Chiral gauge anomalies on Noncommutative $\mathbb{R}^4$

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We discuss the noncommutative counterparts of chiral gauge theories and compute the associated anomalies.

1. Introduction

The chiral anomalies [1] of quantum field theory are of the utmost importance in particle physics [2] and mathematics [3]. In this paper we reappraise them in the framework of quantum field theory on noncommutative manifolds, the latest new hunting ground in both field theory —see [4,5,6] and string theory —see [7,8] and the other references in this last paper. We shall be concerned mainly with fermions on noncommutative $\mathbb{R}^4$, coupled to background $U(N)$ gauge fields. It is good training to start by considering the axial anomaly; there some of our results overlap with the recent calculations by Ardalan and Sadooghi [9]. Then we proceed to compute the “dangerous” anomaly associated to chiral fermions in nonabelian gauge fields. Variants of the Fujikawa method [10,11] will be found to be ideally suited for both kinds of computations.

2. The ABJ anomaly on noncommutative $\mathbb{R}^{2n}$

Noncommutative $\mathbb{R}^{2n}$ is characterized by the Moyal product of functions

$$f \star g(x) := \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} f(x+s)g(x+t)e^{is\omega t} \, ds \, dt,$$

where $\omega$ denotes a real antisymmetric matrix, that in this paper will be taken constant. In general we omit reference to $\omega$ in the notation. Several straightforward mathematical properties of that operation will be used in the sequel without further mention, in particular the Leibniz rule

$$\partial_\mu (f \star g) = \partial_\mu f \star g + f \star \partial_\mu g.$$

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that follows from derivation under the integral sign, and the fact that
\[
\int_{\mathbb{R}^2n} f \star g(x) \, dx = \int_{\mathbb{R}^2n} f(x)g(x) \, dx,
\]
inducing cyclicity in the integral of the Moyal product of several functions. Note also that if \( f, g \) decay quickly at infinity, this is the case for \( f \star g \) as well [12].

Consider a \( U(1) \) gauge field \( A_\mu \) on noncommutative \( \mathbb{R}^{2n} \). Its infinitesimal gauge variation is given by \( \delta_\theta A_\mu(x) = \partial_\mu \theta(x) - i[A_\mu, \theta](x) \), with \( \theta(x) \) denoting the gauge parameter and \([.,.]\) the Moyal commutator or bracket. Let \( \psi \) denote an Euclidean Dirac fermion field on noncommutative \( \mathbb{R}^{2n} \), that we take massless. The fact that the following identity should hold for any two \( U(1) \) gauge transformations of \( \psi \)
\[
(\delta_{\theta_1}, \delta_{\theta_2} - \delta_{\theta_2} \delta_{\theta_1}) \psi = -i\delta_{[\theta_1, \theta_2]} \psi,
\]
leads uniquely to the following permissible types of gauge transformations for \( \psi \):
\[
a) \delta_\theta \psi = i\theta \star \psi, \quad b) \delta_\theta \psi = -i\psi \star \theta, \quad c) \delta_\theta \psi = -i[\psi, \theta].
\]
The previous transformations of the fermion field give rise, respectively, to the following covariant derivatives
\[
a) D_\mu \psi = \partial_\mu \psi - iA_\mu \star \psi, \quad b) D_\mu \psi = \partial_\mu \psi + i\psi \star A_\mu, \quad c) D_\mu \psi = \partial_\mu \psi - i[A_\mu, \psi].
\]
These yield three types of Dirac \(-i\bar{D}\) operators twisted with the \( U(1) \) connection on noncommutative \( \mathbb{R}^{2n} \), namely the given by
\[
\begin{align*}
a) \bar{D} \psi &= \bar{\partial} \psi - i\gamma_\mu A_\mu \star \psi, \\
b) \bar{D} \psi &= \bar{\partial} \psi + i\gamma_\mu \psi \star A_\mu, \\
c) \bar{D} \psi &= \bar{\partial} \psi - i\gamma_\mu [A_\mu, \psi].
\end{align*}
\]
The latter we exclude from our considerations, as we interested in the case that the local gauge action occurs in the fundamental representation of the fermions; in the formula the symbols \( \gamma_\mu \) denote of course the Euclidean gamma matrices, which satisfy \( \{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu} \) and \( \gamma_\mu^\dagger = \gamma_\mu \), and the slash notation has its usual meaning.

