REFLEXIVE POLYTOPES ARISING FROM PERFECT GraphS

TAKAYUKI HIBI AND AKIYOSHI Tsuchiya

Abstract. Reflexive polytopes form one of the distinguished classes of lattice polytopes. Especially reflexive polytopes which possess the integer decomposition property are of interest. In the present paper, by virtue of the algebraic technique on Gröbner bases, a new class of reflexive polytopes which possess the integer decomposition property and which arise from perfect graphs will be presented. Furthermore, the Ehrhart $\delta$-polynomials of these polytopes will be studied.

Background

The reflexive polytope is one of the keywords belonging to the current trends on the research of convex polytopes. In fact, many authors have studied reflexive polytopes from viewpoints of combinatorics, commutative algebra and algebraic geometry. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related with mirror symmetry (see, e.g., [2, 3]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence ([17]) and all of them are known up to dimension 4 ([16]). Moreover, every lattice polytope is a face of a reflexive polytope ([7]). We are especially interested in reflexive polytopes with the integer decomposition property, where the integer decomposition property is particularly important in the theory and application of integer programming [23 §22.10]. A lattice polytope which possesses the integer decomposition property is normal and very ample. These properties play important roles in algebraic geometry. Hence to find new classes of reflexive polytopes with the integer decomposition property is one of the most important problems. For example, the following classes of reflexive polytopes with the integer decomposition property are known:

- Centrally symmetric configurations ([20]).
- Reflexive polytopes arising from the order polytopes and the chain polytopes of finite partially ordered sets ([11] [13] [14] [15]).
- Reflexive polytopes arising from the stable sets polytopes of perfect graphs ([21]).

2010 Mathematics Subject Classification. 13P10, 52B20.

Key words and phrases. reflexive polytope, integer decomposition property, perfect graph, Ehrhart $\delta$-polynomial, Gröbner basis.
Following the previous work [21], the present paper discusses a new class of reflexive polytopes which possess the integer decomposition property and which arise from perfect graphs.

Acknowledgment. The authors would like to thank anonymous referees for reading the manuscript carefully. The second author is partially supported by Grant-in-Aid for JSPS Fellows 16J01549.

1. Perfect graphs and reflexive polytopes

Recall that a lattice polytope is a convex polytope all of whose vertices have integer coordinates. We say that a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension $d$ is reflexive if the origin of $\mathbb{R}^d$ belongs to the interior of $\mathcal{P}$ and if the dual polytope

$$\mathcal{P}' = \{ x \in \mathbb{R}^d : \langle x, y \rangle \leq 1, \forall y \in \mathcal{P} \}$$

is again a lattice polytope. Here $\langle x, y \rangle$ is the canonical inner product of $\mathbb{R}^d$. A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ possesses the integer decomposition property if, for each integer $n \geq 1$ and for each $a \in n\mathcal{P} \cap \mathbb{Z}^d$, where $n\mathcal{P} = \{ na : a \in \mathcal{P} \}$, there exist $a_1, \ldots, a_n$ belonging to $\mathcal{P} \cap \mathbb{Z}^d$ with $a = a_1 + \cdots + a_n$.

Let $G$ be a finite simple graph on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G)$ the set of edges of $G$. (A finite graph $G$ is called simple if $G$ possesses no loop and no multiple edge.) A subset $W \subset [d]$ is called stable if, for all $i$ and $j$ belonging to $W$ with $i \neq j$, one has $\{i, j\} \notin E(G)$. We remark that a stable set is often called an independent set. A clique of $G$ is a subset $W \subset [d]$ which is a stable set of the complementary graph $\overline{G}$ of $G$. The chromatic number of $G$ is the smallest integer $t \geq 1$ for which there exist stable set $W_1, \ldots, W_t$ of $G$ with $[d] = W_1 \cup \cdots \cup W_t$. A finite simple graph $G$ is said to be perfect ([4]) if, for any induced subgraph $H$ of $G$ including $G$ itself, the chromatic number of $H$ is equal to the maximal cardinality of cliques of $H$. The complementary graph of a perfect graph is perfect ([4]).

Let $e_1, \ldots, e_d$ denote the standard coordinate unit vectors of $\mathbb{R}^d$. Given a subset $W \subset [d]$, we may associate $\rho(W) = \sum_{j \in W} e_j \in \mathbb{R}^d$. In particular $\rho(\emptyset)$ is the origin $0_d$ of $\mathbb{R}^d$. Let $S(G)$ denote the set of stable sets of $G$. One has $\emptyset \in S(G)$ and $\{i\} \in S(G)$ for each $i \in [d]$. The stable set polytope $\mathcal{Q}_G \subset \mathbb{R}^d$ of $G$ is the $(0, 1)$-polytope which is the convex hull of $\{ \rho(W) : W \in S(G) \}$ in $\mathbb{R}^d$. One has $\dim \mathcal{Q}_G = d$.

Given lattice polytopes $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^d$ of dimension $d$, we introduce the lattice polytopes $\Gamma(\mathcal{P}, \mathcal{Q}) \subset \mathbb{R}^d$ and $\Omega(\mathcal{P}, \mathcal{Q}) \subset \mathbb{R}^{d+1}$ with

$$\Gamma(\mathcal{P}, \mathcal{Q}) = \text{conv}\{\mathcal{P} \cup (-\mathcal{Q})\},$$

$$\Omega(\mathcal{P}, \mathcal{Q}) = \text{conv}\{ (\mathcal{P} \times \{1\}) \cup (-\mathcal{Q} \times \{-1\}) \}.$$

It is natural to ask when $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ is reflexive and when $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ is reflexive, where $G_1$ and $G_2$ are finite simple graphs on $[d]$. The former is completely solved by [21] and the later is studied in this paper. In fact,
Theorem 1.1. Let $G_1$ and $G_2$ be finite simple graphs on $[d]$.

