WEYL GROUPS AND CLUSTER STRUCTURES OF FAMILIES OF LOG CALABI-YAU SURFACES

YAN ZHOU

Abstract. Given a generic Looijenga pair \((Y, D)\) together with a toric model \(\rho : (Y, D) \to (\overline{Y, \overline{D}})\), one can construct a seed \(s\) such that the corresponding \(X\)-cluster variety \(X_s\) can be viewed as the universal family of the log Calabi-Yau surface \(U = Y \setminus D\). In cases where \((Y, D)\) is positive and \(X_s\) is not acyclic, we describe the action of the Weyl group of \((Y, D)\) on the scattering diagram \(D_s\). Moreover, we show that there is a Weyl group element \(w\) of order 2 that either agrees with or approximates the Donaldson-Thomas transformation \(DT_{X_s}\) of \(X_s\). As a corollary, \(DT_{X_s}\) is cluster. In positive non-acyclic cases, we also apply the folding technique as developed in [YZ] and construct a maximally folded new seed \(s'\) from \(s\). The \(X\)-cluster variety \(X_{s'}\) is a locally closed subvariety of \(X_s\) and corresponds to the maximally degenerate subfamily in the universal family. We show that the action of the special Weyl group element \(w\) on \(D_s\) descends to \(D_{s'}\) and permutes distinct subfans in \(D_{s'}\), generalizing the well-known case of the Markov quiver.

1. Introduction

1.1. Background. In the 90’s, Lusztig did seminal work on total positivity in reductive groups and canonical bases in quantum groups ([Lu90, Lu94]). Cluster algebras ([FZ02]) were invented by Fomin and Zelevinsky in 2000 as an algebraic and combinatorial framework to study total positivity and Lusztig’s dual canonical bases in semisimple Lie groups. Remarkably, cluster structures also appear in many seemingly unrelated areas in mathematics like Teichmüller theory, Poisson geometry, quiver representations, quantum field theory, discrete dynamical systems, combinatorics..., and thus serve as potential bridges between these various areas.

In 2014, Gross, Hacking, Keel and Kontsevich in [GHKK18] proposed a new framework to understand cluster algebras via mirror symmetry. Mirror symmetry originated from string theory and reveals surprising duality between complex geometry and symplectic geometry on pairs of Calabi-Yau varieties that are mirror dual to each other. Calabi-Yau varieties are complex manifolds with a holomorphic volume form that is unique up to scaling. Cluster varieties, varieties obtained by gluing algebraic tori together via a special class of birational maps called mutations which are encoded by cluster combinatorics, are examples of open Calabi-Yau varieties called log Calabi-Yau varieties.

Using combinatorial gadgets from enumerative geometry like scattering diagrams and broken lines, Gross, Hacking, Keel and Kontsevich construct canonical bases for cluster algebras consisting of elements called theta functions. This new framework suggests that cluster varieties from representation theory, like double Bruhat cells, Grassmanians, base affine spaces, should give canonical bases of representations intrinsic to the geometry of their mirror dual varieties. Indeed, the geometric realization of broken lines and theta functions are done in [KY] for smooth log Calabi-Yau varieties containing a Zariski open dense torus, which include the affine closure of double Bruhat cells.

As we expect that canonical bases are intrinsic to the geometry of log Calabi-Yau varieties and should be independent of cluster structures, a natural question arises: Does
there exist a cluster variety with two non-equivalent cluster structures, and yet either one yields the same canonical basis? For precisely what we mean by ‘non-equivalent’ cluster structures, see Definition 4.11. In [YZ], we made a first step towards a positive answer to such question - Yes, there exists a cluster variety with two non-equivalent cluster structures.

The idea of [YZ] is to use distinct subfans in scattering diagrams to construct non-equivalent cluster structures. The standard cluster structure of a cluster variety corresponds to a subfan structure in the scattering diagram called the cluster complex. When the Donaldson-Thomas transformation (DT) [GS16] of a cluster variety is rational but not cluster, i.e., not a composition of a sequence of cluster mutations, the scattering diagram will have at least two distinct subfans and they will provide, at the very least, two non-equivalent cluster atlases (Definition 4.11). If we further show that these two cluster atlases agree up to codimension, then we get an example of a cluster variety with two non-equivalent cluster structures.

The goal of this paper is to construct a class of cluster varieties with non-equivalent cluster atlases generalizing the example we give in [YZ], using the geometry of Looijenga pairs [LS1] and the folding techniques we developed in [YZ] in the context of scattering diagrams.

1.2. Looijenga pairs and rank 2 cluster varieties. A Looijenga pair $(Y, D)$ is a smooth projective surface $Y$ together with a connected nodal singular curve $D \in | -K_Y |$. By the adjunction formula, $p_a(D) = 1$. So $D$ is either an irreducible rational curve with a single node or a cycle of smooth rational curves. Since $K_Y + D$ is trivial, the complement $U = Y \setminus D$ is a log Calabi-Yau surface. Write $D = D_1 + \cdots + D_r$ where $D_i (1 \leq i \leq r)$ is irreducible. Let

$$D^\perp := \{ \alpha \in \text{Pic}(Y) \mid \alpha \cdot [D_i] = 0 \text{ for all } i \}$$

and

$$T_{(D^\perp)^\ast} := (D^\perp)^\ast \otimes \mathbb{C}^\ast.$$ 

By the Global Torelli Theorem for Looijenga pairs (cf. Theorem 1.8 of [GHK12]), we could view $T_{(D^\perp)^\ast}$ as the period domain of the Looijenga pair $(Y, D)$. We say $(Y, D)$ is positive if the intersection matrix given by $(D_i \cdot D_j)$ is not negative semidefinite. We say a Looijenga pair $(Y, D)$ is generic if $Y$ does not contain a smooth rational curve with self-intersection $-2$ and disjoint from $D$. In this paper, unless we say otherwise, we assume that our Looijenga pair is generic. We say a birational morphism $\rho : (Y, D) \to (\overline{Y}, \overline{D})$ is a toric model for $(Y, D)$ if $\overline{Y}$ a smooth projective toric surface and $D$ is the strict transform of the toric boundary $\overline{D}$ of $\overline{Y}$.

By rank 2 cluster varieties, we mean cluster varieties whose seeds have exchange matrices of rank 2. In [GHK15], the relationship between Looijenga pairs and rank 2 cluster varieties is explored. Given a generic Looijenga pair $(Y, D)$ together with a toric model $\rho : (Y, D) \to (\overline{Y}, \overline{D})$, one can construct a skew-symmetric cluster seed $\mathbf{s}$ using the combinatorial data of the exceptional curves of the toric model. Let $K$ be the kernel of the skew-symmetric form of $\mathbf{s}$. This kernel $K$ has codimension 2 and therefore the corresponding $\mathcal{X}$-cluster variety $\mathcal{X}_\mathbf{s}$ is a rank 2 cluster variety. By Theorem 5.5 of [GHK15], there is an isomorphism $K \cong D^\perp$. The space $\mathcal{X}_\mathbf{s}$ fibers over $T_K^\ast \cong T_{(D^\perp)^\ast}$ and the generic fibers in the family $\mathcal{X}_\mathbf{s} \to T_{(D^\perp)^\ast}$, up to codimension 2, are log Calabi-Yau surfaces deformation equivalent to $U := Y \setminus D$ (see subsection 2.3 and section 4 of [GHK15] for more details). Moreover, we can view $\mathcal{X}_\mathbf{s}$ as the universal family of $U$ (cf. Theorem 2.10 and Theorem 5.5 of [GHK15]). This relationship goes the other way, that is, we could start with a skew-symmetric seed $\mathbf{s}$ of rank 2 and construct the corresponding Looijenga pair. To get
rid of the skew symmetric condition, we could allow singularities for our Looijenga pairs. Given these established relationships between Looijenga pairs and rank 2 cluster varieties, the main theme of this paper is to understand the theory of rank 2 cluster varieties from the geometry of Looijenga pairs. As we will see later, the very first application along this line is the well-known example of cluster varieties associated to the Markov quiver, the main example we explored in [YZ].

Since the theory of acyclic cluster varieties is well-understood, in this paper, we will focus on the cases where the corresponding rank 2 cluster variety associated to a Looijenga pair \((Y, D)\) is not acyclic, or equivalently, where \(D^\pm \simeq D_n (n \geq 4)\) or \(E_n (n = 6, 7, 8)\) (cf. Proposition 2.11). In positive non-acyclic cases, we also denote the corresponding \(\mathcal{X}\)-cluster variety by \(\mathcal{X}_{D^\pm}\) when we want to emphasize the particular deformation type of \(U = Y \setminus D\).

1.3. Main results. Let \(\mathcal{D}_s\) be the scattering diagram associated to a seed \(s\) constructed from a toric model for a positive, non-acyclic Looijenga pair. A natural question arises: Does the action of the Weyl group of \((Y, D)\) on \(\mathcal{X}_s\) preserve the cluster complex in \(\mathcal{D}_s\)? The answer is positive (Proposition 4.2): The Weyl group acts faithfully on the scattering diagram as a subgroup of the cluster modular group and in particular, preserves the cluster complex.

The action of the Weyl group on \(\mathcal{D}_s\) enables us to have a simple, geometric understanding of the Donaldson-Thomas transformation \(DT_{\mathcal{X}_{D^\pm}}\) of \(\mathcal{X}_{D^\pm}\) (for details of the Donaldson-Thomas transformations of cluster varieties, see [GS16]). In particular, in subsection 4.2 and subsection 4.3 we show that there exists a special element \(w\) of order 2 in the Weyl group, such that \(w\) either agrees with or approximates \(DT_{\mathcal{X}_Y}\). Thus, we prove the following theorem:

**Theorem 1.1.** (Theorem 4.3, Theorem 4.7) The Donaldson-Thomas transformation of \(\mathcal{X}_{D^\pm}\) for \(D^\pm \simeq D_n (n \geq 4)\) or \(E_n (n = 6, 7, 8)\) is cluster, that is, it can be factored as compositions of cluster transformations.

Another main goal of this paper is to apply the folding technique we developed in [YZ] to families of log Calabi-Yau surfaces. Let \(s_0\) be a general initial seed, not necessarily constructed from a Looijenga pair. Suppose \(s_0\) has certain symmetry (cf. subsection 3.3), then recall that in [YZ] we showed that there is a quotient construction called folding that produces a new seed \(\overline{s}_0\). Moreover, this quotient construction extends to the level of scattering diagrams and we can recover \(\overline{\mathcal{D}_{s_0}}\) from \(\mathcal{D}_{s_0}\). The cluster complex \(\Delta_{s_0}^\pm\) in \(\mathcal{D}_{s_0}\), when restricted to \(\overline{\mathcal{D}_{s_0}}\) can break into distinct subfans. Denote by \(\overline{\Delta_{s_0}^\pm}\) the restriction of \(\Delta_{s_0}^\pm\) in \(\overline{\mathcal{D}_{s_0}}\). In particular, \(\overline{\Delta_{s_0}^\pm}\) always contains the cluster complex \(\overline{\Delta_{s_0}^-}\). Denote by \(A_{\text{prin}, s_0}\) (resp. \(A_{\text{prin}, \overline{s}_0}\)) the cluster variety of \(A_{\text{prin}}\)-type associated to \(s_0\) (resp. \(\overline{s}_0\)). If two subfans in \(\overline{\Delta_{s_0}^\pm}\) are distinct, i.e., they have trivial intersection, the wall-crossing between them will be non-cluster, but we can nevertheless build a variety \(A_{\text{prin}, \overline{\Delta_{s_0}^\pm}}\) using all chambers in \(\overline{\Delta_{s_0}^\pm}\).

In cases where \((Y, D)\) is positive non-acyclic, if \(s\) is the seed constructed from \((Y, D)\), there is a way to maximally fold \(s\) to a new seed \(\overline{s}\). The \(\mathcal{X}\)-cluster variety \(\mathcal{X}_{\overline{s}}\) corresponds to the maximally degenerate subfamily of the universal family of \(U = Y \setminus D\) (cf. subsection 6.3). The action of the special Weyl group element \(w\) we use to factorize \(DT\) into cluster transformations restricts to an action on the scattering diagram \(\overline{\mathcal{D}_{\overline{s}}}\) associated to \(\overline{s}\) and permutes distinct subfans in \(\overline{\mathcal{D}_{\overline{s}}}\).

\(^1\)See Remark 4.6 for the difference between \(w\) and \(DT_{\mathcal{X}_{s_0}}\) when \(n \geq 5\) and Appendix A for the difference between \(w\) and \(DT_{\mathcal{X}_{\overline{s}_0}}\).
Theorem 1.2. (Theorem [4,12]) The scattering diagram $D_s$ has two distinct subfans corresponding to $\Delta^+_s$ and $\Delta^-_s$ respectively. The action of Weyl group element $w$ on $D_s$ descends to $D_s$ and interchanges $\Delta^+_s$ and $\Delta^-_s$. In particular, if we build a variety $\tilde{A}_{\text{prin},s}$ using both $\Delta^+_s$ and $\Delta^-_s$, then $\Delta^+_s$ and $\Delta^-_s$ provide non-equivalent atlases of cluster torus charts for $\tilde{A}_{\text{prin},s}$.

