The classifying space of the 1+1 dimensional free $G$-cobordism category

Carlos Segovia*

June 9, 2021

Abstract

For a finite group $G$, we define the free $G$-cobordism category in dimension two. We show there is a one-to-one correspondence between the connected components of its classifying space and the abelianization of $G$. Also, we find an isomorphism of its fundamental group onto the direct sum $\mathbb{Z} \oplus H_2(G)$, where $H_2(G)$ is the integral 2-homology group, and we study the classifying space of some important subcategories. If $G$ is a finite abelian group, this classifying space has the homotopy type of the product $G \times X^G \times T^{r(G)}$, where $X^G$ is a simply connected infinite loop space and $T^{r(G)}$ is the product of $r(G)$ circles. We give an explicit expression for the number $r(G)$. Finally, we present some results about the classification of $G$-topological quantum field theories in dimension two.

Introduction

Let $\mathcal{S}^G$ denote the 1+1 dimensional free $G$-cobordism category. The objects of $\mathcal{S}^G$ are disjoint unions of principal $G$-bundles over the circle. They are represented by finite sequences $(x_1, x_2, \ldots, x_n)$, $x_i \in G \cup \{0\}$, where $x_i \in G$ denotes the monodromy of a principal $G$-bundle over the circle and $x_i = 0$ denotes the empty $G$-bundle. When $x_i$ is equal to the neutral $1 \in G$, we say that we have trivial monodromy. The morphisms of $\mathcal{S}^G$ are cobordisms of principal $G$-bundles over oriented surfaces up to diffeomorphism (with some assumption for the incoming boundary).

Composition of two cobordisms is given by the gluing of their collars for a common boundary. The gluing of two manifolds has a smooth structure well defined up to

---

*Instituto de Matemáticas UNAM-Oaxaca, csegovia@matem.unam.mx
diffeomorphism, which extends to the gluing of two principal \( G \)-bundles producing the appropriate cobordism.

Associated to \( \mathcal{S}^G \) we have a simplicial set \( N\mathcal{S}^G \), called its nerve [Seg68]. The geometric realization of \( N\mathcal{S}^G \) is denoted by \( B\mathcal{S}^G \). We show in Section 1 that the connected components of \( B\mathcal{S}^G \) have the form described in the following theorem.

**Theorem 1.** There is a bijection \( \pi_0(B\mathcal{S}^G) \cong G/[G,G] \) which is compatible with the tensor product of \( \mathcal{S}^G \).

Let \( H_2(G) \) be the integral 2-homology group of \( G \), or the 2-dimensional free bordism of Conner-Floyd [CF64]. In Section 3 we show that the fundamental group of \( B\mathcal{S}^G \) is isomorphic with a direct sum.

**Theorem 2.** There is an isomorphism \( \pi_1(B\mathcal{S}^G) \cong \mathbb{Z} \oplus H_2(G) \).

Therefore, the classifying space \( B\mathcal{S}^G \) has the structure of a grouplike with abelian fundamental group. Thus \( B\mathcal{S}^G \) is homotopic to an infinite loop space, which splits into the product of \( G/[G,G] \) and some connected component of \( B\mathcal{S}^G \). For a finite abelian group \( G \), we are able to show the following in Section 4.

**Theorem 3.** For a finite abelian group \( G \), \( B\mathcal{S}^G \) is homotopic to the product \( G \times X^G \times T^{r(G)} \), where \( X^G \) is a simply connected infinite loop space and \( T^{r(G)} \) is the product of \( r(G) \) circles.

This article is organized as follows: in Section 1 we define the free \( G \)-cobordism category \( \mathcal{S}^G \). In Section 2 we study the classifying spaces of some important subcategories of \( \mathcal{S}^G \). In Section 3 we describe the groupoid of fractions of \( \mathcal{S}^G \) and we find the fundamental group of \( \mathcal{S}^G \). This led us to show that there is a 2-connected map with the \( G \)-equivariant version of the embedded cobordism category of [GMTW]. In Section 4 we study the homotopy type of the classifying space of \( \mathcal{S}^G \) (for \( G \) an abelian group). Finally, in Section 5 we give some results about the classification of \( G \)-topological quantum field theories, and in the Appendix 6 we obtain a concrete formula for the positive number \( r(G) \). Interesting applications are [SW15, SW20] for the sequence \( \{r(\mathbb{Z}_p^n)\}_{n \geq 0} \), with \( p \) a prime number, which appears in [A007581].

1 The free \( G \)-cobordism category

Every principal \( G \)-bundle over the circle is isomorphic to the following construction: attach the ends of \([0,1] \times G\) via multiplication by some element \( x \in G \), i.e., \((0,y)\) is identified with \((1,yx)\) for every \( y \in G \). This construction projects to the circle by restricting to the first coordinate and the action of \( G \) is defined by left multiplication on the second coordinate. Throughout the paper the element \( x \in G \) is called the *monodromy* of the
corresponding principal $G$-bundle over the circle. When $x$ is the neutral element, $1 \in G$, we say that we have trivial monodromy. Additionally, we consider the empty $G$-bundle represented by the zero $0$.

A finite sequence $(x_1, x_2, \cdots, x_n)$, with $x_i \in G \sqcup \{0\}$, $1 \leq i \leq n$, represents the disjoint union of principal $G$-bundles over the circle for $x_i \in G$ and the empty $G$-bundle for $x_i = 0$. In the case $n = 1$, we can take out the parenthesis.

For example, we consider the cyclic group $\mathbb{Z}/\mathbb{Z}_4 = \{1, x, x^2, x^3\}$ and the symmetric group $S_3 = \langle a, b : a^2 = b^2 = (ab)^3 = 1 \rangle$. For $x, x^2 \in \mathbb{Z}_4$ and $a, ab \in S_3$, we obtain the four principal $G$-bundles of Figures 1 and 2 (the dotted lines denote the identifications).

A (free) $G$-cobordism between two principal $G$-bundles $\pi : E \to X$ and $\pi' : E' \to X'$ is a principal $G$-bundle $\epsilon : \Sigma \to M$ with diffeomorphic boundaries $\partial \Sigma = E \sqcup -E'$ and $\partial M = X \sqcup -X'$, which match with the projection maps and the actions of $G$. Two $G$-cobordisms $\epsilon : \Sigma \to M$ and $\epsilon' : \Sigma' \to M'$ define the same class if $M$ and $M'$ are equivalent as cobordisms by a diffeomorphism $\phi : M \to M'$, $\Sigma$ and $\Sigma'$ are equivalent as cobordisms by a $G$-equivariant diffeomorphism $\psi : \Sigma \to \Sigma'$ and we have the commutative diagram

$$
\begin{array}{ccc}
\Sigma & \psi \downarrow & \Sigma' \\
\epsilon \downarrow & & \epsilon' \\
M & \phi \downarrow & M'
\end{array}
$$

(1)

In dimension two we give some important examples of $G$-cobordisms:
1. (Cylinder) for \( x, y \in G \), this is the \( G \)-cobordism from \( x \) to \( y \) with base space the cylinder, where there is an element \( z \in G \) with \( y = zxz^{-1} \). The diffeomorphism identification is given by the Dehn twist. Notice that the conjugation by the neutral \( 1 \in G \) produces the identity morphism.