(a) The following conditions are equivalent:
(i) The lattice polytope $\Gamma(Q_{G_1}, Q_{G_2})$ is reflexive;
(ii) The lattice polytope $\Gamma(Q_{G_1}, Q_{G_2})$ is reflexive and possesses the integer decomposition property;
(iii) Both $G_1$ and $G_2$ are perfect.

(b) The following conditions are equivalent:
(i) The lattice polytope $\Omega(Q_{G_1}, Q_{G_2})$ possesses the integer decomposition property;
(ii) The lattice polytope $\Omega(Q_{G_1}, Q_{G_2})$ is reflexive and possesses the integer decomposition property;
(iii) Both $G_1$ and $G_2$ are perfect.

A proof of part (b) will be achieved in Section 2. It would, of course, be of interest to find a complete characterization for $\Omega(Q_{G_1}, Q_{G_2})$ to be reflexive. For a finite simple graph $G$ on $[d]$, $\Omega(Q_G, Q_G)$ is called the Hansen polytope $\mathcal{H}(G)$ of $G$. This polytope possesses nice properties (e.g., centrally symmetric and 2-level) and is studied in [6, 22]. Especially, in [6], it is shown that if $G$ is perfect, then $\mathcal{H}(G)$ is reflexive. Theorem 1.1 (b) says that $G$ is perfect if and only if the Hansen polytope $\mathcal{H}(G)$ possesses the integer decomposition property.

If $G_1$ and $G_2$ are not perfect, $\Gamma(Q_{G_1}, Q_{G_2})$ may not possess the integer decomposition property (Examples 4.1 and 4.2). Furthermore, if $G_1$ and $G_2$ are not perfect, $\Omega(Q_{G_1}, Q_{G_2})$ may not be reflexive (Examples 4.2 and 4.3).

We now turn to the discussion of Ehrhart $\delta$-polynomials of $\Gamma(Q_{G_1}, Q_{G_2})$ and $\Omega(Q_{G_1}, Q_{G_2})$. Let, in general, $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension $d$. The Ehrhart $\delta$-polynomial of $\mathcal{P}$ is the polynomial

$$\delta(\mathcal{P}, \lambda) = (1 - \lambda)^{d+1} \left[ 1 + \sum_{n=1}^{\infty} \left| n\mathcal{P} \cap \mathbb{Z}^d \right| \lambda^n \right]$$

in $\lambda$. Each coefficient of $\delta(\mathcal{P}, \lambda)$ is a nonnegative integer and the degree of $\delta(\mathcal{P}, \lambda)$ is at most $d$. In addition $\delta(\mathcal{P}, 1)$ coincides with the normalized volume of $\mathcal{P}$, denoted by $\text{vol}(\mathcal{P})$. Refer the reader to [9, Part II] for the detailed information about Ehrhart $\delta$-polynomials. Moreover, in [12], the Ehrhart theory for stable set polytopes is studied.

The suspension of a finite simple graph $G$ on $[d]$ is the finite simple graph $\tilde{G}$ on $[d+1]$ with $E(\tilde{G}) = E(G) \cup \{\{i, d+1\} : i \in [d]\}$. We obtain the following theorem.

Theorem 1.2. Let $G_1$ and $G_2$ be finite perfect simple graphs on $[d]$. Then one has

$$\delta(\Omega(Q_{G_1}, Q_{G_2}), \lambda) = \delta(\Gamma(Q_{\tilde{G}_1}, Q_{\tilde{G}_2}), \lambda) = (1 + \lambda)\delta(\Gamma(Q_{G_1}, Q_{G_2}), \lambda).$$

Thus in particular

$$\text{vol}(\Omega(Q_{G_1}, Q_{G_2})) = \text{vol}(\Gamma(Q_{\tilde{G}_1}, Q_{\tilde{G}_2})) = 2 \cdot \text{vol}(\Gamma(Q_{G_1}, Q_{G_2})).$$
A proof of Theorem 1.2 will be given in Section 3. Even though the Ehrhart \( \delta \)-polynomial of \( \Omega(Q_{G_1}, Q_{G_2}) \) coincides with that of \( \Gamma(Q_{G_1}, Q_{G_2}) \), \( \Omega(Q_{G_1}, Q_{G_2}) \) may not be unimodularly equivalent to \( \Gamma(Q_{G_1}, Q_{G_2}) \) (Example 1.4).

2. Squarefree Gröbner basis

In this section, we prove Theorem 1.1 by using the theory of Gröbner bases and toric ideals. At first, we recall basic materials and notation on toric ideals. Let \( K[t^{\pm 1}, s] = K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, s] \) denote the Laurent polynomial ring in \( d + 1 \) variables over a field \( K \). For an integer vector \( \mathbf{a} = [a_1, \ldots, a_d]^T \in \mathbb{Z}^d \), the transpose of \([a_1, \ldots, a_d], \mathbf{t}^a \)s is the Laurent monomial \( t_1^{a_1} \cdots t_d^{a_d} s \in K[t^{\pm 1}, s] \). Given an integer \( d \times n \) matrix \( \mathbf{A} = [a_1, \ldots, a_n] \), where \( a_j = [a_{1j}, \ldots, a_{nj}]^T \) is the \( j \)th column of \( \mathbf{A} \), then we define the toric ring \( K[\mathbf{A}] \) of \( \mathbf{A} \) as follows:

\[
K[\mathbf{A}] = K[t^{a_1}, \ldots, t^{a_n}] \subset K[t^{\pm 1}, s].
\]

Let \( \mathbb{Z}^d_{\geq 0} \) denote the set of integer column vectors \([a_1, \ldots, a_d]^T\) with each \( a_i \geq 0 \), and let \( \mathbb{Z}^d_{\geq 0} \) denote the set of \( d \times n \) integer matrices \((a_{ij})_{1 \leq i \leq d, 1 \leq j \leq n}\) with each \( a_{ij} \geq 0 \).

