Abstract

In the theory of tractability of multivariate problems one usually studies problems with finite smoothness. Then we want to know which $s$-variate problems can be approximated to within $\varepsilon$ by using, say, polynomially many in $s$ and $\varepsilon^{-1}$ function values or arbitrary linear functionals.

There is a recent stream of work for multivariate analytic problems for which we want to answer the usual tractability questions with $\varepsilon^{-1}$ replaced by $1 + \log \varepsilon^{-1}$. In this vein of research, multivariate integration and approximation have been studied over Korobov spaces with exponentially fast decaying Fourier coefficients. This is work of J. Dick, G. Larcher, and the authors. There is a natural need to analyze more general analytic problems defined over more general spaces and obtain tractability results in terms of $s$ and $1 + \log \varepsilon^{-1}$.

The goal of this paper is to survey the existing results, present some new results, and propose further questions for the study of tractability of multivariate analytic questions.

Keywords: Tractability, Korobov space, numerical integration, $L_2$-approximation.

2010 MSC: 65D15, 65D30, 65C05, 11K45.

1 Introduction

In this paper we discuss algorithms for multivariate integration or approximation of $s$-variate functions defined on the unit cube $[0, 1]^s$. These problems have been studied in a large number of papers from many different perspectives.

The focus of this article is to discuss algorithms for high-dimensional problems defined for functions from certain Hilbert spaces. There exist many results for such algorithms, and much progress has been made on this subject over the past decades. It is the goal of this review to focus on a recent vein of research that deals with function spaces containing analytic periodic functions with exponentially fast decaying Fourier coefficients.

Footnotes:

*P. Kritzer gratefully acknowledges the support of the Austrian Science Fund, Project P23389-N18 and Project F5506-N26, which is part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”.

†F. Pillichshammer is supported by the Austrian Science Fund (FWF) Project F5509-N26, which is part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”.

‡H. Woźniakowski is supported in part by the National Science Foundation.
We present necessary and sufficient conditions that allow us to obtain exponential error convergence and various notions of tractability. We consider algorithms that use finitely many information evaluations. For multivariate integration, algorithms use $n$ information evaluations from the class $\Lambda^{\text{std}}$ of standard information which consists of only function evaluations. For multivariate approximation in the $L_2$-norm, algorithms use $n$ information evaluations either from the class $\Lambda^{\text{all}}$ of all continuous linear functionals or from the class $\Lambda^{\text{std}}$. Since we approximate functions from the unit ball of the corresponding space, without loss of generality we restrict ourselves to linear algorithms that use nonadaptive information evaluations. In all cases, we measure the error by considering the worst-case error setting. For large $s$, it is essential to not only control how the error of an algorithm depends on $n$, but also how it depends on $s$. To this end, we consider the information complexity, $n(\varepsilon, s)$, which is the minimal number $n$ for which there exists an algorithm using $n$ information evaluations with an error of at most $\varepsilon$ for the $s$-variate functions. In all cases considered in this survey, the information complexity is proportional to the minimal cost of computing an $\varepsilon$-approximation since linear algorithms are optimal and their implementation cost is proportional to $n(\varepsilon, s)$.

We would like to control how $n(\varepsilon, s)$ depends on $\varepsilon^{-1}$ and $s$. This is the subject of tractability. In the standard theory of tractability, see [11, 12, 13], weak tractability means that $n(\varepsilon, s)$ is not exponentially dependent on $\varepsilon^{-1}$ and $s$, polynomial tractability means that $n(\varepsilon, s)$ is polynomially bounded in $\varepsilon^{-1}$ and $s$, and strong polynomial tractability means that $n(\varepsilon, s)$ is polynomially bounded in $\varepsilon^{-1}$ independently of $s$.

Typically, $n(\varepsilon, s)$ is polynomially dependent on $\varepsilon^{-1}$ and $s$ for weighted classes of smooth functions. The notion of weighted function classes means that the dependence of functions on successive variables and groups of variables is moderated by certain weights. For sufficiently fast decaying weights, the information complexity depends at most polynomially on $\varepsilon^{-1}$ and $s$; hence we obtain polynomial tractability, or even strong polynomial tractability.

These notions of tractability are suitable for problems with finite smoothness, that is, when functions from the problem space are differentiable only finitely many times. Then the minimal errors $e(n, s)$ of algorithms that use $n$ information evaluations typically enjoy polynomial convergence, i.e., $e(n, s) = O(n^{-p})$, where the factor in the big $O$ notation as well as a positive $p$ may depend on $s$.

The case of analytic or infinitely many times differentiable functions is also of interest. For such classes of functions we would like to replace polynomial convergence by exponential convergence, and study similar notions of tractability in terms of $(1 + \log \varepsilon^{-1}, s)$ instead of $(\varepsilon^{-1}, s)$. By exponential convergence we mean that $e(n, s) = O(q^{O(n^{p})})$ with $q \in (0, 1)$, where the factors in the big $O$ notation as well as a positive $p$ may depend on $s$.

Exponential convergence with various notions of tractability was studied in the papers [4] and [8] for multivariate integration in weighted Korobov spaces with exponentially fast decaying Fourier coefficients. In the paper [2], multivariate $L_2$-approximation in the worst-case setting for the same class of functions was considered.

In this article, we give an overview of recent results on exponential convergence with different notions of tractability such as weak, polynomial and strong polynomial tractability in terms of $1 + \log \varepsilon^{-1}$ and $s$. We also present a few new results and compare conditions which are needed for the standard and new tractability notions.

In Section 2 we give a short overview of $s$-variate problems, describe how we measure
errors, and give precise definitions of various notions of tractability. In Section 3 we introduce the function class under consideration here, which is a special example of a reproducing kernel Hilbert space that was also studied in [2, 4, 8]. In Sections 4 and 5, we provide details on the particular problems of \(s\)-variate numerical integration and \(L_2\)-approximation by linear algorithms. We summarize and give an outlook to some related open questions in Section 6.

2 Tractability

We consider Hilbert spaces \(H_s\) of \(s\)-variate functions defined on \([0,1]^s\), and we assume that there is a family of continuous linear operators \(S_s: H_s \to G_s\) for \(s \in \mathbb{N}\), where \(G_s\) is a normed space.

Later, we will introduce a special choice of a Hilbert space \(H_s\) (cf. Section 3) and study two particular examples of \(s\)-variate problems, namely:

- Numerical integration of functions \(f \in H_s\), see Section 4. In this case, we have \(S_s(f) = \int_{[0,1]^s} f(x) \, dx\) and \(G_s = \mathbb{R}\).
- \(L_2\)-approximation of functions \(f \in H_s\), see Section 5. In this case, we have \(S_s(f) = f\) and \(G_s = L_2([0,1]^s)\).

As already mentioned, without loss of generality, we approximate \(S_s\) by a linear algorithm \(A_{n,s}\) using \(n\) information evaluations which are given by linear functionals from the class \(\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}\). That is,

\[
A_{n,s}(f) = \sum_{j=1}^{n} L_j(f) a_j \quad \text{for all} \quad f \in H_s,
\]

where \(L_j \in \Lambda\) and \(a_j \in G_s\) for all \(j = 1, 2, \ldots, n\). For \(\Lambda = \Lambda^{\text{all}}\) we have \(L_j \in H_s^*\) whereas for \(\Lambda = \Lambda^{\text{std}}\) we have \(L_j(f) = f(x_j)\) for all \(f \in H_s\), and for some \(x_j \in [0,1]^d\). For \(\Lambda^{\text{std}}\), we choose \(H_s\) as a reproducing kernel Hilbert space so that \(\Lambda^{\text{std}} \subset \Lambda^{\text{all}}\).

We measure the error of an algorithm \(A_{n,s}\) in terms of the worst-case error, which is defined as

\[
e(H_s, A_{n,s}) := \sup_{f \in H_s, \|f\|_{H_s} \leq 1} \|S_s(f) - A_{n,s}(f)\|_{G_s},
\]

where \(\|\cdot\|_{H_s}\) denotes the norm in \(H_s\), and \(\|\cdot\|_{G_s}\) denotes the norm in \(G_s\). The \(n\)th minimal (worst-case) error is given by

\[
e(n, s) := \inf_{A_{n,s}} e(H_s, A_{n,s}),
\]

where the infimum is taken over all admissible algorithms \(A_{n,s}\).

