UNIFORM REGULARITY AND VANISHING VISCOSITY LIMIT FOR THE INCOMPRESSIBLE NON-RESISTIVE MHD SYSTEM WITH TRANSVERSE MAGNETIC FIELD

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Dedicated to Professor Shuxing Chen on the occasion of his 80th birthday

Abstract. This paper is concerned with the vanishing viscosity limit for the incompressible MHD system without magnetic diffusion effect in the half space under the influence of a transverse magnetic field on the boundary. We prove that the solution to the incompressible MHD system is uniformly bounded in both conormal Sobolev norm and $L^\infty$ norm in a fixed time interval independent of the viscosity coefficient. As a direct consequence, the inviscid limit from the viscous MHD system to the ideal MHD system is established in $L^\infty$-norm. In addition, the analysis shows that the boundary layer effect is weak because of the transverse magnetic field.

1. Introduction. In this paper, we are concerned with the motion of electrically conducting fluid occupied in the upper half space of $\mathbb{R}^3$ that is governed by the following incompressible non-resistive MHD equations in the region $\{(t,x) : t \in$
\[ [0, T], \mathbf{x} \in \Omega \] with \( \Omega = \{ \mathbf{x} = (x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2, z \in \mathbb{R}_+ \} \):

\[
\begin{aligned}
& \partial_t u^\epsilon + \mathbf{u}^\epsilon \cdot \nabla u^\epsilon + \nabla p^\epsilon - \mathbf{H}^\epsilon \cdot \nabla \mathbf{H}^\epsilon = \epsilon \Delta u^\epsilon, \\
& \partial_t \mathbf{H}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{H}^\epsilon - \mathbf{H}^\epsilon \cdot \nabla \mathbf{u}^\epsilon = 0, \\
& \nabla \cdot \mathbf{u}^\epsilon = 0, \quad \nabla \cdot \mathbf{H}^\epsilon = 0.
\end{aligned}
\tag{1.1}
\]

Here \( \mathbf{u}^\epsilon = (u^\epsilon_1, u^\epsilon_2, u^\epsilon_3) \) and \( \mathbf{H}^\epsilon = (h^\epsilon_1, h^\epsilon_2, h^\epsilon_3) \) represent the velocity and magnetic field respectively, \( p^\epsilon \) is the total pressure, and \( \epsilon \) is a positive viscosity parameter. The initial data is given by

\[
(u^\epsilon, \mathbf{H}^\epsilon)|_{t=0} = (u_0, h_0)(x, y, z).
\tag{1.2}
\]

In the presence of boundary, we impose the no-slip boundary condition on the velocity field:

\[
uu^\epsilon|_{z=0} = 0.
\tag{1.3}
\]

Based on the no-slip boundary condition (1.3) for velocity \( \mathbf{u}^\epsilon \), we do not impose any boundary condition on the magnetic field. In fact, when we consider the classical solutions to the initial-boundary value problem (1.1)-(1.3), by restricting the equations of magnetic field on the boundary, the divergence-free condition of velocity and boundary conditions (1.3) imply that the normal component of magnetic field remains the same, i.e.,

\[
h^\epsilon_3(t, x, y, z) = h^\epsilon_3(0, x, y, z) = h_{0,3}(x, y, z), \quad \text{on} \ z = 0.
\tag{1.4}
\]

In particular, if the magnetic field is parallel to the boundary initially, i.e., the initial normal magnetic field vanishes on the boundary, then the magnetic field on the boundary preserves:

\[
\mathbf{H}^\epsilon(t, x, y, z) = \mathbf{H}^\epsilon(0, x, y, z) = h_0^\epsilon(x, y, z), \quad \text{on} \ z = 0.
\]

In this paper, we are interested in the vanishing viscosity limit of the system (1.1)-(1.3). Without boundary, the well-posedness for the problem (1.1)-(1.2) with fixed viscosity is justified in [1, 3, 5, 6, 9, 10, 11, 18, 19, 24] and references therein. Recently, in the case when both the viscosity and resistivity are the same, the authors in [2, 8] studied the global well-posedness of MHD equations uniformly in viscosity in the presence of homogeneous magnetic background. While in the presence of boundary, the situation becomes more difficult because boundary layers may appear. Under the slip boundary condition, the boundary layer is ‘weak’ and the vanishing viscosity limit of MHD system was studied in [4, 22, 23]. However, when the no-slip boundary condition on the velocity is imposed, ‘strong’ boundary layers may appear in general. Hence, it is difficult to show the uniform regularity even when the vanishing viscosity limit is obtained. For example, when the magnetic field is tangential to the boundary, in the recent papers [12, 13, 14] the authors investigated the behavior of boundary layer for (1.1) in the inviscid limit, and verified the local in time well-posedness of boundary layer profile provided non-degenerate tangential magnetic field is assumed.

In this paper, we are concerned with the effect of transverse magnetic field on the boundary in the vanishing viscosity limit for the problem (1.1)-(1.3). To simplify the calculation, we consider a uniform normal component of initial magnetic field on the boundary:

\[
h_{0,3}(x, y, z)|_{z=0} = 1,
\]

and along with (1.4), it implies

\[
h^\epsilon_3(t, x, y, z) = h_{0,3}(x, y, z) \equiv 1, \quad \text{on} \ z = 0.
\tag{1.5}
\]
We can then introduce the following new unknown for the magnetic field:
\[
\vec{B}' = H' - \vec{e}_z
\]
with \(\vec{e}_z = (0, 0, 1)\), and rewrite the problem (1.1)-(1.3) as:
\[
\begin{aligned}
\partial_t u' + u' \cdot \nabla u' + \nabla p' - B' \cdot \nabla B' - \partial_z B' &= \epsilon \Delta u', \\
\partial_t B' + u' \cdot \nabla B' - B' \cdot \nabla u' - \partial_z u' &= 0, \\
\nabla \cdot u' &= 0, \\
\nabla \cdot B' &= 0,
\end{aligned}
\]  
(1.6)

Thus combining with (1.5) it is noted that for classical solutions to the problem (1.6), the normal component of \(B'\) vanishes on the boundary:
\[
\frac{\partial u'}{\partial n}|_{z=0} = 0.
\]  
(1.7)

Now we turn to study the inviscid limit of the problem (1.6). Formally, by letting \(\epsilon \to 0\) in (1.6) we obtain the following system of ideal MHD equations:
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p - B \cdot \nabla B - \partial_z B &= 0, \\
\partial_t B + u \cdot \nabla B - B \cdot \nabla u - \partial_z u &= 0, \\
\nabla \cdot u &= 0, \\
\nabla \cdot B &= 0,
\end{aligned}
\]  
(1.8)

with the same initial data
\[
(u, B)|_{t=0} = (u_0, B_0).
\]  
(1.9)

To solve the above initial value problem (1.8)-(1.9), we need to impose some suitable boundary conditions that will be explained later.

We will establish in this paper a uniform estimate in some anisotropic conormal Sobolev space for the solution to the problem (1.6). From the definition of conormal Sobolev space, a uniform estimate on the high order Sobolev norm follows immediately. Then it shows that the boundary layer is ‘weak’, which in some sense implies that the transverse magnetic field prevents the strong boundary layer. On the other hand, based on the uniform estimate on the solution to (1.6), it is natural to impose the no-slip boundary condition on the velocity field to the system (1.8)-(1.9) as (1.6),
\[
\frac{\partial u}{\partial n}|_{z=0} = 0.
\]  
(1.10)

Thus, the consistency of boundary conditions between (1.6) and (1.8)-(1.10) exhibits directly the absence of ‘strong’ boundary layer.

We now introduce some notations to be used in this paper. Throughout the paper, we use \(\| \cdot \|\) and \((\cdot, \cdot)\) to denote the \(L^2\) norm and scalar product with respect to the spatial variable respectively. Denote by \(\| \cdot \|_{H^m}\) and \(\| \cdot \|_{L^\infty}\), \(\| \cdot \|_{W^{m,\infty}}, m \in \mathbb{N}\) the standard Sobolev norms in the domain \(\Omega\). Also we use \(\| \cdot \|_{H^m(\partial \Omega)}\) for the standard Sobolev norms of functions restricted on the boundary \(\partial \Omega\). Next, we introduce the conormal Sobolev spaces. Similar to [15, 16, 17, 20, 21], we take the following conormal derivatives of functions depending on \((t, x)\):
\[
Z_0 = \partial_t, \quad Z_1 = \partial_x, \quad Z_2 = \partial_y, \quad Z_3 = \phi(z) \partial_z,
\]
where the weight \(\phi\) is a smooth and bounded function of \(z \in [0, 1]\) satisfying \(\phi|_{z=0} = 0\) and \(\phi'|_{z=0} > 0\), typically, one can choose \(\phi(z) = \frac{1}{1+z^2}\). Then, for \(m \in \mathbb{N}\) we denote the conormal Sobolev spaces
\[
H^m_{co}([0, T] \times \Omega) = \{ f(t, x) : Z^m f \in L^2([0, T] \times \Omega), \ |\alpha| \leq m \},
\]
Then obviously,

\[ \|f(t)\|_{m,\infty}^2 = \sum_{|\alpha| \leq m} \|Z^\alpha f(t, \cdot)\|^2. \]  

(1.11)

Then for the classical solutions of (1.6), it is easy to calculate the functions

\[ (u', B')(0) = 0 \]  

with \( \epsilon > 0 \) and \( T_0 > 0 \) independent of \( \epsilon \), such that for any \( 0 < \epsilon \leq \epsilon_0 \) and \( t \in [0, T_0] \), the following estimate holds:

\[ N_m(t) + \epsilon \left( \|\partial_z B'(t)\|_{m-1}^2 + \|\partial^2_z B'(t)\|_{m-2}^2 \right) \]

\[ + \int_0^t \|\nabla p'(s)\|_{m-1}^2 + \|\partial_z \nabla p'(s)\|_{m-2}^2 \, ds \leq M, \]  

(1.14)

for some positive constant \( M \) depending only on \( M_0 \).

**Remark 1.1.** From the equations in (1.6), it is easy to calculate the functions \( \partial^k_z(u', B')(t)_{t=0}, k \in \mathbb{N} \) which consist of the initial data \((u_0, B_0)\) and their spatial derivatives up to order \( 2k \). Thus, combining with the definition (1.11) with \( t = 0 \), the assumption (1.13) is actually some regularity restriction on the initial data \((u_0, B_0)\) that requires higher spatial regularity on \((u_0, B_0)\).

Based on the uniform estimates in Theorem 1.1, we justify directly the following vanishing viscosity limit for the problem (1.6).

**Theorem 1.2.** Let \((u', B')\) be smooth solutions obtained in Theorem 1.1. Then there is a solution \((u, B)\) to the problem (1.8)-(1.10) defined on \([0, T_0]\) with \( T_0 > 0 \) given in Theorem 1.1, such that

\[ \|(u', B') - (u, B)\|_{L_t^\infty L_x^\infty} + \|\partial_z(u', B') - \partial_z(u, B)\|_{L_t^\infty L_x^\infty} \to 0, \quad \epsilon \to 0. \]  

(1.15)
Remark 1.2. Following the argument given in this paper, actually we can establish the uniform estimates for higher order derivatives of solutions to the problem (1.6), provided the higher regularity and compatibility conditions for the initial data. Therefore, it means that the boundary layer effect can be arbitrarily weak in some sense.

Finally, throughout this paper we use the notations \( x = (x_h, z) \in \mathbb{R}^2 \times \mathbb{R}_+ \), \( \nabla_h = (\partial_x, \partial_y) \), \( \Delta_h = \partial_x^2 + \partial_y^2 \), \( \mathbf{u}^T = (u_h^T, u_z^T) \in \mathbb{R}^2 \times \mathbb{R} \) and \( \mathbf{B}' = (B_h', B_z') \in \mathbb{R}^2 \times \mathbb{R} \). Also we use the notation \( A \lesssim B \) to show \( A \leq CB \) with a generic constant \( C > 0 \) independent of \( \epsilon \). Denote by \([\cdot, \cdot]\) the commutator. \( \mathcal{P}(\cdot) \) stands for a polynomial function which may be different from line to line.

2. Preliminaries. In this section, we present some preliminaries and important properties of the conormal Sobolev spaces. Firstly, we state the generalized Sobolev-Gagliardo-Nirenberg-Morser type inequality for conormal Sobolev space and its proof can be found in [7].

