Construction of Arithmetic Teichmuller Spaces and some applications

Preliminary version for comments

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Heard melodies are sweet, but those unheard
Are sweeter; therefore, ye soft pipes, play on;

John Keats [John Keats]
1 Introduction

§ 1.1 In this note I construct some categories which can be called *Arithmetic Teichmüller Spaces*. This construction is very broadly inspired by Shinichi Mochizuki’s ideas on Anabelian Geometry, *p*-adic Teichmüller theory and his work on the abc-conjecture, but my approach is based on a completely different set of ideas—nevertheless the theory constructed here and in [Joshi, 2021b] arrives at all the principal landmarks of [Mochizuki, 2021a,b,c]. Notably the principle assertion of [Mochizuki, 2021c] is that a suitable Teichmüller Theory (described in [Mochizuki, 2021a,b,c]) provides an alternate way of bounding the Tate parameters of a semi-stable elliptic curve over a number field. Using the theory of the present paper and [Joshi, 2021b], in [Joshi, 2021b, Theorem 10.1.1], I establish a bound similar to *but not the same as* (see the detailed discussion of this in [Joshi, 2021b]) the one asserted in [Mochizuki, 2021c, Corollary 3.12]. *Note I do not claim any Diophantine inequalities here or in [Joshi, 2021b] other than the inequality of [Joshi, 2021b, Theorem 10.1.1] (especially I do not claim the Diophantine bound of [Mochizuki, 2021d]), but the theory of this paper together with [Joshi, 2021b] demonstrates independently of [Mochizuki, 2021a,b,c] that the existence of a suitable Teichmüller Theory provides an alternate way of bounding the Tate parameters of a semi-stable elliptic curve over a number field. A short, descriptive summary (accessible to general mathematicians) of similarities and differences between the two theories ([Joshi, 2021a] and [Mochizuki, 2021a,b,c,d]) is provided *from my perspective* in [Joshi, 2021c] and its parallel reading along with the Introductions of this paper and [Joshi, 2021b] is highly recommended.

Starting with any geometrically connected, smooth, quasi-projective variety \(X/L\) over number field \(L\), I show that there is a natural category, with a very rich structure, which can be called an *Arithmetic Teichmüller Space* which is a product of categories \(\mathcal{J}(X, L_p)\) for each non-trivial valuation \(p\) of \(L\) (properties of \(\mathcal{J}(X, L_p)\) are summarized in § 1.4), associated to the variety. My construction works in any dimension and the category I construct also comes equipped with functors to Mochizuki’s anabelian landscape (here the dimension is one).

The idea of the construction of Arithmetic Teichmüller Spaces is as follows. For the sake of discussion, let me restrict myself to the one dimensional case. I will use the construction of the classical Teichmüller spaces as a model (see § 7.7 or [Lehto, 1987]). To recall the classical construction, fix a connected, hyperbolic Riemann surface \(\Sigma\). Then the classical Teichmüller space \(T_\Sigma\) of \(\Sigma\) is the category of pairs \((\Sigma, f : \Sigma \to \Sigma')\) consisting of a Riemann surface \(\Sigma\) with a quasi-conformal mapping \(f : \Sigma \to \Sigma'\) (strictly speaking, one works with equivalence classes of such pairs, but let us ignore that for the moment). One has a function

\[
(\Sigma, f : \Sigma \to \Sigma') \mapsto \pi_1(\Sigma')(\simeq \pi_1(\Sigma))
\]

to the isomorphism class of discrete group \(\pi_1(\Sigma)\) given using the homeomorphism underlying \(f\). The pair \((T_\Sigma, (\Sigma, f : \Sigma \to \Sigma') \mapsto \pi_1(\Sigma'))\) together with the obvious functor to complex analytic spaces is an example of an *anabelian variation providing* \(\pi_1(\Sigma)\) (see § 7). An anabelian variation providing a pro-discrete group \(\Pi\) should considered as an anabelian place-holder for the notion of a variation of a (mixed) Hodge structures.

Now fix \(X/E\) to be a geometrically connected, smooth, hyperbolic curve over a \(p\)-adic field \(E\). Of course, since I want to do this over \(p\)-adic fields, one immediately encounters difficulties in such an endeavor because one does not have a good notion of a \(p\)-adic quasi-conformal mapping. The first step (borrowing Mochizuki’s idea of anabelomorphic [Joshi, 2020a]) is that one should consider the category whose objects are geometrically connected,
smooth, hyperbolic curves \( Y/E' \) over a \( p \)-adic field \( E' \) and satisfying

\[
\left\{ \left( Y/E' \right) : \text{there exists } \Pi_{Y/E'}^\text{temp} \simeq \Pi_{X/E}^\text{temp} \right\},
\]
and morphisms are morphisms of \( \mathbb{Z} \)-schemes. Here \( \Pi_{Y/E'}^\text{temp} \) is the tempered fundamental group (this notation hides passage to analytifications, see § 3.1). Any \( Y/E' \) satisfying \( \Pi_{Y/E'}^\text{temp} \simeq \Pi_{X/E}^\text{temp} \) is said to be anabelomorphic (more precisely tempered anabelomorphic) to \( X/E \) (this terminology was introduced in [Joshi, 2020a]). This category comes equipped with a function to the isomorphism class of the pro-discrete group \( \Pi_{X/E}^\text{temp} \) given by \( Y/E' \mapsto \Pi_{Y/E'}^\text{temp} \) and the obvious functor to the category of \( \mathbb{Z} \)-schemes and is also an example of an anabelian variation providing \( \Pi_{X/E}^\text{temp} \). It is natural to expect that this provides a “Teichmuller Space.”

Unfortunately, if the absolute Grothendieck conjecture is true for \( p \)-adic curves then \( Y \simeq X \) as \( \mathbb{Z} \)-schemes for every \( Y/E' \) anabelomorphic to \( X/E \), and hence the above category may contain a single \( \mathbb{Z} \)-scheme (up to isomorphism) namely \( X \)!

So the question is how does one resolve this difficulty?

To understand this, note that the classical Teichmuller space \( T_\Sigma \) is a non-trivial anabelian variation providing \( \pi_1(\Sigma) \) because Grothendieck’s conjecture fails (trivially) for hyperbolic Riemann surfaces.

In the \( p \)-adic setting Theorem 3.9.1 and Theorem 3.15.1 show that the Grothendieck conjecture fails for analytic spaces over algebraically closed perfectoid fields with an isometric embedding of \( \mathbb{Q}_p \), and hence a way out of the above conundrum is the following: to build a good \( p \)-adic Teichmuller Theory along the above lines one should take into account the fact that the analytic space \( Y^\text{an}_{C_p} \) has deformations arising from the deformations of the algebraically closed perfectoid overfield \( C_p \) while its fundamental groups

\[
\Pi_{X/E}^\text{temp} \simeq \Pi_{Y/E'}^\text{temp} \supset \Pi_{Y/C_p}^\text{temp}
\]

stays fixed. That the perfectoid field \( C_p \) has deformations, while its tilt \( C_p^\circ \) remains fixed, is a fundamental result of [Kedlaya and Temkin, 2018]. Here \( C_p^\circ \) is the tilt of \( C_p \) in the sense of [Scholze, 2012] and is a perfectoid field of characteristic \( p > 0 \). This result of [Kedlaya and Temkin, 2018] should be regarded as an “equi-tilted” or horizontal deformation; one can also allow the tilt to grow—for example a maximally complete field containing \( C_p \) is a “vertical” deformation of \( C_p \) (see § 3.10 and § 3.11 for a discussion of this). To underscore the importance of the existence of such deformations, suppose that \( X \) is of strict Belyi type (for example \( X \) is a once punctured elliptic curve defined over a number field), then one has \( Y \simeq X \) as schemes over \( \mathbb{Z} \) (Proposition 8.16.1). However the analytic space \( Y^\text{an}_{C_p} \simeq X^\text{an}_{C_p} \) moves as one deforms \( C_p \).

Since these changes in \( Y^\text{an}_{C_p} \) occur because of changes in the topology of the coefficient overfield \( C_p \), I call these changes arithmetic-topological changes. On the other hand if there exists a \( Y/E' \) anabelomorphic to \( X/E \) but \( Y \not\simeq X \) as \( \mathbb{Z} \)-schemes, then \( Y/E' \) must be considered
a truly anabelomorphic change in the geometry (of $X/E$) (since the absolute Grothendieck conjecture is not known for general $p$-adic curves, this possibility may very well exist).

In contrast, for the archimedean valued field $\mathbb{C}$, there are no arithmetic-topological deformations of the valued field $\mathbb{C}$ (by Ostrowski’s Theorem) so the classical Teichmüller construction proceeds by accounting for all the (truly) geometric changes. Let me remark that a classical result of [Nakai, 1959, 1960] provides a ring theoretic characterization of quasi-conformal and conformal mappings of Riemann surfaces in terms of their Royden algebras of functions and this allows us to view classical Teichmüller theory as a variation of Banach structures (see § 8.13). Thus both: the classical (archimedean) Teichmüller theory and the $p$-adic Teichmüller theory (of this paper) can be viewed as arising from variations of Banach algebra structures!

So at any rate to build a $p$-adic Teichmüller space, one should

1. take all geometrically connected, smooth, hyperbolic curves over $p$-adic fields, anabelomorphic to $X/E$, and

2. and for each such $Y/E'$ include all contributions arising from arithmetic-topological changes of the overfield $E' \hookrightarrow \mathbb{C}_p$ into account.

All of this data, of course, provides an anabelian variation providing the pro-discrete group $\Pi_{\text{temp}}^{\text{per}}(X/E)$.

A natural way of working with topological deformations of $\mathbb{C}_p$ (after [Kedlaya and Temkin, 2018], [Fargues and Fontaine, 2018]) is to work with residue fields of closed points (of degree one) of the Fargues-Fontaine curve $\mathcal{Y}_{\mathbb{C}_p, \mathbb{Q}_p}$. In fact more generally, after Theorems 3.9.1, 3.15.1, one should consider all algebraically closed perfectoid fields with isometric embeddings of $\mathbb{Q}_p$ (no requirement that the tilt is $\mathbb{C}_p$). The construction given below pursues this optik to build an arithmetic Teichmüller space $\mathfrak{J}(X, E)$ associated to $X/E$. Considering perfectoid fields with a fixed tilt $F$ provides a category $\mathfrak{J}(X, E)_F$ equipped with the action of $\operatorname{Aut}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_F))$ (Theorem 8.29.1) where $\mathcal{G}$ is the standard Lubin-Tate formal group over $\mathbb{Z}_p$.

Theorem 8.29.1 shows that one has a natural action of the above automorphism group on $\mathfrak{J}(X, E)\mathbb{C}_p$. If $X$ is of strict Belyi type then $\mathfrak{J}(X, E)$ (and also $\mathfrak{J}(X, E)\mathbb{C}_p$) carries a natural action of $\operatorname{Aut}(\Pi_{\text{temp}}^{\text{per}}(X/E))$ (Proposition 8.16.1).

Let me remark that using the Lubin-Tate formal group $\mathcal{G}/\mathbb{Z}_p$ with logarithm given by

$$\sum_{n=0}^{\infty} \frac{T^n}{p^n}$$

these results can also be read in terms of the multiplicative formal group $\hat{\mathbb{G}}_m/\mathbb{Z}_p$ and this provides a way of transcribing these ideas to the multiplicative context of [Mochizuki, 2021a,b,c,d] (see § 9.5). But there are some differences—these are discussed in § 9.10.

Moreover let me remark (see § 10.7 and § 10.8) that if one moves from one perfectoid field, say $\mathbb{C}_p$, to another algebraically closed perfectoid field $K$ with $K^\circ \simeq \mathbb{C}_p^\circ$, the valuations of elements such as $p$ in the two fields (and also valuations of elements of $\hat{\mathbb{Q}}_p$) undergo a dilatation or scaling. This is easily seen from the fact that $K^\circ \simeq \mathbb{C}_p^\circ \simeq (\mathbb{C}_p)^\circ$ induces equivalent norms on $\mathbb{C}_p$ but not equality of norms on $\mathbb{C}_p$ (in general). So the arithmetic Teichmüller space $\mathfrak{J}(X, E)\mathbb{C}_p$ is equipped with a natural action of $\operatorname{Aut}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_{\mathbb{C}_p}))$ which (in general) also provides dilatations on the value group of $\hat{\mathbb{Q}}_p$ (see § 10.8 for an explicit example). In particular as one passes from $Y^{an}/\mathbb{C}_p$ to $Y^{an}/K$ the dilatation of value groups becomes important in comparing degrees of arithmetic line bundles. The presence of dilatations should be considered to be analogous to the presence of dilatations (see [Lehto, 1987, Chapter 2]) in the classical theory of quasi-conformal mappings.

In recent correspondence, Kiran Kedlaya pointed out to me (see § 8.12) every deformation of an analytic space (arising from the analytification of a quasi-projective variety) over a perfectoid field arises from a deformation of the perfectoid field. So the construction given here
is, in the rather precise sense (of [Kedlaya and Liu, 2015] and [Kedlaya and Liu, 2019]), quite optimal.

So view point of this note is quite a natural one and the arithmetic Teichmuller space constructed here is quite canonical and it retains all the algebro-geometric objects as a part of its datum and hence also retains all the anabelian information and naturally admits all these above features.

The theory provided here is quite general and opens the possibility of considering higher genus curves and even higher dimensional Diophantine applications. With this view I provide a construction of a category which I call the global arithmetic Teichmuller space \( \tilde{\mathfrak{T}}(X_L) \) associated to a quasi-projective (hyperbolic) variety \( X \) over a number field \( L \) (see Theorem 8.35.1). This global category is a product of local Arithmetic Teichmuller Spaces (categories) \( \tilde{\mathfrak{T}}(X, L_p) \) for each non-trivial valuation \( p \) of \( L \). If \( p \) is any non-archimedean (resp. \( p \) is an archimedean) place of \( L \) then any two geometric objects of \( \tilde{\mathfrak{T}}(X, L_p) \) have isomorphic tempered (resp. topological) (and hence also étale) fundamental groups. Especially if \( \dim(X) = 1 \) (i.e. \( X \) is a hyperbolic curve) and \( p = \infty \) is an archimedean place of \( L \), then \( \tilde{\mathfrak{T}}(X, L_p) \) is (by design) the classical Teichmuller Space associated to the Riemann surface \( X(L_p) \simeq X(\mathbb{C}) \). So this category \( \tilde{\mathfrak{T}}(X_L) \) straddles the world of algebraic and anabelian geometry simultaneously.

Let me remind the readers that proofs of the geometric Szpiro inequality [Szpiro, 1979], such as those given by [Kim, 1997], [Amorós et al., 2000], [Zhang, 2001] and [Beauville, 2002] take place in the backdrop of the existence of the classical Teichmuller space; [Mochizuki, 2021a,b,c,d] underlies similar considerations, but as far as I understand Teichmuller spaces of arithmetic interest are not explicitly constructed in [Mochizuki, 2021a,b,c,d]; so a construction such as the one given here is, at the very least, desirable.

§ 1.2 Let me now discuss how this theory may be applied to study Diophantine problems (be aware that no Diophantine inequalities are proved in this paper). My approach here is based on a broad reading of Mochizuki’s rubric in [Mochizuki, 2021a,b,c,d], but relies on the theory of arithmetic Teichmuller spaces developed here.

Consider \( X/E \) a smooth, quasi-projective variety, \( x \in X(E) \) is a closed point, \( \mathcal{L} \) a line bundle on \( X \) and suppose \( s \in \Gamma(X, \mathcal{L}) \) is a global section. In Diophantine problems the absolute value \( |s(x)|_E \) often appears as a local contribution to some (Arakelov) height function. From the construction of the arithmetic Teichmuller spaces: one has data \((X^an_{K_y}, \mathcal{L}_{K_y}, s)\) parameterized by the triple \((X/E, E \hookrightarrow K_y)\) where \( y \in \mathcal{Y}_{E,E} \) is a closed point of degree one of a Fargues-Fontaine curve, and consisting of an analytic space \( X^an_{K_y} \), equipped with a line bundle \( \mathcal{L}_{K_y} \) and a section \( s \in \Gamma(X^an_{K_y}, \mathcal{L}_{K_y}) \) all obtained by base change of the data \((X, \mathcal{L}, s)\) to the valued field \( K_y \). In particular one has a function

\[
y \mapsto |s(x)|_{K_y}.
\]

Note that the absolute value \([-| \cdot |_{K_y}\) induces on \( E \subset K_y \) an absolute value which is typically a non-zero power (depending on \( y \)) of \( |s(x)|_E \) and so the absolute values vary as \( y \) varies—in particular that there is a variation is immediate in my approach from ([Fargues and Fontaine, 2018, Proposition 2.2.17]).

That there might be a variation of absolute values at all, and that one might even gain from its existence was first recognized by Mochizuki by purely group theoretic methods. In [Mochizuki, 2021a,b,c,d] Frobenioids and Hodge-Theaters are used for tracking the similar variation of theta function values. Such devices become necessary without the geometric interpretation which is available in my theory.
At any rate one might expect that
\[
\sup_y \left\{ |s(x)|_{K_y} \right\},
\]
as an upper bound for \( |s(x)|_E \). However this is quite naive and not very useful. Instead the idea is to consider lifts of the values \( s(x) \in K_y \) with respect to the canonical surjection of rings \( B \xrightarrow{\eta_{K_y}} K_y \) where \( B \) is the Fréchet algebra, equipped with the family of norms \( \{ |\cdot|_\rho : \rho \in (0, 1) \subset \mathbb{R} \} \), constructed in [Fargues and Fontaine, 2018, Définition 1.6.2]. Such lifts are obviously well-defined up to elements of \( \ker(\eta_{K_y}) \). This allows us to compare lifts of values \( |s(x)|_{K_y} \) in a uniform way with respect to all the norms on \( B \). Since one is dealing with lifts in \( B \) and with non-archimedean norms on \( B \), some lifts may have a higher absolute value than \( |s(x)|_{K_y} \). Hence one might expect that the supremum over all chosen lifts of \( s(x) \subset K_y \) to be strictly larger than the specific absolute value \( |s(x)|_E \) of interest and so one may (under favorable circumstances) hope to gain by doing this exercise.

In practice considering all lifts in \( B \) is not very useful and probably leads to the trivial bound \( \infty \) on the above supremum. So one must consider carefully defined set of lifts in a smaller Banach subspace of \( B \). Specifically one works with lifts of \( s(x) \in K_y \) which form a torsor under the Tate module \( T_y := T_y(\mathcal{O})/\mathcal{O}_{K_y} \subset B^{p=p} (\mathcal{O}) \) is the standard Lubin-Tate group corresponding to the polynomial \( z^p + pz \). Each \( T_y \) is a rank one \( \mathbb{Z}_p \)-module which moves as \( y \) moves in \( \mathbb{Z}_p \). These considerations can be evidently applied to the case (considered in [Mochizuki, 2021a,b,c,d]) of elliptic curves \( (X, \mathcal{L}, \theta) \) consisting of an elliptic curve, suitable line bundle and a chosen section \( \theta \) (a theta function). The \( T_y \) torsors are then the additive analogs of Mochizuki’s multiplicative theta-value monoids. In § 10 these general considerations are applied to study the specific case of elliptic curves with semi-stable reduction over a \( p \)-adic field, and a chosen theta function to study the locus of lifts of theta values \( \tilde{\Theta}_{X,\ell} \subset B^{p=p} \) and at any rate I have established that it makes perfect sense to talk of
\[
|\tilde{\Theta}_{X,\ell}|_\rho = \sup \left\{ |z|_\rho : z \in \tilde{\Theta}_{X,\ell} \subset B \right\}
\]
for the norms \( |\cdot|_\rho \) on \( B \) (for each \( \rho \in (0, 1) \subset \mathbb{R} \)). To see how this relates to Mochizuki’s work (see § 10.27). It seems reasonable to expect that a more sophisticated variant of \( \tilde{\Theta}_{X,\ell} \) is considered in [Mochizuki, 2021a,b,c,d].

There is an important and interesting phenomenon which one sees here. One has a fundamental and non-trivial self-similarity (Theorem 11.1.1) of the perfectoid field \( \mathbb{C}_p \) (see [Matignon and Reversat, 1984] and [Kedlaya and Temkin, 2018]). This implies that the arithmetic Teichmuller space is also self-similar containing many subcategories isomorphic to itself (see Theorem 11.7.1). This is akin to the existence of fundamental domains in classical Teichmuller Theory. The self-similarity propagates to Fargues-Fontaine curve \( \mathcal{C}_{\mathbb{Z},\mathbb{Q}_p} \) (Theorem 11.3.1) and also to the theta value locus \( \tilde{\Theta}_{X,\ell} \) (Theorem 11.8.1). The existence of this fractal suggests that there is an “intrinsic invariance of scale” typically associated with fractals, in the locus of lifts of theta torsion values in \( B^{p=p} \). The problem of measuring this fractal (with respect to a suitable Hausdorff measure) may be optimistically expected to have Diophantine significance.

§ 1.3 For ease of reading, I provide a short summary of ideas in this paper:

(1) In Theorems 3.9.1 and Theorem 3.15.1, using the principle of invariance of the (tempered) fundamental groups under passage from one algebraically closed (complete) extension to another (due in this case to [Lepage, 2010]) and the existence of topologically
distinct perfectoid fields with isometric tilts [Kedlaya and Temkin, 2018], to prove that Grothendieck’s Conjecture is in fact false for analytifications of (smooth) projective varieties (over topologically distinct algebraically closed, perfectoid overfields). This means that one can construct isomorphs of the fundamental group labeled by the geometric spaces i.e. analytifications over (complete) algebraically closed fields which gives rise to the “geometric (tempered) fundamental (sub)group.”

(2) An important consequence is that it is possible for a fixed (smooth, projective) variety over a p-adic field to provide two distinct analytic function theories over topologically distinct algebraically closed, perfectoid overfields.

(3) Especially: a Tate elliptic curve can (and does) have many distinct analytic (theta) function theories (see § 10.6) if one allows the coefficient overfields to vary over algebraically closed perfectoid fields (and such fields arise as the residue fields of (closed degree one) points of suitable Fargues-Fontaine curves).

(4) I discovered (see Theorem 8.29.1(3)) that, in fact, closed degree one points on a suitable Fargues-Fontaine curve can also be moved using topological, $\mathbb{Z}_p$-linear automorphisms of $\mathcal{O}(\mathcal{O}_{C_p})$, where $\mathcal{O}$ is the Lubin-Tate group used to construct the curve (the theory is independent of this chosen Lubin-Tate group) and where $C_p$ is the tilt of $\mathbb{C}_p$ (see [Scholze, 2012, Lemma 3.4]), and this is equivalent to my idea deforming the topological overfield (as above)!

