AN EXPLICIT POLYNOMIAL ANALOGUE OF ROMANOFF’S THEOREM

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Abstract. Given a polynomial $g$ of positive degree over a finite field, we show that the proportion of polynomials of degree $n$, which can be written as $h + g^k$, where $h$ is an irreducible polynomial of degree $n$ and $k$ is a nonnegative integer, has order of magnitude $1/\deg g$.

1. Introduction

Given an integer $a \geq 2$, the celebrated result of Romanoff [22] asserts that a positive proportion of integers can be written in the form $p + a^k$, where $p$ is prime. In the prominent case $a = 2$, this has been made explicit by Pintz [20] who shows that this proportion is at least 0.09368, which improves estimates by several other authors [9, 11, 18].

Lately, there has been a burst of activity in analytic number theory related to polynomials over finite fields, with a wide range of results modeling many important theorems and open conjectures for the integers; see [2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 15, 21, 23] and the references therein. In this area, monic irreducible polynomials play the role of prime numbers, and monic polynomials of degree $n$ over a finite field correspond to integers of approximate size $q^n$, where $q$ is the order of the field.

Motivated by this recent trend, we give an explicit analogue of Romanoff’s Theorem for polynomials over finite fields. Let $\mathbb{F}_q$ be the finite field with $q$ elements. For $g \in \mathbb{F}_q[x]$, let $R(n, g, q)$ denote the number of monic polynomials $f \in \mathbb{F}_q[x]$ of degree $n$, which can be written in the form $f = h + g^k$, where $h$ is a monic irreducible polynomial of degree $n$ and $k$ is a nonnegative integer. Let

$$r(n, g, q) = \frac{R(n, g, q)}{q^n},$$

the proportion of monic polynomials $f$ of degree $n$, which can be written this way. Since there are close to $q^n/n$ choices for $h$, and about
n/\deg g choices for \(k\), one might expect \(r(n, g, q)\) to be approximately of size \(1/\deg g\). This is in fact the case.

**Theorem 1.1.** Let \(\gamma\) denote Euler’s constant and \(\delta = \deg g\). Uniformly for \(g \in \mathbb{F}_q[x]\), \(n \geq 1\), \(\delta \geq 1\), \(q \geq 2\), we have

\[
\frac{1 + \delta/n}{\delta} \geq r(n, g, q) > \frac{(1 - 2q^{-n/2})^2 (1 + \delta/n)^{-1}}{\delta + 8 \frac{q}{q-1} \left(1 + e^{\gamma} \min \left\{5\sqrt{\delta/q}, \frac{\log 6\delta}{\log q}\right\}\right)}.
\]

We now present three straightforward implications of Theorem 1.1, all of which hold uniformly for \(g \in \mathbb{F}_q[x]\), \(n \geq \delta \geq 1\), \(q \geq 2\). First, we note that (1.1) yields the estimate

\[
\frac{1 + \delta/n}{\delta} \geq r(n, g, q) > \frac{(1 + \delta/n)^{-1}}{\delta} \left(1 + O \left(\frac{\log 2\delta}{\delta}\right)\right),
\]

which shows that \(r(n, g, q) \sim 1/\delta\) provided \(\delta \to \infty\) and \(\delta/n \to 0\). Another immediate consequence of (1.1) is

\[
\frac{1 + \delta/n}{\delta} \geq r(n, g, q) > \frac{(1 + \delta/n)^{-1}}{\delta + 8} \left(1 + O \left(\frac{1}{\sqrt{q\delta}}\right)\right),
\]

which gives good bounds for \(r(n, g, q)\) as soon as \(q\) is large. Finally, a few basic observations at the end of Section 4.1 show that (1.1) implies the simple explicit bounds

\[
\frac{2}{\delta} \geq r(n, g, q) > \frac{0.01}{\delta}.
\]

The number 8 in the estimates (1.1) and (1.3) comes directly from the factor 8 in Lemma 3.5, an explicit upper bound, due to Pollack [21], for the number of monic irreducible pairs with a given difference. Any improvement of the constant 8 in Lemma 3.5 would lead immediately to a corresponding improvement in (1.1) and (1.3).

