A BATALIN-VILKOVISKY STRUCTURE ON THE COMPLETE COHOMOLOGY RING OF A FROBENIUS ALGEBRA

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Abstract. We study the existence of a Batalin-Vilkovisky differential on the complete cohomology ring of a Frobenius algebra. We construct a Batalin-Vilkovisky differential on the complete cohomology ring in the case of Frobenius algebras with diagonalizable Nakayama automorphisms.

Introduction

In 1945, Hochschild [11] introduced the Hochschild cohomology group $H^*(A,M)$ of an associative algebra $A$ with coefficients in an $A$-bimodule $M$. In the earlier of the 1960s, Gerstenhaber [8] discovered that there is a rich algebraic structure on Hochschild cohomology ring $H^*(A,A) := \bigoplus_{r \geq 0} H^r(A,A)$. To be more precise, $H^*(A,A)$ has a Gerstenhaber structure, that is, there is a Lie bracket on Hochschild cohomology such that the Lie bracket satisfies the graded Leibniz rule with respect to the cup product. The Lie bracket is also called the Gerstenhaber bracket.

During recent several decades, a new algebraic structure, the so-called Batalin-Vilkovisky (BV) structure, on Hochschild cohomology has been studied. A BV structure gives rise to an operator on Hochschild cohomology which squares to zero and which can generate the Gerstenhaber bracket together with the cup product. Let us remark that BV structure is originally defined for a graded commutative algebra and gives rise to a Lie bracket making the graded commutative algebra together with itself into a Gerstenhaber algebra (cf. [9]). From this context, the existence of BV operator generating the Gerstenhaber bracket is particularly important. It is known that there exists such a BV structure over Hochschild cohomology of certain classes of algebras, such as Calabi-Yau algebras, finite dimensional symmetric algebras, finite dimensional Frobenius algebras whose Nakayama automorphisms are semisimple and so on (cf. [10, 14, 15, 13, 20]). Here, the semisimplicity of the Nakayama automorphism means that it is diagonalizable over the algebraic closure of the underlying field.

In the 1980s, Buchweitz [5] introduced the notion of singularity category in order to provide a framework for Tate cohomology of Gorenstein algebras. In 2015, under this framework, Wang defined the $r$-th Tate-Hochschild cohomology group of $A$ as

$$\text{Ext}^r_{A\otimes_k A^{op}}(A, A) := \text{Hom}_{D_{sg}(A\otimes_k A^{op})}(A, A[r]),$$

where $r$ is arbitrary integer and $D_{sg}(A\otimes_k A^{op})$ is the singularity category of $A \otimes_k A^{op}$ (cf. [21, 22]). Wang discovered in [21] that Tate-Hochschild cohomology $\text{Ext}^*_{A\otimes_k A^{op}}(A, A)$ has a Gerstenhaber structure.
structure equipped with the Yoneda product $\sim_{sg}$ and the Lie bracket $[\cdot,\cdot]_{sg}$. Moreover, he determined in [23] the Tate-Hochschild cohomology of radical square zero algebras and their Gerstenhaber structures for some classes of such algebras.

Many authors have investigated Tate-Hochschild cohomology in case of Frobenius algebras (cf. [4, 7, 16, 17, 21, 22]). One of the first attempts was made by Nakayama [16]. In the 1950s, as an analogy to Tate cohomology for a finite group, Nakayama introduced a complete cohomology groups $\widehat{H}^\ast(A, M)$ of a Frobenius algebra $A$ with coefficients in an $A$-bimodule $M$. Here, complete cohomology groups coincide with Tate-Hochschild cohomology groups (cf. [5]). A complex which is used to compute complete cohomology groups $\widehat{H}^\ast(A, M)$ is a complete complex. Roughly speaking, the complete complex is an unbounded complex having the Hochschild cochain complex $C^\ast(A, M)$ in non-negative degrees and the Hochschild chain complex $C_\ast(A, M_{\nu^{-1}})$ in negative degrees, where $\nu$ is the Nakayama automorphism of the Frobenius algebra $A$. In 1992, Sanada [19] constructed a cup product on complete cohomology groups by means of a diagonal approximation and investigated a periodicity of complete cohomology groups. Recently, Wang have discovered in [22] that there is a graded commutative product, called $\star$-product, on complete cohomology groups such that the complete cohomology ring is isomorphic to the Tate-Hochschild cohomology ring. Furthermore, as an application of BV differential on Hochschild cohomology, he constructed a BV differential on $\widehat{H}^\ast(A, A)$ in the case that $A$ is a finite dimensional symmetric algebra, where the BV differential consists of Tradler’s BV differential [20] and the Connes operator. In particular, the induced Lie bracket on complete cohomology is isomorphic to Wang’s bracket on Tate-Hochschild cohomology.

In this paper, we generalize Wang’s result to the case of finite dimensional Frobenius algebras with diagonalizable Nakayama automorphisms. Namely, the aim of the present paper is to prove the following result:

**Main result.** Let $A$ be a finite dimensional Frobenius $k$-algebra. If the Nakayama automorphism of $A$ is diagonalizable, then the complete cohomology ring $\widehat{H}^\ast(A, A)$ is a BV algebra together with a BV differential consisting of Lambre-Zhou-Zimmermann’s BV differential [14] and (twisted) Connes operator (see Theorem 4.5 and Corollary 4.7).

This paper is organized as follows: In Section 1, we recall some definitions and basic results on Tate-Hochschild cohomology and complete cohomology. Section 2 is devoted to recalling the complete complex of a Frobenius algebra, and then to relating Tate-Hochschild cohomology groups with complete cohomology groups. In Section 3, following Lambre-Zhou-Zimmermann [14], we define a subcomplex of the complete complex associated with the product of eigenvalues of the Nakayama automorphism. We then show that the cohomology groups of the subcomplex has nice properties when the Nakayama automorphism is diagonalizable. In Section 4, we give the main result and the proof. Our proof is different from Lambre-Zhou-Zimmermann [14]. More precisely, our proof uses the Gerstenhaber bracket $[\cdot,\cdot]_{sg}$ defined by Wang [21], while Lambre, Zhou and Zimmermann’s proof makes use of the notions of Tamarkin-Tsygan calculi and calculi with duality. Section 5 contains three examples of BV structures on the complete cohomology rings of some self-injective Nakayama algebras over algebraically closed fields with diagonalizable Nakayama automorphisms.
1. Preliminaries

Throughout this paper, let $k$ be a field, $A$ a finite dimensional, associative and unital $k$-algebra. Let $A^e$ be the enveloping algebra $A \otimes_k A^{\text{op}}$ of $A$. Here we denote by $A^{\text{op}}$ the opposite algebra of $A$. We can identify an $A$-bimodule $M$ with a left (right) $A^e$-module $M$ whose structure is given by $(a \otimes k b^o)m := amb$ ($m(a \otimes k b^o) := bma$) for $m \in M$ and $a \otimes k b^o \in A^e$. For simplicity, we write $\otimes$ for $\otimes_k$ and Hom for Hom$_k$. We denote by $\overline{A}$ the quotient space of $A$ by the subspace $k1_A$ generated by unit $1_A$. Let $\sigma : A \to A$ be an algebra automorphism of $A$ and $\pi : A \to \overline{A}$ the canonical epimorphism of $k$-vector spaces. We denote by $\overline{\pi}$ denote the image of $a \in A$ under the epimorphism $\pi : A \to \overline{A}$. We write $a_{1,m} \in A^{\otimes m}$ for $a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$, $b_{1,n} \in \overline{A}^{\otimes n}$ for $b_1 \otimes \cdots \otimes b_n \in \overline{A}^{\otimes n}$ and $\overline{\sigma c_{1,l}} \in \overline{A}^{\otimes l}$ for $(\sigma(c_1) \otimes \sigma(c_2) \otimes \cdots \otimes \sigma(c_l)) \in \overline{A}^{\otimes l}$ when no confusion occurs.

1.1. Gerstenhaber algebras and Hochschild (co)homology. Let us start with the definition of Gerstenhaber algebras.

**Definition 1.1.** A Gerstenhaber algebra is a graded $k$-module $\mathcal{H}^\bullet = \bigoplus_{r \in \mathbb{Z}} \mathcal{H}^r$ equipped with two bilinear maps: a cup product of degree zero

$$\cup : \mathcal{H}^{|\alpha|} \otimes \mathcal{H}^{|\beta|} \to \mathcal{H}^{|\alpha|+|\beta|}, \quad (\alpha, \beta) \mapsto \alpha \cup \beta$$

and a Lie bracket of degree $-1$, called the Gerstenhaber bracket,

$$[ , ] : \mathcal{H}^{|\alpha|} \otimes \mathcal{H}^{|\beta|} \to \mathcal{H}^{|\alpha|+|\beta|-1}, \quad (\alpha, \beta) \mapsto [\alpha, \beta]$$

such that

(i) $(\mathcal{H}^\bullet, \cup)$ is a graded commutative algebra with unit $1 \in \mathcal{H}^0$, in particular, $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$;

(ii) $(\mathcal{H}^\bullet[1], [ , ])$ is a graded Lie algebra with components $(\mathcal{H}^\bullet[1])^r = \mathcal{H}^{r+1}$, that is,

$$[\alpha, \beta] = -(-1)^{|\alpha|-1(|\beta|-1)}[\beta, \alpha]$$

and

$$(-1)^{|\alpha|-1(|\gamma|-1)}[[\alpha, \beta], \gamma] + (-1)^{|\beta|-1(|\alpha|-1)}[[\beta, \gamma], \alpha]$$

$$+ (-1)^{|\gamma|-1(|\beta|-1)}[[\gamma, \alpha], \beta] = 0;$$

(iii) The Lie bracket $[ , ]$ is compatible with the cup product $\cup$:

$$[\alpha, \beta \cup \gamma] = [\alpha, \beta] \cup \gamma + (-1)^{|\alpha|-1|\beta|} \beta \cup [\alpha, \gamma],$$

where $\alpha, \beta, \gamma$ are homogeneous elements in $\mathcal{H}^\bullet$ and we denote by $|\alpha|$ the degree of a homogeneous element $\alpha$ in $\mathcal{H}^\bullet$.

One of examples of Gerstenhaber algebras is the Hochschild cohomology of a $k$-algebra $A$. There is a projective resolution $\text{Bar}_\bullet(A)$ of $A$ over $A^e$, which is the so-called normalized bar resolution:

$$\cdots \to A \otimes \overline{A}^{\otimes r} \otimes A \xrightarrow{d_r} A \otimes \overline{A}^{\otimes r-1} \otimes A \to \cdots \to A \otimes \overline{A} \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \to 0,$$
where we set
\[
d_r(a_0 \otimes \alpha_{1,r} \otimes a_{r+1}) = a_0 a_1 \otimes \alpha_{2,r} \otimes a_{r+1}
+ \sum_{i=1}^{r-1} (-1)^i a_0 \otimes \alpha_{1,i-1} \otimes \alpha_i \alpha_{i+1} \otimes \alpha_{i+2,r} \otimes a_{r+1}
+ (-1)^r a_0 \otimes \alpha_{1,r-1} \otimes a_r a_{r+1},
\]
\[
d_0(a_0 \otimes a_1) = a_0 a_1.
\]

We denote \(HH(A) := \text{Im } d_r\) for all \(r \geq 0\). Given an \(A\)-bimodule \(M\), consider the complex \(C^\bullet(A, M) := \text{Hom}_A(\text{Bar}_\bullet(A), M)\) with differential \(\text{Hom}_A(d_\bullet, M)\). Note that for any \(r \geq 0\), we have
\[
C^r(A, M) = \text{Hom}_A(\text{Bar}_r(A), M) = \text{Hom}_A(A \otimes A^{\otimes r} \otimes A, M) \cong \text{Hom}(A^{\otimes r}, M).
\]

We identify \(C^0(A, M)\) with \(M\). Thus, the complex \(C^\bullet(A, M)\) is of the form
\[
0 \rightarrow M \xrightarrow{\delta^0} \text{Hom}(A, M) \rightarrow \cdots \rightarrow \text{Hom}(A^{\otimes r}, M) \xrightarrow{\delta^r} \text{Hom}(A^{\otimes r+1}, M) \rightarrow \cdots
\]
whose differentials \(\delta^r\) are defined by
\[
\delta^r(f)(\alpha_{1,r+1}) = a_1 f(\alpha_{2,r+1}) + \sum_{i=1}^r (-1)^{i+1} f(\alpha_{1,i-1} \otimes a_i \alpha_{i+1} \otimes \alpha_{i+2,r+1})
+ (-1)^{r+1} f(\alpha_{1,r}) a_{r+1}
\]
for any \(f \in \text{Hom}(A^{\otimes r}, M)\) and \(\alpha_{1,r+1} \in A^{\otimes r+1}\). Then the \(r\)-th cohomology group
\[
H^r(A, M) := H^r(C^\bullet(A, M), \delta^\bullet)
\]
is said to be the \(r\)-th Hochschild cohomology group of \(A\) with coefficients in \(M\). We will write \(HH^r(A) := H^r(A, A)\). Since \(A\) is projective over \(k\), we get \(H^r(A, M) \cong \text{Ext}^r_A(A, M)\). Namely, Hochschild cohomology groups do not depend on the choice of projective resolution of \(A\). For \(A\)-bimodules \(M\) and \(N\), the cup product
\[
\smile : C^m(A, M) \otimes C^n(A, N) \rightarrow C^{m+n}(A, M \otimes_A N)
\]
is defined by
\[
(\alpha \smile \beta)(\alpha_{1,m+n}) := \alpha(\alpha_{1,m}) \otimes_A \beta(\alpha_{m+1,m+n})
\]
for all \(\alpha \in C^m(A, M), \beta \in C^n(A, N)\) and \(\alpha_{1,m+n} \in A^{\otimes m+n}\). The cup product \(\smile\) induces a well-defined operator
\[
\smile : H^m(A, M) \otimes H^n(A, N) \rightarrow H^{m+n}(A, M \otimes_A N).
\]

The Gerstenhaber bracket in the Hochschild cohomology \(HH^\bullet(A)\) is defined as follows: let \(\alpha \in C^m(A, A)\) and \(\beta \in C^n(A, A)\). We define a \(k\)-bilinear map
\[
[ , ] : C^m(A, A) \otimes C^n(A, A) \rightarrow C^{m+n-1}(A, A)
\]
as
\[
[\alpha, \beta] := \alpha \circ \beta - (-1)^{(m-1)(n-1)} \beta \circ \alpha \in C^{m+n-1}(A, A),
\]
Theorem 1.2
Gerstenhaber proved the following result.

The Hochschild cohomology $HH^*(A)$ equipped with the cup product $\smile$ and the Lie bracket $[\ ,\ ]$ is a Gerstenhaber algebra.

For an $A$-bimodule $M$, consider a complex $C_\bullet(A, M) := M \otimes_{A^e} \text{Bar}_\bullet(A)$ with differential $\text{id}_M \otimes_{A^e} d_\bullet$. Note that for any $r \geq 0$, we have

$$C_r(A, M) = M \otimes_{A^e} \text{Bar}_r(A) = M \otimes_{A^e} (A \otimes \overline{A}^r \otimes A) \cong M \otimes \overline{A}^r.$$

We identify $C_0(A, M)$ with $M$. Thus, the complex $C_\bullet(A, M)$ is of the form

$$\cdots \to M \otimes \overline{A}^{r+1} \overset{\partial_{r+1}}{\to} M \otimes \overline{A}^r \to \cdots \to M \otimes \overline{A} \overset{\partial_1}{\to} M \to 0,$$

where the differentials $\partial_{r+1}$ are defined by

$$\partial_{r+1}(m \otimes \overline{a}_{1,r+1}) = ma_1 \otimes \overline{a}_{2,r+1} + \sum_{i=1}^r (-1)^i m \otimes \overline{a}_{1,i-1} \otimes \overline{a}_{i+1} \otimes \overline{a}_{i+2,r+1}$$

$$+ (-1)^{r+1} a_{r+1} m \otimes \overline{a}_{1,r}$$

for all $m \otimes \overline{a}_{1,r+1} \in M \otimes \overline{A}^{r+1}$. Then the $r$-th homology group

$$H_r(A, M) := H_r(C_\bullet(A, M), \partial_\bullet)$$

is said to be the $r$-th Hochschild homology group of $A$ with coefficients in $M$. We will write $\text{HH}_r(A) := H_r(A, A)$. Since $A$ is projective over $k$, we get $H_r(A, M) \cong \text{Tor}_{r+1}^A(A, M)$, which means that Hochschild homology groups are independent of projective resolutions of $A$.

There is an action of Hochschild cohomology on Hochschild homology, called the cap product. For two $A$-bimodules $M, N$ and $r, p \geq 0$ with $r \geq p$, a $k$-bilinear map

$$\smile : C_r(A, M) \otimes C^p(A, N) \to C_{r-p}(A, M \otimes_A N)$$

is defined by

$$(m \otimes \overline{a}_{1,r}) \smile \alpha := m \otimes_A \alpha(\overline{a}_{1,p}) \otimes \overline{a}_{p+1,r}$$

for all $m \otimes \overline{a}_{1,r} \in C_r(A, M)$ and $\alpha \in C^p(A, N)$. The $k$-bilinear map $\smile$ induces a well-defined operator

$$\smile : H_r(A, M) \otimes H^p(A, N) \to H_{r-p}(A, M \otimes_A N).$$
1.2. **Gorenstein algebras and complete resolutions.** This section is devoted to recalling some basic results on complete resolutions of a module over a Gorenstein algebra. For more details, we refer the reader to [2, 4, 6]. Recall that a finite dimensional algebra $A$ is a *Gorenstein algebra* of Gorenstein dimension $d$ if the injective dimension of $A$, as a right and as a left $A$-module, are equal to $d$. Assume that $A$ is a Gorenstein algebra of Gorenstein dimension $d$. It follows from [2] that any finitely generated left $A$-module $M$ admits a *complete resolution*

$$
\mathcal{T} : \cdots \to T_2 \to T_1 \to T_0 \to T_{-1} \to T_{-2} \to \cdots
$$
satisfying the three conditions:

(i) $\mathcal{T}$ is an exact sequence of finitely generated projective $A$-modules.

(ii) The $A$-dual complex $\text{Hom}_A(\mathcal{T}, A)$ is acyclic.

(iii) There exist a projective resolution $\mathcal{P}$ of $M$ and a chain map $f : \mathcal{T} \to \mathcal{P}$ such that $f_r$ is an isomorphism for $r \geq d$.

Given a left $A$-module $N$ and an integer $r$, the $r$-th cohomology group of the complex $\text{Hom}_A(\mathcal{T}, N)$ is said to be the $r$-th *Tate cohomology group* of $M$ with coefficients in $N$ and is denoted by $\hat{\text{Ext}}_A^r(M, N)$. For any right $A$-module $N$, the $r$-th homology group of the complex $N \otimes_A \mathcal{T}$ is said to be the $r$-th *Tate homology group* of $M$ with coefficients in $N$ and is denoted by $\hat{\text{Tor}}^A_r(M, N)$. It follows from [6] that the Tate (co)homology groups do not depend on the choice of complete resolutions of $M$. The property (iii) implies that we have

$$
\hat{\text{Ext}}_A^r(M, N) \cong \text{Ext}_A^r(M, N), \quad \hat{\text{Tor}}^A_r(M, N) \cong \text{Tor}_r^A(M, N)
$$

for all $r \geq d + 1$.

**Definition 1.3** (Bergh-Jorgensen [4]). Let $A$ be a $k$-algebra such that the enveloping algebra $A^e$ is Gorenstein, $N$ an $A$-bimodule and $r \in \mathbb{Z}$ arbitrary. Then the $r$-th complete cohomology group $\hat{\text{HH}}^r(A, N)$ and the $r$-th complete homology group $\hat{\text{HH}}_r(A, N)$ of $A$ with coefficients in $N$ are defined by

$$
\hat{\text{HH}}^r(A, N) := \hat{\text{Ext}}_{A^e}^r(A, N), \quad \hat{\text{HH}}_r(A, N) := \hat{\text{Tor}}_{A^e}_r(A, N).
$$

We will write $\hat{\text{HH}}^r(A) := \hat{\text{HH}}^r(A, A)$ and $\hat{\text{HH}}_r(A) := \hat{\text{HH}}_r(A, A)$.

In this paper, we only deal with complete cohomology groups. Originally, Bergh-Jorgensen [4] called the complete cohomology groups Tate-Hochschild cohomology groups. However, throughout the paper, we use the term “Tate-Hochschild” for the cohomology groups defined by Wang [21, 22] described in the next subsection. We remark that both of these cohomology groups coincide for an algebra whose enveloping algebra is Gorenstein of Gorenstein dimension zero.

A class of algebras which we will be studying is a class of Frobenius algebras (see Section 2). Note that a Frobenius algebra is Gorenstein of Gorenstein dimension zero. It follows from [4, Corollary 3.3] that the property of being Frobenius algebras is preserved under taking their enveloping algebras, so that the complete cohomology groups $\hat{\text{HH}}^r(A, N)$ of a Frobenius algebra $A$ are defined, and there are isomorphisms

$$
\hat{\text{HH}}^r(A, N) \cong H^r(A, N), \quad \hat{\text{HH}}^r(A) \cong HH^r(A)
$$

for all $r \geq 1$. 

We will develop the theory of BV differentials on the complete cohomology rings of Frobenius algebras and give their examples. For this purpose, we use appropriate complete resolutions.

1.3. Tate-Hochschild cohomology and its Gerstenhaber structure. This section is devoted to recalling Tate-Hochschild cohomology groups and a Gerstenhaber structure on the Tate-Hochschild cohomology. For more details, we refer the reader to [21, Section 3 and 4]. Let us recall that for \( r \geq 0 \), the \( r \)-th (usual) Hochschild cohomology of \( A \) with coefficients in an \( A \)-bimodule \( M \) can be defined as \( \text{H}^r(A, M) = \text{Hom}_{D^b(Ae)}(A, M[r]) \), where \( D^b(Ae) \) is the bounded derived category of finitely generated left \( Ae \)-modules and the suspension functor \([-]\) denotes the degree shift. For any integer \( r \), the \( r \)-th Tate-Hochschild cohomology group of \( A \) is defined by

\[
\text{Ext}^r_{Ae}(A, A) := \text{Hom}_{D_{sg}(Ae)}(A, A[r]),
\]

where \( D_{sg}(Ae) \) is the singularity category of \( Ae \). Recall that \( D_{sg}(Ae) \) is the Verdier quotient of \( D^b(Ae) \) by the full subcategory of \( D^b(Ae) \) consisting of those complexes quasi-isomorphic to bounded complexes of finitely generated projective left \( Ae \)-modules.

