Interplay of non-Hermitian skin effects and Anderson localization in non-reciprocal quasiperiodic lattices

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Non-Hermiticity from non-reciprocal hoppings has been shown recently to demonstrate the non-Hermitian skin effect (NHSE) under open boundary conditions (OBCs). Here we study the interplay of this effect and the Anderson localization in a non-reciprocal quasiperiodic lattice, dubbed non-reciprocal Aubry-André model, and a rescaled transition point is exactly proved. The non-reciprocity can induce not only the NHSE, but also the asymmetry in localized states with two Lyapunov exponents for both sides. Meanwhile, this transition is also topological, characterized by a winding number associated with the complex eigenenergies under periodic boundary conditions (PBCs), establishing a bulk-bulk correspondence. This interplay can be realized by an elaborately designed electronic circuit with only linear passive RLC devices instead of elusive non-reciprocal ones, where the transport of a continuous wave undergoes a transition between insulating and amplifying. This initiative scheme can be immediately applied in experiments to other non-reciprocal models, and will definitely inspire the study of interplay of NHSEs and other quantum/topological phenomena.

Anderson localization (AL) [1] is an old but everlasting research problem in condensed matters, which reveals a mechanism of insulation due to the destructive interference of multiple scattered waves induced by randomness [2, 3]. This fundamental phenomenon has been observed in experiments for electronic spins [4, 5], light [6–9], microwave [10–12], sound [13], and cold atoms [14–16]. In one dimensional (1D) systems, it is well known that any infinitesimal disorder can localize all eigenstates [1, 2]. However, it was found that relaxing the condition of randomness, the AL can also exist in quasiperiodic systems, e.g., Aubry-André (AA) model [17], but with a finite transition point. This quasiperiodicity also has profound connection to topology [18, 19]. The AA model can be mapped to the two dimensional Hofstadter model [20] with an external periodic parameter as a synthetic dimension, and thus realizes the famous Thouless pumping [21, 22].

On the other hand, non-Hermiticity [23] has been studied intensively for years with the aid of the fast development of the topological photonics [23, 30]; it exhibits rich phenomena without Hermitian counterparts, e.g., PT symmetry breaking [31, 32], exceptional points [33–38], etc. Especially, the non-Hermitian topology is attracting special attention for the violation of the conventional bulk-boundary correspondence of Hermitian topological systems, and new ways of topological characterization are needed [39, 42]. Besides the on-site gain/loss, non-reciprocal hoppings can also bring in non-Hermiticity [43, 58] with exotic features, e.g., the topological non-Hermitian skin effect (NHSE) under open boundary conditions (OBCs), which is helpful to understand the breakdown of bulk-boundary correspondence.

Among references, effects of non-Hermiticity on AL have been studied in different contexts [63–72], but the discussion on the interplay of NHSEs and the AL with accompanying topological transitions is still lacking. Thus, natural questions arise: What is the fate of the NHSE and its topology in the presence of quasiperiodic potentials, whether there is a transition inherited from the well-known AL of the Hermitian AA model, and if yes, what is it like? In this paper, we address the above questions in the AA model with non-reciprocal hoppings, dubbed the “non-reciprocal AA model”, and find the transition of NHSEs and AL under OBCs with an analytically proved rescaled transition point. Affected by the non-reciprocity, besides the NHSE under OBCs, the localized states are asymmetric with respect to the localization center, characterized by two Lyapunov exponents on both sides. Meanwhile, this transition is topological, in the sense of the winding number associated with the complex eigenenergies under periodic boundary conditions (PBCs) [52], which can well distinguish the different skin phases and the localized phase under OBCs, establishing a bulk-bulk correspondence. In the end, to demonstrate the interplay, an electronic circuit is elaborately proposed with only linear passive RLC elements, which undoubtedly shows the phase transition through the transport of continuous waves between insulating and amplifying. Due to the lacking of experimental realizations of NHSEs, especially in electronic circuits [73–80], our design is very practicable and can be immediately applied to other non-reciprocal models, and will definitely inspire the study of interplays of NHSEs and other quan-
tum/topological phenomena.