The proof of Theorem 1.1 is modeled after the original paper by Romanoff [22]. A central ingredient in Romanoff’s proof is an upper bound for the series

\[
\sum_{\substack{n \geq 1 \\ \gcd(n, a) = 1}} \frac{\mu^2(n)}{n \ord_n(a)},
\]

where \(\ord_n(a)\) denotes the multiplicative order of \(a\) modulo \(n\), and \(\mu\) is the Möbius function. To obtain an explicit upper bound for the analogous series in the polynomial case, we adapt the simpler strategy of Murty, Rosen and Silverman [19], who give estimates for sums similar to (1.5), as well as analogous results over number fields and abelian
varieties. Kuan [16] further extends the results in [19] to Drinfeld modules.

Theorem 1.1 also applies to non-monic polynomials. Let \( \tilde{R}(n, g, q) \) denote the number of (not necessarily monic) polynomials \( f \in \mathbb{F}_q[x] \) of degree \( n \), which can be written as \( f = h + g^k \), where \( h \) is an irreducible (not necessarily monic) polynomial of degree \( n \) and \( k \) is a nonnegative integer. We define

\[
\tilde{r}(n, g, q) = \frac{\tilde{R}(n, g, q)}{(q - 1)q^n},
\]

which is the proportion of polynomials \( f \) of degree \( n \), which can be written this way.

**Theorem 1.2.** Theorem 1.1 remains valid if \( r(n, g, q) \) is replaced by \( \tilde{r}(n, g, q) \).

The estimates (1.2), (1.3) and (1.4) also hold if \( r(n, g, q) \) is replaced by \( \tilde{r}(n, g, q) \).

2. The upper bound

2.1. Monic polynomials. We start with the upper bound in (1.1), which is quite elementary. Let \( I_q(n) \) be the set of monic irreducible polynomials of degree \( n \) over \( \mathbb{F}_q \). The number denoted by \( I_q(n) = \#I_q(n) \), satisfies (see [17, Theorem 3.25])

\[
\frac{q^n}{n} - \frac{2q^{n/2}}{n} < I_q(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right)q^d \leq \frac{q^n}{n} \quad (n \geq 1).
\]

Since \( f = g^k + h \) and \( h \) both have degree \( n \), we have \( 0 \leq \deg g^k = k\delta \leq n \). In the monic case, there are \( I_q(n) \) choices for \( h \) and at most \( 1 + \lceil n/\delta \rceil \) choices for \( k \), so

\[
R(n, g, q) \leq I_q(n) \left(1 + \left\lceil \frac{n}{\delta} \right\rceil \right) \leq \frac{q^n}{n} \left(1 + \frac{n}{\delta} \right) = \frac{q^n}{\delta} \left(1 + \frac{\delta}{n} \right),
\]

by (2.1).

2.2. Arbitrary polynomials. Similarly, in the non-monic case we have

\[
\tilde{R}(n, g, q) \leq (q - 1)I_q(n) \left(1 + \left\lfloor \frac{n}{\delta} \right\rfloor \right) \leq \frac{(q - 1)q^n}{\delta} \left(1 + \frac{\delta}{n} \right).
\]
3. Auxiliary results

3.1. Bounds of some products. For \( f \in \mathbb{F}_q[x] \), let \(|f| = q^\deg f\). Throughout, \( p \) stands for a monic irreducible polynomial in \( \mathbb{F}_q[x] \). We define

\[
E(f) = \prod_{p|f} \left( 1 + \frac{1}{|p|} \right).
\]

**Lemma 3.1.** Let \( f \in \mathbb{F}_q[x] \). If \( m \geq 1 \) and \( q^m \geq \deg f \), then

\[
E(f) \leq \prod_{\deg p \leq m} \left( 1 + \frac{1}{|p|} \right).
\]

**Proof.** \( E(f) \) is maximized if \( f \) has as many distinct irreducible factors of small degree as possible. Since

\[
\deg \prod_{\deg p \leq m} p = \sum_{1 \leq k \leq m} k I_q(k) \geq \sum_{k|m} k I_q(k) = q^m \geq \deg f,
\]

the result follows. \( \square \)

**Lemma 3.2.** Let \( H_m = \sum_{k=1}^{m} \frac{1}{k} \) be the \( m \)-th harmonic number. For \( m \geq 1 \),

\[
\prod_{\deg p \leq m} \left( 1 + \frac{1}{|p|} \right) < \exp(H_m).
\]

**Proof.** We have

\[
\sum_{\deg p \leq m} \log \left( 1 + \frac{1}{|p|} \right) < \sum_{\deg p \leq m} \frac{1}{|p|} = \sum_{k \leq m} \frac{I_q(k)}{q^k} \leq \sum_{k \leq m} \frac{1}{k} = H_m,
\]

by (2.1). The result follows from exponentiation. \( \square \)

