Dynamics of kink-soliton solutions for $2 + 1$-dimensional sine-Gordon equation

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Abstract

In this paper we study the dynamics of explicit solutions of $2 + 1$-dimensional (2D) sine-Gordon equation. The Darboux transformation is applied to the associated linear eigenvalue problem to construct nontrivial solutions of 2D sine-Gordon equation in terms of ratios of determinants. We obtained a generalized expression for $N$-fold transformed dynamical variable which enables us to calculate explicit expressions of nontrivial solutions. In order to explore the dynamics of kink soliton solutions explicit expressions one- and two-soliton solutions are derived for particular column solutions. Different profiles of kink-kink and, kink and anti-kink interactions are illustrated for a different parameters and arbitrary functions. First-order bound state solution is also displayed in our work.

Keywords: Integrable systems, sine-Gordon equation, Solitons, Darboux transformation

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1 Introduction

A famous hyperbolic nonlinear partial differential equation is the well-known sine-Gordon equation. The 1D sine-Gordon equation (SGE) reads [1]

$$\frac{\partial^2 s}{\partial Y^2} - \frac{\partial^2 s}{\partial T^2} = \sin s,$$

(1.1)

where \(s(Y,T)\) and \(Y,T\) denote real scalar field and, usual spatial and time coordinates, respectively. The 1D SGE is an important integrable equation in the theory of nonlinear integrable equations. Using light-cone coordinates \(y = \frac{1}{2} (Y + T), t = \frac{1}{2} (Y - T)\), the 1D SGE (1.1) can be written as

$$\frac{\partial^2 s}{\partial y \partial t} = \sin s.$$

(1.2)

The 1D SGE (1.1) was first time appeared in 1862 during the study of surfaces with constant negative curvature as the Gauss-Codazzi equation for surfaces of the curvatures [2]. In 1875, a Swedish mathematician Albert Victor Bäcklund proposed a transformation which permits him to construct a new surface with negative curvature from the known [3]. In fact, the Bäcklund transformation is a differential relation between two different solutions (or surfaces) of the SGE. The Bäcklund transformation allows to compute an infinite many solutions of the SGE from simplest known solution. Indeed, the Bäcklund transformation obeys the theorem of permutability which facilitates in the computation of higher-order non-trivial solutions.

In 1938, Yakov Frenkel and Tatiana Kontorova proposed a mathematical model

$$\frac{d^2 s_n}{dt^2} - C (s_{n+1} + s_{n-1} - 2s_n) = \sin s_n,$$

(1.3)

also known as Frenkel-Kontorova (FK) model which describe the structure and dynamics of a crystal lattice near a dislocation [4]. Here \(C\) is the elastic constant. Under the continuum limit equation (1.3) reduces to 1D SGE (1.1).

There has been an increasing interest in the study of SGE due to inherent rich mathematical structures. It is a relativistic 1D integrable equation with several applications in different fields both mathematical and applied sciences. Ablowitz, Kaup, Newell and Segure explored the integrability of the SGE (1.1) by
means of the inverse scattering transform method and obtained explicit solutions \[5\]-\[6\]. The solvability of SGE was explored by using Bäcklund transformation almost one century earlier the discovery of the integrability of this equation via inverse scattering transform method \[3\]. Lamb in 1967 computed multi-soliton solutions of 1D SGE (1.1) by employing Bäcklund transformation \[7\]. The SGE (1.1) also shares various interesting properties, for example, the existence of infinitely many conserved quantities, multi-soliton solutions and Painlevé property. The 1D SGE is an important simplest integrable models like the Korteweg de Vries and the nonlinear Schrödinger (NLS) equations. The SGE has been attracted a great deal of attention in different areas of applied sciences such as propagation of ultra-short pulse in resonant laser medium \[7\], condensed matter physics \[8, 9, 10, 11, 12\], elementary particle physics \[13, 14, 15, 16, 17\], biology \[18, 19, 20, 21, 22\] and fluid mechanics \[23\].

The 1D SGE (1.2) can also be expressed as the integrability condition of the following linear equations \[5\]

\[
\frac{\partial \Phi}{\partial y} = P_1 \Phi, \quad \frac{\partial \Phi}{\partial t} = P_2 \Phi, \quad (1.4)
\]

where \(\Phi(y, t; \lambda) = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)^T\) is a column solution. The matrices \(P_1(y, t; \lambda)\) and \(P_2(y, t; \lambda)\) are given by

\[
P_1(y, t; \lambda) = \left( \begin{array}{cc} \frac{1}{2} \frac{\partial s}{\partial y} & \lambda \\ \lambda & -\frac{1}{2} \frac{\partial s}{\partial y} \end{array} \right), \quad P_2(y, t; \lambda) = \frac{1}{4\lambda} \left( \begin{array}{cc} 0 & e^{is} \\ -e^{-is} & 0 \end{array} \right), \quad (1.5)
\]

where \(\lambda\) is a real/complex-valued spectral parameter and \(i = \sqrt{-1}\). The integrability condition of the linear system (1.4), that is \(\frac{\partial^2 \Phi}{\partial Y \partial T} = \frac{\partial^2 \Phi}{\partial T \partial Y}\) yields the zero-curvature condition \(\frac{\partial P_2}{\partial T} - \frac{\partial P_1}{\partial Y} + [P_1, P_2] = 0 \iff \left[ \frac{\partial}{\partial Y} - P_1, \frac{\partial}{\partial T} - P_2 \right] = 0\), here \([\, , \,]^{-1}\) represents matrix commutator.

