Oriented Borel-Moore homologies of toric varieties

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Abstract

We generalize the Künneth formula for Chow groups obtained in [6] and [7] to an arbitrary OBM-homology theory satisfying descent (e.g. algebraic cobordism) when taking a product with a toric variety. As a corollary we obtain a universal coefficient theorem for the operational cohomology rings. We also give a description for the homology groups of smooth toric varieties, that allows the calculation of algebraic cobordism groups of singular toric varieties. Some computations are carried out.

Contents

1 Introduction 2
   1.1 Outline of the article ................................................. 2
   1.2 Conventions and notations ........................................... 4
   1.3 Acknowledgements .................................................... 4

2 Background 4
   2.1 Oriented Borel-Moore homology theories ............................ 4
   2.2 Formal group laws ................................................... 7
   2.3 Linear and $G$-linear varieties .................................... 8

3 Homology groups of toric varieties 9
   3.1 Review of the construction and basic properties of equivariant groups ... 10
   3.2 A decomposition theorem for smooth toric varieties ..................... 15
   3.3 Line bundles associated to linear forms ................................ 17
   3.4 Application – algebraic cobordism $\Omega_*(X_\Delta)$ of toric varieties ... 20
   3.5 Stanley-Reisner representations and piecewise functions on a fan ....... 23

4 Künneth formula and a universal coefficient theorem 26
   4.1 Künneth Formula ..................................................... 26
   4.2 Universal coefficients for operational cohomology .................... 28
1 Introduction

In [6] and [7] the authors discovered the following surprising result: the operational Chow cohomology rings $\text{opCH}^*(X_\Delta)$ of a \textit{complete} toric variety can be naturally identified with the $\mathbb{Z}$-dual of the usual Chow groups $\text{CH}_* (X_\Delta)$ as Abelian groups. Such well behavedness was not anticipated from the operational cohomology rings, which were originally thought only as a temporary substitute until some sensible cohomology theory comes to take its place.

The proof depends crucially on another result interesting in its own right. Namely, if we are given a toric variety $X_\Delta$, then the natural Künneth map

$$\text{CH}_*(X_\Delta) \otimes_{\mathbb{Z}} \text{CH}_*(Y) \rightarrow \text{CH}_*(X_\Delta \times Y)$$

is an isomorphism for all varieties $Y$. This is a nontrivial result as it implies, among other things, that if $Y$ and $X_\Delta$ are smooth, then all the line bundles on $Y \times X_\Delta$ can be expressed as an exterior product of line bundles on $Y$ and $X_\Delta$, and this expression is unique up to isomorphism (neither of these facts hold in general).

A natural question to ask is do these theorems generalize for other theories such as the equivariant Chow groups $\text{CH}_T^*$ or algebraic cobordism $\Omega_*$. The results would be useful especially in the case of algebraic cobordism, where computations are hard, and the groups are known only in a handful of cases. Structural results on the behavior of these groups are therefore important to improve our understanding.

This paper grew out of the attempts to generalize the two results to more general than $\text{CH}_*$. It turns out that the Künneth formula is the harder one of these, and that the original proof for the universal coefficient theorem goes trough for an arbitrary theory as soon as the Künneth isomorphism property is known. Moreover, the techniques that allow us to generalize the Künneth property also offer a nice explicit description of the algebraic cobordism $\Omega_* (X_\Delta)$ of a smooth toric variety that can be used to determine the algebraic cobordism of a \textit{singular} toric variety. The methods of computations should immediately generalize for other similar theories.

1.1 Outline of the article

We begin with a section summarizing the necessary background material on oriented Borel-Moore homology theories, and on linear varieties. Most of the proofs are skipped, and instead references are given to appropriate texts. In the first real section of this paper, Section [3], we characterize the equivariant homology groups $B_*^T (X_\Delta)$ of a smooth toric variety, where $B_*$ is an oriented Borel-Moore homology theory. The methods are inspired by those in [13] and [14]. The main work is done in the first three subsections. The
equivariant groups $B^*_T$ are shown determine $B_*$, and hence we also obtain a description of $B_*(X_\Delta)$ for $\Delta$ nonsingular.

The description of algebraic cobordism of a smooth toric variety we obtain is nice enough to allow us to effectively carry out the computation of the algebraic cobordism group $\Omega_*(X_\Delta)$ for singular toric varieties in the Section 3.4 using the descent exact sequence of [11]. The section 3.5 merely states some general facts about the representations of the equivariant groups $B^*_T(X_\Delta)$ for nonsingular toric varieties, and connects the results obtained in this paper to previous results. These two sections are independent from the other results of this paper, and can be skipped.

The Section 4 is devoted to proving the Künneth isomorphism property for all oriented Borel-Moore homologies $B_*$ when taking a product with a toric variety, and the universal coefficient theorem that follows from it. The first of these is obtained in 4.1

**Theorem.** Let $X_\Delta$ be a toric variety, and $B_*$ an oriented Borel-Moore homology theory satisfying descent. Now, for any variety $Y$, the Künneth map

$$B_*(X_\Delta) \otimes_{B_*} B_*(Y) \to B_*(X_\Delta \times Y)$$

is an isomorphism.

Interesting examples where this result holds are the torus equivariant Chow groups $\text{CH}_T^*$, algebraic cobordism $\Omega_*$ and the torus equivariant algebraic cobordism $\Omega_T^*$, although we prove the result in far greater generality. Note that here we are not taking the tensor product over the integers $\mathbb{Z}$ but over the ring $B^*$ which is given by the homology group of the point, and is the natural coefficient ring for the theory $B_*$. 

In the section 4.2 we prove the universal coefficient theorem for the operational cohomology rings $\text{op}B^*(X_\Delta)$ when $X_\Delta$ is a complete toric variety. Here we need to assume that $B_*$ has slightly stronger properties than those of an oriented Borel-Moore homology theory, namely, $B_*$ must come with refined l.c.i. pullbacks. This is needed for the construction of the operational cohomology rings.

**Theorem.** Let $X_\Delta$ be a complete toric variety, and $B_*$ an refined oriented Borel-Moore homology theory. Now there is a canonical identification

$$\text{op}B^*(X_\Delta) \cong \text{Hom}_{B_*}(B_*(X_\Delta), B^*).$$

Notice how, much like in the Künneth formula, we are taking the Hom over the coefficient ring of the theory. Again, interesting examples where this result holds are $\text{CH}_T^*$, $\Omega_*$ and $\Omega_T^*$. 

3
1.2 Conventions and notations

All varieties will be over a field $k$ of characteristic zero. Varieties are not assumed to be irreducible, and virtually everything would go through for finite type separated $k$-schemes in the place of varieties. By [12] we can get rid of all the projectivity assumptions of algebraic cobordism, and we will use this throughout the article (e.g., we have proper pushforwards instead of projective ones and so on).

We will denote by $B_*$ a general oriented Borel-Moore homology theory. When restricted to smooth varieties $X$, this gives an oriented cohomology theory $B^*$ in a natural way, where the contravariant functorality is provided by the l.c.i. pullbacks, and the ring structure is provided by the intersection product. Note that cohomology has different grading: $B^*(X) = B_{n-*}(X)$ where $n$ is the dimension of $X$.

By abuse of notation, we will denote by $B^*$ the group of the point $B_*(pt)$, which acts on all the groups $B_*(X)$ by the exterior product. It is the natural coefficient ring for the theory $B_*$. It does not matter whether we take the homological or cohomological grading for $B^*$ as the grading needs never to be explicitly mentioned.

If $G$ is a linear algebraic group, then we will denote by $B^*_G$ the $G$-equivariant version of $B_*$, whose construction is based on Totaro’s approximation scheme for the classifying space $BG$. The construction of equivariant groups was carried out for a very general class of theories in [10] which includes all oriented Borel-Moore homology theories. The groups $B^*_G$ have formally very similar properties to those of $B_*$, and indeed we can usually treat both cases simultaneously.

1.3 Acknowledgements

I would like to thank my advisor Kalle Karu for a patient introduction to algebraic cobordism, and for many helpful discussions where he pointed out multiple potential weaknesses and mistakes.

2 Background

In this section we are going to summarize technical background necessary for the results of this paper. This consist mostly of definitions, and in the case of theorems, most proofs are omitted.

2.1 Oriented Borel-Moore homology theories

Denote by $C$ a category, which is either the category of finite type separated $k$-schemes or the category of finite type separated $k$-schemes with an action by a linear algebraic
group $G$ with $G$-equivariant morphisms. Both these categories have notion of proper and l.c.i. morphisms, as well as the notion of transversality. A $G$-equivariant vector bundle is a vector bundle $V \to X$ whose projection map is $G$-equivariant, and the map of fibres $V_x \to V_{gx}$ is linear for all $g \in G$. A $G$-equivariant projective bundle is taken to be a projectivization of a $G$-equivariant vector bundle.

Denote by $\mathcal{C}'$ the subcategory of $\mathcal{C}$ whose objects are the objects of $\mathcal{C}$, and whose morphisms are the proper morphisms in $\mathcal{C}$. An oriented Borel-Moore homology theory $B_*$ is a covariant functor from $\mathcal{C}'$ to the category of graded Abelian groups satisfying

(D1) (Additivity): the natural map

$$\bigoplus_{i=1}^n B_*(X_i) \to B_* \left( \prod_{i=1}^n X_i \right)$$

is an isomorphism.

(D2) (L.c.i. pullbacks) If $f : X \to Y$ is an l.c.i. morphism of relative dimension $d$, then we have a pullback map

$$B_* (Y) \to B_{*+d}(X).$$

(D3) (External product) An element $1 \in B_0(\text{Spec}(k))$ and a graded bilinear pairing

$$B_*(X) \times B_*(Y) \to B_*(X \times Y)$$

which is associative, commutative and $1$ is a unit element for $\times$.

These axioms allow us to define the first Chern class of a line bundle $\mathcal{L}$ on $X$, which is an operation $B_*(X) \to B_{*-1}(X)$, as the composition $s^*s_*$ where $s$ is the zero section. These are required to satisfy:

(BM1) The l.c.i. pullbacks are contravariantly functoral.

(BM2) Suppose that the Cartesian square

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & Y' \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{g} & Y
\end{array}$$

5
is transverse, \( f \) is l.c.i. and \( g \) is proper. Now, by assumption, \( f' \) is l.c.i. and \( g' \) is proper. In this situation we require that \( f^*g_*=g'_*f'^* \).

