On the equivalence of certain quasi-Hermitian varieties

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**Abstract**
By Aguglia et al., new quasi-Hermitian varieties \(\mathcal{M}_{\alpha,\beta}\) in \(PG(r, q^2)\) depending on a pair of parameters \(\alpha, \beta\) from the underlying field \(GF(q^2)\) have been constructed. In the present paper we study the structure of the lines contained in \(\mathcal{M}_{\alpha,\beta}\) and consequently determine the projective equivalence classes of such varieties for \(q\) odd and \(r = 3\). As a byproduct, we also prove that the collinearity graph of \(\mathcal{M}_{\alpha,\beta}\) is connected with diameter 3 for \(q \equiv 1 \pmod{4}\).

**KEYWORDS**
collineation, Hermitian variety, quasi-Hermitian variety

1 INTRODUCTION

It is a well-known problem in finite geometry to characterize the absolute points of a polarity in terms of their combinatorial properties. In this line of investigation, one of the most celebrated results is Segre's Theorem stating that in a Desarguesian projective plane \(PG(2, q)\) of odd order \(q\) a set \(\Omega\) which has the same number of points, namely, \(q + 1\), and the same intersections with lines as a conic (i.e., 0, 1, or 2) is indeed a conic; see [15].

As the dimension grows, the combinatorics of the intersection with subspaces turns out not to be enough as to characterize the absolute points of a polarity neither in the orthogonal case nor in the unitary one.

The set of the absolute points of a Hermitian polarity of \(PG(r, q^2)\) is a *nonsingular Hermitian variety*.

Quasi-Hermitian varieties of \(PG(r, q^2)\) are a generalization of nonsingular Hermitian varieties defined as follows. Let \(q\) be any prime power and assume \(r \geq 2\); a *quasi-Hermitian variety* of \(PG(r, q^2)\) is a set of points having the same size and the same intersection numbers with hyperplanes as a nonsingular Hermitian variety \(\mathcal{H}(r, q^2)\). In particular, the intersection numbers with hyperplanes of \(\mathcal{H}(r, q^2)\) only take two values thus, quasi-Hermitian varieties are two-character sets; see [8, 9] for an overview of their applications. The Hermitian variety
\( \mathcal{H}(r, q^2) \) can be viewed trivially as a quasi-Hermitian variety; as such it is called the classical quasi-Hermitian variety of \( \text{PG}(r, q^2) \).

For \( r = 2 \), a quasi-Hermitian variety of \( \text{PG}(2, q^2) \) is called a unital or Hermitian arc. Nonclassical unitals have been extensively studied and characterized \([6]\) and many constructions are known; see, for instance, \([4]\). As far as we know, the only known nonclassical quasi-Hermitian varieties of \( \text{PG}(r, q^2), r \geq 3 \) were constructed in \([2, 3, 10, 14]\) and they are not isomorphic among themselves; see \([14]\).

In \([3]\), quasi-Hermitian varieties \( \mathcal{M}_{\alpha, \beta} \) of \( \text{PG}(3, q^2) \) with \( r \geq 2 \), depending on a pair of parameters \( \alpha, \beta \) from the underlying field \( \text{GF}(q^2) \), were constructed. For \( r = 2 \) these varieties are Buekenhout–Metz (BM) unitals, see \([5, 6, 11, 12]\). As such, for \( r \geq 3 \) we shall call \( \mathcal{M}_{\alpha, \beta} \) the BM quasi-Hermitian varieties of parameters \( \alpha \) and \( \beta \) of \( \text{PG}(r, q^2) \). The number of projectively inequivalent BM unitals in \( \text{PG}(2, q^2) \) has been computed in \([5]\) for \( q \) odd and in \([11]\) for \( q \) even. In the present paper we shall enumerate the BM quasi-Hermitian varieties in spaces of the same dimension and order.

Apart from the Introduction, this paper is organized into four sections. In Section 2 we describe the construction of the BM quasi-Hermitian varieties in \( \text{PG}(3, q^2) \) whereas in Section 3 we determine the number of lines of \( \text{PG}(3, q^2) \) through a point of \( \mathcal{M}_{\alpha, \beta} \) which are entirely contained in \( \mathcal{M}_{\alpha, \beta} \). By using this result in Section 4, we prove that the collinearity graph of \( \mathcal{M}_{\alpha, \beta} \) is connected for \( q \equiv 1 \pmod{4} \) (which is the only interesting case, as for \( q \equiv 3 \pmod{4} \) the only lines contained in \( \mathcal{M}_{\alpha, \beta} \) are those of a pencil of (\( q + 1 \))-lines, all contained in a plane). Finally, in Section 5, we prove our main result:

**Theorem 1.1.** Let \( q = p^n \) with \( p \) an odd prime. Then the number \( N \) of projectively inequivalent quasi-Hermitian varieties \( \mathcal{M}_{\alpha, \beta} \) of \( \text{PG}(3, q^2) \) is

\[
N = \frac{1}{n} \left( \sum_{k|n} \Phi \left( \frac{n}{k} \right) p^k \right) - 2,
\]

where \( \Phi \) is the Euler \( \Phi \)-function.

As a byproduct of our arguments, we also obtain a simple way to determine when two quasi-Hermitian varieties are equivalent, see Lemmas 5.5 and 5.6 for the details.

## 2 | PRELIMINARIES

In this section we recall the construction of the BM quasi-Hermitian varieties \( \mathcal{M}_{\alpha, \beta} \) of \( \text{PG}(3, q^2) \) described in \([3]\).

Fix a projective frame in \( \text{PG}(3, q^2) \) with homogeneous coordinates \( (J, X, Y, Z) \), and consider the affine space \( \text{AG}(3, q^2) \) with infinite hyperplane \( \Sigma_{\infty} \) of equation \( J = 0 \). Then, the affine coordinates for points of \( \text{AG}(3, q^2) \) are denoted by \( (x, y, z) \), where \( x = X/J, y = Y/J, \) and \( z = Z/J \). Set

\[
\mathcal{F} = \{(0, X, Y, Z) : X^{q+1} + Y^{q+1} = 0\};
\]
this can be viewed as a Hermitian cone of \( \Sigma_\infty \cong \text{PG}(2, q^2) \) projecting a Hermitian variety of \( \text{PG}(1, q^2) \). Now take \( \alpha \in \text{GF}(q^2)^* \) and \( \beta \in \text{GF}(q^2) \setminus \text{GF}(q) \) and consider the algebraic variety \( B_{\alpha, \beta} \) of projective equation

\[
B_{\alpha, \beta} : Z^qJ^q - ZJ^{2q-1} + \alpha^q(X^{2q} + Y^{2q}) - \alpha(X^2 + Y^2)J^{2q-2} = (\beta^q - \beta)(X^{q+1} + Y^{q+1})J^{-1}.
\]

We observe that

- \( B_\infty := B_{\alpha, \beta} \cap \Sigma_\infty \) is the union of two lines \( \ell_1 : X - \nu Y = 0 = J \) and \( \ell_2 : X + \nu Y = 0 = J \), with \( \nu \in \text{GF}(q^2) \) such that \( \nu^2 + 1 = 0 \) if \( q \) is odd.
- Let \( P_\infty := (0, 0, 0, 1) \). Then, \( \ell_1 \cap \ell_2 = P_\infty \).
- \( B_\infty \subseteq \mathcal{F} \) if \( q \equiv 1 \pmod{4} \) or \( q \) is even.

