Challenges of Matrix Models

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ABSTRACT

Brief review of concepts and unsolved problems in the theory of matrix models.
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1 Introduction

Matrix models appear again and again at the front line of theoretical physics, and every new generation of scientists discovers something new in this seemingly simple subject. It is getting more and more obvious that matrix models capture the very essence of general quantum field theory and provide the crucial representative example for the string theory [1]: matrix models play for string theory the role, which harmonic oscillator plays for quantum mechanics. Essential difference is, however, that we have a nearly exhaustive understanding of harmonic oscillator, while nothing like complete description of matrix models is yet available. Probably, the time is coming to begin a systematic analysis with the goal of building up a theory of matrix model partition functions as the first special functions of string theory [2]. Various applications of matrix model techniques should use these functions as building blocks for formulation of their results, thus separating the physical content of different applications from common mathematical formalism. The goal can be to build up an analogue of the powerful free-field formalism, developed in 1980’s, which allowed to reduce many problems in perturbative string theory [3] (and related issues in representation theory of Kac-Moody algebras, 2d conformal field theory and finite-zone solutions in integrable-systems theory) to almost classical set of special functions: Riemann’s theta-functions [4, 5], associated with Riemann surfaces. The problem is that matrix model partition functions are non-trivial generalization of Riemann’s theta-functions, which were not fully investigated in XIX-th century; therefore the theory should involve essential new ideas, both in physics and in mathematics. This is the reason why the progress in this field – actually known under the nick-name of non-perturbative string theory – is considerably slower. More and more people begin to realize that generic problems of non-perturbative string theory, i.e. the general theory of phase transitions, have a lot in common, and these most interesting common properties are clearly represented at the simplest level of matrix models. Complications, introduced by sophisticated context of particular applications can and should be separated from important content, captured at the matrix model level.

In development of matrix model theory one can distinguish between several stages. In fact, these stages are typical for every chapter of the string theory, and they characterize not so much the history, but the kind of questions that are posed and the level of abstraction in analysis of these questions. Whenever a new problem arises (or is re-addressed again), it is quite useful to understand its place in this hierarchy.

2 Classical period: introduction of models

During the classical period the main task was to study concrete phenomena, involving matrix models. Since in these notes we are not going to discuss applications, we just mention a few crucial theoretical methods, developed at this stage.

The most important idea was to reduce the $N^2$-fold matrix integrals like

$$Z(t|N) = \frac{1}{\text{Vol}(U(N))} \int_{N \times N} d\Phi \exp \left( \sum_k t_k \text{Tr} \Phi^k \right)$$

(1)

to $N$-fold integrals over eigenvalues $\phi_i$ of matrix-valued fields $\Phi = U^\dagger \text{diag}(\phi_i) U$. This idea, closely related in general context to the problem of gauge invariance, is technically based on the possibility to explicitly integrate over "angular variables" in the simplest situations, of which (1) is a representative example. The single known exactly solvable generalization, which goes slightly beyond this example, involves the Itzykson-Zuber integral [7]

$$\int_{N \times N} [dU] \exp(\text{Tr}AU^\dagger BU),$$

(2)
which is used in construction of two important classes of matrix theories: generalized Kontsevich model [8, 9],

\[
Z_{GKM}(L|W) \sim \int_{m \times m} dM \exp[\text{tr} (W(M) + LM)]
\]

(3)

and even further generalized Kazakov-Migdal-Kontsevich model [10, 11],

\[
Z_{GKM^2}(\{L_\mu\}|\{W_\alpha\}) \sim \prod_{\mu,\alpha} \int_{m \times m} dM^{(\alpha)}[dU^{(\alpha \beta)}][dU^{(\alpha \mu)}].
\]

\[
\cdot \exp \left[ \text{tr} \left( W_\alpha(M_\alpha) + M_\alpha U_{\alpha \beta}^\dagger M_\beta U_{\alpha \beta} + L_\mu U_{\alpha \mu}^\dagger M_\alpha U_{\alpha \mu} \right) \right],
\]

(4)

defined for any graph with Hermitian fields \(M_\alpha\) and background fields \(L_\mu\) standing in the vertices and unitary fields \(U_{\alpha \beta}\) and \(U_{\alpha \mu}\) on the links. In more general situations (sometime quite important for applications, e.g. involving non-trivial multi-brane backgrounds), angular integration is highly non-trivial and remains an open problem. The simplest of such difficulties are echoed in the Gribov copies problem in Yang-Mills theory, while treatment of more sophisticated unitary-matrix integrals face problems, typical for adequate treatment of quantum gravity. The class of theories where gauge (angular) variables can be integrated out in effective way is referred to as eigenvalue models [6]. Most of further development concerns eigenvalue models, which until recently remained implicitly synonymous to matrix models.

The next idea of the classical period used the fact, that angular integration usually provides non-trivial, but very special measures on the space of eigenvalues, which is often made from Van-der-Monde determinants [12]. For example, (1) can be transformed into

\[
Z(t|N) = \prod_{i=1}^{N} d\phi_i \exp \left( \sum_k t_k \phi_i^2 \right) \prod_{i<j}^N (\phi_i - \phi_j)^2,
\]

(5)

where \(\prod_{i<j}^N (\phi_i - \phi_j)^2 = (\det_{i,j} \phi_i^{-1})^2\) can be also considered as discriminant of auxiliary polynomial \(P_N(z) = \prod_{i=1}^{N} (z - \phi_i)\). Occurrence of determinants suggested two alternative technical methods of investigation of eigenvalue models: technique of orthogonal polynomials [13] and free-fermion representation [13, 14]. Occurrence of discriminants was not fully exploited yet, though it opens a set of interesting possibilities, both technically and conceptually.

The last idea of the classical period which needs to be mentioned, is the idea of continuum limit, when the matrix size \(N\) tends to infinity. In fact there are infinitely many different continuum limits – relevant for different particular applications, – and only very few have been analyzed so far. However, from the very beginning the main fact was broadly realized: continuum limits of matrix models possess description in terms of Riemann surfaces [15, 16]. Today we know that occurrence of spectral surfaces is a general phenomenon, and they occur already at finite values of \(N\) [2].

## 3 First stringy period: generalizations and hunt for structures

The first stringy period is characterized by the change of interests: from tools to theory. Instead of studying various applications and developing adequate technical methods to answer the problems, posed by these applications, attention gets concentrated on the search and understanding of internal structures. Instead of "study" the main slogan becomes "deform and generalize" – this is the standard string theory method of revealing hidden structures. It is at this stage that the three main inter-related structures were discovered behind eigenvalue matrix models: rich Ward identities, integrability and CFT representations.

### 3.1 Ward identities

Occurrence of Ward identities is the pertinent property of every integral (and thus of quantum mechanics and all its generalizations, like quantum field and string theory): they reflect invariance of the integral under the change of integration variable – an archetypical example of auxiliary field. However, this obvious hidden symmetry manifests itself in a rather sophisticated manner: as relation between various correlators in the theory. String theory normally deals with partition functions: the generating functions of all the correlators, summed up with the coefficients like \(t_k\) in (1), which have the meaning of extra coupling constants, and can be considered as providing the deformation of the bare action. This formalism allows to treat Ward identities as equations for partition function, because the change of integration variables can be compensated by the change of the
coupling constants, if there are many enough [6]. In particular case of the integral (1) the equations are known as Virasoro constraints [17] or loop equations [15],

$$\hat{L}_-(z)Z(t|N) = 0,$$

(6)

$$\hat{L}_-(z) = \sum_{m \geq -1} z^{-m-2}\hat{L}_m = \mathcal{P}_- \left( \sum_{m = -\infty}^{\infty} z^{-m-2}\hat{L}_m \right) = \mathcal{P}_- \left( \frac{(\partial \hat{\phi}(z))^2}{2} \right)$$

$$\hat{\phi}(z) = \sum_{k \geq 0} \left( t_k z^k - \frac{1}{2kz^k} \frac{\partial}{\partial t_k} \right)$$

For the integral (3) the Ward identities can be written in two forms: either as a Gross-Newman equation [18],

$$(W' (\partial / \partial L_{tr}) + L) Z_{GKM}(L|W) = 0,$$  (7)

which generalizes a similar equation [6] for the Itzykson-Zuber integral (2), or as a set of peculiar $W_n$-constraints (where $n = \text{deg } W'(z)$) [19]. A particular case of these equations in the case of $n = 2$ is continuum Virasoro constraint [20, 21],

$$\hat{\mathcal{L}}_{2m} Z_{GKM} \left( L \left| \frac{1}{3} M^3 \right. \right) = 0, \quad m \geq -1,$$

(8)

$$\hat{\mathcal{L}}_{2m} = \frac{1}{2} \sum_{\text{odd } k = 1}^{2m} k t_k \frac{\partial}{\partial t_{k+2m}} + \frac{1}{4} \sum_{\text{odd } k = 1}^{2m-1} \frac{\partial^2}{\partial t_k \partial t_{2m-k}} + \frac{1}{16} \delta_{m,0} + \frac{1}{4} t^2 \frac{\partial}{\partial t_{2m-1}}$$

where $t_k = r_k + \frac{1}{4} \text{tr } L^{-k/n}$ and $r_k = \frac{n}{k(n-k)} \text{res } (W'(z))^{1-k/n} dz = -\frac{2}{3} \delta_{k,3}$. See [6] for detailed discussion of Ward identities for eigenvalue models and [22] for more technicalities, related to generalized Kontsevich model.

### 3.2 Integrability and RG evolution

Integrability of matrix models means that partition functions satisfy not only linear equations (Ward identities), but also bilinear Hirota-type equations,

$$\Delta(Z \otimes Z) = 0.$$  

Technically, the proofs rely upon determinant representations of eigenvalue models, which are, in turn, immediate corollaries of determinant structures of integration measures, see [6] and references therein. However, the true meaning of integrability remains obscure. Several inter-related ideas should be somehow unified to clarify the issue.

First, integrability should express the fact that the system of Ward identities is rich enough: enough to specify the partition function almost unambiguously – up to some easily controllable degrees of freedom, a sort of zero modes with some clear cohomological interpretation. See [23] for the relevant notions of strong and week completeness.

Second, bilinear equations normally involve the matrix size $N$ in non-trivial way: for example, the simplest bilinear equation for partition function (1) – the lowest term of the Toda chain hierarchy – states [24] that

$$Z_N \frac{\partial^2 Z_N}{\partial t_1^2} - \left( \frac{\partial Z_N}{\partial t_1} \right)^2 = Z_{N+1} Z_{N-1}$$

(9)

This means that bilinear equations involve not only variations of the coupling constant $t_k$, but also those of conjugate variables, like $N$. However, in (1) only $N$ – a conjugate variable for $t_0$ (in the sense that $\partial \log Z_N / \partial t_0 = N$) – is present, while there is nothing like conjugate parameters for all other $t$'s. This can suggest that Hirota equations for such restricted partition function are too special to reveal their general structure. In particular, (9) holds only for the simplest phase of the model (1), other phases belong to multi-component generalizations of Toda chain hierarchy.

Third, while coupling constants parameterize the bare action, i.e. the weight of summation over paths in functional integral, the conjugate variables should rather characterize the integration domain (range of integration, boundary conditions, the target space – whatever formalism one prefers to use in introducing them). In this sense conjugate variables are intimately related to generalized renormalization group flows, which are supposed to do exactly the same: describe the dependence of functional integral on the integration domain.
Relation between integrability and renormalization group is one of the main open problems in modern theoretical physics [1, 6, 25]. To get fully related, both these concepts should be considerably modified. To become a pertinent feature of all partition functions, integrability should not be restricted to ordinary \( \tau \)-functions [26], associated with free fermions and single-loop Kac-Moody algebras at level \( k = 1 \). Indeed, \( \tau \)-functions and bilinear Hirota-like equations can be defined for arbitrary Kac-Moody and even more general Lie algebras and quantum groups [27]. However, the theory of such \( \tau \)-functions is not reducible to that of Plucker (free-fermion) determinants and requires the full-scale application of the free-field formalism [3] and more sophisticated determinant formulas. Expressions of this type are expected to arise in the study of unitary matrix integrals, but only first steps are being done in this direction [28, 29].

