Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers-Fisher Equation

Ramya Selvaraj¹ · V.Swaminathan¹ · A.Durga Devi² · K.Krishnakumar¹

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Abstract In this article, we study the Lie point symmetries for the time fractional generalized Burgers-Fisher (GBF) equation. While getting an appropriate combination of symmetries, the time fractional partial differential equation has been transformed to nonlinear fractional ordinary differential equations (ODE) using Erdelyi-Kober differential operator. Furthermore, using power series method, we get the exact solution of the nonlinear fractional GBF equation with the arbitrary nonlinearity.

Keywords Generalized Burgers-Fisher equation · RL fractional derivative · Lie symmetry · Power series

1 Introduction

The fractional differential equation (FDE) plays a vital role in many branches of science and engineering [5,40,41,1,43]. FDE has many applications in the field of magnetism, fluid mechanics, cardiac tissue electrode interface, ultrasonic wave propagation in human cancellous bone, RLC electric circuit, theory of visco elasticity, lateral and longitudinal control of autonomous vehicles, wave propagation in viscoelastic horn, heat transfer, sound waves propagation in rigid porous materials and many more.

In recent years, many authors find solutions of FDEs using various methods such as variational iteration method [28], homotopy perturbation method [30], Adomian decomposition method [29], the first integral method [32,31], the sub-equation method [41,33,34] and so on.

¹ Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India. E-mail: rsramyaselvaraj@gmail.com, mvsnew@gmail.com
² Department of Physics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India.
Although a large number of methods have been developed to solve FDEs \[46,25,26,24,27,35\], in recent years, to reach the exact solutions of nonlinear partial differential equations \[45\], Lie symmetry method is used which is considered as an efficient method. In the beginning of 19th century, the Norwegian mathematician Sophus Lie \[47\] introduced the Lie symmetry analysis.

To find one or several parameter continuous transformations leaving the equation invariant, the Lie symmetry analysis is used. Later, it has been developed by Ovsiannikov \[48\] and many researchers used it to solve various equations \[51,52,53,51,55,57,58,56,51,57,58,56,51,57,58,56,51,57,58,56\].

The time fractional KdV equations using Lie group analysis was explained by Wang and Xu \[59\]. He also explained the invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation \[58\] and Y. W. Zhang \[59\] performed Lie symmetry analysis to the time fractional generalized fifth order KdV equation.

V. Kumar et al \[60\] studied Lie symmetry based analytical and numerical approach for modified Burgers-KdV equation. Furthermore, T. Bakkyaraj and R. Sahadevan \[42,62\] determined the invariant analysis of nonlinear fractional ordinary differential equations with Riemann-Liouville fractional derivative. They also performed group formalism of Lie transformations to time fractional differential equations \[61\].

This work is organized as follows: In section 2, we give some preliminaries on fractional derivatives. In section 3 and 4, we study the Lie point symmetries and symmetry reduction of the GBF equation. In section 5 and 6, we derive the power series solution to find the explicit solutions of the resultant equation and analyze convergence. Physical meaning of the exact solution of the power series is given in section 7. Section 8 ends with conclusion.

2 Preliminaries

Many researchers used several definitions of fractional derivative such as the Caputo \[5,1\], Riemann-Liouville \[5,3,1,4\], the Weyl \[5,4\], the Grunwald-Letnikov \[5,3,1,4\] and the Riesz \[4\]. Among them Caputo and Riemann-Liouville fractional derivatives have been widely used.

The Riemann-Liouville (RL) fractional derivative is used to study the Lie symmetry analysis of fractional differential equations \[6,7,42\]. In this paper, we use some basic definitions.

2.1 Definition

The RL fractional derivative \[6,7,42\] is given by

\[
D^\alpha f(t) = \begin{cases} \frac{d^n f}{dt^n}, & \alpha = n, \\ \frac{d^n}{dt^n} I^{n-\alpha} f(t), & 0 \leq n - 1 < \alpha < n, \end{cases}
\]  

(1)
given by \( n \in \mathbb{N} \), \( I^\mu f(t) \) is the RL fractional integral of order \( \mu \), where

\[
I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s)ds, \mu > 0
\]  

(2)

and \( \Gamma(z) \) is the gamma function.

2.2 Definition

The RL partial fractional derivative is given by

\[
\partial^\alpha_t = \begin{cases} 
\frac{\partial^n f}{\partial t^n}, & \alpha = n, \\
\frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(s,x)ds, & 0 \leq n-1 < \alpha < n, n \in \mathbb{N}.
\end{cases}
\]  

(3)

If it exists, where \( \partial^\alpha_t \) is the partial derivative of integer \( n \) [6,7,42].

