Abstract We investigate a multi-field model of dark energy in this paper. We develop a model of dark energy with two multiple scalar fields: one we consider is a multi-field tachyon and the other is multi-field phantom tachyon scalars. We make an analysis of the system in phase space by considering inverse square potentials suitable for these models. Through the development of an autonomous dynamical system, the critical points and their stability analysis is performed. It has been observed that these stable critical points are satisfied by power-law solutions. Moving on toward the analysis, we can predict the fate of the universe. A special feature of this model is that it affects the equation of state parameter $w$ to alter from being it greater than $-1$ to be less than it during the evolutionary phase of the universe. Thus, it’s all about the phantom divide which turns out to be decisive in the evolution of the cosmos in these models.

1 Introduction

At present, the Universe is undergoing accelerated expansion cosmologically as the astrophysical and cosmological observational data within past 20 years from type Ia supernovae (SNe) and other sources confirm it [1–11]. Spatial curvature being negligibly small $\Omega_K = 0.001 \pm 0.002$ ascribes a spatially flat (Euclidean) geometry to the observable Universe [12]. The accelerated expansion is thought either due to the presence of some exotic form of matter with negative pressure ($\rho = -p$) dubbed as dark energy that necessitates to modify $T_{\mu\nu}$--the RHS of Einstein Field Equation (EFE) or is explained in the framework of $f(R)$ gravity theories that modify LHS of EFE [13–15].

The equation of state (EoS) parameter $w = p/\rho$ characterizing the dark energy has recent observational constraint to be $-1.03 \pm 0.03$ [12] that favors dynamical dark energy models to a certain extent [16]. It describes the nature of dark energy and determines whether the Universe will end as a time-reversed process of the big bang (big crunch) or will expand forever (big freeze). The energy–momentum tensor $T_{\mu\nu}$ in EFE representing the energy density of normal baryonic matter in the Universe succumbs to failure to be responsible for the cosmic accelerated expansion on the base of observational evidence [17]. Cosmological
constant $\Lambda$ when treated as one of the candidates for dark energy, as Einstein predicted [18], encounters the problems of accounting for the extremely fine-tuned value related to it on the one hand and an extraordinarily small value on the other hand. Due to this, the status of $\Lambda$ does not possess amenability and remains unacceptable; however, different proposals to tackle these issues are presented [19–23].

Scalar fields can atone for the fine-tuning and coincidence problems pertaining to $\Lambda$ and might restitute, as well to the insufficiency of the energy–momentum tensor $T_{\mu\nu}$. To investigate the properties of dark energy, there are a large number of dynamical scalar field models like quintessence, K-essence, phantom, quintom, tachyon, phantom tachyon, etc. discussed in the literature, see Refs. [13,24–28]. Quintessence is a scalar field minimally coupled to gravity and evolves dynamically with a canonical kinetic energy term in its Lagrangian and has EoS parameter $w > -1$, see Refs.[23,29–36]. K-essence arises from a non-canonical kinetic energy term, see Refs. [19,36–40,40–42]. All the models with negative kinetic energy term can be considered its subcases. On the other hand, phantom has non-canonical kinetic energy term in its Lagrangian and has EoS parameter $w < -1$. Introduction of negative energies is, however, a significant issue in these fields, see Refs. [43–56].

Neither quintessence nor phantom individually is able to cross over $-1$; however, the unified model developed from the two, known as quintom, see Refs. [57,58], fulfils the requirement. In quintom models, EoS parameter $w$ transitions from $w > -1$ to $w < -1$ as the constraint on dark energy favors it mildly, see Refs. [59–66]. Recently, a model is proposed to resolve Hubble tension using quintom dark energy [67]. In its earliest emergence, tachyon field surfaced in string theory and was used in cosmology afterward to drive inflationary phase in the early universe, see Refs. [68–72]. It stood up for the candidate of dark energy after the expansion was proved to be accelerating, see Refs. [73–93]. It is, nonetheless, significant to note that tachyonic dark energy models require fine tuning larger than quintessence models. Tachyon field with negative kinetic energy term represents its phantom version, see Refs. [94,95] with EoS parameter $w < -1$ as the data mildly favors it. The Lagrangian density for tachyon field and its phantom version is written as $\mathcal{L} = -V(\phi) \sqrt{1 + \varepsilon g^{\mu\nu} \partial_\mu \partial_\nu}$ where $\varepsilon = +1$ is for tachyon and $\varepsilon = -1$ for phantom tachyon.

Although recent data fit gives specific trend to EoS $w$, it does not determine exactly whether the dark energy is quintessence, phantom, quintom, or tachyon. Tachyon models can effectively illustrate the dark energy and have efficient viability. A quintom version of tachyon field was investigated in Ref. [89], where two scalar fields, namely tachyon and phantom tachyon, represent dark energy that drives the late time accelerated expansion. These scalar fields easily allow EoS parameter to change from $w > -1$ to $w < -1$. Theoretically, it is possible that the dark energy consists of more than one scalar field. This allows one to consider $N$ scalar fields by adapting the Lagrangian for $N$ scalar fields $\mathcal{L}_{\xi_1,\ldots,\xi_N}$. Being motivated by the interesting prospects of tachyon-quintom dark energy, we consider a multi-field model incorporating tachyon and phantom tachyon fields to explain the late time acceleration within the context of flat FLRW spacetime in the background. Using inverse square potentials, we study the dynamics of the model in phase space. This article is organized in the following way: Sect. 2 is devoted to the development of the mathematics for the model. It presents the development of autonomous dynamical system that plays a significant role in investigating and understanding the behavior of such models. We draw plots for evolution between the multi-field scalars and the parameters of the EoS $w$ and dark energy $\Omega_{DE}$ as a function of the number of e-folds $N$. In Sect. 3, we perform the analysis of the data and study the stability of the critical points inferring the future evolutionary development of the universe. Our discussion and conclusion are presented in Sect. 4.
2 Development of the mathematics for the model

We take two homogeneous scalar fields, namely multi-field tachyon \( \sum_{i=1}^{n} \xi_i \) and multi-field phantom tachyon \( \sum_{i=1}^{n} \eta_i \), which is known as tachyon quintom [89]. Their corresponding potentials are \( \sum_{i=1}^{n} V(\xi_i) \) and \( \sum_{i=1}^{n} V(\eta_i) \), respectively. In the background, we consider the FLRW universe with four-dimensional flat spacetime. Since we shall use equation of state parameter \( w \) expressed in terms of pressure and density as ratio of the two, we are going to take the fluid whose density is the function of pressure only, \( \rho = \rho(p) \). The barotropic fluid is disseminated through and replenished in the universe with equation \( p = \gamma \rho \) with the condition \( 0 < \gamma \leq 2 \), where \( \gamma \) has different values for dust and radiation. The system of this type will have action of the form

\[
S = \int \left( \frac{M_P^2 R}{2} + \sum_{i=1}^{n} \mathcal{L}_{\xi_i} + \sum_{i=1}^{n} \mathcal{L}_{\eta_i} + \mathcal{L}_m \right) \sqrt{-g} \, d^4x
\]

where

\[
\sum_{i=1}^{n} \mathcal{L}_{\xi_i} = - \sum_{i=1}^{n} V(\xi_i) \sqrt{1 + g^{\mu\nu} \sum_{i=1}^{n} \partial_\mu \xi_i \partial_\nu \xi_i}
\]

and

\[
\sum_{i=1}^{n} \mathcal{L}_{\eta_i} = - \sum_{i=1}^{n} V(\eta_i) \sqrt{1 - g^{\mu\nu} \sum_{i=1}^{n} \partial_\mu \eta_i \partial_\nu \eta_i}
\]

are the scalar field Lagrangian densities, whereas \( \mathcal{L}_m \) represents the Lagrangian density of the matter fields as Lagrangian.

