THE ASYMPTOTIC GEOMETRY OF $G_2$-MONOPOLES

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Abstract. This article investigates the asymptotics of $G_2$-monopoles.

First, we prove that when the underlying $G_2$-manifold is nonparabolic (i.e. admits a positive Green’s function), finite intermediate energy monopoles with bounded curvature have finite mass. The second main result restricts to the case when the underlying $G_2$-manifold is asymptotically conical. In this situation, we deduce sharp decay estimates and that the connection converges, along the end, to a pseudo-Hermitian–Yang–Mills connection over the asymptotic cone.

Finally, our last result exhibits a Fredholm setup describing the moduli space of finite intermediate energy monopoles on an asymptotically conical $G_2$-manifold.

Contents

1. Introduction
2. Preliminaries
3. Consequences of Moser iteration and $\varepsilon$-regularity
4. Finite mass from finite intermediate energy
5. Bochner–Weitzenböck formulas along the end
6. Refined asymptotics in the AC case
7. Boundary data
8. Bogomolny trick for the intermediate energy
9. Decay of linearized solutions
10. Weighted split Sobolev spaces: Fredholm operators
11. Weighted split Sobolev spaces: Sobolev Embeddings and Multiplication Maps
12. Moduli theory
References

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1. Introduction

1.1. Context. An important problem in $G_2$ geometry is to develop methods to distinguish $G_2$-manifolds. This problem can be put in several ways, and recent advances produced invariants able to detect connected components of the moduli space of $G_2$-holonomy metrics [7–9].

Other approaches intended at producing invariants of $G_2$-manifolds aim to produce enumerative theories counting special submanifolds and gauge fields. For example, in [25] Joyce alluded to the possibility of constructing such an enumerative invariant of $G_2$-manifolds by “counting” rigid, compact, and coassociative submanifolds (see also [24]). On the other hand, Donaldson and Segal, in [14], proposed an enumerative invariant of certain (noncompact) $G_2$-manifolds by considering $G_2$-monopoles instead. They further suggest that this might be easier to define and possibly related to a more direct coassociative “count”.

The underlying idea behind this proposal is inspired by Taubes’ $Gr = SW$ Theorem in [42] for 4-dimensional symplectic manifolds. The similarities stem from the fact that the Seiberg–Witten (SW) invariant is obtained from gauge theory while the Gromov-Witten (Gr) invariant is obtained from holomorphic curves, which in a symplectic manifold are calibrated, just like coassociatives in a $G_2$-holonomy manifold are.

The study of $G_2$-monopoles was initiated in [6, 36]. In [37], the third author gave evidence supporting the Donaldson–Segal program by finding families of $G_2$-monopoles parametrized by a positive real number $m$, called the mass, and showed that in the large mass limit these monopoles concentrate along a compact coassociative submanifold.

This paper is the first installment of a series of papers aimed to study the Donaldson and Segal [14] program, that is the relation between $G_2$-monopoles and coassociative submanifolds.

The goal of this article is to show that several of the asymptotic features satisfied by these examples are in fact general phenomena which follow from natural assumptions such as finiteness of the relevant energy. This is a very much needed development in order to justify the choice of function spaces to be used in a satisfactory moduli theory. In the next article in this series the first and third author will be dealing with investigating the bubbling of $G_2$-monopoles along coassociatives [16].

More about the other gauge theoretical approaches for producing invariants of $G_2$-manifolds can, for example, be found in [11, 12, 14, 22, 39, 46].
1.2. **Summary.** Let \((X^7, \varphi)\) be a noncompact, complete, and irreducible \(G_2\)-manifold. We respectively denote by \(g\) and \(*\) the metric and Hodge star operator induced by the \(G_2\)-structure \(\varphi \in \Omega^3(X)\). We also let \(\psi = *\varphi \in \Omega^4(X)\). Given a compact Lie group \(G\) with Lie algebra \(\mathfrak{g}\) and a principal \(G\)-bundle \(P\) over \(X\), we consider pairs \((\nabla, \Phi)\), where \(\nabla\) is a smooth connection on \(P\) with curvature \(F_\nabla \in \Omega^2(X, \mathfrak{g}_P)\) and \(\Phi\) a smooth section of \(\mathfrak{g}_P = P \times_{\text{Ad}} \mathfrak{g}\), called the Higgs field. Such a pair \((\nabla, \Phi)\) is said to be a \(G_2\)-monopole if

\[ *(F_\nabla \wedge \psi) - \nabla \Phi = 0. \quad (1.1) \]

Furthermore, \(G_2\)-monopoles can be seen as (at least formally\(^1\)) critical points of the intermediate energy:

\[ \mathcal{E}^\psi(\nabla, \Phi) = \int_X \left( |F_\nabla \wedge \psi|^2 + |\nabla \Phi|^2 \right) \text{vol}_X. \quad (1.2) \]

Note that \(F_\nabla \wedge \psi\) only contains certain components of \(F_\nabla\) and so the intermediate energy only controls part of the curvature of \(\nabla\). Indeed, the 2-forms on \((X, \varphi)\) split into irreducible \(G_2\)-representations as \(\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}\), with the subscripts accounting for the dimension of the representation. Using this decomposition we can uniquely write \(F_\nabla = F^7_\nabla + F^{14}_\nabla\) and we find that \(F_\nabla \wedge \psi = F^7_\nabla \wedge \psi\). Thus, the intermediate energy only accounts for the “smaller” \(F^7_\nabla\) component of the curvature. Furthermore, under certain technical and refined assumptions on the asymptotic behavior (see Section 1.4 in [36]) it is in fact possible to prove that \(G_2\)-monopoles minimize \(\mathcal{E}^\psi\). In this article we drop such technical hypothesis and replace them by simpler more natural ones such as finiteness of the intermediate energy.

A word must be said about the reason for restricting to noncompact \(G_2\)-manifolds. Indeed, a short computation resulting from applying \(\nabla^*\) to equation (1.1) and using the Bianchi identity, shows that \(\nabla^* \nabla \Phi = 0\) which in turns implies that \(|\Phi|^2\) is subharmonic. Thus, if \(X\) was to be compact then \(|\Phi|\) would be constant and thus \(\nabla \Phi = 0 = F_\nabla \wedge \psi\). In particular, \(\nabla\) is a so-called \(G_2\)-instanton, which is a very interesting equation in itself. However, in this article we focus on “pure” \(G_2\)-monopoles and so we regard the case when \(\nabla \Phi \neq 0\) as being more interesting.

**Notational warnings.**

\(^1\)In fact, formally, \(G_2\)-monopoles are also critical points for the Yang–Mills–Higgs (YMH) energy:

\[ \mathcal{E}(\nabla, \Phi) = \int_X (|F_\nabla|^2 + |\nabla \Phi|^2) \text{vol}_X. \]

Here, we say formally because this energy need not be finite. Indeed, in contrast with the intermediate energy, the YMH energy is infinite for all known irreducible examples.
Throughout this paper, we use \( n = 7 \), the dimension of \( X \), to emphasize how the dimension of the underlying manifold comes into play in the analysis. This is because many of our results hold for more general setups, in particular, for Yang–Mills–Higgs fields in different dimensions.

(2) For comparable quantities \( \alpha \) and \( \beta \), \( \alpha \lesssim \beta \) means that there exists a real number \( c > 0 \), that is independent of the variable relevant in the given context, such that \( \alpha \leq c \beta \).

**Main results.** Recall that a \( G_2 \)-holonomy Riemannian manifold \((X, \varphi)\) is Ricci-flat. Therefore, by the Cheeger–Gromoll splitting theorem [5], any complete, noncompact and irreducible \((X, \varphi)\) has only one end, meaning that \( X - B_r(x) \) has only one connected component for large \( r \gg 1 \). Our first result gives conditions under which monopoles \((\nabla, \Phi)\) have \( |\Phi|\) converging uniformly to a constant along this end. When this is the case, \((\nabla, \Phi)\) is said to have finite mass and the value of the constant to which \( |\Phi|\) converges is called the mass.

**Main Theorem 1** (Finite intermediate energy and bounded curvature implies finite mass).

Let \((X, \varphi)\) be a complete, noncompact and irreducible \( G_2 \)-manifold of bounded geometry which furthermore is nonparabolic\(^2\) (i.e. admits a positive Green’s function). Suppose that \((\nabla, \Phi)\) is a solution\(^3\) to the \( G_2 \)-monopole equation (1.1) with finite intermediate energy (1.2) and bounded\(^4\) \( |F_{\nabla}^{14}| \), i.e. \( |F_{\nabla}^{14}| \in L^\infty(X) \). Then \((\nabla, \Phi)\) has finite mass, i.e. there is a constant \( m \in [0, \infty) \), called the mass of \((\nabla, \Phi)\), such that for any choice of reference point \( o \in X \) one has

\[
\lim_{\text{dist}(x, o) \to \infty} |\Phi(x)| = m. \tag{1.3}
\]

Notice that, as previously mentioned, for a \( G_2 \)-monopole \((\nabla, \Phi)\) the function \( |\Phi|^2 \) is subharmonic. Hence, if \( m = 0 \) in the previous theorem then we have \( \Phi = 0 \) everywhere and so \( \nabla \) is a \( G_2 \)-instanton. Hence, we are primarily interested in the case where \( m \) is positive (see Remark 2.10).

Our second main result gives the asymptotic structure of \( G_2 \)-monopoles on the so called asymptotically conical (AC) \( G_2 \)-manifolds. This is a very interesting class of complete, noncompact and nonparabolic \( G_2 \)-manifolds for which explicit examples are known [3,19], and on which \( G_2 \)-monopoles have already been constructed [37,38]. A \( G_2 \)-manifold

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\(^2\)See Section 4 for definitions and a discussion on the necessity of this last hypothesis.
\(^3\)Here and in what follows, we only consider smooth solutions.
\(^4\)Under these conditions, \( |F_{\nabla}^{14}| \in L^\infty(X) \) actually implies that the whole curvature \( |F_\nabla| \in L^\infty(X) \), cf. Corollary 3.8.
(X, ϕ) is AC if its end is asymptotically isometric to a metric cone \( C = (1, \infty) \times \Sigma, g_C = dr^2 + r^2g_\Sigma \), see Definition 2.4 for the precise definition. In this case, the cross section of the asymptotic cone \( (\Sigma, g_\Sigma) \) comes equipped with a nearly Kähler structure \((ω, J)\) as defined in Definition 2.2. In this situation, a connection \( \nabla \) on a principal G-bundle over \((\Sigma, ω, J)\) is said to be pseudo-Hermitian–Yang–Mills connection, if

\[
F_{\nabla}^{0,2} = 0, \\
ΛF_{\nabla} = 0,
\]

where \( F_{\nabla}^{0,2} \) denotes the \( (0,2) \)-component of the curvature with respect to the almost complex structure \( J \), determined by the nearly Kähler structure and \( ΛF_{\nabla} \) the contraction of the curvature with the fundamental 2-form \( ω \). In the next theorem, we restrict to the case of \( G = SU(2) \).

**Main Theorem 2** (Asymptotics of \( G_2 \)-monopoles on AC manifolds). Let \((X, ϕ)\) be an irreducible AC \( G_2 \)-manifold, with radius function \( r \), and let \((∇, Φ)\) be a solution to the \( G_2 \)-monopole equation (1.1) with structure group \( G = SU(2) \). Suppose that \((∇, Φ)\) has finite intermediate energy (1.2) and is such that \(|F_{\nabla}^{1,4}|\) decays to zero uniformly along the end, i.e. \(|F_{\nabla}^{1,4}(x)| → 0\) as \( r(x) → \infty \). Then, along the end of \((X, ϕ)\),

\[
|∇Φ| ≲ r^{-n-1},
\]

and \(|[Φ, ∇Φ]| + |[Φ, F_{\nabla}]|\) decays exponentially.

Furthermore, if \( r^2|F_{\nabla}^{1,4}| \) is bounded, i.e. \(|F_{\nabla}^{1,4}|\) decays (at least) quadratically, then there is a principal G-bundle \( P_∞ \) over \( Σ \), together with a pair \((∇_∞, Φ_∞)\) such that:

(a) \( Φ_∞ \) is a \( ∇_∞ \)-parallel section of the Adjoint bundle \( g_{P_∞} \) over \( Σ \), and

(b) \( ∇_∞ \) is a pseudo-Hermitian–Yang–Mills connection with respect to the nearly Kähler structure on \( Σ \);

and

\[
(∇, Φ)|_{[R]×Σ} → (∇_∞, Φ_∞),
\]

uniformly as \( R → \infty \).

**Remark 1.1.** Some remarks are now in place.

- \( G_2 \)-monopoles solve the second order equations (2.1a) and (2.1b) (see Lemma 2.7). These are the Euler–Lagrange equations for both the intermediate energy \( E^0 \) and the YMH energy. We also prove analogues of the above main results for general solutions of these equations, see Theorems 4.1, 6.1, and 7.1.

- The decay estimate for \(|∇Φ|\) given above is sharp as proven in Remark 6.2 and as exemplified by the examples in [37] which satisfy all the conditions in Theorem 10.15.
This article also contains several other interesting results on the asymptotic behavior of $G_2$-monopoles. For example, in the conditions of Main Theorem 1, Corollary 3.8 gives uniform decay of all derivatives of both $F_V$ and $V\Phi$ and Corollary 4.4 gives a further general refinement on the asymptotics of $m^2 - |\Phi|^2$, and in the conditions of Main Theorem 2, Corollary 6.10 gives that $m^2 - |\Phi|^2 \sim r^{2-n}$ as $r \to \infty$ and Corollary 6.12 gives $\nabla^{j+1}\Phi \in L^p_r$ for all $p \in [2, 2n]$ and $j \in \mathbb{N}$. Also, as a consequence of these results we use a $G_2$-version of the Bogomolny trick to obtain a topological formula for the intermediate energy of a monopole on an asymptotically conical $G_2$-manifold. Given the importance of this result we state it here.

Along the end of an asymptotically conical $G_2$-manifold $(X^7, \varphi)$, the cohomology class $[\varphi|_{\Sigma_R}]$ obtained by restricting the 4-form $\varphi$ to the links $\Sigma_R \cong \{R\} \times \Sigma$ of the asymptotic cone determine, for $R \gg 1$, a class $\Psi_\infty \in H^4(\Sigma, \mathbb{R})$ called the asymptotic cohomology class, see Definition 2.6. Given a solution $(V, \Phi)$ to the $G_2$-monopole equation (1.1) with structure group $G = SU(2)$ on an asymptotically conical manifold, having nonzero intermediate energy and bounded $r^2|F_V^{14}|$, it follows from Main Theorem 2 that $V_\infty$ is reducible, since $V_\infty \Phi_\infty = 0$ and $\Phi_\infty \neq 0$. Then, $V_\infty$ reduces to a connection on a $U(1) \subseteq SU(2)$-bundle. Such a $U(1)$-bundle determines a complex line bundle $L$ through the standard representation and its first Chern class $\beta = c_1(L) \in H^2(\Sigma, \mathbb{Z})$ is called the monopole class of $(V, \Phi)$, see Definition 8.1. The energy formula which in the $G_2$-setting replaces the Bogomolny trick is the following.

**Corollary 1.2** ($G_2$-analogue of the Bogomolny trick). Let $(X^7, \varphi)$ be an irreducible asymptotically conical $G_2$-manifold, with asymptotic cohomology class $\Psi_\infty \in H^4(\Sigma, \mathbb{R})$. Suppose that $(V, \Phi)$ is a solution to the $G_2$-monopole equation (1.1) with structure group $G = SU(2)$, finite intermediate energy and bounded $r^2|F_V^{14}|$, such that it has mass $m$ and monopole class $\beta \in H^2(\Sigma, \mathbb{Z})$. Then

$$E^\psi(V, \Phi) = 4\pi m(\beta \cup \Psi_\infty, [\Sigma]).$$

For a more general formulation of this result which applies to solutions of the second order equations (2.1a) and (2.1b), see Theorem 8.3.

This energy formula can be applied in specific cases to find vanishing theorems for $G_2$-monopoles. An example of such an application is given in Corollary 8.4 stating that when $\Psi_\infty = 0$—which happens for instance when $H^2(\Sigma, \mathbb{Z})$ is trivial—then there are no $G_2$-monopoles $(V, \Phi)$ with $|F_V^{14}|$ quadratically decaying and finite nonzero intermediate energy.

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\footnote{For example the Bryant–Salamon metric on $\mathbb{R}^4 \times S^3$.}
Section 9 is entirely dedicated to proving decay properties of solutions to the linearized equation. It provides the foundations for the moduli theory developed in the subsequent sections. Namely, it establishes that all the $L^2$ solutions of the linearized equation decay at rate which is compatible with the appropriate Sobolev spaces to be used in Sections 10 to 12.

All $G_2$-monopoles with finite intermediate energy, bounded $r^2|F^1_{\nabla}|$, fixed monopole class and mass $m > 0$ determines (modulo gauge) the same same asymptotic pair, $(\nabla_\infty, \Phi_\infty)$, at infinity; see Remark 8.2. Moreover, for any such monopole $(\nabla, \Phi)$ we have that (in the right gauge) $|\Phi - \Phi_\infty|$ and $|r(\nabla - \nabla_\infty)|$ both decay along the conical end. Furthermore, the asymptotic configuration $(\nabla_\infty, \Phi_\infty)$ has a group of automorphisms isomorphic to $U(1)$ which we call $\Gamma_\infty$. Based on this, in Sections 10 to 12 we develop a moduli theory describing such monopoles. The main result of these sections is stated as Theorem 12.10 which we restate, informally, here as follows:

**Main Theorem 3.** Let $(\nabla, \Phi)$ be a $G_2$-monopole with finite intermediate energy and $|F^1_{\nabla}|$ decaying quadratically as before. Then, there are Banach manifolds $\tilde{\mathcal{B}}^p_{1,\alpha}$, $\mathcal{F}^p_{1,\alpha}$ defined in Section 12 and a $\Gamma_\infty$-invariant (nonlinear) Fredholm map

$$\text{mon} : \tilde{\mathcal{B}}^p_{1,\alpha} \to \mathcal{F}^p_{1,\alpha},$$

with the following significance. The moduli space of $G_2$-monopoles with finite intermediate energy, $|F^1|$ quadratically decaying, and the same monopole class and mass as $(\nabla, \Phi)$ is in bijection with

$$\text{mon}^{-1}(0)/\Gamma_\infty \subseteq \mathcal{B}^p_{1,\alpha}.$$ 

**Comparison with previous work.** In [36] the third author worked under much stronger hypothesis in order to deduce similar results to those of Main Theorem 2. In that reference it is already assumed that: (1) $(\nabla, \Phi)$ has finite mass, i.e. equation (1.3) holds; and (2) the connection $\nabla$ is asymptotic to a connection $\nabla_\infty$, pulled back from the link $\Sigma$ of the asymptotic cone, with $|\nabla - \nabla_\infty| \lesssim r^{-1-\varepsilon}$ for some $\varepsilon > 0$. Under these hypothesis, the existence of $\Phi_\infty$ as in (a) of Main Theorem 2 was then deducted. However, the proof of part (b) and of Corollary 1.2 in [36] uses the additional hypothesis that (3) $||\Phi_\infty, \nabla - \nabla_\infty|| \lesssim r^{-6-\varepsilon}$ for some $\varepsilon > 0$.

The moduli theory developed here is the same as that appearing in [36] which to date had not yet been published in a journal. The work done in the preceding chapters lays different foundations for the development of this moduli theory than that appearing in [36].
Organization. In Section 2 we fix some nomenclature and notations, and derive preliminary identities satisfied by $G_2$-monopoles. Most notably a Bochner–Weitzenböck formula for $\Delta |\nabla \Phi|^2$. Next, in Section 3, we derive very useful consequences of the previous identities via Moser iteration and $\varepsilon$-regularity results, under the hypothesis of finite intermediate energy and bounded curvature. These yield that $|\nabla \Phi|^2$ decays, is in $L^p$ for all $p \in [1, \infty]$, and in case $|F_V|$ decays, we get that $|\nabla^j F_V|$ and $|\nabla^{j+1} \Phi|$ decay for all $j \in \mathbb{N}$.

Section 4 is mainly concerned with a proof of our first main theorem, but in fact proves a considerably stronger result, stated as Theorem 4.1, and further partial refinements. The main tools here are the integrability and decay properties of the previous section, and classical results on harmonic function theory of complete manifolds with nonnegative Ricci curvature, including Green’s function asymptotics and Yau’s gradient inequality, all combined through a strategy inspired by the original work of Taubes in the classical 3-dimensional monopole equation in [23, Chapter IV].

In Section 5 we prove refined Bochner and Weitzenböck type formulas for finite mass monopoles away from the zero set of the Higgs field when the gauge group is $G = \text{SU}(2)$. Using decay hypothesis, we get in particular strong Bochner inequalities sufficiently far along the end of our irreducible $G_2$-manifold, cf. Corollary 5.4 and Lemma 5.5. We then restrict to the AC $G_2$-manifold case in Section 6. The first striking consequence of the Bochner inequalities, together with the maximum principle, is the exponential decay of the $\Phi$-transversal components of $F_V$ and $\nabla \Phi$ in this context, proved in Proposition 6.3. We then move to use a combination of the Agmon identity, Hardy’s inequality and Moser iteration in Sections 6.2 and 6.4 to get a sharp polynomial decay rate of $|\nabla \Phi|$, completing the proof of the first part of our second main result, restated as Theorem 6.1. Then, in Section 7 we use the previous results, together with Uhlenbeck compactness and related techniques to prove the convergence result of the second part of our second main result.

As an application of our second main result, in Section 8 we develop the $G_2$-analogue of the Bogomolny trick which results in the energy formula stated as Theorem 8.3.

We devote Section 9 to the study of the linearized $G_2$-monopole equation and using the same techniques of the previous sections we prove analogous decay results for its solutions.

Finally, in Sections 10 to 12 we develop the moduli theory for $G_2$-monopoles with finite intermediate energy, quadratically decaying curvature, fixed monopole class and mass. The first of these sections defines the relevant Sobolev norms and proves that the linearized monopole equation is Fredholm. The second settles some useful technical results such as multiplication maps which are needed in order to handle the nonlinearities of the monopole equation. Finally, in the third and last of these sections we topologize the
relevant moduli spaces using the Sobolev norms previously defined and prove that the monopole equation yields such a nonlinear Fredholm map. The main result is stated as Theorem 12.10.

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2. Preliminaries

2.1. Notation and conventions. In this article $n = 7$. We prefer to keep the $n$ explicit as this allows us to more easily read the use of several analytic results such as scaling, Moser iteration arguments, Hardy’s inequality etc. Keeping $n$ instead of 7 is also convenient for more easily compared with other monopole theories.

Throughout the text, unless otherwise stated, we assume that $(X^7, \varphi)$ is a complete, noncompact and irreducible $G_2$-manifold. Moreover, given a principal $G$-bundle $P$ over $X$, we always consider smooth configurations $(\nabla, \Phi)$ on $P \to X$. We assume $G$ to be a compact Lie group, and we fix some Ad-invariant metric on the Lie algebra $\mathfrak{g}$ of $G$, which in turn induces a metric on the adjoint bundle $\mathfrak{g}_P$. In particular, when $G = SU(2)$ we fix the metric on $\mathfrak{g}_P$ to be the one induced by the inner product $(a, b) \mapsto -2\text{tr}(ab)$ on $\mathfrak{g} = \mathfrak{su}(2)$.

We let $\Delta := d^*d$ be the Hodge–Laplacian operator on functions of $X$, and $\Delta_\nabla := d_\nabla d_\nabla + d_\nabla^* d_\nabla$ be the covariant Hodge–Laplacian, induced by $\nabla$, acting on $\Omega^k(X, \mathfrak{g}_P)$. We note that $\Delta_\nabla = d_\nabla^* d_\nabla$ and coincides with the rough Laplacian $\nabla^* \nabla$ on $\Omega^0(X, \mathfrak{g}_P)$.
For any $\alpha \in \Omega^k(X, g_P)$ and $\beta \in \Omega^l(X, g_P)$, we define (locally)
\[
[\alpha \wedge \beta] := \sum_{I, J} |I| = k, |J| = l \ [\alpha_I, \beta_J] dx^I \wedge dx^J \in \Omega^{k+l}(X, g_P).
\]
When $k$ or $l$ is zero (one of them is a "scalar"), then drop the wedge from the notation, that is, we write
\[
[\alpha \wedge \beta] = [\alpha, \beta].
\]
We denote by $c > 0$ a generic constant and we write $\alpha \lesssim \beta$ to mean that $\alpha \leq c\beta$.

2.2. Bounded geometry and Moser iteration. We say that $(X, g)$ has bounded geometry if its global injectivity radius, $\text{inj}(X, g) = \inf_{x \in X} \text{inj}_x(X, g)$, is positive (in particular this implies completeness), and the Riemann curvature tensor, together with all of its derivatives, is bounded, that is for each $j \in \mathbb{N}$, there is $c_j > 0$ such that $|\nabla^j \text{Riem}| \leq c_j$.

We now cite a standard Moser iteration type result in the exact manner we need it in this article.

**Proposition 2.1** (Moser iteration, cf. [48, Lemma 10]). Let $B_r(x) \subseteq (X^n, g)$ be a convex geodesic ball and $u : B_r(x) \to \mathbb{R}$ be a smooth nonnegative function satisfying $\Delta u \leq c_0 u$, for some constant $c_0 \geq 0$. Then, there is a constant $c > 0$ depending only on the geometry of $B_r(x)$ such that
\[
\sup_{y \in B_{r/2}(x)} u(y) \leq c \left( c_0^{n/2} r^{-n} \right) \int_{B_r(x)} u \text{vol}_X.
\]

If $(X, g)$ has bounded geometry then the constant $c$ above can be taken to be universal in a way that it does not depend on $x$. In fact, there is $r_0 \in (0, \text{inj}(X, g))$ such that for every $r \in (0, r_0)$, $x \in X$, and any smooth nonnegative function $u : X \to \mathbb{R}$ satisfying $\Delta u \leq c_0 u$ on all of $X$, then
\[
\sup_{y \in B_{r/2}(x)} u(y) \leq \left( c_0^{n/2} r^{-n} \right) \int_{B_r(x)} u \text{vol}_X.
\]

2.3. Asymptotically conical $G_2$-manifolds. Now we give some definitions and notations concerning AC $G_2$-manifolds.

**Definition 2.2.** Given a 6-manifold $\Sigma$, a pair of forms $(\omega, \Omega_1) \in \Omega^2 \oplus \Omega^3(\Sigma, \mathbb{R})$ determine a $\text{SU}(3)$-structure on $\Sigma$ if:

- The $\text{GL}(6, \mathbb{R})$ orbit of $\Omega_1$ is open, with stabilizer a covering of $\text{SL}(3, \mathbb{C})$;
The following compatibility relations hold
\[ \omega \wedge \Omega_1 = \omega \wedge \Omega_2 = 0, \quad \frac{\omega^3}{3!} = \frac{1}{4} \Omega_1 \wedge \Omega_2, \]

where \( \Omega_2 = J\Omega_1 \) and \( J \) denotes the almost complex structure determined by \( \Omega_1 \).

\( g_\Sigma = \omega(\cdot, J \cdot) \) determines a Riemannian metric on \( \Sigma \).

We let \( \Omega \) be the complex volume form on \( (\Sigma, g_\Sigma) \) such that \( \text{Re}(\Omega) = \Omega_1 \) and \( \text{Im}(\Omega) = \Omega_2 \).

Furthermore, if the forms \( (\omega, \Omega) \) satisfy
\[ d\Omega^2 = -2\omega^2 \quad \text{and} \quad d\omega = 3\Omega_1, \]
then \( (\Sigma, g_\Sigma) \) is said to be nearly Kähler.

**Lemma 2.3.** Suppose that \( \Sigma \) is endowed with an \( SU(3) \)-structure determined by \( (\omega, \Omega_1) \). Then the Riemannian cone \( (C(\Sigma) = (1, \infty)_r \times \Sigma, g_C = dr^2 + r^2 g_\Sigma) \) with the \( G_2 \)-structure
\[ q_C = r^2 dr \wedge \omega + r^3 \Omega_1, \quad \psi_C = r^4 \frac{\omega^2}{2} - r^3 dr \wedge \Omega_2, \]
is a \( G_2 \)-manifold if and only if \( (\Sigma^6, g_\Sigma) \) is nearly Kähler.

**Definition 2.4.** We say that a noncompact, complete, \( G_2 \)-manifold \( (X^7, \varphi) \) is asymptotically conical (AC) with rate \( \nu < 0 \) when there exists a compact subset \( K \subseteq X \), a closed nearly Kähler 6-manifold \( (\Sigma, g_\Sigma) \) and a diffeomorphism \( \Upsilon : C(\Sigma) \to X - K \) such that the cone metric \( g_C \) on \( C(\Sigma) \) and its Levi-Civita connection \( \nabla_C \) satisfy:
\[ \left| \nabla^j_C (\Upsilon^* g - g_C) \right|_{g_C} = O(r^{\nu-j}) \quad \text{as} \; r \to \infty, \quad \text{for all} \; j \in \mathbb{N}. \]
The connected components of \( X - K \) are called the ends of \( X \) and \( \Sigma \) is called the link of the asymptotic cone. By a slight abuse of notation we let \( r \) be any positive smooth extension of \( r \circ \Upsilon^{-1}|_{X-K} \) to \( X \) and call \( r \) a radius function. For each \( R > 0 \), we let \( B_R = \{ x \in X : r(x) \leq R \} \), which, for large enough \( R \), is a smooth manifold-with-boundary, with a fixed diffeomorphism type. We also let \( \Sigma_R = \partial B_R \), which is a closed Riemannian 6-manifold.

**Remark 2.5.** Notice that any AC \( G_2 \)-manifold has bounded geometry. Moreover, they have maximal (Euclidean) volume growth, i.e. \( \text{Vol}(B_r(x)) \geq r^n \) (see [44, Corollary 2.18]). Here the word “maximal” is used because Ricci-flatness (implied by \( G_2 \)-holonomy) together with Bishop’s absolute volume comparison theorem gives \( \text{Vol}(B_r(x)) \leq r^n \). In particular, AC \( G_2 \)-manifolds are nonparabolic; indeed, they satisfy equation (4.1) (see Section 4 for more details on nonparabolicity).
Given an asymptotically conical $G_2$-manifold $(X^7, \varphi)$ as in Definition 2.4, it has the property that along the conical end $|\Upsilon^* \psi - \psi_C|_{\psi_C} = O(r^\nu)$ with derivatives. As a consequence, along the conical end there is a 4-form $\eta$ with $|\eta| = O(r^\nu)$ such that $\psi = (\Upsilon^{-1})^* \psi_C + \eta$. Furthermore, as $\psi_C = -\frac{1}{4}d(r^4 \Omega_2)$ we find that the cohomology class in $H^4(\Sigma, \mathbb{R})$ determined by $\eta|_{\Sigma_R}$ and $\psi|_{\Sigma_R}$ agree, i.e.

$$[\eta|_{\Sigma_R}] = [\psi|_{\Sigma_R}].$$

By construction and the homotopy invariance, $[\psi|_{\Sigma_R}]$ is constant for sufficiently large $R$ and for convenience we now name the class in $H^4(\Sigma, \mathbb{R})$ which it represents.

**Definition 2.6.** In case of an AC $G_2$-manifold, as in Definition 2.4, a class $\Psi_\infty \in H^4(\Sigma, \mathbb{R})$ is said to be an asymptotic cohomology class if

$$\Psi_\infty := [\psi|_{\Sigma_R}],$$

for all sufficiently large $R$.

2.4. **A Bochner–Weitzenböck formula.** Here we derive some basic but crucial equations satisfied by $G_2$-monopoles.

**Lemma 2.7.** Let $(\nabla, \Phi)$ be any solution of the $G_2$-monopole equation (1.1) on $P \to X$. Then the pair $(\nabla, \Phi)$ satisfies

$$\Delta_\nabla \Phi = 0, \quad (2.1a)$$

$$d_\nabla^* F_\nabla = [\nabla \Phi, \Phi]. \quad (2.1b)$$

In particular, $\Delta_\nabla F_\nabla = [[F_\nabla, \Phi], \Phi] - [\nabla \Phi \wedge \nabla \Phi]$.

**Proof.** The first equation, $\Delta_\nabla \Phi = 0$ is immediate from applying $d_\nabla^*$ to the $G_2$-monopole equation (1.1) and using the Bianchi identity $d_\nabla F_\nabla = 0$ together with $d\psi = 0$. As for the second equation, we first use the fact that $3F_\nabla^7 = *(*(F_\nabla \wedge \psi) \wedge \psi)$ to compute

$$3d_\nabla^* F_\nabla^7 = *d_\nabla *^2 (\nabla \Phi \wedge \psi) = *([F_\nabla, \Phi] \wedge \psi) = [\nabla \Phi, \Phi].$$

Notice that $3F_\nabla^7 = F_\nabla + *(F_\nabla \wedge \varphi)$ and $3F_\nabla^{14} = 2F_\nabla - *(F_\nabla \wedge \varphi)$. Thus, using the fact that $\varphi$ is closed we find

$$d_\nabla^* F_\nabla = 3d_\nabla^* F_\nabla^7 = \frac{3}{2} d_\nabla^* F_\nabla^{14}.$$

The result follows from inserting this into the equation above.

