Nonresonant entrainment of detuned oscillators induced by common external noise

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We have found that a novel type of entrainment occurs in two nonidentical limit cycle oscillators subjected to a common external white Gaussian noise. This entrainment is anomalous in the sense that the two oscillators have different mean frequencies, where the difference is constant as the noise intensity increases, but their phases come to be locked for almost all the time. We present a theory and numerical evidence for this phenomenon.

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Entrainment is a key mechanism for the emergence of order and coherence in a variety of physical systems consisting of oscillatory elements. It is one of the fundamental themes of nonlinear physics to explore the possible types of entrainments and clarify their fundamental properties. One of the typical entrainment phenomena is that caused by an external periodic signal. Consider two independent limit cycle oscillators of slightly different natural frequencies \(\omega_i\), \(i = 1, 2\). Suppose that these two oscillators are in resonance with the external periodic signal, i.e., \(m\omega_i - n\Omega \simeq 0\), where \(m\) and \(n\) are integers and \(\Omega\) is the external signal frequency. In this case, it is possible that the two oscillators have the same mean frequency \(n\Omega/m\). This type of entrainment can be observed in various systems as diverse as periodically driven electrical circuits, lasers with coherent optical injections, and biological circadian rhythms. Resonant entrainment can be described by using simple dynamical models for phase variables and the essential properties are well understood (e.g., [1]).

Recent physical and numerical experiments have shown that not only a periodic but also a noise-like signal can give rise to entrainment between two independent oscillators [2, 3, 4, 5, 6]. The concept of entrainment of limit cycle oscillators induced by common signals has to be generalized to include the case of noise-like signals. The entrainment by a noise-like signal is a nonresonant one in the sense that there is no resonance relation between the oscillator and the noise. Entrainment between two independent and identical oscillators induced by a common noise signal has already been studied [5]: it has been analytically shown for a wide class of limit cycle oscillators that the phase locking state becomes linearly stable by applying an arbitrary weak Gaussian noise.

However, in real systems, the two oscillators are never identical but slightly detuned. We note that the theory in Ref. [5] does not guarantee the entrainment between nonidentical oscillators at all. As an illustrative example, consider two pairs of oscillators, where one pair has a natural frequency \(\omega_1\) and the other pair has a different natural frequency \(\omega_2\), and suppose that they are subjected to a weak common noise. The theory tells that the phase locking occurs in each pair. However, it does not occur between the two pairs because for a weak noise the mean frequencies of the pair with \(\omega_1\) and that with \(\omega_2\) are still close to \(\omega_1\) and \(\omega_2\), respectively. One might expect that the nonidentical oscillators come to have the same mean frequency and stable phase locking occurs when large enough noise is applied. As we will show, this is not the case. It has not yet been clarified at all what kind of phenomenon happens between nonidentical oscillators. It is necessary to clarify this point for better understanding of real systems.

In this study, we consider a general class of limit cycle oscillators and reveal that a novel type of entrainment occurs between two nonidentical oscillators subjected to a common white Gaussian noise, which we call the nonresonant entrainment. This entrainment is anomalous in the sense that the two oscillators have different mean frequencies and the difference is constant even if the noise intensity increases but their phases come to be locked for almost all the time.

Let \(X_i \in \mathbb{R}^N\) be a state variable vector and consider the equation

\[
\dot{X}_i = F(X_i) + \delta F_1(X_i) + G(X_i)\eta(t), \quad i = 1, 2, \quad (1)
\]

where \(F\) is an unperturbed vector field, \(\delta F_1\) and \(\delta F_2\) are small deviations from it, \(G \in \mathbb{R}^N\) is a vector function, and \(\eta(t)\) is the white Gaussian noise such that \(\langle \eta(t) \rangle = 0\) and \(\langle \eta(t)\eta(s) \rangle = 2D\delta(t-s)\), where \(\langle \cdot \cdot \cdot \rangle\) denotes averaging over the realizations of \(\eta\) and \(\delta\) is Dirac’s delta function. We call the constant \(D > 0\) the noise intensity. The noise-free unperturbed system \(X = F(X)\) is assumed to have a limit cycle with a frequency \(\omega\). We employ the Stratonovich interpretation for the stochastic differential equation [1]. This interpretation allows us to apply the phase reduction method to Eq. (1), which assumes the conventional variable transformations in differential equations.

