Casimir Energy of a Spherical Shell in $\kappa$–Minkowski Spacetime

Hyeong-Chan Kim$^1$, Chaiho Rim$^2$,† and Jae Hyung Yee$^1$‡

$^1$ Department of Physics, Yonsei University, Seoul 120-749, Republic of Korea
$^2$ Department of Physics and Research Institute of Physics and Chemistry, Chonbuk National University, Jeonju 561-756, Korea.

We study the Casimir energy of a spherical shell of radius $a$ in $\kappa$-Minkowski spacetime for a complex field with an asymmetric ordering and obtain the energy up to $O(1/\kappa^2)$. We show that the vacuum breaks particle and anti-particle symmetry if one requires the spectra to be consistent with the blackbody radiation at the commutative limit.

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I. INTRODUCTION

Since Casimir first predicted that the quantum fluctuation of the electromagnetic field would produce an attractive force between two infinite parallel plates in vacuum [1], the Casimir energy has been found to depend on the geometry of the system: The Casimir force is repulsive for a spherical geometry [2]. The Casimir effect has attracted much attention experimentally and theoretically [3]. The effect has now been measured within about the one percent error range and at distances down to tens of nanometers for parallel plates as reported in Ref. 4. The idea has been applied to a wide range of phenomena, from explaining the amazing ability of a geko to walk across the ceilings [5], to a possible way of understanding the Hawking radiation [6], and to stabilizing the radion field for resolving the hierarchy problem in the brane-world scenarios [7].

On the other hand, at short distances of the Planck length scale, the spacetime itself may change its form due to the quantum gravity effect. Especially, $\kappa$-deformed Poincaré algebra (KPA) is introduced [8]. Here, the four momenta commute with each other, but the boost relation is deformed, where $\kappa$ has the role of the deformation parameter. In this dual picture, when $\kappa$ approaches infinity, the deformed Poincaré symmetry reduces to the commutative limit, the ordinary Poincaré symmetry. The deformed realization implies a deformed special relativity that results in a change of the group velocity of the photon. In this respect, doubly special relativity [9] is closely related with this KPA [10] and the deformation parameter $\kappa$ reflect the Planck-scale physics.

After the appearance of the KPA, it was soon realized that the dual picture of the KPA results in a non-commuting spacetime [11]. This non-commuting spacetime is called the $\kappa$-Minkowski spacetime (KMST), in which the rotational symmetry is preserved, but time...
and space coordinates do not commute each other:

\[ [\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa} \hat{x}^i, \quad [\hat{x}^i, \hat{x}^j] = 0, \quad i, j = 1, 2, 3. \]  

The differential structure of the KMST of 4 spacetime dimensions is not realized in 4-dimensional spacetime but needs to be constructed in 5-dimensional spacetime \([12, 13]\). If the corresponding derivative is realized in 5-dimensional momentum space \(P_A (A = 0, 1, 2, 3, 5)\), then the derivatives satisfy the 4-dimensional De Sitter space

\[ (P_0)^2 - \sum_{i=1}^{3} (P_i)^2 - (P_5)^2 = -\kappa^2. \]  

It is noted that \(P_5\) is invariant under the KPA; therefore, if one requires the physical system to preserve the \(\kappa\)-deformed Poincaré symmetry (KPS), then one can restrict oneself to 4-dimensional spacetime, including the derivatives.

Based on this differential structure, the scalar field theory has been formulated \([14, 15]\). The \(\kappa\)-deformation was extended to a curved space with a \(\kappa\)-Robertson-Walker metric and was applied to the cosmic microwave background radiation \([16]\). The effect of the \(\kappa\)-deformation on the blackbody radiation has been studied recently in Ref. 17.

Still, KMST is not understood well, and an interacting (field) theory, including gauge symmetry, needs more elaboration because many particle properties show a non-local nature (See Refs. 18 and 19 and references therein). For a systematic study, one needs to look into KMST and see if KMST field theory allows a reliable vacuum in which a particle picture can be constructed from the vacuum. In this sense, the Casimir energy can provide a useful check on the nature of the vacuum.

As noted above, KMST field theory is constructed on the dual space of KMST through KPS. To do this, one defines a field variable in momentum space as

\[ \phi(x) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \varphi(p). \]  

Here, all the coordinate variables and momenta are treated as commuting variables. Instead, the non-commuting nature of KMST is encoded in the \(*\)-product between field variables. The product of an exponential element is required to satisfy the composition rule \([20]\)

\[ e^{-ip\cdot x} * e^{-iq\cdot x} = e^{-iv(p,q)\cdot x}, \]  

where we will adopt in this paper the composition law corresponding to the asymmetric ordering

\[ v(p, q) = (p^0 + q^0, p e^{-q^0/\kappa} + q) \]  

In this approach, the spacetime variables \(x^\mu\) are treated as commuting with each other and the effect of the original spacetime non-commutativity is given in terms of the homomorphism of the field variables through the \(*\)-product. One can, thus, avoid various conceptual difficulties of spacetime geometry, which arises from the non-commutating nature of the spacetime.

The KPS in the dual picture is the guiding principle to construct the field theory and is applied to the free scalar action explicitly in Ref. 15. The free analogue of massive complex scalar theory is given as

\[ S = \int d^4x \phi^*(x) * [-\partial_\mu * \partial^\mu * -m^2] \phi(x). \]
\( \phi^c(x) \) is the conjugate of the scalar field

\[
\phi^c(x) \equiv \int \frac{d^4p}{(2\pi)^4} e^{3p^0/\kappa} e^{i\vec{p} \cdot \vec{x}} \varphi^\dagger(p) = \int \frac{d^4p}{(2\pi)^4} e^{-i\vec{p} \cdot \vec{x}} \varphi^\dagger(-\vec{p}) ,
\]

where \( \vec{p}^0 = p^0 \) and \( \vec{p} = e^{p^0/\kappa} \vec{p} \), and \( \varphi^\dagger(p) \) denotes the ordinary complex conjugate of \( \varphi(p) \) in momentum space. The measure factor \( e^{3p^0/\kappa} \) and \( \vec{p}^\mu \) are needed to satisfy the KPS.

In momentum space, the action in Eq. (6) is given as

\[
S = \int \frac{d^4p}{(2\pi)^4} e^{3p^0/\kappa} \varphi^\dagger(p) \Delta^{-1}_F(p) \varphi(p) .
\]

The integration measure is KPS invariant, and the “Feynman propagator” is given as

\[
\Delta^{-1}_F(p) = M^2_\kappa(p) \left( 1 + \frac{M^2_\kappa(p)}{4\kappa^2} \right) - m^2 + i\epsilon ,
\]

where \( M^2_\kappa(p) \) is the first Casimir invariant

\[
M^2_\kappa(p) = \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - \vec{p}^2 e^{p_0/\kappa}.
\]

and a small positive real number, \( \epsilon \), is added to avoid the singularity on the real axis of \( p^0 \).

Explicitly, the Feynman propagator is given as

\[
\Delta^{-1}_F(p) = \frac{\kappa^2}{4} e^{2p_0/\kappa}(e^{-p_0/\kappa} - \alpha_+)(e^{-p_0/\kappa} + \alpha_+)(e^{-p_0/\kappa} - \alpha_-)(e^{-p_0/\kappa} + \alpha_-) ,
\]

where

\[
\alpha_\pm = \sqrt{1 + \frac{m^2 - i\epsilon}{\kappa^2}} \pm \sqrt{\frac{m^2 + \vec{p}^2 - i\epsilon}{\kappa^2}}.
\]

The Feynman propagator has the periodic property

\[
\Delta^{-1}_F(p_0 + i\kappa \pi, \vec{p}) = \Delta^{-1}_F(p_0, \vec{p}) ,
\]

and, thus, possesses an infinite number of poles on the complex plane of \( p^0 \). Nevertheless, the real poles provide a stable particle and an anti-particle dispersion relation, and one can study the physical effects of the modified dispersion relation by simply ignoring the unstable modes because the unstable modes decay very quickly after the Planck time has passed. In this spirit, the blackbody spectra has been investigated in Ref. 17 for the massless scalar theory. The massless dispersion relation is given as

\[
\omega^{(\pm)}_p = -\kappa \ln(1 - |\vec{p}|/\kappa) , \quad \omega^{(-)}_p = \kappa \ln(1 + |\vec{p}|/\kappa) ,
\]

where \( \omega^+_p \) corresponds to the particle dispersion relation and \( \omega^-_p \) to the anti-particle’s. It is demonstrated that the thermal fluctuation of the particle is different from that of the anti-particle. The Stephan-Boltzmann law is modified at the order of \( O(1/\kappa) \). However, due to the different thermal behaviors of the particle and the antiparticle, the \( O(1/\kappa) \) effects cancel each other, and the Stephan-Boltzmann law is left with an \( O(1/\kappa^2) \) correction when both the particle and the anti-particle are present.
The effect of the $\kappa$-deformation for the case of two infinite parallel plates on the Casimir effect has been studied in Ref. 21 using the real pole in Eq. (9). The deformed effect was found to be the order $1/\kappa^2$. A similar effect was also shown at the order of $1/\kappa^2$ in Ref. 22 and 23 when a different dispersion relation was used. The different dispersion relation corresponds to a different realization of KPS even though KMST is the same. The two investigations demonstrate that a different realization of KPS may result in a different correction to the physical effect.

