Spectral integration and spectral theory for non-Archimedean Banach spaces.*

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Abstract

Banach algebras over arbitrary complete non-Archimedean fields are considered such that operators may be non-analytic. There are considered different types of Banach spaces over non-Archimedean fields. We have determined the spectrum of some closed commutative subalgebras of the Banach algebra $\mathcal{L}(E)$ of the continuous linear operators on a free Banach space $E$ generated by projectors. The spectral integration of non-Archimedean Banach algebras is investigated. For this a spectral measure is defined. Its several properties are proved. The non-Archimedean analog of Stone theorem is proved. It contains also the case of $C$-algebras $C_\infty(X, K)$. A particular case of a representation of a $C$-algebra with the help of a $L(\hat{A}, \mu, K)$-projection-valued measure is proved. Spectral theorems for operators and families of commuting linear continuous operators on the non-Archimedean Banach space are considered.

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1 Introduction.

This paper is devoted to the non-Archimedean theory of spectral integration with the help of the projection-valued measure. Spectral integration plays very important role in the theory of Banach algebras, theory of operators and has applications to the representation theory of groups and algebras in the classical case of the field of complex numbers $\mathbb{C}$ \cite{7, 12, 16, 20}. There are also several works about non-Archimedean Banach algebra theory, which show that there are substantial differences between the non-Archimedean and classical cases \cite{3, 5, 6, 9, 10, 11, 15, 21, 22, 25, 28}. In the papers \cite{3, 28} analytic operators over $\mathbb{C}_p$ were considered and the Shnirelman integration of analytic functions was used, which differs strongly from the non-Archimedean integration theory related with the measure theory \cite{25}. In the non-Archimedean case spectral theory differs from the classical results of Gelfand-Mazur, because quotients of commutative Banach algebras over a field $K$ by maximal ideals may be fields $F$, which contain $K$ as a proper subfield \cite{25}. In general for each non-Archimedean field $K$ there exists its extension $F$ such that a field $F \neq K$ \cite{4, 27}.

Ideals and maximal ideals of non-Archimedean commutative $E$-algebras (see §5.1.1) and $C$-algebras were investigated in \cite{25, 26}. In the works \cite{5, 6} it was shown the failure of the spectral theory in the non-Archimedean analog of the Hilbert space and it was shown that even symmetry properties of matrices lead to the enlargement of the initial field while a diagonalisation procedure. In the papers \cite{9, 10, 11} were analysed formulas of spectral radius and different notions of spectrum and analysed some aspects of structures of non-Archimedean Banach algebras. In the book \cite{25} and references therein general theory of non-Archimedean Banach algebras and their isomorphisms was considered. It was introduced the notion of $C$-algebras in the non-Archimedean case apart from the classical $C^*$-algebras. There are principal differences in the orthogonality in the Hilbert space over $\mathbb{C}$ and orthogonality in the non-Archimedean Banach space. Therefore, symmetry properties of operators do not play the same role in the non-Archimedean case as in the classical case.

This paper treats another aspects of the non-Archimedean algebra theory and theory of operators. Banach algebras over arbitrary complete non-Archimedean fields are considered such that operators may be non-analytic. There are considered different types of Banach spaces over non-Archimedean
fields. In §§2-4 are considered specific spaces. In §5 are considered general cases.

Let $K$ be a field. A non-Archimedean valuation on $K$ is a function $|\cdot| : K \to \mathbb{R}$ such that:
1. $|x| \geq 0$ for each $x \in K$;
2. $|x| = 0$ if and only if $x = 0$;
3. $|x + y| \leq \max(|x|, |y|)$ for each $x$ and $y \in K$;
4. $|xy| = |x||y|$ for each $x$ and $y \in K$.

The field $K$ is called topologically complete if it is complete relative to the following metric: $\rho(x, y) = |x - y|$ for each $x$ and $y \in K$. A topological vector space $E$ over $K$ with the non-Archimedean valuation may have a norm $\|\cdot\|$ such that its restriction on each one dimensional subspace over $K$ coincides with the valuation $|\cdot|$. If $E$ is complete relative to such norm $\|\cdot\|$, then it is called the Banach space. Such fields and topological vector spaces are called non-Archimedean. An algebra $X$ over $K$ is called Banach, if it is a Banach space as a topological vector space and the multiplication in it is continuous such that $\|xy\| \leq \|x\|\|y\|$ for each $x$ and $y$ in $X$. A finite or infinite sequence $(x_j : j \in \Lambda)$ of elements in a normed space $E$ is called orthogonal, if $\|\sum_{j \in \Lambda} \alpha_j x_j\| = \max(\|\alpha_j x_j\| : j \in \Lambda)$ for each $\alpha_j \in K$ for which $\lim_{j \to \infty} \alpha_j x_j = 0$. We consider the infinite topologically complete field $K$ with the nontrivial non-Archimedean valuation.

A non-Archimedean Banach space $E$ is said to be free if there exists a family $(e_j : j \in I) \subset E$ such that any element $x \in E$ can be written in the form of convergent sum $x = \sum_{j \in I} x_j e_j$, i.e., $\lim_{j \to \infty} x_j e_j = 0$, and $\|x\| = \sup_{j \in I} \|x_j\||e_j||$ (see §2). In §3 ultrametric Hilbert spaces are considered. In §4 we have determined the spectrum of some closed commutative subalgebras of the Banach algebra $\mathcal{L}(E)$ of the continuous linear operators of $E$ generated by projectors.

§5 is devoted to the spectral integration. We introduce another definition of $E$-algebras in §5.1 apart from [26]. In Propositions 5.3 and 5.4 we have proved that they are contained in the class of $E$-algebras and $C$-algebras considered in [25, 26]. In §5.2 a spectral measure is defined. In §§5.6-5.10 its several properties are proved. In §5.12 the non-Archimedean analog of Stone theorem is proved. It contains also the case of $C$-algebras $C_\infty(X, K)$. A particular case of a representation of a $C$-algebra with the help of $L(\hat{A}, \mu, K)$-projection-valued measure is proved in Theorem 5.15. Spectral theorems for
operators and families of commuting linear continuous operators on a non-Archimedean Banach space are considered in §§5.17, 5.18.

2 Free Banach spaces

1. Let $E$ be the free Banach space with an orthogonal base $(e_j : j \in I)$. The topological dual $E'$ of $E$ is a Banach space with respect to the norm defined for $x' \in E'$ by $\|x'\| = \sup_{x \neq 0} \frac{<x',x>}{\|x\|}$. For $x' \in E'$ and $y \in E$, one defines an element $(x' \otimes y)$ of the Banach algebra of continuous linear operators $\mathcal{L}(E)$ on the space $E$ by setting for $x \in E$, $(x' \otimes y)(x) = <x',x>y$ with norm $\|x' \otimes y\| = \|x'\||y\|$. If $E$ is a free Banach space with base $(e_j : j \in I)$, any $u \in \mathcal{L}(E)$ can be written as a pointwise convergent sum $u = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_j \otimes e_i$. Hence $\lim_{i \in I} \alpha_{ij} e_i = 0$ for each $j \in I$. Moreover $\|u\| = \sup_{i,j} |\alpha_{ij}| \|e'_j\| \|e_i\|$.

Notice that $\|e'_j\| = \frac{1}{\|e_j\|}$. Let $\mathcal{L}_0(E) = \{u : u = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}(E) : \lim_{j \in I} \alpha_{ij} e'_j = 0 \text{ for each } i \in I\}$.

1.1. Theorem. $\mathcal{L}_0(E)$ is a closed subalgebra in $\mathcal{L}(E)$ with the unit element of $\mathcal{L}(E)$.

Proof. Let $u, v \in \mathcal{L}_0(E)$, $u = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_j \otimes e_i$ and $v = \sum_{(i,j) \in I \times I} \beta_{ij} e'_j \otimes e_i$, then $\lim_{i \in I} \alpha_{ij} e_i = 0 = \lim_{i \in I} \beta_{ij} e_i$ for each $j \in I$, and $\lim_{i \in I} \alpha_{ij} e_i = 0 = \lim_{i \in I} \beta_{ij} e_i$ for each $j \in I$. One has $u \circ v = \sum_{(i,j) \in I \times I} (\sum_{k \in I} \alpha_{ik} \beta_{kj}) e'_j \otimes e_i$. Let $i \in I$, $\lim_{k \in I} \alpha_{ik} e'_k = 0$, that is, for each $\varepsilon > 0$, there exists $J_{\varepsilon}(i)$ a finite subset of $I$ such that for each $k \notin J_{\varepsilon}(i)$, $\|\alpha_{ik} e'_k\| < \varepsilon$. Hence $\| (\sum_{k \in I} \alpha_{ik} \beta_{kj}) e'_j \| = \| \sum_{k \in J_{\varepsilon}(i)} (\alpha_{ik} \beta_{kj}) e'_j \| + \sum_{k \notin J_{\varepsilon}(i)} (\alpha_{ik} \beta_{kj}) e'_j \| \leq \max_{k \in J_{\varepsilon}(i)} \| \alpha_{ik} e'_k \| \| \beta_{kj} e_k \| \| e'_j \| \| e_i \| \| e'_j \| \| e'_j \|, \sup_{k \notin J_{\varepsilon}(i)} \| \alpha_{ik} \beta_{kj} e'_j \| \leq \max_{k \in J_{\varepsilon}(i)} \| \beta_{kj} e'_j \| \| e_k \| \| e'_j \|, \varepsilon \| v \| \| e'_i \|$. Since $\lim_{j \in I} \| \beta_{kj} e'_j \| = 0$ for each $k \in J_{\varepsilon}(i)$, one has $\lim_{j \in I} ((\sum_{k \in I} \alpha_{ik} \beta_{kj}) e'_j) = 0$ for each $i \in I$, therefore $u \circ v \in \mathcal{L}_0(E)$. The identity map $id$ being given by $id = \sum_{i \in I} e'_i \otimes e_i$, one has $\alpha_{ii} = 1$.
and $\alpha_{ij} = 0$ if $i \neq j$. Therefore $\lim_{i} \alpha_{ij} e_i = 0$ for each $j \in I$, and $\lim_{j} \alpha_{ij} e'_{j} = 0$ for each $i \in I$. Hence $id \in \mathcal{L}_{0}(E)$. Let $u = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_{j} \otimes e_{i}$ be in the closure of $\mathcal{L}_{0}(E)$. For all $\varepsilon > 0$, there exists $u_\varepsilon = \sum_{(i,j) \in I \times I} \alpha_{ij}(\varepsilon) e'_{j} \otimes e_{i} \in \mathcal{L}_{0}(E)$ such that $\|u - u_\varepsilon\| = \text{sup}_{i,j} |\alpha_{ij} - \alpha_{ij}(\varepsilon)||e'_{j}||e_{i}|| < \varepsilon$. Hence for all $i, j \in I$, one has $|\alpha_{ij||e'_{j}||e_{i}| \leq \text{max}(\varepsilon, |\alpha_{ij}(\varepsilon)||e'_{j}||e_{i}|)$. One obtains $\lim_{i} \|\alpha_{ij} e_{i}\| = 0$ for each $j \in I$ and $\lim_{j} \|\alpha_{ij} e'_{j}\| = 0$ for each $i \in I$. Therefore $u \in \mathcal{L}_{0}(E)$ and $\mathcal{L}_{0}(E)$ is closed.

2. Suppose that the orthogonal basis is orthonormal, i.e., $\|e_{j}\| = 1$ for each $j \in I$. Then $u = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_{j} \otimes e_{i} \in \mathcal{L}_{0}(E)$, if and only if $\lim_{i} \alpha_{ij} = 0$ for each $j \in I$ and $\lim_{j} \alpha_{ij} = 0$ for each $i \in I$. Setting for $u = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_{j} \otimes e_{i}$ $u \in \mathcal{L}_{0}(E)$, $u^* = \sum_{(i,j) \in I \times I} \alpha_{ji} e'_{j} \otimes e_{i}$, one sees that $u^* \in \mathcal{L}_{0}(E)$, called the adjoint of $u$. One verifies easily the following.

2.1. Proposition. An element $u \in \mathcal{L}(E)$ has an adjoint $u^*$ if and only if $u \in \mathcal{L}_{0}(E)$. Let $u, v \in \mathcal{L}_{0}(E), \lambda \in K$. One has $(u + \lambda v)^* = u^* + \lambda v^*$; $(u \circ v)^* = v^* \circ u^*$; $u^{**} = u$. Moreover $\|u^*\| = \|u\|$. 

As usually, one says that $u \in \mathcal{L}_{0}(E)$ is normal (respectively unitary) if $u \circ u^* = u^* \circ u$ (respectively $u \circ u^* = id = u^* \circ u$). And $u$ is self-adjoint if $u = u^*$, this is equivalent here to say that the matrix of $u$ is symmetric.

3. Note. (i). One has $\|u\| = \|u^*\|$. However, in general $\|u \circ u^*\| \neq \|u\|^2$. For example, if $I$ is the set of positive integers, and $E$ with orthogonal base $(e_{n} : n \geq 1)$, let $a, b \in K$. The operator $u$ defined by $u(e_{1}) = ae_{1} + be_{2}, u(e_{2}) = be_{1} - ae_{2}, u(e_{3}) = ce_{3}$, and $u(e_{n}) = 0$ for $n \geq 4$. One sees that $u$ is self-adjoint. If $i = \sqrt{-1} \in K$; then taking $b = ia$ and $|c| < |a|$, one sees that $\|u\|^2 = |c|^2 < |a|^2 = \|u\|^2$.