**Lemma 3.3.** For \( m \geq 1 \) we have

\[
\exp(H_m) \leq e + e^\gamma (m - 1) < 1 + e^\gamma m.
\]

**Proof.** The first inequality appears in Batir [8, Cor. 2.2]. The second inequality follows from \( e - e^\gamma = 0.9372... < 1 \). \( \square \)

**Lemma 3.4.** Let \( f \in \mathbb{F}_q[x] \), \( \deg f \geq 2 \). We have

\[
E(f) \leq 1 + e^\gamma \min \left\{ \frac{\deg f}{q}, \frac{\log(\deg f)}{\log q} \right\}.
\]

**Proof.** Let \( \varphi = \deg f \). If \( \varphi \leq q \), we have

\[
E(f) \leq \left( 1 + \frac{1}{q} \right)^{\varphi} \leq \exp \left( \frac{\varphi}{q} \right) \leq 1 + (e - 1)\frac{\varphi}{q} < 1 + e^\gamma \frac{\varphi}{q}.
\]
Since $q/\log q$ is increasing for $q \geq 3$, and $2/\log 2 = 4/\log 4$, we have $\varphi/q \leq \log \varphi/\log q$ if $\varphi \leq q$, unless $(q, \varphi) = (3, 2)$, in which case $E(f) \leq (1 + 1/3)^2 < 1 + e^{\gamma} \log 2/\log 3$. Thus the result holds for $\varphi \leq q$.

If $\varphi = q^m$ for some integer $m \geq 2$, the result follows from combining Lemmas 3.1, 3.2 and 3.3, since $\varphi/q \geq \log \varphi/\log q$ for $\varphi \geq q^2 \geq 4$.

In the remaining case, $q^m < \varphi < q^{m+1}$ for some integer $m \geq 1$. We write

$$\varphi = q^m + \alpha (q^{m+1} - q^m)$$

for some $0 < \alpha < 1$. Since $E(f)$ is maximized if $f$ has as many distinct irreducible factors of small degree as possible, and

$$\deg \prod_{\deg p \leq m} p \geq q^m$$

(see the proof of Lemma 3.1), we have

$$E(f) \leq \prod_{\deg p \leq m} \left(1 + \frac{1}{|p|}\right) \cdot \left(1 + \frac{1}{q^{m+1}}\right) \frac{\varphi - q^m}{q^{m+1}}.$$ 

Taking logarithms and using Lemma 3.2 yields

$$\log(E(f)) \leq H_m + \frac{\alpha}{m + 1}.$$ 

The inequality

$$H_m + \frac{\alpha}{m + 1} \leq \log(1 + e^{\gamma}(m + \alpha))$$

holds for $\alpha = 0, 1$ by Lemma 3.3, and for $0 < \alpha < 1$ it follows from the concavity of the logarithm. As a result,

$$E(f) \leq 1 + e^{\gamma}(m + \alpha).$$

We have

$$\frac{\log \varphi}{\log q} = m + \frac{\log(1 + \alpha(q - 1))}{\log q} \geq m + \alpha,$$

where the last inequality is obvious for $\alpha = 0, 1$, and for $0 < \alpha < 1$ it follows again from the concavity of the logarithm. Consequently,

$$E(f) \leq 1 + e^{\gamma} \frac{\log \varphi}{\log q}.$$ 

Since $\varphi > q$, we have $\varphi/q \geq \log \varphi/\log q$, unless $(q, \varphi) = (2, 3)$, in which case $E(f) \leq (1 + 1/2)^2 < 1 + e^{\gamma}3/2$. This completes the proof. □
3.2. Irreducibility of shifted irreducible polynomials. As before, let \( I_q(n) \) be the set of monic irreducible polynomials of degree \( n \) over \( \mathbb{F}_q \), and let \( \tilde{I}_q(n) \) be the set of arbitrary irreducible polynomials of degree \( n \) over \( \mathbb{F}_q \).

For \( f \in \mathbb{F}_q[x] \), let

\[
A(f, n) = \# \{ h \in \mathbb{F}_q[x] : (h, h + f) \in I_q(n) \times I_q(n) \},
\]

\[
\tilde{A}(f, n) = \# \{ h \in \mathbb{F}_q[x] : (h, h + f) \in \tilde{I}_q(n) \times \tilde{I}_q(n) \}.
\]

The following explicit upper bound for \( A(f, n) \) is due to Pollack \([21, Lemma 2]\).