Recall that \( \Omega^p(A) = \text{Im} d_p \), where \( d_p : \text{Bar}_p(A) \to \text{Bar}_{p-1}(A) \) is the \( p \)-th differential of the normalized bar resolution \( \text{Bar}_*(A) \). We fix an integer \( m \) and put \( I_{(m)} := \{ p \in \mathbb{Z} | p \geq 0, m + p \geq 0 \} \). Consider an inductive system

\[
\left\{ X^{(m)}_p, \; \theta_{m+p,p} : X^{(m)}_p \to X^{(m)}_{p+1} \right\}_{p \in I_{(m)}},
\]

where

\[
X^{(m)}_p = \text{Ext}^{m+p}_{Ae}(A, \Omega^p(A)),
\]

and \( \theta_{m+p,p} : X^{(m)}_p \to X^{(m)}_{p+1} \) is the connecting homomorphism

\[
\theta_{m+p,p} : \text{Ext}^{m+p}_{Ae}(A, \Omega^p(A)) \to \text{Ext}^{m+p+1}_{Ae}(A, \Omega^{p+1}(A)) \tag{1.1}
\]

induced by the short exact sequence

\[
0 \to \Omega^{p+1}(A) \to A \otimes A^{\otimes p} \otimes A \to \Omega^p(A) \to 0.
\]

Here, we regard \( \text{Ext}^{m+p}_{Ae}(A, \Omega^p(A)) \) as \( \text{H}^{m+p}(A, \Omega^p(A)) \), or equivalently, any element of \( \text{Ext}^{m+p}_{Ae}(A, \Omega^p(A)) \) is represented by an element in \( \text{Hom}_k(A^{\otimes m+p}, \Omega^p(A)) \). Note that the inductive system above has the form

\[
\text{Ext}^{m+i}_{Ae}(A, \Omega^p(A)) \xrightarrow{\theta_{m+i,i}} \text{Ext}^{m+i+1}_{Ae}(A, \Omega^{p+1}(A)) \xrightarrow{\theta_{m+i+1,i+1}} \text{Ext}^{m+i+2}_{Ae}(A, \Omega^{p+2}(A)) \to \cdots,
\]

where \( i \geq 0 \) is the least integer such that \( m + i \geq 0 \).

Remark 1.4. Using the explicit description of the connecting homomorphism (1.1) in [21, page 16], we see that, for any \( m \in \mathbb{Z} \) and \( p \in I_{(m)} \), the connecting homomorphism

\[
\theta_{m+p,p} : \text{Ext}^{m+p}_{Ae}(A, \Omega^p(A)) \to \text{Ext}^{m+p+1}_{Ae}(A, \Omega^{p+1}(A))
\]

sends an element \([f] \in \text{Ext}^{m+p}_{Ae}(A, \Omega^p(A))\) represented by \( f \in \text{Hom}_k(A^{\otimes m+p}, \Omega^p(A))\) to the element \([\theta_{m+p,p}(f)] \in \text{Ext}^{m+p+1}_{Ae}(A, \Omega^{p+1}(A))\). Here, \([\theta_{m+p,p}(f)]\) is represented by the \( k \)-linear map

\[
\theta_{m+p,p}(f) : A^{\otimes m+p+1} \to \Omega^{p+1}(A)
\]
taking an element $\overline{a}_{1,m+p+1} \in A^{\otimes m+p+1}$ into

$$(-1)^{m+p} d_{p+1}(f(\overline{a}_{1,m+p}) \otimes \overline{a}_{m+p+1} \otimes 1) \in \text{Im} d_{p+1} = \overline{\Omega}^{p+1}(A),$$

where $d_{p+1} : \text{Bar}_p(A) \to \text{Bar}_p(A)$ is the $(p+1)$-th differential of $\text{Bar}_p(A)$.

**Proposition 1.5** ([21, Proposition 3.1 and Remark 3.3]). For any $m \in \mathbb{Z}$, there is an isomorphism

$$\lim_{p \in \mathbb{N}_0} \text{Ext}^{m+p}_{A^e}(A, \overline{\Omega}^p(A)) \cong \text{Hom}_{\mathcal{D}_{sg}(A^e)}(A, A[m]) = \text{Ext}^{m}_{A^e}(A, A).$$

We now define a Gerstenhaber structure on Tate-Hochschild cohomology defined by Wang ([21]). Let $m, n, p$ and $q$ be integers such that $m, n, p, q \geq 0$. A cup product

$$\smile_{sg}: C^m(A, \overline{\Omega}^p(A)) \otimes C^n(A, \overline{\Omega}^q(A)) \to C^{m+n}(A, \overline{\Omega}^{p+q}(A))$$

is defined by

$$f \smile_{sg} g(\overline{b}_1, m+n) := \Phi_{p+q}(f(\overline{b}_1, m) \otimes_A g(\overline{b}_{m+1}, m+n)),$$

where $f \otimes g \in C^m(A, \overline{\Omega}^p(A)) \otimes C^n(A, \overline{\Omega}^q(A))$ and $\Phi_{p+q} : \overline{\Omega}^p(A) \otimes_A \overline{\Omega}^q(A) \to \overline{\Omega}^{p+q}(A)$ is an isomorphism of $A$-bimodules determined by

$$\Phi_{p+q}(a_0 \otimes \overline{a}_{1,p} \otimes a_{p+1} \otimes_A b_0 \otimes \overline{b}_{1,q} \otimes b_{q+1}) = a_0 \otimes \overline{a}_{1,p} \otimes a_{p+1} b_0 \otimes \overline{b}_{1,q} \otimes b_{q+1}$$

for $a_0 \otimes \overline{a}_{1,p} \otimes a_{p+1} \in \overline{\Omega}^p(A)$ and $b_0 \otimes \overline{b}_{1,q} \otimes b_{q+1} \in \overline{\Omega}^q(A)$, which is given in [22, Lemma 2.6].

Let $m \in \mathbb{Z}_{>0}$, $p \in \mathbb{Z}_{>0}$ and $f \in C^m(A, \overline{\Omega}^p(A))$ and let $\pi : A \to \overline{A}$ be the canonical epimorphism. We set

$$\pi_p^{(l)} := \pi \otimes \text{id}_{A^{\otimes p-1}} \otimes \text{id}_A : A \otimes A^{\otimes p-1} \otimes A \to A \otimes \overline{A}^{\otimes p} \otimes A,$$

$$\pi_p^{(r)} := \text{id}_A \otimes \text{id}_{A^{\otimes p-1}} \otimes \pi : A \otimes A^{\otimes p-1} \otimes A \to A \otimes \overline{A}^{\otimes p},$$

$$\pi_p^{(b)} := \pi \otimes \text{id}_{A^{\otimes p-1}} \otimes \pi : A \otimes A^{\otimes p-1} \otimes A \to A \otimes \overline{A}^{\otimes p+1}$$

and then denote

$$f^{(l)} := \pi_p^{(l)} f, \quad f^{(r)} := \pi_p^{(r)} f, \quad f^{(b)} := \pi_p^{(b)} f.$$

Let $m, n, p$ and $q$ be integers such that $m, n > 0$ and $p, q \geq 0$. We now define a bilinear map

$$[\ , \ ]_{sg} : C^m(A, \overline{\Omega}^p(A)) \otimes C^n(A, \overline{\Omega}^q(A)) \to C^{m+n-1}(A, \overline{\Omega}^{p+q}(A)).$$

as follows: let

$$f \in C^m(A, \overline{\Omega}^p(A)) = \text{Hom}_k(\overline{A}^{\otimes m}, \overline{\Omega}^p(A))$$

and

$$g \in C^n(A, \overline{\Omega}^q(A)) = \text{Hom}_k(\overline{A}^{\otimes n}, \overline{\Omega}^q(A)).$$

We first define a $k$-linear map $f \bullet g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ for each integer $i$ with $1 \leq i \leq m$. Consider the following four $k$-linear maps:

1. $$(\text{id}_{A^{\otimes i-1}} \otimes g^{(b)} \otimes \text{id}_{A^{\otimes m-i}}) : A^{\otimes m+n-1} \to A^{\otimes m+q}$$
   is given by

$$\overline{a}_{1,m+n-1} \mapsto \overline{a}_{1,i-1} \otimes g^{(b)}(\overline{a}_{i,i+n-1}) \otimes \overline{a}_{i+n,m+n-1};$$
We then define a $k$-linear map $f \bullet_i g \in C^{m+n-1}(A, \overline{T}^{p+q}(A))$ by the composition of the above four maps

$$f \bullet_i g := d_{p+q} \circ (\text{id}_A \otimes \text{id}_{\overline{T}^{p+q}} \otimes \text{id}_{A^{-1}}) \circ \left( (f^{(r)} \otimes \text{id}_{A^{-1}}) \otimes \text{id}_{\overline{T}} \right) \circ \left( \text{id}_{\overline{T}} \otimes g^{(b)} \otimes \text{id}_{\overline{T}^{m-i}} \right)$$

for $1 \leq i \leq m$. On the other hand, we assume that $q > 0$. We also define a $k$-linear map $f \bullet_{-i} g \in C^{m+n-1}(A, \overline{T}^{p+q}(A))$ for each integer $i$ with $1 \leq i \leq q$. Consider the following four $k$-linear maps:

1. $g^{(r)} \otimes \text{id}_{\overline{T}^{m-i-1}} : A \otimes \overline{T}^{m+1-n} \rightarrow A \otimes \overline{T}^{n+1-n}$ is given by

   $$\overline{A}_{1,m+n-1} \mapsto g^{(r)}(\overline{A}_{1,n}) \otimes \overline{A}_{n+1,n+m-1}$$

2. $\left( \text{id}_{A} \otimes \text{id}_{\overline{T}^{p+q}} \otimes \text{id}_{A^{-1}} \right) : A \otimes \overline{T}^{p+q} \otimes A \rightarrow A \otimes \overline{T}^{p+q}$ is given by

   $$a_0 \otimes \overline{A}_{1,m+n-1} \mapsto a_0 \otimes \overline{A}_{1,n-1} \otimes f^{(b)}(\overline{A}_{1,n}) \otimes \overline{A}_{n+1,n+m-1};$$

3. $\left( \text{id}_{A} \otimes \text{id}_{\overline{T}^{p+q}} \otimes \text{id}_{A^{-1}} \right) : A \otimes \overline{T}^{p+q} \otimes A \rightarrow A \otimes \overline{T}^{p+q} \otimes A$ is the same as above;

4. $d_{p+q} : A \otimes \overline{T}^{p+q} \otimes A \rightarrow A \otimes \overline{T}^{p+q} \otimes A$ is the same as above.

Then we define a $k$-linear map $f \bullet_{-i} g \in C^{m+n-1}(A, \overline{T}^{p+q}(A))$ by the composition of the above four maps

$$f \bullet_{-i} g := d_{p+q} \circ (\text{id}_A \otimes \text{id}_{\overline{T}^{p+q}} \otimes \text{id}_{A^{-1}}) \circ \left( (f^{(r)} \otimes \text{id}_{A^{-1}}) \otimes \text{id}_{\overline{T}} \right) \circ \left( \text{id}_{\overline{T}} \otimes g^{(b)} \otimes \text{id}_{\overline{T}^{m-i}} \right) \circ (g^{(r)} \otimes \text{id}_{\overline{T}^{m-1}})$$

for $1 \leq i \leq q$. So far, the $k$-linear map $f \bullet_i g \in C^{m+n-1}(A, \overline{T}^{p+q}(A))$ has been defined in the following way:

$$f \bullet_i g = \begin{cases} 
    d_{p+q}((f^{(r)} \otimes \text{id}_{A^{-1}}) \otimes g^{(b)} \otimes \text{id}_{\overline{T}^{m-i}} \otimes 1) & \text{if } 1 \leq i \leq m; \\
    d_{p+q}((\text{id} \otimes \text{id}_{\overline{T}^{q+1}} \otimes f^{(b)} \otimes \text{id}_{\overline{T}^{m-i}}) \otimes 1) & \text{if } q > 0 \text{ and } -q \leq i \leq -1.
\end{cases}$$

Now, we define a $k$-linear map $f \bullet g \in C^{m+n-1}(A, \overline{T}^{p+q}(A))$ by

$$f \bullet g := \begin{cases} 
    \sum_{i=1}^{m} (-1)^{r(m,p,n,q;i)} f \bullet_i g + \sum_{i=1}^{q} (-1)^{s(m,p,n,q;i)} f \bullet_{-i} g & \text{if } q > 0; \\
    \sum_{i=1}^{m} (-1)^{r(m,p,n,q;i)} f \bullet_i g & \text{if } q = 0,
\end{cases}$$
where \( r(m, p; n, q; i) \) and \( s(m, p; n, q; i) \) are determined by
\[
\begin{align*}
    r(m, p; n, q; i) & := p + q + (i - 1)(q - n - 1) \text{ for } 1 \leq i \leq m, \\
    s(m, p; n, q; i) & := p + q + (i - 1)(q - n - 1) \text{ for } 1 \leq i \leq q.
\end{align*}
\]

Finally, we are able to define a \( k \)-linear map \( [f, g]_{sg} \in \text{Ext}_{A_e}^{m+n-1}(A, \overline{\Omega}^{p+q}(A)) \) as
\[
[f, g]_{sg} := f \bullet g - (-1)^{(m-p-1)(n-q-1)} g \bullet f.
\]

Wang [21] showed that the cup product \( \cup_{sg} \) and the bilinear map \([\ , \ ]_{sg}\) induce well-defined operators, still denoted by \( \cup_{sg} \) and \([\ , \ ]_{sg}\), on a graded \( k \)-vector space
\[
\bigoplus_{m \in \mathbb{Z}, p \in \mathbb{Z} \geq 0, m+p \geq 0} \text{Ext}_{A_e}^{m+p}(A, \overline{\Omega}^{p}(A))
\]
with grading
\[
\left( \bigoplus_{m, p} \text{Ext}_{A_e}^{m+p}(A, \overline{\Omega}^{p}(A)) \right)^i = \bigoplus_{l \geq 0, i \geq 0} \text{Ext}_{A_e}^{i+l}(A, \overline{\Omega}^{l}(A))
\]
for \( i \in \mathbb{Z} \), which make it into a Gerstenhaber algebra. Furthermore, he proved that the two induced operators \( \cup_{sg} \) and \([\ , \ ]_{sg}\) are compatible with the connecting homomorphisms \( \theta_{m, p} : \text{Ext}_{A_e}^{m}(A, \overline{\Omega}^{p}(A)) \rightarrow \text{Ext}_{A_e}^{m+1}(A, \overline{\Omega}^{p-1}(A)) \). Therefore, we have the following result.

**Theorem 1.6** ([21, Theorem 4.1]). Let \( A \) be a finite dimensional algebra over a field \( k \). Then the graded \( k \)-vector space
\[
\bigoplus_{m \in \mathbb{Z}, p \in \mathbb{Z} \geq 0} \lim_{\rightarrow l(m)} \text{Ext}_{A_e}^{m+p}(A, \overline{\Omega}^{p}(A))
\]
equipped with the cup product \( \cup_{sg} \) and the Lie bracket \([\ , \ ]_{sg}\) is a Gerstenhaber algebra.

**Remark 1.7.** The Gerstenhaber brackets on \( \bigoplus_{m, p} \text{Ext}_{A_e}^{m+p}(A, \overline{\Omega}^{p}(A)) \) involving elements of degree zero are defined via the connecting homomorphisms
\[
\theta_{0, e} : \text{Ext}_{A_e}^{0}(A, \overline{\Omega}^{e}(A)) \rightarrow \text{Ext}_{A_e}^{1}(A, \overline{\Omega}^{e+1}(A)),
\]
that is, for \( f \in \text{Ext}_{A_e}^{m+p}(A, \overline{\Omega}^{p}(A)) \) and \( \alpha \in \text{Ext}_{A_e}^{0}(A, \overline{\Omega}^{e}(A)) \), we define
\[
[f, \alpha]_{sg} := [f, \theta_{0, q}(\alpha)]_{sg}.
\]

2. Frobenius algebras and complete resolutions

Let \( k \) be a field and \( A \) a finite dimensional \( k \)-algebra of dimension \( d \) over \( k \), and let \( \sigma \) be an algebra automorphism of \( A \). For any \( A \)-bimodule \( M \), we denote by \( M_{\sigma} \) the \( A \)-bimodule which is \( M \) as a \( k \)-vector space and whose \( A \)-bimodule structure is defined by \( a \cdot m \cdot b := am\sigma(b) \) for \( m \in M_{\sigma} \) and \( a, b \in A \). We also denote by \( A^\vee \) a right \( A^e \)-module \( \text{Hom}_{A^e}(A^e A, A^e A^e) \) whose structure is given...
by the multiplication of $A^e$ on the right hand side. Note that we have an isomorphism of right $A^e$-modules

$$A^e \xrightarrow{\nu} (A \otimes A)^A := \left\{ \sum x_i \otimes y_i \mid \sum ax_i \otimes y_i = \sum x_i \otimes ya \text{ for any } a \in A \right\} ; f \mapsto f(1),$$

where a right $A^e$-module structure of $(A \otimes A)^A$ is defined by the multiplication of $A^e$ on the right hand side. Recall that $A$ is a Frobenius algebra if there is an associative and non-degenerate bilinear form $\langle \cdot, \cdot \rangle : A \otimes A \to k$. The associativity means that $\langle ab, c \rangle = \langle a, bc \rangle$ for all $a, b$ and $c \in A$. If $(u_i)_{i=1}^d$ is a $k$-basis of $A$, then there is a $k$-basis $(v_i)_{i=1}^d$ of $A$ such that $\langle v_i, u_j \rangle = \delta_{ij}$ with $\delta_{ij}$ Kronecker’s delta. In such a case, we call $(u_i)_{i=1}^d, (v_i)_{i=1}^d$ dual bases of $A$. There exists an algebra automorphism $\nu$, up to inner automorphism, of $A$ such that $\langle a, b \rangle = \langle b, \nu(a) \rangle$ for all $a, b \in A$, and the automorphism $\nu$ is said to be the Nakayama automorphism of $A$. In fact, we can write both the Nakayama automorphism $\nu$ and its inverse $\nu^{-1}$, explicitly: for $x \in A$,

$$\nu(x) := \sum_{i=1}^d \langle x, v_i \rangle u_i, \quad \nu^{-1}(x) := \sum_{i=1}^d \langle u_i, x \rangle v_i.$$  

Another definition of Frobenius algebras is that $A$ is isomorphic to $D(A)$ as right or as left $A$-modules. Here the left (right) $A$-module structure of $D(A)$ is defined by $(af)(x) := f(ax)$ ($(fa)(x) := f(ax)$) for any $f \in D(A)$ and any $a \in A$. We can see that the bilinear form $\langle \cdot, \cdot \rangle : A \otimes A \to k$ induces an isomorphism of left $A$-modules

$$\phi : A \xrightarrow{\sim} D(A) ; a \mapsto \langle -, a \rangle.$$

Moreover, this isomorphism gives rise to an isomorphism of $A$-bimodules $A_\nu \xrightarrow{\sim} D(A)$.

The first statement of the next lemma appears in [7, Lemma 2.1.35]. However, we prove it again in order to get the explicit form of the isomorphism below.

**Lemma 2.1.** Let $A$ be a finite dimensional Frobenius algebra. With the same notation above, we have the following assertions.

1. There is an isomorphism of right $A^e$-bimodules $A_{\nu^{-1}} \cong A^\nu$.
2. If $(u_i)_{i=1}^d, (v_i)_{i=1}^d$ and $(u'_j)_{j=1}^d, (v'_j)_{j=1}^d$ are two dual bases of $A$, then we have $\sum_i u_i \otimes v_i = \sum_j u'_j \otimes v'_j$.
3. An element $\sum_i u_i \otimes v_i$ of $A \otimes A$ has the following properties:
   - $(i)$ $\sum_i u_i \otimes v_i = \sum_i v_i \otimes \nu^{-1}(u_i) = \sum_i \nu(v_i) \otimes u_i$;
   - $(ii)$ $\sum_i au_i \otimes v_i = \sum_i u_i \otimes \nu^{-1}(b)v_i$ for any $a, b \in A$.

**Proof.** For the statements (2) and (3), consider the composition $\eta : A_{\nu^{-1}} \otimes A \to \text{Hom}_k(A, A)$ of isomorphisms

$$\xymatrix{ A_{\nu^{-1}} \otimes A \ar[r] & D(A) \otimes A \ar[r] & \text{Hom}_k(A, A) \ar[r] & \sum_i x_i \otimes y_i \ar[r] & \sum_i \langle -, x_i \rangle \otimes y_i \ar[r] & \left[ x \mapsto \sum_i \langle x, x_i \rangle y_i \right].}$$

- \(\nu(x) := \sum_{i=1}^d \langle x, v_i \rangle u_i, \quad \nu^{-1}(x) := \sum_{i=1}^d \langle u_i, x \rangle v_i.

- Another definition of Frobenius algebras is that $A$ is isomorphic to $D(A)$ as right or as left $A$-modules. Here the left (right) $A$-module structure of $D(A)$ is defined by $(af)(x) := f(ax)$ ($(fa)(x) := f(ax)$) for any $f \in D(A)$ and any $a \in A$. We can see that the bilinear form $\langle \cdot, \cdot \rangle : A \otimes A \to k$ induces an isomorphism of left $A$-modules

$$\phi : A \xrightarrow{\sim} D(A) ; a \mapsto \langle -, a \rangle.$$

- Moreover, this isomorphism gives rise to an isomorphism of $A$-bimodules $A_\nu \xrightarrow{\sim} D(A)$.