**Lemma 2.8.** For any solution $(\nabla, \Phi)$ of the second order equations (2.1a) and (2.1b), we have

$$\nabla^* \nabla (\nabla \Phi) = [[\nabla \Phi, \Phi], \Phi] - 2 * [F_\nabla \wedge \nabla \Phi]. \quad (2.2)$$
In particular,
\[
\frac{1}{2} \Delta |\nabla \Phi|^2 + |\nabla^2 \Phi|^2 = \langle \nabla \Phi, \nabla^* \nabla (\nabla \Phi) \rangle \\
= -2 \langle \nabla \Phi, * [F_\nabla \wedge \nabla \Phi] \rangle - ||\Phi, \nabla \Phi||^2,
\]
which implies
\[
\frac{1}{2} \Delta |\nabla \Phi|^2 + |\nabla^2 \Phi|^2 + ||\Phi, \nabla \Phi||^2 \lesssim |F_\nabla||\nabla \Phi|^2.
\]

Proof. Using the Ricci-flatness and the Bochner–Weitzenböck formula, we have
\[
\nabla^* (\nabla \Phi) = \Delta \nabla \Phi - * [F_\nabla \wedge \nabla \Phi].
\]
Now, using the second order equations (2.1a) and (2.1b) and the Bianchi identity we compute
\[
\Delta \nabla \Phi = d^* [F_\nabla, \Phi] \\
= [d^* F_\nabla, \Phi] - * [F_\nabla \wedge \nabla \Phi] \\
= [\nabla^* (\nabla \Phi), \Phi] - * [F_\nabla \wedge \nabla \Phi].
\]
Putting equations (2.5) and (2.6) together implies equation (2.2). \(\square\)

2.5. Finite mass configurations. To finish this preliminary section, we introduce the precise definition of finite mass configurations and make a simple but useful remark. Recall that since \((X^7, \varphi)\) is a complete, noncompact and irreducible \(G_2\)-manifold, it follows from the Cheeger–Gromoll splitting theorem that \((X^7, \varphi)\) has only one end.

Definition 2.9. A configuration \((\nabla, \Phi)\) is said to have finite mass if \(|\Phi|\) converges uniformly to a constant \(m \in \mathbb{R}_+\) along the end, i.e. for any choice of reference point \(o \in X\) one has
\[
\lim_{\text{dist}(x, o) \to \infty} |\Phi(x)| = m.
\]
Then the constant \(m\) is called the mass of \((\nabla, \Phi)\).

Remark 2.10. If \((\nabla, \Phi)\) is a solution to the second order equations (2.1a) and (2.1b) then, in particular, \(\Delta \nabla \Phi = \nabla^* \nabla \Phi = 0\) and this implies that
\[
\frac{1}{2} \Delta |\Phi|^2 = \frac{1}{2} d^* d|\Phi|^2 \\
= d^* \langle \Phi, \Phi \rangle \\
= - d\langle *\nabla \Phi, \Phi \rangle \\
= \langle \nabla^* \nabla \Phi, \Phi \rangle - |\nabla \Phi|^2.
\]
\[ = -|\nabla \Phi|^2 \leq 0. \] (2.7)

Thus, the function \(|\Phi|^2\) is subharmonic. When \((\nabla, \Phi)\) also has finite mass \(m \in \mathbb{R}_+\), then by the maximum principle (cf. [23, Chapter VI, Proposition 3.3]) one has either \(|\Phi| \equiv m\) or \(|\Phi| < m\) everywhere on \(X\). Moreover, by the uniform convergence \(|\Phi| \to m\) along the end, one has that \(|\Phi| \geq \frac{m}{2}\) outside a sufficiently large geodesic ball.

3. Consequences of Moser iteration and \(\epsilon\)-regularity

In this section we deduce step by step the consequences that can be taken from the use of Moser iteration and \(\epsilon\)-regularity along the end of \(X\). The final result of the section which concentrates our conclusions and follows from the preceding work is Corollary 3.8.

We start with a simple consequence of Lemma 2.8 using Moser iteration.

**Lemma 3.1.** Let \((\nabla, \Phi)\) be a solution to the second order equations (2.1a) and (2.1b). Then for any \(x \in X\) and \(0 < r < \frac{1}{2} \text{inj}_x(X, g)\),

\[
\sup_{B_r^2(x)} |\nabla \Phi|^2 \leq \left( \left\| F\nabla \right\|_{L^{n/2}(B_r(x))}^{n/2} + r^{-n} \right) \int_{B_r(x)} |\nabla \Phi|^2 \text{vol}_X. \] (3.1)

**Proof.** This follows from a direct application of the inequality (2.4) in Lemma 2.8 with the Moser iteration result stated in Proposition 2.1. \(\square\)

**Corollary 3.2.** Let \((X, \varphi)\) be a complete, noncompact and irreducible \(G_2\)-manifold of bounded geometry. Let \((\nabla, \Phi)\) be a solution to the second order equations (2.1a) and (2.1b). If \(|F\nabla| \in L^\infty(X)\) and \(|\nabla \Phi|^2 \in L^1(X)\), then \(|\nabla \Phi|^2 \in L^\infty(X) \cap L^p(X)\) for all \(p \in [1, \infty)\) and decays uniformly to zero along the end.

**Proof.** Since \((X, \varphi)\) has bounded geometry, there is \(r_0 \in (0, \text{inj}(X, g))\) such that the inequality (3.1) of Lemma 3.1 holds for all \(x \in X\) and \(r \in (0, r_0]\). Given that \(|F\nabla| \in L^\infty(X)\) and \(|\nabla \Phi|^2 \in L^1(X)\) we find that

\[
\|\nabla \Phi\|^2_{L^\infty(X)} \leq \left( \left\| F\nabla \right\|_{L^{\infty}(X)}^{n/2} + r_0^{-n} \right) \int_X |\nabla \Phi|^2 \text{vol}_X < \infty,
\]

hence \(|\nabla \Phi|^2 \in L^\infty(X) \cap L^1(X) \subseteq L^p(X)\) for all \(p \geq 1\). Moreover, since \(|\nabla \Phi|^2 \in L^1(X)\), if \(x_i \to \infty\) then \(\int_{B_{r_0}(x_i)} |\nabla \Phi|^2 \text{vol}_X \to 0\) and thus by inequality (3.1) one gets \(|\nabla \Phi|^2(x_i) \to 0\). This shows that \(|\nabla \Phi|^2\) decays, completing the proof. \(\square\)

**Remark 3.3.** In fact, it is possible to use inequality (3.1) to obtain a (possibly rude) quantification of the \(|\nabla \Phi|^2\) decay, under the hypotheses of Corollary 3.2. For this, fix \(y \in X\) and take a sequence of points \(\{x_i\}_{i \in \mathbb{N}}\) placed along a geodesic ray emanating from \(y\) with \(\text{dist}(x_i, x_{i+1}) = r\). 
for some fixed \( r \in (0, r_0] \). Then, \( \text{dist}(x, y) = ir \to \infty \) and summing inequality inequality (3.1) centered at all points \( x_i \) we find

\[
\sum_{i=1}^{\infty} \sup_{B_{\frac{r}{2}}(x_i)} |\nabla \Phi|^2 \leq \left( \|F_{\nabla}\|_{L^\infty(X)}^{n/2} + r^{-n} \right) \sum_{i=1}^{\infty} \int_{B_r(x_i)} |\nabla \Phi|^2 \text{vol}_X \leq \left( \|F_{\nabla}\|_{L^\infty(X)}^{n/2} + r^{-n} \right) \int_X |\nabla \Phi|^2 \text{vol}_X.
\]

Hence, if \( |\nabla \Phi|^2 \in L^1(X) \) then the series in the left hand side must converge and so

\[
\lim_{i \to \infty} \left( \text{dist}(x_i, y) \sup_{B_{\frac{r}{2}}(x_i)} |\nabla \Phi|^2 \right) = 0.
\]

**Definition 3.4.** Let \( U \subseteq X \) be an open subset. When finite, we define the energy and the intermediate energy of a field configuration \((\nabla, \Phi)\) by the integrals over \( U \) of

\[
e = \frac{1}{2} |F_{\nabla}|^2 + \frac{1}{2} |\nabla \Phi|^2,
\]

\[
e_{\psi} = \frac{1}{2} |F_{\nabla} \wedge \psi|^2 + \frac{1}{2} |\nabla \Phi|^2,
\]

to which we refer as the energy density and intermediate energy density respectively.

Notice that in case the pair \((\nabla, \Phi)\) is a \( G_2 \)-monopole we have \( e_{\psi} = |\nabla \Phi|^2 \) and so the intermediate energy is simply the squared \( L^2(X) \)-norm of \( \nabla \Phi \). In general, it follows from linear algebra that

\[
|F_{\nabla} \wedge \psi|^2 = 3|F_{\nabla}^7|^2.
\]

We now cite the following \( \varepsilon \)-regularity result for the energy density \( e \).

**Proposition 3.5** (\( \varepsilon \)-regularity; cf. [1, Theorem B] and [43, Theorem 1.3]). Let \((X^n, g)\) be a complete oriented Riemannian \( n \)-manifold of bounded geometry, and let \( P \) be a \( G \)-bundle over \( X \) where \( G \) is a compact Lie group. Then there are constants \( \varepsilon_0 = \varepsilon_0(X, g, g) > 0 \) and \( r_0 = r_0(X, g) \in (0, \text{inj}(X, g)) \) with the following significance. Let \((\nabla, \Phi)\) be a solution to the second order equations (2.1a) and (2.1b) on \( P \to X \). If \( x \in X \) and \( 0 < r \leq r_0 \) are such that

\[
r^{-(n-4)} \int_{B_r(x)} e \text{vol}_X < \varepsilon_0,
\]

then

\[
\sup_{B_{\frac{r}{2}}(x)} \left( |\nabla^j F_{\nabla}|^2 + |\nabla^{j+1} \Phi|^2 \right) \leq_j r^{-n-2} \int_{B_r(x)} e \text{vol}_X, \quad \forall j \in \mathbb{N}.
\]  

(3.3)

**Sketch of proof.** The \( C^0 \)-bound from the \( j = 0 \) case of inequality (3.3) is a particular case of [1, Theorem B]. From this bound, for any fixed \( p > n/2 \), by possibly taking smaller \( r_0 \) and \( \varepsilon_0 \) one can make \( \|F_{\nabla}\|_{L^p(B_{\frac{r}{2}}(x))} \) to be smaller than Uhlenbeck’s constant given by
Thus we can find a Coulomb gauge over $B_r(x)$ in which the second order equations (2.1a) and (2.1b) become an elliptic system and standard elliptic estimates apply, implying the inequality (3.3) for all $j \in \mathbb{N}$. □

As a consequence of Proposition 3.5 and Corollary 3.2, we get:

**Corollary 3.6.** Let $(X, \varphi)$ be a complete, noncompact and irreducible $G_2$-manifold of bounded geometry. Let $(\nabla, \Phi)$ be a solution to the second order equations (2.1a) and (2.1b). Suppose that $|\nabla \Phi| \in L^1(X)$. Then one actually has that $|\nabla^j \Phi| \to 0$ uniformly along the end for all $j \in \mathbb{N}$.

**Proof.** By Corollary 3.2 and the decay hypothesis on the curvature we know that $e$ decays uniformly to zero at infinity. Therefore, if $(x_i)$ is a sequence escaping to infinity then $\int_{B_{r_0}(x_i)} e \text{vol}_X \to 0$, so that by inequality (3.3) one has $|\nabla^j \Phi(x_i)| \to 0$. □

Now we turn to the particular case of $G_2$-monopoles. We start with an $\varepsilon$-regularity result for $e_\psi$.

**Proposition 3.7** ($\varepsilon$-regularity for $e_\psi$). Let $(X^7, \varphi)$ be a complete $G_2$-manifold of bounded geometry and $P$ a principal $G$-bundle over $X$, where $G$ is a compact Lie group. Then there are constants $\varepsilon = \varepsilon(X, \varphi, g) > 0$ and $r_0 = r_0(X, \varphi) \in (0, \text{inj}(X, g_{\varphi}))$ with the following significance. Let $(\nabla, \Phi)$ satisfy the $G_2$-monopole equation (1.1). If $x \in X$ and $0 < r \leq r_0$ are such that

$$r^{-(n-4)} \int_{B_r(x)} e_\psi \text{vol}_X < \varepsilon,$$

then

$$\sup_{B_r(x)} e_\psi \lesssim \left( ||F_\nabla^{14}||_{L^\infty(B_r(x))} + r^{-4} \int_{B_r(x)} e_\psi \text{vol}_X + r^4 ||F_\nabla^{14}||_{L^\infty(B_r(x))}^2 \right).$$

**Proof.** First, we note that since $(\nabla, \Phi)$ is a $G_2$-monopole, we have

$$e_\psi = |\nabla \Phi|^2 = |F_\nabla \wedge \varphi|^2 = 3|F_\nabla|^2,$$

where the last equality is valid in general and follows from linear algebra. In particular, it follows from Lemma 2.8 that

$$\Delta e_\psi \lesssim ||F_\nabla^{14}||_{L^\infty(B_r(x))} e_\psi + e_\psi^{3/2} \text{ on } B_r(x).$$
Next, using a well-known almost monotonicity property for the normalized energy in dimensions greater than four, cf. [1, Theorem 2.1], we have
\[ s^{-(n-4)} \int_{B_r(x)} e^\psi \text{vol}_X \lesssim r^{-(n-4)} \int_{B_r(x)} e^\psi \text{vol}_X + r^4 \| F_{\nabla}^{14} \|_{L^\infty(B_r(x))}^2 \] for all \( s \in (0, r] \).

With these observations in mind, the result follows by a standard nonlinear mean value inequality for the Laplacian, which in turn is a consequence of Moser iteration via the so-called “Heinz trick”; e.g. apply [15, Theorem A.3] with the parameters \( d = 4, \tau(r) = r^4 \| F_{\nabla}^{14} \|_{L^\infty(B_r(x))}^2, a \approx 1, a_0 = 0 \) and \( a_1 = \| F_{\nabla}^{14} \|_{L^\infty(B_r(x))} \) (see also [17, Theorem 5.1]). □

Using the same reasoning that allowed us to deduce Corollary 3.6 from Proposition 3.5, we obtain the next result from Proposition 3.7 and Corollary 3.6.

**Corollary 3.8.** Let \((X, \varphi)\) be a complete, noncompact and irreducible G_2-manifold of bounded geometry. Suppose \((A, \Phi)\) is a solution to the G_2-monopole equation (1.1) with finite intermediate energy (1.2) (i.e. \(| \nabla \Phi |^2 \in L^1(X)\) and such that \(| F_{\nabla}^{14} | \in L^\infty(X)\). Then \(| F_{\nabla} | \in L^\infty(X)\) and the function
\[ e^\psi = | \nabla \Phi |^2 \in L^\infty(X) \cap L^p(X) \]
for all \( p \in [1, \infty) \) and decays uniformly to zero at infinity. If furthermore \(| F_{\nabla}^{14} | \) decays uniformly to zero at infinity, then \(| \nabla^j F_{\nabla} | \) and \(| \nabla^{j+1} \Phi | \) decay uniformly to zero at infinity for all \( j \in \mathbb{N} \).

**Proof.** Given that \( e^\psi \in L^1(X) \) and \(| F_{\nabla}^{14} | \in L^\infty(X)\), we can use Proposition 3.7 to conclude that \( e^\psi \in L^\infty(X) \) or, equivalently, that \(| F_{\nabla}^{14} | \in L^\infty(X)\) and therefore \(| F_{\nabla} | \in L^\infty(X)\). Thus, by Corollary 3.2, we get the first part of the desired result. For the second part, note that we already have that \(| F_{\nabla}^{14} | \) decays, so if \(| F_{\nabla}^{14} | \) decays then \(| F_{\nabla} | \) decays. Hence, Corollary 3.6 applies. □

### 4. Finite mass from finite intermediate energy

This section contains the proof of our first main result Main Theorem 1. In fact, we prove a more refined version of that result stated as Theorem 4.1; see also Corollary 4.4.

Let \((X^7, \varphi)\) be a complete, noncompact and irreducible G_2-manifold and let \( P \rightarrow X \) be a principal \( G \)-bundle, where \( G \) is a compact Lie group. Recall from Remark 2.10 that if \((\nabla, \Phi)\) is a finite mass solution to the second order equations (2.1a) and (2.1b) then \(| \Phi |^2 \) is a bounded subharmonic function on \( X \). Moreover, if \(| \Phi |\) is constant then \(| \nabla \Phi |^2 = -\frac{1}{2} \Delta | \Phi |^2 = 0 \). Now, since we are interested in irreducible solutions \((\nabla, \Phi)\), meaning those for which \( \nabla \Phi \neq 0 \), it follows that the existence of such (if any) forces \((X^7, g_{\varphi})\) to support a
nonconstant, upper bounded subharmonic function. It turns out that this last condition is equivalent to the Riemannian manifold \((X^7, g_φ)\) to be nonparabolic, i.e. to support a positive Green’s function (see e.g. [21, Theorem 5.1 (3)]).

Recall that a Green’s function (for the scalar Laplace operator \(Δ = d^*d\)) on \(X\) is a smooth function \(G(x, y)\) defined on \(X \times X \setminus \{(x, x) : x \in X\}\) which is symmetric in the two variables \(x\) and \(y\) and satisfies the following properties: for all \(f \in C^∞_c(X)\) and \(x \in X\) we have

\[
Δ_x \int_X G(x, y)f(y)\text{vol}_X(y) = f(x) \quad \text{and} \quad \int_X G(x, y)Δf(y)\text{vol}_X(y) = f(x).
\]

The existence of a Green’s function on a complete Riemannian manifold was first proved by Malgrange [33] via a nonconstructive argument, and later Li–Tam [29] gave a constructive proof (by a compact exhaustion method) which is particularly important to understand the behavior of such a function. Some manifolds do not admit a positive Green’s function (e.g. \(\mathbb{R}^2\)), while others may admit them (e.g. \(\mathbb{R}^m\) for \(m \geq 3\)). Thus, such an existence property distinguishes the function theory of complete noncompact manifolds.

When the manifold \(X^n\) is nonparabolic (i.e. admits a positive Green’s function), Li–Tam’s construction produces precisely the unique minimal positive Green’s function \(G(x, y) > 0\) and one can read off the following properties of \(G(x, y)\): for all \(x \neq y\) in \(X\),

(i) \(G(x, y) \sim \text{dist}(x, y)^{2-n}\) as \(\text{dist}(x, y) \to 0\) (i.e. the singularity along the diagonal is of the same order as that in \(\mathbb{R}^n\));

(ii) \(G(x, \cdot)\) is harmonic away from \(x\) (in fact, \(G(x, y)\) is superharmonic on \(X\) if we allow \(+\infty\) as a value of the function on the diagonal);

(iii) \(\sup_{X \setminus B_r(x)} G(x, \cdot) = \sup_{y \in \partial B_r(x)} G(x, y) < \infty\) for all \(r > 0\).

In particular, from (i) and (iii) we have that \(G(x, \cdot) \in L^q_{\text{loc}}(X)\) for any \(q < \frac{n}{n-2}\).

Now, it was proved by Varopoulos [45] that a complete Riemannian manifold \(X\) with nonnegative Ricci curvature is nonparabolic if and only if for some (therefore all) \(x \in X\) one has

\[
\int_1^∞ \frac{t}{\text{Vol}(B_t(x))} dt < ∞.
\]  

(4.1)

In fact, if this is the case, Li–Yau [31] proved that, for all \(x \neq y\) in \(X\), the minimal positive Green’s function \(G(x, y)\) on \(X\) must satisfy

\[
C^{-1} \int_{\text{dist}(x, y)}^∞ \frac{t}{\text{Vol}(B_t(x))} dt \leq G(x, y) \leq C \int_{\text{dist}(x, y)}^∞ \frac{t}{\text{Vol}(B_t(x))} dt,
\]  

(4.2)

for some constant \(C = C(n) > 0\) depending only on the dimension of \(X\). In particular, in this case we have

\[
G(x, y) \to 0 \quad \text{as} \quad \text{dist}(x, y) \to ∞.
\]
We also note that the difference between another positive Green’s function and \( G(x,y) \) must be a positive harmonic function. Now, it is a consequence of Yau’s work in [49] that a complete manifold with nonnegative Ricci curvature does not admit any nonconstant nonnegative harmonic functions. Thus, \( G(x,y) \) must be the unique positive Green’s function up to an additive constant.

After recalling the above facts, we are now in position to settle the main consequence of all these preliminary observations. Namely, we find a large class of manifolds for which solutions of equations (2.1a) and (2.1b) with \( \nabla \Phi \in L^2 \) satisfying mild bounded curvature assumptions have finite mass. We state here the detailed version of our main

Main Theorem 1. Its proof is inspired by Taubes’ original work on the standard 3-dimensional Bogomolny equation in [23, Theorem IV.10.3].

**Theorem 4.1.** Let \((X^7, g_\varphi)\) be a complete, noncompact and irreducible \(G_2\)-manifold of bounded geometry, which furthermore is nonparabolic. Let \( G(x, y) \) be the minimal positive Green’s function of the scalar Laplacian on \((X, g_\varphi)\). Let \((\nabla, \Phi)\) be a solution to the second order equations (2.1a) and (2.1b) with \( |\nabla\Phi|^2 \in L^1(X) \). Suppose either that \( |F_\nabla| \in L^\infty(X) \) or that \((\nabla, \Phi)\) is a \(G_2\)-monopole such that \( |F_\nabla^{14}| \in L^\infty(X) \). Finally, let

\[
W(x) = 2 \int_X G(x, \cdot)|\nabla\Phi|^2 \text{vol}_X. \tag{4.3}
\]

Then the function \( w : X \to \mathbb{R}_+ \) defined by equation (4.3) is the unique (smooth) solution to \( \Delta w = 2|\nabla\Phi|^2 \) which decays uniformly to zero at infinity, and there is a constant \( m \geq 0 \) such that

\[
w = m^2 - |\Phi|^2.
\]

In particular, \((\nabla, \Phi)\) has finite mass \( m \).

**Proof.** Since \((X^7, g_\varphi)\) is a complete nonparabolic manifold and \( |\nabla\Phi|^2 \) is a smooth, nonnegative and integrable function, it follows from the result in [35, (proof of) Lemma 2.3], together with elliptic regularity, that \( w \) as defined by equation (4.3) is a smooth solution to the Poisson equation \( \Delta w = 2|\nabla\Phi|^2 \).

We now show that \( w \) decays uniformly to zero at infinity, and therefore is the unique such solution\(^6\). Fix any reference point \( o \in X \). Let \( R, \rho > 0 \) and suppose that \( x \in X \setminus B_{R+\rho}(o) \). Note that, by triangle’s inequality, for any \( y \in B_\rho(x) \) we have \( \text{dist}(o, y) > R \). Thus, setting \( q := \frac{R-\rho}{n-2} \), separating the region of integration into two and using Hölder’s inequality we

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\(^6\)If \( \tilde{w} \) is another smooth solution to \( \Delta \tilde{w} = 2|\nabla\Phi|^2 \) decaying at infinity, then both \( \pm(w - \tilde{w}) \) are harmonic functions which decay at infinity, so that by the maximum principle we must have \( w = \tilde{w} \).
have
\[
0 \leq \frac{1}{2} w(x) = \left( \int_{B_{\rho}(x)} + \int_{X \setminus B_{\rho}(x)} \right) G(x, y) |\nabla \Phi(y)|^2 \text{vol}_X(y)
\leq \|G(x, \cdot)\|_{L^q(B_{\rho}(x))}\|\nabla \Phi\|_{L^2(X \setminus B_{\rho}(o))}^2 + \sup_{y \in X \setminus B_{\rho}(x)} G(x, y) \|\nabla \Phi\|_{L^2(X)}^2.
\]

(4.4)

Here we recall that, by either of the cases proven in Corollaries 3.2 and 3.8, from the assumptions on \( (\nabla, \Phi) \) we have that \( \nabla \Phi \in L^p(X) \) for all \( p \in [2, \infty] \). Moreover, from the discussion preceding the theorem, note that \( G(x, \cdot) \in L^q_{\text{loc}}(X) \) and, since \( (X^7, g_{\varphi}) \) is Ricci-flat, \( G(x, y) \to 0 \) as \( \text{dist}(x, y) \to \infty \). Hence, given \( \varepsilon > 0 \), we can choose \( \rho \gg 1 \) so that the last term in the right-hand side of equation (4.4) is less than \( \varepsilon/4 \), and then we can choose \( R \gg 1 \) so that the first term in the right-hand side of equation (4.4) is less than \( \varepsilon/4 \). This gives \( r := R + \rho > 0 \) such that \( \text{dist}(o, x) > r \) implies \( w(x) < \varepsilon \), showing that \( w \) decays uniformly to zero as we wanted.

Finally, since \( \Delta |\Phi|^2 = -2|\nabla \Phi|^2 \) by equation (2.7), we conclude that \( m^2 := |\Phi|^2 + w \) is a smooth, nonnegative harmonic function on \( X \). Since \( (X^7, g_{\varphi}) \) is Ricci-flat it follows that \( m^2 \) must be a constant (cf. [49]). This completes the proof. \( \square \)

**Remark 4.2.** Note from the above proof that the assumptions of bounded geometry on \( (X^7, \varphi) \) and bounded curvature on \( \nabla \) are made in Theorem 4.1 just to ensure, via Corollaries 3.2 and 3.8, that \( \nabla \Phi \in L^2(X \setminus B_{R}(o)) \).

An interesting easy consequence of Theorem 4.1 is the following.

**Corollary 4.3.** Under the hypothesis of Theorem 4.1, the following inequality holds on \( X \):
\[
|\nabla \Phi|^2 \leq \|F_{\nabla\Phi}\|_{L^\infty(X)} (m^2 - |\Phi|^2).
\]

**Proof.** By Lemma 2.8, we know that there is \( c > 0 \) such that
\[
\Delta |\nabla \Phi|^2 \leq 2c \|F_{\nabla\Phi}\|_{L^\infty(X)} |\nabla \Phi|^2.
\]

On the other hand, we know that \( w = m^2 - |\Phi|^2 \) is a nonnegative function such that \( \Delta w = 2|\nabla \Phi|^2 \). Thus,
\[
\Delta (c \|F_{\nabla\Phi}\|_{L^\infty}w - |\nabla \Phi|^2) \geq 0.
\]

Moreover, by Corollaries 3.2 and 3.8 and Theorem 4.1 we know that both \( |\nabla \Phi| \) and \( w \) decay. Therefore, the desired conclusion follows by the maximum principle. \( \square \)

Now we recall that on a complete, noncompact Riemannian manifold with nonnegative Ricci curvature, an application of the Bishop–Gromov volume comparison theorem and a clever argument by Yau [50] shows that for any \( x \in X \) and \( r > 2 \), the volume of the radius
$r$ ball centered at $x \in X$ satisfies
\[ \text{Vol}(B_1(x))r \leq \text{Vol}(B_r(x)) \leq r^n. \]

We now combine Theorem 4.1 with a polynomial volume growth assumption on $(X^7, g_\varphi)$ to obtain a slight refinement of the asymptotic behavior of $|\Phi|$. For the rest of this section, suppose that $(X^7, \varphi)$ is a complete, noncompact and irreducible $G_2$-manifold satisfying the following: there is $0 \leq k < n - 2$ such that for any $x \in X$ and $r \gg 1$ we have
\[ \text{Vol}(B_r(x)) \sim r^{n-k}. \]

Since $n - k > 2$, it follows that the equation (4.1) holds and therefore $(X^7, g_\varphi)$ is non-parabolic. Furthermore, using equation (4.2) we get that the minimal positive Green’s function satisfies the asymptotic estimate
\[ G(x, y) \sim \text{dist}(x, y)^{-(n-k-2)} \text{ when } \text{dist}(x, y) \gg 1. \]

Another important observation of our volume growth assumption is the following. Let $x \in X$, $\rho$ be the radial coordinate on $T_xX$ and $\lambda : T_xX \to \mathbb{R}$ the function so that
\[ \exp_x^*(\text{vol}_X) = \lambda \, d\rho \wedge \text{vol}_{S^{n-1}}, \]
where $n = 7$. Then the Laplacian comparison theorem, see [47, Proposition 20.7], states that
\[ \partial_\rho \left( \rho^{-(n-1)} \lambda \right) \leq 0, \]
away from the cut locus. As $\rho^{-(n-1)} \lambda$ converges to a constant as $\rho \to 0$ we find that for all $\rho > 0$ we have $\lambda \leq \rho^{n-1}$. Furthermore, from the Bishop–Gromov volume comparison and the Ricci-flatness together with our volume growth assumption we have that $\lambda \sim \rho^{(n-1)-k}$ for $\rho \gg 1$.

Thus, under the above hypotheses on $(X^7, \varphi)$, we have the following consequences.

**Corollary 4.4.** Let $(\nabla, \Phi)$ be a solution to the second order equations (2.1a) and (2.1b) with $|\nabla \Phi|^2 \in L^1(X)$ and suppose either that $|F_\nabla| \in L^\infty(X)$ or that $(\nabla, \Phi)$ is a $G_2$-monopole such that $|F_\nabla^{14}| \in L^\infty(X)$. Then, for any $o \in X$ there is $c > 0$ and $R_0 \gg 1$ such that if $\text{dist}(x, o) \geq R_0$
\[ |\Phi(x)|^2 \leq m^2 - \frac{c}{\text{dist}(o, x)^{n-k-2}}. \]

**Proof.** It follows from Theorem 4.1 that $w = m^2 - |\Phi|^2$ decays to zero, and as $\Delta w = 2|\nabla \Phi|^2$, it is superharmonic. Now, let $G(o, \cdot)$ be the decaying Green’s function with a pole at $o$. As $w$ is bounded, for any $R > 0$ there is $\epsilon > 0$ such that $w|_{\partial B_R(o)} > \epsilon G|_{\partial B_R(o)}$, and as both of
these decay to zero at infinite we find that
\[ \sup_{x \in X - B_R(o)} w > \sup_{x \in X - B_R(o)} e \in G. \]
Thus, recalling that for dist(x,o) \( \gg 1 \), the Green’s function is of order dist(x,o)\(^{(n-k-2)}\), we find from rearranging the above inequality that
\[ |\Phi(x)|^2 \leq m^2 - \frac{c}{\text{dist}(x,o)^n} \]
for some constant \( c > 0 \).

We finish this section by proving a simple result which constrains the asymptotic behavior of \( \nabla \Phi \) (cf. Remark 6.2).

**Lemma 4.5.** Let \((\nabla, \Phi)\) be a solution to the second order equations (2.1a) and (2.1b) with \( 0 \neq |\nabla \Phi|^2 \in L^1(X) \), and \( r : X \to \mathbb{R} \) be a smooth positive radial function, meaning that for \( r \gg 1 \) we have \( C^{-1} \text{dist}(o,\cdot) \leq r(\cdot) \leq C \text{dist}(o,\cdot) \) for some reference point \( o \in X \). Then, there is a sequence of points \( \{x_i\}_{i \in \mathbb{N}} \) with \( r(x_i) \to \infty \) such that if it exists, then
\[ \lim_{i \to \infty} r(x_i)^{n-k-1}(|\partial_r|\Phi|^2|(x_i)|) > 0; \]
in other words \( |\partial_r|\Phi|^2| = O(r^(-(n-k-1))). \)

**Proof.** We prove the contrapositive. Suppose that \( \lim_{r \to \infty} r^{n-k-1}|\partial_r|\Phi|^2| = 0 \), then
\[ \|\nabla \Phi\|^2_{L^2(X)} = \lim_{R \to \infty} \int_{B_R} |\nabla \Phi|^2 \text{vol}_X \]
\[ = \lim_{R \to \infty} \int_{\Sigma_R} \langle \Phi, \ast \nabla \Phi \rangle \]
\[ = \lim_{R \to \infty} \frac{1}{2} \int_{\Sigma_R} \partial_r|\Phi|^2 \text{vol}_{\Sigma_R} \]
\[ \leq \lim_{R \to \infty} \left( R^{n-k-1} \sup_{\Sigma_R} |\partial_r|\Phi|^2| \right). \]
Thus, we find that \( \nabla \Phi = 0. \)

5. **Bochner–Weitzenböck formulas along the end**

In the Bochner–Weitzenböck formulas presented below we assume that the gauge group \( G = SU(2) \) and that we are away from the zeros of \( \Phi \). By Remark 2.10, if \((\nabla, \Phi)\) is a finite
mass solution to the second order equations (2.1a) and (2.1b), then this condition is met sufficiently far out along the end of our complete, noncompact, irreducible $G_2$-manifold $(X^7, \varphi)$.

Whenever $\Phi(x) \neq 0$ for some $x \in X$, we can decompose $g_P = g_P^\parallel \oplus g_P^\perp$ near $x$, by setting
\[ g_P^\parallel = \ker(\text{ad}_{\Phi(x)} : g_P \to g_P), \]
and $g_P^\perp$ its orthogonal complement. Clearly, when $G = SU(2)$ then in fact $g_P^\parallel = \langle \Phi \rangle$ is the real vector subbundle of $g_P$ whose fiber at a point $y$ (near $x$) is the 1-dimensional vector subspace of $(g_P)_y$ generated by $\Phi(y) \neq 0$. In what follows we split any section $\chi$ of $g_P$ defined around $x$ as $\chi = \chi^\parallel + \chi^\perp$, and we note that with the particular choice of metric on $g_P$ induced by $(a, b) \mapsto -2\text{tr}(ab)$ it holds $\|\Phi, \chi\| \geq \|\Phi\|\chi^\perp$. (Of course, for other choices of normalization of the negative of the Cartan–Killing form of $g = \mathfrak{su}(2)$ the inequality holds up to a constant.)

We start by applying this decomposition to refine the standard Bochner–Weitzenböck inequality in Lemma 2.8.

