Wigner functional theory for quantum optics

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Based on the recent derivation of quadrature bases that incorporate the spatiotemporal degrees of freedom [Phys. Rev. A 98, 043841 (2018)], we develop a Wigner functional theory for quantum optics, as an extension of the Moyal formalism. Since the spatiotemporal quadrature bases span the complete Hilbert space of all quantum optical states, it does not require factorization as a tensor product of discrete Hilbert spaces. The Wigner functions associated with such a space become functional operations and are expressed by functional integrals — the functional version of the star product. The resulting formalism enables tractable calculations for scenarios where both spatiotemporal degrees of freedom and particle-number degrees of freedom are relevant. To demonstrate it, we compute examples of Wigner functionals for a few well-known states and operators.

I. INTRODUCTION

Quantum information technology promises to provide secure communication [1], more accurate measurements [2] and more efficient computations [3], among other benefits. However, quantum states are often fragile. The purity and coherence of such states, for instance, are easily lost when such states interact with the environment [4].

To increase the information capacity of quantum systems [5–7] and to improve the security in quantum cryptography [8–10], the states are often prepared in higher dimensional Hilbert spaces. An example is the spatial modes of photons, such as orbital angular momentum (OAM) modes [11,12]. They represent an infinite dimensional Hilbert space. Applications that use such higher dimensional Hilbert spaces are usually implemented in terms of individual photons encoded in terms of their spatial degrees of freedom. Losses and stray photons tend to reduce the signal-to-noise ratio, slowing down the rate at which such systems can operate [13].

A way to overcome the losses and noise issues is to prepare multiphoton states that also incorporate different spatial modes. Such quantum systems are represented in terms of both their spatiotemporal degrees of freedom and particle-number degrees of freedom [14]. They are often rather complex and difficult to analyze. One approach is to duplicate the operator formalism for a single-mode multi-particle system several times to handle several discrete modes. Numerous such implementations exist [15,16]. The result is best applied in cases of Gaussian states that can be represented in terms of a few discrete spatial modes [17].

In a different development, started during the Second World War, it was independently shown by Groenewold [18] and Moyal [19] that quantum mechanics can be successfully formulated without operators. This formulation of quantum mechanics in phase space [20] represents the states and operators by functions of phase space variables (analogues to position and momentum for the harmonic oscillator). Examples of such functions are the quasi-probability distributions that include the Glauber-Sudarshan $P$-distribution [21,22], the Husimi $Q$-distribution [23] and the Wigner distribution [24]. Products of operators are represented by so-called star products of the phase space functions. The Moyal formulation was shown to reproduce all the uncertainty relations associated with quantum mechanics.

One of the challenges initially encountered with the Moyal formulation was how to incorporate other degrees of freedom (apart from the particle-number degrees of freedom) into the formulation. In the case of spin (and other internal symmetries), the problem was overcome with the aid of the Stratonovich-Weyl correspondence [25,26]. For the spatial degrees of freedom, one can use a similar approach [27] or other approaches (see for example [28]), but these again lead to a finite set of discrete spatial modes (and often tend to return to an operator-based approach).

Recently, we found a way to combine the spatiotemporal degrees of freedom and the particle-number degrees of freedom into one comprehensive Hilbert space [29]. It is spanned by spatiotemporal quadrature bases that are generalizations of the quadrature bases associated with only the particle-number degrees of freedom.

In this article, guided by the Moyal formalism, we use the spatiotemporal quadrature bases to develop a formalism that incorporates both the spatiotemporal degrees of freedom and the particle-number degrees of freedom. For this purpose, we choose the Wigner distribution, since they are naturally related to the quadrature bases, however, one could do the same with the other quasi-probability distributions. In this approach, these quadrature bases are used to generalize the standard Wigner distributions to become Wigner functionals. The development parallels the normal theory of Wigner functions (and of the Moyal formalism), showing that most properties can be carried over to the functional formalism. However, analyses now tend to involve functional integrals. Though the expressions may appear familiar, the resemblance is deceptive in that it now incorporates all the spatiotemporal degrees of freedom.

The involvement of functional integrals in the new for-
nalism may create the impression that any analysis that is done with this formalism would be severely complex and often intractable. However, thanks to the close analogy between the well-known Wigner distributions and the Wigner functional approach presented here, such calculations are found to be generally quite tractable. It is true that, apart from some special cases, one can evaluate such functional integrals only when the integrand is in the form of a Gaussian functional. However, with the aid of auxiliary variables, source terms and generating functionals, it is often possible to represent the quantum states and operations in terms of Gaussian functionals, even if the original functional expressions are not of that form.

To demonstrate its usefulness, we use the formalism to compute the Wigner functionals for a few well-known states and operators. For example, we compute a generating functional for the Wigner functionals of fixed-spectrum Fock states. The term fixed-spectrum indicates that all the photons in the state involve the same spectrum of plane waves. Although the Wigner functionals of Fock states are not in Gaussian form, their generating function is in Gaussian form and can therefore be used in calculations involving functional integrals.

The functional integral form of the formalism may suggest a connection between it and the path integral formalism used in quantum field theory \[3\]. However, there are significant differences, especially in terms of the context. Quantum field theory intends to study the fundamental dynamics of nature. Therefore, path integrals formulated for this purpose always contain the exponentiated action for the dynamics. In the current context, the dynamics is rather trivial — free-space propagation of optical fields. It is therefore already built into the dispersion relation that connects the wave vector with the angular frequency. So the integrand of the functional integrals do not include an exponentiated action. On the other hand, the input and output states that are usually considered in particle physics are rather trivial — single excitation of specific sets of particles — whereas quantum optical scenarios usually involve complex states with complex measurements performed on them. Perhaps the most significant difference between the two formalisms is the nature of the functions or paths that are being integrated. In quantum field theory, the fields (paths) represent single excitations of particular types of particles. In the Wigner functional formalism presented here, the functions are associated with the eigenvalue functions of quadrature operators, which cannot be associated with single excitations. Instead, they are parameter functions that incorporate information about all the degrees of freedom in the system. Hence, the formalism presented here should not be confused with quantum field theory.

The paper is organized as follows. In Sec. \[II\] we review the spatiotemporal quadrature basis, together with some background on other aspects that we need in the rest of the paper. The definition of the Wigner functionals and related quantities are discussed in Sec. [\[III\] Some examples of Wigner functionals are computed in Sec. [\[IV\] We provide a discussion in Sec. [\[V\] and end with conclusions in Sec. [\[VI\].

## II. BACKGROUND

### A. Eigenstates of quadrature operators

The quadrature bases in terms of which the Wigner functional formalism for quantum optics is defined, are obtained as eigenstates of the fixed-momentum quadrature operators. These eigenvalue equations are given by

\[
\hat{q}_s(k) |q\rangle = |q\rangle q_s(k),
\hat{p}_s(k) |p\rangle = |p\rangle p_s(k).
\]

Here, \(k\) represents the three-dimensional wave vector and the subscript \(s\) is the spin index. These quadrature operators are directly defined in terms of the creation and annihilation operators \(\hat{a}_s(k)\) and \(\hat{a}_s^\dagger(k)\) that are obtained from the quantization of the electromagnetic field

\[
\hat{q}_s(k) = \frac{1}{\sqrt{2}} \left[ \hat{a}_s(k) + \hat{a}_s^\dagger(k) \right],
\hat{p}_s(k) = \frac{1}{\sqrt{2}} \left[ \hat{a}_s(k) - \hat{a}_s^\dagger(k) \right].
\]

The creation and annihilation operators obey a Lorentz covariant commutation relation, given by

\[
\left[ \hat{a}_s(k_1), \hat{a}_s^\dagger(k_2) \right] = (2\pi)^3 \omega_1 \delta_{s,r} \delta(k_1 - k_2),
\]

where \(\omega_1 = c|k_1|\) is the angular frequency, given in terms of the free-space dispersion relation, and \(\delta_{s,r}\) is the Kronecker delta for the spin indices. The equivalent Lorentz covariant commutation relation for the fixed-momentum quadrature operators reads

\[
[\hat{q}_s(k_1), \hat{p}_r(k_2)] = i(2\pi)^3 \omega_1 \delta_{s,r} \delta(k_1 - k_2).
\]

Although the two quadrature operators \(\hat{q}_s(k)\) and \(\hat{p}_s(k)\) in Eq. (4) are unique operator-valued functions of the wave vector, the eigenvalue functions \(q_s(k)\) and \(p_s(k)\) and their association eigenstates \(|q\rangle\) and \(|p\rangle\) are not unique — there are an infinite number of them. However, for each eigenvalue function there is a unique eigenstate, which is associated with the function as a whole and not with particular function values of the eigenvalue function. For that reason, the eigenstate does not explicitly depend on the value of the wave vector.