2. (Pair of pants) for \( x, y \in G \), this is the \( G \)-cobordism with base space the pair of pants, with entry \((x, y)\) and exit the product \(xy\). We restrict to principal \( G \)-bundles which are a \( G \)-deformation retract\(^1\) of a principal \( G \)-bundle over the wedge \( S^1 \vee S^1 \). We denote this element by \( P_{(x,y)}(x, y) \), with the opposite orientation.

3. (Pair of pants with multiple legs) for \( \hat{x} = (x_1, \cdots, x_n) \), with \( x_i \in G \), this is the \( G \)-cobordism with base space the pair of pants with \( n \)-legs, where the monodromy for the entries is \( \hat{x} \) and for the exit is the product \( \prod_{i=1}^n x_i \). Similarly, we restrict to principal \( G \)-bundles which are a deformation retract of a principal \( G \)-bundle over the wedge \( S^1 \vee \cdots \vee S^1 \). We represent this element by \( P_{\hat{x}} \).

4. (Disk) there is only one \( G \)-cobordism over the disk which is a trivial bundle. We represent this element by \( D : 1 \to 0 \) and similarly, we define the \( G \)-cobordism \( \overline{D} : 0 \to 1 \).

5. (Handlebody) for \( x_i, y_i \in G, 1 \leq i \leq n \), this is the \( G \)-cobordism with base space the two dimensional handlebody of genus \( n \) with one boundary circle. The monodromy for the boundary circle is given by the product \([y_n, x_n] \cdots [y_1, x_1]\).

In Figure 3, Figure 4 and Figure 5 we have pictures of the previous \( G \)-cobordisms. For these pictures, we draw from left to right the direction of our cobordisms. Also every circle is labelled with its monodromy and for the cylinders we include inside the element of the group with which we do the conjugation.

**Definition 4.** The objects of the free \( G \)-cobordism category \( \mathcal{S}^G \) are finite sequences \((x_1, \cdots, x_n), x_i \in G \cup \{0\}\), where \( x_i \in G \) denotes the monodromy of a principal \( G \)-bundle over the circle and \( x_i = 0 \) denotes the empty \( G \)-bundle. The morphisms of \( \mathcal{S}^G \) are \( G \)-cobordisms over oriented surfaces. We assume that every morphism with connected base space and non-empty incoming boundary, i.e., \( \hat{x} = (x_1, \cdots, x_n) \) with \( x_i \neq 0 \) for \( 1 \leq i \leq n \), factorises with the precomposition of the \( G \)-cobordism \( P_{\hat{x}} \) over the pair of pants (with multiple legs), see Figure 6. The morphisms in \( \mathcal{S}^G \) have a well defined composition by the gluing of their collars for a common boundary.

\(^1\)By a \( G \)-deformation retract we indicate that the homotopy is by means of principal \( G \)-bundles.
Figure 3: $G$-cobordisms over the cylinder, the pair of pants and the disc.

\[ x \quad zxz^{-1} \quad x \quad y \quad xy \quad xy \quad y \quad 1 \quad 1 \]

Notice that the $G$-cobordisms given by the cylinders with non trivial conjugation are not allowed morphisms in $\mathcal{S}_G$. However, they are the building blocks in a combinatorial description of the morphisms of $\mathcal{S}_G$.

The category $\mathcal{S}_G$ has a monoidal structure given by the disjoint union $\sqcup : \mathcal{S}_G \times \mathcal{S}_G \to \mathcal{S}_G$, where the unit is the empty $G$-bundle. Furthermore, there is a symmetric structure induced by a natural transformation $c : \sqcup \to \sqcup \circ \tau$ where $\tau$ is the twist functor. This is given by a bunch of crossing identity $G$-cobordisms over cylinders, where $c^2 = \text{id}$.

Now, we show Theorem 1 which says that there is one-to-one correspondence between...
the connected components of $B\mathcal{C}^G$ and the abelianization of $G$, i.e,

$$\pi_0(B\mathcal{C}^G) \cong \frac{G}{[G,G]}.$$ \hfill (2)

Because of the $G$-cobordism over the disc, we can restrict to objects given by finite sequences $(x_1, \cdots, x_n)$, with $x_i \neq 0$. Then the $G$-cobordism over the pair of pants with multiple legs connects $(x_1, \cdots, x_n)$ with its product $x = \prod_{i=1}^{n} x_i$. Due to the $G$-cobordisms over the handlebodies, two elements $x, y \in G$ are connected in $\mathcal{C}^G$ if and only if they differ by an element in $[G,G]$. Notice that the product in $\pi_0(B\mathcal{C}^G)$ induced by the monoidal structure of $\mathcal{C}^G$, agrees with the product of $G/[G,G]$. This is because of the existence of the $G$-cobordism over the pair of pants (with multiple legs).

## 2 Analysis of subcategories

In this section we study the classifying space of the following subcategories of $\mathcal{C}^G$:

1. $\mathcal{C}_0^G$ is the full subcategory of $\mathcal{C}^G$ with only one object given by the empty $G$-bundle.
2. $\mathcal{C}_{>0}^G$ is the subcategory of $\mathcal{C}^G$ with the same objects of $\mathcal{C}^G$ except for the empty $G$-bundle and where for the base space, each connected component of every morphism has non empty incoming boundary and non empty outgoing boundary.
3. $\mathcal{C}_b^G$ is the subcategory of $\mathcal{C}^G$ with the same objects of $\mathcal{C}^G$ and where for the base space, each connected component of every morphism has non empty outgoing boundary.
4. $\mathcal{C}_1^G$ is the full subcategory of $\mathcal{C}_{>0}^G$ with objects given by only one principal $G$-bundle over the circle.

We denote by $\mathcal{C}$, $\mathcal{C}_0$, $\mathcal{C}_b$ and $\mathcal{C}_1$ the cobordism categories associated to the trivial group. Tillmann [Til96] shows the classifying space of $\mathcal{C}_0$ is the infinite torus $T^\infty$. The same is true for $\mathcal{C}_0^G$. 

![Figure 6: A typical morphism in $\mathcal{C}^G$ with connected base space.](image-url)
Proposition 5. The classifying space $B\mathcal{S}_G^0$ is homotopic to the infinite dimensional torus $T^\infty$.

Proof. This is similar to the proof of Tillmann [Til96]. In fact, the monoid $\mathcal{S}_G^0$ is endowed with the structure of an abelian monoid which is infinitely generated and without torsion. Thus this monoid has the form $\mathbb{N}^\infty$ and since the classifying space of $\mathbb{N}$ is the circle, then the classifying space $B\mathcal{S}_G^0$ is the infinite torus.

Tillmann [Til96] defines a functor $\Phi : \mathcal{S}_G^0 \to \mathcal{S}_1$ which is the constant map in objects and each morphism $\Sigma$ with $n$ incoming circles, $c$ connected components, genus $g$ and $m$ outgoing circles, is mapped by $\Phi$ to

$$\Phi(\Sigma) := \frac{1}{2}(m - n - \chi(\Sigma)) = g + m - c.$$ (3)

This represents the unique morphism in $\mathcal{S}_1$ with genus $g + m - c$. Moreover, $\Phi$ is left adjoint to the inclusion $\mathcal{S}_1 \hookrightarrow \mathcal{S}_G^0$ with a natural transformation defined for the object $n$, by the morphism $P_n : n \to 1$ which is the pair of pants with $n$ legs. Thus we have the commutativity of

$$n \xrightarrow{P_n} 1 \quad \Sigma \quad \downarrow \quad g + m - c \quad \downarrow \quad m \xrightarrow{P_m} 1.$$ (4)

Furthermore, the functor $\Phi$ can be extended to $\mathcal{S}_b$. Since the subcategory $\mathcal{S}_1$ is isomorphic to the natural numbers $\mathbb{N}$, the classifying spaces of $\mathcal{S}_G^0$, $\mathcal{S}_b$ and $\mathcal{S}_1$ have the same homotopy type of the circle $S^1$.

