**OSp(4|4) superconformal currents in three-dimensional \(\mathcal{N} = 4\) Chern-Simons quiver gauge theories**

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**Abstract:** We prove explicitly that the general \(D = 3, \mathcal{N} = 4\) Chern-Simons-matter (CSM) theory has a complete \(OSp(4|4)\) superconformal symmetry, and construct the corresponding conserved currents. We re-derive the \(OSp(5|4)\) superconformal currents in the general \(\mathcal{N} = 5\) theory as special cases of the \(OSp(4|4)\) currents by enhancing the supersymmetry from \(\mathcal{N} = 4\) to \(\mathcal{N} = 5\). The closure of the full \(OSp(4|4)\) superconformal algebra is verified explicitly.
1 Introduction and Summary

The $D = 3$, $\mathcal{N} = 4$ Chern-Simons-matter (CSM) theory was first constructed by Gaiotto and Witten (GW) [1], by choosing the gauge groups carefully so that the $\mathcal{N} = 1$ supersymmetry can be promoted to $\mathcal{N} = 4$. By adding twisted hyper-multiplets into the GW theory, the authors of Ref. [2] have been able to construct an $\mathcal{N} = 4$ Chern-Simons quiver gauge theory.

The $\mathcal{N} = 4$ Chern-Simons quiver gauge theories are natural candidates of the dual gauge theories of multi M2-branes. For instance, the authors of [2] constructed a class of $\mathcal{N} = 4$ theories with the closed loop quiver diagram of gauge groups (see also [3]):

$$
\cdots - U(N_{i-1}) - U(N_i) - U(N_{i+1}) - \cdots
$$

(1.1)

(The above quiver diagram is only a part of the full diagram.) This special class of theories have been conjectured to be the dual gauge theories of multi M2-branes in the orbifold $(\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}^2/\mathbb{Z}_q)/\mathbb{Z}_k$ [4]. Here $p$ and $q$ are the numbers of the un-twisted and twisted multiplets, respectively; $k$ is the Chern-Simons level. The corresponding gravity duals were studied in Ref. [4].

By the gauge/gravity duality, the general $\mathcal{N} = 4$ theory is expected to have a full $OSp(4|4)$ superconformal symmetry. That is, the theory possesses an $\mathcal{N} = 4$ super Poincare
symmetry as well as an \( \mathcal{N} = 4 \) superconformal symmetry\(^1\). However, to our knowledge, only the law of the \( \mathcal{N} = 4 \) super Poincare transformations has been derived in the literature \([2]\). (For a 3-algebra approach, see \([5]\).) To fill this gap, in this paper we derive the law of the \( \mathcal{N} = 4 \) superconformal transformations and verify the action is invariant under these transformations. We also derive the conserved supercurrents associated with the \( \mathcal{N} = 4 \) super Poincare transformations and the \( \mathcal{N} = 4 \) superconformal transformations. In other words, we prove that the \( \mathcal{N} = 4 \) theory possesses a complete \( OSp(4|4) \) superconformal symmetry, and derive the full \( OSp(4|4) \) superconformal currents.

We also demonstrate that the law of \( OSp(5|4) \) superconformal transformations and the \( OSp(5|4) \) superconformal currents \([6]\) in the \( \mathcal{N} = 5 \) CSM theory, can be obtained as special cases of the law of the \( OSp(4|4) \) superconformal transformations and the \( OSp(4|4) \) currents of the \( \mathcal{N} = 4 \) theory, by enhancing the \( SU(2) \times SU(2) \) R-symmetry to \( USp(4) \). In our previous work \([6]\), we have showed that the \( OSp(6|4) \) and \( OSp(8|4) \) superconformal transformations and currents \([7, 8]\), associated with the \( \mathcal{N} = 6,8 \) theories, respectively, can be derived as the special cases of the \( OSp(5|4) \) superconformal transformations and currents\(^2\). Hence our approach provides a unified framework for all \( \mathcal{N} \geq 4 \) CMS theories.

To our best knowledge, so far only the closure of the \( \mathcal{N} = 4 \) super Poincare algebra of the \( \mathcal{N} = 4 \) theory has been checked (in the framework of 3-algebra) in the literature \([5]\). Therefore, it is necessary to verify the closure of the full \( OSp(4|4) \) superconformal superalgebra. This is completed in the framework of Lie 2-algebra in Section 4.

The paper is organized as follows. In section 2, we derive the law of \( \mathcal{N} = 4 \) superconformal transformations and the corresponding conserved currents. In Section 3 we show that the \( OSp(5|4) \) superconformal currents can be obtained as special cases of the \( OSp(4|4) \) currents. In Section 4, we check the closure of the full \( OSp(4|4) \) superalgebra. Our conventions and useful identities are summarized in Appendix A. In Appendix B, we review the general \( \mathcal{N} = 4 \) theory. In Appendix C, we present the details of the derivation of the \( \mathcal{N} = 4 \) super Poincare currents.

## 2 \( OSp(4|4) \) Superconformal Currents

In this section we will construct the \( \mathcal{N} = 4 \) superconformal currents, and show that the \( \mathcal{N} = 4 \) theory has a full \( OSp(4|4) \) superconformal symmetry. (The \( \mathcal{N} = 4 \) theory is reviewed in Appendix B, and our conventions are summarized in Appendix A.)

The \( \mathcal{N} = 4 \) super Poincare currents are derived in Appendix C. They are given by

\[
j^I_{\mu} = -i \bar{\psi}^A a^I_{\mu} (\delta \psi)^{Ia} - i \bar{\psi}^A a_{\mu} (\delta \psi)^{Ia}.
\]  

\(^1\)In this paper, the super Poincare transformations will be denoted as \( \delta_1, \delta_2 \sim P_\mu \) with \( P_\mu \) the translations. The superconformal transformations will be denoted as \( \delta_3 \), satisfying \( [\delta_1, \delta_2] \sim K_\mu \) with \( K_\mu \) the special conformal transformations. The full super transformations (containing both \( \delta_1 \) and \( \delta_2 \)) will be called the \( OSp(4|4) \) superconformal transformations.

\(^2\)For a 3-algebra unifying \( \mathcal{N} = 5, 6, 8 \) theories, see \([9–11]\).
where

\[
(\delta \psi)^{Ia'}_A = -\gamma^\mu D_\mu Z_{Ia'}^{a'\mu} B \sigma^I_A B - \frac{1}{3} k_{mn} \tau^{ma'} \nu' Z^{\nu} B^{\mu n \nu} C \sigma^I_A C + k_{mn} \tau^{ma'} \nu' Z^{\nu} B^{\mu n \nu} A \sigma_A B A \sigma^I_A A \tag{2.2}
\]

\[
(\delta \psi)^{Ia}_A = -\gamma^\mu D_\mu Z_{Ia}^{a\mu} B \sigma^I_B B - \frac{1}{3} k_{mn} \tau^{ma} \nu Z^{\nu} B^{\mu n \nu} C \sigma^I_B C + k_{mn} \tau^{ma} \nu Z^{\nu} B^{\mu n \nu} A \sigma_A B A \sigma^I_A A \tag{2.3}
\]

are defined via the super Poincare transformations of the fermionic fields $\delta \psi^{a'}_A = (\delta \psi)^{Ia'}_A e^I$ and $\delta \psi^{a}_A = (\delta \psi)^{Ia}_A e^I$ (see (B.4)). Here $A = 1, 2, \tilde{A} = 1, 2, \text{ and } I = 1, \ldots, 4$ transform in the undotted, dotted, and vector representations of the $SU(2) \times SU(2)$ R-symmetry group, respectively; and $a = 1, \ldots, 2R$ and $a' = 1, \ldots, 2S$ transform in two different quaternionic representations of the gauge group. The scalar fields are denoted as $Z$. The sigma matrices $\sigma^I_A$ are defined in Appendix A.2.1.

Without changing the physical content, one can add a conserved total derivative term into the currents:

\[
\tilde{j}_\mu^I = j_\mu^I + \partial^\nu A^I_{\mu\nu},
\tag{2.4}
\]

with $A^I_{\mu\nu} = -A^I_{\nu\mu}$, since the improved currents are still conserved, i.e. $\partial^\nu \tilde{j}_\mu^I = 0$, and the set of super-charges remain the same:

\[
Q^I = -\int d^2 x j_\mu^I = -\int d^2 x j'_\mu^I.
\tag{2.5}
\]

The $\gamma$-trace $\gamma^\mu j_\mu^I$ measures the violation of scale invariance of the theory [12]. Since the $\mathcal{N} = 4$ theory is invariant under scale transformations, we expect that $\gamma^\mu j_\mu^I$ vanishes by choosing an appropriate $A^I_{\mu\nu}$. To see this, let us first calculate $\gamma^\mu j_\mu^I$:

\[
\gamma^\mu j_\mu^I = -i\partial^\nu [\gamma^\mu \gamma^\alpha \psi^{A\alpha}_a + \tilde{Z}^A_a \gamma^\nu \psi^I_a] \sigma^I_A A \tag{2.6}
\]

This is equivalent to

\[
\gamma^\mu j_\mu^I + \frac{i}{4} [\gamma^\mu, \gamma^\nu] \partial^\nu [\gamma^\alpha (Z^{A\alpha}_a \tilde{\psi}^A_a + \tilde{Z}^A_a \psi^I_a)] \sigma^I_A A = 0.
\tag{2.7}
\]

This equation immediately suggests us to define

\[
\tilde{j}_\mu^I = j_\mu^I + \frac{i}{4} [\gamma^\mu, \gamma^\nu] \partial^\nu [\gamma^\alpha (Z^{A\alpha}_a \tilde{\psi}^A_a + \tilde{Z}^A_a \psi^I_a)] \sigma^I_A A \tag{2.8}
\]

As a result, the improved currents $\tilde{j}_\mu^I$ satisfy $\gamma^\mu \tilde{j}_\mu^I = 0$. In other words, Eq. (2.7) suggests us to set $A^I_{\mu\nu}$ in (2.4) as follows

\[
A^I_{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] (Z^{A\alpha}_a \tilde{\psi}^A_a + \tilde{Z}^A_a \psi^I_a) \sigma^I_A A \tag{2.9}
\]

Notice that $A^I_{\mu\nu}$ is indeed antisymmetric in $\mu$ and $\nu$.