To simplify notation, we shall neglect the spin degrees of freedom and not display the spin indices in the remainder of this paper. It is nevertheless straightforward to reintroduce them if necessary.

The eigenstates in Eq. (4) can be expressed by

\[
|q\rangle = \hat{a}_q^\dagger |\text{vac}\rangle,
|p\rangle = \hat{a}_p^\dagger |\text{vac}\rangle,
\]
in terms of special quadrature creation operators
\[ \hat{a}_q^\dagger = \pi^{-\Omega/4} \exp \left( -\frac{1}{2i}\|q(k)\|^2 + \hat{a}_Q^1 - \hat{a}_R^1 \right), \]
\[ \hat{a}_p^\dagger = 2^{\Omega/2}\pi^{\Omega/4} \exp \left( -\frac{1}{2i}\|p(k)\|^2 + i\hat{a}_p^1 + \hat{a}_R^1 \right), \]  
(6)
where
\[ \hat{a}_Q^1 = \sqrt{2} \int \hat{a}_Q^1(k) q(k) \, dk, \]
\[ \hat{a}_p^1 = \sqrt{2} \int \hat{a}_p^1(k) p(k) \, dk, \]  
(7)
\[ \hat{a}_R^1 = \frac{1}{2} \int \hat{a}_R^1(k) \hat{a}_p^1(k) \, dk, \]
and, for an arbitrary (complex-valued) function \( f(k) \),
\[ ||f(k)||^2 = \int |f(k)|^2 \, dk. \]  
(8)
The quantity \( \Omega \) in Eq. (6) represents a divergent constant. It is given by
\[ \Omega \equiv \int \delta(0) \, dk. \]  
(9)
and represents the cardinality of a countable infinite set \( \Omega = \aleph_0 \). The integration measures in Eqs. (6) and below are given in terms of a simplified notation
\[ dk \equiv \frac{d^3k}{(2\pi)^3\omega}. \]  
(10)
Note that all the wave vector dependences are integrated out in Eq. (6) so that the quadrature bases elements do not explicitly depend on the wave vector.

The quadrature bases obey orthogonality conditions, expressed in terms of Dirac delta functionals
\[ \langle q | q' \rangle = \delta(q - q'), \]
\[ \langle p | p' \rangle = (2\pi)^\Omega \delta(p - p'). \]  
(11)
The square brackets indicate that the quantity is a functional (a function of functions), where \( q \) and \( p \) represent functions. It depends on the entire functions and not on a particular function value of that function. For that reason, we do not show the arguments of the functions inside the square bracket \( \delta(q(k) - q'(k)) \), because the quantity does not explicitly depend on \( k \).

The constant in the expressions in Eqs. (6) and (11) differ from those in Ref. [30]. Their derivations are discussed in Appen. A.

B. Functional integrals

An important quantity is the overlap \( \langle q | p \rangle \), which reads
\[ \langle q | p \rangle = \exp \left[ i \int q(k)p(k) \, dk \right]. \]  
(12)
It appears when expressions are converted from one quadrature basis into another mutually unbiased quadrature basis and thus can act as the kernel of a kind of Fourier transform. These Fourier transforms suggest a functional (or path-integral) approach for any analysis involving the spatiotemporal quadrature bases.

The expressions of functional integrals are in general rather complex. However, since these functional integrals can, apart from some special cases, only be evaluated when their integrands are in Gaussian form, one can simplify the notation. The Gaussian form implies an exponential function with an argument consisting of integrals over some degrees of freedom, typically the three-dimensional wave vectors. The integrands of these integrals are products of functions of the wave vectors. There may be multiple sets of wave vectors that are being integrated. Usually, a given set of wave vectors would appear exactly twice as arguments of functions in each term, thus connecting a pair of functions in the term. We denote such a connection by a binary operator \( \diamond \).

As an example, we introduce the following notation for the inner product between two functions
\[ \langle f, g \rangle \equiv \int f^*(k)g(k) \, dk \equiv f\diamond g, \]  
(13)
which implies that \( \langle q | p \rangle \equiv \exp(iq \diamond p) \). If there is a kernel function involved, we have
\[ f\diamond B \diamond g \equiv \int f^*(k)B(k,k')g(k') \, dk \, dk'. \]  
(14)
Note that it is not equivalent to
\[ f\diamond T \diamond g \neq \int f^*(k)T(k)g(k) \, dk. \]  
(15)

The completeness conditions for the spatiotemporal quadrature bases are represented as functional integrals
\[ \int |q \rangle \langle q| \, \mathcal{D}[q] = \mathbb{1}, \]
\[ \int |p \rangle \langle p| \, \mathcal{D}[p] = \mathbb{1}, \]  
(16)
where \( \mathbb{1} \) is the identity operator for the entire Hilbert space of all quantum optical states. The functional measures in Eq. (16) run over all finite-energy real-valued continuously differentiable functions. The measure for the integral over \( p \) incorporates a normalization constant
\[ \mathcal{D}^\circ[p] = \frac{1}{(2\pi)^\Omega} \mathcal{D}[p]. \]  
(17)
The derivation of the completeness conditions in Eq. (16) is discussed in Appen. B.

Using Eqs. (11), (12), and (16), one can show that
\[ \int \exp(iq_1 \diamond p - iq_2 \diamond p) \, \mathcal{D}^\circ[p] = \delta[q_1 - q_2], \]
\[ \int \exp(-iq_1 \diamond p_1 + iq_2 \diamond p_2) \, \mathcal{D}[q] = (2\pi)^\Omega \delta[p_1 - p_2]. \]  
(18)
Combining these integrals and converting the “variables” (fields) into complex variables, given by
\[ \alpha(k) = \frac{1}{\sqrt{2}} [\bar{q}(k) + ip(k)], \]
one gets
\[ \int \exp \left( \alpha^* \phi - \alpha \phi^* \right) \mathcal{D}[\phi] = (2\pi)^\alpha \delta[\alpha_0], \]
where
\[ \mathcal{D}[\phi] \equiv \mathcal{D}[\bar{q}] \mathcal{D}[p], \]
and
\[ \delta[\alpha_0] \equiv \delta[q_0] \delta[p_0]. \]

The complex-valued function \( \alpha(k) \), defined in Eq. (19), can serve different purposes. It can be regarded as an independent “variable” in the context of functional expressions and thus can become the integration variable in functional integrals. It can also serve as a parameter function, representing for instance the spectral function in fixed-spectrum coherent states. Such parameter functions can also be turned into integration variables (fields) for functional integrals.

