Gevrey Asymptotic
Existence and Uniqueness Theorem

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Abstract

We prove a general Asymptotic Existence and Uniqueness Theorem for singularly perturbed nonlinear first-order complex-analytic systems of differential equations of the form \( h \partial_x f = F(x, h, f) \) in the setting of Gevrey asymptotics as \( h \to 0 \). The unique analytic solution is the Borel resummation of the corresponding formal power series solution.

Keywords: exact perturbation theory, singular perturbation theory, Borel summation, Borel-Laplace theory, asymptotic analysis, Gevrey asymptotics, resurgence, exact WKB analysis, nonlinear ODEs, existence and uniqueness

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§ 1. Introduction

Consider the following first-order singularly perturbed differential equation:

\[ \hbar \partial_x f = F(x, \hbar, f), \quad (1) \]

where \( F \) is an \( N \)-dimensional holomorphic vector function of a single complex variable \( x \), a small complex perturbation parameter \( \hbar \), and the unknown holomorphic \( N \)-dimensional vector function \( f = f(x, \hbar) \). Suppose \( F \) is a polynomial in \( \hbar \), or more generally admits an asymptotic expansion as \( \hbar \to 0 \) in some sector.

The main problem we solve here is to construct canonical exact solutions of (1); i.e., actual holomorphic solutions \( f = f(x, \hbar) \) that are uniquely specified by formal \( \hbar \)-power series solutions \( \hat{f} = \hat{f}(x, \hbar) \) such that \( \hat{f} \) is the asymptotic expansion of \( f \) as \( \hbar \to 0 \). This is a fundamental problem in what may be referred to as exact perturbation theory; i.e., singular perturbation theory supplemented with techniques from resurgent asymptotic analysis.

Thus, we prove an Asymptotic Existence and Uniqueness Theorem for singular perturbation problems of the form (1). We emphasise that our main result is not only about existence but also about uniqueness in a precise sense. In particular, this canonical exact solution \( f \) is the Borel resummation of the formal \( \hbar \)-power series solution \( \hat{f} \). This is a remarkable property that permits one to readily deduce a lot of refined information about the typically highly transcendental solution \( f \) from the explicitly defined formal solution \( \hat{f} \).

1.1. Motivation. Existence and uniqueness theory for first-order ODEs is obviously a very well developed subject which can also be analysed in the presence of a parameter like \( \hbar \) (e.g., [Was76, Theorem 24.1]). However, it gives no information about the behaviour of solutions near \( \hbar = 0 \). Attempting to solve an equation like (1) by expanding it in powers of \( \hbar \) generically leads to divergent power series solutions. This is a typical phenomenon in singular perturbation theory.

The Asymptotic Existence Theorem (e.g., [Was76, Theorem 26.1]) plays a starring role in the classical theory of analytic differential equations. It guarantees that if a differential equation like (1) has a formal \( \hbar \)-power series solution \( \hat{f} \), then under certain assumptions on \( F \) the power series \( \hat{f} \) is the asymptotic expansion as \( \hbar \to 0 \) of an actual solution \( f \). However, this celebrated theory comes with a number of significant disadvantages.

The most striking drawback is the inability to draw any uniqueness conclusion: the obtained actual solution \( f \) is unreservedly not unique. The reason is that asymptotic expansions cannot detect the so-called exponential corrections (i.e., analytic functions with zero asymptotic expansions) and the classical techniques underpinning the Asymptotic Existence Theorem are ill-adapted to the recovery of such exponential corrections. This situation is only made worse by the fact that the classical techniques provide little control on the size of the sectorial domain of definition of the obtained actual solution \( f \) (e.g., see the remark in [Was76, p.144], immediately following Theorem 26.1), rendering it virtually impossible to describe the set of all possible such actual solutions in any reasonable manner.
We reverberate the opinion of Ramis and Sibuya [RS89] that in the theory of complex-analytic differential equations (with or without a complex perturbation parameter), the more appropriate notion of asymptotic expansions is Gevrey asymptotics rather than the more classical theory in the sense of Poincaré. The aforementioned work of Ramis and Sibuya is in connection with solutions of complex-analytic differential equations near an irregular singularity, but the same point of view is apparent in other related subjects including (to only cite a few) the works of Écalle on resurgent functions [Éca84, Éca85] and of Malgrange, Martinet, and Ramis on analytic diffeomorphisms [Mal82, MR83]. In the present context of singular perturbation theory, this point of view is especially reinforced by the abundant success of the exact WKB method [Vor83, Sil85, Vor04, KT05, GMN13, IN14, Nik21a].

1.2. Overview of the Main Result. Let us first give a brief account of what is achieved in this paper without delving into too much detail or generality. Following this discussion, we will state our main result (Theorem 1.4) in full.

For the purposes of this discussion, suppose \( F = F(x, h, y) \) is a polynomial in \( h \) whose coefficients \( F_k = F_k(x, y) \) are holomorphic maps \( \mathbb{C}_x \times \mathbb{C}_y \to \mathbb{C}^N \). The leading-order part in \( h \) of the differential equation (1) is the functional equation \( F_0(x, y) = 0 \). Suppose \( (x_0, y_0) \) is a point such that \( F_0(x_0, y_0) = 0 \) and the Jacobian \( \partial F_0 / \partial y \) is invertible at \( (x_0, y_0) \). Then it is a classical fact (the Formal Existence and Uniqueness Theorem, see Proposition 2.1) that near \( x_0 \) equation (1) has a unique formal \( h \)-power series solution \( \hat{f} = \hat{f}(x, h) \) such that \( f_0(x_0) = y_0 \).

Generically, \( \hat{f} \) is a divergent power series and therefore has no analytic meaning in a direct sense. The main goal of this paper is to promote – in a canonical way – the formal solution \( \hat{f} \) to an exact solution \( f \); i.e., an actual holomorphic solution such that \( \hat{f} \) is the asymptotic expansion of \( f \) as \( h \to 0 \) in some sector \( S \subset \mathbb{C}_h \). In order to achieve this, we require the opening angle of the sector \( S \) is at least \( \pi \); let us say that \( S \) is the right halfplane \( \{ \text{Re}(h) > 0 \} \).

Define a holomorphic invertible \( N \times N \)-matrix near the point \( x_0 \) by

\[
J_0(x) := \frac{\partial F_0}{\partial z} \bigg|_{(x, f_0(x))}.
\]

Let \( \varphi_1, \ldots, \varphi_N \) be the eigenvalues of \( J_0 \) and define the following possibly multivalued locally conformal transformations:

\[
z = \Phi_i(x) := \int_{x_0}^x \varphi_i(t) \, dt.
\]

They are closely related to the Liouville transformation encountered in the WKB analysis of the Schrödinger equation; see, e.g., [Nik21a, §4.1] or [Olv97, §6.1].

Suppose that for each \( i = 1, \ldots, N \), the point \( x_0 \) has a neighbourhood \( W_i \subset \mathbb{C}_x \) which is mapped by \( \Phi_i \) to a horizontal halfstrip \( H = \{ z \mid \text{dist}(z, \mathbb{R}_+) < r \} \) of some thickness \( r > 0 \), where \( \mathbb{R}_+ \subset \mathbb{C}_x \) is the nonnegative real ray. Suppose furthermore that each polynomial coefficient \( F_k(x, y) \) is bounded on each domain \( W_i \) by the eigenvalue \( \varphi_i \). Then, under these simplifying assumptions, the main result of this paper (Theorem 1.4) can be stated as follows.
1.3. Theorem. The differential equation (1) has a canonical exact solution \( f \) near \( x_0 \) which is asymptotic to \( \hat{f} \) as \( h \to 0 \) in the right halfplane. Namely, there is a neighbourhood \( X_0 \subset \mathbb{C}_x \) of \( x_0 \) and a sectorial subdomain \( S_0 \subset S \), also with opening angle \( \pi \), such that (1) has a unique holomorphic solution \( f \) on \( X_0 \times S_0 \) which is uniformly Gevrey asymptotic to \( \hat{f} \) as \( h \to 0 \) in the closed right halfplane:

\[
\hat{f} \simeq f \quad \text{as } h \to 0 \text{ in the closed right halfplane, unif. } \forall x \in X_0 . \tag{4}
\]

Moreover, \( f \) is the uniform Borel resummation of the formal solution \( \hat{f} \); i.e., \( f = \mathcal{S}[\hat{f}] \).

A brief summary of our notations and definitions can be found in Appendix A. More generally, \( F \) need not be a polynomial in \( h \) but rather we assume it admits a Gevrey asymptotic expansion in some section with opening angle \( \pi \). Namely, the main result of this paper in full generality is the following theorem.

1.4. Theorem (Gevrey Asymptotic Existence and Uniqueness Theorem).

Let \( X \subset \mathbb{C}_x \) be a domain and fix a point \( (x_0, y_0) \in X \times \mathbb{C}^N_y \). Let \( S \subset \mathbb{C}_h \) be a sectorial domain with opening \( A = \left( \theta - \frac{\pi}{2}, \theta + \frac{\pi}{2} \right) \) for some \( \theta \). Consider the differential equation (1) where \( F = F(x, h, y) \) is a holomorphic map \( X \times S \times \mathbb{C}^N_y \to \mathbb{C}^N \) which admits a locally uniform asymptotic expansion \( \hat{F}(x, h, y) \) as \( h \to 0 \) along \( A \). Suppose that

(1) \( F_0(x_0, y_0) = 0 \) and the Jacobian \( \partial F_0 / \partial y \) is invertible at \( (x_0, y_0) \).

Let \( \hat{f} \) be the unique formal power series solution with \( f_0(x_0) = y_0 \). Let \( J_0 \) be defined by (2) near \( x_0 \), and let \( \varphi_1, \ldots, \varphi_N \) be its eigenvalues of \( J_0 \). Consider the possibly multivalued locally conformal transformations \( \Phi_i \) defined by (3). In addition, assume the following hypotheses for each \( i = 1, \ldots, N \).

(2) The point \( x_0 \) has a neighbourhood \( \mathcal{W}_i \subset X \) which is locally conformally mapped by \( x \mapsto z = \Phi_i(x) \) onto a horizontal halfstrip \( H := \{ z \mid \dist(z, \mathbb{R}^d_z) < r \} \) of some thickness \( r > 0 \), where \( \mathbb{R}^d_z \subset \mathbb{C}_y \) is the real ray in the direction \( \theta \).

(3) The asymptotic expansion \( \hat{F} \) of \( F \) is valid on \( \mathcal{W}_i \) with Gevrey bounds along the closed arc \( \overline{A} \) with respect to the asymptotic scale \( \varphi_i(x) \), uniformly for all \( x \in \mathcal{W}_i \) and locally uniformly for all \( y \in \mathbb{C}^N_y \):

\[
F(x, h, y) \simeq \hat{F}(x, h, y) \quad \text{as } h \to 0 \text{ along } \overline{A}, \text{ unif. } \forall x \in \mathcal{W}_i . \tag{5}
\]

Then the differential equation (1) has a canonical exact solution \( f \) near \( x_0 \) which is asymptotic to the formal solution \( \hat{f} \) as \( h \to 0 \) in the direction \( \theta \). Namely, there is a domain neighbourhood \( X_0 \subset X \) of \( x_0 \) and a sectorial subdomain \( S_0 \subset S \) with the same opening \( A \) such that (1) has a unique holomorphic solution \( f \) on \( X_0 \times S_0 \) which is uniformly Gevrey asymptotic to \( \hat{f} \) as \( h \to 0 \) along the closed arc \( \overline{A} \):

\[
f(x, h) \simeq \hat{f}(x, h) \quad \text{as } h \to 0 \text{ along } \overline{A}, \text{ unif. } \forall x \in X_0 . \tag{6}
\]

Moreover, \( f \) is the uniform Borel \( \theta \)-resummation of \( \hat{f} \); for all \( (x, h) \in X_0 \times S_0 \),

\[
f(x, h) = \mathcal{S}_0[\hat{f}](x, h) . \tag{7}
\]
1.5. Brief outline of the proof. This paper is devoted entirely to the proof of this theorem. We use the Borel-Laplace method which is briefly reviewed in §A.2. Our proof represents a combination of techniques developed in [Nik20] and [Nik21b] but many of the ideas underpinning all these works originated in [Nik19]. The overall strategy of the proof is as follows.

