FAMILIES OF $A_\infty$ ALGEBRAS AND HOMOTOPY GROUP ACTIONS

EMMA SMITH ZBARSKY

Abstract. We define homotopy group actions in terms of families of $A_\infty$ algebras indexed by a manifold $M$. We give explicit formulae for the $A_\infty$ morphism induced by a path on the manifold and for the $A_\infty$ homotopy corresponding to a pair of homotopic paths. Finally, we compute examples for finite groups and finitely generated free nonabelian groups and determine that every homotopy group action by a finite group is homotopic to a strict group action.

1. Introduction

As a postdoctoral fellow, James Stasheff studied group-like topological spaces in [28] building on work by Sugawara in [30]. He began by defining the concept of an $A_\infty$ space. Initially these ideas were found useful in homotopy theory. Generalizations were constructed including Boardman and Vogt’s machinery of topological PROPs [4], May’s introduction of operads [20] and Adams’ discussion of infinite loop spaces [2]. In the nineties, $A_\infty$ structures were found to have significant presence in deformation theory, topology, and physics, with [10], [7], [29], and [23], while Stasheff’s birthday conference contributed [21]. Building off of Fukaya’s work, Kontsevich conjectured homological mirror symmetry in a talk at the 1994 ICM [15]. Several special cases of homological mirror symmetry have since been proven, notably by Polishchuk and Zaslow in [24], Seidel in [25, 26] and Abouzaid and Smith in [1]. Partial proofs in other cases have been given by Kontsevich and Soibelman in [17], and Fukaya in [8]. In a Fukaya-Seidel category, because the $A_\infty$ structure arises from intersecting Lagrangians on a symplectic manifold it is natural to wonder how group actions on the manifold may affect the $A_\infty$ structure.

1.1. Definitions, conventions, and notation. An $A_\infty$ algebra $A$ consists of a graded $K$-module $V$ together with a sequence of maps $\mu^k_A : V^\otimes k \to V[2-k]$, $k \geq 0$ that satisfy the sequence of relations

$$\sum_{k+r-1=n} \sum_{j=0}^{k-1} (-1)^{\tilde{\Phi}_j} \mu^k_A(a_1 \otimes \cdots \alpha_j \otimes \mu^r_A(a_{j+1} \otimes \cdots \otimes a_{j+r}) \otimes a_{j+r+1} \otimes \cdots \otimes a_n) = 0,$$

for $n \geq 0$, where $\tilde{\Phi}_j = \sum_{i=1}^j (|a_i| - 1)$. Now let $B$ be an $A_\infty$ algebra with underlying $K$-module $W$. An $A_\infty$ morphism $\mathcal{F} : A \to B$ consists of a sequence of maps

$$\mathcal{F}^n : V^\otimes n \to W[1-n], \quad n \geq 1$$

Key words and phrases. cohomology, homotopy, group, $A_\infty$, Lie algebra.

The author was supported in part by a Bell Labs Graduate Research Fellowship.
which satisfy the corresponding sequence of relations

\[
\sum_{s_k \geq 1, 1 \leq r \leq m} \mu^r_k (\mathcal{F}^{s_1}(a_1, \ldots, a_{s_1}), \ldots, \mathcal{F}^{s_r}(a_{n-s_r+1}, \ldots, a_n)) = 
\]

\[
= \sum_{m,j} (-1)^{m,j} \mathcal{F}^{n-m+1}(a_1, \ldots, a_j, \mu^{m}_A(a_{j+1}, \ldots, a_{j+m}), a_{j+m+1}, \ldots, a_n)
\]

where \(1 \leq m \leq n\) and \(0 \leq j \leq n - m\). Given two \(A_\infty\) morphisms \(\mathcal{F} : A \to B\) and \(\mathcal{G} : A \to A\), the composition of the two morphisms is given by the sequence of maps

\[
(\mathcal{F} \circ \mathcal{G})(\bar{a}) = \sum_{n=0}^{n-1} (-1)^{||\mathcal{G}^m||} (\mathcal{F}^n(1 \otimes \mathcal{G}^m \otimes 1 \otimes \mathcal{G}^{m-n}))(\bar{a})
\]

for \(\bar{a} \in V^{\otimes n+m-1}\) and \(||\mathcal{G}^m||\) the shifted degree of the map \(\mathcal{G}^m\). We say that two \(A_\infty\) morphisms \(\mathcal{F}\) and \(\mathcal{G}\) between \(A\) and \(B\) are \(A_\infty\) homotopic if there is a sequence of maps

\[
T^n : V^{\otimes n} \to W[-n], \quad n \geq 1
\]

which satisfy the sequence of relations

\[
\mathcal{F}^n(\bar{a}) - \mathcal{G}^n(\bar{a}) = \sum_{0 \leq j \leq n-r} (-1)^{j} \mu^r_B(\mathcal{F}^{s_1} \otimes \ldots \otimes \mathcal{F}^{s_j} \otimes T^{s_{j+1}} \otimes \mathcal{G}^{s_{j+2}} \otimes \ldots \otimes \mathcal{F}^{s_r})(\bar{a})
\]

where \(\bar{a} = (|a_1| + \ldots + |a_{s_1} + \ldots + a_{s_j}| - s_1 - \ldots - s_j), S = \sum_{k=1}^{j+r} s_k\), and \(\bar{a} \in V^{\otimes n}\). These definitions follow the sign conventions of \([27]\) for \(A_\infty\) objects.

When there is no danger of confusion we shall omit the subscript on the \(A_\infty\) composition maps. We shall always use \(A\) and \(B\) for \(A_\infty\) algebras while \(\mathcal{F}\) and \(\mathcal{G}\) will always be \(A_\infty\) morphisms. Therefore \(\mathcal{F}^k\) will denote the \(k\)th term of \(\mathcal{F}\) and hence be a map \(\mathcal{F}^k : V^{\otimes k} \to V[1-k]\). Such a grading shift means that given \(a_1 \otimes \ldots \otimes a_k \in V^{\otimes k}\), where \(|a_i|\) denotes the grading of \(a_i\), we have

\[
|\mathcal{F}^k(a_1 \otimes \ldots \otimes a_k)| = \left( \sum_{i=1}^{k} |a_i| \right) + 1 - k.
\]

For brevity, we shall write multilinear combinations of multilinear maps with commas rather than tensors and omit the input objects when working with equalities. We shall denote the differential graded algebra (dga) \(V\) together with \(\mu^1_A\) and \(\mu^2_A\) as \(A\).

In Section \([2]\) we define families of \(A_\infty\) algebras indexed by a manifold \(M\) as solutions to the Maurer-Cartan equation on \(\Omega^*(M; \mathfrak{g})\) where \(\mathfrak{g}\) is a particular locally trivial sheaf of Lie algebras on \(M\) built out of the Hochschild cochain complexes of the underlying dgas of the \(A_\infty\) algebras. Theorem \([23]\) gives an explicit formulation for the \(A_\infty\) morphism \(\mathcal{F}_x^y : A_x \to A_y\) for \(x, y \in M\) given a path \(p : I \to M\) connecting \(x\) and \(y\). In Section \([4]\) after showing that differential homotopies correspond to classical homotopies we prove Theorem \([14]\) which gives an explicit form for a differential homotopy relating families of \(A_\infty\) morphisms \(\mathcal{F}_i : A_i \to A_i\).

In Section \([5]\) we perform calculations. The goal is to compute the cohomology of the total complex \(\Omega^*(M; \mathfrak{g})\) in two special cases. After a discussion of sheaf
cohomology we argue that for our purposes we can consider $BG$ as a manifold for $G$ finite or finitely generated free nonabelian. Computation shows that in these cases the spectral sequence collapses at $E^2$ as we discuss in Section 5.3 for a finite group and Section 5.4 for a finitely generated free nonabelian group. This leads up to Theorem 5.7 which states that for a finite group $\Gamma$, every homotopy $\Gamma$ action on an $A_\infty$ algebra $\mathcal{A}$ has class representatives $\mathcal{F}_g : \mathcal{A} \to \mathcal{A}$ for all $g \in \Gamma$ which comprise a strict action.

The author would like to thank Paul Seidel for his insightful questions and suggestions.

2. The differential graded Lie algebra $\Omega^*(M; g)$

Let $V$ be a $\mathbb{Z}$-graded vector space and $M$ a differentiable manifold. Let $\nabla$ denote a local system on $M$ with fibres isomorphic to $V$.

Define

$$g_x = \bigoplus_{n=0}^\infty \text{Hom}(V_x \otimes^n V_x[1-n]).$$

Now $g_x$ is a $\mathbb{Z}$-graded Lie algebra under the Gerstenhaber bracket. Graded Lie algebras are presented to good effect in [9, 11]. We have chosen the indexing so that $g^1_x$ contains the space $\bigoplus_{n=0}^\infty \text{Hom}(V_x \otimes^n V_x[2-n])$ of possible $A_\infty$ structure maps on $V_x$ including $\delta_x^0$. Let $g$ denote the corresponding locally trivial sheaf of Lie algebras on $M$. Denote the algebra of $g$ valued differential forms on $M$ by $\Omega^*(M; g)$.

**Proposition 2.1.** $\Omega^*(M; g)$ is a differential $\mathbb{Z}$-graded Lie algebra with differential $d_\nabla$ and Lie bracket induced by the Gerstenhaber bracket on $g$ and the wedge product of differential forms.