(ii). It should be interesting to characterize the elements of $\mathcal{L}_{0}(E)$ that are normal, unitary. Considering, whenever the base of $E$ is orthonormal, the bilinear form $f$ on $E$ defined by $f(x, y) = \sum_{i \in I} x_{i} y_{i}$, one obtains that the above definition of an adjoint $u^*$ of an element $u \in \mathcal{L}_{0}(E)$ is equivalent to say that $f(u(x), y) = f(x, u^*(y))$ for each $x$ and $y \in E$. In fact, here the adjoint of an operator is its transposition. This example is related to ultrametric Hilbert
spaces.

3 Ultrametric Hilbert spaces.

For the so called ultrametric Hilbert spaces one can also define the adjoint of an operator with respect to an appropriate bilinear symmetric form.

1. Remark and Definition. H. Ochsenius and W.H. Schikhof write in [24] "as a slogan: There are no $p$-adic Hilbert spaces". Nevertheless we shall give a definition of $p$-adic Hilbert spaces [cf. for example, [17, 18] for some fields with infinite rank valuation]. Let $\omega = (\omega_i)_{i \geq 0}$ be a sequence of non-zero elements of $K$. Let us consider the free Banach space $E_\omega = c_0(\mathbb{N}, K, (|\omega_i|)_{i \geq 0}) = \{ x : x = (x_i)_{i \geq 0} \subset K; \lim_{i \to +\infty} |x_i||\omega_i|^\frac{1}{2} = 0 \}$. Then $x = (x_i)_{i \geq 0} \in E_\omega \iff \lim_{i \to +\infty} x_i^2 \omega_i = 0$. Setting $e_i = (\delta_{i,j})_{j \geq 0}$ (Kronecker symbol), one has that $(e_i : i \geq 0)$ is an orthogonal base of $E_\omega : \forall x \in E_\omega$, $x = \sum_{i \geq 0} x_i e_i$ and $\|x\| = \sup_{i \geq 0} |x_i||e_i|| = \sup_{i \geq 0} |x_i||\omega_i|^\frac{1}{2}$, in particular, $|e_i|| = |\omega_i|^\frac{1}{2}$ for each $i \geq 0$. Let $f_\omega : E_\omega \times E_\omega \to K$ be defined by $f_\omega(x, y) = \sum_{i \geq 0} \omega_i x_i y_i$. It is readily seen that $f_\omega$ is a bilinear symmetric form on $E_\omega$, with $|f_\omega(x, y)| \leq \|x\|\|y\|$, i.e., the bilinear form $f_\omega$ is continuous. Moreover $f_\omega$ is non-degenerate, i.e. $f_\omega(x, y) = 0$ for each $y \in E_\omega \implies x = 0$. Furthermore $f_\omega(x, x) = \sum_{i \geq 0} \omega_i x_i^2$ and $f_\omega(e_i, e_j) = \omega_i \delta_{i,j}$ for $i$ and $j \geq 0$. The space $E_\omega$ is called a $p$-adic Hilbert space.

2. Note. (i). It may happen that $|f_\omega(x, x)| < \|x\|^2$ for some $x \in E_\omega$ and even worse, $E_\omega$ contains isotropic elements $x \neq 0$, i.e., $f_\omega(x, x) = 0$.

(ii). Let $V$ be a subspace of $E_\omega$ and $V^\perp = \{ x \in E_\omega : f_\omega(x, y) = 0, \forall y \in V \}$. The fundamental property on subspaces of the classical Hilbert space $H : V = V^\perp \Rightarrow V \oplus V^\perp = H$ fails to be true in the $p$-adic case. This explains the claim of H. Ochsenius and W.H. Schikhof.

3. Remark. A free Banach space $E$ with an orthogonal base $(e_i : i \geq 0)$ can be given a structure of a $p$-adic Hilbert space if and only if there exists $(\omega_i : i \geq 0) \subset K$ such that $\|e_i\| = |\omega_i|^\frac{1}{2}$ for each $i \geq 0$. Furthermore if $K$ contains a square of any of its element, then any $p$-adic Hilbert is isomorphic, in a natural way, to the space $c_0(\mathbb{N}, K)$. 
4. **Note.** Let \( u, v \in \mathcal{L}(E_\omega) \); one has \( u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \) and \( v = \sum_{i,j} \beta_{ij} e'_j \otimes e_i \) with \( \lim_{i \to +\infty} |\alpha_{ij}| \omega_i^{\frac{1}{2}} = 0 = \lim_{i \to +\infty} |\beta_{ij}| \omega_i^{\frac{1}{2}} \) for each \( j \geq 0 \). Furthermore the norm of \( u \in \mathcal{L}(E_\omega) \) is given by \( \|u\| = \sup_{i,j} \frac{|\omega_i|^{\frac{1}{2}}|\alpha_{ij}|}{|\omega_j|^{\frac{1}{2}}} \).

The operator \( v \) is said to be an adjoint of \( u \) with respect to \( f_\omega \iff f_\omega(u(x), y) = f_\omega(x, v(y)) \), for all \( x, y \in E_\omega \). Since \( f_\omega \) is symmetric \( u \) is an adjoint of \( v \).

Since \( f_\omega \) is non degenerate, if an operator \( u \) has an adjoint, this adjoint is unique and will be denoted by \( u^* \). Since \( (e_i : i \geq 0) \) is an orthogonal basis of \( E_\omega \), one has that \( u \) is an adjoint of \( u \) if and only if \( f_\omega((u(e_i), e_j) = f_\omega(e_i, v(e_j)) \), for each \( i \) and \( j \geq 0 \). That is, \( f_\omega(\sum_{k \geq 0} \alpha_{kj} e_k, e_j) = \alpha_{ji} \omega_j = f_\omega(e_i, \sum_{k \geq 0} \beta_{kj} e_k) = \beta_{ij} \omega_i, \quad \forall i, j \geq 0 \iff \beta_{ij} = \omega_i^{-1} \omega_j \alpha_{ji}, \quad \forall i, j \geq 0 \). Furthermore one must have \( \lim_{i \to +\infty} |\beta_{ij}| \omega_i^{\frac{1}{2}} = 0 \) for each \( j \geq 0 \), that is,

\[
\lim_{i \to +\infty} |\omega_i|^{\frac{1}{2}} |\omega_j^{-1}| |\alpha_{ji}| = |\omega_j| \lim_{i \to +\infty} |\omega_i|^{-\frac{1}{2}} |\alpha_{ji}| = 0, \quad \forall j \geq 0.
\]

Hence \( \lim_{i \to +\infty} |\omega_i|^{-\frac{1}{2}} |\alpha_{ji}| = 0 \) for each \( j \geq 0 \). We have proved the following.

5. **Theorem.** Let \( (\omega_i)_{i \geq 0} \subset K^* \) and \( E_\omega = c_0(\mathbb{N}, K, (|\omega_i|^{\frac{1}{2}})_{i \geq 0}) \) be the \( p \)-adic Hilbert space associated with \( \omega \). Let \( u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}(E_\omega) \). Then \( u \) has an adjoint \( v = u^* \in \mathcal{L}(E_\omega) \) if and only if \( \lim_{j \to +\infty} |\omega_j|^{-\frac{1}{2}} |\alpha_{ij}| = 0 \) for each \( i \geq 0 \). In this condition, \( u^* = \sum_{i,j} \omega_i^{-1} \omega_j \alpha_{ji} e'_j \otimes e_i \).

It follows from the above theorem that not any continuous linear operator of \( E_\omega \) has an adjoint: it is another difference with classical Hilbert spaces. Let \( \mathcal{L}_0(E_\omega) = \{ u : u = \sum_{i \geq 0, j \geq 0} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}(E_\omega); \lim_{j \to +\infty} |\omega_j|^{-\frac{1}{2}} |\alpha_{ij}| = 0, \forall i \geq 0 \} \). Let us remind that \( u = \sum_{j} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}(E_\omega) \) is equivalent to

\[
\lim_{i \to +\infty} |\omega_i|^{\frac{1}{2}} |\alpha_{ij}| = 0 \quad \text{for each} \quad j \geq 0.
\]

It is readily seen, as in Theorem 1, that \( \mathcal{L}_0(E_\omega) \) is a closed unitary subalgebra of \( \mathcal{L}(E_\omega) \).

6. **Corollary.** An element \( u \in \mathcal{L}(E_\omega) \) has an adjoint \( u^* \) if and only if \( u \in \mathcal{L}_0(E_\omega) \). Let \( u, v \in \mathcal{L}_0(E_\omega), \lambda \in K \). One has \( (u + \lambda v)^* = u^* + \lambda v^*; (u \circ v)^* = v^* \circ u^*; \quad u^{**} = u \). Moreover \( \|u^*\| = \|u\| \).
Proof. We only prove $\|u^*\| = \|u\|$. Since for $u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}_0(E_\omega)$ one has $\|u\| = \sup_{i,j} \frac{|\omega_i|^{\frac{1}{2}} |\alpha_{ij}|}{|\omega_j|^{\frac{1}{2}}}$ and $u^* = \sum_{i,j} \omega_j \omega_i^{-1} \alpha_{ij} e'_j \otimes e_i$, one obtains $\|u^*\| = \sup_{i,j} \frac{|\omega_i|^{\frac{1}{2}}}{|\omega_j|^{\frac{1}{2}}} |\omega_j| |\alpha_{ij}|^{-1} = \sup_{i,j} \frac{1}{2} \frac{|\omega_j|^{\frac{1}{2}}}{|\omega_i|^{\frac{1}{2}}} |\alpha_{ji}| = \|u\|$.

7. Remark. (i). $u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}_0(E_\omega)$ is self-adjoint, i.e., $u = u^*$ if and only if $\alpha_{ji} = \omega_i \omega_j^{-1} \alpha_{ij}$, for each $i \geq 0$ and each $j \geq 0$.

(ii). Examples of self-adjoint operators on ultrametric Hilbert spaces and study of their spectrum are given in [1, 2, 19, 6].

4 Closed subalgebras generated by projectors.

1. Let $J$ be a subset of $I$ and $E$ be a free Banach space with orthogonal basis $(e_j : j \in I)$. The linear operator $p_J = \sum_{i \in J} e'_i \otimes e_i$ of $E$ belongs to $\mathcal{L}_0(E)$. Let $\mathcal{D} = \{u : u = \sum_{i \in I} \lambda_i e'_i \otimes e_i \in \mathcal{L}_0(E) ; \sup_{i \in I} |\lambda_i| < +\infty\}$. It is clear that $\mathcal{D}$ is isometrically isomorphic to the the algebra of bounded families $\ell^\infty(I, K)$. Let $\text{Hom}_{alg}(\mathcal{D}, K)$ denotes a family of all algebra homomorphisms of $\mathcal{D}$ into $K$. Consider the spectrum $\mathcal{X}(\mathcal{D}) = \text{Hom}_{alg}(\mathcal{D}, K)$ in a topology inherited from the Tihonov topology of the product $K^\mathcal{D}$ of copies of $K$.

1.1. Proposition. (i). An element $u = \sum_{i \in I} \lambda_i e'_i \otimes e_i \in \mathcal{D}$ is an idempotent if and only if there exists $J \subset I$ such that $u = p_J$.

(ii). The spectrum $\mathcal{X}(\mathcal{D})$ is homeomorphic to the subset of ultrafilters on $I : \Phi_c = \{U : U$ is an ultrafilter on $I$, such that for all $u = \sum_{i \in I} \lambda_i e'_i \otimes e_i \in \mathcal{D}$, the limit $\lim_{U} \lambda_i$ exists in $K\}$. 

Proof. (i). Let $u = \sum_{i \in I} \lambda_i e'_i \otimes e_i$; then $u \circ u = u$ if and only if $\sum_{i \in I} \lambda_i^2 e'_i \otimes e_i = \sum_{i \in I} \lambda_i e'_i \otimes e_i$, if and only if $\lambda_i^2 = \lambda_i$ for each $i \in I$, if and only if $\lambda_i = 0$ or $\lambda_i = 1$. Setting $J = \{i : i \in I ; \lambda_i = 1\}$, one has $u = p_J$.