\((BM_3)\) If \( f : X \to Y \) and \( g : X' \to Y' \) are proper, then it follows that also \( f \times g \) is proper, and moreover

\[(f \times g)_*(\alpha \times \beta) = f_*(\alpha) \times g_*(\beta).
\]

If \( f \) and \( g \) are l.c.i. then it similarly follows that \( f \times g \) is l.c.i, and moreover

\[(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta).
\]

\((PB)\) \((Projective bundle formula)\) Let \( q : P \to X \) be a projective bundle of rank \( r \), and denote by \( \xi \) the first Chern class of \( \mathcal{O}(1) \). Now the maps

\[\xi^i q^* : B_{*-n+i}(X) \to B_*(P)\]

induce an isomorphism

\[\bigoplus_{i=0}^r B_{*-n+i}(X) \to B_*(P).
\]

\((EH)\) \((Extended homotopy property)\) Let \( V \to X \) be a vector bundle or rank \( r \) and \( p : E \to X \) a \( V \)-torsor. Now the pullback map

\[p^* : B_*(X) \to B_{*+r}(E)\]

is an isomorphism.

**Remark 2.1.** In the book \cite{15} it is also required that \( B_\ast \) should satisfy a cellular decomposition axiom. As we do not need this for anything, we will omit it from the definition.

**Remark 2.2.** The exterior product makes \( B_\ast(X) \) a \( B^\ast = B_\ast(pt) \)-module for all \( X \). It follows from the requirements imposed on \( \otimes \) that the pushforwards and pullbacks are \( B^\ast \)-linear maps. Moreover, from the commutativity and the associativity assumptions it follows that the exterior product associated to \( X \times Y \) is \( B^\ast \)-bilinear. Hence we have a natural Künneth morphism

\[B_\ast(X) \otimes_B B_\ast(Y) \to B_\ast(X \times Y),\]

which is one of the protagonists of the paper.
Well known examples of such theories include algebraic cobordism \( \Omega_* \) (see [15]) and the Chow groups \( CH_* \) (see [5]). One can refine this definition to require the theory to have refined l.c.i. pullbacks. This means that for a l.c.i. morphism \( f : X \to Y \) or relative dimension \( d \) and any morphism \( g : Y' \to Y \) we get a pullback map \( f^!_g : B_*(Y') \to B_{*-r}(X') \), where \( X' \) is the pullback of \( X \) along \( g \). These refined pullbacks are required to satisfy certain combability conditions, see [5] or [10] for details. Both the Chow groups and algebraic cobordism have refined l.c.i. pullbacks.

A result we are going to use repeatedly is the localization sequence. Namely, if \( i : Z \to X \) is a closed embedding and \( j : U \to X \) is the inclusion of the complement, then the sequence

\[
B_*(Z) \xrightarrow{i_*} B_*(X) \xrightarrow{j^*} B_*(U) \to 0
\]

is exact for all Borel-Moore homology theories (see [15] chapters 4 and 5). One could take this to be one of the axioms for the purposes of this paper, as most of the important proofs depend crucially on it.

Another important exact sequence is the descent sequence, originally discovered in the cases of Chow groups and \( K \)-theory in [8]. Namely, suppose \( \pi : \tilde{X} \to X \) is a proper envelope (any irreducible subvariety of \( X \) is birationally mapped onto by an irreducible subvariety of \( \tilde{X} \)). In many theories the sequence

\[
B_*(\tilde{X} \times_X \tilde{X}) \xrightarrow{p_1^* - p_2^*} B_*(\tilde{X}) \xrightarrow{\pi^*} B_*(X) \to 0
\]

is exact. We say that such an theory satisfies descent. It is not known whether or not this holds for a general Borel-Moore homology theory, but it is known to hold for \( \Omega_* \) by [11], so most all interesting Borel-Moore homologies satisfy descent.

### 2.2 Formal group laws

A formal group law \( F \) is an element of \( A[[x, y]] \) for some commutative ring \( A \), satisfying the following properties:

1. **Neutral element**: \( F(x, 0) = x \) and \( F(0, y) = y \).
2. **Commutativity**: \( F(x, y) = F(y, x) \).
3. **Associativity**: \( F(x, F(y, z)) = F(F(x, y), z) \).

It is immediate from the first two restrictions that \( F \) must be of form

\[
F(x, y) = x + y + \alpha_{11} xy + \alpha_{21} x^2 y + \alpha_{12} xy^2 + \cdots
\]
where $\alpha_{ij} \in A$, and $\alpha_{ij} = \alpha_{ji}$, although there are more complicated necessary relations between the coefficients arising from the associativity axiom.

It is not hard to show that such an $F$ has a formal inverse, i.e., a power series $F_-$ in one variable such that
\[ F(x, F_-(x)) = F(F_-(x), x) = 0. \]
It is often more convenient to denote $F(x, y)$ by $x_F y$ and $F_-(x) = -F x$. Moreover, the repeated addition $x_F \cdots + F x$ can be denoted by $n_{-F} x$. It is not hard to show that these behave as one would expect, i.e., $n_{-F} x + F n_{-F} y = n_{-F} (x + F y)$, $-n_{-F} x = n_{-F} (-x)$ and so on.

There is universal such an group law. Consider the infinitely generated $\mathbb{Z}$-algebra $\mathbb{Z}[a_{11}, a_{21}, a_{12}, \ldots]$ with the minimal relations making the power series
\[ x + y + a_{11} xy + a_{21} x^2 y + a_{12} xy^2 + \cdots \]
a formal group law. The resulting ring is known as the Lazard ring $L$, and it is characterized by the property that there is a unique map from $L \to A$, where $A$ is a commutative ring with a formal group law $F \in A[[x, y]]$, mapping the universal group law to $F$. If we set the degree of $a_{ij}$ to be $i + j - 1$, then all the relations respect grading, so we see that the Lazard ring has a natural grading.

Formal group laws describe the behavior of Chern classes of line bundles in oriented Borel-Moore theories. Namely, if we have line bundles $\mathcal{L}$ and $\mathcal{M}$, then the first Chern class $c_1(\mathcal{L} \otimes \mathcal{M})$ is given by $F(c_1(\mathcal{L}), c_1(\mathcal{M}))$. Usually infinite expressions are not allowed for our theories, so in order to make sense of the formal group law property one must require that the Chern classes are nilpotent in a suitable sense. However, sometimes infinite expressions make perfect sense (namely in equivariant theories), and then we can make sense of the formal group law topologically (i.e., the series will converge to something).

For Chow groups $\text{CH}_*$, the formal group law is known to be the additive formal group law $F(x, y) = x + y$. A more complicated example is the algebraic cobordism $\Omega_*$ – the universal oriented Borel-Moore homology theory – whose formal group law is the universal formal group law over the Lazard ring defined above.

### 2.3 Linear and $G$-linear varieties

In the paper [19] it was shown that the Chow groups satisfy the Künneth formula for products of arbitrary variety and a linear scheme. Unfortunately the proof made use of higher Chow groups, and therefore it cannot be generalized to, say, algebraic cobordism. However, it gives a neat argument for surjectivity of the Künneth morphism for arbitrary $B$, when taking a product with a linear variety, which is a slight generalization of a linear scheme. As we are going to need this result later, we record its proof.

A linear variety is a variety obtained by the following inductive procedure.
1. An affine space $\mathbb{A}^n$ (a $G$-representation) is linear.

2. If we have a closed embedding $Z \hookrightarrow X$ of linear varieties, then the complement $X - Z$ is linear.

3. If we have a filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_r = X$ such that $X_i - X_{i-1}$ is a linear variety, then also $X$ is linear.

**Proposition 2.3.** Let $X$ be a linear variety, and $Y$ an arbitrary variety. Now the Künneth map

$$B_*(X) \otimes_{B_*} B_*(Y) \to B_* (X \times Y)$$

is surjective.

**Proof.** The fact that the map is surjective when taking a product with either $X = \mathbb{A}^n$ or $X$ a $G$-representation $V$ follows from the extended homotopy property and compatibility properties of exterior products and pullbacks.

Assume that we have a closed inclusion $Z \hookrightarrow X$ such that the claim holds for $X - Z$ and $Z$. Now the localization exact sequence yields us a diagram

$$
\begin{array}{c}
B_*(Z) \otimes_{B_*} B_*(Y) \to B_*(X) \otimes_{B_*} B_*(Y) \to B_*(X - Z) \otimes_{B_*} B_*(Y) \to 0 \\
\downarrow \downarrow \downarrow \\
B_*(Z \times Y) \to B_*(X \times Y) \to B_*((X - Z) \times Y) \to 0
\end{array}
$$

where the leftmost and rightmost vertical maps are surjections. From the 4-lemma it follows that also the middle vertical map is surjective. This shows that surjectivity is preserved in the third operation defining linear varieties. Showing that it is preserved in the second operation is similar but easier, so the claim follows.

\[\square\]

## 3 Homology groups of toric varieties

The main purpose of this section is to study the structure of $B_*(X_\Delta)$, where $X_\Delta$ is a toric variety with the natural torus $T$, and $B_*$ is an arbitrary oriented Borel-Moore homology theory. In order to achieve this, we first study equivariant versions $B_*^T$ of these groups, and then connect the results obtained in equivariant case to the nonequivariant case. The results in this section will also form a basis for the Künneth formula and duality results of the fourth section.

Let $N \cong \mathbb{Z}^n$ be a lattice of rank $n$ sitting inside the vector space $N_\mathbb{R}$. A toric variety is given by a fan $\Delta$ consisting of rational strictly convex polyhedral cones in $N_\mathbb{R}$ (see [4] for
details). The dual lattice $\text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ is denoted by $M$, and $M_\mathbb{R}$ denotes the respective dual $\mathbb{R}$-vector space. By $e_1, \ldots, e_n$ we will denote a $\mathbb{Z}$-basis for $N$, and by $e_1^*, \ldots, e_n^*$ we will denote the respective dual basis for $M$.

3.1 Review of the construction and basic properties of equivariant groups

Let $B_*$ be an arbitrary oriented Borel-Moore homology theory. The construction of the torus-equivariant version $B_T^*$ of $B_*$ is based on the approximation scheme of Totaro. This has led to successful study of equivariant Chow groups [13] and algebraic cobordism [13] among other theories. Recently in [10], the construction was carried out in a very general setting of so-called ROBM pre-homology theories satisfying certain conditions. The same construction works for oriented BM-homology theories, and it is the construction we are going to use here. We note that in order to make sense of the formal group laws in the theory $B_T^*$, one is forced to consider it as a topological group, whose topology is given by the filtration naturally arising from the definition. This is because equivariant version of the theory no longer has to satisfy dimension axiom, and hence Chern classes of line bundles may fail to be nilpotent.

The construction begins with a choice of good system of representations for the group $T$. It consists of pairs $(V_i, U_i)$ for all $i \in \mathbb{N}$, where $V_i$ is a $T$-representation, $U_i \subset V_i$ is an invariant open subvariety of $V_i$. Moreover, these are required to satisfy

1. $G$ acts freely on $U_i$, and $U_i$ exists as a quasiprojective $k$-scheme.
2. The representation $V_{i+1}$ splits as a sum $V_i \oplus W_i$.
3. $U_i \oplus W_i \subset U_{i+1}$.
4. Codimension of $V_i - U_i$ in $V_i$ is strictly smaller than that of $V_j - U_j$ in $V_j$ whenever $i < j$.

