Darboux transformation and solitons of a nonlinear PDE in two dimensions with a third-order mixed derivative

FOLKERT MÜLLER-HOISSEN
Institut für Theoretische Physik, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany
folkert.mueller-hoissen@theorie.physik.uni-goettingen.de

Abstract
We explore a nonlinear PDE in two dimensions with a leading mixed third derivative. It may be regarded as describing dynamics of a complex scalar field in one dimension. The PDE admits traveling wave solutions in terms of elementary Jacobi elliptic functions and regular multi-soliton solutions that can be constructed via a binary Darboux transformation, which we derive using bidifferential calculus.

1 Introduction
The main subject of this work is the third-order nonlinear PDE

\[ \left( \frac{f_{xt}}{f} \right)_t + 2 (f^* f)_x = 0, \tag{1.1} \]

where \( f \) is a complex function of two independent real variables \( x \) and \( t \), and \( f^* \) is the complex conjugate of \( f \). A subscript denotes a partial derivative with respect to one of the independent variables. An evident property of (1.1) is the following.

Proposition 1.1. If \( f(x, t) \) solves (1.1), then also \( f(\sigma(x), t) \), with an arbitrary differentiable function \( \sigma(x) \).

This expresses the fact that (1.1) is invariant under coordinate transformations \( x \mapsto \sigma(x) \) in one dimension, and \( f \) can be regarded as a scalar. A generalization of (1.1) to higher dimensions is the system

\[ \frac{\partial}{\partial t} \left( f^{-1} \frac{\partial}{\partial t} \frac{\partial}{\partial x^\mu} f \right) + 2 \frac{\partial}{\partial x^\mu} (f^* f) = 0 \quad \mu = 1, \ldots, m. \tag{1.2} \]

It behaves as the components of a covector (tensor of type (0,1)) under general coordinate transformations in \( m \) dimensions, if \( f \) is a scalar, also depending on a parameter \( t \). This system thus defines dynamics of a scalar field on an \( m \)-dimensional differentiable manifold. Obviously, the following holds.

Proposition 1.2. If \( f(x, t) \) solves (1.1), then \( f(\sigma(x^1, \ldots, x^m), t) \), with an arbitrary differentiable function \( \sigma \) of real independent variables \( x^\mu, \mu = 1, \ldots, m \), solves (1.2).

\footnote{The factor 2 can be eliminated by a rescaling of \( f \). Writing \( f = e^u \), the equation takes the form \( u_{xtt} + (u_x) t + 2 (e^{3u}) x = 0 \). A leading mixed third derivative also appears, for example, in the Benjamin-Bona-Mahony \([1]\) and the Joseph-Egri equation \([2]\).}
\[ \frac{\partial}{\partial t} \left( f^{-1} \frac{\partial}{\partial t} df \right) + 2 d(f^* f) = 0 , \]

where \( d \) is the exterior derivative on the \( m \)-dimensional differentiable manifold.

The real version of (1.1) appeared indirectly in \[3\] (see the commutative version of equation (9) therein), where it has been related to the sharp line self-induced transparency equations and the sine-Gordon equation. We are not aware of any other exploration of (1.1) or appearance in some context. It is unlikely, however, that this equation escaped the attention of mathematicians and scientists so far. Our motivation for this work is based on the simple form of (1.1) and the fact that (1.1) turned out to have complete integrability features, which will be elaborated in Sections 3 and 4.

In Section 2 we list some symmetries of (1.1) and derive real traveling wave solutions of it. Section 3 presents an \( n \)-fold binary Darboux transformation (see, e.g., [4]) for (1.1), which we exploit to find multi-soliton solutions. This includes solitons superposed on a plane wave background. Section 4 presents a derivation of the binary Darboux transformation. Finally, Section 5 contains some concluding remarks.

## 2 Some symmetries of the PDE and traveling wave solutions

(1.1) admits the following symmetry transformations:

- \( x \mapsto \sigma(x) \), see Proposition 1.1
- \( t \mapsto \pm t + \alpha, \alpha \in \mathbb{R} \).
- \( t \mapsto \pm |\beta| t, f \mapsto \beta f, \beta \in \mathbb{C}, \beta \neq 0 \).
- \( f \mapsto e^{i \varphi_0} f, \varphi_0 \in \mathbb{R} \).
- Complex conjugation of \( f \).

In the following, solutions will typically be presented modulo these symmetries.

Let us assume that \( f \) is real and, in some coordinate \( x \), has the form

\[ f(x, t) = f(x \pm c t) , \]

with a real constant \( c > 0 \). Then (1.1) reduces to the ODE

\[ \frac{f''}{f} + \frac{2}{c^2} f^2 = k , \]

with a real constant \( k \). Exclusively in this section, a prime indicates a derivative with respect to the argument of the function \( f \). Solutions of this equation are provided by the Jacobi elliptic functions \( \text{cn} \) and \( \text{dn} \) (see [6], for example). Indeed,

\[ f_{\text{cn}} = \sqrt{c} \sqrt{m} \text{cn} \left( \frac{1}{\sqrt{c}}(x \pm c t) \mid m \right) \]

solves the ODE with \( k = (2m - 1)/c \). We note that

\[ f_{\text{cn}} = \sqrt{c} \text{sech} \left( \frac{1}{\sqrt{c}}(x \pm c t) \right) \quad \text{if} \quad m = 1 . \]

\(^2\)This is equation 7.7 in [5].
We will recover this solitary wave as a single soliton solution in Section 3. Furthermore,

\[ f_{dn} = \sqrt{c} \tanh \left( \frac{1}{\sqrt{c}} (x \pm ct) \right) \]

satisfies the ODE with \( k = (2-m)/c \). We note that

\[ f_{dn} = \sqrt{c} \text{sech} \left( \frac{1}{\sqrt{c}} (x \pm ct) \right) \quad \text{if } m = 1. \]

3 A binary Darboux transformation and soliton solutions

(1.1) will be treated in the following form,

\[ a_t = (f^* f)_x, \quad f_{xt} + 2a f = 0, \quad (3.1) \]

where \( a \) is a real function. This system is invariant under a coordinate transformation \( x \mapsto x' \) if \( a \mapsto a' = (\partial x/\partial x') a \). We next formulate the main result of this work. A derivation and proof is postponed to Section 4.