(ii). Let $\chi$ be a vanishing character of $\mathcal{D}$, that is an algebra homomorphism (necessary continuous) of $\mathcal{D}$ into $K$. For all $J, L \subset I$ one has $p_J \circ p_L = p_{J \cap L}$, hence
$p_J \circ p_{J'} = p_0 = 0$, where $J^c = I \setminus J$. Furthermore $\chi(p_J) = \chi(p_J) \chi(p_J)$ implies that $\chi(p_J) = 0$ or 1. Let $\mathcal{U}_\chi = \{ J : J \subset I; \chi(p_J) = 1 \}$. This family of subsets is an ultrafilter. Indeed, $\emptyset \notin \mathcal{U}_\chi$. If $J \subset L$ with $J \in \mathcal{U}_\chi$, one has $1 = \chi(p_J) = \chi(p_{J-L}) = \chi(p_J) \chi(p_L) = \chi(p_L)$, hence $L \in \mathcal{U}_\chi$. On the other hand, for $J \subset I$, one has $1 = \chi(J) = \chi(p_J) + \chi(p_{J^c}) = \chi(J) + \chi(p_{J^c})$ with $\chi(p_J) = 1$, or 0 and $\chi(p_{J^c}) = 1$, or 0. If $\chi(p_J) = 1$, one has $\chi(p_{J^c}) = 0$, and if $\chi(p_{J^c}) = 1$, one has $\chi(p_J) = 0$. Hence $J \in \mathcal{U}_\chi$ or $J^c \in \mathcal{U}_\chi$. Let $u = \sum_{i \in I} \lambda_i e'_i \otimes e_i \in \mathcal{D}$. Put $\chi(u) = \lambda \in K$; then for all $J \in \mathcal{U}_\chi$, one has $\chi(up_J) = \chi(u) = \lambda \chi(p_J)$. Therefore $\chi(up_J - \lambda p_J) = 0$, i.e., $up_J - \lambda p_J \in \ker \chi$. Set $\phi_{\mathcal{U}_\chi}(u) = \lim_{\mathcal{U}_\chi} |\lambda_i|$. It is well known and readily seen that $\phi_{\mathcal{U}_\chi}$ is a multiplicative semi-norm on $\mathcal{D}$ and that ker $\phi_{\mathcal{U}_\chi} = \{ u : u \in \mathcal{D}; \phi_{\mathcal{U}_\chi}(u) = 0 \}$ is a maximal ideal of $\mathcal{D}$, since $\mathcal{D}$ is isomorphic to $\ell^\infty(I, K)$. On the other hand $|\chi(up_J)| \leq \| up_J \| = \sup_{i \in J} |\lambda_i|$ for each $J \in \mathcal{U}_\chi$. It follows that $|\chi(u)| = |\chi(up_J)| \leq \inf_{J \in \mathcal{U}_\chi} \sup_{i \in J} |\lambda_i| = \phi_{\mathcal{U}_\chi}(u)$. Hence, one has ker $\phi_{\mathcal{U}_\chi} \subset$ ker $\chi$ and ker $\phi_{\mathcal{U}_\chi} = \ker \chi$. Let $J \in \mathcal{U}_\chi$, one deduces from $up_J - \lambda p_J \in \ker \chi = \ker \phi_{\mathcal{U}_\chi}$, that $0 = \phi_{\mathcal{U}_\chi}(up_J - \lambda p_J) = \lim_{\mathcal{U}_\chi} |\lambda_i - \lambda|$. It follows that $\lim_{\mathcal{U}_\chi} \lambda_i = \lambda$ exists in $K$. Moreover, $\chi(u) = \lambda = \lim_{\mathcal{U}_\chi} |\lambda_i|$, and one sees that $\chi = \chi_{\mathcal{U}_\chi}$. Reciprocally, if $\mathcal{U}$ is an ultrafilter on $I$ such that for all $u = \sum_{i \in I} \lambda_i e'_i \otimes e_i \in \mathcal{D}$, one has $\lim_{\mathcal{U}} \lambda_i$ exists in $K$; then setting $\chi_{\mathcal{U}}(u) = \lim_{\mathcal{U}} \lambda_i$, it is readily seen that $\chi_{\mathcal{U}}$ is a character of $\mathcal{D}$. Moreover, for all $J \in \mathcal{U}$ one has $\chi_{\mathcal{U}}(p_J) = \lim_{\mathcal{U}} 1 = 1$, that is $J \in \mathcal{U}_{\mathcal{U}}$ and $\mathcal{U} = \mathcal{U}_{\mathcal{U}_{\mathcal{U}}}$. The theorem is proved if one considers on $\mathcal{X}(\mathcal{D})$ the weak *-topology and on $\Phi_c$ the topology induced by the natural topology on the space of ultrafilters, which is the weakest topology on $\Phi_c$ relative to which the mapping $\lim : \Phi_c \to K$ is continuous.

2. **Remark.** (i). If $K$ is locally compact, then for any bounded family $(\lambda_i)_{i \in I} \subset K$, the limit $\lim_{\mathcal{U}} \lambda_i$ exists in $K$. Therefore, $\Phi_c$ is equal to the entire set of all ultrafilters on $I$ and $\mathcal{X}(\mathcal{D})$ is compact, homeomorphic to the Stone-Čech compactification $\beta(I)$ of the discrete topological space $I$.

(ii). If $K$ is not spherically complete and $I$ is a small set: i.e. the cardinal of $I$ is nonmeasurable, it is well known that the continuous dual of $\ell^\infty(I, K)$ is equal to the space $c_0(I, K)$ of the families converging to zero (cf. [25] Theorem 4.21). Then, one can prove that $\mathcal{X}(\mathcal{D})$ is homeomorphic with $I$. 


3. Note. For $K$ spherically complete, not locally compact, it is interesting to find explicit conditions on an ultrafilter $U$ in such a way that $\lim_{\nu \in U} \lambda_i$ exists for any bounded family $(\lambda_i : i \in I) \subset K$. One can try to use Banach limits, i.e., continuous linear forms on $\ell^\infty(I, K)$ that extend the usual continuous linear form $\lim$ defined on the subspace $c_0(I, K)$ of convergent familiies. Let $(J_\nu : \nu \in \Lambda)$ be a family of subsets of $I$, such that $J_\nu \cap J_\mu = \emptyset$ for $\nu \neq \mu$. Putting $p_\nu = \sum_{i \in J_\nu} e_i' \otimes e_i$, one obtains $p_\nu \circ p_\mu = \delta_{\nu, \mu} p_\nu$, for $\nu \neq \mu$. Hence the subalgebra with the unity $B$ of $L_0(E)$, generated by $(p_\nu : \nu \in \Lambda)$ is equal to $K.id \oplus \left( \bigoplus_{\nu \in \Lambda} K.p_\nu \right)$. Indeed if $u = \alpha_0id + u_1$ and $v = \beta_0id + v_1$ with $u_1 = \sum_{\nu \in \Lambda} \alpha_\nu p_\nu$ and $v_1 = \sum_{\nu \in \Lambda} \beta_\nu p_\nu$ (finite sums), one has $u \circ v = \alpha_0 \beta_0id + \alpha_0v_1 + \beta_0u_1 + u_1 \circ v_1 = \alpha_0 \beta_0id + \sum_{\nu \in \Lambda} (\alpha_0 \beta_\nu + \alpha_\nu \beta_0 + \alpha_\nu \beta_\nu)p_\nu \in B$.

On the other hand, since $u = \alpha_0id + \sum_{\nu \in \Lambda} p_\nu$ with $\Gamma = \{ \nu : \nu \in \Lambda; \alpha_\nu \neq 0 \}$ finite and $I = \left( \bigcup_{\nu \in \Gamma} J_\nu \right) \cup \left( \bigcap_{\nu \in \Gamma} J_\nu^c \right)$ (a partition), one has $u = \alpha_0 \sum_{i \in I} e'_i \otimes e_i + \sum_{\nu \in \Gamma} \alpha_\nu \sum_{i \in J_\nu} e'_i \otimes e_i = \alpha_0 \sum_{i \in \bigcap_{\nu \in \Gamma} J_\nu^c} e'_i \otimes e_i + \sum_{\nu \in \Gamma} \sum_{i \in J_\nu} (\alpha_\nu + \alpha_\nu'e_i' \otimes e_i$. Hence $\|u\| = \max(|\alpha_0|, \max_{\nu \in \Lambda} |\alpha_\nu|)$. In other words $\{id\} \cup \{p_\nu : \nu \in \Lambda\}$ is an orthonormal family in $L_0(E)$.

Proof. Since $\|u\| = \max_{\nu \in \Lambda} |\alpha_\nu|$, and $\max_{\nu \in \Lambda} |\alpha_0 + \alpha_\nu| \leq \max_{\nu \in \Lambda} (|\alpha_0|, \max_{\nu \in \Lambda} |\alpha_\nu|)$, one has $\|u\| = \max_{\nu \in \Lambda} |\alpha_\nu|$. Moreover, $|\alpha_0| \leq \|u\|$. Hence for $\nu \in \Lambda$, one has $|\alpha_\nu| = |\alpha_\nu + \alpha_0 - \alpha_0| \leq \max(|\alpha_\nu + \alpha_0|, |\alpha_0|) \leq \|u\|$. It follows that $\max_{\nu \in \Lambda} |\alpha_\nu| \leq \|u\|$, and Lemma 4 is proved.

5. Lemma. Assume that $(e_i : i \in I)$ is an orthonormal basis of $E$ or $E$ is an ultrametric Hilbert space. Then any $u \in B$ is self-adjoint, i.e., $u^* = u$, and $\|u^2\| = \|u\|^2$.

Proof. That any element of $B$ is self-adjoint is easy to verify. Let $u = \alpha_0id + \sum_{\nu \in \Lambda} \alpha_\nu p_\nu \in B$, one has $u^2 = \alpha_0^2id + \sum_{\nu \in \Lambda} (2\alpha_0\alpha_\nu + \alpha_\nu^2) p_\nu \in B$. Hence $\|u^2\| = \max_{\nu \in \Lambda} (|\alpha_0|^2, 2\alpha_0\alpha_\nu + \alpha_\nu^2) = \max_{\nu \in \Lambda} (\max_{\nu \in \Lambda} |\alpha_0 + \alpha_\nu|)^2 = \|u\|^2$. 

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6. Note. In fact, Lemma 5 is true for $u \in D$. Let $E$ be a free Banach space with orthogonal basis $\{e_i : i \in I\}$. Fix $\pi \in K$ such that $0 < |\pi| < 1$. There exists for any $i \in I$ an integer $n_i \in \mathbb{Z}$ such that $|\pi|^{n_i+1} < \|e_i\| \leq |\pi|^{n_i}$. For $x = \sum_{i \in I} x_i e_i$, one has $\lim x_i \pi^{n_i} = 0$. Hence one defines on $E$ a norm by setting $\|x\|_\pi = \sup_{i \in I} \|x_i\| |\pi|^{n_i}$; this norm is equivalent to $\|\cdot\|$ with $|\pi|\|x\|_\pi \leq \|x\| \leq \|x\|_\pi$. Furthermore, setting for $x = \sum_{i \in I} x_i e_i$ and $y = \sum_{i \in I} y_i e_i \in E$, $f_\pi(x, y) = \sum_{i \in I} \pi^{2n_i} x_i y_i$, one has a continuous, non degenerated, bilinear form on $E$ such that $|f_\pi(x, y)| \leq \|x\|_\pi \|y\|_\pi \leq |\pi|^{-2} \|x\| \|y\|$. Therefore, one obtains on $E$, a structure of ultrametric Hilbert space $E_\pi = (E, \|\cdot\|_\pi, f_\pi)$. Since the norms $\|\cdot\|$ and $\|\cdot\|_\pi$ are equivalent, one has $\mathcal{L}(E) = \mathcal{L}(E_\pi)$ and $\mathcal{L}_0(E) = \mathcal{L}_0(E_\pi)$. The norms on $\mathcal{L}(E)$ induced by $\|\cdot\|$ and $\|\cdot\|_\pi$ are equivalent with $|\pi|\|u\|_\pi \leq \|u\| \leq |\pi|^{-1}\|u\|_\pi$. As in §3.4, one defines the adjoint $u^*$ of $u \in \mathcal{L}(E)$ with respect to $f_\pi$. One obtains the results stated in Theorem 3.5, that is: $u$ admit an adjoint with respect to $f_\pi$ if and only if $u \in \mathcal{L}_0(E)$. Furthermore if $u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}_0(E)$, one has $u^* = \sum_{i,j} \pi^{n_j - n_i} \alpha_{ji} e'_j \otimes e_i$, and $u$ is self-adjoint, i.e. $u^* = u$ if and only if $\pi^{n_i} \alpha_{ij} = \pi^{n_j} \alpha_{ji}$, for all $i, j \in I$.

7. Note. Let $\pi'$ be another element of $K$ such that $0 < |\pi'| < 1$; let also $\{m_i : i \in I\} \subset \mathbb{Z}$ be defined by $|\pi'|^{m_i+1} < \|e_i\| \leq |\pi'|^{m_i}$. Then the adjoint $u^* = \sum_{i,j} \pi^{m_j - m_i} \alpha_{ji} e'_j \otimes e_i$ of $u$ with respect to $f_{\pi'}$ coincides with $u^*$ if and only if $\pi^{n_j - n_i} \alpha_{ji} = \pi^{m_j - m_i} \alpha_{ji}$, for each $i$ and $j \in I$. If this is true for all $u \in \mathcal{L}_0(E)$, one has $\pi^{n_j - n_i} = \pi^{m_j - m_i}$, for $i, j \in I$. Hence, $\frac{\log|\pi|}{\log|\pi'|} = \frac{m_j - m_i}{n_j - n_i} = \frac{m}{n} > 0$ and the sets $(m_j - m_i)_{i \neq j}$ and $(n_j - n_i)_{i \neq j}$ must be finite.

If $J$ is a subset of $I$, the projector $p_J = \sum_{i \in J} e'_i \otimes e_i$ is self-adjoint with respect to any bilinear symmetric form $f_\pi$ and $\|p_J\| = 1 = \|p_J\|_\pi$.

8. Lemma. Let $E$ be a free Banach space with orthogonal basis $\{e_i : i \in I\}$. Defining adjoint of a continuous operator with respect to $f_\pi$, one has that any $u \in \mathcal{B}$ (respectively $D$) is self-adjoint and $\|u^2\| = \|u\|^2$.

