On shared and multiple information

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1 Introduction

The goal of this work is to address three outstanding problems in information theory. **Problem one** is the definition of a non-negative decomposition of the information conveyed by two or more sources about a target variable into the specific contribution of each possible combination of the sources \([1]\). **Problem two** is the definition of a measure of information shared by several sources about the target variable \([1]\). **Problem three** is the definition of a measure of multiple information, that is, the extension of mutual information to more than two variables \([2, 3]\).

We assume that the reader is familiar with Hu and Yeung’s set-theoretic structure of Shannon’s information theory \([2, 4]\) and with the partial information decomposition and partial information diagrams of Williams and Beer \([1]\). Here we briefly summarize the main concepts of these theories.

Hu and Yeung \([2, 4]\) proposed a correspondence between information theory and set theory based on a substitution of symbols. They showed that this correspondence induces a measure (called \(I\)-measure) on the atoms of an information diagram, that is, a Venn diagram in which each variable is assigned a region of size corresponding to its entropy. Figures 1 and 2 illustrate the information diagrams for two and three variables. The problem with the Hu and Yeung approach is that it is not always clear how to interpret the \(I\)-measure for more than two variables, especially given that the \(I\)-measure can be negative.

Williams and Beer \([1]\) proposed a correspondence between set theory and information theory based on the intuitive idea that a number of sources, \(\{X_1, \ldots, X_N\}\), can share information about a target variable \(Y\). They encoded this intuition into a number of axioms that a desirable measure of shared information should satisfy. They showed that any measure of shared information that satisfies these axioms induces a measure (called PI-function) on the atoms of a partial information diagram, that is, a Venn diagram in which each subset \(A \subseteq \{X_1, \ldots, X_N\}\) is assigned a region of size \(I(A; Y)\). Figures 1 and 2 illustrate the partial information diagrams for two and three sources. Williams and Beer proposed that the PI-function has the potential to capture our intuition of synergy, redundancy and unique information. There are currently two issues with the approach proposed by Williams and Beer. The first issue \([5]\) is that the proposed axioms do not identify a unique measure of shared information and, to date, no agreed-upon measure of shared information has been found. The second issue is how to relate PI-diagrams to the information diagrams of Hu and Yeung.

We proceed as follows. We introduce a novel expansion of the Shannon mutual information on singled-out features of the target variable. We call each set of singled-out features a descriptor of the target. To address problem one, we put forward the idea that the choice of the descriptor affects the way in which the sources interact to provide the total information. We build a measure of information shared by the sources about the descriptor and we show that this measure induces a non-negative PI-function. To address problem two, we extend the descriptor-dependent measure of shared information to a measure of information shared by the sources about the target. To address problem three, we show that the proposed measure of shared information allows linking PI-diagrams and information diagrams and allows defining a measure of multiple information that is compatible with both Hu and Yeung, and Williams and Beer set-theoretic correspondences.

1.1 Notations and conventions

We use uppercase letters \((X, Y, \ldots)\) to indicate random variables and lowercase letters \((x, y, \ldots)\) to indicate specific outcomes. \(\mathbb{X}\) denotes the alphabet of \(X\) and \(|\mathbb{X}|\) the cardinality of \(\mathbb{X}\). We denote with \(X = \{X_1, \ldots, X_N\}\) a \(N\)-variate random variable and with \(x = [x_1, \ldots, x_N]\) its outcomes. When there is no confusion we omit the curly braces. We also denote with \(P(X)\) the power-set of \(X\) and with \(P_1(X)\) the set \(P_1(X) = P(X) \setminus \emptyset\).

We denote probability distributions with a capital letter, e.g., \(P(X, Y)\), and values of specific realisations with lower case shorthand, e.g., \(p(x, y)\) for \(P(X=x, Y=y)\).
H(X) denotes the Shannon entropy. $I(X; Y)$ denotes the Shannon mutual information and $I(X; Y \mid Z)$ the conditional mutual information. When there is no ambiguity, we use the shorthand $I(X; Y \mid z)$ for $I(X; Y \mid Z = z)$.

2 Expansion of mutual information on a descriptor of the target variable

Let $P(X, Y)$ be a discrete probability distribution. Without implying causal relationship, we call $X$ the source variable and $Y$ the target variable. Consider a deterministic function $f_1 : \mathcal{Y} \rightarrow \mathcal{Y}^1$. Because $X$ and $Y^1$ are independent given $Y$, we can rewrite Shannon’s mutual information between $X$ and $Y$ \[1\] as follows:

$$I(X; Y) = I(X; Y^1) + \sum_{y^1 \in \mathcal{Y}^1} p(y^1) \cdot I(X; Y \mid y^1)$$ \[1\]

Equation \[1\] corresponds to breaking the mutual information onto different features of the target variable. The idea is that a deterministic function partitions the elements of a discrete random variable into subsets that can be interpreted as a single out feature of $Y$ \[2\]. For example, let the outcomes of $Y$ be objects and let $f_1 : \mathcal{Y} \rightarrow \mathcal{Y}^1$ be the function that singles out the color of an object. Each element of $\mathcal{Y}^1$ is a subset of objects of $\mathcal{Y}$ of a given color. Using Equation \[1\], $I(X; Y)$ can then be split into two parts. The first part is the average of terms of the form $I(X; Y \mid \text{color})$, i.e., the information between $X$ and the elements of $Y$ of a given color. The second part is the information, $I(X; Y^1)$, conveyed by $X$ about the color variable $Y^1$.

We can further single out features from $Y^1$ through a deterministic function $f_2 : \mathcal{Y}^1 \rightarrow \mathcal{Y}^2$, obtaining

$$I(X; Y) = I(X; Y^2) + \sum_{y^2 \in \mathcal{Y}^2} p(y^2) \cdot I(X; Y^1 \mid y^2) + \sum_{y^1 \in \mathcal{Y}^1} p(y^1) \cdot I(X; Y \mid y^1)$$ \[2\]

\[1\]We provide a step by step derivation of Equation \[1\] in Appendix A.1

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**Figure 1:** Examples of information diagrams and partial information diagrams. In information diagrams each variable $X_i$ is assigned a region of size $H(X_i)$. Shannon’s mutual information is visualized as the intersection of two regions of size $H(X_i)$ and $H(X_2)$, panel a. The generalization of mutual information to more than two variables is visualized as the intersection of all regions in the information diagram, panel b. In partial information diagrams each subset $A \subseteq \{X_1, \ldots, X_N\}$ is assigned a region of size $I(A; Y)$, panels c and d.
Resuming our example with colors, let \( f_2 : \mathcal{Y} \rightarrow \mathcal{Y} \) be the function that returns the temperature of a color. We can expand \( I(X; Y) \) as the average of \( I(X; Y^1 | \text{warm}) \) and \( I(X; Y^1 | \text{cool}) \), plus the information, \( I(X; Y^2) \), conveyed by \( X \) about the temperature feature.