The generic functional integral with an integrand in isotropic Gaussian form can be evaluated to give
\[
\int \exp \left( -\alpha^* \phi K^{-1} - \alpha \phi^* \xi - \zeta^* \phi \zeta \right) \mathcal{D}[\phi] \equiv \det{K},
\]
where \( K \) is an invertible kernel, and \( \xi \) and \( \zeta \) are arbitrary complex functions. To be invertible, the kernel must have an inverse \( K^{-1} \), such that
\[ K \circ K^{-1} = \int K(k_1,k)K^{-1}(k,k_2) \, dk = \delta(k_1-k_2). \]
The functional determinant \( \det{K} \) can be expressed as
\[
\det{K} = \exp \left[ \text{tr} \{ \ln(K) \} \right],
\]
where, for an arbitrary kernel function \( H(k_1,k_2) \),
\[
\text{tr} \{ H(k_1,k_2) \} \equiv \int H(k,k) \, dk.
\]
The expression in Eq. (12) indicates that the quadrature bases are related by functional Fourier transforms. As a result, one can express one in terms of the other as functional Fourier integrals
\[
|p\rangle = \int |q\rangle \exp(\iota q \cdot p) \mathcal{D}[q], \quad \mathcal{D}[q] \equiv \int |p\rangle \exp(-\iota q \cdot p) \mathcal{D}[p].
\]
These Fourier relationships, together with the expressions of the eigen-equations in Eq. (11), allow the quadrature operators to be represent in their dual bases by functional derivatives
\[
\hat{p}(k) = \int |q\rangle \left[ -\iota \frac{\delta}{\delta q(k)} \right] \langle q | \mathcal{D}[q], \quad \hat{q}(k) = \int |p\rangle \left[ \iota \frac{\delta}{\delta p(k)} \right] \langle p | \mathcal{D}[p].
\]
The operation of a functional derivative is defined by
\[
\frac{\delta}{\delta \hat{f}(k')} f(k') = \delta(k-k').
\]

C. Fixed-spectrum Fock states

The fixed-spectrum Fock states are defined as
\[ |n_F\rangle = \frac{1}{\sqrt{n!}} \left( \hat{a}_F^\dagger \right)^n |\text{vac}\rangle, \]
in terms of fixed-spectrum creation operators, given by
\[ \hat{a}_F^\dagger = \int \hat{a}^\dagger F(k) \, dk. \]
The angular spectrum \( F(k) \), which is also the Fourier domain wave function, is normalized:
\[ \int |F(k)|^2 \, dk = 1. \]
It ensures that the fixed-spectrum creation and annihilation operators obey a simple commutation relation \( [\hat{a}_F, \hat{a}_F^\dagger] = 1 \) and that the fixed-spectrum Fock states are individually normalized \( \langle n_F | n_F \rangle = 1 \). The inner product between Fock states with different spectra reads
\[ \langle m_F | n_G \rangle = \delta_{mn} (F,G)^n, \]
where \( \langle F,G \rangle \) is defined in Eq. (13). When the annihilation operator in the momentum basis is applied to the fixed-spectrum Fock states, we obtain
\[ \hat{a}(k) |n_F\rangle = |(n-1)_F\rangle F(k) \sqrt{n}. \]
The fixed-spectrum Fock states are eigenstates of the number operator
\[ \hat{n} = \int \hat{a}^\dagger(k) \hat{a}(k) \, dk. \]
Using Eq. (21), one can show that
\[ \hat{n} |n_F\rangle = |n_F\rangle n. \]
D. Fixed-spectrum coherent states

The fixed-spectrum coherent states are defined as eigenstates of the annihilation operator in the momentum basis

$$\hat{a}(k) |\alpha_F\rangle = |\alpha_F\rangle \alpha(k),$$

(37)

where the eigenvalue function \(\alpha(k)\) is an arbitrary complex-valued spectral function. Note that the fixed-spectrum coherent states \(|\alpha_F\rangle\) does not explicitly depend on \(k\). Therefore, there exists a unique fixed-spectrum coherent state for every spectral function \(\alpha(k)\).

The subscript \(F\) in \(|n_F\rangle\), \(|\alpha_F\rangle\) and \(\hat{a}_F^\dagger\) is a reminder that the state or operator contains a fixed spectrum and should not necessarily be seen as a label for the associated complex-valued function. The latter is thus represented as \(\alpha(k)\) and not \(\alpha_F(k)\). Later, where we use different fixed-spectrum coherent states in the same expression, we will use the parameter functions to label the coherent states, instead of \(\alpha_F\).

The fixed-spectrum coherent states can be expressed in terms of displacement operators given by

$$\hat{D}[\alpha_F] \equiv \exp \left( \alpha \circ \hat{a}^\dagger - \alpha^* \circ \hat{a} \right).$$

(38)

The inner product between different fixed-spectrum coherent states can be derived from their displacement operators and reads

$$\langle \alpha_F | \beta_G \rangle = \exp \left( -\frac{1}{2} ||\alpha||^2 - \frac{1}{2} ||\beta||^2 + \langle \alpha, \beta \rangle \right).$$

(39)

As a consequence, it follows that the inner product between a fixed-spectrum coherent state and the vacuum state is

$$\langle \alpha_F | \text{vac} \rangle = \exp \left( -\frac{1}{2} ||\alpha||^2 \right).$$

(40)

Although not orthogonal, the fixed-spectrum coherent states resolve the identity operator, as we’ll show below in Section \[IVB\].

E. Quadrature representation of coherent states

To expand fixed-spectrum coherent states in terms of the spatiotemporal quadrature bases, we use the operators defined in Eq. \[7\] and employ the eigenstate property of the coherent states in Eq. \[57\]. As a result, we obtain

$$\hat{a}_Q |\alpha_F\rangle = |\alpha_F\rangle \sqrt{2q} \circ \alpha_0,$$

$$\hat{a}_R |\alpha_F\rangle = |\alpha_F\rangle \frac{1}{2} \alpha_0 \circ \alpha_0,$$

(41)

where \(\alpha_0\) represents the complex-valued parameter function of the fixed-spectrum coherent state. Therefore,

$$\langle q|\alpha_F\rangle = \pi^{-\Omega/4} \langle \text{vac} | \exp \left( -\frac{1}{2} ||q||^2 + \hat{a}_Q - \hat{a}_R \right) |\alpha_F\rangle =$$

$$= \pi^{-\Omega/4} \exp \left( -\frac{1}{2} ||q||^2 - \frac{1}{4} ||\alpha_0||^2 \right) + \sqrt{2q} \circ \alpha_0 - \frac{1}{2} \alpha_0 \circ \alpha_0).$$

(42)

If we express \(\alpha_0(k)\) in terms of its real and imaginary parts, we, as in Eq. \[19\], obtain

$$\langle q|\alpha_F\rangle = \pi^{-\Omega/4} \exp \left[ -\frac{1}{2} ||q - q_0||^2 \right. + \pi 0 \circ \left( q - \frac{1}{2} q_0 \right).$$

(43)

III. WIGNER FUNCTIONAL THEORY

Here, we develop the formalism for Wigner functionals in quantum optics. To avoid cluttering the notations, we proceed to neglect the spin indices. However, we emphasize that one can incorporate spin into the formalism when necessary. Therefore, the resulting formalism represents all the degrees of freedom of quantum optics.

A. Definition of the Wigner functional

The generic definition of a Wigner functional is

$$W[q, p] \equiv \int \langle q + \frac{1}{2} q' | \hat{A} | q - \frac{1}{2} q' \rangle$$

$$\times \exp(-ip \circ q') D[q'],$$

(44)

where \(\hat{A}\) is an operator on the Hilbert space of all quantum optical states, incorporating both particle-number degrees of freedom and spatiotemporal degrees of freedom. The square brackets in \(W[q, p]\) indicate that the quantity is a functional.

