The $\theta$-exact Seiberg-Witten maps at the $e^3$ order

Josip Trampetic$^{1,2}$ and Jiangyang You$^2$

1. Max-Planck-Institut für Physik, (Werner-Heisenberg-Institut), Föhringer Ring 6, D-80805 München, Germany
2. Rudjer Bošković Institute, Division of Theoretical Physics, P.O.Box 180, HR-10002 Zagreb, Croatia
E-mail: josipt@rex.irb.hr, youjiangyang@gmail.com

Abstract: We study two distinct $\theta$-exact Seiberg-Witten maps of the gauge parameter, gauge field and the gauge field strengths in the noncommutative U(1) quantum field theory and define ambiguities between them. The measures of ambiguities/freedom for the gauge field strength are given in terms of deformation parameters $\kappa_g$ and $\kappa'_g$, for two, and three photon fields, i.e. up to the $e^3$ order, respectively. We discuss field strength as a function of two $\theta$-exact Seiberg-Witten maps, and the $(\kappa_g, \kappa'_g)$-deformation parameter space.

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1. Introduction

The $\theta$-exact Seiberg-Witten (SW) map is an old and new subject in the noncommutative (NC) field theory (NCFT) nowadays. Some results emerged immediately after the map itself was discovered \[1\] was discovered \[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\]. Applications to the perturbative noncommutative quantum field theories (NCQFT) started several years later. Till now it has been shown to be of great value for developing non-trivial variations of the NCQFT from both theoretical and phenomenological perspectives \[20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\]. Recently a systematic construction of the $\theta$-exact SW map on Moyal space, and for arbitrary gauge groups, was proposed as a power series expansion over the coupling constant \[36\], with the antisymmetric deformation parameter $\theta^{ij}$ being a constant. This could trigger many further applications in the near future.

It is long recognized that SW map has plenty of ambiguity/freedom/redundancy. The ambiguity between the NC gauge field expansions \[7, 8, 14, 15\] can be characterized by, by looking at the composition of the first SW map expansion \{\(\Lambda_1(a_\mu, \lambda), A_{1_\mu}(a_\mu)\)\} and the inverse \{\(\lambda_2(A_\mu, \Lambda), a_{2_\mu}(A_\mu)\)\} of the second SW map expansion \{\(A_2(a_\mu, \lambda), A_{2_\mu}(a_\mu)\)\}. Consistency conditions leads to the following equality in the case of the U(1) gauge theory

$$\partial_\mu \lambda_2(A_{1_\mu}(a_\mu), \Lambda_1(a_\mu, \lambda)) = \delta_\lambda a_2(A_{1_\mu}(a_\mu)).$$  \hspace{1cm} (1.1)
It is further pointed out [14, 15] that such composition bears the following general form

\begin{align}
a_2(A_{1\mu}(a_\mu)) &= a_\mu + X_\mu(a_\mu) + \partial_\mu Y(a_\mu), \\
\lambda_2(A_{1\mu}(a_\mu), \Lambda_1(a_\mu, \lambda)) &= \lambda + \delta_\lambda Y(a_\mu),
\end{align}

(1.2)

where \( \delta_\lambda X_\mu(a_\mu) = 0 \).

A large class of SW map ambiguities (field redefinitions) can be constructed iteratively by adding commutative gauge covariant terms at the \( n \)-th power of \( \theta^{ij} \) to the known solution \( A^{(n)}_\mu \) up to the same order [21]. It is also found that such redefinitions contribute to the field strength and action, which could help to cancel certain divergences in the perturbative/quantum loop computations [24, 37, 38, 39, 40, 41]. Yet, it remains an open question how to find a \( \theta \)-exact counterpart for this procedure since the iteration is based solely on the powers of \( \theta^{ij} \). An alternative which extracts some ambiguity directly from the gauge field strength solution at the \( e^2 \) order was suggested in [42].

In this article we compare, up to the cubic order of the coupling constant \( e \), two different \( \theta \)-exact Seiberg-Witten map expansions for the NC U(1) gauge theory: One obtained from Seiberg-Witten differential equation, and the other by inverting an early \( \theta \)-exact inverted SW map solution. We give a closed form expression for the SW map ambiguity between these two maps and show that this ambiguity/freedom could contribute to the field strength. We also extend the procedure in [42] to extract some of the \( \theta \)-exact gauge field strength ambiguities up to the \( e^3 \) order from each of the solutions.

The paper is structured as follows: In the following section we describe two \( \theta \)-exact Seiberg-Witten map solutions up to the \( e^3 \) order. Sections 3 and 4 are devoted to the computation/presentation/discussions of the \( \theta \)-exact SW map ambiguity and of the \( \theta \)-exact gauge field strength at the \( e^3 \) order, respectively. Section 5 is devoted to discussions and conclusions. In this article the capital letters denote noncommutative objects, and the small letters denote commutative objects.

2. Two \( \theta \)-exact Seiberg-Witten expansions up to the \( e^3 \) order

The first powerful method to obtain a \( \theta \)-exact SW map expansion for noncommutative gauge theories on Moyal space is by solving the SW Differential Equations (SWDE) [1, 26]. For the U(1) gauge theory, in terms of the NC gauge parameter (\( \Lambda \)), the NC gauge field \( (A_\mu) \), and the NC gauge field strength \( (F_{\mu\nu}) \), these equations read

\begin{align}
\frac{d}{dt} \Lambda(x) &= -\frac{1}{4} \theta^{ij} \left\{ A_i \star \partial_j \Lambda \right\}, \\
\frac{d}{dt} A_\mu(x) &= \frac{1}{4} \theta^{ij} \left\{ A_i \star \partial_j A_\mu + F_{j\mu} \right\}, \\
\frac{d}{dt} F_{\mu\nu}(x) &= \frac{1}{4} \theta^{ij} \left[ \left\{ F_{\mu i} \star F_{\nu j} \right\} - \left\{ A_i \star \left( D^\star_{\partial j} + \partial_j \right) F_{\mu\nu} \right\} \right],
\end{align}

(2.1)

(2.2)

