The Yannelis–Prabhakar Theorem on Upper Semi-Continuous Selections in Paracompact Spaces: Extensions and Applications

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Abstract: In a 1983 paper, Yannelis–Prabhakar rely on Michael’s selection theorem to guarantee a continuous selection in the context of the existence of maximal elements and equilibria in abstract economies. In this tribute to Nicholas Yannelis, we root this paper in Chapter II of Yannelis’ 1983 Rochester Ph.D. dissertation, and identify its pioneering application of the paracompactness condition to current and ongoing work of Yannelis and his co-authors, and to mathematical economics more generally. We move beyond the literature to provide a necessary and sufficient condition for upper semi-continuous local and global selections of correspondences, and to provide application to five domains of Yannelis’ interests: Berge’s maximum theorem, the Gale-Nikaido-Debreu lemma, the Gale-McKenzie survival assumption, Shafer’s non-transitive setting, and the Anderson-Khan-Rashid approximate existence theorem. The last resonates with Chapter VI of the Yannelis’ dissertation. (136 words)

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He was one of the young founders ... He has always been on the side of angels. He has not sold out to the interests nor bought in to romantic idiocies. He stands up to speak for the public interest as his scholarly findings interpret that interest, letting those who will charm college Deans.\(^1\)

Paul Samuelson (1964) on Abba Lerner

A fresh survey ... requires the reverse of a chronological order. The model provides by itself a prism with which to diffract the paradigms of various brands of neoclassicism; and, self-reflexively, it can serve to help judge the author; the unity of whose scientific vision then becomes visible.\(^2\)

Paul Samuelson (1987) on Piero Sraffa

When we turn to take a retrospective view, we are always looking and looking away at the same time.\(^3\)

W. G. Sebald (2003)

1 Introduction

With the turn of political economy to economic science, portraiture has certainly gone out of fashion – Paul Samuelson may well have been the last great portraitist of our times.\(^4\) A portrait enables an overview of an scholar’s oeuvre, determines what is worth giving salience to, and what needs to be sidelined in a particular work, parts that need darker colours and those that could do with shading out; but the professional use of the portrait extends beyond the one portrayed, and the portrayer(s), not to speak of the art of portraying itself. A portrait sheds important light on the profession and the professional understanding of the subject at the time that it is

\(^1\)Samuelson is referring to ReStud, adding that Lerner’s “contributions as much as anyone’s made its first volumes so exciting.” Without overstating the parallel, Yannelis was one of the young founders of Economic Theory along with Aliprantis, Khan, Majumdar and Cass. In a portrait that relies on, and revolves around, Yannelis’ first publication, Samuelson’s continuation is also relevant: “You can gauge the quality of a scientist by his first paper. When the editors asked me to sing of the man and his work, I was delighted. For like Tom Sawyer, who enjoyed his own funeral, I believe the best wakes to be those in which the guest of honour is present in the full vigour of his powers.” See Chapter (3:183) referring to Chapter 183 of Volume 3) in the Collected Papers; Samuelson (1966-2011). Throughout this work, we conform to this convention in quotations, and shall be relying on on the individual author’s own italicizations.

\(^2\)Samuelson is referring Sraffa’s magnum opus “The Production of Commodities by Means of Commodities.” He continues, “First comes his 1960 book, which has spawned an extensive literature but still needs – if the technology is to be adequately handled – to have Sraffa’s special equalities embedded in the general inequalities–equalities of the 1937 von Neumann model.” The model is the “essentially completed Sraffa–Leontief circulating capital model;” the others are Marx and Ricardo; and Sraffa himself is sighted, along with Dobb, as Ricardo’s editor; see (6: 411), also (6:410 and 416).

\(^3\)See page ix of Sebald’s (2003) “Natural History of Destruction.”

\(^4\)In addition to his portraits of Lerner, Tsuru (3;297) and Solow (7: 534), see Part XVI in Volume 2, Part XVI in Volume 3, Part X in Volume 4, Parts II and VII in Volume 5, Part II in Volume 6 and Parts IX and X in Volume 7. Keynes’ Essays in Biography constitute another worthy exemplar; see Keynes (2010 (1933)).
being drawn. As such, it is useful if well-done and if it does not descend into hagiography. Even if the talents and gifts of Paul Samuelson are out of our reach, the authors feel that one ought nevertheless to try. This is one such attempt: a portrait of the mathematical economist Nicholas C. Yannelis done for his sixty-fifth birthday, surely with affection and admiration, but also with some measured distance. It is in keeping with an eye to the development of mathematical economics since Nicholas Yannelis’ wrote his 1983 Ph.D. dissertation, and his influential 1983 paper with N. Prabhakar, (still somewhat neglected), down to the present as encapsulated in his 2016 paper with Wei He.

With this preamble then out of the way, we can get to work. In order to attain some sense of perspective, the focus of this work shall be on the Yannelis-Prabhakar selection theorem for lower semi-continuous correspondences in the setting of paracompact spaces – it shall chart what the authors see to be progress in the subject of equilibrium existence theory in its Walrasian and Cournotian-Nash modes since this paper was written roughly four decades ago. To be sure, the paper already had its echo in the second chapter of Yannelis’ dissertation, his debut in the public domain, a chapter written in the light and the shade of Mas-Colell’s 1974 dramatic announcement that for “the general equilibrium Walrasian model to be well-defined and consistent (i.e., for it to have a solution), the hypotheses of completeness and transitivity of consumer’s preferences are not needed.” His paper, supplemented by the efforts of Gale, Shafer and Sonnenschein served as the leitmotiv not only for the thesis but also for Yannelis’ work for the major part of the subsequent decade, and still showing traces in his work.

Since our primary focus is on Yannelis and Prabhakar (1983) (henceforth YP-Paper), and since we see it to be rooted in Chapter 1 of the dissertation (henceforth Y-Chapter), we use the remainder of this introduction to outline the chapter, and to frame it in the dissertation as a whole. Towards this end, we begin with a summary statement that the ambitious and synthetic reach of the chapter is already itself evident in its single sentence introduction:

The purpose of this chapter is to develop mathematical tools which can be applied to prove in a very simple and general way the nonemptiness of demand sets, the existence of competitive equilibrium and the existence of a Nash equilibrium.

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5Samuelson (2007) and Stigler (1965), among others, also wrote on this issue; for the former, see Chapter (527, Volume 7) titled “Reflections on How Biographies of Individual Scholars can Relate to a Science’s Biography.” As to the latter, see Khan and Schlee (2002) for a view that strongly dissents from that of Stigler.

6We use the term in a technical sense, as in Kemp (1978), and as we shall have occasion to emphasize in the concluding Section 4, we exclude Yannelis’ more recent work on the economics of information. Kemp writes in the context of linear perspective “[O]ne of the primary characteristics of linear perspective is that it describes precisely the point at which one object occludes part of another more distant form (p.138).”

7The ordering of the authors’ names already alludes to this.

8The relevant references are Mas-Colell (1974), along with Gale and Mas-Colell (1975, 1979) and Shafer (1974, 1976) and Shafer and Sonnenschein (1975).
The chapter relies on Michael’s selection theorem invoked by\textsuperscript{9} [Gale and Mas-Colell (1975)], and on Fan’s 1961 generalization of the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma, again invoked in mathematical economics by\textsuperscript{10} [Gale (1955)]: the first emphasized lower-semi-continuous correspondences and the second, topological vector spaces, emphases that the chapter clearly imbibes. With a quick recording of a fixed point theorem, and its subsequent application to the existence of a Nash equilibrium of a finite normal form game with payoffs generated by non-ordered preferences, it turns to showing the non-emptiness of demand sets in consumer theory, and to an alternative proof of Geistdorfer-Florenzano’s generalization of the Gale-Nikaido-Debreu (GND) lemma. The thrust of the chapter can best be captured in the (following) remark; see (Yannelis, 1983, p. 21).

Although the relationship of open graph and openness of lower and upper section is known, the relationship of open sections with lower semi-continuity is still unknown. Below we examine this relationship.

As far as mathematical economics is concerned, rather than the results achieved, the chapter is pioneering in bringing the relevant tools to the subject, and its sensitivity to what then were regarded as purely technical (i.e. mathematical) subtleties.

However, it may be worthwhile for the reader to note, if only for the historical record, that the 1983 chapter and the subsequent YP-Paper, are largely tangential to the the preoccupation of the Ph.D itself, a dissertation of six substantive chapters collected under the title \textit{Solution Concepts in Economic Theory}. Its explicitly-expressed gratitude to Professors Lionel W. McKenzie and George Metakides hints at its broad thrust:

Professor McKenzie introduced me to General Equilibrium Theory and Professor Metakides to Logic and Nonstandard Analysis.\textsuperscript{11}

The acknowledgement is of interest in that it signals the application of mathematical logic, model theory in particular, to what Yannelis then saw as classical general equilibrium theory.\textsuperscript{12}

\textsuperscript{9}It is worth noting what [Gale and Mas-Colell (1979)] say in this context: “The proof needs no modification because the selection theorem of Michael which is used requires only a lower semi-continuous correspondence;” also see [Gale and Mas-Colell (1977)]. We shall return to this issue of the proof in the sequel.

\textsuperscript{10}Gale writes with reference to the 1954 paper of Arrow-Debreu, “However, where the latter makes use of some rather sophisticated results of algebraic topology, we shall obtain a simple proof of the existence of an equilibrium using a well-known lemma of -elementary combinatorial topology.” This categorization of the mathematical register is surely of interest for subsequent curriculum development in graduate courses in mathematical economics.

\textsuperscript{11}Among others, Emmanuel M. Drandakis and S. A. Ozga are two additional names that deserve mention for the record. Yannelis writes, “I also wish to express my indebtedness to my first teacher of Mathematical Economics, Professor Emmanuel M. Drandakis of the Athens School of Economics. My interest in General Equilibrium Theory was inspired largely by my undergraduate with him. Thanks are also due to the late Dr, S. A. Ozga of the London School of Economics, who as my supervisor, correctly advised me to come to Rochester.”

\textsuperscript{12}We turn to Lionel McKenzie’s influence on Yannelis in Section 4; the reader is also referred to Footnotes 54, 58 and 59 below and the text they footnote.
chapter that we sight here was something for him to get out of the way before his substantive work of the role of a large number of agents in eliminating the convexity assumption in static general equilibrium theory, and in simultaneous-play normal form game theory, could begin. The essential subtext of the dissertation itself is nonstandard analysis: of the five remaining substantive chapters, three explicitly involve infinitesimal analysis, and the other two involve the Shapley-Folkman theorem. Chapter III presents an existence theorem for a “nonstandard exchange economy” with a special subsection in the chapter on the “novelty of a proof based entirely on Q-concepts,” a passing consideration of consequence to subsequent history of this subject that one can trace with some legitimacy to Anderson’s definitive paper. Chapter IV and V continue with a nonstandard exchange economy but with “large” and “small” traders, the first on “non-discriminatory” allocations, and the second on “fair allocations.” Chapters VI and VII turn to questions of asymptotic implementation of the Shapley-value and Walrasian allocations, and answer them as applications of the Shapley-Folkman-Starr theorem.

With these remarks on portraiture, the chapter and the dissertation behind us, we read the YP-paper not as a synecdoche for Yannelis’ oeuvre even only up to the present— the latter is a task surely outside the authors’ own competence, and perhaps outside the scope of a single essay. As such, the outline is simple: there are two parts to what we present: the theory and the applications. The highlights of the first are a framing of the 1983 paper in the light of current work and two equivalence results spelling out necessary and sufficient conditions for local selections that have so far proved elusive. The highlights of the second are four applications dictated by Yannelis’ interests in classical Walrasian general equilibrium theory, both in the dissertation and as they evolved in the corpus of his work so far: Berge’s maximum theorem for utility functions and preferences, a generalization of the GND lemma for discontinuous preferences, an existence result for Shafer’s take on the Walrasian setting, and an existence result for approximate equilibrium for a pure exchange economy with discontinuous preferences.

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13 Yannelis writes, “Note that in contrast to Brown [11] and Khan [29], our proof is based entirely on Q-concepts. This gives us the opportunity to appeal directly to standard theorems using the “transfer principle.” Anderson (1978) and in his subsequent Handbook survey articles explicitly credits Khan-Rashid.