Integrable equations are extensively used in almost all branches of physics such as the field theory, theory of condensed matter physics, plasma physics, optics, geophysics and nuclear and particle physics, as well as in the other branches of science such as chemistry, biology and communications. During the past few decades integrable equations in 1D have been studied from different view points, for example, construction of multi-soliton solutions (by using Bäcklund-Darboux transformation, dressing method, Hirota’s direct method), conserved quantities, Hamiltonian structures, Painlevé analysis \[24, 25, 26\]. Higher dimensional integrable equations have attracted as great deal of attention in recent years. The Nizhnik-Novikov-Veselov equation \[27\] and Davey-Stewartson equation \[28\] are well known 2D generalizations of the NLS.
and the KdV equations, respectively. Recently, Wang et al. [29] have derived 2D SGE from the AKNS system and obtained nontrivial explicit solutions. The 2D integrable equations have studied recent years [30, 31, 32, 33]. As we have already mentioned above that the integrable 1D SGE explains different physical phenomenon, including the propagation of fluxons in a junction between two superconductors (also known as Josephson junction), the motion of coupled pendulum, dislocations in crystals and dynamics of DNA (Deoxyribonucleic acid). Different integrability aspects such as existence of infinitely many conserved quantities, Painlevé property, multi-soliton solutions, Hirota bilinearization of the 1D SGE have been investigated. These applications motivated us to explore the different integrable features of 2D SGE recently studied by Wang et al [29]. The Darboux transformation in an elegant solution generating technique among other solution generating techniques such as inverse scattering transform method, Hirota direct method, Bäcklund transformation and dressing method [24, 25, 26]. In this paper, we study the construction of multi-soliton solutions of 2D SGE by employing Darboux transformation to the associated AKNS scheme. The expression of $N$-fold dynamical variables are expressed in form of determinants. We also obtain explicit expressions of one-, two-soliton solutions. The dynamics of single-soliton (or kink/anti-kink) and different type of interactions of two-soliton solutions such as kink and kink, kink and anti-kink, and breather solutions have been investigated in details for different choices of parameters and arbitrary functions.

The rest of this is organized as follows. In the section 2, we express the 2D SGE as an integrability condition of linear system. In section 3, we define a matrix Darboux transformation and apply to the solution of the linear system and obtain multi-soliton solutions in term of ratio of determinants. In section 4, we obtain explicit expressions of one-, two-kink and breather soliton solutions of 2D SGE. Last section is dedicated for conclusion and open problems.

2 Linear system and Darboux transformation for 2-dimensional sine-Gordon equation

The 2D SGE is given by

\[ \frac{\partial^2 s}{\partial x^2} - \frac{\partial^2 s}{\partial x \partial y} - \frac{\partial^2 s}{\partial x \partial t} + \frac{\partial^2 s}{\partial y \partial t} = \sin s, \tag{2.1} \]
where \( s = s(x, y, t) \) is the field of the ultra-short optical pulses.

The 2D SGE (2.1) is equivalent to the consistency condition of the following matrix-valued linear system

\[
\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x} + F_1 \Phi, \quad \frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial x} + F_2 \Phi.
\]  

(2.2)

The coefficient matrices \( F_1(x, y, t; \lambda) \) and \( F_2(x, y, t; \lambda) \) are defined as

\[
F_1 = \begin{pmatrix}
\frac{i}{2} \left( \frac{\partial s}{\partial y} - \frac{\partial s}{\partial x} \right) \\
\lambda
\end{pmatrix},
F_2 = \frac{1}{4\lambda} \begin{pmatrix} 0 & e^{is} \\
e^{-is} & 0 \end{pmatrix},
\]  

(2.3)

the consistency condition yields (2.1).

Under appropriate transformations the 2D SGE (2.1) can also be reexpressed in the form of 1D SGE. If we define \( \xi = x + y + t, \eta = y, \tau = t \) and \( s = S(\eta, \tau) \), the 2D SGE (2.1) reduces to

\[
\frac{\partial^2 S}{\partial \eta \partial \tau} = \sin S,
\]  

(2.4)

which is the exactly the case of 1D SGE.

The most important solution generating technique is the Darboux transformation [24, 26]. Darboux transformation is a type of gauge transformation such that associated linear eigenvalue problem remains invariant. The one-fold Darboux transformation for the system (2.2) can be defined as

\[
\phi_1[1] = \lambda \phi_2 - \alpha^{(1)} \phi_1, \\
\phi_2[1] = \lambda \phi_1 - \beta^{(1)} \phi_2,
\]  

(2.5)-(2.6)

the unknown coefficients \( \alpha^{(1)} \) and \( \beta^{(1)} \) may be obtained from the following conditions

\[
\phi_1[1]|_{\lambda=\lambda_1, \phi_1=\phi_1^{(1)}, \phi_2=\phi_2^{(1)}} = 0, \\
\phi_2[1]|_{\lambda=\lambda_1, \phi_1=\phi_1^{(1)}, \phi_2=\phi_2^{(1)}} = 0.
\]  

(2.7)-(2.8)

The above conditions allow us to write one-fold transformation (2.5)-(2.6) as

\[
\phi_1[1] = \frac{\det \left( \begin{pmatrix} \lambda \phi_2 \\
\lambda_1 \phi_2^{(1)} \phi_1^{(1)} \\
\phi_1^{(1)} \phi_1 \end{pmatrix} \end{pmatrix}}{\det \left( \begin{pmatrix} \phi_1^{(1)} \\
\phi_2^{(1)} \phi_2 \\
\phi_2^{(1)} \phi_2 \end{pmatrix} \end{pmatrix}},
\]  

(2.9)

\[
\phi_2[1] = \frac{\det \left( \begin{pmatrix} \lambda \phi_1 \\
\lambda_1 \phi_1^{(1)} \phi_2^{(1)} \\
\phi_1^{(1)} \phi_1 \end{pmatrix} \end{pmatrix}}{\det \left( \begin{pmatrix} \phi_1^{(1)} \\
\phi_2^{(1)} \phi_2 \\
\phi_2^{(1)} \phi_2 \end{pmatrix} \end{pmatrix}},
\]  

(2.10)
here \( \phi_1^{(1)} \) and \( \phi_2^{(1)} \) denote the particular solutions to the linear system (2.2). The linear system (2.2) is covariant under the action of Darboux transformation (2.9)-(2.10), that is,

\[
\begin{align*}
\frac{\partial \phi_1^{[1]}}{\partial y} &= \frac{\partial \phi_1^{[1]}}{\partial x} + \frac{i}{2} \left( \frac{\partial s^{[1]}}{\partial y} - \frac{\partial s^{[1]}}{\partial x} \right) \phi_1^{[1]} + \lambda \phi_2^{[1]}, \\
\frac{\partial \phi_2^{[1]}}{\partial y} &= \frac{\partial \phi_2^{[1]}}{\partial x} + \lambda \phi_1^{[1]} - \frac{i}{2} \left( \frac{\partial s^{[1]}}{\partial y} - \frac{\partial s^{[1]}}{\partial x} \right) \phi_2^{[1]}, \\
\frac{\partial \phi_1^{[1]}}{\partial t} &= \frac{\partial \phi_1^{[1]}}{\partial x} + \frac{1}{4} e^{i s^{[1]}} \phi_2^{[1]}, \\
\frac{\partial \phi_2^{[1]}}{\partial t} &= \frac{\partial \phi_2^{[1]}}{\partial x} + \frac{1}{4} e^{-i s^{[1]}} \phi_1^{[1]}. 
\end{align*}
\]