If we denote \( Y^0 = Y \) and rewrite \( I(X; Y^2) \) as \( I(X; Y^2 | Y^3) \), where \( |Y^3| = 1 \), we can write Equation (2) in a compact form

\[
I(X; Y) = \sum_{\ell=1}^{3} \sum_{y' \in \mathcal{Y}} p(y') \cdot I(X; Y^{\ell-1} | y')
\]  

We can generalize Equation (3) to any deterministic Markov chain \( \mathcal{Y} \triangleq Y^0 \rightarrow Y^1 \rightarrow \cdots \rightarrow Y^L \), identified by the deterministic functions \( f_{\ell} : \mathcal{Y}^{\ell-1} \rightarrow \mathcal{Y} \), with \( Y^0 = Y \) and \( |\mathcal{Y}^L| = 1 \). Without loss of generality, we assume \( \mathcal{Y}^\ell \neq \mathcal{Y}^{\ell+1} \) for all \( \ell \). We obtain the expansion of \( I(X; Y) \)

\[
I(X; \mathcal{Y}) = \sum_{\ell=1}^{L} \sum_{y' \in \mathcal{Y}^\ell} p(y') \cdot I(X; Y^{\ell-1} | y') = I(X; Y).
\]

We say that \( \mathcal{Y} \) is a descriptor of \( Y \) and we denote with \( \Omega_\mathcal{Y} \) the set of all possible descriptors of any length. An example of computation of Equation (4) is shown in Figure 2.

We introduce two special descriptors. First, we note that the canonical expression of the Shannon mutual information \( \mathbb{H} \) corresponds to the expansion of \( I(X; Y) \) obtained for the descriptor \( S \triangleq Y^0 \rightarrow Y^1 \), with \( |\mathcal{Y}^1| = 1 \). We call \( S \) the Shannon descriptor. The Shannon descriptor corresponds to considering all possible features of \( Y \) at once. Second, let \( Y = \{Y_1, \ldots, Y_N\} \) we introduce the canonical descriptor \( C_Y = Y^0 \rightarrow \cdots \rightarrow Y^N \) obtained using the set of functions \( f_{\ell}(y^{\ell-1}) = [y_1, \ldots, y_{N-\ell+1}] \triangleq [y_{\ell-1}^{\ell-1}, \ldots, y_{N-\ell+1}^{\ell-1}] \). In other words, given \( [y_1, \ldots, y_N] \in \mathcal{Y} \), each step in the canonical chain removes one dimension, as follows, \( y^1 = f_1(y) = [y_2, \ldots, y_N] \), \( y^2 = f_2(y^1) = [y_3, \ldots, y_N] \), etc.

### 3 Addressing problem one

Let \( X = \{X_1, \ldots, X_N\} \) and let \( A_1, \ldots, A_K \in P_1(X) \) be nonempty and potentially overlapping subsets of \( X \), called sources \( \mathbf{s} \). We propose that the way the sources interact to convey the information about the target variable depends on the choice of the descriptor of \( Y \). To illustrate this idea, we consider example UNQ from \( \mathbf{[8]} \). The zero bits of shared information expected for UNQ are often explained by noticing that we can partition \( Y \) into two subsets. These two subsets are shown in Figure 2a-c and are built so that \( X_1 \) does not explain any of the information conveyed by \( X \) within each of the subsets, \( X_2 \), instead, explains all of this within information. However, \( X_2 \) does not explain any of the information conveyed by \( X \) between the two subsets, while \( X_1 \) explains all of it. \( X_1 \) and \( X_2 \) thus convey complementary information and we expect their shared information to be non-null. This example suggests that any decomposition of the total information into the specific contribution of each possible combination of sources should be a function of the descriptor \( Y \).

To build such decomposition, we follow the same approach used by Williams and Beer for constructing the \( I_{\text{min}} \) measure \( \mathbf{[1]} \). We replace each term in expansion (4) with the minimum information that any source provides about each feature of \( Y \), as singled out by the descriptor \( \mathcal{Y} \)

\[
I(A_1; \ldots; A_K; \mathcal{Y}) \triangleq \sum_{\ell=1}^{L} \sum_{y' \in \mathcal{Y}^\ell} p(y') \cdot \min_{k=1, \ldots, K} \left\{ I(A_k; Y^{\ell-1} | y') \right\}
\]

In Appendix we show that Equation (5) satisfies the Williams and Beer axioms. We also show that the Shannon descriptor maximizes Equation (5).

Following the same approach used by Williams and Beers to prove that the \( I_{\text{min}} \) measure induces a non-negative PI-function \( \mathbf{[1]} \), it is possible to show that Equation (5) induces a non-negative PI-function

\[
\mu(A_1; \ldots; A_K; \mathcal{Y}) = I(A_1; \ldots; A_K; \mathcal{Y}) - \sum_{\ell=1}^{L} \sum_{y' \in \mathcal{Y}^\ell} p(y') \cdot \max_{\{B_1, \ldots, B_H\} \in \{A_1, \ldots, A_K\}^c} \left\{ \min_{n=1, \ldots, H} \left\{ I(B_n; Y^{\ell-1} | y') \right\} \right\}
\]

where \{\( A_1, \ldots, A_K \)^-\} denotes the subsets of \( A(X) := \{a \in P_1(P_1(X)) : \forall A_i, A_j \in a, A_i \not\subset A_j\} \), which are covered by \{\( A_1, \ldots, A_K \)\} according to the redundancy partial order defined in \( \mathbf{[1]} \).

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\(^2\)See also Appendix C from \( \mathbf{[9]} \).
identified by show the termsof the expansions features of The information unique to that is not conveyed by either that is not conveyed by either X_i nor X_j.

\[
I(X; Y) = I(X_1; Y) + I(X_2; Y) - I(X_1; X_2; Y)
\]

\[
P(X, Y) = \begin{array}{ccc}
X_1 & X_2 & Y & P(X, Y) \\
0 & 0 & y_1 & 1/4 \\
0 & 1 & y_2 & 1/4 \\
1 & 0 & y_3 & 1/4 \\
1 & 1 & y_4 & 1/4
\end{array}
\]

Figure 2: Consider the canonical example UNQ from [3], panel (a). Let Y = Y → Y^1 → Y^2 be the descriptor, shown in panels (b)-(d), identified by f_1(y_1) = f_1(y_2) = y_1, f_1(y_3) = f_1(y_4) = y_2, and f_2(y_1) = f_2(y_2) = y_2^2. Panel (b) to (d) show the terms of expansion I(X; Y), I(X_1; Y) and I(X_2; Y), respectively. Notice that I(X; Y^1 | y_1^2) = I(X_1; Y^1 | y_1^2) while I(X_2; Y^1 | y_1^2) = 0. Furthermore, I(X; Y^0 | y_1^2) = I(X_1; Y^0 | y_1^2) and I(X; Y^0 | y_1^2) = I(X_2; Y^0 | y_1^2), while I(X_1; Y^0 | y_1^2) = I(X_2; Y^0 | y_1^2) = 0. We can thus write I(X_1; X_2; Y) as the sum of terms (highlighted in grey) that reflect the specific unique contribution of either X_1 and X_2. The shared information according to Equation (5) is zero and is obtained as the weighted sum of the zero terms in panels (b) and (c). Panels (d) to (f) show the show the terms of the expansions I(X; S), I(X_1; S) and I(X_2; S), for the Shannon descriptor S. We cannot write I(X; S) in terms of unique contributions of X_1 and X_2. We thus expect the shared information to be non null. Equation (5) returns one bit of shared information.