(5) This action of $\text{Aut}_{\mathbb{Z}_p} (\mathcal{O}(\mathcal{O}_{C_p}))$ stems from the fact that the set of closed points of degree one of the Fargues-Fontaine curve $\mathcal{Y}_{C_p, Q_p}$ can be identified with $(\mathcal{O}(\mathcal{O}_{C_p}) - \{0\}) / \mathbb{Z}_p^*$. 

(6) Especially (for a fixed projective variety $X/E$ over a p-adic field $E$) Theorem 3.15.1 and Theorem 8.29.1 together imply that the action of the group $\text{Aut}_{\mathbb{Z}_p} (\mathcal{O}(\mathcal{O}_{C_p}))$ moves the $\mathbb{Q}_p$-isomorphism class of the analytic space $(X \times_E K)^{an}$ (for a perfectoid overfield $K \supset E$ with $K^p = C_p$) by changing the overfield $K$ topologically (in general) while keeping $K^p = C_p$.

(7) For readers familiar with the Geometric Langlands Program over $\mathbb{C}$ as described in [Beilinson and Drinfel’d, 2000], let me remark that the action of $\text{Aut}_{\mathbb{Z}_p} (\mathcal{O}(\mathcal{O}_{C_p}))$ considered here should be thought of as the p-adic analog of the action of the Virasoro Algebra on moduli spaces of marked Riemann surfaces described in the Virasoro uniformization Theorem of [Beilinson and Schechtman, 1988, Section 4] or [Frenkel and Ben-Zvi, 2001, Theorem 17.3.2] (see §8.30 for more on this). This suggests that to me that a suitable version of the Virasoro Uniformization Theorem might hold in the p-adic setting as well.

§ 1.4 Here is the definition and list of properties of Arithmetic Teichmuller Spaces established in this paper. Fix a complete, valued field $E$ and a geometrically connected, smooth, quasi-projective, hyperbolic curve $X$ over $E$ (assuming $\dim(X) = 1$ is not essential). If $E$ has an archimedean valuation then my theory reduces to classical Teichmuller Theory i.e. $\mathcal{J}(X, E)$ is the classical Teichmuller space. The statement given below is Theorem 8.33.1.

So now assume $E$ is a p-adic field. Consider a category, denoted by $\mathcal{J}(X, E)$ defined as follows (see § 8.4):
(1) objects are triples \((Y/E', E' \hookrightarrow K)\) consisting of \(Y/E'\) a geometrically connected, smooth, quasi-projective curve over a \(p\)-adic field \(E', K\) is an algebraically closed perfectoid field with an isometric embedding \(E \hookrightarrow K\) and an isomorphism of the tempered fundamental groups \(\Pi_{Y/E'}^{\text{temp}} \cong \Pi_{X/E}^{\text{temp}}\).

(2) One should think of \((Y/E', E' \hookrightarrow K)\) as providing the morphism of analytic spaces \((Y \times_{E'} K)_{\text{an}} \rightarrow Y_{\text{an}} = (Y/E')_{\text{an}}\).

(3) Note that in the data of the triple \((Y/E', E' \hookrightarrow K)\), the fact that \(K\) is algebraically closed perfectoid field, means one always has a preferred copy of the algebraic closure \(E' \subset E' \hookrightarrow K\), equipped with the induced valuation, to work with when working with \((Y/E', E' \hookrightarrow K)\).

(4) Morphisms between triples will be defined in the obvious way.

Now the properties.

(a) First of all (by § 8.4) for any \((Y/E', E' \hookrightarrow K)\), one has an isomorphism of topological groups

\[
\Pi_{Y/E'}^{\text{temp}} \cong \Pi_{X/E}^{\text{temp}}
\]

of their tempered fundamental groups. More precisely the algebraically closed perfectoid field \(K\) also provides the geometric tempered fundamental subgroup \(\Pi_{Y/K}^{\text{temp}} \hookrightarrow \Pi_{Y/E'}^{\text{temp}}\). So the data \((Y/E', E' \hookrightarrow K)\) provides an isomorph of \(\Pi_{X/E}^{\text{temp}}\) and a preferred geometric subgroup \(\Pi_{Y/K}^{\text{temp}} \hookrightarrow \Pi_{X/E}^{\text{temp}}\).

(b) Hence \(\mathcal{J}(X, E)\) is, an anabelian variation providing \(\Pi = \Pi_{X/E}^{\text{temp}}\). (see § 7.8, § 7.18).

(c) There are forgetful functors (see § 8.18):

(i) \((Y/E', E' \hookrightarrow K) \mapsto Y/\mathbb{Z}\) (i.e. to Schemes/\(\mathbb{Z}\)).

(ii) \((Y/E', E' \hookrightarrow K) \mapsto E'\) (i.e. to \(p\)-adic fields).

(iii) \((Y/E', E' \hookrightarrow K) \mapsto K\) (i.e. to algebraically closed perfectoid fields of characteristic zero).

(iv) \((Y/E', E' \hookrightarrow K) \mapsto K^p\) (i.e. to algebraically closed perfectoid fields of characteristic \(p > 0\)).

(d) There are functors to analytic spaces (see § 8.19)

\[
(Y/E', E' \hookrightarrow K) \mapsto (Y/E')_{\text{an}},
\]

and

\[
(Y/E', E' \hookrightarrow K) \mapsto (Y \times_{E'} K)_{\text{an}}.
\]

(e) There are functors to Mochizuki’s anabelian landscape (see § 9.14):

\[
(Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'}^{\text{temp}} \subset \mathcal{O}_E^\times \subset \mathcal{O}_K,
\]

and also

\[
(Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'}^{\text{temp}} \subset \mathcal{O}_E^\times \subset \mathcal{O}_K,
\]

and similarly

\[
(Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'}^{\text{temp}} \subset \mathcal{O}_E^\times \subset \mathcal{O}_K.
\]

(for this notation see § 9.7).
(f) If \( \dim(X) = 1 \) and \( X \) is of Strict Belyi Type (this condition is defined in [Mochizuki, 2013, Definition 3.5]) then one has an action of \( \text{Aut}(\Pi) \) on \( \mathfrak{J}(X, E) \) (Proposition 8.16.1).

(g) For a fixed algebraically closed, perfectoid field \( F \) of characteristic \( p > 0 \), there are full subcategories \( \mathfrak{J}(X, E) \) consisting of \( (Y/E', E' \hookrightarrow K) \) such that \( K^\flat = F \).

(h) Now fix an algebraically closed perfectoid field \( F \) of characteristic \( p > 0 \), a uniformizer \( \pi \) for \( E \) and let \( \mathcal{G}/\mathcal{O}_E \) be the Lubin-Tate formal group. Then there is a natural action of \( \text{Aut}_E(\mathcal{G}/\mathcal{O}_E) \) on \( \mathfrak{J}(X, E) \) (Theorem 8.29.1). Notably for \( F = \mathbb{C}_p^\flat \) one has a natural action (Corollary 9.11.1) of \( \text{Aut}_{\mathbb{Z}_p}(\mathcal{G}/\mathcal{O}_{\mathbb{C}_p^\flat}) \) on \( \mathfrak{J}(X, E) \).

(i) The category \( \mathfrak{J}(X, E) \) is self-similar (Theorem 11.7.1).

2 Acknowledgments

The theory presented here is inspired by Mochizuki’s work on anabelian geometry and on \( p \)-adic Teichmuller Theory so my intellectual debt to Shinichi Mochizuki cannot be overstated. Notably this paper could not have existed without his work. I thank Peter Scholze for some correspondence on early versions of Theorem 3.9.1 and Theorem 3.15.1. Thanks are due to Yuichiro Hoshi for answering many of my elementary questions about tempered fundamental groups and [Mochizuki, 2021a,b,c,d]; also to Taylor Dupuy for many conversations about [Mochizuki, 2021a,b,c,d]; he and Anton Hilado also provided early versions of his manuscripts [Dupuy and Hilado, 2020b], [Dupuy and Hilado, 2020a] and [Dupuy, 2021]. I thank Kiran Kedlaya for correspondence, comments and encouragement. Another expert with whom I corresponded, has chosen to remain anonymous, and our correspondence is acknowledged here. I thank Jacob Stix for providing a correction to § 4.7 and for answering a question regarding [Schmidt and Stix, 2016]. I also thank Emmanuel Lepage for some correspondence in the context of [Lepage, 2010].

3 The absolute Grothendieck conjecture is false for Berkovich spaces

§ 3.1 All valuations on base fields considered in this paper will be rank one valuations. For the theory of tempered fundamental groups see [André, 2003, André, 2003] or [Lepage, 2010]. As is noted in [André, 2003], tempered fundamental groups are natural in the \( p \)-adic analytic contexts because they capture finite étale coverings and discrete coverings such as those arising from Tate or Mumford Uniformization available in the \( p \)-adic contexts. Berkovich spaces (see [Berkovich, 1990] and [Berkovich, 1993]) will be strictly analytic (and mostly will arise as analytifications of geometrically connected smooth quasi-projective varieties).

§ 3.2 In what follows I will work with algebraically closed, perfectoid fields of characteristic zero. A typical example of such a field is the completed algebraic closure \( \mathbb{C}_p \) of \( \mathbb{Q}_p \). Such fields can also be characterized in many different ways. For the convenience of the readers unfamiliar with perfectoid fields, the following lemma (immediate from [Scholze, 2012, Definition 3.1]), provides a translation of this condition into more familiar hypothesis.

Lemma 3.2.1. Let \( K \) be a valued field and let \( R \subset K \) be the valuation ring and assume that \( |p|_K < 1 \). The following conditions are equivalent:
(1) $K$ is an algebraically closed field, complete with respect to a rank one non-archimedean valuation with residue characteristic $p > 0$.

(2) $K$ is an algebraically closed, perfectoid field.

Proof. A perfectoid field has residue characteristic $p > 0$ and is complete with respect to a rank one valuation. So (2) $\implies$ (1) is trivial. So it is enough to prove that (1) $\implies$ (2). I claim that Frobenius $\phi : R/pR \to R/pR$ is surjective. Let $\bar{x} \in R/pR$ and suppose $x \in R$ is an arbitrary lift of $\bar{x}$. Then as $K$ is algebraically closed, $x^{1/p} \mod pR$ provides a lift of $\bar{x}$. As $K$ is complete with respect to a rank one valuation and Frobenius is surjective on $R/pR$, so $K$ is perfectoid by [Scholze, 2012, Definition 3.1] and by my hypothesis $K$ also algebraically closed. This proves (1) $\implies$ (2).

§ 3.3 For a perfectoid algebraically closed field $K$ as above, one has naturally associated field $K^{♭}$, algebraically closed, perfectoid of characteristic $p > 0$, called the tilt of $K$ and $K$ is called an untilt of $K^{♭}$ (see [Scholze, 2012, Lemma 3.4]).

§ 3.4 Fix an algebraically closed field, perfectoid $F$ of characteristic $p > 0$ (see [Scholze, 2012]). For example readers can simply assume, without any loss of generality, that $F = \mathbb{C}^{♭}_p$ as this case is quite adequate for my purposes.

§ 3.5 By an untilt of $F$, I will mean a perfectoid field $K$, of characteristic zero, with $K^{♭}$ isometric with $F$. Note that by [Scholze, 2012, Proposition 3.8] $K$ is algebraically closed as its tilt $K^{♭} = F$ is algebraically closed (by my hypothesis). If $F = \mathbb{C}^{♭}_p$ then $K^{♭}$ is isometric with $\mathbb{C}^p$. By the theory of [Fargues and Fontaine, 2018] untilts $K$ of $F$ exist and are parametrized by Fargues-Fontaine curves.

§ 3.6 Let $E$ be a $p$-adic field which is fixed for the present discussion. I will work with untilts $K$, of $F$, equipped with continuous embeddings $E \hookrightarrow K$ with the valuation of $K$ providing a valuation on $E$ which is equivalent to the natural $p$-adic valuation on $E$. By [Fargues and Fontaine, 2018] for a given pair $(F, E)$, such fields $K \hookrightarrow E$, exist and are parametrized by Fargues-Fontaine curves (denoted here by $\mathcal{X}_{F,E}$). Without further mention, all untilts $K$ will be assumed to be of this type (for our chosen $p$-adic field $E$).

§ 3.7 Crucial point for this paper is that there exist untilts of $\mathbb{C}^p$ which are not topologically isomorphic. This is the main result of [Kedlaya and Temkin, 2018, Theorem 1.3] (also see [Matignon and Reversat, 1984]). Note that all characteristic zero untilts of $\mathbb{C}^p$ have the cardinality of $\mathbb{C}_p$ and are complete and algebraically closed fields and hence are abstractly isomorphic fields but may not be topologically isomorphic after [Kedlaya and Temkin, 2018, Theorem 1.3] (also see [Matignon and Reversat, 1984]).

§ 3.8 Now fix a geometrically connected, smooth quasi-projective variety $X/E$, with $E$ a $p$-adic field. Let $X^{an}/E$ be the strictly analytic space associated to $X/E$. Let

$$(3.8.1) \quad \Pi^{temp}_{X/E} = \pi^{temp}_1(X^{an}/E)$$

be the tempered fundamental group of the strictly $E$-analytic space associated to $X/E$ in the sense of [André, 2003] or [André, 2003].

(Note that my notation $\Pi^{temp}_{X/E}$ suppresses the passage to the analytification $X^{an}/E$ for simplicity of notation. The theory of (tempered) fundamental groups also requires a choice of base point which will be suppressed from my notation.)
\[ \text{§ 3.9} \] Let \( E'/E \) be a finite extension of \( E \) with a continuous embedding \( E' \hookrightarrow K \) (as \( K \) is algebraically closed, valued field containing \( E \), such \( E' \) exists). One can consider \( X_{E'/E} = X \times_E E'/E \) (similarly \( X_K = X \times_E K \)). Then one has an exact sequence by [André, 2003, Prop. 2.1.8]

\[ 1 \to \Pi^{\text{temp}}_{X_{E'/E}} \to \Pi^{\text{temp}}_{X/E} \to \text{Gal}(E'/E) \to 1. \]

Let \( \overline{E} \subseteq K \) be the algebraic closure of \( E \) contained in \( K \).

By varying \( E' \) over all finite extensions of \( E \hookrightarrow K \) one obtains (see [André, 2003, Section 5.1]) an exact sequence of topological groups:

\[ 1 \to \lim_{\overset\to{E'/E}} \Pi^{\text{temp}}_{X_{E'/E}} \to \Pi^{\text{temp}}_{X/E} \to \text{Gal}(E'/E) \to 1. \]

**Theorem 3.9.1.** Let \( F \) be an algebraically closed perfectoid field of characteristic \( p > 0 \) (for example \( F = \mathbb{C}_p \)). Let \( E \) be a \( p \)-adic field. Let \( K, K_1, K_2 \) be arbitrary untilts of \( F \) with continuous embedding \( E \hookrightarrow K \) (resp. into \( K_1 \) and \( K_2 \)). Let \( \overline{E} \) (resp. \( \overline{E}_1, \overline{E}_2 \)) be the algebraic closure of \( E \) in \( K \) (resp. in \( K_1, K_2 \)). Let \( X/E \) be a geometrically connected, smooth, quasi-projective variety over \( E \). Then one has the following:

1. **a continuous isomorphism**

\[ \Pi^{\text{temp}}_{X/K} \simeq \lim_{E'/E} \Pi^{\text{temp}}_{X_{E'/E}}, \]

where the inverse limit is over all finite extensions \( E' \) of \( E \) contained in \( K \), and a

2. **a short exact sequence of topological groups**

\[ 1 \to \Pi^{\text{temp}}_{X/K} \to \Pi^{\text{temp}}_{X/E} \to G_E \to 1, \]

and

3. **In particular for any two untilts \( K_1, K_2 \) of \( F \), one has a continuous isomorphism**

\[ \Pi^{\text{temp}}_{X/K_1} \simeq \Pi^{\text{temp}}_{X/K_2}. \]

**Proof.** The assertion (1) is true assuming only that \( K \) is a complete algebraically closed field containing \( E \) isometrically and is due to [Lepage, 2010]. My own proof of (1), before I found the assertion in [Lepage, 2010], was by reworking of [André, 2003, Prop. 5.1.1] for any \( K \) algebraically closed perfectoid field, and I was interested in proving (1) because I wanted to establish (3) (whose importance will become clear in Theorem 3.15.1 below). Here I provide an approach to the proof of (1) via the reduction to the principle of invariance of fundamental groups under extension of algebraically closed fields (also due to [Lepage, 2010]), for completeness. So (3) is the new and important observation here—from the point of view of Theorem 3.15.1 below.

Let me remind the reader that my hypothesis on \( K, K_1, K_2 \) imply that \( K, K_1, K_2 \) are algebraically closed and complete with respect to a rank one valuation.

Let me prove (1), this will also lead to (2). Since \( K \) is algebraically closed, it follows that \( K \) contains an algebraic closure \( \overline{E} \) of \( E \). Let \( \overline{E} \subseteq K \) be the closure (with respect to valuation topology of \( K \) of \( E \)).

It is clear that \( \overline{E} \supset \overline{E} \overline{E} \subseteq \overline{E} \) is complete and algebraically closed field and \( \overline{E} \) contains the algebraic closure \( \overline{E} \subseteq K \) of \( E \) contained in \( K \) as a dense subfield. In particular \( \overline{E} \) is the completion of \( \overline{E} \) with respect to the induced valuation. In other words \( \overline{E} \) is a copy of the completion of an
algebraic closure of $E$ (usually denoted $\hat{E}$) equipped with an isometric embedding $\iota : \hat{E} \hookrightarrow K$ with $\iota(\hat{E}) = \hat{E}$. Hence $K/\hat{E}$ is an isometric extension of algebraically closed, complete valued fields (with rank one valuations).

Now one can apply the principle of invariance of fundamental groups under passage to extensions of algebraically closed fields. This principle is well-known for étale fundamental groups of proper varieties (see [Grothendieck, 1971, Exposé X, Corollaire 1.8]). For tempered fundamental groups (and $X$ not necessarily proper) this principle is proved in [Lepage, 2010, Proposition 2.3.2]. Thus applying [Lepage, 2010, Proposition 2.3.2] to the extension $\bar{K}/\hat{E}$ one has an isomorphism of topological groups

$$\Pi^{\text{temp}}_{X, K} \simeq \Pi^{\text{temp}}_{X, \hat{E}}.$$  

On the other hand by [André, 2003, Proposition 5.1.1], as $\hat{E}$ is the completion of the algebraic closure of $\overline{E} \subset K$ of $E$, one has an isomorphism

$$(3.9.2) \quad \Pi^{\text{temp}}_{X/E} \simeq \lim_{\rightarrow} \Pi^{\text{temp}}_{X/E'},$$

and an exact sequence of topological groups

$$1 \to \Pi^{\text{temp}}_{X/K} \simeq \lim_{\rightarrow} \Pi^{\text{temp}}_{X/E'/E} \to \Pi^{\text{temp}}_{X/E} \to \Gal(\overline{E}/E) \to 1.$$  

This proves the assertions (1), (2) as claimed.

Let me now prove (3). The claimed isomorphism $\Pi^{\text{temp}}_{X/K_1} \simeq \Pi^{\text{temp}}_{X/K_2}$ follows from the fact that both the groups can be identified with $\lim_{\rightarrow} \Pi^{\text{temp}}_{X/K_1}$ where the inverse limit is over all finite extensions of $E'/E$ contained in $K_1$ (resp. $K_2$) and the fact that there is an equivalence between categories of finite extensions of $E$ contained in $K_1$ and the category of finite extensions of $E$ contained in $K_2$, since finite extensions of $E$ are given by adjoining roots of polynomials with coefficients in $E$ and this data is independent of the embedding of $E$ in $K_1$ or $K_2$ and moreover any abstract isomorphism of finite extensions of a complete discretely valued field is in fact an isometry–i.e given a finite extension of $E$, $E' \hookrightarrow K_1$ contained in $K_1$, there is an isometry $E' \hookrightarrow K_2$ and vice versa.

§ 3.10 The importance of working with algebraically closed perfectoid fields $K_1$, $K_2$ with isometric tilts $K_1^\flat \simeq K_2^\flat$ (i.e. with untilts of a fixed algebraically closed perfectoid field of characteristic $p > 0$) will become clear from Theorem 8.29.1 which will be proved later. Note that if $K_1$, $K_2$ are arbitrary algebraically closed perfectoid fields, then $K_1^\flat$ and $K_2^\flat$ need not be isometric. A simple example of this is given as follows. Let $\C_p^{\max}$ be a maximally (i.e. spherically) complete extension of $\bar{\Q}_p$, then $\C_p^{\max}$ and $\C_p$ do not have isometric tilts other wise $\C_p^{\flat} \simeq \C_p^{\max\flat}$ is also spherically (i.e. maximally) complete, which is certainly not the case.

§ 3.11 The following comment will also be useful. The field extension $\C_p^{\max}/\C_p$ is of uncountable transcendence degree (both the fields have the same cardinality) and one can construct many algebraically closed, complete subfields between $\C_p^{\max}$ and $\C_p$. Passage to such field extensions should be considered as a “vertical variation” of the algebraically closed perfectoid field because such variations may also involve extension of their tilts $\C_p^{\max\flat}/\C_p^{\flat}$ (also of uncountable transcendence degree). On the other hand [Kedlaya and Temkin, 2018] shows that a there is also a “horizontal” or an “iso-tilted or equi-tilted variation” possible in which the tilts stay fixed isometrically.
§ 3.12 Let $E$ be a $p$-adic field. In [Berkovich, 1990, Section 2.3, Section 3.1], Berkovich constructs the category of analytic spaces over $E$ (or more simply the category of Berkovich spaces over $E$) (a similar theory is also sketched in [Berkovich, 1993]). While I refer the reader to these references for the general case, let me recall what this means in the context I will use. By [Berkovich, 1990, Section 3.1] an analytic space over $E$ is a $K$-analytic space for some valued field $K \supseteq E$ (with a rank one valuation) and equipped with an isometric embedding $E \hookrightarrow K$. Let $X/E$ be a quasi-projective variety and let $X^{an}$ denote analytification of $X/E$ (in the sense of [Berkovich, 1990]). Thus the $K$-analytic space $X^{an}_K = X^{an} \times_E K$ is an analytic space over $E$. Let $K_1, K_2$ be two valued fields (with rank one valuation) containing $E$ (isomorphically). Let $X^{an}_{K_1} = X^{an} \times_E K_1$ and similarly define $X^{an}_{K_2}$. So one has two analytic spaces over $E$. By [Berkovich, 1990, Section 2.3, Section 3.1] one can consider the notion of (iso)morphisms $X^{an}_{K_1} \xrightarrow{\sim} X^{an}_{K_2}$ of analytic spaces over $E$. Specifically, this reduces to defining the notion of (iso)morphisms between affinoid spaces over $E$. This is done as follows (in the notation of [Berkovich, 1990, Section 1.2]): if $(_{\mathcal{M}}(A_i), A_i)$ are $K_i$-affinoid spaces over $E$, for $i = 1, 2$, then an (iso)morphism between them is given by a bounded (iso)morphism, $f : A_1 \xrightarrow{\sim} A_2$, of Banach rings compatible with their structure as normed algebras over the valued field $E$ (and the corresponding continuous (iso)morphism between the semi-norm spectra $\mathcal{M}(A_i)$). In particular if $E \supseteq \mathbb{Q}_p$, then one can consider (iso)morphisms of analytic spaces over $\mathbb{Q}_p$. Thus an isomorphism $X^{an}_{K_1} \xrightarrow{\sim} X^{an}_{K_2}$ of analytic spaces over $\mathbb{Q}_p$ makes sense and is the $p$-adic analytic analog of the notion of isomorphisms of schemes over $\mathbb{Z}$. [Note that this can be obviously formulated more generally, without assuming that $X/E$ is quasi-projective, but I have restricted myself to the case I will use in Theorem 3.15.1 given below.]