**Proposition 2.1.** For the functions \( f, g \in L^\infty([0, T] \times \Omega) \cap H^m_{co}([0, T] \times \Omega) \) with \( m \in \mathbb{N} \), it holds that for any \( \alpha, \beta \in \mathbb{N}^4 \) with \( |\alpha| + |\beta| = m \),

\[
\int_0^T \|Z^\alpha f Z^\beta g(s)\|^2 ds \lesssim \|f\|_{L^\infty_T}^2 \int_0^T \|g(s)\|_{m}^2 ds + \|g\|_{L^\infty_T}^2 \int_0^T \|f(s)\|_{m}^2 ds. \tag{2.1}
\]

Then from the inequality (2.1) it is easy to obtain

**Proposition 2.2.** Let the integer \( m \geq 1 \) and \( \alpha \in \mathbb{N}^4 \) with \( |\alpha| \leq m \). Let \( f, g \in H^m_{co}([0, T] \times \Omega) \cap L^\infty([0, T] \times \Omega) \), it holds

\[
\int_0^T \|Z^\alpha (fg)(t)\|^2 dt \lesssim \|g\|_{L^\infty_T}^2 \int_0^T \|f(t)\|_{m}^2 dt + \|f\|_{L^\infty_T}^2 \int_0^T \|g(t)\|_{m}^2 dt. \tag{2.2}
\]

In addition, if \( g \in W^{1,\infty}_{co}([0, T] \times \Omega) \), then

\[
\int_0^T \|\nabla Z^\alpha g f(t)\|^2 dt \lesssim \sup_{0 \leq t \leq T} \|g(t)\|_{L^\infty_T}^2 \int_0^T \|f(t)\|_{m-1}^2 dt + \|f\|_{L^\infty_T}^2 \int_0^T \|g(t)\|_{m}^2 dt. \tag{2.3}
\]

The following property is related to the classical trace inequality.

**Proposition 2.3.** Let \( f(x) \in L^2(\Omega) \) and \( \partial_x f(x) \in L^2(\Omega) \), then

\[
\|f\|^2_{L^2(\partial \Omega)} \lesssim \|f\| \cdot \|\partial_x f\|. \tag{2.4}
\]

We give the anisotropic Sobolev embedding inequality as follows.

**Proposition 2.4.** Let \( f(x, t) \in H^3_{co}([0, T] \times \Omega) \) and \( \partial_x f(x, t) \in H^2_{co}([0, T] \times \Omega) \), then it holds \( f \in L^\infty([0, T] \times \Omega) \) and

\[
\|f\|^2_{L^\infty_T} \lesssim \|f(0)\|^2 + \|\partial_x f(0)\|^2 + \int_0^T \|f(t)\|^2 \, dt + \|\partial_x f(t)\|^2 dt. \tag{2.5}
\]

The proof of this proposition can be found in [17]. Note that (2.5) allows us to exchange an \( L^\infty \)-in-time norm for an \( L^2 \)-in-time norm at the cost of one order conormal derivative (actually one order time derivative).
Next, to deal with commutator involving conormal derivatives, we introduce some commutator properties which will be used frequently. Note that
\[
[Z^n, \partial_t] = [Z^n, \partial_x] = [Z^n, \partial_y] = 0, \tag{2.6}
\]
but \(Z^3\) does not commute with \(\partial_z\). As in [17], direct calculations by induction show that for the integer \(m \geq 1\), there exist two families of bounded smooth functions \(\{\phi_{k,m}(z)\}_{0 \leq k \leq m-1}\) and \(\{\phi^{k,m}(z)\}_{0 \leq k \leq m}\) depending only on \(\phi(z)\), such that
\[
[Z^m_3, \partial_z] = \sum_{k=0}^{m-1} \phi_{k,m}(z) Z^k_3 \partial_z = \sum_{k=0}^{m-1} \phi^{k,m}(z) \partial_z Z^k_3. \tag{2.7}
\]
Also, there exist two families of bounded smooth functions \(\{(\phi_{1,k,m}(z), \phi_{2,k,m}(z))\}\) and \(\{(\phi^{1,k,m}(z), \phi^{2,k,m}(z))\}\) with \(0 \leq k \leq m-1\) depending only on \(\phi(z)\), such that
\[
[Z^m_3, \partial_z^2] = \sum_{k=0}^{m-1} (\phi_{1,k,m}(z) Z^k_3 \partial_z + \phi_{2,k,m}(z) Z^k_3 \partial_z^2)
= \sum_{k=0}^{m-1} (\phi^{1,k,m}(z) \partial_z Z^k_3 + \phi^{2,k,m}(z) \partial_z^2 Z^k_3). \tag{2.8}
\]
Moreover, for a suitable function \(f\) defined on \(\Omega\) there exists a family of bounded smooth functions \(\{\psi_{i,k,m}(z)\}_{0 \leq k \leq m-1}\) depending only on \(\phi(z)\), such that
\[
[Z^m_3, \phi^{-i}] f = \sum_{k=0}^{m-1} (\psi_{i,k,m}(z) Z^k_3 (\phi^{-i} f)), \quad i = 1, 2. \tag{2.9}
\]
Therefore, one can deduce immediately the following estimates.

**Proposition 2.5.** Let the integer \(m \geq 2\) and \(f(t, x) \in H^m_{\text{co}}([0, T] \times \Omega), \partial_z f(t, x) \in H^{m-1}_{\text{co}}([0, T] \times \Omega)\). Then, for any \(\alpha \in \mathbb{N}^4\) with \(|\alpha| \leq m\)
\[
\|[Z^n, \partial_z] f(t)\| \lesssim \|\partial_z f(t)\|_{m-1}, \tag{2.10}
\]
and then
\[
\sum_{|\alpha| \leq m} \|\partial_z Z^n f(t)\| \lesssim \|\partial_z f(t)\|_{m} \lesssim \sum_{|\alpha| = m} \|\partial_z Z^n f(t)\| + \|\partial_z f(t)\|_{m-1}. \tag{2.11}
\]

The following Moser type inequalities related to commutator estimates will also be used in the analysis.

**Proposition 2.6.** Let the integer \(m \geq 1\) and \(\alpha \in \mathbb{N}^4\) with \(|\alpha| \leq m\), for suitable functions \(f\) and \(g\) defined on \(\Omega\) satisfying \(g|\partial_3 = 0\), the following properties hold.

1. \(
\|g \partial_z f\| \lesssim \|\partial_z g\|_{L^\infty} \|Z_3 f\|. \tag{2.12}
\)

2. Let \(f \in H^{m+1}_{\text{co}}([0, T] \times \Omega) \cap W^{1,\infty}_{\text{co}}([0, T] \times \Omega) \cap L^{\infty}([0, T] \times \Omega)\). Let \(g\) satisfy \(\partial_z g \in H^m_{\text{co}}([0, T] \times \Omega)\). Then it holds
\[
\int_0^T \|Z^n(g \partial_z f)(t)\|^2 dt \lesssim \|\partial_z g\|_{L^\infty_{t,x}}^2 \int_0^T \|f(t)\|_{m+1}^2 dt \leq \sup_{0 \leq t \leq T} \|f(t)\|_{1,\infty}^2 \int_0^T \|\partial_z g(t)\|_{m}^2 dt. \tag{2.13}
\]
3. Let $f \in H^{m}_{co}(0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega).$ Let $g$ satisfy $\nabla g \in H^{m}_{co}([0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega).$ Then it holds

$$
\int_{0}^{T} \|[Z^{\alpha}, g\partial_{z}]f(t)\|^{2}dt
\lesssim \sup_{0 \leq t \leq T} \|\nabla g(t)\|_{L^{T}_{1, \infty}}^{2} \int_{0}^{T} \|f(t)\|_{m}^{2}dt + \sup_{0 \leq t \leq T} \|f(t)\|_{L^{m+1}_{1, \infty}}^{2} \int_{0}^{T} \|\nabla g(t)\|_{m+1}^{2}dt. \tag{2.14}
$$

The proof of this proposition can be found in [15, 16, 17]. By comparing (2.3) with (2.14), we know that by virtue of the zero boundary condition on $g$, (2.14) allows us to exchange the norm $\|\partial_{z} f(t)\|_{m-1}$ to the norm $\|f(t)\|_{m}$ at the cost of one more spatial derivative of $g$. Thanks to the above proposition, we can reduce one order normal derivative when estimating some commutators related to the convection terms.

The following corollary follows from the above propositions that will be used to estimate the convection terms in (1.6).

**Corollary 2.1.** Let the integer $m \geq 1$ and $\alpha \in \mathbb{N}^{4}$ with $|\alpha| \leq m$. Let $f$ be a scalar function, and $\mathbf{v}$ be a vector defined on $\Omega$ satisfying that $\mathbf{v}$ is divergence-free and tangential to the boundary:

$$
\nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = 0.
$$

1. For suitable $f$ and $\mathbf{v}$ it holds

$$
\|\mathbf{v} \cdot \nabla f\| \lesssim \|\mathbf{v}\|_{1, \infty} \|f\|_{1}. \tag{2.15}
$$

2. Suppose that $f \in H^{m+1}_{co}(0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega, \text{ and } \mathbf{v} \in H^{m+1}_{co}([0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega),$ then

$$
\int_{0}^{T} \|Z^{\alpha}(\mathbf{v} \cdot \nabla f)(t)\|^{2}dt
\lesssim \sup_{0 \leq t \leq T} \|\mathbf{v}(t)\|_{L^{T}_{1, \infty}}^{2} \int_{0}^{T} \|f(t)\|_{m+1}^{2}dt + \sup_{0 \leq t \leq T} \|f(t)\|_{L^{m+1}_{1, \infty}}^{2} \int_{0}^{T} \|\mathbf{v}(t)\|_{m+1}^{2}dt. \tag{2.16}
$$

3. If $f \in H^{m}_{co}(0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega, \text{ and } \mathbf{v} \in H^{m+1}_{co}([0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega),$ then

$$
\int_{0}^{T} \|[Z^{\alpha}, \mathbf{v} \cdot \nabla]f(t)\|^{2}dt
\lesssim \sup_{0 \leq t \leq T} \|\mathbf{v}(t)\|_{2, \infty}^{2} \int_{0}^{T} \|f(t)\|_{m}^{2}dt + \|Zf\|_{L^{m+1}_{1, \infty}}^{2} \int_{0}^{T} \|\mathbf{v}(t)\|_{m+1}^{2}dt. \tag{2.17}
$$

4. Suppose that $f \in H^{m}_{co}(0, T] \times \Omega), \text{ and } \partial_{z} f \in H^{m-1}_{co}([0, T] \times \Omega), \text{ and } \mathbf{v} \in H^{m}_{co}(0, T] \times \Omega) \cap W^{1, \infty}_{co}([0, T] \times \Omega),$ then it holds

$$
\int_{0}^{T} \|[Z^{\alpha}, \mathbf{v} \cdot \nabla]f(t)\|^{2}dt
\lesssim \|Z\mathbf{v}\|_{L^{m+1}_{1, \infty}}^{2} \int_{0}^{T} \|f(t)\|_{m}^{2}dt + \|\partial_{z} f(t)\|_{m-1}^{2}dt + \|\nabla f\|_{L^{m+1}_{1, \infty}}^{2} \int_{0}^{t} \|\mathbf{v}(t)\|_{m}^{2}dt. \tag{2.18}
$$

5. If $f \in H^{m+1}_{co}([0, T] \times \Omega) \cap W^{2, \infty}_{co}([0, T] \times \Omega), \text{ and } \mathbf{v} \satisfies the no-slip boundary condition: \mathbf{v}|_{\partial \Omega} = 0, \text{ and } \partial_{z} \mathbf{v} \in H^{m+1}_{co}([0, T] \times \Omega) \cap W^{2, \infty}_{co}([0, T] \times \Omega),$ then

$$
\int_{0}^{T} \|[Z^{\alpha}, \mathbf{v} \cdot \nabla]f(t)\|^{2}dt
\lesssim \|Z\mathbf{v}\|_{L^{m+1}_{1, \infty}}^{2} \int_{0}^{T} \|f(t)\|_{m}^{2}dt + \|\partial_{z} \mathbf{v}\|_{m-1}^{2}dt. \tag{2.19}
$$
\[ \lesssim \sup_{0 \leq t \leq T} \| \partial_z v(t) \|_{2, \infty}^2 \int_0^T \| f(t) \|_{m+1}^2 dt + \sup_{0 \leq t \leq T} \| f(t) \|_{2, \infty}^2 \int_0^t \| \partial_z v(t) \|_{m+1}^2 dt. \]

**Proof.** For the first part, we rewrite

\[ v \cdot \nabla f = v_h \cdot \nabla h f + v_3 \partial_z f, \]

then along with (2.12) it implies

\[ \| v \cdot \nabla f \| \leq \| v_h \|_{L^\infty} \| \nabla h f \| + \| v_3 \|_{L^\infty} \| Z_3 f \|, \]

which yields (2.15) by using the facts that \( \| v_3 \|_{L^\infty} \lesssim \| \partial_z v_3 \|_{L^\infty} \) and \( \partial_z v_3 = -\nabla_h \cdot v_h \).

