Spectral statistic for decaying random potentials

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Abstract. We consider Anderson model $H^\omega = -\Delta + V^\omega$ on $\ell^2(\mathbb{Z}^d)$ with decaying random potential. We study the point process $\xi_{L,\lambda}^\omega$ associated with eigenvalues of $H_{\Lambda_L}^\omega$, the restriction of $H^\omega$ to the finite cube $\Lambda_L$. Our result is that the weak limit points of $\{\xi_{L,\lambda}^\omega\}$ are poisson point processes as $L \to \infty$.

1. Introduction

The Anderson model with decaying randomness is a random Hamiltonian $H^\omega$ on $\ell^2(\mathbb{Z}^d)$ given by

$$H^\omega = -\Delta + V^\omega, \quad \omega \in \Omega.$$ \hfill (1.1)

$\Delta$ is adjacency operator defined by

$$(\Delta u)(n) = \sum_{|m-n|=1} u(m) \forall u \in \ell^2(\mathbb{Z}^d).$$

The random potential $V^\omega$ which is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ by the sequence $\{a_n q_n(\omega)\}_{n \in \mathbb{Z}^d}$ defined by

$$V^\omega = \sum_{n \in \mathbb{Z}^d} a_n q_n(\omega) |\delta_n \rangle \langle \delta_n|.$$ \hfill (1.2)

where $\{\delta_n\}_{n \in \mathbb{Z}^d}$ be the standard basis for $\ell^2(\mathbb{Z}^d)$. Here $\{a_n\}_{n \in \mathbb{Z}^d}$ is a sequence of positive real numbers such that $a_n \to 0$ as $|n| \to \infty$ and $\{q_n\}_{n \in \mathbb{Z}^d}$ are real valued iid random variables with an absolutely continuous probability distribution $\mu$ which has bounded density. We realize $q_n$ as $\omega(n)$ on $(\mathbb{R}^{\mathbb{Z}^d}, B_{\mathbb{R}^{\mathbb{Z}^d}}, \mathbb{P})$, $\mathbb{P} = \otimes \mu$ construct via Kolmogorov theorem we will refer to this probability space as $(\Omega, B, \mathbb{P})$ henceforth.

Henceforth we follow the notation used by Minami [4] as closely as possible.

For any $B \subset \mathbb{Z}^d$ we consider the orthogonal projection $\chi_B$ onto $\ell^2(B)$ and define the matrices

$$H_B^\omega = \left( (\delta_n, H^\omega \delta_m) \right)_{n,m \in B}, \quad G^B(z; n, m) = \langle \delta_n, (H_B^\omega - z)^{-1} \delta_m \rangle, \quad G^B(z) = (H_B^\omega - z)^{-1}.$$ \hfill (1.3)

$$G(z) = (H^\omega - z)^{-1}, \quad G(z; n, m) = \langle \delta_n, (H^\omega - z)^{-1} \delta_m \rangle, \quad z \in \mathbb{C}^+.$$ 

Note that $H_B^\omega$ is the matrix

$$\chi_B H^\omega \chi_B : \ell^2(\mathbb{Z}^d) \to \ell^2(B), \ a.e \ \omega.$$ 

We note that by an assumption on $V^\omega$, the operator $H^\omega$ are self adjoint a.e $\omega$ and have a common core domain consisting of vectors of finite support.

Let $\Lambda_L$ denote the $d$-dimension box of side length given by

$$\Lambda_L = \{(n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d : |n_i| \leq L\} \subset \mathbb{Z}^d.$$ 

We then assume that.

Hypothesis 1.1. (1) The measure $\mu$ is absolute continuous with density satisfies

$$\rho(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ \frac{1}{|x|^\delta} & \text{if } x \leq -1 \text{ forsome, } \delta > 1. \end{cases}$$ \hfill (1.4)
(2) The sequence $a_n$ satisfy $a_n \simeq |n|^{-\alpha}$, $0 \leq \alpha < \frac{1}{2}$.

(3) The pair $(\alpha, \delta)$ are s.t

\[
\beta_L = \sum_{n \in \Lambda_L} |n|^{-\alpha(\delta-1)} = O \left( (2L + 1)^{d-\alpha(\delta-1)} \right).
\]

satisfies

\[
\beta_{L}^{-1}(2L + 1)^{(d-1)} \to 0 \text{ as } L \to \infty.
\]

These are satisfied for example when $1 < \delta < \frac{1}{2}$, so $0 < \alpha(\delta-1) < 1 - \alpha < 1$.

Remark 1.2. We shall show in Appendix that if we take $a_n$ and $\mu$ as in above then the spectrum of $H_{\lambda L}$ is $[-2d, 2d] + \mathbb{R}$ a.e and average spacing of eigenvalues of $H_{\lambda L}^2$ near the energy $\lambda \in (-\infty, -2d)$ is of order $\beta_{L}^{-1}$.

We recall a few facts concerning point processes and refer to [14] for more details. Consider $(\Omega, \mathcal{B}, \mathbb{P})$ as above and $\mathcal{M}_{\mathbb{N}}(\mathbb{R})$ denote the set of all integer valued Randon measure on $\mathbb{R}$. A point process $\xi^\omega$ is a random variable taking value in $\mathcal{M}_{\mathbb{N}}(\mathbb{R})$ given by

$$
\xi^\omega : \Omega \to \mathcal{M}_{\mathbb{N}}(\mathbb{R}) \quad \omega \to \xi^\omega.
$$

and intensity measure $\nu$ of $\xi^\omega$ is defined by

$$
\nu(A) = \mathbb{E}^\omega[\xi^\omega(A)] \text{ for each Borel set } A \subset \mathbb{R}.
$$

Let $\{\xi_n^\omega\}$ be a sequence of point processes on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and $\xi^{\omega'}$ be a point process on a probability space $(\Omega', \mathcal{B}', \mathbb{P}')$. We say $\xi_n^\omega$ converges weakly to $\xi^{\omega'}$ if the following happens

$$
\lim_{n \to \infty} \mathbb{E}^\omega[e^{-\xi_n^\omega(f)}] = \mathbb{E}^{\omega'}[e^{-\xi^{\omega'}(f)}] \forall f \in C_c^+(\mathbb{R}).
$$

where $C_c^+(\mathbb{R})$ be the set of all non-negative function on $\mathbb{R}$ with compact support and

$$
\xi_n^\omega(f) = \int f(x)d\xi_n^\omega(x),
$$

$$
\xi^{\omega}(f) = \int f(x)d\xi^{\omega}(x).
$$

A point process $\xi^\omega$ is said to be Poisson point process with intensity measure $\nu$ if following two condition satisfied:

(i) If $A$ be a bounded borel set of $\mathbb{R}$ then $\xi^\omega(A)$ should follow Poisson distribution with parameter $\nu(A)$ i.e

$$
\mathbb{P}(\omega : \xi^\omega(A) = k) = e^{-\nu(A)} \frac{\nu(A)^k}{k!}, \quad k = 0, 1, 2, \ldots.
$$

(ii) If $A_1, A_2, \ldots, A_n$ are disjoint Borel set of $\mathbb{R}$, then $\xi^\omega(A_1), \xi^\omega(A_2), \ldots, \xi^\omega(A_n)$ are independent random variables.

