Characterizations of discs via weighted means

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Abstract

New theorems characterizing analytically discs in the Euclidean plane $\mathbb{R}^2$ are proved. Weighted mean value properties of solutions to the modified Helmholtz equation and harmonic functions are used for this purpose. The presence of a logarithmic weight diminish coefficients in the mean value identities. A weighted mean is also valid for solutions of the Helmholtz equation.

1 Introduction

In this note, we consider a weighted mean value property of real-valued solutions to the two-dimensional modified Helmholtz equation

$$\nabla^2 v - \mu^2 v = 0, \quad \mu \in \mathbb{R} \setminus \{0\};$$

(1)

$\nabla = (\partial_1, \partial_2)$ is the gradient operator, $\partial_i = \partial/\partial x_i$. This property is used for a new analytic characterization of discs in the Euclidean plane $\mathbb{R}^2$.

Unfortunately, it is not commonly known that solutions of (1) are called panharmonic (or $\mu$-panharmonic) functions by analogy with harmonic functions solving the Laplace equation. This convenient abbreviation coined by Duffin [1] will be used in what follows.

In the extensive survey article [6], the authors treat various mean value properties of harmonic and caloric functions, saying nothing about panharmonic ones. Meanwhile, Duffin [1] derived the mean value identity over circumferences in $\mathbb{R}^2$ for solutions of (1) in 1970. His result is closely related to the obtained in this note, and so it is formulated below, but before that we introduce some notation.

Let $x = (x_1, x_2)$ be a point in $\mathbb{R}^2$, by $D_r(x) = \{ y \in \mathbb{R}^2 : |y - x| < r \}$ we denote the open disc of radius $r$ centred at $x$. The disc is called admissible with respect to a domain $\Omega \subset \mathbb{R}^2$ provided $\overline{D_r(x)} \subset \Omega$, whereas $\partial D_r(x) = S_r(x)$ is called the admissible circumference in this case. If $\Omega$ has a finite Lebesgue measure and a function $f$ is integrable over $\Omega$ (continuous on $\Omega$), then

$$M(f, \Omega) = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx \quad \left( M(f, S_r(x)) = \frac{1}{2\pi r} \int_{S_r(x)} f(y) \, dS_y \right)$$

is its area mean value over $\Omega$ (its mean value over an admissible circumference, respectively); here $|\Omega|$ is the area of $\Omega$.

Now, we are in a position to recall some results related to those obtained in this note. We begin with mean value properties analogous to those valid for harmonic functions; the latter were reviewed in [6].
Theorem 1 (Duffin [1], Kuznetsov [3]). Let $\Omega$ be a domain in $\mathbb{R}^2$. If $v$ is panharmonic in $\Omega$, then
\[ M(v, S_r(x)) = a^o(\mu r) v(x) \quad \text{and} \quad M(v, D_r(x)) = a^*(\mu r) v(x) \quad (2) \]
for every admissible disc $D_r(x)$; here $a^o(t) = I_0(t)$ and $a^*(t) = 2t^{-1}I_1(t)$; $I_\nu$ denotes the modified Bessel function of order $\nu$.

A new proof of the first identity (2) (originally due to Duffin) was obtained in [3] in the $m$-dimensional ($m \geq 2$) setting, whereas the second one was derived in [3] for the first time.

The aim of this note is twofold: (i) to consider weighted mean value properties of a panharmonic and harmonic functions, and to compare them with those without weight; (ii) to prove inverse properties (the term coined in [2] became widely accepted) of the weighted mean value identities. One of them is similar the following theorem, whose $m$-dimensional version was obtained recently.

Theorem 2 (Kuznetsov [4]). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $r > 0$ be such that $\pi r^2 = |\Omega|$. If for some $\mu > 0$ and a point $x_0 \in \Omega$ the identity $v(x_0) a^*(\mu r) = M(v, \Omega)$ holds for every positive function $v$ satisfying equation (1) in $\Omega_r = \Omega \cup \{x \in \partial \Omega : \partial D_r(x)\}$, then $\Omega = D_r(x_0)$.

2 Weighted mean value property

The standard proof of mean value properties for harmonic functions usually starts with the identity
\[ 2\pi w(x) = \int_{D_r(x)} \nabla^2 w(y) \log |x - y| \, dy + \int_{S_r(x)} w(y) \frac{\partial}{\partial n_y} \log |x - y| \, dS_y. \]
for an admissible disc $D_r(x)$. Along with the mean value properties, it also implies
\[ w(x) = \frac{1}{2\pi r} \int_{S_r(x)} w(y) \, dS_y - \frac{1}{2\pi} \int_{D_r(x)} \nabla^2 w(y) \log \frac{r}{|x - y|} \, dy \quad (3) \]
This yields the following weighted version of the second identity (2).