If we regard the common noise as a weak perturbation to the deterministic oscillators and apply the phase reduction method to Eq. (1), we obtain the equation for the phase variable as follows:

\[
\dot{\phi}_i = \omega + \delta \omega_i(\phi_i) + Z(\phi_i)\eta(t), \quad i = 1, 2, \quad (2)
\]
where \( \omega \) is the frequency of the unperturbed oscillator, \( \delta \omega_i \) is the frequency variation due to \( \delta F_i \); \( Z \) is defined by 
\[
Z(\phi) = G(X_0(\phi)) \cdot (\text{grad}_\phi \phi|_{x=x_0(\phi)}),
\]
where \( \phi \) is the phase variable defined by the unperturbed system \( X = F(X) \) and \( X_0(\phi) \) is its limit cycle solution. By definition, \( Z(\phi) \) is a periodic function, i.e., \( Z(\phi) = Z(\phi + 2\pi) \). We assume that \( Z \) is three times continuously differentiable and not a constant. It is also assumed that \( 0 < D/\omega \ll 1 \) to ensure the validity of the phase reduction.

In order to derive the average equation for \( \phi_i \), we translate Eq. (2) into the equivalent Ito stochastic differential equation:
\[
\dot{\phi}_i = \omega + \delta \omega_i(\phi_i) + DZ(\phi_i)Z'(\phi_i) + Z(\phi_i)\eta(t),
\]
where the dash denotes differentiation with respect to \( \phi_i \). In the Ito equation, unlike in Stratonovich formulation, the correlation between \( \phi_i \) and \( \eta \) vanishes. If we subtract Eq. (3) for \( \phi_2 \) from that for \( \phi_1 \) and take the ensemble average, then we have the average equation:
\[
\frac{d}{dt} \langle \phi_1 - \phi_2 \rangle = \langle \delta \omega_1(\phi_1) \rangle - \langle \delta \omega_2(\phi_2) \rangle + D \{ \langle Z(\phi_1)Z'(\phi_1) \rangle - \langle Z(\phi_2)Z'(\phi_2) \rangle \},
\]
where we used the fact \( \langle Z(\phi_i)\eta(t) \rangle = \langle Z(\phi_i) \rangle \langle \eta(t) \rangle = 0 \). Each ensemble average on the right hand side can be evaluated by using the steady probability distribution \( P_t(\phi_1) \) for \( \phi_1 \), which can be obtained from the Fokker-Planck equation for Eq. (3): i.e., \( A(\phi_1) = \int_{\sigma}^{\infty} A(\phi)P_t(\phi)d\phi \), where \( A \) represents a function of \( \phi_i \). The distribution \( P_t \) can be obtained as \( P_t(\phi_1) = 1/2\pi + O(\sigma, D/\omega) \), where \( \sigma_i = \max_{\phi_1<\phi<2\pi} \delta \omega(\phi)/\omega \). Since \( \delta \phi_i \) is small, \( \sigma_i \) is a small parameter. Therefore, \( P_t \) can be approximated by \( P_t \approx 1/2\pi \) for small \( D/\omega \) and we have
\[
\langle \delta \omega_1(\phi_1) \rangle \simeq \frac{1}{2\pi} \int_0^{2\pi} \delta \omega_1(\phi)d\phi \equiv \delta \omega_1,
\]
\[
\langle Z(\phi_1)Z'(\phi_1) \rangle \simeq \frac{1}{2\pi} \int_0^{2\pi} Z(\phi)Z'(\phi)d\phi = 0,
\]
where we used the fact \( Z(0) = Z(2\pi) \). If we substitute Eqs. (5) and (6) into Eq. (4), we have
\[
\frac{d}{dt} \langle \phi_1 - \phi_2 \rangle = \delta \omega_1 - \delta \omega_2.
\]
Since in general \( \delta \omega_1 - \delta \omega_2 \neq 0 \), this equation indicates that the average phase difference increases or decreases in proportion to the time \( t \). In other words, the two oscillators still have different mean frequencies even when a common white Gaussian noise is applied, i.e., \( d(\phi_1)/dt \neq d(\phi_2)/dt \). Intuitively, this result is natural because the white noise has a uniform power spectrum and does not have a characteristic frequency, which could entain the oscillator frequencies.

