lemma E.2.

Figure 3: Illustration of the average MDP with \( L = 3 \).
We now turn our focus to the $\chi^2$-divergence, which we expand as

$$D_{\chi^2}(P_n \parallel Q_n)$$

$$= E_n((s,a,r,s')_{n=1}^\infty)^n \sim Q_n \left[ \left( \frac{1}{|M|} \sum_{M \in M_1} \frac{P_M((s_i,a_i,r_i,s'_i)_{i=1}^n)}{Q_n((s_i,a_i,r_i,s'_i)_{i=1}^n)} \right)^2 \right] - 1$$

$$= E_n((s,a,r,s')_{i=1}^n)^n \sim Q_n \left[ \left( \frac{1}{|M|} \sum_{M \in M_1} \prod_{i=1}^n \mu(s_i, a_i) \mathbb{1}_{\{r_i = R_M(s_i, a_i)\}} P_M(s'_i \mid s_i, a_i) \right)^2 \right] - 1$$

$$= E_n((s,a,r,s')_{i=1}^n)^n \sim Q_n \left[ \left( \frac{1}{|M|} \sum_{M \in M_1} \prod_{i=1}^n \frac{P_M(s'_i \mid s_i, a_i)}{\prod_{i=1}^n P_0(s'_i \mid s_i, a_i)} \right)^2 \right] - 1$$

$$= \frac{1}{|M_1|^2} \sum_{M,M' \in M_1} E_n((s,a,r,s')_{i=1}^n)^n \sim Q_n \left[ \prod_{i=1}^n \frac{P_M(s'_i \mid s_i, a_i) P_{M'}(s'_i \mid s_i, a_i)}{\prod_{i=1}^n P_0(s'_i \mid s_i, a_i)} \right] - 1.$$

where the third equality follows from $R_M(s,a) = R_1(s,a), \forall M \in M, \forall a \in A, \forall s \in S$.

Using an identical calculation, we also have

$$D_{\chi^2}(P_n^2 \parallel Q_n^2) = \frac{1}{|M_2|^2} \sum_{M,M' \in M_2} E_n((s,a)_{i=1}^n)^n \sim P_{\theta} \left[ \frac{P_M(s' \mid s,a) P_{M'}(s' \mid s,a)}{P_{\theta}(s' \mid s,a)} \right]^n - 1.$$

Equipped with these expressions for the $\chi^2$-divergence, the next step in the proof of Lemma E.2 is to upper bound the right-hand side for Eqs. (23) and (24). This is done in Appendix G.3, but before proceeding we require several intermediate technical lemmas.

G.2 Technical Lemmas for Density Ratios

For this section only, we focus on MDPs in $M_1$ and suppress the subscript indexing the subfamily, i.e., we use $\theta$ for $\theta^{(1)}$ and $\alpha$ for $\alpha_1$. Exactly the same calculations apply for $M_2$, which we will use in the next section. To simplify the presentation and re-use lemmas from Appendix D.2 it will be helpful to define the following notation:

$$\alpha = (\alpha_1, \ldots, \alpha_L), \quad \alpha_l = \frac{\gamma^{L-l}\alpha}{1 - (L - 1)\alpha}$$

$$\beta = (\beta_1, \ldots, \beta_L), \quad \beta_l = 1 - \alpha_l$$

Additionally recall that

$$\theta = (\theta_1, \ldots, \theta_L), \quad \theta_l = \frac{\alpha}{1 - (L - 1)\alpha}$$

These vectors parametrize the MDP transitions in the following sense: Let $I \in \mathcal{I}_\theta$ denote the choice of planted states for each layer. Then for $l \in [L]$ we have:

$$s \in I^l : P_{L,a,I}(s,a) = \alpha_l \text{Unif}(\{X\}) + \beta_l \text{Unif}(\{Y\})$$

$$s \not\in I^l : P_{L,a,I}(s,a) = (1 - \theta_l) \text{Unif}(I^{l+1}) + \theta_l \text{Unif}(\{Y\})$$

where $I^{L+1} = \{Z\}$. 37
To state the results compactly, we define
\[ \phi_{\theta, \alpha, \beta}^I := \theta_I^2 \left( \frac{(\beta_I - \alpha_I)^2}{\theta_I (\beta_I - \alpha_I) + 1 - \beta_I} + \frac{\theta_I (\beta_I - \alpha_I) + \alpha_I}{\theta_I (1 - \theta_I)} \right). \] (25)

We also use \( P_I \) to denote \( P_{L, \alpha, I} \).