**Proof.** First, because $\nabla$ is a flat connection we see that $d_\nabla^2 = 0$ [22, Appx C]. Let $L^{m+k} = \Omega^m(M; g^k)$ be the $(m+k)^{th}$ graded component for $m \geq 0$ and $k \in \mathbb{Z}$. Then $d_\nabla(L^{m+k}) \subset \Omega^{m+1}(M; g^{k+1})$. Let $a = (\omega \otimes \alpha) \in \Omega^m(M; g^k)$ and $b = (\theta \otimes \beta) \in \Omega^n(M; g^l)$, then $d_\nabla[a, b] = [d_\nabla(a), b] + (-1)^{|a|}[a, d_\nabla(b)]$ by the Koszul rule of signs where $|a|$ denotes the total degree of $a$. Since the bracket is induced by the Gerstenhaber bracket on $g$ together with the wedge product of differential forms we see that

$$[a, b] = \omega \wedge \theta \otimes [\alpha, \beta]$$

$$= (-1)^{kn} \omega \wedge \theta \otimes \alpha \circ \beta - (-1)^{|\alpha|} \omega \wedge \theta \otimes \beta \circ \alpha$$

$$= \omega \wedge \theta \otimes (-1)^{nk} \alpha \circ \beta - (-1)^{kl+kn} \beta \circ \alpha \in \Omega^{m+n}(M; g^{k+l}).$$

This shows that the bracket is linear with respect to total degree so $[L^i, L^j] \subset L^{i+j}$. It is also clear that the bracket is homogeneous skew-symmetric because

$$-(-1)^{|a|} [a, b] = \omega \wedge \theta \otimes (-1)^{kl+\ell m+mn} \alpha \circ \beta + (-1)^{\ell m+mn} \beta \circ \alpha$$

$$= \theta \wedge \omega \otimes (-1)^{kl+\ell m} \alpha \circ \beta + (-1)^{\ell m} \beta \circ \alpha$$

$$= [b, a] n.$$

Checking the Jacobi identity is a simple computation, so $\Omega^*(M; g)$ is a dg Lie algebra as desired. \hfill \Box
Consider a solution to the Maurer-Cartan equation
\[ d\nabla(\alpha) + \frac{1}{2} [\alpha, \alpha] = 0 \] (5)
where \( \alpha \in \Omega^m(M; g^{1-m}) \), or in other words the total degree of \( \alpha \) is 1. Index the bigraded components of \( \alpha \) so that
\( \alpha^{m,n} \in \Omega^m(M; \text{Hom}(V^\otimes n, V^{2-m-n})) \).
Since \( m \geq 0 \), this gives us a filter to use to understand solutions to (5). When \( m = 0 \), the \( d\nabla(\alpha) \) component cannot contribute, so the \((0, n)\) component of (5) gives an element of \( \text{Hom}(V^\otimes n, V^{2-n}) \) for each \( x \in M \) so that:
\[
0 = \frac{1}{2} \sum_{m+r=n+1} [\alpha^{0,m}, \alpha^{0,r}]
= \frac{1}{2} \sum_{m+r=n+1} \left( \sum_{j=0}^{m-1} (-1)^{j} \alpha^{0,m}(1 \otimes j, \alpha^{0,r}, 1 \otimes n-r-j) \right)
+ \sum_{k=0}^{r-1} (-1)^{k} \alpha^{0,r}(1 \otimes k, \alpha^{0,m}, 1 \otimes n-r-m) \right) \]
\[
= \sum_{m+r=n+1} \sum_{j=0}^{m-1} (-1)^{j} \alpha^{0,m}(1 \otimes j, \alpha^{0,r}, 1 \otimes n-j-r) \]
which is precisely equation (1) on each point of \( M \).

**Definition 2.2.** A solution \( \alpha \in \Omega^*(M; g) \) to the Maurer-Cartan equation (2) with \( |\alpha| = 1 \) gives a family of \( A_\infty \) algebras over \( M \) where \( A_x \) is the \( A_\infty \) algebra over \( x \) for each \( x \in M \) with the \( A_\infty \) structure maps \( \mu^n_x = \alpha^{0,n} \).

**Assumption 2.3.** For ease of computation, we shall henceforth assume that \( \alpha^{0,0} = \alpha^{1,0} = \alpha^{2,0} = 0 \). This means that the curvature of each \( A_\infty \) algebra \( A_x \) is zero as \( \alpha^{0,0} = 0 \) identically.

**Definition 2.4.** For a manifold \( M = K(G, 1) \) with base point \( * \), a family of \( A_\infty \) algebras over \( M \) defines a homotopy group action by \( G \) on the \( A_\infty \) algebra over the base point where \( [\mathcal{F}_g] : A_* \to A_* \) for \( g \in G \) is defined by integrating around a loop corresponding to \( g \) in \( M \).

We define such \( A_\infty \) morphisms in Theorem 3.4 and the homotopies between them in Theorem 4.4.

### 3. \( A_\infty \) Morphisms on \( I \)

Let \( \gamma(t) : [0, 1] \to M \) be a path with \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). By pulling back \( V \) along \( \gamma \) we may calculate on \( I = [0, 1] \). This is clear since to determine \( \mathcal{F} : A_{x_0} \to A_{x_1} \) we shall integrate along \( \gamma \) and any tangent vectors perpendicular to \( \gamma \) will not contribute. After pulling back to \( I \), choose a trivialization compatible with the flat connection \( \nabla \).
If we assume that \( \alpha^{1,1} = 0 \), by integrating the first few levels of the Maurer-Cartan equation (5) we calculate that

\[
\mathcal{F}^1 = \text{Id}, \\
\mathcal{F}^2 = -\int_0^1 \alpha_t^{1,2}, \\
\mathcal{F}^3 = -\int_0^1 \alpha_t^{1,3} - \int_{0 \leq u \leq 1} \alpha_u^{1,2}(\alpha_t^{1,2}, 1) + (-1)^\xi \alpha_u^{1,2}(1, \alpha_t^{1,2}).
\]

From this point forward, we shall not include the signs in our formulae as they all arise from the Gerstenhaber bracket and the Koszul sign conventions. To prove a general formula for the higher \( \mathcal{F}^n \) denote \( \alpha^{1,n} \) as a height 1 tree with \( n \) leaves and a single root. We shall use the notation \((k, m)\)-trees for the sum of all height \( k \) rooted trees with \( m \) leaves where we do allow valance 2 vertices corresponding to \( \alpha^{1,1} \) terms.

**Definition 3.1.** Let \( d \) be chosen so that \((\alpha^{1,1})^{d+1} = 0 \), then for \( p \leq q \in [0, 1] \):

\[
\mathcal{F}^1_{p \rightarrow q} = (0, 1)\text{-tree} + \sum_{i=1}^d \int_{p \leq t_1 \leq \ldots \leq t_i \leq q} (i, 1)\text{-trees},
\]

\[
\mathcal{F}^n_{p \rightarrow q} = \sum_{i=1}^{n-1+d} \int_{p \leq t_1 \leq \ldots \leq t_i \leq q} (i, n)\text{-trees}.
\]

For \( p > q \), simply reverse the inequalities in the integrals.

For example, when \( d = 0 \) we have the initial terms given by (6):

\[
\mathcal{F}^1 = \text{Id}, \\
\mathcal{F}^2 = \int_{1\text{-simplex}},
\]

and

\[
\mathcal{F}^3 = \int_{1\text{-simplex}} + \int_{2\text{-simplex}}.
\]

**Lemma 3.2.** For the \( \mathcal{F}^n \) defined in Definition 3.1, composition follows the rule

\[
\mathcal{F}_{s \rightarrow t} \circ \mathcal{F}_{r \rightarrow s} = \mathcal{F}_{r \rightarrow t},
\]

for \( s, r, t \in I \) where we compose using (3).

**Proof.** First, note that

\[
\mathcal{F}^1_{r \rightarrow t} = \text{Id} + \sum_{i=1}^d \int_{r \leq r_1 \leq \ldots \leq r_i \leq t} (i, 1)\text{-trees},
\]
and

$$(\mathcal{F}_{s \rightarrow t} \circ \mathcal{F}_{r \rightarrow s})^1 = \mathcal{F}_{s \rightarrow t}^1 \circ \mathcal{F}_{r \rightarrow s}^1$$

$$= \left( \text{Id} + \sum_{j=1}^{d} \int_{s \leq s_1 \leq \cdots \leq s_j \leq t} (j, 1)\text{-trees} \right) \circ \left( \text{Id} + \sum_{i=1}^{d} \int_{r \leq r_1 \leq \cdots \leq r_i \leq s} (i, 1)\text{-trees} \right)$$

$$= \text{Id}_{s \rightarrow t} + \text{Id}_{s \rightarrow t} \circ \sum_{i=1}^{d} \int_{r \leq r_1 \leq \cdots \leq r_i \leq s} (i, 1)\text{-trees} + \left( \sum_{i=1}^{d} \int_{s \leq s_1 \leq \cdots \leq s_i \leq t} (i, 1)\text{-trees} \right) \circ \text{Id}_{r \rightarrow s}$$

$$+ \sum_{j+i=2}^{2d} \int_{j \geq 1, i \geq 1} (i+j, 1)\text{-trees}$$

$$= \mathcal{F}_{r \rightarrow t}^1$$

because the last sum is zero for \(j+i > d\). We shall use a similar approach for higher \(n\) and split up the desired result into pieces with mixed components or terms in only one half or the other. The \(n\)th term of \(\mathcal{F}_{r \rightarrow t}\) consists of integrals over the appropriate simplices of all \(n\)-leaved trees. On the other side we have

$$(\mathcal{F}_{s \rightarrow t} \circ \mathcal{F}_{r \rightarrow s})^n = \sum_{i=1}^{n} \sum_{\sum m_j = n} \mathcal{F}_{s \rightarrow t}^i \left( \mathcal{F}_{r \rightarrow s}^{m_1} \otimes \cdots \otimes \mathcal{F}_{r \rightarrow s}^{m_i} \right)$$

$$= \sum_{i=1}^{n} \sum_{\sum m_j = n} \left( \sum_{k=1}^{n-1} \int \ (k, i)\text{-trees} \right) \circ \left( \sum_{j_1=1}^{m_1-1+d} \int \ (j_1, m_1)\text{-trees} \otimes \cdots \otimes \sum_{j_i=1}^{m_i-1+d} \int \ (j_i, m_i)\text{-trees} \right)$$

$$+ \text{Id}_{s \rightarrow t} \left( \mathcal{F}_{r \rightarrow s}^n \right) + \mathcal{F}_{s \rightarrow t}^n \left( \text{Id}_{r \rightarrow s} \otimes \cdots \otimes \text{Id}_{r \rightarrow s} \right)$$