(2.11)

By substitution of scalar functions \( \phi_1^{[1]} \) and \( \phi_2^{[1]} \) from (2.9)-(2.10) in (2.11)-(2.12), we obtain

\[
s^{[1]} = s - 2i \ln \left( \frac{\phi_2^{(1)}}{\phi_1^{(1)}} \right). 
\]

(2.13)

The two-fold Darboux transformation is defined as

\[
\begin{align*}
\phi_1^{[2]} &= \lambda \phi_2^{[1]} - \alpha^{(2)}[1] \phi_1^{[1]} = \lambda^2 \phi_1 - \theta^1 \lambda \phi_2 - \theta^0 \phi_1, \\
\phi_2^{[2]} &= \lambda \phi_1^{[1]} - \beta^{(2)}[1] \phi_2^{[1]} = \lambda^2 \phi_2 - \theta^1 \lambda \phi_1 - \theta^0 \phi_2,
\end{align*}
\]

(2.14)

(2.15)

with \( \theta^1 = \beta^{(1)} + \alpha^{(2)}[1], \ \theta^0 = -\alpha^{(2)}[1] \alpha^{(1)} \) and \( \theta^1 = \alpha^{(1)} + \beta^{(2)}[1], \ \theta^0 = -\beta^{(2)}[1] \beta^{(1)}. \) The pair of unknown coefficients \( \theta^0, \theta^1 \) and \( \theta^0, \theta^1 \) can be determined from the following conditions

\[
\begin{align*}
\phi_1^{[2]}|_{\lambda=\lambda_k, \phi_1=\phi_1^{(k)}, \phi_2=\phi_2^{(k)}} &= 0, \\
\phi_2^{[2]}|_{\lambda=\lambda_k, \phi_1=\phi_1^{(k)}, \phi_2=\phi_2^{(k)}} &= 0,
\end{align*}
\]

(2.16)

(2.17)

for \( k = 1, 2. \) The above conditions reduce to following systems of linear equations

\[
\begin{align*}
\begin{pmatrix} \phi_1^{(1)} \\ \phi_2^{(2)} \end{pmatrix} &\begin{pmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} \\ \lambda_2^{(1)} & \lambda_2^{(2)} \end{pmatrix} \begin{pmatrix} \theta^0 \\ \theta^1 \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} \phi_1^{(1)} \\ \lambda_2^{(2)} \phi_1^{(2)} \end{pmatrix}, \\
\begin{pmatrix} \phi_2^{(1)} \\ \phi_2^{(2)} \end{pmatrix} &\begin{pmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} \\ \lambda_2^{(1)} & \lambda_2^{(2)} \end{pmatrix} \begin{pmatrix} \theta^0 \\ \theta^1 \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} \phi_2^{(1)} \\ \lambda_2^{(2)} \phi_2^{(2)} \end{pmatrix}.
\end{align*}
\]

(2.18)

(2.19)

On substitution of unknown coefficients in system of equations given by (2.14)-(2.15), we have

\[
\phi_1^{[2]} = \frac{\det \begin{pmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} & \phi_1^{(1)} \\ \lambda_1^{(2)} & \lambda_2^{(1)} & \phi_1^{(2)} \\ \lambda_1^{(1)} & \lambda_2^{(2)} & \phi_1^{(3)} \end{pmatrix}}{\det \begin{pmatrix} \lambda_1^{(1)} & \phi_1^{(1)} \\ \lambda_2^{(2)} & \phi_1^{(2)} \end{pmatrix}},
\]

(2.20)
\[ \phi_2[2] = \frac{\det \begin{pmatrix} \lambda^2 \phi_2 & \lambda \phi_1 & \phi_2 \\ \lambda^2 \phi_2^{(1)} & \lambda_1 \phi_1^{(1)} & \phi_2^{(1)} \\ \lambda^2 \phi_2^{(2)} & \lambda_2 \phi_1^{(2)} & \phi_2^{(2)} \end{pmatrix}}{\det \begin{pmatrix} \lambda_1 \phi_1^{(1)} & \phi_2^{(1)} \\ \lambda_2 \phi_1^{(2)} & \phi_2^{(2)} \end{pmatrix}}. \quad (2.21) \]

The two-fold transformed solutions given by (2.20)-(2.21) also satisfy the linear system (2.2), that is,
\[
\begin{align*}
\frac{\partial \phi_1[2]}{\partial y} &= \frac{\partial \phi_1[2]}{\partial x} + \frac{i}{2} \left( \frac{\partial s[2]}{\partial y} - \frac{\partial s[2]}{\partial x} \right) \phi_1[2] + \lambda \phi_2[2], \\
\frac{\partial \phi_2[2]}{\partial y} &= \frac{\partial \phi_2[2]}{\partial x} + \lambda \phi_1[2] - \frac{i}{2} \left( \frac{\partial s[2]}{\partial y} - \frac{\partial s[2]}{\partial x} \right) \phi_2[2], \\
\frac{\partial \phi_1[2]}{\partial t} &= \frac{\partial \phi_1[2]}{\partial x} + \frac{1}{4\lambda} e^{is[2]} \phi_2[2], \\
\frac{\partial \phi_2[2]}{\partial t} &= \frac{\partial \phi_2[2]}{\partial x} + \frac{1}{4\lambda} e^{-is[2]} \phi_1[2].
\end{align*} \quad (2.22)
\]

Using equations (2.20)-(2.21) in equations (2.22)-(2.23), we obtain
\[ s[2] = s + 2i \ln \frac{\det \begin{pmatrix} \lambda_1 \phi_1^{(1)} & \phi_2^{(1)} \\ \lambda_2 \phi_1^{(2)} & \phi_2^{(2)} \end{pmatrix}}{\det \begin{pmatrix} \lambda_1 \phi_1^{(1)} & \phi_2^{(1)} \\ \lambda_2 \phi_1^{(2)} & \phi_2^{(2)} \end{pmatrix}}. \quad (2.24) \]

