The fundamentals of non-singular dislocations in the theory of gradient elasticity: Dislocation loops and straight dislocations

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1. Introduction

Dislocations play an important role in the understanding of many phenomena in solid state physics, materials science, and engineering. They are the primary carriers of crystal plasticity. Dislocations are line defects which can be straight or curved lines. The internal geometry of generally curved dislocations, in deformed crystals is very complex. In the classical theory of dislocation loops and straight dislocations are investigated. Using the theory of gradient elasticity, the non-singular fields which are produced by arbitrary dislocation loops are given. ‘Modified’ Mura, Peach–Koehler, and Burgers formulae are presented in the framework of gradient elasticity theory. These formulae are given in terms of an elementary function, which regularizes the classical expressions, obtained from the Green tensor of the Helmholtz–Navier equation and bi-Helmholtz–Navier equation. Using the mathematical method of Green’s functions and the Fourier transform, exact, analytical, and non-singular solutions were found. The obtained dislocation fields are non-singular due to the regularization of the classical singular fields.

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lengths. The capability of strain gradient theories in capturing size effects is a direct manifestation of the involvement of characteristic lengths. Simplified versions, which are particular cases of Mindlin’s theories, were proposed and used for dislocation modelling. Such simplified gradient elasticity theories are known as gradient elasticity of Helmholtz type (Lazar and Maugin, 2005), with only one material length scale parameter and gradient elasticity of bi-Helmholtz type (Lazar and Maugin, 2006a; Lazar et al., 2006a) which involves two material length scale parameters as new material coefficients. Gradient elasticity is a continuum model of dislocations with core spreading. Non-singular fields of straight dislocations were obtained in the framework of gradient elasticity of Helmholtz type by Gutkin and Afantis (1999); Lazar and Maugin (2005); Lazar and Maugin (2006a); Lazar et al. (2005) and Gutkin (2000); Gutkin (2006) (see also, Gutkin and Ovid’ko, 2004). Surprisingly enough up until now, not a single work has been done which involves two material length scale parameters as new material coefficients. Gradient elasticity theory evolving from Mindlin’s general gradient elasticity theory is called gradient elasticity. A simplified gradient elasticity theories are known as gradient elasticity of bi-Helmholtz type is considered. Simplified versions, which are particular cases of Mindlin’s theories, were proposed and used for dislocation modelling. Such theories may be the presence of dislocations. Dislocations cause self-stresses that means stresses caused without the presence of body forces. The dislocation density tensor is defined in terms of the elastic and plastic distortion tensors as follows (e.g. Kröner, 1958)

\[ \sigma_{ij} = e_{ij}/C_{0} \]  

(7)

and it fulfills the Bianchi identity of dislocations

\[ \partial_{i} \sigma_{ij} = 0, \]  

(9)

which means that dislocations do not end inside the body. Eq. (9) is a ‘conservation’ law and shows that dislocations are source-free fields.

From Eq. (1) it follows that the constitutive equations are

\[ \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \frac{\partial W}{\partial \epsilon_{ij}} = C_{ijkl} \epsilon_{kl} = C_{ijkl} \epsilon_{ij}. \]  

(10)

where \( \sigma_{ij} \) are the components of the Cauchy stress tensor, \( \tau_{ij} \) are the components of the so-called double stress tensor. It can be seen that \( \tau \) is the characteristic length scale for double stresses. Using Eqs. (10) and (11), Eq. (5) can also be written as (Lazar and Maugin, 2005)

\[ W = \frac{1}{2} \sigma_{ij} \epsilon_{ij} + \frac{1}{2} \epsilon^{2} \partial_{k} \sigma_{ij} \partial_{ij}. \]  

(12)

The strain energy density (12) exhibits the symmetry both in \( \sigma_{ij} \) and \( \epsilon_{ij} \) and in \( \partial_{i} \sigma_{ij} \) and \( \partial_{i} \epsilon_{ij} \).

The total stress tensor is given as a combination of the Cauchy stress tensor and the divergence of the double stress tensor

\[ \sigma_{ij}^{D} = \sigma_{ij} - \partial_{k} \tau_{ij} = (1 - \epsilon^{2} \Delta) \sigma_{ij} \]  

(13)

and it fulfills the equilibrium condition for vanishing body forces

\[ \partial_{i} \sigma_{ij}^{D} = 0. \]  

(14)

The stress tensor \( \sigma_{ij}^{D} \) is called in the notation of Jaunzemis (1967) the polarization of the Cauchy stress \( \sigma_{ij} \). Due to gradient elasticity of Helmholtz type, Eq. (13) reduces to an inhomogeneous Helmholtz equation where the total stress tensor is the inhomogeneous piece. As pointed out by Lazar and Maugin (2005); Lazar and Maugin (2006a), the total stress tensor may be identified with the singular classical stress tensor. This identifies that the inhomogeneous Helmholtz equation (13) is in full agreement with the equation
for the stress proposed by Eringen (1983); Eringen (2002) in his theory of nonlocal elasticity of Helmholtz type.

As shown by Lazar and Maugin (2005); Lazar and Maugin (2006a) the following governing equations for the displacement vector, the elastic distortion tensor, the dislocation density tensor, and the plastic distortion tensor can be derived in the framework of gradient elasticity of Helmholtz type

\[
Lu_0 = u_0, \quad (15) \\
L\beta_{ij}' = \beta_{ij}'', \quad (16) \\
LA\gamma = \gamma', \quad (17) \\
L\gamma_i'' = \gamma_i''', \quad (18)
\]

where

\[
L = 1 - \ell^2 \Delta
\]

is the Helmholtz operator. The singular fields \(u_0, \beta_{ij}', \gamma, \) and \(\gamma_i''\) are the sources of the inhomogeneous Helmholtz Equation (15)–(18). The Helmholtz Equation (15) and (16) can be further reduced to Helmholtz–Navier equations

\[
LLu_0 = C_{ijkl} \partial_j \beta_{kl}''', \quad (20) \\
LL\beta_{ij}' = -C_{ijkl} \gamma_{ik} \partial_j \gamma_{lk}'', \quad (21)
\]

where \(L = C_{ijkl} \partial_j \partial_i \) is the differential operator of the Navier equation. For an isotropic material, it reads

\[
L = \mu \delta_{\alpha \Delta} + (\mu + \lambda) \partial_k \partial_k.
\]

In Eqs. (20) and (21) the sources are now the plastic distortion \(\beta_{kl}''',\) and the dislocation density \(\gamma_{ij}''',\) known from classical elasticity.

