Weighted Estimates of the Cayley Transform Method for Boundary Value Problems in a Banach Space

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ABSTRACT
We consider the boundary value problems (BVPs) for linear second-order ODEs with a strongly positive operator coefficient in a Banach space. The solutions are given in the form of the infinite series by means of the Cayley transform of the operator, the Meixner type polynomials of the independent variable, the operator Green function, and the Fourier series representation for the right-hand side of the equation. The approximate solution of each problem is a partial sum of \(N\) summands. We prove the weighted error estimates depending on the discretization parameter \(N\), the distance of the independent variable to the boundary points of the interval, and some smoothness properties of the input data.

1. Introduction

In numerical methods for boundary value problems (BVPs), the error estimate is quite often expressed only through a particular discretization parameter (e.g., a mesh step \(h\), a number \(N\) of summands in a truncated sum of an infinite series, etc.). However, some other features can also be important for the convergence rate. In one such case, when we deal with the Dirichlet boundary condition and therefore an approximate solution is expected to be more precise near the boundary, it is natural to additionally estimate the error through the distance from a point inside the domain to its boundary. The impact of the Dirichlet boundary condition on the accuracy of an approximate solution can be called a boundary effect in the sense of Ref. [1] and is usually evaluated by means of an appropriate weight function. This aspect is investigated in Refs. [2–8] (see also Ref. [9] where a method with linear-logarithmic complexity for solving elliptic problems with rough boundary data or geometry is presented). In a similar way, it is
possible to study the influence of the initial condition and express the error estimate through both a discretization parameter and the proximity of the time variable to the initial point [10–12]. This aspect is of both theoretical and practical interest; however, there are relatively few publications on the subject.

Another facet is connected with the dependence of the accuracy order on differential properties of the input data in the sense of the work by Babenko [13]. Namely, a method is considered to be a method without saturation of accuracy if an increase in smoothness of an exact solution automatically boosts the convergence rate of an approximate solution. It is also expected for a method to be exponentially convergent if input data are infinitely smooth in some sense. Such algorithms are constructed and studied in Refs. [11, 14, 15] and in a number of publications cited therein; however, these papers do not deal with the boundary effect.

Both of these concepts are developed by Gavrilyuk et al. [16], where the proposed approximations of abstract differential equations take into account the boundary effect and do not have saturation of accuracy. More specifically, in the work by Gavrilyuk et al. [16], we consider in a Hilbert space \( H \) the BVP for a linear second-order differential equation:

\[
\frac{d^2 u(x)}{dx^2} - Au(x) = -f(x), \quad x \in (0, 1),
\]
\[
u(0) = 0, \quad u(1) = 0,
\]

where \( u(x) : [0, 1] \to H \) is an unknown function, \( f(x) : (0, 1) \to H \) is a given function, \( A \) is a self-adjoint positive definite operator in \( H \) with the domain \( D(A) \) and the spectral set \( \Sigma(A) \subset [\lambda_0, +\infty), \lambda_0 > 0. \)

The following boundary value problem for Poisson’s equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y), \quad (x, y) \in \Omega = (0, 1)^2,
\]
\[
u(x, y) = 0, \quad (x, y) \in \partial \Omega,
\]

with the operator \( A \) in the Hilbert space \( L_2(0, 1) \) defined by

\[
D(A) = \{ u(y) \in H^2(0, 1) : u(0) = 0, \ u(1) = 0 \}, \quad Au = -\frac{d^2 u}{dy^2},
\]

can be formulated in the abstract setting (1.1).

To give the general idea of our approach, we briefly describe obtaining one of the two approximations for the solution \( u(x) \).

By using the Fourier series representation of the right-hand side \( f(x) \):

\[
f(x) = \sum_{k=1}^{\infty} \sqrt{2} \sin (2k\pi x)f_{s,k} + f_0 + \sum_{k=1}^{\infty} \sqrt{2} \cos (2k\pi x)f_{c,k},
\]  

(1.2)

where
\[
f_{s,k} = \int_0^1 f(x) \sqrt{2} \sin (2k\pi x) dx, \quad f_{c,k} = \int_0^1 f(x) \sqrt{2} \cos (2k\pi x) dx, \quad k = 1, 2, \ldots, \\
f_0 = \int_0^1 f(x) dx,
\]

we present the solution \(u(x)\) in the form
\[
u(x) = \sum_{k=1}^{\infty} \sqrt{2} \sin (2k\pi x) [(2k\pi)^2 I + A]^{-1} f_{s,k} \\
+ (A \sinh \sqrt{A})^{-1} \left\{ \sinh \sqrt{A} - \sinh(\sqrt{A}(1-x)) - \sinh(\sqrt{A}x) \right\} f_0 \\
+ \sum_{k=1}^{\infty} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \sinh^{-1} \sqrt{A} \\
\times \{ \cos (2k\pi x) \sinh \sqrt{A} - \sinh(\sqrt{A}(1-x)) - \sinh(\sqrt{A}x) \} f_{c,k}.
\]

Taking here the partial sum of \(N\) summands in each infinite series, we obtain for \(u(x)\) the approximation \(u_N(x)\) (for another approximation \(u_{N,M}\) and its accuracy order see details in the work by Gavrilyuk et al. [16]).

Next we prove the assertion.

**Theorem 1.1** ([16]). Let \(\sigma > 0, f_0 \in D(A^\sigma)\) and let \(f_{c,k}, f_{s,k}\) in Equation (1.3) satisfy the conditions
\[
\|f_s\|_\sigma = \left( \sum_{k=1}^{\infty} k^{2\sigma} \|f_{s,k}\|^2 \right)^{1/2} < \infty, \quad \|f_c\|_\sigma = \left( \sum_{k=1}^{\infty} k^{2\sigma} \|f_{c,k}\|^2 \right)^{1/2} < \infty.
\]

Then the following weighted estimate holds true:
\[
\left\| \frac{u(x) - u_N(x)}{\min(x, 1-x)} \right\| \leq \frac{C}{N^{\sigma + 1/2}} (\|f_s\|_\sigma + \|f_c\|_\sigma + \|A^\sigma f_0\|),
\]
where \(C > 0\) is a constant independent of \(N\).

The aim of the present article is to extend this study to a more general case of a Banach space and obtain some new results.

This article is organized as follows. In Section 2, we consider the BVP for a linear second-order ODE in a Banach space with a strongly positive operator coefficient, i.e., a densely defined closed linear operator under a certain assumption about its spectrum and resolvent. We introduce some useful notation for spaces and norms and we prove a number of auxiliary inequalities used throughout the article. The exact solution of the BVP is represented in the form of the infinite series involving the Meixner polynomials of the independent variable and the Cayley transform of the operator coefficient. That representation naturally gives rise to the approximate
solution in the form of the partial sum which is then studied under various conditions about smoothness of the input data. Namely, we prove two weighted error estimates depending on both the discretization parameter $N$ (the number of summands in the partial sum) and the distance to the boundary points of the interval. In Section 3, we take a similar approach to the study of the BVP for the inhomogeneous equation. More specifically, we write down the exact solution in the form of the infinite series by the use of the operator Green function and the Fourier series representation of the right-hand side of the equation. The partial sum of this series produces the approximate solution which is investigated under some smoothness assumptions concerning the descending order of the Fourier coefficients. Then we discuss the proven results and make a few final comments.

2. BVP for the homogeneous equation

We consider in a Banach space $E$ the boundary value problem

$$\frac{d^2 u(x)}{dx^2} - Au(x) = 0, \quad x \in (0, 1),$$

$$u(0) = 0, \quad u(1) = u_1,$$  \hspace{1cm} (2.1)

where $u(x) : [0, 1] \to E$ is an unknown vector-valued function, $u_1 \in E$ is a given vector, $A : E \to E$ is a densely defined closed linear operator with the domain $D(A)$, the resolvent set $\rho(A)$, and the spectrum $\sigma(A)$. Let $A$ satisfy the condition (see Ref. [17, p. 69]): there exist constants $\varphi \in (0, \pi/2)$, $\gamma > 0$, $L > 0$ such that

$$\Sigma \equiv \{ z \in \mathbb{C} | \varphi \leq |\arg(z)| \leq \pi \} \cup \{ z \in \mathbb{C} | |z| \leq \gamma \} \subset \rho(A)$$ \hspace{1cm} (2.2)

and

$$||(zI-A)^{-1}|| \leq \frac{L}{1+|z|} \quad \forall z \in \Sigma.$$ \hspace{1cm} (2.3)

It is shown in Ref. [17] that strongly elliptic operators of order $2m$ in $L_p(\Omega)$ for $1 \leq p \leq +\infty$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial \Omega$ produce important examples of such operators.

