A Liouville type result and quantization effects on the system

$$-\Delta u = uJ'(1 - |u|^2)$$ for a potential convex near zero

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Abstract

We consider a Ginzburg-Landau type equation in $\mathbb{R}^2$ of the form $-\Delta u = uJ'(1 - |u|^2)$ with a potential function $J$ satisfying weak conditions allowing for example a zero of infinite order in the origin. We extend in this context the results concerning quantization of finite potential solutions of H.Brezis, F.Merle, T.Rivi`ere from [10] who treat the case when $J$ behaves polynomially near 0, as well as a result of Th. Cazenave, found in the same reference, and concerning the form of finite energy solutions.

Keywords: Ginzburg-Landau type functional, variational problem, quantization of energy, finite energy solutions.

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1 Introduction

We consider in the paper the following system:

$$-\Delta u = uj(1 - |u|^2) \text{ in } \mathbb{R}^2,$$

with $u : \mathbb{R}^2 \to \mathbb{R}^2$, where $j(t) = \frac{dJ}{dt}(t)$ with a $C^2$ functional $J : \mathbb{R} \to [0, \infty)$ satisfying the following conditions:

(H1) $J(0) = 0$ and $J(t) > 0$, $\forall t \in (0, \infty)$.

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\((H2)\) \(\liminf_{t \to -\infty} \frac{J(t)}{\sqrt{|t|}} > K\), for some \(K > 0\).

\((H3)\) \(j(t) < 0\) on \((-\infty, 0]\) and \(j(t) > 0\) on \((0, 1]\).

\((H4)\) There exists \(\eta_0 > 0\) such that
\[
J''(t) = \frac{d^2J}{dt^2}(t) \geq 0, \text{ for } t \in [0, \eta_0).
\]

Solutions to \((1.1)\) are formal critical points of the functional:
\[
E(u) = \int_{\mathbb{R}^2} (|\nabla u(x)|^2 + J(1 - |u(x)|^2)) dx. \tag{1.2}
\]

In this paper we generalize two results from [10] for which one needs to use and to properly adapt some of the ideas from their proofs.

The first result concerns a Liouville-type theorem:

**Theorem 1.** Under assumptions \((H1) \div (H4)\), let \(u : \mathbb{R}^2 \to \mathbb{R}^2\) be a smooth solution of \((1.1)\) satisfying
\[
\int_{\mathbb{R}^2} |\nabla u|^2 dx < +\infty. \tag{1.3}
\]
Then either \(u \equiv 0\) or \(u\) is a constant \(C\) with \(|C| = 1\) on \(\mathbb{R}^2\).

This result appeared in Brezis, Merle and Rivièrè in [10] for the classic Ginzburg-Landau system, namely \(J(t) = t^2\).

Concerning the problem formulated in \(\mathbb{R}^n\) we mention some references in the following lines. In [1], Alikakos showed that if \(u : \mathbb{R}^n \to \mathbb{R}^n\) is a solution of \((1.1)\) such that \(\int \mathbb{R}^n |\nabla u|^2 + \int \mathbb{R}^n J(1 - |u|^2) \leq C\), then \(u\) is a constant. In [13], Farina showed a Liouville-type theorem for the Ginzburg-Landau system which asserts that every solution \(u \in C^2(\mathbb{R}^n, \mathbb{R}^m)\) satisfying \(\int_{\mathbb{R}^n} (1 - |u|^2)^2 < +\infty\) is constant (and of unit norm), provided \(n \geq 4\) \((m \geq 1)\). Moreover, in [14], Farina showed a Liouville-type theorem for a variant of the Ginzburg-Landau system under assumption that the solution \(u\) satisfies \(\int_{\mathbb{R}^n} |\nabla u|^2 < +\infty\) for \(n \geq 2\).

Our second result generalizes the theorem of quantization of the energy obtained by Brezis, Merle and Rivièrè in [10]:

**Theorem 2.** Under assumptions \((H1) \div (H4)\), let \(u : \mathbb{R}^2 \to \mathbb{R}^2\) be a smooth solution of equation \((1.1)\). Then
\[
\int_{\mathbb{R}^2} J(1 - |u|^2) = \pi d^2 \tag{1.4}
\]
for some integer \(d = 0, 1, 2, \ldots, \infty\).
The hypotheses on $J$ include the classic situation $J(t) = t^2$ considered in [10] and include also a large class of functionals $J$ having a zero of infinite order at $t = 0$, such as

$$J(t) = \begin{cases} 
\exp\left(-1/t^k\right) & \text{if } t > 0, \\
0 & \text{if } t = 0, \\
\sqrt{|t|}g(t) & \text{if } t < 0,
\end{cases} \quad (1.5)$$

for some $k \geq 1$ and a $C^2$ positive non increasing function $g : (-\infty, 0] \rightarrow \mathbb{R}$, $g'(0) = g''(0) = 0$. A key point in our approach refers to the proof of the bound $|u| \leq 1$ which does not follow the lines in [10].

Several results regarding the study of Ginzburg-Landau energy are present in literature. Béthuel-Brezis and Hélein were the first to study the asymptotic behaviour of the standard energy of Ginzburg-Landau in bounded domain in [7] and [8]. Generalized vortices in the magnetic Ginzburg-Landau model is considered in [22]. In [21], Sandier showed that every locally minimizing solution of $-\Delta u = u(1 - |u|^2)$ has a bounded potential.

The presence of a nonconstant weight function which is motivated by the problem of pinning of vortices, that is, of forcing the location of the vortices to some restricted sites, are studied in [3], [4], [5], [6], [17], [12] and [20].

In [18], the authors investigated the energy of a Ginzburg-Landau type energy with potential having a zero of infinite order. They showed that the main difference with respect to the usual GL-energy is in the asymptotic of the energy. For $J$ with a zero of infinite order the “energy cost” of a degree-one vortex may be much less than the cost of $2\pi \log \frac{1}{\varepsilon}$ for the GL-functional.

In [16], the authors considered the asymptotic behaviour of minimizing solutions of a Ginzburg-Landau type functional with a positive weight and with convex potential near 0 and they estimated the energy in this case. They also generalized the lower bound for the energy of the Ginzburg-Landau energy of unit vector field given initially by Brezis-Merle-Rivièere in [10] for the case where the potential has a zero of infinite order.

In [14], Farina showed a quantization effect for a variant of the Ginzburg-Landau system under the assumption that $\int_{\mathbb{R}^2} P_h \left(|u|^2\right) < +\infty$, where the potential $P_h$ is given by $P_h(t) = \frac{1}{2} \prod_{j=1}^{h} \left(1 - t^2 - k_j\right)^2$.

Note that starting from the Ginzburg-Landau type problem

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon j(1 - |u_\varepsilon|^2) \quad \text{in } \Omega_\varepsilon,$$  \quad (1.6)

after a blow-up argument we are led to [11].
Problem (1.6), defined in a bounded domain and with potential satisfying conditions (H1), first part of (H3) and (H4) was considered in [18], where the asymptotic behaviour of the minimizers and their energies is described.

2 Preliminary results

In this section we want to prove some results which will be useful in the sequel. Moreover here and after, for every $R > 0$ and $x_0 \in \mathbb{R}^n$, we will denote with $B_R(x_0)$ the set $\{x \in \mathbb{R}^n; |x - x_0| < R\}$, with $B_R$ the set $\{x \in \mathbb{R}^n; |x| < R\}$ and with $S_R$ its boundary. Moreover, for every function $v$, we will denote $v^+(x) = \max\{v(x), 0\}$ and $v^- = -\min\{v, 0\}$.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^n$ be open, not necessarily bounded, connected domain with smooth boundary $\partial \Omega \neq \emptyset$.

Suppose $c(x) \in L^\infty_{\text{loc}}(\Omega)$, $c(x) \geq 0$ in $\Omega$ and $\delta \geq 0$ are given. Let $\rho \in H^2_{\text{loc}}(\Omega)$ be a strong solution to the problem:

\[
\begin{align*}
-\Delta \rho + c(x)\rho &= 0, \quad x \in \Omega, \\
\rho(x) &= \delta \geq 0, \quad x \in \partial \Omega
\end{align*}
\]  

(2.1)

and $\rho$ satisfies the lower bound $\rho(x) \geq \delta$, $\forall x \in \Omega$.

