Anisotropic gaseous models of tidally limited star clusters –
comparison with other methods

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ABSTRACT

We present new models of the evolution and dissolution of star clusters evolving under the combined influence of internal relaxation and external tidal fields, using the anisotropic gaseous model based on the Fokker-Planck approximation, and a new escaper loss cone model. This model borrows ideas from loss cones of stellar distributions near massive black holes, and describes physical processes related to escaping stars by a simple model based on two timescales and a diffusion process. We compare our results with those of direct \(N\)-body models and of direct numerical solutions of the orbit-averaged Fokker-Planck equation. For this comparative study we limit ourselves to idealized single point mass star clusters, in order to present a detailed study of the physical processes determining the rate of mass loss, core collapse and other features of the system’s evolution. With the positive results of our study the path is now open in the future to use the computationally efficient gaseous models for future studies with more realism (mass spectrum, stellar evolution).

Key words: gravitation – methods: numerical – celestial mechanics, stellar dynamics – globular clusters: general

1 INTRODUCTION

Dynamical modelling of globular clusters and other collisional stellar systems (like galactic nuclei, rich open clusters, and rich galaxy clusters) still poses a considerable challenge for both theory and computational requirements (in hardware and software). On the theoretical side the validity of certain assumptions used in statistical modelling based on the Fokker-Planck (henceforth FP) and other approximations is poorly known. Stochastic noise in a discrete \(N\)-body system and the impossibility to directly model realistic particle numbers with the presently available hardware, are a considerable challenge for the computational side.

Detailed comparisons of the results obtained with the different methods for single mass isolated star clusters have been performed (Giersz & Heggie 1994a,b, Giersz & Spurzem 1994, Spurzem & Aarseth 1996, Spurzem 1996). They include theoretical models such as the direct numerical solution of the orbit-averaged 1D FP equation for isotropic systems (Cohn 1980), isotropic (Heggie 1984) and anisotropic gaseous models (henceforth AGM) (Louis & Spurzem 1991, Spurzem 1994) and direct \(N\)-body simulations using standard \(N\)-body codes (\textsc{Nbody}5, Aarseth 1985, Spurzem & Aarseth 1996; \textsc{Nbody}2, Makino & Aarseth 1992; \textsc{Nbody}4, Makino 1996; \textsc{Nbody}6++, Spurzem 1999, Aarseth 1999a,b, 2003). All the cited work, however, only dealt with idealized single-mass models. There are very few attempts yet to extend the quantitative comparisons to more realistic star clusters containing different mass bins or even a continuous mass spectrum (Spurzem & Takahashi 1995, Giersz & Heggie 1996, Gürkan, Freitag & Rasio 2004).

On the side of the FP models there have been two major recent developments. Takahashi (1995, 1996, 1997) has published new FP models for spherically symmetric star clusters, based on the numerical solution of the orbit-averaged 2D FP equation (solving the FP equation for the distribution \(f = f(E, J^2)\) as a function of energy and angular momentum, on an \((E, J^2)\)-mesh). Drukier et al. (1999) have published results from another 2D FP code based on the original Cohn (1979) code. In such 2D FP models anisotropy, i.e. the possible difference between radial and tangential velocity dispersions in spherical clusters is taken into account. Although the late, self-similar stages of core collapse are not affected very much by anisotropy (Louis & Spurzem 1991), intermediate and outer zones of globular clusters, say outside roughly the Lagrangian radius containing 30 % of the total mass, do exhibit fair amounts of anisotropy, in theoretical model simulations as well as according to parameterized model fits (Lupton, Gunn & Griffin 1987). In contrast to AGMs the 2D FP models contain less inherent model approximations; they do not assume a certain form of the heat conductivity and closure relations between the third order moments as in the case of AGM. Furthermore, the latter contains a numerical
constant $\lambda$ (Spurzem 1996), which is of order unity, but its numerical value has to be determined from comparisons with proper FP or $N$-body models.

Secondly, another 2D FP model has been worked out recently for the case of axisymmetric rotating star clusters (Einsel & Spurzem 1999, Kim et al. 2002, Kim, Lee & Spurzem 2004, Fiestas, Spurzem & Kim 2005). Here, the distribution function is assumed to be a function of energy $E$ and the $z$-component of angular momentum $J_z$ only; a possible dependence of the distribution function on a third integral is neglected. As in the spherically symmetric case the neglecting of an integral of motion is equivalent to the assumption of isotropy, here between the velocity dispersions in the meridional plane ($r$ and $z$ directions); anisotropy between that velocity dispersion and that in the equatorial plane ($\phi$-direction), however, is included.

Thirdly, there is an elegant alternative way to generate models of star clusters, which can correctly reproduce the stochastic features of real star clusters, but without really integrating all orbits directly as in an $N$-body simulation. They rely on the FP approximation (and (hitherto) spherical symmetry, but their data structure is very similar to an $N$-body model. These so-called Monte Carlo models were recently redeveloped by Giersz (1996, 1998, 2001), and by Rasio and collaborators (Joshi, Rasio & Portegies Zwart 2000, Watters, Joshi & Rasio 2000, Joshi, Nave & Rasio 2001, Fregeau et al. 2003, Gürkan, Freitag & Rasio 2004). For another approach, reviving Hénon’s superstar method, compare the work by Freitag (Freitag 2000, Freitag & Benz 2001, 2002). The basic idea is to have pseudo-particles, whose orbital parameters are given in a smooth, self-consistent potential. However, their orbital motion is not explicitly followed; to model interactions with other particles like two-body relaxation by distant encounters or strong interactions between binaries and field stars, a position of the particle in its orbit and further free parameters of the individual encounter are picked from an appropriate distribution by using random numbers. A hybrid variant of the Monte Carlo technique combined with a gaseous model has been proposed by Spurzem & Giersz (1996), and applied to systems with a large number of primordial binaries by Giersz & Spurzem (2000) and Giersz & Spurzem (2003). The hybrid method uses a Monte Carlo model for binaries or any other object for which a statistical description, as used by the gaseous model, is not appropriate, due to small numbers of objects or unknown analytic cross sections for interaction processes. The method is particularly useful for investigating evolution of large stellar systems with realistic fraction of primordial binaries, but could also be used in future to include the build-up of massive stars and blue stragglers by stellar collisions, for example.

In the present and near future a wealth of detailed data on globular clusters will become available by e.g. the Hubble Space Telescope and the new 8m-class terrestrial telescopes such as Gemini and the Very Large Telescope (VLT), for extragalactic as well as Milky Way clusters. These data cover luminosity functions and derived mass functions, color-magnitude diagrams, population and kinematical analysis, including binaries and compact stellar evolution remnants, detailed two-dimensional proper motion and radial velocity data, and tidal tails spanning over arcs several degrees wide (Koch et al. 2004). With detailed observational data such as from King et al. (1998), Piotto & Zoccali (1999), Rubenstain & Bailyn (1999), Ibata et al. (1999), Piotto et al. (1999), Grillmair et al. (1999), Shara et al. (1998), Odenkirchen et al. (2001), Hansen et al. (2002), Richer et al. (2002) to mention only the few recently appeared papers), an easily reproducible reliable modelling becomes more important than before. For that purpose a few more input data are urgently required in the models in addition to anisotropy and rotation: a mass spectrum, a tidal field, and the influence of stellar evolution on the dynamical evolution of the cluster.

