MINIMAX RATES FOR STATISTICAL INVERSE PROBLEMS
UNDER GENERAL SOURCE CONDITIONS

LITAO DING AND PETER MATHE

Abstract. We describe the minimax reconstruction rates in linear ill-posed
equations in Hilbert space when smoothness is given in terms of general source
sets. The underlying fundamental result, the minimax rate on ellipsoids, is
proved similarly to the seminal study by D. L. Donoho, R. C. Liu, and B.
MacGibbon, Minimax risk over hyperrectangles, and implications, Ann. Statist.
18, 1990. These authors highlighted the special role of the truncated series es-
timator, and for such estimators the risk can explicitly be given. We provide
several examples, indicating results for statistical estimation in ill-posed prob-
lems in Hilbert space.

1. Introduction

We consider linear operator equations of the form

\[ y^\sigma = Tx + \sigma \xi, \]

where \( T \) is a compact linear operator between Hilbert spaces \( X \) and \( Y \) under Gauss-
ian white noise \( \xi \) and with noise level \( \sigma > 0 \). The compact operator \( T \) has a singular
value decomposition, where \( \{s_n^2\} \) denotes the sequence of eigenvalues of \( T^*T \), ar-
ranged in decreasing order, and \( \{v_n\} \) in \( X \), and \( \{u_n\} \) in \( Y \) are orthonormal systems.
In particular, we have

\[ Tx = \sum_{j=1}^{\infty} s_j \langle x, v_j \rangle u_j, \quad x \in X. \]

Through the singular value decomposition, and letting \( \sigma_k := s_k^{-1} \), \( k = 1, 2, \ldots \),
as well as the coefficients \( \theta_k = \langle x, v_k \rangle \), \( k = 1, 2, \ldots \) we find that the model (II) is
equivalent to the sequence space model

\[ z^\sigma_k = \theta_k + \sigma \sigma_k \xi_k, \quad \xi_k \sim N(0, 1), \]

provided that the solution element \( x \) is in the orthogonal complement of the kernel of
the operator \( T \). Notice that as a consequence of the compactness of the operator \( T \)
we have that \( s_k \to 0 \) as \( k \to \infty \), and hence \( \sigma_k \to \infty \), such that higher number
coefficients are blurred by higher noise level. This is a typical feature of inverse
problems, and it thus requires to regularize the observations \( z^\sigma_k \), \( k = 1, 2, \ldots \).

In order to apply the minimax paradigm for the analysis of statistical inverse
problems we introduce classes of elements \( \theta = (\theta_k)_{k=1}^{\infty} \). Prototypical classes are
given through Sobolev-type ellipsoids.

**Definition 1** (Sobolev-type ellipsoid). For a given increasing sequence \( a = (a_j)_{j=1}^{\infty} \), \( a_j > 0 \), and a constant \( Q < \infty \) we let

\[ \Theta_a(Q) = \left\{ \theta, \sum_{j=1}^{\infty} a_j^2 \theta_j^2 \leq Q^2 \right\}. \]

2010 Mathematics Subject Classification. 65J22; secondary 62G20.
Key words and phrases. statistical inverse problem, general source condition, minimax rate.
The increasing sequence $a_j > 0$ controls the decay of the coefficients of $\theta$. Within the theory of inverse problems it is common to define the ellipsoid relative to the operator $T$ in the following form.

**Definition 2** (General source set). For a continuous non-decreasing function $\varphi$ with $\varphi(0) = 0$ we let

$$A_\varphi = \{ x \in X, \; x = \varphi(T^*T)v, ||v||_2 \leq 1 \}.$$  

These ellipsoids are related as follows. Since for $x \in A_\varphi$ we have that $\theta_j = \langle x, v_j \rangle = \varphi(s_j^2)\langle v, v_j \rangle$, $j \in \mathbb{N}$, we find that

$$||v||^2_2 := \sum_{j=1}^{\infty} \frac{\theta_j^2}{\varphi(s_j^2)}$$

such that $x \in A_\varphi$ implies for $a_j := \frac{Q}{\varphi(s_j^2)}$, $j \in \mathbb{N}$ that $\theta \in \Theta_a(Q)$. More specific examples will be given in Section 3.

