Injectivity of the Heisenberg X-ray Transform

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Abstract

We initiate the study of X-ray tomography on sub-Riemannian manifolds, for which the Heisenberg group exhibits the simplest nontrivial example. With the language of the group Fourier Transform, we prove an operator-valued incarnation of the Fourier Slice Theorem, and apply this new tool to show that a sufficiently regular function on the Heisenberg group is determined by its line integrals over sub-Riemannian geodesics. We also consider the family of taming metrics $g_\epsilon$ approximating the sub-Riemannian metric, and show that the associated X-ray transform is injective for all $\epsilon > 0$. This result gives a concrete example of an injective X-ray transform in a geometry with an abundance of conjugate points.

1 Introduction

Our object of study is the geodesic X-ray transform associated to the sub-Riemannian geometry of the Heisenberg group, which is $\mathbb{H} := \mathbb{C} \times \mathbb{R}$ with the multiplication law

$$(x + iy, t)(u + iv, s) = (x + u + i(y + v), t + s + \frac{1}{2}(xv - yu)),$$

and a metric defined in Section 3.1. $\mathbb{H}$ is the local model for any 3-dimensional sub-Riemannian manifold of contact type, in the same sense that 3-dimensional Euclidean space is the local model for any 3-dimensional Riemannian manifold \cite[Thm. 1]{19}. This property positions $\mathbb{H}$ as the natural homogeneous starting point for studying the integral geometry of contact manifolds, just as Radon first inverted the X-ray transform in $\mathbb{R}^2$. Furthermore, X-ray transforms on symmetric spaces are extensively studied, for example in \cite{10, 13}, and the Heisenberg group arises as the boundary at infinity of complex hyperbolic space in two dimensions \cite{3, 34}.

To a function $f \in L^1(\mathbb{H})$ we associate the function $If$, its X-ray transform, defined by

$$If(\gamma) := \int f(\gamma(s)) \, ds,$$

where the geodesics $\gamma$ will be cast as (projections of) integral curves of the Hamiltonian flow on $T^*\mathbb{H}$ for the degenerate fiber quadratic Hamiltonian later described \cite{16}. Related integral transforms on $\mathbb{H}$ have been studied, for example, by Rubin \cite{29}, and Stichartz \cite{30}, who consider integration over left translates of hyperplanes. We ask whether $If$ determines $f$.

The sub-Riemannian setting, whose general theory is poorly understood, introduces qualitatively new features to this question. For example, fibers of the unit cotangent bundle $U^*\mathbb{H}$ (defined in Section 3.1) are now cylinders, and there is no unique Levi-Civita connection. Thus each fiber of $U^*\mathbb{H}$ possesses a group structure, unlike three dimensional Riemannian manifolds, but there is no canonical splitting of its tangent space into vertical and horizontal components like there is in the Riemannian case as described in \cite{28}.

A standard geometric obstacle to such inverse problems is presented by the presence of conjugate points. In \cite{8, 21}, and \cite{14} the authors show that conjugate points generally inhibit stable inversion of the X-ray transform on Riemannian manifolds, with unconditional loss in two dimensions. Unfortunately, the conjugate points in the Heisenberg group are ubiquitous; the cut locus to any point passes through that point—a feature
Theorem 1. The Heisenberg X-ray transform
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Strichartz proves indirectly that a function on the Heisenberg group may not in general be recovered from
I
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= 0 is
\[ s \mapsto (z, t)\gamma_\lambda(s), \quad \gamma_\lambda(s) = \left( Re^{i(s/R)}, \frac{1}{2}sR \right) \in \mathbb{H}; \quad R = 1/|\lambda|. \] (1)

Using this identification, we may parameterize geodesics by \((z, t, \lambda)\) as above, uniquely modulo the isotropy group \(\Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\}\) stabilizing \(\gamma_\lambda\), and write the X-ray transform concretely as
\[ I\lambda f(z, t, \lambda) := I\lambda f(z, t) := \int_\mathbb{R} f((z, t)\gamma_\lambda(s)) \, ds, \quad f \in C_c(\mathbb{H}). \]

We ignore the case when \(\lambda = 0\) when the geodesics are straight lines. Furthermore, since \(\lambda < 0\) corresponds to a \(\lambda > 0\) geodesic with opposite orientation, we will always take \(\lambda > 0\). In Proposition 3, we prove that
\[ I_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda, d\zeta) \]
with a natural measure on the codomain, is well-defined and bounded. In [30, p. 392], Strichartz proves indirectly that a function on the Heisenberg group may not in general be recovered from its integrals over \(\lambda = 0\) geodesics, but does not consider \(\lambda \neq 0\) geodesics.

We may now state our main result:

**Theorem 1.** The Heisenberg X-ray transform \(I : L^1(\mathbb{H}) \to L^1(\mathcal{G}, d\zeta)\) is injective. In particular, if \(f \in L^1(\mathbb{H})\), and \(I\lambda f = 0\) for all \(\lambda\) in a neighborhood of zero, then \(f = 0\).

We prove this result using harmonic analysis adapted to the group structure, modifying familiar results in Euclidean space. Consider, for example, the Radon and Mean Value Transforms on \(\mathbb{R}^2\):
\[ Rf(s, \theta) := \int_\mathbb{R} f(se^{i\theta} + ite^{i\theta}) \, dt, \quad M^r f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) \, d\theta \] (2)

where, say, \(f \in C_c(\mathbb{R}^2)\). Taking the Fourier Transforms in \(s\) and \(z\), respectively, yields
\[ F_{s \to \sigma} Rf(\sigma, \theta) = \hat{f}(\sigma e^{i\theta}), \quad F_{z \to \zeta} M^r f(\zeta) = J_0(\rho|\zeta|) \hat{f}(\zeta), \]
where \(J_0\) is the zeroth-order Bessel function. These results are known as Fourier Slice Theorems, or Projection Slice Theorems [24]. They reveal that \(R\), though of as a projection onto \(\{\theta\}\), becomes a restriction operator onto the “slice” \(\sigma \to \sigma e^{i\theta}\) in the Fourier domain, and that \(M^r\) becomes a multiplication operator by \(J_0(\rho|\zeta|)\).
when viewed in the Fourier domain. Fourier Slice Theorems exist for more general Radon transforms as well; for example, in [14, 17].

The Radon and Mean Value Transforms can be interpreted as integration over straight lines or magnetic geodesics in Euclidean space. In the case of $\mathbb{H}$—which is a “flat” sub-Riemannian geometry—we prove a corresponding Fourier Slice Theorem for Heisenberg geodesics. We use the operator-valued group Fourier Transform $\mathcal{F}_\mathbb{H}$ associated to the Bargmann-Fock representation $\beta_h$ (defined in equation (7)), which has proven a useful tool, for example, by Nachman in [23] to find the fundamental solution for the wave operator for the Heisenberg Laplacian. The theory of $\mathcal{F}_\mathbb{H}$ is extensively developed in [9, 33]. In particular it has a Plancherel Theorem and Inversion Theorem [7, 8, 33]. In the Fourier domain of $\mathcal{F}_\mathbb{H}$, the Heisenberg X-ray transform is simultaneously a restriction and a multiplication operator:

**Theorem 2** (Heisenberg Fourier Slice Theorem). If $f \in L^1(\mathbb{H})$, then

$$ \left( \mathcal{F}_\mathbb{H}/\Gamma_\lambda (I_\lambda f) \right) (n) = \frac{2\pi}{\lambda} J_n \circ \left( \mathcal{F}_\mathbb{H} (f) \right) (n\lambda^2), \quad \forall n \in \mathbb{Z} \setminus \{0\}, \forall \lambda > 0. $$

The Heisenberg Fourier Slice Theorem is an equality of operators acting on Bargmann-Fock space (originally described in [3]),

$$ \mathcal{H} := \left\{ F \in \text{Hol}(\mathbb{C}) : \frac{1}{\pi} \int_{\mathbb{C}} |F(\zeta)|^2 e^{-|\zeta|^2} d\zeta < \infty \right\}. \quad (3) $$

$J_n : \mathcal{H} \rightarrow \mathcal{H}$ is the operator

$$ J_n F(\zeta) = \frac{1}{2\pi i} \left( \frac{e^{-n/2}}{n} \right)^{-1} \int z^{n-1} e^{-n\zeta/z} F(\zeta + z) dz, \quad (4) $$

where the contour is a circle around the origin oriented counterclockwise. Loosely speaking, the Heisenberg X-ray transform $I$ is “block-diagonalized” in $\lambda$ by the group Fourier Transform, and each block is essentially a multiple of $J_n$.

The classical Fourier Slice Theorem for $R$ in (2) states that knowledge of $Rf$ for a fixed $\theta_0$ determines the Fourier transform $\hat{f}(\zeta)$ for all $\zeta \parallel \theta_0$. Similarly, the Heisenberg Fourier Slice Theorem says that knowledge of $I_\lambda f$ for fixed $\lambda$ determines the group Fourier transform $\mathcal{F}_\mathbb{H} f(h)$, up to multiplication by the operator $J_n$, for all $h \in \lambda^2 \mathbb{Z}^\times$. Therefore, injectivity of $I$ follows once we show that

**Proposition 1.** The map $J_n : \mathcal{H} \rightarrow \mathcal{H}$ is injective whenever $n$ is an odd integer.

Finally, in Section 5 we consider the ray transform $I^\epsilon$ associated to a special family of left-invariant taming metrics $g_\epsilon$ parameterized by $\epsilon > 0$. We prove a Heisenberg Fourier Slice Theorem (Theorem 3) for $g_\epsilon$ geodesics and use it in the same way to show that $I^\epsilon$ is injective (Theorem 4). In [24], the authors prove that the X-ray transform is injective on step-2 Nilpotent Lie groups of rank greater than 3 with a left-invariant metric. However, this result assumes the existence of 2-dimensional totally geodesic surfaces, which excludes the Heisenberg group.