3 Method of Lie symmetry for the time fractional differential equations

Let us consider the time fractional partial differential equation (TF-PDE) having the form

\[
\partial^\alpha_t u = F(t,x,u,u_x,u_{xx},\ldots), \ (0 < \alpha < 1).
\]  

(4)

A one-parameter Lie group of transformations are given by

\[
\tilde{t} = t + \epsilon\zeta(t,x,u) + O(\epsilon^2), \\
\tilde{x} = x + \epsilon\xi(t,x,u) + O(\epsilon^2), \\
\tilde{u} = u + \epsilon\eta(t,x,u) + O(\epsilon^2), \\
\frac{\partial^\alpha \tilde{u}}{\partial t^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta^0(t,x,u) + O(\epsilon^2), \\
\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x} + \epsilon\eta^x(t,x,u) + O(\epsilon^2), \\
\frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(t,x,u) + O(\epsilon^2),
\]

(5)

where

\[
\eta^x = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\zeta), \\
\eta^{xx} = D_x(\eta^x) - u_{xx} D_x(\xi) - u_{xt} D_t(\zeta),
\]  

(6)
The total differential operator \( D_x \) is defined by
\[
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \ldots
\]  
(7)

The associated Lie algebra of symmetries is spanned by vector fields
\[
X = \xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}.
\]  
(8)

The vector field Eq.(8) is a Lie point symmetry of Eq.(3) provided
\[
pr^{\alpha,2}X(\nabla)|_{\nabla=0} = 0.
\]  
(9)

Also, the invariance condition gives
\[
\zeta(t,x,u)|_{t=0} = 0,
\]  
(10)

and the \( \alpha^{th} \) extended infinitesimal related to RL fractional time derivative with Eq.(10) is given by
\[
\eta^0_\alpha = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\zeta)) \frac{\partial^\alpha u}{\partial t^\alpha} + \mu - \sum_{n=1}^{\infty} \binom{\alpha}{n} D^n_t(\xi) D^{\alpha-n}(u_x) + \sum_{n=1}^{\infty} \left[ \binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \left( \binom{\alpha}{n+1} D^{n+1}_t(\zeta) \right) D^{\alpha-n}(u) \right],
\]  
(11)

where
\[
\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m-1} \sum_{r=0}^{k-1} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)} [-u]^r \frac{\partial^m}{\partial t^m} [u^{k-r}] \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial u^k}.
\]  
(12)

Due to the presence of \( \frac{\partial^k \eta}{\partial u^k} \), if the infinitesimal \( \eta \) is linear in \( u \), the expression for \( \mu = 0 \) for \( k \geq 2 \) in Eq.(12).

3.1 Definition

The function \( u = \theta(x,t) \) is an invariant solution of Eq.(3) associated with Eq.(8) such that
1. \( u = \theta(x,t) \) satisfies Eq.(11).
2. \( u = \theta(x,t) \) is an invariant surface of Eq.(5), this means that
\[
\zeta(t,x,\theta) \theta_t + \xi(t,x,\theta) \theta_x = \eta(t,x,\theta).
\]
4 Lie symmetries for time fractional generalized Burgers-Fisher equation

In this work, Lie symmetry method has been presented for time fractional generalized Burgers-Fisher (GBF) equation.

The mathematical modelling of turbulence was explained by a Dutch physicist, Johannus Martinus Burgers, in 1948. A nonlinear equation which is the combination of reaction, convection and diffusion mechanism is called Burgers-Fisher equation.

The GBF equation is used in the field of fluid dynamics. It has also been found in some applications such as gas dynamics, heat conduction, elasticity and so on.

The time fractional GBF equation is given by,

\[ u_\alpha^t + \beta u^\delta u_x - u_{xx} = \gamma u(1 - u^\delta) \]  \hspace{1cm} (13)

where \(0 < \alpha \leq 1\), \(\alpha\) is the order of fractional time derivative and \(\beta, \gamma, \delta\) are arbitrary constants.