14 We leave for future work carried out in the register of the history of the application of nonstandard analysis to economics; in particular, the specific advances Yannelis’ dissertation makes over dissertations of a decade earlier, Khan (1973), Rashid (1976), Lewis (1977), Tal (1978), as well as those of his contemporaries Nomura (1984) and Emmons (1985), and follower Raub (1998).

15 As we shall have occasion to emphasize in the last section of this this paper, we neglect in particular all of Yannelis’ work on the economics of informations and incentives, on co-operative game theory and on decision theory.
2 The Mathematical Contribution

The plan of this part of the work is straightforward: after a conceptual overview of the problem that revolves around Michael’s selection theorem and considers a convex-valued lower semi-continuous correspondence defined on a paracompact domain, we turn to an exegesis of the Yannelis-Prabhakar paper, track how it feeds into work of four decades in both applied mathematics and mathematical economics, and in particular, extend his purview to show what extension it enables.

2.1 The Conceptual Antecedents: Michael-Browder-Fan-Tarafdar

This subsection revolves around four named theorems: a selection theorem, two fixed point theorems and an early result that inaugurated the application of the Hartman-Stampacchia “variational inequalities.” However, rather than the theorems themselves, our emphasis shall be on the underlying connections between the ideas that have been well-understood by the cognoscenti but have never been made explicit. In particular, we make explicit the local-versus-global subtext, and identify neglected contributions of Hakuhara and Tarafdar.

But first things first: we begin with selection theorems and Himmelberg’s review of Parthasarathy (1972), in which the context of the problem that the author’s book addresses is laid out. He writes: .

Let $F : X \rightarrow \mathcal{A}(Y)$ be a function from a set $X$ to the collection of non-empty subsets $\mathcal{A}(Y)$ of $Y$, i.e., for each $x \in X$; $F(x)$ is a non-empty subset of $Y$. A function $f : X \rightarrow Y$ is called a selector for $F$ if $f(x) \in F(x)$ for all $x \in X$. It is interesting and useful to know, given various continuity and measurability conditions on $F$, that $F$ has selectors satisfying similar conditions. This book collects and organizes such theorems and provides examples of their application to stochastic games, and the solution of generalized differential equations.

He justly describes Chapter 1 as a survey of the results of Michael on continuous selections for lower semi-continuous multi-functions and singles out the following characterization of paracompactness in this chapter.

**Definition 1.** A Hausdorff space $X$ is **paracompact** if each open covering of $X$ has an open, locally-finite refinement.

**Theorem (Michael 1956).** Any Hausdorff space $X$ is paracompact if and only if

(i) every open cover of $X$ has a partition of unity subordinated to it,

(ii) every open cover of $X$ has a locally finite partition of unity subordinated to it,
(iii) each lower semi-continuous function $X \to A(Y)$, $Y$ a Banach space, with closed convex values has a continuous selector.\footnote{Himmelberg writes “measurable” instead of “continuous; initially, we thought of keeping this slip in the statement of the theorem for dramatic emphasis on how the measure-theoretic register is intimately involved with the topological, and is always hovering in the background. This is from Himmelberg’s review of the book in Mathematical Reviews available on MathSciNet; in order to minimize references, we simply cite the review number, in this case MR0417368. Note that Steen and Seebach (1978) refer to a $T_1$-space as a Frechet space: we shall rely on Dugundji (1966) and Willard (1970) as our basic references for topological terminology throughout this work.}

In the light of what we are seeing in this essay as the leitmotiv of Yannelis’ contribution to mathematical economics and to economic theory, this is a most fortuitous starting point, and brings out the intimate connection between paracompactness and the selection problem. However, it is worth noting that Dieudonné’s 1944 concept has proved elusive: a generalization of both compact and metrizable spaces, it is also intimately connected to the existence of partitions of unity and the extension problem. In terms of the way we are setting out the narrative here, what is of consequence is that it allows a move from the local to the global.\footnote{As we shall see below, it is this aspect that will be focused on in He and Yannelis (2016) and others. For a reader interested in paracompactness per se, we refer to Michael (2004, 2010, 2011) and move on.} Figure 1 below illustrates commonly used topological properties that are weakening or strengthening on paracompactness.\footnote{For the definitions of these topological concepts, see any of the standard texts such as Dugundji (1966), Munkres (2000), Willard (1970) and Steen and Seebach (1978). For the concept of stratifiability and its relations, see the work of Ćeder and of Borges referenced in Gruenhage (2014).}

![Figure 1: A Framing of Paracompactness](image)

We now turn to a fixed point theorem that serves as another useful point from which to see the work that we report here: from the dissertation chapter via YP-Paper down to the present. Its principal vernacular – selection, paracompactness, lower semi-continuity, infinite-dimensionality – are all signature terms in the trajectory of this work.
Theorem (Browder 1968). Let $X$ be a non-empty, compact and convex subset of a Hausdorff topological vector space, and $F : X \to X$ a correspondence with non-empty and convex values such that for each $y$ in $X$, $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ is open in $X$. Then there exists $x^* \in K$ such that $x^* \in F(x^*)$.

A subtext of both of the results presented above is the postulate of continuity.

Definition 2. Let $X, Y$ be non-empty topological spaces. A correspondence $P : X \to Y$

(i) has the local intersection property if for all $x \in X$ with $P(x) \neq \emptyset$ there exists $y^x \in Y$ and an open neighborhood $U^x$ of $x$ such that $y^x \in P(z)$ for all $z \in U^x$.

(ii) has open fibers if for all $y \in Y$, $\{x \in X \mid y \in P(x)\}$ is open.

(iii) has open graph if for all $x \in X$, $\{(x, y) \in X \times Y \mid y \in P(x)\}$ is open.

(iv) is upper semi-continuous (usc) if $\{x \in X \mid P(x) \subseteq V\}$ is open for all $x \in X$ and all open $V \subseteq Y$.

(v) is lower semi-continuous (lsc) if $\{x \in X \mid P(x) \cap V \neq \emptyset\}$ is open for all $x \in X$ and all open $V \subseteq Y$.

It is by now well-known that the open graph property implies the open fibers property, which implies the local intersection property.

Definition 3. Let $X, Y$ be non-empty topological spaces. A correspondence $P : X \to Y$ is Tarafdar-continuous if there exists an open cover $\{O^y\}_{y \in Y}$ of $\{x \in X \mid P(x) \neq \emptyset\}$ such that $O^y \subseteq P^{-1}(y) = \{x \in X \mid y \in P(x)\}$ for all $y \in Y$.

Next, we show that Tarafdar-continuity and the local intersection property are equivalent.

Proposition 1. Let $X, Y$ be non-empty topological spaces and $P : X \to Y$ a correspondence. Then $P$ is Tarafdar-continuous if and only if it has the local intersection property.

This result highlights that the local intersection property was first introduced in [Tarafdar 1977]. This property also goes by other names in the literature; a terminological cleaning up is overdue.

Proof of Proposition 1. Assume $P$ is Tarafdar-continuous. Then there exists an open covering $\{O^y\}_{y \in Y}$ of $\{x \in X \mid P(x) \neq \emptyset\}$ such that $O^y \subseteq P^{-1}(y) = \{x \in X \mid y \in P(x)\}$ for all $y \in Y$. Pick

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19 Tian and Zhou (1995) refer to it as the transfer open lower sections and Uyanik (2016) calls it the constant neighborhood selection property; see also Mehta and Tarafdar (1987) and Tarafdar (1991). It is worth noting that Mehta and Tarafdar (1987) provide equivalence of various theorems that use versions of local intersection property, but they do not explicitly show the equivalence of the continuity assumptions without the presence of other assumptions on the correspondence and the spaces. The discussion and example above highlight the importance of such consideration.
\(x \in X\) such that \(P(x) \neq \emptyset\). Since there exists some \(y \in Y\) such that \(x \in O^y\), and \(y \in P(z)\) for all \(z \in O^y\), \(P\) has the local intersection property.

Now assume \(P\) has the local intersection property. Then for all \(x\) with \(P(x) \neq \emptyset\), there exists \(y^x \in Y\) and an open neighborhood \(U^x\) of \(x\) such that \(y^x \in P(z)\) for all \(z \in U^x\). Define \(\hat{Y}\) as the collection of all such \(y^x\), i.e., \(\hat{Y} = \{y^x| x \in X, P(x) \neq \emptyset\}\). For any \(y \in \hat{Y}\), let \(X^y = \{x \in X | y^x = y\}\). Define \(O^y = \bigcup_{x \in X^y} U^x\) for all \(y^x \in \hat{Y}\) and \(O^y = \emptyset\) for all \(y \notin \hat{Y}\). It is clear that \(\{O^y\}_{y \in Y}\) is an open covering of \(\{x \in X | P(x) \neq \emptyset\}\) and \(O^y \subseteq P^{-1}(y) = \{x \in X | y \in P(x)\}\) for all \(y \in Y\). Therefore, \(P\) is Tarafdar-continuous. \(\blacksquare\)

Tarafdar-continuity, as expressed above, is equivalent to the version stated in Tarafdar (1977) when the correspondence \(P\) has non-empty values, which is assumed in Tarafdar’s theorem, and indeed in most fixed point and selection theorems. For correspondences with empty values, the version we present above is weaker. That is, if we state the definition by replacing the set \(\{x \in X | P(x) \neq \emptyset\}\) with \(X\), then Tarafdar-continuity will be stronger than the local intersection property for correspondences which have empty values. In this case, the proof of the proposition above shows that local intersection property is implied by Tarafdar-continuity. However, the converse relationship does not hold. In order to see this, let \([0, 1]\) be the unit interval with the usual topological structure and the correspondence \(P : [0, 1] \to [0, 1]\) defined as \(P(x) = 1\) if \(x > 0\) and \(P(0) = \emptyset\). It is clear that \(P\) has open fibers, hence has the local intersection property. However, for all \(y \neq 1\), \(O^y = \emptyset\) has to hold and \(O^1\) can be at most \((0, 1]\). Hence \(\{O^y\}_{y \in [0, 1]}\) cannot yield an open covering of \([0, 1]\). Therefore, \(P\) is not Tarafdar-continuous.

We can now present the fixed point theorem that motivated Tarafdar-continuity\(^{20}\)

**Theorem (Tarafdar 1977).** Let \(X\) be a non-empty, compact and convex subset of a Hausdorff topological vector space. Let \(P : X \to X\) be a Tarafdar-continuous correspondence with non-empty and convex values. Then there exists \(x^* \in K\) such that \(x^* \in P(x^*)\).

So far we have presented as our mathematical antecedents, a selection theorem due to Michael, and two fixed point theorems, one attributable to Browder and the other to Tarafdar. However, as already noted, Yannelis relied on Fan, and we turn to a 1969 result of his that has some consequence for the narrative that we are in the process of developing.

**Definition 4 (Browder 1954).** Let \(X\) be a reflexive complex Banach space and \(X^*\) its dual. A function \(T : X \to X^*\) is **demi-continuous** if it is continuous from the strong topology of \(X\) to the weak topology of \(X^*\).

\(^{20}\)For Tarafdar’s perspective on fixed point theory, see Tarafdar and Chowdhury (2008). Even though Border (1985) is outdated, the reader may find his Chapter 8 on variational inequalities useful, though the reader should bear in mind Yannelis (1985) is not in the book, and even though he cites his YP-Paper, he does not give it quite the importance that it deserves.
Definition 5 (Fan 1969). Let $E$ be a real Hausdorff topological vector space, and let $X \subseteq E$. A correspondence $F : X \rightrightarrows E$ with non-empty values is upper demi-continuous if for every $x \in X$ and any open half-space $H$ in $E$ containing $F(x)$, there is a neighborhood $N$ of $x$ in $X$ such that $F(x') \subseteq H$ for all $x' \in N$.

It is not difficult to see that an usc correspondence is upper demi-continuous. However, an upper demi-continuous correspondence need not be usc; see Aliprantis and Border (2006, Example 17.39, p. 575). The relationship between these concepts is illustrated in Figure 2.

![Figure 2: Continuity Concepts for Correspondences](image)

It is conventional wisdom in the profession that for questions pertaining to existence of solution concepts, fixed point theorems constitute the first line of attack, and the geometric form of the Hahn-Banach theorems typically reserved for their price characterizations. An early result 1969 result of Ky Fan’s diffused such a sharp line and brought the separation theorem to bear on the fixed point theorem, and in doing so, became a direct precursor of the application of the Hartman-Stampacchia variational inequalities. But before the result, the question to which it is an answer; the problem that furnishes the context.