It is shown in [3] that the point set

\[
\mathcal{M}_{\alpha, \beta} := (B_{\alpha, \beta} \setminus \Sigma_\infty) \cup \mathcal{F},
\]

that is, the union of the affine points of \( B_{\alpha, \beta} \) and \( \mathcal{F} \), is a quasi-Hermitian variety of \( \text{PG}(3, q^2) \) for any \( q > 2 \) even or for \( q \) odd and \( 4\alpha^{q+1} + (\beta^q - \beta)^2 \neq 0 \). This is the variety we shall consider in the present paper limited to the case in which \( q \) is odd.

We stress that (1) is not the equation of \( \mathcal{M}_{\alpha, \beta} \). However, any set of points in a finite projective space can be endowed of the structure of an algebraic variety, so we shall speak of the variety \( \mathcal{M}_{\alpha, \beta} \) even if we do not provide an equation for it.

## 3 COMBINATORIAL PROPERTIES OF \( \mathcal{M}_{\alpha, \beta} \)

We first determine the number of lines passing through each point of \( \mathcal{M}_{\alpha, \beta} \) of \( \text{PG}(3, q^2) \), for \( q \) odd. We recall the following (see [13, Corollary 1.24]).

**Lemma 3.1.** Let \( q \) be an odd prime power. The equation

\[
X^q + aX + b = 0
\]

admits exactly one solution in \( \text{GF}(q^2) \) if and only if \( a^{q+1} \neq 1 \). When \( a^{q+1} = 1 \), the aforementioned equation has either \( q \) solutions when \( b^q = a^q b \) or no solution when \( b^q \neq a^q b \).

**Lemma 3.2.** Let \( B_{\alpha, \beta} \) be the projective variety of Equation (1) and \( B_\infty \) be the intersection of the variety \( B_{\alpha, \beta} \) with the hyperplane at infinity \( \Sigma_\infty : J = 0 \) of \( \text{PG}(3, q^2) \).

- If \( q \equiv 1 \pmod{4} \), then, for any affine point \( L \) of \( B_{\alpha, \beta} \) there are exactly two lines contained in \( B_{\alpha, \beta} \) through \( L \); for any point \( L_\infty \in B_\infty \) with \( L_\infty \neq P_\infty \), there are \( q + 1 \) lines of a pencil through \( L_\infty \) contained in \( B_{\alpha, \beta} \). If \( q \equiv 3 \pmod{4} \) then no line of \( B_{\alpha, \beta} \) passes through any affine point of \( B_{\alpha, \beta} \) whereas through a point at infinity of \( B_\infty \) different from \( P_\infty \) there pass only one line contained in \( B_{\alpha, \beta} \).
There are exactly two lines of $B_{\alpha,\beta}$ through $P_{\infty}$ for all odd $q$.

**Proof.** Let $\ell$ be a line of $\text{PG}(3, q^2)$ passing through an affine point of $B_{\alpha,\beta}$. The affine points $P(x, y, z)$ of $B_{\alpha,\beta}$ satisfy the equation:

$$B_{\alpha,\beta} : z^q - z + \alpha^q(x^{2q} + y^{2q}) - \alpha(x^2 + y^2) = (\beta^q - \beta)(x^{q+1} + y^{q+1}).$$

From [3, Section 4], it can be directly seen that the collineation group of $B_{\alpha,\beta}$ acts transitively on its affine points. Thus, we can assume that $\ell$ passes through the origin $O = (1, 0, 0, 0)$ of the fixed frame and hence it has affine parametric equations:

$$\begin{align*}
  x &= m_1 t, \\
  y &= m_2 t, \\
  z &= m_3 t.
\end{align*}$$

with $t$ ranging over $\text{GF}(q^2)$. We study the following system:

$$\begin{align*}
  z^q - z + \alpha^q(x^{2q} + y^{2q}) - \alpha(x^2 + y^2) &= (\beta^q - \beta)(x^{q+1} + y^{q+1}), \\
  x &= m_1 t, \\
  y &= m_2 t, \\
  z &= m_3 t.
\end{align*}$$

As proved in [1, Theorem 4.3] $\ell$ can be contained in $B_{\alpha,\beta}$ only if $m_3 = 0$. Thus assume $m_3 = 0$ and replace the parametric values of $(x, y, z)$ in the first equation of (3). We obtain that

$$\left(t^2\alpha\left(m_1^2 + m_2^2\right)\right)^q - t^2\alpha\left(m_1^2 + m_2^2\right) = t^{q+1}(\beta^q - \beta)(m_1^{q+1} + m_2^{q+1})$$

must hold for all $t \in \text{GF}(q^2)$. Considering separately the cases $t \in \text{GF}(q)$ and $t = \lambda$ with $\lambda \in \text{GF}(q^2) \setminus \text{GF}(q)$ we obtain the following system, $\forall \lambda \in \text{GF}(q^2) \setminus \text{GF}(q)$:

$$\begin{align*}
  \left(\alpha^q\left(m_1^2 + m_2^2\right)\right)^q - \alpha\left(m_1^2 + m_2^2\right) &= (\beta^q - \beta)(m_1^{q+1} + m_2^{q+1}), \\
  \lambda^2\alpha^q\left(m_1^2 + m_2^2\right)^q - \lambda^2\alpha\left(m_1^2 + m_2^2\right) &= \lambda^{q+1}(\beta^q - \beta)(m_1^{q+1} + m_2^{q+1}).
\end{align*}$$

Replacing the first equation in the second, we get

$$\forall \lambda \in \text{GF}(q^2) \setminus \text{GF}(q) : \lambda^2\alpha^q\left(m_1^2 + m_2^2\right)^q(1 - \lambda^{1-q}) = \lambda^2\alpha\left(m_1^2 + m_2^2\right)(1 - \lambda^{q-1}).$$

Observe that $(1 - \lambda^{1-q}) = \frac{\lambda^{q-1} - 1}{\lambda^{q-1}}$. Suppose $m_1^2 + m_2^2 \neq 0$. Then,
\[ \lambda^{2q-2} \alpha^{q-1} \left( m_1^2 + m_2^2 \right)^{q-1} = -\lambda^{q-1}, \]

whence \( \left( \lambda \alpha \left( m_1^2 + m_2^2 \right) \right)^{q-1} = -1 \) for all \( \lambda \in \text{GF}(q^2) \setminus \text{GF}(q) \). This is clearly not possible, as the equation \( X^{q-1} = -1 \) cannot have more than \( q - 1 \) solutions. So \( m_1^2 + m_2^2 = 0 \), which yields \( m_2 = \pm \nu m_1 \) where \( \nu^2 = -1 \). On the other hand, if \( m_2 = \pm \nu m_1 \) and \( q \equiv 1 \pmod{4} \), then \( m_1^{q+1} + m_2^{q+1} = m^{q+1}(1 + \nu^{q+1}) = 0 \), so (4) is satisfied and the lines \( \ell : y - \nu x = z = 0 \) and \( \ell' : y + \nu x = z = 0 \) are contained in \( \mathcal{B}_{\alpha, \beta} \). On the other hand, if \( q \equiv 3 \pmod{4} \), then \( m_1^{q+1} + m_2^{q+1} = 2m^{q+1} \neq 0 \); so (4) is not satisfied and there is no line contained in \( \mathcal{B}_{\alpha, \beta} \).