For a finite abelian group $G$ we define the functor $\Phi^G : \mathcal{S}_G^0 \to \mathcal{S}_1^G$ in objects $(x_1, \ldots, x_n)$ by the product $x = \prod_{i=1}^n x_i$. For the morphisms of $\mathcal{S}_G^0$, we consider first the case where we have a connected base space. Take $\Sigma$ a $G$-cobordism (with connected base space) which goes from $\hat{x} := (x_1, \ldots, x_n)$ to $\hat{y} := (y_1, \ldots, y_m)$. By hypothesis, see Definition 4, the morphism $\Sigma$ factors by a precomposition of the $G$-cobordism over a pair of pants $P_{\hat{x}}$. Therefore, there is a unique $G$-cobordism $\Phi^G(\Sigma)$, from $x = \prod_{i=1}^n x_i$ to $y = \prod_{j=1}^m y_j$, in the subcategory $\mathcal{S}_1^G$, satisfying the following commutative diagram:

$$\begin{array}{ccc}
\hat{x} & \xrightarrow{P_{\hat{x}}} & x \\
\uparrow{\Sigma} & & \downarrow{\Phi^G(\Sigma)} \\
\hat{y} & \xrightarrow{P_{\hat{y}}} & y
\end{array}$$ (5)

A monoidal property (without unit) is obtained for morphisms $\Sigma_1, \ldots, \Sigma_r$, where $\Sigma_i$, $1 \leq i \leq r$, is a $G$-cobordism in $\mathcal{S}_G^0$ with connected base space, which goes from $\hat{x}_i := (x_{i1}, \ldots, x_{i m_i})$ to $\hat{y}_j := (y_{j1}, \ldots, y_{jm_j})$ ($x_i = \prod_{k=1}^{m_i} x_{ik}$ and $y_j = \prod_{k=1}^{m_j} y_{jk}$). Therefore, there
is a unique $G$-cobordism $\Phi^G(\Sigma_1 \sqcup \cdots \sqcup \Sigma_r)$, from $x_1 \cdots x_r$ to $y_1 \cdots y_r$, in the subcategory $\mathcal{S}_1^G$, satisfying the following commutative diagram:

\[
\begin{array}{c}
\xymatrix{
\hat{x}_1 \sqcup \cdots \sqcup \hat{x}_r \ar[r]^{P_{\hat{x}_1\sqcup \cdots \sqcup \hat{x}_r}} \ar[d]_{\Sigma_1 \sqcup \cdots \sqcup \Sigma_r} & x_1 \cdots x_r \\
\hat{y}_1 \sqcup \cdots \sqcup \hat{y}_r \ar[r]^{P_{\hat{y}_1\sqcup \cdots \sqcup \hat{y}_r}} & y_1 \cdots y_r 
}
\end{array}
\]

(6)

It is straightforward to see that $\Phi^G$ is a functor because of the uniqueness of the factorisations, see Figure 7 for an example.

As a consequence, we obtain the following theorem:

**Theorem 6.** The inclusion functor $\mathcal{S}_1^G \hookrightarrow \mathcal{S}_0^G$ has a left adjoint $\Phi^G$.

Likewise, we can extend the functor $\Phi^G$ to the category $\mathcal{S}_b^G$ and Theorem 6 follows. Consequently, the categories $\mathcal{S}_0^G$, $\mathcal{S}_b^G$ and $\mathcal{S}_1^G$ have homotopy equivalent classifying spaces.

Finally, consider the $[G, G]$-graded monoid $\mathcal{M}_G := \{\mathcal{M}_x\}_{x \in [G,G]}$, where $\mathcal{M}_x$ is the collection of $G$-cobordisms over handlebodies with monodromy $x \in G$ over the boundary circle. There are products $\mathcal{M}_{x_1} \times \mathcal{M}_{x_2} \rightarrow \mathcal{M}_{x_1 x_2}$ given by the composition with the $G$-cobordism over the pair of pants $P_{(x_1, x_2)}$. Also, there is an action $\rho_z : \mathcal{M}_x \rightarrow \mathcal{M}_{x z^{-1}}$, $z \in G$, defined by the composition with a $G$-cobordism over the cylinder with entry $x$ and conjugation by $z$. The product is associative (with unit the disc) and twisted commutative and indeed, we mimic the axioms of a $G$-Frobenius algebra \cite{MS06, GS17}.

The importance of the monoid $\mathcal{M}_G$ is that it works as a model for the composition of $\mathcal{S}_1^G$. This is because every morphism in $\mathcal{S}_1^G$, can be written as the gluing of two
$G$-cobordisms, one over a handlebody and the other over a pair of pants. Thus the composition of two elements in $\mathcal{S}_1^G$, is given by the product of their associated $G$-cobordisms over the handlebodies.

An important part is the submonoid associated to the neutral element, given by $\mathcal{M}_1$. This consists of $G$-cobordisms over handlebodies with trivial monodromy over the boundary circle. This monoid is abelian, finitely generated and without torsion. Consequently, $\mathcal{M}_1$ is a finite direct sum $\mathbb{N}^{r(G)}$ for some positive integer $r(G)$. This number $r(G)$ writes as a finite sum $r_1(G) + r_2(G) + \cdots$, where $r_i(G)$ is the number of generators with base space of genus $i$. For instance, $r_1(G)$ coincides with the cardinality of the quotient of the commuting pairs by an action of the special linear group. When $G$ is an abelian finite group we obtain $r(G) = r_1(G)$, which is a multiplicative function and the Appendix 6 we find its exact value. We conclude the section with the following result.

**Proposition 7.** For $G$ a finite abelian group, the subcategory $\mathcal{S}_1^G$ is isomorphic with a finite direct sum $\mathbb{N}^{r(G)}$.

### 3 The fundamental group

The aim of the present section is to show that the fundamental group of $B\mathcal{S}^G$ is the direct sum $\mathbb{Z} \oplus H_2(G)$, where $H_2(G)$ is the integral 2-homology group. In this respect, we need an interpretation of the groupoid of fractions $\mathcal{G}^G := \mathcal{S}^G[(\mathcal{S}^G)^{-1}]$ with the left calculus of fractions $[\text{GZ67}]$. The fundamental group with base space an object $x$ in $\mathcal{S}^G$, is the automorphism group $\text{Hom}_{\mathcal{S}^G}(x,x)$, see Quillen $[\text{Qui73}]$.

An important assumption is the restriction of $\mathcal{S}^G$ to objects which are in the connected component of the empty $G$-bundle. Thus, for every object $\hat{x}$, we can take a morphism $\delta_{\hat{x}}$ from $\hat{x}$ to the empty $G$-bundle, where for simplicity, take it with connected base space.