Associated with the operators $H_{\lambda L}^\omega$ defined as in equation (1.3) and $\lambda \in (-\infty, -2d)$ defined the point process

$$
\xi_{L,\lambda}^\omega = \sum_{x \in \sigma(H_{\lambda L}^\omega)} \delta_{\beta_L(x-\lambda)}.
$$

Our main result is the following theorem.
Theorem 1.3. Consider a subset \( \Lambda_L \) of \( \mathbb{Z}^d \) and suppose the operators \( H^\omega_{\Lambda_L} \) are as in equation (1.3). Assume that there are \( s \in (0, 1), C, R, r > 0 \) such that the matrix elements \( G^\Lambda,\omega(z,n,m) \) satisfy

\[
\mathbb{E}(|G^\Lambda,\omega(z,n,m)|^s) \leq Ce^{-r|n-m|},
\]

whenever \( |n-m| > R \) \( \Im z > 0 \) and \( \Re z \in \mathbb{R} \setminus [-2d,2d] \). Then any limit point \( \xi^\omega \) of \( \xi^\omega_{\Lambda_L,\lambda} \) with \( \lambda < -2d \) is a poisson point process with an absolutely continuous intensity measure.

The assumption of exponential decay of Green’s function namely (1.6) is discuss below in (1.7).

The exponential decay of fractional moment of the Green function for independent random variable was established by Aizenman Michael, Molchanov Stanislav in [1].

In [3] Kirsch-Krishna-Obermeit consider \( H^\omega = -\Delta + V^\omega \) on \( \ell^2(\mathbb{Z}^d) \) with same \( V^\omega \) defined in (1.2) shown that \( \sigma(H^\omega) = \mathbb{R} \) and \( \sigma_c(H^\omega) \subseteq [-2d,2d] \) a.e \( \omega \), under some conditions on \( \{a_n\} \) and \( \mu \) (The density of \( \mu \) should not decay too fast at infinity and \( a_n \) should not decay too fast). For mathematical formulation of above condition on \( a_n \)’s and \( \mu \) we refer Definition 2.1 in [3].

To show the existence of point spectrum outside \( [-2d,2d] \) they verified Simon-Wolf criterion [17, Theorem 3.1.7] by showing exponential decay of the fractional moment of the Green function [3, Lemma 3.2]. This will give whenever \( |n-m| > 2R \) and the energy \( E \) is outside of \( [-2d,2d] \), the spectrum of \( \Delta \) we have

\[
\mathbb{E}^\omega(|G^\Lambda,\omega(E + i\epsilon : n,m)|^s) \leq D_P(n,m)e^{-c\left(\frac{|n-m|}{2}\right)}, \quad E \in \mathbb{R} \setminus [-2d,2d]
\]

where \( \epsilon > 0, 0 < s < 1, c \) is positive constant and \( R \in \mathbb{Z}^+ \). Here \( D_P(n,m) \) is a constant independent of \( E, \epsilon \), but polynomially bounded in \( |n| \) and \( |m| \).

Now the condition (1.6) is guaranteed by the expression (1.7).

In [4] Minami consider the Anderson model with ergodic random potential (see[17]). Then assuming the exponential decay of fractional moment of the Green function he show that if we view the spectral measure of \( H^\omega_{\Lambda_L} \) near \( \lambda \in \mathbb{R} \) given by

\[
\zeta^\omega_{\Lambda_L,\lambda} = \sum_j \delta_{(2L+1)^d(E_j(\Lambda_L)-E)}.
\]

as a point process then we have the following

\[
\zeta^\omega_{\Lambda_L,\lambda} \xrightarrow{\text{weakly}} \zeta^\omega \text{ as } L \to \infty.
\]

where \( \zeta^\omega \) is the poisson point process with intensity measure \( n(\lambda)dx \), \( n(\lambda) \) is density of state at \( \lambda \). Subsequently the poisson statistics was shown for the trees by Aizenman-Warzel in [2]. In a recent result [9] Germinet and Klopp shown the independence of the processes \( \xi^\omega_{\lambda}, \xi^\omega_{\lambda'} \) for distance \( \lambda, \lambda' \).

In [7] Krishna and Dolai they consider the decaying model as in (1.1) and shown that the statistic near the edges in the absolutely continuous spectrum
in dimension $d \geq 3$ is independent of the randomness and agrees with that of the free part.

In this work we present the statistics for the spectrum of $H^\omega$ in $(-\infty, -2d)$ and shown that the statistics is Poisson.

We divide $(-L - 1, L]^d \subset \mathbb{R}^d$ into $N_L^d$ equal cubes $C_p^*\mathbb{Z}^d$ for $p = 1, 2, \cdots, N_L^d$, with side length $2L+1/N_L$ and define

$$C_p = C_p^* \cap \mathbb{Z}^d$$

$$\text{int}(C_p) = \{x \in C_p : \text{dist}(x, \partial C_p) > l_L\}.$$  

where $\{N_L\}_L$ and $\{l_L\}_L$ are both increasing sequences of positive numbers, which we specify latter. For any cube $B \subset \mathbb{Z}^d$ the boundary of $B$ is denoted by $\partial B$ and define by

$$\partial B = \{x \in B : \exists x' \in B^c \text{ such that } |x - x'| = 1\}.$$  

Now define

$$\eta_{C_p, \lambda}^\omega = \sum_j \delta_{\beta_L(E_j(C_p) - \lambda)},$$

$$\eta_{L, \lambda}^\omega = \sum_{p=1}^{N_L^d} \eta_{C_p, \lambda}^\omega$$

So we divide $\Lambda_L$ into disjoint $C_p$ boxes denotes as $C_p$, then $H_{C_p}^\omega$ becomes statistically independent for different $p'$s and for each $L$ the processes $\{\eta_{C_p, \lambda}^\omega, p = 1, 2, \cdots, N_L^d\}_L$ are mutually independent and form a triangular array. We will show that the above independent triangular array also satisfies the following:

For each bounded interval $I \subset \mathbb{R}$

$$\lim_{L \to \infty} \sup_{1 \leq p \leq N_L^d} \mathbb{P}\{\eta_{C_p, \lambda}^\omega(I) > 0\} = 0.$$

With above condition the array $\{\eta_{C_p, \lambda}^\omega : p = 1, 2, \cdots, N_L^d\}_L$ becomes uniformly asymptotically negligible (u.a.n). Now $\eta_{L, \lambda}^\omega$ be the superposition of uniformly asymptotically negligible array, to show $\eta_{L, \lambda}^\omega$ converges weakly to the poisson point process whose intensity measure is absolutely continuous w.r.t the Lebesgue measure, it is enough to show ([14, Theorem 11.2.5]) the following two conditions :

For any bounded interval $I \subset \mathbb{R}$; as $L$ gets large we need to show

$$\sum_{p=1}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq 1) \to c_\lambda|I|; \text{ as } L \to \infty;$$

$$\sum_{p=1}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq 2) \to 0 \text{ as } L \to \infty.$$
Proposition 1.4. Let \( \mathbb{N}_L \) be such that \( \mathbb{N}_L^d \) \( \sum_{p=1}^{N_L^d} \mathbb{P}(\eta^\omega_{C_p,\lambda}(I) \geq 1) = \sum_{p=1}^{N_L^d} \mathbb{E}[\mathbb{P}(\eta^\omega_{C_p,\lambda}(I))] - \sum_{p=1}^{N_L^d} \sum_{j \geq 2} \mathbb{P}(\eta^\omega_{C_p,\lambda}(I) \geq j). \)

we will show the second term of above goes to zero as \( L \) gets large and there is subsequence \( \{L_k\} \) for which

\[
\sum_{p=1}^{N_L^d} \mathbb{E}[\mathbb{P}(\eta^\omega_{C_p,\lambda}(I)) \rightarrow c|I| \text{ as } L_k \rightarrow \infty. 
\]

Therefore using (1.13), (1.14) we get that (1.11) holds for a subsequence of \( \{\eta^\omega_{L,\lambda}\} \).

Finally will show that \( \overline{\xi}_{L,\lambda}^\omega \) and \( \eta^\omega_{L,\lambda} \) have the same limit points for which we need the exponential decay of Green’s function described by (1.7).

In this work we are going to use stone’s formula [15, Theorem VII.13] several times so we recall it here.

