Generating functional of ChPT at one loop for non-minimal operators

May 7, 2014

Andria Agadjanov\textsuperscript{a,b}, Dimitri Agadjanov\textsuperscript{a,b}, Anzor Khelashvili\textsuperscript{b,c} and Akaki Rusetsky\textsuperscript{a}

\textsuperscript{a} Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, Nussallee 14-16, D-53115 Bonn, Germany

\textsuperscript{b} St. Andrew the First-Called Georgian University of the Patriarchate of Georgia, Chavchavadze Ave. 53a, 0162, Tbilisi, Georgia

\textsuperscript{c} Institute of High Energy Physics, Ivane Javakhishvili Tbilisi State University, University St. 9, 109, Tbilisi, Georgia

Abstract

The divergent part of the one-loop effective action in Chiral Perturbation Theory with virtual photons has been evaluated in an arbitrary covariant gauge. The differential operator, that emerges in the functional determinant, is of a non-minimal type, for which the standard heat kernel methods are not directly applicable. Both $SU(2)$ and $SU(3)$ cases have been worked out. A comparison with existing results in the literature is given.

Pacs: 11.10.Gh, 11.30.Rd, 12.39.Fe, 13.40.Ks

Keywords: Chiral symmetries, chiral Lagrangians, electromagnetic interactions, ultraviolet divergences
1 Introduction

Isospin-breaking corrections to the hadronic observables emerge from two intrinsically distinct sources. The so-called “strong” isospin breaking, which is due to the difference of the $u$- and $d$-quark masses, is embedded in the standard framework of Chiral Perturbation Theory (ChPT) from the outset [1, 2]. The electromagnetic isospin breaking, caused by a presence of the virtual photons, can be systematically included by the use of the spurion technique [3] (see also [4–6]). ChPT with photons and leptons, which is the low-energy effective theory of the Standard Model, can be also constructed [7]. The method has been extended to include the baryon sector of ChPT as well [8]. Last but not the least, including virtual photons becomes inevitable, if an attempt is made to describe metastable bound states of hadrons – the so-called hadronic atoms – which are kept together predominantly by the Coulomb force and decay mainly through strong interactions (for a recent review, see, e.g. [9] and references therein). Note, however, that defining the splitting of the hadronic observables into the “purely strong” and “electromagnetic” contributions in ChPT is, in general, an ambiguous procedure due to the renormalization group running of the parameters of the underlying theory [10, 11] (for related work on the subject, see also [12, 13]).

One of the issues, which should be addressed during the construction of the low-energy effective theory of the Standard Model, is the gauge dependence of the parameters of the effective Lagrangian (more precisely, the dependence of these parameters on the gauge fixing of the electromagnetic field). It should be pointed out that, in the matching of the effective theory to the Standard Model, non-perturbative QCD effects are essentially involved, and the study of the gauge dependence can provide valuable information about the relation of the couplings of the effective Lagrangian to the fundamental parameters of the underlying theory. We also expect that such investigation might be helpful for the interpretation of the lattice QCD results including electromagnetic effects, which started to appear recently (see, e.g., [14–18]).

The gauge-dependence of certain $O(e^2 p^2)$ and $O(e^4)$ electromagnetic low-energy constants (LECs) in ChPT has been investigated in the past (see, e.g., [10, 12, 13]) but a systematic analysis of the problem is still missing. One obvious reason for this is that the one-loop generating functional in ChPT up to now has been calculated only in the Feynman gauge. In other gauges, the differential operator, which emerges in the functional determinant, is of a non-minimal type. Standard heat kernel methods\(^1\) are not applicable for such kernels. In the literature, one finds examples of the calculation of the second Seeley-DeWitt coefficient for the non-minimal operators, which determines the UV-divergent part of the effective action at one loop [21–26], see also Ref. [27] for a review and comparison

\(^1\)See, e.g., Ref. [19] for a review. For the extension of the method to the case when fermions are present, see Refs. [20]
of different methods. However, to the best of our knowledge, the general methods
so far have not been applied to the particular problem we are interested in. The
aim of the study, carried out in this brief note, is to close the gap and to present
a calculation of the divergent part of the one-loop effective functional in ChPT
with virtual photons in an arbitrary covariant gauge.

The layout of the paper is as follows. In the section 2 we collect all the relevant
notation and display the \( O(p^4) \) Lagrangian in the 3-flavor ChPT. In section 3 we
write down the one-loop effective action with virtual photons. The calculation of
the divergent part of the functional determinant in the arbitrary covariant gauge
is presented in section 4 and the renormalization is carried out in section 5, where
the divergent part of the LECs both in 3-flavor and 2-flavor cases are displayed.
In the section 6, comparison to the results, available in the literature, has been
carried out. Finally, section 7 contains a short summary of our findings.