While it is easy in principle to include all these in a direct $N$-body simulation, and considerable effort goes in the construction of new hardware and software for that purpose (Hut & Makino 1999, Makino et al. 1997, Makino & Taiji 1998, Aarseth 1999a,b, 2003), the life span of globular clusters extends over tens to hundreds of thousands of crossing times, taken at the half-mass radius, which requires a high accuracy direct $N$-body code for its modelling. Despite the enormous advances in hardware and software efficiency, still modelling, say, of a few hundred thousand particles is very tedious. For parameter studies one has to rely on lower particle numbers and prescriptions to scale the results to larger $N$ (Aarseth & Heggie 1998, Baumgardt 2001). So we still need the fast but approximate theoretical models. Takahashi, Lee & Inagaki (1997) published first results for anisotropic clusters in a tidal field, and Takahashi & Lee (2000) extended their study to multimass clusters. Takahashi & Portegies Zwart (1998, 2000), Portegies Zwart & Takahashi (1999) compared the influence of stellar evolutionary mass loss as measured in $N$-body and FP models. However, still the tidal boundary poses an unsolved problem in the pure point mass case. Another difficulty is that $N$-body models and theory are difficult to match, since an energy cutoff as it is usual in the FP models has not yet been applied in $N$-body simulations (with the exception of one of the models kindly provided by E. Kim for our comparisons, see below); usually direct $N$-body models employ a tidal radius cutoff.

Moreover, in the theoretical models one has to decide which criteria to use for a star to escape. In contrast to isolated clusters energy and angular momentum are not exact integrals of motion in a tidal field, and one should consider more appropriate quantities like the Jacobian. However, even then it is possible that a star satisfying an escape criterion stays for many orbital times close to the cluster and even can be scattered back to the cluster (Baumgardt 2001), so from an observers viewpoint they do not escape immediately; indeed Fukushige & Heggie (2000) propose a new form of an energy dependent escape time scale, which is formally infinite for stars at the tidal energy (note a similar problem in relation to dwarf spheroidal galaxies in tidal fields raised by Kroupa 1998 and Klessen & Kroupa 1998).

In this situation we propose to look at simple cases and simplified models in more detail first. For example it appears to be useful to separate the dynamical effects of stellar evolution from those induced by the interplay of internal relaxation and a tidal field, and to look at a single mass case first. A gaseous model of a star cluster (here used an anisotropic gaseous model or AGM, see http://www.ari.uni-heidelberg.de/gaseous-model) is a very simple tool with which to understand, on the basis of an idealized model, the physical processes acting on star clusters. It has been very successfully used to detect gravothermal oscillations (Bettwieser & Sugimoto 1984) and to discuss effects present in multimass star clusters (Spurzem & Takahashi 1995) and in systems with primordial binaries (Heggie & Aarseth 1992).

Thus in this paper we present a new approach to include a tidal field in AGM of star clusters, which has not been tackled before. The processes of escape and relaxation are treated in a simplified parameterized way and compared to models using an orbit averaged Fokker-Planck equation and direct $N$-body simulations. Also the different parameters of our escaped model are varied to understand their influence on the results. We will demonstrate that the gaseous model helps in the physical understanding of the tidal
2 THE MODELS

The anisotropic gaseous model (AGM) is based on local moments of the stellar velocity distribution function. From the local Fokker-Planck equation (Rosenbluth, McDonald & Judd 1957) moment equations up to second order are taken and closed in third order by a phenomenological heat flux equation (Lynden-Bell & Eggleton 1980, Bettwieser 1983, Heggie 1984, Louis & Spurzem 1991). Diffusion coefficients are determined self-consistently, including the anisotropy of the background distribution of scatterers, but locally without any orbit average (Spurzem & Takahashi 1995). Two free parameters ($\lambda$ scaling the heat flux, and $\lambda_1$ scaling the collisional decay of anisotropy) are determined by quantitative comparison with orbit-averaged Fokker-Planck and direct N-body models, a standard binary heating term due to three-body binary energy equipartition and dynamical friction, as was shown in Spurzem & Takahashi (1995). We present here an enhancement to the AGM method to handle an external tidal field, and also use data of the 2D FP method worked out by Takahashi (1995, 1996, 1997) and N-body data kindly provided by S. Deiters and D.C. Heggie and E. Kim (2003) for comparison purposes.

The equations of AGM are solved numerically by an implicit Henyey method as described in Giersz & Spurzem (1994) or Spurzem (1996). Collisional terms were evaluated locally with self-consistent anisotropic test and background star distributions (Giersz & Spurzem 1994, Spurzem & Takahashi 1995). The energy generation due to binaries is the same as in the FP models, though in the gaseous models it is applied locally, not in an orbit-averaged way.

To describe the process of stars escaping from a cluster in a tidal field we adopt simple approximations which are outlined in the following. In the spherically symmetric AGM full phase space information is reduced to the knowledge of moments of the velocity distribution up to third order (density, bulk radial mass motion, radial and tangential velocity dispersion, radial fluxes of radial and tangential energy, see e.g. Spurzem 1994 or Giersz & Spurzem 1994 for details of the models). Since the mass and energy fluxes drive the quasistatic evolution of the system under relaxation timescales, to first order the most significant moments shaping the velocity distribution function are density and the radial and tangential velocity dispersions.

In the standard tidal cutoff picture a star is considered to be an escapee if its integrals of motion (that of an isolated cluster), energy and angular momentum, fulfill certain criteria related to the tidal radius $r_t$, or tidal cutoff energy $E_t$ of the cluster (Takahashi, Lee & Inagaki 1997). We approximate the real distribution function of the tidally limited cluster in the gaseous model as follows: a tidally unlimited local anisotropic Schwarzschild Boltzmann distribution

$$f(r, v_r, v_\theta, v_\phi) = \frac{\rho}{(2\pi)^{3/2}\sigma_r \sigma_\theta \sigma_\phi} \exp\left(-\frac{(v_r - u)^2}{2\sigma_r^2} - \frac{(v_\theta + v_\phi)^2}{2\sigma_\phi^2}\right)$$

(1)

(where $\rho$ is the mass density, $v_r$, $v_\theta$, $v_\phi$ are the individual stellar velocities in a local Cartesian coordinate system, whose axes are tangential to the radial, polar and azimuthal directions on a sphere, $u$ is the bulk mass transport velocity, $\sigma_r$, $\sigma_\theta$, $\sigma_\phi$ are the radial and tangential velocity dispersion, respectively) is used to compute the fraction of stars $X_e$, which would be in the escaper region if a tidal limit would be imposed on such a cluster by

$$X_e = \int e^{-\phi(t, \Phi)} d\phi \int f d^3v$$

(2)

