Characterizations of Weighted BMO Space and Its Application

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Abstract In this paper, we prove that the weighted BMO space

$$\text{BMO}^p(\omega) = \left\{ f \in L^1_{\text{loc}} : \sup_Q |\chi_Q|^{-1} L^p(\omega)(f - f_Q)\omega^{-1} \chi_Q L^p(\omega) < \infty \right\}$$

is independent of the scale \( p \in (0, \infty) \) in sense of norm when \( \omega \in A_1 \). Moreover, we can replace \( L^p(\omega) \) by \( L^{p, \infty}(\omega) \). As an application, we characterize this space by the boundedness of the bilinear commutators \([b, T]_j \) \((j = 1, 2)\), generated by the bilinear convolution type Calderón–Zygmund operators and the symbol \( b \), from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega^{1-p}) \) with \( 1 < p_1, p_2 < \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). Thus we answer the open problem proposed by Chaffee affirmatively.

Keywords Boundedness, Calderón–Zygmund operators, characterization, commutators, weighted BMO space

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1 Introduction

A locally integrable function \( f \) is said to belong to BMO space if there exists a constant \( C > 0 \) such that for any cube \( Q \subset \mathbb{R}^n \),

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq C,$$

where \( f_Q = \frac{1}{|Q|} \int_Q f(x) dx \) and the minimal constant \( C \) is defined by \( \|f\|_* \).

There are a number of classical results that demonstrate BMO functions are the right collections to do harmonic analysis on the boundedness of commutators. A well-known result of Coifman, Rochberg and Weiss [4] states that the commutator

\([b, T](f) = bT(f) - T(bf)\)

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is bounded on some $L^p$, $1 < p < \infty$, if and only if $b \in \text{BMO}$, where $T$ is the Hilbert transform. Janson extended the result in [7] via the commutators of Calderón–Zygmund operators with smooth homogeneous kernels; Chanillo in [3] did the same for commutators of the fractional integral operator with the restriction that $n - \alpha$ be an even integer. The theory was then extended and generalized to several directions. For instance, Bloom [1] investigated the same result in the weighted setting; Uchiyama [16] extended the boundedness results on the commutator to compactness; Krantz and Li in [10] and [11] have applied commutator theory to give a compactness characterization of Hankel operators on holomorphic Hardy spaces $H^2(D)$, where $D$ is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$. It is perhaps for this important reason that the boundedness of $[b, T]$ attracted one’s attention among researchers in PDEs.

Recently, Chaffee [2] considered the multilinear setting and proved that for $0 \leq \alpha < 2n$, $1 < p_1, p_2 < \infty$, and $1/p_1 + 1/p_2 - \alpha/n = 1/q$, if $q > 1$, then

$$[b, T]_j : L^{p_1} \times L^{p_2} \to L^q \iff b \in \text{BMO} \quad (1.1)$$

for $j = 1, 2$, where $T$ is a bilinear operator of convolution type with a homogeneous kernel of degree $-2n + \alpha$. His proof required the use of Hölder inequality with $q$ and $q'$, and the exponent $q$ must be larger than 1. Thus, he asked

**Problem 1** If $1/2 < q < 1$ and $[b, T]_j$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^q$, is $b$ in BMO space?

At the same time, Wang, Jiang and Pan [18] obtained the similar result as (1.1) for bilinear fractional integral operator. They also asked

**Problem 2** If $\vec{b} = (b_1, b_2)$ and $[\Pi \vec{b}, I_\alpha]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^q$, is $\vec{b} \in \text{BMO} \times \text{BMO}$?

In this paper, we will give an answer of Problem 1 and show that the answer of Problem 2 is affirmative for the case $\vec{b} = (b, b)$ and bilinear Calderón–Zygmund operator. Moreover, we extend the results to weighted case. To state our result, we first give the following denotations.

We recall the definition of $A_p$ weight introduced by Muckenhoupt [13]. For $1 < p < \infty$ and a nonnegative locally integrable function $\omega$ on $\mathbb{R}^n$, $\omega$ is in the Muckenhoupt $A_p$ class if it satisfies the condition

$$[\omega]_{A_p} := \sup_Q \left( \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \omega(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \right) < \infty.$$  

And a weight function $\omega$ belongs to the class $A_1$ if

$$[\omega]_{A_1} := \frac{1}{|Q|} \int_Q \omega(x) \, dx \left( \text{ess sup}_{x \in Q} \omega(x)^{-1} \right) < \infty.$$  

We write $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

Let $\omega \in A_\infty$ and $p \in (0, \infty)$. We let $L^p(\omega)$ be the space of all measurable functions $f$ such that

$$\|f\|_{L^p(\omega)} := \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty.$$
Let $0 < p < \infty$. Given a nonnegative locally integrable function $\omega$, the weighted BMO space $\text{BMO}^p(\omega)$ is defined by the set of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$
\|f\|_{\text{BMO}^p(\omega)} = \sup_{Q} \left( \frac{1}{\omega(Q)} \int_Q |f(y) - f_Q|^p \omega(y)^{1-p} \, dy \right)^{1/p} = \sup_{Q} \frac{1}{\|\chi_Q\|_{L^p(\omega)}} \left\| \frac{(f - f_Q)\chi_Q}{\omega} \right\|_{L^p(\omega)} < \infty,
$$

where $\omega(Q) = \int_Q \omega(x) \, dx$. We write $\text{BMO}^1(\omega) = \text{BMO}(\omega)$ simple. In [6], García-Cuerva proved that if $\omega \in A_1$, $\text{BMO}(\omega) = \text{BMO}^p(\omega)$ for $1 < p < \infty$ with equivalence of the corresponding norms.

**Theorem 1.1** Let $\omega \in A_1$ and $X = L^{p,\infty}(\omega)$ with $1 < p < \infty$. Then

$$
\text{BMO}(\omega) = \text{BMO}_X(\omega)
$$

with equivalence of the corresponding norms.