We will bound the density ratio terms for each layer separately. First we control the \( L \)th layer.

**Lemma G.2.** For any \( I, J \in \mathcal{I}_\theta \), we have
\[
\mathbb{E}_{s \sim \text{Unif}(S^I), \ s' \sim P_0(|s, a|)} \left[ \frac{P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \right] = 1 + \phi_{\theta, \alpha, \beta}^I \cdot \left( \frac{|I| \cap J^I}{\theta_I^2 S_L} - 1 \right).
\]

We omit the proof, which is identical to that of Lemma D.2. Next we turn to intermediate layers.

**Lemma G.3.** For any \( I, J \in \mathcal{I}_\theta \), for any \( l \in [L - 1] \), we have
\[
\mathbb{E}_{s \sim \text{Unif}(S^I), \ s' \sim P_0(|s, a|)} \left[ \frac{P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \right] \leq 1 + \phi_{\theta, \alpha, \beta}^I \cdot \left( \frac{|I| \cap J^I}{\theta_I^2 S_l} - 1 \right) + \left( \frac{|I| \cap J^I}{\theta_I^2 S_{l+1}} - 1 \right).
\]

**Proof.** For any \( I, J \in \mathcal{I}_\theta \), for any \( l \in [L - 1] \), we observe that
\[
\mathbb{E}_{s \sim \text{Unif}(S^I), \ s' \sim P_0(|s, a|)} \left[ \frac{P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \right] = \sum_{s' \in \{X, Y\} \cup (I \cap J^I)} \mathbb{E}_{s \sim \text{Unif}(S^I)} \left[ \frac{P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \right].
\]

To proceed, we calculate the value of the ratio \( \frac{P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \) for each possible choice for \( s \in S^I \) and \( s' \in \{X, Y\} \cup (I \cap J^I) \) in Table 2 below.

| \( s' = X \) | \( s' = Y \) | \( s' \in I \cap J^I \) |
|---|---|---|
| \( s \in I \cap J^I \) | \( 1/\theta_I \) | \( \beta_I / (\theta_I \beta_I + (1 - \theta_I) \alpha_I) \) | \( 0 \) |
| \( s \in (I \cup J^I) \setminus (I \cap J^I) \) | \( 0 \) | \( \beta_I \alpha_I / (\theta_I \beta_I + (1 - \theta_I) \alpha_I) \) | \( 0 \) |
| \( s \notin (I \cup J^I) \) | \( 0 \) | \( \alpha_I / (\theta_I \beta_I + (1 - \theta_I) \alpha_I) \) | \( \beta_I / (\theta_I \beta_I + (1 - \theta_I) \alpha_I) \) |

Table 2: Value of \( \frac{P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \) for all possible pairs \((s, s')\).

Define \( t_l := |I \cap J^I| \). From Lemma D.1, we must have \( t_l \in [(2 \theta_I - 1) S_I, \theta_I S_I] \). We also have \( |I \cup J^I| = |I| + |J^I| - |I \cap J^I| = 2 \theta_I S_I - t_l \). Hence, the event in the first row of Table 2 occurs with probability \( |I \cap J^I| / S_I = t_l / S_I \); the event in the second row occurs with probability \( |(I \cup J^I) \setminus (I \cap J^I)| / S_I = (2 \theta_I S_I - 2 t_l) / S_I \); and the event in the third row occurs with probability \( |S_I \setminus (I \cup J^I)| / S = ((1 - 2 \theta_I) S_I + t_l) / S_I \). Using these values and performing a similar calculation to the one in the proof of Lemma D.2, we obtain
\[
\mathbb{E}_{s \sim \text{Unif}(S^I)} \left[ \frac{\sum_{s' \in \{X, Y\} \cup (I \cap J^I)} P_I(s' \mid s, a)P_J(s' \mid s, a)}{P_0(s' \mid s, a)} \right] = 1 + \phi_{\theta, \alpha, \beta}^I \left( \frac{t_l}{\theta_I^2 S_I} - 1 \right) + \left( 1 - 2 \theta_I + \frac{t_l}{S_I} \right) \frac{\beta_I}{1 - \theta_I} \left( \frac{t_{l+1}}{\theta_I^2 S_{l+1}} - 1 \right),
\]

where the last inequality follows from \((2 \theta_I - 1) S_I \leq t_l \leq \theta_I S_I\) (which implies \(0 \leq 1 - 2 \theta_I + t_l / S_I \leq 1 - \theta_I\)) and \(0 \leq \beta_I \leq 1\).
G.3 Completing the Proof

