GLOBAL DIMENSION OF POLYNOMIAL RINGS IN PARTIALLY COMMUTING VARIABLES

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Abstract

For any free partially commutative monoid $M(E, I)$, we compute the global dimension of the category of $M(E, I)$-objects in an Abelian category with exact coproducts. As a corollary, we generalize Hilbert’s Syzygy Theorem to polynomial rings in partially commuting variables.

Keywords: cohomology of small categories, free partially commutative monoid, trace monoid, Hochschild-Mitchell dimension, noncommutative polynomial ring

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Introduction

In this paper, the global dimension of the category of objects in an Abelian category with the action of free partially commutative monoid is computed. As a corollary, a formula for the global dimension of polynomial rings in partially commuting variables is obtained.

Let $\mathcal{A}$ be any Abelian category. By [1] Chapter XII, §4, extension groups $\text{Ext}^n(A, B)$ are consisted of congruence classes of exact sequences $0 \to B \to C_1 \to \cdots \to C_n \to A$ in $\mathcal{A}$ for $n \geq 1$ and $\text{Ext}^0(A, B) = \text{Hom}(A, B)$. It allows us to define the global dimension of $\mathcal{A}$ by

$$\text{gl.dim.}\mathcal{A} = \sup\{n \in \mathbb{N} : (\exists A, B \in \text{Ob}\mathcal{A}) \text{ } \text{Ext}^n(A, B) \neq 0\}.$$
Here $\mathbb{N}$ is the set of nonnegative integers. (We set $\sup \emptyset = -1$ and $\sup \mathbb{N} = \infty$.) For a ring $R$ with 1, $\text{gl dim } R$ is the global dimension of the category of left $R$-modules.

As it is well known [2, Theorem 4.3.7], for any ring $R$ with 1,

$$\text{gl dim } R[x_1, \ldots, x_n] = n + \text{gl dim } R.$$ 

Moreover, by [3, Theorem 2.1], if $\mathcal{A}$ is any Abelian category with exact coproducts and $\mathcal{C}$ a bridge category, then $\text{gl dim } \mathcal{A}^\mathcal{C} = 1 + \text{gl dim } \mathcal{A}$. It follows that $\text{gl dim } \mathcal{A}^{\mathbb{N}^n} = n + \text{gl dim } \mathcal{A}$ for the free commutative monoid $\mathbb{N}^n$ generated by $n$ elements. We will get one of possible generalizations of this formula. Let $M(E, I)$ be a free partially commutative monoid with a set of variables $E$, where $I \subseteq E \times E$ is an irreflexive symmetric relation assigning the pairs of commuting variables. In this paper, we prove that

$$\text{gl dim } \mathcal{A}^{M(E, I)} = n + \text{gl dim } \mathcal{A}$$

for any Abelian category with exact coproducts where $n$ is the sup of numbers of mutually commuting distinct elements of $E$. For example, if $R[M(E, I)]$ is the polynomial ring in variables $E = \{x_1, x_2, x_3, x_4\}$ with the commuting pairs $(x_i, x_j)$ corresponding to adjacent vertices of the graph demonstrated in Figure 1, then for any ring $R$ with 1 we have $\text{gl dim } R[M(E, I)] = 2 + \text{gl dim } R$.

![Figure 1: Pairs of commuting variables](attachment:image.png)

The free partially commutative monoids have numerous applications in combinatorics and computer sciences [4]. Our interest in their homology groups is concerned with the studying a topology of mathematical models for concurrency [5].

## 1 Cohomology of small categories

Throughout this paper let $\text{Ab}$ the category of Abelian groups and homomorphisms, $\mathbb{Z}$ the additive group of integers, and $\mathbb{N}$ the set of nonnegative
integers or the free monoid with only one generator. For any category $\mathcal{A}$ and a pair $A_1, A_2 \in \text{Ob} \, \mathcal{A}$, denote by $\mathcal{A}(A_1, A_2)$ the set of all morphisms $A_1 \to A_2$.