(2.3)

where the Moyal star(\( \star \))-product with an additional parameter \( t \) being defined as:

\[ (\phi \star_t \psi)(x) = e^{it^{ij} \partial_i \partial_j} \phi(x + \eta)\psi(y + \xi) \bigg|_{\eta, \xi \to 0} \equiv \phi(x) \tilde{e} \theta^{ij} \partial_i \partial_j \psi(x). \]

(2.4)
Note that this parameter \( t \) would be absorbed into the definition of \( \theta^{ij} \) when not needed in the rest of the article. The noncommutative covariant derivative is defined in the following way \( D^i_j = \partial_j - i[A_j \star \cdot] \). By imposing initial conditions \([36]\)

\[
\Lambda_{\text{SWDE}}(x) = e\lambda + O(e^2),
\]

\[(2.5)\]

\[
A_{\mu\text{SWDE}}(x) = e a_{\mu} + O(e^2),
\]

\[(2.6)\]

one can easily solve equations (2.1) and (2.2) at the \( e^2 \) order and obtain the following solutions

\[
\Lambda_{\text{SWDE}}(x) = e\lambda - \frac{e^2}{2} \theta^{ij} a_i \star_2 \partial_j \lambda + O(e^3),
\]

\[(2.7)\]

\[
A_{\mu\text{SWDE}}(x) = e a_{\mu} - \frac{e^2}{2} \theta^{ij} a_i \star_2 (\partial_j a_{\mu} + f_{j\mu}) + O(e^3),
\]

\[(2.8)\]

where the \( \star_2 \)-product is defined as follows

\[
\phi(x) \star_2 \psi(x) = \left. \frac{\sin \left( \frac{\partial \theta \partial \lambda}{2} \right)}{\frac{\partial \theta \partial \lambda}{2}} \phi(x_1) \psi(x_2) \right|_{x_1 = x_2 = x}.
\]

\[(2.9)\]

Then, the next order of the SW differential equations can be written down recursively

\[
\frac{d}{dt} \Lambda^e(x) = \frac{e^3}{8} \theta^{ij} \theta^{kl} \left\{ a_i \star_1 \partial_j (a_k \star_2 \partial_l \lambda) \right\} + \left\{ a_k \star_2 (\partial_l a_i + f_{li}) \star_1 \partial_j \lambda \right\},
\]

\[(2.10)\]

\[
\frac{d}{dt} A^e_{\mu}(x) = \frac{e^3}{8} \theta^{ij} \theta^{kl} \left\{ a_i \star_1 \left( \partial_j (a_k \star_2 (\partial_l a_{\mu} + f_{l\mu})) - 2(f_{jk} \star_2 f_{j\mu} - a_i \star_2 \partial_l f_{j\mu}) \right) \right\}
\]

\[
+ \left\{ (a_k \star_2 (\partial_l a_i + f_{li})) \star_1 (\partial_j a_{\mu} + f_{j\mu}) \right\}.
\]

\[(2.11)\]

The above \( \star_2 \)-product is defined analogous to the \( \star_1 \)-product, i.e.

\[
\phi(x) \star_2 \psi(x) = \left. \frac{\sin \left( t \frac{\partial \theta \partial \mu}{2} \right)}{\frac{\partial \theta \partial \mu}{2}} \phi(x_1) \psi(x_2) \right|_{x_1 = x_2 = x}.
\]

\[(2.12)\]

Note that we do not scale the denominator because it would be canceled with extra \( \theta^{ij} \) in the differential equations \([36]\).

Since equations (2.10) and (2.11), involve only \( \{ f \star_1 (g \star_2 h) \} \) type terms, in accord with technique from \([36]\) one can immediately introduce new generalized \( \star_3 \)-product,

\[
[fgh]_{\star_3} = \int_0^t dt' \left\{ f \star_1' (g \star_2' h) \right\} = \left( \cos \left[ t \left( \frac{\partial_l \theta \partial \mu}{2} + \frac{\partial_i \theta \partial \lambda}{2} - \frac{\partial_i \theta \partial \mu}{2} \right) \right] - 1 \right) \frac{\partial_i \theta \partial \mu}{2} \frac{\partial_i \theta \partial \lambda}{2} \right)
\]

\[
- \cos \left[ \frac{t}{2} \left( \frac{\partial_l \theta \partial \mu}{2} + \frac{\partial_i \theta \partial \lambda}{2} + \frac{\partial_i \theta \partial \mu}{2} \right) \right] - 1 \right) \frac{\partial_i \theta \partial \mu}{2} \frac{\partial_i \theta \partial \lambda}{2} \right)
\]

\[
f \otimes g \otimes h,
\]

\[(2.13)\]
having thus a universal expression. In this notation we find the $\theta$-exact solutions for the SW differential equations up to the $e^3$ order:

$$\Lambda_{\text{SWDE}}(x) = \lambda e - \frac{e^2}{2} \theta^{ij} a_i \star_2 \partial_j \lambda + \frac{e^3}{8} \theta^{ij} \theta^{kl} \left[ a_i \partial_j (a_k \partial_l \lambda) - \partial_i \lambda a_k (\partial_l a_j + f_{ij}) \right]_{xy} + O(e^4), \quad (2.14)$$

$$A_{\mu \text{SWDE}}(x) = a_{\mu} - \frac{e^2}{2} \theta^{ij} a_i \star_2 (\partial_j a_\mu + f_{j\mu}) + \frac{e^3}{8} \theta^{ij} \theta^{kl} \left[ a_i \partial_j (a_k \partial_l a_\mu + f_{j\mu}) \right]_{xy'} - 2 \left[ a_i (f_{jk} f_{l\mu} - a_k \partial_l f_{j\mu}) \right]_{xy'} + \left[ (\partial_j a_\mu + f_{j\mu}) a_k (\partial_l a_i + f_{li}) \right]_{xy'} + O(e^4). \quad (2.15)$$