**Theorem 3.1.** Let \( a_0, f_0 \) be a solution of (3.1). Let \( n \)-component column vectors \( \eta_i, i = 1, 2 \), be solutions of the linear system

\[
\begin{align*}
\Gamma \eta_{1x} &= a_0 \eta_1 + f_{0x}^* \eta_2, \\
\Gamma \eta_{2x} &= -a_0 \eta_2 + f_{0x} \eta_1, \\
\eta_{1t} &= -\frac{1}{2} \Gamma \eta_1 + f_{0t}^* \eta_2, \\
\eta_{2t} &= -\frac{1}{2} \Gamma \eta_2 - f_0 \eta_1,
\end{align*}
\]

(3.2, 3.3)

where \( \Gamma \) is an invertible constant \( n \times n \) matrix satisfying the spectrum condition \( \text{spec}(\Gamma) \cap \text{spec}(\Gamma^\dagger) = \emptyset \). Furthermore, let \( \Omega \) be an invertible solution of the Lyapunov equation

\[ \Gamma \Omega + \Omega \Gamma^\dagger = \eta_1 \eta_1^\dagger + \eta_2 \eta_2^\dagger, \]

(3.4)

where \( \dagger \) denotes Hermitian conjugation (transposition and complex conjugation). Then

\[ a = a_0 - (\eta_1^\dagger \Omega^{-1} \eta_1)_x, \quad f = f_0 - \eta_1^\dagger \Omega^{-1} \eta_2 \]

(3.5)

is also a solution of (3.1). As a consequence, \( f \) solves (1.1). \( \square \)

Without restriction of generality, \( \Gamma \) can be restricted to Jordan normal form. We also note that \( \Omega \) in the preceding theorem is Hermitian (also see (4.18)) and consequently \( \det(\Omega) \) is real. If \( f_0 \) is a regular solution of (1.1) on \( \mathbb{R}^2 \), a solution \( f \) generated via the above theorem can only be singular if \( \Omega \) is not invertible somewhere on \( \mathbb{R}^2 \), i.e., if \( \det(\Omega) \) has a zero.

**Remark 3.2.** (3.2) and (3.3) constitute a “Lax system” for (3.1), since its integrability conditions are equivalent to \( a_0, f_0 \) satisfying (3.1). If \( f_0 \neq 0 \), the first order system (3.3) can be decoupled into

\[ \eta_{1tt} - \frac{f_{0t}}{f_0} \eta_{1t} - \left( \frac{1}{4} \Gamma^2 + \frac{f_{0t}^*}{2f_0} \Gamma - |f_0|^2 \right) \eta_1 = 0, \quad \eta_2 = \frac{1}{f_0^*} (\eta_{1t} + \frac{1}{2} \Gamma \eta_1). \]

If \( f_{0x} \neq 0 \), (3.2) can be decoupled correspondingly,

\[ \eta_{1xx} - \left( \frac{f_{0xx}}{f_{0x}} \right)^* \eta_{1x} - \left[ (a_0^2 + |f_{0x}|^2) \Gamma^{-2} + \left( a_0 - \frac{f_{0x}}{f_{0x}} \Gamma^{-1} \right) \right] \eta_1 = 0, \quad \eta_2 = \frac{1}{f_{0x}^*} (\Gamma \eta_{1x} - a_0 \eta_1). \]

If \( f_0 \neq 0 \), but \( f_{0x} = 0 \), the second of equations (3.1) requires \( a_0 = 0 \). (3.2) then restricts \( \eta_1 \) and \( \eta_2 \) to not depend on \( x \). Since any function independent of \( x \) solves (1.1), such an \( f_0 \) is not a useful “seed” for the binary Darboux transformation in Theorem 3.1.
3.1 Zero seed solutions

If \( f_0 = 0 \), we choose \( a_0 = -1/2 \). The linear system for \( \eta \) then has the solutions

\[
\eta_1 = \exp \left( -\frac{1}{2} \left( \Gamma^{-1} x + \Gamma t \right) \right) v, \quad \eta_2 = \exp \left( \frac{1}{2} \left( \Gamma^{-1} x + \Gamma t \right) \right) w, \tag{3.6}
\]

where \( v, w \) are constant \( n \)-component column vectors. The ansatz

\[
\Omega = e^{-\frac{1}{2} \left( \Gamma^{-1} x + \Gamma t \right)} X e^{-\frac{1}{2} \left( \Gamma^{-1} x + \Gamma^\dagger t \right)} + e^{\frac{1}{2} \left( \Gamma^{-1} x + \Gamma t \right)} Y e^{\frac{1}{2} \left( \Gamma^{-1} x + \Gamma^\dagger t \right)}, \tag{3.7}
\]

with constant \( n \times n \) matrices \( X, Y \), solves (3.4) if

\[
\Gamma X + X \Gamma^\dagger = v v^\dagger, \quad \Gamma Y + Y \Gamma^\dagger = w w^\dagger. \tag{3.8}
\]

According to Theorem 3.1,

\[
f = v^\dagger e^{-\frac{1}{2} \left( \Gamma^{-1} x + \Gamma^\dagger t \right)} \Omega^{-1} e^{\frac{1}{2} \left( \Gamma^{-1} x + \Gamma t \right)} w = \frac{1}{\det(\Omega)} v^\dagger e^{-\frac{1}{2} \left( \Gamma^{-1} x + \Gamma^\dagger t \right)} \text{adj}(\Omega) e^{\frac{1}{2} \left( \Gamma^{-1} x + \Gamma t \right)} w, \tag{3.9}
\]

where \( \text{adj} \) takes the adjugate of a matrix, represents (an infinite set of) exact solutions of (1.1). Here we dropped a global minus sign, since \( f \mapsto -f \) is a symmetry of (1.1). We also note that \( \Gamma \mapsto -\Gamma \) amounts to \( f \mapsto -f \).