The first step is to construct a formal $\hbar$-power series solution $\hat{f}$ of the formal differential equation $\hbar \partial_x \hat{f} = \hat{F}(x, \hbar, \hat{f})$. This is a classical result which we revisit in §2; namely, Proposition 2.1. The formal solution $\hat{f}$ is constructed using the ordinary Holomorphic Implicit Function Theorem at the leading-order in $\hbar$ and then all the higher-order corrections are uniquely determined by an explicit recursion.

Then we want to apply the Borel resummation to $\hat{f}$ to yield $f$. In order to proceed with this programme, we first need to check that the formal Borel transform $\hat{\varphi}$ of $\hat{f}$ is a convergent power series at the origin in the Borel $\xi$-plane. This is done in §3; namely, Proposition 3.1. The bulk of the proof, presented in §4, is to show that $\hat{\varphi}$ admits an analytic continuation $\varphi$ along a ray in the $\xi$-plane which is Laplace-transformable.

To construct $\varphi$, we use the transformations $\Phi_i$ to make a convenient change of variables in order to put the differential equation (1) into a certain standard form which is more amenable to the Borel transform. Applying the Borel transform, we obtain a first-order partial differential equation, albeit nonlinear and with convolution. Nevertheless, this PDE is easy to convert into an integral equation, which we then proceed to solve using the method of successive approximations. To show that this sequence of approximations converges to an actual solution, we give an estimate on the terms of this sequence by employing in an interesting way the ordinary Holomorphic Implicit Function Theorem. This estimate also allows us to conclude that the solution of this PDE has a well-defined Laplace transform and therefore defines a holomorphic solution of our equation in standard form. Undoing all the changes of variables yields the desired exact solution $f$.

1.6. Remarks and discussion. Our constructions employ relatively basic and classical techniques from complex analysis which form the basis for the more modern and sophisticated theory of resurgent asymptotic analysis à la Écalle [Éca85]; see also for instance [Cos09, Sau14, LR16]. We stress that the Borel-Laplace method “is nothing other than the theory of Laplace transforms, written in slightly different variables”, echoing the words of Alan Sokal [Sok80]. As such, we have tried to keep our presentation very hands-on and self-contained, so the knowledge of basic complex analysis should be sufficient to follow.

What we call Gevrey asymptotics is often called 1-Gevrey asymptotics. It is part of an entire hierarchy of asymptotic regularity classes [Ram78, Ram80]; see also [LR16, §1.2]. However, arguments about other Gevrey classes can usually be reduced to arguments about 1-Gevrey asymptotics via a simple fractional transformation in the $\hbar$-space. Therefore, we believe it is not difficult to extend our results to all other Gevrey asymptotic classes. We leave this as a natural open problem.

We emphasise that the asymptotic condition (5) on the holomorphic map $F$ is required to hold over the closed arc $\overline{\mathcal{A}} = [\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$, which is stronger than ordinary
Gevrey asymptotics along an open arc $A$ (see §A.1 or [Nik20, §A.5 and §A.16]). This type of condition is specifically adapted to the Borel-Laplace method, in particular through the application of Nevanlinna’s theorem, see §A.2.

1.7. Notation and conventions. Our notation, conventions, and definitions from Gevrey asymptotics and Borel-Laplace theory are consistent with those given in Appendices A and B in [Nik20]. A brief summary can be found in Appendix A.

Throughout the paper, we fix a complex plane $\mathbb{C}_x$ with coordinate $x$ and another complex plane $\mathbb{C}_\hbar$ with coordinate $\hbar$. We also fix a complex vector space $\mathbb{C}_y^N$ with coordinate $y = (y_1, \ldots, y_N)$ for $N \geq 1$. We write vector components of holomorphic maps as $F = (F^1, \ldots, F^N)$, etc. The symbol $\mathbb{N}$ stands for nonnegative integers $0, 1, 2, \ldots$. We will use boldface letters to denote nonnegative integer index vectors; i.e., $m := (m_1, \ldots, m_N) \in \mathbb{N}^N$, etc., and we put $|m| := m_1 + \cdots + m_N$. Unless otherwise indicated, all sums over unbolded indices $n, m, \ldots$ are taken to run over $\mathbb{N}$, and all sums over boldface letters $n, m, \ldots$ are taken to run over $\mathbb{N}^N$.

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§ 2. Formal Perturbation Theory

The starting point is the following classical result about formal solutions.

2.1. Proposition (Formal Existence and Uniqueness Theorem).
Let $X \subset \mathbb{C}_x$ be a domain and fix a point $(x_0, y_0) \in X \times \mathbb{C}_y^N$. Consider the following formal differential equation in $\hat{f}$:

$$h \partial_x \hat{f} = \hat{F}(x, h, \hat{f}) \quad (8)$$

where $\hat{F}$ is a formal power series in $h$,

$$\hat{F}(x, h, y) = \sum_{k=0}^{\infty} F_k(x, y) h^k \quad (9)$$

whose coefficients $F_k = F_k(x, y)$ are holomorphic maps $X \times \mathbb{C}_y^N \to \mathbb{C}^N$ such that $F_0(x_0, y_0) = 0$ and the Jacobian $\partial F_0 / \partial z$ is invertible at $(x_0, y_0)$.

Then there is a subdomain $X_0 \subset X$ containing $x_0$ such that the differential equation (8) has a unique formal power series solution

$$\hat{f} = \hat{f}(x, h) = \sum_{n=0}^{\infty} f_n(x) h^n \quad (10)$$

with holomorphic coefficients $f_n : X_0 \to \mathbb{C}^N$, which satisfies $f_0(x_0) = y_0$. In fact, all higher-order coefficients $f_k$ are uniquely determined by $f_0$.

In particular, if $S \subset \mathbb{C}_h$ is a sectorial domain at the origin, and $F$ is a holomorphic map $X \times S \times \mathbb{C}_y^N \to \mathbb{C}^N$ which admits $\hat{F}$ as its asymptotic expansion as $h \to 0$ in $S$, uniformly in $x$ and locally uniformly in $y$, then the differential equation $h \partial_x f = F(x, h, f)$ has a unique formal power series solution $y = \hat{f}$ near $x_0$ such that $f_0(x_0) = y_0$. 6
Although this theorem is well-known, we supply its proof below for completeness. The proof amounts to plugging the solution ansatz (10) into the differential equation \( \hbar \partial_x \hat{f} = \hat{F}(x, \hbar, \hat{f}) \) and solving it order-by-order in \( \hbar \). The ‘miracle’ that makes this possible is that at each order in \( \hbar \) this equation is no longer a differential equation because the factor \( \hbar \) in front of \( \partial_x \) eliminates any unknown derivative information.

**Proof.** First, let us note down a few formulas in order to proceed with the calculation. See Part 1.7 for our notational conventions.

**Step 0: Collect some formulas.** Write the double power series expansion of each component \( \hat{F}^i \) as

\[
\hat{F}^i(x, \hbar, y) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{k,m}(x) \hbar^k y^m, \tag{11}
\]

where \( F^i_{k,m} y^m := F^i_{k,m_1, \ldots, m_N} y_{m_1}^{m_1} \cdots y_{m_N}^{m_N} \). In particular, the expansion of the leading-order part \( F_0 \) is

\[
F_0^i(x, y) = \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{0m}(x) y^m. \tag{12}
\]

For every \( m \in \mathbb{N}^N \), we have \( \frac{\partial}{\partial y^m} \hat{f} = \frac{m!}{y^m} \hat{f} \), so the \((i,j)\)-component of the Jacobian matrix \( \frac{\partial \hat{F}_0}{\partial y} \) can be written as

\[
\left[ \frac{\partial F_0^i}{\partial y} \right]_{ij} = \frac{\partial F_0^i}{\partial y_j} = \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{0m}(x) \frac{\partial y^m}{\partial y_j} = \sum_{m=0}^{\infty} \sum_{|m|=m} \frac{m_j!}{y_j^m} F^i_{0m}(x) y^m. \tag{13}
\]

Next, the \( m \)-th power \( \hat{f}^m \) of the power series ansatz (10) expands as follows:

\[
\left( \sum_{n=0}^{\infty} f_n h^n \right)^m \equiv \left( \sum_{n_1=0}^{\infty} f_{n_1} h^{n_1} \right) \cdots \left( \sum_{n_N=0}^{\infty} f_{n_N} h^{n_N} \right)^m \equiv \left( \sum_{j_1 \in \mathbb{N}^m} f_{j_1}^{1j_1} \cdots f_{j_1}^{1j_{n_1}} h^{n_1} \right) \cdots \left( \sum_{j_N \in \mathbb{N}^N} f_{j_N}^{Nj_N} \cdots f_{j_N}^{Nj_{n_N}} h^{n_N} \right) 
= \sum_{n=0}^{\infty} \sum_{|n|=n} \left( \sum_{j_1 \in \mathbb{N}^m} \sum_{j_1 \in j_1 \cdots j_1} f_{j_1}^{1j_1} \cdots f_{j_1}^{1j_{n_1}} \right) \cdots \left( \sum_{j_N \in \mathbb{N}^N} \sum_{j_N \in j_N \cdots j_N} f_{j_N}^{Nj_N} \cdots f_{j_N}^{Nj_{n_N}} \right) h^n.
\]

In these formulas, we have denoted the components of each vector \( j_i \in \mathbb{N}^{m_i} \) by \((j_{i,1}, \ldots, j_{i,m_i})\). Let us introduce the following shorthand notation:

\[
f^m_n := \left( \sum_{j_{i,1} \in \mathbb{N}^{m_i}} f_{j_{i,1}}^{1j_{i,1}} \cdots f_{j_{i,1}}^{1j_{n_1}} \right) \cdots \left( \sum_{j_{N,1} \in \mathbb{N}^{m_N}} f_{j_{N,1}}^{Nj_{N,1}} \cdots f_{j_{N,1}}^{Nj_{n_N}} \right) \tag{14}
\]

We note the following simple but useful identities:

\[
\begin{align*}
\hat{f}_0^0 &= 1; & \hat{f}_0^m &= \hat{f}_0^1 \hat{f}_0^1 \cdots \hat{f}_0^1 \hat{f}_0^m = (\hat{f}_0^1)_{m_1} \cdots (\hat{f}_0^1)_{m_N} \hat{f}_0^m = \hat{f}_0^m \text{ whenever } |n| > 0. \tag{15}
\end{align*}
\]

Using this notation, the formula for \( \hat{f}^m \) can be written much more compactly:

\[
\hat{f}^m = \left( \sum_{n=0}^{\infty} f_n h^n \right)^m = \sum_{n=0}^{\infty} \sum_{|n|=n} f^m_n h^n. \tag{16}
\]
Step 1: Expand order-by-order. Now, we plug the solution ansatz (10) into the differential equation $\hbar \partial_x \hat{f} = \hat{F}(x, \hbar, \hat{f})$. Using (11) and (16), we find:

$$
\sum_{n=0}^{\infty} \partial_x f_n^{i} h^{n+1} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_{km}^{i} h^{k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{m}^{n} h^{n}
$$

$$
\sum_{n=1}^{\infty} \partial_x f_{n-1}^{i} h^{n} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{km}^{i} f_{m}^{n} h^{n} \quad (i = 1, \ldots, N) \quad (17)
$$

We solve this system of equations for $f_n$ order-by-order in $\hbar$.