2 ChPT with virtual photons

This section collects the notations, which will be used in the following. The lowest
order Lagrangian of ChPT with virtual photons in case of three flavors is given by

\[
L_2 = \frac{F_0^2}{4} \langle d_\mu U d^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle + C \langle UQU^\dagger Q \rangle, \tag{2.1}
\]

where \( U \) is a unitary \( 3 \times 3 \) matrix containing eight Goldstone boson fields, the
brackets stand for the traces in flavor space, and

\[
d_\mu U = \partial_\mu U - i R_\mu U + i U L_\mu, \tag{2.2}
\]

\[
R_\mu = v_\mu + a_\mu + A_\mu Q, \quad L_\mu = v_\mu - a_\mu + A_\mu Q, \tag{2.3}
\]

\[
\chi = 2B_0(s + ip). \tag{2.4}
\]

Here \( Q = e \text{ diag}(2/3, -1/3, -1/3) \) is the charge matrix of the quarks, \( s, p, v_\mu, a_\mu \)
are the external scalar, pseudoscalar, vector and axial-vector sources, respectively,
and \( A_\mu \) is the electromagnetic field. The \( O(p^2) \) low-energy constant \( F_0 \) is the
pion decay constant in the chiral limit, \( B_0 \) is related to the quark condensate
and \( C = F_0^4 Z \) describes the \( O(e^2) \) electromagnetic mass splittings of Goldstone
bosons:

\[
M_{\pi^+}^2 - M_{\pi^0}^2 = M_{K^+}^2 - M_{K^0}^2 = 2e^2 F_0^2 Z. \tag{2.5}
\]

The Lagrangian (2.1) should be supplemented by the electromagnetic field La-
grangian

\[
L_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2a} (\partial^\mu A_\mu)^2, \tag{2.6}
\]
where \( a \) denotes the gauge fixing parameter.

The full Lagrangian at next-to-leading order is given in Ref. [3]. It contains strong and electromagnetic terms

\[
\mathcal{L}_4 = L_1 \langle d^\mu U d^\nu U \rangle^2 + L_2 \langle d^\mu U d^\nu U \rangle \langle d^\mu U d^\nu U \rangle + L_3 \langle d^\mu U d^\nu U d^\rho U \rangle + L_4 \langle d^\mu U d^\nu U \rangle \langle \chi U + \chi U^\dagger \rangle \\
+ L_5 \langle d^\mu U d^\nu U (\chi U + U^\dagger \chi) \rangle + L_6 (\chi U + U^\dagger \chi)^2 + L_7 (\chi U - U^\dagger \chi)^2 \\
+ L_8 (\chi U \chi U^\dagger U + \chi U^\dagger \chi U^\dagger) - iL_9 \langle d^\mu U d^\nu U R_{\mu \nu} \rangle + d^\mu U d^\nu U L_{\mu \nu} \\
+ L_{10} \langle R^\mu R^\nu U L_{\mu \nu} \rangle + H_1 \langle R^\mu R^\nu + L^\mu L_{\mu \nu} \rangle + H_2 \langle \chi \chi \rangle \\
+ F_0^2 \left\{ K_1 \langle d^\mu U d^\nu U \rangle \langle Q^2 \rangle + K_2 \langle d^\mu U d^\nu U \rangle \langle QUQU \rangle \\
+ K_3 \langle (d^\mu UQUU \rangle \langle d^\mu UQUU \rangle + \langle d^\mu UQUU \rangle \langle d^\mu UQUU \rangle \rangle \\
+ K_4 \langle d^\mu UQUU \rangle \langle d^\mu UQUU \rangle + K_5 \langle (d^\mu U d^\nu U + d^\mu U d^\nu U)^2 \rangle \\
+ K_6 \langle d^\mu U d^\nu UQUU \rangle + d^\mu U d^\nu UQUU \rangle \\
+ K_7 (\chi U + U^\dagger \chi)^2 + K_8 (\chi U + U^\dagger \chi)^2 \rangle QUQU \rangle \\
+ K_9 (\chi U + U^\dagger \chi)^2 \rangle QUQU \rangle \\
+ K_{10} (\chi U + U^\dagger \chi)^2 QUQU \rangle + (\chi U + U^\dagger \chi)^2 QUQU \rangle \\
+ K_11 (\chi U + U^\dagger \chi)^2 QUQU \rangle + (\chi U + U^\dagger \chi)^2 QUQU \rangle \\
+ K_{12} \langle d^\mu U c^\mu_R Q, Q \rangle U + d^\mu U c^\mu_L Q, Q \rangle U \rangle \\
+ K_{13} \langle c^\mu_R Q, c^\mu_L Q \rangle + K_{14} \langle c^\mu_R Q, c^\mu_L Q \rangle \rangle \\
+ F_0 \left\{ K_{15} \langle QUQU \rangle^2 + K_{16} \langle QUQU \rangle \langle Q^2 \rangle + K_{17} \langle Q^2 \rangle^2 \right\}, \tag{2.7}
\]

where

\[
c^\mu_L Q = -i [I^\mu, Q], \quad I = L, R, \tag{2.8}
\]

\[
R_{\mu \nu} = \partial_\mu R_\nu = \partial_\nu R_\mu = i [R_\mu, R_\nu], \quad L_{\mu \nu} = \partial_\mu L_\nu = \partial_\nu L_\mu = i [L_\mu, L_\nu]. \tag{2.9}
\]