Let \( \theta \) and \( \psi \) be defined by \( \theta = \phi_1 - \phi_2 \) and \( \psi = \phi_1 + \phi_2 - 2\omega t \). The variable \( \theta \) measures the phase difference between the two oscillators. For small \( D \) and \( \delta \omega_1 \), it is expected that \( \phi_i \) still has a mean frequency close to \( \omega \). Therefore, \( \theta \) and \( \psi \) can be regarded as slow variables. If we change the independent variables form \( (t, \phi_1, \phi_2) \) to \( (t, \theta, \psi) \) and perform the time-averaging with respect to \( t \), we can obtain the Fokker-Planck equation corresponding to Eq. (2) as follows:
\[
\frac{\partial Q}{\partial t} = -(\delta \omega_1 - \delta \omega_2)\frac{\partial Q}{\partial \theta} - (\delta \omega_1 + \delta \omega_2)\frac{\partial Q}{\partial \psi} + D\frac{\partial^2}{\partial \theta^2} [2\{\Gamma(0) - \Gamma(\theta)\}Q] + D\frac{\partial^2}{\partial \psi^2} [2\{\Gamma(0) + \Gamma(\theta)\}Q],
\]
where \( Q(t, \theta, \psi) \) is the joint probability distribution and \( \Gamma \) is defined by
\[
\Gamma(\theta) = \frac{1}{2\pi} \int_0^{2\pi} Z(\phi)Z(\phi + \theta)d\phi.
\]
Hereafter we assume the case of \( \delta \omega_1 > \delta \omega_2 \) without loss of generality.

It is in general possible that \( Z \) has a period smaller than \( 2\pi \). Since \( Z \) is not a constant function, we suppose that \( Z(\phi) = Z(\phi + 2\pi/n) \), where \( n \) is a positive integer. Let \( h(\theta) \) be defined by \( h(\theta) = 2\{\Gamma(0) - \Gamma(\theta)\} \). It can be shown that \( h(\theta) \geq 0 \) for any \( \theta \in [0, 2\pi) \). The zero points \( s_m \) of \( h \) are given by \( s_m = 2\pi n/m \), \( m = 0, 1, \ldots, n-1 \), where \( s_0 = 0 \). Equation (5) has the steady solution \( Q_s(\theta) \) such that it is a continuous function of \( \theta \) only and satisfies the two conditions (i) \( Q_s(\theta) = Q_s(\theta + 2\pi) \) and (ii) \( \int_0^{2\pi} Q_s(\theta)d\theta = 1 \). In each interval \( (s_m, s_{m+1}) \), the solution \( Q_s \) can be obtained as follows:
\[
Q_s(\theta) = \frac{\frac{1}{2\pi}}{\int_{\theta}^{\theta + 2\pi} \exp \left[ -\varepsilon \int_{\theta}^{\theta + 2\pi} \frac{1}{h(y)}dy \right] dx,}
\]
where \( \varepsilon = (\delta \omega_1 - \delta \omega_2)/D > 0 \). The right hand side of Eq. (10) has singularities at the zero points of \( h \). The value of \( Q_s \) for each \( s_m \) is given by \( Q_s(s_m) = \lim_{s_{m+1} \to s_m} Q_s(\theta) \). Assume that \( \theta \in (s_m, s_{m+1}) \), i.e., \( \theta \) is an arbitrary regular point. It can be shown that \( \lim_{s_{m+1} \to s} Q_s(\theta) = 0 \) holds due to the factor \( \varepsilon \) in the numerator. This implies that the probability has to concentrates at the singular points \( s_m \), \( m = 0, 1, \ldots, n-1 \) because \( Q_s \) satisfies the condition (ii). Thus, \( Q_s \) in the limit \( \varepsilon \to 0 \) is given by
\[
Q_s(\theta) = \frac{1}{n} \sum_{m=0}^{n-1} \delta(\theta - s_m),
\]
where \( \delta \) is Dirac’s delta function. For small positive \( \varepsilon \), the distribution \( Q_s \) has narrow and sharp peaks at \( \theta = s_m \), \( m = 0, 1, \ldots, n-1 \) while \( Q_s \) is close to zero in the regions other than the neighborhoods of these singular points. The peaks of \( Q_s \) become narrower as \( \varepsilon \) approaches zero. Equation (11) indicates that multiple peaks exist if \( Z \) has a period smaller than \( 2\pi \), i.e., \( n > 1 \). The existence of multiple peaks has been pointed out in the case of identical oscillators [6].
The above profile of $Q_s$ clearly shows that the phase locking states, where $\theta \mod 2\pi \simeq s_m$, are achieved for a large fraction of time during the time evolution when the noise intensity $D$ is relatively large with respect to the mean frequency difference $\delta \omega_1 - \delta \omega_2$; i.e., the nonresonant entrainment occurs. Let $\delta$ be a small positive constant and $U_\delta$ be the $\delta$-neighborhood defined by $U_\delta = \bigcup_{m=0}^{n-1} (s_m - \delta, s_m + \delta)$, where mod $2\pi$ is taken for $s_0 - \delta$. We identify the phase locking state by the condition $\theta \in U_\delta$. As shown by Eq. (7), the present entrainment is not characterized by coincidence of the mean frequencies of the two oscillators. Therefore, as a measure for the entrainment, we introduce the phase locking time ratio $\mu$ defined by