Define

\[
f_\zeta(x) = \text{Im} \frac{1}{x-(\sigma+i\tau)} = \frac{\tau}{(x-\sigma)^2 + \tau^2}, \quad \zeta = \sigma + i\tau \in \mathbb{C}^+. 
\]

\[
\phi_\tau(x) = \frac{1}{\pi} \int_a^b \text{Im} \frac{1}{x-(\sigma+i\tau)} d\sigma \to \frac{1}{2} \text{[c[a.b] + c(a,b)] as } \tau \to 0, 
\]

Now for any \( S \subset \mathbb{Z}^d \) using above we get

\[
\frac{1}{\pi} \int_a^b f_\zeta(H^\omega_S) d\sigma \overset{\text{strong sense}}{\to} \frac{1}{2} [E_{H_S}^\omega[a,b] + E_{H_S}^\omega(a,b)] \text{ as } \tau \to 0. 
\]

Proposition 1.4. Let \( I = [a,b] \subset \mathbb{R} \) be a bounded interval and \( S \subset \mathbb{Z}^d \). Consider \( H^\omega_S = \chi_S \Delta \chi_S + V^\omega \chi_S \) then \( \mathbb{E}^\omega(\langle \delta_n, E^\omega_{H_S}(I) \delta_n \rangle) \leq \|\frac{d\mu}{dx}\|_{\infty} a_n^{-1} |I|, \quad n \in S. \)

Proof: We write \( H^\omega_S \) as \( H^\omega_S = -\Delta + \sum_{n \neq k \in S} a_k q_k(\omega)|\delta_k\rangle \langle \delta_k| + a_n q_n(\omega)|\delta_n\rangle \langle \delta_n| \)

\[
= H^\omega_S^{n/n} + a_n q_n(\omega) |\delta_n\rangle \langle \delta_n|. 
\]

Then from resolvent equation we get

\[
\langle \delta_n, (H^\omega_S - z)^{-1} \delta_n \rangle = \frac{1}{a_n q_n(\omega) + \langle \delta_n, (H^\omega_S^{n/n} - z)^{-1} \delta_n \rangle^{-1}}. 
\]

Take \( z = \sigma + i\tau \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}z > 0\} \) and set the followings

\[
\langle \delta_n, (H^\omega_S^{n/n} - z)^{-1} \delta_n \rangle = c + id, \quad A = a_n^{-1} \frac{c}{c^2 + d^2}, \quad B = a_n^{-1} \frac{d}{c^2 + d^2} > 0, \text{ when } \text{Im}z > 0. 
\]

From these the following estimate is clear

\[
\int_{\mathbb{R}} \text{Im} \langle \delta_n, (H^\omega_S - z)^{-1} \delta_n \rangle d\mu(q_n) = \frac{1}{a_n} \int_{\mathbb{R}} \frac{B}{(x + A)^2 + B^2} d\mu(x) \leq \pi \|\frac{d\mu}{dx}\|_{\infty} a_n^{-1}. 
\]
This estimate implies,

$$\mathbb{E}^{\omega}\left(\text{Im}(\delta_n, (H_S^\omega - \sigma - i\tau)^{-1}\delta_n)\right) \leq \pi \left\| \frac{d\mu}{dx} \right\|_{\infty} a_n^{-1}. $$

Using Fubini theorem it then follows

$$\mathbb{E}^{\omega}\left(\int_a^b \text{Im}(\delta_n, (H_S^\omega - \sigma - i\tau)^{-1}\delta_n) d\sigma\right) \leq \pi \left\| \frac{d\mu}{dx} \right\|_{\infty} a_n^{-1} |I|. $$

Then by Stone’s formula [15, Theorem VII.13] we have

$$\frac{\pi}{2} \mathbb{E}^{\omega}[\langle \delta_n E_{H_S^\omega}(a, b) | \delta_n \rangle + \langle \delta_n E_{H_S^\omega}(a, b) | \delta_n \rangle] \leq \pi \left\| \frac{d\mu}{dx} \right\|_{\infty} a_n^{-1} |I|$$

In particular $\mathbb{E}^{\omega}(\langle \delta_n E_{H_S^\omega}(r) | \delta_n \rangle) = 0, \ r \in \mathbb{R}$. So we have

$$\mathbb{E}^{\omega}(\langle \delta_n, E_{H_S^\omega}(I) | \delta_n \rangle) \leq \left\| \frac{d\mu}{dx} \right\|_{\infty} a_n^{-1} |I|, \ n \in \mathbb{Z}^d. $$

**Remark 1.5.** For any $z \in \mathbb{C}^+$ and any $S \subseteq \mathbb{Z}^d$ a similar calculation as in (1.18) of above proposition will give

$$\mathbb{E}^{\omega}\left[G^S(z; n, n)\right] \leq \pi \left\| \frac{d\mu}{dx} \right\|_{\infty} a_n^{-1}, \ n \in S. $$

**Proposition 1.6.** For any $z \in \mathbb{C}$ with $\text{Im}z > 0$ and $\text{Re} z \in \mathbb{R} \setminus [-2d, 2d]$, we have, setting $\chi_L = \chi_{\Lambda_L}$

$$\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}\left\{\text{Tr}(\chi_L \text{Im}G(z))\right\} = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}\left\{\text{Tr} \text{Im}G^{\Lambda_L}(z)\right\}$$

$$= \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}\left\{\sum_{p=1}^{N_\delta^d} \text{Tr} \text{Im}G^{C_p}(z)\right\}. $$

Proof: For $n \in \text{int}(C_p)$ and $z \in \mathbb{C}^+$ we have the well known perturbation formula, using the resolvent estimate,

$$G^{\Lambda_L}(z; n, n) = G^{C_p}(z; n, n) + \sum_{(m, k) \in \partial C_p} G^{C_p}(z; n, m) G^{\Lambda_L}(z; k, n)$$

where $(m, k) \in \partial C_p$ means $m \in C_p, k \in \mathbb{Z}^d \setminus C_p$ such that $|m - k| = 1$ and $\Lambda_L$ and $C_p$ are already described in introduction. We then estimate,

$$(1.21) \quad \frac{1}{\beta_L} \text{Tr} \text{Im}G^{\Lambda_L}(z) - \frac{1}{\beta_L} \sum_p \text{Tr} \text{Im}G^{C_p}(z) | \leq \frac{1}{\beta_L} \sum_p \sum_{n \in C_p \setminus \text{int}(C_p)} \left\{ \text{Im}G^{C_p}(z; n, n) + \text{Im}G^{\Lambda_L}(z; n, n) \right\}$$

$$+ \frac{1}{\beta_L} \sum_p \sum_{n \in \text{int}(C_p)} \sum_{(m, k) \in \partial C_p} |G^{C_p}(z; n, m)||G^{\Lambda_L}(z; k, n)|$$

$$= A_L + B_L.$$ 

Using (1.19) with $S = \Lambda_L$ or $C_p$ we get

$$\mathbb{E}^{\omega}\left(\text{Im}G^S(z; n, n)\right) \leq \pi \left\| \frac{d\mu}{dx} \right\|_{\infty} a_n^{-1}, \ n \in S. $$
Hence we start the estimate for the average of $A_L$ as,

\[ (1.22) \quad \mathbb{E}^{\omega}(A_L) \leq \frac{2\pi}{\beta_L} \left| \frac{d\mu}{dx} \right|_{\infty} N_{L}^{d} \left( \frac{2L + 1}{N_{L}} \right)^{d-1} l_{L} M_{L}, \quad M_{L} = \max_{n \in \Lambda_{L}} a_{n}^{-1}. \]