When $D^\perp \simeq D_4$, $s$ is the seed given by the ideal triangulation of 4-punctured sphere and $\pi$ is the seed given by the ideal triangulation of 1-punctured torus and the corresponding quiver is the Markov quiver. Thus, we recover the main example we investigated in [YZ].

In the future, it will be interesting to further investigate whether the two atlases provided by $\Delta^\pm_s$ agree up to codimension 2 and yield the same canonical basis.

Acknowledgements. First I want to thank my advisor Sean Keel for suggesting the initial direction of this project. I also want to thank Travis Mandel for clarifying my confusion about rank 2 cluster varieties and for his numerous helpful comments on the initial draft of this paper.

2. Preliminaries of Looijenga pairs

Throughout this paper, we work over the field $\mathbb{C}$.

Definition 2.1. A Looijenga pair $(Y,D)$ is a smooth projective surface $Y$ with a connected singular nodal curve $D \in |-K_Y|$.

By adjunction formula, the arithmetic genus of $D$ is 1. So $D$ is either an irreducible genus 1 curve with a single node or a cycle of smooth rational curves. In this paper, we only consider cases where $D$ is a cycle of smooth rational curves with at least 3 irreducible components.

Let $U = Y \setminus D$. Then $U$ is a log Calabi-Yau surface, that is, there exists a canonical holomorphic volume $\omega$, unique up to scaling, such that for any normal crossing compactification $U \subset Y'$, $\omega$ has at worst a simple pole along each irreducible component of the boundary $D' = Y' \setminus U$.

Definition 2.2. Let $(Y,D)$ be a Looijenga pair. Given a smooth projective toric surface $\overline{Y}$ together with its toric boundary $\overline{D} := Y \setminus \mathbb{G}_m^2$, i.e., $\overline{D}$ is the union of toric divisors of $\overline{Y}$, we say that $(\overline{Y},\overline{D})$ is a toric model for $(Y,D)$ if there exists a birational morphism $\rho : Y \to \overline{Y}$ such that $\rho$ is a blowup at finite number of points in the smooth locus of $\overline{D}$ (we allow infinitely near points) and $D$ is the strict transform of $\overline{D}$. We say $(Y',D')$ is a simple toric blowup of $(Y,D)$ if $Y'$ is the blowup of $Y$ at a nodal point of $D$ and $D'$ is the reduced inverse image of $D$. A toric blowup of $(Y,D)$ is a composition of simple toric blowups.

The following lemma shows that each Looijenga pair has a very elementary construction via a toric model:

Lemma 2.3. (Proposition 1.3 in [GHK11]) Given a Looijenga pair $(Y,D)$, there exists a toric blowup $(Y',D')$ of $(Y,D)$ such that $(Y',D')$ has a toric model.

Since a toric blowup does not change the log Calabi-Yau surface we are looking at, throughout this paper, we assume that a Looijenga pair has a toric model.

Definition 2.4. Given a Looijenga pair $(Y,D)$, an orientation of $D$ is an orientation of the dual graph of $D$. 

4
For the rest of this paper, we always fix an orientation of $D$ and an ordering $D = D_1 + D_2 + \cdots + D_r (r \geq 3)$ of the irreducible components of $D$ compatible with our choice of the orientation.

**Definition 2.5.** Given a Looijenga pair $(Y, D)$ where $D = D_1 + D_2 + \cdots + D_r$, we say $(Y, D)$ is positive if the intersection matrix $(D_i \cdot D_j)$ is not negative semidefinite.

2.1. Period mappings. By Lemma 2.1 of [GHK12], an orientation of $D$ canonically identifies $\text{Pic}^0(D)$ with $\mathbb{G}_m$.

**Definition 2.6.** Let
\[
D^\perp := \{ \alpha \in \text{Pic}(Y) \mid \alpha \cdot [D_i] = 0 \text{ for all } i \}
\]
and
\[
T_{(D^\perp)^*} := \text{Hom} \left( D^\perp, \text{Pic}^0(D) \right) \simeq \text{Hom} \left( D^\perp, \mathbb{G}_m \right).
\]
We call the following homomorphism $\phi_Y \subset T_{(D^\perp)^*}$ the period point of $(Y, D)$:
\[
\phi_Y : D^\perp \rightarrow \text{Pic}^0(D) \simeq \mathbb{G}_m
\]
\[
L \mapsto L \mid_D.
\]
Recall that $U = Y \setminus D$ has a canonical volume form $\omega$, unique up to scaling, with simple poles along $D$. According to the theory of extensions of mixed Hodge structures as developed in [Ca85], the mixed Hodge structure on $H^2(U)$ is classified by periods of $\omega$ over cycles in $H_2(U, \mathbb{Z})$. By Proposition 3.12 of [F], the data of periods of $\omega$ over cycles in $H_2(U, \mathbb{Z})$ are equivalent to the homomorphism $\phi_Y$. Hence the naming “period point” of $\phi_Y$ is justified and $\phi_Y$ describes the mixed Hodge structure of $U$. See Theorem 1.8 of [GHK12] for the full statement of the Global Torelli Theorem for Looijenga pairs.

For the description of moduli stacks of Looijenga pairs via period domains, see section 6 of [GHK12].

2.2. Admissible groups and Weyl groups. Given a Looijenga pair $(Y, D)$, the cone
\[
\{ x \in \text{Pic}(Y)_{\mathbb{R}} \mid x^2 > 0 \}
\]
has two connected components, let $C^+$ be the component containing all ample classes. Given an ample class $H$ in $\text{Pic}(Y)$, let
\[
\tilde{M} := \{ E \in \text{Pic}(Y) \mid E^2 = K_Y \cdot E = -1 \text{ and } E \cdot H > 0 \}
\]
By Lemma 2.13 of [GHK12], $\tilde{M}$ is independent of the ample class we choose. Let $C^{++}$ be the subcone in $C^+$ defined by the inequalities $x \cdot E \geq 0$ for all $E$ in $\tilde{M}$.

**Definition 2.7.** Given a Looijenga pair $(Y, D)$, we define the admissible group of $(Y, D)$, denoted by $\text{Adm}(Y, D)$, to be the subgroup of the automorphism group of the lattice $\text{Pic}(Y)$ preserving the boundary classes $[D_i]$ and the cone $C^{++}$.

**Definition 2.8.** Let $(Y, D)$ be a Looijenga pair.

1. An internal $(-2)$-curve of $(Y, D)$ is a smooth rational curve on $Y$ of self-intersection $(-2)$ and disjoint from $D$.
2. We say $(Y, D)$ is generic if it has no internal $(-2)$-curves.

**Definition 2.9.** Let $(Y, D)$ be a Looijenga pair.

1. We say a class $[\alpha]$ in $\text{Pic}(Y)$ is a root if there is a family $(Y, D) / S$, a path $\gamma : [0, 1] \rightarrow S$ and identifications
\[
(Y, D) \simeq (Y_{\gamma(0)}, D_{\gamma(0)}) \quad \text{and} \quad (Y', D') \simeq (Y_{\gamma(1)}, D_{\gamma(1)})
\]
such that the isomorphism

$$H^2(Y, \mathbb{Z}) \cong H^2(Y', \mathbb{Z})$$

induced by the parallel transport along $\gamma$ sends $[\alpha]$ to an internal $(-2)$-curve $C$ on $Y'$.

(2) Denote the set of roots $(Y, D)$ by $\Phi$. Given $\alpha$ in $\Phi$, we can define the following reflection automorphism $r_\alpha$ of $\text{Pic}(Y)$:

$$r_\alpha : \text{Pic}(Y) \to \text{Pic}(Y)$$

$$\beta \mapsto \beta + \langle \alpha, \beta \rangle \alpha.$$

We define the Weyl group $W(Y, D)$ of $(Y, D)$ to be the subgroup of $\text{Aut}(\text{Pic}(Y))$ generated by reflections with respect to roots in $\Phi$.

By Theorem 5.1 of [GHK12], given a Looijenga pair $(Y, D)$, the Weyl group $W(Y, D)$ is a normal subgroup of the admissible group $\text{Adm}(Y, D)$.

2.3. The relationship to rank 2 cluster varieties. Let $(Y, D)$ be a generic Looijenga pair together with a toric model $\rho: (Y, D) \to (\overline{\text{Y}}, \overline{\text{D}})$. Let $\{E_i\}_{i=1}^n$ be the exceptional configuration corresponding to the given toric model. Let $\Sigma$ be the fan of the toric surface $\overline{\text{Y}}$ in a lattice $\Lambda \cong \mathbb{Z}^2$. Set $N = \mathbb{Z}^n$ with basis $\{e_i\}_{i \in I}$ where $I := \{1, \ldots, n\}$. Let $M$ be the dual lattice of $N$. For each $i \in I$, denote by $\overline{D}_i \subset D$ the irreducible component containing the center of the non-toric blowup with exceptional locus $E_i$. Each $\overline{D}_i$ corresponds to a ray in $\Sigma$ with the primitive generator $w_i$ in $\Lambda$. If we assume that $w_1, \ldots, w_n$ generate the lattice $\Lambda$, then the map

$$\varphi : N \to \Lambda$$

$$e_i \mapsto w_i$$

is surjective. Moreover, if we fix an isomorphism $\bigwedge^2 \Lambda \cong \mathbb{Z}$, then the wedge product on $\Lambda$ pull-back to an integral valued skew-symmetric form $\{\cdot, \cdot\}$ on $N$:

$$\{n_1, n_2\} = \varphi(n_1) \wedge \varphi(n_2).$$

Thus, from the fixed data $(N, \{\cdot, \cdot\}, I)$, we define a seed $s = \{e_i\}_{i \in I}$. Observe that the kernel $K$ of the skew-symmetric form on $N$ has rank equal to $\text{rank} N - 2$, or equivalently, the exchange matrix of $s$ has rank 2. Denote by $\mathcal{X}_s$ the corresponding $\lambda$-cluster variety constructed from the seed $s$. By section 5 of [GHK14], $\mathcal{X}_s$ fibers over $T_{K^*}$ with generic fiber agreeing up to codimension 2 with a log Calabi-Yau surface deformation equivalent to $Y \setminus D$. More precisely, let $\Sigma$ be the fan in $\overline{\mathcal{M}}_{\mathbb{R}}$ consisting of rays generated by $-\{e_i, \cdot\}$. Denote by $\text{TV}(\Sigma)$ the toric variety associated to $\Sigma$. The projection $M \to M/K_{\perp} \cong K^*$ induces a map

$$\overline{\mathcal{X}} : \text{TV}(\Sigma) \to T_{K^*}$$

For each $i$, let $D_{\{-e_i, \cdot\}}$ be the toric divisor corresponding to the ray generated by $-\{e_i, \cdot\}$. Define the closed subscheme

$$Z_i := D_{\{-e_i, \cdot\}} \cap \overline{\text{Y}}(1 + z^{e_i}).$$

Let $\mathcal{Y}_s$ be the blowup of the toric variety $\text{TV}(\Sigma)$ along $\bigcup_{i=1}^n Z_i$ and $\mathcal{D}_s$ the toric boundary of $\mathcal{Y}_s$. Let $\lambda : \mathcal{Y}_s \to T_{K^*}$ be the fibration after the blowup. Then summarize results in section 5 of [GHK15], we have the following theorem:

**Theorem 2.10.** For $\phi \in T_{K^*}$ general, $(\mathcal{Y}_{s, \phi}, \mathcal{D}_{s, \phi})$ is a generic Looijenga pair deformation equivalent to $(Y, D)$ with the period point $\phi$. Moreover, $\mathcal{Y}_{s, \phi} \setminus \mathcal{D}_{s, \phi}$ agrees with $\mathcal{X}_{s, \phi}$ away from codimension 2.
By Theorem 2.10 we could view a generic fiber in $X_s \to T_{K^*}$, away from codimension 2, as a log Calabi-Yau surface deformation equivalent to $U = Y \setminus D$. Moreover, as proved in Theorem 5.5 of \cite{GHKL3}, there is natural isomorphism $K \cong D^{\perp}$ and we can view $X_s$ as the universal family of $U = Y \setminus D$. Hence the title of this paper is justified.

For more detailed description of the relationship between Looijenga pairs and rank 2 $\mathcal{X}$-cluster varieties, see section 5 of \cite{GHKL3} and section 2 of \cite{TM}.