Note that in the work by Gavrilyuk and Makarov [14], a densely defined closed linear operator satisfying Equations (2.2) and (2.3) is called a strongly positive operator. It is proved [14, p. 708] that for $u_1 \in D(A^\sigma)$, $\sigma > 1$, the solution $u(x)$ of problem (2.1) can be represented in the form

$$u(x) \equiv \sinh^{-1}(\sqrt{A}) \sinh(x\sqrt{A}) u_1 = \sum_{k=0}^{\infty} v_k(x) y_k;$$ \hspace{1cm} (2.4)

here
\[ y_k = (I + A)^{-1} A y_{k-1} = [(I + A)^{-1} A]^k u_1, \]  
(2.5)

\((I + A)^{-1} A\) is the Cayley transform of the operator \(A\), and the functions \(v_k(x)\) are defined by the recurrent sequence of the integral equations

\[
v_k(x) = v_{k-1}(x) - \int_0^1 G_0(x, \xi) v_{k-1}(\xi) \, d\xi \quad x \in [0, 1], \quad k = 2, 3, \ldots,
\]

\[
v_0(x) = x, \quad v_1(x) = -\frac{1}{3!} x(1-x^2),
\]

where

\[
G_0(x, \xi) = \begin{cases} x(1-\xi) & \text{if } x \leq \xi, \\ \xi(1-x) & \text{if } \xi \leq x, \end{cases}
\]

is the Green function of the differential operator

\[ L v(x) = -v''(x), \quad x \in (0, 1), \quad u(0) = 0, \quad u(1) = 0. \]

For example, formula (2.6) produces

\[
v_2(x) = \frac{1}{5!} x(1-x^2) \left( -x^2 - \frac{53}{3} \right), \quad v_3(x) = \frac{1}{7!} x(1-x^2) \left( -x^4 - 78x^2 - \frac{1963}{3} \right).
\]

Note that polynomials \(v_k(x)\) are closely connected with the Meixner polynomials \([18]\) and have recently been studied by Makarov \([19]\). The Meixner polynomials play the same role as the Laguerre polynomials in the Cayley transform method for solving the abstract Cauchy problem for a first-order differential equation with a strongly positive operator coefficient \([15]\).

Next, we recall some notation and terminology related to certain classes of vectors from \(E\) (see Ref. \([20]\)). We denote by \(C^\infty(A)\) the set of all infinitely differentiable vector of \(A\): \(C^\infty(A) = \cap_{n=0}^\infty D(A^n)\). It is shown by Gorbachuk and Knyazyuk \([20]\) that if a densely defined closed linear operator \(A\) has at least one regular point, then \(C^\infty(A)\) is dense in \(E\): \(C^\infty(A) = E\).

Let \((m_n)_{n=0}^\infty\) be a nondecreasing sequence of positive numbers and let \(\nu > 0\). We denote by \(C(A, (m_n), \nu)\) the Banach space of vectors \(f\) from \(C^\infty(A)\) with the norm

\[
||f||_{C(A, (m_n), \nu)} = \sup_n \frac{||A^n f||}{\nu^m n_m}.
\]

The class \(C(A, (m_n)) \overset{\text{def}}{=} \bigcup_{\nu > 0} C(A, (m_n), \nu)\) for various \((m_n)\) is discussed in the work by Gorbachuk and Knyazyuk \([20]\). Namely, vectors from \(C(A, (n^n))\) with \(m_n = n^n\) are called analytic for the operator \(A\) \([21]\); vectors from the Gevrey class of Roumieu type \(C(A, (n^{n\beta}))\) with \(m_n = n^{n\beta}, \beta > 1\),
are called ultradifferentiable [22], and vectors from \( C(A, (1)) \) with \( m_n \equiv 1 \) are known as vectors of exponential type [23].

For convenience, in the next four lemmas we prove some auxiliary inequalities which we will need throughout the paper.

**Lemma 2.1.** For a strongly positive operator \( A \) the following estimate holds true:

\[
\left\| \left( I + \frac{A}{j} \right)^j A^{-j} \right\| \leq (L + 1)^j \quad (j \in \mathbb{N}).
\] (2.8)

**Proof.** From Equation (2.3) we have \( \|A^{-1}\| \leq L \). Therefore

\[
\left\| \left( I + \frac{A}{j} \right)^j A^{-j} \right\| = \left\| \sum_{s=0}^{j} \binom{j}{s} \left( \frac{A}{j} \right)^s A^{-j} \right\| = \left\| \sum_{s=0}^{j} \binom{j}{s} \frac{A^{-(j-s)}}{j^s} \right\|
\]

\[
\leq \sum_{s=0}^{j} \binom{j}{s} \frac{\|A^{-(j-s)}\|}{j^s} \leq \sum_{s=0}^{j} \binom{j}{s} \frac{\|A^{-1}\|^{j-s}}{j^s}
\]

\[
\leq \sum_{s=0}^{j} \binom{j}{s} \frac{L^{j-s}}{j^s} = \left( L + \frac{1}{j} \right)^j \leq (L + 1)^j.
\]

\( \square \)

**Lemma 2.2.** For \( n > 0 \) and \( \alpha > 0 \) the following inequality holds true:

\[
\max_{t \geq 1} \left[ \left( 1 - \frac{1}{t} \right)^n t^{-\alpha} \right] \leq \left( \frac{\alpha}{e} \right)^\alpha n^{-\alpha}.
\]

**Proof.** We consider the function \( \varphi(t) = (1 - \frac{1}{t})^n t^{-\alpha} \), \( t \geq 1 \), and find the derivative

\[
\varphi'(t) = (t-1)^{n-1} t^{-\alpha-1} (n + \alpha - \alpha t).
\]

Then we have

\[
\max_{t \geq 1} \varphi(t) = \varphi \left( \frac{n + \alpha}{\alpha} \right)
\]

\[
= \left( 1 - \frac{\alpha}{n + \alpha} \right)^n \left( \frac{n + \alpha}{\alpha} \right)^{-\alpha} = \left( \frac{n}{n + \alpha} \right)^n \alpha^n n^{-\alpha} \leq e^{-\alpha} \alpha^n n^{-\alpha}.
\]

\( \square \)

**Lemma 2.3.** For \( n > \alpha > 0 \) the following inequality holds true:

\[
\max_{t > 0} \left[ \left( \frac{t}{t + 1} \right)^n t^{-\alpha} \right] \leq \alpha^n n^{-\alpha}.
\]

**Proof.** For the function \( \varphi(t) = (\frac{t}{t+1})^n t^{-\alpha} \), \( t > 0 \), we have the derivative

\[
\varphi'(t) = t^{n-\alpha-1} (t + 1)^{-n-1} (n - \alpha - \alpha t),
\]
which gives the estimate
\[
\max_{t \geq 0} \varphi(t) = \varphi\left(\frac{n-x}{n}\right) = \left(\frac{n-x}{n}\right)^n \left(\frac{n-x}{n}\right)^{-x} = \left(\frac{n-x}{n}\right)^{n-x} \approx n^{2n-x^2} < \alpha^2 n^{-x}.
\]

Lemma 2.4. The following estimate holds true:
\[
\max_{t \geq 0} \left[ \frac{t}{\left(\frac{t}{k+1} + 1\right)(t+1)} \right]^k = \left[ \frac{\sqrt{k}}{\left(\frac{1}{\sqrt{k}} + 1\right)(\sqrt{k}+1)} \right]^k \leq ee^{-2\sqrt{k}} \quad (k \in \mathbb{N}).
\]

It is unimprovable in the asymptotic sense:
\[
\left[ \frac{\sqrt{k}}{\left(\frac{1}{\sqrt{k}} + 1\right)(\sqrt{k}+1)} \right]^k \sim ee^{-2\sqrt{k}} \quad \text{as} \quad k \to \infty.
\]