**Lemma 5.1.** Let $(\nabla, \Phi)$ be a solution to the second order equations (2.1a) and (2.1b). Then
\[
\frac{1}{2} \Delta |\nabla \Phi|^2 + |\nabla^2 \Phi|^2 + \|\Phi, \nabla \Phi\|^2 \leq \|(F_\nabla)^\perp\|\|(\nabla \Phi)^\parallel\| + \|(F_\nabla)^\parallel\| \|\nabla \Phi\|^2. \tag{5.1}
\]
If, moreover, $(\nabla, \Phi)$ is a solution to the $G_2$-monopole equation (1.1) then
\[
\frac{1}{2} \Delta |\nabla \Phi|^2 + |\nabla^2 \Phi|^2 + \|\Phi, \nabla \Phi\|^2 \leq \|(F_\nabla^{14})^\perp\|\|(\nabla \Phi)^\parallel\| + \|(F_\nabla^{14})^\parallel\| \|\nabla \Phi\|^2 \tag{5.2}
\]

**Proof.** Recall equation (2.3) from Lemma 2.8. Splitting $\nabla \Phi = (\nabla \Phi)^\parallel + (\nabla \Phi)^\perp$ and similarly for $F_\nabla$ we find
\[
\langle \nabla \Phi, *[(F_\nabla)^\perp \wedge (\nabla \Phi)^\perp] \rangle = \langle \nabla \Phi, *[(F_\nabla)^\perp \wedge (\nabla \Phi)^\perp] \rangle
\]
\[+ \langle \nabla \Phi, *[(F_\nabla)^\parallel \wedge (\nabla \Phi)^\parallel] \rangle + \langle \nabla \Phi, *[(F_\nabla)^\parallel \wedge (\nabla \Phi)^\perp] \rangle
\]
\[= 2\langle (\nabla \Phi)^\parallel, *[(F_\nabla)^\parallel \wedge (\nabla \Phi)^\parallel] \rangle + \langle (\nabla \Phi)^\parallel, *[(F_\nabla)^\parallel \wedge (\nabla \Phi)^\perp] \rangle.
\]

Thus we get
\[
|\langle \nabla \Phi, *[(F_\nabla)^\perp \wedge (\nabla \Phi)^\perp] \rangle| \leq \|(F_\nabla)^\parallel\|\|(\nabla \Phi)^\parallel\| + \|(F_\nabla)^\parallel\|\|\nabla \Phi\|^2,
\]
thus inserting into equation (2.3) yields inequality (5.1).

As for the case when $(\nabla, \Phi)$ satisfies the $G_2$-monopole equation (1.1), note that we have $3 \ast F_\nabla^2 = \nabla \Phi \wedge \psi$; in particular, $\|(F_\nabla)^\parallel\| \leq \|(\nabla \Phi)^\parallel\|$ and $\|(F_\nabla)^\perp\| \leq \|(\nabla \Phi)^\perp\|$. Using these,
the orthogonal decomposition $F_\nabla = F_\nabla^7 + F_\nabla^{14}$ and inserting into inequality (5.1), we get inequality (5.2).

Next we compute Bochner type formulas for both $||\Phi, \nabla \Phi ||^2$ and $||F_\nabla, \Phi ||^2$.

**Lemma 5.2.** Let $(\nabla, \Phi)$ be a solution to the second order equations (2.1a) and (2.1b). Then,

$$\frac{1}{2} \Delta ||\Phi, \nabla \Phi ||^2 + |\nabla [\Phi, \nabla \Phi] |^2 + |\Phi |^2 ||\nabla \Phi ||^2 \leq (F_\nabla)^{||}[\Phi, \nabla \Phi ] || + ||(\nabla \Phi)|| |[\Phi, \nabla \Phi]||.$$

**Proof.** We start by computing each term in $\Delta_\nabla [\Phi, \nabla \Phi] = d_\nabla d_\nabla^* [\Phi, \nabla \Phi] + d_\nabla^* d_\nabla [\Phi, \nabla \Phi]$. First we get

$$d_\nabla^*[\Phi, \nabla \Phi] = -d_\nabla^*([\nabla \Phi \wedge * \nabla \Phi] + [\Phi, d_\nabla \Phi]) = 0,$$

as $[\nabla \Phi \wedge * \nabla \Phi] = 0$ and $d_\nabla \Phi \wedge d_\nabla \Phi = 0$ are both zero. We also have

$$d_\nabla d_\nabla^*[\Phi, \nabla \Phi] = d_\nabla^*([\nabla \Phi \wedge \nabla \Phi] + [\Phi, d_\nabla \Phi])$$

$$= d_\nabla^*([\nabla \Phi \wedge \nabla \Phi] - *d_\nabla [\Phi, \Phi, F_\nabla])$$

$$= d_\nabla^*([\nabla \Phi \wedge \nabla \Phi] - *([\nabla \Phi \wedge [\Phi, F_\nabla]) - *([\Phi, [\nabla \Phi \wedge F_\nabla]) - [\Phi, [\Phi, d_\nabla F_\nabla])$$

Putting these two together we find that

$$\Delta_\nabla [\Phi, \nabla \Phi] = d_\nabla^*[\nabla \Phi \wedge \nabla \Phi] + *([\nabla \Phi \wedge [*F_\nabla, \Phi]) + [\Phi, \Delta_\nabla \nabla \Phi]$$

Now, a short computation shows that $d_\nabla^*[\nabla \Phi \wedge \nabla \Phi] = 2[\nabla_i \nabla_j \Phi, \nabla_i \Phi] e^i$ and so

$$\langle [\Phi, \nabla \Phi], d_\nabla^*[\nabla \Phi \wedge \nabla \Phi] \rangle = 2\langle [\Phi, \nabla_i \Phi], [\nabla_i \nabla_j \Phi, \nabla_i \Phi] \rangle$$

$$= 2\langle [\nabla_i \Phi, [\Phi, \nabla_i \Phi]], \nabla_i \nabla_j \Phi \rangle$$

$$= -2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]], \nabla_i \nabla_j \Phi \rangle - 2\langle [\Phi, [\nabla_i \Phi, \nabla_j \Phi]], \nabla_i \nabla_j \Phi \rangle$$

$$= -2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle - 2\langle [\Phi, [\nabla_i \Phi, \nabla_j \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle$$

$$+ 2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]], \nabla_i \nabla_j \Phi \rangle$$

$$= -2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle - 2\langle [\Phi, [\nabla_i \Phi, \nabla_j \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle$$

$$- 2\langle [\nabla_i \Phi, [\Phi, \nabla_j \Phi]], \nabla_i \nabla_j \Phi \rangle$$

as $[\nabla_i \Phi, \nabla_j \Phi]$ is anti-symmetric in $i, j$. Thus,

$$2\langle [\Phi, \nabla \Phi], d_\nabla^*[\nabla \Phi \wedge \nabla \Phi] \rangle = 4\langle [\nabla_i \Phi, [\Phi, \nabla_i \Phi]], \nabla_i \nabla_j \Phi \rangle$$

$$= -2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle - 2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle$$

$$= -2\langle [\nabla_i \Phi, [\nabla_j \Phi, \Phi]] + [\Phi, [\nabla_i \Phi, \nabla_j \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle$$

$$= 2\langle [\nabla_i \Phi, [\Phi, \nabla_j \Phi]], [\nabla_i \Phi, \nabla_j \Phi] \rangle.$$
We also have
\[ \nabla^* \nabla [\Phi, \nabla \Phi] = \Delta V[\Phi, \nabla \Phi] = \Phi(\nabla [\Phi, \Phi] - \star [\nabla^* \nabla [\Phi, \Phi]]) \\
= [\Phi, \Delta V \nabla \Phi] + \nabla^* [\nabla \Phi \wedge \nabla \Phi] + \star [\nabla \Phi \wedge [\Phi, \nabla \Phi]] - \star [\nabla^* \nabla \Phi \wedge [\Phi, \nabla \Phi]], \]
and using the second order equations again we find \(\Delta V \nabla \Phi = [[\nabla \Phi, \Phi], \Phi] - \star [\nabla^* \nabla \Phi \wedge [\Phi, \nabla \Phi]]\)
and so
\[ \nabla^* \nabla [\Phi, \nabla \Phi] = [\Phi, [[\nabla \Phi, \Phi], \Phi]] + \nabla^* [\nabla \Phi \wedge \nabla \Phi] \\
+ \star [\nabla \Phi \wedge [\Phi, \nabla \Phi]] - \star [\nabla^* \nabla \Phi \wedge [\Phi, \nabla \Phi]] - \star [\nabla \Phi \wedge [\Phi, \nabla \Phi]] - \star [\nabla \Phi \wedge [\Phi, \nabla \Phi]] \\
= [\Phi, [[\nabla \Phi, \Phi], \Phi]] + \nabla^* [\nabla \Phi \wedge \nabla \Phi] - 2[\Phi, \star [\nabla^* \nabla \Phi \wedge [\Phi, \nabla \Phi]]], \]
where in the last inequality we used the (graded) Jacobi identity. Thus,
\[ \Delta \frac{[\Phi, \nabla \Phi]^2}{2} = \langle [\Phi, \nabla \Phi], \nabla^* \nabla [\Phi, \nabla \Phi] \rangle - |\nabla [\Phi, \nabla \Phi]|^2 \\
= \langle [\Phi, \nabla \Phi], [\Phi, [[\nabla \Phi, \Phi], \Phi]] \rangle - 2\langle [\Phi, \nabla \Phi], [\Phi, \star [\nabla^* \nabla \Phi \wedge [\Phi, \nabla \Phi]]] \rangle \\
+ \langle [\Phi, \nabla \Phi], \nabla^* [\nabla \Phi \wedge \nabla \Phi] \rangle - |\nabla [\Phi, \nabla \Phi]|^2 \\
= -2\langle [\nabla \Phi, \Phi], \Phi \rangle^2 + 2\langle [\nabla \Phi, \Phi], [\nabla \Phi, [\Phi, \nabla \Phi]], [\nabla \Phi, \Phi] \rangle - 2\langle [\Phi, \nabla \Phi], [\Phi, \star [\nabla^* \nabla \Phi \wedge [\Phi, \nabla \Phi]]] \rangle \\
- |\nabla [\Phi, \nabla \Phi]|^2. \]
Since \([[[\nabla \Phi, \Phi], \Phi]]^2 \geq |\Phi|^2 [[\Phi, \nabla \Phi]]^2\), we conclude that
\[ \Delta \frac{[\Phi, \nabla \Phi]^2}{2} + |\nabla [\Phi, \nabla \Phi]|^2 + |\Phi|^2 [[\Phi, \nabla \Phi]]^2 \leq \langle (\nabla \Phi)^\perp, [\nabla^* \nabla \Phi] \rangle \langle [\Phi, \nabla \Phi] \rangle \\
+ |\Phi| \langle (\nabla \Phi)^\perp \langle (\nabla \Phi)^\perp \rangle + |\nabla \Phi |^2 \rangle \langle [\Phi, \nabla \Phi] \rangle, \]
with the stated inequality following from noticing that \(|\Phi||(\nabla \Phi)^\perp| \leq \langle [\nabla \Phi, \Phi] \rangle \) and similarly for \((\nabla \Phi)^\perp\). \(\square\)

**Lemma 5.3.** Let \((\nabla, \Phi)\) be a solution to the second order equations (2.1a) and (2.1b). Then,
\[ \frac{1}{2} \Delta [F_{\nabla}, \Phi]^2 + |\nabla [F_{\nabla}, \Phi]|^2 + |\Phi|^2 [[F_{\nabla}, \Phi]]^2 \leq \langle (\nabla \Phi)^\perp, [\nabla^* \nabla [F_{\nabla}, \Phi] \rangle \langle [\Phi, F_{\nabla}] \rangle \\
+ \langle (\nabla \Phi)^\perp \langle |\Phi|^2 |\nabla \Phi |^2 \rangle + |\nabla \Phi |^2 \rangle \langle [\Phi, F_{\nabla}] \rangle \langle [\Phi, F_{\nabla}] \rangle \\
+ |\Phi|^2 |\nabla [F_{\nabla}, \Phi] |\langle (\nabla \Phi)^\perp \rangle \langle [\Phi, F_{\nabla}] \rangle \langle [\Phi, F_{\nabla}] \rangle. \]

**Proof.** We start by computing
\[ \frac{1}{2} \Delta [F_{\nabla}, \Phi]^2 = \langle [F_{\nabla}, \Phi], \nabla^* \nabla [F_{\nabla}, \Phi] \rangle - |\nabla [F_{\nabla}, \Phi]|^2. \tag{5.3} \]
We work out the first term above using the Leibniz rule, the Weiizenböck formula for $F_\Omega$ given by $\nabla^* \nabla F_\Omega = \Delta_\Omega F_\Omega + \text{Riem}(F_\Omega) + (F_\Omega \cdot F_\Omega)$, and the second order equations to get

$$\nabla^* \nabla [F_\Omega, \Phi] = -\nabla_i \nabla_i [F, \Phi]$$

$$= [-\nabla_i \nabla_i F_\Omega, \Phi] - 2[\nabla_i F_\Omega, \nabla_i \Phi] - [F, \nabla_i \nabla_i \Phi]$$

$$= [\Delta_\Omega F_\Omega, \Phi] + \text{Riem}([F_\Omega, \Phi]) + (F_\Omega \cdot [F_\Omega, \Phi]) - 2[\nabla_i F_\Omega, \nabla_i \Phi] + [F_\Omega, \Delta_\Omega \Phi]$$

$$= [[[F_\Omega, \Phi], \Phi], \Phi] - [[\nabla \Phi \wedge \nabla \Phi], \Phi] + \text{Riem}([F_\Omega, \Phi])$$

$$+ (F_\Omega \cdot [F_\Omega, \Phi]) - 2[\nabla_i F_\Omega, \nabla_i \Phi],$$

so that the first term on the right-hand side of (5.3) is

$$\langle [F_\Omega, \Phi], \nabla^* \nabla [F_\Omega, \Phi] \rangle = -|\Phi|^2 |[F_\Omega, \Phi]|^2 - \langle [F_\Omega, \Phi], [[\nabla \Phi \wedge \nabla \Phi], \Phi] \rangle + \langle [F_\Omega, \Phi], \text{Riem}([F_\Omega, \Phi]) \rangle$$

$$+ \langle [F_\Omega, \Phi], (F_\Omega \cdot [F_\Omega, \Phi]) \rangle - 2\langle [F_\Omega, \Phi], [\nabla_i F_\Omega, \nabla_i \Phi] \rangle$$

$$\leq (-|\Phi|^2 + |\text{Riem}| + |F_\Omega|) |[F_\Omega, \Phi]|^2$$

$$+ (|[\nabla \Phi]| + |[\nabla \Phi]|^{-1} + |[\nabla_i F_\Omega, \nabla_i \Phi]|) |[F_\Omega, \Phi]|.$$ 

Now, notice that $|[\Phi]|([\nabla \Phi]|^{-1} \leq |[\nabla \Phi]|$ while

$$|[\nabla_i F_\Omega, \nabla_i \Phi]| \leq |(\nabla_i F_\Omega)| |(\nabla_i \Phi)|^{-1} + |(\nabla F_\Omega)|^{-1} |(\nabla \Phi)|$$

$$\leq |\Phi|^{-1} |(\nabla F_\Omega)| |(\nabla \Phi)| + |\Phi|^{-2} |(\Phi, \nabla_i F_\Omega)| |(\nabla \Phi)|$$

$$\leq |\Phi|^{-1} |(\nabla F_\Omega)| |(\nabla \Phi)| + |\Phi|^{-2} |(\Phi, \nabla_i \Phi, F_\Omega)| - |\Phi, [\nabla_i \Phi, F_\Omega]| |(\nabla \Phi)|$$

$$\leq |\Phi|^{-1} |(\nabla F_\Omega)| |(\nabla \Phi)| + |\Phi|^{-1} |\nabla_i \Phi, F_\Omega| |(\nabla \Phi)|$$

$$+ |\Phi|^{-2} |\nabla_i \Phi, [F_\Omega, \Phi]| + |F_\Omega, [\Phi, \nabla_i \Phi]| |(\nabla \Phi)|$$

$$\leq |\Phi|^{-1} |(\nabla F_\Omega)| |(\nabla \Phi)| + |\Phi|^{-1} |\nabla_i \Phi, F_\Omega| |(\nabla \Phi)|$$

$$+ |\Phi|^{-2} |(\nabla \Phi)||F_\Omega, \Phi| + |F_\Omega||(\Phi, \nabla \Phi)||(\nabla \Phi)|,$$

which upon inserting above gives

$$\langle [F_\Omega, \Phi], \nabla^* \nabla [F_\Omega, \Phi] \rangle + |\Phi|^2 |[F_\Omega, \Phi]|^2 \leq (|\text{Riem}| + |F_\Omega| + |\Phi|^{-2} |\nabla \Phi|| |F_\Omega||^2$$

$$+ (|[\nabla \Phi]| + |\Phi|^{-2} |F_\Omega||(|\nabla \Phi||^2) + |\Phi|^{-1} |(\nabla F_\Omega)||\nabla \Phi|| |F_\Omega||$$

$$+ |\Phi|^{-1} |\nabla_i \Phi, F_\Omega||(|\nabla \Phi|| |F_\Omega||.$$ 

Inserting into equation (5.3) we obtain the inequality in the statement. \hfill \square

We now give the main consequence of the Bochner inequalities proved in this section.

**Corollary 5.4.** Let $(\nabla, \Phi)$ be a solution to the second order equations (2.1a) and (2.1b) with finite mass $m \neq 0$, set $f = ([\nabla \Phi, \Phi], [F_\Omega, \Phi]) \in \Omega^1(g_p) \oplus \Omega^2(g_p)$ and suppose that $|\text{Riem}|, |\nabla \Phi|, |\Phi|$,
$|F_\nabla|$ and $|\nabla F_\nabla|$ decay uniformly to zero at infinity. Then, for every $\delta \in (0, 1]$, there is a sufficiently large compact set $K = K(\delta) \subset X$ outside of which there holds

$$\frac{1}{2} \Delta |f|^2 + (1 - \delta)|\nabla f|^2 \leq -\frac{1}{2}|\Phi|^2 |f|^2. \quad (5.4)$$

Moreover, if $0 < \delta_1 < \delta_2 \leq 1$ then $K(\delta_1) \supset K(\delta_2)$.

**Proof.** Since $(\nabla, \Phi)$ has finite mass $m > 0$, it follows from Remark 2.10 that there is a sufficiently large compact set $K \subset X$ such that

$$|\Phi| \geq \frac{m}{2} > 0, \quad \text{on } X \setminus K.$$ 

Using this and summing the inequalities in Lemmata 5.2 and 5.3 and using Young’s inequality to deal with the mixed terms of the form $||\nabla \Phi| | |F_\nabla, \Phi||$, we get that there is $c > 0$ such that outside $K$ there holds

$$\frac{1}{2} \Delta |f|^2 \leq -|\nabla f|^2 + c \frac{2}{m}|\nabla [F_\nabla, \Phi]| |(\nabla \Phi)\| |[F_\nabla, \Phi]| - |\Phi|^2 |f|^2$$

$$+ c \left( |\mathrm{Riem}| + |F_\nabla| + \frac{4}{m^2} |\nabla \Phi| + |(\nabla \Phi)\| + \frac{4}{m^2} |F_\nabla| (\nabla \Phi)\| + \frac{2}{m} |(\nabla F_\nabla)\| \right) |f|^2. \quad (5.5)$$

Now, by the hypotheses, the functions $|\mathrm{Riem}|$, $|\nabla \Phi|$, $|F_\nabla|$ and $|\nabla F_\nabla|$ decay uniformly to zero at infinity, so by taking $K$ sufficiently large, we can make the whole term inside the parenthesis in the second line of the right-hand side of inequality (5.5) to be less than $\frac{m^2}{16c}$. On the other hand, for any $\delta \in (0, 1]$, Young’s inequality gives

$$c \frac{2}{m^2} |\nabla [F_\nabla, \Phi]| |(\nabla \Phi)\| |[F_\nabla, \Phi]| \leq \delta |\nabla [F_\nabla, \Phi]|^2 + \frac{c^2}{\delta m^2} (\nabla \Phi)\| |F_\nabla, \Phi\|^2$$

$$\leq \delta |\nabla f|^2 + \frac{c^2}{\delta m^2} (\nabla \Phi)\| |f|^2,$$

and again since $|\nabla \Phi|$ decays, by taking a sufficiently large compact subset $K(\delta) \supset K$, we can arrange $\frac{c^2}{\delta m^2} (\nabla \Phi)\| |f|^2 \leq \frac{m^2}{16} \leq \frac{|\Phi|^2}{4}$ outside $K(\delta)$; note that one chooses $K(\delta)$ in such a way that $K(\delta_1) \supset K(\delta_2)$ whenever $0 < \delta_1 < \delta_2 \leq 1$. In conclusion, combining these facts with inequality (5.5), we get the desired inequality (5.4) outside $K(\delta)$.

□

Here we state and prove a further improvement of the inequalities in Lemma 5.1. These are valid only sufficiently far out along the end and follow explicitly making use of the finiteness of the mass.
Lemma 5.5 (Improved Bochner inequalities). In the conditions of Theorem 4.1, the following inequalities hold outside a sufficiently large compact subset:

\[
\Delta |\nabla \Phi|^2 \leq c(F_\nabla |(\nabla \Phi)^\perp|||\nabla \Phi)^\perp| + \left(c(|F_\nabla| - m^2/4)\right)(\nabla \Phi)^\perp|^2,
\]

\[
\Delta |\nabla \Phi|^2 \leq |(F_\nabla)^\perp|||\nabla \Phi)|\nabla \Phi)^\perp|,
\]

\[
\Delta |\nabla \Phi|^2 \leq |(F_\nabla)^\perp|^2||\nabla \Phi|^2.
\]

In case $(\nabla, \Phi)$ is a $G_2$-monopole, then we have:

\[
\Delta |\nabla \Phi|^2 \leq c(F_\nabla^{14}|(\nabla \Phi)^\perp|||\nabla \Phi)^\perp| + \left(c(|F_\nabla^{14}| - m^2/4)\right)(\nabla \Phi)^\perp|^2, \quad (5.7a)
\]

\[
\Delta |\nabla \Phi|^2 \leq |(F_\nabla^{14})^\perp|||\nabla \Phi)|\nabla \Phi)^\perp|, \quad (5.7b)
\]

\[
\Delta |\nabla \Phi|^2 \leq |(F_\nabla^{14})^\perp|^2||\nabla \Phi|^2. \quad (5.7c)
\]

Proof. We do the proof of the monopole case, the other is entirely analogous. Recall from inequality (5.2) that

\[
\Delta |\nabla \Phi|^2 + ||\Phi, \nabla \Phi||^2 \leq |(F_\nabla^{14})^\perp|||\nabla \Phi)|\nabla \Phi)^\perp| + \left(|(\nabla \Phi)| + |(F_\nabla)|\right)(\nabla \Phi)^\perp|^2.
\]

Since, by Theorem 4.1, $(\nabla, \Phi)$ has finite mass $m > 0$, it follows that outside a sufficiently large compact set $K$ one has $|\Phi| \geq m/2$ (see Remark 2.10) and thus

\[
||\Phi, \nabla \Phi||^2 \geq |\Phi|^2|\nabla \Phi)^\perp|^2 \geq m^2/4|\nabla \Phi)^\perp|^2.
\]

Therefore we get inequality (5.7a). Now recall that $|F_\nabla^{14}|$ decays by hypothesis and that $|\nabla \Phi|$ also decays as a consequence of Corollary 3.8. Thus, if $K$ is large enough, then the last term in inequality (5.7a) becomes negative, so that we get inequality (5.7b).

Finally, to show inequality (5.7c), note that we can use Young’s inequality in the form

\[
2|(|F_\nabla^{14})^\perp|||\nabla \Phi)|\nabla \Phi)^\perp| \leq \varepsilon^{-1}|(|F_\nabla^{14})^\perp|^2||\nabla \Phi|^2 + \varepsilon ||\nabla \Phi)^\perp|^2,
\]

with $\varepsilon > 0$ to be fixed later. Then, by inequality (5.7a), we find that

\[
\Delta |\nabla \Phi|^2 \leq c\varepsilon^{-1}|(|F_\nabla^{14})^\perp|^2||\nabla \Phi)|\nabla \Phi)^\perp| + \left(c\varepsilon + c|(|\nabla \Phi)| + c|(|F_\nabla)| - m^2/4\right)||\nabla \Phi)^\perp|^2.
\]

Now choose $\varepsilon \ll m^2$, then given that both $||\nabla \Phi)|, |(|F_\nabla)|$ decay, we conclude that the second term becomes negative so

\[
\Delta |\nabla \Phi|^2 \leq \varepsilon^{-1}|(|F_\nabla^{14})^\perp|^2||\nabla \Phi)|\nabla \Phi)^\perp|^2,
\]

as we wanted. \(\square\)
6. Refined asymptotics in the AC case

In this section, let \((X, \varphi)\) be an (irreducible) AC \(G_2\)-manifold as in Definition 2.4 and \(G = \text{SU}(2)\). Here we prove the first part of our second Main Theorem 2. For the reader’s convenience we restate this here as follows.

**Theorem 6.1.** Let \((X, \varphi)\) be an AC \(G_2\)-manifold and \((\nabla, \Phi)\) satisfy the second order equations (2.1a) and (2.1b) with \(|\nabla\Phi|^2 \in L^1(X)\). Suppose either that \(|F_\nabla|\) decays uniformly along the end or \((\nabla, \Phi)\) is a \(G_2\)-monopole such that \(|F_\nabla^{14}|\) decays uniformly along the end. Then:

(i) the transverse components of \(\nabla\Phi\) and \(F_\nabla\) decay exponentially along the end;
(ii) \(|\nabla\Phi| = O(r^{-(n-1)})\) as \(r \to \infty\).

**Remark 6.2** (The decay of \(|\nabla\Phi|\) in (ii) is sharp). Let \((X, \varphi)\) be an irreducible AC \(G_2\)-manifold and \((\nabla, \Phi)\) satisfy the second order equations (2.1a) and (2.1b) with \(|\nabla\Phi|^2 \in L^1(X)\). Suppose either that \(|F_\nabla|\) is bounded or \((\nabla, \Phi)\) is a \(G_2\)-monopole and \(|F_\nabla^{14}|\) is bounded. Let \(m > 0\) be the mass of \((\nabla, \Phi)\), cf. Theorem 4.1. We show that if \(|\nabla\Phi| \neq 0\), then \((\nabla, \Phi)\) cannot decay faster than as in (ii) of Theorem 6.1, as a consequence of the same argument in Lemma 4.5. Indeed, start noting that since \(\Delta_\nabla \Phi = 0\), it follows from Stokes’ Theorem that

\[
\int_{B_R} |\nabla\Phi|^2 \text{vol}_X = \int_{\Sigma_R} \langle \Phi, \ast \nabla\Phi \rangle.
\]

Thus, if \(|\nabla\Phi| = o(r^{-(n-1)})\) as \(r \to \infty\) then

\[
\lim_{r \to \infty} \langle \Phi, \ast \nabla\Phi \rangle r^{n-1} \leq \lim_{r \to \infty} |\Phi| |\nabla\Phi| r^{n-1} \leq \lim_{r \to \infty} |\nabla\Phi| r^{n-1} = 0;
\]

hence, using \(|\nabla\Phi|^2 \in L^1(X)\),

\[
||\nabla\Phi||^2_{L^2(X)} = \lim_{R \to \infty} \int_{\Sigma_R} \langle \Phi, \ast \nabla\Phi \rangle = 0,
\]

i.e. \(\nabla\Phi = 0\).

6.1. Exponential decay for the transverse components. In this subsection we prove that the components on \(\nabla\Phi\) and \(F_\nabla\) transverse to the Higgs field decay exponentially with \(r\). For the gauge group \(G = \text{SU}(2)\) it is enough to prove that both \([\Phi, \nabla\Phi]\) and \([\Phi, F_\nabla]\) decay exponentially.

When \((X, \varphi)\) is AC note that \(|\text{Riem}|\) decays uniformly along the end. Let \((\nabla, \Phi)\) satisfy the second order equations (2.1a) and (2.1b) with \(|\nabla\Phi|^2 \in L^1(X)\). If \(|F_\nabla|\) decays uniformly along the end or \((\nabla, \Phi)\) is a \(G_2\)-monopole such that \(|F_\nabla^{14}|\) decays uniformly along the end, then we have that \(|\nabla\Phi|, |F_\nabla|\) and \(|F_\nabla^{14}|\) also decay uniformly along the end by
Corollaries 3.6 and 3.8. Furthermore, one has that $|\Phi| \to m$ uniformly along the end by Theorem 4.1. Thus, we are in the conditions of Corollary 5.4, which in turn implies (taking $\delta = 1$) that sufficiently far along the end of $X$ we have that $f = ([\nabla \Phi, [F, \Phi]] \in \Omega^1(g_P) \oplus \Omega^2(g_P)$ satisfies

$$\Delta |f|^2 \leq -\frac{m^2}{4} |f|^2. \quad (6.1)$$

This has the remarkable consequence that the transverse components controlled by $|f|$ decay exponentially along the end. The following gives part (i) of Theorem 6.1.

**Proposition 6.3.** Let $(\nabla, \Phi)$ be AC and assume that the pair $(\nabla, \Phi)$ satisfies the second order equations (2.1a) and (2.1b) and $|\nabla \Phi|^2 \in L^1(X)$. Suppose furthermore that either that $|F| \nabla \Phi|$ decays uniformly along the end or $(\nabla, \Phi)$ is a $G_2$-monopole such that $|F^1_{\nabla}| \Phi|$ decays uniformly along the end. Denote by $m > 0$ the mass of $(\nabla, \Phi)$. Then, there are constants $c > 0$ (depending only on the geometry), $R > 0$ (depending only on the geometry and $m$) and $M > 0$ (depending on $(\nabla, \Phi)$) such that for $r \geq R$ we have

$$|[\nabla \Phi, \Phi]|^2 + |[F \nabla, \Phi]|^2 \leq Me^{-cmr}.$$ 

In particular, for $r \geq R$,

$$|(\nabla \Phi)^\perp|^2 \leq m^{-2} |[\nabla \Phi, \Phi]|^2 \leq m^{-2} Me^{-cmr} \leq e^{-cmr},$$

$$|(F^1_{\nabla})^\perp|^2 \leq |F^1_{\nabla}|^2 \leq m^{-2} |[F, \Phi]|^2 \leq m^{-2} Me^{-cmr} \leq e^{-cmr}.$$ 

**Proof.** Along the end of $X$, let $r$ be the pullback of the radius function from the cone. Then, using the almost isometry to the cone we can write

$$-\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\Sigma + \ldots,$$

where the dots denote lower order terms. Furthermore, if $w(r)$ is a function of $r$ we find that

$$\Delta w = -w''(r) |dr|^2 + w'(r) \Delta r$$

$$= -w''(r) - \frac{n-1}{r} w'(r) + \ldots.$$ 

Thus, let $M > 0$ and $c > 0$ both to be fixed later and set

$$w := Me^{-cmr}.$$ 

Then,

$$\Delta w = \left(-c^2 m^2 + \frac{n-1}{r} cm + \ldots\right) w.$$
It follows that taking $R \gg 1$ sufficiently large, depending on $m$ and the geometry, we can choose $c > 0$ depending only on the geometry\(^7\) such that for $r \geq R$

$$\Delta w \geq -\frac{m^2}{4} w.$$  

Using inequality (6.1), we find that for $r \geq R$

$$\Delta(|f|^2 - w) \leq -\frac{m^2}{4} (|f|^2 - w).$$

Now, both $f$ and $w$ decay to zero along the end, and for $M > 0$ large enough, depending on $(\nabla, \Phi)$, we have $|f|^2 \leq w$ at $r = R$. Therefore, using the inequality above we can apply the maximum principle to $|f|^2 - w$ in the region $r \geq R$ to find that

$$|f|^2 \leq w,$$

within that region. \(\square\)

6.2. \textbf{Bounds from Hardy's inequality.} This section uses Hardy's inequality, the Agmon technique and Moser iteration to prove the following.

**Proposition 6.4.** Under the hypothesis of Theorem 6.1 and for all $\varepsilon > 0$, we have

$$|\nabla \Phi|^2 \leq \frac{C}{\varepsilon} r^{-2(n-2)+\varepsilon}.$$  

We divide the proof of this result into a series of lemmas which we prove below. The concluding proof is given at the end.

For the rest of the paper, let $L \gg 2l > l \gg 1$ and $r_{l,L}$ be the following function:

$$r_{l,L} = \begin{cases} 
0 & \text{on } B_l, \\
2(r-l) & \text{on } B_{2l} - B_l, \\
r & \text{on } B_L - B_{2l}, \\
L & \text{on } X - B_L.
\end{cases} \tag{6.2}$$

Note that $r_{l,L} \in L^\infty_1(X)$ and $dr_{l,L} = \partial_r r_{l,L} \, dr$, with

$$\partial_r r_{l,L} = \begin{cases} 
2 & \text{on } B_{2l} - B_l, \\
1 & \text{on } B_L - B_l, \\
0 & \text{otherwise.} 
\end{cases} \tag{6.3}$$

**Lemma 6.5.** Under the hypotheses of Theorem 6.1, the tensors $\nabla (r \nabla \Phi)$, $r \nabla^2 \Phi$ and $r^\alpha (\nabla \Phi) \perp$ are all square integrable, for all $\alpha > 0$.

\(^7\)Recall that since $(X, \varphi)$ is AC, we have that $-\Delta r \geq (n-1)r^{-1}(1 - O(r^{-\nu'}))$, for some $\nu' > 0.$
Proof. We start claiming that for any real function \( f \in L_1^\infty(X) \), with support in \( X - B_l \), we have the Agmon identity
\[
\|\nabla(f \nabla \Phi)\|_{L^2(X)}^2 = \|df \otimes \nabla \Phi\|_{L^2(X - B_l)}^2 + \int_{X - B_l} \langle f^2 \nabla \Phi, \nabla^* \nabla(\nabla \Phi) \rangle.
\]
Indeed, it follows from Lemma 5.5 and Proposition 6.3, under the hypotheses above, that
\[
\langle \nabla \Phi, \nabla^* \nabla(\nabla \Phi) \rangle \lesssim e^{-cmr} |\nabla \Phi|^2,
\]
which is in \( L^1(X) \) since \(|\nabla \Phi|^2 \in L^1(X)\). Thus, the claim follows by the obvious approximation argument and integration by parts.

Next, we prove the statement. Let \( f = r_l \) as above. Then, since \( r^2 e^{-cmr} \leq 1 \) for \( r \geq l \gg 1 \), we have, for all \( L \) that
\[
\|\nabla(r_l \nabla \Phi)\|_{L^2(X - B_l)}^2 \lesssim \|\nabla \Phi\|_{L^2(X)}^2 + \int_{X - B_l} r_l^2 e^{-cmr} |\nabla \Phi|^2 \text{vol}_X \lesssim \|\nabla \Phi\|_{L^2(X)}^2.
\]
Hence \( \nabla(r \nabla) \in L^2(X) \). Since \( \nabla \Phi \) is also in \( L^2(X) \), we have that
\[
r \nabla^2 \Phi = \nabla(r \nabla) - dr \otimes \nabla \in L^2(X).
\]
Finally, note that Proposition 6.3 immediately implies that \( r^\alpha(\nabla \Phi)^\perp \in L^2(X) \) for all \( \alpha > 0 \), which concludes the proof. 