The corresponding three-dimensional Green tensor of the Helmholtz–Navier equation is defined by

\[
G(R) = \frac{1}{16\pi\mu(1 - v)} \left[ 2(1 - v) \delta_0 \Delta - \partial_i \partial_j \right] A(R), \quad (24)
\]

with

\[
A(R) = R + \frac{2\ell^2}{R} \left(1 - e^{-R/\ell}\right)
\]

and \(R = |x - x'|.\) In the limit \(\ell \to 0,\) the three-dimensional Green tensor of classical elasticity (Mura, 1987; Li and Wang, 2008) is recovered from Eqs. (24) and (25). It is important to note that \(A(R)\) can be written as the convolution of \(R\) and \(G(R)\):

\[
A(R) = R * G(R), \quad (26)
\]

where * denotes the spatial convolution and \(G\) is the three-dimensional Green function of the Helmholtz equation

\[
LG = \delta(x - x').
\]

It reads (Wladimirow, 1971)

\[
G(R) = \frac{1}{4\pi\ell^2} e^{-R/\ell}. \quad (28)
\]

In addition, it holds

\[
\Delta A(R) = -8\pi \delta(x - x'). \quad (29)
\]

The function (25) fulfills the relations

\[
L\Delta A(R) = -8\pi \delta(x - x'), \quad (30) \\
\Delta A(R) = -8\pi G(R), \quad (31) \\
LA(R) = R. \quad (32)
\]

Thus, \(A(R)\) is the Green function of Eq. (30) which is a Helmholtz–bi-Laplace equation.

Using Eqs. (A.3) and (A.4) for the differentiation of Eq. (24), the explicit form of the three-dimensional Green tensor of the Helmholtz–Navier equation is obtained

\[
G(R) = \frac{1}{16\pi\mu(1 - v)} \left[ \frac{\delta_0}{R} \left(3 - 4v\right)(1 - e^{-R/\ell}) \right. \\
+ \frac{1}{R^2} \left(2\ell^2 - R^2 + 2\ell R + 2\ell^2 e^{-R/\ell} \right) \\
+ \frac{2R\ell}{R^2} \left(1 - 2\ell^2 + \frac{6\ell^4}{R^2} + \frac{6\ell^2}{R^2} + \frac{6\ell^4}{R^2} e^{-R/\ell} \right) \right], \quad (33)
\]

which is non-singular. It is worth noting as a check, that Eq. (33) is in agreement with the corresponding expressions derived by Polyzos et al. (2003) and Gao and Ma (2009) using slightly different approaches.

Note, that the Green tensor (33) gives the non-singular displacement field, \(u_i = G_{ij}\) if \(j\) is the constant value of the magnitude of the point force acting at the arbitrary position \(x\) in an infinite body), of the Kelvin point force problem (e.g. Gurtin, 1972; Mura, 1987; Hetnarski and Ignaczyn, 2004) in the framework of gradient elasticity of Helmholtz type. The original solution of a concentrated force in an infinite body in the context of the classical continuum theory of elasticity was given by Kelvin (1882).

2.1. Dislocation loops

In this subsection, the characteristic fields of dislocation loops in the framework of gradient elasticity theory of Helmholtz type are calculated.

For a general (non-planar or planar) dislocation loop \(L,\) the classical dislocation density and the plastic distortion tensors are (e.g. DeWit, 1973a; Kossecka, 1974)

\[
\chi_0'' = b_i \delta_j(L) = b_i \int_L \delta(x - x') dL', \quad (34)
\]

\[
\beta_{ij}''' = -b_i \delta_j(S) = -b_i \int_S \delta(x - x') dS', \quad (35)
\]

where \(b_i\) is the Burgers vector of the dislocation line element \(dL'\) at \(x\) and \(dS'\) is the dislocation loop area. The surface \(S\) is the dislocation surface, which is a cap of the dislocation line \(L.\) \(\delta_j(S)\) is the Dirac delta function for a closed curve \(L\) and \(\delta_j(S)\) is the Dirac delta function for a surface \(S\) with boundary \(L\).

The solution of Eq. (17) can be written as the following convolution integral

\[
x_0 = G * \chi_0'' = b_i \int_L G(R) dL', \quad (36)
\]

where \(G(R)\) denotes the three-dimensional Green function of the Helmholtz equation given by Eq. (28). The explicit solution of the dislocation density tensor for a dislocation loop in gradient elasticity is calculated as

\[
x_0(x) = b_i \int_L \frac{e^{-R/\ell}}{R^2} dL'. \quad (37)
\]

describing a spreading dislocation core distribution. The plastic distortion tensor of a dislocation loop, which is the solution of Eq. (18), is given by the convolution integral

\[
\beta_{ij}' = G * \beta_{ij}'' = -b_i \int_S G(R) dS'. \quad (38)
\]

It reads as
\[
\beta_i^{(p)}(\mathbf{x}) = -\frac{b_i}{4\pi^2} \int e^{-\frac{r}{L}} dS',
\]  

\(39\)

Substituting Eq. (39) in Eq. (8) and using the Stokes theorem, we obtain formula (37).

Using the Green tensor (24), and after a straightforward calculation all the generalizations of the Mura, Peach–Koehler, and Burgers formulae towards gradient elasticity can be obtained. Starting with the elastic distortion tensor of a dislocation loop, the solution of Eq. (21) gives the representation as the following convolution integral

\[
\beta_{im}(\mathbf{x}) = \int e_{im} G_{ik}(R) z_0^{(p)}(\mathbf{x}) dV',
\]

\(40\)

where \(G_{ik} = \partial_k G_i\). Substituting the classical dislocation density tensor of a dislocation loop (34) and carrying out the integration of the delta function, we find the modified Mura formula valid in gradient elasticity

\[
\beta_{im}(\mathbf{x}) = \frac{1}{8\pi} \int e_{im} \left[ (b_i \partial_0 - b_0 \partial_i) \Delta - \left( \partial_0 \delta_i - \partial_i \delta_0 \right) \right] A(R) dL',
\]

\(41\)

Substitute Eqs. (3) and (24) into Eq. (41) and obtain after rearranging terms

\[
\beta_i(\mathbf{x}) = \frac{1}{8\pi} \int e_{im} \left[ (b_i \partial_0 - b_0 \partial_i) \Delta - \left( \partial_0 \delta_i - \partial_i \delta_0 \right) \right] A(R) dL',
\]

\(42\)

Using the identity

\[
\epsilon_{ijm} (b_i \partial_0 - b_0 \partial_i) = \epsilon_{ijk} \epsilon_{klm} \epsilon_{kli} \partial_0 - (\delta_k \partial_i - \delta_i \partial_k) \epsilon_{klm} \partial_0 = \epsilon_{ijkl} \partial_0 = (\epsilon_{ijkl} - \epsilon_{jilk}) / 2 \epsilon_{klm} \partial_0
\]

\(43\)

and the relation

\[
\int f \left[ \epsilon_{ijm} (b_i \partial_0 - b_0 \partial_i) \partial_0 A(R) dL' \right] = \int f \left[ \epsilon_{ijkl} \partial_0 \partial_0 A(R) dL' \right] = \int f \left[ \epsilon_{ijkl} \partial_0 A(R) dL' \right] = - \int f \left[ \epsilon_{ijkl} \partial_0 \partial_0 A(R) dL' \right] = - \int f \left[ \epsilon_{ijkl} \partial_0 A(R) dL' \right],
\]

\(44\)

the non-singular elastic distortion (42) of a dislocation loop becomes

\[
\beta_i(\mathbf{x}) = -\frac{b_i}{4\pi^2} \int \left[ \epsilon_{ijkl} \partial_0 + \epsilon_{ijkl} \partial_0 + \epsilon_{ijkl} \partial_0 \right] A(R) dL',
\]

\(45\)

This is the ‘Mura formula’ for a dislocation loop in gradient elasticity. It is important to note that if Eq. (45) is substituted into (7) and the relation (31) is used, the dislocation density of a dislocation loop (37) is recovered.