Proof. Considering the function \(g(t) = \left[ \frac{t}{\left(\frac{t}{k+1} + 1\right)(t+1)} \right]^k\), \(t \geq 0\), we get
\[
\frac{d \ln g(t)}{dt} = \frac{k(k-t^2)}{k(t+k)(t+1)},
\]
which means that \(g(t)\) takes its maximum at the point \(t = \sqrt{k}\):
\[
\max_{t \geq 0} g(t) = g\left(\sqrt{k}\right) = \left[ \frac{\sqrt{k}}{\left(\frac{1}{\sqrt{k}} + 1\right)(\sqrt{k}+1)} \right]^k = \left(1 + \frac{1}{\sqrt{k}}\right)^{-2k}.
\]

Making use of the inequality
\[
\left(1 + \frac{1}{\sqrt{k}}\right)^{-2k} = \exp\left\{-2k \ln \left(1 + \frac{1}{\sqrt{k}}\right)\right\} \leq \exp\left\{-2k\left(\frac{1}{\sqrt{k}} - \frac{1}{2k}\right)\right\}
\]
\[
= ee^{-2\sqrt{k}}
\]
and the relation
\[
\left(1 + \frac{1}{\sqrt{k}}\right)^{-2k} = \exp\left\{-2k \ln \left(1 + \frac{1}{\sqrt{k}}\right)\right\} = \exp\left\{-2k\left(\frac{1}{\sqrt{k}} - \frac{1}{2k} + o\left(\frac{1}{k}\right)\right)\right\}
\]
\[
= \exp\left\{1 - 2\sqrt{k} + o(1)\right\} \sim ee^{-2\sqrt{k}} \quad \text{as} \quad k \to \infty,
\]
we arrive at the conclusion of the lemma.

Now we can move on to the inequalities for \(v_k(x)\).

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1We write \(f(x) \sim g(x)\) as \(x \to x_0\) provided that \(\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1\).

2We write \(f(x) = o(g(x))\) as \(x \to x_0\) provided that \(\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0\).
Lemma 2.5. For the polynomials $v_k(x), k = 1, 2, \ldots$, the following estimates hold true:

$$\left| \frac{v_k(x)}{\min(x, 1 - x)} \right| \leq \frac{C_1}{k^{(1-\varepsilon_1)/2}}, \quad x \in [0, 1],$$

(2.9)

where $0 < \varepsilon_1 < 1$, $C_1 = \left( \frac{1-\varepsilon_1}{\varepsilon_1} \right)^{1-\varepsilon_1} \frac{2}{\pi^{1+\varepsilon_1}} \zeta(1 + \varepsilon_1)$, and $\zeta(\cdot)$ is the Riemann zeta function;

$$\left| \frac{v_k(x)}{\min(x, 1 - x)} \right| \leq \frac{1}{3}, \quad x \in [0, 1].$$

(2.10)

Proof. We continue $v_k(x)$ oddly onto the interval $[-1, 0]$ and then periodically onto the whole real axis. First, we prove that $v_k(x)$ can be represented in the form

$$v_k(x) = \sum_{p=1}^{\infty} \sqrt{2} a^{(k)}_p \sin (p\pi x), \quad x \in [0, 1],$$

(2.11)

with

$$a^{(k)}_p = \sqrt{2} \int_{0}^{1} v_k(x) \sin (p\pi x) \, dx = \sqrt{2} \frac{(-1)^p}{(p\pi)^3} \left( 1 - \frac{1}{(p\pi)^2} \right)^{k-1}.$$ 

We make use of the mathematical induction method. For $k=1$ formula (2.11) is true since we have

$$v_1(x) = \sum_{p=1}^{\infty} \sqrt{2} a^{(1)}_p \sin (p\pi x), \quad x \in [0, 1],$$

with

$$a^{(1)}_p = \sqrt{2} \int_{0}^{1} v_1(x) \sin (p\pi x) \, dx = -\frac{\sqrt{2}}{6} \int_{0}^{1} x(1-x^2) \sin (p\pi x) \, dx = \frac{\sqrt{2}(-1)^p}{(p\pi)^3}.$$ 

Assuming that formula (2.11) is true for some $k \in \mathbb{N}$, we will prove it for $k + 1$. We have

$$v_{k+1}(x) = v_k(x) - \int_{0}^{1} G_0(x, \xi) v_k(\xi) \, d\xi = \sum_{p=1}^{\infty} \sqrt{2} a^{(k)}_p \left( 1 - \frac{1}{(p\pi)^2} \sin (p\pi x) \right),$$

from where we get
which gives formula (2.11).

Next we prove estimate (2.9). We have

\[ a_p^{(k+1)} = a_p^{(k)} \left(1 - \frac{1}{(p\pi)^2}\right) = \frac{\sqrt{2}(-1)^p}{(p\pi)^3} \left(1 - \frac{1}{(p\pi)^2}\right)^k, \]

Applying Lemma 2.2 for \( n = k-1, \ z = (1-\varepsilon_1)/2, \ k = 2, 3, ..., \ 0 < \varepsilon_1 < 1, \) we have

\[
\frac{v_k(x)}{\min(x, 1-x)} = \left| \sum_{p=1}^{\infty} \frac{1}{(p\pi)^2} \left(1 - \frac{1}{(p\pi)^2}\right)^{k-1} \frac{\sin(p\pi x)}{\min(x, 1-x)} \right|
\]

\[
\leq \sum_{p=1}^{\infty} \frac{2}{(p\pi)^3} \frac{1}{((p\pi)^2)(1-\varepsilon_1)/2} \left(1 - \frac{1}{(p\pi)^2}\right)^{k-1}.
\]

Applying Lemma 2.2 for \( n = k-1, \ z = (1-\varepsilon_1)/2, \ k = 2, 3, ..., \ 0 < \varepsilon_1 < 1, \) we have

\[
\frac{v_k(x)}{\min(x, 1-x)} \leq \sup_{p \in \mathbb{N}} \left| \sum_{p=1}^{\infty} \frac{1}{(p\pi)^2} \left(1 - \frac{1}{(p\pi)^2}\right)^{k-1} \frac{2}{(p\pi)^3} (1 + \varepsilon_1) \right|
\]

\[
\leq \frac{1}{2} \frac{(1-\varepsilon_1)/2}{(k-1)(2-\varepsilon_1)/2} \frac{2}{(1+\varepsilon_1)^{k-1}} (1 + \varepsilon_1)
\]

For \( k = 1 \) inequality (2.9) takes the form

\[
\left| \frac{v_1(x)}{\min(x, 1-x)} \right| \leq C_1, \quad x \in [0, 1],
\]

from where it follows that it is true since

\[
\max_{0 \leq x \leq 1} \left| \frac{v_1(x)}{\min(x, 1-x)} \right| = \max_{0 \leq x \leq 1} \left| -\frac{1}{3} x(1-x^2) \right| = \frac{1}{3} \quad \text{and}
\]

\[
C_1 > \frac{1}{3} \quad \text{for} \quad 0 < \varepsilon_1 < 1.
\]

Thus, formula (2.9) is fulfilled for all \( k \in \mathbb{N}. \)

Now we prove estimate (2.10):

\[
\frac{v_k(x)}{\min(x, 1-x)} = \left| \sum_{p=1}^{\infty} \frac{\sqrt{2}(-1)^p}{(p\pi)^3} \left(1 - \frac{1}{(p\pi)^2}\right)^{k-1} \frac{\sin(p\pi x)}{\min(x, 1-x)} \right|
\]

\[
\leq \sum_{p=1}^{\infty} \frac{2}{(p\pi)^3} \left(1 - \frac{1}{(p\pi)^2}\right)^{k-1} \frac{|\sin(p\pi x)|}{\min(x, 1-x)} \leq \sum_{p=1}^{\infty} \frac{2}{(p\pi)^2} = \frac{1}{3}, \quad k = 1, 2, ....
\]

This completes the proof. \( \square \)

In the following two lemmas we estimated \( ||y_k||. \)
Lemma 2.6. Let \( \sigma > 0, \ 0 < \varepsilon_2 < \min(1, \sigma), \ k > \sigma - \varepsilon_2, \ u_1 \in D(A^k) \). Then the following inequality holds true:

\[
|y_k| \leq \frac{C_2}{k^{(\sigma - \varepsilon_2)/2}} |A^\sigma u_1| \tag{2.12}
\]

with \( C_2 = \frac{L}{\sin(\pi \varepsilon_2)} (\sigma - \varepsilon_2)^{(\sigma - \varepsilon_2)/2} \).