The spatial homogeneity will imply \( \sum_{i=1}^{n} \partial_\lambda \xi_i = \sum_{i=1}^{n} \partial_\lambda \eta_i = 0 \), that is, only spatially homogeneous solutions that evolve in time, \( i.e., \), time-dependent solutions will be considered. The generalized energy densities of the fields result in the form

\[
\rho_{\sum_{i=1}^{n} \xi_i} = \frac{\sum_{i=1}^{n} V(\xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \dot{\xi}_i^2}}
\]

\[
\rho_{\sum_{i=1}^{n} \eta_i} = \frac{\sum_{i=1}^{n} V(\eta_i)}{\sqrt{1 + \sum_{i=1}^{n} \dot{\eta}_i^2}}
\]

and the generalized pressures for both fields

\[
p_{\sum_{i=1}^{n} \xi_i} = - \sum_{i=1}^{n} V(\xi_i) \sqrt{1 - \sum_{i=1}^{n} \dot{\xi}_i^2}
\]

\[
p_{\sum_{i=1}^{n} \eta_i} = - \sum_{i=1}^{n} V(\eta_i) \sqrt{1 + \sum_{i=1}^{n} \dot{\eta}_i^2}
\]

Now the equations of the scalar fields for generalized tachyon and generalized phantom tachyon read in the following forms

\[
\frac{\sum_{i=1}^{n} \ddot{\xi}_i}{1 - \sum_{i=1}^{n} \dot{\xi}_i^2} + 3H \sum_{i=1}^{n} \dot{\xi}_i + \sum_{i=1}^{n} \frac{V(\xi_i)}{\sum_{i=1}^{n} V(\xi_i)} = 0
\]
and
\[ \frac{\sum_{i=1}^{n} \dot{\eta}_i}{1 - \sum_{i=1}^{n} \dot{\eta}_i^2} + 3H \sum_{i=1}^{n} \dot{\eta}_i - \frac{\sum_{i=1}^{n} V(\xi_i)}{\sum_{i=1}^{n} V(\eta_i)} = 0 \] (9)

and from equation of continuity, substituting for \( p_\gamma = (\gamma - 1) \rho_\gamma \), we obtain evolution equation for barotropic perfect fluid
\[ \frac{d\rho}{dt} + 3H (\rho + p) = 0, \] (10)
\[ \Rightarrow \frac{d\rho_\gamma}{dt} + 3H (\rho_\gamma + p_\gamma) = 0 \] (11)
\[ \Rightarrow \frac{d\rho_\gamma}{dt} - 3H \gamma \rho_\gamma = 0 \] (12)

Now in order to find the equation for acceleration, we have to compute first the following two parameters \( \dot{H} \) and \( H^2 \). We know from Friedmann’s Equations that
\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{k}{a^2} \] (13)
\[ \dot{H} + H^2 = \frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) \] (14)

from Eqs. (13) and (14), we have
\[ \dot{H} = -\frac{4\pi G}{3} (\rho + p) \] (15)

Now Eq. (15) can be written for the densities and pressures of the generalized fields considered in the model and for the barotropic pressure and density, that is,
\[ \dot{H} = -\frac{1}{2M_{Pl}^2} \left( \rho \sum_{i=1}^{n} \xi_i + \rho \sum_{i=1}^{n} \eta_i + p \sum_{i=1}^{n} \xi_i + p \sum_{i=1}^{n} \eta_i + \rho_\gamma + p_\gamma \right) \] (16)

Now, substituting the values for the generalized energy densities and pressures, we have
\[ \dot{H} = -\frac{1}{2M_{Pl}^2} \left[ \left( \frac{\sum_{i=1}^{n} V(\xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} + \frac{\sum_{i=1}^{n} V(\eta_i)}{\sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} \right) \right. \\
\[ + \left. \sum_{i=1}^{n} V(\xi_i) \left( 1 - \sum_{i=1}^{n} \xi_i^2 \right) - \sum_{i=1}^{n} V(\eta_i) \left( 1 + \sum_{i=1}^{n} \eta_i^2 \right) \right] \] (17)

after having simplified the above expression, we obtain
\[ \dot{H} = -\frac{1}{2M_{Pl}^2} \left( \frac{\sum_{i=1}^{n} V(\xi_i) \xi_i^2}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} - \frac{\sum_{i=1}^{n} V(\eta_i) \eta_i^2}{\sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} + \gamma \rho_\gamma \right) \] (18)

Now from Eq. (13)
\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{k}{a^2} \]

With \( k = 0 \) for flat universe,
\[ H^2 = \frac{8\pi G}{3} \rho \] (19)
Since energy density $\rho$ is contributed by $\sum_{i=1}^{n} \xi_i$ and $\sum_{i=1}^{n} \eta_i$ and $\rho_{\gamma}$, and for $M_{Pl}^2 = \frac{1}{8\pi G}$, we have

$$H^2 = \frac{1}{3M_{Pl}^2} \rho = \frac{1}{3M_{Pl}^2} \left( \rho\sum_{i=1}^{n} \xi_i + \rho\sum_{i=1}^{n} \eta_i + \rho_{\gamma} \right)$$

(20)

Using Eqs. (4) and (5), we have

$$H^2 = \frac{1}{3M_{Pl}^2} \left( \frac{\sum_{i=1}^{n} V(\xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} + \frac{\sum_{i=1}^{n} V(\eta_i)}{\sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} + \rho_{\gamma} \right)$$

(21)

Now dividing both sides of the above equation by $H^2$,

$$1 = \frac{1}{3H^2M_{Pl}^2} \left( \frac{\sum_{i=1}^{n} V(\xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} + \frac{\sum_{i=1}^{n} V(\eta_i)}{\sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} + \rho_{\gamma} \right)$$

(22)

or

$$1 = \left( \frac{\sum_{i=1}^{n} V(\xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} + \frac{\sum_{i=1}^{n} V(\eta_i)}{\sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} + \frac{\rho_{\gamma}}{3H^2M_{Pl}^2} \right)$$

(23)

From Eq. (23), we are going now to define some parameters which are dimensionless and can facilitate our calculations. Further we will construct dynamical system with the help of these parameters

$$x\sum_{i=1}^{n} \xi_i = \sum_{i=1}^{n} \dot{\xi}_i$$

(24)

$$x\sum_{i=1}^{n} \eta_i = \sum_{i=1}^{n} \dot{\eta}_i$$

(25)

and

$$y\sum_{i=1}^{n} \xi_i = \frac{\sum_{i=1}^{n} V(\xi_i)}{3H^2M_{Pl}^2}$$

(26)

$$y\sum_{i=1}^{n} \eta_i = \frac{\sum_{i=1}^{n} V(\eta_i)}{3H^2M_{Pl}^2}$$

(27)

and

$$z = \frac{\rho_{\gamma}}{3H^2M_{Pl}^2}$$

(28)

Now using the defined parameters from Eqs. (24–28), (23) becomes

$$1 = \left( \frac{y\sum_{i=1}^{n} \xi_i}{\sqrt{1 - x^2\sum_{i=1}^{n} \xi_i}} + \frac{y\sum_{i=1}^{n} \eta_i}{\sqrt{1 + x^2\sum_{i=1}^{n} \eta_i}} + z \right)$$

(29)

For obtaining the equation of acceleration, we divide both sides of Eq. (18) by $H^2$, and making use of the defined parameters from Eqs. (24–28), we get the following simplified form

$$\frac{\dot{H}}{H^2} = -\frac{3}{2} \left( \frac{y\sum_{i=1}^{n} \xi_i x^2\sum_{i=1}^{n} \xi_i}{\sqrt{1 - x^2\sum_{i=1}^{n} \xi_i}} + \frac{y\sum_{i=1}^{n} \eta_i x^2\sum_{i=1}^{n} \eta_i}{\sqrt{1 + x^2\sum_{i=1}^{n} \eta_i}} + yz \right)$$

(30)
Using the value of \( z \) from Eq. (29) in Eq. (30) and performing simplification, we have

\[
\frac{H'}{H} = -\frac{3}{2} \left( -\frac{\sum_{i=1}^{n} \xi_i \left( \gamma - \frac{x_i^2}{\sum_{i=1}^{n} \xi_i} \right)}{\sqrt{1 - \frac{x_i^2}{\sum_{i=1}^{n} \xi_i}}} - \frac{\sum_{i=1}^{n} \eta_i \left( \gamma + \frac{x_i^2}{\sum_{i=1}^{n} \eta_i} \right)}{\sqrt{1 + \frac{x_i^2}{\sum_{i=1}^{n} \eta_i}}} + \gamma \right) \tag{31}
\]

Here \( H' \) denotes the derivative of \( H \) with respect to the logarithm of the scale factor, i.e.,\( \ln a = N \) and \( H = \partial_t \ln a = \frac{dN}{dt} \) and \( H = \frac{dH}{dN} \times \frac{dN}{dt} = H' H \).