Then the following statements hold:

1. Suppose that $\int_\Omega |\nabla \rho|^2 dx < \infty$ and that $n = 2$ if $\Omega$ is unbounded. Then, $\rho(x) \equiv \delta$ in $\Omega$. If $c \not\equiv 0$ then $\delta = 0$.

2. Suppose that $\rho \in L^1(\Omega)$. If $\delta > 0$ or if $\delta = 0$ and $\mu(\Omega) < \infty$, then $\rho(x) \equiv \delta$ and if $\delta > 0$ then $c(x) \equiv 0$.

Proof. (1) Suppose first that $\int_\Omega |\nabla \rho|^2 dx < \infty$.

If $\Omega$ is bounded multiply the equation by $\rho$, integrate on $\Omega$ and obtain:

\[
\int_\Omega |\nabla \rho|^2 dx - \int_{\partial \Omega} \frac{\partial \rho}{\partial \nu} \rho d\sigma + \int_\Omega c(x)\rho^2 dx = 0.
\]

As $\rho|_{\partial \Omega} = \delta$ and $\rho \geq \delta$ in $\Omega$ we have that $\frac{\partial \rho}{\partial \nu} \leq 0$ on $\partial \Omega$. Consequently, from the above equality we deduce that $c(x)\rho \equiv 0$ in $\Omega$ and $\nabla \rho \equiv 0$. The latter implies $\rho \equiv \delta$ in $\Omega$. Now, as $c(x)\rho \equiv 0$, if $c(x) \not\equiv 0$ then necessarily $\delta = 0$.

Suppose now that $\Omega$ is unbounded.
Consider \( \eta : \mathbb{R}^2 \to \mathbb{R} \) a \( C^\infty \) function satisfying:
\[
\eta \geq 0 \text{ in } \mathbb{R}^2, \quad \eta(x) = 1 \text{ for } |x| < \frac{1}{2} \quad \text{and} \quad \eta(x) = 0 \text{ for } |x| > 1,
\]
and for \( R > 0 \) define
\[
\eta_R(x) = \eta \left( \frac{x}{R} \right).
\]
Observe that for \( K = \sup |\nabla \eta| < \infty \) we have
\[
\left\{ \begin{array}{ll}
|\nabla \eta_R(x)| \leq \frac{K}{R}, & \forall x \in \mathbb{R}^2, \\
|\nabla \eta_R(x)| = 0, & \text{for } |x| \notin \left( \frac{R}{2}, R \right).
\end{array} \right. \tag{2.2}
\]
Denote by \( \Omega_R = \Omega \cap B_R \) and by \( A_{\frac{R}{2}, R} = B_R \setminus B_{\frac{R}{2}} \).
Multiply equation (2.1) by \( \eta_R \) and integrating on \( \Omega_R \) to obtain by Green formula:
\[
\int_{\Omega_R} c(x) \rho(x) \eta_R(x) dx - \int_{\partial \Omega \cap B_R} \frac{\partial \rho}{\partial \nu} \eta_R d\sigma = - \int_{\Omega_R} \nabla \rho \cdot \nabla \eta_R dx = - \int_{\Omega \cap A_{\frac{R}{2}, R}} \nabla \rho \cdot \nabla \eta_R dx. \tag{2.3}
\]
Now, by Cauchy-Schwarz inequality, we have
\[
\left| \int_{\Omega \cap A_{\frac{R}{2}, R}} \nabla \rho \cdot \nabla \eta_R dx \right| \leq \left( \int_{\Omega \cap A_{\frac{R}{2}, R}} |\nabla \rho|^2 dx \right)^{\frac{1}{2}} \frac{K}{R} (\mu(\Omega \cap A_{\frac{R}{2}, R}))^{\frac{1}{2}}.
\]
Observe that, as \( \int_{\Omega} |\nabla \rho|^2 dx < \infty \) we have
\[
\lim_{R \to \infty} \int_{\Omega \cap A_{\frac{R}{2}, R}} |\nabla \rho|^2 dx = 0
\]
and, as \( \mu(\Omega \cap A_{\frac{R}{2}, R}) \leq \mu(A_{\frac{R}{2}, R}) = \frac{3\pi R^2}{4} \), passing to the limit for \( R \to \infty \) in (2.3), we obtain:
\[
\int_{\Omega} c(x) \rho(x) dx - \int_{\partial \Omega} \frac{\partial \rho}{\partial \nu} d\sigma = 0.
\]
As \( \rho \) attains its infimum \( \delta \) in any point on \( \partial \Omega \) we have \( \frac{\partial \rho}{\partial \nu} \leq 0 \) on \( \partial \Omega \) and so
\[
c(x) \rho(x) \equiv 0 \text{ in } \Omega.
\]
Consequently \( \rho \) is an harmonic function in \( \Omega \), hence it attains its minimum \( \rho = \delta \) on \( \partial \Omega \) and by Hopf maximum principle
\[
\rho \equiv \delta \text{ in } \Omega.
\]
(2). Assume now that $\rho \in L^1(\Omega)$.

Observe that if $\delta > 0$ then $\Omega$ has finite Lebesgue measure $\mu(\Omega) < +\infty$ (for $\delta = 0$ this is an hypothesis). Indeed, this is a consequence of the fact that $\rho \in L^1(\Omega)$ and $\rho \geq \delta$ in $\Omega$ and thus the constant $\delta$ is in $L^1$, which is $\mu(\Omega) < \infty$.

We distinguish two cases: $\Omega$ bounded and $\Omega$ unbounded.

If $\Omega$ is bounded multiply the equation by $\rho$, integrate on $\Omega$ and obtain:

$$
\int_{\Omega} |\nabla \rho|^2 \, dx - \int_{\partial \Omega} \frac{\partial \rho}{\partial \nu} \rho \, d\sigma + \int_{\Omega} c(x) \rho^2 \, dx = 0.
$$

As $\rho|_{\partial \Omega} = \delta$ and $\rho \geq \delta$ in $\Omega$ we have that $\frac{\partial \rho}{\partial \nu} \leq 0$ on $\partial \Omega$. Consequently, from the above equality we deduce that $c(x) \rho \equiv 0$ in $\Omega$, which means $c(\cdot) \equiv 0$, and $\nabla \rho \equiv 0$, which implies $\rho \equiv \delta$ in $\Omega$.

Suppose now that $\Omega$ is an unbounded domain.

For $R > 0$ denote by $\Omega_R = \Omega \cap B_R$. Denote by $\varphi : \overline{B_1} \to \mathbb{R}$ the function

$$
\varphi(x) = \frac{1 - |x|^2}{2n},
$$

which is in fact the solution to the boundary value problem

$$
\begin{cases}
-\Delta \varphi = 1 & \text{in } B_1, \\
\varphi = 0 & \text{on } \partial B_1.
\end{cases}
$$

Observe that

$$
|\nabla \varphi(x)| \leq C_1 := \frac{1}{n} \text{ for } x \in B_1
$$

and on the boundary the normal derivative $\frac{\partial \varphi}{\partial \nu} \equiv -C_1$. Denote by $\varphi_R : B_R \to \mathbb{R}$ the function defined as

$$
\varphi_R(x) = \varphi \left( \frac{x}{R} \right),
$$

which is the solution to the boundary value problem

$$
\begin{cases}
-\Delta \varphi = \frac{1}{R^2} & \text{in } B_R, \\
\varphi = 0 & \text{on } S_R,
\end{cases}
$$

and satisfies

$$
|\nabla \varphi_R(x)| \leq \frac{C_1}{R} \text{ in } B_R, \frac{\partial \varphi_R}{\partial \nu} \equiv -\frac{C_1}{R} \text{ on } S_R.
$$

Denote by

$$
\Gamma_1^R = \Omega \cap S_R, \Gamma_2^R = \partial \Omega \cap B_R,
$$
such that

\[ \partial \Omega_R = \Gamma_1^R \cup \Gamma_2^R. \]

Observe that, as \( \Omega \) is unbounded and connected, there exists \( R_0 > 0 \) such that for \( R \geq R_0 \) both parts of \( \partial \Omega_R \) are nonempty, i.e. \( \Gamma_1^R \neq \emptyset \) and \( \Gamma_2^R \neq \emptyset \).