**Lemma 3.5.** Let \( n \geq 1 \) and let \( f \in \mathbb{F}_q[x], f \neq 0, 0 \leq \deg f < n \). Then

\[
A(f, n) \leq \frac{8q^n}{n^2} \prod_{p \mid f} \left( 1 - \frac{1}{|p|} \right)^{-1}.
\]

It is convenient to estimate the last product in terms of \( E(f) \).

**Lemma 3.6.** Let \( n \geq 1 \) and let \( f \in \mathbb{F}_q[x], f \neq 0, 0 \leq \deg f < n \). Then

\[
A(f, n) \leq \frac{8q^n}{n^2(1 - 1/q)} E(f) \quad \text{and} \quad \tilde{A}(f, n) \leq \frac{8q^{n+1}}{n^2} E(f).
\]

**Proof.** The bound for \( A(f, n) \) follows from Lemma 3.5 and

\[
\prod_{p \mid f} \left( 1 - \frac{1}{|p|} \right)^{-1} < \prod_{k \geq 1} \left( 1 - \frac{1}{q^{2k}} \right)^{-I_q(k)} = \sum_{f \in \mathbb{F}_q[x]} \frac{1}{|f|^2} = \frac{1}{1 - 1/q}.
\]

Since

\[
\tilde{A}(f, n) = \sum_{\alpha \in \mathbb{F}_q^\times} A(\alpha^{-1}f, n),
\]

the bound on \( \tilde{A}(f, n) \) follows from the bound on \( A(f, n) \). \( \square \)

4. The lower bounds

4.1. Monic polynomials. The first half of the proof is modeled after Romanoff \([22]\). For \( f \in \mathbb{F}_q[x] \), let

\[
B(f, n) = \# \{ (k_1, k_2) : g^{k_1} - g^{k_2} = f, 0 \leq \delta k_i < n \},
\]

and

\[
C(f, n) = \# \{ (h, k) : h + g^k = f, h \in I_q(n), 0 \leq \delta k < n \}.
\]

We count in two different ways the solutions to \( g^{k_1} - g^{k_2} - h_1 + h_2 = 0 \), where the \( h_i \) are monic irreducible of degree \( n \) and \( 0 \leq \delta k_i < n \). First,
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counting according to $f = g^{k_1} - g^{k_2} = h_1 - h_2$ shows that the number of solutions is $\sum_{f \in \mathbb{F}_q[x]} A(f, n)B(f, n)$. Second, counting according to $f = g^{k_1} + h_2 = g^{k_2} + h_1$, shows that the number of solutions is $\sum_{f \in \mathbb{F}_q[x]} C(f, n)^2$. Thus

$$\sum_{f \in \mathbb{F}_q[x]} C(f, n)^2 = \sum_{f \in \mathbb{F}_q[x]} A(f, n)B(f, n).$$

We have

$$\sum_{f \in \mathbb{F}_q[x]} C(f, n) = I_q(n)\left\lceil \frac{n}{\delta} \right\rceil \geq \frac{q^n}{n} \left( 1 - \frac{2}{q^{n/2}} \right) \frac{n}{\delta} = \frac{q^n}{\delta} \left( 1 - \frac{2}{q^{n/2}} \right).$$

Let $\epsilon(f, n) = 1$ if $C(f, n) \geq 1$ and $\epsilon(f, n) = 0$ otherwise. By the Cauchy–Schwarz inequality we have

$$R(n, g, q) = \sum_{f \in \mathbb{F}_q[x]} \epsilon(f, n)^2 \geq \frac{\left( \sum_{f \in \mathbb{F}_q[x]} C(f, n) \right)^2}{\sum_{f \in \mathbb{F}_q[x]} C(f, n)^2} \geq \frac{q^{2n}\delta^{-2} \left( 1 - 2q^{-n/2} \right)^2}{\sum_{f \in \mathbb{F}_q[x]} A(f, n)B(f, n)}.$$  

(4.1)

We need an upper bound for the last denominator. From Lemma 3.6 we have

$$\sum_{f \in \mathbb{F}_q[x]} A(f, n)B(f, n) \leq I_q(n)\left\lceil \frac{n}{\delta} \right\rceil + \frac{8q^n}{n^2(1 - 1/q)} \sum_{\epsilon \neq 0} E(f)B(f, n).$$

(4.2)