Now, from Eq. (29)

\[
\Omega_{DE} = 1 - \frac{1}{3} \left( \rho_c H^{-2} M_{pl}^{-2} \right) \tag{32}
\]

where we used Eq. (28) and for

\[
\frac{\sum_{i=1}^{n} \xi_i}{\sqrt{1 - \frac{x_i^2}{\sum_{i=1}^{n} \xi_i}}} + \frac{\sum_{i=1}^{n} \eta_i}{\sqrt{1 + \frac{x_i^2}{\sum_{i=1}^{n} \eta_i}}} = \Omega_{DE}
\]

The parameter \( \Omega_{DE} \) weighs out the energy density of dark energy as a fraction of the critical density \( \rho_c. \) The EoS \( w = \frac{p}{\rho} \) of dark energy for the system of multi-field scalars is given by

\[
w = \frac{\rho \sum_{i=1}^{n} \xi_i + P \sum_{i=1}^{n} \eta_i}{\rho \sum_{i=1}^{n} \xi_i + P \sum_{i=1}^{n} \eta_i} \tag{33}
\]

Using Eqs. (4–7), we obtain

\[
w = \frac{-\sum_{i=1}^{n} V (\xi_i) \sqrt{1 - \sum_{i=1}^{n} \xi_i^2} - \sum_{i=1}^{n} V (\eta_i) \sqrt{1 + \sum_{i=1}^{n} \eta_i^2}}{\sum_{i=1}^{n} V (\xi_i) \sqrt{1 - \sum_{i=1}^{n} \xi_i^2} + \sum_{i=1}^{n} V (\eta_i) \sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} \tag{34}
\]

dividing and multiplying by \( \frac{8\pi G}{3H^2} \),

\[
w = \frac{\frac{8\pi G}{3H^2} \left( -\sum_{i=1}^{n} V (\xi_i) \sqrt{1 - \sum_{i=1}^{n} \xi_i^2} - \sum_{i=1}^{n} V (\eta_i) \sqrt{1 + \sum_{i=1}^{n} \eta_i^2} \right)}{\sum_{i=1}^{n} V (\xi_i) \sqrt{1 - \sum_{i=1}^{n} \xi_i^2} + \sum_{i=1}^{n} V (\eta_i) \sqrt{1 + \sum_{i=1}^{n} \eta_i^2}} \tag{35}
\]

after simplification, we get

\[
w = \frac{-\sum_{i=1}^{n} V (\xi_i) \sqrt{1 - \frac{x_i^2}{\sum_{i=1}^{n} \xi_i}} - \sum_{i=1}^{n} V (\eta_i) \sqrt{1 + \frac{x_i^2}{\sum_{i=1}^{n} \eta_i}}}{\sum_{i=1}^{n} V (\xi_i) \sqrt{1 - \frac{x_i^2}{\sum_{i=1}^{n} \xi_i}} + \sum_{i=1}^{n} V (\eta_i) \sqrt{1 + \frac{x_i^2}{\sum_{i=1}^{n} \eta_i}}} \tag{36}
\]

Now we develop an autonomous dynamical system. We use inverse square potentials to study the dynamics of our model constructed for tachyon and phantom tachyon. Inverse square potential renders the similar role to tachyon field as does the exponential potential to the standard scalar field. These potentials have been used extensively in the study of tachyon models and allow to develop the dynamical system [73, 74, 79, 80, 83, 84, 92]. From Eqs. (8), (24) and with inverse square potential defined for multi-field scalars \( \sum_{i=1}^{n} \xi_i \) as

\[
\sum_{i=1}^{n} V (\xi_i) = M^2 \sum_{i=1}^{n} \xi_i \sum_{i=1}^{n} \xi_i^{-2}
\]
for evolution of the system. Here \( M_{\sum_{i=1}^{n} \xi_i}^2 \) is the mass scale of multi-field scalars \( \sum_{i=1}^{n} \xi_i \).

\[
x' \sum_{i=1}^{n} \xi_i = \frac{d \left( x \sum_{i=1}^{n} \xi_i \right)}{dN} = -3 \left( 1 - x^2 \sum_{i=1}^{n} \xi_i \right) \left( x \sum_{i=1}^{n} \xi_i - \sqrt{\lambda \sum_{i=1}^{n} \xi_i \sum_{i=1}^{n} \xi_i} \right)
\]

(37)

From Eqs. (26) and (31) with using inverse square potential \( \sum_{i=1}^{n} V(\xi_i) = M_{\sum_{i=1}^{n} \xi_i}^2 \sum_{i=1}^{n} \xi_i^{-2} \), we find

\[
y' \sum_{i=1}^{n} \xi_i = \frac{d \left( y \sum_{i=1}^{n} \xi_i \right)}{dN} = 3y \sum_{i=1}^{n} \xi_i \left( \frac{y - x^2 \sum_{i=1}^{n} \xi_i}{\sqrt{1 - x^2 \sum_{i=1}^{n} \xi_i}} - \frac{y \sum_{i=1}^{n} \eta_i \left( y + x^2 \sum_{i=1}^{n} \eta_i \right)}{\sqrt{1 + x^2 \sum_{i=1}^{n} \eta_i}} \right)
\]

(38)

where

\[
\lambda \sum_{i=1}^{n} \xi_i = \frac{4M_{pl}^2}{3M_{\sum_{i=1}^{n} \xi_i}^2}
\]

(39)

Now, from Eqs. (9) and (25) with inverse square potential defined for multi-field scalars \( \sum_{i=1}^{n} \eta_i \) as

\[
V \left( \sum_{i=1}^{n} \eta_i \right) = M_{\sum_{i=1}^{n} \eta_i}^2 \sum_{i=1}^{n} \eta_i^{-2}
\]

for evolution of the system. Here \( M_{\sum_{i=1}^{n} \eta_i}^2 \) is the mass scale for multi-field scalars \( \sum_{i=1}^{n} \eta_i \).

\[
x' \sum_{i=1}^{n} \eta_i = \frac{d \left( x \sum_{i=1}^{n} \eta_i \right)}{dN} = -3 \left( 1 + x^2 \sum_{i=1}^{n} \eta_i \right) \left( x \sum_{i=1}^{n} \eta_i + \sqrt{\lambda \sum_{i=1}^{n} \eta_i \sum_{i=1}^{n} \eta_i} \right)
\]

(40)

From Eqs. (27) and (31) with using inverse square potential \( V(\sum_{i=1}^{n} \eta_i) = M_{\sum_{i=1}^{n} \eta_i}^2 \sum_{i=1}^{n} \eta_i^{-2} \), we find

\[
y' \sum_{i=1}^{n} \eta_i = \frac{d \left( y \sum_{i=1}^{n} \eta_i \right)}{dN} = 3y \sum_{i=1}^{n} \eta_i \left( \frac{y \sum_{i=1}^{n} \eta_i \left( y - x^2 \sum_{i=1}^{n} \eta_i \right)}{\sqrt{1 - x^2 \sum_{i=1}^{n} \eta_i}} - \frac{y \sum_{i=1}^{n} \eta_i \left( y + x^2 \sum_{i=1}^{n} \eta_i \right)}{\sqrt{1 + x^2 \sum_{i=1}^{n} \eta_i}} \right)
\]

(41)

where

\[
\lambda \sum_{i=1}^{n} \eta_i = \frac{4M_{pl}^2}{3M_{\sum_{i=1}^{n} \eta_i}^2}
\]

(42)
3 Stability of the model

Where the fixed points $x' = \sum_{i=1}^{n} \xi_i$, $y' = \sum_{i=1}^{n} \eta_i$, $x' = \sum_{i=1}^{n} \xi_i$ and $y' = \sum_{i=1}^{n} \eta_i$ diminish to zero, the existence of these critical points $x(\sum_{i=1}^{n} \xi_i)_{crit}$, $y(\sum_{i=1}^{n} \eta_i)_{crit}$, $x(\sum_{i=1}^{n} \eta_i)_{crit}$ and $y(\sum_{i=1}^{n} \eta_i)_{crit}$ corresponds there. These critical points have been calculated and are listed in Table 1.

Now from Eq. (31), we determine the solutions of the self-similar nature, that is,

$$\frac{H'}{H} = \frac{\dot{H}}{H^2} = -\frac{3}{2} \left( -\frac{y \sum_{i=1}^{n} \xi_i (\gamma - x^2 \sum_{i=1}^{n} \xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} - \frac{y \sum_{i=1}^{n} \eta_i (\gamma - x^2 \sum_{i=1}^{n} \eta_i)}{\sqrt{1 - \sum_{i=1}^{n} \eta_i^2}} + \gamma \right) \quad (43)$$

This corresponds to an expanding universe such that $a(t)$, the scale factor scales like $a(t) \propto t^P$, where

$$P = \frac{2}{3} \left( -\frac{y \sum_{i=1}^{n} \xi_i (\gamma - x^2 \sum_{i=1}^{n} \xi_i)}{\sqrt{1 - \sum_{i=1}^{n} \xi_i^2}} - \frac{y \sum_{i=1}^{n} \eta_i (\gamma - x^2 \sum_{i=1}^{n} \eta_i)}{\sqrt{1 - \sum_{i=1}^{n} \eta_i^2}} + \gamma \right) \quad (44)$$