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  3. An element $\sum_i u_i \otimes v_i$ of $A \otimes A$ has the following properties:
     - $(i)$ $\sum_i u_i \otimes v_i = \sum_i v_i \otimes \nu^{-1}(u_i) = \sum_i \nu(v_i) \otimes u_i$;
     - $(ii)$ $\sum_i au_i \otimes v_i = \sum_i u_i \otimes \nu^{-1}(b)v_i$ for any $a, b \in A$.

- **Proof.** For the statements (2) and (3), consider the composition $\eta : A_{\nu^{-1}} \otimes A \to \text{Hom}_k(A, A)$ of isomorphisms

$$\xymatrix{ A_{\nu^{-1}} \otimes A \ar[r] & D(A) \otimes A \ar[r] & \text{Hom}_k(A, A) \ar[r] & \sum_i x_i \otimes y_i \ar[r] & \sum_i \langle -, x_i \rangle \otimes y_i \ar[r] & \left[ x \mapsto \sum_i \langle x, x_i \rangle y_i \right].}$$

- \(\nu(x) := \sum_{i=1}^d \langle x, v_i \rangle u_i, \quad \nu^{-1}(x) := \sum_{i=1}^d \langle u_i, x \rangle v_i.$$
Since \( x = \sum_i \langle x, u_i \rangle v_i \) for any dual bases \((u_i)_{i=1}^d, (v_i)_{i=1}^d\) of \( A \) and any \( x \in A \), the statements (2) and (3) follow from the injectivity of \( \eta \). On the other hand, we define

\[
\varphi : A_{\nu^{-1}} \to A^\vee; \quad x \mapsto \left[ a \mapsto \sum_i a u_i \nu(x) \otimes v_i \right],
\]

\[
\psi : A^\vee \to (A \otimes A)^A \to A_{\nu^{-1}}; \quad \alpha \mapsto \alpha(1_A) = \sum_i x_i \otimes y_i \mapsto \sum_i \langle 1, x_i \rangle y_i.
\]

Then we get \( \varphi \) is a right \( A^e \)-module homomorphism. Indeed, if \( x \in A_{\nu^{-1}} \) and \( a \otimes b^r \in A^e \), then we have \( \varphi(x (a \otimes b^r)) = \sum_i u_i \nu(bx) a \otimes v_i = \sum_i u_i \nu(x) a \otimes bv_i = (\sum_i u_i \nu(x) \otimes v_i) (a \otimes b^r) \). One can easily check that \( \varphi \psi = \text{id}_{A^\vee} \) and \( \psi \varphi = \text{id}_{A_{\nu^{-1}}} \).

As we remarked in Section 1.2, if \( A \) is a Frobenius algebra \( A \), then so is the enveloping algebra \( A^e \). In particular, \( A^e \) is a Gorenstein algebra of Gorenstein dimension zero. Therefore, \( A \) has a complete resolution over \( A^e \). Note that we may take a projective resolution of \( A \) over \( A^e \) as the non-negative part of the complete resolution. In fact, Nakayama [16] constructed a complete resolution \( \mathcal{T} \) of \( A \) in the following way: we set \( \mathcal{T}_r := \text{Bar}_r(A) = A \otimes \bar{A}^{\otimes r} \otimes A \) for every \( r \geq 0 \) and \( \mathcal{T}_{-s} := D(\text{Bar}_{s-1}(A))_{\nu^{-1}} \) for each \( s \geq 1 \). Then we get an augmented exact sequence

\[
\cdots \to \mathcal{T}_r \xrightarrow{d_r} \mathcal{T}_{r-1} \to \cdots \to \mathcal{T}_0 \xrightarrow{d_0} \mathcal{T}_{-1} \to \cdots \xrightarrow{d_{-s}} \mathcal{T}_{-s-1} \to \mathcal{T}_{-s} \xrightarrow{D(d_0)} A \xrightarrow{\varphi} D(A)_{\nu^{-1}} \to \cdots
\]

where we put

\[
D(d_0)(f) = fd_0 \quad (f \in D(A)_{\nu^{-1}}), \quad d'_0 = D(d_0) \phi d_0, \quad d_{-s}(g) = gd_s \quad (g \in \mathcal{T}_{-s}).
\]

Sanada [19, Lemma 1.1] proved that for any \( A \)-bimodule \( M \) and any integer \( r \), there is an isomorphism between \( \text{Hom}_{A^e}(\mathcal{T}_r, M) \) and \( M_{\nu^{-1}} \otimes_{A^e} \mathcal{T}_{r-1} \) which is natural in \( M \), so that each \( \mathcal{T}_r \ (r \in \mathbb{Z}) \) is projective over \( A^e \). Furthermore, this isomorphism gives an isomorphism between the cochain complex \( \text{Hom}_{A^e}(\mathcal{T}, M) \) and the chain complex \( M_{\nu^{-1}} \otimes_{A^e} \mathcal{T} \). Therefore, the following cochain complex \((D^\bullet(A, M), \delta_r^\bullet)\) has the same cohomology groups as \( \text{Hom}_{A^e}(\mathcal{T}, M) \):

\[
\cdots \to C_2(A, M_{\nu^{-1}}) \xrightarrow{\delta_r} C_1(A, M_{\nu^{-1}}) \xrightarrow{\delta_{r-1}} M_{\nu^{-1}} \xrightarrow{\mu} M \xrightarrow{\delta_0} C^1(A, M) \xrightarrow{\delta_1} C^2(A, M) \to \cdots,
\]

where we define \( \mu : M_{\nu^{-1}} \to M \) by \( \mu(m) := \sum_i d u_i m v_i \) for \( m \in M \) and set

\[
D^r(A, M) = \begin{cases} C^r(A, M) & \text{if } r \geq 0, \\ C_{-r-1}(A, M_{\nu^{-1}}) & \text{if } r \leq -1, \end{cases} \quad \delta_r^\bullet = \begin{cases} \delta_r & \text{if } r \geq 0, \\ \mu & \text{if } r = -1, \\ \partial_{-r-1} & \text{if } r \leq -2. \end{cases}
\]

We give the explicit forms of the 0-th and \((-1)\)-th cohomology groups as follows:

\[
\widetilde{\text{HH}}^0(A) \cong M^A/N_A(M), \quad \widetilde{\text{HH}}^{-1}(A) = N_A M/I_A(M),
\]
where we set

\[ M^A := \{ m \in M \mid am = ma \text{ for all } a \in A \}, \]

\[ N_A(M) := \text{Im}(\mu) = \left\{ \sum_i u_imv_i \mid m \in M \right\}, \]

\[ N_A^0M := \{ m \in M \mid \sum_i u_imv_i = 0 \}, \]

\[ I_A(M) := \left\{ \left( \sum_i (m_i\nu^{-1}(a_i) - a_im_i) \right) \text{ (finite sum)} \mid a_i \in A, m_i \in M \right\}. \]

Note that for any \( x \in A, \sum_i u_ixv_i = 0 \) holds if and only if \( \sum_i u_i\nu(x)v_i = 0 \) holds.

**Remark 2.2.** If \( M = A \), then \( \widehat{\text{HH}}^0(A) \) and \( \widehat{\text{HH}}^{-1}(A) \) are appeared in the following exact sequence:

\[ 0 \to \text{Ext}^{r-1}_{A^e}(A, A) \to A_{\nu-1} \otimes_{A^e} A \xrightarrow{\eta} \text{Hom}_{A^e}(A, A) \to \text{Ext}^0_{A^e}(A, A) \to 0, \]

where the morphism \( \eta(x \otimes_{A^e} a)(b) = \sum_i b_i\nu(x)a_\nu \).

Suppose that \( A \) is a finite dimensional self-injective \( k \)-algebra. Recall that \( A \) is a self-injective algebra if \( A \) is injective as a left and as a right \( A \)-module. Note that the enveloping algebra \( A^e \) is also a self-injective algebra. Observe that if \( A \) is a self-injective algebra, then all of the connecting homomorphisms (1.1)

\[ \theta_{m+p,p} : \text{Ext}^{m+p}_{A^e}(A, \overline{\Omega}^p(A)) \to \text{Ext}^{m+p+1}_{A^e}(A, \overline{\Omega}^{p+1}(A)) \]

are isomorphisms except for the case \( m+p = 0 \), so that we have an isomorphism \( \text{Ext}^{r+p}_{A^e}(A, \overline{\Omega}^p(A)) \cong \text{Ext}^{r}_{A^e}(A, A) \) for all \( r, p \in \mathbb{Z} \) such that \( p \geq 0 \) and \( r + p > 0 \). We need modification for the inductive system \( \{ X_p^{(m)}, \theta_{m+p,p} \}_{p \in I(m)} \) defined in Section 2.3. Let us recall that

\[ \lim_{p \in I(m)} \text{Ext}^{m+p}_{A^e}(A, \overline{\Omega}^p(A)) \]

is the inductive limit of the inductive system \( \{ X_p^{(m)}, \theta_{m+p,p} \}_{p \in I(m)} \) of which the term \( X_p^{(m)} \) is defined by \( X_p^{(m)} := \text{Ext}^{m+p}_{A^e}(A, \overline{\Omega}^p(A)) \) and whose morphism \( \theta_{m+p,p} \) is the connecting homomorphism \( \text{Ext}^{m+p}_{A^e}(A, \overline{\Omega}^p(A)) \to \text{Ext}^{m+p+1}_{A^e}(A, \overline{\Omega}^{p+1}(A)) \). Consider another inductive system

\[ \{ Y_p^{(m)}, \varphi_{m+p,p} \}_{p \in I(m)} \]

of which the term \( Y_p^{(m)} \) is the same as \( X_p^{(m)} \) and whose morphism \( \varphi_{m+p,p} \) is given by

\[ \varphi_{m+p,p} := \begin{cases} (-1)^{m+i}\theta_{m+i,i} & \text{if } p = i, \\ (-1)^m\theta_{m+p,p} & \text{if } p > i, \end{cases} \]

where an integer \( i \geq 0 \) is the least one belonging to \( I(m) \). Then we can readily see

\[ \lim_{p \in I(m)} Y_p^{(m)} \cong \lim_{p \in I(m)} \text{Ext}^{m+p}_{A^e}(A, \overline{\Omega}^p(A)). \]
We will utilize the inductive system \( \{ X_p^{(m)}, \varphi_{m+p,p} \}_{p \in \mathbb{N}} \) instead of \( \{ X_p^{(m)}, \theta_{m+p,p} \}_{p \in \mathbb{N}} \) and denote

\[
\varphi_{m+p,p}^q := \varphi_{m+p+q-1,p+q-1} \circ \cdots \circ \varphi_{m+p,p} : \text{Ext}_{A^e}(A, \Omega^q(A)) \to \text{Ext}_{A^e}(A, \Omega^{p+q}(A)).
\]

Note that \( \varphi_{m+p,p}^1 = \varphi_{m+p,p} \).

The following is a special case of [5, Corollary 6.4.1], which says that we have a description of the Tate-Hochschild cohomology by using Ext and Tor for a self-injective algebra.

**Proposition 2.3.** Let \( A \) be a finite dimensional self-injective \( k \)-algebra. Denote \( A^y = \text{Hom}_{A^e}(A, A^e) \).

Then we have the following.

1. \( \text{Ext}_{A^e}^r(A, A) \cong \text{Ext}_{A^e}^r(A, A) \) for all \( r \geq 1 \).
2. \( \text{Ext}_{A^e}^r(A, A) \cong \text{Tor}_{A^e}^{r-1}(A, A^y) \) for all \( r \geq 2 \).
3. There exists an exact sequence of \( k \)-vector spaces

\[
0 \to \text{Ext}_{A^e}^{-1}(A, A) \to A^y \otimes_{A^e} A \xrightarrow{\eta} \text{Hom}_{A^e}(A, A) \to \text{Ext}_{A^e}^0(A, A) \to 0,
\]

where the morphism \( \eta \) is given by \( \eta((\sum_i x_i \otimes y_i) \otimes_{A^e} a)(b) = \sum_i b x_i a y_i \).

4. \( \text{Ext}_{A^e}^0(A, A) = \text{Hom}_{A^e}(A, A) \), which is the set of \( A \)-bimodule homomorphisms from \( A \) to \( A \) modulo those homomorphisms passing through projective \( A \)-bimodules.

In particular, for \( r \geq 2 \) and \( p \geq 1 \),

\[
\kappa_{-1,p} : \text{Ext}_{A^e}^{-1}(A, A) = \text{Ker}(\eta) \xrightarrow{\sim} \text{Ext}_{A^e}^p(A, \Omega^{p+1}(A)) \cong \text{Ext}_{A^e}^{-1}(A, A),
\]

\[
\varphi_{0,0}^p : \text{Ext}_{A^e}^0(A, A) = \text{Coker}(\eta) \xrightarrow{\sim} \text{Ext}_{A^e}^p(A, \Omega^p(A)) \cong \text{Ext}_{A^e}^0(A, A),
\]

\[
\kappa_{r-1,p} : \text{Tor}_{A^e}^{r-1}(A, A^y) \xrightarrow{\sim} \text{Ext}_{A^e}^p(A, \Omega^{p+1}(A)) \cong \text{Ext}_{A^e}^{-r}(A, A)
\]

are defined, on the \((co)\)chain level, as

\[
\kappa_{-1,p}(\alpha \otimes_{A^e} a)(\bar{b}_{1,p}) = \sum_i d_{p+1}(x_i a \otimes \bar{y}_i \otimes \bar{b}_{1,p} \otimes 1);
\]

\[
\varphi_{0,0}^p(f)(\bar{b}_{1,p}) = d_p(f(1) \otimes \bar{b}_{1,p} \otimes 1),
\]

\[
\kappa_{r-1,p}(\alpha \otimes_{A^e} \bar{a}_{1,r-1})(\bar{b}_{1,p}) = \sum_i d_{r+1}(x_i \otimes \bar{a}_{1,r-1} \otimes \bar{y}_i \otimes \bar{b}_{1,p} \otimes 1),
\]

where we write \( \alpha(1) = \sum_i x_i \otimes y_i \). We denote \( \varphi_{0,0}^1 \) by \( \varphi_{0,0} \).

The third isomorphisms \( \kappa_{r-1,p} : \text{Tor}_{A^e}^{r-1}(A, A^y) \xrightarrow{\sim} \text{Ext}_{A^e}^{-r}(A, A) \) in Proposition 2.3 are given by Wang [21, Remark 6.3].

Algebras which we are interested in are Frobenius algebras, which are self-injective algebras. Using Remark 2.2 and Lemma 2.1, we obtain the following consequence of Proposition 2.3.

**Corollary 2.4.** Let \( A \) be a finite dimensional Frobenius \( k \)-algebra. Then there is an isomorphism \( \text{Ext}_{A^e}^r(A, A) \cong \text{HH}^r(A) \) for all \( r \in \mathbb{Z} \).

Assume that \( A \) is a finite dimensional Frobenius algebra. Then the \( A \)-bimodule isomorphism \( A_{r-1} \cong A^y \) gives an isomorphism of complexes between \( \mathcal{D}^\bullet(A, A) \) and the complex \( C^\bullet(A, A) \) defined by Wang [22]

\[
\cdots \to C_2(A, A^y) \xrightarrow{\theta_2} C_1(A, A^y) \xrightarrow{\theta_1} A^y \xrightarrow{\mu} A \xrightarrow{\delta_0} C_1(A, A) \xrightarrow{\delta_1} C_2(A, A) \to \cdots
\]
whose negative part is the Hochschild chain complex \((C_\bullet(A, A^\vee), \partial_\bullet)\) and of which the non-negative part is the Hochschild cochain complex \((C^\bullet(A, A), \delta^\bullet)\). Here the map \(\mu : A^\vee \to A\) is defined by the multiplication of \(A\), that is, \(\mu(\alpha) = \sum_i x_i y_i\) for \(\alpha \in A^\vee\) with \(\alpha(1) = \sum_i x_i \otimes y_i\).

Moreover, Wang [22, Section 6.2] defined a product on \(C^\bullet(A, A)\), called \(*\)-product, which extends the cup product on \(C^\bullet(A, A)\) and the cap product between \(C^\bullet(A, A)\) and \(C_\bullet(A, A^\vee)\). Although the \(*\)-product is not associative on \(C^\bullet(A, A)\) in general, the \(*\)-product induces a graded commutative and associative product on \(H^\bullet(C^\bullet(A, A))\). The following is the product

\[
* : \mathcal{D}^\bullet(A, A) \otimes \mathcal{D}^\bullet(A, A) \to \mathcal{D}^\bullet(A, A)
\]

on \(\mathcal{D}^\bullet(A, A)\) via the isomorphism \(\mathcal{D}^\bullet(A, A) \cong C^\bullet(A, A)\): let \(f \in C^m(A, A), g \in C^n(A, A)\) and \(\alpha = a_0 \otimes \overline{p}_{1, p} \in C_p(A, A_{\nu^{-1}}), \beta = b_0 \otimes \overline{q}_{1, q} \in C_q(A, A_{\nu^{-1}})\).

(1) \((m, n \geq 0)\) \(* : C^m(A, A) \otimes C^n(A, A) \to C^{m+n}(A, A)\) is given by

\[
f \star g := f \circ g ;
\]

(2) \((m \geq 0, p \geq 0, p \geq m)\)

(i) \(\star : C^m_p(A, A_{\nu^{-1}}) \otimes C^n(A, A) \to C^{m-p}(A, A_{\nu^{-1}})\) is given by

\[
\alpha \star f := \alpha \circ f = a_0 \nu^{-1}(f(\overline{p}_{1, m})) \otimes \overline{p}_{m+1, p} ;
\]

(ii) \(\star : C^m(A, A) \otimes C^p(A, A_{\nu^{-1}}) \to C^{m-p}(A, A_{\nu^{-1}})\) is given by

\[
f \star \alpha := f(\overline{p}_{p-m+1, p}) a_0 \otimes \overline{p}_{1, p-m} ;
\]

(3) \((m \geq 0, p \geq 0, p < m)\)

(i) \(\star : C^m(A, A) \otimes C^p(A, A_{\nu^{-1}}) \to C^{m-p-1}(A, A)\) is given by

\[
(f \star \alpha)(\overline{p}_{1, m-p-1}) := \sum_i f(\overline{p}_{1, m-p-1} \otimes u_i \nu(a_0) \otimes \overline{p}_{1, p}) v_i ;
\]

(ii) \(\star : C^p(A, A_{\nu^{-1}}) \otimes C^m(A, A) \to C^{m-p-1}(A, A)\) is given by

\[
(\alpha \star f)(\overline{p}_{1, m-p-1}) := \sum_i u_i \nu(a_0) f(\overline{p}_{1, p} \otimes \overline{p}_{i} \otimes \overline{p}_{1, m-p-1}) ;
\]

(4) \((p, q \geq 0)\) \(\star : C^p(A, A_{\nu^{-1}}) \otimes C^q(A, A_{\nu^{-1}}) \to C^{p+q+1}(A, A_{\nu^{-1}})\) is given by

\[
\alpha \star \beta := \sum_i v_i b_0 \otimes \overline{b}_{1, q} \otimes u_i \nu(a_0) \otimes \overline{a}_{1, p} .
\]

Dual bases of \(A\) are used in our definition of \(*\)-product, but Lemma 2.1 (2) shows that the \(*\)-product does not depend the choice of dual bases of \(A\).

We summarize the results in the following proposition.

**Proposition 2.5** ([22, Lemma 6.2, Propositions 6.5 and 6.9]). Let \(A\) be a finite dimensional Frobenius algebra. Then the \(*\)-product is compatible with the differential \(\hat{d}\) of the complex \(D(A, A)\). Moreover, the induced product on \(\hat{HH}^\bullet(A)\), still denoted by \(*\), is graded commutative and associative. In particular, \(\hat{HH}^\bullet(A)\) equipped with \(*\) is isomorphic to \(\hat{Ext}_A^\bullet(A, A)\) as graded algebras.
3. Decomposition of complete cohomology associated with the spectrum of the Nakayama automorphism

Let us recall the subcomplexes of the (co)chain Hochschild complexes defined in [14]. Let \( A \) be a (not necessarily Frobenius) \( k \)-algebra and let \( \sigma \) be an algebra automorphism of the algebra \( A \). Let \( \Lambda \) be the set of eigenvalues of \( \sigma \). Assume that \( \Lambda \subset k \). We have \( 0_A \not\in \Lambda \) and \( 1_A \in \Lambda \) because \( \sigma \) is a ring homomorphism. Let \( \tilde{\Lambda} := \langle \Lambda \rangle \) be the submonoid of \( k^\times \) generated by \( \Lambda \). We denote by \( A_\lambda \) the eigenspace \( \text{Ker} (\sigma - \lambda \text{id}) \) associated with an eigenvalue \( \lambda \in \Lambda \). For \( \lambda \in \Lambda \), we write \( \overline{A}_\lambda = A_\lambda \) for \( \lambda \neq 1 \) and \( \overline{A}_1 = A_1/(k \cdot 1_A) \) for \( \lambda = 1 \), and for every \( \mu \in \tilde{\Lambda} \) and every integer \( r \geq 0 \), we put

\[
C_r^{(\mu)}(A, A_\sigma) := \bigoplus_{\mu \in \Lambda, \prod_{i=1}^r = \mu} A_{\mu_0} \otimes \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_r},
\]

\[
C_r^\sigma(A, A) := \bigl\{ f \in C^r(A, A) \bigl| f(\overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_r}) \subset A_{\mu_1 \cdots \mu_r}, \text{ for any } \mu_i \in \Lambda \bigr\}.
\]

Since \( \sigma(xy) = \sigma(x)\sigma(y) \) for \( x, y \in A \), we have \( A_\lambda \cdot A_\lambda' \subset A_{\lambda \lambda'} \) for \( \lambda, \lambda' \in \Lambda \). It is understood that \( A_{\lambda \lambda'} = 0 \) when \( \lambda \lambda' \notin \Lambda \). Then these subspaces \( C_r^{(\mu)}(A, A_\sigma) \) and \( C_r^\sigma(A, A) \) are compatible with the differentials \( \partial^\sigma \) and \( \delta^\sigma \) of the complexes \((C_\bullet(A, A_\sigma), \partial^\sigma)\) and \((C_\bullet(A, A), \delta^\sigma)\), respectively. Thus, we obtain subcomplexes \((C_\bullet(A, A_\sigma), \partial^\mu)\) and \((C_\bullet(A, A), \delta^\mu)\). Then we put

\[
H_r^{(\mu)}(A, A_\sigma) := H_r(C_\bullet^{(\mu)}(A, A_\sigma), \partial^\mu),
\]

\[
H_r^\sigma(A, A) := H_r(C_\bullet^\sigma(A, A), \delta^\sigma).
\]

Hence for all \( r \geq 0 \), we get morphisms of \( k \)-vector spaces \( H_r^{(\mu)}(A, A_\sigma) \rightarrow H_r(A, A_\sigma) \) and \( H_r^\sigma(A, A) \rightarrow \text{HH}_r(A) \). Kowalzing and Krähmer [13] defined a \( k \)-linear map

\[
B_r^\sigma : C_r(A, A_\sigma) \rightarrow C_{r+1}(A, A_\sigma)
\]

by

\[
B_r^\sigma(a_0 \otimes \overline{a}_{1,r}) = \sum_{i=1}^{r+1} (-1)^i a_0 \otimes \overline{a}_i \otimes \cdots \otimes \overline{a}_r \otimes \overline{\sigma(a_{i-1})} \otimes \cdots \otimes \overline{\sigma(a_{r-1})}.
\]

Let \( T : C_r(A, A_\sigma) \rightarrow C_r(A, A_\sigma) \) be the \( k \)-linear map defined by

\[
T(a_0 \otimes \overline{a}_{1,r}) = \sigma(a_0) \otimes \overline{\sigma(a_1)} \otimes \cdots \otimes \overline{\sigma(a_r)}.
\]

A direct calculation shows that \( \partial_{r+1} B_r^\sigma - B_r^\sigma \partial_r = (-1)^{r+1}(\text{id} - T) \) for all \( r \geq 0 \).

