Linked Cluster Expansions Beyond Nearest Neighbour Interactions: Convergence and Graph Classes

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August 27, 2018

Abstract

We generalize the technique of linked cluster expansions on hypercubic lattices to actions that couple fields at lattice sites which are not nearest neighbours. We show that in this case the graphical expansion can be arranged in such a way that the classes of graphs to be considered are identical to those of the pure nearest neighbour interaction. The only change then concerns the computation of lattice imbedding numbers. All the complications that arise can be reduced to a generalization of the notion of free random walks, including hopping beyond nearest neighbour. Explicit expressions for combinatorical numbers of the latter are given. We show that under some general conditions the linked cluster expansion series have a non-vanishing radius of convergence.

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1 Introduction

Linked cluster expansions are series expansions about completely disordered lattice systems [1, 2]. Under very general conditions on the action the high temperature series can be shown to have a non-vanishing radius of convergence [3]. In many cases the singularity closest to the origin can be related to the physical singularity of the free energy, that is, the phase transition. To achieve valuable information from the series representations one needs to compute high orders (for most recent reference cf. [4, 5, 6, 7]). This is best done by means of graph theoretical methods. Usually the expansion is restricted to contact terms that couple fields located at nearest neighbour lattice sites. On the hypercubic lattice, which we refer to in the following, this leads to a considerable reduction of the number of graphs. The most striking property in this respect is that they have to have an even number of lines in every loop.

Here we discuss the question of how to take care of interactions beyond nearest neighbours in an optimal way such that the classes of graphs have to be modified minimally. This implies to keep them reliably small in number. As will be shown below, the graphs of nearest neighbour interactions are sufficient for the general case also. The only complication that arises can be absorbed completely by a generalization of random walks, for which explicit expressions are available.

Let $\Lambda$ denote a $D$-dimensional hypercubic lattice, either infinite $\Lambda = \mathbb{Z}^D$ or finite $\Lambda = \times_{i=0}^{D-1} (\mathbb{Z}/L_i \mathbb{Z})$, with all $L_i$ even (for convenience) and with periodic boundary conditions imposed. To be specific, here we discuss models that are described by the partition function

$$Z(J, v) = \int \prod_{x \in \Lambda} dN \Phi(x) \exp (-S(\Phi, v) + \sum_{x \in \Lambda} \sum_{a=1}^{N} J_a(x)\Phi_a(x)), \quad (1)$$

where $\Phi$ denotes a real, $N$-component scalar field, and $J$ are external sources. The action is assumed to be of the form

$$S(\Phi, v) = \sum_x V(\Phi(x)) + \frac{1}{2} \sum_{x \neq y \in \Lambda} \sum_{a,b=1}^{N} \Phi_a(x)v_{ab}(x, y)\Phi_b(y), \quad (2)$$

with the lattice site action $V(\Phi)$, supposed to be $O(N)$ invariant and appropriately bounded from below.

Truncated correlation functions are obtained by differentiation of the generating functional

$$W(J, v) = \ln Z(J, v),$$

$$W^{(2n)}_{a_1 \ldots a_{2n}}(x_1, \ldots, x_{2n}|v) = \left< \Phi_{a_1}(x_1) \cdots \Phi_{a_{2n}}(x_{2n}) \right>^c, \quad (3)$$

$$= \frac{\partial^{2n}}{\partial J_{a_1}(x_1) \cdots \partial J_{a_{2n}}(x_{2n})} W(J, v) \bigg|_{J=0},$$

$$\partial^{2n}$$
Susceptibilities are defined as zero momentum correlations such as
\[ \delta_{a,b} \chi^2 = \sum_x \langle \Phi_a(x)\Phi_b(0) \rangle^c \]
\[ \delta_{a,b} \mu^2 = \sum_x (\sum_{i=0}^{D-1} x_i^2) \langle \Phi_a(x)\Phi_b(0) \rangle^c . \]

The linked cluster expansion is the Taylor expansion of the generating functional \( W(J, v) \) with respect to \( v(x, y) \),
\[ W(J, v) = \left( \exp \sum_{x,y} \sum_{a,b} v_{ab}(x,y) \frac{\partial}{\partial v_{ab}(x,y)} \right) W(J, \hat{v}) \bigg|_{\hat{v}=0}. \]

It generalized in the obvious way to connected correlation functions according to (3). Multiple derivatives of \( W \) with respect to \( v(x, y) \) are managed by the identity
\[ \frac{\partial W}{\partial v_{ab}(x, y)} = \frac{1}{2} \left( \frac{\partial^2 W}{\partial J_a(x)\partial J_b(y)} + \frac{\partial W}{\partial J_a(x)} \frac{\partial W}{\partial J_b(y)} \right). \]

In most cases under consideration, the hopping term of the action is assumed to couple fields only at nearest neighbour lattice sites, that is,
\[ v_{ab}(x, y) = \begin{cases} 2\kappa \delta_{a,b}, & x = y \pm \hat{\mu} \text{ for some } \mu, \\ 0, & \text{otherwise}. \end{cases} \]

\( \hat{\mu} \) denotes the unit vector in the \( \mu \)th direction, \( \mu = 0, \ldots, D - 1 \). Correlation functions and susceptibilities become series in the hopping parameter \( \kappa \), with coefficients depending on the topology of the lattice and the interaction \( V \). If \( V \) satisfies appropriate bounds, the series can be shown to have a non-zero radius of convergence, and convergence is uniform in volume [3].

In order to manage the complexities which increase rapidly with increasing order, one is led in a natural way to a graphical device. Truncated correlation functions get a representation as a sum over connected graphs, each one endowed with the appropriate weight. In this description a line of a graph represents a nearest neighbour contact term. It implies the constraint that the two attached vertices have to be placed at nearest neighbour lattice sites. In this way, selflines are excluded, but otherwise no additional exclusion constraints hold. This leads to considerable simplifications and fast methods to compute weights of graphs. In particular, on the hypercubic lattice, for the weight of a graph to be non-vanishing, every occurring loop in this graph is subject to the constraint that it has an even number of lines. In turn, the number of graphs can be reduced in a considerable way. A graph with \( L \) lines contributes to the order \( \kappa^L \).

The simplifications just mentioned do no more apply for more general contact terms, that is, if \( v \) couples fields at lattice sites with larger separations. A line
still represents an interaction \( v \) but otherwise no nearest neighbour constraint holds anymore. At the first sight this amounts to a considerable enlargement of the class of graphs to be considered. In turn all the algorithms developed for the construction of graphs to high orders have to be extended as well.