To estimate $B_L$ we use Cauchy-Schwarz inequality

\[ (1.23) \quad B_{L} = \frac{1}{\beta_{L}} \sum_{p} \sum_{n \in \text{int}(C_{p})} \sum_{(m,k) \in \partial C_{p}} |G_{L}^{C}(z;n,m)||G_{L}^{A}(z;k,n)| \]

\[ = \frac{1}{\beta_{L}} \sum_{p} \sum_{n \in \text{int}(C_{p})} \sum_{(m,k) \in \partial C_{p}} |G_{L}^{A}(z;k,n)||G_{L}^{C}(z;n,m)|^{1-s}|G_{L}^{C}(z;n,m)|^{s} \]

Now $m \in \partial C_{p}$ and $n \in \text{int}(C_{p})$ so we have $|n - m| > l_{L} > R$ for large enough $L$. From Aizenman-Molchanov [1, Theorem 2.1], Kirsch-Krishna-Obermeit [3, Lemma 3.2] and (1.7) we have

\[ (1.24) \quad \mathbb{E}^{\omega}|G_{L}^{C}(z;n,m)|^{s} \leq Ce^{-rl_{L}}. \]

We also have

\[ |G_{L}^{A}(z;k,n)|^{1-s} \leq \frac{1}{|Imz|^{1-s}} \quad \text{and} \quad |G_{L}^{A}(z;k,n)| \leq \frac{1}{|Imz|}, \]

therefore from (1.23) and (1.24) we get

\[ (1.25) \quad \mathbb{E}^{\omega}(B_{L}) \leq \frac{1}{\beta_{L}|Imz|^{2-s}N_{L}^{d} \left( \frac{2L + 1}{N_{L}} \right)^{d} \left( \frac{2L + 1}{N_{L}} \right)^{d-1} l_{L} e^{-rl_{L}}. \]

Take $N_{L} = (2L + 1)^{\epsilon}$ and $l_{L} = \gamma \ln(2L + 1)$, sufficiently large $\delta$, namely

\[ \gamma > \frac{1}{\epsilon} [d(2 - \epsilon) - 1 + \epsilon], \]

then we get from (1.25)

\[ (1.26) \quad \mathbb{E}^{\omega}(B_{L}) \leq \frac{1}{\beta_{L}|Imz|^{2-s}(2L + 1)^{d(2-s)-1+\epsilon})^{-r\gamma} \gamma \ln(2L + 1). \]

From (1.5) we have $\beta_{L} \simeq (2L + 1)^{d-\alpha(\delta-1)}$ and $M_{L} = \max_{n \in \Lambda_{L}} a_{n}^{-1} \simeq (2L + 1)^{\alpha}$

using the estimate of $\beta_{L}$ and $M_{L}$ in (1.22) we get

\[ (1.27) \quad \mathbb{E}^{\omega}(A_{L}) \leq 2\pi \left| \frac{d\mu}{dx} \right|_{\infty} (2L + 1)^{-d+\alpha(\delta-1)} N_{L}^{-1}(2L + 1)^{(d-1)} \gamma \ln(2L + 1)(2L + 1)^{\alpha} \]

\[ \simeq (2L + 1)^{-1-\epsilon+\alpha\delta} \gamma \ln(2L + 1), \quad (0 < \alpha\delta < 1 \text{ given in (1.5)}) \]

\[ \simeq \gamma(2L + 1)^{-(1-\alpha\delta+\epsilon)} \ln(2L + 1), \quad (1 - \alpha\delta + \epsilon > 0) \]

Now if we chose $0 < \epsilon < 1$ then form above and (1.26) we get

\[ \mathbb{E}^{\omega}(A_{L}) + \mathbb{E}^{\omega}(B_{L}) \longrightarrow 0 \text{ as } L \rightarrow \infty. \]
which implies,

\[(1.28) \quad \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega \left\{ \text{Tr} \text{Im} G^{A_L}(z) \right\} = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega \left\{ \sum_p \text{Tr} \text{Im} G^{C_p}(z) \right\}.
\]

Again using the resolvent equation for \(G(z; n, n) = (\delta_n, (H^\omega - z)^{-1}\delta_n)\) we get

\[G(z; n, n) = G^{C_p}(z; n, n) + \sum_{(m, k) \in \partial C_p} G^{C_p}(z; n, m)G(z; k, n)\]

Now

\[(1.29) \quad \frac{1}{\beta_L} \text{Tr}(\chi_L \text{Im} G(z)) - \frac{1}{\beta_L} \sum_p \text{Tr} \text{Im} G^{C_p}(z) \leq \frac{1}{\beta_L} \sum_p \sum_{n \in C_p \setminus \text{int}(C_p)} \left\{ \text{Im} G^{C_p}(z; n, n) + \text{Im} G(z; n, n) \right\}
\]

\[\quad + \frac{1}{\beta_L} \sum_p \sum_{n \in \text{int}(C_p)} \sum_{(m, k) \in \partial C_p} |G^{C_p}(z; n, m)||G(z; k, n)|\]

\[= C_L + D_L.
\]

Estimating \(C_L, D_L\) as we did for \(A_L\) and \(B_L\) we get,

\[(1.30) \quad \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega \left\{ \text{Tr} (\chi_L \text{Im} G(z)) \right\} = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega \left\{ \sum_p \text{Tr} \text{Im} G^{C_p}(z) \right\}
\]

From (1.28) and (1.30) we can conclude the proposition.

**Remark 1.7.** If \(I \subset \mathbb{R} \setminus [-2d, 2d]\) is a bounded interval then using Stone’s formula [15] Theorem VII.13] and the above proposition we get

\[(1.31) \quad \lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in A_L} \mathbb{E}^\omega(\langle \delta_n, E_{H^\omega} (I) \delta_n \rangle) = \lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in A_L} \mathbb{E}^\omega(\langle \delta_n, E_{H^\omega_{A_L}} (I) \delta_n \rangle)
\]

\[= \lim_{L \to \infty} \frac{1}{\beta_L} \sum_p \sum_{n \in C_p} \mathbb{E}^\omega(\langle \delta_n, E_{H^\omega_{C_p}} (I) \delta_n \rangle)
\]

**Remark 1.8.** Let \(z = z_L = \lambda + \beta^{-1}(\sigma + i\tau), \quad \tau > 0, \quad \lambda \in \mathbb{R} \setminus [-2d, 2d]\) then \(|\text{Im} z_L|\beta_L = \tau\). The inequality (1.26) will take the following form

\[(1.33) \quad \mathbb{E}^\omega(\langle \delta_n, E_{H^\omega}(I) \delta_n \rangle) \leq \frac{\beta_L^{-1-s}}{\tau^{2-s}} (2L + 1)^{(d(2-\epsilon)-1+\epsilon)-r\gamma} \ln(2L + 1)
\]

\[= \frac{\beta_L^{-1-s}}{\tau^{2-s}} (2L + 1)^{(1-s)(d-\alpha(\delta-1))+d(2-\epsilon)-1+\epsilon}-r\gamma \ln(2L + 1)
\]

Now if we chose \(\gamma\) large enough and using (1.27) and (1.21) we get

\[(1.34) \quad \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega \left\{ \text{Tr} \text{Im} G^{A_L}(z_L) - \sum_p \text{Tr} \text{Im} G^{C_p}(z_L) \right\} = 0.
\]
The next proposition is due to Minami[4].

**Proposition 1.9.** Let $A$ be the set of all functions of the form

$$f(x) = \sum_{k=1}^{n} \frac{c_k \tau}{(x - \sigma_k)^2 + \tau^2}, \quad n \geq 1, \quad \tau > 0 \text{ and } c_k > 0, \quad \sigma_k \in \mathbb{R}.$$  

Then $A$ is dense in $L_+^1(\mathbb{R})$, the set of all non-negative function in $L^1(\mathbb{R})$.