Renormalization group theory also requires considerable generalizations, of which we put especially emphasize on two. First, as already mentioned, it should allow arbitrary variations of integration domain, not just a one-parameter cut-off procedure: renormalization group should study the changes of the shape of integration domain, not just of its volume. This seems to be already a widely accepted generalization. Second, renormalization group theory should be made applicable to the study of fractal structures, which often reproduce themselves (exactly or approximately) at different scales. This means that the theory should allow non-trivial, periodic and perhaps even chaotic, renormalization group flows [30, 31], and this should not contradict the obvious uni-directional nature of these flows, generated by integrating out degrees of freedom. This entropy-like feature of renormalization group is usually expressed through the \( c \)-theorem [32], but in the case of systems with infinitely many degrees of freedom the \( c \)-function can easily have angular nature, and monotonic decrease of \( c \) does not contradict the existence of periodic motions. Besides the obvious example of self-similar fractals, today we already have some field-theory examples of such behaviour [33] – but only at the level of flows. It is not so easy to provide examples of partition functions (not just flows) with such properties, and the reason for this is simple: we usually consider as solvable the systems where answers can be expressed through conventional special functions, and these – almost by definition – have oversimplified branching structure. We do not possess any language to describe more general – and more realistic situations. At the same time such language should exist, because what seems to be sophisticated (fractal and/or chaotic) phase structure is in fact a highly ordered and pure algebraic pattern [34], not so much different from the algebraic-geometry background under the theory of conventional special functions. The jump between order (integrability) and chaos (infinite-tree structure of bifurcations or phase transitions) can be not so big, and most probably the careful study of partition functions already at the matrix-model level will reveal the deep inter-relation (duality) between the two concepts.

This duality should play a big role in landscape theory, which studies the distributions of various algebro-geometric quantities on moduli spaces (either of coupling constants or on the dual space of different branches) and their interplay with renormalization group flows. The crucial hidden double-loop structure, which is supposed to become the basis of such theory [35], should be seen already at the level of matrix models. For old and new attempts to apply – the yet undeveloped – landscape theory to phenomenology see [1] and [36] respectively.

### 3.3 Relation to conformal theory

The third structure behind eigenvalue matrix models is that of 2\( d \) conformal field theories: partition functions of matrix models coincide with certain correlators in conformal models. This fact is of course related with the free-fermion and KP \( \tau \)-function representations of simplest partition functions, but it was quickly realized that this is simple: we usually consider as solvable the systems where answers can be expressed through conventional special functions, and these – almost by definition – have oversimplified branching structure. We do not possess any language to describe more general – and more realistic situations. At the same time such language should exist, because what seems to be sophisticated (fractal and/or chaotic) phase structure is in fact a highly ordered and pure algebraic pattern [34], not so much different from the algebraic-geometry background under the theory of conventional special functions. The jump between order (integrability) and chaos (infinite-tree structure of bifurcations or phase transitions) can be not so big, and most probably the careful study of partition functions already at the matrix-model level will reveal the deep inter-relation (duality) between the two concepts.

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solution of continuum Virasoro constraint (8), but this problem appears surprisingly complicated and still remains open, see [39, 40] for important but still insufficient steps towards its solution.

4 Transcendental period: absolutization of structures

The transcendental or second stringy period is characterized by absolutization of structures, revealed at the previous stage. This means that the logic is inverted: now the structures are given, and we look for generic objects, possessing such structures – expecting in advance that some of these objects can appear different from original matrix integrals. Today we are mostly at this stage, at least in the theory of matrix models, and only the first careful steps are being done in attempts to understand the hidden meaning of the structures that we observed in applications – and continue to observe again and again in new physical systems when they come to our attention.

Discussion of generic CFT correlators as well as of generic $\tau$-functions is – old and important – but still not a very constructive problem. It lies at the very core of landscape theory, but existing theoretical methods are too weak to address the problem directly. The structure which can be rather effectively analyzed by available tools or by their straightforward modifications is that of Ward identities. Working on this structure one can also hope to develop stronger methods, applicable to analysis of the other two structures. To mention just one open question, illustrating the degree of ignorance in this field, it remains unclear to what extent the Virasoro constraints (6) and (8) per se, without explicit integral representations like (1), imply integrable structure of partition function.

4.1 Ward identities

Defining partition function as solution to Ward identities is a typical $D$-module-style definition, i.e. basically a problem from linear algebra, since Ward identities are linear equations (this is what makes examination of this structure much simpler than integrable one, related to quadratic equations). Since we do not expect any mysteries in linear algebra (or, better to say, we think that we already know all of them), the theory building is straightforward, but by no means trivial: we know what to search for, but exhaustive and explicit solution can be quite tedious. We are not yet at the stage when solution of a linear-algebra problem, especially infinite-dimensional, and all its properties can be predicted a priori.

Normally, the set of questions one asks about the solution consists of several groups.

4.1.1 Domain of definition

First of all, one should decide what class of functions one wants the solution to belong to. Basically, there are two principal alternatives: formal series and globally defined functions. However, even if one chooses formal series, it is still necessary to specify, where singularities are allowed to occur. The simplest alternative is between series in positive powers of a variable (singularities are allowed at infinity of the Riemann sphere) and in all integer powers (singularities allowed at infinity and at zero). However, there are many more possibilities: singularities can be allowed at other points (then negative powers of $[z - a]$ can be allowed), the underlying bare Riemann surface need not be a sphere (then certain fractional powers of $[z - a]$ are allowed) etc. More than this, different variables can have different kinds of singularities and we have infinitely many such variables. In different applications different requirements are imposed, and one and the same quantity – unique partition function = generic solution to Ward identities – can show up in absolutely different way: different branches are relevant for different applications. What can look like a singularity from the point of view of particular application, can be nothing but a branching point or phase transition to another branch, which acquires a natural interpretation in terms of an absolutely different application. Once again this emphasizes that the problem of partition functions – even one particular partition function, one particular matrix model $D$-module,– is almost undistinguishable from the central problem of the string theory [1], which can be formulated as a search of unique universal partition function (a universal object of quantum field theory), of which all other thinkable partition functions are particular branches and sub-families. Returning closer to the Earth, even speaking about formal series one – explicitly or implicitly – imposes certain requirements on allowed singularities of solutions, which often do not follow from the Ward identities themselves. Actually, the "rich" Ward identities, that we spoke about in the previous section, fix everything but the behaviour at singularities, they are rich enough to fully constrain the dependence on the choice of the action (on coupling constants), but not rich enough to fix conjugate dependencies, e.g. boundary conditions at possible singularities: in certain sense they fix exactly one half of possible freedom. This is a typical thing for the equation on a wave function in quantum mechanics to do (constrain $q$-dependence but ignore $p$, or in other words, there are many solutions of the same Shroedinger equation, differing, say, by the values of energy): this observation is a starting point on the road, leading to
identification of Ward identities with Shroedinger-type equation and the partition function with a (string-field theory) wave function.

Convenient way to speak about emerging freedom is in terms of "globally defined" functions: to fix it one can ask to solve Ward identities for functions on Riemann sphere or on torus or anywhere else. The problem is that in most interesting cases – and matrix model partition functions are not an exception – singularities are unavoidable. Moreover, small relaxations of compactness, like "functions with simple poles", which were enough for the formalism of free fields (and thus in dealing with arbitrary 2d conformal models), now are not sufficient: τ-functions are different from conformal blocks because they have essential singularities, even if we take those defined on punctured Riemann surfaces of finite genus. Actually, things are even worse: interesting partition functions require this genus to be infinite, if at all defined. Therefore this language – though widely used – by itself does not provide unambiguous classification of solutions.

4.1.2 Generating functions

The next question – after the set of allowed functions is somehow specified – is how to represent the answer. There are few chances that the full answer will be any known special function: as already stated, the most important thing is that matrix models produce new special functions. What is important, however, these new functions are not too new, they are just one step forward more involved than the known special functions, namely the Riemann’s theta-functions, which occupy the previous level of complexity. We expect that all kinds of simplifications of matrix model partition functions reduce them to quantities, already expressible through Riemann’s theta functions. What can be these simplifications? Instead of considering a generating function of all correlators, one can select some one-parametric family, and such reduced generating function can be a candidate. One can take one or another limit (say, one or another continuum limit: naive, double-scaling, fixed genus, ...). One can take a ratio of different branches (a kind of monodromy matrix) or some more sophisticated combination of those.

4.1.3 Different realizations

As usual for a $D$-module, one can look for solutions of linear equations in different forms.

Integral formulas like (1) are particular examples of one possible – integral – representation (or realization, to avoid confusion with group-theory representations) of solutions. For integral realizations the Ward identities play a role of Picard-Fucks equations: solutions are periods of the form which is converted into a full derivative by application of the corresponding operator (which generates Ward identity). Of course, in such situation one can integrate along any closed contour and different non-homological contours give rise to different solutions. Therefore, when one calls eq.(1) Hermitian one matrix model, “Hermitian” actually refers to the measure, $d\Phi = \prod_{i,j=1}^{N} d\Phi_{ij}$ (dictated, in its turn, by the norm $|d\Phi|^2 = \text{Tr} \ |\delta\Phi|^2$), but not to the integration contour: when one defines partition function as solution of the Virasoro constraints, there is no reason to integrate in (1) over the contour $\Phi_{ij} = \Phi_{ji}$, associated with Hermitian matrices. Moreover, since $D$-module is defined by linear equations, the superposition principle is applicable, and a sum of any two solutions is again a solution. If matrix integrals like (1) with different $N$ satisfy the same Ward identities like Virasoro constraints (6), then linear combinations of integrals with different $N$ still are solutions. This means that the matrix size $N$ can be also interpreted as characteristic of integration contour, moreover, this characteristic is not obligatory a positive integer, but can be also negative, rational and even complex-valued.

Differences between possible integral realizations of one and the same partition function are not exhausted by freedom in the choice of integration contours, even with extended interpretation of the term "contour". Partition function can be represented by absolutely different classes of matrix models. The simplest important example is provided by the model (1): it can be represented not only by (1), but also by a Gaussian integral [41] from Kontsevich family (3) with $t_k = \frac{1}{k} \text{tr} L^{-k}$ [42, 6]:

$$Z(t|N) \sim Z_{GKM} \left( L \left|\frac{1}{2} M^2 + N \log M \right| \sim \int_{m \times m} dM (\det M)^N \exp \left[ \text{tr} \left( \frac{1}{2} M^2 + LM \right) \right] \right)$$

In this representation analytic continuation to non-integer values of $N$ is even more straightforward. Again, linear combinations of solutions with different values of $N$, are still solutions to the Virasoro constraints (6), and all of them should be considered as particular branches of the 1-matrix model partition function. This provides additional information about the problem of conjugate variables: $N$ itself can serve as one of them, or the coefficients of above-mentioned linear combinations can – changing the former for the latter is a kind of Fourier transform in the space of conjugate variables.

Possibility to represent one and the same partition function by members of two different matrix model families, by (1) and by (11), is a typical example of duality between different families of quantum field theories. One more duality follows from existence of CFT representations for the same partition function, it relates these
matrix models to 2d free scalar theory and – through one more duality – to free fermions. These dualities illustrate the fact that the nature and the number of integration variables can differ in the most radical way, still partition functions coincide: in absolutely different theories one can find families of identical correlators. Once again, this is an illustration of the general stringy idea: *everything is the same*, one can fully describe one theory (phase) in terms of another. Matrix models provide a nice framework for testing this principle and can help to transform it into constructive and reliable procedure.

### 4.1.4 Limiting procedures and asymptotics

Different realizations of partition functions are adequate or at least convenient for descriptions of different phases of the system (branches of partition function), or at least different classes of phases. As usual, phases become pronouncedly different in particular asymptotics. Systematic approach implies that all kinds of asymptotics should be investigated. In particular, in the case of matrix models, all kinds of continuum limits should be examined, not only naive or 't Hooft’s or double-scaling. Obviously, all kinds of multi-cut solutions and all kinds of multi-scaling limits require attention, but this is only the beginning: by no means all possible asymptotics are exhausted in this way, and also there is a lot of interesting beyond continuum limits. Unfortunately, not too much is known about this variety of limits and nothing like classification of asymptotics is available (this was a trivial thing in the case of one variable and becomes a pretty sophisticated issue in the case of infinitely many variables).

There are two different possibilities to perform limiting procedures: at the level of defining equations (linear Ward identities or bilinear Hirota-style equations) and at the level of particular realizations. Of course, one and the same limit can look very different in different, though equivalent, representations. Because the subject is under-investigated, the set of known realizations is poor and many potentially interesting asymptotics either do not attract attention or are difficult to handle. We mention just two old problems, which still remain unresolved.

In the class of generalized Kontsevich models (3) and (4) one naturally distinguishes [28, 11] between **Kontsevich phase** – the asymptotics of large $L$ – and the **characters phase** – the asymptotics of small $L$. In these phases the "time-variables" $t_k = r_k + \frac{1}{r_k} \text{tr} L^{-k/n}$ with $n = \deg W(z)$ are respectively close to $r_k = \frac{n}{k(n-k)} \text{res} (W'(z))^{-1-k/n} dz$ and to infinity. Kontsevich phase corresponds, at least through the relation (11), to perturbative phase of the models like (1), while the characters phase is the strong coupling limit of these models. In this sense (11) is an example of $S$-duality, interchanging weak and strong coupling phases. Actually, these phases are closely related to the strong and weak coupling limits of Yang-Mills theory and thus to confinement problem. However, this problem was never exhaustively analyzed even at the level of matrix models – despite the existence of Kazakov-Migdal-Kontsevich family (4), which – in variance with realizations like (1) – provides an effective tool for studying both asymptotics and interpolate between them (for these models Kontsevich phase is the WKB asymptotics, while the characters phase is perturbative limit). A systematic analysis, like suggested in [2, 43, 44] for (1) continues to wait for its time for Kontsevich and for more general unitary matrix models.