Here, the index $e$ at the integral denotes an integration over escape velocity space; for example in case of an energy criterion we have

$$v_r^2 + v_\theta^2 + v_\phi^2 > v_{esc}^2 \equiv 2\Phi(t, \Phi(r))$$

(3)

as escape condition with the potential $\Phi$, given at the tidal radius ($r_t = (M/3M_G)^{1/3}R_G$, where $M$ is the total cluster mass, $M_G$ is the parent galaxy mass and $R_G$ is the distance between the galaxy and cluster centers), $v_{esc}$ the escape velocity and $\Phi(r)$ is the potential at the distance $r$. For an apocentre criterion, taking into account energy and angular momentum conservation (at distance $r$ and apocentre distance set to $r_t$), it is straightforward to obtain from Eq. 4

$$\frac{v_r^2}{v_{esc, r}^2} + \frac{(v_\theta^2 + v_\phi^2)}{v_{esc, t}^2} > 1$$

(4)

as a condition for escape, where we have used the convenient definitions $v_{esc, r} \equiv v_{esc}$ and $v_{esc, t} \equiv v_{esc} \cdot r_t^3/(r_t^2 - r^2)$. The integral in Eq. 2 is a 3D-integral over an unbounded domain with an ellipsoidal boundary. In order to solve it we first integrate for the non-escaping fraction of stars $X_{ne}$ (which are inside the ellipsoidal boundary), which is a bound integration domain, and due to the normalization of the velocity distribution function we can determine $X_e = 1 - X_{ne}$. Fortunately, the integral for $X_{ne}$ can be solved analytically by using error functions and some similar kind of integrals, including Dawson’s integral, for which a routine can be obtained from Numerical Recipes (Press et al. 1986). In the Appendix some more detail of this derivation is given. Here, we show the result, using abbreviations:

$$a \equiv \frac{v_{esc, r}}{\sqrt{2}\sigma_r}, b \equiv \frac{v_{esc, t}}{\sqrt{2}\sigma_t}$$

(5)

and getting the end result

$$X_e = 1 - \exp\left(\frac{G}{a}ight) = \frac{\exp(-b^2G)}{a} \begin{cases} \frac{\exp(-b^2H)}{H} & a > b \\ \frac{\exp(-b^2)}{H} & a < b \end{cases}$$

(6)

with the definitions of $G^2 \equiv a^2 - b^2$ and $H^2 \equiv b^2 - a^2$. The special function $I(x)$ is related to Dawson’s integral and defined in the Appendix. Similarly to $X_e$ we also compute the fraction of energy of the stellar system (radial and tangential) belonging to the escaper space by

$$X_r = \int e^{-\phi(r, \Phi)} f d^3v \int f d^3v$$

(7)

$$X_t = \int (v_\theta^2 + v_\phi^2) f d^3v \int (v_\theta^2 + v_\phi^2) f d^3v$$

(8)
If \( X_e = X_t = X_c \) all escaping stars would have the same average specific energy as the non-escaping ones. Generally this should not be the case, since escaping stars would tend to have higher specific energies, in such case the difference between \( X_e, X_t \) and \( X_c \) tells us something about the specific energy of the escapers as compared to the non-escaping stars. Again the values are determined by first integrating over the bounded domain of the non-escaping stars and getting the complement due to the normalization. Our results from the Appendix are:

\[
X_e = 1 - 2 \text{Erf}(a) + \begin{cases} 
2a^4 \exp(-b^2) \frac{\text{Erf}(G)}{G^3} a > b \\
2a^4 \exp(-b^2) \frac{J(H)}{H^3} a < b 
\end{cases} (9)
\]

\[
X_t = 1 - \text{erf}(a) + \begin{cases} 
\frac{a(1 + b^2) \exp(-b^2) \text{Erf}(G)}{G^3} a > b \\
\frac{a(1 + b^2) \exp(-b^2) \frac{J(G)}{G}}{H^3} a < b 
\end{cases} (10)
\]

where \( \text{erf}, \text{Erf}, I, \) and \( J \) are the error function and specially defined generalizations of it (see Appendix).

In a realistic case, however, the distribution function will be different from a Schwarzschild-Boltzmann function and the escape fraction of velocity space will be populated by a few stars only, which are on their way out to leave the cluster. Therefore, we assume that the density of stars \( \rho_e \) prone to escape in reality is smaller than \( X_e, \rho \) by a factor \( k < 1 \), which will be referred to as filling factor. Hence we have \( \rho_e \equiv k X_e \rho \), and for the radial and tangential energies \( p_r e \equiv k X_r p_r \) and \( p_{\theta e} \equiv k X_{\theta} p_{\theta} \), where \( p_r = \rho \sigma r^2 \) and \( p_{\theta} = \rho \sigma \theta^2 \). Such procedure can be seen in close connection to the ansatz of King’s models (King 1966), which just use a lowered Maxwellian to model the distribution function of a tidally limited cluster. Then our ansatz for mass and energy loss of the cluster is

\[
\frac{\Delta \rho}{\Delta t} = -\frac{\rho_e}{\alpha_{\text{cross}}} (11)
\]

\[
\frac{\Delta p_r}{\Delta t} = -\frac{\rho_e}{\alpha_{\text{cross}}} (12)
\]

\[
\frac{\Delta p_{\theta}}{\Delta t} = -\frac{\rho_e}{\alpha_{\text{cross}}} (13)
\]

where \( t_{\text{cross}} \) denotes a crossing time to reach the tidal radius with the radial escape velocity, and \( \alpha \) is a free parameter with which one can describe the unknown process of removal of escaping stars from the cluster.

To complete our model the time evolution of the filling factor \( k \) has to be described. We are doing this in a close analogy to the King’s models (King 1966) and Amaro-Seoane, Freitag, & Spurzem (2004) for stars to be swallowed by a central black hole. The process which brings stars into the escaper region is two-body relaxation, so to the first order we think that the timescale \( t_{\text{in}} \) to refill the “loss cone” (which is here the escaper region of velocity space) is assumed to be

\[
t_{\text{in}} \equiv \beta t_{\text{rx}},
\]

where \( t_{\text{rx}} \) is the local relaxation time. We keep a free parameter \( \beta \) because some details of the process, e.g. to what extent it is a true diffusion process, remain unclear at the moment. The timescale for stars to leave the loss-cone is

\[
t_{\text{out}} \equiv \alpha t_{\text{cross}}.
\]

Hence we have at each radius \( r \) an approximate “diffusion” equation describing how stars enter and leave the escaper region of velocity space:

\[
\frac{d \rho_e}{dt} = -\frac{\rho_e}{t_{\text{out}}} + \frac{(1-k) \rho X_e}{t_{\text{in}}} (16)
\]

We put the term “diffusion” for this process in quotation marks here, because the underlying physical process is angular momentum diffusion in stellar systems, which is properly described only in a 2D Fokker-Planck model, using proper diffusion coefficients. The diffusion process tends to establish isotropy, i.e. an equal distribution of angular momenta - distribution function does not depend on angular momentum across the loss-cone and in its vicinity in velocity space. Here, we model this process in very simplified way by the ansatz that the diffusion term is proportional to the difference in densities inside (low angular momentum) and outside (high angular momentum) the loss cone. So, if \( k = 1 \), we have an isotropic equilibrium and the term vanishes, if \( k < 1 \) the diffusion refills the loss cone with a rate proportional to \( 1/t_{\text{in}} \), where \( t_{\text{in}} \) is the standard local relaxation time times a factor \( \beta \) of order unity. The reader is also referred to Amaro-Seoane, Freitag, & Spurzem (2004), where a similar concept is used for star accretion onto a massive central black hole.