Multiply now the equation satisfied by \( \rho \) with \( \varphi_R \), integrate on \( B_R \) and use Green formula to obtain:

\[
0 = \int_{\Omega_R} (-\Delta \rho(x)) \varphi_R(x) \, dx + \int_{\Omega_R} c(x) \rho(x) \varphi_R(x) \, dx = \\
= \int_{\Omega_R} (-\Delta \varphi_R(x)) \rho(x) \, dx + \int_{\partial \Omega_R} \left( \rho \frac{\partial \varphi_R}{\partial \nu} - \varphi_R \frac{\partial \rho}{\partial \nu} \right) \, d\sigma + \int_{\Omega_R} c(x) \rho(x) \varphi_R(x) \, dx = \\
= \frac{1}{R^2} \int_{\Omega_R} \rho(x) \, dx - \frac{C_1}{R} \int_{\Gamma_1^R} \rho \, d\sigma + \delta \int_{\Gamma_2^R} \frac{\partial \varphi_R}{\partial \nu} \, d\sigma - \int_{\Gamma_2^R} \varphi_R \frac{\partial \rho}{\partial \nu} \, d\sigma + \int_{\Omega_R} c(x) \rho(x) \varphi_R(x) \, dx.
\]

Observe that \( \frac{\partial \rho}{\partial \nu} \leq 0 \) on \( \Gamma_2^R \) and \( c(x) \rho(x) \varphi_R(x) \geq 0 \) in \( \Omega \) and thus:

\[
0 \leq \frac{1}{R^2} \int_{\Omega_R} \rho(x) \, dx + \int_{\Gamma_2^R} \varphi_R \left[ -\frac{\partial \rho}{\partial \nu} \right] \, d\sigma + \int_{\Omega_R} c(x) \rho(x) \varphi_R(x) \, dx = \\
= \delta \int_{\Gamma_2^R} \left[ -\frac{\partial \varphi_R}{\partial \nu} \right] \, d\sigma + \frac{C_1}{R} \int_{\Gamma_1^R} \rho \, d\sigma = I_1 + I_2. \tag{2.4}
\]

Observe now that

\[
I_1 = \delta \int_{\Gamma_2^R} \left[ -\frac{\partial \varphi_R}{\partial \nu} \right] \, d\sigma = \delta \int_{\partial \Omega_R} \left[ -\frac{\partial \varphi_R}{\partial \nu} \right] \, d\sigma - \delta \int_{\Gamma_1^R} \left[ -\frac{\partial \varphi_R}{\partial \nu} \right] \, d\sigma = \\
= \delta \int_{\Omega_R} \left[ -\Delta \varphi_R \right] \, dx - \delta \frac{C_1}{R} \int_{\Gamma_1^R} \, d\sigma = \\
= \frac{\delta}{R^2} \mu(\Omega_R) - \delta \frac{C_1}{R} \int_{\Gamma_1^R} \, d\sigma.
\]

Inserting this into (2.4) we obtain

\[
0 \leq \frac{1}{R^2} \int_{\Omega_R} \rho(x) \, dx + \int_{\Gamma_2^R} \varphi_R \left[ -\frac{\partial \rho}{\partial \nu} \right] \, d\sigma + \int_{\Omega_R} c(x) \rho(x) \varphi_R(x) \, dx = I_1 + I_2 = \\
= \frac{C_1}{R} \int_{\Gamma_1^R} (\rho - \delta) \, d\sigma + \frac{\delta}{R^2} \mu(\Omega_R). \tag{2.5}
\]

First, observe that, as \( \mu(\Omega) < \infty \),

\[
\lim_{R \to +\infty} \frac{1}{R^2} \mu(\Omega_R) = 0. \tag{2.6}
\]
Then, as $\rho \in L^1(\Omega)$ and $\mu(\Omega) < +\infty$

$$\int_{\Omega} (\rho(x) - \delta) dx = \int_0^\infty \int_{\Gamma_R} (\rho - \delta) d\sigma dR < \infty.$$ 

Thus, there exists a sequence $R_m \to +\infty$ such that

$$\lim_{R_m \to +\infty} \int_{\Gamma_{R_m}} (\rho - \delta) d\sigma = 0. \quad (2.7)$$

Using $(2.6)$, $(2.7)$ we obtain, by passing to the limit for $R_m \to +\infty$ in $(2.5)$, that

$$0 = \liminf_{R_m \to \infty} \left[ \frac{1}{R_m^2} \int_{\Omega_{R_m}} \rho(x) dx + \int_{\Gamma_{R_m}} \varphi_{R_m} \left[ -\frac{\partial \rho}{\partial \nu} \right] d\sigma + \int_{\Omega_{R_m}} c(x) \rho(x) \varphi_{R_m}(x) dx \right].$$

By Fatou Lemma we obtain from here that

$$0 = \int_{\partial \Omega} \varphi(0) \left[ -\frac{\partial \rho}{\partial \nu} \right] d\sigma + \int_{\Omega} c(x) \rho(x) \varphi(0) dx. \quad (2.8)$$

A first consequence is that

$$c(x) \rho(x) \equiv 0 \text{ in } \Omega,$$

and thus $\rho$ is harmonic in $\Omega$ and

$$\frac{\partial \rho}{\partial \nu} \equiv 0 \text{ on } \partial \Omega.$$

As on $\partial \Omega$ the function $\rho$ attains its minimum $\rho = \delta$, by Hopf maximum principle

$$\rho \equiv \delta \text{ in } \Omega.$$

**Proposition 2.1.** Let $u$ be a classical solution to the problem

$$-\Delta u = u j(1 - |u|^2) \text{ in } \mathbb{R}^2.$$

Under the assumption (H1), (H2), if

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx < \infty$$

or if

$$\int_{\mathbb{R}^2} J(1 - |u|^2) dx < \infty,$$

then

$$|u(x)| \leq 1, \forall x \in \mathbb{R}^2.$$
Proof. Denote by 
\[ w = 1 - |u|^2, \]
and by \( w^+ = \max\{w, 0\}, w^- = -\min\{w, 0\} \) such that \( w = w^+ - w^- \).

Suppose, by contradiction that \( \sup |u| > 1 \).

For \( \mu > 0 \) denote by \( A_\mu \) the set
\[ A_\mu = \{ x \in \mathbb{R}^2 : w^-(x) = \mu \}. \]

By Sard’s lemma we know that almost all \( \mu > 0 \) in the image of \( w^- \) are regular values. So, we may choose such a \( \mu \) and consequently \( A_\mu \) is a smooth (not necessarily connected) manifold in \( \mathbb{R}^2 \) and
\[ A_\mu = \partial \Omega_\mu, \quad \Omega_\mu = \{ x \in \mathbb{R}^2 : w^-(x) > \mu \}. \]

In \( \Omega_\mu \) we have \( w^- = |u|^2 - 1 \) and \( |u| > \sqrt{1 + \mu} \).

In \( \Omega_\mu \) we may write locally \( u = \rho(x) e^{i\psi(x)} \); the phase \( \psi \) is defined locally up to an additive integer multiple of \( 2\pi \) but \( \nabla \psi \) is defined globally. By separating the real and the imaginary parts, we deduce that the equation satisfied by \( \rho \) is
\[ -\Delta \rho + (|\nabla \psi|^2 - j(-w^-)) \rho = 0 \text{ in } \Omega_\mu, \]
with \( c(x) := (|\nabla \psi|^2 - j(-w^-)) \geq 0 \text{ in } \Omega_\mu \).

Suppose first that \( \int_{\mathbb{R}^2} |\nabla u|^2 dx < \infty \).

