Tensor product extension of entanglement witnesses and their connection with measurement device independent entanglement witness.

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Abstract

We provide a new extension of entanglement witnesses for $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ systems. Our construction preserves the properties of indecomposability and spanning property of entanglement witnesses. We show how our concept of extended entanglement witnesses is connected with the idea of measurement device independent entanglement witnesses.

Keywords: Quantum information, Entangled states, Entanglement witnesses

1. Introduction

Detection of entanglement is one of the fundamental problems in quantum information theory\cite{1}. It is well known that it is extraordinarily hard to check whether for a given density matrix describing a quantum state of the complex system is separable or entangled. There exist several operational criteria which allow us to detect quantum entanglement (see e.g.\cite{2}). One of the most famous criteria is based on the partial transposition: if a state $\rho$ is separable then its partial transposition $\rho^T = (I \otimes T)\rho$ is positive\cite{3}. States which are positive under partial transposition are called PPT states. It is easy to see that each separable state is necessarily PPT but the converse is not always true. The most general approach to characterize quantum entanglement uses the notion of an entanglement witness (EW)\cite{3,5,6}. It turns out that a state is entangled if and only if it is detected by some EW\cite{3}. There was a considerable effort in constructing and analyzing the structure of EWs\cite{7}–\cite{20}. However, there is no general method to construct such objects. An entanglement witness which detects a maximal set of entanglement is said to be optimal, as was introduced in Ref.\cite{7}. Unfortunately, there is no complete characterization of indecomposable optimal EW and only very few examples of optimal indecomposable optimal EW are available in the literature. Optimal EWs are of primary importance since to perform complete classification of quantum states of a bipartite system it is enough to use only optimal EWs. In the present paper we provide some extension of EWs for $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ systems which preserve crucial properties, namely indecomposability and optimality. We also show how our concept is connected with the idea of measurement device independent entanglement witnesses (MDIEW)\cite{22}–\cite{23}. The latter are the witnesses that are robust against misalignment of the measurement setup or even the degrees of freedom of the system entanglement of which is to be tested. We prove that our extensions provide alternative way of explanation of the nature of MDIEW.

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2. Main results

First, we introduce some basic terminology of entanglement witnesses and states used in this letter. Let \( \mathcal{H} \) be a separable complex Hilbert space and \( \mathcal{B}(\mathcal{H}) \) be the algebra of all bounded linear operators on \( \mathcal{H} \). We can define the set:

\[
S(\mathcal{H}) = \{ \rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq 0, \, \text{Tr}\rho = 1 \},
\]

i.e. the set of all states on \( \mathcal{H} \). If \( \mathcal{H} \) and \( \mathcal{K} \) are finite dimensional, a state in the bipartite composition system \( \rho \in S(\mathcal{H} \otimes \mathcal{K}) \) is said to be separable if it can be written as \( \rho = \sum_{i=1}^{k} p_i \rho_i \otimes \sigma_i \), where \( \rho_i \) and \( \sigma_i \) are states on \( \mathcal{H} \) and \( \mathcal{K} \), respectively, and \( p_i \) are positive numbers with \( \sum_{i=1}^{k} p_i = 1 \). Otherwise, \( \rho \) is said to be inseparable or entangled.

One of the most general approaches to characterize quantum entanglement uses a notion of an entanglement witness. The term of entanglement witness was introduced first time in [5]. Entanglement witness allows us to detect quantum states without full information about this state. Every entanglement witness detects something [7], since it detects in particular the projector on the subspace corresponding to the negative eigenvalues. Let us recall the definition of entanglement witness.

**Definition 1.** A Hermitian operator \( W_{AB} \in \mathcal{B}(\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}) \) defined on a tensor product \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) is called an EW iff

\[
\text{Tr}(W_{AB}\sigma_{AB}) \geq 0,
\]

for all separable states \( \sigma_{AB} \in S(\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}) \), and here exists an entangled state \( \rho_{AB} \in S(\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}) \) such that

\[
\text{Tr}(W_{AB}\rho_{AB}) < 0,
\]

(one says that \( \rho_{AB} \) is detected by \( W_{AB} \)).

