On \((1 + \varepsilon)\)-approximate problem kernels for the Rural Postman Problem

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Abstract. Given a graph \(G = (V, E)\) with edge weights \(\omega: E \to \mathbb{N} \cup \{0\}\) and a subset \(R \subseteq E\) of edges, the Rural Postman Problem (RPP) is to find a closed walk \(W^*\) of minimum weight \(\omega(W^*)\) containing all edges of \(R\). We prove that RPP is \(\text{WK}[1]\)-complete parameterized by the number and cost \(d = \omega(W^*) - \omega(R) + |W^*| - |R|\) of edges traversed additionally to the required ones, that is, presumably cannot be polynomial-time reduced to solving instances of size \(\text{poly}(d)\). In contrast, denoting by \(b \leq 2d\) the number of vertices incident to an odd number of edges of \(R\) and by \(c \leq d\) the number of connected components formed by the edges in \(R\), we show how to reduce any RPP instance \(I\) to an RPP instance \(I'\) with \(2b + O(c/\varepsilon)\) vertices in \(O(n^3)\) time so that any \(\alpha\)-approximate solution for \(I'\) gives an \(\alpha(1+\varepsilon)\)-approximate solution for \(I\), for any \(\alpha \geq 1\) and \(\varepsilon > 0\). That is, we provide a polynomial-size approximate kernelization scheme (PSAKS) and make first steps towards a PSAKS for the parameter \(c\).

Keywords: Eulerian extension; capacitated arc routing; lossy kernelization; above-guarantee parameterization; NP-hard problem; parameterized complexity

1 Introduction

In the framework of lossy kernelization [17, 33], we study trade-offs between the provable effect of data reduction and the provably achievable solution quality for the following classical vehicle routing problem [35].

Problem 1.1 (Rural Postman Problem, RPP).

Input: A graph \(G = (V, E)\) with \(n\) vertices, edge weights \(\omega: E \to \mathbb{N} \cup \{0\}\), and a multiset \(R\) of required edges of \(G\).

Task: Find a closed walk \(W^*\) in \(G\) containing each edge of \(R\) and minimizing the total weight \(\omega(W^*)\) of the edges on \(W^*\).

We call any closed walk containing each edge of \(R\) an RPP tour. We will also consider the decision variant of RPP, where one additionally gets a \(k \in \mathbb{N}\) at the input and the task is to decide whether there is an RPP tour \(W\) of cost \(\omega(W) \leq k\).
RPP has direct applications in snow plowing, street sweeping, meter reading [7, 15], vehicle depot location [21], drilling, and plotting [20, 23]. The undirected version occurs especially in rural areas, where service vehicles can operate in both directions even on one-way roads [12]. Moreover, RPP is a special case of the Capacitated Arc Routing Problem (CARP) [22] and used in all “route first, cluster second” algorithms for CARP [1, 6, 39], which are notably the only ones with proven approximation guarantees [4, 27, 40]. Improved approximations for RPP automatically lead to better approximations for CARP.

Unfortunately, containing the metric Traveling Salesman Problem as a special case, RPP is APX-hard [28]. While there is a folklore polynomial-time 3/2-approximation, we aim for $(1 + \varepsilon)$-approximations for all $\varepsilon > 0$. Since finding such approximations typically requires exponential time, we present data reduction rules for this task. Their effectivity depends on the desired approximation factor.

1.1 Our contributions and outline of this paper

In Section 2, we introduce basic notation. In Section 3, we prove basic structural properties of optimal RPP solutions.

In Section 4, we prove that data reduction for RPP is hard when required to maintain optimal solvability. To this end, we employ the recently introduced concept of WK[1]-hardness [26]: it is conjectured that, if a problem is WK[1]-hard with respect to some parameter $k$, then it has no Turing kernel of size $\text{poly}(k)$, that is, it cannot be polynomial-time reduced to solving instances of size $\text{poly}(k)$.

Theorem 1.2. RPP is WK[1]-complete parameterized by the minimum $\omega(W^*) - \omega(R) + |W^*| - |R|$ among the optimal solutions $W^*$, where WK[1]-hardness holds even in complete graphs with metric edge weights 1 and 2.

Note that, herein, $\omega(W^*) - \omega(R) + |W^*| - |R|$ measures the number and cost of the deadheading edges traversed additionally to the required ones.

In contrast to Theorem 1.2, in Section 5, we show that RPP is effectively preprocessable if one is interested in $(1 + \varepsilon)$-approximations.

Theorem 1.3. For any $\varepsilon > 0$, any RPP instance $(G, R, \omega)$ can be reduced to an RPP instance $(G', R', \omega')$ in $O(n^3 + |R|)$ time such that

(i) the number of vertices in $G'$ is $2b + O(c/\varepsilon)$,
(ii) the number of required edges is $|R'| \leq 4b + O(c/\varepsilon)$,
(iii) the maximum edge weight with respect to $\omega'$ is $O((b + c)/\varepsilon)$,
(iv) any $\alpha$-approximate solution for $I'$ for some $\alpha \geq 1$ can be transformed into an $\alpha(1 + \varepsilon)$-approximate solution for $I$ in polynomial time,

where $b$ is the number of vertices of $G$ incident to an odd number of edges in $R$ and $c$ is the number of connected components formed by the edges in $R$.

Theorems 1.2 and 1.3 complement each other since $|W^*| - |R| \geq \max\{b/2, c\}$ (see Section 3.3). Notably, the $\alpha$-approximate solution for $I'$ in Theorem 1.3 may be
obtained by any means, for example exact algorithms or heuristics. Thus, Theorem 1.3 can be used to speed up expensive heuristics without much loss in the solution quality. In terms of the recently introduced concept of lossy kernelization [33], Theorem 1.3 yields a polynomial-size approximate kernelization scheme (PSAKS).

1.2 Related work

Classical complexity. RPP is strongly NP-hard [19, 32], its special case with \( R = E \) is the polynomial-time solvable Chinese Postman problem [10, 11]. Containing the metric Traveling Salesman Problem as a special case, RPP is APX-hard [28]. There is a folklore polynomial-time 3/2-approximation (we refer to arc routing surveys [5, 15] for a detailed algorithm description).

Parameterized complexity. Dorn et al. [9] showed a \( O(4^d \cdot n^3) \)-time algorithm for the directed RPP, where \( d = |W^*| - |R| \) is the minimum number of deadheading arcs in an optimal solution \( W^* \). It can be easily adapted to the undirected RPP. Sorge et al. [37] showed a \( O(4^{c \log b^2} \text{poly}(n)) \)-time algorithm for the directed RPP, where \( c \) is the number of (weakly) connected components induced by the required arcs in \( R \) and \( b = \sum_{v \in V} |\text{indeg}(v) - \text{outdeg}(v)| \). It is not obvious whether this algorithm can be adapted to the undirected RPP maintaining its running time. Gutin et al. [25] showed a randomized algorithm that solves the directed and undirected RPP in \( f(c) \text{poly}(n) \) time if edge weights are bounded polynomially in \( n \). The existence of a deterministic algorithm with this running time is open [5, 25, 38].

Exact kernelization. RPP can easily be reduced to an equivalent instance with \( 2|R| \) vertices [5]. Using a theorem of Frank and Tardos [18] like Etscheid et al. [16], from this one gets a so-called problem kernel of size polynomial in the number of required edges. In contrast, Sorge et al. [37] showed that, unless the polynomial-time hierarchy collapses, the directed RPP has no problem kernel of size polynomial in the number of deadheading arcs. This result is strengthened by our Theorem 1.2, which shows even WK[1]-hardness, also of the directed RPP.

Lossy kernelization. Due to the kernelization hardness of many problems, recently the concept of approximate kernelization has gained increased interest [17, 33]. In this context, Eiben et al. [13] called for finding connectivity-constrained problems that do not have polynomial-size kernels but \( \alpha \)-approximate polynomial-size kernels. Our Theorems 1.2 and 1.3 exhibit that RPP is such a problem. Among the so far few known lossy kernels [13, 14, 30, 31, 33], our Theorem 1.3 stands out since it shows a time and size efficient PSAKS, which is a property previously observed only in results of Krithika et al. [30]. Moreover, Theorem 1.3 is apparently the first lossy kernelization result for parameters above lower bounds, whose study has been initiated by Razgon and O’Sullivan [36].
2 Preliminaries

2.1 Sets and multisets

By $\mathbb{N}$ we denote the set of natural numbers including zero. For two multisets $A$ and $B$, $A \uplus B$ is the multiset obtained by adding the multiplicities of elements in $A$ and $B$. By $A \setminus B$ we denote the multiset obtained by subtracting the multiplicities of elements in $B$ from the multiplicities of elements in $A$. Finally, given some weight function $\omega: A \to \mathbb{N}$, the weight of a multiset $A$ is $\omega(A) := \sum_{e \in A} \nu(e)\omega(e)$, where $\nu(e)$ is the multiplicity of $e$ in $A$.