**Theorem 1.2** Let $\omega \in A_1$ and $0 < r < 1$. Then

$$
\text{BMO}(\omega) = \text{BMO}^r(\omega)
$$

with equivalence of the corresponding norms.

**Remark 1.3** In the unweighted setting, Strömberg in [15] showed that for $0 < s \leq \frac{1}{2}$, $p > 0$, there exists a constant $C$ such that

$$
\gamma^{1/p} \|f\|_{\text{BMO}^s} \leq \|f\|_{\text{BMO}^p} \leq C \|f\|_{\text{BMO}^s},
$$

where

$$
\|f\|_{\text{BMO}^s} = \sup_{Q} \inf_{c} \inf_{t \geq 0} \{ t \geq 0 : \|f(x) - c\| > t \} < s|Q|.
$$

Recall that bilinear singular integral operator $T$ is a bounded operator which satisfies

$$
\|T(f_1, f_2)\|_{L^p} \leq C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}},
$$

for some $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ and the function $K$, defined off the diagonal $y_0 = y_1 = y_2$ in $(\mathbb{R}^n)^{2+1}$, satisfies the conditions as follow:

1. The function $K$ satisfies the size condition.

$$
|K(x, y_1, y_2)| \leq \frac{C}{(|x - y_1| + |x - y_2|)^{2n}};
$$

2. The function $K$ satisfies the regularity condition. For some $\gamma > 0$, if $|y_1 - y'_1| \leq \frac{1}{2} \max \{|x - y_1|, |x - y_2|\},$

$$
|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq \frac{C|y_1 - y'_1|^{\gamma}}{(|x - y_1| + |x - y_2|)^{2n+\gamma}};
$$
if \(|y_2 - y'_2| \leq \frac{1}{\gamma} \max\{|x - y_1|, |x - y_2|\},
\]

\[|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq \frac{C|y_2 - y'_2|}{(|x - y_1| + |x - y_2|)^{2n+\gamma}}.
\]

Then we say \(K\) is a bilinear Calderón–Zygmund kernel. If \(x \notin \text{supp}f_1 \cap \text{supp}f_2\), then \(T(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y_1, y_2)f_1(y_1)f_2(y_2)dy_1dy_2\).

The linear commutators are defined by

\[ [b, T]_1(f_1, f_2)(x) := b(x)T(f_1, f_2)(x) - T(bf_1, f_2)(x), \]

and

\[ [b, T]_2(f_1, f_2)(x) := b(x)T(f_1, f_2)(x) - T(f_1, bf_2)(x). \]

The iterated commutator is defined by

\[ [[\Pi b, T]](f_1, f_2)(x) := [b_2, [b_1, T]](f_1, f_2)(x). \]

In this paper, we say that an operator is of “convolution type” if the kernel \(K(x, y_1, y_2)\) is actually of the form \(K(x - y_1, x - y_2)\). The applications of Theorems 1.1 and 1.2 as follows.

**Theorem 1.4**  Let \(\omega \in A_1\), \(\vec{b} = (b, b)\) and \(T\) be a bilinear convolution type operator defined by

\[ T(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - y_1, x - y_2)f_1(y_1)f_2(y_2)dy_1dy_2\]

for all \(x \notin \text{supp}f_1 \cap \text{supp}f_2\), where \(K\) is a bilinear Calderón–Zygmund kernel and such that for any cube \(Q \subset \mathbb{R}^{2n}\) with \(0 \notin Q\), the Fourier series of \(\frac{1}{K}\) is absolutely convergent. For \(1 < p_1, p_2 < \infty\) with \(1/p = 1/p_1 + 1/p_2\), the following statements are equivalent:

(a1) \(b \in \text{BMO}(\omega)\);

(a2) There exists a positive constant \(C\) such that for \(j = 1, 2,\)

\[\| [b, T]_j(f_1, f_2) \cdot \omega^{-1} \|_{L^p(\omega)} \leq C \| f_1 \|_{L^{p_1}(\omega)} \| f_2 \|_{L^{p_2}(\omega)} .\]

(a3) There exists a positive constant \(C\) such that for \(j = 1, 2,\)

\[\| [b, T]_j(f_1, f_2) \cdot \omega^{-1} \|_{L^{p, \infty}(\omega)} \leq C \| f_1 \|_{L^{p_1}(\omega)} \| f_2 \|_{L^{p_2}(\omega)} .\]

(a4) There exists a positive constant \(C\) such that

\[\| [[\Pi b, T]](f_1, f_2) \cdot \omega^{-2} \|_{L^p(\omega)} \leq C \| f_1 \|_{L^{p_1}(\omega)} \| f_2 \|_{L^{p_2}(\omega)} .\]

(a5) There exists a positive constant \(C\) such that

\[\| [[\Pi b, T]](f_1, f_2) \cdot \omega^{-2} \|_{L^{p, \infty}(\omega)} \leq C \| f_1 \|_{L^{p_1}(\omega)} \| f_2 \|_{L^{p_2}(\omega)} .\]

Specially, if \(\omega(x) \equiv 1\), we have

**Corollary 1.5**  Let \(\vec{b} = (b, b)\) and \(T\) be a bilinear convolution type operator defined by

\[ T(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - y_1, x - y_2)f_1(y_1)f_2(y_2)dy_1dy_2\]

for all \(x \notin \text{supp}f_1 \cap \text{supp}f_2\), where \(K\) is a bilinear Calderón–Zygmund kernel and such that for any cube \(Q \subset \mathbb{R}^{2n}\) with \(0 \notin Q\), the Fourier series of \(\frac{1}{K}\) is absolutely convergent. For \(1 < p_1, p_2 < \infty\) with \(1/p = 1/p_1 + 1/p_2\), the following statements are equivalent:
Throughout this paper, the letter $L$ may change from one occurrence to another.

**Problem A** Let $\vec{b} = (b_1, b_2)$ with $b_1 \neq b_2$ and $[\Sigma \vec{b}, T] := [b_1, T]_1 + [b_2, T]_2$. If $[\Sigma \vec{b}, T]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^q$ for $j = 1, 2$; $[\Pi \vec{b}, T]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^p$; $[\Pi \vec{b}, T]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^{p,\infty}$.