Another type of $\theta$-exact Seiberg-Witten map expansion [26] was obtained by inverting the solutions from [13], hence producing the so called Non-SW Differential Equation (NSWDE) solutions for the NC gauge parameter and the NC gauge field, respectively

$$\Lambda_{\text{NSWDE}}(x) = \lambda e - \frac{e^2}{2} \theta^{ij} a_i \star_2 \partial_j \lambda + \frac{e^3}{8} \theta^{ij} \theta^{kl} \left[ \frac{1}{2} (a_k \star_2 (\partial_l a_i + f_{li}) \right]_{xy} \star_2 \partial_j \lambda + \frac{1}{2} a_i \star_2 \partial_j (a_k \star_2 \partial_l \lambda) - \frac{e^3}{8} \theta^{ij} \theta^{kl} \left[ \partial_k \partial_l \lambda a_j a_l + \partial_k \lambda a_i \partial_l a_j \right]_{xy} + O(e^4), \quad (2.16)$$

$$A_{\mu \text{NSWDE}}(x) = a_{\mu} - \frac{e^2}{2} \theta^{ij} a_i \star_2 (\partial_j a_\mu + f_{j\mu}) + \frac{e^3}{8} \theta^{ij} \theta^{kl} \left[ \frac{1}{2} (a_k \star_2 (\partial_l a_i + f_{li}) \right]_{xy} \star_2 (\partial_j a_\mu + f_{j\mu}) + a_i \star_2 (\partial_j (a_k \star_2 (\partial_l a_\mu + f_{j\mu}) - \partial_{\mu} (a_k \star_2 \partial_l a_j + f_{j\mu}) - \frac{1}{2} a_i \star_2 (\partial_k a_j \star_2 \partial_l a_\mu) \right]_{xy} + \frac{e^3}{8} \theta^{ij} \theta^{kl} \left[ a_i \partial_k a_\mu (\partial_j a_l + f_{jl}) - \partial_k \partial_l a_\mu a_j a_l - 2 \partial_k a_\mu \partial_l a_j a_l \right]_{xy} + O(e^4). \quad (2.17)$$

At the $e^2$ order this gives the same solutions as in equations (2.14) and (2.15), however the $e^3$ order starts to show some difference.

The above completely symmetric $\star_3$-product is defined as follows [3]

$$[f(x)g(x)h(x)]_{\star_3} = \left[ \sin \left( \frac{\partial x_1 \partial x_2}{2} \right) \sin \left( \frac{\partial x_1 \partial x_3}{2} \right) \right]_{x_1=x} f(x_1)g(x_2)h(x_3) + \{ 1 \leftrightarrow 2 \}. \quad (2.18)$$

3. The $\theta$-exact Seiberg-Witten map ambiguity at the $e^3$ order

In this section we compare two $\theta$-exact SW maps up to the $e^3$ order given in section 2. Following the arguments in [14, 17], we consider the composition of one of the SW map and the inverse of the other. Now, since (2.16) and (2.17) were derived from $\theta$-exact inverted
SW map expansion in [5], we choose to use the following original inverted map here,
\[
\lambda_{\text{NSWDE}}(A_\mu, \Lambda) = \Lambda + \frac{1}{2} \theta^{ij} \left( A_i \ast_2 \partial_j \Lambda + \theta^{kl} \left[ \partial_i \partial_k \Lambda A_j A_l + \partial_i \Lambda A_k \partial_j A_l \right]_{*3} \right) + \mathcal{O}(A^3) \Lambda, \tag{3.1}
\]
\[
a_{\mu\text{NSWDE}}(A_\mu) = A_\mu + \frac{1}{2} \theta^{ij} \left( A_i \ast_2 \left( \partial_j A_\mu + F_{j\mu} \right) \right) + \theta^{kl} \left[ - A_i \partial_k A_\mu \left( \partial_j A_l + F_{jl} \right) + \partial_i \partial_k A_\mu A_j A_l + \partial_k A_i \partial_j A_l A_\mu \right]_{*3} + \mathcal{O}(A^4). \tag{3.2}
\]

Expanding the compositions \(\lambda_{\text{NSWDE}}(A_{\mu\text{SWDE}}(a_\mu), \Lambda_{\text{SWDE}}(a_\mu, \lambda))\) and \(a_{\mu\text{NSWDE}}(A_{\mu\text{SWDE}}(a_\mu))\) up to the \(e^3\) order, we have found
\[
\lambda_{\text{NSWDE}}(A_{\mu\text{SWDE}}(a_\mu), \Lambda_{\text{SWDE}}(a_\mu, \lambda)) = e \lambda(x) + \Lambda_{\text{SWDE}}(x) - \Lambda_{\text{NSWDE}}(a_\mu, \lambda) + \mathcal{O}(e^4), \tag{3.3}
\]
\[
a_{\mu\text{NSWDE}}(A_{\mu\text{SWDE}}(a_\mu)) = e a_\mu(x) + A_{\mu\text{SWDE}}(x) - A_{\mu\text{NSWDE}}(a_\mu) + \mathcal{O}(e^4). \tag{3.4}
\]

Note that the \(e^2\) order vanishes as expected. Equations (1.2) and (1.3) in the introduction then indicate that the following expressions represent the solutions to the SWDE - NSWDE differences at the \(e^3\) order:
\[
A_{\mu\text{SWDE}}^e(x) - A_{\mu\text{NSWDE}}^e(x) = X^e(x) + \partial_\mu Y^e(x), \tag{3.5}
\]
\[
\Lambda_{\text{SWDE}}^e(x) - \Lambda_{\text{NSWDE}}^e(x) = \delta_\lambda Y^e(x). \tag{3.6}
\]

In order to retrieve the above solutions, that is to find the explicit forms of \(X^e(x)\) and \(Y^e(x)\), we first Fourier transform \(A_{\mu}^e(x)\) into a momentum space quantity \(\hat{A}_\mu(p, q, k)\):
\[
\hat{A}_\mu(p, q, k) = -\frac{e^3}{8} \left[ \hat{a}_\mu(k) \left( (\hat{a}(p) \theta q)(\hat{a}(q) \theta k) M_1 + (\hat{a}(p) \theta \hat{a}(q))(q \theta k) M_2 
+ (\hat{a}(p) \theta \hat{a}(q))(\hat{a}(q) \theta k) M_3 \right) + k_\mu \left( (\hat{a}(p) \theta \hat{a}(q))(q \theta \hat{a}(k)) M_4 
+ (\hat{a}(p) \theta \hat{a}(q))(\hat{a}(q) \theta \hat{a}(k)) M_5 + (\hat{a}(p) \theta \hat{a}(k))(\hat{a}(q) \theta \hat{a}(k)) M_6 \right) \right]. \tag{3.7}
\]