The coefficients \( L_i, H_i, K_i \) cancel the UV divergences, arising from the divergent part of the one-loop effective action

\[
L_i = \Gamma_i \lambda + L_i^\dagger (\mu),
\]

\[
H_i = \Delta_i \lambda + H_i^\dagger (\mu),
\]

\[
K_i = \Sigma_i \lambda + K_i^\dagger (\mu),
\]

\[
\lambda = \frac{\mu^{d-4}}{16 \pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left( \Gamma'(1) + \ln 4 \pi + 1 \right) \right\}, \tag{2.10}
\]
where \( d \) is the number of space-time dimensions and \( \mu \) denotes the scale of dimensional regularization. The aim of the present work is to determine the gauge-dependent part of \( \Sigma_i \). The quantities \( \Gamma_i, \Delta_i \), as well as \( \Sigma_i \) at \( a = 1 \) are already available in the literature.

### 3 One-loop effective action

The one-loop generating functional for the connected Green functions is given by

\[
e^{iZ(v,a,s,p)} = \int dU dA_{\mu} e^{i\int dx(L_2 + L_4)},
\]

where the integral is evaluated in the semi-classical approximation. To this end, we expand the fields \( U(x), A_{\mu}(x) \) around the solutions of the classical equations of motion \( \bar{U}, \bar{A}_{\mu} \):

\[
U = u e^{i\xi/F_0} u = u \left( 1 + i \frac{\xi}{F_0} - \frac{1}{2} \frac{\xi^2}{F_0^2} + \cdots \right) u \\
A_{\mu} = \bar{A}_{\mu} + \epsilon_{\mu},
\]

where \( \bar{U} = u^2 \) and \( \xi \) is a traceless hermitian matrix, \( \xi = \sum_a \xi^a \lambda^a \) (here, \( \lambda^a \) denote the Gell-Mann matrices).

Next, we substitute this expansion in the action functional in Eq. (3.2) and retain those terms, stemming from \( L_2 \), which are at most quadratic in \( \xi, \epsilon_{\mu} \) (at this order, the fields in \( L_4 \) can be replaced by the classical solutions). The calculations are conveniently done in Euclidean space. The Euclidean action functional then becomes

\[
S_E = \int dx(L_2 + L_4) = \int dx(\tilde{L}_2 + \tilde{L}_4) + \frac{1}{2} \int dx \eta_A D^{AB} \eta_B,
\]

where the Lagrangians \( \tilde{L}_2, \tilde{L}_4 \) are obtained from \( L_2, L_4 \) after continuation to Euclidean space and the substitution \( U, A_{\mu} \to \bar{U}, \bar{A}_{\mu} \). The fluctuations are collected in a single vector

\[
\eta_A = (\xi_a, \epsilon_{\mu}) = (\xi_1, \ldots, \xi_8, \epsilon_0, \ldots, \epsilon_3).
\]

The differential operator \( D \) is defined as:

\[
D = D_0 + \omega,
\]

\[
D_0 = \begin{pmatrix}
-\Box & -\Box \delta^{ab} \\
0 & -\Box \delta^{a\rho} + (1 - \frac{1}{d}) \partial^a \partial^\rho
\end{pmatrix},
\]

\[
\omega = -\{Y_{\mu}, \partial_{\mu}\} - Y_{\mu} Y_{\mu} + \Lambda,
\]

\[
(3.5)
\]
with
\[ Y_\mu = \begin{pmatrix} \Gamma_{\mu}^{ab} & X_\mu^{a\rho} \\ X_\mu^{\rho a} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \sigma^{ab} & \frac{1}{2}\gamma^{a\rho} \\ \frac{1}{2}\gamma^{\rho a} & \rho \delta^{\rho a} \end{pmatrix}. \]

The elements of these matrices are given by the expressions:
\[
\Gamma_{\mu}^{ab} = -\frac{1}{2}\langle[\lambda^a, \lambda^b]\Gamma_\mu\rangle,
X_\mu^{a\rho} = -X^{\rho a} = X^a \delta^\rho, \quad X^a = -\frac{F_0}{4}\langle H_L \lambda^a\rangle,
\sigma^{ab} = -\frac{1}{2}\langle[\Delta_\mu, \lambda^a][\Delta_\mu, \lambda^b]\rangle + \frac{1}{4}\langle\{\lambda^a, \lambda^b\}\sigma\rangle - \frac{F_0^2}{4}\langle H_L \lambda^a\rangle\langle H_L \lambda^b\rangle
- \frac{C}{8F_0^2}\{\langle[H_R + H_L, \lambda^a][H_R - H_L, \lambda^b]\rangle + a \leftrightarrow b\},
\gamma^{a\rho} = \gamma^{\rho a} = F_0\langle[H_R, \Delta_\rho] + \frac{1}{2}D^\rho H_L\rangle \lambda^a,
\rho = \frac{3}{8}F_0^2\langle H_L^2\rangle. \tag{3.7}
\]