While Equation (5) cannot be interpreted as a desirable measure of information shared by the sources about the target variable—because it is not univocally identified by Y—is, it can be interpreted as a measure of the information shared by the sources about the descriptor. Accordingly, we propose to interpret Equation (6) as the descriptor-dependent contribution of each possible combination of the sources to the total information. For K = 2, Equation (6) provides measures of descriptor-dependent redundancy, synergy, and unique information with simple and intuitive interpretations. The redundancy \( \mu(A_1: A_2; Y) \) is the minimum information conveyed by A_1 and A_2 about the features of Y singled-out by Y. The information unique to A_1, \( \mu(A_1; Y) \), is the information about the singled-out feature of Y conveyed by A_1 beyond the information conveyed by X_2. Finally the synergistic information, \( \mu(A_1, A_2; Y) \) is the information about the singled-out features of Y that is not conveyed by either X_1 nor X_2.
4 Addressing problem two

We build a measure of the information shared by the sources about the target variable from Equation (5) by considering the minimum of Equation (5) over all possible descriptors of $Y$, as follows

\[ I(A_1: \ldots : A_K; Y) \triangleq \min_{Y \in \Omega_L} \left\{ I(A_1: \ldots : A_K; Y) \right\} \]  

(7)

The minimization removes the dependency of the shared information measure on the descriptor. We can thus interpret Equation (10) as a measure of information shared about the target variable.

Equation (10) satisfies several properties that have been proposed to be desirable in a measure of information shared about $Y$. First, Equation (10) returns the expected values of shared information for the canonical examples from the literature\[^{5, 8, 10, 11}\]. Second, Equation (10) is non-negative. Third, Equation (10) satisfies the Williams and Beer axioms.\[^{7}\] Fourth, Equation (10) satisfies the identity property. Fifth, Equation (10) satisfies the additivity property. Sixth, Equation (10) also satisfies the Blackwell property.\[^{5}\] Seventh, Equation (10) satisfies the combined secret sharing property. Eighth, Equation (10) is guaranteed to be non-negative, a property which has been called local positivity.\[^{15}\] For $K \geq 2$ a descriptor that minimizes Equation (10) for all choices of $\{A_1, \ldots, A_K\}$ does not necessarily exist. Equation (10), however, is thus not guaranteed to satisfy local positivity for $K > 2$. This result is in agreement with the fact that there can be no measures of shared information that satisfies the Williams and Beer axioms, the identity property and local-positivity for $K > 2$.\[^{15}\] In other words, for $K > 2$ we might not be able to generate a PI-diagram in which all intersections can be interpreted as information shared about the sources by the target variable.

The size of $\Omega_L$ grows according to the rate of Bell numbers. Computing Equation (10) proves prohibitive for $|\mathcal{Y}| > 6$ on a normal personal computer. In appendix A.3 we show that it is possible to considerably reduce the computation burden by restricting the domain of the minimization to the set $\Omega_{LE}$ of the descriptors satisfying $|f_{\mathcal{Y}}^{-1}(Y')| \leq 2$ for all $Y' \in \mathcal{Y}'$ and all $l' = 1, \ldots, L$. In other words

\[ I(A_1: \ldots : A_K; Y) = \min_{Y \in \Omega_{LE}} \left\{ I(A_1: \ldots : A_K; Y) \right\} \]  

(10)

5 Addressing problem three

The correspondence between Shannon's information measures and information diagrams for two variables has led to hypothesize the existence of a generalization of the Shannon mutual information to more than two variables.\[^{2}\] Figure 3. Currently, no agreed-upon measure of multiple information has been identified.\[^{3}\] Based on the information-diagram correspondence, we propose that a measure, $I(X_1; \ldots; X_N)$, of multiple information should satisfy the following properties:

1. **Non-negativity**: $I(X_1; \ldots; X_N) \geq 0$.
2. **Symmetry**: $I(X_1; \ldots; X_N)$ is invariant to permutations of $X_1, \ldots, X_N$.
3. **Monotonicity**: $I(X_1; \ldots; X_{N-1}) \geq I(X_1; \ldots; X_N)$.

\[^{3}\]See Appendix A.4 for a description of these examples and their decompositions.

\[^{4}\]See Appendix A.5 for proofs of the Williams and Beer properties, the identity property, the Blackwell property, and the combined secret sharing property.

\[^{5}\]For the case $K = 2$, see the proof of the identity property, Appendix A.5.
4. **Self-information:** $I(X_1; \ldots; X_N)$ reduces to the Shannon mutual information and the Shannon entropy for $N = 2$ and $N = 1$, respectively.

5. **Blackwell property:** If $X_{N-1} = f(X_N)$ then $I(X_1; \ldots; X_N) = I(X_1; \ldots; X_{N-1})$, Figure 3b. This also implies $I(X_1; \ldots; X_N) = H(X_1)$ if $X_1, \ldots, X_N$ form a Markov chain $X_N \rightarrow X_{N-1} \rightarrow \cdots \rightarrow X_1$, Figure 3f.

6. **Shannon property:** If $\bar{n} \in \{1, \ldots, N\}$ exists such that $p(x_{\bar{n}}, x_n) = p(x_{\bar{n}}) \cdot p(x_n)$ for all $\bar{n} \neq n$ and all $x \in X$, then $I(X_1; \ldots; X_N) = 0$, Figure 3d. This also implies $I(X_1; \ldots; X_N) = 0$ if $p(x_1, \ldots, x_N) = p(x_1) \cdot \cdots \cdot p(x_N)$ for all $x \in X$, Figure 3d. However, $I(X_1; \ldots; X_N) = 0$ does not imply $p(x_{n_1}, x_{n_2}) = p(x_{n_1}) \cdot p(x_{n_2})$ for any $n_1, n_2 = 1, \ldots, N$, Figure 3d.