For \(n = 0\), we consider algorithms that do not use information evaluations and therefore we use \(A_{0,s} \equiv 0\). The error of \(A_{0,s}\) is called the initial (worst-case) error and is given by

\[
e(0, s) := \sup_{f \in H_s, \|f\|_{H_s} \leq 1} \|S_s(f)\|_{G_s} = \|S_s\|.
\]
When studying algorithms $A_{n,s}$, we do not only want to control how their errors depend on $n$, but also how they depend on the dimension $s$. This is of particular importance for high-dimensional problems. To this end, we define, for $\varepsilon \in (0,1)$ and $s \in \mathbb{N}$, the information complexity by

$$n(\varepsilon, s) := \min \{n : e(n, s) \leq \varepsilon\}$$

as the minimal number of information evaluations needed to obtain an $\varepsilon$-approximation to $S_s$. In this case, we speak of the absolute error criterion. Alternatively, we can also define the information complexity as

$$n(\varepsilon, s) := \min \{n : e(n, s) \leq \varepsilon e(0, s)\},$$

i.e., as the minimal number of information evaluations needed to reduce the initial error by a factor of $\varepsilon$. In this case we speak of the normalized error criterion.

The examples considered in this paper have the convenient property that the initial errors are one, and the absolute and normalized error criteria coincide. For problems for which the initial errors are not one, the results for the absolute and normalized error criteria may be quite different; we refer the interested reader to the monographs [11, 12, 13] for further details.

The subject of tractability deals with the question how the information complexity depends on $\varepsilon^{-1}$ and $s$. Roughly speaking, tractability means that the information complexity lacks a certain disadvantageous dependence on $\varepsilon^{-1}$ and $s$.

The standard notions of tractability were introduced in such a way that positive results were possible for problems with finite smoothness. In this case, one is usually interested in when $n(\varepsilon, s)$ depends at most polynomially on $\varepsilon^{-1}$ and $s$. The following notions have been frequently studied. We say that we have:

(a) The curse of dimensionality if there exist positive $c, \tau$ and $\varepsilon_0$ such that

$$n(\varepsilon, s) \geq c (1 + \tau)^s \quad \text{for all} \quad \varepsilon \leq \varepsilon_0 \text{ and infinitely many } s.$$

(b) Weak Tractability (WT) if

$$\lim_{s + \varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, s)}{s + \varepsilon^{-1}} = 0 \quad \text{with} \quad \log 0 = 0 \text{ by convention}.$$

(c) Polynomial Tractability (PT) if there exist non-negative numbers $c, \tau_1, \tau_2$ such that

$$n(\varepsilon, s) \leq c s^{\tau_1} (\varepsilon^{-1})^{\tau_2} \quad \text{for all} \quad s \in \mathbb{N}, \quad \varepsilon \in (0,1).$$

(d) Strong Polynomial Tractability (SPT) if there exist non-negative numbers $c$ and $\tau$ such that

$$n(\varepsilon, s) \leq c (\varepsilon^{-1})^{\tau} \quad \text{for all} \quad s \in \mathbb{N}, \quad \varepsilon \in (0,1).$$

The exponent $\tau^*$ of strong polynomial tractability is defined as the infimum of $\tau$ for which strong polynomial tractability holds.
It turns out that many multivariate problems defined over standard spaces of functions suffer from the curse of dimensionality. The reason for this negative result is that for standard spaces all variables and groups of variables are equally important. If we introduce weighted spaces, in which the importance of successive variables and groups of variables is monitored by corresponding weights, we can vanquish the curse of dimensionality and obtain weak, polynomial or even strong polynomial tractability depending on the decay of the weights. Furthermore, this holds for weighted spaces with finite smoothness. We refer to [11, 12, 13] for the current state of the art in this field of research.

However, the particular weighted function space we are going to define in Section 3 is such that its elements are infinitely many times differentiable and even analytic. Therefore, it is natural to demand more of the \( n \)th minimal errors \( e(n, s) \) and of the information complexity \( n(\varepsilon, s) \) than for those cases where we only have finite smoothness.

To be more precise, we are interested in obtaining exponential or uniform exponential convergence of the minimal errors \( e(n, s) \) for problems with unbounded smoothness. We now explain how these notions are defined. By exponential convergence we mean that there exist functions \( q(\cdot) \) and \( p(\cdot) \) such that

\[
e(n, s) \leq C(s) q(s)^{n^{p(s)}} \quad \text{for all} \quad s, n \in \mathbb{N}.
\]

Obviously, the functions \( q(\cdot) \) and \( p(\cdot) \) are not uniquely defined. For instance, we can take an arbitrary number \( q \in (0, 1) \), define the function \( C_1(s) \) as

\[
C_1(s) = \left( \frac{\log q}{\log q(s)} \right)^{1/p(s)} ,
\]

and then

\[
C(s) q(s)^{n^{p(s)}} = C(s) q^{(n/C_1(s))^{p(s)}} .
\]

We prefer to work with the latter bound which was also considered in [2, 8].

We say that we achieve exponential convergence (EXP) for \( e(n, s) \) if there exist a number \( q \in (0, 1) \) and functions \( p, C, C_1 : \mathbb{N} \rightarrow (0, \infty) \) such that

\[
e(n, s) \leq C(s) q^{(n/C_1(s))^{p(s)}} \quad \text{for all} \quad s, n \in \mathbb{N} .
\]

If (1) holds we would like to find the largest possible rate \( p(s) \) of exponential convergence defined as

\[
p^*(s) = \sup \{ p(s) : p(s) \text{ satisfies (1)} \} .
\]

We say that we achieve uniform exponential convergence (UEXP) for \( e(n, s) \) if the function \( p \) in (1) can be taken as a constant function, i.e., \( p(s) = p > 0 \) for all \( s \in \mathbb{N} \). Similarly, let

\[
p^* = \sup \{ p : p(s) = p > 0 \text{ satisfies (1)} \} \text{ for all } s \in \mathbb{N}
\]

denote the largest rate of uniform exponential convergence.

Exponential convergence implies that asymptotically, with respect to \( \varepsilon \) tending to zero, we need \( \mathcal{O}(\log^{1/p(s)} \varepsilon^{-1}) \) information evaluations to compute an \( \varepsilon \)-approximation. However, it is not clear how long we have to wait to see this nice asymptotic behavior especially for large \( s \). This, of course, depends on how \( C(s), C_1(s) \) and \( p(s) \) depend
on \( s \), and it is therefore near at hand to adapt the concepts (b)–(d) of tractability to exponential error convergence. Indeed, we would like to replace \( \varepsilon^{-1} \) by \( 1 + \log \varepsilon^{-1} \) in the standard notions (b)–(d), which yields new versions of weak, polynomial, and strong polynomial tractability. The following new tractability versions (e), (f), and (g) were already introduced in [2, 4, 8]. We use a new kind of notation in order to be able to distinguish (b)–(d) from (e)–(g). We say that we have:

(e) \textit{Exponential Convergence-Weak Tractability (EC-WT)} if

\[
\lim_{s+\log \varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, s)}{s + \log \varepsilon^{-1}} = 0 \quad \text{with} \quad \log 0 = 0 \quad \text{by convention.}
\]

(f) \textit{Exponential Convergence-Polynomial Tractability (EC-PT)} if there exist non-negative numbers \( c, \tau_1, \tau_2 \) such that

\[
n(\varepsilon, s) \leq c s^{\tau_1} (1 + \log \varepsilon^{-1})^{\tau_2} \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0, 1).
\]

(g) \textit{Exponential Convergence-Strong Polynomial Tractability (EC-SPT)} if there exist non-negative numbers \( c \) and \( \tau \) such that

\[
n(\varepsilon, s) \leq c (1 + \log \varepsilon^{-1})^{\tau} \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0, 1).
\]

The exponent \( \tau^* \) of EC-SPT is defined as the infimum of \( \tau \) for which EC-SPT holds.

Let us give some comments on these definitions. First, we remark that the use of the prefix EC (exponential convergence) in (e)–(g) is motivated by the fact that EC-PT (and therefore also EC-SPT) implies exponential convergence (cf. Theorem [4]). Also EC-WT implies that \( e(n, s) \) converges to zero faster than any power of \( n^{-1} \) as \( n \) goes to infinity, i.e., for any \( \alpha > 0 \) we have

\[
\lim_{n \to \infty} n^\alpha e(n, s) = 0. \tag{2}
\]

This can be seen as follows. Let \( \alpha > 0 \) and choose \( \delta \in (0, \frac{1}{\alpha}) \). For a fixed dimension \( s \), EC-WT implies the existence of an \( M = M(\delta) > 0 \) such that for all \( \varepsilon > 0 \) with \( \log \varepsilon^{-1} > M \) we have

\[
\frac{\log n(\varepsilon, s)}{\log \varepsilon^{-1}} < \delta \iff n(\varepsilon, s) < \varepsilon^{-\delta}.
\]

This implies that for large enough \( n \in \mathbb{N} \) we have \( e(n, s) < n^{-1/\delta} \). Hence, we have \( n^\alpha e(n, s) < n^{\alpha-1/\delta} \to 0 \) as \( n \to \infty \).