We know that for a given operator \( W_{AB} \) it is extremely hard to check whether it is an entanglement witness or not. Here, we shall give a method to construct a new witness by the use of the known witness.

**Theorem 1.** If \( W_{AB} \in \mathcal{B}(\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}) \) is an entanglement witness, then so is \( W_{A'ABB'} \in \mathcal{B}(\mathcal{C}^{d_{A'}} \otimes \mathcal{C}^{d_A} \otimes \mathcal{C}^{d_{B'}} \otimes \mathcal{C}^{d_B}) \) for any \( d_A, d_B = 1, \ldots, d \leq \infty \), where

\[
W_{A'ABB'} = P_{A'} \otimes W_{AB} \otimes P_{B'},
\]

and \( P_{A'} \in \mathcal{B}(\mathcal{C}^{d_{A'}}), P_{B'} \in \mathcal{B}(\mathcal{C}^{d_{B'}}) \) are any positive semidefinite operators.

**Proof.** We need to show that \( \text{Tr}(W_{A'ABB'}\sigma_{A'ABB'}) \geq 0 \), for any separable state \( \sigma_{A'ABB'} \in S(\mathcal{C}^{d_{A'}} \otimes \mathcal{C}^{d_A} \otimes \mathcal{C}^{d_{B'}} \otimes \mathcal{C}^{d_B}) \). We use a spectral decomposition of operators \( P_{A'} = \sum_{i=1}^{d_{A'}} \alpha_i E_i \) and \( P_{B'} = \sum_{i=1}^{d_{B'}} \beta_i F_i \) where \( \alpha_i, \beta_i \geq 0 \).
\[
\text{Tr}(W_{A'BB'}\sigma_{A'ABB'}) = \text{Tr}(P_{A'} \otimes W_{AB} \otimes P_{B'} \sigma_{A'ABB'}) \\
= \text{Tr}\left( \sum_{i=1}^{d_{A'}} \lambda_i E_i \otimes W_{AB} \otimes \sum_{j=1}^{d_{B'}} \beta_j F_j \sigma_{A'ABB'} \right) \\
= \sum_{i=1}^{d_{A'}} \sum_{j=1}^{d_{B'}} \alpha_i \beta_j \text{Tr}(E_i \otimes W_{AB} \otimes F_j \sigma_{A'ABB'}) \\
= \sum_{i=1}^{d_{A'}} \sum_{j=1}^{d_{B'}} \alpha_i \beta_j \text{Tr}(\{I_{A'} \otimes W_{AB} \otimes I_{B'}\}(E_i \otimes I_{AB} \otimes F_j) \sigma_{A'ABB'}(E_i \otimes I_{AB} \otimes F_j)) \\
= \sum_{i=1}^{d_{A'}} \sum_{j=1}^{d_{B'}} \alpha_i \beta_j \text{Tr}(W_{ABB} \text{Tr}_{A'B'} [(E_i \otimes I_{AB} \otimes F_j) \sigma_{A'ABB'}(E_i \otimes I_{AB} \otimes F_j)]) \\
= \sum_{i=1}^{d_{A'}} \sum_{j=1}^{d_{B'}} \alpha_i \beta_j \text{Tr}(W_{ABB} \sigma_{A'ABB'}^i) \geq 0,
\]

where \(\sigma_{A'ABB'}^i\) are some separable states in \(S(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})\). In the next to the last line we use the fact that local operations do not change property of separability. Which ends the proof.

Recall that entanglement witnesses \(W_{AB}\) are decomposable if \(W_{AB} = Q_1 + Q_2^T\), where \(Q_1, Q_2 \geq 0\) and \(Q^T\) denotes the partial transposition, otherwise we say that they are indecomposable. It is clear from the definition that a decomposable EW cannot detect an entangled PPT state, therefore such EW is useless in the search for bound entangled states.