Consider the subset of morphisms of those endomorphisms in $\mathcal{S}^G$ which are the disjoint union of identity $G$-cobordisms over cylinders and $G$-cobordisms over closed surfaces. Denote this subset of morphisms by $\mathcal{S}_0^G$. Notice $\mathcal{S}_0^G$ admits a calculus of left fractions $[\text{GZ67}]$. Therefore, the category of fractions $\mathcal{S}^G[(\mathcal{S}_0^G)^{-1}]$ can be described by roofs

\[
(id_{\hat{x}} \sqcup \gamma, \Sigma) = \begin{array}{c}
\hat{x} \\
\hat{y}
\end{array} \xleftarrow{\Sigma} \begin{array}{c}
\hat{x} \\
\hat{y}
\end{array}
\]

\[
\xleftarrow{id_{\hat{x}} \sqcup \gamma}
\]

where $\Sigma : \hat{x} \to \hat{y}$ is a morphism in $\mathcal{S}^G$ and $\gamma$ is a $G$-cobordism over closed surfaces.

However, the category $\mathcal{S}^G[(\mathcal{S}_0^G)^{-1}]$ is not enough to obtain the groupoid of fractions $\mathcal{G}^G$. For this purpose, we consider the quotient by an equivalence relation which is motivated as follows: the functor $\mathcal{S}^G \to \mathcal{S}^G[(\mathcal{S}_0^G)^{-1}]$ sends a morphisms $\Sigma : \hat{x} \to \hat{y}$


Figure 8: The identification $\text{id}_1 \sqcup (D \circ D) \sim \overline{D} \circ D$.

Figure 9: Property in the category of fractions.

To the roof $(\text{id}_x, \Sigma)$. Take the gluing along $\hat{y}$ of $\Sigma$ and $\delta_y$. This is represented by $\Sigma' = \Sigma \sqcup_{\hat{y}} \delta_y$ which is a morphism from $\hat{x}$ to the empty $G$-bundle. Denote by $\overline{\Sigma'} : 0 \to \hat{x}$ the $G$-cobordism obtained by changing the orientation of $\Sigma'$. We can obtain a left inverse for $(\text{id}_x, \Sigma)$ after we impose some identification. The proposal for the left inverse is the roof $(\text{id}_{\hat{y}} \sqcup (\Sigma' \circ \overline{\Sigma'}), \overline{\Sigma'} \circ \delta_{\hat{y}})$. In fact, the composition of $(\text{id}_x, \Sigma)$ followed by $(\text{id}_{\hat{y}} \sqcup (\Sigma' \circ \overline{\Sigma'}), \overline{\Sigma'} \circ \delta_{\hat{y}})$ is

$$(\text{id}_x \sqcup (\Sigma' \circ \overline{\Sigma'}), \overline{\Sigma'} \circ \Sigma') \ .$$

Therefore, we need to impose the identification

$$\text{id}_x \sqcup (\Sigma' \circ \overline{\Sigma'}) \sim \overline{\Sigma'} \circ \Sigma'.$$ (9)

Take the equivalence relation generated by the identifications (9) using the composition and the monoidal structure of $\mathcal{J}^G$ and denote the quotient category by $\mathcal{J}_G^\sim$.

In Figure 8 we illustrate (9) for the disc $D : 1 \to 0$ and the reverse disc $\overline{D} : 0 \to 1$. As an application we obtain that in the category of fractions we have some sort of cancellation between $G$-cobordisms over handlebodies and spheres, see Figure 9.

In the following proposition we outline the missing details to calculate the category of fractions.

**Proposition 8.** There is an isomorphism $\mathcal{J}^G \cong \mathcal{J}_G^\sim[(\mathcal{J}_0^G)^{-1}]$.

**Proof.** Now we show that the left inverse of $(\text{id}_x, \Sigma)$ is also right inverse. For this purpose, we consider the composition of $(\text{id}_{\hat{y}} \sqcup (\Sigma' \circ \overline{\Sigma'}), \overline{\Sigma'} \circ \delta_{\hat{y}})$ followed by $(\text{id}_x, \Sigma)$. This is equal
to
\[(\alpha, \beta) := (\text{id}_y \sqcup (\Sigma' \circ \Sigma), (\Sigma \circ \Sigma) \sqcup (\delta_y \circ \delta_y)) \, . \tag{10}\]
This is equivalent to the identity morphism because we can take a third roof
\[(\epsilon, \xi) := (\text{id}_y \sqcup (\delta_y \circ \delta_y), (\Sigma \circ \Sigma) \sqcup (\delta_y \circ \delta_y)^{\pm 2}) \, , \tag{11}\]
and we have the commutative diagram
\[\begin{array}{ccc}
\hat{y} & \xrightarrow{\epsilon} & \hat{y} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\hat{y} & \xrightarrow{id_y} & \hat{y} \\
\end{array} \quad \begin{array}{ccc}
\hat{y} & \xleftarrow{\xi} & \hat{y} \\
\downarrow{\rho} & & \downarrow{id_y} \\
\hat{y} & \xleftarrow{id_y} & \hat{y} \\
\end{array} \quad \begin{array}{ccc}
\hat{y} & \xrightarrow{\rho} & \hat{y} \\
\downarrow{\beta} & & \downarrow{id_y} \\
\hat{y} & \xleftarrow{id_y} & \hat{y} \\
\end{array} \]
\[\text{(12)}\]

It is possible to show that the inverse for \((\text{id}_y, \Sigma)\) does not depend on the morphism \(\delta_y\).
The universal properties of the category of fractions and the quotient category implies the proposition. \(\square\)

**Theorem 9.** \(\pi_1(B\mathcal{F}^G) = \mathbb{Z} \oplus H_2(G)\).

**Proof.** The fundamental group \(\pi_1(B\mathcal{F}^G)\) can be identified with the automorphism group \(\text{Hom}_{\mathcal{F}_G}(0,0)\), where 0 denotes the trivial \(G\)-bundle. Proposition 8 implies that it is enough to consider the group completion of the monoid with elements the roofs of the form \((\text{id}_0, \Sigma)\), where \(\Sigma\) is a morphism in the quotient category \(\mathcal{F}_G^G\). There is a normal form for the roof \((\text{id}_0, \Sigma)\) which considers the identification (9) for the disc and the reverse disc, see Figure 8. In fact, every time we find an essential simple curve \(\gamma\) in the base space of \(\Sigma\) with trivial monodromy, we cut \(\Sigma\) along \(\gamma\) and we fill with discs the two boundary circles produced by the cutting. This automatically extends to trivial bundles over the two discs since we start with trivial monodromy for \(\gamma\). Denote by \(c_\gamma(\Sigma)\) the \(G\)-cobordism obtained by this procedure of cutting \(\Sigma\). Thus we have the equality for roofs \((\text{id}_0, \Sigma) = (\text{id}_0 \sqcup S, c_\gamma(\Sigma))\) where \(S\) is the unique \(G\)-cobordism over the sphere. We continue with this process of cutting along every essential simple curve with trivial monodromy, keeping track of the number of times by adding an \(S\) in the first coordinate of the roof. This process ends in a finite number of \(k\) steps, for some \(k \in \mathbb{N}\), obtaining a subset of generators \(\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_s\) of the 2-dimensional integral homology group \(H_2(G)\). Thus for every \(\Gamma_i, 1 \leq i \leq s\), there is no more essential simple curve in the base space with trivial monodromy. The representation of the generators of \(H_2(G)\) through elimination of trivial monodromy can be seen in [Mil40, DS21]. As a consequence, we obtain a roof \((\text{id}_0 \sqcup kS, lS \sqcup \Gamma)\), where \(kS\) and \(lS\) are the disjoint union of \(k\) and \(l\) copies of \(S\). The assignment \((l - k, \Gamma)\) gives a well defined map to \(\mathbb{Z} \oplus H_2(G)\). This extends to the group completion and gives the required isomorphism. \(\square\)

\[^2\text{A closed curve is called essential if it is not homotopic to a point, puncture, or a boundary component.}\]
Finally, we can see $\mathcal{F}^G$ as an equivalent category with circles and surfaces equipped with maps to $BG$. The topological category of cobordisms equipped with maps to $BG$ is a special case of the category studied in [GMTW] with tangential structure $\theta : BG \times BSO(2) \to BSO(2)$. Bökstedt-Svane [BST14] provide a description of the fundamental group in terms of what they call chimera relations. We deduce in the next paragraph these relations using the identification (4). This means that the canonical functor from the topological category to the discrete category $\mathcal{F}^G$ is a 2-connected map in classifying spaces.