**Proposition 3.1** ([14, Propositions 2.1, 2.2 and 2.5]). The following three assertions hold.

1. For every \( 1 \neq \mu \in \tilde{\Lambda} \) and every \( r \geq 0 \), we get

\[
H_r^{(\mu)}(A, A_\sigma) = 0.
\]

2. For all \( r \geq 0 \), the restriction of the map \( B_r^\sigma : C_r(A, A_\sigma) \rightarrow C_{r+1}(A, A_\sigma) \) to the subspaces \( C_r^{(1)}(A, A_\sigma) \) induces a twisted Connes operator

\[
B_r^\sigma : H_r^{(1)}(A, A_\sigma) \rightarrow H_{r+1}^{(1)}(A, A_\sigma),
\]

and it satisfies \( B_r^\sigma B_{r+1}^\sigma = 0 \).
(3) If \( \sigma \) is diagonalizable, then we have
\[
H_r(A, A_\sigma) \cong H_r^{(1)}(A, A_\sigma)
\]
for \( r \geq 0 \).

The following is an easy consequence of Proposition 3.1.

**Corollary 3.2.** If the algebra automorphism \( \sigma \) of \( A \) is diagonalizable, then so is its inverse \( \sigma^{-1} \). Furthermore, if this is the case, then we have two twisted Connes operators
\[
B^\sigma_r : H_r^{(1)}(A, A_\sigma) \to H_{r+1}^{(1)}(A, A_\sigma), \quad B^{-\sigma}_r : H_r^{(1)}(A, A_{\sigma^{-1}}) \to H_{r+1}^{(1)}(A, A_{\sigma^{-1}}).
\]

From now on, we assume \( A \) to be a Frobenius algebra with the Nakayama automorphism \( \nu \) of \( A \). Let \( \Lambda = \{ \lambda_1, \ldots, \lambda_t \} \) be the set of distinct eigenvalues of the Nakayama automorphism \( \nu \). Suppose that \( \Lambda \subset k \). Let \( \hat{\Lambda} := \langle \Lambda \rangle \) be the submonoid of \( k^* \) generated by \( \Lambda \). For any \( \mu \in \hat{\Lambda} \), we define a subspace \( \mathcal{D}^r_{(\mu)}(A, A) \) of \( \mathcal{D}^r(A, A) \) in the following way: for any \( \mu \in \hat{\Lambda} \),
\[
\mathcal{D}^r_{(\mu)}(A, A) := \begin{cases} C^r_{(\mu)}(A, A) & \text{if } r \geq 0, \\ C^{-r}_{(\mu)}(A, A_{\nu^{-1}}) & \text{if } r \leq -1. \end{cases}
\]

**Lemma 3.3.** For any \( \mu \in \hat{\Lambda} \), the subspaces \( \mathcal{D}^r_{(\mu)}(A, A) \) of \( \mathcal{D}^r(A, A) \) are compatible with the differentials \( \hat{d}^r \) of the complex \( (\mathcal{D}^r(A, A), \hat{d}^r) \).

**Proof.** It is sufficient to show that \( \hat{d}^{-1}(\mathcal{D}^{-1}_{(\mu)}(A, A)) \subset \mathcal{D}^0_{(\mu)}(A, A) \). If \( x \in A_\mu = \mathcal{D}^{-1}_{(\mu)}(A, A) \), then we have
\[
\nu(\hat{d}^{-1}(x)) = \sum_i \nu(u_i)\nu(x)\nu(v_i) = \sum_{i,j} \langle u_i, v_j \rangle u_j \cdot \nu(x)\nu(v_i)
\]
\[
= \sum_j u_j \nu(x) \left( \sum_i \langle u_i, v_j \rangle v_i \right) = \sum_j u_j \nu(x)\nu(\nu^{-1}(v_j))
\]
\[
= \sum_j u_j \nu(x)v_j.
\]
Since \( 0 = (\nu - \mu \text{id})(x) = \nu(x) - \mu x \), we get
\[
(\nu - \mu \text{id})(\hat{d}^{-1}(x)) = \nu(\hat{d}^{-1}(x)) - \mu \hat{d}^{-1}(x) = \sum_j u_j \nu(x)v_j - \mu \sum_j u_j xv_j = 0.
\]
Therefore, we have \( \hat{d}^{-1}(x) \in \mathcal{D}^0_{(\mu)}(A, A) \). \( \Box \)

From Lemma 3.3, we obtain a subcomplex \( (\mathcal{D}^r_{(\mu)}(A, A), \hat{d}^r_{(\mu)}) \) of \( (\mathcal{D}^r(A, A), \hat{d}^r) \). Put
\[
\widetilde{\text{HH}}^r_{(\mu)}(A) := H^r(\mathcal{D}^r_{(\mu)}(A, A), \hat{d}^r_{(\mu)})
\]
for all \( r \in \mathbb{Z} \). Hence we have a morphism of \( k \)-vector spaces \( \widetilde{\text{HH}}^r_{(\mu)}(A) \to \widetilde{\text{HH}}^r(A) \) for \( r \in \mathbb{Z} \). Before starting with the next proposition, let us recall a well-known duality between Hochschild
cohomology and Hochschild homology: there is an isomorphism \( \Theta : D(C_r(A, A_\nu)) \to C^*(A, A) \) given by

\[
D(C_r(A, A_\nu)) = \text{Hom}(A_\nu \otimes_A A \otimes A^\otimes r, A) \\
\cong \text{Hom}_A(A \otimes A^\otimes r, \text{Hom}(A_\nu, k)) \\
\cong \text{Hom}_A(A \otimes A^\otimes r, A, A) = C^r(A, A),
\]

where \( r \geq 0 \) and the second isomorphism is induced by \( A_\nu \cong D(A) \). Then \( \Theta : D(C_r(A, A_\nu)) \to C^r(A, A) \) is a morphism of complexes and hence induces a duality \( D(H_r(A, A_\nu)) \cong \text{HH}^r(A) \). In fact, we can write \( \Theta : D(C_r(A, A_\nu)) \to C^r(A, A) \) and its inverse \( \Theta^{-1} : C^r(A, A) \to D(C_r(A, A_\nu)) \) as follows:

\[
\Theta : D(C_r(A, A_\nu)) \to C^r(A, A); \quad \psi \mapsto \left[ \overline{b}_{1,r} \mapsto \sum_j \psi(u_j \otimes \overline{b}_{1,r})v_j \right], \\
\Theta^{-1} : C^r(A, A) \to D(C_r(A, A_\nu)); \quad f \mapsto \left[ a_0 \otimes \overline{a}_{1,r} \mapsto \langle f(\overline{a}_{1,r}), a_0 \rangle \right].
\]

**Proposition 3.4.** Let \( A \) be a finite dimensional Frobenius \( k \)-algebra. If the Nakayama automorphism \( \nu \) of \( A \) is diagonalizable, then three statements hold.

1. The isomorphism \( \Theta : D(C_r(A, A_\nu)) \to C^*(A, A) \) induces an isomorphism of complexes

\[
D(C^*_r(A, A_\nu)) \cong C^*_{(\nu-1)}(A, A)
\]

for all \( \mu \in \hat{\Lambda} \).

2. For \( r \in \mathbb{Z} \) and \( \mu \neq 1 \in \hat{\Lambda} \), we get

\[
\text{HH}^r_{(\mu)}(A) = 0.
\]

3. For each \( r \in \mathbb{Z} \), there exists an isomorphism of \( k \)-vector spaces

\[
\text{HH}^r_{(1)}(A) \cong \text{HH}^r(A).
\]

**Proof.** It follows from Lemma 3.5 below that the inverse of each eigenvalue \( \lambda \in \Lambda \) is also an eigenvalue of the Nakayama automorphism \( \nu \) of \( A \). Since \( A \) is the (finite) direct sum of the eigenspaces \( A_{\lambda_1}, \ldots, A_{\lambda_t} \), we have \( D(C_r(A, A_\nu)) \cong \bigoplus_{\mu \in \hat{\Lambda}} D(C^*_r(\mu)(A, A_\nu)) \) for all \( r \geq 0 \). For the first statement, it is sufficient to show that the inverse \( \Theta^{-1} : C^*(A, A) \to D(C^*_r(A, A_\nu)) \) induces an isomorphism \( C^*_r(\mu)(A, A) \cong D(C^*(\mu^{-1})(A, A_\nu)) \). Since \( \Theta^{-1}(f) \in D(C_r(A, A_\nu)) \) is a non-zero map for \( 0 \neq f \in C^*_r(A, A) \), there exist \( \mu' \in \hat{\Lambda} \) and \( a_0 \otimes \overline{a}_{1,r} \in C^*_r(A, A_\nu) \) such that \( \langle f(\overline{a}_{1,r}), a_0 \rangle \neq 0 \), so that we get \( \langle \mu \mu' - 1 \rangle \langle f(\overline{a}_{1,r}), a_0 \rangle = 0 \) and hence \( \mu' = \mu^{-1} \). As a result, we have shown that if \( \lambda \in \hat{\Lambda} \) with \( \lambda \neq \mu^{-1} \), then the restriction of \( \Theta^{-1}(f) \) to \( C^*_r(\lambda)(A, A_\nu) \) is the zero map. Thus, we have a monomorphism

\[
\Theta^{-1}_{(\mu)} := \Theta^{-1}|_{C^*_r(A, A_\nu)} : C^*_r(A, A) \to D(C^*_{(\mu^{-1})}(A, A_\nu)).
\]
Furthermore, we get $\Theta_{(-\mu)}^{-1}$ is surjective. Indeed, for any $\psi \in D(C_{r}(\mu^{-1})(A, A_\nu))$, there exists $f \in C_r(A, A)$ such that $\psi = \Theta_{(-\mu)}^{-1}(f)$. Let $\mu_1, \ldots, \mu_r \in \Lambda$ and $\overline{b}_{1, r} \in \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_r}$. It follows from $A = \bigoplus \Lambda \lambda_i$ and $\psi|_{C_r(\lambda_i)(A, A_\nu)} = 0$ for all $\lambda \neq \mu^{-1}$ that

$$\langle f(\overline{b}_{1, r}), a \rangle = \langle \nu(f(\overline{b}_{1, r})), \nu(a) \rangle = \langle \nu(f(\overline{b}_{1, r})), (\mu_1 \cdots \mu_r)^{-1} \mu^{-1}a \rangle$$

for any $a \in A$. Consequently, we get $\nu(f(\overline{b}_{1, r})) = \mu_1 \cdots \mu_r f(\overline{b}_{1, r})$ and hence $f \in C_{r}(\mu)(A, A)$. This shows that $\Theta_{(-\mu)}^{-1} : C_{r}(\mu)(A, A) \to D(C_{r}(\mu^{-1})(A, A_\nu))$ is surjective.

For the second statement, let $r$ be an integer and $\mu \in \hat{\Lambda}$ such that $\mu \neq 1$. In the case $r \leq -2$, the desired result is a consequence of Proposition 3.1 (1). If $r \geq 1$, then the first statement (1) and Proposition 3.1 (1) imply that there is an isomorphism

$$\widehat{HH}_{(\mu)}^r(A) = H^r(\mu)(A, A) \cong D(H^r_{(\mu^{-1})}(A, A_\nu)).$$

We also have $\widehat{HH}_{(\mu)}^0(A) = 0$ and $\widehat{HH}_{(\mu)}^0(\mu) = 0$ because $\widehat{HH}_{(\mu)}^0(A) \leq H^0_{(\mu)}(A, A_{\nu^{-1}})$, and $\widehat{HH}_{(\mu)}^0(\mu)$ is a quotient space of $H^0_{\mu}(A, A)$.

For the last statement, let $r$ be an integer. For the case $r \leq -2$, the desired result is a consequence of Proposition 3.1 (3). If $r \geq 1$, then the first statement (1) and Proposition 3.1 (1) yield that there are isomorphisms

$$\widehat{HH}^r(A) = HH^r(A) \cong D(H_r(A, A_\nu)) \cong D(H^1_{(\mu)}(A, A_\nu)) \cong H^1_{(\mu)}(A, A) = \widehat{HH}^r_{(1)}(A).$$

Since $A_{\nu^{-1}} = \bigoplus \Lambda \lambda_i$ as $k$-vector spaces, the differential $\widehat{d}^{-1}$ can be decomposed as $\widehat{d}^{-1} = [\widehat{d}^{-1}_\lambda : A_{\lambda_i} \to A]$ is the restriction of $\widehat{d}^{-1}$ to $A_{\lambda_i}$. Then we have

$$\widehat{HH}^{-1}(A) \cong \bigoplus_{1 \leq i \leq \ell} \widehat{HH}^{-1}_{(\lambda_i)}(A) = \widehat{HH}^{-1}_{(1)}(A).$$

Similarly, we have $\widehat{HH}^0(A) \cong \widehat{HH}^0_{(1)}(A)$. This completes the proof. \hfill \square

**Lemma 3.5** ([14, Lemma 3.5]). Let $A$ be a finite dimensional Frobenius $k$-algebra with the Nakayama automorphism $\nu$ diagonalizable. Then we have the following statements.

1. For any $\lambda \in \Lambda$, its inverse $\lambda^{-1}$ belongs to $\Lambda$.

2. The isomorphism of $A$-bimodules $A_{\nu} \cong D(A)$ induces an isomorphism of vector spaces $A_{\lambda} \cong D(A_{\lambda^{-1}})$ for any $\lambda \in \Lambda$.

Suppose that the Nakayama automorphism $\nu$ is diagonalizable. For each $\lambda_i \in \Lambda = \{\lambda_1, \ldots, \lambda_\ell\}$, we denote by $m_i$ its algebraic multiplicity. Then we have a $k$-basis $(u_{i, m_i}^{\lambda_j})$ of the eigenspace $A_{\lambda_i}$ associated with $\lambda_i$. Thus $d$-tuple $(u_{i, m_i}^{\lambda_j}, \ldots, u_{m_i}^{\lambda_j}, u_{1}^{\lambda_j}, \ldots, u_{l}^{\lambda_j})$ forms a $k$-basis of $A$, and we obtain its dual basis $(v_{1}^{\lambda_j}, v_{m_i}^{\lambda_j}, \ldots, v_{1}^{\lambda_j}, \ldots, v_{l}^{\lambda_j})$ of $A$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$. It follows from Lemma 3.5 and $\langle v_{k}^{\lambda_j}, u_{i}^{\lambda_j} \rangle = \delta_{ij} \delta_{kl}$ that the dual basis vectors $v_{1}^{\lambda_j}, \ldots, v_{l}^{\lambda_j}$ belong to $A_{\lambda^{-1}_i}$ for each $\lambda_i$. We fix the dual bases $(u_{i, m_i}^{\lambda_j}, \ldots, u_{m_i}^{\lambda_j})$ of $A$. For simplifying the notation, we will write $(u_1, \ldots, u_d)$ and $(v_1, \ldots, v_d)$ for $(u_i^{\lambda_j})_{i, j}$ and $(v_i^{\lambda_j})_{i, j}$ when there is no danger of confusion.

**Proposition 3.6.** Let $A$ be a finite dimensional Frobenius $k$-algebra with the Nakayama automorphism $\nu$ diagonalizable. For any $\mu, \mu' \in \hat{\Lambda}$, $* : D^*(A, A) \otimes D^*(A, A) \to D^*(A, A)$ induces the restrictions $*_{\mu, \mu'} : D^*_\mu(A, A) \otimes D^*_{\mu'}(A, A) \to D^*_{\mu\mu'}(A, A)$.
Proof. We only show that the $\ast$-product $\ast$ restricts to the subcomplexes in the cases (3) (i). Proofs of the other cases are similar to the proof of the case (3) (i). Let $\mu, \mu' \in \hat{\Lambda}$ be arbitrary and $m, p \in \mathbb{Z}$ such that $m \geq 0, p \geq 0$ and $p > m$, and let $f \in C^m_{(\nu)}(A, A)$ and $\alpha = a_0 \otimes \overline{a}_{1, p} \in A_{\nu_0} \otimes \overline{A}_{\nu_1} \otimes \cdots \otimes \overline{A}_{\mu_m} \subset C^m_{(\mu')} (A, A_{\nu_1})$ with $\prod \mu_i' = \mu'$. We claim that

$$(f \ast \alpha)(\overline{b}_{1, m-p-1}) = \sum_{1 \leq i \leq t, 1 \leq j \leq m_i} f(\overline{b}_{1, m-p-1} \otimes u_j^\lambda \nu(a_0) \otimes \overline{a}_{1, p}) v_j^\lambda \in A_{\mu_1' \mu_2' \cdots \mu_{m-p-1}}$$

holds for any $\overline{b}_{1, m-p-1} \in \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_{m-p-1}}$, where the $\mu_i$ are elements of $\Lambda$. Indeed, we have

$$\nu(\sum_{i,j} f(\overline{b}_{1, m-p-1} \otimes u_j^\lambda \nu(a_0) \otimes \overline{a}_{1, p}) v_j^\lambda)$$

$$= \sum_{i,j} \nu(f(\overline{b}_{1, m-p-1} \otimes u_j^\lambda \nu(a_0) \otimes \overline{a}_{1, p})) \nu(v_j^\lambda)$$

$$= \sum_{i,j} \mu_1' \cdots \mu_{m-p-1} \lambda_i \lambda_j^0 f(\overline{b}_{1, m-p-1} \otimes u_j^\lambda \nu(a_0) \otimes \overline{a}_{1, p}) \lambda_j^{-1} v_j^\lambda$$

$$= \mu_1' \mu_2' \cdots \mu_{m-p-1} \sum_{i,j} f(\overline{b}_{1, m-p-1} \otimes u_j^\lambda \nu(a_0) \otimes \overline{a}_{1, p}) v_j^\lambda$$

and therefore $f \ast \alpha \in C^m_{(\mu')} (A, A)$. \hfill \Box

Put $\ast_1 := \ast_{1,1} : D_{(1)}^* (A, A) \otimes D_{(1)}^* (A, A) \to D_{(1)}^* (A, A)$. Then we have the following result.

Corollary 3.7. Let $A$ be a finite dimensional Frobenius $k$-algebra. Then $(\overline{HH}_{(1)}^* (A), \ast_1)$ is a graded commutative algebra. Furthermore, if the Nakayama automorphism $\nu$ of $A$ is diagonalizable, then $(\overline{HH}_{(1)}^* (A), \ast_1)$ is isomorphic to $(\overline{HH}^* (A), \ast)$ as graded algebras.

4. BV Structure on the Complete Cohomology

Let us recall the definition of Batalin-Vilkovisky algebras.

Definition 4.1. A graded commutative algebra $(\mathcal{H}^* = \bigoplus_{r \in \mathbb{Z}} \mathcal{H}^r, \cdot)$ with $1 \in \mathcal{H}^0$ is a Batalin-Vilkovisky algebra (BV algebra, for short) if there exists an operator $\Delta_\ast : \mathcal{H}^* \to \mathcal{H}^{*-1}$ of degree $-1$ such that:

(i) $\Delta_{r-1} \Delta_r = 0$ for any $r \in \mathbb{Z}$;
(ii) $\Delta_0(1) = 0$;
(iii) For homogeneous elements $\alpha, \beta$ and $\gamma$ in $\mathcal{H}^*$,

$$\Delta(\alpha \cdot \beta \cdot \gamma) = \Delta(\alpha \cdot \beta \cdot \gamma) + (-1)^{|\alpha|} \alpha \cdot \Delta(\beta \cdot \gamma)$$

$$+ (-1)^{|\beta|+|\alpha|} \beta \cdot \Delta(\alpha \cdot \gamma) - \Delta(\alpha) \cdot \beta \cdot \gamma$$

$$- (-1)^{|\alpha|} \alpha \cdot \Delta(\beta) \cdot \gamma - (-1)^{|\alpha|+|\beta|} \alpha \cdot \beta \cdot \Delta(\gamma),$$

where $|\alpha|$ denotes the degree of a homogeneous element $\alpha \in \mathcal{H}^*$. 
Remark 4.2. For each BV algebra \((H^\bullet, \llcorner, \Delta)\), we can associate a graded Lie bracket \([\ , \ ]\) of degree \(-1\) as
\[
[\alpha, \beta] := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|} \left( (-1)^{|\alpha|+1} \Delta(\alpha \llcorner \beta) + (-1)^{|\alpha|} \Delta(\alpha) \llcorner \beta + \alpha \llcorner \Delta(\beta) \right),
\]
where \(\alpha, \beta\) are homogeneous elements of \(H^\bullet\). The equation is said to be the BV identity. It follows from [9, Proposition 1.2] that the bracket \([\ , \ ]\) above makes \((H^\bullet, \llcorner, [\ , \ ])\) into a Gerstenhaber algebra.