Proof. It is the same as in Lemma 5. Since for any $u = \alpha_0 id + \sum_{\nu \in \Lambda} \alpha_\nu p_\nu \in \mathcal{L}_0(E)$
In $\mathcal{B}$ one has $\|u\| = \max_{\nu \in \Lambda_0} |\alpha_\nu|$, i.e. $\{id, p_\nu : \nu \in \Lambda\}$ is an orthonormal family in $\mathcal{L}_0(E)$, one sees that the closure $\mathcal{A} = \overline{\mathcal{B}}$ of $\mathcal{B}$ is the subspace of $\mathcal{L}_0(E)$ of all elements $u$ which can be written in the unique form of summable families $u = \alpha_0id + \sum_{\nu \in \Lambda} \alpha_\nu p_\nu$ with $\alpha_0, \alpha_\nu \in K$ and $\lim_{\nu} \alpha_\nu = 0$. It is readily seen that $\mathcal{A}$ is a closed unitary subalgebra of $\mathcal{L}_0(E)$, contained in $\mathcal{D}$, such that any element $u$ of $\mathcal{A}$ is self-adjoint. Moreover for the pointwise convergence, one has $u = \alpha_0id + \sum_{i \in J_\nu} e_i' \otimes e_i + \sum_{\nu \in \Lambda} \sum_{i \in J_\nu} (\alpha_0 + \alpha_\nu)e_i' \otimes e_i$. Hence, if $\bigcap_{\nu \in \Lambda} J_\nu = \emptyset$, one has $u = \sum_{\nu \in \Lambda} \sum_{i \in J_\nu} (\alpha_0 + \alpha_\nu)e_i' \otimes e_i$ and $id = \sum_{\nu \in \Lambda} p_\nu$.

9. Example. If $\Lambda = I$ and $J_i = \{i\}$ for each $i \in I$, one has $\mathcal{A} = \{\alpha_0id + \sum_{i \in I} \alpha_i e_i' \otimes e_i : \alpha_i \in K, \lim_{i \in I} \alpha_i = 0\}$. As an element of $\mathcal{D}$ any $u \in \mathcal{A}$ is in the form $u = \sum_{i \in I} a_i e_i' \otimes e_i$ with $\lim_{i \in I} a_i = \alpha_0$ exists in $K$.

10. Proposition. (i) Any element $u$ of the Banach algebra $\mathcal{A}$ with the unit element is self-adjoint with respect to any bilinear symmetric form $f_\pi$ and $\|u^2\| = \|u\|^2$.

(ii) The spectrum $\mathcal{X}(\mathcal{A}) = \text{Hom}_{alg}(\mathcal{A}, K)$ of $\mathcal{A}$, equipped with the weak*-topology, is homeomorphic to the Alexandroff compactification of the discrete space $\Lambda$.

Proof. The first part is an easy consequence of Lemma 8. Let $\chi \in \mathcal{X}(\mathcal{A})$, then $\chi$ is a continuous linear form with norm $\|\chi\| = 1$. Furthermore, one has $\chi(id) = 1$ and $\chi(p_\nu p_\mu) = \chi(p_\nu)\chi(p_\mu) = \delta_{\nu,\mu}\chi(p_\nu)$, for $\nu, \mu \in \Lambda$. It follows that for any $\nu \in \Lambda$, one has $\chi(p_\nu) = 1$ or $\chi(p_\nu) = 0$. Hence: (a) there exists $\nu \in \Lambda$ such that $\chi(p_\nu) = 1$ and $\chi(p_\mu) = 0$ for $\mu \neq \nu$, or (b) $\chi(p_\nu) = 0$ for all $\nu \in \Lambda$. In the case (a) one puts $\chi = \chi_\nu$ and in the case (b), $\chi = \chi_0$. One verifies that for $u = \alpha_0id + \sum_{\nu \in \Lambda} \alpha_\nu p_\nu \in \mathcal{A}$, one has $\chi_0(u) = \alpha_0$ and $\chi_\nu(u) = \alpha_0 + \alpha_\nu$, $\nu \in \Lambda$. It follows that $\mathcal{X}(\mathcal{A}) = \{\chi_0, \chi_\nu : \nu \in \Lambda\}$ and $\mathcal{X}(\mathcal{A})$ is in a bijective correspondence with the set $\Lambda_0 = \Lambda \cup \{0\}$. Let $W(\chi; \varepsilon, u_1, \ldots, u_n) = \{\eta : \eta \in \mathcal{X}(\mathcal{D}) ; \|\chi(u_j) - \eta(u_j)\| < \varepsilon, u_j \in \mathcal{A}, 1 \leq j \leq n\}$ be a fundamental neighborhood of $\chi \in \mathcal{X}(\mathcal{A})$ for the weak*-topology. Since for $u_j = \alpha_0id + \sum_{\nu \in \Lambda} \alpha_{\mu j} p_\nu \in \mathcal{A}$, one has $\lim_{\mu \in \Lambda} \alpha_{\mu j} = 0$, there exists a finite subset $\Gamma_\varepsilon$ of $\Lambda$, such that for any $\mu \not\in \Gamma_\varepsilon$, one has $|\alpha_{\mu j}| < \varepsilon$ for each $1 \leq j \leq n$. If $\chi = \chi_\nu, \nu \in \Lambda$, one has for
1 \leq j \leq n, \mu \in \Lambda, \chi_{\nu}(u_j) - \chi_{\mu}(u_j) = \alpha_{\nu j} - \alpha_{\mu j}. Choosing (u_j : 1 \leq j \leq n) such that \varepsilon_{\nu} = \min_{1 \leq j \leq n} |\alpha_{\nu j}| > 0, there exists \Gamma_{\nu} \subset \Lambda, \Gamma_{\nu} finite such that 
|\alpha_{\nu j}| < \varepsilon_{\nu}, for 1 \leq j \leq n and for all \mu \notin \Gamma_{\nu}. Hence |\alpha_{\mu j}| < |\alpha_{\nu j}| and 
|\alpha_{\nu j} - \alpha_{\mu j}| = |\alpha_{\nu j}| \geq \varepsilon_{\nu}, for 1 \leq j \leq n and \mu \notin \Gamma_{\nu}. Therefore, if \varepsilon < \varepsilon_{\nu}, one has 
W(\chi_{\nu}; \varepsilon, u_1, \ldots, u_n) = \{\chi_{\nu}\}, that is \{\chi_{\nu}\} is open in \mathcal{X}(\mathcal{A}). Hence 
\{\chi_{\nu} : \nu \in \Lambda\} is a discrete subset of \mathcal{X}(\mathcal{A}). On the other hand, if \chi = \chi_{0}, one has \chi_0(u_j) - \chi_{\mu}(u_j) = -\alpha_{\mu j}. Hence for \varepsilon > 0, there exists a finite subset 
\Gamma_{\varepsilon} of \Lambda such that for \mu \notin \Gamma_{\varepsilon}, one has
|\chi_0(u_j) - \chi_{\mu}(u_j)| = |\alpha_{\mu j}| < \varepsilon for each 1 \leq j \leq n. In other words, 
W(\chi_0; \varepsilon, u_1, \ldots, u_n) = \{\chi_{\mu} : \mu \notin \Gamma_{\varepsilon}\}. Furthermore \chi_0 = \lim_{\mu \in \Lambda} \chi_{\mu} in \mathcal{X}(\mathcal{A}) for the weak*-topology. It follows that 
\mathcal{X}(\mathcal{A}) is weak*-compact. Consider on \Lambda_0 = \Lambda \cup \{0\} the topology such that 
\Lambda is a discrete subset of \Lambda_0 and the neighborhoods of 0 are \mathcal{W}_1(0) = \Lambda_0 \setminus \Gamma, where \Gamma \subset \Lambda is finite. It becomes clear that \Lambda_0 is homeomorphic to the 
Alexandroff compactification of the discrete space \Lambda. Identifying \mathcal{X}(\mathcal{A}) with 
\Lambda_0 one concludes the proof of the proposition.

11. Let \mathcal{C}(\mathcal{X}(\mathcal{A}), K) be the K-Banach algebra of the continuous functions 
f on the compact space \mathcal{X}(\mathcal{A}) with values in K. It is readily seen that 
f \in \mathcal{C}(\mathcal{X}(\mathcal{A}), K) is defined by the family (f(\chi_{\nu}) : \nu \in \Lambda_0) \subset K such that 
\lim_{\nu \in \Lambda} f(\chi_{\nu}) = f(\chi_0). Hence \mathcal{C}(\mathcal{X}(\mathcal{A}), K) is isometrically isomorphic to the 
algebra \mathcal{C}_v(\Lambda_0, K) = \{a : a = (a_\nu : \nu \in \Lambda_0) \subset K; \lim_{\nu \in \Lambda} a_\nu = a_0\} on 
\mathcal{C}_v(\Lambda_0, K), one considers the usual multiplication defined pointwise and the 
norm (a_\nu : \nu \in \Lambda_0) = \sup_{\nu \in \Lambda_0} |a_\nu|.

11.1. Corollary. The Banach algebra \mathcal{A} with the unit element is isometrically isomorphic to the algebra 
\mathcal{C}_v(\Lambda_0, K) = \{a : a = (a_\nu)_{\nu \in \Lambda_0} \subset K; \lim_{\nu \in \Lambda} a_\nu = a_0\}.

Proof. Let \mathcal{G} : \mathcal{A} \rightarrow \mathcal{C}(\mathcal{X}(\mathcal{A}), K) be the Gelfand transform: 
\mathcal{G}(u)(\chi) = \chi(u). As usual, \mathcal{G} is continuous. Since for \nu \in \Lambda, 
\alpha_0 id + \sum_{\nu \in \Lambda} \alpha_\nu p_\nu \in \mathcal{A}, one has 
\chi_0(u) = \alpha_0 and \chi_{\nu}(u) = \alpha_0 + \alpha_\nu, \nu \in \Lambda, one obtains \|u\| = 
\max_{\nu \in \Lambda}(|\chi_0(u)|, \sup_{\nu \in \Lambda} |\chi_{\nu}(u)|) = \sup_{\nu \in \Lambda} |\chi(u)|. Hence, \|\mathcal{G}(u)\| = \|u\|. Furthermore, 
\mathcal{G}(id)(u) = 1, i.e. \mathcal{G}(id) = f_0 the constant function equal to 1. On 
the other hand, for \nu \in \Lambda, \mathcal{G}(p_\nu)(\chi) = 1 if \chi = \chi_{\nu} and 0 otherwise. Hence 
setting for \nu \in \Lambda, f_\nu : \mathcal{X}(\mathcal{A}) \rightarrow K such that 
f_\nu(\chi_{\nu}) = \delta_{\nu,\mu}, \mu \in \Lambda, one has 
\mathcal{G}(p_\nu) = f_\nu. Let \nu \in \Lambda, \alpha_0 id + \sum_{\nu \in \Lambda} \alpha_\nu p_\nu \in \mathcal{A}, one has 
\mathcal{G}(u) = \alpha_0 f_0 + \sum_{\nu \in \Lambda} \alpha_\nu f_\nu.
Since any $f \in \mathcal{C}(\mathcal{X}(\mathcal{A}), \mathcal{K})$ can be written in the unique convergent sum $f = f(\chi_0)f_0 + \sum_{\nu \in \Lambda} (f(\chi_\nu) - f(\chi_0))f_\nu$ with $\lim_{\nu \in \Lambda}(f(\chi_\nu) - f(\chi_0)) = 0$, one has $f = \mathcal{G}(u)$ with $u = f(\chi_0)id + \sum_{\nu \in \Lambda} (f(\chi_\nu) - f(\chi_0))p_\nu$. Hence, $\mathcal{G}$ is surjective. Together with $\|\mathcal{G}(u)\| = \|u\|$, the Corollary is proved.

5 Spectral integration.

1. Suppose that $X$ and $Y$ are Banach spaces over a topologically complete non-Archimedean field $\mathcal{K}$ with a non-trivial valuation and $\mathcal{L}(X, Y)$ denotes the Banach space of bounded linear operators $E : X \rightarrow Y$ supplied with the operator norm: $\|E\| := \sup_{x \neq 0} \|Ex\|_Y/\|x\|_X$. For $X = Y$ we denote $\mathcal{L}(X, Y)$ simply by $\mathcal{L}(X)$. Let $X$ and $Y$ be isomorphic with the Banach spaces $c_0(\alpha, \mathcal{K})$ and $c_0(\beta, \mathcal{K})$ and let they be supplied with the standard orthonormal bases $\{e_j : j \in \alpha\}$ in $X$ and $\{q_j : j \in \beta\}$ in $Y$ respectively, where $c_0(\alpha, \mathcal{K}) := \{x = (x^j : j \in \alpha) | x^j \in \mathcal{K}, \text{ such that for each } \epsilon > 0 \text{ a set } \{j : |x^j| > \epsilon\} \text{ is finite }\}$ with a norm $\|x\| := \sup_{j} |x^j|_\mathcal{K}$, $\alpha$ and $\beta$ are ordinals (it is convenient due to Kuratowski-Zorn Lemma). Then each operator $E \in \mathcal{L}(X, Y)$ has its matrix realisation $E_{j,k} := q_k^* E e_j$, which may be infinite, where $q_k^* \in Y^*$ is a continuous $\mathcal{K}$-linear functional $q_k^* : Y \rightarrow \mathcal{K}$ corresponding to $q_k$ under the natural embedding $Y \hookrightarrow Y^*$ associated with the chosen basis, $Y^*$ is a topologically conjugated or dual space of $\mathcal{K}$-linear functionals on $Y$, $q_k^* (q_l) = \delta_k^l$. Therefore, to each $E \in \mathcal{L}(X, Y)$ there corresponds an adjoint operator $E^* \in \mathcal{L}(Y^*, X^*)$. By a transposed operator $E^t$ we mean a restriction $E^*|_Y$, where $Y$ is embedded into $Y^*$ such that $E^t_{j,k} = E_{k,j}$ for each $j \in \alpha$ and $k \in \beta$.