## 3 Preliminaries

### 3.1 Heisenberg Geometry

We define the sub-Riemannian metric on $\mathbb{H}$ by declaring the left-invariant vector fields

$$ X = \partial_x - \frac{1}{2} y \partial_t \quad Y = \partial_y + \frac{1}{2} x \partial_t \quad (5) $$

to be orthonormal, and the length of $T = \partial_t$ to be infinite. Then any finite length smooth path in $\mathbb{H}$ must be tangent to the nonintegrable distribution $\mathcal{D}_q := \text{Span}\{X_q, Y_q\}, q \in \mathbb{H}$. We call such a path horizontal. The length of a horizontal path equals the length of its projection to the plane by the map

$$ \pi(x, y, t) = (x, y). $$
A minimizing Heisenberg geodesic is a shortest horizontal path joining two points of $\mathbb{H}$. That any two points in $\mathbb{H}$ are connected by a horizontal path is guaranteed by Chow’s Theorem and the fact that $\mathcal{D}$ satisfies the Hörmander condition (i.e. $\mathcal{D}$ is bracket-generating).

The fiber quadratic Hamiltonian $H: T^*\mathbb{H} \to \mathbb{R}$ given in canonical coordinates by

$$H(x, y, t, p_x, p_y, p_t) = \frac{1}{2} \left( (p_x - \frac{1}{2} y p_t)^2 + (p_y + \frac{1}{2} x p_t)^2 \right),$$

(6)

generates the Heisenberg geodesics. By ‘generate’ we mean that any solution to Hamilton’s equations for $H$ projects, via the canonical projection $T^*\mathbb{H} \to \mathbb{H}$, to a sub-Riemannian geodesic, and conversely, all Heisenberg geodesics arise this way [22, Sec 1.5]. If we want geodesics parameterized by arclength we only take solutions for which $H = 0$. Indeed

$$\dot{\lambda} = -\frac{\partial H}{\partial t} = 0,$$

so that $\lambda := p_t$ is a constant of motion. If we interpret $\lambda$ as the charge $e$ of a particle, then $H$, viewed as a Hamiltonian on $T^*\mathbb{R}^2$, is the Hamiltonian for a particle of charge $e = \lambda$ travelling in the plane under the influence of a constant unit strength magnetic field. These solutions are well-known and easy to derive [22, p. 12]. When $H = 1/2$ they are circles of radius $R = 1/|\lambda|$ for $\lambda \neq 0$, and lines when $\lambda = 0$. See eq (11) for a concrete representation of all geodesics with $\lambda \neq 0$.

### 3.2 The Group Fourier Transforms

We start by giving a brief description of the representation theory of the Heisenberg group. A more detailed discussion can be found in [6]. Denote by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on Bargmann-Fock space, defined in [3]. For each $h \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the map (motivated in Section 6.4)

$$\beta_h : \mathbb{H} \to \mathcal{U}(\mathcal{H})$$

given by

$$\beta_h(z, t)F(\zeta) := e^{2iht - \sqrt{\hbar} |z|^2} F(\zeta + \sqrt{\hbar}z), \quad h > 0,$$

(7)

and $\beta_h(z, t) = \beta_{|h|}(z, -t)$ for $h < 0$, is a strongly continuous unitary representation of the Heisenberg group on $\mathcal{H}$. Moreover, it is known that these representations are irreducible, and by the Stone-von Neumann Theorem, up to unitary equivalence, these are all of the irreducible unitary representations on $\mathbb{H}$ that are nontrivial on the center of $\mathbb{H}$ [6].

We define the group Fourier Transform of an integrable function on $\mathbb{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. The Heisenberg Fourier Transform of $f \in L^1(\mathbb{H})$ is the operator-valued function

$$\mathcal{F}_\mathbb{H}(f) : \mathbb{R}^* \to \mathcal{B}(\mathcal{H})$$

$$\mathcal{F}_\mathbb{H}(f)(h) := \int_{\mathbb{H}} f(q) \beta_h(q)^* dq$$

where the integral is taken in the Bochner sense [33, p. 11]. Think of $h$ as a semi-classical parameter.

**Remark 1.** Many authors define $\mathcal{F}_\mathbb{H}$ alternatively with the Schrödinger representations. Our definition seems more natural for studying the X-ray transform due to the simplicity of $\mathcal{F}_\mathbb{H}$, and is equivalent by conjugation with a unitary intertwining map; the choice is largely a personal preference. We also normalize the representations $\beta_h$ in such a way that they all act on the same space $\mathcal{H}$, rather than a family of spaces parameterized by $h \in \mathbb{R}^*$, as in [6].

If $f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$, then $\mathcal{F}_\mathbb{H}(f)(h)$ is a Hilbert-Schmidt operator on $\mathcal{H}$ [8, 9]. Let $S_2$ denote the space of Hilbert-Schmidt operators on $\mathcal{H}$, and define the Hilbert Space $L^2(\mathbb{R}^*, S_2; d\mu) = L^2(S_2)$ via the inner product

$$\langle A, B \rangle_{L^2(S_2)} := \int_{\mathbb{R}^*} \text{tr} (A(h)B(h)^*) \, d\mu(h), \quad d\mu = \pi^{-2}|h|dh.$$
Theorem (Plancherel Theorem). If \( f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H}) \), then \( ||f||_{L^2(\mathbb{H})} = ||\mathcal{F}_\mathbb{H}f||_{L^2(S_2)} \).

Theorem (Fourier Inversion Theorem). If \( f \in S(\mathbb{H}) \), Schwartz space on \( \mathbb{R}^3 \), then

\[
 f(q) = \int_{\mathbb{R}^2} \text{tr} (\delta_h(q) \mathcal{F}_\mathbb{H} f(h)) \, d\mu(h), \quad q \in \mathbb{H}.
\]

Thus \( \mathcal{F}_\mathbb{H} \) extends to an isometry from \( L^2(\mathbb{H}) \) into \( L^2(S_2) \). In fact, it is onto as well. Furthermore if \( f \in L^1(\mathbb{H}) \), then by convolving \( f \) with an approximation of identity, we may use \( \mathcal{F}_\mathbb{H} \) to prove \( \mathcal{F}_\mathbb{H} \) is injective on \( L^1(\mathbb{H}) \).

While the definition above is sufficient for our purposes, we remark that \( \mathcal{F}_\mathbb{H} \) has been extended to much more general classes of function such as tempered distributions \( [2] \). In \( [3, 32] \), and much more generally in \( [18] \) the authors use the group Fourier transform to develop theory of pseudo-differential operators.

Finally, \( \Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\} \), where \( R = 1/\lambda \), is a discrete subgroup of the center of \( \mathbb{H} \). Since \( \beta_h(z, t) = e^{2i\pi h} \beta_h(z, 0) \), the representation \( \beta_h \) descends to the so-called reduced Heisenberg group \( \mathbb{H}/\Gamma_\lambda \) if and only if \( h \in \lambda^2 \mathbb{Z}^* \). To a function \( g \in L^1(\mathbb{H}/\Gamma_\lambda) \), we associate the reduced Fourier Transform, defined as

\[
 \mathcal{F}_{\mathbb{H}/\Gamma_\lambda}(g) : \mathbb{Z}^* \to \mathcal{B}(\mathcal{H})
\]

\[
 \mathcal{F}_{\mathbb{H}/\Gamma_\lambda}(g)(n) := \int_{\mathbb{H}/\Gamma_\lambda} g(q) \beta_{n\lambda^2}(q)^* \, dq,
\]

where \( \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \).

Remark 2. The reduced Fourier Transform defined above is not invertible unless we also consider the representations \( (z, t) \mapsto e^{i\xi}; \xi \in \mathbb{C} \), which are trivial on the center, in the definition. (Indeed, if \( \partial_t f(z, t) = 0 \), then \( \mathcal{F}_{\mathbb{H}/\Gamma_\lambda}f = 0 \).) This extension is not necessary for our purposes.

4 Proof of Theorems 1 and 2

4.1 The space of geodesics

Recall that \( \mathbb{H} \) acts transitively on \( \mathcal{G}_\lambda \) on the left. Since

\[
 (0, \pi R^2)\gamma_\lambda(s) = \left( Re^{is/R}, R\frac{s + 2\pi R}{2}\right) = \gamma_\lambda(s + 2\pi R); \quad R = 1/\lambda,
\]

the subgroup \( \Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\} \) stabilizes \( \mathcal{G}_\lambda \). Upon fixing \( \gamma_\lambda \), we have the identification

\[
 \mathcal{G}_\lambda \cong \mathbb{H}/\Gamma_\lambda
\]

\[
 (z, t)\gamma_\lambda \mapsto (z, t)\Gamma_\lambda.
\]

When \( \lambda = 1 \), we omit subscripts and write \( \Gamma = \Gamma_1 \).

Let \( d(z, t)\Gamma_\lambda \equiv dx \wedge dy \wedge dt \) be the Haar measure on \( \mathbb{H}/\Gamma_\lambda \), and let \( \mathcal{G}_\lambda \) inherit a multiple of the Haar measure \( d\mathcal{G}_\lambda := \lambda dx \wedge dy \wedge dt \), normalized to satisfy \( [14] \). Furthermore, let \( d\mathcal{G} := \lambda e^{-\lambda x} dx \wedge dy \wedge dt \wedge d\lambda \), chosen to ensure boundedness in Proposition 3.