A diagram $\mathcal{C} \to \mathcal{A}$ is a functor from a small category $\mathcal{C}$ to a category $\mathcal{A}$. Given a small category $\mathcal{C}$ we denote by $\mathcal{A}^{\mathcal{C}}$ the category of diagrams $\mathcal{C} \to \mathcal{A}$ and natural transformations. For $A \in \text{Ob} \, \mathcal{A}$, let $\Delta_{\mathcal{C}} \, A$ (shortly $\Delta A$) denote a diagram $\mathcal{C} \to \mathcal{A}$ with constant values $A$ on objects and $1_A$ on morphisms.

In this section, we recall some results from the cohomology theory of small categories.

### 1.1 Homology groups of a nerve

Recall a definition of a nerve of the category and properties of homology groups of simplicial sets. We refer the reader to [1] and [6] for the proofs.

#### 1.1.1 A nerve of the category

Let $\mathcal{C}$ be a small category. Its nerve $N_\ast \mathcal{C}$ is the simplicial set in which $N_n \mathcal{C}$ consists of all sequences of composable morphisms $c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n$ in $\mathcal{C}$ for $n > 0$ and $N_0 \mathcal{C} = \text{Ob} \, \mathcal{C}$. For $n > 0$ and $0 \leq i \leq n$, boundary operators $d_i^n : N_n \mathcal{C} \to N_{n-1} \mathcal{C}$ acts as

$$d_i^n(c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} c_n) = c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\hat{\alpha}_i} \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} c_n.$$ 

Here $c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_i} \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} c_n \in N_{n-1} \mathcal{C}$ is the $(n-1)$-fold sequence obtained from $c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n$ for $0 < i < n$ by substitution the morphisms $c_i \xrightarrow{\alpha_i} c_{i+1}$ by their composition $c_{i-1} \xrightarrow{\alpha_{i-1}} \xrightarrow{\alpha_i} c_{i+1}$. The map $d_0^n$ removes $\alpha_1$ with $c_0$ and $d_n^n$ removes $\alpha_n$ with $c_n$. Degeneracy operators $s_i^n : N_n \mathcal{C} \to N_{n+1} \mathcal{C}$ insert in $c_0 \to \cdots \to c_n$ the identity morphism $c_i \to c_i$ for every $0 \leq i \leq n$.

#### 1.1.2 Homology groups of simplicial sets

Let $X$ be a simplicial set given by boundary operators $d_i^n$ and degeneracy operators $s_i^n$ for $0 \leq i \leq n$. Consider a chain complex $C_\ast(X)$ of free Abelian groups $C_n(X)$ generated by the sets $X_n$ for $n \geq 0$. Differentials $d_n : C_n(X) \to C_{n-1}(X)$ are defined on the basis elements $x \in X_n$ by $d_n(x) = \sum_{i=0}^n (-1)^i d_i^n(x)$. Let $C_n(X) = 0$ for $n < 0$. The groups $H_n(X) = \text{Ker} \, d_n / \text{Im} \, d_{n+1}$ are called $n$-th homology groups of the simplicial
set $X$. The groups $H_n(X)$ are isomorphic to $n$-th singular homology groups of the geometric realization of $X$ by the Eilenberg theorem [6, Appl. 2].

1.1.3 Cohomology of a category with coefficients in an Abelian group

For a small category $\mathcal{C}$, let $H_n(\mathcal{C})$ denote the $n$-th homology group of the nerve $N_*\mathcal{C}$. For a simplicial set $X$ and an Abelian group $A$, cohomology groups $H^n(X, A)$ are defined as cohomology groups of the complex $\text{Hom}(C_*(X), A)$. Let $\mathcal{C}$ be a small category. We introduce its cohomology groups $H^n(\mathcal{C}, A)$ with coefficients in $A$ as $H^n(N_*(\mathcal{C}), A)$. It follows from [1, Chapter III, Theorem 4.1] that there is the following exact sequence (Universal Coefficient Theorem)

$$0 \rightarrow \text{Ext}(H_{n-1}(\mathcal{C}), A) \rightarrow H^n(\mathcal{C}, A) \rightarrow \text{Hom}(H_n(\mathcal{C}), A) \rightarrow 0$$

1.2 Cohomology of categories with coefficients in diagrams

Recall the definition and properties of right derived functors $\lim^{\leftarrow} : \text{Ab}^{\mathcal{C}} \rightarrow \text{Ab}$ of the limit functor.