Another abounded problem would be just a small paragraph in this systematic analysis, still it is quite important by itself. This is the problem of "double-scaling" limit [45] of (1), which is one of the simplest non-naive large-$N$ asymptotics of $Z(t|N)$, when $N \to \infty$ together with certain special adjustment of $t$-variables. For investigation of this limit one can make use of various techniques, of which the most important are two: taking limit of loop equations (Virasoro constraints) (6) [46] and exploiting the identity (11) and taking limit within the family of Kontsevich models [42, 6]. In these ways one can argue that

$$\lim_{d.s.} Z(t|N) \sim \lim_{d.s.} Z_{\text{GKM}} \left( L \frac{1}{2} M^2 + N \log M \right) = Z_{\text{GKM}}^{2} \left( \tilde{L} \frac{1}{3} M^3 \right)$$  \hspace{1cm} (12)

where at the l.h.s. $t_{2k+1} = \frac{1}{2k+1} \text{tr} L^{-2k-1} = 0$ and $t_{2k} = \frac{1}{2k} \text{tr} L^{-2k}$, and the quantity at the r.h.s. (expressed through $\tilde{t}_{2k+1} = \frac{1}{2k+1} \text{tr} \tilde{L}^{-k-1/2}$) has a special name of **Kontsevich τ-function** $\tau_{K}(\tilde{t})$ and is an increasingly important matrix-model and string-theory special function. However, even reliable derivation of the principally important result (12) and exact relation between $t$, $L$ and $\tilde{t}$, $\tilde{L}$ is not available, nothing to say about corrections to this formula (describing what happens when one *approaches* the limit) or its generalizations. Of special importance among such generalizations are the limits, when $Z(t|N) \to \tau_{K}^{2n}(\tilde{t})$ with $n > 1$, because of their obvious relation to Givental-style-decomposition formulas [47, 2, 40] and their potential role in applications. It goes without saying that various non-perturbative asymptotics of $\tau_{K}$ should be investigated: this is the next natural task after such program is put on track for the basic special function $Z(t|N)$. 

7
4.2 Integrability

Integrability is believed [1, 6] to be the pertinent property of quantum partition functions, reflecting the fact that they are results of integration, which erases most of initial information (roughly speaking, almost all forms are exact), leaving only cohomological variables. Therefore integrability is intimately related [48] to topological-cohomological-holographic-stringy theories and all these kinds of ideas should be considered together. However, despite many efforts, spent on investigation of particular models, we are still far from formulating clear concepts. Partly this is because examples are dictated more by applications than by the internal logic of the theory, and they provide somewhat chaotic flow of information. There are already several things that we seem to know, but their complete list and relative significance remain obscure.

4.2.1 Hirota equations

The first in the list are bilinear Hirota-style equations, which establish contact with Lie algebra theory [27]. Bilinearity is usually related to the properties of comultiplication and to the basic relation \( \Delta(g) = g \otimes g \) for group elements [49, 50]. Connes-Kreimer theory [50]-[52] implies, that quantum partition functions respect these properties, even if they are defined through Feynman diagrams, without explicit reference to functional integrals, and identify the underlying algebraic structure with (generalized) renormalization group. As already mentioned, deeper understanding of relation between integrability and renormalization groups remains among the main open problems of the theory.

4.2.2 Moduli space of solutions

The second big problem is understanding, classification and control over the freedom in solving Hirota equations. We mentioned already, that these bilinear equations restrict the dependence on coupling constants \( t_k \), and the freedom remains in the conjugate/dual variables, like zero-modes, or boundary conditions, or choices of vacua, or holographic data, or whatever else name and analogy one prefers to use in connection with one’s favorite application. The problem is to find not just a name, but adequate language to speak about this data. Suggestive example is provided by original Hirota equations [53], associated with KP hierarchy: there the freedom (or at least a part of it) can be interpreted in terms of Riemann surfaces. Different KP \( \tau \)-functions, at least from the class of the finite-zone solutions, differ by the choice of Krichever data \( \Sigma \) [54]: a complex curve, a point on it and holomorphic coordinates in the vicinity of a point. Given this data, the KP \( \tau \)-function is fixed to be

\[
\tau(t|\Sigma) = e^{t Q(t)} \theta \left( \sum_k \bar{B}_k t_k | T_{ij} \right),
\]

where \( T_{ij} \) is the period matrix of the Riemann surface, \( \bar{B}_k = \oint \Omega_k \) and \( \Omega_k(\zeta) = (\zeta^{-(k+1)} + O(1)) \, d\zeta \) are meromorphic 1-forms with vanishing \( A \)-periods. If ansatz (13) is substituted into Hirota equations, they become equivalent to Shottky condition for the period matrix \( T_{ij} \) [55], i.e. require \( \theta(\bar{z}|T_{ij}) \) to be Riemannian theta function, not just arbitrary Abelian one. Rather straightforwardly, the simplest – rational and solitonic – KP \( \tau \)-functions can be treated as particular cases of (13) with Riemann surface of genus zero. All remaining, non-finite-zone KP \( \tau \)-functions are more sophisticated and can be considered as associated with infinite-genus Riemann surfaces, though exact meaning is not yet ascribed to these words. Clearly, there are many different kinds of ”infinite-genus” \( \tau \)-functions. Even the principal example of Kontsevich \( \tau \)-function \( \tau_K(t) \), which belongs to this infinite-genus family, is not well understood and investigated.

Anyhow, example (13) remains the main reference point in the theory of integrable systems. It underlies the belief that the main language for description of dual/conjugate variables to times (coupling constants) should be that of Hodge theory, different \( \tau \)-functions should be parameterized by data, encoded in spectral surfaces, not obligatory 2-dimensional. Thus integrability is believed to be a unification of group theory and algebraic geometry, with certain combinatorial flavor, already coming from the studies of adjacent and clearly related subjects [34, 50, 51]. At the same time, even despite these ambitious perspectives, too much remains obscure in the general structure of integrability theory, see [56]-[71] for various important constituents which it should finally unify. All this makes natural the fastly increasing attention to integrability concepts, but decisive conceptual breakthrough is still to come.

4.2.3 Seiberg-Witten theory, Whitham integrability and WDVV equations

The third problem is the search for an adequate language, which can help to merge Hodge theory with Lie algebra structures. There are various approaches. The most obvious is to study cohomological (topological) field and string theories [72] and find traces of integrability there. This program is very successful in dealing with various examples, but general concepts appear very hard to extract from it. Among such concepts definitely are the
WDVV equations [73, 74] and Batalin-Vilkovisky formalism [75, 76]. Alternative approach, from integrability side usually starts from (13), directly or implicitly, and notes that theta-function in that formula is oscillating, therefore one can distinguish between fast (oscillating) and slow variables. Then, as usual, one can take an average over fast variables and consider an effective theory induced on the space of slow variables. Since, according to (13), the fast variables are exactly the times \(t_k\), the slow variables of the emerging effective theory should be exactly the dual variables, that we are interested in. This idea, known as Whitham theory, has obvious contacts with those of renormalization and renormalization group, and not-surprisingly appears to have immediate applications. While absolutely un-developed in general, it has a solid very well formulated chapter: the Krichever-Hitchin-Seiberg-Witten theory, describing explicit construction of peculiar special functions – Seiberg-Witten prepotentials – from the analogue of Krichever data, associated with peculiar Seiberg-Witten families of spectral surfaces (the best known examples are Calabi-Yau spaces and special families of Riemann surfaces). The prepotential \(F(S_k)\) is defined [77]-[79] by two equations:

\[
S_k = \int_{A_k} \Omega_{SW},
\]

\[
\frac{\partial F}{\partial S_k} = \int_{B_k} \Omega_{SW}, \tag{14}
\]

where \(\Omega_{SW}(z)\) is a linear combination of \(\Omega_k(z)\) in (13) and holomorphic 1-differentials \(\omega(z)\) (with non-vanishing \(A\)-periods), possessing a special-geometry property that \(\delta \Omega_{SW}\) is a symplectic form on the ”sheaf” of spectral surfaces over the moduli space. In practice this condition states that \(\delta \Omega_{SW}\) is holomorphic (has no poles) if the variation is taken along the moduli space of Seiberg-Witten curves. Canonical system of contours \(\{A_k, B_k\}\) in (14) consists of all non-contractable cycles (including those around resolved singularities of \(\Omega_{k}\)) and their duals, see [79] for details. The variables \(S_i\) define a sort of flat coordinates on the moduli space.

It appears that the so defined prepotential always satisfies the (appropriately generalized) WDVV equations [74], namely for matrices \(\left(\hat{F}_{ij}\right)_{jk} = \frac{\partial^3 F}{\partial S_i \partial S_j \partial S_k}\) made out of prepotential’s third derivatives,

\[
\hat{F}_{ij}\hat{F}_{jk}^{-1}\hat{F}_{ki} = \hat{F}_{ik}\hat{F}_{kj}^{-1}\hat{F}_{ji} \tag{15}
\]

for any triple \((i, j, k)\). Moreover, this system of equations for infinitely-large matrices often survives reduction to finite-dimensional matrices when only \(S\)-variables, associated with ordinary non-contractable contours, are taken into account. This reasons why such reductions exist are far from obvious, and so is the entire theory of WDVV equations. Technically, it relies upon residue formulas for moduli derivatives of period matrices, but conceptually it involves reduction theory of non-trivial algebra of forms and is still very unsatisfactory. Additional puzzle is that reduced WDVV equations are violated, at least when applied naively, in the case of elliptic Calogero system (see the third paper in ref.[74]). Other elliptic examples [80, 71] were not yet analyzed in the context of WDVV equations.

At the same time the theory of WDVV equations can appear even more important than it seems. Today this set of equations is the only candidate for a definition of a prepotential in internal terms, which does not refer to explicit construction procedure. Thus WDVV equations can probably be promoted to the status of definition of the prepotential and acquire the status, similar to the one that Hirota equations have for \(\tau\)-functions. (Refering to its possible origin as a Whitham average of (13) – or, more, accurately, of the underlying Hirota equations, – prepotential is often called Whitham or quasiclassical \(\tau\)-function, and the above-mentioned problem is to find the internal definition of quasiclassical integrability – a very important theoretical challenge.) The main problem with WDVV equations from this point of view is that they describe only ”spherical prepotentials”: from the study of topological theories and related algebraic constructions [81] we know that there is a whole hierarchy of prepotentials, associated with the genus expansion of Yang-Mills and matrix models partition functions, and Seiberg-Witten prepotential is just the zeroth (genus-zero) term of this sequence. Appropriate generalization of WDVV equations for entire sequence is not yet found, for first steps in the cases of genus one and two see [82], and for relation to Batalin-Vilkovsky theory see [83] (to avoid possible confusion, these papers consider only WDVV equations with additional requirement of “unit metric”, which is natural in applications to quantum cohomologies [84] and naive topological models – to ”geometric prepotentials”, – but is not quite straightforward to relax).

### 4.3 CFT representations, Wick theorem and decomposition formulas

The theory of 2d conformal systems [85], once at the front-line of attention in string theory, was suddenly abandoned before an exhaustive theory was formulated, and many problems, including effective description of entire variety of conformal models and equivalencies (dualities) between them, are left unresolved. Partly, this happened because attempts to glue the entire subject together – in the framework of landscape theory [35] –
naturally caused a desire to interpolate between different conformal models, and this unavoidably embeds [32] the world of 2d conformal theories into that of 2d integrable systems – still a very badly understood subject, intimately related to the theory of quantum Kac-Moody algebras [86, 87, 88], which attracts much less attention than it deserves.

As already mentioned, there is no doubt that every class of matrix models have CFT representations, i.e. partition function of a given family of matrix models can be represented as a correlator of some operator in 2d conformal theory. The questions are: what is the way to build this operator – so far it is matter of art rather than a systematic procedure: what kind of information from the much bigger phase space of 2d free fields is lost (projected out) in transition to the matrix models (in particular, even models which are not dual themselves can still produce representations of one and the same matrix model partition function); how the data of matrix model is mapped into the data of conformal model etc. Today we do not know anything similar to answers to any of these questions.

The crucial property of free field (and thus of 2d conformal) theories is straightforward realization of the Wick theorem: all correlators are decomposed into multi-linear combinations of pair correlators (see [89] for a rather fresh discussion of the issue). In principle, for topological and integrable theories naive Wick theorem is not applicable: even after all possible simplifications there are contact terms (and prepotential $F(S)$ is almost never quadratic). In interacting theory transformation to angle-action variables which would explain to what kind of correlators the naive Wick theorem is applicable, is highly non-linear and probably not very useful.