In [21], the concept that \( \mathbf{A} \in \mathbb{Z}^{d \times n}_{\geq 0} \) and \( B \in \mathbb{Z}^{d \times m}_{\geq 0} \) are of harmony is introduced. For an integer vector \( \mathbf{a} = [a_1, \ldots, a_d]^T \in \mathbb{Z}^d \), let \( \mathbf{a}^{(+)} = [a_1^{(+)}, \ldots, a_d^{(+)})^T, \mathbf{a}^{(-)} = [a_1^{(-)}, \ldots, a_d^{(-)}]^T \in \mathbb{Z}^d_{\geq 0} \) where \( a_i^{(+)} = \max\{0, a_i\} \) and \( a_i^{(-)} = \max\{0, -a_i\} \). Note that \( \mathbf{a} = \mathbf{a}^{(+)} - \mathbf{a}^{(-)} \) holds in general. Given \( \mathbf{A} \in \mathbb{Z}^{d \times n}_{\geq 0} \) and \( \mathbf{B} \in \mathbb{Z}^{d \times m}_{\geq 0} \) such that the zero vector \( \mathbf{0}_d = [0, \ldots, 0]^T \in \mathbb{Z}^d \) is a column in each of \( \mathbf{A} \) and \( \mathbf{B} \), we say that \( \mathbf{A} \) and \( \mathbf{B} \) are of harmony if the following condition is satisfied: Let \( \mathbf{a} \) be a column of \( \mathbf{A} \) and \( \mathbf{b} \) that of \( \mathbf{B} \). Let \( \mathbf{c} = \mathbf{a} - \mathbf{b} \in \mathbb{Z}^d \). If \( \mathbf{c} = \mathbf{c}^{(+)} - \mathbf{c}^{(-)} \), then \( \mathbf{c}^{(+)} \) is a column vector of \( \mathbf{A} \) and \( \mathbf{c}^{(-)} \) is a column vector of \( \mathbf{B} \).

Now we prove the following theorem.

**Theorem 2.1.** Let \( \mathbf{A} = [a_1, \ldots, a_n] \in \mathbb{Z}^{d \times n}_{\geq 0} \) and \( \mathbf{B} = [b_1, \ldots, b_m] \in \mathbb{Z}^{d \times m}_{\geq 0} \), where \( a_n = b_m = \mathbf{0}_d \in \mathbb{Z}^d \), be of harmony. Let \( K[\mathbf{x}] = K[x_1, \ldots, x_n] \) and \( K[\mathbf{y}] = K[y_1, \ldots, y_m] \) be the polynomial rings over a field \( K \). Suppose that \( \text{in}_{<, \mathbf{A}}(I_A) \subset K[\mathbf{x}] \)
and \( \text{in}_{<B}(I_B) \subset K[y] \) are squarefree with respect to reverse lexicographic orders \(<_A \) on \( K[x] \) and \(<_B \) on \( K[y] \) respectively satisfying the condition that

- \( x_i \prec_A x_j \) if for each \( 1 \leq k \leq d \) \( a_{ki} \leq a_{kj} \).
- \( x_n \) is the smallest variable with respect to \(<_A \).
- \( y_m \) is the smallest variable with respect to \(<_B \).

Let \([−B, A]^*\) denote the \((d+1) \times (n+m+1)\) integer matrix

\[
\begin{bmatrix}
    -b_1 & \cdots & -b_m & a_1 & \cdots & a_n & 0_d \\
    -1 & \cdots & -1 & 1 & \cdots & 1 & 0
\end{bmatrix}.
\]

Then the toric ideal \( I_{[−B, A]}^* \) of \([−B, A]^*\) possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column \(0_{d+1} \in \mathbb{Z}^{d+1}\) of \([−B, A]^*\).

**Proof.** Let \( I_{[−B, A]^*} \subset K[x, y, z] = K[x_1, \ldots, x_n, y_1, \ldots, y_m, z] \) be the toric ideal of \([−B, A]^*\) defined by the kernel of

\[
\pi^* : K[x, y, z] \to K[[−B, A]^*] \subset K[t_1^{\pm 1}, \ldots, t_{d+1}^{\pm 1}, s]
\]

with \( \pi^*(z) = s \), \( \pi^*(x_i) = t_i^{a_i}t_{d+1}^{-s} \) for \( i = 1, \ldots, n \) and \( \pi^*(y_j) = t_j^{b_j}t_{d+1}^{-s} \) for \( j = 1, \ldots, m \). Assume that the reverse lexicographic orders \(<_A \) and \(<_B \) are induced by the orderings \( x_n <_A \cdots <_A x_1 \) and \( y_m <_B \cdots <_B y_1 \). Let \( <_\text{rev} \) be the reverse lexicographic order on \( K[x, y, z] \) induced by the ordering

\[
z <_\text{rev} x_n <_\text{rev} \cdots <_\text{rev} x_1 <_\text{rev} y_m <_\text{rev} \cdots <_\text{rev} y_1.
\]

In general, for an integer vector \( a = [a_1, \ldots, a_d]^\top \in \mathbb{Z}^d \), we let \( \text{supp}(a) = \{ i : 1 \leq i \leq d, a_i \neq 0 \} \). Set the following:

\[
\mathcal{E} = \{ (i, j) : 1 \leq i \leq n, 1 \leq j \leq m, \text{supp}(a_i) \cap \text{supp}(b_j) \neq \emptyset \}.
\]