In this paper, we calculate the Casimir energy for a sphere of radius $a$ and study the particle and the antiparticle contributions to the vacuum energy. In the commutative spacetime, the Casimir energy is positive for a spherical boundary. It would be interesting how the $\kappa$-deformation alters this Casimir energy and how it affects the vacuum. In Sec. II we illustrate the computational procedure for the Casimir energy with a spherical boundary in the KMST, closely following that of Refs. [2, 24, 25].

In Sec. III and Sec. IV we compute the Casimir energy of the anti-particle mode, $\omega_p^(-)$, in Eq. (13). In Sec. III the Casimir energy is calculated without the measure factor in the momentum space, where the momentum variable is treated as a mere mode-counting parameter. In Sec. IV the Casimir energy is computed, including the measure factor. In Sec. V we summarize the Casimir energies given in Sec. III and Sec. IV and compare the results with the energy of the particle mode, $\omega_p^(+)$, in Eq. (13). We discuss the particle and the anti-particle symmetries of the vacuum and present the ordering effect on the symmetry of the vacuum. Some detailed calculations are given in the appendices: The calculation of the divergent part, $E_0$, is given in the App. A of the $O(1/\kappa)$ correction in App. B and of the $O(1/\kappa^2)$ correction in App. C.

II. CASIMIR ENERGY OF A SPHERICAL SHELL

In this section, we present an idea on how to calculate the Casimir energy of a massless scalar field in $\kappa$-Minkowski spacetime for a spherical shell of radius $a$. The Casimir energy is the zero point vacuum energy of massless scalar fields. The massless modes are are given in Eq. (13). We note that the particle mode $\omega_p^+$ is defined when $|p| < \kappa$ whereas the antiparticle mode $\omega_p^-$ is defined for all momentum. The Feynman propagator in Eq. (11) turns out to provide an additional pole on the real axis when $|p| > \kappa$,

$$p_0^{(3)} = -\kappa \ln(|p|/\kappa - 1), \quad (14)$$

in addition to the two modes

$$p_0^{(+)} = \omega_p^{(+)}, \quad p_0^{(-)} = -\omega_p^{(-)}. \quad (15)$$

Thus, all three real modes contribute to the Casimir energy

$$E_c = \frac{\hbar}{2} \sum_{i=+, -, 3} \sum_p \omega_p^{(i)} \quad (16)$$

where $\omega_p^{(3)}$ refers to the mode related with $p_0^{(3)}$ whose relation is not fixed yet. The difficulty lies in that the value of $p_0^{(3)}$ ranges from $+\infty$ to $-\infty$ in contrast with $p_0^{(+)}$ and
Thus, we will divide the Casimir energy into two parts, \( E_c = E_c(A) + E_c(P) \), where

\[
E_c^{(A)} = \frac{\hbar}{2} \sum_p \omega_p^{(-)}, \quad E_c^{(P)} = \frac{\hbar}{2} \sum_{i=+3} \sum_p \omega_p^{(i)}
\]  

so that the momentum ranges from \(-\infty\) to \(\infty\). In the momentum configuration, the vacuum energy will be represented as

\[
E_c^{(A)} = \frac{\hbar}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p^{(-)} e^{\alpha p_0^{(-)} / \kappa},
\]

\[
E_c^{(P)} = \frac{\hbar}{2} \left( \int_{|p|<\kappa} \frac{d^3p}{(2\pi)^3} \omega_p^{(+)} e^{\alpha p_0^{(+)} / \kappa} + \int_{|p|>\kappa} \frac{d^3p}{(2\pi)^3} \omega_p^{(3)} e^{\alpha p_0^{(3)} / \kappa} \right),
\]

where \(\alpha = 3\) from the \(\kappa\)-Poincaré invariant measure.

Since there is an ambiguity in \(E_c^{(P)}\), we will consider \(E_c^{(A)}\) first. One can find the momentum mode contribution in a spherical shell for \(p_0^{(-)} = -\hbar \omega_p^{(-)}\) by using the wave equation in coordinate space:

\[
-\nabla^2 \psi(\vec{x},t) = \lambda^2 \psi(\vec{x},t),
\]

where \(p^2 = \hbar^2 \lambda^2\). It is noted that in this dual (momentum space) picture, the KMST effect is entirely encoded in the dispersion relation, Eq. \[13\], through the *-product, the spacetime coordinates are treated as commuting variables, and, thus, the ordinary quantum mechanical tool can be employed without conceptual difficulty, which chiefly arises from the spacetime non-commutativity. Especially, one can separate the time and the space coordinates in the wave function \(\psi(x,t) = \phi(x) e^{-i\omega p t}\) and to arrive at the eigenvalue equation, by using spherical symmetry,

\[
\left\{ r^2 \frac{d^2}{dr^2} + 2r \frac{d}{dr} + \left[ \lambda^2 r^2 - l(l+1) \right] \right\} \phi_l(r) = 0.
\]

There is a \((2l + 1)\)-fold degeneracy in the eigenvalues \(\lambda\). Explicitly, the solution is given by the spherical Bessel functions:

\[
\phi(r) = \begin{cases} 
  j_l(\lambda r) & \text{for } r < a \\
  A_l j_l(\lambda r) + B_l n_l(\lambda r) & \text{for } r > a,
\end{cases}
\]

where the regularity is imposed at \(r = 0\) and \(A_l\) and \(B_l\) are constants to be determined by prescribing the correct asymptotic behavior at large \(r\).

At this stage, one can follow the usual trick to impose the boundary conditions \[2, 24, 25\]. At \(r = a\), one imposes the Dirichlet boundary condition

\[
 j_l(\lambda a) = 0, \quad A_l j_l(\lambda a) + B_l n_l(\lambda a) = 0.
\]

In addition, to find the asymptotic behavior at \(r \to \infty\), one may conveniently regularize the exterior modes by enclosing the entire system within another concentric sphere of radius \(R \gg a\). The boundary condition at large \(R\), \(A_l j_l(\lambda R) + B_l n_l(\lambda R) = 0\), gives the phase

\[
\tan \delta_l = \frac{B_l}{A_l} = \tan(\lambda R - \frac{l\pi}{2}).
\]
To accommodate the boundary condition at $r = a$ for the modes inside and outside, one may define an analytic function

$$\tilde{f}_i(z) = f^{(1)}_i(z)f^{(2)}_i(z),$$

(25)

with $f^{(1)}_i(z) = j_i(z)$ and $f^{(2)}_i(z) = j_i(z) + \tan \delta_i(z)n_i(z)$, where $z = \lambda r$. Then, the boundary condition in Eq. (23) is written as $\tilde{f}_i(z_n) = 0$, where $z_n = \lambda_n a$ and $\lambda_n$ is the quantized value of $\lambda$ due to the spherical boundary. One uses the Cauchy theorem to write the sum of analytic functions

$$\sum_i \zeta(x_i) = \frac{1}{2\pi i} \oint_C \zeta(z) \frac{\tilde{f}_i(z)}{f_i(z)} \, dz = \frac{1}{2\pi i} \oint_C dz \zeta(z) \frac{d}{dz} \log \tilde{f}_i(z),$$

(26)

where $x_i$'s are isolated zeros of $\tilde{f}_i(z)$ within a closed contour $C$. The Casimir energy given by the sum of the vacuum modes,

$$E^{(A)}_c = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \left( l + \frac{1}{2} \right) \omega(a; z_n) e^{-\alpha \omega(a; z_n)/\kappa},$$

(27)

is then written as

$$E^{pre}_c(a) = \frac{1}{2\pi i} \sum_{l=0}^{\infty} \left( l + \frac{1}{2} \right) \oint_C dz e^{-\alpha \omega(a; z)/\kappa} \omega(a; z) \frac{d}{dz} \ln f_i(z),$$

(28)

where $\omega(a; z) = \kappa \ln(1 + z/(\kappa a))$ and

$$f_i(z) = 2 \zeta \tilde{f}_i(z) = 2 \zeta^2 f^{(1)}_i(z)f^{(2)}_i(z).$$

(29)

Here, we introduced a pole at $z = 0$ without changing the value of the integral in Eq. (28), noting that $\omega(a; 0) = 0$. This freedom allows one to replace $\tilde{f}_i(z)$ by $f_i(z)$.

The expression of the Casimir energy in Eq. (28) needs a few comments. In general, the expression does not converge when summing over the modes. It is noted that the high-frequency mode grows rapidly as the momentum becomes large. Thus, in general, one has to regularize the expression first and find a mean to find the finite contribution. To regularize, we conveniently introduce an infinitesimal positive parameter $\sigma$ and write the energy as

$$E^{\sigma}_c(a) = \frac{1}{2\pi i} \sum_{l=0}^{\infty} \left( l + \frac{1}{2} \right) \oint_C dz e^{-\sigma z - \omega(a; z)/\kappa} \omega(a; z) \frac{d}{dz} \ln f_i(z).$$

(30)

The factor $e^{-\sigma z}$ plays the role of a cutoff and suppresses the high-frequency contributions to the Casimir energy. In our case, however, the presence of the measure factor provides a natural cut-off effect already. Nonetheless, we will carry the $\sigma$ for future convenience when the case of $\alpha = 0$ is considered for comparison.