The goal of the present study is to establish minimax rates for statistical estimation over such sets of elements. Let $\hat{\theta} := \hat{\theta}(z)$ be any estimator based on the data $z$. Its RMS-error on the class $\Theta_a$ is then given as

$$e(\hat{\theta}, \Theta_a, \sigma) := \sup_{\theta \in \Theta_a} (\mathbb{E}[\|\theta - \hat{\theta}\|^2_2 ]^{1/2}. $$

Above, we highlight that the error depends on the underlying noise level $\sigma$. The minimax error on a class $\Theta_a$ is consequently given as

$$e(\Theta_a, \sigma) := \inf_{\hat{\theta}} e(\hat{\theta}, \Theta_a, \sigma). $$

We will denote by $e_T(\Theta_a, \sigma)$ the minimax errors when restricted to truncation estimators, only. It will be handy to introduce the following quantity

$$\rho_n^2 := \sum_{j=1}^{n} \frac{1}{s_j^2}, \quad n \in \mathbb{N}. $$

The main result presented in this note is the following

**Theorem.** Let $\Theta_a$ be as in (4), and let $s_j$, $j = 1, 2, \ldots$ denote the singular numbers of the operator $T$. Then for $\sigma > 0$ we have that

$$e(\Theta_a, \sigma) \leq \inf_D \left( \frac{Q^2}{\sigma_D^2 + \sigma_D^2} \right)^{1/2} \leq 2.2e(\Theta_a, \sigma) $$

We formulate the following consequence for source sets.

**Corollary.** Consider the source set (5) from above. Then

$$e^2(A_\varphi, \sigma) \leq \inf_D \left\{ \varphi^2(s_D^2) + \sigma_D^2 \right\} \leq 4.84e^2(A_\varphi, \sigma).$$

This consequence allows to explicitly compute the minimax rates for statistical inverse problems in Hilbert space for a variety of index functions $\varphi$ and decay rates of singular numbers $s_j$, $j \in \mathbb{N}$ in subsequent examples.

2. **Proof of the theorem**

The main portion in this note is to provide the details for proving the main result. The fundamental underlying principle is comprised in three observations.

1. The error of the truncated (at $D$-th summand) series estimator upper bounds the minimax error.

2. The intermediate inf in (9) is the error of the (best) truncated series estimator.
(3) Up to a factor 2.22 the (squared) error of the truncated series estimator is best possible.

The first assertion is trivial. For the second assertion let us introduce the truncated series estimator, given observations $z_j^\sigma$, $j \in \mathbb{N}$, as

$$\hat{\theta}_n(z^\sigma) := \sum_{j=1}^{n} z_j^\sigma v_j. \tag{11}$$

**Remark 1.** The above estimator corresponds to the spectral cut-off estimator for the original data $y^\sigma$. In these terms we find that

$$\hat{\theta}_n(z^\sigma) = \sum_{j=1}^{n} \frac{1}{s_j} y_j^\sigma v_j.$$

**Lemma 1.** We have that

$$e^2(\hat{\theta}_n, \Theta_\alpha, \sigma) = Q^2/a_{n+1}^2 + \sigma^2 \rho_n^2.$$ 

**Proof.** We derive the bias-variance decomposition for the estimator as

$$E[\|\theta - \hat{\theta}_n(z^\sigma)\|^2_X] = \sum_{j=1}^{n} E[(\theta_j - z_j^\sigma)^2] + \sum_{j=n+1}^{\infty} \theta_j^2 = \sigma^2 \sum_{j=1}^{n} \frac{1}{s_j^2} + \sum_{j=n+1}^{\infty} \theta_j^2. $$

Thus, uniformly over the class $\Theta_\alpha$ we find that

$$\sup_{\theta \in \Theta_\alpha} E[\|\theta - \hat{\theta}_n(z^\sigma)\|^2_X] = \sigma^2 \sum_{j=1}^{n} \frac{1}{s_j^2} + \sup_{\theta \in \Theta_\alpha} \sum_{j=n+1}^{\infty} \theta_j^2.$$ 

Now we observe that

$$\sup_{\theta \in \Theta_\alpha} \sum_{j=n+1}^{\infty} \theta_j^2 = \sup_{\theta \in \Theta_\alpha} \sum_{j=n+1}^{\infty} a_j^2 \theta_j^2 a_j^{-2} \leq \frac{Q^2}{a_{n+1}^2},$$

from the monotonicity assumptions for $\alpha$. But the upper bound is attained for the element

$$\theta_0^j := \begin{cases} \frac{Q}{a_{n+1}}, & j = n + 1 \\ 0, & \text{else.} \end{cases}$$