If the operator is a density operator \(\hat{\rho}\), the resulting Wigner functional represents a quantum state. The density operator can also be represented as a density ‘matrix’, which we refer to as a density functional

$$\rho \left[ q + \frac{1}{2} q', q - \frac{1}{2} q' \right] \equiv \langle q + \frac{1}{2} q' | \hat{\rho} | q - \frac{1}{2} q' \rangle.$$

(45)

In the case of a pure state, the density functional becomes a product of a wave functional \(\psi[q] = \langle q | \psi \rangle\) and its complex conjugate

$$\rho \left[ q + \frac{1}{2} q', q - \frac{1}{2} q' \right] = \psi \left[ q + \frac{1}{2} q' \right] \psi^* \left[ q - \frac{1}{2} q' \right].$$

(46)

B. Functional Weyl transformation

The inverse process whereby the density functional in either of the quadrature bases is reproduced from the Wigner functional is represented by a generalization of the Weyl transformation. For the \(q\)-basis, we have

$$\rho[q, q'] = \int W \left[ \frac{1}{2}(q + q'), p \right] \exp[ip \circ (q - q')] D^\circ[p].$$

(47)

A similar expression applies for the \(p\)-basis. The generalized Weyl transformation can also be used to reproduce the density operator

$$\hat{\rho} = \int |q + \frac{1}{2} q' \rangle W[q, p] \exp(ip \circ q') \langle q - \frac{1}{2} q' |$$

$$\times D^\circ[p] D[q, q'].$$
It then follows that the trace of the density operator is represented by the functional integral of the associated Wigner functional
\[ \text{tr}\{\hat{\rho}\} = \int W[\alpha] \mathcal{D}\alpha = 1, \] (49)
where we used Eqs. (19) and (21) to express it in terms of \(\alpha\)'s, instead of \(q\)'s and \(p\)'s.

C. Wigner functional for products of operators

The Wigner functional for the product of two operators can be obtained by computing the Wigner functional for the product of these operators when they are expressed in terms of Weyl transformations Eq. (18). The result is a functional integral over the Wigner functionals of these operators:
\[ W_{\hat{A}\hat{B}}[q,p] = 2^{2\Omega} \int W_{\hat{A}}[q - q_1, p - p_1] W_{\hat{B}}[q - q_2, p - p_2] \times \exp[i2(q_1 \circ p_2 - q_2 \circ p_1)] \times \mathcal{D}[q_1, q_2] \mathcal{D}[p_1, p_2]. \] (50)
The result is the equivalent of the star-product for Wigner functionals [20]. For the product of three operators, the functional integral expression is
\[ W_{\hat{A}\hat{B}\hat{C}}[\alpha] = \int W_{\hat{A}}[\frac{1}{2}(\alpha_a + \alpha_b + \alpha_c)] W_{\hat{B}}[\alpha_a] \times W_{\hat{C}}[\frac{1}{2}(\alpha_a - \alpha_b + \alpha_c)] \exp[(\alpha^* - \alpha_a^*) \circ \alpha_b - \alpha_b^* \circ (\alpha - \alpha_a)] \mathcal{D}[\alpha_a, \alpha_b], \] (51)
where we used Eqs. (19) and (21).

D. Characteristic functional

The characteristic functional is the functional Fourier transform of the Wigner functional
\[ \chi[\xi, \zeta] = \int \exp(i\xi \circ \zeta - i\xi \circ q) W[q, p] \mathcal{D}[q] \mathcal{D}[p]. \] (52)
The Wigner functional is obtained from the characteristic functional via the inverse functional Fourier transform
\[ W[q, p] = \int \exp(i\xi \circ q - ip \circ \zeta) \chi[\xi, \zeta] \mathcal{D}[\xi] \mathcal{D}[\zeta]. \] (53)

E. Probability distribution from the Wigner functional

As a quasi-probability distribution over the functional phase space, the Wigner functional of a state does not qualify as a true probability density. However, one can compute a probability density from it by integrating either over \(p\) or \(q\) (or any linear combination of \(p\) and \(q\)). Integrating the Wigner functional over \(p\), we obtain
\[ \int W[q, p] \mathcal{D}[p] = \int \langle q + \frac{i}{2}q' \mid \hat{\rho} \mid q - \frac{i}{2}q' \rangle \delta[q'] \mathcal{D}[q'] = \langle q \mid \hat{\rho} \mid q \rangle = \rho[q, q]. \] (54)

Hence, we recover the diagonal of the density tensor, which represents the probabilities. To perform the integration over \(q\), we first need to insert identities resolved in the \(p\)-basis
\[ \int W[q, p] \mathcal{D}[q] = \int \langle q + \frac{i}{2}q' \mid p_1 \rangle \langle p_1 \mid \hat{\rho} \mid p_2 \rangle \langle p_2 \mid q - \frac{i}{2}q' \rangle \times \exp(-ip \circ q') \mathcal{D}[p_1, p_2] \mathcal{D}[q, q'] \]
\[ = \langle p \mid \hat{\rho} \mid p \rangle = \rho[p, p]. \] (55)

If we integrate these probability distributions over the remaining variable, we obtain 1, thanks to the normalization.

IV. EXAMPLES OF WIGNER FUNCTIONALS

A. Fixed-spectrum coherent state

To obtain the Wigner functional for a fixed-spectrum coherent state, we substitute \(\hat{\rho} \rightarrow |\alpha_F\rangle \langle \alpha_F|\) into Eq. (44)
\[ W[q, p] = \int \langle q + \frac{i}{2}q' \mid \alpha_F \rangle \langle \alpha_F \mid q - \frac{i}{2}q' \rangle \times \exp(-ip \circ q') \mathcal{D}[q’]. \] (56)
The expressions for the two inner products are obtained from Eq. (50). After substituting them into Eq. (56) and evaluating the functional integral over \(q'\), we obtain
\[ W[\alpha] = N_0 \exp\left(-2|\alpha - \alpha_0|^2\right), \] (57)
where \(N_0\) is a normalization constant, and \(\alpha_0(k)\) is the parameter function of the fixed-spectrum coherent state.

The normalization constant \(N_0\) can be obtained by keeping track of the constants during the calculation. It can also be determined by imposing the requirement that the state is normalized, as in Eq. (49). Both ways lead to
\[ N_0 = 2^\Omega. \] (58)

B. Completeness of fixed-spectrum coherent state

Here, we consider the completeness of the fixed-spectrum coherent state, using the expression of their Wigner functionals, given in Eq. (57) with Eq. (58). In
terms of the Weyl representation, the density operator for the fixed-spectrum coherent state is given by
\[
\hat{\rho}_\alpha = |\alpha\rangle \langle \alpha| \\
= 2\Omega \int |q + \frac{i}{2}q'| \exp \left(-2||\alpha - \alpha_0||^2\right) \\
\times \exp(ip \diamond q') \langle q - \frac{i}{2}q' | \mathcal{D}^\alpha[p] \mathcal{D}[q,q'],
\]
where \(\alpha\) is given in Eq. \(\text{(40)}\).