Furthermore we note, as in [2, 4], that if (1) holds then

\[
n(\varepsilon, s) \leq \left[ C_1(s) \left( \frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p(s)} \right] \quad \text{for all} \quad s \in \mathbb{N} \text{ and } \varepsilon \in (0, 1). \tag{3}
\]

Moreover, if (3) holds then

\[
e(n + 1, s) \leq C(s) q^{(n/C_1(s)) p(s)} \quad \text{for all} \quad s, n \in \mathbb{N}.
\]

This means that (1) and (3) are practically equivalent. Note that \( 1/p(s) \) determines the power of \( \log \varepsilon^{-1} \) in the information complexity, whereas \( \log q^{-1} \) affects only the multiplier of \( \log^{1/p(s)} \varepsilon^{-1} \). From this point of view, \( p(s) \) is more important than \( q \).
In particular, EC-WT means that we rule out the cases for which \( n(\varepsilon, s) \) depends exponentially on \( s \) and \( \log \varepsilon^{-1} \).

For instance, assume that (1) holds. Then uniform exponential convergence (UEXP) implies EC-WT if

\[
C(s) = \exp(\exp(o(s))) \quad \text{and} \quad C_1(s) = \exp(o(s)) \quad \text{as} \quad s \to \infty.
\]

These conditions are rather weak since \( C(s) \) can be almost doubly exponential and \( C_1(s) \) almost exponential in \( s \).

The definition of EC-PT (and EC-SPT) implies that we have uniform exponential convergence with \( C(s) = e \) (where \( e \) denotes \( \exp(1) \)), \( q = 1/e \), \( C_1(s) = c s^{\tau_1} \) and \( p = 1/\tau_2 \).

We briefly mention a recent paper [14], where a new notion of weak tractability is defined similarly to EC-WT. Namely, let \( \kappa \geq 1 \). Then it is required that

\[
\lim_{s+\log \varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, s)}{s + \lfloor \log \varepsilon^{-1} \rfloor^\kappa} = 0 \quad \text{with} \quad \log 0 = 0 \quad \text{by convention.} \tag{4}
\]

Obviously, for \( \kappa = 1 \) this is the same as EC-WT. However, for \( \kappa > 1 \) the condition on WT is relaxed. This is essential and leads to new results for linear unweighted tensor product problems.

In the following sections, we are going to discuss a special choice of \( H_s \) and study the problems of \( s \)-variate integration and \( L_2 \)-approximation.

### 3 A weighted Korobov space of analytic functions

In this article, we choose for the Hilbert space \( H_s \) a weighted Korobov space of periodic and smooth functions, which is probably the most popular kind of space used to analyze periodic functions. Such Korobov spaces can be defined via a reproducing kernel (for general information on reproducing kernel Hilbert spaces, see [1]) of the form

\[
K_s(x, y) = \sum_{h \in \mathbb{Z}^s} \rho_h \exp(2\pi i h \cdot (x - y)) \quad \text{for all} \quad x, y \in [0, 1]^s \tag{5}
\]

with the usual dot product

\[
h \cdot (x - y) = \sum_{j=1}^s h_j (x_j - y_j),
\]

where \( h_j, x_j, y_j \) are the \( j \)-th components of the vectors \( h, x, y \), respectively. Furthermore, \( i = \sqrt{-1} \). The nonnegative \( \rho_h \) for \( h \in \mathbb{Z}^s \), which may also depend on \( s \) and other
parameters, are chosen such that \( \sum_{h \in \mathbb{Z}^s} \rho_h < \infty \). This choice guarantees that the kernel \( K_s \) is well defined, since
\[
|K_s(x, y)| \leq K_s(x, x) = \sum_{h \in \mathbb{Z}^s} \rho_h < \infty.
\]
Obviously, the function \( K_s \) is symmetric in \( x \) and \( y \) and it is easy to show that it is also positive definite. Therefore, \( K_s(x, y) \) is indeed a reproducing kernel. The corresponding Korobov space is denoted by \( H(K_s) \).

The smoothness of the functions from \( H(K_s) \) is determined by the decay of the \( \rho_h \)'s. A very well studied case in literature is for Korobov spaces of finite smoothness \( \alpha \). Here \( \rho_h \) is of the form
\[
\rho_h = r_{\alpha, \gamma}(h),
\]
where \( \alpha > 1 \) is a real, \( \gamma = (\gamma_1, \gamma_2, \ldots) \) is a sequence of positive reals, and for \( h = (h_1, \ldots, h_s) \) we have
\[
r_{\alpha, \gamma}(h) = \prod_{j=1}^{s} r_{\alpha, \gamma_j}(h_j),
\]
with \( r_{\alpha, \gamma}(0) = 1 \) and \( r_{\alpha, \gamma}(h) = \gamma|h|^{-\alpha} \) whenever \( h \neq 0 \).

Hence the \( \rho_h \)'s decay polynomially in the components of \( h \). The parameter \( \alpha \) guarantees the existence of some partial derivatives of the functions and the so-called weights \( \gamma \) model the influence of the different components on the variation of the functions from the Korobov space. More information can be found in [11, Appendix A.1].

The idea of introducing weights stems from Sloan and Woźniakowski and was first discussed in [16]. For multivariate integration defined over weighted Korobov spaces of smoothness \( \alpha \), algorithms based on \( n \) function evaluations can obtain the best possible convergence rate of order \( \mathcal{O}(n^{-\alpha/2+\delta}) \) for any \( \delta > 0 \). Under certain conditions on the weights, weak, polynomial or even strong polynomial tractability in the sense of (b)–(d) can be achieved. We refer to [11] and the references therein and to the recent survey [3] for further details.

Besides the case of finite smoothness, Korobov spaces of infinite smoothness were also considered. In this case, the \( \rho_h \)'s decay to zero exponentially fast in \( h \). Multivariate integration and \( L_2 \)-approximation for such Korobov spaces have been analyzed in [2, 4, 8].

To model the influence of different components we use two weight sequences
\[
a = \{a_j\}_{j \geq 1} \quad \text{and} \quad b = \{b_j\}_{j \geq 1}.
\]
In order to guarantee that the kernel that we will introduce in a moment is well defined we must assume that \( a_j > 0 \) and \( b_j > 0 \). In fact, we assume a little more throughout the paper, namely that with the proper ordering of variables we have
\[
0 < a_1 \leq a_2 \leq \cdots \quad \text{and} \quad b_* = \inf b_j > 0.
\]
(6)

Let \( a_* = \inf a_j \) which is \( a_1 \) in our case.

Fix \( \omega \in (0, 1) \) and put in (5)
\[
\rho_h = \omega_h := \omega \sum_{j=1}^{s} a_j |h_j|^{b_j} \quad \text{for all} \quad h = (h_1, h_2, \ldots, h_s) \in \mathbb{Z}^s.
\]
(7)

8
For this choice of $\rho_h$ we denote the kernel in (5) by $K_{s,a,b}$. We suppress the dependence on $\omega$ in the notation since $\omega$ will be fixed throughout the paper and $a$ and $b$ will be varied. Note that $K_{s,a,b}$ is well defined since

$$\sum_{h \in \mathbb{Z}} \omega_h = \prod_{j=1}^{s} \left(1 + 2\sum_{h=1}^{\infty} \omega_{aj} h^j\right) \leq \left(1 + 2\sum_{h=1}^{\infty} \omega_{ah} h^s\right)^s < \infty.$$ 

The last series is finite by the comparison test because $a_s > 0$ and $b_s > 0$.

The Korobov space with reproducing kernel $K_{s,a,b}$ is denoted by $H(K_{s,a,b})$. Clearly, functions from $H(K_{s,a,b})$ are infinitely many times differentiable, see [4], and they are even analytic as shown in [2, Proposition 2].

