PIECEWISE CONTINUITY
OF FUNCTIONS DEFINABLE OVER
HENSELIAN RANK ONE VALUED FIELDS

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Abstract. Consider a Henselian rank one valued field \( K \) of equi-
characteristic zero along with the language \( \mathcal{L}^P \) of Denef–Pas. Let
\( f : A \rightarrow K \) be an \( \mathcal{L}^P \)-definable (with parameters) function on a
subset \( A \) of \( K^n \). We prove that \( f \) is piecewise continuous; more
precisely, there is a finite partition of \( A \) into \( \mathcal{L}^P \)-definable locally
closed subsets \( A_1, \ldots, A_s \) of \( K^n \) such that the restriction of \( f \) to
each \( A_i \) is a continuous function.

1. Introduction

Consider a Henselian rank one valued field \( K \) of equicharacteristic
zero along with the language \( \mathcal{L}^P \) of Denef–Pas, which consists of three
sorts: the valued field \( K \)-sort, the value group \( \Gamma \)-sort and the residue
field \( k \)-sort. The only symbols of \( \mathcal{L}^P \) connecting the sorts are the
following two maps from the main \( K \)-sort to the auxiliary \( \Gamma \)-sort and
\( k \)-sort: the valuation map \( v \) and an angular component map \( ac \) which
is multiplicative, sends 0 to 0 and coincides with the residue map on
units of the valuation ring \( R \) of \( K \). The language of the \( K \)-sort is
the language of rings; that of the \( \Gamma \)-sort is any augmentation of the
language of ordered abelian groups (with \( \infty \)); finally, that of the \( k \)-
sort is any augmentation of the language of rings. Throughout the
paper the word ”definable” means ”definable with parameters”. We
consider \( K^n \) with the product topology, called the \( K \)-topology on \( K^n \).
Let \( \mathbb{P}^1(K) \) stand for the projective line over \( K \).

The main purpose is to prove the following

Theorem 1.1. Let \( A \subset K^n \) and \( f : A \rightarrow \mathbb{P}^1(K) \) be an \( \mathcal{L}^P \)-definable
function in the three-sorted language of Denef–Pas. Then \( f \) is piecewise
continuous, i.e. there is a finite partition of \( A \) into \( \mathcal{L}^P \)-definable locally
closed subsets \( A_1, \ldots, A_s \) of \( K^n \) such that the restriction of \( f \) to each
\( A_i \) is continuous.

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We immediately obtain

**Corollary 1.2.** The conclusion of the above theorem holds for any \( \mathcal{L}^P \)-definable function \( f : A \to K \).

The proof of Theorem 1.1 relies on two basic ingredients. The first one is concerned with a theory of algebraic dimension and decomposition of definable sets into a finite union of locally closed definable subsets, recalled in the next section. It was established by van den Dries \cite{1} for certain expansions of rings (and Henselian valued fields, in particular) which admit quantifier elimination and are equipped with a topological system. The second one is the closedness theorem from our paper \cite{3}, Theorem 3.1. Let us mention that the latter paper is devoted to geometry over Henselian rank one valued fields and i.a. to the results achieved in our joint article \cite{2} about hereditarily rational functions on real and \( p \)-adic varieties. Section 3 provides the proof of the main result (Theorem 1.1).

2. Definable sets over Henselian valued fields

Consider an infinite integral domain \( D \) with quotient field \( K \). One of the fundamental concepts introduced by van den Dries \cite{1} is that of a topological system on a given expansion \( \mathcal{D} \) of a domain \( D \) in a language \( \mathcal{L} \). That concept incorporates both Zariski-type and definable topologies. We remind the reader that it consists of a topology \( \tau_n \) on each set \( D^n \), \( n \in \mathbb{N} \), such that:

1) For any \( n \)-ary \( \mathcal{L}_D \)-terms \( t_1, \ldots, t_s \), \( n, s \in \mathbb{N} \), the induced map 
\[ D^n \ni a \mapsto (t_1(a), \ldots, t_s(a)) \in D^s \]
is continuous.

2) Every singleton \( \{a\}, a \in D \), is a closed subset of \( D \).

3) For any \( n \)-ary relation symbol \( R \) of the language \( \mathcal{L} \) and any sequence \( 1 \leq i_1 < \ldots < i_k \leq n \), \( 1 \leq k \leq n \), the two sets 
\[ \{(a_{i_1}, \ldots, a_{i_k}) \in D^k : \mathcal{D} \models R((a_{i_1}, \ldots, a_{i_k})^\&), a_{i_1} \neq 0, \ldots, a_{i_k} \neq 0\} \]
\[ \{(a_{i_1}, \ldots, a_{i_k}) \in D^k : \mathcal{D} \models \neg R((a_{i_1}, \ldots, a_{i_k})^\&), a_{i_1} \neq 0, \ldots, a_{i_k} \neq 0\} \]
are open in \( D^k \); here \( (a_{i_1}, \ldots, a_{i_k})^\& \) denotes the element of \( D^n \) whose \( i_j \)-th coordinate is \( a_{i_j}, j = 1, \ldots, k \), and whose remaining coordinates are zero.