**Proof.** For estimating \( |y_k| \), we use Equation (2.5) and apply integration along the path \( \Gamma = \Gamma_+ \cup \Gamma_- \) consisting of two rays on the complex plane:

\[
\Gamma_\pm = \{ z \in \mathbb{C} | z = \rho e^{\pm i\varphi}, \ \rho \in [0, +\infty) \}. \tag{2.13}
\]

Then we have

\[
|y_k| = \left\| (I + A)^{-1} A^k u_1 \right\| = \left\| \frac{1}{2\pi i} \int_\Gamma \left( \frac{z - \sigma z(1 - A)^{-1} A^\sigma u_1}{1 + z} \right)^k \right\|
\]

Making use of the relations

\[
|z| = |\rho e^{\pm i\varphi}| = \rho, \quad |dz| = |d (\rho e^{\pm i\varphi})| = |e^{\pm i\varphi} d\rho| = d\rho,
\]

\[
\left| \frac{z}{1 + z} \right|^2 = \frac{\rho e^{\pm i\varphi}}{1 + \rho e^{\pm i\varphi}} = \frac{\rho^2}{1 + 2\rho \cos \varphi + \rho^2} \leq \frac{\rho^2}{1 + \rho^2},
\]

we get

\[
|y_k| \leq \frac{L}{\pi} \int_0^{+\infty} \left( \frac{\rho^2}{1 + \rho^2} \right)^{k/2} \frac{\rho^{-\sigma}}{1 + \rho} d\rho \left| A^\sigma u_1 \right|
\]

\[
= \frac{L}{\pi} \int_0^{+\infty} \left( \frac{\rho^2}{1 + \rho^2} \right)^{k/2} (\rho^2)^{-(\sigma - \varepsilon_2)/2} \frac{d\rho}{\rho^{\varepsilon_2}(1 + \rho)} \left| A^\sigma u_1 \right|
\]

\[
\leq \frac{L}{\pi} \sup_{t > 0} \left( \frac{t^2}{1 + t} \right)^{k/2} (\rho^2)^{-(\sigma - \varepsilon_2)/2} \frac{\pi}{\sin(\pi \varepsilon_2)} \left| A^\sigma u_1 \right|.
\]

Applying Lemma 2.3 with \( n = \frac{k}{2} \) and \( \alpha = \frac{\sigma - \varepsilon_2}{2} \), we obtain the inequality

\[
|y_k| \leq \frac{L}{\pi} \left( \frac{\sigma - \varepsilon_2}{k} \right)^{(\sigma - \varepsilon_2)/2} \frac{\pi}{\sin(\pi \varepsilon_2)} \left| A^\sigma u_1 \right|,
\]

and therefore Equation (2.12) is proved.

Lemma 2.7. Let \( u_1 \in E \) satisfy the condition \( u_1 \in D(A^k) \ \forall k \in \mathbb{N} \). Then the following estimate holds true:
\[ \| y_k \| \leq \frac{Le}{\sqrt{2}} e^{-2\sqrt{k}} \left( \frac{\sqrt{k} + 1}{\sqrt{k}} \right)^2 \| u_1 \|_{C(A, (1), \nu)}, \quad \nu = \frac{\cos \varphi}{L + 1}. \quad (2.14) \]

**Proof.** Taking representation (2.5) into consideration and integrating along the path \( \Gamma = \Gamma_+ \cup \Gamma_- \) (see Equation (2.13)), we have

\[
\| y_k \| = \left\| \left( I + A \right)^{-1} A^k u_1 \right\| = \left\| \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{z}{1 + z} \right)^k \left( 1 + \frac{z}{k} \right)^{-k} (zI - A)^{-1} (I + \frac{A}{k})^k u_1 dz \right\| \leq \frac{1}{2\pi} \int_{\Gamma} \left| \frac{z}{1 + z} \right|^k \left| 1 + \frac{z}{k} \right| \frac{L}{1 + |z|} \left| dz \right| \left\| (I + \frac{A}{k})^k u_1 \right\| = \frac{L}{\pi} \int_{0}^{+\infty} \frac{\rho \cos \varphi}{\sqrt{1 + 2 \rho \cos \varphi + \rho^2 \sqrt{1 + 2 \frac{k}{k^2} \cos \varphi + \frac{\rho^2}{k^2}}} d\rho \cos^{-k} \varphi \left\| (I + \frac{A}{k})^k A^{-k} A^k u_1 \right\|.
\]

Next we make use of the relations

\[
\sqrt{1 + 2 \rho \cos \varphi + \rho^2} \geq 1 + \rho \cos \varphi, \quad \sqrt{1 + 2 \frac{k}{k^2} \cos \varphi + \frac{\rho^2}{k^2}} \geq \sqrt{1 + \frac{\rho^2}{k^2}} \geq \sqrt{2 \frac{k}{k^2}} \geq \left| dz \right| = \left| d \left( \rho e^{i\varphi} \right) \right| = \left| e^{i\varphi} d\rho \right| = d\rho
\]

and Lemma 2.1. Then

\[
\| y_k \| \leq \frac{L\sqrt{k}}{\pi \sqrt{2}} \int_{0}^{+\infty} \frac{\rho \cos \varphi}{\left( 1 + \rho \cos \varphi \right)^{1 + \frac{\rho \cos \varphi}{k}}} \left( 1 + \frac{\rho \cos \varphi}{k} \right)^{k-1} d\rho \cos^{-k} \varphi \left\| \left( I + \frac{A}{k} \right)^k A^{-k} \right\| \left\| A^k u_1 \right\| \leq \frac{L\sqrt{k}}{\pi \sqrt{2}} \max_{t > 0} \left[ \frac{t}{(1 + t)(1 + t)} \right]^{k-1} \int_{0}^{+\infty} \frac{d\rho}{\rho (1 + \rho)} \cos^{-k} \varphi (L + 1)^k \left\| A^k u_1 \right\| \leq \frac{L\sqrt{k}}{\sqrt{2}} \left[ \frac{\sqrt{k}}{(1 + \sqrt{k})(1 + \frac{1}{\sqrt{k}})} \right]^k \left( \frac{\sqrt{k} + 1}{\sqrt{k}} \right)^2 \left\| A^k u_1 \right\| \frac{2}{\nu^k}.
\]

Applying here Lemma 2.4 and using norm (2.7), we obtain the estimate (2.14).

We approximate the exact solution \( u(x) \) of BVP (2.1) by the partial sum of series (2.4):
\[ u_N(x) = \sum_{k=0}^{N} v_k(x)y_k. \] (2.15)

In the next two theorems, we study the error \( u(x) - u_N(x) \) under various assumptions about smoothness of the vector \( u \).

**Theorem 2.1.** Let \( u_1 \in D(A^{\sigma}), \sigma > 1 \). Then the accuracy of the approximate solution (2.15) is characterized by the weighted estimate

\[
\left\| \frac{u(x) - u_N(x)}{\min(x, 1 - x)} \right\| \leq \frac{C}{N^{(\sigma-1-\varepsilon)/2}} \| A^\sigma u_1 \|, \quad x \in [0, 1] \quad (N \geq \sigma - 1),
\]

(2.16)

where \( \varepsilon > 0 \) is an arbitrary small number and \( C > 0 \) is a constant independent of \( N \).