Next we improve the above result to allow higher powers of the radius function. We make use of the following Hardy type inequality.

**Lemma 6.6** (Hardy’s Inequality, cf. [10, Proposition 3.7]). Let \((X, \varphi)\) be an AC G2-manifold with rate \( \nu < 0 \). Then there is a constant \( C_H > 0 \) such that for any function \( \xi \in H^1 \) with support in \( X - B_l \), one has
\[
\|\nabla \xi\|_{L^2(X - B_l)}^2 \geq \left(\frac{n - 2}{2}\right)^2 \|r^{-1} \xi\|_{L^2(X - B_l)}^2 - C_H \|r^{-1 + \nu} \xi\|_{L^2(X - B_l)}^2.
\]

**Lemma 6.7.** Under the hypotheses of Theorem 6.1, there is a constant \( C > 0 \), such that for all \( \epsilon > 0 \)
\[
\|r^{\frac{n-4-\epsilon}{2}} \nabla \Phi\|_{L^2(X - B_l)}^2 \leq \frac{C}{\epsilon} \|\nabla \Phi\|_{L^2(X - B_l)}^2,
\]
Moreover both \( \nabla(r^\alpha \nabla \Phi) \) and \( r^\alpha \nabla \nabla \Phi \) are square integrable as long as \( \alpha < \frac{n-2}{2} = \frac{5}{2} \).
Proof. It suffices to consider the case where $\alpha > 1$ by Lemma 6.5. If $f \in L^\infty_1(X)$ is a function with support in $X - B$, then (cf. proof of Lemma 6.5)

$$\|\nabla(f \Phi)\|_{L^2(X - B)}^2 \leq \|df \otimes \nabla \Phi\|_{L^2(X - B)}^2 + c \int_{X - B} f^2 e^{-cmr} |\nabla \Phi|^2 \text{vol}_X. \quad (6.6)$$

On the other hand, by Lemma 6.5 we may apply Hardy’s inequality (6.4) to the function $f|\nabla \Phi|$ and using Kato’s inequality this gives us

$$\frac{(n-2)^2}{4} \|r^{-1} f \nabla \Phi\|_{L^2(X - B)}^2 - C_H \|r^{-1+\nu} f \nabla \Phi\|_{L^2(X - B)}^2 \leq \|\nabla(f \Phi)\|_{L^2(X - B)}^2.$$

The combination of these two inequalities gives

$$\frac{(n-2)^2}{4} \|r^{-1} f \nabla \Phi\|_{L^2(X - B)}^2 \leq \|df \otimes \nabla \Phi\|_{L^2(X - B)}^2 + C_H \|r^{-1+\nu} f \nabla \Phi\|_{L^2(X - B)}^2 + c \int_{X - B} f^2 e^{-cmr} |\nabla \Phi|^2 \text{vol}_X. \quad (6.7)$$

Let now $f = r_{l,l}^\alpha$ from equation (6.2). Using $|dr| = 1 + O(r^\nu)$ and equation (6.3), we get that

$$|dr_{l,l}^\alpha| \leq 2\alpha r^{-1} \chi([l,2l]) + \alpha r_{l,l}^\alpha r^{-1} \chi([2l,l]) + O(r_{l,l}^\alpha r^{-1+\nu} \chi([l,l])\chi([l,l]),
$$

and thus, we can rearrange inequality (6.7) as

$$\left(\frac{(n-2)^2}{4} - \alpha^2\right) \|r_{l,l}^\alpha \nabla \Phi\|_{L^2(X - B)}^2 \leq \|r^{-1+\nu} r_{l,l}^\alpha \nabla \Phi\|_{L^2(X - B)}^2 + \int_{X - B} r_{l,l}^{2\alpha} e^{-cmr} |\nabla \Phi|^2 \text{vol}_X + 1^{2(\alpha-1)} \|\nabla \Phi\|_{L^2(B_{2l}-B)}^2.$$

The right hand side is finite and bounded independent of $L$, as long as $\alpha \leq 1 - \nu > 1$. Thus, if also $\alpha < \frac{n-2}{2} = \frac{5}{2}$, then

$$\|r_{l,l}^\alpha \nabla \Phi\|_{L^2(X - B)}^2 \leq \frac{C}{\frac{n-2}{2} - \alpha} \|\nabla \Phi\|_{L^2(X - B)}^2. \quad (6.8)$$

This works for any $\alpha < \min\left(1 - \nu, \frac{5}{2}\right)$. If $\nu \leq -\frac{3}{2}$, then for all $\alpha < \frac{5}{2}$, we immediately have inequality (6.8).\(^8\)

On the other hand, if $\nu \in \left(-\frac{3}{2}, 0\right]$, we can still prove inequality (6.8) for all $\alpha < \frac{n-2}{2} = \frac{5}{2}$ by iterating the following argument, finitely many times: Start with $\alpha_0 := 1$, from which obtain that $\|r_{l,l}^{\alpha_0} \nabla \Phi\|_{L^2(X)}^2 < \infty$. Then, for all $k \geq 1$, let

$$\alpha_k := \min\left(1 - k\nu, \frac{5}{2}\right) \leq \alpha_k - 1 - \nu,$$

\(^8\)Note that, all known examples have $\nu \leq -3$.\]
and the sequence $\alpha_k$ reaches $\frac{5}{2}$ in finitely many steps, and then becomes constant. Thus we get, by the same argument
\[
\left( \frac{(n-2)^2}{4} - \alpha_k^2 \right) \| r^{-1+\alpha_k} \nabla \Phi \|_{L^2(X-B_l)}^2 \leq \| r^{-1+\alpha_k} \nabla \Phi \|_{L^2(X-B_l)}^2 + \| \nabla \Phi \|_{L^2(X-B_l)}^2 + \| \nabla \Phi \|_{L^2(X-B_l)}^2.
\]
Thus an induction proves inequality (6.8) with $\alpha = \alpha_k$, for all $k \in \mathbb{N}$. Writing $\varepsilon = 5-2\alpha$ gives inequality (6.5).

Finally, similar to Lemma 6.5, using inequality (6.6) again, with $f = r_{l,L}^\alpha$, we get
\[
\| \nabla (r_{l,L}^\alpha \nabla \Phi) \|_{L^2(X-B_l)}^2 \leq \| dr_{l,L}^\alpha \otimes \nabla \Phi \|_{L^2(X-B_l)}^2 + c \int_{X-B_l} r_{l,L}^\alpha e^{-cmr} |\nabla \Phi|^2 \text{vol}_X
\]
\[
\leq \| r_{l,L}^{\alpha-1} \nabla \Phi \|_{L^2(X-B_l)}^2 + \| r_{l,L}^{\alpha+1} r_{l,L}^\alpha \nabla \Phi \|_{L^2(X-B_l)}^2 + \| \nabla \Phi \|_{L^2(X-B_l)}^2.
\]
Now the right hand side is bounded for all $L$ when $\alpha < \frac{n-2}{2}$. Thus $\nabla (r^\alpha \nabla \Phi)$ is square integrable, and hence so is $r^\alpha \nabla^2 \Phi = \nabla (r^\alpha \nabla \Phi) - \alpha r^{\alpha-1} dr \otimes \nabla \Phi$. \hfill \Box

Now we are ready to prove Proposition 6.4.

**Proof of Proposition 6.4.** Pick $x \in X$ and let $R = \frac{1}{4} \min(r(x), \text{inj}_x(X,g)) \approx r(x)$. It follows from Proposition 6.3 and inequality (5.7c) that
\[
\Delta |\nabla \Phi|^2 \leq e^{-cmr} |\nabla \Phi|^2.
\]
For all $\varepsilon > 0$, Moser iteration gives
\[
|\nabla \Phi(x)|^2 \leq R^{-n} \int_{B_R(x)} |\nabla \Phi|^2 \text{vol}_X \leq r(x)^{-2(n-2)+\varepsilon} \int_{B_R(x)} |r^{(n-4)/2-\varepsilon} \nabla \Phi|^2 \text{vol}_X,
\]
where in the last inequality we used that $R \approx r(x)$. By Lemma 6.7, the last integral in the right-hand side of equation (6.9) can be bounded by $\frac{C}{\varepsilon}$, yielding the stated result. \hfill \Box

6.3. **Bounds from an improved Hardy inequality.** This section follows the same strategy of the previous except that we combine the previously obtained bound with an improved Hardy-inequality which holds for $H^1$-functions supported along an end $X-B_l$. We summarize the main result as follows.

**Proposition 6.8.** Under the hypothesis of Theorem 6.1, there are constants $c_1, c_2 > 0$ such that for all $t \in (0,1)$ and $\alpha < \frac{n-1}{2}$, we have
\[
\| r^{\alpha-1/2} \nabla \Phi \|_{L^2(X)}^2 \leq c_1 + c_2 (n-1-2\alpha)^{-1}.
\]
In particular,
\[ |\nabla \Phi|^2 = O(r^{-2(n-1)+\epsilon}) \quad \text{as } r \to \infty \] (6.11)
for all \( \epsilon > 0 \).

We start with the statement of the improved Hardy inequality.

**Lemma 6.9** (Improved Hardy’s Inequality as in [10]). Let \((X, \varphi)\) be an AC \(G_2\)-manifold with rate \( \nu < 0 \). Then, there is a constant \( C_H > 0 \) such that for all \( \xi \in H^1 \) with support in \( X - B_l \)
\[ \| \nabla \partial_r \xi \|^2_{L^2(X-B_l)} \geq \left( \frac{n-2}{2} \right)^2 \| r^{-1} \xi \|^2_{L^2(X-B_l)} - C_H \| r^{-1+\nu} \xi \|^2_{L^2(X-B_l)} \] (6.12)

The proof of this result is exactly the same as that in [10] and so we jump into the proof of the main result of this section.

**Proof of Proposition 6.8.** Let \( r_{l,L} \) as in equation (6.2) and \( \alpha > 0 \), to be determined later. Applying the improved Hardy inequality (6.12) to the function \( r_{l,L}^\alpha \sqrt{r} |\nabla \Phi| \) and using Kato’s inequality gives us
\[ \frac{(n-2)^2}{4} \| r_{l,L}^\alpha r^{-1/2} |\nabla \Phi| \|^2_{L^2(X-B_l)} \leq \| \nabla \partial_r (r_{l,L}^\alpha \sqrt{r} |\nabla \Phi|) \|^2_{L^2(X-B_l)} + C_H \| r_{l,L}^\alpha r^{-1/2+\nu} |\nabla \Phi| \|^2_{L^2(X-B_l)} \] (6.13)
which we now combine with the previously used strategy, now with the goal of proving Proposition 6.8.

Note that there is a number \( C = C(I, \nabla, \Phi) \), such that
\[ \| r^{\alpha-1/2} |\nabla \Phi| \|^2_{L^2(X)} = C + \lim_{L \to \infty} \| r_{l,L}^{\alpha-1/2} |\nabla \Phi| \|^2_{L^2(X)} \]
so it is enough to show inequality (6.10) with the left-hand side replaced by \( \| r_{l,L}^{\alpha-1/2} |\nabla \Phi| \|^2_{L^2(X)} \).

First we prove that for all \( \alpha < \frac{n-1}{2} \), \( \| r_{l,L}^{\alpha-1/2} |\nabla \Phi| \|^2_{L^2(X)} \) stays bounded as \( L \to \infty \). We use proof by contradiction. Thus let us assume that \( \alpha \) is such that \( \alpha < \frac{n-1}{2} \) and
\[ \lim_{L \to \infty} \| r_{l,L}^{\alpha-1/2} |\nabla \Phi| \|^2_{L^2(X)} = \infty. \] (6.14)

Let \( \nabla^\Sigma \) denote covariant differentiation, using the connection \( \nabla \), in the directions along the kernel of \( dr \), so that \( \nabla = \partial_r \otimes dr + \nabla^\Sigma \). Note that \( \nabla^\Sigma r = 0 \), and thus \( \nabla^\Sigma r_{l,L} = 0 \). Then, using the computation in the proof of Lemma 6.5, we have
\[ \| \nabla (r_{l,L}^\alpha \sqrt{r} |\nabla \Phi|) \|^2_{L^2(X-B_l)} = \| \nabla \partial_r (r_{l,L}^\alpha \sqrt{r} |\nabla \Phi|) \|^2_{L^2(X-B_l)} + \| r_{l,L}^\alpha \sqrt{r} \nabla^\Sigma |\nabla \Phi| \|^2_{L^2(X-B_l)} \]
\[ \lesssim \| d (r_{l,L}^\alpha \sqrt{r}) \otimes |\nabla \Phi| \|^2_{L^2(X-B_l)} + \int_{X-B_l} r_{l,L}^{2\alpha} re^{-cmr} |\nabla \Phi|^2 \text{vol}_X. \] (6.15)
Combining inequalities (6.13) and (6.15), we get

\[
\frac{(n - 2)^2}{4} \left\| r_{i,L}^{a} r^{-1/2} \nabla \Phi \right\|_{L^2(B_{2l} - B_l)}^2 + \left\| r_{i,L}^{a} \nabla \nabla \Phi \right\|_{L^2(B_{2l} - B_l)}^2 \leq \left\| d(r_{i,L}^{a} \nabla r) \nabla \Phi \right\|_{L^2(X)}^2 + \left\| r_{i,L}^{a} r^{-1/2 + \nu} \nabla \Phi \right\|_{L^2(B_{2l} - B_l)}^2 + O(1),
\]

where \( O(1) \) corresponds to the rightmost term of the right hand side of inequality (6.15), which is bounded independently of \( L \) for any fixed \( a \).

Now let

\[
t_L = t_L(l, \alpha) := \frac{\left\| r_{i,L}^{a} r^{-1/2} \nabla \Phi \right\|_{L^2(B_{2l} - B_l)}^2}{\left\| r_{i,L}^{a} r^{-1/2} \nabla \Phi \right\|_{L^2(B_{2l} - B_l)}^2} \in (0, 1),
\]

\[
t = t(l, \alpha) := \limsup_{L \to \infty} t_L \in (0, 1].
\]

Using equations (6.2) and (6.17), we get, as \( L \to \infty \), that

\[
\left\| d(r_{i,L}^{a} \nabla r) \nabla \Phi \right\|_{L^2(X)}^2 = \left( (a^2 + \alpha)t_l + \frac{1}{4} \right) \left\| r_{i,L}^{a} r^{-1/2} \nabla \Phi \right\|_{L^2(X)}^2 + O(1),
\]

where this \( O(1) \) term depends only on \( l, \alpha \) and the \( L^2(X) \)-norm of \( \nabla \Phi \) in \( B_{2l} - B_l \); in particular, for any fixed \( \alpha \), it remains bounded as \( L \to \infty \).

Next, we obtain a lower bound for the term \( \left\| r_{i,L}^{a} \nabla \nabla \Phi \right\|_{L^2(X)}^2 \) appearing in the left hand side of inequality (6.16). First, we compute

\[
0 = \int_{X - B_l} d(r_{i,L}^{2a} |\nabla \Phi|^2 i_{\partial r} (\text{vol}_X)) = \int_{X - B_l} \left( \partial_r (r_{i,L}^{2a}) - r_{i,L}^{2a} \Delta r |\nabla \Phi|^2 + r_{i,L}^{2a} \partial_r (|\nabla \Phi|^2) \right) \text{vol}_X.
\]

The first equality follows from an extension of Stokes’ theorem due to [20], which states that for an orientable complete Riemannian \( n \)-manifold \( (M^n, g) \), one has \( \int_M d\gamma = 0 \) provided that \( |\gamma||d\gamma| \in L^1(X) \). Here \( \gamma := r_{i,L}^{2a} |\nabla \Phi|^2 i_{\partial r} (\text{vol}_X) \) satisfies these conditions. In the second equality, we use the fact that \( r_{i,L}|_{B_l} = 0 \) and \( d(i_{\partial r} \text{vol}_X) = -(\Delta r) \text{vol}_X \). Using equation (6.3), we can further expand and rearrange equation (6.19), to get that there is \( C_1 = O(\alpha t_{i,L}^{2a} |\nabla \Phi|^2_{L^2(B_{2l} - B_l)}) \), bounded independently of \( L \), such that

\[
\int_{X - B_l} \left( \alpha t_{i,L}^{2a} r^{-1} - \frac{1}{2} r_{i,L}^{2a} \Delta r |\nabla \Phi|^2 \right) \text{vol}_X = - \int_{X - B_l} r_{i,L}^{2a} \langle \nabla \Phi, \nabla \partial_r \nabla \Phi \rangle \text{vol}_X + C_1.
\]
Let us rewrite the right hand side of equation (6.20) first. Using a normal frame, whose first element is $\partial_r$, and the harmonicity of $\Phi$, we show that

$$-\langle \nabla \Phi, \nabla_{\partial_r} \nabla \Phi \rangle = -\langle \nabla_{\partial_r} \Phi, \nabla \Phi \rangle - \sum_{i=2}^{n} \langle \nabla_i \Phi, \nabla_{\partial_r} \nabla_i \Phi \rangle$$

$$= \sum_{i=2}^{n} \left( \langle \nabla_{\partial_r} \Phi, \nabla_i \nabla_i \Phi \rangle - \langle \nabla_i \Phi, \nabla_i \nabla_{\partial_r} \Phi + [F_{ri}, \Phi] \rangle \right)$$

$$\leq \left| \nabla_{\partial_r} \Phi \right| \left( \sum_{i=2}^{n} |\nabla_i \nabla_i \Phi|^{2} \right)^{\frac{1}{2}} + \left| \nabla^\Sigma \Phi \right| \left( \sum_{i=2}^{n} |\nabla_i \nabla_{\partial_r} \Phi|^{2} \right)^{\frac{1}{2}} - \sum_{i=2}^{n} \langle \nabla_i \Phi, [F_{ri}, \Phi] \rangle.$$
Proposition 6.3

Thus, using also Proposition 6.3, for the right hand side of equation (6.20), we get that (for another $C_2 > 0$, bounded independently of $L$ or $\alpha$)

$$- \int_{X-B_l} r_{l,L}^2 \langle \nabla \Phi, \nabla \rho, \nabla \Phi \rangle \, \text{vol}_X \leq \sqrt{n-1} r_{l,L}^{\alpha} r^{-1/2} \|\nabla \Phi\|_{L^2(X)} \|r_{l,L}^\alpha \sqrt{r} \nabla^2 \Phi\|_{L^2(X)} + C_2. \quad (6.21)$$

Next we estimate the left hand side of equation (6.20). Recall that for some $\nu' > 0$, the radial function $r$ satisfies

$$\frac{n-1}{r} - O(r^{-1-\nu'}) \leq -\Delta r \leq \frac{n-1}{r},$$

with the lower bound following from the assumption that $(X, \varphi)$ is AC and the upper bound from the Laplacian comparison theorem together with the Ricci-flatness. Using this, we get (for some $C_3 > 0$) that

$$\int_{X-B_l} \left( \alpha t_L r_{l,L}^{2\alpha} r^{-1} - \frac{1}{2} r_{l,L}^2 \Delta r \|\nabla \Phi\|^2 \right) \, \text{vol}_X \geq \left( \alpha t_L + \frac{n-1}{2} - C_3 l^{-\nu'} \right) \|r_{l,L}^\alpha r^{-1/2} \nabla \Phi\|_{L^2(X)}^2. \quad (6.22)$$

Combining equation (6.20) and inequalities (6.21) and (6.22), we get

$$\sqrt{n-1} r_{l,L}^{\alpha} r^{-1/2} \|\nabla \Phi\|_{L^2(X)} \|r_{l,L}^\alpha \sqrt{r} \nabla^2 \Phi\|_{L^2(X)} + C_2 \geq \left( \alpha t_L + \frac{n-1}{2} - C_3 l^{-\nu'} \right) \|r_{l,L}^\alpha r^{-1/2} \nabla \Phi\|_{L^2(X)}^2.$$ 

Let us subtract $C_2$, divide by $\sqrt{n-1} r_{l,L}^{\alpha} r^{-1/2} \|\nabla \Phi\|_{L^2(X)}$, and square to get

$$\|r_{l,L}^\alpha \sqrt{r} \nabla^2 \Phi\|_{L^2(X)}^2 \geq \frac{1}{n-1} \left( \alpha t_L + \frac{n-1}{2} - C_3 l^{-\nu'} \right)^2 \|r_{l,L}^\alpha r^{-1/2} \nabla \Phi\|_{L^2(X)}^2 - \frac{2C_2 \left( \alpha t_L + \frac{n-1}{2} - C_3 l^{-\nu'} \right)}{\sqrt{n-1}} + \frac{1}{(n-1) \|r_{l,L}^\alpha r^{-1/2} \nabla \Phi\|_{L^2(X)}^2}.$$

By equation (6.14), the last term is bounded above for large $L$, and thus we get

$$\frac{1}{n-1} \left( \alpha t_L + \frac{n-1}{2} - C_3 l^{-\nu'} \right)^2 \|r_{l,L}^\alpha r^{-1/2} \nabla \Phi\|_{L^2(X)}^2 \leq \|r_{l,L}^\alpha \sqrt{r} \nabla^2 \Phi\|_{L^2(X)}^2 + O(1). \quad (6.23)$$
Hence inserting equation (6.18) and inequality (6.23) into inequality (6.16), gives

\[
\left(\frac{(n-2)^2}{4} - \left(\alpha^2 + \alpha t + \frac{1}{4}\right) + \frac{1}{n-1} \left(\alpha t + \frac{n-1}{2} - C_3 l^{-\nu'}\right)^2\right)\|r_{l,L}^{\alpha} r^{-1/2} \nabla \Phi\|_{L^2(X)}^2 \\
\leq \|r_{l,L}^{\alpha} r^{-1/2+\nu} \nabla \Phi\|_{L^2(X)}^2 + O(1). \tag{6.24}
\]

Thus \(r_{l,L}^{\alpha} r^{-1/2} \nabla \Phi\) is bounded in \(L^2(X)\) independently of \(L\), as long as \(r^{\alpha-1/2+\nu} \nabla \Phi \in L^2(X)\) and the quantity in the parentheses above is positive, this last condition being equivalent in the \(L \to \infty\) limit to

\[
\left(\alpha t + \frac{c l^{-\nu'}}{n-1}\right)^2 < \frac{(n-1)^2}{4} + \frac{nc^2 l^{-2\nu'}}{(n-1)^2} - c(n-1)l^{-\nu'}.
\]

Since \(t \in (0, 1]\), the above inequality is definitely satisfied for \(\alpha = \frac{n-1}{2} - O(l^{-\nu'/2}).\) For such \(\alpha\), in case \(\nu \leq -1\), by Lemma 6.7 we do have that \(r^{\alpha-1/2+\nu} \nabla \Phi \in L^2(X)\) and therefore that \(r^{\alpha-1/2} \nabla \Phi \in L^2(X)\). In case \(-1 < \nu < 0\) we can still achieve the desired integrability by a finite iteration of the above argument, in the same way as we did in the final part of the proof of Lemma 6.7. This finishes the proof by contradiction argument and, in fact, yields inequality (6.10).

Combining this with Moser iteration (as in the proof of Proposition 6.4), we get the bound

\[
|\nabla \Phi|^2 \leq C r^{-2(n-1)+O(l^{-\nu'/2})}.
\]

Using this, we can bootstrap the previous computation, since for \(l\) large (but finite), we can now replace inequality (6.22) with

\[
\int_{X-B_l} \left(\alpha t r_{l,L}^{2\alpha} r^{-1} - \frac{1}{2} r_{l,L}^{2\alpha} A r\right)|\nabla \Phi|^2 \right) \text{vol}_X \geq \left(\alpha t + \frac{n-1}{2}\right) \|r_{l,L}^{\alpha} r^{-1/2} \nabla \Phi\|_{L^2(X)}^2 \\
- c\|r_{l,L}^{\alpha} r^{-(1+\nu')/2} \nabla \Phi\|_{L^2(X)}^2,
\]

as long as \(\alpha < \frac{n-1}{2}\). After going through the remaining steps in an analogous way, we get the improved inequality

\[
\left(\frac{(n-2)^2}{4} - \left(\alpha^2 + \alpha t + \frac{1}{4}\right) + \frac{1}{n-1} \left(\alpha t + \frac{n-1}{2}\right)^2\right)\|r_{l,L}^{\alpha} r^{-1/2} \nabla \Phi\|_{L^2(X)}^2 \leq O(1),
\]

which implies that, as long as \(\alpha < \frac{n-1}{2}\), we have

\[
\|r^{\alpha-1/2} \nabla \Phi\|_{L^2(X)}^2 \leq c_1 + \lim_{L \to \infty} \|r_{l,L}^{\alpha} r^{-1/2} \nabla \Phi\|_{L^2(X)}^2 \leq c_1 + c_2 (n - 1 - 2\alpha t)^{-1}.
\]

Using Moser iteration again, together with \(t \in (0, 1]\), we get that on \(X-B_l\) and for all \(\varepsilon > 0\)

\[
|\nabla \Phi| \leq \frac{C_l}{\varepsilon} r^{-(n-1)+\varepsilon},
\]
which proves equation (6.11) and thus concludes the proof. \qed

6.4. The final estimate. In this section we finish the proof of Theorem 6.1, giving a proof of its part (ii).

We start by recalling inequality (6.10) stated in Proposition 6.8. This states that there are positive constants \( c_1 \) and \( c_2 \) such that

\[
\| r^{n-1/2} \nabla \Phi \|_{L^2(X)}^2 \leq c_1 + c_2 (n - 1 - 2\alpha)^{-1}.
\]

Let \( i \in \mathbb{N} \) be such that \( i \gg c_1 + c_2 \) and choose \( \alpha = \frac{n-1}{2} - \frac{1}{t} \) and \( t = 1 - \frac{1}{7} \). Then we have

\[
\int_X |r^{n-2} \nabla \Phi|^2 \text{vol}_X \leq i,
\]

and given that in \( \{ x \in X : 1 \leq r(x) \leq R_i \} \) we have \( r^{-1/i} \geq R^{-1} \) we find that

\[
\int_{\{ x \in X : 1 \leq r(x) \leq R_i \}} |r^{n-2} \nabla \Phi|^2 \text{vol}_X \leq iR.
\]

This may be also regarded as the average of the \( L^2 \)-norm of \( r^{n-2} \nabla \Phi \) on the \( i \)-cylinders \( C_k = \{ x \in X : R_k \leq r(x) \leq R_{k+1} \} \) for \( k = 0, 1, \ldots, i - 1 \), which is therefore uniformly bounded independently of \( i \). As a consequence, there must be an infinite sequence of cylinders \( \{ C_{ij} \}_j \) with \( i_j \not\to \infty \) for which

\[
\int_{C_{ij}} |\nabla \Phi|^2 \text{vol}_X \leq Rr^{-(n-2)}.
\]

Then, using Moser iteration in a ball \( B_{\rho}(x_{ij}) \subseteq C_{ij} \) gives that

\[
|\nabla \Phi(x_{ij})|^2 \leq \rho^{\rho-n} \int_{B_{\rho}(x_{ij})} |\nabla \Phi|^2 \text{vol}_X \leq \rho^{-n} Rr^{-(n-2)}.
\]

In case, \( R > 2 \) we can set \( \rho = R_i^{i_j-1} \). Then, \( R^2 r^{-1} \geq \rho^{-1} \geq R^{-1} \) in \( C_{ij} \) and the Moser iteration above yields a bound at all \( x_{ij} \) in the sub-cylinders \( C'_{ij} = \{ x \in X : R^i_{ij} + R^i_{j-1} \leq r(x) \leq R^i_{j+1} - R^i_{j-1} \} \subseteq C_{ij} \) of the form

\[
\sup_{x_{ij} \in C'_{ij}} |\nabla \Phi(x_{ij})|^2 \leq r^{-n}(R^2 - 1)^n R^{-(n-2)} \leq R(R^2 - 1)^n r^{-2(n-1)}.
\] (6.25)

We have thus obtained a sequence of cylinders \( C'_{ij} \) going off to infinity and where the inequality (6.25) holds. As a consequence of this, if it was not true that \( |\nabla \Phi|^2 = O(r^{-2(n-1)}) \) for \( r \gg 1 \), there must a sequence \( y_i \) with \( r(y_i) \not\to \infty \) at which \( r^{2(n-1)}|\nabla \Phi|^2 \) attains a local
maxima and
\[ \limsup_{i \to \infty} r(y_i)^{2(n-1)}|\nabla \Phi(y_i)|^2 = \infty. \]

From the condition that \( r^{2(n-1)}|\nabla \Phi|^2 \) attains a local maxima at the \( y_i \) we find, by differentiating in the \( r \)-direction, that at \( y_i \)
\[ 2(n-1)|\nabla \Phi|^2 + r \partial_r(|\nabla \Phi|^2) = 0. \]

On the other hand, the second derivative test at \( y_i \) yields
\[ 0 \leq \Delta (r^{2(n-1)}|\nabla \Phi|^2) \]
\[ = -(2n-2)(3n-4)r^{2(n-2)}|\nabla \Phi|^2 - (2n-2)r^{2n-3} \partial_r(|\nabla \Phi|^2) + r^{2(n-1)}\Delta |\nabla \Phi|^2 + \ldots, \]
with the \( \ldots \) denotes lower order terms in comparison to \( r^{2(n-2)}|\nabla \Phi|^2 \). Inserting the first order equation above we find that
\[ 0 \leq -(2n-2)(3n-4)r^{2(n-1)}|\nabla \Phi|^2 + (2n-2)r^{2(n-1)}|\nabla \Phi|^2 + r^{2(n-1)}\Delta |\nabla \Phi|^2 + \ldots \]
which we can rewrite as
\[ 2(n-1)(n-2)r^{-2}|\nabla \Phi|^2 \leq \Delta |\nabla \Phi|^2 + \ldots. \]

Thus, using the improved Bochner inequality in Lemma 5.5 we find that
\[ |\nabla \Phi|^2 \leq r^2|(F\nabla)^-|^2)|\nabla \Phi||^2 + o(|\nabla \Phi|^2). \]

Given that \( (F\nabla)^- \) decays exponentially, as shown in Proposition 6.3, this is impossible unless \( |\nabla \Phi(y_i)| = 0 \) which would contradict \( r^{2(n-1)}|\nabla \Phi|^2(y_i) \not\to \infty \). This completes the proof of Theorem 6.1.

6.5. **Asymptotic decay rate of** \( m^2 - |\Phi|^2 \). In this brief subsection we state and prove an easy consequence of part (ii) of Theorem 6.1, giving the precise polynomial asymptotic decay rate of \( m^2 - |\Phi|^2 \) on this AC case.

**Corollary 6.10.** Under the hypothesis of Theorem 6.1, let \( m \) be the mass of \( (\nabla, \Phi) \), given by Theorem 4.1. Then along the end of \( X \) we have
\[ m^2 - |\Phi|^2 \sim r^{2-n}. \]

**Proof.** By Corollary 4.4 and Remark 2.5, we already know that \( m^2 - |\Phi|^2 \geq r^{2-n} \). To prove that \( m^2 - |\Phi|^2 \leq r^{2-n} \), we first recall from the result of Theorem 4.1 that
\[ (m^2 - |\Phi|^2)(x) = 2 \int_X G(x,y)|\nabla \Phi|^2(y)\text{vol}_X(y), \]
and then conclude by using the sharp decay estimate $|\nabla \Phi|^2 \leq r^{-(n-1)}$ of part (ii) of Theorem 6.1 combined with the Green's function behavior $G(x,y) \sim \text{dist}(x,y)^{2-n}$, as in the argument in [44, (first part of the proof of) Theorem 2.30].

6.6. Bounds on higher order covariant derivatives of $\Phi$. In this subsection, we assume the hypothesis of Theorem 6.1. The arguments here follow from an inductive procedure based on [40, Section 2].

**Lemma 6.11.** $\nabla^{j+1} \Phi \in L^2(X)$ for all $j \in \mathbb{N}$.

**Proof.** By assumption $\nabla \Phi \in L^2(X)$, and by Lemma 6.5 we know that $\nabla^2 \Phi \in L^2(X)$. We now induct. Suppose that $\nabla^i(\nabla \Phi) \in L^2(X)$ for all $i \leq j$. Recall from either Corollaries 3.6 and 3.8 that $\nabla^i F_\nabla, \nabla^{i+1} \Phi \in L^\infty(X)$ for all $i$, as well as $||\Phi||_{L^\infty} \leq m$ by Theorem 4.1 and Remark 2.10. Let $\{f_k : X \rightarrow [0,1]\}$ be a sequence of uniformly $C^l$-bounded functions, with $\lim_{k \rightarrow \infty} f_k = 1$ pointwise. Differentiating the Bochner identity of equation (2.2) $j$ times and taking the $L^2(X)$-inner product with $f_k^2 \nabla^j(\nabla \Phi)$ gives:

$$0 = \langle \nabla^j(\nabla^* \nabla + 2 \cdot [F_\nabla \wedge \cdot] + \text{ad}(\Phi)^2(\cdot))(\nabla \Phi), f_k^2 \nabla^j(\nabla \Phi) \rangle_{L^2(X)}$$

$$\geq ||\nabla^j(\nabla \Phi)||^2_{L^2(X)} - ||d f_k \otimes \nabla^j(\nabla \Phi)||^2_{L^2(X)} - \sum_{0 \leq i \leq j} c_i ||\nabla^i(\nabla \Phi)||^2_{L^2(X)},$$

for constants $c_i > 0$ that are determined by the sup norms of $\nabla^i F_\nabla, \nabla^{i+1} \Phi, i \leq j$, the mass $m \geq 0$ of $(\nabla, \Phi)$, and the other coefficients of $[\nabla^j, \nabla^*]$. Taking the limit as $k \rightarrow \infty$ gives $\nabla^{j+1}(\nabla \Phi) \in L^2(X)$. \hfill $\Box$

**Corollary 6.12.** $\nabla^{j+1} \Phi \in L^p_1(X)$ for all $p \in [2, 2n]$ and $j \in \mathbb{N}$.

**Proof.** Using Kato's inequality we get that if $\nabla^{j+2} \Phi \in L^p(X)$, then $d||\nabla^{j+1} \Phi|| \in L^p(X)$. Hence, using Lemma 6.11, we get $||\nabla^{j+1} \Phi|| \in L^2_1(X)$ for all $j \in \mathbb{N}$. Now iterate this argument using the Sobolev Embeddings $L^p_1(X) \hookrightarrow L^{np}_{\frac{np}{np-p}}(X)$ valid for all $p < n$, together with Hölder's inequality. \hfill $\Box$

7. Boundary data

In this section we prove the second part of Main Theorem 2.