Take $\Sigma_i, i \in \{1, 2, 3, 4\}$, $G$-cobordisms from $X$ to the empty $G$-bundle. We obtain the following relations for roofs:

\[(\text{id}_0, \Sigma_3 \circ \Sigma_1 \sqcup \Sigma_4 \circ \Sigma_2) = (\Sigma_1 \circ \Sigma_1 \sqcup \Sigma_2 \circ \Sigma_2, (\Sigma_3 \circ \Sigma_1 \circ \Sigma_2) \sqcup (\Sigma_4 \circ \Sigma_1 \circ \Sigma_2))\]

\[= (\Sigma_1 \circ \Sigma_1 \sqcup \Sigma_2 \circ \Sigma_2, (\Sigma_3 \circ \Sigma_2 \sqcup \Sigma_2 \circ \Sigma_1) \sqcup (\Sigma_4 \circ \Sigma_1 \sqcup \Sigma_1 \circ \Sigma_2))\]

\[= (\Sigma_1 \circ \Sigma_1 \sqcup \Sigma_2 \circ \Sigma_2, (\Sigma_3 \circ \Sigma_2 \sqcup \Sigma_2 \circ \Sigma_1) \sqcup (\Sigma_2 \circ \Sigma_1 \circ \Sigma_1 \circ \Sigma_2))\]

\[= (\text{id}_0, \Sigma_3 \circ \Sigma_2 \sqcup \Sigma_4 \circ \Sigma_1)\]

### 4 Multiplicative structure

In this section the category $\mathcal{F}^G$ is restricted to objects which are in the connected component of the empty $G$-bundle. This category continues to be a symmetric monoidal category with the disjoint union and the crossing identity $G$-cobordism over cylinders.

The subcategory $\mathcal{F}^G_{\geq 0}$ from Section 2, now restricted to the component of the empty $G$-bundle, does not have a unit for the monoidal structure. However, there is a way to get a unit using the trivial $G$-bundle over the circle and constructing a quoting category $\mathcal{F}^G_{\leq 0}$ as follows: first of all, principal $G$-bundles with connected base space admit a connected sum which is a $G$-equivariant extension of the usual connected sum of surfaces. For the product $\mathcal{F}^G_{\geq 0} \times \mathbb{Z}$, take morphisms $\gamma_i, i \in \{1, 2\}$, in $\mathcal{F}^G_{\geq 0}$, each one with connected base space, with the relation $(\gamma_1 \sqcup \gamma_2, n) \sim (\gamma_1\sharp \gamma_2, n + 1)$, where $\gamma_1\sharp \gamma_2$ denotes the connected sum of the morphisms $\gamma_1$ and $\gamma_2$. Consider the equivalence relation in $\mathcal{F}^G \times \mathbb{Z}$ generated by these identifications using the composition and the monoidal structure and denote the quotient category by $\mathcal{F}^G_{\geq 0}$. In Figure 10, we illustrate the triangle diagram for the unit axiom. Thus $\mathcal{F}^G_{\geq 0}$ satisfies to be a symmetric monoidal category.

We have a symmetric monoidal functor $\Psi : \mathcal{F}^G \to \mathcal{F}^G_{\geq 0}$, which for objects $(x_1, \cdots, x_n)$ with $x_i \neq 0$, $1 \leq i \leq n$, the functor $\Phi^G$ is the identity. As one might expect the empty $G$-bundle is sent to the trivial $G$-bundle over the circle, i.e., $0 \to 1$. Denote by $S$ the unique $G$-cobordism over the sphere. For the morphisms, $\Psi$ is defined first for the disc $D : 1 \to 0$ and the reverse disc $\overline{D}$, by $\Psi(D) = (\text{id}_1, 1)$ and $\Psi(\overline{D}) = (\text{id}_1, 0)$, where $\text{id}_1$ is the identity over the trivial $G$-bundle over the circle. Now, for $\Sigma$ a morphism in $\mathcal{F}^G$ from $(x_1, \cdots, x_n)$
to \((y_1, \cdots, y_m)\), with \(x_i, y_j \neq 0, i \in \{1, \cdots, n\}\) and \(j \in \{1, \cdots, m\}\), we consider that the base space of \(\Sigma\) has \(k\) connected components, so we can write \(\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k\), where \(\Sigma_l, l \in \{1, \cdots, k\}\), is the \(G\)-cobordism over the \(l\) connected component. Thus we take \(\Psi(\Sigma) = (\Sigma_1 \# \cdots \# \Sigma_k, k - 1)\). Using the composition of \(\mathcal{J}^G\) we have a well defined functor \(\Psi\) and it is not so difficult to see that the functor is symmetric and monoidal.

**Proposition 10.** There is a symmetric monoidal functor \(\Psi : \mathcal{J}^G \rightarrow \tilde{\mathcal{J}}^G_{>0}\).

Similarly as in Section 2, the subcategory \(\mathcal{J}_1^G \subset \mathcal{J}^G_{>0}\) defines the subcategory \(\tilde{\mathcal{J}}_1^G\). Again, the inclusion \(\tilde{\mathcal{J}}_1^G \rightarrow \tilde{\mathcal{J}}^G_{>0}\) has a left adjoint denoted by

\[
\tilde{\Phi}^G : \tilde{\mathcal{J}}^G_{>0} \rightarrow \tilde{\mathcal{J}}_1^G.
\]

The natural transformation is provided by the \(G\)-cobordism over the pair of pants with multiple legs. Thus for \(\Sigma : \hat{x} \rightarrow \hat{y}\) a morphism in \(\tilde{\mathcal{J}}^G_{>0}\), we obtain the following commutative square

\[
\begin{array}{ccc}
\hat{x} & \xrightarrow{P_{\Sigma}} & x \\
\Sigma & \downarrow & \phi^G(\Sigma) \\
\hat{y} & \xrightarrow{P_{\Sigma}} & y
\end{array}
\]

The extension of the functor \(\Phi^G\) to the whole category \(\mathcal{J}^G\) is defined by the following composition:

\[
\begin{array}{ccc}
\mathcal{J}^G & \xrightarrow{\Psi} & \tilde{\mathcal{J}}^G_{>0} \\
\Phi^G & \xrightarrow{\tilde{\Phi}^G} & \tilde{\mathcal{J}}_1^G
\end{array}
\]

For the rest of the section assume that \(G\) is an abelian group.
Proposition 11. For a finite abelian group $G$, the canonical map $\mathcal{S}^G_1 \to \tilde{\mathcal{S}}^G_1$ is a homotopy equivalence in classifying spaces.