Each Dirac operator yields a fermionic action:
\[
S_{U(1)} = -i \int d^{2n}x \bar{\psi} \bar{D} \psi.
\]
To apply Fujikawa’s method to the computation of the ABJ anomaly for the quantum theory defined by the previous fermionic action, one begins by introducing chiral local transformations for the Dirac operators introduced in equation (1). These transformations read
\[
\begin{align*}
a) \delta^5_\theta \psi &= i\theta \star \gamma_5 \psi, \\
b) \delta^5_\theta \psi &= -i\gamma_5 \psi \star \theta,
\end{align*}
\]
respectively. The symbol \( \gamma_5 \) denotes the product \((-i)^n \prod_{i=1}^{2n} \gamma_\mu \).

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Let us consider first the theory defined by case a) of (1). Then, the variation of the action in formula (2) under the chiral transformations a) of (3) reads

$$\delta \theta SU(1) = \int d^2 x \theta (D_\mu (A) j^5_\mu),$$

where

$$j^5_\mu := \psi_\beta \tilde{\psi}_\alpha (\gamma_\mu \gamma_5)_{\alpha\beta}, \quad D_\mu (A) j^5_\mu = \partial_\mu j^5_\mu - i[A_\mu, j^5_\mu]. \quad (4)$$

To compute the change of the path integral fermionic measure under the chiral transformations a) of equation (3), one expands the fermionic fields $\psi$ and $\tilde{\psi}$ in an orthonormal base of eigenfunctions, $\{\varphi_n\}$, of the Dirac operator defined in a) of equation (1):

$$\psi(x) = \sum_n a_n \varphi_n(x), \quad \tilde{\psi} = \sum_n \tilde{b}_n \tilde{\varphi}_n(x).$$

The coefficients $a_n$ and $b_n$ are anticommuting variables. Now, the standard calculation of the functional derivative of the determinant of the change of variables in the path integral expression of the partition function leads to

$$\langle D_\mu (A) j^5_\mu (x) \rangle = i A(x), \quad \text{where} \quad A(x) = 2 \sum_m \gamma_5 \alpha \beta (\varphi_{n\beta} \tilde{\varphi}_{n\alpha})(x). \quad (5)$$

The sum $A(x)$ is not well-defined and to define it we first regularize it in a gauge invariant way with the help of the heat kernel associated with the Dirac operator at hand. Hence,

$$A(x) = \lim_{\Lambda \to \infty} \frac{1}{2} \sum_m \gamma_5 \alpha \beta \left( \left( e^{\frac{\psi^2}{\Lambda^2}} \varphi_n \right)_{\beta} \tilde{\varphi}_{n\alpha}(x) \right).$$

By expanding $\varphi_n$ in plane waves, one can recast $A(x)$ into the form

$$A(x) = \lim_{\Lambda \to \infty} \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left( \gamma_5 \left( e^{\frac{\psi^2}{\Lambda^2}} e^{ipx} \right) \right) \left( e^{ipx} \right).$$

A little computation yields

$$A(x) = \lim_{\Lambda \to \infty} \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left( \gamma_5 \sum_m \left( \left[ (ip_\mu + \partial_\mu - iA_\mu \right) \right] \frac{1}{m! \Lambda^{2m}} F_{\mu\nu} \right) \left( e^{ipx} \right),$$

where $\text{tr}$ denotes trace over the Dirac matrices, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ and $I$ denotes the unit function on $\mathbb{R}^{2n}$.

Since $e^{ipx} \star e^{-ipx} = I$, one finally obtains that

$$A(x) = 2 \frac{1}{(4\pi)^n} \frac{1}{n!} \varepsilon^{\mu_1 \cdots \mu_{2n}} F_{\mu_1 \mu_2} \tilde{F}_{\mu_3 \mu_4} \cdots \tilde{F}_{\mu_{2n-1} \mu_{2n}} \quad (6)$$
which is a \(*\)-deformation of the standard result for commutative \(\mathbb{R}^{2n}\). Substituting the previous result back in equations (5), we conclude that the \(U(1)\) axial current \(j_5^\mu(x)\), which is classically (covariantly) conserved, is not conserved at the quantum level, i.e., there is an anomaly.