Similarly, three-fold Darboux transformation is defined as
\[
\begin{align*}
\phi_1[3] &= \lambda \phi_2[2] - \alpha^{(3)}[2] \phi_1[2] = \lambda^3 \phi_2 - \theta^2 \lambda^2 \phi_1 - \theta^1 \lambda \phi_2 - \theta^0 \phi_1, \\
\phi_2[2] &= \lambda \phi_1[2] - \beta^{(3)}[2] \phi_2[2] = \lambda^3 \phi_1 - \theta^2 \lambda^2 \phi_2 - \theta^1 \lambda \phi_1 - \theta^0 \phi_2.
\end{align*} \quad (2.25, 2.26)
\]

The set of unknown coefficients \( \{\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5\} \) can be determined from the following conditions
\[
\begin{align*}
\phi_1[3] |_{\lambda=\lambda_k, \phi_1=\phi_k^{(1)}, \phi_2=\phi_k^{(2)}} &= 0, \\
\phi_2[3] |_{\lambda=\lambda_k, \phi_1=\phi_k^{(1)}, \phi_2=\phi_k^{(2)}} &= 0
\end{align*} \quad (2.27, 2.28)
\]

for \( k = 1, 2, 3 \). The above conditions can also be written as
\[
\begin{pmatrix} \phi_1^{(1)} \\ \phi_1^{(2)} \\ \phi_1^{(3)} \end{pmatrix} \begin{pmatrix} \lambda_1 \phi_1^{(1)} \\ \lambda_2 \phi_1^{(2)} \\ \lambda_3 \phi_1^{(3)} \end{pmatrix} \begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 \phi_1^{(1)} \\ \lambda_2^3 \phi_2^{(2)} \\ \lambda_3^3 \phi_2^{(3)} \end{pmatrix}, \quad (2.29)
\]
After substitution of unknown coefficients the expression of three-fold transformation (2.25)-(2.26) can also be expressed as ratio of determinants

\[
\phi_1[3] = \frac{\det \begin{pmatrix}
\lambda_1^3 & \lambda^2 \phi_1 & \lambda \phi_2 \\
\phi_1^{(1)} & \lambda_1^2 & \lambda_1 \phi_2^{(1)} \\
\phi_2^{(1)} & \lambda_2 \phi_2^{(1)} & \lambda_1 \phi_2^{(2)} \\
\phi_3^{(1)} & \lambda_3 \phi_2^{(1)} & \lambda_3 \phi_2^{(2)} \\
\end{pmatrix}}{\det \begin{pmatrix}
\lambda_1^3 & \lambda^2 \phi_1 & \lambda \phi_2 \\
\phi_1^{(1)} & \lambda_1^2 & \lambda_1 \phi_2^{(1)} \\
\phi_2^{(1)} & \lambda_2 \phi_2^{(1)} & \lambda_1 \phi_2^{(2)} \\
\phi_3^{(1)} & \lambda_3 \phi_2^{(1)} & \lambda_3 \phi_2^{(2)} \\
\end{pmatrix}}
\]

(2.31)

\[
\phi_2[3] = \frac{\det \begin{pmatrix}
\lambda_1^3 & \lambda^2 \phi_2 & \lambda \phi_1 \\
\phi_1^{(1)} & \lambda_1^2 & \lambda_1 \phi_2^{(1)} \\
\phi_2^{(2)} & \lambda_2 \phi_2^{(2)} & \lambda_2 \phi_2^{(2)} \\
\phi_3^{(2)} & \lambda_3 \phi_2^{(2)} & \lambda_3 \phi_2^{(2)} \\
\end{pmatrix}}{\det \begin{pmatrix}
\lambda_1^3 & \lambda^2 \phi_2 & \lambda \phi_1 \\
\phi_1^{(1)} & \lambda_1^2 & \lambda_1 \phi_2^{(1)} \\
\phi_2^{(2)} & \lambda_2 \phi_2^{(2)} & \lambda_2 \phi_2^{(2)} \\
\phi_3^{(2)} & \lambda_3 \phi_2^{(2)} & \lambda_3 \phi_2^{(2)} \\
\end{pmatrix}}
\]

(2.32)

Three-fold transformed solutions \(\phi_1[3]\) and \(\phi_2[3]\) satisfy the linear system (2.2) such as

\[
\begin{align*}
\frac{\partial \phi_1[3]}{\partial y} &= \frac{\partial \phi_1[3]}{\partial x} + \frac{i}{2} \left( \frac{\partial s[3]}{\partial y} - \frac{\partial s[3]}{\partial x} \right) \phi_1[3] + \lambda \phi_2[3], \\
\frac{\partial \phi_2[3]}{\partial y} &= \frac{\partial \phi_2[3]}{\partial x} + \lambda \phi_1[3] - \frac{i}{2} \left( \frac{\partial s[3]}{\partial y} - \frac{\partial s[3]}{\partial x} \right) \phi_2[3], \\
\frac{\partial \phi_1[3]}{\partial t} &= \frac{\partial \phi_1[3]}{\partial x} + \frac{1}{4\lambda} e^{is[3]} \phi_2[3], \\
\frac{\partial \phi_2[3]}{\partial t} &= \frac{\partial \phi_2[3]}{\partial x} + \frac{1}{4\lambda} e^{-is[3]} \phi_1[3].
\end{align*}
\]

(2.33)

Using equations (2.31)-(2.32) in (2.33)-(2.34), we obtain

\[
\begin{align*}
\frac{\partial \phi_1[3]}{\partial y} &= \frac{\partial \phi_1[3]}{\partial x} + \frac{i}{2} \left( \frac{\partial s[3]}{\partial y} - \frac{\partial s[3]}{\partial x} \right) \phi_1[3] + \lambda \phi_2[3], \\
\frac{\partial \phi_2[3]}{\partial y} &= \frac{\partial \phi_2[3]}{\partial x} + \lambda \phi_1[3] - \frac{i}{2} \left( \frac{\partial s[3]}{\partial y} - \frac{\partial s[3]}{\partial x} \right) \phi_2[3], \\
\frac{\partial \phi_1[3]}{\partial t} &= \frac{\partial \phi_1[3]}{\partial x} + \frac{1}{4\lambda} e^{is[3]} \phi_2[3], \\
\frac{\partial \phi_2[3]}{\partial t} &= \frac{\partial \phi_2[3]}{\partial x} + \frac{1}{4\lambda} e^{-is[3]} \phi_1[3].
\end{align*}
\]