Proof. In this case both $\mathcal{S}^G_1$ and $\tilde{\mathcal{S}}^G_1$ have only one object given by the trivial $G$-bundle over the circle. Moreover, they are commutative monoids and the canonical map $\mathcal{S}^G_1 \to \tilde{\mathcal{S}}^G_1$ has a unique comma category which we show is a filtered one. An important fact is that the property illustrated in Figure 9 happens only if $x$ is the neutral element. This means that the $G$-cobordism $S$ over the sphere cancels one genus in the base space of a $G$-cobordism represented by a trivial $G$-bundle. Therefore, for $\gamma, \gamma'$ two morphisms in $\tilde{\mathcal{S}}^G_1$ we can eliminate in both morphisms the disjoint union of $G$-cobordisms over the sphere by composing with adequate $G$-cobordisms represented by trivial $G$-bundles. This implies the first property of a filtered category since both monoids are commutative. The second property of a filtered category follows using the unique decomposition $\gamma = \gamma_{\text{Ntr}} \sqcup \gamma_{\text{tr}}$ of a morphism in $\tilde{\mathcal{S}}^G_1$, where the $G$-cobordism $\gamma_{\text{tr}}$ is represented by a trivial $G$-bundle and it is maximal in the sense that $\gamma_{\text{Ntr}}$ is represented by a non-trivial $G$-bundle. Since the classifying space of a filtered category is a contractible space, then by the work of Quillen [Qui73], we obtain the proposition.

For example in the case when $G$ is the trivial group, the map $\mathcal{S}^G_1 \to \tilde{\mathcal{S}}^G_1$ reduces to the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ as in the work of Tillmann [Til96].

In what follows, we study the homotopy type of the classifying space of whole category $\mathcal{S}^G$.

Theorem 12. For a finite abelian group $G$, every connected component of the free $G$-cobordism category has the homotopy type of the product $X^G \times T^{r(G)}$ where $X^G$ is a simply connected infinite loop space and $T^{r(G)}$ is the product of $r(G)$ circles.

Proof. We have $\Psi$ and $\tilde{\Phi}^G$ are symmetric monoidal functors and hence the induced maps on classifying spaces are maps of infinite loop spaces. Thus if we denote by $X^G$ the homotopy fiber of $\Phi^G = \tilde{\Phi}^G \circ \Psi$, then it is an infinite loop space. Consider the composition $\mathcal{S}^G_1 \hookrightarrow \mathcal{S}^G \xrightarrow{\Phi^G} \tilde{\mathcal{S}}^G$ which coincides with the map of monoids $\mathcal{S}^G_1 \to \tilde{\mathcal{S}}^G_1$ which by Proposition 11 it is a homotopy equivalence in classifying spaces. Thus we obtain a section for $\Phi^G$ and we end the proof of the theorem.

We end this section with a categorical description of the space $X^G$ for a finite abelian group $G$. Proposition 7 says that the subcategory $\mathcal{S}^G_1$ is of the form $\mathbb{N}^{r(G)}$ for some positive number $r(G)$. We can extend $\Phi^G$ to the group completion of $\tilde{\mathcal{S}}^G$. Thus we can apply the Theorem B of Quillen [Qui73], and we obtain the homotopy groups $\pi_* X^G = \pi_* (1/\Phi^G)$ where $1/\Phi^G$ is the unique comma category of $\Phi^G$.
5 Applications to $G$-TQFT’s

First, consider functors $F : \mathcal{I}^G \to \mathcal{D}$ with $\mathcal{D}$ a groupoid. The universal property of the groupoid of fractions $\mathcal{I}^G[\mathcal{I}^G^{-1}]$ associates a unique functor $\overline{F} : \mathcal{I}^G[\mathcal{I}^G^{-1}] \to \mathcal{D}$, satisfying the commutative diagram

$$
\begin{array}{ccc}
\mathcal{I}^G & \xrightarrow{P_{\mathcal{I}^G}} & \mathcal{I}^G[\mathcal{I}^G^{-1}] \\
F \downarrow & & \downarrow \overline{F} \\
\mathcal{D} & & \\
\end{array}
$$

Similarly, the groupoid $\mathcal{I}^G[\mathcal{I}^G^{-1}]$ can be considered as the union of its connected components given by $G/[G, G]$. Any two connected components are related by the $G$-cobordisms over the pair of pants. Therefore, $F$ is determined by the restriction to some connected component plus the image of the $G$-cobordisms over the pair of pants. Furthermore, from Section 4 there is a factorisation of the localization functor $P_{\mathcal{I}^G}$ as follows

$$
\begin{array}{ccc}
\mathcal{I}^G & \xrightarrow{P_{\mathcal{I}^G}} & \mathcal{I}^G[\mathcal{I}^G^{-1}] \\
\Psi \downarrow & & \\
\tilde{\mathcal{I}}^G >_0 & & \\
\end{array}
$$

Denote by $T$ the functor $\tilde{\mathcal{I}}^G >_0 \to \mathcal{I}^G[\mathcal{I}^G^{-1}]$ and set $F' = \overline{F} \circ T$. In Section 4 we have an adjunction between the categories $\tilde{\mathcal{I}}^G >_0$ and $\tilde{\mathcal{I}}^G_1$, which was used to define the functor $\Phi^G : \mathcal{I}^G \to \tilde{\mathcal{I}}^G_1$. For $\Sigma$ a morphism in $\mathcal{I}^G$, we consider its image $\Psi(\Sigma) : \hat{x} \to \hat{y}$, with $x = \prod x_i$ and $y = \prod y_j$. This adjunction has the following commutative diagram

$$
\begin{array}{ccc}
\hat{x} & \xrightarrow{P_{\hat{x}}} & x \\
\Psi(\Sigma) \downarrow & & \downarrow \Phi^G(\Sigma) \\
\hat{y} & \xrightarrow{P_{\hat{y}}} & y \\
\end{array}
$$

Therefore, the functor $F : \mathcal{I}^G \to \mathcal{D}$ is completely determined as follows:

$$
F(\Sigma) = F'(P_{\hat{y}})^{-1} \circ F'(\Phi^G(\Sigma)) \circ F'(P_{\hat{x}}),
$$

As an application we consider an abelian group $\mathcal{D}$. We recover the case of Tillmann [Til96] by setting $a_0 = F(S^2)$ and $c_n = F(P_n)$. Thus for $\Sigma : n \to m$ the functor $F$ is given by

$$
F(\Sigma) = c_n - c_m + a_0(\Phi(\Sigma)),
$$

where $b_n + c_n = b_1$ with $b_n = F(D_n)$ for $D_n : 0 \to n$ the disjoint union of $n$ disks.
Second, we consider symmetric monoidal functors $F: \mathcal{S}^G \rightarrow \text{Vect}_\mathbb{C}$, where $\text{Vect}_\mathbb{C}$ is the category of finite dimensional complex vector spaces. They are currently called $G$-topological quantum field theories [MS06]. A folk result states a correspondence between every symmetric monoidal functor $F: \mathcal{S}^G \rightarrow \text{Vect}_\mathbb{C}$ and a $G$-Frobenius algebra [MS06, GS17]. Briefly, a $G$-Frobenius algebra is a $G$-graded vector space $A = \bigoplus_{x \in G} A_x$, where $A_x$ is a finite dimensional $\mathbb{C}$-vector space, with $G$-graded associative products $A_x \otimes A_y \rightarrow A_{xy}$ which are twisted commutative, also there is a trace $\theta: A_1 \rightarrow \mathbb{C}$ which is non degenerate and we have an action $\rho: G \rightarrow \text{Aut}(A)$ of algebra automorphisms and some other axioms.