2.4. Admissible groups and cluster modular groups. Now let $s = (e_i)_{i \in I}$ be an initial seed constructed from a toric model $\rho: (Y, D) \to (\mathcal{Y}, \mathcal{T})$ for a generic Looijenga pair $(Y, D)$. Let $\{E_i\}_{i \in I}$ be the exceptional configuration of $\rho$. Let $v_0$ be the root of $\mathcal{T}_s$. Given any $v$ in $\mathcal{T}_s$, let $\mathcal{Y}_{s,v}$ be the blowup of toric variety $TV(\Sigma_{s,v})$ as described in subsection 2.3. By section 5 of \cite{GHKL3}, the birational map

$$\mu_{v_0,v}: T_{M,s} \dashrightarrow T_{M,s,v}$$

extends to a birational map between blowups of toric varieties,

$$\mu_{v_0,v}: \mathcal{Y}_s \dashrightarrow \mathcal{Y}_{s,v}$$

such that for $\phi \in T_{K^*}$ general, $\mu_{v_0,v}$ restrict to a biregular isomorphism $\mathcal{Y}_{s,\phi}/\mathcal{D}_{s,\phi} \to \mathcal{Y}_{s,v,\phi}/\mathcal{D}_{s,v,\phi}$. Let $\{E_{i,\phi}\}_{i \in I}$ be the exceptional configuration of

$$\rho_{\phi}: (\mathcal{Y}_{s,\phi}, \mathcal{D}_{s,\phi}) \to (\mathcal{Y}_s, \mathcal{T}_s)$$

Now suppose $s_v$ is isomorphic to $s$ and $\mu_{v_0,v}^t: \mathcal{M}_s \to \mathcal{M}_s^{\mathbb{R}}$ restricts to an isomorphism between fans $\Sigma_s \to \Sigma_{s,v}$. Then the biregular isomorphism

$$\mu_{v_0,v}: \mathcal{Y}_{s,\phi}/\mathcal{D}_{s,\phi} \to \mathcal{Y}_{s,v,\phi}/\mathcal{D}_{s,v,\phi}$$

extends to a biregular isomorphism $\mu_{v_0,v}: \mathcal{Y}_{s,\phi} \to \mathcal{Y}_{s,v,\phi}$. Let $\gamma: s \to s_v$ be a seed isomorphism compatible with the isomorphism between fans $\mu_{v_0,v}^t: \Sigma_s \to \Sigma_{s,v}$, i.e., for any $i \in I$,

$$\mu_{v_0,v}^t(\{-e_i, :\}) = \{-\gamma(e_i), :\}$$

The isomorphism

$$\begin{array}{ccc}
T_{M,s,v} & \xrightarrow{\gamma^{-1}} & T_{M,s} \\
\downarrow & & \downarrow \\
T_{K^*} & \xrightarrow{\gamma^{-1}} & T_{K^*}
\end{array}$$

induced by $\gamma^{-1}$ extends to an isomorphism

$$\begin{array}{ccc}
\mathcal{Y}_{s,v} & \xrightarrow{\gamma^{-1}} & \mathcal{Y}_s \\
\downarrow & & \downarrow \\
T_{K^*} & \xrightarrow{\gamma^{-1}} & T_{K^*}
\end{array}$$

Now consider the isomorphism

$$\gamma^{-1} \circ \mu_{v_0,v}: (\mathcal{Y}_{s,\phi}, \mathcal{D}_{s,\phi}) \to (\mathcal{Y}_{s,\gamma^{-1} \phi}, \mathcal{D}_{s,\gamma^{-1} \phi})$$

Let $\{E_{i,\gamma^{-1} \phi}\}_{i \in I}$ be the exceptional configuration of

$$\rho_{\gamma^{-1} \phi}: (\mathcal{Y}_{s,\gamma^{-1} \phi}, \mathcal{D}_{s,\gamma^{-1} \phi}) \to (\mathcal{Y}_s, \mathcal{T}_s)$$

This is the geometric tropicalization using the min-plus convention. Fock-Goncharov tropicalization in \cite{GHKK18} uses the max-plus convention. For more details on this two tropicalizations, see section 2 of \cite{GHKK18}.
Since $\gamma^{-1} \circ \mu_{v_0, v}$ induces trivial map on the boundary,
\[ \{ E_i, \phi \}_{i \in I} := \{ (\gamma^{-1} \circ \mu_{v_0, v})^* (E_i, \gamma^{-1} \phi) \}_{i \in I} \]
is another exceptional configuration of $\rho$ and there exists an unique element $\delta$ in the admissible group $\text{Adm}(Y, D)$ that maps $\{ E_i, \phi \}_{i \in I}$ to $\{ E_i, \delta \}_{i \in I}$. Thus, we can identify $\delta$ with $\gamma^{-1} \circ \mu_{v_0, v}$ as an element in the cluster modular group.

2.5. **Classification of positive Looijenga pairs.** The following proposition follows from Theorem 4.4 and Theorem 4.5 of [TM].

**Proposition 2.11.** Let $(Y, D)$ be a generic positive Looijenga pair. Then $X_{Y, D}$ is non-acyclic, that is, it cannot be constructed from an acyclic seed if and only if
\[ D^\perp \cong D_n \ (n \geq 4) \text{ or } E_n \ (n = 6, 7, 8). \]

Since the theory of acyclic cluster varieties are well-understood, for the rest of this paper, we will focus only on the cases where $D^\perp \cong D_n \ (n \geq 4)$ or $E_n \ (n = 6, 7, 8)$.

3. **The Geometry of Folding**

In this section, we first briefly review basics of scattering diagrams that arise in cluster theory. Readers familiar with the Gross-Siebert program could skip this part. Then we recall the folding technique developed in [YZ] in the context of cluster scattering diagrams. In the last subsection [3.3] we review the geometric meaning of the folding procedure for $X$-cluster varieties associated to log Calabi-Yau surfaces.

3.1. **Cluster scattering diagrams.** Let $N$ be a lattice of rank $n$ with dual lattice $M$ and a skew-symmetric $\mathbb{Z}$-bilinear form $\{,\} : N \times N \to \mathbb{Z}$. Let $I = \{1, 2, \cdots, n\}$ be the index set. Fix an initial seed $s$, i.e., $s = (e_i)_{i=1}^n$ is a labelled basis of $N$. For simplicity, we assume that we do not have frozen variables for the entire paper. In general when we fix the lattice $N$, we also fix a sublattice $N^0 \subset N$ and positive integers $d_i$ for $i \in I$ such that \{d_i e_i | i \in I\} is a basis for $N^0$. As in [YZ], we assume that the original seed $s$ is simply-laced, that is, $d_i = 1$ for all $i \in I$.

Define
\[ N^+ = \{ \sum_{i \in I} a_i e_i \mid a_i \in \mathbb{N}, \sum a_i > 0 \} \]
and the following algebraic torus over $\mathbb{C}$:
\[ T_M = M \otimes_{\mathbb{Z}} \mathbb{G}_m = \text{Hom}(N, \mathbb{G}_m) = \text{Spec}(\mathbb{C}[N]). \]
The skew-symmetric form $\{,\}$ on $N$ gives the algebraic torus $T_M$ a Poisson structure:
\[ \{ z^n, z^{n'} \} = \{ n, n' \} z^{n+n'}. \]

Let $g = \bigoplus_{n \in N^+} g_n$ where $g_n = \mathbb{C} \cdot z^n$. The Poisson bracket $\{,\}$ on $\mathbb{C}[N]$ endows $g$ with a Lie bracket $[,] : [f, g] = -\{f, g\}$. By [2.1] it is clear that $g$ is graded by $N^+$, i.e., $[g_{n_1}, g_{n_2}] \subset g_{n_1+n_2}$ and $g$ is skew-symmetric for $\{,\}$ on $N$, i.e., $[g_{n_1}, g_{n_2}] = 0$ if $\{n_1, n_2\} = 0$.

Define the following linear functional on $N$:
\[ d : N \to \mathbb{Z} \]
\[ \sum_{i \in I} a_i e_i \mapsto \sum_{i \in I} a_i. \]

For each $k \in \mathbb{Z}_{\geq 0}$, there is an Lie subalgebra $g^>k := \bigoplus_{n \in N^+ | d(n) > k} g_n$ of $g$ and a $N^+$-graded nilpotent Lie algebra $g^\leq k := g / g^>k$. We denote the unipotent Lie group corresponding to
\( \mathfrak{g}^{\leq k} \) as \( G^{\leq k} \). As a set, it is in bijection with \( \mathfrak{g}^{\leq k} \), but the group multiplication is given by the Baker-Campbell-Hausdorff formula. Given \( i < j \), we have canonical homomorphism:

\[
\mathfrak{g}^{\leq j} \to \mathfrak{g}^{\leq i}, \quad G^{\leq j} \to G^{\leq i}.
\]

Define the pro-nilpotent Lie algebra and its corresponding pro-unipotent Lie group by taking the projective limits:

\[
\hat{\mathfrak{g}} = \lim_{\longrightarrow} \mathfrak{g}^{\leq k}, \quad \hat{G} = \lim_{\longrightarrow} G^{\leq k}.
\]

We have canonical bijections:

\[
\exp : \mathfrak{g}^{\leq k} \to G^{\leq k}, \quad \exp : \hat{\mathfrak{g}} \to \hat{G}.
\]

Given an element \( n_0 \in N^+ \), there is an Lie subalgebra \( \mathfrak{g}||_{n_0} := \bigoplus_{k \in \mathbb{Z}_{>0}} \mathfrak{g}_{k-n_0} \) of \( \mathfrak{g} \) and its corresponding Lie subgroup \( G||_{n_0} := \exp(\mathfrak{g}||_{n_0}) \) of \( G \). By our assumption that \( \mathfrak{g} \) is skew-symmetric for \( \{,\} \), \( \mathfrak{g}||_{n_0} \) and therefore \( G||_{n_0} \) are abelian.

**Definition 3.1.** A wall in \( M_\mathbb{R} \) for \( N^+ \) and \( \mathfrak{g} \) is pair \((\mathfrak{d}, g_\mathfrak{d})\) such that

1) \( g_\mathfrak{d} \) is contained in \( G||_{n_0} \) for some primitive element \( n_0 \in N^+ \)

2) \( \mathfrak{d} \) is contained in \( n_0^- \subset M_\mathbb{R} \) and is a convex rational polyhedral cone of codimension 1.

**Definition 3.2.** A scattering diagram \( \mathcal{D} \) for \( N^+ \) and \( \mathfrak{g} \) is a collection of walls such that for every order \( k \) in \( \mathbb{Z}_{\geq 0} \), there are only finitely many walls \((\mathfrak{d}, g_\mathfrak{d})\) in \( \mathcal{D} \) with non-trivial image of \( g_\mathfrak{d} \) under the projection map \( G \to G^{\leq k} \).

Given a wall \((\mathfrak{d}, g_\mathfrak{d})\) such that \( \mathfrak{d} \) is contained in \( n_0^- \) for a primitive element \( n_0 \in N^+ \), we say that \((\mathfrak{d}, g_\mathfrak{d})\) is incoming if \( \{n_0, \cdot\} \) is contained in \( \mathfrak{d} \). Otherwise, we say that \((\mathfrak{d}, g_\mathfrak{d})\) is outgoing.

**Theorem 3.3.** (Theorem 1.12 and Theorem 1.21 of [GHKK18]) The equivalence class of a consistent scattering diagram is determined by its set of incoming walls. Vice versa, let \( \mathcal{D}_{\text{in}} \) be a scattering diagram whose only walls are full hyperplanes, i.e., are of the form \((n_0^+, g_{n_0})\) where \( n_0 \) is a primitive element in \( N^+ \). Then there is a scattering diagram \( \mathcal{D} \) satisfying:

1) \( \mathcal{D} \) is consistent,

2) \( \mathcal{D} \supset \mathcal{D}_{\text{in}} \),

3) \( \mathcal{D}\backslash \mathcal{D}_{\text{in}} \) consists only of outgoing walls.

Moreover, \( \mathcal{D} \) satisfying these three properties is unique up to equivalence.

Define the following set of initial walls associated to the seed \( s = (e_i)_{i \in I} \):

\[
\mathcal{D}_{s, \text{in}} = \{(e_i^+, \exp(-\text{Li}_2(-z^{e_i})) | i \in I\}
\]

where \( \text{Li}_2 \) is the dilogarithm function \( \text{Li}_2(x) = \sum_{k \geq 1} \frac{x^k}{k^2} \). By Theorem 3.3, there is a consistent scattering diagram \( \mathcal{D}_s \), unique up to equivalence, whose set of initial walls is \( \mathcal{D}_{s, \text{in}} \). It is the cluster scattering diagram associated to the seed \( s \). Slightly different from [GHKK18], we consider a single scattering diagram with wall-crossing automorphisms in the Lie group. By letting the Lie group acting on different completed rings, we get wall-crossings for both \( \mathcal{X} \) and \( \mathcal{A}_{\text{prin}} \) spaces.