Observe that in the set \( \Omega_\mu \) we have
\[ \nabla u = \nabla \rho e^{i\psi} + i\rho \nabla \psi e^{i\psi}, \]
so \( |\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \psi|^2 \geq |\nabla \rho|^2 \). Consequently,
\[ \int_{\Omega_\mu} |\nabla \rho|^2 dx < \infty. \]

Lemma 2.1 tells that necessarily \( \rho \equiv \sqrt{1 + \mu} \text{ in } \Omega_\mu \) but this contradicts the definition of \( \Omega_\mu = \{ x \in \mathbb{R}^2 : |u(x)| > \sqrt{1 + \mu} \}. \)

Suppose now that \( \int_{\mathbb{R}^2} J(1 - |u|^2) dx < \infty \).

This implies that
\[ \int_{\Omega_\mu} J(-w^-(x)) dx < \infty. \]
By (H2) we have
\[
\liminf_{t \to -\infty} \frac{J(t)}{\sqrt{-t}} > 0.
\]
So, in \(\Omega_\mu\) we have \(J(-w^-) = J(1 - |u|^2) \geq \varepsilon |u|\) for some \(\varepsilon > 0\) small enough. We deduce that \(\rho \in L^1(\Omega_\mu)\).

Clearly, \(\rho > \sqrt{1 + \mu}\) in \(\Omega_\mu\) and \(\rho = \sqrt{1 + \mu}\) on \(\partial \Omega_\mu\). The Lemma 2.1 applied to \(\rho\) says that \(\rho \equiv \sqrt{1 + \mu}\) in \(\Omega_\mu\), but this is a contradiction with the definition of \(\Omega_\mu\).

Consequently, we get \(|u| \leq 1\) in \(\mathbb{R}^2\).

**Remark 2.1.** For the Ginzburg-Landau system the estimate \(|u| \leq 1\) should hold true without any additional assumption on the energy terms. Indeed, this was proved by Hervé and Hervé for \(u: \mathbb{R}^2 \to \mathbb{R}^2\), see [15] and later generalized by Brezis for \(u: \mathbb{R}^n \to \mathbb{R}^k\) using an argument, based on the Keller-Osserman theory, see [9].

### 3 Proof of Theorem 1

The proof of Theorem 1 will be performed into two steps.

**Step 1.** By (1.1) we get
\[
\Delta |u|^2 = 2|\nabla u|^2 + 2u \Delta u = 2|\nabla u|^2 - 2|u|^2 j(1 - |u|^2) \quad \text{in} \quad \mathbb{R}^n
\]

hence
\[
|u|^2 j(1 - |u|^2) = |\nabla u|^2 - \frac{1}{2} \Delta |u|^2. \tag{3.1}
\]

Let \(\eta \in C^\infty(\mathbb{R}^n, [0, 1])\) satisfy \(\eta(x) = 1\) for \(|x| \leq 1\), and \(\eta(x) = 0\) for \(|x| \geq 2\). We set \(\eta_h = \eta \left(\frac{x}{h}\right)\). Multiplying (3.1) by \(\eta_h\) and integrating over \(\mathbb{R}^2\) (which is effectively an integral over \(B_{2h}\)), we have
\[
\int_{\mathbb{R}^2} |u|^2 j(1 - |u|^2) \eta_h dx = \int_{\mathbb{R}^2} |\nabla u|^2 \eta_h dx + \int_{\mathbb{R}^2} u \nabla u \nabla \eta_h dx = \int_{\mathbb{R}^2} |\nabla u|^2 \eta_h dx + \int_{h \leq |x| < 2h} u \nabla u \nabla \eta_h dx. \tag{3.2}
\]

For the last term, using that \(|u| \leq 1\) and Cauchy inequality, we have the estimate:
\[
\left| \int_{h \leq |x| < 2h} u \nabla u \nabla \eta_h dx \right| \leq C \int_{h \leq |x| < 2h} |\nabla u|^2 \frac{1}{h} dx \leq C \left( \int_{h \leq |x| < 2h} |\nabla u|^2 dx \right)^{\frac{1}{2}} \to 0 \quad \text{for} \quad h \to +\infty,
\]

and the last convergence is motivated by \(|\nabla u| \in L^2(\mathbb{R}^2)|. \]
Consequently, from (3.2) passing to the limit for \( h \to +\infty \), we find
\[
\int_{\mathbb{R}^2} |u|^2 j \left( 1 - |u|^2 \right) \, dx = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx < +\infty.
\] (3.3)

Let us define the set \( B = \{ x \in \mathbb{R}^2 : z_1 \leq |u(x)| \leq z_2 \} \) with \( z_1 \) and \( z_2 \) such that
\[
1 - \frac{\eta_0}{4} < z_1 < z_2 < 1 - \frac{\eta_0}{8},
\] (3.4)
where \( \eta_0 \) is given in \((H_4)\).

We claim that \( B \) is bounded. Indeed, arguing by contradiction, let us suppose that there exists a sequence \( (x_k)_k \subset B \) such that \( \|x_k\| \) goes to +\( \infty \) as \( k \) goes to +\( \infty \).

Let us fix \( R_0 = \frac{\eta_0}{4} + z_1 - 1 \) where \( \|\nabla u\|_{\infty} = M \). Then, by mean value theorem, for every \( x \in B_{R_0} (x_k) \), we get
\[
|u(x)| \leq M |x_k - x| + |u(x_k)| \leq \frac{\eta_0}{4} + z_1 - 1 + z_2
\] (3.5)
and
\[
|u(x)| \geq |u(x_k)| - M |x_k - x| \geq z_1 - \frac{\eta_0}{4} - z_1 + 1 = 1 - \frac{\eta_0}{4}.
\] (3.6)
Moreover, for every \( x \in B_{R_0} (x_k) \) by Proposition 2.1 and (3.6) we have
\[
1 - |u(x)|^2 = (1 - |u(x)|)(1 + |u(x)|) \leq 2 \frac{\eta_0}{4} = \frac{\eta_0}{2}.
\] (3.7)

By Proposition 2.1, (3.4) and (3.5) we have
\[
1 - |u(x)|^2 \geq (1 - |u(x)|) \geq \left( 2 - \frac{\eta_0}{4} - z_1 - z_2 \right) > 0.
\] (3.8)

Hence, by using (3.7), (3.8), assumption \((H_4)\) and (3.8), we get
\[
\int_{B_{R_0} (x_k)} |u|^2 j \left( 1 - |u|^2 \right) \, dx \geq |B_{R_0} (x_k)| \left( 1 - \frac{\eta_0}{4} \right)^2 j \left[ \left( 2 - \frac{\eta_0}{4} - z_1 - z_2 \right) \right].
\] (3.9)

By (3.3), for \( k \) large enough, there exists \( R \) depending only on \( M, \eta_0, z_1 \) and \( z_2 \) such that
\[
\int_{B_{R_0} (x_k)} |u|^2 j \left( 1 - |u|^2 \right) \, dx \leq \int_{|x| > R} |u|^2 j \left( 1 - |u|^2 \right) \, dx
\] (3.10)
Since we know by (3.3) that \( \int_{\mathbb{R}^2} |u|^2 j \left( 1 - |u|^2 \right) \, dx < \infty \) we conclude that
\[
\lim_{\|x_k\| \to +\infty} \int_{B_{R_0} (x_k)} |u|^2 j \left( 1 - |u|^2 \right) \, dx = 0
\]
and this contradicts (3.9).
Step 2. As the set $B$ is bounded, by (3.3) as in [10] we get either
\[
\int_{\mathbb{R}^2} |u|^2 dx < +\infty \tag{3.11}
\]
or
\[
\int_{\mathbb{R}^2} j \left( 1 - |u|^2 \right) dx < +\infty. \tag{3.12}
\]
When (3.12) holds, by (H1), (H4), Proposition 2.1 and (1.3) we can apply the result of [1] and conclude.

More precisely, as the set $B$ is bounded, it means that for some $R > 0$ big enough one has for $|x| > R$ either $|u| > 1 - \eta_0/8$ or $|u| < 1 - \eta_0/4$. If we are in this situation we have $J(t) < tj(t)$, by convexity property (H4), hence $\int_{\mathbb{R}^2} J(1 - |u|^2) dx < \infty$ and then we can apply the result in [1].

