Four questions for quantum-classical hybrid theory

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Abstract. Four questions are discussed which may be addressed to any proposal of a quantum-classical hybrid theory which concerns the direct coupling of classical and quantum mechanical degrees of freedom. In particular, we consider the formulation of hybrid dynamics presented recently in Ref. [1]. This linear theory differs from the nonlinear ensemble theory of Hall and Reginatto, but shares with it to fulfill all consistency requirements discussed in the literature, while earlier attempts failed. – Here, we additionally ask: Does the theory allow for superposition, separable, and entangled states originating in the quantum mechanical sector? Does it allow for “Free Will”, as introduced, in this context, by Diósi [2]? Is it local? Does it provide hints for the emergence of quantum mechanics from dynamics beneath?

PACS numbers: 03.65.Ca, 03.65.Ta

1. Introduction
The hypothetical direct coupling of quantum mechanical and classical degrees of freedom – “hybrid dynamics” – departs from quantum mechanics. It has been researched at length for practical as well as theoretical reasons. In particular, the standard Copenhagen interpretation entails the unresolved measurement problem which, together with the fact that quantum mechanics needs interpretation, in order to be operationally well defined, may indicate that it deserves amendments. It has been recognized early on that a theory which dynamically bridges the quantum-classical divide should have an impact on the measurement problem [3] as well as on attempts to describe consistently the interaction between quantum matter and classical spacetime [4].

Numerous works have appeared, in order to formulate hybrid dynamics in a satisfactory way. Generally, they showed deficiencies in one or another respect. Which has led to various no-go theorems in view of a list of desirable properties or consistency requirements, see, for example, Refs. [5, 6]:

- Conservation of energy.
- Conservation and positivity of probability.
- Separability of quantum and classical subsystems in the absence of their interaction, recovering the correct quantum and classical equations of motion.
- Consistent definitions of states and observables; existence of a Lie bracket structure on the algebra of observables that suitably generalizes Poisson and commutator brackets.

Main parts of author’s talk at “Heinz von Foerster Congress / Emergent Quantum Mechanics” (Vienna, Nov. 2011) are in Ref. [1]; questions are considered here which were raised afterwards or in related correspondence.
• Existence of canonical transformations generated by the observables; invariance of the classical sector under canonical transformations performed on the quantum sector only and vice versa.

• Existence of generalized Ehrenfest relations (i.e. the correspondence limit) which, for bilinearly coupled classical and quantum oscillators, are to assume the form of the classical equations of motion (“Peres-Terno benchmark” test [7]).

These issues have been reviewed in recent works by Hall and Reginatto where they have introduced the first theory of hybrid dynamics that conforms with the points listed above [8, 9, 10]. Their ensemble theory is based on configuration space, which requires a certain nonlinearity of the action functional from which it is derived and entails effects that might allow to falsify this proposal experimentally.

We have proposed an alternative theory of hybrid dynamics based on notions of phase space [1]. This is partly motivated by work on related topics of general linear dynamics and classical path integrals [11, 12] and extends work by Heslot, who demonstrated that quantum mechanics can entirely be rephrased in the language and formalism of classical analytical mechanics [13]. Introducing unified notions of states on phase space, observables, canonical transformations, and a generalized quantum-classical Poisson bracket, this has led to an intrinsically linear hybrid theory, which fulfills the above consistency requirements.

Objects that somehow reside between classical and quantum mechanics have been described recently also in a statistical theory, based on very different premises than the hybrid theories considered here [14]. It would be interesting to uncover any relation, if it exists.

In any case, it must be emphasized that it is also of great practical importance to better understand quantum-classical hybrids appearing in quantum mechanical approximation schemes. These typically address many-body systems or interacting fields, which are naturally separable into quantum and classical subsystems, for example, representing fast and slow degrees of freedom (keywords: Born-Oppenheimer approximation, mesoscopic systems, “semiclassical quantum gravity”); for references see Ref. [1].

Furthermore, concerning the hypothetical emergence of quantum mechanics from some coarse-grained deterministic dynamics (see, for example, Refs. [15, 16, 17] with numerous references to earlier work), the quantum-classical backreaction problem seems to appear in a new form, namely regarding the interplay of fluctuations among underlying deterministic and emergent quantum mechanical degrees of freedom. Which can be rephrased succinctly as the question: “Can quantum mechanics be seeded?”

Besides constructing the quantum-classical hybrid formalism and showing how it conforms with the above consistency requirements, we discussed the possibility to have classical-environment induced decoherence, quantum-classical backreaction, a deviation from the Hall-Reginatto proposal in presence of translation symmetry, and closure of the algebra of hybrid observables [1].

Presently, we add four questions to accompany the above consistency requirements:

• Does the hybrid theory allow for the superposition principle, for separable and for entangled states originating in the quantum mechanical sector?

• Does it allow for “Free Will”, a notion introduced, in this context, by Diósi [2]?

• Does it describe a local interaction of classical and quantum mechanical subsystems which evolve in spacetime?

• Does it provide hints concerning the study of emergence of quantum mechanics from deterministic dynamics beneath?

which may be addressed to any model intended to describe quantum-classical hybrids.
The remainder of the paper is organized as follows. In Section 2., we collect some of the results of Ref. [1], which will be useful in the following. In Sections 3. to 6., we discuss, if not answer, the questions just listed with respect to the hybrid theory proposed in Ref. [1], followed by concluding remarks in Section 7.

2. Linear quantum-classical hybrid dynamics – a summary

In the following two subsections, we will present some important results drawn from the exposition of classical Hamiltonian mechanics and its generalization incorporating quantum mechanics by Heslot [13]. This will form the starting point of our brief review of the hypothetical direct coupling between quantum and classical degrees of freedom, as developed in Ref. [1]; the reader familiar with the earlier derivations may directly pass to Section 3.

2.1. Classical mechanics

The evolution of a classical object is described with respect to its $2n$-dimensional phase space, which is identified as its state space. A real-valued regular function on the state space defines an observable, i.e., a differentiable function on this smooth manifold.

Darboux’s theorem shows that there always exist (local) systems of so-called canonical coordinates, commonly denoted by $(x_k, p_k)$, $k = 1, \ldots, n$, such that the Poisson bracket of any pair of observables $f, g$ assumes the standard form [18]:

$$\{f, g\} = \sum_k \left( \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial x_k} \right). \tag{1}$$

This is consistent with $\{x_k, p_l\} = \delta_{kl}$, $\{x_k, x_l\} = \{p_k, p_l\} = 0$, $k, l = 1, \ldots, n$, and reflects the bilinearity, antisymmetry, derivation-like product formula, and Jacobi identity which define a Lie bracket operation, $f, g \rightarrow \{f, g\}$, mapping a pair of observables to an observable.