Non-reciprocal AA model. The Hamiltonian of the non-reciprocal AA model [Fig. 1(a)] reads

$$\hat{H} = \sum_n (J_R |n+1\rangle \langle n| + J_L |n\rangle \langle n+1| + \Delta_n |n\rangle \langle n|),$$

where $J_{R(L)}$ is the right (left)-hopping amplitude, and $\Delta_n = 2\Delta \cos(2\pi \beta n)$ is an on-site quasiperiodic potential with $\Delta$, without loss of generality, set positive and $\beta$ usually taken to be an irrational number, say, the inverse of the golden ratio $(\sqrt{5} - 1)/2$ for infinite systems. For finite systems with site number $N = F_n + 1$, where $F_n$ is the $n$th Fibonacci number, because $\lim_{n \to \infty} F_n / F_{n+1} = (\sqrt{5} - 1)/2$, we usually take the rational number $\beta = F_n / F_{n+1}$, preserving the quasiperiodicity. For simplicity, we restrict the hoppings to be positive, which can be parameterized as $J_R = J e^{-\alpha}$, $J_L = J e^{\alpha}$ with $J > 0$ and $\alpha$ both real, unless mentioned otherwise. The non-reciprocity of hoppings ($\alpha \neq 0$) leads to the non-Hermiticity of the model, different from the non-Hermitian models based on the on-site gain/loss.

It is well known that, in the Hermitian case ($\alpha = 0$), AL occurs at $\Delta / J = 1$ for infinite systems due to the self-duality [17]: The extended states for $\Delta / J < 1$ become exponentially localized when $\Delta / J > 1$ with the form $|\psi\rangle \propto \sum_n e^{-\eta|n-n_0|} |n\rangle$, where $n_0$ is the index of the localization center, and $\eta = \ln(\Delta / J) > 0$ is the Lyapunov exponent, i.e., the inverse of the decaying length.

Deviated from the Hermitian limit, the transition should be extended to the non-reciprocal case ($\alpha \neq 0$). To catch a glimpse of the non-reciprocity effect on the transition, we can quickly look into the two limits of the non-reciprocal AA model, because the sites are decoupled, the non-Hermitian matrix $\hat{h}$ becomes a Hermitian one,

$$\hat{h} = \begin{pmatrix} J_1 & \Delta & J & \Delta_2 & J & \cdots & \Delta_{N-1} & J & \Delta_N \end{pmatrix}.$$

As a demonstration, we calculate the averaged inverse participation ratios (IPRs) over all right eigenstates of $\hat{H}$ under OBCs,

$$\text{IPR} = \frac{1}{N} \sum_{s=1}^{N} \text{IPR}_s = \frac{1}{N} \sum_{s=1}^{N} \sum_n |\langle n|\psi_s\rangle|^4 / \langle \psi_s | \psi_s \rangle^2,$$
Finite-size scaling analysis for the minimum IPR, $\Delta_0/J$ profiles of the eigenstates of 10th lowest $|E|$ in (a), showing the NHSE and the AL at $\Delta_0/J = 0.5$ and 3, respectively. (d) Finite-size scaling analysis for the minimum IPR, $\Delta_0/J$ (circles), of different lengths with the linear fitting (line), showing the asymptotic value $1.647 \pm 0.001$ when $N \to \infty$. 

where $|\psi_s\rangle$ is the $s$th right eigenstate of $\hat{H}$. A state with IPR = 1 is completely localized at a single site, while it is homogeneously distributed through all sites with IPR = $1/N$. Different from the extended phase with small IPRs of the Hermitian case, the skin phase should have larger values due to its boundary-localization nature. Therefore, the transition point should correspond to the most extended case, i.e., the smallest IPR. As expected, a deep dive at $\sim 1.56$ is found in Fig. 2(a), close to the theoretically predicted $e^{\alpha=0.5} \approx 1.65$ under consideration of the finite size effect, which is verified by the finite-size scaling analysis in Fig. 2(d). Figures 2(b) and 2(c) typically show the skin mode, which is exponentially decaying from one boundary, and the asymmetrically localized mode with different decaying lengths on both sides, respectively.

**Periodic boundary conditions.**—Because of the breakdown of the conventional bulk-boundary correspondence, the behaviors under PBCs and OBCs should be much different. However, the insensitivity of the localized states to the boundaries hints that the onset of AL under both boundary conditions should be identical. This judgment is numerically verified in Fig. 3(a): A steep rise of IPR around $e^{\alpha}$. Different from OBCs, the IPR keeps low prior to the transition due to the lack of the localized skin modes [Fig. 3(b)], while the localized states possess the same feature as OBCs [Fig. 3(c)].

