Delocalization in an open one-dimensional chain in an imaginary vector potential

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We present first results for the transmittance, $T$, through a 1D disordered system with an imaginary vector potential, $ih$, which provide a new analytical criterion for a delocalization transition in the model. It turns out that the position of the critical curve on the complex energy plane (i.e. the curve where an exponential decay of $\langle T \rangle$ is changed by a power-law one) is different from that obtained previously from the complex energy spectra. Corresponding curves for $(\ln T)$ are also different. This happens because of different scales of the exponential decay of one-particle Green’s functions (GF) defining the spectra and many-particle GF governing transport characteristics, and reflects higher-order correlations in localized eigenstates of the non-Hermitian model.

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Non-Hermitian models with disorder have attracted a great deal of attention, e.g. in context of open quantum-mechanical (or optical) systems connected to reservoirs or random walks in disordered media, or Gaussian ensembles of non-Hermitian random matrices. The discovery by Hatano and Nelson of a delocalization transition in a simple Anderson model with an imaginary vector potential has led to a new burst of activity in this area and also focused research interest on this particular model and related problems.

The numerical results on the complex spectra of the model have been confirmed analytically, both for the zero-$D$ and 1$D$ cases. The fact that the transition into complex spectra in this model is indeed a delocalization transition has been convincingly illustrated by direct numerical investigations of the product of left and right eigenfunctions corresponding to the same complex eigenvalue. This product, which should enter any physical observable, has shown to be delocalized, in spite of the fact that eigenfunctions by themselves still look localized even in complex spectra. It demonstrates that the description of the localization-delocalization transition may depend on the choice of the localization criterion applied, in contrast to the standard Hermitian model. This makes it interesting to look at some analytical criterion of the transition.

One of the best ways to address this problem is to calculate the transmittance $T$ or, equivalently, the Landauer conductance. Short of a direct analytical calculation of the wavefunction distribution, an analytical dependence of $T$ on the system size $L$ provides the most straightforward criterion of the delocalization transition. It allows one to investigate how the localization length $\xi$ depends on the particle energy and how it diverges in approaching the transition. It is well known that the transmittance through a 1D Hermitian disordered wire decays as

$$T = a(L) \exp \left[ -\frac{L}{\xi} \right], \quad a(L) \propto L^{-3/2}, \quad (1)$$

indicating that all the wavefunctions are localized with the localization length $\xi$ (depending on energy). The existence of the delocalization transition should be manifested by the divergence in $\xi$ at the mobility edge.

In this Letter, we present the analytical results for the transmittance $T$ which confirm the existence of the delocalization transition in the 1D model introduced in. However, properties of the transition and even the position of the mobility edge on the complex energy plane turn out to be different from those expected from the knowledge of the complex spectra alone. Before describing this, note that $T$ can only be analyzed for an open system. If one considered a system with free boundaries, the imaginary vector potential, $h$, could be wiped out by a gauge transformation. We consider a disordered sample attached to ideal leads, with the logarithmic derivative of the wave function being continuous at the boundaries. Although $h$ can be eliminated from the sample by the gauge transformation, the requirement that the wave functions remain finite in the leads imposes a nontrivial constraint on the class of allowed functions, similar to that imposed by the periodic boundary conditions.

Although the open geometry for the model with an imaginary vector potential could not be directly related to the problem of depinning of flux lines in a superconductor with columnar defects, which was the original motivation for this model, the subsequent interest has arisen due to a general character of the observed hypersensitivity of a disordered system to the presence of even a small non-Hermiticity. Thus we believe that the analytical description in the case of the open geometry can be very useful, even if not directly related to the original model of depinning.

The model is described by the Schrödinger equation

$$\mathcal{H}\psi(x) = \left[ -\left( \frac{d}{dx} - h \right)^2 + \nu(x) \right] \psi(x) = z \psi(x). \quad (2)$$

Here $\psi(x) \equiv \psi_R(x)$ is the right eigenfunction of $\mathcal{H}$ corresponding to the complex energy $z$; in this model the left eigenfunction at the same energy is $\psi_L(x; h) = \psi_R^*(x; -h)$. The random potential, $\nu(x)$, is chosen to be Gaussian white-noise with zero mean and variance $u^2$. It vanishes.
in the ideal leads attached to the sample at \( x = 0 \) and \( x = -L \). The imaginary vector potential, \( i\hbar \), is homogeneous inside the sample and make take a different constant value (not necessarily 0) in the leads. The logarithmic derivative of \( \psi(x) \) is continuous at the boundaries.