Than from equations (2.14) to (2.17) we read out the above coefficients: \(M_1, ..., M_6\), for the SWDE and the NSWDE cases, respectively:
\[
M_{1\text{SWDE}} = 4f_{*3'} \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2} \right) + 4f_{*3'} \left( -\frac{q \theta k}{2}, -\frac{p \theta k}{2}, -\frac{p \theta q}{2} \right),
\]
\[
M_{2\text{SWDE}} = -\frac{1}{2} M_{1\text{SWDE}},
\]
\[
M_{3\text{SWDE}} = 4f_{*3'} \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2} \right),
\]
\[
M_{4\text{SWDE}} = -3f_{*3'} \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2} \right) - 2f_{*3'} \left( -\frac{q \theta k}{2}, -\frac{p \theta k}{2}, -\frac{p \theta q}{2} \right),
\]
\[
M_{5\text{SWDE}} = -2f_{*3'} \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2} \right) - f_{*3'} \left( -\frac{q \theta k}{2}, -\frac{p \theta k}{2}, -\frac{p \theta q}{2} \right),
\]
\[
M_{6\text{SWDE}} = 2f_{*3'} \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2} \right) + f_{*3'} \left( \frac{q \theta k}{2}, -\frac{p \theta k}{2}, -\frac{p \theta q}{2} \right),
\]

\[ -5 - \]
\[
M_{1\text{NSWDE}} = 8 f_{x_2} \left( \frac{p\theta q}{2} \right) f_{x_2} \left( \frac{(p + q)\theta k}{2} \right) + 8 f_{x_2} \left( \frac{q\theta k}{2} \right) f_{x_2} \left( \frac{p\theta (q + k)}{2} \right) - 8 f_{x_3} \left( \frac{p\theta q}{2}, \frac{p\theta k}{2}, \frac{q\theta k}{2} \right), \\
M_{2\text{NSWDE}} = -\frac{1}{2} M_{1\text{NSWDE}}, \\
M_{3\text{NSWDE}} = 8 f_{x_2} \left( \frac{p\theta q}{2} \right) f_{x_2} \left( \frac{p\theta (q + k)}{2} \right) - 4 f_{x_3} \left( \frac{p\theta q}{2}, \frac{p\theta k}{2}, \frac{q\theta k}{2} \right), \\
M_{4\text{NSWDE}} = -4 f_{x_2} \left( \frac{p\theta q}{2} \right) f_{x_2} \left( \frac{(p + q)\theta k}{2} \right) - 6 f_{x_2} \left( \frac{q\theta k}{2} \right) f_{x_2} \left( \frac{p\theta (q + k)}{2} \right) + 8 f_{x_3} \left( \frac{p\theta q}{2}, \frac{p\theta k}{2}, \frac{q\theta k}{2} \right), \\
M_{5\text{NSWDE}} = 2 f_{x_2} \left( \frac{p\theta q}{2} \right) f_{x_2} \left( \frac{(p + q)\theta k}{2} \right) + 4 f_{x_2} \left( \frac{q\theta k}{2} \right) f_{x_2} \left( \frac{p\theta (q + k)}{2} \right), \\
M_{6\text{NSWDE}} = -4 f_{x_2} \left( \frac{q\theta k}{2} \right) f_{x_2} \left( \frac{p\theta (q + k)}{2} \right) - 2 f_{x_2} \left( \frac{p\theta k}{2} \right) f_{x_2} \left( \frac{q\theta (p + k)}{2} \right).
\]

The above functions \( f_{x_2}(a) \), \( f_{x_3}(a, b, c) \), and \( f_{x_3'}(a, b, c) \), are defined respectively:

\[
f_{x_2}(a) = \frac{\sin a}{a}, \\
f_{x_3}(a, b, c) = \frac{\sin b \sin(a + b)}{(a + b)(b + c)} + \frac{\sin c \sin(a - c)}{(a - c)(b + c)}, \\
f_{x_3'}(a, b, c) = \frac{\cos(a + b - c) - 1}{(a + b - c)c} - \frac{\cos(a + b + c) - 1}{(a + b + c)c}.
\]

From equations (3.8) and (3.9) we observe that under the permutation symmetry \( q \leftrightarrow k \):

\[
M_{5(N)\text{SWDE}} = -M_{6(N)\text{SWDE}},
\]

which indicate that the \( M_{5(N)\text{SWDE}} \) and \( M_{6(N)\text{SWDE}} \) (and part of the \( M_{4(N)\text{SWDE}} \)) contributions could be made equivalent to appropriate infinitesimal NC gauge transformation(s)

\[
A^{e^3}_{\mu(N)\text{SWDE}}(x) = A^{e^3}_{\mu(N)\text{SWDE}}(x) + \delta_{\Xi}^{e^3}_{\mu(N)\text{SWDE}} A^{e^3}_{\mu(N)\text{SWDE}}(x) \\
= A^{e^3}_{\mu(N)\text{SWDE}}(x) + \partial_{\mu} \Xi^{e^3}_{(N)\text{SWDE}}(x) - i \left[ \Xi^{e^3}_{(N)\text{SWDE}}(x) \ast A^{e^3}_{\mu(N)\text{SWDE}}(x) \right].
\]

The above gauge parameters \( \Xi^{e^3}_{(N)\text{SWDE}}(x) \) have been found explicitly for both cases, the SWDE and the NSWDE,

\[
\Xi^{e^3}_{\text{SWDE}}(x) = -\frac{e^3}{8} \theta^{ij} \theta^{kl} \left( 2 [a_i \partial_j a_k a_l]_{*3} + [a_i \partial_j a_k a_l]_{*3} \right) + O(e^4), \\
\Xi^{e^3}_{\text{NSWDE}}(x) = -\frac{e^3}{4} \theta^{ij} \theta^{kl} \left( 2 (a_i \ast 2 \partial_j a_k) \ast 2 a_l + (a_i \ast 2 (\partial_j a_k \ast 2 a_l) \right) + O(e^4).
\]
respectively. The rest of the gauge field $A_{\mu(N)SWDE}^{te}(x)$ bear the following form in the momentum space:

$$
\tilde{A}_{\mu(N)SWDE}^{te} = -\frac{e^3}{8} \left[ \tilde{a}_\mu(k) \left( (\tilde{a}(p)\theta q) (\tilde{a}(q)\theta k) M_{1(N)SWDE} + (\tilde{a}(p)\theta \tilde{a}(q)) (q\theta k) M_{2(N)SWDE} \\
+ (\tilde{a}(p)\theta k) (\tilde{a}(q)\theta k) M_{4(N)SWDE} \right) + k_\mu \left( (\tilde{a}(p)\theta \tilde{a}(q)) (q\theta \tilde{a}(k)) M'_{4(N)SWDE} \right) \right],
$$