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21 Note that an open half-space $H$ in $E$ is a set of the form $H = \{x \in E | \psi(x) > r\}$, where $\psi$ is a continuous linear form on $E$, not identically zero, and $r$ is a real number. For details consider Tarafdar and Chowdhury (2008).

22 For the definition of quasi upper semi-continuity, see Podczeck (1997). The idea is the same as Hausdorff upper semi-continuity presented in Sun (1997). This concepts goes back to Aubin and Ekeland (1984), see Yannelis (1990) for details. Moreover, for a comprehensive discussion of the relationship between different continuity postulates in economic theory, see Khan and Uyanık (2019a).

23 As we shall see below, McCabe (1981) and Yannelis (1985) were the firsts to breach this line in their proofs of the GND lemma. Mas-Colell’s proof of the existence of the bargaining set furnishes another example.

24 We refer the reader for a comprehensive discussion of this subject and its application to mathematical economics to Hartman and Stampacchia (1966), Tarafdar and Chowdhury (2008), and McLean (2020) for a state of the art treatment. We shall return to this subject in Section 3.2 below.

25 As McLean (2020) writes “The Ky Fan equilibrium problem (aka the Ky Fan Inequality) is a fundamental result of non-linear analysis with myriad applications in optimization theory, fixed point theory, mathematical economics and game theory to name just a few.” Also see Footnote 26 below, and the text that it footnotes.
Ky Fan Equilibrium Problem: Given a set $X$ and a function $f : X \times X \to \mathbb{R}$, find $\bar{x} \in X$ such that $f(\bar{x}, x) \leq 0$ for each $x \in X$.

This problem can be seen in a slightly general setting.

A Variational Inequality: Given two sets $X$ and $Y$ and a function $f : X \times Y \to \mathbb{R}$, find $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for each $y \in Y$.

However, we can now consider a concretization,

A Variational Inequality for TVS: Let $X$ be a locally convex topological vector space with $X^*$ its dual space, $K \subseteq X$ a non-empty compact convex set, a continuous mapping $T : K \to X^*$ find $\bar{x} \in X$ such that $\langle T(\bar{x}), \bar{x} - y \rangle \leq 0$ for each $x \in X$.

Browder (1967) offers a generalization of the fixed point theorems of Schauder and Tychonoff and its reviewer writes that they are consequences of a theorem that solves the above problem, and that the proof of this Browder’s theorem “uses a continuous partition of unity in a fashion reminiscent of M. Hukuhara’s elegant proof of the Schauder theorem.”

We can now present

Theorem (Fan 1969). Let $X$ be a non-empty compact convex set in a Hausdorff topological vector space $E$. Let $F, G$ be two upper demi-continuous set-valued mappings defined on $X$ such that:

(i) For each $x \in X$, $F(x)$ and $G(x)$ are non-empty subsets of $E$.

(ii) For every $x \in X$, there exist three points $y \in X$, $u \in F(x), v \in G(x)$ and a real number $\lambda > 0$ such that $y - x = \lambda(u - v)$.

Then there exists a point $\bar{x} \in X$ for which $F(\bar{x})$ and $G(\bar{x})$ cannot be strictly separated by a closed hyperplane, i.e., we cannot find a continuous linear form $\psi$ on $E$ and a real number $r$ such that $\psi(x) < r$ for $x \in F(\bar{x})$ and $\psi(y) > r$ for $y \in G(\bar{x})$.

Readers may find the following example useful in understanding assumption (ii) in Fan’s theorem. Let $E = \mathbb{R}^2$ with the usual linear and topological properties and $X = [0, 1]^2$. Define correspondences $F, G$ as $F(x) = \{y \in E | 0.5 < y_1 = y_2 \leq 1\}$ and $G(x) = \{y \in E | 0 \leq y_1 = y_2 \leq 0.5\}$ for all $x \in X$. Pick $x = (1, 1)$. For any $u \in F(x), v \in G(x)$ and $\lambda > 0$, $\lambda(u - v) = (a, a), a \in (0, 0.5)$. It is easy to see that for all $y \in X$, $(y - x)_i \leq 0$ for some index $i = 1, 2$. Therefore, assumption (ii) fails. Note

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26For Belluce’s review, see MR223944 of Browder’s paper.
also that $F(x)$ and $G(x)$ cannot be strictly separated by a closed hyperplane for all $x \in X$. As you see in this example, as illustrated in Figure 3, the algebraic boundary of $X$ plays a crucial role. In fact, for any other point in $X$, assumption (ii) is automatically satisfied; see Fan (1969, p. 237).  

2.2 The Initiation: Yannelis-Prabhakar (1983)

Thanks to Mas-Colell (1974), non-ordered preferences were very much in the neoclassical air in the decade since 1974, and with this as our point of departure, we begin the exegesis of YP-Paper with the contemporaneous dissertation chapter is solely in the register of Walrasian equilibrium theory with a “weak form of Walras’ law”: the commonality lies in his invocation and use of Fan’s generalization, as in the chapter.

We now turn to the pièce de résistance of this tribute, and single out the following result in YP-Paper. To the authors knowledge, this was the first time that the notion of a paracompact set was used in mathematical economics.

**Theorem (Yannelis-Prabhakar 1983).** Let $X$ be a paracompact Hausdorff space and $Y$ be a topological vector space. Suppose $P : X \rightarrow Y$ is a non-empty and convex valued correspondence with open fibers. Then $P$ has a continuous selector.

Presents the proof. (1) The Brouwer fixed point theorem, in its classical single-valued form. (2) The existence for a finite covering of a compact space of a partition of unity subordinated to this covering. Indeed, one can sight Michael and Browder as the two mathematicians. As it happened there was an additional set of lecture notes published two years earlier in which Michael has his own overview of the problem.

The composition of the paper is straightforward: titled “Existence of maximal elements and equilibria in linear topological spaces,” it departs from the chapter in a move beyond (i) the finiteness assumption and in allowing an infinite number of commodities and a countably infinite number of agents, (ii) Walrasian equilibrium theory to the abstract economy setting of Debreu (1952) and Shafer and Sonnenschein (1975), and (iii) Fan and Michael to an explicit invocation of the Browder (1968) and his open-fiber assumption as well as that of lower semi-continuity of a correspondence. We say that the correspondence $E : D \rightarrow \mathbb{R}^n$ is upper demi-continuous if

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27 We return to demi-continuity in Section 3.2 on the generalizations of the Gale-Nikaido-Debreu Lemma.

28 In the sequel, we shall simply refer to YP-Paper as the “paper,” and to Chapter 2 of Yannelis’ dissertation as the “chapter.”

29 For a detailed reference and discussion on weak and strong versions of Walras’ Law see Maskin and Roberts (2008) and Cornet (2020), and see also McCabe (1981), Yannelis (1985), Mehta and Tarafdar (1987), Uyanik (2016) and He and Yannelis (2017).
for each open half-space $B$ in $H \in \mathbb{R}^n$, $E^+[H] = p : E(p) \subseteq E$ is open in $D$,” a definition due to Browder (1964) and Fan (1969).  

In terms of a more detailed listing, after the preliminary first and second sections, the chapters’ echo is clearest in Sections 3 and 4 that are devoted to selection and fixed point theorems: a lot of the definitions are simply transcribed directly from the chapter. However, it is in the elimination of the the section on the GND lemma, and transmutation of a game into an abstract economy that Sections 5 and 6 constitute the pièce de résistance of the paper. Section 7 lists examples of topological vector spaces, and is more a reflection of the professional unfamiliarity with functional analysis in the infinite-dimensional mode, even eleven years after Bewley’s 1972 paper.

In terms of a bald statement, the Yannelis-Prabhakar paper is pioneering on several counts: (i) its invocation of the lower semi-continuity of a preference correspondence, (ii) in its systematic use of Michael’s selection theorem, (iii) in its assumption of the paracompactness of an action set, and (iv) in its topological sensitivity in the dropping of the a locally-convex adjective, and considering only topological vector space. But all these are micro–criteria; at the macro level, the biggest contribution of the YP-Paper and the Y-Chapter is its resolute move from finite to infinite-dimensions, one that was to lead only two years later in Yannelis (1985) an infinite-dimensional version of the GND.

2.3 The State of the Art: He-Yannelis (2016) and Cornet (2020)

Our focus in this subsection is to identify the traces of the 1983 work in the statement and understanding of the subject as of now: what is it in current and ongoing work that would be new to the 1983 authors, as in both the Y-Chapter and the YP-Paper as far as the selection theorems and their applications are concerned? We begin with the property encapsulated in the following definition due to He and Yannelis (2016).

**Definition 6.** Let $X, Y$ be non-empty subsets of a topological vector space and $P : X \rightarrow Y$ a correspondence. The correspondence $P$ has the **continuous inclusion property** if for all $x \in X$ such that $P(x) \neq \emptyset$, there exists an open neighborhood $U^x$ of $x$ and an usc correspondence $F^x : X \rightarrow Y$ which has non-empty, convex and closed values and $F^x(y) \subseteq P(y)$ for all $y \in U^x$.  

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30For closed-valued correspondences this definition is equivalent to what Geistdoerfer-Florenzano calls *upper hemicontinuity*.” – we’ll have more to say on this in the next section.

31 Given the importance that the authors are attaching attach to the YP-Paper, they feel it is incumbent on them to be crystal-clear about how it relates to the antecedent literature, and in particular to Browder (1968): they invite the reader to compare the proof of Theorem 3.3 in the YP-Paper and the proof of Theorem 1 in Browder (1968). The reader should also note Footnote 4 in the YP-Paper for its reference to Brower’s work.

32See Footnote 45 for further elaboration.
We provide five different continuity postulates on correspondences in Definition 2. For topological vector spaces, any of the five conditions above, in the presence of some other assumptions which are standard in fixed point and selection theorems, implies the continuous inclusion property; see He and Yannelis (2016) for details. The reader should note well how the notion of a local selection function from a correspondence has been naturally uplifted to the idea of obtaining a local selection correspondence from a correspondence. Now with this notion at hand, He-Yannelis unify the fixed point theorems of Browder and Kakutani-Fan-Glicksberg for locally convex Hausdorff topological vector spaces in the following way.

**Theorem (He-Yannelis 2017).** Let $K$ be a non-empty, convex and compact subset of a Hausdorff locally convex topological vector space and $T : X \twoheadrightarrow X$ a correspondence with non-empty, convex values and have the continuous inclusion property. Then there exists $x^* \in K$ such that $x^* \in T(x^*)$.

As is well-understood, the Kakutani-Fan-Glicksberg fixed point theorem assumes the relevant mapping to be usc, whereas the Browder theorem assumes that the mapping has open fibers, and the above theorem is a unification simply because the continuous inclusion property is weaker than either of these assumptions. Neither assumption implies the other. In order to see that upper semi-continuity does not imply the open fiber condition, consider the following example. Let $[0, 1]$ has the usual topology and $P : [0, 1] \twoheadrightarrow [0, 1]$ is defined as $P(x) = \{x\}$ for all $x \in [0, 1]$. It is clear that $P$ is usc and it does not have open fibers. In order to see that the open fiber condition does not imply upper semi-continuity, consider the following example. Let $[0, 1]$ has the usual topology and $Q : [0, 1] \twoheadrightarrow [0, 1]$ is defined as $Q(x) = \{x' | x' > x\} \cup \{1\}$ for all $x \in [0, 1]$. It is clear that $Q$ has open fibers and is not usc.

The continuous inclusion property assumes that the relevant correspondence has ‘nice’ local selections. As mentioned in Section 2.2, a special form of local selections, which assumes the local selections are constant, was first presented by Tarafdar (1977) in the context of a fixed point theorem and variational inequalities. Here we are focussing on the post-Tarafdar period; see for example Mehta and Tarafdar (1987) and Wu and Shen (1996) for fixed point theorems that are based on constant selections and Park (1999) for fixed point theorems that are based on single-valued continuous selections. The fixed point theorem above not only generalizes and unifies the fixed point theorems of Browder and Kakutani-Fan-Glicksberg but also those of furnished in Tarafdar, Wu-Shen and Park.

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33 Analogous versions of this property have been assigned different names: correspondence security in Barelli and Soza (2009), Barelli and Meneghel (2013), continuous security in Reny (2016a), Carmona and Podczeck (2016) and continuous neighborhood selection property in Uyanık (2016). The last work provides a fixed point theorem based on this weak continuity assumption; see Uyanık (2016, Definition 22 and Theorem 9, pp. 93–94).