For a split torus $T$ of rank $n$ an example of good system of representations would be $((\mathbb{A}^{i+1})^n, ((\mathbb{A}^{i+1} - 0)^n))$, where the $j^{th}$ coordinate of the torus acts diagonally on the $j^{th}$ copy of $\mathbb{A}^{i+1}$. The quotient in this case would be $(\mathbb{P}^i)^n$.

The equivariant group $B_T^*(X)$ of a $T$-variety is given by the limit

$$\lim_{\leftarrow \mathbb{N}} B_{*+ni}(U_i \times T \times X)$$

where the limit is taken in each degree separately. If we denote by $B_*$ the ring $B_*(pt)$ of a point (which is the natural coefficient ring of the theory), then it follows from the projective bundle formula that $B_*(([\mathbb{P}^i]^n))$ is a ring isomorphic to $B_*[\xi_1, \ldots, \xi_n]/(\xi_i^{i+1}, \ldots, \xi_n^{i+1})$. 10
where $\xi_j$ corresponds to the first Chern class of the pullback of the tautological line bundle $\pi_j^*\mathcal{O}(-1)$ (having therefore degree $-1$), and $\pi_j$ is the $j^{th}$ projection $(\mathbb{P}^2)^n \to \mathbb{P}^1$. The limit is the so called graded power series algebra $B^*[[\xi_1, \ldots, \xi_n]]$, i.e., the ring consisting of $B^*$-linear combinations of degree $k$ homogeneous pieces, which are power series of form

$$\sum_{i_1, \ldots, i_n} b_{i_1, \ldots, i_n} \xi_1^{i_1} \cdots \xi_n^{i_n}$$

where $b_{i_1, \ldots, i_n} \in B_{k+i_1+\cdots+i_n}(pt)$ (note the homological grading). If $B_*(pt)$ vanishes in high degrees, then this coincides by the polynomial algebra generated by $\xi_i$ over $B^*$.

Let $\mathcal{L}$ be the line bundle defined by

$$\mathcal{L} = \pi_1^*\mathcal{O}(-a_1) \otimes \pi_2^*\mathcal{O}(-a_2) \otimes \cdots \otimes \pi_n^*\mathcal{O}(-a_n).$$

By the formal group law axiom of the original theory $B_*$, the first Chern class of $\mathcal{L}$ is given by the expression

$$a_1 \cdot_F \xi_1 +_F a_2 \cdot_F \xi_2 +_F \cdots +_F a_n \cdot_F \xi_n$$

where $+_F$ is an additive shorthand for the formal group operation, and $\cdot_F$ is a shorthand for repeated formal addition. Depending on the formal group law of the theory, this expression may be complicated, which makes changing bases inconvenient at times.

**Equivariant exterior product and OBM-homology structure**

We recall that the exterior product of $B_*$ is can be extended to the equivariant theory $B_*^T$ by the limit of compositions

$$B_{j+n}(U_i \times X) \times B_{k+n}(U_i \times Y) \xrightarrow{T} B_{j+k+2n}((U_i \times X) \times (U_i \times Y)) \xrightarrow{T} B_{j+k+n}((X \times Y))$$

where the first map is the ordinary exterior product, and the second map is l.c.i. pullback induced by the diagonal map $U_i \to U_i \times U_i$. The equivariant exterior product satisfies similar formal properties as the original exterior product. For example, it is associative and commutative (and hence bilinear over the ring of the point), and it works well with various operations of the homology theory such as pullbacks and pushforwards.

It is sometimes useful to have a slightly different way of thinking for the action of the product, i.e., what does it mean to multiply by $\xi_j$ an element of $B_*^T(X)$. Recall that
\(\xi_j \in B^T_s(pt)\) arises as the limit of the first Chern classes of \(\pi_j^*O(-1)\) on \((\mathbb{P}^i)^n\) as \(n \to \infty\).

On the other hand, we might pull back \(\pi_j^*O(-1)\) via the map \(U_i \times^T X \to (\mathbb{P}^i)^n\) and taking the first Chern classes of the pulled back bundles, and this would yield an operation on \(B^T(X) \to B^T_{s-1}(X)\). It easily follows from the basic compatibility properties of Chern classes that the operation we just constructed coincides with the multiplication by \(\xi_i\) as constructed previously.

**Remark 3.1.** If \(B_*\) was an oriented Borel-Moore homology theory, then so is \(B^G_*\) with the obvious restriction to \(G\)-equivariant morphisms. This follows more or less directly from [10], where it was shown that if \(B_*\) was a refined oriented Borel-Moore pre-homology theory, then so is \(B^G_*\). Note that we can do everything even if \(B_*\) was already an equivariant theory. Now we just need to assume that the \(G\)-actions on the varieties are compatible with the "old" action, namely \(G\) must act by equivariant morphisms.

**First properties of the equivariant groups**

We begin listing properties of \(B^T_*\) that are more or less direct from the definition.

**Proposition 3.2.** Let \(X\) be a \(k\)-scheme with a trivial \(T\)-action. Now
\[
B^T_*(X) = B^*_T \otimes_{B_*} B_*(X).
\]

**Proof.** As the action of \(T\) on \(X\) is trivial, we can identify \(X \times^T U_i\) with \(X \times \mathbb{P}^i\). Using the projective bundle formula, we see that
\[
B^*[\xi_1, \ldots, \xi_n]/(\xi_1^{i_1+1}, \ldots, \xi_n^{i_n+1}) \otimes_{B_*} B_*(X) \to B_{*+ni}(U_i \times^T X)
\]
is an isomorphism. The limit of the left groups is going to be linear combinations of power series of form
\[
\sum_{i_1, \ldots, i_n} \xi_1^{i_1} \cdots \xi_n^{i_n} b_{i_1, \ldots, i_n}
\]
where \(i_1, \ldots, i_n\) run over all the natural numbers, and
\[
b_{i_1, \ldots, i_n} \in B_{k+i_1+\ldots+i_n}(X)
\]
for fixed \(k\). This coincides with the completed tensor product of the two, where \(B_*(X)\) is taken to have the trivial filtration.

Another result easily proven, which is a special case of more general Morita isomorphism principle, is the following.

**Proposition 3.3.** Let \(X\) be any \(T\)-scheme. Now \(B^{T}_{*+n}(T \times X)\), where the product variety has the diagonal action, is naturally isomorphic to \(B_*(X)\).

12
Proof. The map \( T \times X \to T \times X_t \) defined by \( (t, x) \mapsto (t, t^{-1}x) \) identifies \( T \times X \) with the original action on \( X \) with \( T \times X_t \), where \( X_t \) is \( X \) with the trivial \( T \)-action. Therefore for all \( U_i \) we have isomorphisms

\[
U_i^T \times (T \times X) \cong U_i^T \times (T \times X_t) = (U_i^T \times T) \times X_t = U_i \times X_t.
\]

Because the \( T \)-action does not affect the groups \( B_* \), and as \( U_i = (A_i^{i+1} - 0)^n \), we are done if we can show that

\[
B_{*+i+1}((A_i^{i+1} - 0) \times X) \cong B_*(X).
\]

From the extended homotopy property we can deduce that the smooth pullback map \( B_*(X) \to B_{*+i+1}(A_i^{i+1} \times X) \) is an isomorphism. On the other hand, the first map in the localization sequence

\[
B_*(X) \xrightarrow{s*} B_*(A_i^{i+1} \times X) \to B_*((A_i^{i+1} - 0) \times X) \to 0
\]

is zero because \( A_i^{i+1} \times X \) is a trivial vector bundle over \( X \), \( s^*s_* \) corresponds to its top Chern class (which vanishes), and because \( s^* \) is an isomorphism. This shows that the pullback map \( B_*(X) \to B_{*+i+1}((A_i^{i+1} - 0) \times X) \) is an isomorphism, and we obtain an isomorphism

\[
B_*(X) \to B_{*+n(i+1)}(U_i \times X)
\]

given by the pullback associated to the natural projection. \( \square \)

Remark 3.4. Suppose \( B_* \) was already a \( G \)-equivariant theory. Nothing stops us from forming a \( T \)-equivariant theory out of that with the procedure described above, although the torus \( T \) must act on the varieties by \( G \)-morphisms. To get a sensible theory one must choose the \( G \)-action on the approximations \( U_i \) of \( ET \) to be the trivial ones. It is not hard to check that the above theorem holds in this setting as well if \( T \) has a trivial \( G \)-action. Note, however, that if either the \( U_i \) or \( T \) had a nontrivial \( G \)-action, then the above theorem might not hold, as the bundles would "lose" their triviality.

For a \( T \)-representation \( W \), the maps \( U_i \times^T W \to U_i/T \) are possibly nontrivial vector bundles. For example, if \( W \) is the one dimensional representation with action given by the formula

\[
(\lambda_1, ..., \lambda_n)w = \lambda_j w,
\]

then it is easy to see that the bundle \( U_i \times^T W \to U_i/T \) corresponds to \( \pi_j^*O(-1) \) (this can be seen from the transition maps). Therefore the first equivariant Chern class of \( W \) is \( \xi_j \). The following result abuses this phenomenon to give a nice geometric interpretation for certain quotients.
Proposition 3.5. Let $W$ be a $T$-representation of dimension $n$. Then $B^T_{*+n}((W-0) \times X)$ is the quotient of $B^T_*(X)$ by the image of the top equivariant Chern class of $W$. Moreover, the Künneth morphism

$$B^T_*(W-0) \otimes_{B^*_T} B^T_*(X) \to B^T_*((W-0) \times X)$$

is an isomorphism.

Proof. As the maps $U_i \times^T (W \times X) \to U_i \times^T X$ are vector bundles, the Künneth isomorphism

$$B^T_*(W) \otimes_{B^*_T} B^T_*(X) \to B^T_*(W \times X)$$

follows from the extended homotopy property. Indeed, it is easy to show using only the basic properties of Borel-Moore exterior product that taking the exterior product with the fundamental class of a vector bundle exactly coincides with the associated pullback morphism.