This means, that if $X = Y$ and $E^t = E$, then $E$ is called a symmetric operator. For $X = Y = c_0(\alpha, \mathcal{K})$ there is an inclusion $E^*(Y) \subset X^*$. Since $X^* = \ell^\infty(\alpha, \mathcal{K})$, then $\|x\|_X = \|x\|_{X^*}$ for each $x \in X$. Since $\|E\| = \sup_{j,k} |E_{j,k}|$, then $\|E\| = \|E^*\|$ and $\|E\| = \|E^t\|$. If $A, E \in \mathcal{L}(X)$ and $E = A^t$, then $A$ and $E$ belong to the closed subalgebra $\mathcal{L}_0(X)$ (see §2).

1.1. Now let $A$ be an abstract Banach algebra over a field $\mathcal{K}$, which is complete relative to a norm $\|\cdot\|_\mathcal{K}$ in it. We say that $A$ is with an operation of transposition $a \mapsto a^t$ for each $a \in A$ if the following conditions $(\alpha - \delta)$ are satisfied:

$$ (\alpha) \ (a + b)^t = a^t + b^t; $$
(\beta) (\lambda a)^t = \lambda a^t;  
(\gamma) (ab)^t = b^ta^t;  
(\delta) (a')^t = a'' = a \text{ for each } a, b \in A \text{ and each } \lambda \in K.

Let \( A \) be an algebra over \( K \), which satisfies the following conditions (i - iii):

(i) \( A \) is a Banach algebra
(ii) with the operation of transposition \( a \mapsto a^t \),
(iii) \( \|a'a\| = \|a\|^2 \) for each \( a \in A \) while evaluation of norms.

Then such algebra is called an \( E \)-algebra.

Without condition (iii) it is called the \( T \)-algebra. If instead of (iii) it is satisfied the following condition:

(iv) \( \|a'a\| = \|a^2\| \) then \( A \) is called a \( S \)-algebra.

For each \( E \)-algebra we have \( \|a\|^2 = \|aa^t\| \leq \|a\|\|a^t\| \), hence \( \|a\| \leq \|a^t\| \)
and also \( \|a^t\| \leq \|(a')^t\| = \|a\| \), consequently, \( \|a\| = \|a^t\| \).

1.2. Evidently, \( \mathcal{L}_0(X) \) is the \( T \)-algebra. Each \( C \)-algebra is at the same time the \( E \)-algebra (see also §5.6), since for each singleton \( x \in X \) a closed subalgebra \( C(\{x\}, K) \) is isomorphic with \( K \) and the restriction of transposition on \( C(\{x\}, K) \) gives \( f^t(x) = f(x) \) for each \( f \in C_\infty(X, K) \).

Let \( A_1, \ldots, A_n \) be linear operators. Then the equation \( \sum_{j=1}^n \lambda_j A_j, \sum_{k=1}^n \mu_k A_k \) = \( \sum_{j<k} (\lambda_j \mu_k - \lambda_k \mu_j)(A_j A_k - A_k A_j) = 0 \) for each \( \lambda_j \) and \( \mu_k \in K \) is equivalent to \( [A_j, A_k] = 0 \) for each \( j < k \). In view of §§IV.6, VII.7, VIII.2 [13] for each \( n \in \mathbb{N} \) there are pairwise commuting matrices of the size \( m \times m \) such that for sufficiently large \( m > n \) they in addition can be found non-diagonal (non-reducible to diagonal form by transformations, \( U_j A_j U_j^{-1} \), where \( U_j \) are invertible matrices), since in view of Theorem VIII.7.2 [13] a number of linearly independent matrices, which commute with the given matrix \( A \) is defined by the following formula: \( n = n_1 + 3n_2 + \ldots + (2t - 1)n_t \), where \( n_1, n_2, \ldots, n_t \) are degrees of non-constant invariant polynomials \( i_1(\lambda), i_2(\lambda), \ldots, i_t(\lambda) \) and \( n = n_1 + n_2 + \ldots + n_t \) is a size of the square \( n \times n \) matrix \( A \). This can be done by suitable choices of Jordan forms of matrices over \( K \). From this it follows, that in \( \mathcal{L}_0(c_0(\omega_0, K)) \) for each \( n \in \mathbb{N} \) there always exist \( n \) pairwise commuting operators such that they are not reducible to the diagonal form by adjoint transformations \( U_j A_j U_j^{-1} \). This produces examples of \( T \)-algebras.

When in the (finite case) Jordan form \( |\lambda_{j,k}| > 1 \) and \( \|I - U_j\| < 1 \) for each \( j = 1, \ldots, n \), where \( \lambda_{j,k} \) are diagonal elements of the Jordan normal forms of \( A_j \), then each \( A_j \) satisfy condition (iii) together with \( A_j^t \). We take the case \( U_j = U_1 =: U \) for each \( j \). Let in addition \( A_j \) be pairwise commutative
Theorem 4.7.2 [14]). There are also cases, when it may be over
example, \(|\lambda_{j,k}|^a > |b\lambda_{j,k}|^c\) for each \(a > c > 0\) and \(b, c \in \mathbb{Z}\). As it follows from § VIII.7.2 [13] these \(A_j\) can be
chosen such that \(sp_K\{I, A_j, A_j^2, ..., A_j^n\}\) does not contain \(A_i\) for each \(l \neq j\).

The construction of \(S\)-algebras can be done analogously and more lightly, since Condition (iii) is replaced by Condition (iv).

Let \(A\) be an \(E\)-algebra with a \(K\)-linear isometry \(Y : A \to A\) such that \(Y(ab) = Y(b)Y(a)\) and \(Y^t(a) = Y(a^t)\) for each \(a, b \in A\). Then \(Y(A)\) is the \(E\)-algebra.

There are general constructions of Banach algebras also. In particular we can take a free Banach algebra \(A\) generated by a set \(J\). This means, that \(A\) is a completion of \(sp_K\{a_1...a_n : a_1, ..., a_n \in J, n \in \mathbb{N}\}\) with the definite order of letters \(a_1, ..., a_n\) in each word \(w = a_1a_2...a_n\), when neighbouring elements \(a_j\) and \(a_{j+1}\) are distinct in \(J\). There exists a norm on \(sp_K\{a_1...a_n : a_1, ..., a_n \in J, n \in \mathbb{N}\}\) such that \(\|ab\| \leq \|a\| \|b\|\) for each \(a, b \in A\). For example, \(\|w\| = 1\) for each word \(w = a_1...a_n\) with \(a_1, ..., a_n \in J\), \(\|c_1w_1 + ... + c_mw_m\| = \max_{1 \leq j \leq m} |c_j|_{K}\) for different words \(w_1, ..., w_m\) with \(c_j \in K\) for each \(j = 1, ..., m\). Then for \(Y : A \to A\) preserving a closed ideal \(V\) we can consider the quotient mapping \(\bar{Y} : \bar{A} \to \bar{A}\), where \(\bar{A} = A/V\) and \(Y\) on \(A\) is defined by \(Y\) on \(J\) due to the continuous extension.

Another example is the following. For a subset \(J\) of symmetric (that is, \(a^t = a\) for each \(a \in J \subset \mathcal{L}_0(X)\)) pairwise commuting elements (that is, \(ab = ba\) for each \(a\) and \(b \in J \subset \mathcal{L}_0(X)\)) let \(A := cl\{sp_K\{\prod_{i=1}^n a_i^{n_i} : 0 \leq n_i \in \mathbb{Z}, m \in \mathbb{N}, a_i \in J\}\}\), where \(a^0 := I\) is the unit operator on \(X\). Then such \(A\) is the \(T\)-algebra. Since \(\|a^t\| = \|a\|^2\) for each \(a \in A\), then it is the \(S\)-algebra.

Another example of an \(E\)-algebra is the algebra of diagonal operators in \(\mathcal{L}_0(X)\). Then each \(E\)-algebra is certainly a \(S\)-algebra and each \(S\)-algebra is a \(T\)-algebra (see also Lemmas 4.5, 4.8 and Proposition 4.10.(i)). Above were constructed more interesting examples of \(E\)-algebras and \(S\)-algebras.

In general diagonal form of an algebra is unnecessary for the spectral theory. Moreover, there are well-known theorems, when Lie algebras (in particular of finite square \(m \times m\) matrices over \(C_p\)) can be reduced simultaneously to the upper triangular form by one transformation \(UAU^{-1}\) (see K. Iwasawa Theorem 4.7.2 [14]). There are also cases, when it may be over \(K\). Using
limits such cases can be spread on subalgebras of $L_0(X)$.

It will be shown below that for the spectral integration it is sufficient to consider $C$-algebras.

2. Let $A$ and $B$ be two $E$-algebras over the same field $K$, an algebraic homomorphism $T : A \to B$ is called a $t$-representation of $A$ in $B$, if $T_{a^t} = (T_a)^t$ for each $a \in A$. The reducing ideal $\mathcal{T}$ of $A$ is defined as the intersection of the kernels of all $t$-representations of $A$. $\mathcal{T}$ is also called the $t$-radical. If $\mathcal{T} = 0$, then $A$ is called reduced (or $t$-simple).

Let $\|a\|_t := \sup_{T \in \Psi} \|Ta\|$ for a reduced algebra $A$, where $\Psi := \Psi_A$ denotes the family of all $t$-representations of $A$. Since $A$ is reduced, then $\|a\|_t \neq 0$ for each $A \ni a \neq 0$. Such $\|*\|_t$ is called a $E$-norm of $A$.

The $E$-algebra obtained by completing $A/\mathcal{T}$ by its $E$-norm is called the $E$-completion of $A$ and is denoted by $A_t$. Denote by $\pi : A \to A/\mathcal{T}$ the natural $t$-homomorphism of $A$ into $A_t$ such that $\pi(a) = a + \mathcal{T}$ for each $a \in A$. Then the map $T \mapsto T' = T \circ \pi$ is a bijective correspondence between the set of all $t$-representations $T$ of $A_t$ and the set of all $t$-representations $T'$ of $A$. This correspondence preserves closed stable subspaces, non-degeneracy, bounded intertwining operators, isometric equivalence and Banach direct sums.

3. Let $A$ be a commutative Banach $T$-algebra and $A^+$ denotes the Gelfand space of $A$, that is, $A^+ = Sp(A)$, where $Sp(A)$ was defined in Ch. 6 [25]: it is the set of all nonzero algebra homomorphisms $\phi : A \to K$ topologized as the subset of $K^A$. Every $x \in A$ induces a function $G_x : Sp(A) \to K$ by $G_x(\phi) := \phi(x)$, where $\phi \in Sp(A)$, $G_x$ is called the Gelfand transform of $x$, $G$ is called the Gelfand transfromation. Then it is defined the spectral norm $\|x\|_{sp} := \sup_{\phi \in Sp(A)} |G_x(\phi)|$ of $x \in A$. If $Sp(A) = \emptyset$, then $\|x\|_{sp} := 0$ for each $x \in A$. We denote by $\hat{A}$ the closed subset of $A^+$ consisting of those $\phi \in A^+$ for which $\phi(a^t) = \phi(a)$ for each $a \in A$, $\phi \in A$ is called symmetric, if $\phi \in \hat{A}$. Let $C_\infty(A, K)$ be the same space as in [25]. For a locally compact $E$ the space $C_\infty(E, K)$ is a subspace of the space $BUC(E, K)$ of bounded uniformly continuous functions $f : E \to K$ such that for each $\epsilon > 0$ there exists a compact subset $V \subset E$ for which $|f(x)| < \epsilon$ for each $x \in E \setminus V$. When $E$ is not locally compact and has an embedding into $B(K, 0, 1)^\gamma$ such that $E \cup \{x_0\} = cl(E)$ we put $C_\infty(E, K) := \{f \in C(E, K) : \lim_{x \to x_0} f(x) = 0\}$, where $B(X, x, r) := \{y \in X : d(x, y) \leq r\}$ is a ball in the metric space $(X, d)$, $cl(E)$ is taken in $B(K, 0, 1)^\gamma$, $\gamma$ is an ordinal, $x_0 \in B(K, 0, 1)^\gamma$.

3.1. Definition (see also Ch. 6 in [25]). A commutative Banach algebra
A is called a C-algebra if it is isomorphic with $C_\infty(X, K)$ for a locally compact Hausdorff hereditarily disconnected space $X$, where $f + g$ and $fg$ are defined pointwise for each $f, g \in C_\infty(X, K)$.

3.2. Proposition. The reducing ideal $\mathcal{T}$ of $A$ consists of those $a \in A$ such that $\hat{a}(\phi) = 0$ for each $\phi \in \hat{A}$. The equation

\[(i) \ F(\pi(a)) = \hat{a} |_{\hat{A}}\]
determines an isometric $t$-isomorphism $F$ of $A_t$ onto $C_\infty(\hat{A}, K)$, when $K$ is a locally compact field.

