FRAGMENTATION NORM AND RELATIVE QUASIMORPHISMS

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ABSTRACT. We prove that manifolds with complicated enough fundamental group admit measure-preserving homeomorphisms which have positive stable fragmentation norm with respect to balls of bounded measure.

1. Introduction

Homeomorphisms of a connected manifold $M$ can be often expressed as compositions of homeomorphisms supported in sets of a given cover of $M$. This is known as the fragmentation property. In this paper we are interested in groups $G \subseteq \text{Homeo}_0(M, \mu)$ of compactly supported measure-preserving homeomorphisms which satisfy the fragmentation property with respect to topological balls of measure at most one. Given $f \in G$, its fragmentation norm $\|f\|_{\text{frag}}$ is defined to be the smallest $n$ such that $f = g_1 \cdots g_n$ and each $g_i$ is supported in a ball of measure at most one. Thus the fragmentation norm is the word norm on $G$ associated with the generating set consisting of maps supported in balls as above. We are also interested in the stable fragmentation norm defined by $\lim_{k \to \infty} \frac{\|f^k\|}{k}$. The existence of an element with positive stable fragmentation norm implies that the diameter of the fragmentation norm is infinite. We say that the fragmentation norm on $G$ is stably unbounded if $G$ has an element with positive stable fragmentation norm. In general, a group $G$ is called stably unbounded if it admits a stably unbounded conjugation invariant norm; see Section 2 for details.

The main result of the paper provides conditions under which the existence of an essential quasimorphism (see Section 2.1) on the fundamental group of $M$ implies the existence of an element of $G$ with a positive

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stable fragmentation norm. Then we specify this abstract result to the following concrete cases.

**Theorem 1.1 (Homeomorphisms).** Let $M$ be a connected PL-manifold equipped with a Lebesgue measure $\mu$. Let $G$ be the commutator subgroup of the group $\text{Homeo}_0(M, \mu)$ of compactly supported measure-preserving homeomorphisms of $M$. If $\pi_1(M)$ has trivial center and admits an essential quasimorphism then the fragmentation norm on $G$ is stably unbounded. [See Section 4.1]

**Theorem 1.2 (Diffeomorphisms).** Let $M$ be a connected differentiable manifold equipped with a volume form $\mu$. Let $G$ be the commutator subgroup of the group $\text{Diff}_0(M, \mu)$ of compactly supported volume-preserving diffeomorphisms of $M$. If $\pi_1(M)$ has trivial center and admits an essential quasimorphism then the fragmentation norm on $G$ is stably unbounded. [See Section 4.2]

**Remark 1.3.** The assumption on the trivial center can be weakened to saying that the point evaluation map $G \to M$ induces the trivial homomorphism on the fundamental group; see Section 2.2 for more details.

**Theorem 1.4.** Let $(M, \omega)$ be a symplectic manifold. If $\pi_1(M)$ admits an essential quasi-morphism then the fragmentation norm on the group $\text{Ham}(M, \omega)$ is stably unbounded. [See Section 4.3]

The following are simple examples of manifolds for which the stable unboundedness of the fragmentation norm is a new result.

**Example 1.5.** Let $M = \mathbb{R}^3 \setminus (L_1 \cup L_2)$, where $L_i$ are disjoint lines. Then the commutator subgroup of $\text{Diff}_0(M, \mu)$ has stably unbounded fragmentation norm, and in particular is stably unbounded. ♦

**Example 1.6.** Let $M = \mathbb{R}^4 \setminus (P_1 \cup P_2)$, where $P_i$ are disjoint planes, or $M = \mathbb{R}^2 \setminus \{x, y\}$. Suppose that $M$ is equipped with the standard symplectic form $\omega$ induced from $\mathbb{R}^4$, $(\mathbb{R}^2)$ respectively. Then $\text{Ham}(M, \omega)$ has stably unbounded fragmentation norm. ♦