Define

\[
\mathcal{C}_s^+ = \{m \in M_\mathbb{R} \mid m \, |_{N^+} > 0\}, \quad \mathcal{C}_s^- = \{m \in M_\mathbb{R} \mid m \, |_{N^+} \leq 0\}.
\]

By Theorem 2.13 of [GHKK18], \( \mathcal{C}_s^+ \) and \( \mathcal{C}_s^- \) are two distinguished chambers in \( \mathcal{D}_s \). Moreover, \( \mathcal{C}_s^+ \) (resp. \( \mathcal{C}_s^- \)) is a maximal cone of a simplicial fan in \( \mathcal{D}_s \) which we denote by \( \Delta_s^+ \) (resp. \( \Delta_s^- \)). We call \( \Delta_s^+ \) the cluster complex of \( \mathcal{D}_s \). For more details on the structure of the cluster complex and its description via tropicalization, see section 2 of [GHKK18].
3.2. Summary of the folding procedure. Suppose $\Pi$ is a subgroup of the symmetric group $S_n$ acting on $I$ satisfies the following condition:

For any indices $i, j \in I$, for any $\pi_1, \pi_2$ in $\Pi$,

\[
\{e_i, e_j\} = \{e_{\pi_1 \cdot i}, e_{\pi_2 \cdot j}\}.
\]

Then we can apply the folding procedure to construct a new seed $\bar{s}$. For any $i \in I$, denote by $\Pi_i$ the orbit of $i$ under the action of $\Pi$. Let $T$ be the set indexing the orbits of $I$ under the action of $\Pi$. Let $N$ be the lattice with basis $\{e_{\Pi_i} \mid i \in T\}$. Define a $\mathbb{Z}$-valued skew-symmetric bilinear form $\{\cdot, \cdot\}$ on $N$ such that $\{e_{\Pi_i}, e_{\Pi_j}\} = \{e_i, e_j\}$. By condition 3.3, the skew-symmetric form on $N$ is well-defined. There is a natural quotient homomorphism of lattices $q : N \to \overline{N}$ that sends $e_i$ to $e_{\Pi_i}$. Now we have the following fixed data:

- The lattice $\overline{N}$ with the $\mathbb{Z}$-valued skew-symmetric bilinear form $\{\cdot, \cdot\}$.
- The index set $T$ parametrizing the set of orbits of $I$ under the action of $\Pi$.
- For each $\Pi_i \in T$, a positive integer $d_{\Pi_i} = |q^{-1}\{e_{\Pi_i}\}|$.

We define the following new seed:

\[
\bar{s} = (e_{\Pi_i} \mid \Pi_i \in T).
\]

Let $N^\circ \subset N$ be the sublattice such that $\{d_{\Pi_i} e_{\Pi_i} \mid i \in T\}$ is a basis. Unlike $[\text{GHKK18}]$, we do not impose the condition that the greatest common divisor of $\{d_{\Pi_i}\}_{i \in T}$ need to be 1. Notice that the new seed $\bar{s}$ may no longer be simply laced since $d_{\Pi_i}$ may not be 1. The seed $\bar{s}$ defines the exchange matrix $\bar{\tau}$ such that $\bar{\tau}_{\Pi_i \Pi_j} = \{e_{\Pi_i}, e_{\Pi_j}\}d_{\Pi_j}$. Take

\[
\overline{N^+} = \big\{ \sum_{\Pi_i \in T} a_{\Pi_i} e_{\Pi_i} \mid a_{\Pi_i} \geq 0, \sum a_{\Pi_i} > 0 \big\}.
\]

Let $\bar{g} \subset \mathbb{C}[\overline{N}]$ be the $\mathbb{C}$-vector space with basis $z^n, \bar{n} \in \overline{N^+}$. Define a Lie bracket on $\bar{g}$ as follows:

\[
[z^{e_{\Pi_i}}, z^{e_{\Pi_j}}] = -\{e_{\Pi_i}, e_{\Pi_j}\}z^{e_{\Pi_i} + e_{\Pi_j}}.
\]

Define $\tilde{q} : \mathfrak{g} \to \overline{\mathfrak{g}}$ by $z^n \mapsto z^n$ where $\bar{n}$ is the image of $n$ under the surjective quotient homomorphism $q : N \to \overline{N}$. Since

\[
[q(z^n), \tilde{q}(z^{n'})] = -\{n, n'\}z^{n+n'} = -\{n, n'\}z^{n+n'} = \tilde{q}\left([z^n, z^{n'}]\right),
\]

$\tilde{q}$ is a surjective graded Lie algebra homomorphism.

As in the case of $\mathfrak{g}$, we can complete $\bar{g}$ with respect to its grading. Let $\overline{G}$ be the corresponding pro-nilpotent group after the completion. Then $\tilde{q}$ induces a graded Lie group homomorphism from $G$ to $\overline{G}$.

By Theorem 3.3, the seed $\bar{s}$ gives rises to a consistent scattering diagram $\mathcal{D}_\bar{s}$ whose set of incoming walls are defined as follows:

\[
\mathcal{D}_{\bar{s}, \text{in}} = \left\{ (e_{\Pi_i}^+, \exp(-d_{\Pi_i}Li_2(-z^{e_{\Pi_i}}))) \mid \Pi_i \in T \right\}.
\]

Denote by $\overline{M}$ the dual basis of $\overline{N}$ and $\overline{N^*}$ the sublattice of $\overline{N}$ with basis $\{d_{\Pi_i} e_{\Pi_i} \mid i \in T\}$. Let $\overline{\mathfrak{t}}^*$ be the dual lattice of $\overline{N}$. Let

\[
q^* : \overline{\mathfrak{t}}^* \hookrightarrow \mathfrak{t}^*
\]

the embedding induced by the quotient homomorphism $q : N \to \overline{N}$. One of the main technical result in $[\text{YZ}]$ is that the quotient construction of folding extends to the level of scattering diagrams:

**Theorem 3.4.** (cf. Construction 2.19, Theorem 2.20, and Corollary 2.21 of $[\text{YZ}]$) The scattering diagram $\mathcal{D}_\bar{s}$ can be obtained from $\mathcal{D}_s$ via a quotient construction.
For more details of the folding construction in the context of scattering diagrams and cluster theory and its application to the Fock-Goncharov canonical basis conjecture, see section 2 and 3 of \cite{YZ}.

For the rest of this subsection, we prove some lemmas that will be used later. Let $\Sigma_s$ (resp. $\Sigma_s^*$) the oriented tree parametrizing all seeds mutationally equivalent to $s$ (resp. $s^*$).

Given $k$ in $I$, let $l_1, \ldots, l_{|\Pi_k|}$ be an enumeration of elements in $|\Pi_k|$.

**Lemma 3.5.** Let $\mathbf{s} = (e_i)_{i \in I} = \mu_1 \circ \cdots \circ \mu_{|\Pi_k|}(s)$. Then $\mathbf{s}$ does not depend on the choice of ordering of elements in $|\Pi_k|$. With respect to the new seed $\mathbf{s}'$, the action of $\Pi$ still satisfies the condition for folding, that is, for any $\pi_1, \pi_2$ in $\Pi$, 

\[ \{e_{\pi_1,i}, e_{\pi_2,j}\} = \{e_i, e_j\}. \]

Moreover, $\mu_{|\Pi_k|}(\mathbf{s}) = (q(e'_i))_{i \in I}$ where $q : N \to N$ is the quotient map of lattice sending $e_i$ to $e_{\Pi_k}$.

**Proof.** First observe that for any $1 \leq a \leq |\Pi_k|$, $e'_a = -e_a$. Therefore for any $1 \leq a, b \leq |\Pi_k|$, $(e'_a, e'_b) = 0$. If $i, j \in I$ are not in $\Pi_k$, then for any $\pi_1, \pi_2$ in $\Pi$,

\[ e_{\pi_1,i} = e_{\pi_2,i} + \sum_{a=1}^{[\Pi_k]} [(e_{\pi_1,i}, e_{\pi_1,a})] + [e_{\pi_1,i}] + \sum_{a=1}^{[\Pi_k]} e_{\pi_2,a}, \quad \epsilon = 1, 2. \]

Therefore,

\[ \{e'_{\pi_1,i}, e'_{\pi_2,j}\} = \{e_{\pi_1,i} + \{e_i, e_k\} + \sum_{a=1}^{[\Pi_k]} e_{\pi_2,a} + \{e_j, e_k\} + \sum_{a=1}^{[\Pi_k]} e_{\pi_1,a}\} = \{e_i, e_j\}. \]

If $i \in I$ is not in $\Pi_k$, then for any $1 \leq a \leq |\Pi_k|$, 

\[ \{e'_{\pi_1,i}, e'_{\pi_2,a}\} = \{e_{\pi_1,i} + \{e_i, e_k\} + \sum_{a=1}^{[\Pi_k]} e_{\pi_2,a} - e_{\pi_1,a}\} = \{e_i, -e_k\} = \{e_i, e'_k\}. \]

Hence the action of $\Pi$ on $I$ with respect to the new seed $\mathbf{s}'$ still satisfies the folding condition. It remains to prove the second part of the lemma. Notice that for any $1 \leq a \leq |\Pi_k|$, $q(e'_a) = -e_{\Pi_k} = \mu_{|\Pi_k|}(e_{\Pi_k})$. If $i \in I$ is not in $\Pi_k$, then for any $\pi$ in $\Pi$,

\[ q(e'_{\pi,i}) = q\left[e_{\pi,i} + \{e_i, e_k\} + \sum_{a=1}^{[\Pi_k]} e_{\pi,a}\right] = \mu_{|\Pi_k|}(e'_{\Pi_k}). \]

\[ \square \]

**Definition 3.6.** Denote by $[\mathbf{s}]$ the set of seeds mutationally equivalent to $\mathbf{s}$. Given $k$ in $I$, by Lemma 3.5 there is a well-defined composition of mutations

\[ (3.4) \quad \mu_{|\Pi_k|} = \prod_{k' \in \Pi_k} \mu_{k'}. \]

We call such compositions of mutations $\mu_{|\Pi_k|}$ a $\Pi$-constrained mutation. Moreover, also by Lemma 3.5 the new seed $\mu_{|\Pi_k|}(\mathbf{s})$ is an element in $[\mathbf{s}]$ and $\Pi$-constrained mutation is also well-defined for $\mu_{|\Pi_k|}(\mathbf{s})$. We denote by $[\mathbf{s}]^\Pi \subset [\mathbf{s}]$ the set of seeds related to $\mathbf{s}$ by $\Pi$-constrained mutations.

Given $\mathbf{s}_k \in [\mathbf{s}]$, let $G^v$ be the $g$-matrix of the seed $\mathbf{s}_k$. We say $G^v$ is $\Pi$-invariant if for any $\pi$ in $\Pi$, for any $k \in I$, $\pi \cdot G^v_k = G^v_{\pi k}$. We say the chamber $C^+_v$ in $\Delta^+_v$ is $\Pi$-invariant if $G^v$ is. In particular, for any $\pi$ in $\Pi$, $\pi(C^+_v) = C^+_v$ under the action of $\pi$ on $\mathfrak{D}_s$ (cf. section 2.2 of \cite{YZ}).
Lemma 3.7. Let $s' = s_w$ be a seed in $[s]^{iso}$. Suppose $C^+_w \in \Delta^+_s$ is $\Pi$-invariant. Then given any $v'$ in $\Sigma_s' \subset \Sigma_s$ such that $s_{v'}$ is contained in $[s]^\Pi$, $C^+_v \in \Delta^+_s$ is $\Pi$-invariant. In particular, $C^+_v \cap q^*(\overline{M_R})$ is of full dimension in $q^*(\overline{M_R})$.

Proof. It suffices to prove for the case where $s_{v'} = \mu_{\Pi k}(s_w)$. Indeed, since $s_w$ is isomorphic to $s$, up to permuting of indices, we could assume that $s_w$ and $\Pi$ satisfy the folding condition 3.3. Let $G^w$, $C^w$ be the $g$-matrix and $c$-matrix associated to $C^+_w$ respectively. By sign coherence of $c$-vectors, all entries of $C^+_w$ either $\geq 0$ or $\leq 0$.

Case 1: All entries of $C^+_w \geq 0$. By the $\Pi$-invariance of $C^+_w$, the same holds for any $k' \in \Pi k$. By the mutation formula for $g$-vectors, for any $k' \in \Pi k$,

$$G^w_{k'} = -G^w_k + \sum_{i=1}^n \left[ -\epsilon^w_{ik'} \right] + G^w_i.$$ 

Let $k' = \pi \cdot k$. Then it follows from the $\Pi$-invariance of $G^w$ that

$$\pi \cdot G^w_k = -\pi \cdot G^w_k + \sum_{i=1}^n \left[ -\epsilon^w_{ik} \right] \pi \cdot G^w_i = -G^w_{\pi k} + \sum_{i=1}^n \left[ -\epsilon^w_{ik'} \right] G^w_{\pi i} = -G^w_{k'} + \sum_{i=1}^n \left[ -\epsilon^w_{ik'} \right] G^w_{\pi i}.$$ 

Since for any $i \in I$,

$$\epsilon^w_{ik'} = \{ e_i, e_{k'} \} = \{ e_{\pi i}, e_{k'} \} = \epsilon(\pi \cdot i)_{k'},$$

we conclude that

$$\pi \cdot G^w_{k'} = -G^w_{k'} + \sum_{i=1}^n \left[ -\epsilon^w_{ik'} \right] G^w_{\pi i} = -G^w_{k'} + \sum_{i=1}^n \left[ -\epsilon^w_{(\pi i)k'} \right] G^w_{\pi i} = G^w_{k'}. $$

Therefore $G^w_{k'}$ and hence $C^+_w$ is $\Pi$-invariant.