34 The content of the last two paragraphs is simply essentially copied from Uyanık (2016, pp. 95–96).
Browder (1968) proved his theorem for Hausdorff topological vector spaces. The theorem above is a generalization of Browder’s theorem for locally convex Hausdorff topological vector spaces. A natural question arises at this stage whether the He-Yannelis result is valid for topological vector spaces that are not necessarily locally convex or Hausdorff. In the context that Browder’s work, it is a source of satisfaction that Tarafdar (1977, Theorem 1) showed that the following full generalization is possible.

**Theorem (Tarafdar 1977).** Let $X$ be a non-empty, convex and compact subset of a Hausdorff topological vector space and $P : X \rightarrow X$ a correspondence with non-empty, convex values and have the local intersection property. Then there exists $x^* \in X$ such that $x^* \in P(x^*)$.

Note that Tarafdar’s theorem pertains to Hausdorff topological vector spaces, and Balaj and Muresan (2005, Theorem 6) and Uyanık (2016, Theorem 10) generalized it to show that the Hausdorff separation axiom is redundant.

Next, we turn to the antecedent literature. The continuity postulate is the standard technical assumption in order to obtain existence of an equilibrium. Earlier work assumes the (weak) preference relation is complete transitive and has closed sections, which is equivalent to closed graph assumption, and also to the openness of the graph of the asymmetric part of the weak relation. However, when completeness and/or transitivity is relaxed, one needs to be careful as to which continuity assumption is to be used. For non-ordered preferences, Fan (1961) uses open graph assumption in order to obtain a maximal element in a tvs. Sonnenschein (1971) replaces it with the open fibers assumption. Mas-Colell (1974), Gale and Mas-Colell (1975) and Shafer and Sonnenschein (1975) use open graph assumption in order to obtain the existence of an equilibrium in games and economies. Y-Chapter and YP-Paper replace it with open fibers for topological vector spaces and with lower semi-continuity for finite dimensional topological vector spaces. See Bergstrom (1992) for an exposition. Finally, there is an alternative literature started with Borglin and Keiding (1976) and Toussaint (1984), where the traces can be found in the proof of the main result of Gale and Mas-Colell (1975), moving in the opposite direction and obtaining a majorization, a “nice” covering relation. In this branch of the literature, any maximal element under the new covering relation is required to be a maximal element under the original preference relation. This is not the direction we pursue in this paper. The following

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35 Tarafdar (1977) generalized Browder’s theorem by assuming correspondences satisfies a continuity property which we show in Proposition 1 is equivalent to the local intersection property. Also, Wu and Shen (1996) proved a version of this theorem for locally convex Hausdorff topological vector spaces; see also Tarafdar and Chowdhury (2008, Section 4.7) and Prokopovych (2011) for recent treatments. The authors thank Rich McLean for pointing out Balaj and Muresan (2005) after the first draft of this manuscript was circulated. We invite the reader to check that the methods of proof of Balaj-Muresan and Uyanik embody different ideas.

36 On the one hand they are not equivalent, on the other hand assuming too much continuity may imply completeness and transitivity; see Khan and Uyanik (2019b).

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table provides a portrait of the selection theorems and their applications. Its objective is not to discuss the priority of the results but simply to connect.

| Continuity Assumption          | Continuous Function               |
|-------------------------------|-----------------------------------|
| Open Graph                    | Gale-Mas-Colell (1975), Yannelis (1983) |
| Open Fibers                   | Browder (1968), Yannelis-Prabhakar (1983) |
| Lower Semi-continuous         | Michael (1956), Yannelis-Prabhakar (1983) |
| Constant Local Selections     | Tarafdar (1977), Mehta-Tarafdar (1987) |
| Continuous Local Selections   | Tian-Zhou (1995), Wu-Shen (1996), Uyanik (2016) |
| Upper Semi-Continuous Local Selections | Park (1999) |
| Upper Semi-Continuous         | Barelli-Meneghel (2013), He-Yannelis (2016, 2017), Carmona-Podczeck (2016), Reny (2016), Uyanik (2016), McLean (2020), Cornet (2020) |

Table 1: Continuous Selection Theorems

In the remaining part of this subsection we engage with the definitive analysis of Cornet (2020) on the Gale-Nikaido-Debreu lemma and frame it in the light of the antecedent work of McCabe (1981), Yannelis (1985), Mehta and Tarafdar (1987), Uyanık (2016) and He and Yannelis (2017). We do it not only for its own intrinsic interest but also as a preparatory step for the generalization of the Gale-Nikaido-Debreu lemma presented here and in Section 3.2. We begin with a prescient statement made more than 25 years ago in a consideration of Lionel McKenzie’s work on the existence of Walrasian equilibrium.

Khan (1993, pp. 34–35) is worth quoting at some length:

The conjunction of Brouwer’s fixed point theorem and the separating hyperplane theorem that is at the heart of McKenzie’s 1959 proof raises the question whether his technique can be used to give an alternative proof of the (so-called) Debreu-Gale-Nikaido lemma. Such a proof has been offered in several recent contributions. The basic idea is simple.

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37 See also Shafer and Sonnenschein (1975, 1976, 1982), Tarafdar (1977, 1991), Mehta and Tarafdar (1987), Uyanik (2016), Moore (1975, 2007), McKenzie (2002), Baye et al. (1993), Shafer and Sonnenschein (1975), Geller (1986), Tian and Zhou (1992), Park (1999, 2011), Ding and Park (2002), Askoura and Godet-Thobie (2006) and Scalzo (2015) for results on continuous selections and existence of an equilibrium in games and economics.

38 See Terry Tao’s 2019 survey, Denton, Parke, Tao and Zhang (2019, Figure 1), of a basic identity in linear algebra for an example of result in mathematics which has been proved several times independently.

39 Khan continues, “If the excess demand set does not intersect with the negative orthant at any price, the two sets, being closed and convex, can be strictly separated. Consider the set of these normalized separating hyperplanes, one set for each price we began with. This correspondence has a continuous selection and hence a fixed point. But this furnishes a contradiction to Walras’ law. The point of contact with McKenzie’s 1959 proof is that he, too, is selecting continuously a unique hyperplane and this selection is obtained, as in McCabe, by construction. Mehta and Tarafdar use the more powerful Tarafdar fixed point theorem, and Yannelis offers, and relies on, a more sophisticated selection theorem.” The reader can pick up this discussion in Section 4 below.
Cornet (2020, p.2) provides a state-of-the-art reading of the literature on the Gale-Nikaido-Debreu lemma; he notes that the classical proofs of this lemma use either a fixed point theorem, or K-K-M lemma; see for example Debreu (1982), Geistdoerfer-Florenzano (1982); Florenzano and Le Van (1986); Krasa and Yannelis (1994). McCabe (1981) and Yannelis (1985) provided a simple alternative proof by using the separating hyperplane theorem and a fixed point theorem. They introduced a ‘nice’ mapping Browder-McCabe-Yannelis map\(^{40}\) which assigns for each price \(p\), the set of prices at which the value of the excess demand is strictly greater than zero. It also allows them to use a weaker version of Walra’s law. McCabe (1981) assumed the excess demand correspondence is use which implies the Browder-McCabe-Yannelis map has open fibers. Later, Mehta and Tarafdar (1987) extended Yannelis (1985) by directly imposing a weaker continuity assumption on the Browder-McCabe-Yannelis map, rather than the excess demand correspondence.

In addition to the GND lemma, Cornet writes an equivalence result that shows that the generalization of the GND lemma is equivalent to Kakutani’s fixed point theorem. He wrote “The proof of Theorem 1 relies on Kakutani’s theorem and is in fact equivalent to it, as shown in Remark 2.”\(^{41}\) We now turn to his principle result.

**Theorem (Cornet-2020).** Let \(P \subseteq \mathbb{R}^\ell\) be a nondegenerate, closed, convex, cone, with vertex 0. Let \(Z\) be a correspondence, from \(\text{co}[P \cap S]\) to \(\mathbb{R}^\ell\), with non-empty, convex, compact values, satisfying the continuous inclusion property and Walras’ law on \(P \cap S\). Then there exists \(p^* \in \text{co}[P \cap S]\) such that \(Z(p^*) \cap P^0 \neq \emptyset\).

As Cornet (2020) points out, he limits himself to finite dimensional spaces; we remind the reader that Yannelis (1985) and He and Yannelis (2017) are infinite dimensional.\(^{42}\) Cornet’s ideas and proofs are novel, an elaboration on this is needed; especially his Claim 2 and Lemma 2. Cornet pioneering steps follow the path originally laid out in McKenzie (1954) and delineated in Khan (1993). These steps are of profound consequences for the extension of the Walrasian general equilibrium theory to increasing returns to scale production technologies in which Cornet is a pioneering scholar with significant contributions.\(^{43}\) In his Remark 1, Cornet identifies the joint continuity of a function that maps price and excess demand to the real line. He mentions that this property “may not hold in infinite dimensional spaces, hence the importance of considering

\(^{40}\)The Browder-McCabe-Yannelis map is related to variational inequalities and the traces can be found in Browder (1967, 1968), which relies on Hakuhara; see Footnote 25. In particular, the mapping is given as the \(N\) correspondence in Browder (1968, p. 287) and in Browder (1967, p. 287), as the \(\Psi\) correspondence in McCabe (1981, p. 168) and as the \(F\) correspondence in Yannelis (1985, p. 598). All obtain a continuous selection of the relevant correspondence by appealing to partition of unity, either directly or indirectly, to obtain a fixed point.

\(^{41}\)For an elaboration of this interesting contribution to the mathematics literature on fixed point theory, see Cornet (2020, pp. 2 and 11) and Majumdar (2009, Section 2.1).

\(^{42}\)For this constant drum beat on infinite-dimensionality, see Footnotes 32 and 45 and the text they footnote.

\(^{43}\)See Khan (1987), Cornet (1988), Vohra (1992) and their references.
the weak version of Walras’ law” which we return in Section 3.2 below. What the reader should appreciate that Cornet is also not working on the price simplex but also, as in Debreu (1956) on a cone.\footnote{Cornet (2020, p. 2) wrote “[…] we allow for the cone \(P\) of admissible prices to be a non-degenerate closed, convex, cone of vertex 0 in the finite dimensional commodity space \(\mathbb{R}^\ell\).” He also essentially reproduced this sentence in his abstract.}

Following the pioneering work of Browder-Fan, Yannelis (1985) uses demi-continuity in the generalization of the GND lemma to infinite dimensional commodity spaces. We reproduce the result for the reader’s convenience.

**Theorem (Yannelis-1985).** Let \(X\) be a Hausdorff locally convex linear topological space, \(C\) a closed, convex cone of \(X\) having an interior point \(e\), \(C^*=\{p\in X^*: p\cdot x<0 \text{ for all } x\in C\}\neq\{0\}\) the dual cone of \(C\) and \(A=\{p\in C^*: p\cdot e=-1\}\). Let \(\zeta: \Delta \to 2^X\) be a correspondence such that:

(i) \(\zeta\) is upper demi-continuous in the weak* topology,

(ii) \(\zeta(p)\) is convex non-empty and compact for all \(p\in \Delta\),

(iii) for all \(p\in \Delta\) there exists \(x\in \zeta(p)\) such that \(p\cdot x\leq 0\).

Then there exists \(p^*\in \Delta\) such that \(\zeta(p^*)\cap C\neq\emptyset\).

Podczeck (1997) comments on this theorem as follows.

Note that Theorem 3.1 in Yannelis (1985) only requires \(\langle p, z \rangle\leq 0\) to hold for some \(z\in \zeta(p)\) but not for all \(z\in \zeta(p)\). As shown by our constructions, this allows to replace demand correspondences by objects with enough continuity properties. In this respect this theorem offers, in particular, a powerful tool to overcome the problem that the wealth map may not be jointly continuous in an infinite dimensional setting.\footnote{We do not want to be overly defensive about this: as emphasized by Halmos and others it is this dialectical interplay between the finite and the infinite that leads to a synthetic development for the subject.}

We send the reader to these papers and move on.