Recall that a symmetric algebra \(A\) is a Frobenius algebra with a non-degenerate bilinear form \(\langle \ , \ \rangle : A \otimes A \to k\) satisfying \(\langle a, b \rangle = \langle b, a \rangle\) for all \(a, b \in A\). Wang gave the following result for symmetric algebras.

**Theorem 4.3** ([21, Corollary 6.21]). Let \(A\) be a finite dimensional symmetric \(k\)-algebra. Then the complete cohomology ring \((\widetilde{HH}(A), \ast)\) is a BV algebra together with an operator \(\tilde{\Delta}_\ast : \widetilde{HH}(A) \to \widetilde{HH}^{-1}(A)\) defined by
\[
\tilde{\Delta}_r = \begin{cases} 
\Delta_r & \text{if } r \geq 1, \\
0 & \text{if } r = 0, \\
(-1)^r B_{r-1} & \text{if } r \leq -1,
\end{cases}
\]
where \(B_r\) is the Connes operator defined by
\[
B_r(a_0 \otimes \overline{a}_1, r) = \sum_{i=1}^{r+1} (-1)^{ir} 1 \otimes \overline{a}_i, r \otimes \overline{a}_0 \otimes \overline{a}_{1, i-1}
\]
for any \(a_0 \otimes \overline{a}_1, r \in C^r(A, A_{v-1})\), and \(\Delta_r\) defined in [20] is the dual of the Connes operator \(B_{r-1}\), which is equivalent to saying that \(\Delta_r\) is given by a formula
\[
\langle \Delta_r(f)(\overline{a}_1, r-1), a_r \rangle = \sum_{i=1}^r (-1)^{i(r-1)} \langle f(\overline{a}_i, r-1 \otimes \overline{a}_r \otimes \overline{a}_{1, i-1}), 1 \rangle
\]
for any \(f \in C^r(A, A)\). In particular, the restrictions \(\widetilde{HH}^{>0}(A)\) and \(\widetilde{HH}^{<0}(A)\) are BV subalgebras of \(\widetilde{HH}(A)\).

**Remark 4.4.** Let \(A\) be a finite dimensional symmetric \(k\)-algebra. It follows from Remark 4.2 that the BV differential \(\hat{\Delta}\) in Theorem 4.3 gives rise to a Lie bracket \([\ , \ ]\) (of degree \(-1\)) defined by
\[
\{\alpha, \beta\} := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|} \left( (-1)^{|\alpha|+1} \hat{\Delta}(\alpha \llcorner \beta) + (-1)^{|\alpha|} \hat{\Delta}(\alpha) \llcorner \beta + \alpha \llcorner \hat{\Delta}(\beta) \right)
\]
for any homogeneous elements \(\alpha, \beta \in \widetilde{HH}(A)\). Moreover, the Gerstenhaber algebra \((\widetilde{HH}(A), \ast, \{\ , \ \})\) is isomorphic to \((\text{Ext}_{A^e}(A, A), \ast_{sg}, [\ , \ ]_{sg})\) as Gerstenhaber algebras.

In the rest of this section, we will show the following result on Frobenius algebras whose Nakayama automorphisms are diagonalizable.

**Theorem 4.5.** Let \(A\) be a finite dimensional Frobenius \(k\)-algebra. If the Nakayama automorphism \(\nu\) is diagonalizable, then the graded commutative ring \((\widetilde{HH}^\bullet_{(1)}(A), \ast_1)\) is a BV algebra together with
an operator $\hat{\Delta}_r : \hat{\mathbb{H}}(1)_{(1)}(A) \to \hat{\mathbb{H}}(1)_{(1)}^{-1}(A)$ defined by

$$\hat{\Delta}_r = \begin{cases} \Delta^r & \text{if } r \geq 1, \\ 0 & \text{if } r = 0, \\ (-1)^i B^{-1}_{r-1} & \text{if } r \leq -1, \end{cases}$$

where $B^{-1}_{r-1}$ is the twisted Connes operator defined by

$$B^{-1}_{r-1}(a_0 \otimes \bar{a}_{1,r}) = \sum_{i=1}^{r+1}(-1)^{ir}1 \otimes \bar{a}_{i,r} \otimes \bar{a}_0 \otimes \nu^{-1}(a_1) \otimes \cdots \otimes \nu^{-1}(a_{i-1})$$

for any $a_0 \otimes \bar{a}_{1,r} \in C_r(A, A_{\nu^{-1}})$, and $\Delta^r$ defined in [14] is the dual of the twisted Connes operator $B^{-1}_{r-1}$, which is equivalent to saying that $\Delta^r$ is given by a formula

$$\langle \Delta^r(f)(\bar{a}_{1,r-1}), a_r \rangle = \sum_{i=1}^{r}(-1)^{(r-1)}\langle f(\bar{a}_{i,r-1} \otimes \bar{a}_r \otimes \nu(a_1) \otimes \cdots \otimes \nu(a_{i-1})), 1 \rangle$$

for any $f \in C^r(A, A)$. In particular, the restrictions $\hat{\mathbb{H}}_{(1)}^{>0}(A)$ and $\hat{\mathbb{H}}_{(1)}^{<0}(A)$ are BV subalgebras of $\hat{\mathbb{H}}_{(1)}^*(A)$.

**Remark 4.6.** Each of the components $\Delta^r$ and $B^{-1}_{r-1}$ is defined on the chain level. Corollary 3.2 and Lemma 3.5 imply that we can lift the two components $\Delta^r$ and $B^{-1}_{r-1}$ to the cohomology level when we restrict them to $D^*_r(A, A)$.

Using the isomorphism $\hat{\mathbb{H}}_{(1)}^*(A) \cong \hat{\mathbb{H}}^*(A)$ appeared in Corollary 3.7, we have our main result.

**Corollary 4.7.** Let $A$ be a finite dimensional Frobenius $k$-algebra. If the Nakayama automorphism of $A$ is diagonalizable, then the complete cohomology ring $\hat{\mathbb{H}}^*(A)$ is a BV algebra.

In order to prove Theorem 4.5, we claim that the bilinear map

$$\{ \ , \ \} : \hat{\mathbb{H}}^m_{(1)}(A) \otimes \hat{\mathbb{H}}^n_{(1)}(A) \to \hat{\mathbb{H}}^{m+n-1}_{(1)}(A) \ (m, n \in \mathbb{Z})$$

defined by

$$\{ \alpha, \beta \} := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|} (-1)^{|\alpha|+1} \hat{\Delta}(\alpha \star_1 \beta) + (-1)^{|\alpha|} \hat{\Delta}(\alpha) \star_1 \beta + \alpha \star_1 \hat{\Delta}(\beta)$$

for any $\alpha \otimes \beta \in \hat{\mathbb{H}}^m_{(1)}(A) \otimes \hat{\mathbb{H}}^n_{(1)}(A)$ commutes with the Gerstenhaber bracket

$$[\ , \ ]_{sg} : \text{Ext}^m_{Ae}(A, A) \otimes \text{Ext}^n_{Ae}(A, A) \to \text{Ext}^{m+n-1}_{Ae}(A, A).$$

By considering whether $m + n - 1$ is negative or not together with Figure 1 and using the anti-commutativity of the Gerstenhaber bracket $[\ , \ ]_{sg}$, we see that it suffices to show our claim for a pair $(m, n)$ of integers $m \leq n$ satisfying one of the following conditions:

1. $(m, n)$ is on the lines $m = 0$ or $n = 0$.
2. $(m, n)$ belongs to the regions I, IV, V or VI.
Thus our claim can be divided into the five cases Propositions 4.8, 4.9, 4.10 and 4.11 and Remark 4.12. In particular, Propositions 4.8, 4.9, 4.10 and 4.11 prove our claim for the pairs in the regions VI, V, IV and I, respectively. Further, we consider the case (1) in Remark 4.12. Among the four propositions, we prove only the first one. We also remark that, in the following propositions, the appearing integers \( m \) and \( n \) are independent of the above argument.

**Proposition 4.8.** Let \( A \) be a finite dimensional Frobenius \( k \)-algebra with the Nakayama automorphism \( \nu \) of \( A \) diagonalizable, and let \( m, n \) be integers such that \( m > n \geq 1 \), so \( m - n - 1 \geq 0 \). Then we have the following commutative diagram.

\[
\begin{array}{c}
\bigotimes
\end{array}
\]

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\begin{array}{c}
\bigotimes
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\[
\begin{array}{c}
\bigotimes
\end{array}
\]
holds in \(\text{Ext}_A^m(A,\Omega^{n+1}(A))\) for \(f \in \text{Ker} \delta_{(1)}^m\) and \(z := a_0 \otimes \overline{a}_{1,n-1} \in \text{Ker} \delta_{n-1}^{(1)}\). Denote by \(\overline{f}\) the composition of \(f : \overline{A}^{\otimes m} \to A\) with the canonical epimorphism \(\pi : A \to \overline{A}\). For the right hand side of the formula (4.1), we have, for \(\overline{b}_{1,m} \in \overline{A}^{\otimes m}\),

\[
\varphi_{m-n-0,0}^{n+1}(\{f, z\})(\overline{b}_{1,m}) = (-1)^{mn+n+1} \varphi_{m-n-0,0}^{n+1}(\Delta^\nu(f \otimes z))(\overline{b}_{1,m}) + (-1)^{mn+n} \varphi_{m-n-1,0}^{n+1}(\Delta^\nu(f) \otimes z)(\overline{b}_{1,m}) + (-1)^{mn+m} \varphi_{m-n-1,0}^{n+1}(f \otimes B^\nu(z))(\overline{b}_{1,m})
\]

\[
= \sum_{j,k} \sum_{i=1}^{n-m} (-1)^{i(n-m)+1+n+1} d((u_k \nu a_0 f(\overline{a}_{1,n-1} \otimes \overline{a}_k \otimes \overline{b}_{i,m-n-1} \otimes \overline{a}_j \otimes \overline{v} \overline{b}_{1,i-1}), 1)v_j \otimes \overline{b}_{m-n,m} \otimes 1
\]

\[
+ \sum_{j,k} \sum_{i=1}^{n-m} (-1)^{i(m+1)} d((u_j \nu a_0 f(\overline{a}_{i,n-1} \otimes \overline{a}_k \otimes \overline{b}_{i,m-n-1} \otimes \overline{a}_j \otimes \nu^{-1} a_{1,i-1}), 1)v_k \otimes \overline{b}_{m-n,m} \otimes 1
\]

\[
+ \sum_{j,k} \sum_{i=1}^{m-n} (-1)^{(i+n)(m+1)} d((u_j \nu a_0 f(\overline{b}_{i,m-n-1} \otimes \overline{a}_k \otimes \nu^{-1} a_{1,n-1} \otimes \nu a_j \otimes \nu \overline{b}_{1,i-1}), 1)v_k \otimes \overline{b}_{m-n,m} \otimes 1
\]

\[
+ \sum_{j} \sum_{i=1}^{m-n} (-1)^{(m+i)(n+1)} d((f(\overline{b}_{1,m-n-1} \otimes \overline{a}_i \otimes \overline{a}_{i,n-1} \otimes \nu^{-1} a_{1,i-1})v_j \otimes \overline{b}_{m-n,m} \otimes 1
\]

in \(\Omega^{n+1}(A)\). On the other hand, for the left hand side of the formula (4.1), we get

\[
[f, \kappa_{n-1,1}(z)](\overline{b}_{1,m}) = (f \bullet \kappa_{n-1,1}(z) - (-1)^{(m-1)(-n-1)} \kappa_{n-1,1}(z) \bullet f)(\overline{b}_{1,m})
\]

\[
= \sum_{j} \sum_{i=1}^{m-n} (-1)^{(i+1)(n+1)} d((f(\overline{b}_{i-1} \otimes \overline{a}_j \nu a_0 \otimes \overline{a}_{1,n-1} \otimes \overline{v}_j \otimes \overline{b}_{i,m-n-1}) \otimes \overline{b}_{m-n,m} \otimes 1)
\]

\[
+ \sum_{j} \sum_{i=m-n+1}^{m} (-1)^{(i+1)(n+1)} d((f(\overline{b}_{i-1} \otimes \overline{a}_j \nu a_0 \otimes \overline{a}_{1,m-i}) \otimes \overline{a}_{m-i+1,n-1} \otimes \overline{v}_j \otimes \overline{b}_{i,m} \otimes 1)
\]

\[
+ \sum_{j} \sum_{i=1}^{n} (-1)^{(i+1)(n+1)} d((u_j \nu a_0 \otimes \overline{a}_{i-1} \otimes f(\overline{a}_{i,n-1} \otimes \overline{v}_j \otimes \overline{b}_{1,m-i-1}) \otimes \overline{b}_{m-n+i,m} \otimes 1)
\]

\[
(4.3)
\]

We will transform \([f, \kappa_{n-1,1}(z)](\overline{b}_{1,m})\) to \(\varphi_{m-n-1,0}^{n+1}(\{f, z\})(\overline{b}_{1,m})\) in \(\Omega^{n+1}(A)\), using some boundaries.
First, we deform (4.2). A direct calculation shows

\[
\sum_{j, k} \sum_{i=1}^{n-m} (-1)^{(i-m+1)+n+1} d\left(\langle u_{i,j} \nu a_{0} f(\overline{a}_{1,n-1} \otimes \nu_{k} \otimes \overline{b}_{i,m-n-1} \otimes \overline{v}_{j} \otimes \nu_{b_{1,i-1}}), 1\rangle v_{j} \otimes \overline{b}_{m-n,m} \otimes 1\right) \\
+ \sum_{j, k} \sum_{i=1}^{m-n} (-1)^{(i+n)(m+1)} d\left(\langle u_{j} \nu a_{0} f(\overline{b}_{i,m-n-1} \otimes \nu_{k} \otimes \nu_{a_{1,n-1}} \otimes \nu_{v_{j}} \otimes \nu_{b_{1,i-1}}), 1\rangle v_{k} \otimes \overline{b}_{m-n,m} \otimes 1\right)
\]

\[
= \sum_{j} \sum_{i=1}^{m-n-2} \sum_{l=i+2}^{m-n} (-1)^{(i-1)+n+1} \varphi_{m-n-1}^{-1} \left(\delta\left(\langle \sum_{j, k} \langle f(\nu_{id_{A}^{\otimes i-1}} \otimes \overline{u}_{j} \nu a_{0} \otimes \overline{\nu}_{1,n-1} \otimes \nu_{v_{j}}, 1\rangle v_{k} \otimes \overline{b}_{1,m} \right) \\
+ \sum_{j, k} \sum_{i=1}^{m-n} (-1)^{(i+1)(m+n)} \varphi_{m-n-1}^{-1} \left(\delta\left(\langle \sum_{j, k} \langle f(\nu_{id_{A}^{\otimes i-1}} \otimes \overline{\nu}_{k} \otimes \nu_{v_{j}}, 1\rangle v_{k} \otimes \overline{b}_{1,m} \right)
\right) \\
+ \sum_{j} \sum_{i=1}^{m-n} (-1)^{(i+1)(n+1)} d\left(\langle \overline{b}_{j, i-1} \otimes \nu_{j} \nu a_{0} \otimes \overline{\nu}_{1,n-1} \otimes \nu_{v_{j}} \otimes \overline{b}_{i,m-n-1} \otimes \overline{b}_{m-n,m} \otimes 1\right),
\]

where the \(k\)-linear map \(t : \overline{A}^{\otimes m-n-2} \to \overline{A}^{\otimes m-n-2}\) is given by \(t(\overline{b}_{1,m-n-2}) = \overline{b}_{2,m-n-2} \otimes \overline{b}_{1}\) for \(\overline{b}_{1,m-n-2} \in \overline{A}^{\otimes m-n-2}\). In particular, we have \(t^{i-1}(\overline{b}_{1,m-n-2}) = \overline{b}_{i,m-n-2} \otimes \overline{b}_{1,i-1}\). Note that the two maps

\[
\varphi_{m-n-1, 0}^{-1} \left(\delta\left(\langle \sum_{j, k} \langle f(\nu_{id_{A}^{\otimes i-1}} \otimes \overline{\nu}_{1,n-1} \otimes \nu_{v_{j}}, 1\rangle v_{k} \otimes \overline{b}_{1,m} \right) \\
+ \sum_{j, k} \sum_{i=1}^{m-n} (-1)^{(i+1)(n+1)} d\left(\langle \overline{b}_{j, i-1} \otimes \nu_{j} \nu a_{0} \otimes \overline{\nu}_{1,n-1} \otimes \nu_{v_{j}} \otimes \overline{b}_{i,m-n-1} \otimes \overline{b}_{m-n,m} \otimes 1\right),
\]
are zero in $\text{Ext}^m_{\mathcal{A}}(A, \mathcal{O}^{i+1}(A))$. Hence, we have

\[
\sum_j \sum_{i=1}^{m-n} (-1)^{(n+1)(i+1)} d(f(\bar{b}_{1,i-1} \otimes u_j \nu a_0 \otimes \bar{a}_{1,n-1} \otimes \bar{v}_j \otimes \bar{b}_{i,m-n-1} \otimes \bar{b}_{m-n,m} \otimes 1)
\]

\[
= \sum_j \sum_{i=1}^{m-n-2} \sum_{m-n} (-1)^{i(m+1)+(n+1)l} \varphi_{m-n-1,0}^{n-1} \left( \delta \left( \sum_{j,k} f(i_d \otimes i_l^{-1} \otimes u_j \nu a_0 \otimes \bar{a}_{1,n-1} \otimes \bar{v}_j \otimes id^{\otimes m-n-l} \otimes \bar{u}_k \otimes \bar{v}^{\otimes (-1)} \otimes 1) \otimes i^{\otimes -1} \right) \right) \left( \bar{b}_{1,m} \right)
\]

\[
+ \sum_j \sum_{i=1}^{m-n-1} (-1)^{i(m+1)+(n+1)(i+1)} \varphi_{m-n-1,0}^{n-1} \left( \delta \left( \sum_{j,k} f(u_j \nu a_0 \otimes \bar{a}_{1,n-1} \otimes \bar{v}_j \otimes id^{\otimes m-n-i-1} \otimes \bar{u}_k \otimes \bar{v}^{\otimes (-1)} \otimes 1) \otimes i^{\otimes -1} \right) \right) \left( \bar{b}_{1,m} \right)
\]

\[
+ \sum_{j,k} \sum_{i=1}^{n-m} (-1)^{i(n-m)+n+1} d((u_k \nu a_0 f(\bar{u}_{1,n-1} \otimes \bar{v}_k \otimes \bar{b}_{1,m-n-1} \otimes \bar{b}_{i,m-n-1} \otimes \nu^{-1} a_{1,n-1} \otimes \nu^{-1} a_{1,n-1} \otimes \nu^{-1} a_{1,n-1} \otimes 1) \otimes \bar{b}_{m-n,m} \otimes 1)
\]

\[
+ \sum_{j,k} \sum_{i=1}^{n-m} (-1)^{i(n+m)+m+1} d(u_j \nu a_0 (f(\bar{b}_{1,m-n-1} \otimes \bar{u}_k \otimes \bar{v}_j \otimes \nu^{-1} a_{1,n-1} \otimes \nu^{-1} a_{1,n-1} \otimes \nu^{-1} a_{1,n-1} \otimes 1) \otimes \bar{b}_{m-n,m} \otimes 1).
\]

Secondly, we deform (4.3). A direct calculation shows

\[
\sum_j \sum_{i=0}^{n-1} \sum_{i=0}^{n} (-1)^{(n+1)(i+1)} \otimes \nu^{-1} a_{n-i,1,n-1} \otimes \nu^{-1} a_{n-i,1,n-1} \otimes \nu^{-1} a_{n-i,1,n-1} \otimes 1) \left( \bar{b}_{1,m} \right)
\]

\[
= \sum_j \sum_{i=m-n+1}^{m-n+1} (-1)^{(n+1)(i+1)} d(f(\bar{b}_{1,i-1} \otimes u_j \nu a_0 \otimes \bar{a}_{1,m-i} \otimes \bar{b}_{i,m} \otimes 1)
\]

\[
+ \sum_j \sum_{i=1}^{n} (-1)^{(m+i)(n+1)} d(f(\bar{b}_{1,m-n-1} \otimes \bar{u}_j \otimes \bar{v}_i \otimes \nu^{-1} a_{1,n-1} \otimes \nu^{-1} a_{1,n-1} \otimes \nu^{-1} a_{1,n-1} \otimes 1) \otimes \bar{b}_{m-n,m} \otimes 1)
\]

\[
+ \sum_j \sum_{i=0}^{m-1} (-1)^{(m+1)(i+1)} d(f(\bar{b}_{1,m-n+i} \otimes \bar{u}_j \otimes \nu^{-1} a_{i+1,n-1} \otimes \nu^{-1} a_{i+1,n-1} \otimes \nu^{-1} a_{i+1,n-1} \otimes \nu^{-1} a_{i+1,n-1} \otimes 1) \otimes \bar{b}_{m-n-i+1} \otimes 1).
\]
Hence, we have

\[\sum_{j} \sum_{i=m-n+1}^{n} (-1)^{(n+1)(i+1)} d(f(\overline{b}_{1,i-1} \otimes \overline{u}_{j} \nu \overline{a}_{0} \otimes \overline{a}_{1,m-i}) \otimes \overline{a}_{m-i+1,n-1} \otimes \overline{v}_{j} \otimes \overline{b}_{i,m} \otimes 1)\]

\[= \sum_{j} \sum_{i=0}^{n-1} \sum_{l=i+1}^{n} (-1)^{n(m+i+1)+(n+1)(l-i+1)} \delta(d(f(id_{\overline{A}}^{m-n+i-1} \otimes \overline{u}_{j} \otimes \overline{a}_{n+i-l+1,n-1} \otimes \overline{a}_{0} \otimes \nu^{-1}a_{1,i-1}) \otimes \overline{b}_{1,m})\]

\[+ \sum_{j} \sum_{i=1}^{n} (-1)^{(m+i)(n+1)} d(f(\overline{b}_{1,m-n-1} \otimes \overline{u}_{j} \otimes \overline{a}_{i,n-1} \otimes \overline{a}_{0} \otimes \nu^{-1}a_{1,i-1}) \otimes \overline{b}_{m-n,m} \otimes 1)\]

\[+ \sum_{j} \sum_{i=0}^{n-1} (-1)^{n(m+1)+1} d(f(\overline{b}_{1,m-n+i} \otimes \overline{u}_{k} \otimes \nu^{-1}a_{i+1,n-1} \nu a_{0} \otimes \overline{a}_{1,i} \otimes \overline{v}_{k} \otimes \overline{b}_{m-n+i+1,m} \otimes 1)\]