Suppose, we have to readjust the filling factor \( k \) of escaper space for some model at time \( t \) and radial shell \( r \) during the numerical solution of the gaseous model equations. Then we can consider locally \( \rho \) and \( X_e \) as constant, and find after dividing a factor \( \rho X_e \) out of the above equation

\[
\frac{dk}{dt} = -k \frac{t_{\text{out}}}{t_{\text{in}}} + \frac{(1-k)}{t_{\text{in}}} (17)
\]

It can be solved directly only if \( t_{\text{in}} \) and \( t_{\text{out}} \) are held fixed. But we use this solution only to advance \( k(t) \) from one time step to another, then reinitialising it with new values of \( t_{\text{in}} \) and \( t_{\text{out}} \). If the timestep is small enough such that \( t_{\text{in}} \) and \( t_{\text{out}} \) do not change much this is a good approximation.

\[
k(t) = k_0 \exp\left(-\frac{K_0(t - t_0)}{t_{\text{out}}/t_{\text{in}}}ight) + \frac{t_{\text{out}}}{t_{\text{in}} K_0} \left(1 - \exp\left(-\frac{K_0(t - t_0)}{t_{\text{out}}/t_{\text{in}}}ight)\right) (18)
\]

Here \( k_0 \equiv k(t_0) \) and \( K_0 \equiv 1 + t_{\text{out}}/t_{\text{in}} \); for \( t \to \infty \) we find a stationary solution \( k_{\infty} \equiv 1/(1 + t_{\text{in}}/t_{\text{out}}) \). In our numerical models we compute the new filling factor \( k = k(r) \) according to Eq. [18] in every radial shell between the time steps of the Henyey iteration by setting \( k_0 \equiv k(t_0) \) and \( t \equiv t_0 + \Delta t \) with the model time step \( \Delta t \). For small time steps a linear approximation to the exponential function would be sufficient, which would again lead to linearly discretised form of Eq. [17] since the computational costs are small compared to the Henyey iteration we prefer to be on the safe side and use the full exponential function expression Eq. [18] for the recomputation of \( k \).

Note, some special meanings of the parameters \( \alpha \) and \( \beta \). Choosing for example a very small \( \alpha \ll 1 \) is equivalent to an immediate removal of escaping stars from the system, as e.g. Chernoff & Weinberg (1990) and other FP models usually treat the escapers. If \( \alpha \) is of the order of one it means that we allow for some time before the actual removal of the stars in a complex tidal gravitational field really takes place. If \( \beta \ll 1 \) the loss cone region is very quickly refilled (practically all the time is full), and if \( \beta \gg 1 \) the loss cone is not refilled.

This simplified diffusion and escape model, described just by
the two time-scales $t_{\text{out}}$ and $t_{\text{in}}$, coming with the corresponding two parameters $\alpha$ and $\beta$, is being used only in those gaseous model shells where $E < E_t$, i.e. the entire shell is still bound to the system and its total energy is smaller than the tidal energy. For shells whose energy as a whole is lifted above the tidal energy $E_t$ we follow a prescription originally proposed by Lee & Ostriker (1987, LO87). They point out that stars at the tidal energy need very long time to actually escape from the cluster, and only if their energy is higher than that, they will asymptotically escape with a time scale proportional to the crossing time at the tidal radius. The ansatz of LO87 for the evolution of the phase space distribution function $f = f(E, J^2)$ is

$$
\frac{\partial f}{\partial t} = -\alpha_{\text{FP}} f \left[ 1 - \left( \frac{|E|}{|E_t|} \right)^{3/2} \frac{4\pi}{3} G \rho_{\text{av}} \right].
$$

Note that we have added here the modulus of energies $E$, $E_t$ because our sign convention for the energy ($E < 0$) is different from that of LO87. $\rho_{\text{av}}$ denotes the average density of the cluster, so the last term is actually inversely proportional to a crossing time at the tidal radius. Portegies Zwart & Takahashi (1999) find that such a model provides a good match between FP models and direct $N$-body simulation. According to their choice of words, there is no “crisis” in Fokker-Planck models due to the $N$-dependence of the dissolution time, provided the parameter $\alpha_{\text{FP}}$ is adjusted to a value of order unity. LO87 varied $\alpha_{\text{FP}}$ from 0.2 to 5.0, while Takahashi & Portegies Zwart (2000) in an extensive multi-mass study for realistic globular clusters (with mass spectrum and stellar evolution) claim that $\alpha_{\text{FP}} = 2.5$ is the best value. We will discuss the role of $\alpha_{\text{FP}}$ in our models presented here later; physically $\alpha_{\text{FP}}$ brings in another time scale of mass loss (rather than the relaxation time), so its role is most prominent for small particle numbers, where both time scales are comparable. If $N$ is larger, typically 32000 or more, the crossing time is small compared to the relaxation time, so the latter completely determines the overall evolution of the system. But if in the course of the tidal evolution of a star cluster the particle number drops, at some moment the terms of LO87 will become important, and so the time of final dissolution of any cluster will depend on $\alpha_{\text{FP}}$.

In the anisotropic gaseous model we implement the dynamical mass loss of LO87 simply by applying a mass loss term in the densities corresponding to Eq. 19 if in a radial shell there is $E > E_t$:

$$
\frac{\partial \rho}{\partial t} = -\alpha_{\text{FP}} \rho \left[ 1 - \left( \frac{|E|}{|E_t|} \right)^{3/2} \frac{4\pi}{3} G \rho_{\text{av}} \right].
$$

The tidal energy $E_t$ is determined in the gaseous model simply by $E_t = GM(t)/r_t(t)$, where $r_t$ results from the standard condition that the average density of the entire cluster remains constant, and $M(t)$ is the time-dependent total mass ($r_t(t) = (M(t)/M_0)^{1/3}r_t(0)$, where $M_0$ and $r_t(0)$ are the initial total mass and tidal radius, respectively).

At any time step we redetermine the tidal radius from the present mass of the cluster, and remove all shells which fall outside of the newly determined tidal radius. This leads typically to a small zone where $E > E_t$, but still $r < r_t$ in which the LO87 dynamical mass loss procedure applies. Sometimes, in the very late phases of rapid mass loss before dissolution it can happen that shells inside the tidal radius have even positive energy ($E > 0$). If this happens the shell is immediately removed, as if it would lie outside the tidal radius.