Then, let us suppose that (3.11) holds. Acting as in [10], we would multiply equation in problem (1.1) by $\left( \sum x_i \frac{\partial u}{\partial x_i} \right) \eta_h$, where $\eta_h$ is as in Step 1. It is easy to see that
\[
\int_{\mathbb{R}^2} (\Delta u) \left( \sum x_i \frac{\partial u}{\partial x_i} \right) \eta_h dx \to 0,
\]
hence
\[
\int_{\mathbb{R}^2} uj \left( 1 - |u|^2 \right) \left( \sum x_i \frac{\partial u}{\partial x_i} \right) \eta_h dx \to 0 \tag{3.13}
\]
when $h$ goes to $+\infty$.

On the other hand we observe that
\[
\int_{\mathbb{R}^2} uj \left( 1 - |u|^2 \right) \left( \sum x_i \frac{\partial u}{\partial x_i} \right) \eta_h dx = -\frac{1}{2} \int_{\mathbb{R}^2} \sum x_i \left( \frac{\partial}{\partial x_i} J \left( 1 - |u|^2 \right) \right) \eta_h dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \sum x_i \frac{\partial}{\partial x_i} \left[ J(1) - J \left( 1 - |u|^2 \right) \right] \eta_h dx = -\frac{1}{2} \int_{\mathbb{R}^2} \left[ J(1) - J \left( 1 - |u|^2 \right) \right] \eta_h dx \tag{3.14}
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^2} \left[ J(1) - J \left( 1 - |u|^2 \right) \right] x \cdot \nabla \eta_h dx.
\]

By the mean value theorem applied to function $J$, there exists $\vartheta \in (0, 1)$ such that
\[
J(1) - J \left( 1 - |u|^2 \right) = j \left( 1 - \vartheta |u|^2 \right) |u|^2.
\]
Then (3.14) becomes
\[
\int_{\mathbb{R}^2} uj \left( 1 - |u|^2 \right) \left( \sum x_i \frac{\partial u}{\partial x_i} \right) \eta_h dx + \frac{1}{2} \int_{\mathbb{R}^2} j \left( 1 - \vartheta |u|^2 \right) |u|^2 x \cdot \nabla \eta_h dx
\]
\[
= -\int_{\mathbb{R}^2} j \left( 1 - \vartheta |u|^2 \right) |u|^2 \eta_h dx. \tag{3.15}
\]
By Proposition 2.1 we have \((1 - \vartheta |u|^2) \in (0, 1)\), then by the regularity of function \(j\) and properties of function \(\eta_h\), acting as in [10], by (3.11), we easily get
\[
\int_{\mathbb{R}^2} j (1 - \vartheta |u|^2) |u|^2 x \cdot \nabla \eta_h dx \to 0
\]  
(3.16)
when \(h\) goes to \(+\infty\). By (3.13), (3.15), (3.16) we obtain
\[
- \int_{\mathbb{R}^2} j (1 - \vartheta |u|^2) |u|^2 \eta_h dx \to - \int_{\mathbb{R}^2} j (1 - \vartheta |u|^2) |u|^2 dx = 0
\]
as \(h\) goes to \(+\infty\), which by \((H2)\) directly implies \(u = 0\). Then the theorem is completely proved.

4 Proof of Theorem 2
Throughout this section, \(u\) will be a smooth solution of problem (1.1) satisfying
\[
\int_{\mathbb{R}^2} J \left(1 - |u|^2\right) dx < \infty.
\]  
(4.1)
Let us prove some results which will be useful in the sequel.

Proposition 4.1 (Pohozaev identity). Let \(u\) be a smooth solution of problem (1.1). Then for every \(r > 0\), it holds
\[
\int_{S_r} \frac{\partial u}{\partial \nu} \left(x \cdot \nabla \right) \Delta u dx = \int_{S_r} \frac{\partial u}{\partial \tau} \left(x \cdot \nabla \right) u dx - \int_{S_r} \nabla (x \cdot \nabla u) \nabla u dx
\]
and
\[
\int_{B_r} \left(x \cdot \nabla \right) u j \left(1 - |u|^2\right) dx = - \frac{1}{2} \int_{B_r} \left(x \cdot \nabla \right) J \left(1 - |u|^2\right) dx
\]
(4.3)
Moreover, since $|\nabla u|^2 = \left| \frac{\partial u}{\partial \nu} \right|^2 + \left| \frac{\partial u}{\partial \tau} \right|^2$, by (4.3) and (4.4) we have

$$- \frac{r}{2} \int_{S_r} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma + \frac{r}{2} \int_{S_r} \left| \frac{\partial u}{\partial \tau} \right|^2 \, d\sigma = \int_{B_r} J(1 - |u|^2) \, dx - \frac{r}{2} \int_{S_r} J(1 - |u|^2) \, d\sigma. \quad (4.5)$$

Finally, by (4.3), (4.4) and (4.5), we obtain (4.2).

Now we prove the following result.

**Proposition 4.2.** Assume (H1), (H2) and (4.1). Let $u$ be a smooth solution of problem (1.1). Then

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} < +\infty, \quad (4.6)$$

$$|u(x)| \to 1, \quad \text{as} \quad |x| \to \infty. \quad (4.7)$$

and

$$\int_{B_R} |\nabla u|^2 \, dx \leq CR, \quad (4.8)$$

where $C$ is a positive constant independent of $R$.

**Proof.** First of all we prove (4.6).

To this aim, for any $y \in \mathbb{R}^2$, let us denote by $B_2(y)$ the ball of radius 2 of $\mathbb{R}^2$ and consider the following equation

$$- \Delta u = u j(1 - |u|^2) \quad \text{in} \quad B_2(y). \quad (4.9)$$

By interior $W^{2,p}$ estimates, with $p > 2$ for the solution of the equation (4.9), by Proposition 2.1 and by regularity of $j$, there exists a constant $C$ independent of $y$ such that

$$\|u\|_{W^{2,p}(B_1(y))} \leq C \left( \|u\|_{L^\infty(B_2(y))} + \|uj(1 - |u|^2)\|_{L^p(B_2(y))} \right) \leq C.$$

Finally, by using the Sobolev embedding for $p > 2$, we have $W^{2,p}(B_1(y)) \subset C^1(\overline{B_1(y)})$, hence (4.6).

Let us prove (4.7). To this aim we suppose that it were not true, hence there exists a sequence $|x_n| \to \infty$ such that $|u(x_n)| \leq 1 - \delta$ for some $\delta > 0$. Let us consider the ball $B_{\frac{M}{2\delta}}(x_n)$ where $M = \|\nabla u\|_{L^\infty}$. Then, by mean value theorem, we get

$$|u(x) - u(x_n)| \leq M|x - x_n| \leq \frac{M \delta}{2M} = \frac{\delta}{2}. \quad (4.10)$$

Then

$$|u(x)| = |u(x) - u(x_n)| + |u(x_n)| \leq \frac{\delta}{2} + 1 - \delta = 1 - \frac{\delta}{2}. \quad (4.10)$$
By Proposition 2.1 \(|u(x)|^2 \leq |u(x)|^2\) and by (4.10), we get \(1 - |u(x)|^2 \geq \frac{\delta}{2}\). By (H2), function \(J\) is increasing on \((0, 1]\) hence \(J\left(1 - |u(x)|^2\right) \geq J\left(\frac{\delta}{2}\right)\) and

\[
\int_{B_{\frac{\delta}{2M}\|x_n\|}} J\left(1 - |u(x)|^2\right) \, dx \geq J\left(\frac{\delta}{2}\right) \frac{\delta^2}{4M^2}.
\] (4.11)

By (4.11) there exists \(R_0\) such that

\[
\int_{|x|>R_0} J\left(1 - |u(x)|^2\right) \, dx < J\left(\frac{\delta}{2}\right) \frac{\delta^2}{4M^2}.
\] (4.12)

As \(||x_n|| \rightarrow \infty\) there exists a ball \(B_{\frac{\delta}{2M}\|x_n\|} \subset \mathbb{R}^2 \setminus B_{R_0}\), then (4.11) and (4.12) are clearly in contradiction.