For now, let us also focus on a single MDP subfamily $M_1$ and suppress the family indices associated with $\alpha$ and $(\theta,\alpha,\beta)$. As above the same calculations apply to $M_2$. To keep notation compact, for any $d \in \Delta(S \times A)$, define

$$DR_{M,M'}(d) := \mathbb{E}_{(s,a) \sim d, \ s' \sim P_0(\cdot | s,a)} \left[ \frac{P_M(s' | s,a)P_M'(s' | s,a)}{P_0^2(s' | s,a)} \right].$$

Consider any $M, M' \in M_1$. For any $\pi : S \to \Delta(A)$, by Lemmas G.2 and G.3, we have

$$\sum_{l=1}^L \frac{1}{2^l} DR_{M,M'}(\text{Unif}(S') \times \pi_0) \leq \sum_{l=1}^L \frac{1}{2^l} + \sum_{l=1}^L \frac{1}{2^l} \phi_{\theta,\alpha,\beta} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) + \sum_{l=2}^L \frac{1}{2^{l-1}} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) + \sum_{l=1}^L \frac{1}{2^l} \phi_{\theta,\alpha,\beta} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) \leq \frac{1}{2 L} + \sum_{l=1}^L \frac{1}{2^l} \phi_{\theta,\alpha,\beta} + \frac{2}{2^l} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right). \tag{26}$$

Note that $P_M(\cdot | s,a)$ and $P_M'(\cdot | s,a)$ differ from $P_0(\cdot | s,a)$ only when $(s,a) = (s,2)$ or $s \in \{s\} \cup S^1 \cup \cdots \cup S^L$, so, recalling the value of $\mu$, we have

$$\mathbb{E}_{(s,a) \sim \mu, s' \sim P_0(\cdot | s,a)} \left[ \frac{P_M(s' | s,a)P_M'(s' | s,a)}{P_0^2(s' | s,a)} \right] = \frac{1}{8} \sum_{l=1}^L \frac{1}{2^l} \mathbb{E}_{(s,a) \sim \mu} \mathbb{E}_{\text{Unif}(S') \times \pi_0} \left[ \frac{P_M(s' | s,a)P_M'(s' | s,a)}{P_0^2(s' | s,a)} \right] = \frac{1}{8} \sum_{l=1}^L \frac{1}{2^l} \phi_{\theta,\alpha,\beta} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 + \sum_{l=1}^L \phi_{\theta,\alpha,\beta} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right),$$

where the first inequality follows from (26). As a result, we have

$$\frac{1}{|M_1|^2} \sum_{M,M' \in M_1} \left( \mathbb{E}_{(s,a) \sim \mu, \ s' \sim P_0(\cdot | s,a)} \left[ \frac{P_M(s' | s,a)P_M'(s' | s,a)}{P_0^2(s' | s,a)} \right] \right)^n \leq \left( \prod_{l=1}^L \frac{1}{\theta_l^2 S_l} \right)^2 \sum_{l_1,\ldots,l_L | I_l \cap I'_l | = l} \left( \sum_{l=1}^L \phi_{\theta,\alpha,\beta} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) \right)^n \leq \mathbb{E}_{\text{Hyper}(\theta_l S_l, \theta_l S_l), \forall l \in [L]} \left[ \sum_{l=1}^L \phi_{\theta,\alpha,\beta} \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) \right]^n, \tag{27}$$

where $\text{Hyper}(-,\cdot,\cdot)$ denotes the hypergeometric distribution (cf. Lemma D.6 for background).