The symmetric part of the elastic distortion tensor (45) gives the elastic strain tensor of a dislocation loop

\[
\epsilon_i(\mathbf{x}) = -\frac{b_i}{8\pi} \int \left[ \left( \frac{1}{2} \epsilon_{ijkl} \partial_0 + \frac{1}{2} \epsilon_{ijkl} \partial_0 + \epsilon_{ijkl} \partial_0 \right) \partial_0 A(R) dL',
\]

\(46\)

The elastic dilation of a dislocation loop is nothing but the trace of the elastic strain (46)

\[
\epsilon_i(\mathbf{x}) = \frac{1}{8\pi} \int \epsilon_{ijkl} \partial_0 A(R) dL',
\]

\(47\)

The elastic rotation vector is defined as the skew-symmetric part of the elastic distortion tensor \(\omega_i = \frac{1}{2} \epsilon_{ijkl} \partial_0 j\) and reads

\[
\omega_i(\mathbf{x}) = -\frac{b_i}{8\pi} \int \left[ \delta_0 \partial_i - \frac{1}{2} \partial_0 \partial_i \right] A(R) dL',
\]

\(48\)

Using the constitutive relation (10) with Eq. (3), the non-singular stress field produced by a dislocation loop is found

\[
\sigma_i(\mathbf{x}) = \frac{\mu b_i}{8\pi} \int \left[ \epsilon_{ijkl} \partial_0 + \epsilon_{ijkl} \partial_0 + \epsilon_{ijkl} \partial_0 \right] \partial_0 A(R) dL',
\]

\(49\)

which can be interpreted as the Peach–Koehler formula within the framework of gradient elasticity. One may verify that the stress is divergence-less, \(\partial_i \sigma_i = 0\). The double stress tensor of a dislocation loop is easily obtained if Eq. (49) is substituted into Eq. (11).

The solution of Eq. (20) is the following convolution integral

\[
\beta_i(\mathbf{x}) = -\frac{1}{8\pi} \int_{-\infty}^{\infty} C_{ikj} G_{jkl}(R) \beta_{lm}^{(p)}(\mathbf{x}) dV',
\]

\(50\)

Substituting the classical plastic distortion of a dislocation loop (35) into Eq. (50) gives the modified Volterra formula valid in gradient elasticity

\[
u_i(\mathbf{x}) = \int b_i C_{ikl} G_{jkl}(R) dS',
\]

\(51\)

Substituting Eqs. (3) and (24) into Eq. (51) and rearranging terms yield

\[
u_i(\mathbf{x}) = -\frac{b_i}{8\pi} \int \left[ \delta_0 \partial_i - \epsilon_{ijkl} \partial_0 \partial_0 + \frac{1}{2} \epsilon_{ijkl} \partial_0 \partial_0 \right] A(R) dS',
\]

\(52\)

Except the first term of Eq. (52), we apply the Stokes theorem in order to obtain line integrals with

\[
\int \left[ \delta_0 \partial_i - \epsilon_{ijkl} \partial_0 \partial_0 + \frac{1}{2} \epsilon_{ijkl} \partial_0 \partial_0 \right] A(R) dS' = -\int \frac{\epsilon_{ijkl} \partial_0 \partial_0 A(R) dS'}{2},
\]

\(53\)

and

\[
\int \left[ \delta_0 \partial_i - \epsilon_{ijkl} \partial_0 \partial_0 + \frac{1}{2} \epsilon_{ijkl} \partial_0 \partial_0 \right] A(R) dS' = -\int \frac{\epsilon_{ijkl} \partial_0 \partial_0 A(R) dS'}{2},
\]

\(54\)

In this way, the key-formula for the non-singular displacement vector in gradient elasticity is found

\[
u_i(\mathbf{x}) = \frac{b_i}{8\pi} \int \Delta_0 A(R) dS' + \frac{b_i}{8\pi} \int \left[ \delta_0 \partial_0 - \frac{1}{2} \epsilon_{ijkl} \partial_0 \partial_0 \right] A(R) dS',
\]

\(55\)

which is the Burgers formula in the framework of gradient elasticity of Helmholtz type. Eq. (55) determines the displacement field of a single dislocation loop. The Eqs. (45)–(55) are straightforward, simple, and closely resemble the singular solutions of classical elasticity theory. In the limit \(r \to 0\), the classical expressions are recovered in Eqs. (45)–(55). The expressions (45), (49), and (55) retain most of the analytic structure of the classical Mura, Peach–Koehler, and Burgers formulae. The expressions (45)–(55) are given in terms of the elementary function \(A(R)\) given in Eq. (25), instead of the classical expression \(R\). The explicit expressions can be obtained by simple substitution of the formulae for the derivatives of \(A\) given in Eqs. (A.2)–(A.6). It is important to note that Eqs. (45)–(55) are non-singular due to the regularization of the classical singular expressions (see Appendix A). As an example, we substitute Eqs. (A.5) and (A.6) into Eq. (49) and obtain the explicit expression for the stress tensor.
The Green function (59) gives the non-singular displacement field \( u_i = -G_{i\alpha}f \) of a line force with the magnitude \( f \) calculated by Lazar and Maugin (2006b) in the framework of gradient elasticity.

