Sinusoidal excitations in reduced Maxwell-Duffing model

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Sinusoidal wave solutions are obtained for reduced Maxwell-Duffing equations describing the wave propagation in a non-resonant atomic medium. These continuous wave excitations exist when the medium is initially polarized by an electric field. Other obtained solutions include both mono-frequency and cnoidal waves.

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An atomic medium in general conditions is modelled by N-level atoms. In the two-level resonant approximation, the system is characterized by the Bloch equations, which is inaccurate in several physical situations [1, 2], like dense atomic media and systems involving three or more level atoms. Thus the resonant model needs to be generalized and extended to the non-resonant scenario. One of the well studied approaches is to consider the response of the medium as weakly nonlinear. Such situation leads to the Duffing oscillator model, where the nonlinear response of the medium is assumed to be cubic. This is the simplest generalization of the Lorentz model.

On the other hand, Maxwell wave equation, for a linearly polarized light, allows propagation in both the directions. However, this can be approximated to unidirectional wave propagation when anharmonic contribution to the polarization is very small. As a result, the wave equation reduces from second order to a first order equation. For a non-resonant medium, this approximation results in the reduced Maxwell-Duffing model (RMD).

Different excitations in non-resonant atomic media are currently attracting considerable attention, because of their relevance to ultra-short regime. Detailed reviews of various aspects of non-linear excitations in atomic media can be found in [3, 4]. In case of two level atoms, the localized soliton solutions of the Maxwell-Bloch equations explained the physical phenomenon of self-induced transparency [5]. In the same system, general cnoidal waves have also been found as exact solution [6, 7, 8]. It was observed [9] that, these waves can be naturally excited in the presence of relaxation. Such shape preserving Jacobi elliptic pulse train solutions have been experimentally observed [10]. More general periodic solutions in multi-level systems have also been reported [11].

In case of Maxwell-Duffing model, a class of exact localized soliton solutions have been recently obtained [12]. We present here mono frequency, sinusoidal wave excitations for RMD system. This excitation exists only in the presence of a polarizing background. General cnoidal wave solutions are found both with and without background.

Below a brief summary of the reduced Maxwell-Duffing model is presented after which a procedure to find exact solutions of this system through a fractional linear transform is outlined. We then present the novel sinusoidal wave solutions including single frequency and cnoidal waves.

Reduced Maxwell-Duffing Model:

The propagation of electromagnetic waves in a medium is described by the wave equation:

\[ \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P}{\partial t^2}, \]  

where \( P \) is polarization of the medium. For unidirectional wave propagation, the above equation can be reduced to a first order equation,

\[ \frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} = -\frac{2\pi}{c} \frac{\partial P}{\partial t}. \]  

In the Duffing oscillators model, the nonlinear response of the medium is cubic. The corresponding equation for the motion of electrons in the presence of an external electric field is given by

\[ \frac{\partial^2 X}{\partial t^2} + \omega_0^2 X + \kappa_3 X^3 = \frac{e}{m} E. \]  

Here \( X \) represents the displacement of an electron from its equilibrium position, \( \omega_0 \) is the oscillator frequency, \( \kappa_3 \) is anharmonicity constant, and \( m \) is the effective mass of the electron of charge \( e \). The medium polarization is defined as \( P = neX \), where \( n \) is the number density of the oscillators in the medium.