§ 3.13 Let $X/E$ be a geometrically connected, projective variety over a $p$-adic field and $E \hookrightarrow K$ an isometric embedding into a complete valued field with a rank one valuation. Then one has the (projective) analytic spaces $X^{an}/E$ and $X^{an}/K$. Projectivity (though not essential for my argument) ensures, by [Berkovich, 1990, Chap 3], that a number of adjectives which may be applied to an analytic space, can be applied to both of these spaces: both are proper (hence separated, so quasi-separated), strict, good, compact (hence quasi-compact) i.e. covered by a finite number of affinoid open subsets and the construction below applies to analytic spaces which enjoy some of these properties (but not necessarily projectivity).

By definition of an analytic space, $X^{an}_K$ is equipped with an atlas of affinoid opens. This data can be used to equip $X^{an}_K$ with a sheaf of Banach algebras $\mathcal{O}_{X^{an}_K}$ (to be precise this means that for any quasi-compact open, $U \subset X^{an}_K$, the algebra $\mathcal{O}_{X^{an}}(U)$ is a Banach algebra which is functorial in such $U$ with the following properties: (1) if $U = \mathcal{M}(A)$ is an affinoid open then $\mathcal{O}_{X^{an}}(U) = A$ and (2) if $U$ is any quasi-compact open with $U = \cup_i U_i$ a finite cover by affinoids then $\mathcal{O}_{X^{an}}(U) \to \prod_i \mathcal{O}_{X^{an}}(U_i)$ is a closed embedding of Banach algebras (note that the Banach norms provided in this construction are not claimed to be unique (locally) but equivalent). This construction is detailed in [Temkin, 2015, Section 3.3.2, Section 4.1.2]. The important point here is not the sheaf itself, but the fact that the spaces of local analytic functions acquire a Banach space structure, which agrees with the norm on constant functions i.e. on our field $K$, in a manner that is compatible with gluing of local analytic functions and independent of the gluing data. The most succinct way of expressing all this is to say that one has a sheaf of Banach algebras $\mathcal{O}_{X^{an}_K}$ on $X^{an}_K$ for a suitable Grothendieck topology on $X^{an}_K$. This implies, in particular, that the ring $\Gamma(X^{an}_K, \mathcal{O}_{X^{an}_K})$ of global analytic functions on $X^{an}_K$ is naturally a Banach algebra.

§ 3.14 Let me briefly sketch a proof of the above claims. Readers familiar with the construction of such a sheaf may skip this paragraph. By definition, an analytic space is equipped with
an atlas of affinoid open subsets and some gluing data and the analytic space can be equipped with a Grothendieck topology using this data i.e. one restricts the notion of open subsets for the purpose of constructing sheaves. The construction of $\mathcal{O}_{X^K_\mathcal{A}}$ uses this datum. The key tool in the construction of $\mathcal{O}_{X^K_\mathcal{A}}$ is Tate’s Acyclicity Theorem [Temkin, 2015, 3.3.2.1]. The following general facts about Banach algebras and Banach modules over Banach algebras will be useful to remember:

1. If $A$ is any $K$-affinoid algebra then $A$ is equipped with a norm (and even a power multiplicative norm, if one assumes additionally that $A$ is reduced, which is certainly true in the case which I am concerned with here [Bosch et al., 1984, 6.2.4, Theorem 1], but the existence of some norm on $A$ can always be inferred from the Gauss norm), equipping $A$ with a structure of a Banach algebra (i.e. $A$ is complete with respect to this norm), and any two norms on $A$ are equivalent and moreover the restriction of this norm to $K \hookrightarrow A$ is the valuation norm on $K$. If $K \supseteq E$ is a complete valued subfield, then one can think of a $K$-affinoid algebra as an $E$-Banach algebra.

2. If $A, B$ are Banach $E$-algebras then any $E$-linear homomorphism $f : A \to B$ is continuous if and only if it is bounded [Bosch et al., 1984, 2.1.8, Proposition 2].

3. Product of $E$-Banach algebras $A_1, \ldots, A_n$, is also a $E$-Banach algebra with the obvious definition of a norm.

The sheaf $\mathcal{O}_{X^K_\mathcal{A}}$ is constructed as follows (see [Temkin, 2015, Section 3.3.2, Section 4.1.2]). First consider the case of an affinoid open set. If $U = \mathcal{M}(A) \subset X^K_\mathcal{A}$ is an affinoid open subset then $\mathcal{O}_{X^K_\mathcal{A}}(U) = A$ is evidently a Banach algebra over $E$. If $U$ is covered by a finite number of affinoid opens $U = \bigcup_{i=1}^n U_i$, with $U_i = \mathcal{M}(A_i)$ and $U_i \cap U_j = \mathcal{M}(A_{i,j})$ then using Tate’s Acyclicity Theorem [Bosch et al., 1984, 8.2.1, Theorem 1] or [Temkin, 2015, 3.3.2.1]) one obtains the equality of Banach algebras $A = \ker(\prod A_i \to \prod A_{i,j})$ so one can indeed define the sheaf $\mathcal{O}_{X^K_\mathcal{A}}$ using the rule $\mathcal{O}_{X^K_\mathcal{A}}(U) = A$ on affinoids (provided in the atlas and the net of affinoids defining the analytic space $X^K_\mathcal{A}$). Moreover, Tate’s acyclicity theorem also shows, that this gives a sheaf of Banach algebras on $U = \mathcal{M}(A)$ which is independent of the choice of the covering.

Now suppose $U \subseteq X^K_\mathcal{A}$ is an arbitrary quasi-compact open subset of $X^K_\mathcal{A}$. Choose a finite covering of $U = \{U_i\}_{i=1}^n$ by affinoids with $U_i = \mathcal{M}(A_i)$, with $U_i \cap U_j = \mathcal{M}(A_{i,j})$. Then $\mathcal{O}_{X^K_\mathcal{A}}(U)$ is the equalizer of the two restriction arrows

$$\prod A_i \longrightarrow \prod_{i,j} A_{i,j}.$$  

So $\mathcal{O}_{X^K_\mathcal{A}}(U)$ is closed in the product and hence carries a natural structure of Banach algebra. This is independent of the choice of the covering: any two such covers of $U$ have a common refinement and provide isomorphisms between the three possible Banach structures on $\mathcal{O}_{X^K_\mathcal{A}}(U)$ via [Bosch et al., 1984, 2.8.1 Banach’s Open Mapping Theorem]. So one gets independence of coverings and also natural compatibility of the Banach norms on $\mathcal{O}_{X^K_\mathcal{A}}(U)$.

Since $X^K_\mathcal{A}$ is covered by a finite number of affinoids, this constructs $\mathcal{O}_{X^K_\mathcal{A}}$ as a sheaf of $E$-Banach algebras for the Grothendieck topology of $X^K_\mathcal{A}$ given by the net of compact analytic domains in $X$. Moreover, affinoid locally on $X^K_\mathcal{A}$, the norm on $\mathcal{O}_{X^K_\mathcal{A}}$, on constant functions $K$, coincides with the valuation. The construction of the sheaf of Banach algebras $\mathcal{O}_{X^K_\mathcal{A}}$ is functorial for morphisms of analytic spaces described above. Moreover one also sees from this local description that the ring of global functions $\Gamma(X^K_\mathcal{A}, \mathcal{O}_{X^K_\mathcal{A}}) = H^0(X^K_\mathcal{A}, \mathcal{O}_{X^K_\mathcal{A}})$, on $X^K_\mathcal{A}$, is a Banach $E$-algebra and on the constant functions, this norm is equivalent to the one given by the valuation.
§ 3.15  Now let me prove the following important observation:

**Theorem 3.15.1.** Let $X/E$ be a geometrically connected, smooth projective variety. Let $K_1, K_2$ be two untilts of $\mathbb{C}_p$ which contain $E$. Suppose that $K_1, K_2$ are not topologically isomorphic. Then

1. one has an isomorphism of topological groups
   \[ \Pi_{temp}^{X/K_1} \simeq \Pi_{temp}^{X/K_2}, \]
2. but the analytic spaces $X^{an}/K_1$ and $X^{an}/K_2$ are not isomorphic as analytic spaces over $\mathbb{Q}_p$ (in the sense of §3.12).
3. In particular the absolute Grothendieck conjecture fails in the category of Berkovich spaces over perfectoid fields of characteristic zero.

**Proof.** After Theorem 3.9.1, only (2) needs to be proved as $(2) \implies (3)$. The hypothesis of Theorem 3.15.1 are non-vacuous–by [Kedlaya and Temkin, 2018], fields $K_1, K_2$ exist.

Assume that $X/E, K_1, K_2$ are as in my hypothesis and that $X$ is geometrically connected, smooth and projective over $E$. Suppose, if possible, that $X^{an}/K_1$ and $X^{an}/K_2$ are isomorphic as analytic spaces over $\mathbb{Q}_p$. Then one has a bounded isomorphism of Banach rings
\[ K_1 \simeq H^0(X^{an}/K_1, O_{X^{an}/K_1}) \simeq H^0(X^{an}/K_2, O_{X^{an}/K_2}) \simeq K_2. \]

I claim that this is in fact an isomorphism of valued fields. Write $K_i^\circ$ (resp. $K_i^\circ$) for the respective subrings of power bounded elements of $K_1$ (resp. $K_2$) (for the respective norms). By [Bosch et al., 1984, 1.2.5, Proposition 4], the above isomorphism induces an isomorphism of $K_1^\circ \simeq K_2^\circ$. Further, as the norm $|−|_{K_i}$ on $K_i$ arises from the valuation of these fields, so one sees that the norms are power-multiplicative. This implies, by [Bosch et al., 1984, 1.3.1, Proposition 4], that one has the equality
\[ K_i^\circ = \{x \in K_i : |x|_{K_i} \leq 1\}, \]
i.e. $K_i^\circ$ is the valuation subring of $K_i$ and thus the valued fields $K_i$ have isomorphic valuation rings and hence $K_1$ and $K_2$ are therefore isomorphic as valued fields. Thus one has arrived at a contradiction. 

**Remark 3.15.2.** As an aside let me remark that the proof of [Kedlaya and Temkin, 2018] (also see [Matignon and Reversat, 1984, Théorème 2 and §3 Remarque 2]) provides an uncountable collection of perfectoid fields $K_1, K_2$ with tilts isometric to $\mathbb{C}_p$ and such that $K_1, K_2$ are not topologically isomorphic.

§ 3.16  Let me introduce some terminology from [Joshi, 2020a]. I will say that two geometrically connected varieties $X/E$ and $X'/E'$ over fields $E, E'$ are anabelomorphic (resp. tempered anabelomorphic) if one has a topological isomorphism of their étale fundamental groups (resp. tempered fundamental groups if $E, E'$ are $p$-adic fields):
\[ \alpha : \pi_1(X/E) \simeq \pi_1(X'/E') \text{ resp. } \alpha : \Pi_{temp}^{X/E} \simeq \Pi_{temp}^{X'/E'}, \]

and in this situation I will write $\alpha : X/E \leftrightarrow X'/E'$ for this anabelomorphism (resp. tempered anabelomorphism). I will say that an anabelomorphism (resp. tempered anabelomorphism) $\alpha : X/E \leftrightarrow X'/E'$ is a strict anabelomorphism (resp. strict tempered anabelomorphism) if
$X/E$, $X'/E'$ are anabelomorphic but not isomorphic (resp. anabelomorphic but not isomorphic analytic spaces).

In this terminology, Theorem 3.15.1 says there exist perfectoid fields $K, K' \supset E$ and a strict tempered anabelomorphism

\[(3.16.1) \quad X^{an}/K \leftrightarrow X^{an}/K'.\]

Note that anabelomorphism defines an equivalence relation on geometrically connected, smooth, quasi-projective varieties and it makes perfect sense to talk about the anabelomorphism class of a variety.

Let me remark that in [Joshi, 2020a] I show that anabelomorphy of $p$-adic fields changes important invariants of $p$-adic fields such as discriminants and more importantly it also impacts geometric invariants of varieties such as minimal discriminants of elliptic curves.

Another notion introduced in [Joshi, 2020a] is that of amphoricity: a quantity, a property or an algebraic structure associated to $X/E$ is said to be amphoric if it is an invariant of the anabelomorphism class of $X/E$.

§ 3.17 As is well-known from [Mochizuki, 1997], a $p$-adic field $E$ is not amphoric [Mochizuki, 1997] i.e. $G_E$ does not determine the isomorphism class of $E$, but as was shown in [Mochizuki, 1997], $G_E$ equipped with its upper numbering ramification filtration $G^\bullet_E$ determines $E$. Now let $X/E$ be a geometrically connected, smooth, quasi-projective variety. Let $\Pi = \Pi^{temp}_{X/E}$ and let $\Pi \supset \overline{\Pi}$ be the geometric tempered fundamental group. Mochizuki has shown that the quotient $\Pi_{X/E} / G_E$ is amphoric [Mochizuki, 2004, Lemma 1.3.8] and hence its kernel $\overline{\Pi} \subset \Pi$ is an amphoric subgroup i.e. determined by the isomorphism class of $\Pi = \Pi^{temp}_{X/E}$. In the light of this and Theorem 3.9.1 and Theorem 3.15.1 one can ask the following question:

**Question 3.17.1.** Let $K$ be a complete, algebraically closed valued field containing an isometric embedding of $E$. Is there some filtration by normal subgroups $\Pi^\bullet_X \subset \Pi_X$ which determines the pair of analytic spaces $(X^{an}_E, X^{an}_K)$ up to an isomorphism?

Mochizuki’s Theorem that $\overline{\Pi}$ is amphoric should be considered as the analog of the assertion (of [Mochizuki, 1997]) that the inertia subgroup $I_E \subset G_E$ is amphoric. If $G \simeq G_E$ is an isomorph of $G_E$ then one may equip $G$ with many different inertia filtrations corresponding to anabelomorphisms $G_E \simeq G \simeq G_{E'}$. Similarly if $\Pi$ is an isomorph of $\Pi_{X/E}$ then above suggests remarkably that there are many different filtrations $\Pi^\bullet$ each corresponding to an algebraically closed, complete valued field $K \supset E$. At least when $X/E$ is a hyperbolic curve this question is quite reasonable.

## 4 Untilts of tempered fundamental groups

§ 4.1 The results of the preceding section can be applied to the problem of producing labeled isomorphs of the tempered fundamental groups. A simple example of the labeling problem is the following: let $G$ be a topological group isomorphic to the absolute Galois group of some $p$-adic field. In this case one can ask if there are any distinguishable elements in the topological isomorphism class of $G$ with the distinguishing features serving as labels.

§ 4.2 For $G$ as above the answer is simple: there is a distinguishable collection of isomorphs of $G$, labeled by the $p$-adic fields $E$, i.e. $G_E \simeq G$ as $p$-adic fields $E$, serving as labels for isomorphs of $G$ and the labels are distinguishable by their topological isomorphism class and
so the label $E$ in $G_E$ is correspond to distinguishable geometric/topological datum of the $p$-adic field $E$. Moreover the main theorem of [Mochizuki, 1997] also asserts that in fact the geometric label $K$ corresponds to an algebraic substructure of $G$ (“the upper numbering ramification filtration”) which provides the distinguishability.

§ 4.3 Now consider the labeling problem for the topological group $\Pi = \Pi_{X/E}$ for some hyperbolic curve $X$ over some $p$-adic field $E$. So one may again ask: is it possible to provide isomorphs of $\Pi$ which are labeled by geometrically/topologically distinguishable labels?

§ 4.4 Theorem 3.9.1 provides a solution to this problem. It shows that there exist isomorphs of tempered fundamental groups which arise from topologically distinct geometric data $X/E$ and an algebraically closed, complete field $K \supset E$. The labels provided by these theorems arise from untilts of a fixed perfectoid, algebraically closed field of characteristic $p > 0$ and so I call these isomorphs of $\Pi$, untilts of the fundamental group of $\Pi$.

§ 4.5 By an untilt of the tempered fundamental group $\Pi = \Pi_{X/E}$ with respect to an untilt $K$ of $F$, I mean the tempered group $\Pi_{X/K}$ together with this short exact sequence
\[ 1 \to \Pi_{X/K} \to \Pi_{X/E} \to G_E \to 1, \]
and I write
\[ \Pi_{X/E;K} \]
for this datum. Theorem 3.15.1 asserts that $\Pi_{X/K}$ is labelled by topologically and geometrically distinguished label $X/K$ especially as by Theorem 3.9.1 and Theorem 3.15.1 one knows that if $K_1, K_2$ are two untilts of $F$ which are not topologically isomorphic (and hence non-isometric) then the spaces $X^\text{an}/K_1$ and $X^\text{an}/K_2$ are not be isomorphic rigid analytic spaces. So while $\Pi_{X/E;K_1} \simeq \Pi_{X/E;K_2}$ are isomorphic as tempered groups, these arise from possibly distinct geometric spaces. So the terminology of untilting makes sense.

Note that the labeling also provides a algebraic substructure of $\Pi$ namely the normal subgroup $\Pi_{X/K} \subset \Pi_{X/E}$ (the subgroup provides the labeling).

§ 4.6 In particular one has the following corollary:

**Corollary 4.6.1.** Let $X/E$ be a geometrically connected, smooth quasi-projective variety over a $p$-adic field $E$. Then the natural function
\[ K \longmapsto \Pi_{X/E;K} \]
from the set of inequivalent untilts of $\mathbb{C}_p$ to the topological isomorphism class of $\Pi_{X/E}$ provides a distinguished collection of distinctly labeled isomorphs
\[ \left\{ \Pi_{X/E;K_x} : x \in \mathcal{B}_{E,E} \text{ a closed point with residue field } K_x \right\} \]
of the tempered fundamental group $\Pi_{X/E}$.

§ 4.7 The above consideration can be applied to étale fundamental groups of geometrically connected, smooth quasi-projective varieties as follows. Let $X/E$ be a geometrically connected, smooth, quasi-projective variety over a $p$-adic field $E$. Then one has a natural homomorphism ([André, 2003, Proposition 4.4.1], [André, 2003, Section 2.1.4]):
\[ \Pi_{X/E} \to \pi_1(X/E), \]
which is injective if $\dim(X) = 1$, and in any dimension its image is dense and moreover $\pi_1(X/E)$ is the profinite completion
\[ \widehat{\Pi_{X/E}} = \pi_1(X/E). \]
§ 4.8 Let $K$ be an untilt of $F$. I define

$$\pi_1(X/E)_K = \widehat{\Pi}_{X/E, K}^{\text{temp}},$$

and call $\pi_1(X/E)_K$ the untilt of the étale fundamental group $\pi_1(X/E)$ corresponding to the untilt $K$ (of $F \simeq K^\circ$). Thus one has the notion of untilts of $\pi_1(X/E)$.

**Corollary 4.8.1.** Let $X/E$ be a geometrically connected, smooth quasi-projective variety over a $p$-adic field $E$. Then the natural function

$$K \mapsto \pi_1(X/E)_K$$

from the set of inequivalent untilts of $F$ to the topological isomorphism class of $\pi_1(X/E)$ provides a distinguished collection of distinctly labeled isomorphs

$$\{\pi_1(X/E)_{K_x} : x \in \mathcal{X}_{F, E} \text{ a closed point with residue field } K_x\}$$

of the étale fundamental group $\pi_1(X/E)$.

§ 4.9 I have used perfectoid algebraically closed fields as a set of distinguishing labels for the isomorphs of fundamental groups produced here. There is in fact a bigger indexing set:

**Corollary 4.9.1.** Let $E$ be a $p$-adic field, $X/E$ a geometrically connected, smooth, quasi-projective variety over $E$. Consider the set of topological isomorphism classes of algebraically closed, complete valued fields $K \supset E$ (isometric inclusions):

$$\mathcal{K}_E = \left\{ K : E \subset K, K = \widehat{K} \right\}.$$

Then there is a natural function $K \mapsto \pi_1(X/E)_K$ from $\mathcal{K}_E$ to the topological isomorphism class of the profinite group $\pi_1(X/E)$ given by considering the tempered fundamental group associated to the datum $(X, E \hookrightarrow K)$.

**Remark 4.9.2.** There is a further aspect of this result which should be pointed out. One should view elements $K \in \mathcal{K}_E$ as providing a topological variation of ambient field structure $K \supset E$ while keeping internal field structure of $E$ unchanged. Such variations exist because, unlike the number field case, $p$-adic fields, even complete algebraically closed fields such as $\mathbb{C}_p$, are quite far from being topologically rigid. This is in complete contrast with the archimedean case, where by the well-known theorem of Ostrowski [Bourbaki, 1985, Chap. 6, §6, Théorème 2], one knows that the only algebraically closed field complete with respect to an archimedean valuation is isometric to $\mathbb{C}$. To put Ostrowski’s Theorem differently: Any two algebraically closed, archimedean perfectoid fields (i.e. fields which are algebraically closed and complete with respect to an archimedean valuation) are isometric (and also isometric with $\mathbb{C}$) and hence such fields are topologically rigid.

5 Untilts of fundamental groups of Riemann surfaces

§ 5.1 Let me point out that there is a complex analytic analogue of the theory of untilting of fundamental groups which is outlined above. Let $\Pi = \pi_1^{\text{top}}(X)$ be the topological fundamental group of a connected Riemann surface $X$, which one assumes to be hyperbolic to avoid trivialities. Then consider all connected Riemann surfaces $X'$ of with the same genus and number of punctures as $X$ and whose topological fundamental group $\pi_1^{\text{top}}(X') \simeq \Pi$. 