Now, take \( L_\infty = (0, a, b, c) \in \mathcal{B}_\infty \setminus \{P_\infty\} \); hence \( a^2 + b^2 = 0 \) and \( a, b \neq 0 \). Let \( r \) be a line through \( L_\infty \). We may assume that \( r \) has (affine) parametric equations

\[
\begin{cases}
x = l + at, \\
y = m + bt, \\
z = n + ct,
\end{cases}
\]

where \( t \) ranges over \( \text{GF}(q^2) \). Assume also that \( (l, m, n) \) are the affine coordinates of a point in \( \mathcal{B}_{\alpha, \beta} \), that is,

\[ n^q - n + \alpha q (m^{2q} + l^{2q}) - \alpha (m^2 + l^2) = (\beta^q - \beta)(l^{q+1} + m^{q+1}). \] (5)

Now, \( r \) is contained in \( \mathcal{B}_{\alpha, \beta} \) if and only if \( q \equiv 1 \pmod{4} \) and the following condition holds:

\[ c + 2\alpha (al + bm) + (\beta - \beta^q)(al^{q} + bm^{q}) = 0. \] (6)

Since \( b = \nu a \) where \( \nu^2 = -1 \) and \( \nu \in \text{GF}(q) \), setting \( k = l + \nu m \) Equation (6) becomes

\[ c + 2\alpha k + a(\beta - \beta^q)k^q = 0. \] (7)

From Lemma 3.1, the above equation has exactly one solution if and only if

\[ (2\alpha)^{q+1} \neq (\beta - \beta^q)^{q+1}. \] (8)

Considering that \( 2 \in \text{GF}(q) \) and \( (\beta - \beta^q)^q = (\beta^q - \beta) \), we obtain that (8) is equivalent to

\[ 4\alpha^{q+1} + (\beta^q - \beta)^2 \neq 0, \]

which holds true.

Let \( \bar{k} \) be the unique solution of (7). Since \( \bar{k} = l + \nu m \), we find \( q^2 \) pairs \( (l, m) \) satisfying (6). For any fixed pair \( (l, m) \), because of (5), there are \( q \) possible values of \( n \). Thus we obtain that the number of affine lines through the point \( P \) and contained in \( \mathcal{B}_{\alpha, \beta} \) is \( q^2 q^{2} / q^2 = q \).

Furthermore, the \( q + 1 \) lines through \( L_\infty \) lie on the plane of (affine) equation: \( x + \nu y = \bar{k} \). The theorem follows. \( \square \)
Lemma 3.3. If \( q \equiv 1 \pmod{4} \) then, for any point \( P \in B_{\alpha,\beta} \setminus B_{\infty} \) there are two lines \( r_1(P) \) and \( r_2(P) \) through \( P \) contained in \( B_{\alpha,\beta} \) such that \( r_1(P) \cap (E \setminus \{P\}) \neq \emptyset \).

Proof. As we already know, \( B_{\infty} \) is the union of two lines \( \ell_1 \) and \( \ell_2 \), through the point \( P_{\infty} \). By considering (4), we see that the point at infinity of the two lines through the origin \( P \in O_{\alpha,\beta} \), are one on \( \ell_1 \) and one on \( \ell_2 \). The semilinear automorphism group \( G \) of \( \mathcal{M}_{\alpha,\beta} \) is transitive on its affine points, see [3] and maps lines into lines. Also, by Lemma 3.2 follows directly that \( G \) must fix the hyperplane at infinity. If \( q \equiv 3 \pmod{4} \), the point \( P_{\infty} \) is the only point of \( B_{\alpha,\beta} \) incident with two lines therein contained. So \( P_{\infty} \) must be stabilized by \( G \). If \( q \equiv 1 \pmod{4} \), we see that \( P_{\infty} \) is the only point at infinity incident with just two lines of the variety, while the remaining points at infinity are incident with \( q + 1 \) lines. So, again \( P_{\infty} \), which is the only point of \( B_{\infty} \) which is on no affine line of \( B_{\alpha,\beta} \), is fixed by \( G \). It follows that for each affine point \( P \) we have that one of the lines intersects with \( \ell_1 \) and the other with \( \ell_2 \). □

Theorem 3.4. Let \( \mathcal{M}_{\alpha,\beta} \) be the BM quasi-Hermitian variety described in (2).

• If \( q \equiv 1 \pmod{4} \) then through each affine point of \( \mathcal{M}_{\alpha,\beta} \) there pass two lines of \( \mathcal{M}_{\alpha,\beta} \) whereas through a point at infinity of \( \mathcal{M}_{\alpha,\beta} \) on the union of the two lines \( \ell_1 \cup \ell_2 \) there pass \( q + 1 \) lines of a pencil contained in \( \mathcal{M}_{\alpha,\beta} \); finally through a point at infinity incident with just two lines of the variety, while the remaining points at infinity are incident with \( q + 1 \) lines. So, again \( P_{\infty} \), which is the only point of \( B_{\infty} \) which is on no affine line of \( B_{\alpha,\beta} \), is fixed by \( G \). It follows that for each affine point \( P \) we have that one of the lines intersects with \( \ell_1 \) and the other with \( \ell_2 \).

• Through the point \( P_{\infty} \) there are always \( q + 1 \) lines contained in \( \mathcal{M}_{\alpha,\beta} \).

Proof. We observe that the affine points of \( \mathcal{M}_{\alpha,\beta} \) are the same as those of \( B_{\alpha,\beta} \), whereas the set \( \mathcal{F} \) of points at infinity of \( \mathcal{M}_{\alpha,\beta} \) consists of the points \( P = (0, x, y, z) \) such that \( x^{q+1} + y^{q+1} = 0 \). Furthermore, \( B_{\infty} = \ell_1 \cup \ell_2 \) is contained in \( \mathcal{F} \) if \( q \equiv 1 \pmod{4} \). Hence, from Lemma 3.2 we get the result. □

4 | CONNECTED GRAPHS FROM \( \mathcal{M}_{\alpha,\beta} \) IN PG(3, \( q^2 \), \( q \equiv 1 \pmod{4} \))

Let \( \mathcal{V} \) be an algebraic variety in PG\((n - 1, q_2)\) or, more in general, just a set of points and suppose that \( \mathcal{V} \) contains some projective lines. Then we can define the collinearity graph of \( \mathcal{V} \), say \( \Gamma(\mathcal{V}) = (\mathcal{P}, \mathcal{E}) \) as the graph whose vertices \( \mathcal{P} \) are the points of \( \mathcal{V} \) and such that two points \( P \) and \( Q \) are collinear in \( \Gamma(\mathcal{V}) \) if and only if the line \( \langle P, Q \rangle \) is contained in \( \mathcal{V} \).