The localization sequence yields the following commutative diagram:

$$
\begin{array}{c}
B^T_*(pt) \otimes_{B^*_T} B^T_*(X) \longrightarrow B^T_*(W) \otimes_{B^*_T} B^T_*(X) \to B^T_*(W-0) \otimes_{B^*_T} B^T_*(X) \to 0 \\
\cong \quad \cong \\
B^T_*(X) \longrightarrow B^T_*(W \times X) \longrightarrow B^T_*((W-0) \times X) \longrightarrow 0
\end{array}
$$

which gives the Künneth-formula for $(W-0) \times X$ by 5-lemma. To prove the claim, it thus suffices to consider the localization sequence

$$B^T_*(pt) \xrightarrow{i^*} B^T_*(W) \to B^T_*(W-0) \to 0.$$ 

As the zero-section pullback $i^*$ is an isomorphism, and as $i^*i_*$ corresponds to the top equivariant Chern class of the bundle $W$, we can identify $B^T_*(W-0)$ with the quotient of $B^T_*(pt)$ by the image of $c^T_n(W)$ together with a degree shift. $\square$

As an immediate corollary, we obtain a generalization of analogue of a statement in [13] for the algebraic cobordism $\Omega_*:

Corollary 3.6. The natural surjection $B^T_* \to B^*$, obtained by setting $\xi_1, \ldots, \xi_n$ to be zero, gives an isomorphism

$$B^* \otimes_{B^*_T} B^T_*(X) \xrightarrow{\sim} B_*(X).$$
Proof. We already know that $B^T_{r+n}(T \times X) = B_*(X)$, where $n$ is the rank of the torus $T$. Moreover, $T = (W_1 - 0) \times \cdots \times (W_n - 0)$ where $W_i$ are the standard one dimensional coordinate representations of $T$, and hence by the previous lemma taking the product with $T$ corresponds algebraically to setting the variables $\xi_i$ to be zero. But setting $\xi_1, ..., \xi_n = 0$ is exactly how one obtains the natural map $B^T_* \to B^*$, so we are done.

Therefore the equivariant groups $B^T_*$ determine the ordinary groups $B_*$. If the $T$-action is trivial, this does not help much, but as we shall see soon, a natural action can help very much in determining the structure.

3.2 A decomposition theorem for smooth toric varieties

Now we turn our attentions to toric varieties. Throughout this section $X_\Delta$ will denote a toric variety with the natural torus $T$. The following lemma will provide a basis for a decomposition theorem:

Lemma 3.7. Let $X_\Delta$ be a nonsingular toric variety, and let $\sigma \in \Delta$ be a maximal cone (so that the orbit $O_\sigma$ is closed). Now the inclusion $i : O_\sigma \to X_\Delta$ induces an injection

$$i_* : B^T_*(O_\sigma) \to B^T_*(X_\Delta).$$

Proof. Without loss of generality we may assume that the cone $\sigma$ is generated by $e_1, ..., e_r$ so that the open set $U_\sigma$ corresponding to $\sigma$ is

$$\text{Spec}(k[x_1, ..., x_r, x_{r+1}^\pm, ..., x_n^\pm]) = \mathbb{A}^r \times O_\sigma.$$

Denote by $j$ the obvious inclusion $\sigma \to U_\sigma = W \times O_\sigma$ where $W$ is the $T$ representation with action

$$(\lambda_1, ..., \lambda_n)(w_1, ..., w_r) = (\lambda_1 w_1, ..., \lambda_r w_r)$$

We know that $j^* j_*$ corresponds to the equivariant top Chern class of $W$. As $W$ splits into the direct sum of the natural coordinate representations $V_1 \oplus \cdots \oplus V_r$, the top Chern class of $W$ is just $c_1(V_1) \cdots c_1(V_r)$, i.e., multiplication by $\xi_1 \cdots \xi_r$. On the other hand, $B^T_*(O_\sigma)$ is isomorphic to $B^T_*/(\xi_{r+1}, ..., \xi_n)$ with a shift, from which we conclude that the Chern class of $W$ acts injectively on it, and hence the map $j_*$ must be injective as well.

We can use this to show that also $i_*$ is injective. Denote by $u$ the natural open inclusion $U_\sigma \to X_\Delta$. Now the transverse Cartesian square

$$\begin{array}{ccc}
O_\sigma & \xrightarrow{j} & U_\sigma \\
\downarrow{1} & & \downarrow{u} \\
O_\sigma & \xrightarrow{i} & X_\Delta
\end{array}$$
tells us that \( j_* = u^* i_* \), and hence the injectivity of \( i_* \) follows from that of \( j_* \).

We record an immediate corollary.

**Corollary 3.8.** Let \( i : Z \rightarrow X_\Delta \) be a closed equivariant embedding to a smooth toric variety \( X_\Delta \), i.e., \( Z \) is a closed subvariety that is a union of orbits. Now the induced proper pushforward map \( i_* : B^T_*(Z) \rightarrow B^T_*(X_\Delta) \) is injective.

**Proof.** We proceed by induction on the number of orbits in \( Z \), case 0 being trivial. Let \( O \) be a minimal orbit in \( Z \). As \( Z \) is closed inside \( X_\Delta \), we see that \( O \) is a minimal orbit inside \( X_\Delta \) as well; denote by \( U \) and \( V \) the open complements of \( O \) in \( X_\Delta \) and \( Z \) respectively.

The previous lemma yields us the diagram

\[
\begin{array}{c}
0 \rightarrow B^T_*(O) \rightarrow B^T_*(Z) \rightarrow B^T_*(V) \rightarrow 0 \\
| \quad 1 \quad | \quad | \\
0 \rightarrow B^T_*(O) \rightarrow B^T_*(X_\Delta) \rightarrow B^T_*(U) \rightarrow 0
\end{array}
\]

where the rightmost vertical map is injective by induction. Thus the claim follows as a simple application of 5-lemma.

We can use the Lemma 3.7 to arrive at the following decomposition result. Suppose we have a nonsingular toric variety \( X_\Delta \). We can remove all the cones from \( \Delta \) one by one by choosing an arbitrary maximal cone and removing its interior. By the previous result, at each step we have the short exact sequence

\[
0 \rightarrow B^T_*(O_\sigma) \rightarrow B^T_*(X_\Delta) \rightarrow B^T_*(X_\Delta') \rightarrow 0
\]

where \( B^T_*(O_\sigma) \) is isomorphic to shifted copy of \( B^*_T/(\xi'_1, \ldots, \xi'_r) \), where \( \xi'_j \) are the first Chern classes of the line bundles corresponding to linear forms orthogonal to \( \sigma \). (For more details, see the Section 3.3 following this section).

This is very much in line of the structural results obtained in [7] for Chow groups of toric varieties. We can define a map

\[
\bigoplus_{\sigma \in \Delta} (V_\sigma)_{B^*_T} \rightarrow B^T_*(X_\Delta),
\]

where the map \( (V_\sigma)_{B^*_T} \rightarrow B^T_*(X_\Delta) \) is sends \( b \in B^*_T \) to \( b \times i_*[1_{V_\sigma}] \), where \( i \) is the closed immersion \( V_\sigma \rightarrow X_\Delta \). This map corresponds to the decomposition of \( B^T_*(X_\Delta) \) achieved in the previous paragraph, so especially the map is surjective.
We understand the generators of the equivariant homology group $B^*_T(X)$ over the coefficient ring $B_T$, so we are left with the task of characterizing the relations. From the decomposition we can almost immediately conclude that the relations will be generated by those of form
\[
\xi'[V_\tau] = \sum_{\sigma \supset \tau} b_\sigma[V_\sigma],
\]
where $\xi'$ is a Chern class of a line bundle associated to linear form orthogonal to $\tau$, $\sigma$ runs over all the cones of $\Delta$ containing $\tau$ and $b_\sigma \in B_T$. In order to say more, we need to look more closely at the line bundles associated to linear forms.

### 3.3 Line bundles associated to linear forms

Let us have a linear form $m = a_1 e^*_1 + \cdots + a_n e^*_n \in M$. To any such form, we may associate a line bundle
\[
\mathcal{L}_{m,i} = \pi^*_i O(-a_1) \otimes \cdots \otimes \pi^*_n O(-a_n),
\]
where $\pi_j$ is the $j^{th}$ projection $(\mathbb{P}^i)^n \to \mathbb{P}^i$. Passing to limit $i \to \infty$, its Chern class defines an element $\xi_m \in B_T$. This section is devoted to showing that these elements behave as we would assume them to behave. Note that $\xi_i$ is the Chern class of the line bundle associated to $e^*_i$. We note that this defines a map from the dual lattice $M$ to the topological group of the elements of the ideal generated by $\xi_1, \ldots, \xi_n$ considered with the formal group operation $+F$.

**Proposition 3.9.** Let $m_1, \ldots, m_r$ be linearly independent forms. Now the $B^*$-algebra generated by $\xi_{m_i}$ is the free $B^*$-algebra $B^*[\xi_{m_1}, \ldots, \xi_{m_r}]$.

Moreover, if $m_i$ generate all the integral forms, then the quotient of $B_T$ by the ideal generated by the $\xi_{m_i}$ is naturally identified with $B^*$.

**Proof.** Let $m_i = a^i_1 e^*_1 + \cdots + a^i_n e^*_n$, where $a^i_j \in \mathbb{Z}$. By the properties of formal group law $F$ of the theory $B_*$, we see that
\[
c_1(\mathcal{L}_{m_i}) = a^i_1 \cdot_F \xi_1 + \cdots + a^i_n \cdot_F \xi_n
= a^i_1 \xi_1 + \cdots + a^i_n \xi_n + \mathcal{O}(\text{quadratic in } \xi_i).
\]

Using this, one can show that the algebra generated by the respective Chern classes in $B^*((\mathbb{P}^i)^n) = B^*[\xi_1, \ldots, \xi_n]/(\xi_1^{j+1}, \ldots, \xi_n^{j+1})$ only satisfy the obvious relations $\xi_i^{j+1} = 0$.

For the second claim, we first observe from the formal group law that $\xi_{m_i}$ is always contained in the ideal generated by $\xi_1, \ldots, \xi_n$, and hence the natural map $B_T \to B^*$ descends to a map $B_T/(\xi_{m_1}, \ldots, \xi_{m_r})$. But as $m_i$ generate, $x_j$ can be expressed as an integral sum of $m_i$, and hence $\xi_j$ can be expressed as a formal sum of $\xi_{m_1}, \ldots, \xi_{m_r}$. From this it follows that $\xi_j$ is contained in the ideal $(\xi_{m_1}, \ldots, \xi_{m_r})$, and the claim follows. □
The following lemma connects one dimensional representations of $T$ with the $\xi_i$.

**Lemma 3.10.** Let $V$ be an one dimensional representation of $T$ with the action

$$(\lambda_1, ..., \lambda_n)v = \lambda_1^{a_1} \cdots \lambda_n^{a_n}v.$$  

Now the first equivariant Chern class of $V$ equals $a_1 \cdot F \xi_1 + F \cdots + F a_n \cdot F \xi_n$.