**Example 1.7.** Let $M$ be the Klein Bottle with a point removed equipped with a Lebesgue measure. Then the group of compactly supported measure-preserving homeomorphisms generated by maps supported in balls of bounded area has stably unbounded fragmentation norm. ♦

**Remark 1.8.** Lanzat [8] and Monzner-Vichery-Zapolsky [10] showed that the fragmentation norm on groups of Hamiltonian diffeomorphisms
of certain non-compact symplectic manifolds is stably unbounded. Stable unboundedness of the Hofer norm on the group \( \text{Ham}(\mathbb{R}^2 \setminus \{x, y\}) \) was proved by Polterovich-Sieburg [12]. For \( M \) compact Theorem 1.2 was proven by Polterovich, see [11, Section 3.7].

**Remark 1.9.** Unboundedness of a group is usually proven with the use of unbounded quasimorphisms. When \( M \) is not compact it is not known whether its groups of transformations admit unbounded quasimorphisms. Our proofs involve a construction of relative quasimorphisms on groups in question. We would like to note that several constructions of relative quasimorphisms in symplectic geometry appeared in [4, 10, 7].

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2. Preliminaries

2.1. **Definitions.** Let \( G \) be a group. A function \( \nu : G \to [0, \infty) \) is called a **conjugation-invariant norm** if it satisfies the following conditions for all \( g, h \in G \):

1. \( \nu(g) = 0 \) if and only if \( g = 1_G \)
2. \( \nu(g) = \nu(g^{-1}) \)
3. \( \nu(gh) \leq \nu(g) + \nu(h) \)
4. \( \nu(ghg^{-1}) = \nu(h) \).

Let \( \psi : G \to \mathbb{R} \) be a function. The **stabilization** of \( \psi \) is a function

\[
\overline{\psi}(g) = \lim_{n \to \infty} \frac{\psi(g^n)}{n},
\]

provided that the above limit exists for all \( g \in G \).

A norm \( \nu \) is called **stably unbounded** if there exists \( g \in G \) with positive stabilization: \( \overline{\nu}(g) > 0 \). A group \( G \) is called (stably) **unbounded** if it admits a (stably) unbounded conjugation-invariant norm. For more information about these notions see [2].
A function $\psi : G \to \mathbb{R}$ is called a \textit{quasimorphism} if there exists a real number $C \geq 0$ such that
\[ |\psi(gh) - \psi(g) - \psi(h)| \leq C \]
for all $g, h \in G$. The infimum of such $C$'s is called the \textit{defect} of $\psi$ and is denoted by $D_\psi$. A quasimorphism $\psi$ is called \textit{homogeneous} if
\[ \psi(g^k) = k\psi(g) \]
for all $k \in \mathbb{Z}$ and all $g \in G$. The stabilization of a quasimorphism is homogeneous [3]. A homogeneous quasimorphism is called \textit{essential} if it is nontrivial and it is not a homomorphism.

A function $\psi : G \to \mathbb{R}$ is called a \textit{relative quasimorphism} with respect to a conjugation-invariant norm $\nu$ if there exists a positive constant $C$ such that for all $g, h \in G$
\[ |\psi(gh) - \psi(g) - \psi(h)| \leq C \min\{\nu(g), \nu(h)\}. \]

For more information about quasimorphisms and their connections to different branches of mathematics, see [3].

2.2. \textbf{The setup and assumptions.}

(1) Let $M$ be a topological manifold admitting a triangulation and let $z \in M$ be a base-point. For each $x \in M$ let $\gamma_x : [0, 1] \to M$ be a path from $z$ to $x$.

We assume that the paths $\gamma_x$ are of bounded lengths on compact subsets. More precisely, we choose an auxiliary length metric on $M$ so that for every compact subset $K \subset M$ there exists a constant $C_K > 0$ such that $\ell(\gamma_x) \leq C_K$ for every $x \in K$. Here $\ell(\gamma)$ denotes the length of a path $\gamma$.