When \( \mathcal{V} \) is a (nondegenerate) quadric or Hermitian variety, the graph \( \Gamma(\mathcal{V}) \) has a very rich structure for it is strongly regular and admits a large automorphism group; this has been widely investigated; see [7, Chapter 2, 16].

More in general, the properties of the graph \( \Gamma(\mathcal{V}) \) provide insight on the geometry of \( \mathcal{V} \) since any automorphism of \( \mathcal{V} \) is also naturally an automorphism of \( \Gamma(\mathcal{V}) \), even if the converse is not true in general.
Lemma 4.1. Let \( \mathcal{V} \) be an algebraic variety containing some lines and let \( \mathcal{V}_\infty = \mathcal{V} \cap \Sigma_\infty \) where \( \Sigma_\infty \) is a hyperplane of \( \text{PG}(n - 1, q^2) \). If the graph \( \Gamma(\mathcal{V}_\infty) \) is connected and through each point of \( \mathcal{V} \) there passes at least one line of \( \mathcal{V} \) then the collinearity graph \( \Gamma(\mathcal{V}) \) is connected and its diameter \( d(\Gamma(\mathcal{V})) \) is at most \( d(\Gamma(\mathcal{V}_\infty)) + 2 \).

Proof. Each line of \( \mathcal{V} \) has at least a point at infinity hence, given two points \( P \) and \( Q \) there exists a path from \( P \) to a point at infinity \( P' \) and from \( Q \) to another point at infinity \( Q' \) and finally a path consisting of points in \( \mathcal{V}_\infty \) from \( P' \) to \( Q' \).

Let \( \mathcal{M}_{\alpha,\beta} \) be as in (2).

Theorem 4.2. If \( q \equiv 1 \pmod{4} \), then the graph \( \Gamma(\mathcal{M}_{\alpha,\beta}) \) is connected and its diameter is 3.

Proof. We recall that \( B_{\alpha,\beta} \subseteq \mathcal{M}_{\alpha,\beta}, B_{\alpha,\beta} \setminus B_{\infty} = \mathcal{M}_{\alpha,\beta} \setminus \Sigma_\infty \) and that \( B_{\infty} \) splits in the union of the two distinct lines \( \ell_1, \ell_2 \) through \( P_{\infty} \). In particular, \( \Gamma(B_{\infty}) \) is a connected graph of diameter 2. Take now two points \( P, Q \in B_{\alpha,\beta} \). If \( P, Q \in B_{\infty} \), then we have \( d(P, Q) \leq 2 \) and there is nothing to prove. Suppose now \( P \in B_{\alpha,\beta} \setminus B_{\infty} \) and \( Q \in B_{\infty} \). Suppose \( Q \in \ell_1 \). Then, from Lemma 3.3 we can consider a point \( P' = \eta(P) \cap \ell_1 \) where \( \eta(P) \) is one of the two lines through \( P \) which is contained in \( B_{\alpha,\beta} \). If \( P' = Q \), then \( d(P, Q) = 1 \); otherwise \( d(P, Q) = 2 \).

Take now \( P, Q \in B_{\alpha,\beta} \setminus B_{\infty} \). Then, again from Lemma 3.3, the lines \( \eta(P) \) and \( \eta(Q) \) meet \( \ell_1 \). Let \( P' = \eta(P) \cap \ell_1 \) and \( Q' = \eta(Q) \cap \ell_1 \). If \( P' = Q' \), then \( d(P, Q) \leq 2 \); otherwise \( d(P, Q) \leq 3 \). We now show that there are pairs of points in \( \mathcal{M}_{\alpha,\beta} \) which are at distance 3. Take \( P \in B_{\alpha,\beta} \setminus B_{\infty} \) and \( Q \in M_{\alpha,\beta} \setminus B_{\alpha,\beta} \). Then, \( Q \) is not collinear with any affine point by construction; also \( Q \) is not collinear with \( P_i := \ell_i \cap \eta(P), i = 1, 2 \). So, the shortest paths from \( P \) to \( Q \) are of the form \( P\ell_i P_i Q \). It follows that \( d(P, Q) = 3 \) and thus the diameter of the graph is 3.

5 \ MAIN RESULT

In this section we show that the arguments of [5] for classifying BM unitals in \( \text{PG}(2, q^2) \) can be extended to BM quasi-Hermitian varieties in \( \text{PG}(3, q^2), q \) odd. We keep all previous notations.

Two BM quasi-Hermitian varieties \( \mathcal{M}_{\alpha,\beta} \) and \( \mathcal{M}_{\alpha',\beta'} \) of \( \text{PG}(3, q^2) \) are projectively equivalent if there exists a semilinear collineation \( \psi \in \text{PG}(4, q^2) \) such that \( \psi(\mathcal{M}_{\alpha,\beta}) = \mathcal{M}_{\alpha',\beta'} \).

Lemma 5.1. Let \( \psi \) be a semilinear collineation of \( \text{PG}(3, q^2), q \) odd, such that \( \psi(\mathcal{M}_{\alpha,\beta}) = \mathcal{M}_{\alpha',\beta'} \) where \( \mathcal{M}_{\alpha,\beta} \) and \( \mathcal{M}_{\alpha',\beta'} \) are two BM quasi-Hermitian varieties. Then \( \psi \) fixes \( P_{\infty} \) and stabilizes \( \Sigma_{\infty} \). Also, if \( q \equiv 1 \pmod{4} \) then \( \psi(B_{\alpha,\beta}) = B_{\alpha',\beta'} \).

Proof. First, we show that \( \psi \) fixes \( P_{\infty} \) for \( q \equiv 3 \pmod{4} \). From Theorem 3.4 we have that \( P_{\infty} \) is the only point of the two varieties contained in \( q + 1 \) lines and hence \( \psi(P_{\infty}) = P_{\infty} \). Furthermore, we observe that \( \Sigma_{\infty} \) is the plane through \( P_{\infty} \) meeting both \( \mathcal{M}_{\alpha,\beta} \) and \( \mathcal{M}_{\alpha',\beta'} \) in \( q^3 + q^2 + 1 \) points which are on the \( q + 1 \) lines through \( P_{\infty} \). All of the \( q^3 - q^2 \) points of \( \mathcal{M}_{\alpha,\beta} \) and \( \mathcal{M}_{\alpha',\beta'} \) lying on exactly one line contained in the respective variety are in this plane, and these points also span \( \Sigma_{\infty} \). So also \( \Sigma_{\infty} \) is left invariant by \( \psi \).
Now assume \( q \equiv 1 \pmod{4} \); from Theorem 3.4, for each point in \( \ell_1 \cup \ell_2 \) there pass \( q + 1 \) lines of the quasi-Hermitian varieties however \( P_\infty \) is the only point on \( \ell_1 \cup \ell_2 \) such that the other \( q - 1 \) lines through it are not incident with other lines of the two varieties, hence we again obtain \( \psi(P_\infty) = P_\infty \). In this case \( B_{\alpha,\beta} \subseteq M_{\alpha,\beta} \). Since \( \psi(\Sigma_\infty) = \Sigma_\infty \), we have

\[
\psi(B_{\alpha,\beta}\setminus \Sigma_\infty) = \psi(M_{\alpha,\beta}\setminus \Sigma_\infty) = M_{\alpha',\beta'}\setminus \Sigma_\infty = B_{\alpha',\beta'}\setminus \Sigma_\infty,
\]

that is, \( \psi \) stabilizes the affine part of \( B_{\alpha,\beta} \).