Next, the amount of vacuum energy without a spherical boundary is subtracted from this expression to obtain the net vacuum energy due to a sphere of radius $a$. To do this, one calculates the energy for a large sphere of radius $\eta R$ ($\eta$ is a finite number on the order of 1 so that $a < \eta R < R$) and subtracts the result from the expression in Eq. (30):

$$E^{(A)}_c(a) = \lim_{R \to \infty, \sigma \to 0} \left( E^{\sigma}_c(a) - E^{\sigma}_c(\eta R) \right).$$

(31)
To compute the Casimir energy in Eq. (30), we take the contour $C$ for the integration, as shown in Fig. 1, which can be conveniently broken into three parts: a circular segment $C_{\Lambda}$, and two straight line segments $\Gamma_1$ and $\Gamma_2$. Since the $\Gamma$ contours are oriented at a nonzero angle $\phi$ with respect to the imaginary axis, it follows that the contribution to $C_{\Lambda}$ (especially when $\alpha = 0$) is bounded by $\exp(-\sigma \Lambda \sin \phi)$, where $\Lambda$ is the radius of the circular arc. Since the logarithm in the Casimir energy grows at most algebraically, it follows that the contribution to $C_{\Lambda}$ vanishes exponentially in the limit of large $\Lambda$, provided that $\phi \neq 0$.

Along $\Gamma_1$, setting the coordinate $z = iye^{-i\phi}$ with $y$ real leads to

$$\tan \delta = \tan(iye^{-i\phi} R/a - \frac{l\pi}{2}) \rightarrow i$$

for sufficiently large $R \gg a$ and

$$f_{l}^{(1)}(z) = \sqrt{\frac{\pi}{2z}} J_{\nu}(z) = e^{i\pi\nu/2} \sqrt{\frac{\pi}{2z}} I_{\nu}(ye^{-i\phi}),$$

$$f_{l}^{(2)}(z) = \sqrt{\frac{\pi}{2z}} H_{\nu}^{(1)}(z) = -e^{-i\pi\nu/2} (\frac{2i}{\pi}) \sqrt{\frac{\pi}{2z}} K_{\nu}(ye^{-i\phi}),$$

with $\nu = l + 1/2$. Thus, on $\Gamma_1$, one has $f_l(z) = \lambda_{\nu}(ye^{-i\phi})$, where

$$\lambda_{\nu}(y) = 2y I_{\nu}(y) K_{\nu}(y).$$

The contribution from $\Gamma_2$ is the complex conjugate of the $\Gamma_1$ contribution. This gives the Casimir energy

$$E_{c}^{(A)}(a) = \lim_{R \rightarrow \infty, \sigma \rightarrow 0, \phi \rightarrow 0} \sum_{l} \left(E_{l}^{reg}(a) - E_{l}^{reg}(\eta R)\right),$$

where

$$E_{l}^{reg}(r) = \frac{\kappa \nu R}{\pi} \int_{0}^{\infty} dy e^{-\sigma ye^{-i\phi}} ig(r, iye^{-i\phi}) \frac{d}{dy} \log \lambda_{\nu}(ye^{-i\phi}),$$
with
\[ g(r, z) = \frac{\log \left( 1 + \frac{z}{\kappa r} \right)}{(1 + \frac{z}{\kappa r})^\alpha}. \tag{38} \]

To sum up the angular momentum modes, one may conveniently use the large-\(\nu\) behavior of the Bessel function. After shifting \(y \to \nu y\) in the integration,
\[ E_{\text{reg}}(r) = \frac{\kappa \nu}{\pi} \Re \int_0^\infty dy i e^{-i\nu ye^{-i\phi}} g(r, i\nu ye^{-i\phi}) \frac{d}{dy} \log \lambda_{\nu}(\nu ye^{-i\phi}), \tag{39} \]
one uses the large-order series expansion of the Bessel function \[26\] for \(\nu \gg 1:\]
\[ \log \lambda_{\nu}(\nu y) = \sum_{n=0}^\infty q_n(y) \nu^{2n}. \tag{40} \]
\(q_n(y)\) is a function of \(O(y^{-2n})\) for large \(y\), whose explicit forms for \(n = 0, 1, 2\) are given in Eq. (A3). This manipulation results in the Casimir energy
\[ E_c(a) = \sum_{n=0}^\infty \left( \mathcal{E}_n(\sigma, a) - \mathcal{E}_n(\sigma, \eta R) \right), \tag{41} \]
where
\[ \mathcal{E}_n(\sigma, r) = \sum_l \frac{\kappa}{\pi \nu^{2n-1}} \Re \int_0^\infty dy i \left( e^{-i\sigma ye^{-i\phi}} g(r, i\nu ye^{-i\phi}) \right) \frac{d}{dy} q_n(ye^{-i\phi}). \tag{42} \]
with the limits \(R \to \infty, \sigma \to 0,\) and \(\phi \to 0\) being taken at the end.

This decomposition of the Casimir energy in Eq. (41) is useful in taking care of the divergent structure in the \(1/\kappa\) expansion. First, one can be convinced that \(E_0(a)\) vanishes because the integration gives only a pure imaginary contribution, as shown in Appendix [A]. (A similar conclusion can be made using the zeta function regularization as in Ref. 24.) The rest of the terms with \(n \geq 1\) are finite even when the limits \(\sigma \to 0\) and \(\phi \to 0\) are taken before the summation over \(l\) and integration over \(y\). Thus, the finite Casimir energy is simplified as
\[ E_c(a) = \sum_{n=1}^\infty \lim_{R \to \infty} \left( \mathcal{E}_n(a) - \mathcal{E}_n(\eta R) \right), \tag{43} \]
where \(\mathcal{E}_n(r)\) is summed up with angular momentum contributions
\[ \mathcal{E}_n(r) = \frac{1}{r} \sum_l \frac{B_n(\nu, r)}{\nu^{2n-2}}; \quad B_n(\nu, r) = \frac{1}{\pi} \int_0^\infty dy q_n(y) G(\frac{\nu y}{\kappa r}). \tag{44} \]
Here, integration by parts is used, and \(G(x)\) is an even function of \(x:\)
\[ G(x) = \begin{cases} \frac{1+x^2}{(1-2x^2+3x^4)\log(1+x^2)-12(1-x^2)x}\tan^{-1}x & \text{for } \alpha = 0 \\ \frac{1-x^2}{(1-6x^2+x^4)(1+3x^2)} & \text{for } \alpha = 3. \tag{45} \end{cases} \]
As \(\kappa \to \infty, G(\frac{\nu y}{\kappa a}) \to 1.\) In this commutative limit, one may have \(B_n(\nu, r) \to B_n^{(0)}(\nu, r),\)
\[ \mathcal{E}_n(r) \to \mathcal{E}_n^{(0)}(r) = \frac{1}{r} \sum_l \frac{B_n^{(0)}(\nu, r)}{\nu^{2n-2}}. \tag{46} \]
and $E_c(a) \rightarrow E_c^{(0)}(a)$, whose expression is exactly the same as the one given in Ref 27,

$$E_c^{(0)}(a) = \frac{0.002819}{a}. \quad (47)$$

Then, the higher-order terms in $1/\kappa$ are given as

$$\Delta \mathcal{E}_n(r) = \mathcal{E}_n(r) - \mathcal{E}_n^{(0)}(r) = \frac{1}{r} \sum_l \frac{\Delta B_l(\nu,r)}{l^{2n-2}}, \quad (48)$$

where

$$\Delta B_l(\nu,r) \equiv B_l(\nu,r) - B_l^{(0)}(\nu,r) = \frac{1}{\pi} \int_0^\infty dy q_n(y) \left[ G\left(\frac{\nu y}{\kappa r}\right) - 1 \right]. \quad (49)$$

The correction terms are considered in the next two sections.

It is obvious from Eqs. (44) and (45) that $E_c(a)$ is independent of the sign of $\kappa$. Thus, one may expect the particle and the anti-particle to give the same contributions to the Casimir energy. However, there arises a subtle point due to the presence of the branch cut in the particle dispersion relation $\omega^{+}_{p}$. This will be carefully investigated in the last section.

### III. CASIMIR ENERGY FOR THE $\alpha = 0$ CASE

Let us consider the $\alpha = 0$ case in this section. This case neglects the KPS invariant measure in the integration, but is simpler than the non-zero $\alpha$ case and provides an informative structure in the systematic calculation in the $1/\kappa$ series expansion.

The higher-order contribution of $B_n(\nu)$ is given in Eq. (49), whose explicit expression is given as

$$\Delta B_l(\nu,r) = -\frac{\nu^2}{\pi (\nu^2)^2} \int_0^\infty dy q_n(y) \frac{y^2}{1 + \nu^2 y^2}. \quad (50)$$

The details of the calculations of $\Delta \mathcal{E}_1(r)$ and $\Delta \mathcal{E}_2(r)$ are given in Appendix E. A large sphere of radius $\eta R$ only gives a non-trivial contribution to $\Delta \mathcal{E}_1(\eta R)$; $\Delta \mathcal{E}_{n \geq 2}(\eta R)$ vanishes as $R \rightarrow \infty$. The finite correction terms $\Delta \mathcal{E}_1(r)$ and $\Delta \mathcal{E}_2(r)$ are of the order of $O(1/|\kappa|)$ and are given as

$$\Delta \mathcal{E}_1(a) = \frac{1}{a} \sum_{l=0}^\infty \left( \Delta B_l(\nu,a) - \Delta B_l(\nu,\eta R) \right) = -\frac{1}{a} \left( \frac{1}{384 |\kappa| a} + O\left(\frac{1}{(\kappa a)^3}\right) \right),$$

$$\Delta \mathcal{E}_2(a) = \frac{1}{\nu^2} \sum_{l=0}^\infty \frac{\Delta B_2(\nu,a)}{\nu^2} = \frac{1}{a} \left( \frac{1}{256 |\kappa| a} + O\left(\frac{1}{(\kappa a)^3}\right) \right). \quad (51)$$

Here, we put the absolute value notation to $\kappa$, even though $\kappa$ is positive, to emphasize that $\mathcal{E}_n(a)$ is independent of the sign of $\kappa$.