Case 2: All entries of $C^+_w \leq 0$. Then we have

$$C^w_{k'} = -G^w_k + \sum_{i=1}^n \left[ \epsilon^w_{ik} \right] + G^w_i.$$ 

Similar to Case 1, we again conclude that $C^+_w$ is $\Pi$-invariant.

\[ \square \]

3.3. Folding and degenerate loci of families of log Calabi-Yau surfaces. Let $(Y, D)$ be a generic Looijenga pair together with a toric model $\rho : (Y, D) \rightarrow (\overline{Y}, \overline{D})$. Then as described in subsection 2.3, $(Y, D)$ together with the toric model gives rise to seed data $(N, \{\cdot, \cdot\}, s)$. Notice that folding construction naturally applies in this situation. Given $i, j \in I$, if $\{ e_i \} = \{ e_j \}$, then $E_i$ and $E_j$ meet the same boundary component and we can naturally fold $e_i$ and $e_j$ together. More generally, let $\Pi$ be a subgroup of the symmetric group $S_n$ generated by involutions such that for any $\pi \in \Pi$,

$$\{ e_i, \cdot \} = \{ e_{\pi i}, \cdot \}.$$ 

Then the action of $\Pi$ satisfies the folding condition and gives us a folded seed $\overline{s}$. Moreover, we can identify $\Pi$ as a subgroup of the Weyl group $(Y, D)$. Indeed, given $i \in I$ and $j \in \Pi \setminus \{ i \}$, under the identification $D^+ \cong K$ given in Theorem 5.5 in [GHKL], $e_i - e_j$ corresponds the root $E_i - E_j$. Let $\pi$ be the involution $(ij)$ in $\Pi$. Then the induced action
of $\pi$ on $K \simeq D^\perp$ agrees with the reflection $r_{E_i - E_j}$ with respect to the root $E_i - E_j$ in $W(Y, D)$.

Let $s$ be a seed obtained from a generic Looijenga pair $(Y, D)$ with toric model $\rho : (Y, D) \to (\overline{Y}, \overline{D})$ and $\overline{s}$ the seed folded from $s$ via the action of a group $\Pi$ on the index set $I$. Notice that folding procedure does not change the rank of the exchange matrix, so the $X$-cluster variety associated to $\overline{s}$ is still a rank 2 cluster variety. Moreover, $X_s$ is a locally closed cluster subvariety of $X_s$ and corresponds to a degenerate subfamily of the universal family of $U = Y \setminus D$. To see this, let $K$ be the skew-symmetric form on $N$.

Let $\overline{Y}$ be the blowup of the toric variety given by $\overline{s}$. Given a general $\overline{\phi}$ in $T_{\overline{s}, \phi}$, $X_{\overline{s}, \phi}$ up to codimension 2 agrees with $\overline{Y}_{\overline{s}, \phi}$, which is blowup of of $\overline{Y}$ at a collection of distinct points $p_{i1}, \ldots, p_{in}$ where $p_i \in \overline{D}_i$ has multiplicity $|\Pi_i|$. For each $i$ such that $|\Pi_i| > 1$, the weighted blow-up at $p_{ii}$ gives rise to a du Val singularity of type $A_{|\Pi_i| - 1}$ in $\overline{Y}_{\overline{s}, \phi}$. Therefore, we could view $X_{\overline{s}}$ as a degenerate subfamily of the universal family of $U$ such that generic members in this subfamily have singularities specified by the combinatorial data of the folding from $s$ to $\overline{s}$.

**Example 3.8.** Consider $(\mathbb{P}^2, \overline{D})$ where $\overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ is a triangle of lines in $\mathbb{P}^2$. On each $\overline{D}_i$, we blow up at two distinct points in the smooth locus of $\overline{D}$ and obtain two exceptional curves $E_{2i-1}$ and $E_{2i}$. The new pair $(Y, D)$ we get is a cubic surface whose anticanonical cycle $D$ is the strict transform of $\overline{D}$ via the blowup $\rho : (Y, D) \to (\overline{Y}, \overline{D})$. Let $l$ be the class of the pull-back of a line in $\mathbb{P}^2$ via $\rho$. We have $D^\perp \simeq D_4$ and the Weyl group of $(Y, D)$ is the Weyl group $W(D_4)$ of $D_4$. Moreover, $e_1 - e_2, e_3 - e_4, e_5 - e_6, e_1 + e_2 + e_3$ form a basis of $K$ which under the identification $K \simeq D^\perp$ correspond to roots $E_1 - E_2, E_3 - E_4, E_5 - E_6, l - E_1 - E_2 - E_3$. These roots form a simple root system in $D_4$. Let $\Pi$ be the group generated by involutions $(12), (34), (56)$. Then $\Pi$ can be identified as the subgroup generated by reflections with respect to roots $E_1 - E_2, E_3 - E_4, E_5 - E_6$. Let $s$ be the seed given by $(Y, D)$ and $\overline{s}$ be the folded seed given by the action of $\Pi$ on the index set. Then given a very general $\overline{\phi} \in T_{\overline{s}, \phi}$, $X_{\overline{s}, \phi}$ up to codimension 2 agrees with a singular cubic surface having the following description. We blow up at an interior point $p_{i, \overline{\phi}}$ in $\overline{D}_i$ twice for each $i$. Since we assume that $\overline{\phi}$ is general, $p_{1, \overline{\phi}}, p_{2, \overline{\phi}}, p_{3, \overline{\phi}}$ are not collinear and therefore we could blow down the three $(-2)$-curves simultaneously and get a singular cubic surface with three ordinary double points.

### 4. The action of Weyl groups on scattering diagrams: positive cases

Throughout this section, we will focus on positive Looijenga pairs $(Y, D)$ such that the corresponding $X$-cluster variety is non-acyclic. By Proposition 2.4.11 if $(Y, D)$ is positive and non-acyclic, then $D^\perp \simeq D_n (n \geq 4)$ or $E_n (n = 6, 7, 8)$. By Theorem 4.4 of [1M], in these cases, we could assume that $(Y, D)$ has a toric model $\rho : (Y, D) \to (\mathbb{P}^2, \overline{D})$ where $\overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ is the toric boundary and $D = D_1 + D_2 + D_3$ is the strict transform of $\overline{D}$. Denote by $l$ the class of the pullback of a line in $\mathbb{P}^2$ via $\rho$. Let $\{E_i\}_{i \in I}$ be the exceptional configuration of the toric model $\rho : (Y, D) \to (\mathbb{P}^2, \overline{D})$ as given on the left side of Figure 4.1 and $s = (e_i)_{i \in I}$ the seed constructed from this toric model as described in subsection 2.2. We construct irreducible root systems $D_n (n \geq 4)$ or $E_n (n = 6, 7, 8)$ explicitly as shown on the right side of the same figure.
Figure 4.1. \((Y,D)\) when \(D^\perp \cong D_n (n \geq 4)\) or \(E_n (n = 6, 7, 8)\)

4.1. **Fundamental example: Blowing up \(\mathbb{P}^2\) at three points.** Consider the seed \(s\) given following exchange matrix

\[
\epsilon = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.
\]

The exchange matrix we use here follows the convention of [FZ07]. To agree with the convention of [GHKK18], one need to replace \(\epsilon\) with \(-\epsilon\). The two papers use different conventions to define \(X\)-cluster varieties. When we define the \(X\)-cluster varieties, we follows the convention of [GHKK18], but when we do calculations we follow the convention of [FZ07] because we use Sage, which follows [FZ07], to do all computations.
The element $e_1 + e_2 + e_3$ generates the kernel $K$ of the skew-symmetric form. The corresponding $\mathcal{X}$-cluster variety has the fibration $\mathcal{X} \to T_{K^*}$ such that a generic fiber, up to codimension 2, is a log Calabi-Yau surface obtained by blowing up $\mathbb{P}^2$ at a smooth point on each toric boundary divisor $\mathcal{D}_i$ and then deleting the strict transform $D$ of $\mathcal{D}_i$. After the identification of $K$ with $D^\perp$, $e_1 + e_2 + e_3$ corresponds to the root $l = E_1 - E_2 - E_3$ where $l$ is the pullback of a line in $\mathbb{P}^2$ via the blowup and $E_i$s are the three exceptional divisors. Let

$$s' = \mu_1 \circ \mu_3 \circ \mu_2 \circ \mu_1(s).$$

The seed $s' = (e'_i)_{i \in I}$ has exchange matrix

$$\epsilon' = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

and its $g$-matrix is

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

So up to seed isomorphisms, $s$ and $s'$ are related by the Donaldson-Thomas transformation. Under $\mu_{s,s'}^{\mathcal{D}} : M_{\mathcal{D}} \to M_{\mathcal{D}}$ where $\mu_{s,s'}^{\mathcal{D}}$ is the geometric tropicalization of $\mu_{s,s}^{\mathcal{D}} : T_{M,s} \to T_{M,s'}$, $\{e_1, \cdot \}$ is mapped to $\{-e'_1, \cdot \}$, $\{-e'_2, \cdot \}$ to $\{-e'_3, \cdot \}$ and $\{-e'_3, \cdot \}$ to $\{-e'_2, \cdot \}$. Thus we define the seed isomorphism

$$\gamma : s' \to s, \quad e'_1 \mapsto e_1, \quad e'_2 \mapsto e_3, \quad e'_3 \mapsto e_2.$$

Let $\delta = \gamma \circ \mu_1 \circ \mu_3 \circ \mu_2 \circ \mu_1$. Then $\delta$ is an element in the cluster modular group of $X_s$ and it induces a birational automorphism of $T_{N,s}$ such that

$$\delta^*(z^{e_1}) = -z^{e_1} \frac{F_1}{F_2}, \quad \delta^*(z^{e_2}) = -z^{e_2} \frac{F_2}{F_3}, \quad \delta^*(z^{e_3}) = -z^{e_3} \frac{F_3}{F_1}.$$

where $F_i = f(z^{e_i}, z^{e_{i+2}})$ with $f(z_1, z_2) = 1 + z_1 + z_1 z_2$. Split $N$ as $N/K \oplus K$ via the following section:

$$N/K \hookrightarrow N$$

$$\begin{cases} e_1 \mapsto \epsilon_1 \\ e_3 \mapsto \epsilon_3 \end{cases}.$$

Then the above splitting induces an isomorphism of lattices $M \simeq K^+ \oplus K^*$ and hence an isomorphism between toric varieties

$$TV(\Sigma_s) \simeq \mathbb{P}^2 \times \mathbb{C}^*.$$

Thus, we could write $\delta$ as a birational automorphism of toric variety $\mathbb{P}^2 \times \mathbb{C}^*$:

$$\delta : \mathbb{P}^2 \times \mathbb{C}^* \dashrightarrow \mathbb{P}^2 \times \mathbb{C}^*$$

$$\left([z^{e_1} : 1 : z^{e_3}], \alpha \right) \mapsto \left([\alpha^{-1} \epsilon_1^{e_1} \epsilon_3^{e_3} \frac{F_3}{F_1} : 1 : z^{e_1} \frac{F_2}{F_1}], \alpha^{-1} \right)$$

where $\alpha = z^{e_1+e_2+e_3}$, $\frac{F_1}{F_1} = 1 + z^{e_1} + z^{e_1+e_3}$, $\frac{F_2}{F_2} = 1 + \alpha z^{e_1-e_3} + \alpha z^{e_3}$ and $\frac{F_3}{F_3} = 1 + z^{e_3} + \alpha z^{-e_3}$. Homogenize our equation by setting $z^{e_1} = \frac{X_1}{X_2}$ and $z^{-e_3} = \frac{X_3}{X_2}$, we get

$$[(X_1 : X_2 : X_3), \alpha] \mapsto [(\alpha^{-1} X_1 (X_1 + X_2 + \alpha X_3) : X_2 (X_1 + X_2 + X_3) : X_3 (X_1 + \alpha X_2 + \alpha X_3))$$

$$: X_3 (X_1 + \alpha X_2 + \alpha X_3)].$$
For $\alpha \neq 1$, $\delta$ restricts to a birational automorphism of $\mathbb{P}^2$:

$$\delta_{\alpha} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$$

$$[X_1 : X_2 : X_3] \mapsto [(\alpha^{-1}X_1(X_1 + X_2 + \alpha X_3) : X_2(X_1 + X_2 + X_3) : X_3(X_1 + \alpha X_2 + \alpha X_3)]$$

We can minimally resolve $\delta_{\alpha}$ as shown in the following diagram, where $\rho_1 : Y_\alpha \to \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at $p_1 = [0 : 1 : -1]$, $p_2 = [1 : 0 : -\alpha^{-1}]$ and $p_3 = [1 : -1 : 0].$

To understand the morphism $\rho_2 : Y_\alpha \to \mathbb{P}^2$, we can look at the inverse image of each boundary divisor of $\mathbb{P}^2$ under $\delta_{\alpha}$, as shown in the following figure. Let $L_{ij}$ ($1 \leq i < j \leq 3$) be the line in $\mathbb{P}^2$ joining $p_i$ and $p_j$. Then $L_{123}$ is given by the equation $X_1 + X_2 + \alpha X_3 = 0$, $L_{13}$ by the equation $X_1 + X_2 + X_3 = 0$ and $L_{12}$ by the equation $X_1 + \alpha X_2 + \alpha X_3 = 0$. After blowing up at $p_1$, $p_2$, and $p_3$, the strict transform of $L_{123}, L_{13}, L_{12}$ gives another exceptional configuration of $Y_\alpha$ and $\rho_2$ is the morphism that blows down this exceptional configuration. Therefore $\delta$ coincides with the reflection with respect to the root $l - E_1 - E_2 - E_3$.