Another big difference is the presence of imaginary eigenenergies [Fig. 3(d)]; the emergence of corner entries in $h$ invalidates the similarity to a Hermitian matrix.

This feature is intimately related to the phase transition if we are reminded that the localized states are insensitive to the boundaries and thus have the real eigenenergies: The complexity-reality transition of the eigenenergies coincides with the AL. Using this tie, we may establish a bulk-bulk correspondence between systems under OBCs and PBCs through a winding number with respect to the complex eigenenergies.

**Winding number.**—The conventional winding number cannot be used here because the chiral symmetry is broken by the on-site quasiperiodic potential $51, 60$. Thus, we consider the ring chain with a magnetic flux $-\Phi$ penetrating through the center, yielding

$$\hat{H}(\Phi) = \hat{H} + J_R e^{-i\Phi} |1\rangle \langle N| + J_L e^{i\Phi} |N\rangle \langle 1|, \quad (6)$$

and the winding number is defined as $52$

$$\nu = \frac{1}{2\pi i} \int_0^{2\pi} d\Phi \partial_\Phi \ln \det \hat{H}(\Phi) = \frac{1}{2\pi} \int_0^{2\pi} \partial_\Phi \theta(\Phi) d\Phi, \quad (7)$$

where $\theta(\Phi)$ is the argument of $\det \hat{H}(\Phi)$. Apparently, $\nu = 0$ for the localized phase on account of the reality of the spectrum.

Figure 3(e) show numerically how $\theta(\Phi)$ changes with $\Phi$ from 0 to $2\pi$ in the three phases of Fig. 3(b), and the corresponding winding numbers are obtained. The phase boundaries can alternatively be determined by analyzing the asymptotical behavior of $\det \hat{H}(\Phi)$ (See Supplemental Material). As a result, the chirality of the winding number can exactly tell the left/right-skin phases $\nu = \pm 1$.
and the localized phase ($\nu = 0$) under OBCs. Different from the conventional bulk-boundary correspondence, where edge states under OBCs can be predicted by a topological invariant defined under PBCs, here we establish a bulk-bulk correspondence, where the behavior of bulk states under OBCs can be predicted by a topological invariant defined under PBCs.

Electronic circuit’s realization.–We propose a driven RLC electronic circuit for the non-reciprocal AA model under OBCs, as shown in Fig. (a), where inductors with inductances $L_n = Lg^{-n}$ and $l_n = Lg^{-n}[2\Delta(\cos 2\pi\beta n + 1)]^{-1}$, capacitors with capacitance $C_n = Cg^n$, and resistors with resistance $R_n = Rg^{-n}$ are all linear passive elements with positive free parameters, $L, C, R, \text{and } g$. The leftmost node is grounded for an open boundary while the other is connected to a voltage source of a continuous wave, $V_s(t) = V_c \sin(\Omega t)$ with driving frequency $\Omega$.

Without resistors, the intrinsic eigenfrequencies $\omega$ can be obtained by grounding the rightmost node instead of the source. Based on the Kirchhoff’s current law, the corresponding eigenvalue equation reads,

$$V_{n+1} + gV_{n+1} - \Delta_n V_n = \left(f - \frac{\omega^2}{\omega_0^2}\right) V_n,$$

(8)

where $V_n$ is the amplitude of the voltage $V_n(t)$ on node $n$, $f = 1 + g + 2\Delta$, and $\omega_0 = 1/\sqrt{LC}$. Rewritten in matrix form, $\mathcal{H}V = EV$, where $V = [(V_n)]^T$ is a column vector and $E = f - \omega^2/\omega_0^2$ is the eigenvalue, $\mathcal{H}$ is just the matrix representation of the non-reciprocal AA model under OBCs with $JL = g$ and $JR = 1$. Notably, this classical circuit can only have real $E$, which is consistent with the previous proof. Figure (b) shows the intrinsic eigenfrequencies $\omega/\omega_0$ versus $\Delta/J$ with $J = \sqrt{g}$ and $\alpha = (\ln g)/2$.