Let us consider Eq. (13) is invariant with respect to Eq. (5), we have that

\[ \tilde{u}_\alpha^t + \beta \tilde{u}_x^\delta \tilde{u}_x - \tilde{u}_{xx} = \gamma \tilde{u}(1 - \tilde{u}^\delta), \]  \hspace{1cm} (14)

such that \(\tilde{u} = u(x, t)\) satisfies Eq. (13). Using Eq. (5) in Eq. (14), we get the invariant equation

\[ \eta_0^\alpha + \beta \delta u^{\delta - 1} \eta u_x + \beta u^\delta \eta_x^\delta - \eta_{xx} - \gamma \eta_x + \gamma (\delta + 1) u^\delta \eta = 0. \]  \hspace{1cm} (15)

Applying the values of \(\eta_0^\alpha, \eta_x\) and \(\eta_{xx}\) given in Eq. (6) and Eq. (11) into Eq. (15) and then isolating coefficients in partial derivatives with respect to \(x\) and power of \(u\), we get

\[ \partial_\alpha^t \eta - u \beta^\delta \eta u_x - \gamma \eta_x + \gamma (\delta + 1) u^\delta \beta^\delta \eta_x = 0, \]  \hspace{1cm} (16)

\[ \left( \frac{\alpha}{n} \right) \partial_\alpha^t (\eta) - \left( \frac{\alpha}{n + 1} \right) D^{n+1}(\zeta) = 0, n = 1, 2, ... \]

\[ \xi_u = \zeta_u = \zeta_x = \zeta_t = 0, \]

\[ \eta_{uu} = \zeta_{uu} = \zeta_{uu} = 0. \]

Solving the over determining equations, we get:

\[ \zeta = k_1 + xak_2, \xi = 2t(\delta - 2)k_2, \eta = -\alpha uk_3, \]

where \(k_1\) and \(k_2\) are arbitrary constants. Thus infinitesimal symmetry group for Eq. (13) is spanned by the two vector fields

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2t \frac{\partial}{\partial t} + x \alpha \delta \frac{\partial}{\partial x} - u_0 \frac{\partial}{\partial u}. \]  \hspace{1cm} (17)
In particular, the symmetry $X_2$ possess the similarity transformation and similarity variable as follows:

$$k_1 = xt^{\frac{2}{\alpha}}, \quad k_2 = ut^{\frac{\beta}{\alpha}},$$

and this yields

$$u = t^{\frac{2}{\alpha}} f(\xi), \quad \xi = xt^{\frac{1}{\alpha}}.$$

In Eq. (19), $f$ is an arbitrary function of $\xi$. Using Eq. (19), Eq. (13) is transformed to a special nonlinear ODE of fractional order as mentioned in the following theorem 1.

**Theorem 1** The similarity transformation $u(x, t) = t^{\frac{2}{\alpha}} f(\xi)$ along with the similarity variable $\xi = xt^{\frac{1}{\alpha}}$ reduces Eq. (13) to the nonlinear fractional ODE of the form,

$$\left(P^{1-\frac{\alpha(1+2\delta)}{\delta}} f\right) (\xi) + \beta f^{\delta} f_\xi - u_{xx} - \gamma u(1 - u^\delta) = 0,$$

where the Erdelyi-Kober fractional differential operator (EK-FDO) is defined as

$$\left(P^{\frac{\beta}{\alpha}} f\right) (\xi) = \prod_{j=0}^{n-1} \left(\xi + j - \frac{1}{\beta} \frac{d}{d\xi}\right) (K^{\frac{\delta}{\alpha}, n} f) (\xi),$$

where

$$(K^{\frac{\delta}{\alpha}, n} f) (\xi) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_{0}^{\infty} (u - 1)^{\alpha-1} u^{-(\delta + \alpha)} f(\xi u^{\alpha}) du, & \alpha > 0, \\ f(\xi), & \alpha = 0, \end{cases}$$

is the Erdelyi-Kober fractional integral operator (EK-FIO).

**Proof** Let $n - 1 < \alpha < n$, $n = 1, 2, 3, ...$. Then the RL fractional derivative for the similarity transformation of Eq. (19) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[ \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t - s)^{n-\alpha-1} s^{\frac{\alpha}{\delta}} f(xs^{\frac{\beta}{\alpha}}) ds \right].$$

Let $v = \frac{1}{s}, ds = -\frac{1}{v^2} dv$. Thus, Eq. (24) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[ \frac{1}{\Gamma(n-\alpha)} \int_{1}^{\infty} (v - 1)^{n-\alpha-1} v^{-(\alpha+1)} \left(\frac{\beta}{\delta} f(\xi v^{\alpha})\right) dv \right].$$

