A Comment on “Free energy fluctuations in Ising spin glasses”, by T. Aspelmeier and M.A. Moore

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We show that there is no need to modify the Parisi replica symmetry breaking ansatz, by working with $R$ steps of breaking and solving exactly the discrete stationarity equations generated by the standard “truncated Hamiltonian” of spin glass theory.
1. Introduction

In a quite recent work Aspelmeier and Moore \[1\] have considered the sample-to-sample free energy fluctuations in finite dimensional spin glasses via the replica method. To that effect they reconsider higher order terms in the replica number \(n\) and they conclude that the Parisi symmetry breaking scheme \[2\] does not give the correct answer for these higher order terms. Finally they propose a modified symmetry breaking scheme that resolves the problem.

What we set out to do here is the following. Starting from the same truncated Hamiltonian (AM.3) we solve exactly the discrete stationarity equations for \(R\) steps of replica symmetry breaking, namely we obtain the \(R + 1\) values of \(q_{\alpha\beta}\) indexed by their overlap values \(q_0, q_1, ..., q_R\) (together with \(q_{\alpha\alpha} \equiv q_{R+1} = 0\)) and the \(R\) values of Parisi box sizes \(p_1, p_2, ..., p_R\) together with the two fixed boundary values \(p_0 = n\) and \(P_{R+1} = 1\). As a result, we find two families (a), (b) of solutions associated with two possible values of \(q_0\), namely, letting \(g = w/(2y)\)

(a) \(q_0 = \frac{3n}{2}g\)

In this case the corresponding free energy is identical to the Kondor \[3\] result

\[
nf^{(a)}(n) = nf - \frac{9n^6}{640}wg^3
\]  

(1.1)

(b) \(q_0 = 0\)

The free energy is now larger

\[
nf^{(b)}(n) = nf
\]  

(1.2)

The (b) solution is therefore the appropriate one to choose, both solutions having a non-negative Hessian spectrum when \(R \to \infty\). Among the family of solutions (b) with \(q_0 = 0\) and free energy \(f^{(b)}\), we will pick a reference solution with a set of values \(q_t, p_t, t = 1, 2, ..., R\). All the other solutions will be shown elsewhere \[4\] to correspond to a (discrete) reparametrization for large \(R\). With that set of values, we proceed and compute the contribution to fluctuations, with a result that matches for \(R \to \infty\) the Aspelmeier and Moore ones \[1\]. We thereby establish that there is indeed no need for modifying the Parisi replica symmetry breaking scheme.
2. Solution of the stationarity equations

The stationarity equations are derived from the free energy functional

\[ n f = - \sum_{t=0}^{R+1} \left\{ (p_t - p_{t+1}) \left[ \frac{\tau}{2} q_t^2 + \frac{u}{12} q_t^4 \right] + \left( \frac{1}{p_t} - \frac{1}{p_{t-1}} \right) \frac{w}{6} q_t^3 \right\} \]  

(2.1)

where we have used the replica Fourier transform \( \hat{q} \) of \( q \):

\[ \hat{q}_k = \sum_{t=k}^{R+1} p_t (q_t - q_{t-1}) = \sum_{t=k}^{R} p_t (q_t - q_{t-1}) - q_R \]  

(2.2)

Combining the stationarity equations, we obtain in the end

\[ gp_t = \frac{1}{2} (q_t + q_{t-1}) \quad t = 1, 2, ..., R \]  

\[ (q_t - q_{t-1})^2 = (q_{t-1} - q_{t-2})^2 = ... = (q_1 - q_0)^2 \quad t = 1, 2, ..., R \]  

(2.3)

Here we concentrate on the particular reference solution such that

\[ q_t - q_{t-1} = q_{t-1} - q_{t-2} = ... = q_1 - q_0 = \frac{q_R - q_0}{R} \]  

(2.4)

which leads to

\[ q_t = q_0 + (q_R - q_0) \frac{t}{R} \quad t = 0, 1, ..., R \]  

\[ gp_t = q_0 + (q_R - q_0) \frac{2t - 1}{2R} \quad t = 1, 2, ..., R \]  

(2.5)

The new \( q_R \) is monotonous except for its last step \((q_{R+1} = 0)\), and \( p_t \) is monotonous except for its first step (when \( p_0 \) is kept fixed at a value \( n \neq 0 \)).