Furthermore \( B_\infty = B_{\alpha,\beta} \cap \Sigma_\infty \) consists of the union of the two lines, \( \ell_1 \) and \( \ell_2 \). Observe also that the lines through the affine points of \( M_{\alpha,\beta} \) are also lines of \( B_{\alpha,\beta} \) (see Theorem 3.4) and, in particular they are incident either \( \ell_1 \) or \( \ell_2 \). This is equivalent to say that the points of \( \ell_1 \cup \ell_2 \) different from \( P_\infty \) are exactly the points of \( \Sigma_\infty \) through which there pass some affine lines of \( M_{\alpha,\beta} \). This implies that

\[
\psi(\ell_1 \cup \ell_2) = \ell_1 \cup \ell_2
\]

and, consequently

\[
\psi(B_{\alpha,\beta}) = \psi(B_{\alpha,\beta}\setminus \Sigma_\infty) \cup \psi(\ell_1 \cup \ell_2) = (M_{\alpha',\beta'}\setminus \Sigma_\infty) \cup (\ell_1 \cup \ell_2) = B_{\alpha',\beta'}.
\]

\[
\square
\]

**Theorem 5.2.** Suppose \( q \equiv 1 \pmod{4} \). Let \( G \) be the group of collineations \( G = \text{Aut}(\Gamma(M_{\alpha,\beta})) \subseteq \text{PGL}(4, q^2) \) and \( \Gamma \) the group of graph automorphisms \( \Gamma = \text{Aut}(\Gamma(M_{\alpha,\beta})) \). Then the sets

- \( \Omega_0 := \{P_\infty\} \);
- \( \Omega_1 \) consisting of the points at infinity of \( B_{\alpha,\beta} \) different from \( P_\infty \);
- \( \Omega_2 := M_{\alpha,\beta}\setminus \Sigma_\infty \)

are all stabilized by both \( G \) and \( \Gamma \). Furthermore, \( \Omega_3 = M_{\alpha,\beta}\setminus B_{\alpha,\beta} \) is an orbit for \( \Gamma \).

**Proof.** By [3, Section 4], we know that there is a subgroup of \( G \) which is transitive on the affine points of \( M_{\alpha,\beta} \), that is, on \( \Omega_2 \). By Lemma 5.1, any collineation in \( G \) must stabilize the plane \( \Sigma_\infty \); so any element of \( G \) maps points of \( \Omega_2 \) into points of \( \Omega_2 \) and \( \Omega_2 \) is an orbit of \( G \). Also by Lemma 5.1, \( \Omega_0 := \{P_\infty\} \) is fixed by any \( \gamma \in G \). So we have that the points at infinity of \( B_{\alpha,\beta}\setminus \{P_\infty\} \), as well as the points of \( M_{\alpha,\beta}\setminus B_{\alpha,\beta} \), are the union of orbits. Let \( \ell_1, \ell_2 \) be the two lines of \( B_{\alpha,\beta} \) at infinity. Using Lemma 3.3, we see that \( G \) is transitive on \( \Omega_1 = (\ell_1 \cup \ell_2)\setminus \{P_\infty\} \). Indeed, for any two points \( P, Q \in \ell_1 \setminus \{P_\infty\} \), by Lemma 3.2, there are points \( P_0, Q_0 \in \Omega_2 \) such that \( r_1(P_0) \cap \Sigma_\infty = \{P\} \) and \( r_1(Q_0) \cap \Sigma_\infty = \{Q\} \).

Since \( G \) is transitive on \( \Omega_2 \), there is \( \gamma \in G \) such that \( \gamma(P_0) = Q_0 \). It follows that \( \gamma((r_2(P_0) \cap \Sigma_\infty) \cup \{P\}) = (r_2(Q_0) \cap \Sigma_\infty) \cup \{Q\} \). If \( \gamma(P) = Q \), then we are done. Otherwise, consider the element \( \theta : (J, X, Y, Z) \rightarrow (J, X, -Y, Z) \) of \( G \). Observe that \( \theta(r_2(Q_0)) \cap \Sigma_\infty = r_2(Q_0) \cap \Sigma_\infty \). Hence, \( \theta r_2(P) = Q \). Also, \( \theta(\ell_1) = \ell_2 \); so it follows that \( \Omega_1 := (\ell_1 \cup \ell_2)\setminus \{P_\infty\} \) is an orbit of \( G \).

Since \( \Gamma \) contains \( G \), the orbits of \( \Gamma \) are possibly unions of orbits of \( G \). However, observe that the points of \( \Omega_3 \) are the only points of \( M_{\alpha,\beta} \) which are on exactly one line of \( M_{\alpha,\beta} \) through the point \( P_\infty \). So these points must be permuted among each other also by \( \Gamma \).

The same argument shows that \( \Omega_0 \) is also an orbit for \( \Gamma \). Now, consider the points of \( \Omega_2 \). They are the points of \( B_{\alpha,\beta}\setminus \Omega_0 \) incident with exactly two lines, while the points of \( \Omega_1 \)
are incident with more than two lines. So $\mathcal{G}$ cannot map a vertex in $\Omega_2$ into a vertex in $\Omega_1$ and these orbits are distinct.

Put $\Gamma := \Gamma(\mathcal{M}_{a,\beta})$. Observe that the graph $\Gamma \setminus \{P_\infty\}$ is the disjoint union of $\Gamma(\Omega_3)$ and $\Gamma(\Omega_1 \cup \Omega_2)$. In turn, $\Gamma(\Omega_3)$ consists of the disjoint union $K_1 \cup K_2 \cup \cdots \cup K_{q-1}$ of $q-1$ copies of the complete graph on $q^2$ elements. Write $\{v_j^l\}_{j=1, \ldots, q^2}$ for the list of vertices of $K_i$ with $i = 1, \ldots, q-1$.