For the second part, note that

\[ Z^\alpha (v \cdot \nabla f) = \sum_{\beta \leq \alpha} C^\beta \alpha (Z^{\alpha - \beta} v_h \cdot Z^\beta \nabla h f + Z^{\alpha - \beta} v_3 \cdot Z^\beta \partial_z f). \]

First, we handle \( Z^{\alpha - \beta} v_h \cdot Z^\beta \nabla h f \) by using (2.1) and get

\[ \int_0^T \| Z^{\alpha - \beta} v_h \cdot Z^\beta \nabla h f(t) \|_{2, \infty}^2 dt \lesssim \| v_h \|_{L^\infty, \infty}^2 \int_0^T \| f(t) \|_{m+1}^2 dt + \| f(t) \|_{2, \infty}^2 \int_0^t \| v_h(t) \|_{m+1}^2 dt. \]

For \( Z^{\alpha - \beta} v_3 \cdot Z^\beta \partial_z f \), by virtue of (2.7) and (2.9) we use the techniques in [15, 17] to obtain

\[
\int_0^T \| Z^{\alpha - \beta} v_3 \cdot Z^\beta \partial_z f(t) \|_{2, \infty}^2 dt \\
\lesssim \int_0^T \| Z^{\alpha - \beta} v_3 \cdot Z^\beta \partial_z f(t) \|_{2, \infty}^2 dt \\
\leq \| v_3 \|_{L^\infty, \infty}^2 \int_0^T \| Z_3 f(t) \|_{m+1}^2 dt + \| Z_3 f \|_{2, \infty}^2 \int_0^t \| v_3(t) \|_{m+1}^2 dt \\
\lesssim \sup_{0 \leq t \leq T} \| v(t) \|_{2, \infty}^2 \int_0^T \| f(t) \|_{m+1}^2 dt + \| Z_3 f \|_{2, \infty}^2 \int_0^t \| v(t) \|_{m+1}^2 dt. \quad (2.20)
\]

Thus, combining the above two inequalities gives

\[ \int_0^T \| Z^\alpha (v \cdot \nabla f(t)) \|_{2, \infty}^2 dt \\
\lesssim \sup_{0 \leq t \leq T} \| v(t) \|_{2, \infty}^2 \int_0^T \| f(t) \|_{m+1}^2 dt + \sup_{0 \leq t \leq T} \| f(t) \|_{2, \infty}^2 \int_0^t \| v(t) \|_{m+1}^2 dt, \]

and we obtain (2.16).

The proof of (2.17) is similar as for (2.16), and we omit it for brevity.

To prove (2.18), note that

\[ [Z^\alpha, v \cdot \nabla] f = \sum_{\beta \leq \alpha, \beta \leq |\alpha|-1} C^\beta \alpha Z^{\alpha - \beta} v \cdot Z^\beta \nabla f + v \cdot [Z^\alpha, \nabla] f \]

\[ = \sum_{\beta \leq \alpha, \beta \leq |\alpha|-1} C^\beta \alpha \left( Z^{\alpha - \beta} v_h \cdot Z^\beta \nabla h f + Z^{\alpha - \beta} v_3 \cdot Z^\beta \partial_z f \right) + v_3 [Z^\alpha, \partial_z] f. \]

By using \( |\alpha - \beta| \geq 1 \) and (2.1) one has

\[ \int_0^T \| Z^{\alpha - \beta} v \cdot Z^\beta \nabla f(t) \|_{2, \infty}^2 dt \]
\[ \| Z \mathbf{v} \|^2_{L^\infty_t L^\infty_x} \int_0^T \| \nabla f(t) \|^2_{m-1} dt + \| \nabla f \|^2_{L^\infty_t L^\infty_x} \int_0^T \| Z \mathbf{v} \|^2_{m-1} dt. \]  

(2.22)

Recall (2.7), by virtue of (2.15) it implies

\[ \int_0^T \| v_3 [Z^\beta, \partial_z] f(t) \|^2 dt \]

\[ \lesssim \int_0^T \sum_{|\gamma| \leq m-1} \| v_3 \partial_z Z^\gamma f(t) \|^2 dt \lesssim \int_0^T \| \partial_z v_3(t) \|^2_{L^\infty_x} \| Z^\gamma Z f(t) \|^2 dt \]

\[ \lesssim \sup_{0 \leq t \leq T} \| \nabla_h \cdot \mathbf{v}_h(t) \|^2_{1, \infty} \int_0^T \| f(t) \|^2_m dt. \]  

(2.23)

Applying the above inequality to (2.21) and combining with (2.23) yield

\[ \int_0^T \| [Z^\alpha, \mathbf{v} \cdot \nabla] f(t) \|^2 dt \]

\[ \lesssim \| Z \mathbf{v} \|^2_{L^\infty_t L^\infty_x} \int_0^T \| f(t) \|^2_m + \| \partial_z f(t) \|^2_{m-1} dt + \| \nabla f \|^2_{L^\infty_t L^\infty_x} \int_0^T \| \mathbf{v} \|^2_m dt, \]

so that we obtain (2.18).

To prove (2.19), as (2.21) we write

\[ [Z^\alpha, \mathbf{v} \cdot \nabla] \partial_z f = \sum_{\beta \leq \alpha, |\beta| \leq |\alpha|-1} C_{\alpha}^\beta (Z^{\alpha-\beta} \mathbf{v}_h \cdot Z^\beta \partial_z \nabla_h f + Z^{\alpha-\beta} v_3 \cdot Z^\beta \partial_z Z^\gamma f) + v_3 [Z^\alpha, \partial_z] \partial_z f. \]

As \( \mathbf{v} \mid_{\partial \Omega} = 0 \), similar to (2.20), one can obtain

\[ \int_0^T \| Z^{\alpha-\beta} \mathbf{v}_h \cdot Z^\beta \partial_z \nabla_h f(t) \|^2 dt \]

\[ \lesssim \| Z(\frac{\mathbf{v}_h}{\phi(z)}) \|^2_{L^\infty_t L^\infty_x} \int_0^T \| \nabla_h f(t) \|^2_m dt + \| \nabla_h f \|^2_{L^\infty_t L^\infty_x} \int_0^T \| Z(\frac{\mathbf{v}_h}{\phi(z)}) \)(t) \|^2_m dt \]

\[ \lesssim \sup_{0 \leq t \leq T} \| \partial_z \partial_z \mathbf{v}(t) \|^2_{1, \infty} \int_0^T \| f(t) \|^2_m + \| \nabla_h f \|^2_{L^\infty_t L^\infty_x} \int_0^T \| \partial_z \mathbf{v}(t) \|^2_{m+1} dt. \]  

(2.24)

By (2.8) it follows

\[ \int_0^T \| Z^{\alpha-\beta} v_3 \cdot Z^\beta \partial_z Z^\gamma f(t) \|^2 dt \]

\[ \lesssim \int_0^T \sum_{|\gamma| \leq |\beta|} \| Z^{\alpha-\beta} v_3 \cdot \partial_z Z^\gamma f(t) \|^2 dt + \int_0^T \sum_{|\gamma| \leq |\beta|-1} \| Z^{\alpha-\beta} v_3 \cdot \partial_z Z^\gamma f(t) \|^2 dt \]

\[ \cong J_1 + J_2. \]

Similar to (2.20), we can estimate \( J_2 \) as follows.

\[ J_2 \lesssim \| \mathbf{v}_3 \|^2_{L^\infty_t L^\infty_x} \int_0^T \| f(t) \|^2_m dt + \| f \|^2_{L^\infty_t L^\infty_x} \int_0^T \| (\frac{v_3}{\phi}) \)(t) \|^2_m dt \]

\[ \lesssim \| \partial_z v_3 \|^2_{L^\infty_t L^\infty_x} \int_0^T \| f(t) \|^2_m dt + \| f \|^2_{L^\infty_t L^\infty_x} \int_0^T \| \partial_z v_3(t) \|^2_m dt. \]

To estimate \( J_1 \), note that from \( \mathbf{v} |_{\partial \Omega} = 0 \) and \( \nabla \cdot \mathbf{v} = 0 \), it holds \( \partial_z v_3 |_{\partial \Omega} = 0 \) and

\[ |v_3(t, x)| \lesssim \phi^2(z) \| \partial_z^2 v_3 \|^2_{L^\infty_x} \lesssim \phi^2(z) \| \partial_z \nabla_h \mathbf{v} \|^2_{L^\infty_x}. \]
Then, by using $\phi^2 \partial^2_z = Z^2 - \phi' Z_3$ and (2.9) it holds
\[
J_1 \lesssim \int_0^T \sum_{|\gamma| \leq |\beta|, |\delta| \leq |\alpha - \beta|} \|Z^{\delta} (\frac{v_3}{\phi^2}) \cdot (Z_3^2 - \phi' Z_3) Z^\gamma f(t)\|^2 dt
\]
\[
\lesssim \|Z(\frac{v_3}{\phi^2})\|_{L^\infty_t L^p_t} \int_0^t \|Z_3^2 f(t)\|^2_{m-1} + \|Z_3 f(t)\|^2_{m-1} dt
\]
\[
+ \left( \|Z_3^2 f\|_{L^2_t L^\infty_x} + \|Z_3 f\|_{L^2_t L^\infty_x} \right) \int_0^T \|Z(\frac{v_3}{\phi^2})\|^2_{m-1} dt
\]
\[
\lesssim \sup_{0 \leq t \leq T} \|\partial_z \phi(t)\|_{2,\infty}^2 \int_0^T \|f(t)\|^2_{m+1} dt + \sup_{0 \leq t \leq T} \|f(t)\|_{2,\infty}^2 \int_0^T \|\partial_z \phi(t)\|^2_{m+1} dt.
\]
Thus, we obtain
\[
\int_0^T \|Z^\beta v_3 \cdot Z^{\alpha - \beta} \partial^2_z f(t)\|^2 dt \tag{2.25}
\]
\[
\lesssim \sup_{0 \leq t \leq T} \|\partial_z \phi(t)\|_{2,\infty}^2 \int_0^T \|f(t)\|^2_{m+1} dt + \sup_{0 \leq t \leq T} \|f(t)\|_{2,\infty}^2 \int_0^T \|\partial_z \phi(t)\|^2_{m+1} dt.
\]
Combining (2.24) with (2.25) yields (2.19). The proof of the corollary is completed. \]}

3. The a priori estimates. In this section, we establish the key a priori estimate for classical solution of (1.6), from which one can prove Theorem 1.1 immediately.

**Theorem 3.1.** Under the assumptions of Theorem 1.1, for a smooth solution $(u^\epsilon, B^\epsilon)$ defined on $[0, T]$ of the problem (1.6), there exists a sufficiently small $\epsilon_0 > 0$, such that for any $0 < \epsilon < \epsilon_0$ and $t \in [0, T]$ the following a priori estimates hold:

\[
N_m(t) + \epsilon (\|\partial_z B^\epsilon(t)\|^2_{m-1} + \|\partial^2_z B^\epsilon(t)\|^2_{m-2}) + \int_0^t \|\nabla p^\epsilon\|^2_{m-1} + \|\partial_z \nabla p^\epsilon(s)\|^2_{m-2} ds
\]
\[
\lesssim \mathcal{P}(M_0) + (t + \epsilon)\mathcal{P}(N_m(t)),
\]
where $\mathcal{P}(\cdot)$ is a polynomial.

We divide the proof of Theorem 3.1 into several steps in the following.