We now study the stability around the critical points given in Table 1 for which we consider small perturbations $\delta x \sum_{i=1}^{n} \xi_i$, $\delta y \sum_{i=1}^{n} \eta_i$, $\delta x \sum_{i=1}^{n} \eta_i$, about the critical points $x(\sum_{i=1}^{n} \xi_i)_{crit}$, $y(\sum_{i=1}^{n} \xi_i)_{crit}$, $x(\sum_{i=1}^{n} \eta_i)_{crit}$ and $y(\sum_{i=1}^{n} \eta_i)_{crit}$, respectively, such that

$$x(\sum_{i=1}^{n} \xi_i)_{crit} \rightarrow x(\sum_{i=1}^{n} \xi_i)_{crit} + \delta x \sum_{i=1}^{n} \xi_i$$

$$y(\sum_{i=1}^{n} \xi_i)_{crit} \rightarrow y(\sum_{i=1}^{n} \xi_i)_{crit} + \delta y \sum_{i=1}^{n} \xi_i$$

$$x(\sum_{i=1}^{n} \eta_i)_{crit} \rightarrow x(\sum_{i=1}^{n} \eta_i)_{crit} + \delta x \sum_{i=1}^{n} \eta_i$$

$$y(\sum_{i=1}^{n} \eta_i)_{crit} \rightarrow y(\sum_{i=1}^{n} \eta_i)_{crit} + \delta y \sum_{i=1}^{n} \eta_i$$

When we substitute the above small perturbations around the critical points in Eqs. (37–38) and (40–41), these lead to the following equations in matrix form which represents differential equations of the first order.

$$\begin{pmatrix}
\delta x \\ \delta y \\ \delta x \\ \delta y
\end{pmatrix}_{\sum_{i=1}^{n} \xi_i} = X \begin{pmatrix}
\delta x \\ \delta y \\ \delta x \\ \delta y
\end{pmatrix}_{\sum_{i=1}^{n} \eta_i} \quad (45)$$

where the $X$ represents matrix depending upon the critical points $x(\sum_{i=1}^{n} \xi_i)_{crit}$, $y(\sum_{i=1}^{n} \xi_i)_{crit}$, $x(\sum_{i=1}^{n} \eta_i)_{crit}$ and $y(\sum_{i=1}^{n} \eta_i)_{crit}$.

The dependence of the matrix $X$ on $x(\sum_{i=1}^{n} \xi_i)_{crit}$, $y(\sum_{i=1}^{n} \xi_i)_{crit}$, $x(\sum_{i=1}^{n} \eta_i)_{crit}$ and $y(\sum_{i=1}^{n} \eta_i)_{crit}$ is clear. Now the general solution for the evolution of linear perturbations can be expressed in the following way using the eigenvalues a, b, c and d of the matrix $X$. Thus, it is the nature of the eigenvalues upon which stability around the fixed points depends.

$$\delta x \sum_{i=1}^{n} \xi_i = v_{11} e^{aN} + v_{12} e^{bN} + v_{13} e^{cN} + v_{14} e^{dN} \quad (46)$$

$$\delta y \sum_{i=1}^{n} \xi_i = v_{21} e^{aN} + v_{22} e^{bN} + v_{23} e^{cN} + v_{24} e^{dN} \quad (47)$$

$$\delta x \sum_{i=1}^{n} \eta_i = v_{31} e^{aN} + v_{32} e^{bN} + v_{33} e^{cN} + v_{34} e^{dN} \quad (48)$$

$$\delta y \sum_{i=1}^{n} \eta_i = v_{41} e^{aN} + v_{42} e^{bN} + v_{43} e^{cN} + v_{44} e^{dN} \quad (49)$$
Table 1: Enlisting the critical points

| Sr.no. | \( x(\sum_{i=1}^{n} \xi_i)^{\text{Cr}} \) | \( y(\sum_{i=1}^{n} \xi_i)^{\text{Cr}} \) | \( x(\sum_{i=1}^{n} \eta_i)^{\text{Cr}} \) | \( y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}} \) | Existence Status |
|--------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|-----------------|
| II     | 0                              | 0                              | \(-\sqrt{\frac{\lambda}{\sum_{i=1}^{n} \xi_i}} y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}}\) | \(\frac{1}{2} \left(\sqrt{\frac{\lambda^2}{\sum_{i=1}^{n} \eta_i}} y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}} + 4 - \lambda(\sum_{i=1}^{n} \eta_i)\right)\) | \(\sum_{j=0}^{2} y_j\) |
| III    | \(\pm 1\)                      | 0                              | 0                               | 0                               | \(\sum_{j=0}^{2} y_j\) |
| IV     | \(\pm 1\)                      | 0                              | \(-\sqrt{\frac{\lambda}{\sum_{i=1}^{n} \xi_i}} y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}}\) | \(\frac{1}{2} \left(\sqrt{\frac{\lambda^2}{\sum_{i=1}^{n} \eta_i}} y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}} + 4 + \lambda(\sum_{i=1}^{n} \eta_i)\right)\) | \(\sum_{j=0}^{2} y_j\) |
| V      | 1                              | 0                              | \(\frac{1}{\lambda(\sum_{i=1}^{n} \xi_i)}\) | 0                               | \(\sum_{j=1}^{2} y_j\) |
| VI     | \(-1\)                         | \(\frac{1}{\lambda(\sum_{i=1}^{n} \xi_i)}\) | \(-\sqrt{\frac{\lambda^2}{\sum_{i=1}^{n} \eta_i}} y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}}\) | \(\frac{1}{2} \left(\sqrt{\frac{\lambda^2}{\sum_{i=1}^{n} \eta_i}} y(\sum_{i=1}^{n} \eta_i)^{\text{Cr}} + 4 + \lambda(\sum_{i=1}^{n} \eta_i)\right)\) | \(\sum_{j=1}^{2} y_j\) |
| VII    | \(\sqrt{\psi}\)               | \(\frac{\gamma}{\lambda(\sum_{i=1}^{n} \xi_i)}\) | 0                               | 0                               | \(\sum_{j=0}^{2} y_j\) |
| VIII   | \(\sqrt{\frac{\lambda^2}{\sum_{i=1}^{n} \xi_i}} y(\sum_{i=1}^{n} \xi_i)^{\text{Cr}}\) | \(\frac{1}{2} \left(\sqrt{\frac{\lambda^2}{\sum_{i=1}^{n} \xi_i}} y(\sum_{i=1}^{n} \xi_i)^{\text{Cr}} + 4 - \lambda(\sum_{i=1}^{n} \xi_i)\right)\) | 0                               | 0                               | \(\sum_{j=0}^{2} y_j\) |
The points $h$ and $k$ are given below

\[
h = \frac{3}{4} \left[ (\gamma - 2) \lambda \sum_{i=1}^{n} \xi_i + \sqrt{16 \lambda \sum_{i=1}^{n} \xi_i^2 \gamma^2 (\gamma + 1) + \lambda^2 \sum_{i=1}^{n} \xi_i^2 (17 \gamma^2 - 20 \gamma + 4)} \right]
\]

and

\[
k = \frac{3}{4} \left[ (\gamma - 2) \lambda \sum_{i=1}^{n} \xi_i + \sqrt{16 \lambda \sum_{i=1}^{n} \xi_i^2 \gamma^2 (\gamma + 1) + \lambda^2 \sum_{i=1}^{n} \xi_i^2 (17 \gamma^2 - 20 \gamma + 4)} \right]
\]

Copeland et al. [84] and Guo et al. [93] have shown that the real parts of all eigenvalues might be negative for the remaining stable points. Table 2 lists all the eigenvalues and the status of their stability. A fixed critical point I gives a solution with fluid domination; the points II and III represent a phantom tachyon and tachyon domination in the solutions, respectively. A two-field dominated solution is manifested by the fixed critical point IV. For the $\gamma$ to remain unity, the fixed critical points V and VI owe their existence to it. The energy densities $\rho_{\gamma}^{x_0}$ and $\rho_{\gamma}$ show a decrease at the same rate in the point VII when the point VIII indicates a solution where tachyon field energy dominates.

The above diagrams in Fig. 3 illustrate the plot of the evolution of the equation of state parameter $w$ and dark energy parameter $DE$ cosmologically against the growth of e-folding number $N$. It can be seen that the evolutionary path cosmologically bends toward the point II which is a fixed stability point in Table 1 in the model. A comparison between the plots of Fig. 1 and the values of the critical points listed in Table 1 can clarify the situation further (Fig. 2).