**Definition 4.1.** Given a Looijenga pair $(Y, D)$ and a root $\alpha$ in $D^\perp$, denote by $r_{\alpha}$ the reflection automorphism of $\text{Pic}(Y)$ with respect to the root $\alpha$.

Throughout the rest of this section, $(Y, D)$ will be a generic Looijenga pair with $D^\perp \simeq D_n(n \geq 4)$ or $E_n(n = 6, 7, 8)$ and $s$ the seed constructed from the toric model as shown in Figure 4.1. We also use $X_{D_n}(n \geq 4)$ or $X_{E_n}(n = 6, 7, 8)$ to denote the corresponding $X$-cluster variety when we want to emphasize the particular deformation type of $U = Y \setminus D$. Let $D_{s}$ be the scattering diagram corresponding to the seed $s$. Since $s$ is skew-symmetric,
\( A'_n \cong X_s \). Hence the scattering diagram \( \mathcal{D}_s \) lives on \( X_s(\mathbb{R}^T) \). Moreover, we have the following proposition:

**Proposition 4.2.** The Weyl group \( W(Y,D) \) of \( (Y,D) \) acts faithfully on the scattering diagram \( \mathcal{D}_s \) as a subgroup of the cluster modular group. In particular, its action preserves the cluster complex in \( \mathcal{D}_s \).

**Proof.** For simple roots of the form \( E_{i_1} - E_{i_2} \) where \( E_{i_1} \) and \( E_{i_2} \) are two distinct exceptional curves along the same boundary component of \( D \), we have \( \{ e_{i_1}^+, e_{i_2}^+ \} \). The reflection \( r_{E_{i_1} - E_{i_2}} \) acts on the scattering diagram linearly by interchanging the dual basis elements \( e_{i_1}^* \) and \( e_{i_2}^* \). For simple roots of the form \( l - E_{i_1} - E_{i_2} - E_{i_3} \) where \( E_i \) is an exceptional curve along \( D \), we have seen in the fundamental example that the reflection with respect to these roots is also elements in the cluster modular group. Since the simple roots form a root basis for \( D^\perp \), \( W(Y,D) \) is generated by reflections with respect to simple roots. Hence we conclude that the action of \( W(Y,D) \) preserves the cluster complex. \( \square \)

Here is the outline for the rest of this section. In subsection 4.2 and subsection 4.3 we will find a special element \( w_1 \) in the Weyl group of \( W(D^\perp) \) with \( D^\perp \cong D_6(n \geq 4) \) or \( E_6(n = 6, 7, 8) \) such that the action of \( w \) on \( X_{D^\perp} \) is close to the Donaldson-Thomas transformation in the cluster modular group. As a result, we could approximate \( DT(X_{D^\perp}) \) by \( w \) and factorize \( DT(X_{D^\perp}) \) into cluster transformations since the action of \( w \) on \( X_{D^\perp} \) is cluster. For details of the Donaldson-Thomas transformations of cluster varieties, see [GS16]. In subsection 4.4 we apply the folding procedure as described in section 3 to \( s \), \( X_s \) and \( \mathcal{D}_s \). The main result will be Theorem 4.12.

### 4.2. Factorizations of DT-transformations as cluster transformations for \( X_{D_n} \) (\( n \geq 4 \)).

Suppose \( D^\perp \cong D_n(n \geq 4) \). For any \( 5 \leq i \leq n + 2 \), denote by \( w_i \) the composition of reflections

\[
\text{w}_i = r_{1-E_1-E_3-E_i} \circ r_{1-E_2-E_4-E_i}.
\]

Let \( s_w = (e_{i_1}^w)_{i \in I} \) be the seed we get after performing the following sequence of mutations

\[
[5, 1, 3, 2, 4, 5, 6, 1, 3, 2, 4, 6, \ldots \, n + 2, 1, 3, 2, 4, n + 2] \quad (n \geq 4).
\]

We denote this sequence of mutations by \( W_n \). Let \( \gamma : s_w \to s \) be the seed isomorphism such that if \( n \) is even, \( \gamma(e_{i_1}^w) = e_{i_1} \) for all \( i \) and if \( n \) is odd, \( \gamma(e_{i_1}^w) = e_{i_1}^+ \). \( \gamma(e_{i_2}^w) = e_{i_2} \), for all \( i \geq 5 \). It follows from the mutation formula for a single reflection of

\[
r_{1-E_1-E_2-E_3} \quad (e_1 \in \{ 1, 2 \}, \ e_2 \in \{ 3, 4 \} \text{ and } e_3 \in \{ 5, 6, \ldots , n + 1, n + 2 \})
\]

that the action of

\[
\text{w}_5 \circ \text{w}_6 \circ \cdots \circ \text{w}_{n+2}
\]

is given by \( \gamma \circ W_n \).

**Lemma 4.3.** For any \( 5 \leq i, j \leq n + 2 \) (\( n \geq 4 \)), \( w_i \) and \( w_j \) commute with each other. In particular, the Weyl group element \( w_5 \circ w_6 \circ \cdots \circ w_{n+2} \) is of order 2.

**Proof.** For any \( 5 \leq i \leq n + 2 \), since

\[
\langle l - E_1 - E_3 - E_i, l - E_2 - E_4 - E_i \rangle = 0,
\]

the reflections \( r_{l-E_1-E_3-E_i} \) and \( r_{l-E_2-E_4-E_i} \) commute. Therefore, \( w_i \) is of order 2 for any \( 5 \leq i \leq n + 2 \). For any \( 5 \leq i, j \leq n + 2 \) with \( i \neq j \),

\[
w_jw_iw_j = (w_j \circ r_{l-E_1-E_3-E_i} \circ w_j) (w_j \circ r_{l-E_2-E_4-E_i} \circ w_j)
\]

\[\text{Since cluster transformations are unique up to seed automorphisms, } w \text{ is unique up to compositions by seed automorphisms of } s, \text{ which are generated by reflections by simple roots of the form } E_{i_1} - E_{i_2} \text{ where } E_{i_1} \text{ and } E_{i_2} \text{ are two distinct exceptional curves along the same boundary component of } D.\]
After the sequence of mutations 

Lemma 4.4. Therefore, \( w_i \) and \( w_j \) commute for any \( 5 \leq i, j \leq n + 2 \). Together with that \( w_i \) is of order 2 for any \( 5 \leq i \leq n + 2 \), we conclude that \( w_5 \circ w_6 \circ \cdots \circ w_{n+2} \) is of order 2. \( \square \)

Let \( \Pi \) be the subgroup of the symmetric group \( S_{n+2} \) generated by involutions 
\[
(1 \ 2), (3 \ 4), (5 \ 6), (6 \ 7), \ldots, (n + 1 \ n + 2)
\]
and \( \Pi' \) the subgroup of \( \Pi \) generated by involutions \( (1 \ 2) \) and \( (3 \ 4) \).

**Lemma 4.4.** After the sequence of mutations \( W_n \) (4.2), the \( c \)-matrix \( C^w \) of \( s_w \) is of the form 
\[
\begin{pmatrix}
M_n & -I_{n-2} \\
-I_{n-2} & -I_{n-2}
\end{pmatrix}
\]
where \( M_n \) is a \( 4 \times 4 \) matrix and \( I_{n-2} \) is the identity matrix of rank \( n - 2 \).

**Proof.** We prove this statement by induction on \( n \). When \( n = 4 \), after the sequence of mutations \( W_4 \), the \( c \)-matrix is 
\[
\begin{pmatrix}
-1 & -1 \\
-1 & -1 \\
-1 & -1 \\
-1 & -1
\end{pmatrix}
\]
So the statement holds for the base case. Suppose for \( n = k(k \geq 4) \), the statement is true. When \( n = k + 1 \), by induction hypothesis, after the sequence of mutations \( W_k \), the \( c \)-matrix is of the form 
\[
C^\alpha = \begin{pmatrix}
M_k & -I_{k-2} & * \\
* & -I_{k-2} & * \\
* & * & *
\end{pmatrix}
\]
Here \( M_k \) is a \( 4 \times 4 \) matrix and we do not know yet what the last column and the last row of the \( c \)-matrix \( C^\alpha \) are. We claim that the last row of \( C^\alpha \) has 1 at the last entry and 0 at all other entries; the last column of \( C^\alpha \) has 1 at the first four and the last entry and 0 at all other entries. Indeed, by the mutation formula for \( c \)-vectors, since we have not mutated in the \( (k + 3) \)-th direction yet, the last row of \( C^\alpha \) must have all but the last entries equal to 0. Denote the \( i \)-th-column of the \( c \)-matrix \( C^\alpha \) by \( C^\alpha_i \). To see that the last column \( C^\alpha_{k+3} \) is of the desired form, recall that for the initial seed \( s = (e_i)_{i \in I} \), under the identification \( D^\perp \cong K \) as given in Theorem 5.5 of [GHK13], \( e_{k+3} - e_5 \) is identified with the root \( E_{k+3} - E_5 \) and the root 
\[
w_5 \circ w_6 \circ \cdots \circ w_{k+2}(E_{k+3} - E_5) = 2l - \sum_{i=1}^{5} E_i - E_{k+3}
\]
is identified with \( \sum_{i=1}^{5} e_i + e_{k+3} \) in \( K \). So we know that 
\[
C^\alpha_{k+3} - C^\alpha_5 = \sum_{i=1}^{5} e_i + e_{k+3}.
\]
Since \( C^\alpha_5 = -e_5 \), we know that \( C^\alpha_{k+3} = e_1 + e_2 + e_3 + e_4 + e_{k+3} \). Thus, if we continue to perform the last reflection \( w_{k+3} \), the \( c \)-matrix will be of the form 
\[
C^\beta = \begin{pmatrix}
M_{k+1} & -I_{k-2} & * \\
* & -I_{k-2} & *
\end{pmatrix}
\]

18
Again, we consider the action of the Weyl group on \( D^\perp \). The root

\[
  w_5 \circ w_6 \circ \cdots \circ w_{k+2} \circ w_{k+3}(E_{k+3} - E_5) = E_5 - E_{k+3}
\]
is identified with \( e_5 - e_{k+3} \), so

\[
  C_{k+3}^\beta - C_5^\beta = e_5 - e_{k+3}.
\]

Since \( C_5^\beta = -e_5 \), we have \( C_{k+3}^\beta = -e_{k+3} \). Hence the \( c \)-matrix is of the desired form for \( n = k + 3 \). Thus the induction is complete. \( \square \)

**Theorem 4.5.** The Donaldson-Thomas transformation of \( X_{D_n} \) (\( n \geq 4 \)) is cluster.

**Proof.** By Lemma 3.7 after the sequence of mutations as given in 4.2 the \( c \)-matrix \( C^w \) is of the form

\[
  C^w = \begin{pmatrix}
    M_n & -I_{n-2}
  \end{pmatrix}
\]

where \( M_n \) is a \( 4 \times 4 \) matrix and \( I_{n-2} \) is the identity matrix of rank \( n - 2 \). By the relationship between \( g \)-matrix and \( c \)-matrix as recorded in Lemma 5.12 of [GHKK18], we know that the \( g \)-matrix of \( s_w \) is of the form

\[
  G^w = \begin{pmatrix}
    M'_n & -I_{n-2}
  \end{pmatrix}
\]

where \( M'_n \) is a \( 2 \times 2 \) matrix and \( I_{n-2} \) is the identity matrix of rank \( n - 2 \). Observe that the mutation sequence \( W_n \) is equivalent to the mutation sequence

\[
  W'_n : [5, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 6, \ldots, n + 2, 1, 2, 3, 4, n + 2]
\]

that is, \( W_n \) and \( W'_n \) give rise to the same \( c \)-matrix. Let \( \Pi' \) be the subgroup of the symmetric group \( S_{n+2} \) generated by involutions \( (1 \ 2) \) and \( (3 \ 4) \). Since \( W_n' \) is \( \Pi' \)-constrained, by Lemma 3.7 \( G^w \) is \( \Pi' \)-invariant and hence \( M'_n \) is \( \Pi' \)-invariant. Let \( C^+_w \) be the cluster chamber corresponding to the seed \( s_w \). Now, to show that Donaldson-Thomas transformation for \( X_{D_n} \) is cluster, it remains to show that the chamber \( C^+_w \) and and the chamber \( C^-_w \) are mutationally equivalent. Since \( M'_n \) is \( \Pi' \)-invariant, by first projecting to the first 4 coordinates of \( M'_n \) and then applying the folding procedure, we are reduced to the case of Kronecker 2-quiver. Then the mutational equivalence between \( C^+_w \) and \( C^-_w \) follows from the fact that the cluster complex of the scattering diagram of the Kronecker 2-quiver fill the whole plane except for a single ray. \( \square \)