For $f \in H(K_{s,a,b})$ we have

$$f(x) = \sum_{h \in \mathbb{Z}^s} \hat{f}(h) \exp(2\pi i h \cdot x)$$

for all $x \in [0,1]^s$, where $\hat{f}(h) = \int_{[0,1]^s} f(x) \exp(-2\pi i h \cdot x) \, dx$ is the $h$th Fourier coefficient of $f$. The inner product of $f$ and $g$ from $H(K_{s,a,b})$ is given by

$$\langle f, g \rangle_{H(K_{s,a,b})} = \sum_{h \in \mathbb{Z}^s} \hat{f}(h) \overline{\hat{g}(h)} \omega_h^{-1},$$

and the norm of $f$ from $H(K_{s,a,b})$ by

$$\|f\|_{H(K_{s,a,b})} = \left(\sum_{h \in \mathbb{Z}^s} |\hat{f}(h)|^2 \omega_h^{-1}\right)^{1/2} < \infty.$$ 

Define the functions

$$e_h(x) = \exp(2\pi i h \cdot x) \omega_h^{1/2}$$

for all $x \in [0,1]^s$. Then $\{e_h\}_{h \in \mathbb{Z}^s}$ is a complete orthonormal basis of the Korobov space $H(K_{s,a,b})$.

4 Integration in $H(K_{s,a,b})$

In this section we study numerical integration, i.e., we are interested in numerical approximation of the values of integrals

$$I_s(f) = \int_{[0,1]^s} f(x) \, dx$$

for all $f \in H(K_{s,a,b}).$

Using the general notation from Section 2 we now have $S_s(f) = I_s(f)$ for functions $f \in H_s = H(K_{s,a,b})$, and $G_s = \mathbb{C}$.

We approximate $I_s(f)$ by means of linear algorithms $Q_{n,s}$ of the form

$$Q_{n,s}(f) := \sum_{k=1}^{n} q_k f(x_k),$$

9
where coefficients $q_k \in \mathbb{C}$ and sample points $x_k \in [0,1)^s$. If we choose $q_k = 1/n$ for all $k = 1, 2, \ldots, n$ then we obtain so-called quasi-Monte Carlo (QMC) algorithms which are often used in practical applications especially if $s$ is large. For recent overviews of the study of QMC algorithms we refer to [3, 5, 9].

The $n$th minimal worst-case error is given by

$$e^{\text{int}}(n, s) = \inf_{q_k, x_k, k=1,2,\ldots,n} \sup_{f \in H(K_s, a, b)} \{I_s(f) - \frac{n}{\sum_{k=1}^{n} q_k f(x_k)}\}.$$

It is well known, see for instance [12, 17], that

$$e^{\text{int}}(n, s) = \inf_{x_k, k=1,2,\ldots,n} \sup_{f \in H(K_s, a, b), f(x_k)=0, k=1,2,\ldots,n} |I_s(f)|. \quad (9)$$

For $n = 0$, the best we can do is to approximate $I_s(f)$ simply by zero, and

$$e^{\text{int}}(0, s) = \|I_s\| = 1 \quad \text{for all} \quad s \in \mathbb{N}.$$ 

Hence, the integration problem is well normalized for all $s$.

We now summarize the main results regarding numerical integration in $H(K_s, a, b)$. Here and in the following, we will be using the notational abbreviations

\[
\begin{array}{cccc}
\text{EXP} & \text{UEXP} \\
\text{WT} & \text{PT} & \text{SPT} \\
\text{EC-WT} & \text{EC-PT} & \text{EC-SPT}
\end{array}
\]

to denote exponential and uniform exponential convergence, and weak, polynomial and strong polynomial tractability in terms of (b)–(d) and (e)–(g). We now state relations between these concepts as well as necessary and sufficient conditions on $a$ and $b$ for which these concepts hold. As we shall see, in the settings considered in this paper, many conditions for obtaining these concepts are equivalent.

We first state a theorem which describes conditions on the weight sequences $a$ and $b$ to obtain exponential (EXP) and uniform exponential (UEXP) convergence. This theorem is from [2, 8].

**Theorem 1** Consider integration defined over the Korobov space $H(K_s, a, b)$ with weight sequences $a$ and $b$ satisfying (6).

- EXP holds for all considered $a$ and $b$ and

  $$p^*(s) = \frac{1}{B(s)} \quad \text{with} \quad B(s) := \sum_{j=1}^{s} \frac{1}{b_j}.$$ 

- UEXP holds iff $b$ is such that

  $$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$ 

  If so then $p^* = 1/B$. 

10
Theorem 1 states that we always have exponential convergence. However, a necessary and sufficient condition for uniform exponential convergence is that the weights $b_j$ go to infinity so fast that $B := \sum_{j=1}^{\infty} b_j^{-1} < \infty$, with no extra conditions on $a_j$ and $\omega$. The largest exponent $p$ of uniform exponential convergence is $1/B$. Hence for small $B$ the exponent $p$ is large. For instance, for $b_j = j^{-2}$ we have $B = \pi^2/6$ and $p^* = 6/\pi^2 = 0.6079 \ldots$

Next, we consider standard notions of tractability, (b)–(d). They have not yet been studied for the Korobov space $H(K_{s,a,b})$ and therefore we need to prove the next theorem.

**Theorem 2** Consider integration defined over the Korobov space $H(K_{s,a,b})$ with weight sequences $a$ and $b$ satisfying (6). For simplicity, assume that $A := \lim_{j \to \infty} a_j \log j$ exists.

- **SPT holds if** $A > \frac{1}{\log \omega - 1}$. In this case the exponent $\tau^*$ of SPT satisfies
  $$\tau^* \leq \min \left( 2, \frac{2}{A \log \omega - 1} \left( 1 + \frac{1}{A \log \omega - 1} \right) \right).$$

  On the other hand, if we have SPT with exponent $\tau^*$, then $A \geq \frac{1}{\tau^* \log \omega - 1}$.

- **PT holds if there is an integer $j_0 \geq 2$ such that**
  $$\frac{a_j}{\log j} \geq \frac{1}{\log \omega - 1} \quad \text{for all} \quad j \geq j_0.$$

- **WT holds if** $\lim_{j \to \infty} a_j = \infty$.

**Proof.** It is well known that integration is no harder than $L_2$-approximation for the class $\Lambda_{std}$. For the Korobov class the initial errors of integration and approximation are 1. Therefore the corresponding notions of tractability for approximation imply the same notions of tractability for integration. From Theorem 5, presented in the next section, we thus conclude SPT, PT and WT also for integration. The second bound on the exponent $\tau^*$ of SPT also follows from Theorem 5. It remains to prove that $\tau^* \leq 2$. It is known, see, e.g., [12, Theorem 10.4], that

$$[\epsilon^{\text{int}}(n, s)]^2 \leq \frac{1}{n} \int_{[0,1]^s} K_{s,a,b}(x,x) \, dx \leq \frac{1}{n} \prod_{j=1}^{s} \left( 1 + 2 \sum_{k=1}^{\infty} \omega^{\alpha_j h^k} \right).$$

It is shown in the proof of Theorem 5 (with $\tau = 1$) that $A > 0$ implies the existence of $C \in (0, \infty)$ such that $2 \sum_{h=1}^{\infty} \omega^{\alpha_j h^k} \leq C \omega^{\alpha_j}$. Therefore

$$[\epsilon^{\text{int}}(n, s)]^2 \leq \frac{1}{n} \prod_{j=1}^{s} \left( 1 + C \sum_{j=1}^{\infty} \omega^{\alpha_j} \right) \leq \frac{1}{n} \exp \left( C \sum_{j=1}^{s} \omega^{\alpha_j} \right).$$

Note that for $j \geq 2$ we have $\omega^{\alpha_j} = j^{-a_j (\log \omega - 1) / \log j}$. Since $A \geq 1/(\log \omega - 1)$ for large $j$ we conclude that $\omega^{\alpha_j} \leq j^{-\beta}$ with $\beta \in (1, A \log \omega - 1)$. Hence $\sum_{j=1}^{\infty} \omega^{\alpha_j} < \infty$ and $\epsilon(n, s) \leq \varepsilon$.
for \( n = O(\varepsilon^{-2}) \) with the factor in the big \( O \) notation independent of \( s \). This implies SPT with the exponent at most 2.