3.1. Conormal energy estimates.

**Proposition 3.1.** For any integer $m \geq 0$, the smooth solution $(u^\epsilon, B^\epsilon)$ of (1.6) on $[0, T]$ satisfies that for any $t \in [0, T]$,

\[
\|(u^\epsilon, B^\epsilon)(t)\|^2_m + \epsilon \int_0^t \|\nabla u^\epsilon(s)\|^2_m ds \tag{3.2}
\]
\[
\lesssim \|(u^\epsilon, B^\epsilon)(0)\|^2_m + \int_0^t \|\partial_z p^\epsilon(s)\|^2_{m-1} ds
\]
\[
+ \left(1 + \|(u^\epsilon, B^\epsilon)\|^2_{L^2_t L^\infty_x} \right) \int_0^t \|(u^\epsilon, B^\epsilon)(s)\|^2_m + \|\partial_z (u^\epsilon, B^\epsilon)(s)\|^2_{m-1} ds.
\]

**Proof.** We apply the conormal derivatives $Z^\alpha, |\alpha| \leq m$ to the equations of (1.6), and obtain that

\[
\begin{align*}
\partial_t Z^\alpha u^\epsilon + (u^\epsilon \cdot \nabla) Z^\alpha u^\epsilon + \nabla Z^\alpha p^\epsilon - (B^\epsilon \cdot \nabla) Z^\alpha B^\epsilon - \partial_z Z^\alpha B^\epsilon - \epsilon \Delta Z^\alpha u^\epsilon
\end{align*}
\]
\[
= C^\alpha_1 + C^\alpha_2 + C^\alpha_3,
\]
\[
\partial_t Z^\alpha B^\epsilon + (u^\epsilon \cdot \nabla) Z^\alpha B^\epsilon - (B^\epsilon \cdot \nabla) Z^\alpha u^\epsilon - \partial_z Z^\alpha u^\epsilon = C^\alpha_4,
\]

(3.3)
with
\[
\begin{align*}
C_1^α &= -[Z^α, \nabla]p^e, & C_2^α &= ε[Z^α, \Delta]u^e, \\
C_3^α &= -[Z^α, u^e \cdot \nabla]u^e + [Z^α, B^e \cdot \nabla]B^e + [Z^α, \partial_z]B^e, \\
C_4^α &= -[Z^α, u^e \cdot \nabla]B^e + [Z^α, B^e \cdot \nabla]u^e + [Z^α, \partial_z]u^e.
\end{align*}
\]

Multiplying (3.3) by \((Z^α u^e, Z^α B^e)\) and integrating the resulting equations over \([0, t] \times Ω\), one obtains that by integration by parts,
\[
\begin{align*}
&\frac{1}{2} \|(Z^α u^e, Z^α B^e)(t)\|^2 - \frac{1}{2} \|(Z^α u^e, Z^α B^e)(0)\|^2 + ε \int_0^t \|\nabla Z^α u^e(s)\|^2 ds \\
&= \int_0^t \int_{Ω} Z^α p^e (\nabla \cdot Z^α u^e) dx ds + \int_0^t \int_{Ω} (C_1^α + C_2^α + C_3^α) \cdot Z^α u^e dx dt \\
&+ \int_0^t \int_{Ω} C_4^α \cdot Z^α B^e dx dt. \tag{3.4}
\end{align*}
\]

Here we have used the divergence-free conditions \(\nabla \cdot u^e = 0\) and the boundary conditions \(Z^α u^e|_{z=0} = 0\). In the following we estimate the terms on the right-hand side of (3.4).

Firstly, from the divergence-free condition \(\nabla \cdot u^e = 0\), one has
\[
\nabla \cdot Z^α u^e = -[Z^α, \partial_z]u^e_g.
\]

It holds by virtue of (2.10) that
\[
\|\nabla \cdot Z^α u^e\| \lesssim \|\partial_z u^e_g\|_{m-1} = \|\nabla h \cdot u^e_h\|_{m-1} \lesssim \|u^e\|_m,
\]

and then
\[
|\int_0^t \int_{Ω} Z^α p^e (\nabla \cdot Z^α u^e) dx ds| \leq \int_0^t \|Z^α p^e(s)\| \cdot \|\nabla \cdot Z^α u^e(s)\| ds \\
\lesssim \int_0^t \|\partial_z p^e(s)\|_{m-1} \|u^e(s)\|_m ds. \tag{3.5}
\]

Similarly, for \(C_1^α\) note that
\[
C_1^α = -[Z^α, \nabla]p^e = -[Z^α, \partial_z]p^e \hat{e}_z,
\]

which along with (2.10) implies
\[
|\int_0^t \int_{Ω} C_1^α \cdot Z^α u^e dx ds| \leq \int_0^t \|C_1^α\| \cdot \|Z^α u^e(s)\| ds \\
\lesssim \int_0^t \|\partial_z p^e(s)\|_{m-1} \|u^e(s)\|_m ds. \tag{3.6}
\]

Secondly, applying (2.8) to \(C_2^α\) and by virtue of (2.11) one can obtain that
\[
|\int_0^t \int_{Ω} C_2^α \cdot Z^α u^e dx ds| \lesssim ε \int_0^t \|C_2^α, Z^α u^e\| ds + ε \int_0^t \|\partial_z u^e(s)\|_{m-1} \|u^e(s)\|_m ds,
\]

where
\[
\tilde{C}_2^α = \sum_{|β| ≤ m-1} ψ_β(z) \partial^2 Z^β u^e
\]

with \(ψ_β(z)\) depending only on \(φ(z)\). Then, by integration by parts and the boundary conditions \(Z^α u^e|_{z=0} = 0\) it yields
\[
|\tilde{C}_2^α, Z^α u^e| \lesssim \|\partial_z Z^α u^e(s)\| + \|Z^α u^e(s)\| \sum_{|β| ≤ m-1} \|\partial_z Z^β u^e(s)\|,
\]
where we have used (2.11) in the second inequality. Thus,
\[
\left| \int_0^t \int_\Omega C^\alpha_2 \cdot Z^\alpha u^\prime dxds \right| \lesssim \epsilon \int_0^t \| \partial_2 u^\prime(s) \|_{m-1} \| \partial_2 u^\prime(s) \|_m + \| u^\prime(s) \|_m ds, \quad (3.7)
\]
Thirdly, we consider \( C^3_3 \). By (2.18) it follows
\[
\int_0^t \| [Z^\alpha, u^\prime \cdot \nabla] u^\prime(s) \|^2 ds \lesssim \| u^\prime \|^2_{W^{1,\infty}} \int_0^t \| u^\prime(s) \|^2_m + \| \partial_2 u^\prime(s) \|^2_{m-1} ds.
\]
Similarly, by virtue of (1.7), that is, \( b_\Delta |_{z=0} = 0 \), one can obtain
\[
\int_0^t \| [Z^\alpha, \mathcal{B}^\prime \cdot \nabla] \mathcal{B}^\prime(s) \|^2 ds \lesssim \| \mathcal{B}^\prime \|^2_{W^{1,\infty}} \int_0^t \| \mathcal{B}^\prime(s) \|^2_m + \| \partial_2 \mathcal{B}^\prime(s) \|^2_{m-1} ds.
\]
Moreover, by (2.10) it is easy to get
\[
\int_0^t \| [Z^\alpha, \partial_2 \mathcal{B}^\prime(s) \|^2 ds \lesssim \int_0^t \| \partial_2 \mathcal{B}^\prime(s) \|^2_{m-1} ds.
\]
Therefore, we obtain that from the above three estimates,
\[
\int_0^t \| C^3_3 \|^2 ds \lesssim (1 + \| (u^\prime, B^\prime) \|^2_{W^{1,\infty}}) \int_0^t \| (u^\prime, B^\prime) \|^2_m + \| \partial_2 (u^\prime, B^\prime)(s) \|^2_{m-1} ds,
\]
which implies that
\[
\left| \int_0^t \int_\Omega C^\alpha_3 \cdot Z^\alpha u^\prime dxds \right| \lesssim \frac{1}{2} \int_0^t \| C^3_3 \|^2 ds + \frac{1}{2} \int_0^t \| Z^\alpha u^\prime(s) \|^2 ds \lesssim (1 + \| (u^\prime, B^\prime) \|^2_{W^{1,\infty}}) \int_0^t \| (u^\prime, B^\prime) \|^2_m + \| \partial_2 (u^\prime, B^\prime)(s) \|^2_{m-1} ds. \quad (3.8)
\]
In the same way, one can obtain
\[
\left| \int_0^t \int_\Omega C^\alpha_4 \cdot Z^\alpha \mathcal{B}^\prime dxds \right| \lesssim (1 + \| (u^\prime, B^\prime) \|^2_{W^{1,\infty}}) \int_0^t \| (u^\prime, B^\prime) \|^2_m + \| \partial_2 (u^\prime, B^\prime)(s) \|^2_{m-1} ds. \quad (3.9)
\]
Thus, substituting (3.5)-(3.9) into (3.4), it holds
\[
\| (Z^\alpha u^\prime, Z^\alpha \mathcal{B}^\prime)(t) \|^2 + \epsilon \int_0^t \| \nabla Z^\alpha u^\prime(s) \|^2 ds \lesssim \| (Z^\alpha u^\prime, Z^\alpha \mathcal{B}^\prime)(0) \|^2 + \int_0^t \| \partial_2 p^\prime(s) \|_{m-1} \| u^\prime(s) \|_m ds \]
\[
+ \left( 1 + \| (u^\prime, B^\prime) \|^2_{W^{1,\infty}} \right) \int_0^t \| (u^\prime, B^\prime) \|^2_m + \| \partial_2 (u^\prime, B^\prime)(s) \|^2_{m-1} ds \]
\[
+ \epsilon \int_0^t \| \partial_2 u^\prime(s) \|_m \| \partial_2 u^\prime(s) \|_{m-1} dt.
\]
We take the sum of the above inequalities with \( \alpha, |\alpha| \leq m \), and use (2.11) and the Young’s inequality to absorb the term \( \epsilon \| \partial_z u' \|_m \) in the last term on the right-hand side. Finally we obtain

\[
\|(u', B')(t)\|_m^2 + \epsilon \int_0^t \| \nabla u'(s) \|_m^2 ds \\
\lesssim \|(u', B')(0)\|_m^2 + \epsilon \int_0^t \| \partial_s p'(s) \|_{m-1} \| u'(s) \|_m ds \\
+ \left( 1 + \|(u', B')\|_{W^{1,\infty}}^2 \right) \int_0^t \|(u', B')(s)\|_m^2 + \| \partial_z (u', B')(s) \|_{m-1}^2 ds,
\]

and this completes the proof of the proposition. \( \square \)

3.2. Normal derivative estimates. Thanks to the structure of equations in (1.6), we are able to establish the conormal estimates for high order normal derivatives. Denote by

\[
Q_m(t) = \begin{cases} 
1 + \sup_{0 \leq s \leq t} \left( \|(u', B')(s)\|_m^2 + \epsilon^2 \| \partial_z u'(s) \|_{L^\infty}^2 \right), & m = 0, \\
1 + \sup_{0 \leq s \leq t} \left\{ \sum_{k=0}^m \| \partial_z^k (u', B')(s) \|_{L^\infty}^2 + \epsilon^2 \| \partial_z^{m+1} u'(s) \|_{L^\infty}^2 \right\}, & m \geq 1.
\end{cases}
\]

**Proposition 3.2.** For any integers \( k, q \geq 0 \), it holds that for the smooth solution \((u', B')\) of (1.6) on \([0, T]\),

\[
\epsilon \| \partial_z^{k+1} B'(t) \|_q^2 + \int_0^t \| \partial_z^{k+1} (u', B')(s) \|_q^2 + \epsilon^2 \| \partial_z^{k+2} u'(s) \|_q^2 ds \\
\lesssim \epsilon \| \partial_z^{k+1} B'(0) \|_q^2 + \int_0^t \| \partial_z^{k+1} \nabla p'(s) \|_q^2 ds \\
+ Q_k(t) \int_0^t \sum_{i=0}^k \| \partial_z^i (u', B')(s) \|_{L^q}^2 + \epsilon^2 \| \partial_z^{k+1} u'(s) \|_{L^q}^2 ds,
\]

for any \( t \in [0, T] \).

We divide the proof of the above a priori estimates into two parts.

3.2.1. **Estimates for normal derivatives of** \( u' \). We rewrite the equations for \( B' \) in (1.6) as follows

\[
\partial_z u' = \partial_z B' + u' \cdot \nabla B' - B' \cdot \nabla u',
\]

from which we establish the following estimates for normal derivatives of \( u' \).

**Lemma 3.1.** For any integers \( k, q \geq 0 \), the smooth solution \((u', B')\) of (1.6) on \([0, T]\) satisfies:

\[
\int_0^t \| \partial_z^{k+1} u'(s) \|_q^2 ds \lesssim Q_k(t) \int_0^t \sum_{i=0}^k \| \partial_z^i (u', B')(s) \|_{L^q}^2 ds, \quad \forall t \in [0, T].
\]

**Proof.** We apply the conormal derivatives \( Z^\alpha \partial_z^k \) with \(|\alpha| \leq q\) to the equation (3.12), and obtain

\[
Z^\alpha \partial_z^{k+1} u' = \partial_t Z^\alpha \partial_z^k B' + Z^\alpha (u' \cdot \nabla \partial_z^k B' + B' \cdot \nabla \partial_z^k u') + C^\alpha,
\]

where $C^α = 0$ if $k = 0$, and
\[
C^α = \sum_{i=1}^{k} C_k^i \left\{ Z^α (\partial_z^i u^* \cdot \nabla \partial_z^{k-i} B^*) - Z^α (\partial_z^i B^* \cdot \nabla \partial_z^{k-i} u^*) \right\},
\]
if $k \geq 1$.