**Remark 4.6.** It follows from the proof of the above theorem that, for \( n \geq 5 \), the Weyl group element \( w \) and \( DT_{X_{D_n}} \) differs only by a finite alternating sequence of mutations 1, 2 and 3, 4. In fact, one can show that if \( n \geq 5 \) and \( n \) is odd, \( w \) and \( DT_{X_{D_n}} \) differs by

\[
  \begin{bmatrix}
    1, 2 & 3, 4, 1, 2, & 3, 4, 1, 2, & \cdots & 3, 4, 1, 2
  \end{bmatrix}
\]

and if \( n \geq 5 \) and \( n \) is even, \( w \) and \( DT_{X_{D_n}} \) differs by

\[
  \begin{bmatrix}
    3, 4, 1, 2, & 3, 4, 1, 2, & 3, 4, 1, 2
  \end{bmatrix}
\]

4.3. **Factorizations of DT-transformations as cluster transformations for \( X_{B_n} \) (\( n = 6, 7, 8 \)).
4.3.1. $D^\perp \simeq E_6$. For $D^\perp \simeq E_6$, let
\begin{align*}
\alpha_1 &= 2l - E_1 - E_2 - E_3 - E_4 - E_6 - E_7 \\
\alpha_2 &= 2l - E_1 - E_2 - E_3 - E_5 - E_6 - E_8 \\
\alpha_3 &= 2l - E_1 - E_2 - E_4 - E_5 - E_7 - E_8.
\end{align*}
Notice that for $1 \leq i, j \leq 3$, $r_{\alpha_i}$ and $r_{\alpha_j}$ commute. For each $r_{\alpha_i}$, we could further factor $r_{\alpha_i}$ as reflections with respect to simple roots. For example we could factor $r_{\alpha_1}$ as
\begin{align*}
  r_{\alpha_1} &= r_{1-E_1-E_3-E_6} \circ r_{1-E_1-E_4-E_7} \circ r_{1-E_2-E_3-E_6} \circ r_{1-E_2-E_4-E_7}.
\end{align*}
The Weyl group element
\begin{align*}
  r_{\alpha_1} \circ r_{\alpha_2} \circ r_{\alpha_3}
\end{align*}
differs from the DT-transformation of $\mathcal{X}_{E_6}$ by two mutations $[1,2]$. See Appendix A for an explicit mutation sequence for $\text{DT} \mathcal{X}_{E_6}$.

4.3.2. $D^\perp \simeq E_7$. For $D^\perp \simeq E_7$, let
\begin{align*}
\alpha_1 &= 2l - E_1 - E_2 - E_3 - E_4 - E_6 - E_7 \\
\alpha_2 &= 2l - E_1 - E_2 - E_3 - E_5 - E_6 - E_8 \\
\alpha_3 &= 2l - E_1 - E_2 - E_4 - E_5 - E_6 - E_9 \\
\alpha_4 &= 2l - E_1 - E_3 - E_4 - E_5 - E_8 - E_9 \\
\alpha_5 &= 2l - E_1 - E_2 - E_3 - E_5 - E_7 - E_9 \\
\alpha_6 &= 2l - E_1 - E_2 - E_4 - E_5 - E_7 - E_8.
\end{align*}
Notice that for $1 \leq i, j \leq 6$, $r_{\alpha_i}$ and $r_{\alpha_j}$ commute. The Weyl group element
\begin{align*}
  \prod_{i=1}^{6} r_{\alpha_i}
\end{align*}
agrees with $\text{DT} \mathcal{X}_{E_7}$. See Appendix A for an explicit mutation sequence for $\text{DT} \mathcal{X}_{E_7}$.

4.3.3. $D^\perp \simeq E_8$. For $D^\perp \simeq E_8$, let
\begin{align*}
\alpha &= 6l - 3(E_1 + E_2) - 2(E_3 + E_4 + E_5) - (2E_6 + E_7 + E_8 + E_9 + E_{10})
\end{align*}
\begin{align*}
\beta_1 &= 2l - E_1 - E_2 - E_3 - E_4 - E_7 - E_8 \\
\beta_2 &= 2l - E_1 - E_2 - E_4 - E_5 - E_7 - E_9 \\
\beta_3 &= 2l - E_1 - E_2 - E_3 - E_5 - E_7 - E_{10} \\
\beta_4 &= 2l - E_1 - E_2 - E_3 - E_4 - E_9 - E_{10} \\
\beta_5 &= 2l - E_1 - E_2 - E_4 - E_5 - E_8 - E_{10} \\
\beta_6 &= 2l - E_1 - E_2 - E_3 - E_5 - E_8 - E_9.
\end{align*}
Notice that for any $1 \leq i, j \leq 6$, $r_{\beta_i}$ and $r_{\beta_j}$ commute and $r_{\alpha}$ commutes with $r_{\beta_i}$ for any $i$. We could factor $r_{\alpha}$ as
\begin{align*}
  r_{\alpha} &= r_{\gamma_1} \circ r_{\gamma_2} \circ r_{\delta_1} \circ r_{\delta_2}
\end{align*}
where for $\epsilon \in \{1,2\}$,
\begin{align*}
  \gamma_\epsilon &= 3l - \sum_{i=1}^{8} E_i - E_\epsilon \\
  \delta_\epsilon &= 3l - \sum_{i=1}^{6} E_i - (E_9 + E_{10}) - E_\epsilon.
\end{align*}
We could factor $r_{\gamma_i}$ and $r_{\delta_j}$ into compositions of reflections with respect to simple roots. For example, we could factor $r_{\gamma_1}$ as

$$r_{\gamma_1} = r_{1-E_1-E_5-E_8} \circ r_{2-E_1-E_2-E_3-E_4-E_5} \circ r_{1-E_1-E_5-E_8}$$

and we know how to further factor $r_{2-E_1-E_2-E_3-E_4-E_5}$. The Weyl group element

$$r_\alpha \circ \prod_{i=1}^{6} r_{\beta_i}$$

agrees with $\DT_{X_{E_n}}$. See Appendix A for an explicit mutation sequence for $\DT_{X_{E_8}}$.

Combine results in this subsection, we have the following theorem:

**Theorem 4.7.** The Donaldson-Thomas transformation of $X_{E_n}$ $(n = 6, 7, 8)$ is cluster.

**Corollary 4.8.** The Donaldson-Thomas transformation of $A_{\text{prin}, s}$ is cluster.

**Proof.** This follows immediately from Theorem 4.5, Theorem 4.7 and the definition of $\DT_{A_{\text{prin}, s}}$ (cf. [CS16]). \qed

**4.4. Application of folding to cases where** $D^1 \cong D_n$ $(n \geq 4)$ or $E_n(n = 6, 7, 8)$. Let us recall some basic definitions of cluster structures for log Calabi-Yau varieties.

**Definition 4.9.** Given a log Calabi-Yau variety $V$, if there exists a cluster variety $V' = \bigcup_{w \in \Sigma_0} T_{L,w}$ of type $A, \mathcal{X}$ or $A_{\text{prin}}$, together with a birational morphism

$$\iota : V' \to V$$

such that for any pair of adjacent seeds $s_w \leftrightarrow s_{w'}$ in $\Sigma_0$, the restriction of $\iota$ to $T_{L,w} \cup_{\mu \in \Phi^+} T_{L,w}$ is an open embedding into $V$, then we say that the cluster torus charts $T_{L,w}$ together with their embeddings $\iota_{s_w} : T_{L,w} \to V$ into $V$ provide an *atlas of cluster torus charts* for $V$. If $\iota$ is an isomorphism outside strata of codimension at least 2 in the domain and range, then we say that the atlas of cluster torus charts $\{(T_{L,w}, \iota_{s_w})_{w \in \Sigma_0}\}$ provides a *cluster structure* for $V$. In this case, the codimension 2 condition guarantees that $V$ and the cluster variety $V'$ have the same ring of regular functions.

**Remark 4.10.** In the above definition, by calling $V$ a variety, we are assuming that $V$ is an integral, separated scheme over $\mathbb{C}$, though possibly not of finite type. As shown in [GHK15], $A$ and $A_{\text{prin}}$ spaces are always separated while $X$ spaces are in general not (cf. Theorem 3.14 and Remark 4.2 of [GHK15]). In this light, calling the $X$ spaces $X$ cluster varieties can be very misleading, though the terminology has already been accepted in the cluster literature. In the above definition, if $V'$ is of type $A$ or $A_{\text{prin}}$, we could just assume that $\iota$ is an open embedding. If $V'$ is of type $X$, by requiring only the restriction of $\iota$ to any pair of adjacent cluster tori to be an open embedding into $V$, we allow certain loci in $V'$ to get identified when mapped into $V$ and therefore get around the issue of non-separateness of $V'$.

**Definition 4.11.** Given a log Calabi-Yau variety $V$ and two atlases of cluster torus charts

$$\mathcal{T}_1 = \{(T_{L,w}, \iota_{s_w})_{w \in \Sigma_0}\}, \quad \mathcal{T}_2 = \{(T_{L',w}, \iota_{s_w})_{v \in \Sigma_0'}\}$$

for $V$, we say $\mathcal{T}_1$ and $\mathcal{T}_2$ are *non-equivalent atlases* if for any $w$ in $\Sigma_0$, there is no $v$ in $\Sigma_0'$ such that the embeddings $\iota_{s_w} : T_{L,w} \to V$ and $\iota_{s_v} : T_{L',v} \to V$ have the same image in $V$. Given two non-equivalent atlases of cluster torus charts for $V$, we say they provide *non-equivalent cluster structures* for $V$ if both atlases cover $V$ up to codimension 2.
Now, we are ready to apply the folding technique to families of log Calabi-Yau surfaces. For $D^\perp \simeq D_n (n \geq 4)$, let $\Pi$ be the subgroup of the symmetric group $S_{n+2}$ generated by involutions
\[(1 \, 2), \ (3 \, 4), \ (5 \, 6), (6 \, 7), \cdots , (n+1 \, n+2).\]
For $D^\perp \simeq E_n (n = 6, 7, 8)$, let $\Pi$ be the subgroup of the symmetric group $S_{n+2}$ generated by involutions
\[(1 \, 2), \ (3 \, 4), (45), (6 \, 7), (7 \, 8), \cdots , (n+1 \, n+2)\]
Let $\mathfrak{s}$ be the folded seed via the symmetry of $\Pi$ on $s$, then for $D^\perp \simeq D_n (n \geq 4)$, $\mathfrak{s}$ has the exchange matrix equal to
\[
\begin{pmatrix}
0 & -2 & 2 \\
2 & 0 & -2 \\
-(n-2) & n-2 & 0
\end{pmatrix}
\]
and for $D^\perp \simeq E_n (n = 6, 7, 8)$, $\mathfrak{s}$ has the exchange matrix equal to
\[
\begin{pmatrix}
0 & -2 & 2 \\
3 & 0 & -3 \\
-(n-3) & n-3 & 0
\end{pmatrix}
\]
The $\mathcal{X}$-cluster variety $\mathcal{X}_{\mathfrak{s}}$ is a locally closed cluster subvariety of $\mathcal{X}_s$. By subsection 3.3 $\mathcal{X}_{\mathfrak{s}}$ is a maximal degenerate 1-parameter subfamily of $\mathcal{X}_s$.

As we have seen in subsection 4.2 and subsection 4.3, there exists a special element $\mathfrak{w}$ of order 2 in the Weyl group of $(Y,D)$ that either coincides with the Donaldson-Thomas transformation of $\mathcal{X}_s \simeq \mathcal{X}_{D^\perp}$ or closely relates to it. For $D^\perp \simeq D_n (n \geq 4)$, $\mathfrak{w}$ is given by \ref{4.3} for $D^\perp \simeq E_6$, $E_7$, and $E_8$, $\mathfrak{w}$ is given by \ref{4.3} \ref{4.7} and \ref{4.10} respectively.

**Theorem 4.12.** The scattering diagram $\mathfrak{D}_{\mathfrak{s}}$ has two distinct subfans corresponding to $\Delta^+_{\mathfrak{s}}$ and $\Delta^-_{\mathfrak{s}}$ respectively. The action of Weyl group element $\mathfrak{w}$ on $\mathfrak{D}_{\mathfrak{s}}$ descends to $\mathfrak{D}_{\mathfrak{s}}$ and interchanges $\Delta^+_{\mathfrak{s}}$ and $\Delta^-_{\mathfrak{s}}$. In particular, if we build a variety $\bar{A}_{\text{prim},\mathfrak{s}}$ using both $\Delta^+_{\mathfrak{s}}$ and $\Delta^-_{\mathfrak{s}}$ as in Theorem 1.7 of \cite{YZ}, then $\Delta^+_{\mathfrak{s}}$ and $\Delta^-_{\mathfrak{s}}$ provide non-equivalent atlases of cluster torus charts for $\bar{A}_{\text{prim},\mathfrak{s}}$.

