A master identity for Horadam numbers

Kunle Adegoke
kunle.adegoke@yandex.com
Department of Physics and Engineering Physics, Obafemi Awolowo University, 220005 Ile-Ife, Nigeria

Abstract
We derive an identity involving Horadam numbers. Numerous new identities as well as those found in the existing literature are subsumed in this single identity.

1 Introduction
Our aim in writing this paper is to prove the presumably new identity

\[ u_{r-s}w_{n+m} = u_{m-r}w_{n+r} - q^{r-s}u_{m-r}w_{n+s}, \]

(H)

where \( r, s, n \) and \( m \) are arbitrary integers and \((u_j(p, q))_{j \in \mathbb{Z}}\) and \((w_j(a, b; p, q))_{j \in \mathbb{Z}}\) are the Lucas sequence of the first kind and the Horadam sequence, respectively, with complex parameters \( p \) and \( q \). Summation identities invoked by identity (H) will also be derived. Numerous new identities as well as those derived by Horadam [2] and later researchers are subsumed in the single identity (H).

The Horadam sequence \((w_n) = (w_n(a, b; p, q))\) is defined by the recurrence relation

\[ w_0 = a, \ w_1 = b; \ w_n = pw_{n-1} - qw_{n-2} (n \geq 2), \]

(1.1)

where \( a, b, p \) and \( q \) are arbitrary complex numbers, with \( p \neq 0 \) and \( q \neq 0 \).

Two important cases of \((w_n)\) are the Lucas sequences of the first kind, \((u_n(p, q)) = (w_n(0, 1; p, q))\), and of the second kind, \((v_n(p, q)) = (w_n(2, p; p, q))\); so that

\[ u_0 = 0, \ u_1 = 1; \ u_n = pu_{n-1} - qu_{n-2}, (n \geq 2); \]

(1.2)

and

\[ v_0 = 2, \ v_1 = p; \ v_n = pv_{n-1} - qv_{n-2}, (n \geq 2). \]

(1.3)

The most well-known Lucas sequences are the Fibonacci sequence, \((f_n) = (u_n(1, -1))\) and the sequence of Lucas numbers, \((l_n) = (v_n(1, -1))\). The Fibonacci numbers, \( f_n \), and the Lucas numbers, \( l_n \) are defined by:

\[ f_0 = 0, \ f_1 = 1, \ f_n = f_{n-1} + f_{n-2} (n \geq 2) \]

(1.4)
and

\[ l_0 = 2, \ l_1 = 1, \ l_n = l_{n-1} + l_{n-2} \ (n \geq 2). \]  \hspace{1cm} (1.5)

Denote by \( \alpha \) and \( \beta \), the zeros of the characteristic polynomial \( x^2 - px + q \) for the Horadam sequence and the associated Lucas sequences. Then

\[ \alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \ \beta = \frac{p - \sqrt{p^2 - 4q}}{2}, \]  \hspace{1cm} (1.6)

\[ \alpha + \beta = p, \ \alpha - \beta = \sqrt{p^2 - 4q} \text{ and } \alpha \beta = q. \]  \hspace{1cm} (1.7)

The difference equations (1.2), (1.3) and (1.1) are solved by the Binet-like formulas

\[ u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \ v_n = \alpha^n + \beta^n, \]  \hspace{1cm} (1.8)

and

\[ w_n = b \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) - a \alpha \beta \left( \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right). \]  \hspace{1cm} (1.9)

Thus,

\[ w_n = bu_n - a \alpha \beta u_{n-1}. \]  \hspace{1cm} (1.10)

It also follows that

\[ u_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) (\alpha^n + \beta^n) = u_n v_n. \]  \hspace{1cm} (1.11)

Setting \( x = \alpha \) and \( y = \beta \) in each of the following algebraic identities:

\[ \frac{x^m - y^m}{x - y} (x^n + y^n) = \frac{x^{n+m} - y^{n+m}}{x - y} - x^m y^m \frac{x^{n-m} - y^{n-m}}{x - y}, \]  \hspace{1cm} (1.12)

\[ (x - y)^2 \frac{x^m - y^m}{x - y} - \frac{x^n y^n}{x - y} = x^{n+m} + y^{n+m} - x^m y^m (x^{n-m} + y^{n-m}), \]  \hspace{1cm} (1.13)

\[ (x^m + y^m) \frac{x^n - y^n}{x - y} = \frac{x^{n+m} - y^{n+m}}{x - y} + x^m y^m \frac{x^{n-m} - y^{n-m}}{x - y}; \]  \hspace{1cm} (1.14)

and

\[ (x^m + y^m) (x^n + y^n) = x^{n+m} + y^{n+m} + x^m y^m (x^{n-m} + y^{n-m}), \]  \hspace{1cm} (1.15)

we find the following multiplication formulas for Lucas sequences:

\[ u_m v_n = u_{n+m} - q^m u_{n-m}, \]  \hspace{1cm} (1.16)

\[ (p^2 - 4q) u_m u_n = v_{n+m} - q^m v_{n-m}, \]  \hspace{1cm} (1.17)

\[ v_m u_n = u_{n+m} + q^m u_{n-m}, \]  \hspace{1cm} (1.18)

and

\[ v_m v_n = v_{n+m} + q^m v_{n-m}. \]  \hspace{1cm} (1.19)

More properties of Lucas sequences can be found in the book by Ribenboim [3, Chapter 1]. The Mathworld [4] and Wikipedia [5] articles are also good sources of information on the subject, with many references to useful materials. The books by Koshy [6] and by Vajda [7] are excellent reference materials on Fibonacci numbers and Lucas numbers.
Extension of the definition of \( w_n \) to negative subscripts is provided by writing the recurrence relation as \( w_{-n} = (pw_{-n+1} - w_{-n+2})/q \). Using the Binet-like formulas (1.8) and (1.9), it is readily established that

\[
u_{-n} = -u_n/q^n, \quad v_{-n} = v_n/q^n. \tag{1.20}\]

From (1.10) and (1.20), it follows that

\[
w_{-n} = \frac{w_{-n}}{w_n} w_n = \left(\frac{(ap - b)u_n - aq u_{n-1}}{bu_n - aq u_{n-1}}\right) q^{-n} w_n. \tag{1.21}\]

We require the following very general summation identities.