It remains to show the necessary condition for SPT with exponent \( \tau^* \). First of all we show the estimate

\[
e^{\text{int}}(s, s) \geq \frac{\omega^{a_\tau}}{\sqrt{1 + \omega^{2a_\tau}}} \quad \text{for all} \quad s \in \mathbb{N}.
\]

Let \( h^{(0)} = (0, 0, \ldots, 0) \in \mathbb{Z}^s \). For \( j = 1, 2, \ldots, s \), let

\[
h^{(j)} = (0, 0, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^s \quad \text{with 1 on the \( j \)th place}.
\]

For \( h \in \mathbb{Z}^s \), let

\[
c_h(x) = \exp \left( 2\pi i \sum_{j=1}^{s} h_j x_j \right) \quad \text{for all} \quad x \in [0, 1]^s.
\]

For \( j = 0, 1, \ldots, s \), note that \( c_{h^{(j)}}(x) = \exp(2\pi i x_j) \) and

\[
c_{h^{(j)}}(x) \overline{c_{h^{(k)}}(x)} = c_{h^{(j)}-h^{(k)}}(x).
\]

Consider the function

\[
f(x) = \sum_{j=0}^{s} \alpha_j c_{h^{(j)}}(x) \quad \text{for all} \quad x \in [0, 1]^s
\]

for some complex numbers \( \alpha_j \).

We know that adaption does not help for the integration problem. Suppose that we sample functions at \( s \) nonadaptive points \( x_1, x_2, \ldots, x_s \in [0, 1]^s \). We choose numbers \( \alpha_j \) such that

\[
f(x_j) = 0 \quad \text{for all} \quad j = 1, 2, \ldots, s.
\]

This corresponds to \( s \) homogeneous linear equations in \( (s+1) \) unknowns. Therefore there exists a nonzero solution \( \alpha_0, \alpha_1, \ldots, \alpha_s \) which we may normalize such that

\[
\sum_{j=0}^{s} |\alpha_j|^2 = 1.
\]

Let

\[
g(x) = \overline{f(x)} f(x) = \sum_{j,k=0}^{s} \alpha_j \overline{\alpha_k} c_{h^{(j)}-h^{(k)}}(x) \quad \text{for all} \quad x \in [0, 1]^s.
\]

Clearly, \( g(x_j) = 0 \) for all \( j = 1, 2, \ldots, s \). Since \( I_s(c_{h^{(j)}-h^{(k)}}) = 0 \) for \( j \neq k \) and 1 for \( j = k \) we obtain

\[
I_s(g) = \sum_{j=0}^{s} |\alpha_j|^2 = 1.
\]

Now it follows from (9) that

\[
e^{\text{int}}(s, s) \geq I_s \left( \frac{g}{\|g\|_{H(K_s,a,b)}} \right) = \frac{1}{\|g\|_{H(K_s,a,b)}}.
\]
This is why we need to estimate the norm of $g$ from above. Note that

$$\|g\|_{H(K_{s,a,b})}^2 = \langle g, g \rangle_{H(K_{s,a,b})} = \left\langle \sum_{j_1,k_1=0}^{s} \alpha_{j_1} \alpha_{k_1} \langle C_{h^{(j_1)}_-h^{(k_1)}}, \sum_{j_2,k_2=0}^{s} \alpha_{j_2} \alpha_{k_2} \langle C_{h^{(j_2)}_-h^{(k_2)}}, H(K_{s,a,b}) \right\rangle$$

$$= \sum_{j_1,k_1,j_2,k_2=0}^{s} \alpha_{j_1} \alpha_{j_2} \alpha_{k_1} \alpha_{k_2} \langle C_{h^{(j_1)}_-h^{(k_1)}}, C_{h^{(j_2)}_-h^{(k_2)}}, H(K_{s,a,b}) \rangle.$$

For $h^{(j_1)} - h^{(k_1)} \neq h^{(j_2)} - h^{(k_2)}$ we have

$$\langle C_{h^{(j_1)}_-h^{(k_1)}}, C_{h^{(j_2)}_-h^{(k_2)}}, H(K_{s,a,b}) \rangle = 0,$$

whereas for $h^{(j_1)} - h^{(k_1)} = h^{(j_2)} - h^{(k_2)}$ we have

$$\langle C_{h^{(j_1)}_-h^{(k_1)}}, C_{h^{(j_2)}_-h^{(k_2)}}, H(K_{s,a,b}) \rangle = \omega^{-1}_{h^{(j_1)}_-h^{(k_1)}}.$$

Therefore it is enough to consider

$$h^{(j_1)} - h^{(k_1)} = h^{(j_2)} - h^{(k_2)}.$$

Suppose first that $j_1 \neq k_1$. Then $h^{(j_1)} - h^{(k_1)} = h^{(j_2)} - h^{(k_2)}$ implies that $j_2 = j_1$ and $k_2 = k_1$ and

$$\omega^{-1}_{h^{(j_1)}_-h^{(k_1)}} = \omega^{-a_{j_1} - a_{k_1}}.$$

On the other hand, if $j_1 = k_1$ then $h^{(j_1)} - h^{(k_1)} = h^{(0)}$ which implies that $j_2 = k_2$ and

$$\omega^{-1}_{h^{(j_1)}_-h^{(k_1)}} = 1.$$

Therefore

$$\|g\|_{H(K_{s,a,b})}^2 = \sum_{j_1,k_1,j_2,k_2=0}^{s} \alpha_{j_1} \alpha_{j_2} \alpha_{k_1} \alpha_{k_2} \omega_{h^{(j_1)}_-h^{(k_1)}}$$

$$= \sum_{j=0}^{s} \sum_{k=0}^{s} |\alpha_j|^2 |\alpha_k|^2 \omega_{-a_{j} - a_{k}} + \sum_{j=0}^{s} |\alpha_j|^2 \sum_{j_2=0}^{s} |\alpha_{j_2}|^2$$

$$= \sum_{j=0}^{s} |\alpha_j|^2 \omega_{-a_{j}} \left( -|\alpha_j|^2 \omega_{-a_{j}} + \sum_{k=0}^{s} |\alpha_k|^2 \omega_{-a_{k}} \right) + 1$$

$$\leq \left( \sum_{j=0}^{s} |\alpha_j|^2 \omega_{-a_{j}} \right)^2 + 1 \leq \omega^{-2a} + 1.$$
and thus (10) is shown.

Assume that we have SPT with the exponent \( \tau^* \). This means that for any positive \( \delta \) there exists a positive number \( C_{\delta} \) such that

\[
n(\varepsilon, s) \leq C_{\delta} \varepsilon^{-(\tau^* + \delta)} \quad \text{for all} \quad \varepsilon \in (0, 1), \ s \in \mathbb{N}.
\]

Let \( n = n(\varepsilon) := \lfloor C_{\delta} \varepsilon^{-(\tau^* + \delta)} \rfloor \). Then

\[
e^{\int n(\varepsilon), s} \leq \varepsilon \quad \text{for all} \quad s \in \mathbb{N}.
\]

Taking \( s = n(\varepsilon) \), we conclude from (10) that

\[
\omega a s \sqrt{1 + \omega 2 a s} \leq e^{\int (s, s)} \leq \varepsilon,
\]

which implies

\[
(1 - \varepsilon^2) \omega 2 a s \leq \varepsilon^2.
\]

Taking logarithms this means that

\[
\frac{a_s}{\log \varepsilon^{-1}} \geq \frac{1 + o(1)}{\log \omega^{-1}} \quad \text{as} \quad \varepsilon \to 0.
\]

Since \( \log \varepsilon^{-1} = (1 + o(1))(\tau^* + \delta)^{-1} \log s \) we finally have

\[
A = \lim_{s \to \infty} \frac{a_s}{\log s} \geq \frac{1}{(\tau^* + \delta) \log \omega^{-1}}.
\]

Since \( \delta \) can be arbitrarily small, the proof is completed. \( \square \)

We stress that for integration we only know sufficient conditions on \( a \) and \( b \) for the standard notions PT and WT. Obviously, it would be welcome to find also necessary conditions and verify if they match the conditions presented in the last theorem. For SPT we have a sufficient condition and a necessary condition, but there remains a (small) gap between these. Again, it would be welcome to find matching sufficient and necessary conditions for SPT. Note that it may happen that \( A = \infty \). This happens when \( a_j \)'s go to infinity faster than \( \log j \). In this case, the exponent of SPT is zero. This means that for any positive \( \delta \), no matter how small, \( n(\varepsilon, s) = \mathcal{O}(\varepsilon^{-\delta}) \) with the factor in the big \( \mathcal{O} \) notation independent of \( s \). We also stress that the conditions on all standard notions of tractability depend only on \( a \) and are independent of \( b \).

Finally, we have a result regarding the EC notions of tractability, (d)–(f). The subsequent theorem follows by combining the findings in [8] and [2, Section 9].