**Theorem 7.1.** Let $(X, \varphi)$ be an irreducible AC $G_2$-manifold and $P \rightarrow X$ be a principal SU(2)-bundle. Assume that $(\nabla, \Phi)$ satisfies the second order equations (2.1a) and (2.1b) with $\nabla \Phi \in L^2(X)$, and either $|F_\nabla|$ decays quadratically along the end or $(\nabla, \Phi)$ is a $G_2$-monopole such that $|F_\nabla^{14}|$ decays quadratically along the end.
Then there is a principal \( SU(2) \)-bundle, \( P_\infty \to \Sigma \) and a pair \( (\nabla_\infty, \Phi_\infty) \), where \( \nabla_\infty \) is a connection on \( P_\infty \) and \( \Phi_\infty \) is a \( \nabla_\infty \)-parallel section of the adjoint bundle \( g_{P_\infty} \) over \( \Sigma \), such that \( (\nabla, \Phi)|_{\Sigma_R} \) converges uniformly to \( (\nabla_\infty, \Phi_\infty) \) as \( R \to \infty \). Moreover, if \( (\nabla, \Phi) \) is a \( G_2 \)-monopole, then \( \nabla_\infty \) is a pseudo-Hermitian–Yang–Mills connection on \( P_\infty \) with respect to the nearly Kähler structure on \( \Sigma \).

We prove the several statements in this theorem as a result of three Lemmata stated and proved below.

Lemma 7.2 (Existence of \( \nabla_\infty \)). Under the hypotheses of Theorem 7.1, there is a connection \( \nabla_\infty \) on a bundle over the link \( \Sigma \) such that \( \nabla|_{\Sigma_R} \) converges uniformly to \( \nabla_\infty \) as \( R \to \infty \).

Proof. By hypothesis, we know that

\[
|F|_g^2 \leq r^{-4}.
\]

For the \( G_2 \)-monopole case, in which we merely assume quadratic decay of \( F_\nabla^{14} \), the quadratic decay of the full \( F_\nabla \) follows from Theorem 6.1 (ii):

\[
3|F_\nabla^2|_g^2 = |\nabla\Phi|^2_g \leq r^{-2(n-1)}.
\]

Now consider the cylinders \( C_R = \{ x \in X : r(x) \in [R, eR] \} \) with the conical metric \( g_C \) which for large \( R \) approximates well the \( G_2 \)-metric \( g \). Then, we rescale it by \( r^{-2} \) to obtain the cylindrical metric

\[
h = r^{-2}g_C = dt^2 + g_{\Sigma},
\]

where \( t = \log(r) \). With respect to this translation invariant metric we can identify all the cylinders \( C_R \) with \( (C = [0, 1]_t \times \Sigma, h) \). As from the above, we have

\[
|\nabla\Phi|^2_h \leq e^{-2nt} \quad \& \quad |F_\nabla|^2_h \leq 1,
\]

we find that the restriction of \( \nabla_i = \nabla|_{C_i} \) seen as a connection over \( C \) has uniformly bounded curvature with respect to \( h \). Thus, Uhlenbeck’s compactness results [43] apply and by possibly passing to a subsequence, \( A_i \) convergences modulo gauge, as \( i \to \infty \), to a well defined connection \( \nabla_\infty \) on \( C \).

We must now argue that such a connection is unique and does not depend on the subsequence. For that consider \( \nabla_i \) on \( C_i \) written in radial gauge with respect to \( r \), i.e. \( \nabla_i = a_i(r) \) with \( a_i(\cdot) \) a 1-parameter family of connections over \( \Sigma \) parametrized by \( r \in [R, eR] \), then \( F_{\nabla_i} = dr \wedge \partial_r a_i(r) + F_{a_i}(r) \) where \( F_{a_i}(r) \) is the curvature of \( a_i(r) \) over \( \{ r \} \times \Sigma \). Using this, we
find $|\partial_r \nabla_i|_g \lesssim |F_{\nabla_i}|_r \lesssim r^{-2}$ and so
\[
\int_R^e |\partial_r \nabla_i|_g \, dr \lesssim R^{-1},
\]
which decays as $R \to \infty$. This not only shows that the limit
\[
\nabla_\infty = \lim_{r \to \infty} \nabla(r),
\]
exists, as proves it is independent of the coordinate $r$ and so a pullback from a connection over $\Sigma$. Thus, it agrees with the connection $\nabla_\infty$ obtained as the uniform limit of the $\nabla_i$ which is therefore pullback from $\Sigma$. □

**Lemma 7.3** ($\nabla_\infty$ is pseudo-Hermitian–Yang–Mills). In case $(\nabla, \Phi)$ is a G$_2$-monopole, the connection $\nabla_\infty$ from Lemma 7.2 is pseudo-Hermitian–Yang–Mills, i.e. it satisfies
\[
\Lambda F_{\nabla_\infty} = 0 = F_{\nabla_\infty}^{0,2},
\]
with respect to the nearly Kähler structure on $\Sigma$ induced by the conical G$_2$-structure.

**Proof.** With respect to the metric $g_C$ and the coordinate $r$ we can view
\[
(C, g_C) = \left( [1, e], \Sigma, dr^2 + r^2 g_\Sigma \right),
\]
that is, $g_C$ is a conical metric on the fixed cylinder $C$. This metric has holonomy G$_2$ and $\varphi_C = dr \wedge \omega + \text{Re}(\Omega)$ with $(\omega, \text{Re}(\Omega))$ determining the nearly Kähler structure on the cone. Hence, we can decompose the curvature of $\nabla_\infty$ as
\[
F_{\nabla_\infty} = F_{\nabla_\infty}^7 + F_{\nabla_\infty}^{14},
\]
according to types with respect to $\varphi_C$.

On the other hand, as $|F_{\nabla_\infty}^7|_{h} \lesssim e^{-2nt} \to 0$ uniformly, we conclude that any $\nabla_\infty$ must satisfy
\[
F_{\nabla_\infty}^7 = 0.
\]
Now, write $\nabla_\infty$ in radial gauge in $C$, i.e. as $\nabla_\infty = a_\infty(r)$ with $a_\infty(\cdot)$ a 1-parameter family of connections over $\Sigma$ parametrized by $r \in [1, e]$. Then, its curvature can be written as
\[
F_{\nabla_\infty} = F_{a_\infty} + dr \wedge \partial_r a_\infty.
\]
Then, the condition $F_{\nabla_\infty}^7 = 0$ can equivalently be written as $F_{\nabla_\infty} \wedge \psi_C = 0$ where $\psi_C = \frac{\omega^2}{2} - dr \wedge \text{Im}(\Omega)$, which then yields
\[
\partial_r a_\infty \wedge \frac{\omega^2}{2} = F_{a_\infty} \wedge \text{Im}(\Omega),
\]
\[ F_{a_{\infty}} \wedge \frac{\omega^2}{2} = 0. \]

The result follows from observing that \( \partial_r a_{\infty} = 0. \)

Lemma 7.4 (Existence of \( \Phi_{\infty} \)). Under the hypotheses of Theorem 7.1, there is a \( \nabla_{\infty} \)-parallel section \( \Phi_{\infty} \) of the adjoint bundle \( g_{P_{\infty}} \) over \( \Sigma \) such that \( \Phi(R) = \Phi_{|\Sigma_R} \) converges uniformly to \( \Phi_{\infty} \) as \( R \to \infty \).

Proof. Consider \( \Phi_R = \Phi_{|C_R} \) where \( C_R = \{ x \in X : \log(R) \leq t(x) \leq \log(R) + 1 \} \) is equipped with the cylindrical metric \( h \) introduced in the proof of Lemma 7.2. Then, translating the coordinate \( t \), we consider \( \Phi_R \) as a 1-parameter family of Higgs fields in the fixed cylinder \( C = [0,1]_t \times \Sigma \) with the fixed metric \( h \). Then, Theorem 6.1 (ii) implies

\[ |\nabla \Phi|^2 \lesssim e^{-2nt}, \]

for \( r \in [R,eR] \), i.e. \( t \in [\log(R),\log(R) + 1] \). Thus, translating this back to analyze the sequence \( \Phi_R \) in the cylinder \( C \) we have

\[ |\nabla \Phi_R|^2 \lesssim R^{-2(n-1)}, \]

which converges to zero as \( R \to \infty \). This together with \( |\Phi_R| \lesssim m \) shows that \( \Phi_R \to \Phi_{\infty} \) uniformly over \( C \) with \( \nabla_{\infty} \Phi_{\infty} = 0 \). In particular, \( \partial_t \Phi_{\infty} = 0 \) and so \( \Phi_{\infty} \) is independent of \( t \), or \( r \), and so is pulled back from \( \Sigma \). \( \square \)

8. Bogomolny trick for the intermediate energy

In this section we use the outcome of our second main result Main Theorem 2 to deduce a formula for the intermediate energy of critical point configurations, showing in particular that \( G_2 \)-monopoles minimize such functional. Before stating the main result we recall a few basic features of our setup. Suppose \((X,\varphi)\) is an (irreducible) AC \( G_2 \)-manifold. Then for all sufficiently large \( r \gg 1 \), the cohomology class \( [\Upsilon^*\psi|_{[r]\times \Sigma}] \in H^4(\Sigma,\mathbb{R}) \) is independent of \( r \). In Definition 2.6 we have called this the asymptotic cohomology class of \((X,\varphi)\) and we denote it by \( \Psi_{\infty} \).

Now let \((\nabla,\Phi)\) be as in Theorem 7.1, with finite mass \( m \) (by Theorem 4.1), and suppose that \( m \neq 0 \). Then \((\nabla,\Phi)\) converges along the end to \((\nabla_{\infty},\Phi_{\infty})\) with \( \Phi_{\infty} \neq 0 \) and \( \nabla_{\infty} \Phi_{\infty} = 0 \). Hence, assuming \( G = SU(2) \), the holonomy of \( \nabla_{\infty} \) reduces to \( U(1) \subseteq SU(2) \) and \( \nabla_{\infty} \) is reducible to a connection on a \( U(1) \)-subbundle \( Q_{\infty} \) of \( P_{\infty} \). It follows from \( SU(2) \) representation theory that \( g_{P_{\infty}} \otimes \mathbb{C} \cong \mathbb{C} \oplus L \oplus L^* \), for a complex line bundle \( L \to \Sigma \). We now define what the higher dimensional analog of what in 3 dimensions is known as the monopole charge.
Definition 8.1. Given a $G_2$-monopole $(\nabla, \Phi)$ on an asymptotically conical manifold as above, the class $\beta = c_1(L) \in H^2(\Sigma, \mathbb{Z})$ is called a monopole class of $(\nabla, \Phi)$.

Remark 8.2. Given a monopole class $\beta$, there is a unique pseudo-Hermitian–Yang–Mills connection on a complex line bundle $L$ with $c_1(L) = \beta$, see in [18, Remark 3.25]. This is precisely the unique connection whose curvature is the harmonic representative of $-2\pi i \beta$. The connection $\nabla_\infty$, to which $\nabla$ is asymptotic, is induced by this unique connection.

We are now in position to state our energy formula.

Theorem 8.3. Let $(X^7, \varphi)$ be an irreducible AC $G_2$-manifold, with asymptotic cohomology class $\Psi_\infty \in H^4(\Sigma, \mathbb{R})$ (cf. Definition 2.6), and $P \to X$ be a principal $SU(2)$-bundle. Suppose that $(\nabla, \Phi)$ is a configuration satisfying the second order equations (2.1a) and (2.1b) with $\nabla \Phi \in L^2(X)$, and either $|F_\nabla|\,|\,d \nabla|$ decays quadratically along the end or $(\nabla, \Phi)$ is a $G_2$-monopole such that $|F_\nabla|$ decays quadratically along the end. Let $m$ be the mass of $(\nabla, \Phi)$, assume that $m \neq 0$, and let $\beta \in H^2(\Sigma, \mathbb{Z})$ be the monopole class of $(\nabla, \Phi)$. If $(\nabla, \Phi)$ has finite intermediate energy $\mathcal{E}^\psi(\nabla, \Phi) < \infty$, then

$$\mathcal{E}^\psi(\nabla, \Phi) = 4\pi m \langle \beta \cup \Psi_\infty, [\Sigma] \rangle + \frac{1}{2} ||F_\nabla \wedge \psi - \ast \nabla \Phi||^2_{L^2(X)}.$$

(8.1)

In particular, when $(\nabla, \Phi)$ is a $G_2$-monopole, then

$$\mathcal{E}^\psi(\nabla, \Phi) = 4\pi m \langle \beta \cup \Psi_\infty, [\Sigma] \rangle.$$

Proof. Let $U \subseteq X$ be a precompact open set with smooth boundary $\partial U$. Then the intermediate energy of $(\nabla, \Phi)$ over $U$ can be written as

$$\mathcal{E}_U^{\psi}(\nabla, \Phi) := \frac{1}{2} \left( ||F_\nabla \wedge \psi||_{L^2(U)}^2 + ||\nabla \Phi||_{L^2(U)}^2 \right) = \frac{1}{2} ||F_\nabla \wedge \psi - \ast \nabla \Phi||_{L^2(U)}^2 + \langle F_\nabla \wedge \psi, \ast \nabla \Phi \rangle_{L^2(U)},$$

and

$$\langle F_\nabla \wedge \psi, \ast \nabla \Phi \rangle_{L^2(U)} = \int_U \langle F_\nabla \wedge \psi \wedge \nabla \Phi \rangle = \int_U d(\langle F_\nabla \wedge \psi \rangle) = \int_{\partial U} \langle F_\nabla \wedge \psi \rangle,$$

where we have used the Bianchi identity $d F_\nabla = 0$ and the fact that $d \psi = 0$ on the second equality and Stokes’ theorem in the last. Thus

$$\mathcal{E}_U^{\psi}(\nabla, \Phi) = \int_{\partial U} \langle F_\nabla \wedge \psi \rangle + \frac{1}{2} ||F_\nabla \wedge \psi - \ast \nabla \Phi||_{L^2(U)}^2.$$

Now, the finiteness of the intermediate energy implies that the function

$$f(R) := \mathcal{E}^{\psi}_{B_R}(\nabla, \Phi) = \int_{\Sigma_R} \langle F_\nabla \wedge \psi \rangle + \frac{1}{2} ||F_\nabla \wedge \psi - \ast \nabla \Phi||_{L^2(B_R)}^2$$
is bounded, nondecreasing and converges to the intermediate energy of \((\nabla, \Phi)\). Thus, we are done if we show that
\[
\lim_{R \to \infty} \int_{\Sigma_R} \langle \Phi, F_\Sigma \rangle \wedge \psi = 4\pi m(c_1(L) \cup \Psi_\infty, [\Sigma]). \tag{8.2}
\]

Let \(\Psi_\infty\) denote the asymptotic cohomology class of \([\psi|_{\Sigma_R}]\) as in Definition 2.6 and \(\psi_\infty\) its harmonic representative. Then, we can write \(\psi|_{\Sigma_R} = \psi_\infty + d\gamma\) for some \(|\gamma| = O(R)|\) with respect to the conical metric. Using this we find,
\[
\int_{\Sigma_R} \langle \Phi, F_\Sigma \rangle \wedge d\gamma \leq \int_{\Sigma_R} \langle \nabla \Phi \wedge F_\Sigma \rangle \wedge \gamma \leq \text{Vol}_{\text{SU}^2}([R] \times \Sigma) \sup_{r=R} |\nabla \Phi|/|F_\Sigma|/|\gamma| \leq R^{-1},
\]
where we have used the fact that \(|\nabla \Phi| = O(r^{-6})|\) by Theorem 6.1, \(|F_\Sigma| = O(r^{-2})|\) (cf. proof of Lemma 7.2) and \(|\gamma| = O(r)|\). Hence,
\[
\lim_{R \to \infty} \int_{\Sigma_R} \langle \Phi, F_\Sigma \rangle \wedge \psi = \lim_{R \to \infty} \int_{\Sigma_R} \langle \Phi, F_\Sigma \rangle \wedge \psi_\infty = \int_{\Sigma} \langle \Phi_\infty, F_{\Psi_\infty} \rangle \wedge \psi_\infty. \tag{8.3}
\]

In the case at hand we have \(G = \text{SU}(2),|\) and as \(\Phi_\infty \neq 0|\) while \(\nabla_\infty \Phi_\infty = 0|\), the bundle \(P_\infty\) reduces from \(\text{SU}(2)\) to \(\text{U}(1) \subseteq \text{SU}(2)\) and thus \(g_{P_\infty} = \mathbb{R} \oplus L\), for a complex line bundle, \(L\), such that the complex rank 2 vector bundle associated with the standard representation can be written as \(P_\infty \times_{\text{SU}(2)} \mathbb{C}^2 \cong L \oplus L^*\). Using this splitting one writes
\[
\Phi_\infty = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix}, \quad \text{and} \quad F_{\nabla_\infty} = \begin{pmatrix} F_L & 0 \\ 0 & -F_L \end{pmatrix},
\]
with \(F_L \in -2\pi ic_1(L) \in H^2(\Sigma, -2\pi i\mathbb{Z})|\) and \(m = |\Phi_\infty|\) is the mass of \((\nabla, \Phi)\). Then, inserting into equation (8.3) we find equation (8.2) as we wanted. \(\square\)

Under the hypothesis of Theorem 8.3, the energy formula (8.1) writes the intermediate energy as a sum of a topological/geometrical term, determined by the asymptotic geometry of the configuration and the manifold, and a quantity which is always greater than or equal to zero, with equality if and only if \((\nabla, \Phi)\) is a \(G_2\)-monopole. Thus showing, in particular, that keeping both the monopole class \(\beta\) and the mass \(m\) fixed, \(G_2\)-monopoles minimize the intermediate energy amongst such configurations.

This result also has interesting applications to the existence of \(G_2\)-monopoles on certain asymptotically conical \(G_2\)-manifolds. The following corollary is adapted from [37], where it was applied using a different hypothesis on the asymptotic behavior of the \(G_2\)-monopoles. Thus our next result may be regarded as a slight improvement on the aforementioned result in [37].
Corollary 8.4. Let \((X^7, \varphi)\) be an irreducible AC \(G_2\)-manifold, with vanishing asymptotic cohomology class \(\Psi_\infty \in H^4(\Sigma, \mathbb{R})\), e.g. if \(H^2(\Sigma, \mathbb{Z})\) vanishes as is the case for the Bryant–Salamon metric on \(\mathbb{R}^4 \times S^3\) for which \(\Sigma = S^3 \times S^3\). Then, there are no irreducible \(G_2\)-monopoles \((\nabla, \Phi)\) on \((X^7, \varphi)\) with \(|F^1_{\nabla}|\) quadratically decaying and finite nonzero intermediate energy.

Proof. In this situation, the energy formula from Theorem 8.3 applies and we find that the intermediate energy vanishes. \(\square\)

9. Decay of linearized solutions

In this section we make further use of the methods of the previous sections, and give decay estimates to finite energy solutions to the linearization of the \(G_2\)-monopole equation, with appropriate gauge fixing.

9.1. The linearized equation. First we compute the (formal) linearization of the \(G_2\)-monopole equation (1.1).

Let the configuration space be \(C_P = \text{Conn}_P \times \Omega^0_{g_p}\), that is, the space of (Sobolev) pairs of connections on \(P\) and sections of \(g_p\). Consider the (smooth) maps

\[
\text{mon}^\pm : C_P \to \Omega^1_{g_p};
\]

\[
(\nabla, \Phi) \mapsto * (F_\nabla \wedge \psi) \pm \nabla \Phi,
\]

that, in case of a \(G_2\)-monopole, satisfy

\[
\text{mon}^-(\nabla, \Phi) = 0,
\]

\[
\text{mon}^+(\nabla, \Phi) = 2\nabla \Phi.
\]

For simplicity we set \(\text{mon} := \text{mon}^-\) for the rest of the paper.

Since \(C_P\) is an affine space—in fact, an affine Hilbert manifold—the tangent spaces of \(C_P\) at any point can canonically be identified with \(\Omega^1_{g_P} \oplus \Omega^0_{g_P}\), as Hilbert spaces. We can now compute the first derivative of \(\text{mon}\). If \(X = (a, \phi) \in T_{(\nabla, \Phi)} C_P\), then

\[
\left( T_{(\nabla, \Phi)} \text{mon} \right)(a, \phi) = * (d_\nabla a \wedge \psi) - \nabla \phi - [a, \Phi].
\]

We also impose the standard, Coulomb type gauge fixing condition that we only consider tangent vectors \((a, \phi) \in T_{(\nabla, \Phi)} C_P\) that are perpendicular to the gauge orbit through \((\nabla, \Phi)\), which amounts to the following PDE:

\[
d_\nabla a + [\phi, \Phi] = 0.
\]
We can organize equations (9.1) and (9.2) into a single equation. Let
\[
\mathcal{D}(a, \phi) = \big( \ast(d\nabla a \wedge \psi) - \nabla \phi, -d\nabla^\ast a \big),
\]
\[
q(a, \phi) = ([\Phi, a], [\Phi, \phi]),
\]
and let \( \mathcal{L} = \mathcal{D} + q \). Then equations (9.1) and (9.2) are equivalent to
\[
\mathcal{L}(a, \phi) = 0. \tag{9.4}
\]
We call equation (9.4) (or equations (9.1) and (9.2) together) the linearized \( G_2 \)-monopole equation.

9.2. A priori properties of the linearized \( G_2 \)-monopole equation. First we prove two Weitzenböck type formulas, one for \( \mathcal{L} \) and one for \( \mathcal{L}^\ast \).

Lemma 9.1. Let \((\nabla, \Phi) \in C_p \) be any pair. For any \((a, \phi) \in T_{(\nabla, \Phi)}C_p \) smooth pair let us define
\[
I_+(a, \phi) = 2 \ast([\ast F_\nabla] \wedge a) - [\text{mon}^+(\nabla, \Phi), \ast[\text{mon}^+(\nabla, \Phi) \wedge a]),
\]
\[
I_-(a, \phi) = -2 \ast(\phi \wedge [F_\nabla \wedge a]) - [\text{mon}^-(\nabla, \Phi), \ast[\text{mon}^-(\nabla, \Phi) \wedge a]).
\]

Then we have
\[
\mathcal{L}^\ast = \nabla^\ast + I_+ - q^2,
\]
\[
\mathcal{L} \mathcal{L}^\ast = \nabla^\ast + I_- - q^2.
\]

Proof. We start by noticing that \( \mathcal{L} = \mathcal{D} + q \) and \( \mathcal{L}^\ast = \mathcal{D} - q \), then we can write
\[
\mathcal{L}^\ast \mathcal{L} = \mathcal{D}^2 + [\mathcal{D}, q] - q^2,
\]
\[
\mathcal{L} \mathcal{L}^\ast = \mathcal{D}^2 - [\mathcal{D}, q] - q^2.
\]

Let us first rewrite \( \mathcal{D}^2 \):
\[
\mathcal{D}^2(a, \phi) = \mathcal{D} \big( \ast(d\nabla a \wedge \psi) - \nabla \phi, -d\nabla^\ast a \big)
\]
\[
= \big( \ast(d\nabla (\ast(d\nabla a \wedge \psi) - \nabla \phi) \wedge \psi) - \nabla \big( -d\nabla^\ast a \big), -d\nabla^\ast (\ast(d\nabla a \wedge \psi) - \nabla \phi) \big)
\]
\[
= \big( d\nabla d\nabla^\ast a + \ast(\ast(d\nabla a \wedge \psi)) \wedge \psi \big) - \ast \left( \big( d\nabla^2 \phi \big) \wedge \psi, \nabla^\ast \nabla \phi - \ast \left( \big( d\nabla^2 a \big) \wedge \psi \right) \right)
\]
\[
= \big( d\nabla d\nabla^\ast a + \ast(\ast(d\nabla a \wedge \psi)) \wedge \psi \big) - \ast \left( \big( F_\nabla \wedge \psi, \phi \big), \nabla^\ast \nabla \phi - \ast \left( \big( F_\nabla \wedge a \big) \wedge \psi \right) \right)
\]
Let \( d_{\nabla}^2 := \Pi_7 \circ d\nabla : \Omega^1 \to \Omega^2_7 \), where \( \Pi_7 : \land^2 \to \land^2_7 \) is the pointwise orthogonal projection. Since for 2-form, \( \omega \), we have that \( \ast(\omega \wedge \psi) = \omega^\ast \), we get that
\[
\ast \big( d\nabla (\ast(d\nabla a \wedge \psi)) \wedge \psi \big) = 3d\nabla d_{\nabla}^2 a - d_{\nabla}^2 d\nabla a = \ast \big( (F_\nabla \wedge \psi) \wedge a \big).
Using also \( *(F_\nabla \wedge \psi) = \nabla \Phi \) and the Jacobi identity, we get that
\[
D^2(a, \phi) = (d_\nabla d_\nabla a + d_\nabla^* d_\nabla a - *(F_\nabla \wedge \phi) \wedge a) - [\nabla \Phi, \phi], \nabla^* \nabla \Phi + *(\nabla \Phi) \wedge a). \]

Now recall the Weitzenböck identity for bundle-valued 1-forms on a Ricci-flat manifold:
\[
d_\nabla d_\nabla a + d_\nabla^* d_\nabla a = \nabla^* \nabla a + *(\nabla \nabla \Phi) \wedge a. \]

Combining these we get
\[
D^2(a, \phi) = (\nabla^* \nabla a + *(F_\nabla - F_\nabla \wedge \phi) \wedge a) - [\nabla \Phi, \phi], \nabla^* \nabla \Phi + *(F_\nabla \wedge \psi) \wedge a). \]

Let us now compute \([D, q]\).
\[
[D, q](a, \phi) = D([\Phi, a], [\Phi, \phi]) - q(\Phi, d_\nabla a \wedge \psi - \nabla \Phi, d_\nabla a)
= \left(\Phi, *(d_\nabla([a] \wedge \psi) - \nabla([\Phi, a]), -d_\nabla([\phi, a])\right)
- \left([\Phi, *(d_\nabla a \wedge \psi) - \Phi, \nabla a], [\Phi, -d_\nabla a]\right)
= \left(\Phi, *(d_\nabla a \wedge \psi) - [\Phi, \nabla a], [\Phi, -d_\nabla a]\right)
= *(\nabla \Phi \wedge \psi) \wedge a - [\nabla \Phi, \phi], [\nabla \Phi \wedge a]. \]

Thus, after some simplification, we get
\[
(D^2 \pm [D, q] - \nabla^* \nabla)(a, \phi) =
\left(\Phi, *(\pm \nabla \Phi \wedge \psi + F_\nabla - F_\nabla \wedge \phi) \wedge a) - [\text{mon}\,(\nabla \Phi, \phi), *\text{mon}\,(\nabla \Phi \wedge \phi)\right]. \]

Using the \(G_2\)-monopole equation (1.1) and the following identity
\[
\omega \wedge \psi = 2 * \omega^7 - \omega^{14}, \tag{9.5} \]
that holds for any \(\omega = \omega^7 + \omega^{14} \in \Lambda^2 X = \Lambda^2_7 X \oplus \Lambda^2_{14} X\), we can rewrite \(I_{\pm}(a, \phi)\) as
\[
(D^2 \pm [D, q] - \nabla^* \nabla)(a, \phi) = \left(*\left((-1 \pm 3 * F_\nabla^7 + 2 * F_\nabla^{14}) \wedge a) - [\text{mon}\,(\nabla \Phi, \phi), *\text{mon}\,(\nabla \Phi \wedge \phi)\right)\right), \]
which completes the proof (after using equation (9.5) again in negative sign case). \(\square\)

Using Moser iteration, we get the following immediate corollary.

**Corollary 9.2.** Let \((\nabla, \Phi) \in C_P\) be a pair, such that \(r^2(|F_\nabla| + |\nabla \Phi|) \in L^\infty(X)\). Let \(V = (a, \phi) \in T_{(\nabla, \Phi)} C_P\) be in the \(L^2(X)\)-kernel of either \(L\) or \(L^*\). Then \(r^n |V|^2 \in L^\infty(X)\).
Proof. By elliptic regularity, \( V \) is smooth. Using the results of Lemma 9.1, we get (in both cases) and outside of \( B_R \) (for \( R \) large enough) that
\[
|V(x)|^2 \leq C \frac{r}{r^2} |V|^2.
\]
In fact, we have faster than quadratic decay for the coefficient, but that does not change what follows. We can now use Moser iteration (cf. Proposition 2.1). Since the injectivity radius satisfies \( \text{inj}(x) \geq cr(x) \), we have (with a potentially different constant) that
\[
|V(x)|^2 \leq C r(x)^n \int_{B_{r(x)/2}(x)} |V|^2 \text{vol}_X,
\]
which completes the proof. □

9.3. Decay properties of solutions to the linearized equation.

**Theorem 9.3.** Let \((\nabla, \Phi) \in C_p \) be a pair, such that \( r^2(|F| + |\nabla \Phi|) \in L^\infty(X) \). Let \((a, \phi) \in T_{(\nabla, \Phi)} C_p \) be in the kernel of either \( L \) or \( L^* \), \( V = (a, \phi) \), and \( |V|^2 = |a|^2 + |\phi|^2 \). Then for all \( \varepsilon > 0 \), we have that \( r^{2(n-1)}|V|^2 \in L^\infty(X) \).

**Proof.** To prove the claim, we now bound the integrals \( \int_{B_{r(x)}(x)} |V|^2 \text{vol}_X \) in inequality (9.6), using the improved Hardy’s inequality (6.12) in a similar way as in inequalities (6.13) and (6.15). More concretely, we show that for all \( \alpha < \frac{n-1}{2} \):
\[
\| r^{\alpha-\frac{1}{2}} V \|^2_{L^2(X)} \leq C \frac{n-1-2\alpha}{n-1} \| V \|^2_{L^2(X)}.
\]
We again use inequality (6.12) together with Kato’s inequality, but now with \( f = r_{l,L}^\alpha r^{-1/2} |V| \), to get
\[
\frac{(n-2)^2}{4} \| r_{l,L}^\alpha r^{-1/2} V \|^2_{L^2(X-B_l)} + \| r_{l,L}^\alpha \sqrt{r} \nabla \Sigma \nabla \Phi \|^2_{L^2(X-B_l)} \leq \| \nabla (r_{l,L}^\alpha \sqrt{r} \nabla \Phi) \|^2_{L^2(X-B_l)}
\]
\[
+ C_H \| r_{l,L}^{\alpha} r^{1/2+\varepsilon} \nabla \Phi \|^2_{L^2(X-B_l)}.
\]
From here we can proceed almost identically to proof of Proposition 6.8 (in fact, the computations are now easier as the equations are linear, and we already have the decay result for \((\nabla, \Phi)\) by Main Theorem 2) to get that \( r^{2(n-1)-\varepsilon} |V|^2 \in L^\infty(X) \).

Then, as in Section 6.4, one can show that, in fact, \( r^{2(n-1)} |V|^2 \in L^\infty(X) \). □
10. Weighted split Sobolev spaces: Fredholm operators

Let $(X, g)$ be an asymptotically conical spin manifold. In this section we construct Sobolev spaces suitable for a Fredholm theory for $G_2$-monopoles. In that setting, the gauge fixed linearization of the $G_2$-monopoles equation $L = D + q$, as defined in Section 9.1, is the sum of twisted Dirac type operator $D$ and a potential $q$. Such operators are usually called Callias type operators \[4\] and are well known to be related to the deformation theory of monopoles in other dimensions \[27, 28\]. Hence, in this section we work with a slightly more general setup than what is needed and the Sobolev spaces constructed are those suitable to analyze Callias type operators.

Let $S$ be the spinor bundle of $(X, g)$, i.e. the vector bundle associated with the standard Spin$(n)$ representation, and $E$ an auxiliary metric vector bundle which in the monopole case should be thought of as being $g_P$. Denote the tensor product of these two bundles by $S_E := S \otimes E$ and equip it with a connection $\nabla$ and a metric $\langle \cdot, \cdot \rangle$ both induced by counterparts on $E$ and $S$ (equipped with the metric and connection induced by $g$ and the Levi–Civita connection). This section culminates in Theorem 10.12 with the proof that $L$ is Fredholm when operating between some suitable and carefully constructed Banach spaces.

10.1. Nondegenerate potential and standard Sobolev spaces. We start by analyzing the case when the potential $q$ is pointwise invertible at infinity and prove that, in this situation, the operator $L = D + q$ is Fredholm between standard Sobolev spaces. This is the setting worked out in \[2, 27\] where a formula for the index in a quite general setup is given. Here we give a short proof that $L$ is Fredholm along the lines motivated by \[13, 41\]. We begin by studying the model situation on a cone which we then extend to the AC setting. Before proceeding, we introduce a little notation. Let $\epsilon$ be a smooth positive function on a metric cone $C = ((1, \infty) \times \Sigma, g_C = dr^2 + r^2 g_\Sigma)$ decaying with all derivatives as $r \to \infty$, i.e. such that \[10.1\]

$$\lim_{r \to \infty} |r^i \nabla^i \epsilon(r)| = 0.$$ 

Proposition 10.1. In the metric cone as above and $L_C = D_C + q_C$ as before with $q_C$ parallel and bounded by bellow, i.e. $\nabla(q_C(f)) = q_C(\nabla f)$ and $|q_C(f)|^2 \geq c|f|^2$ for some constant $c > 0$ and all $f \in \Omega^0(C, S_E)$. Furthermore, suppose there is a Weitzenböck formula

$$L_C^* L_C = \nabla^* \nabla + W + q_C^* q_C,$$
with \( W \) satisfying \( |W(f)| \leq \varepsilon^2(r)|f| \) for some function \( \varepsilon(r) > 0 \) as in equation (10.1). Then, the following inequality holds

\[
\|f\|_{L_1^2(X)}^2 \leq \|\mathcal{L}_C f\|_{L_1^2(X)}^2 + \|\varepsilon(r)f\|_{L_1^2(X)}^2, \tag{10.2}
\]

for all compactly supported \( f \).