(4.5)

Finally, we deform (4.3). A direct calculation shows

\[\sum_{j} \sum_{i=0}^{n-1} \sum_{l=i+1}^{n} (-1)^{n(m+i+1)+(n+1)(l-i+1)} \delta(d(u_{j} \nu a_{0} \otimes \overline{a}_{1,i-1} \otimes f(\nu^{-1}a_{n+i-l+1,n-1} \otimes \overline{v}_{j})\]

\[\otimes \overline{u}_{k} \otimes \overline{a}_{i+1,n-l+1}, 1) \otimes \overline{v}_{k} \otimes id_{\overline{A}}^{n-i} \otimes 1)) (\overline{b}_{1,m})\]

\[= \sum_{j} \sum_{i=1}^{n} (-1)^{(m+1)} d(u_{j} \nu a_{0} \otimes \overline{a}_{1,i-1} \otimes f(\overline{a}_{i,n-1} \otimes \overline{v}_{j} \otimes \overline{b}_{1,m-n+i-1} \otimes \overline{b}_{m-n+i,m} \otimes 1)\]

\[+ \sum_{j,k} \sum_{i=1}^{n} (-1)^{i(m+1)+1} d(u_{j} \nu a_{0} (f(\overline{a}_{i,n-1} \otimes \overline{v}_{j} \otimes \overline{b}_{1,m-n-1} \otimes \overline{u}_{k} \otimes \nu^{-1}a_{1,i-1}), 1) \nu k\]

\[\otimes \overline{b}_{m-n,m} \otimes 1)\]

\[+ \sum_{j} \sum_{i=0}^{n-1} (-1)^{m(n+1)+i} d(f(\overline{b}_{1,m-n+i} \otimes \overline{u}_{k} \otimes \nu^{-1}a_{i+1,n-1} \nu a_{0} \otimes \overline{a}_{1,i} \otimes \overline{v}_{k} \otimes \overline{b}_{m-n+i+1,m} \otimes 1).\]
Thus, we get
\[
\sum_{j} \sum_{i=1}^{n} (-1)^{(i+1)} d(u_j \nu a_0 \otimes \tau_{1,i-1} \otimes \overline{f}(\tau_{i,n-1} \otimes \tau_j \otimes \overline{b}_{1,m-n+i-1}) \otimes \overline{b}_{m-n+i,m} \otimes 1)
\]
\[=
\sum_{j} \sum_{i=0}^{n-1} \sum_{l=i+1}^{n} (-1)^{n(m+i+1)+(n+1)(l-i+1)} \delta(d(u_j \nu a_0 \otimes \nu^{-1}a_{1,i} \otimes \langle f(\nu^{-1}a_{n+i-l+1,n} \otimes \tau_j \\
\otimes \text{id} \otimes \nu^{-1}a_{n+i-l+1,n}) \rangle) \otimes \overline{f}(\nu^{-1}a_{1,i-1}, 1) \overline{\nu}(k) \otimes \text{id} \otimes 1)) (\overline{b}_{1,m})
\]
\[+ \sum_{j,k} \sum_{i=1}^{n} (-1)^{(i+1)} d(u_j \nu a_0 \langle f(\tau_{i,n-1} \otimes \tau_j \otimes \overline{b}_{1,m-n-1} \otimes \overline{\nu}(\nu^{-1}a_{1,i-1}), 1 \rangle \nu a_0 \otimes \tau_{1,i} \otimes \overline{\nu}(\nu^{-1}a_{1,i-1}) \otimes \overline{b}_{m-n+i+1,m} \otimes 1).
\]
\[
(4.6)
\]
Combining (4.5), (4.6) and (4.4), we obtain a formula
\[
[f, \kappa_{n-1,1}(z)]_{sg} (\overline{b}_{1,m}) + \delta(*) (\overline{b}_{1,m}) + \varphi^{n-1}_{m-n-1,0}(\delta(*))(\overline{b}_{1,m}) = \varphi^{n+1}_{m-n-1,0}(\{f, z\})(\overline{b}_{1,m})
\]
in $\overline{\Omega}^{n+1}(A)$ for all $\overline{b}_{1,m} \in \overline{A}^n$ and therefore
\[
[f, \kappa_{n-1,1}(z)]_{sg} = \varphi^{n+1}_{m-n-1,0}(\{f, z\})
\]
in $\text{Ext}^m_{A^e}(A, \overline{\Omega}^{n+1}(A))$ for $f \in \text{Ker} \delta^{m}_{n-1}$ and $z = a_0 \otimes \overline{a}_{1,n-1} \in \text{Ker} \delta^{(1)}_{n-1}$. This completes the proof of the statement. ☐

**Proposition 4.9.** Let $A$ be a finite dimensional Frobenius $k$-algebra with the Nakayama automorphism $\nu$ of $A$ diagonalizable, and let $m, n$ be integers such that $n \geq m \geq 1$, so $m - n - 1 < 0$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\widehat{\text{HH}}^m_{(1)}(A) \otimes \widehat{\text{HH}}^{-n}_{(1)}(A) & \{, \} & \widehat{\text{HH}}^{m-n}_{(1)}(A) \\
\text{Ext}^m_{A^e}(A, A) \otimes \text{Tor}^A_{n-1}(A, A_{\nu^{-1}}) & \cong & \text{Tor}^A_{n-m}(A, A_{\nu^{-1}}) \\
\text{id} \otimes \kappa_{n-1,1} & \cong & \kappa_{n-m,m} \\
\text{Ext}^m_{A^e}(A, A) \otimes \text{Ext}^1_{A^e}(A, \overline{\Omega}^{n+1}(A)) & \{, \} & \text{Ext}^m_{A^e}(A, \overline{\Omega}^{n+1}(A)) \\
\cong & & \cong \\
\text{Ext}^m_{A^e}(A, A) \otimes \text{Ext}^{-n}_{A^e}(A, A) & \{, \} & \text{Ext}^m_{A^e}(A, A),
\end{array}
\]
where \( \{ , \} : \mathcal{D}^{m}_{(1)}(A, A) \otimes \mathcal{D}^{-n}_{(1)}(A, A) \to \mathcal{D}^{m-n-1}_{(1)}(A, A) \) is defined by
\[
\{ f, z \} = (-1)^{|\mu|+|\nu|+|z|} \left( (-1)^{|\nu|+1} B^{\nu-1}(f *_1 z) + (-1)^{|\mu|} \Delta^{\nu}(f) *_1 z + (-1)^{|\nu|} f *_1 B^{\nu-1}(z) \right)
\]
for \( f \otimes z \in \mathcal{D}^{m}_{(1)}(A, A) \otimes \mathcal{D}^{-n}_{(1)}(A, A) \).

**Proposition 4.10.** Let \( A \) be a finite dimensional Frobenius \( k \)-algebra with the Nakayama automorphism \( \nu \) of \( A \) diagonalizable, and let \( m, n \) be integers such that \( m \geq 1 \) and \( n \geq 1 \). Then we have the following commutative diagram:

\[
\begin{array}{c}
\hat{\text{HH}}_{(1)}^{-m}(A) \otimes \hat{\text{HH}}_{(1)}^{-n}(A) \\
\downarrow \cong \\
\text{Tor}_{m-1}^{A^e}(A, A_{\nu-1}) \otimes \text{Tor}_{n-1}^{A^e}(A, A_{\nu-1}) \\
\downarrow \cong \\
\text{Ext}_{A^e}^{1}(A, \hat{\Omega}^{m+1}) \otimes \text{Ext}_{A^e}^{1}(A, \hat{\Omega}^{n+1}) \\
\downarrow \cong \\
\text{Ext}_{A^e}^{-m}(A, A) \otimes \text{Ext}_{A^e}^{-n}(A, A) \\
\downarrow \cong \\
\text{Ext}_{A^e}^{-m-n-1}(A, A)
\end{array}
\]

where \( \{ , \} : \mathcal{D}^{m}_{(1)}(A, A) \otimes \mathcal{D}^{-n}_{(1)}(A, A) \to \mathcal{D}^{m-n-1}_{(1)}(A, A) \) is defined by
\[
\{ w, z \} = (-1)^{|w|+|z|+|w|+|z|} \left( (-1)^{|\nu|+1} B^{\nu-1}(w *_1 z) + B^{\nu-1}(w) *_1 z + (-1)^{|\nu|} w *_1 B^{\nu-1}(z) \right)
\]
for \( w \otimes z \in \mathcal{D}^{m}_{(1)}(A, A) \otimes \mathcal{D}^{-n}_{(1)}(A, A) \).

The following is a consequence of Lambre-Zhou-Zimmermann.

**Proposition 4.11 ([14, Corollary 3.8]).** Let \( A \) be a finite dimensional Frobenius \( k \)-algebra with the Nakayama automorphism \( \nu \) of \( A \) diagonalizable, and let \( m, n \) be integers such that \( m > 0 \) and \( n > 0 \). Then we have the following commutative diagram:

\[
\begin{array}{c}
\hat{\text{HH}}_{(1)}^{m}(A) \otimes \hat{\text{HH}}_{(1)}^{n}(A) \\
\downarrow \cong \\
\text{Ext}_{A^e}^{m}(A, A) \otimes \text{Ext}_{A^e}^{n}(A, A) \\
\downarrow \cong \\
\text{Ext}_{A^e}^{m+n-1}(A, A)
\end{array}
\]
where $[ , ]$ is the Gerstenhaber bracket on Hochschild cohomology and $\{ , \} : D^{m}_{(1)}(A, A) \otimes D^{n}_{(1)}(A, A) \to D^{m+n-1}_{(1)}(A, A)$ is defined by
\[
\{ f, g \} = (-1)^{|f||g|+|f|+|g|} \left( (-1)^{|f|+1} \Delta^{\nu}(f \star_{1} g) + (-1)^{|f|} \Delta^{\nu}(f) \star_{1} g + f \star_{1} \Delta^{\nu}(g) \right) = (-1)^{|f||g|+|f|+|g|} \left( (-1)^{|f|+1} \Delta(f \star_{1} g) + (-1)^{|f|} \Delta(f) \star_{1} g + f \star_{1} \Delta(g) \right)
\]
for $f \otimes g \in D^{m}_{(1)}(A, A) \otimes D^{n}_{(1)}(A, A)$.

**Remark 4.12.** We have to consider the case of either $m = 0$ or $n = 0$. If $m \geq 0$ and $n = 0$, then we will prove that there is a commutative diagram
\[
\begin{array}{ccc}
\text{HH}^{m}_{(1)}(A) \otimes \text{HH}^{0}_{(1)}(A) & \xrightarrow{\{ , \}} & \text{HH}^{m-1}_{(1)}(A) \\
\downarrow \cong & & \downarrow \cong \\
\text{Ext}^{m}_{A^{e}}(A, A) \otimes \text{HH}^{0}_{(1)}(A) & \xrightarrow{\text{id} \otimes \varphi_{0,0}} & \text{Ext}^{m-1}_{A^{e}}(A, A) \\
\downarrow \cong & & \downarrow \cong \\
\text{Ext}^{m}_{A^{e}}(A, A) \otimes \text{Ext}^{1}_{A^{e}}(A, \Omega^{1}_{(1)}(A)) & \xrightarrow{[ , ]_{sg}} & \text{Ext}^{m}_{A^{e}}(A, \Omega^{1}_{(1)}(A)) \\
\downarrow \cong & & \downarrow \cong \\
\text{Ext}^{m}_{A^{e}}(A, A) \otimes \text{Ext}^{0}_{A^{e}}(A, A) & \xrightarrow{[ , ]_{sg}} & \text{Ext}^{m-1}_{A^{e}}(A, A),
\end{array}
\]
where the vertical isomorphism $\varphi_{0,0} : \text{HH}^{0}_{(1)}(A) \to \text{Ext}^{1}_{A^{e}}(A, \Omega^{1}_{(1)}(A))$ is defined in Proposition 2.3 and $\{ , \}$ is defined by
\[
\{ f, g \} = (-1)^{|f|} \left( (-1)^{|f|+1} \Delta^{\nu}(f \star_{1} g) + (-1)^{|f|} \Delta^{\nu}(f) \star_{1} g \right)
\]
for $f \otimes g \in D^{m}_{(1)}(A, A) \otimes D^{n}_{(1)}(A, A)$. We must show that
\[
\varphi_{m-1,0}([f \otimes g]) = ([f \otimes g](\text{id} \otimes \varphi_{0,0}))(f \otimes g)
\]
in $\text{Ext}^{m}_{A^{e}}(A, \Omega^{1}_{(1)}(A))$ for $f \otimes g \in \text{Ker} \delta^{m}_{(1)} \otimes \text{Ker} \delta^{0}_{(1)}$. A direct calculation shows that we have
\[
[f, \varphi_{0,0}(g)]_{sg} = \varphi_{m-1,0}([f, g])
\]
as maps, where $[ , ]$ is the Gerstenhaber bracket on Hochschild cohomology. It follows from [14, Corollary 3.8] that $[f, g] = -\Delta^{\nu}(f \star_{1} g) + \Delta^{\nu}(f) \star_{1} g$ in $\text{Ext}^{m-1}_{A^{e}}(A, A)$. As a result, we obtain a formula in $\text{Ext}^{m}_{A^{e}}(A, \Omega^{1}_{(1)}(A))$:
\[
[f, \varphi_{0,0}(g)]_{sg} = \varphi_{m-1,0}([f, g]) = \varphi_{m-1,0}(\Delta^{\nu}(f \star_{1} g) + \Delta^{\nu}(f) \star_{1} g) = \varphi_{m-1,0}(\{ f, g \})
\]
Similarly, one can prove our claim in the other case $m = 0$ and $n \geq 0$.

We are now able to prove Theorem 4.5.
Proof of Theorem 4.5. It follows from Propositions 4.8, 4.9, 4.10 and 4.11 and Remark 4.12 that we have the following commutative diagram

\[
\begin{array}{ccc}
\widehat{HH}_{(1)}^m(A) \otimes \widehat{HH}_{(1)}^{-n}(A) & \xrightarrow{\{ , \}} & \widehat{HH}_{(1)}^{m-n-1}(A) \\
\cong & & \cong \\
\text{Ext}_{kQ/I}^m(A, A) \otimes \text{Ext}_{kQ/I}^{-n}(A, A) & \xrightarrow{[ , ]_{sg}} & \text{Ext}_{kQ/I}^{m-n-1}(A, A),
\end{array}
\]

where \( m, n \) are arbitrary integers. Since \((\text{Ext}_{kQ/I}^*(A, A), \sim_{sg}, [ , ]_{sg})\) is a Gerstenhaber algebra, we have

\[
[f, g \sim_{sg} h]_{sg} = [f, g]_{sg} \sim_{sg} h + (-1)^{|f|(|g|+1)}g \sim_{sg} [f, h]_{sg}
\]

for arbitrary homogeneous elements \( f, g \) and \( h \in \text{Ext}_{kQ/I}^*(A, A) \). Since we have proved that \([ , ]_{sg}\) commutes with \( \{ , \} \), using the defining formula for \( \{ , \} \) and the formula

\[
\{f, g \triangleright h\} = (-1)^r \left((-1)^{|f|+1}\hat{\Delta}(f \triangleright g \triangleright h) + (-1)^{|f|}\hat{\Delta}(f) \triangleright g \triangleright h + f \triangleright \hat{\Delta}(g \triangleright h)\right)
\]

with \( r = |f|(|g|+|h|)+|f|+|g|+|h| \), we obtain

\[
\hat{\Delta}(f \triangleright g \triangleright h) = \hat{\Delta}(f \triangleright g) \triangleright h + (-1)^{|f|}f \triangleright \hat{\Delta}(g \triangleright h) + (-1)^{|g|}g \triangleright \hat{\Delta}(f \triangleright h)
\]

\[- \hat{\Delta}(f) \triangleright g \triangleright h1) = (-1)^{|f|+|g|}f \triangleright g \triangleright h1 \triangleright \hat{\Delta}(g \triangleright h)
\]

for arbitrary homogeneous elements \( f, g \) and \( h \in \widehat{HH}_{(1)}(A) \). Finally, by the definition of the operator \( \hat{\Delta} \), we get \( \hat{\Delta}^2 = 0 \) and \( \hat{\Delta}_0(1) = 0 \).

\[\square\]

Remark 4.13. Recall that the Nakayama automorphism \( \nu \) of \( A \) is semisimple if the map \( \nu \otimes \text{id}_k : A \otimes \overline{k} \to A \otimes \overline{k} \) is diagonalizable over the algebraic closure \( \overline{k} \) of \( k \). The results of Lambre-Zhou-Zimmermann [14, Section 4] and an easy calculation imply that the complete cohomology ring of a Frobenius algebra is a BV algebra when the Nakayama automorphism is semisimple.

5. Examples

Throughout this section, we assume that \( k \) is an algebraically closed field whose characteristic \( \text{char} \ k \) is \( p \). Lambre-Zhou-Zimmermann [14] showed that there are many examples of Frobenius algebras with diagonal Nakayama automorphisms. This section is devoted to computing the graded commutative ring structure and the BV structure of the complete cohomology for three certain self-injective Nakayama algebras whose Nakayama automorphisms are diagonalizable. Lambre-Zhou-Zimmermann [14] also gave an useful and combinatorial criterion to check that the Nakayama automorphism is diagonalizable: let \( A = kQ/I \) be the algebra given by a quiver with relations. Let \( Q_0 \) be the set of vertices in \( Q \). It is well-known that we can choose a \( k \)-basis \( B \) of \( A \) such that \( B \) contains a \( k \)-basis of the socle of the right regular \( A \)-module \( A \). Suppose that \( A \) is a Frobenius algebra. It follows from [12, Proposition 2.8] that we can construct an associative and non-degenerate bilinear form \( \langle , \rangle : A \otimes A \to k \) by defining \( \langle a, b \rangle := \text{tr}(ab) \) for \( a, b \in A \), where \( \text{tr} : A \to k \) is given by

\[
\text{tr}(p) = \begin{cases} 
1 & \text{if } p \in B \cap \text{soc } A_A, \\
0 & \text{otherwise}.
\end{cases}
\]
Suppose that $B$ satisfies two additional conditions:

(i) For any two paths $p, q \in B$, there exist a path $r \in B$ and a constant $\lambda \in k$ such that $p \cdot q = \lambda r$ in $A$;

(ii) For every path $p \in B$, there uniquely exists a path $p' \in B$ such that $0 \neq p \cdot p' \in \text{soc}A_A$.

**Criterion 5.1** ([14, Criterion 5.1]). Under the situation above, assume that $k$ is an algebraically closed field of characteristic zero or of characteristic $p$ larger than the number of arrows of $Q$. Then the Nakayama automorphism of $A$ associated with the bilinear form $\langle \ , \ \rangle : A \otimes A \to k$ given above is diagonalizable over $k$.

Suppose that $A = kQ/I$ is a self-injective Nakayama algebra. It is known that the ordinary quiver $Q$ of $A$ is a cyclic quiver with $|Q_0| = s$, and an admissible ideal $I$ of $kQ$ is of the form $R^N_Q$, where $R_Q$ is the arrow ideal of $kQ$ and $N \geq 2$. Obviously, we can take a $k$-basis $B$ of $A$ consisting of paths contains a $k$-basis of $\text{soc}A_A$. Since any indecomposable projective $A$-module is uniserial, $B$ satisfies the two condition (i) and (ii). Hence, we can rewrite Criterion 5.1 as follows:

**Criterion 5.2.** Let $A = kQ/R^N_Q$ be a self-injective Nakayama algebra. If the characteristic of $k$ is zero or $p$ larger than the number of arrows of $Q$, then the Nakayama automorphism of $A$ is diagonalizable over $k$.

**Remark 5.3.** If $A = kQ/R^N_Q$ is a self-injective Nakayama algebra, then the exponent $N$ does not affect Criterion 5.2, and only the number of arrows of $Q$ is important.

We will compute BV algebras of Nakayama algebras $A = kQ/R^N_Q$ with $|Q_0| = s$ for three cases.