1.2.1 Definition of cohomology of categories with coefficients in diagrams

Let $\mathcal{C}$ be a small category. For every family $\{A_i\}_{i \in I}$ of Abelian groups we consider the direct product $\prod_{i \in I} A_i$ as the Abelian group of maps $\varphi : I \rightarrow \bigcup_{i \in I} A_i$ such that $\varphi(i) \in A_i$ for all $i \in I$.

For any functor $F : \mathcal{C} \rightarrow \text{Ab}$, consider the sequence of Abelian groups

$$C^0(\mathcal{C}, F) = \prod_{c_0 \in \text{Ob} \mathcal{C}} F(c_0), \ldots, C^n(\mathcal{C}, F) = \prod_{c_0, \ldots, c_n} F(c), \ldots$$
and homomorphisms $\delta^n : C^n(\mathcal{C}, F) \to C^{n+1}(\mathcal{C}, F)$ defined by

$$(\delta^n \varphi)(c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} \tilde{c}_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}) + (-1)^{n+1} F(c_n \xrightarrow{\alpha_{n+1}} c_{n+1})(\varphi(c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} c_n)).$$

Let $C^n(\mathcal{C}, F) = 0$ for $n < 0$. The equalities $\delta^{n+1} \delta^n = 0$ hold for all integer $n$. The obtained cochain complex will be denoted by $C^\ast(\mathcal{C}, F)$. Abelian groups $H^n(\mathcal{C}^\ast(\mathcal{C}, F)) = \ker \delta^{n+1} / \text{im} \delta^n$ are called cohomology groups of the small category $\mathcal{C}$ with coefficients in a diagram $F$ and denoted by $\varprojlim_n^\mathcal{C} F$.

It follows from [6, Appl. 2, Prop. 3.3] by the substitution $A = \text{Ab}^{\text{op}}$ that the functors $\varprojlim_n^\mathcal{C}$ are $n$-th right satellites of $\varprojlim^\mathcal{C} : \text{Ab}^\mathcal{C} \to \text{Ab}$. Since the category $\text{Ab}^\mathcal{C}$ has enough injectives, the functors $\varprojlim_n^\mathcal{C}$ are isomorphic to right derived of the limit functor.

1.2.2 Cohomology of categories without retractions

A morphism $\alpha : a \to b$ $\mathcal{C}$ is a retraction if there exists a morphism $\beta : b \to a$ such that $\alpha \beta = 1_b$.

**Proposition 1.1.** [7, Prop. 2.2] If a small category $\mathcal{C}$ does not contain nonidentity retractions, then for any diagram $F : \mathcal{C} \to \text{Ab}$, the groups $\varprojlim_n^\mathcal{C} F$ are isomorphic to the homology groups of the subcomplex $C^\ast_+(\mathcal{C}, F) \subseteq C^\ast(\mathcal{C}, F)$ composed of the products

$$C^n_+(\mathcal{C}, F) = \prod_{c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} c_n} F(c_n), \quad n \geq 0,$$

where indices run the sequences $c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n$ such those $\alpha_i \neq \text{id}_{c_i}$ for all $1 \leq i \leq n$.

**Corollary 1.2.** If a small category $\mathcal{C}$ does not contain nonidentity retractions and the length $m$ of every sequence of nonidentity morphisms $c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m$ is not greater than $n$, then $\varprojlim_k^\mathcal{C} F = 0$ for $k > n$.