**Theorem 2.** If \(W_{AB}\) is indecomposable entanglement witness, then so is \(W_{A'ABB'}\).

**Proof.** If \(W_{AB}\) is indecomposable entanglement witness then there exists the state \(\rho_{AB} \in S(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})\) PPT such that \(\text{Tr}(W_{AB} \rho_{AB}) < 0\). Let us consider the extended state \(\tilde{\rho}_{A'ABB'} = \tilde{P}_{A'} \otimes \rho_{AB} \otimes \tilde{P}_{B'} \in S(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_B'})\), where \(\tilde{P}_{A'} \in B(\mathbb{C}^{d_A'})\) and \(\tilde{P}_{B'} \in B(\mathbb{C}^{d_B'})\) are positive semidefinite operators. To prove this theorem we use the below fact

**Fact 1.** If \((I_{d_A} \otimes T_{d_B}) \rho_{AB} \geq 0\), then \((I_{d_A} \otimes T_{d_B} \otimes d_{B'}) \tilde{\rho}_{A'ABB'} \geq 0\).

To prove this fact we use decomposition of \(\rho_{AB} = \sum_{i,j=1}^{d_A} e_{ij} \otimes \rho_{ij}\). If \(\rho_{AB}^F = (I_{d_A} \otimes T_{d_B}) \rho_{AB} = \sum_{i,j=1}^{d_A} e_{ij} \otimes \rho_{ij}^F \geq 0\) then we have

\[
\rho_{A'ABB'}^F = (I_{d_A} \otimes T_{d_B}) \tilde{\rho}_{A'ABB'} \\
= (I_{d_A} \otimes T_{d_B}) \tilde{P}_{A'} \otimes \sum_{i,j=1}^{d_A} e_{ij} \otimes \rho_{ij} \otimes \tilde{P}_{B'} \\
= \tilde{P}_{A'} \otimes \sum_{i,j=1}^{d_A} e_{ij} \otimes \rho_{ij}^F \otimes \tilde{P}_{B'}^T, \\
= \tilde{P}_{A'} \otimes \rho_{AB}^F \otimes \tilde{P}_{B'}^T \geq 0,
\]

because \(\tilde{P}_{B'}^T \geq 0\), and we obtain

\[
\text{Tr}(W_{AABB'} \rho_{A'ABB'}) = \text{Tr}\left(\tilde{P}_{A'} \tilde{P}_{A'}\right) \text{Tr}(W_{AB} \rho_{AB}) \text{Tr}\left(\tilde{P}_{B'} ^T \tilde{P}_{B'}\right) < 0,
\]

which means that \(W_{AABB'}\) is an indecomposable entanglement witness. Which ends the proof.
Given an entanglement witness $W_{AB}$ one defines a set of all entangled states in $H_{AB}$ detected by $W_{AB}$

$$D_{W_{AB}} = \{ \rho_{AB} \in S (H_{AB}) \mid \text{Tr} (\rho_{AB} W_{AB}) < 0 \} .$$

Suppose now that we are given two entanglement witnesses $W_1$ and $W_2$ in $H_{AB}$.

**Definition 2.** We call $W_1$ finer than $W_2$ if $D_{W_1} \supseteq D_{W_2}$. $W$ is called optimal if there is no other entanglement witness which is finer than $W$.

It is clear that optimal witnesses are sufficient to detect all entangled states. The Authors of Ref. [7] formulated the following sufficient condition for the optimality. For a given entanglement witness of $W_{AB} \in B(\mathbb{C}^d_A \otimes \mathbb{C}^d_B)$ we define the set

$$P_{W_{AB}} = \{ |\phi_A \otimes \psi_B \rangle \in H_A \otimes H_B | \text{Tr} (W_{AB} |\phi_A \otimes \psi_B \rangle \langle \phi_A \otimes \psi_B|) = 0 \} .$$

We say that $W_{AB}$ possesses a spanning property if $\text{span} P_W = H_{AB}$.