When driving the system, the transport of continuous waves in different phases can be detected; the introduction of resistors, as seen in the following, is for system to quickly stabilize. The inhomogeneous equation with dimensionless parameters reads

$$\frac{d^2}{dt^2} \mathcal{V}(\tau) + \frac{d}{dt} \mathcal{V}(\tau) - (\mathcal{H} - f) \mathcal{V}(\tau) = \mathcal{V}_e \sin \Omega \tau,$$

(9)

where $\gamma = \frac{1}{\sqrt{\frac{\tau}{\omega_0^2}}} > 0$, $\tau = \omega_0 t$, and $\mathcal{V}_e = (0, ..., 0, V_e)^T$. The ‘$\sim$’ over the frequency hereafter means the frequency is dimensionless in unit of $\omega_0$. The solution is

$$\mathcal{V}(\tau) = \sum_s \mathcal{V}_s \left[ e^{-\gamma \tau/2} (c_s \cos \lambda_s \tau + d_s \sin \lambda_s \tau) \right. + \mathcal{W}_s^T \mathcal{V}_e \left( a_s \cos \Omega \tau + b_s \sin \Omega \tau \right),$$

(10)

where $a_s = \frac{\gamma \tilde{\Omega}}{\sqrt{\gamma^2 \tilde{\Omega}^2 + (\Omega^2 - \gamma^2/2)^2}}$, $b_s = \frac{\Omega^2 - \gamma^2/2}{\gamma^2 \tilde{\Omega}^2 + (\Omega^2 - \gamma^2/2)^2}$, $\lambda_s = \sqrt{\gamma^2 \tilde{\Omega}^2 + (\Omega^2 - \gamma^2/2)^2}$, and $(c_s, d_s)$ are coefficients determined by initial conditions. $\mathcal{V}_s$ and $\mathcal{W}_s^T$ are $s$th right and left eigenvectors of $\mathcal{H}$, respectively, satisfying $\mathcal{W}_s^T \mathcal{V}_{s'} = \delta_{ss'}$.

Note that if $\mathcal{V}_e$ is accumulated to one boundary, $\mathcal{W}_s$ is to the other, because $\mathcal{W}_s$ is the right eigenvector of $\mathcal{H}^T$. Thus, to detect the left skin modes, the source should be connected to the right end for the possible large overlap $\mathcal{W}_s^T \mathcal{V}_e$. In Eq. (10), the first part in the square brackets is the general solution, which, due to the resistance, will decay in a long time limit and thus, the effect of initial conditions can be ignored; the second part is one specific solution, which is stable, oscillating with the driving frequency. Moreover, if $\gamma \ll 1$, the system is resonant when $\Omega \approx \omega_s$ with a large value of $a_s$ and vanishing $b_s$, unless the overlap $\mathcal{W}_s^T \mathcal{V}_e$ is zero, and the corresponding right eigenvector $\mathcal{V}_e$ can be picked out.

The IPR of the time-averaged voltage vector, $\overline{\mathcal{V}} = \frac{1}{T} \int_{-T}^{T} |\mathcal{V}(\tau)| d\tau$ with $T = 2\pi/\Omega$ in $\tau \rightarrow \infty$ limit, is shown in Fig. (c), where a deep dive at $\sim 1.59$ is close to the transition point. Figures (d) and (e) plot the typical transports in both phases at $\alpha = 0.5$: In the skin phase, due to the existence of left-skin modes, the continuous wave is resonantly transferred and accumulated to the left boundary; while in the localized phase, because of the small overlap $\mathcal{W}_s^T \mathcal{V}_e$, the wave is confined to the right boundary without resonance. If the input is from the left boundary, the existence of right-skin modes at $\alpha = -0.5$ will benefit the transport from left to the right. This indicates that NHSEs can enhance the wave transport and may be useful in applications. This initiative realization of the non-reciprocity by circuits can be immediately applied to other non-reciprocal models, e.g., the non-reciprocal Su-Schrieffer-Heeger model.