### 2.4 A Necessary and Sufficient Condition for USC Selections

In order to study the existence of an equilibrium, one does not need full continuity: a nice selection of the relevant preference relation, or mapping, is sufficient. It is this observation that motivates the results in the literature, and in line with it, recent results on the existence of an equilibrium in games and in economies require preferences to have “nice” selections instead of having an open graph or open fibers; see \(\text{He and Yannelis (2016)}\) for a most recent illustration. To be sure, a nice selection of a correspondence is not a new idea. Michael (1951, 1956) provides sufficient conditions for the existence of a continuous selection of a correspondence.
Gale and Mas-Colell (1975), Y-Chapter and YP-Paper bring this result to economics and provide results on existence of a maximal element and an equilibrium in games and economies with possibly discontinuous preferences. In this line of literature, recent works assume weaker conditions such as constant local selection, continuous local selections and finally upper semi-continuous local selections. Our selection result provides a necessary and sufficient condition for the existence of such nice selections, hence a complement to these results.

**Definition 7.** Let $E, X$ be two topological spaces and $M : E \to X$ be a correspondence. We say $M$ is closed if its graph, $\text{Gr}M$, is closed in $E \times X$.

Although the upper semi-continuity and closedness of a correspondence are distinct topological properties, they are equivalent under some suitable assumption on the range of the correspondence or the correspondence itself; see (Aliprantis and Border 2006, Section 17.2).

We next define a selection and a local selection of a correspondence.

**Definition 8.** Let $E, X$ be two topological spaces, $M : E \to X$ a correspondence and $E_M = \{e \in E | M(e) \neq \emptyset\}$.

(i) A selection of $M$ is a correspondence $\psi : E \to X$ such that $\text{Gr}\psi \subseteq \text{Gr}M$ and $\psi(e) \neq \emptyset$ for all $e \in E_M$.

(ii) A local selection of $M$ at $a \in E$ is a pair $(V^a, \psi^a)$ of an open neighborhood $V^a$ of $a$ and a correspondence $\psi^a : V^a \to X$ such that $\text{Gr}\psi^a \subseteq \text{Gr}M$ and $\psi^a(e) \neq \emptyset$ for all $e \in V^a \cap E_M$.

We say a local selection $(V^a, \psi^a)$ is closed if $\text{Gr}\psi^a$ is closed in the subspace $V^a \times X$.

(iii) We say $M$ has local selections if it has a local selection at all $a \in E_M$.

We now introduce a new continuity assumption on a correspondence. It is considerably weaker than both upper semi-continuity and closed graph properties.

**Definition 9.** Let $E, X$ be two topological spaces. A correspondence $M : E \to X$ has the neighborhood selection property (NSP) if every $a \in E_M$ has an open neighborhood $V^a$ such that for all $e \in V^a$ and all $x \not\in M(e)$ there exists an open neighborhood $U$ of $(e, x)$ in $V^a \times X$ such that $\{e'\} \times M(e') \nsubseteq U$ for all $(e', x') \in U \cap (E_M \times X)$.

Michael (1956) proves a selection theorem for lsc and convex-valued correspondences whose range is a separable Banach space. YP-Paper proved a version of this theorem for convex-valued correspondences which have open fibers whose range is a topological vector space. In their result the range is a more general space but their continuity assumption is stronger.

The selection has also convex-valued in the literature. We do not assume a linear structure on the space, hence convexity is not relevant. In $\mathbb{R}^n$, however, convex hull of an usc correspondence is usc, hence suitable convexity assumptions on preferences, convex-valued and usc selections exist.
Note that if a correspondence is closed, then in the definition above \((\{e'\} \times M(e')) \cap U^{(e,x)} = \emptyset\) for all \((e', x') \in U^{(e,x)}\). Therefore, any closed correspondence satisfies the NSP. However, the converse argument is not true – the first example below illustrates this. The second example illustrates a correspondence which does not satisfy the NSP.

**Example 1.** Let \(E = X = [0, 1]\) be endowed with the usual metric. The correspondence \(M : E \rightarrow X\) is defined as \(M(e) = (e, 1)\) if \(e < 1\) and \(M(1) = \emptyset\). It is clear that \(M\) is not closed.

The correspondence \(\xi : E \rightarrow X\) defined as \(\xi(e) = \{0.5 + 0.5e\}\) if \(e < 1\) and \(\xi(1) = \emptyset\) is a selection of \(M\) which is closed in \([0, 1] \times [0, 1]\). Moreover, the correspondence \(M\) has closed local selections and has the NSP.

**Example 2.** Let \(E = X = [0, 1]\) be endowed with the usual metric. The correspondence \(M : E \rightarrow X\) is defined as \(M(e) = 1\) if \(e\) is rational and \(M(e) = 0\) otherwise. It is clear that \(M\) has no closed (local) selections since \(M\) has singleton values and its graph is not closed. It is easy to see that \(M\) does not have the NSP.

Before we present our results, it is useful to observe that one can equivalently define the NSP as follows.

**Observation 1.** Let \(E, X\) be two topological spaces and \(M : E \rightarrow X\) a correspondence. We call \(M\) has the neighborhood selection property (NSP) if \(M\) has a local selection \((V^a, \psi^a)\) at each \(a \in E\) with \(M(a) \neq \emptyset\) such that for all \((e, x) \in V^a \times X\), \(x \notin M(e)\) implies there exists an open neighborhood \(U^{(e,x)}\) of \((e, x)\) in \(V^a \times X\) such that \(x' \notin \psi^a(e')\) for all \((e', x') \in U^{(e,x)}\).

Note that no topological assumption is imposed on the correspondence \(\{\psi^a\}\), hence it does not necessarily be closed. In order to see this note that for those \((e', x') \notin \psi^a\) which satisfy \((e', x') \notin M\), the existence of \(U^{(e,x)}\) implies that \(U^{(e,x)} \cap \psi^a = \emptyset\). However, for those \((e', x') \notin \psi^a\) which satisfy \((e', x') \in M\), we do not have open neighborhoods. Hence, \(\psi^a\) need not be a closed correspondence.

The selection properties in the literature assume the existence of a closed (local) selection. The following proposition provides necessary and sufficient conditions for the existence of a closed local selection of a correspondence.

**Proposition 2.** Let \(E, X\) be two topological spaces and \(M : E \rightarrow X\) a correspondence. Then, the following are equivalent.

(i) \(M\) has closed local selections.

(ii) \(M\) has the NSP.

If \(M(E)\) is compact and Hausdorff, then the local selections are usc.
Proof of Proposition 2. Let $E, X$ be two topological spaces and $M : E \to X$ a correspondence.

Sufficiency. Pick $a \in E$ such that $M(a) \neq \emptyset$. Assume $(V^a, \xi^a)$ is a closed local selection of $M$ at $a$. Then $\xi^a$ has a closed graph in $V^a \times X$, $\text{Gr}\xi^a \subseteq \text{Gr}M$ and $\xi^a(e) \neq \emptyset$ for all $e \in E_M$. Pick $(e, x) \in V^a \times X$ with $x \notin M(e)$. Then, $(e, x) \notin \text{Gr}\xi^a$. Since $(\text{Gr}\xi^a)^c$ is open in $V^a \times X$, there exists an open neighborhood $U^{(e,x)}$ of $(e, x)$ in $V^a \times X$ such that $x' \notin \xi^a(e')$ for all $(e', x') \in U^{(e,x)}$. Therefore, $M$ has the NSP.

Necessity. Assume $M$ has the NSP. Pick $a \in E_M$. Observation 1 implies that $M$ has a local selection $(V^a, \psi^a)$ at $a$ such that for all $(e, x) \in V^a \times X$, $x \notin M(e)$ implies there exists an open neighborhood $U^{(e,x)}$ of $(e, x)$ in $V^a \times X$ such that $x' \notin \psi^a(e')$ for all $(e', x') \in U^{(e,x)}$. Define $U^{(a,X)} = \{U^{(e,x)}|e \in V^a, x \notin M(e)\}$ and $U^{(a,X)} = \bigcup U^{(a,X)}$.

We next define a closed local selection $\xi^a : V^a \to X$ of $M$ at $a$ as $\text{Gr}(\xi^a) = (U^{(a,X)})^c$. Then $\xi^a$ has a closed graph in $V^a \times X$, hence it is a closed correspondence. It is clear that $(U^{(a,X)})^c \subseteq \text{Gr}M$, hence $\xi^a(e) \subseteq M(e)$ for all $e \in V^a$. Assume $\xi^a(e_0) = \emptyset$ for some $e_0 \in V^a \cap E_M$. By construction of $\xi^a$, for all $x \in X$, $(e_0, x) \in U$ for some $U \in U^{(a,X)}$. Then, $x \notin \psi^a(e_0)$ for all $x \in X$, i.e., $\psi^a(e_0) = \emptyset$ while $M(e_0) \neq \emptyset$. This furnishes us a contradiction with $\psi^a$ being a local selection of $M$ at $a$. Therefore, $\xi^a$ is a closed local selection of $M$ at $a$. This completes the necessity argument.

Finally, the upper semi-continuity of the local selections $(V^a, \xi^a)$ defined on the subspace $V^a$ follows from the Closed Graph Theorem [Aliprantis and Border 2006, Theorem 17.11].

Proposition 2 is on the local selections of a correspondence. The next result extends the local selections to global selections under the paracompactness hypothesis. This extension of “nice” local properties to a “nice” global property can be found in the literature for topological vector spaces; see for example Michael (1956, Theorem 3.1”), Yannelis and Prabhakar (1983, Theorem 3.1), Wu and Shen (1996, Theorem 1) and Barelli and Soza (2009, Lemma A. 2). The following proposition shows that when only the topological properties are relevant, this extension result is true for purely topological spaces. Moreover, the following theorem also relates it to the NSP which is the novel contribution of this paper.

Proposition 3. Let $E, X$ be two topological spaces and $M : E \to X$ a correspondence such that $E_M$ is paracompact and Hausdorff. Let $M|_{E_M}$ denote the restriction of $M$ on the subspace $E_M$. Then, the following are equivalent.

(i) $M|_{E_M}$ has closed local selections.

(ii) $M|_{E_M}$ has the NSP.

48Every regular and Lindelof space is paracompact and Hausdorff, hence we can replace the latter with the former assumptions (Munkres, Theorem 41.5, p. 257). Similarly, every subspace of a metric space is paracompact and Hausdorff.
(iii) $M|_{E_M}$ has a closed selection.

If $M(E)$ is compact and Hausdorff, then the selections are usc.

Proof of Proposition 3. Let $E, X$ be two topological spaces and $M : E \rightarrow X$ be a correspondence such that $E_M$ is paracompact and Hausdorff. The proof of $M|_{E_M}$ has closed local selections if and only if it has the NSP follows from Proposition 2. It remains to prove that $M|_{E_M}$ has a closed selection if and only if it has closed local selections.

Sufficiency. Assume $\xi : E_M \rightarrow X$ is a closed selection of $M|_{E_M}$. Then, $\xi$ has a closed graph in $E_M \times X$, non-empty values and $\xi(e) \subseteq M(e)$ for all $e \in E_M$. Therefore $(E_M, \xi)$ is a closed local selection of $M|_{E_M}$ for all $a \in E_M$.

Necessity. For all $a \in E_M$, assume $(V^a, \xi^a)$ is a closed local selection of $M|_{E_M}$ at $a$. We now construct a closed selection $\xi$ of $M|_{E_M}$. Note that the collection $V^a = \{V^a\}_{a \in E_M}$ is an open cover of $E_M$. Since $E_M$ is paracompact and Hausdorff, $V$ has a closed, locally finite refinement $K = \{K^a\}_{a \in D}$ which covers $E_M$ (Michael, 1953, Lemma 1), i.e., $K$ is a covering of $E_M$ consisting of closed sets such that every element of $K$ is a subset of some element of $V$, and each $a \in E_M$ has an open neighborhood which intersects only finitely many elements of $K$. It follows from $K^a \times X$ is closed in $E_M \times X$ for all $\alpha \in D$ and the union of a locally finite collection of closed set is closed (Munkres, 2000, Lemma 39.1) that the correspondence $\xi : E_M \rightarrow X$ defined as

$$\text{Gr}\xi = \bigcup_{\alpha \in D} \text{Gr}\xi^a \cap (K^a \times Y)$$

has a closed graph in the subspace $E_M \times X$. Since $\xi(a) \subseteq M(a)$ for all $a \in E_M$, $M|_{E_M}$ has a closed selection.

Finally, the upper semi-continuity of the selections defined on the subspace $E_M$ follows from the Closed Graph Theorem (Aliprantis and Border, 2006, Theorem 17.11).