**Proposition 1.** [7] An entanglement witness possessing a spanning property is optimal.

Note that the spanning property is not a necessary condition for optimality of the EW since the Choi map \[21, 23\] gives rise to an optimal entanglement witness that has no spanning property.

Consider now an indecomposable EW $W_{AB}$ and define a set

$$D_{W_{AB}}^{PPT} = \{ \rho_{AB} \in S (H_{AB}) \mid \text{Tr} (\rho_{AB} W_{AB}) < 0 \text{ and } \rho_{AB} \geq 0 \} ,$$

i.e. a set of PPT entangled states detected by $W_{AB}$.

**Definition 3.** We call $W_1$ nd-finer than $W_2$ if $D_{W_1}^{PPT} \supseteq D_{W_2}^{PPT}$. $W$ is called nd-optimal if there is no other entanglement witness which is nd-finer than $W$.

We see that nd-optimality is stronger than optimality.

**Proposition 2.** [7] An entanglement witness is nd-optimal if and only if both $W_{AB}$ and $W_{AB}^F$ are optimal.

Endowed with this information we can formulate the following theorem.

**Theorem 3.** If $W_{AB} \in B (H_A \otimes H_B)$ (respectively $W_{AB}^F \in B (H_A \otimes H_B)$) entanglement witness possess a spanning property, then so is $W_{A'ABB'} \in B (H_{A'} \otimes H_{BB'})$ (respectively $W_{A'ABB'}^F \in B (H_{A'} \otimes H_{BB'})$).

**Proof.** We proof this theorem in two steps:

1. If $W_{AB}$ possess a spanning property, then so is $W_{A'ABB'}$ (see [1]).

   We know that there exists the set of vectors

   $$\{ |\phi_A \otimes \psi_B \rangle \in H_A \otimes H_B | \text{Tr} (W_{AB} |\phi_A \otimes \psi_B \rangle \langle \phi_A \otimes \psi_B|) = 0 \}$$

   spans Hilbert space $H_A \otimes H_B$, because $W_{AB}$ possess a spanning property. Let $\{ |e_A \rangle \}$ denote an orthonormal basis in $H_A'$ and respectively $\{ |f_{B'} \rangle \}$ in $H_{B'}$, then we can easily construct the new set $\{ |e_{A'} \otimes \phi_A \otimes \psi_B \rangle \otimes e_{B'} \langle f_{B'} \rangle \}$ which spans Hilbert space $H_{A'A'} \otimes H_{BB'}$. Now we need to show that for any vector from this set and $W_{A'ABB'}$ the condition $\text{Tr} (W_{A'ABB'} |e_{A'} \otimes \phi_A \otimes \psi_B \rangle \otimes f_{B'} \langle e_{A'} \otimes \phi_A \otimes \psi_B \rangle) = 0$ is satisfied. Let us denote $E = |e_{A'} \rangle \langle e_{A'}|$, $F = |f_{B'} \rangle \langle f_{B'}|$ and $P_{\phi \otimes \psi} = |\phi_A \otimes \psi_B \rangle \langle \phi_A \otimes \psi_B|$. Indeed, we have

   $$\text{Tr} (W_{A'ABB'} E \otimes P_{\phi \otimes \psi} \otimes F) = \text{Tr} (P_{A'} E) \text{ Tr} (W_{AB} P_{\phi \otimes \psi}) \text{ Tr} (P_{B'} F) = 0,$$