(2.34)

Using equations (2.31)-(2.32) in (2.33)-(2.34), we obtain

\[
s[3] = s - 2i \ln \begin{pmatrix}
\lambda_1^3 & \lambda_1 \phi_1^{(1)} & \lambda \phi_2^{(1)} \\
\phi_1^{(1)} & \lambda_1^2 & \lambda_1 \phi_2^{(1)} \\
\phi_2^{(2)} & \lambda_2 \phi_2^{(2)} & \lambda_1 \phi_2^{(2)} \\
\phi_3^{(2)} & \lambda_3 \phi_2^{(2)} & \lambda_3 \phi_2^{(2)} \\
\end{pmatrix}
\]

(3.5)

\[
s[3] = s - 2i \ln \begin{pmatrix}
\lambda_1^3 & \lambda_1 \phi_1^{(1)} & \lambda \phi_2^{(1)} \\
\phi_1^{(1)} & \lambda_1^2 & \lambda_1 \phi_2^{(1)} \\
\phi_2^{(2)} & \lambda_2 \phi_2^{(2)} & \lambda_1 \phi_2^{(2)} \\
\phi_3^{(2)} & \lambda_3 \phi_2^{(2)} & \lambda_3 \phi_2^{(2)} \\
\end{pmatrix}
\]

(3.5)
In what follows, we would like to generalize our results for $N$-times iteration of Darboux transformation. For this purpose, first take $N = 2K$, the Darboux transformation can be factorize as

$$
\phi_1[2K] = \lambda^{2K} \phi_1 - \theta^{2K-1} \lambda^{2K-1} \phi_2 - \ldots - \theta^1 \lambda \phi_2 - \theta^0 \phi_1,
$$

$$
\phi_2[2K] = \lambda^{2K} \phi_2 - \theta^{2K-1} \lambda^{2K-1} \phi_1 - \ldots - \theta^1 \lambda \phi_1 - \theta^0 \phi_2,
$$

where the unknown coefficients can be computed from the following conditions

$$
\phi_1[2K]|_{\lambda=\lambda_k, \phi_1=\phi_1^{(k)}, \phi_2=\phi_2^{(k)}} = 0,
$$

$$
\phi_2[2K]|_{\lambda=\lambda_k, \phi_1=\phi_1^{(k)}, \phi_2=\phi_2^{(k)}} = 0,
$$

for $k = 1, 2, \ldots, 2K$. The above conditions reduce to following systems of linear equations

$$
\begin{pmatrix}
\phi_1^{(2K-1)} & \lambda_1 \phi_1^{(1)} & \ldots & \lambda_{2K-2} \phi_1^{(1)} & \lambda_{2K-1} \phi_1^{(1)} \\
\phi_2^{(2K-1)} & \lambda_2 \phi_2^{(1)} & \ldots & \lambda_{2K-2} \phi_2^{(1)} & \lambda_{2K-1} \phi_2^{(1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_1^{(2K-1)} & \lambda_{2K-1} \phi_2^{(2K-1)} & \ldots & \lambda_{2K-2} \phi_2^{(2K-1)} & \lambda_{2K-1} \phi_2^{(2K-1)} \\
\phi_2^{(2K-1)} & \lambda_{2K} \phi_2^{(2K-1)} & \ldots & \lambda_{2K-2} \phi_2^{(2K-1)} & \lambda_{2K-1} \phi_2^{(2K-1)} \\
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_1^{(2K-1)} \\
\phi_2^{(2K-1)}
\end{pmatrix}
= \begin{pmatrix}
\theta^0 \\
\theta^1 \\
\vdots \\
\theta_{2K-2} \\
\theta_{2K-1}
\end{pmatrix},
$$

$$
\begin{pmatrix}
\phi_1^{(1)} & \lambda_1 \phi_1^{(1)} & \ldots & \lambda_{2K-2} \phi_1^{(1)} & \lambda_{2K-1} \phi_1^{(1)} \\
\phi_2^{(1)} & \lambda_2 \phi_2^{(1)} & \ldots & \lambda_{2K-2} \phi_2^{(1)} & \lambda_{2K-1} \phi_2^{(1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_1^{(1)} & \lambda_{2K-1} \phi_2^{(1)} & \ldots & \lambda_{2K-2} \phi_2^{(1)} & \lambda_{2K-1} \phi_2^{(1)} \\
\phi_2^{(1)} & \lambda_{2K} \phi_2^{(1)} & \ldots & \lambda_{2K-2} \phi_2^{(1)} & \lambda_{2K-1} \phi_2^{(1)} \\
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_1^{(1)} \\
\phi_2^{(1)}
\end{pmatrix}
= \begin{pmatrix}
\theta^0 \\
\theta^1 \\
\vdots \\
\theta_{2K-2} \\
\theta_{2K-1}
\end{pmatrix}.
$$

Using the values of unknown coefficients the $2K$-fold transformation becomes (2.36)-(2.37) as

$$
\phi_1[2K] = \det \begin{pmatrix}
\lambda^{2K} \phi_1 & \lambda^{2K-1} \phi_2 & \ldots & \lambda \phi_2 & \phi_1 \\
\lambda_1 \phi_1^{(1)} & \lambda_1^{2K-1} \phi_2^{(1)} & \ldots & \lambda_1 \phi_2^{(1)} & \phi_1^{(1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{2K} \phi_1^{(2K-1)} & \lambda_{2K}^{2K-1} \phi_2^{(2K-1)} & \ldots & \lambda_{2K} \phi_2^{(2K-1)} & \phi_1^{(2K-1)} \\
\lambda_{2K} \phi_1^{(2K-1)} & \lambda_{2K}^{2K-1} \phi_2^{(2K-1)} & \ldots & \lambda_{2K} \phi_2^{(2K-1)} & \phi_1^{(2K-1)} \\
\end{pmatrix},
$$