3 Applications to Walrasian Equilibrium Theory

In this section we present four applications of the results exposed in the above section to underscore the wide-range applications of Yannelis’ selection theorem. three propositions on Berges theorem, a generalization of the GND lemma and another on the Browder-McCabe-Yannelis map, an existence result in Shafers setting of a non-transitive consumer, and finally, a generalization the approximation result of Anderson-Khan-Rashid to preferences that are not necessarily continuous. We emphasize that like Cornet (2020), we remain in a finite-dimensional setting.
3.1 On Berge’s Maximum Theorem

There are two major applications of Berge’s maximum theorem: (i) existence of an equilibrium and (ii) the structure of the equilibrium correspondence. For the latter, the upper semi-continuity of the entire argmax correspondence is a desirable property. However, as mentioned in Section 2.4, for the existence of an equilibrium, a nice selection is sufficient. Berge’s theorem shows that if a function \( u \) is continuous on \( E \times X \) and the constraint correspondence \( F : E \rightrightarrows X \) is both upper and lsc and has non-empty, compact values, then the argmax correspondence has non-empty and compact values, and is usc. The literature on the generalization of the Berge’s maximum theorem is preoccupied with the upper semi-continuity of the argmax correspondence. Walker (1979) provides a general version of Berge’s theorem for preferences (and for abstract correspondences). Dutta and Mitra (1989) weaken continuity of the objective function but impose order theoretic and convexity assumptions on it. Tian and Zhou (1995) provide a necessary and sufficient condition for the upper semi-continuity and the non-empty-valuedness of the argmax correspondence.

In this subsection we present three propositions weakening the continuity assumptions of the Berge’s maximum theorem, one for functions and the other two for correspondences, which provide sufficient conditions on the existence of an usc selection for the argmax correspondence.

We start with functions. Let \( E, X \) be two topological spaces, \( F : E \rightrightarrows X \) a correspondence and \( u : E \times X \rightarrow \mathbb{R} \) a function. Define \( M : E \rightarrow X \) as

\[
M^u(e) = \arg\max_{x \in F(e)} u(e, x).
\]

**Definition 10.** Let \( E, X \) be two topological spaces, \( F : E \rightrightarrows X \) a correspondence and \( u : E \times X \rightarrow \mathbb{R} \) a function. We say \( u \) has the continuous selection property with respect to \( F \) if every \( a \in E \) has an open neighborhood \( V^a \) such that for all \( (e, x) \in \text{Gr} F \cap (V^a \times X) \) with \( x \notin M^u(e) \), there exist an open neighborhood \( U^{(e,x)} \) of \( (e, x) \) in \( V^a \times X \) and \( y \in X \) such that for all \( (e', x') \in U^{(e,x)} \), \( u(e', y) \geq u(e', x') \) and \( (e', y) \in \text{Gr} F \setminus U^{(e,x)} \).

It is easy to observe that the continuous selection property is considerably weaker than those used in Tian and Zhou (1995, Definitions 7, 8, 9), severally, and collectively, hence the following generalizes Tian and Zhou (1995, Theorem 3).

**Proposition 4.** Let \( E, X \) be two topological spaces, \( E \) paracompact and Hausdorff, \( F : E \rightrightarrows X \) a correspondence with non-empty and compact values and \( u : E \times X \rightarrow \mathbb{R} \) a function that has

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49 See also Morgan and Scalzo (2007) for a recent generalization of Berge’s theorem. Moreover, in this paper we explicitly focus on topological generalizations of Berge’s theorem. There is a literature on measure-theoretic version of Berge’s theorem which we do not pursue; see Castaing and Valadier (1977) and Khan (1985) for details.
the continuous selection property with respect to \( F \). Then \( M^u \) has non-empty values and a closed selection. If, in addition, \( F(E) \) is compact and Hausdorff, then the selection is usc.

Proof of Proposition 4. First, it is easy to observe that the non-emptiness of \( M^u(e) \) for all \( e \in E \) follows from \[ \text{Tian and Zhou (1995, Theorem 1).} \) Second, since \( u \) has the continuous selection property with respect to \( F \), every \( a \in E \) has an open neighborhood \( V^a \) such that for all \((e, x) \in GrF \setminus (V^a \times X)\) with \( x \notin M^u(e) \), there exist an open neighborhood \( U^{(e, x)} \) of \((e, x)\) in \( V^a \times X \) and \( y \in X \) such that for all \((e', x') \in U^{(e, x)}, u(e', y) \geq u(e', x') \) and \((e', y) \in GrF \setminus U^{(e, x)}\). Now pick \((e', x') \in U^{(e, x)}\). If \( y \in M^u(e') \), then \( \{e'\} \times M^u(e') \notin U^{(e, x)} \). If \( y \notin M^u(e') \), then pick \( \hat{y} \in M^u(e') \). It follows from \( u(e', \hat{y}) > u(e', y) \geq u(e', x') \) that \( \{e'\} \times M^u(e') \notin U^{(e, x)} \). Therefore, \( M^u \) has the NSP. Then Proposition 3 imply that \( M^u \) has a closed selection, and when \( F(E) \) is compact and Hausdorff, then this selection is usc.

Note that Proposition 4 suggests that the continuous selection property provided in Definition 10 allows the argmax correspondence to have a graph which is not closed. Hence, it does not rule out the possibility that a point \( x \) is not a maximal element at some parameter \( e \), yet all of its neighborhoods contains maximal elements.

We next turn to the generalization of Berge’s theorem for correspondences. Let \( E, X \) be two topological spaces, \( F : E \to X \) and \( P : E \times X \to X \) be two correspondences. Define a correspondence \( M^P : E \to X \) as

\[
M^P(e) = \{x \in F(e) \mid P(e, x) \cap F(e) = \emptyset\}.
\]

We say the correspondence \( P \) is irreflexive if \( x \notin P(e, x) \) for all \((e, x) \in E \times X \), and fully transitive if for all \( e \in E \) and all \( x, y, z \in X \), \( y \notin P(e, x) \) and \( z \notin P(e, y) \) implies \( z \notin P(e, x) \).

Definition 11. Let \( E, X \) be two topological spaces, \( F : E \to X \) a correspondence and \( P : E \times X \to X \) a function. We say \( P \) has the continuous selection property with respect to \( F \) if every \( a \in E \) has an open neighborhood \( V^a \) such that for all \((e, x) \in GrF \setminus (V^a \times X)\) with \( x \notin M^P(e) \), there exist an open neighborhood \( U^{(e, x)} \) of \((e, x)\) in \( V^a \times X \) and \( y \in X \) such that for all \((e', x') \in U^{(e, x)}, x' \notin P(e', y) \) and \((e', y) \in GrF \setminus U^{(e, x)}\).

Analogous to the discussion above, the continuous selection property (for correspondences) is weaker than those used in \[ \text{Tian and Zhou (1995, Definitions 10, 11, 12), several, and collectively, hence the following result weakens the continuity assumption of Tian and Zhou (1995, Theorem 4).} \]

\[ \text{See Zhou and Tian (1992) and Khan and Uyanık (2019b) for a detailed discussion on different transitivity postulates. Note that the irreflexivity and full transitivity of a correspondence are not unusual properties if we interpret the graph of} \ P(e, \cdot) \text{as a binary relation on} \ X. \]
Proposition 5. Let $E, X$ be two topological spaces, $E$ paracompact and Hausdorff, $F : E \to X$ a correspondence with non-empty and compact values and $P : E \times X \to X$ an irreflexive and fully transitive correspondence that has the continuous selection property with respect to $F$. Then $M^P$ has non-empty values and a closed selection. If, in addition, $F(E)$ is compact and Hausdorff, then the selection is usc.

Proof of Proposition 5. First, it is easy to observe that the non-emptiness of $M^P(e)$ for all $e \in E$ follows from Zhou and Tian (1992, Theorem 1). Note that it follows from Khan and Yañnik (2019b, Proposition 2) that $P$ is transitive, i.e., for all $e \in E$ and all $x, y, z \in X$, $z \in P(e, y)$ and $y \in P(e, x)$ implies $z \in P(e, x)$. Second, since $P$ has the continuous selection property with respect to $F$, every $a \in E$ has an open neighborhood $V^a$ such that for all $(e, x) \in \text{Gr} F \cap (V^a \times X)$ with $x \notin M^P(e)$, there exist an open neighborhood $U^{(e, x)}$ of $(e, x)$ in $V^a \times X$ and $y \in X$ such that for all $(e', x') \in U^{(e, x)}$, $x' \notin P(e', y)$ and $(e', y) \in \text{Gr} F \setminus U^{(e, x)}$.

Now pick $(e', x') \in U^{(e, x)}$. If $y \in M^P(e')$, then $\{e'\} \times M^P(e') \notin U^{(e, x)}$. If $y \notin M^P(e')$, then there exists $z \in F(e')$ such that $z \in P(e', y)$. If $z \notin M^P(e')$, then transitivity and irreflexivity of $P$ imply $\{e'\} \times M^P(e') \notin U^{(e, x)}$. Otherwise, there exists $z' \in F(e')$ such that $z' \in P(e', z)$. Now pick $\hat{y} \in M^P(e')$. Then, it follows from full transitivity of $P$ that $\hat{y} \in P(e', z)$ (otherwise $z' \notin P(e', \hat{y})$ and $\hat{y} \notin P(e', z)$ yield a contradiction). Then transitivity of $P$, $\hat{y} \in P(e', z)$ and $z \in P(e', y)$ imply $\hat{y} \in P(e', y)$. Hence transitivity and irreflexivity of $P$ imply $\{e'\} \times M^P(e') \notin U^{(e, x)}$. Therefore, $M^a$ has the NSP. Then Proposition 3 implies that $M^a$ has a closed selection, and when $F(E)$ is compact and Hausdorff, then this selection is usc.

It is well-known in mathematical economics that without the transitivity assumption, topological assumptions on a choice set and on a preference relation defined on it are not enough to guarantee the existence of a maximal element. However, with the added linear structure on the choice set and a suitable convexity assumption on preferences, the existence of a maximal element is guaranteed. The following result provides a maximum theorem with a weak continuity assumption on preferences which is not necessarily complete or transitive but satisfies a convexity assumption.

Proposition 6. Let $E$ be a non-empty topological space, $X$ a non-empty Hausdorff locally convex topological vector space, $F : E \to X$ a closed correspondence with non-empty, convex and compact values, $P : E \times X \to X$ a correspondence with $x \notin \text{co} P(e, x)$ for all $(e, x) \in E \times X$, and the correspondence $\psi : E \times X \to X$ defined as $\psi(e, x) = P(e, x) \cap F(e)$ has the continuous inclusion property. Then $M^P$ is a closed correspondence with non-empty values. If, in addition, $F(E)$ is compact, then $M^P$ is usc.

Proof of Proposition 6. It follows from He and Yannelis (2017, Corollary 1) that for each $e \in E$, $M^P(e) \neq \emptyset$. It follows from $\psi$ having the continuous inclusion property that $P$ has quasi-transfer
open lower sections in \((e, x)\) with respect to \(F\) \cite{Tian and Zhou 1995} Definition 10). Then the argument in the proof of \cite{Tian and Zhou 1995} Theorem 4 shows that \(M^P\) is closed. Finally, when \(F(E)\) is compact, the upper semi-continuity of \(M^P\) follows from the Closed Graph Theorem \cite{Aliprantis and Border 2006} Theorem 17.11).

3.2 The Gale-Nikaido-Debreu Lemma

Inspired by the development in discontinuous games and economies prior to the work of \cite{Cornet 2020}, in this subsection we present a generalization of the GND lemma by imposing a weak continuity assumption on Browder-McCabe-Yannelis map – the continuous inclusion property. \cite{McCabe 1981} Theorem 1 and \cite{Yannelis 1985} Theorem 3.1) assumed the excess demand correspondence is usc, \cite{He and Yannelis 2017} Theorem 4) and \cite{Cornet 2020} Theorems 1 and 2) assume it has the continuous inclusion property, hence the Browder-McCabe-Yannelis map has open fibers. \cite{Mehta and Tarafdar 1987} Theorem 8), on the other hand, impose a direct assumption on the Browder-McCabe-Yannelis map, and assumed that it has the local intersection property. Proposition \[7\] below assumes a weaker continuity assumption on the Browder-McCabe-Yannelis map, hence generalizes these results as well as previous results cited in these papers for finite dimensional commodity spaces.\[51\] Moreover, we show that a continuity assumption on the excess demand correspondence which put restrictions only on the downward jumps, hence weaker than upper semi-continuity, which implies the Browder-McCabe-Yannelis map has the continuous inclusion property.