   it holds because of property of $W_{AB}$.
2. If $W_{AB}^T$ possess a spanning property, then so is $W_{ABB'}^T$. To prove this we use the same way like before. We know that there exist the set of vectors
\[
\left\{ \tilde{\phi}_A \otimes \tilde{\psi}_B \right\} \in \mathcal{H}_A \otimes \mathcal{H}_B [\text{Tr}(W_{AB}^T | \tilde{\phi}_A \otimes \tilde{\psi}_B \rangle \langle \tilde{\phi}_A \otimes \tilde{\psi}_B |) = 0 \}
\]
spans Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, because $W_{AB}^T$ possess a spanning property. Let $\{|\tilde{c}_A\rangle\}$ denote an orthonormal basis in $\mathcal{H}_A$ and respectively $\{|\tilde{f}_{B'}\rangle\}$ in $\mathcal{H}_{B'}$. We can construct the new set $\left\{ \tilde{\phi}_{A'} \otimes \tilde{\phi}_A \otimes \tilde{\psi}_B \otimes \tilde{f}_{B'} \right\}$ which spans Hilbert space $\mathcal{H}_{A'} \otimes \mathcal{H}_{B'B'}$. We need to show that for any vector from this set and $W_{ABB'}^T$, the condition $\text{Tr}(W_{ABB'}^T \tilde{E} \otimes \tilde{P}_{\phi \phi} \otimes \tilde{F}) = 0$ is satisfied, where $\tilde{E} = |\tilde{e}_{A'}\rangle \langle \tilde{e}_{A'}|$, $\tilde{F} = |\tilde{f}_{B'}\rangle \langle \tilde{f}_{B'}|$, and $P_{\phi \phi} = \left| \tilde{\phi}_A \otimes \tilde{\psi}_B \right\rangle \left\langle \tilde{\phi}_A \otimes \tilde{\psi}_B |$.

Theorem 4. (nontrivial extension) There are the states $\rho_{A'ABB'} \in S(C^{d_{A'}} | d_A \otimes C^{d_{B'}} | d_B)$ which are detected by $W_{ABB'}^T$, but the states $S(C^{d_A} \otimes C^{d_B}) \ni \rho_{AB} = Tr_{A'B'}(\rho_{A'ABB'})$ are not detected by $W_{AB}^T$.

**Proof.** We will give explicit example of these states. Let us consider Choi witness of the form:

$$W_{AB} = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (4)$$

We extend this witness to the form (for convenience we will take $d_{A'} = 1$, $d_{B'} = 2$) $W_{ABB'} = W_{AB} \otimes P_{B'}$, where $P_{B'} \in \mathcal{B}(C^2)$. Let us consider the state $\rho_{ABB'} \in S(C^3 \otimes C^3 \otimes C^2) \simeq S(C^3 \otimes C^6)$ of the following form:

$$\rho_{ABB'} = \frac{1}{(a_{11} + a_{22} + b_{11} + b_{22})} \sum_{i,j=1}^3 |i\rangle \langle j| \otimes \rho_{ij}, \quad (5)$$

where:

$$\rho_{ij} = (S^{i-1} \otimes I_2) X (S^{j-1} \otimes I_2)^\dagger, \quad (6)$$

$S$ is the shift operator defined by $S |k\rangle = |k+1\rangle$, and

$$X = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_{11} & b_{12} \\
0 & 0 & 0 & 0 & b_{21} & b_{22}
\end{pmatrix}. \quad (7)$$
and
\[ \rho_{ij} = |i\rangle\langle j| \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad i \neq j. \] (8)

The state \( \rho_{ABB'} \) is positive semidefinite if and only if
\[ a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \succeq 0 \] and
\[ b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \succeq 0. \] It is easy to show that
\[ \text{Tr}(W_{ABB'}\rho_{ABB'}) = \frac{3}{(\text{Tr}(a + b))} \text{Tr}(P_{B'}(b - a)) \] (9)
and
\[ \text{Tr}(W_{ABPAB}) = \frac{3}{(\text{Tr}(a + b))} \text{Tr}(b - a), \] (10)

where \( \rho_{AB} = \text{Tr}_{B'}(\rho_{ABB'}) \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3) \). Let us choose for example \( P_{B'} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), \( a_{21} = a_{12} \) (\( b_{21} = b_{12} \)) and \( b_{11} = a_{11} \) and \( b_{22} = a_{22} \). For this particular state we get
\[ \text{Tr}(W_{ABB'}\rho_{ABB'}) < 0 \text{ if } b_{12} \neq a_{12} \] and
\[ \text{Tr}(W_{ABPAB}) = 0 \] (In fact the state \( \rho_{AB} \) is separable if and only if \( a_{11} = 0 \) and \( a_{22} = 0 \)). Which ends the proof.