If \( c = a_i - b_j \) with \((i, j) \in \mathcal{E}\), then it follows that \( c^{(+)} \neq a_i \) and \( c^{(-)} \neq b_j \). Since \( A \) and \( B \) are of harmony, we know that \( c^{(+)} \) is a column of \( A \) and \( c^{(-)} \) is a column of \( B \). It follows that \( f = x_iy_j - x_ky_{\ell} \neq 0 \) belongs to \( I_{[−B, A]^*} \), where \( c^{(+)} = a_k \) and \( c^{(-)} = b_{\ell} \). Then since for each \( 1 \leq c \leq d \), \( a_{ck} \leq a_{ci} \), one has \( x_k <_A x_i \) and \( \text{in}_{<\text{rev}}(f) = x_iy_j \). Hence

\[
\{ x_iy_j : (i, j) \in \mathcal{E} \} \subset \text{in}_{<\text{rev}}(I_{[−B, A]^*}).
\]

Moreover, it follows that \( x_ny_m - z^2 \in I_{[−B, A]^*} \) and \( x_ny_m \in \text{in}_{<\text{rev}}(I_{[−B, A]^*}) \). We set

\[
\mathcal{M} = \{ x_ny_m \} \cup \{ x_iy_j : (i, j) \in \mathcal{E} \} \cup \mathcal{M}_A \cup \mathcal{M}_B \subset \text{in}_{<\text{rev}}(I_{[−B, A]^*}),
\]

where \( \mathcal{M}_A \) (resp. \( \mathcal{M}_B \)) is the minimal set of squarefree monomial generators of \( \text{in}_{<A}(I_A) \) (resp. \( \text{in}_{<B}(I_B) \)). Let \( \mathcal{G} \) be a finite set of binomials belonging to \( I_{[−B, A]^*} \) with \( \mathcal{M} = \{ \text{in}_{<\text{rev}}(f) : f \in \mathcal{G} \} \).

Now, we prove that \( \mathcal{G} \) is a Gröbner base of \( \text{in}_{<\text{rev}}(I_{[−B, A]^*}) \) with respect to \( <_{\text{rev}} \).

By the following fact ([19](0.1), p. 1914) on Gröbner bases, we must prove the
following assertion: If \( u \) and \( v \) are monomials belonging to \( K[x, y, z] \) with \( u \neq v \) such that \( u \notin (\text{in}_<(g) : g \in \mathcal{G}) \) and \( v \notin (\text{in}_<(g) : g \in \mathcal{G}) \), then \( \pi^*(u) \neq \pi^*(v) \).

Suppose that there exists a nonzero irreducible binomial \( g = u - v \) belonging to \( I_{[-B, A]^*} \) such that \( u \notin (\text{in}_<(g) : g \in \mathcal{G}) \) and \( v \notin (\text{in}_<(g) : g \in \mathcal{G}) \). Write

\[
  u = \left( \prod_{p \in P} x_p^{i_p} \right) \left( \prod_{q \in Q} y_q^{j_q} \right), \quad v = z^\alpha \left( \prod_{p' \in P'} x_{p'}^{i_{p'}} \right) \left( \prod_{q' \in Q'} y_{q'}^{j_{q'}} \right),
\]

where \( P \) and \( P' \) are subsets of \([n] \), \( Q \) and \( Q' \) are subsets of \([m] \), \( \alpha \) is a nonnegative integer, and each of \( i_p, j_q, i_{p'}, j_{q'} \) is a positive integer. Since \( g = u - v \) is irreducible, one has \( P \cap P' = Q \cap Q' = \emptyset \). Furthermore, by the fact that each of \( x, y_j \) with \((i, j) \in \mathcal{E}\) can divide neither \( u \) nor \( v \), it follows that

\[
  \left( \bigcup_{p \in P} \text{supp}(a_p) \right) \cap \left( \bigcup_{q \in Q} \text{supp}(b_q) \right) = \left( \bigcup_{p' \in P'} \text{supp}(a_{p'}) \right) \cap \left( \bigcup_{q' \in Q'} \text{supp}(b_{q'}) \right) = \emptyset.
\]

Hence, since \( \pi^*(u) = \pi^*(v) \), it follows that

\[
  \sum_{p \in P} i_p a_p = \sum_{p' \in P'} i_{p'} a_{p'}, \quad \sum_{q \in Q} j_q b_q = \sum_{q' \in Q'} j_{q'} b_{q'}.
\]

Let \( \xi = \sum_{p \in P} i_p, \xi' = \sum_{p' \in P'} i_{p'}, \nu = \sum_{q \in Q} j_q, \) and \( \nu' = \sum_{q' \in Q'} j_{q'} \). Then \( \xi + \nu = \xi' + \nu' + \alpha \). Since \( \alpha \geq 0 \), it follows that either \( \xi \geq \xi' \) or \( \nu \geq \nu' \). Assume that \( \xi > \xi' \). Then

\[
  h = \prod_{p \in P} x_p^{i_p} - x_n^{\xi - \xi'} \left( \prod_{p' \in P'} x_{p'}^{i_{p'}} \right)
\]

belongs to \( I_A \) and \( I_{[-B, A]^*} \). If \( h \neq 0 \), then \( \text{in}_<(h) = \text{in}_{\text{rev}}(h) = \prod_{p \in P} x_p^{i_p} \) divides \( u \), a contradiction. Hence \( P = \{n\} \) and \( Q = \emptyset \). If \( \xi = \xi' \), then the binomial

\[
  h_0 = \prod_{p \in P} x_p^{i_p} - \prod_{p' \in P'} x_{p'}^{i_{p'}}
\]