By a same argument we also have the following result.

**Corollary 1.6** Let $\vec{b} = (b, b)$ and $I_\alpha$ be a bilinear fractional integral operator defined by

$$I_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(y_1)f_2(y_2)}{|x-y_1| + |x-y_2|^{2n-\alpha}} dy_1 dy_2.$$  

For $0 < \alpha < 2n$, $1 < p_1, p_2 < \infty$ with $1/q = 1/p_1 + 1/p_2 - \alpha/n$, the following statements are equivalent:

(c1) $b \in \text{BMO}$;
(c2) $[b, I_\alpha]_j$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^q$ for $j = 1, 2$;
(c3) $[b, I_\alpha]_j$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^{q,\infty}$ for $j = 1, 2$;
(c4) $[\Pi \vec{b}, I_\alpha]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^q$;
(c5) $[\Pi \vec{b}, I_\alpha]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^{q,\infty}$.

Finally, two open problems will be given.

**Problem B** Let $\vec{b} = (b_1, b_2)$ with $b_1 \neq b_2$. If $[\Pi \vec{b}, T]$ is a bounded operator from $L^{p_1} \times L^{p_2}$ to $L^q$, is $\vec{b}$ in $\text{BMO} \times \text{BMO}$?

## 2 Main Lemmas

Throughout this paper, the letter $C$ denotes constants which are independent of main variables and may change from one occurrence to another. $Q(x, r)$ denotes a cube centered at $x$, with side length $r$, sides parallel to the axes.

For $X = L^{q_2,\infty}(\omega)$, it is clear that $\text{BMO}^{q_2}(\omega)$ is contained in $\text{BMO}_X(\omega)$ and $\| \cdot \|_{\text{BMO}_X(\omega)} \leq \| \cdot \|_{\text{BMO}^{q_2}(\omega)} \leq \| \cdot \|_{\text{BMO}^{q_1}(\omega)}$ if $1 < q_2 \leq q_1 < \infty$. However, for $1 < q_1 < q_2 < \infty$, one has the reverse inequality as follows.

**Lemma 2.1** Let $1 < q_1 < q_2 < \infty$, $\omega \in A_\infty$ and $X = L^{q_2,\infty}(\omega)$. Then $\text{BMO}_X(\omega)$ is contained in $\text{BMO}^{q_2}(\omega)$ and $\| \cdot \|_{\text{BMO}^{q_2}(\omega)} \leq C \| \cdot \|_{\text{BMO}_X(\omega)}$.

**Proof** Let $f \in \text{BMO}_X(\omega)$. Given a fixed cube $Q \subset \mathbb{R}^n$, it is easy to see that $\| \chi_Q \|_{L^{q_1,\infty}(\omega)} = \omega(Q)^{1/q}$, then for any $\lambda > 0$,

$$\frac{1}{\omega(Q)^{1/q_2}}(\lambda^{q_2} \omega \{ x \in Q : |f(x) - f_Q| > \lambda \omega(x) \})^{1/q_2} \leq \| f \|_{\text{BMO}_X(\omega)};$$

that is,

$$\omega \{ x \in Q : |f(x) - f_Q| > \lambda \omega(x) \} \leq \| f \|_{\text{BMO}_X(\omega)}^{q_2} \omega(Q) \lambda^{-q_2}.$$

Choose

$$N = \| f \|_{\text{BMO}_X(\omega)} \left( \frac{q_1}{q_2 - q_1} \right)^{1/q_2}.$$
Thus,
\[
\int_Q |f(x) - f_Q|^{q_1} \omega(x)^{1-q_1} \, dx = \int_Q \left( \frac{|f(x) - f_Q|}{\omega(x)} \right)^{q_1} \omega(x) \, dx \\
= q_1 \int_0^\infty \lambda^{q_1-1} \omega \{ x \in Q : |f(x) - f_Q| > \lambda \omega(x) \} \, d\lambda \\
\leq q_1 \int_0^N \lambda^{q_1-1} \omega(x) \, d\lambda + q_1 \int_N^\infty \lambda^{q_1-1} \||f||_{BMO}^{q_2} \omega(x) |Q| \lambda^{-q_2} \, d\lambda \\
= \omega(Q) N^{q_1} + \frac{q_1}{q_2 - q_1} ||f||_{BMO}^{q_2} \omega(Q) N^{q_1 - q_2},
\]
which gives
\[
\left( \frac{1}{\omega(Q)} \int_Q |f(y) - f_Q|^{q_1} \omega(x)^{1-q_1} \, dy \right)^{1/q_1} \leq 2 \left( \frac{q_1}{q_2 - q_1} \right)^{1/q_2} ||f||_{BMO}^{q_2} \omega(Q).
\]
Then
\[
||f||_{BMO}^{q_1} \omega \leq 2 \left( \frac{q_1}{q_2 - q_1} \right)^{1/q_2} ||f||_{BMO}^{q_2} \omega
\]
and the lemma follows.

Let $\omega \in A_1$ and $d\mu(x) = \omega(x) \, dx$. For $0 < r < \infty$, we set
\[
||f||_{BMO_r(\omega)} = \sup_{Q \subset \mathbb{R}^n} \inf_{c \in \mathbb{R}} \left\{ \frac{1}{\mu(Q)} \int_Q \left( \frac{|f(x) - c|}{\omega(x)} \right)^r \, d\mu(x) \right\}^{1/r},
\]
and $BMO_r(\omega) = \{ f \in L_{loc} : ||f||_{BMO_r(\omega)} < \infty \}$.