Also, each vertex of $\Gamma(\Omega_3 \cup \{P_\infty\})$ is collinear with $P_\infty$. Let $S_{q^2}$ be the symmetric group on $q^2$ elements, and consider its action on $\Gamma$ given by

$$\forall \xi \in S_{q^2} : \xi(v_l^j) := v_l^j(\xi)$$

if $v_l^j \in K_i$ and fixing all remaining vertices. Obviously $\hat{S}_{q^2} < \mathcal{G}$ and $\hat{S}_{q-1}$ is transitive on $K_i$. Let $S_{q-1}$ be the symmetric group on $\{1, \ldots, q-1\}$ and consider its action on $\Gamma$ given by

$$\forall \sigma \in S_{q-1} : \sigma(v_l^j) := v_l^j(\sigma), j = 1, \ldots, q^2$$

and all the remaining vertices of $\Gamma$ are fixed. We also have $\hat{S}_{q-1} < \mathcal{G}$ and $\hat{S}_{q-1}$ permutes the sets $K_i$ for $i = 1, \ldots, q-1$. By construction, we see that the wreath product $\hat{S}_q \hat{S}_{q-1}$ is a subgroup of $\mathcal{G}$, it acts naturally on $\Gamma$, fixes all vertices not in $\Omega_3$ and acts transitively on $\Omega_3$. It follows that $\mathcal{G}$ is transitive on $\Omega_3$. \hfill $\square$

Remark 5.3. It can be easily seen that the automorphism group of $\Gamma := \Gamma(\mathcal{M}_{a,\beta})$ is in general much larger than the subgroup of collineations stabilizing $\mathcal{M}_{a,\beta}$. In particular the elements of $\hat{S}_q \hat{S}_{q-1}$ are not, in general, collineations. For instance, in the case $q = 5$ with $\alpha = \beta = \varepsilon$ where $\varepsilon$ is a primitive element of $GF(25)$, root of $x^2 - x + 2$ in $GF(5)$, the group $\mathcal{G}$ has order $2^65^5$, while $\mathcal{G}$ has order $2^{10}3^24^25^37^211^13^417^19^423^4$. In this case also $\mathcal{G}$ is transitive on $\Omega_3$.

Lemma 5.4. If $\mathcal{M}_{a,\beta}$ and $\mathcal{M}_{a',\beta'}$ are two projectively equivalent BM quasi-Hermitian varieties then there is a semilinear collineation $\phi : \mathcal{M}_{a,\beta} \rightarrow \mathcal{M}_{a',\beta'}$ of the following type:

$$\phi(j, x, y, z) = (j^\sigma, x^\sigma, y^\sigma, z^\sigma)M, \quad \text{where}$$

$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & -b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$\sigma \in \text{Aut}(GF(q^2))$, $a \in GF(q) \setminus \{0\}$, $b, c \in GF(q^2)$, $b^2 + c^2 \neq 0$ and if $b \neq 0 \neq c$ then $c = \lambda b$ with $\lambda \in GF(q) \setminus \{0\}$ such that $\lambda^2 + 1 \neq 0$.

Proof. By Lemma 5.1, $\phi$ fixes the point $P_\infty$ and stabilizes $\Sigma_\infty$. As the automorphism group of $\mathcal{M}_{a,\beta}$ is transitive on its affine points, we can also assume that $\phi(1, 0, 0, 0) = (1, 0, 0, 0)$. \hfill $\square$
More in detail, let $G'$ be the collineation group of $\mathcal{M}_{\alpha',\beta'}$ fixing $P_\infty$, leaving $\mathcal{F}_P^\infty$ invariant and transitive on the affine points of $\mathcal{M}_{\alpha',\beta'}$. If $\phi(1,0,0,0) \neq (1,0,0,0)$ we can consider the collineation $\phi' \in G'$ mapping $\phi(1,0,0,0)$ to $(1,0,0,0)$ and then we replace $\phi$ by $\phi\phi'$. This implies that $\phi$ has the following form up to scalar multiple:

$$\phi(j,x,y,z) = (j^\sigma,x^\sigma,y^\sigma,z^\sigma) \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & d \\ 0 & e & f & g \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\sigma \in \text{Aut}(\text{GF}(q^2))$, $a, b, c, d, e, f, g \in \text{GF}(q^2)$ and $a \neq 0 \neq bf - ce$.

Since $(1,0,0,c)$, belongs to $\mathcal{M}_{\alpha,\beta}$ if and only if $c \in \text{GF}(q)$, it follows that $\phi(1,0,0,c) = (a,0,0,c) \in \mathcal{M}_{\alpha',\beta'}$ implies $ca^{-1} \in \text{GF}(q)$, and thus $a \in \text{GF}(q)^*$. Now we observe that the affine plane $Y = 0$ has in common with $\mathcal{M}_{\alpha,\beta}$ the points $(1,x,0,z)$ for which $-\alpha x^2 + \beta x^{q+1} - z \in \text{GF}(q)$; so, $a^{-1}(-\alpha x^2 + \beta x^{q+1} - z) \in \text{GF}(q)$. Thus, suppose that $(1,x,0,z) \in \mathcal{M}_{\alpha,\beta}$; we have $\phi(1,x,0,z) \in \mathcal{M}_{\alpha',\beta'}$ and therefore

$$(\alpha^\sigma - \alpha'(b^2 + c^2)/a)x^{2\sigma} - (\beta^\sigma - \beta'(b^{q+1} + c^{q+1})/a)x^{(q+1)} - dx^\sigma \in \text{GF}(q), \quad (9)$$

as $\sigma$ stabilizes $\text{GF}(q)$. Let $\eta \in \text{GF}(q^2) \setminus \text{GF}(q)$ such that $\eta^2$ is a primitive element of $\text{GF}(q)$. Considering $x^\sigma = 1, -1, \eta, -\eta, 1 + \eta$ in (9), we get

$$d = 0,$$

$$\alpha^\sigma - \alpha'(b^2 + c^2)/a = 0, \quad (10)$$

$$\beta^\sigma - \beta'(b^{q+1} + c^{q+1})/a \in \text{GF}(q).$$

Similarly if we consider the affine points in common between the plane $X = 0$ and $\mathcal{M}_{\alpha,\beta}$, arguing as before, we obtain

$$g = 0,$$

$$\alpha^\sigma - \alpha'(e^2 + f^2)/a = 0,$$

$$\beta^\sigma - \beta'(e^{q+1} + f^{q+1})/a \in \text{GF}(q). \quad (11)$$

In particular,

$$b^2 + c^2 = e^2 + f^2 \neq 0. \quad (12)$$

Also, since $\beta' \notin \text{GF}(q)$,

$$b^{q+1} + c^{q+1} = e^{q+1} + f^{q+1} \neq 0. \quad (13)$$

Now we recall that a generic point $(1,x,y,z) \in \mathcal{M}_{\alpha,\beta}$ if and only if $\phi(1,x,y,z) \in \mathcal{M}_{\alpha',\beta'}$. On the other hand,
\[(1, x, y, z) \in M_{\alpha, \beta} \Leftrightarrow -\alpha(x^2 + y^2) + \beta(x^{q+1} + y^{q+1}) - z \in GF(q).\]

Since \(a \in GF(q) \setminus \{0\}\) and \(\sigma\) stabilizes \(GF(q)\), the former equation is equivalent to

\[
a^{-1}\{-\alpha^{\sigma}(x^{2\sigma} + y^{2\sigma}) + \beta^{\sigma}[x^{\sigma(q+1)} + y^{\sigma(q+1)}] - z^{\sigma}\} \in GF(q).
\]

(14)