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§ 5.2 The assignment $X' \mapsto \pi_{1}^{top}(X') \simeq \Pi$ provides a function from the isomorphism classes of connected, hyperbolic Riemann surfaces of genus $g$ and with $n$ punctures to the isomorphism class of the group $\Pi$. Then $\pi_{1}^{top}(X')$ is an untilt of $\Pi$ with the complex structure of $X'$ serving as a geometrically distinguishable feature of this copy of $\Pi$.

§ 5.3 Now assume that $K$ is a number field and $X/K$ is a hyperbolic, geometrically connected smooth quasi-projective curve. In [Tamagawa, 1997], [Mochizuki, 1996] it has been shown that the genus $g$ of $X$ and the number of punctures on $X$ is amphoric i.e. determined by the isomorphism class of the topological group $\pi_{1}^{et}(X/K)$. So one can fix $g, n$.

§ 5.4 An untilt of $\pi_{1}^{et}(X)$ at $\infty$ (here $\infty$ is short for “at archimedean primes”) is a pair consisting of an embedding $K \hookrightarrow \mathbb{C}$ and a Riemann surface $X'$, of genus $g$ and with $n$ punctures, such that $\hat{\pi}_{1}^{top}(X') \simeq \pi_{1}(X/K)$, where $\hat{\cdot}$ denotes the profinite completion. An untilt of $\Pi = \pi_{1}(X/K)$ at $\infty$ will be labeled $\Pi_{K \hookrightarrow \mathbb{C}, X'}$. Two untilts of $\pi_{1}(X/K)$ at $\infty$ are equivalent if the the two embeddings of $K \hookrightarrow \mathbb{C}$ are equivalent (in the obvious sense) and the two corresponding Riemann surfaces are isomorphic.

§ 5.5 Thus one has the following tautology:

**Proposition 5.5.1.** Fix a profinite group $\Pi \simeq \pi_{1}(X/K)$ with $X/K$ a geometrically connected, smooth, hyperbolic curve over a number field $K$ with no real embeddings. Then the equivalence classes of untilts of $\Pi$ at $\infty$ are in bijection with

$$\widetilde{\text{Hom}}(K, \mathbb{C}) \times \mathcal{M}_{g,n},$$

where $\widetilde{\text{Hom}}(K, \mathbb{C})$ is the set of equivalence classes of embeddings of $K \hookrightarrow \mathbb{C}$ and $\mathcal{M}_{g,n}$ is the moduli stack of Riemann surfaces of genus $g$ with $n$ punctures.

**Remark 5.5.2.** Owing to the topological rigidity of algebraically closed fields complete with respect to an archimedean absolute value, forced by Ostrowski’s Theorem (see Remark 4.9.2), one could say that untilts of topological fundamental groups at $\infty$ (i.e. at archimedean primes) can arise only from the existence of geometric anabelian variations of the underlying objects.

**Remark 5.5.3.** Readers familiar with the classical Szpiro inequality (for surfaces fibered over curves) and its several different proofs (see [Szpiro, 1979], [Kim, 1997], [Amorós et al., 2000], [Zhang, 2001], [Beauville, 2002]), may notice that the above proposition provides a unified way of viewing these proofs as taking place over the “space of untilts.” More precisely the “space of untilts” provides the geometric Kodaira-Spencer classes which underly these proofs. In the subsequent sections this analogy will be extended to the $p$-adic context i.e. I will construct an arithmetic Teichmuller space in the $p$-adic context.

6 An aside on hyperbolic varieties

§ 6.1 This section may be skipped on the initial reading. Reader may simply work with hyperbolic curves instead. The discussion of hyperbolic varieties of dimension bigger than one is included only to illustrate my point that the construction provided in this paper works in higher dimensions.

§ 6.2 Let $L$ be a number field. After replacing $L$ by a finite extension one can assume that $L$ has no real archimedean places. This assumption will be in force throughout the rest of this section and let $X/L$ be a geometrically connected, smooth projective variety over $L$.  

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§ 6.3 Let $E \supset \mathbb{Q}$ be any complete valued field, which is either an archimedean or a non-archimedean with a rank one valuation inducing a $p$-adic valuation on $\mathbb{Q}$, and $\hat{E}$ be its completed algebraic closure. I will say that $X/E$ is a hyperbolic variety if the analytic space $X^{an}/\hat{E}$, is a (Brody) hyperbolic variety (see [Lang, 1986] for the archimedean case, [Javanpeykar and Vezzani, 2018] for the non-archimedean case).

§ 6.4 If $\dim(X) = 1$ then $X/E$ is hyperbolic (in the above sense) if and only if $X \times_E \mathbb{C}$ is a hyperbolic Riemann surface.

§ 6.5 Let me say that for an initial reading the following familiar case of $\dim(X) = 1$ is more than adequate.

§ 6.6 For concrete applications, beyond $\dim(X) = 1$, one may also need to assume that $X/E$ is a $K(\pi, 1)$-space in some suitable sense.

§ 6.7 As an aside let me say that I have used the case of hyperbolic varieties here because of their relevance in Diophantine Geometry via Lang’s Conjecture [Lang, 1986] and [Faltings, 1991]. One can also use other related hypothesis as a substitute for the hyperbolic hypothesis, for example, one can work with groupless varieties instead of hyperbolic varieties (see [Javanpeykar, 2020] for other related hypothesis which may be used here instead).

§ 6.8 An important property in many anabelian considerations is the following: a profinite group $\Pi$ is said to be a slim profinite group (or simply $\Pi$ is slim) if every open subgroup of $\Pi$ has trivial center. By [Mochizuki, 2004, Def. 0.1 and Remark 0.1.3], $\Pi$ is slim if and only if the centralizer of any open subgroup of $\Pi$ is trivial.

§ 6.9 If $X/E$ is a hyperbolic curve over a $p$-adic field or a finitely generated field $E$ then $\pi_1(X/E)$ is slim [Mochizuki, 2004, Corollary 1.3.3 and Lemma 1.3.1].

§ 6.10 Hyperbolic and Anabelian geometry should be related through the following optimistic, but natural, conjecture:

Conjecture 6.10.1. Let $X/L$ be any smooth, quasi-projective and hyperbolic variety over a finitely generated field $L$ over $\mathbb{Q}$. Assume that $\pi_1(X/L) \neq 1$ is a slim profinite group. Then for any smooth, quasi-projective hyperbolic variety $Y/L$, one has

$$\text{Isom}_L(Y, X) \simeq \text{Isom}^{\text{ab}}_G(\pi_1(Y/L), \pi_1(X/L)).$$

§ 6.11 For $\dim(X) = 1$ the slimness of $\pi_1(X/E)$ is well-known (see § 6.9), and hence in $\dim(X) = 1$, this conjecture is the celebrated anabelian conjecture of Grothendieck, proved by [Mochizuki, 1996], [Tamagawa, 1997]. Let me remark that there exist hyperbolic varieties with non-slim étale fundamental groups (see [Ihara and Nakamura, 1997]). My construction of Arithmetic Teichmüller spaces does not make any use of slim-ness assumption.

§ 6.12 As an addendum to [Ihara and Nakamura, 1997], which will not be used in the rest of this paper, let me remark that if $A/E$ is a simple abelian variety over a number field $E$, and suppose $D \subset A$ over $E$ is an irreducible, smooth, ample divisor, then by [Debarre, 1995, Theorem 4.1], one has $\pi_1(D) \simeq \pi_1(X)$. In particular $\pi_1(D)$ is not slim. But as $A$ is simple, $D$ is not a translate of an a sub-abelian variety of $A$, and hence by [Lang, 1986], $D$ is hyperbolic. So this provides another class of examples of non-slim hyperbolic varieties complementing the non-slim examples of [Ihara and Nakamura, 1997]. Note that by [Faltings, 1991, Theorem 1], $D$ has finitely many rational points. In particular this discussion suggests that while the slimness hypothesis in the above conjecture, while essential for anabelian techniques, may not be relevant for Diophantine problems!
§ 6.13 The rationale for this conjecture (apart from the fact that it is true in dimension one) is the following (elementary) anabelian property of a hyperbolic variety over a finitely generated field.

**Proposition 6.13.1.** Let $X/E$ be a geometrically connected, smooth, hyperbolic variety over a finitely generated field $E$ of characteristic zero or a $p$-adic field. Then there exists a basis of Zariski open subsets $\{U\}$ of $X$ such that

1. every $U \neq \emptyset$ in this basis is hyperbolic, and
2. for every pair $U, V$ nonempty opens in this basis one has
   $$\text{Isom}_E(U, V) \simeq \text{Isom}^{\text{out}}_{G_E}(\pi_1(U/E), \pi_1(V/E)),$$
3. and for every $U \neq \emptyset$, $\pi_1(U/E)$ is slim.

**Proof.** From [Schmidt and Stix, 2016, Corollary 1.7], if $E$ is finitely generated field and by [Hoshi, 2014, Theorem C], if $E$ is a $p$-adic field, it follows that a basis of Zariski open sets satisfying property (2) exist. From [Lang, 1986] it is immediate that every non-empty open subvariety of $X$ is hyperbolic. The last assertion is proved by induction: for $\dim(U) = 1$, $U$ is a hyperbolic curve and so the slim-ness property is immediate from the aforementioned results. For $\dim(U) > 1$ one uses induction on dimension and the proofs of [Schmidt and Stix, 2016] or [Hoshi, 2014] from the fibration structure $U$ is equipped with by the construction of this basis. \qed

§ 6.14 In [Lang, 1986], Serge Lang has conjectured that any hyperbolic variety $X$ defined over a number field $E$ has a finite number of $E$-rational points.

7 Anabelian variations providing $\Pi$

§ 7.1 I want to present some elementary considerations which will prove useful in understanding the problem of constructing arithmetic Teichmuller spaces using the tempered fundamental group.

§ 7.2 Fix a pro-discrete group $\Pi$ (for example $\Pi = \Pi^{\text{temp}}_{X/E}$). Let $\mathcal{P}r \circ \mathcal{D}_\Pi$ be the isomorphism class of the pro-discrete group $\Pi$, and for each pro-discrete group $H \in \mathcal{P}r \circ \mathcal{D}_\Pi$, let $\hat{H}$ be its profinite completion. Let $\mathcal{S}\mathcal{C}
\mathcal{H}_\mathcal{Z}$ be the category of schemes. Let $\mathcal{A}n_{\mathbb{Q}_p}$ (resp. $\mathcal{A}n_{\mathbb{C}}$) be the category of $\mathbb{Q}_p$-analytic spaces in the sense of Theorem 3.9.1 (resp. $\mathbb{C}$-analytic spaces).

§ 7.3 Let $S$ be one of categories $\mathcal{S}\mathcal{C}
\mathcal{H}_\mathcal{Z}, \mathcal{A}n_{\mathbb{C}}, \mathcal{A}n_{\mathbb{Q}_p}$. Suppose $\mathcal{C}$ is a category and $\Pi$ is a fixed pro-discrete group. I will say that $\mathcal{C}$ is an ananabelian variation providing $\Pi$ with base $S$ if the following conditions are satisfied:

1. For every $V$ in $\mathcal{C}$ there exists an isomorphism of pro-discrete groups $\alpha_V : \Pi_V \simeq \Pi$ i.e. one is given a function
   $$\text{ob}(\mathcal{C}) \rightarrow \mathcal{P}r \circ \mathcal{D}_\Pi$$
   written $V \mapsto \Pi_V$ from the class of objects of $\mathcal{C}$ to the isomorphism class $\mathcal{P}r \circ \mathcal{D}_\Pi$ of $\Pi$.

2. There is a functor $\mathcal{C} \rightarrow S$ denoted $V \mapsto [V] \in S$. In this case $[V]$ called the scheme (resp. $\mathcal{C}$-analytic space, $\mathbb{Q}_p$-analytic space) underlying $V \in \mathcal{C}$.
§ 7.4 I will often simply say “anabelian variation providing \( \Pi \)” instead of “anabelian variation providing \( \Pi \) with base \( S \)” . Hopefully there will be no confusion. The categories constructed here will come with functors to all the three values of \( S \).

§ 7.5 I will say that an anabelian variation providing \( \Pi \) is a trivial anabelian variation providing \( \Pi \) if any pair of objects \( V, V' \in \mathcal{C} \) are isomorphic. Obviously one is interested in constructing non-trivial anabelian variations providing \( \Pi \).

§ 7.6 Suppose \( \mathcal{C} \) is a non-trivial anabelian variation providing \( \Pi \). Then the function \( V \mapsto \Pi_V \) can be thought of as providing labeled isomorphs of \( \Pi \) and the function \( \text{ob}(\mathcal{C}) \to \mathcal{D}_\Pi \) \( (V \mapsto \Pi_V) \) will be called the labeling function of \( \mathcal{C} \).

§ 7.7 The classical Teichmuller Space of any connected, hyperbolic Riemann surface \( \Sigma \) is a non-trivial anabelian variation providing \( \Pi \), the topological fundamental group of \( X \). Indeed suppose \( \Sigma \) a connected, hyperbolic Riemann surface with \( \Pi = \pi_1(\Sigma) \) be its étale fundamental group and consider the Teichmuller space \( T_\Sigma \) of \( \Sigma \) [Lehto, 1987, Chapter V]. Indeed one may think of \( T_\Sigma \) as a category consisting of pairs \( (\Sigma, f : \Sigma \to \Sigma') \) where \( f \) is a quasi-conformal mapping of \( \Sigma \) onto a Riemann surface \( \Sigma' \). Then one has \( (\Sigma, f) \mapsto \pi_1(\Sigma') \cong \pi_1(\Sigma) = \Pi \); and as any connected hyperbolic Riemann surface is obviously \( \mathbb{C} \)-analytic space so one has an obvious functor to \( T_\Sigma \to \mathfrak{sl}_n_\mathbb{C} \) given by \( (\Sigma, f : \Sigma \to \Sigma') \mapsto \Sigma' \). Hence \( T_\Sigma \) is an anabelian variation providing \( \Pi \) with base \( \mathfrak{sl}_n_\mathbb{C} \).

§ 7.8 I will say that an anabelian variation providing \( \Pi \) with base \( S \) is a geometric anabelian variation providing \( \Pi \) with base \( S \) if there exists \( V, V' \in \mathcal{C} \) such that \( [V] \neq [V'] \) (in \( S \)).

§ 7.9 Obviously any geometric anabelian variation providing \( \Pi \) is non-trivial. Hence if one constructs geometric anabelian variations providing \( \Pi \) then one automatically gets a non-trivial anabelian variation providing \( \Pi \).

§ 7.10 The moduli \( \mathcal{M}_g/\mathbb{C} \) stack of smooth, proper curves of genus \( g \geq 2 \) over \( \mathbb{C} \) is an example of a geometric anabelian variation providing \( \Pi \). Indeed it is clear that this is an anabelian variation providing \( \Pi = \pi_1^{\text{top}}(X(\mathbb{C})) \) is the topological fundamental group of the Riemann surface \( X(\mathbb{C}) \) where \( X/\mathbb{C} \) is any smooth curve, proper curve of genus \( g \). This is also a geometric anabelian variation providing \( \Pi \) because one can obviously find two non-isomorphic smooth, proper curves of genus \( g \) over \( \mathbb{C} \). Such a pair of curves cannot be isomorphic as \( \mathbb{Z} \)-schemes as well. In particular \( \mathcal{M}_g \) is a non-trivial anabelian variation providing \( \Pi \).

§ 7.11 These examples should convince the reader that, when a non-trivial (or even a geometric) anabelian variation providing \( \Pi \) exists, then it can serve as an anabelian stand-in for a variation of (mixed) Hodge structures.

§ 7.12 In some presence the sense of this sort of a structure (i.e. an anabelian variation providing \( \Pi \) with base \( S \)) should be understood as a manifestation of Kodaira-Spencer classes! Constructing similar structures in the \( p \)-adic setting leads to an \( p \)-adic Teichmuller Landscape or an \( p \)-adic Teichmuller Theory presented here. Assembling such data for each valuation of a number field, leads to a global Arithmetic Teichmuller Landscape or an Arithmetic Teichmuller Theory (also presented here) in which one can hope to contemplate applications to global Diophantine problems as is done in [Mochizuki, 2021a,b,c,d].

§ 7.13 Now suppose \( E \) is a \( p \)-adic field and \( X/E \) is a geometrically connected, smooth quasi-projective variety over \( E \). To construct a Teichmuller space associated to \( X/E \) one should try and construct an anabelian variation providing \( \Pi_{X/E}^{\text{comp}} \).
§ 7.14 Simplest way to do this is take all geometrically connected, smooth hyperbolic curves $Y/E'$ equipped with an anabelomorphism $\Pi_{\text{temp}}^{Y/E'} \simeq \Pi_{\text{temp}}^{X/E}$. Let me call this the minimal anabelian prescription for a Teichmuller space for $X/E$.

§ 7.15 Recall that the absolute Grothendieck Conjecture for hyperbolic curves over $p$-adic fields asserts that for any pair of $Y/E'$ and $X/E$ anabelomorphic, geometrically connected, smooth curves over $p$-adic fields one has an isomorphism $Y \simeq X$ as $\mathbb{Z}$-schemes. Hence unfortunately, the minimal anabelian prescription for constructing a Teichmuller space associated to $X/E$ may not always yield a geometric anabelian variation providing $\Pi$ over $\text{ch}_{\mathbb{Z}}$ and so such a variation also fails to be non-trivial over $\text{ch}_{\mathbb{C}}$.

§ 7.16 For clarity let me say that the absolute Grothendieck conjecture for geometrically connected, smooth, hyperbolic curves of Belyi type (this class includes curves of strict Belyi type) over $p$-adic fields has been proved in [Mochizuki, 2007, Corollary 2.3]. But the conjecture remains open in general and in [Mochizuki, 2004, Remark 1.3.5.1] it has even been suggested that the conjecture is false in general. On the other hand in [Mochizuki, 1999] Mochizuki has also proved the relative Grothendieck conjecture over $p$-adic fields.

§ 7.17 As an aside let me remark that recently I have proved in [Joshi, 2020b], that the absolute Grothendieck conjecture does fail over $p$-adic fields for Fargues-Fontaine curves ([Fargues and Fontaine, 2018]) over $p$-adic fields—but these examples are not of finite type.

§ 7.18 Theorem 3.9.1 and Theorem 3.15.1 suggest that in the $p$-adic setting that a second type of variation providing can $\Pi$ exist because Grothendieck conjecture does fail to hold if the valued field is complete and algebraically closed.

§ 7.19 Specifically if the minimal prescription is augmented by the data of a complete and algebraically perfectoid closed fields then one can construct an anabelian variation providing $\Pi$ which is non-trivial (using Theorem 3.15.1 or [Kedlaya and Temkin, 2018]). Since this arises from variation of the coefficient algebraically closed, perfectoid overfields, I call this an arithmetic-topological anabelian variation (of the geometric overfields) providing $\Pi$.

§ 7.20 An important point of this paper is that Arithmetic-topological anabelian variation providing $\Pi$ is always present in the non-archimedean setting and even in all dimensions (by Theorem 3.9.1 and Theorem 3.15.1). The existence of the arithmetic-topological anabelian variation arises from the failure of the Grothendieck conjecture in an appropriate category.

§ 7.21 As was noted in Remark 4.9.2, in the archimedean case, the geometric base field $\mathbb{C}$, as an algebraically closed, complete archimedean valued field, is rigid; and hence in the archimedean case arithmetic-topological anabelian variation providing $\Pi$ does not exist; but one does have a geometric anabelian variation providing $\Pi$ which arises from the existence of Riemann surfaces with fundamental group isomorphic to $\Pi$. Further note that the Grothendieck conjecture also fails (trivially) for Riemann surfaces.

§ 7.22 At any rate, in both, the archimedean and the non-archimedean settings, the existence of (some) distinguishable isomorphs of $\Pi$ is a consequence of the failure of the Grothendieck conjecture in some appropriate category.

§ 7.23 The construction of subsequent sections follows this strategy to construct an arithmetic Teichmuller space associated to $X/E$. 

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8 Construction of Arithmetic Teichmuller spaces

§ 8.1 With this prelude on hyperbolic varieties, let me return to the construction of an Arithmetic Teichmuller Spaces. The general considerations presented here are unaffected if the reader assumes that \( \dim(X) = 1 \).

§ 8.2 Let me also remark that I will make use of [Fargues and Fontaine, 2018]–which is quite extensive, and so I have provided accurate references within it. However for initial reading, readers may also consult other, shorter surveys of loc. cit. such as [Fargues and Fontaine, 2014], [Fargues and Fontaine, 2012], [Morrow, 2019].

§ 8.3 The assumption that \( X \) be hyperbolic variety is not essential (but introduced with a view on higher dimensional applications). Reader may simply assume \( \dim(X) = 1 \) and that \( X \) is hyperbolic in the usual sense namely \( 2g - 2 + n \geq 1 \).

§ 8.4 Let \( X/E \) be a geometrically connected, hyperbolic, smooth and quasi-projective variety over a field \( E \).

If \( E \) is a \( p \)-adic field then \( \mathcal{J}(X, E) \) is a category whose objects are defined by:

\[
\mathcal{J}(X, E) = \left\{ (Y/E', E' \hookrightarrow K) : \begin{array}{l}
Y/E' \text{ hyp. geom. con. smooth over } \mathcal{O}_E \text{ field } E' \text{ and, } \\
\Pi^{\text{temp}}_{Y/E', K} \simeq \Pi^{\text{temp}}_{X/E} \text{ and } \dim(Y) = \dim(X)
\end{array} \right\}
\]

and morphisms between these objects will be isomorphisms of the triples.

Now suppose that \( E \) is an archimedean complete local field. So \( E \simeq \mathbb{R} \) or \( E \simeq \mathbb{C} \) and at any rate \( E \hookrightarrow \mathbb{C} \). The category \( \mathcal{J}(X, E) \) is defined as follows:

\[
\mathcal{J}(X, E) = \left\{ (Y/E, E \hookrightarrow \mathbb{C}, \alpha) : \begin{array}{l}
Y/E \text{ a geom. connected, hyperbolic variety} \\
\text{with an orientation preserving homeomorphism } Y(\mathbb{C}) \xrightarrow{\alpha} X(\mathbb{C}) \\
\text{so } \pi^\text{top}_1(Y(\mathbb{C})) \simeq \pi^\text{top}_1(X(\mathbb{C}))
\end{array} \right\}
\]

Again, morphisms between triples will be isomorphisms of the triples.