The dominant contribution of $\Delta \mathcal{E}_{n \geq 3}(a)$ is considered in Appendix F

$$\Delta \mathcal{E}_n(a) = \frac{1}{\pi a} \sum_l \frac{1}{l \nu^{2n-2}} \int_0^\infty dy q_n(y) \left( G\left(\frac{\nu y}{\kappa a}\right) - G(0) \right). \quad (52)$$
and its summation is expressed as
\[
\sum_{n \geq 3} \Delta \mathcal{E}_n(a) = E_c^{(2)}(a) + E_c^{(3)}(a), \tag{53}
\]
\[
E_c^{(2)}(a) = -\frac{1}{\pi a} \sum_{n \geq 3} \sum_l \frac{1}{\nu^{2n-2}} \int_0^\infty dy q_n(y) \left( \frac{\nu y}{\kappa a} \right)^2,
\]
\[
E_c^{(3)}(a) = \frac{1}{\pi a} \sum_{n \geq 3} \sum_l \frac{1}{\nu^{2n-2}} \int_0^\infty dy q_n(y) \frac{\left( \frac{\nu y}{\kappa a} \right)^4}{1 + \left( \frac{\nu y}{\kappa a} \right)^2}.
\]

Noting \( q_n(y) = O(y^{-2n}) \), one can confirm that \( E_c^{(2)}(a) \) and \( E_c^{(3)}(a) \) are of the orders of \( O(1/\kappa^2) \) and \( O(1/\kappa^3) \), respectively. \( E_c^{(2)}(a) \) is calculated with the help of numerics. The first 10 angular momentum modes are obtained numerically and are shown to converge to the asymptotic expression for large \( l \). This allows one to find the numerical value accurately, whose value is given in Eq. \( \text{(C8)} \):
\[
E_c^{(2)} = \frac{1}{a} \left( -0.000545 \right). \tag{54}
\]

Combining all the terms, we have
\[
E_c^{(4)}(a) = \frac{1}{a} \left( 0.002819 + \frac{1}{768(|\kappa|a)} - 0.000545 \frac{(\kappa a)^2}{(\kappa a)^2} + O \left( \frac{1}{\kappa} \right)^3 \right). \tag{55}
\]

IV. CASIMIR ENERGY FOR THE \( \alpha = 3 \) CASE

We now take into account the measure effect. To find \( \Delta \mathcal{E}_n(r) \), one needs to take care of the non-rational function \( G(x) \) given in Eq. \( \text{(15)} \),
\[
G(x) = \frac{(1 - 6x^2 + x^4)(1 - 3 \log(1 + x^2)) - 12(1 - x^2)x \tan^{-1} x}{(1 + x^2)^4}.
\]

In this case, \( G(x) \) does not allow easy summation over \( l \). To estimate \( \Delta \mathcal{E}_n(r) \), we note that \( \log(1 + x^2) \) and \( x \tan^{-1} x \) satisfy \( 0 \leq \log(1 + x^2) \leq x \tan^{-1} x \leq x^2 \), with the equalities holding only at \( x = 0 \). Thus, one can estimate the range of \( G(x) \):
\[
G_{\min}(x) \leq G(x) \leq G_{\max}(x), \tag{56}
\]
\[
G_{\min}(x) = \frac{2 - 39x^2 + 2x^4 - 3x^6}{2(1 + x^2)^4},
\]
\[
G_{\max}(x) = \frac{1 - 6x^2 + 22x^4}{(1 + x^2)^4}.
\]

If we get the contribution to the order of \( O(1/\kappa) \), we only have to consider \( \Delta \mathcal{E}_1(r) \) and \( \Delta \mathcal{E}_2(r) \) because \( \Delta \mathcal{E}_{n \geq 3}(r) \) contributes to the \( O(1/\kappa^2) \), as seen in the previous section. Suppose we use \( G_{\max} \) to evaluate \( O(1/\kappa) \):
\[
\mathcal{E}_1(r) + \mathcal{E}_2(r) = \frac{1}{r} \sum_{l=0}^{\infty} \frac{1}{\pi} \int_0^\infty dy \max G_{\max}(by) \left[ q_1(y) + \nu^{-2} q_2(y) \right]
\]
\[
= -\frac{1}{r} \left( \frac{63\kappa r}{1024} - \frac{35\pi^2}{65536} - \frac{43}{192288 \kappa r} + O \left( \frac{1}{\kappa r} \right)^3 \right).
\]
FIG. 2: $G$ function (black curve). The dashed curves denote $G_{\text{max}}$ and $G_{\text{min}}$, respectively. The gray curves denote a systematic approximation of $G(x)$ by using Eqs. (60) and (61).

Subtracting $\mathcal{E}_1(\eta R)$ from $\mathcal{E}_1(a)$, we find that the $O(\kappa)$ contribution goes away and that a finite contribution is obtained as $R \to \infty$. The $\kappa$-independent term is already contained in Eq. (47), and the $1/\kappa$ contribution is obtained as

$$ E_{c\ B1}^{(1)} = \sum_{i=1}^{2} \left( \Delta \mathcal{E}_i(a) - \Delta \mathcal{E}_i(\eta R) \right) = \frac{1}{a} \frac{43}{192288\kappa a} \approx \frac{1}{a} \frac{0.00022}{\kappa a}, \quad (57) $$

which will give a lower bound to $E_{c\ B1}^{(1)}(a)$.

Suppose we use $G_{\text{min}}$ to evaluate $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$. We then have

$$ \mathcal{E}_1(r) + \mathcal{E}_2(r) = \frac{1}{r} \sum_{l=0}^{\infty} \frac{1}{\pi} \int_{0}^{\infty} dy G_{\text{min}}(by) \left[ q_1(y) + \nu^{-2} q_2(y) \right] $$

$$ = \frac{1}{r} \left( \frac{63\kappa r}{1024} + \frac{35\pi^2}{65536} + \frac{133}{12288\kappa r} + O \left( \frac{1}{\kappa r} \right)^3 \right). $$

This gives an upper bound to the $1/\kappa$ contribution:

$$ E_{c\ B2}^{(1)} = \sum_{i=1}^{2} \left( \Delta \mathcal{E}_i(a) - \Delta \mathcal{E}_i(\eta R) \right) = \frac{1}{a} \left( \frac{133}{12288\kappa a} \right) \approx \frac{1}{a} \left( \frac{0.01082}{\kappa a} \right). \quad (58) $$

Comparing the results in Eqs. (57) and (58), we have lower and upper bounds on the $1/\kappa$ contribution to $E_{c\ B1}^{(1)}(a)$, respectively,

$$ E_{c\ B1}^{(1)} < E_{c\ B1}^{(1)}(a) < E_{c\ B2}^{(1)} \quad (59) $$

One may find a good approximate value of $E_{c\ B1}^{(1)}(a)$ if one finds a good approximation of
$G(x)$ in a quotient form. To do this, one may approximate $\log(1 + x^2)$ as

$$f_1(x) = x^2,$$
$$f_2(x) = \frac{1 + x^2/2}{1 + x^2} \ f_1(x),$$
$$f_3(x) = \frac{1 + 2x^2 + 5x^4/6}{(1 + x^2)^2} \ f_2(x),$$
$$f_4(x) = \frac{1 + 3x^2 + 3x^4 + 5x^6/6}{(1 + x^2)^3} \ f_3(x),$$
$$f_5(x) = \frac{1 + 4x^2 + 6x^4 + 4x^6 + 13x^8/15}{(1 + x^2)^4} \ f_4(x),$$

and $x \tan^{-1} x$ as

$$h_1(x) = x^2,$$
$$h_2(x) = \frac{1 + 2x^2/3}{1 + x^2} \ h_1(x),$$
$$h_3(x) = \frac{1 + 2x^2 + 13x^4/15}{(1 + x^2)^2} \ h_2(x),$$
$$h_3(x) = \frac{1 + 3x^2 + 3x^4 + 277x^6/315}{(1 + x^2)^3} \ h_3(x),$$
$$h_5(x) = \frac{1 + 4x^2 + 6x^4 + 4x^6 + 859x^8/945}{(1 + x^2)^4} \ h_4(x),$$

and so on. The approximate functions $f_n(x)$ and $h_n(x)$ agree with $\log(1 + x^2)$ and $x \tan^{-1} x$, respectively, up to $O(x^{2n})$ for small $x$. In addition, one can show that the bounded values of $\frac{f_n(x) - \log(1 + x^2)}{(1 + x^2)^4}$ and $\frac{h_n(x) - x \tan^{-1} x}{(1 + x^2)^3}$ improve as $n$ increases for the whole integration range of $x$. For example, $\frac{f_n(x) - \log(1 + x^2)}{(1 + x^2)^4}$ is bounded by 0.0036024 when $n = 2$, and as one uses higher $n$, the bounded value decreases by around $(1/2)^n$. The same thing holds for $h_n(x)$.