Properties 2-4 are reminiscent of the Williams and Beer axioms for shared information [11]. This correspondence suggests that it might be possible to derive a measure of multiple information from a measure of shared information. To this aim we note that, for any measure of shared information that satisfies the identity property, the PI-diagram for two sources, Figure 4b, reduces to to the information diagram for two variables when $Y = X$, Figure 4c. Based on this correspondence, we propose to define multiple information $I(X_1; \ldots; X_N)$, as follows

$$I(X_1; \ldots; X_N) \triangleq I(X_1; \ldots; X_N; X_1, \ldots, X_N)$$

(11)

Local-positivity, symmetry, monotonicity, self-information and the Blackwell property follow from the properties of Equation (10). In Appendix A.6 we also show that Equation (11) satisfies the Shannon property.

**Figure 3:** Information diagrams for three variables. a) Standard representation. Each variable is assigned a region of size corresponding to its entropy. The multiple mutual information is visualized as the intersection of the three regions. b) If $X_2 = f(X_3)$ we expect $I(X_1; X_2; X_3) = I(X_1; X_2)$. c) If $X_2 = f(X_3)$ and $X_1 = f'(X_3)$ we expect $I(X_1; X_2; X_3) = H(X_1)$. d) If $p(x_1, x_2, x_3) = p(x_1) \cdot p(x_2) \cdot p(x_3)$ and $p(x_2, x_3) = p(x_2) \cdot p(x_3)$ we expect $I(X_1; X_2; X_3) = 0$. e) If $p(x_1, x_2, x_3) = p(x_1) \cdot p(x_2) \cdot p(x_3)$ we expect $I(X_1; X_2; X_3) = 0$. f) We expect that it is possible to have $I(X_1; X_2; X_3) = 0$ even when $I(X_1; X_2), I(X_2; X_3), I(X_1; X_3) > 0$. 

6
6 Discussion

The measures of descriptor-dependent and descriptor-independent shared information proposed in this work are a direct extension of the approach proposed by Williams and Beer. Instead of minimizing the terms of a point-wise decomposition, we minimize the terms of a novel decomposition of mutual information, Equation (4). Unlike point-wise decompositions \[17\], Equation (4) does not attempt to decompose mutual information into singled-out outcomes of the target variable. Instead, Equation (4) decomposes mutual information onto singled-out features of the target variable. Mutual information is intrinsically a non-point-wise measure, as epitomized by the fact that mutual information is null whenever the alphabet of any of the two arguments has cardinality one. While the terms of point-wise decompositions cannot be interpreted in terms of Shannon’s information quantities, the terms of Equation (4) are themselves mutual informations.

We proposed that, for more than two sources, problem one should be disentangled from problem two. The problem of quantifying the information conveyed specifically by a collection of sources depends on the choice of the descriptor of the target variable. Instead, shared information is descriptor-independent. Our approach allows reconciling our intuition that the information conveyed specifically by a collection of sources should be non-negative with the results from \[16\] that a non-negative measure of shared information is not compatible with the partial information decomposition. An important open research question will be to identify the sufficient conditions that \[P(X,Y), N \geq 2\] must satisfy to ensure that a single descriptor exists which minimizes equation (10) for all choices of the collection of sources.

To our knowledge, our measure of shared information is the only proposed measure satisfying the ten highlighted properties. It is easy to show that our measure does not satisfy left monotonicity \[15\]. We propose that no measure of shared information exists, which satisfies left monotonicity and is compatible with the accepted values of shared information for the canonical examples. This is because distribution \[AND\] can be obtained from distribution \[UNQ\] through a transformation of the realizations of the target. However the information shared by \[X_1\] and \[X_2\] in \[AND\] is expected to be higher than that in \[UNQ\]. This also implies that no measure of shared information exists which is compatible with the expected breakdown for the canonical examples and which satisfies strong symmetry and the left chain rule \[15\].

We proposed a new measure of multiple information. We note that among the measures of multiple information proposed in the literature \[3\] ours and McGill’s interaction information \[18, 2\] are compatible with the set-theoretic intuition of multiple information that we derive from the information diagrams. However, unlike interaction information \[1, 2\], our measure of multiple information is guaranteed to be non-negative.

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A Supplemental information and supporting proofs

A.1 Step-by-step derivation of Equation (1)

Theorem 1. Consider $P(X, Y)$ discrete and $f : \mathcal{Y} \rightarrow \mathcal{Y}^1$ deterministic. We have $I(X; Y) = I(X; Y^1) + I(X; Y \mid Y^1)$.

Proof. Remember that any deterministic function $f : \mathcal{Y} \rightarrow \mathcal{Y}^1$ partitions the elements of $\mathcal{Y}$ into subsets that correspond to the element of the $\mathcal{Y}^1$. That means that for $y^1 \in \mathcal{Y}^1$ we have $y^1 \subseteq \mathcal{Y}$. From the definition of the Shannon mutual information, we have

$$I(X; Y) = \sum_{x \in X} \sum_{y \in Y} p(x, y) \cdot \log \frac{p(x, y)}{p(x) \cdot p(y)} = H(X) + \sum_{x \in X} \sum_{y \in Y} p(x, y) \cdot \log \frac{p(x, y)}{p(y)}$$

(a) $= H(X) + \sum_{x \in X} \sum_{y \in y^1} \sum_{y \in Y} p(x, y) \cdot \log p(x, y) \cdot \log p(x, y) \cdot p(y) \cdot p(y) = H(X) + \sum_{x \in X} \sum_{y \in Y^1} p(x, y) \cdot \log \frac{p(x, y \mid y^1)}{p(y \mid y^1)}$

(b) $= H(X) - H(X \mid Y^1) + I(X; Y \mid Y^1) = I(X; Y^1) + I(X; Y \mid Y^1)$

c

where in (a) $r(y, Y^1)$ denotes the number of elements of $Y^1$ that include $y$. For (b) we used the relationship $p(x, y) / p(y) = p(x, y \mid y^1) / p(y \mid y^1)$. In (c) we added and subtracted $H(X \mid Y^1)$.