For the completeness condition, we consider the operator defined by
\[
\hat{L} = \int |\alpha_0\rangle \langle \alpha_0| \mathcal{D}[\alpha_0]
\]
\[
= 2\Omega \int |q + \frac{i}{2}q'| \exp \left(-2||\alpha - \alpha_0||^2\right) \exp(ip \diamond q') \\
\times \langle q - \frac{i}{2}q' | \mathcal{D}^\alpha[p] \mathcal{D}[q,q'] \mathcal{D}[\alpha_0].
\]
We can show that
\[
\int \exp \left(-2||\alpha - \alpha_0||^2\right) \mathcal{D}[\alpha_0] = \pi^\Omega.
\]
Hence
\[
\hat{L} = (2\pi)^\Omega \int |q + \frac{i}{2}q'| \exp(ip \diamond q') \\
\times \langle q - \frac{i}{2}q' | \mathcal{D}^\alpha[p] \mathcal{D}[q,q']
\]
\[
= (2\pi)^\Omega \int |q + \frac{i}{2}q'| \delta[q'] \langle q - \frac{i}{2}q' | \mathcal{D}[q,q']
\]
\[
= (2\pi)^\Omega \int \langle q | q | \mathcal{D}[q] = (2\pi)^\Omega \mathbb{1},
\]
where we used the completeness condition of the \(q\)-basis given in Eq. \(\text{(40)}\). The factor \((2\pi)^\Omega\) indicates that the fixed-spectrum coherent states are severely overcomplete. The completeness condition for the fixed-spectrum coherent states can be expressed as
\[
\mathbb{1} = \int \langle \alpha | \langle \alpha | \mathcal{D}^\alpha.[
\]
Note that this result is different from the one obtained in \(\text{(50)}\). The reason is discussed in Sec. \(\text{V}\).

C. Coherent state assisted calculation

It is often convenient to employ coherent states in the computation of the Wigner functionals. Inserting identity operators, resolved in terms of coherent states, as in Eq. \(\text{(50)}\), into Eq. \(\text{(44)}\), we obtain
\[
W_{\hat{A}}[q,p] = \int \langle q + \frac{i}{2}q' | \alpha_0\rangle \langle \alpha_1 | \hat{A} | \alpha_2 \rangle \langle \alpha_2 | q - \frac{i}{2}q' | \mathcal{D}[q'] \mathcal{D}^\alpha[\alpha_1, \alpha_2],
\]
where \(\hat{A}\) is an arbitrary operator, and \(\alpha_1\) and \(\alpha_2\) are the parameter functions associated with the fixed-spectrum coherent states, serving as integration variables. Next, we substitute Eq. \(\text{(53)}\) into the result and evaluate the functional integral over \(q'\) to obtain
\[
W_{\hat{A}}[\alpha] = \mathcal{N}_0 \int \exp \left(-2||\alpha||^2 + 2\alpha^* \diamond \alpha_1 + 2\alpha_2^* \diamond \alpha_1ight) \\
\times \langle \alpha_1 | \hat{A} | \alpha_2 \rangle \mathcal{D}^\alpha[\alpha_1, \alpha_2],
\]
where \(\mathcal{N}_0\) is given in Eq. \(\text{(58)}\). It now remains to evaluate the overlap of the operator \(\hat{A}\) by the two coherent states and to perform the functional integrations over \(\alpha_1\) and \(\alpha_2\) to obtain the Wigner functional for \(\hat{A}\).

D. Fixed-spectrum Fock states

Next, we use the coherent state assisted approach to compute the Wigner functionals for the fixed-spectrum Fock states, defined in Eq. \(\text{(30)}\). The overlap between such a Fock state and an arbitrary fixed-spectrum coherent state is
\[
\langle n_F | \alpha_G \rangle = \exp \left(-\frac{1}{2}||\alpha||^2\right) \frac{1}{\sqrt{n!}}((F,\alpha_G))^n,
\]
where we used Eqs. \(\text{(31)}, \text{(34)}, \text{(37)}\) and \(\text{(40)}\). Hence,
\[
\langle \alpha_1 | n_F \rangle \langle n_F | \alpha_2 \rangle = \exp \left(-\frac{1}{2}||\alpha_1||^2 - \frac{1}{2}||\alpha_2||^2\right) \\
\times \frac{1}{n!}((\alpha_1,F)\langle F,\alpha_2\rangle)^n.
\]

One can simplify the expression by representing it as a generating functional
\[
\mathcal{K} = \sum_n \eta^n \langle \alpha_1 | n_F \rangle \langle n_F | \alpha_2 \rangle \\
= \exp \left(-\frac{1}{2}||\alpha_1||^2 - \frac{1}{2}||\alpha_2||^2 + \eta\langle \alpha_1, F \rangle \langle F, \alpha_2 \rangle\right),
\]
where \(\eta\) is an auxiliary parameter, such that
\[
\langle \alpha_1 | n_F \rangle \langle n_F | \alpha_2 \rangle = \eta^n \mathcal{K}_{\eta=0}.
\]

Substituting \(\langle \alpha_1 | \hat{A} | \alpha_2 \rangle \rightarrow \mathcal{K}\) into Eq. \(\text{(55)}\), we obtain a generating functional for the Wigner functionals of the Fock states, expressed as a functional integral
\[
\mathcal{W}(\eta) = \sum_n \eta^n W_n[\alpha] \\
= \mathcal{N}_0 \int \exp \left(-2||\alpha||^2 + 2\alpha^* \diamond \alpha_1 + 2\alpha_2^* \diamond \alpha_1ight) \\
- ||\alpha_1||^2 - ||\alpha_2||^2 - \alpha_2^* \diamond \alpha_1 + \eta\alpha_1^* \diamond FF^* \diamond \alpha_2 \mathcal{D}^\alpha[\alpha_1, \alpha_2].
\]

With the aid of Eq. \(\text{(53)}\), one can evaluate the functional integrals over \(\alpha_1\) and \(\alpha_2\). The result reads
\[
\mathcal{W}(\eta) = \frac{\mathcal{N}_0}{\det(1 + \eta FF^*)} \exp \left[-2\alpha^* \diamond \alpha_1 + 4\eta\alpha^* \diamond FF^* \diamond (1 + \eta FF^*)^{-1} \diamond \alpha_1\right].
\]
The inverse of the kernel is obtained by assume it has the form
\[(1 + \eta FF^*)^{-1} = 1 + AFF^*, \tag{72}\]
where \(A\) is unknown. It then follows that
\[(1 + \eta FF^*) \circ (1 + AFF^*) = 1, \tag{73}\]
which implies that
\[A = \frac{-\eta}{1 + \eta}, \tag{74}\]
where we used the fact that \(FF^* \cdot FF^* = FF^*\), thanks to Eq. (32). It also allows us to simplify the determinant using Eq. (23) \[\det \{1 + \eta FF^*\} = 1 + \eta. \tag{75}\]

As a result, we obtain
\[W(\eta) = \frac{N_0}{1 + \eta} \exp \left(-2|\alpha|^2 + \frac{4\eta}{1 + \eta}|(F, \alpha)|^2\right), \tag{76}\]
where the normalization constant \(N_0\) is given in Eq. (38), which we obtained by computing the trace in Eq. (49). Comparing the result in Eq. (76) with the generating function for Laguerre polynomials,
\[L(\nu) = \frac{1}{1 - \nu} \exp \left(-\frac{x\nu}{1 - \nu}\right) = \sum_n \nu^n L_n(x), \tag{77}\]
where \(L_n(x)\) is the \(n\)-th order Laguerre polynomial, one finds that the Wigner functionals for the fixed-spectrum Fock states are of the form
\[W_{|\alpha\rangle}[\alpha] = (-1)^n N_0 L_n\left(4|(F, \alpha)|^2\right) \exp \left(-2|\alpha|^2\right). \tag{78}\]