**Example 1.1.** Let $\Theta$ be the category with Ob $\Theta = \{a, b\}$ and Mor $\Theta = \{1_a, 1_b, a \xrightarrow{\alpha_1} b, a \xrightarrow{\alpha_2} b\}$. It follows from Proposition [7, 2] that for any diagram $F : \Theta \to \text{Ab}$ and $n > 1$, the groups $\varprojlim_n^\Theta F$ equal 0.
For any Abelian group $A$, $\varprojlim_n^\Delta A \cong H^n(\mathcal{C}, A)$.

**Lemma 1.3.** Let $\Theta^n$ be the $n$-th power of the category $\Theta$ for $n \geq 1$. The functors $\varprojlim_n^k$ equal 0 for all $k > n$. For any Abelian group $A$, there is an isomorphism $\varprojlim_n^\Delta A \cong A$.

**Proof.** The first assertion follows from Corollary [1.2]. Since the geometric realization of the nerve of $\Theta^n$ is the $n$-dimensional torus, $H_k(\Theta^n) \cong \mathbb{Z}^\binom{n}{k}$ for $0 \leq k \leq n$. Here $\binom{n}{k}$ is the binomial coefficients. Universal Coefficient Theorem for the cohomology groups of the nerve of $\Theta^n$ gives $H_n(\Theta^n, A) \cong A$. □

### 1.2.3 Strongly coinitial functors

A small category $\mathcal{C}$ is *acyclic* if $H_n(\mathcal{C}) = 0$ for all $n > 0$ and $H_0(\mathcal{C}) = \mathbb{Z}$. Let $S : \mathcal{C} \to \mathcal{D}$ be a functor from a small category to an arbitrary category. For any $d \in \text{Ob} \mathcal{D}$, a *fibre* (or comma-category) $S/d$ is the category whose objects are given by pairs $(c, \alpha)$ where $c \in \text{Ob}(\mathcal{C})$ and $\alpha \in \mathcal{D}(S(c), d)$. Morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ in $S/d$ are triples $(f, \alpha_1, \alpha_2)$ with $f \in \mathcal{C}(c_1, c_2)$ satisfying $\alpha_2 \circ S(f) = \alpha_1$. If $S$ is a full embedding $\mathcal{C} \subseteq \mathcal{D}$, then $S/d$ is denoted by $\mathcal{C}/d$.

**Definition 1.2.** A functor $S : \mathcal{C} \to \mathcal{D}$ between small categories is called strongly coinitial if $S/d$ is acyclic for each $d \in \mathcal{D}$.

**Lemma 1.4** (Oberst). Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. If $S : \mathcal{C} \to \mathcal{D}$ be a strongly coinitial functor, then the canonical homomorphisms $\varprojlim_n^\Delta F \to \varprojlim_n^\Delta FS$ are isomorphisms for all $n \geq 0$.

**Proof.** It follows from the opposite assertion [8, 2.3] for the functors $S^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ and $F^{\text{op}} : \mathcal{D}^{\text{op}} \to \text{Ab}^{\text{op}}$. □

### 1.3 Cohomological dimension of a small category

Let $\mathbb{N}$ be the set of nonnegative integer numbers. We will be consider it as the subset of $\{-1\} \cup \mathbb{N} \cup \{\infty\}$ ordered by $-1 < 0 < 1 < 2 < \cdots < \infty$.

**Definition 1.3.** Cohomological dimension $\text{cd} \mathcal{C}$ of a small category $\mathcal{C}$ is the sup in $\{-1\} \cup \mathbb{N} \cup \{\infty\}$ of the set $n \in \mathbb{N}$ for which the functors $\varprojlim_n^\Delta : \text{Ab}^{\mathcal{C}} \to \text{Ab}$ are not equal 0.
It follows from Lemma 1.3 that \( \text{cd } \Theta^n = n \). Lemma 1.4 gives the following

**Corollary 1.5.** If there exists a strongly coinitial functor \( S : \mathcal{C} \to \mathcal{D} \) between small categories, then \( \text{cd } \mathcal{C} \geq \text{cd } \mathcal{D} \).