Using the approximate functions $f_n(x)$ and $h_n(x)$, one can integrate and sum over $l$ to get

$$E_{cn}^{(1)} = \frac{1}{a} \left( \frac{D_n}{\kappa \tau} \right),$$

where $D_1 = 0.009114$, $D_2 = 0.009094$, $D_3 = 0.009100$, $D_4 = 0.009106$, and $D_5 = 0.009109$. From these approximated results, we have

$$E_c^{(1)} = \frac{1}{a} \left( \frac{D}{\kappa \tau} \right),$$

with $D \cong 0.00911$, which value is close to the upper bound $E_{cB2}^{(1)}$ in Eq. (58).

We may find the $O(1/\kappa^2)$ contribution by summing over $E_{n \geq 3}(r)$. This contribution is easily read from Eq. (C8) by using the coefficient $G_1 = -47/2$:

$$E_c^{(2)} = \frac{0.001713}{\pi a} \frac{G_1}{(\kappa a)^2} = \frac{1}{a} \left( -\frac{0.01281}{(\kappa a)^2} \right).$$

(63)
From this consideration, we conclude that the Casimir energy is given by

$$E_{c,\alpha=3}(a) = \frac{1}{a} \left( 0.002819 + \frac{0.00911}{|\kappa|a} - \frac{0.01281}{(\kappa a)^2} + O(\frac{1}{|\kappa| a^3}) \right). \quad (64)$$

It is noted that the sign of the first-order term allows the Casimir force to be more repulsive than that of the commutative result. The first-order term is stronger than it is in the case where the measure factor is neglected.

V. SUMMARY AND DISCUSSION

We have evaluated the Casimir energy in $\kappa$-Minkowski spacetime when the massless scalar anti-particle mode satisfies the Dirichlet boundary condition at a spherical boundary of radius $a$. The boundary condition is incorporated using the Cauchy integration. The scalar theory is used in the $*$-product formalism and is required to satisfy the $\kappa$-deformed Poincaré symmetry in momentum space to avoid the conceptual difficulty due to the non-commutative nature of the time and the space coordinates.

The Casimir energy is regulated by the introduction of a cut-off function (for the case when the integration measure is neglected), and a geometry independent term is subtracted to find the spherical geometric effect. The Casimir energy which respects the $\kappa$-deformed Poincaré invariance is given as

$$E_{c,\alpha=3}(a) = \frac{1}{a} \left( 0.002819 + \frac{0.00911}{|\kappa|a} - \frac{0.01281}{(\kappa a)^2} + O(\frac{1}{|\kappa| a^3}) \right).$$

On the other hand, if one regards the momentum in the integration of Eq. (18) as a mere mode-counting parameter and neglects the integration measure (i.e., $\alpha = 0$ case), the Casimir energy is given by

$$E_{c,\alpha=0}(a) = \frac{1}{a} \left( 0.002819 + \frac{1}{768(|\kappa|a)} - \frac{0.00545}{(\kappa a)^2} + O\left(\frac{1}{|\kappa| a^3}\right) \right).$$

This shows that the $\kappa$-deformed Poincaré invariant measure affects an physical values such as the Casimir energy. In addition, the $\kappa$-deformed spacetime seems to give an additional positive contribution at long distances and to provide an attractive contribution at short distances around $\kappa a \approx O(1)$.

The Casimir energy is an even function of $\kappa$ and is independent of the sign of $\kappa$. On the other hand, $\kappa \rightarrow -\kappa$ changes the energy, $\omega_p(-) \rightarrow \omega_p(\cdot)$, in Eq. (13). This seems to suggest there is a particle and an anti-particle symmetry in the vacuum. However, if one tries to compute the Casimir energy by using the positive mode $\omega_p(\cdot)$, one encounters a branch-cut at $\lambda = \kappa$.

The presence of the branch-cut suggests that one needs to include another mode $\tilde{\rho}_0 = -\kappa \ln(z/(\kappa a) - 1)$, which appears as a new real pole in Eq. (14) in the Feynman propagator (11). Suppose one consider two contour integrals, I and II. I consists of 4 components in Fig. 3, $\Gamma$ along the imaginary axis at $z = 0$, $C$ along the branch-cut at $z = \kappa a - \epsilon + iy$, and the rest at $z = \pm i\infty$ between $z = 0$ and $z = \kappa a$. II consists of 2 components, $D$ along the branch-cut at $z = \kappa a + \epsilon + iy$ and $E$ along the large half circle.
Contour integration I is defined as

\[ I = \oint dz e^{-\sigma z + \frac{\alpha_0(a; z)}{\kappa} \ln f_i(z)} \left( \frac{d}{dz} \ln f_i(z) \right), \quad (65) \]

where \( p_0(a; z) = -\kappa \ln (1 - z/\kappa a) \) and \( \sigma \) is introduced to regularize the integral. This integration is written as

\[ I_{\Gamma} + I_C = \sum_{z_n < \kappa a} p_0(a; z_n) e^{-\sigma z_n + \alpha_0(a; z_n)/\kappa}, \quad (66) \]

where \( I_{\Gamma} \) and \( I_C \) denote the integrations along segments \( \Gamma \) and \( C \), respectively. Due to the regularization, the integration along \( z = \pm i\infty \) vanishes.

Contour integration II is defined as

\[ II = \oint dz e^{-\sigma z + \frac{\alpha_0(a; z)}{\kappa} \ln f_i(z)} \left( \frac{d}{dz} \ln f_i(z) \right), \quad (67) \]

where \( \tilde{p}_0(a; z) = -\kappa \ln (z/\kappa a - 1) \). This integration gives the relation

\[ II = II_D = \sum_{z_n > \kappa a} \tilde{p}_0(a; z_n) e^{-\sigma z_n + \alpha_0(a; z_n)/\kappa} \quad (68) \]

because \( II_E \) vanishes.

On the other hand, \( I_C \) and \( II_D \) are written as

\[ I_C = -\kappa \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{-\sigma(k+iy) \ln \left( \frac{-iy}{\kappa a} \right)} \left( \frac{d}{dz} \ln f_i(z) \right) \bigg|_{z=\kappa+iy}, \quad (69) \]

\[ II_D = +\kappa \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{-\sigma(k+iy) \ln \left( \frac{i\bar{y}}{\kappa a} \right)} \left( \frac{d}{dz} \ln f_i(z) \right) \bigg|_{z=\kappa+iy}. \quad (70) \]
One may have the following relation between the integrations:

$$
\Pi_D = -(-1)^{\alpha} I_C + B, \quad (71)
$$

$$
B = \mp \kappa \pi (i)^{\alpha+1} \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{-\sigma(k+iy)} \left. \frac{1}{(\frac{y}{\kappa a})^\alpha} \frac{d}{dz} \ln f_l(z) \right|_{z=\kappa+iy}, \quad (72)
$$

where $B$ appears due to the branch-cut and its sign $\mp$ depends on the branch-cut position, which may lie either on the upper half plane or on the lower half plane.

When $\alpha = 0$, the branch-cut contribution, $B$, can be understood if one considers an integration from a discrete mode $z_n$:

$$
J = \int_{-\infty}^{\infty} dy \left. \frac{e^{-\sigma z}}{z-z_n} \right|_{z=\kappa+iy} = e^{-\sigma \kappa} \int_0^{\infty} dy \Re \frac{e^{-i\sigma y}}{\kappa + iy - z_n}
$$

$$
= 2e^{-\sigma \kappa} \int_0^{\infty} dy \frac{(\kappa - z_n) \cos(\sigma y) - y \sin(\sigma y)}{(\kappa - z_n)^2 + y^2}. \quad (73)
$$

Since $\cos(\sigma y)$ and $\sin(\sigma y)$ are oscillating functions, one can put $\sigma \to 0$ before the integration. In this case, one can see that $J$ is real and is evaluated as $\pi$, which is independent of $z_n$. Thus, each mode’s contribution is independent of $a$ and goes away when the contribution of the radius $\eta R$ is subtracted. Thus, $B$ is imaginary, but does not contribute to the Casimir energy when $\alpha = 0$.

When $\alpha = 3$, one has an integration from the branch-cut contribution:

$$
K = \int_{-\infty}^{\infty} dy \left. \left( \frac{\kappa a}{y} \right)^3 \frac{e^{-\sigma z}}{z-z_n} \right|_{z=\kappa+iy}
$$

$$
= ie^{-\sigma \kappa} \int_0^{\infty} dy \left( \frac{\kappa a}{y} \right)^3 \frac{(\kappa - z_n) \sin(\sigma y) + y \cos(\sigma y)}{(\kappa - z_n)^2 + y^2}
$$

$$
\to i \left( \frac{\kappa a}{\kappa - z_n} \right)^3 \int_0^{\infty} dy \frac{1}{y^2 \left( 1 + \frac{y^2}{y^2} \right)}. \quad (74)
$$

In this case, $\sigma$ is not effective in regulating the theory, and $K$ not only diverges due to the singularity at $y = 0$ but also depends on each discrete mode $z_n$. This non-vanishing branch-cut contribution makes $B$ real. However, $B$ has a sign ambiguity, which is not physical because the real world should not depend on the branch-cut’s position. To make the branch-cut independent, one may average the branch-cut’s contribution to get rid of the branch-cut’s arbitrariness. As a result, $B$ vanishes.