Finally, in order to prove (4.8), let us multiply (1.1) by \(u\) and integrate over \(B_R\). We obtain

\[
\int_{B_R} |\nabla u|^2 \, dx = \int_{S_R} \frac{\partial u}{\partial \nu} u \, d\sigma + \int_{B_R} |u|^2 j \left(1 - |u|^2\right) \, dx,
\]

where \(\nu\) denotes the outward normal to \(B_R\). It is easy to show that

\[
\left| \int_{S_R} \frac{\partial u}{\partial \nu} u \, d\sigma \right| \leq 2M\pi R,
\] (4.13)

then it remains to prove

\[
\int_{B_R} |u|^2 j \left(1 - |u|^2\right) \, dx \leq CR
\] (4.14)

for some constant \(C\) independent of \(R\). For this purpose let us observe that by Proposition 2.1 and Cauchy-Schwarz inequality we have

\[
\int_{B_R} |u|^2 j \left(1 - |u|^2\right) \, dx \leq \sqrt{\pi R} \left(\int_{B_R} J^2 \left(1 - |u|^2\right) \, dx\right)^{\frac{1}{2}}.
\] (4.15)

Now we observe that

\[
J^2(t) \leq CJ(t) \quad \forall t \in [0, 1].
\] (4.16)

Indeed as \(J''\) is a continuous function, there exists a positive constant \(M\) such that

\[
J''(t) \leq M \forall t \in [0, 1].
\]

Multiplying both members of previous inequality by \(J'(t)\) which is nonnegative by (H2), we obtain

\[
J''(t)J'(t) \leq MJ'(t) \forall t \in [0, 1]
\]

which is

\[
\frac{1}{2} \frac{d}{dt} \left(J'(t)^2\right) \leq MJ'(t) \quad \forall t \in [0, 1].
\]
Taking into account that $J(0) = J'(0) = 0$ and integrating both members we get
\[
(J'(t))^2 \leq 2MJ(t) \forall t \in [0,1],
\]
hence (4.10) holds. By (4.13), (4.15), (4.16) and Proposition 2.1 we obtain (4.18). □

By previous results, $\deg(u, S_R)$ is well defined for $R$ large (see [10]) and we denote $d = |\deg(u, S_R)|$. By virtue of (4.17), there exists $R_0 > 0$, such that
\[
|u(x)| \geq \frac{3}{4}, \text{ for } |x| = R \geq R_0 \tag{4.17}
\]
and a smooth single-valued function $\psi(x)$, defined for $|x| \geq R_0$, such that
\[
u(x) = \rho(x)e^{i(d\varphi + \psi(x))}, \tag{4.18}
\]
where $\rho = |u|$. If we denote
\[
\varphi(x) = d\varphi(x) + \psi(x), \tag{4.19}
\]
then $\varphi(x)$ is well defined and smooth locally on the set $|x| \geq R_0$.

Now we are able to prove the following result:

**Proposition 4.3.** Assume (H2) and (H3). Let $u$ be a smooth solution of problem (1.1) written as in (4.18). Then
\[
\lim_{R \to +\infty} \frac{1}{\log R} \int_{B_R \setminus B_{R_0}} |\nabla \psi|^2 \, dx = 0 \tag{4.20}
\]
and
\[
\lim_{R \to +\infty} \frac{1}{\log R} \int_{B_R \setminus B_{R_0}} |\nabla \rho|^2 \, dx = 0 \tag{4.21}
\]

**Proof.** By putting (4.18) in (1.1) we get
\[
-\Delta u = -(\Delta \rho) e^{i\varphi} - 2i(\nabla \rho \nabla \varphi) e^{i\varphi} - \rho e^{i\varphi} \left(i\Delta \varphi - |\nabla \varphi|^2\right) = \rho e^{i\varphi} j \left(1 - \rho^2\right).
\]
Separating the real and imaginary parts we obtain
\[
\rho \Delta \varphi + 2\nabla \rho \nabla \varphi = 0 \text{ for } |x| \geq R_0 \tag{4.22}
\]
\[
-\Delta \rho + \rho |\nabla \varphi|^2 = \rho j \left(1 - \rho^2\right) \text{ for } |x| \geq R_0. \tag{4.23}
\]
Let us observe that equation (4.22) can be written as
\[
\text{div} \left(\rho^2 \nabla \varphi\right) = 0 \text{ for } |x| \geq R_0. \tag{4.24}
\]
Let $V(x)$ be the vector field in $\mathbb{R}^2 \setminus \{0\}$ defined by

$$V(r \cos \theta, r \sin \theta) = (-\sin \theta, \cos \theta).$$

By (4.19) we have

$$\nabla \varphi = d\nabla \theta + \nabla \psi = \frac{d}{r} V + \nabla \psi. \quad (4.25)$$

Then, combining (4.24) and (4.25) we have

$$\text{div} \left( \rho^2 \left( \frac{d}{r} V + \nabla \varphi \right) \right) = 0 \quad \text{for} \ |x| \geq R_0. \quad (4.26)$$

Now we want to prove that

$$\int_{S_R} \rho^2 \frac{\partial \psi}{\partial \nu} d\sigma = 0. \quad (4.27)$$

To this aim let us consider the vector field $D = (u \wedge u_x, u \wedge u_y)$ which is well defined and smooth on $\mathbb{R}^2$. Note that by equation (1.1)

$$\text{div} D = u \wedge \Delta u = 0 \quad (4.28)$$

and by integrating (4.28) over $B_R$ we have

$$\int_{S_R} D \cdot \nu d\sigma = 0 \quad \forall R \geq R_0. \quad (4.29)$$

On the other hand, we have

$$D = \rho^2 \nabla \varphi \quad \text{for} \ |x| \geq R_0, \quad (4.30)$$

so (4.25) and the fact that $V \cdot \nu = 0$ on $S_R$ give (4.27).

Now, we want to prove (4.20). To this aim let us pose

$$\psi_R = \frac{1}{2\pi R} \int_{S_R} \psi d\sigma.$$

Multiplying (4.24) by $\psi - \psi_R$ and integrating over $A_R = B_R \setminus B_{R_0}$, we obtain

$$\int_{A_R} \rho^2 \left( \frac{d}{r} V + \nabla \psi \right) \nabla \psi d\sigma = \int_{S_{R_0}} \rho^2 \left( \frac{d}{r} V \cdot \nu + \frac{\partial \psi}{\partial \nu} \right) (\psi - \psi_R) d\sigma$$

$$- \int_{S_{R_0}} \rho^2 \left( \frac{d}{r} V \cdot \nu + \frac{\partial \psi}{\partial \nu} \right) (\psi - \psi_R) d\sigma. \quad (4.31)$$

As $V \cdot \nu = 0$, by (4.27) we get

$$\int_{S_{R_0}} \rho^2 \left( \frac{d}{r} V \cdot \nu + \frac{\partial \psi}{\partial \nu} \right) (\psi - \psi_R) d\sigma = \int_{S_{R_0}} \rho^2 \frac{\partial \psi}{\partial \nu} \psi d\sigma = C \quad (4.32)$$
where $C$ is a constant independent of $R$. We also observe that
\[
\int_{A_R} \frac{d}{dr} V \cdot \nabla \psi \, dx = \int_{A_R} \frac{d}{dr} \nabla \psi \, dx = 0.
\] (4.33)

By (4.31), (4.32) and (4.33) we have
\[
\int_{A_R} \rho^2 |\nabla \psi|^2 \, dx \leq \int_{S_R} \left| \frac{\partial \psi}{\partial \nu} \right| |\psi - \psi_R| \, d\sigma + \int_{A_R} (1 - \rho^2) \frac{d}{r} |\nabla \psi| \, d\sigma + C.
\] (4.34)

Cauchy-Schwarz inequality implies
\[
\left| \int_{S_R} \frac{\partial \psi}{\partial \nu} (\psi - \psi_R) \, d\sigma \right| \leq \left( \int_{S_R} \left| \frac{\partial \psi}{\partial \nu} \right|^2 \, d\sigma \right)^{\frac{1}{2}} \left( \int_{S_R} |\psi - \psi_R|^2 \, d\sigma \right)^{\frac{1}{2}}.
\]