Theorem 3. Let $\Omega$ be a domain in $\mathbb{R}^2$. If $v$ is panharmonic in $\Omega$, then
\[ a(\mu r) v(x) = \frac{1}{\pi r^2} \int_{D_r(x)} v(y) \log \frac{r}{|x - y|} \, dy, \quad a(t) = \frac{2 |I_0(t) - 1|}{t^2}. \quad (4) \]
for every admissible disc $D_r(x)$.

Proof. Substituting $v$ into (3) and taking into account the first identity (2) and equation (1) in the first and second terms, respectively, on the right-hand side, we obtain
\[ 2v(x) = 2a^o(\mu r) v(x) - \frac{(\mu r)^2}{\pi r^2} \int_{D_r(x)} v(y) \log \frac{r}{|x - y|} \, dy, \]
after multiplying by two both sides. Now (4) follows by rearranging. $\square$

The behaviour of the logarithmic weight in the area mean value identity (4) is quite simple: it is a positive function of $y$ within $D_r(x)$, growing from zero attained for $y \in S_r(x)$ to infinity as $|x - y| \to 0$, and is negative for $y \notin D_r(x)$. 

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In view of the behaviour of $I_0$, one obtains that
\[ a(0) = \lim_{t \to +0} a(t) = 1/2, \]
and $a(t)$ increases monotonically from this value to infinity similar to $a^\mu(t)$. Moreover,
\[ a^\mu(t) - a(t) = 2 [t I_1(t) - I_0(t) + 1]/t^2 > 0 \quad \text{for all } t \in [0, \infty). \]
An immediate consequence of (4) and the fact that $a(t) > 1/2$ for $t > 0$ is the following.

**Corollary 1.** Let $\Omega$ be a domain in $\mathbb{R}^2$, and let $v$ be a $\mu$-panharmonic in $\Omega$ for some $\mu > 0$. If $v \geq 0$ does not vanish identically in $\Omega$, then
\[ \frac{1}{2} v(x) < \frac{1}{\pi r^2} \int_{D_r(x)} v(y) \log \frac{r}{|x - y|} \, dy \quad (5) \]
for every admissible disc $D_r(x)$.

**Remark 1.** In the limit $\mu \to +0$, one obtains the Laplace equation from (1), whereas identity (4) turns into
\[ \frac{1}{2} v(x) = \frac{1}{\pi r^2} \int_{D_r(x)} v(y) \log \frac{r}{|x - y|} \, dy, \quad (6) \]
where $x$ is a point of a bounded domain $\Omega$ and $D_r(x)$ is an admissible disc. Therefore, it is reasonable to conjecture that (4) constitutes a weighted mean value identity for a harmonic function $v$. To the best author’s knowledge, this identity has not been proven yet.

**Remark 2.** According to the first identity (2), every nonnegative, $\mu$-panharmonic function is subharmonic. In view of Corollary 1 and Remark 1, one might expect that inequality (5) holds for nonnegative subharmonic functions which do not vanish identically.

### 3 Characterizations of discs

The following analogue of Theorem 2 is based on weighted means of positive panharmonic functions.

**Theorem 4.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $r > 0$ be such that $|\Omega| \geq \pi r^2$. If for a point $x_0 \in \Omega$ and some $\mu > 0$ the weighted mean value identity
\[ a(\mu r) v(x_0) = \frac{1}{|\Omega|} \int_{\Omega} v(y) \log \frac{r}{|x_0 - y|} \, dy \quad (7) \]
holds for every positive function $v$ satisfying equation (1) in $\Omega_r = \Omega \cup \bigcup_{x \in \partial \Omega} D_r(x)$, then $\Omega = D_r(x_0)$.

Prior to proving Theorem 4, we notice that the radially symmetric function
\[ V(x) = I_0(\mu|x|), \quad x \in \mathbb{R}^2, \]
monotonically increases from one to infinity as $|x|$ goes from zero to infinity. Also, it solves equation (1) in $\mathbb{R}^2$; indeed, the Poisson’s integral for $I_0$ (see [7], p. 223) yields that
\[ V(x) = \frac{2}{\pi} \int_0^1 \frac{\cosh(\mu|x|s)}{(1 - s^2)^{1/2}} \, ds, \]
which is easy to differentiate, thus verifying (1).
Proof of Theorem 4. Without loss of generality, we suppose that the domain \( \Omega \) is located so that \( x_0 \) coincides with the origin. Let us show that the assumption \( \Omega \neq D_r(0) \) leads to a contradiction. For this purpose we consider bounded open sets \( G_i = \Omega \setminus D_r(0) \) (nonempty by the assumption about \( \Omega \) and \( r \)) and \( G_e = D_r(0) \setminus \Omega \) (possibly empty).