Applying EK-FDO Eq. (23) in Eq. (25), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[ \frac{1}{\Gamma(n-\alpha)} \left(\frac{\beta}{\delta} f(\xi)\right) (\xi) \right].$$
In order to simplify the right hand side of Eq. (26), let us consider the relation
\[ \xi = xt^{-\frac{\alpha}{2}}, \varphi \in (0, \infty), \]
we acquire
\[ t \frac{\partial}{\partial t} \varphi(\xi) = t x (\frac{-\alpha}{2}) t^{-\frac{\alpha}{2} - 1} \varphi(\xi) = -\frac{\alpha}{2} x \xi \frac{\partial}{\partial \xi} \varphi(\xi). \quad (27) \]

Hence,
\[ \frac{\partial^n}{\partial t^n} \left[ t^n - \frac{\alpha(1 + 2\delta)}{2\delta} \left( K^{1-\frac{\alpha}{2}, n-\alpha} f \right)(\xi) \right] \]
\[ = \frac{\partial^{n-1}}{\partial t^{n-1}} \left[ \frac{\partial}{\partial t} \left( t^{n-\frac{\alpha(1+2\delta)}{2\delta}} \left( K^{1-\frac{\alpha}{2}, n-\alpha} f \right)(\xi) \right) \right] \]
\[ = \frac{\partial^{n-1}}{\partial t^{n-1}} \left[ t^{n-\frac{\alpha(1+2\delta)}{2\delta} - 1} \left( n - \frac{\alpha}{2\delta} + \frac{\alpha}{2} \xi \frac{\partial}{\partial \xi} \left( K^{1-\frac{\alpha}{2}, n-\alpha} f \right)(\xi) \right) \right]. \quad (28) \]

Processing repeatedly for \( n - 1 \) times, we get
\[ \frac{\partial^n}{\partial t^n} \left[ t^n - \frac{\alpha(1 + 2\delta)}{2\delta} \left( K^{1-\frac{\alpha}{2}, n-\alpha} f \right)(\xi) \right] \]
\[ = t^{-\frac{\alpha(1+2\delta)}{2\delta} - 1} \left( 1 - \frac{\alpha(1 + 2\delta)}{2\delta} + j \frac{\alpha}{2\delta} \xi \frac{\partial}{\partial \xi} \left( K^{1-\frac{\alpha}{2}, n-\alpha} f \right)(\xi) \right). \quad (29) \]

Applying EKF-FDO Eq. (21) in Eq (29), we have
\[ \frac{\partial^n u}{\partial t^n} \left[ t^{n-\frac{\alpha(1+2\delta)}{2\delta}} \left( K^{1-\frac{\alpha}{2}, n-\alpha} f \right)(\xi) \right] = t^{-\frac{\alpha(1+2\delta)}{2\delta} - 1} \left( p^{1-\frac{\alpha(1+2\delta)}{2\delta}} \alpha f \right)(\xi). \quad (30) \]

Using Eq. (30) into Eq. (26), we obtain
\[ \frac{\partial^n u}{\partial t^n} = t^{-\frac{\alpha(1+2\delta)}{2\delta} - 1} \left( p^{1-\frac{\alpha(1+2\delta)}{2\delta}} \alpha f \right)(\xi). \quad (31) \]

Thus Eq. (13) becomes
\[ \left( p^{1-\frac{\alpha(1+2\delta)}{2\delta}} \alpha f \right)(\xi) + \beta f^4 f_\xi - f \xi \xi - \gamma f(1 - f^3) = 0. \quad (32) \]

### 5 Explicit power series solutions

In this section, we obtain the power series method for the resultant equation
Eq. (32) which has the arbitrary nonlinearity \( \delta \). Furthermore, we analyze
Substituting Eqs. (33), (34) and (35) into Eq. (32), we get

\[ f' = \sum_{n=0}^{\infty} (n+1)b_n \xi^n, \]  
\[ f'' = \sum_{n=0}^{\infty} (n+2)(n+1)b_n \xi^n. \]

Comparing coefficients in Eq. (37), when \( n = 0 \), we have

\[ f(\xi) = \sum_{n=0}^{\infty} b_n \xi^n, \]  
\[ f'(\xi) = \sum_{n=0}^{\infty} (n+1)b_n \xi^n, \]  
\[ f''(\xi) = \sum_{n=0}^{\infty} (n+2)(n+1)b_n \xi^n. \]