### 5.1. The case $s = 2, N = 2$.

Let $Q$ be a quiver

$$
1 \xrightarrow{\alpha_1} 2.
$$

Consider the algebra $A := kQ/R^2_Q$. Thus, $A$ is a self-injective Nakayama algebra and, moreover, a truncated algebra. It follows from Criterion 5.2 that the Nakayama automorphism of $A$ is diagonalizable if and only if char $k \neq 2$. Thus, we suppose that char $k \neq 2$. Note that we need the assumption on char $k$ only if we construct BV differential. However, we assume that char $k \neq 2$ in advance. We denote by $e_i$ the primitive idempotent of $A$ corresponding to a vertex $i$ of $Q$ such that $e_i \alpha_i e_{i+1} = \alpha_i$ holds, where we regard the subscripts $i$ of $e_i$ and $\alpha_i$ modulo 2. Take a $k$-basis $B = (u_1, u_2, u_3, u_4) = (e_1, e_2, \alpha_1, \alpha_2)$ of $A$. Clearly, it contains a $k$-basis $\{\alpha_1, \alpha_2\}$ of $\text{soc}A_A$. We hence get an associative and non-degenerate bilinear form $\langle \ , \ \rangle : A \otimes A \to k$ and the dual basis $B^* = (v_1, v_2, v_3, v_4) = (\alpha_2, \alpha_1, e_1, e_2)$ of $A$ such that $\langle v_i, u_j \rangle = \delta_{ij}$, where $\delta_{ij}$ denotes Kronecker’s delta. Under the basis $B$, the representation matrix of the Nakayama automorphism $\nu$ of $A$ is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$
and is similar to a diagonal matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Moreover, we have a decomposition \( A = A_1 \oplus A_{-1} \) of \( A \) by two \( k \)-vector spaces
\[
A_1 = \text{Ker} (\nu - \text{id}) = k 1_A \oplus k (\alpha_1 + \alpha_2),
\]
\[
A_{-1} = \text{Ker} (\nu + \text{id}) = k (e_1 - e_2) \oplus k (\alpha_1 - \alpha_2).
\]

Let us recall that a set \( \{ Ae_i \otimes e_j A \mid i, j \in Q_0 \} \) is a complete set of pairwise non-isomorphic finitely generated indecomposable projective \( A \)-bimodules, and we denote by \( P(i, j) \) the indecomposable projective \( A \)-bimodule \( Ae_i \otimes e_j A \). It follows from [3] that a minimal projective resolution \( \mathcal{P}_\bullet \) of \( A \) as an \( A \)-bimodule is an exact sequence
\[
\cdots \rightarrow P_{2r+1} \xrightarrow{\phi_{2r+1}} P_{2r} \xrightarrow{\phi_{2r}} P_{2r-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} A \rightarrow 0,
\]
where
\[
\begin{aligned}
P_n := \begin{cases}
P(1, 2) \oplus P(2, 1) & \text{if } n \text{ is odd}, \\
P(1, 1) \oplus P(2, 2) & \text{if } n \text{ is even}
\end{cases}
\end{aligned}
\]
and \( A \)-bimodule homomorphisms \( \phi_s : P_s \rightarrow P_{s-1} \) are defined as follows:
\[
\begin{aligned}
\phi_0 (e_i \otimes e_i) &= e_i; \\
\phi_{2r} (e_i \otimes e_i) &= e_i \otimes \alpha_{i+1} + \alpha_i \otimes e_i; \\
\phi_{2r+1} (e_i \otimes e_{i+1}) &= \alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i.
\end{aligned}
\]
For a finite dimensional \( k \)-vector space \( V \) and a \( k \)-basis \( B \) of \( V \), given a basis vector \( p \in B \), we denote by \( p^* \) the \( k \)-linear map \( V \rightarrow k \) sending \( q \in B \) to 1 if \( q = p \) and to 0 otherwise. Applying the exact functor \( D = \text{Hom}(-, k) \) to \( \mathcal{P}_\bullet \) and twisting each term of \( D(\mathcal{P}_\bullet) \) by the automorphism \( \nu^{-1} \) on the right hand side, we get an exact sequence \( D(\mathcal{P}_\bullet)_{\nu^{-1}} \) as follows:
\[
0 \rightarrow D(A)_{\nu^{-1}} \xrightarrow{D(\phi_0)} \cdots \rightarrow D(P_{2r-1})_{\nu^{-1}} \xrightarrow{D(\phi_{2r})} D(P_{2r})_{\nu^{-1}} \xrightarrow{D(\phi_{2r+1})} D(P_{2r+1})_{\nu^{-1}} \rightarrow \cdots,
\]
where
\[
D(P_n)_{\nu^{-1}} = \begin{cases}
A(\alpha_2 \otimes \alpha_2)^* A \oplus A(\alpha_1 \otimes \alpha_1)^* A & \text{if } n \text{ is odd}, \\
A(\alpha_2 \otimes \alpha_1)^* A \oplus A(\alpha_1 \otimes \alpha_2)^* A & \text{if } n \text{ is even}
\end{cases}
\]
and \( A \)-bimodule homomorphisms \( D(\phi_s) : D(P_s)_{\nu^{-1}} \rightarrow D(P_s)_{\nu^{-1}} \) are defined as follows:
\[
\begin{aligned}
D(\phi_0) ((-1, 1_A)) &= \alpha_1 (\alpha_2 \otimes \alpha_1)^* + (\alpha_2 \otimes \alpha_1)^* \alpha_1 + \alpha_2 (\alpha_1 \otimes \alpha_2)^* + (\alpha_1 \otimes \alpha_2)^* \alpha_2; \\
D(\phi_{2r}) ((\alpha_i \otimes \alpha_i)^*) &= \alpha_{i+1} (\alpha_i \otimes \alpha_{i+1})^* + (\alpha_{i+1} \otimes \alpha_i)^* \alpha_i; \\
D(\phi_{2r+1}) ((\alpha_i \otimes \alpha_{i+1})^*) &= (\alpha_i \otimes \alpha_i)^* \alpha_{i+1} - \alpha_{i+1} (\alpha_i \otimes \alpha_{i+1})^*.
\end{aligned}
\]
Therefore, we obtain an exact sequence $X_\bullet$

\[
\cdots \rightarrow P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\mu} D(P_0)_{\nu=1} \xrightarrow{D(\phi_1)} D(P_1)_{\nu=1} \xrightarrow{D(\phi_2)} D(P_2)_{\nu=1} \rightarrow \cdots
\]

of which the composition $\mu$ is defined by

\[
\mu(e_i \otimes e_i) = \alpha_i \otimes e_i + \alpha_{i+1} \otimes e_{i+1}
\]

and whose term $P_n$ is of degree $n \geq 0$. Observe that there are $A$-bimodule isomorphisms

\[
D(P(i, j))_{\nu=1} = D(\nu^{-1} A e_i \otimes e_j A) \cong \text{Hom}(e_j A, D(\nu^{-1} A e_i))
\]

\[
\cong D(e_j A) \otimes D(A e_i)_{\nu=1} \cong A e_{j+1} \otimes e_{i+1} A_{\nu=1}
\]

\[
\cong A e_{j+1} \otimes e_i A = P(j + 1, i),
\]

where the fourth isomorphism is induced by the $A$-bimodule isomorphism $A_{\nu} \cong D(A)$ and the fact that $\nu = \nu^{-1}$. Since $A^e$ is injective as an $A$-bimodule, the contravariant functor $\text{Hom}_{A^e}(-, A^e)$ is exact, so that the exact sequence $X_\bullet$ is a complete resolution of $A$. Before applying the functor $\text{Hom}_{A^e}(-, A)$ to $X_\bullet$, we notice that there are isomorphisms

\[
\text{Hom}_{A^e}(D(P(i, j))_{\nu=1}, A) \cong \text{Hom}_{A^e}(D(P(i, j)), D(A)) \cong D(A \otimes_{A^e} D(P(i, j)))
\]

\[
\cong \text{Hom}_{A^e}(A, P(i, j)) \cong \text{Hom}_{A^e}(A, A^e) \otimes_{A^e} P(i, j)
\]

\[
\cong A_{\nu=1} \otimes_{A^e} P(i, j)
\]

for any $i, j \in Q_0$. Using these isomorphisms, we have the following commutative diagram with exact rows:

\[
\begin{array}{c}
\text{Hom}_{A^e}(D(P_2)_{\nu=1}, A) \xrightarrow{\text{Hom}(D(\phi_2), A)} \text{Hom}_{A^e}(D(P_{2r})_{\nu=1}, A) \xrightarrow{\text{Hom}(D(\phi_{2r}), A)} \text{Hom}_{A^e}(D(P_{2r-1})_{\nu=1}, A)\\
A_{\nu=1} \otimes_{A^e} P_{2r+1} \xrightarrow{id \otimes_{A^e} \phi_{2r+1}} A_{\nu=1} \otimes_{A^e} P_{2r} \xrightarrow{id \otimes_{A^e} \phi_{2r}} A_{\nu=1} \otimes_{A^e} P_{2r-1},
\end{array}
\]

where the $A$-bimodules $A_{\nu=1} \otimes_{A^e} P_\ast$ are given by

\[
A_{\nu=1} \otimes_{A^e} P_{2r} = k(e_2 \otimes_{A^e} e_1 \otimes e_2) \oplus k(e_1 \otimes_{A^e} e_2 \otimes e_1);
\]

\[
A_{\nu=1} \otimes_{A^e} P_{2r+1} = k(\alpha_2 \otimes_{A^e} e_1 \otimes e_1) \oplus k(\alpha_1 \otimes_{A^e} e_2 \otimes e_2),
\]

and the $k$-linear maps $id \otimes_{A^e} \phi_\ast$ are given by

\[
id \otimes_{A^e} \phi_{2r}(\alpha_i \otimes_{A^e} e_i \otimes e_i) = 0;
\]

\[
id \otimes_{A^e} \phi_{2r+1}(e_i \otimes_{A^e} e_{i+1} \otimes e_i) = \alpha_i \otimes_{A^e} e_i \otimes e_i - \alpha_{i+1} \otimes_{A^e} e_{i+1} \otimes e_{i+1}.
\]

Hence, the complex $\text{Hom}_{A^e}(X_\bullet, A)$ can be identified with a complex

\[
\cdots \rightarrow A_{\nu=1} \otimes_{A^e} P_1 \xrightarrow{id \otimes \phi} A_{\nu=1} \otimes_{A^e} P_0 \xrightarrow{\text{Hom}(\mu, A)} \text{Hom}_{A^e}(P_0, A) \xrightarrow{\text{Hom}(\phi_1, A)} \text{Hom}_{A^e}(P_1, A) \rightarrow \cdots
\]
of which the remaining terms and differentials are given by

\[ \text{Hom}_{A^e}(P_n, A) \cong \begin{cases} e_1Ae_2 \oplus e_2Ae_1 = k \alpha_1 \oplus k \alpha_2 & \text{if } n \text{ is odd}, \\ e_1Ae_1 \oplus e_2Ae_2 = k e_1 \oplus k e_2 & \text{if } n \text{ is even}; \end{cases} \]

\[ \text{Hom}_{A^e}(\phi_{2r+1}, A)(e_i) = \alpha_{i+1} - \alpha_i; \]

\[ \text{Hom}_{A^e}(\mu, A)(\alpha_i \otimes_A e_i \otimes e_i) = 0 \]

and whose term \( \text{Hom}_{A^e}(P_n, A) \) is of degree \( n \geq 0 \).

Therefore, the complete cohomology groups \( \widehat{\text{HH}}(A) \) are given as follows: for \( n \geq 0 \)

\[
\widehat{\text{HH}}^n(A) \begin{cases} k \alpha_1 & \text{if } n \text{ is odd,} \\ k \alpha_1 & \text{if } n \text{ is even;} \end{cases}
\]

\[
\widehat{\text{HH}}^{-n}(A) \begin{cases} k \alpha_1 \otimes_A e_1 \otimes e_1 & \text{if } n \text{ is odd,} \\ k \alpha_1 \otimes_A e_2 \otimes e_1 + e_2 \otimes A \otimes_A e_1 \otimes e_2 & \text{if } n > 0 \text{ is even.} \end{cases}
\]

Observe that we have \( \widehat{\text{HH}}^0(A) = \text{HH}^0(A) \) and \( \widehat{\text{HH}}^{-1}(A) = H_0(A, A_{\nu-1}) \).

From now on, we fix a \( k \)-basis

\[ (u_1, u_2, u_3, u_4) = (1, \alpha_1 + \alpha_2, e_1 - e_2, \alpha_1 - \alpha_2) \]

of \( A \) consisting of eigenvectors associated with the eigenvalues of the diagonalizable Nakayama automorphism \( \nu \) of \( A \). Then we have its dual basis

\[ (v_1, v_2, v_3, v_4) = ((1/2)(\alpha_1 + \alpha_2), 1/2, 1/2(\alpha_1 - \alpha_2), (1/2)(e_1 - e_2)) \]

of \( A \). Following [1], we will construct comparison morphisms between the minimal projective resolution \( P_* \) and the normalized bar resolution \( \text{Bar}_*(A) \) of \( A \) (cf. [18] for monomial algebras in general). Let \( F_0 \) be the canonical inclusion \( P_0 \hookrightarrow A \otimes A \), and for each \( n > 0 \), we define \( F_n : P_n \rightarrow A \otimes A \otimes A^\otimes n \otimes A \) in the following way: if \( n = 2r \), then let

\[ F_{2r}(e_i \otimes e_i) = 1 \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes 1, \]

where \( \overline{\alpha}_i \) and \( \overline{\alpha}_{i+1} \) appear each other. If \( n = 2r + 1 \), then let

\[ F_{2r+1}(e_i \otimes e_{i+1}) = 1 \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \overline{\alpha}_i \otimes 1. \]

On the other hand, let \( G_0 \) be the canonical projection \( A \otimes A \rightarrow P_0 \), and for each \( n > 0 \), \( G_n : A \otimes A \otimes A \rightarrow P_n \) is given as follows: if \( n = 2r \), then let

\[ G_{2r}(w) = \begin{cases} e_i \otimes e_i & \text{if } w = 1 \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes 1, \\ 0 & \text{otherwise}. \end{cases} \]

If \( n = 2r + 1 \), then let

\[ G_{2r+1}(w) = \begin{cases} e_i \otimes e_{i+1} & \text{if } w = 1 \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \overline{\alpha}_i \otimes 1, \\ 0 & \text{otherwise.} \end{cases} \]
One can easily check that $F$ and $G$ are comparison morphisms. Using these comparison morphisms and the definition of the $*$-product $*$, we have the following result.

**Proposition 5.4.** For every $i \in \mathbb{Z}$, the $n$-th complete cohomology group $\widehat{\text{HH}}^n(A)$ of $A$ is of dimension one, and the complete cohomology ring $(\widehat{\text{HH}}^*(A), *)$ is isomorphic to

$$k[\alpha, \beta, \gamma]/(\alpha \gamma - 1, \beta^2)$$

with $|\alpha| = 2, |\beta| = 1$ and $|\gamma| = -2$, where $\alpha$, $\beta$ and $\gamma$ correspond to $\overline{1}_A \in \overline{\text{HH}}^2(A)$ in (5.1), $\overline{\alpha_1} \in \overline{\text{HH}}(A)$ in (5.1) and $e_1 \otimes e_2 \in \overline{\text{HH}}^{-2}(A)$ in (5.2), respectively.

**Remark 5.5.** As we have seen before, the complete cohomology groups $\overline{\text{HH}}^n(A)$ with $n \geq 0$ of $A$ coincide with the Hochschild cohomology groups $\text{HH}^n(A)$ of $A$. Hence, the Hochschild cohomology ring $(\text{HH}^*(A), \sim)$ of $A$ is a subring of the complete cohomology ring $(\widehat{\text{HH}}^*(A), *)$.

**Remark 5.6.** We have another description of the complete cohomology ring above as follows:

$$k[\alpha, \beta, \alpha^{-1}]/(\beta^2)$$

where $|\alpha| = 2, |\beta| = 1$ and $|\alpha^{-1}| = -2$. Therefore, we will write $\alpha^{-1}$ for $\gamma$.

Following our main result, we now construct a BV operator $\widehat{\Delta}_i : \widehat{\text{HH}}^i(A) \to \widehat{\text{HH}}^{i-1}(A)$ for all $i \in \mathbb{Z}$. It follows from Proposition 5.4 that

$$\widehat{\text{HH}}^{2l}(A) = k\alpha^l \quad \text{and} \quad \widehat{\text{HH}}^{2l+1}(A) = k\beta\alpha^l$$

for all $l \in \mathbb{Z}$. Note that the number of the generators contained in the basis element of $\widehat{\text{HH}}^i(A)$ is at least 3 except for $-4 \leq i \leq 4$. Thus, one can use the operators $\widehat{\Delta}_i : \widehat{\text{HH}}^i(A) \to \widehat{\text{HH}}^{i-1}(A)$ for $-4 \leq i \leq 4$ and the formulas in Definition 4.1 to obtain the remaining operators $\widehat{\Delta}_* : \widehat{\text{HH}}^*(A) \to \widehat{\text{HH}}^{*}(A)$. From this point of view, it suffices to construct $\widehat{\Delta}_i$ only for $i = -4, -2, -1, 1, 2, 3, 4$. We will show a way of constructing $\widehat{\Delta}_1$ and $\widehat{\Delta}_{-1}$. The others can be constructed in a similar way. Let us recall that every complete cohomology group has a decomposition associated with the product of eigenvalues and in particular, except for the cohomology associated with the product of eigenvalues equal to $1_A$, the other vanish. Moreover, the BV operator defined on the chain level can be lifted to the cohomology level when we restrict it to the subcomplex associated with the product of eigenvalues equal to $1_A$.

We first compute $\widehat{\Delta}_1 : \overline{\text{HH}}^1(A) \to \overline{\text{HH}}^0(A)$. Consider a diagram

$$\begin{array}{ccc}
\text{Hom}_{A^e}(P_1, A) & \xrightarrow{\widehat{\Delta}_1} & \text{Hom}_{A^e}(P_0, A) \\
\text{Hom}_{A^e}(G_1, A) \downarrow & & \downarrow \text{Hom}_{A^e}(F_0, A) \\
\text{Hom}(A, A) \otimes A & \cong & \cong \\
D(A_A \otimes A) & \xrightarrow{D(R^1_B)} & D(A_A).
\end{array}$$
We know $\hat{HH}^1(A) = k\pi_1$ and hence deal with only $\alpha_1$. Put
\[ f_{\alpha_1} := \text{Hom}_{A^e}(G_1, A)(\alpha_1), \quad f_{u_2} := \text{Hom}_{A^e}(G_1, A)(u_2), \quad f_{u_4} := \text{Hom}_{A^e}(G_1, A)(u_4). \]
Namely, each of $f_{\alpha_1}, f_{u_2}$ and $f_{u_4}$ sends $\overline{x} \in A$ with $x \in B$ to
\[ f_{\alpha_1}(\overline{x}) = \begin{cases} \alpha_1 & \text{if } \overline{x} = \overline{\alpha_1}, \\ 0 & \text{otherwise}, \end{cases} \quad f_{u_2}(\overline{x}) = \begin{cases} \alpha_1 & \text{if } \overline{x} = \overline{\alpha_1}, \\ \alpha_2 & \text{if } \overline{x} = \overline{\alpha_2}, \\ 0 & \text{otherwise}, \end{cases} \quad f_{u_4}(\overline{x}) = \begin{cases} \alpha_1 & \text{if } \overline{x} = \overline{\alpha_1}, \\ -\alpha_2 & \text{if } \overline{x} = \overline{\alpha_2}, \\ 0 & \text{otherwise}. \end{cases} \]
Then we have $f_{\alpha_1} = (1/2)f_{u_2} + (1/2)f_{u_4}$, $f_{u_2} \in C_1^{(1)}(A, A)$ and $f_{u_4} \in C_1^{(-1)}(A, A)$. Since it is sufficient to only consider the image of $(1/2)f_{u_2}$, a direct computation shows that
\[ \hat{\Delta}_1(\beta) = \hat{\Delta}_1(\alpha_1) = (1/2)\pi_A = 1/2 \]
in $\hat{HH}^0(A)$. On the other hand, consider a diagram
\[
\begin{array}{ccc}
A_{\nu-1} \otimes_A P_0 & \xrightarrow{\hat{\Delta}_{-1}} & A_{\nu-1} \otimes_A P_1 \\
\downarrow \text{id} \otimes_A F_0 & & \downarrow \text{id} \otimes_A G_1 \\
A_{\nu-1} & \xrightarrow{-B_{\nu-1}^0} & A_{\nu-1} \otimes \overline{A}. 
\end{array}
\]
We know that $\hat{HH}^{-1}(A) = k\alpha_1 \otimes A^e e_2 \otimes e_1$ holds and hence handle $\alpha_1 \otimes A^e e_2 \otimes e_1$. The element $(\text{id} \otimes_A F_0)(\alpha_1 \otimes_A e_2 \otimes e_1) = \alpha_1$ can be decomposed as $\alpha_1 = (1/2)u_2 + (1/2)u_4$ in $A_{\nu-1}$, where $u_2 \in C_0^{(1)}(A, A_{\nu-1})$ and $u_4 \in C_0^{(-1)}(A, A_{\nu-1})$. Thus, a direct calculation gives us the formula
\[ \hat{\Delta}_{-1}(\alpha_1 \otimes_A e_2 \otimes e_1) = (1/2) e_1 \otimes_A e_2 \otimes e_1 + e_2 \otimes_A e_1 \otimes e_2 \]
in $\hat{HH}^{-2}(A)$. Thus, we have $\hat{\Delta}_{-1}(\alpha^{-1}\beta) = (-1/2)\alpha^{-1}$. Combining the formulas in Definition 4.1, we have the following result.

**Proposition 5.7.** The nonzero BV differentials $\hat{\Delta}_*$ on $\hat{HH}^\bullet(A)$ are
\[ \hat{\Delta}_{2n+1}(\alpha^n\beta) = ((2n+1)/2)\alpha^n \]
with $n \in \mathbb{Z}$. In particular, the nonzero Gerstenhaber brackets are induced by
\[ \{\alpha, \beta\} = \alpha, \quad \{\beta, \alpha^{-1}\} = \alpha^{-1}. \]

**Remark 5.8.** Since the non-negative part $\hat{HH}^{\geq 0}(A)$ of the complete cohomology $\hat{HH}^\bullet(A)$ is the Hochschild cohomology of $A$, the non-negative BV differential $\hat{\Delta}_{\geq 0}$ gives rise to a BV differential on the Hochschild cohomology ring of $A$, which means that there is a non-trivial example for our main theorem and for the theorem of Lambre-Zhou-Zimmermann [14, Theorem 4.1].