**Proof.** The proof is a simple matter of looking at the transition maps of the line bundles

$$U_i \times T V \rightarrow U_i/T = (\mathbb{P}^i)^n$$

to notice that they correspond to those of the line bundle

$$\mathcal{L}_{m,i} = \pi_1^*O(-a_1) \otimes \cdots \otimes \pi_n^*O(-a_n),$$

from which the claim follows by earlier results. $\square$

We also record a statement whose proof is essentially contained in that of the Lemma 3.7.

**Proposition 3.11.** Let $\sigma$ be a nonsingular cone having a lattice basis $v_1, ..., v_r$ in $N_R \cong \mathbb{R}^n$. Now

$$B_T^r(O_\sigma) \cong B_T^r/(\xi_{m,r+1}, ..., \xi_{m,n}),$$

with an appropriate degree change, where $O_\sigma$ is the (nonclosed) orbit corresponding to $\sigma$, and $m_i$ form an integral basis for the linear forms $m \in M$ such that $m\sigma = 0$.

**Proof.** Extend $m_{r+1}, ..., m_n$ to an integral basis $m_1, ..., m_n$ of $M$. Now the open set $U_\sigma$ corresponding to the cone $\sigma$ is naturally identified with

$$\text{Spec}(k[\chi^{m_1}, ..., \chi^{m_r}, \chi_{m_{r+1}}, ..., \chi_{m_n}])$$

by construction. The torus orbit $O_\sigma$ is given as the vanishing set of the ideal generated by $\chi_{m_1}, ..., \chi_{m_r}$. It splits as the product

$$\text{Spec}(k[\chi^{m_{r+1}}]) \times \cdots \times \text{Spec}(k[\chi^{m_n}])$$

of $T$-varieties, so in order to prove claim it is enough by the previous lemma and 3.5 to look at the $T$-action on $U = \text{Spec}(k[\chi^m])$, where $m$ is a linear form $a_1 e_1^* + \cdots + a_n e_n^*$, and to make sure that it coincides with

$$(\lambda_1, ..., \lambda_n)u = \lambda_1^{a_1} \cdots \lambda_n^{a_n}.$$  

But as this action arises from the map of $k$-algebras

$$k[\chi^m] \rightarrow k[x_1^{\pm 1}, ..., x_n^{\pm 1}] \otimes_k k[\chi^m]$$

$$\chi^m \mapsto x_1^{a_1} \cdots x_n^{a_n} \otimes \chi^m,$$

this is certainly true, and hence we are done. $\square$
The elements $\xi_i$ and divisors of $X_\Delta$

In order to make geometric sense about what it means when we multiply by the elements $\xi_i$, we will express them with the help of divisors of $X_\Delta$. This is made easy by the happy accident that the intermediate approximations $U_N \times^T X_\Delta$ turn out to be toric varieties. (We replaced $i$ with $N$ to make the notation in the following less painful).

Recall that $U_N = (\mathbb{A}^{N+1} - 0)^n$ where the $i^{th}$ coordinate of $T$ acts diagonally on the $i^{th}$ copy of $\mathbb{A}^{N+1} - 0$. One immediately observes that this is a toric variety: it is given by the fan one obtains from the standard $N + 1$-cone $\langle e_0, ..., e_N \rangle_{\mathbb{R}_{\geq 0}}$ in $\mathbb{R}^{N+1}$ after removing the interior of the maximal cone. Moreover, the product of toric varieties is given by the product of respective fans, so $U_N \times X_\Delta$ is given by the product fan inside the $n(N + 2)$ dimensional vector space, whose basis we are going to denote by

$$e_0^1, ..., e_N^1; e_0^2, ..., e_N^2; ..., e_0^n, ..., e_N^n; e_1, ..., e_n,$$

where $e_j^i$ is the $j^{th}$ basis vector for the space corresponding to the $i^{th}$ copy of $\mathbb{A}^{N+1} - 0$, and $e_i$ are the basis vectors for the space corresponding to $X_\Delta$. In order to get the reduced product $V_N \times^T X_\Delta$, we will need take the quotient under the action of $T$, and as the action of $T$ on $V_N \times X_\Delta$ is taken to be the antidiagonal one, this corresponds to taking the quotient by the subspace generated by the vectors

$$e_1 - (e_0^1 + ... + e_N^1), ..., e_n - (e_0^n + ... + e_N^n).$$

By choosing the images of

$$e_0^1, ..., e_{N-1}^1, e_0^2, ..., e_{N-1}^2, ..., e_0^n, ..., e_{N-1}^n, e_1, ..., e_n$$

to form a basis for the quotient space, we see that we get a fan that is almost just the product fan corresponding to $\mathbb{P}^N \times \cdots \times \mathbb{P}^N \times X_\Delta$, but the "back rays" corresponding to the images of $e_N^i$, are not just simply $-(\mathfrak{c}_0 + ... + \mathfrak{c}_{N-1})$, but instead $\mathfrak{c}_i - (\mathfrak{c}_0 + ... + \mathfrak{c}_{N-1})$. Thus we see that our space is an $X_\Delta$-bundle over $(\mathbb{P}^N)^n$.

From the last observation, one immediately gets the identity

$$-D_i = \sum_{\rho \in \Delta} \langle e_i^*, v_{\rho} \rangle D_{\rho},$$

in the divisor class group of $U_N \times^T X_\Delta$, using standard results of toric varieties. Here $D_i$ is the divisor class associated to the ray generated by $\mathfrak{c}_i - (\mathfrak{c}_0 + ... + \mathfrak{c}_{N-1})$, $\rho$ runs over all the rays of $\Delta$, $v_\rho$ is the primitive lattice vector generating $\rho$, and $D_\rho$ is the divisor associated to the ray $\rho$.

This result is readily interpreted in the context of sheaves as saying that

$$\pi_i^* \mathcal{O}(-1) = \mathcal{O}(D_{\rho_1}) \otimes \langle e_i^*, v_{\rho_1} \rangle \otimes \cdots \otimes \mathcal{O}(D_{\rho_r}) \otimes \langle e_i^*, v_{\rho_r} \rangle,$$

where $\pi_i^*$ is the pullback map.
where $\rho_i$ are the rays of $\Delta$ enumerated in some order. This gives the following identity of Chern classes:

$$\xi_i = \langle e^*_{i}, v_{\rho_1} \rangle \cdot F c_1(D_{\rho_1}) + F \cdots + F \langle e^*_{i}, v_{\rho_1} \rangle \cdot F c_1(D_{\rho_r})$$

where $c_1(D_{\rho_i})$ is a shorthand for $c_1(O(D_{\rho_i}))$. To connect this with the previous decomposition theorem, let $V_\tau$ be an orbit closure corresponding to the fan $\tau$, and let $m$ be a form orthogonal to $\tau$. Now we get the relation

$$\xi_m = \langle m, v_{\rho_1} \rangle \cdot F c_1(D_{\rho_1}) + \cdots + F \langle m, v_{\rho_1} \rangle \cdot F c_1(D_{\rho_r}),$$

and intersecting this with $V_\tau$, it gives us

$$\xi_m[V_\tau] = \left( \langle m, v_{\rho_1} \rangle c_1(D_{\rho_1}) + \cdots + \langle m, v_{\rho_1} \rangle c_1(D_{\rho_r}) \right) \cap [V_\tau] = \sum_{\sigma \supset \tau} \langle m, n_\sigma \rangle [V_\sigma],$$

These generate all the relations over $B_T$.

**Remark 3.12.** One can easily read off the description of the Chow groups of toric varieties achieved in [7] from this description, although one gets it immediately only for smooth toric varieties. As the formal group law of $\text{CH}_*$ is the additive one, the relations we get for the equivariant Chow groups are

$$\xi_m[V_\tau] = \left( \langle m, v_{\rho_1} \rangle c_1(D_{\rho_1}) + \cdots + \langle m, v_{\rho_1} \rangle c_1(D_{\rho_r}) \right) \cap [V_\tau] = \sum_{\sigma \supset \tau} \langle m, n_\sigma \rangle [V_\sigma],$$

where $\sigma$ runs over all the cones $\sigma$ of one dimension higher than $\tau$ containing $\tau$, and $n_\sigma$ is the primitive generator for the ray of $\sigma$ not in $\tau$. Passing to nonequivariant case, we obtain

$$\sum_{\sigma \supset \tau} \langle m, n_\sigma \rangle [V_\sigma] = 0$$

exactly corresponding to the relations given in [7]. We note, however, that here we fully described the $T$-equivariant Chow groups as well, at least for smooth toric varieties. The presentation in the singular case should follow easily from the descent exact sequence, and the proof left as an exercise for the reader.

### 3.4 Application – algebraic cobordism $\Omega_* (X_\Delta)$ of toric varieties

In this subsection we show the power of the description for homology groups of toric varieties, by showing through examples how it is possible to determine the algebraic cobordism $\Omega_* (X_\Delta)$ for an arbitrary toric variety $X_\Delta$. There does not seem to be a single nice formula
for all the relations as in the case of Chow groups, but the computation is still more or less mechanical. The main idea is to take a toric resolution \( X_\Delta \rightarrow X_\Delta \) of \( X_\Delta \), compute the algebraic cobordism of the smooth toric variety \( X_\Delta \), and then use the descent sequence of \([11]\) to determine \( \Omega_\ast(X_\Delta) \).

We recall here some aspects of the main result of \([11]\) in greater detail. For a toric resolution \( \pi : X_\Delta \rightarrow X_\Delta \), the induced proper pushforward \( \Omega_\ast(X_\Delta) \rightarrow \Omega_\ast(X_\Delta) \) is surjective, and the kernel is generated by the kernel of the map \( Z_\ast(X_\Delta) \rightarrow Z_\ast(X_\Delta) \) of cobordism cycles. These relations were enough to be able to define a map \( s : \Omega_\ast(X_\Delta) \rightarrow \Omega_\ast(X_\Delta) \) such that \( \pi s = 1 \). Here \( \Omega_\ast(X_\Delta) \) is \( \Omega_\ast(X_\Delta) \) with extra relations.

In our context, the distinguished lifting map has an easy description. Namely, for each cone \( \sigma \) in \( \Delta \) we pick one cone \( \tilde{\sigma} \) in \( \tilde{\Delta} \) whose orbit closure maps birationally to that of \( \sigma \). The image of the class of \( \sigma \) in \( \Omega_\ast(X_\Delta) \) will be \( \tilde{\sigma} \) together with some extra stuff that is supported on lower dimension. Thus, in order to make the map to \( \Omega_\ast(X_\Delta) \) well defined, we need simply to add relations between cones in \( \tilde{\Delta} \) that lie in a subdivision of a single cone of \( \Delta \). If the cones have orbit closures that are isomorphic over \( X_\Delta \), then they can simply be identified with each other (this is the case that happens in the following computations), but in the general case when they are merely birational, they differ by something that is supported on lower dimensional orbit closures (higher dimensional cones).