The equation for the ABJ anomaly for the Dirac operator \(b\) of formula (1) can be obtained from the previous expressions by doing first the replacements \(*_\omega \rightarrow *_{-\omega}, A_\mu \rightarrow -A_\mu\) and then undoing them. One then obtains

\[
\langle D_\mu(A)j_5^\mu \rangle = -2i \frac{(-1)^n}{(4\pi)^n} \frac{1}{n!} \varepsilon^{\mu_1 \cdots \mu_{2n}} F_{\mu_1 \mu_2} \ast F_{\mu_3 \mu_4} \ast \cdots \ast F_{\mu_{2n-1} \mu_{2n}},
\]

where now

\[
j_5^\mu = \bar{\psi}_\beta \gamma_\mu \gamma_5 \psi_\alpha, \quad \text{and} \quad D_\mu(A)j_5^\mu = \partial_\mu j_5^\mu + i[j_5^\mu, A_\mu].
\]

The field strength tensor \(F_{\mu \nu}\) keeps its previous definition. Notice that the chiral currents defined in equations (4) and (8) differ only by a sign in the commutative limit (recall the Grassmann character of the classical spinor field); not so on noncommutative Euclidean space.

Now, it will not come unexpectedly that if the gauge group \(U(1)\) is replaced by \(U(N)\), the singlet currents corresponding to equations (4) and (8) satisfy, respectively, the following anomalous equations:

\[
a) \quad \langle \partial_\mu j_5^\mu - \sum_{i,j} (A_\mu_{ij} \ast j_5^\mu_{ji} - j_5^\mu_{ij} \ast A_\mu_{ji}) \rangle = 2 \frac{(-i)^{n-1}}{(4\pi)^n} \frac{1}{n!} \varepsilon^{\mu_1 \cdots \mu_{2n}} \text{Tr} F_{\mu_1 \mu_2} \ast F_{\mu_3 \mu_4} \ast \cdots \ast F_{\mu_{2n-1} \mu_{2n}},
\]

\[
b) \quad \langle \partial_\mu j_5^\mu + \sum_{i,j} (A_\mu_{ij} \ast j_5^\mu_{ji} - j_5^\mu_{ij} \ast A_\mu_{ji}) \rangle = 2 \frac{i^{n-1}}{(4\pi)^n} \frac{1}{n!} \varepsilon^{\mu_1 \cdots \mu_{2n}} \text{Tr} F_{\mu_1 \mu_2} \ast F_{\mu_3 \mu_4} \ast \cdots \ast F_{\mu_{2n-1} \mu_{2n}},
\]

where \(A_\mu\) is the \(U(N)\) gauge field (taken to be antihermitian), \(i, j\) are \(U(N)\) indices, \(\text{Tr}\) is the trace on \(U(N)\) and \(j_5^\mu\) and \(j_5^\mu_{ij}\) in \(a)\) and \(b)\) are given by

\[
a) \quad j_5^\mu = \sum_i \bar{\psi}_\beta i \ast \psi_\alpha (\gamma_\mu \gamma_5)_{\alpha \beta}, \quad j_5^\mu_{ij} = \bar{\psi}_\beta i \ast \psi_\alpha_j (\gamma_\mu \gamma_5)_{\alpha \beta},
\]

\[
b) \quad j_5^\mu = \sum_i \bar{\psi}_\alpha i \ast \psi_\beta (\gamma_\mu \gamma_5)_{\alpha \beta}, \quad j_5^\mu_{ij} = \bar{\psi}_\alpha i \ast \psi_\beta_j (\gamma_\mu \gamma_5)_{\alpha \beta},
\]

respectively. Note that equations (6),(7) and (9) would result from a “naïve” \(*\)-deformation of their counterparts on commutative Euclidean space. Again, cases \(a)\) and \(b)\) essentially match in the commutative limit.

We will now turn our attention to the index of the Dirac operators \(a)\) and \(b)\) of equation (1) with a \(iA_\mu\) replaced with a \(U(N)\) gauge field \(A_\mu\). We can readily adapt the
computational method displayed above to carry out a physicist’s computation of the index. The results read

\[ a) \text{ index } (-i\mathcal{D}) = \frac{i^n}{(4\pi)^n n!} \int d^{2n} x \varepsilon^{\mu_1 \cdots \mu_{2n}} \text{Tr} F_{\mu_1 \mu_2} \star F_{\mu_3 \mu_4} \cdots \star F_{\mu_{2n-1} \mu_{2n}}, \]

\[ b) \text{ index } (-i\mathcal{D}) = \frac{(-i)^n}{(4\pi)^n n!} \int d^{2n} x \varepsilon^{\mu_1 \cdots \mu_{2n}} \text{Tr} F_{\mu_1 \mu_2} \star F_{\mu_3 \mu_4} \cdots \star F_{\mu_{2n-1} \mu_{2n}}, \]

respectively, for cases \( a) \) and \( b) \). Note the nonlocal character of the index density in general.