Finally, we are left with the relations

$$
\Pi_D^{reg} = -(-1)^{\alpha} I_C^{reg} \quad (75)
$$

and

$$
I_\Gamma = \sum_{z_n < \kappa a} p_0(a; z_n) e^{\alpha p_0(a; z_n)/\kappa} + \sum_{z_n > \kappa a} (-1)^{\alpha} \tilde{p}_0(a; z_n) e^{\alpha \tilde{p}_0(a; z_n)/\kappa}. \quad (76)
$$

These expressions hint that one needs to identify $\omega_p^{(3)}$ as

$$
\omega_p^{(3)} = (-1)^{\alpha} P_0^{(3)} \quad (77)
$$
where $\alpha = 0$ or 3 and

\[
E^{(P)}_c = \frac{\hbar}{2} \left( \int_{|p|<\kappa} \frac{d^3p}{(2\pi)^3} \omega_p^{(+)}/\kappa + \int_{|p|>\kappa} \frac{d^3p}{(2\pi)^3} \omega_p^{(3)} \right)
\]

This definition gives $\kappa \to -\kappa$ symmetry. Noting that $\Gamma$ is the same as the one obtained from the anti-particle contribution in the contour integrations $\Gamma_1$ and $\Gamma_2$ in Fig. 1, we have particle and anti-particle symmetry of the vacuum $E^{(P)}_c = E^{(A)}_c$. This demonstrates that the high-momentum (HM) mode $\omega_p^{(3)}$, which exists only when the momentum is greater than $\kappa$, will make the particle and the anti-particle contribution to the Casimir energy equal. In other words, the vacuum respects the particle and the anti-particle symmetry.

However, this is in a serious contradiction with the result from the thermal response calculation. Even though the blackbody radiation [17] and the vacuum in an acceleration frame [13] distinguish the particle and the anti-particle responses at the order of $O(1/\kappa)$, one needs to avoid the HM mode because in the presence of the high-momentum mode, a particle with a small energy may have a low and a high momentum at the same time, and HM can spoil the commutative result because of the thermal density of the HM contribution which is the order of $O(\kappa^2 T^2)$ [17] when $\kappa \to \infty$ ($O(\kappa^2 T^2)$ if one uses the relation in Eq. (77)). Thus, from the physics at the $\kappa \to \infty$ limit, one naturally should avoid the HM mode. This fact can also be seen in the dispersion relation for the high-momentum mode in Eq. (14), which does not have the proper commutative limit. In this asymmetric ordering case, therefore, the vacuum breaks the particle and the anti-particle symmetry.

Thus, if one imposes the particle and the anti-particle symmetry on the vacuum, then one needs to modify the theory. Even though the KMST is unique, the KPS is not: Depending on the ordering of the kernel of the Fourier transformation, the $\kappa$-Poincaré algebra is differently realized. There have been attempts [14, 17, 20, 28, 29] to construct the star-product of field theories. If the exponential kernel function of the Fourier transformation is ordered symmetrically, then the Casimir invariant given in Eq. (10) changes into

\[
M^2_s(p) = \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - p^2,
\]

thus, the dispersion relation of the massless field changes into

\[
\omega_p = 2\kappa \ln \left( \left| \frac{p}{2\kappa} \right| \sqrt{1 + \left( \frac{p^2}{4\kappa^2} \right)} \right) = -2\kappa \ln \left( -\left| \frac{p}{2\kappa} \right| \sqrt{1 + \left( \frac{p^2}{4\kappa^2} \right)} \right)
\]

instead of the one given in Eq. (13).

This massless mode has $\kappa \to -\kappa$ invariance and the particle and antiparticle dispersion relation is simply $p_0 = \pm \omega_p$. In this case, because there is no branch-cut ambiguity, one can expect the particle and the antiparticle symmetry of the vacuum and may construct a field theory with $\kappa$-deformed Poincaré symmetry on the symmetric vacuum.

On the other hand, the HM mode is also known to appear in particle and anti-particle spectra and to spoil the $\kappa \to \infty$ limit [17]. Thus, one has to restrict the on-shell spectra and construct the field theory based on this observation. This restriction can be imposed as $M^2_s(p) \geq 0$ ($M^2_s(p) = 0$ for the massless case and $M^2_s(p) > 0$ for the massive case), which respects the $\kappa$-deformed Poincaré symmetry. The details of this investigation will be provided in a separate paper.
In addition to the vacuum symmetry, the ordering is well known to affect the Casimir energy from the studies of Refs. 21-23 for the case of two infinite parallel plates. The deformed effect is seen at the order $1/\kappa^2$, but the contributions are drastically different. The symmetric ordering deformation gives a more attractive effect $[22, 23]$ whereas the asymmetric ordering deformation reduces the attraction and can result in a stable configuration at a certain range of $\kappa a$ $[21]$. The convergence of the $1/\kappa$ expansion is considered in Ref. 22 where the $1/\kappa$ expansion might turn out to be an asymptotic series expansion rather than a converging series expansion. It is not clear yet how these results $[21, 22, 23]$ will change if the KPS measure is incorporated. The structure of the higher-order series expansion is to be studied carefully in this spherical geometry also and is beyond the scope of this paper.

We remark in passing that the angular momentum summations of $B_1(\nu, a)$ and $B_1(\nu, \eta R)$ are finite and are $O(\kappa)$, as seen in Eq. (57), even though we put the regularization $\sigma \to 0$ before the integration and summation. In the commutative limit, however, the terms $B_1(\nu, a)$ and $B_1(\nu, \eta R)$ become infinity and cannot be evaluated without a proper regularization.

Finally, suppose one considers the early Universe and takes the Casimir energy as one of the main radiation sources to the Universe after the inflationary regime because the excitation modes decay away, but the Casimir energy is just the vacuum energy and might survive during the inflation. Then, at the final regime of density fluctuations, the Casimir energy may leave some effect on the global structure of our Universe. Note that the Casimir energy of a sphere measures the finite-size-corrected energy with respect to the infinite-size vacuum energy and is given as $O(1/a^4)$. In addition, one can confirm that most of the finite-size Casimir energy comes from the lower part of the $l$ modes, about 90% of the contribution comes from $l = 0$ to 4. The $l = 0$ mode is the angular-independent contribution, and the $l = 1$ mode can be removed by the motion of observer. Therefore, the $l = 2$ mode would be the most relevant mode in the cosmological sense, and it remains to be seen if its $\kappa$-deformed correction can be detected at the large scale of the present Universe.

APPENDIX A: EVALUATION OF THE DIVERGENT PART $\mathcal{E}_0$

The regularized angular momentum mode of the Casimir energy is given in Eq. (37):

$$E_l^{\text{reg}}(a) = -\frac{\kappa\nu}{\pi} \Re \int_0^\infty dy e^{-i\sigma y e^{-i\phi}} i g(a, i y e^{-i\phi}) \frac{d}{dy} \log \lambda_{\nu}(y e^{-i\phi})$$

$$= -\frac{\kappa\nu}{\pi} \Re \int_0^\infty dy e^{-i\sigma y e^{-i\phi}} i g(a, i \nu y e^{-i\phi}) \frac{d}{dy} \log \lambda_{\nu}(y e^{-i\phi}), \quad (A1)$$

where we rescale $y$ as $\nu y$ so that we can use the the explicit large-order behavior of the Bessel function $[26]$. For large $\nu$, the large-order behavior of $\lambda_{\nu}(\nu y)$ is given as

$$\log \lambda_{\nu}(\nu y) \equiv \log \left(2\nu y I_{\nu}(\nu y)K_{\nu}(\nu y)\right) = \sum_{n=0}^{\infty} \frac{q_n(y)}{\nu^{2n}}, \quad (A2)$$
where

\[ q_0(y) = \frac{1}{2} \log \frac{y^2}{1 + y^2}, \quad (A3) \]

\[ q_1(y) = \frac{y^2}{8(1 + y^2)^2} \left( 1 - \frac{5}{1 + y^2} \right), \]

\[ q_2(y) = \frac{y^2}{64(1 + y^2)^2} \left( 13 - \frac{271}{1 + y^2} + \frac{791}{(1 + y^2)^2} - \frac{565}{(1 + y^2)^3} \right), \]

and \( q_{n \geq 1}(y) \) is \( O(y^{-2n}) \) for large \( y \). The Casimir energy is rewritten in terms of the large-order behavior as

\[ E_c(a) = \sum_{n=0}^{\infty} \left( E_n(\sigma, a) - E_n(\sigma, \eta R) \right), \quad (A4) \]

\[ E_n(\sigma, r) = -\sum_{l=0}^{\infty} \frac{\kappa}{\pi \nu^{2n-1}} \Re \int_0^\infty dy ie^{-i\nu ye^{-i\phi}} g(r, i\nu ye^{-i\phi}) \frac{1}{(1 + y^2 e^{-2i\phi})(y^2 e^{-2i\phi})} q_n(ye^{-i\phi}), \quad (A5) \]

where the limits \( R \to \infty, \sigma \to 0, \phi \to 0 \) are to be taken at the end.