By (3.14), one has
\[
\int_0^t \| Z^α \partial_z^{k+1} u^*(s) \|^2 \, ds \\
\leq \int_0^t \| \partial_τ Z^α \partial_z^k B^*(s) \|^2 \, ds + \int_0^t \| Z^α (u^* \cdot \nabla \partial_z^k B^*) (s) \|^2 \, ds + \| Z^α (B^* \cdot \nabla \partial_z^k u^*) (s) \|^2 \, ds \\
+ \int_0^t \| C^α (s) \|^2 \, ds. \tag{3.15}
\]

For the second term on the right-hand side of (3.15), as $u_0^*|_{z=0} = b_0^*|_{z=0} = 0$, we apply (2.16) to obtain
\[
\int_0^t \| Z^α (u^* \cdot \nabla \partial_z^k B^*) (s) \|^2 + \| Z^α (B^* \cdot \nabla \partial_z^k u^*) (s) \|^2 \, ds \\
\leq \sup_{0 \leq s \leq t} \| u^*(s) \|^2 \int_0^t \| \partial_z^k B^*(s) \|^2_{q+1} \, ds + \sup_{0 \leq s \leq t} \| \partial_z^k u^*(s) \|^2_{q+1} \int_0^t \| B^*(s) \|^2_{q+1} \, ds \\
+ \sup_{0 \leq s \leq t} \| B^*(s) \|^2_{q+1} \int_0^t \| \partial_z^k u^*(s) \|^2_{q+1} \, ds + \sup_{0 \leq s \leq t} \| \partial_z^k u^*(s) \|^2_{q+1} \int_0^t \| B^*(s) \|^2_{q+1} \, ds \\
\leq \sup_{0 \leq s \leq t} \| \partial_z^k (u^*, B^*) (s) \|^2_{q+1} \int_0^t \| (u^*, B^*) (s) \|^2_{q+1} \, ds. \tag{3.16}
\]

To estimate $C^α$, as $k, i \geq 1$ we use the following form by virtue of divergence-free conditions:
\[
Z^α (\partial_z^i u^* \cdot \nabla \partial_z^{k-i} B^*) = Z^α (\partial_z^i u^* \cdot \nabla \partial_z^{k-i} B^*) - Z^α (\partial_z^{k-i} (\nabla_h \cdot u^*) \partial_z^{k-i} B^*). \]

Applying (2.1) to the above equality yields
\[
\sum_{i=1}^{k} C_k^i \int_0^t \| Z^α (\partial_z^i u^* \cdot \nabla \partial_z^{k-i} B^*) (s) \|^2 \, ds \\
\leq \sum_{i=1}^{k} \left\{ \| \partial_z^{k-i} u^*_h \|^2_{L^∞_{r,x}} \int_0^t \| \partial_z^{k-i} B^*(s) \|^2_{q+1} \, ds + \| \partial_z^{k-i} B^* \|^2_{L^∞_{r,x}} \int_0^t \| \partial_z^{k-i} u^*_h (s) \|^2_{q+1} \, ds \\
+ \| \partial_z^{k-i} u^*_h \|^2_{L^∞_{r,x}} \int_0^t \| \partial_z^{k-i+1} B^*(s) \|^2_{q+1} \, ds + \| \partial_z^{k-i+1} B^* \|^2_{L^∞_{r,x}} \int_0^t \| \partial_z^{k-i+1} u^*_h (s) \|^2_{q+1} \, ds \right\} \\
\leq \sum_{i=1}^{k} \left( \| \partial_z^{k-i} (u^*, B^*) \|^2_{L^∞_{r,x}} \int_0^t \| \partial_z^{k-i} (u^*, B^*) (s) \|^2_{q+1} \, ds \right). \tag{3.17}
\]

Thus one can obtain
\[
\int_0^t \| C^α (s) \|^2 \, ds \leq \sum_{i=1}^{k} \left( \| \partial_z^{k-i} (u^*, B^*) \|^2_{L^∞_{r,x}} \int_0^t \| \partial_z^{k-i} (u^*, B^*) (s) \|^2_{q+1} \, ds \right). \tag{3.17}
\]
We substitute (3.16) and (3.17) into (3.15) and obtain
\[
\int_0^t \| Z^\alpha \partial_x^{k+1} u'(s) \|^2 ds \\
\lesssim \sum_{i=0}^k \left\{ \left( 1 + \sup_{0 \leq s \leq t} \| \partial_x^{k-i} (u^e, B') (s) \|_{1, \infty}^2 \right) \right\} \int_0^t \| \partial_x^i (u', B') (s) \|^2_{q+1} ds.
\]
Then (3.13) follows by summing the above inequalities for \(|\alpha| \leq q\). \qed

3.2.2. **Estimates for normal derivatives of** \(B'\). Inspired by the previous process, we rewrite the equations for \(u'\) in (1.6) as follow:
\[
\partial_t B' + \epsilon \partial_x^2 u' = \partial_t u' + u' \cdot \nabla u' - B' \cdot \nabla B' + \nabla p' - \epsilon \Delta_h u'.
\] (3.18)
However, it is not straightforward to gain the estimates for high order derivatives of \(u'\) because of the presence of \(\epsilon \partial_x^2 u'\) which arises an uncontrolled term for \(u'\) with one order higher normal derivative. The key ingredient to overcome this difficulty is to absorb the terms involving the highest order normal derivatives of \(u'\) with the help of the equations of \(B'\). Actually we have the following lemma.

**Lemma 3.2.** For any integers \(q \geq 0\), the smooth solution \((u', B')\) of (1.6) on \([0, T]\) satisfies that for every \(t \in [0, T]\),
\[
\epsilon \| \partial_x B'(t) \|^2 + \int_0^t \| \partial_x B'(s) \|^2 ds + \epsilon^2 \| \partial_x^2 u'(s) \|^2 ds \\
\lesssim \epsilon \| \partial_x B'(0) \|^2 + \int_0^t \| \nabla p'(s) \|^2 ds + Q_0(t) \int_0^t \| (u', B') (s) \|^2_{q+1} + \epsilon^2 \| \nabla u'(s) \|^2_{q+1} ds.
\] (3.19)
**Proof.** We divided the proof into the following three parts.

(1). We apply the conormal derivatives \(Z\alpha\) with \(|\alpha| \leq q\) to the equation (3.18) and obtain
\[
Z^\alpha \partial_x B' + \epsilon Z^\alpha \partial_x^2 u' = \partial_t Z^\alpha u' + Z^\alpha \nabla p' - \epsilon Z^\alpha \Delta_h u' + Z^\alpha (u' \cdot \nabla u' - B' \cdot \nabla B').
\] (3.20)
We take the \(L^2\) norm over \([0, t] \times \Omega\) on both sides of (3.20) to obtain
\[
\int_0^t \| Z^\alpha \partial_x B'(s) \|^2 ds + \epsilon \int_0^t \| Z^\alpha \partial_x^2 u'(s) \|^2 ds + 2\epsilon \int_0^t \int_{\Omega} Z^\alpha \partial_x B' \cdot Z^\alpha \partial_x^2 u' dx ds \\
\lesssim \int_0^t \| \partial_t Z^\alpha u'(s) \|^2 + \| Z^\alpha \nabla p'(s) \|^2 + \epsilon^2 \| Z^\alpha \Delta_h u'(s) \|^2 ds \\
+ \int_0^t \| Z^\alpha (u' \cdot \nabla u') (s) \|^2 + \| Z^\alpha (B' \cdot \nabla B') (s) \|^2 ds.
\] (3.21)
It is easy to get
\[
\int_0^t \| \partial_t Z^\alpha u'(s) \|^2 + \| Z^\alpha \nabla p'(s) \|^2 + \epsilon^2 \| Z^\alpha \Delta_h u'(s) \|^2 ds \\
\lesssim \int_0^t \| \nabla p'(s) \|^2 + \| u'(s) \|^2_{q+1} + \epsilon^2 \| \nabla_h u'(s) \|^2_{q+1} ds.
\]
We use (2.16) to obtain
\[
\int_0^t \| Z^\alpha (u' \cdot \nabla u') (s) \|^2 + \| Z^\alpha (B' \cdot \nabla B') (s) \|^2 ds
\]
\[
\lesssim \sup_{0 \leq s \leq t} \| (u^r, B^r)(s) \|_{1, \infty}^{2} \int_{0}^{t} \| (u^r, B^r)(s) \|_{q+1}^{2} ds.
\]

Substituting the above two inequalities into (3.21) yields

\[
\begin{aligned}
\int_{0}^{t} \| Z^\alpha \partial_z B^r(s) \|^{2} ds + \epsilon^{2} \int_{0}^{t} \| Z^\alpha \partial_z^2 u^r(s) \|^{2} ds + 2\epsilon \int_{0}^{t} \int_{\Omega} Z^\alpha \partial_z B^r \cdot Z^\alpha \partial_z^2 u^r dx ds \\
\lesssim \left( 1 + \sup_{0 \leq s \leq t} \| (u^r, B^r)(s) \|_{1, \infty}^{2} \right) \int_{0}^{t} \| (u^r, B^r)(s) \|_{q+1}^{2} ds \\
+ \int_{0}^{t} \| \nabla p^r(s) \|_{u}^{2} + \epsilon^{2} \| \nabla_h u^r(s) \|_{q+1}^{2} ds.
\end{aligned}
\]  

(3.22)

(2). Now we need to estimate the last term on the left-hand side of (3.22). For this, we apply the operator $Z^\alpha \partial_z$ to the equations for $B^r$ in (1.6), and obtain

\[
\partial_t Z^\alpha \partial_z B^r - Z^\alpha \partial_z^2 u^r + (u^r \cdot \nabla) Z^\alpha \partial_z B^r = - [Z^\alpha, u^r \cdot \nabla] \partial_z B^r + Z^\alpha (-\partial_z u^r \cdot \nabla B^r + B^r \cdot \nabla \partial_z u^r + \partial_z B^r \cdot \nabla u^r). 
\]  

(3.23)

Multiplying (3.23) by $2\epsilon Z^\alpha \partial_z B^r$ and integrating the resulting equations over $[0, t] \times \Omega$, one has by integration by parts,

\[
\epsilon \| Z^\alpha \partial_z B^r(t) \|^{2} - \epsilon \| Z^\alpha \partial_z B^r(0) \|^{2} - 2\epsilon \int_{0}^{t} \int_{\Omega} Z^\alpha \partial_z B^r \cdot Z^\alpha \partial_z^2 u^r dx ds \\
\leq \frac{1}{2} \int_{0}^{t} \| Z^\alpha \partial_z B^r(s) \|^{2} ds + C\epsilon^{2} \int_{0}^{t} \sum_{j=1}^{4} \| C_j(s) \|^{2} ds,
\]

(3.24)

where

\[
C_1 = [Z^\alpha, u^r \cdot \nabla] \partial_z B^r, \quad C_2 = Z^\alpha (\partial_z u^r \cdot \nabla B^r), \\
C_3 = Z^\alpha (B^r \cdot \nabla \partial_z u^r), \quad C_4 = Z^\alpha (\partial_u B^r \cdot \nabla u^r).
\]

In the following, we estimate $\int_{0}^{t} \| C_j(s) \|^{2} ds, j = 1, 2, 3$. Firstly, note that by $\nabla \cdot u^r = 0$ and $u^r|_{z=0} = 0$, it implies

\[
\partial_z u^r|_{z=0} = 0. 
\]

(3.25)

Thus we use (2.19) to estimate $C_1$ and obtain

\[
\int_{0}^{t} \| C_1(s) \|^{2} ds \leq \sup_{0 \leq s \leq t} \| \partial_z u^r(s) \|_{1, \infty}^{2} \int_{0}^{t} \| B^r(s) \|_{q+1}^{2} ds + \sup_{0 \leq s \leq t} \| B^r(s) \|_{1, \infty}^{2} \int_{0}^{t} \| \partial_z u^r(s) \|_{q+1}^{2} ds.
\]

(3.26)