A very important question is what substitutes the naive Wick theorem for generic partition functions. As we know, generic partition functions are not at all in general position in the space of all functions, they satisfy many constraints, like linear Ward identities, bilinear Hirota equations, highly non-linear WDVV equations. Thus we can expect that above question about Wick theorem can make sense. Of course, one can try to interpret as avatars of the Wick theorem all of above-mentioned relations on partition functions. Remarkably, there is also a much more profound candidate: decomposition formulas. It appears that partition functions can often be represented as poly-linear combinations of simpler building blocks, like Gaussian matrix models and Kontsevich $\tau$-functions $\tau_K$. So far this fact, which we call Givental-style decomposition, was explicitly formulated in restricted number of cases [47, 2, 40], but we believe that this is just the tip of the iceberg, and further work will confirm the universality and important role of decomposition phenomenon: a probable substitute of Wick theorem for generic partition functions.

5 Towards exhaustive theory of 1-matrix model: the Dijkgraaf-Vafa theory

Dijkgraaf-Vafa theory [90]-[104] deserves special attention, because for the first time an application was found, which requires understanding of a whole variety of phases of a single matrix model, namely the model (1). Despite it is still not the entire variety – only peculiar phases are considered interesting, where partition function is consistent with the genus expansion, – it is still a very big step forward, and it stimulated fast progress in matrix model theory. In this section we briefly characterize some directions of this progress As everywhere in these notes, we concentrate on pure theoretical issues and ignore the – somewhat very interesting – results in applications to Yang-Mills, quantum gravity and model building.

5.1 Genus expansion

Genus expansion, i.e. the t’Hooft’s $1/N$ expansion [105], attracts constant attention since the early studies of matrix models [15]. While it has clear meanings both in diagram technique (genus expansion for fat graphs) and in WKB approach to the integrals like (1), specification of partition functions possessing genus expansion among generic solutions of Ward identities [17] is not straightforward and looks somewhat artificial. As explained in [2, 44], it involves several steps.

First of all, by rescaling of all time-variables $t_k \rightarrow \frac{1}{g} t_k$ one introduces the new parameter $g$. It appears in Virasoro constraints (6) as a coefficient $g^2$ in front of the double-derivative terms. Second, solution of these Ward identities is looked for among rather special functions, such that

$$g^2 \log Z(t/g|N) = \sum_{p \geq 0} g^{2p} F^{(p)}(t|gN)$$

is a series in non-negative powers of $g^2$, provided the coefficients $F^{(p)}(t|gN)$ of this expansion depend on the t’Hooft’s coupling constant $S = gN$. Though consistent with (6), this is actually a tricky requirement. It implies that $Z$ itself is a series in all integer, positive and negative, powers of $g^2$, just coefficients in front of negative powers are strongly correlated. Moreover, there is no $N$ in Ward identities (6). $N$ can be introduced
as $S = gN = \partial(g^2 \log Z)/\partial t_0$, but this requirement breaks superposition principle: once it is imposed, the sum of two solutions is not a solution (this is nearly obvious: a sum of two exponentials is not an exponential). This means that genus expansion can be at best a property of the elements of a linear basis in the space of solutions, while generic element of this space does not respect it. In other words, genus expansion can be a property of particular integrals, like (1), but there is no way to impose this requirement on linear combinations of integrals with different $N$, which will still be solutions of the Virasoro constraints.

There are at least two ways to deal with this problem (the obvious possibility to ignore it, as is usually done, is not counted). First, one can look for the ways to define special basises – the problem is to define them in invariant way, by specifying their properties, not by explicit construction, like matrix integrals. Second, one can try to formulate genus expansion without explicit reference to $N$ – this road leads to introduction of **check operators** [44].

### 5.2 Gaussian and non-Gaussian partition functions

Requirement of genus expansion is not enough to specify the phase/branch of partition function completely, it regulates only dependence on the common scaling factor of all couplings $t_k$ and says nothing about their ratios. The next step [2] is to select combinations of $t$-variables which can appear in denominators. A way to do this is to make a shift, $t_k \rightarrow t_k - T_k$, and consider $W(\phi) = \sum_k T_k \phi^k$ as a **bare action** and $t_k \phi^k$ as perturbations. In other words, the branch

$$Z_W(t|N) = \frac{1}{\text{Vol}(U(N))} \int_{\mathbb{R}^{N \times N}} d\Phi \exp \left( -\text{Tr} W(\Phi) + \sum_k t_k \text{Tr} \Phi^k \right)$$

of partition function is defined as a formal series in non-negative powers of $t_k$, while $W(z)$, i.e. $T_k$’s, are allowed to appear in denominator. Once $W(z)$ is introduced, one immediately distinguishes between the Gaussian (with $W(\phi) = \frac{M}{2} \phi^2$) and non-Gaussian phases.

Next, it turns out that while the variable $N$ is naturally introduced in the Gaussian phase, in non-Gaussian case the situation is more sophisticated: there are naturally $n = \deg W'(z)$ parameters like $N$: the phase is still split into a $n$-parametric family of phases. Additional variables can be interpreted as numbers of eigenvalues concentrated near the $n$ extrema (maxima and minima) of the bare potential $W(\phi)$, a more reliable interpretation is as integration constants in solution of shifted (by $W(z)$) Virasoro constraints. See [2] for further details.

Gaussian partition function can be represented as

$$Z_G^{(M)}(t|S|g) = \exp \left( \frac{1}{g^2} \left( -ST_0 - \frac{M}{2} S^2 \right) \right) \exp \left( \sum_{p \geq 0} g^{2p-2} F_G^{[p]}(t|S) \right)$$

and information about the $t$-dependence of prepotentials $F_G^{[p]}(t|S)$ can be represented in terms of **multi-densities**, the generating functions of $m$-point correlators, which – for given $p$ and $m$ – are ordinary poly-differentials on the Riemann sphere.

Multi-densities can be defined in different ways (for different families of correlators) [2], one of the ways consistent with genus expansion and with the free-field representation (6) of Virasoro constraints makes use of the operator $\hat{\nabla}(z) = 2d\phi(z) = \sum_{k=0}^{\infty} \frac{dz}{y_G^k} d\phi^k$:  

$$\left[ \hat{\nabla}(z_1) \cdots \hat{\nabla}(z_m) \left( g^2 \log Z_W(t|\bar{S}|g) \right) \right] \big|_{t=0} = \rho^{(1|m)}(z_1, \ldots, z_m|\bar{S}|g) = \sum_{p \geq 0} g^{2p} \rho^{(p|m)}(z_1, \ldots, z_m|\bar{S})$$

The first *Gaussian* multi-densities are:

$$\rho_G^{(0|1)}(z) = \frac{z - y_G(z)}{2},$$

$$\rho_G^{(1|1)}(z) = \frac{\nu}{y_G} = -\frac{y_G^{\nu}}{4y_G},$$

$$\rho_G^{(2|1)}(z) = \frac{5}{16} \left( \frac{y_G^{\nu}}{y_G^2} \right)^2 - \frac{1}{8y_G} \partial^2 \left( \frac{y_G^{\nu}}{y_G} \right) - \frac{1}{8} \frac{y_G^{\nu}}{y_G},$$

$$\rho_G^{(0|2)}(z_1, z_2) = \frac{1}{2(z_1 - z_2)^2} \left( \frac{z_1 z_2 - 4\nu}{y_G(z_1) y_G(z_2)} - 1 \right) = -\frac{1}{2y_G(z_1)} \frac{\partial y_G(z_1) - y_G(z_2)}{z_1 - z_2} = -\frac{1}{2y_G(z_2)} \frac{\partial y_G(z_1) - y_G(z_2)}{z_1 - z_2}.$$
As explained in the previous section, this Givental-style formula is an example of general phenomenon, gener-

\[\rho_G^{(1,2)}(z_1, z_2) = \nu \frac{y_G}{y_G'(z_1)y_G(z_2)} \left( z_1 z_2 (5z_1^4 + 4z_1^3 z_2 + 3z_1^2 z_2^2 + 4z_1 z_2^3 + 5z_2^4) + 
+ 4\nu [z_1^4 - 13z_1 z_2(z_1^2 + z_2^2)] + 16\nu^2 (-z_1^2 + 13z_1 z_2 - z_2^2) + 320\nu^3 \right) = 
\frac{1}{y_G} \left( \left( \frac{4}{2y_G} y_G'' - \frac{1}{2y_G} \frac{1}{2} \frac{\partial}{\partial z_2} (z_1 - z_2) + 
\frac{\partial}{\partial z_2} \frac{1}{2y_G^2} y_G'' \right) \left( \frac{1}{2y_G} \frac{1}{2} \frac{\partial}{\partial z_2} (z_1 - z_2) \right) \right),
\]

We assumed here that \( M = 1 \), then \( y_G^2 = z^2 - 4\nu \) and \( \nu = S = gN \).

Non-Gaussian are somewhat more sophisticated, they are actually made from Riemann theta-functions for a peculiar Seiberg-Witten family of hyperelliptic complex curves. For example, the \((p, m) = (0, 2)\)-density is the 2-point correlator on such surfaces:

\[\rho_W^{(0,2)}(z_1, z_2) = d_z d_{z_2} E(z_1, z_2), \tag{21}\]

where \( E(z_1, z_2) \sim \frac{\phi(z_1) \phi(z_2)}{\theta(z_1 - z_2)} \) is the prime form [5, 3].

### 5.3 Decomposition formulas

Rewriting (17) in terms of the eigenvalues and then separating the eigenvalues into sets, associated with different extrema of \( W(\phi) \), one can deduce [93] a decomposition formula [2] for non-Gaussian \( Z_W \):

\[Z_W \sim \prod_{i=1}^{n} \frac{e^{-N_i W(\alpha_i) \Vol(U(N_i))}}{\Vol(U(N))} \prod_{i < j} \alpha_{ij}^{2N_i N_j} \hat{\mathcal{O}}_{ij} \prod_{i=1}^{n} \hat{\mathcal{O}}_i \prod_{i=1}^{n} Z_G^{(M_i)}(\ell_i^{(i)} | N_i) \tag{22}\]

with \( \alpha_i \) denoting the \( n \) roots of

\[W'(z) = \sum_{k=0}^{n+1} kT_k z^{k-1} = (n + 1)T_{n+1} \prod_{i=1}^{n} (z - \alpha_i), \]

\( \alpha_{ij} = \alpha_i - \alpha_j \), the frequencies \( M_i = W''(\alpha_i) = (n + 1)T_{n+1} \prod_{j \neq i} \alpha_{ij} \), and operators

\[\hat{\mathcal{O}}_{ij} = \exp \left( \sum_{k,l=0}^{\infty} (-1)^k \frac{(k + l - 1)!}{k! l!} \frac{\partial}{\partial t^{(i)}_k} \frac{\partial}{\partial t^{(j)}_l} \right), \]

\[\hat{\mathcal{O}}_i = \exp \left( - \sum_{k=4}^{\infty} \frac{\partial^k W(\alpha_i)}{k!} \frac{\partial}{\partial t^{(i)}_k} \right) \]

As explained in the previous section, this Givental-style formula is an example of general phenomenon, generalizing the Wick theorem from free fields and conformal theories to matrix models and non-trivial integrable systems.

Eq. (22) requires (and deserves) deep investigation. Today almost nothing is known about it. It is unclear how (22) follows directly from Virasoro constraints, without explicit use of matrix integrals. It is unclear what
is exact implication of the relation (12) for Gaussian functions, standing at the r.h.s. of (22). It is clear that the combination of (12) and (22) should represent $Z_W$ as some operator acting on $\tau_K^{2n}$, but exact formula is unavailable (Greg Moore in [39] and especially Ivan Kostov in [40] came very close to the answer, but it still escapes).

Eq. (22) can be also considered as a definition of a linear basis in the space of solutions of Virasoro constraints, which is formed by the functions, respecting the genus expansion. Arbitrary solution can be represented as a linear combination,

$$Z_W(t) = \int_{\mathcal{S}} \mu(\tilde{S}) Z_W(t|\tilde{S}), \quad (23)$$

with arbitrary measure $\mu(\tilde{S})$.

### 5.4 Check-operators and $t$-evolution operator

According to their definition (19), multi-densities do not depend on the time variables $t_k$, only on $T_0, \ldots, T_{n+1}$ and $\tilde{S}$. Actually, dependence on $T_n$ and $T_{n+1}$ is fixed by (6), and multi-densities depend on $2n$ variables: $n$ of them are $T$’s and $n$ are $S$’s. Therefore one can ask for a procedure which defines multi-densities directly in terms of these variables, without an intermediate introduction of – infinitely many – auxiliary (from the point of view of multi-densities theory) variables $t_k$. This problem is solved, at least conceptually, in [2, 44] in terms of check-operators.