where
\[
D_\mu H_L = \partial_\mu H_L + [\Gamma_\mu, H_L]
\Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{1}{2}iu^\dagger \tilde{R}_\mu u - \frac{1}{2}iu\tilde{L}_\mu u^\dagger,
\tilde{R}_\mu = v_\mu + a_\mu + Q\tilde{A}_\mu, \quad \tilde{L}_\mu = v_\mu - a_\mu + Q\tilde{A}_\mu,
\Delta_\mu = \frac{1}{2}u^\dagger d_\mu \bar{U} u^\dagger = -\frac{1}{2}ud_\mu \bar{U} u^\dagger,
H_R = u^\dagger Qu + uQu^\dagger,
H_L = u^\dagger Qu - uQu^\dagger,
\sigma = \frac{1}{2}(u^\dagger \chi u^\dagger + u\chi^\dagger u). \tag{3.8}
\]

Thus, the Euclidean generating functional at one loop is given by
\[
Z(v, a, s, p) = \int dx (\mathcal{L}_2 + \mathcal{L}_4) + \frac{1}{2} \ln \det D, \tag{3.9}
\]

where all quantities are to be evaluated at the classical solutions $\bar{U}(x), \bar{A}_\mu(x)$. The determinant of the operator $D$ requires renormalization, since it contains
divergences of one-loop graphs. As mentioned before, these divergences should be absorbed by the counterterms, contained in \( \hat{\mathcal{L}}_4 \), see Eq. (2.10).

Note that, if \( a = 1 \), the expression in Eq. (3.5) turns into the standard expression (see, e.g., [3, 6]). The differential operator, emerging there, is of a minimal type. The UV-divergent part thereof can be found in a straightforward manner by using the well-known expression

\[
\frac{1}{2} \ln \det D \bigg|_{a=1} = -\lambda \int dx \, \text{tr} \left( \frac{1}{12} Y_{\mu\nu} Y_{\mu\nu} + \frac{1}{2} \Lambda^2 \right) + \text{UV-finite part},
\]

(3.10)

where “tr” means the trace in the multi-index \( A = (a, \mu) \) and

\[
Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu + [Y_\mu, Y_\nu].
\]

(3.11)

4 The case \( a \neq 1 \)

In general, when \( a \) is not equal to 1, the differential operator in Eq. (3.5) is of a non-minimal type and one has to resort to a different method for calculating the determinant. In particular, in analogy to Ref. [28], we evaluate the UV-divergent part of the determinant by means of a straightforward expansion in powers of \( \omega \), see Eq. (3.5)

\[
\frac{1}{2} \ln \det(D_0 + \omega) = \frac{1}{2} \ln \det D_0 + \frac{1}{2} \text{Tr}(D_0^{-1} \omega) - \frac{1}{4} \text{Tr}(D_0^{-1} \omega D_0^{-1} \omega) - \frac{1}{8} \text{Tr}(D_0^{-1} \omega D_0^{-1} \omega D_0^{-1} \omega) + \text{UV-finite part},
\]

(4.1)

where the symbol “Tr” stands for the trace in both coordinate and matrix indices. It is easy to truncate this expression so as to retain only those terms that contain UV divergences. If \( a = 1 \), the divergent part of the expression above reproduces the already known result, see Eq. (3.10).

The matrix elements of the operator \( D_0^{-1} \) have the following form:

\[
\langle x | (D_0^{-1})^{ab} | y \rangle = -\delta^{ab} \Delta(x - y),
\]

\[
\langle x | (D_0^{-1})^{\rho\sigma} | y \rangle = -\delta^{\rho\sigma} \Delta(x - y) + \Delta^{\rho\sigma}(x - y),
\]

\[
\langle x | (D_0^{-1})^{ab} | y \rangle = 0, \quad \langle x | (D_0^{-1})^{a} y \rangle = 0,
\]

(4.2)

where

\[
\Delta(x - y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik(x-y)}}{-k^2},
\]

\[
\Delta^{\rho\sigma}(x - y) = (a - 1) \int \frac{d^d k}{(2\pi)^d} \frac{k^\rho k^\sigma}{k^4} e^{-ik(x-y)}.
\]

(4.3)
The matrix elements of the operator $\omega$ are given by

$$
\langle y | \omega^{AB} | x \rangle = -2 Y^{AB}_\mu (y) \frac{\partial}{\partial y_\mu} \delta(x - y) + c^{AB}(x) \delta(x - y),
$$

$$
c(x) = -\partial_\mu Y_\mu - Y_\mu Y_\mu + \Lambda.
$$

(4.4)