### 2.2.2. Edge dislocation

Now the plane strain problem of an edge dislocation is investigated. The Green tensor of the plane strain problem in gradient elasticity of Helmholtz type is derived as (see Eq. (B.17))

\[
G_{i\alpha}(R) = -\frac{1}{8\pi\mu(1-\nu)} \frac{3-4\nu}{R^2 - \frac{3}{R}} \delta_{i\alpha} \left\{ (3-4\nu) \frac{R_{\alpha}}{R^2} - 2 \delta_{i\alpha} \frac{R_{\beta}}{R^2} + 2 \frac{R R_{\alpha} R_{\beta}}{R^4} \right\} K_0(R/\ell) + \frac{2}{\ell^2} \left( \delta_{i\alpha} + \delta_{i\beta} + \delta_{i\gamma} - 4 \frac{R R_{\alpha} R_{\beta}}{R^4} \right) \left( 2 \frac{\ell^2}{K_0} - K_2(R/\ell) \right) - 4 \left( \frac{1}{1-\nu} \delta_{i\alpha} \frac{R_{\beta}}{R^2} - 2 \frac{R R_{\alpha} R_{\beta}}{R^4} \right) K_1(R/\ell). \tag{64}
\]

Substituting Eq. (64) and the dislocation density of a strain edge dislocation along \( z \) axis with Burgers vector \( b_y = b_y(\delta x, \delta y) \) into Eq. (40), the elastic distortion of an edge dislocation is obtained. Eventually, the non-vanishing components of the elastic distortion of an edge dislocation are calculated as

\[
\beta_{xy} = -\frac{b_y}{4\pi(1-\nu)} K_2(R/\ell) - \frac{2}{\ell^2} \left( \frac{3}{1} - 2 \frac{y^2 - 2y^2}{r^2} - 3x^2 \right) K_1(R/\ell). \tag{65}
\]

\[
\beta_{yx} = \frac{b_y}{4\pi(1-\nu)} K_2(R/\ell) + \frac{2}{\ell^2} \left( \frac{3}{1} - 2 \frac{y^2 - 2y^2}{r^2} - 3x^2 \right) K_1(R/\ell). \tag{66}
\]

\[
\beta_{xx} = -\frac{b_x}{4\pi(1-\nu)} K_2(R/\ell) - \frac{2}{\ell^2} \left( \frac{3}{1} - 2 \frac{y^2 - 2y^2}{r^2} - 3x^2 \right) K_1(R/\ell). \tag{67}
\]

\[
\beta_{yy} = \frac{b_x}{4\pi(1-\nu)} K_2(R/\ell) + \frac{2}{\ell^2} \left( \frac{3}{1} - 2 \frac{y^2 - 2y^2}{r^2} - 3x^2 \right) K_1(R/\ell). \tag{68}
\]

which are non-singular and agree with the formulae given by Lazar (2003) and Lazar and Maugin (2006a). In the limit \( \ell \to 0 \), the classical expressions given by DeWit (1973b) are recovered in Eqs. (61) and (62).

The Green function (59) gives the non-singular displacement field \( u_i = -G_{i\alpha}f \) of a line force with the magnitude \( f \) calculated by Lazar and Maugin (2006b) in the framework of gradient elasticity.
as \( R_c \approx 6\ell \). If \( \ell \approx 0.4a \), where \( a \) denotes the lattice parameter, it is adopted as proposed by Eringen (1983), the dislocation core radius is \( R_c \approx 2.5a \). Using \( \ell \approx 0.4a \), the internal length reduces to \( \ell \approx 1.97 \text{ Å} \) for lead (Pb) with \( a \approx 4.95 \text{ Å} \).

Note that the two-dimensional Green function (63) gives the non-singular displacement field, \( u_i = -\partial G_{i\ell} / \partial \ell \) of a line force with magnitude \( f \) calculated by Lazar and Maugin (2006b) in the framework of gradient elasticity.

3. Gradient elasticity of bi-Helmholtz type

In this section, gradient elasticity theory of higher order is considered. Gradient elasticity theory of higher order was originally introduced by Mindlin (1965); Mindlin (1972) (see also, Jauzemis, 1967; Wu, 1992; Agiasofitou and Lazar, 2009). Mindlin’s theory of second strain gradient elasticity involves for isotropic materials, in addition to the two Lamé constants, sixteen additional material constants. These constants produce four characteristic length scales.

A simple and robust gradient elasticity of higher order which is called gradient elasticity theory of bi-Helmholtz type was introduced by Lazar et al. (2006a) and Lazar and Maugin (2006a) and successfully applied to the problems of straight dislocations (Lazar et al., 2006a; Lazar and Maugin, 2006a), straight disclinations (Deng et al., 2007) and point defects (Zhang et al., 2006). Lazar et al., 2006a and Lazar and Maugin, 2006a have shown that all state quantities are non-singular. By means of this second order gradient theory it is possible to eliminate not only the singularities of the strain and stress tensors, but also the singularities of the double stress tensors at the dislocation line. In general, all fields calculated in the theory of gradient elasticity of bi-Helmholtz type are smoother than those calculated by gradient elasticity theory of Helmholtz type. In general, there are a two main motivations for the use of gradient elasticity of bi-Helmholtz type: a consistent regularization of all state quantities, and a more realistic modelling of dispersion relations. A simple higher-order gradient theory in order to investigate dislocation loops should be used. The theory of gradient elasticity of bi-Helmholtz type is the gradient version of nonlocal elasticity of bi-Helmholtz type (Lazar et al., 2006b).

The strain energy density of gradient elasticity theory of bi-Helmholtz type for an isotropic, linearly elastic material has the form (Lazar et al., 2006a)

\[
W = \frac{1}{2} C_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_2 C_{ijkl} \rho_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_2^2 C_{ijkl} \rho_{ijkl} \rho_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_4 \rho_{ijkl} \rho_{ijkl} \rho_{ijkl} \rho_{ijkl},
\]

where \( \ell_1 = \ell, \ell_2 \) is another characteristic length scale and \( C_{ijkl} \) is given in (3). Due to the symmetry of \( C_{ijkl} \), Eq. (66) is equivalent to

\[
W = \frac{1}{2} C_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_2 C_{ijkl} \rho_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_2^2 C_{ijkl} \rho_{ijkl} \rho_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_4 \rho_{ijkl} \rho_{ijkl} \rho_{ijkl} \rho_{ijkl}.
\]

In addition to the constitutive Eqs. (10) and (11) another one is present in such a higher-order gradient theory,

\[
\tau_{ijkl} = \frac{\partial W}{\partial \rho_{ijkl}} = \frac{\partial W}{\partial \rho_{ijkl}} = \ell_2 C_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_4 \rho_{ijkl} \rho_{ijkl},
\]

where \( \tau_{ijkl} \) is called the triple stress tensor. It can be seen that \( \ell_2 \) is the characteristic length scale for double stresses. On the other hand, \( \ell_1 \) is the characteristic length scale for double stresses. Using Eqs. (10), (11), and (71) Eq. (70) can also be written as (Lazar et al., 2006a)

\[
W = \frac{1}{2} C_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_2 \rho_{ijkl} \rho_{ijkl} + \frac{1}{2} \ell_4 \rho_{ijkl} \rho_{ijkl} \rho_{ijkl}.
\]

The strain energy density (72) exhibits the symmetry in \( \rho_{ijkl} \) and \( \rho_{ijkl} \), in \( \partial \rho_{ijkl} \) and \( \partial \rho_{ijkl} \), and in \( \partial \rho_{ijkl} \) and \( \partial \rho_{ijkl} \). The condition for non-negative strain energy density, \( W \geq 0 \), gives

\[
\ell_1^2 \geq 0, \quad \ell_2^2 \geq 0, \quad \ell_4 \geq 0.
\]

in addition to \( (3\mu + 2\lambda) \geq 0 \) and \( \mu \geq 0 \).