4.2 Simplification to the reduced X-ray transform

The dilation map, \( \delta_\lambda(z, t) := (\lambda z, \lambda^2 t) \), is an automorphism of the Heisenberg group for \( \lambda \neq 0 \). Furthermore,

\[
 \delta_\lambda : \Gamma_\lambda \ni (0, k\pi \lambda^{-2}) \mapsto (0, k\pi) \in \Gamma,
\]

so \( \delta_\lambda : \mathbb{H}/\Gamma_\lambda \to \mathbb{H}/\Gamma \) is well-defined. Denote by \( \delta_\lambda^* \) the pullback relation defined on functions:

\[
 \delta_\lambda^* : L^1(\mathbb{H}) \to L^1(\mathbb{H})
\]

\[
 \delta_\lambda^* : L^1(\mathbb{H}/\Gamma) \to L^1(\mathbb{H}/\Gamma_\lambda)
\]

\[
 \delta_\lambda^* (\gamma_\lambda g((z, t)\Gamma_\lambda)) = g((\lambda z, \lambda^2 t)\Gamma).
\]

Remark 3. In the sequel, we write any function \( g : \mathbb{H}/\Gamma_\lambda \to \mathbb{C} \) as \( g(z, t) \), in place of \( g((z, t)\Gamma_\lambda) \), understanding that the \( t \) variable is taken mod \( \pi \lambda^{-2} \).
The dilation map $\delta_\lambda$ is relevant because it is a conformal map for the sub-Riemannian metric (with constant conformal factor $\lambda$). Consequently, we have the following homogeneity of the ray transform:

**Proposition 2** (Homogeneity of $I$). For $f \in C_c(\mathbb{H})$,

$$I_\lambda f(z,t) = (1/\lambda)\delta_\lambda^* \left( I_{1/\lambda} f \right) (z,t). \quad (11)$$

**Proof.** Note that dilation preserves geodesics but rescales their speed:

$$\delta_1 \gamma_1(s) = \gamma_1(s/\lambda). \quad (12)$$

Then

$$\delta_\lambda^* \left( I_{1/\lambda} f \right) (z,t) = I_1 \left( \delta_{1/\lambda}^* f \right) (\lambda z, \lambda^2 t)$$

$$= \int \delta_{1/\lambda}^* f \left( (\lambda z, \lambda^2 t) \gamma_1(s) \right) ds$$

$$= \int f \left( \delta_1 \gamma_1(\lambda z, \lambda^2 t) \right) ds, \quad \text{because } \delta_\lambda \in \text{Aut}(\mathbb{H}),$$

$$= \int f \left( (z,t) \gamma_1(s/\lambda) \right) ds, \quad \text{by (12),}$$

$$= \lambda \int f \left( (z,t) \gamma_1(s) \right) ds = \lambda I_\lambda f(z,t).$$

Next, we exploit the periodic symmetry of Heisenberg geodesics to reduce the X-ray transform to one period.

**Proposition 3.** For any $\lambda > 0$, $I_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda)$ is well-defined, bounded, and factors in the following way:

$$L^1(\mathbb{H}) \xrightarrow{I_\lambda} L^1(\mathcal{G}_\lambda) \xrightarrow{P_\lambda} L^1(\mathbb{H}/\Gamma_\lambda) \xrightarrow{I^\text{red}} L^1(\mathcal{G}, \text{d}g),$$

where the maps which we call *Poisson Summation* and the *reduced X-ray transform* are given by

$$P_\lambda f(z,t) = \sum_{k \in \mathbb{Z}} f \left( z, t + k\pi R^2 \right), \quad I^\text{red}_\lambda g(z,t) = \int_0^{2\pi R} g \left( (z,t) \gamma_\lambda(s) \right) ds; \ R = 1/\lambda.$$ 

Furthermore, $I : L^1(\mathbb{H}) \to L^1(\mathcal{G}, \text{d}g)$ is well-defined and bounded.

**Proof.** By homogeneity (11), and since pullback by $\delta_\lambda$ is bounded in the above $L^1$ spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write $P$ and $I^\text{red}$. The map

$$C_c(\mathbb{H}) \ni f \mapsto \int_{\mathbb{H}/\Gamma} Pf(z,t) d(z,t)$$

is a left-invariant positive linear functional on $C_c(\mathbb{H})$. By uniqueness of the Haar measure on $\mathbb{H}$ (which is just the Lebesgue measure), and the Riesz-Representation theorem, $\exists c > 0$ such that

$$\int_{\mathbb{H}/\Gamma} Pf(z,t) d(z,t) \Gamma = c \int_{\mathbb{H}} f(z,t) d(z,t), \quad (13)$$

and one may check that $c = 1$ (see [7] Thm. 2.49) for the general statement). So in particular, $||Pf||_{L^1(\mathbb{H}/\Gamma)} \leq ||f||_{L^1(\mathbb{H})}$. 

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For $g \in C_c(\mathbb{H}/\Gamma)$,

$$||I_{red}g||_{L^1(\mathcal{G}_1)} = \int_{\mathcal{G}_1} |I_{red}g(z,t)|d\mathcal{G}_1$$

$$= \int_{\mathbb{H}/\Gamma} \left| \int_0^{2\pi} g((z,t)\gamma_1(s)) \, ds \right| d(z,t)\Gamma$$

$$\leq \int_{\mathbb{H}/\Gamma} \int_0^{2\pi} |g((z,t)\gamma_1(s))| \, |d(z,t)\Gamma| \, ds$$

$$= \int_{\mathbb{H}/\Gamma} \int_0^{2\pi} \gamma_1(s)^{-1} \, |d((z,t)\gamma_1(s)^{-1})\Gamma| \, ds$$

$$= \int_{\mathbb{H}/\Gamma} \int_0^{2\pi} |g((z,t))| \, |d(z,t)\Gamma| \, ds,$$

since $\mathbb{H}/\Gamma$ is unimodular,

$$= 2\pi||g||_{L^1(\mathbb{H}/\Gamma)}.$$

Thus $P$ and $I_{red}$ extend to $L^1$ bounded maps. Given $f \in C_c(\mathbb{H})$, since $Pf \in C_c(\mathbb{H}/\Gamma)$ and

$$I_{red}Pf(z,t) = \int_0^{2\pi} \sum_{k \in \mathbb{Z}} f((z,t+k\pi)\gamma_1(s)) \, ds = \int_0^{2\pi} \sum_{k \in \mathbb{Z}} f((z,t)\gamma_1(s+2\pi k)) \, ds,$$

by (9),

$$= \sum_{k \in \mathbb{Z}} \int_0^{2\pi} f((z,t)\gamma_1(s+2\pi k)) \, ds = \sum_{k \in \mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} f((z,t)\gamma_1(s)) \, ds = I_1f(z,t),$$

we have $||I_1f||_{L^1(\mathcal{G}_1)} \leq 2\pi||f||_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0, 2\pi] \ni s$. Therefore $I_1$ extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_1)$. In particular one may check, using (11), that $||I_1f||_{L^1(\mathcal{G}_1)} = ||I_1f||_{L^1(\mathcal{G}_1)} \leq 2\pi||f||_{L^1(\mathbb{H})}$.

Finally, for $f \in L^1(\mathbb{H})$, we have

$$||If||_{L^1(\mathcal{G})} := \int_{\mathcal{G}} |If(z,t,\lambda)| \, d\mathcal{G}$$

$$= \int_{\mathbb{H}/\Gamma} \int_0^{2\pi} |I_{\lambda}f(z,t)| \, d\mathcal{G}_\lambda e^{-\lambda} \, d\lambda$$

$$\leq 2\pi||f||_{L^1(\mathbb{H})} \int_0^{2\pi} e^{-\lambda} \, d\lambda = 2\pi||f||_{L^1(\mathbb{H})}.$$

as desired. \qed

**Remark 4.** The reduced X-ray transform $I_{red}: L^1(\mathbb{H}/\Gamma) \to L^1(\mathcal{G}_1)$ is not injective. In fact, if

$$g(z,t) = z^2 e^{-|z|^2} e^{4it},$$

then $I_{red}g = 0$. In Appendix 6.1, we give essentially a Singular Value Decomposition of $I_{red}$.

**Remark 5.** From these computations, we may also deduce a sub-Riemannian Santaló formula:

$$\int_{\mathcal{G}_\lambda} I_{\lambda}f(z,t) \, d\mathcal{G}_\lambda = 2\pi \int_{\mathbb{H}} f(z,t) \, d(z,t), \quad f \in L^1(\mathbb{H}).$$

This is an example of a Santaló formula like those proven in [27], but without the latter’s restriction to the “reduced unit cotangent bundle.”

### 4.3 Lemmas on the group Fourier Transform

We now prove a few general properties of the group Fourier Transform. The first is a Poisson Summation formula for $\mathbb{H} \to \mathbb{H}/\Gamma$ - a quick consequence of the classical version. The author has not found a reference for this version, but does not believe it is new.
Lemma 1 (Poisson Summation Formula). If \( f \in L^1(\mathbb{H}) \), then
\[
\mathcal{F}_{\mathbb{H}/\Gamma} (Pf)(n) = \mathcal{F}_{\mathbb{H}}(f)(n), \quad \forall n \in \mathbb{Z}^*.
\]

Proof. For \( F, G \in \mathcal{H} \),
\[
\langle \mathcal{F}_{\mathbb{H}/\Gamma} (Pf)(n)F,G \rangle = \int_{\mathbb{H}/\Gamma} \sum_{k \in \mathbb{Z}} f(z,t+k\pi) \langle \beta_n(z,t)^*F,G \rangle_{\mathcal{H}} d(z,t) = \int_{\mathbb{H}/\Gamma} \sum_{k \in \mathbb{Z}} f(z,t+k\pi) \langle \beta_n(z,t+k\pi)^*F,G \rangle_{\mathcal{H}} d(z,t) = \int_{\mathbb{H}} f(z,t) \langle \beta_n(z,t)^*F,G \rangle_{\mathcal{H}} d(z,t),
\]
where the third equality follows from \( (13) \), and the fact that \( f(z,t) \langle \beta_n(z,t)^*F,G \rangle_{\mathcal{H}} \in L^1(\mathbb{H}) \) by the Cauchy-Schwartz inequality. Since \( F \) and \( G \) were arbitrary, the identity follows from the definition of the Bochner integral.

Next, we observe how the Fourier Transforms behave with respect to dilations.

Lemma 2 (Dilation Property). Fix \( \lambda > 0 \).