Now it is possible to construct the new currents

\[
\tilde{j}_\mu^I = x \cdot \gamma \tilde{j}_\mu^I.
\tag{2.10}
\]
Using $\gamma^\mu j^I_\mu = 0$ and $\partial^\mu j^I_\mu = 0$, it is easy to prove that the new currents are also conserved: $\partial^\mu s^I_\mu = 0$. The corresponding conserved supercharges are defined as follows:

$$S^I = - \int d^2 x s^I_0,$$

(2.11)

If we impose the equal-time commutators

$$\{\bar{\psi}_A^{\dagger}(t, \vec{x}), \psi_B^a(t, \vec{x})\} = -\delta^A_B \delta^a_0 \delta^2(x - x'),$$

$$\{\bar{\psi}_A^{\dagger}(t, \vec{x}), \psi_B^a(t, \vec{x})\} = -\delta^A_B \delta^a_0 \delta^2(x - x'),$$

$$[\Pi^A_a(t, \vec{x}), \bar{Z}^\dagger_B(t, \vec{x})] = -i\delta^A_B \delta^a_0 \delta^2(x - x'),$$

$$[\Pi^A_a(t, \vec{x}), \bar{Z}^\dagger_B(t, \vec{x})] = -i\delta^A_B \delta^a_0 \delta^2(x - x'),$$

$$[\Pi^a_m(t, \vec{x}), A^\dagger_n(t, \vec{x})] = -i\delta^a_m \delta^2(x - x'),$$

(2.12)

where $\Pi^A_a(t, \vec{x}) = D_0 \bar{Z}^\dagger_A(t, \vec{x})$, $\Pi^A_a(t, \vec{x}) = D_0 \bar{Z}^\dagger_A(t, \vec{x})$, and $\Pi^a_m(t, \vec{x}) = \epsilon^{00}k_{mp}a^A(t, \vec{x})$, then the superconformal variation of an arbitrary field $\Phi$ can be defined as

$$\delta_\eta \Phi = [-i\eta^I S^I, \Phi],$$

(2.13)

Using the above equation and the commutation relations (2.12), one can readily derive the law of $\mathcal{N} = 4$ superconformal transformations (we will prove that $[\delta_{\eta_1}, \delta_{\eta_2}] \sim K_\mu$ in Section 4):

$$\delta_\eta Z^a_A = i(x \cdot \gamma \eta A \bar{A}) \psi^a_A,$$

$$\delta_\eta Z^a'_A = i(x \cdot \gamma \eta A \bar{A}) \psi^a_A,$$

$$\delta_\eta \psi^a_A = -\gamma^\mu D_\mu Z^a_B(x \cdot \gamma \eta A \bar{B}) - \frac{1}{3} k_{mn} \tau^{ma} b Z^b_{A\bar{B}}^n C(x \cdot \gamma \eta A \bar{C})$$

$$+ k_{mn} \tau^{ma} b Z^b_{A\bar{B}}^n C(x \cdot \gamma \eta A \bar{C}) - \eta^A \bar{Z}^a_A,$$

$$\delta_\eta \psi^a_A = -\gamma^\mu D_\mu Z^a_B(x \cdot \gamma \eta A \bar{B}) - \frac{1}{3} k_{mn} \tau^{ma} b Z^b_{A\bar{B}}^n C(x \cdot \gamma \eta A \bar{C})$$

$$+ k_{mn} \tau^{ma} b Z^b_{A\bar{B}}^n C(x \cdot \gamma \eta A \bar{C}) - \eta^A \bar{Z}^a_A,$$

$$\delta_\eta A^m_A = i(x \cdot \gamma \eta A \bar{B}) \gamma m_{\mu A} + i(x \cdot \gamma \eta A \bar{B}) \gamma m_{\mu A}.$$

(2.14)

The set of parameters $\eta^A_B = \eta^I \sigma^I_A \bar{B}$ ($I = 1, \ldots, 4$) obey the reality conditions

$$\eta^A_B = -\epsilon^{BC} \epsilon_{AB} \eta^C_B.$$

(2.15)

We now must prove that $\mathcal{N} = 4$ action (B.4) is invariant under the transformations (2.14). Notice that the $\mathcal{N} = 4$ superconformal transformations (2.14) can be obtained by replacing $e^I$ by $x \cdot \gamma^I$ in the $\mathcal{N} = 4$ super Poincaré transformations (B.5) and adding the two additional terms

$$\delta^a_I \psi^a_A = -\eta^A \bar{Z}^a_A$$

and

$$\delta^a_I \psi^a_A = -\eta^A \bar{Z}^a_A.$$

(2.16)

into the transformations of the fermion fields $\psi^a_A$ and $\psi^a_A$, respectively. (In a similar way, the $\mathcal{N} = 6, 8$ superconformal transformations can be obtained from the $\mathcal{N} = 6, 8$ super
Poincare transformations \([7, 8]\). So the superconformal variation of the action can be calculated as follows:

\[
\delta_\eta S = (\delta S)_{x \rightarrow x \cdot \gamma_\eta} + \delta'_{\eta} S
\]

\[
= \int d^3 x (-j_\mu^I) \partial^\mu (x \cdot \gamma_\eta^I) + \delta'_{\eta} S,
\]

where \((\delta S)_{x \rightarrow x \cdot \gamma_\eta}\) is the quantity obtained by replacing \(\epsilon^I\) in \(\delta_\epsilon S\) (see (C.1)) by \(x \cdot \gamma_\eta^I\), and \(\delta'_{\eta} S\) is the super variation of the action under the transformations (2.16). A direct calculation gives

\[
\delta'_{\eta} S = -\int d^3 x [(\gamma^\mu j_\mu^I) \eta^I].
\]

Substituting the above equation into (2.17), we find that the action is indeed invariant under the transformations (2.14), i.e.

\[
\delta_\eta S = 0.
\]

3 Unifying \(OSp(4|4)\) and \(OSp(5|4)\) Superconformal Currents

In this section we will demonstrate that the \(OSp(5|4)\) superconformal currents, associated with the \(\mathcal{N} = 5\) theory, can be derived as special cases of the \(OSp(4|4)\) ones by enhancing the \(SU(2) \times SU(2)\) R-symmetry to \(USp(4)\).

In Ref. [13], it was showed that if the twisted and untwisted multiplets furnish the same representations of the gauge group, i.e. if \(\tau_{ab}^m = \tau_{a'b'}^m\), the \(\mathcal{N} = 4\) supersymmetry will be enhanced to \(\mathcal{N} = 5\) automatically. Specifically, if

\[
\tau_{ab}^m = \tau_{a'b'}^m,
\]

it is possible to embed the \(SU(2) \times SU(2)\) R-symmetry group into \(USp(4)\) by combining the \(\mathcal{N} = 4\) twisted and untwisted multiplets to form the \(\mathcal{N} = 5\) multiplets:

\[
Z_A^a = \begin{pmatrix} Z_A^a \\ Z_A^\dagger \\ \end{pmatrix}, \quad \psi_A = \begin{pmatrix} \psi_A^a \\ \psi_A^{\dagger} \\ \end{pmatrix}.
\]

In the left hand sides of the above equations, \(A = 1, \ldots, 4\) transforms in the fundamental representation of \(USp(4)\), while in the right hand sides, \(A = 1, 2\) and \(\tilde{A} = 1, 2\) transform in the undotted and dotted representations of \(SU(2) \times SU(2)\). We hope this will not cause any confusion. In terms of these \(\mathcal{N} = 5\) fields, the reality conditions (B.1) become

\[
\bar{Z}_A^a = \omega^{AB} \omega_{ab} Z_B^b, \quad \bar{\psi}_A^a = \omega^{AB} \omega_{ab} \psi_B^b,
\]

where the antisymmetric tensor \(\omega_{AB}\) of \(USp(4)\) is defined as the charge conjugate matrix (see Appendix A.2.2):

\[
\omega^{AB} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon^{AB} \end{pmatrix}.
\]
In the RHS, \( A, B, \hat{A}, \) and \( \hat{B} \) run from 1 to 2. Using (3.1)–(3.4), the authors of Ref. [13] have been able to promote the \( N = 4 \) Lagrangian (B.4) to the manifestly \( USp(4) \)-invariant \( N = 5 \) Lagrangian:

\[
\mathcal{L} = \frac{1}{2}(-D_\mu \bar{Z}^A \partial^\mu Z^a_A + i\bar{\psi}_a A^\mu \partial_\mu \psi_a^A) - \frac{i}{2} \omega^{AB} \omega^{CD} k_{mn}(\mathcal{J}^m_{AC} \mathcal{J}^n_{BD} - 2J^m_{AC} \mathcal{J}^n_{DB}) + \frac{1}{2} \epsilon^{\mu\nu\lambda} (k_{mn} A^m_{\mu} \partial_\nu A^n_\lambda + \frac{1}{3} C_{mn} A^m_{\mu} A^n_\lambda ) + \frac{1}{30} C_{mn} \partial \kappa_{\alpha} (\tau^m_{\alpha} \tau^n_{\beta} - \tau^m_{\beta} \tau^n_{\alpha}) \bar{Z}_{AB} Z_{aA} \bar{Z}_{B}^b \mu C^a C_B.
\]  

(3.5)

The “momentum map” and “current” operators are defined as

\[
\mu^m_{AB} \equiv \tau^m_{ab} \bar{Z}_{A} Z_{B}, \quad \mathcal{J}^m_{AB} \equiv \tau^m_{\alpha A} \psi_{A} \psi_{B},
\]

(3.6)

The \( N = 5 \) action is invariant under the \( USp(4) \) R-symmetry transformations

\[
\delta_R \psi_a^A = \frac{1}{2} \epsilon_{IJ} \Sigma_{AB} \psi_a^B, \quad \delta_R Z_a^A = \frac{1}{2} \epsilon_{IJ} \Sigma_{AB} Z_a^B,
\]

(3.7)

where \( \epsilon_{IJ} = -\epsilon_{JI} \) are set of parameters \( I, J = 1, \ldots, 5 \), and the \( USp(4) \) generators \( \Sigma_{AB} \) are defined by (A.21).