Graph theory. We generally consider multigraphs $G = (V, E)$ with a set $V(G) := V$ of vertices, a multiset $E(G) := E$ over $\{(u, v) \mid u, v \in V\}$ of (undirected) edges, and edge weights $\omega: E \to \mathbb{N}$. Graphs are allowed to have loops and parallel edges. For a multiset $R$ of edges, we denote by $V(R)$ the set of their incident vertices.

Paths and cycles. A walk from $v_0$ to $v_\ell$ in $G$ is a sequence $w = (v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell)$ such that $e_i$ is an edge with end points $v_{i-1}$ and $v_i$ for each $i \in \{1, \ldots, \ell\}$. If $v_0 = v_\ell$, then we call $w$ a closed walk. If all vertices on $w$ are pairwise distinct, then $w$ is a path. If only its first and last vertex coincide, then $w$ is a cycle. By $E(w)$ we denote the multiset of edges on $w$. The length of walk $w$ is its number $|w| := \ell = |E(w)|$ of edges. The weight of walk $w$ is $\omega(w) := \sum_{i=1}^{\ell} \omega(e_i)$. An Euler tour for $G$ is a closed walk that traverses each edge of $G$ exactly as often as it is present in $G$. A graph is Eulerian if it allows for an Euler tour.

Connectivity and blocks. Two vertices $u, v$ of $G$ are connected if there is a path from $u$ to $v$ in $G$. A connected component of $G$ is a maximal subgraph of $G$ in which the vertices are mutually connected. A vertex $v$ of $G$ is a cut vertex if removing $v$ and its incident edges increases the number of connected components of $G$. A biconnected component or block of $G$ is a maximal subgraph without cut vertices.

Edge- and vertex-induced subgraphs. For a subset $U \subseteq V$ of vertices, the subgraph $G[U]$ of $G = (V, E)$ induced by $U$ consists of the vertices of $U$ and all edges of $G$ between them (respecting multiplicities). For a multiset $R$ of edges of $G$, $G(R) := (V(R), R)$ is the graph induced by the edges in $R$. For a walk $w$, we also denote $G(w) := G(E(w))$. Note that $G(R)$ and $G(w)$ do not contain isolated vertices yet might contain edges with a higher multiplicity than $G$ and, therefore, are not necessarily sub(multi)graphs of $G$.

2.2 Kernelization

Kernelization is a notion of provably effective and efficient data reduction [24, 29] from parameterized complexity theory [8].

Definition 2.1 (parameterized problem). Instances $(x, k) \in \Sigma^* \times \mathbb{N}$ of parameterized (decision) problems $L \subseteq \Sigma^* \times \mathbb{N}$ consist of input $x$ and parameter $k$.
**Definition 2.2 (Kernelization).** Let \( L \subseteq \Sigma^* \times \mathbb{N} \) be a parameterized problem. A kernelization is an algorithm that maps any instance \((x, k) \in \Sigma^* \times \mathbb{N}\) to an instance \((x', k') \in \Sigma^* \times \mathbb{N}\) in \(\text{poly}(|x| + k)\) time such that

(i) \((x, k) \in L \iff (x', k') \in L'\), and

(ii) \(|x'| + k' \leq f(k)\) for some computable function \(f\).

We call \((x', k')\) the problem kernel and \(f\) its size.

A generalization of problem kernels are Turing kernels, where one is allowed to generate multiple reduced instances instead of a single one.

**Definition 2.3 (Turing kernelization).** Let \( L \subseteq \Sigma^* \times \mathbb{N} \) be a parameterized problem. A Turing kernelization for \(L\) is an algorithm \(A\) that decides \((x, k) \in L\) in polynomial time given access to an oracle that answers \((x', k') \in L\) in constant time for any \((x', k') \in \Sigma^* \times \mathbb{N}\) with \(|x'| + k' \leq f(k)\), where \(f\) is an arbitrary function called the size of the Turing kernel.

**Theorem 1.2** means that the decision variant of RPP presumably neither allows for ordinary nor Turing kernels of polynomial size. Instead, we will prove polynomial-size approximate kernelization schemes [33].

**Definition 2.4 (polynomial-size approximate kernelization scheme).** A polynomial-size approximate kernelization scheme (PSAKS) for an optimization problem \(L\) with parameter \(k\) consists of two algorithms: for each constant \(\varepsilon > 0\),

(i) the first algorithm reduces an instance \(I\) of \(L\) to an instance \(I'\) of size \(\text{poly}(k)\) in polynomial time,

(ii) the second algorithm turns any \(\alpha\)-approximate solution for \(I'\) into an \(\alpha \cdot (1 + \varepsilon)\)-approximate solution for \(I\) in polynomial time.

**Kernelization hardness.** Parameterized problems that are \(\text{WK}[1]\)-complete do not have problem kernels of polynomial size unless the polynomial-time hierarchy collapses and are conjectured not to have Turing kernels of polynomial size either [26]. An archetypal \(\text{WK}[1]\)-complete problem is the following [26]:

**Problem 2.5 (NDTM Halting).**

**Input:** A nondeterministic Turing machine \(M\) and an integer \(t\).

**Parameter:** \(t \log |M|\).

**Question:** Does \(M\) halt in \(t\) steps on the empty input string?

The class \(\text{WK}[1]\) can now be defined as the class of all parameterized problems reducible to NDTM Halting using the following type of reduction.

**Definition 2.6 (polynomial parameter transformation).** A polynomial parameter transformation (PPT) of a parameterized problem \(L \subseteq \Sigma^* \times \mathbb{N}\) into a parameterized problem \(L' \subseteq \Sigma^* \times \mathbb{N}\) is an algorithm that maps any instance \((x, k)\) to an instance \((x', k')\)
(i) in \( \text{poly}(|x| + k) \) time such that
(ii) \((x, k) \in L \iff (x', k') \in L' \) and
(iii) \( k' \in \text{poly}(k) \).

**Definition 2.7 (WK[1]-complete).** WK[1] is the class of parameterized problems PPT-reducible to NDTM Halting. A parameterized problem \( L \) is WK[1]-hard if every problem in WK[1] is PPT-reducible to \( L \). It is WK[1]-complete if it is WK[1]-hard and contained in WK[1].

Notably, since PPT-reducibility is a transitive relation, to prove WK[1]-hardness of a problem \( L \), it is enough to PPT-reduce on WK[1]-hard problem to \( L \).

**3 Solution structure**

In this section, we prove fundamental properties of optimal solutions to RPP. To make these hold, we first establish the triangle inequality in Section 3.1. In Section 3.2, we translate RPP to the problem of finding Eulerian extensions. In Section 3.3, we derive inequalities to bound parts of optimal solutions.
3.1 Triangle inequality

We will assume that the weight function satisfies the triangle inequality:

**Proposition 3.1 ([3]).** In $O(n^3)$ time, an RPP instance $(G, R, \omega)$ can be turned into an RPP instance $(G', R, \omega')$ such that

- $G'$ is a complete graph,
- $\omega'$ satisfies the triangle inequality, and
- any $\alpha$-approximate RPP tour for $(G, R, \omega')$ can be turned into an $\alpha$-approximate RPP tour for $(G, R, \omega)$ in polynomial time.

**Remark 3.2.** Proposition 3.1 holds in particular for $\alpha = 1$ and does not increase the number of connected components of $G(R)$, the number of odd-degree vertices of $G(R)$, the number and cost of deadheading edges of an optimal RPP tour. Thus, it is sufficient to prove Theorems 1.2 and 1.3 for RPP with triangle inequality. We will henceforth assume that the input graph is complete and satisfies the triangle inequality.

3.2 Edge-minimizing Eulerian extensions

Consider any RPP tour $W$ for an RPP instance $(G, R, \omega)$. Then $G(W)$ is an Eulerian supergraph of $G(R)$ whose total edge weight is $\omega(W)$. Moreover, any Eulerian supergraph $G(W')$ of $G(R)$ yields an RPP tour for $(G, R, \omega)$ of total weight $\omega(W')$. Thus, RPP tours one-to-one correspond to Eulerian extensions [38]:

**Definition 3.3 (Eulerian extension, edge-minimizing).** An Eulerian extension (EE) for an RPP instance $(G, R, \omega)$ is a multiset $S$ of edges such that $G(R \cup S)$ is Eulerian. We say that an Eulerian extension $S$ is edge-minimizing if there is no Eulerian extension $S'$ with $|S'| < |S|$ and $\omega(S') \leq \omega(S)$.