We also assume that the paths depend continuously on $x$ on a set of full measure. This can be done as follows. Let $\epsilon > 0$ be a small number. Join the base-point with the barycenter of a top dimensional simplex by a path of length close to the distance between the base-point and the barycenter and concatenate it with paths joining continuously the barycenter with each point in the interior of the simplex.

(2) Let $\mu$ be a Lebesgue measure on $M$ and let $G \subseteq \text{Homeo}_0(M, \mu)$ be a connected subgroup of the connected group of compactly-supported measure-preserving homeomorphisms of $M$. 
(3) Let \( \text{ev}_z : G \to M \) be the evaluation at the base-point defined by \( \text{ev}(f) = f(z) \). We say that \( G \) has trivial evaluation if the homomorphism

\[
\text{ev}_* : \pi_1(G, 1) \to \pi_1(M, z)
\]

induced by the evaluation map is trivial. For example, this is the case if \( \pi_1(M, z) \) has trivial center.

(4) Let \( \psi : \pi_1(M, z) \to \mathbb{R} \) be a nontrivial homogeneous quasimorphism. Let \( \Psi : G \to \mathbb{R} \) be defined by

\[
\Psi(f) = \int_M \psi([f_x]) \mu,
\]

where \( f_x \) is a loop represented by the concatenation \( \gamma_x \cdot \{f_t\} \cdot \gamma_f(x) \). Since the support of \( f \) is compact and the length of paths \( \gamma_x \) are bounded on compact sets, the length of \([f_x]\) is bounded by some constant independent of \( x \in \text{supp}(f) \). It follows that the function \( x \mapsto \psi([f_x]) \) is bounded. Moreover, if \( G \) has trivial evaluation then \([f_x]\) does not depend on the isotopy \( \{f_t\} \). Thus \( \Psi(f) \) is well-defined.

(5) Let \( \mathcal{B} \) be the set of all subsets \( B \subset M \) which are homeomorphic to the \( n \)-dimensional Euclidean unit closed ball \( B^n \subset \mathbb{R}^n \) and of measure at most one:

\[
\mathcal{B} = \{ B \subset M \mid B \cong B^n \text{ and } \mu(B) \leq 1 \}.
\]

Notice that the group of all measure-preserving homeomorphisms of \( M \) acts on the set \( \mathcal{B} \).

(6) Assume that the group \( G \) has the fragmentation property with respect to the family \( \mathcal{B} \). This means that for every \( f \in G \) there exist \( g_1, \cdots, g_n \in G \) such that

1. \( f = g_1 \cdots g_n \), and
2. \( \text{supp}(g_i) \subset B_{g_i} \in \mathcal{B} \).
3. each \( g_i \) is isotopic to the identity through an isotopy supported in \( B_{g_i} \); that is, there exists an isotopy \( \{g^t_i\} \) such that \( g^0_i = 1 \) and \( g^1_i = g_i \) and \( \text{supp}(g^t_i) \subset B_{g_i} \).

(7) The fragmentation norm on \( G \) associated with \( \mathcal{B} \) is defined by

\[
\|f\|_{\text{frag}} = \min \{ n \in \mathbb{N} \mid f = g_1 \cdots g_n \},
\]

where \( g_i \)'s are as in the previous item. Notice that it is a conjugation invariant norm.
2.3. **The main technical result.** Before stating the theorem we summarize the assumptions we need.

**Assumptions:**

- $M$ is a topological $n$-dimensional manifold admitting a triangulation.
- $\mu$ is a Lebesgue measure on $M$.
- $B$ is the set of topological balls in $M$ of measure at most 1.
- $G \subseteq \text{Homeo}_0(M, \mu)$ is a connected group with trivial evaluation.
- $G$ has fragmentation property with respect to $B$.