(3.17)

with $M'_{4(N)SWDE}$ coefficients satisfying:

$$
M'_{4(N)SWDE} = -M_{1(N)SWDE}.
$$

(3.18)

Meanwhile, a $p \leftrightarrow q$ permutation symmetry of the $(\tilde{a}(p)\theta k)(\tilde{a}(q)\theta k)$ term, leads to $M'_{3(N)SWDE}$:

$$
2M'_{3(N)SWDE} = M_{3(N)SWDE}[p, q, k] + M_{3(N)SWDE}[q, p, k],
$$

(3.19)

which further simplifies (3.17).

Finishing all above transformations we now examine the remaining difference

$$
W_\mu(x) = A_{\mu(N)SWDE}^{te}(x) - A_{\mu(NSWDE)}^{te}(x),
$$

in momentum space:

$$
\tilde{W}_\mu = \tilde{A}_{\mu(N)SWDE}^{te} - \tilde{A}_{\mu(NSWDE)}^{te} = e^3 \left[ \tilde{a}_\mu(k) \left( (\tilde{a}(p)\theta q) (\tilde{a}(q)\theta k) \tilde{W}_1 - \frac{1}{2} (\tilde{a}(p)\theta \tilde{a}(q)) (q\theta k) \tilde{W}_1 \\
+ (\tilde{a}(p)\theta k) (\tilde{a}(q)\theta k) \tilde{W}_3 \right) - k_\mu \left( (\tilde{a}(p)\theta \tilde{a}(q)) (q\theta \tilde{a}(k)) \tilde{W}_1 \right) \right],
$$

(3.21)

where

$$
\tilde{W}_1 = \frac{p\theta k}{2} \left[ \left( \frac{p\theta q}{2} - \frac{p\theta k}{2} + \frac{q\theta k}{2} \right) f_1 \left( \frac{p\theta q}{2}, \frac{p\theta k}{2}, \frac{q\theta k}{2} \right) + \frac{p\theta q}{2} \frac{p\theta k}{2} \frac{q\theta k}{2} f_2 \left( \frac{p\theta q}{2}, \frac{p\theta k}{2}, \frac{q\theta k}{2}; 0 \right) \right],
$$

(3.22)

with functions $f_1(a, b, c)$ and $f_2(a, b, c; n)$ having generally complicated structures

$$
f_1(a, b, c) = \frac{1}{4} \left( \cos(a + b + c) \cos(a + b - c) \cos(a - b + c) \cos(a - b - c) \right) \right) \right) \right),
$$

(3.23)

$$
f_2(a, b, c; n) = \frac{e^{2n} \sin a \sin b \cos c}{ab(b^2 - c^2)(c^2 - a^2)} + \frac{b^{2n} \sin a \cos b \sin c}{ac(a^2 - b^2)(b^2 - c^2)} + \frac{a^{2n} \cos a \sin b \sin c}{bc(a^2 - b^2)(c^2 - a^2)}. \quad (3.24)
$$

One can observe that functions $f_1$ and $f_2$ are both even completely symmetric under any permutation over $a, b$ and $c$. This enables us to express the relevant part of the difference
between two, \( \theta \)-exact, SW maps in terms of field strengths via two new generalized entirely symmetric 3-products \( \circ_1 \) and \( \circ_2(n) \):

\[
[fgh]_{\circ_1}(x) = \int e^{-i(p+q+k)x} \tilde{f}(p) \tilde{g}(q) \tilde{h}(k) f_1 \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2} \right), \\
[fgh]_{\circ_2(n)}(x) = \int e^{-i(p+q+k)x} \tilde{f}(p) \tilde{g}(q) \tilde{h}(k) f_2 \left( \frac{p \theta q}{2}, \frac{p \theta k}{2}, \frac{q \theta k}{2}; n \right).
\]

(3.25)

(3.26)

After applying the inverse Fourier transformation on (3.21), we have found the following expression for \( W_\mu(x) \) in terms of \( \circ_1 \) and \( \circ_2(n) \) 3-products:

\[
W_\mu(x) = -e^3 g^{ij} g^{kl} \theta p^aq^rs \left( [\partial_r f_{ip} f_{jk} \partial_s \partial_q f_{\mu}]_{\circ_1} + [\partial_r f_{ip} f_{kj} \partial_q \partial_s f_{\mu}]_{\circ_1} + [\partial_p f_{ri} \partial_q f_{jk} \partial_s f_{\mu}]_{\circ_1} \\
+ \theta^{ab} \theta^{cd} [\partial_p \partial_a f_{ri} \partial_q \partial_c f_{jk} \partial_d f_{\mu}]_{\circ_2(0)} + \partial_{\mu} \left( [\partial_p f_{ri} \partial_q f_{jk} \partial_s a_l]_{\circ_1} + 2[\partial_p \partial_r a_i \partial_q \partial_j a_k \partial_s a_l]_{\circ_1} \\
- \frac{1}{3} \theta^{ab} \theta^{cd} (3[\partial_p \partial_a \partial_i a_r \partial_q \partial_c \partial_j a_k \partial_s \partial_d a_l]_{\circ_2(0)} - [\partial_p \partial_a \partial_i a_r \partial_q \partial_c \partial_k a_j \partial_s \partial_d a_l]_{\circ_2(0)}) \right) \right).
\]

(3.27)