3.1.1 Simple multi-soliton solutions

These are obtained by choosing \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n) \), where \( \gamma_i \neq -\gamma_j, \ i, j = 1, \ldots, n \). The solutions of the Lyapunov equations (3.8) are then the Cauchy-like matrices

\[
X = \left( \begin{array}{c} v_i v_j^* \\ \gamma_i + \gamma_j \end{array} \right), \quad Y = \left( \begin{array}{c} w_i w_j^* \\ \gamma_i + \gamma_j \end{array} \right).
\]

\( n = 1 \). In this case, (3.9) becomes

\[
f = 2 \text{Re}(\gamma) v^* w \left( |v|^2 e^{-\gamma^{-1} x + \gamma t} + |w|^2 e^{\gamma^{-1} x + \gamma^* t} \right)^{-1}.
\]

If \( \gamma, v, w \) are real, this can be rewritten, up to a global sign, as

\[
f = \gamma \text{sech}(\gamma^{-1} x + \gamma t + \alpha), \tag{3.10}
\]

where \( \alpha = \ln(|w/v|) \). This arbitrary constant appears since our PDE is autonomous. The solitary wave is similar to that of the modified KdV (mKdV) equation and the bright soliton of the NLS equation.

\( n = 2 \). (3.9) yields

\[
f = \frac{1}{2 \text{Re}(\gamma_1) (\gamma_1 + \gamma_2^*) \det(\Omega)} e^{-\text{Re}(\gamma_1) (t \gamma_1 + x/|\gamma_1|^2) + \text{Im}(\gamma_2) (t \gamma_2 + x/|\gamma_2|^2)} \left( (\gamma_2^* - \gamma_1^*) |v_1|^2 v_2^* w_2 + (\gamma_1 + \gamma_2^*) |v_1|^2 w_2^* e^{2 \text{Re}(\gamma_1) (t \gamma_1 + x/|\gamma_1|^2)} - 2 \text{Re}(\gamma_1) |v_1|^2 w_2 |w_2|^2 e^{(\gamma_1 + \gamma_2^*) t + (\gamma_1^{-1} + \gamma_2^{-1}) x} \right)
\]

\[
+ \frac{1}{2 \text{Re}(\gamma_2) (\gamma_1^* + \gamma_2)} e^{-\text{Re}(\gamma_2) (t \gamma_2 + x/|\gamma_2|^2) + \text{Im}(\gamma_1) (t \gamma_1 + x/|\gamma_1|^2)} \left( (\gamma_1^* - \gamma_2^*) |v_2|^2 v_1^* v_2^* w_1 \right)
\]
\[(\gamma_1^* + \gamma_2)v_1^* w_1 |w_2|^2 e^{2 \Re (\gamma_2)(t+x)/|\gamma_2|^2} - 2 \Re (\gamma_2)v_2^* |w_1|^2 w_2 e^{(\gamma_1^* + \gamma_2) t + (\gamma_1^{-1} + \gamma_2^{-1}) x},\]

where

\[
\det(\Omega) = e^{-\Re(\gamma_1+\gamma_2) t - \Re(\gamma_1^{-1} + \gamma_2^{-1}) x} \left( |\gamma_1 + \gamma_2|^2 |v_2 w_1 e^{\gamma_1 t + \gamma_1^{-1} x} - v_1 w_2 e^{\gamma_2 t + \gamma_2^{-1} x}|^2 + |\gamma_1 - \gamma_2|^2 \left| v_1 v_2^* + w_1 w_2^* e^{(\gamma_1 + \gamma_2)^* t + (\gamma_1^{-1} + \gamma_2^{-1}) x} \right|^2 \right),
\]

which we were able to express in an explicitly non-negative form.

**Proposition 3.3.** The 2-soliton solution is regular if \(\gamma_1, \gamma_2 \neq 0, \gamma_1 \neq \gamma_2, \gamma_1 \neq -\gamma_2^*, \{v_1, w_1\} \neq \{0\}\) and \(\{v_2, w_2\} \neq \{0\}\).

**Proof.** Assuming \(v_2 w_1 w_2 \neq 0\), for a zero of \(\det(\Omega)\) we would need

\[
\left| v_1 w_2 \right|^2 \left| v_1 w_2 \right|^2 = \frac{v_1 w_2}{w_1 w_2^*},
\]

where \(\phi_k = \gamma_k t + \gamma_k^{-1} x, k = 1, 2\). Hence

\[
e^{2 \Re (\phi_2)} = -\frac{|v_2|^2}{|w_2|^2},
\]

which contradicts the positivity of the real exponential function. If any of \(v_2, w_1, w_2\) is zero, a zero of \(\det(\Omega)\) is only possible if either \(v_1\) and \(w_1\), or \(v_2\) and \(w_2\) are zero, which we excluded.

**Example 3.4.** Choosing \(n = 2, \gamma_1 = 1, \gamma_2 = 2\) and \(v_1 = v_2 = w_1 = w_2 = 1\), (3.9) yields the special real 2-soliton solution

\[
f = 6 \frac{\cosh(2t + \frac{1}{2} x) - 2 \cosh(t + x)}{\cosh(3t + \frac{3}{2} x) + 9 \cosh(t - \frac{1}{2} x) - 8}.
\]

A plot is shown in Fig. 1.

---

**Figure 1:** Plot of the real 2-soliton solution in Example 3.4.
3.1.2 Solitons associated with Jordan block data

In contrast to simple solitons, solutions determined by (3.9), with $\Gamma$ chosen as a non-diagonal Jordan block, depend also rationally on $x$ and $t$. Let $n = 2$ and

$$\Gamma = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}. $$

The solution of (3.8) is then

$$X = \frac{1}{4 \text{Re}(\gamma)^2} \left( \frac{|v_2|^2}{\text{Re}(\gamma)} + 2 \text{Re}(\gamma) |v_1|^2 - v_1 v_2^* - v_1^* v_2 \left( 2 \text{Re}(\gamma) v_1 - v_2 \right) v_2^* \right),$$

from which $Y$ is obtained by replacing $v$ with $w$. From (3.7) we obtain the following components of the $2 \times 2$ matrix $\Omega = (\Omega_{ij})$,

$$\Omega_{11} = \frac{1}{8 |\gamma|^4 \text{Re}(\gamma)} \left( 4 |\gamma|^4 |w_1|^2 + 2 |w_2|^2 \left[ \text{Re}(\gamma^2) x - |\gamma|^4 t + \frac{|\gamma|^4}{\text{Re}(\gamma)} \right] + |w_2|^2 |x - \gamma^2 t|^2 \right.$$  

\[ -4 \text{Re} \left[ \gamma^2 w_1 w_2^* (x - \gamma^2 t + \frac{-\gamma^2 t}{\text{Re}(\gamma)}) \right] e^{\text{Re}(\gamma)(t + x/|\gamma|^2)} \]