We are interested when $F$ is morphism inverting, hence $A_x \cong \mathbb{C}$ for any $x \in G$. The algebraic structure is called discrete torsion determined by Turaev [Tur10] as follows:

$$\mathbb{C}_b(G) := \bigoplus_{x \in G} \mathbb{C} \times \{x\},$$

where for a basis $e_x$ in each $A_x$ and $b \in H^2(G, \mathbb{C}^*)$ a 2-cocycle, the product is $e_x \cdot e_y = b(x, y) \cdot e_{xy}$. There is a commutative diagram

$$\begin{array}{ccc}
\mathcal{S}^G & \xrightarrow{P_{\mathcal{S}^G}} & \mathcal{S}^G[\mathcal{S}^G^{-1}] \\
\downarrow F & & \downarrow \mathcal{P} \\
\text{Vect}_\mathbb{C} & & \\
\end{array}$$

Thus $F$ is completely determined by the restriction to the automorphism group of the empty $G$-bundle in $\mathcal{S}^G[\mathcal{S}^G^{-1}]$, plus the images of a generating set of the monoid $\mathcal{M}_G$ from Section 2. This induces a representation of the fundamental group $\mathbb{Z} \oplus H_2(G) \rightarrow \mathbb{C}^*$ from Section 3.

Finally, for any $G$-topological quantum field theory $F: \mathcal{S}^G \rightarrow \text{Vect}_\mathbb{C}$, we consider the subset of morphism $\mathcal{S}_0^G$ used in Section 3. This consists of those endomorphisms in $\mathcal{S}^G$ which are the disjoint union of identity $G$-cobordisms over cylinders and $G$-cobordisms over closed surfaces. Similarly, as in the case of Tillman [Til96], any $G$-cobordism over a closed surface can be written as a double construction given by a $G$-cobordism over the gluing $M \sqcup_X \overline{M}$, where $M$ is a surface with boundary $X$ and $\overline{M}$ is $M$ with opposite orientation. The manifold $M$ is found observing that any $G$-cobordism over a closed surface has a decomposition in terms of $G$-cobordisms over pair of pants and cylinders, see Figure 11 for an example. Then we cut the surface in two halves given by the front and the back side observing they have the same $G$-cobordisms but with different orientation. This implies that any $G$-cobordism over a closed surface is assigned to a non-zero complex number by the non-degeneracy of the inner product. Consequently, we obtain that every $G$-topological quantum field theory $F: \mathcal{S}^G \rightarrow \text{Vect}_\mathbb{C}$ factors through $\mathcal{S}^G[\mathcal{S}_0^G^{-1}]$. 

16
6 Appendix

Let $G$ be a finite abelian group. In Section 2 and 4 we gave a positive integer number $r(G)$ which is the cardinality of the quotient of $G \times G$ by the action of the special linear group $\text{Sl}(2,\mathbb{Z})$ generated by the identifications

$$(x, y) \sim (x, x + y) \text{ and } (x, y) \sim (y, -x).\quad (23)$$

This number is the sum $r(G) = c(G) + e(G)$, where $c(G)$ is the number of cyclic subgroups of $G$ and $e(G)$ is some error correction. Consider two finite groups $G_1$ and $G_2$ with relative prime orders, i.e., $(|G_1|, |G_2|) = 1$. Since the number of cyclic groups is a multiplicative function, it is not difficult to show that the number $r(G)$ is also a multiplicative function.

In order to find the exact value for any finite abelian group we take $p$ prime and $l, n$ positive integers and we consider the group $\mathbb{Z}_p^n$. Take the product $\mathbb{Z}_p^n \times \mathbb{Z}_p^n$ with the action of $\text{Sl}(2,\mathbb{Z})$ and divide the set of orbits into three disjoint sets:

i) The classes are given by pairs $(x, y) \in \mathbb{Z}_p^n \times \mathbb{Z}_p^n$ such that $(x, y) = (0, 0) \mod p$.

Since we have the exact sequence

$$0 \longrightarrow \mathbb{Z}_p^{l-1} \times_p \mathbb{Z}_p^l \longrightarrow \mathbb{Z}_p^l \mod_p \longrightarrow \mathbb{Z}_p \longrightarrow 0.\quad (24)$$

Thus the number of orbits is $r(\mathbb{Z}_p^{n-1})$.

ii) The classes are given by pairs $(x, y) \in \mathbb{Z}_p^n \times \mathbb{Z}_p^n$ which are equivalent to a pair $(x', y')$ with either $x' \equiv 0 \mod p$ or $y' \equiv 0 \mod p$. The number of these classes is, for $n > 2$,

$$A_p(l, n) = p^{(l-2)(n-1)} \left[ p^{n-1} + (p^{2n-3} - p^{n-2}) \sum_{i=0}^{l-2} p^{i(n-2)} \right] \left[ \begin{array}{c} n \\ 1 \end{array} \right]_p,$$  \quad (25)

$$= \left[ p^{(l-1)(n-1)} + \frac{p^{n-1} - 1}{p^{n-1} - p} \left( p^{(l-1)(2n-3)} - p^{(l-1)(n-1)} \right) \right] \left[ \begin{array}{c} n \\ 1 \end{array} \right]_p, \quad (26)$$

In addition, $A_p(l, 2) = (lp^{l-1} - (l - 1)p^{l-2}) (p + 1)$ and $A_p(l, 1) = 1$. 

17
iii) The classes are given by pairs \((x, y) \in \mathbb{Z}_p^l \times \mathbb{Z}_p^n\) which are in the complement of the cases i) and ii). The number of these classes is

\[
B_p(l, n) = \frac{p^{l-1}(2n-3)(p-1)}{2} \binom{n}{2}_p,
\]

and \(B_p(l, 1) = 0\).

**Proposition 13.** For \(n > 2\), \(r(\mathbb{Z}_p^n) - r(\mathbb{Z}_p^{n-1})\) has the formula

\[
\left[\frac{p^{l-1}(2n-3) - p^{l-1}(n-1)}{p^{l-1}(2n-3) - p^{l-1}(n-1)}\right] \frac{n}{1}_p + \frac{p^{l-1}(2n-3)(p-1)}{2} \binom{n}{2}_p,
\]

and \(r(\mathbb{Z}_p^n) - r(\mathbb{Z}_p^{n-1}) = (lp^{l-1} - (l-1)p^{l-2})(p+1) + p^{l-1}(p-1)\).

Thus we obtain the following result.