§ 8.5 For archimedean valued fields, I am keeping the definition of morphisms a bit flexible–especially for higher dimensional applications. Already for curves, one would like to work with quasi-conformal mappings.

§ 8.6 Two important points to be noted from the definition in the \( p \)-adic case:

(1) For every triple \( (Y/E', E' \hookrightarrow K) \in \mathcal{J}(X, E) \) one has an anabelomorphism

\[ \Pi^{\text{temp}}_{Y/E'} \simeq \Pi^{\text{temp}}_{X/E}. \]

(2) For every triple \( (Y/E', E' \hookrightarrow K) \), since \( K \) is algebraically closed perfectoid field, one always has a preferred copy of the algebraic closure of \( E \) to work with, namely the algebraic closure of \( E' \) contained in \( K \).

§ 8.7 For many practical reasons (which will be clarified later on), the category \( \mathcal{J}(X, E) \) is too big. One can consider the following categories. Let \( E \) be \( p \)-adic field and \( F \) be an algebraically closed perfectoid field of characteristic \( p > 0 \). Then let

\[
\mathcal{J}(X, E)_F = \left\{ (Y/E', E' \hookrightarrow K, t : K^t \simeq F) : \begin{array}{l}
Y/E' \text{ hyp. geom. con. smooth over } \mathcal{O}_E \text{ field } E' \text{ and, } \\
K \text{ alg. closed perfectoid field }, K^t \text{ isometric to } F \text{ and } \\
\Pi^{\text{temp}}_{Y/E', K} \simeq \Pi^{\text{temp}}_{X/E} \text{ and here } \dim(Y) = \dim(X)
\end{array} \right\}
\]
§ 8.8 For archimedean primes, additional restrictions may also be needed: for example for the case of compact Riemann surfaces, every homeomorphism class $Y(C) \cong X(C)$ also contains at least one quasi-conformal mapping. This is not true in the non-compact case and one may need to replace ‘homeomorphism’ in the above definition to ‘quasi-conformal mapping’ to get Teichmuller spaces at archimedean primes.

§ 8.9 Let $X/L$ be a geometrically connected, hyperbolic, smooth and quasi-projective variety over a number field $L$ and let $p$ denote a non-archimedean prime of $L$ and let $\infty_1, \ldots, \infty_n$ be all the archimedean primes of $L$. Let $L_p$ (resp. $L_\infty$) be the completions of $L$ at $p$ (resp. $\infty$). By an *local arithmetic-geometric anabelian variation of fundamental group of $X/E$* at $p$ is the following collection:

1. If $p$ is a non-archimedean place of $L$, then let
   \[ \mathcal{J}(X, L, p) = \mathcal{J}(X, L_p), \quad \text{and} \]
   \[ \mathcal{J}(X, L, p)_F = \mathcal{J}(X, L_p)_F. \]
2. If $p = \infty_i$ for some $i$ (i.e. $p = \infty_i$ is an archimedean place of $L$) then let
   \[ \mathcal{J}(X, L, \infty_i) = \mathcal{J}(X, L_{\infty_i}). \]

Similar definition can be made of $\mathcal{J}(X, L, p)_F$ and for $p = \infty_i$ one takes $\mathcal{J}(X, L, p)_F = \mathcal{J}(X, L, \infty_i)$ purely for notational symmetry.

§ 8.10 If $p$ is an archimedean place than $\mathcal{J}(X, L, p)$ contains only geometric anabelian variations providing $\Pi$ (see § 4.9.2). If $p$ is non-archimedean and if the Grothendieck conjecture does fail for $X/L_p$, i.e. there exists $Y/L_p'$ such that $\Pi_{Y/L_p'}^\temp \cong \Pi_{X/L_p}^\temp$ with $Y$ not isomorphic to $X$ (over $\mathbb{Z}$) then one even has a geometric anabelian variation providing $\Pi_{X/L_p}^\temp$ (and one has $(Y/E L_p', L_p' \hookrightarrow K) \in \mathcal{J}(X, L, p)$).

§ 8.11 Let $X/E$ be a geometrically connected, smooth, quasi-projective hyperbolic variety over a field $E$ which is either a $p$-adic field or $E = \mathbb{C}$. Then one has $\mathcal{J}(X, E) \neq \emptyset$ and in fact

1. if $E$ is a non-archimedean then $\mathcal{J}(X, E)$ contains a subset which is in bijection with topological isomorphism classes of perfectoid fields $K \supset E$ with $K^\circ = \mathbb{C}_p^\circ$.
2. if $E$ is archimedean then $\mathcal{J}(X, E)$ contains a set which is in bijection with the set of hyperbolic varieties $Y/\mathbb{C}$ and (orientation preserving) homeomorphisms $Y(\mathbb{C}) \cong X(E)$.

**Proof.** If $E$ is archimedean then this is clear from the constructions of Section 5. So assume $E$ is non-archimedean. By Theorem 3.15.1 one may take $Y = X$ and thus this is always non-empty as it contains triples $(X, E, K)$ with $K \supset E$ running through perfectoid fields with $K^\circ = \mathbb{C}_p^\circ$ considered up to topological isomorphisms. So the assertion is clear. \(\square\)

§ 8.12 In recent correspondence, Kiran Kedlaya pointed out to me the following consequence of [Kedlaya and Liu, 2019, Proposition 8.8.9]: *every deformation of an analytic space (arising from the analytification of a quasi-projective variety) over a perfectoid field arises from a deformation of the perfectoid field*. So the idea of moving the algebraically closed perfectoid field (considered here) is, in a rather precise sense (of [Kedlaya and Liu, 2019]), optimal. The above analogy with [Beilinson and Schechtman, 1988] suggests (to me) that a suitable version of the Virasoro Uniformization Theorem might hold in the $p$-adic setting as well.
§ 8.13 This remark will not be used in the rest of the paper, but let me point out another important aspect of my construction which readers will find useful. Classical Teichmüller Spaces also admits a purely Banach ring theoretic description. This rests on a remarkable (Banach) ring theoretic characterization of quasi-conformality and conformality of Riemann surfaces due to [Nakai, 1959, 1960]. Let $\Sigma, \Sigma'$ be connected Riemann surfaces (open or closed). Let $\mathcal{R}_\Sigma$ be the Royden algebra of $\Sigma$. Recall (from [Nakai, 1960]) that the Royden algebra $\mathcal{R}_\Sigma$ of a connected Riemann surface $\Sigma$ is the $C^*$-algebra of all complex valued functions $f : \Sigma \to \mathbb{C}$ satisfying the following three properties:

1. $f$ is absolutely continuous on $\Sigma$ in the Tonelli sense,
2. $|f|$ is bounded on $\Sigma$, and
3. the Dirichlet integral $D[f] = \int \int_\Sigma |\text{grad}(f)| \, dx \, dy < \infty$.

Then $\mathcal{R}_\Sigma$ is a Banach algebra (see [Nakai, 1960]) with respect to the norm given by

$$|f|_\Sigma = \sup_\Sigma |f| + \sqrt{D[f]}.$$

Recall from [Nakai, 1959, 1960] that this is a Banach algebra equipped with several different topologies other than its norm-topology. The main theorem of [Nakai, 1959, 1960] asserts that two Riemann surfaces $\Sigma, \Sigma'$ are quasi-conformal if and only if the Royden algebra $\mathcal{R}_\Sigma \simeq \mathcal{R}_{\Sigma'}$ are (topologically) isomorphic and $\Sigma$ and $\Sigma'$ are conformally equivalent if and only if $\mathcal{R}_\Sigma \simeq \mathcal{R}_{\Sigma'}$ is an isomorphism of normed algebras. Thus quasi-conformality and conformality of Riemann surfaces has a purely Banach algebra theoretic characterization.

In particular by [Nakai, 1959, 1960] one can describe the classical Teichmüller space $T_\Sigma$ as the collection of all Riemann surfaces whose Royden algebras are isomorphic to $\mathcal{R}_\Sigma$ and one can view $T_\Sigma$ as arising from variation of the Banach structure of $\mathcal{R}_\Sigma$ i.e. view $\mathcal{R}_\Sigma$ as being a fixed Royden algebra with possibly different normed algebra structures arising from pull-back via quasi-conformal mappings $\Sigma \to \Sigma'$.

The constructions of the present paper can be viewed in a similar manner: suppose $X/E$ is a geometrically connected, smooth hyperbolic curve over a $p$-adic field. Assume in addition that $X/E$ is of strict Belyi type, so that if $Y/E'$ is anabelomorphic to $X/E$ then $Y \simeq X$ as $\mathbb{Z}$-schemes. So in the theory presented here the sheaf of rings $\mathcal{O}_X$ remains fixed while the sheaf of Banach algebras $\mathcal{O}_{X_{\mathbb{C}_p}}$ moves as one deforms the valued field $\mathbb{C}_p$.

§ 8.14 For greater flexibility and with a view to applications, it is useful to consider variants of the construction of $\mathfrak{J}(X, E)$. Let $\Sigma$ be a (finite) set of geometric or arithmetic conditions one can impose on the data $(Y/E', E' \hookrightarrow K)$. Let

$$\mathfrak{J}_\Sigma(X, E) = \{ (Y/E', E' \hookrightarrow K) \in \mathfrak{J}(X, E) \text{ and } (Y/E', E' \hookrightarrow K) \text{ satisfies } \Sigma \}.$$

§ 8.15 One important example of $\Sigma$ (from the point of view of [Mochizuki, 2021a,b,c,d]) is

$$\Sigma = \{ \dim(Y) = 1 \text{ and } Y \text{ is of strict Belyi Type} \},$$

(for the definition of Strict Belyi Type see [Mochizuki, 2013, Definition 3.5]). For this $\Sigma$, I will write:

$$\mathfrak{J}_{SB}(X, E) = \{ (Y/E', E' \hookrightarrow K) \in \mathfrak{J}(X, E) : \dim(Y) = 1 \text{ and } Y/E' \text{ is of Strict Belyi Type} \}.$$
§ 8.16 The following is immediate from Mochizuki’s proof of the Absolute Grothendieck Conjecture (see [Mochizuki, 2007, Corollary 2.12] for hyperbolic curves over $p$-adic fields of Strict Belyi Type) and the construction of $J_{SB}(X, E)$:

**Proposition 8.16.1.** Let $E$ be a $p$-adic field and $X/E$ be a geometrically connected, smooth, quasi-projective, hyperbolic curve of Strict Belyi Type over $E$. Then

1. For every $(Y/E', E' \hookrightarrow K) \in J_{SB}(X, E)$ one has an isomorphism of schemes (over $\mathbb{Z}$) $Y \simeq X$.

2. Hence one has a natural action of $\text{Aut}(\Pi)$ on $J_{SB}(X, E)$ via its action on $\text{Isom}(Y, X) \simeq \text{Isom}^{\text{out}}(\pi_1(Y), \pi_1(X))$.

3. However two such triples $(Y/E', E' \hookrightarrow K), (Y''/E'', E'' \hookrightarrow K') \in J_{SB}(X, E)$ may not be isomorphic in general.

§ 8.17 This paragraph is not used in the rest of the paper. For this paragraph assume $E, E'$ are $p$-adic fields and $X/E$ (resp. $Y/E'$) is a geometrically connected, smooth hyperbolic curve over $E$ (resp. over $E'$) such that $\alpha : \Pi_{X/E}^{\text{temp}} \rightarrow \Pi_{Y/E'}^{\text{temp}}$ i.e. $X/E$ and $Y/E'$ are anabelomorphic hyperbolic curves over $p$-adic fields, then by [Mochizuki, 2004, Lemma 1.3.8] it follows that $\alpha$ induces an isomorphism of topological groups $G_{E'} \iso G_E$ i.e. the $p$-adic fields $E$ and $E'$ are (necessarily) anabelomorphic $p$-adic fields. For properties of anabelomorphic $p$-adic fields see [Joshi, 2020a] and its bibliography.

§ 8.18 Let $E$ be a $p$-adic field and let $X/E$ be a geometrically connected, smooth, quasi-projective variety over $E$. Let me remark that one can consider $\mathfrak{J}(X, E)$ as a category equipped with the following forgetful functors:

1. $\mathfrak{J}(X, E) \to \text{Schemes}/\mathbb{Z}$ given by 
   $$(Y/E', E' \hookrightarrow K) \mapsto Y.$$ 

2. $\mathfrak{J}(X, E) \to \text{Finite separable extensions of } \mathbb{Q}_p$ given by 
   $$(Y/E', E' \hookrightarrow K) \mapsto E'.$$

3. $\mathfrak{J}(X, E) \to \text{Algebraically Closed Perfectoid fields } \supset \mathbb{Q}_p$ which is given by 
   $$(Y/E', E' \hookrightarrow K) \mapsto K.$$

4. $\mathfrak{J}(X, E) \to \text{Algebraically Closed Perfectoid fields of char. } p > 0$ which is given by 
   $$(Y/E', E' \hookrightarrow K) \mapsto K^\flat.$$ 

Thus $\mathfrak{J}(X, E)$ is a category over each of these base categories which are targets of the above functors.
§ 8.19 There are two functors to the category of analytic spaces given by the rules

\[ (Y/E', E' \hookrightarrow K) \mapsto Y^{an}, \]

and

\[ (Y/E', E' \hookrightarrow K) \mapsto (Y \times_{E'} K)^{an}. \]

§ 8.20 Let me now come to an important property of \( \mathcal{J}(X, E)_F \). The key tool in proving this will be [Fargues and Fontaine, 2018]. So let me begin with the following remark.

§ 8.21 Let \( E \) be a \( p \)-adic field and let \( \pi \) be a uniformizer for its ring of integers, let \( q \) be the cardinality of the residue field \( \mathcal{O}_E/\pi \) of \( E \). By a Lubin-Tate formal group \( \mathcal{G} \) over \( \mathcal{O}_E \), I mean a formal group constructed by [Lubin and Tate, 1965], using some polynomial \( Q(T) \in \mathcal{O}_E[T] \) satisfying the following two hypothesis of [Lubin and Tate, 1965]:

1. \( Q(T) = \pi T + O(T^2), \) and
2. \( Q(T) = T^q \mod \pi. \)

By [Lubin and Tate, 1965] the formal groups determined by two such polynomials is naturally isomorphic. By the results of [Fargues and Fontaine, 2018, Chapter 1, 2] especially [Fargues and Fontaine, 2018, Proposition 2.1.7], the \( \mathcal{O}_E \)-algebra \( \mathcal{W}_{\mathcal{O}_E}(\mathcal{O}_F) \) required in the construction of the Fargues-Fontaine curves is independent of the choice of the Lubin-Tate polynomial \( Q(T) \) used to define \( \mathcal{G} \) and in particular these constructions are independent of the choice of the Lubin-Tate group \( \mathcal{G} \).

§ 8.22 Associated to a Lubin-Tate \( \mathcal{O}_E \)-formal group \( \mathcal{G} \) over \( \mathcal{O}_E \), is a \( \pi \)-divisible group over \( \mathcal{O}_E \), and its special fibre (over the residue field of \( \mathcal{O}_E \)) and also a \( \pi \)-divisible formal \( \mathcal{O}_E \)-module. I will pass between these objects whenever needed (to invoke results of [Fargues and Fontaine, 2018]), but beware that I will notationally conflate all of these objects as \( \mathcal{G} \). Hopefully there will be no confusion.

§ 8.23 I will also use \( \mathcal{G} \) for the special fiber of \( \mathcal{G}/\mathcal{O}_E \). I hope that readers will be able to unravel the usage from the context (in [Fargues and Fontaine, 2018], the special fiber is denoted by \( \mathcal{G}_k \) where \( k \) is the residue field of \( F \)). This means for example where Fargues and Fontaine write \( \mathcal{G}_k(\mathcal{O}_F) \), I will simply write \( \mathcal{G}(\mathcal{O}_F) \). By [Fargues and Fontaine, 2018, Proposition 4.4.1] \( \mathcal{G}(\mathcal{O}_F) \) is naturally a Banach space over \( E \).

§ 8.24 Suppose \( E \) is a \( p \)-adic field and \( \mathcal{O}_E \) its ring of \( p \)-adic integers, let \( K \) be an algebraically closed perfecFT field and let \( \mathcal{G} \) be a Lubin-Tate group or a \( \mathcal{O}_E \)-formal group equipped with an homomorphism \( \mathcal{O}_E \to \text{End}_{\mathcal{O}_K}(\mathcal{G}) \). Let \( \pi \in \mathcal{O}_E \) be a uniformizer. Let

\[ \widehat{\mathcal{G}(\mathcal{O}_K)} = \lim_{\text{null. by } \pi} \mathcal{G}(\mathcal{O}_K), \]

this is naturally an \( E \)-vector space (and hence also an \( \mathcal{O}_E \)-module). Note that \( \widehat{\mathcal{G}(\mathcal{O}_K)} \) is denoted as \( X(\mathcal{G})(\mathcal{O}_K) \) [Fargues and Fontaine, 2018, Chap IV].

§ 8.25 Let me explicate this for \( E = \mathbb{Q}_p, F = \mathbb{C}_p^p \) and \( K = \mathbb{C}_p \), here \( \mathcal{G}(\mathbb{C}_p) \simeq \mathcal{G}(\mathbb{C}_p^p) \). In this case \( \mathcal{G}(\mathbb{C}_p) = \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \) is considered as a \( \mathbb{Z}_p \)-module via the Lubin-Tate action in which the endomorphism \( p \) acts on \( \mathcal{G}(\mathbb{C}_p) \) by the endomorphism \( z \mapsto z^p + pz \) of \( \mathcal{G}(\mathbb{C}_p) \) and \( \widehat{\mathcal{G}(\mathbb{C}_p)} \) is the \( \mathbb{Q}_p \)-vector space obtained by formally inverting the Lubin-Tate action of \( p \) on the group \( \mathcal{G}(\mathbb{C}_p) \simeq \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \).

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§ 8.26 Let $E$ be a $p$-adic field and let $F$ be an algebraically closed perfectoid field of characteristic $p > 0$. Recall that in [Fargues and Fontaine, 2018, Chapter 2] the construction of the Fargues-Fontaine curve proceeds via the construction of an auxiliary curve, denoted in loc. cit by $Y_{F,E}$ (resp. $X_{F,E}$) (or simply by $Y$ if the choice of $F, E$ is unambiguous) and denoted here by $\mathcal{Y}_{F,E}$ (resp. $\mathcal{X}_{F,E}$). More precisely $\mathcal{Y}_{F,E}$ is constructed as an adic space (but I will not use this fact here).

§ 8.27 Of particular interest to us are the sets of closed points of $|\mathcal{Y}_{F,E}|$ (resp. $|\mathcal{X}_{F,E}|$) of closed (classical Tate points) of $\mathcal{Y}_{F,E}$ (resp. closed points of degree one of $\mathcal{X}_{F,E}$). The curve $\mathcal{Y}_{F,E}$ is equipped with a natural Frobenius $\varphi : \mathcal{Y}_{F,E} \rightarrow \mathcal{Y}_{F,E}$. In the discussion which follows I will habitually conflate $\mathcal{X}_{F,E}$ and $\mathcal{Y}_{F,E}$ with $|\mathcal{X}_{F,E}|$ (resp. $|\mathcal{Y}_{F,E}|$).

§ 8.28 By [Fargues and Fontaine, 2018, Théorème 6.5.2(4)], one has a canonical identification

$$|\mathcal{Y}_{F,E}|/\varphi^Z \simeq |\mathcal{Y}_{F,E}|,$$

given by $y \mapsto \{\varphi^n(y) : n \in \mathbb{Z}\}$. It is standard that $\mathcal{Y}_{F,E}/\varphi^Z \rightarrow \mathcal{X}_{F,E}$ is in fact a morphism of adic spaces which provides the above identification on points.

§ 8.29 With this preparation let me prove the following:

**Theorem 8.29.1.** Let the notation and assumptions be as in the previous paragraph. Assume $K$ is an algebraically closed perfectoid field with $K^p = F$.

1. The isomorphism class of the topological $\mathcal{O}_E$-module $\hat{\mathcal{O}}(\mathcal{O}_K)$ is independent of $K$ more precisely, there is a natural homeomorphism of $\mathcal{O}_E$-modules

$$\hat{\mathcal{O}}(\mathcal{O}_K) \simeq \mathcal{G}(\mathcal{O}_F),$$

(in fact this is an isomorphism of Banach $E$-vector spaces).

2. The isomorphism class of the topological $\mathcal{O}_E$-module $\mathcal{G}(\mathcal{O}_F)$ is independent of the choice of the Lubin-Tate $\mathcal{O}_E$-formal group $\mathcal{G}$.

3. There is a natural action of the group $\text{Aut}_{\mathcal{O}_E}(\mathcal{G}(\mathcal{O}_F))$, of topological automorphisms of the $\mathcal{O}_E$-module $\mathcal{G}(\mathcal{O}_F) \simeq \mathcal{G}(\mathcal{O}_K)$, on the set of closed points of degree one of the Fargues-Fontaine curve $\mathcal{Y}_{F,E}$, arising from the natural identification

$$|\mathcal{Y}_{F,E}| = (\mathcal{G}(\mathcal{O}_F) - \{0\}) / \mathcal{O}_E^*.$$

4. Let $y \in \mathcal{Y}_{F,E}$ be a closed point of degree one. Then one has an action of $\text{Aut}_{\mathcal{O}_E}(\mathcal{G}(\mathcal{O}_F))$ on closed points of degree one of $\mathcal{X}_{F,E}$ via mapping

$$\{\varphi^n(y) : n \in \mathbb{Z}\} \mapsto \{\varphi^n(\sigma(y)) : n \in \mathbb{Z}\}.$$

5. Thus given any topological $\mathcal{O}_E$-linear automorphism $\sigma : \mathcal{G}(\mathcal{O}_F) \rightarrow \mathcal{G}(\mathcal{O}_F)$, and a closed point $y \in \mathcal{Y}_{F,E}$ of degree one, with residue field $K_y$, there is a perfectoid algebraically closed field $\sigma(K) = K_{\sigma(y)}$ with isometries $K^p \simeq F \simeq K^p_{\sigma(y)}$ and $E$ embeds in both $K_y, K_{\sigma(y)}$.