Writing $B(f, n)$ as a sum over $k_1, k_2$, and changing the order of summation, we obtain

$$\sum_{f \in \mathbb{F}_q[x]} A(f, n)B(f, n) \leq \frac{q^n}{n} \left( \frac{n}{\delta} + 1 \right) + \frac{16q^n}{n^2(1 - 1/q)} \sum_{0 \leq k_1 < k_2 < n/\delta} E(g^{k_2} - g^{k_1}) \leq \frac{q^n}{\delta} \left( 1 + \frac{\delta}{n} \right) + \frac{16q^nE(g)}{n^2(1 - 1/q)} \sum_{1 \leq k \leq n/\delta} \left( \frac{n}{\delta} + 1 - k \right) E(g^k - 1),$$
where we put \( k = k_2 - k_1 \). To estimate the last sum, we write

\[
\sum_{1 \leq k \leq n/\delta} \left( \frac{n}{\delta} + 1 - k \right) E(g^k - 1)
\]

\[
= \sum_{1 \leq k \leq n/\delta} \left( \frac{n}{\delta} + 1 - k \right) \prod_{p \mid (g^k - 1)} \left( 1 + \frac{1}{|p|} \right)
\]

\[
= \sum_{1 \leq k \leq n/\delta} \left( \frac{n}{\delta} + 1 - k \right) \sum_{f \mid (g^k - 1)} \frac{\mu^2(f)}{|f|}
\]

\[
= \sum_{\gcd(f,g) = 1} \frac{\mu^2(f)}{|f|} \sum_{1 \leq k \leq n/\delta} \left( \frac{n}{\delta} + 1 - k \right)
\]

\[
\leq \frac{n^2(1 + \delta/n)}{2\delta^2} \sum_{\gcd(f,g) = 1} \frac{\mu^2(f)}{|f| \ord_f(g)}
\]

where the sums are over monic polynomials \( f \) and \( \ord_f(g) \) denotes the multiplicative order of \( g \) modulo \( f \). We have shown that

\[
(4.3) \quad \sum_{f \in \mathbb{F}_q[x]} A(f, n) B(f, n) \leq \frac{q^n}{\delta} \left( 1 + \frac{\delta}{n} \right) \left( 1 + \frac{8E(g)S(g)}{\delta(1 - 1/q)} \right),
\]

where

\[
S(g) = \sum_{\gcd(f,g) = 1} \frac{\mu^2(f)}{|f| \ord_f(g)} = \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{\ord_f(g) = \ell} \frac{\mu^2(f)}{|f|}.
\]

As in [19], we use Abel summation to estimate the last sum. We define

\[
T_g(\ell) = \sum_{\ord_f(g) = \ell} \frac{\mu^2(f)}{|f|}
\]

and consider the function

\[
H_g(x) = \sum_{\ell \leq x} T_g(\ell) = \sum_{\ord_f(g) \leq x} \frac{\mu^2(f)}{|f|}
\]

\[
\leq \sum_{f \mid \prod_{\ell \leq x} (g^\ell - 1)} \frac{\mu^2(f)}{|f|} = E \left( \prod_{\ell \leq x} (g^\ell - 1) \right).
\]
Since $\prod_{\ell \leq x} (g^\ell - 1)$ and $g$ are relatively prime, we have

$$E(g)H_g(x) \leq E(g)E\left(\prod_{\ell \leq x} (g^\ell - 1)\right) = E\left(g \prod_{\ell \leq x} (g^\ell - 1)\right).$$

For $x \geq 1$, define

$$z = \deg \left(g \prod_{\ell \leq x} (g^\ell - 1)\right) = \delta \left(1 + \frac{|x|x + 1}{2}\right) \geq 2.$$

Lemma 3.4 shows that

$$E(g)H_g(x) \leq 1 + e^\gamma \min \left\{\frac{z}{q}, \frac{\log z}{\log q}\right\},$$

which implies $\lim_{x \to \infty} H_g(x)/x = 0$. Abel summation yields

$$E(g)S(g) = E(g) \int_1^\infty \frac{H_g(x)}{x^2} dx \leq 1 + e^\gamma \int_1^\infty \frac{\log(\delta(1 + |x|x + 1)/2)) dx}{x^2} < 1 + e^\gamma \frac{\log \delta + 1.771}{\log q} < 1 + e^\gamma \frac{\log 6\delta}{\log q}.\tag{4.5}$$