**Lemma 2.2** Let $0 < r < 1$, $\omega \in A_1$ and $d\mu(x) = \omega(x) \, dx$. Suppose $||f||_{BMO_r(\omega)} = 1$ and for each cube $Q$ let $c_Q$ be the value which minimizes $\int_Q (|f(x) - c|/\omega(x))^r \, d\mu(x)$. Then
\[
\mu \left( \left\{ x \in Q : \frac{|f(x) - c_Q|}{\omega(x)} > t \right\} \right) \leq c_1 e^{-c_2 t} \mu(Q),
\]
where $c_1$ and $c_2$ are positive constants.

**Proof** Take any cube $Q$, and write $E_Q = \{ x \in Q : |f(x) - c_Q|/\omega(x) > t \}$. Then
\[
\mu(E_Q) \leq \int_{E_Q} \frac{|f(x) - c_Q|^r}{t^r \omega(x)^r} \, d\mu(x) \\
\leq \frac{1}{t^r} \frac{\mu(Q)}{\mu(Q)} \int_Q \frac{|f(x) - c_Q|^r}{\omega(x)^r} \, d\mu(x) \\
\leq \frac{1}{t^r} \mu(Q).
\]
Write $F_1(t) = 1/t^r$, then
\[
\mu(E_Q) \leq F_1(t) \mu(Q).
\]

Let $s > 1$ and $t \in (0, \infty)$ such that $2^{q_1 + 1} s^{1/r} \mu(\omega)_{A_1} \leq t$. Fixing a cube $Q_0$, there is a Calderon–Zygmund decomposition of disjoint cubes $\{Q_j\}$ such that $Q_j \subset Q_0$ and
\[
(i) \quad s^r \leq \frac{1}{\mu(Q_j)} \int_{Q_j} \left( \frac{|f(x) - c_{Q_0}|}{\omega(x)} \right)^r \, d\mu(x) \leq 2^n s^r,
(ii) \quad \frac{|f(x) - c_{Q_0}|}{\omega(x)} \leq s \text{ for } x \in (\bigcup_j Q_j)^c.
\]
Since \( \omega \in A_1 \) and \( x \in Q \), then \( \frac{1}{\omega(x)} \leq \frac{[\omega]_{A_1}}{\mu(Q_j)}. \) By (i) and \( 0 < r < 1, \)
\[
\int_{Q_j} \left( \frac{|f(y) - c_{Q_0}|}{\omega(y)} \right)^r \omega(y)^r dy = \int_{Q_j} \left( \frac{|f(y) - c_{Q_0}|}{\omega(y)} \right)^r \omega(y)^r d\mu(y) \\
\leq \left( \frac{[\omega]_{A_1}}{\mu(Q_j)} \right)^{1-r} \int_{Q_j} \left( \frac{|f(y) - c_{Q_0}|}{\omega(y)} \right)^r d\mu(y) \\
\leq 2^n s^r [\omega]_{A_1}^{1-r} |Q_j|^{1-r} \mu(Q_j)^r.
\]

Notice that
\[
\int_{Q_j} \left( \frac{|f(x) - c_{Q_0}|}{\omega(x)} \right)^r d\mu(x) \leq \int_{Q_j} \left( \frac{|f(x) - c_{Q_0}|}{\omega(x)} \right)^r d\mu(x),
\]
which implies that
\[
\int_{Q_j} \left( \frac{|f(y) - c_{Q_0}|}{\omega(y)} \right)^r \omega(y)^r dy \leq 2^n s^r [\omega]_{A_1}^{1-r} |Q_j|^{1-r} \mu(Q_j)^r.
\]

Therefore,
\[
\left( \frac{|c_{Q_j} - c_{Q_0}|}{\omega(x)} \right)^r = \frac{\omega(x)^{-r}}{|Q_j|} \int_{Q_j} |c_{Q_j} - c_{Q_0}|^r dy \\
\leq \frac{1}{|Q_j| \omega(x)^r} \int_{Q_j} \left( \frac{|f(y) - c_{Q_0}|}{\omega(y)} \right)^r \omega(y)^r dy \\
+ \frac{1}{|Q_j| \omega(x)^r} \int_{Q_j} \left( \frac{|f(y) - c_{Q_0}|}{\omega(y)} \right)^r \omega(y)^r dy \\
\leq 2^{n+1} s^r [\omega]_{A_1}^{1-r} \left( \frac{\mu(Q_j)}{|Q_j| \omega(x)} \right)^r \\
\leq 2^{n+1} s^r [\omega]_{A_1}.
\]

From the fact that \( [\omega]_{A_1} \geq 1 \), we have \( t > s \). By (i) and (ii), we have
\[
\mu(E_{Q_0}) = \sum_j \mu\left( \left\{ x \in Q_j : \frac{|f(x) - c_{Q_0}|}{\omega(x)} > t \right\} \right) \\
\leq \sum_j \mu\left( \left\{ x \in Q_j : \frac{|f(x) - c_{Q_0}|}{\omega(x)} + \frac{|c_{Q_j} - c_{Q_0}|}{\omega(x)} > t \right\} \right) \\
\leq \sum_j \mu\left( \left\{ x \in Q_j : \frac{|f(x) - c_{Q_0}|}{\omega(x)} > t - \frac{2^{n+1}}{s^r} [\omega]_{A_1}^{1/r} \right\} \right) \\
\leq \sum_j F_1(t - 2^{n+1} s^r [\omega]_{A_1}^{1/r}) \cdot \mu(Q_j) \\
\leq F_1(t - 2^{n+1} s^r [\omega]_{A_1}^{1/r}) \sum_j \frac{1}{s^r} \int_{Q_j} \left( \frac{|f(x) - c_{Q_0}|}{\omega(x)} \right)^r d\mu(x) \\
\leq \frac{F_1(t - 2^{n+1} s^r [\omega]_{A_1}^{1/r})}{s^r} \int_{Q_0} \left( \frac{|f(x) - c_{Q_0}|}{\omega(x)} \right)^r d\mu(x) \\
\leq \frac{F_1(t - 2^{n+1} s^r [\omega]_{A_1}^{1/r})}{s^r} \mu(Q_0).
\]
Let
\[ F_2(t) = \frac{F_1(t - \frac{n+1}{r}s[\omega]^{1/r}_{A_1})}{s^r}. \]
Continuing this process indefinitely, we obtain for any \( k \geq 2 \),
\[ F_k(t) = \frac{F_{k-1}(t - \frac{n+1}{r}s[\omega]^{1/r}_{A_1})}{s^r}, \]
and
\[ \mu(E_{Q_0}) \leq F_k(t)\mu(Q_0). \]