A.2 Properties of Equation (5)

Theorem 2 (Williams and Bear axioms). Equation (5) satisfies the following properties:

1. Symmetry: $I(A_1; \ldots : A_K; \mathcal{Y})$ does not depend on the order of $A_1, \ldots, A_K$
2. Self-redundancy: $I(A; A; \mathcal{Y}) = I(A; \mathcal{Y})$
3. Monotonicity: $I(A_1; \ldots : A_K; \mathcal{Y}) \leq I(A_1; \ldots : A_{K-1}; \mathcal{Y})$

Proof. Property one follows from the fact that the minimum operator $\min\{a_1, \ldots, a_K\}$ does not depend on the order of the elements of $\{a_1, \ldots, a_K\}$. Property two follows from the fact that equation $I(A; A; \mathcal{Y})$ reduces to decomposition (4) for any choice of descriptor of $\mathcal{Y}$. We define

$$\mathcal{Y}' \triangleq \arg\min_{\mathcal{Y} \in \Omega_\mathcal{Y}} \left\{ I(A_1; \ldots : A_K; \mathcal{Y}) \right\} \text{ and } \mathcal{Y}'' \triangleq \arg\min_{\mathcal{Y} \in \Omega_\mathcal{Y}} \left\{ I(A_1; \ldots : A_{K-1}; \mathcal{Y}) \right\}.$$

Because $\min\{a_1, \ldots, a_K\} \leq \min\{a_1, \ldots, a_{K-1}\}$ for any set $\{a_1, \ldots, a_K\}$, we have

$$I(A_1; \ldots : A_K; Y) = I(A_1; \ldots : A_K; \mathcal{Y}') \leq I(A_1; \ldots : A_K; \mathcal{Y}'') \leq I(A_1; \ldots : A_{K-1}; \mathcal{Y}'') = I(A_1; \ldots : A_{K-1}; \mathcal{Y})$$

This property three.

We can now prove the following theorem on the upper bound of Equation (5).

Theorem 3. For any $\mathcal{Y} \in \Omega_\mathcal{Y}$ the relationship $I(A_1; \ldots : A_K; \mathcal{Y}) \leq I(A_1; \ldots : A_K; S)$ holds.

Proof. Suppose that $I(A_1; \ldots : A_K; S) < I(A_1; \ldots : A_K; \mathcal{Y})$. By construction, $I(A_1; \ldots : A_K; S) = I(A_k; S) = I(A_k; Y)$ for some $k \in \{1, \ldots, K\}$. Because $I(A_k; Y) = I(A_k; \mathcal{Y})$, we have $I(A_k; \mathcal{Y}) < I(A_1; \ldots : A_K; \mathcal{Y})$ which violates the monotonicity rule.

A.3 Refining a descriptor reduces the information shared by the sources

Let $\mathcal{Y} = Y_0^0 \frac{f_1}{f_0} \ldots \frac{f_{L-1}}{f_0} Y_{L-0} Y_0^1 \frac{f_e}{f_0} \ldots \frac{f_e}{f_e} Y^1 \frac{f_{L}}{f_L} \ldots \frac{f_{L}}{f_L} Y^L \in \Omega_\mathcal{Y}$. We build a new descriptor $\mathcal{Y}^* \in \Omega_\mathcal{Y}$ from $\mathcal{Y}$, as follows: $\mathcal{Y}^* = Y^0 \frac{f_1}{f_1} \ldots \frac{f_{L-1}}{f_{L-1}} Y_{L-0}^* Y^1 \frac{f_{L}}{f_{L}} \ldots \frac{f_{L}}{f_{L}} Y^L$ with $f_e \sigma f_b = f_e$. We say that $\mathcal{Y}^*$ is a refinement of $\mathcal{Y}$. An example of refinement is shown in Figure 3.

Theorem 4 (Refining a descriptor reduces the information shared by the sources). Let $\mathcal{Y}^*$ be a refinement of $\mathcal{Y} \in \Omega_\mathcal{Y}$, then

$$I(A_1; \ldots : A_K; \mathcal{Y}^*) \leq I(A_1; \ldots : A_K; \mathcal{Y}).$$
Theorem 4 guarantees that \( \mathcal{Y}' \) is a refinement of \( \mathcal{Y} \). Figure 5: Example of refinement of a descriptor. \( \mathcal{Y}' \) belongs to \( \Omega^2_2 \), while \( \mathcal{Y} \) only belongs to \( \Omega_Y \).

**Proof.** Let \( \hat{k} \in \arg \min_{k=1,...,K} \{ I(A_k; Y^{\ell-1} | y') \} \). Applying Equation (1) we have

\[
I(A_k; Y^{\ell-1} | y') = I(A_k; Y^* | y') + \sum_{y^{\ell-1} \in f_k(y')} I(A_k; Y^{\ell-1} | y^{\ell-1}, y') \\
\geq \min_{k=1,...,K} \{ I(A_k; Y^* | y') \} + \sum_{y^{\ell-1} \in f_k(y')} \min_{k=1,...,K} \{ I(A_k; Y^{\ell-1} | y^{\ell-1}, y') \}
\]

By construction we have

\[
I(A_1: ...: A_K; \mathcal{Y}) - I(A_1: ...: A_K; \mathcal{Y}^*) = \\
I(A_k; Y^{\ell-1} | y') - \left( \min_{k=1,...,K} \{ I(A_k; Y^* | y') \} + \sum_{y^{\ell-1} \in f_k(y')} \min_{k=1,...,K} \{ I(A_k; Y^{\ell-1} | y^{\ell-1}, y') \} \right) \geq 0
\]

The theorem is proved. \( \square \)

Theorem 4 has the following implications.

**Corollary 4.1.** For any \( \mathcal{Y} \in \Omega_Y \) we have

\[
I(A_1: ...: A_K; \mathcal{Y}) \leq I(A_1: ...: A_K; S)
\]

**Proof.** The proof follows from the fact that any \( \mathcal{Y} \in \Omega_Y \) with \( \mathcal{Y} \neq S \) can be viewed as a refinement of \( S \). \( \square \)

**Corollary 4.2.** The following equality holds

\[
\min_{\mathcal{Y} \in \Omega_Y} \left\{ I(A_1: ...: A_K; \mathcal{Y}) \right\} = \min_{\mathcal{Y} \in \Omega_Y} \left\{ I(A_1: ...: A_K; \mathcal{Y}') \right\}
\]

**Proof.** Let \( \mathcal{Y} \in \arg \min_{\mathcal{Y} \in \Omega_Y} \left\{ I(A_1: ...: A_K; \mathcal{Y}) \right\} \) with \( \mathcal{Y} \notin \Omega^2_2 \). We can always build \( \mathcal{Y}^* \in \Omega^2_2 \) by refining \( \mathcal{Y} \). Theorem 4 guarantees that \( \mathcal{Y}^* \in \arg \min_{\mathcal{Y} \in \Omega_Y} \left\{ I(A_1: ...: A_K; \mathcal{Y}') \right\} \). \( \square \)