An economy \(\mathcal{E} = (Z, \zeta)\) with \(m\) commodities is defined as follows: \(Z \subseteq \mathbb{R}^m\) is the set of possible excess demands which is non-empty, convex and compact, and \(\zeta : \Delta \to Z\) the excess demand correspondence where \(\Delta = \{p \in \mathbb{R}^m_+ \mid \sum_{k=1}^m p_k = 1\}\) is the set of prices. Let \(\Omega = \{x \in \mathbb{R}^m \mid x \leq 0\}\). The \textit{Browder-McCabe-Yannelis map} is a correspondence \(\Psi : \Delta \to \Delta\) defined as

\[\Psi(p) = \{q \in \Delta : q \cdot \zeta(p) > 0\}\]

\textbf{Proposition 7.} Let \(\mathcal{E} = (Z, \zeta)\) be the economy defined above such that

(i) the Browder-McCabe-Yannelis map \(\Psi\) has the continuous inclusion property,

(ii) \(\zeta\) has non-empty, convex and closed values,

(iii) for each \(p \in \Delta\) there exists \(x \in \zeta(p)\) such that \(p \cdot x \leq 0\).

Then there exists \(p^* \in \Delta\) such that \(\zeta(p^*) \cap -\Omega \neq \emptyset\).

\textbf{Proof of Proposition 7.} Assume the conclusion of Proposition \[7\] is false. Then \(\zeta(p) \cap -\Omega = \emptyset\) for each \(p \in \Delta\). Also, since \(\zeta(p)\) is non-empty, compact, convex set and \(-\Omega\) is a closed, convex set,
there exists \( q \in \Delta \) that strictly separates \( \zeta(p) \) and \( -\Omega \) for each \( p \in \Delta \). Hence, the correspondence \( \Psi \) has non-empty values. It is clear that \( \Psi \) has convex values. And since \( \Psi \) is assumed to have the continuous inclusion property, Theorem (He-Yannelis) implies there exists \( p^* \in \Delta \) such that \( p^* \in \Psi(p^*) \), i.e. \( p^* \cdot x > 0 \) for all \( x \in \zeta(p^*) \). This furnishes a contradiction with (iii).

In Proposition 7, we assume the Browder-McCabe-Yannelis map \( \Psi \) has the continuous inclusion property which is merely a technical assumption. Now, we propose a weak continuity assumption on the excess demand correspondence which imposes a restriction only on downward jumps and implies that \( \Psi \) has the continuous inclusion property.

**Definition 12.** Let \( X, Y \) be two non-empty subsets of \( \mathbb{R}^m \) and \( P : X \to Y \) a correspondence. \( P \) is continuous from below (cfb) at \( x \in X \) if for each open half-space \( H \) containing the closure of \( P(x) \), there exists an open neighborhood \( U \) of \( x \) such that for all \( x' \in U \) and all \( z' \in P(x') \), there exists \( z \in H \) such that \( z \leq z' \). The correspondence \( P \) is cfb if it is cfb at each \( x \in X \).

It is easy to see that cfb is weaker than upper semi-continuity. Now, let \( X, Y \) be non-empty subsets of \( \mathbb{R}^m \), \( P : X \to Y \) a correspondence, and \( \pi \) the usual projection map. Define \( \pi^k P : X \to \mathbb{R} \) as \( \pi^k P(x) = \pi^k(P(x)) \) and \( \underline{x}^k = \min \pi^k P(x) \) for each \( k = 1, \ldots, m \).

**Definition 13.** Let \( X, Y \) be non-empty subsets of \( \mathbb{R}^m \) and \( P : X \to Y \) a correspondence. \( P \) is weakly continuous from below (wcfb) at \( x \in X \) if (i) \( K = \{ k : \underline{x}^k > 0, k = 1, \ldots, m \} \neq \emptyset \) implies \( \pi^k P \) is cfb at \( x \) for some \( k \in K \), and (ii) \( K = \emptyset \) implies \( P \) is cfb at \( x \). The correspondence \( P \) is wcfb if it is wcfb at each \( x \in X \).

Note that, if at some price \( p \), the excess demand of at least one commodity is positive and its excess demand remains positive in a neighborhood of \( p \), then this is sufficient for the excess demand correspondence to be wcfb at \( p \), irrespective of the behavior of the excess demand of other commodities around \( p \).

**Proposition 8.** Let \( E = (Z, \zeta) \) be the economy defined above such that \( \zeta \) is wcfb and has non-empty, closed values. Then the Browder-McCabe-Yannelis map \( \Psi \) has the continuous inclusion property.

**Proof of Proposition 8.** First, it follows from \( Z \) is compact and \( \zeta \) has non-empty and closed values that \( \pi^k \zeta \) has non-empty and compact values, and hence, \( \underline{p}^k = \min \pi^k \zeta(p) \) is well defined for each \( k = 1, \ldots, m \). Now pick \( p \in \Delta \) such that \( \Psi(p) \neq \emptyset \). Then, there exists \( q \in \Delta \) such that \( q \cdot \zeta(p) > 0 \).

First, assume for some \( k = 1, \ldots, m \), \( \underline{p}^k > 0 \) and \( \pi^k \zeta \) is cfb at \( p \). Since \( \zeta(p) \) and \( Z \) are compact, \( \pi^k \zeta(p) \times \pi^{\neg k}(Z) \) is compact. Then, there exists \( \bar{q} \in \Delta \) that strictly separates \( \pi^k \zeta(p) \times \pi^{\neg k}(Z) \) and \( -\Omega \). And, since \( \pi^k \zeta \) is cfb at \( p \in \Delta \), for sufficiently small \( \varepsilon > 0 \), there exists an open...
neighborhood $U^p$ of $p$ such that for all $p' \in U^p$, $\bar{q}$ still strictly separates $\{\pi^k \zeta(p) - \varepsilon\} \times \pi^{-k}(Z)$. Then $\bar{q} \cdot \zeta(p') > 0$ for all $p' \in U^p$. Therefore, $\Psi$ has the local intersection property at $p$. Second, assume for each commodity $k = 1, \ldots, m$, $\nu^k \leq 0$. Then $\zeta$ is cfb at $p$. And since $q \cdot \zeta(p) > 0$, $q$ determines an open half space $H^q$ in $\mathbb{R}^m$ containing $\zeta(p)$. And since $\zeta$ is cfb, there exists an open neighborhood $U^p$ of $p$ such that for all $p' \in U^p$ and all $z' \in \zeta(p')$, there exists $z \in H^q$ such that $z \leq z'$. Therefore, $q \cdot \zeta(p') > 0$ for all $p' \in U^p$, hence $\Psi$ has the local intersection property at $p$.

Therefore, it follows from $\Psi$ has the local intersection property at all $p \in \Delta$ that $\Psi$ has the continuous inclusion property.

We end this section by providing some examples of aggregation of correspondences. In economies, existence of an equilibrium does not require well-behaved individual demand correspondences – well-behaved aggregate demand correspondence is enough as Uzawa suggested to Debreu (1959, note 1, p88) for production sets: individual demands as functions of prices are allowed to jump. As long as the aggregate demand has closed local selections, then we can have an (approximate) equilibrium; see Subsection 3.4 below.

**Definition 14.** Let $E, X$ be topological space. We call a correspondence $F : E \rightarrow X$ jumps at $e \in E$ if $F$ has no closed local selection at $e$.

**Example 1.** Let $E = X = \mathbb{R}_+$ and $F^i(e) = \{0, k\}, k > 0$, for $e \neq 1$ and $F^i(1) = \{k/n\}$ for all $i = 1, \ldots, n$, $n > 1$. Since $F^i$ jumps at 1 for all $i$, it follows from Proposition 3 that each $F^i$ does not have an usc selection. Although the aggregate correspondence $F = \sum^n_{i=1} F^i$ is not usc, it has an use selection $\zeta$ defined as $\zeta(e) = \{k\}$ for all $e \in E$.

**Example 2.** Let $E = \{p \in \mathbb{R}_+^2 \mid 0 < p \leq 1\}$ denote the set of all strictly positive prices. Consider the following two-agent, two-good pure-exchange economy. Let $X = \mathbb{R}_+^2$ be the consumption set of both agents. The endowments are $\omega^A = \omega^B = (1, 1)$. The preference of agents are such that their demand correspondences are as follows:

$$x^A(p) = \begin{cases} \left\{\left(\frac{p_1+p_2}{p_1}, 0\right)\right\}, & \text{if } p_1 \leq p_2 \\ \left\{\left(0, \frac{p_1+p_2}{p_2}\right)\right\}, & \text{if } p_1 > p_2 \end{cases}$$

and

$$x^B(p) = \left\{\left(\frac{p_1+p_2}{p_1}, 0\right), \left(0, \frac{p_1+p_2}{p_2}\right)\right\}.$$ 

Agent $A$ strictly prefers good 1 over good 2 if its price is lower, however, he shifts to good 2 when the first commodity is more expensive. Agent $B$ does not like to mix. It is clear that agent $A$’s demand correspondence jumps at prices $p$ such that $p_1 = p_2$, hence it has no usc selection. However, the aggregate demand correspondence has an usc selection $\zeta(p) = \left\{\left(\frac{p_1+p_2}{p_1}, \frac{p_1+p_2}{p_2}\right)\right\}$.  

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3.3 On Shafer’s Non-Transitive Consumer

In the existence of a Walrasian equilibrium, two main approaches have been used in economic theory: simultaneous optimization and excess demand. The former was introduced by Debreu (1952) and used in Arrow and Debreu (1954) while the latter was introduced in Gale-Debreu-Nikaido and used in Debreu (1959). The former was picked up by Shafer and Sonnenschein (1975) and YP-Paper; see Debreu (1982) for a detailed comparison. Without the transitivity assumption, the excess demand approach may not work since it is not guaranteed that the excess demand correspondence is convex valued; see Sonnenschein (1977). However, if one guarantees that a nice selection exists, than this method can still work. In a recent work, Scapparone (2015) uses extension of preferences in order to obtain a nice selection of the demand correspondence. Gale and Mas-Colell (1975) use a similar extension result about half a century ago in order to obtain existence of an equilibrium in an economy.

In this subsection we provide a simple economy where the demand correspondence is a function, and hence convex valuedness is not an issue. The commodity space is $X$ be non-empty, convex and compact subset of $\mathbb{R}^k_+$ and $I$ denotes a finite set of individuals. For each agent $i$, let $\succsim_i \subseteq X \times X$ denote her preference relation and $e_i \in \mathbb{R}^m_+$ her the endowment. The set of prices is $\Delta = \{p \in \mathbb{R}^m_+ | 0 \leq p \leq 1\}$. Define the budget correspondence of agent $i$ as $B_i(p) = \{x \in \mathbb{R}^m_+ | px \leq pe_i\}$. Assume the preference relation of each agent $i$ satisfies the following assumptions:

(i) the correspondence $\psi_i : \Delta \times X$ defined as $\psi_i(p, x) = B_i(p) \cap P_i(x)$ has the continuous inclusion property, where $P_i(x) = \{y \in X : y \succsim_i x\}$,

(ii) strong convexity, i.e., $x \succsim_i y$ and $z \succsim_i y, x \neq z$ imply $\lambda x + (1 - \lambda)z \succsim_i z$ for all $x, y, z \in X$ and all $\lambda \in (0, 1)$.

For each $i \in I$ and $p \in \Delta$, the excess demand correspondence of individual $i$ is

$$d_i(p) = \{x - e_i | x \in \mathbb{R}^m_+, px \leq pe_i, y \succsim_i x \Rightarrow py > pe_i\} = \{x - e_i | x \in B_i(p), \psi_i(p, x) = \emptyset\}.$$ 

Proposition 9. Given the exchange economy just described, the excess demand correspondence is a continuous function.