The components of the above matrix can be written in the following form

$$
\langle y | \omega^{ab} | x \rangle = -2 \Gamma^{ab}_\mu (y) \frac{\partial}{\partial y_\mu} \delta(x - y) + b^{ab}(y) \delta(x - y),
$$

$$
\langle y | \omega^{a\sigma} | x \rangle = -2 X^a(y) \frac{\partial}{\partial y_\sigma} \delta(x - y) + b^{a\sigma}(y) \delta(x - y),
$$

$$
\langle y | \omega^{\rho b} | x \rangle = -2 X^b(x) \frac{\partial}{\partial x_\rho} \delta(x - y) + b^{\rho b}(x) \delta(x - y),
$$

$$
\langle y | \omega^{\rho\sigma} | x \rangle = \delta^{\rho\sigma} b(y) \delta(x - y),
$$

(4.5)

where

$$
b^{ab} = -\partial_\mu \Gamma^{ab}_\mu - \Gamma^{ac}_\mu \Gamma^{cb}_\mu + 4 X^a X^b + \sigma^{ab},
$$

$$
b^{a\sigma} = b^{\sigma a} = -\partial_\sigma X^a - \Gamma^{ac}_\mu X^c + \frac{1}{2} \gamma^{a\sigma},
$$

$$
b = X^a X^a + \rho.
$$

(4.6)

From Eq. (4.1) after rather voluminous calculations one obtains

$$
\frac{1}{2} \ln \det(D_0 + \omega) = \frac{1}{2} \ln \det(D_0 + \omega) \bigg|_{a=1} + \lambda \int dx \left( \frac{1}{2} S_1 - \frac{1}{4} S_2 + \frac{1}{6} S_3 - \frac{1}{8} S_4 \right) + \text{UV-finite part},
$$

(4.7)

where the first term is given by Eq. (3.10), and

$$
S_1 = 0,
$$

$$
S_2 = (a - 1) \left[ -8 \partial_\mu X^a b^{a\mu}_\mu - b^{a\mu}_\mu b_a^{\mu\mu} \right] - (2(a - 1) + (a - 1)^2) \left[ 2b^2 \right],
$$

$$
S_3 = (a - 1) \left[ -24 X^a b^{ab} X^b - 12 X^a \Gamma^{ab}_\mu b^{\mu b}_\mu \right] - (2(a - 1) + (a - 1)^2) \left[ 24 b X^a X^a \right],
$$

$$
S_4 = (a - 1) \left[ 32 X^a \Gamma^{ac}_\mu \Gamma^{cb}_\mu X^b \right] - (2(a - 1) + (a - 1)^2) \left[ 64 X^a X^a X^b X^b \right].
$$

(4.8)

Note that $S_1, \ldots, S_4$ correspond to the loop with one, two, three, four external legs, see Eq. (4.1). The quantity $S_1$ vanishes in the dimensional regularization, because it contains a no-scale integral.
Putting things together and using Eq. (4.6), we obtain

\[
S = \frac{1}{2} S_1 - \frac{1}{4} S_2 + \frac{1}{6} S_3 - \frac{1}{8} S_4 \\
= (a - 1) \left[ 2b^a_\mu (\partial_\mu X^a + \Gamma^{ab}_\mu X^b) + \frac{1}{4} f^a_\mu b^a_\mu - 16X^a X^b B^b - 4X^a X^b \sigma^{ab} \right] \\
+ \frac{1}{2} (2(a - 1) + (a - 1)^2) \left[ \rho - 3X^a X^a \right]^2. 
\]

(4.9)

This is our final expression for the one-loop determinant in an arbitrary covariant gauge.

5 Renormalization

At the next step, we continue Eq. (4.9) back to Minkowski space and substitute explicit expressions, given in Eq. (3.7). Carrying out the summation over flavor indices, in the three-flavor case one obtains:

\[
\text{div } Z_{\text{one loop}} = \text{div } Z_{\text{one loop}}^{a=1} + \lambda(a - 1) \frac{F_0^2}{2} \int dx \left\{ \langle [H_L, \Delta]\rangle^2 - \langle [H_R, \Delta]\rangle^2 \right\} \\
+ \left\{ \langle \sigma H_L^2 \rangle - \frac{Z F_0^2}{2} \langle [H_R, H_L] \rangle^2 - 2 \langle [H_R, \Delta]\rangle \langle G^\mu \rangle - \frac{3}{4} \langle G^\mu G^\mu \rangle \right\},
\]

(5.1)

where we have used the relation

\[
D_\mu H_L = [H_R, \Delta] + G^\mu, \quad G^\mu = u^\dagger c_{R\mu} Q u - u c_{L\mu} Q u^\dagger. 
\]

(5.2)

It should be pointed out that the terms with \((a - 1)^2\) have completely cancelled in the final expression.