To end this section we will make some comments regarding the structure of equations (9). The classical counterpart of this equation runs

\[ \partial_\mu \hat{j}^5_\mu = - \sum_{i,j} (A_{\mu ij} \star \hat{j}^5_{\mu ji} - \hat{j}^5_{\mu ji} \star A_{\mu ji}), \] (11)

This is a consequence of the fact that the Lagrangian

\[ \mathcal{L}(x) = i \left( \bar{\psi}_i \gamma_5 \psi_i \right) + i \left( \bar{\psi}_i \gamma_5 \psi_j \right), \] (12)

is invariant under the following global abelian chiral transformations

\[ \psi_j' = e^{i\theta \gamma_5} \psi_j, \quad \bar{\psi}_j' = \bar{\psi}_j e^{i\theta \gamma_5}; \] (13)

as can be seen by employing Noether’s theorem modified so that the noncommutativity of the Moyal product is taken into account. Note that \( \mathcal{L} \) yields (the fermion fields are always Grassmann variables) the action in equation (2) for the group \( U(N) \). Now, equation (11) can be traded for a simpler equation if instead of the current \( \hat{j}^5_\mu \), defined in \( a) \) of equation (10), one uses the current

\[ \hat{j}^5_\mu = \sum_i (\bar{\psi}_\beta i \star \psi_\alpha i \gamma_5)_{\alpha\beta}. \]

Indeed, taking into account that

\[ \hat{j}^5_\mu = j^5_\mu - \left[ (\bar{\psi}_\beta i, \psi_\alpha i) + (\gamma_5)_{\alpha\beta} \right], \]

where \( [f, g]_+ = f \star g + g \star f \), and imposing the equation of motion of the fields, one readily shows that

\[ \partial_\mu \hat{j}^5_\mu = \partial_\mu j^5_\mu - \sum_{i,j} (A_{\mu ij} \star j^5_{\mu ji} - j^5_{\mu ji} \star A_{\mu ji}). \] (14)

Hence, equations (11) and (14) lead to

\[ \partial_\mu \hat{j}^5_\mu = 0, \]

i.e., the current \( \hat{j}^5_\mu \) is classically conserved. This conservation can be derived, by using Noether’s theorem, from the invariance of the Lagrangian

\[ \hat{\mathcal{L}}(x) = -i \bar{\psi}_\alpha i \star (\partial \psi_i) + i \bar{\psi}_\alpha i \star (\bar{A}_{ij} \star \psi_j), \] (15)
under the global transformations defined in equation (13).

Notice that both $\mathcal{L}$ and $\hat{\mathcal{L}}$, defined in equations (12) and (15), respectively, yield the same action since they differ by a Moyal bracket (i.e., a total derivative) of the fields and its derivatives:

$$\hat{\mathcal{L}} = \mathcal{L} + i[(\partial \psi_i)_\alpha - (\mathfrak{A}_{ij} \star \psi_j)_\alpha, \bar{\psi}_\alpha].$$

Putting it all together we conclude that if the equations of motion hold at the quantum level the right hand side of $a)$ of equation (9) can be recast into the form

$$\langle \partial_\mu \hat{\psi}_5 \rangle = 2 \frac{(-i)^{n-1}}{(4\pi)^n} \frac{1}{n!} \varepsilon^{\mu_1 \cdots \mu_{2n}} \text{Tr} \, F_{\mu_1 \mu_2} \star F_{\mu_3 \mu_4} \star \cdots \star F_{\mu_{2n-1} \mu_{2n}}.$$

Similar arguments can be put forward for case $b)$ of equation (9).