**Theorem 2.1.** Let $M$, $\mu$, $B$ and $G$ be as above. If there exists a homogeneous quasimorphism $\psi: \pi_1(M) \to \mathbb{R}$ such that the homogenization $\overline{\Psi}$ is nonzero then the fragmentation norm on $G$ is stably unbounded.

**Remark 2.2.** Notice that the assumption on the trivial evaluation rules out one-dimensional manifolds.

In order to apply Theorem 2.1 to a concrete group $G$ we need to verify that the function $\overline{\Psi}: G \to \mathbb{R}$ is nonzero. It is done with the use of various push maps. These applications are presented in Section 4.

3. PROOF OF THEOREM 2.1

**Lemma 3.1.** The function $\Psi: G \to \mathbb{R}$ is a relative quasimorphism with respect to the fragmentation norm.

**Proof.** Let $f, g \in G$ and let $\{f_t\}$ and $\{g_t\}$ be isotopies from the identity to $f = f_1$, and to $g = g_1$ respectively. Denote by $\{f_t\} * \{g_t\}$ the concatenation of $\{g_t\}$ and $\{f_t \circ g\}$, so that it is an isotopy from the identity to $fg$. In what follows these isotopies are used to represent $[f_{g(x)}]$, $[g_x]$ and $[(fg)_x]$ respectively. Let us discuss several cases:

1. If $x \notin \bigcup_{t \in [0,1]} \text{supp}(g_t)$, then $[(fg)_x] = [f_{g(x)}]$ and $[g_x]$ is the identity element of $\pi_1(M, z)$. It follows that for all such $x$ we have
   \[
   \psi((fg)_x) - \psi([f_{g(x)}]) - \psi([g_x]) = 0.
   \]

2. If $x \notin \bigcup_{t \in [0,1]} g^{-1}(\text{supp}(f_t))$, then $[(fg)_x] = [g_x]$ and $[f_{g(x)}]$ is the identity element of $\pi_1(M, z)$. It follows that for all such $x$ we have
   \[
   \psi((fg)_x) - \psi([f_{g(x)}]) - \psi([g_x]) = 0.
   \]
Thus for every \( x \notin \left( \bigcup_{t \in [0,1]} \text{supp}(g_t) \right) \cap \left( \bigcup_{t \in [0,1]} g_t^{-1}(\text{supp}(f_t)) \right) \) we have that
\[
\psi([fg_x]) - \psi([f_{g(x)}]) - \psi([g_x]) = 0.
\]
It follows from the definition of the fragmentation norm that we may choose \( f_t \) and \( g_t \) so that
\[
\mu(\bigcup_{t \in [0,1]} \text{supp}(g_t)) \leq \|g\|_{\text{frag}} \quad \text{and} \quad \mu(\bigcup_{t \in [0,1]} g_t^{-1}(\text{supp}(f_t))) \leq \|f\|_{\text{frag}}.
\]
Let \( U = \left( \bigcup_{t \in [0,1]} \text{supp}(g_t) \right) \cap \left( \bigcup_{t \in [0,1]} g_t^{-1}(\text{supp}(f_t)) \right) \). The two inequalities above imply that \( \mu(U) \leq \min\{\|f\|_{\text{frag}}, \|g\|_{\text{frag}}\} \) and we obtain that
\[
|\Psi(fg) - \Psi(f) - \Psi(g)| \leq \int_M \psi([fg_x]) - \psi([f_{g(x)}]) - \psi([g_x]) \mu(U) \\
\leq \int_U D_{\psi} \mu \\
\leq D_{\psi} \min\{\|f\|_{\text{frag}}, \|g\|_{\text{frag}}\},
\]
where \( D_{\psi} \) is the defect of the quasimorphism \( \psi \). This shows that \( \Psi \) is a relative quasi-morphism with respect to the fragmentation norm. \( \square \)

**Lemma 3.2.** Let \( \Psi : G \to \mathbb{R} \) be the function defined in Section 2.2 (4). Its homogenization \( \overline{\Psi} : G \to \mathbb{R} \) is well defined and invariant under conjugations.