Finally, we add \( (\partial_{\mu} \Xi_{SWDE}^e(x) - \partial_{\mu} \Xi_{NSWDE}^e(x)) \) back to the \( W_\mu(x) \) and obtain explicit solutions to the equations (3.3) and (3.4):

\[
X^e_\mu(x) = -e^3 g^{ij} g^{kl} \theta p^aq^rs \left( [\partial_r f_{ip} f_{jk} \partial_s \partial_q f_{\mu}]_{\circ_1} + [\partial_r f_{ip} f_{kj} \partial_q \partial_s f_{\mu}]_{\circ_1} + [\partial_p f_{ri} \partial_q f_{jk} \partial_s f_{\mu}]_{\circ_1} \\
+ \theta^{ab} \theta^{cd} [\partial_p \partial_a f_{ri} \partial_q \partial_c f_{jk} \partial_d f_{\mu}]_{\circ_2(0)} \right),
\]

(3.28)

\[
Y^e_\mu(x) = \frac{e^3}{8} g^{ij} g^{kl} \left( 2[a_i \partial_j a_k a_l]_{\ast, \ast, \ast, \ast} + [a_i \partial_j a_k a_l]_{\ast, \ast, \ast, \ast, \ast} - 4 (a_i \ast_2 \partial_j a_k \ast_2 a_l) + 2a_i \ast_2 (\partial_j a_k \ast_2 a_l) \right) \\
- e^3 g^{ij} g^{kl} \theta p^aq^rs \left( [\partial_p f_{ri} \partial_q f_{jk} \partial_s a_l]_{\circ_1} + 2[\partial_p \partial_r a_i \partial_q \partial_j a_k \partial_s a_l]_{\circ_1} \\
- \frac{1}{3} \theta^{ab} \theta^{cd} (3[\partial_p \partial_a \partial_i a_r \partial_q \partial_c \partial_j a_k \partial_s \partial_d a_l]_{\circ_2(0)} - [\partial_p \partial_a \partial_i a_r \partial_q \partial_c \partial_k a_j \partial_s \partial_d a_l]_{\circ_2(0)}) \right).
\]

(3.29)

Simple structure for the gauge field difference between two SW maps at the \( e^3 \) order (3.3) leads to the following result for the gauge field strength difference/comparison:

\[
F^e_{\mu tSWDE}(x) - F^e_{\mu tNSWDE}(x) = \partial_{\mu} X^e_\mu(x) - \partial_{\nu} X^e_\nu(x).
\]

(3.30)

The \( Y^e_\mu(x) \) terms drop out and leave (3.30) containing only pure \( U(1) \) gauge field strengths. Therefore, the expression (3.30) is manifestly gauge invariant, as (3.28).

4. The \( \theta \)-exact gauge field strength ambiguities up to the \( e^3 \) order

Besides discussion in the previous section, the gauge field strength SW map expansion possesses certain ambiguity/freedom within itself [42]. The \( A_{\mu SWDE}(x) \) from (2.15), and
\(A_{\mu \text{NSWDE}}(x)\) from (2.17), lead to the same noncommutative \(U(1)\) gauge field strength expansion up to the \(e^2\) order
\[
F_{\mu \nu}(x) = e f_{\mu \nu} + e^2 \theta^{ij} \left( f_{\mu i} \star_2 f_{\nu j} - a_i \star_2 \partial_j f_{\mu \nu} \right) + \mathcal{O}(e^3).
\] (4.1)

At the \(e^2\) order, the general consistency condition for the gauge field strength
\[
\delta \lambda F_{\mu \nu} = i [\lambda \star F_{\mu \nu}],
\] (4.2)
becomes
\[
\delta \lambda F_{\mu \nu}^{e^2} = ie^2 [\lambda \star f_{\mu \nu}].
\] (4.3)

Examining the gauge field strength (4.1) we find that consistency condition (4.3) is fulfilled solely through the term \(-e^2 \theta^{ij} a_i \star_2 \partial_j f_{\mu \nu}\), because of the following relation between the \(\star_2\)-product and the \(\star\)-commutator
\[
[\phi \star \psi] = i \theta^{ij} \partial_i \phi \star_2 \partial_j \psi.
\] (4.4)

This observation promotes us to put an arbitrary parameter \(\kappa_g\) in front of the other term \(e^2 \theta^{ij} f_{\mu i} \star_2 f_{\nu j}\) in equation (4.1) since this does not break the \(e^2\) order consistency condition (4.3). Such a procedure leads to the \(\kappa_g\)-deformed gauge field strength up to the \(e^2\) order
\[
F_{\mu \nu}(x)_{\kappa_g} = e f_{\mu \nu} + e^2 \theta^{ij} \left( \kappa_g f_{\mu i} \star_2 f_{\nu j} - a_i \star_2 \partial_j f_{\mu \nu} \right) + \mathcal{O}(e^3).
\] (4.5)

To extend such a procedure to higher order we must handle the effect of \(\kappa_g\) in the \(e^3\) order consistency relation, i.e. solving the equality
\[
\delta \lambda F_{\mu \nu}^{e^3}(x)_{\kappa_g} = ie \left( [\Lambda^{e^2} \star f_{\mu \nu}] + [\lambda \star F_{\mu \nu}(x)_{\kappa_g}] \right)
= ie \left( [\Lambda^{e^2} \star f_{\mu \nu}] + [\lambda \star e^2 \theta^{ij} (\kappa_g f_{\mu i} \star_2 f_{\nu j} - a_i \star_2 \partial_j f_{\mu \nu})] \right).
\] (4.6)

We start by observing that both SW maps in section 2 would satisfy the \(e^3\) order consistency relation (4.6) when \(\kappa_g = 1\), i.e. without the \(\kappa_g\)-deformation. Then identify, within the undeformed noncommutative field strength \(F_{\mu \nu(N)\text{SWDE}}(x)\), those terms relevant to the to-be-deformed term \(i \theta^{ij} [\lambda \star f_{\mu i} \star_2 f_{\nu j}]\) and make them \(\kappa_g\)-proportional. We also search for possible \(\kappa_g\)-unrelated freedom/ambiguity in the undeformed noncommutative field strength.