Compatibility with the Poisson bracket structure restricts general transformations $G$ of the state space to so-called canonical transformations which do not change the physical properties of the object under study; e.g., a translation, a rotation, a change of inertial frame, or evolution in time. Such $G$ induces a change of an observable, $f \rightarrow G(f)$, and is an automorphism of the state space compatible with its Poisson bracket structure. Due to the Lie group structure of the set of canonical transformations, it is sufficient to consider infinitesimal transformations generated by the elements of the corresponding Lie algebra. Then, an infinitesimal transformation $G$ is canonical, if and only if for any observable $f$ the map $f \rightarrow G(f)$ is given by $f \rightarrow f' = f + \{f, g\} \delta \alpha$, with some observable $g$, the so-called generator of $G$, and $\delta \alpha$ an infinitesimal real parameter.

Thus, for the canonical coordinates, in particular, an infinitesimal canonical transformation amounts to:

$$x_k \rightarrow x'_k = x_k + \frac{\partial g}{\partial p_k} \delta \alpha, \tag{2}$$

$$p_k \rightarrow p'_k = p_k - \frac{\partial g}{\partial x_k} \delta \alpha, \tag{3}$$

employing the Poisson bracket given in Eq. (1).

This analysis shows the fundamental relation between observables and generators of infinitesimal canonical transformations in classical Hamiltonian mechanics.

2.2. Quantum mechanics

An important achievement of Heslot’s work is the realization that the previous analysis can be generalized and applied to quantum mechanics; this concerns the dynamical aspects as well as the notions of states, canonical transformations, and observables.
To begin with, we recall that the Schrödinger equation and its adjoint can be derived from a well-known action principle [1]. Which shows immediately that they have the character of Hamiltonian equations of motion.

To this must be added the normalization condition for the state vector $|\Psi\rangle$:

$$C := \langle \Psi(t) | \Psi(t) \rangle \overset{!}{=} \text{constant} \equiv 1 \ ,$$

which is an essential ingredient of the probability interpretation associated with state vectors. Adding here that state vectors that differ by an unphysical constant phase are to be identified, we are reminded that the quantum mechanical state space is formed by the rays of an underlying Hilbert space, i.e., forming a complex projective space.

### 2.2.1. The oscillator representation

Quantum mechanical evolution can be described by a unitary transformation, $|\Psi(t)\rangle = \hat{U}(t - t_0) |\Psi(t_0)\rangle$, with $U(t - t_0) = \exp[-i\hat{H}(t - t_0)]$, which formally solves the Schrödinger equation. It follows immediately that a stationary state, i.e., $|\psi_i(t)\rangle = \exp[-iE_i(t - t_0)]|\psi_i(t_0)\rangle \equiv \exp[-iE_i(t - t_0)]|\phi_i\rangle$. Henceforth, we assume a denumerable set of eigenstates of the Hamilton operator.

With the Hamiltonian character of the underlying equation(s) of motion, the harmonic motion suggests to introduce what we called oscillator representation. We consider the expansion of any state vector with respect to a complete orthonormal basis, $\{|\Phi_i\rangle\}$:

$$|\Psi\rangle = \sum_i |\Phi_i\rangle (X_i + iP_i)/\sqrt{2} \ ,$$

where the generally time dependent expansion coefficients are written in terms of real and imaginary parts, $X_i, P_i$. Employing this expansion, allows to evaluate what will serve as the Hamiltonian function, i.e., $\hat{H} := \langle \Psi | \hat{H} | \Psi \rangle$:

$$\hat{H} = \frac{1}{2} \sum_{i,j} \langle \Phi_i | \hat{H} | \Phi_j \rangle (X_i - iP_i)(X_j + iP_j) =: \hat{H}(X_i, P_i) \ .$$

Choosing especially the set of energy eigenstates, $\{|\phi_i\rangle\}$, as basis, we obtain:

$$\hat{H}(X_i, P_i) = \sum_i E_i (P_i^2 + X_i^2) \ ,$$

hence the name oscillator representation. The reasoning leading to this result indicates that $(X_i, P_i)$ can play the role of canonical coordinates in the description of a quantum mechanical object and its evolution with respect to the state space.

This interpretation is substantiated by the fact that the Schrödinger equation is recovered by evaluating $|\Psi\rangle = \sum_i |\Phi_i\rangle (X_i + iP_i)/\sqrt{2}$ according to Hamilton’s equations of motion, using the Hamiltonian function of Eq. (6) or (7). Furthermore, the constraint of Eq. (4) becomes:

$$C(X_i, P_i) = \frac{1}{2} \sum_i (X_i^2 + P_i^2) \overset{!}{=} 1 \ .$$

Thus, the vector with components given by the canonical coordinates $(X_i, P_i)$, $i = 1, \ldots, N$, is constrained to the surface of a $2N$-dimensional sphere with radius $\sqrt{2}$. This constraint presents a major difference to classical Hamiltonian mechanics.
Similarly as in Subsection 2.1., it is natural to introduce also here a **Poisson bracket** for any two observables on the **spherically compactified state space**, i.e. real-valued regular functions \(F,G\) of the coordinates \((X_i, P_i)\):

\[
\{F,G\} = \sum_i \left( \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial X_i} \right) .
\]

Then, the Hamiltonian acts as the generator of time evolution of any observable \(O\):

\[
\frac{dO}{dt} = \partial_t O + \{O, \mathcal{H}\} ,
\]

and one verifies that the constraint, Eq. (8), is conserved under the Hamiltonian flow:

\[
\frac{dC}{dt} = \{C, \mathcal{H}\} = 0 .
\]

It remains to demonstrate the **compatibility** of the notion of **observable** introduced here – as in classical mechanics – with the one adopted in quantum mechanics. This concerns, in particular, the implementation of **canonical transformations** and the role of observables as their generators.

### 2.2.2. Canonical transformations and quantum observables

The Hamiltonian function has been introduced as observable in the Eq. (6) which provides a direct relation to the corresponding quantum observable, namely the expectation value of the self-adjoint Hamilton operator. This is indicative of the general structure to be discussed now.