We fix a constant \( t > 0 \). If
\[ k \cdot 2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s < t \leq (k + 1) \cdot 2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s \]
for some \( k \geq 1 \), thus
\[
\mu(E_{Q_0}) \leq \mu\left( \left\{ x \in Q_0 : \frac{|f(x) - c_{Q_0}|}{\omega(x)} > t \right\} \right)
\leq \mu\left( \left\{ x \in Q_0 : \frac{|f(x) - c_{Q_0}|}{\omega(x)} > k \cdot 2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s \right\} \right)
\leq F_k(k \cdot 2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s)\mu(Q_0)
= F_1(2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s)\mu(Q_0)
= \frac{1}{2^{n+1}[\omega]^{1/r}_{A_1}} s^{k-1} k^r \mu(Q_0)
\leq \frac{e^{-kr \log s}}{2^{n+1}[\omega]^{1/r}_{A_1}} \exp \left( -\frac{tr \log s}{2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1}} \right) \mu(Q_0)
\]
Since \(-k \leq 1 - \frac{t}{2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s}\). If \( t \leq 2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s \), then use the trivial estimate
\[ \mu(E_{Q_0}) \leq \mu(Q_0) \leq e^{-e} 2^{\frac{n+1}{r}}[\omega]^{1/r}_{A_1} s \mu(Q_0). \]
Recall that \( s \) is any real number larger than 1. Choosing \( s = e \), this yields
\[ \mu\left( \left\{ x \in Q : \frac{|f(x) - c_Q|}{\omega(x)} > t \right\} \right) \leq c_1 e^{-ct} \mu(Q), \]
for some positive constants \( c_1 \) and \( c_2 \), which proves the inequality of Lemma 2.2. \( \Box \)

**Lemma 2.3** Let \( \omega \in A_1 \) and \( 0 < r < 1 \). Then
\[ \text{BMO}^r(\omega) = \text{BMO}_r(\omega). \]

The norms are mutually equivalent.

**Proof** By Lemma 2.2 and the homogeneity of \( || \cdot ||_{\text{BMO}_r(\omega)} \), we obtain that for any \( f \in \text{BMO}_r(\omega) \),
\[ \omega\left( \left\{ x \in Q : \frac{|f(x) - c_Q|}{\omega(x)} > t \right\} \right) \leq c_1 \exp \left( -\frac{c_2 t}{||f||_{\text{BMO}_r(\omega)}} \right) \omega(Q). \]
This gives us
\[
\frac{1}{\omega(Q)} \int_Q |f(x) - c_Q| \omega(x) dx = \frac{1}{\omega(Q)} \int_0^\infty \omega\left( \left\{ x \in Q : \frac{|f(x) - c_Q|}{\omega(x)} > t \right\} \right) dt \\
\leq \frac{1}{\omega(Q)} \int_0^\infty c_1 \exp\left(-\frac{c_2 t}{\|f\|_{\text{BMO},(\omega)}} \right) \omega(Q) dt \\
\leq C \|f\|_{\text{BMO},(\omega)}.
\]

Therefore,
\[
\frac{1}{\omega(Q)} \int_Q \left( \frac{|f(x) - f_Q|}{\omega(x)} \right)^r \omega(x) dx \leq \frac{1}{\omega(Q)} \int_Q \left( \frac{|f(x) - c_Q|}{\omega(x)} \right)^r \omega(x) dx \\
+ \frac{1}{\omega(Q)} \int_Q \left( \frac{|c_Q - f_Q|}{\omega(x)} \right)^r \omega(x) dx \\
\leq \|f\|_{\text{BMO},(\omega)}^r + \left( \frac{1}{\omega(Q)} \int_Q |f(x) - c_Q| dx \right)^r \\
\leq C \|f\|_{\text{BMO},(\omega)}^r.
\]

Conversely, \( \| \cdot \|_{\text{BMO},(\omega)} \leq \| \cdot \|_{\text{BMO}^r(\omega)} \) is obvious. Thus, the equivalence of \( \| \cdot \|_{\text{BMO},(\omega)} \) and \( \| \cdot \|_{\text{BMO}^r(\omega)} \) is shown. \( \square \)

Standard real analysis tools as the maximal function \( M(f) \), the weighted maximal function \( M_\omega(f) \) and the sharp maximal function \( M^s(f) \) carry over to this context, namely,
\[
M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy; \\
M_\omega(f)(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy; \\
M^s(f)(x) = \sup \inf_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c| dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.
\]

A variant of weighted maximal function and sharp maximal operator \( M_{\omega,s}(f)(x) = (M_\omega(f^s))^{1/s} \) and \( M_{\delta}^s(f)(x) = (M^s(f)(x))^{1/\delta} \), which will become the main tool in our scheme.

The following relationships between \( M_\delta \) and \( M_{\delta}^s \) to be used is a version of the classical ones due to Fefferman and Stein [5].

**Lemma 2.4** Let \( 0 < p, \delta < \infty \) and \( \omega \in A_\infty \). There exists a positive \( C \) such that
\[
\int_{\mathbb{R}^n} (M_\omega f(x))^p \omega(x) dx \leq C \int_{\mathbb{R}^n} (M_{\omega,s}^p f(x))^p \omega(x) dx,
\]
for any smooth function \( f \) for which the left-hand side is finite.