**Theorem 3** Consider integration defined over the Korobov space \( H(K_s, a, b) \) with weight sequences \( a \) and \( b \) satisfying (a). Then the following results hold:

- EC-PT (and, of course, EC-SPT) implies UEXP.
- We have

\[
\begin{align*}
EC-WT & \iff \lim_{j \to \infty} a_j = \infty, \\
EC-WT + UEXP & \iff B < \infty \quad \text{and} \quad \lim_{j \to \infty} a_j = \infty.
\end{align*}
\]
The following notions are equivalent:

\[ EC-PT \iff EC-PT+EXP \iff EC-PT+UEXP \iff EC-SPT \iff EC-SPT+EXP \iff EC-SPT+UEXP. \]

- \( EC-SPT+UEXP \) holds iff \( b_j^{-1} \)'s are summable and \( a_j \)'s are exponentially large in \( j \), i.e.,

\[ B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \quad \text{and} \quad \alpha^* := \liminf_{j \to \infty} \frac{\log a_j}{j} > 0. \]

Then the exponent \( \tau^* \) of \( EC-SPT \) satisfies

\[ \tau^* \in \left[ B, B + \min \left( B, \frac{\log 3}{\alpha^*} \right) \right]. \]

In particular, if \( \alpha^* = \infty \) then \( \tau^* = B \).

Theorem 3 states that \( EC-PT \) implies \( UEXP \) and hence \( B < \infty \). The notion of \( EC-PT \) is therefore stronger than the notion of uniform exponential convergence. \( EC-WT \) holds if and only if the \( a_j \)'s tend to infinity. This holds independently of the weights \( b \) and independently of the rate of convergence of \( a \) to infinity. As already shown, this implies that (2) holds. Furthermore, \( EC-WT+UEXP \) holds if additionally \( B < \infty \). Hence for \( \lim_j a_j = \infty \) and \( B = \infty \), \( EC-WT \) holds without \( UEXP \). It is a bit surprising that the notions of \( EC \)-tractability with uniform exponential convergence are equivalent. Necessary and sufficient conditions for \( EC-SPT \) with uniform exponential convergence are \( B < \infty \) and \( \alpha^* > 0 \). The last condition means that \( a_j \)'s are exponentially large in \( j \) for large \( j \).

5 \( L_2 \)-approximation in \( H(K_{s,a,b}) \)

Let us now turn to approximation in the space \( H(K_{s,a,b}) \). We study \( L_2 \)-approximation of functions from \( H(K_{s,a,b}) \). This problem is defined as an approximation of the embedding from the space \( H(K_{s,a,b}) \) to the space \( L_2([0,1]^s) \), i.e.,

\[ \text{EMB}_s : H(K_{s,a,b}) \to L_2([0,1]^s) \quad \text{given by} \quad \text{EMB}_s(f) = f. \]

In terms of the notation in Section 2, \( S_s(f) = \text{EMB}_s(f) = f \) for \( f \in H(K_{s,a,b}) \), and \( G_s = L_2([0,1]^s) \).

Without loss of generality, see again [11, 17], we approximate \( \text{EMB}_s \) by linear algorithms \( A_{n,s} \) of the form

\[ A_{n,s}(f) = \sum_{k=1}^{n} a_k L_k(f) \quad \text{for} \quad f \in H(K_{s,a,b}), \quad (11) \]

where each \( a_k \) is a function from \( L_2([0,1]^s) \) and each \( L_k \) is a continuous linear functional defined on \( H_s \) from a permissible class \( \Lambda \) of information, \( \Lambda \in \{ \Lambda^{\text{all}}, \Lambda^{\text{std}} \} \). Since \( H(K_{s,a,b}) \) is a reproducing kernel Hilbert space, function evaluations are continuous linear functionals and therefore \( \Lambda^{\text{std}} \subseteq \Lambda^{\text{all}} \).
Let $e^{L_2\text{-app},\Lambda}(n, s)$ be the $n$th minimal worst-case error,

$$e^{L_2\text{-app},\Lambda}(n, s) = \inf_{A_{n,s}} e^{L_2\text{-app}}(H(K_{s,a,b}), A_{n,s}),$$

where the infimum is taken over all linear algorithms $A_{n,s}$ of the form (11) using information from the class $\Lambda \in \{\Lambda^{all}, \Lambda^{std}\}$. For $n = 0$ we simply approximate $f$ by zero, and the initial error is

$$e^{L_2\text{-app},\Lambda}(0, s) = \|EMB_s\| = \sup_{f \in H(K_{s,a,b})} \|f\|_{L_2([0,1]^s)} = 1.$$

This means that also $L_2$-approximation is well normalized for all $s \in \mathbb{N}$.

Let us now outline the main results regarding $L_2$-approximation in $H(K_{s,a,b})$. Again, we start with results on EXP and UEXP. The following result was proved in [2].

**Theorem 4** Consider $L_2$-approximation defined over the Korobov space $H(K_{s,a,b})$ with weight sequences $a$ and $b$ satisfying (6). Then the following results hold for both classes $\Lambda^{all}$ and $\Lambda^{std}$:

- **EXP** holds for all considered $a$ and $b$ with
  $$p^*(s) = \frac{1}{B(s)} \quad \text{with} \quad B(s) := \sum_{j=1}^{s} \frac{1}{b_j}.$$

- **UEXP** holds iff $a$ is an arbitrary sequence and $b$ is such that
  $$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$

  If so then $p^* = 1/B$.

Note that the conditions are the same as for the integration problem in Theorem 1. Hence the comments following Theorem 1 also apply for approximation. Beyond that it is interesting that we have the same conditions for $\Lambda^{all}$ and $\Lambda^{std}$, although the class $\Lambda^{std}$ is much smaller than the class $\Lambda^{all}$.

We now address conditions on the weights $a$ and $b$ for the standard concepts of tractability. This has not yet been done before for $\omega_h$ of the form (7), and therefore we need to prove the next theorem.

**Theorem 5** Consider $L_2$-approximation defined over the Korobov space $H(K_{s,a,b})$ with arbitrary sequences $a$ and $b$ satisfying (6). Assume for simplicity that

$$A := \lim_{j \to \infty} \frac{a_j}{\log j}$$

exists. Then the following results hold:

For $\Lambda^{all}$ we have:
• $SPT \iff A > 0$.

In this case, the exponent of $SPT$ is

$$[\tau^{\text{all}}]^* = \frac{2}{A \log \omega - 1}.$$

• $PT \iff SPT$.

• $WT$ holds for all considered $a$ and $b$.

For $\Lambda^{\text{std}}$ we have:

• $SPT$ holds if $A > 1/(\log \omega - 1)$. In this case, the exponent $[\tau^{\text{std}}]^*$ satisfies

$$[\tau^{\text{all}}]^* \leq [\tau^{\text{std}}]^* \leq [\tau^{\text{all}}]^* + \frac{1}{2}([\tau^{\text{all}}]^*)^2 < [\tau^{\text{all}}]^* + 2.$$  

On the other hand, if we have $SPT$ with exponent $[\tau^{\text{std}}]^*$, then $A \geq \frac{1}{[\tau^{\text{std}}]^* \log \omega - 1}$.

• $PT$ holds if there is an integer $j_0 \geq 2$ such that

$$\frac{a_j}{\log j} \geq \frac{1}{\log \omega - 1} \quad \text{for all} \quad j \geq j_0.$$

• $WT$ holds if $\lim_{j \to \infty} a_j = \infty$.

Proof. Consider first the class $\Lambda^{\text{all}}$.