In this letter we show that the arising complications do not have any influence on the graphical expansions. Hopping parameter expansions for lattice models with pair interactions over larger distances can be cast into graph classes identical to those already known from pure nearest neighbour interactions. This implies that the same algorithms to generate all contributing graphs apply to this case. The only complication that arises is that the notion of free random walks assigned to chains of vertices with two neighbours have to be generalized appropriately.

In the second part of this paper we will generalize the proof given in [3], namely that the radius of convergence is non-vanishing for linked cluster expansions of models with nearest neighbour interactions, to the present case. It will be shown that under some general conditions the connected Green functions and susceptibilities are analytic functions in the hopping parameters for a large class of pair interactions.

2 Graphical expansion.

2.1 Some basic graph theory

A graph is a sequence of two objects and two maps
\[
\Gamma = (\mathcal{L}_\Gamma, \mathcal{B}_\Gamma, E_\Gamma, \Phi_\Gamma),
\]
with \( \mathcal{L}_\Gamma \) and \( \mathcal{B}_\Gamma \neq \emptyset \) disjoint sets, the internal lines and vertices of \( \Gamma \), respectively. \( E_\Gamma \) is a map
\[
E_\Gamma : \mathcal{B}_\Gamma \rightarrow \{0, 1, 2, \ldots \},
\]
\[
v \rightarrow E_\Gamma(v), \tag{9}
\]
that assigns to every vertex \( v \) the number of external lines \( E_\Gamma(v) \) attached to it. The number of external lines of \( \Gamma \) is given by \( E_\Gamma = \sum_{v \in \mathcal{B}_\Gamma} E_\Gamma(v) \). Finally, \( \Phi_\Gamma \) is the incidence relation that assigns internal lines to their endpoint vertices. We consider lines as being unoriented, so \( \Phi_\Gamma \) maps onto unoriented pairs of vertices
\[
\Phi_\Gamma : \mathcal{L}_\Gamma \rightarrow (\mathcal{B}_\Gamma \times \mathcal{B}_\Gamma). \tag{10}
\]
In order to stay more general we do not exclude selflines in this definition. External vertices are those that have external lines attached,
\[
\mathcal{B}_{\Gamma, ext} = \{ v \in \mathcal{B}_\Gamma \mid E_\Gamma(v) \neq 0 \}, \tag{11}
\]
whereas internal vertices don’t, \( \mathcal{B}_{\Gamma, int} = \mathcal{B}_\Gamma \setminus \mathcal{B}_{\Gamma, ext} \). For every pair of vertices \( v, w \in \mathcal{B}_\Gamma \), \( m(v, w) \) denotes the number of common lines of \( v \) and \( w \), i.e. the number of elements of
\[
\mathcal{M}(v, w) = \{ l \in \mathcal{L}_\Gamma \mid \Phi_\Gamma(l) = (v, w) \}, \tag{12}
\]
\[ m(v, w) = |\mathcal{M}(v, w)|. \]

\( w \) is called a neighbour of \( v \) if \( m(v, w) \neq 0 \), and

\[ \mathcal{N}(v) = \{ w \in B_{\Gamma} \mid m(v, w) \neq 0 \} \]  

is the set of neighbours of \( v \), with \( \mathcal{N}(v) = |\mathcal{N}(v)| \) its number of elements. For this definition, multiple lines are not count according to their multiplicity. For every integer \( n \), \( n \)-vertices are those that have precisely \( n \) neighbours,

\[ B_{\Gamma}^{(n)} = \{ v \in B_{\Gamma} \mid \mathcal{N}(v) = n \}. \]  

(14)

An \( n \)-vertex has at least \( n \) internal lines attached. The number of total lines, both internal and external ones, attached to a vertex \( v \) is denoted by

\[ l(v) := \sum_{w \in B_{\Gamma}} m(v, w) + E_{\Gamma}(v). \]

If \( l(v) \) is even for every \( v \in B_{\Gamma} \), \( \Gamma \) is called even. In particular, then, \( \Gamma \) has an even number of external lines. A vertex \( v \in B_{\Gamma} \) is called weak \( n \)-vertex, \( v \in B_{\Gamma, weak}^{(n)} \), if \( \mathcal{N}(v) = l(v) = n \). A weak vertex does not have any external line attached.

The following topological notions will be used below. A path is an ordered non-empty sequence \( p = (w_1, \ldots, w_n) \) of vertices with \( m(w_i, w_{i+1}) \neq 0 \) for all \( i = 1, \ldots, n - 1 \). \( p \) is called a path in \( \Gamma \) from \( w_1 \) to \( w_n \) of length \( n - 1 \). If in addition \( w_n = w_1 \), \( p \) is called a loop, with \( n - 1 \) its number of lines. If for every pair of vertices \( v, w \in B_{\Gamma}, v \neq w \), there is a path from \( v \) to \( w \), the graph \( \Gamma \) is called connected. A path \( p = (w_1, w_2, \ldots, w_n) \) is called a maximal weak 2-chain if \( w_i \in B_{\Gamma, weak}^{(2)} \) for all \( i = 2, \ldots, n - 1 \), but \( w_1, w_n \notin B_{\Gamma, weak}^{(2)} \).

\[ \Phi_2 \circ \phi_2 = \Phi_1 \circ \phi_1, \]

\[ E_2 \circ \phi_1 = E_1, \]  

(16)

Figure 1: Example of a maximal weak 2-chain of length 4. The 2-chain has 3 vertices belonging to \( B_{\Gamma, weak}^{(2)} \) and hence 4 lines.

Two graphs

\[ \Gamma_1 = (\mathcal{L}_1, B_1, E_1, \Phi_1), \Gamma_2 = (\mathcal{L}_2, B_2, E_2, \Phi_2) \]  

are called (topologically) equivalent if there are two maps \( \phi_1 : B_1 \to B_2 \) and \( \phi_2 : \mathcal{L}_1 \to \mathcal{L}_2 \), such that

\[ \Phi_2 \circ \phi_2 = \Phi_1 \circ \phi_1, \]

\[ E_2 \circ \phi_1 = E_1, \]  

(16)
where $\circ$ means decomposition of maps, and

$$\tilde{\varphi}_1 : \mathcal{B}_1 \times \mathcal{B}_1 \rightarrow \mathcal{B}_2 \times \mathcal{B}_2$$

$$\tilde{\varphi}_1(v, w) = (\varphi_1(v), \varphi_1(w)).$$

(17)

A symmetry of a graph $\Gamma = (\mathcal{L}, \mathcal{B}, E, \Phi)$ is a pair of maps $\phi_1 : \mathcal{B} \rightarrow \mathcal{B}$ and $\phi_2 : \mathcal{L} \rightarrow \mathcal{L}$, such that

$$\Phi \circ \phi_2 = \tilde{\phi}_1 \circ \Phi,$$

$$E \circ \phi_1 = E.$$  

(18)

The number of those maps is called the symmetry number of $\Gamma$.

