THEORIES OF ANYTHING

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Abstract. We suggest a generalization of $\pi_0$ for topological groupoids, which encodes incidence relations among the strata of the associated quotient object, and argue for its utility by example, starting from the orbit categories of the theory of compact Lie groups.

One of the points of this note is that Thom’s theory of structurally stable forms fits quite nicely with the categorical theory of databases developed recently by D. Spivak; the other is that the stratifications studied by Thom are closely related to the phase transitions studied in physics, and that the generalization of $\pi_0$ proposed here may be useful in their study: in particular, in organizing our understanding of the scaling laws which naturally accompany such phenomena, in the theory of condensed matter, biology, finance . . . .

1. Introduction

Many difficulties in mathematical classification involve degenerate objects: the theory of conic sections is a classical example, and geometric invariant theory has evolved powerful techniques to deal with such questions. A central issue, in topological terms, is that an action of a reasonable group $G$ on a nice space $X$ can quite easily fail to have a Hausdorff quotient space.

This paper is a rough and intuitive sketch. It proposes encoding the incidence structure of such bad quotients in a database category $\Phi_0$ which generalizes the classical notion of the set $\pi_0$ of components of a topological space. It presents some examples, and argues for the hope of a coherent framework to accommodate them; it is thus mostly about what physicists call phenomenology.

We focus on smooth transformation groupoids $[X/G]$, perhaps not finite-dimensional, though the stratified spaces of Thom [29] (and more general topological groupoids [12, 23, 30]) share many features with this class of objects. An archetypical example is Arnol’d’s classification of isolated singularities of holomorphic functions; the bulk of this paper [§3] is concerned

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with that case, which illustrates particularly clearly that (in physical examples) degeneracy relations among orbits can lead to the existence of scaling laws near phase transitions.

Acknowledgements: This work began in conversations with David Spivak; I will try to explain below why the ‘phase diagrams’ defined here fit into his thinking [26, 27] about databases. I also owe thanks to José Manuel Gomez and John Lind, for patient conversations about equivariant topology, and, at a different coherence length, to Ben Mann, Simon Levin, Andrew Salch, Jim Stasheff, and Abel Wolman III.

2. Transformation groupoids and orbit categories

2.1 If $G$ is a compact Lie group, there is a topological category whose objects are its closed subgroups, e.g. $H_0, H_1, \ldots$, with the spaces

$$G = \text{Maps}(G/H_0, G/H_1) := \text{Mor}(H_0, H_1)$$

of equivariant maps as morphism objects. It is equivalent to the topological category $\mathcal{O}(G)$ with the (totally disconnected [10, 13]) space of conjugacy classes of closed subgroups as its space of objects, and

$$\text{Mor}_0(H_0, H_1) := \text{Trans}_G(H_0, H_1)/H_0$$

as morphisms; where

$$\text{Trans}_G(H_0, H_1) = \{g \in G \mid gH_0g^{-1} \subset H_1\}$$

is the ‘transporter’ of $H_0$ to $H_1$ in $G$. Thus

$$\text{Aut}_0(H) = N_G(H)/H := W_G(H)$$

is a kind of Weyl group. The space of subgroups of $G$ is filtered by dimension, and the space of morphisms from $H_0$ to $H_1$ will be empty if $\dim H_0 > \dim H_1$.

Many of the constructions of this paper have analogs for locally compact groups, using the Chabauty topology [5].

Definition $\mathcal{O}_0(G)$ is the (topological) category with conjugacy classes of closed subgroups of $G$ as objects, and the discrete spaces

$$\text{Mor}_{0_0}(H_0, H_1) := \pi_0\text{Mor}(H_0, H_1)$$

as morphism objects.