If \( f \in L^1(\mathbb{H}) \), then
\[
\mathcal{F}_{\mathbb{H}}(\delta^\lambda f)(h) = \lambda^{-4} \mathcal{F}_{\mathbb{H}}(f)(h/\lambda^2), \quad \forall h \in \mathbb{R}^*.
\]

And if \( g \in L^1(\mathbb{H}/\Gamma) \), then
\[
\mathcal{F}_{\mathbb{H}/\Gamma} (\delta^\lambda g)(n) = \lambda^{-4} \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n), \quad \forall n \in \mathbb{Z}^*.
\]

We expect the above exponent of \( \lambda \) because the homogeneous dimension of the Heisenberg group is 4.

Proof.
\[
\mathcal{F}_{\mathbb{H}}(\delta^\lambda f)(h) = \int_{\mathbb{H}} f(\lambda z, \lambda^2 t) \beta_h(z,t)^* d(z,t) = \lambda^{-4} \int_{\mathbb{H}} f(z,t) \beta_h(\lambda^{-1} z, \lambda^{-2} t)^* d(z,t) = \lambda^{-4} \int_{\mathbb{H}} f(z,t) \beta_h/\lambda^2 (z,t)^* d(z,t) = \lambda^{-4} \mathcal{F}_{\mathbb{H}}(f)(h/\lambda^2),
\]
and the proof of for \( \mathcal{F}_{\mathbb{H}/\Gamma} \) is nearly identical.

4.4 Proof of Theorem 2

The reduced X-ray transform \( I^{\text{red}} \) is equivariant with respect to left translation by \( \mathbb{H} \) in the sense that
\[
I^{\text{red}} \left( L_{(w,s)}^* g \right)(z,t) = \int_0^{2\pi} L_{(w,s)}^* g \left( (z,t) \gamma_1(\theta) \right) d\theta = \int_0^{2\pi} g \left( (w,s) (z,t) \gamma_1(\theta) \right) d\theta = I^{\text{red}} g \left( (w,s) (z,t) \right) = \left( L_{(w,s)}^* I^{\text{red}} g \right)(z,t).
\]

Thus, \( I^{\text{red}} \) is a convolution operator. In fact, if we define the compactly supported distribution \( \kappa \in \mathcal{E}'(\mathbb{H}/\Gamma) \) by
\[
\kappa(g) := \int_0^{2\pi} g \left( \gamma_1(\theta)^{-1} \right) d\theta \quad \text{then} \quad I^{\text{red}} g = \kappa * g, \quad \text{where} \quad f * g(z,t) := \int_{\mathbb{H}/\Gamma} f \left( (z,t)(w,s)^{-1} \right) g(w,s) d(w,s) d\Gamma.
\]

Therefore, by an analogous Paley-Wiener theory \( (33) \) Ch.1], we expect \( \mathcal{F}_{\mathbb{H}/\Gamma}(\kappa)(n) \in \mathcal{B}(\mathcal{H}) \), and \( \mathcal{F}_{\mathbb{H}/\Gamma} (I^{\text{red}} g)(n) = \mathcal{F}_{\mathbb{H}/\Gamma}(\kappa)(n) \circ \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n) \). The next proposition makes this heuristic explicit.

Proposition 4. If \( g \in L^1(\mathbb{H}/\Gamma) \), then for all \( n \in \mathbb{Z}^* \),
\[
\mathcal{F}_{\mathbb{H}/\Gamma} (I^{\text{red}} g)(n) = (2\pi) J_n \circ \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n).
\]

with \( J_n \) defined in \( (3) \).
Proof.

\[ \mathcal{F}_{\mathbb{H}/\Gamma} (I^{\text{red}}g) (n) := \int_{\mathbb{H}/\Gamma} \int_{0}^{2\pi} g ((z,t) \gamma_1(s)) \beta_n(z,t)^* dsd(z,t) \Gamma \]
\[ = \int_{0}^{2\pi} \int_{\mathbb{H}/\Gamma} g(z,t) \beta_n ((z,t) \gamma_1(s))^{-1} dsd(z,t) \Gamma, \quad \text{since } \mathbb{H}/\Gamma \text{ is unimodular}, \]
\[ = \int_{0}^{2\pi} \beta_n(\gamma_1(s)) ds \int_{\mathbb{H}/\Gamma} g(z,t) \beta_n(z,t)^* dsd(z,t) \Gamma \]
\[ = (2\pi) \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma} (g)(n), \]
\]

where the “multiplier”
\[ \mathcal{J}_n := \frac{1}{2\pi} \int_{0}^{2\pi} \beta_n(\gamma_1(s)) ds \quad (15) \]
is the same as \[ \mathcal{I} \].

Remark 6. \( \mathcal{J}_n \) is similar to the “representation integral” considered in \[ [15] \], though \( s \mapsto \beta_n(\gamma_1(s)) \) is not a homomorphism. Such integration of representations over geodesics also appear in \[ [12] \], where the authors used the Principal Series representations of \( SL(2,\mathbb{R}) \) to show that the normal operator \( I^* I \) associated to the X-ray transform on constant negative curvature surfaces is a nontrivial function of the Laplace-Beltrami operator.

Together with Proposition \[ \ref{prop:HeisenbergFourierSliceTheorem} \] these imply the Heisenberg Fourier Slice Theorem:

Proof of Theorem \[ \ref{thm:HeisenbergFourierSliceTheorem} \] Let \( f \in L^1(\mathbb{H}) \), \( \lambda > 0 \) and \( n \in \mathbb{Z}^* \). By Proposition \[ \ref{prop:HeisenbergFourierSliceTheorem} \] and \[ \ref{prop:HeisenbergFourierSliceTheorem} \], we have
\[
\mathcal{F}_{\mathbb{H}/\Gamma} (I_1 f)(n) = \mathcal{F}_{\mathbb{H}/\Gamma} (I^{\text{red}} P f)(n) = (2\pi) \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma} (P f)(n) = (2\pi) \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}} (f)(n). \quad (16)
\]

Exploiting homogeneity of \( I \),
\[
\mathcal{F}_{\mathbb{H}/\Gamma^l} (I_\lambda f)(n) = \lambda^{-1} \mathcal{F}_{\mathbb{H}/\Gamma^l} \left( \delta^*_\lambda I_1 \left( \delta^*_\lambda f \right) \right)(n), \quad \text{Proposition } \ref{prop:HeisenbergFourierSliceTheorem} \\
= \lambda^{-5} \mathcal{F}_{\mathbb{H}/\Gamma^l} \left( I_1 \left( \delta^*_\lambda f \right) \right)(n), \quad \text{Lemma } \ref{lem:HeisenbergFourierSliceTheorem} \\
= 2\pi \lambda^{-5} \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}} \left( \delta^*_\lambda f \right)(n), \quad \text{by } \ref{lem:HeisenbergFourierSliceTheorem}, \\
= 2\pi \lambda^{-5} \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}} (f)(n \lambda^2), \quad \text{Lemma } \ref{lem:HeisenbergFourierSliceTheorem}
\]
as desired. \( \square \)

Remark 7. In the special case when \( n = 0 \) or \( h = 0 \), the group Fourier Transforms are qualitatively different; they are the Euclidean Fourier transform in the \( z \) variable (the precise sense in which this limiting behavior occurs is formalized by Geller in \[ [8] \]). In this case, the Fourier Slice theorem takes the form
\[
\mathcal{I}(I_\lambda f)(\lambda \zeta, 0) = (2\pi / \lambda) J_0(\lambda |\zeta|) \widehat{f}(\lambda \zeta, 0); \quad \forall \lambda > 0, \ f \in L^1(\mathbb{H}),
\]
where \( J_0 \) is the classical Bessel function of order zero, and
\[
\widehat{f}(\zeta, 0) = \int_{\mathbb{C}} \int_{\mathbb{R}} f(z,t) e^{-i \zeta z} dt dz, \ f \in L^1(\mathbb{H}), \quad \widehat{g}(\zeta, 0) = \int_{\mathbb{C}} \int_{0}^{\pi} g(z,t) e^{-i \zeta z} dt dz; \ g \in L^1(\mathbb{H}/\Gamma^l).
4.5 Proof of Theorem

We now make use of the Heisenberg Fourier Slice theorem to prove injectivity of $I$. First, we describe an important class of functions which are the cylindrical harmonics of the Heisenberg group.

With respect to the standard orthonormal basis $\omega_k = \zeta^k / \sqrt{k!}$, $k = 0, 1, \ldots$, of $\mathcal{H}$, the matrix coefficients of the Bargmann-Fock representation, (7), $M^n_{jk}(z, t) := \langle \beta_n(z, t) \omega_j, \omega_k \rangle_{\mathcal{H}}$ are given for $h > 0$ via a brute force computation by

$$M^h_{jk}(z, t) = \begin{cases} \sqrt{\frac{h}{j!}} \left( + \sqrt{h} \right)^{j-k} L^k_{j-k} (h|z|^2) e^{-h|z|^2/2e^{2ith}} & j \geq k \\ \sqrt{\frac{h}{j!}} \left( - \sqrt{h} \right)^{k-j} L^k_{j-k} (h|z|^2) e^{-h|z|^2/2e^{2ith}} & j \leq k \end{cases},$$

and $M^h_{jk}(z, t) = M^{|h|}(\pi, -t)$ for $h < 0$ (see Appendix 5.5 for conversion between Folland’s [6] p. 64 and our conventions).

Here $L_j^{(\alpha)}(x)$ is the generalized Laguerre polynomial, defined recursively by

$$L_0^{(\alpha)}(x) = 1$$
$$L_1^{(\alpha)}(x) = 1 + \alpha - x$$

$$(j+1)L_{j+1}^{(\alpha)}(x) = (2j+1+\alpha-x)L_j^{(\alpha)}(x) - (j+\alpha)L_{j-1}^{(\alpha)}(x).$$

The following mild generalization of (15) will be useful for subsequent computations.