**Lemma 1** ([1, Lemma 1]). Let \((X_n)\) and \((Y_n)\) be any two sequences such that \(X_n\) and \(Y_n\), \(n \in \mathbb{Z}\), are connected by a three-term recurrence relation \(hX_n = f_1X_{n-c} + f_2Y_{n-d}\), where \(h\), \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(n\), and \(c\) and \(d\) are integers. Then, the following identity holds for integer \(k\):

\[
f_2 \sum_{j=0}^{k} f_1^{k-j} h^j Y_{n-ke-d+cj} = h^{k+1} X_n - f_1^{k+1} X_{n-(k+1)c}. \tag{1.22}\]

**Lemma 2** ([1, Lemma 2]). Let \((X_n)\) be any arbitrary sequence, where \(X_n\), \(n \in \mathbb{Z}\), satisfies a three-term recurrence relation \(hX_n = f_1X_{n-c} + f_2X_{n-d}\), where \(h\), \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(n\), and \(c\) and \(d\) are integers. Then, the following identities hold for integer \(k\):

\[
f_2 \sum_{j=0}^{k} f_1^{k-j} h^j X_{n-ke-d+cj} = h^{k+1} X_n - f_1^{k+1} X_{n-(k+1)c}, \tag{1.23}\]

\[
f_1 \sum_{j=0}^{k} f_2^{k-j} h^j X_{n-kd-c+cj} = h^{k+1} X_n - f_2^{k+1} X_{n-(k+1)d}, \tag{1.24}\]

and

\[
h \sum_{j=0}^{k} (-1)^j f_2^{k-j} f_1 X_{n-(d-c)k+c+(d-c)j} = (-1)^k f_1 f_1^{k+1} X_n + f_2^{k+1} X_{n-(d-c)(k+1)}. \tag{1.25}\]

**Lemma 3** ([1, Lemma 3]). Let \((X_n)\) be any arbitrary sequence. Let \(X_n\), \(n \in \mathbb{Z}\), satisfy a three-term recurrence relation \(hX_n = f_1X_{n-c} + f_2X_{n-d}\), where \(h\), \(f_1\) and \(f_2\) are non-vanishing complex functions, not dependent on \(n\), and \(c\) and \(d\) are integers. Then,

\[
\sum_{j=0}^{k} \binom{k}{j} f_2^{k-j} f_1 X_{n-dk+(d-c)j} = h^k X_n, \tag{1.26}\]

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} f_2^{k-j} h^j X_{n+(c-d)k+d+} = (-1)^k f_1^k X_n \tag{1.27}\]

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} f_1^{k-j} h^j X_{n+(d-c)k+c+j} = (-1)^k f_2^k X_n, \tag{1.28}\]

for \(k\) a non-negative integer.
Lemma 4. Let \((X_n)\) and \((Y_n)\) be any two sequences such that \(X_n\) and \(Y_n\), \(n \in \mathbb{Z}\), are connected by a three-term recurrence relation \(hX_n = f_1X_{n-c} + f_2Y_{n-d}\), where \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(n\), and \(c\), \(d\) and \(k\) are integers. Then,

\[
X_nX_{n-c(k+1)}f_2\sum_{j=0}^{k} h^{k-j}f_1^j \frac{Y_{n-d-ck+ cj}}{X_{n-ck+ cj}X_{n-c-ck+ cj}} = h^{k+1}X_n - f_1^{k+1}X_{n-c(k+1)}. \tag{1.28}
\]

Lemma 5. Let \((X_n)\) be any arbitrary sequence. Let \(X_n\), \(n \in \mathbb{Z}\), satisfy a three-term recurrence relation \(hX_n = f_1X_{n-c} + f_2X_{n-d}\), where \(f_1\) and \(f_2\) are non-vanishing complex functions, not dependent on \(n\), and \(c\), \(d\) and \(k\) are integers. Then, the following identities hold for arbitrary integers \(n\), \(c\), \(d\) and \(k\) for which the summand is not singular in the summation interval:

\[
X_nX_{n-c(k+1)}f_2\sum_{j=0}^{k} h^{k-j}f_1^j \frac{X_{n-d-ck+ cj}}{X_{n-ck+ cj}X_{n-c-ck+ cj}} = h^{k+1}X_n - f_1^{k+1}X_{n-c(k+1)}, \tag{1.29}
\]

\[
X_nX_{n-d(k+1)}f_1\sum_{j=0}^{k} h^{k-j}f_2^j \frac{X_{n-c-dk+ dj}}{X_{n-dk+ dj}X_{n-d- dk+ dj}} = h^{k+1}X_n - f_2^{k+1}X_{n-d(k+1)}, \tag{1.30}
\]

and

\[
X_nX_{n-(d-c)(k+1)}h\sum_{j=0}^{k} (-1)^j f_1^{k-j}f_2^j \frac{X_{n+c-(d-c)k+(d-c)j}}{X_{n-(d-c)k+(d-c)j}X_{n-c+(d-c)k+(d-c)j}} = f_1^{k+1}X_n + (-1)^k f_2^{k+1}X_{n-(d-c)(k+1)}. \tag{1.31}
\]