### A.4 Two-sources examples

In Section 4 we showed that for the case \( X = \{X_1, X_2\} \) the terms, \( \mu(X_1: X_2; Y), \mu(X_1; Y), \mu(X_2; Y), \mu(X_1, X_2; Y) \), of the PI-function induced by Equation (10) are guaranteed to be non-negative. Williams and Beer proposed that for a desirable measure of shared information we should be able to interpret these terms as the redundant information, the information unique to \( X_1 \), the information unique to \( X_2 \) and the synergistic information about \( Y \), respectively. In Table 1 we show that Equation (10) satisfies the interpretation proposed by Williams and Beer by summarizing the values obtained with our decomposition for the canonical examples from the literature [5][8][10][11].
To analyze the geometrical properties of the examples in Table 1 and their decomposition we introduce the following graphical convention to visualize the probability distributions \( P(X, Y) \) for \( X = \{X_1, X_2\} \), Figure 6. We denote with shapes \{\( \circ, \bullet, \times, +, \ldots \)} the elements of \( \mathcal{Y} \). We denote with points in \( \mathbb{R}^2 \) the elements of \( \mathcal{X} \). At each position \( x \in \mathcal{Y} \) we draw the shape of those elements \( y \in \mathcal{Y} \) for which \( p(x, y) > 0 \). Next to each shape we indicate the value \( p(x, y) \). For example, \( \bigcirc^{0.3} \) indicates \( p(x, \bigcirc) = 0.3 \). If \( P(X, Y) \) is constant for all \( x \in \mathcal{X} \) and all \( y \in \mathcal{Y} \), we omit the indication of the probability value, for convenience. If \( x = [x_1, x_2], x' = [x'_1, x'_2] \in \mathcal{X} \) are such that \( x_n = x'_n \), for \( n = 1 \) or \( n = 2 \), we highlight this by connecting the two points in \( \mathbb{R}^2 \) with a line.

Distribution RDN, Figure 6, is the archetype of redundant information \[8\]. Both \( X_1 \) and \( X_2 \) are isomorphic to \( X \) and thus convey the same information about \( Y \) than \( X \) itself. Our decomposition returns one bit of information shared by \( X_1 \) and \( X_2 \). All other terms of the breakdown have zero information, Table 1.

We can transform the redundancy in RDN into information unique to \( X_1 \) by imagining of moving the two points in RDN so that they become aligned parallel to the first coordinate axis, Figure 6b. We call UNQ1 this distribution in which \( X_1 \) is still isomorphic to \( X \) but all information available in \( X \) is “masked” to \( X_2 \). Our decomposition returns one bit of information unique to \( X_1 \) and zero information for all other elements of the breakdown for this example, Table 1. If we align the two points in RDN parallel to the second coordinate axis, Figure 6c, we obtain the complementary distribution UNQ2 with one bit of information unique to \( X_2 \), Table 1.

We propose that UNQ1 and UNQ2 are the “building blocks” of unique information. Any distribution \( P(X_1, X_2, Y) \) in which either \( X_1 \) and \( X_2 \) convey unique information must include some variation of the maskings between different outcomes of \( Y \) observed in UNQ1 or UNQ2. For example, IMPERFECT RDN, Figure 6d, can be thought of as “leaking” part of the redundant information RDN into information unique to \( X_2 \) by means of masking some of the target information available in \( X \) to \( X_1 \). Our decomposition provides 0.93 bit of redundancy and 0.07 bit of information unique to \( X_2 \), Table 1. Example UNQ, Figure 6, is the archetype of unique information. Using the descriptor previously shown in Figure 2, our decomposition provides one bit of unique information both for \( X_1 \) and for \( X_2 \) and no redundant or synergistic information, Table 1.

We can generate a distribution with synergistic information by means of combining UNQ1 and UNQ2 to obtain example SYN shown in Figure 6f. An alternative way to derive SYN is to mirror either UNQ1 or UNQ2 along an imaginary axis with slope \( \pi/4 \) (or \( -\pi/4 \)) and random intercept. For SYN we cannot devise a descriptor of \( Y \) such that either \( X_1 \) or \( X_2 \) can convey the same information as \( Y \) for each single-out features of \( Y \). Our decomposition assigns half bit of synergistic information and half bit of redundant information, Table 1. We propose that the two maskings involving the same two realizations of \( Y \) in SYN are the “building block” of synergistic information. A special case of SYN is the distribution CORNER shown in Figure 6f. This distribution is of particular interest because it is found in all canonical examples of synergy discussed below. Our decomposition breaks the 0.92 bit information of CROSSING into 0.25 bit of redundant information and 0.67 bit of synergistic information, Table 1.

![Table 1: Values of total information, \( I(X_1, X_2; Y) \), redundant information, \( \mu(X_1; X_2; Y) \), information unique to \( X_1 \), \( \mu(X_1; Y) \), information unique to \( X_2 \), \( \mu(X_2; Y) \), and synergistic information, \( \mu(X_1, X_2; Y) \), for some canonical examples from the literature \[8\],[10],[3],[11]. The distribution of the examples considered are shown in Figure 6 together with the descriptors that minimize Equation \[24\]. The two bottom lines show the values of shared information for two special conditions considered in \[5\]. The second condition corresponds to the identity property.](image)
The XOR distribution, Figure 6, is the archetype of synergistic information [8] and can be thought of as the combination of four CORNER elements. According to our breakdown, all information conveyed jointly by \( X_1 \) and \( X_2 \) about \( Y \) is synergistic, Table 1 as expected.

Two further classic example of synergistic distribution are AND and SUM [8], Figures 6 and 7. Equation (6) breaks the 0.81 bit of total information of AND into 0.5 bit of synergistic and approximately 0.31 bits of redundant information, Table 1. For SUM, our decomposition breaks the 1.5 bit of total information into 1 bit of synergistic and 0.5 bits of redundant information, Table 1.

Our decomposition also allocates the three bits of information in examples DYADIC and TRIADIC in a way that reflects the different generative structures of the two systems [11]. Example DYADIC, Figure 6, consists of two identical UNQ structures [11]. Our decomposition returns one bit of information for both unique terms, Table 1. Examples TRIADIC and RDNXOR, Figures 6 and 7, both consist of two XOR-like structures [11]. Our decomposition breaks the two bits of information of these examples into one bit of synergistic and one bit of redundant information, Table 1.

For RDNUNQXOR [8], Figure 6, our decomposition provides one bit of information for each term of the breakdown, Table 1 as expected.

Finally, for a proof that Equation (10) satisfies \( I(X_1;X_2;X_1) = I(X_1;X_2) \) and \( I(X_1;X_2;[X_1,X_2]) = I(X_1;X_2) \), see the proof of the identity property, Section A.5.