Secondly, with the help of (3.25) we apply (2.16) to $C_2$ and obtain

\[
\int_{0}^{t} \| C_2(s) \|^{2} ds \leq \sup_{0 \leq s \leq t} \| \partial_z u^r(s) \|_{1, \infty}^{2} \int_{0}^{t} \| B^r(s) \|_{q+1}^{2} ds + \sup_{0 \leq s \leq t} \| B^r(s) \|_{1, \infty}^{2} \int_{0}^{t} \| \partial_z u^r(s) \|_{q+1}^{2} ds.
\]

(3.27)

Similarly, as $b_3|_{z=0} = 0$ one has

\[
\int_{0}^{t} \| C_3(s) \|^{2} ds \leq \sum_{j=1}^{4} \int_{0}^{t} \| C_j(s) \|^{2} ds.
\]

(3.28)
Next, we estimate \( C_4^* \). By virtue of \( \nabla \cdot B^* = 0 \), we rewrite \( C_4^* \) as

\[
C_4^* = Z^\alpha (\partial_z B_h^* \cdot \nabla_h u^* - (\nabla_h \cdot B_h^*) \partial_z u^*).
\]

Thanks to the no-slip boundary condition, one has \( \nabla_h u^* |_{z=0} = 0 \) and then, it implies that by virtue of (2.13),

\[
\int_0^t \| Z^\alpha (\partial_z B_h^* \cdot \nabla_h u^*) (s) \|^2 ds \leq \| \partial_z \nabla_h u^* \|_{L^\infty_{t,x}} \int_0^t \| B^* (s) \|^2_{q+1} ds + \sup_{0 \leq s \leq t} \| B^* (s) \|^2_{1,\infty} \int_0^t \| \partial_z \nabla_h u^* (s) \|^2_q ds \leq \| \partial_z u^* (s) \|^2_{l,\infty} \int_0^t \| B^* (s) \|^2_{q+1} ds + \sup_{0 \leq s \leq t} \| B^* (s) \|^2_{1,\infty} \int_0^t \| \partial_z u^* (s) \|^2_{q+1} ds.
\]

By using (2.2), one has

\[
\int_0^t \| Z^\alpha (\nabla_h B_h^*) \partial_z u^* (s) \|^2 ds \leq \| \nabla_h B_h^* \|_{L^\infty_{t,x}} \int_0^t \| \partial_z u^* (s) \|^2_{l,\infty} \int_0^t \| \nabla_h B_h^* (s) \|^2_q ds.
\]

Combining the above two inequalities yields

\[
\int_0^t \| C_4^* (s) \|^2 ds \leq \| \partial_z u^* (s) \|^2_{l,\infty} \int_0^t \| B^* (s) \|^2_{q+1} ds + \sup_{0 \leq s \leq t} \| B^* (s) \|^2_{1,\infty} \int_0^t \| \partial_z u^* (s) \|^2_{q+1} ds.
\]

Now, we substitute (3.26), (3.27), (3.28) and (3.31) into (3.24) to get

\[
\epsilon \| Z^\alpha \partial_z B^* (t) \|^2 - \epsilon \| Z^\alpha \partial_z B^* (0) \|^2 - 2\epsilon \int_0^t \| Z^\alpha \partial_z B^* \cdot Z^\alpha \partial_z u^* \| dxdy ds \\
\leq \frac{1}{2} \int_0^t \| Z^\alpha \partial_z B^* (s) \|^2 ds + C \epsilon^2 \left( \sup_{0 \leq s \leq t} \| \partial_z u^* (s) \|^2_{2,\infty} \int_0^t \| B^* (s) \|^2_{q+1} ds + \sup_{0 \leq s \leq t} \| B^* (s) \|^2_{2,\infty} \int_0^t \| \partial_z u^* (s) \|^2_{q+1} ds \right).
\]

(3) Adding (3.21) and (3.32) together yields

\[
\epsilon \| Z^\alpha \partial_z B^* (t) \|^2 + \int_0^t \| Z^\alpha \partial_z B^* (s) \|^2 + \epsilon^2 \| Z^\alpha \partial_z^2 u^* (s) \|^2 ds \\
\leq \epsilon \| Z^\alpha \partial_z B^* (0) \|^2 + \int_0^t \| \nabla p^* (s) \|^2_q ds + \epsilon^2 (1 + \sup_{0 \leq s \leq t} \| B^* (s) \|^2_{2,\infty}) \int_0^t \| \nabla u^* (s) \|^2_{q+1} ds + \left( 1 + \sup_{0 \leq s \leq t} \| (u^*, B^*) (s) \|^2_{1,\infty} + \epsilon^2 \sup_{0 \leq s \leq t} \| \partial_z u^* (s) \|^2_{2,\infty} \right) \int_0^t \| (u^*, B^*) (s) \|^2_{q+1} ds.
\]

Then summing up the above inequality about \( |\alpha| \leq q \) gives the desired estimate (3.19). \( \square \)

We continue the above process to establish the estimates for higher order normal derivatives of \( B^* \).
Lemma 3.3. For any integers $k \geq 1$ and $q \geq 0$, the smooth solution $(u', B')$ of (1.6) on $[0, T]$ satisfies:

$$
\|z^{k+1} B'(t)\|_q^2 + \int_0^t \|z^{k+1} B'(s)\|_q^2 + \epsilon^2 \|z^{k+2} u'(s)\|_q^2 ds
\
\lesssim \epsilon \|z^{k+1} B'(0)\|_q^2 + \int_0^t \|z^k \nabla p'(s)\|_0^2
\
+ Q_k(t) \sum_{i=0}^k \|z^i (u', B')(s)\|_{q+1}^2 + \epsilon^2 \|z^k \nabla u'(s)\|_{q+1}^2,
$$

(3.33)

for every $t \in [0, T]$.

Proof. We prove this lemma in an analogous way as the previous lemma 3.2, and divide the proof into the following three parts.

(1) We apply the conormal derivatives $Z^\alpha z^k$ with $k \geq 1$, $|\alpha| \leq q$ to the equations (3.18), and obtain

$$
Z^\alpha z^{k+1} B' + \epsilon Z^\alpha z^{k+2} u' = \partial_t Z^\alpha z^k u' + Z^\alpha (u' \cdot \nabla z^k u' - B' \cdot \nabla z^k B')
\
+ C_1^\alpha + Z^\alpha z^k \nabla p' - \epsilon Z^\alpha \Delta z^k u',
$$

(3.34)

where

$$
C_1^\alpha = \sum_{i=1}^k C_i^\alpha \left\{ Z^\alpha (z^i u' \cdot \nabla z^{k-i} u') - Z^\alpha (z^i B' \cdot \nabla z^{k-i} B') \right\}.
$$

We take the $L^2$ norm over $[0, t] \times \Omega$ on both sides of (3.34) to obtain

$$
\int_0^t \|Z^\alpha z^{k+1} B'(s)\|_0^2 ds + \epsilon^2 \int_0^t \|Z^\alpha z^{k+2} u'(s)\|_0^2 ds
\
+ 2\epsilon \int_0^t \int_\Omega Z^\alpha z^{k+1} B' \cdot Z^\alpha z^{k+2} u' dx ds
\
\lesssim \int_0^t \|\partial_t Z^\alpha z^k u'(s)\|_0^2 ds + \int_0^t \|Z^\alpha (u' \cdot \nabla z^k u') (s)\|_0^2 + \|Z^\alpha (B' \cdot \nabla z^k B') (s)\|_0^2 ds
\
+ \int_0^t \|C_1^\alpha (s)\|_0^2 + \|Z^\alpha z^k \nabla p'(s)\|_0^2 + \epsilon^2 \|Z^\alpha \Delta z^k u'(s)\|_0^2 ds.
$$

(3.35)

By noting that

$$
\int_0^t \|Z^\alpha z^k \nabla p'(s)\|_0^2 + \epsilon^2 \|Z^\alpha \Delta z^k u'(s)\|_0^2 ds \lesssim \int_0^t \|\partial_t z^k \nabla p'(s)\|_q^2 + \epsilon^2 \|z^k \nabla u'(s)\|_{q+1}^2 ds,
$$

the other terms on the right hand side of (3.35) can be estimated similar to the proof of Lemma 3.1. Hence, one has

$$
\int_0^t \|Z^\alpha z^{k+1} B'(s)\|_q^2 ds + \epsilon^2 \int_0^t \|Z^\alpha z^{k+2} u'(s)\|_q^2 ds
\
+ 2\epsilon \int_0^t \int_\Omega Z^\alpha z^{k+1} B' \cdot Z^\alpha z^{k+2} u' dx ds
\
\lesssim \int_0^t \|\partial_t z^k \nabla p'(s)\|_q^2 + \epsilon^2 \|z^k \nabla u'(s)\|_{q+1}^2 ds
\
+ \sum_{i=0}^k \left\{ \left( 1 + \sup_{0 \leq s \leq t} \|z^{k-i} (u', B')(s)\|_{1, \infty} \right) \int_0^t \|z^i (u', B')(s)\|_{q+1}^2 ds \right\}.
$$

(3.36)
(2). Now it remains to control the last term on the left hand side of (3.36). For this, we apply the operator $Z^\alpha \partial_z^{k+1}$ to the equations for $\mathbf{B}'$ in (1.6), and obtain

$$
\partial_t Z^\alpha \partial_z^{k+1} \mathbf{B}' - Z^\alpha \partial_z^{k+2} \mathbf{u}' + (\mathbf{u}' \cdot \nabla) Z^\alpha \partial_z^{k+1} \mathbf{B}' = C^2 + C^3,
$$

where

$$
C^2 = -[Z^\alpha, \mathbf{u}' \cdot \nabla] \partial_z^{k+1} \mathbf{B}' - C^2_k \mathbf{B}' + Z^\alpha (\partial_z \mathbf{u}' \cdot \nabla) \partial_z^{k+1} \mathbf{B}' + Z^\alpha (\mathbf{B}' \cdot \nabla) \partial_z^{k+1} \mathbf{u}' + Z^\alpha (\partial_z \mathbf{B}' - \nabla \partial_z^{k+1} \mathbf{u}')
$$

and

$$
C^3 = -k \sum_{t=1}^{k+1} \{ Z^\alpha (\partial_z \mathbf{u}' \cdot \nabla) \partial_z^{k+1-i} \mathbf{B}' \} + \sum_{t=1}^{k+1} \{ Z^\alpha (\partial_z \mathbf{B}' - \nabla \partial_z^{k+1-i} \mathbf{u}') \}.
$$

Multiplying (3.37) by $2 \epsilon Z^\alpha \partial_z^{k+1} \mathbf{B}'$ and integrating the resulting equations over $[0, t] \times \Omega$, one has by integration by parts,

$$
\epsilon \| Z^\alpha \partial_z^{k+1} \mathbf{B}'(t) \|^2 - \epsilon \| Z^\alpha \partial_z^{k+1} \mathbf{B}'(0) \|^2 + 2 \epsilon \int_0^t \int_\Omega Z^\alpha \partial_z^{k+1} \mathbf{B}' \cdot Z^\alpha \partial_z^{k+2} \mathbf{u}' dx ds
$$

$$
\leq \frac{1}{2} \int_0^t \| Z^\alpha \partial_z^{k+1} \mathbf{B}'(s) \|^2 ds + C \epsilon^2 \int_0^t \| C^2(s) \|^2 + \| C^3(s) \|^2 ds.
$$

Note that one can estimate the four terms in $C^2$ similar to (3.26), (3.27), (3.28) and (3.31) respectively. Actually, it holds that

$$
\int_0^t \| C^2(s) \|^2 ds \leq 3 B \int_0^t \| \partial_z \mathbf{u}'(s) \|^2 + \| \partial_z \mathbf{B}'(s) \|^2 + \| \partial_z^{k+1} \mathbf{u}'(s) \|^2 + \| \partial_z^{k+1} \mathbf{B}'(s) \|^2 ds.
$$

Then, similar to (3.17) we can estimate $C^3$ as follows:

$$
\int_0^t \| C^3(s) \|^2 ds \leq \| \mathbf{B}'(t) \|_{H^\infty} \int_0^t \| \partial_z^{k+1} \mathbf{u}'(s) \|^2 + \| \partial_z^{k+1} \mathbf{B}'(s) \|^2 ds + \sum_{i=1}^k \int_0^t \| \partial_z^{k+1-i} \mathbf{u}'(s) \|^2 + \| \partial_z^{k+1-i} \mathbf{B}'(s) \|^2 ds.
$$

Plugging (3.39) and (3.40) into (3.38), one has

$$
\epsilon \| Z^\alpha \partial_z^{k+1} \mathbf{B}'(t) \|^2 - \epsilon \| Z^\alpha \partial_z^{k+1} \mathbf{B}'(0) \|^2 + 2 \epsilon \int_0^t \int_\Omega Z^\alpha \partial_z^{k+1} \mathbf{B}' \cdot Z^\alpha \partial_z^{k+2} \mathbf{u}' dx ds
$$

$$
\leq \frac{1}{2} \int_0^t \| Z^\alpha \partial_z^{k+1} \mathbf{B}'(s) \|^2 ds + C \epsilon^2 \int_0^t \| \mathbf{B}'(s) \|^2_{q+1} ds
$$

$$
+ C \epsilon^2 \sum_{i=1}^k \left( \left[ 1 + \sup_{0 \leq s \leq t} \| \partial_z^{k+1-i} \mathbf{u}'(s) \|_{H^\infty} \right] \int_0^t \| \partial_z^{k+1-i} \mathbf{B}'(s) \|^2_{q+1} ds \right).
$$

(3) Now, by combining (3.36) and (3.41) and taking the sum over $|\alpha| \leq q$, it follows that

$$
\epsilon \| \partial_z^{k+1} \mathbf{B}'(t) \|^2 + \int_0^t \| \partial_z^{k+1} \mathbf{B}'(s) \|^2 ds + \epsilon^2 \int_0^t \| \partial_z^{k+2} \mathbf{u}'(s) \|^2 ds
$$
This completes the proof of the lemma.