The set of equivalence classes of connected graphs with $E$ external and $L$ internal lines is henceforth denoted by $\mathcal{G}_E(L)$, and

$$\mathcal{G}_E := \bigcup_{L \geq 0} \mathcal{G}_E(L).$$

(19)

If we impose the additional constraint that every loop has to have an even number of lines, we get the sets $\mathcal{G}^e_E(L)$ and $\mathcal{G}^e_E$.

In practice one introduces further classes such as one-particle (or better to say: one-line) irreducible ones. Because this is of no importance for the following, we do not introduce them here. Also, just for simplicity we discuss the one-component model only. The generalizations are straightforward.

### 2.2 Weights

The graphical representation of the linked cluster expansion of susceptibilities is an expansion in term of equivalence classes of connected graphs as defined above. For every such equivalence class we need exactly one representative.

Every graph $\Gamma$ represents a number, which is called its weight. If $\Gamma$ is connected and has $E$ external lines, it contributes this weight to the susceptibility $\chi_E$. For the one-component field models, it is computed along the following lines. Only even graphs need to be considered, otherwise the weight vanishes.

1. All the vertices of $\Gamma$ are placed at lattice sites. No exclusion principle holds, i.e. any number of vertices can be placed at the same site. By assumption, the model lives on the lattice $\Lambda$, which satisfies translation invariance. An arbitrary selected vertex is located at a fixed lattice site, avoiding a volume factor. A priori, all the other vertices of the graph are placed arbitrarily.

2. Every internal line of $\Gamma$ is assigned a factor $v(x, y)$, where $x, y \in \Lambda$ are the lattice sites its endpoints are placed at.
3. The sum is taken over all possible placements of the vertices at lattice sites (except for the fixed vertex). The amount of computational work can be reduced considerably by taking into account the particular form of \( v(x, y) \) from the very beginning.

4. A factor \( v_{2n}^c \) for every vertex \( v \in B_\Gamma \), where
\[
v_{2n}^c = W^{(2n)}(0, \ldots, 0|v = 0),
\]
 cf. Eqn. (20), and \( 2n \) is the sum of internal and external lines attached to \( v \).

5. Two final factors complete the weight. The first one is given by \( (S(\Gamma))^{-1} \), where \( S(\Gamma) \) is the symmetry number of \( \Gamma \). It has the product representation
\[
S(\Gamma) = S_P(\Gamma) \cdot \prod_{(v, w) \in B_\Gamma \times B_\Gamma} m(v, w)!,
\]
 where the integer number \( S_P(\Gamma) \) corresponds to the symmetries of \( \Gamma \) under permutations of vertices. The latter product is due to the exchange of multiple lines between vertices. The other final factor counts the various enumerations of the external lines,
\[
\frac{E!}{\prod_{v \in B_\Gamma} E_\Gamma(v)!}.
\]

Weighted susceptibilities such as the moment \( \mu_2 \) are described by the same diagrams as unweighted ones. The only difference is that they have the appropriate weight factors inserted. This, of course, must be done before the sum over the lattice imbeddings is carried out.

3  Reorganization of graphs.

Let us repeat that \( \Lambda \) denotes the \( D \)-dimensional hypercubic lattice \( \Lambda = \times_{i=0}^{D-1} \mathbb{Z}/L_i \mathbb{Z} \), with all \( L_i \) even, possibly infinity, and with periodic boundary conditions imposed. In terms of the classes of graphs introduced in the last section, truncated correlation functions and in particular susceptibilities allow for a representation as sum over graphs
\[
\chi_E = \sum_{\Gamma \in \mathcal{G}_E} w(\Gamma),
\]
 with corresponding weight \( w(\Gamma) \). At this point we are not concerned with questions of convergence. Our interest here is in combinatorical rearrangements to arbitrary large but finite order. If desired we limit the discussion to propagators with \( v(x, y) = 0 \) if \( ||x - y|| > R \) for convenient \( R > 0 \), where
\[
||x - y|| = \sum_{i=0}^{D-1} \inf_{n \in \mathbb{Z}} |x_i - y_i + nL_i|,
\]
and to graphs $\Gamma \in \mathcal{G}_E$ with a limited number of lines. The limitation is not necessary for a convergence proof of the linked cluster expansion. This will be discussed in the last section of this work. We also mention in passing that the discussion below works also for more restricted classes of graphs such as 1-vertex irreducible or 1-particle irreducible ones.

In the following we restrict attention to pair interactions that respect the euclidean lattice symmetries. A convenient notion in this respect is given by

**Definition 3.1** A signature $p$ of length $l \in \mathbb{N}$ is a collection of $D$ non-negative integers $l_i \in \mathbb{N} \cup \{0\}$,

$$p = (l_0, \ldots, l_{D-1}),$$

satisfying $\sum_{i=0}^{D-1} l_i = l$.

Collections of integers are unordered sequences, i.e. permutations of elements in $p$ represent the same signature. For a signature $p$ we write $l_p$ for its length, and the set of ordered $D$-tupels originating from $p$ is denoted by $P_D(p)$.

We may represent a signature $p = (l_0, \ldots, l_{D-1})$ as a path connecting two sites $x, y$ on the $D$-dimensional lattice. The path is given by \{(x_0), (x_1, x_2), \ldots, (x_{D-1}y)\}, $x_{i+1} = x_i + l_i e_{\hat{i}}$, $i \in \{0, \ldots, D-1\}$, $e_i \in \{-1, 1\}$, $x_0 = x$, $x_D = y$, where $\hat{i}$ is the unit vector in $i$-direction.