As in the \(n = 1\) example, we shall need to include identities to transport values to the appropriate endpoints. This allows us to increase the depth of a tree freely. Now, let \(j\) be the maximum depth of the \(\{j_i\}\) of any fixed configuration. Extend the other trees with identities to the same depth so we can combine the terms. This is a finite process because we do not modify the maximum-depth tree so the depth of the combined tree is simply \(j+k\). We shall also use the fact that any tree deeper than \(n-1+d\) is automatically 0. Thus, we can continue the above equation as
Corollary 3.3. Given a path \( p : I \rightarrow M \), the strict inverse of \( \mathcal{F}_{p(t)} \) is \( \mathcal{F}_{p(1-t)} \) because \( \mathcal{F}_{r-s} \circ \mathcal{F}_{s-r} = \mathcal{F}_{s-s} = \text{Id} \).

Theorem 3.4. When \( \alpha^{1,1} = 0 \), the maps \( \mathcal{F}^k \) defined in Definition 3.1 are \( A_{\infty} \) morphisms from \( A_p \) to \( A_q \).

Lemma 3.2 demonstrates that the proposed morphism maps in Theorem 3.4 compose as required for \( A_{\infty} \) morphisms in \( \mathcal{F}^k \). Using this composition property, we shall apply results from differential equations to argue that our choice is correct.

Lemma 3.5. Using the \( \mathcal{F}^k \) in Theorem 3.4, define

\[
\Delta_{r,s}^n := \sum_{i=2}^{n} \sum_{k_j=n} \mu_s^i \left( \mathcal{F}_{r,s}^k \otimes \cdots \otimes \mathcal{F}_{r,s}^{k_1} \right) - \sum_{i=2}^{n} \sum_{j=0}^{n-i} (-1)^j \mathcal{F}_{r,s}^{n-i+1} \left( 1 \otimes j \otimes \mu_s^i \otimes 1 \otimes n-(i+j) \right).
\]

Then

\[
\Delta_{r,s}^n = 0 \quad \forall s, n.
\]

Notice that \( \Delta_{r,s}^n \) is precisely a measurement of the failure of our proposed maps to satisfy the \( n \)-th level of the \( A_{\infty} \)-morphism relations [2].

Proof. First, we determine the initial conditions. Clearly, \( \Delta_{r,r}^n = 0 \) as \( \mathcal{F}^1 = \text{Id} \) and all higher order terms \( \mathcal{F}^k \) are zero because they involve integrating over sets of measure zero. Now

\[
\frac{\partial}{\partial s} \Delta_{r,s}^n \bigg|_{s=r} = \frac{\partial \mu_s^i}{\partial s} \bigg|_{s=r} + \sum_{k=2}^{n-1} \sum_{j=0}^{n-k} (-1)^j \mu_r^{n-k+1} \left( 1 \otimes j \otimes \frac{\partial \mathcal{F}_{r-s}^k}{\partial s} \bigg|_{s=r} \right) \otimes 1 \otimes n-j-k
\]

\[
- \sum_{k=2}^{n-1} \sum_{j=0}^{n-k} (-1)^j \frac{\partial \mathcal{F}_{r-s}^k}{\partial s} \bigg|_{s=r} \left( 1 \otimes j \otimes \mu_r^{n-k+1} \otimes 1 \otimes k-j-1 \right)
\]

\[
= \frac{\partial \mu_r^i}{\partial s} \bigg|_{s=r} + \sum_{k=2}^{n-1} \left[ \alpha_r^{0,n-k+1}, \alpha_r^{1,k} \right] = 0
\]
where we used (5) at the point \( x = r \) on \( I \) and the fact that

\[
\frac{\partial \mathcal{F}^k_{r,s}}{\partial s} \bigg|_{s=r} = \left\{ \begin{array}{ll} \frac{\partial}{\partial s} \left( \int^s_0 \alpha^1_r \cdot dt + \mathcal{O}((s-r)^2) \right) \bigg|_{s=r} = \alpha^1_r & \text{if } k \geq 2 \\ \frac{\partial}{\partial s} \text{Id} \bigg|_{s=r} = 0 & \text{if } k = 1. \end{array} \right.
\]

(8)

Determine a differential equation defining \( \Delta^k_{r,s} \) in terms of \( \Delta_{0,s} \).

\[
\Delta^k_{0,s} = \sum_{i=2}^{n} \mu^i_s \left( \mathcal{F}^k_{0,s} \otimes \cdots \otimes \mathcal{F}^k_{0,s} \right)
\]

\[- \sum_{i=2}^{n} \sum_{j=0}^{n-i-1} (-1)^j \mathcal{F}^k_{r,s} \mathcal{F}^{n-i+1} \left( 1 \otimes \mu^i_0 \otimes 1 \otimes \cdots \right)
\]

\[= \sum_{i=2}^{n} \sum_{j=0}^{n-i-1} \mu^i_s \left( \sum_{M_t=k} \mathcal{F}_{r,s}^t \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_t}_{0,r} \right) \otimes \cdots \right)
\]

\[\cdots \sum_{M_t=k} \mathcal{F}_{r,s}^t \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_t}_{0,r} \right)
\]

\[\cdots \sum_{M_t=k} \mathcal{F}_{r,s}^t \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_t}_{0,r} \right)
\]

where we use the notation \( M_k = \sum_{i=1}^{k} m_i \). Therefore,

\[\Delta^k_{0,s} = \sum_{k=1}^{n} \frac{\partial \Delta^k_{r,s}}{\partial s} \left( \sum_{M_k=n} \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_k}_{0,r} \right) \right)
\]

\[+ \sum_{k=1}^{n} \sum_{M_k=n} \sum_{\ell=1}^{k} \frac{\partial \mathcal{F}^k_{r,s}}{\partial s} \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_\ell-1}_{0,r} \otimes \Delta^k_{r,s} \otimes \mathcal{F}^{m_{\ell+1}}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_k}_{0,r} \right)
\]

Thus

\[
\frac{\partial \Delta^k_{0,s}}{\partial s} = \sum_{k=1}^{n} \frac{\partial \Delta^k_{r,s}}{\partial s} \left( \sum_{M_k=n} \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_k}_{0,r} \right) \right)
\]

\[+ \sum_{k=1}^{n} \sum_{M_k=n} \sum_{\ell=1}^{k} \frac{\partial \mathcal{F}^k_{r,s}}{\partial s} \left( \mathcal{F}^{m_1}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_{\ell-1}}_{0,r} \otimes \Delta^k_{r,s} \otimes \mathcal{F}^{m_{\ell+1}}_{0,r} \otimes \cdots \otimes \mathcal{F}^{m_k}_{0,r} \right)
\]

since all the other terms are independent of \( s \) and hence simply disappear. Now \( r \) is simply a variable so we can choose \( r = s \). After applying (4) and (6) we have

\[
\frac{\partial \Delta^k_{0,s}}{\partial s} = \sum_{k=2}^{n} \sum_{M_k=n} \sum_{\ell=1}^{k} \alpha^{1,k}_s \left( \mathcal{F}^{m_1}_{0,s} \otimes \cdots \otimes \mathcal{F}^{m_{\ell-1}}_{0,s} \otimes \Delta^k_{0,s} \otimes \mathcal{F}^{m_{\ell+1}}_{0,s} \otimes \cdots \otimes \mathcal{F}^{m_k}_{0,s} \right).
\]

Since \( k \geq 2 \), we see that \( m_{\ell} < n \). We also know by explicit computation that \( \Delta^2_{r,s} = 0 \) and \( \Delta^3_{r,s} = 0 \). By the multilinearity of \( \alpha^{1,k}_s \), we therefore know that

\[
\frac{\partial \Delta^k_{0,s}}{\partial s} = 0 \text{ for } n = 2, 3, 4.
\]

Now we can induct on \( n \). Say we know that \( \Delta^k_{0,s} = 0 \) for all \( k < n \) and hence that \( \frac{\partial \Delta^k_{0,s}}{\partial s} = 0 \) for \( k \leq n \). Using the initial condition that \( \Delta^0_{0,0} = 0 \), by the existence and uniqueness of linear ordinary differential equations,
it is clear that $\Delta^0_{r,s} = 0$ for all $s$. However, there was nothing special about our choice of 0 as the initial point so by translating by $r$, $\Delta_{r,s} = 0$ for all $r, s \in I$. □

Proof of Theorem 3.4. By Lemma 3.2 the manipulations that we took to prove Lemma 3.5 were valid. Therefore, $F_{r,s}$ is an $A_\infty$ morphism from $A_p$ to $A_q$. □

Note that while we assumed $\alpha^{1,1} = 0$ in Theorem 3.4, we did so to eliminate the possibility of infinite recursions with Stokes’ Theorem leading to infinite depth trees with valence two vertices in Definition 3.1. We may relax the assumption that $\alpha^{1,1} = 0$ by instead requiring a finite descending filtration on $\Omega^*(M; g)$ with the condition that $\alpha^{1,1}$, and any non-invariant component of $\alpha^{0,1}$, decrease the degree. This will give us a nilpotency restriction on $\alpha^{1,1}$ thereby fixing a $d < \infty$ for Definition 3.1.