Using (3.1)–(3.4), one can rewrite the \( N = 4 \) super Poincare currents (2.8) as

\[
\bar{j}_\mu^I = -i \partial \bar{\psi}_a^A \gamma_\mu (\delta \psi)^I_a + \frac{i}{4} [\tau_{\mu}, \gamma_\nu] \partial \bar{\psi}_a^A \Gamma^I_a \psi_{B} (A = I, J, K, L), \quad (I = 1, \ldots, 4)
\]

(3.8)

where

\[
(\delta \psi)^I_a = \gamma^\mu D_\mu Z_c^a \Gamma^I_a - \frac{1}{3} k_{mn} \tau^m_{ab} \omega^{BC} Z_b^m \psi_{D} \partial \psi_{C} \Gamma^I_a D + \frac{2}{3} k_{mn} \tau^m_{ab} \omega^{BC} Z_b^m \psi_{D} \partial \psi_{C} \Gamma^I_a D, \quad (I = 1, \ldots, 4)
\]

(3.9)

with

\[
\Gamma_{AB}^I = \begin{pmatrix} 0 & \sigma^I \\ \sigma^{I \dagger} & 0 \end{pmatrix} \quad (I = 1, \ldots, 4)
\]

(3.10)

(See Appendix A.2.2.) However, due to the \( USp(4) \) R-symmetry, there is a fifth conserved supercurrent. To see this, let consider for example the fourth supercurrent \( \bar{j}_\mu^4 \). Under the \( USp(4) \) R-symmetry transformations (3.7), it becomes

\[
\bar{j}_\mu^4 \rightarrow \bar{j}_\mu^4 + \delta_R \bar{j}_\mu^4
\]

(3.11)

\[
\delta_R \bar{j}_\mu^4 = \frac{1}{2} K_L (\tau^{KL})^A \Gamma^I_a \bar{j}^I_a \mu (J, K, L = 1, \ldots, 5)
\]

(3.12)

where \( (\tau^{KL})^A \) are set of \( SO(5) \) matrices. Since the action (3.5) is manifestly \( USp(4) \)-invariant, the transformed forth supercurrent must be conserved as well:

\[
\partial \mu (\bar{j}_\mu^4 + \delta_R \bar{j}_\mu^4) = 0.
\]

(3.13)

This implies that

\[
\partial \mu (\delta_R \bar{j}_\mu^4) = 0.
\]

(3.14)
Combining (3.14) and (3.12) gives

$$\partial^\mu \tilde{j}^I_\mu = 0, \quad (I = 1, \ldots, 5)$$

(3.15)

where

$$\tilde{j}^I_\mu = -i\tilde{\psi}^A_\mu \gamma_\mu (\delta \psi) f_a^A + \frac{i}{2}[\gamma_\mu, \gamma_\nu] \partial^\nu (Z^a_A \bar{\psi}^B_\nu \Gamma^I_\nu A), \quad (I = 1, \ldots, 5.)$$

(3.16)

where $\Gamma^5$ is defined by equation (A.16). In this way, we have obtained the $N = 5$ super Poincare currents from the $N = 4$ supercurrents. Similarly, the $N = 5$ superconformal currents

$$z^I_\mu = \gamma \cdot x^I_\mu, \quad (I = 1, \ldots, 5)$$

(3.17)

can be derived from the $N = 4$ superconformal currents.

In summary, our approach provides a unified framework for constructing the $OSp(4|4)$ and $OSp(5|4)$ superconformal currents. Furthermore, since the $OSp(6|4)$ and $OSp(8|4)$ superconformal currents of the $N = 6$ and $N = 8$ can be derived as the special cases of the $OSp(5|4)$ currents of the $N = 5$ theory [6], our approach actually provided a unified framework for constructing all conserved superconformal currents of the $N \geq 4$ theories.

4 Closure of the $OSp(4|4)$ Superconformal algebra

In the literature, only the closure of the super Poincare algebra of the $N = 4$ Chern-Simons quiver gauge theory has been explicitly checked (in the framework of 3-algebra) [5]. In this section, we will verify the closure of the full $OSp(4|4)$ superconformal superalgebra in the framework of Lie 2-algebra.

We begin by considering the scalar fields of the untwisted multiplets. The commutator of two super Poincare transformations acting on the scalar fields gives [5]

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] Z^a_A = v^a_1 D_\mu Z^a_A + \tilde{\Lambda}^a_{1b} Z^b_A,$$

(4.1)

where

$$v^a_1 = -2i\epsilon^I_1 \gamma^a \epsilon^I_1,$$

(4.2)

$$\tilde{\Lambda}^a_{1b} = \Lambda^a_{1m} k_{mn} \tau^{na}_{b},$$

(4.3)

$$\Lambda^a_{1m} = -2i(\epsilon^I_1 \epsilon^J_2) (\mu^m_{AB} \sigma^{IJAB} + \mu^m_{AB} \sigma^{IJAB}).$$

(4.4)

Here $I = 1, \ldots, 4$ transforms in the fundamental representation of $SO(4)$, and the two $SU(2) \times SU(2)$ matrices are defined as follows

$$\sigma^{IAB} = \frac{1}{4}(\sigma^I \sigma^J - \sigma^J \sigma^I)_{AB}, \quad \tilde{\sigma}^{IAB} = \frac{1}{4}(\sigma^I \sigma^J - \sigma^J \sigma^I)_{AB}.$$
The set of $SO(4)$ matrices $\sigma^I$ are defined in Appendix A.2.1.

Replacing $\epsilon^I_1$ in (4.1) by $x \cdot \eta^I_1$ and adding
\begin{equation}
\delta^I_{\eta_1} \psi^a_B = -\eta^I_{1B} C Z^a_C \tag{4.6}
\end{equation}
into the variation of the fermionic fields\(^4\), we obtain
\begin{equation}
[\delta_{\eta_1}, \delta_{\eta_2}] Z^a_A = \psi^a_2 D_\mu Z^a_A + \tilde{\Lambda}^b_A Z^b_B - i \epsilon_{2A} \eta_{1B}^\dagger C Z^a_C, \tag{4.7}
\end{equation}
where
\begin{equation}
\psi^a_2 = -2i \epsilon^I_2 \gamma^\mu (\gamma \cdot x \eta^I_1) \tag{4.8}
\end{equation}
\begin{equation}
\tilde{\Lambda}^a_{2b} = \Lambda^m_{2} k_{mn} \tau^m a_b, \tag{4.9}
\end{equation}
\begin{equation}
\Lambda^m_{2} = 2i x^\mu (\epsilon^I_2 \eta^I_1) (\mu^m_{AB} \sigma^{IJAB} + \mu^m_{AB} \tilde{\sigma}^{IJAB}). \tag{4.10}
\end{equation}

Notice that (4.7) is a gauge covariant equation, which can be simplified to give
\begin{equation}
[\delta_{\eta_1}, \delta_{\eta_2}] Z^a_A = -2i (\eta^I_1 \epsilon^I_2)(x^\mu \partial_\mu + \frac{1}{2}) Z^a_A + (\tilde{\Lambda}^a_{2b} + \psi^a_2 A^m \tau^m a_b) Z^b_B. \tag{4.11}
\end{equation}
The first three lines are the scale, Lorentz, and R-symmetry transformations, respectively: these transformations suggest that $\delta_{\eta_1}$ is the superconformal transformation. The last line is a gauge transformation. The first line indicates that the dimension of the scalar field is $\frac{3}{2}$, as expected.

Similarly, replacing $\epsilon^I_2$ in (4.7) by $x \cdot \eta^I_2$ and adding
\begin{equation}
\delta^I_{\eta_2} \psi^a_B = -\eta^I_{2B} C Z^a_C \tag{4.12}
\end{equation}
into the variation of the fermionic fields, we obtain
\begin{equation}
[\delta_{\eta_1}, \delta_{\eta_2}] Z^a_A = \psi^a_3 D_\mu Z^a_A + \tilde{\Lambda}^b_A Z^b_A - i (x \cdot \eta_{2A} \eta^B_1) \eta_{1B}^\dagger C Z^a_C + i (x \cdot \eta_{1A} \eta^B_2) \eta_{2B}^\dagger C Z^a_C, \tag{4.13}
\end{equation}
where
\begin{equation}
\psi^a_3 = -2i[(\gamma \cdot x \eta^I_2) \gamma^\mu (\gamma \cdot x \eta^I_1)] \tag{4.14}
\end{equation}
\begin{equation}
\tilde{\Lambda}^a_{3b} = \Lambda^m_{3} k_{mn} \tau^m a_b, \tag{4.15}
\end{equation}
\begin{equation}
\Lambda^m_{3} = -2i[(\gamma \cdot x \eta^I_2) (\gamma \cdot x \eta^I_1)] (\mu^m_{AB} \sigma^{IJAB} + \mu^m_{AB} \tilde{\sigma}^{IJAB}) \tag{4.16}
\end{equation}
\begin{equation}
= 2i x^2 (\eta^I_1 \eta^I_2) (\mu^m_{AB} \sigma^{IJAB} + \mu^m_{AB} \tilde{\sigma}^{IJAB}).
\end{equation}

\(^4\)The relation between $\delta_{\eta_1}$ and $\delta_{\eta_2}$ has been discussed in Section 2.
We observe that (4.13) is a gauge covariant equation. One can convert (4.13) into the form

\[ [\delta_\eta_1, \delta_\eta_2] Z^a_\mu = 2(\eta_1 \gamma^\nu \eta_2) [(-2ix_\nu x^\sigma \partial_\sigma + ix^2 \partial_\nu) Z^a_\mu - ix_\nu Z^a_\mu] + (\Lambda^a_{\nu b} + v^a_\nu A^a_{\nu b}) Z^b_\mu. \]  

(4.17)

The first line is the standard special conformal variation: this again shows that \( \delta_\eta \) is indeed the superconformal transformation, as also suggested by Eq. (4.11). (The second line of (4.17) is a gauge transformation.)