We choose new variables \( \tau = z/l, x = \omega_0(t - z/c) \) and normalize the independent variables as

\[ \dot{e} = E/A_0, q = X/X_0. \]  

In terms of the new variables, Eq. (3) then takes the form

\[ \frac{\partial^2 q}{\partial x^2} + q + 2\mu q^3 = \ddot{e}, \]  

where \( 2\mu = \kappa_3 X_0^2/e_0^2 \) and \( A_0 = m\omega_0^2 X_0/e = m\omega_0^2 e^{-1}(2\mu/\kappa_3)^{1/2} \). \( X_0 \) can be expressed as \( X_0 = (2\mu\omega_0^2/|\kappa_3|)^{1/2} \). Similarly Eq. (4) is transformed to

\[ \frac{\partial e}{\partial \tau} = \frac{-\partial q}{\partial x}. \]  

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Here \( l \) is defined as

\[
l^{-1} = 2\pi ne^2/\left(m\omega_0\right) = \omega_p^2/2\omega_0,
\]

where \( \omega_p = (4\pi ne^2/m)^{1/2} \) is the plasma frequency. Eq. (5) and (6) together are called reduced Maxwell-Duffing equations. For finding propagating solutions, one defines

\[
\eta = x - \tau/\alpha,
\]

where \( \alpha \) is related to the velocity of the pulse. Eq. (6) can be integrated with respect to the single variable \( \eta \) to yield

\[
\tilde{e}(t, x) = \alpha q(t, x) + C,
\]

where \( C \) is a constant, which signifies the background electric field, when electron amplitude \( q \) goes to zero. Non-linear equation of motion in Eq. (5) then takes the form:

\[
\frac{d^2q}{d\eta^2} + (1 - \alpha)q + 2\mu q^3 = C.
\]

For a wave propagating in the right direction, \( \alpha > 1 \) and \( \mu > 0 \), which is considered below. The other propagation direction can be likewise studied.

\[
A^3g + 2D(AD - B) + Ae - C = 0,
\]

\[
3A^2Bg + AD(1 + 2\epsilon) + B(\epsilon - 1) - 3CD = 0,
\]

\[
3AB^2g + AD^2(\epsilon - 1) + BD(1 + 2\epsilon) - 3C\epsilon = 0,
\]

\[
B^3g + BD^2\epsilon - C\epsilon = 0.
\]

It should be pointed out that \( cn(\eta, 0) = \cos(\eta) \). One can see from the above FT (Eq. (12)) that \( AD = B \) implies only a constant or trivial solution and is not considered here. An \( (AD - B) \) factor can be taken out of
all the conditions by tacitly using the first consistency in Eq. (13). Other conditions were used to solve for the unknowns $A$, $B$ and $D$. The solution is expressed as

$$q(\eta) = \frac{A + B \cos(\eta)}{1 + D \cos(\eta)},$$

(17)

where, $A$, $B$ and $D$ respectively are

$$A = \pm \frac{(\epsilon + 2)}{\sqrt{3g(1-\epsilon)}}, \quad B = -\sqrt{\frac{-1(1+2\epsilon)}{6g}},$$

and

$$D = \pm \sqrt{\frac{-1(1+2\epsilon)}{2(1-\epsilon)}}.$$  

(18)

After solving the solution parameters, the source term or the constant electric field part can be determined from Eq. (13):

$$C = -\frac{(1+2\epsilon)}{3} \sqrt{\frac{1-\epsilon}{3g}}$$

(19)

As has been mentioned earlier, for wave propagation in the right direction, $\mu > 0$ and $\alpha > 1$, implying $g > 0$ and $\epsilon < 0$ respectively. The parameter values exhibited in Eq. (13) yield the domain of the solutions: $g > 0$ and $\epsilon < -1/2$, where $A$, $B$, $D$ are real and $C$ is a positive quantity. It is worth pointing out that all the solutions are non-singular in nature. This is because the magnitude of $D$ is less than unity. The solutions are depicted in Fig. 1(a) for $\mu = 0.5$ with $\alpha = 1.6$. The dotted line corresponds to the solution for positive value of $D$ and the solid line is for the negative one. The first plot is with a background, i.e., $A \neq 0$.

ii) The consistency conditions, (13) support solution for $A = 0$ if the source is non-zero. In this case, $\epsilon = -2$ and the solution is written as

$$q(\eta) = \pm \frac{1}{\sqrt{g}} \left(\frac{\cos(\eta)}{\sqrt{2} \pm \cos(\eta)}\right).$$