3. The Nonabelian Anomaly on Noncommutative $\mathbb{R}^4$

In this section we discuss the lack of gauge invariance of the effective action, $\Gamma[A]$, of Weyl fermions coupled to $\text{U}(N)$ gauge fields on noncommutative $\mathbb{R}^4$. Let $\psi_R$ be a right-handed (say) fermion, i.e., $\psi_R = P_+ \psi$, $\psi$ being a Euclidean Dirac fermion, and $P_+ = \frac{1}{2} (1 + \gamma_5)$. In noncommutative $\mathbb{R}^4$ there are two basic infinitesimal gauge transformations of $\psi_R$ under the gauge group $\text{U}(N)$, namely

$$a) \ (\delta_\theta \psi_R)_i = \theta_{ij} \star \psi_R^j, \quad b) \ (\delta_\theta \psi_R)_i = -\psi_R^j \star \theta_{ji}, \quad (16)$$

where $\theta_{ij} = -\theta_{ji}^*$ and $i, j = 1, \ldots, N$ are the $\text{U}(N)$ matrix indices. To each of these gauge transformations there is associated a Dirac operator twisted with $\text{U}(N)$ gauge field. These Dirac operators act on $\psi_R$ as follows

$$a) \ (\hat{D} \psi_R)_i = (\partial \psi_R)_i - \mathfrak{A}_{ij} \star \psi_R^j, \quad b) \ (\hat{D} \psi_R)_i = (\partial \psi_R)_i + \gamma_\mu \psi_R^j \star A_{\mu ji}, \quad (17)$$

where the gauge field $A_\mu$ is an antihermitian $N \times N$ matrix. Each Dirac operator give rise to a classical action which can be written in the form

$$S = \int d^4 x \, \bar{\psi} \hat{\Psi}(A)_+ \psi, \quad (18)$$

where $\hat{\Psi}(A)_+ = \hat{\Psi}(A)P_+$. Since $\hat{\Psi}(A)_+P_+$ has not a well-defined eigenvalue problem, the partition function obtained from $S$ in equation (18) cannot be defined as the determinant of $\hat{\Psi}_+$. Hence, we cannot express $\Gamma[A]$, the effective action of the right-handed fermion $\psi_R$ in the $\text{U}(N)$ background field $A_\mu$, in terms of the determinant of $\hat{\Psi}(A)_+$. This is an unwelcomed feature of the action (18). In a classic paper [13] Alvarez-Gaumé and Ginsparg taught us how to define $\Gamma[A]$ in terms of a determinant. Following these authors we shall replace $\hat{\Psi}(A)_+$ in equation (18) with a non-hermitian operator $\hat{D}(A)$ —or simply $\hat{D}$— defined as follows

$$a) \ (\hat{D} \psi)_i = \partial \psi_i - \mathfrak{A}_{ij} P_+ \star \psi_j, \quad b) \ (\hat{D} \psi)_i = \partial \psi_i + \gamma_\mu P_+ \psi_j \star A_{\mu ji}. \quad (19)$$
Notice that the operators in a) and b) correspond, respectively, to the Dirac operator in a) and b) of (17). The operators in a) and b) of equation (19) have a formally well-defined eigenvalue problem since they are still elliptic operators (the noncommutative character of the space does not modify the principal symbol of these operators as regards to their counterparts in commutative Euclidean space), with compact inverse, though they are not hermitian. Taking into account that in equation (19) only the right-handed degrees of freedom couple to the gauge field, we then define $\Gamma[A]$ as follows

$$e^{-\Gamma[A]} = \int d\bar{\psi}d\psi e^{-\int d^4x \bar{\psi}i\hat{D}(A)\psi} = \det i\hat{D}(A). \quad (20)$$

We next compute the variation of $\Gamma[A]$ as defined by equation (20) under gauge transformations; we shall give a detailed discussion only for $\hat{D}(A)$ as given by a) in equation (19). The results for case b) can be easily retrieved from the results for case a) by making the appropriate changes. It can be readily seen that the exponential factor in the integrand of equation (20) is invariant under the gauge transformations

$$\delta_\theta A_{\mu ij} = \partial_\mu \theta_{ij} - [A_\mu, \theta]_{ij}, \quad (\delta_\theta \psi)_i = \theta_{ij} \psi_{Rj}, \quad (\delta_\theta \bar{\psi})_i = -\bar{\psi}_{Lj} \theta_{ji}, \quad (21)$$

where $\psi_L = \frac{i}{2}(1 - \gamma_5)\psi$. Hence, $\Gamma[A]$ can fail to be gauge invariant only if the fermionic measure fails to do so.