A subcategory \( \mathcal{D} \subseteq \mathcal{C} \) is said to be *closed* if \( \mathcal{D} \) is a full subcategory containing the domain for any morphism whose codomain is in \( \mathcal{D} \).

**Corollary 1.6.** Let \( \mathcal{D}_j \subseteq \mathcal{D} \) be a family of closed subcategories for all \( j \in J \). If the inclusion \( \bigcup_{j \in J} \mathcal{D}_j \subseteq \mathcal{D} \) is strongly coinitial, then \( \text{cd } \mathcal{D} = \sup_{j \in J} \{ \text{cd } \mathcal{D}_j \} \).

**Proof.** For \( c \in \text{Ob } \mathcal{C} \), let \( \mathcal{C}_c \subseteq \mathcal{C} \) be denote a full subcategory which consists of \( c' \in \text{Ob } \mathcal{C} \) having morphisms \( c' \to c \). It follows from [9, Corollary 7] that the equality \( \text{cd } \mathcal{C} = \sup_{c \in \text{Ob } \mathcal{C}} \text{cd } \mathcal{C}_c \) holds. Consequently \( \sup_{j \in J} \{ \text{cd } \mathcal{D}_j \} = \text{cd } \bigcup_{j \in J} \mathcal{D}_j \leq \text{cd } \mathcal{D} \). Since the inclusion \( \bigcup_{j \in J} \mathcal{D}_j \subseteq \mathcal{D} \) is strongly coinitial, the equality follows from Corollary 1.5. \( \square \)

## 2 Dimension of a free partially commutative monoids

We will prove the main results. We compute the Baues-Wirsching dimension of a free partially commutative monoids and show a formula for the global dimension of the category of objects with actions of a free partially commutative monoid. We prove that for any graded \( R[M(E,I)] \)-module, there exists a free resolution.

### 2.1 Cohomological dimension of the factorization category

We consider the category of factorization of a small category, although we apply it for the case of the small category is a monoid.

#### 2.1.1 The category of factorizations

Let \( \mathcal{C} \) be a small category. Objects of the *category of factorizations* \( \mathcal{F} \mathcal{C} \) are all morphisms of \( \mathcal{C} \). For any \( \alpha, \beta \in \text{Ob } \mathcal{F} \mathcal{C} = \text{Mor } \mathcal{C} \), the set of morphisms \( \alpha \to \beta \) consists from all pairs \( (f, g) \) of morphisms in \( \mathcal{C} \) satisfying
\(g \circ \alpha \circ f = \beta\). The composition of \(\alpha \xrightarrow{(f_1, g_1)} \beta\) and \(\beta \xrightarrow{(f_2, g_2)} \gamma\) is defined by \(\alpha \xrightarrow{(f_1 \circ f_2, g_2 \circ g_1)} \gamma\). The identity of an object \(a \rightarrow b\) of \(\mathcal{F}\mathcal{C}\) equals \(\alpha \xrightarrow{(1_a, 1_b)} \alpha\).

### 2.1.2 Baues-Wirsching dimension

A natural system of Abelian groups on \(\mathcal{C}\) is any functor \(F : \mathcal{F}\mathcal{C} \rightarrow \text{Ab}\). Baues and Wirsching introduce cohomology groups \(H^n(\mathcal{C}, F)\) of \(\mathcal{C}\) with coefficients in a natural system \(F\) and have proved that these groups are isomorphic to \(\varprojlim_n F\mathcal{C}\). The Baues-Wirsching dimension \(\text{Dim } \mathcal{C}\) is the cohomological dimension of \(\mathcal{F}\mathcal{C}\).

**Example 2.1.** Let \(\mathbb{N} = \{1, a, a^2, \ldots\}\) be the free monoid generated by one element. It easy to see that the inclusion \(\Theta_a \subseteq \mathcal{F}\mathbb{N}\) of the full subcategory with the objects \(\text{Ob } \Theta_a = \{1, a\}\) is strongly coinitial. The subcategory \(T_a\) is closed in \(\mathcal{F}\mathbb{N}\). It is isomorphic to \(\Theta\) from Example 1.1. Consequently \(\text{Dim } \mathbb{N} = 1\).