**Lemma 2.5** Let \( \omega \in A_1 \) and \( b \in \text{BMO}(\omega) \). Then, there exists a constant \( C \) such that
\[
M_{\frac{1}{2}}^s([b, T]_1(f_1, f_2))(x) \leq C\|b\|_{\text{BMO}(\omega)} \omega(x) M(T(f_1, f_2)(x)) \\
+ C\|b\|_{\text{BMO}(\omega)} \omega(x) M_{\omega,s}(f_1)(x) M(f_2)(x), \tag{2.1}
\]
\[
M_{\frac{1}{2}}^s([b, T]_2(f_1, f_2))(x) \leq C\|b\|_{\text{BMO}(\omega)} \omega(x) M(T(f_1, f_2)(x)) \\
+ C\|b\|_{\text{BMO}(\omega)} \omega(x) M(f_1)(x) M_{\omega,s}(f_2)(x), \tag{2.2}
\]
Characterizations of Weighted BMO Space and Its Application

and

\[ M^2_{1/3}(\|M_1^b, T\|_2(f_1, f_2))(x) \leq C \omega(x)^2 \|b\|^2_{BMO(\omega)} M(T(f_1, f_2))(x) \]

\[ + C \omega(x) \|b\|_{BMO(\omega)} M_{1/2}(\|b, T\|_1(f_1, f_2))(x) \]

\[ + C \omega(x) \|b\|_{BMO(\omega)} M_{1/2}(\|b, T\|_2(f_1, f_2))(x) \]

\[ + C \omega(x)^2 \|b\|^2_{BMO(\omega)} M_{\infty,s}(f_1)(x) M_{\infty,s}(f_2)(x), \tag{2.3} \]

for any \(1 < s < \infty\) and bounded compact supported functions \(f_1, f_2\).

**Proof** We only prove (2.1) and the proof of (2.2) and (2.3) are very similar to that of (2.1). Let \(Q := Q(x, r)\) be a cube and \(x \in Q\). Then,

\[
\left( \frac{1}{|Q|} \int_Q \|b, T\|_1(f_1, f_2)(y)\|1/2 - |c|^{1/2}|dz \right)^2
\leq C \left( \frac{1}{|Q|} \int_Q \|b, T\|_1(f_1, f_2)(y) - c\|^{1/2}|dz \right)^2
\leq C \left( \frac{1}{|Q|} \int_Q \|b, T\|_1(f_1, f_2)(y) - c\|^{1/2}|dz \right)^2
\]

\[+ \left( \frac{1}{|Q|} \int_Q \|b, T\|_1(f_1, f_2)(y) - c\|^{1/2}|dz \right)^2 =: A_1 + A_2,\]

where \(\lambda = b_Q\).

We first consider the term \(A_1\). By Hölder inequality, we obtain that

\[ A_1 = \left( \frac{1}{|Q|} \int_Q \|b, T\|_1(f_1, f_2)(y) - c\|^{1/2}|dz \right)^2 \leq C \|b\|_{BMO(\omega)} \frac{\omega(|Q|)}{|Q|} \left( \frac{1}{|Q|} \int_Q \|b, T\|_1(f_1, f_2)(y) - c\|^{1/2}|dz \right)^2 \leq C \omega(x) \|b\|_{BMO(\omega)} M(T(f_1, f_2))(x). \]

Let us consider next the term \(A_2\). Let

\[ \Omega_0 = \{ (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n : |x_0 - y_1| + |x_0 - y_2| \leq 2\sqrt{nr} \} \]

and for \(k \geq 1\),

\[ \Omega_k = \{ (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n : 2^{k+1}\sqrt{nr} \geq |x_0 - y_1| + |x_0 - y_2| > 2^k\sqrt{nr} \}. \]

We write

\[ A_2 \leq \left( \frac{1}{|Q|} \int_Q \int_{\Omega_0} (b(y_1) - \lambda)K(z - y_1, z - y_2)f_1(y_1)f_2(y_2)dy_1dy_2 \right)^{1/2}|dz \right)^2 \]

\[+ \left( \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Omega_0} (b(y_1) - \lambda)K(z - y_1, z - y_2)f_1(y_1)f_2(y_2)dy_1dy_2 - c \right)^{1/2}|dz \right)^2 \]

\[= : A_{21} + A_{22}. \]

It is obvious that \(\Omega_0 \subset 4\sqrt{nr}Q \times 4\sqrt{nr}Q\). We write \(f^0 = f_1 \chi_{4\sqrt{nr}Q}\). By Kolmogorov inequality and the fact that \(T\) is bounded from \(L^1 \times L^1\) to \(L^{1/2, \infty}\), we get

\[ A_{21} \leq \frac{1}{|Q|^2} \|T((b - b_Q)f^0, f^0)\|_{L^{1/2, \infty}}. \]
for any 1 < s < ∞ and bounded compact supported functions f_1, f_2.

Therefore,
\[
A_{22} \leq C \sum_{k=1}^{\infty} \frac{r^\gamma}{Q} \int_Q \int_{\Omega_k} \frac{|b(y_1) - \lambda||f_1(y_1)||f_2(y_2)|}{|z - y_1| + |z - y_2|^{2n+\gamma}} dy_1 dy_2 dz
\leq C \sum_{k=1}^{\infty} \left( \frac{1}{2k} \right)^\gamma \frac{1}{2kQ^2} \int_{2k+1}^{\infty} \int_{2k+1}^{\infty} \frac{|b(y_1) - \lambda||f_1(y_1)||f_2(y_2)|}{|z - y_1| + |z - y_2|^{2n+\gamma}} dy_1 dy_2 dz
\leq C \omega(x) \omega(x) M_{\omega, s}(x) M_{\omega, s}(x) M_{\omega, s}(x) M_{\omega, s}(x).
\]

Collecting our estimates, we have shown that
\[
M_{\frac{3}{2}}^s([b, T]_{1}(f_1, f_2))(x) \leq C \|b\|_{\text{BMO}^s(\omega)} \omega(x) M(T(f_1, f_2))(x)
+ C \|b\|_{\text{BMO}^s(\omega)} \omega(x) M_{\omega, s}(f_1)(x) M_{\omega, s}(f_2)(x)
\]
for any 1 < s < ∞ and bounded compact supported functions f_1, f_2.

\[
\square
\]

3 Proof of Theorem 1.1–Theorem 1.4

Proof of Theorem 1.1 Let 1 < p < ∞, ω ∈ A_1 and X = L^{p, \infty}(ω). By Lemma 2.1, we have
\[
\|\cdot\|_{\text{BMO}(\omega)} \leq C \|\cdot\|_{\text{BMO}_X(\omega)}.
\]

From the fact that BMO(ω) = BMO^p(ω) and \|\cdot\|_{\text{BMO}^p(\omega)} \leq \|\cdot\|_{\text{BMO}(\omega)}, it follows that
\[
\|\cdot\|_{\text{BMO}_X(\omega)} \leq C \|\cdot\|_{\text{BMO}(\omega)}.
\]

Thus we complete the proof of Theorem 1.1.