6. In particular $\text{Aut}_{\mathcal{O}_E}(\mathcal{G}(\mathcal{O}_F))$ acts naturally on $\mathfrak{X}(X, E)_F$ via

$$\{Y, E', K\} \mapsto \{\sigma(Y), \sigma(E), \sigma(K)\} \quad \text{for all } \sigma \in \text{Aut}_{\mathcal{O}_E}(\mathcal{G}(\mathcal{O}_F)).$$
Proof. Before proceeding to the proofs let me remark that items (1) and (2) are due to \[\text{[Fargues and Fontaine, 2018]}\] and I include them here for completeness. The assertion (1) is \[\text{[Fargues and Fontaine, 2018, Proposition 4.5.11]}\] (what I have denoted as \(\mathcal{G}(O_K)\)) is denoted by \(X(\mathcal{G})(O_K)\) in loc. cit.). The independence from the choice of the \(O_E\)-formal Lubin-Tate group \(\mathcal{G}\) is clear from \[\text{[Lubin and Tate, 1965]}\] as the Lubin-Tate \(O_E\)-formal group over \(O_E\) is unique up to isomorphism by \[\text{[Lubin and Tate, 1965]}\].

The identification of \(|\mathcal{G}_{F,E}|\) with the \((\mathcal{G}(O_F) - \{0\})/O_E^*\) is \[\text{[Fargues and Fontaine, 2018, Proposition 2.1.10]}\] and hence for any \(\sigma \in \text{Aut}_{O_E}(\mathcal{G}(O_F))\), \(\sigma\) is evidently a bijection on \((\mathcal{G}(O_F) - \{0\})/O_E^*\). Thus the claim (3) is immediate.

The proof of (4) is now clear now that (3) has been established.

To prove (5) it suffices to prove that the residue fields of closed points of \(\mathcal{G}_{F,E}\) are algebraically closed, perfectoid with tilts isometric with \(F\). This is immediate from \[\text{[Fargues and Fontaine, 2018, Corollaire 2.2.22]}\]. Finally one takes \(\sigma(E') \subset K_{\sigma(y)}\) to be the finite extension \(\sigma(E')/Q_p\) contained in \(K_{\sigma}\) corresponding to \(E'/Q_p\). More precisely, by the primitive element theorem (\[\text{[Lang, 2002]}\]), there exists an \(\alpha \in E'\) such that \(E' = E(\alpha)\) where \(f(\alpha) = 0\) for some monic irreducible polynomial \(f(T) \in E[T]\) and \(\sigma(E')\) is the (unique) finite extension corresponding to this data in \(K_{\sigma} \supset E\). Since \(E'\) and \(\sigma(E')\) are isomorphic field extensions of \(Q_p\) and both are complete and discretely valued, this is an isomorphism of discretely valued fields (by \[\text{[Schmidt, 1933]}\]). Finally \(\sigma(Y)/\sigma(E')\) is the pull-back of \(Y/E'\) along the isomorphism (of discretely valued fields) \(E' \to \sigma(E')\). This completes the proof.

§ 8.30 For readers familiar with the Geometric Langlands Program over \(C\), let me remark that the action of \(\text{Aut}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_{C})\) considered here is the \(p\)-adic analog of the action of the Virasoro Algebra on moduli spaces of marked Riemann surfaces described in the Virasoro uniformization Theorem \[\text{[Beilinson and Schechtman, 1988, Section 4]}\], \[\text{[Beilinson and Drinfeld, 2000]}\], \[\text{[Frenkel and Ben-Zvi, 2001]}\]. In the Geometric Langlands setting of \[\text{[Beilinson and Drinfeld, 2000]}\], the Virasoro algebra plays a fundamental role and manifests itself via the action of the group scheme \(\mathbb{C} \subset R \mapsto \text{Aut}_{cont}(R((T)))\). As is described in \[\text{[Beilinson and Schechtman, 1988, Section 4]}\] or \[\text{[Frenkel and Ben-Zvi, 2001, Theorem 17.3.2]}\], this action also changes complex structures of marked Riemann surfaces (in general). As has been noted above, \(\text{Aut}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_{C})\)) acts by changing the analytic structure of \((X \times E K)^{an}\) and hence must be considered as the \(p\)-adic analog of the Virasoro action in the complex setting. As was remarked in \[\text{[Beilinson and Schechtman, 1988]}\], the Virasoro uniformization Theorem complements the Teichmuller Uniformization.

§ 8.31 An important consequence of this is that topological \(O_E\)-linear automorphisms of \(\mathcal{G}(O_F)\) can be used to change the ring structures in the sense of Theorem 3.15.1.

§ 8.32 Let me also record the following useful corollary in the special case of hyperbolic curves of strict Belyi Type.

Corollary 8.32.1. Let \(E\) be a \(p\)-adic field. Let \(X/E\) be a geometrically connected, smooth, quasi-projective, hyperbolic curve over \(E\) of strict Belyi type. Then there is a natural action of \(\text{Out}(\Pi)\) where \(\Pi = \Pi^{an}_{X/E}\) on \(\mathcal{A}_{SB}(X, E)\).

Proof. This is immediate from Proposition 8.16.1: for any \((Y/E', E' \hookrightarrow K)\) one has an isomorphism of schemes \(Y \simeq X\).

§ 8.33 The results of the preceding paragraphs can be assembled into the following theorem:

Theorem 8.33.1. Let \(E\) be a \(p\)-adic field, let \(X/E\) be a geometrically connected, smooth, quasi-projective variety over \(E\). Then there exists a category \(\mathcal{A}(X, E)\), called the \(p\)-adic Teichmuller Space associated to \(X/E\) with the following properties:
(1) objects of $\mathfrak{Z}(X, E)$ are triples $(Y/E', E' \hookrightarrow K)$ consisting of $Y/E'$ a geometrically connected, smooth, quasi-projective curve over a $p$-adic field $E'$, $K$ is an algebraically closed perfectoid field with an isometric embedding $E \hookrightarrow K$ and an isomorphism of the tempered fundamental groups $\Pi_{Y/E'}^{\text{temp}} \simeq \Pi_{X/E}^{\text{temp}}$.

(2) Morphisms between triples will be defined in the obvious way.

The $p$-adic Teichmüller Space $\mathfrak{Z}(X, E)$ has the following properties:

(a) The category $\mathfrak{Z}(X, E)$ is an anabelian variation providing $\Pi = \Pi_{X/E}^{\text{temp}}$ (see §7.8, §7.18) i.e. for any $(Y/E', E' \hookrightarrow K)$, one has an isomorphism of topological groups

$$\Pi_{Y/E'}^{\text{temp}} \simeq \Pi_{X/E}^{\text{temp}}.$$

(b) There are forgetful functors (see §8.18):

(i) $(Y/E', E' \hookrightarrow K) \longmapsto Y/\mathbb{Z}$ (i.e. to Schemes/$\mathbb{Z}$).

(ii) $(Y/E', E' \hookrightarrow K) \longmapsto E'$ (i.e. to $p$-adic fields).

(iii) $(Y/E', E' \hookrightarrow K) \longmapsto K$ (i.e. to algebraically closed perfectoid fields of characteristic zero and residue characteristic $p > 0$).

(iv) $(Y/E', E' \hookrightarrow K) \longmapsto K^\flat$ (i.e. to algebraically closed perfectoid fields of characteristic $p > 0$).

(c) There are functors to analytic spaces (see §8.19)

$$(Y/E', E' \hookrightarrow K) \longmapsto Y^\text{an},$$

and

$$(Y/E', E' \hookrightarrow K) \longmapsto (Y \times_{E'} K)^\text{an}.$$

(d) There are functors to Mochizuki’s anabelian landscape (see §9.14): (one uses the given perfectoid field to compute algebraic closures)

$$(Y/E', E' \hookrightarrow K) \longmapsto \Pi_{Y/E'}^{\text{temp}} \curvearrowright \mathcal{O}_E^\times \subset \mathcal{O}_K,$$

and also

$$(Y/E', E' \hookrightarrow K) \longmapsto \Pi_{Y/E'}^{\text{temp}} \curvearrowright \mathcal{O}_E^{\times \mu} \subset \mathcal{O}_K^{\times} / \mu(K),$$

and similarly

$$(Y/E', E' \hookrightarrow K) \longmapsto \Pi_{Y/E'}^{\text{temp}} \curvearrowright \mathcal{O}_E^\varphi \subset \mathcal{O}_K^\varphi.$$

(e) If $\dim(X) = 1$ and $X$ is of Strict Belyi Type (this condition is defined in [Mochizuki, 2013, Definition 3.5]) then one has an action of $\text{Aut}(\Pi)$ on $\mathfrak{Z}(X, E)$ (Proposition 8.16.1).

(f) For a fixed algebraically closed, perfectoid field $F$ of characteristic $p > 0$, there are categories $\mathfrak{Z}(X, E)_F$ consisting of $(Y/E', E' \hookrightarrow K)$ such that $K^\flat = F$.

(g) Now fix an algebraically closed perfectoid field $F$ of characteristic $p > 0$, a uniformizer $\pi$ for $E$ and let $\mathcal{G}/\mathcal{O}_E$ be the Lubin-Tate formal group. Then there is a natural action of $\text{Aut}_{\mathcal{O}_E}(\mathcal{G}/\mathcal{O}_F)$ on $\mathfrak{Z}(X, E)_F$ (Theorem 8.29.1). Notably for $F = \mathbb{C}_p^\flat$, one has a natural action (Corollary 9.11.1) of

$$\text{Aut}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_{C_p})) \simeq \text{Aut}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_{C_p}))$$

on $\mathfrak{Z}(X, E)_{\mathbb{C}_p}$. 

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The category \( \tilde{J}(X, E)_{\mathbb{C}^p} \) is self-similar (Theorem 11.7.1).

**Proof.** The only assertion which remains to be proved is the last claim that \( \tilde{J}(X, E)_{\mathbb{C}^p} \) is self-similar and this is Theorem 11.7.1 and will be proved in the next section.

§ 8.34 Let us now assemble these constructions when one wants to work over a number field \( L \). Let \( X/L \) be a geometrically connected, smooth, quasi-projective variety over a number field \( L \) with no real embeddings (I will make this restriction to avoid notational complexity). Then the adelic tempered fundamental group of \( X/L \) is the group (equipped with product topology):

\[
\tilde{\Pi}^\text{temp}_{X/L} = \prod_{0 \neq p \in \text{Spec}(\mathcal{O}_L)} \Pi^\text{temp}_{X/L_p} \times \prod_{i=1}^{n} \pi_1^{\text{top}}(X/L_{\infty_i}).
\]

§ 8.35 In the global i.e. number situation, the local \( p \)-adic Teichmuller Spaces can be assembled into a global category.

**Theorem 8.35.1.** Let \( X/L \) be a geometrically connected, smooth, quasi-projective, hyperbolic variety over a number field \( L \). Assume \( L \) has no real embeddings. Then there exists a category \( \tilde{J}(X/L) \), called the Arithmetic Teichmuller Space associated to \( X/L \) which has the following properties:

1. \( \tilde{J}(X/L) \) is given as a product category:
   \[
   \tilde{J}(X/L) = \prod_p \tilde{J}(X, L_p)
   \]
   where \( p \) runs over all the inequivalent, non-trivial valuations of \( L \) and where \( \tilde{J}(X, L_p) \) is the \( p \)-adic Teichmuller Space associated to \( X/L_p \) constructed in Theorem 8.33.1.

2. \( \tilde{J}(X/L) \) is an anabelian variation providing the adelic tempered fundamental group

\[
\Pi^\text{temp}_{X/L} = \prod_{0 \neq p \in \text{Spec}(\mathcal{O}_L)} \Pi^\text{temp}_{X/L_p} \times \prod_{i=1}^{n} \pi_1^{\text{top}}(X/L_{\infty_i}).
\]

9 Relationship to Mochizuki’s Anabelian Landscape

§ 9.1 This section will not be used in the rest of the paper but readers of [Mochizuki, 2021a,b,c,d] may find it useful. Let me now show how Arithmetic Teichmuller Theory of preceding sections comes equipped with functors to the Anabelian Landscape considered in [Mochizuki, 2021a,b,c,d]. Theorem 8.29.1 and its corollaries. provides a concrete way of understanding this relationship.

§ 9.2 Let me begin by remarking that Arithmetic Teichmuller Theory of this paper is designed to be fully compatible with [Mochizuki, 2021a,b,c,d]. Notably as the Question 1.1.1 was initially raised in the context of [Mochizuki, 2021a,b,c,d], the solution to Question 1.1.1 presented in this paper is equally applicable to the context of [Mochizuki, 2021a,b,c,d].
§ 9.3 Let me remark that in [Mochizuki, 2021a,b,c,d] Mochizuki works with multiplicative groups as the anabelian approach considered in loc. cit. is inherently multiplicative. On the other hand [Fargues and Fontaine, 2018] Fargues-Fontaine work with additive i.e. Lubin-Tate group i.e. the Fargues-Fontaine approach (to [Fargues and Fontaine, 2018]) is necessarily additive (as opposed to being multiplicative). In the next few paragraphs I provide a translation between the two. This allows one to construct functors from \( \mathfrak{g}(X, E) \) to Mochizuki’s anabelian landscape. In passing let me remark that the resolution of Question (1.1.1) presented in Theorem 3.9.1 and Theorem 3.15.1 can also be applied to [Mochizuki, 2021a,b,c,d] via the functors described here. However there are some important differences between the two approaches—these are discussed in § 9.10.

§ 9.4 For the multiplicative description let me fix some notations. Let \( E = \mathbb{Q}_p \), let \( \mathcal{G}/\mathbb{Z}_p \) be the Lubin-Tate formal group with formal logarithm given by \( \sum_{n=0}^{\infty} \frac{T^n}{p^n} \), \( F = \mathbb{C}_p \). Let \( \overline{E} = \mathbb{Q}_p \) be the algebraic closure of \( E = \mathbb{Q}_p \) in \( \mathbb{C}_p \).

§ 9.5 The Artin-Hasse Exponential provides the following:

**Lemma 9.5.1.** Let \( \mathcal{G} \) be the Lubin-Tate formal group over \( \mathbb{Z}_p \) with logarithm \( \sum_{n=0}^{\infty} \frac{T^n}{p^n} \). Let \( \text{Exp}_{AH}(T) \) be the Artin-Hasse exponential function. Then the homomorphism \( a \mapsto \text{Exp}_{AH}(a) \) provides a natural isomorphism of topological \( \mathbb{Z}_p \)-modules

\[
\text{Exp}_{AH} : \mathcal{G}(\mathcal{O}_{\mathbb{C}_p}) \simeq \hat{G}_m(\mathcal{O}_{\mathbb{C}_p})
\]

and hence also of

\[
\hat{G}(\mathcal{O}_{\mathbb{C}_p}) \simeq \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}) \simeq \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}).
\]

**Proof.** See [Fargues and Fontaine, 2018, Example 4.4.7].

§ 9.6 Let \( \hat{G}_m/\mathbb{Z}_p \) be the multiplicative formal group. Then one has for the multiplicative formal \( \hat{G}_m \) one has

\[
(9.6.1) \quad \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}) = 1 + \mathcal{O}_{\mathbb{C}_p}
\]

where \( 1 + \mathcal{O}_{\mathbb{C}_p} \subset \mathcal{O}^\times_{\mathbb{C}_p} \) is the subgroup of units congruent to 1 modulo the maximal ideal \( \mathcal{m}_{\mathbb{C}_p} \subset \mathcal{O}_{\mathbb{C}_p} \), and one also has from this that

\[
(9.6.2) \quad \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}) = \left\{ (x_n)_{n \in \mathbb{Z}} : x_n \in \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}) = 1 + \mathcal{m}_{\mathbb{C}_p}, x_n^p = x_n, \forall n \in \mathbb{Z}\right\}
\]

This fits into an exact sequence of \( \mathbb{Q}_p \)-Banach spaces ([Fargues and Fontaine, 2018, Proposition 4.5.14])

\[
0 \to T_p(\hat{G}_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}) \xrightarrow{\log_{\hat{G}_m}} \mathbb{C}_p \to 0,
\]

where \( T_p(\hat{G}_m) \) is the \( p \)-adic Tate-module of \( \hat{G}_m \) (note that \( T_p(\hat{G}_m) \) is a rank one free \( \mathbb{Z}_p \)-module computed using the \( p \)-power roots of unity contained in \( \mathbb{C}_p \)), and where \( \log_{\hat{G}_m} \) is the logarithm of the formal group \( \hat{G}_m \). Explicitly \( \log_{\hat{G}_m} \) is given in terms of the \( p \)-adic logarithm as follows ([Fargues and Fontaine, 2018, Proof of Proposition 4.5.9]). Let \( x = (x_n)_{n \in \mathbb{Z}} \in \hat{G}_m(\mathcal{O}_{\mathbb{C}_p}) \) then

\[
\log_{\hat{G}_m}(x) = \log(x_0),
\]

where

\[
\log : \mathcal{O}_{\mathbb{C}_p}^\times \to \mathbb{C}_p
\]

is the \( p \)-adic logarithm.
§ 9.7 For a valued field $K \supset \mathbb{Q}_p$ let where

$$\mathcal{O}_K^* = \mathcal{O}_K^*/\mu(K)$$

be the (topological) subgroup of roots of unity in $K$ and write

$$\mathcal{O}_K^{*\mu} = \mathcal{O}_K^*/\mu(K)$$

for the quotient of $\mathcal{O}_K^*$ by $\mu(K)$. Let $\mathcal{O}_K^p = \mathcal{O}_K - \{0\}$ be the multiplicative monoid of non-zero elements of $\mathcal{O}_K$. Both these notations were introduced and used extensively in [Mochizuki, 2021a,b,c,d].

§ 9.8 Let $\mu(\mathbb{C}_p) \subset \mathbb{C}_p^*$ be the subgroup of roots of unity contained in $\mathbb{C}_p$. Then one has the exact sequence of topological groups

$$0 \to \mu(\mathbb{C}_p) \to \mathcal{O}_{\mathbb{C}_p}^* \xrightarrow{\log} \mathbb{C}_p \to 0.$$  

§ 9.9 In [Mochizuki, 2021a,b,c,d] especially [Mochizuki, 2021c], Mochizuki works with

$$\mathcal{O}_{\mathbb{Q}_p}^{*\mu} = \mathcal{O}_{\mathbb{Q}_p}^*/\mu(\mathbb{Q}_p).$$

This is one key difference between the theory of the present paper and Mochizuki’s work. Note that

**Lemma 9.9.1.** The inclusions $\mathcal{O}_{\mathbb{Q}_p}^* \subset \mathcal{O}_{\mathbb{C}_p}^*$ and $\mathcal{O}_{\mathbb{Q}_p}^{*\mu} \subset \mathcal{O}_{\mathbb{C}_p}^{*\mu}$ are dense inclusions.

**Proof.** The first assertion is standard and it implies the second assertion as $\mu(\mathbb{Q}_p) = \mu(\mathbb{C}_p)$. \(\square\)

**Lemma 9.9.2.** One has an exact sequence of topological $G_{\mathbb{Q}_p}$-modules

$$1 \to \mu_p(\mathbb{Q}_p) \to (1 + m_{\mathbb{Q}_p}) \to \mathcal{O}_{\mathbb{Q}_p}^{*\mu} \to 1.$$  

**Proof.** Let $\mu'(\mathbb{Q}_p) \subset \mu(\mathbb{Q}_p)$ (resp. $\mu_p(\mathbb{Q}_p) \subset \mu(\mathbb{Q}_p)$) be the subgroup of roots of unity with orders coprime to $p$ (resp. the subgroup of roots of unity of order a power of $p$). Then one has

$$\mu'(\mathbb{Q}_p) \times \mu_p(\mathbb{Q}_p) \simeq \mu(\mathbb{Q}_p).$$

Note that for every $n \geq 1$,

$$1 - \zeta_p^n \equiv 0 \mod m_{\mathbb{Q}_p}.$$  

So any $p$-power root of unity is contained in the group of $1$-units $1 + m_{\mathbb{Q}_p}$ and hence

$$\mu_p(\mathbb{Q}_p) \subset 1 + m_{\mathbb{Q}_p}.$$  

Hence

$$\mathcal{O}_{\mathbb{Q}_p}^* = \mu'(\mathbb{Q}_p) \times (1 + m_{\mathbb{Q}_p}),$$

and hence by definition

$$\mathcal{O}_{\mathbb{Q}_p}^*/\mu(\mathbb{Q}_p) \simeq (1 + m_{\mathbb{Q}_p})/\mu_p(\mathbb{Q}_p) \simeq \mathcal{O}_{\mathbb{Q}_p}^{*\mu},$$

and this provides the asserted exact sequence. \(\square\)
§ 9.10 Let me explain the key difference between the theory described here and that of [Mochizuki, 2021a,b,c,d]. In [Mochizuki, 2021a,b,c,d] Mochizuki works with the pair \( G_E \) and its action on \( \mathcal{O}_{\mathbb{Q}_p}^X \), that is with

\[
G_E \curvearrowright \mathcal{O}_{\mathbb{Q}_p}^X.
\]

In [Mochizuki, 2021a,b,c,d], roughly speaking, algorithms of Anabelian Reconstruction Theory, automorphisms of \( G_E \curvearrowright \mathcal{O}_{\mathbb{Q}_p}^X \), the theory of log-link and theta-link are used to produced variation in the data of arithmetic line bundles.

The present paper can also be read in the multiplicative context using the isomorphism (§ 9.5)

\[
G_E \curvearrowright \widehat{\mathcal{G}_m}(\mathcal{O}_{\mathbb{Q}_p}) \simeq G_E \curvearrowright \widehat{\mathcal{G}_m}(\mathcal{O}_{\mathbb{C}_p}),
\]

and variation of the data of arithmetic line bundles arises from existence Arithmetic Teichmuller Spaces (Theorem 8.33.1) which arise from existence of deformations of analytic structure of \( X_{\mathbb{C}_p}^{an} \) via deformations of \( \mathbb{C}_p \).

§ 9.11 Hence one has the following corollary of Theorem 8.29.1 and 9.5.1:

**Corollary 9.11.1.** There is a natural action of \( \text{Aut}_{\mathbb{Z}_p}(\mathfrak{J}(\mathcal{O}_{\mathbb{C}_p})) \simeq \text{Aut}_{\mathbb{Z}_p}(\widehat{\mathcal{G}_m}(\mathcal{O}_{\mathbb{C}_p})) \) on the closed points \( |\mathfrak{J}_{\mathbb{C}_p,\mathbb{Q}_p}| \) of \( \mathfrak{J}_{\mathbb{C}_p,\mathbb{Q}_p} \), which provides an action of \( \text{Aut}_{\mathbb{Z}_p}(\widehat{\mathcal{G}_m}(\mathcal{O}_{\mathbb{C}_p})) \) on \( \mathfrak{J}(X, E)_{\mathbb{C}_p} \).