**Definition 1.** For $n \in \mathbb{Z}^*$, let

$$\mathcal{J}_n(r) := \frac{1}{2\pi} \int_0^{2\pi} \beta_n(\mathrm{re}^{i\theta}, \theta/2) \, d\theta, \quad r > 0.$$  

In particular, $\mathcal{J}_n(1) = \mathcal{J}_n$, defined in [5].

**Proposition 5** (SVD of $\mathcal{J}_n(r)$). For every $n \in \mathbb{Z}^*$ and $r > 0$, the operator $\mathcal{J}_n(r) : \mathcal{H} \rightarrow \mathcal{H}$ is bounded in the operator-norm topology. Furthermore, $\mathcal{J}_{-n}(r) = \mathcal{J}_n(r)$, and, with respect to the orthonormal basis $\{\omega_j = \zeta^j / \sqrt{j!} : j = 0, 1, 2, \ldots\}$ of $\mathcal{H}$, we have

$$\mathcal{J}_n(r) \omega_j = \sqrt{j!/(j+n)!} (nr^2)^{n/2} e^{-nr^2/2} L_n^{(\alpha)}(nr^2) \omega_{j+n}, \quad \forall j \in \mathbb{N}, \ n > 0.$$

**Proof.** $\mathcal{J}_n(r) : \mathcal{H} \rightarrow \mathcal{H}$ is bounded in the operator-norm topology for any $n \in \mathbb{Z}^*$ since

$$||\mathcal{J}_n(r)||_{op} \leq \frac{1}{2\pi} \int_0^{2\pi} ||\beta_n(\mathrm{re}^{i\theta}, \theta/2)||_{op} \, d\theta = 1.$$  

Note that, for $n \in \mathbb{Z}^*$,

$$\mathcal{J}_{-n} = \frac{1}{2\pi} \int_0^{2\pi} \beta_n(e^{-i\theta}, -\theta/2) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \beta_n(e^{i\theta}, \theta/2) \, d\theta = \mathcal{J}_n.$$

For $n > 0$,

$$\langle \mathcal{J}_n(r) \omega_j, \omega_k \rangle_H = \frac{1}{2\pi} \int_0^{2\pi} (\beta_n(\mathrm{re}^{i\theta}, \theta/2) \omega_j, \omega_k)_{\mathcal{H}} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} M^n_{jk}(\mathrm{re}^{i\theta}, \theta/2) \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{i(j-k+n)\theta} \, d\theta M^n_{jk}(r, 0)$$

$$= \delta(j-k+n)M^n_{jk}(r, 0)$$

$$= M^n_{j+k+n}(r, 0),$$

(21)
in which case,

\[ J_n(r) \omega_j = M^n_{j,j+n}(r,0) \omega_{j+n}, \]

and, by \[17\]

\[ M^n_{j,j+n}(r,0) = \sqrt{\frac{j!}{(j+n)!}} (n^m)^{n/2} e^{-nr^2/2} L_j^{(n)}(m^r^2), \quad n > 0. \tag{22} \]

\[ \square \]

**Corollary 1.** Let \( r > 0 \) and \( n \in \mathbb{Z}^* \) be fixed. Since \( J_n(r) \) is bounded, it is injective if and only if \( L_j^{(n)}(m^r^2) \) is nonzero for all \( j \in \mathbb{N} \).

**Proof of Proposition 1.** Given \( n \in 2\mathbb{Z} + 1 \), by Corollary \[11\] the operator \( J_n \) is injective if and only if the sequence \( \{L_j^{(n)}(n)\}_{j=0}^{\infty} \) is nonvanishing.

Set \( a_j^{(n)} = j! L_j^{(n)}(n) \in \mathbb{Z} \). Then \( a_0^{(n)} = a_1^{(n)} = 1 \), and by \[18\],

\[ a_j^{(n)} = (2j + 1)a_j^{(n)} - j(j + n)a_{j-1} \equiv a_j^{(n)} \pmod{2} \]

since \( n \) is odd. Therefore \( a_j^{(n)} \equiv a_0^{(n)} = 1 \pmod{2} \) for all \( j = 0, 1, 2, \ldots \). In particular, \( L_j^{(n)}(n) = a_j^{(n)}/j! \neq 0 \) for \( j \in \mathbb{N} \). Therefore \( J_n \) is injective whenever \( n \) is an odd integer. \[ \square \]

**Remark 8.** We know what \( J_2 \) is not injective since \( L_2^{(2)}(2) = 0 \). However, the author is not currently aware of a general statement characterizing all \((j, n) \in \mathbb{N} \times \mathbb{N}^* \) for which \( L_j^{(n)}(n) = 0 \). While knowing this is not essential for proving injectivity of \( I \), it would provide more ways to invert \( I \). This is because the space of geodesics is four dimensional, and so we only need a subset of the overdetermined data to reconstruct \( f \) from \( If \).

The proof of Theorem \[11\] is now almost immediate.

**Proof of Theorem 1.** Suppose \( I_{\lambda} f = 0 \) for all \( \lambda \in (0, \eta) \), where \( \eta > 0 \). By the Heisenberg Fourier Slice Theorem,

\[ 0 = J_n \circ \mathcal{F}_\mathbb{H}(f)(n\lambda^2), \quad \forall n \in \mathbb{Z}^*, \forall \lambda \in (0, \eta). \]

By Proposition \[11\]

\[ 0 = \mathcal{F}_\mathbb{H}(f)(n\lambda^2), \quad \forall n \in 2\mathbb{Z} + 1, \forall \lambda \in (0, \eta). \tag{23} \]

In which case

\[ 0 = \mathcal{F}_\mathbb{H}(f)(h), \quad \forall h \in \bigcup_{n \in 2\mathbb{Z} + 1} n(0, \eta^2) = \mathbb{R}^*. \]

Therefore \( f = 0 \) by the Fourier Inversion theorem for \( \mathcal{F}_\mathbb{H} \). \[ \square \]

## 5 X-ray Transform for the taming metric \( g_\epsilon \)

We use the same machinery to prove injectivity of the X-ray transform associated to the family of left-invariant taming metric on \( \mathbb{H} \).

Consider the family of left-invariant Riemannian metrics for \( \epsilon > 0 \):

\[ g_\epsilon := dx^2 + dy^2 + (1/\epsilon)^2 \Theta^2, \]

where \( \Theta := dt - \frac{1}{2} (x dy - y dx) \) is a contact form for the Heisenberg distribution \( \mathcal{D} \), defined in Section \[8.1\] Geodesics of \( (\mathbb{H}, g_\epsilon) \) converge uniformly to the sub-Riemannian geodesics as \( \epsilon \to 0 \), \[5\] p. 33. The explicit expression for \( g_\epsilon \) geodesics is derived in \[5\] Sec. 2.4.4. We record the exponential map for \( g_\epsilon \) in \[41\].

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Proof. This is essentially the same proof as 11: we will assume that we have chosen a fixed $\epsilon > 0$.

Let $G^\epsilon$ be the set of geodesics for $g_e$ without orientation, and $G^\epsilon_\lambda$ be the subset of geodesics having charge $\lambda$ (which is still a constant of motion). Geodesics with $\lambda \neq 0$ still project to circles in the plane, and those with $\lambda = 0$ project to lines; they only differ from sub-Riemannina geodesics by an $\epsilon$-dependent velocity in the $T = \partial_t$ direction. Left translation by any element $(z, t) \in \mathbb{H}$ is a $g_e$-isometry, and so $\mathbb{H}$ acts on $G^\epsilon$ by pointwise left multiplication. This action does not change the value of $\lambda$ and is transitive when $\lambda \neq 0$.

We choose a particular geodesic $\gamma^\epsilon_\lambda$ to be the one whose projection to the plane is a unit-speed circular path of radius $R = 1/|\lambda|$, and parameterize the set of $g_e$ geodesics having charge $\lambda$ by

$$ s \to (z, t)\gamma^\epsilon_\lambda(s), \quad \gamma^\epsilon_\lambda(s) = \left(Re^{is/R}, \frac{\sqrt{R^2 + 2\epsilon^2}}{2R}\right) \in \mathbb{H}; \quad R = 1/\lambda. \quad (24) $$

Remark 9. To avoid quantifying $\epsilon$ in every proposition of this section, with the exception of Theorems 3 and 4 we will assume that we have chosen a fixed $\epsilon > 0$.

Let $G^\epsilon$ be the set of geodesics for $g_e$ without orientation, and $G^\epsilon_\lambda$ be the subset of geodesics having charge $\lambda$ (which is still a constant of motion). Geodesics with $\lambda \neq 0$ still project to circles in the plane, and those with $\lambda = 0$ project to lines; they only differ from sub-Riemannina geodesics by an $\epsilon$-dependent velocity in the $T = \partial_t$ direction. Left translation by any element $(z, t) \in \mathbb{H}$ is a $g_e$-isometry, and so $\mathbb{H}$ acts on $G^\epsilon$ by pointwise left multiplication. This action does not change the value of $\lambda$ and is transitive when $\lambda \neq 0$.

We choose a particular geodesic $\gamma^\epsilon_\lambda$ to be the one whose projection to the plane is a unit-speed circular path of radius $R = 1/|\lambda|$, and parameterize the set of $g_e$ geodesics having charge $\lambda$ by

$$ s \to (z, t)\gamma^\epsilon_\lambda(s), \quad \gamma^\epsilon_\lambda(s) = \left(Re^{is/R}, \frac{\sqrt{R^2 + 2\epsilon^2}}{2R}\right) \in \mathbb{H}; \quad R = 1/\lambda. \quad (24) $$

Remark 10. The geodesics described by (24) are not arclength parameterized; indeed, $g_e(\gamma^\epsilon_\lambda(s), \gamma^\epsilon_\lambda(s)) = 1 + \epsilon^2\lambda^2$. Instead, we insist that their projections to the plane are unit-speed.