The total stress tensor reads now

\[
\sigma_{ijkl}^0 = \sigma_{ijkl} - \partial \tau_{ijkl} + \partial \rho_{ijkl}. \]

In absence of body forces, the equation of equilibrium has the following form

\[
\partial \sigma_{ijkl}^0 = \partial \left( \sigma_{ijkl} - \partial \tau_{ijkl} + \partial \rho_{ijkl} \right) = 0.
\]

Using Eqs. (11) and (71), the total stress tensor (74) can be written

\[
\sigma_{ijkl}^0 = L \sigma_{ijkl},
\]

where the differential operator \( L \) is given by

\[
L = (1 - \ell_1^2 \Delta + \ell_2^2 \Delta^2) - (1 - \ell_1^2 \Delta)(1 - \ell_2^2 \Delta).
\]

Due to its structure as a product of two Helmholtz operators, the differential operator (77) is called bi-Helmholtz operator.

An important point, is the question concerning the mathematical character of the length scales \( c_1 \) and \( c_2 \). Mindlin, 1965 (see also, Mindlin, 1972; Wu, 1992) pointed out that the conditions for non-negative \( W \) supply no indications of the character, real or complex, of the characteristic lengths. Mindlin (1965) and Wu (1992) have treated the characteristic lengths as if they were real and positive. They also pointed out that a complex character of the lengths is equally admissible. The character, real or complex, of the lengths dictates the behaviour of the field variables. In the theory of gradient elasticity of bi-Helmholtz type the condition for the character, real or complex, of the length scales \( c_1 \) and \( c_2 \) can be obtained from the condition if the argument of the square root in Eqs. (78) and (79) is positive or negative. Thus, \( c_1 \) and \( c_2 \) are real if

\[
\ell_1^2 - 4 \ell_2^2 \geq 0,
\]

and \( c_1 \) and \( c_2 \) are complex if

\[
\ell_1^2 - 4 \ell_2^2 < 0.
\]

If the lengths \( c_1 \) and \( c_2 \) are complex, then the behaviour of the solutions of the field quantities would be oscillatory. In this case, the far-field behaviour of the strain and stress fields of dislocations would not agree with the classical behaviour. The limit from gradient elasticity of bi-Helmholtz type to gradient elasticity of Helmholtz type is: \( c_2 \rightarrow 0, \ell_2 \rightarrow 0 \) and \( c_1 \rightarrow \ell_1 \). If \( c_1 \) is complex, then also \( \ell_1 \) becomes complex what would be rather strange. Thus, a real character of the length scales \( c_1 \) and \( c_2 \) seems to be more realistic and more physical. In addition, Zhang et al., 2006 determined, in an atomistic calculation, the length scales \( c_1 \) and \( c_2 \) for graphene. In what follows, the length scales \( c_1 \) and \( c_2 \) will be treated as if they are real and positive.

The Green tensor of the bi-Helmholtz–Navié equation is calculated as (see Eq. (B.27))

\[
G_{ij}(R) = \frac{1}{16\pi\mu(1 - v)} \left[ 2(1 - v) \delta_{ij} \Delta - \partial \rho_{ijkl} \right] A(R).
\]
where the elementary function (25) is changed to

$$A(R) = R + \frac{2(c_1^2 + c_2^2)}{R} - \frac{2}{c_1^2 - c_2^2} \frac{1}{R} (c_1^2 e^{-c_1 R} - c_2^2 e^{-c_2 R}).$$

Eq. (85) is the Green function of the three-dimensional bi-Helmholtz–bi–Laplace equation. It is worth noting that the Green tensor (84) with (85) is in agreement with the corresponding expression derived by Zhang et al. (2006). On the other hand, the Green function of the bi-Helmholtz equation is given by (e.g. Lazar et al., 2006b)

$$G(R) = \frac{1}{4\pi(c_1^2 - c_2^2) R} (e^{-c_1 R} - e^{-c_2 R}).$$

Eq. (86) is the Green function of the anti-plane strain problem in gradient elasticity of bi-Helmholtz type (77) appears in Eqs. (15)–(18).

If we use Eqs. (A.9) and (A.10) for the differentiation of Eq. (84), the expressions obtained by Lazar and Maugin (2006a).

The Green function (90) gives the non-singular displacement field $u_i = G_{ij}(x)$ of the Kelvin point force problem, in the framework of gradient elasticity of bi-Helmholtz type.

### 3.1. Dislocation loops

The calculation of the characteristic fields of a dislocation loop in gradient elasticity of bi-Helmholtz type, is analogous to the technique used in gradient elasticity of Helmholtz type. The only difference in the results is that now the Green function (86) and the elementary function (85) of bi-Helmholtz type enter the characteristic fields of a dislocation loop. In gradient elasticity of bi-Helmholtz type, the distortion density tensor (36) and the plastic distortion tensor (38) are given in terms of the Green function of bi-Helmholtz type (86). Thus, they are calculated as

$$x_0(x) = \frac{b_i}{4\pi(c_1^2 - c_2^2)} \int \frac{e^{-c_1 R} - e^{-c_2 R}}{R} dL_i,$$

$$x_0(x) = -\frac{b_i}{4\pi(c_1^2 - c_2^2)} \int \frac{e^{-c_1 R} - e^{-c_2 R}}{R} dS_i.$$

In the limit $R \to 0$, the integrands of Eqs. (88) and (89) are non-singular at the dislocation line in contrast to the corresponding ones, Eqs. (37) and (39), calculated in gradient elasticity of Helmholtz type. On the other hand, the elastic distortion tensor (45), the elastic strain tensor (46), the elastic dilatation (47), the elastic rotation vector (48), the stress tensor (49), and the displacement vector (55) are given in terms of the elementary function (85) and only (85) has to be substituted in these formulae. The explicit formulae are not reproduced. The only difference between the fields of a dislocation loop in gradient elasticity of bi-Helmholtz type, and of Helmholtz type is that the Green function of bi-Helmholtz type (86) and the elementary function (85) have to be substituted instead of the Green function of Helmholtz type (28) and the elementary function (17). For the derivatives of the function (85), Eqs. (A.7)–(A.12) can be substituted into the corresponding formulae. The characteristic fields of a dislocation loop in gradient elasticity of bi-Helmholtz type retain all the analytical tensor structure of the corresponding classical formulae.

The triple stress tensor of a dislocation loop is easily obtained if the stress tensor $\sigma_{ij}$ is substituted into Eq. (71). In gradient elasticity of bi-Helmholtz type the fields produced by a dislocation loop are smoother than those predicted by gradient elasticity of Helmholtz type.