**proof:** Define

$$\phi_\tau(x) = \frac{1}{\pi} \int_a^b \frac{\tau}{(x - \sigma)^2 + \tau^2} d\sigma, \quad \tau > 0,$$

Then we have

$$\phi_\tau(x) = \frac{1}{\pi} \int_a^b \frac{\tau}{(x - \sigma)^2 + \tau^2} d\sigma = \frac{1}{\pi} \left[ \tan^{-1} \frac{b - x}{\tau} - \tan^{-1} \frac{a - x}{\tau} \right]$$

If $a < x < b$ then we have

(1.35) \quad \phi_\tau(x) \longrightarrow \chi_{(a,b)} \text{ as } \tau \to 0.

It is easy to see that

(1.36) \quad \int_\mathbb{R} \phi_\tau(x) dx = b - a = \int_\mathbb{R} \chi_{(a,b)} dx \Rightarrow \int_\mathbb{R} (\phi_\tau(x) - \chi_{(a,b)}) dx = 0.

So the equation (1.35) and (1.36) implies $\phi_\tau(x) \xrightarrow{L^1} \chi_{(a,b)}$ as $\tau \to 0$. So our lemma will be over if we construct a sequence $\{\phi_{n,\tau}\}_n$ in $A$ such that

(1.37) \quad \phi_{n,\tau} \xrightarrow{L^1} \phi_\tau \text{ as } n \to \infty.

Define

$$\phi_{n,\tau}(x) = \frac{1}{\pi} \frac{(b - a)}{n} \sum_{k=0}^{n-1} \frac{\tau}{(x - \sigma_{k,n})^2 + \tau^2}, \quad \sigma_{k,n} = a + \frac{(b - a)k}{n}.$$  

Since $\phi_{n,\tau}(x)$ is the Riemann sum of the integral $\frac{1}{\pi} \int_a^b \frac{\tau}{(x - \sigma)^2 + \tau^2} d\sigma = \phi_\tau(x)$ so we have

$$\phi_{n,\tau}(x) \xrightarrow{\text{pointwise}} \phi_\tau(x) \text{ as } n \to \infty.$$

To show the the above convergence in $L^1(\mathbb{R})$ we have to do the following.

(1.38) \quad \Psi_{n,k}(x, \sigma) = \frac{\tau}{(x - \sigma)^2 + \tau^2} - \frac{\tau}{(x - \sigma_{k,n})^2 + \tau^2}

$$= \frac{(\sigma - \sigma_{k,n})(2x - \sigma - \sigma_{k,n})}{((x - \sigma)^2 + \tau^2)((x - \sigma_{k,n})^2 + \tau^2)}.$$

Now $[a, b] = \bigcup_{k=0}^{n-1} [\sigma_{k,n}, \sigma_{k+1,n}]$, if $\sigma \in [\sigma_{k,n}, \sigma_{k+1,n}]$ then $|\sigma - \sigma_{k,n}| \leq \frac{1}{n}$, Now using the integrability of the functions $\frac{(2x - \sigma - \sigma_{k,n})}{((x - \sigma)^2 + \tau^2)((x - \sigma_{k,n})^2 + \tau^2)}$ we have a constant $C = C(\tau)$ such that

$$\int_\mathbb{R} \Psi_{n,k}(x, \sigma) dx \leq \frac{C}{n}, \forall \ k.$$
Now using the above calculations we get
\[
\int_{\mathbb{R}} |\phi_{\tau}(x) - \phi_{n,\tau}(x)|dx = \frac{1}{\pi} \sum_{k=0}^{n} \int_{\sigma_{k,n}}^{\sigma_{k+1,n}} \left( \int_{\mathbb{R}} |\Psi_{n,k}(x)|dx \right) d\sigma \leq \frac{1}{\pi} \frac{C}{n}.
\]

So from above we have
\[
\phi_{n,\tau} \xrightarrow{L^1} \phi_{\tau} \text{ as } n \to \infty.
\]

Hence the Proposition.

**Lemma 1.10.** The weak convergence of \( \{\xi_{\ell,\lambda}\} \) and \( \{\eta_{\ell,\lambda}\} \) are equivalent, i.e.
\[
\lim_{L \to \infty} \mathbb{E}^{\omega}[e^{-\xi_{\ell,\lambda}(f)}] = \lim_{L \to \infty} \mathbb{E}^{\omega}[e^{-\eta_{\ell,\lambda}(f)}], \quad \forall f \in C_c^+(\mathbb{R})
\]
where \( \nu(f) = \int_{\mathbb{R}} f(x)d\nu(x) \) for any measure \( \nu \).

**proof:** Since the set of functions \( f_{\zeta}(x) := \text{Im} \frac{1}{x-\sigma-i\tau}, \quad \zeta = \sigma + i\tau \in \mathbb{C}^+ \)
are dense in \( L_1^+(\mathbb{R}) \), the set of all non-negative continuous functions on \( L^1(\mathbb{R}) \)
(Proposition 1.9). Since \( C_c^+(\mathbb{R}) \subset L^1(\mathbb{R}) \) so to prove the lemma it is enough
to verify (1.40) for such functions \( f_{\zeta} \).

Now using the fact \( |e^{-x} - e^{-y}| < |x - y| \) to verify (1.40) we have to show
\[
\lim_{L \to \infty} \mathbb{E}^{\omega}[|\xi_{\ell,\lambda}(f_{\zeta}) - \eta_{\ell,\lambda}(f_{\zeta})|] = 0
\]
We set \( z_L = \lambda + \beta^{-1}_L(\sigma + i\tau) \) then,
\[
\mathbb{E}^{\omega}[|\xi_{\ell,\lambda}(f_{\zeta}) - \eta_{\ell,\lambda}(f_{\zeta})|] = \frac{1}{\beta_L} \mathbb{E}^{\omega}\left(\left|\text{Tr} \text{Im} G^{\lambda L}(z_L) - \sum_p Tr \text{Im} G^{\lambda p}(z_L)\right|\right).
\]

Now (1.13) will follow from (1.34). Hence the lemma.

**Remark 1.11.** Since the set of functions \( f_{\zeta}(x) := \text{Im} \frac{1}{x-\sigma-i\tau}, \quad \zeta = \sigma + i\tau \in \mathbb{C}^+ \)
are dense in \( L_1^+(\mathbb{R}) \), so from (1.13) we have the following
For any bounded interval \( I \subset \mathbb{R} \)
\[
\lim_{L \to \infty} \mathbb{E}^{\omega}(\xi_{\ell,\lambda}(I)) = \lim_{L \to \infty} \mathbb{E}^{\omega}(\eta_{\ell,\lambda}(I)).
\]

**Lemma 1.12.** There exist a set \( B \subseteq (-\infty, -2d) \) with \( |B| > 0 \) such that for each \( \lambda \in B \) the following is true.
For any bounded interval \( I \) there exist a positive constant \( c_\lambda \) such that
\[
\lim_{L \to \infty} \mathbb{E}^{\omega}[\eta_{\ell,\lambda}(I)] = c_\lambda |I|, \quad \lambda \in B,
\]
for some subsequence \( \{L_k\} \) of \( \{L\} \).