Our model of mass loss in a tidal field can be determined by in total three parameters: $\alpha$ and $\beta$, which describe the time scales of the loss cone description for stars entering and leaving the escape

| Table 1. Parameters of the initial models |
|------------------------------------------|
| Name | N   | $W_0$ | $\alpha$ | $\beta$ | $\alpha_{\text{FP}}$ | k - trh |
| AGM-1 | 128000 | 3    | 1    | 1    | 1    | No - No |
| AGM-2 | 64000  | 3    | 1    | 1    | 1    | No - No |
| AGM-3 | 128000 | 3    | 1    | 1    | 1    | No - No |
| AGM-4 | 64000  | 3    | 1    | 1    | 1    | No - No |
| AGM-5 | 64000  | 3    | 1    | 1    | 1    | No - No |
| AGM-6 | 64000  | 3    | 1    | 1    | 1    | No - No |
| AGM-7 | 32000  | 3    | 1    | 1    | 1    | No - No |
| AGM-8 | 32000  | 6    | 1    | 1    | 1    | No - No |
| AGM-9 | 32000  | 6    | 1    | 0.5  | 1    | No - Yes |
| AGM-10| 64000  | 2.5  | 0.5  | 1    | 1    | No - Yes |
| AGM-11| 32000  | 6    | 5    | 0.5  | 1    | No - Yes |
| AGM-12| 32000  | 6    | 1    | 1    | 1    | No - Yes |
| AGM-13| 32000  | 6    | 2.5  | 1    | 1    | No - Yes |
| AGM-14| 32000  | 6    | 5    | 1    | 1    | No - Yes |
| AGM-15| 32000  | 6    | 1    | 2    | 1    | No - Yes |
| AGM-16| 32000  | 6    | 2.5  | 2    | 1    | No - Yes |
| AGM-17| 32000  | 6    | 5    | 2    | 1    | No - Yes |
| AGM-18| 32000  | 6    | 1    | 1    | 0    | No - No |
| AGM-19| 32000  | 6    | 1    | 0.2  | 1    | No - No |
| AGM-20| 32000  | 6    | 1    | 1    | 2.5  | No - No |
| AGM-21| 32000  | 6    | 1    | 5    | 1    | No - No |
| AGM-22| 32000  | 6    | 1    | 0.5  | 1    | Yes - No |
| AGM-23| 32000  | 6    | 1    | 1    | 1    | Yes - No |
| AGM-24| 32000  | 6    | 1    | 1    | 2    | Yes - No |
| AGM-25| 32000  | 6    | 1    | 1    | 1    | No - No |
| AGM-26| 16000  | 3    | 1    | 1    | 1    | No - No |
| AGM-27| 16000  | 6    | 1    | 1    | 1    | No - No |
| AGM-28| 16000  | 6    | 1    | 2    | 1    | No - No |
| AGM-29| 16000  | 9    | 1    | 1    | 1    | No - No |
| AGM-30| 5000   | 3    | 1    | 1    | 1    | No - No |
| AGM-31| 5000   | 6    | 1    | 1    | 1    | No - No |
| AGM-32| 5000   | 6    | 1    | 1    | 0    | No - No |
| AGM-33| 5000   | 6    | 1    | 1    | 2.5  | No - No |
| AGM-34| 5000   | 6    | 1    | 1    | 1    | Yes - No |
| AGM-35| 5000   | 9    | 1    | 1    | 1    | No - No |
| AGM-36| 1000   | 3    | 1    | 1    | 1    | No - No |
| AGM-37| 1000   | 6    | 1    | 1    | 1    | No - No |
| AGM-38| 1000   | 9    | 1    | 1    | 1    | No - No |
| IFP-1 | 32000  | 3    | 1    | 1    | 1    | No - No |
| IFP-2 | 32000  | 6    | 1    | 1    | 1    | No - No |
| IFP-3 | 1000   | 3    | 1    | 1    | 1    | No - No |
| IFP-4 | 1000   | 6    | 1    | 1    | 1    | No - No |
| AFP-1 | 5000   | 6    | 1    | 1    | 1    | No - No |

AGM - anisotropic gaseous models, IFP - isotropic Fokker-Planck models, AFP - anisotropic Fokker-Planck models, Nbody - N-body models, N - initial number of stars in the model, $W_0$ - concentration parameter of the King model, $\alpha$ and $\beta$ - parameters describing the process of removal of escaping stars and replenishing the escape region, respectively (see text - discussion of Eqs. 11 - 16), $\alpha_{\text{FP}}$ - parameter of Eq. 20 (see text), k - full or in equilibrium loss cone of escaping stars (No - equilibrium, Yes - full), trh - initial or actual half mass relaxation time used in diffusion equation (No - initial, Yes - actual) - see text.
loss cone, and $\alpha_{FP}$, which scales dynamical mass loss terms at the outer boundary.

Loss terms according to Eqs. [11] [12] [13] and [20] are applied as additional terms to the gaseous model equations during the Henyey iteration. It turns out that both the standard loss terms of Eqs. [11] [12] [13] as well as the LO87 term Eq. [20] are necessary to get a reasonable fit of mass loss in a cluster in a tidal field as compared with direct N-body as well as FP models. The filling degree $k(r)$ can be either set to a constant initial value of $k = 0$ or $k = 1$ according to empty or full loss cones, or the stationary value $k_{\infty}$ can be used which generally depends on $r$. For most models discussed in this paper we used the stationary value $k_{\infty}$ initially, see Table 1. It is interesting to note that in an unpublished study of Vicary (1997) using a Lagrangian gaseous model (Heggie 1984) the same qualitative behaviour of mass loss was found as here using a much simpler description based upon the loss of mass shells across the tidal boundary only. Unfortunately further details of that model are not available now.

3 INITIAL SETUP

All models discussed in the paper are described by idealized single mass star clusters under the influence of external tidal field. The initial positions and velocities of all stars were drawn from a King model. The set of initial King models were characterized by $W_0 = 3, 6$ and 9, number of stars by $N = 1000, 5000, 16000, 32000, 64000$ and 128000. The range of parameters $\alpha, \beta$ and $\alpha_{FP}$ are 1, 2, 5, 5, 0.5, 1, 2 and 0, 0.2, 0.5, 1, 2, 5, 5, respectively. The most natural values for parameters are: $\alpha = 1$ $\beta = 1$ and $\alpha_{FP} = 1$, which means that the time scales for the processes characterized by these parameters are described exactly by the local crossing and relaxation times. The initial models are described in Table 1. The data for anisotropic Fokker-Planck model (AFP) and N-body model for particle number $N = 5000$ were kindly provided by Kim (2003), and N-body data for $N = 16000$ by Heggie and Deiters.

To properly describe, in AGM, the process of mass removal connected with loss cone effect and removal of unbounded shells it was necessary to increase the spatial resolution of the model, particularly close to the tidal boundary. Removal of stars or shells from the gaseous model is a big challenge for the numerical algorithm, when the boundary of the model is closed. Enhanced mass loss generates errors of density, velocity, energy and mass distributions, which lead to uncontrolled error in the total mass and energy of the system. To reduce the error to the acceptable level (a few percent) we decided to use, instead of the logarithmically equidistant grid-points, the mesh, which resolution was enhanced towards the tidal boundary. After some numerical experiments the number of shells was chosen to: 541, 783, 967 for $W_0 = 3, 6, 9$, respectively. This guarantee that mass and energy errors was always less than 7%. Unfortunately, the code became less efficient. The time of calculations is linearly proportional to the number of shells.