In the following, we will concentrate on finding minimum-weight Eulerian extensions rather than RPP tours and exploit that a graph without isolated vertices is Eulerian if and only if it is connected and balanced:

**Definition 3.4 (balanced).** A vertex is balanced if it has even degree. A graph is balanced if each of its vertices is balanced.

Thus, solving RPP reduces to finding a minimum-weight set $S$ of edges such that $G(R \cup S)$ is connected and balanced. Since an Euler tour in the Eulerian graph $G(R \cup S)$ is computable in linear time using Hierholzer’s algorithm, we can easily recover an RPP tour from an Eulerian extension.

**Proposition 3.5.** Let $(G, R, \omega)$ be an RPP instance.

(i) From any RPP tour $W$ for $(G, R, \omega)$, one can compute an Eulerian extension $S$ of cost $\omega(W) = \omega(R) + \omega(S)$ in time linear in $|W|$.

(ii) From any Eulerian extension $S$ for $(G, R, \omega)$, one can compute an RPP tour $W$ of cost $\omega(W) = \omega(R) + \omega(S)$ in time linear in $|R| + |S|$.
Fig. 3.1. Proof that the bound given in Lemma 3.8 is tight: $G(R)$ has $c = 4$ connected components and $2c - 2 = 6$ vertices are incident to the Eulerian extension.

Assuming the triangle inequality, any RPP tour can be shortcut so as not to contain vertices that are not incident to required edges.

**Observation 3.6.** Any edge-minimizing Eulerian extension $S$ for an RPP instance $(G, R, \omega)$ satisfies $V(S) \subseteq V(R)$.

The following lemma, in particular, shows that no edge-minimizing Eulerian extension contains required edges between balanced vertices.

**Lemma 3.7.** An edge-minimizing Eulerian extension $S$ for an RPP instance $(G, R, \omega)$ does not contain any edge $\{u, v\}$ such that $u$ and $v$ belong to the same connected component of $G(R)$ and such that $u$ is balanced.

**Proof.** Towards a contradiction, assume that $\{u, v\} \in S$. Since $u$ is balanced in $G(R)$ and $G(R \cup S)$, $S$ additionally contains an edge $\{u, w\}$ (possibly, $v = w$). Then $S' \setminus \{\{u, v\}, \{u, w\}\} \cup \{\{v, w\}\}$ satisfies $|S'| < |S|$ and also is an Eulerian extension: the balance of $u$, $v$ and $w$ is the same in $G(R \cup S)$ and $G(R \cup S')$, and $u$ still is connected to $v$ in $G(R \cup S')$ since $u$ and $v$ belong to the same connected component of $G(R)$. Finally, using the triangle inequality, $\omega(S') \leq \omega(S)$, contradicting the fact that $S$ is edge-minimizing. $\square$

**Lemma 3.8.** Let $(G, R, \omega)$ be an RPP instance and $c$ be the number of connected components of $G(R)$. At most $2c - 2$ balanced vertices in $G(R)$ are incident to edges of an edge-minimizing Eulerian extension and this bound is tight.

**Proof.** Let $S$ be an edge-minimizing Eulerian extension for $(G, R, \omega)$ and $T \subseteq S$ be an inclusion-minimal subset such that $G(R \cup T)$ is connected. Then $|T| = c - 1$ and $S \setminus T$ is an edge-minimizing Eulerian extension for $(G, R \cup T, \omega)$. Thus, by Observation 3.6, $V(S \setminus T) \subseteq V(R \cup T)$. Combining this with Lemma 3.7, $S \setminus T$ does not contain any edges incident to balanced vertices of $G(R \cup T)$. The only vertices that might be balanced in $G(R)$ but not in $G(R \cup T)$ are the at most $2c - 2$ end points of edges in $T$. In the worst case, all of them are incident to edges in $S$. Figure 3.1 shows that the bound is tight. $\square$

**Remark 3.9.** The following lemma shows that an edge-minimizing Eulerian extension contains exactly one edge incident to each unbalanced vertex of $G(R)$ and either no or two edges incident to each balanced vertex of $G(R)$.

**Lemma 3.10.** Each vertex $v \in V$ is incident to at most two edges of an edge-minimizing Eulerian extension $S$ for an RPP instance $(G, R, \omega)$. 
Fig. 3.2. Illustration of the proof of Lemma 3.10. Snaked edge indicates a $u_2$-$u_3$ path.

Proof. Towards a contradiction, assume that $S$ contains $e_i = \{u_i, v\}$ for $i \in \{1, 2, 3\}$. Obviously, $S' = (S \setminus \{e_1, e_2\}) \cup \{\{u_1, u_2\}\}$ satisfies $|S'| < |S|$. Moreover, $\omega(S') \leq \omega(S)$ follows from the triangle inequality. We argue that $S'$ is an Eulerian extension, contradicting the choice of $S$. The proof is illustrated in Figure 3.2.

The balance of $v, u_1, u_2,$ and $u_3$ is the same in $G(R \uplus S)$ and $G(R \uplus S')$. It remains to show that the vertices $v, u_1, u_2, u_3$ are connected in $G(R \uplus S')$. To this end, observe that $G(R \uplus S)$ is Eulerian and thus contains two edge-disjoint paths between $u_2$ and $u_3$. At most one of these paths, namely $(u_2, v, u_3)$, is lost in $G(R \uplus S')$. Thus, $G(R \uplus S')$ contains the edges $\{v, u_3\}$, $\{u_1, u_2\}$, and a path between $u_2$ and $u_3$. \qed

3.3 Inequalities

Throughout this work, we will use the following notation.

Definition 3.11. In the context of an RPP instance $(G, R, \omega)$, we denote by

- $R$ – the set of required arcs,
- $c$ – the number of connected components in $G(R)$,
- $b$ – the number of imbalanced vertices in $G(R)$,
- $W^*$ – a minimum-weight RPP tour with a minimum number of edges,
- $D$ – an minimum-weight edge-minimizing Eulerian extension for $(G, R, \omega)$,
- $T$ – a minimum-weight set of edges such that $G(R \uplus T)$ is connected, of minimum cardinality,
- $M$ – a minimum-weight set of edges such that $G(R \uplus M)$ is balanced, of minimum cardinality.

Lemma 3.12. The following relations hold:

\[
\begin{align*}
\omega(W^*) &= \omega(R) + \omega(D), \quad (3.1) \\
|W^*| &= |R| + |D|, \quad (3.5) \\
\omega(M) &\leq \omega(D), \quad (3.2) \\
2b &\leq |M| \leq |D|, \quad (3.6) \\
\omega(T) &\leq \omega(D), \quad (3.3) \\
c - 1 &\leq |T| \leq |D|, \quad (3.7) \\
\omega(D) &\leq \omega(M) + 2\omega(T), \quad (3.4) \\
|D| &\leq |M| + 2|T|, \quad (3.8)
\end{align*}
\]

where $|S| \leq |M| + 2|T|$ holds for any edge-minimizing Eulerian extension $S$. \hfill $\Box$
Fig. 4.1. Illustration for the proof of Lemma 4.1. Thick solid edges are required. Thin dashed edges are a colorful cycle and, at the same time, an Eulerian extension.

Proof. (3.1) and (3.5) follow from Proposition 3.5. (3.2) and (3.6) follow by choice of $M$ and the fact that, since we assume the triangle inequality, $M$ is simply a minimum-weight perfect matching on the $b$ imbalanced vertices in $G(R)$ [11]. (3.3) and (3.7) follow by choice of $T$. (3.4) follows from the fact that $G(R \cup M)$ is balanced and adding each edge of $T$ twice to it does not change the balance of vertices, yet connects the graph. We now derive (3.8). Consider any edge-minimizing Eulerian extension $S$.

- By Lemma 3.8, a set $X$ of at most $2c - 2$ balanced vertices in $G(R)$ are incident to edges of $S$.
- By Remark 3.9, $S$ contains exactly one edge incident to each imbalanced vertex in $G(R)$ and exactly two edges incident to each vertex in $X$.

Thus, by the handshaking lemma, we get $2|X| + b = 2|S|$. Therefore, $|S| = |X| + b/2 \leq 2c - 2 + |M| = 2|T| + |M|$. \hfill \qed

4 Hardness of kernelization for Rural Postman

In this section, we prove Theorem 1.2. We first show WK[1]-hardness in Lemma 4.1, then we show containment in WK[1] in Lemma 4.3. Then Theorem 1.2 immediately follows from Lemmas 4.1 and 4.3 using (3.1) and (3.5).