4. $A_\infty$ homotopies on $M = I \times I$

4.1. Classical homotopies. Following [12, Ch. X], [27, Ch. I] we can algebraically define a (classical) homotopy between $F_0 : A \to B$ and $F_1 : A \to B$ to be given by the strictly commutative diagram

![Diagram](https://via.placeholder.com/150)

where $I$ is considered as the quiver

$\bullet_{u_0} \xymatrix{ & \bullet_{h} \ar[r]^{\bullet_{u_1}} & }$

with the relations

$$
|u_0| = |u_1| = 0, \quad |h| = 1 \\
u_0^2 = u_0 \quad u_1^2 = u_1 \quad h^2 = 0 \\
hu_0 = h \quad u_0h = 0 = hu_1 \quad u_1h = h.
$$

(9)

We then impose a differential $\partial$ according to the rules:

$$
\partial u_0 = h = -\partial u_1 \quad \partial h = 0.
$$

(10)

Note that $I$ is a differential graded algebra (dga). This is clear by checking the interaction of $\partial$ with the relations in equations (9) and (10) as well as the Leibnitz rule.

4.2. Differential homotopies. By extrapolating the relevant characteristics of the classical picture, we can define differential homotopies on a family of $A_\infty$ morphisms between $A$ and $B$ indexed by $t \in [0, 1]$ by the commutative diagram of $A_\infty$
Proposition 4.2. Differential homotopies give rise to classical homotopies.
Proof. We must show strict commutativity of the following diagram

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{e_{v_0}} & \mathbb{R} \\
\downarrow & & \downarrow \\
\mathbb{R} & \xrightarrow{e_{v_1}} & \mathbb{R} \\
\phi \downarrow & & \Omega^*(\{0,1\}) \\
\end{array}
$$

for a quasi-isomorphism $\Phi$. We shall denote $\Omega^*(\{0,1\})$ by $A$ for ease of notation. We proceed by defining $\Phi^1$ inductively as we require $\Phi$ to be an $A_\infty$ morphism satisfying (2). For $a, b \in A^0 = \Omega^0([0,1])$, define $\Phi^1$ by

$$
\Phi^1(a + b dt) = a(0)u_0 + a(1)u_1 + \left( \int_0^1 b dt \right) h.
$$

Thus it is clear that $\Phi^1(fg) = \Phi^1(f)\Phi^1(g)$ for $f, g \in A^1$. Therefore, we need to have a $\Phi^2$ that will cancel the mixed terms

$$
\Phi^1(a(t)b(t)dt) = \left( \int_0^1 a(t)b(t)dt \right) h
$$

and

$$
\Phi^1(a(t))\Phi^1(b(t)dt) = \left( a(1) \int_0^1 b(t)dt \right) h
$$

exactly without disturbing the equality of our earlier compositions. Recall that $\mu^2_A$ is just normal multiplication of forms and $\mu^1_A$ is differentiation of forms. Also, $\mu^1_I$ is the linear extension of our multiplication table given in equation (9) and $\mu^1_I = \partial$ as constructed in equation (10). Hence, the only possible input for which $\Phi^2$ should be nonzero is $(f(t)dt, g(t)dt)$ as all others would contribute to equations that are already satisfied. Let

$$
\Phi^2(f(t)dt, g(t)dt) := \left( \int_{0 \leq t \leq s \leq 1} f(s)g(t)dsdt \right) h. \quad (11)
$$

Now, putting all of this together, we require

$$
\Phi^1(\mu^2_A(f(t), g(t)dt)) + \Phi^2(\mu^1_A \otimes 1 + 1 \otimes \mu^2_A)(f(t), g(t)dt) = \mu^1_I(\Phi^2)(f(t), g(t)dt) + \mu^2_I(\Phi^1 \otimes \Phi^1)(f(t), g(t)dt)
$$

or in other words,

$$
\left( \int_0^1 f(t)g(t)dt \right) h + \Phi^2 \left( \frac{\partial f(t)}{\partial t} dt, g(t)dt \right) + 0 = 0 + \left( \int_0^1 g(t)dt \right) h.
$$

However, by Stokes’ Theorem and (11), this is satisfied. The check that

$$
\Phi^1(g(t)dt f(t)) + \Phi^1(g(t)dt)\Phi^1(f(t)) = \Phi^2(1 \otimes \mu^2_A + \mu^2_A \otimes 1)(g(t)dt, f(t))
$$

is similar. Thus, we may define

$$
\Phi^k(a_1(t), \ldots, a_k(t)) = \begin{cases} 
\left( \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} a_1(t_1) \cdots a_k(t_k) \right) h & \text{if } a_i(t) \in A^1 \forall i \\
0 & \text{if any } a_i \in A^0 \end{cases} \quad (12)
$$

In the $A_\infty$ morphism relations (2), all terms involving $\mu^n, n \geq 3$ drop out, and the term $\mu^1_I(\Phi^{k+1})$ drops out by $\partial h = 0$ any time $\Phi^{k+1}$ is nonzero. Hence, the only terms which are nonzero on the left (LHS) have exactly one 0-form entry and the
only nonzero terms on the right (RHS) are those involving $\Phi^1$ and $\Phi^k$. If we have more than one 0-form, then the LHS is also zero because of our definitions of any $\Phi^n, n > 1$ since we have explicitly checked $\Phi^2$. On the other hand, if there are no 0-form entries, then the LHS is zero and the RHS is $O(h^2) = 0$. Therefore, the relations reduce to:

$$\Phi^{k+1} \left( \sum_{j=0}^{k} 1^\otimes j \otimes \mu_A^1 \otimes 1^{k-j} \right) + \Phi^k \left( \sum_{j=0}^{k-1} 1^\otimes j \otimes \mu_A^2 \otimes 1^{k-1-j} \right) = \mu_i^2 \left( \Phi^1 \otimes \Phi^k + \Phi^k \otimes \Phi^1 \right)$$

where, for $f_j \in A^0$, the possible arguments are

$$(f_1(t)dt, \cdots, f_{i-1}(t)dt, f_i(t), f_{i+1}(t)dt, \cdots, f_k+1(t)dt), \quad i = 1, \cdots, k + 1.$$ 

Thus, on the left we have

$$LHS = \left( \int_{0 \leq t_{k+1} \leq \cdots \leq t_1} f_1(t_1) \cdots f_i(t_i) \cdots f_{k+1}(t_{k+1})dt_1 \cdots dt_i \cdots dt_{k+1} \right) + \left( \int_{0 \leq t_{k+1} \leq \cdots \leq t_1} f_1(t_1) \cdots f_i(t_i) \cdots f_{k+1}(t_{k+1})dt_1 \cdots dt_i \cdots dt_{k+1} \right) + \left( \int_{0 \leq t_{k+1} \leq \cdots \leq t_1} f_1(t_1) \cdots f_{i-1}(t_i-1) f_i(t_i) \cdots f_{k+1}(t_{k})dt_1 \cdots dt_i \cdots dt_{k+1} \right) + \left( \int_{0 \leq t_{k+1} \leq \cdots \leq t_1} f_1(t_1) \cdots f_i(t_i) f_{i+1}(t_i) \cdots f_{k+1}(t_{k+1})dt_1 \cdots dt_i \cdots dt_{k+1} \right) h$$

which agrees with the RHS in all cases. □

**Remark 2.** Since $\Phi : \Omega^*([0,1]) \to I$ is a quasi-isomorphism, it is true that differential homotopies correspond to classical homotopies by inverting $\Phi$, but we shall not do the necessary calculations here.