For the computational units the standard N-body units (Heggie & Mathieu 1986) were used: total mass $M = 1$, $G = 1$ and initial total energy of the cluster equal to $-1/4$. In all presented in this paper figures the unit of time is expressed in terms of the initial half-mass relaxation time, which for single mass system and N-body units is equal to (Spitzer 1987):

$$t_{rh0} = \frac{0.138 N_0 r_{h0}^{3/2}}{\ln (\gamma N_0)},$$  \(21\)

where $N_0, r_{h0}$ are initial number of stars and half-mass radius.

![Figure 1](image)

**Figure 1.** a: Central density of initial King model $W_0 = 6$ as a function of time for gaseous models with 32000 particles, constant $\alpha = 1.0$ and varying $\beta = 0.5, 1.0, 2.0$ as indicated in the key. Increasing $\beta$ leads monotonically to increasing lifetimes of the cluster, because the mass loss time scale is larger. respectively. The value of coefficient in the Coulomb logarithm is taken to be $\gamma = 0.11$ (Giersz and Heggie 1994a).

4 RESULTS

Fig. 1a shows the effect of varying $\beta$ on the evolution of the central density in the gaseous model with $N = 32000$ particles in the case of constant $\alpha = 1.0$. First, one can see that the evolution of the system already in pre-collapse varies for different $\beta$; the reason is that the smaller $\beta$ the quicker refilling of the loss cone region and the higher mass loss from the system. In other words, if stars are removed from the cluster at a certain multiple of the crossing time, this process has a different speed relative to relaxation for each $N$. With constant $\alpha$ Fig. 1b shows, that a variation of $\beta$ just increases or decreases the efficiency of the escape process in a constant amount for each particle number. The differences in the dissolution time are even greater than in the collapse time. This is connected with the fact that the dynamical mass loss described by Eq. 20 is most prominent for the small particle numbers - models with smaller $\beta$ and higher mass loss.

In contrast to that Figs. 2 show the effect of varying $\alpha$ with constant $\beta$. There is almost no effect of changing $\alpha$ on collapse, and a small effect on dissolution times. The rate of mass loss is only slightly faster for small $\alpha$ than for larger $\alpha$ (see Fig. 2b). This is consistent with the picture that the rate of mass loss strongly depends on the rate of refilling the loss cone region. Star first have to be scattered by the relaxation process to the loss cone region and then on the crossing time scale escape from the system.

From Figs. 1 and 2 one can see that the diffusion time scale $t_{\infty} = \beta t_{rs}$, i.e. the parameter $\beta$ has a significant influence on the
core collapse time and the final dissolution time, by changing the mass loss rate. If the mass loss rate is larger, the core collapse accelerates, since the actual half-mass relaxation time becomes smaller than the initial one, and obviously the final dissolution time is different. We have used \( \alpha_{FP} = 1.0 \) for all models, unless stated otherwise.

Figs. 3 demonstrate the role of \( \alpha_{FP} \) – it hardly changes the early evolution, however, the onset of the final dramatic dissolution phase can be influenced by it. In the extreme case, where \( \alpha_{FP} \to 0 \) the final dissolution is taking extremely long time. This is clearly an unphysical case, as has already been stated by LO87 and Takahashi & Portegies Zwart (1998, 2000), Portegies Zwart & Takahashi (1999). We also show one example, where we have started initially with full loss cones, i.e. the phase space part in the escaper region is fully populated for all regions of the system. It leads to a quick mass loss in the beginning, due to draining of the full loss cone in a dynamical time, which cannot be resolved in the figure, and thereafter we end up with a somewhat faster evolution. In all other models we start with a stationary filling degree of the loss cone, \( k_\infty \).

Now, we turn to comparison of our models with direct \( N \)-body results. There are only few models available, and most of them are not published, or only partly published. We use here one 16000 \( N \)-body simulation kindly provided by S. Deiters and D.C. Heggie, for \( W_0 = 6 \), and another one, using 5000 particles, kindly provided by E. Kim (2003).

Comparing these model results with those of our AGM (Figs. 4) we find that the natural choice of \( \alpha = 1.0, \beta = 1.0 \) and \( \alpha_{FP} = 1.0 \) provides a fairly good match of the mass loss between \( N \)-body and AGM for both particle numbers. However, the core collapse times differ by a non-negligible amount. The increase of the parameter \( \beta \) to values much larger then one will not solve the problem, because this will destroy very good agreement for the mass loss rate. It should be stressed here the good agreement in the mass loss rate between \( N \)-body and AGM models. This shows the adopted simplified model for the mass loss from the AGM describes well the complicated process of mass loss from the stellar systems. Finally, it should be stressed that the noisier appearance of the \( N \)-body model with 16000 particles as compared to that with 5000 (see Fig. 4a) is connected with the fact that the larger model undergoes gravothermal like oscillations, whereas the smaller one shows a smooth post-collapse evolution. However, in the collapse phase, as expected, the smaller \( N \)-body model is noisier.

Now, we show a comparison of our AGM results with isotropic 1D Fokker-Planck results obtained by K. Takahashi’s code. The results are shown in Figs. 5, 6. One sees that now, the overall agreement between anisotropic gaseous and isotropic FP models for different King model concentration parameter \( W_0 = 3, 6 \) is quite good for the standard set of parameters: \( \alpha = 1, \beta = 1, \alpha_{FP} = 1 \), in particular regarding the mass loss, but less good for the collapse time. The small adjustment of these parameters will give a better agreement of the mass loss rate, but still will not change much the picture for the collapse time. It seems that generally tidally limited anisotropic gaseous models intrinsically show a too fast collapse time. AGM predicts a collapse time of about 60% of the \( N \)-body one. It will be discussed further below in connection with Fig. 7 that this may be due to an incorrect distribution of tidal mass loss across radial shells in our model compared to the \( N \)-body results (nevertheless we get obviously the total mass loss rate well). So, taking for example too much mass out of the core would change the collapse time incorrectly. Note that comparing individual \( N \)-body models (like we do here) with AGM creates further deviations in core collapse time (see for a single \( N = 10000 \) \( N \)-body run Spurzem & Aarseth 1996), because, as Giersz & Heggie (1994a,b) and Giersz & Spurzem (1994) have shown, a proper ensemble average of direct \( N \)-body models makes the convergence to AGM results reliable. In the cited papers we find that AGM pre-

**Figure 1 – continued b:** As Fig. 1a, but for the total mass of the system.

**Figure 2. a:** Central density of initial King model \( W_0 = 6 \) as a function of time for gaseous models with 32000 particles, constant \( \beta = 1.0 \) and varying \( \alpha = 1.0, 2.5, 5.0 \) as indicated in the key.
dicts core collapse times in a generally very good agreement with $N$-body models.