Let us now consider the gauge fields [5]:

\[ [\delta_\epsilon_1, \delta_\epsilon_2] A^m_\mu = v^\nu_\epsilon F^m_{\nu \mu} - D_\mu A^m_\mu \]

\[ + v^\nu_\epsilon \{ F^m_{\nu \mu} - \varepsilon_{\mu \nu \lambda}[(Z^a_\lambda D^\lambda Z^b_\nu \gamma^{ab} - \frac{i}{2} \bar{\psi}_B \gamma^{ab} \psi_B) \gamma^m_{ab} + (Z^a_\lambda D^\lambda \bar{Z}^b_\nu - \frac{i}{2} \bar{\psi}_B \gamma^{ab} \psi_B) \tau^m_{ab}]\}, \]

(4.18)

where the second and third lines are the equations of motion (EOM) for the gauge fields. Eq. (4.18) can be written as

\[ [\delta_\epsilon_1, \delta_\epsilon_2] A^m_\mu = v^\nu_\epsilon F^m_{\nu \mu} - (\epsilon_2 \epsilon_1^T) D_\mu (A^m_{\mu AB} \sigma^{IJAB} + \mu^m_{\mu AB} \bar{\sigma}^{IJAB}) + \text{EOM}. \]

(4.19)

Applying the replacement \( \epsilon_1 \to x \cdot \gamma \eta_1 \) to (4.19), and taking account of Eq. (4.6), we obtain the equation

\[ [\delta_\eta_1, \delta_\eta_2] A^m_\mu = v^\nu_\eta F^m_{\nu \mu} - D_\mu A^m_\mu + \text{EOM}, \]

(4.20)

which can be converted into

\[ [\delta_\eta_1, \delta_\eta_2] A^m_\mu = -2i(\eta_1^T \eta_2^T) (x^\mu \partial_\mu + 1) A^m_\mu \]

\[ + 2i(\eta_1^T \gamma^{\rho \sigma} \eta_2^T) [-i(x_\mu \partial_\sigma - x_\sigma \partial_\mu) A^m_\mu + (S_{\rho \sigma})_{\mu \nu} A^m_{\nu}] \]

\[ - D_\mu (A^m_\mu + v^m_\mu A^m_\mu) \]

+ EOM, \]

(4.21)

with \( S_{\rho \sigma} \) the \( SO(1,2) \) matrices:

\[ (S_{\rho \sigma})_{\mu \nu} = i(\eta_{\mu \rho} \delta^\nu_{\sigma} - \eta_{\mu \sigma} \delta^\nu_{\rho}). \]

(4.22)

It can be seen that the first three lines of (4.21) are the scale, Lorentz, and gauge transformations, respectively. From the first line we read off that the dimension of \( A^m_\mu \) is 1.

We now would like to evaluate \([\delta_\eta_1, \delta_\eta_2] A^m_\mu \). To do so, we first rewrite Eq. (4.20) as

\[ [\delta_\eta_1, \delta_\eta_2] A^m_\mu = v^\nu_\eta F^m_{\nu \mu} - (\epsilon_2 \gamma_\nu \eta_2^T) D_\mu [\sigma^{IJAB} - 2i x^{\nu} (\mu^m_{\nu AB} \sigma^{IJAB} + \mu^m_{\nu AB} \bar{\sigma}^{IJAB})] + \text{EOM} \]

(4.23)

Applying the replacement \( \epsilon_2 \to x \cdot \gamma \eta_2 \) to (4.23), and using Eq. (4.12), we obtain the equation

\[ [\delta_\eta_1, \delta_\eta_2] A^m_\mu \]

\[ = v^\nu_\eta F^m_{\nu \mu} - D_\mu A^m_\mu \]

\[ = 2(\eta_1^T \gamma^\nu \eta_2) [(-2ix_\nu x^\rho \partial_\rho + ix^2 \partial_\nu) A^m_\mu - 2ix_\nu A^m_\mu - 2x^\rho (S_{\rho \nu})_{\mu \sigma} A^m_{\sigma}] \]

\[ - D_\mu (A^m_\mu + v^m_\mu A^m_\mu) \]

+ EOM. \]

(4.24)

(4.25)
The first line of (4.25) is indeed the special conformal transformation, while the second line is a gauge transformation.

Let us now recall the fermion supersymmetry transformation in Ref. [5]:

\[
\begin{align*}
\delta \psi^a_A &= \gamma^\mu D_\mu \psi^a_A + \tilde{\Lambda}^a_{1b} \psi^b_A \\
- \frac{i}{2} & (\epsilon^1_1 \epsilon^1_2 \epsilon^1_2 - \epsilon^1_2 \epsilon^1_2) E^a_A \\
- \frac{1}{2} v_\nu \gamma^\nu E^a_A,
\end{align*}
\]

(4.26)

where

\[
0 = E^a_A = \gamma^\mu D_\mu \psi^a_A + \tau^a_{mb} (Z^b_{AB} j^m_{AB} - \psi^C_{\bar{A}} \psi^{\bar{C}}_B + 2Z^b_{AB} j^m_{AB} - \psi^C_{\bar{A}} \psi^{\bar{C}}_B)
\]

(4.27)

are equations of motion for the fermionic fields.

Using the same trick for evaluating the scalar and gauge fields, we obtain

\[
\begin{align*}
[\delta \eta_1, \delta \eta_2] \psi^a_A &= [v^a_\nu D_\nu \psi^a_A - i(\psi^a_C \gamma^\mu \eta_1 \epsilon^1_2 \psi^{\bar{C}}_B (\gamma_\mu \epsilon^1_2 \psi^a_A)] + i \eta_1^{\dagger} B (\epsilon^1_2 \gamma^\nu \psi^{b}_B) \\
+ \tilde{\Lambda}^a_{2b} \psi^b_A \\
+ \text{EOM},
\end{align*}
\]

(4.28)

Using the Fierz transformations (A.5), one can recast the above equation as

\[
\begin{align*}
[\delta \eta_1, \delta \eta_2] \psi^a_A &= -2i(\eta_1^{I} \epsilon^1_2) (x^\mu \partial_\mu + 1) \psi^a_A \\
+ 2(\eta_1^{I} \gamma^\mu \epsilon^1_2) (x_\mu \partial_\nu - x_\nu \partial_\mu + i \gamma^\mu \nu) \psi^a_A \\
+ 2i(\eta_1^{I} \epsilon^1_2) \sigma^{IJ} \dot{A}_B \psi^a_B \\
+ (\tilde{\Lambda}^a_{2b} + v^a_\nu A_\mu \tau^a_{mb}) \psi^b_A \\
+ \text{EOM}.
\end{align*}
\]

(4.29)

The first three lines are the scale, Lorentz, and R-symmetry transformations, while the fourth line is a gauge transformation. The first line indicates that the dimension of the spinor field is 1.

The commutator of two superconformal transformations acting on the fermionic fields gives

\[
\begin{align*}
[\delta \eta_1, \delta \eta_2] \psi^a_A &= 2(\eta_1^{I} \gamma^\mu \eta_2^{I}) [(-2ix_\nu x^\mu \partial_\mu + ix^2 \partial_\nu) Z^a_A + (-2ix_\nu - 2x^\mu \gamma^\mu \nu) \psi^a_A] \\
+ (\tilde{\Lambda}^a_{2b} + v^a_\nu A_\mu \tau^a_{mb}) \psi^b_A \\
+ \text{EOM}.
\end{align*}
\]

(4.30)

The first line is nothing but the special conformal transformation of the fermionic fields, and the second line is a gauge transformation. Also, Eq. (4.30) can recast as the following gauge invariant form:

\[
\begin{align*}
[\delta \eta_1, \delta \eta_2] \psi^a_A &= 2(\eta_1^{I} \gamma^\mu \eta_2^{I}) [(-2ix_\nu x^\mu D_\mu + ix^2 D_\nu) Z^a_A + (-2ix_\nu - 2x^\mu \gamma^\mu \nu) \psi^a_A] \\
+ \tilde{\Lambda}^a_{2b} \psi^b_A + \text{EOM}.
\end{align*}
\]

(4.31)
The transformations of the scalar and spinor fields of the twisted multiplets have similar expressions of that of the untwisted multiplets. First of all, the $N = 4$ super Poincare algebra is closed (on-shell) on the twisted multiplets [5]:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]Z_A^{a'} = v_1^\mu D_\mu Z_A^{a'} + \tilde{\Lambda}_1^{a'} b' Z_A^{a'},$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_A^{a'} = v_1^\mu D_\mu \psi_A^{a'} + \tilde{\Lambda}_1^{a'} \psi_A^{b'} + \text{EOM},$$

where $\Lambda_1^{a'} b' = k_{mn} \Lambda_1^{a} \tau^{a} b'$, and the equations of motion for the spinor fields are given by

$$0 = F_A^{a'} = \gamma^{\mu} D_\mu \psi_A^{a'} + \tau_m a' b' (Z_B^{j m B} A - \mu_{A C} \bar{\psi}^{C b'} + 2 Z_B^{b'} j^{m A} \bar{\Lambda}^a)$$

Secondly, taking advantage of the strategy for calculating the transformations of the fields of the untwisted multiplets, we find that the rest of the $OSp(4|4)$ superalgebra is also closed (on-shell) on the scalar and fermionic fields of the twisted multiplets:

$$[\delta_{\eta_1}, \delta_{\eta_2}]Z_A^{a'} = -2i(\eta_1^I \gamma^I_j (x^\mu \partial_\mu + \frac{1}{2})Z_A^{a'}$$

$$+ 2(\eta_1^I \gamma^{\mu \nu} \eta_2^I) (x_\nu \partial_\mu - x_\mu \partial_\nu)Z_A^{a'}$$

$$+ 2i(\eta_1^I \gamma^I_j \sigma^{IJ} \tilde{A}^I \tilde{A}^J) Z_B^{a'}$$

$$+ (\Lambda_2^{a'} b' + v_2^\mu A^m \tau_m a' b')Z_A^{a'},$$

$$[\delta_{\eta_1}, \delta_{\eta_2}]\psi_A^{a'} = -2i(\eta_1^I \gamma^I_j (x^\mu \partial_\mu + 1)\psi_A^{a'}$$

$$+ 2(\eta_1^I \gamma^{\mu \nu} \eta_2^I) (x_\nu \partial_\mu - x_\mu \partial_\nu + i \gamma_{\mu \nu})\psi_A^{a'}$$

$$+ 2i(\eta_1^I \gamma^I_j \sigma^{IJ} \tilde{A}^I \tilde{A}^J) \psi_B^{a'}$$

$$+ (\Lambda_2^{a'} b' + v_2^\mu A^m \tau_m a' b')\psi_A^{a'}$$

$$+ \text{EOM},$$

$$[\delta_{\eta_1}, \delta_{\eta_2}]Z_A^{b'} = 2(\eta_1^I \gamma^I_j \eta_2^I) ((-2ix_\nu x^\mu \partial_\mu + ix^2 \partial_\nu)Z_A^{b'} - ix_\nu Z_A^{b'})$$

$$+ (\Lambda_3^{a'} b' + v_3^\mu A^m \tau_m a' b')Z_A^{b'},$$

and

$$[\delta_{\eta_1}, \delta_{\eta_2}]\psi_A^{b'} = 2(\eta_1^I \gamma^I_j \eta_2^I) ((-2ix_\nu x^\mu D_\mu + ix^2 D_\nu)Z_A^{b'} + (-2ix_\nu - 2x^\mu \gamma_{\mu \nu})\psi_A^{b'}$$

$$+ \Lambda_3^{a'} \psi_A^{b'} + \text{EOM.}$$

5 Acknowledgement

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A Conventions and Useful Identities