**proof:** From the definition of \( \eta_{\ell,\lambda} \) and \( f_{\zeta}(x) \) as they are given in (1.3) and (1.13) we have
\[
\mathbb{E}^{\omega}[\eta_{\ell,\lambda}(f_{\zeta})] = \frac{1}{\beta_L} \sum_{p=1}^{N_L} \mathbb{E}^{\omega}\left(\text{Tr} \text{Im}(G^{\lambda p}(z_L))\right), \quad z_L = \lambda + \beta^{-1}_L(\sigma + i\tau).
\]
Now we have from (1.34)

\begin{equation}
\lim_{L \to \infty} \frac{1}{\beta L} \sum_{p=1}^{N_L} \mathbb{E}^\omega (\text{Tr} \text{Im}(G^p(z_L))) = \lim_{L \to \infty} \frac{1}{\beta L} \mathbb{E}^\omega (\text{Tr} \text{Im}(G^{\lambda L}(z_L))).
\end{equation}

we also have,

\begin{equation}
\frac{1}{\beta L} \mathbb{E}^\omega (\text{Tr} \text{Im}(G^{\lambda L}(z_L))) = \frac{1}{\beta L} \mathbb{E}^\omega (\text{Im}(G^{\lambda L}(z_L; n, n)))
\end{equation}

\[ = \int_{\mathbb{R}} \frac{\beta^{-1}_L \tau}{(x - \lambda - \beta^{-1}_L \sigma)^2 + (\beta^{-1}_L \tau)^2} d\nu_L(x). \]

where the measure $\nu_L$ is given by

\[ \nu_L(.) = \frac{1}{\beta L} \sum_{n \in \Lambda_L} \mathbb{E}^\omega ((\delta_n, E_{H^L}(.)\delta_n)). \]

From Proposition 1.4 we can claim that $\nu_L$ is absolute continuous w.r.t Lebesgue measure, let $h_L$ be the density of $\nu_L$, i.e.

\[ d\nu_L(x) = h_L(x)dx. \]

Using the upper bound in Lemma (1.14) of the Appendix we have for any $M = -2d - \epsilon < -2d$,

\begin{equation}
\nu_L(-\infty, M) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta - 1)}} = \frac{1}{(\delta - 1)(-2d - M)^{(\delta - 1)}}.
\end{equation}

Let $I = [a, b] \subset (-\infty, -2d)$ with $|I| > 4d$, set $a = -2d - \epsilon_1$, $b = -2d - \epsilon_2$ such that $\epsilon_1 - \epsilon_2 > 4d$. Using the lower bound in Lemma (1.14) of the Appendix we get

\begin{equation}
\nu_L(I) \geq \frac{1}{(\delta - 1)} \left[ \frac{1}{(4d + \epsilon_2)^{(\delta - 1)}} - \frac{1}{\epsilon_1^{(\delta - 1)}} \right] > 0.
\end{equation}

So we conclude that whenever $I = [a, b] \subset (-\infty, -2d)$ with $|I| > 4d$ there exist a constant $C > 0$, independent of $L$, such that

\begin{equation}
\nu_L(I) \geq C.
\end{equation}

From Theorem 1.4.16 of [17] and (1.47) we get

\begin{equation}
\lim_{\tau \to 0} \frac{1}{\beta L} \mathbb{E}^\omega (\text{Tr} \text{Im}(G^{\lambda L}(z_L))) = \lim_{\tau \to 0} \int_{\mathbb{R}} \frac{\beta^{-1}_L \tau}{(x - \lambda - \beta^{-1}_L \sigma)^2 + (\beta^{-1}_L \tau)^2} d\nu_L(x) = h_L(\lambda + \beta^{-1}_L \sigma).
\end{equation}

Set

\[ f_L(\lambda) = h_L(\lambda + \beta^{-1}_L \sigma). \]

Then for bounded $\sigma$ large $L$, there is an $M < -2d$ such that $M + \beta^{-1}_L \sigma < M_1 < -2d$. Therefore (1.48) gives

\begin{equation}
\int_{-\infty}^{M} f_L(\lambda) d\lambda = \int_{-\infty}^{M} h_L(\lambda + \beta^{-1}_L \sigma) d\lambda = \int_{-\infty}^{M + \beta^{-1}_L \sigma} h_L(\lambda) d\lambda = \nu_L(-\infty, M + \beta^{-1}_L \sigma) \leq \nu_L(-\infty, M_1) < \infty.
\end{equation}

Suppose on a subset $S \subset (-\infty, -2d)$ of positive Lebesgue measure

\begin{equation}
\int_{-\infty}^{M} f_L(\lambda) d\lambda \to \infty \text{ as } L \to \infty.
\end{equation}
Then monotone convergence theorem implies
\[ \int_{-\infty}^{M} f_L(\lambda) d\lambda \geq \int_{S} f_L(\lambda) d\lambda \to \infty \text{ as } L \to \infty. \]
which is a contradiction to (1.52). So (1.53) is false, hence.
\[ \lim_{L \to \infty} f_L(\lambda) < \infty \text{ a.e } \lambda \in (-\infty, -2d). \]
We note that if \( S \subset (-\infty, -2d) \) with \( |S| > 4d \), then the interval \( [\inf S, \sup S] \) has length \( > 4d \). Therefore if \( f_L(\lambda) \to 0 \) a.e for \( \lambda \in S \) for any subset \( S \subset (-\infty, -2d) \) with \( |S| > 4d \), an argument similar to the one for \( \lim \) gives that \( \lim_{L \to \infty} f_L(\lambda) \neq 0 \) a.e \( \lambda \in S \).
So we can have \( B \subseteq (-\infty, -d) \) with \( |B| > 0 \) such that for each \( \lambda \in B \) there exist a subsequence \( \{L_k\} \) of \( \{L\} \) such that
\[ f_{L_k}(\lambda) = h_{L_k}(\lambda + \beta_{L_k}^{-1}\sigma) \to c_\lambda > 0 \text{ as } L_k \to \infty. \]
Finally Stone’s formula together with (1.45), (1.46), (1.51) and (1.54) gives if \( \lambda \in B \) then
\[ \lim_{L_k \to \infty} \mathbb{E}^{\omega}[\eta_{L_k,\lambda}(I)] = c_\lambda |I|, \lambda \in B. \]
Hence the lemma.

Consider the measures
\[ (1.55) \quad \mu_n(\cdot) = \mathbb{E}^{\omega}(\langle \delta_n, E_{H^{\omega}}(\cdot) \delta_n \rangle). \]
Now Proposition [1.4] implies \( \mu_n \) are absolute continuous measures, Let this densities be denoted by \( f_n \). Define
\[ g_L(x) = \sum_{n \in \Lambda_L} f_n(x). \]
Then we have the lemma.

**Lemma 1.13.** For almost all \( x \in \mathbb{R} \)
\[ (1.56) \quad g_L(x) \to \infty \text{ as } L \to \infty. \]

**Proof:** If we take \( H^{\omega} \) as in (1.1) then in [3] Krisch-krishna-Obermeit proved that \( \sigma(H^{\omega}) = \mathbb{R} \) a.e \( \omega \). So for any interval \( I \subseteq \mathbb{R} \) we have
\[ Tr(E_{H^{\omega}}(I)) = \infty \text{ a.e } \omega \]
Hence
\[ Tr(E_{H^{\omega}}(B)) = \infty \text{ a.e } \omega, \text{ whenever } |B| > 0 \]
From above and using monotonicity of \( Tr(\chi_{\Lambda_L} E_{H^{\omega}}(B)) \) we get
\[ (1.57) \quad \lim_{L \to \infty} \mathbb{E}^{\omega}(Tr(\chi_{\Lambda_L} E_{H^{\omega}}(B))) = \infty \text{ whenever } |B| > 0. \]
Now
\[ E^\omega(Tr(\chi_{\Lambda_L} E_{H^\omega}(B))) = \sum_{n \in \Lambda_L} E^\omega(\langle \delta_n, E_{H^\omega}(B) \delta_n \rangle) = \int_B \sum_{n \in \Lambda_L} f_n(x) dx = \int_B g_L(x) dx \]

Let \( \lim_{L \to \infty} g_L(x) = g(x) \). Since \( g_L \) is monotone increasing,
\[ \int_B g(x) dx = \lim_{L \to \infty} \int_B g_L(x) dx = \lim_{L \to \infty} E^\omega(Tr(\chi_{\Lambda_L} E_{H^\omega}(B))) \]