2.2 The topological category $[X/G]$ defined by an action

$$G \times X \rightarrow X$$

of $G$ on a space $X$ has elements $x_0, x_1, \cdots \in X$ as objects, with

$$\text{Mor}_{[X/G]}(x_0, x_1) := \{g \in G \mid gx_0 = x_1\}$$
as morphisms; in particular, if \( X \) is at all nice, the isotropy group
\[
\text{Aut}_{[X/G]}(x) = \{ g \in G \mid gx = x \} := \text{Iso}(x)
\]
of \( x \) is closed, and
\[
\text{Fix}(X) : H \mapsto \{ x \in X \mid hx = x \ \forall h \in H \}
\]
defines a contravariant functor from closed subgroups of \( G \) to spaces. Taking components defines a presheaf
\[
H \mapsto \pi_0 \text{Fix}(H) : \mathcal{O}_0(G) \to \text{(Sets)}.
\]
**Definition:** The pullback (or coend [19])
\[
\Phi_0[X/G] \longrightarrow \text{(Sets)}
\]
\[
\pi_0 \mathcal{O}_0(G) \longrightarrow \pi_0 \text{Fix}(X) \quad \text{(Sets)}
\]
defines Grothendieck’s category of elements for the functor \( \pi_0 \circ \text{Fix}(X) \). Its objects are pairs \((c, H)\), consisting of (conjugacy classes of) closed subgroups \( H \), together with a component \( c \) of \( \text{Fix}(H) \); thus
\[
\text{Aut}_\Phi(c, H) = \{ g \in \pi_0 W_G(H) \mid gc = c \}.
\]
This defines a functor \( \Phi_0 \) from transformation groupoids to (Boolean) topological categories, which I’d like to call the **phase diagram** [25] of the groupoid.

**Example** \( \Phi_0[1/G] = \mathcal{O}_0(G) \) is the discretized orbit category defined above, and the forgetful functor
\[
\Phi_0[X/G] \to \Phi_0[1/G]
\]
presents its domain as a category fibered over its target.

2.3 In fact \([X/G]\) can be expressed as the (topological) category of elements of the \( G \)-space \( X \), regarded as a functor from \([1/G]\) to topological spaces. It can similarly be identified with the category of elements of the presheaf
\[
\text{Fix}(X) : \mathcal{O}(G) \to \text{(Spaces)};
\]
which implies the existence of a functor
\[
[X/G] \to \Phi_0[X/G]
\]
generalizing the classical map \( X \to \pi_0 X \).

Classically, \( \pi_0 \) is adjoint
\[
\text{Maps}_{\text{Spaces}}(X, S_{\text{discrete}}) = \text{Maps}_{\text{Sets}}(\pi_0 X, S)
\]
to the inclusion of sets into spaces, and it would be very useful to characterize \( \Phi_0 \) in similar terms. Peter May observes [p.c., but cf [20]] that the category of finite topological spaces is Cartesian closed; it is a little small for
our purposes, but one can ask if the inclusion of the category of profinite topological categories into some reasonable larger class of topological categories might possess a (2-)adjoint.

2.4 When $X$ is a manifold, and $G$ acts smoothly on it, I’ll refer to $[X/G]$ as a smooth transformation groupoid. In the finite-dimensional case, the slice theorem [14, 18] implies that any $x \in X$ has a $G$-invariant open neighborhood $S(x)$ equivariantly diffeomorphic to

$$N_x \times_{\text{Iso}(x)} G,$$

where $N_x$ is essentially [2 I §1.7] the linear representation of $\text{Iso}(x)$ defined by the cokernel of the tangent map to the inclusion of the $G$-orbit through $x$ into the total space $X$. A morphism

$$\alpha \in \text{Mor}_\Phi((x_0, H_0), (x_1, H_1))$$

defines (the germ of) an embedding

$$S(x_0) \subset S(x_1)$$

and hence (the conjugacy class of) a monomorphism

$$N_{x_0} \to N_{x_1},$$

equivariant with respect to the embedding $H_0 \to H_1$ defined by the adjacency $\alpha$.

Definition I’ll call the resulting functor

$$N : \Phi_0[X/G] \to \text{Vect}$$

(with values in the category of real vector spaces and monomorphisms), the degeneracy quiver of $[X/G]$.

3. Some motivating examples

3.1 Arnol’d’s category of (stable equivalence classes of) isolated singularities of holomorphic functions:

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is the germ of a holomorphic function, its associated algebra $\mathfrak{o}_f$ is the quotient of the local ring $\mathbb{C}\{x_1, \ldots, x_n\}$ of functions holomorphic at 0 by the (Jacobian) ideal of $f$ generated by its partial derivatives $\partial f / \partial x_i$.