belongs to \( I_A \) and \( I_{[-B, A]^*} \). Moreover, if \( h_0 \neq 0 \), then either \( \prod_{p \in P} x_p^{i_p} \) or \( \prod_{p' \in P'} x_{p'}^{i_{p'}} \) must belong to \( \text{in}_<(I_A) \) and \( \text{in}_{\text{rev}}(I_{[-B, A]^*}) \). This contradicts the fact that each of \( u \) and \( v \) can be divided by none of the monomials belonging to \( \mathcal{M} \). Hence \( h_0 = 0 \) and \( P = P' = \emptyset \). Similarly, \( Q = \{m\} \) and \( Q' = \emptyset \), or \( Q = Q' = \emptyset \). Hence we know that \( g = x_n^k y_m^\ell - z^\alpha \), where \( k \) and \( \ell \) are nonnegative integers. Since \( u \) cannot be divided by \( x_n y_m \), it follows that \( g = 0 \), a contradiction. Therefore, \( \mathcal{G} \) is a Gröbner base of \( \text{in}_{\text{rev}}(I_{[-B, A]^*}) \) with respect to \( <_{\text{rev}} \).

Now, we recall the following lemma.

**Lemma 2.2** ([12] Lemma 1.1]). Let \( P \subset \mathbb{R}^d \) be a lattice polytope of dimension \( d \) such that the origin of \( \mathbb{R}^d \) is contained in its interior and \( P \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \). Suppose that any integer point in \( \mathbb{Z}^{d+1} \) is a linear integer combination of the integer
points in $\mathcal{P} \times \{1\}$ and there exists an ordering of the variables $x_{i_1} < \cdots < x_{i_n}$ for which $a_{i_j} = 0_d$ such that the initial ideal $\text{in}_{<}(I_A)$ of the toric ideal $I_A$ with respect to the reverse lexicographic order $<$ on the polynomial ring $K[x_1, \ldots , x_n]$ induced by the ordering is squarefree, where $A = [a_1, \ldots , a_n]$. Then $\mathcal{P}$ is a reflexive polytope which possesses the integer decomposition property.

By Theorem 2.1 and this lemma, we obtain the following corollary.

**Corollary 2.3.** Work with the same situation as in Theorem 2.1. Let $\mathcal{P} \subset \mathbb{R}^{d+1}$ be the lattice polytope of dimension $d + 1$ with

$$\mathcal{P} \cap \mathbb{Z}^{d+1} = \left\{ \begin{bmatrix} a_1 \\ 1 \end{bmatrix}, \ldots , \begin{bmatrix} a_n \\ 1 \end{bmatrix}, \begin{bmatrix} -b_1 \\ -1 \end{bmatrix}, \ldots , \begin{bmatrix} -b_m \\ -1 \end{bmatrix}, 0_{d+1} \right\}. $$

Suppose that $0_{d+1} \in \mathbb{Z}^{d+1}$ belongs to the interior of $\mathcal{P}$ and any integer point in $\mathbb{Z}^{d+2}$ is a linear integer combination of the integer points in $\mathcal{P} \times \{1\}$. Then $\mathcal{P}$ is a reflexive polytope which possesses the integer decomposition property.

Recall that an integer matrix $A$ is compressed ([18], [24]) if the initial ideal of the toric ideal $I_A$ is squarefree with respect to any reverse lexicographic order.

Finally, we prove Theorem 1.1.

**Proof of Theorem 1.1.** For a finite simple graph $G$ on $[d]$, let $A_{S(G)}$ be the matrix whose columns are those $\rho(W)$ with $W \in S(G)$. If $W \in S(G)$, then each subset of $W$ is also a stable set of $G$. This means that $S(G)$ is a simplicial complex on $[d]$. Hence it is easy to show that $A_{S(G_1)}$ and $A_{S(G_2)}$ are of harmony. Moreover, for any perfect graph $G$, $A_{S(G)}$ is compressed ([18] Example 1.3 (c)). Let $\mathcal{P} \subset \mathbb{R}^{d+1}$ be the convex hull of $\{ \pm(a_1 + e_{d_1}), \ldots , \pm(a_d + e_{d_1}), \pm e_{d_1} \}$. Then it follows that $0_{d+1} \in \mathbb{Z}^{d+1}$ belongs to the interior of $\mathcal{P}$ and any integer point in $\mathbb{Z}^{d+2}$ is a linear integer combination of the integer points in $\mathcal{P}$. Moreover, we have $\mathcal{P} \subset \Omega(Q_{G_1}, Q_{G_2})$. This implies that $0_{d+1} \in \mathbb{Z}^{d+1}$ belongs to the interior of $\Omega(Q_{G_1}, Q_{G_2})$ and any integer point in $\mathbb{Z}^{d+2}$ is a linear integer combination of the integer points in $\Omega(Q_{G_1}, Q_{G_2}) \times \{1\}$. On the other hand, one has

$$\Omega(Q_{G_1}, Q_{G_2}) \cap \{ [a_1, \ldots , a_{d+1}]^T \in \mathbb{R}^{d+1} : a_{d+1} = 0 \} = \frac{1}{2}(Q_{G_1} - Q_{G_2}) \times \{0\}. $$

Since $\frac{1}{2}(Q_{G_1} - Q_{G_2}) \cap \mathbb{Z}^d = \{0_d\}$, we obtain

$$\Omega(Q_{G_1}, Q_{G_2}) \cap \mathbb{Z}^{d+1} = \left\{ \begin{bmatrix} a \\ 1 \end{bmatrix} : a \in Q_{G_1} \cap \mathbb{Z}^d \right\} \cup \left\{ \begin{bmatrix} -b \\ -1 \end{bmatrix} : b \in Q_{G_2} \cap \mathbb{Z}^d \right\} \cup \{0_{d+1}\}. $$

Hence, by Corollary 2.3 if $G_1$ and $G_2$ are perfect, $\Omega(Q_{G_1}, Q_{G_2})$ is a reflexive polytope which possesses the integer decomposition property.