Step 2: Leading-order part. First, at order $n = 0$, equation (17) yields:

$$
0 = \sum_{m=0}^{\infty} \sum_{|m|=m} F_{0m}^{i}(x) f_{0}^{m} \quad (i = 1, \ldots, N).
$$

(18)

Comparing with (12), these equations are simply the components of the equation $F_0(x, f_0) = 0$. By the Holomorphic Implicit Function Theorem, there is a domain $X_0 \subset X$ containing $x_0$ such that there is a unique holomorphic map $f_0 : X_0 \to \mathbb{C}^N$ that satisfies $F_0(x, f_0(x)) = 0$ and $f_0(x_0) = y_0$. In fact, the domain $X_0$ can be chosen so small that the Jacobian $\partial F_0 / \partial y$ remains invertible at $(x, y) = (x_0, f_0(x_0))$ for all $x \in X_0$. Thus, we can define a holomorphic invertible $N \times N$-matrix $J_0$ on $X_0$ by

$$
J_0(x) := \left. \frac{\partial F_0}{\partial y} \right|_{(x, f_0(x))}
$$

(19)

The $(i, j)$-component of $J_0$ is:

$$
[J_0]_{ij} = \left. \frac{\partial F_0^i}{\partial y_j} \right|_{(x, f_0(x))} = \sum_{m=0}^{\infty} \sum_{|m|=m} \frac{m_i}{f_0^j} F_{0m}^{i} f_{0}^{m}.
$$

(20)

Step 3: Next-to-leading-order part. For clarity, let us also examine equation (17) at order $n = 1$. First, let us note that if $|n| = 1$, then $n = (0, \ldots, 1, \ldots, 0)$ with the only 1 in some position $j$, in which case notation (14) reduces to:

$$
f_{n}^{m} = (f_{0}^{j})^{m_1} \cdots \left( m_j f_{1}^{j} \right) (f_{0}^{j})^{m_j} \cdots (f_{N}^{j})^{m_N} = \frac{m_i}{f_0^j} f_{0}^{m} f_{1}^{j}.
$$

(21)

Then at order $n = 1$, equation (17) comprises two main summands corresponding to $k = 0$ and $k = 1$, which simplify using identities (20) and (21):

$$
\partial_x f_0^i = \sum_{m=0}^{\infty} \sum_{|m|=m} \sum_{|n|=1} F_{0m}^{i} f_{m}^{n} + \sum_{m=0}^{\infty} \sum_{|m|=m} F_{1m}^{i} f_{0}^{m} f_{1}^{j}.
$$

$$
\partial_x f_0^i = \sum_{j=1}^{N} \sum_{m=0}^{\infty} \frac{m_i}{f_0^j} F_{0m}^{i} f_{0}^{m} f_{1}^{j} + \sum_{m=0}^{\infty} \sum_{|m|=m} F_{1m}^{i} f_{0}^{m}.
$$

$$
\partial_x f_0^i = \sum_{j=1}^{N} [J_0]_{ij} f_{1}^{j} + \sum_{m=0}^{\infty} \sum_{|m|=m} F_{1m}^{i} f_{0}^{m}.
$$

(22)
Observe that the blue term is nothing but the $i$-th component of the vector $J_0 f_1$. Since $J_0$ is an invertible matrix, multiplying the system of $N$ equations (22) on the left by $J_0^{-1}$, we solve uniquely for a holomorphic vector $f_1$ on $X_0$.

**Step 4:** **Inductive step.** Suppose now that $n \geq 1$ and we have already solved equation (17) for holomorphic vectors $f_0, f_1, \ldots, f_{n-1}$ on $X_0$. Similar to (21), we have that if $n = (0, \ldots, n, \ldots, 0)$ with the only nonzero entry in some position $j$, then

$$f_i^n = (f_0^i)^{m_1} \cdots (m_j f_0^i)(f_0^i)^{m_j-1} \cdots (f_0^N)^{m_N} = \frac{m_j}{f_0^i} f_i^0 f_0^i f_0^i \cdots f_0^i f_0^i f_0^i. \quad (23)$$

Then at order $n$ in $\hbar$, we first separate out the $k = 0$ summand from which we then take out all the terms with $n = (0, \ldots, n, \ldots, 0)$, and simplify using (20) and (23):

$$\partial_x f_{n-1}^i = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{|n|=n-k} \sum_{|m|=m} F_{km} f_n^m, \quad \sum_{m=0}^{\infty} \left( \sum_{|n|=n} \sum_{|m|=m} F_{0m} f_n^m + \sum_{k=1}^{n} \sum_{|n|=n-k} \sum_{|m|=m} F_{km} f_n^m \right), \quad \sum_{j=1}^{N} \sum_{m=0}^{\infty} \frac{m_j}{f_0^i} f_0^m f_0^m f_0^m f_0^m f_0^m + \sum_{n=0}^{\infty} \sum_{|n|=n} \sum_{|m|=m} F_{0m} f_n^m + \sum_{k=1}^{n} \sum_{|n|=n-k} \sum_{|m|=m} F_{km} f_n^m \right), \quad \sum_{j=1}^{N} [J_0]_{ij} f_n^i + \sum_{m=0}^{\infty} \left( \sum_{|n|=n} \sum_{|m|=m} F_{0m} f_n^m + \sum_{k=1}^{n} \sum_{|n|=n-k} \sum_{|m|=m} F_{km} f_n^m \right).$$

The term in blue is nothing but the $i$-th component of the vector $J_0 f_n$. Observe that the remaining part of this expression involves only the already-known components of the lower-order vectors $f_0, \ldots, f_{n-1}$. Therefore, since $J_0$ is invertible, multiplying this system of $N$ equations on the left by $J_0^{-1}$, we can solve uniquely for the holomorphic vector $f_n$ on $X_0$.

**§ 3. Gevrey Regularity of the Formal Solution**

Now we show that the formal Borel transform of the formal solution $\hat{f}$ is a convergent power series in the Borel variable $\xi$; that is, the coefficients $f_n$ essentially grow at most like $n!$.

**3.1. Proposition (Gevrey Formal Existence and Uniqueness Theorem).**

Assume all the hypotheses of Proposition 2.1 and suppose in addition that the power series $\hat{F}$ is locally uniformly Gevrey on $X \times \mathbb{C}_N^N$. Then the formal power series solution $\hat{f}$ is locally uniformly Gevrey on $X_0$. In particular, the formal Borel transform

$$\hat{\varphi}(x, \xi) = \hat{B}(\hat{f})(x, \xi) := \sum_{n=0}^{\infty} \frac{1}{n!} f_{n+1}(x) \xi^n \quad (24)$$

is a convergent power series in $\xi$, locally uniformly for all $x \in X_0$. Concretely, if $X_0 \subset X$ is any subset where all eigenvalues of $J_0$ are bounded from below and such that there
are real constants $A, B > 0$ satisfying $|F_k(x, y)| \leq AB^k k!$ for all $k \geq 0$, uniformly for all $x \in X_0$ and all $y$ in some finite-radius ball in $\mathbb{C}_y^N$, then there are constants $C, M > 0$ such that

$$|f_k(x)| \leq C M^k k! \quad \forall x \in X_0, \forall k.$$  \hspace{1cm} (25)

**Proof.** Let $X_0 \subset X$ be such that all the eigenvalues of the invertible holomorphic matrix $J_0$ from (19) are bounded from below.

**Step 1: Preliminary transformation.** Let $K_0 := \text{diag}(\varphi_1, \ldots, \varphi_N)$ be the diagonal matrix of eigenvalues of $J_0$ and let $P_0 = P_0(x)$ be a holomorphic invertible matrix that diagonalises $J_0$; i.e.,

$$P_0 J_0 P_0^{-1} = K_0.$$  \hspace{1cm} (26)

Consider the change of the unknown variable $\hat{f} \rightarrow \hat{g}$ given by the formula

$$\hat{f} = f_0 + h f_1 + h P_0^{-1} \hat{g}.$$  \hspace{1cm} (27)

We argue that it transforms the formal differential equation (8) into one of the form

$$h K_0^{-1} \partial_x \hat{g} - \hat{g} = h \hat{G}(x, h, \hat{g})$$  \hspace{1cm} (28)

or, written in components for $i = 1, \ldots, N$,

$$h \varphi_i^{-1} \partial_x \hat{g}^i - \hat{g}^i = h \hat{G}^i(x, h, \hat{g}),$$  \hspace{1cm} (29)

where

$$\hat{G} = \hat{G}(x, h, w) := \sum_{k=0}^{\infty} G_k(x, w) h^k$$ \hspace{1cm} (30)

is a formal power series in $h$ with holomorphic coefficients $G_k : X_0 \times \mathbb{C}_w^N \rightarrow \mathbb{C}_w^N$, which is Gevrey uniformly for all $x \in X_0$ and locally uniformly for all $w \in \mathbb{C}_w^N$.

Indeed, substituting (27) into the formal equation $h \partial_x \hat{f} = \hat{F}(x, h, \hat{f})$, we find:

$$h \partial_x f_0 + h^2 \partial_x f_1 + h^2 (\partial_x P_0^{-1}) \hat{g} + h^2 P_0^{-1} \partial_x \hat{g} = \hat{F}(x, h, f_0 + h f_1 + h P_0^{-1} \hat{g})$$ \hspace{1cm} (31)

At the leading-order in $h$, the righthand side is simply $F_0(x, f_0)$ which is zero since $f_0$ is a leading-order solution. Next, we argue that the next-to-leading-order part of $\hat{F}(x, h, f_0 + h f_1 + h w)$ is $\partial_x f_0 + J_0 w = \partial_x f_0 + P_0^{-1} K_0 P_0 w$, so that the expression

$$\hat{F}(x, h, f_0 + h f_1 + h P_0^{-1} \hat{g}) - h \partial_x f_0 - h P_0^{-1} K_0 \hat{g}$$

is of order at least $2$ in $h$. In other words, subtracting the term $P_0^{-1} K_0 \hat{g}$ from both sides of (31) and rearranging leads to the equation

$$h^2 P_0^{-1} \partial_x \hat{g} - h P_0^{-1} K_0 \hat{g} = \hat{F}(x, h, f_0 + h f_1 + h P_0^{-1} \hat{g}) - h \partial_x f_0 - h P_0^{-1} K_0 \hat{g} - h^2 \partial_x f_1 - h^2 (\partial_x P_0^{-1}) \hat{g} = h^2 K_0^{-1} P_0 \hat{G}(x, h, \hat{g}),$$

where $\hat{G}$ is defined by this equality. Evidently,

$$\left[ F(h, f_0 + h f_1 + h w) \right]^{O(h)} = F_1(f_0) + \left[ F_0(f_0 + h f_1 + h w) \right]^{O(h)}.$$  \hspace{1cm} (32)
The $i$-th component of $F_1(f_0)$ is easy to write down:

$$F_1^i(f_0) = \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{1m} f_0^m .$$  \hfill (33)