4.3. **Mapping from $I \times I$ to $I$.**

**Theorem 4.3.** Let $\alpha^{*,*}$ be a solution of equation (4.3) on $I \times I$. We can collapse $I \times I$ to $I$ by applying the standard projection onto the first element. We shall identify the first interval with $0 \leq s \leq 1$ and the second with $0 \leq t \leq 1$. Then we can construct a solution $\hat{\alpha}^{m,n} \in \Omega^m(I, \text{Hom}(\Omega^*(I,V))^{\otimes n}, \Omega^*(I,V))$ on the submanifold $I$ as follows:

$$\hat{\alpha}^{0,n} = \delta_{1,n} d + \alpha^{0,n} + \iota_\partial \alpha^{1,n} dt$$

$$\hat{\alpha}^{1,n} = \iota_\partial \alpha^{1,n} ds + (1 \otimes \iota_\partial + \iota_\partial \otimes 1) \alpha^{2,n} dt$$

$$\hat{\alpha}^{m,n} = 0 \quad \text{for all } m \geq 2$$

where $d$ is the differential on $\Omega^*(I,V)$ considered as a constant function of $s$, and $\delta$ is the Kronecker delta function.
Proof. We must check that \( \hat{\alpha}^{k,l} \) is a solution to (5). Let us begin with the lowest level when \( k = 0 \).

\[
\sum_{m+r=n+1} \frac{1}{2} \left[ \hat{\alpha}^{0,m}, \hat{\alpha}^{0,r} \right] = \sum_{m+r=n+1} \frac{1}{2} \left[ \left[ \delta_1 m \delta_1, c_1 \right] + \left[ \alpha^{0,m}, \delta_1, c_1 \right] + \left[ \iota_{\partial}, \alpha^{1,m} dt, \delta_1, c_1 \right] 
+ \left[ \delta_1, m \iota_{\partial}, \alpha^{1,r} dt \right] + \left[ \delta_1 m \delta_1, a^{0,r} \right] + \left[ \alpha^{0,m}, a^{0,r} \right] 
+ \left[ \alpha^{0,m}, \iota_{\partial}, \alpha^{1,r} dt \right] + \left[ \iota_{\partial}, \alpha^{1,m} dt, \alpha^{0,r} \right] + \left[ \iota_{\partial}, \alpha^{1,m} dt, \iota_{\partial}, \alpha^{1,r} dt \right] \right) \right) 
= \sum_{m+r=n+1} \left[ \delta_1 m \delta_1, a^{0,r} \right] + \left[ \iota_{\partial}, \alpha^{1,m} dt, \alpha^{0,r} \right] 
= \iota_{\partial} \left( d\varphi \alpha^{0,n} + \sum_{m+r=n+1} \left[ \alpha^{1,m}, \alpha^{0,r} \right] \right) dt 
= 0.
\]

Now consider the component of (5) that lies in \( \Omega^1(I) \).

\[
\frac{\partial}{\partial s} (\hat{\alpha}^{0,n})ds + \sum_{m+n=1} \left[ \alpha^{0,m}, \hat{\alpha}^{1,r} \right] 
= \frac{\partial (\alpha^{0,n} + \iota_{\partial}, \alpha^{1,n})}{\partial s} ds + \frac{\partial (\iota_{\partial}, \alpha^{1,n})}{\partial t} ds dt + \sum_{r+m=n+1} \left[ \left[ \alpha^{0,m}, \iota_{\partial}, \alpha^{1,r} \right] ds \right] 
+ \left[ \iota_{\partial}, \alpha^{1,m} dt, \iota_{\partial}, \alpha^{1,r} ds \right] + \left[ \alpha^{0,m}, (1 \otimes \iota_{\partial} + \iota_{\partial} \otimes 1) \alpha^{2,r} dt \right] 
= \iota_{\partial} \left( d\varphi \alpha^{0,n} + \sum_{m+r=n+1} \left[ \alpha^{0,m}, \alpha^{1,r} \right] \right) ds 
+ (1 \otimes \iota_{\partial} + \iota_{\partial} \otimes 1) \left( d\varphi \alpha^{1,n} + \sum_{m+r=n+1} \left[ \alpha^{0,m}, \alpha^{2,r} \right] + \frac{1}{2} \left[ \alpha^{1,m}, \alpha^{1,r} \right] \right) 
= 0.
\]

All further levels of the Maurer-Cartan equation are zero by dimensionality. \( \square \)

4.4. \( A_\infty \) homotopies. Let \( \alpha^{m,n} \) be a solution to the Maurer-Cartan equation on the square \( I \times I \) with \( \alpha^{1,1} = 0 \). Since we presume this square is a pullback of two homotopic paths on \( N \), we require that the \( A_\infty \) structure be constant for the edges \( s = 0 \) and \( s = 1 \). Let \( \mathcal{F}_t \) be a family of \( A_\infty \) morphisms given by Theorem 3.4 using paths along constant \( t \) between the \( A_\infty \) algebras \( A_{s=0} \) and \( A_{s=1} \) on the square. Now \( \hat{\alpha}^{0,1} = d \) and \( \hat{\alpha}^{1,1} = (1 \otimes \iota_{\partial} + \iota_{\partial} \otimes 1) \alpha^{2,1} \) are both nilpotent of order 2.

**Theorem 4.4.** Define \( \mathcal{G}^n \) using the \( \hat{\alpha}^{1,k} \) terms in Definition 3.4 with \( d = 1 \). Then \( \mathcal{G} \) defines a differential homotopy with respect to the \( \mathcal{F}_t \).

**Proof.** First, we show that \( \mathcal{G} \) is an \( A_\infty \) morphism from \( \Omega^*(I; A_{s=0}) \) to \( \Omega^*(I; A_{s=1}) \). Then we shall show that there is a natural injection of \( A_{s=0} \) into \( \Omega^*(I; A_{s=0}) \).
By definition, \( \hat{G}^n = \mathcal{F}^n + \mathcal{G}^n \) where \( \text{Im}(\mathcal{G}^n) \in \Omega^1(I; A_{s=1}) \). As in Lemma 3.5 define

\[
\hat{\Delta}_{r,s}^n := \sum_{i=2}^{n} \sum_{k_j = n} \mu_i \left( \mathcal{G}^{k_1}_{r,s} \otimes \cdots \otimes \mathcal{G}^{k_i}_{r,s} \right) - \sum_{i=2}^{n} \sum_{j=0}^{n-i} (-1)^j \mathcal{G}^{n-i+1}_{r,s} \left( 1 \otimes \mu_i \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \right)
\]

\[
= \Delta_{r,s}^n + \sum_{i=2}^{n} \sum_{j=0}^{n-i} \left( \mu_i \left( \mathcal{F}^{k_1}_{r,s} \otimes \cdots \otimes \mathcal{G}^{k_i}_{r,s} \otimes \cdots \otimes \mathcal{F}^{k_j}_{r,s} \right) - \mathcal{G}^{n-i+1}_{r,s} \left( 1 \otimes \mu_i \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \right) \right)
\]

(13)

Since \( \Delta_{r,s}^n = 0 \) by Lemma 3.5 we see that \( \hat{\Delta}_{r,s}^n \in \Omega^1(I; A_{s=1}) \). However in this case,

\[
\frac{\partial \hat{G}^k}{\partial s} \bigg|_{s=r} = \hat{\alpha}^1_{r,k},
\]

so we also have

\[
\frac{\partial \hat{\Delta}_{0,s}^n}{\partial s} = \sum_{k=1}^{n} \sum_{\ell=1}^{k} \hat{\alpha}^1_{s,k} \left( \mathcal{G}^{m_1}_{0,s} \otimes \cdots \otimes \mathcal{G}^{m_{\ell-1}}_{0,s} \otimes \hat{\Delta}_{0,s}^n \otimes \hat{\Delta}^{m_{\ell+1}}_{0,s} \otimes \cdots \otimes \mathcal{G}^{m_j}_{0,s} \right)
\]

As before, proceed by induction. We know that

\[
\hat{\Delta}_{0,s} = \alpha^0_{1,s}(\mathcal{G}^1_{r,s}) - \mathcal{G}^1_{r,s}(\alpha^0_{r,1}) = 0.
\]

Assume that \( \hat{\Delta}_{0,s}^n = 0 \) for \( k < n \). For \( k = n \), after using (13) we are thus left with

\[
\frac{\partial \hat{\Delta}_{0,s}^n}{\partial s} = \hat{\alpha}^{1,n}_{s,n} = 0.
\]

Therefore, by the same ODE argument used to show Lemma 3.5 \( \hat{\Delta}_{r,s}^n = 0 \) so \( \mathcal{G} \) is an \( A_{\infty} \) morphism.

Now consider the diagram:

where the map \( \iota \) maps elements of \( A \) to constant maps to \( A \). All higher order terms of \( \iota \) as an \( A_{\infty} \) map are 0. By definition of \( \mathcal{G} \) it is clearly a commutative diagram, so per the discussion in Section 4.2 \( \phi = \mathcal{G} \circ \iota \) and therefore \( \mathcal{G} \) gives rise to a differential homotopy \( \mathcal{G}^k(\iota \otimes^k) \). \( \square \)
5. Examples

Let $E^*_{p,q}$ be the cohomological spectral sequence determined by the bigraded complex on $Ω^p(M;Hom(A⊗^q A[1-q]))$ with $d_h = [α^{q,2};]$ and $d_v = d_v$. It follows that $E^2_{p,q} = H^p_{dH}(M; HH^q(A,A))$. We shall compute the cohomology of the total complex in several cases where the $E^2$ term collapses.

Since de Rham cohomology is only defined for globally defined differential forms on $M$ we shall consider de Rham cohomology with local coefficients as a version of sheaf cohomology. Using the fact that any two cohomology theories on $M$ with coefficients in sheaves of $R$-modules over $M$ are uniquely isomorphic, see for example [51, p. 184], we may use the cohomology theory that best fits our circumstances. Subsequently, we will construct a manifold which is homotopy equivalent to $BG$ for any finite group $G$. This will allow us to transfer the computation of $E^2_{p,q}$ into group cohomology where it will clearly collapse. Finally, we shall compute the example of $E^2_{p,q}$ for $M$ homotopy equivalent to a wedge product of circles, i.e. $BG$ for $G$ a free group with a finite number of generators.