Lemma 4.1. RPP is WK[1]-hard parameterized by $\omega(D) + |D|$ even in complete graphs with metric edge weights one and two.

To prove Lemma 4.1, we provide a polynomial parameter transformation from the following known WK[1]-complete parameterized problem [26].

Problem 4.2 (Multicolored Cycle).
Input: An undirected graph $G = (V, E)$ with a vertex coloring $c: V \rightarrow \{1, \ldots, k\}$.
Parameter: $k$.
Question: Is there a cycle in $G$ containing exactly one vertex of each color?

Proof (of Lemma 4.1). Let $I := (G, c)$ with a graph $G = (V, E)$ and a vertex $k$-coloring $c: V \rightarrow \{1, \ldots, k\}$ be an instance of Multicolored Cycle. For $i \in$
\{1, \ldots, k\}, denote \(V_i := \{v \in V \mid c(v) = i\}\). Now consider the RPP instance \(I' = (G', R, \omega)\), illustrated in Figure 4.1: \(G' = (V, E')\) is a complete graph, the set \(R\) contains a cycle on the vertices in \(V_i\) for each \(i \in \{1, \ldots, k\}\), and

\[
\omega: E' \to \mathbb{N}, e \mapsto \begin{cases} 
1 & \text{if } e \in E \cup R, \\
2 & \text{otherwise}.
\end{cases}
\]

Note that, since all edge weights \(\omega\) are one and two, \(\omega\) is metric. Moreover, since \(G'(R)\) is balanced, from (3.4) and (3.8), we get \(|D| + \omega(D) \leq 2|T| + 2\omega(T) \in O(k)\). We show that \(I\) is a yes-instance if and only if \(I'\) has an RPP tour of cost \(\omega(R) + k = |R| + k\), which, by Proposition 3.5, is equivalent to having an Eulerian extension \(S\) of cost \(\omega(S) \leq k\).

\((\Rightarrow)\) Let \(S\) be a multicolored cycle in \(G\). Since \(G'(R)\) is a disjoint union of cycles, \(G'(R)\) is balanced. Since \(S\) is a cycle, \(G'(R \uplus S)\) is also balanced. Since \(S\) contains one vertex of each color, \(G'(R \uplus S)\) is additionally connected. Thus, \(S\) is an Eulerian extension for \((G', R, \omega)\). Since \(S\) consists of edges of \(G\), we conclude \(\omega(S) = |S| = k\).

\((\Leftarrow)\) Let \(S\) be an edge-minimizing Eulerian extension with \(\omega(S) \leq k\) for \((G', R, \omega)\). Since \(G'(R)\) and \(G'(R \uplus S)\) are balanced, so is \(G'(S)\). Since \(G'(R \uplus S)\) is connected and \(G'(S)\) is balanced, \(S\) contains at least two edges incident to a vertex in \(V_i\) for each \(i \in \{1, \ldots, k\}\). Thus, since \(\omega(S) \leq k\), \(G'(S)\) has to contain exactly \(k\) edges, all of weight one, and exactly one vertex of \(V_i\) for each \(i \in \{1, \ldots, k\}\), that is, \(k\) vertices. Since \(G'(S)\) is balanced, it follows that \(G'(S)\) is a collection of cycles whose color sets do not intersect. Thus, if \(G'(S)\) was not connected, then \(G'(R \uplus S)\) would not be either. We conclude that \(G'(S)\) is connected, that is, a single cycle containing exactly one vertex of each color. By Lemma 3.7, none of the edges in \(S\) are in \(R\). Since all of them have weight one, they are in \(G\). It follows that \(S\) forms a multicolored cycle in \(G\). \(\square\)

Having shown \(WK[1]\)-hardness in Lemma 4.1, we now show containment in \(WK[1]\), concluding the proof of Theorem 1.2. Note that we showed hardness for the parameter \(|D| + \omega(D)|\), whereas containment we show even for the smaller parameter \(|D| + \log \omega(D)|\). This means that, if any problem in \(WK[1]\) turns out to have a polynomial-size Turing kernel, then there will be a Turing kernel for RPP with size polynomial even in \(|D| + \log \omega(D)|\).

**Lemma 4.3.** RPP parameterized by \(|D| + \log \omega(D)|\) is contained in \(WK[1]\).

**Proof.** We prove a polynomial-parameter transformation from RPP parameterized by \(|D| + \log \omega(D)|\) to NDTM Halting (Problem 2.5). By Remark 3.2, it is sufficient to reduce RPP with triangle inequality. To this end, we construct a number \(t \in \mathbb{N}\) and a nondeterministic Turing machine \(M\) that, given an empty input string, has a computation path halting within \(t\) steps if and only if a given RPP instance \(I = (G, R, \omega)\) on a graph \(G = (V, E)\) with \(n\) vertices and triangle inequality has an RPP tour of given cost \(\omega(R) + k\), that is, an Eulerian extension of cost \(k\). For the polynomial-parameter transformation to be correct,
we will ensure \( t \log |\mathcal{M}| \in \text{poly}(d_1 + \log d_2) \), where \( d_1 := |\mathcal{M}| + 2|T| \leq 3|D| \) and \( d_2 := \omega(M) + 2\omega(T) \leq 3\omega(D) \) by Lemma 3.12.

If \( k \geq d_2 \), then, by (3.4) and (3.8), \( I \) is a yes-instance and we simply return \( t = 1 \) and a Turing machine \( \mathcal{M} \) of constant size that immediately halts. Thus, we henceforth assume

\[
 k < d_2. \tag{4.1}
\]

By (3.8), there is an optimal Eulerian extension of at most \( d_1 \) edges for \((G, R, \omega)\). Thus, if \( d_1 \leq \log n \), then we optimally solve \( I \) in polynomial time [9] and return \( t = 1 \) and a Turing machine of constant size that immediately halts or never halts in dependence of whether \( I \) is a yes-instance. Thus, we henceforth assume

\[
 \log n < d_1. \tag{4.2}
\]

By (4.1), edges \( e \in E \) with cost \( \omega(e) \geq d_2 \) will not be part of the sought Eulerian extension of cost \( k \), thus we lower their weight to \( d_2 \) and henceforth assume

\[
 \omega(e) \leq d_2 \quad \text{for all } e \in E. \tag{4.3}
\]

We now construct Turing machine \( \mathcal{M} \). The Turing machine has vertices of the input graph as alphabet. The state names of \( \mathcal{M} \) encode the incidence matrix of \( G \), the weight \( \omega(e) \) of each edge \( e \in E \) in binary, and, for each vertex \( v \in V \), the number of its connected component in \( G(R) \). Turing machine \( \mathcal{M} \) uses three tapes: on the edge tape, it guesses at most \( d_1 \) edges, on the balancing tape, it records how the balance of the initially \( O(d_1) \) imbalanced vertices (by (3.6)) changes by adding guessed edges, on the connection tape, it records which of the initially \( O(d_1) \) connected components of \( G(R) \) (by (3.7)) gets connected by the guessed \( d_1 \) edges. The program of Turing machine \( \mathcal{M} \) is as follows. On empty input, at most \( d_1 \) times:

1. In constant time, guess an edge of \( G \) (encoded in the state name) and append it to the edge tape.
2. Update the balancing tape in \( \text{poly}(d_1) \) steps because there are only \( O(d_1) \) vertices on it.
3. Update the connection tape in \( \text{poly}(d_1) \) steps because there are only \( O(d_1) \) component names on it.
4. If the last guessed edge was \( \{u, v\} \) such that \( v \) is balanced in \( G(R) \), then next guess an edge of the form \( \{v, w\} \), so that \( v \) is kept balanced.

If, after at most \( d_1 \) guessed edges, the computation does not reach a configuration where all initially imbalanced vertices are balanced and all components of \( G(R) \) are connected, then \( \mathcal{M} \) goes into an infinite loop. Otherwise, in \( \text{poly}(d_1) \) steps, we reached such a configuration and it remains to check whether the guessed edges have cost at most \( k \). To this end, \( \mathcal{M} \) can write down the weights of the at most \( d_1 \) guessed edges in binary, sum them up, and compare them to \( k \) in \( \text{poly}(d_1 + \log d_2) \) steps because of (4.1) and (4.3). If their cost is more than \( k \), then \( \mathcal{M} \) goes into an infinite loop. Otherwise, \( \mathcal{M} \) stops. Observe that each
computation path of $\mathcal{M}$, if it terminates, then it does so within $t$ steps for some $t \in \text{poly}(d_1 + \log d_2)$.