Substituting Eqs. (33), (34) and (35) into Eq. (32), we get

\[ \sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{\alpha(1+25)}{25})}{\Gamma(2 - \frac{\alpha(1+25)+\frac{na}{2}}{25} + \alpha)} b_n \xi^n + \beta \left( \sum_{n=0}^{\infty} b_n \xi^n \right) \delta \sum_{n=0}^{\infty} (n+1)b_n \xi^n \]

\[ - \sum_{n=0}^{\infty} (n+2)(n+1)b_n \xi^n - \gamma \sum_{n=0}^{\infty} b_n \xi^n + \gamma \left( \sum_{n=0}^{\infty} b_n \xi^n \right)^{\delta+1} = 0. \]  

Comparing coefficients in Eq. (37), when \( n = 0 \), we have

\[ b_2 = \frac{1}{2} \left[ \frac{\Gamma(2 - \frac{\alpha(1+25)}{25})}{\Gamma(2 - \frac{\alpha(1+25)+\frac{na}{2}}{25} + \alpha)} b_0 + \beta b_1 \xi b_0 - \gamma b_0 + \gamma \xi^{\delta+1} \right]. \]  

When \( n \geq 1 \), we have the recurrence relations among the coefficients become

\[ b_{n+2} = \frac{1}{(n+1)(n+2)} \sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{\alpha(1+25)}{25})}{\Gamma(2 - \frac{\alpha(1+25)+\frac{na}{2}}{25} + \alpha)} b_n \]

\[ + \beta \left( \sum_{k_1=0}^{k_1} \sum_{k_2=0}^{k_2} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-1}} b_{k_1} b_{k_2} \cdots b_{k_{\delta-1}} b_{n-k_1-k_2-\cdots-k_{\delta-1}} \right) \]
The power series solution of Eq.(32) can be written in the form:

\[ b_{k_1-k_2}(n-k_1+1)b_{(n-k_1+1)} - \gamma b_n + \gamma \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_{k_1-1}=0}^{k_{k_2-1}} \sum_{k_{k_2-1}=0}^{k_{k_2-k_1}} b_{k_1} b_{k_{k_1-1}-k_1} b_{k_{k_2-1}-k_2} \right) \cdot (39) \]

The power series solution of Eq.(32) can be written in the form:

\[ f(\xi) = b_0 + b_1 \xi + \sum_{n=0}^{\infty} b_{n+2} \xi^{n+2} \]

\[ = b_0 + b_1 \xi + \sum_{n=0}^{\infty} \left( \frac{1}{(n+1)(n+2)} \right) \left[ \sum_{n=0}^{\infty} \Gamma(2 - \frac{\alpha(1+3\delta)}{2\delta} + \frac{an}{2} + \alpha) b_n \right. \]

\[ + \left. \beta \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_{k_1-1}=0}^{k_{k_2-1}} \sum_{k_{k_2-1}=0}^{k_{k_2-k_1}} b_{k_1} b_{k_{k_1-1}-k_1} b_{k_{k_2-1}-k_2} \right) \right] \cdot (40) \]

Consequently, we acquire exact power series solution of Eq.(32) as follows:

\[ u(x, t) = b_0 t^{-\alpha} + b_1 x t^{-\alpha} \]

\[ + \sum_{n=0}^{\infty} \left( \frac{1}{(n+1)(n+2)} \right) \left[ \sum_{n=0}^{\infty} \Gamma(2 - \frac{\alpha(1+3\delta)}{2\delta} + \frac{an}{2} + \alpha) b_n \right. \]

\[ + \left. \beta \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_{k_1-1}=0}^{k_{k_2-1}} \sum_{k_{k_2-1}=0}^{k_{k_2-k_1}} b_{k_1} b_{k_{k_1-1}-k_1} b_{k_{k_2-1}-k_2} \right) \right] x^{n+2} t^{-\frac{\alpha(n+3)}{2}} \cdot (41) \]
6 Convergence analysis

In this section, convergence of the power series solution of Eq. (41) will be presented. Consider Eq. (42), we can write

\[
|b_{n+2}| \leq \left\{ \frac{|\Gamma(2 - \frac{\alpha(1+2\beta)}{2\delta} + \frac{m\alpha}{2})|}{|\Gamma(2 - \frac{\alpha(1+2\beta)}{2\delta} + \frac{m\alpha}{2} + \alpha)|} |b_n| + |\beta| \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} |b_{k_{\delta}}| \right) \\
|b_{k_{\delta-1} - k_{\delta}}| \sum_{i=1}^{k_{\delta-1}} \cdots \sum_{i=1}^{k_{\delta-2}} \sum_{i=1}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \right) \\
+ |\gamma| \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \right) \right\}.
\]