5.2. **The case $s = 3, N = 2$.** Let $Q$ be a quiver
\[
\begin{array}{c} 1 \\
\downarrow \alpha_1 \\strut \\
\downarrow \alpha_3 \\
\downarrow \alpha_2 \\
3 \\
\end{array}
\]
\[ 1 \quad 2 \]

\[ \xrightarrow{\alpha_3} \]

\[ \xrightarrow{\alpha_1} \]

\[ \xrightarrow{\alpha_2} 2 \]

\[ \xrightarrow{\alpha_1} \]

\[ 3 \]

\[ \xrightarrow{\alpha_3} \]

\[ \xrightarrow{\alpha_2} 2 \]

\[ \xrightarrow{\alpha_1} \]

\[ 3 \]
and $A$ the algebra $kQ/R_Q$. It follows from Criterion 5.2 and the fact that a primitive root of a polynomial $x^3 - 1$ is not equal to $1 \in k$ when $\text{char } k = 2$ that the Nakayama automorphism of $A$ is diagonalizable if and only if $\text{char } k \neq 3$. Hence we assume that $\text{char } k \neq 3$. We see that $A$ is a self-injective Nakayama algebra of which the representation matrix of the Nakayama automorphism $\nu$ is

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

under a $k$-basis $(e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3)$ of $A$. This matrix is similar to a diagonal matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 & \omega^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega^2
\end{pmatrix}
$$

where the element $\omega \in k$ is one of roots of a polynomial $x^2 + x + 1$. Moreover, we can decompose $A = A_1 \oplus A_\omega \oplus A_{\omega^2}$, where

$$
A_1 = \text{Ker } (\nu - \text{id}) = k1_A \oplus k\left( \sum_{i=1}^{3} \alpha_i \right),
$$

$$
A_\omega = \text{Ker } (\nu - \omega \text{id}) = k(\omega^2 e_1 + \omega e_2 + e_3) \oplus k(\omega^2 \alpha_1 + \omega \alpha_2 + \alpha_3),
$$

$$
A_{\omega^2} = \text{Ker } (\nu - \omega^2 \text{id}) = k(\sum_{i=1}^{3} \omega^i e_i) \oplus k(\sum_{i=1}^{3} \omega^i \alpha_i).
$$

Let $l \geq 0$ be an integer. In a similar way to the first example, we have a complete resolution of $A$ as follows:

$$
\cdots \xrightarrow{\phi_2} P_2 \xrightarrow{\phi_1} P_1 \xrightarrow{\phi_0} P_0 \xrightarrow{\mu} D(P_0)_{\nu^{-1}} \xrightarrow{D(\phi_1)} D(P_1)_{\nu^{-1}} \xrightarrow{D(\phi_2)} D(P_2)_{\nu^{-1}} \xrightarrow{\cdots} \xrightarrow{\phi_0} A \xrightarrow{\cong} D(A)_{\nu^{-1}} \xrightarrow{D(\phi_0)} \cdots
$$
where each of the $P_n$ and the $D(P_n)_{\mu^{-1}}$ is given by

$$P_n = \begin{cases} \bigoplus_{i=1}^3 P(i, i) & \text{if } n = 3l, \\ \bigoplus_{i=1}^3 P(i, i + 1) & \text{if } n = 3l + 1, \\ \bigoplus_{i=1}^3 P(i, i + 2) & \text{if } n = 3l + 2, \end{cases}$$

$$D(P_n)_{\mu^{-1}} = \begin{cases} \bigoplus_{i=1}^3 A(\alpha_i \otimes \alpha_{i+1})^*A & \text{if } n = 3l, \\ \bigoplus_{i=1}^3 A(\alpha_i \otimes \alpha_{i+2})^*A & \text{if } n = 3l + 1, \\ \bigoplus_{i=1}^3 A(\alpha_i \otimes \alpha_i)^*A & \text{if } n = 3l + 2, \end{cases}$$

each $A$-bimodule homomorphism $\phi_\nu : P_\nu \to P_{\nu-1}$ given by

$$\phi_{\nu+1}(e_i \otimes e_{i+1}) = \alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i; \quad \phi_{\nu+2}(e_i \otimes e_{i+2}) = e_i \otimes \alpha_{i+1} + \alpha_i \otimes e_{i+2};$$

$$\phi_{\nu+3}(e_i \otimes e_i) = \alpha_i \otimes e_i - e_i \otimes \alpha_i; \quad \phi_{\nu+4}(e_i \otimes e_{i+1}) = e_i \otimes \alpha_i + \alpha_i \otimes e_{i+1};$$

$$\phi_{\nu+5}(e_i \otimes e_{i+2}) = \alpha_i \otimes e_{i+2} - e_i \otimes \alpha_{i+1}; \quad \phi_{\nu+6}(e_i \otimes e_i) = e_i \otimes \alpha_i + \alpha_i \otimes e_i,$$

each $A$-bimodule homomorphism $D(\phi_\nu) : (P_{\nu-1})_{\mu^{-1}} \to (P_{\nu})_{\mu^{-1}}$ given by

$$D(\phi_{\nu+1})(\alpha_i \otimes \alpha_{i+1}) = (\alpha_i + \alpha_{i+1})^*\alpha_i - \alpha_i^*; \quad D(\phi_{\nu+2})(\alpha_i \otimes \alpha_{i+2}) = (\alpha_i + \alpha_{i+2})^*\alpha_i - \alpha_{i+1}^*;$$

$$D(\phi_{\nu+3})(\alpha_i \otimes \alpha_i) = (\alpha_i + \alpha_i)^*\alpha_i - \alpha_{i+1}^*; \quad D(\phi_{\nu+4})(\alpha_i \otimes \alpha_{i+1}) = (\alpha_i + \alpha_{i+1})^*\alpha_i - \alpha_i^*;$$

$$D(\phi_{\nu+5})(\alpha_i \otimes \alpha_{i+2}) = (\alpha_i + \alpha_{i+2})^*\alpha_i - \alpha_{i+1}^*; \quad D(\phi_{\nu+6})(\alpha_i \otimes \alpha_i) = (\alpha_i + \alpha_i)^*\alpha_i - \alpha_{i+1}^*;$$

the $A$-bimodule homomorphism $\phi_0 : P_0 \to A$ given by the multiplication of $A$, the $A$-bimodule homomorphism $D(\phi_0) : (A)_{\mu^{-1}} \to (P_0)_{\mu^{-1}}$ given by

$$D(\phi_0)\langle-1, 1\rangle = \sum_{i=1}^3 \alpha_i + \alpha_{i+1}^*\alpha_i + (\alpha_i + \alpha_{i+1}^*)\alpha_i,$$

and the composition $\mu : P_0 \to (P_0)_{\mu^{-1}}$ given by

$$\mu(e_i \otimes e_i) = \alpha_i^\nu \alpha_i^\nu + (\alpha_i + \alpha_i)\alpha_i - \alpha_{i+1}.$$

A complex which is used to compute complete cohomology groups $\widehat{HH}^\nu(A)$ is a complex

$$\cdots \to A_{\nu^{-1}} \otimes A^\nu P_1 \xrightarrow{id \otimes \phi_1} A_{\nu^{-1}} \otimes A^\nu P_0 \xrightarrow{\text{Hom}(\mu, A)} \text{Hom}_A(P_0, A) \xrightarrow{\text{Hom}(\phi_0, A)} \text{Hom}_A(P_1, A) \to \cdots$$

of which the terms and the nonzero differentials of the non-negative part are determined by

$$\text{Hom}_A(P_n, A) \cong \begin{cases} \bigoplus_{i=1}^3 e_i A e_i & \text{if } n = 3l, \\ \bigoplus_{i=1}^3 e_i A e_{i+1} & \text{if } n = 3l + 1, \\ 0 & \text{if } n = 3l + 2, \end{cases}$$

$$\text{Hom}(\phi_{\nu+1}, A)(e_i) = \alpha_i + \alpha_i; \quad \text{Hom}(\phi_{\nu+4}, A)(e_i) = \alpha_i - \alpha_i.$$
and that of the negative part are given by

\[ A^*_{n-1 \otimes A^*} P_n = \begin{cases} 
0 & \text{if } n = 3l, \\
\bigoplus_{i=1}^3 \alpha_{i+1} \otimes A^* e_i \otimes e_{i+1} & \text{if } n = 3l + 1, \\
\bigoplus_{i=1}^3 e_{i+2} \otimes A^* e_i \otimes e_{i+2} & \text{if } n = 3l + 2,
\end{cases} \]

\[
\text{id} \otimes \phi_{6l+2}(e_{i+2} \otimes A^* e_i \otimes e_{i+2}) = \alpha_{i+2} \otimes A^* e_{i+1} \otimes e_{i+2} + \alpha_{i+1} \otimes A^* e_i \otimes e_{i+1};
\]

\[
\text{id} \otimes \phi_{6l+5}(e_i \otimes A^* e_{i+1} \otimes e_i) = \alpha_{i+2} \otimes A^* e_{i+1} \otimes e_{i+2} - \alpha_{i+1} \otimes A^* e_i \otimes e_{i+1}.
\]

Here the term \( \text{Hom}_{A^*}(P_n, A) \) is of degree \( n \geq 0 \). Note that the two morphisms \( \text{Hom}(\phi_{6l+4}, A) \) and \( \text{id} \otimes \phi_{6l+2} \) are isomorphisms when \( \text{char } k \neq 2 \). We can see that the complete cohomology groups \( \widehat{\text{HH}}^*(A) \) of \( A \) are divided into two cases: for \( l \geq 0, \)

(1) \( \text{char } k \neq 2, 3 \)

\[
\widehat{\text{HH}}^n(A) = \begin{cases} 
k \frac{1}{A} & \text{if } n \equiv 0 \pmod{6}, \\
k \frac{1}{\alpha} & \text{if } n \equiv 1 \pmod{6}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\widehat{\text{HH}}^{-n}(A) = \begin{cases} 
k \alpha_2 \otimes A^* e_1 \otimes e_2 & \text{if } n \equiv 5 \pmod{6}, \\
k \sum_{i=1}^3 e_{i+2} \otimes A^* e_i \otimes e_{i+2} & \text{if } n \equiv 0 \pmod{6}, n \geq 1, \\
0 & \text{otherwise},
\end{cases}
\]

(2) \( \text{char } k = 2 \)

\[
\widehat{\text{HH}}^n(A) = \begin{cases} 
k \frac{1}{A} & \text{if } n = 3l, \\
k \frac{1}{\alpha} & \text{if } n = 3l + 1, \\
0 & \text{if } n = 3l + 2;
\end{cases}
\]

\[
\widehat{\text{HH}}^{-n}(A) = \begin{cases} 
0 & \text{if } n = 3l + 1, \\
k \alpha_2 \otimes A^* e_1 \otimes e_2 & \text{if } n = 3l + 2, \\
k \sum_{i=1}^3 e_{i+2} \otimes A^* e_i \otimes e_{i+2} & \text{if } n = 3l + 3.
\end{cases}
\]

As can be seen, the complete cohomology groups have the period six if \( \text{char } k \neq 2, 3 \) and the period three if \( \text{char } k = 2 \). We omit the constructions of two comparison morphisms between the minimal projective resolution and the normalized bar resolution of \( A \). However, they are constructed in a similar way to the first example. We have the graded commutative ring structure and the BV structure on the complete cohomology of \( A \).

**Proposition 5.9.** If \( \text{char } k \neq 2, 3 \), then the complete cohomology ring \( \widehat{\text{HH}}^*(A, \star) \) is isomorphic to

\[ k[\alpha, \beta, \alpha^{-1}] / \langle \beta^2 \rangle \]

where \( |\alpha| = 6, |\beta| = 1 \) and \( |\alpha^{-1}| = -6 \). Further, if this is the case, then the nonzero BV differentials \( \widehat{\Delta}_* \) on \( \widehat{\text{HH}}^*(A) \) are

\[
\widehat{\Delta}_{6l+1}(\alpha^l \beta) = ((6l + 1)/3) \alpha^l, \quad \widehat{\Delta}_{6l-5}(\alpha^{-l-1} \beta) = ((-6l - 5)/3) \alpha^{-l-1}
\]
with \( l \geq 0 \). In particular, the nonzero Gerstenhaber brackets are induced by
\[
\{\alpha, \beta\} = 2\alpha, \quad \{\beta, \alpha^{-1}\} = 2\alpha^{-1}.
\]

**Proposition 5.10.** If \( \text{char } k = 2 \), then the complete cohomology ring \( \widehat{\text{HH}}^*(A, \ast) \) is isomorphic to
\[
k[\alpha, \beta, \alpha^{-1}]/\langle \beta^2 \rangle
\]
where \(|\alpha| = 3, |\beta| = 1 \) and \(|\alpha^{-1}| = -3\). Further, if this is the case, then the nonzero BV differentials on \( \widehat{\text{HH}}^*(A) \) are
\[
\hat{\Delta}_{6l+1}(\alpha^{2l}\beta) = \alpha^{2l}, \quad \hat{\Delta}_{3l-2}(\alpha^{-l-1}\beta) = \alpha^{-l-1}
\]
with \( l \geq 0 \). In particular, the nonzero Gerstenhaber brackets are induced by
\[
\{\alpha, \beta\} = \alpha.
\]

5.3. **The case** \( s = 3, N = 3 \). Let \( Q \) be a quiver
\[
\begin{array}{ccc}
\alpha_3 & \rightarrow & \alpha_1 \\
\alpha_2 & \downarrow & \\
3 & \leftarrow & 2
\end{array}
\]
and \( A \) the algebra \( kQ/R_Q^3 \). It follows from Criterion 5.2 and Remark 5.3 that the Nakayama automorphism of \( A \) is diagonalizable if and only if \( \text{char } k \neq 3 \). Hence, we assume that \( \text{char } k \neq 3 \). We see that \( A \) is a self-injective Nakayama algebra of which the representation matrix of the Nakayama automorphism \( \nu \) is
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
under a \( k \)-basis \( (e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1) \) of \( A \). This matrix is similar to a diagonal matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2
\end{pmatrix}
\]
where the element $\omega \in k$ is one of roots of a polynomial $x^2 + x + 1$. Moreover, we can decompose $A = A_1 \oplus A_\omega \oplus A_{\omega^2}$, where

$$A_1 = \text{Ker} (\nu - \text{id}) = k 1_A \oplus k \left( \sum_{i=1}^{3} \alpha_i \right) \oplus k \left( \sum_{i=1}^{3} \alpha_i \alpha_{i+1} \right),$$

$$A_\omega = \text{Ker} (\nu - \omega \text{id}) = k \left( \sum_{i=1}^{3} \omega^i e_i \right) \oplus k \left( \sum_{i=1}^{3} \omega^i \alpha_i \right) \oplus k \left( \sum_{i=1}^{3} \omega^i \alpha_i \alpha_{i+1} \right),$$

$$A_{\omega^2} = \text{Ker} (\nu - \omega^2 \text{id})$$

$$= k (\omega^2 e_1 + \omega e_2 + e_3) \oplus k (\omega^2 \alpha_1 + \omega \alpha_2 + \alpha_3) \oplus k (\omega^2 \alpha_1 \alpha_2 + \omega \alpha_2 \alpha_3 + \alpha_3 \alpha_1).$$

In a similar way to the first example, we have a complete resolution of $A$ as follows:

$$\cdots \xrightarrow{\phi_2} P_2 \xrightarrow{\phi_1} P_1 \xrightarrow{\phi_0} P_0 \xrightarrow{\mu} D(P_0)_{\nu^{-1}} \xrightarrow{D(\phi_1)} D(P_1)_{\nu^{-1}} \xrightarrow{D(\phi_2)} D(P_2)_{\nu^{-1}} \xrightarrow{D(\phi_0)} A \xrightarrow{\cong} D(A)_{\nu^{-1}} \cdots$$

where each of the $P_n$ and the $D(P_n)_{\nu^{-1}}$ is given by

$$P_n = \begin{cases} \bigoplus_{i=1}^{3} P(i, i + 1) & \text{if } n \text{ is odd}, \\ \bigoplus_{i=1}^{3} P(i, i) & \text{if } n \text{ is even}; \end{cases}$$

$$D(P_n)_{\nu^{-1}} = \begin{cases} \bigoplus_{i=1}^{3} A(\alpha_i \alpha_{i+1} \otimes \alpha_i \alpha_{i+1})^* A & \text{if } n \text{ is odd}, \\ \bigoplus_{i=1}^{3} A(\alpha_i \alpha_{i+1} \otimes \alpha_i \alpha_{i+3})^* A & \text{if } n \text{ is even}, \end{cases}$$

each $A$-bimodule homomorphism $\phi_s : P_s \to P_{s-1}$ given by

$$\phi_{2r+1} (e_i \otimes e_{i+1}) = \alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i;$$

$$\phi_{2r} (e_i \otimes e_i) = e_i \otimes \alpha_{i+1} \alpha_{i+2} + \alpha_i \otimes \alpha_i + \alpha_i \alpha_{i+1} \otimes e_i,$$

each $A$-bimodule homomorphism $D(\phi_s) : D(P_{s-1})_{\nu^{-1}} \to D(P_s)_{\nu^{-1}}$ given by

$$D(\phi_{2r+1})((\alpha_i \alpha_{i+1} \otimes \alpha_i \alpha_{i+3})^*)$$

$$= (\alpha_{i+2} \alpha_i \otimes \alpha_{i+2} \alpha_i)^* \alpha_{i+1} - \alpha_{i+1} (\alpha_i \alpha_{i+1} \otimes \alpha_i \alpha_{i+1})^*;$$

$$D(\phi_{2r})((\alpha_i \alpha_{i+1} \otimes \alpha_i \alpha_{i+1})^*)$$

$$= \alpha_{i+2} \alpha_i + \alpha_{i+3} \alpha_{i+1} \alpha_i \alpha_{i+2} \alpha_i^* + \alpha_{i+2} (\alpha_{i-1} \alpha_i \otimes \alpha_{i+1} \alpha_{i+2})^* \alpha_{i-2} + (\alpha_{i-2} \alpha_{i-1} \otimes \alpha_{i} \alpha_{i+1})^* \alpha_{i-3} \alpha_{i-2},$$

each $A$-bimodule homomorphism $\phi_0 : P_0 \to A$ given by the multiplication of $A$, the $A$-bimodule homomorphism $D(\phi_0) : D(A)_{\nu^{-1}} \to D(P_0)_{\nu^{-1}}$ given by

$$D(\phi_0)(\langle -1 \rangle) = \sum_{i=1}^{3} (\alpha_i \alpha_{i+1} (\alpha_{i-2} \alpha_{i-1} \otimes \alpha_i \alpha_{i+1})^* + \alpha_{i+1} (\alpha_{i-2} \alpha_{i-1} \otimes \alpha_i \alpha_{i+1})^* \alpha_i + (\alpha_{i-2} \alpha_{i-1} \otimes \alpha_i \alpha_{i+1})^* \alpha_{i-3} \alpha_{i-2}),$$
and the composition \( \mu : P_0 \to D(P_0)_{\nu-1} \) given by
\[
\mu(e_i \otimes e_i) = \alpha_i \alpha_{i+1}(\alpha_{i-2} \alpha_{i-1} \otimes \alpha_i \alpha_{i+1})^* + (\alpha_{i-1} \alpha_i \otimes \alpha_{i+1} \alpha_{i+2})^* \alpha_{i-2} \alpha_{i-1} \\
+ \alpha_i(\alpha_{i-3} \alpha_{i-2} \otimes \alpha_{i-1} \alpha_i)^* \alpha_{i-4}.
\]
Moreover, a complex which is used to compute complete cohomology groups is a complex
\[
\cdots \to A_{\nu-1} \otimes_{A^e} P_1 \xrightarrow{id \otimes \phi_1} A_{\nu-1} \otimes_{A^e} P_0 \\
\xrightarrow{\text{Hom}(\mu, A)} \text{Hom}_{A^e}(P_0, A) \xrightarrow{\text{Hom}(\phi_1, A)} \text{Hom}_{A^e}(P_1, A) \to \cdots
\]
of which the terms and the nonzero differentials are determined by
\[
\text{Hom}_{A^e}(P_n, A) \cong \begin{cases} 
\bigoplus_{i=1}^{3} k \alpha_i & \text{if } n \text{ is odd,} \\
\bigoplus_{i=1}^{3} k e_i & \text{if } n \text{ is even;}
\end{cases}
\]
\[
A_{\nu-1} \otimes_{A^e} P_n = \begin{cases} 
\bigoplus_{i=1}^{3} k e_{i+1} \otimes_{A^e} e_i \otimes e_{i+1} & \text{if } n \text{ is odd,} \\
\bigoplus_{i=1}^{3} k \alpha_i \otimes_{A^e} e_i \otimes e_i & \text{if } n \text{ is even;}
\end{cases}
\]
\[
\text{Hom}(\phi_{2r+1}, A)(e_i) = \alpha_{i+1} - \alpha_i;
\]
\[
id \otimes \phi_{2r+1}(e_i \otimes_{A^e} e_{i+1} \otimes e_i) = \alpha_i \otimes_{A^e} e_i \otimes e_i - \alpha_{i+1} \otimes_{A^e} e_{i+1} \otimes e_i + 1
\]
and whose term \( \text{Hom}_{A^e}(P_n, A) \) is of degree \( n \geq 0 \). Therefore, we have, for \( n \geq 0 \),
\[
\widehat{\text{HH}}^n(A) = \begin{cases} 
k \alpha_1 & \text{if } n \text{ is odd,} \\
k \Gamma_A & \text{if } n \text{ is even;}
\end{cases}
\]
\[
\widehat{\text{HH}}^{-n}(A) = \begin{cases} 
k \alpha_1 \otimes_{A^e} e_1 \otimes e_1 & \text{if } n \text{ is odd,} \\
k \sum_{i=1}^{3} e_{i+1} \otimes_{A^e} e_i \otimes e_{i+1} & \text{if } n > 0 \text{ is even.}
\end{cases}
\]
We omit the description of comparison morphisms between the minimal projective resolution and the normalized bar resolution of \( A \), because it is not easy to write the two comparison morphisms. However, a direct calculation shows the graded commutative ring structure and the BV structure on the complete cohomology of \( A \).

**Proposition 5.11.** The complete cohomology ring \( (\widehat{\text{HH}}^*(A), *) \) is isomorphic to
\[
k[\alpha, \beta, \alpha^{-1}] / \langle \beta^2 \rangle
\]
where \( |\alpha| = 2, |\beta| = 1 \) and \( |\alpha^{-1}| = -2 \). Moreover, the nonzero BV differentials on \( \widehat{\text{HH}}^*(A) \) are
\[
\widehat{\Delta}_{2l+1}(\alpha^l \beta) = ((3l + 2)/3) \alpha^l, \quad \widehat{\Delta}_{2l-1}(\alpha^{-l-1} \beta) = \begin{cases} 
(-1/3) \alpha^{-1} & \text{if } l = 0, \\
((-3l - 2)/3) \alpha^{-l-1} & \text{if } l \neq 0
\end{cases}
\]
with \( l \geq 0 \). In particular, the nonzero Gerstenhaber brackets are induced by
\[
\{\alpha, \beta\} = \alpha, \quad \{\beta, \alpha^{-1}\} = \alpha^{-1}.
\]
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