**Proof.** For compactly supported \( f \) one can integrate by parts and use the Weitzenböck formula in the statement

\[
\|\mathcal{L}_C f\|_{L_1^2(X)}^2 = \langle \mathcal{L}_C^* \mathcal{L}_C f, f \rangle_{L^2(X)} = \|\nabla f\|_{L_1^2(X)}^2 + \langle W(f), f \rangle_{L^2(X)} + \|q(f)\|_{L^2(X)}^2
\]

and passing the term \( \|\varepsilon(r)f\|_{L_1^2(X)}^2 \) to the left hand side yields the inequality (10.2). \( \square \)

We now go from the conical case to the asymptotically conical one.

**Lemma 10.2.** Let \( \varepsilon : X \to \mathbb{R}_+ \) be smooth function satisfying (10.1) and \( \varepsilon^{-1}L^2(X) = \{ f \mid \varepsilon f \in L^2 \} \). Then the embedding \( L_1^2(X) \hookrightarrow \varepsilon^{-1}L^2(X) \) is compact.

**Proof.** The Banach space \( \varepsilon^{-1}L^2(X) \) can equally be defined as the completion of the smooth compactly supported sections in the norm \( \|f\|_{\varepsilon^{-1}L^2} = \|\varepsilon f\|_{L^2(X)} \). Then, we consider a generic sequence \( \{f_i\} \subseteq L_1^2 \) satisfying \( \|f_i\|_{L_1^2(X)} = 1 \), which we show must subsequentially converges in \( \varepsilon^{-1}L^2(X) \).

Since \( \|f_i\|_{L_1^2(X)}^2 = 1 \), there is a subsequence converging to a weak limit \( f \in L_1^2(X) \) satisfying \( \|f\|_{L_1^2(X)}^2 \leq 1 \). We now show that this subsequence converges strongly in \( \varepsilon^{-1}L^2(X) \) to \( f \). In the following computation let \( B_R = B_R(x_0) \). Then,

\[
\|\varepsilon(f_i - f)\|_{L_1^2(X)}^2 = \|\varepsilon(f_i - f)\|_{L_1^2(B_R)}^2 + \|\varepsilon(f_i - f)\|_{L_1^2(X-B_R)}^2 \leq c_1\|f_i - f\|_{L_1^2(B_R)}^2 + \varepsilon^2(R)\|f_i - f\|_{L_1^2(X-B_R)}^2 \leq c_1\|f_i - f\|_{L_1^2(B_R)}^2 + 4\varepsilon^2(R), \tag{10.3}
\]

where in the last inequality we used

\[
\|f_i - f\|_{L_1^2(X-B_R)}^2 \leq \|f_i - f\|_{L_1^2(X-B_R)}^2 \leq 2\|f_i\|_{L_1^2(X-B_R)}^2 + 2\|f\|_{L_1^2(X-B_R)}^2 \leq 4.
\]

The second term in equation (10.3) is \( 4\varepsilon^2(R) \) and, by increasing \( R \), can be taken to be arbitrarily small. Regarding the first one \( \|f_i - f\|_{L_1^2(B_R)}^2 \), since the embedding \( L_1^2(B_R) \hookrightarrow L^2(B_R) \) is compact, \( f_i \) does converge strongly to \( f \) in \( L^2(B_R) \) and the term \( \|f_i - f\|_{L_1^2(B_R)}^2 \) can also be made arbitrarily small by increasing \( i \). \( \square \)
Lemma 10.3. Let $\mathcal{L} : \Omega^0(X,S_E) \to \Omega^0(X,S_E)$ be modeled on a conical operator $\mathcal{L}_C$ as in Proposition 10.1. Then, there is a function $\varepsilon$ as in (10.1) such that the inequality
\[
\|f\|_{L^2_1(X)}^2 \leq \|\mathcal{L}f\|_{L^2_1(X)}^2 + \|f\|_{c_2 L^2_1(X)}^2
\]
holds for any $f \in L^2_1(X)$.

Proof. We prove this inequality by putting together two similar inequalities computed over: the interior of $X$; and its ends. As in the proof of the previous lemma, let $R \gg 1$ and $B_R := B_R(x_0)$. The ellipticity of $L$ on $B_{R+1}$ implies that
\[
\|f\|_{L^2_1(B_{R+1})}^2 \leq \|\mathcal{L}f\|_{L^2(B_{R+2})}^2 + \|f\|_{L^2(B_{R+2})}^2
\]
for all $f$. This deals with the interior of $X$ and we now focus with what happens at its ends, i.e., on $X - B_R$. Notice that by possibly increasing $R$, we can assume that $X - B_R$ is quasi isometric to the metric cone and there is a model conical operator $\mathcal{L}_C$ on the cone satisfying the hypothesis in Proposition 10.1, and $\mathcal{L} - \mathcal{L}_C = O(r^{-1-\delta})$ for some $\delta > 0$. Hence, it follows from Proposition 10.1 that
\[
\|f\|_{L^2_1(X - B_R)}^2 \leq \|\mathcal{L}f\|_{L^2(X - B_R)}^2 + \|\varepsilon f\|_{L^2(X - B_R)}^2
\]
for $\varepsilon' = \max\{r^{-1-\delta}, \varepsilon\}$. The last step is to put this together (10.5) and (10.6). For this, fix a bump function $\varphi_R$ be a supported on $B_{R+1}$ which equals 1 on $B_R$, then
\[
\|f\|_{L^2_1(X)}^2 = \|f\|_{L^2_1(B_R)}^2 + \|f\|_{L^2_1(X - B_R)}^2
\]
which is simply (10.4). \qed

Corollary 10.4. Let $\mathcal{L} : L^2_1(X) \to L^2(X)$ be as in Lemma 10.3. Then $\mathcal{L}$ has closed range and finite dimensional kernel.

Proof. To prove that the kernel is finite dimensional we prove that the unit ball in the kernel is compact. So let $\{f_i\} \subseteq \ker(D)$ be a sequence with $\|f_i\|_{L^2_1(X)}^2 = 1$. From Lemma 10.2, the embedding $L^2_1(X) \hookrightarrow \varepsilon^{-1} L^2(X)$ is compact and so there is a subsequence $f_i$, which converges strongly in $\varepsilon^{-1} L^2(X)$ to some $f \in \ker(D) \cap \varepsilon^{-1} L^2(X)$. But then, the inequality (10.4) gives $\|f_i - f\|_{L^2_1(X)}^2 \leq c_2 \|f_i - f\|_{L^2(X)}^2$.$\to 0$, and so $f_i$ does converge to $f$ strongly in $L^2_1(X)$.
Next we prove that the image is closed, for that it is enough to prove that there is a constant $c > 0$, such that for all $f \in (\ker(\mathcal{L}))^\perp \cap L^2_1(X)$

$$\|\mathcal{L} f\|_{L^2_1(X)} \geq c\|f\|_{L^2_1(X)}^2.$$  

Suppose not, then there is a sequence $\{f_i\} \subseteq (\ker(\mathcal{L}))^\perp \cap L^2_1(X)$ with $\|\mathcal{L} f_i\|_{L^2_1(X)} \to 0$ and $\|f_i\|_{L^2_1(X)} = 1$. There is a weak limit $f \in L^2_1(X)$ such that $\mathcal{L} f = 0$ and from Lemma 10.2, the limit $f$ is strong in $\mathcal{L}^* L^2(X)$. In fact $f = 0$ since by assumption it is the limit of the $f_i$’s which are in the orthogonal complement to the kernel. Then inequality (10.4) gives

$$1 = \|f_i\|_{L^2_1(X)}^2 \leq \|\mathcal{L} f_i\|_{L^2_1(X)}^2 + \|\mathcal{E} f_i\|_{L^2_1(X)}^2$$

as the first term in the right hand side vanishes, while the second one converges to zero this is a contradiction. \hfill \blacksquare

**Corollary 10.5.** Let $\mathcal{L} : L^2_1(X) \to L^2(X)$ be as in Lemma 10.3, then it is a Fredholm operator.

**Proof.** We have shown in Corollary 10.4 that $\mathcal{L}$ has finite dimensional kernel and closed image. Hence, the only thing missing to prove that it is Fredholm is that the cokernel is finite dimensional. As $\text{coker}(\mathcal{L}) \cong \ker(\mathcal{L}^*) \cap L^2(X)$ one just needs to prove that this later one is finite dimensional. Since $\mathcal{L}^* = \mathcal{D} + \mathcal{Q}$, it is also modeled on an operator as in the hypothesis of Proposition 10.1 and therefore satisfies an inequality as in (10.4). Using such an inequality, one concludes that

$$\|f\|_{L^2_1(X)} \leq \|\mathcal{D} f\|_{L^2(X)} \leq \|f\|_{L^2(X)},$$

for all $f \in \ker(\mathcal{L}^*) \cap L^2(X)$, and so $\ker(\mathcal{L}^*) \cap L^2(X) \hookrightarrow L^2_1(X)$ and applying Corollary 10.4 to $\mathcal{L}^*$ proves that its kernel in $L^2_1(X)$ is finite dimensional. \hfill \blacksquare

For completeness we now see that any such $\mathcal{L}$ is also a Fredholm operator when considered as an operator on higher derivative Sobolev spaces.

**Proposition 10.6.** Let $k \in \mathbb{N}$ and $\mathcal{L} : \Omega^0(X, \mathcal{S}_E) \to \Omega^0(X, \mathcal{S}_E)$ be modeled on a conical operator $\mathcal{L}_C$ as in Proposition 10.1. Then $\mathcal{L} : L^2_{k+1}(X) \to L^2_k(X)$ is a Fredholm operator.

**Proof.** If one can prove an inequality of the form

$$\|f\|_{L^2_{k+1}(X)}^2 \leq \|\mathcal{L} f\|_{L^2_k(X)}^2 + \|\mathcal{E} f\|_{L^2_k(X)}^2$$  \hspace{1cm} (10.7)

for both $\mathcal{L}$ and $\mathcal{L}^*$ and some $\mathcal{E}$ as in equation (10.1), then the result follows. Indeed, one can repeat all of the steps done before with $L^2(X)$ replaced by $L^2_k(X)$ and $L^2_1(X)$ replaced by $L^2_{k+1}(X)$. We start by noticing that the operator $\mathcal{L}$ can be extended to act on sections of $\mathcal{L}^* \mathcal{T} \mathcal{X} \otimes \mathcal{S}_E$. Then, the Weitzenböck formulas for $\mathcal{L}^* \mathcal{L}$ and $\mathcal{L} \mathcal{L}^*$ have a further contribution...
coming from the Riemannian curvature, which actually vanishes in the Ricci flat case. In general, the manifold is AC and this algebraic term decays and it can be bounded from above by an $\varepsilon$ function as in equation (10.1), so one can assume these Weitzenböck formulas are as in Proposition 10.1. To establish the inequality, notice that

$$
\|f\|_{L^2_{k+1}(X)}^2 \leq \|f\|_{L^2_k(X)}^2 + \|\nabla f\|_{L^2_k(X)}^2
$$

(10.8)

and arguing by induction one can assume equation (10.7) to be true for $k$ replaced by $j < k$, and we now prove the case $j = k$. Then, using the induction hypothesis and inequality (10.8)

$$
\|f\|_{L^2_{k+1}(X)}^2 \leq \left(\|\mathcal{L}f\|_{L^2_k(X)}^2 + \|\nabla\mathcal{L}f\|_{L^2_{k-1}(X)}^2\right) + \left(\|\varepsilon f\|_{L^2_k(X)}^2 + \|\varepsilon \nabla f\|_{L^2_{k-1}(X)}^2\right).
$$

(10.9)

Notice that $\varepsilon \nabla f = \nabla(\varepsilon f) - (d\varepsilon) \otimes f$. Moreover, since $\varepsilon$ satisfies equation (10.1), there is some other function $\varepsilon_1$ still decaying as in equation (10.1) and so that $|\varepsilon| + |d\varepsilon| \leq \varepsilon_1$. So one can bound the above terms in the second pair of parentheses by $\|\varepsilon_1 f\|_{L^2_k(X)}^2$. To bound from above the terms in the first bracket in inequality (10.9), let $\{e_i\}$ be an orthonormal frame at $p \in X$ such that $\nabla e_i = 0$ at $p$. Then at $p$

$$
L(\nabla_j f) = \mathcal{D}\nabla_j f + q(\nabla_j f) = \sum_i e_i \nabla_i \nabla_j f + q(\nabla_j f)
$$

$$
= \sum_i \left(\nabla_j (e_i \nabla_i f) + e_i F_{\nabla}(e_i, e_j)(f) + \nabla_j q(f) - (\nabla_j q)(f)\right)
$$

$$
= \nabla_j (\mathcal{L} f) + \sum_i e_i F_{\nabla}(e_i, e_j)(f) - (\nabla_j q)(f),
$$

where $F_{\nabla}$ is the curvature of $\nabla$. As $\mathcal{L}$ is modeled in a conical operator as in Proposition 10.1 for which $q$ is parallel, we must have that $\nabla q$ and $F_{\nabla}$ decay and are therefore bounded above by some $\varepsilon_2$ as in equation (10.1). From this it immediately follows that

$$
\|\mathcal{L}\nabla f\|_{L^2_{k+1}(X)}^2 \leq \|\nabla\mathcal{L} f\|_{L^2_k(X)}^2 + \|\varepsilon_2 f\|_{L^2_{k+1}(X)}^2.
$$

which together with the previous bound $\|\varepsilon f\|_{L^2_k(X)}^2 + \|\varepsilon \nabla f\|_{L^2_{k-1}(X)}^2 \leq \|\varepsilon_1 f\|_{L^2_k(X)}^2$ gives the inequality (10.9) for some $\varepsilon \geq \varepsilon_1 + \varepsilon_2$. \qed

10.2. Vanishing potential and weighted Sobolev spaces. A Dirac-type operator $\mathcal{D}$ on an AC manifold is not Fredholm for the standard Sobolev spaces. Lockhart–McOwen [32] and Marshall [34] have both constructed suitable weighted versions of the standard Sobolev spaces on which the Dirac operator is Fredholm. In this subsection we recall the definition of these spaces and state the resulting Fredholm property.
Let $\alpha \in \mathbb{R}$ and $p, k \in \mathbb{N}_+$, the Lockhart–McOwen [32] weighted norms $\|\cdot\|_{L^p_{k,\alpha}(X)}$ of a smooth compactly supported twisted Dirac spinor $f \in \Gamma(X, S_E)$ are inductively defined by
\[
\|f\|_{L^p_{k,\alpha}(X)} = \|\nabla f\|_{L^p_{k-1,\alpha-1}(X)} + \|f\|_{L^p_{0,\alpha}(X)},
\]
and $\|f\|_{L^p_{k,\alpha}(X)} = \int_X |r^{-\alpha} r^p r^{-n} \text{vol}_X f|$. 

**Definition 10.7** (Lockhart–McOwen weighted Sobolev spaces; cf. [32]). The Lockhart–McOwen weighted Sobolev spaces with weight $\alpha \in \mathbb{R}$, $L^p_{k,\alpha}(X)$, are the norm-completion of the smooth compactly supported functions with respect to the norm in equation (10.10).

The next result states that the twisted Dirac operator $\mathcal{D}$ is Fredholm for these Sobolev spaces. This is a standard result, stated for example in [26, 34]. Alternatively, this theorem follows by translating all the setup into the cylindrical setting and using the results in [13, 32]. In fact, the results in [13] also prove that the model operator on a cone admits a right inverse in this case.

**Theorem 10.8.** Let $\mathcal{D}$ be a Dirac type operator on an AC manifold as above. Then, there is a discrete set of weights $K(\mathcal{D})$ such that for all $\alpha \in K(\mathcal{D})$ and $k \in \mathbb{N}$, the operator
\[
\mathcal{D} : L^2_{k+1,\alpha+1}(X) \to L^2_{k,\alpha}(X),
\]
is Fredholm, and
\[
L^2_{k,\alpha} = \mathcal{D}(L^2_{k+1,\alpha+1}(X)) \oplus W_\alpha(X),
\]
where $W_\alpha \cong \ker(\mathcal{D})_{-\alpha-n}$. Moreover, if $\alpha \geq -\frac{n}{2}$ equality holds as $\ker(\mathcal{D})_{-\alpha-n} \subseteq L^2_{k,\alpha}(X)$.

10.3. **Mixed situation and split Sobolev spaces.** This subsection analyses the case when $q$ is asymptotically degenerate but does not vanish identically. More precisely, we consider the following situation.

**Hypothesis 10.9.** Suppose that outside a compact set $K$, there is a splitting
\[
S_E = S_E^\parallel \oplus S_E^\perp
\]
such that:
- $q$ vanishes on $S_E^\parallel$ and is nondegenerate on $S_E^\perp$.
- $q$ is asymptotically parallel, meaning that $|r^j \nabla q|$ is a function as $\varepsilon$ in equation (10.1). In particular, there is $q_C$ on some $S_{E_\infty}$ such that $\Upsilon S_E \cong S_{E_\infty}$.
- $\nabla$ is modeled at infinity on a connection $\nabla_\infty$ on $S_{E_\infty}$ pulled back from the link of the asymptotic cone. In particular, $\mathcal{D}$ is modeled on a conical operator $\mathcal{D}_C$ as in Proposition 10.1.
\[ \nabla_\infty \text{ vanishes on } S_{E_\infty}^\parallel \text{ and is irreducible on } S_{E_\infty}^\perp. \]

From now on, we always assume that these hypotheses are satisfied.

**Remark 10.10.** This is the relevant case for finite mass, irreducible, \( G_2 \)-monopoles with gauge group \( G = SU(2) \) as we are considering. Indeed, in that case a \( G_2 \)-monopole \( (\nabla, \Phi) \) satisfies \( |\Phi| \to m > 0 \) at infinite and \( q = \text{ad}_\Phi \). Therefore, as already mentioned in the discussion opening Section 5, outside a large compact set, we can split

\[ g_P = g_P^\parallel \oplus g_P^\perp, \]

where \( g_P^\parallel = \ker(\text{ad}_\Phi) \). This in turn induces a splitting on \( S_{g_P} \) as in equation (10.11).

In this situation we consider a mixed type family of Sobolev spaces which along the end agree with the standard ones on \( S_{E}^\perp \) and with the weighted ones on \( S_{E}^\parallel \). Given \( s \in \Omega^0(X, S_E) \) be supported along the end where equation (10.11) is valid, then the respective components of \( s \) be referred as \( s^\parallel, s^\perp \).

With this discussion in mind, we now define the relevant function spaces.

**Definition 10.11.** Let \( \alpha \in \mathbb{R} \) and \( k \in \mathbb{N} \). Define the \( H_{k,\alpha} \)-norm of a compactly supported \( s \in \Omega^0(X, S_E) \) as

\[ \|s\|_{H_{k,\alpha}(X)}^2 = \|s\|_{L_k^2(K)}^2 + \|s\|_{L_{k,\alpha}(X-K)}^2 + \|s^\perp\|_{L_{k}^2(X-K)}^2, \]

and the spaces \( H_{k,\alpha}(X) \) as the completion of the smooth compactly supported sections in this norm.

**Theorem 10.12.** Let \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). Then, there is a discrete set \( \mathcal{K}(\mathcal{D}) \subseteq \mathbb{R} \) such that for \( \alpha \not\in \mathcal{K}(\mathcal{D}) \), the operator

\[ \mathcal{L} : H_{k+1,\alpha+1}(X) \to H_{k,\alpha}(X), \]

is Fredholm. In particular, there exist parametrices \( P_L, P_R : H_{k,\alpha} \to H_{k+1,\alpha+1} \), such that

\[ \mathcal{L}P_R = 1 + S_R, \quad P_L \mathcal{L} = 1 + S_L, \tag{10.12} \]

with \( S_R : H_{k+1,\alpha+1}(X) \to H_{k+1,\alpha+1}(X) \) and \( S_L : H_{k,\alpha}(X) \to H_{k,\alpha}(X) \) compact operators.

**Proof.** The statement is true if \( X \) is replaced by a cone, as in that case \( \mathcal{L} \) consists on the direct sum of two operators as in the previous two subsections which are therefore Fredholm. As the direct sum of Fredholm operators is Fredholm the result holds on a model cone.

The general case follows from a standard procedure, which constructs global parametrices by gluing those obtained for the model operators. This is illustrated below, in the construction of a global right parametrix \( P_R \).
Let \( V_0 = X - K \) and \( K \subseteq \bigcup_{i=1}^{N} V_i \) be an open cover of \( X \), such that there are local right inverses \( P_i \) to the operator \( L \), defined on some slightly larger open sets \( U_i \) containing \( V_i \). Moreover, suppose \( K \) is big enough, so that on \( U \), the operator \( L \) is modeled on some conical operator \( L_C = \mathcal{D}_C + q_C \) with \( \nabla q_C = 0 \). Let \( \{ \beta_0, \ldots, \beta_N \} \) be a partition of unity subordinate to this cover.

First, notice that one can change the operator \( L \) over \( V_0 \) so that it is exactly conical as \( L_C \). In fact, this amounts to subtract to \( L \) the operator \( T(s) = \beta_0(Ls - L_C(\beta_0 s)) \), which is a compact operator

\[
T : H_{k+1,a+1}(X) \to H_{k,a}(X),
\]

and the Fredholm property is not affected by perturbations by compact operators. Then there is a parametrix \( P_0 \) constructed for \( L_C \) satisfying the conditions in equation (10.12) with \( L \) replaced by \( L_C \). We now glue \( P_0 \) with the local inverses \( P_i \). Following the approach in page 95 of [13] we define the candidate for a global parametrix as

\[
P_R = \sqrt{\beta_0} P_0 \sqrt{\beta_0} + \sum_{i \in I} \sqrt{\beta_i} P_i \sqrt{\beta_i},
\]

and notice that even though \( P_0 \) and the \( P_i \)'s are not globally defined the expression above is. To check that \( P_R \) is indeed a right parametrix for \( L \) we compute

\[

L P_R(s) = \sigma(d\sqrt{\beta_0}) P_0 \sqrt{\beta_0} s + \sum_{i \in I} \sigma(d\sqrt{\beta_i}) P_i \sqrt{\beta_i} s + \sqrt{\beta_0} L P_0 \sqrt{\beta_0} s + \sum_{i \in I} \sqrt{\beta_i} L P_i \sqrt{\beta_i} s,
\]

where \( \sigma \) denotes the higher order symbol of \( L \) (which coincides with that of \( \mathcal{D} \)). The term in the first line is a compact operator \( T' : H_{k,a} \to H_{k,a} \) and, again, does not affect the Fredholm property. This follows from the fact that it is supported on a compact set where the derivatives of the \( \beta \)'s are non vanishing. Moreover, over this compact set, by elliptic regularity one can control the \( L^2(X) \) norms of the derivatives of \( P_0 s \) and \( P_i s \) in terms of the \( L^2(X) \) norms of \( s \). For the last two terms one can use \( L P_0 = I + S_0 + T'' \) for some compact operators \( S_0 \) and \( T'' \) over \( V_0 \); and \( L P_i = I \) over \( V_i \) to obtain

\[

L P_R(s) = T'''(s) + \sqrt{\beta_0}(I + S_0)\sqrt{\beta_0} s + \sum_{i \in I} \beta_i s
\]

\[

= s + T''(s) + \sqrt{\beta_0} S_0(\sqrt{\beta_0} s),
\]

for \( T''' = T' + T'' \). Moreover since the last term is supported on the conical end where it agrees with \( S_0 \), which is a compact operator on these function spaces, the operator \( T'' + \sqrt{\beta_0} S_0 \sqrt{\beta} \) is compact and this proves that \( P_R \) is a right parametrix for \( L \).
10.4. **The case when** $p > 2$. This subsection extends Theorem 10.12 from $p = 2$ to $p > 2$. The upshot is Theorem 10.15 below. The relevant function spaces for the general situation are the ones in Definition 10.11 but constructed with $p > 2$.

**Definition 10.13.** For $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_+$ and $p \geq 2$ define the spaces $H^p_{k,\alpha}(X)$ to be the completion of the smooth compactly supported sections with respect to the norm given by

$$
\|s\|_{H^p_{k,\alpha}(X)} = \|s\|_{L^p(K)} + \|s\|_{L^p_{k,\alpha}(X-K)} + \|s\|_{L^p_{X-K}}
$$

where $K \subseteq X$ is a large compact set outside of which the splitting $S_E = S_E^\parallel \oplus S_E^\perp$ is well defined.

**Remark 10.14.** Notice that $H^2_{k,\alpha}(X) = H_{k,\alpha}(X)$ in the notation from the previous section.

The main result of this section is the following.

**Theorem 10.15.** Let $k \in \mathbb{N}$ and $p \geq 2$, there is a discrete set $K(\mathcal{D}) \subseteq \mathbb{R}$ such that for $\alpha \notin K(\mathcal{D})$ and $\alpha \geq -n/2$

$$
\mathcal{L} : H^p_{k+1,\alpha+1}(X) \rightarrow H^p_{k,\alpha}(X),
$$

is a Fredholm operator.

This is proven by showing that the parametrices $P_R, P_L$ obtained for $p = 2$ in Theorem 10.12 extend to bounded operators with $S_R, S_L$ compact operators when regarded as operators on the relevant $H^p_{k,\alpha}$-spaces with $p > 2$.

**Proposition 10.16.** Let $\alpha \geq -n/2$ be such that $\alpha \notin K(\mathcal{D})$ and $-n - \alpha \notin K(\mathcal{D})$. Then, the parametrices $P_R$ and $P_L$ for $\mathcal{L}$ obtained in Theorem 10.12, extend to bounded operators

$$
P_R, P_L : H^p_{0,\alpha}(X) \rightarrow H^p_{1,\alpha+1}(X),
$$

such that

$$
P_R \mathcal{L} = 1 + S_R, \text{ and } P_L \mathcal{L} = 1 + S_L,
$$

with $S_R : H^p_{0,\alpha}(X) \rightarrow H^p_{0,\alpha}(X)$ and $S_L : H^p_{1,\alpha+1}(X) \rightarrow H^p_{1,\alpha+1}(X)$ compact operators.

**Proof.** Consider the restricted operator

$$
\mathcal{L}|_{(\ker(\mathcal{L}) \cap H^p_{0,\alpha+1}(X))^{\perp L^2(X)}} : (\ker(\mathcal{L}) \cap H^p_{0,\alpha+1}(X))^{\perp L^2(X)} \rightarrow \text{im}(\mathcal{L}|_{(\ker(\mathcal{L}) \cap H^p_{0,\alpha+1}(X))^{\perp L^2(X)}}).
$$

Any complement to $\text{im}(\mathcal{L}|_{(\ker(\mathcal{L}) \cap H^p_{0,\alpha+1}(X))^{\perp L^2(X)}})$ in $H^p_{0,\alpha}(X)$ is isomorphic to the kernel of the adjoint operator $\mathcal{L}^*$ in the dual space $H^p_{0,-n-\alpha}(X)$. In the range when $\alpha \geq -n/2$ we have $H^p_{0,-n-\alpha}(X) \subseteq H^p_{0,\alpha}(X)$ and so we have

$$
\text{im}(\mathcal{L}|_{(\ker(\mathcal{L}) \cap H^p_{0,\alpha+1}(X))^{\perp L^2(X)}}) = (\ker(\mathcal{L}^*) \cap H^p_{0,-n-\alpha}(X))^{\perp L^2(X)}.
$$
Having this said, we now lay out how the operators $S_L$ and $S_R$ are constructed in order to then prove that they are compact. The parametrix $P_L$ in the statement is obtained by constructing a left inverse to $L|_{(\ker(\mathcal{D}) \cap H_{0, a+1}(X))^{1, 2}(X)}$, then $S_L$ is minus the projection onto $\ker(L) \cap H_{0, a+1}(X)$, which is finite dimensional as $L$ is Fredholm for $p = 2$ due to Theorem 10.12. In the same way, $P_R$ is obtained by constructing a right inverse to $L$ as an operator onto $(\ker(L^*) \cap H_{0, -n-a})^{1, 2}(X)$ and so $S_R$ is minus the projection onto $\ker(L^*) \cap H_{0, -n-a}(X)$ which is finite dimensional as $L^* = \mathcal{D} - q$ is Fredholm for $p = 2$. As projections into finite dimensional subspaces are compact operators, both $S_R$ and $S_L$ are compact.

Next, we turn to the proof that the parametrices $P_R, P_L$ do extend to bounded operators from $H^p_{0, a}(X)$ to $H^p_{0, a+1}(X)$. Let us choose a big compact geodesic ball $K$, and then we have:

(1) Over the big compact set $K \subseteq X$, the spaces $H^p_{k, a}(X)$ can be taken to agree with the usual $L^k_p(X)$ ones. We then fix a finite open cover $\{V_i\}_{i \in I}$, where the standard Calderon–Zygmund inequalities hold. These are

$$\|\nabla g\|_{L^p(V_i)} \leq \| \mathcal{L} g \|_{L^p(V_i)} + \| g \|_{L^p(V_i)}$$

$$\| g \|_{L^p(V_i)} \leq \| \mathcal{L} g \|_{L^p(V_i)} + \| g \|_{L^2(V_i)}$$

where $V_i'' \supset V_i' \supset V_i$ are open. The reason why we chose to arrange them in this manner is that they can now be combined into

$$\| g \|_{L^p(V_i)} \leq \| \mathcal{L} g \|_{L^p(V_i''')} + \| g \|_{L^p(V_i''')}$$

Inserting $g = P_R f$ into the above inequality and using $\mathcal{L} P_R = 1 + S_R$, gives

$$\| P_R f \|_{L^p(V_i)} \leq \| \mathcal{L} P_R f \|_{L^p(V_i')} + \| P_R f \|_{L^2(V_i'')}$$

$$\leq \| f \|_{L^p(V_i'')} + \| S_R f \|_{L^p(V_i'')} + \| P_R f \|_{L^2(V_i'')}$$

Then the fact that $P_R$ is bounded for $p = 2$, $L^p(V_i') \subseteq L^2(V_i'')$ for $p \geq 2$, and $S_R$ is compact and hence bounded for $p \geq 2$ combine to further give

$$\| P_R f \|_{L^p(V_i)} \leq \| f \|_{L^p(V_i'')}$$

thus showing that $P_R : L^p(K) \to L^p(K)$ is bounded.

(2) In the noncompact end $X - K$, the operator $\mathcal{L} = \mathcal{D}_C + q$ is modeled on a conical operator $\mathcal{L}_C = \mathcal{D}_C + q_C$ as in Hypothesis 10.9. The rest of the proof requires Lemmata 10.17 and 10.21 below. For now assume these hold, then from Lemma 10.21 one can use the alternative form of the $H^p_{1, a+1}(X)$ norm

$$\| g \|_{H^p_{1, a+1}(X)} = \| \mathcal{L} g \|_{H^p_{0, a}(X)} + \| g \|_{H^p_{0, a+1}(X)}$$
Insert into this $g = P_R f$ with $f \in H^p_{0,\alpha}(X)$ and use $LP_R = 1 + S_R$, gives
\[
\|P_R f\|_{H^p_{1,\alpha+1}(X)} = \|f + S_R f\|_{H^p_{0,\alpha}(X)} + \|P_R f\|_{H^p_{0,\alpha+1}(X)}.
\]
By using the generalized Young inequality and the fact that $S_R : H^p_{0,\alpha}(X) \to H^p_{0,\alpha}(X)$ is compact, the first term can be bounded above by a term of the form $\|f\|_{H^p_{0,\alpha}(X)}$.

As for the second term, it is guaranteed by Lemma 10.21 that it can equally be bounded above by $\|f\|_{H^p_{0,\alpha}(X)}$, thus showing that the parametrix $P_R : H^p_{0,\alpha}(X) \to H^p_{1,\alpha+1}(X)$ is bounded.

Combining these two pieces finishes the proof of Proposition 10.16.

\[\square\]

The rest of this section focuses on proving Lemmata 10.17 and 10.21 used in the proof of Proposition 10.16.

**Lemma 10.17.** The norm $H^p_{k+1,\alpha+1}(X)$ is equivalent to the norm $\| \cdot \|$ defined by
\[
\|f\| = \|\mathcal{L} f\|_{H^p_{k,\alpha}(X)} + \|f\|_{H^p_{0,\alpha+1}(X)}.
\]

**Proof.** The result follows from induction in $k$. We only do the first step with $k = 1$ as the general one is similar. In this $k = 1$ case it is enough to show that $\|f\|^p$ can be bounded from above and below by fixed multiples of $\|f\|_{H^p_{1,\alpha+1}(X)}$.