Multi-densities are not arbitrary functions of their $2n$ variables, the Ward identities (6) express all of them in terms of a single function of $n+1$ variables, $T_0, \ldots, T_{n-1}$ and $g$ [2], which we call bare prepotential. The choice of bare prepotential is actually the choice of particular branch of partition function. Note, that this means that there are not just infinitely many branches, they form a continuous variety – and this is a property of every reasonable partition function and of anything obtained by application of evolution operators (for which the functional integral is a particular realization), see [34] for explanation, how continuous phase structure emerges from discrete bifurcations (phase transitions), and for discussion of the adequate formalism. This set of questions is very important, in particular it can help to understand the relevant topology and may be even the metric structure on the space of phases, what is the principal problem in landscape theory. Development of such formalism for matrix models is a task for the future, today we know how to proceed within a given phase, when bare prepotential is somehow specified and the question of relative importance of different choices is not addressed.

#### 5.4.1 Independent variables and bare prepotential

To understand what it means to specify the bare prepotential, we consider immediate corollary of (6) with $t_k \to \frac{1}{g} (t_k - \bar{T}_k)$ for partition function at vanishing times $t_k$. It is given by [44]:

$$Z(T|g)|_{t=0} = \int dk z(k|\eta_2, \ldots, \eta_n|g^2) e^{\frac{1}{g} (kx - k^2w)} \quad (24)$$

with an arbitrary function $z$ of $n$ arguments $(k, \eta_2, \ldots, \eta_n)$ and $g^2$. Here the $L_{-1}$-invariant variables are used,

$$w = \frac{1}{n+1} \log T_{n+1}, \quad \eta_k = T_{n+1}^{-\frac{k^2}{n}} (T_n^k + \ldots), \quad x = T_0 + \ldots \sim \eta_{n+1} \quad (25)$$

In particular,

$$\eta_2 = \left( T_n^2 - \frac{2(n+1)}{n} T_{n-1} T_{n+1} \right) T_{n+1}^{-\frac{2}{n+1}},$$

$$\eta_3 = \left( T_n^3 - \frac{3(n+1)}{n} T_{n-1} T_n T_{n+1} + \frac{3(n+1)^2}{n(n-1)} T_{n-2} T_{n+1}^2 \right) T_{n+1}^{-\frac{3}{n+1}},$$

$$\eta_4 = \left( T_n^4 - \frac{4(n+1)^2}{n} T_{n-1} T_n^2 T_{n+1} + \frac{8(n+1)^3}{n(n-1)} T_{n-2}^2 T_{n+1}^2 - \frac{8(n+1)^3}{n(n-1)(n-2)} T_{n-3} T_{n+1}^3 \right) T_{n+1}^{-\frac{4}{n+1}},$$

$$\ldots$$

$$\eta_k = \left( T_n^k + \frac{k(k-2)}{n} \sum_{l=1}^{k-1} (-1)^l \frac{(n+1)^l (n-l)!}{(k-l-1)!} T_{n-l} T_{n-l}^{-1} T_{n+1}^l \right) T_{n+1}^{-\frac{k}{n+1}},$$

The variable $x$ is obtained from $\eta_{n+1}$ by normalization and it is the only variable which contains $T_0$. At the same time only $T_0$ appears in double-derivative item of $L_0$, thus in (24) the $L_0$-constraint links the $x$- and $w$-dependencies. Of course, generic (24) does not possess genus expansion, i.e. does not guarantee that the
bare prepotential \( F(T|g) = g^2 \log Z(T|g) \), is expanded in non-negative powers of \( g^2 \): as explained above, this requirement is somewhat artificial from the perspective of \( D \)-module theory. Still, a simple ansatz makes things consistent (it is not quite clear if this ansatz is absolutely necessary for genus expansion to exist): if

\[
S = \frac{\partial F}{\partial T_0} = \text{const}
\]

i.e. is independent of \( T_0, \ldots, T_{n+1} \) and \( g \), then

\[
z(k|\eta_2, \ldots, \eta_n|g^2) = H(\eta_2, \ldots, \eta_n|S|g^2) \; \delta(k - S)
\]

and

\[
F = g^2 \log Z = Sx + \frac{S^2}{n+1} \log T_{n+1} + g^2 \log H(\eta_2, \ldots, \eta_n|S|g^2)
\]

where \( H \) is an arbitrary function of \( n + 1 \) variables. Genus expansion occurs, if we request \( g^2 \log H \) is expanded in non-negative powers of \( g^2 \).

Introduction of the other \( \vec{S} \)-variables can be considered as ingenious version of Fourier-Radon transform with the help of Dijkgraaf-Vafa partition functions (22), which converts \( H(\eta_2, \ldots, \eta_n) \) into an arbitrary measure \( \mu(\vec{S}) \) of the \( \vec{S} \)-variables in (23) [2].

### 5.4.2 Check-operator multi-densities

Other \( \hat{L}_k \)-constraints with \( k > 0 \) express \( t \)-dependencies of \( Z(t|T) \) through \( T \)-dependencies. In particular, they allow to represent multi-densities (both Gaussian and non-Gaussian) by action of \( T \)-dependent operators on partition function (24). Such operators, involving only derivatives with respect to \( T \)-variables, are named check-operators in [44], to distinguish them from hat-operators, acting on \( t \)-variables. The task is to express multi-densities, defined by application of hat-operators to the full prepotential \( F(t|T|g) \), through check-operators applied to the bare prepotential \( F(T) = F(t = 0|T|g) \). These check operators will predictably be more sophisticated than hat-operators \( \nabla(z) \), but instead they can be applied to the independent (free) function \( F(T) \).

Remarkably, the problem of building check-operator multi-densities appears essentially equivalent to the problem of Gaussian multi-densities [44]. Though this fact is not fully proved – and even adequately formulated – yet, it is not too surprising: the needed check-operators are universal, in certain sense they do not depend on the phase, thus they should be restorable from information, available in any phase, including the Gaussian one (once again we encounter the main idea of string theory). Anyhow, modulo some details concerning ordering prescriptions, the first check-operators can be just read from formulas (20) for Gaussian multi-densities [44]:

\[
\hat{\rho}_W^{(0|1)}(z|g) = \frac{W'(z) - \tilde{y}(z|g)}{2},
\]

\[
\hat{\rho}_W^{(1|1)}(z|g) = -\frac{1}{4g^2} \tilde{y}'' ,
\]

\[
\hat{\rho}_W^{(2|1)}(z|g) = \frac{5}{16} \left( \frac{\tilde{y}''}{\tilde{y}^2} \right)^2 - \frac{1}{8} \frac{\partial^2}{\tilde{y}^2} \left( \frac{\tilde{y}''}{\tilde{y}^2} \right) - \frac{1}{8} \frac{\tilde{y}'''}{\tilde{y}} ,
\]

\[
\hat{\rho}_W^{(2|2)}(z_1, z_2|g) = -\frac{1}{2\tilde{y}(z_1|g)} \frac{\partial}{\partial z_2} \left( \frac{\partial \tilde{y}(z_1|g) - \tilde{y}(z_2|g)}{z_1 - z_2} \right),
\]

\[
\hat{\rho}_W^{(1|2)}(z_1, z_2|g) = \frac{1}{y_1} \left( \frac{4}{4y_1^2} \tilde{y}''_1 - \frac{1}{2y_1} \tilde{y}'' \right) \frac{1}{2y_1} \frac{\partial}{\partial z_2} \left( \frac{\partial \tilde{y}_1 - \tilde{y}_2}{z_1 - z_2} \right) + \frac{\partial}{\partial z_2} \left( \frac{1}{4y_1^2} \tilde{y}''_1 - \frac{1}{4y_1^2} \tilde{y}'' \right) + \frac{1}{y_1} \left( -\frac{1}{4y_1^2} \tilde{y}'_1 + \frac{1}{2y_1} \frac{\partial}{\partial z_2} \left( \frac{\partial \tilde{y}_1 - \tilde{y}_2}{z_1 - z_2} \right) \right),
\]

\[
\hat{\rho}_W^{(0|3)}(z_1, z_2, z_3|g) = \frac{1}{y_1} \left( \frac{1}{2y_1} \left( \frac{\partial \tilde{y}_1 - \tilde{y}_2}{z_1 - z_2} \right) \frac{1}{2y_1} \left( \frac{\partial \tilde{y}_1 - \tilde{y}_3}{z_1 - z_3} \right) + \frac{\partial}{\partial z_2} \left( \frac{1}{2y_1} \left( \frac{\partial \tilde{y}_1 - \tilde{y}_3}{z_1 - z_3} \right) - \frac{1}{2y_2} \left( \frac{\partial \tilde{y}_2 - \tilde{y}_3}{z_2 - z_3} \right) \right),
\]

\[
+ \frac{\partial}{\partial z_2} \left( \frac{1}{2y_1} \left( \frac{\partial \tilde{y}_1 - \tilde{y}_3}{z_1 - z_3} \right) - \frac{1}{2y_2} \left( \frac{\partial \tilde{y}_2 - \tilde{y}_3}{z_2 - z_3} \right) \right) + \frac{\partial}{\partial z_3} \left( \frac{1}{2y_2} \left( \frac{\partial \tilde{y}_2 - \tilde{y}_3}{z_2 - z_3} \right) - \frac{1}{2y_3} \left( \frac{\partial \tilde{y}_3 - \tilde{y}_3}{z_3 - z_3} \right) \right),
\]
\[
\frac{\partial}{\partial z_3} \frac{1}{z_3 - z_1} \left( \frac{1}{2\tilde{y}_1} \left( \frac{\partial}{\partial z_2} \tilde{y}_1 - \tilde{y}_2 \right) - \frac{1}{2\tilde{y}_3} \left( \frac{\partial}{\partial z_2} \tilde{y}_2 - \tilde{y}_3 \right) \right). 
\]

(29)

Here
\[
\tilde{y}^2(z|g) = W'(z)^2 + 4g^2 \tilde{R}_W(z) \quad \text{and} \quad \tilde{R}_W(z) = \sum_{a,b=0} \left( a + b + 2 \right) T_a + b + 2 z^a \frac{\partial}{\partial T_b} 
\]

(30)

The check-multidensities are defined to satisfy
\[
K_W^{(m)}(z_1, \ldots, z_m|g) Z_W(T|g) = K_W^{(m)}(z_1, \ldots, z_m|g) Z_W(T|g) = \left[ \tilde{\nabla}(z_1) \ldots \tilde{\nabla}(z_m) Z_W(t|T|g) \right]_{t=0}
\]

for full correlators \(K_W^{(m)}(z_1, \ldots, z_m|g)\), which are derivatives of \(Z(t)\), not of its logarithm, while for connected \(\rho_W^{(m)}(z_1, \ldots, z_m|g)\)
\[
(Z_W(T|g))^{-1} \rho_W^{(m)}(z_1, \ldots, z_m|g) Z_W(T|g) \neq \rho_W^{(m)}(z_1, \ldots, z_m|g) = \tilde{\nabla}(z_1) \ldots \tilde{\nabla}(z_m) \left( g^2 \log Z_W(T|g) \right)
\]

Relation between the full and connected correlators is provided by straightforward – though heavily looking – combinatorial formula [44]
\[
K_W^{(m)}(z_1, \ldots, z_m|g) = \sum_{\sigma} \sum_{k=1}^m \sum_1^\infty m_k \sum_0^\infty g^{\nu_1+\ldots+\nu_k-\nu} \left[ \frac{1}{\nu_1!(m_1)!^{\nu_1} \ldots \nu_k!(m_k)!^{\nu_k}} \rho_W^{(p_1|m_1)}(z_{\sigma(1)}, \ldots, z_{\sigma(m_1)}) \right]
\]
\[
\cdot \rho_W^{(p_2|m_2)}(z_{\sigma(m_1+1)}, \ldots, z_{\sigma(m_2)}) \ldots \rho_W^{(p_{\nu-1}|m_{\nu-1})}(z_{\sigma(m-\nu+1)}, \ldots, z_{\sigma(m)}) \right), \quad \text{ (31)}
\]

which serves as generalization of the Wick formula from the case when the only connected correlators are 2-point (note that this simplification does not occur even in Gaussian phases of matrix models beyond naive continuum limit). A similar expression relates the check-operators \(\tilde{K}\) and \(\tilde{\rho}\), in particular,
\[
K_W^{(1)}(z|g) = \sum_{p=0}^{\infty} g^{2p-2} \tilde{\rho}^{(p)(1)}(z|g)
\]

(32)

Note also, that in variance with (20), expressions in (29) explicitly contain \(g^2\).

Actually, once the check-operators multi-densities are introduced, one can forget about the genus-expansion constraint: operators can be applied to any bare partition function. Instead of being a constraint on partition functions, genus expansion requirement dictates becomes a selection criterion for a basis in check-operators space. The next step is to build up a theory of the \textbf{t-evolution operator} \(\tilde{U}(t)\), which generates the \(t\)-dependence of partition function:
\[
Z(t|T) = \tilde{U}(t) Z(0|T).
\]

(33)

Its existence is almost obvious, see [2, 44], but properties and explicit realizations remain to be found.