The proof of the Lemma is complete. $\square$

**Remark 2.** The estimator $\hat{\theta}_n$, which selects the first $n$ coordinates, is the best among all truncation estimators which use $n$ coordinates. Suppose that a truncation estimator, say $\hat{\theta}_P$ uses a set $P \subset \{1, 2, \ldots, n\}$, $|P| = n$. Then its squared risk (uniform on $\Theta_\alpha$) is given as $\sup_{\theta \in \Theta_\alpha} \sum_{j \in P} \theta_j^2 + \frac{\omega^2}{2} \sum_{j \in P} \frac{1}{\theta_j^2}$. Obviously, the variance term is minimal for the initial segment. If $k := \min P^c \leq n$, then similar to above, we let $\theta_0$ have only the $k$th component different from zero with $\theta_0^k := Q/a_k$. Then the uniform squared bias is upper bounded by its value at $\theta_0^k$, and this gives $Q^2/a_k^2 \geq Q^2/a_{n+1}^2$, from monotonicity. Therefore, consideration may be restricted to the initial segment, only.

We turn to proving the final assertion from above, and this is the crucial part for the overall proof. To this end we use arguments from the seminal study [4]. In Corollary (to Theorem 10) ibid. these authors establish that for orthosymmetric, compact, convex and quadratically convex sets $\Theta$ the squared risk of the best
truncated series estimator is less than 4.44 times the squared minimax risk over $\Theta$. Ellipsoids as the set $\Theta_n$ are prototypical examples of such sets.

However, in that study the model was the sequence space model, similar to [3], with i.i.d noise $\xi_i$, but with $\sigma_i \equiv 1$. Thus their argument does not cover statistical inverse problems (with compact operator $T$ as in [1]). Therefore, we (briefly) recall the arguments used in the study [4].

The proof can be divided as follows. Let $\Theta(r) := \{ \theta, |\theta_i|^2 \leq r_i \} \subset \Theta_n$ ($r_i \to 0$) be any hyperrectangle.

1. Plainly we have that
   \[ e(\Theta(r), \sigma) \leq e(\Theta_n, \sigma) \leq e_T(\Theta_n, \sigma). \]

2. The reasoning in [4] starts with considering 1d-subproblems, and there we cannot distinguish between regression or inverse estimation problems. It is shown in [4] Thm. 2 that for 1d-problems best nonlinear estimators cannot perform better that $\sqrt{2.22}$ times best truncation estimators, or the best linear estimator.

3. This extends to hyperrectangles, as shown in § 3 ibid, because the minimax risk is a Bayes risk, and the worst Bayes prior is of product type. Specifically, as shown in [4] Prop. 8 we find that $e_T(\Theta(r), \sigma) \leq \sqrt{2.22} e(\Theta(r), \sigma)$, thus, together with (12) we already find
   \[ e_T(\Theta(r), \sigma) \leq \sqrt{2.22} e_T(\Theta_n, \sigma). \]

4. In view of Lemma 1 the assertion of the theorem follows once we can prove the next result, similar to [4] Thm. 10 (Notice that $\sqrt{2.22 + \sqrt{2}} \leq 2.2$).

**Lemma 2.** For ellipsoids we have that
\[ e_T(\Theta_n, \sigma) \leq \sqrt{2} \sup \{ e_T(\Theta(r), \sigma), \Theta(r) \subset \Theta_n \}. \]

**Proof.** Arguing as in [4] for ellipsoids the error is controlled on the positive orthant $\Theta_+$. We consider the following functional, which expresses the minimax risk of a truncated series estimator on the hyperrectangle $\Theta(r)$.

\[ J(r) := \sum_{i=1}^{\infty} \min \{ r_i, \sigma^2 \sigma_i^2 \}, \quad r \in \Theta_+^2. \]

This functional is concave (as a sum of concave functions), and it is continuous, such that on the compact set $\Theta_+^2$ it attains it’s maximum value, say at $r^*$. This maximality property yields that $J(r) \leq J(r^*)$, $r \in \Theta_+^2$. We introduce the sets $Q := \{ i, r_i^* = \sigma^2 \sigma_i^2 \}$, and the $P := \{ i, r_i^* \geq \sigma^2 \sigma_i^2 \}$. The latter will be finite, since $r_i^* \to 0$ and $\sigma^2 \sigma_i^2 \to \infty$. With this notation we can derive, similarly as in the original study [3], the explicit form of the Gateaux derivative of the functional $J$ at $r^*$ as

\[ D_{r^*} J(h) = \sum_{i \in P} h_i - \sum_{i \in Q} (h_i)_-, \quad h := r - r^* \in \Theta_+^2. \]