To prove the upper bound, we use $\mathcal{L} f = \mathcal{D} f + q(f)$ and the generalized version of Young’s inequality
\[
\|f\| \leq \|\mathcal{D} f\|_{H^p_{0,\alpha}(X)} + \|q(f)\|_{H^p_{0,\alpha}(X)} + \|f\|_{H^p_{0,\alpha+1}(X)}.
\]
As $|\mathcal{D}| \leq |\nabla f|$, the first term above has an upper bound of the form $\|\nabla f\|_{H^p_{0,\alpha}(X)}$. Furthermore, $|q(f)| \leq |f^{\perp}|$ and given the the weights do not affect the $f^{\perp}$ component we also have $\|f^{\perp}\|_{H^p_{0,\alpha}(X)} = \|f^{\perp}\|_{H^p_{0,\alpha+1}(X)}$. Combining these, we find
\[
\|f\| \lesssim \|\nabla f\|_{H^p_{0,\alpha}(X)} + \|f\|_{H^p_{0,\alpha+1}(X)} = \|f\|_{H^p_{1,\alpha+1}(X)}.
\]
To prove the lower bound on $\|f\|^p$, we need to establish an inequality as
\[
\|f\|_{H^p_{1,\alpha+1}(X)} \leq \|\mathcal{L} f\|_{H^p_{0,\alpha}(X)} + \|f\|_{H^p_{0,\alpha+1}(X)}.
\]
We prove this by splitting it into two cases when $f$ is either in $S^\|_E$ or $S^\perp_E$. In the first case, i.e. $f \in S^\|_E$, we have $\mathcal{L} f = \mathcal{D} f$ and $\|f\|_{H^p_{k,\alpha}(X)} = \|f\|_{L^p_{k,\alpha}(X)}$ and given that $\mathcal{D}$ is an elliptic asymptotically conical operator modeled on $\mathcal{D}_C$, there is an inequality
\[
\|f\|_{L^p_{1,\alpha+1}(X)} \leq \|\mathcal{D} f\|_{L^p_{0,\alpha}(X)} + \|f\|_{L^p_{0,\alpha+1}(X)}.
\]
This is immediate from a change of coordinates to the cylindrical setting and the result of Lockhart–McOwen in [32]. On the other hand, if \( f \in S^1 \), \( \| f \|_{H^p_{k,a}(X)} = \| f \|_{L^p_k(X)} \) and to bound \( \| \nabla f \|_{L^p(X)} \) by above we use the fact that \( L^p(X) = L^p_{0,-n/p}(X) \). This gives

\[
\| \nabla f \|_{L^p(X)} \leq \| \nabla f \|_{L^p(X)} + \| r^{-1} f \|_{L^p(X)} = \| f \|_{L^p_{1,-n/p+1}(X)}.
\]

Then, using the weighted inequality (10.14), for the case \( \alpha = -\frac{2}{p} \), gives

\[
\| \nabla f \|_{L^p(X)} \leq \| \partial f \|_{L^p(X)} + \| r^{-1} f \|_{L^p(X)} \leq \| \mathcal{L} f \|_{L^p(X)} + \| f \|_{L^p(X)} + \| r^{-1} f \|_{L^p(X)} \leq \| \mathcal{L} f \|_{L^p(X)} + \| f \|_{L^p(X)},
\]

where in the second inequality we used \( \partial f = \mathcal{L} f - q(f) \) and the fact that \( q \) is bounded. The inequality (10.13) is now immediate from summing these two components.

We use the above result in the analysis of a mixed norm designed to interpolate between the \( L^p \)-norm along the radial direction and the \( L^2 \)-norm along the “closed” directions.

**Definition 10.18.** Let \( R \gg 1 \) be such that the splitting \( S_E = S_E^+ \oplus S_E^- \) is well defined in the complement of \( B_R \). We define the intermediate norm \( \| \cdot \|_{H^{(p,2)}_{0,a}(X)} \) by

\[
\| f \|_{H^{(p,2)}_{0,a}(X)}^p = \| f \|_{L^p_2(K)}^p + \int_{R} (r^{-\alpha p-n} \| f \|_{L^2_2(S_r)}^p + \| f \|_{L^2_2(S_r)}) r^{-(n-1)p-2} dr,
\]

where the \( L^2 \)-norms on the right hand side are with respect to the induced metric on \( S_r \) defined as \( Y(\{r\} \times \Sigma) \).

**Lemma 10.19.** Let \( p \geq 2 \) and \( \alpha \in \mathbb{R} \), then

\[
\| f \|_{H^{(p,2)}_{0,a}(X)} \leq \| f \|_{H^p_{0,a}(X)},
\]

for all \( f \in H^p_{0,a}(X) \).

**Proof.** When \( f \) is supported in \( K \) the inequality follows from the standard embedding \( L^p(K) \hookrightarrow L^2(K) \) and so we need only prove the case when \( f \) is supported in \( X - K \). Using again that for \( p \geq 2 \) and over compact sets, the \( L^p \)-norm is stronger than the \( L^2 \)-norm. In fact, over \( B_1 \subseteq \mathbb{R}^k \) we have \( \| f \|_{L^2(B_1)} \leq \| f \|_{L^p(B_1)} \), and then, by scaling, we get

\[
\| f \|_{L^2(B_r)} \leq r^{\frac{p-2}{2}} \| f \|_{L^p(B_r)},
\]

for all positive \( r \). Applying this scaling behavior of the \( L^p \) norms, we find that

\[
\| f \|_{L^2_2(S_r)} \leq r^{(n-1)p-2} \| f \|_{L^p(X)},
\]
Inserting this into the definition of the $H^{(p,2)}_{0,\alpha}$-norm above gives an upper bound with respect to the $H^p_{0,\alpha}$-norm.

\[ \square \]

**Lemma 10.20.** Let $p \geq 2$ and $\alpha \in \mathbb{R}$, then there is an inequality

\[ \|P_R f\|_{H^{(p,2)}_{0,\alpha+1}(X)} \leq \|f\|_{H^{(p,2)}_{0,\alpha}(X)}, \]

for all $f \in H^p_{0,\alpha}(X)$. Moreover, combining this with Lemma 10.19 we find

\[ \|P_R f\|_{H^{(p,2)}_{0,\alpha+1}(X)} \leq \|f\|_{H^p_{0,\alpha}(X)}. \]

**Proof.** As asserted in the statement, it is enough to prove the first inequality. Recall that $P_R$ is a bounded operator for $p = 2$, i.e. from $H_{0,\alpha}(X)$ to $H_{1,\alpha+1}(X)$ and so we need only prove the inequality for $f$ supported along the conical end. As in the previous proof, it is convenient to separately prove inequalities for the components of $f$ in $S_E^\parallel$ and $S_E^{\perp}$.

We start with the case when $f \in S_E^\parallel$, then the $H^p_{k,\alpha}$-norm is the standard Lockhart–McOwen weighted $L^p_{k,\alpha}$-norm, and by changing coordinates to $t = \log(r)$, the statement that $P_R$ is bounded from $H_{0,\alpha}(X) = L^2_{0,\alpha}(X)$ into $H_{0,\alpha+1}(X) = L^2_{0,\alpha+1}(X)$ reads

\[ \int_{\log(R)}^\infty \|e^{-t} f\|_{L^2_{\Sigma, g_{\Sigma}}}^2 e^{-2\alpha t} dt \leq \int_{\log(R)}^\infty \|f\|_{L^2_{\Sigma, g_{\Sigma}}}^2 e^{-2\alpha t} dt. \]

Equivalently, this statement can be formulated as saying that for all $T > \log(R)$, the assignment $e^{-\alpha t} f \mapsto e^{-(\alpha+1)T} (P_R f(T)$ gives rise to a bounded map

\[ M_\alpha(T) : L^2((\log(R), \infty), L^2_{\Sigma, g_{\Sigma}}) \to L^2_{\Sigma, g_{\Sigma}}, \]

whose operator norm integrable in the parameter $T \in (\log(R), \infty)$.

Still in the cylindrical setting, the fact that $f \in H^{(p,2)}_{0,\alpha}(X)$ means that

\[ e^{-\alpha t -(n-1)\frac{p-2}{2p}} f(t) \in L^p((\log(R), \infty), L^2_{\Sigma, g_{\Sigma}}). \]

Hence, we can use the fact that the family $M_\alpha(\cdot)$ has integrable operator norm and the map $L^1 \times L^p \hookrightarrow L^p$ is bounded along $(\log(R), \infty) \times L^2_{\Sigma, g_{\Sigma}}$ to prove that

\[ \|e^{-(n-1)\frac{p-2}{2p}} T (M_\alpha e^{-\alpha t} f(T))\|_{L^p_{\Sigma}}^p \leq \|M_\alpha(T)\|_p \|e^{-\alpha t -(n-1)\frac{p-2}{2p}} f(t)\|_{L^p_{\Sigma}}^p. \]

Since $\|M_\alpha(T)\|_{L^1} \leq 1$ is bounded, we can change coordinates back to $r$ and obtain

\[ \|P_R f\|_{H^{(p,2)}_{0,\alpha+1}(X)} \leq C\|f\|_{H^{(p,2)}_{0,\alpha}(X)}. \]

This proves that $P_R : H^{(p,2)}_{0,\alpha}(X) \to H^{(p,2)}_{0,\alpha+1}(X)$ is bounded for those components in $S_E^\parallel$. 

64
We now turn to the situation when \( f \in \mathcal{S}^1_E \) in which case the \( H^p_{k,\alpha} \) norm is the standard \( L^p_k \) one. The statement that \( P_R \) is bounded from and into \( L^2(X) \) can equivalently be stated in the cylindrical setting, as follows. Using the measure \( e^{nt}dt \) on \((\log(R), \infty)\), and all \( T > \log(R) \), the assignment \( f \mapsto (P_Rf)(T) \) gives a bounded map
\[
P_R(T) : L^2((\log(R), \infty), L^2(\Sigma, g_\Sigma)) \to L^2(\Sigma, g_\Sigma),
\]
and this \( T \)-parametrized family has integrable operator norm. Then, given \( f \in H^{(p,2)}_{0,\alpha} \) which in the cylindrical setting means
\[
e^{-(n-1)\frac{p^2}{2p}} f(t) \in L^p((\log(R), \infty), L^2(\Sigma, g_\Sigma)),
\]
using the measure \( e^{nt}dt \) on \((\log(R), \infty)\). Proceeding as before and combining the map \( L^1 \times L^p \hookrightarrow L^p \) with the fact that the family \( P_R(T) \) has integrable operator norm gives
\[
\|e^{-(n-1)\frac{p^2}{2p}} T (P_Rf(t))(T)\|^p_{L^p} \leq \|P_R(T)\|^p_{L^2} \|e^{-(n-1)\frac{p^2}{2p}} t e^{-at} f(t)\|^p_{L^p},
\]
with \( \|P_R(T)\|^p_{L^1} \leq 1. Back to the conical world this statement gets translated into
\[
\|P_Rf\|^2_{H^{(p,2)}_{0,\alpha+1}} \leq \|f\|^2_{H^{(p,2)}_{0,\alpha}},
\]
proving the statement for those components in \( \mathcal{S}^1_E \).

The complete statement is obtained by putting together these two separate cases. \( \square \)

**Lemma 10.21.** There is an inequality of the form
\[
\|P_Rf\|_{H^p_{0,\alpha+1}(X)} \leq \|f\|_{H^p_{0,\alpha}(X)},
\]
which holds for all \( f \in H^p_{0,\alpha}(X) \).

**Proof.** Again, in this case it is enough to prove the statement for the case when \( f \) is supported along the conical end. Recall the \( H^p_{0,\alpha}(X) \) norm in Definition 10.13. It is convenient to have it rewritten as a sum of \( L^p \)-norms on conical annuli going off along end as follows
\[
\|g\|^p_{H^p_{0,\alpha+1}(U)} = \int_{R} \left( r^{-(\alpha+1)p-n} \|g\|^p_{L^p(\Sigma)} + \|g^{-\perp}\|^p_{L^p(\Sigma^\perp)} \right) dr
\approx \sum_{k \geq 1} \left( R^{-k((\alpha+1)p+n)} \|g\|^p_{L^p(C_k)} + \|g^{-\perp}\|^p_{L^p(C_k^\perp)} \right),
\]
where \( \approx \) above denotes an equivalence of norms (which is straightforward to check) and \( C_k = (R^k, R^{k+1}) \times \Sigma \) equipped with the conical metric \( g_C = dr^2 + r^2 g_\Sigma = r^2 \left( \frac{dr^2}{r^2} + g_\Sigma \right) \). Notice that the conical annulus \( C_{k+1} \) is obtained from \( C_k \) by scaling with a factor of \( R > 1 \). As usual, it is convenient to separate into components.
First, one focuses on the components in \( S_E^\| \). Over the bounded annulus \( C_1 \), the standard Calderon–Zygmund inequalities give \( \| g \|_{L^p(C_k)} \lesssim \| Dg \|_{L^p(C_k)}^p + \| g \|_{L^2(C_k)}^p \), where \( C'_1 \supset C_1 \) is a slightly larger annulus in the cone. This inequality is not scale invariant and scaling it gives

\[
\| g \|_{L^p(C_k)} \lesssim R^{kp} \| Dg \|_{L^p(C_k)}^p + R^{-nk^p \frac{p-2}{2}} \| g \|_{L^2(C_k)}^p,
\]

and in this component \( \mathcal{D} = \mathcal{L} \). Moreover, since \( p > 2 \), \( R^{-nk^p \frac{p-2}{2}} \lesssim R^{-(n-1)k^p \frac{p-2}{2}} \) and

\[
\| g \|_{L^2(C_k)}^p \lesssim \int_{R^k} \| g \|_{L^2(\Sigma_r)}^p \, dr.
\]

Then, the inequality above further gives

\[
\| g \|_{L^p(C_k)}^p \lesssim R^{kp} \| \mathcal{L}g \|_{L^p(C_k)}^p + R^{-(n-1)k^p \frac{p-2}{2}} \int_{R^{k-1}}^{2R^{k+1}} \| g \|_{L^2(\Sigma_r)}^p \, dr
\]

\[
\lesssim R^{kp} \| \mathcal{L}g \|_{L^p(C_k)}^p + \int_{R^{k-1}}^{2R^{k+1}} r^{-(n-1)k^p \frac{p-2}{2}} \| g \|_{L^2(\Sigma_r)}^p \, dr.
\]

Inserting this into equation (10.15) and recalling so that \( g \in S_E^\| \), gives

\[
\| g \|_{H^p_{0, \alpha+1}(X)}^p \lesssim \sum_{k \geq 1} R^{-k((\alpha+1)p+n)} R^{kp} \| \mathcal{L}g \|_{L^p(C_k)}^p + \int_{R}^{\infty} r^{-(\alpha+1)p-n} \| g \|_{L^2(\Sigma_r)}^p \, dr
\]

\[
\lesssim \| \mathcal{L}g \|_{H^p_{0, \alpha}(X)}^p + \| g \|_{H^p_{0, 2}(\Sigma_1)}^p.
\]

Insert into this inequality \( g = P_R f \), then by using \( LP_R = I + S_R \), the fact that \( S_R \) is bounded from and into \( H_{0, \alpha}^p(X) \) and Lemma 10.20, gives

\[
\| P_R f \|_{H^p_{0, \alpha+1}(X)}^p \lesssim \| f + S_R f \|_{H^p_{0, \alpha+1}(X)}^p + \| P_R f \|_{H^p_{0, \alpha+1}(X)}^p \lesssim \| f \|_{H^p_{0, \alpha+1}(X)}^p.
\]

Next, we turn to those components in \( S_E^\perp \), recall that for these the map \( q : \Omega^0(\text{End}(S_E)) \) is bounded below, i.e. \( |q(g)| \geq c|g| \), for some \( c > 0 \) independent of \( g \in S_E^\perp \). Then in any \( C_k \), we have the inequalities

\[
\| g \|_{L^p(C_k)}^p \lesssim \| q(g) \|_{L^p(C_k)}^p \lesssim \| \mathcal{L}g \|_{L^p(C_k)}^p + \| Dg \|_{L^p(C_k)}^p.
\]

Moreover, as \( \mathcal{D} : L^p_1(C_1) \rightarrow L^p(C_1) \) is bounded we find, after rescaling,

\[
\| Dg \|_{L^p(C_k)}^p \leq c_1 (\| \nabla g \|_{L^p(C'_k)}^p + R^{-p} \| g \|_{L^p(C'_k)}^p).
\]
and given that $\mathcal{D}$ and $\mathcal{L}$ are elliptic, the (rescaled) standard Calderon–Zygmund inequality gives
\[ \|\nabla g\|^p_{L^p(C_k)} \lesssim \|\mathcal{D} g\|^p_{L^p(C_k')} + R^{-nk\frac{p-2}{2}} \|g\|^p_{L^2(C_k')}, \]
where $C'_k \subseteq C_k \supseteq C_k''$ are nested annuli. Combining these yield
\[ \|\mathcal{D} g\|^p_{L^p(C_k)} \lesssim \|\mathcal{L} g\|^p_{L^p(C_k')} + R^{-nk\frac{p-2}{2}} \|g\|^p_{L^2(C_k')} + R^{-pk} \|\mathcal{L} g\|^p_{L^p(C_k')}, \]
and inserting this back into inequality (10.16) gives
\[ \|g\|^p_{L^p(C_k)} \lesssim \|\mathcal{L} g\|^p_{L^p(C_k')} + R^{-nk\frac{p-2}{2}} \|g\|^p_{L^2(C_k')} + R^{-pk} \|\mathcal{L} g\|^p_{L^p(C_k')} \tag{10.17} \]
Moreover since $p > 2$ also in this case $R^{-nk\frac{p-2}{2}} \lesssim R^{-(n-1)k\frac{p-2}{2}}$ and one can dominate the second term in the right above by
\[ \int_{R^{k+1}}^{R^{k+1+1}} \|g\|^p_{L^2(\Sigma_r)} R^{-(n-1)\frac{p-2}{2}} dr, \]
which is for components in $S_{\mathcal{E}}^\perp$ the $H^{(p,2)}_{0,\alpha+1}$ norm. Then, inserting inequality (10.17) into the norm in (10.15) for $g \in S_{\mathcal{E}}^\perp$ gives
\[ \|g\|^p_{H^0_{0,\alpha+1}} \leq \sum_{k \geq 1} \|\mathcal{L} g\|^p_{L^p(C_k')} + \int_{R^k}^{\infty} \|g\|^p_{L^2(\Sigma_r)} R^{-(n-1)\frac{p-2}{2}} dr + R^{-p} \int_{R^k}^{\infty} \|g\|^p_{L^p(\Sigma_r)} dr \]
\[ \lesssim \|\mathcal{L} g\|^p_{H^0_{0,\alpha}(X)} + \|g\|^p_{H^{(p,2)}_{0,\alpha+1}(X)} + R^{-p} \|\mathcal{L} g\|^p_{H^0_{0,\alpha+1}(X)'} \]
and for $R \gg 1$ we can reabsorb the last term into the first yielding
\[ \|g\|^p_{H^0_{0,\alpha+1}(X)} \lesssim \|\mathcal{L} g\|^p_{H^0_{0,\alpha}(X)} + \|g\|^p_{H^{(p,2)}_{0,\alpha+1}(X)'} \]
and notice that the weights $\alpha$ here are irrelevant but are introduced in order to use the appropriate notation. Following a similar strategy as in the previous case let $g = P_R f$ in the inequality above. Then, using $\mathcal{L} P_R = 1 + S_R$, that $S_R$ is bounded on $H^p_{0,\alpha}(X)$ and Lemma 10.20 gives $\|P_R f\|^p_{H^0_{0,\alpha+1}(X)} \leq \|f\|^p_{H^0_{0,\alpha}(X)'}$. The general result follows immediately from combining the inequalities in the two components.

**Remark 10.22.** Recall that restricted to the components in $S_{\mathcal{E}}^\perp$, the operator $\mathcal{L}$ coincides with the Dirac operator $\mathcal{D}$. Furthermore, along this component $H^p_{k,\alpha}(X) = L^p_{k,\alpha}(X)$ are the Lockhart–McOwen spaces. The results obtained in this subsection, when restricted to these components, also follow from the standard Lockhart–McOwen theory which could have been used instead; cf. [32]. In fact, that was done when using inequality (10.14) in the proof of Lemma 10.17.
However, such an inequality follows from the standard Calderón–Zygmund inequality:
\[ \| \nabla g \|_{L^p(C_1)}^p \leq \| Lg \|_{L^p(C_1)}^p + \| g \|_{L^p(C_1)'}^p \]
after rescaling the annulus \( C_1 \) to any other annulus, \( C_k \), in a similar fashion to what we did in Lemma 10.21.

11. Weighted split Sobolev spaces: Sobolev Embeddings and Multiplication Maps

Denote by \( L^p_{k,a}(X) \) the weighted spaces introduced in Definition 10.7. In order to develop a moduli theory for \( G_2 \)-monopoles, we need to understand these spaces, in particular, to be able to handle the nonlinearities of the equations. The most relevant of these properties is the one stated in Proposition 11.7 below. Its proof requires a combination of Lemmata 11.1 to 11.3.

**Lemma 11.1. (Weighted Hölder Inequality)** Let \( \beta, \gamma \in \mathbb{R} \) and \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{q} \), then the multiplication property \( L^{s_1}_{0,\beta}(X) \times L^{s_2}_{0,\gamma}(X) \hookrightarrow L^q_{0,\gamma+\beta}(X) \) holds. In particular, if \( \gamma \leq 0 \), then \( L^{s_1}_{0,\beta}(X) \times L_{0,\gamma}^{s_2}(X) \hookrightarrow L^q_{0,\beta}(X) \).

**Proof.** Let \( f \in L^{s_1}_{0,\beta}(X), g \in L^{s_2}_{0,\gamma}(X) \), then using the definition of the weighted norms, rearranging the exponents and the usual Hölder inequality shows
\[
\| fg \|_{L^q_{0,\gamma+\beta}(X)} = \| r^{-\frac{n}{q} - \gamma - \beta} fg \|_{L^q(X)}
\]
\[
= \| (r^{-\frac{n}{s_1} - \beta} f) (r^{-\frac{n}{s_2} - \gamma} g) \|_{L^q(X)}
\]
\[
\leq \| r^{-\frac{n}{s_1} - \beta} f \|_{L^{s_1}(X)} \| r^{-\frac{n}{s_2} - \gamma} g \|_{L^{s_2}(X)}
\]
\[
= \| f \|_{L^{s_1}_{0,\beta}(X)} \| g \|_{L^{s_2}_{0,\gamma}(X)},
\]
thus showing the first statement. In particular, when \( \gamma \leq 0 \), then \( L^q_{0,\gamma+\beta}(X) \hookrightarrow L^q_{0,\beta}(X) \). \( \square \)

**Lemma 11.2.** Let \( p_2 > p_1 \) and \( \gamma_2 > \gamma_1 \), then for all \( s \in [p_1, p_2] \) and \( \gamma \geq \max_{i=1,2} \left( \frac{n}{p_i} - \frac{n}{s} + \gamma_i \right) \), there is an inclusion \( L_{0,\gamma_1}^{p_1}(X) \cap L_{0,\gamma_2}^{p_2}(X) \hookrightarrow L_{0,\gamma}^s(X) \).

**Proof.** First one notices that since \( p_1 < s < p_2 \), then for all \( g \in L^{p_1}(X) \cap L^{p_2}(X) \), it holds that \( \| g \|_{L^s(X)} \leq \| g \|_{L^{p_1}(X)} + \| g \|_{L^{p_2}(X)} \). Let \( f \in L_{0,\gamma_1}^{p_1}(X) \), then
\[
\| f \|_{L_{0,\gamma}^s(X)} = \| r^{-\gamma - \frac{n}{s}} f \|_{L^s(X)}
\]
\[
\leq \| r^{-\gamma - \frac{n}{s}} f \|_{L^{p_1}(X)} + \| r^{-\gamma - \frac{n}{s}} f \|_{L^{p_2}(X)}
\]
\[
= \| f \|_{L_{0,\gamma_1}^{p_1}(X)} + \| f \|_{L_{0,\gamma_2}^{p_2}(X)},
\]

Since $\gamma \geq \max\{\frac{n}{p_i} - \frac{n}{s} + \gamma_i\}$, one has $\gamma + \frac{n}{s} - \frac{n}{p_i} \geq \gamma_i$ for $i = 1, 2$ and
\[ \|f\|_{L^p_{0,\gamma}(\mathcal{X})} \leq \|f\|_{L^p_{0,\gamma_1}(\mathcal{X})} + \|f\|_{L^p_{0,\gamma_2}(\mathcal{X})}. \]

Lemma 11.3. Let $\beta \in \mathbb{R}$, $p \in \left[\frac{n}{2}, n\right]$ and $k \in \mathbb{N}_+$. Then, the following hold

- $L^p_k(\mathcal{X}) = \bigcap_{i=0}^k L^p_{i,\frac{-n}{p}+i}(\mathcal{X})$
- $L^q_{k+1,\beta}(\mathcal{X}) \hookrightarrow L^q_{k,\beta}(\mathcal{X})$, for $q \leq \frac{np}{n-p}$.
- $L^p_{k+1,\text{loc}}(\mathcal{X}) \hookrightarrow C^{k-1}_\text{loc}(\mathcal{X})$ and $L^p_{k+1,\beta}(\mathcal{X}) \hookrightarrow C^k_\beta(\mathcal{X})$.

Proof. The first bullet is a consequence of the definition of the $L^p_k$-norms in (10.10). The case $k = 0$ amounts to $\|f\|_{L^p_{0,\gamma}(\mathcal{X})} = \|r^{-n/p+n/p}f\|_{L^p(\mathcal{X})}$ and the general case follows from an induction argument. Here we simply do the case $k = 1$ with the remaining general following from a similar case. Write for the norm in the right hand side
\[ \|f\|^p_{L^p_{0,\gamma}(\mathcal{X})} + \|f\|^p_{L^p_{1,\gamma}(\mathcal{X})} = \|f\|^p_{L^p_{0}(\mathcal{X})} + \|r^{-1}f\|^p_{L^p(\mathcal{X})} + \|\nabla f\|^p_{L^p(\mathcal{X})} \]
\[ = \|f\|^p_{L^p_{1}(\mathcal{X})} + \|r^{-1}f\|^p_{L^p(\mathcal{X})} \leq 2\|f\|^p_{L^p_{1}(\mathcal{X})}, \]
while clearly
\[ \|f\|^p_{L^p_{0,\gamma}(\mathcal{X})} + \|f\|^p_{L^p_{1,\gamma}(\mathcal{X})} \geq \|f\|^p_{L^p_{1}(\mathcal{X})}. \]
The remaining two bullets are particular instances of the standard weighted Sobolev embedding theorems (Theorem 4.17 in [34]). To apply them one just needs to check that $1 - \frac{n}{p} > -\frac{n}{q}$ and $k + 1 - \frac{n}{p} \geq k - 1$.

Lemma 11.4. Let $p \geq n/2$, $\beta \leq -1$ and $\xi \in \Omega^0(X, V)$ with $\nabla \xi \in L^p_{1,\beta}(\mathcal{X})$. Then $\lim_{r \to \infty} \xi(r)$ exists and is equal to a $\nabla \xi$-parallel continuous section $\xi_\infty$ of $S_{E_\infty}$.

Proof. Along the conical end $K - X$ we write
\[ \nabla \xi = \frac{\partial \xi}{\partial r} \otimes dr + \nabla \xi^\perp + \frac{\partial \xi^\perp}{\partial r} \otimes dr + \nabla \xi^\perp, \]
and as the summands are linearly independent as sections of $\Lambda^1 \otimes S_E$, each of them has its norm bounded by that of $\nabla \xi$. Since $\nabla \xi_\infty$ is irreducible on $S^{\perp}_{E_\infty}$ we find that along the conical end so is $\nabla$ on $S^{\perp}_E$ component. Hence, there is a Poincaré type inequality on the slices $\Sigma_1 = \mathcal{Y}(\{1\} \times \Sigma)$, which can be written as $\|\xi^\perp\|_{L^p(\Sigma_1)} \leq \|\nabla \xi^\perp\|_{L^p(\Sigma_1)}$. Scaling this inequality gives
\[ \|\xi^\perp\|_{L^p(\Sigma_1)} \leq r\|\nabla \xi^\perp\|_{L^p(\Sigma_1)} \leq r\|\nabla \xi\|_{L^p(\Sigma_1)}. \]

\[69\]
on each $\Sigma_r = Y((r) \times \Sigma)$. This together with the hypothesis that $\nabla \xi \in L^p_{k,\beta}$ shows that
\[
\int_1^\infty r^{-(\beta+1)p} \|\nabla \xi\|^p_{L^p(\Sigma_r)} \frac{dr}{r^n} \leq \int_1^\infty cr^{-\beta p} \|\nabla \xi\|^p_{L^p(\Sigma_r)} \frac{dr}{r^n} < \infty.
\]
Scaling the metric on $(1, \infty) \times \Sigma$ to the cylindrical metric $r^{-2}g = dt^2 + g_\Sigma$, where $t = \log(r)$, this implies that as $t \to \infty$, all three $e^{-tp(\beta+1)} \xi^\perp$, $e^{-tp(\beta+1)} \nabla \xi^\perp$ and $e^{-tp(\beta+1)} \nabla \nabla \xi^\perp$ converge in the $L^p$-norm to zero, over the intervals $(t, t+1) \times \Sigma$, equipped with the cylindrical metric $dr^2 + g_\Sigma$. Since $-(\beta+1) \geq 0$, one concludes that as $t \to \infty$, $\xi$ converges to zero in $L^p_{k}(X)$ over these intervals equipped with the fixed cylindrical metric. Using, the Sobolev embedding $L^p_{2}(X) \hookrightarrow C^0(X)$, which holds for $p \geq n/2$, one concludes that $\xi^\perp$ converges uniformly to zero.

For the other component, i.e. $\xi^\parallel$, one has $|\partial_r \xi^\parallel| \leq |\nabla \xi|$ and using this together with the Hölder inequality into
\[
\int_1^\infty \left| \frac{\partial \xi^\parallel}{\partial r} \right| \, dr \leq \int_1^\infty r^{-\beta p} |\nabla \xi|^p \, dr \int_1^\infty r^{\beta p'} \, dr,
\]
where $p' = p/(p - 1)$ is the conjugate exponent. The first integral is bounded above by $\|\nabla \xi\|^p_{L^p_{0,\beta}(X)}$. The second one is $\int_1^\infty r^{\beta p'} \, dr$ and since $\beta \leq -1 < 1/p - 1 = (1-p)/p$ one concludes this integral is finite. It follows that there is a limit $\xi_\infty$ to which $\xi^\parallel$ converges. \hfill \Box

**Remark 11.5.** Using the same proof as in Lemma 11.4 but replacing the Sobolev embedding $L^p_{2}(X) \hookrightarrow C^0(X)$ over $\Sigma^{n-1}$ by the Sobolev embedding $L^p_{k}(X) \hookrightarrow C^0(X)$, which holds for $k > \frac{n}{2}$, leads to the following conclusion. If $k > \frac{n}{2}$ and $\xi \in \Omega^0(X, V)$ with $r^{j-1} \nabla^j \xi \in L^2(X)$ for all $1 \leq j \leq k$. Then $\xi$ converges to a $\nabla_\infty$ parallel section $\xi_\infty$ of $S_{E_\infty}$.

**Corollary 11.6.** Let $k \in \mathbb{N}$ and $p \geq \frac{n}{2}$. Then, for any $\xi \in \Omega^k(X, g_p)$ with $r \nabla_0 \xi \in H^p_{k,\beta+1}(X)$ we have $\xi^\perp \in L^p_{k+1}(X)$. Furthermore, in case $\beta < -1 + \frac{1}{p}$, then $\xi$ converges to a $\nabla_\infty$ parallel section $\xi_\infty$ of $S_{E_\infty}$.

**Proof.** Since $r \nabla_0 \xi \in H^p_{k,\beta+1}(X)$, one knows $r \nabla_0 \xi^\perp \in L^p_{k}(X)$ and the same proof as that of the beginning of Lemma 11.4 shows that $\xi^\perp \in L^p_{k+1}(X)$ and converges to zero as $r \to \infty$. The other component follows from the fact that $r \nabla_0 \xi^\parallel \in L^p_{k,\beta+1}(X)$ is equivalent to $\nabla_0 \xi^\parallel \in L^p_{k,\beta}(X)$. Then, one can repeat the final part of the proof of Lemma 11.4 and notice that the argument there using Hölder’s inequality works for $\beta < -1 + \frac{1}{p}$. \hfill \Box

As the conclusion of the previous lemmata we finally arrive at the main result of this subsection.
Proposition 11.7. Let \( p \in \left[ \frac{n}{2}, n \right), \alpha = 1 - n/p \), and \( N(\cdot, \cdot) \) a bilinear map satisfying 
\[
N(S_E^\parallel, S_E^\parallel) = 0, \quad N(S_E^\parallel, S_E^\perp) \subseteq S_E^\perp, \quad \text{and} \quad N(S_E^\perp, S_E^\perp) \subseteq S_E^\parallel.
\]
Then, \( [\cdot, \cdot] \) gives rise to a continuous multiplication map 
\[
N(\cdot, \cdot) : H_{1,\alpha}^p(X) \times H_{1,\alpha}^p(X) \leftrightarrow H_{0,\alpha-1}^p(X).
\]

Proof. Let \( \chi, \xi \in H_{1,\alpha}^p(X) \) and \( q = \frac{np}{n-p} \), then by definition \( \chi^\parallel, \xi^\parallel \in L_{1,\alpha}^p(X) \), which using the embedding in the second bullet of Lemma 11.3 lie in \( L_{0,\alpha}^q(X) \). On the other hand, by the first bullet in that same Lemma we find that \( \chi^\perp, \xi^\perp \in L_{1,\alpha}^p(X) = L_{0,-n/p}^p(X) \cap L_{1,-n/p+1}^p(X) \), and again by the second bullet \( L_{1,-n/p+1}^p(X) \subseteq L_{0,-n/p+1}^q(X) \). In conclusion,
\[
\chi^\parallel, \xi^\parallel \in L_{0,\alpha}^p \cap L_{0,\alpha}^q(X), \chi^\perp, \xi^\perp \in L_{0,-n/p}^p \cap L_{0,-n/p}^q(X).
\]

By the hypothesis, the term \( N(\chi^\parallel, \xi^\parallel) \) vanishes and
\[
N(\chi, \xi) = N(\chi^\perp, \xi^\perp) + (N(\chi^\parallel, \xi^\perp) + N(\chi^\perp, \xi^\parallel)),
\]
with the first term lying in \( S_E^\parallel \) while both the second and third lie in \( S_E^\perp \). So it is enough to show that \( N(\chi^\perp, \xi^\perp) \in L_{0,\alpha-1}^p(X) \) and \( N(\chi^\parallel, \xi^\perp) \in L_{1,\alpha}^p(X) = L_{1,-n/p}^p(X) \).