Referring to Refs. [1, 13] for all details, we summarize here the main points:

- **A) Compatibility of unitary transformations and Poisson structure.** – The canonical transformations discussed in Section 2.1. represent automorphisms of the classical state space which are compatible with the Poisson brackets. In quantum mechanics automorphisms of the Hilbert space are implemented by unitary transformations. This implies a transformation of the canonical coordinates here, *i.e.*, of the expansion coefficients \((X_i, P_i)\) introduced in Eq. (5). Analysing this, one finds that the fundamental Poisson brackets remain invariant under unitary transformations. Thus, **unitary transformations on Hilbert space are canonical transformations on the** \((X, P)\) **state space.**

- **B) Self-adjoint operators as observables.** – Any infinitesimal unitary transformation \(\hat{U}\) can be generated by a self-adjoint operator \(\hat{G}\), such that:

\[
\hat{U} = 1 - i\hat{G}\delta\alpha ,
\]

which will lead to the quantum mechanical relation between an observable and a self-adjoint operator. In fact, straightforward calculation shows that in the present case we have:

\[
X_i \rightarrow X'_i = X_i + \frac{\partial \langle \Psi | \hat{G} | \Psi \rangle}{\partial P_i} \delta\alpha ,
\]

\[
P_i \rightarrow P'_i = P_i - \frac{\partial \langle \Psi | \hat{G} | \Psi \rangle}{\partial X_i} \delta\alpha .
\]

Then, the relation between an observable \(G\), defined in analogy to Section 2.1., and a self-adjoint operator \(\hat{G}\) can be inferred from Eqs. (13)–(14) within a few steps:

\[
G(X_i, P_i) = \langle \Psi | \hat{G} | \Psi \rangle ,
\]
by comparison with the classical result. We find that a real-valued regular function $G$ of the state is an observable, if and only if there exists a self-adjoint operator $\hat{G}$ such that Eq. (15) holds. This implies that all quantum observables are quadratic forms in the $X_i$’s and $P_i$’s, which are essentially fewer than in the corresponding classical case; see, however, Subsection 3.2 for the generalization necessitated by interacting quantum-classical hybrids.

- C) Commutators as Poisson brackets. – Relation (15) between observables and self-adjoint operators together with the Poisson bracket (9) allow to demonstrate the important result:

$$\{F,G\} = \langle \Psi | \frac{1}{i} [\hat{F}, \hat{G}] | \Psi \rangle ,$$

(16)

with both sides of the equality considered as functions of the variables $X_i, P_i$ and with the commutator defined as usual. This shows that the commutator is a Poisson bracket with respect to the $(X, P)$ state space and relates the algebra of observables, in the sense of the classical construction of Section 2.1., to the algebra of self-adjoint operators in quantum mechanics.

- D) Normalization, phase arbitrariness, and admissible observables. – Coming back to the normalization condition $\langle \Psi | \Psi \rangle \equiv 1$, which compactifies the state space, cf. the constraint Eq. (8), it must be preserved under infinitesimal canonical transformations, since it belongs to the structural characteristics of the state space. This leads to a constraint on admissible observables, which turns out to be compatible with the requirement that observables $G$ are invariant under an infinitesimal phase transformation $|\Psi\rangle \rightarrow |\Psi\rangle \cdot \exp(i\delta \theta)$, with constant $\delta \theta$. Conversely, assuming this phase invariance of observables, we recover that Hilbert space vectors differing by a constant phase are indistinguishable and represent the same physical state.

We note that any observable $G$ with an expansion as in Eq. (6) automatically satisfies the invariance requirements of item D). Explicit calculation shows:

$$\{\mathcal{C}, G\} = \sum_j \left( \frac{\partial G}{\partial P_j} X_j - \frac{\partial G}{\partial X_j} P_j \right) = 0 ,$$

(17)

assuming that:

$$G(P_i, X_i) = \langle \Psi | \hat{G} | \Psi \rangle = \frac{1}{2} \sum_{i,j} G_{ij} (X_i - iP_i)(X_j + iP_j) ,$$

(18)

and where $G_{ij} = \langle \Phi_i | \hat{G} | \Phi_j \rangle = G_{ji}^*$, for a self-adjoint operator $\hat{G}$.

In conclusion, quantum mechanics shares with classical mechanics an even dimensional state space, a Poisson structure, and a related algebra of observables. Yet it differs essentially by a restricted set of observables and the requirements of phase invariance and normalization, which compactify the underlying Hilbert space to the complex projective space formed by its rays.

2.3. Quantum-classical Poisson bracket and separability

The result of aligning classical and quantum mechanics in the way summarized above suggests to introduce a generalized Poisson bracket for observables defined on the Cartesian product state space of CL (classical) and QM (quantum mechanical) sectors of a hybrid:

$$\{A, B\}_\times := \{A, B\}_{\text{CL}} + \{A, B\}_{\text{QM}}$$

$$:= \sum_k \left( \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial x_k} \right) + \sum_i \left( \frac{\partial A}{\partial X_i} \frac{\partial B}{\partial P_i} - \frac{\partial A}{\partial P_i} \frac{\partial B}{\partial X_i} \right) ,$$

(20)

for any two observables $A, B$. It is bilinear and antisymmetric, leads to a derivation-like product formula and obeys the Jacobi identity.
Let us say an observable “belongs” to the CL (QM) sector, if it is constant with respect to the canonical coordinates of the QM (CL) sector. Then, the generalized Poisson bracket has the additional important properties:

- It reduces to the Poisson brackets introduced in Eqs. (1) and (9), respectively, for pairs of observables that belong either to the CL or the QM sector.
- It reduces to the appropriate one of the former brackets, if one of the observables belongs only to either one of the two sectors.
- It reflects the separability of CL and QM sectors, since \( \{ A, B \}_x = 0 \), if \( A \) and \( B \) belong to different sectors.

The physical relevance of separability is this: If a canonical transformation is performed on the QM (CL) sector only, then all observables that belong to the CL (QM) sector remain invariant.

3. Superposition, separable and entangled states from the QM sector?

Whether our description of quantum-classical hybrids incorporates the possibility of superposition, separable, and entangled states originating in the QM sector can be answered exactly, in the absence of a hybrid coupling, as we have shown [1]. This continues to hold, as long as the dynamical variables pertaining to the CL sector can be considered as external parameters for the QM sector. However, further comment is necessary when the QM-CL hybrid is fully dynamical and we will come to this in due course.

3.1. Definition and interpretation of hybrid states

It may be useful to look from a different angle at the question how different kinds of quantum states enter the description of hybrids. We recall that the hybrid density \( \rho \) has been introduced in Ref. [1] as the expectation with respect to a given state vector of a self-adjoint, positive semi-definite, trace normalized density operator \( \hat{\rho} \):

\[
\rho(x_k, p_k; X_i, P_i) := \langle \Psi | \hat{\rho}(x_k, p_k) | \Psi \rangle = \frac{1}{2} \sum_{i,j} \rho_{ij}(x_k, p_k)(X_i - iP_j)(X_j + iP_j),
\]

using the oscillator expansion, Eq. (5), and \( \rho_{ij}(x_k, p_k) := \langle \Phi_i | \hat{\rho}(x_k, p_k) | \Phi_j \rangle = \rho_{ij}^\ast(x_k, p_k) \).

It describes a quantum-classical hybrid ensemble by a real-valued, positive semi-definite, normalized, and possibly time dependent regular function, the probability distribution \( \rho \), on the Cartesian product state space canonically coordinated by \( 2(n + N) \)-tuples \((x_k, p_k; X_i, P_i); \) the variables \( x_k, p_k, k = 1, \ldots, n \) and \( X_i, P_i, i = 1, \ldots, N \) are reserved for the CL and QM sector, respectively.