One might think that a further adjustment of the gaseous model parameters (such as the general conductivity $\lambda$, see e.g. Giersz & Spurzem (1994), which scales the collapse time in isolated models without mass loss) would improve the situation. After all, the standard value of $\lambda = 0.4977$, corresponding to $C = 0.104$ in the models of Giersz & Heggie (1994a), has been never checked for the case of tidally limited systems. In a larger number of further numerical experiments we find, however, that there is a non-trivial relation between the value of $\lambda$, the core collapse time, and the value of $\beta$, which changes the mass loss rate, and thus also the core collapse time (see Figs. 1a,b). It was impossible to find a better combination of $\lambda$ and $\beta$, which both reproduce the mass loss and core collapse behaviour more accurately. We concluded that the optimal combination of parameters is the use of the standard value of $\lambda = 0.4977$ together with $\beta = 1$; it provides good agreement of mass loss rates, but an error in the core collapse times has to be tolerated at this point. One possible reason is a still different origin of escapers in the AGM and the $N$-body model. Fig. 7 shows the cumulative number of escapers as a function of the radius at which the last scattering took place lifting them to escape energy, comparing AGM and $N$-body for 5000 bodies, at a time close to the initial model. While both models show that escapers can origin from near the core or half-mass radius, it is clear that the AGM underestimates the rate of escapers from the halo. The larger mass loss from the core the faster the system evolution. We do not intend here to further adjust some gaseous models parameters for this effect, because this is beyond the scope of this paper.

One can see from Fig. 8 that the overall escape rate is very well modelled here in the AGM, by looking at the half-mass time (time at which the model will contain half of its initial mass) as a function of the initial number of stars in comparison to published $N$-body results. The scaling is proportional to $t_{\text{rh}}^{3/4}$ for low $N$; stan-

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**Figure 2** – continued b: As Fig. 2a, but for the total mass of the system.

**Figure 3** – continued b: As Fig. 3a, but for the total mass of the system.

**Figure 3.** a: Central density of initial King model $W_0 = 6$ as a function of time for gaseous models with 32000 particles, constant $\beta = 1.0$ and $\alpha = 1.0$. Here we vary the dynamical mass loss constant $\alpha_{FP}$ as indicated in the key. The * symbol denotes a case started with full loss cones, $k = 1$ (see text).
standard theory suggests instead a scaling proportional to $t_{\alpha \beta}$, if potential escapers are immediately removed. Only for large $N$ this scaling is achieved, while for lower $N$ (below 50,000) it turns over to the flatter slope, which is in very good agreement with the theory and $N$-body results presented by Baumgardt (2001). We point out that in the gaseous model gravothermal oscillations in a tidally limited, mass-losing system are observed for the first times, and they are suppressed by the increasing mass loss towards the end of the cluster life time. The timestep in the FP model was chosen large enough to suppress oscillation for the reasons of computational time, so there is no conflict between the results of the two methods. Gaseous models are computationally cheap, even in the anisotropic tidally limited case. A typical model for the two methods. Gaseous models are computationally cheap, even in the anisotropic gaseous models with 5000 (5k) and 16000 (16k) particles, compared to the corresponding direct $N$-body models, see keys.

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Figure 4. a: Central density of initial King model $W_0 = 6$ as a function of time for gaseous models with 5000 (5k) and 16000 (16k) particles, comparing data with the corresponding direct $N$-body models, see keys.

We have derived suitable model equations and their numerical solutions for star clusters in a tidal field with mass loss in the framework of the anisotropic gaseous models (AGM, Louis & Spurzem 1991, Spurzem 1994, Giersz & Spurzem 1994). The equations properly model the $N$-dependence of the mass loss as discussed by Takahashi & Portegies Zwart (1998, 2000), Portegies Zwart & Takahashi (1999); since escapers are not removed immediately from the system but with a time scale of order of the local crossing time, small systems retain their escapers for a longer time with respect to their relaxation time scale than large clusters. Our results reproduce the evolution of the mass loss of the cluster in good agreement with Fokker-Planck (FP) and direct $N$-body models; some discrepancy in the core collapse time (gaseous model collapses too fast by some 40%) has to be accepted at the present time. We think that a possible reason for this problem is the different radial distribution of the origin of escapers in the clusters between AGM and direct $N$-body model, as seen in Fig. 7. An improvement of this is beyond the scope of this paper. Still our gaseous models provide an excellent tool to model the global mass loss behaviour with a very efficient code (see Fig. 8). Cluster evolution in tidal fields proceeds in three main phases according to our results. First, the pre-collapse evolution, which is only slightly modified by the mass loss as compared to the isolated model. Second, the steady mass loss phase in post-collapse, which in the framework of the gaseous model can be described by a simplified diffusion - escape picture, and drives the system steadily to smaller and smaller mass. At some time the mass is small enough to lift more and more stars across the tidal energy, and a runaway mass loss sets in, which is properly described by the mass loss formula of Eq. 4. While in the FP model the diffusion in energy and angular momentum is naturally included (which had to be added as an additional feature in the gaseous model), the final runaway mass loss is not well described by both models, even not in the FP model,

5 CONCLUSION

We have derived suitable model equations and their numerical solutions for star clusters in a tidal field with mass loss in the framework of the anisotropic gaseous models (AGM, Louis & Spurzem 1991, Spurzem 1994, Giersz & Spurzem 1994). The equations properly model the $N$-dependence of the mass loss as discussed by Takahashi & Portegies Zwart (1998, 2000), Portegies Zwart & Takahashi (1999); since escapers are not removed immediately from the system but with a time scale of order of the local crossing time, small systems retain their escapers for a longer time with respect to their relaxation time scale than large clusters. Our results reproduce the evolution of the mass loss of the cluster in good agreement with Fokker-Planck (FP) and direct $N$-body models; some discrepancy in the core collapse time (gaseous model collapses too fast by some 40%) has to be accepted at the present time. We think that a possible reason for this problem is the different radial distribution of the origin of escapers in the clusters between AGM and direct $N$-body model, as seen in Fig. 7. An improvement of this is beyond the scope of this paper. Still our gaseous models provide an excellent tool to model the global mass loss behaviour with a very efficient code (see Fig. 8). Cluster evolution in tidal fields proceeds in three main phases according to our results. First, the pre-collapse evolution, which is only slightly modified by the mass loss as compared to the isolated model. Second, the steady mass loss phase in post-collapse, which in the framework of the gaseous model can be described by a simplified diffusion - escape picture, and drives the system steadily to smaller and smaller mass. At some time the mass is small enough to lift more and more stars across the tidal energy, and a runaway mass loss sets in, which is properly described by the mass loss formula of Eq. 4. While in the FP model the diffusion in energy and angular momentum is naturally included (which had to be added as an additional feature in the gaseous model), the final runaway mass loss is not well described by both models, even not in the FP model,
unless it is completed by the term of Eq. 19. In the gaseous model it is possible to switch off the diffusion across the tidal energy; when doing so, we find that the total mass evolution is extremely slow, practically halted, even if the mass loss term Eq. 20 is included. Only the presence of steady mass loss caused by diffusion leads to conditions where the runaway dissolution of the cluster finally can take place. Gaseous models have here again shown their ability as excellent tools to analyze the physical processes going on in the evolution of heat conducting spheres, as a model for relaxing star clusters.

As shown in Fig. 8, we find that half-mass times ($t_{\text{half}}$, time at which the model will contain half of its initial mass) scale proportional to $t^{3/4}_{rh}$. Standard theory predicts to $t_{\text{half}} \propto t_{rh}$, if potential escapers are immediately removed. Our result using the AGM, however, is in excellent agreement with an improved theory and $N$-body results presented by Baumgardt (2001), who points out that there is a difference in time between the moment when the star reaches an energy necessary to escape and the moment when it leaves the cluster. Potential escapers can be scattered back to non-escaping energy during this time and become bound again. This process is responsible for a deviation from the standard scaling and leads to $t_{\text{half}} \propto t^{3/4}_{rh}$ for particle numbers at or smaller than 50000. This good agreement with $N$-body results additionally confirms the way in which the escapers are treated in AGM and shows that AGM can be used with success in simulations of tidally limited star clusters.