Proof of Theorem 1.2 Let f ∈ BMO^p(ω). In the proof of Lemma 2.3, we have shown that
\[
\frac{1}{\omega(Q)} \int_Q \|f(x) - c_Q\| dx \leq C \|f\|_{\text{BMO}^p(\omega)}.
\]

Therefore,
\[
\frac{1}{\omega(Q)} \int_Q \|f(x) - f_Q\| dx \leq \frac{2}{\omega(Q)} \int_Q \|f(x) - c_Q\| dx
\leq C \|f\|_{\text{BMO}^p(\omega)} \leq C \|f\|_{\text{BMO}^p(\omega)}.
\]
As a result, \( \|f\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}^r} \). The opposite inequality is a consequence of Hölder inequality, then the equivalence of \( \|f\|_{\text{BMO}} \) and \( \|f\|_{\text{BMO}^r} \) is shown.

Proof of Theorem 1.4 (a1) ⇒ (a2): Since \( \omega \in A_1 \), then \( \omega^{1-p} \in A_\infty \). By Lemma 2.4 and Lemma 2.5 with \( 1 < s < \min\{p_1, p_2\} \), from a standard argument that we can obtain

\[
\|\sum_b, T\|(f_1, f_2)\omega^{-1}\|_{L^p(\omega)} = \|\sum_b, T\|(f_1, f_2)\|_{L^p(\omega^{1-p})} \\
\leq M_1\|\sum_b, T\|(f_1, f_2)\|_{L^p(\omega^{1-p})} \\
\leq C\|\sum_b, T\|(f_1, f_2)\|_{L^p(\omega^{1-p})} \\
\leq C\|b\|_{\text{BMO}(\omega)}\|M(T(f_1, f_2))\|_{L^p(\omega)} \\
+ C\|b\|_{\text{BMO}(\omega)}\|M(f_1(x)M_{\omega,s}(f_2))\|_{L^p(\omega)} \\
\leq C\|b\|_{\text{BMO}(\omega)}\prod_{i=1}^2\|f_i\|_{L^{p_i}(\omega)}.
\]

We observe that to use the Fefferman–Stein inequality, one needs to verify that certain terms in the left-hand side of the inequalities are finite. We can assume that \( f_1, f_2 \) are bounded functions with compact support, applying a similar argument as in [12, pp. 32–33] and Fatou’s lemma, one gets the desired result.

(a2) ⇒ (a3) is obvious.

(a3) ⇒ (a1): Let \( z_0 \in \mathbb{R}^n \) such that \( |(z_0, z_0)| > 2\sqrt{n} \) and let \( \delta \in (0, 1) \) small enough. Take \( B = B((z_0, z_0), \delta \sqrt{2n}) \subset \mathbb{R}^{2n} \) be the ball for which we can express \( \frac{1}{K} \) as an absolutely convergent Fourier series of the form

\[
\frac{1}{K(y_1, y_2)} = \sum_j a_j e^{iv_j \cdot (y_1, y_2)}, \quad (y_1, y_2) \in B,
\]

with \( \sum_j |a_j| < \infty \) and we do not care about the vectors \( v_j \in \mathbb{R}^{2n} \), but we will at times express them as \( v_j = (v_j^1, v_j^2) \in \mathbb{R}^n \times \mathbb{R}^n \).

Set \( z_1 = \delta^{-1}z_0 \) and note that

\[
(|y_1 - z_1|^2 + |y_2 - z_1|^2)^{1/2} < \sqrt{2n} \Rightarrow (|\delta y_1 - z_0|^2 + |\delta y_2 - z_0|^2)^{1/2} < \delta \sqrt{2n}.
\]

Then for any \( (y_1, y_2) \) satisfying the inequality on the left, we have

\[
\frac{1}{K(y_1, y_2)} = \frac{\delta^{-2n}}{K(\delta y_1, \delta y_2)} = \delta^{-2n} \sum_j a_j e^{i\delta v_j \cdot (y_1, y_2)}.
\]

Let \( Q = Q(x_0, r) \) be any arbitrary cube in \( \mathbb{R}^n \). Set \( \tilde{z} = x_0 + rz_1 \) and take \( Q' = Q(\tilde{z}, r) \subset \mathbb{R}^n \).

So for any \( x \in Q \) and \( y_1, y_2 \in Q' \), we have

\[
\left| \frac{x - y_1}{r} - z_1 \right| \leq \left| \frac{x - x_0}{r} \right| + \left| \frac{y_1 - \tilde{z}}{r} \right| \leq \sqrt{n}, \quad \left| \frac{x - y_2}{r} - z_1 \right| \leq \left| \frac{x - x_0}{r} \right| + \left| \frac{y_2 - \tilde{z}}{r} \right| \leq \sqrt{n},
\]

which implies that

\[
\left( \left| \frac{x - y_1}{r} - z_1 \right|^2 + \left| \frac{x - y_2}{r} - z_1 \right|^2 \right)^{1/2} \leq \sqrt{2n}.
\]