Furthermore, expanding \( \hat{\rho} \) in terms of its eigenstates, \( \hat{\rho} = \sum_j w_j |j\rangle \langle j| \), one obtains:

\[
\rho(x_k, p_k; X_i, P_i) = \sum_j w_j(x_k, p_k) \text{Tr}(|j\rangle \langle j|) \langle \Psi | \Psi \rangle |j\rangle \langle j|)
\]

\[
= \sum_j w_j(x_k, p_k) |\langle j| \Psi \rangle|^2,
\]

with \( 0 \leq w_j \leq 1 \) and \( \sum_j |\langle j| \Psi \rangle|^2 = 1 \). This suggests that \( \rho(x_k, p_k; X_i, P_i) \), when properly normalized, is the probability density to find in the hybrid ensemble the QM state \( |\Psi\rangle \), parametrized by \( X_i, P_i \) through Eq. (5), together with the CL state given by a point in phase space, specified by the coordinates \((x_k, p_k)\).
Clearly, this interpretation does not depend on whether \( \hat{\rho} \) stands for any pure state, which might be a coherent superposition of other pure states in the Hilbert space of the QM sector, or for a mixed state, i.e., a probabilistic superposition of pure states.

Note, in particular, how a probabilistic mixture of two densities, \( \rho := p\rho^{(1)} + (1 - p)\rho^{(2)} \), with \( 0 \leq p \leq 1 \), can be represented according to Eq. (23):

\[
\rho(x_k, p_k; X_i, P_i) = \sum_j \left( p\omega_j(x_k, p_k) + (1 - p)\omega_j(x_k, p_k) \right) |\langle j|\Psi \rangle|^2 ,
\]

which is consistent with the interpretation given. – It must be emphasized that our definition of the density, Eq. (21), is not suitable to ask questions like “What is the probability to find in the hybrid ensemble a given QM mixed state?” It does not contain sufficient information for such, more general purposes.

However, the observations made allow us to address the cases of separable and entangled states relevant for a composite quantum system as part of a hybrid. For simplicity, we treat a bi-partite QM sector; the generalization for multi-partite systems follows by induction. – Generally, two QM systems A and B are defined to be separable, if their state can be prepared as a statistical mixture of tensor product states [19]:

\[
\rho^{(AB)} = \sum_l w_l \rho_l^{(A)} \otimes \rho_l^{(B)} , \quad \text{with} \quad \sum_l w_l = 1 , \quad w_l \geq 0 .
\]

If this is not the case, A and B are said to be in an entangled state. If A and B are separable, then there exist at most classical but no quantum correlations between them. For a pure state, this is equivalent to the statement that it is separable if and only if it is of the form \( |\Psi^{(AB)} \rangle = |\Psi^{(A)} \rangle \otimes |\Psi^{(B)} \rangle \); here, the sum in Eq. (25) has only one nonvanishing term.

Considering separable states, generally, we can adapt Eqs. (22)–(23) to the case at hand by expanding each one of the self-adjoint factors \( \hat{\rho}_l^{(A)} \) and \( \hat{\rho}_l^{(B)} \) in the definition, Eq. (25), in terms of its eigenstates, e.g., \( \hat{\rho}_l^{(A)} = \sum_{i,j} w_{ij}^{(A)} |A; ji \rangle \langle A; ji| \). Inserting these expansions, the Eq. (23) is replaced by:

\[
\rho(x_k, p_k; X_i, P_i) = \sum_{i,j,j'} w_{ij}^{(A)}(x_k, p_k) w_{j'i'}^{(B)}(x_k, p_k) |\langle j'i'|\Psi \rangle|^2
\]

\[
= \sum_{j,j'} w_{ij}^{(A)}(x_k, p_k) w_{j'i'}^{(B)}(x_k, p_k) |\langle j'|\Psi \rangle|^2 ,
\]

where \( |ji',j \rangle := |A; ji \rangle \otimes |B; j' \rangle \), and the second equality holds for an overall pure state with only one term contributing to the sum over \( l \), hence \( l \)-dependence is suppressed; note that each one of the weights \( w \) has the meaning of a probability. The interpretation here is an obvious generalization of the one following Eq. (23).

Next, we distinguish pure and mixed entangled states of the QM composite system. – The case of pure entangled states can be easily fit into the above considerations. To the bi-partite (finite-dimensional) Hilbert space, in which a pure state has a given tensor product structure, corresponds a factorized algebra of QM observables. It has been shown recently that this situation is unitarily equivalent to one where the Hilbert space is unstructured, provided the algebra of observables is suitably rearranged (“Tailored Observables Theorem”) [20]. Thus, the tensor product structure can be effectively eliminated and the entanglement properties shifted to the observables, and vice versa. With an unstructured Hilbert space, we are back to the situation discussed in the context of Eqs. (22)–(23). In this way, in our formulation, pure entangled states from the QM sector can naturally occur in a QM-CL hybrid.
Mixed states present a more subtle situation. This is immediately obvious if one looks at the limiting case of a totally mixed state, described by \( \hat{\rho} = N^{-1} \sum_{i=1}^{N} |\Phi_i\rangle \langle \Phi_i| \equiv N^{-1} 1_N \), where \( \{\Phi_i\} \) stands for any complete orthonormal basis of the N-dimensional Hilbert space of the composite. By definition, cf. Eq. (25), this state is separable for all (factorized) representations, and thus can be handled as before. By continuity, this holds for states in a sufficiently small neighbourhood of the totally mixed state. However, it turns out that the possibility (or not) to switch between entangled and separable forms of the state, by tailoring the algebra of observables, which we needed for our above arguments, still cannot be ascertained for mixed states in all generality. Criteria to assess this situation are presently under investigation, see Ref. [21], with references to earlier work therein. In conclusion, for mixed entangled states an intuitive interpretation of the hybrid density, along the lines sketched above, is still missing. Yet we are confident that this will work out eventually.

### 3.2. Evolution of hybrid densities

Turning to dynamics, we proposed a straightforward generalization of Hamiltonian mechanics (see also Section 4.), which we employ here in the form of a Liouville equation, in order to describe the evolution of hybrid ensembles [1]. This is based on Liouville’s theorem and the generalized Poisson bracket defined in Eqs. (19)–(20) and assumes the compact form:

\[
-\partial_t \rho = \{\rho, \mathcal{H}_\Sigma\}_x ,
\]

with \( \mathcal{H}_\Sigma \equiv \mathcal{H}_\Sigma(x_k, p_k; X_i, P_i) \) and:

\[
\mathcal{H}_\Sigma := \mathcal{H}_{CL}(x_k, p_k) + \mathcal{H}_{QM}(X_i, P_i) + \mathcal{T}(x_k, p_k; X_i, P_i) ,
\]

which defines the relevant Hamiltonian function, including a hybrid interaction. We require \( \mathcal{H}_\Sigma \) to be an observable, last not least to have a meaningful notion of conserved energy.