E. Wigner functional for the number operator

Wigner functionals are not only associated with quantum states — they can also represent operators. We now use the coherent state assisted approach to obtain a Wigner functional for the number operator, defined in \[33\]. For this purpose, we compute
\[
\langle \alpha_1 | \hat{\hat{n}} | \alpha_2 \rangle = \langle \alpha_1, \alpha_2 \rangle \exp \left(-\frac{i}{2}||\alpha_1||^2 - \frac{i}{2}||\alpha_2||^2 + (\alpha_1, \alpha_2)\right), \tag{79}\]
where we used \[39\]. One can simplify the expression by representing it as a generating function
\[\mathcal{G} = \exp \left(-\frac{i}{2}||\alpha_1||^2 - \frac{i}{2}||\alpha_2||^2 + J(\alpha_1, \alpha_2)\right), \tag{80}\]
such that
\[\langle \alpha_1 | \hat{n} | \alpha_2 \rangle = \partial_J \mathcal{G}|_{J=1}. \tag{81}\]
We substitute \(\langle \alpha_1 | \hat{A} | \alpha_2 \rangle \rightarrow \mathcal{G}\) into Eq. (32) and evaluate all the functional integrals, to obtain
\[W(J) = N_0 \int \exp \left(-2|\alpha|^2 + 2\alpha^* \cdot \alpha_1 - ||\alpha_1||^2 - ||\alpha_2||^2 + 2\alpha_2^* \cdot \alpha - \alpha_2^* \cdot \alpha_1 + J\alpha_2^* \cdot \alpha_2\right) \mathcal{D}^\alpha[\alpha_1, \alpha_2] = \frac{N_0}{(1 + J)^{\eta}} \exp \left[-2 \left(\frac{1}{1 + J}\right) \alpha^* \cdot \alpha\right]. \tag{82}\]
Finally, we evaluate the derivative with respect to \(J\) and set \(J = 1\), to get
\[W_{\hat{n}}[\alpha] = \alpha^* \cdot \alpha - \frac{\Omega}{2}. \tag{83}\]

F. Displacement operator

Next, we consider the displacement operator given in Eq. (83). Using the Baker-Campbell-Hausdorff formula
\[\exp \left(\hat{X} + \hat{Y}\right) = \exp \left(-\frac{i}{2}[\hat{X}, \hat{Y}]\right) \exp(\hat{X}) \exp(\hat{Y}), \tag{84}\]
which assumes \([\hat{X}, [\hat{X}, \hat{Y}]] = [\hat{Y}, [\hat{X}, \hat{Y}]] = 0\), to separate the displacement operator into a product of exponential operators, we obtain
\[\hat{D}[\alpha_0] = \exp \left(-\frac{i}{2}||\alpha_0||^2\right) \exp(\hat{\alpha}_0) \exp(-\hat{\alpha}_0). \tag{85}\]
For the coherent state assisted approach, we compute the overlap
\[
\langle \alpha_1 | \hat{D}[\alpha_0] | \alpha_2 \rangle = \exp \left(-\frac{i}{2}||\alpha_0||^2 + \alpha_1^* \cdot \alpha_0 - \frac{i}{2}||\alpha_1||^2 - \frac{i}{2}||\alpha_2||^2 + \alpha_1^* \cdot \alpha_2 - \alpha_2^* \cdot \alpha_1 + J\alpha_1^* \cdot \alpha_2 - \alpha_2^* \cdot \alpha_1\right). \tag{86}\]
Then we substitute it into Eq. (33) to obtain a functional integral expression for the Wigner functional of the displacement operator
\[W_{\hat{D}} = N_0 \int \exp \left(-2|\alpha|^2 + 2\alpha^* \cdot \alpha + 2\alpha^*_i \cdot \alpha - ||\alpha_1||^2 - ||\alpha_2||^2 + \alpha_1^* \cdot \alpha_2 + \alpha_2^* \cdot \alpha_1 + J\alpha_1^* \cdot \alpha_2 - \alpha_2^* \cdot \alpha_1\right) \mathcal{D}^\alpha[\alpha_1, \alpha_2]. \tag{87}\]
After evaluating the functional integrals over \(\alpha_1\) and \(\alpha_2\), we obtain a familiar form:
\[W_{\hat{D}}[\alpha; \alpha_0] = \exp \left(\alpha^* \cdot \alpha_0 - \alpha_0^* \cdot \alpha\right). \tag{88}\]
One can use the Wigner functional expression for the product of three operators in Eq. (51) to obtain a general expression for the Wigner functional of an arbitrary state after displacement operators are applied to it:
\[W_{\hat{D}\hat{D}^\dagger}[\alpha] = \int \exp(\alpha^*_b \cdot \alpha_0 - \alpha^*_0 \cdot \alpha_b) \times \exp([\alpha^* - \alpha^*_0] \cdot \alpha_b - \alpha^*_0 \cdot (\alpha - \alpha_0)] \times W_{\hat{D}^\dagger}[\alpha_b] \mathcal{D}^\alpha[\alpha_0, \alpha_b]. \tag{89}\]

---

1 Expand the ln(\cdot) to all orders in \(\eta\) and evaluate the trace over each term. The traces over the \(F\)’s all evaluate to 1, leaving a series that can be re-summed to a ln(\cdot), which is then removed by the exponential function.
There are no quadratic terms for $\alpha_b$ in the exponent. Hence, the functional integration over $\alpha_b$ produces a Dirac delta functional

$$W_{D\hat{\rho}D}[\alpha] = (2\pi)^\Omega \int W_{\hat{\rho}}[\alpha_a] \delta(\alpha_a - \alpha + \alpha_0) \mathcal{D}[\alpha_a]$$

$$= W_{\hat{\rho}}[\alpha - \alpha_0]. \quad (90)$$

As expected, the effect of the displacement operation on an arbitrary Wigner functional is a shift in its argument.

V. DISCUSSION

It is a mark of the power of the notation of a new formalism when it leads to expressions that are almost identical to those that exclude the generalization incorporated into the new formalism. While the notation may alleviate the complexities in its use, its uncanny resemblance could be misleading — causing one to confuse the more general expressions for those of the simpler case. Therefore, it is necessary to emphasize the difference in meaning.

Consider for example the Wigner function of the coherent state incorporating only the particle-number degree of freedom

$$W(\alpha; \alpha_0) = 2 \exp \left(-2|\alpha - \alpha_0|^2\right), \quad (91)$$

and compare it to the Wigner functional of the fixed-spectrum coherent state incorporating both the spatiotemporal and particle-number degrees of freedom

$$W[\alpha; \alpha_0] = 2^{\Omega} \exp \left(-2|\alpha - \alpha_0|^2\right). \quad (92)$$

While the two expression may seem very similar, they represent different content. In Eq. (91), $\alpha$ and $\alpha_0$ represent a complex variable and a complex parameter. But in Eq. (92), they represent functions. That is why the former is a Wigner function, while the latter is a Wigner functional. The argument of the former contains the modules square of the difference between the complex values. On the other hand, the argument of the latter contains an integral over the space of wave vectors that computes the magnitude of the difference between complex functions. Due to the integral, the functional dependences of the complex functions are removed — integrated out. Therefore, the Wigner functional does not explicitly depend on the wave vector. However, it does depend on the complex function as a whole — the defining characteristic of a functional.

One can also compare the current formalism with the symplectic formalism. The latter represents the spatial degrees of freedom in terms of a finite number of discrete modes. The implication is that the complete Hilbert space of quantum optical states is stratified into a tensor product of discrete Hilbert spaces, each representing one spatial mode. Practical calculations usually require a truncation to a finite number of such Hilbert spaces. A coherent state in this formalism is represented by

$$W[\mathbf{Q}; \mathbf{Q}_0] = 2^M \exp \left[-2(\mathbf{Q} - \mathbf{Q}_0)^T J (\mathbf{Q} - \mathbf{Q}_0)\right], \quad (93)$$

where $\mathbf{Q}$ is a vector consisting of $M$ pairs of quadrature variables, one pair for each of the $M$ Hilbert spaces; $\mathbf{Q}_0$ is a vector of the associated parameters and $J$ is a symplectic matrix that maintains the correct multiplications among the quadrature variables. Although there are many applications where the latter formalism has been used successfully, such cases invariably truncate the set of discrete spatial modes to a finite number. Therefore, it does not represent the complete Hilbert space of all quantum optical states, as represented by the Wigner functional formalism.