Observing that for $i \in Q$ and $h_i < 0$ we have
\[ (h_i)_- = -h_i = r_i^*-r_i \leq \sigma^2 \sigma_i^2, \]
and using that $D_{r^*} J(h) \leq 0$ (maximality property) we arrive at
\[ \sum_{i \in P} r_i < \sum_{i \in P} r_i^* + \sigma^2 \sum_{i \in Q} \sigma_i^2, \quad r \in \Theta_+^2. \]
Consider the truncation estimator $\hat{\theta}^*$ for $\Theta_a$ given by $\hat{\theta}^* = z^\sigma X_{(i \in P)}$. It has the following (squared) risk:

$$\mathbb{E}[\|\hat{\theta}^* - \theta\|^2] = \sum_{i \notin P} \theta_i^2 + \sigma^2 \sum_{i \in P} \sigma_i^2.$$  

Since by definition $Q \subset P$, and with $r_i := \theta_i^2$ this can be upper bounded by

$$\mathbb{E}[\|\hat{\theta}^* - \theta\|^2] \leq \sum_{i \notin P} r_i^* + 2\sigma^2 \sum_{i \in P} \sigma_i^2 \leq 2J(r^*).$$

Therefore, $e^2_T(\Theta_a, \sigma) \leq \mathbb{E}[\|\hat{\theta}^* - \theta\|^2] \leq 2J(r^*)$. Since, by construction of $r^*$ we have that $J(r^*) \leq \sup \{e_T(\Theta(r), \sigma), \Theta(r) \subset \Theta_a\}$ the proof of the lemma is complete. 

\[ \square \]

3. Discussion and examples

Minimax rates for statistical inverse problems were established in many studies. The seminal study is [7]. An update of the subsequent studies was given in the survey [2], which will be used for comparison below. Two studies are related to questions in the present study.

The study [1] provides a detailed analysis of statistical inverse problems (in a slightly more general framework). Theorem 8 ibid. actually asks for the relation between the minimax error and best error bounds for spectral cut-off. Assumptions are given where the error of a certain specific regularized estimator is not worse than the error obtained by spectral cut-off. In our approach we do not rely on any specific way of obtaining estimators of the unknown element.

The authors in [6] discuss, among many other things, the relation between ellipsoids and source sets as means of regularity conditions. Theorem 4.1 ibid. asserts the minimax rate for power type decay of singular numbers of the operator $T$ and power type increase of the sequence $a$ in the ellipsoid [3]. One specific focus is on the concept of maxisets, which is an interesting approach, but which is beyond the focus of the present study.

Previous minimax rates on general source sets were given in [9], but restricted to operator monotone index functions, only. These index functions are limited to low smoothness, and the result was obtained by an application of Pinsker's study [10].

Next, we relate the results to similar results as known in ‘classical’ inverse problems and to results from non-parametric statistical testing. For classical inverse problems, when the noise obeys $\|\xi\|_Y \leq 1$, then the minimax rate of recovery is related to the modulus of continuity. We skip details and refer to the study [8].

As can be seen from those results, the corresponding minimax rate is (up to the constant $\sqrt{2}$) given by $\inf_D \left( \frac{\alpha^2}{\sigma_{D+1}^2} + \sigma^2 \right)^{1/2}$, which, again is attained by a truncated series estimator. Looking the the second summand above we immediately conclude that for exponentially decaying singular numbers the rates for ‘classical’ and statistical inverse problems are the same.

In non-parametric statistical testing such estimators also play a similar role, and (the square of) the minimum separation radius was given in [5, Prop. 3] as $\inf_D \max \left\{ \frac{\alpha^2}{\sigma_{D+1}^2}, \sigma^2 \left( \sum_{j=1}^D \frac{1}{\sigma_j^2} \right)^{1/2} \right\}$. Additional constants, determined by the prescribed errors of the first and second kind appear. Still, the truncated series estimator plays a prominent role. The major differences can be seen from the significance of the fourth powers instead of the second ones, such that ‘estimation is harder than testing’ again, for exponentially decaying singular numbers the rates for testing and estimation coincide.
The following examples highlight the minimax rates in statistical inverse problems in prototypical situations. The corresponding rates are also seen in [2, Tbl. 1].

**Definition 3** (ill-posedness of the operator). We call the operator $T$ *moderately ill-posed* if its singular numbers decay at a power type rate $s_j \asymp j^{-p}$, $j = 1, 2, \ldots$ for some $p > 0$. It is called *severely ill-posed* if $s_j \asymp e^{-pj}$, $j = 1, 2, \ldots$.

**Definition 4** (solution smoothness). The solution smoothness, expressed in terms of $\Theta_a$ is said to be *moderate* if $a_j \asymp j^\kappa$. It is called *analytic* in case $a_j \asymp e^{\kappa j}$.