3. Extended entanglement witnesses and measurement device independent entanglement wit- nesses

Here we shall show that there is a natural connection of an extended EW of the form \( W_{A'ABB'} = P_{A'} \otimes W_{AB} \otimes P_{B'} \), for \( P_{A'}, P_{B'} \geq 0 \) and the measurement device independent entanglement witnesses (MDIEW). First we shall recall the scheme for measurement device independent entanglement witnesses in the spirit of [22] which is built upon the previous idea of Buscemi [23].

Let us recall the concept of MDIEW represents an operator, or in broader sense quantum operation that allows us to detect entanglement of a given quantum state \( \rho_{AB} \) in the case when (i) the measurement device is not fully controlled and may be misaligned (ii) even the dimensionality of Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) associated with the state \( \rho_{AB} \) is uncontrollable. The point (ii) should be understood in the sense that the experimentalists assume that there is some Hilbert space associated with the state \( \rho_{AB} \); however, due to misalignment of the measurement device it happens that it may couple with other degrees of freedom of the states. The extremal, purely hypothetical example would be the situation when Alice and Bob expect to measure polarisation entanglement of the two photons, but their local measurement devices in reality are coupled to other degree of freedom like angular momentum or frequency, and neither of the observers knows about it. In this situation, there is danger that the physical state of the new (angular momentum) degrees of freedom is separable, but due to the nature of the coupling eventually the false information about the presence of entanglement might be reported in the sense that the overall statistical mean value might be negative. The MDIEW-s are just designed to avoid this situation. They must not report false entanglement if the actual (not hypothetical, or expected one) physical state is separable. The needed element of the corresponding action of the observers is a full control of some degrees of freedom of local apparatus or systems. In Figure 1, we have the setup in which local by controlled objects are the prepared states \( \sigma_{A}, \sigma_{B} \), their Hilbert spaces are known to Alice and Bob also in the sense that local POVM’s (represented in Figure 1 as boxes) couple the local parts of the system in the state \( \rho_{AB} \) exactly to the degrees of freedom corresponding to the known Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \). However, the nature of this coupling is asymmetric: we know that it couples something known (physical degrees associated with the known states \( \sigma_{A}, \sigma_{B} \)) to those (associated with the physical degrees of the state \( \rho_{AB} \)) that are assumed by the observers to be the same as the previous ones but in practice it does not need to be so. In that sense the observers expect that \( \mathcal{H}_A \) and \( \mathcal{H}_B \) \( \cong \mathcal{H}_A \) and \( \mathcal{H}_B \) in the physical sense, i.e. they describe systems with the same degrees of freedom eg. photon polarisation while this may not be true - this corresponds to the property (ii) above. The property (i) which is direct misalignment, says that it may even happen that the assumption about the physical character of the degrees
of freedom of the state $\rho_{AB}$ may be correct (i.e. they may really be polarized, as the observers expect); however, the coupling with the polarization of the states $\sigma_{A'}$, $\sigma_{B'}$ may be not good. For instance this may happen when the Hong-Ou-Mandel (H-O-M) interferometer coupling the polarisation of local Alice photon $A'$ with the polarisation of the incoming photon $A$ works badly. For completeness, let us recall here that double click in the standard H-O-M interferometer results in the event, when the two incoming polarizations of the two photons were projected jointly onto the maximally entangled state $\Psi_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ which, however, with the help of the polarization rotators may be modified to encompass projection onto the entangled state $\Psi_+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (see (27) for the experimental application of the interferometer in the experiment implementing nonlinear entanglement witness associated with entropic inequalities).