Remark 3.13. Before we begin doing explicit computations, it is useful to record some tricks that may make the calculation easier. Consider the map \( X_\Delta \rightarrow X_\Delta \) between smooth toric varieties induced by blowing up an orbit closure \( V_\sigma \subset X_\Delta \). By \([3,8]\) we have the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & B^T_\ast(E) & \longrightarrow & B^T_\ast(X_\Delta) & \longrightarrow & B^T_\ast(U) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^T_\ast(V_\sigma) & \longrightarrow & B^T_\ast(X_\Delta) & \longrightarrow & B^T_\ast(U) & \longrightarrow & 0 \\
& & & & & & \text{1} & & \\
\end{array}
\]

giving us a Mayer-Vietoris type sequence

\[
0 \rightarrow B^T_\ast(E) \rightarrow B^T_\ast(V_\sigma) \oplus B^T_\ast(X_\Delta) \rightarrow B^T_\ast(X_\Delta) \rightarrow 0,
\]

where the injectivity of the left vertical maps is another application of \([3,8]\). As we are dealing with smooth blowups, the exceptional divisor \( E \) is a projective bundle of rank \( r \) over \( V_\sigma \), and hence the map \( B^T_\ast(E) \rightarrow B^T_\ast(V_\Delta) \) has a section induced by the smooth pullback and applying the first Chern class of \( \mathcal{O}_E(1) \) \( r \) times. Therefore we get the following exact sequence

\[
0 \rightarrow \bigoplus_{i=1}^r B^T_{\ast-i}(V_\sigma) \rightarrow B^T_\ast(X_\Delta) \rightarrow B^T_\ast(X_\Delta) \rightarrow 0.
\]
If the above sequence splits, then we obtain

\[ B_*(\hat{X}_\Delta) \cong B_*(X_\Delta) \oplus \bigoplus_{i=1}^{r} B_*(V_\sigma) \]

simply by tensoring with \( B^* \) over \( B^*_T \).

**Example 3.14.** Consider the complete fan \( \Delta \) spanned by the rays \( \tau_1 = \langle 1, 0 \rangle \), \( p_n = \langle -1, n \rangle \), \( \tau_3 = \langle -1, 0 \rangle \) and \( q_m = \langle -1, m \rangle \). We obtain a resolution \( \hat{\Delta} \) by adding the rays \( \tau_2 = \langle 0, 1 \rangle \), \( \tau_3 = \langle 0, -1 \rangle \), \( p_i \langle -1, i \rangle \) and \( q_j = \langle -1, j \rangle \) for \( i = 1 \ldots n - 1 \) and \( j = 1 \ldots m \).

In order to compute \( \Omega_*(\hat{X}_\Delta) \), we first note that the relations generated by the rays of the fan simply identify all the maximal cones with each other. Hence we are left with the representation

\[
\Omega_*(\hat{X}_\Delta) = \langle s, \tau_1, \tau_2, \tau_3, p_1, \ldots, p_n, q_1, \ldots, q_m, \sigma \mid \\
\tau_1 - F (\tau_3 + F p_1 + F + \cdots + F p_n + F q_1 + F \cdots + F q_m) = 0, \\
p_1 + F 2 \cdot F p_2 + F \cdots + F n \cdot F p_n - F (q_1 + F 2 \cdot F q_2 + F \cdots + F m \cdot F q_m) = 0 \rangle.
\]

In order to arrive at \( \Omega_*(X_\Delta) \), we must add additional relations

\[
\tau_2 = \tau_4 = p_1 = \cdots = p_{n-1} = q_1 = \cdots = q_{m-1} = [\mathbb{P}^1] \times pt = -a_{11} \sigma,
\]

and as

\[
p_1 + F 2 \cdot F p_2 + F \cdots + F n \cdot F p_n = \sum_{i=1}^{n} ip_i - \sum_{i=1}^{n-1} i(i - 1) - \frac{n(n - 1)}{2} + \sum_{i=1}^{n-1} i(i + 1)\]

\[
= \sum_{i=1}^{n} ip_i + \frac{n(n - 1)}{2}
\]

\[
= np_n,
\]

(first linear terms, then two terms from self intersections, and finally term coming from intersections between consecutive \( p_i \)) and as \( p_i \) and \( q_j \) don’t intersect, we arrive at the description

\[
\Omega_*(X_\Delta) = \langle s, \tau_3, p_n, q_m, \sigma \mid np_n - mq_m = 0 \rangle.
\]

**Example 3.15.** Consider next \( \Delta \) to be the fan over the cube, as in the end of Chapter 2 of [4]. The cube has vertices at points \((\pm 1, \pm 1, \pm 1)\) and we consider this as a rational polytope in the lattice generated by the vertices of the cube. The fan is the fan whose cones are generated by the faces, edges and vertices of the cube. As a toric resolution \( X_\Delta \to X_\Delta \)
we subdivide each face of the cube diagonally, in order to obtain a fan corresponding to $\mathbb{P}^3$ blown up at the four $T$-fixed points.

As $X_\Delta$ it is obtained by blowing up 4-points on the $\mathbb{P}^3$, it follows from the earlier remark that

$$\Omega_*(X_\Delta) = \langle s, \tau, \sigma, \rho, \tau_1, \sigma_1, ..., \tau_4, \sigma_4 \rangle,$$

where the first four basis elements come from the $\mathbb{P}^3$ before blowup, and the next eight are the new elements introduced by the four blowups. In order to compute $\Omega_*(X_\Delta)$ we need to set the all the classes corresponding to the two dimensional cones of $\mathbb{P}^3$ to equal $[\mathbb{P}^1] \times pt = -a_{11} \rho$. Note that these classes are no longer equal in $\Omega_*(X_\Delta)$, even though they would be in $\mathbb{P}^3$, and hence we need more relations than just setting $\sigma = -a_{11} \rho$. A straightforward computation shows that the "extra" relations merely identify $\sigma_1, ..., \sigma_4$ with each other, and hence

$$\Omega_*(X_\Delta) = \langle s, \tau, \tau_1, \tau_2, \tau_3, \tau_4, \sigma_1, \rho \rangle.$$ 

Note that there are 5 basis vectors of degree 2 but only 1 of degree 1 showing how the symmetry between degrees one would expect between degrees for smooth complete toric varieties (by the results of the next section) breaks down for nonsimplicial toric varieties.

### 3.5 Stanley-Reisner representations and piecewise functions on a fan

Let us take another look at the relations of $B^*_T(X_\Delta)$ for a nonsingular toric variety $X_\Delta$ found in the end of the Section 3.3. If we consider this as a $B^*_T$-algebra where the product is taken to be the intersection product, then the relations can clearly be written as

$$\xi_i = \langle e_i^*, v_{\rho_1} \rangle \cdot_f \rho_1 + \cdots + \rho_r \langle e_i^*, v_{\rho_1} \rangle \cdot_f \rho_r,$$

where $i$ ranges from 1 to $n$ (the dimension of $X_\Delta$), and $\rho_i$ ranges over all the rays of the fan $\Delta$. Thus, if we consider the natural cohomology ring $B^*_T(X_\Delta) = B^*\mathbb{T}(X_\Delta)$ of a smooth variety $X_\Delta$, we obtain the following result.

**Theorem 3.16.** The ring $B^*_T(X_\Delta)$ is isomorphic to the graded power series ring $B^*[\rho_1, ..., \rho_r]$ modulo the Stanley-Reisner relations

$$\rho_{i_1} \cdots \rho_{i_j} = 0$$

whenever $\rho_{i_1}, ..., \rho_{i_j}$ do not span a cone in $\Delta$. 

23
Proof. Certainly the Stanley-Reisner ring $R$ maps to $B_T^*(X_{\Delta})$. Consider a monomial of the form
\[ \rho_1^{n_1} \cdots \rho_j^{n_j}, \]
where the exponents are strictly positive, and the rays span a cone $\sigma$, which we are going to call the support of the monomial. Hence we are interested in evaluating
\[ \rho_1^{n_1-1} \cdots \rho_j^{n_j-1} \sigma, \]
which is essentially a problem of self intersection. As the fan is nonsingular, we may choose a linear form $m$ such that $\langle m, \rho_i \rangle = 1$, and $\langle m, \rho_{i_2} \rangle = \cdots = \langle m, \rho_{i_r} \rangle = 0$. Now we can use the relations between the divisors to move $\rho_i$ to some power series including other rays $\rho$ that are not in the cone $\sigma$ and $-F_\rho$. We now make a choice of such $m_\rho, \sigma$ for all $\rho$ and for all $\sigma$, and fix it for the rest of the proof. It is clear that for a cone $\sigma$, all the forms $m_\rho, \sigma$ are linearly independent.

Suppose now that we have a nonzero element of the graded Stanley-Reisner ring. We are going to show that it determines a nonzero element in $B_T^*(X_{\Delta})$. Without loss of generality, we may assume that the power series is homogeneous, i.e., it is of form
\[ \sum_{i_1, \ldots, i_n} b_{i_1, \ldots, i_n} \rho_1^{i_1} \cdots \rho_r^{i_r} \]
where $b_{i_1, \ldots, i_r} \in B_k - (i_1 + \cdots + i_r)(pt)$ for some fixed $k$. Choose a minimal cone $\sigma$ that supports something nontrivial, and take the homogeneous term
\[ H'(\rho_{i_1}, \ldots, \rho_{i_j}) \]
of minimal degree as a polynomial in $\rho_i$ (so the degree of the base ring is ignored) that is supported exactly on $\sigma$. Let
\[ H(\rho_{i_1}, \ldots, \rho_{i_j}) = H'(\rho_{i_1}, \ldots, \rho_{i_j})/(\rho_{i_1} \cdots \rho_{i_j}) \]
so that $H'(\rho_{i_1}, \ldots, \rho_{i_j}) = H(\rho_{i_1}, \ldots, \rho_{i_j})\sigma$. Using the choices of $m_\rho, \sigma$, we see that
\[ H(\rho_{i_1}, \ldots, \rho_{i_j}) \sigma = H \left( \sum_l \langle m_{\rho_{i_1}, e_l} \rangle \xi_l, \ldots, \sum_l \langle m_{\rho_{i_j}, e_l} \rangle \xi_l \right) \sigma \]
modulo things that are either of higher degree or supported on some other cone. This nonzero term cannot be killed by anything by minimality assumptions, and therefore the map from $R$ is injective. The proof of surjectivity is of the same spirit, and is left to the reader. \[\square\]
This extends well known results in the cases of Chow-groups, K-theory and algebraic cobordism, and shows that in all of these cases the reason of the Stanley-Reisner ring appearing is the same.