Another difference between the expressions in Eqs. (91) and (92) is the normalization constant. In the former case, the constant is a finite number. In the latter case, it becomes a divergent constant that one can associate with the cardinality of the space. If $\Omega$ is associated with the cardinality of countable infinity, then $2^{\Omega}$ would represent the cardinality of the continuum. It is inevitable that functional integrals would produce such divergent constants. However, when these functional integrals are employed to compute the predicted results of measurements, one expects to obtain finite quantitative results. Therefore, the divergent constants must cancel. According to cardinal arithmetic, all divergent constants of the same cardinality are formally equal. However, unless one can keep careful track of these constants, their cancellation may hide finite numbers that are important for the correct quantitative predictions. For this reason, we retain the precise form of the divergent constants, be it $\pi^{-2\Omega/4}$, $2^{\Omega}$, $(2\pi)^\Omega$, or whatever else, even though all these constants are formally equal.

Although the current development is a generalization of the Wigner functions, one can also consider generalizations of other quasi-distributions, such as the Glauber-Sudarshan $P$-distribution [21, 22] or the Husimi $Q$-distribution [23]. However, the development of these generalizations and the transformations that would convert one into another is beyond the scope of this paper.

For the development of the formalism, we have avoided the use of any results based on the fixed-spectrum Fock states. In [30], the completeness condition for the fixed-spectrum coherent states was derived with the aid of the cardinality of countable infinity, then $2^\Omega$ would represent the cardinality of the continuum. It turns out that the assumed form of that generalized completeness condition [Eq. (19) in [30]] lacks a normalization constant on the right-hand side. To illustrate the problem, we consider the simplest case [Eq. (16) in [30]], given by

$$\int F(\mathbf{k}_1) F^*(\mathbf{k}_2) \mathcal{D}[F] = (2\pi)^2 \omega_1 \delta(\mathbf{k}_1 - \mathbf{k}_2). \quad (94)$$
We multiply both sides by $(2\pi)^3\omega_1 \delta(k_1 - k_2)$ and integrate over $k_1$ and $k_2$. The left-hand side becomes

$$
\int \left[ \int |F(k)|^2 \, dk \right] \mathcal{D}[F] = \int \mathcal{D}[F],
$$

thanks to Eq. (82). It produces the volume of the function space, which equals the cardinality of the continuum. The right-hand side, on the other hand, gives

$$
\int \delta(0) \, d^3k_1 = \Omega,
$$

which is associated with the cardinality of countable infinity. Clearly, the two sides cannot be equal, unless a normalization constant proportional to the cardinality of infinity. Clearly, the two sides cannot be equal, unless a normalization constant proportional to the cardinality of the continuum is multiplied on the right-hand side. For the generalized completeness condition [Eq. (19) in [30]], this constant would also depend on the generalized completeness condition [Eq. (19) in [30]].

The existence of a complete orthogonal basis for the spatiotemporal quadrature bases in Ref. [30] neglected the orthogonality constants. Here we consider the derivation more carefully and provide slightly different expressions for these orthogonality conditions.

We start with the overlap between elements of the dual quadrature bases [cf. Eqs. (88) in Ref. [30]]

$$
\langle q | p \rangle = \frac{V_0 W_0}{2^{3/2}} \exp \left( \frac{1}{2} ||q||^2 + \frac{1}{2} ||p||^2 + i q \cdot p \right). \tag{A1}
$$

While specific expressions were assumed in Ref. [30] for $V_0$ and $W_0$, respectively, to obtain the desired expression in Eq. (12), one only needs to specify their product to be

$$
V_0 W_0 = 2^{3/2} \exp \left( -\frac{1}{2} ||q||^2 - \frac{1}{2} ||p||^2 \right). \tag{A2}
$$

Their $q$- and $p$-dependences require that they are of the form

$$
V_0 = \kappa_q \exp \left( -\frac{1}{2} ||q||^2 \right)
$$

$$
W_0 = \kappa_p \exp \left( -\frac{1}{2} ||p||^2 \right), \tag{A3}
$$

where $\kappa_q$ and $\kappa_p$ are constants to be determined.

Next, we consider the overlap between two quadrature basis elements, expressed as a limit [cf. Eqs. (92) and (93) in Ref. [30], with some improvements]

$$
\langle q | q' \rangle = \kappa_q^2 \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^{3/2}} \exp \left( -\frac{1}{2\epsilon} ||q - q'||^2 \right). \tag{A4}
$$

It then follows that, unless $||q - q'||^2 = 0$, the limit would produce zero. It is tempting to conclude that

$$
\langle q | q' \rangle = \Lambda_q \delta(q - q'), \tag{A5}
$$

where $\Lambda_q$ is an unknown orthogonality constant. However, to be a Dirac delta orthogonality constant. However, to be a Dirac delta functional, the quantity must satisfy the requirement

$$
\int W[q] \langle q | q' \rangle \mathcal{D}[q] = \Lambda_q W[q'], \tag{A6}
$$

where $W[q]$ is an arbitrary functional of $q$. Using the

 VI. CONCLUSION

The existence of a complete orthogonal basis for the full Hilbert space of quantum optical states, which incorporates all the degrees of freedom associated with photonic states, allows one to formulate powerful tools to analyze quantum optical systems. Since the complete orthogonal basis is a quadrature basis, the natural choice of such a formalism is a generalization of the well-known Wigner distribution formalism.

Here, the Wigner functional formalism is presented, based on the spatiotemporal quadrature basis — the $q$-basis. The result demonstrates a clear analogy between the functional formalism and the well-known Wigner function, characteristic function and Weyl transform. Even the star-product is reproduced in a similar form.

We used the functional formalism to compute some examples of Wigner functionals. These examples include: the Wigner functional for fixed-spectrum coherent states, a generating functional for the Wigner functionals of fixed-spectrum Fock states and the Wigner functionals for the number operator and the displacement operator.

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Appendix A: Orthogonality of spatiotemporal quadrature bases

The derivation of the orthogonality conditions of the spatiotemporal quadrature bases in Ref. [30] neglected the orthogonality constants. Here we consider the derivation more carefully and provide slightly different expressions for these orthogonality conditions.
expression in Eq. (A2), we have

\[
\int W[q] \langle q | q' \rangle \mathcal{D}[q] \\
= \kappa^2 \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^{1/2}} \int W[q] \exp \left( -\frac{||q - q'||^2}{2\epsilon} \right) \mathcal{D}[q] \\
= \kappa^2 \lim_{\epsilon \to 0} \int W[\sqrt{2\epsilon}q_0 + q'] \exp \left( -||q_0||^2 \right) \mathcal{D}[q_0] \\
= \kappa^2 q^{3/2} W[q']. \tag{A7}
\]

Here, we first shifted the \(q\)-variable and then absorbed the \(\epsilon\)-factor into \(q\), which then emerge from the measure, causing it to cancel the \(\epsilon\)-factor in front. At the same time, the argument of the functional became independent of the integration field. In the end, the result has the form required for a Dirac delta functional and it gives the expression for the orthogonality constant \(\Lambda_q = \kappa^2 q^{3/2} \Omega_q/2\).