By using the equations of motion, the equation (5.1) can be simplified to

\[
\text{div } Z_{\text{one loop}} = \text{div } Z_{\text{one loop}}^{a=1} + \lambda(a - 1) F_0^2 \int dx \left\{ \frac{1}{4} \langle (\chi U^\dagger + U \chi^\dagger + \chi^\dagger U + \bar{U}^\dagger \chi) Q^2 \rangle \\
- \frac{1}{4} \langle (\chi U^\dagger + U \chi^\dagger) Q \bar{U} Q \bar{U}^\dagger + (\chi^\dagger U + \bar{U}^\dagger \chi) Q U^\dagger Q^\dagger \rangle \\
+ \frac{1}{4} \langle (\chi \bar{U}^\dagger - \bar{U} \chi^\dagger) Q \bar{U} Q \bar{U}^\dagger + (\chi^\dagger \bar{U} - \bar{U}^\dagger \chi) Q U^\dagger Q^\dagger \rangle \\
+ \frac{1}{2} \langle d_\mu \bar{U} [c^\mu_R Q, Q] \bar{U} + d_\mu \bar{U} [c^\mu_L Q, Q] \bar{U}^\dagger \rangle + \frac{3}{4} \langle c^\mu_R Q \bar{U} c_{R\mu} Q \bar{U}^\dagger \rangle \\
- \frac{3}{8} \langle c^\mu_R Q c_{R\mu} Q + c^\mu_L Q c_{L\mu} Q \rangle \right\}. 
\]

(5.3)
The UV divergences in the electromagnetic LECs can be directly read off from the expression above. Using the result of Ref. [3], obtained for \( a = 1 \), we get (see also Eq. (A.7) from Ref. [6]):

\[
\begin{align*}
\Sigma_1 &= \frac{3}{4}, \\
\Sigma_4 &= 2Z, \\
\Sigma_7 &= 0, \\
\Sigma_{10} &= \frac{1}{4} + \frac{3}{8} Z - \frac{1}{4} (1 - a), \\
\Sigma_{13} &= \frac{3}{4} (1 - a), \\
\Sigma_{16} &= -3 - \frac{3}{2} Z - Z^2, \\
\sigma_1 &= -\frac{27}{5} - \frac{1}{5} Z, \\
\sigma_2 &= 2Z, \\
\sigma_3 &= -\frac{3}{4}, \\
\sigma_4 &= 2Z, \\
\sigma_5 &= -\frac{1}{4} - \frac{1}{5} Z + \frac{1}{4} (1 - a), \\
\sigma_6 &= \frac{1}{4} + 2Z - \frac{1}{4} (1 - a), \\
\sigma_7 &= 0, \\
\sigma_8 &= \frac{1}{8} Z + \frac{1}{4} (1 - a), \\
\sigma_9 &= \frac{1}{4} + \frac{1}{2} (1 - a), \\
\sigma_{11} &= -\frac{3}{8} (1 - a), \\
\sigma_{12} &= \frac{3}{2} - \frac{12}{5} Z + \frac{84}{25} Z^2, \\
\sigma_{13} &= -3 - \frac{3}{2} Z - \frac{12}{5} Z^2, \\
\sigma_{14} &= \frac{3}{2} + 3Z + 12Z^2.
\end{align*}
\]  

(5.4)

In case of two flavors, the first term of Eq. (5.3) can be further simplified by using the following identity

\[
\langle (\chi \bar{U}^\dagger + \bar{U} \chi^\dagger + \chi^\dagger \bar{U} + \bar{U}^\dagger \chi) Q^2 \rangle
- \langle (\chi \bar{U}^\dagger + \bar{U} \chi^\dagger) Q \bar{U} Q^\dagger + (\chi^\dagger \bar{U} + \bar{U}^\dagger \chi) Q \bar{U} Q^\dagger \rangle = -\langle \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \rangle \langle Q^2 \rangle + \langle \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \rangle \langle Q \bar{U} Q^\dagger \rangle
\]  

(5.5)

Using the effective Lagrangian, given in Ref. [6], one reads off the divergent parts of the LECs in the two-flavor case as well

\[
k_i = \lambda \sigma_i + k_i^r(\mu), \quad i = 1, \ldots, 14.
\]  

(5.6)

The equations (5.4) and (5.7) represent the main result of the present paper. Note also that, through the renormalization group equations, these equations define the gauge dependence of the scale-dependent part of the LECs \( K^r_i(\mu) \) and \( k_i^r(\mu) \), respectively.

### 6 Comparison to existing calculations

The dependence of some of the LECs on the gauge parameter has been studied in Refs. [10, 12, 13]. We have explicitly checked that, in all cases, our results agree with those from Refs. [10, 12, 13].
Furthermore, as a useful check, we have verified that the scale-dependent part of the combinations of the renormalized electromagnetic LECs, which appear in the expressions for the pion, kaon and \(\eta\)-meson masses in 3-flavor ChPT [3],