**Proof.** If \( M \) is a closed manifold then Polterovich proved that \( \overline{\Psi} \) is a homogeneous quasi-morphism [11, Section 3.7]. In particular, it is invariant under conjugation. Since \( \overline{\Psi} \) is evaluated on a compactly supported homeomorphism, there exists a compact subset of \( M \) containing this support and the base-point and Polterovich’s argument implies the statement. \( \square \)

**Lemma 3.3.** The homogenization \( \overline{\Psi} \) is Lipschitz with respect to the fragmentation norm. More precisely,
\[
|\overline{\Psi}(f)| \leq 3 \ D_{\psi} \ |f|_{\text{frag}},
\]
for every \( f \in G \).

**Proof.** Let \( f \in G \) be such that \( |f|_{\text{frag}} = n \). This means that there exist \( g_1, \ldots, g_n, h_1, \ldots, h_n \in G \) such that \( f = g_1 \cdots g_n \) and each \( g_i \) is...
supported in a ball $h_i(B)$. Then for $n > 1$ we have
\[ |\Psi(f) - \Psi(g_1) - \ldots - \Psi(g_n)| \leq \sum_{i=1}^{n-1} |\Psi(g_1 \ldots g_{i+1}) - \Psi(g_1 \ldots g_i) - \Psi(g_{i+1})| \]
\[ \leq \sum_{i=1}^{n-1} D_\psi < nD_\psi, \]
where the second inequality comes from the chain of inequalities at the end of the proof of Lemma 3.1. It follows from the adaptation of the proof of Lemma 2.21 in [3] to our case that for each $g \in G$ one has
\[ |\Psi(g) - \Psi(g)| \leq D_\psi \mu(\text{supp}(g)) \leq D_\psi \|g\|_{\text{Frag}}, \]
This yields to the following inequalities
\[ |\Psi(f) - \Psi(g_1) - \ldots - \Psi(g_n)| \]
\[ \leq |\Psi(f) - \Psi(f)| + \sum_{i=1}^{n} |\Psi(g_i) - \Psi(g_i)| + |\Psi(f) - \Psi(g_1) - \ldots - \Psi(g_n)| \]
\[ \leq D_\psi \|f\|_{\text{Frag}} + \sum_{i=1}^{n} D_\psi + nD_\psi = 3nD_\psi. \]
Note that each $g_i$ is supported in a ball, hence for each $x \in M$ the length of $[(g^p_i)_x] \in \pi_1(M, z)$ is bounded by some constant independent of $p$. It follows that for each $i$
\[ (1) \quad \Psi(g_i) = \int_M \lim_{p \to \infty} \frac{\psi([(g^p_i)_x])}{p} \mu = 0. \]
Combining (1) with previous inequalities we get
\[ (2) \quad |\Psi(f)| \leq 3nD_\psi = 3D_\psi \|f\|_{\text{Frag}}, \]
which shows that $\Psi$ is Lipschitz with respect to the fragmentation norm. \hfill \Box

Proof of Theorem 2.1. Recall that the hypothesis says that there exists a non-trivial homogeneous quasimorphism $\psi: \pi_1(M) \to \mathbb{R}$ such that the relative quasimorphism $\Psi$ is nontrivial. Take $f \in G$ such that $\Psi(f) \neq 0$.