Finite intersections of closed sets of the form 
\[ \{a \in D^n : t(a) = 0\}, \]
where $t$ is an $n$-ary $L_D$-term, will be called *special closed subsets* of $D^n$. Finite intersections of open sets of the form

$$\{ a \in D^n : t(a) \neq 0 \},$$

$$\{ a \in D^n : D \models R((t_{i_1}(a), \ldots, t_{i_k}(a))^k), t_{i_1}(a) \neq 0, \ldots, t_{i_k}(a) \neq 0 \}$$

or

$$\{ a \in D^n : D \models \neg R((t_{i_1}(a), \ldots, t_{i_k}(a))^k), t_{i_1}(a) \neq 0, \ldots, t_{i_k}(a) \neq 0 \},$$

where $t, t_{i_1}, t_{i_k}$ are $L_D$-terms, will be called *special open subsets* of $D^n$. Finally, an intersection of a special open and a special closed subsets of $D^n$ will be called a *special locally closed* subset of $D^n$. Every quantifier-free $L$-definable set is a finite union of special locally closed sets.

Suppose now that the language $L$ extends the language of rings and has no extra function symbols of arity $> 0$ and that an $L$-expansion $D$ of the domain $D$ under study admits quantifier elimination and is equipped with a topological system such that every non-empty special open subset of $D$ is infinite. These conditions ensure that $D$ is algebraically bounded and algebraic dimension defines a dimension function on $D$ ([1, Proposition 2.15 and 2.7]). Algebraic dimension is the only dimension function on $D$ whenever, in addition, $D$ is a non-trivially valued field and the topology $\tau_1$ is induced by its valuation. Then, for simplicity, the algebraic dimension of an $L$-definable set $E$ will be denoted by $\dim E$.

Now we recall the following two basic results from [1, Proposition 2.17 and 2.23]:

**Proposition 2.1.** Every $L$-definable set is a finite union of intersections of Zariski closed with special open sets and, a fortiori, a finite union of locally closed sets.

**Proposition 2.2.** Let $E$ be an $L$-definable subset of $D^n$ and $\partial E := E \setminus \overline{E}$ denote its frontier. Then

$$\text{alg.dim}(\partial E) < \text{alg.dim}(E).$$

It is not difficult to strengthen the former proposition as follows.

**Corollary 2.3.** Every $L$-definable set is a finite disjoint union of locally closed sets.

Quantifier elimination due to Pas [5, Theorem 4.1] (and more precisely, elimination of $K$-quantifiers) enables translation of the language of Denef–Pas on $K$ into a language described above equipped with the topological system wherein $\tau_n$ is the $K$-topology, $n \in \mathbb{N}$. Indeed, we must augment the language of valued rings by adding extra relation
symbols for the inverse images under the valuation and angular component map of relations on the value group and residue field. More precisely, we must add the names of sets of the form
\[ \{ a \in K^n : (v(a_1), \ldots, v(a_n)) \in P \} \]
and
\[ \{ a \in K^n : (\overline{a} a_1, \ldots, \overline{a} a_n) \in Q \}, \]
where \( P \) and \( Q \) are definable subsets (in the auxiliary sorts of the Denef–Pas language) of \( \Gamma^n \) and \( \mathbb{k}^n \), respectively.

Summing up, the foregoing results apply in the case of Henselian non-trivially valued fields with the three-sorted language of Denef–Pas.

3. Proof of the main theorem

Consider an \( \mathcal{L}^P \)-definable function \( f : A \to \mathbb{P}^1(K) \) and its graph
\[ E := \{ (x, f(x)) : x \in A \} \subset K^n \times \mathbb{P}^1(K). \]
We shall proceed with induction with respect to the dimension
\[ d = \dim A = \dim E \]
of the source and graph of \( f \). By Corollary 2.3, we can assume that the graph \( E \) is a locally closed subset of \( K^n \times \mathbb{P}^1(K) \) of dimension \( d \) and that the conclusion of the theorem holds for functions with source and graph of dimension \(< d \).

Let \( F \) be the closure of \( E \) in \( K^n \times \mathbb{P}^1(K) \) and \( \partial E := F \setminus E \) be the frontier of \( E \). Since \( E \) is locally closed, the frontier \( \partial E \) is a closed subset of \( K^n \times \mathbb{P}^1(K) \) as well. Let
\[ \pi : K^n \times \mathbb{P}^1(K) \longrightarrow K^n \]
be the canonical projection. Then, by virtue of the closedness theorem ([1], Theorem 3.1), the images \( \pi(F) \) and \( \pi(\partial E) \) are closed subsets of \( K^n \). Further,
\[ \dim F = \dim \pi(F) = d \]
and
\[ \dim \pi(\partial E) \leq \dim \partial E < d; \]
the last inequality holds by Proposition 2.2. Putting
\[ B := \pi(F) \setminus \pi(\partial E) \subset \pi(E) = A, \]
we thus get
\[ \dim B = d \quad \text{and} \quad \dim (A \setminus B) < d. \]
Clearly, the set
\[ E_0 := E \cap (B \times \mathbb{P}^1(K)) = F \cap (B \times \mathbb{P}^1(K)) \]
is a closed subset of \( B \times \mathbb{P}^1(K) \) and is the graph of the restriction
\[
f_0 : B \longrightarrow \mathbb{P}^1(K)
\]
of \( f \) to \( B \). Again, it follows immediately from the closedness theorem that the restriction
\[
\pi_0 : E_0 \longrightarrow B
\]
of the projection \( \pi \) to \( E_0 \) is a definably closed map. Therefore \( f_0 \) is a continuous function. But, by the induction hypothesis, the restriction of \( f \) to \( A \setminus B \) satisfies the conclusion of the theorem, whence so does the function \( f \). This completes the proof of Theorem 1.1. \( \Box \)

Finally, let us mention that every \( L^P \)-definable continuous function \( f : A \rightarrow K \) on a closed bounded subset \( A \) of \( K^n \) is Hölder continuous, as proven in our recent paper \([4]\).

References

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