The conventions and identities of this appendix are adopted from Ref. [6].
A.1 Spinor Algebra

In 1 + 2 dimensions, the gamma matrices are defined as

\[(\gamma_\mu)^\alpha_\gamma (\gamma_\nu)^\beta_\gamma + (\gamma_\nu)^\alpha_\gamma (\gamma_\mu)^\beta_\gamma = 2\eta_{\mu\nu}\delta^\alpha_\beta.\] (A.1)

For the metric we use the \((-++,++)\) convention. The gamma matrices in the Majorana representation can be defined in terms of Pauli matrices: \((\gamma_\mu)^\alpha_\beta = (i\sigma_2, \sigma_1, \sigma_3)\), satisfying the important identity

\[(\gamma_\mu)^\alpha_\gamma (\gamma_\nu)^\beta_\gamma = \eta_{\mu\nu}\delta^\alpha_\beta + \varepsilon_{\mu\nu\lambda}(\gamma^\lambda)^\alpha_\beta.\] (A.2)

We also define \(\varepsilon^{\mu\nu\lambda} = -\varepsilon_{\mu\nu\lambda}\). So \(\varepsilon_{\mu\nu\lambda}\varepsilon^{\nu\lambda\rho} = -2\delta^\rho_\mu\). We raise and lower spinor indices with an antisymmetric matrix \(\varepsilon^\alpha_\beta = -\varepsilon^\alpha_\beta\), with \(\varepsilon_{12} = -1\). For example, \(\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\) and \(\gamma^\mu_\alpha = \epsilon_\beta(\gamma^\mu)^\alpha_\beta\), where \(\psi_\beta\) is a Majorana spinor. Notice that \(\gamma^\mu_\alpha = (\mathbb{1}, -\sigma^3, \sigma^1)\) are symmetric in \(\alpha\) and \(\beta\). A vector can be represented by a symmetric bispinor and vice versa:

\[A_{\alpha\beta} = A_\mu\gamma^\mu_\alpha_\beta, \quad A_\mu = -\frac{1}{2}\gamma_\alpha_\beta A_{\alpha\beta}.\] (A.3)

We use the following spinor summation convention:

\[\psi\chi = \psi^\alpha\chi_\alpha, \quad \psi\gamma_\mu\chi = \psi^\alpha(\gamma_\mu)^\alpha_\beta\chi_\beta,\] (A.4)

where \(\psi\) and \(\chi\) are anti-commuting Majorana spinors. In 1 + 2 dimensions the Fierz transformations are

\[(\lambda\chi)\psi = \frac{1}{2}(\lambda\psi)\chi - \frac{1}{2}(\lambda\gamma_\mu\psi)\gamma^\mu\chi.\] (A.5)

There is another useful identity:

\[(\psi_1\psi_2)\psi_3 + (\psi_2\psi_3)\psi_1 + (\psi_3\psi_1)\psi_2 = 0,\] (A.6)

where \(\psi_1, \psi_2,\) and \(\psi_3\) are arbitrary spinors. Finally, we define

\[\gamma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu].\] (A.7)

A.2 SO(4) and SO(5) Gamma Matrices

A.2.1 SO(4) Gamma Matrices

We define the 4 sigma matrices as

\[\sigma^I_A B = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{1}),\] (A.8)

by which one can establish a connection between the \(SU(2) \times SU(2)\) and \(SO(4)\) group. These sigma matrices satisfy the following Clifford algebra:

\[\sigma^I_A \bar{C} \sigma^J_A B + \sigma^J_A \bar{C} \sigma^I_A B = 2\delta^I_J \delta^A_B,\] (A.9)

\[\sigma^I_A C \sigma^J_A B + \sigma^J_A C \sigma^I_A B = 2\delta^I_J \delta^A_B.\] (A.10)
We use antisymmetric matrices
\[
\epsilon_{AB} = -\epsilon_{AB}^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\hat{A}\hat{B}} = -\epsilon_{\hat{A}\hat{B}}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
(A.11)
to raise or lower un-dotted and dotted indices, respectively. For example,
\[
\sigma^I_{\hat{I}} A B = -\epsilon_{\hat{A}\hat{B}} \sigma^I C \quad \text{and} \quad \sigma^I_{\hat{I}} A B = -\epsilon_{\hat{B}\hat{A}} \sigma^I C.
\]
(A.12)

The antisymmetric matrices \(\epsilon_{AB}\) and \(\epsilon_{\hat{A}\hat{B}}\) satisfy the identities
\[
\epsilon_{AB} \epsilon_{CD} = - (\delta_A^C \delta_B^D - \delta_A^D \delta_B^C),
\]
(A.13)
\[
\epsilon_{\hat{A}\hat{B}} \epsilon_{\hat{C}\hat{D}} = - (\delta_{\hat{A}}^\hat{C} \delta_{\hat{B}}^\hat{D} - \delta_{\hat{A}}^\hat{D} \delta_{\hat{B}}^\hat{C}).
\]
(A.14)

A.2.2 \(SO(5)\) Gamma Matrices

Using (A.8), we define the first four \(SO(5)\) gamma matrices as \(^5\)
\[
\Gamma^I A B = \begin{pmatrix} 0 & \sigma^I \\ \sigma^I & 0 \end{pmatrix} \quad (I = 1, \ldots, 4),
\]
(A.15)
and define the fifth \(SO(5)\) gamma matrix as
\[
\Gamma_5 A B = (\Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4)_A B.
\]
(A.16)
Notice that \(\Gamma^I A B\) \((I = 1, \ldots, 5)\) are Hermitian, satisfying the Clifford algebra
\[
\Gamma^I A C \Gamma^J C + \Gamma^J A C \Gamma^I C = 2 \delta^{IJ} \delta_A^B.
\]
(A.17)
We use an antisymmetric matrix \(\omega_{AB} = -\omega^{AB}\) to lower and raise indices; for instance
\[
\Gamma^{IAB} = \omega^{AC} \Gamma^I C B.
\]
(A.18)
It can be chosen as the charge conjugate matrix:
\[
\omega^{AB} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon^{\hat{A}\hat{B}} \end{pmatrix}.
\]
(A.19)
(Recall that \(A\) and \(\hat{B}\) of the RHS run from 1 to 2.)

By the definition (A.15) and the convention (A.18), the gamma matrix \(\Gamma^I\) is antisymmetric and traceless, and satisfies a reality condition
\[
\Gamma^{IAB} = -\Gamma^{IAB}, \quad \Gamma^I_A = 0 \quad \text{and} \quad \Gamma^{I\hat{A}} = \Gamma^{IAB} = \omega^{AC} \omega^{BD} \Gamma^I C D.
\]
(A.20)
The \(USp(4)\) generators are defined as
\[
\Sigma^{IJ}_A B = \frac{1}{4} [\Gamma^I, \Gamma^J]_A B.
\]
(A.21)

\(^5\)To avoid introducing too many indices, we still use the capital letters \(A, B, \ldots\) to label the \(USp(4)\) indices, and use \(I\) to label a fundamental index of \(SO(4)\). We hope this will not cause any confusion.
B  A review of the $\mathcal{N} = 4$ theory

In this appendix, we review the general $\mathcal{N} = 4$ theory constructed in [2]. This theory was constructed by generalizing the $\mathcal{N} = 4$ GW theory [1] to include twisted multiplets. So the theory contains both twisted and un-twisted multiplets. Both the twisted and untwisted multiplets are required to furnish quaternionic representations of the gauge group. Generally speaking, the quaternionic representation furnished by the twisted multiplets is not equivalent to the representation furnished by the untwisted multiplets.

We denote the untwisted and twisted multiplets by $(Z^a_A, \psi^a_A)$ and $(Z^a'_{\dot{A}}, \psi^a'_{\dot{A}})$, respectively. Here $A, \dot{A} = 1, 2$ transform in the two-dimensional representation of the $SU(2) \times SU(2)$ R-symmetry group; $a = 1, \ldots, 2R$ transforms in a quaternionic representation of the gauge group, and $a' = 1, \ldots, 2S$ transforms in another quaternionic representation of the gauge group. The corresponding quaternionic forms are denoted as $\omega_{ab}$ and $\omega_{a'b'}$, satisfying $\omega_{ab} \omega_{bc} = \delta^a_c$ and $\omega_{a'b'} \omega_{b'c'} = \delta^a_{a'}$. We use the antisymmetric tensors $\omega$ to lower or raise the indices; for instance, $\tau_{ab}^m = \omega_{ac} \tau^{mc} b$, where $\tau^{mc} b$ is a set of representation matrices of the Lie algebra of gauge symmetry. The un-twisted multiplets $(Z^a_A, \psi^a_A)$ and twisted multiplets $(Z^a'_{\dot{A}}, \psi^a'_{\dot{A}})$ satisfy the reality conditions:

\[
\begin{align*}
\bar{Z}^a_A &= \omega_{ab} A^b B^A, \\
\bar{Z}^a_{\dot{A}} &= \omega_{a'b'} \epsilon^{A'B'} B'^{A'}.
\end{align*}
\]  

where $\epsilon^{A'B'}$ and $\epsilon^{AB}$ are antisymmetric forms of the $SU(2) \times SU(2)$ R-symmetry group (see Appendix A.2.1).