If $q \geq 4\delta$, we use $z = 2\delta$ for $1 \leq x < 2$, and $z \leq \delta(1 + x(x + 1)/2) \leq \delta x^2$ for $x \geq 2$. With (4.4) we get

$$E(g)S(g) = E(g) \int_1^\infty \frac{H_g(x)}{x^2} dx \leq 1 + e^\gamma \int_1^2 \frac{2\delta dx}{q x^2} + e^\gamma \int_2^{\sqrt{q/\delta}} \frac{\delta x^2 dx}{q x^2} + e^\gamma \int_{\sqrt{q/\delta}}^\infty \frac{\log(\delta x^2) dx}{\log q x^2} \leq 1 + e^\gamma \left(\delta/q + \sqrt{\delta}/q - 2\delta/q + \frac{2 + \log q}{\sqrt{q/\delta}\log q}\right) < 1 + 5e^\gamma \sqrt{\delta}/q.$$

This estimate also holds if $4\delta > q \geq 2$, because in that case it follows from (4.5) and

$$\frac{5\sqrt{\delta}}{\log 6\delta} > 2\frac{\sqrt{6\delta}}{\log 6\delta} > \frac{\sqrt{q}}{\log q}.$$

To summarize, we have shown that

$$E(g)S(g) < 1 + e^\gamma \min \left\{5\sqrt{\delta}/q, \frac{\log 6\delta}{\log q}\right\}.$$
After inserting this estimate into (4.3), the lower bound in (1.1) follows from (4.1).

It remains to establish the lower bound in (1.4). If \( n/\delta \leq 50 \), this follows from \( \widetilde{R}(n, g, q) \geq I_q(n) \) and the estimate (2.1). Hence we may assume \( n > 50\delta \). Define

\[
\alpha(q, \delta) = 1 + \frac{8}{\delta(1 - 1/q)} \left( 1 + e^{7 \log 6 \log q} \right).
\]

Note that \( \alpha(q, \delta) \) is decreasing in \( q \). For \( q = 2 \), it is decreasing in \( \delta \). Hence \( \alpha(q, \delta) \leq \alpha(2, 1) < 91 \) for all \( q \geq 2, \delta \geq 1 \). Using this estimate in (1.1) establishes the lower bound in (1.4).

4.2. Arbitrary polynomials. For the proof of Theorem 1.2, it remains to show that the lower bound in (1.1) also applies to \( \widetilde{r}(n, g, q) \).

We only indicate which changes are needed to adapt the proof of Theorem 1.1. Replace \( A(f, n) \) by \( \widetilde{A}(f, n) \), and \( C(f, n) \) by

\[
\widetilde{C}(f, n) = \# \{ (h, k) : h + g^k = f, h \in \mathcal{I}_q(n), 0 \leq \delta k < n \}.
\]

Since \( \sum_{f \in \mathbb{F}_q[x]} \widetilde{C}(f, n) = (q - 1) \sum_{f \in \mathbb{F}_q[x]} C(f, n) \), the analogue of (4.1) is

\[
\widetilde{R}(n, g, q) \geq \frac{\left( \sum_{f \in \mathbb{F}_q[x]} \widetilde{C}(f, n) \right)^2}{\sum_{f \in \mathbb{F}_q[x]} \widetilde{A}(f, n) B(f, n)} = \frac{(q - 1)^2 \left( \sum_{f \in \mathbb{F}_q[x]} C(f, n) \right)^2}{\sum_{f \in \mathbb{F}_q[x]} A(f, n) B(f, n)}.
\]

From Lemma 3.6 we have

\[
\sum_{f \in \mathbb{F}_q[x]} \widetilde{A}(f, n) B(f, n) \leq (q - 1) I_q(n) \left[ \frac{n}{\delta} \right] + \frac{8q^n(q - 1)}{n^2(1 - 1/q)} \sum_{f \neq 0} E(f) B(f, n),
\]

which is the same as the right-hand side of (4.2) multiplied by \((q - 1)\). Consequently, all subsequent upper bounds for \( \sum_{f \in \mathbb{F}_q[x]} A(f, n) B(f, n) \) become valid upper bounds for \( \sum_{f \in \mathbb{F}_q[x]} \widetilde{A}(f, n) B(f, n) \), after multiplying by \((q - 1)\). It follows that the lower bound for \( r(n, g, q) \) derived from (4.1) is also a valid lower bound for \( \widetilde{r}(n, g, q) \).

The reasoning at the end of Section 4.1 for the lower bound in (1.4) also applies to \( \widetilde{r}(n, g, q) \).

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