Another well known representation for the equivariant cohomology rings of a smooth variety are in terms of the global sections of a sheaf of functions (of some kind) on a fan. Examples include $T$-equivariant Chow rings, which are described in terms of polynomial functions on fan, and $T$-equivariant algebraic cobordism, where we have graded power series on the fan. These kinds of representations have the advantage of usually extending to describe the operational cohomology rings of arbitrary toric varieties, as is evident from the work done in [1], [18] and [10]. We can obtain similar description for arbitrary theory $B_T^*$, at least in the smooth case.

We quickly recall what we mean by functions on a fan. Consider a smooth toric variety $X_\Delta$. Now the inclusions of the orbits $O_\sigma \to X_\Delta$ are l.c.i. for all cones $\sigma \in \Delta$, and hence we get a natural map

$$i : B_T^*(X) \to \prod_\sigma B_T^*(O_\sigma)$$

induced by the l.c.i. pullbacks. We can think $B_T^*(O_\sigma)$ as a stalk of a sheaf at the cone $\sigma$, and by the description of the equivariant homologies of $B_T^*(O_\sigma)$ we obtained earlier, we see that for a inclusion $\tau \subset \sigma$ of fans, we get a surjective restriction morphism

$$B_T^*(O_\sigma) \to B_T^*(O_\tau)$$

and the basic functorality properties of these restriction morphisms imply that the rings at cones glue together to give a sheaf on the fan, where the open sets are taken to be the subfans. We call this the sheaf of graded power series on the fan $\Delta$.

**Theorem 3.17.** Let everything be as above. Now the map $i$ identifies $B_T^*(X_\Delta)$ with the global sections of the sheaf of graded power series on $\Delta$.

**Proof.** The proof is mostly formal from the Stanley-Reisner description. Consider at first a single l.c.i. pullback

$$B_T^*(X_\Delta) \to B_T^*(O_\sigma)$$

which, we recall, is a ring homomorphism. Let $\sigma$ be spanned by rays $\tau_{i_1}, \ldots, \tau_{i_j}$, and note that $B_T^*(O_\sigma)$ can be identified as the graded power series algebra of $\tau_1, \ldots, \tau_r$ over the base ring $B^*$. I claim that the pullback of any monomial including any other ray $\tau$ must be zero. This is because the support of such a monomial does not meet the orbit $O_\sigma$ (although it might meet its closure $V_\sigma$). This proves the injectivity of $i$, and in fact, we see that the pullback to minimal orbits would have been injective already.

Let us then have an element $(f_\sigma)_{\sigma \in \Delta}$, where $f_\sigma \in B_T^*(O_\sigma)$, which corresponds to a global section of the sheaf of graded power series, i.e., this collection respects the restriction
maps. It is clear that we can always find an element of the Stanley-Reisner ring pulling back to this collection, finishing the proof. □

This description should easily generalize to the operational cohomology of an arbitrary toric variety using the techniques of [18], and perhaps assuming some extra compatibility conditions on the theory.

4 Künnefeld formula and a universal coefficient theorem

In this section, we generalize the results of [7] concerning Künnefeld formula and operational cohomology rings to arbitrary Borel-Moore homology theories satisfying certain extra assumptions. This will be fairly straightforward after all the work in the previous section. Throughout the section $B_*$ will denote an oriented Borel-Moore homology theory.

4.1 Künnefeld Formula

The purpose of this subsection is to prove the following:

**Theorem 4.1.** Let $B_*$ be an oriented Borel-Moore homology theory satisfying descent, and let $X_\Delta$ be a toric variety. Now the Künnefeld morphism

$$B_*(X_\Delta) \otimes_{B_*} B_*(Y) \rightarrow B_*(X_\Delta \times Y)$$

is an isomorphism for all varieties $Y$. (We say that $X_\Delta \times Y$ satisfies Künnefeld formula in $B_*$.)

The proof is based to reducing this question to the $T$-equivariant case, where $T$ is the natural torus acting on $X_\Delta$. We begin with the case $X_\Delta$ nonsingular.

**Lemma 4.2.** Let $X_\Delta$ be a smooth toric variety, and let $Y$ be a variety with a trivial $T$-action. Now $X_\Delta \times Y$ satisfies Künnefeld formula in $B_*$.

**Proof.** By 3.5 this holds if there is only one cone in $\Delta$, i.e., $X_\Delta$ is a torus. In the general case let $\sigma \in \Delta$ be a maximal cone, and consider the inclusion $i : O_\sigma \times Y \hookrightarrow X_\Delta \times Y$. As in the proof of 3.7 we assume $\sigma$ to be generated by the standard lattice vectors $e_1, \ldots, e_r$ so that $i^*i_*$ coincides with multiplication by $\xi_1 \cdots \xi_r$, and

$$B^*_T(O_\sigma \times Y) \cong B^*_T/(\xi_{r+1}, \ldots, \xi_n) \otimes_{B^*_T} B^*_T(Y).$$

But as the action of $T$ on $Y$ is trivial, we know by 3.2 that $B^*_T(Y)$ is the completed tensor product $B^*_T \otimes_{B_*} B_*(Y)$. It is therefore clear that multiplication by $\xi_1 \cdots \xi_r$ is injective in $B^*_T(O_\sigma \times Y)$, and therefore $i_*$ must be an injection.

Now localization exact sequence gives us the commuting diagram

26
\[
\begin{array}{c}
B_*^T(O_\sigma) \otimes_{B_*^T} B_*^T(Y) \rightarrow B_*^T(X_\Delta) \otimes_{B_*^T} B_*^T(Y) \rightarrow B_*^T(X_{\Delta'}) \otimes_{B_*^T} B_*^T(Y) \rightarrow 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & B_*^T(O_\sigma \times Y) & \rightarrow & B_*^T(X_\Delta \times Y) & \rightarrow & B_*^T(X_{\Delta'} \times Y) & \rightarrow & 0 \\
\end{array}
\]

with exact rows, where the leftmost vertical map is known to be an isomorphism by \ref{3.7} and the rightmost vertical map is known to be an isomorphism by induction on the number of cones in $\Delta$. It follows from 5-lemma that also the middle vertical arrow is an isomorphism.

To generalize the previous result to singular varieties, we need to assume that the theory $B_*$ satisfies descent (this holds for $\Omega_*$ by \cite{11}, and therefore it holds for almost any sensible oriented BM-homology). It is clear that if $B_*$ satisfies descent, then so does the equivariant version $B_*^T$.

**Lemma 4.3.** Let $B_*$ be as above, $X_\Delta$ an arbitrary toric variety, and $Y$ a variety with a trivial $T$-action. Now $X_\Delta \times Y$ satisfies Künneth formula in $B_*^T$.

**Proof.** Pick a toric resolution $X_{\tilde{\Delta}}$ of $X_\Delta$, and denote $X = X_\Delta$, $\tilde{X} = X_{\tilde{\Delta}}$. By descent assumption, we have the commutative diagram

\[
\begin{array}{cccccc}
B_*^T(\tilde{X} \times_X \tilde{X}) \otimes_{B_*^T} B_*^T(Y) & \rightarrow & B_*^T(\tilde{X}) \otimes_{B_*^T} B_*^T(Y) & \rightarrow & B_*^T(X) \otimes_{B_*^T} B_*^T(Y) & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
B_*^T((\tilde{X} \times_X \tilde{X}) \times Y) & \rightarrow & B_*^T(\tilde{X} \times Y) & \rightarrow & B_*^T(X \times Y) & \rightarrow & 0 \\
\end{array}
\]

Where the middle vertical map is known to be isomorphism by the previous lemma. In order to show that the rightmost vertical map is an isomorphism, it is enough to show that the leftmost vertical map is a surjection.

As $\tilde{X}$ and $X$ are toric varieties and the map $\tilde{X} \rightarrow X$ equivariant, we see that $\tilde{X} \times_X \tilde{X}$ has a filtration by tori. Indeed, if $\tilde{O}$ is an orbit in $\tilde{X}$ and $O$ is its image in $X$, the restricted map $\tilde{O} \rightarrow O$ is essentially just the projection $(\alpha_1, ..., \alpha_r) \mapsto (\alpha_1, ..., \alpha_s)$, and hence $\tilde{O} \times_{\tilde{O}} O$ is isomorphic to a torus (although this is no longer necessarily a single $T$ orbit). Thus $\tilde{X} \times_X \tilde{X}$ is a $T$-linear scheme. We know that a product with a linear variety has a surjective Künneth map by \ref{2.3} so we are done.

Bringing the theorem back to the ordinary case is now easy.
Proof of 4.1. Let $X$ be a toric variety with torus $T$, and let $Y$ be an arbitrary variety. Now we know that the equivariant Künneth map

$$B^*_{B^*}(X) \otimes_{B^*_T} B^*_T(Y) \to B^*_{B^*_T}(X \times Y)$$

is an isomorphism. This isomorphism is preserved after tensoring both sides with $B^*$ considered as a $B^*_T$-algebra in the natural way. On the right hand side, this tensor product equals $B_*(X \times Y)$, and on the left hand side, we get

$$B^* \otimes_{B^*_T} (B^*_X(T) \otimes_{B^*_T} B^*_T(Y)) = (B^* \otimes_{B^*_T} B^*_X(T)) \otimes_{B^*_T} B^*_T(Y) = B_*(X) \otimes_{B^*} B_*(Y).$$

This proves the claim.

Remark 4.4. Note how we get the Künneth formula

$$B^*_X(T) \otimes_{B^*_T} B^*_T(Y) = B^*_X(T) \otimes_{B^*} B_*(Y)$$

in equivariant theory only if we assume $Y$ to have a trivial $T$-action. This can be remedied quite easily: even if we are already working equivariantly over some group $G$, we can still pass to the $T$-equivariant case. Note that the action of $T$ must behave well with respect to whatever action we already had on the varieties: the action of $T$-must commute with that of $G$. This is certainly the case if we set $T$ to act on $Y$ trivially, and if the actions of $G$ and $T$ on $X$ commute. Hence, in this situation we recover the Künneth formula, which we are recording as a theorem to be explicit.

Theorem 4.5. Let $G$ be a linear algebraic group, and let it act on a toric variety $X$ in such a way that the action commutes with the natural $T$-action. Now for any $G$-variety $Y$, the Künneth map

$$B^*_G(X) \otimes_{B^*_G} B^*_G(Y) \to B^*_G(X \times Y)$$

is an isomorphism.