\[
\begin{align*}
C_1 &= 6K^r_1 + 6K^r_2 + 5K^r_5 + 5K^r_6 - 6K^r_7 - 15K^r_8 - 5K^r_9 - 23K^r_{10} - 18K^r_{11}, \\
C_2 &= K^r_8, \\
C_3 &= 12K^r_1 + 12K^r_2 - 18K^r_3 + 9K^r_4 + 10K^r_6 - 12K^r_7 - 12K^r_8 \\
&\quad - 10K^r_9 - 10K^r_{10}, \\
C_4 &= 3K^r_7 + 9K^r_9 + K^r_{10}, \\
C_5 &= 6K^r_1 + 6K^r_2 + 5K^r_5 + 5K^r_6 - 6K^r_7 - 24K^r_8 - 2K^r_9 - 20K^r_{10} - 18K^r_{11}, \\
C_6 &= 3K^r_7 + 3K^r_9 + K^r_5 - K^r_6 - 3K^r_7 - K^r_8 - K^r_{10}, \\
C_7 &= 12K^r_1 + 12K^r_2 - 6K^r_3 + 3K^r_4 + 6K^r_5 + 6K^r_6 - 12K^r_7 - 12K^r_8 \\
&\quad - 4K^r_9 - 4K^r_{10}, \quad (6.1)
\end{align*}
\]

does not depend on the gauge parameter \(a\). The same statement holds for the combinations of the LECs, which appear in the \(\pi K\) scattering amplitude [29]. Below, for illustration, we list some of these combinations:

\[
\begin{align*}
C_8 &= 9(M^2_\pi + 2M^2_K)K^r_8 - M^2_\pi K^r_9 + (17M^2_\pi + 18M^2_K)K^r_{10} + 18(M^2_\pi + M^2_K)K^r_{11}, \\
C_9 &= K^r_7 + K^r_9 + 12K^r_{10} - 6K^r_{11}, \\
C_{10} &= 18K^r_3 - 9K^r_4 - 12K^r_8 + 2K^r_9 - 34K^r_{10} - 36K^r_{11}. \quad (6.2)
\end{align*}
\]

Similar checks have been carried out in case of the 2-flavor ChPT, considering the combinations of LECs, appearing in the pion masses and the \(\pi\pi\) scattering amplitudes [5, 6].

In Ref. [30], the matching of the 2- and 3-flavor electromagnetic LECs has been carried out. For illustration, consider one of the relations from Ref. [30]:

\[
k^r_5 = \frac{6}{5}K^r_7 + \frac{1}{5}K^r_8 + \frac{4}{9}K^r_9 - \frac{1}{5}K^r_{10} - \frac{1}{10}Z\frac{1}{32\pi^2}(\ln\frac{M^2_K}{\mu^2} + 1), \quad (6.3)
\]

where \(M_K\) stands for the kaon mass (at this order in \(e\), there is no difference between charged and neutral kaon masses). Note that only the loops with the particles containing \(s\)-quark(s) (kaons and \(\eta\)) contribute to the matching conditions. Consequently, since the photon loop is absent, the matching condition should not contain the gauge parameter \(a\) and be, therefore, gauge invariant.

Inserting Eqs. (5.4) and (5.7), we have checked that this is indeed the case for the scale-dependent part of all relations listed in Ref. [30].
Finally, we wish to comment on the result of Ref. [22] (the divergent coefficients in 4 dimensions are given in Ref. [27], which contains a compilation of the earlier results). It turns out that the formulae displayed there are not well suited for the direct comparison with our expressions. However, it should be still pointed out that the divergent coefficients from table 2 of Ref. [27] contain logarithmic dependence on the gauge parameter. It is clear that such a dependence can never arise in our framework. Note also that such a logarithmic dependence on the gauge parameter arises from the photon mass term in the propagator and is thus of the infrared origin. Its appearance in the $\beta$-functions looks counterintuitive to us. Further, the logarithmic contributions in Ref. [12], which were mentioned in Ref. [27], arise in various correlators and not in the LECs. Consequently, these (scale independent) contributions can not be identified with those from table 2 of Ref. [27].

7 Summary

In this paper, we calculate the divergent part of the one-loop effective action in ChPT with virtual photons in an arbitrary covariant gauge. Except in the Feynman gauge, the differential operator in the action is of a non-minimal type, for which the standard technique, based on the heat kernel expansion, is not directly applicable. Instead, we have resorted to a straightforward perturbative expansion of the determinant. The final result for the divergent part of the effective action is given in Eqs. (4.7), (4.8) and (4.9). The divergent parts of the LECs in the 3- and 2-flavor ChPT are listed in Eqs. (5.4) and (5.7), respectively.

Acknowledgments: The authors thank J. Gasser for suggesting to investigate this problem and valuable comments. We would like to thank B. Ananthanarayan for current interest into the work and helpful discussions. We thank B. Kubis, U.-G. Meißner and B. Moussallam for the interesting discussions. This work is partly supported by the EU Integrated Infrastructure Initiative HadronPhysics3 Project under Grant Agreement no. 283286. We also acknowledge the support by DFG (CRC 16, “Subnuclear Structure of Matter” and CRC 110, “Symmetries and the Emergence of Structure in QCD”) and by the Shota Rustaveli National Science Foundation (Project DI/13/02). This research is supported in part by Volkswagenstiftung under contract no. 86260.