Discussion and conclusion.–The phase diagram in Fig. (b) is obtained for positive hoppings. For general com-
plex hoppings with arbitrary phases $\phi_{R(L)}$, an identical phase diagram is found numerically. Although no proper way to relate it to the positive-hopping case due to the effective net flux between each two nearest-neighbor sites, the special case satisfying $\phi_R + \phi_L = n\pi$ ($n \in \mathbb{Z}$) can be proved exactly by the duality. We note that this transformation can map the non-reciprocal model to the AA model with complex on-site potentials, which, in the new basis, shares a similar AL but has no topological NHSEs. The details can be seen in Supplemental Material.

For the circuit’s realization, typically the element values can be taken as $L \sim \mu\text{H}$, $C \sim \mu\text{F}$, and $R \sim \text{k\Omega}$, i.e., $\omega_0 = 1/\sqrt{LC} \sim \text{kHz}$, which is accessible in usual circuit experiments. For typical non-reciprocal hoppings, say $\alpha = 0.2$ and thus $g = e^{0.4} \approx 1.49$, the element values can still drop in almost the same orders for $N = 10$ sites with $L_n \sim \mu\text{H}$ to $\mu\text{H}$, $C_n \sim \mu\text{F}$, and $R_n \sim \text{k\Omega}$.

In summary, we have revealed the interplay of NHSEs and AL in the non-reciprocal AA model with accompanying topologies, and obtained analytically the exact phase diagram. Especially, an elegant experimental scheme with electronic circuits has been proposed, demonstrating a transport transition from insulating to amplifying.

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In this Supplemental Material, we present the duality of the nonreciprocal AA model, the calculation of the winding number, and the discussion of the general case with complex hoppings.

Duality

That the non-reciprocal AA model can be transformed to the AA model with a complex on-site potential, i.e., the duality, can work in two cases: 1) Under PBCs with $\beta = p/N$, where $p \in \mathbb{Z}$; 2) Under OBCs with $N \to \infty$, because these two cases can ensure that the transformed k-space is closed by the following Fourier transform.

Firstly, let’s deal with Hamiltonian (10). By a gauge transformation $|n\rangle \to e^{-i\Phi n/N}|n\rangle$, Hamiltonian (6) becomes

$$H(\Phi) = \sum_n \left[ J_R e^{-i\Phi} |n+1\rangle \langle n| + J_L e^{i\Phi} |n\rangle \langle n+1| \right] + \Delta_n |n\rangle \langle n|,$$

(11)

Then, a Fourier transform, $|n\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-i2\pi\beta kn}|k\rangle$, can further change it to the k-space,

$$H(\Phi) = \sum_k \left[ \Delta(|k+1|k| + |k|k+1|) + J_k(\Phi) \langle k\rangle \langle k| \right],$$

(12)

where $J_k(\Phi) = 2J[\cosh \alpha \cos(2\pi\beta k + \Phi/F) - i \sinh \alpha \sin(2\pi\beta k + \Phi/N)]$. Note that the quasimomentum is $2\pi\beta k$, not the index $k$; The hopping term actually couple the two quasimomenta with difference $2\pi\beta$. Due to the PBCs, the quasimomentum should satisfy $2\pi\beta k = 2\pi m/N$, i.e., $k = m/\beta N$, where $m \in \mathbb{Z}$. To make the Hilbert space closed, we can just set $\beta = p/N$, and thus, $k + 1 = (m + p)/p$ corresponds to another quasimomentum index in the same Hilbert space, if considering the periodicity of the Brillouin zone. In this sense, the two dual models, Eqs. (11) and (12), are equivalent with identical energy spectra.

Secondly, consider the Hamiltonian (11) under OBCs with infinite length, i.e., $N \to \infty$. The dual Hamiltonian in k-space has the same form as Eq. (12) with only the difference that $\Phi = 0$ and the boundaries are open. When $J_R = J_L = J$, i.e., $\alpha = 0$, the dual Hamiltonians have the same form and thus det $h'(\Delta/J) = \det h'(J, \Delta)$, i.e., $J^N \det h'(\Delta/J) = \Delta^N \det h'(J/J)$. Note that $\det h = \det h'$ because of their similarity, we have the relation that $\det h(\Delta/J) = (\Delta/J)^N \det h(J/J)$. We have noted that Ref. [68] numerically gives the condition for the AL of the on-site complex AA model (12), $|J/J \cdot \cos \alpha| + |J/J \cdot \sinh \alpha| = 1$, i.e., $\Delta/J = e^{i\alpha}$, which is consistent with our result in the main text.
Calculation of the winding number