### A.5 Properties of Equation (10)

**Theorem 5** (Williams and Beer axioms). Equation (10) satisfies the following properties:

1. Symmetry: \( I(A_1; \ldots; A_K; Y) \) does not depend on the order of \( A_1, \ldots, A_K \)
2. Self-redundancy: \( I(A; A; Y) = I(A; Y) \)
3. Monotonicity: \( I(A_1; \ldots; A_K; Y) \leq I(A_1; \ldots; A_{K-1}; Y) \)

**Proof.** Property one follows from the fact that the minimum operator \( \min\{a_1, \ldots, a_k\} \) does not depend on the order of the elements of \( \{a_1, \ldots, a_k\} \). Property two follows from the fact that equation \( I(A; A; Y) \) reduces to decomposition (4) for any choice of descriptor of \( Y \). We define

\[
\mathcal{Y}' \triangleq \arg\min_{\mathcal{Y} \in \Omega_Y} \left\{ I(A_1; \ldots; A_K; \mathcal{Y}) \right\} \quad \text{and} \quad \mathcal{Y}'' \triangleq \arg\min_{\mathcal{Y} \in \Omega_Y} \left\{ I(A_1; \ldots; A_{K-1}; \mathcal{Y}) \right\}.
\]

Because \( \min\{a_1, \ldots, a_k\} \leq \min\{a_1, \ldots, a_{k-1}\} \) for any set \( \{a_1, \ldots, a_k\} \), we have

\[
I(A_1; \ldots; A_K; Y) = I(A_1; \ldots; A_K; \mathcal{Y}') \leq I(A_1; \ldots; A_K; \mathcal{Y}'') \leq I(A_1; \ldots; A_{K-1}; \mathcal{Y}'') = I(A_1; \ldots; A_{K-1}; Y)
\]

This proves property three.

**Theorem 6** (Identity property). Equation (10) satisfies \( I(X_1;X_2;[X_1,X_2]) = I(X_1;X_2) \).

**Proof.** In general, we have

\[
I(X_1;X_2;Y) = \min_{\mathcal{Y} \in \Omega_Y} \left\{ I(X_1;X_2;\mathcal{Y}) \right\} = \min_{\mathcal{Y} \in \Omega_Y} \left\{ \sum_{\ell=1}^{L} \sum_{y} p(y') \cdot \min_{k=1,2} \left\{ I(A_k; Y^{\ell-1} | y') \right\} \right\} =
\]

\[
\overset{(a)}{=} \min_{\mathcal{Y} \in \Omega_Y} \left\{ I(X_1;\mathcal{Y}) + I(X_2;\mathcal{Y}) - \sum_{\ell=1}^{L} \sum_{y} p(y') \cdot \max_{k=1,2} \left\{ I(A_k; Y^{\ell-1} | y') \right\} \right\} =
\]

\[
\overset{(b)}{=} I(X_1; Y) + I(X_2; Y) + \min_{\mathcal{Y} \in \Omega_Y} \left\{ - \sum_{\ell=1}^{L} \sum_{y} p(y') \cdot \max_{k=1,2} \left\{ I(X_k; Y^{\ell-1} | y') \right\} \right\} =
\]

\[
\overset{(c)}{=} I(X_1; Y) + I(X_2; Y) - \max_{\mathcal{Y} \in \Omega_Y} \left\{ \sum_{\ell=1}^{L} \sum_{y} p(y') \cdot \max_{k=1,2} \left\{ I(X_k; Y^{\ell-1} | y') \right\} \right\} =
\]

\[
\overset{(d)}{=} I(X_1; Y) + I(X_2; Y) - U(X_1;X_2;Y)
\]

(12)

where (a) follows from the maximum-minimums identity, (b) from the fact that \( I(X_1; Y) \) and \( I(X_2; Y) \) do not depend on the choice of the descriptor, (c) from the fact that mutual information is non-negative, and (d) from the definition of union information, Equation (8).
Figure 6: Graphical representation of the probability distributions of the canonical examples in Table 6. The distributions are shown using the following graphical convention. We denote with shapes \{\bigcirc, \bullet, \times, +, \ldots\} the elements of \(Y\). We denote with points in \(\mathbb{R}^2\) the elements of \(X\). At each position \(x \in Y\) we draw the shape of those elements \(y \in Y\) for which \(p(x, y) > 0\). Next to each shape we indicate the value \(p(x, y)\). For example, \(\mathbb{0}^{0.3}\) indicates \(p(x, \bigcirc) = 0.3\). If \(P(X, Y)\) is constant for all \(x \in X\) and all \(y \in Y\), we omit the indication of the probability value, for convenience. If \(x = [x_1, x_2], x' = [x'_1, x'_2] \in X\) are such that \(x_n = x'_n\) for \(n = 1\) or \(n = 2\), we highlight this by connecting the two points in \(\mathbb{R}^2\) with a line. For each example, we show the descriptors of \(Y\) that provides the shared information according to Equation (10) using shaded areas. The outcomes of \(Y^1\) are highlighted in light-grey. The outcomes of \(Y^2\) are highlighted in middle-dark-grey. The outcomes of \(Y^3\) are highlighted in darker-grey.
By construction \( U(X_1; X_2; Y) \leq I(X_1, X_2; Y) \). Furthermore, \( U(X_1; X_2; C_Y) = I(X_1, X_2; Y) \leq U(X_1; X_2; Y) \leq I(X_1, X_2; Y) \). Substituting \( U(X_1; X_2; Y) = I(X_1, X_2; Y) \) in equation (12) provides \( I(X_1; X_2; Y) = I(X_1; Y) + I(X_2; Y) - I(X_1, X_2; Y) = I(X_1; X_2) \). The theorem is proved. □

The same strategy can also be used to show that \( I(X_1; X_2; X_1) = I(X_1; X_2) \).

**Theorem 7** (Additivity property). Assume that \( \{A_1, A_2, Y\} \) is independent of \( \{\hat{A}_1, \hat{A}_2, \hat{Y}\} \). Equation (10) satisfies

\[
I(\{A_1, \hat{A}_1\} : \{A_2, \hat{A}_2\} : \{Y, \hat{Y}\}) = I(A_1 : A_2 : Y) + I(\hat{A}_1 : \hat{A}_2 : \hat{Y})
\]

**Proof.** From Equation (12) we have \( I(A_1 : A_2 : Y) = I(A_1 : Y) + I(A_2 : Y) - I(A_1, A_2 ; Y) \) and \( I(\hat{A}_1 : \hat{A}_2 : \hat{Y}) = I(\hat{A}_1 : \hat{Y}) + I(\hat{A}_2 : \hat{Y}) - I(\hat{A}_1, \hat{A}_2 ; \hat{Y}) \), and also

\[
I(\{A_1, \hat{A}_1\} : \{A_2, \hat{A}_2\} : \{Y, \hat{Y}\}) = I(A_1, \hat{A}_1 : Y, \hat{Y}) + I(A_2, \hat{A}_2 : Y, \hat{Y}) - I(A_1, \hat{A}_1, A_2, \hat{A}_2 ; Y, \hat{Y})
\]

\[
= I(A_1 : Y) + I(\hat{A}_1 : \hat{Y}) + I(A_2 : Y) + I(\hat{A}_2 : \hat{Y}) - I(A_1, A_2 ; Y) - I(\hat{A}_1, \hat{A}_2 ; \hat{Y}).
\]

The theorem is proved. □

**Theorem 8** (Blackwell property). Equation (10) satisfies \( I(X_1 : \ldots : X_N ; Y) = I(X_1 : \ldots : X_{N-1} ; Y) \) if \( X_N = f(X_n) \) for some \( n \in \{1, \ldots, N\} \).