**Proof.** Let $C^+_w$ the cluster chamber that $C^+_s$ is mapped to under the action of $\mathfrak{w}$ on $\mathfrak{D}_{\mathfrak{s}}$. As we have seen in subsection 4.2 and subsection 4.3 the cluster chamber $C^+_w$ is $\Pi$-invariant. By Lemma 4.7 $C^+_w$ corresponds to a chamber $C^+_w$ in $\mathfrak{D}_{\mathfrak{s}}$. Since $C^+_w$ and $C^-_w$ are related by $\Pi$-constrained mutations, $\bar{C}^+_w$ and $\bar{C}^-_w$ belong to the same subfan $\Sigma_{\text{res}}$ which coincides $\Delta^-_{\mathfrak{s}}$. Let $U = Y \setminus D$. Then under the set-theoretic identification of $\mathcal{X}_s(\mathbb{R}^T)$ with $M_\mathfrak{R}$ as given by the seed $\mathfrak{s}$, $U^{\text{trop}}(\mathbb{R})$ can be identified with the two-dimensional subspace of $M_\mathfrak{R}$ defined by the equations
\[
\langle e_1 - e_2, \cdot \rangle = 0, \langle e_3 - e_4, \cdot \rangle = 0,
\langle e_5 - e_6, \cdot \rangle = \langle e_6 - e_7, \cdot \rangle \cdots = \langle e_{n+1} - e_{n+2}, \cdot \rangle = 0,
\langle e_1 + e_3 + e_5, \cdot \rangle = 0
\]
if $D^\perp \simeq D_n (n \geq 4)$ and by the equations
\[
\langle e_1 - e_2, \cdot \rangle = 0, \langle e_3 - e_4, \cdot \rangle = \langle e_4 - e_5, \cdot \rangle = 0,
\langle e_6 - e_7, \cdot \rangle = \langle e_7 - e_8, \cdot \rangle \cdots = \langle e_{n+1} - e_{n+2}, \cdot \rangle = 0,
\langle e_1 + e_3 + e_6, \cdot \rangle = 0
\]
if $D^\perp \simeq E_n$ $(n = 6, 7, 8)$. Thus via the embedding $q^\ast : M^\ast \mapsto \overline{M}^\ast$, $U^{\text{trop}}(\mathbb{R})$ can be identified as the hyperplane in $\overline{M}^\ast$ given by the equation

\[ \langle e_{11} + e_{13} + e_{15}, \cdot \rangle = 0 \]

if $D^\perp \simeq D_n$ $(n \geq 4)$ and by the equation

\[ \langle e_{11} + e_{13}, \cdot \rangle = 0 \]

if $D^\perp \simeq E_n$ $(n = 6, 7, 8)$. By Theorem 4.5 of [11], $U^{\text{trop}}(\mathbb{R})$ intersects with $\Delta^+_s$ only at the origin. Thus, $U^{\text{trop}}(\mathbb{R})$ intersects with $\Delta^+_s$ and $\Delta^-_s$ only at the origin. Since the chamber $C^+_s$ of $\Delta^+_s$ and the chamber $C^-_s$ of $\Delta^-_s$ live in different half spaces separated by $U^{\text{trop}}(\mathbb{R})$ as a hyperplane in $\overline{M}^\ast$, we conclude that $\Delta^+_s$ and $\Delta^-_s$ are two distinct subfans in $D^\ast$.

It follows from Proposition 3.4 and Proposition 3.6 of [12] that the action of Weyl group element $w$ on $D^\ast$ descends to $D^\ast$. Since the induced action of $w$ on $D^\ast$ maps $\Delta^+_s$ to $\Delta^-_s$ and $w$ is of order 2, we conclude that it interchanges $\Delta^+_s$ and $\Delta^-_s$. The last statement of the theorem follows from Theorem 1.7 of [12].

Let us end by the final remark that there are other, potentially more interesting, ways to apply the folding procedure in positive non-acyclic cases. For example when $D^\perp \simeq D_{2m}$ $(m \geq 2)$, for each $1 \leq l \leq m - 1$, denote by $\alpha_l$ the root $2l - E_1 - E_2 - E_3 - E_4 - E_{2l+3} - E_{2l+4}$. Observe that we have the following relation in the Weyl group

\[ r_{\alpha_l} = w_{2l+3} \circ w_{2l+4} \]

where $w_l$ is defined as in [11]. Let $\mathcal{P}$ be the power set of $\{1, \cdots , l\}$. Given $\varpi = \{i_1, \cdots , i_h\} \in \mathcal{P}$, let $s_{\varpi}$ be the seed after performing the sequence of reflections

\[ r_{\alpha_{i_1}} \circ \cdots \circ r_{\alpha_{i_h}}. \]

Since the subgroup of the Weyl group generated by $r_{\alpha_l}$ is abelian, $C^+_{\varpi}$ is independent of the choice of ordering of elements in $\varpi$. We have the following conjecture:

**Conjecture 4.13.** For any two distinct elements $\varpi, \varpi'$ in $\mathcal{P}$, $\Delta^+_{\varpi, \Pi}$ and $\Delta^+_{\varpi', \Pi}$ has trivial intersection.

The above conjecture is based on the numerical evidence that the chambers in $\Delta^+_{\varpi, \Pi}$ is contained in the region in $\mathcal{X}^{\text{trop}}_{D_{2m}}$ cut out by the equation

\[ (z^{e_1 + e_3 + e_5})^T \geq 0, (z^{e_1 + e_3 + e_5})^T \geq 0, \cdots , (z^{e_1 + e_3 + e_{2m+1}})^T \geq 0 \]

while given any nonempty subset $\varpi$ in $\mathcal{P}$, $C^+_{\varpi}$ is outside the above region.

**Appendix A. Mutation sequences for $\text{DT}_X^{\mathcal{X}_{\mathcal{E}_6}} (n = 6, 7, 8)$ in SAGE code.**

In the ClusterSeed package of SAGE, the indexing for cluster seeds starts from 0 instead of 1. Thus we need to shift mutation sequences given in the paper by $-1$ in SAGE code.

**A.1. Mutation sequence for $\text{DT}_X^{\mathcal{X}_{\mathcal{E}_6}}$.** In the following SAGE code, the first three lines of the mutation sequence give reflections $r_{\alpha_i} (i = 1, 2, 3)$ as defined in [11]. The two mutations $[0, 1]$ in the last line of the mutation sequence is the difference between the Weyl group element $w = r_{\alpha_1} \circ r_{\alpha_2} \circ r_{\alpha_3}$ and DT_{X_{\mathcal{E}_6}}.

sage: $S =$ ClusterSeed(matrix([[0,0,-1,-1,1,1,1],[0,0,-1,-1,1,1,1], ....: [1,1,0,0,-1,-1,1,1,1,0,-1,1,1,1,0,0,0,0,0,-1,1,1,1,0,0,0,0,0]]));

sage: $S$.mutate([0,2,5,3,6,0,1,2,5,3,6,1]);
An alternative, shorter mutation sequence for DT_{E_6} is as follows:

```sage
sage: S = ClusterSeed(matrix([[0,0,-1,-1,-1,1,1,1], [0,0,-1,-1,-1,1,1,1], ....: [1,1,0,0,0,-1,-1,-1], [1,1,0,0,0,-1,-1,-1], [1,1,0,0,0,-1,-1,-1], ....: [-1,-1,1,1,1,0,0,0], [-1,-1,1,1,1,0,0,0], [-1,-1,1,1,1,0,0,0]]));
sage: S.mutate([0,2,5,3,6,4,7,0,1,2,5,3,6,4,7,1]);
sage: S.g_matrix()
```

```
                0 1 0 0 0 0 0 0
0 1 0 0 0 0 0 0
-1 -1 0 0 0 -1 0 0
-1 -1 0 0 0 0 -1 0
-1 -1 0 0 0 0 0 0
0 0 -1 0 0 0 0 0
0 0 0 -1 0 0 0 0
0 0 0 0 -1 0 0 0
```

A.2. **Mutation sequence for DT_{E_7}**. In the following SAGE code, each line of the mutation sequence gives one of the reflections $r_{\alpha_i}$ ($i = 1, 2, \ldots, 6$) as defined in 4.6. In this case, the Weyl group element $w = \prod_{i=1}^{6} r_{\alpha_i}$ agrees with DT_{E_7}.

```sage
sage: S = ClusterSeed(matrix([[0,0,-1,-1,-1,1,1,1,1], [0,0,-1,-1,-1,1,1,1,1], ....: [1,1,0,0,0,-1,-1,-1,-1], [1,1,0,0,0,-1,-1,-1,-1], [1,1,0,0,0,-1,-1,-1,-1], ....: [-1,-1,1,1,1,0,0,0,0], [-1,-1,1,1,1,0,0,0,0], [-1,-1,1,1,1,0,0,0,0], [-1,-1,1,1,1,0,0,0,0]]));
sage: S.mutate([0,2,3,5,6,0,1,2,3,5,6,1]);
sage: S.use_fpolys(False);
sage: S.mutate([0,2,3,5,6,0,1,2,3,5,6,1]);
```

```sage
                -1 0 0 0 0 0 0 0
0 -1 0 0 0 0 0 0
0 0 -1 0 0 0 0 0
0 0 0 -1 0 0 0 0
0 0 0 0 -1 0 0 0
0 0 0 0 0 -1 0 0
0 0 0 0 0 0 -1
```

```sage
```

```
```
A.3. **Mutation sequence for** $\text{DT}_{\mathcal{X}_{\mathfrak{g}_8}}$. In the following SAGE code, the first four lines of the mutation sequence give the reflection $r_\alpha$ as defined in 4.8. The next six lines give reflections $r_{\beta_i}(i = 1, 2, \cdots, 6)$ as defined in 4.9. In this case, the Weyl group element $w = r_\alpha \circ \prod_{i=1}^6 r_{\beta_i}$ also agrees with $\text{DT}_{\mathcal{X}_{\mathfrak{g}_8}}$.

```python
sage: S = ClusterSeed(matrix([[0,0,-1,-1,-1,-1,1,1,1,1],[0,0,-1,-1,-1,-1,1,1,1,1],
....: [1,1,0,0,-1,-1,-1,-1,-1,-1],[1,1,0,0,-1,-1,-1,-1,-1,-1],
....: [-1,-1,1,1,0,0,0,0,0,0],[-1,-1,1,1,0,0,0,0,0,0],[-1,-1,1,1,0,0,0,0,0,0],
....: [-1,-1,1,1,0,0,0,0,0,0]])

sage: S.mutate([0,4,7,0]);S.mutate([0,2,3,5,6,0,1,2,3,5,6,1]);S.mutate([1,4,7,1]);

sage: S.use_fpolys(False);

sage: S.mutate([0,2,3,7,8,0,1,2,3,7,8,1]);

sage: S.mutate([0,2,4,6,8,0,1,2,4,6,8,1]);

sage: S.mutate([0,3,4,6,7,0,1,3,4,6,7,1]);

sage: S.g_matrix()
```
\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

References

[Ca85] J. Carlson, *The one-motif of an algebraic surface*, Compos. Math., 56 (1985) 271–314.

[FZ02] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15, 2 (2002), 497–529 (electronic).

[FZ07] S. Fomin and A. Zelevinsky, *Cluster algebras. IV. Coefficient*, Compos. Math. 143 (2007), 112–164.

[F] R. Friedman, *On the geometry of anticanonical pairs*, preprint arXiv:1502.02560v2.

[FG06] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci., 103 (2006) 1–211.

[GS16] A. Goncharov and L. Shen, *Donaldson-Thomas transformations of moduli spaces of G-local systems*, preprint arXiv:1602.06479.

[GHK11] M. Gross, P. Hacking, and S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publ. Math. Inst. Hautes Études Sci. 122 (2015), 65168. arXiv:1106.4977

[GHK12] M. Gross, P. Hacking, and S. Keel, *Moduli of surfaces with an anti-canonical cycle*, Compos. Math., 151(2), 265-291. doi:10.1112/S0010437X14007611.

[GHK15] M. Gross, P. Hacking, and S. Keel, *Birational geometry of cluster algebras*, Algebraic Geometry 2 (2) (2015) 137–175.

[GHKK18] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical Bases for Cluster Algebras*, J. Amer. Math. Soc., 31 (2018), 497-608, arXiv: 1411.1394.

[KY] S. Keel, Y. Yu, *The Frobenius structure theorem for affine log Calabi-Yau varieties containing a torus*, preprint arXiv:1908.09861v1.

[L81] E. Looijenga, *Rational surfaces with an anticanonical cycle*. Ann. of Math. (2) 114 (1981), no. 2, 267-322.

[Lu90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990), no. 2, 447498.

[Lu94] G. Lusztig, *Total positivity in reductive groups*, Lie Theory and Geometry: In Honor of B. Kostant, Progr. in Math. 123, Birkhäuser, 1994, 531-568.

[TM] T. Mandel, *Classification of rank 2 cluster varieties*, SIGMA 15 (2019), 042, arXiv: 1407.6241v5.

[YZ] Y. Zhou, *Cluster structures and subfans in scattering diagrams*, SIGMA 16 (2020), 013, 35 pages, arXiv: 1901.04166v3.