Proof. We have $\sup_{\phi \in \hat{A}} |\hat{a}(\phi)| \leq \|\pi(a)\|_t$ for each $a \in A$. Then we take a $t$-representation $T$ of $A$ and $B := \text{cl}_{\|\cdot\|_{\text{range}(T)}}$, hence $B$ is a commutative $E$-algebra. To finish the proof of Proposition 3 we need the following.

4. Proposition. Let $A$ be a commutative $E$-algebra as in §1: $A = \text{cl} \left( sp_K \{ \prod_{i=1}^{m} a_i^{n_i} : 0 \leq n_i \in \mathbb{Z}, a_i \in J, a^t = a, m \in \mathbb{N} \} \right)$, then $\hat{A} = A^+$. Furthermore, the Gelfand transform map $a \mapsto \hat{a}$ is an isometric isomorphism of $A$ onto $C_\infty(\hat{A}, K)$, when $K$ is a locally compact field.

Proof. In view of Corollaries 6.13, 6.14 and 6.17 [25] it is sufficient to show that $A^+ = \hat{A}$, since $A$ is isomorphic with $C_\infty(Sp(A), K)$, where $Sp(A) = \hat{A}$. If $a^t = a \in A$ and $\phi \in A^+$, then $\phi(a^t) = \phi(a) \in K$. If $1 \notin A$, then $\phi$ extends to a $t$-homomorphism of the $S$-algebra $A_1$ obtained by adjoining the unit 1 to $A$, since it is possible to consider $X \oplus K$ (see, for example, about adjoining of 1 in §VI.3.10 [12] and Ch. 6 [25]). Since $K$ is locally compact, $A_1$ is isomorphic with $C(\alpha Y, K)$, where $\alpha Y = Sp(A) \cup \{0\}$ is a one-point (Alexandroff) compactification of $sp(A)$ (see Observation 6.2 in [25]). Indeed, $\|a\| = \sup_{a \in A^+} |\chi(a)|$. Let $\phi(a) = r \in K$, $b := a + z1$, where $z \in K$. From $\|\phi\| \leq 1$ it follows, that $|\phi(b' b)|_p = |(r + z)^2|_p \leq \|b^2\|$ and $\|b^2\| = \|a a + z a^t + z a + z^2 1\| \leq \max(\|a^2\|, \|z\|_p \|a\|, \|z^2\|)$. Then there exists $0 < \epsilon < (\|a^2\|)^{1/2}$ such that for each $|z|_p < \epsilon$: $|\phi(b' b)|_p \leq \|a^2\|$, consequently, $\phi$ has the continuos extension on $A_1$.

If $\phi \in A^+$ and $a = b + c \in A$ with $b^t = b$ and $c^t = -c \in A$, then $\phi(a^t) = \phi(b) - \phi(c)$. If $\phi(a^t) = -\phi(a)$ for each $a \in A$, then $\phi(b) = 0$ for each $b = b^t \in A$. If $\phi(a^t) = \phi(a)$ for each $a \in A$, then $\phi(c) = 0$ for each $c^t = -c \in A$. The operation of transposition $a \mapsto a^t$ is continuous in $A$. Let $\phi \in A^+$ and $\phi \neq 0$. Therefore, for each $\phi(a) \neq 0$ we have $\phi(a^t) \neq 0$, since $a^{tt} = a$. Whence $\phi(a^t) = \lambda_\phi \phi(a)$ for each $a \in A$ such that $\phi(a) \neq 0$, since $\text{coker}_K \phi$ is one-dimensional, where $0 \neq \lambda_\phi \in K$. We have $\phi((a^t a)^n) = \lambda_\phi^n \phi(a)^{2n}$. Since $\|a^t a\| = \|a^2\|$, $a^{tt} = a$ and $\phi$ is the continuous multiplicative
linear functional, then $|\lambda_\phi| = 1$. On the other hand, $\lambda_\phi(ab) = \phi(a'b') = \lambda_\phi^2 \phi(a) \phi(b) = \lambda_\phi \phi(ab)$, hence $\lambda_\phi = 1$, where there are $a$ and $b \in A$ such that $\phi(ab) \neq 0$. Therefore, $\phi^t = \phi$, where $\phi^t(a) := \phi(a^t)$ for each $a \in A$. Consequently, $A = A^\perp$.

Continuation of the proof of Proposition 3.2. In view of §2 there exists $\psi \in B$ such that $|\psi(T_a)| = ||T_a||$, since $\psi' : a \mapsto \psi(T_a) \in \hat{A}$, hence $||T_a|| = |\psi'(a)| \leq \sup_{\phi \in \hat{A}} |\hat{a}(\phi)|$, consequently, $||\pi(a)||_t \leq \sup_{\phi \in \hat{A}} |\hat{a}(\phi)|$. Therefore, $||\pi(a)||_t = \sup_{\phi \in \hat{A}} |\hat{a}(\phi)|$, hence the map $F$ defines the isometric $t$-isomorphism of $A_t$ into $C_\infty(SP(A), K)$. The range of $F$ is a $T$-subalgebra of $C_\infty(\hat{A}, K)$, which automatically separates points of $\hat{A}$, consequently, by the Kaplansky theorem $cl \ range(F) = C_\infty(\hat{A}, K)$ (see §A.A [27]).

5. Let $H = c_0(\alpha, K)$, where $K$ is a topologically complete field. A strong operator topology in $\mathcal{L}(H, Y)$ (see §1) is given by a base $V_{c,E;x_1,...,x_n} := \{Z \in \mathcal{L}(H, Y) : \sup_{1 \leq j \leq n} \|(E - Z)x_j\|_Y < \epsilon\}$, where $0 < \epsilon$, $E \in \mathcal{L}(H, Y)$, $x_j \in H$; $j = 1, ..., n$; $n \in \mathbb{N}$. Let also $X$ be a topological space with the small inductive dimension $ind(X) = 0$. An $H$-projection-valued measure on an algebra $\mathcal{L}$ of subsets of $X$ is a function $P$ on $\mathcal{L}$ assigning to each $A \in \mathcal{L}$ a projection $P(A)$ on $H$ and satisfying the following conditions:

(i) $P(X) = 1_H$,

(ii) for each sequence $\{A_n : n = 1, ..., k\}$ of pairwise disjoint sets in $\mathcal{L}$ there are projections $P(A_n)$ such that $P(A_n)P(A_l) = 0$ for each $n \neq l$ and $P(\bigcup_{n=1}^k A_n) = \sum_{n=1}^k P(A_n)$,

(iii) if $A \subset \mathcal{L}$ is shrinking and $\cap A = \emptyset$, then $\lim_{A \in A} P(A) = 0$, where the convergence on the right hand side is unconditional in the strong operator topology and the sum is equal to the projection onto the closed linear span over $K$ of $\{\text{range}(P(A_n)) : n = 1, ..., k\}$ such that $P(\emptyset) = 0$, $k \in \mathbb{N}$.

If $\eta \in H^*$ and $\xi \in H$, then $A \mapsto \eta(P(A)\xi)$ is a $K$-valued measure on $\mathcal{L}$. The case of a $\sigma$-algebra $\mathcal{L}$ and of $k = \infty$ in (ii) is unnecessary for the subsequent consideration and it will not be used, but it may be considered as a particular case. The $\sigma$-additive case leads to the restriction that each measure $\eta(P(A)\xi)$ is atomic, when $K$ is spherically complete (see Chapter 7 in [25]).

Then by the definition $P(A) \leq P(B)$ if and only if $\text{range}(P(A)) \subset \text{range}P(B)$. There are many projection operators on $H$, but for $P$ there is chosen some such fixed system.

A subset $A \subset X$ is called $P$-null if there exists $B \in \mathcal{L}$ such that $A \subset B$ and $P(B) = 0$, $A$ is called $P$-measurable if $A \Delta B$ is $P$-null, where $A \Delta B :=$
\((A \setminus B) \cup (B \setminus A)\). A function \( f : X \to K \) is called \( P \)-measurable, if \( f^{-1}(D) \) is \( P \)-measurable for each \( D \) in a field \( Bcc(K) \) of clopen subsets of \( K \). It is essentially bounded, if there exists \( k > 0 \) such that \( \{ x : |f(x)| > k \} \) is \( P \)-null. \( \| f \|_\infty \) is by the definition the infimum of such \( k \). Then \( F := sp_k \{Ch_B : B \in L\} \) is called the space of simple functions, where \( Ch_B \) denotes the characteristic function of \( B \). The completion of \( F \) relative to \( \| * \|_\infty \) is the Banach algebra \( L_\infty(P) \) under the pointwise multiplication.

There exists a unique linear mapping \( I : F \to \mathcal{L}(H) \) by the following formula:

(iv) \( I(\sum_{i=1}^{n} \lambda_i Ch_{B_i}) = \sum_{i=1}^{n} \lambda_i P(B_i) \), where \( n \in N, B_i \in L, \lambda_i \in K \). Since

(v) \( \| I(f) \| = \| f \|_\infty \), then \( I \) extends to a linear isometry (also called \( I \)) of \( L_\infty(P) \) onto \( \mathcal{L}(H) \).

If \( f \in L_\infty(P) \), then the operator \( I(f) \) in \( \mathcal{L}(H) \) is called the spectral integral of \( f \) with respect to \( P \) and is denoted

(vi) \( \int_X f(x) P(dx) := I(f) \).

Evidently properties (I – III, V, VI) from \( \$II.11.8 \) [12] are transferable onto the case considered here. These and another properties of the spectral integral are as follows.

5.1. Propositions. (I). \( \int_X f(x) P(dx) = \int_X g(x) P(dx) \) if and only if \( f \) and \( g \) differ only on a \( P \)-null set.

(II). \( \int_X f(x) P(dx) \) is linear in \( f \).

(III). \( \int_X f(x) g(x) P(dx) = (\int_X f(x) P(dx))(\int_X g(x) P(dx)) \) for each \( f \) and \( g \in L_\infty(P) \).

(V). \( \| \int_X f(x) P(dx) \| = \| f \|_\infty \).

(VI). If \( A \in L \), then \( \int_X Ch_A(x) P(dx) = P(A) \), in particular \( \int_X P(dx) = P(X) = 1_H \).

(VII). For each pair \( \xi \in H \) and \( \eta^* \in H^* \), let \( \mu_{\xi,\eta}(A) := \eta^*(P(A)\xi) \) for each \( A \in L \). If \( E = \int_X f(x) P(dx) \) then \( \eta^*(E\xi) = \int_X f(x) \mu_{\xi,\eta}(dx) \).

(VIII). If \( A \in L \), then \( P(A) \) commutes with \( \int_X f(x) P(dx) \), where \( e_1^* := e_i^* \) such that \( e_1^*(e_j) = \delta_{ij} \).

An \( H \)-projection-valued measure \( P \) on the algebra \( L \) containing an algebra \( Bcc(X) \) of clopen (closed and open at the same time) subsets of \( X \) is called an \( H \)-projection-valued tight measure on \( X \). We call \( P \) regular if

(vii) \( P(A) = \sup \{ P(C) : C \subseteq A \text{ and } C \text{ is compact} \} \) for each \( A \in L \), where \( sup \) is the least closed subspace of \( H \) containing range \( P(C) \) and to it corresponds projector on this subspace. Indeed, \( P(A)H \) is closed in \( H \), since \( P^2(A) = P(A) \). Therefore,
(vi) \( P(A) = \inf \{ P(U) : U \text{ is open and } U \supset A \} = I - \sup \{ P(C) : C \subset X \setminus A \text{ and } C \text{ is compact} \} \), hence

(ix) the infimum corresponds to the projection on \( \cap_{U \supset A} U \) is open \( P(U)H \).

A measure \( \mu : L \to K \) is called regular, if for each \( \epsilon > 0 \) and each \( A \in L \) with \( \| A \|_{\mu} < \infty \) there exists a compact subset \( C \subset A \) such that \( \| A \setminus C \|_{\mu} < \epsilon \). Since \( \| P(X) \| = 1 \), then \( \| \mu_{\xi,\eta} \| \leq \| \xi \|_{H} \| \eta \|_{H^*} \). For the space \( H \) over \( K \) measures \( \mu_{\xi,\eta} \) on locally compact \( X \) are tight for each \( \xi,\eta \) in a subset \( J \subset H \to H^* \) separating points of \( H \) if and only if \( P \) is defined on \( L \supset Bco(X) \); \( P \) is regular if and only if \( \mu_{\xi,\eta} \) are regular for each \( \xi,\eta \in J \) due to Conditions (vi) and (ix). We can restrict our consideration by \( \mu_{\xi,\eta} \) instead of \( \mu_{\xi,\eta} \) with \( \xi,\eta \in spK \), since 2\( \mu_{\xi,\eta} = \mu_{\xi(\cdot)\eta,\xi(\cdot)\eta} - \mu_{\xi,\eta} = \mu_{\eta,\eta} \).

By the closed support of an \( H \)-projection-valued tight measure \( P \) on \( X \) we mean the closed set \( D \) of all those \( x \in X \) such that \( P(U) \neq 0 \) for each open neighbourhood \( U \) of \( x \), \( supp(P) := D \).