Next, we observe that \(\phi(1, x, y, z) = (1, \frac{bx + ey}{a}, \frac{cx + fy}{a}, \frac{z}{a})\) and this point belongs to \(M_{\alpha', \beta'}\) if and only if

\[
a^{-1}\{-\alpha'\left(\frac{(bx^\sigma + ey^\sigma)^2}{a} + \frac{(cx^\sigma + fy^\sigma)^2}{a}\right)
+ \beta'\left(\frac{(bx^\sigma + ey^\sigma)^{q+1}}{a} + \frac{(cx^\sigma + fy^\sigma)^{q+1}}{a}\right) - z^\sigma\} \in GF(q).
\]

(15)

From (14) and (15), we get that for all \((1, x, y, z) \in M_{\alpha, \beta}\) the following holds:

\[
\alpha^{\sigma}(x^{2\sigma} + y^{2\sigma}) - \alpha'\left(\frac{(bx^\sigma + ey^\sigma)^2}{a} + \frac{(cx^\sigma + fy^\sigma)^2}{a}\right)
+ \beta'\left(\frac{(bx^\sigma + ey^\sigma)^{q+1}}{a} + \frac{(cx^\sigma + fy^\sigma)^{q+1}}{a}\right) - \beta^{\sigma}[x^{\sigma(q+1)} + y^{\sigma(q+1)}]
\in GF(q),
\]

that is, using (12) and (13),

\[
-\alpha'[2x^\sigma y^\sigma (be + cf)] + \beta'[(b^q e + c^q f)x^{\sigma q}y^\sigma + (be^q + cf^q)x^\sigma y^{\sigma q}] \in GF(q).
\]

(16)

We are going to prove that \(b^q e + c^q f = 0\). Thus, let \(v \in GF(q^2)\) be any solution of \(X^{q+1} = -1\). The semilinear collineation \(\phi\) has to leave invariant the Hermitian cone \(F\), that is, \(\phi(0, x, vx, z) \in F\), and because of the first equation in (13) this means

\[
(b^q e + c^q f)v^\sigma + (be^q + cf^q)v^{\sigma q} = 0
\]

for any of the \(q + 1\) different solutions of \(X^{q+1} = -1\). If \((b^q e + c^q f) \neq 0\) then the equation \((b^q e + c^q f)X + (b^q e + c^q f)^q X^q = 0\) would have more than \(q\) solutions which are impossible. Thus,

\[
b^q e + c^q f = 0
\]

(17)

and since \(\alpha' \notin GF(q)\) (16) gives

\[
be + cf = 0.
\]

(18)

Since \(\det(M) \neq 0\), it cannot be \(ce = 0 = bf\), so either \(c \neq 0 \neq e\) or \(b \neq 0 \neq f\). Thus, from (12) and (18) we also get \((e, f) = (c, -b)\) or \((e, f) = (-c, b)\). Thus from (17) we also obtain
\[ b^9c - bc^q = 0. \tag{19} \]

Hence if \( b \neq 0 \neq c \) then \( c = \lambda b \) where \( \lambda \in \text{GF}(q) \) and \( \lambda^2 + 1 \neq 0 \). So the lemma follows. \( \Box \)

From the previous lemma, taking into account conditions from (10) to (11), we get that if \( M_{\alpha,\beta} \) and \( M_{\alpha',\beta'} \) are projectively equivalent, then

\[(\alpha', \beta') = (a\delta/ (b^2 + c^2), a\beta^q/ (b^{q+1} + c^{q+1}) + u)\tag{20}\]

for some \( \sigma \in \text{Aut} \left( \text{GF}(q^2) \right) \), \( a \in \text{GF}(q)^* \), \( u \in \text{GF}(q) \), \( b, c \in \text{GF}(q^2) : b^2 + c^2 \neq 0 \) and if \( b \neq 0 \neq c \) then \( c = \lambda b \) with \( \lambda \in \text{GF}(q) \setminus \{0\} \). Conversely, if condition (20) holds, there is a semilinear collineation \( M_{\alpha,\beta} \rightarrow M_{\alpha',\beta'} \); so \( M_{\alpha,\beta} \) and \( M_{\alpha',\beta'} \) are projectively equivalent.

In this case we write \( (\alpha, \beta) \sim (\alpha', \beta') \) where \( \sim \) is in particular an equivalence relation on the ordered pairs \( (\alpha, \beta) \in \text{GF}(q^2)^2 \) such that \( 4\alpha^{q+1} + (\beta^q - \beta)^2 \neq 0 \).

**Lemma 5.5.** Let \( M_{\alpha,\beta} \) be a BM quasi-Hermitian variety of PG(3, \( q^2 \)), \( q \) odd and \( \varepsilon \) be a primitive element of \( \text{GF}(q^2) \). Then, there exists \( \alpha' \in \text{GF}(q^2) \setminus \{0\} \) such that \( M_{\alpha,\beta} \) is projectively equivalent to \( M_{\alpha,\beta'} \).

**Proof.** Write \( \beta = \beta_0 + \varepsilon \beta_1 \), with \( \beta_0, \beta_1 \in \text{GF}(q) \) and \( \beta_1 \neq 0 \). Then, there exists \( b \in \text{GF}(q^2) \setminus \{0\} \), such that \( \beta_1 / b^{q+1} = 1 \). Therefore \( (\alpha, \beta) \sim (\alpha / b^2, \beta / b^{q+1} - \beta_0 / b^{q+1}) = (\alpha / b^2, \varepsilon) \). \( \Box \)

In light of the previous lemma, to determine the equivalence classes of BM quasi-Hermitian varieties it is enough to determine when two varieties \( M_{\alpha,\varepsilon} \) and \( M_{\alpha',\varepsilon} \) are equivalent. This is done in the following.

**Lemma 5.6.** Let \( q = p^n \) be an odd prime, \( \varepsilon \) be a primitive element of \( \text{GF}(q^2) \), \( M_{\alpha,\varepsilon} \), and \( M_{\alpha',\varepsilon} \) be two BM quasi-Hermitian varieties of PG(3, \( q^2 \)). Put

\[ \delta(\alpha) := \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^{q+1}}. \]

Then, \( M_{\alpha,\varepsilon} \) is projectively equivalent to \( M_{\alpha',\varepsilon} \) if and only if there exist \( \sigma \in \text{Aut} \left( \text{GF}(q^2) \right) \) such that

\[ \delta(\alpha') = \delta(\alpha)^{\sigma}. \]

**Proof.** First we observe that for all \( \alpha \in \text{GF}(q^2) \setminus \{0\} \) such that \( 4\alpha^{q+1} + (\varepsilon^q - \varepsilon)^2 \neq 0 \)

\[ \delta(\alpha) := \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^{q+1}} \]

belongs to \( \text{GF}(q) \setminus \{0, -1\} \). Conversely, given any \( \delta \in \text{GF}(q) \setminus \{0, -1\} \) we can generate some BM quasi-Hermitian varieties \( M_{\alpha,\varepsilon} \), by choosing \( \alpha \) to be any solution of \( 4\delta \alpha^{q+1} = (\varepsilon^q - \varepsilon)^2 \). In fact, it turns out that \( (\varepsilon^q - \varepsilon)^2 + 4\alpha^{q+1} \neq 0 \). Furthermore, let \( \alpha_1 \) and \( \alpha_2 \) be any two such solutions. Then there exists \( k \) such that \( \alpha_2 = \varepsilon^k(q-1)\alpha_1 \). On the other hand, \( (\alpha_1, \varepsilon) \sim (\alpha_1, \varepsilon^{q+1} - \varepsilon^{q+1}) = (\alpha_1, \varepsilon^{q+1}) \). By repeating this process \( k \) times, we see
\((\alpha_1, \varepsilon) \sim (\alpha_1 \varepsilon^k (q-1), \varepsilon) = (\alpha_2, \varepsilon)\).