$$
\phi_2[2K] = \det \begin{pmatrix}
\lambda^{2K} \phi_1 & \lambda^{2K-1} \phi_2 & \ldots & \lambda \phi_2 & \phi_1 \\
\lambda_1 \phi_1^{(1)} & \lambda_1^{2K-1} \phi_2^{(1)} & \ldots & \lambda_1 \phi_2^{(1)} & \phi_1^{(1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{2K} \phi_1^{(2K-1)} & \lambda_{2K}^{2K-1} \phi_2^{(2K-1)} & \ldots & \lambda_{2K} \phi_2^{(2K-1)} & \phi_1^{(2K-1)} \\
\lambda_{2K} \phi_1^{(2K-1)} & \lambda_{2K}^{2K-1} \phi_2^{(2K-1)} & \ldots & \lambda_{2K} \phi_2^{(2K-1)} & \phi_1^{(2K-1)} \\
\end{pmatrix}.
$$
The $2K$-fold transformed solutions given by (2.42)-(2.43) satisfy the linear system (2.2)

$$\phi_2[2K] = \det \begin{pmatrix}
\lambda^{2K}\phi_2 & \lambda^{2K-1}\phi_1 & \cdots & \lambda\phi_1 & \phi_2 \\
\lambda^{2K} & \lambda^{2K-1} & \cdots & \lambda & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda^{2K-1}\phi_2^{(2K-1)} & \lambda^{2K-1}\phi_1^{(2K-1)} & \cdots & \lambda\phi_1^{(2K-1)} & \phi_2^{(2K-1)} \\
\lambda^{2K}\phi_2^{(2K)} & \lambda^{2K-1}\phi_1^{(2K)} & \cdots & \lambda\phi_1^{(2K)} & \phi_2^{(2K)} \\
\end{pmatrix}$$

(2.43)

The $2K$-fold transformed solutions given by (2.42)-(2.43) satisfy the linear system (2.2)

$$\frac{\partial\phi_1[2K]}{\partial y} = \frac{\partial\phi_1[2K]}{\partial x} + \frac{i}{2} \left( \frac{\partial s[2K]}{\partial y} - \frac{\partial s[2K]}{\partial x} \right) \phi_1[2K] + \lambda\phi_2[2K],$$

$$\frac{\partial\phi_2[2K]}{\partial y} = \frac{\partial\phi_2[2K]}{\partial x} + \lambda\phi_1[2K] - \frac{i}{2} \left( \frac{\partial s[2K]}{\partial y} - \frac{\partial s[2K]}{\partial x} \right) \phi_2[2K],$$

(2.44)

$$\frac{\partial\phi_1[2K]}{\partial t} = \frac{\partial\phi_1[2K]}{\partial x} + \frac{1}{4\lambda} e^{-i\lambda[2K]} \phi_2[2K],$$

$$\frac{\partial\phi_2[2K]}{\partial t} = \frac{\partial\phi_2[2K]}{\partial x} + \frac{1}{4\lambda} e^{-i\lambda[2K]} \phi_1[2K].$$

(2.45)

From the covariance of the Darboux transformation, one can obtain $2K$-fold transformed dynamical variable as

$$s[2K] = s + 2i\ln \det \begin{pmatrix}
\lambda^{2K-1}\phi_1^{(1)} & \lambda^{2K-2}\phi_2^{(1)} & \cdots & \lambda\phi_1^{(1)} & \phi_2^{(1)} \\
\lambda^{2K-1} & \lambda^{2K-2} & \cdots & \lambda & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda^{2K-1}\phi_1^{(2K-1)} & \lambda^{2K-2}\phi_2^{(2K-1)} & \cdots & \lambda\phi_1^{(2K-1)} & \phi_2^{(2K-1)} \\
\lambda^{2K}\phi_1^{(2K)} & \lambda^{2K-1}\phi_2^{(2K)} & \cdots & \lambda\phi_1^{(2K)} & \phi_2^{(2K)} \\
\end{pmatrix}.$$

(2.46)

Similarly, for $N = 2K + 1$, we have

$$\phi_1[2K+1] = \lambda^{2K+1}\phi_2 - \theta^{2K}\lambda^{2K}\phi_1 - \cdots - \theta^{1}\lambda\phi_2 - \theta^{0}\phi_1,$$

(2.47)
\[
\phi_2^{[2K+1]} = \lambda^{2K+1} \phi_1 - \theta^{2K} \lambda^{2K} \phi_2 - \ldots - \theta^1 \lambda \phi_1 - \theta^0 \phi_2,
\]

with
\[
\phi_1^{[2K+1]}|_{\lambda=\lambda_k, \phi_1=\phi_1^{(k)}, \phi_2=\phi_2^{(k)}} = 0,
\]
\[
\phi_2^{[2K+1]}|_{\lambda=\lambda_k, \phi_1=\phi_1^{(k)}, \phi_2=\phi_2^{(k)}} = 0,
\]

for \(k = 1, 2, \ldots, 2K+1\). The above conditions can also be expressed in matrix notation as

\[
\begin{pmatrix}
\phi_1^{(1)} & \lambda_1 \phi_1^{(1)} & \ldots & \lambda_2^{K-1} \phi_1^{(1)} & \lambda_2^{K} \phi_1^{(1)} \\
\phi_2^{(2)} & \lambda_2 \phi_2^{(2)} & \ldots & \lambda_2^{K-1} \phi_2^{(2)} & \lambda_2^{K} \phi_2^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_1^{(2K)} & \lambda_{2K} \phi_1^{(2K)} & \ldots & \lambda_{2K}^{2K-1} \phi_1^{(2K)} & \lambda_{2K}^{2K} \phi_1^{(2K)} \\
\phi_2^{(2K+1)} & \lambda_{2K+1} \phi_2^{(2K+1)} & \ldots & \lambda_{2K+1}^{2K-1} \phi_2^{(2K+1)} & \lambda_{2K+1}^{2K} \phi_2^{(2K+1)}
\end{pmatrix}
\begin{pmatrix}
\theta^0 \\
\theta^1 \\
\vdots \\
\theta^{2K-1} \\
\theta^{2K}
\end{pmatrix}
= \begin{pmatrix}
\lambda_1^{2K+1} \phi_1^{(1)} \\
\lambda_2^{2K+1} \phi_2^{(1)} \\
\vdots \\
\lambda_{2K}^{2K+1} \phi_1^{(2K)} \\
\lambda_{2K+1}^{2K+1} \phi_2^{(2K+1)}
\end{pmatrix}.
\]