By Lemma D.6, for any $l \in [L]$, the event

$$E_l := \{ t_l \geq (\theta_l + \epsilon_l)\theta_l S_l \},$$

happens with probability at most $\exp(-2\epsilon_l^2 \theta_l S_l)$. Hence, the event

$$E_{\text{bad}} := \{ \exists l \in [L], t_l \geq (\theta_l + \epsilon_l)\theta_l S_l \} = \bigcup_{l=1}^L E_l$$

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happens with probability at most \( \sum_{l=1}^{L} \exp(-2\epsilon_l^2 \theta_l S_l) \). Conditional on \( E_{\text{clean}} := E_{\text{bad}} \), i.e., the complement of \( E_{\text{bad}} \), we have

\[
\left( 1 + \sum_{l=1}^{L} \phi_{l,\alpha,\beta} / 2^l \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) \right)^n \leq \left( 1 + \sum_{l=1}^{L} \phi_{l,\alpha,\beta} / 2^l \left( \frac{1}{\theta_l} - 1 \right) \right)^n \leq \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n.
\]

Here we are using the bound \( \phi_{l,\alpha,\beta} \leq \frac{1}{1-\theta_l} \), which follows from Lemma D.3. On the other hand, under \( E_{\text{bad}} \), we have

\[
\left( 1 + \sum_{l=1}^{L} \phi_{l,\alpha,\beta} / 2^l \left( \frac{t_l}{\theta_l^2 S_l} - 1 \right) \right)^n \leq \left( 1 + \sum_{l=1}^{L} \phi_{l,\alpha,\beta} / 2^l \left( \frac{1}{\theta_l} - 1 \right) \right)^n \leq \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n,
\]

where the first inequality follows from \( t_l \leq \theta_l S_l \). Hence we have

\[
\mathbb{E}_{t_l \sim \text{Hyper}(\theta_l S_l, \theta_l S_l, \theta_l S_l), \forall l \in [L]} \left[ \left( 1 + \sum_{l=1}^{L} \frac{\phi_{l,\alpha,\beta} / 2^l \left( t_l S_l - 1 \right)}{\theta_l^2 S_l} \right)^n \right] \leq \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n + \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n \cdot \mathbb{P}_{t_l \sim \text{Hyper}(\theta_l S_l, \theta_l S_l, \theta_l S_l), \forall l \in [L]} (E_{\text{bad}})
\]

\[
\leq \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n + \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n \cdot \sum_{l=1}^{L} \exp(-2\epsilon_l^2 \theta_l S_l)
\]

\[
= \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n + \sum_{l=1}^{L} \exp \left( n \log \left( 1 + \sum_{j=1}^{L} \frac{1}{2^j \theta_j (1 - \theta_j)} \right) - 2\epsilon_l^2 \theta_l S_l \right)
\]

\[
= \left( 1 + \sum_{l=1}^{L} \frac{1}{2^l \theta_l (1 - \theta_l)} \right)^n + \sum_{l=1}^{L} \exp \left( n \sum_{j=1}^{L} \frac{1}{2^j \theta_j (1 - \theta_j)} - 2\epsilon_l^2 \theta_l S_l \right)
\]

Combining Eqs. (23), (27) and (28) (note that we are focusing on \( M_1 \)), we have

\[
D_{\chi^2}(\mathbb{P}_n \| \mathbb{Q}_n) \leq \inf_{\epsilon_l \in (0, \theta_l^2 S_l) \forall l \in [L]} \left\{ \left( 1 + \sum_{l=1}^{L} \frac{\epsilon_l}{2^l \theta_l (1 - \theta_l)} \right)^n + \sum_{l=1}^{L} \exp \left( n \sum_{j=1}^{L} \frac{1}{2^j \theta_j (1 - \theta_j)} - 2\epsilon_l^2 \theta_l S_l \right) \right\} - 1,
\]

Let \( \epsilon \in (0, 1/2) \) be an arbitrary constant. We set \( \epsilon_l = 2c \cdot \frac{(1 - \theta_l) \theta_l}{n} \) (which belongs to \( (0, \theta_l^2 S_l) \) because \( \epsilon_l < \theta_l \) since \( n \geq 1 \) and \( \theta_l S_l \geq 1 \) by assumption) for all \( l \in [L] \). Then we have

\[
\left( 1 + \sum_{l=1}^{L} \frac{\epsilon_l}{2^l (1 - \theta_l) \theta_l} \right)^n \leq \left( 1 + \frac{c}{n} \right)^n \leq e^c \leq 1 + 2c,
\]