\[ + \left( 4 |\gamma|^4 |v_1|^2 - 2 |v_2|^2 \left[ \text{Re}(\gamma^2) x - |\gamma|^4 t - \frac{|\gamma|^4}{\text{Re}(\gamma)} \right] + |v_2|^2 |x - \gamma^2 t|^2 \right) \]

\[ + 4 \text{Re} \left[ \gamma^2 v_1 v_2^* (x - \gamma^2 t - \frac{\gamma^2 t}{\text{Re}(\gamma)}) \right] e^{-\text{Re}(\gamma)(t + x/|\gamma|^2)} \right),

$$\Omega_{12} = \Omega_{21} = \frac{1}{4 \gamma^2 \text{Re}(\gamma)^2} \left( \left[ \text{Re}(\gamma) |v_2|^2 (x - \gamma^2 t) - \gamma^2 (v_2 - 2 \text{Re}(\gamma) v_1) v_2 \right] e^{-\text{Re}(\gamma)(t + x/|\gamma|^2)} \right.$$  

\[ - \left[ \text{Re}(\gamma) |w_2|^2 (x - \gamma^2 t) + \gamma^2 (w_2 - 2 \text{Re}(\gamma) w_1) w_2^* \right] e^{\text{Re}(\gamma)(t + x/|\gamma|^2)} \right),

$$\Omega_{22} = \frac{1}{2 \text{Re}(\gamma)} \left( |v_2|^2 e^{-\text{Re}(\gamma)(t + x/|\gamma|^2)} + |w_2|^2 e^{\text{Re}(\gamma)(t + x/|\gamma|^2)} \right).$$

If we restrict our considerations to solutions with real data, i.e., $\gamma \in \mathbb{R}$ and $v_k, w_k \in \mathbb{R}, k = 1, 2$, then (3.9) yields

$$f = \frac{1}{4 \gamma^2 \text{det}(\Omega)} \left( [-\gamma^2 v_1 v_2^* w_2 + \gamma^2 v_1^2 w_2 + v_2^2 w_2 (\gamma + \gamma^2 t)] e^{-(t + x/\gamma^2) x} \right.$$  

\[ + [-\gamma^2 v_2 w_1^* v_2^2 + v_2^2 w_2^* (\gamma + \gamma^2 t)] e^{-(t + x/\gamma^2) x} \right) \tag{3.11}

with

$$\text{det}(\Omega) = \frac{1}{16 \gamma^6} \left( 2 \gamma^2 v_2^2 w_2^2 + 4 |v_2| w_2 (x - \gamma^2 t) + \gamma^2 (v_1 w_2 - v_2 w_1)^2 \right.$$  

\[ + \gamma^2 v_2^4 e^{-\frac{\gamma^2}{2} (x + \gamma^2 t)} + \gamma^2 w_2^4 e^{-\frac{\gamma^2}{2} (x + \gamma^2 t)} \right),$$

which has been cast into an explicitly non-negative form. These solutions are regular if $\gamma \neq 0$ and if either $v_2$ or $w_2$ is different from zero.

**Example 3.5.** Choosing $\gamma = v_1 = v_2 = w_1 = w_2 = 1$, (3.11) takes the form

$$f = 4 \cos h (x + t) + (x - t) \sin h (x + t) \left( \frac{1}{1 + 2 (x-t)^2 + \cosh(2(x+t))} \right).$$

A plot is shown in Fig. 2.
Corresponding results will be obtained if $\Gamma$ is chosen as an $m \times m$ Jordan block with $m > 2$. More generally, based on (3.9), solutions can be worked out with $\Gamma$ being composed of different Jordan blocks.

### 3.2 Solutions with a plane wave background

Choosing as the “seed” $f_0$ the plane wave solution

$$f_0 = Ce^{i(x-t)} ,$$

with a complex constant $C \neq 0$, (3.1) determines $a_0 = -1/2$ and the linear system (3.2), (3.3) can be decoupled to

$$\eta_{1xx} + i \eta_{1x} - \Gamma^{-2}(|C|^2 + \frac{1}{4} - \frac{i}{2} \Gamma) \eta_1 = 0 ,$$

$$\eta_{1tt} - i \eta_{1t} + (|C|^2 - \frac{1}{4} \Gamma^2 - \frac{i}{2} \Gamma) \eta_1 = 0 ,$$

$$\eta_2 = \frac{i}{C^*} e^{i(x-t)} (\Gamma \eta_{1x} + \frac{1}{2} \eta_1) = \frac{1}{C^*} e^{i(x-t)} (\eta_{1t} + \frac{1}{2} \Gamma \eta_1)$$

(cf. Remark 3.2), from which we obtain

$$\eta_1 = e^{-\frac{1}{2} i (x-t)} \left( e^{-\frac{1}{2} (\Gamma^{-1} x + it) R} v + e^{\frac{1}{2} (\Gamma^{-1} x + it) R} w \right) ,$$

$$\eta_2 = e^{\frac{i}{2} i (x-t)} \left( e^{-\frac{1}{2} (\Gamma^{-1} x + it) R} \tilde{v} + e^{\frac{1}{2} (\Gamma^{-1} x + it) R} \tilde{w} \right) ,$$

with constant $n$-component column vectors $v$ and $w$, and

$$\tilde{v} = \frac{1}{2 C^*} (\Gamma + i (I - R)) v , \quad \tilde{w} = \frac{1}{2 C^*} (\Gamma + i (I + R)) w .$$