Next, we prove that if $G_1$ is not perfect, then $\Omega(Q_{G_1}, Q_{G_2})$ does not possess the integer decomposition property. Assume that $G_1$ is not perfect and $\Omega(Q_{G_1}, Q_{G_2})$ possesses the integer decomposition property. By the strong perfect graph theorem
\( \text{Since } 2 < \frac{2\ell + 2}{\ell} \leq 3, \ a \in 3\Omega(Q_{G_1}, Q_{G_2}). \) Hence there exist \( a_1, a_2, a_3 \in \Omega(Q_{G_1}, Q_{G_2}) \cap \mathbb{Z}^{d+1} \) such that \( a = a_1 + a_2 + a_3. \) Then we may assume that \( a_1, a_2 \in Q_C \times \{1\} \) and \( a_3 = 0_{d+1}. \) However, since the maximal cardinality of the stable sets of \( C \) in [2\( \ell \) + 1] equals \( \ell, \) a contradiction.

Suppose that \( G_1 \) possesses an odd antihole \( C \) such that the length of \( C \) equals 2\( \ell \) + 1, where \( \ell \geq 2 \) and we regard \( C \) as a finite graph on \([d] \). Similarly, we may assume that the edge set of \( C \) in [2\( \ell \) + 1] equals \( \ell, \) a contradiction.

Therefore, if \( \Omega(Q_{G_1}, Q_{G_2}) \) possesses the integer decomposition property, then \( G_1 \) and \( G_2 \) are perfect, as desired.

\[ \square \]

3. Ehrhart \( \delta \)-polynomials

In this section, we consider the Ehrhart \( \delta \)-polynomials and the volumes of the polytopes \( \Omega(Q_{G_1}, Q_{G_2}) \) and \( \Gamma(Q_{G_1}, Q_{G_2}) \), in particular, we prove Theorem \[1.2\]. Let \( \mathcal{P} \subseteq \mathbb{R}^d \) be a lattice polytope of dimension \( d \) with \( \mathcal{P} \cap \mathbb{Z}^d = \{a_1, \ldots, a_n\} \). Set \( A = [a_1, \ldots, a_n] \). We define the toric ring \( K[\mathcal{P}] \) and the toric ideal \( I_{\mathcal{P}} \) of \( \mathcal{P} \) by \( K[A] \) and \( I_A \). In order to prove Theorem \[1.2\], we use the following facts.

- If \( \mathcal{P} \) possesses the integer decomposition property, then the Ehrhart polynomial \( n\mathcal{P} \cap \mathbb{Z}^d \) of \( \mathcal{P} \) is equal to the Hilbert polynomial of the toric ring \( K[\mathcal{P}] \).
Let $S$ be a polynomial ring and $I \subset S$ be a graded ideal of $S$. Let $<$ be a monomial order on $S$. Then $S/I$ and $S/\text{in}_<(I)$ have the same Hilbert function (see [3, Corollary 6.1.5]).

Now, we prove the following theorem.

**Theorem 3.1.** Work with the same situation as in Theorem 2.1. Let $\mathcal{P} \subset \mathbb{R}^d$ be the lattice polytope with $\mathcal{P} \cap \mathbb{Z}^d = \{a_1, \ldots, a_n\}$ and $\mathcal{Q} \subset \mathbb{R}^d$ the lattice polytope with $\mathcal{Q} \cap \mathbb{Z}^d = \{b_1, \ldots, b_m\}$. Suppose that any integer point in $\mathbb{Z}^{d+1}$ is a linear integer combination of the integer points in $\Gamma(\mathcal{P}, \mathcal{Q}) \times \{1\}$, any integer point in $\mathbb{Z}^{d+2}$ is a linear integer combination of the integer points in $\Omega(\mathcal{P}, \mathcal{Q}) \times \{1\}$,

$$\Gamma(\mathcal{P}, \mathcal{Q}) \cap \mathbb{Z}^d = \{a_1, \ldots, a_{n-1}, -b_1, \ldots, -b_{m-1}, 0_d\}$$

and

$$\Omega(\mathcal{P}, \mathcal{Q}) \cap \mathbb{Z}^{d+1} = \left\{ \begin{bmatrix} a_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} a_n \\ 1 \end{bmatrix}, \begin{bmatrix} -b_1 \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} -b_m \\ -1 \end{bmatrix}, 0_{d+1} \right\}.$$

Then we obtain

$$\delta(\Omega(\mathcal{P}, \mathcal{Q}), \lambda) = (1 + \lambda)\delta(\Gamma(\mathcal{P}, \mathcal{Q}), \lambda).$$

In particular,

$$\text{vol}(\Omega(\mathcal{P}, \mathcal{Q})) = 2 \cdot \text{vol}(\Gamma(\mathcal{P}, \mathcal{Q})).$$

**Proof.** Set $\mathcal{R} = \text{conv}\{\Gamma(\mathcal{P}, \mathcal{Q}) \times \{0\}, \pm e_{d+1}\}$. Then it follows from [3, Theorem 1.4] that $\delta(\mathcal{R}, \lambda) = (1 + \lambda)\delta(\Gamma(\mathcal{P}, \mathcal{Q}), \lambda)$. Moreover, by [21, Theorem 1.1] and Theorem 2.1, $\mathcal{R}$ and $\Omega(\mathcal{P}, \mathcal{Q})$ possess the integer decomposition property. Hence we should show that $K[\mathcal{R}]$ and $K[\Omega(\mathcal{P}, \mathcal{Q})]$ have the same Hilbert function.