### 3.2. Straight dislocations

In this subsection, the modified Mura Equation (40) is used for gradient elasticity of bi-Helmholtz type. The technique of Green functions is used in order to determine the non-singular elastic distortion of straight dislocations.

#### 3.2.1. Screw dislocation

The Green function of the anti-plane strain problem in gradient elasticity of bi-Helmholtz type is the Green function of the two-dimensional bi-Helmholtz–Laplace equation and is given by (see Eq. (B.34))

$$G_{ij}(R) = -\frac{1}{2\pi R} \left\{ \gamma_e + \ln R + \frac{1}{c_1^2 - c_2^2} \left[ c_i K_0(R/c_1) - c_i K_0(R/c_2) \right] \right\}.$$

If Eq. (91) and $x_{ij}^* = b_i \delta(x) \delta(y)$ are substituted into Eq. (40), the elastic distortion produced by a screw dislocation with Burgers vector $b_i$ is obtained

$$\beta_{ij} = -\frac{b_i y}{2\pi R^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2} \left[ c_i K_1(R/c_1) - c_i K_1(R/c_2) \right] \right\},$$

$$\beta_{xy} = \frac{b_i x}{2\pi R^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2} \left[ c_i K_1(R/c_1) - c_i K_1(R/c_2) \right] \right\},$$

where $r = \sqrt{x^2 + y^2}$. Eqs. (92) and (93) are in agreement with the expressions obtained by Lazar andMaugin (2006a).

The Green function (90) gives the non-singular displacement field $u_i = -G_{ij}(x)$ of a line force with the magnitude $f_i$ in the framework of gradient elasticity of bi-Helmholtz type.

#### 3.2.2. Edge dislocation

The plane stress problem of an edge dislocation is now investigated. The Green tensor of the plane strain problem in gradient elasticity of bi-Helmholtz type is found as (see Eq. (B.30))

$$G_{ij}(R) = -\frac{1}{2\pi R} \left\{ \gamma_e + \ln R + \frac{1}{c_1^2 - c_2^2} \left[ c_i^2 K_0(R/c_1) - c_i^2 K_0(R/c_2) \right] \right\} + \frac{1}{16\pi(1-\nu)} \partial_i \partial_j \left\{ R^2 (\gamma_e + \ln R) + 4(c_i^2 + c_j^2) (\gamma e + \ln R) \right\} + \frac{4}{c_1^2 - c_2^2} \left[ c_i^2 K_0(R/c_1) - c_i^2 K_0(R/c_2) \right].$$

The gradient of the Green tensor Eq. (94) is calculated as
If Eq. (95) and $\alpha_{\beta} = b_\beta \delta(x)\delta(y)$ are substituted into Eq. (40), the non-vanishing components of the elastic distortion of an edge dislocation are found as

$$
C_{ij}(R) = -\frac{1}{8\pi \mu (1-\nu)} \left[ (3-4\nu) \beta_{ij} \frac{R}{R^2} \delta_{R_i R_j} \delta_{R_k R_l} - 2 \frac{R_i R_j}{R^4} \delta_{R_i R_j} \delta_{R_k R_l} \right] R^2 + 2 \frac{R_i R_j}{R^4} \delta_{R_i R_j} \delta_{R_k R_l} + 2 \frac{R_i R_j}{R^4} \delta_{R_i R_j} \delta_{R_k R_l} \right]
$$

$$
+ \frac{2}{R^4} \left[ \delta_{R_i R_j} + \delta_{R_k R_l} \right] \frac{1}{R^4} \left[ c_i K_2(R/c_1) - c_i K_2(R/c_2) \right]
$$

$$
\times \frac{2(c_i^2 + c_j^2)}{R^4} \left[ c_i K_2(R/c_1) - c_i K_2(R/c_2) \right] \left[ c_i K_2(R/c_1) - c_i K_2(R/c_2) \right]
$$

$$
- \frac{4(1-\nu) \delta_{R_k R_l}}{R} - 2 \frac{R_i R_j}{R^4} \delta_{R_i R_j} \delta_{R_k R_l} + 2 \frac{R_i R_j}{R^4} \delta_{R_i R_j} \delta_{R_k R_l} + 2 \frac{R_i R_j}{R^4} \delta_{R_i R_j} \delta_{R_k R_l} \right]
$$

$$
\left[ c_i K_2(R/c_1) - c_i K_2(R/c_2) \right] \left[ c_i K_2(R/c_1) - c_i K_2(R/c_2) \right]
$$

which are in agreement with the formulae given by Lazar and Maugin (2006a).

The two-dimensional Green function (94) gives the non-singular displacement field, $u_i = -C_{ij}f_j$ of a line force with magnitude $f_j$ calculated in the framework of gradient elasticity of bi-Helmholtz type.

4. Conclusions

Non-singular dislocation fields are presented in the framework of gradient elasticity. The technique of Green functions is used. The Green tensors of all relevant partial differential equations of generalized Navier type were calculated. For the first time, the elastic distortion, plastic distortion, stress, displacement, and dislocation density of a closed dislocation loop, using the theories of gradient elasticity of Helmholtz type and of bi-Helmholtz type were calculated. Straight dislocations using Green tensors were revisited. Such generalized continuum theories allow dislocation core spreading in a straightforward manner. In classical dislocation theory the dislocation function is a Dirac delta function, $\delta(x)$, without core spreading. In the non-singular approaches by Cai et al. (2006) and Lazar, presented in the present paper, the dislocation spreading functions are $w$ and $G$, respectively (see Table 1). In the theory of gradient elasticity all formulae are closed in contrast to the theory of Cai et al. (2006) where the spreading function $w$ is determined in a sophisticated way in order to obtain $R_s = [R^2 + a^2]^{1/2}$.

Due to the use of simplified theories of gradient elasticity, the dislocation fields retain most of the analytical structure of the classical expressions for these quantities but remove the singularity at the dislocation core due to the mathematical regularization of the classical singular expressions. In gradient elasticity of Helmholtz type, the characteristic length $\ell$ takes into account the information from atomistic calculations as discussed in this paper. In a straightforward manner, the length $\ell$ determines the dislocation core radius. Therefore, in gradient elasticity it is not necessary to introduce an artificial core-cutoff radius. It should be mentioned that the characteristic lengths which arise in first strain gradient elasticity (e.g. Maraganti and Sharma, 2007; Shodja and Tehranchi, 2010) and in second strain gradient elasticity (e.g. Zhang et al., 2006; Shodja et al., 2012) have been recently computed using atomistic approaches.

The obtained results can be used in computer simulations and numerics of dislocation cores, discrete dislocation dynamics, and arbitrary 3D dislocation configurations. The results can be implemented in dislocation dynamics codes (finite element implementation, technique of fast numerical sums), and compared to atomistic models (e.g. Choniem et al., 1999; Li and Wang, 2008).