When we substitute $\frac{1}{3}$, the value of $\lambda^{\sum_{i=1}^{n} \eta_i}$ in point II the stable critical point, i.e.,

\[
x_0^{(\sum_{i=1}^{n} \xi_i)_{crit}} = 0, \quad y_0^{(\sum_{i=1}^{n} \eta_i)_{crit}} = 0, \quad x^{(\sum_{i=1}^{n} \eta_i)_{crit}} = -\sqrt{\frac{\lambda^{\sum_{i=1}^{n} \eta_i} \gamma (\sum_{i=1}^{n} \eta_i)}{2}}
\]

\[
y^{(\sum_{i=1}^{n} \eta_i)_{crit}} = \frac{\sqrt{\lambda^{2} (\sum_{i=1}^{n} \eta_i) + 4 + \lambda (\sum_{i=1}^{n} \eta_i)}}{2}
\]

We obtain the values of these points $x^{(\sum_{i=1}^{n} \xi_i)_{crit}} = 0, \quad y^{(\sum_{i=1}^{n} \xi_i)_{crit}} = 0, \quad x^{(\sum_{i=1}^{n} \eta_i)_{crit}} = -0.627285$ and

\[
y^{(\sum_{i=1}^{n} \eta_i)_{crit}} = \frac{\sqrt{\lambda^{2} (\sum_{i=1}^{n} \eta_i) + 4 + \lambda (\sum_{i=1}^{n} \eta_i)}}{2} = 1.180460
\]

which shows consistency with plotting in Fig. 1. Further, with the help of Eq. (34), at the fixed point II, we possess $w = \frac{p}{\rho} = -1 - x^{(\sum_{i=1}^{n} \eta_i)_{crit}} = -1.39367$ which also shows consistency with the plots drawn in Fig. 3.

In Table 1, we would like to describe the initial values of the points $x^{\sum_{i=1}^{n} \xi_i}, \quad y^{\sum_{i=1}^{n} \xi_i}, \quad x^{\sum_{i=1}^{n} \eta_i}, \quad y^{\sum_{i=1}^{n} \eta_i}$ would evolve toward stability. If these are not the values of unstable points in the model granted the condition $1 - x^{(\sum_{i=1}^{n} \xi_i)} > 0$ does not get violated as physical constraint. When the values of the points $x^{\sum_{i=1}^{n} \xi_i}, \quad y^{\sum_{i=1}^{n} \xi_i}, \quad x^{\sum_{i=1}^{n} \eta_i}, \quad y^{\sum_{i=1}^{n} \eta_i}$ deviate slightly by a quantity amounting $\delta$ from the values of $x^{(\sum_{i=1}^{n} \xi_i)_{crit}}, y^{(\sum_{i=1}^{n} \xi_i)_{crit}}, x^{(\sum_{i=1}^{n} \eta_i)_{crit}}, y^{(\sum_{i=1}^{n} \eta_i)_{crit}}$ from Eqs. (46–49), it can be envisaged that $\delta$ may be larger despite diminishing.
Table 2 Enlisting the eigenvalues and the status of stability

| Labels | a     | b                     | c                     | d                     | Status of stability   |
|--------|-------|-----------------------|-----------------------|-----------------------|-----------------------|
| I      | $-3$  | $3\gamma$            | $-3$                  | $3\gamma$            | Unstable              |
| II     | $-3$  | $-3\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | $-3 - \frac{3}{2}\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | $-3\gamma - 3\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | Stable                |
| III    | $6$   | $3\gamma$            | $-3$                  | $3\gamma$            | Unstable              |
| IV     | $6$   | $-3\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | $-3 - \frac{3}{2}\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | $-3\gamma - 3\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | Unstable              |
| V      | $0$   | $-\frac{3}{2}$       | $-3$                  | $3$                   | Unstable              |
| VI     | $6\left(1 + x^2(\sum_{i=1}^{n} \eta_i)_{cr t}\right)$ | $\frac{3}{2}\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | $-3 - \frac{3}{2}\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | $-3\gamma - 3\lambda^2(\sum_{i=1}^{n} \eta_i)_{cr t}$ | Unstable              |
| VII    | $h$   | $k$                   | $-3$                  | $3\gamma$            | Unstable              |
| VIII   | $3\left(-\gamma + x^2(\sum_{i=1}^{n} \xi_i)_{cr t}\right)$ | $\frac{3}{2}\left(-2 + x^2(\sum_{i=1}^{n} \xi_i)_{cr t}\right)$ | $-3$                  | $3\lambda^2(\sum_{i=1}^{n} \xi_i)_{cr t}$ | Unstable              |
Fig. 1  The figure shows how does the growth of equation of state parameter $w$ and the parameter of dark energy density $\Omega_{DE}$ occurs with evolution of the number of e-folds $N$ and for $\gamma=1$ and $\lambda \sum_{i=1}^{n} \xi_{i}$ and $\lambda \sum_{i=1}^{n} \eta_{i}$ as 0.33.

4 Final remarks and summary

The analysis shows that the model does not indicate sensitivity to the kinetic energy density of under consideration multi-field scalars initially. It has been shown that there exists a stable unique critical point during the analysis of background spatially flat universe in the phase space. We make its comparison with the tachyon model altogether.

The scale for sum-masses $M \sum_{i=1}^{n} \xi_{i}$ of the multi-field scalars $\sum_{i=1}^{n} \xi_{i}$ should be enough larger [84] than $M_{pl}$ in case of dark energy of multi-field tachyon with inverse square potential of the type $\sum_{i=1}^{n} V(\xi_{i}) = M_{pl}^{2} \sum_{i=1}^{n} \xi_{i} \sum_{i=1}^{n} \xi_{i}^{-2}$ in order to meet the late time accelerated
Fig. 2 The above figures indicate the behavior of general points of scalar multi-fields as the number of e-folds $N$ evolve. The points $\sum_{i=1}^{n} x_i = \xi_i$ and $\sum_{i=1}^{n} y_i = \xi_i$, $\sum_{i=1}^{n} x_i = \eta_i$ and $\sum_{i=1}^{n} y_i = \eta_i$ develop gradually as the function of e-folding number $N$ for $\gamma = 1$ and $\lambda \sum_{i=1}^{n} \xi_i$ and $\lambda \sum_{i=1}^{n} \eta_i$ both with assigned a value equivalent to 0.33.

Fig. 3 The diagrams demonstrating the evolution of equation of state parameter $w$ and the development of the parameter of dark energy $\Omega_{DE}$ in the very early Universe and their development during its behavior in late time accelerated expansion where $\gamma = 1$ and $\lambda \sum_{i=1}^{n} \xi_i$ and $\lambda \sum_{i=1}^{n} \eta_i$ both take on the value 0.33.
expansion of the cosmos, i.e., \( a(t) \propto [\text{time}]^{1/2} \left( \frac{M_{\Lambda} - \xi_{i}}{M_{\Lambda}} \right) \). This huge mass pushes the solutions toward dense energy regions where even general theory of relativity fails; therefore, the potential \( \sum_{i=1}^{n} V(\xi_{i}) = M_{\Lambda}^{2} (\sum_{i=1}^{n} \xi_{i})^{2} \) where \( \rho \) lies between 0 and 2. The phantom tachyon fields \( \sum_{i=1}^{n} \eta_{i} \) in our model take the responsibility of this late time acceleration with equation of state parameter \( w_{\sum_{i=1}^{n} \xi_{i}} < -1 \). From Eqs. (37–38), it becomes clear that the value of \( M_{\Lambda} \sum_{i=1}^{n} \xi_{i} \) is not still smaller as required by the recent observational constraints. This is due to the reason for \( \lambda_{\sum_{i=1}^{n} \xi_{i}} = \frac{4}{3} \frac{M_{\Lambda}^{2}}{M_{\Lambda}} \) being larger; therefore, \( 1 - \sum_{i=1}^{n} \xi_{i}^{2} \) falls in risk of non-positive behavioral increment.

In our multi-field model of tachyon and phantom tachyon, there is only one stable critical point, namely II, whose value does not rest on the value of \( \gamma \); on the other hand in reference to [92], it is shown that in tachyon model of dark energy the sole source of dark energy is tachyon and it has three critical stable points whose existence hinges upon \( \gamma \). Further, we have seen the values of scalar fields \( \sum_{i=1}^{n} \xi_{i} \) and \( \sum_{i=1}^{n} \eta_{i} \) are zero at critical points in our model of tachyon and phantom tachyon, but in tachyon model of dark energy singly, the values of both fields at critical points can be nonzero also as depicted in Table 1. The values of \( x(\sum_{i=1}^{n} \xi_{i})_{\text{crit}}, y(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) and \( \rho(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) at the critical points are fixed with the value of \( x(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) becoming nonzero, and if its value is zero, then \( y(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) turns out to be zero; this is obvious from Eq. (40); it further ensues the impossibility that \( x(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) and \( \rho(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) are zero, i.e.,

\[
\rho(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} = \frac{\sum_{i=1}^{n} V(\xi_{i})_{\text{crit}}}{\sqrt{1 - (\sum_{i=1}^{n} \xi_{i}^{2})_{\text{crit}}}} = \frac{3y(\sum_{i=1}^{n} \xi_{i})_{\text{crit}}}{M_{\Lambda}^{2} H^{2}}
\]

\[
\Rightarrow w(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} = -1 + \left( \sum_{i=1}^{n} \xi_{i}^{2} \right)_{\text{crit}} = -1 + x^{2}(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \geq -1 \quad \text{(54)}
\]

and

\[
\rho(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} = \frac{\sum_{i=1}^{n} V(\eta_{i})_{\text{crit}}}{\sqrt{1 - (\sum_{i=1}^{n} \eta_{i}^{2})_{\text{crit}}}} = \frac{3y(\sum_{i=1}^{n} \eta_{i})_{\text{crit}}}{M_{\Lambda}^{2} H^{2}}
\]

\[
\Rightarrow w(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} = -1 - \left( \sum_{i=1}^{n} \eta_{i}^{2} \right)_{\text{crit}} = 1 - x^{2}(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} < -1 \quad \text{(55)}
\]

