Center Vortices, Instantons, and Confinement

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We study the relation between center vortices and instantons in lattice QCD.

1. Motivation

One approach to understanding the physical origin of confinement and chiral symmetry breaking is to attempt to capture the essential physics in a much smaller number of degrees of freedom by projecting from a maximal abelian or maximal center gauge. For simplicity, we consider $SU(2)$ and denote a link variable transformed by a gauge transformation $g$ by

$$ U^g_{\mu}(x) = g(x) U_{\mu}(x) g(x + \mu) $$

with $\sum a^2 = 1$. Maximal abelian gauge corresponds to calculating

$$ \text{Max}_{\{g\}} \sum_{x,\mu} \text{tr} \left( U^g_{\mu}(x) \sigma_3 U^{g\dagger}_{\mu}(x) \sigma_3 \right) $$

and the abelian projection is performed by truncating $a_1$ and $a_2$ and normalizing to obtain a $U(1)$ matrix. Although results are ambiguous because of Gribov ambiguities associated with local minima, the projected results generally reproduce the proper string tension.

Maximal center gauge corresponds to calculating

$$ \text{Max}_{\{g\}} \sum_{x,\mu} \left( \text{Tr} \, U^g_{\mu}(x) \right)^2 = \text{Max}_{\{g\}} \sum_{x,\mu} 4 (a_0^2)_{x,\mu} $$

and center projection is performed by truncating $a_1$, $a_2$, and $a_3$ and normalizing the resulting $Z(2)$ variable to $\pm 1$. The properties of center projection have been discussed widely by Greensite and collaborators and Reinhardt and collaborators. The set of negative links defines a set of center vortices, corresponding to closed lines in three dimensions and closed surfaces in four dimensions. Vortices may also be thought of as Dirac strings connecting monopoles.

For subsequent reference, it is useful to visualize a vortex sheet generated by a single negative link in the $z$ direction from $(0,0,0,0)$ to $(0,0,1,0)$. The vortex sheet is the surface of the cube dual to this link, that is, the unit cube in $(x,y,t)$ centered at the origin and having $z = \frac{1}{2}$. If one cuts this four-dimensional surface at time $t = 0$ and observes it in the three-dimensional $x,y,z$ space, it corresponds to a square loop piercing the center of each of the plaquettes with Wilson loop $-1$ that share the negative-$z$ link. If two adjacent links in the $z$ direction are $-1$, the cube grows to a rectangular parallelepiped, and in general surfaces can grow by stacking the dual cubes together. Returning to our single cube dual to a $z$ link, we note for example that the $x$-$t$ surface corresponds to a $y$-$z$ plaquette and thus corresponds to $B_x \sim F_{yz}$. In general, the vortex sheet in the $ij$ plane corresponds to $\epsilon_{ijkl} F_{kl}$. By suitably choosing the time direction in euclidean space to either be in or perpendicular to the plane of the sheet, we can think of the vortex sheets as being sheets of magnetic or electric fields. Hence, a single smoothly bending vortex sheet corre-
sponds locally to either \( \vec{E} \) or \( \vec{B} \), but can never generate \( \vec{E} \cdot \vec{B} \). The simplest way to generate lumps of \( \vec{E} \cdot \vec{B} \), corresponding to the topological charge density one expects from instantons, is to have an intersection of two mutually perpendicular sheets. For example, a sheet in the \( x-y \) plane at fixed \((t_0, z_0)\) with \( F_{xt} \sim E_z \) intersecting a sheet in the \( z-t \) plane at fixed \((x_0, y_0)\) with \( F_{xy} \sim B_z \) yields a point of \( \vec{E} \cdot \vec{B} \) at \((x_0, y_0, z_0, t_0)\). If the vortices have finite thickness, the resulting lump of \( \vec{E} \cdot \vec{B} \) will have finite size.

Lattice phenomenology provides several indications that the reduced degrees of freedom subsequent to center projection still embody the essence of confinement and chiral symmetries that the reduced degrees of freedom.

When the contribution of a lattice gluon dimension and periodic in the other two dimensions and twisted boundary conditions to prevent decay to the vacuum yield stationary vortex solutions that are exponentially localized in two dimensions and periodic in the other two dimensions. When the contribution of a lattice gluon configuration to a Wilson loop \( W_n \) is labeled by the number \( n \) of projected vortices that would pierce the loop, one observes that \( \langle W_2 \rangle/\langle W_0 \rangle \to 1 \) and \( \langle W_1 \rangle/\langle W_0 \rangle \to -1 \) for large loops. This suggests that there is a finite size object in the actual configuration that, when enclosed by a large enough loop, behaves just like an idealized “thin vortex”. In addition, at finite temperature, the vortices appear to become aligned in the time direction, so that they pierce spatial loops yielding an area law. Finally, de Forcrand and D’Elia have studied the effect of removing center vortices. For each link \( U_\mu(x) \) one determines \( Z_\mu(x) \) by center projection and defines the modified distribution \( U'_\mu(x) = Z_\mu(x)U_\mu(x) \), which projects to the trivial configuration. Not only does one lose confinement, but in addition \( \langle \bar{\psi}\psi \rangle \to 0 \) and the topological charge \( \langle Q \rangle \to 0 \). That \( \langle Q \rangle \to 0 \) is understandable, since \( U \) must be far from 1 some-

where for nontrivial topology and projection leads to the trivial sector. That \( \langle \bar{\psi}\psi \rangle \to 0 \) implies via the Banks Cacher relation that the zero modes and thus all the instantons and antiinstantons must also be removed.