4.2 Universal coefficients for operational cohomology

We are now ready to prove the universal coefficient theorem. Throughout this subsection, $B_*$ is going to denote a ROBM-homology theory, i.e., an oriented Borel-Moore homology theory with refined l.c.i. pullbacks (see [10] for more details). Again $\Omega_*$ is an example of such a theory, as is proved in [15] and [16], so almost any oriented BM-homology that naturally occurs also has refined pullbacks. For any such theory, we may construct the operational bivariant group, and especially we get the operational cohomology theory op$B_*$. The main purpose of this section is to prove the following theorem:
Theorem 4.6. Let $B_*$ be a ROBM-homology theory, and let $X$ be a proper variety having the property that the Künneth formula holds for $X \times Y$ for all $Y$. Now there is a natural isomorphism

$$\text{op}B^*(X) \cong \text{Hom}_{B^*}(B_*(X), B^*).$$

The proof is formally the same proof as the one in [6]. Note that the $B^*$-module $\text{Hom}_{B^*}(B_*(X), B^*)$ has a natural grading as the Hom-module of graded modules over a graded ring. In order to have grading that coincides with the usual cohomological grading, we set $\text{Hom}_{B^*}(B_*(X), B^*)$ to consist of degree preserving $B^*$-linear morphisms $B_{s+k}(X) \to B_s(pt)$. Before embarking on the proof, we quickly review the definition of operational cohomology groups.

Review on operational cohomology groups

Here we recall the construction of operational cohomology groups. Let $X$ be any variety. Now an operational cohomology class $c \in \text{op}B^*(X)$ consists of morphisms

$$c \equiv c_{Y \to X} : B_*(Y) \to B_*(Y)$$

for any morphism $Y \to X$. Moreover, these maps are required to satisfy the following compatibility axioms:

1. Given maps $Y' \xrightarrow{f} Y \to X$, where $f$ is proper, the diagram

$$\begin{array}{c}
B_*(Y') \xrightarrow{c} B_*(Y') \\
\downarrow f_* \hspace{1cm} \downarrow f_* \\
B_*(Y) \xrightarrow{c} B_*(Y)
\end{array}$$

commutes, i.e., operational classes commute with proper pushforward.

2. Given maps $Y' \xrightarrow{f} Y \to X$, where $f$ is smooth, the diagram

$$\begin{array}{c}
B_*(Y) \xrightarrow{c} B_*(Y) \\
\downarrow f_* \hspace{1cm} \downarrow f_* \\
B_*(Y') \xrightarrow{c} B_*(Y')
\end{array}$$

commutes, i.e., operational classes commute with smooth pullbacks.

3. If we have morphisms $Y \to X$, and $Y' \to Z$, and an l.c.i. map $i : Z' \to Z$ inducing a cartesian square
then the induced diagram

\[ \begin{array}{ccc}
B_*(Y') & \xrightarrow{c} & B_*(Y) \\
\downarrow i! & & \downarrow i!
\end{array} \]

commutes, i.e., operational classes commute with refined l.c.i. pullbacks.

(C4) If we have maps $Y \times Z \to Y \to X$, where the first map is the canonical projection, then

\[ c(\alpha \times \beta) = c(\alpha) \times \beta \]

in $B_*(Y \times Z)$, i.e., operational cohomology classes are compatible with the exterior product. This also shows that the maps $c$ are linear over the coefficient ring $B^*$ of the theory.

Two of the three bivariant operations don’t make sense when only talking about cohomology. First of all, one defines the bivariant product to simply be the composition of two bivariant classes. Moreover, for any morphism $f : X' \to X$ one can define the operational pullback

\[ f^* : \text{op}B^*(X) \to \text{op}B^*(X') \]

simply by setting

\[ (f^*c)|_{X' \to X'} = c|_{X' \to X'} \cdot f^* \cdot X'. \]

One readily verifies that these operations produce bivariant classes.

**Proof of the theorem 4.6**

We are now ready to prove the main theorem of this subsection. Let $X$ be a proper variety. We define the *Kronecker duality map*

\[ \text{op}B^*(X) \to \text{Hom}_{B^*}(B_*(X), B^*) \]

as the composition

\[ B_*(X) \xrightarrow{\pi} B_*(pt) = B^*, \]

where $\pi : X \to pt$ is the structure morphism. We begin with a simple observation.
Lemma 4.7. Let $X$ be a proper variety satisfying the Künneth isomorphism criterion in 4.6. Now the Kronecker duality map is an injection.

Proof. Let $Y \to X$ be a map, $\Gamma : Y \to X \times Y$ be the graph embedding, and let $c$ be an operational cohomology class. Now we have the following diagram

\[
\begin{array}{c c c c}
B_*(Y) & \longrightarrow & B_*(Y) & \\
\downarrow \Gamma_* & & \downarrow \Gamma_* & \\
B_*(X) \otimes_{B^*} B_*(Y) & \longrightarrow & B_*(X) \otimes_{B^*} B_*(Y) & \longrightarrow \pi_{2*} \ B_*(Y)
\end{array}
\]

where $\pi_2$ is the projection $X \times Y \to pt \times Y = Y$ inducing the proper pushforward

\[
\pi_{2*} = 1 \otimes \pi_* : B_*(X) \otimes_{B^*} B_*(Y) \to \pi_* : B_*(pt) \otimes_{B^*} B_*(Y) = B^*(Y),
\]

by the basic compatibility properties of the external product with pushforwards. Moreover, by the operational cohomology axiom $C_4$, the map

\[
c_{X \times Y \to X} : B_*(X) \otimes_{B^*} B_*(Y) \to B_*(X) \otimes_{B^*} B_*(Y)
\]

coincides with $c_1 \otimes 1$, where $c_1$ is the map corresponding to the identity map $X \to X$.

As $c$ is given by the composition

\[
B_*(Y) \xrightarrow{\Gamma_*} B_*(X) \otimes_{B^*} B_*(Y) \to B_*(Y),
\]

where the latter morphism is induced by the image of $c$ in the Kronecker map. As $\Gamma_*$ does not depend on $c$, we have shown that the class $c$ only depends on its image in the duality map, and we are done. \hfill \Box

Remark 4.8. In the above proof, we did not in fact use the Künneth isomorphism requirement in its full strength. Indeed, it would have been enough to require the Künneth morphism to be surjective. This is to make sure that the proper pushforward $\Gamma_*$ of an element $\beta \in B_*(Y)$ is of the form

\[
\Gamma_*(\beta) = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,
\]

and therefore its image in $\pi_{2*} c$ is completely determined by the image of $c$ in the Kronecker morphism.
In order to finish the proof of 4.6, it is enough to prove that any $B^*$-linear map $\psi : B_*(X) \to B^*$ gives rise to an operational cohomology class via the formula

$$c_{Y \to X} := B_*(Y) \xrightarrow{\Gamma} B_*(X) \otimes_{B^*} B_*(Y) \xrightarrow{\psi \otimes 1} B_*(Y).$$

The image of this operational class would be the original linear map $\psi$, which follows from the commutativity of the diagram

\[
\begin{array}{ccc}
B_*(X) & \xrightarrow{\Delta} & B_*(X) \otimes_{B^*} B_*(X) \\
& \downarrow{1} & \downarrow{1 \otimes \pi_*} \\
B_*(X) \otimes_{B^*} B_*(pt) & \xrightarrow{\psi \otimes 1} & B_*(pt) \otimes_{B^*} B_*(pt)
\end{array}
\]

and hence the Kronecker morphism would be surjective. Note that this is where we require the Künneth morphism to be an isomorphism: if we cannot say that $B_*(X \times Y) = B_*(X) \otimes_{B^*} B_*(Y)$, then we cannot define the function $\psi \otimes 1$.

**Proof of Theorem 4.6.** We have to verify that the axioms $C_1$-$C_4$ are satisfied for a collection of morphism defined as above.

$(C_1)$ Let $f : Y' \to Y$ be a proper morphism. Now we have the induced diagram

\[
\begin{array}{ccc}
B_*(Y') & \xrightarrow{\Gamma} & B_*(X) \otimes_{B^*} B_*(Y') \\
& \downarrow{f_*} & \downarrow{1 \otimes f_*} \\
B_*(Y) & \xrightarrow{\Gamma} & B_*(X) \otimes_{B^*} B_*(Y)
\end{array}
\]

The two small squares commute, and hence the big square commutes, proving $(C_1)$. One proves the axioms $(C_2)$ and $(C_3)$ the same way, using the fact that smooth pullbacks and refined l.c.i. pullbacks commute with proper pushforwards.

$(C_4)$ Let us have maps $Y \times Z \to Y' \to X$, where the first map is the canonical projection. The graph embedding $Y \times Z \to Y \times Y \times Z$ equals $\Gamma \times 1_Z$, where $\Gamma$ equals the graph embedding of $Y \to X \times Y$. Thus map associated to $Y \times Z \to X$ is given by

$$B_*(Y \times Z) \xrightarrow{(\Gamma \times 1)_*} B_*(X) \otimes_{B^*} B_*(Y \times Z) \xrightarrow{\psi \otimes 1} B_*(Y \times Z)$$

Now

$$(\Gamma \times 1)_*(\alpha \times \beta) = \Gamma_*(\alpha) \times \beta,$$
and

$$(\psi \otimes 1)(\Gamma_*(\alpha) \times \beta) = (\psi \otimes 1)\Gamma_*(\alpha) \times \beta,$$

so the collection of maps we have defined satisfies $(C_4)$.

This identification is functoral in the following sense. Suppose we have a morphism $f : X \to Y$ of proper varieties. Now there are two kinds of pullbacks one can think about in this situation. First of all, we have the usual operational pullbacks $f^* : \text{opB}^*(Y) \to \text{opB}^*(X)$. On the other hand, the morphism $f_*$ is proper, so you have the pullback

$$f^* : \text{Hom}_{B^*}(B_*(Y), B^*) \to \text{Hom}_{B^*}(B_*(X), B^*)$$

induced by the proper pushforward $B_*(X) \to B_*(Y)$. These work well with the Kronecker duality map, namely:

**Proposition 4.9.** The two pullbacks above commute with the Kronecker morphism.

**Proof.** Let $c \in \text{opB}^*(Y)$ be an operational cohomology class. It is enough to show that the Kronecker image of $f^*c$ coincides with $\psi \circ f$, where $\psi$ is the Kronecker image of $c$. This follows directly from the commutativity of the diagram

\[
\begin{array}{ccccccccc}
B_*(X) & \xrightarrow{\Gamma_*} & B_*(Y) \otimes_{B^*} B_*(X) & \xrightarrow{\psi \otimes 1} & B_*(X) & \xrightarrow{\pi_X^*} & B^* \\
\downarrow{f_*} & & \downarrow{1 \otimes f_*} & & \downarrow{f_*} & & \downarrow{1} \\
B_*(Y) & \xrightarrow{\Delta_*} & B_*(Y) \otimes_{B^*} B_*(Y) & \xrightarrow{\psi \otimes 1} & B_*(Y) & \xrightarrow{\pi_Y^*} & B^* 
\end{array}
\]

where the top row is by constructions of operational pullback and Kronecker map just the Kronecker image of $f^*c$, and the bottom row is similarly the Kronecker image of $c$.

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