Summarising, the observers must control the preparation of the states $\sigma_{A'}$, $\sigma_{B'}$, and be sure that that comes into the local device and is labeled by $A$ and $B$ is coupled just to those states. This is the control that allows us to detect entanglement of $\rho_{AB}$ in the measurement device independent i.e. in such a way that false report about entanglement will never occur.

The above description was presented also in a more mathematical way [22]. To recall the description suppose that there is a witness $W_{AB}$ which detects entanglement of the state $\rho_{AB}$. The crucial idea of [22] is that to represent the witness as a linear combination of quantum states from Figure 1, namely:

$$W_{AB} = \sum_{s,t} \beta_{st} \sigma_{A'}^{sT} \otimes \sigma_{B'}^{tT}. \quad (11)$$

Suppose now, that the observers use the measuring devices with two inputs $\mathcal{H}_A$, $\mathcal{H}_{A'}$ (and $\mathcal{H}_B$, $\mathcal{H}_{B'}$ respectively), and claim that the device performs entangling the von Neuman measurements $\{P_{A'A}, P_{A'B}^\perp\}$ and $\{P_{B'B}, P_{B'B}^\perp\}$, where $P_{A'A} = \langle \Psi_{A'A}^- | \Psi_{A'A}^- \rangle$ with $|\Psi_{A'A}^-\rangle = \frac{1}{\sqrt{d_A-1}} \sum_{k=0}^{d'-1} |k\rangle |k\rangle$, and $P_{B'B}^\perp$ in full analogy.

Note that if both observers report the result "0" which corresponds to local projections onto $P_{A'A}^0 \equiv P_{A'A}^+$, $P_{B'B}^0 \equiv P_{B'B}^+$ then the joint probability with the inputs $\sigma^s$, $\sigma^t$ is

$$P(a = 0, b = 0 | st) \overset{def}{=} \text{Tr} \left( \rho_{AB} \otimes \sigma_{A'}^{sT} \otimes \sigma_{B'}^{tT} P_{A'A}^0 \otimes P_{B'B}^0 \right). \quad (12)$$
Combining this with the formula (11) gives the mean value of the entanglement witness

\[ \mathcal{W} = \langle W_{AB} \rangle = \sum_{s,t} \beta_{st} \mathcal{P}(a = 0, b = 0 \mid st). \]  

(13)

with the standard mean value \( \langle W_{AB} \rangle = Tr(\rho_{AB} W_{AB}) \) of the witness observable \( W_{AB} \) is defined by the formula (11). The major claim of the [22] is that instead of the intended von Neumann measurements the local observers perform, due to misalignment, arbitrary POVM's \( \{ \tilde{P}^{0}_{A'A}, \tilde{P}^{i}_{A'A} \}, \{ \tilde{P}^{0}_{BB'}, \tilde{P}^{i}_{BB'} \} \) \( (0 \leq \tilde{P}^{0}_{A'A} \leq I_{A'A}, 0 \leq \tilde{P}^{0}_{BB'} \leq I_{BB'}, \tilde{P}^{0}_{A'A} + \tilde{P}^{1}_{A'A} = I_{A'A}, \tilde{P}^{0}_{BB'} + \tilde{P}^{1}_{BB'} = I_{BB'}) \) and at the same time, the dimensions of \( \mathcal{H}_{A}, \mathcal{H}_{B} \) and the degrees of freedom they represent (associated with the state \( \rho_{AB} \)) are uncontrolled, then still the negative value of the quantity

\[ \tilde{\mathcal{W}} \equiv \sum_{s,t} \beta_{st} \tilde{\mathcal{P}}(a = 0, b = 0 \mid st), \]  