2 The master identity and some consequences

Theorem 1. The following identity holds for arbitrary integers \(r\), \(s\), \(n\) and \(m\):

\[
u_{r-s}w_{n+m} = u_{m-s}w_{n+r} - q^{r-s}u_{m-r}w_{n+s}.
\]

Proof. Since both sequences \((u_n)\) and \((w_n)\) have the same recurrence relation, we choose a basis set in \((u_n)\) and express the numbers from \((w_n)\) in this basis. Let

\[
w_{n+m} = \lambda_1u_{m-s} + \lambda_2u_{m-r}, \tag{2.1}
\]

where \(r\), \(s\), \(n\) and \(m\) are arbitrary integers and the coefficients \(\lambda_1\) and \(\lambda_2\) are to be determined. Setting \(m = r\) and \(m = s\), in turn, gives

\[
w_{n+r} = \lambda_1u_{r-s}, \quad w_{n+s} = \lambda_2u_{s-r}. \tag{2.2}
\]

Multiplying through identity (2.1) by \(u_{r-s}u_{s-r}\) gives

\[
u_{r-s}u_{s-r}w_{n+m} = \lambda_1u_{r-s}u_{s-r}u_{m-s} + \lambda_2u_{s-r}u_{r-s}u_{m-r}. \tag{2.3}
\]

Thus, we find

\[
u_{r-s}u_{s-r}w_{n+m} = u_{s-r}u_{m-s}w_{n+r} + u_{r-s}u_{m-r}w_{n+s}
\]

\[
= u_{s-r}u_{m-s}w_{n+r} - q^{r-s}u_{s-r}u_{m-r}w_{n+s};
\]

so that identity (H) is satisfied identically if \(r = s\) and numerically if \(r \neq s\). 

\]
Since \( r, s, m \) and \( n \) are arbitrary, identity (H) also implies the following identities:

\[
\begin{align*}
ur-s w_{n+m} &= u_{n-s} w_{m+r} - q^{r-s} u_{n-r} w_{m+s}, \\
u_r s w_{n+m} &= u_{n+r} w_{m-s} - q^{r-s} u_{n+s} w_{m-r}
\end{align*}
\]

and

\[
u_r-s w_{n+m} = u_{m+r} w_{n-s} - q^{r-s} u_{m+s} w_{n-r}.
\]

In the particular cases of Lucas sequences, identities (H), (F), (G) and (J) read:

\[
\begin{align*}
ur-s u_{n+m} &= u_{m-s} u_{n+r} - q^{r-s} u_{n-r} u_{n+s}, \\
u_r-s u_{n+m} &= u_{n-s} u_{m+r} - q^{r-s} u_{n-r} u_{m+s}, \\
u_r-s u_{n+m} &= u_{n+r} u_{m-s} - q^{r-s} u_{n+s} u_{m-r}, \\
u_r-s u_{n+m} &= u_{m+r} u_{n-s} - q^{r-s} u_{m+s} u_{n-r};
\end{align*}
\]

and

\[
\begin{align*}
ur-s v_{n+m} &= u_{m-s} v_{n+r} - q^{r-s} u_{n-r} v_{n+s}, \\
u_r-s v_{n+m} &= u_{n-s} v_{m+r} - q^{r-s} u_{n-r} v_{m+s}, \\
u_r-s v_{n+m} &= u_{n+r} v_{m-s} - q^{r-s} u_{n+s} v_{m-r}, \\
u_r-s v_{n+m} &= u_{m+r} v_{n-s} - q^{r-s} u_{m+s} v_{n-r}.
\end{align*}
\]

Identities (H), (F), (G) and (J) have unlimited consequences. We list a few.

**Corollary 2.** The following identities hold for integers \( n, m, j, r, s \) and \( t \):

\[
r = 0, \ s = -m \ in \ (H) \implies v_m w_n = w_{n+m} + q^m w_{n-m},
\]

\[
v_n w_n = w_{2n} + q^n a,
\]

\[
r = 0, \ s = -m \ in \ (F) \implies u_m w_n = u_n w_m - q^m a u_{n-m},
\]

\[
r = 1, \ s = 0 \ in \ (H) \implies w_{n+m} = u_m w_{n+1} - q u_{m-1} w_n,
\]

\[
m \to -m \ in \ (2.15) \implies q^m w_{n-m} = u_{m+1} w_n - u_m w_{n+1},
\]

\[
(2.15) - (2.16) \implies w_{n+m} - q^m w_{n-m} = u_m (w_{n+1} - qw_{n-1}),
\]

\[
w_{n+m} = u_n w_{m+1} - q u_{n-1} w_m,
\]

\[
w_{n+m} = u_{m-j} w_{n+j+1} - q u_{m-j-1} w_{n+j},
\]

\[
w_{n+m} = u_{n-j} w_{m+j+1} - q u_{n-j-1} w_{m+j},
\]

\[
w_{2n} = u_n w_{n+1} - q u_{n-1} w_n,
\]

\[
w_{2n} = u_{n+1} w_n - q u_n w_{n-1},
\]

\[
w_{2n-1} = u_{n+1} w_{n-1} - q u_n w_n,
\]

\[
w_{2n-1} = u_n w_n - q u_{n-1} w_{n-1},
\]

\[
r = 0, \ s = m - n \ in \ (H) \implies u_{n-m} w_{n+m} = u_n w_n - q^{n-m} u_m w_m,
\]

\[
r = 0, \ s = -m \ in \ (F) \implies u_{n-m} w_{n+m} = w_{2n-m} w_m - q^{n-m} u_n w_{2m-n},
\]