\footnote{We thank B. Ananthanarayan and B. Moussallam for the discussions on this issue.}
References

[1] S. Weinberg, Physica A 96 (1979) 327.

[2] J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142;
    J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 465.

[3] R. Urech, Nucl. Phys. B 433 (1995) 234 [hep-ph/9405341].

[4] H. Neufeld and H. Rupertsberger, Z. Phys. C 71 (1996) 131
    [hep-ph/9506448].

[5] U.-G. Meißner, G. Müller and S. Steininger, Phys. Lett. B 406 (1997) 154
    [Erratum-ibid. B 407 (1997) 454] [hep-ph/9704377].

[6] M. Knecht and R. Urech, Nucl. Phys. B 519 (1998) 329 [hep-ph/9709348].

[7] M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C 12 (2000) 469 [hep-ph/9909284];
    V. Cirigliano, M. Knecht, H. Neufeld and H. Pichl, Eur. Phys. J. C 27 (2003) 255 [hep-ph/0209226];
    V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C 23 (2002) 121 [hep-ph/0110153].

[8] U.-G. Meißner and S. Steininger, Phys. Lett. B 419 (1998) 403 [hep-ph/9709453];
    G. Müller and U.-G. Meißner, Nucl. Phys. B 556 (1999) 265 [hep-ph/9903375];
    N. Fettes and U.-G. Meißner, Nucl. Phys. A 693 (2001) 693 [hep-ph/0101030].

[9] J. Gasser, V. E. Lyubovitskij and A. Rusetsky, Phys. Rept. 456 (2008) 167
    [arXiv:0711.3522 [hep-ph]].

[10] J. Gasser, A. Rusetsky and I. Scimemi, Eur. Phys. J. C 32 (2003) 97 [hep-ph/0305260].

[11] A. Rusetsky, PoS CD 09 (2009) 071 [arXiv:0910.5151 [hep-ph]].

[12] B. Moussallam, Nucl. Phys. B 504 (1997) 381 [hep-ph/9701400].

[13] B. Ananthanarayan and B. Moussallam, JHEP 0406 (2004) 047 [hep-ph/0405206].
[14] A. Torok, S. Basak, A. Bazavov, C. Bernard, C. DeTar, E. Freeland, W. Freeman and S. Gottlieb et al., PoS LATTICE 2010 (2010) 127; S. Basak, A. Bazavov, C. Bernard, C. DeTar, E. Freeland, W. Freeman, J. Foley and S. Gottlieb et al., arXiv:1301.7137 [hep-lat].

[15] A. Portelli et al. [Budapest-Marseille-Wuppertal Collaboration], PoS LATTICE 2010 (2010) 121 [arXiv:1011.4189 [hep-lat]].

[16] T. Blum, R. Zhou, T. Doi, M. Hayakawa, T. Izubuchi, S. Uno and N. Yamada, Phys. Rev. D 82 (2010) 094508 [arXiv:1006.1311 [hep-lat]].

[17] N. Ukita [PACS-CS Collaboration], PoS LATTICE 2011 (2011) 144 [arXiv:1111.6380 [hep-lat]].

[18] Sz. Borsanyi et al. [Budapest-Marseille-Wuppertal Collaboration], arXiv:1306.2287 [hep-lat].

[19] D. V. Vassilevich, Phys. Rept. 388 (2003) 279 [hep-th/0306138].

[20] H. Neufeld, J. Gasser and G. Ecker, Phys. Lett. B 438 (1998) 106 [hep-ph/9806436];
H. Neufeld, Eur. Phys. J. C 7 (1999) 355 [hep-ph/9807425].

[21] I. G. Avramidi, J. Math. Phys. 36 (1995) 1557 [gr-qc/9403035].

[22] V. P. Gusynin and V. V. Kornyak, Nucl. Instrum. Meth. A 389 (1997) 365.

[23] V. P. Gusynin, Phys. Lett. B 225 (1989) 233;
V. P. Gusynin and E. V. Gorbar, Phys. Lett. B 270 (1991) 29;
V. P. Gusynin, E. V. Gorbar and V. V. Romankov, Nucl. Phys. B 362 (1991) 449.

[24] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. 119 (1985) 1.

[25] P. I. Pronin and K. V. Stepanyantz, Nucl. Phys. B 485 (1997) 517 [hep-th/9605206].

[26] G. M. Shore, Annals Phys. 137 (1981) 262.

[27] B. Ananthanarayan, J. Phys. A 41 (2008) 415402 [arXiv:0808.2781 [hep-th]].

[28] J. Schweizer, JHEP 0302 (2003) 007 [hep-ph/0212188].

[29] A. Nehme and P. Talavera, Phys. Rev. D 65 (2002) 054023 [arXiv:hep-ph/0107299].

[30] C. Haefeli, M. A. Ivanov and M. Schmid, Eur. Phys. J. C 53 549 (2008) [arXiv:0710.5432 [hep-ph]].