Combining Lemmas 3.1 with 3.3 we are in the position to verify Proposition 3.2.

**Proof of Proposition 3.2.** We combine (3.13) with (3.33) to get that for \( k \geq 1, \)

\[
\epsilon \| \partial_x^{k+1} B'(0) \|_q^2 + \int_0^t \| \partial_x^{k+1} \nabla p'(s) \|_q^2 + \epsilon^2 \| \partial_x^{k+1} u'(s) \|_{q+1}^2 ds
\]

\[
+ \epsilon^2 \sup_{0 \leq s \leq t} \| \nabla u'(s) \|_{1, \infty} \int_0^t \| \partial_x^{k+1} u'(s) \|_{q+1}^2 ds
\]

\[
+ \left( 1 + \sup_{0 \leq s \leq t} \| \partial_x^k (u', B') (s) \|_{1, \infty}^2 + \epsilon^2 \sup_{0 \leq s \leq t} \| \partial_x^{k+1} u'(s) \|_{q+1}^2 \right) \int_0^t \| (u', B') (s) \|_{q+1}^2 ds
\]

\[
+ \sum_{i=1}^k \left\{ \left( 1 + \sup_{0 \leq s \leq t} \| \partial_x^{k-i} (u', B') (s) \|_{1, \infty}^2 + \epsilon^2 \sup_{0 \leq s \leq t} \| \partial_x^{k+1-i} (u', B') (s) \|_{2, \infty}^2 \right) \right\} \times \int_0^t \| \partial_x^i (u', B') (s) \|_{q+1}^2 ds.
\]

This completes the proof of the lemma.

In fact, to prove the above Proposition, we can first follow the process in [15] to decompose

\[ p^* = p_1^* + p_2^*, \]
where \( p_1 \) solves
\[
\begin{aligned}
\Delta p_1 &= \nabla \cdot F, \quad \text{in } \Omega, \\
\partial_z p_1|_{z=0} &= 0,
\end{aligned}
\quad (4.3)
\]
and \( p_2 \) solves
\[
\begin{aligned}
\Delta p_2 &= 0, \quad \text{in } \Omega, \\
\partial_z p_2|_{z=0} &= -\nabla_h \cdot (\epsilon \partial_z u_h' + B_h')(0),
\end{aligned}
\quad (4.4)
\]

Then as in [15], we can get the explicit representation of the solutions to (4.3) and (4.4) in Fourier space. More precisely, by taking the Fourier transform in the \((x, y)\) variable, one can obtain
\[
\hat{p}_1(\xi, z) = \int_0^\infty G_\xi(z, z') \hat{F}(\xi, z') dz', \quad \hat{p}_2(\xi, z) = e^{-|\xi|z} \left( \frac{i\xi}{|\xi|}, \epsilon \partial_z u_h' + B_h' \right)(\xi, 0),
\quad (4.5)
\]
where \( \xi = (\xi_1, \xi_2) \) and
\[
G_\xi(z, z') = \begin{cases} 
- e^{-|\xi|z'} \cosh(|\xi|z) 
\left( \frac{i\xi}{|\xi|}, 1 \right), & z < z', \\
- e^{-|\xi|z} \left( \frac{i\xi}{|\xi|} \cosh(|\xi|z'), -\sinh(|\xi|z') \right), & z > z'.
\end{cases}
\]

Firstly, it was proved in [15] that \( \|\nabla p_1\| \lesssim \|F\|^2 \), and
\[
\|\nabla p_1\|_q \lesssim \|F\|_q + \|\nabla \cdot F\|_{q-1}, \quad \|\partial_z^2 p_1\|_{q-1} \lesssim \|\nabla \cdot F\|_{q-1}, \quad \forall q \geq 1. \quad (4.6)
\]

Then, by using \( \partial_z^2 p_1 = \nabla \cdot F - \Delta_h p_1 \) it is easy to obtain that
\[
\|\partial_\xi \nabla p_1\|_q \lesssim \|\partial_\xi p_1\|_{q+1} + \|\partial_\xi^2 p_1\|_q \lesssim \|F\|_{q+1} + \|\nabla \cdot F\|_q,
\]
and for \( k \geq 2 \),
\[
\|\partial_\xi^k \nabla p_1\|_q \lesssim \|\partial_\xi^{k-1} (\partial_\xi p_1)\|_{q+1} + \|\partial_\xi^{k-1} (\nabla \cdot F - \Delta_h p_1)\|_q \lesssim \|\partial_\xi^{k-1} (\nabla \cdot F)\|_q + \|\partial_\xi^{k-1} \nabla p_1\|_{q+1}.
\]

Thus, by induction one has
\[
\|\partial_\xi^k \nabla p_1\|_q \lesssim \sum_{i=0}^{k-1} \|\partial_\xi^i (\nabla \cdot F)\|_{q+k-i-1} + \|F\|_{q+k}, \quad k \geq 1. \quad (4.7)
\]

Next, for \( p_2 \) we have from (4.5),
\[
\nabla p_2(\xi, z) = e^{-|\xi|z} \left( \frac{i\xi}{|\xi|} \cdot (\epsilon \partial_z u_h' + B_h') \right)(\xi, 0) \left( \frac{i\xi}{|\xi|}, -1 \right).
\]

Then by the Plancherel identity, this implies
\[
\|\nabla p_2\|_{H^k}^2 \lesssim \|\epsilon \partial_z u_h' + B_h' \|_{H_{k+\frac{1}{2}}^*}^2 \lesssim \epsilon^2 \|\partial_\xi^2 u_h'\|_k \|\partial_\xi u_h'\|_k + \|\partial_\xi B_h'\|_k \|B_h'\|_k,
\]
where we have used the trace inequality (2.4) in the second inequality. Thus, one has that for any integers \( k, q \geq 0 \)
\[
\|\partial_\xi^k \nabla p_2\|_q^2 \lesssim \epsilon^2 \|\partial_\xi^2 u_h'\|_{k+q} \|\partial_\xi u_h'\|_{k+q} + \|\partial_\xi B_h'\|_{k+q} \|B_h'\|_{k+q}. \quad (4.8)
\]

To sum up, we are now ready to conclude the proof of Proposition 4.1.
Proof of Proposition 4.1. By virtue of (4.6) and (4.7),
\[ \|\nabla p^t_1\|_{m-1} + \|\partial_t \nabla p^t_1\|_{m-2} \lesssim \|F\|_{m-1} + \|\nabla \cdot F\|_{m-2}. \]
Recall that \( F = -u' \cdot \nabla u' + B' \cdot \nabla B' \), with (2.16), we get
\[ \int_0^t \|F(s)\|_{m-2}^2 ds \lesssim \sum_{0 \leq s \leq t} \|\nabla (u', B')(s)\|_{1, \infty}^2 \int_0^t \|(u', B')(s)\|_{m-1}^2 ds. \]
To estimate \( \nabla \cdot F \), by noting that \( \nabla \cdot u' = \nabla \cdot B' = 0 \), we have
\[ \nabla \cdot F = \sum_{i=1}^2 \{- (\partial_i u' \cdot \nabla) u'_i + (\partial_i B' \cdot \nabla) b'_i\} + [- (\partial_2 u' \cdot \nabla) u'_3 + (\partial_2 B' \cdot \nabla) b'_3] \triangleq I_1 + I_2, \]
with \( \partial_1 = \partial_x \) and \( \partial_2 = \partial_y \). We first use (2.16) to estimate \( I_1 \) as
\[ \int_0^t \|I_1\|_{m-2}^2 ds \lesssim \sum_{0 \leq s \leq t} \|\nabla h(u', B')(s)\|_{1, \infty}^2 \int_0^t \|(u', B')(s)\|_{m-1}^2 ds + \sup_{0 \leq s \leq t} \|(u', B')(s)\|_{2, \infty}^2 \int_0^t \|\nabla_h(u', B')(s)\|_{m-1}^2 ds \]
\[ \lesssim \sup_{0 \leq s \leq t} \|(u', B')(s)\|_{2, \infty}^2 \int_0^t \|(u', B')(s)\|_{m}^2 ds. \]
For \( I_2 \), we rewrite
\[ I_2 = - (\partial_i u'_i \cdot \nabla) u'_3 + (\partial_2 B' \cdot \nabla) b'_3 - (\partial_2 u'_3)^2 + (\partial_2 b'_3)^2 \]
\[ = - \nabla h u'_3 \cdot \partial_x u'_i + \nabla h b'_3 \cdot \partial_x B'_3 - (\nabla h \cdot u'_3)^2 + (\nabla h \cdot B'_3)^2 \triangleq I_{2,1} + I_{2,2}. \]
Then by (2.13) and divergence-free conditions, we have
\[ \int_0^t \|I_{2,1}\|_{m-2}^2 ds \lesssim \|\partial_2 \nabla_h (u'_3, b'_3)\|_{L^\infty_t} \int_0^t \|(u'_3, B'_3)(s)\|_{m-1}^2 ds \]
\[ + \sup_{0 \leq s \leq t} \|(u'_3, B'_3)(s)\|_{2, \infty}^2 \int_0^t \|\partial_2 \nabla_h (u'_3, b'_3)(s)\|_{m-1}^2 ds \]
\[ \lesssim \sup_{0 \leq s \leq t} \|(u', B')(s)\|_{2, \infty}^2 \int_0^t \|(u', B')(s)\|_{m}^2 ds. \]
Also, by (2.2) one has
\[ \int_0^t \|I_{2,2}\|_{m-2}^2 ds \lesssim \|\nabla_h (u'_3, B'_3)\|_{L^\infty_t} \int_0^t \|\nabla_h (u'_3, B'_3)(s)\|_{m-2}^2 ds \]
\[ \lesssim \sup_{0 \leq s \leq t} \|(u', B')(s)\|_{2, \infty}^2 \int_0^t \|(u', B')(s)\|_{m-1}^2 ds. \]
Combining the above three estimates yields
\[ \int_0^t \|I_{2}\|_{m-2}^2 ds \lesssim \sup_{0 \leq s \leq t} \|(u', B')(s)\|_{2, \infty}^2 \int_0^t \|(u', B')(s)\|_{m}^2 ds, \]
which along with (4.10) implies
\[ \int_0^t \|\nabla \cdot F(s)\|_{m-2}^2 ds \lesssim \sup_{0 \leq s \leq t} \|(u', B')(s)\|_{2, \infty}^2 \int_0^t \|(u', B')(s)\|_{m}^2 ds. \]
Thus, it follows from (4.9) and (4.11) that
\[
\int_0^t \| \nabla p_1^*(s) \|_{m-1}^2 + \| \partial_z \nabla p_1^*(s) \|_{m-2}^2 ds \\
\lesssim \sup_{0 \leq s \leq t} \| (u^*, B^*)'(s) \|_{2, \infty}^2 \int_0^t \| (u^*, B^*)'(s) \|_{m}^2 ds.
\] (4.12)

On the other hand, one has by (4.8) that
\[
\| \nabla p_2^* \|_{m-1}^2 + \| \partial_z \nabla p_2^* \|_{m-2}^2 \\
\lesssim \epsilon^2 \| \partial_z^2 u_h^* \|_{m-1} \| \partial_z u_h' \|_{m-1} + \| \partial_z B_h^* \|_{m-1} \| B_h' \|_{m-1},
\] (4.13)
which along with (4.12) gives (4.2). And this completes the proof. \( \square \)