Here $R$ is a matrix square root satisfying

$$R^2 = (I - i \Gamma)^2 + 4 |C|^2 I .$$

Inserting the ansatz

$$\Omega = e^{-\frac{1}{2} (\Gamma^{-1} x + it) R} X_1 e^{-\frac{1}{2} R^t (\Gamma^{-1} x - it)} + e^{-\frac{1}{2} (\Gamma^{-1} x + it) R} Y e^{\frac{1}{2} R^t (\Gamma^{-1} x - it)}$$

$$+ e^{\frac{1}{2} (\Gamma^{-1} x + it) R} Y^t e^{-\frac{1}{2} R^t (\Gamma^{-1} x - it)} + e^{\frac{1}{2} (\Gamma^{-1} x + it) R} X_2 e^{\frac{1}{2} R^t (\Gamma^{-1} x - it)} ,$$
with constant $n \times n$ matrices $X_1, X_2, Y$, in (3.4), requires 
\[
\Gamma X_1 + X_1 \Gamma^\dagger = vv^\dagger + \tilde{v} \tilde{v}^\dagger, \quad \Gamma X_2 + X_2 \Gamma^\dagger = ww^\dagger + \tilde{w} \tilde{w}^\dagger, \quad \Gamma Y + Y \Gamma^\dagger = vv^\dagger + \tilde{v} \tilde{v}^\dagger.
\]
If $\Gamma$ is diagonal, the solutions of these Lyapunov equations are given by the Cauchy-like matrices
\[
X_1 = \left( \frac{v_i v_j^* + \tilde{v}_i \tilde{v}_j^*}{\gamma_i + \gamma_j^*} \right), \quad X_2 = \left( \frac{w_i w_j^* + \tilde{w}_i \tilde{w}_j^*}{\gamma_i + \gamma_j^*} \right), \quad Y = \left( \frac{v_i w_j^* + \tilde{v}_i \tilde{w}_j^*}{\gamma_i + \gamma_j^*} \right).
\]
After inserting these expression in that for $\Omega$, it remains to compute the inverse matrix $\Omega^{-1}$ in order to find an explicit form of the new solution given by (3.5).

For $n = 1$, we find
\[
\Omega = \frac{1}{2} \text{Re}(\gamma) \left( (\sqrt{2}|C|)^2 e^{-\text{Re}(\gamma)|x|+\text{Im}(\gamma)|t|} + (\sqrt{2}|C|)^2 e^{\text{Re}(\gamma)|x|+\text{Im}(\gamma)|t|} \right) + \text{Re}\left( (\sqrt{2}|C|)^2 e^{-\text{Re}(\gamma)|x|+\text{Im}(\gamma)|t|} \right)
\]
\[
= \frac{1}{2} \text{Re}(\gamma) \left( v e^{-\frac{1}{2}((\gamma/\sqrt{2})|x|+\sqrt{2}|C|)|t|} + w e^{\frac{1}{2}((\gamma/\sqrt{2})|x|+\sqrt{2}|C|)|t|} + |\bar{v} e^{-\frac{1}{2}((\gamma/\sqrt{2})|x|+\sqrt{2}|C|)|t|} + \bar{w} e^{\frac{1}{2}((\gamma/\sqrt{2})|x|+\sqrt{2}|C|)|t|} \right),
\]
where
\[
\tilde{v} = \frac{1}{\sqrt{2}|C|} (1 - i \gamma - \gamma^{-1}) v, \quad \tilde{w} = \frac{1}{\sqrt{2}|C|} (1 - i \gamma + \gamma^{-1}) w,
\]
and $r$ is a square root of $(1 - i \gamma)^2 + 4|C|^2$. Then we have
\[
f = C e^{i(x-t)} - \frac{1}{\Omega} e^{i(x-t)} e^{-\text{Re}(\gamma)|x|+\text{Im}(\gamma)|t|} \left( \tilde{v} + \tilde{w} e^{(\gamma/\sqrt{2})|x|+\sqrt{2}|C|)|t|} \right) \left( \bar{v} e^{i\gamma|t|} + \bar{w} e^{(\gamma/\sqrt{2})|x|+\sqrt{2}|C|)|t|} \right).
\]
We note that $f = -C e^{-i(x-t)}$ if $\gamma = 2|C| - i$. Fig. 3 shows a plot of the real part of the above solution for specified data.

Figure 3: Plot of the real part of $-f$ for a single soliton on a plane wave background. Here we chose the data $\gamma = 2, C = v = w = 1$.

Fig. 4 shows a plot of the real part of a solution with two solitons on the plane wave background.

4 Derivation of the binary Darboux transformation

4.1 Binary Darboux transformations in bidifferential calculus

A graded associative algebra is an associative algebra $\Omega = \bigoplus_{r \geq 0} \Omega^r$ over a field $\mathbb{K}$ of characteristic zero, where $A := \Omega^0$ is an associative algebra over $\mathbb{K}$ and $\Omega^r, \ r \geq 1$, are $A$-bimodules such that
Figure 4: Plot of the real part of $f$ for the solution in Section 3.2 with $n = 2$, diagonal $\Gamma$, and the data $\gamma_1 = 2$, $\gamma_2 = 4$, $C = v_1 = v_2 = w_1 = w_2 = 1$.

$\Omega^r \Omega^s \subseteq \Omega^{r+s}$. Elements of $\Omega^r$ will be called $r$-forms. A bidifferential calculus is a unital graded associative algebra $\Omega$, supplied with two $K$-linear graded derivations $d, \overline{d} : \Omega \to \Omega$ of degree one (hence $d\Omega^r \subseteq \Omega^{r+1}$, $\overline{d}\Omega^r \subseteq \Omega^{r+1}$), and such that

$$d^2 = \overline{d}^2 = d\overline{d} + \overline{d}d = 0.$$  \hspace{1cm} (4.1)

We refer the reader to [7] for an introduction to this structure and an extensive list of references.

**Theorem 4.1.** Given a bidifferential calculus, let 0-forms $\Delta, \Gamma$ and 1-forms $\kappa, \lambda$ satisfy

$$\overline{d}\Delta + [\lambda, \Delta] = (d\Delta) \Delta, \quad \overline{d}\lambda + \lambda^2 = (d\lambda) \Delta, \quad \overline{d}\Gamma - [\kappa, \Gamma] = \Gamma d\Gamma, \quad \overline{d}\kappa - \kappa^2 = \Gamma d\kappa.$$  \hspace{1cm} (4.2)

Let 0-forms $\theta$ and $\eta$ be solutions of the linear equations

$$\overline{d}\theta = A\theta + (d\theta) \Delta + \theta \lambda, \quad \overline{d}\eta = -\eta A + \Gamma d\eta + \kappa \eta,$$  \hspace{1cm} (4.3)

where the 1-form $A$ satisfies

$$dA = 0, \quad \overline{d}A = A^2.$$  \hspace{1cm} (4.4)

Furthermore, let $\Omega$ be an invertible solution of the linear system

$$\Gamma \Omega - \Omega \Delta = \eta \theta, \quad \overline{d}\Omega = (d\Omega) \Delta - (d\Gamma) \Omega + \kappa \Omega + \Omega \lambda + (d\eta) \theta.$$  \hspace{1cm} (4.5)

Then

$$A' := A - d(\theta \Omega^{-1} \eta)$$  \hspace{1cm} (4.7)

also solves (4.4).