We now proceed to get the fluctuation contribution as in (AM.5).
3. Fluctuations: the Replicon sector

We have as in (AM.9)

\[ n \delta f_{Rep} = \frac{V}{2} \int \frac{d^Dp}{(2\pi)^D} I_{Rep}(p) \]  

(3.1)

where

\[ I_{Rep}(p) = n \sum_{r=0}^{R} \sum_{k,l=r+1}^{R+1} \mu(r;k,l) \log (p^2 + \lambda(r;k,l)) \]  

(3.2)

Here the Replicon eigenvalue \( \lambda \) is

\[ \lambda(r;k,l) = -2\tau - w\hat{q}_k - w\hat{q}_l - 2yq_r^2 \]  

(3.3)

The multiplicity \( \mu(r;k,l) \) \[6\] is given by

\[ \mu(r;k,l) = \frac{1}{2}(p_r - p_{r+1}) \mu(k) \mu(l) \]  

(3.4)

where

\[ \mu(k) = \begin{dcases} 
\frac{1}{p_k - \frac{1}{p_{k-1}}} & k > r+1 \\
\frac{1}{p_{r+1}} & k = r+1 
\end{dcases} \]  

(3.5)

We note that \( p_0 \) is absent from \( \hat{q} \), since even if the index \( k, l \) were allowed to take the value 0, it would appear in the vanishing combination \( p_0q_0 \). The \( p_0 \) dependence can therefore only arise from the multiplicity. Collecting the terms in \( p_0 \) we get

\[ I_{Rep} = \frac{n}{2}p_0 \sum_{k,l=1}^{R+1} \mu(k) \mu(l) \log (p^2 + \lambda(0;k,l)) + \frac{n}{2} \sum_{r=1}^{R} p_r \sum_{k,l=r+1}^{R+1} \log \left( \frac{p^2 + \lambda(r;k,l)}{p^2 + \lambda(r-1;k,l)} \right) \]  

(3.6)

With the use of (2.3)-(2.7) and (3.1)-(3.3) we get

\[ \lambda(r;k,l) = -2E(q_R) + g\left(\frac{q_R}{R}\right)^2 \left(\frac{1}{2}(k^2 + l^2) - r^2 - (k + l) + 1\right) \]

\[ = g\left(\frac{q_R}{R}\right)^2 \left(\frac{k}{R}\right)^2 + \left(\frac{l}{R}\right)^2 - r^2 - (k + l - \frac{5}{6}) \]  

(3.7)

We note that the lowest (Replicon) eigenvalue is given by

\[ \lambda(r;r+1,r+1) = -\frac{1}{6}\left(\frac{q_R}{R}\right)^2 \quad r = 0, 1, 2, ..., R \]  

(3.8)

hence we find an instability, except in the limit \( R \to \infty \) where this is suppressed. All other eigenvalues are positive. We thus have in the Parisi limit \( R + 1 \) zero modes arising from the negative eigenvalues (3.8). In that limit, one has

\[ I_{Rep} = \frac{n}{2} \int_0^1 x dx \partial_x \left\{ \int_0^1 \frac{dk}{k} \partial_k \int_0^1 \frac{dl}{l} \partial_l \log \left( p^2 + g\left(\frac{q_R}{R}\right)^2 (k^2 + l^2 - 2x^2) \right) \right\} \]

\[ + \frac{n^2}{2} \int_0^1 \frac{dk}{k} \partial_k \int_0^1 \frac{dl}{l} \partial_l \log \left( p^2 + g\left(\frac{q_R}{R}\right)^2 (k^2 + l^2) \right) \]  

(3.9)

which coincides with (AM.12).
4. Fluctuations: the longitudinal-anomalous (LA) sector