To expand the term $[F_0(f_0 + hf_1 + hw)]^{O(h)}$, consider first the following calculation:

$$(f_0 + h(f_1 + w))^m = \left(f_1^i + w_i\right)^m (f_1^j + w_j)^m \cdots (f_1^n + w_n)^m$$

$$= \left( \sum_{i, j, \ldots} \left( \begin{array}{c} m_1 \\ i_1, j_1 \\ \ldots \\ i_N, j_N \end{array} \right) f_0^{i_1} f_1^{j_1} \cdots f_0^{i_N} f_1^{j_N} \right)^m$$

$$= \sum_{j \in \mathbb{N}} \left( \begin{array}{c} m_1 \\ i_1, j_1 \\ \ldots \\ i_N, j_N \end{array} \right) f_0^j f_1^j = (f_1 + w)^j h^j \cdot$$

We are only interested in the $|j| = 1$ part of this sum. This means $j = (0, \ldots, 1, \ldots, 0)$; i.e., for each $k = 1, \ldots, N$, we have $j_k = 1$, $i_k = m_k - 1$, and $j_{k'} = 0$, $i_{k'} = m_1$ for all $k' \neq k$. Since $(m_{k-1}, 1) = m_k$ and $(m_{k'}, 0) = 1$, the coefficient of $h$ in the above expression simplifies as follows:

$$\sum_{k=1}^{N} \frac{m_k}{f_0^k} f_0^m (f_1 + w_k) .$$

Therefore, continuing (32) and using the above calculation together with (20) and (33), we find for every $i = 1, \ldots, N$:

$$\left[ F_1^i(h, f_0 + hf_1 + hw) \right]^{O(h)} = \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{1m} f_0^m + \sum_{k=1}^{N} \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{0m} m_k f_0^m (f_1 + w_k)$$

$$= \sum_{m=0}^{\infty} \sum_{|m|=m} F^i_{1m} f_0^m + \sum_{k=1}^{N} [J_0] f_1^k \sum_{k=1}^{N} [J_0] w_k .$$

Using (22), it is now clear this expression is the $i$-th component of $\partial_x f_0 + J_0 w$.

**Step 2: Solve the transformed equation.** Equation (28) has a unique formal power series solution

$$\hat{g} = \hat{g}(x, h) = \sum_{n=0}^{\infty} g_n(x) h^n .$$  \hfill (34)

Moreover, $g_0 \equiv 0$ and all the higher-order coefficients $g_n$ are given by the following recursive formula: for every $i = 1, \ldots, N$,

$$g_{n+1}^i = \varphi_i^{-1} \partial_x g_n^i - \sum_{k=0}^{n} \sum_{m_1=0}^{n-k} \sum_{m_n=-n-k}^{n} C_{km}^i g_m^m ,$$

where

$$g_m^m := \left( \sum_{j_1=0}^{n_1} \frac{g_{j_1}^1}{1 \cdots g_{j_1}^1} \right) \cdots \left( \sum_{j_N=0}^{n_N} \frac{g_{j_N}^N}{1 \cdots g_{j_N}^N} \right) ,$$  \hfill (36)

and $C_{km}^i$ are the coefficients of the formal series $\varphi_i^{-1}$.
and where \( G_{km}^i = G_{km}^i(x) \) are the coefficients of the double power series expansion

\[
\tilde{G}^i(x, h, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|m|=m} G_{km}^i(x) h^k w^m.
\]  

(37)

Indeed, similar to the computation in the proof of Proposition 2.1, we plug the solution ansatz (34) into the double power series expansion (37) of \( \tilde{G}^i \). The fact that \( g_0 \equiv 0 \) is obvious, and the righthand side of equation (28) expands as follows:

\[
\tilde{G}^i(x, h, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|m|=m} G_{km}^i(x) h^k w^m.
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|m|=m} G_{km}^i(x) h^k w^m.
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|m|=m} G_{km}^i(x) h^k w^m.
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|m|=m} G_{km}^i(x) h^k w^m.
\]

where in the last step we noticed that all terms with \( m > |n| = n - k \) are zero because \( g_0 \equiv 0 \); cf. (36). So we obtain (35).

**Step 3: Reduce problem to the transformed equation.** Since \( f_0, f_1 \) and \( P_0 \) are necessarily bounded on any compactly contained subset of \( \mathbb{X}_0 \), it is sufficient to show that the formal solution \( \tilde{g} \) of (28) is locally uniformly Gevrey on \( \mathbb{X}_0 \). Let \( \mathbb{D}_R \subset \mathbb{X}_0 \) be any sufficiently small disc of some radius \( R > 0 \), such that there are constants \( A, B > 0 \) that give the following bounds: for all \( i = 1, \ldots, N \), all \( k, m \in \mathbb{N} \), all \( m \in \mathbb{N}^N \) with \( |m| = m \), and all \( x \in \mathbb{D}_R \),

\[
|G_{km}^i(x)| \leq \rho_m AB^{k+m} k! \quad \text{and} \quad |\varphi_i^{-1}(x)| \leq A.
\]  

(38)

where \( \rho_m \) is a normalisation constant defined by

\[
\frac{1}{\rho_m} := \sum_{|m|=m} 1 = (m+N-1)_{N-1}.
\]  

(39)

It will be convenient for us to assume without loss of generality that \( A \geq 3 \) and \( R < 1 \). We shall prove that the solution \( \tilde{g} \) is a uniformly Gevrey power series on any compactly contained subset of \( \mathbb{D}_R \). In fact, we will prove something a little bit stronger as follows. For any \( r \in (0, R) \), denote by \( \mathbb{D}_r \subset \mathbb{D}_R \) the concentric subdisc of radius \( r \). Then our assertions follow from the following claim.

**Main Claim.** There is a real constant \( M > 0 \) such that, for all \( r \in (0, R) \),

\[
|g_{n+1}^i(x)| \leq M^{n+1} \delta^{-n} n!.
\]  

(40)

for all \( n \in \mathbb{N} \), all \( i = 1, \ldots, N \), and uniformly for all \( x \in \mathbb{D}_r \), where \( \delta := R - r \). (The constant \( M \) is independent of \( r, x, n \), but may depend on \( R, A, B \).) In particular, for any \( r \in (0, R) \), the power series \( \tilde{g} \) is uniformly Gevrey on \( \mathbb{D}_r \).

It remains to prove this claim. The bound (40) will be demonstrated in two main steps. First, we will recursively construct a sequence \( \{M_n\}_{n=0}^{\infty} \) of nonnegative real
numbers such that for all \( n \in \mathbb{N} \), all \( i = 1, \ldots, N \), all \( r \in (0, R) \), and all \( x \in \mathbb{D}_r \), we have the bound
\[
\left| g_{n+1}^i(x) \right| \leq M_{n+1} \delta^{-n} n! .
\] (41)
Then we will show that there is a constant \( M > 0 \) (independent of \( r \)) such that \( M_n \leq M^n \) for all \( n \).

**Step 4:** **Construction of** \( \{M_n\}_{n=0}^\infty \). Let \( M_0 := 0 \). Now we use induction on \( n \) and formula (35), which is more convenient to rewrite as follows:
\[
g_{n+1}^i = \varphi_i^{-1} \partial_x g_n^i - \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{|m|=m} \sum_{|n|=n} C_{km}^i g_n^m .
\] (42)
Notice that \( g_n^m = 0 \) whenever \( m = |m| > |n| = n - k \), so this expression really is the same as (35).

**Step 4.1: Inductive Hypothesis.** Assume that we have already constructed non-negative real numbers \( M_0, \ldots, M_n \) such that, for all \( i = 1, \ldots, N \), all \( j = 0, \ldots, n-1 \), all \( r \in (0, R) \), and all \( x \in \mathbb{D}_r \), we have the bound
\[
\left| g_{n+1}^i(x) \right| \leq M_{j+1} \delta^{-j} j! .
\] (43)

**Step 4.2: Bounding the Derivative.** In order to derive an estimate for \( g_{n+1}^i \), we first need to estimate the derivative term \( \partial_x g_n^i \), for which we use Cauchy estimates as follows. We claim that for all \( r \in (0, R) \) and all \( x \in \mathbb{D}_r \),
\[
|\partial_x g_n^i(x)| \leq A |M_n| \delta^{-n} n! ,
\] (44)
where \( \delta = R - r \). Indeed, for every \( r \in (0, R) \), define
\[
\delta_n := \delta \frac{n}{n+1} \quad \text{and} \quad r_n := R - \delta_n .
\]
Then inequality (43) holds in particular with \( j = n - 1 \) and \( r = r_n \). Thus, for all \( x \in \mathbb{D}_{r_n} \), we find:
\[
|g_n^i(x)| \leq M_n \delta_n^{1-n} (n-1)! = M_n \delta^{1-n} \frac{n}{n+1} \left( \frac{n+1}{n} \right)^n (n-1)! \leq A M_n \delta^{-n} n! \frac{\delta}{n+1} .
\]
Here, we have used the estimate \((1 + 1/n)^n \leq e \leq A\). Finally, notice that for every \( x \in \mathbb{D}_r \), the closed disc centred at \( x \) of radius \( r_n - r = (R - \delta_n) - (R - \delta) = \delta - \delta_n = \frac{\delta}{n+1} \) is contained inside the disc \( \mathbb{D}_{r_n} \). Therefore, Cauchy estimates imply (44).

**Step 4.3: Bounding \( g_n^m \).** Let us estimate each \( g_n^m \) separately using formula (36):
\[
|g_n^m| \leq \left( \sum_{j_1 \in \mathbb{N}^{m_1}} |g_{j_1,1}^1| \cdots |g_{j_1,m_1}^1| \right) \cdots \left( \sum_{j_N \in \mathbb{N}^{m_N}} |g_{j_N,1}^N| \cdots |g_{j_N,m_N}^N| \right) \leq \left( \sum_{j_1 \in \mathbb{N}^{m_1}} |M_{j_1,1} \cdots M_{j_1,m_1}| \right) \cdots \left( \sum_{j_N \in \mathbb{N}^{m_N}} |M_{j_N,1} \cdots M_{j_N,m_N}| \right) \delta^{-n} (|n| - |m|)! ,
\]
where we repeatedly used the inequality \( i!j! \leq (i + j)! \). Introduce the following
shorthand:

\[
M_n^m := \left( \sum_{j_1 \in \mathbb{N}^{m_1}} M_{j_1,1} \cdots M_{j_1,m_1} \right) \cdots \left( \sum_{j_N \in \mathbb{N}^{m_N}} M_{j_N,1} \cdots M_{j_N,m_N} \right)
\]  \hspace{1cm} (45)

Then the estimate for \( g_n^m \) becomes simply \(|g_n^m| \leq M_n^m (|n| - |m|)!\).