To compute the change of the fermionic measure, $d\mu = d\bar{\psi}d\psi$ under the chiral gauge transformations in equation (21), we shall define it as done in reference [13]. One introduces first right, $\{\varphi_n\}$, and left, $\{\chi_n^\dagger\}$, eigenfunctions of $i\hat{D}(A)$ in a) of equation (19), defined as follows

$$i\hat{D}(A)\varphi_n = \lambda_n \varphi_n, \quad (i\hat{D}(A))^\dagger \chi_n = \lambda_n^* \chi_n, \quad \int d^4x \chi_m^\dagger(x)\varphi_m(x) = \delta_{nm}. \quad (22)$$

Then one expands $\psi$ and $\bar{\psi}$ in terms of these eigenfunctions

$$\psi(x) = \sum_n a_n \varphi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \chi_n^\dagger(x);$$

so that the fermionic measure reads $d\mu = \prod_n d\bar{b}_n da_n$. The variation of this measure under the gauge transformations in equation (21) is given by

$$A(\theta, A) = -\sum_n \int d^4x (\chi_n^\dagger \star \theta_{ij} \star \gamma_5 \varphi_n)(x). \quad (22)$$

We end up with the following formal equation

$$\delta_\theta \Gamma[A] = A(\theta, A),$$
which yields the variation under a gauge transformation of the effective action as defined in equation (20). However, the right hand side of equation (22) is not well-defined and needs regularization. We redefine $A$ always following reference [13]

$$A(\theta, A) = -\int d^4x \lim_{\Lambda \to \infty} \sum_m \chi_n^{\dagger} \gamma_{n\alpha i} \star \theta_{ij} \gamma_{5\alpha \beta} e^{-\frac{\lambda^2}{\Lambda^2} \varphi_n} \beta_j$$

Expanding $\varphi_n$ and $\chi_n^{\dagger}$ in plane waves, we recast the expression (23) into the form

$$A(\theta, A) = -\int d^4x \theta_{ij}(x) \lim_{\Lambda \to \infty} \int \frac{d^4p}{(2\pi)^4} \text{tr} \gamma_5 \left( \left( e^{\frac{\lambda^2}{\Lambda^2} \varphi_n} \right)_{ij} \star e^{-ipx} \right).$$

A little algebra turns the previous equation into the following one

$$A(\theta, A) = -\int d^4x \theta_{ij}(x) \lim_{\Lambda \to \infty} \int \frac{d^4p}{(2\pi)^4} \left[ \text{tr} \left( P_+ \left( e^{\frac{\lambda^2}{\Lambda^2} \varphi_n} \right)_{ij} \star e^{-ipx} \right) \right] - \text{tr} \left( P_- \left( e^{\frac{\lambda^2}{\Lambda^2} \varphi_n} \right)_{ij} \star e^{-ipx} \right).$$

Here $i\not{D}$ denotes the Dirac operator introduced in equation (17) a). That leads to

$$A(\theta, A) = -\int d^4x \theta_{ij}(x) \lim_{\Lambda \to \infty} \int \frac{d^4p}{(2\pi)^4} \left[ \text{tr} \left( P_+ \left( e^{\frac{\lambda^2}{\Lambda^2} \varphi_n} \right)_{ij} \star e^{-ipx} \right) \right] - \text{tr} \left( P_- \left( e^{\frac{\lambda^2}{\Lambda^2} \varphi_n} \right)_{ij} \star e^{-ipx} \right).$$

Here $C(\Lambda, A, \theta)$ stands for an even parity funcional which is a polynomial —in the Moyal product— of the gauge field and its derivatives, and it is quadratically divergent with $\Lambda$. This even parity contribution gives rise to a renormalization of the effective action $\Gamma[A]$.

The odd parity contribution to $A(\theta, A)$ is the intrinsic form of the nonabelian anomaly in noncommutative $\mathbb{R}^4$. It cannot be expressed as the gauge variation of an integrated polynomial in the Moyal product of the fields and its derivatives as long as the $U(N)$ groups come in direct sums of the fundamental representation. We note here that this is the case in the Connes-Lott and Chamseddine-Connes noncommutative geometry formulations of
the Standard Model [14]. We close this section by remarking that the anomaly we have obtained is the so-called consistent anomaly, for it satisfy the Wess-Zumino consistency conditions [15]  

$$\delta_{\lambda,\theta}(\theta, A) - \delta_{\theta}(\lambda, A) = -A([\lambda,\theta], A),$$  

where $[\lambda,\theta]_{ij} = \lambda_{ik} \ast \theta_{kj} - \theta_{ik} \ast \lambda_{kj}$. One can of course pass to the gauge covariant formulation by adding to the nonabelian chiral current a suitable non-covariant polynomial in $A$.