**Proof.** The property follows from the data processing inequality. □

**Theorem 9** (Combined secret sharing property). Equation (10) satisfies the combined secret sharing property

\[
I(A_1 : \ldots : A_K ; S_1, \ldots, S_L) = H([S_\zeta : A_1, \ldots, A_K \in A_1])
\]

where \( A_1, \ldots, A_K \) are access structures of a combination of \( L \) perfect secret sharing schemes.

**Proof.** The probabilistic independence of the secrets (13) implies

\[
H([S_\zeta : A_1, \ldots, A_K \in A_1]) = \sum_{S_\zeta : A_1, \ldots, A_K \in A_1} H(S_\zeta).
\]

Denote \( S = \{S_\zeta : A_1, \ldots, A_K \in A_1\} \), we have

\[
I(A_1 : \ldots : A_K ; S_1, \ldots, S_L) \geq I(A_1 : \ldots : A_K ; S_1, \ldots, S_L) \geq (a)
\]

\[
\geq I(S ; S_1, \ldots, S_L) = H(S)
\]

(14)

where (a) follows from the monotonocity property and (b) from the Blackwell property, given that \( I(A_k ; S) = H(S) \) for \( k = 1, \ldots, K \), implies \( f_k(A_k) = S \) exists for each \( k \). For the canonical descriptor \( C_{\{S_1, \ldots, S_L\}} = Y^0 \rightarrow \cdots \rightarrow Y^L \) we have \( I(A_k ; Y^{\ell-1} \mid Y^\ell) = I(A_k ; S_\zeta \mid S_{\zeta+1}, \ldots, S_L) = I(A_k ; S_\zeta) \). Thus \( min_{k=1, \ldots, K} \{I(A_k ; Y^{\ell-1})\} \) is equal to \( H(S_\zeta) \) if \( A_1, \ldots, A_K \in A_L \), and is zero otherwise. This also implies that \( I(A_1 : \ldots : A_K ; \hat{Y}) = H(S) \). The theorem is proved. □

**A.6 Proof of the properties of Equation (11)**

**Theorem 10** (Non-negativity). Equation (11) satisfies \( I(X_1 ; \ldots ; X_N) \geq 0 \).

**Proof.** The property follows from the non-negativity of Equation (10). □

**Theorem 11** (Symmetry). Equation (11) is invariant to permutations of \( X_1, \ldots, X_N \).

**Proof.** The property follows from the symmetry of Equation (10). □

**Theorem 12** (Monotonicity). Equation (11) satisfies \( I(X_1 ; \ldots ; X_{N-1}) \geq I(X_1 ; \ldots ; X_N) \) for all \( P(X_1, \ldots, X_N) \).

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Refer to (13) for a definition of the relevant quantities and notations.
To prove property three consider the counterexample in Table 2 for the case $N = 3$.

| $X_1$ | $X_2$ | $X_3$ | $P(X,Y)$ | $Y^0$ | $Y^1$ | $Y^2$ |
|-------|-------|-------|----------|-------|-------|-------|
| 0     | 0     | 1     | 1/4      | 0     | 0     | 0     |
| 1     | 0     | -1    | 1/4      | 1     | 0     | 0     |
| 0     | 1     | 0     | 1/4      | 2     | 1     | 0     |
| 1     | -1    | 0     | 1/4      | 3     | 1     | 0     |

Table 2: Counterexample demonstrating property three of the Shannon property for the case $N = 3$.

Proof. We have $I(X_1; \ldots; X_N) = I(X_1; \ldots; X_N; X_1, \ldots, X_N) \leq I(X_1; \ldots; X_{N-1}; X_1, \ldots, X_N)$ from the monotonicity of Equation (10). We note that, in general, $\Omega_{X_1, \ldots, X_{N-1}} \subseteq \Omega(X_1, \ldots, X_N)$. The minimization in Equation (10) thus guarantees that $I(X_1; \ldots; X_{N-1}; X_1, \ldots, X_N) \leq I(X_1; \ldots; X_{N-1})$.

Theorem 13 (Blackwell property). Equation (11) satisfies the following properties

1. $I(X_1; \ldots; X_N) = I(X_1; \ldots; X_{N-1})$ if $X_{N-1} = f(X_N)$ for some $f : X_N \to X_{N-1}$;
2. $I(X_1; \ldots; X_N) = H(X_1)$ if $X_1, \ldots, X_N$ form a Markov chain $X_N \to X_{N-1} \to \cdots \to X_1$

Proof. Property one follows from Equation (10) and from the signal processing inequality. Property two follows directly from property one.

Theorem 14 (Shannon property). Equation (11) satisfies the following properties

1. $\exists \bar{n} \in \{1, \ldots, N\} : p(x_{\bar{n}}, x_n) = p(x_{\bar{n}}) \cdot p(x_n)$ for all $n \neq \bar{n}$ and all $x \in X \implies I(X_1; \ldots; X_N) = 0$.
2. $p(x_1, \ldots, x_N) = p(x_1) \cdot \cdots \cdot p(x_N)$ for all $x \in X \implies I(X_1; \ldots; X_N) = 0$.
3. $I(X_1; \ldots; X_N) = 0 \implies p(x_{n_1}, x_{n_2}) = p(x_{n_1}) \cdot p(x_{n_2})$ for all $n_1, n_2 = 1, \ldots, N$

Proof. To prove property one, without loss of generality thanks to the symmetry of Equation (11), we assume $\bar{n} = N$. We consider $Y = X$. For the canonical descriptor $C_Y = Y^0 \to \cdots \to Y^N$ we obtain

$$
\min_{n=1, \ldots, N} \left\{ I(X_n; Y^{n-1} \mid y^n) \right\} = 0, \quad n = 1, \ldots, N - 1, \quad \forall y^n \in Y^n
$$

Furthermore

$$
\min_{n=1, \ldots, N} \left\{ I(X_n; Y^{N-1} \mid y^N) \right\} = \min_{n=1, \ldots, N} \left\{ I(X_n; Y^{N-1}) \right\} = \min_{n=1, \ldots, N} \left\{ I(X_n, X_N) \right\} = I(X_n; X_N) = 0
$$

This proves the first property. Property two is follows directly from property one.

To prove property three consider the counterexample in Table 2 for the case $N = 3$. We obtain $I(X_2; Y^0 \mid Y^1 = 0) = I(X_3; Y^0 \mid Y^1 = 1) = I(X_1; Y^1 \mid Y^2) = 0$, thus $I(X_1; X_2; X_3) = 0$. However $I(X_1; X_2) = 1, I(X_1; X_3) = 0.5, I(X_2; X_3) = 1$. 

\[ \square \]