Let \( s(x) = \text{sgn}(\int_{Q'} (b(x) - b(y))dy) \). Then

\[
|b(x) - b_{Q'}| = s(x)(b(x) - b_{Q'})
\]
\[
= \frac{s(x)}{|Q|^2} \int_{Q'} \int_{Q'} (b(x) - b(y_1)) dy_1 dy_2.
\]

Setting
\[
g_j(y_1) = e^{-i \frac{x}{2} \cdot y_1} \chi_{Q'}(y_1),
\]
\[
h_j(y_2) = e^{-i \frac{x}{2} \cdot y_2} \chi_{Q'}(y_2),
\]
\[
m_j(x) = e^{i \frac{x}{2} \cdot (x, x)} \chi_{Q}(x) s(x),
\]

which shows that
\[
|b(x) - b_{Q'}| = s(x) \frac{2n \delta - 2n}{|Q|^2} \int_{Q'} \int_{Q'} \frac{b(x) - b(y_1)}{|x - y_1|^2 + |x - y_2|^2} dy_1 dy_2.
\]

If \( p > 1 \), we have the following estimate
\[
\frac{\lambda}{\omega(Q)^{1/p}} \omega \left( x \in Q : \frac{|b(x) - b_{Q'}|}{\omega(x)} > \lambda \right)^{1/p}
\]
\[
= \frac{\lambda}{\omega(Q)^{1/p}} \omega \left( x \in Q : \frac{|b(x) - b_{Q'}|}{\omega(x)} > \lambda \right)^{1/p}
\]
\[
\leq \frac{\lambda}{\omega(Q)^{1/p}} \omega \left( x \in Q : \sum_j |a_j|||\Pi^1 T|(g_j, h_j)(x)|_{L^p, \infty(\omega)} > \lambda \right)^{1/p}
\]
\[
\leq \frac{C}{\omega(Q)^{1/p}} \sum_j |a_j|||\Pi^1 T|(g_j, h_j)||_{L^p, \infty(\omega)}
\]
\[
\leq C \sum_j |a_j|.
\]

We write
\[
\|b\|_{\text{BMO}_*(\omega)} := \sup_Q \sup_{\lambda > 0} \frac{\lambda}{\omega(Q)^{1/p}} \omega \left( x \in Q : \frac{|b(x) - b_{Q'}|}{\omega(x)} > \lambda \right)^{1/p},
\]
then \( \|b\|_{\text{BMO}_*(\omega)} \leq C \sum_j |a_j| \). The same estimate as lemma 2.1, we conclude that
\[
|b_Q - b_{Q'}| \leq \frac{1}{|Q|} \int_Q |b(x) - b_{Q'}| dx
\]
\[
\leq \frac{\omega(Q)}{|Q|} \|b\|_{\text{BMO}_*(\omega)}
\]
\[
\leq C \frac{\omega(Q)}{|Q|} \sum_j |a_j|.
\]

By the definition of \( A_1 \) weights, we concluded that \( \omega(Q) \leq |Q| \omega(x) \), which implies that for any cube \( Q \) and \( \lambda > 0 \),
\[
\frac{\lambda}{\omega(Q)^{1/p}} \omega \left( x \in Q : \frac{|b(x) - b_Q|}{\omega(x)} > \lambda \right)^{1/p}
\]
\[
\leq \frac{\lambda}{\omega(Q)^{1/p}} \omega \left( x \in Q : \frac{|b(x) - b_{Q'}|}{\omega(x)} > \lambda \right)^{1/p}
\]
This shows that $b \in \text{BMO}_X(\omega)$ with $X = L^{p,\infty}$; that is, the symbol $b$ belongs to $\text{BMO}(\omega)$.

If $p \leq 1$, choose $q \in (0, p)$. By the fact that $L^{p,\infty}(\omega) \subseteq M^p_q(\omega)$ in [17, Corollary 2.3] (see also [9, Lemma 1.7] for the unweighted case), $M^p_q(\omega)$ stands for the weighted Morrey spaces; that is, for $0 < q < p < \infty$,

$$M^p_q(\omega) = \left\{ f \in L^q_{\text{loc}} : \|f\|_{M^p_q} = \sup_Q \frac{1}{\omega(Q)^{1/q-1/p}} \left( \int_Q |f(y)|^q \omega(y) dy \right)^{1/q} < \infty \right\}.$$  

Therefore,

$$\inf_c \frac{1}{\omega(Q)} \int_Q \left( \frac{|b(x) - c|}{\omega(x)} \right)^q \omega(x) dx \right)^{1/q} \leq \left( \frac{1}{\omega(Q)} \int_Q \left( \frac{|b(x) - b_Q|}{\omega(x)} \right)^q \omega(x) dx \right)^{1/q} \leq \left( \frac{C}{\omega(Q)} \int_Q \sum_j |a_j|[b, T_1(g_j, h_j)](x) \omega(x)^{-1} \right)^{1/q} \leq C\omega(Q)^{-1/p} \sum_j |a_j||[b, T_1(g_j, h_j)\omega^{-1}]\|_{M^p_q(\omega)} \leq C\omega(Q)^{-1/p} \sum_j |a_j||[b, T_1(g_j, h_j)\omega^{-1}]\|_{L^{p,\infty}(\omega)} \leq C \sum_j |a_j|.$$  

Thus showing that $b \in \text{BMO}_q(\omega)$. The desired result follows from here.

By the inequality (2.3) in Lemma 2.4 and the same argument as (a1) $\Rightarrow$ (a2), we can obtain that (a1) $\Rightarrow$ (a4). It is easy to see that (a4) $\Rightarrow$ (a5). The proof of (a5) $\Rightarrow$ (a1) follows the method that of (a3) $\Rightarrow$ (a1) except replacing (3.1) by

$$|b(x) - b_{Q'}|^2 = s(x)^2(b(x) - b_{Q'})^2$$

$$= \frac{s(x)^2}{|Q'|^2} \int_{Q'} \int_{Q'} (b(x) - b(y_1)) (b(x) - b(y_2)) dy_1 dy_2.$$  

Therefore, we complete the proof of Theorem 1.4. \hfill \Box

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