An important advantage of Hamiltonian flow and a general property of the Liouville equation in this context is [18]:

- The normalization and positivity of the probability density \( \rho \) are conserved in the presence of a genuine hybrid interaction. Therefore, the interpretation of \( \rho \) as a probability density remains valid.

However, the simple relation between \( \rho \) as a function of the QM "phase space" variables \( X_i, P_i \) and an expectation of a self-adjoint density operator \( \hat{\rho} \), as employed so far, does not continue to hold generally under hybrid evolution. As pointed out in Section 5.4 of Ref. [1], the oscillator expansion of observables, such as in the second of Eqs. (21), has to be replaced by a more general form, appropriate for what we named almost-classical observables, cf. below.

This comes about, since the "classical part" of the bracket, \( \{A, B\}_CL \), can generate terms which do not qualify as observable with respect to the QM sector; here we assume that \( A \) and \( B \) are both hybrid observables, as defined before. Such terms are of the form:

\[
\frac{1}{4} \sum_{i,i',j,j'} \{A_{ij}, B_{i'j'}\}_CL(X_i - iP_i)(X_j + iP_j)(X_{i'} - iP_{i'})(X_{j'} + iP_{j'})
\]

\[
= \sum_{i',j',j} \langle \Psi | \Phi_i \rangle \langle \Psi | \Phi_{i'} \rangle \langle A_{ij}, B_{i'j'} \rangle CL \langle \Phi_j | \Psi \rangle \langle \Phi_{j'} | \Psi \rangle ,
\]

where we used the oscillator expansion, Eq. (5), and:

\[
\{A_{ij}, B_{i'j'}\}_CL = \sum_k \left( \frac{\partial A_{ij}}{\partial x_k} \frac{\partial B_{i'j'}}{\partial p_k} - \frac{\partial A_{ij}}{\partial p_k} \frac{\partial B_{i'j'}}{\partial x_k} \right) ,
\]
since, for example, $A \equiv A(x_k, p_k; X_i, P_i) = \sum_{i,j} A_{ij}(x_k, p_k) (X_i - iP_i)(X_j + iP_j)$, cf. Eq. (18).

In this way, evolution of hybrid observables, of the density $\rho$ in particular, can induce a structural change: while continuing to be CL observables, they do not remain QM observables (quadratic forms in $X_i$’s and $P_i$’s). They fall outside of the product algebra generated by the observables to which we confined ourselves in Section 2. Therefore, if we wish to maintain the formal consistency of our scheme, we have to assume:

- The algebra of hybrid observables is closed under the QM-CL Poisson bracket operation, implemented by $\{ , \}_\times$.

Thus, the product algebra of CL and QM observables is replaced by its larger closure. This amounts to a physical hypothesis, as we shall discuss in the remainder of this section.

Referring to the phase space coordinates $(X_i, P_i)$, we define an almost-classical observable as a real-valued regular function of pairs of factors like $(X_i - i P_i)(X_j + i P_j)$, such as in the left-hand side of Eq. (30), subject to the constraint: $C(X_i, P_i) = \frac{1}{2} \sum_i (X_i^2 + P_i^2)^\dagger = 1$.

This normalization constraint, cf. Eq. (8), is preserved under the evolution, since $\{C, H_\Sigma\}_\times = 0$, in the presence of QM-CL hybrid interaction. Furthermore, consistently with the closure of the enlarged algebra of observables, we find:

$$\{C(X_i, P_i), G(x_k, p_k; X_i, P_i)\}_\times = \{C(X_i, P_i), G(x_k, p_k; X_i, P_i)\}_{QM} = 0 \ ,$$

where $G(x_k, p_k; X_i, P_i)$ stands for any almost-classical observable, including QM observables as a special case, of course. This follows, since the explicit form of the generalized observables, as on the left-hand side of Eq. (30), or generalizations including additional pairs of factors like $(X_i - i P_i)(X_j + i P_j)$, leads to a sum of terms, each one vanishing as if it stemmed from a QM observable, which commute with the constraint represented by $C$ under the Poisson bracket.

According to this definition, members of the complete algebra of hybrid observables, generally, are classical with respect to coordinates $(x_k, p_k)$ and almost-classical with respect to coordinates $(X_i, P_i)$, which can be restated as:

- QM observables (quadratic forms in phase space coordinates) form a subset of almost-classical observables which, in turn, form a subset of classical observables (real-valued regular functions of phase space coordinates), cf. Section 2.

Physical consequences of this enlarged “classical × almost-classical algebra” for interacting QM-CL hybrids are illustrated by the following Gedankenexperiment.

Consider a quantum together with a classical object subject to a transient hybrid interaction. As long as the hybrid interaction is ineffective, both objects evolve independently according to Schrödinger’s and Hamilton’s equations, respectively. However, once they form an interacting hybrid, the corresponding phase space density changes from a factorized form, in absence of any initial correlation, to become an almost-classical/classical hybrid observable. Most likely, the density maintains such a mixed character, even when the hybrid interaction ceases.

This outcome contradicts naive expectation that quantum and classical objects evolve separately in quantum and classical ways, when they no longer interact. – Two possibilities come to mind. Either persistence of the almost-classical/classical character is a physical effect accompanying QM-CL hybrids, if they exist. Or our description needs to be augmented with a reduction mechanism by which evolving observables return to standard QM or CL form, following a hybrid interaction. Both possibilities seem quite interesting in their own right.

Presently, we take the enlargement of the QM algebra of observables, induced by a hybrid interaction, as “first hint” that features of QM-CL hybrids might be relevant for an understanding of how QM emerges. One would like to understand how a large algebra of classical observables (functions on phase space) is reduced, possibly via almost-classical observables at an intermediary stage, to a smaller QM algebra (linear operators on Hilbert space) for an object that becomes “quantized”. 
4. Does hybrid dynamics allow for “Free Will”?  
In order to address this question, we recall the generalized Hamiltonian equations of motion derived earlier [1]. They reflect the dynamical structure underlying hybrid evolution, according to our proposal, and correspond in detail to the Liouville equation of the previous section.

We consider hybrid systems described by a generic classical Hamiltonian function and a quantum mechanical Hamiltonian function, respectively:

\[
\mathcal{H}_{\text{CL}} := \sum_k \frac{p_k^2}{2} + v(x_k), \tag{33}
\]

\[
\hat{\mathcal{H}}_{\text{QM}} := \frac{\hat{p}^2}{2} + V(\hat{X}), \tag{34}
\]

where \(v(x_k) = v(x_1, \ldots, x_n)\) and \(V\) denote relevant potentials and all masses are set equal to one, for simplicity. Furthermore, there is a self-adjoint hybrid interaction operator \(I(x_k, p_k; \hat{X}, \hat{P})\), invoking symmetrical (Weyl) ordering of the noncommuting operators \(\hat{X}\) and \(\hat{P}\). By Eq. (15), this gives rise to the following Hamiltonian function \(\mathcal{H}_\Sigma\):

\[
\mathcal{H}_\Sigma = \sum_k \frac{p_k^2}{2} + v(x_k) + \langle \Psi | \left( \frac{\hat{p}^2}{2} + V(\hat{X}) \right) | \Psi \rangle + \langle \Psi | \hat{I}(x_k, p_k; \hat{X}, \hat{P}) | \Psi \rangle
\]

\[
= : \mathcal{H}_{\text{CL}}(x_k, p_k) + \mathcal{H}_{\text{QM}}(X_i, P_i) + I(x_k, p_k; X_i, P_i), \tag{35}
\]

when evaluated in a pure state |\(\Psi\rangle\), invoking the oscillator representation of Eq. (5). With these definitions in place, one finds the equations of motion by the rules of Hamiltonian dynamics.