If the Milnor number $\mu(f) := \dim_{\mathbb{C}} \mathfrak{o}_f$ is finite, $f$ defines an isolated singularity at 0 [2 I §1.4]. Let $\mathcal{A}_n$ denote the space of such function-germs, and let $\mathcal{G}_n$ be the group (under composition) of germs of biholomorphic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, which acts on $\mathcal{A}_n$ by left composition.
The transformation groupoid \([\mathcal{A}_n/\mathcal{G}_n]\) is smooth but infinite-dimensional; but since the Milnor numbers of its objects are finite, their orbits are determined by the behavior of some representative jet [2 I §1.5]. The resulting groupoid therefore looks in many ways like a global quotient defined by an action of a locally compact group. The stabilization map
\[
f \mapsto f + x_{n+1}^2 : \mathbb{C}\{x_1, \ldots, x_n\} \to \mathbb{C}\{x_1, \ldots, x_{n+1}\}
\]
[2 I §1.3] defines Arnol’d’s big stack
\[
\lim_{n \to \infty} [\mathcal{A}_n/\mathcal{G}_n] := [\mathcal{A}/\mathcal{G}]
\]
of (stable equivalence classes of) isolated singularities of holomorphic functions. It is naturally stratified by orbit codimension; the modality
\[
m(f) := (\mu(f) - 1) - \text{codim } f
\]
is an important related (subtle) invariant.

Arnol’d and his school have worked out the structure of this stack, for \(m < 3\); their results seem to me a plausible candidate for an olog in the sense of [27], ie a kind of annotated database. In particular, the stratification of \([\mathcal{A}/\mathcal{G}]\) up to codimension ten is well-understood; see for example the adjacency diagrams in [2 I §2].

3.2 The moduli stack of one-dimensional formal groups is how I got into this [22 §1.4]; it is a big arithmetic stack (ie defined over Spec \(\mathbb{Z}\)), with a nontrivial but very simple stratification (isomorphic to \(\mathbb{N}\), regarded as a poset with its usual order) at each prime. Formal groups of higher dimension are of course much more complicated . . .

3.3 Knots (and links . . .) can be studied similarly, as topological groupoids defined by a space
\[
X := \text{Imm}(S^1, S^3)
\]
of immersions, under the action of the group
\[
G := \text{Diff}(S^1) \times \text{Diff}(S^3)
\]
of diffeomorphisms of the domain and range; the theory of finite-type invariants comes from its stratification by self-intersection number. Knot tables (cf eg [3, 29]) are another natural class of examples of mathematical ologs.

3.4.1 Similarly, Riemann surfaces seem to fit well enough into some such framework; but formulating the moduli problem as a global quotient is not so easy [15], so I’ll leave this example aside; however,

3.4.2 The topological groupoid of Riemannian metrics on a manifold, under the action of its group of diffeomorphisms, is the natural configuration space for general relativity [9] (just as the analogous stacks of connections on principal bundles are relevant to gauge theory). Classical solutions of the
Einstein equations tend to have large symmetry groups, and thus represent quite singular points of this moduli stack . . .

3.5 Finally, Thom’s notion [30 §2.1] of a structurally stable form lies behind this whole essay. A cartoon version of his program asks for the structure of the big stack of everything in the world, modulo the pseudogroup defined by their local isomorphisms. The related theory of stratified sets, among other things, is one of his creations (see further §5.2), but in his day the language of (higher) category theory was only beginning to emerge. One of the points of this note is the close compatibility of that language (and with Spivak’s theory of databases) with Thom’s thinking.

- A very natural, highly nontrivial example of a Thomist olog, dear to my heart, is the stratification of the space of everything in the world defined by the Bantu noun-class system [6, 8]). The ‘Leitfaden’ in Serre’s Corps locaux [p 13] is another striking example . . .