Explicitly this is given as follows: For \( \sigma \in \text{Aut}_{\mathbb{Z}_p}(\widehat{\mathcal{G}_m}(\mathcal{O}_{\mathbb{C}_p})) \) and any closed point of degree one \( y \in |\mathfrak{J}_{\mathbb{C}_p,\mathbb{Q}_p}| \) one has the associations:

\[
(Y/E', E' \leftrightarrow K_y) \xrightarrow{\Pi_{Y/E':K_y}^{\text{temp}} \rightarrow \Pi_{Y/E':K_{\sigma(y)}}^{\text{temp}}} (Y/E', E' \leftrightarrow K_{\sigma(y)})
\]

and on labeled fundamental groups by

\[
\Pi_{Y/E':K_y}^{\text{temp}} \rightarrow \Pi_{Y/E':K_{\sigma(y)}}^{\text{temp}}.
\]

§ 9.12 Let me now show that Theorem 3.9.1 and Theorem 3.15.1 provide functors to the Anabelian Landscape of [Mochizuki, 2021a,b,c,d]. Notably these theorems show that there are geometrically distinguishable isomorphs of the tempered fundamental groups. The arithmetic Teichmüller space \( \mathfrak{J}(X, E) \) constructed in Theorem 8.33.1 includes all hyperbolic curves with tempered fundamental group topologically isomorphic to that of a given hyperbolic curve \( X/E \).

§ 9.13 Let \( Y/E' \) be a smooth, quasi-projective variety over a \( p \)-adic field \( E' \) and let \( \overline{E'} \) be an algebraic closure of \( E' \). This provides us a surjection \( \Pi_{Y/E':K_y}^{\text{temp}} \rightarrow G_{E'} \). Then one has an action of \( \Pi_{Y/E':K_y}^{\text{temp}} \) on \( \mathcal{O}_{\overline{E'}}^X \) via the surjection \( \Pi_{Y/E':K_y}^{\text{temp}} \rightarrow G_{E'} \) and the tautological action of \( G_{E'} \) on \( \mathcal{O}_{\overline{E'}}^X \subset \overline{E'} \). Following Mochizuki notation scheme in [Mochizuki, 2021a], I will write this data as

\[
\Pi_{Y/E':K_y}^{\text{temp}} \curvearrowright \mathcal{O}_{\overline{E'}}^X.
\]

Similarly one can also consider other related monoids such as \( \mathcal{O}_{\overline{E'}}^{X,\mu} \) equipped with its action \( G_{E'} \curvearrowright \mathcal{O}_{\overline{E'}}^{X,\mu} \) and \( \Pi_{Y/E':K_y}^{\text{temp}} \curvearrowright \mathcal{O}_{\overline{E'}}^{X,\mu} \).

§ 9.14 Consider an arbitrary triple \( (Y/E', E' \leftrightarrow K) \in \mathfrak{J}(X, E) \). Then as \( K \) is an algebraically closed, perfectoid field of characteristic zero, one can consider the algebraic closure \( \overline{E'} \subset K \) (as a valued fields) of \( E' \leftrightarrow K \). Thus the data of our triple provides us with a preferred
algebraic closure of $E'$ to work with. With this prelude one can define functors from $\mathcal{J}(X, E)$ to Mochizuki’s anabelian landscape constructed in [Mochizuki, 2021a,b,c,d] is given by

\[(9.14.1) \quad (Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'; K}^{\text{temp}} \mathcal{O}_E^\times,\]

where $\Pi_{Y/E'; K}^{\text{temp}} \mathcal{O}_E^\times$ means that the field $K$ is used to compute the algebraic closure of $E'$, and the action of $\Pi_{Y/E'; K}^{\text{temp}}$ on $\mathcal{O}_E^\times$ through the quotient $\Pi_{Y/E'}^{\text{temp}} \to G_{E'}$ is computed using the algebraic closure $\overline{E'} \subset K$.

Similarly one has

\[(9.14.2) \quad (Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'; K}^{\text{temp}} \mathcal{O}_E^\times,\]

and

\[(9.14.3) \quad (Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'; K}^{\text{temp}} \mathcal{O}_E^\times,\]

and

\[(9.14.4) \quad (Y/E', E' \hookrightarrow K) \mapsto \Pi_{Y/E'; K}^{\text{temp}} \mathcal{O}_E^\times.\]

Note that one has an isomorphism of topological groups $\Pi_{Y/E'; K}^{\text{temp}} \simeq \Pi_{X/E}^{\text{temp}}$.

§ 9.15 This remark will not be used in the rest of the paper, but readers of [Mochizuki, 2021a,b,c,d] will find it useful. In [Mochizuki, 2021a,b,c,d] Mochizuki considers the notion of a prime strip. Various versions of prime strips used there are summarized in the table [Mochizuki, 2021a, Fig II.2, page 6]. Prime strips can be readily constructed in the theory of this paper: suppose $(Y/E, E \hookrightarrow K)$ is an object of the sort considered here and $\overline{E} \subset K$ is the algebraic closure of $E$ in $K$, and following Mochizuki write $\mathcal{O}_E^\times = \mathcal{O}_E - \{0\}$ for the multiplicative monoid of non-zero elements of $\mathcal{O}_E$ then one has the prime strip (in Mochizuki’s notation) $\mathcal{F} = \Pi_{Y/E}^{\text{temp}} \mathcal{O}_E^\times$ (where $K$ reminds us that I am using $K \supset E$ to compute the algebraic closure $\overline{E}$ of $E$). Similarly $\mathcal{F}^\times := G_E \mathcal{O}_E^\times$, etc. In fact one sees by the results of this paper and from this discussion that there exist many distinctly labeled primes strips $\mathcal{F}^\times : = G_E \mathcal{O}_E^\times$. The translation of the table in loc. cit. in the notation of the present paper can be readily obtained in this manner.

§ 9.16 Theorem 8.29.1 provides the following action on the data $\Pi_{Y/E'}^{\text{temp}} \mathcal{O}_E^\times$. Let $\sigma : \hat{\mathcal{G}}_m(\mathcal{O}_p) \to \hat{\mathcal{G}}_m(\mathcal{O}_p)$ be an automorphism of topological groups. Let $(Y/E', E' \hookrightarrow K) \in \mathcal{J}(X, E)_{\mathcal{C}_p}$ be an arbitrary object. Then $\sigma$ provides the following action on the pair

\[\Pi_{Y/E'}^{\text{temp}} \mathcal{O}_E^\times \sigma(K) \mathcal{O}_E^\times.\]

10 Applications to Elliptic curves

§ 10.1 This section is based on Mochizuki’s ideas in [Mochizuki, 2021a,b,c,d] but from the point of view of this paper. In this section the general strategy of § 1.2 will be applied in the specific context of elliptic curves with a view to Diophantine applications along the lines of [Mochizuki, 2021a,b,c,d] (beware that no Diophantine inequalities are claimed in this paper).
§ 10.2 Fix a perfectoid field \( F \) of characteristic \( p > 0 \). Fix a \( p \)-adic field \( E \). Let \( y \in \mathcal{Y}_{F,E} \) be a closed point of degree one. Let \( K_y \) be the residue field of \( y \). This gives us a prime ideal \( \mathfrak{p}_y \subset W_{\mathcal{O}_E}(\mathcal{O}_F) \). I will write

\[
\eta_y : W_{\mathcal{O}_E}(\mathcal{O}_F) \to W_{\mathcal{O}_E}(\mathcal{O}_F)/\mathfrak{p}_y \simeq \mathcal{O}_{K_y}
\]

for the quotient homomorphism (see [Fargues and Fontaine, 2018, Chap 2, 2.2.2]) and write

\[
\eta_{K_y} : W_{\mathcal{O}_E}(\mathcal{O}_F)[1/\pi] \to K_y.
\]

for the extension of \( \eta_y \) to \( W_{\mathcal{O}_E}(\mathcal{O}_F)[1/\pi] \). Note that in [Fargues and Fontaine, 2018] and other \( p \)-adic Hodge-Theory literature, this homomorphism is usually denoted by \( \theta \). I will reserve the letters \( \theta, \vartheta, \Theta, \Theta \) for theta functions which will appear later.

§ 10.3 For an element \( z \in K_y \) it makes sense to consider the set of lifts \( \eta_{K_y}^{-1}(z) \) of \( z \in K_y \) to \( W_{\mathcal{O}_E}(\mathcal{O}_F) \). Such a lift is not uniquely defined and evidently the difference between any two chosen lifts of \( z \) in \( W_{\mathcal{O}_E}(\mathcal{O}_F) \) lives in \( \ker(\eta_{K_y}) \).

§ 10.4 Now let us consider applications of the results of the preceding sections to elliptic curves. Let me begin by elaborating a useful consequence of Theorem 3.15.1.

§ 10.5 Let \( E \) be a \( p \)-adic field (note that this is usually not the notational convention in the theory of elliptic curves). Let \( C/E \) be an elliptic curve over \( E \). Let us assume that \( C/E \) has split multiplicative reduction over \( E \) i.e. \( C \) is a Tate elliptic curve over \( E \). Consider two algebraically closed perfectoid fields \( K_1, K_2 \supset E \). Then \( C/K_1 \) and \( C/K_2 \) are both Tate elliptic curves over \( K_1 \) and \( K_2 \) respectively. By Tate’s Theorem, both are uniformized by Tate parameters \( q_{K_1} \in K_1^\ast \) and \( q_{K_2} \in K_2^\ast \) respectively. By the theory of \( p \)-adic \( \theta \)-functions (see [Roquette, 1970]), the function field of the analytic space \( C_{K_1}^{an} \) (resp. \( C_{K_2}^{an} \)) is described in terms of \( \theta \)-functions.

§ 10.6 In the notation and assumptions of the above paragraph, assume \( K_1 \) and \( K_2 \) are not topologically isomorphic. Then Theorem 3.15.1 asserts that the analytic spaces \( C_{K_1}^{an} \) and \( C_{K_2}^{an} \) are not isomorphic. So the function theory of \( \theta \)-functions on \( C_{K_1}^{an} \) and \( C_{K_2}^{an} \) looks quite different even though both the analytic spaces arise from the same geometric object (namely \( C/E \)). In the subsequent paragraphs, one would like to compare these two different “function-theoretic snapshots” of \( X/E \).

§ 10.7 Let me stress an important point here. As one moves from one perfectoid field, say \( K_1 = \mathbb{C}_p \), to another algebraically closed perfectoid field \( K_2 \) with \( K_1^\ast = \mathbb{C}_p^\ast \simeq K_2^\ast \) the valuations of elements such as \( p \) in these two fields (and also valuations of elements of \( \mathbb{Q}_p \)) undergo a dilatation or scaling. This is easily seen from the fact that \( K_1^\ast \simeq \mathbb{C}_p^\ast \simeq K_2^\ast \) induces equivalent norms on \( \mathbb{C}_p \) but not equality of norms on \( \mathbb{C}_p \) (in general). So the arithmetic Teichmuller space \( \mathfrak{Z}(X,E)_{\mathbb{C}_p} \) is equipped with a natural action of \( Aut_{\mathbb{C}_p}(\mathcal{O}(\mathcal{O}^\circ_{\mathbb{C}_p})) \) which (in general) also provides dilatations on the value group of \( \mathbb{Q}_p \). In particular if one passes from \( Y^{an}/K_1 \) to \( Y^{an}/K_2 \) such a dilatation of value groups becomes important in comparing degrees of arithmetic line bundles in diophantine problems involving \( C \) as one passes from \( \mathbb{C}_p \) to \( K \). The presence of dilatations should be considered to be analogous to the presence of dilatations in the classical theory of quasi-conformal mappings ([Lehto, 1987]).

§ 10.8 Let me provide an explicit example of this phenomenon of dilatation of value groups as one moves amongst untilts of \( \mathbb{C}_p \) (or any algebraically closed perfectoid field \( F \) of characteristic \( p > 0 \)). Let \( K_1 = \mathbb{C}_p \) and let \( t = p^{\eta} = (p, \sqrt[p]{p}, \sqrt[p^2]{p}, \ldots) \in \mathbb{C}_p \) providing us an identification
$\mathbb{C}_p = \overline{\mathbb{F}_p(t)}$. Let $r > 0$ be an element of the value group of $\mathbb{C}_p$. Then consider the element $a = t^r \in \mathbb{F}_p((t))$. Then applying [Fargues and Fontaine, 2018, 2.2.23] to the pair $(F = \mathbb{C}_p^v, a = t^r)$ one gets the algebraically closed, perfectoid field

$$K_2 = \frac{W(\mathcal{O}_{\mathbb{C}_p^v})[1/p]}{([t^r] - p)}$$

with

$$K_2^1 \simeq \mathbb{C}_p^v,$$

but the induced isomorphisms on the tilts $K_2^1 \simeq \mathbb{C}_p^v \simeq K_2^2$ provide norms on $\mathbb{C}_p^v$ differing by a dilatation by factor $r$ on the value groups. In general $K_1$ and $K_2$ may not be topologically isomorphic and hence these two fields have distinct arithmetic-topological structures and by Theorem 3.9.1 and Theorem 3.15.1 this change of ring structure can be propagated to geometry!

§ 10.9 Let $X = C - \{O\}$ where $C/E$ is an elliptic curve with split multiplicative reduction over $E$ and $O \in C(E)$ be the origin of the group law. Let $\ell \geq 5$ be a prime and suppose that all the $\ell^2$ torsion points $C[\ell] \subset C(E)$. Then as $C[\ell]$ is a closed subscheme of $C$. Note that $X \cap C[\ell] = C[\ell] - \{O\}$ where $O$ is the origin of the group law of $C$. Let $K_y \supset E$ be the residue field of $y \in \mathfrak{g}_{F,E}$. Let $f \in \Gamma(X^{an}/E, \mathcal{O}_{X^{an}}) \subset \Gamma(X^{an}/K_y, \mathcal{O}_{X^{an}})$ be a non-constant holomorphic function on $X$. Since $C$ is a Tate elliptic curve one can describe analytic functions on $C$ quite explicitly in terms of theta-functions (see [Roquette, 1970]). In practice this will be a suitably normalized (and uniquely determined) $\theta$-function but its precise form is irrelevant for the moment. For compatibility with the strategy adopted in [Mochizuki, 2021a,b,c,d] it is enough to choose this theta-function with the theta-function chosen in [Mochizuki, 2009, Proposition 1.4] and I will certainly do this one needs to make a choice of the theta-function. For a closed, classical point $x \in X$ (for example $x \in X(\overline{E})$), one can evaluate $f$ at $x$. So it makes sense to talk about its value $f(x) \in K_y$. By this I mean evaluation of $f$ at a closed classical point $x$ of the analytic space $X^{an}/K_y$. In particular one can consider the set of lifts of $\eta^{-1}_{K_y}(f(x))$ of $f(x) \in K_y$ as $y \in \mathfrak{g}_{F,E}$ is allowed to vary.

§ 10.10 In the above paragraph I consider values of functions, but more generally this discussion can be extended to values of sections of line bundles. Let $X/E$ be a geometrically connected, smooth, quasi-projective variety and $M$ be a line bundle on $X$ then for each closed point of degree one $y \in \mathfrak{g}_{F,E}$, one has the analytic space $X^{an}_{K_y}$ and a line bundle $M$ obtained by extension of scalars $E \rightarrow K_y$. By an argument similar to that given for the sheaf of analytic functions in § 3, $\Gamma(X^{an}_{K_y}, M)$ is a Banach space ([Ducros et al., 2015, 3.3.4]). If $X/E$ is proper then $\Gamma(X/E, M)$ is a finite dimensional $E$-vector space and $\Gamma(X^{an}_{K_y}, M) \simeq \Gamma(X/E, M) \otimes_E K_y$ is a Banach space of finite dimension over $K_y$. Notably if $s \in \Gamma(X/E, M)$ is a global section of $M$, then it makes perfect sense to talk about its value $s(P) \subset K_y$ of $s$ at a closed classical point $P \in X(E') \subset X(K_y)$ for any finite extension $E'/E$ contained in $K_y$. Of special interest to us is the absolute value $|s(P)|_{K_y}$ as $y$ varies.

§ 10.11 For the chosen non-constant holomorphic function $f$ as above, I will write Let

$$(f)_{X,\ell} = \{ z \in W_{\mathcal{O}_E}(\mathcal{O}_F) : \eta_{K_y}(z) \in f(C[\ell] - \{O\}) \subset K_y, \text{ for some } y \in \mathfrak{g}_{F,E} \}$$

and refer to $(f)_{X,\ell}$ as the $\ell$-torsion value locus of $f$. 

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§ 10.12 By [Fargues and Fontaine, 2018, Corollaire 1.4.15] one knows that \( W_{\mathcal{E}}(\mathcal{O}_F) \) is equipped with a family of non-archimedean norms \( |·|_\rho \) for each \( \rho \in [0, 1] \subset \mathbb{R} \) and \( W_{\mathcal{E}}(\mathcal{O}_F) \) is complete and separated with respect to these norms. Notably the topology on \( W_{\mathcal{E}}(\mathcal{O}_F) \) given by the norm \( |·|_0 \) is the \( \pi \)-adic topology on \( W_{\mathcal{E}}(\mathcal{O}_F) \). So it follows that elements of \( (f)_X,\ell \) can be compared with respect to the norms \( |·|_\rho \).

§ 10.13 At any rate I have thus established the following assertion inspired by [Mochizuki, 2021a,b,c,d].

**Theorem 10.13.1.** Let \( C/E \) be an elliptic curve over \( E \) with good or semi-stable reduction over \( E \). Let \( f \in \Gamma(X, \mathcal{O}_{C/E}) \) be a non-constant holomorphic function. Let \( \ell \) be a prime number. Then

1. The values of \( f \) as \( C[\ell] \cap X \), viewed by their lifts to \( W_{\mathcal{E}}(\mathcal{O}_F) \), are not uniquely determined i.e subject to natural indeterminacies.

2. These values are not comparable in the perfectoid fields \( K \) which is their natural locale of existence.

3. However their lifts are comparable in \( W_{\mathcal{E}}(\mathcal{O}_F) \) as it is a ring equipped with a family of norms \( |·|_\rho \) and especially with respect to the norm \( |·|_0 \) which induces the \( \pi \)-adic topology on \( W_{\mathcal{E}}(\mathcal{O}_F) \).

§ 10.14 This theorem is still a bit impractical to work with because the kernel of \( \eta_{K_Y} \) is still too big to deal with. Let me now demonstrate how this problem can be resolved.

This will require us to work with a related, and a more complicated ring, denoted by \( B \), in [Fargues and Fontaine, 2018, Chap 2]. I will not recall the details of the construction of this ring, but providing the following version of the results of [Fargues and Fontaine, 2018] required for my constructions. Let \( F \) be an algebraically closed perfectoid field of characteristic \( p > 0 \). Let

\[
\mathcal{E}_F = W_{\mathcal{E}}(F)[1/\pi] \supset W_{\mathcal{E}}(\mathcal{O}_F)[1/\pi] \supset W_{\mathcal{E}}(\mathcal{O}_F).
\]

\[
B^b = \left\{ \sum_{n \geq -\infty} [x_n] \pi^n \in \mathcal{E}_F : \sup_n |x_n| < \infty \right\} \supset W_{\mathcal{E}}(\mathcal{O}_F).
\]

The ring \( B^b \) is equipped with a family of non-archimedean, multiplicative norms \( |·|_\rho \) with \( \rho \in [0, 1] \). The ring \( B \) is the completion of \( B^b \) with respect to the multiplicative norms \( |·|_\rho \) for all \( \rho \in [0, 1] \) (see [Fargues and Fontaine, 2018, Définition 1.6.2, Proposition 1.4.9])

**Theorem 10.14.1.** The ring \( B \) has the following properties:

1. \( B \) is a Fréchet algebra with respect to the family of non-archimedean norms indexed by \( \rho \in [0, 1] \subset \mathbb{R} \).

2. \( B \supset W_{\mathcal{E}}(\mathcal{O}_F) \) with the norms on \( B \) inducing the norms \( |·|_\rho \) on \( W_{\mathcal{E}}(\mathcal{O}_F) \) for each \( \rho \in [0, 1] \subset \mathbb{R} \).

3. \( B \) is equipped with an action of Frobenius \( \varphi : B \to B \) which is continuous and bijective.

§ 10.15 Of special interest to us is the \( E \)-Banach subspace \( B^{\pi = \pi} \subset B \). By [Fargues and Fontaine, 2018, Proposition 4.1.5], all the norms \( |·|_\rho \) for \( \rho \in [0, 1] \), on \( B \) provide the same topology (of a Banach \( E \)-vector space) on \( B^{\pi = \pi} \). Let me now demonstrate how this subspace intervenes in the problem of understanding the theta-value locus.
§ 10.16 Let \( \mathcal{Y}_{F,E} \) be the Fargues-Fontaine curve corresponding to the datum \( (F = \mathbb{C}_p, E) \). Let \( y \in \mathcal{Y}_{F,E} \) be a closed classical point of degree one. with residue field \( K_y \). Then by [Fargues and Fontaine, 2018, Théorème 6.4.1], there is an exact sequence of \( E \)-linear continuous mappings of Banach vector spaces

\[
(10.16.1) \quad 0 \to T\gamma \otimes_{O_E} E \to B^{q=\pi} \overset{\eta_{K_y}}{\longrightarrow} K_y \to 0,
\]

where \( B^{q=\pi} \to K_y \) is the restriction to \( B^{q=\pi} \) of the natural extension of \( \eta_{K_y} : W_{\mathcal{O}_E}(\mathcal{O}_F)[1/\pi] \to K_y \) to \( B \to K_y \) and where \( T\gamma \) is a rank one \( \mathcal{O}_E \)-module naturally identified with the Tate-module of the Lubin-Tate \( \mathcal{G} \). As \( y \in \mathcal{Y}_{F,E} \) moves, so move the one-dimensional subspace \( T\gamma \otimes E \subset B^{q=\pi} \) and the field \( K_y \).

§ 10.17 Let me make the above exact sequence explicit so one has an explicit description of \( T\gamma \). Let \( \mathcal{G} \) be a Lubin-Tate \( \mathcal{O}_E \)-formal group. Let \( F \) be an algebraically closed perfectoid field of characteristic \( p > 0 \). Then the isomorphism class of \( \mathcal{G}(\mathcal{O}_F) \) is independent of \( \mathcal{G} \). Let \( K_y \) be a perfectoid field with \( K^\flat_y = F \). Then one has a natural identification [Fargues and Fontaine, 2018, Chap 4, Proposition 4.4.6]

\[
(10.17.1) \quad \mathcal{G}(\mathcal{O}_F) \simeq B^{q=\pi}
\]

and the above exact sequence can be identified with the following exact sequence of Banach \( E \)-vector spaces ([Fargues and Fontaine, 2018, Propositions 4.5.6, 4.5.11 and 4.5.14])

\[
(10.17.2) \quad 0 \longrightarrow T\pi(\mathcal{G}) \otimes E \longrightarrow \widehat{\mathcal{G}(\mathcal{O}_{K_y})} \simeq \mathcal{G}(\mathcal{O}_F) \longrightarrow K_y \longrightarrow 0,
\]

where \( T\pi(\mathcal{G}) \simeq T_y \) is the \( \pi \)-adic Tate module of the \( \mathcal{O}_E \)-formal \( \mathcal{G} \) (considered as a formal group over \( \mathcal{O}_{K_y} \)) and \( T\pi(\mathcal{G}) \otimes E \simeq T_y \) is a one-dimensional \( E \)-vector space.