5. **Proof of Theorem 3.1.** We are now ready to prove Theorem 3.1. After that it is not difficult to verify Theorem 1.1 via the standard arguments. Firstly from Proposition 3.2 one has
\[
\epsilon \| \partial_z B^*(t) \|_{m-1}^2 + \int_0^t \| \partial_z (u^*, B^*)'(s) \|_{m-1}^2 + \epsilon^2 \| \partial_z^2 u^* \|_{m-1}^2 ds \\
\lesssim \epsilon \| \partial_z B^*(0) \|_{m-1}^2 + \int_0^t \| \nabla p'(s) \|_{m-1}^2 ds + Q_0(t) \int_0^t \| (u^*, B^*)'(s) \|_{m}^2 ds.
\] (5.1)

Then, substituting (4.2) into (5.1) yields
\[
\epsilon \| \partial_z B^*(t) \|_{m-1}^2 + \int_0^t \| \partial_z (u^*, B^*)'(s) \|_{m-1}^2 + \epsilon^2 \| \partial_z^2 u^* \|_{m-1}^2 ds \\
+ \int_0^t \| \nabla p'(s) \|_{m-1}^2 + \| \partial_z \nabla p'(s) \|_{m-2}^2 ds \\
\lesssim \epsilon \| \partial_z B^*(0) \|_{m-1}^2 + \int_0^t \epsilon^2 \| \partial_z^2 u_h^* \|_{m-1} \| \partial_z u_h' \|_{m-1} + \| \partial_z B_h^* \|_{m-1} \| B_h' \|_{m-1} ds \\
+ Q_0(t) \int_0^t \| (u^*, B^*)'(s) \|_{m}^2 + \epsilon^2 \| \nabla u^* \|_{m}^2 ds.
\] (5.2)

It along with Young’s inequality implies that for sufficiently small \( \epsilon \),
\[
\epsilon \| \partial_z B^*(t) \|_{m-1}^2 + \int_0^t \| \partial_z (u^*, B^*)'(s) \|_{m-1}^2 + \epsilon^2 \| \partial_z^2 u^* \|_{m-1}^2 ds \\
+ \int_0^t \| \nabla p'(s) \|_{m-1}^2 + \| \partial_z \nabla p'(s) \|_{m-2}^2 ds \\
\lesssim \epsilon \| \partial_z B^*(0) \|_{m-1}^2 + Q_0(t) \int_0^t \| (u^*, B^*)'(s) \|_{m}^2 + \epsilon^2 \| \nabla u^* \|_{m}^2 ds.
\] (5.3)
We substitute (5.3) into (3.2) to obtain
\[
\| (\mathbf{u}^r, \mathbf{B}^r)(t) \|_{m}^{2} + \int_{0}^{t} \epsilon \| \nabla \mathbf{u}^r(s) \|_{m}^{2} ds \\
\lesssim \| (\mathbf{u}^r, \mathbf{B}^r)(0) \|_{m}^{2} + \epsilon \| \partial_2 \mathbf{B}^r(0) \|_{m-1}^{2} Q_0(t) \\
+ Q_1(t)(1 + Q_0(t)) \int_{0}^{t} \| (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \epsilon^2 \| \nabla \mathbf{u}^r(s) \|_{m}^{2} ds,
\]
(5.4)
where we have used the fact
\[
1 + \| (\mathbf{u}^r, \mathbf{B}^r) \|_{W^{1,\infty}_{1}} \leq Q_1(t).
\]
Similarly, plugging (5.3) into (5.2) gives
\[
\epsilon \| \partial_2^2 \mathbf{B}^r(t) \|_{m-2}^{2} + \int_{0}^{t} \| \partial_2^2 (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m-2}^{2} ds + \epsilon^2 \| \partial_2^3 \mathbf{u}^r(s) \|_{m-2}^{2} ds \\
\lesssim \epsilon \| \partial_2^2 \mathbf{B}^r(0) \|_{m-2}^{2} + \epsilon \| \partial_2 \mathbf{B}^r(0) \|_{m-1}^{2} Q_1(t) \\
+ Q_1(t)(1 + Q_0(t)) \int_{0}^{t} \| (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \epsilon^2 \| \nabla \mathbf{u}^r(s) \|_{m}^{2} ds.
\]
(5.5)
Therefore, by recalling from (1.12) that
\[
N_m(t) = \sup_{0 \leq s \leq t} \| (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \int_{0}^{t} \| \partial_2 (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m-1}^{2} + \| \partial_2^2 (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m-2}^{2} ds \\
+ \epsilon \int_{0}^{t} \| \nabla \mathbf{u}^r(s) \|_{m}^{2} ds + \epsilon^2 \int_{0}^{t} \| \partial_2^2 \mathbf{u}^r(s) \|_{m-1}^{2} + \| \partial_2^3 \mathbf{u}^r(s) \|_{m-2}^{2} ds,
\]
we combine (5.3)-(5.5) to obtain
\[
N_m(t) + \epsilon \| (\partial_2 \mathbf{B}^r(t)) \|_{m-1}^{2} + \| \partial_2^2 \mathbf{B}^r(t) \|_{m-2}^{2} \int_{0}^{t} \| \nabla \mathbf{u}^r(s) \|_{m}^{2} + \| \nabla \mathbf{v}^r(s) \|_{m}^{2} ds \\
\lesssim \| (\mathbf{u}^r, \mathbf{B}^r)(0) \|_{m}^{2} + \epsilon \| \partial_2^2 \mathbf{B}^r(0) \|_{m-2}^{2} + \epsilon \| \partial_2 \mathbf{B}^r(0) \|_{m-1}^{2} Q_1(t) \\
+ Q_1(t)(1 + Q_0(t)) \int_{0}^{t} \| (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \epsilon^2 \| \nabla \mathbf{u}^r(s) \|_{m}^{2} ds \\
\lesssim \| (\mathbf{u}^r, \mathbf{B}^r)(0) \|_{m}^{2} + \epsilon \| \partial_2^2 \mathbf{B}^r(0) \|_{m-2}^{2} + \epsilon \| \partial_2 \mathbf{B}^r(0) \|_{m-1}^{2} Q_1(t) \\
+ Q_1(t)(1 + Q_0(t)) \cdot (t + \epsilon) N_m(t).
\]
(5.6)
Now it remains to establish the \(L^\infty\) estimates on \(Q_0(t)\) and \(Q_1(t)\). By (2.5) it hold for \(q \geq 0\),
\[
\sup_{0 \leq s \leq t} \| f(s) \|_{q+\infty} \lesssim \| f(0) \|_{q+2} + \| \partial_s f(0) \|_{q+1} + \int_{0}^{t} \| f(s) \|_{q+3} + \| \partial_s f(s) \|_{q+2} ds.
\]
Thus, recall the definition of \(Q_k, k = 0, 1\) given in (3.10), one has that
\[
Q_0(t) \leq Q_1(t) \\
\lesssim 1 + \| (\mathbf{u}^r, \mathbf{B}^r)(0) \|_{m}^{2} + \| \partial_2 (\mathbf{u}^r, \mathbf{B}^r)(0) \|_{m}^{2} + \| \partial_2^2 (\mathbf{u}^r, \mathbf{B}^r)(0) \|_{m}^{2} + \epsilon^2 \| \partial_2^3 \mathbf{u}^r(0) \|_{m}^{2} \\
+ \int_{0}^{t} \| (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \| \partial_2 (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \| \partial_2^2 (\mathbf{u}^r, \mathbf{B}^r)(s) \|_{m}^{2} + \epsilon^2 \| \partial_2^3 \mathbf{u}^r(s) \|_{m}^{2} ds.
\]
(5.7)
Thus, we know that for \(m \geq 6\),
\[
Q_0(t) \leq Q_1(t) \leq C(M_0) N_m(t)
\]
Finally we combine (5.6) with (5.7) to obtain

\[ N_m(t) + \epsilon \left( \| \partial_z \mathbf{B}'(t) \|_{m-1}^2 + \| \partial_z^2 \mathbf{B}'(t) \|_{m-2}^2 \right) + \int_0^t \| \nabla p' \|_{m-1}^2 + \| \partial_z \nabla p'(s) \|_{m-2}^2 ds \]

\[ \lesssim \mathcal{P}(M_0) + (t + \epsilon) \mathcal{P}(N_m(t)). \]  

(5.8)

And this completes the proof of the a priori estimate (3.1).

REFERENCES

[1] H. Abidi and P. Zhang, On the global solution of a 3-D MHD system with initial data near equilibrium, Commun. Pure Appl. Math., 70 (2017), 1509–1561.

[2] Y. Cai and Z. Lei, Global well-posedness of the incompressible magnetohydrodynamics, Arch. Ration. Mech. Anal., 228 (2018), 969–993.

[3] J. Y. Chemin, D. S. McCormick, J. C. Robinson and J. L. Rodrigo, Local existence for the non-resistive MHD equations in Besov spaces, Adv. Math., 286 (2016), 1–31.

[4] Q. Duan, Y. Xiao and Z. Xin, On the vanishing dissipation limit for the incompressible MHD equations on bounded domains, Preprint, 2020.

[5] C. Fefferman, D. McCormick, J. Robinson and J. Rodrigo, Higher order commutator estimates and local existence for the non-resistive MHD equations and related models, J. Funct. Anal., 267(2014), 1035–1056.

[6] C. Fefferman, D. McCormick, J. Robinson and J. Rodrigo, Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces, Arch. Ration. Mech. Anal., 223 (2017), 677–691.

[7] O. Guès, Problème mixte hyperbolique quasi-linéaire caractéristique, Commun. Partial Differ. Equ., 15 (1990), 599–645.

[8] L. B. He, L. Xu and P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, Ann. PDE, 5 (2018), 105pp.

[9] Z. Lei, On axially symmetric incompressible magnetohydrodynamics in three dimensions, J. Differential Equations, 259 (2015), 3202–3215.

[10] J. Li, W. Tan and Z. Yin, Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces, Adv. Math., 317 (2017), 786–798.

[11] F. Liu, L. Xu, P. Zhang, Global small solutions of 2-D incompressible MHD system, J. Differential Equations, 259 (2015), 5440–5485.

[12] C.-J. Liu, D. Wang, F. Xie and T. Yang, Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces, J. Funct. Anal., 279 (2020), 108637, 45pp.

[13] C.-J. Liu, F. Xie and T. Yang, MHD boundary layers in Sobolev spaces without monotonicity, I. Well-posedness theory, Commun. Pure Appl. Math., 72 (2019), 63–121.

[14] C.-J. Liu, F. Xie and T. Yang, Justification of Prandtl ansatz for MHD boundary layer, SIAM J. Math. Anal., 51 (2019), 2748–2791.

[15] N. Masmoudi and F. Rousset, Uniform regularity for the Navier-Stokes equation with Navier boundary condition, Arch. Ration. Mech. Anal., 203 (2012), 529–575.

[16] N. Masmoudi and F. Rousset, Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations, Arch. Ration. Mech. Anal., 223 (2017), 301–417.

[17] M. Paddick, The strong inviscid limit of the isentropic compressible Navier-Stokes equations with Navier boundary conditions, Discret. Contin. Dyn. Syst., 36 (2016), 2673–2709.

[18] X. Ren, J. Wu, Z. Xiang, Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, J. Funct. Anal., 267 (2014), 503–541.

[19] R. Wan, On the uniqueness for the 2D MHD equations without magnetic diffusion, Nonlinear Anal. Real World Appl., 30 (2016), 32–40.

[20] Y. Wang, Uniform regularity and vanishing dissipation limit for the full compressible Navier-Stokes system in three dimensional bounded domain, Arch. Ration. Mech. Anal., 221 (2015), 4123–4191.

[21] Y. Wang, Z. P. Xin and Y. Yong, Uniform regularity and vanishing viscosity limit for the compressible Navier-Stokes with general Navier-slip boundary conditions in 3-dimensional domains, SIAM J. Math. Anal., 47 (2015), 4123–4191.

[22] D. Wei and Z. Zhang, Global well-posedness of the MHD equations in a homogeneous magnetic field, Anal. PDE, 10 (2017), 1361–1406.
[23] Y. L. Xiao, Z. P. Xin and J. H. Wu, Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition, *J. Funct. Anal.*, **257** (2009), 3375–3394.

[24] L. Xu, P. Zhang, Global small solutions to three-dimensional incompressible magnetohydrodynamical system, *SIAM J. Math. Anal.*, **47** (2015), 26–65.

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