**Proof.** For \( 0 < \varepsilon_1 < 1, 0 < \varepsilon_2 < 1, 1 + \varepsilon_1 + \varepsilon_2 < \sigma \) and \( N + 1 > \sigma - \varepsilon_2 \), the assumptions of Lemma 2.5 and Lemma 2.6 are fulfilled. Then we get

\[
\left\| \frac{u(x) - u_N(x)}{\min(x, 1 - x)} \right\| = \left\| \sum_{k=N+1}^{\infty} \frac{v_k(x)}{\min(x, 1 - x)}y_k \right\| \leq \sum_{k=N+1}^{\infty} \left\| \frac{v_k(x)}{\min(x, 1 - x)} \right\| \| y_k \|
\]

\[
\leq \sum_{k=N+1}^{\infty} \frac{C_1}{k(1-\varepsilon_1)/2} \frac{C_2}{k(\sigma-\varepsilon_2)/2} \| A^\sigma u_1 \| \leq C_1 C_2 \int_{N}^{+\infty} \frac{dx}{x^{(1+\sigma-\varepsilon_1-\varepsilon_2)/2}} \| A^\sigma u_1 \|
\]

\[
= \frac{2C_1 C_2}{\sigma - 1 - \varepsilon_1 - \varepsilon_2 N^{(\sigma-1-\varepsilon_1-\varepsilon_2)/2}} \| A^\sigma u_1 \|
\]

from where estimate (2.16) easily follows. \qed

**Theorem 2.2.** Let \( u_1 \in C(A, (1), \nu) \) with \( \nu = \frac{\cos \alpha}{L+1} \). Then the accuracy of the approximate solution (2.15) is characterized by the weighted estimate

\[
\left\| \frac{u(x) - u_N(x)}{\min(x, 1 - x)} \right\| \leq C e^{-\sqrt{N+1}} (N + 1)^{1/2 - \varepsilon} \| u_1 \|_{C(A, (1), \nu)}, \quad x \in [0, 1] \quad (N \in \mathbb{N}),
\]

(2.17)

where \( \varepsilon > 0 \) is an arbitrary small number and \( C > 0 \) is a constant independent of \( N \).

**Proof.** Applying Lemma 2.5 and Lemma 2.7, we have

\[
\left\| \frac{u(x) - u_N(x)}{\min(x, 1 - x)} \right\| = \left\| \sum_{k=N+1}^{\infty} \frac{v_k(x)}{\min(x, 1 - x)}y_k \right\| \leq \sum_{k=N+1}^{\infty} \left\| \frac{v_k(x)}{\min(x, 1 - x)} \right\| \| y_k \|
\]

\[
\leq \sum_{k=N+1}^{\infty} \frac{C_1}{k^{1-\varepsilon_1}/2} \sqrt{2} e^{-2\sqrt{k}} \frac{(\sqrt{k} + 1)^2}{\sqrt{k}} \| u_1 \|_{C(A, (1), \nu)}
\]

\[
\leq \frac{C_1 L e^{-\sqrt{N+1}}}{\sqrt{2}} (N+1)^{(1-\varepsilon_1)/2} \sum_{k=1}^{\infty} e^{-\sqrt{k}} \frac{(\sqrt{k} + 1)^2}{\sqrt{k}} \| u_1 \|_{C(A, (1), \nu)}.
\]

(2.18)
Denoting by $S$ the sum of the convergent number series: $S = \sum_{k=1}^{\infty} e^{-\sqrt{k}} \left( \frac{\sqrt{k+1}}{\sqrt{k}} \right)^2 = 8.152349342\ldots$, and putting $C = \frac{C_1L\sqrt{\pi}}{\sqrt{2}}$, we arrive at inequality (2.17).

### 3. BVP for the inhomogeneous equation

In this section, we consider the BVP

$$
\frac{d^2 u(x)}{dx^2} - Au(x) = -f(x), \quad x \in (0, 1),
$$

$$
u(0) = 0, \quad u(1) = 0,
$$

with the operator $A$ satisfying the same conditions as in Section 2.

To write down the solution $u(x)$ in a convenient way, we make use of the Fourier series representation (1.2) and (1.3) for the right-hand side $f(x)$. Applying the operator Green function

$$G(x, \zeta; A) = (\sqrt{A}\sinh\sqrt{A})^{-1} \begin{cases} \sinh(\sqrt{A}x)\sinh(\sqrt{A}(1-\zeta)) & \text{if } x \leq \zeta, \\ \sinh(\sqrt{A}\zeta)\sinh(\sqrt{A}(1-x)) & \text{if } \zeta \leq x, \end{cases}
$$

we modify $u(x)$ as follows:

$$u(x) = \int_0^1 G(x, \xi; A)f(\xi) \, d\xi$$

$$= (\sqrt{A}\sinh\sqrt{A})^{-1}\sinh(\sqrt{A}x)\int_0^x \sinh(\sqrt{A}\xi)f(\xi) \, d\xi$$

$$+ (\sqrt{A}\sinh\sqrt{A})^{-1}\sinh(\sqrt{A}x)\int_0^1 \sinh(\sqrt{A}(1-\zeta))f(\xi) \, d\xi$$

$$= \sum_{k=1}^{\infty} \sqrt{2} \sin(2k\pi x) \left[ (2k\pi)^2 I + A \right]^{-1} f_{s,k}$$

$$+ A \sinh(\sqrt{A})^{-1} \left\{ \sinh(\sqrt{A} - \sinh(\sqrt{A}(1-x)) - \sinh(\sqrt{A}x) \right\} f_0$$

$$+ \sum_{k=1}^{\infty} \sqrt{2} \left[ (2k\pi)^2 I + A \right]^{-1} \sinh^{-1}(\sqrt{A})$$

$$\times \left\{ \cos(2k\pi x)\sinh(\sqrt{A} - \sinh(\sqrt{A}(1-x)) - \sinh(\sqrt{A}x) \right\} f_{c,k},
$$

(3.2)

Bearing Equation (2.4) in mind, we then rewrite Equation (3.2) in the form
\[ u(x) = \sum_{k=1}^{\infty} \sqrt{2} \sin (2k\pi x) [(2k\pi)^2 I + A]^{-1} f_{s,k} \]

\[ + A^{-1} \left\{ I - \sum_{j=0}^{\infty} [v_j(1-x) + v_j(x)] \left[ (I + A)^{-1} A \right]^j \right\} f_0 \]

\[ + \sum_{k=1}^{\infty} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \left\{ \cos (2k\pi x) I - \sum_{j=0}^{\infty} [v_j(1-x) + v_j(x)] \left[ (I + A)^{-1} A \right]^j \right\} f_{c,k}. \]

(3.3)

This representation yields the following approximation:

\[ u_{N,M}(x) = \sum_{k=1}^{N} \sqrt{2} \sin (2k\pi x) [(2k\pi)^2 I + A]^{-1} f_{s,k} \]

\[ + A^{-1} \left\{ I - \sum_{j=0}^{M} [v_j(1-x) + v_j(x)] \left[ (I + A)^{-1} A \right]^j \right\} f_0 \]

\[ + \sum_{k=1}^{N} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \left\{ \cos (2k\pi x) I - \sum_{j=0}^{M} [v_j(1-x) + v_j(x)] \left[ (I + A)^{-1} A \right]^j \right\} f_{c,k}. \]

(3.4)

Next we study the accuracy of \( u_{N,M}(x) \). To this end, we introduce the error \( u(x) - u_{N,M}(x) \) as the sum:

\[ u(x) - u_{N,M}(x) = \sum_{k=1}^{5} D_k, \]

(3.5)

where

\[ D_1 = \sum_{k=N+1}^{\infty} \sqrt{2} \sin (2k\pi x) [(2k\pi)^2 I + A]^{-1} f_{s,k}, \]

\[ D_2 = -\sum_{j=M+1}^{\infty} [v_j(1-x) + v_j(x)] A^{-1} \left[ (I + A)^{-1} A \right]^j f_0, \]

\[ D_3 = \sum_{k=N+1}^{\infty} \sqrt{2} \cos (2k\pi x) - 1 [(2k\pi)^2 I + A]^{-1} f_{c,k}, \]

\[ D_4 = -\sum_{k=1}^{\infty} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \sum_{j=M+1}^{\infty} [v_j(1-x) + v_j(x)] \left[ (I + A)^{-1} A \right]^j f_{c,k}, \]

\[ D_5 = -\sum_{k=N+1}^{\infty} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \sum_{j=1}^{M} [v_j(1-x) + v_j(x)] \left[ (I + A)^{-1} A \right]^j f_{c,k}. \]

(3.6)
Now, we prove some error estimates for approximation (3.4) under certain assumptions about smoothness of $f(x)$ in terms of smoothness of the coefficients $f_0, f_{c,k}, f_{s,k}$ in Equation (1.3).