We recall the following Poincaré inequality
\[
\int_{S_R} |\psi - \psi_R|^2 \, d\sigma \leq R^2 \int_{S_R} |\nabla \tau \psi|^2 \, d\sigma.
\] (4.35)

Therefore by (4.35) and Young inequality we have
\[
\left| \int_{S_R} \frac{\partial \psi}{\partial \nu} (\psi - \psi_R) \, d\sigma \right| \leq \frac{R}{2} \int_{S_R} \left| \frac{\partial \psi}{\partial \nu} \right|^2 \, d\sigma + \frac{R}{2} \int_{S_R} |\nabla \tau \psi|^2 \, d\sigma = \frac{R}{2} \int_{S_R} |\nabla \psi|^2 \, d\sigma.
\] (4.36)

By (4.36) and (4.34) we obtain
\[
\int_{A_R} \rho^2 |\nabla \psi|^2 \, dx \leq \frac{R}{2} \int_{S_R} |\nabla \psi|^2 \, d\sigma + \int_{A_R} (1 - \rho^2) \frac{d}{r} |\nabla \psi| \, dx + C.
\] (4.37)

Remember that $A_R = B_R \setminus B_{R_0}$. Given $0 < \epsilon < \frac{1}{4}$ we choose $R_0$ big enough such that
\[
1 - \epsilon \leq \rho^2(x) \leq 1, \text{ for } |x| \geq R_0.
\]

Denote by
\[
f(R) = \int_{A_R} |\nabla \psi|^2 \, dx.
\] (4.38)

whose derivative is
\[
f'(R) = \int_{S_R} |\nabla \psi|^2 \, d\sigma
\]

and by (4.38) and (4.25) satisfies the estimate
\[
f(R) \leq CR.
\] (4.39)

Thus (4.37) becomes:
\[
(1 - \epsilon) f(R) \leq \frac{R}{2} f'(R) + \int_{A_R} (1 - \rho^2) \frac{d}{r} |\nabla \psi| \, dx + C.
\] (4.40)
We estimate the second term in (4.40). By Young inequality we obtain
\[
\int_{A_R} (1 - \rho^2) \frac{d}{r} |\nabla \psi| \, dx \leq \frac{d^2}{\epsilon} \int_{A_R} (1 - \rho^2)^2 \, dx + \epsilon \int_{A_R} |\nabla \psi|^2 \, dx = \frac{d^2}{\epsilon} \int_{A_R} (1 - \rho^2)^2 \, dx + \epsilon f(R).
\]
and thus (4.40) implies
\[
(1 - 2\epsilon) f(R) \leq \frac{R}{2} f'(R) + \frac{d^2}{\epsilon} \int_{A_R} (1 - \rho^2)^2 \, dx + C.
\]
(4.41)

Observe, by l'Hospital rule that
\[
\lim_{R \to \infty} \frac{1}{\log R} \int_{A_R} (1 - \rho^2)^2 \, dx = \lim_{R \to \infty} \frac{1}{R} \int_{S_R} (1 - \rho^2)^2 \, d\sigma = 0,
\]
(4.43)

because \( \lim_{|x| \to \infty} \rho(x) = 1 \). Consequently we may rewrite (4.42) in the form
\[
f(R) \leq \frac{R}{\beta} f'(R) + F(R),
\]
(4.44)

where \( \beta = 2(1 - 2\epsilon) > 1 \) and \( F(R) = \frac{d^2}{\epsilon(1 - 2\epsilon)} \int_{A_R} (1 - \rho^2)^2 \, dx + \frac{C}{1 - 2\epsilon} \) which, by (4.43) satisfies
\[
\lim_{R \to \infty} \frac{F(R)}{\log R} = 0.
\]
(4.45)

Denoted by
\[
g(R) = \frac{f(R)}{\log R},
\]
we want to prove
\[
\lim_{R \to \infty} g(R) = 0.
\]

We plug the expression for \( g \) in (4.44) and obtain:
\[
g(R) \leq g'(R) \frac{R \log R}{\beta \log R - 1} + \frac{\beta F(R)}{\beta \log R - 1} = \frac{g'(R)}{\alpha(R)} + H(R),
\]
(4.46)

where we denoted by \( \alpha(R) = \frac{\beta \log R - 1}{R \log R} \) and by \( H(R) = \frac{\beta F(R)}{\beta \log R - 1} \). Denote by \( \tilde{\beta} = \frac{\beta + 1}{2} > 1 \) and observe that
\[
\frac{\tilde{\beta}}{R} \leq \alpha(R) \leq \frac{\beta}{R},
\]
(4.47)

for \( R > \tilde{R} \) big enough.

Observe that, by (4.45), \( \lim_{R \to \infty} H(R) = 0 \). Hence, given \( \mu > 0 \) there exists \( R(\mu) > \tilde{R} \) such that \( H(R) < \mu \) for all \( R > R(\mu) \). Consequently, by (4.46) we have for \( R > R(\mu) \)
\[
\alpha(R) (g(R) - \mu) \leq g'(R) = (g(R) - \mu)',
\]
(4.48)
Given \( R > \theta > R(\mu) \) we multiply (4.48) by \( \exp \left( - \int_{\theta}^{R} \alpha(\tau) d\tau \right) \) and find that the function

\[
R \to (g(R) - \mu) \exp \left( - \int_{\theta}^{R} \alpha(\tau) d\tau \right)
\]
is increasing and thus, using also the linear growth of \( f \) i.e. (4.39), we find

\[
(g(\theta) - \mu) \leq (g(R) - \mu) \exp \left( - \int_{\theta}^{R} \alpha(\tau) d\tau \right) \leq \left( \frac{CR}{\log R} - \mu \right) \exp \left( - \int_{\theta}^{R} \alpha(\tau) d\tau \right). \tag{4.49}
\]

By (4.47) we have

\[
\exp \left( - \int_{\theta}^{R} \alpha(\tau) d\tau \right) \leq \left( \frac{\theta}{R} \right)^{\beta},
\]

which plugged into (4.49) gives

\[
(g(\theta) - \mu) \leq \left( \frac{CR}{\log R} + \mu \right) \left( \frac{\theta}{R} \right)^{\beta} \to 0 \ \text{for} \ R \to +\infty. \tag{4.50}
\]

We found that for \( \theta > R(\mu) \) one has \( 0 \leq g(\theta) \leq \mu \). As \( \mu > 0 \) may be chosen arbitrarily small, we obtain that

\[
\lim_{R \to \infty} g(R) = 0,
\]

that is (4.20).

Now we want to prove (4.21). To this aim let us consider the function \( \eta \) as in the proof of Theorem 2 and set \( \eta_{R}(x) = \eta \left( \frac{x}{R} \right) \). Multiplying (4.23) by \((1 - \varrho) \eta_{R}\) and integrating over \( \mathbb{R}^2 \setminus B_{R_0} \), we obtain

\[
\int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \rho|^2 \eta_{R} dx = \int_{\mathbb{R}^2 \setminus B_{R_0}} (1 - \rho) \nabla \rho \nabla \eta_{R} dx + \int_{\mathbb{R}^2 \setminus B_{R_0}} \rho (1 - \rho) |\nabla \varphi|^2 \eta_{R} dx
\]

\[
- \int_{S_{R_0}} \frac{\partial \rho}{\partial \nu} (1 - \rho) \eta_{R} d\sigma - \int_{\mathbb{R}^2 \setminus B_{R_0}} \rho (1 - \rho) j (1 - \rho^2) \eta_{R} dx. \tag{4.51}
\]

Now we estimate each term in (4.51). By definition of \( \eta_{R} \) and as \((1 - \rho^2) \leq 1\), we have

\[
\int_{\mathbb{R}^2 \setminus B_{R_0}} (1 - \rho) \nabla \rho \nabla \eta_{R} dx = -\frac{1}{2} \int_{\mathbb{R}^2 \setminus B_{R_0}} \nabla (1 - \rho)^2 \nabla \eta_{R} dx = \frac{1}{2} \int_{\mathbb{R}^2 \setminus B_{R_0}} (1 - \rho)^2 \Delta \eta_{R} dx
\]

\[
= \frac{1}{2} \int_{B_{2R} \setminus B_{R_0}} (1 - \rho)^2 \Delta \eta_{R} dx \leq \frac{C}{R^2} \int_{B_{2R} \setminus B_{R_0}} (1 - \rho)^2 dx \leq C. \tag{4.52}
\]