\[
m = 0 \ in \ (2.26) \implies q^n w_{-n} = aw_n - w_n,
\]
use of (2.27) in (2.12) \[ v_n w_m - a q^m v_{n-m} = w_{n+m} - q^m w_{n-m}, \]  
(2.12) \times (2.28) \[ w_{n+m}^2 - q^{2m} w_{n-m}^2 = v_m w_n (v_n w_m - a q^m v_{n-m}), \]  
s = -r in (H) \[ u_2 r w_{n+m} = u_{m+r} w_{n+r} = q^{2r} u_{m-r} w_{n-r}, \]  
m \rightarrow -m in (3.2) \[ q^{m-r} u_2 r w_{n-m} = u_{m+r} w_{n+r} = u_{m-r} w_{n+r}, \]  
u_2 r w_{2n} = u_{n+r} w_{n+r} - q^{2r} u_{n-r} w_{n-r}, \]  
u_2 r w_{2n-1} = u_{n+r} w_{n+r-1} - q^{2r} u_{n-r} w_{n-r-1}, \]  
r = 1 in (2.30) \[ pw_{n+m} = u_{n+1} w_{n+1} - q^2 u_{n-1} w_{n-1}, \]  
pw_{n+m} = u_{n+1} w_{n+1} - q^2 u_{n-1} w_{n-1}, \]  
pw_{2n} = u_{n+1} w_{n+1} - q^2 u_{n-1} w_{n-1}, \]  
pw_{2n-1} = u_{n+1} w_{n+1} - q^2 u_{n-1} w_{n-2}, \]  
m = 0, \ r = t + s in (H) \[ u_1 w_n = u_s w_{n+t} - q^t u_{s-t} w_{n-s}, \]  
m = 0, \ r = t + s in (F) \[ u_1 w_n = u_{n-s} w_{t+s} - q^t u_{n-t} w_{s}, \]  
m = 0, \ r = t + s in (G) \[ u_1 w_n = u_{n+s} w_{n-t} - q^t u_{s} w_{s-t}, \]  
m = 0, \ r = t + s in (J) \[ u_1 w_n = u_{t+s} w_{n-s} - q^t u_{s} w_{s-t}, \]  
The identities in Corollary 2 have interesting implications for the Lucas sequences. The following list is far from being exhaustive.

**Corollary 3.** The following identities hold for integers \( n, m, r, s \) and \( t \):

\[ w_n = v_n \text{ in (2.13)} \implies v_n^2 = v_{2n} + 2 q^n, \]  
\[ w_n = v_n \text{ in (2.14)} \implies u_n v_m - u_m v_n = 2 q^m u_{n-m}, \]  
\[ w_n = v_n, \ m = 0 \text{ in (2.18)} \implies v_n = pu_n - 2 q u_{n-1}, \]  
\[ u_{n+m} = u_{m+1} w_{n+1} - q u_{m-1} w_{n}, \]  
\[ v_{n+m} = u_{m+1} w_{n+1} - q u_{m-1} w_{n}, \]  
\[ u_{2m-1} = u_2^2 - q u_{2m-2}, \]  
\[ v_{2m-1} = u_2^2 - q u_{2m-2}, \]  
\[ w_n = u_n \text{ in (2.25)} \implies u_{n-m} u_{n+m} = u_n^2 - q^{n-m} u_m^2, \]  
\[ w_n = v_n \text{ in (2.25)} \implies u_{n-m} v_{n+m} = u_{2n} - q^{n-m} u_{2m}, \]  
\[ \text{(1.17) and } w_n = v_n \text{ in (2.28)} \implies v_n v_m - (p^2 - 4 q) u_m u_n = 2 q^m v_{n-m}, \]  
\[ w_n = u_n \text{ in (2.32)} \implies u_2 r u_{n+m} = u_{n+r} u_{m+r} - q^{2r} u_{m-r} u_{n-r}, \]  
\[ w_n = v_n \text{ in (2.32)} \implies u_2 r v_{n+m} = u_{n+r} v_{m+r} - q^{2r} u_{m-r} v_{n-r}, \]  
\[ u_2 r u_{2n} = u_{n+r} u_{2n} - q^{2r} u_{n-r}^2, \]  
\[ u_2 r v_{2n} = u_{n+r} v_{2n} - q^{2r} u_{n-r}^2, \]  
\[ pu_{2n} = u_{n+1}^2 - q^2 u_{n-1}^2, \]  
\[ pv_{2n} = u_{2n+1} - q^2 u_{2n-1}, \]  
\[ w_n = u_n \text{ in (2.39)} \implies u_t u_n = u_s u_{n+t} - q^t u_{s-t} u_{n-s}, \]  
\[ w_n = v_n \text{ in (2.39)} \implies u_t v_n = u_s v_{n+t} - q^t u_{s-t} v_{n-s}, \]  
\[ s = 0, \ w_n = v_n \text{ in (2.41)} \implies u_n v_t + u_t v_n = 2 u_{n+t}, \]  
\[ m = t \text{ in (2.44) } \times (2.61) \implies u_t^2 v_n^2 = u_t^2 v_n^2 = 4 q^t u_{n+t} u_{n-t}, \]  
\[ p^2 u_n^2 - v_n^2 = 4 q u_{n+1} u_{n-1}. \]
3 Summation identities involving binomial coefficients

**Theorem 4.** The following identities hold for positive integer $k$ and arbitrary integers $r,$ $s,$ $n,$ $m$:

\[ \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} \binom{k}{j} u^{j}_{m-s} u^{k-j}_{m-r} u_{m-(m-s)k+(r-s)j} = (-1)^k u^k_{r-s} u_n, \]  

(3.1)  

\[ \sum_{j=0}^{k} q^{(r-s)j} \binom{k}{j} u^{j}_{r-s} u^{k-j}_{m-s} u_{n-(r-s)k+(m-s)j} = q^{(r-s)k} u^k_{m-s} u_n, \]  

(3.2)  