Theorem 3.1. Let the following conditions be fulfilled:

\[ \sigma > 0, \quad f_0 \in D(A^\sigma), \quad f_{c,k} \in D(A^\sigma) \quad \forall k \in \mathbb{N}, \]

\[ ||f_s||_\sigma = \left\{ \sum_{k=1}^{\infty} k^{\sigma+1} ||f_{s,k}||^2 \right\}^{1/2} < \infty, \quad ||f_c||_\sigma = \left\{ \sum_{k=1}^{\infty} k^{\sigma+1} ||f_{c,k}||^2 \right\}^{1/2} < \infty, \]

\[ ||f_c||_{A^\sigma} = \sum_{k=1}^{\infty} ||A^\sigma f_{c,k}|| < \infty. \]

Then the accuracy of the approximate solution (3.4) is characterized by the weighted estimate

\[ \left| \frac{u(x) - u_{N,N}(x)}{\min(x, 1-x)} \right| \leq \frac{C}{N^{(\sigma - \varepsilon)/2}} \left( ||f_s||_\sigma + ||f_c||_\sigma + ||A^\sigma f_0|| + ||f_c||_{A^\sigma} \right), \tag{3.7} \]

\[ x \in [0, 1] \quad (N \geq \sigma), \]

where $\varepsilon > 0$ is an arbitrary small number and $C > 0$ is a constant independent of $N$.

Proof. We focus now on estimating each summand in Equation (3.6). Throughout the whole proof, we use the integration path (2.13). We will also need the relations

\[ \left| (2k\pi)^2 + z \right| = \left| (2k\pi)^2 + \rho e^{\pm i\phi} \right| = \sqrt{(2k\pi)^4 + 2(2k\pi)^2 \rho \cos \phi + \rho^2} \]

\[ \geq \sqrt{(2k\pi)^4 + \rho^2} \geq 2(2k\pi)^2 \rho \geq 2k\pi \sqrt{2\rho}, \]

\[ \left| (2k\pi)^2 + z \right| = \left| (2k\pi)^2 + \rho e^{\pm i\phi} \right| = \sqrt{(2k\pi)^4 + 2(2k\pi)^2 \rho \cos \phi + \rho^2} \geq \rho, \]

\[ \left| \frac{z}{1+z} \right|^2 = \left| \frac{\rho e^{\pm i\phi}}{1 + \rho e^{i\phi}} \right|^2 = \frac{\rho^2}{1 + 2\rho \cos \phi + \rho^2} \leq \frac{\rho^2}{1 + \rho^2} \tag{3.8} \]

and the Cauchy–Bunyakovsky–Schwarz inequality for number series.
Then for $D_1$ we get
\[
\left\| \frac{D_1}{\min(x, 1-x)} \right\| = \left\| \sum_{k=N+1}^{\infty} \frac{\sqrt{2} \sin(2k\pi x)}{\min(x, 1-x)} [(2k\pi)^2 I + A]^{-1} f_{x,k} \right\| 
\leq \sum_{k=N+1}^{\infty} \frac{\sqrt{2} \sin(2k\pi x)}{\min(x, 1-x)} \left\| \frac{1}{2\pi i} \int \frac{1}{(2k\pi)^2 + z} (zI - A)^{-1} f_{x,k} dz \right\|
\leq \sum_{k=N+1}^{\infty} \frac{\sqrt{22k\pi}}{2\pi} \int \frac{1}{|z|^2 + 1} |dz| \left\| f_{x,k} \right\|
\leq \frac{L}{\pi} \sum_{k=N+1}^{\infty} \int_0^{+\infty} \frac{d\rho}{\sqrt{\rho}(1+\rho)} \left\| f_{x,k} \right\| = \frac{L}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{k^{1/2}} k^{(\sigma+1)/2} \left\| f_{x,k} \right\|
\leq L \left\{ \sum_{k=N+1}^{\infty} k^{1/2} \left\| f_{x,k} \right\|^2 \right\}^{1/2} \leq L \left\{ \int_{N}^{\infty} \frac{dx}{x^{\sigma+1}} \right\}^{1/2} \left\| f_{x} \right\| = \frac{L}{\sqrt{\sigma}} \left\| f_{x} \right\| (N \in \mathbb{N}, \ \sigma > 0).
\]

(3.9)

To estimate $D_2$, we assume that $0 < \varepsilon_1 < 1$, $0 < \varepsilon_3 < 1$, $\varepsilon_1 + \varepsilon_3 < \sigma$, $M > \sigma - \varepsilon_3$. These conditions make it possible to apply Lemma 2.5 and Lemma 2.3 for $n = j/2$ and $\alpha = (1 + \sigma - \varepsilon_3)/2$. We have
\[
\left\| \frac{D_2}{\min(x, 1-x)} \right\| = \left\| - \sum_{j=M+1}^{\infty} \frac{\nu_l(1-x) + \nu_r(x)}{\min(x, 1-x)} A^{-1} [(I + A)^{-1} A] f_0 \right\|
\leq \sum_{j=M+1}^{\infty} \frac{\nu_l(1-x) + \nu_r(x)}{\min(x, 1-x)} \left\| \frac{1}{2\pi i} \int \frac{z^{-(1+\sigma)} (zI - A)^{-1} A f_0 dz}{1 + |z|} \right\|
\leq \frac{2C_1}{2\pi i} \sum_{j=M+1}^{\infty} \frac{1}{j^{(1-\varepsilon_1)/2}} \int_0^{+\infty} \left( \frac{\rho^2}{1 + \rho^2} \right)^{j/2} (\rho^2)^{-(1+\sigma-\varepsilon_3)/2} \frac{d\rho}{\rho^{(1+\sigma)} (1+\rho)} \left\| A^\sigma f_0 \right\|
\leq \frac{2C_1 L}{\sin(\varepsilon_3)} \left( 1 + \sigma - \varepsilon_3 \right)^{(1+\sigma-\varepsilon_3)/2} \sum_{j=M+1}^{\infty} \frac{1}{j^{(1+\sigma-\varepsilon_3)/2}} \left\| A^\sigma f_0 \right\|
\leq \frac{2C_1 L}{\sin(\varepsilon_3)} \left( 1 + \sigma - \varepsilon_3 \right)^{(1+\sigma-\varepsilon_3)/2} \int_{M}^{+\infty} \frac{dx}{x^{(2+\sigma-\varepsilon_1)/2}} \left\| A^\sigma f_0 \right\|
\leq \frac{C_3}{M^{(\sigma-\varepsilon_1)/2}} \left\| A^\sigma f_0 \right\|
\]
with $C_3 = \frac{4C_1 L}{\sin(\varepsilon_3)} \left( 1 + \sigma - \varepsilon_3 \right)^{(1+\sigma-\varepsilon_3)/2}$. 
Next, since $D_3$ and $D_1$ are very much alike, we obtain

$$
\left\| \frac{D_3}{\min(x, 1 - x)} \right\| = \left\| \sum_{k=N+1}^{\infty} \frac{\sqrt{2}[\cos{(2k\pi x)}-1]}{\min(x, 1 - x)} [ (2k\pi)^2 I + A ]^{-1} f_{c,k} \right\|
\leq \frac{L}{\sigma N^{\sigma/2}} \| f_\sigma \| (N \in \mathbb{N}, \quad \sigma > 0).
$$

(3.11)