**Proposition 2.1.** For any integer \(n \geq 1\), \(\text{Dim } \mathbb{N}^n = n\).

**Proof.** Consider the full subcategory \(\Theta_a^n \subseteq \mathcal{F}\mathbb{N}^n\) with objects \((a^{\varepsilon_1}, \ldots, a^{\varepsilon_n})\) where \(\varepsilon_i \in \{0, 1\}\) for all \(1 \leq i \leq n\). It not hard to see that it is isomorphic to \(\Theta^n\) and the fibre of the inclusion over \((a^{k_1}, \ldots, a^{k_n}) \in \mathbb{N}^n\) is isomorphic to the product \(\Theta_a/a^{k_1} \times \cdots \times \Theta_a/a^{k_n}\). Since \(H_i(\Theta_a/a^k) = 0\) for \(i > 0\) and \(H_0(\Theta_a/a^k) \cong \mathbb{Z}\), it follows that the category \(\mathcal{F}\mathbb{N}^n\) contains the strongly coinitial subcategory \(\Theta_a^n\), which is isomorphic to \(\Theta^n\). It is clear that \(\Theta_a^n\) is closed in \(\mathcal{F}\mathbb{N}^n\). Hence, \(\text{Dim } \mathbb{N}^n = \text{cd } \Theta^n = n\). \(\square\)

### 2.2 The dimension of a free partially commutative monoid

This subsection is devoted to computing the Baues-Wirsching dimension of free partially commutative monoids.

#### 2.2.1 The independence graph

Let \(E\) be a set and \(I \subseteq E \times E\) an irreflexive symmetric binary relation on \(E\). Monoid given by a generating set \(E\) and relations \(ab = ba\) for all \((a, b) \in I\) is called free partially commutative and denoted by \(M(E, I)\).

The pair \((E, I)\) may be considered as a simple independence graph of \(M(E, I)\) with the set of vertices \(E\) and edges \(\{a, b\}\) for all pairs \((a, b) \in I\).
Figure 2: The independence graph

It is shown in Figure 2 the independence graph of the monoid given by the generators \( E = \{a, b, c, d, e\} \) and relations \( ab = ba, \ bc = cb, \ cd = dc, \ ad = da, \ ae = ea, \ dc = ed \).

The \textit{clique number} \( \omega(E, I) \) of a simple graph with vertices \( E \) and edges \( I \) is the sup of cardinalities of its finite complete subgraphs. If \((E, I)\) contains complete graphs \( K_n \) for all \( n \in \mathbb{N} \), then \( \omega(E, I) = \infty \).

For example, the clique number of the graph in Figure 2 is equal to 3.

2.2.2 Computing the dimensions of free partially commutative monoids

Let \( V \) be the set of maximal cliques of the independence graph of \( M(E, I) \). (These cliques may be infinite.) For example, the set \( V \) for the graph in Figure 2 consists of the sets \( \{a, b\}, \ \{b, c\}, \ \{c, d\}, \ \{a, d, e\} \). Let \( E_v \) be the set of vertices belonging to a clique \( v \) and \( M(E_v) \) the submonoid of \( M(E, I) \) generated by \( E_v \). It is clear that \( M(E_v) \) are commutative monoids. The category of factorization \( \mathcal{F} M(E_v) \) is a closed subcategory of \( \mathcal{F} M(E, I) \).

**Lemma 2.2.** [11] The inclusion \( \bigcup_{v \in V} \mathcal{F} M(E_v) \subseteq \mathcal{F} M(E, I) \) is strongly coinitial.