We consider hopping propagators of the form

$$v(x, y) = \sum_{p \in \mathcal{S}} 2\kappa_p v^p(x, y),$$

where $\mathcal{S}$ is a finite set of signatures $p$ of lengths $l_p \geq 1$, and with

$$v^p(x, y) = \sum_{l \in P_D(p)} \sum_{e_0, \ldots, e_{D-1} = \pm 1} \delta^T_{x-y, \sum_{i=0}^{D-1} l_i e_i \hat{i}}.$$

$\delta^T$ denotes the Kronecker delta on the torus, that is,

$$\delta^T_{x,y} = 1 \text{ if and only if } \inf_{n \in \mathbb{Z}} |x_j - y_j + nL_j| = 0 \text{ for all } j = 0, \ldots, D - 1,$$

and $\delta^T_{x,y} = 0$ in all other cases. Eqn. \((27)\) denotes a propagator of a particular geometric profile. We assume that every $p \in \mathcal{S}$ of the form \((25)\) satisfies for all $j = 0, 1, \ldots, D - 1$

$$l_j < \min_{i=0,\ldots,D-1} \frac{L_i}{2}.$$  

In particular, then, we have for $p \in \mathcal{S}$ of length $l_p$

$$v^p(x, y) = 0 \text{ if } ||x - y|| \neq l_p.$$  

The assumption made here that $\mathcal{S}$ is a finite set is just for convenience. Sufficient convergence conditions that allow $\mathcal{S}$ to be infinite are discussed in the next section.
Our aim is to cast (23) in such a form that we have to sum over graphs only that we have met already for the case of pure nearest neighbour interactions. That is,

\[ \chi_E = \sum_{\Gamma \in G_E} w_{\text{ev}}(\Gamma), \]

with appropriate weight \( w_{\text{ev}}(\Gamma) \). Toward this end, let \( \tilde{G}_E \) denote the set of pairwise inequivalent graphs with \( E \) external lines and no weak 2-vertices, that is,

\[ \tilde{G}_E = \left\{ \Gamma \in G_E | \mathcal{B}_{\Gamma, \text{weak}} = \emptyset \right\}. \]

(30)

We obtain from (23)

\[ \chi_E = \sum_{\Gamma \in \tilde{G}_E} \tilde{w}(\Gamma), \]

(31)

where the weight is computed as outlined in the last section, the only modification being that a line now represents, instead of \( v(z, w) \), a factor

\[ P_{z \rightarrow w}((\kappa_p)_{p \in S}, \tilde{v}_2) = \sum_{m \geq 1} \sum_{x_1, \ldots, x_{m-1} \in \Lambda} v(z, x_1) v(x_1, x_2) \cdots v(x_{m-1}, w) \left( \frac{\partial}{\partial \kappa_p} \right)^m \]

\[ = \frac{1}{\tilde{v}_2} \left( 1 - \frac{\partial}{\partial \kappa_p} \right)^{-1} (z, w). \]

(32)

The next step is to decompose \( P_{z \rightarrow w} \) in terms of random walks of given length in units of lattice links. This is formulated as

**Lemma 3.1** For the hopping propagator \( v \) given by (20), we have

\[ P_{z \rightarrow w}((\kappa_p)_{p \in S}, \tilde{v}_2) = \sum_{l \geq 1} P_{z \rightarrow w}^l((\kappa_p)_{p \in S}, \tilde{v}_2), \]

(33)

with

\[ P_{z \rightarrow w}^l((\kappa_p)_{p \in S}, \tilde{v}_2) = \sum_{m=1}^{l} \left( \frac{\partial}{\partial \kappa_p} \right)^m \sum_{p_1, \ldots, p_m \in S, \left( \sum_{p_i=1}^m p_i = l \right)} \sum_{x_1, \ldots, x_{m-1} \in \Lambda} \prod_{i=1}^m v^{p_i}(z, x_1) \cdots v^{p_m}(x_{m-1}, w) \left( \frac{\partial}{\partial \kappa_p} \right)^m \]

\[ = \sum_{(\tau_p)_{p \in S}} \left( \frac{\partial}{\partial \kappa_p} \right)^{\sum_{p \in S} n_p - 1} \mathcal{N}_{z \rightarrow w}^\tau \prod_{p \in S} (2\kappa_p)^{n_p}, \]

(34)

where we have set \( x_0 = z \), and with non-negative integers

\[ \mathcal{N}_{z \rightarrow w}^\tau = \left( \prod_{p \in S} \frac{1}{n_p!} \frac{\partial^{n_p}}{\partial^{n_p} \kappa_p} \right) \left( 1 - \sum_{p \in S} \kappa_p^{n_p} \right)^{-1} (z, w) \bigg|_{\kappa_p = 0}. \]

(36)
Furthermore
\[ P_{z \to w}^l((\kappa_p)_{p \in S}, v_2^c) \neq 0 \] only if \( l - ||z - w|| \geq 0 \) and even. \quad (37)

Proof: Inserting (26) into the right hand side of the first equality of (32) and collecting the sum over all products of propagators \( v^p \) of fixed total length \( l = \sum l_{p_i} \), we obtain (33), with \( P_{z \to w}^l \) as given by (34). Furthermore, (35) follows immediately from the Taylor expansion of
\[ P_{z \to w}((\kappa_p)_{p \in S}, v_2^c) = 1 \]
\[ \circ v_2 (1 - \circ v_2 (2 \sum_{p \in S} \kappa_p) v^p) (z, w). \]

Finally, for every term of (34) we have, with \( x_0 = z \) and \( x_m = w \)
\[ ||z - w|| = \| \sum_{i=1}^{m-1} (x_i - x_{i-1}) \| \leq \sum_{i=1}^{m-1} ||x_i - x_{i-1}|| = \sum_{i=1}^{m-1} l_{p_i} = l, \]
and we have used (28) and the triangle inequality for the norm (24). For \( j = 0, \ldots, D - 1 \),
\[ \inf_{n \in \mathbb{Z}} |z_j - w_j + n L_j| = \inf_{n \in \mathbb{Z}} |\sum_{i=1}^{m-1} (x_i - x_{i-1})_j + n L_j| \]
\[ = \sum_{i=1}^{m-1} \left( \inf_{n \in \mathbb{Z}} |(x_i - x_{i-1})_j + n L_j| \right) + \text{even number}. \]

Hence,
\[ ||z - w|| = \sum_{j=0}^{D-1} \inf_{n \in \mathbb{Z}} |z_j - w_j + n L_j| \]
\[ = \sum_{i=1}^{m-1} ||x_i - x_{i-1}|| + \text{even number} \]
\[ = l + \text{even number}, \]
that is, (37). This completes the proof. \( \square \)

Inserting the decomposition (33) into (31), we obtain the desired representation (29). The property (37) ensures that only those graphs contribute which have an even number of lines in each of their loops.