Proof of Proposition 9. Proposition 6 implies that $d_i$ is used with non-empty values. It follows (ii) that $M^a$ is singleton valued, hence it is continuous as a function. ■

52Of course this excludes McKenzie’s and Negishi-Moore’s methods; see McKenzie (1981, 2002) for a discussion.
53It is of interest to explore if these two augmentations/extensions are relevant. Bergstrom (1992, Definition 15) implies that Gale and Mas-Colell’s (1975) augmented preference is equivalent to Scapparone’s!
McKenzie used excess demand correspondence in the presence of non-ordered, non-convex preferences. As mentioned above, the excess demand correspondence may not have a convex selection, see for example Sonnenschein (1977), but who ask us to use the excess demand derived by using the original correspondence. McKenzie worked with an extension of the original preference relation and then by some careful and delicate arguments obtained the equilibrium; see also Khan (1993). McKenzie (1981, p. 824) had already stressed the importance of excess-demand approach:

I think there are advantages to the use of the demand function, or correspondence, in proofs of existence, both for mathematical power and for understanding the proof. I will show how the demand correspondence may be used in a mapping of the Cartesian product of the price simplex and the social consumption set into itself whose fixed points are competitive equilibria even in the absence of the survival assumption. This will hold us in good stead for the result of the next section.

3.4 On Approximate Equilibrium: Starr-Arrow-Hahn

In this subsection, we provide a result on the existence of an approximate equilibrium. It weakens the continuity assumption in Anderson, Khan and Rashid (1982). Consider the following exchange economy. The commodity space is $\mathbb{R}^m_+$ and $I = \{1, \ldots, n\}$ denotes a finite set of individuals $n \geq m$. The endowment of agent $i$ is $e_i \in \mathbb{R}^m_+$, the set of prices is $\Delta = \{p \in \mathbb{R}^m_+ | 0 < p \leq 1\}$ and the budget correspondence of agent $i$ is $B_i(p) = \{x \in \mathbb{R}^m_+ : px \leq pe_i\}$. Let $\mathcal{P}$ denote the set of preferences $\succ_i \subseteq \mathbb{R}^{2m}_+$, i.e. binary relations on $\mathbb{R}^m_+$ satisfying

(i) **continuity**: the correspondence $P_i : \Delta \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ defined as $P_i(p, x) = \{y \in \mathbb{R}^m_+ : y \succ_i x\}$ has the continuous selection property with respect to $B_i$ for all $i \in I$,

(ii) **full transitivity**: $x \nRightarrow y$, $y \nRightarrow z \Rightarrow x \nRightarrow z$,

(iii) **irreflexivity**: $x \nRightarrow x$,

as defined in Subsection 3.1.

An exchange economy is a map $\mathcal{E} : I \to \mathcal{P} \times \mathbb{R}^m_+$. For each $i \in I$, $\mathcal{E}(i) = (\succ_i, e_i)$ assigns agent $i$ a preference relation $\succ_i$ and endowment $e_i$. Given $i \in I$ and $p \in \Delta$, the excess demand correspondence continues, “At the equilibrium the budget sets will not be empty, the demand correspondences will be well defined and upper semi-continuous, but these conditions need not be satisfied for non-equilibrium prices. We will avoid the difficulties posed by this possibility by using an extension of the demand correspondence which reduces to the original correspondence whenever the original correspondence is well defined and non-empty. The extended demand correspondence will be well defined and non-empty for all price vectors.” in this connection also see Footnotes 58 and 59 below and the text they footnote.

54. McKenzie continues, “At the equilibrium the budget sets will not be empty, the demand correspondences will be well defined and upper semi-continuous, but these conditions need not be satisfied for non-equilibrium prices. We will avoid the difficulties posed by this possibility by using an extension of the demand correspondence which reduces to the original correspondence whenever the original correspondence is well defined and non-empty. The extended demand correspondence will be well defined and non-empty for all price vectors.” in this connection also see Footnotes 58 and 59 below and the text they footnote.

55. In the result presented in this section, it is possible to weaken full transitivity by replacing it with the transitivity, or acyclicity, of $\succ$. This can be achieved by strengthening the continuity assumptions on the preference and budget correspondences; see for example Tian and Zhou (1995) Proposition 2 and Theorem 4) for sufficient conditions to obtain a non-empty valued, upper semi-continuous argmax correspondence.
correspondence of individual $i$ is
\[ d_i(p) = \{ x - e_i \mid x \in \mathbb{R}_+^m, \ px \leq pe_i, \ y_i x \Rightarrow py > pe_i \}. \]

And the excess demand correspondence $D : P \rightarrow \mathbb{R}_+^m$ is defined as $D(p) = \sum_{i \in I} d_i(p)$. Let $\|x\|$ denote $\sum_{l=1}^{m} |x^l|$. Also, for $1 \leq n' \leq n$, let $E_{n'} = (1/n') \max_{S \subseteq A, |S| = n'} \| \sum_{i \in S} e_i \|$, the norm of the average endowment of the $n'$ norm-best endowed agents, and let $E = E_m$, the norm of the economy-wide average endowment.

**Proposition 10.** Given the exchange economy just described, there exist $p \in \Delta$ and $z_i \in d_i(p)$ for all agent $i$ such that
\[ \frac{1}{n} \sum_{l=1}^{m} \max \left\{ \sum_{i \in I} z_i^l, 0 \right\} \leq 2 \sqrt{\frac{m}{n} E_m E}. \]

**Proof of Proposition 10.** It follows from Proposition 5 that the excess demand correspondence has non-empty values and a closed selection $\zeta$. The rest of the proof is identical to the proof of Geller’s (1986) theorem (which uses a construction slightly different from the proof of the theorem in Anderson, Khan and Rashid (1982)) except that the demand correspondence is replaced by the selection $\zeta$.

## 4 Open Questions and Alternative Directions

We now come to the end of this essay, and towards the completion of our portrait of Nicholas Yannelis: by necessity it is an incomplete completion. For one thing, we exclude the economics of information from our purview and do not even cite his 1991 paper that has proved so influential.

To the extent that a portrait is a metonymy for a proper name, Kripke’s (1972) “necessary and sufficient conditions that will work for a term like reference” are intimately tied with naming and necessity, and with what a proper name represents. Kripke writes, “I want to present a better picture than the picture presented by the received views.”

It is in this way that we would like the work above to be read. Indeed, when one thinks of the name Nicholas Yannelis, one thinks of Lionel McKenzie and Gerard Debreu, on the one (substantive) hand, and of infinite-dimensions, of continuous selections, of non-convex correspondences on the (technical) other. In this concluding section, we briefly consider these signature terms.

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56 For the statement, see page 92 and for the context of the statement, see pages 91-97, and more generally the text to the entry “Proper names, correct picture of reference.” To doubleback to footnote Kemp advocates the “historian’s version of Occam’s razor” regarding representation and perspective: “In attempting to resolve this impasse the historian has, I believe, no choice but to adopt the simplest, clearest, most direct solution which is compatible with the evidence, asking rigorously at each stage if his solution is necessary and essential, no matter how intellectually attractive it may be (p. 143).”
We begin with the technical register: with infinite-dimensionality. The reader has already noted perhaps a dissonance in the essay in that Section 2 on the mathematical contribution is constantly striving for a view of the subject that is not obscured by the finite-dimensional limitation, be it of the cardinality of the space of agents or the space of commodities or actions at their disposal. This “striving” is a subtext of the 1983 dissertation. And yet, our Section 3 on the applications to Walrasian equilibrium theory are all set in the context of a finite-dimensional commodity space, even though we have referred in passing to Yannelis (1985) and He and Yannelis (2016, 2017) in the context of the GND lemma. But surely more remains to be done, and it is this is one direction that is clearly opened by our consideration. But there is also another affiliated investigation that derives from the fact that in infinite dimensional spaces, the convex hull of a closed set and that of a compact set is not necessarily closed. This slips from infinite-dimensionality to non-convexity, again a preoccupation of the dissertation and the work that followed it. And indeed in the context of selection, when the correspondence has non-convex values, one may explore the direction of ensuring nice selections from correspondences with “star-shaped” values, as in Mas-Colell (1979) and Mitra (1991).

Turning from the ideas to the names, the signature terms for Lional McKenzie, if projected to the register of “classical general equilibrium theory” are two: irreducibility and the survival assumption. As regards the first, it is best to let him speak in his own words (McKenzie, 1999, p. 374):

One of my chief contributions to general equilibrium, following a suggestion from David Gale, is the concept of irreducibility for a competitive economy. Loosely speaking, irreducibility means that the economy cannot be divided into two groups where one group has nothing to offer the other group which has value for it. This replaces the assumption used by Arrow and Debreu that everyone owns a factor that is always able to increase the output of a good which is always desired in larger quantities by everyone. This assumption implies irreducibility, but it is rather implausible.

As to the “survival assumption” and to the words of (McKenzie, 1981, p. 823):

Perhaps the most dramatic innovation since 1959 is the discovery that the survival assumption [...] can be dispensed with in the presence of the other assumptions, in particular in the presence of Assumption (6) that the economy is irreducible.

And as he was to recapitulate in (McKenzie, 1999, p. 374):

Footnotes 32 and 45 above, and the text that they footnote, concern these references.

McKenzie continues, “This idea was first defined and used in my paper on the existence of equilibrium published in 1959, where various generalizations were made of the theory announced by Arrow and Debreu and myself in 1954 and by me in 1956.” We send the interested reader to this text and to Khan (2020), and also to Footnote 54.
It was first seen from the work of John Moore (1975) that irreducibility made this assumption unnecessary, although he did not call attention to the generalization. He applied a fixed point theorem to a mapping of a set of normalized utility possibility vectors into itself in the manner of Negishi (1960) and Arrow and Hahn (1971). This suggested that the survival assumption for isolated consumers had been needed only because the mappings were defined by demand functions in the commodity and price spaces. I was able to confirm this in my presidential address to the Econometric Society (1981).

In the introduction to this portrait, we had occasion to refer to the impact of McKenzie and Metakides on Yannelis’ 1983 dissertation; we round off the portrait it by considering Gerard Debreu’s impact on Yannelis’ subsequent oeuvre. In this connection, one can begin by singling out Debreu’s 1952 social existence theorem: the concept of an abstract economy has its origin in this pioneering paper. Following, Shafer and Sonnenschein (1975), Khan and Vohra (1984) and Toussaint (1984), it gets taken up in Yannelis (1987, 2009) and Kim et al. (1989): as delineated above, these papers have found application and extension in applied mathematics. However, the issues are a little more subtle as far as applications in economics are concerned. The difference between an economy and a game is by now a staple of undergraduate courses: what is of interest and not as equally appreciated is the difference between between an abstract economy in the sense of Debreu-Shafer-Sonnenschein and a game. In his Ely lecture, Arrow (1994) goes into the difference.

The current formulation of methodological individualism is game theory. In a game, each agent chooses one among a set of strategies available to him or her ... In the usual formulations, the set of available strategies is fixed, independent of the choices of others, and all the interactions among players are embodied in the payoff functions. The choice of actions is totally individualistic.

The point is that in an abstract economy, the actions are not independent: the very notion of an abstract economy was instrumentally motivated by the problem of the existence of a Walrasian equilibrium with each of the individual action sets depending on the actions of the Walrasian auctioneer. If the motivation is different, one has to work to show, as does Reny (2016c) for example, that there are circumstances when an abstract economy can induce a games whose equilibria are the same as that of the economy, and hence is a suitable vehicle for applied

59 He did so by “showing that demand functions based on the pseudo-utility functions of Shafer with some rather difficult indirect arguments allowed a commodity space approach to the existence proof where survival for isolated individuals is not assumed. The new approach was essential in order to achieve the further generalization of the existence theorem without individual survival to the case of intransitive preferences, since in the absence of transitivity the utility functions do not exist. Thus, a mapping in the space of utility vectors is not available.” McKenzie (2002, p. 208) added “Moreover it has been shown in the activities model (McKenzie 1981) that the survival assumption can be dropped when irreducibility is assumed. However, the argument is too involved to be introduced here.” Also see Footnotes 54 and 58
work. It is then another interesting open problem to apply the results obtained in Section 3.3 and 3.4 to investigate the existence of an approximate equilibrium with externalities.

5 Bibliography

This is a bibliography in two parts: the first provides detailed references to the chapter numbers cited in the text in keeping with the convention spelt out in Footnote; the second collects the references to a technical and substantive economic literature cited in the text;

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Khan is grateful to Kalyan Chatterjee who first discussed these issues as a room-mate at a conference in Bangalore in the early nineties. Reny (2016c) renames an abstract economy to be an abstract game. For a comprehensive discussion of these issues, in so far as they relate to the history of economic thought, see Ghosh et al. (2020).

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