**Theorem 2.3.** \( \text{Dim} M(E, I) = \omega(E, I) \).

**Proof.** For every subset of mutually commuting elements \( \{e_1, \ldots, e_n\} \subseteq E \), the full subcategory \( \mathcal{F} M(\{e_1, \ldots, e_n\}) \) is closed in \( \mathcal{F} M(E, I) \). Hence, the equality is true in the case of \( \omega(E, I) = \infty \). The subcategories \( \mathcal{F} M(E_v) \) are closed in \( \mathcal{F} M(E, I) \). It follows from Lemma 2.2 that we can use Corollary 1.6. We get \( \text{cd} \mathcal{F} M(E, I) = \sup_{v \in V} \{\text{cd} \mathcal{F} M(E_v)\} \). If \( E_v \) are finite, then we get the assertion \( \text{Dim} M(E, I) = \omega(E, I) \) by Proposition 2.1. \( \square \)
2.3 The generalized syzygy theorem

In this subsection, we prove the main theorem.

2.3.1 The global dimension of the category of \( M(E, I) \)-objects

By [12, Corollary 13.4'] for any small category \( C \) and Abelian category with exact coproducts, there exists the inequality \( \text{gl dim } \mathcal{A} \leq \text{dim } C + \text{gl dim } \mathcal{A} \). Here \( \text{dim} \) is the Hochschild-Mitchell dimension. We will show that if \( C = M(E, I) \), then the equality holds.

**Theorem 2.4.** Let \( \mathcal{A} \) be an Abelian category with exact coproducts. Then \( \text{gl dim } \mathcal{A}^{M(E,I)} = \omega(E, I) + \text{gl dim } \mathcal{A} \).

**Proof.** For any finite subset \( E' \subseteq E \), the submonoid generated by \( E' \) is cancellative [4]. It follows that \( M(E, I) \) is cancellative and \( \text{Dim } M(E, I) \) is equal to Hochschild-Mitchell dimension \( \text{dim } M(E, I) \) [7, Theorem 3.1]. Consequently \( \text{gl dim } \mathcal{A}^{M(E,I)} \leq \omega(E, I) + \text{gl dim } \mathcal{A} \). For each finite subset of mutually commuting elements \( S \subseteq E \) there exists a retraction \( M(E, I) \to M(S) \). It follows by [3, Corollary 1.4] that \( \text{gl dim } \mathcal{A}^{M(E,I)} \geq \text{gl dim } \mathcal{A}^{M(S)} = |S| + \text{gl dim } \mathcal{A} \). Since \( \text{dim } M(E, I) \) is equal to sup of cardinalities \( |S| \) of finite subsets \( S \subseteq E \) of mutually commuting elements, we get \( \text{gl dim } \mathcal{A}^{M(E,I)} \geq \omega(E, I) + \text{gl dim } \mathcal{A} \). \( \square \)

2.3.2 Graded syzygies

Let \( R \) be a ring with 1. The monoid ring has the natural graduation \( R[M(E, I)] = \bigoplus_{n \in \mathbb{Z}} R[M(E, I)]_n \) by \( R \)-modules \( R[M(E, I)]_n = \{ r\mu : r \in R, \mu \in M(E, I), \mu| = n \} \). In particular \( R[M(E, I)]_0 = R \). Let \( R[M(E, I)]_n = 0 \) for all \( n < 0 \). The ring \( R \) with 1 is called projective free if any projective \( R \)-module is free. By [13, §8.7, Corollary 2] and Theorem 2.4 we get:

**Corollary 2.5.** Let \( M(E, I) \) be a free partially commutative monoid and \( R \) projective free ring with \( \text{gl dim } R = n < \infty \). If there is the maximal number \( m < \infty \) of mutually commuting distinct elements of \( E \), then for each bounded below \( \mathbb{Z} \)-graded \( R[M(E, I)] \)-module \( A \), there exists an exact sequence of \( \mathbb{Z} \)-graded \( R[M(E, I)] \)-modules and \( \mathbb{Z} \)-graded homomorphisms of degree 0

\[ 0 \to F_{n+m} \to F_{n+m-1} \to \cdots \to F_0 \to A \to 0, \]

with free bounded below \( \mathbb{Z} \)-graded \( R[M(E, I)] \)-modules \( F_0, F_1, \ldots, F_{n+m} \).
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