For the correctness of the polynomial-parameter transformation, it remains to show $t \log |\mathcal{M}| \in \text{poly}(d_1 + \log d_2)$. Since $t \in \text{poly}(d_1 + \log d_2)$, it remains to show that $\log |\mathcal{M}| \in \text{poly}(d_1 + \log d_2)$. The graph $G$ can be hard-coded in Turing machine $\mathcal{M}$ with alphabet $V$ using $\text{poly}(n)$ symbols. The encoded edge weights have total size $\text{poly}(n + \log d_2)$ by (4.3). Its program therefore has size $\text{poly}(n + d_1 + \log d_2)$. Thus,

$$\log |\mathcal{M}| = \log \text{poly}(n + d_1 + \log d_2) \in \text{poly}(\log n + d_1 + \log d_2),$$

which, by (4.2) is $\text{poly}(d_1 + \log d_2)$. $\square$

5 Approximate kernelization schemes for Rural Postman

In Section 4, we have seen that provably effective and efficient data reduction for RPP is hard when one requires exact solutions. In this section, we show effective data reduction rules that only slightly decrease the solution quality. Indeed, we will proof Theorem 1.3. To this end, in Sections 5.1 to 5.3, we present three data reduction rules. In Section 5.4, we then show how to apply these rules to obtain a polynomial-size approximate kernelization scheme (PSAKS) of size $2^b + O(c/\varepsilon)$, proving Theorem 1.3. Finally, in Section 5.5, we discuss some problems that one faces when trying to improve it to a PSAKS of size $O(c)$.

5.1 Removing vertices non-incident to required edges

We can simply delete vertices that are not incident to required edges [5].

Reduction Rule 5.1. Let $(G, R, \omega)$ be an RPP instance with triangle inequality. Delete all vertices that are not incident to edges in $R$.

Since by Observation 3.6, no edge-minimizing Eulerian extension uses vertices outside of $V(R)$, the following proposition is immediate.

Proposition 5.2. Reduction Rule 5.1 turns an RPP instance $(G, R, \omega)$ into an RPP instance $(G', R, \omega)$ such that

– any edge-minimizing Eulerian extension for $(G, R, c)$ is one for $(G', R, c)$ and
– any Eulerian extension for $(G', R, c)$ is one for $(G, R, c)$.

5.2 Reducing the number of required edges

In this section, we present a data reduction rule to shrink the set of required edges. This will be crucial since other data reduction rules only reduce the number of vertices, yet may leave the the multiset of required edges between unbounded.
Reduction Rule 5.3. Let \((G, R, \omega)\) be an instance of RPP and \(C\) be a cycle in \(G(R)\) such that \(G(R \setminus C)\) has the same number of connected components as \(G(R)\), then delete the edges of \(C\) from \(R\).

Lemma 5.4. Using Reduction Rule 5.3, one can in \(O(|R|)\) time compute a set \(R' \subseteq R\) of required edges with the following properties.

(i) Any Eulerian extension for \((G, R', \omega)\) is one for \((G, R, \omega)\) and vice versa.
(ii) The number of edges in each connected component of \(G(R')\) with \(k\) vertices is at most \(\max\{1, 2k - 2\}\).

Proof. We apply Reduction Rule 5.3 as follows. For \(i \in \{1, \ldots, c\}\), let \(R_i \subseteq R\) be the set of required edges in the \(i\)-th connected component of \(G(R)\). In \(O(|R_i|)\) time, one can compute a depth-first search tree \(T_i\) of \(G(R_i)\), which is a spanning tree of \(G(R_i)\). Now we remove all cycles from \(G(R_i \setminus T_i)\) as follows. We start a depth-first search on \(G(R_i \setminus T_i)\). Whenever we meet a vertex \(v\) a second time, we backtrack to the previous occurrence of \(v\), deleting all visited edges from the graph on the way. This procedure removes all cycles from \(G(R_i \setminus T_i)\) and looks at each edge of \(R_i \setminus T_i\) at most twice, thus works in \(O(|R_i|)\) time.

(i) Any two vertices are connected in \(G(R)\) if and only if they are connected in \(G(R')\). Moreover, the balance of each vertex is the same in \(G(R)\) and \(G(R')\).

(ii) Each component of \(G(R')\) with \(k = 1\) vertex has one edge (a loop). Each component of \(G(R')\) with \(k > 1\) vertices consists of \(k - 1\) edges of a spanning tree \(T_i\) for some \(i \in \{1, \ldots, c\}\) and at most \(k - 1\) additional edges, otherwise they would contain a cycle. \(\square\)

5.3 Reducing the number of balanced vertices

In this section, we present a data reduction rule that removes balanced vertices. To this end, we introduce an operation that allows us to remove balanced vertices while maintaining the balance of their neighbors.

First, the following lemma in particular shows that removing a balanced vertex with all its incident edges changes the balance of an even number of vertices. This allows us to restore their original balance by adding a matching to the set of required edges, not increasing the total weight of required edges. This will be crucial to prove that our reduction rules maintain approximation factors.

Lemma 5.5. Let \(\Gamma = (V, E)\) be a multigraph, \(\omega: \{\{u, v\} \mid u, v \in V\} \rightarrow \mathbb{N}\) satisfy the triangle inequality, and \(F\) be an even-cardinality submultiset of edges incident to a common vertex \(v \in V\). Then

(i) The set \(U \subseteq V \setminus \{v\}\) of vertices incident to an odd number of edges of \(F\) has even cardinality.
(ii) For any matching \(M_v\) in the complete graph on \(U\), \(\omega(M_v) \leq \omega(F)\) and \(|M_v| \leq |F|\).
On $(1 + \varepsilon)$-approximate problem kernels for the Rural Postman Problem

Proof. (i) Any graph, in particular $\Gamma(F)$, has an even number of odd-degree vertices. Since $|F|$ is even, $v$ is not one of them.

(ii) Let $e_i := \{x_i, y_i\}$ for $i \in \{1, \ldots, |M_v|\}$ be the edges of $M_v$. Then there are pairwise edge-disjoint paths $p_i := (x_i, v, y_i)$ for $i \in \{1, \ldots, |M_v|\}$ in $\Gamma(F)$. Thus

$$\omega(M_v) = \sum_{i=1}^{\left| M_v \right|} \omega(e_i) \leq \sum_{i=1}^{\left| M_v \right|} \omega(p_i) \leq \omega(F).$$

Finally, $|M_v| \leq |U| \leq |F|$. \hfill $\Box$

We now use Lemma 5.5 to define an operation that allows us to remove a balanced vertex from $G(R)$. It is illustrated in Figure 5.1.

**Definition 5.6 (vertex extraction).** Let $(G, R, \omega)$ be an RPP instance with $\omega$ satisfying the triangle inequality, $v$ be a vertex that

- is balanced in a connected component of $G(R)$ with at least three vertices and
- not a cut vertex of $G(R)$ or contained in exactly two blocks of $G(R)$,

and let $R_v \subseteq R$ be the required edges incident to $v$. The result of extracting $v$ is a set $R'$ constructed as follows:

(a) If $v$ is not a cut vertex of $G(R)$, then $R' = (R \setminus R_v) \cup M_v$, where $M_v$ is any perfect matching on the set of vertices incident to an odd number of edges of $R_v$.

(b) If $v$ is a cut vertex of $G(R)$ contained in exactly two blocks $A$ and $B$ of $G(R)$, then $R' = (R \setminus R_v) \cup M_v \cup \{\{a, b\}\}$, where $a$ is a neighbor of $v$ in $A$, $b$ is a neighbor of $v$ in $B$, and $M_v$ is any perfect matching on the set of vertices incident to an odd number of edges of $R_v \setminus \{\{a, v\}, \{b, v\}\}$.

**Lemma 5.7.** Let $(G, R, \omega)$ be an RPP instance and $R'$ be the result of extracting a balanced vertex $v$ of $G(R)$. Then the following properties hold.

(i) $V(R') = V(R) \setminus \{v\}$.

(ii) $\omega(R') \leq \omega(R)$ and $|R'| \leq |R|$. 

![Fig. 5.1. Illustration of Definition 5.6(a). Only required edges are shown. Thick edges on the right are the added matching $M_v$.](image)
(iii) Each vertex of \(G(R')\) is balanced if and only if it is balanced in \(G(R)\).
(iv) Two vertices of \(G(R')\) are connected if and only if they are so in \(G(R)\).
(v) Any multiset \(S\) of edges with \(V(S) \subseteq V(R')\) is an Eulerian extension for \((G, R', \omega)\) if and only if it is one for \((G, R, \omega)\).