For large $N$ (32000) we observe gravothermal oscillations (Bettwieser & Sugimoto 1984, Goodman 1987, Makino 1996) during the steady mass loss phase in post-collapse. We observe for the first time the suppression of post-collapse gravothermal oscillations by a critically increasing mass loss at the end of the cluster life time (see Fig. 4). Note that earlier isotropic FP models of Drukier,
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Figure 6 – continued b: As Fig. 6a, but for the total mass of the system.

Figure 7. Cumulative number of escapers as a function of radius, at which the last scattering occurred, which lifted the star to higher than escape energy. We compare for 5000 particles, close to the initial $W_0 = 6$, the results of AGM and direct $N$-body. Also the actual core and half-mass radii are indicated.

Figure 8. Time at which the total cluster mass is equal to half of the initial mass for $W_0 = 3$, $W_0 = 6$ and $W_0 = 9$ as a function of initial number of particles AGM models with $\alpha = 1, \beta = 1$ and $a_{FP} = 1.0$.

Fahlman & Richer (1992) and Drukier (1993) also observed cases of gravothermal oscillations in tidally limited clusters, but did not follow the cluster evolution to full dissolution. The choice of the timestep in the FP model shown here was large enough to suppress the oscillations.

The difference between collapse times for AGM and $N$-body and FP models remains unsolved at the present code version. We did extensive parameter studies, many more than actually reported in this paper, to find a better optimal combination of parameters used to describe the mass loss in AGM ($a_{FP}, \alpha, \beta$), sometimes even also varying the standard scaling parameter for the conductivity $\lambda \equiv 0.4977$ (in Heggie’s models it is $C \equiv 2T\sqrt{\pi\lambda/10}$, see e.g. Giersz & Spurzem (1994). The result is that the standard parameters give the best agreement for all other properties between $N$-body, FP and AGM models, just the core collapse time remains too short in AGM. It should be noted that also the agreement between FP models and $N$-body models is not perfect regarding the collapse time. We speculate that the radial origin of escapers as described above may be the main reason for the discrepancy.

Note that our anisotropic gaseous models (AGM) have been used in their multi-mass form already with 50 components in Gürkan, Freitag, & Rasio (2004) and in Boily et al. (2005) with 15 components for studies of mass segregation. We want to study the future our models with the improved tidal boundary in connection with more realistic systems with a stellar mass spectrum (cf. e.g. the collaborative experiment, Heggie et al. 1998) and stellar evolution, and do a further comparison with a set of $N$-body models, possibly improving also the remaining problems in AGM.
APPENDIX

For clarity we have defined the following integrals in analogy to the error function (shown first):

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \]

\[ \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x t^2 \exp(-t^2) dt \]

\[ I(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(t^2) dt \]

\[ J(x) = \frac{2}{\sqrt{\pi}} \int_0^x t^2 \exp(t^2) dt \]

Note the relations

\[ \text{Erf}(x) = \frac{1}{2} \text{erf}(x) - \frac{x}{\sqrt{\pi}} \exp(-x^2) \]

\[ J(x) = -\frac{1}{2} I(x) + \frac{x}{\sqrt{\pi}} \exp(x^2) \]

\[ I(x) = \frac{2}{\sqrt{\pi}} \exp(x^2) \mathcal{D}(x) \]

where \( \mathcal{D}(x) \) is Dawson’s Integral defined as

\[ \mathcal{D}(x) = \exp(-x^2) \int_0^x t^2 \exp(t^2) dt \]

While for the standard error function we use the intrinsic function provided by the standard Fortran compilers, Dawson’s integral has to be taken from the Numerical Recipes, Chapter 6.10 (Press et al. 1986). Dawson’s Integral vanishes for \( x \to 0 \). Since the standard method given in the Recipes involves exponential functions this is not good for small argument values, therefore for \( |x| < 0.2 \) Dawson’s integral is in the given recipe evaluated by a Taylor series up to order \( x^5 \). In our expressions for \( X_e, X_r, \) and \( X_t \) we have similarly ill-behaved functions, namely \( \text{erf}(x)/x, \text{Erf}(x)/x^3, I(x)/x, \) and \( J(x)/x^3 \). All these expressions have to be taken for the arguments \( G \equiv \sqrt{b^2 - a^2}, b > a \) or \( H \equiv \sqrt{a^2 - b^2}, a > b \), and they approach zero for \( b \to a \). In our numerical computation of these functions we therefore also use the following series expansions for \( |x| < 0.2 \):

\[ I(x) \cdot \frac{\text{erf}(x)}{x} \approx \frac{2}{\sqrt{\pi}} \left( 1 \pm \frac{1}{3} x^2 \left( 1 \pm \frac{3}{10} x^2 \left( 1 \pm \frac{5}{21} x^2 \right) \right) \right) \]

\[ J(x) \cdot \frac{\text{Erf}(x)}{x^3} \approx \frac{2}{\sqrt{\pi}} \left( 1 \pm \frac{1}{3} x^2 \left( 1 \pm \frac{5}{14} x^2 \left( 1 \pm \frac{7}{27} x^2 \right) \right) \right) \]

The \( \pm \) sign should be taken as \( + \) for \( I, J \) (involving \( \exp(\pm t^2) \)) and as \( - \) for \( \text{erf}, \text{Erf} \) (involving \( \exp(-t^2) \)).

While the above Taylor series are completely well behaved for \( x \to 0 \), it is nevertheless instructive to look at the asymptotic forms for \( X_e, X_r, \) and \( X_t \) which are obtained from the following asymptotic equalities for \( x \to 0 \):

\[ I(x) \cdot \frac{\text{erf}(x)}{x} \sim \frac{2}{\sqrt{\pi}} \exp(\pm x^2) \]

\[ J(x) \cdot \frac{\text{Erf}(x)}{x^3} \sim \frac{2}{\sqrt{\pi}} \exp(\pm x^2) \]

The use of \( \pm \) is to be understood as above. With \( b = a \) we get the results

\[ X_e = \text{erf}(a) - \frac{2}{\sqrt{\pi}} \cdot a \cdot \exp(-a^2) \]

\[ X_r = \text{erf}(a) - \frac{2}{\sqrt{\pi}} \cdot a \left( 1 \pm \frac{5}{21} a^2 \right) \cdot \exp(-a^2) \]

\[ X_t = \text{erf}(a) - \frac{2}{\sqrt{\pi}} \cdot a \left( 1 \pm \frac{5}{21} a^2 \right) \cdot \exp(-a^2) = X_e \]

This same result is obtained approaching \( b = a \) from both sides \( (b < a, b > a) \), and it reaffirms that in the case of an isotropic velocity distribution and equal escape velocities in both the radial and tangential direction (i.e. using an energy criterion for escape) we have isotropy for the energy of the escaping stars.

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