In the non-Hermitian case considered, we must find \( T \) as a function of complex energy \( z \). Then the localization-delocalization transition would reveal itself via the divergence of \( \xi(z) \) at certain values of \( z \) (the mobility edge). Furthermore, for the values of \( z \) at which the wavefunctions are delocalized, the transmittance is expected to have a non-exponential dependence on \( L \).

However, solving the standard scattering problem for the Hamiltonian \( \mathcal{H} \), we find that \( T \) always remains exponential as in Eq. (2), albeit with \( \xi \) increasing as a function of \( \hbar \). On the face of it, this result is quite surprising. It means that, were one able to have an experimental realization of a 1D wire in the presence of the imaginary vector potential, measuring a current through it would not reveal an insulator-to-metal transition, at least as a function of the wire length. The reason for this is rather simple: the energies of incident plane waves on the complex \( z \) plane do not overlap with the energies of the delocalized states. Although these states exist, they make only an exponentially small contribution to the transmittance. Therefore, we need to generalize the scattering technique to be able to detect the delocalized states.

We will define the scattering amplitudes, \( t \) and \( r \), via the asymptotics of the wave function:

\[
\psi(x) = \begin{cases} 
  t e^{ik_+ x} e^{i\hbar(x+L)} + r e^{-ik_- (x+L)}, & x > 0; \\
  e^{ik_- x} - \theta(\hbar - \kappa) e^{-i \kappa x}, & x \leq 0
\end{cases}
\]  

(3)

We suppose, for a moment, that the imaginary vector potential \( \hbar \) is the same both inside and outside the sample. Then the generalized wave-vector, \( k_\pm \), is defined by

\[
k_\pm = \sqrt{z} \equiv i\hbar \equiv k \mp i(\kappa - \hbar),
\]

with \( k \) and \( \kappa \) being the real and imaginary parts of \( \sqrt{z} \) (a branch with \( \kappa > 0 \) is chosen). Both the incident and transmitted waves are not divergent at \( x \to \infty \) only provided that \( k_+ \) is real, i.e., on the curve \( S_0 \) in the \( \kappa \)-plane,

\[
S_0 : \quad \kappa = \hbar \quad \iff \quad \Re z = (3\Im z/2\hbar)^2 - \hbar^2,
\]

(4)

which happens to be also the DoS support curve in the absence of the impurity potential \( \nu \). The DoS support inside the sample (\( \nu \neq 0 \)) is, however, entirely different.

To illustrate this, we first consider a simple case of the one-particle Green function (GF), \( G(z) \), which is the GF of the Schrödinger equation (2). A straightforward calculation in the absence of the disorder, \( \nu(x) = 0 \), yields

\[
G_0(x,0;z) = \frac{1}{2i\sqrt{\pi}} \times \begin{cases} 
  \theta(\kappa - \hbar)e^{i k_+ x} & x \geq 0 \\
  e^{-i k_- x} - \theta(\hbar - \kappa)e^{i k_+ x} & x \leq 0
\end{cases}
\]

(5)

As a function of \( z \), \( G_0 \) is discontinuous at \( \kappa = \hbar \) which means that all the eigenvalues lie on the curve \( S_0 \) in the \( z \)-plane defined by Eq. (4). This can easily be seen from the well known formula for the density of states (DoS), \( \nu(z) = \pi^{-1} \partial \nu G(0,0;z) \). Thus, the DoS support lies on the parabola along the \( \Re z \) axis on the \( z \)-plane.

In the presence of the disorder it is straightforward to show that in the quasi-classical regime, \( k \gg \kappa, \ell^{-1}, \hbar \), the ensemble-averaged GF of Eq. (2) remains of the same form as \( G_0 \), Eq. (5), with the only change

\[
k_\pm \to k_\pm \pm i/2\ell(k),
\]

(6)

the same as for the Hermitian problem, with \( \ell \equiv \ell(k) = 2k^2/\nu^2 \) being a mean free path. The discontinuity line of the GF defines the DoS support inside the sample:

\[
S_1 : \quad \kappa = \max\{h - 1/2\ell(k), 0\}.
\]

(7)