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and
\[ D_{N^2}(\mathbb{P}^1_n \| \mathbb{Q}^1_n) \leq 2c + \sum_{l=1}^{L} \exp \left( n \sum_{j=1}^{L} \frac{1}{2j+1} - 8c^2 \frac{(1 - \theta_j)^2 \theta_j^3}{n^2} S_l \right). \]

In particular, whenever \( S_l \geq \frac{n^3}{4c^2(1 - \eta)^2 \min_{j \in [L]} \theta_j} \), we have
\[ D_{N^2}(\mathbb{P}^1_n \| \mathbb{Q}^1_n) \leq 2c + \exp \left( n \sum_{j=1}^{L} \frac{1}{2j+1} - 2n \frac{1}{\min_{j \in [L]} \theta_j} \right) \leq 2c + \exp(-n). \]

Since \( \theta_l = \frac{1}{1 - (l-1)\alpha} \) and the parameter \( \alpha \in [\frac{1}{2L}, \frac{1}{L+1}] \) for the MDP family \( M_l \), we have \( \theta_l \in [\frac{1}{2L}, \frac{1}{L}] \) for all \( l \in [L] \). Setting \( c = 1/10 \). Whenever \( n \geq 5 \) and \( S - 5 \geq 3200n^3L^6 \), we have \( S_l \geq \frac{S - 5}{L_{\text{div}}} (2L + 1 - l)(L + 2 - l) > 1600n^3L^4 \) for all \( l \in [L] \) (recall that \( L_{\text{div}} \leq 4L^3 \)), and hence
\[ D_{N^2}(\mathbb{P}^1_n \| \mathbb{Q}^1_n) \leq \frac{1}{5} + \exp(-n) \leq \frac{1}{4}. \]

Using the same calculation, whenever \( n \geq 5 \) and \( S - 5 \geq 800n^3L^6 \), it holds that
\[ D_{N^2}(\mathbb{P}^2_n \| \mathbb{Q}^2_n) \leq \frac{1}{5} + \exp(-n) \leq \frac{1}{4}. \]

Combining the above two inequalities with (22), we have \( D_{TV}(\mathbb{P}^1_n, \mathbb{P}^2_n) \leq 1/2 + n/(8 \cdot 2^L) \), which proves the lemma.

\[ \square \]

**H Proofs of Proposition E.1 and Proposition E.2**

**Proof of Proposition E.1.** Since for all states in \( S \setminus \{s\} \) the two actions in \( \mathcal{A} \) have identical effects, we have \( Q^\pi(s, a) = Q^\pi(s, a) \) for all \( (s, a) \in S \times \mathcal{A} \) and for all \( \pi : S \to \Delta(\mathcal{A}) \). Hence we only need to show \( Q_M^\pi = f_1 \) for all \( M \in \mathcal{M}_1 \) and \( Q_M^\pi = f_2 \) for all \( M \in \mathcal{M}_2 \).

Consider an arbitrary \( M = M_{L, \alpha, w} \in \mathcal{M} \). First, for any self-looping terminal state \( s \in \{W, X, Y, Z\} \), we have

\[ V_M^*(s) = Q_M^*(s, a) = \sum_{h=0}^{\infty} \gamma^h R_{L, \alpha, w}(s, a) = \begin{cases} w, & s = W, \\ 1, & s = X, \\ 0, & s = Y, \\ \frac{\alpha}{1 - L \alpha}, & s = Z. \end{cases} \]