To be compatible with the $\mathcal{N} = 4$ supersymmetry, the representation matrices $\tau_{ab}$ and $\tau_{a'b'}^m$ of the gauge group are required to satisfy the fundamental identities

\[
\begin{align*}
k_{mn} \tau_{(ab)}^m \tau_{(d'c)}^n = 0, & \quad k_{mn} \tau_{(a'b')}^m \tau_{(d'c')}^n = 0. 
\end{align*}
\]

Following Ref. [1], we define the “momentum maps” and “currents” as

\[
\begin{align*}
\mu_{AB}^m &\equiv \tau_{ab}^m Z^a_A Z^b_B, \quad j_{AB}^m &\equiv \tau_{a'b'}^m Z^a'_{\dot{A}} Z^b'_{\dot{B}} + \mu_{\dot{A}B}^m, \\
\mu_{\dot{A}B}^m &\equiv \tau_{a'b'}^m Z^a'_{\dot{A}} Z^b'_{\dot{B}}, \quad j_{\dot{A}B}^m &\equiv \tau_{a'b'}^m Z^a'_{\dot{A}} Z^b'_{\dot{B}}.
\end{align*}
\]

Denoting the gauge fields as $A^m_A$, the Lagrangian of the $\mathcal{N} = 4$ CSM theory reads [2, 3]

\[
\mathcal{L} = \frac{1}{2} \epsilon^{\mu\nu\lambda}(k_{mn} A^m_A \partial_\nu A^n_A + \frac{1}{3} C_{mnp} A^m_A A^n_A A^p_A)
\]

\[
\quad + \frac{1}{4} (-D_\mu Z^a_A D^\mu Z^a_A - D_\mu Z^\dot{a}_{\dot{A}} D^\mu Z^\dot{a}_{\dot{A}} + i \tilde{\psi}_a A^\mu + i \tilde{\psi}_{\dot{a}} A^\mu)
\]

\[
\quad + \frac{i}{2} k_{mn} (j_{AB}^m j_{AB}^n A_{\lambda}^m A_{\lambda}^n - j_{AB}^m j_{AB}^n A_{\lambda}^m A_{\lambda}^n)
\]

\[
\quad + \frac{i}{2} k_{mn} (\mu_{\dot{A}B}^m \tau_{a'b'} A^a_{\dot{A}} A^b_B + \mu_{AB}^m \tau_{a'b'} A^a_B A^b_{\dot{A}})
\]

\[
\quad + \frac{1}{24} C_{mnp} (\mu_{AB}^m A_{\mu}^\nu B_{\mu}^\rho C_{\nu}^\sigma_A + \mu_{\dot{A}B}^m \mu_{\dot{A}B}^\nu A_{\mu}^\rho C_{\nu}^\sigma_B)
\]

\[
\quad + \frac{1}{4} k_{mnkd} ((\tau_{a'b'}^m)_{ab} Z^a_A Z^b_B + (\tau_{a'\dot{b}}^m)_{a'\dot{b}} Z^a_{\dot{A}} Z^b'_{\dot{B}}) C_{\mu}^A A_{\mu}^\nu B_{\mu}^\rho C_{\nu}^\sigma_B).
\]
We have used the invariant form $k_{ms}$ on the Lie algebra of the gauge group to lower the indices of the structure constants: $C_{mpq} = k_{ms} k_{mq} C^{sq}$. (We will also define $\tau_m^a b = k_{mn} \tau^n a b$.)

The $\mathcal{N} = 4$ super Poincare transformations are given by

\[
\begin{align*}
\delta \epsilon_A &= i \epsilon_A \hat{A} \psi^a_A, \\
\delta \epsilon_A &= i \epsilon A^B \psi^a_A, \\
\delta \epsilon_A &= -\gamma^B D_\mu Z_{\mu}^a \epsilon_B^A - \frac{1}{3} \kappa_{mn} \tau^{ma} \psi Z_{nB}^a \epsilon_B^A - \epsilon_C \hat{C}_A^B \\
\delta \epsilon_A &= -\gamma^B D_\mu Z_{\mu}^a \epsilon_B^A - \frac{1}{3} \kappa_{mn} \tau^{ma} \psi Z_{nB}^a \epsilon_B^A + \epsilon_C \hat{C}_A^B.
\end{align*}
\]  

(B.5)

Here the parameters $\epsilon_A^B = (\epsilon^I_A)^B$ ($I = 1, \ldots, 4$) obey the reality conditions

\[\epsilon^I_A^B = -\epsilon^C_B \epsilon_A^C \epsilon_B^C.\]  

(B.6)

The fundamental identities (B.2) can be converted into certain Jacobi identities of two superalgebras $G$ and $G'$ admitting quaternionic structures [1]; and the Lie algebra of the gauge symmetry is the bosonic parts of $G$ and $G'$. More concretely, the Lie algebra of the gauge group is given by the bosonic subalgebra of the superalgebra $G$ [1]

\[
\left[ M^a, M^b \right] = f^a_{vw} M^w, \quad \left[ M^a, Q_a \right] = -\tau_{ab} \omega^{bc} Q_c, \quad \left\{ Q_a, Q_b \right\} = \tau_{ab} k_{uw} M^u,
\]

(B.7)

and the bosonic subalgebra of the superalgebra $G'$

\[
\left[ M^a', M^b' \right] = f^a'_{vw} M^w, \quad \left[ M^a', Q_a' \right] = -\tau_{a'b'} \omega^{b'c'} Q_c', \quad \left\{ Q_a', Q_b' \right\} = \tau_{a'b'} k_{uw'} M^{w'}.
\]

(B.8)

In order that there are physical interactions between the twisted and untwisted multiplets, the bosonic parts of the superalgebras $G$ and $G'$ are required to share at least one simple factor or $U(1)$ factor [2, 3]. More precisely, decomposing $M^u$ and $M^{u'}$ as $M^u = (M^a, M^g)$ and $M^{u'} = (M^{a'}, M^{g'})$, with $M^g$ the common generators, then the Lie algebra of the gauge symmetry is spanned by the generators [2, 3]

\[
M^m = (M^a, M^g, M^{a'}).
\]

(B.9)

In accordance with the decompositions of $M^u$ and $M^{u'}$, we may decompose the quadratic forms and structure constants as follows

\[
k_{uv} = (k_{ab}, k_{gh}), \quad k_{u'v'} = (k_{a'b'}, k_{gh}), \quad (B.10)
\]

\[
f^a_{vw} = (f^{ab}_{\gamma}, f^{gh}_{\gamma}), \quad f^a'_{vw} = (f^{a'b'}_{\gamma}, f^{gh}_{\gamma}).
\]

(B.11)

Let us now put (B.7) and (B.8) together:

\[
\left[ M^m, M^m \right] = C_{mp} M^p, \quad \left[ M^m, Q_a \right] = -\tau_{ab} \omega^{bc} Q_c, \quad \left[ M^m, Q_a' \right] = -\tau_{a'b'} \omega^{b'c'} Q_{c'},
\]

\[
\left\{ Q_a, Q_b \right\} = \tau_{ab} k_{mn} M^m, \quad \left\{ Q_a', Q_b' \right\} = \tau_{a'b'} k_{mn} M^m,
\]

(B.12)
where
\[ C^{mn}_p = (f^{\alpha\beta\gamma}, f^{fgh}, f^{\alpha'\beta'\gamma'}), \quad (B.13) \]
\[ k_{mn} = (k_{\alpha\beta}, k_{gh}, k_{\alpha'\beta'}). \quad (B.14) \]

It can be seen that the fundamental identities (B.2) are equivalent to the \( Q_a Q_b Q_c \) and \( Q_a' Q_b' Q_c' \) Jacobi identities of (B.13) [1]. The classification of the gauge groups can be found in [2, 3].

If the twisted and untwisted multiplets form the same representation of gauge group\(^6\), the \( N = 4 \) supersymmetry can be enhanced to \( N = 5 \) [13].

C Derivation of the \( N = 4 \) Super Poincare Currents

In this Appendix, we will derive the \( N = 4 \) super Poincare currents by using the standard Noether method.

The super Poincare variation of the action (B.4) must take the form
\[ \delta \epsilon S = \int d^3 x (-j^I_\mu) \partial^\mu \epsilon^I, \quad (C.1) \]
if we allow the set of parameters \( \epsilon^I (I = 1, \ldots, 4) \) to depend on the spacetime coordinates \( x^\mu \), since the action is invariant under the supersymmetry transformations (B.5) for constants \( \epsilon^I \). If the equations of motion are obeyed, the right hand side of (C.1) must vanish; Integrating by parts, we obtain
\[ \partial^\mu j^I_\mu = 0. \quad (C.2) \]

To derive \( j^I_\mu \), let us first calculate the super-variation of the Chern-Simons term in (B.4):
\[ \delta \epsilon L_{CS} = \frac{1}{2} \epsilon^{\mu\nu\lambda} k_{mn} \partial_\nu (A^m_\mu \delta A^n_\lambda) 
+ \epsilon_\mu A^\lambda \gamma^{\mu\nu} (j^mA_A - j^m_A A) k_{mn} F^n_{\mu\nu}. \quad (C.3) \]

The variations of the kinematic terms of the Lagrangian (B.4) are give by
\[ - \frac{1}{2} \delta \epsilon (D_\mu \bar{Z}^A_\mu D^\mu Z^A_\mu) = - \partial_\mu (i D^\mu \bar{Z}^A_\mu \epsilon_A \gamma^{\mu} \psi_A) 
+ iD^2 \bar{Z}^A_\alpha \epsilon_A \gamma^{\mu} \psi^{A}_\alpha 
+ iD^\mu \bar{Z}^B_\alpha \tau_m a B \epsilon_A \gamma_\mu (j^mA_A - j^m_A A), \quad (C.4) \]
\[ - \frac{1}{2} \delta \epsilon (D_\mu \bar{Z}^A_\alpha \gamma^{\mu} Z^A_\mu) = - \partial_\mu (i D^\mu \bar{Z}^A_\alpha \epsilon_A \gamma^{\mu} \psi^{A}_\alpha) 
+ iD^2 \bar{Z}^{A'}_\alpha \epsilon_A \gamma^{\mu} \psi^{A'}_\alpha 
+ iD^\mu \bar{Z}^B_\alpha \gamma^{\mu} \tau_m a' B \epsilon_A \gamma_\mu (j^mA_A - j^m_A A), \quad (C.5) \]