6. We fix a locally compact totally disconnected Hausdorff space \( X \) and a Banach space \( H \) over \( K \) and let \( T : C_{00}(X,K) \to \mathcal{L}(H) \) be a linear continuous map from the \( C \)-algebra \( C_{00}(X,K) \) of functions \( f : X \to K \) such that:

(i) \( T_{fg} = T_{f}T_{g} \) for each \( f \) and \( g \in C_{00}(X,K) \),

(ii) \( T_{1} = 1 \) for compact \( X \).

In general \( C_{00}(X,K) \) can be considered as the \( E \)-algebra if define \( f^{t} := f \) for each \( f \in C_{00}(X,K) \), so we can put \( T_{f}^{t} = T_{f} \), but the latter equality will not be used.

From this definition it follows, that \( \| T \| \leq 1 \), since \( T_{fn} = T_{f}^{n} \) for each \( n \in \mathbb{Z} \) and \( f \in C_{00}(X,K) \). If \( X \) is locally compact and is not compact, then \( X_{\infty} := X \cup \{ x_{\infty} \} \) be its one-point Alexandroff compactification. Each \( f \in C(X_{\infty},K) \) can be written just in one way in the form \( f = \lambda 1 + g \), where \( g \in C_{00}(X,K) \) and \( 1 \) is the unit function on \( X_{\infty} \). Therefore, we can extend \( T : C_{00}(X,K) \to \mathcal{L}(H) \) to a linear map \( T' : C(X_{\infty},K) \to \mathcal{L}(H) \) by setting \( T'_{\lambda 1 + g} = \lambda 1_{H} + T_{g} \) such that \( T'_{1} = 1_{H} \).

Therefore, \( f \mapsto \eta^{*}(T_{f} \xi) =: \tilde{\mu}_{\xi,\eta}(f) \) is a continuous \( K \)-linear functional on \( C_{00}(X,K) \), where \( \xi \in H \) and \( \eta^{*} \in H^{*} \). In view of Theorems 7.18 and 7.22 [25] about correspondence between measures and continuous linear functionals (the non-Archimedean analog of the F. Riesz representation theorem) there exists the unique measure \( \mu_{\xi,\eta} \in M(X) \) such that

(I) \( \eta^{*}(T_{f} \xi) = \int_{X} f(x) \mu_{\xi,\eta}(dx) \) for each \( f \in C_{00}(X,K) \). In the case \( T_{f}^{t} = T_{f} \) we have \( \mu_{\xi,\eta} = \mu_{\eta,\xi} \), when \( \xi,\eta \in H \). Since \( T_{1} = I \), then \( \mu_{\xi,\eta}(X) = \)}
for each $A \in \mathbf{L}$ we have $\|A\|_{\mu_{\xi,\eta}} \leq \|\xi\| \|\eta\| \sup_{f \neq 0} \|T_f\| \leq \|\xi\| \|\eta\|$. Since $H$ considered as a subspace of $H^*$ separates points in $H$, then for each $A \in \mathbf{L}$ there exists the unique linear operator $P(A) \in \mathcal{L}(H)$ such that:

$$
\text{(II) } \|P(A)\| \leq 1 \text{ and } \eta^*(P(A)\xi) = \mu_{\xi,\eta}(A), \text{ since } \mu_{\xi,\eta}(A) \text{ is a continuous bilinear } \mathbf{K}\text{-valued functional by } \xi \text{ and } \eta \in H. \text{ From the existence of the } H\text{-projection-valued measure in the case of locally compact } X \text{ we get a projection-valued measure } P' \text{ on } X_\infty \text{ such that }
$$

$$
\text{(III) } T'_f = \int_{X_\infty} f(x) P'(dx) \text{ for each } f \in C(X_\infty, \mathbf{K}).
$$

7. **Lemma.** For each $A$ and $B \in \mathbf{L}$:

(i) $P(A \cap B) = P(A)P(B) = P(B)P(A)$.

**Proof.** For each $g \in C(\infty, X, \mathbf{K})$ and $\xi, \eta \in H$ let $\nu_g(dx) := g(x)\mu_{\xi,\eta}(dx)$. For each $f$ and $g \in C(\infty, X, \mathbf{K})$ we have: $\int_X f(x)\mu_{T_g\xi,\eta}(dx) = \eta^*(T_f T_g\xi) = \eta^*(T_f g) = \int_X f(x)g(x)\mu_{\xi,\eta}(dx) = \int_X f(x)\nu_g(dx)$, consequently, $\nu_g = \mu_{T_g\xi,\eta}$. For a fixed $A \in \mathbf{L}$ let $\rho(B) := \mu_{\xi,\eta}(A \cap B)$ for each $B \in \mathbf{L}$. Therefore, $\rho$ is a tight measure on $X$: $\rho \in M(X)$, where $M(X)$ denotes the set of all tight measures on $X$. For each $g \in C(\infty, X, \mathbf{K})$ there are equalities:

$$
\int_X g(x)\rho(dx) = \int_A g(x)\mu_{\xi,\eta}(dx) = \nu_g(A) = \int_X g(x)\mu_{P(A)\xi,\eta}(dx).
$$

Then for each $B \in \mathbf{L}$ we get:

$$
\eta^*(P(A \cap B)\xi) = \mu_{\xi,\eta}(A \cap B) = \rho(B) = \mu_{P(A)\xi,\eta}(B) = \eta^*(P(A)P(B)\xi).
$$

The elements $\xi$ and $\eta \in H$ were arbitrary, hence $P(A \cap B) = P(A)P(B)$.

Interchanging $A$ and $B$ we get the conclusion of this lemma.

8. **Corollary** For each $A \in \mathbf{L}$ we have $P^2(A) = P(A)$ and $P(A)$ is a projection operator such that $P(X) = I$. If $A \cap B = \emptyset$, $A$ and $B \in \mathbf{L}$, then $P(A)P(B) = 0$.

9. **Proposition.** If Conditions 5.(i – iii) are satisfied, then $P$ is the unique regular $H$-projection-valued tight measure on $X$ and $T_f = \int_X f(x)P(dx)$ for each $f \in C(\infty, X, \mathbf{K})$.

**Note.** Such integral is called the spectral integral.

**Proof.** Let $\{A_n : n \in \mathbf{N}\}$ be a sequence of pairwise disjoint subsets of $X$, $A_n \in \mathbf{L}$. Since $X$ is locally compact, then the spectral integral defined in §5 as the limit of certain finite sums exists. By Corollary 8 $P(A_n)$ are pairwise orthogonal projectors. Put $Q = \sum_n P(A_n)$. Then for each $\xi$ and $\eta \in H$ we have:
\[ \eta^*(Q\xi) = \sum_n \eta^*(P(A_n)\xi) = \sum_n \mu_{\xi,\eta}(A_n) = \mu_{\xi,\eta}(\bigcup_n A_n) = \eta^*(P(\bigcup_n A_n)\xi), \]

consequently, \( P(\bigcup_n A_n) = \sum_n P(A_n) \) and \( P \) is an \( H \)-projection-valued measure. Since \( X \) is locally compact, then each measure \( \mu_{\xi,\eta} \) is tight and regular (see Theorem 7.6 in [25]), hence \( P \) is regular (see §5). Take \( f \in C_\infty(X,K) \) and form the spectral integral \( E = \int_X f(x)P(dx) \). For each \( \xi, \eta \in H \) we have \( \eta^*(E\xi) = \int_X f(x)\mu_{\xi,\eta}(dx) = \eta^*(T_f\xi) \), consequently, \( E = T_f \). In view of Equality 6.([III]) we have a regular \( H \)-projection valued measures both in the case of compact and non-compact locally compact \( X \).

It remains to verify the uniqueness of \( P \). Suppose there exists another regular \( H \)-projection-valued tight measure on \( X \) with the same properties. Put \( \mu_{\xi,\eta}(A) = \eta^*(P(A)\xi) \), \( \nu_{\xi,\eta}(A) = \eta^*(Q(A)\xi) \) for each \( A \in \mathcal{L} \), where \( \xi \) and \( \eta \in H \). Then \( \int_X f(x)\mu_{\xi,\eta}(dx) = \int_X f(x)\nu_{\xi,\eta}(dx) \) for each \( f \in C_\infty(X,K) \), hence \( \mu_{\xi,\eta} = \nu_{\xi,\eta} \) for each \( \xi, \eta \in H \), consequently, \( P(A) = Q(A) \) for each \( A \in \mathcal{L} \).

10. **Corollary.** The relation \( T_f = \int_X f(x)P(dx) \) for each \( f \in C_\infty(X,K) \) sets a one-to-one correspondence between the set of all regular \( H \)-projection-valued tight measures \( P \) on \( X \) and the set of all continuous linear maps \( T : C_\infty(X,K) \to L(H) \), which satisfy conditions 5.(i–iii).

11. **Note.** A particular case of \( H = C_\infty(X,K) \) for locally compact totally disconnected Hausdorff space \( X \) and \( T_f = f \) for each \( f \in C_\infty(X,K) \) can be considered independently from the given above and it is the following. Each such \( f \) is a limit of a certain sequence by \( n \in \mathbb{N} \) of finite sums \( \sum_j f(x_{j,n})Ch_{V_{j,n}}(x) \), where \( \{V_{j,n} : j \in \Lambda_n\} \) is a finite partition of \( X \) into the disjoint union of \( V_{j,n} \) clopen in \( X \), \( x_{j,n} \in V_{j,n}, \Lambda_n \subset \mathbb{N} \), since Range \( (f) \) is bounded. If take \( P(V) = Ch_V \) for each \( V \in \mathcal{L} \), then \( T_a g = \lim_{n \to \infty} \sum_{j} f(x_{j,n})Ch_{V_{j,n}}(x)g = \int_X f(x)P(dx)g \) for each \( g \in H \), so there is the bijective correspondence between elements \( a \in A \) of a \( C \)-algebra \( A \) realised as \( C_\infty(X,K) \) with \( X = Sp(A) \) and their spectral integral representations. It can be lightly seen that \( P(V_1 \cap V_2) = Ch_{V_1\cap V_2} = Ch_{V_1}Ch_{V_2} = P(V_1)P(V_2) = P(V_2)P(V_1) \) for each \( V_j \in \mathcal{L} \). If \( \{V_j : V_j \in \mathcal{L}, j \in \mathbb{N}\} \) is a disjoint family, then \( P(\bigcup_j V_j)g = Ch_{\bigcup_j V_j}g = \sum_j Ch_{V_j}g = \sum_j P(V_j)g \) for each \( g \in H \). Also \( P(\emptyset)H = Ch_\emptyset H = \{0\} \) and \( P(X)g = Ch_Xg = g \) for each \( g \in H \). Therefore, \( P \) is indeed an \( H \)-projection-valued tight measure.

Suppose now that \( X \) is not locally compact, for example, \( X = c_0(\omega_0,S) \) with an infinite residue class field \( k \) of a non-Archimedean infinite field \( S \) with non-trivial valuation. Then there are \( f \in C_\infty(X,K) \) for which convergence of
finite or even countable or of the cardinality \( \text{card} (k) \) (which may be greater or equal to \( \text{card} (\mathbb{R}) \)) sums \( \sum_j f(x_{j,n})\text{Ch}_{V_{j,n}} \) becomes a problem for a disjoint family \( \{V_{j,n} : j \} \) of clopen in \( X \) subsets, since \( \|\text{Ch}_{V_{j,n}}\|_{C(X,K)} = 1 \) for each \( j \) and \( n \).

12. Theorem. (The non-Archimedean analog of the Stone theorem.) Let \( A \) be a commutative Banach \( C \)-algebra over a locally compact field \( K \). If \( P \) is a regular \( H \)-projection-valued tight measure on \( \hat{A} \) (see §§6,9), then the equation

\[
(i) \quad T_a = \int_{\hat{A}} \hat{a}(\phi) P(d\phi) \quad \text{for each } a \in A \text{ defines a non-degenerate representation of } A \text{ in } H. \quad \text{Conversely, each non-degenerate representation } T \text{ of } A \text{ on a Banach space } H \text{ determines a unique regular } H \text{-projection-valued tight measure } P \text{ on } \hat{A} \text{ such that } (i) \text{ holds.}
\]

Proof. The right side of (i) is the spectral integral. Let \( P \) be a regular \( H \)-projection-valued tight measure on \( \hat{A} \). By Corollary 10 \( T' : f \to \int_X f(x)P(dx) \) is a non-degenerate representation of \( C_\infty(\hat{A},K) \) on \( H \). By Proposition 5.2 the map \( a \mapsto \hat{a}|_A \) is a homomorphism of \( A \) onto a dense subset of a subalgebra of \( C_\infty(\hat{A},K) \) such that the map \( T : a \mapsto T'|_{(\hat{a}|_A)} = \int_{\hat{A}} \hat{a}P(d\hat{a}) \) is a non-degenerate representation of \( A \).