The analysis of $D_4$ is similar to that of $D_2$. We assume once again that $0 < \varepsilon_1 < 1$, $0 < \varepsilon_3 < 1$, $\varepsilon_1 + \varepsilon_3 < \sigma$, $M > \sigma - \varepsilon_3$ and therefore apply Lemma 2.5 and Lemma 2.3 for $n = j/2$ and $\alpha = (1 + \sigma - \varepsilon_3)/2$. Thus we have

$$
\left\| \frac{D_4}{\min(x, 1 - x)} \right\| = \left\| - \sum_{k=1}^{\infty} \frac{\sqrt{2}}{(2k\pi)^2 I + A}^{-1} \sum_{j=M+1}^{\infty} \frac{v_j(1-x) + v_j(x)}{\min(x, 1 - x)} [(I + A)^{-1} A]^{j} f_{c,k} \right\|
\leq \sum_{k=1}^{\infty} \sum_{j=M+1}^{\infty} \frac{\sqrt{2}}{2\pi j(1-\varepsilon_3)/2} \left\| \frac{\sqrt{2}}{2\pi i} \int \frac{z (z-\sigma)}{(2k\pi)^2 + z} (zI-A)^{-1} A^{\sigma} f_{c,k} \, dz \right\|
\leq 2\sqrt{2}C_1 L \sum_{k=1}^{\infty} \sum_{j=M+1}^{\infty} \frac{1}{j(1-\varepsilon_3)/2} \sup_{t>0} \left[ \left( \frac{t}{1+t} \right)^{j/2} t^{-(1+\sigma-\varepsilon_3)/2} \right] \int_0^{+\infty} \frac{d\rho}{\rho^3 (1+\rho)} \| A^{\sigma} f_{c,k} \|
\leq 2\sqrt{2}C_1 L \sin\left( \frac{\pi \varepsilon_3}{3} \right) (1 + \sigma - \varepsilon_3)^{1+\sigma-\varepsilon_3}/2 \sum_{j=M+1}^{\infty} \frac{1}{j(1+\sigma-\varepsilon_3)/2} \sum_{k=1}^{\infty} \| A^{\sigma} f_{c,k} \|
\leq 2\sqrt{2}C_1 L \sin\left( \frac{\pi \varepsilon_3}{3} \right) (1 + \sigma - \varepsilon_3)^{1+\sigma-\varepsilon_3}/2 \int_0^{+\infty} \frac{dx}{x^{(2+\sigma-\varepsilon_3)/2}} \| A^{\sigma} f_{c,k} \| = \frac{\sqrt{2}C_3}{M(\sigma-\varepsilon_3)^{1/2}} \| A^{\sigma} f_{c,k} \|
$$

(3.12)

with the constant $C_3$ defined in Equation (3.10).
And finally, we evaluate the summand $D_5$:

\[
\left\| \frac{D_5}{\min(x, 1-x)} \right\| = \left\| - \sum_{k=N+1}^{\infty} \sqrt{2} \left( (2k\pi)^2 I + A \right)^{-1} \sum_{j=1}^{M} \frac{v_j(1-x) + v_j(x)}{\min(x, 1-x)} (I + A)^{-1} f_{c,k} \right\|
\]

\[
\leq \sum_{k=N+1}^{\infty} \sum_{j=1}^{M} \frac{2\sqrt{2}}{\min(x, 1-x)} \left\| 2\pi i \right\| \left( \frac{z}{1+z} \right)^{j} \left\| \frac{1}{(2k\pi)^2 + z} \right\| (2iI-A)^{-1} f_{c,k} \right\| dz
\]

\[
= \frac{LM}{3\pi^2} \sum_{k=N+1}^{\infty} \frac{1}{k} \left\| f_{c,k} \right\| \sum_{j=1}^{M} \left\| \int_{0}^{\infty} \frac{d\rho}{\sqrt{\rho(1+\rho)}} = \frac{LM}{3\pi} \sum_{k=N+1}^{\infty} \frac{1}{k} \left\| f_{c,k} \right\| \right. \\
\left. \right. = \frac{LM}{3\pi} \sum_{k=N+1}^{\infty} \frac{1}{k^{(1+\sigma)/2}} \left\| f_{c,k} \right\| = \frac{LM}{3\pi} \left\{ \sum_{k=N+1}^{\infty} \frac{1}{k^{1+\sigma}} \right\}^{1/2} \left\{ \sum_{k=N+1}^{\infty} k^{1+\sigma} \left\| f_{c,k} \right\|^2 \right\}^{1/2}
\]

\[
\leq \frac{LM}{3\pi} \left\{ \int_{N}^{\infty} \frac{dx}{x^{1+\sigma}} \right\}^{1/2} \left\| f_{c} \right\|_{o} = \frac{LM}{3\pi \sqrt{2 + \sigma N^{1+\sigma}/2}} \left\| f_{c} \right\|_{o} (N \in \mathbb{N}, \ \sigma > -2).
\]

Estimate (3.7) easily follows from inequalities (3.9)–(3.13) for $M = N$. □

In the next theorem, we investigate error (3.6) under another set of assumptions concerning smoothness of $f(x)$.

**Theorem 3.2.** Let the following conditions hold true:

\[
\nu = \cos \frac{\varphi}{L}, \quad f_0 \in C(A, (1), \nu), \quad f_{c,k} \in C(A, (1), \nu) \quad \forall k \in \mathbb{N},
\]

\[
\left\| f_{c} \right\|_{A} \text{ def } = \left\{ \sum_{k=1}^{\infty} \left\| f_{c,k} \right\|_{C(A, (1), \nu)}^2 \right\}^{1/2} < \infty,
\]

\[
\left\| f_{c} \right\|_{\infty} \text{ def } = \sum_{k=1}^{\infty} e^{k} \left\| f_{c,k} \right\| < \infty, \quad \left\| f_{c} \right\|_{\infty} \text{ def } = \sum_{k=1}^{\infty} e^{k} \left\| f_{c,k} \right\| < \infty.
\]

Then the accuracy of the approximate solution (3.4) is characterized by the weighted estimate

\[
\left\| u(x) - u_{N,N}(x) \right\| \leq \frac{Ce^{-\sqrt{N+1}}}{(N+1)^{1/2-\epsilon}} \left( \left\| f_{c} \right\|_{\infty} + \left\| f_{c} \right\|_{\infty} + \left\| f_{0} \right\|_{C(A, (1), \nu)} + \left\| f_{c} \right\|_{A} \right),
\]

\[
x \in [0, 1] \quad (N \in \mathbb{N}),
\]

(3.14)

where $\epsilon > 0$ is an arbitrary small number and $C > 0$ is a constant independent of $N$.

**Proof.** Except for the summand $D_2$, we use the integrations path (2.13). For the sake of brevity, some obvious calculations which are similar to those in Theorem 3.1 and Lemma 2.7 will be omitted.
Relying on Equation (3.9), we evaluate $D_1$ as follows:

$$
\left\| \frac{D_1}{\min(x, 1-x)} \right\| = \left\| \sum_{k=N+1}^{\infty} \frac{\sqrt{2} \sin(2k\pi x)}{\min(x, 1-x)} \left[ (2k\pi)^2 I + A \right]^{-1} f_{s,k} \right\|
$$

$$
\leq L \sum_{k=N+1}^{\infty} \|f_{s,k}\| = L \sum_{k=N+1}^{\infty} e^{-k} \|f_{s,k}\| \tag{3.15}
$$

To estimate $D_2$, we take the integration path $\tilde{\Gamma}$ consisting of two rays and a circle arc:

$$
\tilde{\Gamma} = \tilde{\Gamma}_- \cup \tilde{\Gamma}_+ \cup \Gamma_\gamma, \quad \tilde{\Gamma}_{\pm} = \{ z \in \mathbb{C} | z = \rho e^{\pm i\phi}, \ \rho \in [\gamma, +\infty) \},
$$

$$
\Gamma_\gamma = \{ z \in \mathbb{C} | z = \gamma e^{i\theta}, \ \theta \in [-\varphi, \varphi] \},
$$

with $dz = d(\rho e^{\pm i\phi}) = e^{\pm i\phi} d\rho$ for $\tilde{\Gamma}_{\pm}$ and $dz = d(\gamma e^{i\theta}) = i\gamma e^{i\theta} d\theta$ for $\Gamma_\gamma$. Then

$$
\left\| \frac{D_2}{\min(x, 1-x)} \right\| = \left\| - \sum_{j=M+1}^{\infty} \frac{\nu_j(1-x) + \nu_j(x)}{\min(x, 1-x)} A^{-1} [(I + A)^{-1}]^j f_0 \right\|
$$

$$
\leq \sum_{j=M+1}^{\infty} \frac{2C_1}{2\pi j(1-\epsilon_1)/2} \left\| \int_{\tilde{\Gamma}} \frac{1}{|z|} \left| \frac{z}{1+|z|} \right| \left( \frac{1+|z|}{1+|z|} \right)^{-j} L \frac{dz}{|z|} \right\| \left( \frac{I + A}{j} \right)^j f_0
$$