Let us consider \( E_0(\sigma, a) \) in detail. \( E_0(\sigma, a) \) is divergent when \( \sigma \to 0 \) before summing over \( l \). Thus, one needs to evaluate this term with non-vanishing \( \sigma \):

\[ E_0(\sigma, a) = -\frac{\kappa}{\pi} \sum_{l=0}^{\infty} \Re \int_0^\infty dy ye^{-2i\phi \nu ye^{-i\phi} - i\phi} (a, i\nu ye^{-i\phi}) \frac{1}{(1 + y^2 e^{-2i\phi})(y^2 e^{-2i\phi})}. \quad (A5) \]

Formally, one can write

\[ E_0(\sigma, a) = -\frac{\kappa}{\pi} \Re g(a, -\frac{\partial}{\partial \sigma}) \left( -\frac{\partial}{\partial \sigma} \right) \int_0^\infty dy e^{-i\phi} \sum_{l=0}^{\infty} e^{-i\nu ye^{-i\phi}} \frac{1}{(1 + y^2 e^{-2i\phi})(y^2 e^{-2i\phi})} \]

\[ = \frac{\kappa}{2\pi} \Re g(a, -\frac{\partial}{\partial \sigma}) \left( -\frac{\partial}{\partial \sigma} \right) \int_0^\infty dy e^{-i\phi} \frac{i}{\sin(\frac{\sigma ye^{-i\phi}}{2})} \frac{1}{(1 + y^2 e^{-2i\phi})(y^2 e^{-2i\phi})}. \quad (A6) \]

The integral can be done using a change of variable \( y \to ye^{i\phi} \) because the angular integral vanishes at \( \infty \). Then, the line integral is finite and becomes pure imaginary, and the real part vanishes. (One can be convinced that the integration near \( y = 0 \) is finite from Eq. (A5) directly.) This allows one to ignore \( E_0(\sigma, a) \) and \( E_0(\sigma, \eta R) \) completely.

**APPENDIX B: \( E_1(a) \) AND \( E_2(a) \) WHEN \( \alpha = 0 \)**

\( E_n(a) \) in Eq. (44) for \( \alpha = 0 \) is given as

\[ E_n(r) = \frac{1}{r} \sum_{l} \frac{B_n(\nu, r)}{\nu^{2n-2}}, \quad B_n(\nu, r) = \frac{1}{\pi} \int_0^\infty dy \frac{q_n(y)}{1 + \frac{\nu^2 ye^{2i\phi}}{\sin^2(\frac{\sigma ye^{-i\phi}}{2})}}. \quad (B1) \]

In this appendix, we evaluate \( E_1(r) \) and \( E_2(r) \) in two different ways. One is to sum over \( l \) first and to evaluate the integration later. The other way is to integrate first and to sum later. Both ways provide useful viewpoints.
Let us consider
\[ E_1(r) = \frac{1}{r} \sum_l B_1(\nu, r), \quad B_1(\nu, r) = \frac{1}{\pi} \int_0^\infty dy \frac{q_1(y)}{1 + \frac{\nu^2 y^2}{(\kappa r)^2}}. \]  

Using the summation result
\[ \sum_l \frac{1}{1 + \frac{\nu^2 y^2}{(\kappa r)^2}} = \frac{\pi kr}{2y} \tanh(\pi kr / y), \]
one has
\[ E_1(r) = \frac{\kappa}{2} \int_0^\infty \frac{dy}{y} q_1(y) \tanh\left(\frac{\pi kr}{y}\right). \]  

This integration is not convergent and is subtracted by \( E_1(\eta R) \):
\[ E_1(a) - E_1(\eta R) = \frac{\kappa}{2} \int_0^\infty dy q_1(y) \left\{ \tanh\left(\frac{\pi \kappa a}{y}\right) - 1 \right\} \]
\[ = \frac{\kappa}{16} \left(\frac{1}{\pi \kappa a}\right)^2 \int_0^\infty d\xi \frac{\xi}{1 + \left(\frac{\xi}{\pi \kappa a}\right)^2} \left(1 - \frac{5(\xi/\pi \kappa a)^2}{1 + \left(\frac{\xi}{\pi \kappa a}\right)^2}\right) \left(\tanh \xi - 1\right) \]
\[ = -\frac{1}{\kappa a^2} \frac{1}{384} \left(1 - \frac{21}{20} \left(\frac{1}{\kappa a}\right)^2 + O\left(\frac{1}{\kappa a}\right)^4\right), \]  

where \( R \to \infty \) is taken. It is to be noted that the limiting procedure is taken for the case \( \kappa > 0 \). If one considers the case with \( \kappa < 0 \), one has to use the absolute value of \( \kappa \).

Likewise, for \( E_2(r) \), one has
\[ E_2(r) = \frac{1}{r} \sum_l B_2(\nu, r), \quad B_2(\nu, r) = \frac{1}{\pi} \int_0^\infty dy \frac{q_2(y)}{1 + \frac{\nu^2 y^2}{(\kappa r)^2}}. \]  

The summation over \( l \) gives
\[ \sum_l \frac{1}{\nu^2} \left(1 + \frac{\nu^2 y^2}{(\kappa r)^2}\right) = \frac{\pi^2}{2} \left(1 - \frac{y}{\kappa \pi r} \tanh \left(\frac{\pi kr}{y}\right)\right), \]
and one has
\[ E_2(r) = \frac{\pi}{2r} \int_0^\infty d\frac{y}{y} q_2(y) \left(1 - \frac{y}{\kappa \pi r} \tanh \left(\frac{\pi kr}{y}\right)\right). \]  

This integration is convergent, and \( E_2(\eta R) \) vanishes as \( R \to \infty \):
\[ E_2(a) = \frac{\pi}{2a} \int_0^\infty d\xi \frac{\xi^2 q_2(1/\xi)}{\xi^2} \left(1 - \frac{\tanh(\pi \kappa a \xi)}{\kappa \pi a \xi}\right) \]
\[ = \frac{\pi^2}{2a} \frac{35}{32768} + \frac{1}{\kappa a^2} \left(\frac{1}{256} - \frac{13}{3072} \left(\frac{1}{\kappa a}\right)^2 + O\left(\frac{1}{\kappa a}\right)^4\right). \]  

Now one may, instead, use integration first and get
\[ B_1(\nu, a) = -\frac{1 + \frac{11 \nu^2}{\kappa a} + \frac{11 \nu^4}{\kappa a^3}}{128(1 + \frac{\kappa a \nu}{\kappa a} + \frac{\kappa a^2}{\kappa a})^3}, \]
\[ B_2(\nu, a) = \frac{35 + \frac{211 \nu^2}{\kappa a} + \frac{562 \nu^4}{\kappa a^2} + \frac{425 \nu^6}{\kappa a^3} \kappa a^4}{128(1 + \frac{\kappa a \nu}{\kappa a})^3}. \]
Suppose one expands $B_1(\nu,a)$ and $B_2(\nu,a)$ in $1/(\kappa a)$:

\[
B_1(\nu) = -\frac{1}{128} - \frac{1}{16 \kappa a} + \frac{27}{128} \frac{\nu^2}{(\kappa a)^2} + \cdots,
\]

\[
B_2(\nu) = \frac{35}{32768} + \frac{37}{32768} \frac{\nu^2}{(\kappa a)^2} + \cdots.
\]

In this expansion, there is no $1/(\kappa a)$ term in $B_2(\nu)$. However, as can be seen above in Eq. (B7), the summation of $B_2(\nu)/\nu$ over $l$ contains a $1/(\kappa a)$ term. This implies that the naive series expansion in $1/(\kappa a)$ is not valid. One finds that there are nontrivial contributions at $\nu \sim \kappa a$ and that the large-order form of $l \sim \kappa a$ contributes to the summation to result in $O(1/(\kappa a))$:

\[
\begin{align*}
\mathcal{E}_1(a) - \mathcal{E}_1(\eta R) &= \frac{1}{a} \sum_{l=0}^{\infty} \left( B_1(\nu, a) - B_1(\nu, \eta R) \right) \\
&= -\frac{\kappa^2 a}{128} \left[ 11 \psi^{(1)}(12) + 5 \kappa a \psi^{(2)}(12) + \frac{\eta R}{a} (a \to \eta R) \right] \\
&= -\frac{1}{384} \frac{1}{\kappa a^2} \left( 1 + O \left( \frac{1}{\kappa a} \right)^2 \right), \quad \text{(B9)}
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}_2(a) &= \frac{1}{a} \sum_{l=0}^{\infty} \frac{B_2(\nu)}{\nu^2} \\
&= \frac{1}{32768a} \left[ \frac{35 \pi^2}{2} - 35 \psi^{(1)}(\frac{1}{2} + \kappa a) + 35 \kappa a \psi^{(2)}(\frac{1}{2} + \kappa a) \\
&\quad - 127 (\kappa a)^2 \psi^{(3)}(\frac{1}{2} + \kappa a) - 226 (\kappa a)^3 \psi^{(4)}(\frac{1}{2} + \kappa a) - \frac{113}{3} (\kappa a)^4 \psi^{(5)}(\frac{1}{2} + \kappa a) \right] \\
&= + \frac{1}{a} \left( \frac{\pi^2}{2} \frac{35}{32768} + \frac{1}{256 \kappa a} + O \left( \frac{1}{\kappa a} \right)^3 \right), \quad \text{(B10)}
\end{align*}
\]

where $\psi^{(n)}(z)$ is the poly-Gamma function.