\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{j}_{r-s} u^{k-j}_{m-s} u_{n+(r-s)k+(m-s)j} = q^{(r-s)k} u^k_{m-s} u_n, \]  

(3.3)  

\[ \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} \binom{k}{j} u^{j}_{m+r} u^{k-j}_{m+s} u_{n-(m+r)k+(r-s)j} = (-1)^k u^k_{r-s} u_n, \]  

(3.4)  

\[ \sum_{j=0}^{k} q^{(r-s)j} \binom{k}{j} u^{j}_{r-s} u^{k-j}_{m+s} u_{n-(r-s)k+(m+s)j} = q^{(r-s)k} u^k_{m+s} u_n, \]  

(3.5)  

and

\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{j}_{r-s} u^{k-j}_{m+s} u_{n+(r-s)k+(m+s)j} = q^{(r-s)k} u^k_{m+s} u_n. \]  

(3.6)

**Proof.** To derive identities (3.1) – (3.3), write identity (H) as

\[ u_{r-s} w_n = u_{m-s} w_{n-(m-r)} - q^{r-s} u_{m-r} w_{n-(m-s)}; \]

identify $h = u_{r-s}, f_1 = u_{m-s}, f_2 = -q^{r-s} u_{m-r}, X_n = u_n, c = m - r, d = m - s$ and use these in Lemma 3. Identities (3.4) – (3.6) are obtained from identities (3.1) – (3.3) by interchanging $r$ and $-s$ and $s$ and $r$.

Particular cases of identities (3.1) – (3.6) are the following summation identities involving only numbers from Lucas sequence of the first kind:

\[ \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} \binom{k}{j} u^{j}_{m-s} u^{k-j}_{m-r} u_{n-(m-s)k+(r-s)j} = (-1)^k u^k_{r-s} u_n, \]  

(3.7)  

\[ \sum_{j=0}^{k} q^{(r-s)j} \binom{k}{j} u^{j}_{r-s} u^{k-j}_{m-r} u_{n-(r-s)k+(m-r)j} = q^{(r-s)k} u^k_{m-s} u_n, \]  

(3.8)  

\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{j}_{r-s} u^{k-j}_{m-s} u_{n+(r-s)k+(m-s)j} = q^{(r-s)k} u^k_{m-s} u_n, \]  

(3.9)  

\[ \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} \binom{k}{j} u^{j}_{m+r} u^{k-j}_{m+s} u_{n-(m+r)k+(r-s)j} = (-1)^k u^k_{r-s} u_n, \]  

(3.10)
\[ \sum_{j=0}^{k} q^{(r-s)j} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m+r)} = q^{(r-s)k} u_{m+r}^{k} u_{n} \quad (3.11) \]

and

\[ \sum_{j=0}^{k} (-1)^{j} q^{(r-s)(k-j)} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m+r)} = q^{(r-s)k} u_{m+s}^{k} u_{n} \quad (3.12) \]

and the corresponding identities involving numbers from Lucas sequences of both kinds:

\[ \sum_{j=0}^{k} (-1)^{j} q^{(r-s)(k-j)} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m+s)} = (-1)^{k} u_{r-s}^{k} v_{n} \quad (3.13) \]

\[ \sum_{j=0}^{k} q^{(r-s)j} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m-s)} = q^{(r-s)k} u_{m-s}^{k} v_{n} \quad (3.14) \]

\[ \sum_{j=0}^{k} (-1)^{j} q^{(r-s)(k-j)} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m-r)} = q^{(r-s)k} u_{m-r}^{k} v_{n} \quad (3.15) \]

\[ \sum_{j=0}^{k} (-1)^{j} q^{(r-s)(k-j)} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m+r)} = (-1)^{k} u_{r-s}^{k} v_{n} \quad (3.16) \]

\[ \sum_{j=0}^{k} q^{(r-s)j} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m+s)} = q^{(r-s)k} u_{m+r}^{k} v_{n} \quad (3.17) \]

and

\[ \sum_{j=0}^{k} (-1)^{j} q^{(r-s)(k-j)} \binom{k}{j} u_{r-s}^{j} u_{m-s}^{k-j} u_{n-(r-s)k+(m+s)} = q^{(r-s)k} u_{m+s}^{k} v_{n} \quad (3.18) \]

4 Summation identities not involving binomial coefficients

**Theorem 5.** The following identities hold for \( r, s, m, n \) and \( k \) arbitrary integers:

\[ u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} w_{m+s}^{j} w_{m+r}^{j} w_{n-(r-s)k+m+s+(r-s)} = q^{(s-r)k} u_{n} w_{m+r}^{k+1} - q^{r-s} u_{n-(r-s)(k+1)} w_{m+s}^{k+1} \quad (4.1) \]

and

\[ u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} w_{m-r}^{j} w_{m-s}^{j} w_{n-(r-s)k+m-r+(r-s)} = q^{(s-r)k} u_{n} w_{m-s}^{k+1} - q^{r-s} u_{n-(r-s)(k+1)} w_{m-r}^{k+1} \quad (4.2) \]