We start by analyzing the term \( N(\chi^\perp, \xi^\perp) \), by using Lemma 11.1 twice in the forms
\[
L_{0,-n/p}^p(X) \times L_{0,-n/p}^p(X) \subseteq L_{0,-2n/p}^p(X), \quad \text{and} \quad L_{0,-n/p+1}^q(X) \times L_{0,-n/p+1}^q(X) \subseteq L_{0,-2n/p+2}^q(X).
\]

Then, \( N(\chi^\perp, \xi^\perp) \in L_{0,-n/p+2}^q(X) \cap L_{0,-n/p+2}^q(X) \) and using Lemma 11.2 with \( p_1 = p/2, \gamma_1 = -2n/p, p_2 = q/2, \gamma_2 = -2n/p + 2 \) and \( s = p \) gives that \( N(\chi^\perp, \xi^\perp) \in L_{0,\alpha-1}^p(X) \) for all \( \alpha \) such that
\[
\alpha - 1 \geq \max \left\{ \frac{2n - n}{p} - \frac{2n}{p} - 2n, \frac{2n}{q} - \frac{2n}{p} + 2 \right\} = \frac{n}{p}.
\]

Next, we turn to the terms in \( S_E^\perp \). For this and apply again Lemma 11.1 twice, now in the form
\[
L_{0,\alpha}^q(X) \times L_{0,-n/p+1}^q(X) \subseteq L_{0,\alpha-n/p+1}^q(X), \quad \text{and} \quad L_{0,\alpha}^q(X) \times L_{0,-n/p}^q(X) \subseteq L_{0,\alpha-n/p}^q(X).
\]

Then \( N(\chi^\parallel, \xi^\parallel), N(\chi^\perp, \xi^\perp) \in L_{0,\alpha-n/p+1}^q(X) \cap L_{0,\alpha-n/p+1}^q(X) \) and now we use Lemma 11.2 with \( p_1 = np/(2n - p), \gamma_1 = \alpha - n/p, p_2 = q/2, \gamma_2 = \alpha - n/p + 1 \) and \( s = p \), which gives that \( [\chi^\parallel, \xi^\parallel], [\chi^\perp, \xi^\perp] \in L_{0,\alpha-n/p}^p(X) \). At this point we mention that the condition to apply Lemma 11.2 is that
\[
\max \left\{ \frac{2n - p - n}{np} - \frac{n}{p}, \frac{2n}{q} - \frac{n}{p} + \alpha - n/p + 1 \right\} = \frac{n}{p} \leq \frac{n}{q}.
\]
which holds by the condition that $p \geq n/2$ arises. One must remark that this condition is further required for the Sobolev embeddings in Lemma 11.3 to hold and the condition that $p < n$ is required in order for $p_1 = np/(2n - p) < p$ and lemma Lemma 11.2 to apply in the second case above.

\[\square\]

12. Moduli theory

12.1. A discussion of possible moduli spaces of $G_2$-monopoles. In this section we discuss a few possible possibilities for the moduli space of $G_2$-monopoles on an asymptotically conical $G_2$-manifolds with finite mass and fixed monopole class.

Recall that a finite mass monopole $(\nabla, \Phi)$ with monopole class $\alpha \in H^2(\Sigma, \mathbb{Z})$ is modeled at infinity on a reducible pair $(\nabla_\infty, \Phi_\infty)$ on $P_\infty \to \Sigma$ as in Theorem 7.1 or Main Theorem 2. We fix a framing at infinity

$$\eta : \mathbb{Y}^*P|_{X-K} \to \pi^*P_\infty,$$

where $\mathbb{Y}$ is as in Definition 2.4 and $\pi : C(\Sigma) \to \Sigma$ denotes the projection to the second factor. Let $[(\nabla_\infty, \Phi_\infty)]$ denote the gauge equivalence class of this pair and define

$$\Gamma_\infty = \{g \in \text{Aut}(P_\infty) \mid g \cdot (\nabla_\infty, \Phi_\infty) = (\nabla_\infty, \Phi_\infty)\},$$

$$\gamma_\infty = \{\xi \in \Gamma(g P_\infty) \mid \nabla_\infty \xi = 0 = [\xi, \Phi_\infty]\}.$$

Then $\Gamma_\infty$ are the gauge transformations of $P_\infty$ which preserve the boundary data and $\gamma_\infty$ its Lie algebra. Furthermore, we define $G$ to be the group of continuous gauge transformations, which have a limit $g_\infty = \lim_{r \to \infty} g(r) \in G_\infty$. It comes equipped with an evaluation map $ev : G \to G_\infty$ which associates to $g \in G$ its limit at infinity. Using the framing (12.1), we can define

$$G(0) := \text{ker}(ev) \subseteq \Gamma := \text{ev}^{-1}(\Gamma_\infty) \subseteq G.$$

There are two possible approaches to setting up the moduli theory:

1. Consider gauge equivalence classes of pairs $(\nabla, \Phi)$ on $P$ that are asymptotic to a pair in $[(\nabla_\infty, \Phi_\infty)]$.
2. Fix the representative $(\nabla_\infty, \Phi_\infty) \in [(\nabla_\infty, \Phi_\infty)]$ and consider pairs $(\nabla, \Phi)$ asymptotic to this representative modulo the action of $\Gamma \subseteq G$.

The automorphism group of the boundary data $\Gamma_\infty$ is isomorphic to a subgroup $H \subseteq SU(2)$, i.e. it is either trivial or isomorphic to $U(1)$.

Remark 12.1. Recall that $g_{P_\infty} \cong \mathbb{R} \oplus L^2(X)$, where $L$ is a line bundle over $\Sigma$. However, if $(\nabla, \Phi)$ is irreducible, then $||\Phi||_{L^2} \neq 0$ and the energy formula in Theorem 8.3 shows that $\alpha = -2\pi i c_1(L)$.
must be nontrivial in which case \( H \cong U(1) \). Tracing through the definitions, this isomorphism can be seen more explicitly as follows.

- If \( g \in \text{Aut}(P_\infty) \) and \( g \cdot \Phi_\infty = \Phi_\infty \), then one can write \( g = e^{if\Phi_\infty} \), for some \( f \in C^\infty(\Sigma, \mathbb{R}/\mathbb{Z}) \). Moreover, if \( g \) is further supposed to preserve the connection then it must be constant, this gives an isomorphism \( \Gamma_\infty \cong U(1) \).

- If \( \xi \in g_{P_\infty} \) and \([\xi, \Phi_\infty] = 0\), then \( \xi = f\Phi_\infty \) for \( f \in C^\infty(\Sigma, \mathbb{R}) \) and if \( \nabla_\infty \xi = 0 \) then \( f \) must be constant. This gives an isomorphism \( \gamma_\infty \cong \mathbb{R} \).

It is also useful to consider a slightly larger moduli space which fibers over these ones with fiber \( \Gamma_\infty \). Then, consider the moduli space of configurations to be those pairs \((\nabla, \Phi)\) which are asymptotic to \((\nabla_\infty, \Phi_\infty)\) modulo the action of \( \mathcal{G}(0) \). Any implementation of this idea gives a moduli space of configurations, which fibers over the previous ones with fiber \( \Gamma_\infty \cong H \).

Remark 12.2. There is also one other way of constructing such a moduli space which comes with the framing \( \eta \) incorporated in the definition at the expense of considering a slightly larger gauge group. Consider triples \((\nabla, \Phi, \eta)\) of configurations and a framing \( \eta \) modulo the action of \( \Gamma \). Here \( \Gamma \) acts on the framing in a nontrivial way and this is what accounts for increasing the gauge group from \( G(0) \) to \( \Gamma \).

Let \((\nabla_0, \Phi_0)\) be a connection and an Higgs field on \( P \) which along the conical end, \( X - K \), is asymptotic to pullbacks of \((\nabla_\infty, \Phi_\infty)\) via the framing \( \eta \) as in (12.1). The adjoint action of \( \Phi_0 \) induces a splitting of \( g_P \) along the conical end as in the beginning of Section 5

\[
g_P|_{X - K} \cong g_P^\| \oplus g_P^\perp,
\]

where \( g_P^\| = \text{ker}(\text{ad}_{\Phi_0}) \) and \( g_P^\perp \) its orthogonal. So one can uniquely split sections \( \chi \in \Omega^k(X - K, g_P) \) as \( \chi = \chi^\| + \chi^\perp \), for \( \chi^\| \in \Omega^k(X - K, g_P^\|) \) and \( \chi^\perp \in \Omega^k(X - K, g_P^\perp) \).

12.2. Moduli of Configurations. This subsection defines and constructs moduli spaces of configurations \((\nabla, \Phi)\) with fixed mass and monopole class. So we fix \((\nabla_0, \Phi_0)\) and construct \( H^P_{k,\alpha} \)-spaces as in Definition 10.11 using this pair and

\[
S_E = (\Lambda^1 \oplus \Lambda^0) \otimes g_P,
\]

with the decomposition into \( \| \) and \( \perp \) being that induced on \( g_P \) along the conical end. The upshot of this subsection is Theorem 12.9 which gives the moduli space of configurations the structure of a smooth Banach manifold. Then the boundary conditions defined by a finite mass monopole are preserved in

\[
A^P_{k,\alpha} = \{ \nabla = \nabla_0 + a \mid a \in H^P_{k,\alpha}(X) \}, \quad H^P_{k,\alpha} = \{ \Phi = \Phi_0 + \phi \mid \phi \in H^P_{k,\alpha}(X) \}.
\]
Now we define the configuration space of $G_2$-monopoles.

**Definition 12.3.** Let $p > 0$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then, we define

$$C^p_{k, \alpha} := A^p_{k, \alpha} \times H^p_{k, \alpha},$$

which we refer to as the space of configurations.

The induced topology of these spaces, in principle, depends on the background configuration $(\nabla_0, \Phi_0)$ and on $p, k,$ and $\alpha$. A gauge transformation $g \in \mathcal{G} \cap L^p_{k+1, \text{loc}}(X)$ acts on a configuration $(\nabla_0 + a, \Phi_0 + \phi)$ via

$$\left(\nabla_0 + g(\nabla_0 g^{-1}) + gag^{-1}, \Phi_0 + (g\Phi_0 g^{-1} - \Phi_0) + g\phi g^{-1}\right),$$

and two configurations in $C^p_{k, \alpha}$ be considered equivalent if related by such a $g \in \mathcal{G} \cap L^p_{k+1, \text{loc}}(X)$. We now describe the necessary setup in order to view this equivalence relation as generated by the action of a Banach Lie Group. Set

$$G^p_{k, \alpha} := \{ g \in L^p_{k+1, \text{loc}}(X) \mid r\nabla_0 g \in H^p_{k, \alpha+1}(X)\},$$

$$\text{Lie}(G^p_{k, \alpha}) := \{ \xi \in \Omega^0(X, g^P) \mid r\nabla_0 \xi \in H^p_{k, \alpha+1}(X)\},$$

which we topologize as follows: The pointwise exponential defines a map, $\exp$, is a now map from $\text{Lie}(G^p_{k, \alpha})$ to $G^p_{k, \alpha}$. For each $\epsilon > 0$ define

$$V_\epsilon = \{ \xi \in \text{Lie}(G^p_{k, \alpha}) \mid ||r\nabla_0 \xi||_{H^p_{k, \alpha+1}(X)} \leq \epsilon \},$$

and let the topology on $G^p_{k, \alpha}$ be generated by the image under the exponential of the open sets $V_\epsilon \subseteq \text{Lie}(G^p_{k, \alpha})$ together with their translations.

**Proposition 12.4.** Let $p \in [\frac{n}{2}, n)$, $\alpha = -\frac{n}{p} + 1$, then the following hold

1. With the topology defined above $G^p_{1, \alpha}$ is a Banach Lie group with Lie algebra $\text{Lie}(G^p_{1, \alpha})$.
2. If one further supposes that $p < \frac{n+1}{2}$, then there is a surjective evaluation homomorphism $\text{ev} : G^p_{1, \alpha} \to \Gamma_\infty$, with derivative $\text{dev} : \text{Lie}(G^p_{1, \alpha}) \to \gamma_\infty$.
3. $G^p_{1, \alpha}$ acts smoothly in $C^p_{1, \alpha}$.

**Proof.** Start by noticing that if $g \in G^p_{1, \alpha}$, then $g \in L^p_{2, \text{loc}}(X)$ and since one is working in a range where $p \geq \frac{n}{2}$, the third bullet in Lemma 11.3 applies and $g \in C^0_{\text{loc}}(X)$, i.e. these gauge transformations are continuous.

1. To prove that $G^p_{1, \alpha}$ is a Banach Lie group we must show that pointwise multiplication and inversion are well defined. Then, with the above topology $\text{Lie}(G^p_{1, \alpha})$ is its Lie algebra.
Lemma 11.3 shows that (Lemma 11.2 guarantees that Lemma 11.1)

(a) To prove that multiplication is well defined let \( g, h \in G_{1,\alpha}^p \), i.e. \( r\nabla_0 g, r\nabla_0 h \in H_{1,\alpha+1}^p (X) \) and one needs to show that

\[
r\nabla_0 (gh) = r(\nabla_0 g)h + rg\nabla_0 h \in H_{1,\alpha+1}^p (X),
\]

for all \( l \leq k \). The gauge transformations are continuous and \( r\nabla_0 h \in H_{1,\alpha+1}^p (X) \), so it follows that \( rg\nabla_0 h \in H_{1,\alpha+1}^p (X) \) and the same applies to \( r(\nabla_0 g)h \). Alternatively one uses the Sobolev embedding in the second bullet of Lemma 11.3, which gives

\[
\begin{align*}
r\nabla_0 h &\in L_{1,\alpha+1}^p (X) \subseteq L_0^{q,\alpha+1} (X), \\
r\nabla_0 h &\in L_{1,-n/p+1}^p (X) \subseteq L_0^{q,\alpha+1} (X), \\
\end{align*}
\]

and using \( \alpha = 1 - n/p \),

\[
r\nabla_0 h, r\nabla_0 g \in L_{0,-n/p+1}^p (X) \cap L_0^{q,\alpha+1} (X).
\]

and the multiplication map in Lemma 11.1 guarantees that

\[
r\nabla_0 g \nabla_0 h \in L^p (X) \subseteq H_{0,\alpha}^p (X).
\]

(b) To prove \( g^{-1} \in G_{1,\alpha}^p \), notice that \( \nabla_0 g^{-1} = -g^{-1}(\nabla_0 g)g^{-1} \). Then proceeding as before, separating terms and using \( g, g^{-1} \in C_{loc}^0 (X) \) and Lemmata 11.1 and 11.2 yield \( r\nabla_0 g^{-1} \in H_{1,\alpha+1}^p (X) \).

(2) Let \( g \in G_{1,\alpha}^p \), then \( r\nabla_0 g \in H_{1,\alpha+1}^p (X) \), i.e. \( (\nabla_0 g)_{\perp} \in L_{1,\alpha}^p (X) \) and \( L_{0,1-n/p}^p (X) \) and \( (\nabla_0 g)^\perp \in L_{1,-n/p}^p (X) \). Then, Lemma 11.4 shows that \( (\nabla_0 g)^\perp \to 0 \), but the same does not apply to the other component. However, the last part of the argument in that Lemma can be used and we now repeat it here. Notice that \( \nabla_0 g \in L_{0,-n/p+1}^p (X) \), then Hölder’s inequality gives

\[
\int_1^\infty |\frac{\partial g}{\partial r}|^p dr \leq \int_1^\infty |r^{n/p-1} \nabla_0 g|^p dr \int_1^\infty r^{p/(p-1)(1-n/p)} dr.
\]

The first integral is bounded above by \( \|\nabla_0 g\|_{L_{0,-n/p+1}^p (X)}^p \) while the second is finite if and only if \( p < \frac{n+1}{2} \). Hence in this case this proves the existence of \( g_{\infty} \in G_{\infty} \) such that \( g \to g_{\infty} \) and \( \nabla_0 g_{\infty} = 0 \) (i.e. \( g_{\infty} \in \Gamma_{\infty} \)). Using a bump function it is straightforward to check that the evaluation maps, given by taking the limits, are surjective.

(3) To check the action of \( G_{1,\alpha}^p \) on \( C_{1,\alpha}^p \) is well defined, one needs to prove that

\[
g(\nabla_0 g^{-1}) + g \Phi_{0} g^{-1}, \quad \text{and} \quad (g \Phi_{0} g^{-1} - \Phi_{0}) + g \phi g^{-1}
\]
Lemma 11.3, used in the same way as before, show which is a Banach Lie subgroup of $\mathfrak{g}$ involving $\gamma$ and $\delta$ in $\mathfrak{g}$ are in $\mathcal{G}$. For the terms $gag^{-1}$, $g\phi g^{-1}$ and $g\nabla_0 g^{-1} = -(\nabla_0 g)g^{-1}$ notice that $(a,\phi) \in H^p_{1,\alpha}(X)$, $g \in C^0$ as it is in $L^p_{2,\text{loc}}(X)$, and $r\nabla_0 g \in H^p_{1,\alpha+1}(X)$. Then, repeating the arguments in the proof of the first item proves that these are $H^p_{1,\alpha}(X)$. One is now left with analyzing $(g\Phi_0 g^{-1} - \Phi_0)$, for which one requires again the conclusion of the second item, above. Namely that if $g \in \mathcal{G}$ and $\xi \in \mathcal{L}(\mathcal{G})$ are such that $r\nabla_0 g, r\nabla_0 \xi \in H^p_{1,\alpha+1}(X)$, then $g, \xi$ converge to limits $g_\infty \in \Gamma_\infty$ and $\xi_\infty \in \gamma_\infty$. Moreover, one has $\xi_\perp \in L^2(X)$ by Corollary 11.6 and writing $g = e^\xi$ gives

$$g\Phi_0 g^{-1} - \Phi_0 = \sum_{k=1}^\infty \frac{1}{k!} \text{ad}(\xi)^k(\Phi_0),$$

and the multiplication map in Lemma 11.3, used in the same way as before, show that the $k \geq 2$ terms are in $H^p_{1,\alpha}(X)$ if and only if $[\xi, \Phi_0] \in H^p_{1,\alpha}(X)$. Along the conical end, we can write $[\xi, \Phi_0] = [\xi_\perp, \Phi_0]$ and since $\Phi_0$ is smooth and bounded and $\xi_\perp \in L^p_2(X)$ it is indeed true that $[\xi, \Phi_0] \in H^p_{1,\alpha}(X)$. First, the convergence of the series above is follows from $||[\xi, \Phi_0]| \leq |\xi_\perp|$ which converges to zero as $r \to \infty$. It is therefore bounded and the convergence of the series is guaranteed by the term $\frac{1}{k!}$.

Conversely, if $(\nabla, \Phi)$ and $g \cdot (\nabla, \Phi)$ are both in $C^p_{1,\alpha}$ are related by an $L^p_{2,\text{loc}}(X)$ gauge transformation $g = e^\xi$, then actually $e^\xi \in \mathcal{G}_{1,\alpha}$ one rewinds the previous arguments. First, the fact that $[\xi, \Phi] = [\xi_\perp, \Phi_0] + \ldots \in L^2_2(X) \subseteq L^1_1(X)$ implies $r\nabla_0 \xi_\perp \in L^p_1(X)$. Second, the fact that $g^{-1}(\nabla_0 g) = \nabla_0 \xi \in H^p_{1,\alpha}(X)$ implies that $r\nabla_0 \xi \perp \in L^p_1(X)$. Put these two together to conclude that $r\nabla_0 \xi \in H^p_{1,\alpha+1}(X)$ and so $g \in \mathcal{G}^p_{1,\alpha}$.

Using the second item in this proposition, we define

$$\mathcal{G}^p_{k,\alpha}(0) := \ker(\text{ev}),$$

which is a Banach Lie subgroup of $\mathcal{G}^p_{k,\alpha}$ consisting of the gauge transformations converging to the identity along the end. For $p \in [n/2, n)$ and $\alpha = 1 - n/p$ its Lie algebra is the Lie subalgebra of $\text{Lie}(\mathcal{G}^p_{k,\alpha})$ consisting of those sections which decay, i.e.

$$\text{Lie}(\mathcal{G}^p_{k,\alpha}(0)) = H^p_{k+1,\alpha+1}(X, \mathfrak{g}_p).$$

**Lemma 12.5.** Let $\beta \in \mathbb{R} - \{-1\}$ and $(\nabla, \Phi) \in C^p_{k,\beta}$. Denote by $\delta^\ast_\nabla$ the formal $L^2(X)$ adjoint of the operator $\nabla$ and for all $\beta$ extend $\nabla, \nabla^\ast$ to operators

$$\nabla, (\nabla^\delta_\nabla)^\ast : L^p_{k+1,\beta+1}(X, \mathfrak{g}_p) \to L^p_{k,\beta}(X, T^*X \otimes \mathfrak{g}_p).$$
Then, for $\beta \neq -1$, there is a constant $c > 0$ and an inequality $\|\nabla \eta\|_{L^p_{0,\beta}(X)} \geq c \|\eta\|_{L^p_{0,\beta+1}(X)}$, and so a decomposition

$$L^p_{k,\beta}(X, T^*X \otimes g_P) = \ker(d^*_r) \cap L^p_{k,\beta}(X) \oplus \text{im}(\nabla).$$

(12.2)

Proof. For all $p, k, \beta$ the map $r^{-\beta-1} : L^p_{k,\beta+1}(X) \to L^p_{k,1}(X)$, which multiplies a section by $r^{-\beta-1}$, is an isomorphism of Banach spaces. Conjugation with it gives then an equivalence of linear operators

$$L^p_{k+1,\beta+1} \xrightarrow{\nabla} L^p_{k,\beta} \quad \downarrow \quad \downarrow$$

$$L^p_{k+1,0} \xrightarrow{\nabla^\beta} L^p_{k,-1}$$

with

$$\nabla^\beta = r^{-(\beta+1)} \circ \nabla \circ r^{\beta+1} = (\beta + 1) \frac{dr}{r} + \nabla.$$

For simplicity we only present the proof of the $p = 2$ case, as in this case it is easy to complete squares. As $K$ is compact and $\nabla$ is irreducible on $K$, one can combine inequalities of Kato and Poincaré to get

$$\|\nabla \eta\|_{L^2(K)} \geq c_1 \|\eta\|_{L^2(K)},$$

for some $c_1 > 0$ and all $\eta$ compactly supported in the interior of $K$. Moreover, as $r$ is bounded on $K$, this holds equally well for $\nabla^\beta = \nabla$. Then one needs to prove a similar inequality for a section $\eta$ which is supported on the conical end $X - B_R$ one writes $\eta = \eta^\parallel + \eta^\perp \in L^2_{1,0}(X)$ and splitting $\nabla^\beta \eta$ into orthogonal components we compute

$$\|\nabla^\beta \eta\|_{L^2_{0,-1}(X-K)}^2 = \int_R^\infty \frac{dr}{r} \int_\Sigma |r\nabla^\beta \eta|^2 \text{vol}_\Sigma$$

$$= \int_R^\infty \int_\Sigma \left( |\nabla_0 \eta|^2 + \frac{\beta + 1}{r} \eta^\parallel + \eta^\perp \right)^2 \text{vol}_\Sigma$$

$$\geq \int_R^\infty \int_\Sigma \left( \frac{(\beta + 1)^2}{r} |\eta|^2 + r|\nabla_0 \eta|^2 \right) \text{vol}_\Sigma.$$
To handle this let \( \Sigma_r = \mathcal{Y}([r] \times \Sigma) \), then the irreducibility of the connection \( \nabla_\infty \) on \( g_P^\perp \), gives a Poincaré type inequality, which after scaling is \( \| \nabla_\infty \eta \|_{L^2(\Sigma_r)}^2 \geq c_2 r^{-2} \| \eta \|_{L^2(\Sigma_r)}^2 \) for some constant \( c_2 > 0 \). Moreover, as the connection \( \nabla_0 \) is asymptotic to \( \nabla_\infty \) one can assume the same inequality holds for \( \nabla_0 \) and large \( r \). Inserting this into the previous inequality yields

\[
\| \nabla_\beta \eta \|_{L^2(\nabla_0, X-K)}^2 \geq \int \int_{\Sigma} \left( (\beta + 1)^2 \frac{c_2 + (\beta + 1)^2}{r} \right) \text{vol}_\Sigma \geq (1 + \beta)^2 \| \eta \|_{L^2(\nabla_0, X-K)}^2.
\]

Combining this with the similar inequality one has on \( K \), gives the inequality in the first item of the statement.

As a consequence of this Poincaré type inequality \( \nabla_\beta \) has closed image and the decomposition in the theorem follows. Recall that the operator \( \nabla_\beta \) above is equivalent to \( \nabla : L^2_{1,\beta+1}(X) \to L^2_{0,\beta}(X) \), so this one has closed image. Then the same is true for \( \nabla : L^p_{k+1,\beta+1}(X) \to L^p_{k,\beta}(X), \) which gives the decomposition (12.2). Using the weighted inner product \( \langle \cdot, \cdot \rangle_{L^2_0(X)} \) one can identify a copy of cokernel of \( d_A \) with the orthogonal complement, i.e. the kernel of its adjoint

\[
(\nabla_\beta)^* = r^{2(\beta+1)n} \circ \nabla^* \circ r^{-2\beta-n} = (2\beta + n) r \partial_r + \nabla^*.
\]

\[\square\]

**Remark 12.6.** The proof above gives a bound \( \| \nabla \eta \|_p \geq c \| \eta \|_{p+1} \) with an explicit constant \( c = |1 + \beta| \). For \( \beta = -n/2 \) this gives back Hardy’s inequality

\[
\| \nabla \eta \|_{L^2(X)}^2 \geq \left( \frac{n-2}{2} \right)^2 \| \eta \|_{L^2(X)}^2.
\]

Actually this gives the best possible constant on any asymptotically Euclidean manifold.

**Corollary 12.7.** For \( \beta \neq -1 \), the operator

\[
\mathcal{L}_0 : H^P_{k+1,\beta+1}(X, g_P) \to H^P_{k,\beta}(X, (\Lambda^0 \oplus \Lambda^1) \otimes g_P), \tag{12.3}
\]

\[
\xi \mapsto (-\nabla \xi, [\xi, \Phi]),
\]

has closed image. Using the notation \( H^P_{k,\beta}(X) \) for the right hand side in equation (12.3), there is an orthogonal decomposition

\[
H^P_{k,\beta}(X) = \ker(\mathcal{L}_0^*) \oplus \text{im}(\mathcal{L}_0).
\]

Where \( \mathcal{L}_0(a, \phi) = -\nabla^* a + [\Phi, \phi] \).
Proof. This proof combines the inequality in the previous Lemma 12.5 with \( |\Phi, \xi| \geq c|\xi, \perp| \), which holds sufficiently far out along the end. As \( \|L_0(\xi)\|_{H_{0,\beta}^2(X)} = \|\nabla \xi\|_{H_{0,\beta}^2(X)} + \|\Phi, \xi\|_{H_{0,\alpha}^2(X)}^2 \) we immediately conclude that \( L_0 \) has closed image. \( \square \)

Definition 12.8. A configuration \((\nabla, \Phi)\) is said to be irreducible if \( \ker(L_0) = 0 \).

Theorem 12.9. Let \( p \in [n/2, n) \) and \( \alpha = 1 - n/p \). Then, the quotient spaces

\[
\tilde{B}_{1,\alpha}^p = C_{1,\alpha}^p/G_{1,\alpha}^p(0), \text{ and } B_{1,\alpha}^p = C_{1,\alpha}^p/G_{1,\alpha}^p,
\]
inherit the structure of Banach manifolds with the property that

\[
\mathcal{B}_{1,\alpha}^p = \mathcal{B}_{1,\alpha}^p/\Gamma_\infty.
\]
Moreover, the subset \( (B_{1,\alpha}^p)^* \subseteq B_{1,\alpha}^p \) consisting of the image of the irreducible configurations is a smooth Banach manifold.

Proof. To prove that \( \tilde{B}_{1,\alpha}^p = C_{1,\alpha}^p/G_{1,\alpha}^p(0) \) is a Banach manifold one constructs local slices to the action of \( G_{1,\alpha}^p(0) \) using the inverse function theorem. Then these slices can be used as charts for \( \tilde{B}_{1,\alpha}^p \). Let \( \epsilon > 0 \) and define the slice candidates as

\[
T_{(\nabla, \Phi), \epsilon} = \{(a, \phi) \in H_{1,\alpha}^p(X) \mid \nabla^* a - [\Phi_0, \phi] = 0, \|(a, \phi)\|_{H_{1,\alpha}^p(X)} < \epsilon \}.
\]
Then, in order to prove that these are actual slices one needs to show that the map

\[
h : T_{(\nabla, \Phi), \epsilon} \times G_{1,\alpha}^p(0) \to C_{1,\alpha}^p,
\]
which for \( g = e^{g} \) sufficiently close to the identity, sends \((a, \phi), g\) to the gauge equivalent configuration

\[
h((a, \phi), g) = g \cdot (\nabla + a, \Phi + \phi),
\]
is an isomorphism onto an open set around \((A, \Phi)\). This can be proved using the inverse function theorem, by simply showing that the derivative

\[
dh = \text{id} \oplus L_0 : (\ker(L_0^*) \cap H_{1,\alpha}^p(X)) \oplus H_{1,\alpha}^p(X) \to H_{1,\alpha}^p(X)
\]

\[
((a, \phi), \xi) \mapsto (\nabla \xi + a, [\xi, \Phi] + \phi),
\]
is an isomorphism. This is a direct consequence of Corollary 12.7. There is still the extra action of \( \Gamma_\infty \) on \( C_{1,\alpha}^p \) and one can quotient out by its action to obtain the full quotient \( B_{1,\alpha}^p = C_{1,\alpha}^p/G_{1,\alpha}^p \). Moreover, away from reducible configurations the action of \( G_{1,\alpha}^p \) is free and so \( (\tilde{B}_{1,\alpha}^p)^* \) is smooth. \( \square \)

12.3. Moduli of Monopoles. In this short subsection we finally prove the theorem establishing the main Fredholm setup describing the moduli space of \( G_2 \)-monopoles; see
Theorem 12.10.
Fix $p \in [n/2, n)$ and $\alpha = 1 - n/p \notin \mathcal{K}(D)$, then Theorem 10.15 applies to the linear map $L = D + q$ defined in equation (9.4) as the gauge fixed linearized monopole equation. Using this, we now show that the moduli space of $G_2$-monopoles can be described as a quotient of the zero set of a $\Gamma_\infty$-invariant Fredholm section of the Banach space bundle

$$\mathcal{F}^p_{1,\alpha} = C^p_{1,\alpha} \times_{g^p_{1,\alpha}(0)} H^p_{0,\alpha-1}(X, \Lambda^*X \otimes g_P),$$

over the Banach manifold $\tilde{B}^p_{1,\alpha}$. Notice that sections of this bundle are in one-to-one correspondence with $\mathcal{G}^p_{1,\alpha}(0)$-equivariant maps from $C^p_{1,\alpha} \to H^p_{0,\alpha-1}(X, \Lambda^*X \otimes g_P)$ and, as we see in the proof of the next result, the monopole equation is precisely given by the map

$$\text{mon} : C^p_{1,\alpha} \to H^p_{0,\alpha-1}(X, \Lambda^*X \otimes g_P),$$

defined by

$$\text{mon}(\nabla, \Phi) = * (F_{\nabla} \wedge \psi) - \nabla \Phi,$$

which is invariant by the action of the gauge transformations $\mathcal{G}^p_{1,\alpha} \supset \mathcal{G}^p_{1,\alpha}(0)$. We have now everything in place to prove the main theorem in this section.

**Theorem 12.10.** Let $G = SU(2)$ and $p \in [n/2, n)$ such that $\alpha = 1 - n/p \notin \mathcal{K}(D)$. Then, there is a $\Gamma_\infty$-invariant Fredholm section

$$\text{mon} : \tilde{B}^p_{1,\alpha} \to \mathcal{F}^p_{1,\alpha},$$

of the bundle $\mathcal{F}^p_{1,\alpha} \to \tilde{B}^p_{1,\alpha}$ such that the moduli space of $G_2$-monopoles is in bijection with

$$\text{mon}^{-1}(0) / \Gamma_\infty \subseteq \tilde{B}^p_{1,\alpha}.$$

**Proof.** The monopole moduli space is the zero locus of the section $\text{mon}$ and we must now show this is a section of $\mathcal{F}^p_{1,\alpha} \to \tilde{B}^p_{1,\alpha}$. To achieve this we write $(\nabla, \Phi) \in C^p_{1,\alpha}$ as $(\nabla_0 + a, \Phi_0 + \phi)$ with $(a, \phi) \in H^p_{1,\alpha}(X)$. Then, using $\text{mon}(\nabla_0, \Phi_0) = 0$

$$\text{mon}(\nabla, \Phi) = \text{mon}(\nabla_0, \Phi_0) + \left( T_{(\nabla_0, \Phi_0)} \text{mon} \right)(a, \phi) + N((a, \phi), (a, \phi)) \quad (12.4)$$

where $\left( T_{(\nabla_0, \Phi_0)} \text{mon} \right)(a, \phi)$ is as in equation (9.1) and $N((a, \phi), (a, \phi))$ is a quadratic term in $(a, \phi)$ which can be written as

$$N((a, \phi), (a, \phi)) = \frac{1}{2} * ([a \wedge a] \wedge \psi) - [a, \phi].$$

This satisfies the conditions of Proposition 11.7 and so the right hand side of equation (12.4) lies in $H^p_{0,\alpha-1}(X)$. We can define $\text{mon}^{-1}(0)$ inside $\tilde{B}^p_{1,\alpha}$ using the local slices constructed in Theorem 12.9 which modeled on the kernel of $\mathcal{L}^0_{1,\alpha}$. Equivalently, we may instead construct such a local model for $\text{mon}^{-1}(0)$ in the quotient $\tilde{B}^p_{1,\alpha}$ by considering instead the zero locus.
of the joint map $\text{mon} + \mathcal{L}_0^\ast$. Its linearization

$$\left( T_{(V,\Phi)}\text{mon} \right) \oplus \mathcal{L}_0^\ast,$$

is precisely the map $\mathcal{L} = \mathcal{D} + q$ defined in equation (9.4). For these $p,k,\alpha$ Theorem 10.15 applies and the map $\mathcal{L} : H_{p,\alpha}^0(X) \to H_{p,\alpha-1}^0(X)$ is therefore Fredholm. □

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83