$$\mu = \lim_{T \to \infty} \frac{T_L}{T},$$

where $T_L$ represents the total time length for which $\theta \in U_\delta$ happens during the period $T$. This ratio can also be expressed in terms of $Q_s$ by $\mu = \int_{U_\delta} Q_s(\theta) d\theta$, where the integral is taken over the set $U_\delta$. Equation (11) shows that $\mu \to 1$ in the limit $\epsilon = (\delta \omega_1 - \delta \omega_2)/D \to 0$.

A phase locking state cannot continue for the infinite time but phase slips have to happen during the periods such that $\theta \not\in U_\delta$ because the two mean frequencies $d(\phi_1)/dt$ and $d(\phi_2)/dt$ are different. Equation (4) indicates that the mean frequency difference is given by the constant $\delta \omega_1 - \delta \omega_2$. This implies that the average number of phase slips, which happen in a unit time interval, does not become small but remains constant even for relatively large $D$ compared with $\delta \omega_1 - \delta \omega_2$. In other words, the average interslip interval remains constant. On the other hand, the probability for $\theta \not\in U_\delta$ decreases and converges to zero as $D$ increases: i.e., the phases come to be locked for almost all the time. These two facts imply that a single phase slip completes more rapidly: i.e., the time needed for one phase slip decreases and converges to zero as $D$ increases. We emphasize that the above mentioned behavior is a remarkable feature of the nonresonant entrainment. This behavior is very different from that of resonant entrainment by a periodic signal, where the average interslip interval diverges and the mean frequencies becomes identical as the signal intensity approaches the critical value for entrainment.

In order to demonstrate the above analytical results, we show numerical results for an example described by the Stratonovich stochastic differential equations

$$\dot{\phi}_i = \omega_i + \sin(\phi_i) \eta(t), \quad i = 1, 2,$$

where $\omega_i$, $i = 1, 2$ are slightly different constants.

Figure 1(a) shows the mean frequency difference $\Delta \omega$ vs. $D$. It is clearly shown that $\Delta \omega$ is constant and independent of $D$. This result coincides with the analytical result of Eq. (7). The steady distribution $P_i(\phi_i)$ is approximately given by $P_i(\phi_i) \simeq (1/2\pi)[1 + (D/2\omega_i) \sin(2\phi_i)]$ for this example. This shows that the assumption $P_i(\phi_i) \simeq 1/2\pi$ is reasonable for small $D$ used in the numerical calculations. Thus, the result of Eq. (7) holds.