Therefore using (1.57) and (1.58) we have
\[ \int_B g(x) dx = \infty \text{ if } |B| > 0. \]

Suppose the claim of the lemma is not true, then there exist a set \( A \) with positive Lebesgue measure \( |A| > 0 \) such that
\[ \lim_{L \to \infty} g_L(x) = g(x) < \infty \quad \forall \ x \in A, \ |A| > 0 \]

Set
\[ K_l = \{ x \in \mathbb{R} : l \leq g(x) < l + 1 \}, \ l = 0, 1, 2, \ldots \]

Then we have
\[ A = \bigcup_{l=0}^{\infty} (A \cap K_l) \]

Since \( |A| > 0 \) then for some \( l \) say \( l_0 \) we have \( |A \cap K_{l_0}| > 0 \), since Lebesgue measure is sigma finite w.l.o.g we can assume \( 0 < |A \cap K_{l_0}| < \infty \). Now we have
\[ \int_{A \cap K_{l_0}} g(x) dx < (l_0 + 1)|A \cap K_{l_0}| < \infty, \ where \ 0 < |A \cap K_{l_0}| < \infty. \]

which is contradiction to (1.59). Hence the lemma.

Proof of the theorem 1.3

For any bounded interval \( I \subset \mathbb{R} \) we have
\[ \mathbb{P}(\eta^\omega_{C_p,\lambda}(I) \geq 1) = E^\omega[\eta^\omega_{C_p,\lambda}(I)] - \sum_{j \geq 2} \mathbb{P}(\eta^\omega_{C_p,\lambda}(I) \geq j) \]

(1.61)
$$\sum_{j \geq 2} \mathbb{P}(\eta_{C_p,\lambda}^j(I) \geq j) = \sum_{j=2}^{\infty} (j-1) \mathbb{P}(\eta_{C_p,\lambda}^j(I) = j)$$

$$\leq \sum_{j=2}^{\infty} j(j-1) \mathbb{P}(\eta_{C_p,\lambda}^j(I) = j)$$

$$= \mathbb{E}^{\nu}[\{\eta_{C_p,\lambda}^j(I)\} \{\eta_{C_p,\lambda}^j(I) - 1\}]$$

$$= \mathbb{E}^{\nu} \left[ Tr\left( E_{H_{C_p}}(\lambda + \beta^{-1}L) \right) \left( Tr\left( E_{H_{C_p}}(\lambda + \beta^{-1}L) \right) - 1 \right) \right]$$

$$\leq \left( \pi \rho_{\infty}^{A_L} |J||C_p| \right)^2$$

Here we use the estimate given in Combes-Germinet-Klein [8, Theorem 2.3].

Where

$$J = \lambda + \beta^{-1}L$$

$$\rho_{\infty}^{A_L} := \max_{j \in \Lambda_L} \|\rho_j\|_{\infty}$$

$$\rho_j$$ is the bounded density of the random variable $a_j q_j$, where $q_n$ are iid random variables with distribution $\mu$ and $\frac{d\mu}{dx} = \rho$. Then we have

$$\rho_j(x) = a_n^{-1} \rho(a_n^{-1} x), \ |\rho_j|_{\infty} = a_n^{-1} \|\rho\|_{\infty}.$$

$$\rho_{\infty}^{A_L} = \max_{j \in \Lambda_L} \left\{ a^{-1}_j \|\rho\|_{\infty} \right\} = M_L \|\rho\|_{\infty}, \ M_L = \max_{j \in \Lambda_L} a^{-1}_j$$

Now we have

$$\rho_{\infty}^{A_L} |J||C_p| = M_L \|\rho\|_{\infty} \frac{|I|}{\beta_L} \left( \frac{2L + 1}{N_L} \right)^d$$

So we get

$$\sum_{j \geq 2} \mathbb{P}(\eta_{C_p,\lambda}^j(I) \geq j) \leq \pi^2 \|\rho\|_{\infty}^2 |I|^2 \left( \frac{M_L(2L + 1)^d}{\beta_L N_L^d} \right)^2.$$

Our Hypothesis (3) implies that $\beta_L \approx (2L + 1)^{d-\alpha(\delta-1)}$.

Take $N_L = (2L + 1)^{\epsilon}, \ 2\alpha \delta < \epsilon < 1$ ($0 < \delta < \frac{1}{2\alpha}$) then we have from (1.62)

$$\sum_{j \geq 2} \mathbb{P}(\eta_{C_p,\lambda}^j(I) \geq j) \leq \pi^2 \|\rho\|_{\infty}^2 |I|^2 \left( \frac{L^{\alpha}(2L + 1)^d}{(2L + 1)^{d-\alpha(\delta-1)}(2L + 1)^{de}} \right)^2 \approx \frac{1}{I^{2(\epsilon - \alpha \delta)}}.$$
Now,

\[ (1.64) \quad \mathbb{E}^\omega [\eta_{C_p, \lambda}(f_\xi(x))] = \mathbb{E}^\omega \left[ \eta_{C_p, E}^\omega \left( \text{Im} \left( \frac{1}{x - (\sigma + i\tau)} \right) \right) \right] \quad \text{(using (1.15))} \]

\[ = \frac{1}{\beta_L} \mathbb{E}^\omega [\text{TrIm} G^{C_p}(\lambda + \beta_L^{-1}(\sigma + i\tau))] \]

\[ = \frac{1}{\beta_L} \mathbb{E}^\omega [\text{TrIm} G^{C_p}(z_L)], \quad z_L = \lambda + \beta_L^{-1}(\sigma + i\tau) \]

\[ = \frac{1}{\beta_L} \sum_{n \in C_p} \mathbb{E}^\omega [\text{Im} G^{C_p}(z_L; n, n)] \]

\[ \leq \pi \| \frac{dt}{dx} \|_\infty \frac{1}{\beta_L} \sum_{n \in C_p} a_n^{-1} \]

using (1.19) for the last inequality.

Using (1.16) and \( I = [a, b] \) we get

\[ (1.66) \quad \mathbb{E}^\omega [\eta_{C_p, \lambda}^\omega(I)] \leq \pi \| \frac{dt}{dx} \|_\infty \frac{|I|}{\beta_L} \sum_{n \in C_p} a_n^{-1} \]

\[ \leq \pi \| \frac{dt}{dx} \|_\infty \frac{|I|}{\beta_L} \left( \frac{2L + 1}{N_L} \right)^d M_L \]

\[ \approx \pi \| \frac{dt}{dx} \|_\infty \frac{1}{L^{d\epsilon - \alpha\delta}} \]

As we have

\[ \beta_L^{-1} \approx (2L + 1)^{-d + \alpha(\delta - 1)}, \quad N_L = (2L + 1)^\epsilon, \quad M_L \approx L^{\alpha}, \quad d\epsilon > 2\alpha\delta. \]

(1.63), (1.67) and (1.61) together will give

\[ (1.68) \quad \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq 1) \to 0 \quad \text{as } L \to \infty. \]

Since \( \eta_{C_p, \lambda}^\omega \) is integer valued, equation (1.10) verified.