**Proof.** To prove identity (4.1), write identity (H) as

\[ w_{m+r} u_{n} = q^{r-s} w_{m+s} u_{n-(r-s)} + u_{r-s} w_{n+m+s} \quad (4.3) \]

identify \( h = w_{m+r}, f_{1} = q^{r-s} w_{m+s}, f_{2} = u_{r-s} \), \( X_{n} = u_{n}, Y_{n} = w_{n+m+s}, c = r-s \) and \( d = 0 \) and use these in Lemma 1. Identity (4.2) is obtained from identity (4.1) through the transformation \( r \rightarrow -s, s \rightarrow -r \). \( \square \)
Theorem 6. The following identities hold for arbitrary integers \( r, s, n, m \) and \( k \):

\[
-q^{r-s}u_{m-r} \sum_{j=0}^{k} u_{m-s}^{k-j} u_{r-s}^{j} w_{n-(m-r)k-(m-s)+(m-r)j} = u_{r-s}^{k+1} w_{n} - u_{m-s}^{k+1} w_{n-(m-r)(k+1)},
\]

\[
(-1)^k u_{m-s} \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} u_{m-r}^{k-j} u_{r-s}^{j} w_{n-(m-s)k-(m-r)+(m-s)j} = u_{r-s}^{k+1} w_{n} - (-1)^k k+1 q^{(r-s)(k+1)} u_{m-r}^{k+1} w_{n-(m-s)(k+1)},
\]

\[
u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} u_{m-r}^{k-j} u_{r-s}^{j} w_{n-(r-s)k+(m-r)+(r-s)j} = q^{(s-r)k} u_{m-s}^{k+1} w_{n} - q^{r-s} u_{m-r}^{k+1} w_{n-(r-s)(k+1)},
\]

\[
-q^{r-s} u_{m+s} \sum_{j=0}^{k} u_{m+r}^{k-j} u_{r-s}^{j} w_{n-(m+s)k-(m-r)+(m+s)j} = u_{r-s}^{k+1} w_{n} - u_{m+s}^{k+1} w_{n-(m-s)(k+1)},
\]

\[
(-1)^k u_{m+r} \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} u_{m+s}^{k-j} u_{r-s}^{j} w_{n-(m+r)k-(m-s)+(m+r)j} = u_{r-s}^{k+1} w_{n} - (-1)^k k+1 q^{(r-s)(k+1)} u_{m+s}^{k+1} w_{n-(m-r)(k+1)}.
\]

and

\[
u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} u_{m+s}^{k-j} u_{r-s}^{j} w_{n-(r-s)k+(m+s)+(r-s)j} = q^{(s-r)k} u_{m+r}^{k+1} w_{n} - q^{r-s} u_{m+s}^{k+1} w_{n-(r-s)(k+1)}.
\]

Proof. In Lemma 2, with \( X_n = w_n \), use the \( h, f_1, f_2, c \) and \( d \) obtained in the proof of Theorem 4.

In particular, we have the following summation identities involving only numbers from Lucas sequence of the first kind:

\[
-q^{r-s} u_{m-r} \sum_{j=0}^{k} u_{m-s}^{k-j} u_{r-s}^{j} w_{n-(m-r)k-(m-s)+(m-r)j} = u_{r-s}^{k+1} w_{n} - u_{m-s}^{k+1} w_{n-(m-r)(k+1)},
\]

\[
(-1)^k u_{m-s} \sum_{j=0}^{k} (-1)^j q^{(r-s)(k-j)} u_{m-r}^{k-j} u_{r-s}^{j} w_{n-(m-s)k-(m-r)+(m-s)j} = u_{r-s}^{k+1} w_{n} - (-1)^k k+1 q^{(r-s)(k+1)} u_{m-r}^{k+1} w_{n-(m-s)(k+1)},
\]

\[
u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} u_{m-r}^{k-j} u_{r-s}^{j} w_{n-(r-s)k+(m-r)+(r-s)j} = q^{(s-r)k} u_{m-s}^{k+1} w_{n} - q^{r-s} u_{m-r}^{k+1} w_{n-(r-s)(k+1)}.
\]
\[-q^{r-s}u_{m+s} \sum_{j=0}^{k} u_{m+r}^{k-j} u_{r-s}^{j} u_{n-(m+s)k-(m+r)+(m+s)j} \]
\[= u_{r-s}^{k+1} u_{n} - u_{m+r}^{k+1} u_{n-(m+s)(k+1)} , \] (4.13)

\[-(1)^{k} u_{m+r} \sum_{j=0}^{k} (-1)^{j} q^{r-s}\sum_{j=0}^{m+r} u_{r-s}^{j} u_{n-(m+s)k-(m+r)+(m+r)j} \]
\[= u_{r-s}^{k+1} u_{n} - (-1)^{k+1} q^{r-s}\sum_{j=0}^{k} u_{m+r}^{k+1} u_{n-(m+r)(k+1)} , \] (4.14)

and

\[q^{(s-r)}u_{m+s} \sum_{j=0}^{k} q^{(s-r)j} u_{m+r}^{k-j} u_{r-s}^{j} u_{n-(m+s)(r-s)k-(m+r)+(r-s)j} \]
\[= q^{(s-r)k} u_{m+r}^{k+1} u_{n} - q^{r-s} u_{m+s}^{k+1} u_{n-(r-s)(k+1)} , \] (4.15)

and the corresponding results involving numbers from Lucas sequences of both kinds:

\[-q^{r-s}u_{m-r} \sum_{j=0}^{k} u_{m-s}^{k-j} u_{r-s}^{j} u_{n-(m-r)k-(m-s)+(m-r)j} \]
\[= u_{r-s}^{k+1} u_{n} - u_{m-s}^{k+1} u_{n-(m-r)(k+1)} , \] (4.16)

\[-(1)^{k} u_{m-s} \sum_{j=0}^{k} (-1)^{j} q^{r-s}\sum_{j=0}^{m-s} u_{r-s}^{j} u_{n-(m-s)k-(m-r)+(m-s)j} \]
\[= u_{r-s}^{k+1} u_{n} - (-1)^{k+1} q^{r-s}\sum_{j=0}^{k} u_{m-s}^{k+1} u_{n-(m-s)(k+1)} , \] (4.17)

\[u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} u_{m-r}^{k-j} u_{r-s}^{j} u_{n-(m-r)(r-s)k-(m+r)+(r-s)j} \]
\[= q^{(s-r)k} u_{m-r}^{k+1} u_{n} - q^{r-s} u_{m-s}^{k+1} u_{n-(r-s)(k+1)} . \] (4.18)