**APPENDIX C: $1/\kappa^2$ CORRECTION**

In this appendix, we evaluate the dominant contribution of $\Delta \mathcal{E}_{n \geq 3}(a)$ to the Casimir energy:

\[
\sum_{n \geq 3} \Delta \mathcal{E}_n(a) = \frac{1}{a} \sum_{n \geq 3, l} \frac{\Delta B_n(\nu, a)}{\nu^{2n-2}} = \frac{1}{a} \sum_{n \geq 3, l} \frac{1}{\nu^{2n-2}} \int_0^\infty dy q_n(y) \left( G \left( \frac{\nu y}{\kappa a} \right) - G(0) \right), \quad \text{(C1)}
\]

where $G(0) = 1$. Noting that $q_n(y) = O(y^{-2n})$, one may divide the integral as

\[
\begin{align*}
\int_0^\infty dy q_n(y) \left( G \left( \frac{\nu y}{\kappa a} \right) - G(0) \right) &= \int_0^\infty dy q_n(y) \left( G \left( \frac{\nu y}{\kappa a} \right) - 1 - \left( \frac{\nu y}{\kappa a} \right)^2 G_1 \right) \\
&\quad + \int_0^\infty dy q_n(y) \left( \left( \frac{\nu y}{\kappa a} \right)^2 G_1 \right), \quad \text{(C2)}
\end{align*}
\]
where \( G_1 = \frac{1}{2} \frac{d^2}{dy^2} G(y) \bigg|_{y=0} \). We use the fact that the odd derivative of \( G(y) \) at \( y = 0 \) vanishes.

From this decomposition, one may put the summation as

\[
\sum_{n \geq 3} \Delta E_n(a) = E_c^{(2)}(a) + E_c^{(3)}(a),
\]

\[
E_c^{(2)}(a) = \frac{G_1}{\pi a} \sum_{n \geq 3} \sum_l \frac{1}{\nu^{2n-2}} \int_0^\infty dy \, q_n(y) \left( \frac{\nu y}{\kappa a} \right)^2,
\]

\[
E_c^{(3)}(a) = \frac{1}{\pi a} \sum_{n \geq 3} \sum_l \frac{1}{\nu^{2n-2}} \int_0^\infty dy \, q_n(y) \left( G \left( \frac{\nu y}{\kappa a} \right) - 1 - \left( \frac{\nu y}{\kappa a} \right)^2 G_1 \right),
\]

whose contribution turns out to be convergent and is order of \( O(1/\kappa^2) \) and \( O(1/\kappa^3) \), respectively.

| \( l \) | \( J(l) \) | \( J_{\text{asymp}}(l) \) |
|---|---|---|
| 0 | 0.00102501 | 0.00344 |
| 1 | 0.000287343 | 0.000382667 |
| 2 | 0.000122372 | 0.00013776 |
| 3 | 0.0000661443 | 0.0000702857 |
| 4 | 0.0000410683 | 0.0000425185 |
| 5 | 0.0000278738 | 0.0000284628 |
| 6 | 0.000020120 | 0.0000203787 |
| 7 | 0.0000151907 | 0.0000153067 |
| 8 | 0.0000118677 | 0.000011917 |
| 9 | 9.52338 \times 10^{-6} | 9.54017 \times 10^{-6} |
| 10 | 7.80878 \times 10^{-6} | 7.80952 \times 10^{-6} |

TABLE I: Comparison of the values \( J(l) \) with the corresponding values of the asymptotic \( J_{\text{asymp}}(l) \).

To evaluate \( E_c^{(2)}(a) \), one notes that

\[
E_c^{(2)}(a) = \frac{G_1}{\pi a} \left( \frac{1}{\kappa a} \right)^2 \sum_{l \geq 0} J(l),
\]

where

\[
J(l) \equiv \nu^4 \int_0^\infty dy \, y^2 \sum_{n \geq 3} \frac{q_n(y)}{\nu^{2n}}
\]

\[
= \nu^4 \int_0^\infty dy \, y^2 \left[ \log \lambda_{\nu}(\nu y) - q_0(y) - \frac{q_1(y)}{\nu^2} - \frac{q_2(y)}{\nu^4} \right].
\]

The asymptotic form for large \( l, l \gg 1 \), is proportional to \( 1/\nu^2 \):

\[
J_{\text{asymp}}(l) \simeq 0.000861/(l + 1/2)^2.
\]
The numerical values of $J(l)$ are presented in Table I. Comparing this with the value of $J_{\text{asymp}}(l)$ for given $l$, one notices that $J(l)$ converges very fast to $J_{\text{asymp}}(l)$. Thus, we find that the summed value, with the help of $\sum_{l=0}^{\infty} (l + 1/2)^{-2} = \pi^2/2$ for the large $l$ contribution as

$$J_1 = \sum_{l=0}^{\infty} J(l) \simeq 0.001713,$$

and we have $E_c^{(2)}$

$$E_c^{(2)} = \frac{0.001713 \ G_1}{\pi a \ (\kappa a)^2}.$$

Note that $G_1 = -1$ when $\alpha = 0$, and $G_1 = -47/2$ when $\alpha = 3$.

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[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).
[2] T. H. Boyer, Phys. Rev. 174, 1764 (1968); B. Davis, J. Math. Phys. 13, 1324 (1972); R. Balian and B. Duplantier, Ann. Phys. (N.Y.) 112, 165 (1978); K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (N.Y.) 115, 388 (1978).
[3] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. 353, 1 (2001); K. A. Milton, J. Phys. A: Math. Gen. 37, R209 (2004) and references therein.
[4] F. Chen, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Phys. Rev. A69, 022117 (2004).
[5] K. Autumn, M. Sitti, Y. A. Liang, A. Peattie, W. W. Hansen, S. Spongber, T. W. Kenny, R. Fearing, J. N. Israelachvili, and R. J. Full, Proc. Natl. Acad. Sci. USA 99, 12252 (2002); K. Autumn, Am. Sci. 94, 124 (2006).
[6] T. H. Boyer, Phys. Rev. D21, 2137 (1980); D. W. Sciama, P. Candelas, and D. Deutsch, Adv. Phys. 30, 327 (1981); S. Hacyan, A. Sarmiento, G. Cocho, and F. Soto, Phys. Rev. D32, 914 (1985).
[7] J. Garriga, O. Pujolas, and T. Tanaka, Nucl. Phys. B605, 4922 (2001); J. Garriga and A. Pomarol, Phys. Lett. B560, 91 (2003).
[8] J. Lukierski, A. Nowicki, H. Ruegg, and V. N. Tolstoy, Phys. Lett. B264, 331 (1991); S. Majid and H. Ruegg, Phys. Lett. B329, 189 (1994);
[9] G. Amelino-Camelia, Phys. Lett. B510, 255 (2001); Int. J. Mod. Phys. D11, 35 (2002).
[10] N. R. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, Phys. Lett. B522, 133 (2001); J. Kowalski-Glikman and S. Nowak, Phys. Lett. B539, 126 (2002)

[11] S. Majid and H. Ruegg, Phys. Lett. B334, 348 (1994).

[12] A. Sitarz, Phys. Lett. B349, 42 (1995); C. Gomera, P. Kosiński, and P. Maślanka, J. Math. Phys. 37, 5820 (1996).

[13] C. Gonera, P. Kosiński, and P. Maślanka, J. Math. Phys. 37, 5820 (1996).

[14] P. Kosiński, J. Lukierski, and P. Maślanka, Phys. Rev. D62, 025004 (2000).

[15] H.-C. Kim, J. H. Yee, and C. Rim, Phys. Rev. D75, 045017 (2007).

[16] H.-C. Kim, J. H. Yee, and C. Rim, Phys. Rev. D72, 103523 (2005).

[17] H.-C. Kim, C. Rim, and J. H. Yee, Phys. Rev. D76, 105012 (2007).

[18] M. Daszkiewicz, J. Lukierski and M. Woronowicz, [arXiv:0708.1561 [hep-th]].

[19] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, and J. Wess, Class. Quant. Grav. 22, 3511 (2005).

[20] L. Freidel, J. Kowalski-Glikman, and S. Nowak, Phys. Lett. B648, 70 (2007).

[21] S. Nam, H. Park, and Y. Seo, J. Korean. Phys. Soc. 42, 467 (2003).

[22] J. P. Bowes and P. D. Jarvis, [arXiv:gr-qc/9602016]

[23] M. V. Cougo-Pinto, C. Farina, and J. F. M. Mendes, Phys. Lett. B529, 256 (2002).

[24] V. V. Nesterenko and I. G. Pirozhenko, Phys. Rev. D 57, 1284 (1998).

[25] M. E. Bowers and C. R. Hagen, Phys. Rev. D 59, 025007 (1998).

[26] M. Abramwitz and I. A. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, Washington, D.C., 1964).

[27] C. M. Bender and K. A. Milton, Phys. Rev. D 50, 6547 (1994); A. Romeo, Phys. Rev. D 52, 7308 (1995).

[28] J. Lukierski, H. Ruegg, and W. Zakrzewski, Ann. Phys. 243, 90-116 (1995).

[29] A. Agostini, G Amelino-Camelia, and F. D’Andrea, Int. J. Mod. Phys. A 19, 5187 (2004); A. Agostini, F. Lizzi, and A. Zampini, Mod. Phys. Lett. A 17, 2105 (2002).