Hence we end up with the same class of graphs as for the case of pure nearest neighbour interactions. The weight of a graph is computed as described in the last section with the following exception only. Every maximal weak 2-chain of a graph contributes a factor \( P_{z \to w}^l((\kappa_p), v_2^c) \) to the lattice imbedding number, where \( l \) is the
length of the chain and \( w, z \) the lattice sites where its two endpoint vertices are placed at. We notice that for even \( l \) neither these endpoint vertices have to be different nor the lattice sites \( z \) and \( w \). If \( l = 1 \), \( P^l_{z \rightarrow w} \) just imposes the nearest neighbour constraint on \( z \) and \( w \).

If we choose the hopping parameters \( (\kappa_p | p \in \mathcal{S}) \) according to

\[
\kappa_p = c_p \kappa^l_p,
\]

for all signatures \( p \in \mathcal{S} \), with the \( c_p \) being fixed, every graph \( \Gamma \in \mathcal{G}^E \) with \( L \) lines that contributes to (29) has a weight proportional to \( \kappa^L \). In this case, every graph contributes to the same order of \( \kappa \) as it would for the pure nearest neighbour interaction.

Translation invariance amounts to Fourier transform. With

\[
\int_k := \prod_{i=0}^{D-1} \left( \frac{1}{L_i} \sum_{k_i = \frac{2\pi}{L_i} n_i} \right)
\]

we have

\[
\sum_{\epsilon_0, \ldots, \epsilon_{D-1} = \pm 1} \delta^T_{\epsilon_0, \ldots, \epsilon_{D-1}} = \int_k e^{ik \cdot x} \prod_{i=0}^{D-1} (2 \cos k_i n_i).
\]

The random walk numbers \( \mathcal{N}^{(n_p | p \in \mathcal{S})}_{z \rightarrow w} \) are then given by

\[
\mathcal{N}^{(n_p | p \in \mathcal{S})}_{z \rightarrow w} = \left( \frac{\prod_p n_p!}{\prod_p n_p!} \int_k e^{ik \cdot (z - w)} \prod_{p \in \mathcal{S}} \left( \sum_{l \in P^l_{D(p)}} \prod_{i=0}^{D-1} (2 \cos k_i l_i) \right)^{n_p} \right) \n_p.
\]

Similar as for the pure nearest neighbour interactions, this representation can be used to compute the random walk combinatorial integers \( \mathcal{N}^{(n_p | p \in \mathcal{S})}_{z \rightarrow w} \) explicitly.

As an illustrative example we supplement the nearest neighbour couplings by next-to-nearest neighbour interactions. In this case, we have 3 signatures, \( \mathcal{S} = \{p_1, p_2, p_3\} \), with

\[
p_1 = (1, 0, 0) \quad p_2 = (1, 1, 0) \quad p_3 = (2, 0, 0)
\]

written here for \( D = 3 \) dimensions. A maximal weak 2-chain of length \( l \) as part of a graph implies a factor \( P^l_{z \rightarrow w}((\kappa_p | p \in \mathcal{S}), \nu^c_2) \), cf. (34),(35), with \( z, w \) the lattices sites its end vertices are placed at. In order to get lattice imbedding numbers, one needs to compute the random walk numbers \( \mathcal{N}^{(n_1, n_2, n_3)}_{z \rightarrow w} \) for all nonnegative integers \( n_1, n_2, n_3 \) with

\[
l = n_1 + 2(n_2 + n_3).
\]

In Table 1 we give all the numbers for \( z = w \) and chain length \( l = 16 \).
Table 1: Example of next-to-nearest neighbour combinatorical random walk numbers $N_{z\rightarrow w}^{(n_1,n_2,n_3)}$ on the 3-dimensional hypercubic lattice, infinitely extended in all directions. The numbers are for $z = w$ and total chain length $l = 16$. The pure nearest neighbour case gives $N_{0\rightarrow 0}^{(16,0,0)} = 27770358330$.

| $n_1$ | $n_2$ | $n_3$ | $N_{0\rightarrow 0}^{(n_1,n_2,n_3)}$ | $n_1$ | $n_2$ | $n_3$ | $N_{0\rightarrow 0}^{(n_1,n_2,n_3)}$ |
|-------|-------|-------|-------------------------------|-------|-------|-------|-------------------------------|
| 14    | 1     | 0     | 137889591840                  | 4     | 4     | 2     | 20325816000                   |
| 14    | 0     | 1     | 56502916470                  | 4     | 3     | 3     | 11722233600                   |
| 12    | 2     | 0     | 277007722992                  | 4     | 2     | 4     | 38905272000                   |
| 12    | 1     | 1     | 227358907776                  | 4     | 1     | 5     | 6785856000                    |
| 12    | 0     | 2     | 48295573326                  | 4     | 0     | 6     | 54299700                      |
| 10    | 3     | 0     | 288364684608                  | 2     | 7     | 0     | 437724000                     |
| 10    | 2     | 1     | 355550656032                  | 2     | 6     | 1     | 1266904800                    |
| 10    | 1     | 2     | 151142397120                  | 2     | 5     | 2     | 1617174720                    |
| 10    | 0     | 3     | 21791715660                  | 2     | 4     | 3     | 1175005440                    |
| 8     | 4     | 0     | 165353348160                  | 2     | 3     | 4     | 509120640                     |
| 8     | 3     | 1     | 272212617600                  | 2     | 2     | 5     | 142067520                     |
| 8     | 2     | 2     | 173721477600                  | 2     | 1     | 6     | 18506880                     |
| 8     | 1     | 3     | 50101286400                  | 2     | 0     | 7     | 1610280                      |
| 8     | 0     | 4     | 5517544230                  | 0     | 8     | 0     | 4038300                      |
| 6     | 5     | 0     | 51139740960                  | 0     | 7     | 1     | 13305600                     |
| 6     | 4     | 1     | 105386339520                  | 0     | 6     | 2     | 20049120                     |
| 6     | 3     | 2     | 89715669120                  | 0     | 5     | 3     | 17015040                     |
| 6     | 2     | 3     | 38870314560                  | 0     | 4     | 4     | 9918720                      |
| 6     | 1     | 4     | 8536207680                  | 0     | 3     | 5     | 2938660                      |
| 6     | 0     | 5     | 761454540                    | 0     | 2     | 6     | 1028160                      |
| 4     | 6     | 0     | 7704774000                  | 0     | 1     | 7     | 0                             |
| 4     | 5     | 1     | 19072972800                  | 0     | 0     | 8     | 44730                       |

4 Convergence of the linked cluster expansion

It was shown in [3] that the linked cluster expansion for connected Green functions and susceptibilities have a non-zero radius of convergence for the case of nearest neighbour couplings, i.e. $v^p = 0$ for all signatures with length $l_p > 1$. We will generalize the proof given in [3] to hopping propagators which contain terms beyond nearest neighbour interactions.