Now (2) yields
\[ \lim_{k \to \infty} \frac{\|f^k\|_{\text{Frag}}}{k} \leq \lim_{k \to \infty} \frac{|\Psi(f^k)|}{3kD_\psi} = \lim_{k \to \infty} \frac{k|\Psi(f)|}{3kD_\psi} = \frac{|\Psi(f)|}{3D_\psi} > 0, \]
which proves that the fragmentation norm is stably unbounded. \hfill \Box
4. Applications of Theorem 2.1

4.1. The commutator subgroup of the group of homeomorphisms. Let $S^1 \times D^n - 1$, where $D^n - 1$ is a closed $(n-1)$-dimensional Euclidean disc of radius $1 + \epsilon$, where $\epsilon > 0$ is an arbitrarily small number. Let $\varphi_s: D^n - 1 \rightarrow \mathbb{R}$, for $s \in [0,1]$ be a family of smooth functions supported away from the boundary such that it is equal to $s$ on a disc of radius $1$. Let $f_s: S^1 \times D^n - 1 \rightarrow S^1 \times D^n - 1$ be defined by $f_s(t, z) = (t + \varphi_s(z), z)$. It is straightforward to verify that $f_s$ preserves the standard product Lebesgue measure. Let $\gamma: S^1 \rightarrow M$ be an embedded loop and let $\tilde{\gamma}: S^1 \times D^n - 1 \rightarrow M$ be an embedding such that $\tilde{\gamma}(t, 0) = \gamma(0)$. We define the associated push-map $f_\gamma: M \rightarrow M$ by

$$f_\gamma(x) = \begin{cases} 
\tilde{\gamma} \circ f_1 \circ \tilde{\gamma}^{-1}(x) & \text{for } x \in \tilde{\gamma}(S^1 \times D^n - 1) \\
\text{Id} & \text{otherwise.}
\end{cases}$$

Changing the parameter $s$ defines an isotopy from the identity of $f_\gamma$ through measure preserving homeomorphisms. We choose the constant $\epsilon$ suitably small so that the value $\Psi(f_\gamma)$ is arbitrarily close to $\psi(\gamma)\mu(\text{supp}(f_\gamma))$.

Let $G = [\text{Homeo}_0(M, \mu), \text{Homeo}_0(M, \mu)]$ be the commutator subgroup of the group of compactly supported measure-preserving homeomorphisms of $M$. Since it is contained in the kernel of the flux homomorphism, it has the fragmentation property. If $\dim M \geq 3$ then it is equal to the kernel of flux [5]. If $\dim M = 2$ then there is only an inclusion and the equality is an open problem to the best of our knowledge. For more information about the flux homomorphism see [1, Section 3]. Theorem 1.1 is a consequence of the following result.

**Proposition 4.1.** If $\psi: \pi_1(M) \rightarrow \mathbb{R}$ is an essential quasimorphism then the fragmentation norm on $G$ is stably unbounded.

**Proof.** For simplicity we denote elements of the fundamental group $\pi_1(M, z)$ and their representing loops by the same Greek letters. Since $\psi$ is an essential quasimorphism, there exist $\alpha, \beta \in \pi_1(M)$ such that

$$|\psi(\alpha) - \psi(\alpha\beta) + \psi(\beta)| = a > 0.$$

Let $\alpha, \beta: S^1 \rightarrow M$ be embedded based loops representing the above elements of the fundamental group. If $\dim M \geq 3$ then such loops exists for all elements of the fundamental group for dimensional reasons. If $\dim M = 2$ then if $\pi_1(M)$ admits an essential quasimorphism then $\pi_1(M)$ either non-abelian free or the surface group of higher genus (up to index two if $M$ is non-oriented). In this case $\pi_1(M)$ has abundance
of essential quasimorphisms and we can choose $\psi$ and embedded loops $\alpha, \beta$ which satisfy the above requirement.