**STEP 4.4: INDUCTIVE STEP.** Now we can finally estimate \( g_{n+1}^i \) using formula (42):

\[
|g_{n+1}^i| \leq |\varphi_i^{-1} \partial_2 g_n^i| + \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{|m|=|n|=n-k} |G_{km}^i| \cdot |g_n^m|
\]

\[
\leq A^2 M_n \delta^{-n} n! + \sum_{k=0}^{n} \sum_{m=0}^{\infty} \sum_{|m|=|n|=n-k} \rho_m A B^k m! M_n^m \delta^{-n} (n - k - m)!
\]

\[
\leq A^2 \left( M_n + \sum_{k=0}^{n} B^k \sum_{m=0}^{\infty} \sum_{|m|=|n|=n-k} \rho_m B^m M_n^m \right) \delta^{-n} n!. 
\]

Thus, we can define

\[
M_{n+1} := A^2 \left( M_n + \sum_{k=0}^{n} B^k \sum_{m=0}^{\infty} \sum_{|m|=|n|=n-k} \rho_m B^m M_n^m \right)
\]  \hspace{1cm} (46)

**STEP 5: CONSTRUCTION OF \( M \).** To see that \( M_n \leq M^m \) for some \( M > 0 \), we argue as follows. Consider the following pair of power series in an abstract variable \( t \):

\[
\tilde{p}(t) := \sum_{n=0}^{\infty} M_n t^n \quad \text{and} \quad Q(t) := \sum_{m=0}^{\infty} B^m t^m.
\]  \hspace{1cm} (47)

Notice that \( \tilde{p}(0) = M_0 = 0 \) and that \( Q(t) \) is convergent. We will show that \( \tilde{p}(t) \) is also convergent. The key is the observation that they satisfy the following equation, which was found by trial and error:

\[
\tilde{p}(t) = A^2 \left( t \tilde{p}(t) + t Q(t) Q(\tilde{p}(t)) \right) = A^2 \left( t \tilde{p}(t) + t Q(t) \sum_{m=0}^{\infty} B^m \tilde{p}(t)^m \right).
\]  \hspace{1cm} (48)

**STEP 5.1: VERIFICATION.** In order to verify this equality, we rewrite the power series \( Q(t) \) in the following way:

\[
Q(t) = \sum_{n=0}^{\infty} \sum_{|m|=|n|} \rho_m B^m t^m
\]

where \( t^m := t^{m_1} \cdots t^{m_N} = t^m \). Then (48) is straightforward to check directly by substituting the power series \( \tilde{p}(t) \) and \( Q(t) \) and comparing the coefficients of \( t^{n+1} \) using the defining formula (46) for \( M_{n+1} \). Indeed, using the notation introduced in (45), we find:

\[
\tilde{p}(t)^m = \tilde{p}(t)^{m_1} \cdots \tilde{p}(t)^{m_N} = \left( \sum_{n_1=0}^{\infty} M_{n_1} t^{n_1} \right)^{m_1} \cdots \left( \sum_{n_N=0}^{\infty} M_{n_N} t^{n_N} \right)^{m_N}
\]
\[ \left( \sum_{n_1=0}^{\infty} \sum_{|j_1|=n_1} \frac{M_{j_1,1} \cdots M_{j_{n_1},1} t^{n_1}}{j_1!} \right) \cdots \left( \sum_{n_N=0}^{\infty} \sum_{|j_N|=n_N} \frac{M_{j_{n_1},1} \cdots M_{j_{n_N},1} t^{n_N}}{j_N!} \right) \]

Then the bracketed expression on the right-hand side of (48) expands as follows:

\[ t\hat{p}(t) + t \left( \sum_{k=0}^{\infty} B_k t^k \right) \left( \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_m B^m (\hat{p}(t))^m \right) \]

\[ = \sum_{n=0}^{\infty} M_n t^{n+1} + t \left( \sum_{k=0}^{\infty} B_k t^k \right) \left( \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_m B^m \sum_{n=0}^{\infty} M_n^m t^n \right) \]

\[ = \sum_{n=0}^{\infty} M_n t^{n+1} + t \sum_{n=0}^{\infty} B_k C_n t^n \]

which matches with (46).

**Step 5.2: Implicit Function Theorem Argument.** Now, consider the following holomorphic function in two complex variables \((t, p)\):

\[ H(t, p) := -p + A^2 tp + A^2 tQ(t)Q(p) . \]

It has the following properties:

\[ H(0, 0) = 0 \quad \text{and} \quad \frac{\partial H}{\partial p} \bigg|_{(t,p)=(0,0)} = -1 \neq 0 . \]

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic function \(p(t)\) near \(t = 0\) such that \(p(t) = 0\) and \(H(t, p(t)) = 0\). Thus, \(\hat{p}(t)\) must be the convergent Taylor series expansion at \(t = 0\) for \(p(t)\), and so its coefficients grow at most exponentially: i.e., there is a constant \(M > 0\) such that \(M_n \leq M^n\).

\[ \text{§ 4. Exact Perturbation Theory} \]

Finally, we are ready to give the proof of our main result.

**Proof of Theorem 1.4.** Uniqueness of the solution \(f\) is an easy consequence of the asymptotic property (6). Indeed, suppose \(f'\) is another such solution. Then their difference \(f - f'\) is a holomorphic map \(X_0 \times S_0 \rightarrow C^r\) which is uniformly Gevrey asymptotic to the zero map as \(h \to 0\) along the closed arc \(\Lambda\) of opening angle \(\pi\). By
Nevanlinna’s Theorem ([Nev18, pp.44-45]; see also [Nik20, Theorem B.11]), there can only be one holomorphic function on $S_0$ (namely, the constant function 0) which is Gevrey asymptotic to 0 as $h \to 0$ along $\overline{A}$, so $f - f'$ must be the zero map.

What remains is to construct the solution $f$ with all the desired properties.

**Step 0: Strategy and setup.** The strategy to construct $f$ is as follows. First, we make a preliminary simplifying transformation to put the differential equation (1) into a standard form. An application of the Borel transform induces a first-order PDE which, after a coordinate transformation, can be easily rewritten as an integral equation. Most of the hard work is then concentrated on solving this integral equation, which we do using the method of successive approximations. The final step is to apply the Laplace transform.

Via a simple rotation in the $h$-plane, we can assume without loss of generality that $\theta = 0$. Let us also immediately restrict our attention to a Borel disc in the $h$-plane; i.e., without loss of generality, assume that $S = \{ h \mid \text{Re}(1/h) > 1/d \}$ for some diameter $d > 0$. Let $W := W_1 \cup \cdots \cup W_N$.

**Step 1: Preliminary transformation.** Let $K_0 := \text{diag}(\varphi_1, \ldots, \varphi_N)$ be the diagonal matrix of eigenvalues of $J_0$ and let $P_0 = P_0(x)$ be a holomorphic matrix that diagonalises $J_0$; i.e.,

$$P_0 J_0 P_0^{-1} = K_0.$$  

Consider the change of the unknown variable $f \mapsto g$ given by the formula

$$f = f_0 + hf_1 + hP_0^{-1}g.$$  

We argue that it transforms the differential equation (1) into one of the form

$$hK_0^{-1} \partial_x g - g = hG(x, h, g)$$  

or, written in components for $i = 1, \ldots, N$,

$$h\varphi_i^{-1} \partial_x g^i - g^i = hG^i(x, h, g)$$  

where $G = G(x, h, w)$ is a holomorphic map $W \times S \times \mathbb{C}_w^N \to \mathbb{C}^N$ which admits a Gevrey asymptotic expansion

$$G(x, h, w) \simeq G(x, h, w) \quad \text{as } h \to 0 \text{ along } \overline{A},$$  

uniformly for all $x \in W$ and and locally uniformly for all $w$. The argument here is identical to Step 1 in the proof of Proposition 3.1 (see page 10).

**Step 2: The multi-Liouville transformation.** Recall the $N$ (possibly multivalued) local coordinate transformations $\Phi_i : W_i \to H$ given by (3). For each $i = 1, \ldots, N$, consider the (possibly multivalued) local biholomorphisms

$$\Phi_{(i)} := \Phi_1^{-1} \circ \Phi_i : W_i \longrightarrow W_1.$$  

Restrict $x$ to the halfstrip $W_1$. Then (after possibly choosing a branch cut or lifting to a universal cover) each new independent variable $x_i := \Phi_{(i)}^{-1}(x)$ gives a local
coordinate on the halfstrip \( W_i \). For each \( i = 1, \ldots, N \), make another change of the unknown variable \( g \) to \( g_{(i)} \) by \( g_{(i)} = \Phi_{(i)}^* g_i \); i.e., by the formula \( g_{(i)}(x_i) = g(x) \). Note that chain rule gives \( \partial_x g_{(i)}(x) = \partial_x g_{(i)}(x_i) \), so the \( i \)-th equation (52), written in the new independent variable \( x_i \) (i.e., pulling this equation back by \( \Phi_{(i)} \)), becomes:

\[
h \varphi_i^{-1} \partial_x g_{(i)} - g_i = hG^i(x_i, h; g_{(i)}) \tag{55}\]

The advantage of this point of view is that if we now pull the system (55) back by the (single-valued) holomorphic map

\[
(\Phi_1^{-1}, \ldots, \Phi_N^{-1}) : H \rightarrow W_1 \times \cdots \times W_N, \tag{56}
\]

we obtain the following coupled system of \( N \) nonlinear ordinary differential equations on the horizontal halfstrip \( H \) for \( s = (s^1, \ldots, s^N) \) with \( s^i = s^i(z, h) \):

\[
h \partial_s s - s = hA(z, h, s), \tag{57}
\]

where \( A^i(z, h, w) := G^i(x_i(z), h, w) \) for \( x_i(z) = \Phi_1^{-1}(z) \), and the unknown variables \( s, g_{(i)} \), and \( g \) are related by \( s = (\Phi_1^{-1})^* g_{(i)} = (\Phi_1^{-1})^* g \).

**Step 3: Expansion.** Each component \( A^i \) of \( A \) can be expressed as a uniformly convergent multipower series in the components \( w_1, \ldots, w_N \) of \( w \):

\[
A^i(z, h, w) = \sum_{m=0}^{\infty} \sum_{|m|=m} A^i_m(z, h) w^m, \tag{58}
\]

where \( A^i_m w^m := A^i_{m_1 \cdots m_N} w_1^{m_1} \cdots w_N^{m_N} \). It is convenient to separate the \( m = 1 \) term from the sum:

\[
A^i(z, h, w) = A^i_0 + \sum_{m=1}^{\infty} \sum_{|m|=m} A^i_m(z, h) w^m, \tag{59}
\]

where \( 0 = (0, \ldots, 0) \). Then the system of equations (57) can be written as

\[
h \partial_s s^i - s^i = hA^i_0 + h \sum_{m=1}^{\infty} \sum_{|m|=m} A^i_m(z, h) s^m, \tag{60}
\]

**Step 4: The analytic Borel transform.** Let \( a^i_m = a^i_m(z) \) be the \( h \)-leading-order part of \( A^i_m \) and let \( \alpha^i_m(z, \xi) := \mathcal{B}[A^i_m](z, \xi) \). By assumption, there is some \( \varepsilon > 0 \) such that each \( \alpha^i_m \) is a holomorphic function on \( H \times \Xi \), where

\[
\Xi := \{ \xi \mid \text{dist}(\xi, \mathbb{R}_+) < \varepsilon \},
\]

with uniformly at most exponential growth at infinity in \( \xi \), and such that

\[
A^i_m(z, h) = a^i_m(z) + \mathcal{O}[\alpha^i_m](z, h) \tag{61}
\]

for all \( (z, h) \in H \times S \) provided that the diameter \( d \) of \( S \) is sufficiently small.

Dividing each equation (60) by \( h \) and applying the analytic Borel transform, we obtain the following system of \( N \) coupled nonlinear partial differential equations.
with convolution:
\[
\partial_z \sigma^i - \partial_\xi \sigma^i = \alpha^i_0 + \sum_{m=1}^{\infty} \sum_{|m|=m} \left( a^i_m \sigma^m + \alpha^i_m \ast \sigma^m \right),
\]
(62)
where \( \sigma^m := (\sigma^1)^{m_1} \ast \cdots \ast (\sigma^N)^{m_N} \) and the unknown variables \( s^i \) and \( \sigma^i \) are related by \( \sigma^i = B[s^i] \) and \( s^i = L[\sigma^i] \).