\[-q^{r-s}u_{m+s} \sum_{j=0}^{k} u_{m+s}^{k-j} u_{r-s}^{j} u_{n-(m+s)k-(m+r)+(m+s)j} \]
\[= u_{r-s}^{k+1} u_{n} - u_{m+s}^{k+1} u_{n-(m+s)(k+1)} , \] (4.19)

\[-(1)^{k} u_{m+r} \sum_{j=0}^{k} (-1)^{j} q^{r-s}\sum_{j=0}^{m+r} u_{r-s}^{j} u_{n-(m+s)k-(m+r)+(m+r)j} \]
\[= u_{r-s}^{k+1} u_{n} - (-1)^{k+1} q^{r-s}\sum_{j=0}^{k} u_{m+r}^{k+1} u_{n-(m+r)(k+1)} , \] (4.20)

and

\[u_{r-s} \sum_{j=0}^{k} q^{(s-r)j} u_{m+r}^{k-j} u_{r-s}^{j} u_{n-(m+r)(r-s)k-(m+s)+(r-s)j} \]
\[= q^{(s-r)k} u_{m+r}^{k+1} u_{n} - q^{r-s} u_{m+s}^{k+1} u_{n-(r-s)(k+1)} . \] (4.21)
5 Summation identities involving reciprocals

Theorem 7. The following identities hold for values of \( r, s, m, n, k \) for which the summand is non-singular in the summation interval:

\[
\begin{align*}
\sum_{j=0}^{k} q^{(r-s)j} u_{n-s}^{k-j} w_{m+s}^{j} w_{n+m+s-(r-s)k+(r-s)j} = u_{n} w_{m+r}^{k+1} - q^{(r-s)(k+1)} u_{n-s} w_{m+s},
\end{align*}
\]

(5.1)

\[
\begin{align*}
\sum_{j=0}^{k} q^{(r-s)j} u_{n-s}^{k-j} w_{m+r}^{j} w_{n+m+r-(r-s)k+(r-s)j} = u_{n} w_{m+r}^{k+1} - q^{(r-s)(k+1)} u_{n-s} w_{m+s},
\end{align*}
\]

(5.2)

Proof. In Lemma 4, make the identification \( X_n = u_n \) and \( Y_n = w_{n+m+s} \) and use the \( f_1, f_2, h, c \) and \( d \) obtained in the proof of Theorem 5.

Particular cases of identities (5.1) and (5.2) are the following:

\[
\begin{align*}
\sum_{j=0}^{k} q^{(r-s)j} u_{n-s}^{k-j} w_{m+s}^{j} w_{n+m+s-(r-s)k+(r-s)j} = u_{n} w_{m+r}^{k+1} - q^{(r-s)(k+1)} u_{n-s} w_{m+s},
\end{align*}
\]

(5.3)

\[
\begin{align*}
\sum_{j=0}^{k} q^{(r-s)j} u_{n-s}^{k-j} w_{m+r}^{j} w_{n+m+r-(r-s)k+(r-s)j} = u_{n} w_{m+r}^{k+1} - q^{(r-s)(k+1)} u_{n-s} w_{m+s},
\end{align*}
\]

(5.4)

and

\[
\begin{align*}
\sum_{j=0}^{k} q^{(r-s)j} u_{n-s}^{k-j} w_{m+s}^{j} w_{n+m+s-(r-s)k+(r-s)j} = u_{n} w_{m+r}^{k+1} - q^{(r-s)(k+1)} u_{n-s} w_{m+s},
\end{align*}
\]

(5.5)

\[
\begin{align*}
\sum_{j=0}^{k} q^{(r-s)j} u_{n-s}^{k-j} w_{m+r}^{j} w_{n+m+r-(r-s)k+(r-s)j} = u_{n} w_{m+r}^{k+1} - q^{(r-s)(k+1)} u_{n-s} w_{m+s},
\end{align*}
\]

(5.6)

Theorem 8. The following identities hold for values of \( r, s, m, n, k \) for which the summand is non-singular in the summation interval:

\[
\begin{align*}
-q^{(r-s)} u_{m-r} w_{n} w_{n-(m-r)(k+1)} = u_{r-s}^{k+1} - u_{m-s}^{k+1} u_{n-(m-r)(k+1)},
\end{align*}
\]

(5.7)

\[
\begin{align*}
u_{m-s} w_{n} w_{n-(m-s)(k+1)} = u_{r-s}^{k+1} - u_{m-s}^{k+1} u_{n-(m-s)(k+1)},
\end{align*}
\]

(5.8)
\[ u_{r-s}w_n w_{n-(r-s)(k+1)} \sum_{j=0}^{k} q^{(r-s)j} u_{m-s}^{k-j} u_{m-r}^{j} w_{n+m-r-(r-s)k+(r-s)j} \]
\[ = u_{m-s}^{k+1} w_n - q^{(r-s)(k+1)} u_{m-r}^{k+1} w_{n-(r-s)(k+1)} ; \]  \hspace{1cm} (5.9) 

\[-q^{r-s} u_{m+s} w_n w_{n-(m+s)(k+1)} \sum_{j=0}^{k} q^{r-s} u_{m+s-r}^{k-j} w_{n-(m+s)k+(m+s)j} \]
\[ = u_{m-s}^{k+1} w_n - q^{r-s}(k+1) u_{m-s}^{k+1} w_{n-(m+s)(k+1)} ; \]  \hspace{1cm} (5.10) 