• From [11, Theorem 5.2] it follows that $SPT$ for $\Lambda^{\text{all}}$ is equivalent to the existence of a number $\tau > 0$ such that

$$C_{\text{SPT},\tau} := \sup_s\left( \sum_{h \in \mathbb{Z}^s} \omega_h^{\tau} \right)^{1/\tau} < \infty.$$

Note that

$$\sum_{h \in \mathbb{Z}^s} \omega_h^{\tau} = \prod_{j=1}^s \left( 1 + 2 \sum_{h=1}^{\infty} \omega^{\tau a_j h^b j} \right) = \prod_{j=1}^s \left( 1 + 2 \omega^{\tau a_j} \sum_{h=1}^{\infty} \omega^{\tau a_j (h^b - 1)} \right).$$

We have

$$1 \leq \sum_{h=1}^{\infty} \omega^{\tau a_j (h^b - 1)} \leq \sum_{h=1}^{\infty} \omega^{\tau a_j (h^b - 1)} =: A_\tau.$$

We can rewrite $A_\tau$ as

$$A_\tau = \sum_{h=1}^{\infty} h^{-x_h},$$

where $x_1 = 1$ and for $h \geq 2$ we have

$$x_h = \tau a_s (\log \omega - 1) \frac{h^b}{\log h}. $$
Since \( \lim_{h} x_h = \infty \) the last series is convergent and therefore \( A_\tau < \infty \). This proves that
\[
\sum_{h \in \mathbb{Z}^s} \omega_h^\tau = \prod_{j=1}^{s} (1 + 2A(\tau) \omega^{\tau a_j}) \quad \text{with} \quad A(\tau) \in [1, A_\tau].
\]
This implies that
\[
\sup_s \left( \sum_{h \in \mathbb{Z}^s} \omega_h^\tau \right)^{1/\tau} = \prod_{j=1}^{\infty} \left(1 + 2A(\tau) \omega^{\tau a_j}\right)^{1/\tau} < \infty \iff \sum_{j=1}^{\infty} \omega^{\tau a_j} < \infty.
\]
We now show that
\[
\sum_{j=1}^{\infty} \omega^{\tau a_j} < \infty \quad \text{for some} \quad \tau \quad \text{iff} \quad A > 0.
\]
Indeed, for \( j \geq 2 \) we can write \( \omega^{\tau a_j} = j^{-y_j} \) with
\[
y_j = \tau \log \frac{a_j}{\log j}.
\]
If \( A > 0 \) then for an arbitrary positive \( \delta \) we can choose \( \tau \) such that \( y_j \geq 1 + \delta \) for sufficiently large \( j \) and therefore the series
\[
\sum_{j=1}^{\infty} \omega^{\tau a_j} = \omega^{\tau a_1} + \sum_{j=2}^{\infty} j^{-y_j}
\]
is convergent.

If \( A = 0 \) then independently of \( \tau \) the series \( \sum_{j=1}^{\infty} \omega^{\tau a_j} \) is divergent. Indeed, then \( \lim_j y_j = 0 \) and for an arbitrary positive \( \delta \leq 1 \) and \( \tau \) we can choose \( j(\delta, \tau) \) such that \( y_j \in (0, \delta) \) for all \( j \geq j(\delta, \tau) \) and
\[
\sum_{j=1}^{\infty} \omega^{\tau a_j} \geq \sum_{j=j(\delta, \tau)}^{\infty} j^{-\delta} = \infty,
\]
as claimed. This proves that SPT holds iff \( A > 0 \).

Furthermore, [11, Theorem 5.2] states that the exponent of SPT is \( 2\tau^* \), where \( \tau^* \) is the infimum of \( \tau \) for which \( C_{\text{SPT}, \tau} < \infty \). In our case, it is clear that we must have \( \tau \geq (1 + \delta)/(A - \delta \log \omega^{-1}) \) for arbitrary \( \delta \in (0, A) \). This completes the proof of this point.

To show that PT is equivalent to SPT, it is obviously enough to show that PT implies SPT. According to [11, Theorem 5.2], PT for \( \Lambda \) is equivalent to the existence of numbers \( \tau > 0 \) and \( q \geq 0 \) such that
\[
C_{\text{PT}} := \sup_s \left( \sum_{h \in \mathbb{Z}^s} \omega_h^\tau \right)^{1/\tau} s^{-q} < \infty.
\]
This means that
\[
\log \sum_{h \in \mathbb{Z}^s} \omega_h^\tau \leq \tau (\log C_{\text{PT}} + q \log s). \quad (12)
\]
We now turn to the class $\Lambda^{\text{std}}$ some $C > 1$. In fact, we also have quasi-polynomial tractability, i.e., $\text{WT}$ due to \cite[Theorem 5.5]{11}. To obtain $\text{PT}$ we use \cite[Theorem 26.13]{13} which states that polynomial tractabilities

$$
\sum_{h \in \mathbb{Z}^s} \omega_h^T = \log \prod_{j=1}^s \left(1 + 2A(\tau)\omega^{\tau_{aj}}\right) = \sum_{j=1}^s \log \left(1 + 2A(\tau)\omega^{\tau_{aj}}\right).
$$

Assume that $A = 0$. Suppose first that $a_j$’s are uniformly bounded. Then $\sum_{h \in \mathbb{Z}^s} \omega_h^T$ is of order $s$ which contradicts the inequality \cite{12}. Assume now that $\lim_{j} a_j = \infty$. Then $\log \sum_{h \in \mathbb{Z}^s} \omega_h^T$ is of order $\sum_{j=2}^s \omega^{\tau_{aj}} = \sum_{j=2}^s j^{-y_j}$. Since $\lim_{j} y_j = 0$ we have for $\delta \in (0,1)$, as before, $j^{-y_j} \geq j^{-\delta}$ for large $j$. This proves that $\sum_{j=2}^s j^{-\delta} \approx \int_2^s x^{-\delta} \, dx$ is of order $s^{1-\delta}$ which again contradicts the inequality \cite{12}. Hence, $A > 0$ and we have SPT.

- We now show $\text{WT}$ for all $a$ and $b$ with $a_s, b_s > 0$. We have

$$
\omega_h = \omega_s \sum_{j=1}^s a_j |h_j|^{b_j} \leq \omega_s, \quad \omega_h := \omega^a \sum_{j=1}^s |h_j|^{b_s}.
$$

Note that for $h = 0$ we have $\omega_h = \omega_s, h = 1$. This shows that the approximation problem with $\omega_h$ is not harder than the approximation problem with $\omega_s, h$. The latter problem is a linear tensor product problem with the univariate eigenvalues of $W_1 = \text{EMB}_1^* \text{EMB}_1 : H(K_{1,a,b}) \to H(K_{1,a,b})$ given by

$$
\lambda_1 = 1, \quad \lambda_{2j} = \lambda_{2j+1} = \omega_s, j \geq 1.
$$

Clearly, $\lambda_2 < \lambda_1$ and $\lambda_j$ goes to zero faster than polynomially with $j$. This implies $\text{WT}$ due to \cite[Theorem 5.5]{11}.

We now turn to the class $\Lambda^{\text{std}}$.

- For $A > 1/(\log \omega^{-1})$ we have $[\tau^{\text{all}}] > 2$. From \cite[Theorem 26.20]{13} we get SPT for $\Lambda^{\text{std}}$ as well as the bounds on $[\tau^{\text{std}}]$. The necessary condition for SPT with exponent $[\tau^{\text{std}}]$ follows from Theorem \cite{2}.

- To obtain $\text{PT}$ we use \cite[Theorem 26.13]{13} which states that polynomial tractabilities for $\Lambda^{\text{std}}$ and $\Lambda^{\text{all}}$ are equivalent if trace($W_s$) = $O(s^q)$ for some $q \geq 0$, where trace($W_s$) is the sum of the eigenvalues of the operator

$$
W_s = \text{EMB}_s^* \text{EMB}_s : H(K_{s,a,b}) \to H(K_{s,a,b}).
$$

In our case, $W_s$ is given by

$$
W_s f = \sum_{h \in \mathbb{Z}^s} \omega_h \langle f, e_h \rangle_{H(K_{s,a,b})} e_h
$$

with $e_h$ given by \cite{8}. The eigenpairs of $W_s$ are $(\omega_h, e_h)$ since

$$
W_s e_h = \omega_h e_h = \omega \sum_{j=1}^s a_j |h_j|^{b_j} e_h \quad \text{for all} \quad h \in \mathbb{Z}^s
$$

\footnote{In fact, we also have quasi-polynomial tractability, i.e., $n(\varepsilon, s) \leq C \exp((t + \log \varepsilon^{-1})(1 + \log s))$ for some $C > 0$ and $t \approx 2/(a_s \log \omega^{-1})$, see \cite{7}.}
and hence
\[
\text{trace}(W_s) = \prod_{j=1}^{s} (1 + 2A(1)\omega^{a_j}) \leq \exp \left( 2A(1) \sum_{j=1}^{s} \omega^{a_j} \right).
\]

Due to the assumption \(a_j / \log j \geq 1/(\log \omega^{-1})\) for \(j \geq j_0\) we have \(\omega^{a_j} \leq j^{-1}\) for \(j \geq j_0\). Therefore there is a positive \(C\) such that
\[
\text{trace}(W_s) \leq C \exp \left( A(1) \sum_{j=j_0}^{s} j^{-1} \right) \leq C s^{A(1)}.
\]

This proves that PT for \(\Lambda_{\text{std}}\) holds iff PT for \(\Lambda_{\text{all}}\) holds. As we already proved, the latter holds iff \(A > 0\). The assumption on \(a_j\) implies that \(A \geq 1/(\log \omega^{-1}) > 0\).