5.1. Cohomology with local coefficients. A theorem of Eilenberg in [4] tells us that $H^*(X;E) ≅ H^*_{eq}(X;E)$ where $H^*_{eq}$ are the equivariant cohomology groups and $\tilde{X}$ is the universal covering space of $X$ with $π_1(X)$ as covering transformations left operating on $E$. As noted in [31, 45], when $X$ is a $K(G,1)$, $\widetilde{K}(G,1)$ is acyclic and so the augmented cellular chain complex is a free resolution of $R$. Equivariant cohomology is that defined on the equivariant cochains

$$C^q_{eq}(\widetilde{BG};E) = \{ f ∈ C^q(\widetilde{BG};E) \mid δf(σgσ^{-1}) = g(δσ)(σq+1) \} \cong \text{Hom}_G(C^n_{eq}(\widetilde{BG});E)$$

for $g ∈ G$ and $σq+1$ a $(q + 1)$-simplex. Therefore, noting that $C^*_{eq}(\widetilde{BG})$ is a chain complex that resolves $R$, we see that

$$C^*_{eq}(\widetilde{BG};E) \cong H^*_{eq}(\widetilde{BG};E) \cong \text{Hom}_G(C^*_{eq}(\widetilde{BG});E) = C^*(G;E),$$

and thus

$$H^*(BG;E) \cong H^*(G,E).$$

5.2. Making $BG$ a manifold. Let $Γ$ be a finite group. First, recall that every finite group $Γ$ of order $k$ embeds in the unitary group $U(k)$. This follows by noting that every finite group has a faithful representation in $GL(k)$ given by the permutation representation and that every finite subgroup of $GL(k)$ is conjugate to a subgroup of $U(k)$ (see, for example [4, §9.2]). Now $U(k)$ is a compact Lie group and certainly a manifold. The Grassmannian $G(k,n+k) = U(n+k)/(U(n) × U(k))$ is a smooth compact manifold and $BU(k) = \lim_{n→∞} G(k,n+k)$. The group $U(k)$ acts freely on $U(n+k)$ and on $U(n+k)/U(n)$, so $Γ \subset U(k)$ also acts freely on both spaces. Consider the space $U(n+k)/U(n)$ as the space of orthonormal families of $k$ vectors in $C^{n+k}$. Thus we have a fibre bundle

$$\xymatrix{ U(n-1+k)/U(n-1) \ar[r] \ar[d] & U(n+k)/U(n) \ar[d] \ar[r] & S^{2(n+k)} \ar[d] \ar[l] \ar[r] \ar[l] & U(n+k)/U(n) \ar[d] \ar[r] & S^{2(n+k)-1}.}$$
Therefore, again taking \( n \to \infty \), we see that \( EU(k) = \lim_{n \to \infty} U(n + k)/U(n) \) is contractible and has a free \( \Gamma \) action so \( EU(k)/\Gamma \) is a classifying space for \( \Gamma \) (see, for example [14]).

For brevity, let us denote submanifold \( U(n + k)/U(n) \times \Gamma \subset G(k, n + k) \) by \( M^n \).

Therefore, we have constructed a series of manifolds

\[
M^1 \hookrightarrow M^2 \hookrightarrow M^3 \hookrightarrow \ldots \hookrightarrow M^k \hookrightarrow \ldots
\]

using the natural inclusions with the property that the limit space \( EU(k)/\Gamma = \bigcup_{i=1}^{\infty} M^i \) has the same homotopy type as \( BG \).

We want to be able to talk about \( \Omega^*(BG) \), and thus need a notion of a path. Define “a path in \( BG \)” as a path in \( M^i \) for some \( i \). Then

\[
\Omega^*(BG) = \{ \theta_i \in \Omega^*(M^i)_{i \geq 0} \mid \theta_i|_{M^{i-1}} = \theta_{i-1} \}.
\]

This is an inverse limit system which trivially satisfies the Mittag-Leffler condition because \( M^{i-1} \subset M^i \) [13] p. 191). We can define a differential and a wedge product on these differential forms. Given a differential form \( \theta \in \Omega^*(BG) \), define

\[
d\theta = \lim_i d\theta_i \in \Omega^*(BG).
\]

Similarly, given two differential forms \( \omega = \lim_i \omega_i \) and \( \eta = \lim_i \eta_i \) in \( \Omega^*(BG) \), define the wedge product

\[
\omega \wedge \eta = \lim_i (\omega_i \wedge \eta_i) \in \Omega^*(BG).
\]

With these maps, we can consider the de Rham cohomology of \( BG \).

Given a suitable topology on \( BG \), we know that

\[
H^*_{dR}(BG; \mathbb{E}) \cong H^*_\Delta(BG; \mathbb{E}).
\]

Thus we can compute cohomologies using the simplicial cohomology of \( BG \) with twisted coefficients in \( HH^*(A, A) \) where we consider \( BG \) as the standard simplicial complex generated from the universal cover \( E\Gamma \). In this case, there is one vertex, \( * \), in \( BG \). The simplices of \( BG \) can be described using the bar notation \( [g_1g_2\cdots g_n] \). In this notation, the boundary simplices of \( [g_1g_2\cdots g_n] \) are \( [g_2\cdots g_n], [g_1\cdots g_n-1] \), and \( [g_1\cdots g_i g_{i+1}\cdots g_n] \) for \( i = 1, \ldots, n-1 \). Since this is independent of topology, we shall not specify one.

5.3. Finite groups. Let \( \Gamma \) be a finite group. Construct the manifold \( BG \) as in Section 5.2. Let \( A \) be an \( A \) local system on \( BG \) where \( A \) is a real vector space that is an associative algebra under the multiplication map \( \mu^2 = \alpha^{0,2} \). Consider the total complex \( \Omega^*(BG; \text{Hom}(A^{\otimes n}, A[1-n])) \) with differential \( d_{cF} + [\alpha^{0,2}, \cdot] \). As a spectral sequence with \( E^2_{m,n} = \Omega^m(BG; \text{Hom}(A^{\otimes n}, A[1-n])) \), we have

\[
E^2_{m,n} = H^m_{dR}(BG; HH^n(A, A)) = H^m_{\text{simp}}(BG; HH^n(A, A)).
\]

By the discussion above, it is sufficient to calculate \( H^*(\Gamma; HH^n(A, A)) \). However, by [15] pg. 117), because \( HH^n(A, A) \) is always a real vector space and therefore a divisible abelian group with no elements of finite order

\[
H^p(\Gamma; HH^n(A, A)) = \begin{cases} HH^n(A, A)^\Gamma & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}
\]

Thus, \( E^2 \) degenerates to a single nonzero column so \( E^2 \cong E^\infty \).
5.4. **Finitely generated free group.** Let $G$ be a finitely generated nonabelian free group with $r$ generators, let $S$ be a set with $r$ elements and let $Y = \vee_{s \in S} S^1_s$. Then $Y$ is clearly a $K(G, 1)$ because

$$\pi_1(Y) = G$$

and

$$\pi_k(Y) = \bigoplus_{s \in S} \pi_k(S^1_s) = 0 \text{ for } k \geq 2.$$  

It is also straightforward to see that we may consider $Y$ as a manifold with only one inflationary step of the type used in Section 5.2. Therefore we again have

$$H^p(Y; HH^q(A, A)) \cong H^p(G; HH^q(A, A))$$

which we can compute using the resolution:

$$0 \rightarrow R \Gamma \rightarrow R \rightarrow 0$$

where $R \Gamma$ has basis $t_s$ corresponding to the oriented 1-simplex mapping to $S^1_s$, $R$ has basis $x$ corresponding to the base point and $\partial(t_s) = (1 - g_s)x$ because we must translate the endpoints of $\Delta^1$ to the same point before summing.

Now the cohomology $H^*(G; HH^q(A, A))$ is the cohomology of the complex

$$HH^q(A, A) \rightarrow \bigoplus_{s \in S} HH^q(A, A)_s \rightarrow 0 \rightarrow \cdots,$$

with $(\delta u) = \bigoplus_{s \in S} (1 - g_s)u$. Therefore,

$$H^p(G; HH^q(A, A)) = \begin{cases} HH^q(A, A)^G & \text{if } p = 0 \\ \text{Coker}(\delta) & \text{if } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $E^2_{p,q}$ has two nonzero columns, but that still means that $E^2_{p,q} \cong E^\infty_{p,q}$.

6. **Transferring Maurer-Cartan solutions between $CC^*(A, A)^\Gamma$ and $\Omega^*(B\Gamma; g)$**

**Proposition 6.1.** Let $\Gamma$ be a finite group. Then

$$HH^*(A, A)^\Gamma \cong H^*(g^\Gamma)$$

**Proof.** First, note that $H^*(g^\Gamma)$ is isomorphic to $H^*(CC^*(A, A)^\Gamma)$ so we shall use them interchangeably. It is clear that $H^q(g^\Gamma) \subseteq HH^q(A, A)^\Gamma$, so it only remains to check the opposite inclusion. Let $[f] \in HH^q(A, A)^\Gamma$ for $f \in CC^q(A, A)$ closed. Consider the element

$$\overline{f} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} gf.$$

Now, $\overline{f}$ is a closed element of $CC^q(A, A)^\Gamma$ by construction and because $[f]$ represents a $\Gamma$-invariant class it is clear that $[\overline{f}] = [f] \in HH^q(A, A)^\Gamma$. Now consider changing $f$ by a coboundary $d\omega$. The averaging process then shows that

$$\overline{f} + d\omega = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} gf + gd\omega = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} gf + d\omega$$

since $g\mu^2 = \mu^2(g \otimes g)$ and therefore $[\overline{f}] = [\overline{f} + d\omega]$ so we can also choose the coboundary representatives in $CC^{q-1}(A, A)^\Gamma$.