**Proof.** Clearly, we have $dA' = 0$. Using (4.4), we obtain

$$\overline{d}A' - A'^2 = \overline{d}(\theta \Omega^{-1} \eta) + A d(\theta \Omega^{-1} \eta) + \theta \Omega^{-1} \eta A - d(\theta \Omega^{-1} \eta) d(\theta \Omega^{-1} \eta).$$

\footnote{Under suitable assumptions for $\Delta$ and $\Gamma$, these equations arise as integrability conditions of the linear system and “adjoint linear system” given in (4.3), by use of (4.2). In any case, the integrability conditions are satisfied if (4.2) and (4.4) hold.}

\footnote{The equation obtained by acting with $\overline{d}$ on (4.6) is satisfied as a consequence of the preceding equations.}
With the help of the linear equations (4.3) and (4.6), we find
\[ \tilde{d}(\theta \Omega^{-1}\eta) = A \theta \Omega^{-1}\eta - \theta \Omega^{-1}\eta A + (d\theta) \Delta \Omega^{-1}\eta + \theta \Omega^{-1} \Gamma \eta - \theta \Omega^{-1}(d\eta) \theta \Omega^{-1}\eta. \]

Eliminating \( \Gamma \) using (4.5), it becomes
\[ \tilde{d}(\theta \Omega^{-1}\eta) = A \theta \Omega^{-1}\eta - \theta \Omega^{-1}\eta A + d(\theta \Delta \Omega^{-1}\eta) + \theta \Omega^{-1}\eta d(\theta \Omega^{-1}\eta). \]

Inserting this in our first equation leads to \( \tilde{d}A' - A'^2 = 0. \)

The preceding theorem, and also the result stated next, remain true if the ingredients are matrices of forms with dimensions chosen in such a way that the required products are all defined. Furthermore, it will be sufficient to have the maps \( d \) and \( \tilde{d} \) defined on those matrices that appear in the theorem, but not necessarily on the whole of \( \Omega. \)

**Corollary 4.2.** Let (4.2) hold and (4.3) with \( A = d\phi \), where the 0-form \( \phi \) is a solution of
\[ \tilde{d}d\phi = d\phi d\phi. \] (4.8)

If \( \Omega \) is an invertible solution of (4.5) and (4.6), then
\[ \phi' = \phi - \theta \Omega^{-1}\eta + K, \] (4.9)

where \( K \) is any \( d \)-constant (i.e., \( dK = 0 \)), solves the same equation.

The result in Corollary 4.2 can be regarded as a reduction of that in Theorem 4.1. Corollary 4.2 has been used in many previous applications of bidifferential calculus, see in particular [7, 8, 9]. Here we provided short proofs of the above general results. Below, we will use Corollary 4.2 to deduce Theorem 3.1.

### 4.2 An application

Let \( \mathcal{A} \) be a unital associative algebra over \( \mathbb{C} \), where the elements are allowed to depend on real variables \( x, t \). We choose
\[ \Omega = \mathcal{A} \otimes \bigwedge \mathbb{C}^2, \] (4.10)

where \( \bigwedge \mathbb{C}^2 \) is the exterior algebra of the vector space \( \mathbb{C}^2 \). It is then sufficient to define \( d \) and \( \tilde{d} \) on (matrices over) \( \mathcal{A} \), since they extend in an evident way to (matrices over) \( \Omega \), treating elements of \( \bigwedge \mathbb{C}^2 \) as \( d \)- and \( \tilde{d} \)-constants.

Let \( \xi_1, \xi_2 \) be a basis of \( \bigwedge^1 \mathbb{C}^2 \). For each \( m \in \mathbb{N} \), let \( J_m \) be a constant \( m \times m \) matrix over \( \mathcal{A} \). For an \( m \times n \) matrix \( F \) over \( \mathcal{A} \), let
\[ dF = F_x \xi_1 + \frac{1}{2}(J_m F - F J_n) \xi_2, \quad \tilde{d}F = \frac{1}{2}(J_m F - F J_n) \xi_1 + F_t \xi_2 \]