\[ u_{m+r} w_n w_{n-(m+r)(k+1)} \sum_{j=0}^{k} (-1)^j q^{r-s} u_{m+r-s}^{k-j} w_{n-(m+r)k+(m+r)j} \]
\[ = u_{m-s}^{k+1} w_n - (-1)^k q^{r-s}(k+1) u_{m-s}^{k+1} w_{n-(m+r)(k+1)} ; \]  \hspace{1cm} (5.11) 

\[ -q^{r-s} u_{m-r} u_{n-(m-r)(k+1)} \sum_{j=0}^{k} q^{r-s} u_{m-s}^{k-j} u_{m-r}^{j} w_{n+(m-r)k+(m-r)j} \]
\[ = u_{m-s}^{k+1} w_n - q^{r-s}(k+1) u_{m-r}^{k+1} w_{n-(m-r)(k+1)} ; \]  \hspace{1cm} (5.12) 

and 

\[ u_{r-s} w_n w_{n-(r-s)(k+1)} \sum_{j=0}^{k} q^{r-s} u_{m+s}^{k-j} u_{m-r}^{j} w_{n+(r-s)k+(r-s)j} \]
\[ = u_{m-s}^{k+1} w_n - q^{r-s}(k+1) u_{m-s}^{k+1} w_{n-(r-s)(k+1)} ; \]  \hspace{1cm} (5.13) 

\[ u_{m-s} u_n u_{n-(m-s)(k+1)} \sum_{j=0}^{k} (-1)^j q^{r-s} u_{m-r}^{k-j} u_{m+s-r}^{j} w_{n+(m-s)k+(m-s)j} \]
\[ = u_{m-r}^{k+1} w_n - (-1)^k q^{r-s}(k+1) u_{m-r}^{k+1} w_{n-(m-s)(k+1)} ; \]  \hspace{1cm} (5.14) 

\[ u_{r-s} u_n u_{n-(r-s)(k+1)} \sum_{j=0}^{k} q^{r-s} u_{m-s}^{k-j} u_{m-r}^{j} w_{n+(r-s)k+(r-s)j} \]
\[ = u_{m-s}^{k+1} w_n - q^{r-s}(k+1) u_{m-r}^{k+1} w_{n-(r-s)(k+1)} ; \]  \hspace{1cm} (5.15) 

\[ -q^{r-s} u_{m+s} u_n u_{n-(m+s)(k+1)} \sum_{j=0}^{k} q^{r-s} u_{m-r}^{k-j} u_{n+m-s}^{j} w_{n+(m+s)k+(m+s)j} \]
\[ = u_{m-r}^{k+1} w_n - q^{r-s}(k+1) u_{n+m-r}^{k+1} w_{n-(m+s)(k+1)} ; \]  \hspace{1cm} (5.16) 

\[ u_{m+r} u_n u_{n-(m+r)(k+1)} \sum_{j=0}^{k} (-1)^j q^{r-s} u_{m+s}^{k-j} u_{m-r}^{j} w_{n+(m+r)k+(m+r)j} \]
\[ = u_{m-s}^{k+1} w_n - (-1)^k q^{r-s}(k+1) u_{m-s}^{k+1} w_{n-(m+r)(k+1)} ; \]  \hspace{1cm} (5.17) 

Proof. In Lemma 5, make the identification \( X_n = u_n \) and use the \( f_1, f_2, h, c \) and \( d \) obtained in the proof of Theorem 6.

In particular, we have the summation identities involving only numbers from Lucas sequence of the first kind:
and

\[ u_{r-s}u_m u_{n-(r-s)(k+1)} \sum_{j=0}^{k} q_{m+r}^{(r-s)j} u_{m+r}^{k-j} u_{m+s}^{j} u_{n+m+s-(r-s)k+(r-s)j} \]

\[ = u_{m+s}^{k+1} u_n - q^{(r-s)(k+1)} u_{m+s}^{k+1} u_{n-(r-s)(k+1)} ; \]

and the corresponding identities involving numbers from Lucas sequences of both kinds:

\[ -q^{r-s} u_{m-r} v_n v_{n-(m-r)(k+1)} \sum_{j=0}^{k} q_{m-s}^{(r-s)j} u_{m-s}^{k-j} u_{m-r}^{j} v_{n-(m-r)k+(m-r)j} v_{n-(m-r)-(m-r)k+(m-r)j} \]

\[ = u_{m+s}^{k+1} v_n - q^{(r-s)(k+1)} u_{m-r}^{k+1} v_{n-(m-r)(k+1)} ; \]

\[ u_{r-s} v_n v_{n-(r-s)(k+1)} \sum_{j=0}^{k} q_{m+r}^{(r-s)j} u_{m+r}^{k-j} u_{m-s}^{j} v_{n-m+s-(r-s)k+(r-s)j} \]

\[ = u_{m+s}^{k+1} v_n - q^{(r-s)(k+1)} u_{m-r}^{k+1} v_{n-(r-s)(k+1)} ; \]

\[ -q^{r-s} u_{m+s} v_n v_{n-(m+r)(k+1)} \sum_{j=0}^{k} q_{m+r}^{(r-s)j} u_{m+r}^{k-j} u_{m+s}^{j} v_{n-(m+r)k+(m+r)j} v_{n-(m+r)-(m+r)k+(m+r)j} \]

\[ = u_{m+s}^{k+1} v_n - q^{(r-s)(k+1)} u_{m-s}^{k+1} v_{n-(m+r)(k+1)} ; \]

and

\[ u_{r-s} v_n v_{n-(r-s)(k+1)} \sum_{j=0}^{k} q_{m+r}^{(r-s)j} u_{m+r}^{k-j} u_{m+s}^{j} v_{n-m+s-(r-s)k+(r-s)j} \]

\[ = u_{m+s}^{k+1} v_n - q^{(r-s)(k+1)} u_{m-r}^{k+1} v_{n-(r-s)(k+1)} . \]

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