We do not suppose here that there are only a finite number of terms in the hopping propagator eq. (26). The non-local hopping parameter may connect arbitrary lattice sites. $S$ is the set of all signatures $p$ of length greater or equal to one.
Let us define a hopping parameter $\kappa$ and constants $c_p$ for all signatures $p$ by

$$\kappa_p = c_p \kappa^{l_p}.$$  

$l_p$ denotes the length of signature $p$. The constants $c_p$ have to be chosen such that the series on the right hand side of eq. (26) is convergent. The next lemma gives a sufficient condition on the coefficients $c_p$.

**Lemma 4.1** Suppose that there exists a positive constant $\kappa > 0$ such that for all signatures $p$

$$|c_p| \leq (2^{-1-\eta\kappa^{-1}})^{l_p} \kappa,$$

where $\eta > 0$ is a constant obeying

$$\sum_{y: 0 \neq y \in \Lambda} 2^{-\eta \|y\|} \leq 4D 2^{-D}.$$  

Then a bound for the non-local part of the action is given by

$$\left| \frac{1}{2} \sum_{x,y \in \Lambda} \Phi(x)v(x,y)\Phi(y) \right| \leq 4D|\kappa| \sum_{x \in \Lambda} |\Phi(x)|^2$$

for all complex $\kappa$, $|\kappa| \leq \kappa$ and all $\Phi \in \mathcal{H}(\Lambda)$ (=Hilbert space of square summable functions on $\Lambda$).

**Proof**: The number of all signatures $p$ with length $l_p = l$ is bounded by

$$|\{ p \text{ signature} \mid l_p = l \}| \leq 2^{D+l}.$$  

Then the left hand side of inequality (42) is bounded by

$$\sum_{x,y \in \Lambda} \sum_{l_p = |x-y|} |c_p| |\kappa|^{l_p} |\Phi(x)|^2.$$  

Using supposition (40) the bound becomes

$$|\kappa| \sum_{x,y \in \Lambda} \sum_{l_p = |x-y|} 2^{-l_p} 2^{-\eta \|x-y\|} |\Phi(x)|^2.$$  

By the bounds (43) and (41) the left hand side of inequality (42) is estimated by

$$2^D |\kappa| \sum_{x,y \in \Lambda} 2^{-\eta \|x-y\|} |\Phi(x)|^2.$$  

Using supposition (41) we obtain the assertion.  

The convergence of the linked cluster expansion is presented in the following theorem for a symmetric translation invariant scalar field theory on an infinite lattice. A modification of the proof for $N$-component models or models on a torus is straightforward. Also the restriction to translation invariance and $\mathbb{Z}_2$-symmetry is of no relevance for the convergence proof.
**Theorem 4.1** Let a translation invariant and $\mathbb{Z}_2$-symmetric model be defined on the lattice $\Lambda = \mathbb{Z}^D$ by the partition function in eq. (1). Suppose that the supposition of lemma 4.1 is valid and there exist a positive $c > 0$ and a real $\delta$ such that for all $\Phi \in \mathbb{R}$

$$V(\Phi) \geq c\Phi^2 - \delta. \quad (45)$$

Then there exists a positive constant $\kappa_*$ such that for all $x, y \in \Lambda$ the 2-point Green function $W^{(2)}(x, y)$ is an analytic function in $K(\kappa_*):= \{ \kappa \in \mathbb{C} \mid |\kappa| < \kappa_* \}$. Furthermore, there exist a constant $\alpha > 0$ and for all $\kappa \in K(\kappa_*)$ there is $m(\kappa) > 0$ such that for all $x, y \in \Lambda$

$$|W^{(2)}(x, y)| \leq e^{\alpha q^{(2)}_V \delta_{xy}} + \alpha \exp\{-m\|x - y\|\}, \quad (46)$$

where

$$q^{(2)}_V := \frac{\int d\Phi \Phi^2 \exp\{-V(\Phi)\}}{\int d^N\Phi \exp\{-V(\Phi)\}}. \quad (47)$$

Furthermore, for $\kappa \to 0$,

$$m(\kappa) = O(\ln |\kappa|) > 0. \quad (48)$$

For $m > 0$ the series in the definition (4) of the 2-point susceptibility $\chi_2$ is convergent. Thus, using a well-known theorem for analytic functions, the above theorem implies that the 2-point susceptibility $\chi_2$ exists and is an analytic function for $\kappa$ in $K(\kappa_*):= \{ \kappa \in \mathbb{C} \mid |\kappa| < \kappa_* \}$ for some $\kappa_* > 0$.

Define a positive number

$$q_V := \int d\Phi \exp\{-V(\Phi)\} \quad (49)$$

and for all finite subsets $X$ of $\Lambda$ the partition function

$$Z(X|J, \kappa) = q_V^{-|X|} \int \prod_{x \in X} d\Phi(x) \exp\{-S(X|\Phi, \kappa) + \sum_{x \in X} J(x)\Phi(x)\}, \quad (50)$$

where the action is

$$S(X|\Phi, \kappa) = \sum_{x \in X} V(\Phi(x)) - \frac{1}{2} \sum_{x, y \in X} \Phi(x)v(x, y)\Phi(y),$$

and $|X|$ denotes the number of lattice sites of $X$. The following lemma states that the partition functions are well-defined if the local interaction $V$ obeys the stability bound (15).

**Lemma 4.2** Suppose that there exist positive $c > 0$ and real $\delta$ such that the stability bound for the local interaction eq. (45) is valid. Then we have for all $\kappa \in \mathbb{C}$ and $\epsilon > 0$ obeying $4d|\kappa| + \epsilon < c$, and for all subsets $X \subseteq \Lambda$

$$S(X|\Phi, \kappa) \geq \epsilon \sum_{x \in X} \Phi^2(x) - \delta|X|,$$

and the integral on the right hand side of Eq. (50) is convergent.
This follows from the stability bound (45) and Lemma 4.1.