5.5 Seiberg-Witten theory

The Dijkgraaf-Vafa theory is intimately related to Seiberg-Witten theory.
5.5.1 Genus-zero level

For detailed discussion of this relation in the particular case of Dijkgraaf-Vafa partition functions per se, i.e. with requirements that genus expansion exists and only genus-zero contributions (spherical prepotentials) are considered, see [96]-[100]. Here are the main points of this analysis. The smallest moduli space of Seiberg-Witten structure unifies all phases with all the bare potentials \( W(z) \) of a fixed degree \( n \). It has (complex) dimension \( 2n \), and \( S \)-variables form the set of \( 2n \) flat moduli of Seiberg-Witten structure, defined with the help of \( \Omega_{SW} = \sqrt{W(z)^2 - 4f_W(z)dz} \), where \( f_W(z) = -\bar{R}_W(z)F(0)(T) \) is a polynomial of degree \( n - 1 \), made from arbitrary function \( F(0)(T) \) of \( n \) variables \( T_0, \ldots, T_{n-1} \) (and dependence on \( T_n \) and \( T_{n+1} \) prescribed by \( L_1 \) and \( L_0 \)-constraints). The space of all functions of \( n \) variables can – in the framework of Fourier-Radon transforms – be parameterized by arbitrary functions of \( n \) other parameters, and \( \mathcal{S} \) can play the role of such parameters no better, no worse than any other set of variables. What distinguishes \( \mathcal{S} \) is that they are flat Seiberg-Witten moduli, i.e. the symplectic form on the sheaf of Seiberg-Witten curves is

\[
\delta \Omega_{SW} = \omega(z) \wedge \delta S_i + \Omega_k(z) \wedge \delta T_k
\]

and

\[
\mathcal{S} = \oint_\mathcal{A} \Omega_{SW}, \quad T_k = \text{res}_\infty z^k \Omega_{SW}
\]

The CIV-DV prepotential \([90, 91]\) is defined by

\[
\frac{\partial \mathcal{F}_{CIV}}{\partial \mathcal{S}} = \oint_\mathcal{B} \Omega_{SW}, \quad \frac{\partial \mathcal{F}_{CIV}}{\partial T_k} = \frac{1}{k} \text{res}_\infty z^{-k} \Omega_{SW}
\]

it can be explicitly calculated term by term and possesses pronounced group-theoretical structure \([98, 2]\), which still awaits its adequate interpretation and explanation. No reasonable generic formulas for this prepotential are yet found, despite it is a genus-zero quantity, and despite it can be deduced in at least two dual ways: from (36) and from (22). Moreover, even direct relation between these equivalent representations is not yet established.

Also important is that there are various differently looking forms for the same items in \( \mathcal{F}_{CIV} \), their equivalence follows from peculiar identities \([99]\), which are simple to prove, but not so simple to understand.

5.5.2 WDVV equations

In \([96, 99, 100]\) the proof is given of the WDVV equations (15) for the CIV-DV prepotential. This is an additional check that \( 2n \) moduli \( T \) and \( S \) form complete set, at least at the level of genus-zero prepotentials. In \([97]\) a less obvious observation is reported: sometime this set can be reduced further, for example, by elimination of one of the 4 moduli in the case of \( n = 2 \), however, the meaning of this observation remains obscure.

The analogues of the CIV-DV prepotential for higher genera can also be defined because of its relation to Ward identities (6) – this makes the situation different from generic Seiberg-Witten theory, where the problem of lifting from genus zero to all genera remains very important, but unsolved. This makes it possible to ask, if higher genus prepotentials satisfy the (properly generalized) WDVV equations, for example the ones implied by studies of topological models \([82]\). This problem was not seriously attacked yet.

5.5.3 Towards quantum Seiberg-Witten theory of check multi-densities

The next natural question in the theory of check operators concerns definition of \( S \)-variables. Since in particular phases they are periods of \( \rho^{(0)}_W \), it is natural to treat \( \mathcal{S} \)-variables as operators, defined as periods of \( \tilde{\rho}^{(0)}_W \). Moreover, under certain conditions the \( A \)-periods of higher-genus one-point densities are vanishing, and one can even consider the periods of \( \tilde{\rho}^{(1)} \) or

\[
\tilde{S}_i = \frac{g^2}{4\pi i} \oint_\mathcal{A}_i \tilde{K}^{(1)}(z).
\]

This is a very perspective direction of research. Among other, obvious and not-so-obvious, questions is development of the full-scale version of Seiberg-Witten theory for check-operator quantities, i.e. a quantization of Seiberg-Witten structure. Introduction of check operators with the goal to deal with arbitrary bare prepotentials in a uniform way is a nice illustration of string "third-quantization" procedure, which implies that in order to study families of models one actually needs to quantize a representative of the family. An example of the simplest relation in this quantum Seiberg-Witten theory is provided by the quantum version of (14) \([44]\):

\[
\left[ \frac{g^2}{4\pi i} \oint_\mathcal{A} \tilde{K}, \quad \frac{g^2}{2} \oint_{\mathcal{B}} \tilde{K} \right] = g^2 \delta_{ij}.
\]
probably, with no corrections at the r.h.s.

The check-operator $A$-and $B$-periods in (38) are equal respectively to $\tilde{S}_i = \tilde{S}_i^{(0)} + g^2 \tilde{S}_i^{(1)} + O(g^4)$ and $\tilde{\Pi}_i^{(0)} + O(g^2)$, where [98]

$$
\tilde{S}_i^{(0)} = \frac{g^2}{M_i} \tilde{R}_i, \quad \tilde{\Pi}_i^{(0)} = \tilde{W}(\alpha_i),
$$

and $\alpha_i = 1, \ldots, n$ are the roots of $W'(z)$, operators $\tilde{R}_i = \tilde{R}_W(\alpha_i)$,

$$
M_i = W''(\alpha_i) = (n + 1)T_{n+1} \prod_{j \neq i} \alpha_{ij},
$$

and $\alpha_{ij} = \alpha_i - \alpha_j$. Note that in the leading order the operator $B$-periods and genus-1 contributions to $A$-periods do not contain $T$-derivatives, thus non-trivial are only commutators, involving $\tilde{S}_i^{(0)}$. These commutators can be easily deduced from two general formulas from [44],

$$
[\tilde{R}_W(x), W(z)] = \frac{W'(x) - W'(z)}{x - z}, \quad [\tilde{R}_W(x), \tilde{R}_W(z)] = (\partial_x - \partial_z) \frac{\tilde{R}_W(x) - \tilde{R}_W(z)}{x - z},
$$

(40)

together with

$$
\frac{\partial \alpha_j}{\partial T_k} = -\frac{k \alpha_j^{k-1}}{M_j},
$$

(41)

from [99], which implies straightforwardly:

$$
[\tilde{R}_W(x), \alpha_j] = \sum_{a, b = 0} (a + b + 2)T_{a+b+2} x^a \frac{\partial \alpha_j}{\partial T_b} = -\frac{1}{M_j} \left. \left( \frac{W'(x) - W'(z)}{x - z} \right) \right|_{z = \alpha_j}
$$

and, since $W'(\alpha_i) = W'(\alpha_j) = 0$,

$$
[\tilde{R}_i, \alpha_j] = [\tilde{R}_W(\alpha_i), \alpha_j] = \frac{1}{\alpha_{ij}}
$$

(42)

Eqs. (40) need to be supplemented by (42), because in commutators of periods (39) the arguments $\alpha_i$ also depend on $T_k$ and their derivatives should be taken into account. Thus instead of (40) we need:

$$
[\tilde{R}_i, M_j] = [\tilde{R}_W(\alpha_i), W''(\alpha_j)] = \left. \left( \frac{W'(x) - W'(z)}{x - z} \right) \right|_{z = \alpha_j} + W'''(\alpha_j) \left[ \tilde{R}_i, \alpha_j \right] = -\frac{2M_j}{\alpha_{ij}^2},
$$

(43)

and

$$
[\tilde{R}_i, \tilde{R}_j] = [\tilde{R}_W(\alpha_i), \tilde{R}_W(\alpha_j)] = (\partial_x - \partial_z) \frac{\tilde{R}_W(x) - \tilde{R}_W(z)}{x - z} \bigg|_{x = \alpha_i, z = \alpha_j} +
$$

$$
+ \left[ \tilde{R}_W(\alpha_i), \alpha_j \right] \tilde{R}_W'(\alpha_j) - \left[ \tilde{R}_W(\alpha_j), \alpha_i \right] \tilde{R}_W'(\alpha_i) = \frac{2}{\alpha_{ij}^2} (\tilde{R}_i - \tilde{R}_j)
$$

(44)

Now we have everything to check, that

$$
\left[ \tilde{S}_i^{(0)}, \tilde{S}_j^{(0)} \right] = \left[ \frac{g^2}{M_i} \tilde{R}_i, \frac{g^2}{M_j} \tilde{R}_j \right] = \frac{g^4}{M_i M_j} \left( \left[ \tilde{R}_i, \tilde{R}_j \right] - \frac{1}{M_j} \left[ \tilde{R}_i, M_j \right] \tilde{R}_j + \frac{1}{M_i} \left[ \tilde{R}_j, M_i \right] \tilde{R}_i \right) = 0
$$

Direct generalization of (43) states, that

$$
[\tilde{R}_i, \partial^m W(\alpha_j)] = -m! \sum_{k=0}^{m-2} \frac{1}{(k + 1)!} \frac{\partial^{k+2} W(\alpha_j)}{\alpha_{ij}^{m-k}},
$$

in particular,

$$
[\tilde{R}_i, \frac{\partial^4 W \partial^2 W - (\partial^3 W)^2}{(\partial^2 W)^3}(\alpha_j)] = -\frac{24}{M_j \alpha_{ij}^2},
$$

so that

$$
\left[ \tilde{S}_i^{(0)}, g^2 \tilde{S}_i^{(1)} \right] = -\frac{3}{M_i M_j \alpha_{ij}^2} = \left[ \tilde{S}_j^{(0)}, g^2 \tilde{S}_i^{(1)} \right]
$$
is symmetric under the permutation $i \leftrightarrow j$. Since also \[ g^2 S^{(1)}_i, g^2 S^{(1)}_j = 0 \text{ and } g^2 S^{(1)}_i, \tilde{\Pi}^{(0)}_j = 0, \] we finally obtain:
\[
\left[ S^{(0)}_i + g^2 S^{(1)}_i, S^{(0)}_j + g^2 S^{(1)}_j \right] = 0,
\]
\[
\left[ \tilde{\Pi}^{(0)}_i, \tilde{\Pi}^{(0)}_j \right] = [W(\alpha_i), W(\alpha_j)] = 0,
\]
\[
\left[ \tilde{\xi}^{(0)}_i + g^2 \tilde{\xi}^{(1)}_i, \tilde{\xi}^{(0)}_j + g^2 \tilde{\xi}^{(1)}_j \right] = \left[ \frac{g^2}{M_i^j} \tilde{R}_i, W(\alpha_j) \right] = \frac{g^2}{W^{(0)}(\alpha_i)} \frac{W'(\alpha_i) - W'(\alpha_j)}{\alpha_{ij}} = g^2 \delta_{ij}
\]
(45)

Note that in the last formula there is no need to vary the argument $\alpha_j$, because this variation gets multiplied by $W'(\alpha_j) = 0$. When $j = i$ one needs to apply l'Hopital rule to resolve the 0/0 ambiguity, and it provides the non-vanishing answer. This ends the proof of (38). It is an open question, if there are higher order corrections $\sim O(g^4)$ to the r.h.s. of that formula.

Given (38), of special interest is this kind of quantization of the WDVV equations. At least in principle, such approach opens a possibility to unify WDVV equations for different genera, the spherical equations from \cite{[73, 74]} with higher genus equations from \cite{[82, 83]}, and thus provide a new deep connection with geometry of the moduli space of punctured Riemann surfaces.

6 Neoclassical period: back to concrete results and back to physics?

The story about the challenges of matrix models is nearly infinite and can be continued further and further. We, however, draw a line here. The only brief comment that deserves being made at the end of these notes, is that the transcendental period, when most attention is concentrated on non-obvious structures underlying the subject and thus on high abstractions, is never the final stage of development of a theory. Sooner or later, deep insights from transcendental studies open a possibility to solve practical problems and, in our case, derive concrete formulas with the left hand side and the right hand side. Remarkably, some of such results are already starting to emerge. Despite they are rather week, it is principally important that they appear, it means that the theory is on the right track and new knowledge continues to have many non-trivial intersections with the old – what is a necessary feature of a healthy research project.