(also see [3, 10, 11]). Then \( d \) and \( \tilde{d} \) satisfy the Leibniz rule on a product of matrices, and the conditions in (4.1) are satisfied. In the linear systems (4.3) we choose a \( 2 \times n \) matrix \( \theta \) and an \( n \times 2 \) matrix \( \eta \). Then \( A \) has to be a \( 2 \times 2 \) matrix of 1-forms. Writing
\[ A = A_1 \xi_1 + A_2 \xi_2, \quad \kappa = \kappa_1 \xi_1 + \kappa_2 \xi_2, \quad \lambda = \lambda_1 \xi_1 + \lambda_2 \xi_2, \]
with $2 \times 2$ matrices (over $A$) $A_1$ and $A_2$. (4.3) reads
\[
\frac{1}{2} (J_2 \theta - \theta J_n) = A_1 \theta + \theta_x \Delta + \theta \lambda_1, \quad \theta_t = A_2 \theta + \frac{1}{2} (J_2 \theta - \theta J_n) \Delta + \theta \lambda_2,
\]
\[
\frac{1}{2} (J_n \eta - \eta J_2) = -\eta A_1 + \Gamma \eta_x + \kappa_1 \eta, \quad \eta_t = -\eta A_2 + \frac{1}{2} \Gamma (J_n \eta - \eta J_2) + \kappa_2 \eta.
\]
Choosing
\[
\kappa_1 = \frac{1}{2} J_n, \quad \kappa_2 = -\frac{1}{2} \Gamma J_n, \quad \lambda_1 = -\frac{1}{2} J_n, \quad \lambda_2 = \frac{1}{2} J_n \Delta,
\]
the latter system simplifies to
\[
\frac{1}{2} J_2 \theta = A_1 \theta + \theta_x \Delta, \quad \theta_t = A_2 \theta + \frac{1}{2} J_2 \theta \Delta,
\]
\[
\frac{1}{2} \eta J_2 = \eta A_1 - \Gamma \eta_x, \quad \eta_t = -\eta A_2 - \frac{1}{2} \Gamma \eta J_2,
\]
which does not involve $J_n$ with $n \neq 2$ anymore. The conditions in (4.2) boil down to
\[
\Delta_x = \Delta_t = 0, \quad \Gamma_x = \Gamma_t = 0,
\]
so that $\Delta$ and $\Gamma$, which are $n \times n$ matrices over $A$, have to be constant. (4.6) becomes
\[
\Omega_x \Delta = -\eta_x \theta, \quad \Omega_t = -\frac{1}{2} \eta J_2 \theta.
\]
In addition, the $n \times n$ matrix $\Omega$ has to satisfy (4.5). Choosing $J_2 = \text{diag}(1, -1)$, where $1$ stands for the identity element of $A$, and writing
\[
\phi = \begin{pmatrix} p & f \\ q & -\tilde{p} \end{pmatrix},
\]
elaborating Corollary 4.2 we have
\[
A = d\phi = \begin{pmatrix} p_x & f_x \\ q_x & -\tilde{p}_x \end{pmatrix} \xi_1 + \begin{pmatrix} 0 & f \\ -q & 0 \end{pmatrix} \xi_2,
\]
and (4.8) takes the form
\[
\phi_{xt} = \frac{1}{2} \begin{bmatrix} [J, \phi] \end{bmatrix},
\]
also see [3]. This results in the system
\[
f_{xt} = f - p_x f - f \tilde{p}_x, \quad q_{xt} = q - q p_x - \tilde{p}_x q, \quad p_{xt} = (f q)_x, \quad \tilde{p}_{xt} = (q f)_x.
\]
Introducing
\[
a := p_x - \frac{1}{2} 1, \quad \tilde{a} := \tilde{p}_x - \frac{1}{2} 1,
\]
it reads
\[
f_{xt} = -a f - f \tilde{a}, \quad q_{xt} = -q a - \tilde{a} q, \quad a_t = (f q)_x, \quad \tilde{a}_t = (q f)_x.
\]
The constraint
\[
q = \pm f^\dagger, \quad a^\dagger = a, \quad \tilde{a}^\dagger = \tilde{a},
\]
reduces the last system to
\[
f_{xt} = -a f - f \tilde{a}, \quad a_t = \pm (f f^\dagger)_x, \quad \tilde{a}_t = \pm (f^\dagger f)_x.
\]
Remark 4.3. Instead, the reduction \( q = \pm f, \tilde{a} = a \) leads to
\[
f_{xt} = -a f - f a, \quad a_t = \pm (f^2)_x.
\]
Choosing the upper sign, this system may be regarded, as has been suggested in \[3\] (see equation (9) therein), as a matrix version of the self-induced transparency (SIT) equations.

4.3 The commutative case

Let \( \mathcal{A} \) be the commutative algebra of functions on \( \mathbb{R}^2 \). Then we have
\[
\text{tr}(\theta \Omega^{-1} \eta) = \text{tr}(\eta \Omega^{-1}) = \text{tr}((\Gamma \Omega - \Omega \Delta) \Omega^{-1}) = \text{tr}(\Gamma) - \text{tr}(\Delta),
\]
where \( \Gamma \) and \( \Delta \) shall now be \( n \times n \) matrices over \( \mathbb{C} \), \( \theta \) and \( \eta \) of size \( k \times n \) and \( n \times k \), respectively. Hence
\[
\text{tr}(d(\theta \Omega^{-1} \eta)) = \text{tr}(\theta \Omega^{-1} \eta)_x \eta_1 = (\text{tr}(\Gamma) - \text{tr}(\Delta))_x \eta_1 = 0.
\]
As a consequence,
\[
\text{tr} A = 0
\]
is a reduction that is consistent with the solution-generating method of Theorem 4.1. The latter reduction means \( \tilde{a} = a \), so that (4.11) becomes
\[
a_t = (f q)_x, \quad f_{xt} = -2a f, \quad q_{xt} = -2a q.
\] (4.12)

Since we guaranteed form-invariance of \( A \) under the transformation given by Corollary 4.2, writing
\[
\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \eta = (\eta_1, \eta_2),
\]
we have
\[
d\phi' = A' = A - d(\theta \Omega^{-1} \eta) = \begin{pmatrix} a' + \frac{1}{2} f' \\ q' \end{pmatrix} \eta_1 + \begin{pmatrix} 0 \\ -\frac{1}{2} f' \end{pmatrix} \eta_2
\]
\[
= \begin{pmatrix} a' + \frac{1}{2} (\theta_1 \Omega^{-1} \eta_1)_x \\ q - \theta_2 \Omega^{-1} \eta_2 \end{pmatrix} \eta_1 + \begin{pmatrix} f - \theta_1 \Omega^{-1} \eta_2 \\ -q - \theta_1 \Omega^{-1} \eta_1 \end{pmatrix} \eta_2,
\]
and hence
\[
a' = a - (\theta_1 \Omega^{-1} \eta_1)_x = a + (\theta_2 \Omega^{-1} \eta_2)_x, \quad f' = f - \theta_1 \Omega^{-1} \eta_2, \quad q' = q - \theta_2 \Omega^{-1} \eta_1,
\]
satisfy the same equations as \( a, f, q \). Collecting the main results, we arrive at the following theorem, which expresses a binary Darboux transformation for the system (4.12).

Theorem 4.4. Let \( a_0, f_0, q_0 \) be a solution of (4.12). Let \( \theta_i \) and \( \eta_i, i = 1, 2 \), be solutions of the linear system
\[
\begin{align*}
\theta_{1x} \Delta &= -a_0 \theta_1 - f_{0x} \theta_2, & \quad \theta_{2x} \Delta &= a_0 \theta_2 - q_{0x} \theta_1, \\
\theta_{1t} &= \frac{1}{2} \theta_1 \Delta + f_0 \theta_2, & \quad \theta_{2t} &= -\frac{1}{2} \theta_2 \Delta - q_0 \theta_1, \\
\Gamma \eta_{1x} &= a_0 \eta_1 + q_{0x} \eta_2, & \quad \Gamma \eta_{2x} &= -a_0 \eta_2 + f_{0x} \eta_1, \\
\eta_{1t} &= \frac{1}{2} \Gamma \eta_1 + q_0 \eta_2, & \quad \eta_{2t} &= \frac{1}{2} \Gamma \eta_2 - f_0 \eta_1,
\end{align*}
\]
where $\Delta$ and $\Gamma$ are invertible constant $n \times n$ matrices. Let $\Omega$ be an invertible solution of the linear equations

\[
\Gamma \Omega - \Omega \Delta = \eta_1 \theta_1 + \eta_2 \theta_2 ,
\]

\[
\Omega x \Delta = -\eta_1 x \theta_1 - \eta_2 x \theta_2 , \quad \Omega_t = -\frac{1}{2} \eta_1 \theta_1 + \frac{1}{2} \eta_2 \theta_2 .
\]

Then

\[
a = a_0 - (\theta_1 \Omega^{-1} \eta_1)_x , \quad f = f_0 - \theta_1 \Omega^{-1} \eta_2 , \quad q = q_0 - \theta_2 \Omega^{-1} \eta_1 ,
\]

constitutes also a solution of (4.12).