It is clear from Eq. (54) that \( H^{2} \) becomes non-increasing for nonzero \( y(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) at the points which are fixed, with \( \rho(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) also non-increasing. The same is not inapplicable from Eq. (55) for \( y(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) and \( \rho(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) making one of the two critical points \( y(\sum_{i=1}^{n} \xi_{i})_{\text{crit}}, y(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) equal to zero. Therefore \( y(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) is set to zero for \( \rho(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) as non-increasing and \( \rho(\sum_{i=1}^{n} \eta_{i})_{\text{crit}} \) as increasing. Moreover, if \( 1 - x^{2}(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} > 0 \) does not get violated and for \( y(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) being zero, the critical point \( x(\sum_{i=1}^{n} \xi_{i})_{\text{crit}} \) also becomes zero. In
tachyon model of dark energy, the speed of sound \( c_s^2 \) is described by the following expression

\[
c_s^2 = \frac{p \sum_{i=1}^{n} \xi_i \frac{1}{2} \left( \sum_{i=1}^{n} \partial_\mu \xi_i \right)^2}{\rho \sum_{i=1}^{n} \xi_i \frac{1}{2} \left( \sum_{i=1}^{n} \partial_\mu \xi_i \right)^2} = 1 - \sum_{i=1}^{n} \xi_i^2 \leq 0 \tag{56}
\]

To investigate whether a ghost dark energy model is stable or the otherwise, we use the theory of perturbation. In case the model is unstable, we can find ghost modes for ghost instability therein. In the energy density of the background, we consider a perturbation of very small size and investigate to observe whether it will collapse or will grow cosmologically. We can write from the linear theory of perturbation

\[
\rho (x, t) = \rho (t) + \delta \rho (x, t);
\]

here \( \rho (t) \) represents the energy density in the background that remains unperturbed. Now from Eq. (10) or from covariant divergence of energy–momentum tensor \( \nabla_i T^{ij} = 0 \), we obtain the following equation

\[
\delta (\partial_{tt} \rho) = c_s^2 \nabla^2 \delta \rho (x, t); \tag{57}
\]

here \( c_s^2 \) is described as the term with speed of sound squared and is expressed as \( \frac{dp}{d\rho} \).

Two solutions \( \delta \rho = \delta \rho_0 \exp \left( i \vec{k} \cdot \vec{x} \pm i\omega t \right) \) come out, where for the case when \( c_s^2 > 0 \), the first solution comes out to be oscillatory wave solution \( \delta \rho = \delta \rho_0 \exp \left( i \vec{k} \cdot \vec{x} - i\omega t \right) \) which demonstrates propagation mode for the perturbations of energy density. When the second solution is \( \delta \rho = \delta \rho_0 \exp \left( i \vec{k} \cdot \vec{x} + i\omega t \right) \) for \( c_s^2 < 0 \). This describes that the density perturbation would grow out because in this case the oscillation frequency comes out to be purely imaginary. This propagation mode as growing perturbation marks the possibility for ghost instabilities to emerge. The unwanted consequence of \( c_s^2 < 0 \) is principally interpreted as the growth of amplitude exponentially for the modes with short wavelengths. The robust coupling scale being light during \( c_s^2 < 0 \) excludes the validity range of an effective field theory for the dangerous short-wavelength modes to incorporate. The occurrence of ghost instability is related to the field having negative kinetic energy term. In the field theories, these are usually followed by the violation of Null Energy Condition (NEC) in the development of singularity-free cosmological models in effective field theories. The problem of ghosts is resolved by considering the theory of Galileon, its generalized forms and the theories beyond it see Ref. [96] and the references therein. Gradient instability arises when the field has a negative momentum squared and is related to negative speed of sound squared. This may lead to perturbations grow exponentially. Some spatial operators cure this instability. However, it was shown that tachyon dark energy models have speed of sound squared \( c_s^2 \) to be positive.

Therefore, these models are supposed to be stable against small perturbations [97, 98]. The speed of sound squared is to be found

\[
c_s^2 = \frac{\dot{c}_3 - ac_2}{ac_1} \tag{58}
\]

from the quadratic action of scalar perturbations where \( c_2 = M_p^2 f (t) \) and the action is [96]

\[
S_\xi^2 = \int \left[ c_1 \dot{\xi}^2 - \frac{\left( \dot{c}_3 - c_2 \right)}{a^2} \left( \frac{\partial \xi}{a} \right)^2 + \frac{c_4}{a^2} \left( \frac{\partial^2 \xi}{a} \right)^2 - \frac{16\lambda (t)}{M_p^2 a^6} \left( \frac{\partial^2 \xi}{a} \right)^2 \right] a^3 d^4 x \tag{59}
\]
The condition to avoid the ghost instability is $c_1 > 0$ and the gradient instability is avoided by the condition $\dot{c}_3 - ac_2 > 0$. Similarly, from quadratic action of tensor perturbations

$$S_T^2 = \frac{M_p^2}{8} \int \left( \gamma_{\mu\nu}^2 - c_T^2 \left( \frac{\gamma_{\mu\nu,\rho}}{a^2} \right)^2 \right) a^3 Q_T d^4x$$

we have $Q_T = f + 2 \left( \frac{m_T}{M_p} \right)^2$ and $c_T^2 = \frac{f}{Q_T}$. The conditions for avoiding the gradient and ghost instabilities for tensor modes are $c_T^2 > 0$ and $Q_T > 0$, respectively.

In relation to Eq. (56), we explain that due to the presence of under-root in the Lagrangian density, the difference term $1 - \sum_{i=1}^n \dot{\xi}_i^2$ will be non-positive accordingly, and it keeps as a consequence, the pressure and energy to remain in the realm of real; therefore, a positive sound speed is attributed to the homogeneous perturbations which owes stability. We can put to use independent sound speed of each component of the multi-fields tachyon and phantom tachyon in our case here to give a description to the model. It is, however, notable that Xia et al. [99] and Kodama et al. [100] have shown the use of effective sound speed because using of two or more independent components of sound does not coincide with the present juncture of the constraints of dark energy. The effective sound speed for larger $N$ in the case when the energy density is considered as a fraction of dark energy density of phantom tachyon, when $\Omega_{1 \sum_{i=1}^n \eta_i}$ approaches to unity, is

$$c_s^2 = \frac{p_{\sum_{i=1}^n \eta_i} \frac{1}{2} (\sum_{i=1}^n \partial_\mu \eta_i)^2}{\rho_{\sum_{i=1}^n \eta_i} \frac{1}{2} (\sum_{i=1}^n \partial_\mu \eta_i)^2} = 1 + \sum_{i=1}^n \dot{\eta}_i^2 > 1$$

for the effective density in the Lagrangian of Eq. (2). It can be interpreted as the perturbations of the scalar field in the background that can move with a speed larger than the speed of light in the preferred and privileged frame of reference where the field is existing homogeneously in the background. In concluding remarks, we investigated a dark energy model consisting of multi-field tachyon and multi-field phantom tachyon known as the multi-field tachyon-quintom model. During evolution of the universe, the equation of state parameter $w$ in $p = w \rho$ alters to $w < -1$ from $w > -1$ in this model. Inverse square potentials are used in the development of the autonomous system for doing the analysis in phase space where we found stable points that have power-law solutions. The analysis of spatially flat background universe of FLRW metric manifests the existence of a unique critical point which is compared with the tachyon dark energy model. We observed that neither multi-field tachyon nor the multi-field phantom tachyon showed sensitivity to the kinetic energy of initial conditions. It happens approximately when the e-folding number is near to ten and the variation in multi-field tachyons by the order of magnitude four is still observed coinciding with the observations conducted in the recent past.