We mention here just three results of this kind, all concern the theory of CIV-DV prepotentials and all establish non-trivial relations with free fields on Riemann surfaces: the previous-level theory, from which one starts from when departing into the world of integrability and matrix models. The first result \cite{[104]} identifies 1-loop correction to the CIV-DV prepotential with certain combination of determinants in some – yet unidentified conformal theory. The second result \cite{[40]} provides additional evidence in a form of a CFT representation (not fully polished yet) of the all-genus matrix-model partition function $Z_W$. As expected, this representation actually constructs the partition function by a Givental-style procedure from $2n$ ($n = \text{deg } W'(z)$) Kontsevich $tau$-functions $\tau_K$, obtained by the application of star operators \cite{[39]} at $2n$ points of the bare spectral curve (Riemann sphere), which become ramification points of the hyperelliptic spectral curve, associated with $Z_W$. The third result \cite{[43]} formulates a kind of a perturbation theory for correlation functions of $2n$ operators inserted at these points and represents the partition function as a combinatorial ”quantum field theory”, essentially of Chern-Simons or, better, Batalin-Vilkovisky type. We consider this small set of preliminary results as a clear manifestation that all the ideas, discussed during the transcendental period can indeed be brought together and the head can catch the tail: the seemingly transcendental ideas will finally form a dense network of knowledge, not a shaky road leading into nowhere...

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References

[1] A.Morozov, Sov.Phys.Usp. 35 (1992) 671-714 (Usp.Fiz.Nauk 162 83-176), http://ellib.itep.ru/mathphys/people/morozov/92ufn-e1.ps & 92ufn-e2.ps
[2] A.Alexandrov, A.Mironov and A.Morozov, Int. J. Mod. Phys. **A19** (2004) 4127, hep-th/0310113

[3] V.Knizhnik, Sov.Phys.Usp. **32** (1989) 945 (Usp.Fiz.Nauk **159** 454), http://ellib.itep.ru/mathphys/people/knizhnik/kniz1.ps.gz, kniz2.ps.gz & kniz3.ps.gz; Phys. Lett. **B180** (1986) 247; Comm. Math. Phys. **112** (1987) 567; D.Lebedev and A.Morozov, Nucl. Phys. **B302** (1988) 342, http://ellib.itep.ru/mathphys/people/morozov/88_ap302.ps; A.Morozov, Nucl. Phys. **B303** (1988) 342-372, http://ellib.itep.ru/mathphys/people/morozov/88-29.ps; A.Gerasimov, A.Marshakov, A.Morozov, M.Olshanetsky and S.Shatashvili, Int. J. Mod. Phys. **A5** (1990) 2495, http://ellib.itep.ru/mathphys/people/morozov/89-69.ps, 89-70.eps, 89-72.eps & 89-74.ps; R.Kallosh and A.Morozov, Int. J. Mod. Phys. **A3** (1988) 1943-1958, http://ellib.itep.ru/mathphys/people/morozov/87-29.ps; D.Lebedev and A.Morozov, Nucl. Phys. **B302** (1988) 342, http://ellib.itep.ru/mathphys/people/morozov/88_ap302.ps; A.Morozov, Nucl. Phys. **B303** (1988) 343-372, http://ellib.itep.ru/mathphys/people/morozov/87-207.ps; A.Morozov and A.Perelomov, *Complex Geometry and String Theory*, Encyclopedia of Mathematical Sciences (ed.G.M.Khenkin), Springer-Verlag, **54** (1993) 197-280, http://ellib.itep.ru/mathphys/people/morozov/89-92.ps, /89-99.ps & /89-107.ps

[4] D.Mumford, *Tata Lectures on Theta*, Progress in Mathematics, vol.28, Birkhauser, 1983, 1984

[5] J.Fay, *Theta Functions on Riemann Surfaces*, LNM, #352, 1973

[6] A.Morozov, Sov.Phys.Usp. **37** (1994) 1 (Usp.Fiz.Nauk **164** 3-62), hep-th/9303139; hep-th/9502091; A.Mironov, Int. J. Mod. Phys. **A9** (1994) 4335, hep-th/9312212; Phys. of Particles and Nuclei **33** (2002) 537

[7] Harish-Chandra, Am. J. Math. **79** (1957) 87; C.Itzykson and J.-B.Zuber, J.Math.Phys. **21** (1980) 411; J.Duistermaat and G.Heckman, Invent.Math. **69** (1982) 259

[8] M.Kontsevich, Funk. Anal. Prilozh. **25** (1991) v.2, p.50; Comm. Math. Phys. **147** (1992) 1; M.Adler and P. van Moerbeke, Comm.Math.Phys. **147** (1992) 25; P.Di Francesco, C.Itzykson and J.-B.Zuber, Comm. Math. Phys. **151** (1993) 193

[9] S.Kharchev, A.Marshakov, A.Mironov and A.Zabrodin, Nucl. Phys. **B380** (1992) 181; Phys. Lett. **B275** (1992) 311; S.Kharchev, hep-th/9810091

[10] V.Kazakov and A.Migdal, Nucl. Phys. **B397** (1993) 214

[11] S.Kharchev, A.Marshakov, A.Mironov and A.Morozov, Int. J. Mod. Phys. **A10** (1995) 2015, hep-th/9312210

[12] E.P.Wigner, Ann.Math. **53** (1951) 36; F.J.Dyson, J.Math.Phys. **3** (1962) 140

[13] M.Mehta, *Random Matrices*, 2nd ed., Acad. Press., N.Y., 1991; E.Brezin, C.Itzykson, G.Parisi and J.-B.Zuber, Comm. Math. Phys. **59** (1978) 35; D.Bessis, C.Itzykson and J.-B.Zuber, Adv. Appl. Math. **1** (1980) 109

[14] D.Gross and E.Witten, Phys.Rev. **D21** (1980) 446; T.Eguchi and H.Kawai, Phys.Rev.Lett. **48** (1982) 1063; P.Di Francesco, P.Ginsparg and J.Zinn-Justin, J. Phys. Rep. **254** (1995) 1

[15] A.Migdal, Phys. Rep. **102** (1980) 199; J.Ambjorn, J.Jurkiewicz and Yu.Makeenko, Phys. Lett. **B251** (1990) 517; J.Ambjorn, L.Chekhov, C.F.Kristjansen and Yu.Makeenko, Nucl. Phys. **B404** (1993) 127, erratum in Nucl. Phys. **B449** (1995) 681, hep-th/9302014; G.Akemann, Nucl. Phys. **B482** (1996) 403, hep-th/9606004

[16] K.Demeterfi, N.Deo, S.Jain and C.-I.Tan, Phys.Rev. **D42** (1990) 4105-4122; C.Crnkovic and G.Moore, Phys. Lett. **B257** (1991) 322; F.David, Phys. Lett. **B302** (1993) 403, hep-th/9212106; G.Bonnet, F.David and B.Eynard, J.Phys. **A33** (2000) 6739, cond-mat/0003324
[17] A.Mironov and A.Morozov, Phys. Lett. B252 (1990) 47-52, http://ellib.itep.ru/mathphys/people/moro-
zov/90-252.ps;
F.David, Mod. Phys. Lett. A5 (1990) 1019;
J.Ambjorn and Yu.Makeenko, Mod. Phys. Lett. A5 (1990) 1753;
H.Itoyama and Y.Matsuo, Phys. Lett. B255 (1991) 202
[18] D.Gross ans M.Newman, Nucl. Phys. B380 (1992) 168, hep-th/9112069
[19] A.Marshakov, A.Mironov and A.Morozov, Mod. Phys. Lett. A7 (1992) 1345-1360, hep-th/9201010
[20] M.Fukuma, H.Kawai and R.Nakayama, Int. J. Mod. Phys. A6 (1991) 1385;
R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl. Phys. B348 (1991) 435
[21] E.Witten, in Proc.of the XX-th Int. Conf. on Diff. Geom. Methods in Theor. Phys., Vol.1,2 (1991) 176-216,
World Sci. Publ., River Edge, NJ, 1992, Eds. S.Catto and A.Rocha;
A.Marshakov, A.Mironov and A.Morozov, Phys. Lett. B274 (1992) 280
[22] S.Kharchev, A.Marshakov, A.Mironov and A.Morozov, Mod. Phys. Lett. A8 (1993) 1047-1062, Theor. Math. Phys. 95 (1993) 571-582, hep-th/0208046
[23] A.Marshakov, A.Mironov and A.Morozov, Phys. Lett. B490 (2000) 173-179, hep-th/0005280
[24] A.Gerasimov, A.Marshakov, A.Mironov and A.Orlov, Nucl. Phys. B357 (1991) 565
[25] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov and A.Morozov, Phys. Lett. B355 (1995) 466, hep-th/9505035;
A.Morozov, hep-th/9903087.
[26] M.Jimbo, T.Miwa and M.Sato, Publ. RIMS 14 (1978) 223, 15 (1979) 201, 577, 871, 1531
[27] A.Gerasimov, S.Khoroshkin, D.Lebedev, A.Mironov and A.Morozov, Int. J. Mod. Phys. A10 (1995) 2589-2614, hep-th/9405011;
A. Mironov, A. Morozov and L. Vinet, Theor.Math.Phys. 100 (1995) 890-899 (Teor.Mat.Fiz. 100 (1994) 119-131), hep-th/9312213
[28] A.Mironov, A.Morozov and G.Semenoff, Int. J. Mod. Phys. A11 (1996) 5031, hep-th/9404005
[29] A.Morozov, Mod. Phys. Lett. A7 (1992) 3503-3508, hep-th/9209074;
S.Shatashvili, Commun. Math. Phys. 154 (1993) 421-432, hep-th/9209083;
L.D. Paniak Nucl. Phys. B553 (1999) 583-600, hep-th/9902089;
P. Zinn-Justin and J.-B. Zuber, J.Phys. A36 (2003) 3173-3194, math-ph/0209019;
M. Bertola and B. Eynard, J.Phys. A36 (2003) 7733-7750, hep-th/0303161;
S. Aubert, C.S. Lam, J.Math.Phys. 44 (2003) 6112-6131, math-ph/0307012; J.Math.Phys. 45 (2004) 3019-3039, math-ph/0405036
[30] G.t'Hooft, hep-th/0208054;
A.Morozov and A.Niemi, Nucl.Phys. B666 (2003) 311-336, hep-th/0304178
[31] M.Tierz, hep-th/0308121;
E.Goldfain, Chaos, Solitons and Fractals (2004);
S.Franco (MIT, LNS), Y.H.He, C.Herzog and J.Walcher, Phys.Rev. D70 (2004) 046006, hep-th/0402120;
J.I.Latorre, C.A. Lutken, E. Rico and G. Vidal, quant-ph/0403248;
J.Gaite, J.Phys. A37 (2004) 10409-10420, hep-th/0404212;
T. Oliynyk, V. Suneeta and E. Woolgar, hep-th/0410001.
[32] A.Zamolodchikov, JETP Lett. 43 (1986) 730; Sov. J. Nucl. Phys. 46 (1987) 1090
[33] P.F.Bedaque, H.-W.Hammer and U.van Kolck, Phys.Rev.Lett. 82 (1999) 463, nucl-th/9809025;
D.Bernard and A.LeClair, Phys.Lett. B512 (2001) 78, hep-th/0103096;
S.D.Glazek and K.G.Wilson, Phys.Rev. D47 (1993) 4657, hep-th/9203088;
A.LeClair, J.M.Roman and G.Sierra, Phys.Rev. B69 (2004) 20505, cond-mat/0211338; Nucl.Phys. B675 (2003) 584-606, hep-th/0301042; B700 (2004), 407-435, hep-th/0312141;
E.Braaten and H.-W.Hammer, cond-mat/0303249
[34] V.Dolotin and A.Morozov, hep-th/0501235
[35] A.Gerasimov, D.Lebedev and A.Morozov, Int. J. Mod. Phys. A6 (1991) 977-988, http://ellib.itep.ru/mathphys/people/morozov/90-04.ps; A.Morozov, Mod. Phys. Lett. A6 (1991) 1525-1532, http://ellib.itep.ru/mathphys/people/morozov/90-50.ps

[36] N.Arkani-Hamed, S.Dimopoulos and S.Kachru, hep-th/0501082

[37] S. Kharchev, A. Marshakov, A. Mironov and S. Pakuliak, Nucl. Phys. B404 (1993) 717, hep-th/9208044; A.Mironov and S.Pakuliak, Theor. Math. Phys. 95 (1993) 604-625 (Teor. Mat. Fiz. 95 (1993) 317-340)

[38] I.Kostov, Phys. Lett. B297 (1992) 74-81; J.Alfaro and I.Kostov, hep-th/9604011

[39] G.Moore, Comm.Math.Phys. 133 (1990) 261-304; Prog. Theor. Phys. Suppl. 102 (1990) 255-286

[40] I.Kostov, Conformal Field Theory Techniques in Random Matrix Models, preprint SPHT/t98/112, 1998

[41] L.Chekhov and Yu.Makeenko, Phys. Lett. B278 (1992) 271

[42] S.Kharchev, A.Marshakov, A.Mironov and A.Morozov, Nucl. Phys. B397 (1993) 339, hep-th/9203043.