**Step 5: The integral equation.** The principal part of the PDE (62) has constant coefficients, so it is easy to rewrite it as an equivalent integral equation as follows. Consider the holomorphic change of variables
\[
(z, \xi) \xrightarrow{T} (\zeta, t) := (z + \xi, \xi) \quad \text{and its inverse} \quad (\zeta, t) \xrightarrow{T^{-1}} (z, \xi) = (\zeta - t, t).
\]
Explicitly, for any function \( \alpha = \alpha(z, \xi) \) of two variables,
\[
T^* \alpha(z, \xi) := \alpha(T(z, \xi)) = \alpha(z + \xi, \xi) \quad \text{and} \quad T_t \alpha(\zeta, t) := \alpha(T^{-1}(\zeta, t)) = \alpha(\zeta - t, t).
\]
Note that \( T^* T_t \alpha = \alpha \). Under this change of coordinates, the differential operator \( \partial_z - \partial_\xi \) transforms into \( -\partial_t \), and so the lefthand side of (62) becomes \( -\partial_t (T^* \sigma^i) \).
Integrating from 0 to \( t \), imposing the initial condition \( \sigma^i(z, 0) = a^i_0(z) \), and then applying \( T^* \), we convert the system of PDEs (62) into the following system of integral equations:
\[
\sigma^i = a^i_0 - T^* \int_0^t T_t \left( \alpha^i_0 + \sum_{m=1}^{\infty} \sum_{|m|=m} \left( a^i_m \sigma^m + \alpha^i_m \ast \sigma^m \right) \right) du .
\]
More explicitly, this integral equation reads as follows:
\[
\sigma^i(z, \xi) = a^i_0(z) - \int_0^\xi \left[ \alpha^i_0(z + \xi - u, u) + \sum_{m=1}^{\infty} \sum_{|m|=m} \left( a^i_m(z + \xi - u, u) \cdot \sigma^m(z + \xi - u, u) + (\alpha^i_m \ast \sigma^m)(z + \xi - u, u) \right) \right] du .
\]
Here, the integration is done along a straight line segment from 0 to \( \xi \). Note also that the convolution products are with respect to the second argument; i.e.,
\[
(\alpha \ast \alpha')(t_1, t_2) = \int_0^{t_2} \alpha(t_1, t_2 - y) \alpha'(t_1, y) dy ,
\]
\[
(\alpha \ast \alpha' \ast \alpha'')(t_1, t_2) = \int_0^{t_2} \alpha(t_1, t_2 - y) \int_0^y \alpha'(t_1, t_2 - y') \alpha''(t_1, y') dy' dy .
\]
Introduce the following notation: for any function \( \alpha = \alpha(z, \xi) \) of two variables,

\[
I[\alpha](z, \xi) := -T^* \int_0^t T_s \alpha \, du = - \int_0^\xi \alpha(z + \xi - u, u) \, du = \int_0^\xi \alpha(z + t, \xi - t) \, dt .
\]

where as before the integration path is the straight line segment connecting 0 to \( \xi \). Then the system of integral equations (63) can be written more succinctly as follows:

\[
\sigma^1 = \sigma_0^1 + I \left[ \alpha_0^1 + \sum_{m=1}^\infty \sum_{\|m\|=m} \left( a_m^1 \sigma^m + a_0^1 \sigma^*m \right) \right] .
\]

**Step 6: Method of successive approximations.** To solve this system, we use the method of successive approximations. To this end, define a sequence of holomorphic maps \( \{ \sigma_n = (\sigma_1^n, \ldots, \sigma_N^n) : H \times \Xi \to \mathbb{C}^N \}_{n=0}^\infty \), as follows: for each \( i = 1, \ldots, N \), let

\[
\sigma_i^0 := a_i^0 \quad \text{and} \quad \sigma_i^1 := I \left[ \alpha_0^1 + \sum_{|m|=1} a_m^i \sigma_0^m \right] ,
\]

and for all \( n \geq 2 \),

\[
\sigma_i^n := I \left[ \sum_{m=1}^n \sum_{|m|=m} \left( a_m^i \sum_{|n|=n-m} \sigma_n^m + a_0^i \sum_{|n|=n-m-1} \sigma_0^m \right) \right] .
\]

Here, we have introduced the following notation: for any \( n, m \in \mathbb{N}^N \),

\[
\sigma_n^m := \left( \sum_{|j_1|=n_1} \sigma_1^{m_1} \cdots \sigma_1^{m_{N-1}} \right) \cdots \left( \sum_{|j_N|=n_N} \sigma_N^{m_1} \cdots \sigma_N^{m_J} \right) .
\]

Let us also note the following simple but useful identities:

\[
\sigma_0^0 = 1 ; \quad \sigma_0^n = 0 \quad \text{whenever} \quad |n| > 0 ;
\]

\[
\sigma_0^m = (\sigma_0^1)^{m_1} \cdots (\sigma_0^N)^{m_N} = \frac{1}{(m-1)!} \sigma_0^m \sigma_0^m .
\]

The technical crux of the argument is the following claim.

**Main Technical Claim.** Fix any \( r_0 \in (0, r) \) and let \( H_0 := \{ z \mid \text{dist}(z, \mathbb{R}_+) < r_0 \} \subset H \). Let \( \varepsilon \) (which is the thickness of the halfstrip \( \Xi \)) be so small that \( \varepsilon < r - r_0 \). Then the infinite series

\[
\sigma(z, \xi) := \sum_{n=0}^\infty \sigma_n(z, \xi)
\]

defines a holomorphic solution of the system of integral equations (65) on the domain

\[
H := \{ (z, \xi) \in H \times \Xi \mid z + \xi \in H \}
\]

where it has uniformly at-most-exponential growth at infinity in \( \xi \); more precisely, there are constants \( D, K > 0 \) such that

\[
|\sigma(x, \xi)| \leq De^{K|\xi|} \quad \forall (x, \xi) \in H .
\]
Furthermore, the formal Borel transform
\[ \hat{s}(z, \xi) = \hat{\mathcal{B}}[\hat{s}](z, \xi) = \sum_{n=0}^{\infty} \frac{1}{n!} s_{n+1}(z)\xi^n \] (73)
of the formal solution $\hat{s}(z, h)$ of (57) is the Taylor series expansion of $\sigma$ at $\xi = 0$. In particular, $\sigma$ is a well-defined holomorphic solution on $H_0 \times \Xi \subset H$ where it satisfies the exponential estimate (72).

Before justifying this claim, let us explain how it implies our main theorem.

**Step 7: Laplace transform.** Assuming the Main Technical Claim, only one step remains in order to complete the proof of Theorem 1.4, which is to apply the Laplace transform to $\sigma$:
\[ s(z, h) := L[\sigma](z, h) = \int_{0}^{+\infty} e^{-\xi/h}\sigma(z, \xi)\,d\xi . \] (74)

Thanks to the exponential estimate (72), this Laplace integral is uniformly convergent for all $z \in H_0$ provided that $\Re(1/h) > K$. Thus, if we take $d_0 \in (0, d]$ strictly smaller than $1/K$, then formula (74) defines a holomorphic solution of the differential equation (57) on the domain $H_0 \times S_0$ where $S_0 := \{ h \mid \Re(1/h) > 1/d_0 \}$. Furthermore, Nevanlinna’s Theorem ([Nev18, pp.44-45]; see also [Nik20, Theorem B.11]) implies that $s$ admits a uniform Gevrey asymptotic expansion on $H_0$ as $h \to 0$ along $\overline{A}$, and this asymptotic expansion is necessarily the formal solution $\hat{s}$.

Undoing all the changes of variables we made at the beginning of the proof, we define a holomorphic solution $g_{(i)}$ of (55) for $x_i \in W'_i := \Phi_i^{-1}(H_0)$ and $h \in S_0$ by
\[ g_{(i)}(x_i, h) := \Phi_i^* s(x_i, h) = \Phi_i^* L[\sigma](x_i, h) = \int_{0}^{+\infty} e^{-\xi/h}\sigma(\Phi_i(x_i), \xi)\,d\xi , \] (75)
and consequently a holomorphic solution $g$ of (52) on $W'_1 \times S_0$ given by
\[ g(x, h) = \int_{0}^{+\infty} e^{-\xi/h}\sigma(\Phi_1(x), \xi)\,d\xi \] (76)
This yields a holomorphic solution of the original differential equation (1) on $W'_1 \times S_0$ defined by $f = f_0 + hf_1 + hP_0^{-1}g$ with all the desired properties. This completes the proof of Theorem 1.4 assuming the Main Technical Claim.

Let us now prove the Main Technical Claim.

**Step 8: Solution Check.** First, assuming that the infinite series $\sigma$ is uniformly convergent for all $(z, \xi) \in H$, we verify that it satisfies the integral equation (65) by direct substitution. Thus, the righthand side of (65) becomes:
\[ a_{n}^{i} + L \left[ \alpha_{n}^{i} + \sum_{m=1}^{\infty} \sum_{|m|=m} a_{m}^{n} \left( \sum_{n=0}^{\infty} \sigma_{n} \right)^{m} + \sum_{m=1}^{\infty} \sum_{|m|=m} \alpha_{m}^{i} \ast \left( \sum_{n=0}^{\infty} \sigma_{n} \right)^{m} \right] . \] (77)
Using the notation introduced in (68), the \( m \)-fold convolution product of the infinite series \( \sigma \) expands as follows:

\[
\left( \sum_{n=0}^{\infty} \sigma_n \right)^m = \left( \sum_{n_1=0}^{\infty} \sigma_{n_1}^1 \right) \ast \cdots \ast \left( \sum_{n_N=0}^{\infty} \sigma_{n_N}^N \right)^{m_N}
\]

\[
= \sum_{n_1=0}^{\infty} \sum_{[j_1]=n_1} \sum_{j_1 \in \mathbb{N}^{1,m_1}} \sigma_{j_1}^1 \ast \cdots \ast \sigma_{j_{1,m_1}}^1 \ast \cdots \ast \left( \sum_{n_N=0}^{\infty} \sum_{[j_N]=n_N} \sum_{j_N \in \mathbb{N}^{N,m_N}} \sigma_{j_N}^N \ast \cdots \ast \sigma_{j_{N,m_N}}^N \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{|m|=n} \sigma_{m,n}^m
\]

Use this to rewrite the blue terms in (77), separating out first the \( m = 1 \) part and then the \( (m, n) = (1, 1) \) part using the identity (70):

\[
\sum_{m=1}^{\infty} \sum_{|m|=m} a_i^m \left( \sum_{n=0}^{\infty} \sigma_n \right)^m
\]

\[
= \sum_{|m|=1} a_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m + \sum_{n=2}^{\infty} \sum_{|m|=m} a_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m
\]

\[
= \sum_{|m|=1} a_i^m \sigma_0^m + \sum_{|m|=1} a_i^m \sum_{n=1}^{\infty} \sum_{|n|=n} \sigma_n^m + \sum_{m=2}^{\infty} \sum_{|m|=m} a_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m
\]

Substituting this back into (77) and using (66), we find:

\[
\sigma_0^i + \sigma_1^i + I \left[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{|n|=n} \sigma_n^m + \sum_{m=2}^{\infty} \sum_{|n|=n} \sum_{|m|=m} a_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m + \sum_{m=1}^{\infty} \sum_{|n|=n} \sum_{|m|=m} \alpha_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m \right]
\]