We define the X-ray transform associated to the taming metric $g$ by

$$ I^*f(z, t, \lambda) := I^*_\lambda f(z, t) := \int f((z, t)\gamma^\epsilon_\lambda(s)) \, ds, \quad f \in C_c(\mathbb{H}). \quad (25) $$

Note that

$$ \gamma^\epsilon_\lambda(s + 2\pi R) = \gamma^\epsilon_\lambda(s)(0, \pi R^2 + 2\pi \epsilon^2). \quad (26) $$

Therefore the isotropy group of $\gamma^\epsilon_\lambda$ for the action of $\mathbb{H}$ by left translation on $G^\epsilon_\lambda$ is

$$ \Gamma^\epsilon_\lambda := \{(0, k\pi(R^2 + 2\epsilon^2)) \in \mathbb{H} : k \in \mathbb{Z}\}. $$

We have the identification

$$ G^\epsilon_\lambda \cong \mathbb{H}/\Gamma^\epsilon_\lambda \quad (z, t)\gamma^\epsilon_\lambda \mapsto (z, t)\Gamma^\epsilon_\lambda. $$

Remark 11. Again, when $\lambda = 1$, we omit subscripts and write $\Gamma^\epsilon = \Gamma^1_\lambda$. We will also write $g(z, t)$, for any function $g: \mathbb{H}/\Gamma^\epsilon_\lambda \to \mathbb{C}$, in place of $g((z, t)\Gamma^\epsilon_\lambda)$.

Let $d(z, t)\Gamma^\epsilon_\lambda \cong dx \wedge dy \wedge dt$ be the Haar measure on $\mathbb{H}/\Gamma^\epsilon_\lambda$, and let $G^\epsilon_\lambda$ inherit a multiple of the Haar measure $dG^\epsilon_\lambda := \lambda dx \wedge dy \wedge dt$. Furthermore, let $dG^\epsilon := \lambda e^{-\lambda} dx \wedge dy \wedge dt \wedge d\lambda$.

Note the homogeneity of geodesics with respect to dilation:

$$ \delta^{1/\lambda}_{1/\lambda} \gamma^\epsilon_\lambda(s) = \gamma^\epsilon_\lambda(s/\lambda); \quad R = 1/\lambda. \quad (27) $$

Proposition 6 (Homogeneity of $I^*$). For $f \in C_c(\mathbb{H})$, we have

$$ I^*_\lambda(f)(z, t) = \lambda^{-1} \delta^{1/\lambda}_{1/\lambda} I^\epsilon_\lambda \left(\delta^{1/\lambda}_{1/\lambda} f\right)(z, t). \quad (28) $$

Proof. This is essentially the same proof as [11].
Furthermore, in virtually the same way as Proposition we reduce the X-ray transform $I^c$ to one period:

**Proposition 7.** For any $\lambda > 0$, $I^c_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda)$ is well-defined, bounded, and factors in the following way:

\[
\begin{array}{ccc}
L^1(\mathbb{H}) & \xrightarrow{I^c_\lambda} & L^1(\mathcal{G}_\lambda) \\
\downarrow P^c_\lambda & & \nearrow P^c_\lambda \text{red} \\
L^1(\mathbb{H}/\Gamma_\lambda) & & I^c_\lambda \text{red}
\end{array}
\]

where

\[
P^c_\lambda f(z,t) = \sum_{k \in \mathbb{Z}} f(z, t + k\pi(R^2 + 2\epsilon^2)), \quad P^c_\lambda \text{red} g(z,t) := \int_0^{2\pi R} g((z,t)\gamma^c_\lambda(s)) \, ds; \quad R = 1/\lambda.
\]

Furthermore, $I^c : L^1(\mathbb{H}) \to L^1(\mathcal{G}, d\mathcal{G}^c)$ is well-defined and bounded.

**Proof.** By homogeneity \(28\), and since pullback by $\delta_\lambda$ is bounded in the above $L^1$ spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write $P^c$ and $I^c_{\text{red}}$.

For exactly the same reason as \(13\), $P^c$ maps $C_c(\mathbb{H})$ to $C_c(\mathbb{H}/\Gamma^c)$, and

\[
\int_{\mathbb{H}/\Gamma^c} P^c f(z,t) \, d(z,t) \Gamma^c = \int_{\mathbb{H}} f(z,t) \, d(z,t).
\]

(30)

So in particular, $\|P^c f\|_{L^1(\mathbb{H}/\Gamma^c)} \leq \|f\|_{L^1(\mathbb{H})}$.

For $g \in C_c(\mathbb{H}/\Gamma^c)$,

\[
\|I^c_{\text{red}} g\|_{L^1(\mathcal{G}_1^c)} = \int_{\mathcal{G}_1^c} |I^c_{\text{red}} g(z,t)\, d\mathcal{G}^c_1
\]

\[
= \int_{\mathbb{H}/\Gamma^c} \left| \int_0^{2\pi} g((z,t)\gamma^c_1(s)) \, ds \right| d(z,t) \Gamma^c
\]

\[
\leq \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^c} |g((z,t)\gamma^c_1(s))| \, d(z,t) \Gamma^c ds
\]

\[
= \int_{\mathbb{H}/\Gamma^c} \int_0^{2\pi} |g((z,t))| \, d((z,t)\gamma^c_1(s))^{-1} \Gamma^c ds
\]

\[
= \int_{\mathbb{H}/\Gamma^c} |g(z,t)| \, d(z,t) \Gamma^c ds,
\]

since $\mathbb{H}/\Gamma^c$ is unimodular,

\[
= 2\pi \|g\|_{L^1(\mathbb{H}/\Gamma^c)}.
\]

Thus $P^c$ and $I^c_{\text{red}}$ extend to $L^1$ bounded maps. Given $f \in C_c(\mathbb{H})$, since $P f \in C_c(\mathbb{H}/\Gamma^c)$ and

\[
I^c_{\text{red}} P^c f(z,t) = \int_0^{2\pi} \sum_{k \in \mathbb{Z}} f((z,t + k\pi(1 + 2\epsilon^2))\gamma^c_1(s)) \, ds = \int_0^{2\pi} \sum_{k \in \mathbb{Z}} f((z,t)\gamma^c_1(s + 2\pi k)) \, ds, \quad \text{by \(26\),}
\]

\[
= \sum_{k \in \mathbb{Z}} \int_0^{2\pi} f((z,t)\gamma_1(s + 2\pi k)) \, ds = \sum_{k \in \mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} f((z,t)\gamma_1(s)) \, ds = I^c f(z,t),
\]

we have $\|I^c f\|_{L^1(\mathcal{G}_1^c)} \leq 2\pi \|f\|_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0, 2\pi] \ni s$. Therefore $I^c_\lambda$ extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_\lambda)$. In particular one may check, using \(28\), that $\|I^c_\lambda f\|_{L^1(\mathcal{G}_\lambda)} = \|I^c f\|_{L^1(\mathcal{G}_1^c)} \leq 2\pi \|f\|_{L^1(\mathbb{H})}$. 

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Finally we have, for $f \in L^1(\mathbb{H})$,
\[
|f||L^1(\mathbb{H})| := \int_{G^\vee} |I_f(z, t, \lambda)|dG^\vee
\]
\[
\geq \int_0^\infty \int_{G^\vee} |I_f(z, t, \lambda)|dG^\vee e^{-\lambda}d\lambda
\]
\[
\leq 2\pi ||f||L^1(\mathbb{H}) \int_0^\infty e^{-\lambda}d\lambda = 2\pi ||f||L^1(\mathbb{H}).
\]
as desired. 

**Remark 12.** From these computations, may also deduce a Santaló formula for $g_\epsilon$:
\[
\int_{G^\vee} I_f^\epsilon(z, t)dG^\vee = 2\pi \int_{\mathbb{H}} f(z, t)d(z, t), \ f \in L^1(\mathbb{H}).
\]
which refines the usual Santaló formula.

We note a Poisson summation formula for $P^\epsilon$:

**Lemma 3.** For $f \in L^1(\mathbb{H})$,
\[
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(P^\epsilon f)(n) = \mathcal{F}_{\mathbb{H}}(f) \left( \frac{n}{1+2\epsilon^2} \right), \ \forall n \in \mathbb{Z}^*.
\]  

**Proof.** This is just a rescaling of Lemma 4. Observe that $\Gamma^\epsilon = (1+2\epsilon^2)\Gamma$. Using Lemma 2 with $\lambda = 1/\sqrt{1+2\epsilon^2}$, and noting that $\delta_{\sqrt{1+2\epsilon^2}P^\epsilon f} = P^1\delta_{\sqrt{1+2\epsilon^2}\Gamma}$, we are done. 

**Lemma 4.** For $g \in L^1(\mathbb{H}/\Gamma^\epsilon)$, $\lambda > 0$,
\[
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(\delta^\lambda_\epsilon g) = \lambda^{-4} \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n), \ \forall n \in \mathbb{Z}^*.
\]  

**Proof.** Observe that $\Gamma^\epsilon = \lambda^{-2}(1+2\epsilon^2)\Gamma$, and $\Gamma^\epsilon\lambda = (1+2\epsilon^2\lambda^2)\Gamma$. Then apply Lemma 2. 