$\square$
Lemma 6.2. Let $M$ be a manifold with a basepoint. Consider $CC^*(A, A)^{\pi_1(M)}$ as a differential graded Lie algebra under the Hochschild differential and the Gerstenhaber bracket. Then there is a dg Lie map $\eta : CC^*(A, A)^{\pi_1(M)} \to \Omega^*(M; g)$ defined by $\eta(f) \mapsto \{\text{the constant 0-form with value } f \text{ at the basepoint}\}$ and extending this to all of $g$ by parallel transport.

Proof. There are no difficulties with nontrivial loops because $f \in CC^*(A, A)^{\pi_1(M)}$. Also, $\eta$ clearly commutes with the differentials because the DeRham differential on $\Omega^*(M; g)$ is zero on constant forms leaving only the Hochschild differential in each case and $\eta$ commutes with the bracket because the wedge of constant 0-forms is a constant 0-form and will not change any signs because it is of even degree. □

Corollary 6.3. When $M = B\Gamma$ for a finite group $\Gamma$, then $CC^*(A, A)^\Gamma$ is quasi-isomorphic to $\Omega^*(M; g)$.

Proof. The map $\eta$ constructed in Lemma 6.2 induces a map

$$\eta : H^*(CC^*(A, A)^\Gamma) \to H^*(\Omega^*(M; g)),$$

but by Section 5.3 we know that $H^*(\Omega^*(M; g)) \cong HH^*(A, A)^\Gamma$ and by Proposition 6.1 we know that $H^*(CC^*(A, A)^\Gamma) = HH^*(A, A)^\Gamma$. □

Let $\mathfrak{h} \subset CC^*(A, A)^\Gamma$ be the sub-dg Lie algebra of functions with negative internal degree in $A$, i.e.

$$\mathfrak{h} = \{f \in CC^*(A, A)^\Gamma \mid f \in \text{Hom}(V^\otimes n, V[-n]), n \geq 1\}.$$ There is a natural decreasing filtration $L_k \mathfrak{h}$ for $k \geq 1$ given by

$$L_k \mathfrak{h} = \{f \in CC^*(A, A)^\Gamma \mid f \in \text{Hom}(V^\otimes n, V[-n]), n \geq k\}.$$ Thus, $L_1 \mathfrak{h} = \mathfrak{h}$ and $\mu^2 \in \text{Hom}(V \otimes V, V[0])$ so for $f \in \text{Hom}(V^\otimes k, V[-m])$ we have

$$[\mu^2, f] \in \text{Hom}(V^{k+1}, V[-m])$$

and thus $d(L_n \mathfrak{h}) \subset L_{n+1} \mathfrak{h}$. Lastly, $[\mathfrak{h}^m, \mathfrak{h}^n] \subset \mathfrak{h}^{m+n}$ by the additivity of degrees so if we include a formal degree 0 parameter $\hbar$ so that $F_\hbar \mathfrak{h} = L_\hbar \mathfrak{h} \hbar^k$, then $\mathfrak{h}$ with filtration $F_\hbar$ is a filtered pronilpotent dg Lie algebra.

Let $\omega \in \Omega^*(M; g)$ be, similarly, the sub-dg Lie algebra of functions with negative internal degree strictly less than $-1$. Our checks in Proposition 2.4 show that

$$L_k \omega = \{\Omega^*(M; \text{Hom}(L^\otimes k, V[-k]))\}, \quad k \geq 1$$

and $F_\hbar \omega = L_\hbar \omega \hbar^k$ make $\omega$ a filtered pronilpotent dg Lie algebra.

Definition 6.4. For a dg Lie algebra, let $MC(\mathfrak{f})$ denote the set of solutions to the Maurer-Cartan equation in $\mathfrak{f}$. An element $\alpha \in MC(\mathfrak{f})$ will be called a Maurer-Cartan element.

Lemma 6.5. Let $\mathfrak{h}$ and $\mathfrak{g}$ be filtered pronilpotent dg Lie algebras. Suppose that $\Phi : \mathfrak{h} \to \mathfrak{g}$ is a filtered quasi-isomorphism between them which means that $\Phi$ induces quasi-isomorphisms of chain complexes $F_r \mathfrak{h} / F_{r+1} \mathfrak{h} \to F_r \mathfrak{g} / F_{r+1} \mathfrak{g}$ for any $r$. Then $\Phi$ induces a bijection between equivalence classes of Maurer-Cartan elements.

Proof. This is a special case of the isomorphism between deformation functors of Section 4.4 of [16]. A similar formulation in terms of filtered dglas can be found in Lemma 2.2 of [25]. □
Corollary 6.6. The map $\eta$ defined in Lemma 6.3 is a filtered quasi-isomorphism between $\mathfrak{h} \subset CC^*(A, A)^{\Gamma}$ and $\mathfrak{w} \subset \Omega^*(\mathfrak{M}; \mathfrak{g})$. Thus the map $\alpha \mapsto \eta(\alpha)$ induces a bijection between equivalence classes of Maurer-Cartan elements.

Let $\alpha \in MC(\mathfrak{h})$ be a Maurer-Cartan element. Then, since $\alpha \in CC^1(A, A)^{\Gamma} \cap \mathfrak{h}$ we see that $\alpha = \sum_{k=1}^{\infty} \alpha^{k+2} h^k$ where $\alpha^k \in \text{Hom}(V^\otimes k, V[2-k])$. Likewise, we have Maurer-Cartan elements $\hat{\alpha} \in \mathfrak{w}$. The total degree one constraint in $\Omega^*(\mathfrak{M}; \mathfrak{g})$ requires that $\hat{\alpha}^{m,n} = 0$ for $m + n \leq 2$ if a solution is to be in $\mathfrak{w}$. Therefore, $\hat{\alpha}^{0,0} = 0, \hat{\alpha}^{0,1} = 0, \hat{\alpha}^{0,2} = 0, \hat{\alpha}^{1,1} = 0$, and $\hat{\alpha}^{2,0} = 0$. In particular, Maurer-Cartan elements in $\mathfrak{w}$ will satisfy all the assumptions that we applied in Section 6.1.

6.1. Homotopies of Maurer-Cartan elements. There is a natural Lie algebra homomorphism from $\mathfrak{w}^0$ to the space of affine vector fields on $\mathfrak{w}^1$ which associates to $\gamma \in \mathfrak{w}^0$ the infinitesimal gauge transformation

$$\alpha \mapsto -d_\mathfrak{w} \gamma + [\gamma, \alpha].$$

Using the Baker-Campbell-Hausdorff formula, and the fact that we are working in pronilpotent dg Lie algebras there is a group action on the set of Maurer-Cartan elements by $\exp(\mathfrak{w})$. For $\gamma \in \mathfrak{w}^0$ and $\alpha \in MC(\mathfrak{w})$, let us denote the action of $\exp(\gamma)$ on $\alpha$ by $\circ$ while the infinitesimal action of $\gamma$ on $\alpha$ is simply denoted by $\circ$.

Let $\alpha_0$ and $\alpha_1$ be two equivalent Maurer-Cartan elements in $\mathfrak{w}^1$. We shall construct a homotopy between them. Since $\alpha_0$ is equivalent to $\alpha_1$, there exists a $\gamma \in \mathfrak{w}^0$ so that

$$\alpha_1 = \exp(\gamma) \circ \alpha_0 = \alpha_0 - d_\mathfrak{w} \gamma + [\gamma, \alpha_0] - \frac{1}{2!} [\gamma, d_\mathfrak{w} \gamma] + \frac{1}{2!} [\gamma, [\gamma, \alpha_0]] + \cdots.$$

For any element $\gamma \in \mathfrak{w}^0$, it follows that $t \gamma \in \mathfrak{w}^0$ for $t \in [0, 1]$. Consider the one parameter family of Maurer-Cartan elements

$$\alpha_t = \exp(t \gamma) \circ \alpha_{t_0}.$$

This family has the property that the derivative at each point $t \in [0, 1]$ is defined in terms of $\gamma$ and the value at the point $\alpha_t$.

$$\frac{\partial \alpha_t}{\partial t} = \gamma \circ (\exp(t \gamma) \circ \alpha_0) = -d_\mathfrak{w} \gamma + [\gamma, \alpha_t].$$

Consider the space $\Omega^*(I_t; \mathfrak{w})$ as a dg Lie algebra with differential $D = \frac{\partial}{\partial t} dt + d_\mathfrak{w}$ and the bracket induced by the bracket on $\mathfrak{w}$ and the wedge product of forms. Then $\alpha_t + \gamma dt$ is a Maurer-Cartan element because

$$D(\alpha_t + \gamma dt) + \frac{1}{2} [\alpha_t + \gamma dt, \alpha_t + \gamma dt] = d_\mathfrak{w} (\alpha_t) + \frac{1}{2} [\alpha_t, \alpha_t]$$

$$+ \left( \frac{\partial \alpha_t}{\partial t} + d_\mathfrak{w} (\gamma) + [\alpha_t, \gamma] \right) dt.$$

As the bracket is induced, the first line is 0 because we know that $\alpha_t$ is a Maurer-Cartan element in $\mathfrak{w}$ for all $t$. Likewise, the second line is zero because

$$\frac{\partial \alpha_t}{\partial t} + d_\mathfrak{w} \gamma + [\alpha_t, \gamma] = -d_\mathfrak{w} \gamma + [\gamma, \alpha_t] + d_\mathfrak{w} \gamma - (-1)^{||\gamma||} ||\alpha_t|| [\gamma, \alpha_t] = 0.$$
6.2. **Map from** \(\Omega^*(I; w) \to \Omega^*(I \times M; g^\text{neg})\). Define \(g^\text{neg} \subset g\) to be all elements of \(g\) with negative internal degree. Thus, we can recall that \(\Omega^*(M; g^\text{neg}) = w\). We wish to say that the map
\[
\iota : \Omega^*(I; w) \to \Omega^*(I \times M; g^\text{neg})
\]
will allow us to transfer Maurer-Cartan elements to homotopies of Maurer-Cartan elements. There is clearly a map \(\pi : \Omega^*(I \times M; g^\text{neg}) \to \Omega^*(I; w)\) given by restricting forms on \(I \times M\) with values in \(g^\text{neg}\) to forms on \(I\) with values in \(w\).