Let us observe that by (4.25) we can write

\[
|\nabla \varphi|^2 \leq 2 \left( \frac{d^2}{r^2} + |\nabla \psi|^2 \right). \tag{4.53}
\]
Then by definition of $\eta_R$, (4.17) and (4.53), we obtain
\[
\int_{\mathbb{R}^2 \setminus B_R} \rho (1 - \rho) |\nabla \varphi|^2 \eta_R dx \leq 2 \int_{B_{2R} \setminus B_R} \rho (1 - \rho) \left( \frac{d^2}{r^2} + |\nabla \psi|^2 \right) \eta_R dx
\]
\[
\leq 2d^2 \int_{B_{2R} \setminus B_R} \frac{\rho(1 - \rho)}{r^2} dx + 2 \int_{B_{2R} \setminus B_R} \rho (1 - \rho) |\nabla \psi|^2 dx
\]
\[
\leq 2d^2 \int_{B_{2R} \setminus B_R} \frac{\rho (1 - \rho^2)}{(1 + \rho) r^2} dx + 2 \int_{B_{2R} \setminus B_R} |\nabla \psi|^2 dx
\]
\[
\leq C_1 \int_{B_{2R} \setminus B_R} \frac{(1 - \rho^2)}{r^2} dx + C_2 \int_{B_{2R} \setminus B_R} |\nabla \psi|^2 dx,
\]
where $C_1$ and $C_2$ being constants independent of $R$.

By (4.43) we have that
\[
\lim_{R \to \infty} \frac{1}{\log R} \int_{\mathbb{R}^2 \setminus B_R} \frac{(1 - \rho^2)}{r^2} dx = 0,
\]
which combined with (4.20) gives that
\[
\lim_{R \to \infty} \frac{1}{\log R} \int_{\mathbb{R}^2 \setminus B_R} \rho (1 - \rho) |\nabla \varphi|^2 \eta_R dx = 0.
\]

Finally, by (H2), we can assert that
\[
- \int_{\mathbb{R}^2 \setminus B_R} \rho (1 - \rho) j (1 - \rho^2) \eta_R dx \leq 0.
\]

By putting (4.52), (4.56) and (4.57) in (4.51) and as $\eta_R(x) = 1$ in $B_R$, we find
\[
\lim_{R \to \infty} \frac{1}{\log R} \int_{B_R \setminus B_{2R}} |\nabla \rho|^2 dx \leq \lim_{R \to \infty} \frac{1}{\log R} \int_{\mathbb{R}^2 \setminus B_R} |\nabla \rho|^2 \eta_R dx = 0,
\]
which is (4.21).

### 4.1 Quantization of the energy

The final step is to prove
\[
\int_{\mathbb{R}^2} J (1 - \rho^2) dx = \pi d^2.
\]

Let us consider
\[
E = \int_{\mathbb{R}^2} J (1 - \rho^2) dx,
\]
\[
E(r) = \int_{B_r} J (1 - \rho^2) dx.
\]
We have that
\[
\lim_{r \to +\infty} E(r) = E,
\]
\[
(4.60)
\]
By integrating (4.2) for \( r \in (0, R) \) we get
\[
\int_{B_R} \left| \frac{\partial u}{\partial \nu} \right|^2 \, dx + 2 \int_0^R \frac{E(r)}{r} \, dr = \int_{B_R} \left| \frac{\partial u}{\partial r} \right|^2 \, dx + E(R).
\]
By dividing previous equality by \( \log R \), we obtain
\[
\frac{1}{\log R} \int_{B_R} \left| \frac{\partial u}{\partial \nu} \right|^2 \, dx + \frac{2}{\log R} \int_0^R \frac{E(r)}{r} \, dr = \frac{1}{\log R} \int_{B_R} \left| \frac{\partial u}{\partial r} \right|^2 \, dx + \frac{1}{\log R} E(R).
\]
(4.61)
We observe that for \( r > R_0 \)
\[
\left| \frac{\partial u}{\partial \nu} \right|^2 = \left| \frac{\partial \rho}{\partial \nu} \right|^2 + \rho^2 \left| \frac{\partial \varphi}{\partial \nu} \right|^2 \leq |\nabla \rho|^2 + |\nabla \psi|^2
\]
and by (4.25)
\[
\left| \frac{\partial u}{\partial \tau} \right|^2 = \left| \frac{\partial \rho}{\partial \tau} \right|^2 + \rho^2 \left| \frac{\partial \varphi}{\partial \tau} \right|^2 = \left| \frac{\partial \rho}{\partial \tau} \right|^2 + \rho^2 \left( \frac{d}{r} + \frac{\partial \psi}{\partial \tau} \right)^2.
\]
Hence
\[
\left| \frac{\partial u}{\partial \tau} \right|^2 - \frac{d^2}{r^2} = \left| \frac{\partial \rho}{\partial \tau} \right|^2 - (1 - \rho^2) \frac{d^2}{r^2} + \left| \frac{\partial \psi}{\partial \tau} \right|^2 - 2(1 - \rho^2) \frac{d}{r} \frac{\partial \psi}{\partial \tau} + 2 \frac{d}{r} \frac{\partial \psi}{\partial \tau}.
\]
(4.63)
By integrating on \( B_R \setminus B_{R_0} \) and using (4.33) we easily get
\[
\int_{B_R \setminus B_{R_0}} \left( \frac{\partial u}{\partial \tau} - \frac{d^2}{r^2} \right) \, dx = \int_{B_R \setminus B_{R_0}} \left( \left| \frac{\partial \rho}{\partial \tau} \right|^2 + \left| \frac{\partial \psi}{\partial \tau} \right|^2 \right) \, dx - d^2 \int_{B_R \setminus B_{R_0}} \frac{1 - \rho^2}{r^2} \, dx
\]
\[-2d \int_{B_{R_0}} \frac{1 - \rho^2}{r} \frac{\partial \psi}{\partial \tau} \, dx.
\]
(4.64)
By Proposition 2.1 and Cauchy-Schwarz inequality, (4.64) becomes
\[
\int_{B_R \setminus B_{R_0}} \left( \left| \frac{\partial u}{\partial \tau} \right|^2 - \frac{d^2}{r^2} \right) \, dx \leq \int_{B_R \setminus B_{R_0}} \left( |\nabla \rho|^2 + |\nabla \psi|^2 \right) \, dx + d^2 \int_{B_R \setminus B_{R_0}} \frac{1 - \rho^2}{r^2} \, dx + 2d \sqrt{\frac{\log R}{R_0}} \left( \int_{B_R \setminus B_{R_0}} |\nabla \psi|^2 \, dx \right)^{\frac{1}{2}}.
\]
(4.65)
Combining (4.20), (4.21), (4.55), and (4.65), we have
\[
\lim_{R \to +\infty} \frac{1}{\log R} \int_{B_R \setminus B_{R_0}} \left( \frac{\partial u}{\partial \tau} \right)^2 - \frac{d^2}{r^2} \, dx = 0,
\]
and so
\[
\lim_{R \to +\infty} \frac{1}{\log R} \int_{B_R} \left( \frac{\partial u}{\partial \tau} \right)^2 \, dx = 2\pi d^2. \tag{4.66}
\]
Moreover, by (4.20), (4.21) and (4.62) it results
\[
\lim_{R \to +\infty} \frac{1}{\log R} \int_{B_R \setminus B_{R_0}} \left( \frac{\partial u}{\partial \nu} \right)^2 \, dx = 0. \tag{4.67}
\]
Finally, from (4.60), (4.66) and (4.67), by passing to the limit as \( R \to +\infty \) in (4.61), by (4.1) we obtain
\[
E = \pi d^2; \tag{4.68}
\]
which is (4.59).
Hence Theorem 2 is completely proved.

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