The equations of motion for the CL observables \(x_k, p_k\) are:

\[
\dot{x}_k = \{x_k, \mathcal{H}_\Sigma\}_x = p_k + \partial_{p_k} I(x_k, p_k; X_i, P_i), \tag{36}
\]

\[
\dot{p}_k = \{p_k, \mathcal{H}_\Sigma\}_x = -\partial_{x_k} v(x_k) - \partial_{x_k} I(x_k, p_k; X_i, P_i). \tag{37}
\]

Similarly, we obtain for the QM variables \(X_i, P_i\), which are not observables:

\[
\dot{X}_i = \{X_i, \mathcal{H}_\Sigma\}_x = \partial_{P_i} \mathcal{H}_{\text{QM}}(X_j, P_j) + \partial_{P_j} I(x_k, p_k; X_j, P_j), \tag{38}
\]

\[
= E_i P_i + \partial_{P_i} I(x_k, p_k; X_j, P_j), \tag{39}
\]

\[
\dot{P}_i = \{P_i, \mathcal{H}_\Sigma\}_x = -\partial_{X_i} \mathcal{H}_{\text{QM}}(X_j, P_j) - \partial_{X_j} I(x_k, p_k; X_j, P_j) \tag{40}
\]

\[
= -E_i X_i - \partial_{X_i} I(x_k, p_k; X_j, P_j). \tag{41}
\]

where Eqs. (39) and (41) follow, if the oscillator expansion is performed with respect to the stationary states of \(\hat{H}_{\text{QM}}\), cf. Eqs. (6)–(7).

Notably, the Eqs. (36), (37) together with Eqs. (38), (40), or together with Eqs. (39), (41), form a closed set of \(2(n + N)\) deterministic equations, where \(n\) denotes the number of CL degrees of freedom and \(N\) the dimension of the QM Hilbert space (assumed denumerable, if not finite). Earlier we contrasted this exact result with generalized Ehrenfest equations obtained for the hybrid model [1], which we will employ in Section 5.

Following Diósi [2], someone might ask “Do I have the freedom ("Free Will") to measure one of the CL observables, say \(z\), and, conditioned on its value, to perturb the subsequent hybrid evolution in such a way that the following reasonable properties are guaranteed?”:

- i) the variable \(z\) is a smooth real function of time;
• ii) the CL and QM variables, as well as the density $\rho$ (with its statistical interpretation), coexist and depend on each other;

• iii) the perturbation at time $t_0$ affects the QM-CL hybrid in a causal way, i.e., it has no effect at earlier times $t < t_0$.

The classical variable is called tangible, in this case.

In the present hybrid theory the CL observables pertain to a classical object with its own dynamics, in the absence of interaction with a quantum object; they are not thought to derive somehow from measurements performed on this or an auxiliary quantum object, unlike in all models discussed by Diósi. So measuring $z$ could be the result of a coupling to another CL object, which serves as apparatus.

In this case, it seems clear that property i) holds, provided the perturbation applied to the hybrid, in the form of some $\delta_z(t)$, added to $z$ for $t \geq t_0$, is sufficiently smooth; the effect on the equations of motion, Eqs. (36)–(41), is simply to change initial conditions of the subsequent evolution. Similarly, properties ii) and iii) hold as a property of the Hamiltonian dynamics considered; consequently, this applies for the density $\rho$, evolving according to the equivalent Liouville equation, Eq. (28), as we discussed in Subsection 3.2.

We conclude that there is “Free Will” and classical observables are tangible in our theory.

5. Does locality remain intact in hybrids?

Let us consider the Ehrenfest relations for hybrids [1]. Here we look at the case of bilinearly coupled oscillators, which yields a very simple version of these equations and still illustrates the issue of locality. – The following discussion applies as well to the general equations of motion of Section 4, as we shall see shortly.

Earlier we obtained the closed set of equations determining the CL observables $x_k, p_k$ and QM coordinate and momentum observables $X, P$, defined by:

$$X(X_i, P_i) := \langle \Psi | \hat{X} | \Psi \rangle , \quad P(X_i, P_i) := \langle \Psi | \hat{P} | \Psi \rangle ,$$ (42)

following our construction, cf. Section 2., and assuming here a set of CL oscillators coupled bilinearly to one QM oscillator, as defined by:

$$\mathcal{H}_{\text{CL}} := \sum_k \left( \frac{1}{2m_k} p_k^2 + \frac{m_k \omega_k^2}{2} x_k^2 \right) ,$$ (43)

$$\hat{H}_{\text{QM}} := \frac{1}{2M} \hat{P}^2 + \frac{M \Omega^2}{2} \hat{X}^2 ,$$ (44)

$$\hat{I} := \hat{X} \sum_k \lambda_k x_k .$$ (45)

The constants $m_k, M, \omega_k, \Omega$, and $\lambda_k$ denote masses, frequencies, and couplings, respectively. – For this hybrid system, the general equations of motion and Ehrenfest relations indeed reduce to a simple closed set of equations:

$$\dot{x}_k = \frac{1}{m_k} p_k ,$$ (46)

$$\dot{p}_k = -m_k \omega_k^2 x_k - \lambda_k X ,$$ (47)

$$\dot{X} = \frac{1}{M} P ,$$ (48)

$$\dot{P} = -M \Omega^2 X - \sum_k \lambda_k x_k .$$ (49)
We observe that the backreaction of QM on CL subsystem appears, as if the CL subsystem was coupled to another CL oscillator. For a more general discussion of backreaction of QM on CL subsystems, see Ref. [1], in particular, concerning the role of quantum fluctuations.

However, as defined above, the QM observables are expectations and we have explicitly, with \( \hat{X}(q) = q|q\rangle \), for example:

\[
X(X_i, P_i) = \sum_{i,j} \langle X_i - iP_i|X_j + iP_j \rangle \int dq \langle \Phi_i(q)|q\rangle \Phi_j(q) \int dq \langle \Phi_j(q)|q\rangle \Phi_i(q) ,
\]

or, more generally, for hybrid interactions that include typical position measurement interactions:

\[
\mathcal{I}(x_k, p_k; X_i, P_i) = \sum_{i,j} \langle X_i - iP_i|X_j + iP_j \rangle \int dq \Phi_i^*(q)|q\rangle \Phi_j(q) \mathcal{I}(x_k, p_k; q) ,
\]

with \( \{|\Phi_j\rangle\} \) representing any complete orthonormal set of states. According to quantum theory, if the QM subsystem of the hybrid is evolving in spacetime, the integration over the variable \( q \) is understood as an integration over all space. Thus, the coupling in Eqs. (35) or (45) is generally highly nonlocal indeed.