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Appendix A. Appendix: A and its derivatives

In gradient elasticity theory, the stress tensor, the elastic distortion tensor, the elastic strain tensor, and the displacement vector of a dislocation loop are given in terms of derivatives of the elementary function $A$.

A.1. Helmholtz type

For gradient elasticity of Helmholtz type, the elementary function $A$ is given by

$$
A = R + \frac{2\ell^2}{R} (1 - e^{-r/\ell})
$$

Higher-order derivatives of $A$ are given by the following set of equations
\[ A_i = \frac{R_i}{R} \left[ 1 - \frac{2\ell^2}{R^2} (1 - e^{-R/r_i}) + \frac{2\ell}{R} e^{-R/r_i} \right], \]

where \( R_i = x_i - x'_i. \)

\[ A_{ij} = \delta_{ij} \frac{R_i}{R} \left[ 1 - \frac{2\ell^2}{R^2} (1 - e^{-R/r_i}) + \frac{2\ell}{R} e^{-R/r_i} \right] - \frac{R R_i}{R^2} \left[ 1 - \frac{6\ell^2}{R^2} (1 - e^{-R/r_i}) + \left(2 + \frac{6\ell}{R}\right) e^{-R/r_i} \right]. \]

\[ A_{iik} = \frac{2}{R} (1 - e^{-R/r_i}), \]

\[ A_{ijk} = -\frac{\delta_{ik} R_i + \delta_{jk} R_j + \delta_{ji} R_i}{R^3} \left[ 1 - \frac{6\ell^2}{R^2} (1 - e^{-R/r_i}) + \left(2 + \frac{6\ell}{R}\right) e^{-R/r_i} \right] + \frac{3R R_i R_j}{R^3} \left[ 1 - \frac{10\ell^2}{R^2} (1 - e^{-R/r_i}) + \left(4 + \frac{10\ell}{R} + \frac{2R}{3\ell}\right) e^{-R/r_i} \right] \]

and

\[ A_{iikk} = -\frac{2R_i}{R^2} \left[ 1 - \frac{1}{c_1 - c_2} \left(c_1^2 e^{-R/c_1} - c_2^2 e^{-R/c_2} \right) \right]. \]

The expressions (A.7)–(A.12) are non-singular. In the limit \( c_2 \to 0 \)
and \( c_1 = \ell, \) Eqs. (A.7)–(A.12) reduce to Eqs. (A.1)–(A.6).

**Appendix B. Green tensors of generalized Navier equations**

The following notation is used for the \( n \)-dimensional Fourier transform (Gueffand and Chilov, 1962)

\[ \tilde{f}(k) \equiv \mathcal{F}_n[f(r)] = \int_{-\infty}^{\infty} f(r) e^{ikr} dr, \quad (B.1) \]

\[ f(r) \equiv \mathcal{F}_n^{-1}[\tilde{f}(k)] = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \tilde{f}(k) e^{-ikr} dk. \quad (B.2) \]

We have (Wladimirow, 1971; Nowacki, 1986)

\[ \mathcal{F}_n^{-1} \left[ \frac{1}{k^2} \right] = -\frac{1}{2\pi} \left( \gamma_k + \ln \sqrt{x^2 + y^2} \right), \quad (B.3) \]

\[ \mathcal{F}_n^{-1} \left[ \frac{1}{k^3} \right] = \frac{\text{Re}}{4\pi \sqrt{x^2 + y^2 + z^2}}, \quad (B.4) \]

\[ \mathcal{F}_n^{-1} \left[ \frac{1}{k^4} \right] = \frac{1}{8\pi} \left( \pi^2 \gamma_k + \ln \sqrt{x^2 + y^2} \right), \quad (B.5) \]

\[ \mathcal{F}_n^{-1} \left[ \frac{1}{k^6} \right] = \frac{1}{8\pi} \sqrt{x^2 + y^2 + z^2}, \quad (B.6) \]

\[ \mathcal{F}_n^{-1} \left[ \frac{1}{k^8} \right] = \frac{1}{2\pi} K_0 \left( \sqrt{x^2 + y^2 / c^2} \right), \quad (B.7) \]

\[ \mathcal{F}_n^{-1} \left[ \frac{1}{k^{10}} \right] = \frac{1}{4\pi \sqrt{x^2 + y^2 + z^2}} \exp \left( -\sqrt{x^2 + y^2 + z^2 / c^2} \right), \quad (B.8) \]

**B.1. Green tensor of the Helmholtz–Navier equation**

The Green tensor of the Helmholtz–Navier equation is defined by

\[ (1 - \ell^2 \Delta) (\mu \delta_{ik} \partial + (\lambda + \mu) \partial \partial) \tilde{G}_0(r) = -\delta_{ij} \delta(r). \quad (B.9) \]

The Fourier transform of Eq. (B.9) reads

\[ (1 + \ell^2 k^2) (\mu \delta_{ik} k^2 + (\lambda + \mu) k k) \tilde{G}_0(k) = \delta_{ij}, \quad (B.10) \]

where \( \lambda = 2\mu v/(1 - 2v) \) and \( v \) is Poisson’s ratio. The Fourier transformed Green tensor is found as

\[ \tilde{G}_0(k) = \frac{1}{\mu} \left[ \frac{\delta_{ij} k}{k^2} - \frac{1}{2(1 - v)} k k \right] \frac{1}{1 + \ell^2 k^2}. \quad (B.11) \]

Using partial fractions and the inverse Fourier transform, we find

\[ \mathcal{F}_n^{-1} \left[ \frac{k k}{k^4 (1 + \ell^2 k^2)} \right] = -\partial_i \partial_j \mathcal{F}_n^{-1} \left[ \frac{k^4}{k^4 (1 + \ell^2 k^2)} \right] \]

\[ = -\partial_i \partial_j \mathcal{F}_n^{-1} \left[ \frac{1}{k^4} - \frac{\ell^2}{k^2 + \ell^2} \right] \]

\[ = \frac{\partial_i \partial_j}{8\pi} \left( \frac{r^2 + \ell^2}{r^2} e^{-r/\ell} \right). \quad (B.12) \]

and
\[ F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c^2 k^2)} \right) = -\Delta F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c^2 k^2)} \right), \]

(B.13)

The Fourier transform of Eq. (B.18) reads

\[ F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c^2 k^2)} \right) = -\Delta F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c^2 k^2)} \right) \]

and

(B.15)

and

\[ \frac{1}{k^2(1 + c^2 k^2)} = F^{-1}_{(1)} \left( \frac{1}{k^2 + \frac{\nu}{\mu}} \right) = -\frac{1}{2\pi} \left( \gamma_k + \ln r + K_0(r/\ell) \right). \]

(B.16)

the two-dimensional Green tensor of the Helmholtz–Navier equation is obtained as

\[ G_0(r) = -\frac{1}{2\pi \mu} \gamma_k \{ \gamma_k + \ln r + K_0(r/\ell) \} \]

\[ + \frac{1}{16\pi \mu(1 - \nu)} \partial_\ell \partial_\ell \{ r^2 (\gamma_k + \ln r) + 4\ell^2 (\gamma_k + \ln r + K_0(r/\ell)) \}. \]

(B.17)

where \( r = \sqrt{x^2 + y^2} \).