Proof. (i) First, assume that \(R'\) was obtained according to Definition 5.6(a). Let \(R_v \subseteq R\) be the required edges incident to \(v\), \(U \subseteq V \setminus \{v\}\) be the set of vertices incident to an odd number of edges of \(R_v\), and \(W \subseteq V \setminus \{v\}\) be those incident to a positive even number of edges of \(R_v\). Obviously, \(V(R') \subseteq V(R) \setminus \{v\}\) and \(V(R') \supseteq V(R) \setminus (U \cup W \setminus \{v\})\). Moreover, \(V(R') \supseteq U\) since \(R'\) is obtained from \(R\) by removing \(R_v\) and adding at least one edge incident to each vertex of \(U\). Moreover, \(V(R') \supseteq W\): since \(v\) is in a connected component of \(G(R)\) with at least three vertices but not a cut vertex, the vertices in \(W\) are incident to edges in \(R \setminus R_v\). These are retained in \(G(R')\).

Now, if \(R'\) was obtained according to Definition 5.6(b), then \(v\) is in a connected component with at least three vertices but not a cut vertex of \(G(R)\). Thus, the same argument as in the previous paragraph works with \(R \supseteq \{a, b\}\) in place of \(R\) and \(R_v \setminus \{a, v\}, \{b, v\}\) in place of \(R_v\).

(ii)–(iv) If \(R'\) was obtained according to Definition 5.6(a), then (ii) and (iii) follow from Lemma 5.5 applied to \(\Gamma = G(R)\) and \(F = R_v\), whereas (iv) is clear (observe that \(v\) is not in \(G(R')\) in this case). Now, consider the case when \(R'\) was obtained according to Definition 5.6(b). Let \(R_1 := (R \setminus \{a, v\}, \{b, v\})\). Then, (ii)–(iv) hold for \(R_1\) in place of \(R'\) (\(v\) may be absent from \(G(R_1)\)). Now, observe that \(R' = (R_1 \setminus (R_v \setminus \{a, v\}, \{b, v\})) \cup M_v\). Thus, (ii) and (iii) follow from Lemma 5.5 with \(\Gamma = G(R_1)\) and \(F = R_v \setminus \{a, v\}, \{b, v\}\) (note that \(|F|\) is even since \(|R_v|\) is even). Finally, (iv) follows since \(v\) is not a cut vertex of \(G(R_1)\).

(v) We show that \(G(R \supseteq S)\) is connected and balanced if and only if \(G(R \supseteq S)\) is.

Connectivity. By (iv), two vertices of \(V(R')\) are connected in \(G(R')\) if and only if they are connected in \(G(R)\). Since \(V(S) \subseteq V(R') \subseteq V(R)\) by (i), two vertices in \(V(R') = V(R' \supseteq S)\) are connected in \(G(R' \supseteq S)\) if and only if they are connected in \(G(R \supseteq S)\). By (i), the only vertex of \(G(R \supseteq S)\) that is absent from \(G(R' \supseteq S)\) is \(v\), which is not isolated in \(G(R \supseteq S)\) since it is not isolated in \(G(R)\).

Balance. By (iii), each vertex in \(V(R')\) is balanced in \(G(R')\) if and only if it is balanced in \(G(R)\). Since \(V(S) \subseteq V(R') \subseteq V(R)\) by (i), each vertex in \(V(R') = V(R' \supseteq S)\) is balanced in \(G(R' \supseteq S)\) if and only if it is balanced in \(G(R \supseteq S)\). By (i), the only vertex in \(G(R \supseteq S)\) that is absent from \(G(R' \supseteq S)\) is \(v\). If so, then \(v \notin V(S)\) and \(v\) is balanced in \(G(R \supseteq S)\) because it is balanced in \(G(R)\).

We can now turn Definition 5.6 into a data reduction rule. Its parameter \(\gamma \in \mathbb{Q}\) allows a trade-off between aggressivity and introduced error.

Reduction Rule 5.8. Let \((G, R, \omega)\) be an RPP instance with \(G = (V, E)\), \(\omega\) satisfying the triangle inequality, and \(\gamma \in \mathbb{Q}\). Let \(C_i\) be the vertices in connected component \(i \in \{1, \ldots, c\}\) of \(G(R)\) and \(B_i \subseteq C_i\) be an inclusion-maximal set of vertices such that, for each \(u, v \in B_i\) with \(u \neq v\), one has \(\omega(\{u, v\}) > \gamma\). Finally, let

\[B := \bigcup_{i=1}^{c} B_i.\]
Now, initially let \( R' := R \) and, as long as \( G(R') \) contains a vertex \( v \in V \setminus B \) that can be extracted using Definition 5.6, replace \( R' \) by the result of extracting \( v \).

We now analyze the effectivity and error of Reduction Rule 5.8.

**Lemma 5.9.** Let \( (G, R, \omega) \) be an RPP instance with \( \omega \) satisfying the triangle inequality. Then, Reduction Rule 5.8 in \( O(n^3) \) time yields a multiset \( R' \) of edges such that

(i) \( \omega(R') \leq \omega(R) \) and \( V(R') \subseteq V(R) \).

(ii) Any multiset \( S \) of edges with \( V(S) \subseteq V(R') \) is an Eulerian extension for \( (G, R', \omega) \) if and only if it is one for \( (G, R, \omega) \).

(iii) Any edge-minimizing Eulerian extension \( S \) for \( (G, R, \omega) \) can be turned into an Eulerian extension \( S' \) for \( (G, R', \omega) \) such that \( \omega(S') \leq \omega(S) + 2\gamma \cdot (2b - 2) \).

(iv) \( G(R') \) contains at most \( 2b + 2c + 4\omega(R)/\gamma \) vertices.

**Proof.** (i) and (ii) follow from Lemma 5.7 since \( R' \) is the result of a sequence of vertex extractions.

(iii) We turn \( S \) into an Eulerian extension \( S' \) with \( V(S') \subseteq V(R') \) and then apply (ii). First, since \( S \) is edge-minimizing and \( \omega \) satisfies the triangle inequality, by Observation 3.6, \( V(S) \subseteq V(R) \). By Reduction Rule 5.8, the vertices in \( X := V(R) \setminus V(R') \) are not in \( B \) and, thus, for each \( v \in X \cap C_i \), we find a vertex \( v' \in B_i \) such that \( \omega(v, v') \leq \gamma \). Note that \( v' \in V(R') \). Since each vertex in \( X \) is balanced in \( G(R) \), by Remark 3.9, each vertex \( v \in X \cap V(S) \) is incident to exactly two edges \( \{v, u\} \) and \( \{v, w\} \) of \( S \) (possibly, \( u = w \)). Since \( \{v, v'\} \subseteq C_i \), \( S' := (S \setminus \{\{v, u\}, \{v, w\}\}) \cup \{v', u\} \cup \{v', w\} \) is also an Eulerian extension for \( (G, R, \omega) \). Moreover, \( \omega(S') \leq \omega(S) + 2\gamma \). Doing this replacement for each \( v \in X \cap V(S) \), we finally obtain an Eulerian extension \( S' \) for \( (G, R, \omega) \) with \( V(S') \subseteq V(R') \) and \( \omega(S') \leq \omega(S) + 2\gamma \cdot |X \cap V(S)| \). Since each vertex in \( X \) is balanced in \( G(R) \), by Lemma 3.8, \( |X \cap V(S)| \leq 2c - 2 \). Finally, by (ii), \( S' \) is an Eulerian extension for \( (G, R', \omega) \).

(iv) The vertices of \( G(R') \) can be partitioned into \( X \uplus Y \uplus Z \), where \( X \) are imbalanced in \( G(R') \), \( Y \) are balanced and in \( B \), and \( Z \) are balanced but not in \( B \).