(78)

The goal is to show that the integral in (78) is equal to \( \sum_{n \geq 2} \sigma_n^i \). Focus on the expression inside the integral:

\[
\sum_{|m|=1}^{\infty} \sum_{n=1}^{\infty} \sum_{|n|=n} \sigma_n^m + \sum_{m=2}^{\infty} \sum_{|n|=n} \sum_{|m|=m} a_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m + \sum_{m=1}^{\infty} \sum_{|n|=n} \sum_{|m|=m} \alpha_i^m \sum_{n=0}^{\infty} \sum_{|n|=n} \sigma_n^m
\]

Shift the summation index \( n \) up by 1 in the black sum, by \( m \) in the blue sum, and by \( m+1 \) in the green sum:

\[
\sum_{|m|=1}^{\infty} \sum_{n=2}^{\infty} \sum_{|n|=n-1} \sum_{|m|=m} \sigma_n^m + \sum_{m=2}^{\infty} \sum_{|n|=n} \sum_{|m|=m} \sigma_n^m + \sum_{m=1}^{\infty} \sum_{|n|=n-1} \sum_{|m|=m} \sum_{n=0}^{\infty} \sigma_n^m + \sum_{m=1}^{\infty} \sum_{|n|=n} \sum_{n=m+1}^{\infty} \sigma_n^m
\]

Notice that all terms in the blue sum with \( n < m \) are zero, so we can start the summation over \( n \) from \( n = 2 \) (which is the lowest possible value of \( m \)) without
altering the result. Similarly, all terms in the green sum with \( n < m + 1 \) are zero, so we may as well start from \( n = 2 \). The black sum is left unaltered. Thus, we get:

\[
\sum_{|m|=1}^{\infty} a_m^i \sum_{|n|=n-1}^{\infty} \sigma_n^m + \sum_{m=2}^{\infty} a_m^i \sum_{|n|=n-1}^{\infty} \sum_{n=2}^{\infty} a_m^i \sum_{|n|=n-m}^{\infty} \sigma_n^m + \sum_{m=1}^{\infty} a_m^i \sum_{|n|=n-m}^{\infty} \sum_{n=2}^{\infty} \sigma_n^m
\]

The advantage of this way of expressing the sums is that we can now interchange the summations over \( m \) and \( n \) to obtain:

\[
\sum_{n=2}^{\infty} \left\{ \sum_{|m|=1}^{\infty} a_m^i \sum_{|n|=n-1}^{\infty} \sigma_n^m + \sum_{m=2}^{\infty} a_m^i \sum_{|n|=n-m}^{\infty} \sum_{n=2}^{\infty} a_m^i \sum_{|n|=n-m-1}^{\infty} \sigma_n^m \right\}.
\]

Observe that the black sum fits well into the blue sum over \( m \) to give the \( m = 1 \) term. So we get:

\[
\sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \sum_{|m|=m}^{\infty} \left\{ a_m^i \sum_{|n|=n-m}^{\infty} \sigma_n^m + a_m^i \sum_{|n|=n-m-1}^{\infty} \sigma_n^m \right\}.
\]

Finally, notice that both sums are empty for \( m > n \), so we get:

\[
\sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \sum_{|m|=m}^{\infty} \left\{ a_m^i \sum_{|n|=n-m}^{\infty} \sigma_n^m + a_m^i \sum_{|n|=n-m-1}^{\infty} \sigma_n^m \right\}.
\]

The sum over \( m \) is precisely the expression inside the integral in (67) defining \( \sigma_n^i \). This shows that \( \sigma \) satisfies the integral equation (65).

**Step 9: Convergence.** Now we show that \( \sigma \) is a uniformly convergent series on \( H \) and therefore defines a holomorphic map \( H \rightarrow \mathbb{C}^N \). In the process, we also establish the estimate (72).

Let \( B, C, L > 0 \) be such that for all \((z, \xi) \in H, \) all \( j = 1, \ldots, N, \) and all \( m \in \mathbb{N}^N, \)

\[
|a_m^i(z)| \leq \rho_mC B^m \quad \text{and} \quad |a_m^i(z, \xi)| \leq \rho_mC B^m e^{L|\xi|}, \tag{79}
\]

where \( m = |m| \) and \( \rho_m \) is the normalisation constant (39). We claim that there are constants \( D, M > 0 \) such that for all \((z, \xi) \in H \) and all \( n \in \mathbb{N}, \)

\[
|\sigma_n^i(z, \xi)| \leq D M^n \frac{|\xi|^n}{n!} e^{L|\xi|}. \tag{80}
\]

If we achieve (80), then the uniform convergence and the exponential estimate (72) both follow at once because

\[
|\sigma^i(z, \xi)| \leq \sum_{n=0}^{\infty} |\sigma_n^i(z, \xi)| \leq \sum_{n=0}^{\infty} D M^n \frac{|\xi|^n}{n!} e^{L|\xi|} \leq D e^{(M+L)|\xi|}.
\]

To demonstrate (80), we proceed in two steps. First, we construct a sequence of positive real numbers \( \{M_n\}^{\infty}_{n=0} \) such that for all \( n \in \mathbb{N} \) and all \((z, \xi) \in H, \)

\[
|\sigma_n^i(z, \xi)| \leq M_n \frac{|\xi|^n}{n!} e^{L|\xi|}. \tag{81}
\]

We will then show that there are constants \( D, M \) such that \( M_n \leq D M^n \) for all \( n. \)
Step 9.1: Construction of \( \{M_n\} \). We can take \( M_0 := C \) and \( M_1 := C(1 + BM_0) \) because \( \sigma^0_i = a^i_0 \) and

\[
|\sigma^1_i| \leq \int_0^\xi \left( |\alpha^i_0| + \sum_{|m|=1} |a^i_m||\sigma^m_0| \right) |dt| \leq \int_0^\xi \left( Ce^{L|\xi|} + C^2B\rho_1 \sum_{|m|=1} 1 \right) |dt| \\
\leq C(1 + BM_0) \int_0^\xi e^{Ls} ds \leq C(1 + BM_0)|\xi|e^{L|\xi|},
\]

where in the final step we used Lemma A.10. Now, let us assume that we have already constructed the constants \( M_0, \ldots, M_{n-1} \) such that \( |\sigma^k_i| \leq M_k |\xi|^k e^{L|\xi|} \) for all \( k = 0, \ldots, n-1 \) and all \( i = 1, \ldots, N \). Then we use formula (67) together with Lemma A.10 and Lemma A.11 in order to derive an estimate for \( \sigma_n \).

First, let us write down an estimate for \( \sigma^m_n \) using formula (68). Thanks to Lemma A.11, we have for each \( i = 1, \ldots, N \) and all \( n_i, m_i \):

\[
\sum_{j_i \in \mathbb{N}^{n_i}} \left| \sigma^i_{j_1} \cdots \sigma^i_{j_{m_i}} \right| \leq \sum_{j_i \in \mathbb{N}^{n_i}} \left| M_{j_i, 1} \cdots M_{j_i, m_i} \right| \sum_{|n_i| = n_i, m_i} \frac{|\xi|^{n_i + m_i - 1}}{(n_i + m_i - 1)!} e^{L|\xi|}.
\]

Then, for all \( n, m \in \mathbb{N}^N \),

\[
|\sigma^m_n| \leq M^m_n \frac{|\xi|^{n + |m| - 1}}{(n + |m| - 1)!} e^{L|\xi|}, \tag{82}
\]

where \( M^m_n \) is the shorthand introduced in (45). Therefore, formula (67) gives the following estimate:

\[
|\sigma^m_n| \leq \int_0^\xi \left| \sum_{|m|=|n|} \sum_{|n|=n-m} \left| \rho_{n} \sum_{|n|=n-m} M^m_n + \rho_{n} C B^m \sum_{|n|=n-m} M^m_n \right| \right| |dt| \\
\leq \sum_{|m|=|n|} \sum_{|n|=n-m} \left\{ \rho_{n} C B^m \sum_{|n|=n-m} M^m_n \right\} \left( \sum_{|n|=n-m} M^m_n \right) |\xi|^{n-1} e^{L|\xi|} |dt| \\
\leq \sum_{|m|=|n|} \sum_{|n|=n-m} \left\{ \rho_{n} C B^m \sum_{|n|=n-m} M^m_n \right\} \left( \frac{|\xi|^n}{n!} e^{L|\xi|} \right)
\]

Thus, this expression allows us to define the constant \( M_n \) for \( n \geq 2 \). In fact, a quick glance at this formula reveals that it can be extended to \( n = 0, 1 \) by defining

\[
M_n := \sum_{m=0}^n \rho_{n} C B^m \sum_{|m|=m} \left( \sum_{|n|=n-m} M^m_n \right) + \sum_{|m|=m} M^m_n \quad \forall n \in \mathbb{N}. \tag{83}
\]

Indeed, if \( m = 0 \), then the two sums inside the brackets can only possibly be nonzero when \( n = 0 \), in which case the second sum is empty and the first sum is 1, so we recover \( M_0 = C \). Likewise, if \( n = 1 \), then the \( m = 0 \) term is \( 0 + C \) and the \( m = 1 \) term is \( CBM_0 + 0 \), so again we recover the constant \( M_1 \) defined previously.
Step 9.2: Bounding $M_n$. To see that $M_n \leq DM^n$ for some $D, M > 0$, consider the following two power series in an abstract variable $t$:

$$\hat{p}(t) := \sum_{n=0}^{\infty} M_n t^n \quad \text{and} \quad Q(t) := \sum_{m=0}^{\infty} CB^m t^m . \quad (84)$$

Notice that $Q(t)$ is convergent and $Q(0) = C = M_0$. We will show that $\hat{p}(t)$ is also a convergent power series. The key observation is that $\hat{p}$ satisfies the following functional equation:

$$\hat{p}(t) = (1 + t)Q(t\hat{p}(t)) . \quad (85)$$

This equation was found by trial and error. In order to verify it, we rewrite the power series $Q(t)$ in the following way:

$$Q(t) = \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_mC B^m t^m \quad (86)$$

Then (85) is straightforward to verify by direct substitution and comparing the coefficients of $t^n$ using the defining formula (83) for $M_n$. Thus, the righthand side of (85) expands as follows:

$$(1 + t) \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_mC B^m \left( \sum_{n=0}^{\infty} M_n t^n \right)^m$$

$$= (1 + t) \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_mC B^m \left( \sum_{n_1=0}^{\infty} M_{n_1} t^{n_1} \right)^m \cdots \left( \sum_{n_N=0}^{\infty} M_{n_N} t^{n_N} \right)^m$$

$$= (1 + t) \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_mC B^m \sum_{n=0}^{\infty} \sum_{|n|=n-m} M_n^m t^{n+m}$$

$$= (1 + t) \sum_{m=0}^{\infty} \sum_{|m|=m} \rho_mC B^m \sum_{n=0}^{\infty} \sum_{|n|=n-m} M_n^m t^{n+m}$$

In the final equality, we once again noticed that both sums inside the curly brackets are zero whenever $m > n$.