This nonlocality has been held as a deficiency against mean-field semiclassical theories and the present theory, as discussed so far, shares this problem.

However, several remarks are in order here. – First of all, we may introduce a localizing factor and redefine the hybrid interaction:

\[
\mathcal{I}(x_k, p_k; X_i, P_i) := \langle \Psi|\hat{I}(x_k, p_k; \hat{X})\delta(\hat{X} - x_k\hat{1})|\Psi\rangle
\]

\[
= \frac{1}{2} \sum_{i,j} (X_i - iP_i)(X_j + iP_j) \int dq \Phi_i^*(q)\Phi_j(q) \mathcal{I}(x_k, p_k; x_k) .
\]

For the bilinear interaction of Eq. (45), for example, we obtain here by localization:

\[
\mathcal{I}(x_k; X_i, P_i) = \sum_k \lambda_k x_k \cdot x_k |\Psi(x_k)|^2
\]

\[
= \frac{1}{2} \sum_{i,j,k} \lambda_k x_k \cdot x_k \Phi_i^*(x_k)\Phi_j(x_k)(X_i - iP_i)(X_j + iP_j) ,
\]

where the couplings \( \lambda_k \) include a suitable dimensional factor, as compared to before.

This definition of the hybrid interaction is consistent with our formal framework and solves the locality issue. It describes the interaction in terms of the CL coordinates, however, weighted by appropriate wave function factors. The latter reflect that the QM subsystem is generally not localized. Of course, more general forms of the localizing factor are possible and can be extended to field theories as well.

By localizing the hybrid interaction in this way, we assume ad hoc that the QM subsystem is probed by the CL subsystem at its location.

Due to the nonlinear features of the localization, the generalized Ehrenfest relations obtain additional terms, as compared to our earlier results [1], which we will not explore here.

Instead, we would like to draw attention to an aspect of locality of hybrid dynamics, in particular of the general equations of motion, Eqs. (36)–(41). This will turn out to be independent of whether localization, as discussed above, is present or not.
Suppose a physicist lacking any knowledge of quantum mechanics were presented with these equations (plus the normalization constraint, Eq. (8)). – We know that these equations present independent CL and QM sectors, in the absence of a hybrid interaction. – However, he/she would naturally interpret them to describe the dynamics of a composite CL object, with part of its phase space compactified (due to the constraint). Looking at it this way, he/she finds a perfectly local dynamics. In fact, our knowledge of nonlocal features can be traced to the definition of the canonical coordinates and momenta $X_i, P_i$, introduced by the oscillator representation, Eq. (5), since: $X_i/\sqrt{2} = \text{Re} \int dq \Phi_i^*(q)\Psi(q)$ and $P_i/\sqrt{2} = \text{Im} \int dq \Phi_i^*(q)\Psi(q)$. Therefore, spatially nonlocal (and probabilistic) features enter only by reference to the QM wave function.

We tentatively conclude that the spatial interpretation of QM phenomena is, in some sense to be better understood, of a secondary character and take this as “second hint” that features of QM-CL hybrids might be relevant for an understanding of how QM emerges.

### 6. Are there hints for the emergence of quantum mechanics?

Our presentation of QM-CL hybrid dynamics has been based on suitably re-presenting the QM sector in the generalized analytical mechanics framework incorporating quantum mechanics, which was outlined by Heslot [13]. This has allowed us, in particular, to generate the hybrid evolution by a generalized Poisson bracket and to have a unified QM-CL description of states and observables.

As we discussed in Sections 3. and 5., we have found at least two hints that our present description may have already something to say about dynamics beyond quantum, classical, and QM-CL hybrid mechanics, indicating structures that lead us beneath quantum mechanics.

A caveat is in order here. Dynamics from which quantum mechanics could emerge by some coarse-graining process for large scales will not necessarily have the Hamiltonian structure that we invoked. It could be of a much more general, less structured kind, such as based on discrete deterministic automata [15].

However, the present work could help to identify deviations from quantum mechanics, which seemingly works perfectly on the scales accessible to present-day experiments. For this purpose, the Hamiltonian dynamics here can serve as a model, in order to parametrize such deviations.

It has been repeatedly pointed out by V.I. Man’ko and collaborators that classical states may differ widely from what could be obtained as the $\hbar \to 0$ limit of quantum mechanical ones. Furthermore, they show that all states can be classified by their “tomograms” as either CL or QM, CL and QM, and neither CL nor QM [22, 23]. Yet, in order to explain these “Man’ko classes of states”, some unknown dynamics seems missing.

In this context, we find the “first hint” of Section 3. very interesting, namely that a consistent hybrid description forces an enlarged algebra of observables upon us, as compared to the quantum mechanical one. More specifically, the QM observables obtain through a shrinking of the algebra of observables, via intermediary almost-classical observables, from the largest algebra of classical observables. This seems to complement and underline the findings by Man’ko in our approach, where all states are represented in a suitable phase space. We have also emphasized the spherical compactification of QM state space, which is absent classically.

It would be most interesting to find underlying dynamical reasons for such structural change that occurs when a CL object interacts with a QM object. This might help to explain how quantum mechanics emerges in a deterministic world.

Furthermore, a “second hint”, concerning the relevance of QM-CL hybrids for an understanding of how QM emerges, is coming from Section 5. There, we have seen that the hybrid dynamics might appear as a perfectly local deterministic scheme, if it were not for seemingly added-on interpretation of the QM sector with the help of a wave function defined on its to-be configuration space. Let us assume that spatial notions are somehow related to how gravity enters the picture. Then, leaving the latter aside at this time, one might devise a rather
general dynamical framework, possibly incorporating dissipation in the form of an attractor mechanism, already alluded to in Refs. [24, 25, 26, 27], which drives classical through hybrid to quantum mechanics.

7. Concluding remarks
We have discussed four questions in these notes which can be addressed to any proposal of a quantum-classical hybrid theory, in addition to the consistency requirements listed in the introduction, which, in turn, have been widely studied by others. Presently, we considered these questions in the context of the hybrid theory of Ref. [1], which meets all consistency requirements and can be seen as a generalization of proposals incorporating mean field theory.

We emphasize that here the classical sector is not necessarily meant to present an approximation for some of the quantum mechanical degrees of freedom in a fully quantum mechanical multi-partite system. Rather, we have continued to study here, as well as earlier, whether such a hybrid model as ours can stand formally on its own and meet precisely all the consistency requirements posed.