The time evolution of the phase difference $\theta = \phi_1 - \phi_2$ is shown for three different values of $D$ in Fig. 1(b), where $\omega_1 = 1$ and $\omega_2 = 0.98$. These results clearly show that the phases are locked near $\theta \simeq 2\pi n$, $n \in \mathbb{Z}$ and the
phase slips occur intermittently. It should be noted that the time needed for a single phase slip becomes smaller as D increases. This observation is in agreement with the analytical result.

The probability distribution $Q_s(\theta)$ is shown in Fig. 1(c) for three different values of $D$, where $\omega_1 = 1$ and $\omega_2 = 0.98$. The analytical results of Eq. (10) are also shown for the corresponding values of $\varepsilon = (\omega_1 - \omega_2)/D$. It is seen that $Q_s$ is close to the uniform distribution for small $D$ or large $\varepsilon$. In contrast, the distribution has a sharp peak near $\theta = 0$ for large $D$ or small $\varepsilon$. The peak in $Q_s$ becomes narrower and its position becomes closer to $\theta = 0$ as $D$ increases. This agrees with the previous theory since the zero point of $h(\theta)$ is only $\theta = 0$ in this example and thus the theory tells that $Q_s$ has a peak only at $\theta = 0$. It is also seen that the peak is not centered at $\theta = 0$ but shifted to the positive direction: i.e., the phase $\phi_1$ of the larger natural frequency oscillator is kept advanced with respect to $\phi_2$ even in the phase locking state. The inset of Fig. 1(c) shows that the phase locking time ratio $\mu$ monotonically increases and approaches unity with increasing $D$. Figure 1(c) clearly demonstrates that the phases are locked for a larger fraction of the time as $D$ increases.

In order to validate the theory based on the phase reduction method, we carried out numerical experiments for the Stuart-Landau (SL) oscillator

$$\dot{\psi}_j = (1 + ic_j)\psi_j - |\psi_j|^2\psi_j - \eta(t), \quad j = 1, 2,$$

where $\psi_j \in \mathbb{C}$ and $c_j = 1 + \delta \omega_j$ is a real constant. This is reduced to the phase model $\dot{\phi}_j = 1 + \delta \omega_j + \sin(\phi_j)\eta(t)$, where $\phi_j$ is the appropriately defined phase variable.

In Fig. 2 the numerically obtained distribution $Q_s(\theta)$ is shown for three different values of $D$, where $\delta \omega_1 = 0$ and $\delta \omega_2 = -0.02$. The analytical results obtained from the corresponding phase model are also shown for the corresponding values of $\varepsilon = (\delta \omega_1 - \delta \omega_2)/D$. A sharp peak of $Q_s$ appears near $\theta = 0$. It becomes narrower and approaches $\theta = 0$ as $D$ increases. Agreement between the numerical and analytical results is excellent, especially in small $D$ region, where the phase reduction method gives a good approximation. The inset shows the mean frequency difference $\Delta \omega = d(\phi_1)/dt - d(\phi_2)/dt$ plotted as a function of $D$ for the same $\delta \omega_1$ and $\delta \omega_2$. It is clearly shown that $\Delta \omega$ does not depend on $D$ and its constant value is given by $\Delta \omega = \delta \omega_1 - \delta \omega_2$. This behavior also agrees with the theory. The agreements in the behaviors of $Q_s$ and $\Delta \omega$ validate the theory based on the phase model.

In conclusion, we have found the nonresonant entrainment between two nonidentical limit cycle oscillators subjected to a common external white Gaussian noise. Wee explained this phenomenon by using a phase model and presented numerical evidence for a particular phase model and the SL oscillator. The nonresonant entrainment is anomalous in the sense that the two oscillators have different mean frequencies, where the difference is independent of the noise intensity and the average interslip interval is constant, while their phases come to be locked for almost all the time for relatively large noise. It is expected that similar entrainment occurs for various noise-like signals having broad continuous power spectra.

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