We calculate the winding number (17) of Hamiltonian (9). In matrix form, it can be rewritten as

\[ \hat{H}_\Phi = \sum_{mn} h_{mn}(\Phi) |mn\rangle \langle mn|, \]

where \( h_{mn}(\Phi) \) is the entry of the following matrix,

\[ h(\Phi) = \begin{pmatrix} \Delta_1 & J_L & J_{RE}^{-i\Phi} \\ J_R & \Delta_2 & J_L \\ & \ddots & \ddots \\ J_L & J_{\Phi} & \Delta_{N-1} & J_L \\ J_{RL} & \Delta_N & J_R \end{pmatrix}. \]

The key to calculate the winding number is the determinant of \( h(\Phi) \). Mathematically, we have

\[
\det h(\Phi) = -(J_L)^N e^{i\Phi} - (J_R)^N e^{-i\Phi} + P = -2(-J)^N (\cosh \alpha N \cos \Phi + i \sinh \alpha N \sin \Phi) + P,
\]

where \( P = \det h' - J^2 \det u' \) with \( h' \) being defined in Eq. (2) in the main text and \( u' \) is a submatrix with \((N-2)\) dimension of \( h' \) by removing the first and last row and column. Apparently, \( P \) is real.

Because the winding number (17) reveals how \( \det \hat{H}(\Phi) \) evolves with respect to \( \Phi \) from 2 to \( 2\pi \) in the complex plain, we can rewrite the winding number with the aid of the sign operators

\[
\nu = \frac{1}{2} \sum_i \text{sgn}[x(\Phi_i)] \cdot \text{sgn}\left[ \frac{dy(\Phi_i)}{d\Phi} \right],
\]

where \( x = \text{Re}[\det h(\Phi)] = P - 2(-J)^N \cosh \alpha N \cos \Phi \) and \( y = \text{Im}[\det h(\Phi)] = -2(-J)^N \sinh \alpha N \sin \Phi \). \( \Phi_i \) is \( i \)th solution of \( y(\Phi) = 0 \). Here are two solutions \( \Phi_1 = 0 \) and \( \Phi_2 = \pi \). Therefore, we have

\[
\nu = \frac{(-1)^N \text{sgn}(\alpha)}{2} \left[ \text{sgn}(P + 2(-J)^N \cosh \alpha N) \right.
\]

\[
- \text{sgn}(P - 2(-J)^N \cosh \alpha N) \]

\[
= \text{sgn}(\alpha) \theta(2J \cosh \alpha N - |P|).
\]

The transition point is determined by

\[
|P| = 2J \cosh \alpha N \approx J e^{i|\alpha|N},
\]

i.e.,

\[
\mathcal{P} \equiv \sqrt{|P|} \approx J e^{i|\alpha|},
\]

where the squiggly equal sign is for the large \( N \) limit. To calculate \( P \), we can expand it as

\[
P = \sum_{n=0}^{[N/2]} c_{N-2n}(-J)^{2n} (2\Delta)^{N-2n} + \text{Res.},
\]

with

\[
c_{N-2n} = \sum_{j_s = j_{s-1}+1}^{L} \prod_{i=1}^{N} \cos(2\pi i),
\]

where \([N/2]\) means the nearest integer less than \( N/2 \), and “Res.” is the residual if \( N/2 \) is not an integer. 

For the coefficient \( c_N = \prod_{i=1}^{N} \cos(2\pi i) \), we have

\[
\lim_{N \to \infty} \text{Im} c_N = \lim_{N \to \infty} \sum_{i=1}^{N} \text{Im} \cos(2\pi i) = N \int_0^1 \text{Im} \cos(2\pi N x) dx = \frac{1}{2\pi} \mathcal{L}(2\pi N) \approx -N \ln 2
\]

where

\[
\mathcal{L}(x) = - \int_0^x \ln \cos(x') dx'
\]

\[
= x \ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \sin(2kx) \frac{k}{k^2}.
\]