The DoS support lines inside and outside the sample (with the same imaginary vector potential, \( i\hbar \), in both regions), Eqs. (4) and (7), do not overlap at any disorder. If we put \( h = 0 \) outside the sample, then all states are on the line \( \kappa = 0 \) which only intersects \( S_1 \) at a single point. In order to detect delocalized states inside the sample, \( k_+ \) in the incident wave, Eq. (6), should be allowed to take on any complex value. To tune \( k_+ \), one could formally apply the imaginary potential \( i\hbar \nu \) outside of the sample. Then, by changing \( h_0 \), one would shift the DoS support line (4) in such a way that it will intersect with \( S_1 \) (see Fig. 1). Note that any physical quantity, like conductance \( \propto T \), would be defined by the ratio of the wave functions on the sample boundaries, and thus will be totally unaffected by \( h_0 \).

One would expect that \( S_1 \) also defines scattering properties of the disordered system, in particular the transmittance \( T \) and thus the mobility edge where \( \Re z \to \infty \). Indeed, Eq. (4) defines a parabola on the \( z \) plane which is an exact counter-part of the bubble-like curve analytically found in [3] for the lattice variant of the model. However, the situation turns out to be not that simple. What we have found is that the transmittance cumulants, \( \langle T^n \rangle \), are characterized by the whole set of the critical curves, \( C_n \) (see Fig. 1).

![FIG. 1. Phase diagram in \( \sqrt{z} \)-plane: the DoS support lines for a pure system, \( S_0 \), and a disordered system, \( S_1 \); the critical curves for \( \langle \ln T \rangle, S_2, \langle T \rangle, C_1; \ell(k) = 2k^2/\nu^2 \), Eq. (6).](image-url)
This multitude of the critical curves is due to the absence of self-averaging in 1D \([13]\). It is \(\ln T\) which is normally-distributed in the Hermitian case. The longnormal distribution of \(T\) leads to all the moments \(T^n\) to have the same decay exponent \(-L/\xi\) (which is not scaled with \(n\)), where \(\xi\) is the localization length. It is the two-particle GF which decays \(\propto e^{-L/\xi}\) and thus contains the information about the localization, while the decay of the one-particle GF, Eq. (1), is only due to the loss of the momentum direction in elastic scattering, which is the same in any dimensionality. The self-averaging \(\ln T\) is governed by its own exponent different from that in the one-particle GF by substituting the transport scattering time, \(\ell_{tr} = 2\ell\), for \(\ell\). In the presence of \(i\hbar\), the exponential decay is replaced by \(e^{-\hbar/\xi/2L}\) for \(T^n\), or by \(e^{(h-1/2\ell L)}\) for the one-particle GF and \((h-1/2\ell_{tr})L\) for \(\ln T\). Then for \(h > 1/\xi\) (or \(1/2\ell\), \(1/2\ell_{tr}\), respectively) no small readjustment of localized states could prevent a mismatching of boundary conditions for a closed system (or an exponential increase in one of the leads for an open system). This defines a set of critical curves, Fig. 1, which are different for all these quantities.

Now we outline our procedure of rigorous analytical derivation of these results. By introducing vector \(\Psi\),

\[
\Psi = \left(\begin{array}{c} \psi_+ \\ \psi_- \end{array}\right), \quad \psi_\pm = \pm \frac{1}{2\sqrt{|x|}} \frac{d\psi}{dx} + ik_x \psi, \tag{8}
\]

we reduce the Schrödinger equation (3) to the first-order matrix equation:

\[
\frac{d\Psi}{dx} = \mu \Psi, \quad \mu(x) = h + i\sqrt{\sigma_3 + \frac{v(x)}{2i\sqrt{2}}(\sigma_3 + i\sigma_2)}, \tag{9}
\]

where \(\sigma_i\) are the Pauli matrices. The corresponding transfer-matrix, \(m\), is defined by the relation

\[
\Psi(x) = m(x, x')\Psi(x') \tag{10}
\]

and obeys the following first order differential equation

\[
\frac{dm(x, x')}{dx} = \mu(x) \cdot m(x, x'), \tag{11}
\]

which is subject to the initial condition \(m(x, x) = 1\).