**Theorem 14.** For a prime number \(p\) and \(l, n\) positive integers, the number \(r(\mathbb{Z}_p^n)\) has the following value

\[
r(\mathbb{Z}_p^n) = 1 + a_p(l, n) \binom{n}{1}_p + \frac{(p^{l(2n-3)} - 1)(p-1)}{p^{2n-3} - 1} \binom{n}{2}_p,
\]

where, for \(n \neq 1,2\), we have the formula

\[
a_p(l, n) = \frac{p^{l(n-1)} - 1}{p^{n-1} - 1} + \frac{p^{n-1} - 1}{p^{n-1} - p} \left[\frac{p^{l(2n-3)} - 1}{p^{2n-3} - 1} - \frac{p^{l(n-1)} - 1}{p^{n-1} - 1}\right].
\]

with \(a_p(l, 2) = lp^{l-1}\) and \(a_p(l, 1) = l\).

The formulas for \(A_p(l, n)\) and \(B_p(l, n)\) can be deduced as follows:

**Lemma 15.** We obtain

\[
A_p(1, n) = \binom{n}{1}_p,
\]

and every class has exactly \(p^2 - 1\) elements.

**Proof.** An element in \(\mathbb{Z}_p^n \times \mathbb{Z}_p^n\) is considered as a \(2 \times n\) matrix. The elements of \(A_p(1, n)\) represent the matrices with rank one, so we obtain a vector in \(\mathbb{Z}_p^n\). Two of these vectors are identified if they represent the same one dimensional space in \(\mathbb{Z}_p^n\) and every class has \(p^2 - 1\) elements. \(\square\)
Proposition 16. For $n > 2$, there is the formula

$$A_p(l, n) = p^{(l-2)(n-1)} \left[ p^{n-1} + (p^{2n-3} - p^{n-2}) \sum_{i=0}^{l-2} p^{i(n-2)} \right] \left[ \begin{array}{c} n \\ 1 \end{array} \right]_p,$$

and $A_p(l, 2) = (lp^{l-1} - (l-1)p^{l-2}) \left[ \begin{array}{c} n \\ 1 \end{array} \right]_p$ and $A_p(l, 1) = 1$.

Proof. We can show by induction on $l$, that the number of pairs $(x,y)$ in $\mathbb{Z}_p^l \times \mathbb{Z}_p^l$ which are equivalent to pairs $(x',y')$ with only one coordinate equivalent to $0 \mod p$, are divided in classes

$$\left[ \begin{array}{c} n \\ 1 \end{array} \right]_p p^{(l-2)(n-1)} (p^{n-1}, (p^{2n-3} - p^{n-2}), p^{n-2}(p^{2n-3} - p^{n-2}), \ldots, p^{(l-2)(n-2)}(p^{2n-3} - p^{n-2})),$$

where each class of the vector has the following number of elements

$$(p^2 - 1)p^{2(l-1)} (1, p^2, \ldots, p^{l-1}).$$

The sum of (32) gives the number $A_p(l, n)$. Notice that the base induction is precisely the Lemma 15.

Proposition 17. There are the identities:

a) $B_p(1, n) = (p - 1) \left[ \begin{array}{c} n \\ 2 \end{array} \right]_p$ and $B_p(l, 1) = 0$.

b) $B_p(l, 2) = p^l - p^{l-1}$.

c) $B_p(l, n) = p^{2n-3}B_p(l - 1, n)$.

Proof. In [SW13, p. 6], the author with Winklmeier showed that

$$r(\mathbb{Z}_p^n) = 1 + \left[ \begin{array}{c} n \\ 1 \end{array} \right]_p + (p - 1) \left[ \begin{array}{c} n \\ 2 \end{array} \right]_p.$$

In this case, we have $r(\mathbb{Z}_p^n) = 1 + A_p(1, n) + B_p(1, n)$, so the item a) follows by Lemma 15.

For item b), notice the relation $B_p(l, 2) = pB_p(l - 1, 2)$, which is an implication of the form of the classes. In fact, the number $B_p(l, 2)$ consists of classes formed by elements in $\mathbb{Z}_p^2 \times \mathbb{Z}_p^2$ of the form $((j, 0), (0, j))$, with $j \in \mathbb{Z}_{p^l} = \{0, 1, \ldots, p^l - 1\}$ and $j \neq 0 \mod p$. The number of this elements has the recurrence of multiplying by $p$ every time we add a unit to the index $l$. Since $B_p(1, 2) = p - 1$, then it follows the formula $B_p(l, 2) = p^l - p^{l-1}$.
Finally, for the item c) apply mod $p$ to the elements of $B_p(l, n)$ and then we obtain $(p - 1) \left\lfloor \frac{n}{2} \right\rfloor_p$ elements given in $B_p(1, n)$. The number $B_p(1, n)$ counts the elements given by $2 \times n$ matrices of rank two. Using the exact sequence (24), the fiber by the map mod $p$ is the space $\mathbb{Z}_p^{-1}$ for $2n - 3$ coordinates, where we subtract three because the matrix have to be of rank two. Therefore, we have

$$B_p(l, n) = p^{l-1}(2n-3)(p - 1) \left\lfloor \frac{n}{2} \right\rfloor,$$

which proves the recurrence $B_p(l, n) = p^{2n-3}B_p(l - 1, n)$. \hfill \square

**Corollary 18.** There is the formula

$$B_p(l, n) = p^{l-1}(2n-3)(p - 1) \left\lfloor \frac{n}{2} \right\rfloor.$$  \hspace{2cm} (36)

**Acknowledgement**

The author is supported by cátedras CONACYT and Proyecto CONACYT ciencias básicas 2016, No. 284621.

**References**

[BS14] Marcel Bökstedt and Anne Marie Svane, *A geometric interpretation of the homotopy groups of the cobordism category*, Algebr. Geom. Topol. 14 (2014), no. 3, 1649–1676.

[CF64] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33 Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964.

[DS21] Emilio Domínguez and Carlos Segovia, *Extending finite free group actions on surfaces*, https://arxiv.org/abs/2012.02464, January 2021.

[GMTW] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss, *The homotopy type of the cobordism category*, Acta Mathematica 202, no. 2, 195–23.

[GS17] Ana González and Carlos Segovia, *G-topological quantum field theory*, Bol. Soc. Mat. Mex. 23 (2017), 439–456.

[GZ67] Peter Gabriel and Michel Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag New York Inc., 1967.
[Mil40] G. A. Miller, *Subgroups of the groups whose order are below thirty*, Proceedings of the National Academy of Sciences of the United Stated of America 26(8) (1940), 500–502.

[MS06] Gregory W. Moore and Graeme Segal, *D-branes and K-theory in 2D topological field theory*, http://arxiv.org/abs/hep-th/0609042, September 2006.

[Qui73] Daniel Quillen, *Higher algebraic K-theory: I*, in: *Algebraic K-theory i*, Lect. Notes Math. 341 (1973), 77–139.

[Seg68] Graeme Segal, *Classifying spaces and spectral sequences*, Ins.Hautes Études Sci. Publ. Math. 34 (1968), 105–112.

[SW15] Carlos Segovia and Monika Winklmeier, *On the density of certain languages with $p^2$ letters*, Electron. J. Combin. 22 (2015), no. 3, 10.

[SW20] ______, *Calculating the dimension of the universal embedding of the symplectic dual polar space using languages*, Electron. J. Combin. 27 (2020), no. 4, 28.

[Til96] Ulrike Tillmann, *The classifying space of the 1+1 dimensional cobordism category*, J. für die reine und angewandte Mathematik 479 (1996), 67–75.

[Tur10] Vladimir Turaev, *Homotopical field theory in dimension 2 and group-algebras*, European Mathematical Society (EMS) 10 (2010).