By Lemma 5.7(iii), the vertices in \( X \) are imbalanced in \( G(R) \) also. Thus,

\[
|X| \leq b. \tag{5.1}
\]

We next analyze \( |Y| \). For \( i \in \{1, \ldots, c\} \), let \( R_i \subseteq R \) be the edges between vertices in \( C_i \), \( T_i^* \) be a tree of least weight in \( G(R_i) \) connecting all vertices in \( B_i \), \( T_i \) be a minimum-weight spanning tree in \( G[B_i] \), and \( H_i \) be a minimum-weight Hamiltonian cycle in \( G[B_i] \). Doubling all edges of \( T_i^* \) yields a closed walk in \( G(R_i) \) containing the vertices in \( B_i \). Using the triangle inequality of \( \omega \), it can be shortcut...
to a Hamiltonian cycle in $G[B_i]$. Thus, $\omega(T_i) \leq \omega(H_i) \leq 2\omega(T_i^*)$.\footnote{That is, $T_i$ is the folklore 2-approximation of a Steiner tree with terminals $B_i$ in $G[R_i]$.} We thus get

\[(|B_i| - 1)\gamma = \sum_{e \in E_i} \gamma < \sum_{e \in E_i} \omega(e) = \omega(T_i) \leq 2\omega(T_i^*) \leq 2\omega(R_i) \quad \text{and thus}
\]

\[|Y| \leq |B| = \sum_{i=1}^{c} |B_i| < \sum_{i=1}^{c} \left( \frac{2\omega(R_i)}{\gamma} + 1 \right) = 2\omega(R)/\gamma + c. \quad (5.2)\]

Finally, we analyze $|Z|$. Definition 5.6 is not applicable to any vertex $v \in Z$, since it would have been removed by Reduction Rule 5.8. Thus, $v$ is a cut vertex contained in at least three blocks of $G[R']$ or its connected component of $G[R']$ consists of only two vertices. To analyze $|Z|$, for each $i \in \{1, \ldots, c\}$, consider $X_i := X \cap C_i$, $Z_i := Z \cap C_i$, the set $R'_i \subseteq R'$ of edges between vertices in $C_i$, and the block-cut tree $T_i$ of $G[R'_i]$; the vertices of $T_i$ are the cut vertices and the blocks of $G[R'_i]$ and there is an edge between a cut vertex $v$ and a block $A$ of $G[R'_i]$ in $T_i$ if $v$ is contained in $A$. Then either $|Z_i| \leq 2$ or the vertices in $Z_i$ have degree at least three in $T_i$. Therefore, $T_i$ has at most $|X_i| + |Y_i|$ leaves. Since a tree with $\ell$ leaves has at most $\ell - 1$ vertices of degree three, $|Z_i| \leq \max\{2, |X_i| + |Y_i| - 1\}$. Thus,

\[|Z| = \sum_{i=1}^{c} |Z_i| \leq |X| + \sum_{i=1}^{c} |Y_i| = |X| + |Y|. \quad (5.3)\]

Combining (5.1), (5.2), (5.3), and that $|V(R')| = |X| + |Y| + |Z|$, (iv) follows.

We finally analyze the running time of Reduction Rule 5.8. For $i \in \{1, \ldots, c\}$, all sets $C_i$ and $B_i$ can be computed in $O(n^2)$ time. Also the blocks of $G[R']$ required by Definition 5.6 are computable in $O(n^2)$ time using depth-first search. Thus, in $O(n)$ time, we can find a vertex $v$ to which Definition 5.6 applies. Vertex $v$ can then be extracted in $O(n)$ time since the matchings $M_i$ in Definition 5.6 can be chosen arbitrarily, that is, in particular greedily in $O(n)$ time, and the blocks can be recomputed in $O(n^2)$ time. Finally, we extract at most $n$ vertices. 

\[\square\]

5.4 A polynomial-size approximate kernelization scheme for the parameter $b + c$ (proof of Theorem 1.3)

This section proves Theorem 1.3. We describe how to transform a given RPP instance $I$ and $\varepsilon > 0$ into an RPP instance $I'$ such that any $\alpha$-approximate solution for $I'$ can be transformed into an $\alpha(1+\varepsilon)$-approximate solution for $I$. Due to Proposition 3.1, we assume that $I = (G, R, \omega)$ has been preprocessed in $O(n^3)$ time so as to satisfy the triangle inequality.

**Shrinking the graph.** Choose $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Apply Reduction Rule 5.8 with

\[\gamma = \frac{\varepsilon_1 \cdot c(R)}{4c - 4}. \quad (5.4)\]
which, by Lemma 5.9, in $O(n^3)$ time gives an instance $(G, R_1, \omega)$ with
\[ |V(R_1)| \leq 2b + 2c + \frac{16c - 16}{\varepsilon_1}. \] (5.5)

To $(G, R_1, \omega)$ we apply Reduction Rule 5.3, which, by Lemma 5.4, in $O(|R|)$ time gives an instance $(G, R_2, \omega)$ with
\[ R_2 \subseteq R_1 \quad \text{and} \quad |R_2| \leq 4b + 4c + \frac{32c - 32}{\varepsilon_1}. \] (5.6)

Finally, applying Reduction Rule 5.1 to $(G, R_2, \omega)$ in linear time yields an instance $(G_2, R_2, \omega)$ such that
\[ |V(G_2)| \leq |V(R_2)| \leq |V(R_1)|. \] (5.7)

**Shrinking edge weights.** Since $G(R \uplus T)$ is connected, due to the triangle inequality of $\omega$, each edge $e = \{u, v\}$ of $G$, and thus of its subgraph $G_2$, satisfies $\omega(e) \leq \omega(R) + \omega(T)$. Moreover, by Lemma 3.12, any edge-minimizing Eulerian extension for $(G_2, R_2, \omega)$, has at most $|M| + 2|T| = b/2 + 2c - 2$ edges. Thus, we can apply Lemma 2.8 with $\beta = \omega(R) + \omega(T)$ and $N = |R_2| + b/2 + 2c - 2$ to $(G_2, R_2, \omega)$ to get an instance $(G_2, R_2, \omega_2)$ such that for all edges $e$,
\[ \omega(e) \leq \frac{|R_2| + b/2 + 2c - 2}{\varepsilon_2}. \] (5.8)

In Lemma 2.8, set $\mathcal{F}$ just contains all vectors $x$ that encode RPP tours $W$ induced by edge-minimizing Eulerian extensions for $(G_2, R_2, \omega)$ (it has a 1 for each edge of $G_2$ in $W$ and a 0 for each edge of $G_2$ not in $W$). We finally return $(G_2, R_2, \omega_2)$, whose construction takes $O(n^3 + |R|)$ time, as required by Theorem 1.3.

**Kernel size analysis.** The returned instance satisfies Theorem 1.3(i) due to (5.5) and (5.7), (ii) due to (5.6), and (iii) due to (5.8).

**Approximation factor analysis.** It remains to prove Theorem 1.3(iv), that is, that we can lift an $\alpha$-approximate solution for $(G_2, R_2, \omega_2)$ to an $\alpha(1 + \varepsilon)$-approximate solution for $(G, R, \omega)$.

An optimal RPP tour for $(G, R, \omega)$ has cost $\omega(W^*) = \omega(R) + \omega(D)$ by (3.1), where $D$ is a minimum-cost Eulerian extension. By Lemma 5.9(iii) and (5.4), there is an Eulerian extension $D'$ for $(G, R_1, \omega)$ with
\[ \omega(D') \leq \omega(D) + 2\gamma(2c - 2) = \omega(D) + \varepsilon_1 \cdot \omega(R). \] (5.9)

By Lemma 5.4, $D'$ is an Eulerian extension for $(G, R_2, \omega)$ and, by Proposition 5.2, for $(G_2, R_2, \omega)$. Then $D'$ is also an Eulerian extension for $(G_2, R_2, \omega_2)$. Thus, an optimal RPP tour for $(G_2, R_2, \omega_2)$ has cost at most $\omega_2(R_2) + \omega_2(D')$. By
Proposition 3.5, an $\alpha$-approximate solution for $(G_2, R_2, \omega_2)$, can be turned into an Eulerian extension $S$ such that
\[
\omega_2(R_2) + \omega_2(S) \leq \alpha(\omega_2(R_2) + \omega_2(D')).
\] (5.10)

By Proposition 5.2, $S$ is an Eulerian extension for $(G, R_2, \omega)$. By Lemma 5.4, $S$ is an Eulerian extension for $(G, R_1, \omega)$, and by Lemma 5.9, it is one for $(G, R, \omega)$, since $V(S) \subseteq V(G_2) = V(R_2) \subseteq V(R_1) \subseteq V(R)$. Thus, by Proposition 3.5, $S$ can be turned into a RPP tour of cost $\omega(R) + \omega(S)$ for $(G, R, \omega)$. We analyze this cost. By (5.10) and Lemma 2.8 with $\beta = \omega(R) + \omega(T)$,
\[
\omega(R_2) + \omega(S) \leq \alpha(\omega(R_2) + \omega(D')) + \varepsilon_2(\omega(R) + \omega(T)).
\]

Using $\omega(R_2) \leq \omega(R_1) \leq \omega(R)$ from Lemmas 5.4 and 5.9, and $\alpha \geq 1$, we get
\[
\omega(R) + \omega(S) \leq\]
\[
\leq \alpha(\omega(R) + \omega(D')) + \varepsilon_2(\omega(R) + \omega(D))
\]
\[
\leq \alpha(\omega(R) + \omega(D) + \varepsilon_1 \omega(R)) + \varepsilon_2(\omega(R) + \omega(D))
\]
\[
\leq \alpha(1 + \varepsilon_1 + \varepsilon_2)(\omega(R) + \omega(D)) = \alpha(1 + \varepsilon)\omega(W^*)
\]
using (3.1).