Taking into account that the values of \(\Psi\) at the boundaries, \(\Psi^T(-L) = (1, r)\) and \(\Psi^T(0) = (t, 0)\) are related via Eq. (13) by \(m(-L, 0)\), we express the reflection and transmission amplitudes in Eq. (8) as follows

\[
t = \frac{1}{m_{11}(-L, 0)}, \quad r = \frac{m_{21}(-L, 0)}{m_{11}(-L, 0)}. \tag{12}
\]

From Eq. (11) we can extract two coupled equations for the scattering amplitudes \(r\) and \(t\):

\[
\frac{dr}{dL} = 2i\sqrt{z}r + \frac{v(-L)}{2i\sqrt{z}}(1 + r)^2, \tag{13}
\]

\[
\frac{dt}{dL} = (h + i\sqrt{z})t + \frac{v(-L)}{2i\sqrt{z}}t(1 + r),
\]

which obey the boundary conditions at the right end of the sample, \(r(0) = 0\) and \(t(0) = 1\). By performing the ensemble averaging, one can easily derive equations for the mixed moments \((T^n R^m)\). As \(T + R \neq 1\) in the non-Hermitian case, \(R \equiv |r|^2\) cannot be simply excluded. For our purposes, it is sufficient to consider only conditional averages, \(P_n(L) = (T^n(L, R)\delta(R(L) - R))\) so that the moments \((T^n)\) could be found by integrating over all \(R\). As the reflectance \(R\) is an auxiliary quantity, it is convenient to introduce a new variable, \(\chi\), by \(R = \tanh^2(\chi/2)\). In the absence of the imaginary potential, \(L/\chi\) is equal to the localization length \([13]\). The conditional probability \(P_n(\tau, \chi)\) in new variables (with \(\tau \equiv L/2\ell\)) obeys the following Fokker-Planck equation,

\[
\frac{\partial P_n}{\partial \tau} = \frac{\partial}{\partial \chi} \left[ \frac{\partial}{\partial \chi} + \frac{\partial \Omega_n}{\partial \chi} \right] P_n + V_n P_n, \tag{14}
\]

where the initial condition is \(P_n(\tau = 0) = \delta(\chi)\), and

\[
V_n(\chi) \equiv n^2 \frac{4}{2} - n - 4n(\kappa - h)\ell, \tag{15}
\]

\[
\Omega_n(\chi) \equiv 4\kappa\ell \cos h - \ln \sinh h + 4n \ln \cosh h, \tag{15}
\]

which is derived from Eqs. (13) for \(k \gg \kappa, h, \ell^{-1}\) in a way similar to that for the imaginary scalar potential \([4]\). The moments of \(T\) can be found by the integration:

\[
\langle T^n(\tau) \rangle = \int_0^\infty d\chi P_n(\chi, \tau). \tag{16}
\]

As \(\ln T\) is well known to be a self-averaging quantity in 1D, the Lyapunov exponent,

\[
\lambda = -\lim_{L \to \infty} \frac{\ln T}{L}, \quad \langle \ln T \rangle = \lim_{n \to 0} \ln(T^n)/n, \tag{17}
\]

found via the standard ‘replica’ trick, gives the ‘best’ representation of the inverse localization length.

Equation (14) cannot be solved exactly and we map it onto the imaginary-time Schrödinger equation:

\[
-\frac{\partial \Phi_n}{\partial \tau} = -\frac{\partial^2 \Phi_n}{\partial \chi^2} + U_n(\chi)\Phi_n. \tag{18}
\]

Here \(\Phi_n(\chi, \tau) = P_n(\chi, \tau) \exp[\Omega_n(\chi)]\), and the effective potential \(U_n \equiv (\Omega_n')^2/4 - \Omega_n''/2 - V_n\) reduces to

\[
U_n(\chi) = \frac{1}{4} - \frac{1}{4\sinh^2 h} - 4n\kappa\ell
+ 4(\kappa\ell)^2 \sinh^2 h + 4\kappa\ell(n - 1) \cosh h. \tag{19}
\]

Although the eigenfunctions \(\Phi_n^\dagger(\chi)\) of Eq. (18), where \(\Phi_n(\chi, \tau) \equiv \sum \Phi_n^\dagger(\chi) e^{-\int_0^\tau \dot{\chi}}\), cannot be found exactly, the form of the effective potential, Eq. (19), makes possible to find the long-\(\tau\) limit of \(\Phi_n(\chi, \tau)\), and thus of \(P_n(\chi, \tau)\). A sharp increase in \(U_n\) at \(\chi \gtrsim \ln(1/\kappa\ell)\) makes the spectrum of Eq. (18) discrete, with a gap of order 1 separating
the ground and excited states. Therefore, only the lowest

eigenstates contribute to $\Phi_n(x, \tau)$ and thus to $\langle T^n \rangle$ for