(14)

reports entanglement of \( \rho_{AB} \) (or, equivalently, the value is always nonegative for separable state \( \rho_{AB} \)), if only Alice and Bob can fully control preparation of \( \sigma_{A}^{s} \) and \( \sigma_{B}^{t} \) which means that they control the physical degrees of freedom associated with \( \mathcal{H}_{A}, \mathcal{H}_{B} \). Here the conditional probability corresponds to the '0' outcomes (represented by the operators \( \tilde{P}^{0}_{A'A}, \tilde{P}^{0}_{BB'} \) introduced above) of some POVM's (may be uncontrolled by the experimentalists) which couples the correctly prepared states \( \sigma_{A}, \sigma_{B} \) to the experimentally investigated state \( \rho_{AB} \). Also it should be stressed that we denoted, in full analogy to the formula (12)

\[ \tilde{\mathcal{P}}(a = 0, b = 0 \mid st) \overset{def}{=} Tr \left( \rho_{AB} \otimes \sigma_{A}^{s} T \otimes \sigma_{B}^{t} T \tilde{P}^{0}_{A'A} \otimes \tilde{P}^{0}_{BB'} \right) \]  

(15)

with the only difference that - unlike in (12) - here the operators \( \tilde{P}^{0}_{A'A}, \tilde{P}^{0}_{BB'} \) are no longer projectors but just elements of some local POVM-s.

Here we will interpret the result of [22] in terms of our extension of entanglement witnesses. Namely we will show that the quantity (14) is always positive for any separable states \( \rho_{AB} \) because then it can be actually represented as mean value of the convex combination of some extended witnesses on the unnormalized product states.

Consider any separable state \( \rho_{AB} = \sum_{i} p_{i} \rho_{A}^{i} \otimes \rho_{B}^{i} \) (uncontrolled). As a result of the previous section we have the family of extended entanglement witnesses

\[ W^{i}_{A'ABB'} = \rho_{A}^{i} \otimes \mathcal{W}_{AB} \otimes \rho_{B'}^{i}, \]

with \( \rho_{A}^{i}, \rho_{B}^{i} \geq 0 \). As such they have positive mean values on any product of two positive operators

\[ Tr \left( W^{i}_{A'ABB'} X_{A'A} \otimes Y_{BB'} \right) \geq 0, \]

with \( X_{A'A}, Y_{BB'} \geq 0 \). In particular we may put in their places the POVM elements considered above i.e. assume \( X_{A'A} = \tilde{P}^{0}_{A'A}, Y_{BB'} = \tilde{P}^{0}_{BB'} \), getting it this way

\[ Tr \left( W^{i}_{A'ABB'} \tilde{P}^{0}_{A'A} \otimes \tilde{P}^{0}_{BB'} \right) \geq 0. \]

Taking further (with help of the probabilities \{p_{i}\} defining separable \( \rho_{AB} \)) the convex combination and remembering the notation (15) it is easy to check that

\[ \tilde{\mathcal{W}} = \sum_{s,t} \beta_{st} \tilde{\mathcal{P}}(a = 0, b = 0 \mid st) = \sum_{i} p_{i} Tr \left( W^{i}_{A'ABB'} \tilde{P}^{0}_{A'A} \otimes \tilde{P}^{0}_{BB'} \right) \geq 0. \]

This means that if such a convex combination were negative, then the state \( \rho_{AB} \) must have been entangled. This fact is a crucial property of MDIEW: since quantum entanglement is a resource we would not like be informed about its presence when actually it is not there and that is what MDIEW guarantees us to avoid.
4. Conclusions

We have provided product extension of entanglement witnesses and shown that they inherit the properties of indecomposability and spanning property (hence optimality) from the original entanglement witnesses. We also have shown that the structure of such extension is naturally present in the mathematics of the scheme of measurement device independent entanglement witnessing of a given entanglement state. It should be stressed that the general construction of product extension of the entanglement witness and its relation to measurement device independent entanglement witnesses can be easily generalised to the case of multipartite case. It is interesting to analyse the optimality concept for the multi-partite version, since it has not been developed yet (see [22]).

There is a natural question whether the concept presented here has any relation to the general scheme of quantum cryptography which requires quantum states with composed local systems (see [3] and references therein).

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