Now, consider the following holomorphic function in two variables $(t,p)$:

$$F(t,p) := -p + (1 + t)Q(tp) . \quad (87)$$

It has the following properties:

$$F(0,C) = 0 \quad \text{and} \quad \frac{\partial P}{\partial p} \bigg|_{(t,p)=(0,C)} = -1 \neq 0 .$$

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic
function $p(t)$ near $t = 0$ such that $p(0) = C$ and $F(t, p(t)) = 0$. Therefore, $\hat{p}(t)$ must be the convergent Taylor series expansion of $p(t)$ at $t = 0$, so its coefficients grow at most exponentially: i.e., there are constants $D, M > 0$ such that $M_n \leq DM^n$. This completes the proof of the Main Technical Claim and hence of Theorem 1.4. □

## Appendix A. Background Information

### § A.1. Gevrey Asymptotics

A.1. A **sectorial domain** at the origin in $\mathbb{C}_h$ is a simply connected domain $S \subset \mathbb{C}_h^* = \mathbb{C}_h \setminus \{0\}$ whose closure $\overline{S}$ in the real-oriented blowup $[\mathbb{C}_h : 0]$ intersects the boundary circle $\overline{S}^1$ in a closed arc $\overline{A} \subset S^1$ with nonzero length. The open arc $A$ is called the **opening** of $S$, and its length $|A|$ is called the **opening angle** of $S$. A **Borel disc** of **diameter** $R > 0$ is the sectorial domain $S = \{ h \in \mathbb{C}_h \mid \text{Re}(1/h) > 1/R \}$. Its opening is $A = (-\pi, \pi)$. Likewise, a Borel disc bisected by a direction $\theta \in \mathbb{S}^1$ is the sectorial domain $S = \{ h \in \mathbb{C}_h \mid \text{Re}(e^{i\theta}/h) > 1/R \}$. Its opening is $A = (\theta - \pi/2, \theta + \pi/2)$.

A.2. A holomorphic function $f(h)$ on a sectorial domain $S$ is admits a power series $\hat{f}(h)$ as its **asymptotic expansion as** $h \to 0$ **along** $A$ (or as $h \to 0$ **in** $S$) if, for every $n \geq 0$ and every compactly contained subarc $A_0 \Subset A$, there is a sectorial subdomain $S_0 \subset S$ with opening $A_0$ and a real constant $C_{n,0} > 0$ such that

$$\left| f(h) - \sum_{k=0}^{n-1} f_k h^k \right| \leq C_{n,0} |h|^n \quad (88)$$

for all $h \in S_0$. The constants $C_{n,0}$ may depend on $n$ and the opening $A_0$. If this is the case, we write

$$f(h) \sim \hat{f}(h) \quad \text{as } h \to 0 \text{ along } A. \quad (89)$$

If the constants $C_{n,0}$ in (88) can be chosen uniformly for all compactly contained subarcs $A_0 \Subset A$ (i.e., independent of $A_0$ so that $C_{n,0} = C_n$ for all $n$), then we write

$$f(h) \sim \hat{f}(h) \quad \text{as } h \to 0 \text{ along } \overline{A}. \quad (90)$$

A.3. We also say that the holomorphic function $f$ admits $\hat{f}$ as its **Gevrey asymptotic expansion as** $h \to 0$ **along** $A$ if the constants $C_{n,0}$ in (88) depend on $n$ like $C_0 M_0^n n!$. More explicitly, for every compactly contained subarc $A_0 \Subset A$, there is a sectorial domain $S_0 \subset S$ with opening $A_0 \Subset A$ and real constants $C_0, M_0 > 0$ which give the bounds

$$\left| f(h) - \sum_{k=0}^{n-1} f_k h^k \right| \leq C_0 M_0^n n! |h|^n \quad (91)$$

for all $h \in S_0$ and all $n \geq 0$. In this case, we write

$$f(h) \simeq \hat{f}(h) \quad \text{as } h \to 0 \text{ along } A. \quad (92)$$

If in addition to (91), the constants $C_0, M_0$ can be chosen uniformly for all $A_0 \Subset A$, then we will write

$$f(h) \simeq \hat{f}(h) \quad \text{as } h \to 0 \text{ along } \overline{A}. \quad (93)$$
A.4. A formal power series \( \hat{f}(\hbar) = \sum f_n \hbar^n \) is a Gevrey power series if there are constants \( C, M > 0 \) such that for all \( n \geq 0 \),
\[
|f_n| \leq CM^n n!. \tag{94}
\]

A.5. All the above definitions translate immediately to cover vector-valued holomorphic functions on \( S \) by using, say, the Euclidean norm in all the above estimates.

§ A.2. Borel-Laplace Theory

A.6. Let \( \Xi_\theta := \{ \xi \in \mathbb{C} \xi : |\text{dist}(\xi, e^{i\theta} \mathbb{R}_+) < \varepsilon \} \), where \( e^{i\theta} \mathbb{R}_+ \) is the real ray in the direction \( \theta \). Let \( \varphi = \varphi(\xi) \) be a holomorphic function on \( \Xi_\theta \). Its Laplace transform in the direction \( \theta \) is defined by the formula:
\[
\mathcal{L}_\theta[\varphi](x, \hbar) := \int_{e^{i\theta} \mathbb{R}_+} \varphi(x, \xi) e^{-\xi/\hbar} d\xi. \tag{95}
\]

When \( \theta = 0 \), we write simply \( \mathcal{L} \). Clearly, \( \varphi \) is Laplace-transformable in the direction \( \theta \) if \( \varphi \) has at-most-exponential growth as \( |\xi| \to +\infty \) along the ray \( e^{i\theta} \mathbb{R}_+ \). Explicitly, this means there are constants \( A, L > 0 \) such that for all \( \xi \in \Xi_\theta \),
\[
|\varphi(\xi)| \leq Ae^{L|\xi|}. \tag{96}
\]

A.7. The convolution product of two holomorphic functions \( \varphi, \psi \) is defined by the following formula:
\[
\varphi \ast \psi(\xi) := \int_0^\xi \varphi(\xi - y) \psi(y) dy, \tag{97}
\]
where the path of integration is a straight line segment from 0 to \( \xi \).

A.8. Let \( f \) be a holomorphic function on a Borel disc \( S = \{ \hbar \in \mathbb{C} \hbar : \text{Re}(e^{i\theta}/\hbar) > 1/R \} \).

The (analytic) Borel transform (a.k.a., the inverse Laplace transform) of \( f \) in the direction \( \theta \) is defined by the following formula:
\[
\mathfrak{B}_\theta[\varphi](x, \xi) := \frac{1}{2\pi i} \oint_{\partial S'} f(x, \hbar) e^{\xi/\hbar} d\hbar/\hbar^2, \tag{98}
\]
where the integral is taken along the boundary of any Borel disc
\[
S' = \{ \hbar \in \mathbb{C} \hbar : \text{Re}(e^{i\theta}/\hbar) > 1/R' \} \subset S
\]
of strictly smaller diameter \( R' < R \), traversed anticlockwise (i.e., emanating from the singular point \( \hbar = 0 \) in the direction \( \theta - \pi/2 \) and reentering in the direction \( \theta + \pi/2 \)). When \( \theta = 0 \), we write simply \( \mathfrak{B} \).

The fundamental fact that connects Gevrey asymptotics and the Borel transform is the following (cf. [Nik20, Lemma B.5]). If \( f = f(\hbar) \) is a holomorphic function defined on a sectorial domain \( S \) with opening angle \( |A| = \pi \) and \( f \) admits Gevrey asymptotics as \( \hbar \to 0 \) along the closed arc \( \overline{A} \), then the analytic Borel transform \( \varphi(\xi) = \mathfrak{B}_\theta[f](\xi) \) defines a holomorphic function on a tubular neighbourhood \( \Xi_\theta \) of some thickness \( \varepsilon > 0 \). Moreover, its Laplace transform in the direction \( \theta \) is well-defined and satisfies \( \mathcal{L}_\theta[\varphi] = f \).
A.9. Similarly, for a power series \( \hat{f}(h) \), the (formal) Borel transform is defined by

\[
\hat{\varphi}(\xi) = \hat{\mathcal{B}}[\hat{f}](\xi) := \sum_{k=0}^{\infty} \varphi_k \xi^k \quad \text{where} \quad \varphi_k := \frac{1}{k!} f_{k+1} .
\]

The fundamental fact that connects Gevrey power series and the formal Borel transform is the following (cf. [Nik20, Lemma B.8]). If \( \hat{f} \) is a Gevrey power series, then its formal Borel transform \( \hat{\varphi} \) is a convergent power series in \( \xi \). Furthermore, a Gevrey power series \( \hat{f}(h) \) is called a Borel summable series in the direction \( \theta \) if its convergent Borel transform \( \hat{\varphi}(\xi) \) admits an analytic continuation \( \varphi(\xi) = \text{AnCont}_\theta[\hat{\varphi}](\xi) \) to a tubular neighbourhood \( \Xi_\theta \) of the ray \( e^{i\theta} \mathbb{R}_+ \) with at-most-exponential growth in \( \xi \) at infinity in \( \Xi_\theta \). If this is the case, the Laplace transform \( \text{L}_\theta[\varphi](h) \) is well-defined and defines a holomorphic function \( f(h) \) on some Borel disc \( S \) bisected by the direction \( \theta \), and we say that \( f(h) \) is the Borel resummation in direction \( \theta \) of the formal power series \( \hat{f}(h) \), and we write

\[
f(h) = \mathcal{S}_\theta[\hat{f}(h)](h) .
\]

If \( \theta = 0 \), we write simply \( \mathcal{S} \). Expressly, we have the following formulas:

\[
\mathcal{S}_\theta[\hat{f}(h)](h) = \mathcal{L}_\theta[\varphi](h) = \mathcal{L}_\theta[\text{AnCont}_\theta[\hat{\varphi}]](h) .
\]

Thus, Borel resummation \( \mathcal{S}_\theta \) can be seen as a map from the set of (germs of) holomorphic functions \( f \) on \( S \) with \( |A| = \pi \) satisfying (93) to the set of Borel summable power series. One of the most fundamental theorems in Gevrey asymptotics and Borel-Laplace theory is a theorem of Nevanlinna ([Nev18, pp.44-45])\(^1\), which says that this map \( \mathcal{S}_\theta \) is invertible and its inverse is the asymptotic expansion \( \tilde{a} \).

§ A.3. Some Useful Elementary Estimates

Here, for reference, we collect some elementary estimates used in this paper. Their proofs are straightforward (see [Nik20, Appendix C.4]).

A.10. Lemma. For any \( R \geq 0 \), any \( L \geq 0 \), and any nonnegative integer \( n \),

\[
\int_0^R \frac{r^n}{n!} e^{LR} dr \leq \frac{R^{n+1}}{(n+1)!} e^{LR} .
\]

A.11. Lemma. Let \( j_1, \ldots, j_m \) be nonnegative integers and put \( n := j_1 + \cdots + j_m \). Let \( f_{j_1}, \ldots, f_{j_m} \) be holomorphic functions on \( \Xi := \{ \xi \mid \text{dist}(\xi, \mathbb{R}_+) < \varepsilon \} \) for some \( \varepsilon > 0 \). If there are constants \( M_{j_1}, \ldots, M_{j_m}, L \geq 0 \) such that

\[
|f_{j_i}(\xi)| \leq M_{j_i} \frac{|\xi|^{j_i}}{j_i!} e^{L|\xi|} \quad \forall \xi \in \Xi ,
\]

then their total convolution product satisfies the following bound:

\[
|f_{j_1} \ast \cdots \ast f_{j_m}(\xi)| \leq M_{j_1} \cdots M_{j_m} \frac{|\xi|^{n+m-1}}{(n+m-1)!} e^{L|\xi|} \quad \forall \xi \in \Xi .
\]

\(^1\)It was rediscovered and clarified decades later by Sokal [Sok80]; see also [Mal95, p.182], [LR16, Theorem 5.3.9], as well as [Nik20, §B.3].
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