Thus, the questions raised at present are meant to further illuminate the contents and limitations of our or similar models.

The first question essentially asks, whether the proposed formalism is general enough to incorporate all possible quantum states, including superpositions, entangled or separable pure and mixed states. Which we have answered positively, with the exception of mixed entangled states, where the necessary tailoring of observables has not yet been achieved in quantum mechanics proper in all generality, cf. Ref. [21]. We have interpreted (Section 3.) the encountered necessity of an enlarged algebra of almost-classical observables as a “first hint” that hybrid dynamics might offer a glimpse beyond quantum mechanics.

The second question concerns the control of a hybrid system by external classical means, obviously a question of great practical importance. Indeed, our proposal allows for this and for “Free Will”, in particular, in the sense defined by Diósi [2].

The third question, whether locality is not violated by the proposed hybrid dynamics, as in other proposals related to mean field theory, has been discussed in some detail. Only if we invoke an ad hoc localization of the relevant interactions, this seems to be guaranteed. On the other hand, leaving out the interpretation of the quantum mechanical sector related to a wave function on configuration space, we have argued that here we may have an interesting indication of dynamics – a “second hint” – that transcends the usual quantum mechanical framework.

In Section 5., we outlined what might be taken forward from quantum-classical hybrid dynamics tackling the emergence of quantum mechanics from a deterministic theory beneath, which comprises the fourth question. Based on the formal developments of Ref. [1] and further discussion here, we seem to get closer to be able to study these issues in some detailed models.

Acknowledgments
It is a pleasure to thank L. Diósi, P. Hajicek, M.J.W. Hall, and M. Reginatto for asking related questions and G. Grössing and co-organizers for inviting me to the “Heinz von Foerster Congress / Emergent Quantum Mechanics”. ²

References
[1] Elze H-T 2011 Linear dynamics of quantum-classical hybrids *Preprint* arXiv:1111.2276
[2] Diósi L 2011 Classical-quantum coexistence: a ‘Free Will Test’ *Preprint* arXiv:1202.2472 (this volume)

² After completion of this work, we have noticed Ref. [28], where issues of Section 3 are discussed in different ways. A stringent requirement of statistical consistency is introduced which is, however, based on the assumption that its validity extends from QM to quantum-classical hybrid dynamics.
[3] Sherry T N and Sudarshan E C G 1978 Interaction between classical and quantum systems: A new approach to quantum measurement. I. Phys. Rev. D 18 4580; do. II. 1979 Phys. Rev. D 20 857
[4] Boucher W and Traschen J 1988 Semiclassical physics and quantum fluctuations Phys. Rev. D 37 3522
[5] Caro J and Salcedo L I 1999 Impediments to mixing classical and quantum dynamics Phys. Rev. A 60 842
[6] Diósi L, Gisin N and Strunz W T 2000 Quantum approach to coupling classical and quantum dynamics Phys. Rev. A 61 022108
[7] Peres A and Terno D R 2001 Hybrid classical-quantum dynamics Phys. Rev. A 63 022101
[8] Hall M J W and Reginatto M 2005 Interacting classical and quantum ensembles Phys. Rev. A 72 062109
[9] Hall M J W 2008 Consistent classical and quantum mixed dynamics Phys. Rev. A 78 042104
[10] Reginatto M and Hall M J W 2009 Quantum-classical interactions and measurement: a consistent description using statistical ensembles on configuration space J. Phys.: Conf. Ser. 174 012038 (Preprint arXiv:0905.2948)
[11] Elze H-T, Gambarotta G and Vallone F 2011 General linear dynamics – quantum, classical or hybrid J. Phys.: Conf. Ser. 306 012010 (Preprint arXiv:1103.5589)
[12] Elze H-T, Gambarotta G and Vallone F 2011 A path integral for classical dynamics, entanglement, and Jaynes-Cummings model at the quantum-classical divide Int. J. Qu. Inf. (IJQI) 9 Suppl. 1 203-224 (Preprint arXiv:1006.1569)
[13] Heslot A 1985 Quantum mechanics as a classical theory Phys. Rev. D 31 1341
[14] Wetterich C 2009 Zwitters: particles between quantum and classical Preprint arXiv:0911.1261
[15] ’t Hooft G 2010 Classical cellular automata and quantum field theory Int. J. Mod. Phys. A 25 No. 23 4385-4396; do. 2009 Preprint arXiv:0908.3408
[16] Elze H-T 2009 Does quantum mechanics tell an atomistic spacetime? J. Phys.: Conf. Ser. 174 012009 (Preprint arXiv:0906.1101); do. 2006 Deterministic models of quantum fields J. Phys.: Conf. Ser. 33 399 (Preprint arXiv:gr-qc/0512016)
[17] Adler S I 2005 Quantum Mechanics as an Emergent Phenomenon (Cambridge, UK: Cambridge U. Press)
[18] Arnold V I 1978 Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics, Vol. 60 (Berlin: Springer-Verlag), Part III
[19] Diósi L 2007 A Short Course in Quantum Information Theory Lect. Notes Phys. 713 (Berlin, Heidelberg: Springer)
[20] Harshman N L and Ranade K S 2011 Observables can be tailored to make any pure state entangled (or not) Phys. Rev. A 84 012303 (Preprint arXiv:1102.0955)
[21] Thirring W, Bertlmann R A, Köhler P and Narnhofer H 2011 Entanglement or separability: The choice of how to factorize the algebra of the density matrix Preprint arXiv:1106.3047
[22] Man’ko O V and Man’ko V I 2004 Classical mechanics is not the $\hbar \rightarrow 0$ limit of quantum mechanics J. Russ. Laser Res. 25(5) 477 (Preprint arXiv:quant-ph/0407183)
[23] Chernega V N and Man’ko V I 2011 System with classical and quantum subsystems in tomographic probability representation Preprint
[24] Blasone M, Jizba P and Vitiello G 2001 Dissipation and quantization Phys. Lett. A 287 205 (Preprint arXiv:hep-th/0007138)
[25] ’t Hooft G 2007 Emergent quantum mechanics and emergent symmetries AIP Conf. Proc. 957 154-163 (Preprint arXiv:0707.4568)
[26] Elze H-T 2008 Note on the existence theorem in “Emergent quantum mechanics and emergent symmetries” J. Phys. A.: Math. Theor. 41 304020 (Preprint arXiv:0710.2765); do. 2009 The attractor and the quantum states Int. J. Qu. Inf. (IJQI) 7 83 (Preprint arXiv:0806.3408)
[27] Blasone M, Jizba P, Scardigli F and Vitiello G 2009 Dissipation and quantization for composite systems Phys. Lett. A 373 4106 (Preprint arXiv:0905.4078)
[28] Salcedo L L 2012 Statistical consistency of quantum-classical hybrids Phys. Rev. A 85 022127 (Preprint arXiv:1201.4237)