Next, for \( l = L, \ldots, 1 \), for any \( l \)-th layer intermediate state \( s \in S^l \), by the Bellman optimality equation, we have

\[ V_M^+(s) = Q_M^+(s, a) = R_{L, \alpha, w}(s, a) + \gamma \mathbb{E}_{s' \sim P_{L, \alpha, w}(s, a)}[V_M^*(s')], \]

for \( s \in I^l \)
\[ = \begin{cases} 0 + \gamma \left[ \gamma^{L-l} \alpha V_M^+(X) + 0 \right], & s \in I^l, \\ 0 + \gamma \left[ \gamma^{L-l} \alpha \mathbb{E}_{s' \sim \text{Uniform}(I^{l+1})} V_M^+(s') + 0 \right], & s \in I^l, \end{cases} \]
\[ = \begin{cases} \gamma \gamma^{L-l} \alpha, & s \in I^l, \\ \gamma \gamma^{L-l} \alpha, & s \in I^l, \end{cases} \]
\[ = \frac{\gamma \gamma^{L-l} \alpha}{1 - \gamma (1 - L \alpha)}. \]

For the initial state \( s \), we have
\[ Q_M^+(s, 1) = R_{L, \alpha, w}(s, 1) + \gamma [V_M^+(W)] = \frac{\gamma w}{1 - \gamma}. \]
\[ Q_M^*(s, 2) = R_{b, \alpha, w}(s, 2) + \gamma \mathbb{E}_{s' \sim P_{L, \alpha, w}^*(s, 2)}[V_M^*(s')] = \frac{\gamma V_{\alpha}}{1 - \gamma} \]

Therefore, \( Q_M^* = f_1 \) if \( M \in \mathcal{M}_1 \), and \( Q_M^* = f_2 \) if \( M \in \mathcal{M}_2 \).

**Proof of Proposition E.2.** We now verify the concentrability condition (1).

Consider any \( M \in \mathcal{M}_1 \). For any \( (s, a) \in S \times \mathcal{A} \), we have

\[
\sup_{\nu \text{ is admissible}} \nu(s, a) \leq \begin{cases} 
1, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, \\
\frac{1}{2} \cdot \frac{1}{2}, & \text{if } s = Z, a \in \mathcal{A}, \\
\frac{1}{2} \cdot \frac{1}{2} S_l, & \text{maximized when } h = 1 \\
\frac{1}{2^2} \cdot \frac{1}{2^2} S_l, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, l \in [L], \\
\max \left\{ \frac{1}{2} \cdot \frac{1}{2^2} S_l, \frac{1}{2} \cdot \frac{1}{2^2} \frac{1}{2^{l-1}} \frac{1 - (l - 1) \alpha_2}{(l - 2) \alpha_2} \frac{1 - (l - 1) \alpha_1}{1 - (l - 1) \alpha_1} \frac{1}{\theta_l(1) S_l} \right\}, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, 2 \leq l \leq L, \text{ maximized over } h = 1, 2
\end{cases}
\]

Recall the definition of \( \mu \) in Appendix E.3. We have

\[
\mu(s, a) \geq \begin{cases} 
\frac{1}{16} \cdot \frac{1}{2}, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, \\
\frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2}, & \text{if } s = Z, a \in \mathcal{A}, \\
\frac{1}{8} \cdot \frac{1}{2} S_l \cdot \frac{1}{2}, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, l \in [L], \\
\frac{1}{8} \cdot \frac{1}{2} S_l \cdot \frac{1}{2}, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, \text{maximized when } h = 1 \\
\frac{1}{2} \cdot \frac{1}{2} S_l \cdot \frac{1}{2}, & \text{if } s \in \mathcal{S}, a \in \mathcal{A}, 2 \leq l \leq L
\end{cases}
\]

Combining the above two inequalities, we have

\[
\sup_{\nu \text{ is admissible}} \frac{\|\nu\|}{\|\mu\|} \leq \min_{2 \leq l \leq L} \frac{1}{16} \cdot \frac{1}{2^2} \frac{1}{2^{l-1}} \frac{1 - (l - 1) \alpha_2}{(l - 2) \alpha_2} \frac{1 - (l - 1) \alpha_1}{1 - (l - 1) \alpha_1} \frac{1}{\theta_l(1) S_l} = \frac{16 (1 - \alpha_2)}{\alpha_2} \leq 32 L
\]

where the last inequality follows from \( \alpha_2 \geq 1/(2L) \).

Similarly, consider any \( M \in \mathcal{M}_2 \), we have

\[
\sup_{\nu \text{ is admissible}} \frac{\|\nu\|}{\|\mu\|} \leq 32 L
\]

We conclude that the construction satisfies concentrability with \( C_{\text{conc}} \leq 32 L \).