In contrast to the convergence proof of [3] given for nearest-neighbour couplings we need here a more general notion of polymers. Here we call all finite nonempty subsets $X$ of $\Lambda$ polymers. Define Mayer activities for polymers $X$ by

$$M(X|J, \kappa) := -\delta_{1,|X|} + \sum_{n \geq 1} (-1)^{n-1} (n-1)! \sum_{X=\sum_{i=1}^{n} Q_i} \prod_{i=1}^{n} Z(Q_i|J, \kappa).$$

(51)

Then, the Mayer Montroll equation for the connected 2-point Green function reads (cf. [8] for a proof), for all $x, y \in \Lambda$,

$$W^{(2)}(x, y) = \sum_{P: Q \text{ polymer}} \sum_{n \geq 0} \sum_{P_1, \ldots, P_n \in K(\Lambda)} \frac{\partial^2 M(Q|J, \kappa)}{\partial J(x) \partial J(y)} |\partial J=P|_{J=0}$$

(52)

where we used the following definitions

$$K(X) := \{ P = \{ P_1, \ldots, P_n \} | P_i \subseteq X, P_a \cap P_b = \emptyset, \forall a \neq b, n \in \mathbb{N} \cup \{0\} \}$$

and $P_0 := \{ \{ x \} | x \in Q \}$ in the sum on the right hand side of eq. (52). Furthermore, for a set $P = \{ P_1, \ldots, P_n \}$ consisting of polymers $P_n$ we have defined

$$\text{Conn}(P) := \{ P' \in K(\Lambda) | \forall P' \in P' \exists a \in \{1, \ldots, n\} : P' \cap P_a \neq \emptyset \}$$

and used the abbreviation

$$(-M)^{P_1 + \cdots + P_n} := \prod_{i=1}^{n} (-M(P_i|J, \kappa)).$$

The following lemma presents the conditions under which the Mayer Montroll expansion eq. (52) is convergent and exponentially bounded (cf. [9]). For a proof see [3].

**Lemma 4.3** Suppose that there exist positive constants $\alpha, \kappa_\ast > 0$ and a function $m(\kappa) > 0$ such that

$$\sum_{P: y \in P, |P| \geq 2} |M(P|J = 0, \kappa)| \exp\{\alpha|P|\} \leq \frac{\alpha}{2}$$

(53)

and

$$\sum_{P: y \in P} \left| \frac{\partial^2 M(P|J, \kappa)}{\partial J(x) \partial J(y)} \right|_{J=0} \exp\{\alpha|P|\} \leq e^{\alpha q^2_\kappa} \delta_{xy} + \frac{\alpha}{2} \exp\{-m\|x-y\|\},$$

(54)
for all $x, y \in \Lambda$ and $|\kappa| \leq \kappa_*$, where
\[
q_V^{(2)} := \frac{\int d^N \Phi \Phi^2 \exp\{-V(\Phi)\}}{\int d^N \Phi \exp\{-V(\Phi)\}}.
\] (55)

Then the series expansion eq. (52) is convergent and
\[
|W^{(2)}(x, y)| \leq e^\alpha q_V^{(2)} \delta_{xy} + \alpha \exp\{-m \|x - y\|\}. \tag{56}
\]

As $\kappa \to 0$,
\[
m(\kappa) = O(\ln |\kappa|).
\]

The next lemma is the core of the convergence proof. It states sufficient conditions for the supposition of Lemma 4.3 to hold.

**Lemma 4.4** Suppose that the stability bound (45) and the supposition of lemma 4.1 are valid. There exist positive constants $\alpha > 0$ and $\kappa_* > 0$ and a positive function $m(\kappa) > 0$ such that inequalities (53) and (54) are valid for all $|\kappa| \leq \kappa_*$. Furthermore, for $\kappa \to 0$,
\[
m(\kappa) = O(\ln |\kappa|).
\]

**Proof:** For the proof we will use the tree graph formula (57) for Mayer activities. Let $X$ be a polymer, $|X| \geq 2$. Then
\[
M(X|J, \kappa) = \sum_{\tau: \tau \in T(X)} \int \prod_{x \in X} d\Phi(x) \exp\{\sum_{x \in X} (-V(\Phi(x)) + J(x)\Phi(x))\} \prod_{(xy) \in \tau} \left(\Phi(x)v(x, y)\Phi(y)\right) \exp\left\{\frac{1}{2} \sum_{x, y \in X, x \neq y} l_{\tau}^{\min}(x, y) \Phi(x)v(x, y)\Phi(y)\right\}. \tag{57}
\]

In the following we will explain the notations used in formula (57). $T(X)$ denotes the set of all tree graphs (graph containing no loops) with lines $(xy)$, $x \in X$ and vertex set $X$. Furthermore,
\[
l_{\tau}^{\min}(x, y) := \min \{t_l| \text{ path connecting } x \text{ and } y \text{ and containing link } l \in \tau\}.
\]

Using the stability bound (45), definition (49) of the constant $q_V$, Lemma 4.2 and the tree graph formula (57) we obtain
\[
|M(X|J = 0, \kappa)| \leq \sum_{\tau: \tau \in T(X)} \left(\prod_{(xy) \in \tau} |v(x, y)|\right) q_V^{-|X|} e^{|X|} \\
\epsilon^{-\sum_{x \in X} (d_r(x) + 1)} \prod_{x \in X} \left(d_r(x) + 1\right)! \tag{58}
\]
where

\[ d_\tau(x) := |\{ l \in \tau | \text{line } l \text{ emerges from vertex } x \}| \]

is the number of lines in the tree graph \( \tau \) which are connected to vertex \( x \). For the estimation of the fields in the term \( \prod_{x \in \tau} \Phi(x) v(x, y) \Phi(y) \) we have used the stability bound (45) and

\[
2 \int_0^\infty dx \exp\{-\epsilon x^2\} x^n = \epsilon^{-\frac{n+1}{2}} \left( \frac{n+1}{2} \right)!,
\]

and we write \( z! = \Gamma(z+1) \) for \( z \geq 0 \).