Consider the push maps $f_\alpha, f_\beta$ such that their support have equal measures and $\overline{\Psi}(f_\alpha)$ is arbitrarily close to $\psi(\alpha)\mu(\text{supp}(f_\alpha))$ and similarly for $f_\beta$. By a computation similar to (2) in the proof of Lemma 3.1 it follows that

$$|\overline{\Psi}(f_\alpha) - \overline{\Psi}(f_\alpha f_\beta) + \overline{\Psi}(f_\beta)| \geq a\mu(\text{supp}(f_\alpha) \cap \text{supp}(f_\beta)) - \delta > 0,$$

where $\delta > 0$ is a constant which can be made arbitrarily small by a suitable choice of the push maps. This shows that the relative quasimorphism $\overline{\Psi}: \text{Homeo}_0(M, \mu) \to \mathbb{R}$ is nonzero and it is not a homomorphism. It follows that it must be nontrivial on the commutator subgroup $G$ and the statement follows from a direct application of Theorem 2.1.

\[\square\]

4.2. The commutator subgroup of $\text{Diff}_0(M, \mu)$. The fragmentation property is due to Thurston; see Banyaga [1, Lemma 5.1.2]. The proof of Theorem 1.2 is analogous to the above proof for homeomorphisms in the sense that the push map is obtained from the same maps

$$f_s: S^1 \times D^{n-1} \to S^1 \times D^{n-1}$$

which are transplanted to $M$ via differentiable maps. Then the application of Proposition 4.1 is the same.

4.3. The group of Hamiltonian diffeomorphisms. Notice that in this case we don’t assume that the fundamental group has trivial center. This is because it follows from the proof of Arnold’s conjecture that $\text{Ham}(M, \omega)$ has trivial evaluation [9, Exercise 11.28]. The fragmentation property is due to Banyaga [1, page 110].

Remark 4.2. The proof of Arnold’s conjecture in full generality is more complicated than initially thought; see discussion in [6] and references therein.

Here the strategy is the same but the construction of the push map needs slightly more care. This is done as follows.

Proof of Theorem 1.4. Let $T = S^1 \times [-1, 1] \times D^{2n-2}$ be equipped with the product of an area form on the annulus and the standard symplectic form on the disc. The coordinate on $S^1$ is denoted by $x$, on $[0, 1]$ by $y$ and a point in the disc by $z$. So the symplectic form is $dx \wedge dy + \omega_0$. Let $\varphi: D^{2n-2} \to \mathbb{R}$ be a non-negative function supported in the interior
of the disc and equal to 1 on a disc arbitrarily close to $D^{2n-2}$. Let $H: T \to \mathbb{R}$ be defined by $H(x, y, z) = y f(y) \varphi(z)$. Then

$$dH = \varphi(f + y f') dy + y f d\varphi$$

and the corresponding Hamiltonian vector field is given by

$$X_H(x, y, z) = \varphi(z) (f(y) + y f'(y)) \partial_x + Z(y, z),$$

where $Z$ is a suitable vector field on the disc which depends on the coordinate $y$. If $f$ is the standard bell-shaped function then the equation $f(y) + y f'(y) = 0$ has two solutions $\pm y_0$ and the expression $f(y) + y f'(y) > 0$ for $y \in (-y_0, y_0)$. We restrict the vector field $X_H$ to the subset $S^1 \times (-y_0, y_0) \times D^{2n-2}$ and extend it by zero to the rest of $T$. Notice that this vector field is symplectic but not Hamiltonian and it points in the non-negative direction of $\partial_x$.

The rest of the proof is the same as in the case of homeomorphisms. That is, we choose the classes $\alpha, \beta \in \pi_1(M)$ for which

$$\psi(\alpha \beta) \neq \psi(\alpha) \psi(\beta).$$

Then we choose their embedded representatives and choose their small tubular neighborhoods symplectically diffeomorphic to $T$ with possibly rescaled summands of the symplectic form ([9, Exercise 3.37]). We transplant the above symplectic flows to create $f_\alpha, f_\beta \in \text{Symp}_0(M, \omega)$ and the same argument shows that $\overline{\Psi}$ is nontrivial and that it is not a homomorphism and hence it is nontrivial on the commutator subgroup of the symplectic diffeomorphisms group which is the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms.

\[ \square \]

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