B.2. Green function of the Helmholtz–Laplace equation

For the anti-plane strain problem, the Green tensor of the Navier–Helmholtz–Laplace equation reduces to the Green function of the two-dimensional Helmholtz–Laplace equation which is defined by

\[ (1 - \ell^2 \Delta) \Delta G_{zz}(r) = -\frac{1}{\mu} \delta(\mathbf{x}). \]

(B.18)

The Fourier transform of Eq. (B.18) reads

\[ F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c^2 k^2)} \right) = -\Delta F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c^2 k^2)} \right) \]

\[ (1 - c_1^2 \Delta)(1 - c_2^2 \Delta)(\mu \partial_\ell \Delta + (\mu + \nu) \partial_\ell \partial_\ell) G_0(r) = -\delta(\mathbf{x}) \].

(B.22)

The Fourier transform of Eq. (B.22) reads

\[ (1 + c_1^2 k^2)(1 + c_2^2 k^2) \left( \mu \delta_k k^2 + (\mu + \nu) k \right) \bar{G}_0(k) = \delta_y. \]

(B.23)

The Fourier space Green tensor is

\[ \bar{G}_0(k) = \frac{\delta_y}{\mu} - \frac{1}{2(1 - \nu)} \frac{k k}{k^2 + (1 - \nu) k^2} \left( \frac{c_1^2}{k^2 + \frac{c_1^2}{c_2^2}} - \frac{c_2^2}{k^2 + \frac{c_2^2}{c_1^2}} \right). \]

(B.24)

Using

\[ F^{-1}_{(1)} \left( \frac{k k}{k^2(1 + c_1^2 k^2)(1 + c_2^2 k^2)} \right) \]

\[ = -\partial_\ell \partial_\ell F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c_1^2 k^2)(1 + c_2^2 k^2)} \right) \]

\[ = -\partial_\ell \partial_\ell F^{-1}_{(1)} \left( \frac{1}{k^2} \left( \frac{c_1^2}{k^2 + \frac{c_1^2}{c_2^2}} - \frac{c_2^2}{k^2 + \frac{c_2^2}{c_1^2}} \right) \right) \]

\[ = -\partial_\ell \partial_\ell \left( \frac{r^2 (\gamma_k + \ln r) + 4\ell^2 (\gamma_k + \ln r + K_0(r/\ell))}{16\pi \mu(1 - \nu)} \right) \]

\[ - \frac{2}{c_1^2 - c_2^2} \left( \frac{c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)}{r} \right). \]

(B.25)

the three-dimensional Green tensor of the bi-Helmholtz–Navier equation is found as

\[ G_0(r) = \frac{1}{16\pi \mu(1 - \nu)} \left[ (1 - \nu) \gamma_y + \ln r + \frac{1}{c_1^2 - c_2^2} [c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)] \right]. \]

(B.27)

where \( r = \sqrt{x^2 + y^2 + z^2} \).

In two dimensions, we use the formulae

\[ F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c_1^2 k^2)(1 + c_2^2 k^2)} \right) \]

\[ = -\Delta F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c_1^2 k^2)(1 + c_2^2 k^2)} \right) \]

\[ = -\frac{1}{4\pi} \left( \gamma_k + \ln r + \frac{1}{c_1^2 - c_2^2} [c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)] \right) \]

(B.28)

and

\[ F^{-1}_{(1)} \left( \frac{k k}{k^2(1 + c_1^2 k^2)(1 + c_2^2 k^2)} \right) \]

\[ = -\partial_\ell \partial_\ell F^{-1}_{(1)} \left( \frac{1}{k^2(1 + c_1^2 k^2)(1 + c_2^2 k^2)} \right) \]

\[ = -\partial_\ell \partial_\ell F^{-1}_{(1)} \left( \frac{1}{k^2} \left( \frac{c_1^2}{k^2 + \frac{c_1^2}{c_2^2}} - \frac{c_2^2}{k^2 + \frac{c_2^2}{c_1^2}} \right) \right) \]

\[ = -\frac{4}{c_1^2 - c_2^2} \left( \frac{c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)}{r} \right) \]

(B.29)

B.3. Green tensor of the bi-Helmholtz–Navier equation

The Green tensor of the bi-Helmholtz–Navier equation is defined by

\[ \bar{G}_0(k) = \frac{\delta_y}{\mu} - \frac{1}{2(1 - \nu)} \frac{k k}{k^2 + (1 - \nu) k^2} \left( \frac{c_1^2}{k^2 + \frac{c_1^2}{c_2^2}} - \frac{c_2^2}{k^2 + \frac{c_2^2}{c_1^2}} \right). \]
Eventually, the two-dimensional Green tensor of the bi-Helmholtz–Navier equations is obtained as

$$G_{ij}(r) = -\frac{1}{2\mu r} \delta_{ij} \left( \gamma_i + \ln r + \frac{1}{c_1 - c_2} \left( c_1 K_0(r/c_1) - c_2 K_0(r/c_2) \right) \right)$$

$$+ \frac{1}{16\pi \mu (1 - \nu)} \delta_{ij} \left( \bar{R}^2 \left( \gamma_i + \ln r \right) + 4 \left( c_1^2 + c_2^2 \right) \gamma_i + \ln r \right)$$

$$+ \frac{4}{c_1 - c_2} \left( c_1 K_0(r/c_1) - c_2 K_0(r/c_2) \right).$$

(B.30)

B.4. Green function of the bi-Helmholtz-Laplace equation

For the anti-plane strain problem, the Green tensor of the Navier–Helmholtz equation reduces to the Green function of the two-dimensional bi-Helmholtz-Laplace equation which is defined by

$$(1 - c_1 \Delta)(1 - c_2 \Delta) G_{ij}(r) = -\frac{1}{\mu} \delta(x).$$

(B.31)

The Fourier transform of Eq. (B.31) reads

$$(1 + c_1^2 k^2)(1 + c_2^2 k^2) \tilde{G}_{ij}(k) = \frac{1}{\mu} \delta(k).$$

(B.32)

The Fourier transformed Green function is given by

$$\tilde{G}_{ij}(k) = \frac{1}{\mu k^2 (1 + c_1^2 k^2)(1 + c_2^2 k^2)}.$$

(B.33)

Using Eq. (B.28), the two-dimensional Green function is obtained as

$$G_{ij}(r) = -\frac{1}{2\mu r} \left( \gamma_i + \ln r + \frac{1}{c_1 - c_2} \left( c_1 K_0(r/c_1) - c_2 K_0(r/c_2) \right) \right).$$

(B.34)

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