(20)

which is plotted in Fig. 1(b) for $\mu = 0.5$ with $\alpha = 3.0$.

iii) Although a wide class of localized pulse propagation is analyzed in (12) for different initial boundary conditions, for completeness we indicate the same for $m = 1$ in the fractional transform. This is expressed as

$$q(\eta) = \frac{A + B \text{sech}(\eta)}{1 + D \text{sech}(\eta)},$$

(21)

where

$$A = \pm \sqrt{-\frac{(1+\epsilon)}{3g}}, \quad D = \pm \sqrt{-\frac{2(1+\epsilon)}{3g(1-2\epsilon)}}, \quad \text{and} \quad B = \pm \sqrt{\frac{2(\epsilon-2)^2}{3g(1-2\epsilon)}},$$

with $C^2 = -(1+\epsilon)(1-2\epsilon)^2/(27g)$. This solution exists for $g > 0$ and $\epsilon < -1$. For $\epsilon = 2$, the solution goes to the one, with $B = 0$. All the solutions are non-singular except for $B = 0$ and $D = -\sqrt{2}$, which implies a singular one, signifying self focusing effect.

iv) We now analyze the nature of the solutions in the absence of a polarizing background: $C = 0$. The resultant dynamical equation is the real part of the well known non-linear Schrödinger equation. In this case, the periodic solution for the electron amplitude is of quadratic fractional type ($\delta = 2$): $A = \sqrt{-8/g}$, $B = 0$, $D = -2$ and $\epsilon = 4$. The solution is singular one, implying an instability in the electron amplitude or the self focussing of the electric field. This happens for any value of $\mu < 0$ and for $\alpha = -3$.

v) In addition to the above mentioned mono-frequency solutions, there are other conoidal wave solutions. As an example, when $m = 1/2$ the consistency conditions are

$$A^2 + D(AD - B) + Ae - C = 0,$$

$$3A^2Bg + (2AD + B)e - 3CD = 0,$$

$$3AB^2g + D(AD + 2B)e - 3CD^2 = 0,$$

$$B^3g + BD^2e + (AD - B) - CD^3 = 0.$$
These equations lead to,

\[ A = \sqrt{\frac{F}{6g^2}}, \quad B = -\frac{\epsilon}{g} \sqrt{\frac{F}{3g^2}}, \quad D = \sqrt{\frac{F}{2g}}, \]

and

\[ C = \frac{1}{3\sqrt{6}} \sqrt{-\frac{\epsilon}{g} \left( \frac{4\epsilon^2 - F}{\sqrt{E}} \right)} , \]

where \( E = 9 + 2\epsilon^2 - 3\sqrt{9 + 4\epsilon^2} \) and \( F = \sqrt{9 + 4\epsilon^2} - 3 \). \( E \) and \( F \) are positive for all \( \epsilon \neq 0 \). These conoidal waves exist in the domain \( \epsilon < 0 \) and \( g > 0 \). Solutions are pictured in Fig. 2 where the variations of the electron amplitude with \( g \) and \( \epsilon \) are displayed. As \( \mu \) increases the amplitudes diminish for a fixed value of \( \alpha = 6 \) (Fig. 2(a)). Fig. 2(b) shows the nature of the solutions with \( \alpha \) for certain value of \( \mu = 2.5 \).

In conclusion, novel periodic solutions have been obtained for the reduced Maxwell-Duffing model, which are unidirectional, sinusoidal and of mono-frequency character. These solutions complement the localized soliton solutions known in the literature and occur only when a back ground electric field is present. General cnoidal wave solutions are found both with and without back ground fields. It will be interesting to study the nature of the waves when Maxwell equation is modified by a non-local dispersion term [14]. Polarization properties of the propagating modes should also be investigated in this non-linear medium [15], as also the nature of the waves when slowly varying amplitude approximation is not valid [16].

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