$$
\leq \frac{2C_1 L}{\pi} \sum_{j=M+1}^{\infty} \frac{1}{j(1-\epsilon_1)/2} \left\{ \int_0^{\infty} \frac{\rho \cos \phi}{\sqrt{1 + 2 \rho \cos \phi + \rho^2}} \frac{d\rho}{\rho(1+\rho)} \right\}
$$

$$
+ \left\{ \int_0^{\infty} \frac{\gamma \cos \theta}{\sqrt{1 + 2 \gamma \cos \theta + \gamma^2}} \frac{d\theta}{1+\gamma} \right\} \frac{\rho \cos \phi}{(1 + \rho \cos \phi)(1 + \rho^2 \cos \phi)} \left( \frac{I + A}{j} \right)^j A^{-j} ||A_f||_0
$$

$$
\leq \frac{2C_1 L}{\pi} \sum_{j=M+1}^{\infty} \frac{1}{j(1-\epsilon_1)/2} \left\{ \int_0^{\infty} \frac{\rho \cos \phi}{\sqrt{1 + 2 \rho \cos \phi + \rho^2}} \frac{d\rho}{\rho(1+\rho)} \right\}
$$

$$
+ \left\{ \int_0^{\infty} \frac{\gamma \cos \theta}{\sqrt{1 + 2 \gamma \cos \theta + \gamma^2}} \frac{d\theta}{1+\gamma} \right\} \frac{\rho \cos \phi}{(1 + \rho \cos \phi)(1 + \rho^2 \cos \phi)} \left( \frac{I + A}{j} \right)^j A^{-j} ||A_f||_0
$$

$$
\leq 2C_1 L \sum_{j=M+1}^{\infty} \frac{1}{j(1-\epsilon_1)/2} \max_{t \geq 0} \left\{ \frac{t}{(1+t)(1+j)} \right\} \left\{ \int_0^{\infty} \frac{d\rho}{\rho(1+\rho)} \right\}
$$

$$
\times \cos^{-j} \phi (L + 1)^j ||A_f||_0.\]
Applying here Lemma 2.4, we obtain

\[
\left\| \frac{D_2}{\min(x, 1-x)} \right\| \leq \frac{2C_1L}{\pi} \left( \ln \frac{1+\gamma}{\gamma} + \frac{\varphi}{1+\gamma} \right) \sum_{j=M+1}^{\infty} e e^{-\sqrt{j}} \|f_0\|_{C(A(1), \nu)} (3.16)
\]

with \( C_4 = \frac{2C_1L\tilde{S}_e}{\pi} \left( \ln \frac{1+\gamma}{\gamma} + \frac{\varphi}{1+\gamma} \right), \) where \( \tilde{S} \) is the sum of the convergent number series \( \tilde{S} = \sum_{j=1}^{\infty} e^{-\sqrt{j}} = 1.670406818 \ldots \)

The summand \( D_3 \) is estimated in the same way as \( D_1 \):

\[
\left\| \frac{D_3}{\min(x, 1-x)} \right\| = \left\| \sum_{k=N+1}^{\infty} \left[ \sqrt{2} \cos (2k\pi x) - 1 \right] \left[ (2k\pi)^2 I + A \right]^{-1} f_{e,k} \right\| 
\]

\[
\leq L e^{-\sqrt{N+1}} \|f_e\|_{\infty} (3.17)
\]

Next, we have

\[
\left\| \frac{D_4}{\min(x, 1-x)} \right\| = \left\| - \sum_{k=1}^{\infty} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \sum_{j=M+1}^{\infty} \frac{v_j(1-x) + v_j(x)}{\min(x, 1-x)} (I + A)^{-1} A^{j} f_{e,k} \right\|
\]

\[
\leq \sum_{k=1}^{\infty} \sum_{j=M+1}^{\infty} \left[ \frac{v_j(1-x) + v_j(x)}{\min(x, 1-x)} \right] \left[ (2k\pi)^2 I + A \right]^{-1} \frac{v_j(1-x) + v_j(x)}{\min(x, 1-x)} (I + A)^{-1} A^{j} f_{e,k} \right\|
\]

\[
\leq \frac{2\sqrt{2C_1L}}{\pi} \left\| \sum_{k=1}^{\infty} \sum_{j=M+1}^{\infty} \left[ \frac{1}{(2k\pi)^2 + z} \right] \left[ \cos^{-1} \left( \frac{1+\rho}{1+\rho \cos \varphi} \right) \right]^{j} d\rho \right\| (3.18)
\]

\[
\leq \frac{2\sqrt{2C_1L}}{\pi} \sum_{j=M+1}^{\infty} e^{-\sqrt{j}} \sum_{k=1}^{\infty} \left[ \frac{1}{(2k\pi)^2 + z} \right] \left[ \cos^{-1} \left( \frac{1+\rho}{1+\rho \cos \varphi} \right) \right]^{j} d\rho \right\| (3.18)
\]

\[
\leq \frac{C_5 e^{-\sqrt{M+1}}}{\sqrt{6}} \|f_e\|_{A^{\infty}} (3.18)
\]

with the constant \( C_5 = \frac{C_1L\tilde{S}_e}{\sqrt{6}} \) and \( \tilde{S} \) described in Equation (3.16).
At last, it remains to consider $D_5$. Taking into account Equation (3.13), we get

$$
\left\| \frac{D_5}{\min(x, 1-x)} \right\| = \left\| - \sum_{k=N+1}^{\infty} \sqrt{2} [(2k\pi)^2 I + A]^{-1} \sum_{j=1}^{M} v_j (1-x) + v_j(x) \frac{1}{\min(x, 1-x)} [(I + A)^{-1} A]^{1/2} f_{c,k} \right\| \\
\leq \frac{LM}{3\pi} \sum_{k=N+1}^{\infty} \frac{1}{k} ||f_{c,k}|| = \frac{LM}{3\pi} \sum_{k=N+1}^{\infty} e^{-k} \sum_{k=1}^{\infty} e^k ||f_{c,k}|| \\
\leq \frac{LM e^{-(N+1)}}{3\pi(N + 1)} \sum_{k=1}^{\infty} e^k ||f_{c,k}|| \leq C_6 e^{-\sqrt{N+1}} \frac{e}{\sqrt{N+1}} |f_i|_{\infty}
$$

(3.19)

with the constant $C_6 = \frac{L}{3\pi}$.

Now estimates (3.15)–(3.19) with $M = N$ easily lead to the assertion of the theorem.

4. Conclusion

To summarize, we make some general comments on the results proved above.

In Theorems 2.1 and 2.2 for the homogeneous equation and in Theorems 3.1 and 3.2 for the inhomogeneous one, the boundary effect is evaluated through the weight function $\min(x, 1-x)$ which characterizes the distance from the boundary points of the interval $[0, 1]$.

Both estimate (2.16) and estimate (3.7) are power-dependent on the parameter $\sigma$. The first one indicates that if $\sigma$ increases (meaning that differential properties of $u_1$ improve), the convergence rate of the approximate solution $u_N(x)$ automatically gets higher. Therefore, method (2.15) is a method without saturation of accuracy. The same is true of the second estimate. Namely, when $\sigma$ increases (implying that the Fourier coefficients decay more quickly, i.e., $f(x)$ has better smoothness), the convergence rate of the approximate solution $u_{N,N}(x)$ accordingly goes up. So, method (3.4) does not have saturation of accuracy either.

Estimate (2.17) shows that method (2.15) has the exponential rate of convergence provided that $u_1$ is a vector of the exponential type in the sense of Ref. [23]. In a similar way, estimate (3.14) indicates that method (3.4) has the exponential rate of convergence provided vectors $f_0$, $f_{c,k}, f_{c,k}, k = 1, 2, \ldots$, in the Fourier representation of $f(x)$ have appropriate exponential rates of decay. Both of these methods can be called super-exponentially convergent.

It follows from estimate (3.7) that the complexity of algorithm (3.4) in order to arrive at the accuracy $\varepsilon > 0$ is $N \approx \varepsilon^{2/(\alpha - \sigma)}$, where $\sigma > 0$ and $\alpha$ is an arbitrarily small positive number.
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