Taking into account that \( V(0) = 1 \), we write (5) for \( V \) as follows:

\[
|\Omega| a(\mu r) = \int_{\Omega} V(y) \log \frac{r}{|y|} \, dy, 
\]

(8)

Since identity (2) holds for \( V \) over \( D_r(0) \), we write it in the same way:

\[
\pi r^2 a(\mu r) = \int_{D_r(0)} V(y) \log \frac{r}{|y|} \, dy. 
\]

(9)

Subtracting (9) from (8), we obtain

\[
[|\Omega| - \pi r^2] a(\mu r) = \int_{G_i} V(y) \log \frac{r}{|y|} \, dy - \int_{G_e} V(y) \log \frac{r}{|y|} \, dy. 
\]

Here the difference on the right-hand side is negative. Indeed, \( V > 0 \) everywhere, whereas \( \log(r/|y|) < 0 \) on \( G_i \neq \emptyset \), because \( |y| > r \) there. Hence, the first term is negative. If \( G_e \neq \emptyset \), then the second integral is positive because \( \log(r/|y|) > 0 \) on \( G_e \), where \( |y| < r \). On the other hand, the expression on the left-hand side is nonnegative. The obtained contradiction proves the theorem.

Remark 3. Comparing Theorems 2 and 4, we observe two points worth mentioning.

First, Theorem 4 is essentially two-dimensional, whereas the \( m \)-dimensional (\( m \geq 2 \)) version of Theorem 2 is proved in [4].

Second, the common feature of both theorems is that their proofs involve the function \( V \).

It occurs that discs are characterized in the same way via harmonic functions provided identity (6) is true for them.

Theorem 5. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain, and let \( r > 0 \) be such that \( |\Omega| \geq \pi r^2 \). Suppose that identity (6) is true for functions harmonic in \( \Omega_r \). If for a point \( x_0 \in \Omega \) the weighted mean value identity

\[
\frac{1}{2} v(x_0) = \frac{1}{|\Omega|} \int_{\Omega} v(y) \log \frac{r}{|x_0 - y|} \, dy
\]

holds for every positive function \( v \) harmonic in \( \Omega_r \), then \( \Omega = D_r(x_0) \).

To prove this assertion one has to repeat literally the proof of Theorem 4, but using \( H(x) \equiv 1 \) instead of \( V(x) \). Identity (6) is valid for \( H(x) \) as one readily finds by a direct calculation.

4 Weighted mean of solutions to the Helmholtz equation

For real-valued solutions of the Helmholtz equation

\[
\nabla^2 u + \lambda^2 u = 0, \quad \lambda \in \mathbb{R} \setminus \{0\},
\]

(10)

the mean value identities analogous to those in Theorem 1 are valid. Of course, \( I_0 \) and \( I_1 \) must be replaced by the Bessel functions \( J_0 \) and \( J_1 \), respectively, in the formulae for \( a^* \) and \( a^* \).
see [3], pp. 675 and 677. However, the inverse mean value property analogous to Theorem 2 is more complicated for solutions of (11), because a restriction on the domain’s size is imposed; see [5], Remark 2.1. It is required since the function $U(x) = J_0(\lambda|x|)$, used in the proof instead of $V(x)$, is monotonic only for $\lambda|x| \in (0, j_{1.1})$; here $j_{1.1}$ is the first positive zero of $J_1$.

Therefore, it is interesting to find out how the logarithmic weight changes the mean value identity for discs in this case, and whether it allows to improve the inverse property obtained in [5]. It is clear that minor changes in the proof of Theorem 3 yield the following.

**Theorem 6.** Let $\Omega$ be a domain in $\mathbb{R}^2$. If $u$ is a solution of (10) in $\Omega$, then

$$\tilde{a}(\lambda r) u(x) = \frac{1}{\pi r^2} \int_{D_r(x)} u(y) \log \frac{r}{|x - y|} \, dy, \quad \tilde{a}(t) = \frac{2 \left[1 - J_0(t)\right]}{t^2},$$

(11)

for every admissible disc $D_r(x)$.

The behaviour of $\tilde{a}$ is as follows: $\tilde{a}(0) = \lim_{t \to +0} \tilde{a}(t) = 1/2$, whereas $\tilde{a}(t)$ asymptotes zero as $t \to +\infty$ decreasing nonmonotonically, but remaining positive. The latter property of $\tilde{a}(t)$ distinguishes it from $2t^{-1}J_1(t)$—the coefficient in the identity for discs with the mean value without weight. The latter coefficient has infinitely many zeros.

**Remark 4.** As in Remark 1, the Laplace equation results from (10) in the limit $\lambda \to +0$, whereas identity (11) turns into

$$\frac{1}{2} u(x) = \frac{1}{\pi r^2} \int_{D_r(x)} u(y) \log \frac{r}{|x - y|} \, dy,$$

thus confirming the conjecture made in Remark 1 that this equality constitutes a weighted mean value identity for harmonic functions.

Thus, the role of weight is essential. However, it is easy to establish that the restriction on the size of domain, under which the inverse theorem analogous to that obtained in [5] is valid, is even stronger when identity (11) is used, and so it has no advantage.

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