The results for multiple tachyon and phantom tachyon fields are presented graphically, in which slightly changed behavior of the parameters involved than single field. At the fixed point II in Table 2, we obtained $w = \frac{p}{\rho} = -1 - \frac{1}{2} \left( \sum_{i=1}^n \eta_i \right) c_{rr} = -1.39367$ which also shows consistency with the plots drawn in Fig. 3 and has minute difference with single scalar field. The evolution of equation of state parameter $w$ and dark energy parameter $\Omega_{DE}$ as function of number of e-folds $N$ is plotted beginning with the initial condition $-10$. In accordance with it, the initial values of the fields have slight variations which impacts the overall results very lightly. Different initial values of the scalar fields have convergence toward a track that evolves in common with all for $N \geq 2$. From this point onwards, the values of the fields
become slightly larger; however, the convergence of these fields with distinct and different initial values stays constant and does not change.

References

1. A.G. Riess, A.V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P.M. Garnavich, J. Tonry, Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron. J. 116(3), 1009 (1998)
2. S. Perlmutter, G. Aldering, G. Goldhaber, R.A. Knop, P. Nugent, P.G. Castro, Supernova cosmology project. Measurements of and from 42 high-redshift supernovae. Astrophys. J. 517(2), 565 (1999)
3. A.G. Riess, P.E. Nugent, R.L. Gilliland, B.P. Schmidt, J. Tonry, M. Dickinson, S. Veilleux, The farthest known supernova: support for an accelerating universe and a glimpse of the epoch of deceleration. Astrophys. J. 560(1), 49 (2001)
4. J.L. Tonry, B.P. Schmidt, B. Barris, P. Candia, P. Challis, A. Clocchiatti, N.B. Suntzeff, Cosmological results from high-z supernovae. Astrophys. J. 594(1), 1 (2003)
5. P. Astier, R. Pain, Observational evidence of the accelerated expansion of the universe. C. R. Phys. 20(6), 2059 (2009)
6. Y. Yang, Y. Gong, The evidence of cosmic acceleration and observational constraints. J. Cosmol. Astropart. Phys. 2020(06), 059 (2020)
7. A.G. Riess, The expansion of the Universe is faster than expected. Nat. Rev. Phys. 1(1), 10–12 (2020)
8. D. Rubin, J. Heitlauf, Is the expansion of the universe accelerating? All signs still point to yes: a local dipole anisotropy cannot explain dark energy. Astrophys. J. 894(1), 68 (2020)
9. M. Tegmark, M.A. Strauss, M.R. Blanton, K. Abazajian, S. Dodelson, H. Sandvik, D.G. York, Cosmological parameters from SDSS and WMAP. Phys. Rev. D 69(10), 103501 (2004)
10. B.D. Sherwin, J. Dunkley, S. Das, J.W. Appel, J.R. Bond, C.S. Carvalho, E. Wollack, Evidence for dark energy from the cosmic microwave background alone using the Atacama Cosmology Telescope lensing measurements. Phys. Rev. Lett. 107(2), 021302 (2011)
11. Z.L. Tu, J. Hu, F.Y. Wang, Probing cosmic acceleration by strong gravitational lensing systems. Mon. Not. R. Astron. Soc. 484(3), 4337–4346 (2019)
12. N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, G. Roudier, Planck 2018 results- VI. Cosmological parameters. Astron. Astrophys. 641, A6 (2020)
13. J. Yoo, Y. Watanabe, Theoretical models of dark energy. Int. J. Mod. Phys. D 21(12), 1230002 (2012)
14. H. Wei, S.N. Zhang, How to distinguish dark energy and modified gravity? Phys. Rev. D 78(2), 023011 (2008)
15. V. Faraoni, S. Capozziello, Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics (Springer, Berlin, 2011)
16. C.J. Feng, X.H. Zhai, X.Z. Li, Multi-pole dark energy. Chin. Phys. C 44(10), 105103 (2020)
17. E.J. Copeland, M. Sami, S. Tsujikawa, Dynamics of dark energy. Int. J. Mod. Phys. D 15(11), 1753–1935 (2006)
18. A. Einstein, Cosmological Considerations in the General Theory of Relativity, Math. Phys., 1917, 142–152 (1917)
19. C. Armendariz-Picon, V. Mukhanov, P.J. Steinhardt, Dynamical solution to the problem of a small cosmological constant and late-time cosmic acceleration. Phys. Rev. Lett. 85(21), 4438 (2000)
20. P.J.E. Peebles, B. Ratra, The cosmological constant and dark energy. Rev. Mod. Phys. 75(2), 559 (2003)
21. G.Z. Kang, D.S. Zhang, L. Jun, H.S. Zong, Fine tuning problem of the cosmological constant in a generalized Randall–Sundrum model. Chin. Phys. C 44(12), 125102 (2020)
22. J.J. Zhang, C.C. Lee, C.Q. Geng, Observational constraints on running vacuum model. Chin. Phys. C 43(2), 025102 (2019)
23. P.J.E. Peebles, B. Ratra, Cosmology with a time-variable cosmological ‘constant’. Astrophys. J. 325, L17–L20 (1988)
24. E.J. Copeland, M. Sami, S. Tsujikawa, Dynamics of dark energy, hep-th/0603057
25. M. Sami, Models of dark energy, in The Invisible Universe: Dark Matter and Dark Energy, (Springer, Berlin, Heidelberg, 2007), pp. 219–256
26. C.A.I. Rong-Gen, On theoretical models of dark energy. Chin. Phys. C 31(9), 827–834 (2007)
27. S. Bahamonde, C.G. Böhmer, S. Carloni, E.J. Copeland, W. Fang, N. Tamanini, Dynamical systems applied to cosmology dark energy and modified gravity. Phys. Rep. 775, 1–122 (2018)
28. M. Li, X.D. Li, S. Wang, Y. Wang, Dark energy: a brief review. Front. Phys. 8(6), 828–846 (2013)
29. B. Ratra, P.J. Peebles, Cosmological consequences of a rolling homogeneous scalar field. Phys. Rev. D 37(12), 3406 (1988)
30. P.J. Steinhardt, A quintessential introduction to dark energy. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 361(1812), 2497–2513 (2003)
31. Z.K. Guo, N. Ohta, Y.Z. Zhang, Parametrization of quintessence and its potential. Phys. Rev. D 72(2), 023504 (2005)
32. Z.K. Guo, N. Ohta, Y.Z. Zhang, Parametrizations of the dark energy density and scalar potentials. Mod. Phys. Lett. A 22(12), 883–890 (2007)
33. J.Z. Qi, M.J. Zhang, W.B. Liu, Dynamical evolution of quintessence cosmology in a physical phase space. Int. J. Theor. Phys. 55(8), 3672–3681 (2016)
34. Hughes, J. The quintessential dark energy theory: quintessence (2019)
35. I. Zlatev, L. Wang, P.J. Steinhardt, Quintessence, cosmic coincidence, and the cosmological constant. Phys. Rev. Lett. 82(5), 896 (1999)
36. T. Barreiro, E.J. Copeland, N.A. Nunes, Quintessence arising from exponential potentials. Phys. Rev. D 61(12), 127301 (2000)
37. T. Chiba, T. Okabe, M. Yamaguchi, Kinetically driven quintessence. Phys. Rev. D 62(2), 023511 (2000)
38. C. Armendariz-Picon, T. Damour, V.I. Mukhanov, k-inflation. Phys. Lett. B 458(2–3), 209–218 (1999)
39. C. Armendariz-Picon, V. Mukhanov, P.J. Steinhardt, Essentials of k-essence. Phys. Rev. D 63(10), 103510 (2001)
40. Y. Rong-Jia, G. Xiang-Ting, Observational constraints on purely kinetic k-essence dark energy models. Chin. Phys. Lett. 26(8), 089501 (2009)
41. S. Sur, S. Das, Multiple kinetic k-essence, phantom barrier crossing and stability. J. Cosmol. Astropart. Phys. 2009(01), 007 (2009)
42. R.J. Scherrer, Purely kinetic k-essence as unified dark matter. Phys. Rev. Lett. 93(1), 011301 (2004)
43. M. Sami, A. Toporensky, Phantom field and the fate of the universe. Mod. Phys. Lett. A 19(20), 1509–1517 (2004)
44. J.M. Cline, S. Jeon, G.D. Moore, The phantom menaced: constraints on low-energy effective ghosts. Phys. Rev. D 70(4), 043543 (2004)
45. M. Sami, A. Toporensky, P.V. Tretjakov, S. Tsujikawa, The fate of (phantom) dark energy universe with string curvature corrections. Phys. Lett. B 619(3–4), 193–200 (2005)
46. A. Vikman, Can dark energy evolve to the phantom? Phys. Rev. D 71(2), 023515 (2005)
47. J. Kujat, R.J. Scherrer, A.A. Sen, Phantom dark energy models with negative kinetic term. Phys. Rev. D 74(8), 083513 (2006)
48. W. Wen-Fu, S. Zheng-Wei, T. Bin, Exact solution of phantom dark energy model. Chin. Phys. B 19(11), 119801 (2010)
49. R.R. Caldwell, M. Kamionkowski, N.N. Weinberg, Phantom energy: dark energy with w1 causes a cosmic doomsday. Phys. Rev. Lett. 91(7), 071301 (2003)
50. W. Hu, Crossing the phantom divide: dark energy internal degrees of freedom. Phys. Rev. D 71(4), 071301 (2005)
51. S.M. Carroll, M. Hoffman, M. Trodden, Can the dark energy equation-of-state parameter w be less than 1? Phys. Rev. D 68(2), 023509 (2003)
52. Z.K. Guo, Y.Z. Zhang, Interacting phantom energy. Phys. Rev. D 71(2), 023501 (2005)
53. K.J. Ludwick, The viability of phantom dark energy: a review. Mod. Phys. Lett. A 32(28), 1730025 (2017)
54. R.R. Caldwell, M. Kamionkowski, N.N. Weinberg, Phantom energy and cosmic doomsday (2003). arXiv preprint arXiv:astro-ph/0302506
55. W. Fang, H. Tu, J. Huang, C. Shu, Dynamical system of scalar field from 2-dimension to 3-D and its cosmological implications. Eur. Phys. J. C 76(9), 1–12 (2016)
56. R.J. Scherrer, Phantom dark energy, cosmic doomsday, and the coincidence problem. Phys. Rev. D 71(6), 063519 (2005)
57. H. Štefančič, Dark energy transition between quintessence and phantom regimes: an equation of state analysis. Phys. Rev. D 71(12), 124036 (2005)
58. B. Feng, X. Wang, X. Zhang, Dark energy constraints from the cosmic age and supernova. Phys. Lett. B 607(1–2), 35–41 (2005)
59. Z.K. Guo, Y.S. Piao, X. Zhang, Y.Z. Zhang, Cosmological evolution of a quintom model of dark energy. Phys. Lett. B 608(3–4), 177–182 (2005)
60. X.F. Zhang, H. Li, Y.S. Piao, X. Zhang, Two-field models of dark energy with equation of state across-1. Mod. Phys. Lett. A 21(03), 231–241 (2006)
61. Y.F. Cai, M. Li, J.X. Lu, Y.S. Piao, T. Qiu, X. Zhang, A string-inspired quintom model of dark energy. Phys. Lett. B 651(1), 1–7 (2007)
62. B. Feng, M. Li, Y.S. Piao, X. Zhang, Oscillating quintom and the recurrent universe. Phys. Lett. B \textbf{634}(2-3), 101–105 (2006)