\( ^6 \)This can be achieved by letting both twisted and untwisted multiplets to take the representation of the bosonic subalgebra of \( G \) (B.7); this representation is furnished by the set of fermionic generators \( Q_a \).
and

$$\frac{1}{2} \delta \left(i \bar{\psi}_a \gamma^\mu D_\mu \psi_A^a\right) = -i \frac{1}{2} \partial_\mu \left(\bar{\psi}_a \gamma^\mu \delta \psi_A^a\right)$$

$$+ i \bar{\psi}_a \gamma^\mu \left(\delta \psi_A^a\right) \nu_{ab} \gamma_{\mu}^b \partial_\mu \epsilon^I$$

$$- i \epsilon_A \bar{\psi}_a D Z_a^\mu \epsilon^I A \mu_j A \mu^m A \mu \nu F_{\nu I}^m$$

$$+ i \left(\epsilon_A \bar{\psi}_a \gamma_\mu^a \right) m_{ab} \partial_\mu \left(\bar{\psi}_a \gamma_\mu^a \right)$$

$$- (\epsilon_A \bar{\psi}_a \gamma_\mu^a \left(\bar{\psi}_a \gamma_\mu^a \right) \left(\bar{\psi}_a \gamma_\mu^a \right)$$

\[ (C.7) \]

We have used the shorthand

$$\left(\delta \psi_\mu A^I_A \right) = - \gamma^\mu D_\mu \bar{Z}_A^\mu \sigma^I_{AB}$$

in (C.6), and used the shorthand

$$\left(\delta \psi_\mu A^I_A \right) = - \gamma^\mu D_\mu \bar{Z}_B^\mu \sigma^I_{AB}$$

in (C.7).

Let us now consider the Yukawa terms. To simplify the expressions, we define

$$\nu_{AB}^m = \tau_{ab}^m \bar{\psi}_A^b \psi_B^a$$

$$\nu_{AB}^m = \tau_{ab}^m \bar{\psi}_A^a \psi_B^b$$

$$\left(\tau^m \tau^m \right)_{[ab]} = \frac{1}{2} \left(\tau^m \tau^m \right)_{[ab]}$$

$$\left(\tau^m \tau^m \right)_{[a'b']} = \frac{1}{2} \left(\tau^m \tau^m \right)_{[a'b']}$$

The Yukawa terms are given by the third and fourth lines of (B.4). Their super variations read

$$\frac{1}{2} \delta \left(j_{AB} \bar{m} \right)$$

$$= \frac{1}{2} \left(\epsilon_A \bar{\psi}_a \gamma^a \right) m_{ab} \partial_\mu \left(\bar{\psi}_a \gamma_\mu^a \right)$$

$$+ \frac{1}{2} \left(\epsilon_A \bar{\psi}_a \gamma_\mu^a \right) \left(\tau^m \tau^m \right)_{[ab]} Z_B^A Z_B^B \mu_{A}^n B$$

\[ (C.11) \]
\[-\frac{i}{2} \delta_s (j^{\mu}^{AB} J^{m}_{AB} j^{m}_{AB}) = i \left[ i (\epsilon_{A} \dot{A} \psi_{A}^{\dot{A}}) \tau_{m a} \psi_{B}^{\dot{m}} (\psi_{B}^{\dot{m}} - j^{m}_{AB}) \right. \\
\left. - (\gamma^{\mu} D_{\mu} Z_{\dot{A}}^{\dot{A}} \epsilon_{A} \dot{A}) j^{m}_{AB} \tau_{m a} \psi_{B}^{\dot{m}} Z_{\dot{A}}^{\dot{A}} \right. \\
\left. - \frac{1}{4} (\epsilon_{A} \dot{A} j^{m}_{AB}) C_{m n p} \mu^{B} \frac{pB}{C} \mu^{nCN} \right. \\
\left. + \frac{1}{2} (\epsilon_{A} \dot{A} j^{m}_{AB} B) C_{m n p} \mu^{B} A \mu^{nA} \right. \\
\left. + \frac{1}{2} (\epsilon_{A} \dot{A} j^{m}_{AB} B) (\tau_{m} \tau_{n}) [a] \psi_{B}^{\dot{m}} Z_{\dot{A}}^{\dot{A}} Z_{\dot{A}}^{\dot{A}} \right], \quad (C.12)\]

\[2i \delta_s (j^{m}_{AB} j^{B}_{AB}) = -2i \left[ i (\epsilon_{A} \dot{A} \psi_{A}^{\dot{A}}) \tau_{m a} \psi_{B}^{\dot{m}} (\psi_{B}^{\dot{m}} - j^{m}_{AB}) \right. \\
\left. - (\gamma^{\mu} D_{\mu} Z_{\dot{A}}^{\dot{A}} \epsilon_{A} \dot{A}) j^{m}_{AB} \tau_{m a} \psi_{B}^{\dot{m}} Z_{\dot{A}}^{\dot{A}} \right. \\
\left. - \frac{1}{4} (\epsilon_{A} \dot{A} j^{m}_{AB}) C_{m n p} \mu^{B} \frac{pB}{C} \mu^{nCA} \right. \\
\left. + \frac{1}{2} (\epsilon_{A} \dot{A} j^{m}_{AB} B) C_{m n p} \mu^{B} A \mu^{nA} \right. \\
\left. + \frac{1}{2} (\epsilon_{A} \dot{A} j^{m}_{AB} B) (\tau_{m} \tau_{n}) [a] \psi_{B}^{\dot{m}} Z_{\dot{A}}^{\dot{A}} Z_{\dot{A}}^{\dot{A}} \right], \quad (C.13)\]

\[\frac{i}{2} \delta_s (\mu^{AB} \tau_{a} \psi_{A}^{\dot{A}} \psi_{B}^{\dot{A}}) = - (\epsilon_{A} \dot{A} j^{m}_{AB}) \psi_{B}^{\dot{m}} \\
\left. - i (\gamma^{\mu} D_{\mu} Z_{\dot{A}}^{\dot{A}} \epsilon_{A} \dot{A}) \psi_{B}^{\dot{m}} \tau_{a} \psi_{B}^{\dot{m}} \mu^{AB} \right. \\
\left. - \frac{i}{6} (\epsilon_{A} \dot{A} j^{p}_{AB} B) C_{m n p} \mu^{B} A \mu^{nmAB} \right. \\
\left. - \frac{i}{3} (\epsilon_{A} \dot{A} \psi_{B}^{\dot{A}} \tau_{a} \psi_{B}^{\dot{m}}) [a] \psi_{B}^{\dot{m}} \mu^{AB} \right. \\
\left. + \frac{i}{2} (\epsilon_{A} \dot{A} j^{p}_{AB} C_{m n p} \mu^{B} A \mu^{BC} \right. \\
\left. + i (\epsilon_{A} \dot{A} \psi_{B}^{\dot{A}} \tau_{a} \psi_{B}^{\dot{m}}) [a] \psi_{B}^{\dot{m}} \mu^{ABBC}, \quad (C.14)\]
and

\[
\frac{i}{2} \delta_c (\mu_{AB}^m \nu_{AB}^m) = - (\epsilon_A \lambda^m_B \lambda^m_A) \nu_{mA} \delta_B - i(\gamma_\mu \pi^a_{\lambda_B} \lambda^B \tau_{mab} \mu_{AB}^m
\]

\[
\begin{aligned}
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \mu_{AB}^m C_{mnp} \mu_{AB}^m \\
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \tau_{mB} \mu_{AB}^m \\
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \tau_{B} \mu_{AB}^m \\
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \tau_{mB} \mu_{AB}^m \\
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \mu_{AB}^m C_{mnp} \mu_{AB}^m \\
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \mu_{AB}^m C_{mnp} \mu_{AB}^m \\
&- i(\epsilon_A \lambda^m_B \lambda^m_A) \mu_{AB}^m C_{mnp} \mu_{AB}^m.
\end{aligned}
\]

(C.15)

The potential of the theory is given by the last two line of (B.4). Its supersymmetry transformation reads

\[
\delta_c \mathcal{L}_{\text{pot}} = \frac{i}{4} C_{mn} \epsilon_A \lambda^m_B \lambda^m_A \mu_{AB} \mu_{AB} \mu_{AB} \mu_{AB}.
\]

(C.16)

In deriving (C.16), we have used the identity

\[
C_{mnp} \tau_{m}^{n} \tau_{n}^{p} \tau_{ab}^{p} = (\tau_{mn}^{a}) \tau_{m}^{n} \tau_{n}^{a} + (\tau_{mn}^{b}) \tau_{m}^{n} \tau_{n}^{b} + (\tau_{mn}^{c}) \tau_{m}^{n} \tau_{n}^{c} + (\tau_{mn}^{d}) \tau_{m}^{n} \tau_{n}^{d}.
\]

(C.17)

Eq. (C.17) was derived [6] by using the identity \( k_{mn} \tau_{m}^{n} \tau_{n}^{p} = 0 \). There is a similar identity for the primed representation matrices \( \tau_{m}^{n} \tau_{n}^{p} \tau_{ab}^{p} \):

\[
C_{mnp} \tau_{m}^{n} \tau_{n}^{p} \tau_{ab}^{p} = (\tau_{mn}^{a}) \tau_{m}^{n} \tau_{n}^{a} + (\tau_{mn}^{b}) \tau_{m}^{n} \tau_{n}^{b} + (\tau_{mn}^{c}) \tau_{m}^{n} \tau_{n}^{c} + (\tau_{mn}^{d}) \tau_{m}^{n} \tau_{n}^{d}.
\]

(C.18)

Combining every thing, we obtain the variation of the action

\[
\delta_c S = \int d^4x [i \bar{\psi}_a \gamma^\mu (\delta \psi)_A \mu_{AB} \nabla^\mu \nabla^\mu \epsilon_A + i \bar{\psi}_a \gamma^\mu (\delta \psi)_A \mu_{AB} \nabla^\mu \nabla^\mu \epsilon_A + O(D\mu) + O(D\mu') + O(D\mu') + O(D\mu')]
\]

(C.19)

where we have dropped the total derivative terms. We have denoted the terms containing \( D\mu_j \) as \( O(D\mu_j) \), where \( D, \mu, \) and \( j \) stand for the covariant derivative, “momentum map” and “current” operators (see (B.3)), respectively. The other terms have similar meanings.
For instance, $O(\nu j)$ stands for the terms containing the “current” operator $j$ and the quantity $\nu$ defined by the first equation of (C.10).