Now, use the same notation as in the proof of Theorem 2.1. Then we have

$$\frac{K[x, y, z]}{\text{in}_<(M_{\Omega(\mathcal{P}, \mathcal{Q})})} = \frac{K[x, y, z]}{(M)}.$$

Set

$$a'_i = \begin{cases} \begin{bmatrix} a_i \\ 1 \end{bmatrix}, & 1 \leq i \leq n - 1, \\ e_{d+1}, & i = n, \\ 0_{d+1}, & i = n + 1, \end{cases}$$

and

$$b'_j = \begin{cases} \begin{bmatrix} b_j \\ 0 \end{bmatrix}, & 1 \leq j \leq m - 1, \\ e_{d+1}, & j = m, \\ 0_{d+1}, & j = m + 1. \end{cases}$$

Then it is easy to show that $A' = [a'_1, \ldots, a'_{n+1}]$ and $B' = [b'_1, \ldots, b'_{m+1}]$ are of harmony. Moreover, $\text{in}_{<_{B'}}(I_{A'}) \subset K[y_1, \ldots, y_{m+1}]$ and $\text{in}_{<_{A'}}(I_{A'}) \subset K[x_1, \ldots, x_{n+1}]$ are squarefree with respect to reverse lexicographic orders $<_{A'}$ on $K[x_1, \ldots, x_{n+1}]$ and $<_{B'}$ on $K[y_1, \ldots, y_{m+1}]$ induced by the orderings $x_{n+1} <_{A'} x_n <_{A'} \cdots <_{A'} x_1$ and $y_{m+1} <_{B'} y_m <_{B'} \cdots <_{B'} y_1$. Now, we introduce the following:

$$\mathcal{E}' = \{ (i, j) : 1 \leq i \leq n, 1 \leq j \leq m, \text{supp}(a'_i) \cap \text{supp}(b'_j) \neq \emptyset \}.$$
Then we have $E' = E \cup \{(n, m)\}$. Let $M_{A'}$ (resp. $M_{B'}$) be the minimal set of squarefree monomial generators of $\text{in}_{< A'}(I_{A'})$ (resp. $\text{in}_{< B'}(I_{B'})$). Then it follows that $M_{A'} = M_A$ and $M_{B'} = M_B$. This says that $M = E' \cup M_{A'} \cup M_{B'}$. By the proof of [21, Theorem 1.1], we obtain $\text{in}_{\text{rev}}(I_R) = (M) \subset K[x, y, z]$. Hence it follows that $K[x, y, z]$ in $\text{in}_{\text{rev}}(\Omega(P, Q)) = K[x, y, z]$ in $\text{in}_{\text{rev}}(I_R)$.

Therefore, $K[R]$ and $K[\Omega(P, Q)]$ have the same Hilbert function, as desired. □

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. For any finite simple graph $G$ on $d$, we have $S(\hat{G}) = S(G) \cup \{d + 1\}$. Hence it follows that $\Gamma(G_1, G_2) = \text{conv}\{\Gamma(G_1, G_1) \times \{0\}, \pm e_{d+1}\}$. Therefore, by Theorem 3.1, we obtain

$$\delta(\Omega(G_1, G_2), \lambda) = \delta(\Gamma(G_1, G_2), \lambda) = (1 + \lambda)\delta(\Gamma(G_1, G_2), \lambda),$$

as desired. □

4. Examples

In this section, we give some curious examples of $\Gamma(G_1, G_2)$ and $\Omega(G_1, G_2)$. At first, the following example says that even though $G_1$ and $G_2$ are not perfect, $\Omega(G_1, G_2)$ may be reflexive.

Example 4.1. Let $G$ be the finite simple graph as follows:

$$G:$$

\[ \text{Namely, } G \text{ is a cycle of length 5. Then } G \text{ is not perfect. Hence } \Gamma(G, G) \text{ is not reflexive. However, } \Omega(G, G) \text{ is reflexive. In fact, we have} \]

$$\delta(\Gamma(G, G), \lambda) = 1 + 15\lambda + 60\lambda^2 + 62\lambda^3 + 15\lambda^4 + \lambda^5,$$

$$\delta(\Omega(G, G), \lambda) = 1 + 16\lambda + 75\lambda^2 + 124\lambda^3 + 75\lambda^4 + 16\lambda^5 + \lambda^6.$$

Moreover, $\Gamma(G, G)$ possesses the integer decomposition property, but $\Omega(G, G)$ does not possess the integer decomposition property.
For this example, $\Gamma(Q_G, Q_G)$ possesses the integer decomposition property. Next example says that if $G_1$ and $G_2$ are not perfect, $\Gamma(Q_{G_1}, Q_{G_2})$ may not possess the integer decomposition property.

**Example 4.2.** Let $G$ be a finite simple graph whose complementary graph $\overline{G}$ is as follows:

![Graph](image)

Then $G$ is not perfect. Hence $\Gamma(Q_G, Q_G)$ is not reflexive. However, $\Omega(Q_G, Q_G)$ is reflexive. Moreover, in this case, $\Gamma(Q_G, Q_G)$ and $\Omega(Q_G, Q_G)$ do not possess the integer decomposition property.

For any finite simple graph $G$ with at most 6 vertices, $\Omega(Q_G, Q_G)$ is always reflexive. However, in the case of finite simple graphs with more than 6 vertices, we obtain a different result.