[43] B.Eynard, JHEP 0411 (2004) 031, hep-th/0407261

[44] A.Alexandrov, A.Mironov and A.Morozov, hep-th/0412099, hep-th/0412205

[45] E.Brezin and V.Kazakov, Phys. Lett. B236 (1990) 144; M.Douglas and S.Shenker, Nucl. Phys. B335 (1990) 635; D.Gross and A.Migdal, Phys.Rev.Lett. 64 (1990) 127

[46] Yu.Makeenko, A.Marshakov, A.Mironov and A.Morozov, Nucl. Phys. B356 (1991) 574-628, http://ellib.itep.ru/mathphys/people/morozov/mmmm.ps

[47] A.Givental. math.AG/0009067; J.S.Song and Y.S.Song, J.Math.Phys. 45 (2004) 4539-4550, hep-th/0103254; A.Alexandrov, J.Math.Phys. 44 (2003) 5268-5278, hep-th/0205261

[48] A.Morozov, Talk at INTAS-RFBR School, September 98, hep-th/9810031

[49] A.Morozov and L.Vinet, Int. J. Mod. Phys. A13 (1998) 1651, hep-th/9409093

[50] A.Gerasimov, A.Morozov and K.Selivanov, Int. J. Mod. Phys. A16 (2001) 1531-1558, hep-th/0005053

[51] D.Kreimer, Adv. Theor. Math. Phys. 2 (1998) 303-334, hep-th/9810022; A.Connes and D.Kreimer, Comm. Math. Phys. 199 (1998) 203-242; Lett. Math. Phys. 48 (1999) 85-96, hep-th/9904044; JHEP 9909024, hep-th/9909126; Comm. Math. Phys. 210 (2000) 249-273, hep-th/9912092; hep-th/0003188; Annales Henri Poincare 3 (2002) 411-433, hep-th/0201157; A.Connes and M.Marcolli, math.NT/0409306; hep-th/0003188; Annales Henri Poincare 3 (2002) 411-433, hep-th/0201157; A.Connes and M.Marcolli, math.NT/0409306; hep-th/0411114

[52] D.Malyshev, JHEP 0205 (2002) 013, hep-th/0112146; hep-th/0408230

[53] R.Hirota, Topics in Current Physics 17 (1980) 157

[54] I.Krichever, Russ. Math. Surveys 32 (1977) v.6 p.185 (Usp.Mat.Nauk 32 (1977) v.6 p.183)

[55] S.Novikov, unpublished; B.Dubrovin, Sov.Math.Usp. 36 (1981) v.2 p.11-80; T.Shiota, Characterization of Jacobian Varieties in Terms of Soliton Equations, Harvard University, 1984; M.Mulase, J.Diff.Geom. 19 (1984) 403-430; Invent.Math. 92 (1988) 1-46

[56] A.Marshakov, M.Martellini and A.Morozov, Phys. Lett. B418 (1998) 294-302, hep-th/9706050

[57] A.Mironov and A.Morozov, Phys. Lett. B524 (2002) 217-226, hep-th/0107114

[58] V.Kazakov, Phys. Lett. B237 (1990) 212; I.Kostov, Phys. Lett. B238 (1990) 181; J. de Boer, A.Sinkovics, E.Verlinde and J.-T. Yee, JHEP 0403 (2004) 023, hep-th/0312135; C.Johnson, hep-th/0408049
[76] E.Witten, Phys.Rev. D46 (1992) 5467-5473, hep-th/9208027; Phys.Rev. D47 (1993) 3405-3410, hep-th/9210065; hep-th/9306122; M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Comm. Math. Phys. 165 (1994) 311-428, hep-th/9309140; Nucl. Phys. B405 (1993) 279-304, hep-th/9302103; W.Siegel, Phys. Lett. B151 (1984) 391; hep-th/0107094 B.Zwiebach Nucl. Phys. B390 (1993) 33, hep-th/9206084; N. Berkovits and W. Siegel, Nucl. Phys. B505 (1997) 139-152, hep-th/9703154; N.Berkovits, JHEP 0004 (2000) 018, hep-th/0001035; JHEP 0409 (2004) 047, hep-th/0406055; N.Berkovits and P.Howe, Nucl. Phys. B635 (2002) 75-105, hep-th/0112160; A.Gorodentsev and A.Losev, lectures at ITEP-DIAS School, 2004.

[77] N.Seiberg and E.Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087; Nucl. Phys. B431 (1994) 484, hep-th/9408099; A.Klemm, W.Lerche, S.Theisen and S.Yankielowicz, Phys. Lett. B344 (1995) 169, hep-th/9411048; E.Martinec and N.Warner, Nucl. Phys. B459 (1996) 97-112, hep-th/9509161; T.Eguchi and S.Yang, Mod. Phys. Lett. A11 (1996) 131-138, hep-th/9510183; R.Donagi and E.Witten, Nucl. Phys. B460 (1996) 299, hep-th/9510101; Eric D'Hoker, I.M. Krichever, D.H. Phong, Nucl.Phys. B489 (1997) 179-210, 211-222, hep-th/9609041, hep-th/090145; hep-th/0212313; R.Carroll, hep-th/9712110, hep-th/9802130, hep-th/9804086, hep-th/9905010; A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, Nucl. Phys. B527 (1998) 690-716, hep-th/9802007

[78] B.Dubrovin, I.Krichever and S.Novikov, Integrable Systems I, VINITI, Dynamical Systems 4 (1985) 179; I.Krichever, Comm. Math. Phys. 143 (1992) 415, hep-th/9205110; B.Dubrovin, Nucl. Phys. B379 (1992) 627; T.Nakatsu and K.Takasaki, Mod. Phys. Lett. A11 (1996) 417, hep-th/9509162

[79] H.Itoyama and A.Morozov, Nucl. Phys. B477 (1996) 855, hep-th/9511125; Nucl. Phys. B491 (1997) 529, hep-th/9512161; hep-th/9601168; A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, Nucl. Phys. B527 (1998) 690-716, hep-th/9802007

[80] A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, hep-th/9604078

[81] A. Losev, G. Moore, N. Nekrasov and S. Shatashvili, Nucl.Phys.Proc.Suppl. 46 (1996) 130-145, hep-th/9509151; hep-th/9511185; Nucl. Phys. B484 (1997) 196-222, hep-th/9606082; A. Losev, N. Nekrasov and S. Shatashvili, hep-th/9908204; Class. Quant. Grav. 17 (2000) 1181-1187, hep-th/9911099; N.Nekrasov, Adv. Theor. Math. Phys. 7 (2004) 831-864, hep-th/0206161, hep-th/0306211; R.Flume and R. Poghossian, Int. J. Mod. Phys. A18 (2003) 2541, hep-th/0208176; A.Losev, A.Marshakov and N.Nekrasov, hep-th/0302191; N.Nekrasov and A.Okounkov, hep-th/0306238; N.Nekrasov, S.Shadchin, Comm. Math. Phys. 252 (2004) 359-391, hep-th/0404225

[82] E.Getzler, Comm. Math. Phys. 163 (1994) 473-489, hep-th/9305013; alg-geom/9612004; math.AG/9801003

[83] A.Losev and S.Shadrin, preprint ITEP-TH-67/04

[84] M.Kontsevich and Yu.Manin, Comm.Math.Phys. 164 (1994) 525

[85] A.Belavin, A.Polyakov and A.Zamolodchikov, Nucl. Phys. B241 (1984) 333; V.Dotsenko, Proceedings of ITEP Winter School, 12 (1985) v.3 p.90-140

[86] S.Khoroshkin and V.Tolstoy, hep-th/9406194; G.Felder, hep-th/9412207; S. Khoroshkin, D. Lebedev and S. Pakuliak, q-alg/9702002; A.Varchenko and G.Felder, q-alg/9704005

[87] V.Bazhanov, S.Lukyanov and A.Zamolodchikov, Comm. Math. Phys. 177 (1996) 381-398, hep-th/9412229; Comm. Math. Phys. 190 (1997) 247-278, hep-th/9604044; Comm. Math. Phys. 200 (1999) 297-324, hep-th/9805008; Adv.Theor.Math.Phys. 7 (2004) 711-725, hep-th/0307108

[88] S. Kharchev, D. Lebedev and M. Semenov-Tian-Shansky, Comm. Math. Phys. 225 (2002) 573-609, hep-th/0102180; A.Gerasimov, S.Kharchev and D.Lebedev, math.QA/0204206; math.QA/0402112; A. Gerasimov, S. Kharchev, D. Lebedev and S. Oblezin, math.AG/0409031
[89] H. W. Braden, A. Mironov and A. Morozov, Phys. Lett. B514 (2001) 293-298, hep-th/0105169
[90] F. Cachazo, K. Intriligator and C. Vafa, Nucl. Phys. B603 (2001) 3, hep-th/0103067; F. Cachazo and C. Vafa, hep-th/0206017
[91] R. Dijkgraaf and C. Vafa, Nucl. Phys. B644 (2002) 3, hep-th/0206255; Nucl. Phys. B644 (2002) 21, hep-th/0207106; hep-th/0208048
[92] L. Chekhov and A. Mironov, Phys. Lett. B552 (2003) 293, hep-th/0209085
[93] A. Klemm, M. Marino and S. Theisen, JHEP 0303 (2003) 051, hep-th/0211216
[94] A. Gorsky, Phys. Lett. B554 (2003) 185-189, hep-th/0210281
[95] F. Cachazo, M. Douglas, N. Seiberg and E. Witten, JHEP 0212 (2002) 071, hep-th/0211170; F. Cachazo, N. Seiberg and E. Witten, JHEP 0302 (2003) 042, hep-th/0301006; JHEP 0304 (2003) 01, hep-th/0303207
[96] H. Itoyama and A. Morozov, Nucl. Phys. B657 (2003) 53, hep-th/0211245
[97] H. Itoyama and A. Morozov, Phys. Lett. B555 (2003) 287, hep-th/0211259
[98] H. Itoyama and A. Morozov, Progr. Theor. Phys. 109 (2003) 433, hep-th/0212032
[99] H. Itoyama and A. Morozov, Int. J. Mod. Phys. A18 (2003) 5889, hep-th/0301136
[100] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, Phys. Lett. B562 (2003) 323, hep-th/0301071
[101] T. Eguchi and Y. Sugawara (Tokyo U.), JHEP 0305 (2003) 063, hep-th/0305050
[102] S. Naculich, H. Schnitzer and N. Wyllard, JHEP 0301 (2003) 015, hep-th/0211254; I. Bena, R. Roiban and R. Tata, Nucl. Phys. B679 (2004) 168-188, hep-th/0211271; B. Feng, Nucl. Phys. B661 (2003) 113-138, hep-th/0212010; Phys.Rev. D68 (2003) 025010, hep-th/0212274; K. Ohta, JHEP 0302 (2003) 057, hep-th/0212025; S. Seki, Nucl. Phys. B661 (2003) 257-272, hep-th/0212079; I. Bena, S. de Haro and R. Roiban, Nucl. Phys. B664 (2003) 45-58, hep-th/0212083; C. Hofman, JHEP 0310 (2003) 022, hep-th/0212095; H. Suzuki, JHEP 0303 (2003) 036, hep-th/0212121; Y. Demasure and R. Janik, Nucl. Phys. B661 (2003) 153-173, hep-th/0212212; C. Ahn and S. Nam (Kyung Hee U.), Phys. Lett. B562 (2003) 141-146, hep-th/0212231; A. Mironov, Fortschr. Phys. 51 (2003) 781-786, hep-th/0301196; R. Roiban, R. Tata and J. Walcher, Nucl. Phys. B665 (2003) 211-235, hep-th/0301217; Y. Ookouchi and Y. Watabiki, Mod. Phys. Lett. A18 (2003) 1113-1126, hep-th/0301226; C. Ahn and Y. Ookouchi, JHEP 0303 (2003) 010, hep-th/0302150; D. Berenstein, JHEP 0306 (2003) 019, hep-th/0303033; H. Itoyama and H. Kanno, Phys. Lett. B573 (2003) 227-234, hep-th/0304184; Nucl. Phys. B686 (2004) 155-164, hep-th/0312306; M. Matone and I. Mazzucato, JHEP 0307 (2003) 015, hep-th/0305225; M. Alishahiha, J. de Boer, A. Mosaffa and J. Wijnhout, JHEP 0309 (2003) 066, hep-th/0308120; S. Aoyama and T. Masuda, JHEP 0403 (2004) 072, hep-th/0309232; R. Argurio, G. Ferretti and R. Heise, Int. J. Mod. Phys. A19 (2004) 2015-2078, hep-th/0311066; M. Gomez-Reino, JHEP 0406 (2004) 051, hep-th/0405242; P. B. Ronne, hep-th/0408103; K. Fujiwara, Hiroshi Itoyama and M. Sakaguchi, hep-th/0409060
[103] M. Matone, Nucl. Phys. B656 (2003) 78-92, hep-th/0212253; A. Dymarsky and V. Pestun, Phys.Rev. D67 (2003) 125001, hep-th/0301135
[104] L. Chekhov, hep-th/0401089
[105] G. t’ Hooft, Nucl. Phys. B72 (1974) 461