This means in the limit \( N \to \infty \), \( c_N \sim 2^{-N} \). In the same way, \( c_{N-2} \sim 2^{-(N-2)} \). Thus, using Eq. (20), we have

\[
\mathcal{P} = J \left[ c_N \left( \frac{2\Delta}{J} \right)^N - c_{N-2} \left( \frac{2\Delta}{J} \right)^{N-2} + \ldots \right]^{1/N}.
\]

For \( \Delta/J \leq 1 \), \( \lim_{N \to \infty} \mathcal{P} = J \), and thus \( \nu = \text{sgn}(\alpha) \), while for \( \Delta/J > 1 \), \( \lim_{N \to \infty} \mathcal{P} = \Delta \) and thus \( \nu = \text{sgn}(\alpha) \theta(J e^{i|\alpha|} - \Delta) \), that is, when \( e^{i|\alpha|} < \Delta/J \), \( \nu = 0 \) and when \( e^{i|\alpha|} > \Delta/J \), \( \nu = \text{sgn}(\alpha) \).

Phase diagrams for other cases

In the main text, we paid attention to the typical case of positive \( J_L \) and \( J_R \) in Hamiltonian (11). Here we show that the general case is related to this special case, and thus share the same transition point on AL.

The Hamiltonian with arbitrary complex hoppings reads

\[
\hat{H}_{\text{gei}} = \sum_n (J_{RE} e^{i\phi_R} |n+1\rangle\langle n| + J_L e^{i\phi_L} |n\rangle\langle n+1| + \Delta_n |n\rangle\langle n|),
\]

where \( J_{RL} > 0 \) and \( \Delta_n \) keep the same definitions as in Hamiltonian (11) of the main text, and \( \phi_{RL} \) is the arbitrary argument of the corresponding hopping. To reveal the relation between the general case of hoppings and the positive case, we do the following gauge transformation,
which does not change the energy spectrum,
\[
\hat{U} \hat{H}_{gel} \hat{U}^{-1} = e^{i \frac{\phi_R + \phi_L}{2} n} \sum_n \bigg( \Delta_n e^{-i \frac{\phi_R + \phi_L}{2} |n\rangle \langle n|} 
+ J_R |n + 1\rangle \langle n| + J_L |n\rangle \langle n + 1| \bigg),
\]
(28)
where \( \hat{U} \) is a unitary operator defined by \( \hat{U} |n\rangle = e^{i \frac{\phi_L - \phi_R}{2} n} |n\rangle \). Except for the overall phase and the phase of on-site terms, the above transformed phase Hamiltonian is similar to Hamiltonian (1).

Specifically, when \( \phi_R + \phi_L = 2n\pi \) (\( n \in \text{integer} \)), we have
\[
\hat{H}_{gel} = (-1)^n \hat{U}^{-1} \hat{H} \hat{U}
\]
(29)
where \( \hat{H} \) is just the Hamiltonian (1) in the main text. Apparently, the phase boundaries of this case is identical to the real-hopping case with only the eigenenergy \( E \) becoming \( (-1)^n E \). Note that for odd \( n \), the minus sign of on-site terms in Eq. (28) can be absorbed to the cosine terms in \( \Delta_n \) by shifting a phase, which makes no difference for the infinite chain.

For the general case, we cannot find a relation to the positive real-hopping case, which can be understood by noting that the right and left hoppings generally generate a net flux, \( \phi_L + \phi_R \), for each two nearest-neighbor sites, as there seems a coil in between with a magnetic field through it, and thus, the phase cannot be gauged away. However, the phase diagrams seem the same by our numerical calculation, which can also be characterized by the winding number, as shown in Fig. 5.

FIG. 5. Phase transition for \( (\phi_R, \phi_L) = (0, \pi/2) \). (a) IPR v.s. \( \Delta /J \) for \( \alpha = 0.5 \) under PBCs. Insets: The profiles of the eigenstates of 10th lowest \( |E| \), showing the NHSE and the AL at \( \Delta /J = 0.5 \) and 3, respectively. (b) \( \theta(\Phi) \) for \( \alpha = \Delta /J = 0.5 \) (solid red), \( \alpha = -\Delta /J = -0.5 \) (dashed blue), and \( \alpha = 0, \Delta /J = 3 \) (dash-dotted black), which correspond to \( \nu = +1, -1, \text{ and } 0 \), respectively. The calculation is carried on with \( N = 89 \) and \( \beta = 55/89 \).