As before, $I^\epsilon\text{red}$ is a convolution operator by a compactly supported distribution. We compute its Fourier multiplier:

**Proposition 8.** For $g \in L^1(\mathbb{H}/\Gamma^\epsilon)$,
\[
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(I^\epsilon\text{red} g)(n) = 2\pi \mathcal{F}_{\mathbb{H}} \left( \frac{1}{\sqrt{1+2\epsilon^2}} \right) \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n), \ \forall n \in \mathbb{Z}^*.
\]  

**Proof.**
\[
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(I^\epsilon\text{red} g)(n) = \int_{\mathbb{H}/\Gamma^\epsilon} \int_0^{2\pi} g((z, t)\gamma_1^\epsilon(s)) \beta_n/(1+2\epsilon^2)(z, t)^*d\theta(z, t)\Gamma^\epsilon
\]
\[
\geq \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} g(z, t) \beta_n/(1+2\epsilon^2) ((z, t)\gamma_1^\epsilon(s)^{-1})^* d(z, t)\Gamma^\epsilon d\theta
\]
\[
= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} g(z, t) \beta_n/(1+2\epsilon^2) \circ \beta_n/(1+2\epsilon^2)(z, t)^* d(z, t)\Gamma^\epsilon d\theta
\]
\[
= \int_0^{2\pi} \beta_n/(1+2\epsilon^2)(\gamma_1^\epsilon(s)) d\theta \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n).
\]
\[
= 2\pi \mathcal{F}_{\mathbb{H}} \left( \frac{1}{\sqrt{1+2\epsilon^2}} \right) \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n).
\]
We may now prove the Heisenberg Fourier Slice Theorem for $g_c$:

**Theorem 3** ($g_c$ Heisenberg Fourier Slice Theorem). If $f \in L^1(\mathbb{H})$, and $\epsilon > 0$ then

$$\mathcal{F}_{\mathbb{H}}(I^*_\lambda f)(n) = (2\pi/\lambda) \mathcal{J}_n \left( \frac{1}{\sqrt{1+2\epsilon^2 \lambda^2}} \right) \circ \mathcal{F}_{\mathbb{H}}(f) \left( \frac{n\lambda^2}{1+2\epsilon^2 \lambda^2} \right), \quad \forall n \in \mathbb{Z}^*, \forall \lambda > 0.$$

**Proof.** Combining Proposition 7 and Proposition 8

$$\mathcal{F}_{\mathbb{H}}(I^*_\lambda f) = 2\pi \mathcal{J}_n \left( \frac{1}{\sqrt{1+2\epsilon^2}} \right) \circ \mathcal{F}_{\mathbb{H}}(f) \left( \frac{n}{1+2\epsilon^2} \right). \quad (33)$$

Now, exploiting homogeneity of $I^*$,

$$\mathcal{F}_{\mathbb{H}}(I^*_\lambda f)(n) = \lambda^{-1} \mathcal{F}_{\mathbb{H}}(I^*_{\lambda^2} \left( \delta^*_\lambda I^*_1 \left( \left( \delta^*_1/\lambda f \right) \right) \right))(n), \quad \text{by Proposition 6}$$

$$= \lambda^{-5} \mathcal{F}_{\mathbb{H}}(I^*_{\lambda^5} \left( I^*_1 \left( \delta^*_1/\lambda f \right) \right))(n), \quad \text{by Lemma 4}$$

$$= 2\pi \lambda^{-5} \mathcal{J}_n \left( \frac{1}{\sqrt{1+2\epsilon^2 \lambda^2}} \right) \circ \mathcal{F}_{\mathbb{H}}(\delta^*_1/\lambda f) \left( \frac{n}{1+2\epsilon^2 \lambda^2} \right), \quad \text{by (33)}.$$

$$= (2\pi/\lambda) \mathcal{J}_n \left( \frac{1}{\sqrt{1+2\epsilon^2 \lambda^2}} \right) \circ \mathcal{F}_{\mathbb{H}}(f) \left( \frac{n\lambda^2}{1+2\epsilon^2 \lambda^2} \right), \quad \text{by Lemma 4}.$$

$\square$

**Proposition 9.** Let $\epsilon > 0$ and $n \in \mathbb{Z}^*$ be fixed. Then $\mathcal{J}_n \left( \frac{1}{\sqrt{1+2\epsilon^2 \lambda^2}} \right) : \mathcal{H} \to \mathcal{H}$ is injective for almost all $\lambda > 0$.

**Proof.** Set $r = \frac{1}{\sqrt{1+2\epsilon^2 \lambda^2}}$. By Corollary 10, the operator $\mathcal{J}_n(r)$ is injective if and only if $nr^2$ is not a zero of $L_j^{(n)}$ for any $j \in \mathbb{N}$. Since there are only countably many such zeros, the proposition follows. $\square$

**Theorem 4.** For all $\epsilon > 0$, the Heisenberg taming X-ray transform $I^* : L^1(\mathbb{H}) \to L^1(\mathbb{G}^*, dg^*)$ is injective. In particular, if $f \in L^1(\mathbb{H})$ and $I^*_\lambda f = 0$ for all $\lambda$ in a neighborhood of zero, then $f = 0$.

Suppose, $I^*_\lambda(f) = 0$ for all $\lambda \in (0, \eta)$, where $\eta > 0$. Then by Theorem 4 and Proposition 9

$$0 = \mathcal{F}_{\mathbb{H}}(f) \left( \frac{n\lambda^2}{1+2\epsilon^2 \lambda^2} \right)$$

for almost all $\lambda \in (0, \eta)$, and all $n \in \mathbb{Z}^*$. Let $A$ be the set of all such $\lambda \in (0, \eta)$, and $B = \{ \lambda^2/(1+2\epsilon^2 \lambda^2) : \lambda \in A \}$. Then in other words

$$0 = \mathcal{F}_{\mathbb{H}}(f)(h) \quad \forall h \in \bigcup_{n \in \mathbb{Z}} B.$$ 

Since $B$ has full-measure on the interval $(0, \frac{\pi^2}{1+2\epsilon^2 \eta^2})$, we know $\mathcal{F}_{\mathbb{H}}f = 0$ almost everywhere. Therefore $f = 0$ by the Fourier Inversion Theorem.

6 **Appendix**

6.1 SVD of $I^{\text{red}}|_{L^2(\mathbb{H}/\Gamma)}$

While not strictly necessary for our main result, the computation in Proposition 9 also gives us the SVD $I^{\text{red}}$ when restricted to a specific subspace. Here, similarly with [20], we implicitly exploit the fact that $I^{\text{red}}$ intertwines the Heisenberg Laplacian on $\mathbb{H}$ with another differential operator on $\mathbb{H}/\Gamma$, for which the functions $M_{j_k}^h, h \in \mathbb{R}^*$, and $M_{j_k}^n, n \in \mathbb{Z}^*$, are eigenfunctions, respectively.
Consider the subspaces of \( L^2(\mathbb{H}/\Gamma) \)
\[
L^2(\mathbb{C}) \cong \{ f \in L^2(\mathbb{H}/\Gamma) : f(z, t) = f(z, 0), \forall(z, t) \in \mathbb{H}/\Gamma \}
\]
\[
^0L^2(\mathbb{H}/\Gamma) := \{ f \in L^2(\mathbb{H}/\Gamma) : \int_0^\pi f(z, t) dt = 0, \forall z \in \mathbb{C} \}.
\]

**Lemma 5.** We have the orthogonal decomposition
\[
L^2(\mathbb{H}/\Gamma) \cong L^2(\mathbb{C}) \oplus ^0L^2(\mathbb{H}/\Gamma).
\]

**Proof.** If \( f \in L^2(\mathbb{H}/\Gamma) \), let \( f_0(z, t) := \frac{1}{2} \int_0^\pi f(z, t) dt \) and \( g = f - f_0 \), then \( f_0 \in L^2(\mathbb{C}) \) and \( g \in ^0L^2(\mathbb{H}/\Gamma) \).

Furthermore, for arbitrary \( f_0 \in L^2(\mathbb{C}) \), and \( g \in ^0L^2(\mathbb{H}/\Gamma) \),
\[
\int_{\mathbb{H}/\Gamma} f_0(z, t)g(z, t)d(z, t) \Gamma = \int_{\mathbb{C}} f_0(z) \int_0^\pi g(z, t) dt dz = 0.
\]
\( \square \)

In what follows, set
\[
\psi^0_{jk} := \frac{\sqrt{2\pi}}{2\pi} M^0_{jk}; \quad j, k \in \mathbb{N}, n \in \mathbb{Z}^*
\]
for \( M^0_{jk} \) defined in \([11]\). The functions \( \psi^0_{jk} \) for \( n \in \mathbb{Z}^* \) and \( j, k \in \mathbb{N} \) form an orthonormal basis for \(^0L^2(\mathbb{H}/\Gamma)\).
(See \([33], \text{Ch. 4}\), where the author uses slightly different notation.)

**Theorem.** Let \( f \in L^2(\mathbb{H}/\Gamma) \) such that \( \int_0^\pi f(z, t) dt = 0 \), then we have
\[
f = \sum_{n \in \mathbb{Z}^*} \sum_{j, k = 0}^{\infty} \langle f, \psi^0_{jk} \rangle_{L^2(\mathbb{H}/\Gamma)} \psi^0_{jk}
\]
where the series converges in \( L^2 \). See \([33], \text{Ch. 4}\)

The next result is a consequence of the fact that \( I^\text{red} \) is convolution by a compactly supported distribution, but we proceed explicitly.

**Proposition 10.**
\[
I^\text{red} : L^2(\mathbb{H}/\Gamma) \to L^2(\mathbb{H}/\Gamma)
\]
is well-defined and bounded.

**Proof.** For \( g \in C_c(\mathbb{H}/\Gamma) \),
\[
|I^\text{red}g(z, t)| \leq \int_0^{2\pi} |g((z, t)(e^{i\theta}, \theta/2))| d\theta \leq 2\pi |g((z, t)(e^{i\theta_0}, \theta_0/2))|, \quad \text{for some } \theta_0 \in [0, 2\pi].
\]

Then
\[
||I^\text{red}g||_2^2(\mathbb{H}/\Gamma) = \int_{\mathbb{H}/\Gamma} |I^\text{red}g(z, t)|^2 d(z, t) \Gamma
\]
\[
= (2\pi)^2 \int_{\mathbb{H}/\Gamma} |g((z, t)(e^{i\theta_0}, \theta_0/2))|^2 d(z, t) \Gamma
\]
\[
= (2\pi)^2 \int_{\mathbb{H}/\Gamma} |g((z, t))|^2 d(z, t) \Gamma, \quad \text{by left translation,}
\]
so \( I^\text{red} \) extends to a bounded function on from \( L^2(\mathbb{H}/\Gamma) \) to itself. \( \square \)
Proposition 11. $I_{\text{red}}$ preserves the orthogonal decomposition in Lemma 5 i.e.

$$I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{C})} : L^2(\mathbb{C}) \to L^2(\mathbb{C})$$

$$I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{H}/\Gamma)} : 0^0 L^2(\mathbb{H}/\Gamma) \to 0^0 L^2(\mathbb{H}/\Gamma).$$

Furthermore, the restriction $I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{C})}$ is essentially $2\pi$ times the Mean Value Transform $M^1$.