63. Z.K. Guo, Y.S. Piao, X. Zhang, Y.Z. Zhang, Two-field quintom models in the $w$ $w$ plane. Phys. Rev. D \textbf{74}(12), 127304 (2006)

64. Y.F. Cai, H. Li, Y.S. Piao, X. Zhang, Cosmic duality in quintom universe. Phys. Lett. B \textbf{646}(4), 141–144 (2007)

65. Y.F. Cai, T. Qiu, X. Zhang, Y.S. Piao, M. Li, Bouncing universe with quintom matter. J. High Energy Phys. 2007(10), 071 (2007)

66. S. Mishra, S. Chakraborty, Dynamical system analysis of quintom dark energy model. Eur. Phys. J. C \textbf{78}(11), 1–9 (2018)

67. S. Panpanich, P. Burikham, S. Ponglertsakul, L. Tannukij, Resolving Hubble tension with quintom dark energy model. Chin. Phys. C \textbf{45}(1), 015108 (2021)

68. A. Mazumdar, S. Panda, A. Perez-Lorenzana, Assisted inflation via tachyon condensation. Nucl. Phys. B \textbf{614}(1–2), 101–116 (2001)

69. Y.S. Piao, R.G. Cai, X. Zhang, Y.Z. Zhang, Assisted tachyonic inflation. Phys. Rev. D \textbf{66}(12), 121301 (2002)

70. Z.K. Guo, Y.S. Piao, R.G. Cai, Y.Z. Zhang, Inflationary attractor from tachyonic matter. Phys. Rev. D \textbf{68}(4), 043508 (2003)

71. L. Kofman, A. Linde, Problems with tachyon inflation. J. High Energy Phys. 2002(07), 004 (2002)

72. S. Ashoke, Remarks on tachyon driven cosmology, String Theory Cosmol., pp. 70–75 (2005)

73. T. Padmanabhan, Accelerated expansion of the universe driven by tachyonic matter. Phys. Rev. D \textbf{66}(2), 021301 (2002)

74. E.J. Copeland, A.R. Liddle, D. Wands, Exponential potentials and cosmological scaling solutions. Phys. Rev. D \textbf{57}(8), 4686 (1998)

75. G.W. Gibbons, Cosmological evolution of the rolling tachyon. Phys. Lett. B \textbf{537}(1–2), 1–4 (2002)

76. J.G. Hao, X.Z. Li, Reconstructing the equation of state of the tachyon. Phys. Rev. D \textbf{66}(8), 087301 (2002)

77. G.W. Gibbons, Thoughts on tachyon cosmology. Class. Quantum Gravity \textbf{20}(12), S321 (2003)

78. L.R. Abramo, F. Finelli, Cosmological dynamics of the tachyon with an inverse power-law potential. Phys. Lett. B \textbf{575}(3–4), 165–171 (2003)

79. J.S. Bagla, H.K. Jassal, T. Padmanabhan, Cosmology with tachyon field as dark energy. Phys. Rev. D \textbf{67}(6), 063504 (2003)

80. Z.K. Guo, Y.S. Piao, Y.Z. Zhang, Cosmological scaling solutions and multiple exponential potentials. Phys. Lett. B \textbf{568}(1–2), 1–7 (2003)

81. V. Gorini, A. Kamenshchik, U. Moschella, V. Pasquier, Tachyons, scalar fields, and cosmology. Phys. Rev. D \textbf{69}(12), 123512 (2004)

82. M. Sami, P. Chingangbam, T. Qureshi, Cosmology with rolling tachyon. Pramana \textbf{62}(3), 765–770 (2004)

83. G. Calcagni, A.R. Liddle, Tachyon dark energy models: dynamics and constraints. Phys. Rev. D \textbf{74}(4), 043528 (2006)

84. E.J. Copeland, M.R. Garousi, M. Sami, T. Qureshi, Cosmology with rolling tachyon. Pramana \textbf{69}(12), 123512 (2004)

85. A. Singha, A. Sangwan, H.K. Jassal, Low redshift observational constraints on tachyon dark energy. J. Cosmol. Astropart. Phys. \textbf{2019}(04), 047 (2019)

86. S.G. Shi, Y.S. Piao, C.F. Qiao, Cosmological evolution of a tachyon-quintom model of dark energy. J. Cosmol. Astropart. Phys. \textbf{2009}(04), 027 (2009)

87. A. Frolov, L. Kofman, A. Starobinsky, Prospects and problems of tachyon matter cosmology. Phys. Lett. B \textbf{545}(1–2), 8–16 (2002)

88. A. Feinstein, Power-law inflation from the rolling tachyon. Phys. Rev. D \textbf{66}(6), 063511 (2002)

89. J.M. Aguirregabiria, R. Lazkoz, Tracking solutions in tachyon cosmology. Phys. Rev. D \textbf{69}(12), 123502 (2004)

90. Z.K. Guo, Y.Z. Zhang, Cosmological scaling solutions of multiple tachyon fields with inverse square potentials. J. Cosmol. Astropart. Phys. \textbf{2004}(08), 010 (2004)

91. J.G. Hao, X.Z. Li, Phantom with Born–Infeld-type Lagrangian. Phys. Rev. D \textbf{68}(4), 043501 (2003)
95. A. Sheykhi, M.S. Movahed, E. Ebrahimi, Tachyon reconstruction of ghost dark energy. Astrophys. Space Sci. 339(1), 93–99 (2012)
96. Y. Cai, Y. Wan, H.G. Li, T. Qiu, Y.S. Piao, The effective field theory of nonsingular cosmology. J. High Energy Phys. 2017(1), 90 (2017)
97. V. Gorini, A. Kamenshchik, U. Moschella, V. Pasquier, A. Starobinsky, Stability properties of some perfect fluid cosmological models. Phys. Rev. D 72(10), 103518 (2005)
98. H.B. Sandvik, M. Tegmark, M. Zaldarriaga, I. Waga, The end of unified dark matter? Phys. Rev. D 69(12), 123524 (2004)
99. J.Q. Xia, Y.F. Cai, T.T. Qiu, G.B. Zhao, X. Zhang, Constraints on the sound speed of dynamical dark energy. Int. J. Mod. Phys. D 17(08), 1229–1243 (2008)
100. H. Kodama, M. Sasaki, Cosmological perturbation theory. Prog. Theor. Phys. Suppl. 78, 1–166 (1984)