4. ADJACENCY AND PHASE TRANSITIONS

4.1 Arnol’d’s simple singularities (whose orbits have no moduli, ie $m = 0$, eg with codimension $\leq 7$ [2 II §2.4]) are worth further discussion. [They extend Thom’s classification [30 §3.2, 5.2-5.4] of elementary catastrophes.] In this class of examples, the nerve of the groupoid $[\mathcal{N}_f/\text{Iso}(f)]$ is the classifying space of a generalized braid group [2 II §5.3] and the adjacency maps of §1.4 above can be described in terms of inclusions of Dynkin diagrams [2 II §5.9, 2 §6 (Fig 39)].

Functions with simple singularities have nice quasihomogeneous normal forms, which endow their associated algebras with canonical Euler derivations [2 II §5.7]. A degeneration $\alpha : f \mapsto g$ (ie with adjacent orbits [2 I §2.7]) defines a homomorphism

$$(o_f, \mathcal{D}_f) \rightarrow (o_g, \mathcal{D}_g)$$

of differential algebras; its cokernel $\mathcal{N}^f_g$ can be identified with that of the map defined in §2.4. In particular, when the codimension jump from $f$ to $g$ is one, we can think of $\mathcal{N}^f_g$ as generated by $\mathcal{D}_g$ (modulo $\mathcal{D}_f$).

4.2.1 Physical systems are often analyzed in Morse-theoretic terms, by defining a suitable Lagrangian (action) functional of on some space $X$ of states, perhaps invariant under some (large) symmetry group $G$ (eg as in gauge theory). Such models can be formulated in terms of the gradient flow associated to the Lagrangian function on the associated quotient object; and in interesting cases these quotients are not manifolds, but are instead stratified sets.
These stratifications can be described in terms of (‘spontaneously’) broken symmetry: that is, the states of the system may have more (or less) symmetry than the equations of motion themselves \cite{30 §6.1 C, E}. In this conceptualization, phase transitions correspond to arrows

\[(x_0, H_0) \rightarrow (x_1, H_1) \in \Phi_0[X/G]\]

which identify \(H_0\) with a proper subgroup of \(H_1\). In particular, when \(\dim H_1 > \dim H_0\), the normal bundle \(N_{x_0}^{x_1}\) parametrizes the loss or gain in infinitesimal symmetry associated to the transition \(x_0 \rightarrow x_1\).

A classical theorem (perhaps better: a classical principle) of Noether associates a conserved quantity to a continuous symmetry of a mechanical system; conservation of momentum, for example, is a consequence of translation invariance, and angular momentum is similarly related to rotational invariance. In the case above we expect to see, not a conserved quantity but rather the appearance of an order parameter \(\eta\) \cite[1 p 449, 16]{1}, which vanishes as \(H_0 \rightarrow H_1\). In this sense \(N_{x_1}^{x_0}\) is the Lie algebra generated by \(\partial/\partial \eta\) (when \(\dim \mathbb{C} \dim \text{Lie} H_1/\text{Lie} H_0 = 1\)).

4.2.2 For adjacent simple singularities in \([\mathcal{A}/\mathcal{G}]\), the derivation \(\partial/\partial \eta\) can be identified with the relative Euler operator above. In classical systems with \(g \in \mathcal{A}\) as potential function, this generates an action of \(\mathbb{G}_m \cong \mathbb{C}^\times\): the renormalization group \cite[§5.1]{4, 17} associated to the phase transition.

Eigenfunctions of Euler operators satisfy scaling laws, eg like those satisfied by ‘fat-tailed’ probability distributions \cite{11, 17}; see further §5.4. In fact

Rings of functions of observables defined near phase transitions are naturally graded by scaling laws.

This is familiar in physics, but may have broader applications, eg to the complex systems encountered in fields such as biology and finance. The second point of this paper is to suggest that constructions such as \(\Phi_0\) may be useful in organizing our thinking about these scaling laws.

5. Afterthoughts

5.0 Jim Stasheff reminds me of Thom’s dictum, that ‘Une théorie qui explique tout n’explique rien’. The working title of these notes was ‘Phase diagrams for big stacks’; it is an attempt, not at a theory of anything, but of a theory of theories of anything.