(42)

It is well known from the properties of \( \Gamma \), it is easily found that

\[
\frac{|\Gamma(2 - \frac{\alpha(1+2\beta)}{2\delta} + \frac{m\alpha}{2} + \alpha)|}{|\Gamma(2 - \frac{\alpha(1+2\beta)}{2\delta} + \frac{m\alpha}{2})|} < 1 \text{ for arbitrary } n.
\]

Thus Eq. (42) can be written as

\[
|b_{n+2}| \leq M \left\{ |b_n| + \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \right) \\
|b_{k_{\delta-1} - k_{\delta}}| \sum_{i=1}^{k_{\delta-1}} \cdots \sum_{i=1}^{k_{\delta-2}} \sum_{i=1}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \right) \\
+ |\gamma| \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} |b_{k_{\delta}}| |b_{k_{\delta-1} - k_{\delta}}| \right) \right\}
\]

(43)

where \( M = \max\{|1 - |\gamma||, |\gamma|, |\beta|\} \).

Consider another power series of the form

\[
G(\xi) = \sum_{n=0}^{\infty} q_n \xi^n.
\]

(44)

Let \( q_i = |b_i|, i = 0, 1, 2 \). Then we have

\[
q_{n+2} \leq M \left\{ q_n + \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} q_{k_{\delta}} q_{k_{\delta-1} - k_{\delta}} \right) \right. \\
q_{k_{\delta-1} - k_{\delta}} \cdots q_{k_2 - k_1} q_{k_1 - k_2} q_{k_2 - k_1} \left( q_{n-k_{\delta}+1} \right) \\
+ \left( \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\delta-1}=0}^{k_{\delta-2}} \sum_{k_{\delta}=0}^{k_{\delta-1}} q_{k_{\delta}} q_{k_{\delta-1} - k_{\delta}} \right) \right\}
\]

(45)
\[ q_{k_2-2} - q_{k_2-1} \cdots q_{k_2} q_{k_1} - q_{k_2} q_{n-k_1} \} \text{.} \tag{45} \]

Therefore it is easily seen that \(|q_n| \leq b_n, n = 0, 1, 2, \ldots\).

On the other hand, the series \( G(\xi) = \sum_{n=0}^{\infty} q_n \xi^n \) is majorant series of Eq. (42). We next show that the series \( G(\xi) \) has positive radius of convergence. By simple calculation, we have that

\[
G(\xi) = q_0 + q_1 \xi + M \left\{ \sum_{n=0}^{\infty} q_n + \left( \sum_{n=0}^{\infty} \sum_{k_1=0}^{k_2} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_n} q_{k_2} q_{k_1} - q_{k_2} q_{n-k_1} \right) + \left( \sum_{n=0}^{\infty} \sum_{k_1=0}^{k_2} \sum_{k_2=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_n} q_{k_2} q_{k_1} - q_{k_2} q_{n-k_1} \right) \right\} \xi^{n+2} \text{.} \tag{46} \]

Consider an implicit functional system with respect to the independent variable \( \xi \) as follows:

\[
G(\xi, G) = G - q_0 - q_1 \xi - M \left\{ \xi^2 G + \xi G^2 (G - q_0) + \xi^2 G^3 + 1 \right\} \text{.} \tag{47} \]

Since \( G \) is analytic in a neighbourhood of \((0, q_0)\) where \( G(0, q_0) = 0 \) and \( \frac{\partial G}{\partial \xi} (0, q_0) \neq 0 \). Then by implicit function theorem [2], one can see that the series \( G(\xi) = \sum_{n=0}^{\infty} q_n \xi^n \) is analytic in neighbourhood of the point \((0, q_0) = 0 \) and with a positive radius. This implies that Eq. (41) converges in a neighbourhood of the point \((0, q_0) = 0 \).

7 Conclusion

In this paper, to generate all symmetries, we applied Lie group method for the time fractional generalized Burgers-Fisher equation using RL fractional derivative. Using symmetries, we reduced the time fractional generalized Burgers-Fisher equation to the nonlinear fractional ODE. Then the power series has been obtained to get an explicit solution for the nonlinear fractional ODE of the generalized Burgers-Fisher equation with arbitrary nonlinearity and analyzed the convergence also. To illustrate the physical meaning of the exact solution, some plots are given with suitable parameters values.

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