Now equation (1.12) is direct consequence of the following

\[ (1.69) \quad \sum_{p=1}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq 2) \leq \sum_{p=1}^{N_L^d} \sum_{j=2}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq j) \]

\[ \approx N_L^d \frac{1}{L^{2(\epsilon - \alpha\delta)}} = \frac{1}{L^{(d\epsilon - 2\alpha\delta)}}, \quad \text{(using (1.63)).} \]

We have

\[ (1.70) \quad \sum_{p=1}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq 1) = \sum_{p=1}^{N_L^d} \mathbb{E}^\omega [\eta_{C_p, \lambda}^\omega(I)] - \sum_{p=1}^{N_L^d} \sum_{j=2}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq j) \]

\[ = \mathbb{E}^\omega [\eta_{C_p, \lambda}^\omega(I)] - \sum_{p=1}^{N_L^d} \sum_{j=2}^{N_L^d} \mathbb{P}(\eta_{C_p, \lambda}^\omega(I) \geq j) \]

Now using Lemma 1.12 and (1.70) we can claim the following for a subsequence \( \{L_k\} \) of \( L \).
We take $H^\omega$, $\mu$, $\rho$ and $\{a_n\}$ as in equation (1.1) and Hypothesis 1.1. Let $x < 0$ and $\epsilon > 0$ such that $x + \epsilon < 0$ then for large enough $|n| \geq M$ we have $a_n^{(x+\epsilon)} \leq -1$ since $a_n^{-1} \to \infty$ as $|n| \to \infty$. Then we have for $|n| \geq M$

\[
\mu \left( \frac{1}{a_n}(x - \epsilon, x + \epsilon) \right) = \int_{a_n^{-1}(x-\epsilon)}^{a_n^{-1}(x+\epsilon)} \rho(t)dt = a_n^{(\delta-1)} \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^\delta}dt.
\]

Therefore,

\[
\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n}(x - \epsilon, x + \epsilon) \right) \geq \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^\delta}dt \sum_{|n| \geq M} a_n^{(\delta-1)} = \infty
\]

since $\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \to \infty$ as $L \to \infty$ (using 1.5).

Then from [3, Definition 2.1] we see that

\[
M = \cap_{k \in \mathbb{Z}^+} (a_kn - \text{supp}) = \mathbb{R}^-.
\]

Therefore [3, Corollary 2.5] will ensure the existence of spectrum of $H^\omega$ outside $[-2d, 2d]$. In particular we have

\[
\sigma_{\text{ess}}(H^\omega) = [-2d, 2d] + \mathbb{R}^- \text{ and } \sigma_c(H^\omega) \subseteq [-2d, 2d] \text{ a.e } \omega.
\]

Define

\[
N_L^\omega(\lambda) = \# \{ j; E_j \leq \lambda, E_j \in \sigma(H_{\Lambda_L}^\omega) \}, H_{\Lambda_L}^\omega = \chi_{\Lambda_L} H^\omega \chi_{\Lambda_L}
\]

Set

\[
\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)}.
\]

Now we will prove the following lemma.

**Lemma 1.14.** If $E = -2d - \epsilon < -2d$ for some $\epsilon > 0$ then we have

\[
\frac{1}{(\delta-1)(4d+\epsilon)^{(\delta-1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{P}(N_L^\omega(E)) \leq \frac{1}{(\delta-1)e^{(\delta-1)}}.
\]

**Proof:** Define

\[
A_{L,\pm}^\omega = \pm 2d + \sum_{n \in \Lambda_L} a_n q_n(\omega)P_{\delta_n}
\]

Define

\[
N_{L,\pm}^\omega(E) = \# \{ j; E_j \leq E, E_j \in \sigma(A_{L,\pm}^\omega) \}
\]
Since $\sigma(\Delta) = [-2d, 2d]$ it is easy to see the following operator inequality
\begin{equation}
A_{L,-}^\omega \leq H_{\Lambda_L}^\omega \leq A_{L,+}^\omega.
\end{equation}
where $H_{\Lambda_L}^\omega$ is given by
\[H_{\Lambda_L}^\omega = \chi_{\Lambda_L} \Delta \chi_{\Lambda_L} + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{\delta_n}.
\]

Simple application of the min-max principle [18, Theorem 6.44] will provide
\begin{equation}
N_{\omega, L}(E) \leq N_{\omega, L}^\omega(E) \leq N_{\omega, L}(E).
\end{equation}
Now the spectrum $\sigma(A_{L,\pm}^\omega)$ of $A_{L,\pm}^\omega$ consists of only eigen values and is given by
\[\sigma(A_{L,\pm}^\omega) = \{ n \in \Lambda_L : \pm 2d + a_n q_n(\omega) \}
\]
Let $E < -2d$ with $E = -2d - \epsilon$ for some $\epsilon > 0$.
\[N_{\omega, L}(E) = \# \{ n \in \Lambda_L : -2d + a_n q_n(\omega) \leq -2d - \epsilon \}
\]
\[= \# \{ n \in \Lambda_L : q_n(\omega) \in (-\infty, -a_n^{-1} \epsilon]\}
\]
\[= \sum_{n \in \Lambda_L} \chi_{\{\omega : q_n(\omega) \in (-\infty, -a_n^{-1} \epsilon]\}}
\]
Since $q_n$ are i.i.d so if we take expectation of both side of above we get
\begin{equation}
\mathbb{E}(N_{\omega, L}^\omega(\lambda)) = \sum_{n \in \Lambda_L} \mu(\lambda, -a_n^{-1} \epsilon]
\end{equation}
\[= \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx
\]
since $a_n^{-1} \to \infty$ as $|n| \to \infty$ and $\epsilon > 0$ so there exist a $M \in \mathbb{N}$ such that
\[a_n^{-1} \epsilon > 1, -a_n^{-1} \epsilon < -1 \forall |n| > M.
\]
So from (1.75) for large enough $L$, we get
\begin{equation}
\mathbb{E}(N_{\omega, L}(E)) = \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx
\end{equation}
\[= \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx + \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx.
\]
Since $\# \{ n \in \mathbb{Z}^d : |n| \leq M \} \leq (2M + 1)^d$ then we have
\begin{equation}
\sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx \leq (2M + 1)^d \int_{-\infty}^{-1} \rho(x) dx
\end{equation}
\[= (2M + 1)^d \int_{-\infty}^{-1} \frac{1}{|x|^{\delta}} dx
\]
\[= \frac{(2M + 1)^d}{(\delta - 1)}, \delta > 1 \text{ is given.}
\]
If we take \( \beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \) then \( \beta_L \to \infty \) as \( L \to \infty \) and we have from above

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx = 0.
\]

Now

\[
\sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx = \sum_{n \in \Lambda_L, |n| > M} a_n^{-1} \int_{-\infty}^{-\epsilon} \rho(a_n^{-1} t) dt
\]

\[
= \sum_{n \in \Lambda_L, |n| > M} a_n^{-1} \int_{-\infty}^{-\epsilon} \frac{1}{|a_n^{-1} t|} dt
\]

\[
= \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)} \int_{-\infty}^{-\epsilon} \frac{1}{|t|^{\delta}} dt
\]

\[
= \frac{\epsilon^{\delta-1}}{1 - \delta} \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)}, \delta > 1
\]

This will give

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx = \frac{\epsilon^{1-\delta}}{\delta - 1}.
\]

Using (1.79) and (1.81) in (1.70) we get

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}_\omega(N^-_{\omega,L}(\lambda)) = \frac{\epsilon^{1-\delta}}{\delta - 1} = \frac{1}{(\delta - 1) \epsilon^{(\delta-1)}} > 0.
\]

Now a similar calculation with \( \mathbb{E}_\omega(N^+_{\omega,L}(\lambda)) \) will give

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}_\omega(N^+_{\omega,L}(\lambda)) = \frac{(4d + \epsilon)^{1-\delta}}{\delta - 1} = \frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta-1)}} > 0.
\]

Now using (1.82) and (1.83), from (1.74) we can conclude the following

\[
0 < \frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta-1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}_\omega(N^+_{\omega,L}(\lambda)) \leq \frac{1}{(\delta - 1) \epsilon^{(\delta-1)}} < \infty.
\]

Hence the lemma.

So the above lemma tells us that the average spacing of eigenvalues of \( H_{\Lambda_L}^\omega \) near a energy \( \lambda < -2d \) is of order \( \beta_L^{-1} \).

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