Thus \(\delta(\alpha_1) = \delta(\alpha_2)\) implies that \(\mathcal{M}_{\alpha_1, \varepsilon}\) is projectively equivalent to \(\mathcal{M}_{\alpha_2, \varepsilon}\). Hence, to determine the number \(N\) of projectively inequivalent BM quasi-Hermitian varieties we need to count the number of "inequivalent" \(\delta \in \text{GF}(q) \setminus \{0, -1\}\).

Now, given two BM quasi-Hermitian varieties \(\mathcal{M}_{\alpha, \varepsilon}\) and \(\mathcal{M}_{\alpha', \varepsilon}\) and setting
\[
\delta = \delta(\alpha) = \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^q + 1}, \quad \delta' = \delta(\alpha') = \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha'^q + 1},
\]
we have to show that \(\mathcal{M}_{\alpha, \varepsilon} \sim \mathcal{M}_{\alpha', \varepsilon}\) if and only if \(\delta' = \delta^\sigma\) for some \(\sigma \in \text{Aut}(\text{GF}(q^2))\).

First, suppose that \(\mathcal{M}_{\alpha, \varepsilon}\) and \(\mathcal{M}_{\alpha', \varepsilon}\) are equivalent, that is, \((\alpha', \varepsilon) \sim (\alpha, \varepsilon)\). This is true if and only if
\[
\alpha' = \frac{\alpha^2 a}{b^2 + c^2}, \quad \varepsilon = \frac{ae^\sigma}{b^{q+1} + c^{q+1}} + u,
\]
for some \(\sigma \in \text{Aut}(\text{GF}(q^2)), a \in \text{GF}(q) \setminus \{0\}, u \in \text{GF}(q), b, c \in \text{GF}(q^2)\) such that the conditions in the thesis of Lemma 5.4 hold.

Then
\[
\delta' = (b^2 + c^2)^{q+1} \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^2(\alpha^\sigma)^{q+1}}, \quad \delta^\sigma = (b^{q+1} + c^{q+1})^2 \frac{(\varepsilon^{q} - \varepsilon)^2}{4\alpha^{2}(\alpha^{\sigma})^{q+1}}.
\]

We observe that
\[
(b^2 + c^2)^{q+1} = (b^{q+1} + c^{q+1})^2. \tag{21}
\]

In fact, if either \(b = 0\) or \(c = 0\), then (21) is trivially satisfied and there is nothing further to prove. Otherwise, a direct manipulation yields that (21) is equivalent to
\[
\frac{b^{q-1}}{c^{q-1}} + \frac{c^{q-1}}{b^{q-1}} = 2.
\]
This gives \(\frac{b^{q-1}}{c^{q-1}} = 1\), which is always true, since (19) holds. Because of (21) then \(\delta' = \delta^\sigma\).

Conversely, suppose that \(\delta' = \delta^\sigma\) for some \(\sigma\). Then we observe that \((\alpha, \varepsilon) \sim (\alpha^\sigma, \varepsilon^{\sigma})\). Furthermore \((\alpha^2, \varepsilon^2) \sim (\alpha^\sigma/b^2, \varepsilon)\) where \(\varepsilon^2 = b_1 \varepsilon + b_0\) with \(b_1/b^{q+1} = 1\) for a suitable \(b \in \text{GF}(q^2) \setminus \{0\}\), as seen in the proof of Lemma 5.5.

Thus we have that
\[
\delta(\alpha^2/b^2) = (\varepsilon^q - \varepsilon)^2(b^2)^{q+1}/4(\alpha^\sigma)^{q+1}
\]
\[
= (b^2)^{q+1}[(\alpha^\sigma)^q - b_0] - (\varepsilon^\sigma - b_0)]^2/(4(\alpha^\sigma)^{q+1}(b^{q+1})^2)
\]
\[
= [(\varepsilon^q - \varepsilon)^2]/4(\alpha)^{(q+1)} = \delta^\sigma = \delta'.
\]
Hence,

\[(\alpha', \varepsilon) \sim (\alpha^2/b^2, \varepsilon) \sim (\alpha^2, \varepsilon^3) \sim (\alpha, \varepsilon)\]

\[\square\]

**Conjecture 5.7.** We conjecture that Lemma 5.6 holds for all odd \( r \geq 3 \), as the conditions on the coefficients \( \alpha, \beta \) are the same and the block structure of the matrices representing the classes should be analogous to that of Lemma 5.4. For \( r \) even the algebraic conditions on \( \alpha \) and \( \beta \) to construct quasi-Hermitian varieties are different, see [3].

**Theorem 5.8.** Let \( q = p^n \) with \( p \) an odd prime. Then the number \( N \) of projectively inequivalent BM quasi-Hermitian varieties \( \mathcal{M}_{\alpha, \beta} \) of \( \text{PG}(3, q^2) \) is

\[
N = \frac{1}{n} \left( \sum_{k \mid n} \Phi \left( \frac{n}{k} \right) p^k \right) - 2,
\]

where \( \Phi \) is the Euler \( \Phi \)-function.

**Proof.** For all \( \delta, \delta' \in \text{GF}(q)\setminus\{0, -1\} \) write \( \delta \sim \delta' \) if and only if \( \delta' = \delta^\sigma \) for some \( \sigma \in \text{Aut}(\text{GF}(q^2)) \). By Lemma 5.6, \( N \) is the number of inequivalent classes \([\delta]\) under \( \sim \). Let \( N_e = |\{\delta \in \text{GF}(p^n)\setminus\{0, -1\} : \delta \text{ is not contained in any smaller subfield of } \text{GF}(q)\}| \). We have

\[
N = \sum_{e \mid n} N_e.
\]

Observing that

\[
\sum_{e' \mid e} N_{e'} = p^e - 2,
\]

denote by \( \mu(x) \) the Möbius function [12]. Then, Möbius inversion gives

\[
N_e = \sum_{e' \mid e} \mu(e') p^{e/e'} - 2 \sum_{e' \mid e} \mu(e').
\]

It follows that

\[
N = \left( \sum_{e \mid n} \frac{1}{e} \sum_{e' \mid e} \mu(e') p^{e/e'} \right) - 2.
\]

Let \( m = e/e' \) be a divisor of \( n \), then the coefficient of \( p^m \) is

\[
\frac{1}{n} \sum_{(e/m) \parallel (n/m)} \mu \left( \frac{e}{m} \right) \frac{n/m}{e/m} = \frac{1}{n} \Phi \left( \frac{n}{m} \right)
\]
and finally

\[ N = \frac{1}{n} \left( \sum_{k/n} \Phi\left(\frac{n}{k}p^k\right) \right) - 2. \]

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