Proof. For $f \in L^2(\mathbb{C})$,

$$I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{C})} f(z,t) = \int_0^{2\pi} f((z,t)(e^{i\theta}, \theta/2)) d\theta = \int_0^{2\pi} f(z + e^{i\theta}, t + \theta/2 + \tfrac{1}{2} \text{Im}(\psi e^{i\theta})) d\theta$$

$$= \int_0^{2\pi} f(z + e^{i\theta}) d\theta = 2\pi M^1 f(z),$$

and so $I_{\text{red}} f \in L^2(\mathbb{C})$.

For $g \in 0^0 L^2(\mathbb{H}/\Gamma)$,

$$\int_0^{\pi} I_{\text{red}} g(z,t) dt = \int_0^{2\pi} g(z + e^{i\theta}, t + \theta/2 + \tfrac{1}{2} \text{Im}(\psi e^{i\theta})) d\theta dt = \int_0^{2\pi} \int_0^{\pi} g(z + e^{i\theta}, t) dtd\theta = 0,$$

so that $I_{\text{red}} g \in 0^0 L^2(\mathbb{H}/\Gamma)$.

We know that $I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{C})} = 2\pi M^1$ has a continuous spectrum (see (2), or Remark 7), so we restrict the reduced X-ray transform to $0^0 L^2(\mathbb{H}/\Gamma)$, where it has a discrete spectrum, and compute the Singular Value Decomposition there.

Theorem 5 (SVD of $I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{H}/\Gamma)}$). Furthermore

$$I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{H}/\Gamma)} \psi_{jk}^n = 2\pi \sqrt{\frac{j!}{(j+n)!}} (n/e)^{n/2} L_j^{(n)}(n) \psi_{j+n,k}^n$$

Proof. Note that, for $g_1, g_2 \in \mathbb{H}$

$$M_j^n (g_1 g_2) = \langle \beta_n (g_1 g_2) \omega_j, \omega_k \rangle = \langle \beta_n (g_1) \circ \beta_n (g_2) \omega_j, \omega_k \rangle = \sum_{l=0}^{\infty} \langle \beta_n (g_1) \omega_l, \omega_k \rangle \langle \beta_n (g_2) \omega_j, \omega_l \rangle = \sum_{l=0}^{\infty} M_j^n (g_2) M_j^n (g_1).$$

Then

$$I_{\text{red}}^{\text{red}}|_{L^2(\mathbb{H}/\Gamma)} \psi_{jk}^n(z,t) = \sqrt{\frac{2\pi}{2\pi}} \int_0^{2\pi} M_j^{2n} ((z,t)(e^{i\theta}, \theta/2)) d\theta$$

$$= \sqrt{\frac{2\pi}{2\pi}} \sum_{l=0}^{\infty} \int_0^{2\pi} M_j^{2n} (e^{i\theta}, \theta/2) M_j^{2n} (z,t) d\theta$$

$$= \sqrt{\frac{2\pi}{2\pi}} \sum_{l=0}^{\infty} \delta(j - l + n) M_j^{2n} (1, 0) M_j^{2n} (z,t),$$

by (21) in Proposition 4

$$= M_j^{2n} + M_j^{2n} (1, 0) \psi_{j+n,k}^n(z,t)$$

$$= 2\pi \sqrt{\frac{j!}{(j+n)!}} (n/e)^{n/2} L_j^{(n)}(n) \psi_{j+n,k}^n(z,t).$$
6.2 Exponential Map for Heisenberg Geodesics

The sub-Riemannian flow maps from the unit cotangent bundle $U^* \mathbb{H} := H^{-1}(\frac{1}{2})$ to itself. We work in the left-trivialization of the unit cotangent bundle: $U^* \mathbb{H} \cong \mathbb{H} \times U(1) \ni (z, t, e^{i\phi}, \lambda)$. The exponential map $\exp : \mathbb{R} \times U^* \mathbb{H} \to \mathbb{H}$ is given in these coordinates by

$$
\exp_{(z,t)} \left( s(e^{i\phi}, \lambda) \right) = (z,t) \begin{pmatrix} e^{i\phi \left( e^{\lambda s} - 1 \right) / \lambda} ; \lambda \sin(e^{\lambda s}) \\ se^{i\phi} ; 0 \end{pmatrix} \right) \begin{pmatrix} \lambda \neq 0 \\ \lambda = 0 \end{pmatrix}.
$$

(see [22, Ch. 1]), which describes the unit-speed geodesic with initial point $(z, t)$ whose projection to the plane is a counterclockwise circle of radius $R = 1/|\lambda|$, and has initial velocity in the direction of $\phi$ if $\lambda > 0$, and $\phi + \pi$ if $\lambda < 0$. If $\lambda = 0$ this projection is a straight line in the direction $\phi$. The geodesics in (1) are obtained by rotations and left translation of (40).

The Riemannian exponential map $\exp^r$ for $g$ is given in the same coordinates by

$$
\exp^r_{(z,t)} \left( s(e^{i\phi}, \lambda) \right) = \exp_{(z,t)} \left( s(e^{i\phi}, \lambda) \right) \left( 0, e^{2\lambda s} \right)
$$

(see [22, Thm. 11.8] for an explanation). Because we are using cylindrical coordinates in the fibers, neither of these exponential maps describe geodesics with initial condition strictly in the $\lambda$ direction. In the case of $g$, these geodesics are fixed points in $\mathbb{H}$, and in the case of $g^r$, these geodesics are integral curves of the Reeb vector field $e^{2\lambda T}$. In both cases, the X-ray transforms are inverted without considering these geodesics.

6.3 Bessel Functions

The classical Bessel function of order $n$ is defined by

$$
J_n(r) := \frac{1}{2\pi i^n} \int_0^{2\pi} e^{ir \cos \theta} e^{-in\theta} d\theta.
$$

6.4 Infinitesimal Representation

Define the complex vector fields on $\mathbb{H}$

$$
Z := \frac{1}{2} (X - iY), \quad \overline{Z} := \frac{1}{2} (X + iY)
$$

where $X$ and $Y$ are given in (5). Then $\beta_h : \mathbb{H} \to \mathcal{U}(\mathcal{H})$ is the unique strongly continuous unitary representation of $\mathbb{H}$ on $\mathcal{H}$ for which, on the level of Lie algebras,

$$
\left( \beta_h \right)_* Z = \sqrt{h} \partial_\zeta, \quad \left( \beta_h \right)_* \overline{Z} = -\sqrt{h} \zeta.
$$

(42)

Note that $\zeta$ and $\partial_\zeta$ are the creation and annihilation operators on $\mathcal{H}$. One may check that

$$
\beta_h(z, t) = e^{2\sqrt{h} t + \sqrt{h} \partial_\zeta - \sqrt{h} \zeta}
$$

is the same as (7).

6.5 Alternate Conventions

Folland [6] defines the Bargmann-Fock representation on the 1-parameter family of Hilbert spaces

$$
\mathcal{H}^h := \left\{ F \in \text{Hol}(\mathbb{C}) : h \int_\mathbb{C} |u(\zeta)|^2 e^{-\pi h |\zeta|^2} d\zeta < \infty \right\}, \quad h > 0,
$$

and $\mathcal{H}^b := \left\{ F : \mathcal{F} \in \mathcal{H}^{h(b)} \right\}$ for $h < 0$.

For $h \in \mathbb{R}^*$ and $\lambda > 0$, the maps

$$
S_\lambda : \mathcal{H}^h \to \mathcal{H}^{h\lambda}; \quad S(F)(\zeta) := F(\sqrt{\lambda} \zeta)
$$

and

$$
c : \mathcal{H}^h \to \mathcal{H}^{-h}; \quad c(F) := \mathcal{F}
$$

are unitary.
are all isometries.

Folland defines the Fock representation, for $h > 0$, as

$$\beta_{h}^{\text{Fol}}(z, t)F(\zeta) := e^{2\pi hit - \pi h\zeta - \pi h|z|^2/2}F(\zeta + z), \quad F \in \mathcal{H}^h$$

and $\beta_{h}^{\text{Fol}}(z, t) = c \circ \beta_{|h|}^{\text{Fol}}(\zeta, -t) \circ c$ for $h < 0$.

Our definition is rescaled so that every $\beta_h$ acts on the same space $\mathcal{H} = \mathcal{H}^{1/\pi}$. Folland’s definition, $\beta_{h}^{\text{Fol}}$, is related to ours via

$$\beta_{h}^{\text{Fol}}(z, t) = S_{\pi h} \circ \beta_{h}(z, t) \circ S^{-1}_{\pi h}, \quad h > 0.$$  

An advantage of this convention is that as $h$ varies, $\beta_h$ varies by precomposition with automorphisms of $\mathbb{H}$:

$$\begin{align*}
\beta_{h}(z, t) &= \beta_{1}(\sqrt{h}z, ht), \quad \text{for } h > 0 \\
\beta_{h}(z, t) &= \beta_{|h|}(\overline{z}, -t), \quad \text{for } h < 0.
\end{align*}$$

Granted, an advantage of Folland’s definition is that the Fourier transform defined with $\beta_{h}^{\text{Fol}}$ does “converge” to the Euclidean Fourier transform as $h \to 0$.

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