The terms containing $D\mu j$ are given by the first term of the third line of (C.4), the fourth line of (C.7), and the second line of (C.11):

$$O(D\mu j) = i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D^a \bar{A}_m B a Z_B^a + \frac{i}{3} \epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a Z_B^a D_B \mu (Z_B^a \bar{A} B^a)$$

$$-i(\gamma D\mu j^A A_A A_A A_A A_A) j_m B \bar{A} m Z_B^a.$$  \hspace{1cm} (C.20)

$O(D\mu' j)$ are given by the first term of the third line of (C.5), the fifth line of (C.7), the seventh line of (C.13), and the second line of (C.15):

$$O(D\mu' j) = +iD^a \bar{A}_m B a \gamma_m j^m A_A \bar{A} B^a (\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a)$$

$$-i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a)$$

$$+2i(\gamma D\mu j^A A_A A_A A_A A_A) j_m B \bar{A} m Z_B^a.$$  \hspace{1cm} (C.21)

$O(D\mu' j')$ are given by the second line of (C.13), the second line of (C.14), the last term of (C.4), and the fifth line of (C.6):

$$O(D\mu' j') = 2i(\gamma D\mu j^A A_A A_A A_A A_A) j_m B \bar{A} m Z_B^a$$

$$-i(\gamma D\mu j^A A_A A_A A_A A_A) j_m B \bar{A} m Z_B^a$$

$$+i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a) - i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a).$$  \hspace{1cm} (C.22)

$O(D\mu' j')$ are given by the last term of (C.5), the fourth line of (C.6), and the second line of (C.12):

$$O(D\mu' j') = -i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D^a \bar{A}_m B a Z_B^a$$

$$+\frac{i}{3} (\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a)$$

$$-i(\gamma D\mu j^A A_A A_A A_A A_A) j_m B \bar{A} m Z_B^a.$$  \hspace{1cm} (C.23)

$O(\nu j)$ are given by the first term of (C.11) and the first term of the last line of (C.7):

$$O(\nu j) = -i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a)$$

$$-i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a).$$  \hspace{1cm} (C.24)

$O(\nu' j)$ are given by the first term of the last line of (C.6), the sixth line of (C.13), and the first line of (C.14):

$$O(\nu' j) = -(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) (\bar{A} B^a \gamma_m j^m A_A \bar{A} B^a)$$

$$+2i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a)$$

$$-i(\epsilon_A \dot{A}_j \gamma_m j^m A_A \bar{A} B^a) D_B \mu (Z_B^a \bar{A} B^a).$$  \hspace{1cm} (C.25)
\(O(\nu j')\) are given by the second term of the last line of (C.7), the first line of (C.13), and the first line of (C.15):

\[
O(\nu j') = (\epsilon_A \xi_{ma} \psi_B^a \bar{\psi}_B^b)(\psi_a^B j^m A) + 2(\epsilon_A \xi_{ma} \psi_B^a \bar{\psi}_B^b)(\psi_a^B j^m A) - (\epsilon_A \xi_{ma} \psi_B^a \bar{\psi}_B^b)\mu_{m\bar{A}} B.
\] (C.26)

\(O(\nu' j')\) are given by the second term of the last line of (C.6) and the first line of (C.12):

\[
O(\nu' j') = (\epsilon_A \xi_{ma} \psi_B^a \bar{\psi}_B^b)(\psi_a^B j^m A) - (\epsilon_A \xi_{ma} \psi_B^a \bar{\psi}_B^b)\mu_{m\bar{A}} B.\] (C.27)

\(O(\mu j)\) are given by the first line of (C.16), and the third line of (C.11):

\[
O(\mu j) = \frac{i}{4} C_{mn\rho} \epsilon_A \xi_{B B A} j^{m} \mu_{n(B C \mu p) A C} - \frac{i}{4} (\epsilon_A \xi_{B B A} j^{m}) C_{mn\rho} \mu_{n B} C \mu^{p C A} = 0.
\] (C.28)

\(O(\mu' j)\) are given by the last line of (C.16), the last two lines of (C.11), the last two lines of (C.13), and the third and fourth lines of (C.15):

\[
O(\mu' j) = -i\epsilon_A \xi_{B B A} j^{m} \mu_{n(B C \mu p) A C} \mu_{nB A} - \frac{1}{2} (\epsilon_A \xi_{B B A} j^{m B}) C_{mn\rho} \mu_{n B} A \mu_{m B A} + \frac{1}{2} (\epsilon_A \xi_{B B A} j^{m B}) (\tau_{m B} n B \mu_{n B} A) - i(\epsilon_A \xi_{B B A} j^{m B}) C_{mn\rho} \mu_{n B} A \mu_{m B A} - i(\epsilon_A \xi_{B B A} j^{m B}) (\tau_{m B} n B \mu_{n B} A) - \frac{1}{6} (\epsilon_A \xi_{B B A} j^{m B}) C_{mn\rho} \mu_{n B} A \mu_{m B A} + i(\epsilon_A \xi_{B B A} j^{m B}) (\tau_{m B} n B \mu_{n B} A) - \frac{1}{3} (\epsilon_A \xi_{B B A} j^{m B}) (\tau_{m B} n B \mu_{n B} A) - \frac{1}{6} (\epsilon_A \xi_{B B A} j^{m B}) C_{mn\rho} \mu_{n B} A \mu_{m B A}.
\] (C.29)

\(O(\mu' j')\) are given by the fourth line of (C.16), and third line of (C.13), the last two lines of (C.14):

\[
O(\mu j') = \frac{i}{2} \epsilon_A \xi_{B B A} j^{m} (\tau_{m B} n B \mu_{n B} A) - \frac{i}{2} (\epsilon_A \xi_{B B A} j^{m B}) C_{mn\rho} \mu_{n B} C \mu_{m C B}.
\] (C.30)
\( \mathcal{O}(\mu \mu' j') \) are given by fifth line of (C.16), the last two lines of (C.12), the forth and fifth lines of (C.13), and the third and fourth lines of (C.14):

\[
\mathcal{O}(\mu \mu' j') = i \epsilon_A A j^m B A (\tau_m \tau_n)_{[ab]} Z^{Ba} Z^{Bn} m_B A + i (\epsilon A j^m B A) C_{mn} \mu^p B A \mu^{m,n} A B
\]

\[
+ \frac{i}{2} (\epsilon A j^m B A) C_{mn} \mu^p B A \mu^{m,n} A B
\]

\[
+ \frac{i}{2} \epsilon_A j^m B A (\tau_m \tau_n)_{[a'b']} Z^{Cu'} Z^{Cu'} Z^{Cn} A \mu^{m,n} A B
\]

\[
+ i \epsilon_A j^m A A j^m B A (\tau_m \tau_n)_{[ab]} Z^{Ba} Z^{Bn} m_B A
\]

\[
- i \epsilon_A j^m A A j^m B A (\tau_m \tau_n)_{[ab]} Z^{Ba} Z^{Bn} m_B A
\]

\[
- \frac{i}{3} \epsilon_A j^m A A j^m B A (\tau_m \tau_n)_{[ab]} Z^{Ba} Z^{Bn} m_B A \mu^{m,n} A B. \tag{C.31}
\]

\( \mathcal{O}(\mu' \mu' j') \) are given by the third line of (C.16), the eighth line of (C.13), and the last two lines of (C.15):

\[
\mathcal{O}(\mu' \mu' j') = \frac{i}{2} \epsilon_A j^b A (\tau_m \tau_n)_{[ab]} Z^{A\mu} m_B C H^{mC} A
\]

\[
+ i (\epsilon A j^M A) C_{mn} \mu^p B A \mu^{m,n} A C
\]

\[
- \frac{i}{2} \epsilon_A j^p A C_{mn} \mu^p B A \mu^{m,n} A C
\]

\[
- i (\epsilon A j^b A) (\tau_m \tau_n)_{[ba]} Z^{A\mu} m_C A \mu^{m,n} A C. \tag{C.32}
\]

\( \mathcal{O}(\mu' \mu' j') \) are given by the second line of (C.16) and the third line of (C.12):

\[
\mathcal{O}(\mu' \mu' j') = \frac{i}{4} C_{mn} \epsilon_A j^m B A \mu^{mn} (\mu^p B A)
\]

\[
- \frac{i}{4} \epsilon_A j^m B A C_{mn} \mu^p B A \mu^{m,n} A C
\]

\[
= 0. \tag{C.33}
\]

Using the identities (B.2), (C.17), (C.18), and the identities in Appendix A, it is not difficult to prove that every quantity of the last three lines of (C.19) vanishes, i.e. \( \mathcal{O}(D \mu j) \cdots \mathcal{O}(\mu' \mu' j') = 0 \), or (C.20)= \cdots = (C.33)= 0. As a result, only the first line of (C.19) remains:

\[
\delta_S = \int d^4 x \left[ i \bar{\psi}_a^A \gamma^\mu (\delta \psi)^{la}_A \partial \mu \epsilon^l + i \bar{\psi}_a^A \gamma^\mu (\delta \psi)^{la}_A \partial \mu \epsilon^l \right]. \tag{C.34}
\]

Comparing (C.34) with (C.1), we are led to the \( N = 4 \) super Poincare currents

\[
J^{A}_\mu = - i \bar{\psi}_a^A \gamma_\mu (\delta \psi)^{la}_A - i \bar{\psi}_a^A \gamma_\mu (\delta \psi)^{la}_A. \tag{C.35}
\]
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