Finally, we introduce the corresponding notion for the statistical problem at hand.

**Definition 5** (ill-posedness of the statistical problem). The statistical problem (1) is called *moderately ill-posed* if the minimax rate of reconstruction (in terms of the noise level $\sigma$) is of power type. It is called *severely ill-posed* if the rate is logarithmic, and it is called *mildly ill-posed* if (up to a logarithmic factor) the minimax rate is linear in the noise level $\sigma$.

The results as obtained from the application of the theorem are shown in Table 1. The number $D_*$ denotes the optimal truncation level, the number which balances both terms in the middle sum in (4). By ‘rate’ we denote the corresponding minimax rate as $\sigma \to 0$.

| $s_j \asymp j^{-p}$ | $a_j \asymp j^\kappa$ | $\varphi(t) = t^{\kappa/(2p)}$ | $D_* = \left(\frac{1}{\kappa}\right)^{1/(\kappa+p+1/2)}$ | $\sigma^{\kappa/(\kappa+p+1/2)}$ | $s_j \asymp e^{-pj}$ | $a_j \asymp e^{\kappa j}$ | $\varphi(t) = \log^{-\frac{1}{\kappa}}(1/t)$ | $D_* = \log(1/\sigma)$ | $\sigma^{\kappa/\left(p+\kappa\right)}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\varphi(t) = \log^{-\frac{1}{\kappa}}(1/t)$ | $D_* = \log(1/\sigma)$ | $\varphi(t) = t^{\kappa/(2p)}$ | $D_* = \frac{1}{\kappa}\log(1/\sigma)$ | $\sigma^{\kappa/\left(p+\kappa\right)}$ |

**Table 1.** Outline of reconstruction rates for Sobolev-type smoothness of the truth $\theta \in \Theta_a$ and power/exponential type decay of the singular numbers.

As we see, we find mild ill-posedness of the problem for moderately ill-posed operator and analytic smoothness. The problem is severely ill-posed for severely ill-posed operator and moderate smoothness. We emphasis that the problem is moderately ill-posed, both for moderate ill-posedness of the operator and power type smoothness, but also for analytic smoothness and severely ill-posed operator. That is why the distinction between ill-posedness of the operator and ill-posedness of the statistical problem is recommended.

**References**

[1] N. Bissantz, T. Hohage, A. Munk, and F. Ruymgaart. Convergence rates of general regularization methods for statistical inverse problems and applications. SIAM J. Numer. Anal., 45(6):2610–2636, 2007.

[2] L. Cavalier. Nonparametric statistical inverse problems. Inverse Problems, 24(3):034004, 19, 2008.

[3] David L. Donoho, Richard C. Liu, and Brenda MacGibbon. Minimax risk for hyperrectangles. Technical Report 123, Dept. Statistics, University of California, Berkeley, 1988.

[4] David L. Donoho, Richard C. Liu, and Brenda MacGibbon. Minimax risk over hyperrectangles, and implications. Ann. Statist., 18(3):1416–1437, 1990.

[5] Béatrice Laurent, Jean-Michel Loubes, and Clément Marteau. Non asymptotic minimax rates of testing in signal detection with heterogeneous variances. Electron. J. Stat., 6:91–122, 2012.
[6] Jean-Michel Loubes and Vincent Rivoirard. Review of rates of convergence and regularity conditions for inverse problems. *International Journal of Tomography and Statistics*, 11:61–82, 06 2009.

[7] Bernard A. Mair and Frits H. Ruymgaart. Statistical inverse estimation in Hilbert scales. *SIAM J. Appl. Math.*, 56(5):1424–1444, 1996.

[8] Peter Mathé and Sergei V. Pereverzev. Geometry of linear ill-posed problems in variable Hilbert scales. *Inverse Problems*, 19(3):789–803, 2003.

[9] Peter Mathé and Sergei V. Pereverzev. Regularization of some linear ill-posed problems with discretized random noisy data. *Math. Comp.*, 75(256):1913–1929, 2006.

[10] M. S. Pinsker. Optimal filtration of square-integrable signals in Gaussian noise. *Problems Inform. Transmission*, 16(2):52–68, 1980.

E-mail address: 13110180018@fudan.edu.cn (LiTao Ding)

School of Mathematical Sciences, Fudan University, Shanghai, China 200433

E-mail address: peter.mathe@wias-berlin.de (Peter Mathé)

Weierstrass Institute, Mohrenstrasse 39, D-10117 Berlin, Germany