Thus, we got an $\alpha(1 + \varepsilon)$-approximation for $(G, R, c)$. □

5.5 Towards a polynomial-size approximate kernelization scheme for the parameter $c$

In the previous section we have shown a polynomial-size approximate kernelization scheme (PSAKS) for RPP parameterized by $b + c$. An obvious question is whether there is a PSAKS for the parameter $c$. Unfortunately, in this work, we leave this question open, yet in the following make some first steps and discuss the difficulties in resolving this question.

To get the PSAKS for $c$, one has to reduce the number of imbalanced vertices in $G(R)$. An obvious idea to do so is adding to $R$ cheap edges of a minimum-weight perfect matching $M$ on imbalanced vertices, since this is optimal if it happens to connect $G(R)$.

Reduction Rule 5.10. Let $(G, R, \omega)$ be an RPP instance with triangle inequality and $\delta \in \mathbb{Q}$. Add to $R$ a subset $M^* \subseteq M$ of edges with
\[
\sum_{e \in M^*} \omega(e) \leq \delta.
\]

Observation 5.11. Let $R' = R \cup M^*$ be obtained by applying Reduction Rule 5.10 to $R$.

(i) There are at most $2(|M| - |M^*|)$ imbalanced vertices in $G(R')$.
(ii) For any Eulerian extension $S'$ for $(G, R', \omega)$, $S = S' \cup M^*$ is an Eulerian extension for $(G, R, \omega)$ and $\omega(R) + \omega(S) = \omega(R') + \omega(S')$. 
(iii) For any Eulerian extension $S$ for $(G, R, \omega)$, $S' = S \uplus M^*$ is an Eulerian extension for $(G, R', \omega)$ with $\omega(S') \leq \omega(S) + \delta$.

We expect that Reduction Rule 5.10 will indeed have some impact in practice when choosing $\delta = \varepsilon(\omega(R) + \omega(M))$, for example. Yet to show a PSAKS, it is unsuitable for two reasons:

1. To reduce the number of imbalanced vertices in $G \langle R \rangle$ to some constant, we have to add all but a constant number of edges of $M$ to $R$, yet, by Observation 5.11(iii), each added edge potentially contributes to the error and thus would merely retain a 2-approximation. Unfortunately, Figure 5.2 shows that the bound given by Observation 5.11(iii) is tight.

2. Reduction Rule 5.10 increases the total weight of required edges. This makes it unusable for a PSAKS, since, in the resulting instance, a solution might be $(1 + \varepsilon)$-approximate merely due to the fact that the lower bound $\omega(R)$ on the solution is sufficiently large (we will use this fact below).

Given the difficulties of showing a PSAKS for $c$, it is tempting to disprove its existence. However, the existing tools for excluding PSAKSes [33] also exclude polynomial-size kernels from which only optimal solutions can be lifted to $(1 + \varepsilon)$-approximate solutions for the input instance. In terms of Fellows et al. [17], these are so-called $(1 + \varepsilon)$-fidelity-preserving kernels and we can easily build a $(1 + \varepsilon)$-fidelity-preserving kernel with size polynomial in $\omega(T)$, which gives such a kernel of size polynomial in $c$ in case that the edge weights are bounded by poly$(c)$. More specifically, we can prove the following.

**Proposition 5.12.** Let $(G, R, \omega)$ be an instance of RPP with triangle inequality.

(i) If $\omega(T) \leq \varepsilon(\omega(R) + \omega(M))$, then one can find a $(1 + 2\varepsilon)$-approximate RPP tour for $(G, R, \omega)$ in polynomial time.

(ii) If $\omega(M) \leq \varepsilon(\omega(R) + \omega(T))$, then $(G, R, \omega)$ has a $(1 + 3\varepsilon)$-fidelity-preserving kernel with $O(c)$ vertices.

(iii) Otherwise, $(G, R, \omega)$ has an (exact) problem kernel with respect to the parameter $\min\{\omega(T)/\varepsilon - \omega(M), \omega(M)/\varepsilon - \omega(T)\}$. 
Proposition 5.12 shows that, in order to exclude PSAKses for RPP parameterized by \( c \), a reduction must use unbounded edge weights, the weights of \( T \), \( M \), and \( R \) may not differ too much (by (i) and (ii)), yet the the weights of \( T \) and \( M \) must not be too close either (by (iii)). Given these restrictions, we conjecture:

**Conjecture 5.13.** RPP has a PSAKS with respect to the parameter \( c \).

We finally prove Proposition 5.12.

**Proof (of Proposition 5.12).** (i) Observe that \( T \cup T \cup M \) is an Eulerian extension for \((G, R, \omega)\). Using Proposition 3.5, it yields an RPP tour of cost

\[
\omega(R) + \omega(M) + 2\omega(T) \leq \omega(R) + \omega(M) + 2\varepsilon(\omega(R) + \omega(M))
\]

\[
\leq (1 + 2\varepsilon)(\omega(R) + \omega(D)) \quad \text{using (3.2)}
\]

\[
= (1 + 2\varepsilon)\omega(W^*) \quad \text{using (3.1)}.
\]

(ii) Let \( R' \) be obtained from \( R \) using Reduction Rule 5.10 with \( \delta = \varepsilon(\omega(R) + \omega(T)) \), that is, \( R' = R \cup M \). In \( G(R') \) all vertices are balanced. Let \( D \) be an optimal Eulerian extension for \((G, R, \omega)\), then, by Observation 5.11, there is an optimal Eulerian extension \( D' \) for \((G, R', \omega)\) with \( \omega(D') \leq \omega(D) + \omega(M) \). Applying Reduction Rule 5.1 after Reduction Rule 5.8 with

\[
\gamma = \frac{\omega(R)}{4c - 4} \cdot \varepsilon
\]

to \((G, R', \omega)\), by Lemma 5.9, yields an instance \((G'', R'', \omega)\) with at most \( 2c + (16c + 16)/\varepsilon \) vertices that has an optimal Eulerian extension \( D'' \) of cost at most

\[
\omega(D'') \leq \omega(D') + 2\gamma(2c - 2) = \omega(D') + \varepsilon \omega(R).
\]

By Proposition 3.5, \( D'' \) gives a RPP tour for \((G, R, \omega)\) with cost

\[
\omega(R') + \omega(D'') \leq \omega(R) + \omega(M) + \omega(D') + \varepsilon \omega(R)
\]

\[
\leq \omega(R) + 2\omega(M) + \omega(D) + \varepsilon \omega(R)
\]

\[
\leq (1 + 3\varepsilon)(\omega(R) + \omega(D)) \quad \text{using (ii)}
\]

\[
= (1 + 3\varepsilon)\omega(W^*) \quad \text{using (3.1)}.
\]

To fully bound the size of the kernel, one can finally reduce the number of required arcs and the weights of \((G'', R'', \omega)\) to size polynomial in \(|G''|\) maintaining optimal RPP tours using Reduction Rule 5.3 and a theorem of Frank and Tardos [18] as described by Etscheid et al. [16].

(iii) Otherwise, one has

\[
\omega(R) \leq \omega(M)/\varepsilon - \omega(T) \quad \text{and} \quad \omega(R) \leq \omega(T)/\varepsilon - \omega(M)
\]

and thus the known \( 2|R| \)-vertex problem kernel [5] for RPP and will be a kernel for both of these parameters. \(\square\)
6 Conclusion

Our main algorithmic contribution is a polynomial-size approximate kernelization scheme for the Rural Postman Problem parameterized by \( b + c \), where \( b \) is the number of vertices incident to an odd number of required edges and \( c \) is the number of connected components formed by the required edges. In future work, we plan to implement the algorithm and to evaluate it on real-world data.

We think that the approach taken by Reduction Rule 5.8, namely reducing all vertices that do not belong to some inclusion-maximal set \( B \) of mutually sufficiently distant vertices, might be applicable to other metric graph problems: it ensures that, for each deleted vertex, some nearby representative in \( B \) is retained.

Notably, this approach does not generalize well to asymmetric distances, so that the main open question besides our Conjecture 5.13 (whether RPP has a polynomial-size approximate kernelization scheme for the parameter \( c \)) is whether the scheme for the parameter \( b + c \) presented in this work can be generalized to the directed Rural Postman Problem. We point out that, using known ideas [4], one can reduce any instance \( I \) of the directed or undirected RPP to an instance \( I' \) with \( c \) vertices in \( O(n^3 \log n) \) time such that any \( \alpha \)-approximation for \( I' \) yields a \((\alpha + 1)\)-approximation for \( I \). Given that undirected RPP is 3/2-approximable, this is interesting only for the directed RPP.

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