We now have

\[ n\delta f_{LA} = \frac{V}{2} \int \frac{d^Dp}{(2\pi)^D} \delta I_{LA}(p) \]

\[ \delta I_{LA}(p) = n \sum_{k=0}^{R+1} \mu(k) \log \det \Delta_k(r,s) \]

\[ \Delta_k(r,s) = \delta_{r,s}^{Kr} - \frac{wgq_{\text{min}}(r,s)}{\Lambda_k^{(r)}} \delta_{s}^{(k-1)} \]

where \( \delta^{Kr} \) denotes the Kronecker delta, while we have

\[ \delta_{s}^{(k-1)} = p_s^{(k-1)} - p_{s+1}^{(k-1)} \]

\[ p_s^{(k-1)} = \begin{cases} p_s & s \geq k - 1 \\ 2p_s & s < k - 1 \end{cases} \]

\[ \Lambda_k^{(r)} = \begin{cases} p^2 + \lambda(r; r + 1, k) & k \geq r + 1 \\ p^2 + \lambda(r; r + 1, r + 1) & k < r + 1 \end{cases} \]

Expanding the determinant \( \Delta_k(r,s) \) yields

\[ \det \Delta_k(r,s) = 1 + \sum_{m=1}^{\infty} (-wg)^n \sum_{0 \leq s_1 < s_2 < \cdots < s_m} \prod_{i=1}^{m} (q_{s_i} - q_{s_{i-1}}) \frac{\delta_{s_i}^{(k-1)}}{\Lambda_{s_i}^{(s_i)}} \]

where we have set \( s_0 \equiv 0 \). In order to have \( p_0 \) occurring in the determinant, i.e. in one of the \( \delta_{s}^{(k-1)} \), we need \( s_i = 0 \) hence the only possible term is \( s_1 = 0 \), but the prefactor \( q_{s_1} - q_{s_0} \equiv q_{s_1} = q_0 \) vanishes and there is no \( p_0 \) contribution form the Log again. The only \( p_0 \) contribution comes from the multiplicity \( n\mu(k) \) which now cannot sustain an \( n^2 \) contribution\(^3\).

5. Conclusion

With no contribution to fluctuations from the LA sector, we conclude that the full answer is given by the contribution from the Replicon sector as of (3.1) and (3.9), thus corroborating the result of \([1]\).

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\(^3\) One may then ask what becomes of the term \( k = 0 \) which has a factor \( n/p_0 \). It is actually given by \( \delta I^0_{LA} = \log \det(\Delta_0(r,s)/\Delta_1(r,s)) \), but with \( \delta_0^{(0)} = \delta_0^{(-1)} - p_0\delta^{Kr}_{s,0} \) whereas \( \Lambda_1(s) = \Lambda_0(s) \). Again, \( \delta I^0_{LA} \) reduces to a contribution \( \sim p_0\delta^{Kr}_{s,0} \) which vanishes with \( q_0 \).
6. Extension to Aspelmeier, Moore, Young calculation

There the authors give an analytic answer to a long standing problem: computing the interface free energy of the Ising spin glass. They show that (part of) the answer is obtained by computing $Z^n Z^m$ and, in the associated free energy, leaving aside the terms in $(n + m)$, $(n + m)^2$ and keeping the terms in $nm$. From (3.6) we clearly see that, leaving aside a $(n + m)^2$ contribution, one is left with a term

$$I_P(p) \equiv I_{Rep}(p) = -nm \sum_{k,l \geq 1} \mu(k)\mu(l)\log(p^2 + \lambda(0; k, l))$$

(replacing the quadratic term of (3.6) and associated with periodic boundary conditions). The fluctuation contributions from the off-diagonal blocks of the Hessian (mixed sector, associated with antiperiodic boundary conditions) is much easier to deal with since it is exactly given by

$$I_{AP}(p) = +nm \sum_{k,l=0}^{R+1} \mu(k)\mu(l)\log(p^2 + \lambda(0; k, l))$$

(eigenvalues and multiplicities matching the $R = 0$ calculation presented earlier by the same authors [7]. The LA sector is represented here respectively by $k = l = 0$ and $k = 0, l \geq 1$ or $k \geq 1, l = 0$, the Replicon sector by $k, l \geq 1$ as in (6.1). But with $\hat{q}_1 = \hat{q}_0$ we now see that the two contributions periodic (6.1) and antiperiodic (6.2) are formally identical but for their sign. This result again matches the one proposed in [8].

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