### 4.1 Gauge field strength from the SWDE solution

The easiest way to determine the gauge field strength corresponding to \(A_{\mu \text{SWDE}}(x)\) is by solving directly the Seiberg-Witten Differential Equation for the gauge field strength
\[
\frac{d}{dt} F_{\mu \nu \text{SWDE}}(x) = \frac{1}{4} \theta^{ij} \left[ 2 \left\{ F_{\mu i} \star f_{\nu j} \right\} - \left\{ A_i \star \left( 2 \partial_j F_{\mu \nu} - i [A_j \star F_{\mu \nu}] \right) \right\} \right],
\] (4.7)
which, at the $e^3$ order yields

$$F_{\mu\nu}^{e^3}_{SWDE}(x) = \frac{e^3}{4} \theta^{ij} \theta^{kl} \left( 2 [f_{\mu k} f_{\nu l} f_{ij}]_{x_{y'}} - 2 [f_{\nu l} a_i \partial_j f_{\mu k}]_{x_{y'}} + 2 [f_{\mu l} f_{\nu i} f_{kj}]_{x_{y'}} - 2 [f_{\mu k} a_i \partial_j f_{\nu l}]_{x_{y'}} - 2 [a_k \partial_l (f_{\mu i} f_{\nu j} - a_i \partial_j f_{\mu l})]_{x_{y'}} - 2 [a_k \partial_l a_i \partial_j f_{\mu l}]_{x_{y'}} + [\partial_l f_{\mu i} a_i (\partial_j a_k + f_{jk})]_{x_{y'}} \right)$$

Employing an identity

$$2 (a + b) f_{s_2} (a + b, c) = a \left( f_{s_2} (a, b, c) + f_{s_2} (-b, -c, a) \right) + b \left( f_{s_2} (a, b, c) + f_{s_2} (-a, c, b) \right), \quad (4.9)$$

we find that the next three terms together satisfy transformation property

$$\delta \lambda \frac{1}{2} \theta^{ij} \theta^{kl} \left( [f_{\mu l} a_i \partial_j f_{\mu k}]_{x_{y'}} + [f_{\mu k} a_i \partial_j f_{\nu l}]_{x_{y'}} + [a_k \partial_l (f_{\mu i} f_{\nu j})]_{x_{y'}} \right) = i \theta^{kl} [\lambda^* f_{\mu k} \star_2 f_{\nu l}], \quad (4.10)$$

thus they are relevant for the $\kappa_{y'}$-deformation at the $e^3$ order.

Among the terms after the second equal sign in the equation (4.8), the first two are manifestly invariant under the commutative gauge transformation and could be a subject to the free variation, thus associated with new deformation (weight) parameter $\kappa'_{y'}$. These together lead us to the $(\kappa_y, \kappa'_{y'})$-deformed expression for the gauge field strength (4.8):

$$F_{\mu\nu}^{e^3}_{SWDE}(x)_{\kappa_y, \kappa'_{y'}} = \frac{e^3}{2} \theta^{ij} \theta^{kl} \left[ \kappa'_{y'} \left( [f_{\mu k} f_{\nu l} f_{ij}]_{x_{y'}} + [f_{\nu l} f_{\mu i} f_{kj}]_{x_{y'}} \right) - \kappa_g \left( [f_{\mu l} a_i \partial_j f_{\mu k}]_{x_{y'}} + [f_{\mu k} a_i \partial_j f_{\nu l}]_{x_{y'}} + [a_k \partial_l (f_{\mu i} f_{\nu j})]_{x_{y'}} \right) + 2 [a_i \partial_j a_k \partial_l f_{\mu l}]_{x_{y'}} + 2 [\partial_l f_{\mu i} a_i \partial_j a_k]_{x_{y'}} + 2 [a_k a_i \partial_l \partial_j f_{\mu l}]_{x_{y'}} - [a_i \partial_k a_j \partial_l f_{\mu l}]_{x_{y'}} - [\partial_l f_{\mu i} a_i \partial_k a_j]_{x_{y'}} \right].$$

(4.11)

### 4.2 Gauge field strength from the NSWDE solution

Next we consider the $e^3$ order $\theta$-exact gauge field strength from $A_{\mu NSWDE}(x)$, which can be
expressed as

\[
F_{\mu\nu}^{e^3}(x) = e^3 \theta^{ij} \theta^{kl} \left[ f_{\mu i} \ast_2 (f_{jk} \ast_2 f_{\nu l}) + f_{l\nu} \ast_2 (f_{jk} \ast_2 f_{\mu i}) - [f_{\mu i} f_{jk} f_{\nu l}]_{s_3} \right. \\
+ \left. \theta^{pq} \theta^{rs} \left[ [f_{\mu i} \partial_j f_{pq} \partial_k \partial_l f_{\nu l}]_{s_2(1)} + \frac{1}{2} \theta^{ab} \theta^{cd} \left[ 2 \partial_{pq} \partial_{r} f_{rs} \partial_{j} \partial_{k} \partial_{l} f_{\nu l} + \partial_{pq} \partial_{r} f_{t} \partial_{j} \partial_{k} \partial_{l} f_{\nu l} + \partial_{pq} \partial_{r} f_{t} \partial_{j} \partial_{k} \partial_{l} f_{\nu l} \right]_{s_2(0)} \right] \right.
\]

\[
\frac{1}{2} \left( a_i \ast_2 (\partial_k a_j \ast_2 \partial_l f_{\mu \nu}) + (a_i \ast_2 \partial_k a_j) \ast_2 \partial_l f_{\mu \nu} - [a_i \partial_k a_j \partial_l f_{\mu \nu}]_{s_3} + [a_i a_k \partial_j \partial_l f_{\mu \nu}]_{s_3} \right),
\]

(4.12)

where we observe that the first two lines does not contribute to \( \delta \lambda F_{\mu\nu}^{e^3} \). Among the rest of the terms, we notice that the first one is compatible with the \( \ast_2 \)-commutator, since:

\[
\delta \lambda \left( - \theta^{ij} \theta^{kl} a_i \ast_2 \partial_j (f_{\mu k} \ast_2 f_{\nu l}) \right) = -i \theta^{kl} \left[ \lambda \ast f_{\mu k} \ast f_{\nu l} \right].
\]

(4.13)

Thus, this term gives the transformation of the fully commutative gauge field strength term \( \theta^{ij} f_{\mu i} \ast_2 f_{\nu j} \) in the first order. Now we see that multiplying equation (4.13) by the \( \kappa_g \) parameter ensures the compatibility at the \( e^3 \) order.