2. Center Projection of a Single Instanton

To understand how the modified configuration removes instantons, we have studied the center projection of a single instanton on a lattice. Projection yields a cluster of parallel \(-1\) links in the center of the instanton in one common direction that we will call time. This breaks the euclidean invariance of the instanton, and yields a vortex sheet that is pure \( E \). The most relevant configurations are a single dual cube, a \( 2^3 \) cube composed of 8 elementary cubes, and a \( 3^3 \) cube composed of 27 elementary cubes. To see how the modified distribution cancels an instanton, note that there must be a shift in group space of \( \pi \) so \( U \sim e^{iE} \to e^{i(E+\pi)} \). Since

\[
Q_{\text{Lat}}^{\text{Lat}} = \frac{1}{4\pi^2} \sum_{a,i} 2E_i^a B_i^a
\]

if we assume that the full shift of \( \pi \) occurs in a single plaquette for an instanton of size \( \rho \) and assume a spherical sheet of radius \( R \),

\[
Q_{\text{Lat}}^{\text{Lat}} = \frac{1}{4\pi^2} \sum_{\text{vortex}} 2(\pi E_i^a)B_i^a = \frac{2\rho^2 R^2}{\rho^2 + R^2}
\]

so that \( \delta Q = 1 \) when \( R = \rho \). For a cube, \( \delta Q = 1.203 \) for \( R = 0.813 \rho \). Hence, if the size of the vortex is \( \sim \rho \), it should cancel \( Q \).

The primary technical problem is local minima corresponding to Gribov ambiguities. We start with a cooled instanton on lattices from \( 16^4 \) to \( 32^4 \), minimize

\[
R_{\{g\}} = \sum_{x,\mu} \left[ 1 - \left( \frac{1}{2} \text{tr} g^i(x)U_\mu(x)g(x+\mu) \right)^2 \right]
\]

successively on each site by matrix diagonalization, and overrelax by applying \( g^\alpha \) with \( \alpha \sim 1.7 \) to 1.8. The local minimum depends upon \( \alpha \). For example, a regular gauge instanton with \( \rho = 4 \) on a \( 16^4 \) lattice yields no vortex for \( \alpha = 1.7 \) and
Figure 1. Spherical average of topological charge density times $r^3$ versus radius $r$. The solid curve denotes a cooled instanton, the crosses indicate the density after removal of a $4^3$ vortex, and the squares and stars denote the density after one and two cooling steps, respectively.

A $2^3$ vortex for $\alpha = 1.8$. Similarly the initial gauge matters. A random gauge transformation at $\alpha = 1.7$ yields a $2^3$ vortex whereas the regular gauge does not. Also, going first to Lorentz gauge can prevent finding a vortex solution. By exploring a variety of gauges and initial conditions, the $2^3$ vortex emerged as the minimum solution for this case.

As expected from the analytic estimate of $\delta Q_{\text{Lat}}^\text{rad}$, for a vortex size $R$ comparable to $\rho$, center projection removes the instantons. Figure 1 shows $r^3$ times the topological charge density as a function of distance for a cooled instanton of size $\rho = 4.7$, the instanton with a vortex of size $R = 4$ removed, and the density after one and two cooling sweeps. Since $\delta Q$ begins localized on a shell, it takes several cooling sweeps to see that the integral averages to zero. After more cooling sweeps, the entire density relaxes to zero. A similar calculation with size $R = 2$ cools back to the original instanton with $Q = 1$.

With hindsight, we believe this single-instanton example is an unrepresentative special case. The problem is that the vortex sheet by itself has no topological charge and thus fails to represent the essential physics of the original configuration.

As argued above we expect lumps of $\vec{E} \cdot \vec{B}$ to arise from intersections of two vortices. Whereas an isolated instanton will not have vortex sheets extending to infinity, it would be natural for vortices to connect neighboring instantons and anti-instantons. Thus, the relevant case to examine is an ensemble of instantons and anti-instantons, and to see whether projection produces intersections with localized instanton-like lumps of $\vec{E} \cdot \vec{B}$. Were this to occur, center vortices would succinctly represent the essential physics of both confinement and chiral symmetry breaking.

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