\footnote{In statistical mechanics, this goes back to Landau’s work in the late 1930’s, where it is related to failure of the partition function to be analytic.}
5.1 (re §2.3) Substantial issues of finiteness, ultimately (point-set) topological [cf eg [12 §2.2]], lie behind the notion of a ‘big stack’; I have left this term undefined in the hope that someone with a better understanding of such things will take up the challenge.

Examples in different contexts have been successfully treated by various methods (eg in terms of pro-algebraic structures in ex’s 3.1-2 above, while Sobolev ‘riggings’ [9] are useful in examples like 3.4.2) and it may be that no easy general formalism accommodates them all; but the existence of slices [23] seems to be a very useful general property. [A slice at an element $x$ of a topological groupoid is presumably a nice neighborhood of its component in its equivalence class; but ‘nice’ will require further specification in infinite-dimensional contexts.]

Finally, the 2-categorical invariance of finiteness conditions seems quite subtle; this brings to mind G. Segal’s remark, that St. Anselm’s ontological argument is basically Zorn’s lemma without chain conditions.

5.2 (re §3.5) A stratified set $X$, eg an algebraic variety, is a union of locally closed subsets $X_i$ indexed by a poset $I$, perhaps endowed with a monotone increasing integer-valued codimension function. The category of elements of the functor

$$i \mapsto \pi_0(X_i) : I \to (\text{Sets})$$

defines a category $\Phi_0(X)$ which (surprisingly) seems to have no name in algebraic geometry.

5.3 (re §4.2.1) In statistical mechanics, the Helmholtz free energy of a system with Langrangian $L$ is also the cumulant generating function

$$\Gamma(\theta) = \log \mathbb{E}(\exp(\theta L))$$

of the random variable $L$. The stationary phase principle for the Feynman-Gibbs partition function

$$Z(\hbar) = \exp \Gamma(i\hbar^{-1}) = \int \exp(iL(x)/\hbar)dx$$

suggests the interest of asymptotic expansions of

$$\Gamma(i\hbar^{-1}) = \log Z(\hbar) \in \mathbb{C}(\hbar),$$

but in one of the simplest cases, ie the Bernoulli process with

$$\Gamma(\theta) = \log(1 - p(1 - e^{\theta})),$$

this is unproductive. On the other hand, in evolutionary contexts the Cramér ‘internal energy’ (defined by the Legendre transform $\Gamma^*$ [21, 24]

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2More precisely, it’s $\theta^{-1}\Gamma(\theta)$, with $\theta^{-1} = -kT$, where $T$ is the temperature and $k$ is Boltzmann’s constant. In population biology $\theta^{-1}$ is the number of organisms [7], and in finance . . .
of \( \Gamma \), which measures the rate of the system’s excursions from the mean 

\[ C(x) := -\Gamma^*(x) = -x \log(1 - p) + \log(1 - p) + S(x) , \]

with

\[ S(x) = -x \log x - (1 - x) \log(1 - x) , \]

and hence

\[ C(ih^{-1}) \sim -ih^{-1} \log(1 - p) + \log h + [\log(1 - p)] - 1 + i\pi/4 \sum_{n \geq 1} \frac{(ih)^n}{n(n + 1)} \ldots \]

[It is in some sense well-known that Planck’s constant \( h \) is not a number. Geometric quantization identifies \( h^{-1} \) with the Chern class of a Hermitian line bundle \((L, \nabla)\) with connection on the phase space \( X \) of a physical system, ie

\[ ih^{-1} \sim [\omega] \in H^2(X, 2\pi i\mathbb{Z}) \subset H^2_{dR}(X, \mathbb{C}) ; \]

from this point of view, \( 2\pi i \) is not a number either: it is a motive...]

5.4 (re §4.2.2) It is therefore not surprising to find abundant non-Gaussian phenomena in the vicinity of phase transitions; this may lie behind the biologists’ intuition, that living organisms colonize, and thrive, in systems on the ‘edge of chaos’. It is worth noting the existence of a robust theory of non-Gaussian but ‘stable’ probability distributions (going back to work of Kolmogorov and Gnedenko in the 1930’s [28]) which need not possess higher moments.

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