**Example 4.3.** Let $G$ be the finite simple graph as follows:

![Graph](image)

Namely, $G$ is a cycle of length 7. Then $G$ is not perfect. Hence $\Gamma(Q_G, Q_G)$ is not reflexive. Moreover, $\Omega(Q_G, Q_G)$ is not reflexive. In fact, we have

$$
\delta(\Gamma(Q_G, Q_G), \lambda) = 1 + 49\lambda + 567\lambda^2 + 1801\lambda^3 + 1799\lambda^4 + 569\lambda^5 + 49\lambda^6 + \lambda^7,
$$

$$
\delta(\Omega(Q_G, Q_G), \lambda) = 1 + 50\lambda + 616\lambda^2 + 2370\lambda^3 + 3598\lambda^4 + 2368\lambda^5 + 618\lambda^6 + 50\lambda^7 + \lambda^8.
$$

Finally, we show that even though the Ehrhart $\delta$-polynomial of $\Omega(Q_{G_1}, Q_{G_2})$ coincides with that of $\Gamma(Q_{\overline{G_1}}, Q_{\overline{G_2}})$, $\Omega(Q_{G_1}, Q_{G_2})$ may not be unimodularly equivalent to $\Gamma(Q_{\overline{G_1}}, Q_{\overline{G_2}})$. 

11
Example 4.4. Let $G$ be the finite simple graph as follows:

$G$:

Namely, $G$ is a $(2, 2, 2)$-complete multipartite graph. Then $G$ is perfect. Hence we know that $\Omega(Q_G, Q_G)$ and $\Gamma(Q_G, Q_G)$ have the same Ehrhart $\delta$-polynomial and the same volume. However, $\Omega(Q_G, Q_G)$ has 54 facets and $\Gamma(Q_G, Q_G)$ has 432 facets. Hence, $\Omega(Q_G, Q_G)$ and $\Gamma(Q_G, Q_G)$ are not unimodularly equivalent. Moreover, for any finite simple graph $G'$ on $\{1, \ldots, 7\}$ except for $\hat{G}$, the Ehrhart $\delta$-polynomial of $\Gamma(Q_{G'}, Q_{G'})$ is not equal to that of $\Omega(Q_G, Q_G)$. This implies that the class of $\Omega(Q_{G_1}, Q_{G_2})$ is a new class of reflexive polytopes.

**References**

[1] C. A. Athanasiadis, $h^*$-vectors, Eulerian polynomials and stable polytopes of graphs, *Electron. J. Combin.*, 11 (2004), 1–13.

[2] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.*, 3 (1994), 493–535.

[3] M. Beck, P. Jayawant, and T. B. McAllister, Lattice-point generating functions for free sums of convex sets, *J. Combin. Theory, Ser. A* 120 (2013), 1246–1262.

[4] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, *Ann. of Math.* 164 (2006), 51–229.

[5] D. Cox, J. Little and H. Schenck, “Toric varieties”, Amer. Math. Soc., 2011.

[6] R. Freij, M. Henze, M. W. Schmitt and G. M. Ziegler, Face numbers of centrally symmetric polytopes from split graphs, *Electron. J. Combin.* 20 (2013), 1–13.

[7] C. Haase and H. V. Melinkov, The Reflexive Dimension of a Lattice Polytope, *Ann. Comb.* 10 (2006), 211–217.

[8] H. Herzog and T. Hibi, “Monomial Ideals”, Graduate Text in Mathematics, Springer, 2011.

[9] T. Hibi, “Algebraic Combinatorics on Convex Polytopes,” Carslaw Publications, Glebe NSW, Australia, 1992.

[10] T. Hibi, Ed., “Gröbner Bases: Statistics and Software Systems,” Springer, 2013.

[11] T. Hibi and K. Matsuda, Quadratic Gröbner bases of twinned order polytopes, *European J. Combin.* 54 (2016), 187–192.

[12] T. Hibi, K. Matsuda, H. Ohsugi and K. Shibata, Centrally symmetric configurations of order polytopes, *J. Algebra* 443 (2015), 469–478.

[13] T. Hibi, K. Matsuda and A. Tsuchiya, Quadratic Gröbner bases arising from partially ordered sets, *Math. Scand.* 121 (2017), 19–25.
[14] T. Hibi, K. Matsuda and A. Tsuchiya, Gorenstein Fano polytopes arising from order polytopes and chain polytopes, arXiv:1507.03221.
[15] T. Hibi and A. Tsuchiya, Facets and volume of Gorenstein Fano polytopes, Math. Nachr. 290 (2017), 2619–2628.
[16] M. Kreuzer and H. Skarke, Complete classification of reflexive polyhedra in four dimensions, Adv. Theor. Math. Phys. 4(2000), 1209–1230.
[17] J. C. Lagarias and G. M. Ziegler, Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, Canad. J. Math. 43(1991), 1022–1035.
[18] H. Ohsugi and T. Hibi, Convex polytopes all of whose reverse lexicographic initial ideals are squarefree, Proc. Amer. Math. Soc. 129 (2001), 2541–2546.
[19] H. Ohsugi and T. Hibi, Quadratic initial ideals of root systems, Proc. Amer. Math. Soc. 130 (2002), 1913–1922.
[20] H. Ohsugi and T. Hibi, Centrally symmetric configurations of integer matrices Nagoya Math. J. 216 (2014), 153-170
[21] H. Ohsugi and T. Hibi, Reverse lexicographic squarefree initial ideals and Gorenstein Fano polytopes, J. Commut. Alg., to appear.
[22] R. Sanyal, A. Werner and G. M. Ziegler On Kalai’s conjectures about centrally symmetric polytopes, Discrete Comput. Geometry 41 (2009), 183 – 198.
[23] A. Schrijver, “Theory of Linear and Integer Programming”, John Wiley & Sons, 1986.
[24] S. Sullivant, Compressed polytopes and statistical disclosure limitation, Tohoku Math. J. 58 (2006), 433 – 445.

(Takayuki Hibi) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN
E-mail address: hibi@math.sci.osaka-u.ac.jp

(Akiyoshi Tsuchiya) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN
E-mail address: a-tsuchiya@ist.osaka-u.ac.jp