Using the values of unknown coefficients the \((2K+1)\)-fold transformation (2.47)-(2.48) can be expressed as

\[
\phi_1^{[2K+1]} = \det \begin{pmatrix}
\lambda_2^{2K+1} \phi_2^{(1)} & \lambda_2^{2K} \phi_1^{(1)} & \ldots & \lambda_2^{2K} \phi_1^{(1)} \\
\lambda_1^{2K+1} \phi_2^{(1)} & \lambda_1^{2K} \phi_1^{(1)} & \ldots & \lambda_1^{2K} \phi_1^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{2K}^{2K+1} \phi_2^{(1)} & \lambda_{2K}^{2K} \phi_1^{(1)} & \ldots & \lambda_{2K}^{2K} \phi_1^{(1)} \\
\lambda_{2K+1}^{2K+1} \phi_2^{(1)} & \lambda_{2K+1}^{2K} \phi_1^{(1)} & \ldots & \lambda_{2K+1}^{2K} \phi_1^{(1)}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\lambda_2^{2K} \phi_1^{(1)} \\
\lambda_2^{2K} \phi_1^{(2)} \\
\vdots \\
\lambda_{2K}^{2K} \phi_1^{(2)} \\
\lambda_{2K+1}^{2K} \phi_1^{(2)}
\end{pmatrix}
\begin{pmatrix}
\lambda_2^{2K+1} \phi_1^{(1)} \\
\lambda_2^{2K+1} \phi_1^{(2)} \\
\vdots \\
\lambda_{2K+1}^{2K+1} \phi_1^{(2)}
\end{pmatrix}.
\]
Similarly the $(2K+1)$-fold transformed solutions given by (2.53)-(2.54) satisfy the linear system (2.2)
\[ \frac{\partial \phi_1[2K+1]}{\partial y} = \frac{\partial \phi_1[2K+1]}{\partial x} + \frac{i}{2} \left( \frac{\partial s[2K+1]}{\partial y} - \frac{\partial s[2K+1]}{\partial x} \right) \phi_1[2K+1] + \lambda \phi_2[2K+1], \]
\[ \frac{\partial \phi_2[2K+1]}{\partial y} = \frac{\partial \phi_2[2K+1]}{\partial x} + \lambda \phi_1[2K+1] - \frac{i}{2} \left( \frac{\partial s[2K+1]}{\partial y} - \frac{\partial s[2K+1]}{\partial x} \right) \phi_2[2K+1], \]
\[ \frac{\partial \phi_1[2K+1]}{\partial t} = \frac{\partial \phi_1[2K+1]}{\partial x} + \frac{1}{4} \lambda e^{-is[2K+1]} \phi_2[2K+1], \]
\[ \frac{\partial \phi_2[2K+1]}{\partial t} = \frac{\partial \phi_2[2K+1]}{\partial x} + \frac{1}{4} \lambda e^{is[2K+1]} \phi_1[2K+1]. \]

Using equations (2.53)-(2.54) in (2.55)-(2.56), we obtain
\[ s[2K+1] = s + 2i \ln \left( \frac{\lambda^{2K+1} \phi_1}{\lambda^{2K} \phi_2} \right) \]
\[ = \lambda^{2K+1} \phi_1 - \lambda^{2K} \phi_2 \cdots \lambda \phi_2 \phi_1 \phi_2. \]

In the following section, we shall derive explicit expressions of first two nontrivial solutions and demonstrate our results graphically for different choices of parameters and arbitrary functions.
In order to compute higher-order nontrivial solutions of the 2D SGE we should start from a trivial solution, i.e., \( s = 0 \). Under this choice the linear system (2.2) becomes

\[
\begin{align*}
\frac{\partial}{\partial y} \left( \phi_1 \right) &= \frac{\partial}{\partial x} \left( \phi_2 \right) + \left( \begin{array}{c}
0 \\
\lambda \\
0
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right), \\
\frac{\partial}{\partial t} \left( \phi_1 \right) &= \frac{\partial}{\partial x} \left( \phi_2 \right) + \left( \begin{array}{c}
0 \\
\frac{1}{\lambda} \\
0
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right).
\end{align*}
\]

Integration of above linear system yields \( \phi_1(x, y; t; \lambda) = Ae^\chi + Be^{-\chi} \), \( \phi_2(x, y; t; \lambda) = Ae^\chi - Be^{-\chi} \),

where \( \chi = \lambda y + \frac{1}{\lambda^2} t + g(x, y, t) \) and \( g(x, y, t) \) is a function of the form \( g(x, y, t) = f(x + y + t) \).

### 3.1 First-order solutions

Substituting \( \phi_1^{(1)} \) and \( \phi_2^{(1)} \), the particular solutions of the linear system at \( \lambda_1 \) in equation (2.13), we get first-order nontrivial solution of 2D SGE given by

\[
s[1] = -2i \ln \left( \frac{A_1 e^{\lambda_1 y + \frac{1}{\lambda^2} t + f(x+y+t)} - B_1 e^{\lambda_1 y - \frac{1}{\lambda^2} t - f(x+y+t)}}{A_1 e^{\lambda_1 y + \frac{1}{\lambda^2} t + f(x+y+t)} + B_1 e^{\lambda_1 y - \frac{1}{\lambda^2} t - f(x+y+t)}} \right),
\]

or

\[
s[1] = -2i \ln \left( \frac{A_1 e^{2\lambda_1 y + \frac{1}{\lambda^2} t + 2f} - B_1}{A_1 e^{2\lambda_1 y + \frac{1}{\lambda^2} t + 2f} + B_1} \right),
\]

is the required explicit expression of first-order nontrivial solution of 2D SGE. Here \( f(x, y, t) \) is an arbitrarily chosen function that satisfies equation (3.1). Profiles of one-kink soliton solution of 2D SGE (2.1) are shown in figures (1)-(5) for \( A_1 = -B_2 = 1 \) and \( A_2 = B_1 = 1 \). Fig. (1) represents a kink soliton plotted for \( \lambda_1 = 0.9 \). Fig. (2) indicates a propagation of one-kink on a parabolic path for \( \lambda_1 = 1.5 \). Figs. (3) illustrates s-shaped kinks for \( \lambda_1 = 2 \). Similarly Fig. 4 represents bending of one-kink from its path during propagation in nonlinear medium whereas Fig. 5 display oscillating one-kink for \( \lambda_1 = 3 \). (All choices of arbitrary function mentioned in Figures captions)