It is also straightforward to notice that, minimally one more additional free variation could be performed in \( F_{\mu\nu}^{e^3}(x) \) via multiplication of the manifestly gauge invariant first two lines of equation (4.12) by new \( \kappa'_g \)-deformation parameter. Finally we obtain the \( (\kappa_g, \kappa'_g) \)-deformed gauge field strength at the \( e^3 \) order

\[
F_{\mu\nu}^{e^3}(x)_{\kappa_g, \kappa'_g} = e^3 \theta^{ij} \theta^{kl} \left[ \kappa'_g \left( f_{\mu i} \ast_2 (f_{jk} \ast_2 f_{\nu l}) + f_{l\nu} \ast_2 (f_{jk} \ast_2 f_{\mu i}) - [f_{\mu i} f_{jk} f_{\nu l}]_{s_3} \right. \\
+ \left. \theta^{pq} \theta^{rs} \left[ [f_{\mu i} \partial_j f_{pq} \partial_k \partial_l f_{\nu l}]_{s_2(1)} + \frac{1}{2} \theta^{ab} \theta^{cd} \left[ 2 \partial_{pq} \partial_{r} f_{rs} \partial_{j} \partial_{k} \partial_{l} f_{\nu l} + \partial_{pq} \partial_{r} f_{t} \partial_{j} \partial_{k} \partial_{l} f_{\nu l} + \partial_{pq} \partial_{r} f_{t} \partial_{j} \partial_{k} \partial_{l} f_{\nu l} \right]_{s_2(0)} \right] \right.
\]

\[
\frac{1}{2} \left( a_i \ast_2 (\partial_k a_j \ast_2 \partial_l f_{\mu \nu}) + (a_i \ast_2 \partial_k a_j) \ast_2 \partial_l f_{\mu \nu} - [a_i \partial_k a_j \partial_l f_{\mu \nu}]_{s_3} + [a_i a_k \partial_j \partial_l f_{\mu \nu}]_{s_3} \right),
\]

(4.14)

From equations (4.3), (4.11), and (4.14) we obtain gauge field strength up to the \( e^3 \) order, for two distinct \( \theta \)-exact SW map solutions (N)SWDE,

\[
F_{\mu\nu}(x) = e f_{\mu\nu}(x) + F_{\mu\nu}^{e^2}(x)_{\kappa_g} + F_{\mu\nu}^{e^3}(x)_{\kappa_g, \kappa'_g},
\]

(4.15)

necessary to compute the pure Yang-Mills gauge action up to the \( e^4 \) order.
5. Comparison with the θ-iterative field redefinition

In the last two sections we examined the SW map ambiguities rising explicitly from known solutions. It remains an interesting question how are these ambiguities compared with the θ-iterative field redefinition procedure in [21]. The relevant term $X_\mu$ in section 3 could in principle be casted as a gauge field redefinition in this sense. The $(\kappa_g, \kappa'_g)$-deformations are somehow different. To see that explicitly we follow the argument from [24], that is, the only relevant gauge field redefinition would be of the form

$$\varphi_\mu = \frac{b}{4} e^{\theta ij} D_\mu f_{ij},$$

which then gives the following gauge field strength correction,

$$\Phi_{\mu \nu} = D_\mu \varphi_\nu - D_\nu \varphi_\mu = \frac{b}{4} e^{\theta ij} f_{\mu \nu} f_{ij}.$$  

It is clear that the $b$ correction to the gauge field strength does not match the $\kappa_g$ correction in the gauge field strength (4.5). The $b$ and $\kappa_g$ corrections are instead connected by the (inter-)actions. The $\kappa_g$-deformation (4.5) from [42] leads to the following interaction terms:

$$S_{\kappa_g}^{e^2 \theta^1} = - \int e^{2 \theta ij} f^{\mu \nu} \left( \kappa_g f_{\mu i} f_{\nu j} - \frac{1}{4} f_{ij} f_{\mu \nu} \right).$$

The $b$ correction (5.2), on the other hand, gives

$$S_{a}^{e^2 \theta^1} = - \int e^{2 \theta ij} f^{\mu \nu} \left( f_{\mu i} f_{\nu j} - \frac{1 + b}{4} f_{ij} f_{\mu \nu} \right).$$

Here we see that both the $\kappa_g$ and/or $a = 1 + b$ present the ratio between two gauge invariant terms in an inverted fashion, therefore they may be regarded as equivalent. It is but not yet clear which $\kappa'_g$ deformation, derived at the $e^3$ order of the gauge field strength in subsections 4.1 and 4.2, would correspond to $\kappa_g$ and/or $a = 1 + b$. Namely, the corresponding $\theta^2$ order ambiguity/freedom terms via the method presented in [21, 24] are not thoroughly studied till now. Thus, at this point any further comparison would be highly plausible.

6. Conclusion

In this article we compare two distinct $\theta$-exact Seiberg-Witten map expansions up to the $e^3$ order; one obtained by solving Seiberg-Witten differential equations $\theta$-exactly, the other by inverting a known SW solution [3]. We manage to determine the ambiguity between these two explicitly and subject it into the standard form given in [14, 15].

At a next stage we extend the $e^2$ order gauge field strength ambiguity parameter $\kappa_g$ to the $e^3$ order. Namely, we have found that part of the $e^3$ order gauge field strength should be multiplied by $\kappa_g$ to keep the consistency condition, while there are another terms which are gauge invariant by themselves, thus they can be varied independently. This promotes the introduction of a second ambiguity/freedom/ratio parameter $\kappa'_g$ (alone side $\kappa_g$), which
is highly nontrivial. We therefore conclude that the SW map expansions for (even) U(1) gauge field at the $e^3$ order possess much profounder structures than the prior order.

Besides its own manifestness, results on the gauge field strength expansion in this paper enables the construction of $\theta$-exact, and $(\kappa_g, \kappa'_g)$-deformed U(1) gauge theory (pure Yang-Mills gauge theory action) up to the four-photon coupling term, which should then lead to completion of the one-loop photon two-point function computation started in [33]. We hope that exploring the extended deformation freedom parameter space $(\kappa_g, \kappa'_g)$ would provide more than enough control over the divergences in this case. The same term should also contribute to the NC phenomenology at extreme energies, for example tree-level NCQED contributions to the $\gamma\gamma \to \gamma\gamma$ scattering amplitude [43].

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