First, observe that \(\gamma \in w^0\) is independent of \(t\). Second, note that \(\alpha_t = \exp(t\gamma) \circ \alpha_0\), and therefore the degree in \(t\) rises with each included term of \(\gamma\). We want \(\gamma \in w^0\), so that means \(\gamma \in \Omega^m(M; g^{-m})\) or in other words, \(\gamma^{0,0} = \gamma^{0,1} = \gamma^{1,0} = 0\). Thus,
\[
\gamma = (\gamma^{0,2} + \gamma^{1,1} + \gamma^{2,0})h +
(\gamma^{0,3} + \gamma^{1,2} + \gamma^{2,1} + \gamma^{3,0})h^2 +
(\gamma^{0,4} + \gamma^{1,3} + \gamma^{2,2} + \gamma^{3,1} + \gamma^{4,0})h^3 + \cdots
\]  

(15)

Let us filter \(\alpha_t = \exp(t\gamma) \circ \alpha_0\) by our filtration \(F_k w\). The first few terms are:
\[
\alpha_t = -d_m(t\gamma) + \alpha_0 \mod F_2 w
\]
\[
\alpha_t = -d_m(t\gamma) + \alpha_0 + t \left[\gamma, \alpha_0\right] - \frac{t^2}{2!} \left[\gamma, d_m \gamma\right] \mod F_3 w
\]
\[
\vdots \quad \vdots \quad \vdots
\]
\[
\alpha_t = -d_m(t\gamma) + \alpha_0 + t \left[\gamma, \alpha_0\right] + \cdots - \frac{t^k}{k!} \left[\gamma, \left[\cdots, \left[\gamma, d_m \gamma\right]\cdots\right]\right] \mod F_{k+1} w
\]

where each term contributes only a finite number of components because of the filtration as shown in (15). Thus, the pronilpotence of \(w\) takes care of the convergence of the map.

**Theorem 6.7.** For a finite group \(G\), every homotopy \(\Gamma\) action on an \(A_\infty\) algebra \(A\) has class representatives \(\mathcal{F}_g : A \to A\) for all \(g \in G\) which comprise a strict action. Therefore, \(\mathcal{F}_g \circ \mathcal{F}_h = \mathcal{F}_{gh}\) and \(\mathcal{F}_e = \text{Id}\).

**Proof.** Let \(p\) be a closed loop in \(M \cong K(\Gamma, 1)\) based at a point \(x\) where \([p] = g \in \Gamma\). The homotopy group action of \(G\) on \(A_x\) is defined by the actions of the generators \(g\) on \(A_x\). In Figure 1 we see that the loop \(p\) defines a cylinder in \(I \times M\).
Figure 1. The correspondence between the cylinder $p \times I$ and the homotopy between $\mathcal{F}_0$ and $\mathcal{F}_1$.

construct $\mathcal{F}_h = \mathcal{G}_{0-1} \circ \mathcal{G}_{q,1} \circ \mathcal{G}_{1,0}$ in the same manner. By construction, it is clear that $\mathcal{F}_g \in [\mathcal{F}_g]$ and $\mathcal{F}_h \in [\mathcal{F}_h]$. Let

$$pq = \begin{cases} p(2s) & 0 \leq s \leq \frac{1}{2} \\ q(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$  

Now by applying Corollary 3.3 to eliminate $\mathcal{G}_{1-0} \circ \mathcal{G}_{0-1}$ and Lemma 3.2 to combine $\mathcal{F}_{p,1} \circ \mathcal{F}_{q,1}$ we have

$$\mathcal{F}_g \circ \mathcal{F}_h = \mathcal{G}_{0-1} \circ \mathcal{F}_{p,1} \circ \mathcal{G}_{1-0} \circ \mathcal{F}_{q,1} \circ \mathcal{G}_{1,0} = \mathcal{G}_{0-1} \circ \mathcal{F}_{p,1} \circ \mathcal{G}_{1,0} = \mathcal{G}_{0-1} \circ \mathcal{F}_{pq,1} \circ \mathcal{G}_{1,0} = \mathcal{F}_{gh} \in [\mathcal{F}_{gh}],$$

as desired. □

References

[1] M. Abouzaid and I. Smith, Homological mirror symmetry for the four torus (2009), available at arXiv:math.SG/0903.3065.
[2] J. F. Adams, Infinite Loop Spaces, Annals of Mathematics Studies, Princeton University Press, 1978.
[3] M. Artin, Algebra, Prentice-Hall, 1991.
[4] J. M. Boardman and R. M. Vogt, Homotopy-everything $H$-spaces, Bull. Amer. Math. Soc. 74 (1968), 1117-1122.
[5] K. Brown, Cohomology of Groups, Graduate Texts in Mathematics, Springer-Verlag, 1982.
[6] S. Eilenberg, Homology of Spaces With Operators, I, Trans. Amer. Math. Soc. 61 (1947), no. 3, 378-417, available at http://www.jstor.org/stable/1990380.
[7] K. Fukaya, Morse homotopy, $A_\infty$-category, and Floer homologies, Proceedings of GARC Workshop on Geometry and Topology ’93 Seoul, 1993, pp. 1-102.
[8] ———, Mirror symmetry of abelian varieties and multi-theta functions, J. Algebraic Geom. 11 (2002), 393–512.
[9] M. Gerstenhaber, The cohomology structure of an associative ring, Annals of Math. 78 (1963), no. 2, 267–288.
[10] E. Getzler and J. D. S. Jones, $A_\infty$-algebras and the cyclic bar complex, Illinois J. Math. 34 (1990), no. 2, 256–283.
[11] W. M. Goldman and J. J. Millson, The deformation theory of representations of fundamental groups of compact Kähler manifolds, Inst. Hautes Études Sci. Publ. Math. 67 (1988), 43–96.
[12] P. Griffiths and J. Morgan, Rational Homotopy Theory and Differential Forms, Progress in Mathematics, vol. 16, Birkhäuser, 1981.
[13] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, Springer-Verlag, 1977.
[14] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[15] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, 1995, pp. 120-139, available at arXiv:alg-geom/9411018.

[16] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216, available at arXiv:q-alg/9709040.

[17] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), 2001, pp. 203–263.

[18] S. Mac Lane, *Homology*, Springer-Verlag, 1975.

[19] M. Markl and S. Shnider, *Associahedra, cellular W-constructions and products of $A_{\infty}$-algebras*, Trans. Amer. Math. Soc. **358** (2006), no. 6, 2353–2372, available at arXiv:math.AT/0312277.

[20] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. 271, Springer, 1972.

[21] J. McCleary (ed.), *Higher Homotopy Structures in Topology and Mathematical Physics*, Contemporary Mathematics, AMS Bookstore, 1999.

[22] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, Princeton University Press, 1974.

[23] M. Penkava and A. Schwarz, *$A_{\infty}$ algebras and the cohomology of moduli spaces*, Lie groups and Lie algebras: E. B. Dynkin’s Seminar, 1995, pp. 91–107, available at arXiv:hep-th/9408064.

[24] A. Polishchuk and E. Zaslow, *Categorical mirror symmetry in the elliptic curve*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), 2001, pp. 275–295.

[25] P. Seidel, *Homological mirror symmetry for the genus two curve* (2008), available at arXiv:math.AG/0812.1171.

[26] P. Seidel, *Homological mirror symmetry for the quartic surface* (2003), available at arXiv:math.SG/0310414.

[27] P. Seidel, *Fukaya Categories and Picard-Lefshetz Theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2008.

[28] J. D. Stasheff, *Homotopy Associativity of $H$-Spaces. I, II*, Trans. Amer. Math. Soc **108** (1963), no. 2, 275–312.

[29] J. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*, Lecture Notes in Mathematics, Springer-Verlag, 1992.

[30] M. Sugawara, *A condition that a space is group-like*, Math. Jour. Okayama Univ. **7** (1957), no. 2, 123-149.

[31] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, Springer-Verlag, 1983.

E-mail address: emmas@math.uchicago.edu

5734 S. University Avenue Chicago, Illinois 60637 USA