Setting \(\kappa_q = \pi^{-\Omega_q/4}\), we get \(\Lambda_\theta = 1\). Following a similar calculation for the \(p\)-basis (deliberately using \(\mathcal{D}[p]\) and not \(\mathcal{D}[p]\)), one obtains

\[
\int W[p] \langle p | p' \rangle \mathcal{D}[p] = \Lambda_p W[p'] = \kappa_p^{2} p^{3/2} W[p']. \tag{A8}
\]

The choice of \(\kappa_q\) and Eq. (A2) gives \(\kappa_p = \pi^{2/2} p^{3/4}\) and \(\Lambda_p = (2\pi)^{1/2}\). Thus we obtain the orthogonality conditions for the spatiotemporal quadrature bases given in Eq. (11) and the constants in the definitions of the spatiotemporal quadrature bases given in Eq. (3).

### Appendix B: Completeness of spatiotemporal quadrature bases

The completeness of spatiotemporal quadrature bases is affected by the constants obtained in Append. A. Here, we rederive the completeness conditions for these quadrature bases, which were derived in Ref. [20]. The expression for the completeness of the functional quadrature bases has the form

\[
\hat{B} = \int \langle q | q \rangle \mathcal{D}[q]. \tag{B1}
\]

Overlapping it on both sides by coherent states, we get

\[
\langle \alpha_1 | \hat{B} | \alpha_2 \rangle = \int \langle \alpha_1 | q \rangle \langle q | \alpha_2 \rangle \mathcal{D}[q] \\
= \pi^{-\Omega/2} \int \exp \left( -\frac{1}{2}||q - q_1||^2 - \frac{1}{2}||q - q_2||^2 \right) \mathcal{D}[q] \\
- i\pi_1 \circ (q - \frac{i}{2}q_1) \\
+ i\pi_2 \circ (q - \frac{i}{2}q_2) \mathcal{D}[q], \tag{B2}
\]

where we used Eq. (13) and defined \(\alpha_1\) and \(\alpha_2\) as in Eq. (19). The functional integration over \(q\) produces a factor \(\pi^{\Omega/2}\), which removes the factor in front. The result, expressed in terms of \(\alpha_1\) and \(\alpha_2\), reads

\[
\langle \alpha_1 | \hat{B} | \alpha_2 \rangle \equiv \exp \left( -\frac{1}{2}||\alpha_1||^2 - \frac{1}{2}||\alpha_2||^2 + \langle \alpha_1, \alpha_2 \rangle \right) \\
= \langle \alpha_1 | \alpha_2 \rangle, \tag{B3}
\]

which follows from Eq. (39). Since \(\langle \alpha_1 \rangle\) and \(\langle \alpha_2 \rangle\) can represent arbitrary coherent states, it then follows that \(\hat{B} \equiv 1\). A similar conclusion applies for the \(p\)-basis. Hence, the spatiotemporal quadrature bases obey the completeness conditions given in Eq. (16).

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[1] N. Gisin and R. Thew, “Quantum communication,” Nature Photon. 1, 165 (2007).
[2] V. Giovannetti, S. Lloyd, and L. Maccone, “Advances in quantum metrology,” Nature Photon. 5, 222 (2011).
[3] A. Steane, “Quantum computing,” Rept. Prog. Phys. 61, 117 (1997).
[4] M. Schlosshauer, “Decoherence, the measurement problem, and interpretations of quantum mechanics,” Rev. Mod. Phys. 76, 1267 (2003).
[5] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, “Classical information capacity of a quantum channel,” Phys. Rev. A 54, 1869 (1996).
[6] S. Lloyd, “Capacity of the noisy quantum channel,” Phys. Rev. A 55, 1613 (1997).
[7] A. S. Holevo, “The capacity of the quantum channel with general signal states,” IEEE Trans. Inf. Theory 44, 269 (1998).
[8] H. Bechmann-Pasquinucci and A. Peres, “Quantum cryptography with 3-state systems,” Phys. Rev. Lett. 85, 3313 (2000).
[9] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, “Security of quantum key distribution using d-level systems,” Phys. Rev. Lett. 88, 127902 (2002).
[10] V. Scarani, H. Bechmann-Pasquinucci, N. J. Cerf, M. Dusek, N. Lütkenhaus, and M. Peev, “The security of practical quantum key distribution,” Rev. Mod. Phys. 81, 1301 (2009).
[11] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, “Orbital angular momentum of light and the transformation of laguerre-gaussian laser mode,” Phys. Rev. A 45, 8185 (1992).
[12] M. Mirhosseini, O. S. Magona-Losaiza, M. N. O’Sullivan, B. Rodenburg, M. Malik, M. P. J. Lavery, M. J. Padgett, D. J. Gauthier, and R. W. Boyd, “High-dimensional quantum cryptography with twisted light,” New J. Phys. 17, 033033 (2015).
[13] E. Waks, K. Inoue, C. Santori, D. Fattal, J. Vuckovic, G. S. Solomon, and Y. Yamamoto, “Secure communication: Quantum cryptography with a photon turnstile,” Nature 420, 762 (2002).
[14] B. C. Sanders, “Review of entangled coherent states,” J. Phys. A: Math. Theor. 45, 244002 (2012).
[15] S. L. Braunstein and P. Van Loock, “Quantum information with continuous variables,” Rev. Mod. Phys. 77, 513 (2005).
[16] G. Adesso, S. Ragy, and A. R. Lee, “Continuous variable quantum information: Gaussian states and beyond,” Open Syst. Inf. Dyn. 21, 1440001 (2014).
[17] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, “Gaussian quantum information,” Rev. Mod. Phys. 84, 621 (2012).
[18] H. J. Groenewold, “On the principles of elementary quantum mechanics,” Physica 12, 405 (1946).
[19] J. E. Moyal, “Quantum mechanics as a statistical theory,” Math. Proc. Camb. Philos. Soc. 45, 99 (1949).
[20] T. L. Curtright and C. K. Zachos, “Quantum mechanics in phase space,” Asia Pacific Physics Newsletter 1, 37 (2012).
[21] R. J. Glauber, “Coherent and incoherent states of the radiation field,” Phys. Rev. 131, 2766 (1963).
[22] E. C. G. Sudarshan, “Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams,” Phys. Rev. Lett. 10, 277 (1963).
[23] K. Husimi, “Some formal properties of the density matrix,” Nippon Sugaku-Buturigakkwan Ki Zai 3 3 Ki 22, 264 (1940).
[24] E. Wigner, “On the quantum correction for thermodynamic equilibrium,” Phys. Rev. 40, 749 (1932).
[25] R. L. Stratonovich, “On distributions in representation space,” Sov. Phys. JETP 4, 891 (1957).
[26] C. Brif and A. Mann, “Phase-space formulation of quantum mechanics and quantum-state reconstruction for physical systems with Lie-group symmetries,” Phys. Rev. A 59, 971 (1999).
[27] T. Tilma, M. J. Everitt, J. H. Samson, W. J. Munro, and K. Nemoto, “Wigner functions for arbitrary quantum systems,” Phys. Rev. Lett. 117, 180401 (2016).
[28] R. P. Rundle, T. Tilma, J. H. Samson, V. M. Dwyer, R. F. Bishop, and M. J. Everitt, “General approach to quantum mechanics as a statistical theory,” Phys. Rev. A 99, 012115 (2019).
[29] F. Krumm, W. Vogel, and J. Sperling, “Time-dependent quantum correlations in phase space,” Phys. Rev. A 95, 063805 (2017).
[30] F. S. Roux, “Combining spatiotemporal and particle-number degrees of freedom,” Phys. Rev. A 98, 043841 (2018).
[31] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley Publishing Company, Reading, Massachusetts, USA, 1995).
[32] U. Abraham and M. Magidor, in Handbook of set theory (Springer, 2010), pp. 1149–1227.