### 3.2 Second-order nontrivial solutions

Substituting \( \phi_1^{(1)} = A_1 e^{\theta_1} + B_1 e^{-\theta_1} \), \( \phi_2^{(1)} = A_2 e^{\theta_1} - B_2 e^{-\theta_1} \), \( \phi_1^{(2)} = A_3 e^{\theta_2} + B_3 e^{-\theta_2} \), \( \phi_2^{(2)} = A_4 e^{\theta_2} - B_4 e^{-\theta_2} \), \( A_1 = -A_3 = -B_2 = B_4 = 1 \) and \( A_2 = A_4^* = B_1 = B_3^* = i \), in equation (2.24), we have,

\[
s[2] = 4 \tan^{-1} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right) \sinh (\theta_1 - \theta_2) \cosh (\theta_1 + \theta_2),
\]

(3.5)
Figure 1: One-kink solution (3.4) of 2D SGE (2.1) for $\lambda_1 = 0.9$ and $f(x+y+t) = x + y + t$ at $x = 0$.

Figure 2: One-kink solution (3.4) of 2D SGE (2.1) for $\lambda_1 = 1.5$ and $f(x+y+t) = (x + y + t)^2$ at $x = 0$. 
Figure 3: One-kink solution (3.4) of 2D SGE (2.1) for \(\lambda_1 = 2\) and \(f(x+y+t) = \frac{x+y+t-(x+y+t)^3}{2}\) at \(x = 0\).

Figure 4: One-kink solution (3.4) of 2D SGE (2.1) for \(\lambda_1 = 2\) and \(f(x+y+t) = (x+y+t)^3\) at \(x = 0\).
Figure 5: One-kink solution (3.4) of 2D SGE (2.1) for \( \lambda_1 = 3 \) and \( f(x+y+t) = \sin(x+y+t) + \cos(x+y+t) \) at \( x = 0 \).

where \( \theta_1 = \lambda_1 y + \frac{1}{4\lambda_1} t + f_1(x + y + t) \) and \( \theta_2 = \lambda_2 y + \frac{1}{4\lambda_2} t + f_2(x + y + t) \).

Kinks display two types of interactions, namely, repulsive and attractive. Interaction of kinks are seeking great attention due to its wide implications in many branches of science such as particle physics and biological sciences. Two-kink solutions usually appear in the problems whose respective potential is controlled by some parameters instead of fields. In kink-kink interactions when two kinks approaches each other, they exert repulsive force on each other and reflect back with an equal but opposite velocities to their initial velocities. Since kink soliton exhibits particle-like characteristics in this way interaction of kink and anti-kink can be perceived as an interaction between particle and anti-particle. In attractive type of interactions, kink and anti-kink attempt to annihilate when they get closer. Interactions of kinks for various choices of arbitrary functions are exhibited in Figs. (6), (7) and (8). Fig. (6) represents an attractive type of interaction of two kinks for \( \lambda_1 = 1 \) and \( \lambda_2 = -1.09 \). Similarly, Figs. (7) and (8) also demonstrate attractive type of interactions obtained for the parameters \( \lambda_1 = 1, \lambda_2 = 0.2 \) and \( \lambda_1 = 1, \lambda_2 = 0.9 \), respectively. Where as Fig. (9) displays a repulsive type of interaction of two kinks for parameters \( \lambda_1 = 0.9 \) and \( \lambda_2 = -0.8 \).

When two kinks fused together in such a way that they may not preserve their shape, consequently a bound state is formed, which is known as breather solutions. An explicit expression of first-order breather is obtained by taking \( \lambda_2 = \lambda_1^* \) in expression (3.5), i.e.,

\[
s[2] = 4 \tan^{-1} \left( \frac{\Re(\lambda_1)}{\Im(\lambda_1)} \frac{\sinh(\theta_1 - \theta_1^*)}{\cosh(\theta_1 + \theta_1^*)} \right). \tag{3.6}
\]

Different profiles of (3.6) are represented in Figs. (10)-(12) for spectral parameter \( \lambda_1 = 0.2 + 0.6i, \lambda_1 = 0.2 + 0.7i, \lambda_1 = 0.2 + 0.7i \) and \( \lambda_1 = 0.5 + 0.5i \) respectively.
Figure 6: Attractive type of interaction of kinks of 2D SGE (2.1) for $f_1 = f_2 = x + y + t$ at $x = 0$.

Figure 7: Attractive type of interaction of kinks of 2D SGE (2.1) for $f_1 = (x + y + t)$ and $f_2 = \sin(x + y + t)$ at $x = 0$. 

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Figure 8: Attractive type of interaction of kinks of 2D SGE (2.1) for $f_1 = \cos(x + y + t)$ and $f_2 = \sin(x + y + t)$ at $x = 0$.

Figure 9: Repulsive type of interaction of kinks of 2D SGE (2.1) for $f_1 = \cos(x + y + t)$ and $f_2 = \sin(x + y + t)$ at $x = 0$. 
Figure 10: Breather solution of 2D SGE (2.1) at $f_1 = 0$ and $f_2 = x + y + t$.

Figure 11: Breather solution of 2D SGE (2.1) at $f_1 = 0$ and $f_2 = \sin(x + y + t)$.
4 Conclusions

In this article, we have derived a general formula for \( N \)-soliton solution of sine-Gordon equation (2.1) in 2 + 1-dimensions from sequential application of Darboux transformation on associated linear eigenvalue problem. Explicit expressions of one- and two-soliton solutions have computed for the our model. In order to illustrate our results we have presented dynamics of single and different profile two-soliton interactions for different choices of spectral parameters as well as for the arbitrary functions. We are expecting that solutions obtained in this article will be helpful in explaining various physical phenomena in higher dimensions, for example, the dynamics of DNA, dynamics of crystalline latices near dislocation etc. There are several interesting directions, for example, one can explore the multi-soliton solutions of some well-known integrable equations such as nonlinear Schrödinger equation, the KdV equation, short pulse equation in higher dimensions. Similarly one can also calculate higher order degenerate solutions of sine-Gordon equation (2.1) in 2 + 1-dimensions. We will address these open problems in near future.

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