• To obtain WT we use \([13, \text{Theorem 26.11}]\). This theorem states that weak tractabilities for classes \(\Lambda_{\text{std}}\) and \(\Lambda_{\text{all}}\) are equivalent if \(\log \text{trace}(W_s) = o(s)\). The proof of Theorem 4 in \([2]\) yields that \(\lim_j a_j = \infty\) implies \(\sum_{j=1}^{s} \omega^{a_j} = o(s)\). Hence,
\[
\log \text{trace}(W_s) \leq \log (\exp (2A(1) o(s))) = o(s),
\]
as needed.

We briefly comment on Theorem 5. For the class \(\Lambda_{\text{all}}\) we know necessary and sufficient conditions on SPT, PT and WT if the limit of \(a_j / \log j\) exists. It is interesting to study the case when the last limit does not exist. It is easy to check that \(A_{\inf} := \liminf_j a_j / \log j > 0\) implies SPT but it is not clear whether SPT implies \(A_{\inf} > 0\).

For the class \(\Lambda_{\text{std}}\) we only know sufficient conditions for PT and WT. It would be of interest to verify if these conditions are also necessary. For SPT, as for multivariate integration, there remains a (small) gap between sufficient and necessary conditions. Again it would be desirable to close this gap.

Finally, we have results regarding the EC-notions of tractability, (e)–(g). The subsequent theorem has been shown in \([2]\).

**Theorem 6** Consider \(L_2\)-approximation defined over the Korobov space \(H(K_{s,a,b})\) with arbitrary sequences \(a\) and \(b\) satisfying (6). Then the following results hold for both classes \(\Lambda_{\text{all}}\) and \(\Lambda_{\text{std}}\):

• **EC-PT** (and, of course, **EC-SPT**) tractability implies uniform exponential convergence, \(\text{EC-PT} \Rightarrow \text{UEXP}\).

• We have
\[
\begin{align*}
\text{EC-WT} & \iff \lim_{j \to \infty} a_j = \infty, \\
\text{EC-WT+UEXP} & \iff B < \infty \text{ and } \lim_{j \to \infty} a_j = \infty.
\end{align*}
\]
The following notions are equivalent:

\[ \text{EC-PT} \iff \text{EC-PT+EXP} \iff \text{EC-PT+UEXP} \]
\[ \iff \text{EC-SPT} \iff \text{EC-SPT+EXP} \iff \text{EC-SPT+UEXP}. \]

• EC-SPT+UEXP holds iff \( b_j^{-1} \)’s are summable and \( a_j \)’s are exponentially large in \( j \), i.e.,

\[ \text{EC-SPT+UEXP} \iff B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \quad \text{and} \quad \alpha^* := \liminf_{j \to \infty} \frac{\log a_j}{j} > 0. \]

Then the exponent \( \tau^* \) of EC-SPT satisfies

\[ \tau^* \in \left[ B, B + \min \left(B, \log \frac{3}{\alpha^*} \right) \right]. \]

In particular, if \( \alpha^* = \infty \) then \( \tau^* = B \).

Again, the conditions are the same as for the integration problem in Theorem 3 and we have the same conditions for \( \Lambda^\text{all} \) and \( \Lambda^\text{std} \). The comments following Theorem 3 apply also for approximation. We remark that the results are constructive. The corresponding algorithms for the class \( \Lambda^\text{all} \) and \( \Lambda^\text{std} \) can be found in [2].

We want to stress that for the class \( \Lambda^\text{std} \) we obtain the results of Theorem 6 by computing function values at grid points with varying mesh-sizes for successive variables. Such grids are also successfully used for multivariate integration in [2, 8]. This relatively simple design of sample points should be compared with the design of (almost) optimal sample points for analogue problems defined over spaces of finite smoothness. In this case, the design is much harder and requires the use of deep theory of digital nets and low discrepancy points, see [5, 10].

It is worth adding that if we use the definition (4) of WT with \( \kappa > 1/b_* \) then it is proved in [14] that WT holds even for \( a_j = a_1 > 0 \) and \( b_j = b_* > 0 \). Hence, the condition \( \lim_j a_j = \infty \) which is necessary and sufficient for EC-WT is now not needed.

## 6 Conclusion and Outlook

The study of tractability with exponential convergence is a new research subject. We presented a handful of results only for multivariate integration and approximation problems defined over Korobov spaces of analytic functions. Obviously, such a study should be performed for more general multivariate problems defined over more general spaces of \( C^\infty \) or analytic functions. It would be very much desirable to characterize multivariate problems for which various notions of tractability with exponential convergence hold. In this survey we presented the notions of EC-WT, EC-PT and EC-SPT. We believe that other notions of tractability with exponential convergence should be also studied. In fact, all notions which were presented for tractability with respect to the pairs \((\varepsilon^{-1}, s)\) can be easily generalized and studied for the pairs \((1 + \log \varepsilon^{-1}, s)\). In particular, the notions of EC-QPT (exponential convergence-quasi polynomial tractability) and EC-UWT (exponential convergence-uniform weak tractability) are probably the first candidates for such
a study. Quasi-polynomial tractability was briefly mentioned in the footnote of Section 5.

Uniform weak tractability generalizes the notion of weak tractability and means that $n(\varepsilon, s)$ is not exponential in $\varepsilon^{-\alpha}$ and $s^\beta$ for all positive $\alpha$ and $\beta$, see [15].

The proof technique used for EC-tractability of integration and approximation is quite different than the proof technique used for standard tractability. Furthermore, it seems that some results are easier to prove for EC-tractability than their counterparts for the standard tractability. In particular, optimal design of sample points seems to be such an example. We are not sure if this holds for other multivariate problems.

We hope that exponential convergence and tractability will be an active research field in the future.

References

[1] N. Aronszajn, Theory of reproducing kernels. Trans. Amer. Math. Soc. 68, 337–404, 1950

[2] J. Dick, P. Kritzer, F. Pillichshammer, H. Woźniakowski. Approximation of analytic functions in Korobov spaces. To appear in J. Complexity, 2014.

[3] J. Dick, F.Y. Kuo, I.H. Sloan. High dimensional integration—the quasi-Monte Carlo way. Acta Numer. 22, 133–288, 2013.

[4] J. Dick, G. Larcher, F. Pillichshammer, H. Woźniakowski. Exponential convergence and tractability of multivariate integration for Korobov spaces. Math. Comp. 80, 905–930, 2011.

[5] J. Dick, F. Pillichshammer. Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.

[6] M. Gnewuch, H. Woźniakowski, Generalized tractability for multivariate problems, Part II: Linear tensor product problems, linear information, unrestricted tractability. Found. Comput. Math. 9, 431–460, 2009.

[7] M. Gnewuch, H. Woźniakowski. Quasi-polynomial tractability. J. Complexity 27, 312–330, 2011.

[8] P. Kritzer, F. Pillichshammer, H. Woźniakowski. Multivariate integration of infinitely many times differentiable functions in weighted Korobov spaces. To appear in Math. Comp., 2014.

[9] F.Y. Kuo, Ch. Schwab, I.H. Sloan. Quasi-Monte Carlo methods for high dimensional integration: the standard (weighted Hilbert space) setting and beyond. ANZIAM J. 53, 1–37, 2011.

[10] H. Niederreiter. Random Number Generation and Quasi-Monte Carlo Methods. SIAM, Philadelphia, 1992.

[11] E. Novak and H. Woźniakowski. Tractability of Multivariate Problems, Volume I: Linear Information. EMS, Zürich, 2008.
[12] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume II: Standard Information for Functionals*. EMS, Zürich, 2010.

[13] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume III: Standard Information for Operators*. EMS, Zürich, 2012.

[14] A. Papageorgiou and I. Petras, A new criterion for tractability of multivariate problems. Submitted, 2013.

[15] P. Siedlecki. Uniform weak tractability. J. Complexity 29, 438–453, 2013.

[16] I.H. Sloan, H. Woźniakowski. When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? J. Complexity 14, 1–33, 1998.

[17] J.F. Traub, G.W. Wasilkowski, and H. Woźniakowski. *Information-Based Complexity*. Academic Press, New York, 1988.