The part quadratic in the gauge potentials in equation (24) can be rewritten as

$$- \frac{1}{96\pi^2} \int d^4x \varepsilon_{\mu_1\mu_2\mu_3\mu_4} \text{Tr} T^a[T^b,T^c] \partial_{\mu_1} \theta^a \left[ A^b_{\mu_2} \ast \partial_{\mu_3} A^c_{\mu_4} - \partial_{\mu_3} A^c_{\mu_4} \ast A^b_{\mu_2} \right]$$

$$- \frac{1}{96\pi^2} \int d^4x \varepsilon_{\mu_1\mu_2\mu_3\mu_4} \text{Tr} T^a \{T^b,T^c\} \partial_{\mu_1} \theta^a \left[ A^b_{\mu_2} \ast \partial_{\mu_3} A^c_{\mu_4} + \partial_{\mu_3} A^c_{\mu_4} \ast A^b_{\mu_2} \right].$$

The first term is a new contribution that does not vanish in the noncommutative case.

4. Conclusions

The form of the nonabelian anomaly in equation (24) is a naïve $\ast$-deformation of the consistent anomaly in commutative Euclidean space. However, this apparently innocuous operation has physical consequences. Indeed, the vanishing of the anomalous contribution from triangle diagrams in commutative euclidean space demands the famous anomaly cancellation condition $\text{Tr} T^a \{T^b,T^c\} = 0$ to hold. The reader is invited to check that the term left after imposing $\text{Tr} T^a \{T^b,T^c\} = 0$ in the anomalous gauge field quadratic part of equation (24) is not (modulo $A^3$) a $\delta_{\theta}$-exact polynomial in the Moyal product of the gauge field and its derivatives. Therefore, in noncommutative $\mathbb{R}^4$ the anomalous contribution from the triangle diagrams (i.e., the contribution in (24) which is quadratic in the gauge fields) will vanish if, and only if, $\text{Tr} T^a T^b T^c = 0$. This is a consequence of the noncommutative character of the Moyal product.

As a further curiosity, notice that the pure $SU(2)$ contribution to the nonabelian anomaly, which vanishes on a commutative $\mathbb{R}^4$, does not vanish for a noncommutative $\mathbb{R}^4$. Recall that the left-handed fermions of the Standard Model are $SU(2)$ doublets.

We conclude that if our theory on noncommutative $\mathbb{R}^4$ has a right-handed fermion transforming under the $U(N)$ group as in $a)$ of equation (16), the only chance of having an anomaly free theory is to have a left-handed fermion transforming in the same manner. We see below that having a right-handed fermion transforming as in $b)$ of equation (16) can be turned into having a left-handed fermion transforming as in $a)$ of equation (16) by means of charge conjugation [16]. The theory will thus be a vector theory. This result has been already established for the group $U(1)$ in [17].

Therefore, to close we shall compute the behaviour under a gauge transformation of the effective action $\Gamma[A]$ when $i\hat{D}$ in equation (20) is given by the elliptic operator in $b)$ of equation (19). This situation corresponds to having a right-handed fermion transforming as given in $b)$ equation (16). One realizes that the situation at hand can be converted into
the previous one by expressing the former in terms of $\star_{-\omega}$ by using $f \star_{\omega} g = g \star_{-\omega} f$ and performing the substitutions $A_{\mu ij} \rightarrow -A_{\mu ji}$. Thus, we can use formula (24) to obtain

$$
\delta_\theta \Gamma[A] = -\frac{1}{24\pi^2} \text{Tr} \int d^4x \varepsilon_{\mu_1\mu_2\mu_3\mu_4} \theta \partial_{\mu_1} [A_{\mu_2} \star A_{\mu_3} A_{\mu_4} - \frac{1}{2} A_{\mu_2} \star A_{\mu_3} \star A_{\mu_4}] + \delta_\theta C(A, A, \theta).
$$

Notice that the intrinsic anomaly given by the previous equation and the intrinsic anomaly given in equation (24) have opposite sign. Hence, a theory with two right-handed fermions transforming, respectively, as defined by $a)$ and $b)$ in equation (16) will be anomaly free. This theory is equivalent to a theory with a right-handed fermion and a left-handed fermion both transforming as given by $a)$ in equation (16): just define $\psi_R = \tilde{\psi}_L^c$, where $c$ stands for charge conjugation, for a right-handed fermion $\tilde{\psi}$ transforming as in $a)$ of equation (16).

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