If we impose the condition

\[
q = \pm f^* ,
\]

the system (4.12) reduces to (3.1), if we choose the plus sign. It remains to implement the above reduction in the solution-generating method.

4.4 The reduction $q = f^*$

Let us set

\[
q = f^* , \quad \theta = \eta^\dagger , \quad \Delta = -\Gamma^\dagger .
\]

Then Theorem 4.4 implies Theorem 3.1

**Proof of Theorem 3.1.** The linear system in Theorem 4.4 reduces to (3.2) and (3.3), by using (4.17), and (4.13) becomes (3.4). If $\Gamma$ and $-\Gamma^\dagger$ have no eigenvalue in common, i.e., $\text{spec}(\Gamma) \cap \text{spec}(-\Gamma^\dagger) = \emptyset$, the Lyapunov equation (3.4) is known to have a unique solution $\Omega$. By taking its conjugate, we can then deduce that

\[
\Omega^\dagger = \Omega ,
\]

which in turn implies that the equations

\[
\Omega x \Gamma^\dagger = \eta_1 x \eta_1^\dagger + \eta_2 x \eta_2^\dagger , \quad \Omega_t = -\frac{1}{2} \eta_1 \eta_1^\dagger + \frac{1}{2} \eta_2 \eta_2^\dagger ,
\]

resulting from (4.14), are satisfied as a consequence of the equations

\[
\Omega x \Gamma^\dagger - \eta x \eta^\dagger + \Gamma \Omega x - \eta \eta^\dagger = 0 , \quad \Gamma (\Omega_t + \frac{1}{2} \eta_1 \eta_1^\dagger - \frac{1}{2} \eta_2 \eta_2^\dagger) + (\Omega_t + \frac{1}{2} \eta_1 \eta_1^\dagger - \frac{1}{2} \eta_2 \eta_2^\dagger) \Gamma^\dagger = 0 ,
\]

obtained by differentiation of (3.4) with respect to $x$, respectively $t$, and using (3.3). Furthermore,

\[
(\eta^\dagger \Omega^{-1} \eta)^\dagger = \eta^\dagger \Omega^{-1} \eta ,
\]

so that

\[
(\eta_1^\dagger \Omega^{-1} \eta_2)^* = \eta_2^\dagger \Omega^{-1} \eta_1 ,
\]

and the last two equations in (4.15) indeed coincide if (4.17) holds. Furthermore, $\eta_1^\dagger \Omega^{-1} \eta_1$ is real, so that $a$ is real if $a_0$ is real.

\footnote{The transformation $t \mapsto i t , x \mapsto i x$ relates the two equations obtained from (4.12) via (4.16).}
5 Conclusion

In this work we explored the nonlinear PDE (1.1). We derived an $n$-fold binary Darboux transformation for this PDE from a universal binary Darboux transformation in bidifferential calculus and exploited it to obtain multi-soliton solutions, also on a plane wave background solution. We derived more generally a binary Darboux transformation for the system (4.12).

Most likely, the multi-soliton solutions admit generalizations in terms of Jacobi elliptic functions. We have seen that there are even two such extensions of the 1-soliton solution.

The plots in this work have been generated using Mathematica [12].

Acknowledgment. I would like to thank Rusuo Ye for reviving my interest in the bidifferential calculus in [3] and for an interesting communication.

References

[1] T.B. Benjamin, J.L. Bona, and J.J Mahony. Model equations for long waves in nonlinear dispersive systems. Phil. Trans. R. Soc. London A, 272:47–78, 1972.

[2] R.I. Joseph and R. Egri. Another possible model equation for long waves in nonlinear dispersive systems. Phys. Lett. A, 61:429–430, 1977.

[3] A. Dimakis, N. Kanning, and F. Müller-Hoissen. Bidifferential calculus, matrix SIT and sine-Gordon equations. Acta Polytechnica, 51:33–37, 2011.

[4] V.B. Matveev and M.A. Salle. Darboux Transformations and Solitons. Springer Series in Nonlinear Dynamics. Springer, Berlin, 1991.

[5] E. Kamke. Gewöhnliche Differentialgleichungen, Lösungsmethoden und Lösungen, volume I. Teubner, Stuttgart, 9th edition, 1977.

[6] M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions. Dover Publications, New York, 1965.

[7] A. Dimakis and F. Müller-Hoissen. Differential calculi on associative algebras and integrable systems. In S. Silvestrov, A. Malyarenko, and M. Rančić, editors, Algebraic Structures and Applications, volume 317 of Springer Proceedings in Mathematics & Statistics, pages 385–410. Springer, 2020.

[8] A. Dimakis and F. Müller-Hoissen. Binary Darboux transformations in bidifferential calculus and integrable reductions of vacuum Einstein equations. SIGMA, 9:009, 2013.

[9] O. Chvartatskyi, A. Dimakis, and F. Müller-Hoissen. Self-consistent sources for integrable equations via deformations of binary Darboux transformations. Lett. Math. Phys., 106:1139–1179, 2016.

[10] A. Dimakis and F. Müller-Hoissen. Solutions of matrix NLS systems and their discretizations: a unified treatment. Inverse Problems, 26:095007, 2010.

[11] A. Dimakis and F. Müller-Hoissen. Bidifferential calculus approach to AKNS hierarchies and their solutions. SIGMA, 6:055, 2010.

[12] Wolfram Research, Inc. Mathematica, Version 13.0. Champaign, IL, 2021.