§ 10.18 There is a version of (10.17.2) which I want to use to define liftings of values. I will assume \( F = \mathbb{C}_p \) from now on until the end of this section and work with \( B^{q=\nu} \subset B \).

Proposition 10.18.1. Let \( K \supset E \) be a characteristic zero unilt of \( F = \mathbb{C}_p \). Let \( \mathcal{G} \) be a Lubin-Tate \( \mathbb{Z}_p \)-formal group over \( \mathbb{Z}_p \). One has an exact sequence of topological \( \mathcal{O}_E \)-modules

\[
0 \to T_p(\mathcal{G}) \to \mathcal{G}(\mathcal{O}_{\mathbb{C}_p}) \overset{pr_{K_y}}{\longrightarrow} \mathcal{G}(\mathcal{O}_K) \to 0.
\]

Moreover one has

\[
\mathcal{G}(\mathcal{O}_K) \simeq \mathfrak{m}_{\mathcal{O}_K}.
\]

Proof. By [Fargues and Fontaine, 2018, 4.5.3 and Prop. 4.5.1] one has the following description of \( \mathcal{G}(\mathcal{O}_F) \). One can identify \( \mathcal{G}(\mathcal{O}_F) \) as limit

\[
\mathcal{G}(\mathcal{O}_F) = \mathcal{G}(\mathcal{O}_K) \underset{p}{\longrightarrow} \mathcal{G}(\mathcal{O}_K) \underset{p}{\longrightarrow} \mathcal{G}(\mathcal{O}_K) \underset{p}{\longrightarrow} \mathcal{G}(\mathcal{O}_K) \underset{p}{\longrightarrow} \cdots
\]

This provides us a projection mapping \( \mathcal{G}(\mathcal{O}_F) \overset{pr_K}{\longrightarrow} \mathcal{G}(\mathcal{O}_K) \simeq \mathfrak{m}_{\mathcal{O}_K} \) which is evidently surjective. This is immediate from the fact that multiplication by \( p \) in \( \mathcal{G} \) corresponds to the endomorphism \( z \mapsto pz + z^p \) of the Lubin-Tate group \( \mathcal{G} \). Using this or the explicit description of \( \mathcal{G}(\mathcal{O}_F) \) [Fargues and Fontaine, 2018, 4.5.3] the kernel of \( pr_K \) can be easily seen to be the Tate module \( T_K(\mathcal{G}) \) and the result is established. \( \square \)
§ 10.19 Now will make some assumptions about \( f \). Assume \( f(\mathbb{C}[\ell] - \{O\}) \subseteq \mathfrak{m}_{K_y} \) for all \( y \in \mathcal{Y}_{F,E} \). In context of [Mochizuki, 2021a,b,c,d] this automatically satisfied as \( \theta \)-values are positive powers of the Tate parameters and hence have absolute value less than one. I will redefine \((f)_{X,\ell}\) as follows. Let

\[
(10.19.1) \quad (f)_{X,\ell} = \left\{ z \in \mathcal{G}(\mathcal{O}_{C_p}) \simeq B^{e=p} : \exists y \in [\mathcal{Y}_{F,E}], \text{pr}_{K_y}(z) \in f(\mathbb{C}[\ell] - \{O\}) \right\}
\]

be the set union (over \( y \in \mathcal{Y}_{F,E} \)) of all the inverse images of \( f(\mathbb{C}[\ell] - \{O\}) \) under the projection \( \text{pr}_{K_y} : \mathcal{G}(\mathcal{O}_E) \to \mathcal{G}(K_y) \) of Proposition 10.18.1. I will refer to \((f)_{X,\ell} \subseteq B^{e=p}\) as the \( \ell \)-value locus of \( f \) in \( B^{e=p} \). Note that by definition if \( z \in (f)_{X,\ell} \) and \( t \in T_{K_y}(\mathcal{G}) \), then \( z + t \in (f)_{X,\ell} \) and \( z + T_{K_y}(\mathcal{G}) \). So \((f)_{X,\ell}\) is a union of \( T_{K_y}(\mathcal{G}) \)-torsors.

§ 10.20 The advantage of working with \( B^{e=p} \) now becomes clear: in the Banach space \( B^{e=p} \), lifts of the \( \theta \)-value sets (i.e. inverse images under \( B^{e=p} \to \mathcal{G}(\mathcal{O}_{K_y}) \)) are defined up to elements of the one dimensional \( \mathcal{O}_E \)-module \( T_y \) (i.e. indeterminacies of \( \theta \)-values arise live in \( T_y \) and arise from the torsion of the Lubin-Tate group). In the \( W_{\mathcal{O}_E}(\mathcal{G}) \) description the kernel of \( \eta_{K_y} \) is an (uncountably) infinite dimensional \( E \)-vector space.

§ 10.21 To further understand the importance of working with a small kernel like \( T_y \), let me remark that in Mochizuki’s Theory multiplicative theory the analog of \( T_y \) is a choice of a distinct Kummer theory and the \( T_y \) torsors constructed above as the additive analogs of Mochizuki’s \( \theta \)-value monoids built using different Kummer theories in [Mochizuki, 2021a,b,c,d]. If \( y_1 \neq y_2 \) in \( \mathcal{Y}_{F,E} \) then the subspaces \( T_{y_1} \) and \( T_{y_2} \) of \( B^{e=p} \) are distinct. So the \((f)_{X,\ell}\) is the collection of all \( \theta \)-value monoids for all the Kummer theories parameterized by \( y \in \mathcal{Y}_{F,E} \).

§ 10.22 So one considers all lifts of \( \theta \)-values and allow \( K_y \) to move with \( y \in \mathcal{Y}_{F,E} \), and one obtains the \textit{locus of \( \theta \)-values} in \( B^{e=p} \) which represents all the possible lifts of the \( \theta \)-values in \( K_y \) for all \( y \in \mathcal{Y}_{F,E} \). So that in the theory proposed here, the \( E \)-Banach space \( B^{e=p} \) serves as the Log-shell. In [Mochizuki, 2021a,b,c,d] all the local calculations take place in Mochizuki’s Log-shell (see [Hoshi, 2019]). One advantage of working with \( B^{e=p} \) is that one can compare valuations in one fixed location in a single, fixed ordered group \( \mathbb{R} \).

§ 10.23 Note that because the norms \(|-|_\rho \) (for \( \rho \in \mathbb{Q}[0,1] \)) on \( B^{e=p} \) are non-archimedean. So one sees that for general lifts \( z_1, z_2 \in (f)_{X,\ell} \) one can have \(|z_1|_\rho > |z_2|_\rho \) and so

\[
|(f)_{X,\ell}|_\rho = \sup \left\{ |z|_\rho : z \in (f)_{X,\ell} \right\}
\]

may exceed the norm of a chosen element. The problem of course is to demonstrate that \(|(f)_{X,\ell}|_\rho \leq \infty \) and non-trivial.

§ 10.24 Let \( f_\theta \) be a certain \( \theta \)-function on \( X \) which will chosen later and in accordance with the choice in [Mochizuki, 2009] for compatibility with [Mochizuki, 2021a,b,c,d]. Let us apply the above discussion to \( f_\theta \) i.e. I redefine the \( \theta \)-torsion locus using lifts into \( B^{e=p} \). Write

\[
(10.24.1) \quad \tilde{\Theta}_{X,\ell} = (f_\theta)_{X,\ell}
\]

and call this the \( \theta \)-torsion locus of \( X \) in \( \mathcal{G} \mathcal{O}_{C_p} \). Thus, explicitly, one has

\[
(10.24.2) \quad \tilde{\Theta}_{X,\ell} = \left\{ z \in B^{e=p} \simeq \mathcal{G}(\mathcal{O}_{C_p}) : \text{pr}_{K_y}(z) \in f_\theta(\mathbb{C}[\ell] - \{O\}) \subseteq K_y \text{ for some } y \in \mathcal{Y}_{F,E} \right\}
\]
§ 10.25 Before proceeding it will be useful to remind oneself of the following added advantage of working with $B^{p=e}$: for every $\rho \in \mathbb{R}$, the norms $|-|_{\rho}$ on $B$ induce the same topology on $B^{p=e}$ ([Fargues and Fontaine, 2018, Proposition 4.1.5]). Of course, the norms need not be all identical, but all the topologies induced by the norms coincide.

§ 10.26 In summary one has the following properties of $\tilde{\Theta}_{X,\ell}$:

1. $\tilde{\Theta}_{X,\ell} \subset B^{p=e}$,

2. For every $z \in \tilde{\Theta}_{X,\ell}$, there exists a closed point of degree one $y \in \mathcal{Y}_{F,E}$, and $x \in C[\ell] - \{O\}$ such that $pr_{K_y}(z) = f_\ell(x) \in K_y$; moreover, $z + T_y \subset \Theta_{X,\ell}$.

3. For each $z, y$ as above, the sets $z + T_y$ are the additive analogs of the multiplicative monoids considered by Mochizuki (in [Mochizuki, 2021a,b,c,d]).

4. Suppose $z_1, z_2 \in \tilde{\Theta}_{X,\ell}$ and suppose that for some $y \in \mathcal{Y}_{F,E}$, one has $pr_{K_y}(z_1) = pr_{K_y}(z_2)$, then the difference $z_1 - z_2 \in T_y \simeq T_p(\mathcal{G})(K_y)$.

In particular the following are well-defined:

\[(10.26.1) \quad \text{diam}(\tilde{\Theta}_{X,\ell}) = \sup \left\{ |z_1 - z_2|_\rho : z_1, z_2 \in \tilde{\Theta}_{X,\ell}, \rho \in \mathbb{R} \cup \{\infty\} \right\} \in \mathbb{R} \cup \{\infty\} \]

\[(10.26.2) \quad \left| \tilde{\Theta}_{X,\ell} \right|_\rho = \sup \left\{ |z|_\rho : z \in \tilde{\Theta}_{X,\ell} \right\} \in \mathbb{R} \cup \{\infty\} \]

The above discussion demonstrates that it makes perfect sense to talk about $\text{diam}(\tilde{\Theta}_{X,\ell})_\rho$ and $\left| \tilde{\Theta}_{X,\ell} \right|_\rho$ in arithmetic Teichmuller Theory presented here.

§ 10.27 In [Mochizuki, 2009, Section 1] described certain Galois cohomology classes associated with a certain theta-function. I want to briefly recall these ideas in order to relate them to the ideas considered here. For this paragraph, $C/E$ is an elliptic curve with semi-stable reduction and $X = C - \{O\}/E$ is the standard elliptic cyclops over $E$. Let $\pi_{X/E}^{\text{temp}}$ be the tempered fundamental group of $X/E$. Suppose $W \to X$ is a finite étale cover of $X$. A point $w \in W$ lying over the point $O \in C$ in a smooth proper compactification of $W$ will be called a cusp of $W$. Fix an identification of the decomposition group $D_w \subset \pi_{X/E}^{\text{temp}}$ with $D_w \simeq G_E$. Let $(X/E, E \hookrightarrow K) \in \mathcal{J}(X, E)$. Let $\pi = \pi_{X/E}^{\text{temp}}$ be the tempered fundamental group of $(X/E, E \hookrightarrow K)$ and let $\overline{\pi}_K := \overline{\pi}_{X/E}^{\text{temp}} = \pi_{X/K}^{\text{temp}}$ be the tempered fundamental group of the analytic space $X_{K}^{an} / K$ (this is the geometric tempered fundamental (sub) group of $\pi$). As mentioned in Section 3, $\overline{\pi}_{X/E}^{\text{temp}}$ fits into the usual exact sequence

$$1 \to \overline{\pi}_K \to \pi \to G_E \to 1$$

where $\overline{E} \subset K$ is the algebraic closure of $E$ computed in the algebraically closed $K$. Note that from the perspective of this paper (and Theorem 3.9.1), there are many analytic spaces which provide the group $\overline{\pi}$ namely $X_{K}^{an}$ for every algebraically closed perfectoid field $K$ and it thus
makes sense to label the copies of $\Pi$ using the label $(X/E, E \hookrightarrow K)$. I will simply label copies by the subscript $K$ for notational simplicity. One has the usual filtrations by subgroups

$$\Pi \supseteq [\Pi, \Pi] \supseteq [\Pi, [\Pi, \Pi]] \supseteq \cdots$$

and similarly for

$$\Pi_K \supseteq [\Pi_K, \Pi_K] \supseteq [\Pi_K, [\Pi_K, \Pi_K]] \supseteq \cdots$$

and the abelianization of $\Pi^b_K$ fits into an exact sequence:

$$1 \to \hat{\mathbb{Z}}(1)_K \to \Pi^b_K \to \hat{\mathbb{Z}}_K \to 0.$$ 

The quotient group $\Delta_K = [\Pi_K, \Pi_K]/[\Pi_K, [\Pi_K, \Pi_K]]$ may be identified with $\hat{\mathbb{Z}}(1)_K$. Let

$$\theta(q, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}h(n+1)} z^{2n+1}$$

where $q$ is the Tate parameter of $C/E$, considered as a $\theta$-function on $X/E$. If $W/E'$ is a finite cover of $X/E$ (which is to be chosen) then one may regard $\theta(q, z)$ as a function on $W$.

In [Mochizuki, 2009, Proposition 1.4] Mochizuki constructs a cohomology class (defined up to a $\mathcal{O}_{E'}$-multiple)

$$\eta_\theta \in H^1(\Pi^{temp}_{W/E'}, \Delta_K) \simeq H^1(\Pi^{temp}_{W/E'}, \hat{\mathbb{Z}}(1)_K),$$

which correspond to the theta-function chosen above. Especially under the “Galois evaluation” i.e. restriction of this class to section $s_w : G_E \hookrightarrow \Pi^{temp}_{W/E;K}$ given by a closed point of $w \in W$, is the value $\xi = \theta(q, z)|_{z=w} \in K$ as a function on $W^p_K$ at this point $w$ in the conventional sense of evaluation of a function at a point. There is also a similar description for cusps of $W$. As I have established earlier $|\xi|_K$ changes as $K$ varies.

This allows us to work with $\theta$-values and notably this ensures compatibility with the formalism of [Mochizuki, 2009].

**Remark 10.27.1.** As far as I understand in [Mochizuki, 2021a,b,c,d], the variation of the theta values $\xi$ (as above) is produced by means of Mochizuki’s Anabelian Reconstruction Algorithms [Mochizuki, 2012, 2013, 2015]. In this paper the variation of the data $\xi$ arises because of the variation of $(X/E, E \hookrightarrow K) \in \mathfrak{J}(X, E)$ i.e. from the existence of the Arithmetic Teichmuller space.

## 11 Self similarity of $\mathbb{C}^b_p$ and its consequences

### § 11.1

Let me begin with the following reformulation of an important result of [Matignon and Reversat, 1984, Théorème 2 and §3 Remarque 2] (and also [Kedlaya and Temkin, 2018]).

**Theorem 11.1.1.** Let $p$ be any prime number, $\mathbb{C}_p = \hat{\mathbb{Q}}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{C}^b_p$ be the tilt of $\mathbb{C}_p$. Then

(1) There exists an isomorphism $\mathbb{C}^b_p \simeq \mathbb{F}_p((x))$.

(2) There exists $y \in \mathbb{F}_p((x))$ such that $\mathbb{F}_p((y)) \subsetneq \mathbb{F}_p((x))$ (more precisely $x \not\in \mathbb{F}_p((y))$).
§ 11.2 By a labeled copy of $C_p$ will mean an identification $C_p \simeq \mathbb{F}_p((t))$ for some variable $t$. I will write $C_{p,t}$ for a copy of $C_p$ labeled by the variable $t$.

§ 11.3 A fundamental consequence of Theorem 11.1.1 is the following:

**Theorem 11.3.1.** The Fargues-Fontaine curve $\mathcal{Y}_{C_p,\mathbb{Q}_p}$ is a self-similar curve. More precisely, for every pair of elements $x, y \in C_p$ as in Theorem 11.1.1, there exists infinitely many strict inclusions

$$|\mathcal{Y}_{C_{p,y},\mathbb{Q}_p}| \hookrightarrow |\mathcal{Y}_{C_{p,x},\mathbb{Q}_p}|$$

arising from the strict isometric inclusions $C_{p,y} \hookrightarrow C_{p,x}$.

**Proof.** Since $|\mathcal{Y}_{C_{p,y},\mathbb{Q}_p}|$ is identified by [Fargues and Fontaine, 2018, Théorème 2.4.1 and Corollaire 2.4.2] with the set of primitive degree one elements of $W(\mathcal{O}_{C_p})$. By [Fargues and Fontaine, 2018, Corollaire 2.2.9] any primitive element of degree one in $W(\mathcal{O}_{C_p})$ can be written, up to multiplication by a unit in $W(\mathcal{O}_{C_{p,y}})$, as

$$[\alpha] - p,$$

for some element $\alpha \in \mathcal{O}_{C_{p,y}}$ with $v(\alpha) > 0$ and $\mathcal{Y}_{C_{p,y},\mathbb{Q}_p} \hookrightarrow \mathcal{Y}_{C_{p,x},\mathbb{Q}_p}$ is given by sending the primitive element $[\alpha] - p \in W(\mathcal{O}_{C_{p,y}})$ to the primitive degree one element $[\alpha] - p \in W(\mathcal{O}_{C_{p,x}})$ and at the level of ideals

$$([\alpha] - p)W(\mathcal{O}_{C_{p,y}}) \longmapsto ([\alpha] - p)W(\mathcal{O}_{C_{p,x}}).$$

One has

$$W(\mathcal{O}_{C_{p,y}}) \subset W(\mathcal{O}_{C_{p,x}}),$$

as $x \notin C_{p,y}$ so $[x] - p \notin W(\mathcal{O}_{C_{p,y}})$ and hence there is a primitive element of degree one of $W(\mathcal{O}_{C_{p,x}})$ which is not contained in the set of primitive elements of degree one of $W(\mathcal{O}_{C_{p,y}})$. So the inclusion of $|\mathcal{Y}_{C_{p,y},\mathbb{Q}_p}| \hookrightarrow |\mathcal{Y}_{C_{p,x},\mathbb{Q}_p}|$ is strict. This proves the assertion. \qed

§ 11.4 Before proceeding it will be useful to understand this self-similarity in terms of Classical Teichmüller Theory. In the classical Teichmüller Theory (i.e. Teichmüller Theory for Riemann Surfaces), the Teichmüller space is tiled by isomorphs of a fundamental domain for the mapping class group or modular group actions. To put it differently the Teichmüller space is equipped with a self-similar tiling (not unique in general).
§ 11.5 So the question arises if there is a theory of fundamental domains in the Arithmetic Teichmüller Theory constructed here. The answer to this question is yes. There is a notion of fundamental domains in Arithmetic Teichmuller Theory and this arises precisely from the fact that that \( \mathbb{C}_p \) is a self-similar valued field i.e. \( \mathbb{C}_p \) (see Theorem 11.1.1 and Theorem 11.3.1).

§ 11.6 Let \( E \) be a \( p \)-adic field and let \( X/E \) be a smooth, geometrically connected, smooth, quasi-projective variety. Let \( \mathfrak{J}^{\text{arith}}(X, E) \) be the category of \( \mathfrak{J}(X, E) \)

\[
\mathfrak{J}^{\text{arith}}(X, E) = \mathfrak{J}(X, E)_{\mathbb{C}_p}
\]

§ 11.7 A fundamental domain for \( \mathfrak{J}^{\text{arith}}(X, E) \) is the full subcategory \( \mathfrak{J}^{\text{arith}}(X, E)_{\mathbb{C}_p,x} \) for some variable \( x \). The following is now a tautology:

Theorem 11.7.1. Let \( X \) be a geometrically connected, smooth, quasi-projective variety over a \( p \)-adic field \( E \).

1. For any non-archimedean prime \( p \), any triple \( (Y/E', E' \hookrightarrow K) \in \mathfrak{J}^{\text{arith}}(X, E) \) belongs to some fundamental domain \( \mathfrak{J}(X, E)_{\mathbb{C}_p,t} \).
2. The identification \( y = x \) (of variables) provides a tautological equivalence of categories

\[
\mathfrak{J}^{\text{arith}}(X, E)_{\mathbb{C}_p,y} \simeq \mathfrak{J}^{\text{arith}}(X, E)_{\mathbb{C}_p,x}.
\]
3. In particular \( \mathfrak{J}^{\text{arith}}(X, E) \) is tiled with copies of the fundamental domains.

§ 11.8 The self-similarity of \( \mathbb{C}_p \) discussed in Section 11 at once implies that the group \( \mathcal{G}(\mathcal{O}_{\mathbb{C}_p}) \) is self-similar and this in turn has the following important consequence whose proof is clear from Theorem 11.1.1 and the preceding definitions and discussion:

Theorem 11.8.1. Assume \( F = \mathbb{C}_p \). Then

1. \( \tilde{\Theta}_{X,\ell} \subset B^{v=p} \simeq \mathcal{G}(\mathcal{O}_F) \).
2. The set \( \tilde{\Theta}_{X,\ell} \) is a self-similar subset of \( B^{v=p} \): more precisely let

\[
\tilde{\Theta}_{X,\ell,t} = \{ z \in \mathcal{G}(\mathcal{O}_F) : \eta_K(z) \in f_\theta(C[\ell] - \{O\}) \subset K \text{ for some perfectoid field with } K^\flat = \mathbb{C}_p \}
\]

Then

\[
\tilde{\Theta}_{X,\ell} = \bigcup_{\mathbb{C}_p,t} \tilde{\Theta}_{X,\ell,t}
\]

where the union runs over all the (isometric) identifications \( \mathbb{C}_p = \mathbb{C}_p \).
3. To put it colloquially, the \( \theta \)-torsion value sets \( \tilde{\Theta}_{X,\ell} \) form a fractal in \( B^{v=p} \simeq \mathcal{G}(\mathcal{O}_F) \).

Proof. After Theorem 11.1.1 it is enough to note that any \( z \in \tilde{\Theta}_{X,\ell} \) lives in some \( \tilde{\Theta}_{X,\ell,t} \) and so the assertion is immediate.
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