$\tau \equiv L/2\ell \gg 1$. It is easy to verify that for $n = 0$ the
ground state has the energy $\varepsilon_0 = 0$ and the eigenfunction

$$
\Phi_0(x) = \frac{e^{-z|\Omega_0(x)|}}{[\int dxe^{-z\Omega_0(x)}]^{1/2}}.
$$

(20)

For $n \ll 1$, by treating the $n$-dependent part of the

potential $[\Omega_0]$ as a perturbation, one finds $\Psi_0^0 \approx \Psi_0$ and

$\varepsilon_0^0 = n[1 + 4(\kappa - h)\ell]$. This leads, via Eq. (17), to exact

$$
\lambda = -\frac{1}{2\ell} \lim_{\tau \to \infty} \lim_{n \to 0} \frac{\varepsilon_0^0}{n} = 2(\kappa - h) + \frac{1}{2\ell}.
$$

(21)

The Lyapunov exponent $\lambda(z)$ vanishes (i.e. the localization

length diverges) on the curve (see Fig. 1)

$$
S_2 : \quad \kappa = h - 1/4(\ell/k).
$$

(22)

This curve, as expected, could be obtained from $S_1$, Eq. (1),

by substituting there $\ell_1 = 2\ell$ for $\ell$. In contrast to $\langle \ln T \rangle$, the moments $\langle T^n \rangle$ cannot be found exactly for $n \geq 1$. However, as $\kappa \ell \ll 1$, approximate

eigenfunctions can be found by separating $\chi \approx 1$ and $\chi \gg \ln(\kappa \ell)^{-1}$ scales and matching appropriate solutions. Thus we find the moments with $n \ll (\kappa \ell)^{-1}$:

$$
\langle T^n(\tau, z) \rangle = c_n (1/\tau + 1/\tau_c)^{3/2} e^{-[\frac{1}{2} - 2nh\ell + \frac{\psi}{2}]} \tau,
$$

(23)

where $\tau_c(n) = \pi^{-2} \ln(1/2\kappa \ell) - \psi(n - 1/2)^2$, $\psi$ is the digamma function, $c_n = (2n - 3)!^2 \pi^{5/2} 2^{1-2n} ((n-1)!)^{-1}$. For moments with $n \gg (\kappa \ell)^{-1}$, the preexponential factor is proportional to $\sqrt{\kappa \ell/n}$, while the exponent becomes $1/4 - 2n\ell(h - \kappa)$. One finds from Eq. (23) that the $n$-th exponent vanishes on the curve $C_n$ (which is drawn for $n = 1$ in Fig. 1 and is qualitatively the same for any $n$):

$$
C_n : \quad \kappa = \frac{1}{2n\ell(k)} e^{-[2n\ell(h) - 1/4]}^{1-1/2}.
$$

(24)

The corresponding transmittance moment on the curve close to the point where $C_n$ hits the $k$-axis on the $k$-$\kappa$ plane, i.e. for $h\ell(k) - 1/8n \ll 1$, is given by

$$
T_n(\tau, k) = c_n \left( \tau^{-1} + 2n\ell(h - 1/4) \right)^{3/2}
$$

(25)

Such a power-law behavior, $\langle T^n \rangle \propto (\ell/L)^{3/2}$, is typical for a critical regime at the metal-insulator transition point. What is unusual is that each moment reaches the criticality at a different point in the $z$ plane.

The existence of a set of different critical curves means that

the delocalization transition, say in $\langle T \rangle$, happens in the point in the $z$ plane where the DoS support for a corresponding closed system, $S_1$, is still on the real axis (Fig. 1) and therefore all products of the left and right eigenfunctions are still localized. This manifests the existence of some higher-order correlations between the localized states, similar to the anti-correlations between the amplitudes of $\psi_R(z)$ and $\psi_L(z)$ which ensures the delocalization of their product on the curve $S_1$, despite each of these function by itself is still ‘localized’, i.e. occupies only a very small part of the sample $[1]$. As the properties of the localized states should not be strongly dependent on boundary conditions, these higher-order correlations between them could also exist for the closed disordered systems with the imaginary vector potential.

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