For \( X = \{ x_1, \ldots, x_n \} \) and \( d_i \in \mathbb{N}, i \in \{ 1, \ldots, n \} \) let \( T(X; d_1, \ldots, d_n) \) be the set of all tree graphs \( \tau \in T(X) \) such that \( d_\tau(x_i) = d_i \) for all \( i \in \{ 1, \ldots, n \} \). Cayley’s Theorem (for a proof cf. [11]) is

\[
|T(X; d_1, \ldots, d_n)| = \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}.
\]

In eq. (58) we replace the sum over tree graphs by

\[
\sum_{\tau: \tau \in T(X)} = \sum_{X: y \in X, |X| \geq 2} \sum_{x_{i=1}^{d_i \geq 1}, d_i = 2(n-1)} \tau \in T(X; d_1, \ldots, d_n).
\]

Using that

\[
\left( \frac{d + 1}{2} \right)! \leq (d-1)!
\]

for all \( d \geq 3 \) and that there exists at least one \( x \in X \) such that \( d_\tau(x) = 1 \), we obtain

\[
\prod_{x \in X} \left( \frac{d_\tau(x) + 1}{2} \right)! \leq \left( \frac{3}{2} \right)! |X|-1 \prod_{x \in X} (d_\tau(x) - 1)!. \quad (60)
\]

Furthermore,

\[
\sum_{X: y \in X, |X| \geq 2} \frac{1}{(n-1)!} \sum_{x_2 \ldots x_n} \left| M(\{y, x_2, \ldots, x_n\}|J = 0, \kappa) \right| \exp\{\alpha n\}. \quad (61)
\]

We have replaced here the sum over all polymers \( X \) with \( |X| \geq 2, y \in X \), by a sum over distinct labelled lattice sites \( x_2, \ldots, x_n \). We have defined \( x_1 := y \). Then a tree \( \tau \in T(\{x_1, \ldots, x_n\}) \) corresponds to a tree \( T(\{1, \ldots, n\}) \) where the vertices are labelled by \( 1, \ldots, n \). We want to estimate the sum over all lattice sites \( x_2, \ldots, x_n \) for a given tree \( \tau \). For that we have to find a bound for

\[
\sum_{\{x_2 \ldots x_n\} \in \tau} \prod_{(ab) \in \tau} |v(x_a, x_b)| \quad (62)
\]
for all tree graphs $\tau \in T(\{1, \ldots, n\})$. This can be done by estimating

$$\sum_{x':x' \in \Lambda} |v(x, x')|.$$ 

To obtain later in this proof the exponential factor $\exp\{-m \|x - y\|\}$ in the bound (54) we find an upper bound for

$$\sum_{x':x' \in \Lambda} |v(x, x')|(\frac{\kappa_*}{|\kappa|})^{\|x-x'\|}.$$ 

Using the suppositions (40) and (41) and (26), we see that for $\kappa$ with $|\kappa| \leq \kappa_* \leq \kappa$ we have

$$\sum_{x':x' \in \Lambda} \prod_{p:l_p=\|x-x'\|} 2^{l_p} 2^{-\eta \|x-x'\|} \left( \frac{\kappa_*}{|\kappa|} \right)^{l_p} \kappa \leq 2^{l_p} 2^{-\eta \|x-x'\|} \left( \frac{\kappa_*}{|\kappa|} \right) \kappa \leq 8D\kappa_*.$$ 

(63)

We have used that $l_p \geq 1$ for all signatures $p$. We sum in expression (62) successively over all $x_2, \ldots, x_n \in \Lambda$ starting with vertices having a maximal distance to the “root” vertex $y = x_1$. This gives a bound

$$\sum_{x_2,\ldots,x_n} \prod_{(ab) \in \tau} |v(x_a, x_b)| \leq (8D|\kappa|)^{n-1}. \tag{64}$$

Thus, using

$$\sum_{d_1,\ldots,d_n \geq 1} \prod_{i=1}^{n} d_i^{q_i} = 2^{(n-1)},$$

Cayley’s theorem Eq. (59), inequalities (60) and (61) and the bound (64), we obtain

$$\sum_{X,y \in X \mid |X| \geq 2} |M(X|J = 0, \kappa)| \exp\{\alpha|X|\} \leq \frac{1}{n-1} (32(\frac{3}{2})! D\kappa_*)^{n-1} q_v^{-\eta \epsilon - \frac{3n-2}{2}} \exp\{((\alpha + \delta)n\}$$

$$= \sum_{n \geq 1} \frac{1}{n} (32(\frac{3}{2})! D\kappa_* q_v^{-\eta \epsilon - \frac{3}{2} \epsilon^{\alpha+\delta}}) e^{\alpha+\delta} q_v^{-1} \epsilon^{-\frac{1}{2}} = -\ln(1 - u) e^{\alpha+\delta} q_v^{-1} \epsilon^{-\frac{1}{2}}, \tag{65}$$

where

$$u := 32(\frac{3}{2})! D\kappa_* q_v^{-\eta \epsilon - \frac{3}{2} \epsilon^{\alpha+\delta}}.$$ 

The proof of the bound of the term on the left hand side of (54) is similar to the proof of inequality (53). To obtain the exponential factor $\exp\{-m \|x - y\|\}$ we use
the fact that in the trees $\tau$ occurring in tree graph formula of $\frac{\partial^2 M(X|J,\kappa)}{\partial J(x)\partial J(y)}$ there exists a path connecting the vertices $x$ and $y$. Then we use the bound

$$\prod_{(ab)\in\tau} |v(x_a, x_b)| \leq (\frac{|\kappa|}{\kappa_\ast})^\|x-y\| \prod_{(ab)\in\tau} |v(x_a, x_b)| (\frac{\kappa_\ast}{|\kappa|})^\|x_a-x_b\|.$$ 

For the sum over all positions $x_2, \ldots, x_n$ for a given tree $\tau$ we use inequality (63). The result is

$$\sum_{X, y \in \mathcal{X}} \frac{\partial^2 M(X|J,\kappa)}{\partial J(x)\partial J(y)}|_{J=0} \exp\{\alpha|X|\}$$

$$\leq -\ln(1-u) e^{\alpha+\delta} \epsilon^\frac{3}{2} \exp\{-m\|x-y\|\},$$

where

$$m = \ln(\frac{\kappa_\ast}{|\kappa|}).$$

This proves the assertion if $\kappa_\ast$ is small enough. □

Lemmata 4.3 and 4.4 prove that the Mayer Montroll equations (52) are absolutely convergent under the assumptions of theorem 4.1. By a standard theorem for analytic functions we see that theorem 4.1 holds.

5 Summary

We have discussed the linked cluster expansion in application to lattice models with general pair interactions beyond nearest neighbour couplings. Under very general conditions, free energy, truncated correlation functions and susceptibilities are uniformly convergent in volume to analytic functions, for sufficiently small hopping parameters.

The graphical expansion is cast in terms of graph classes well known from the pure nearest neighbour interactions on hypercubic lattices. This implies that all the simplifications and computational efforts that are based on the topology of those lattices are fully exploited. The only modification occurs as a generalization of free random walks.

Acknowledgement

A.P. would like to thank the Deutsche Forschungsgemeinschaft for financial support.

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