Conversely, let \( T \) be a non-degenerate representation of \( A \) on \( H \). Then from §1,2 and Proposition 5.2 it follows, that there exists a non-degenerate representation \( T' \) of \( C_\infty(\hat{A},K) \) such that

\[
(ii) \quad T_a = T'|_{(\hat{a}|_A)} \quad \text{for each } a \in A. \quad \text{In view of Proposition 9 there exists a regular } H \text{-projection-valued tight measure } P \text{ on } \hat{A} \text{ satisfying}
\]

\[
(iii) \quad T'f = \int_{\hat{A}} f(x)P(dx) \quad \text{for each } f \in C_\infty(\hat{A},K). \quad \text{Combining } (ii,iii) \text{ we get Formula } (i). \quad \text{Let } Q \text{ be another regular } H \text{-projection-valued tight measure which is also related with the representation } T \text{ by Formula } (i), \text{ then}
\]

\[
(iv) \quad \int_{\hat{A}} \hat{a}(x)Q(dx) = T_a = \int_{\hat{A}} \hat{a}(x)P(dx) \quad \text{for each } a \in A. \quad \text{Due to Proposition 5.2 } \{\hat{a}|_A : a \in A\} \text{ is dense in } C_\infty(\hat{A},K) \text{ with respect to the supremum-norm. Hence from } (iv) \text{ and } §5 \text{ it follows that } \int_{\hat{A}} f(x)P(dx) = \int_{\hat{A}} f(x)Q(dx) \quad \text{for each } f \in C_\infty(\hat{A},K), \text{ consequently, by Proposition 9 } Q = P.
\]

13. Definition. \( P \) from Theorem 12 is called the spectral measure of the non-degenerate representation \( T \) of \( A \).

14. Proposition. Let \( P \) be the spectral measure of the non-degenerate representation \( T \) of a commutative Banach \( C \)-algebra \( A \) over a locally compact field \( K \). If \( \Omega \subset \hat{A} \) and \( \Omega \in L \), then

\[
(i) \quad \text{range } (P(\Omega)) = \bigcup_{\phi \in \Omega} \{ \xi \in H(T) : T_a \xi = \phi(a)\xi \text{ for each } a \in A \}.
\]
Proof. Relation 12.(i) and the definition of the spectral integral show, that if ξ ∈ range \( P(V) \) for each \( V ∈ L \) with \( φ ∈ V \), then \( T_φξ = φ(a)ξ \) for each \( a ∈ A \).

Conversely, suppose that \( T_φξ = φ(a)ξ \) for each \( a ∈ A \) If \( T' \) is the representation of \( A_φ \) isomorphic with \( C_∞(A,K) \) and \( T' \) corresponds to \( T \), then

\[
(ii) \quad T'_φξ = f(φ)ξ \text{ for each } f ∈ C_∞(A,K).
\]

Assume \( ξ \notin \text{range } (P(Ω)) \) and consider a measure \( μ_{ξ,η}(W) := η^∗(P(W)ξ) \) for \( ξ \) and \( η ∈ H \). There exists \( η = ξ ≠ 0 \) such that \( μ_{ξ,ξ} \) is not carried by \( Ω \). Due to regularity of \( μ_{ξ,ξ} \) there exists a compact \( E ⊂ A, E ∈ L, E ⊂ Ω \) such that \( φ ∉ E \) and \( ∥E∥_{μ_{ξ,ξ}} > 0 \). We take \( f ∈ C_∞(A,K) \) which is not equal to zero everywhere on \( E \) and \( f(φ) = 0 \), since \( A \) is the completely regular topological space \( T_{3,5} \) (see Theorem 2.3.11 in [8]). From Formula (ii) it follows, that \( T'_φξ = 0 \). By Chapter 7 in [25] and Formula 5.1.(VII) above there is an inequality:

\[
∥T'_φξ∥ ≥ ∥f∥_{N_{μ_{ξ,ξ}}} ≥ \sup_{x ∈ E}|f(x)|N_{μ_{ξ,ξ}}(x): = ∥f∥_{E}∥N_{μ_{ξ,ξ}},\text{ where } ∥f∥_φ := \sup_{x ∈ E}|f(x)φ(x)| \text{ for } f : X → K \text{ and } φ : X → [0,∞); \quad N_μ(x) := \inf_{U ∈ L, x ∈ U}∥U∥; \quad Aμ := \sup\{∥μ(B)∥ : B ∈ L, B ⊂ A}\text{ for each } A ∈ L. \text{ If } ∥ξ∥_H = 1, \text{ then } ∥T'_φξ∥ = ∥f∥_{N_{μ_{ξ,ξ}}}. \text{ We get a contradiction, consequently, } ξ ∈ \text{range } P(Ω).
\]

15. Let \( A \) be a commutative \( C \)-algebra with the unit \( 1 \) over a locally compact field \( K \) and let \( μ \) be any regular tight measure on \( A \). Let the space \( L(A,μ,K) \) be defined on the algebra \( L \) such that \( L ⊂ Bco(A) \) of \( A \) as in Chapter 7 [25]: it is the completion relative to \( ∥f∥_N_μ \) of the \( K \)-linear space of all step functions, that is, finite linear combinations of characteristic functions of elements of \( L \).

15.1. Theorem. The equation

(i) \( (T_φf)(φ) = φ(a)f(φ) \) for each \( a ∈ A, f ∈ L(A,μ,K) \) and \( φ ∈ A \) defines a non-degenerate representation \( T \) of \( A \) on \( H = L(A,μ,K) \) and the spectral measure \( P \) of \( T \) is given by \( P(W)f = Ch_Wf \) for each \( W ∈ L \) and \( f ∈ H \).

Proof. If \( φ(a) ∈ C(Δ) \), then \( \sup_{φ ∈ A}|φ(a)| < ∞ \), so

\[
\sup_{φ ∈ A}|f(φ)| |φ(a)| N_μ(φ) ≤ ∥f∥_μ∥φ∥_{C(A,K)} ≤ ∥f∥_{N_μ}, \text{ consequently, } \hat{a}f ∈ H \text{ and } T_φf = f_φ(φ)P(dφ)f \text{ due to Theorem 12. Therefore, for } \hat{a} = Ch_W \text{ we have } T_φf = P(W)f \text{ for each } W ∈ Bco(Δ). \text{ Each measure } μ_{ξ,η}(W) = η^∗(P(W)ξ) \text{ has an extension from } Bco(Δ) \text{ on } L \text{ due to}
its regularity, where $\xi$ and $\eta \in H$ (see §5). The family of such measures $\mu_{\xi,\eta}$ characterise $P$ completely, since $H$ is the Banach space of separable type over $K$. Therefore, we have an extension of $P$ on $L$.

16. From the results above it follows that $\text{supp} \ (P) \subset \text{Sp}(A)$. If a representation $T$ is one to one, that is, $\ker T = \{0\}$, then $T$ is called faithful.

16.1. Proposition. The kernel of a non-degenerate representation $T$ of $A$ consists of $a \in A$ such that $\hat{a}$ vanishes everywhere on the spectrum of $T$. Suppose in addition that $A$ is a commutative $C$-algebra over a locally compact field $K$. Then $T$ is faithful if and only if its spectrum is all of $A$.

Proof. A condition $T_a = 0$ is equivalent to $\int_{\hat{A}} \hat{a}(\phi) P(d\phi) = 0$, which is equivalent to $\hat{a}(\phi)|_{\hat{A}} = 0$ due to Theorem 12. Therefore, $\ker T = \{0\}$ is equivalent to $\text{supp} \ T = \hat{A}$.

17. Fix a Banach space $H$ over a non-Archimedean complete field $F$ such that $F \subset C_p$. If $b \in \mathcal{L}(H)$ we write shortly $\text{Sp}(b)$ instead of $\text{Sp}_{\mathcal{L}(H)}(b) := \text{cl}(\text{Sp}(\text{sp}\{b^n : n = 1, 2, 3, \ldots\}))$ (see also [25]). If $A$ is a commutative Banach subalgebra in $\mathcal{L}(H)$, then there exists a quotient mapping $\theta : A \to A/B_A$, where $B_A$ is a closed subalgebra of $A$ such that $B_A = \ker(\| \ast \|_{sp})$ is the kernel of the spectral norm, $B_A := \{x : x \in A; \|x\|_{sp} = 0\}$. Then $\theta(A)$ is the normed algebra, $\theta(A)$ is the subalgebra of $\mathcal{L}(H)/B_{\mathcal{L}(H)}$. Choose a locally compact subfield $K$ in $F$.

17.1. Spectral theorem for operators. Let $b \in \mathcal{L}(H)$. Then there exists a unique $H$-projection-valued tight measure $P$ on $K$ with values in $\mathcal{L}(H)$ with the following properties:

(i) the closed support $D$ of $P$ is bounded in $K$;

(ii) $\theta(b) = \int_K xP(dx)$; also $b = \lambda_b V \int_K xP(dx)$, where $V$ is a continuous operator from $H$ onto its closed $K$-linear subspace such that $|\pi_K| \leq \|V\| \leq |\pi_K|^{-1}$, $\pi_K \in K$, $|\pi_K| = \max\{|x| : x \in K, |x| < 1\}$; $|\lambda_b| = \|b\|$, $\lambda_b \in F$; $V$ is an isometry of $H$ onto its closed $K$-linear subspace for $K = F$;

(iii) if $K = F$, then $D = \text{Sp}(a)$, where $a$ is an auxiliary operator defined by $V$ and $b$.

Moreover, if $S$ is a family of commuting operators, $S \subset \mathcal{L}(H)$, then there exists a unique $H$-projection-valued tight measure $P$ on a locally compact subset $X \subset B(K,0,1)^\circ$ such that for each $b \in S$ there exists a unique $f_b \in C^\infty(X,F)$ for which $\theta(b) = \int_X f_b(x)P(dx)$, $b = V \int_X f_b(x)P(dx)$ and $V$ as above, where $\text{cl} \ X = X \cup \{0\}$ is compact.

Proof. If $\|b\|_{sp} = 0$, then $\text{Sp}(b) = \emptyset$ and this case is trivial with $P(0) =$
In the case of the locally compact field $K$ and $H$ over $K$ we can take $W := cl(b(H))$ and $b = Va$, where $V$ is an operator such that $V(H) = W$, $V_{|W} : W \to W$ is an isometry, $V(H \ominus W) = \{0\}$, $a$ is an operator of $H$ onto $H$ such that $\|a\|_{sp} > 0$ and $a$ is representable as a convergent series of projectors in some basis of $H$, that can be shown by transfinite induction (see [21]). In this case we get $(ii)$. Analogously for commuting algebras of operators.

The field $F$ can be considered as the Banach space over $K$. This means that $F$ supplied with the linear structure over $K$ is isomorphic with $c_0(\beta, K)$ and a corresponding ordinal $\beta$, since $K$ is locally compact and hence spherically complete (see Theorems 5.13 and 5.16 [25]). This isomorphism $\chi : F \to c_0(\beta, K)$ may be non-isometrical. The isomorphism $\chi$ generates the isomorphism of $H$ considered as the Banach space $H_K$ over $K$ with $c_0(\alpha_K, K)$, $\chi : H_K \to c_0(\alpha_K, K)$ with the corresponding ordinal $\alpha_K$. This isomorphism is $K$-linear and it produces an injective continuous $K$-linear embedding $\chi^* : \mathcal{L}(H) \to \mathcal{L}(c_0(\alpha_K, K))$ with continuous $(\chi^*)^{-1}|_{\chi^*(\mathcal{L}(H))}$. The embedding $\chi^*$ is given by the following formula: $\chi^*(a)y := \chi a^{-1}y$ for each $a \in \mathcal{L}(H)$ and $y \in c_0(\alpha_K, K)$. This is the well-known construction of the contraction of a scalar field for a Banach space. In the particular case of $H = F^n$ with $n \in \mathbb{N}$ and if $F$ is a finite algebraic extension of $K$ to each $a \in \mathcal{L}(H)$ there corresponds a finite $n \times n$ matrix, hence $\chi^*(a) \in \mathcal{L}(K^{bn})$.

Suppose there is a representation of a $C$-algebra $C_\infty(X, F)$ with the help of a $c_0(\alpha_K, K)$-projection-valued tight measure $P$ on a locally compact subset $X$ in $K^*$. Then $(\chi^*)^{-1}$ produces from $P$ an $H$-projection-valued tight measure $P_F$, since

$$(\chi^{-1}P(V)\chi)(\chi^{-1}P(W)\chi) = \chi^{-1}P(V \cap W)\chi$$

for each $V$ and $W \in L$. Consequently,

$$\chi^{-1}\int_X g(x)P(dx)\chi z = \int_X (\chi^{-1}g(x)\chi)(\chi^{-1}P(dx)\chi)z$$

for each $z \in H$ and $f \in C_\infty(X, K)$. Therefore, if $\chi^*(a) = \int_X g_a(x)P(dx)$, then $a = \int_X f_a(x)P_F(dx)$, where $g_a \in C_\infty(X, K)$ and hence $\chi^*(g_a) =: f_a \in C_\infty(X, F)$ such that $f_a = g_a$, since the restriction of $\chi$ on $K$ embedded into $F$ is the identity $K$-linear mapping.

From this it follows, that instead of $b$ or $S$ it is sufficient to consider $\chi^*(b)$ or $\chi^*(S)$. Denote $\chi^*(b)$ and $\chi(S)$ simply by $b$ and $S$ respectively. The
operator $b$ or the family $S$ generates a commutative subalgebra $A$ of $\mathcal{L}(H_K)$ generated by $sp_K\{b^n : 0 \leq n \in \mathbb{Z}\}$ or by $sp_K\{a_1^{m_1}...a_n^{m_n} : a_j \in S, n \in \mathbb{N}, 0 \leq m_j \in \mathbb{Z}; j = 1, ..., n\}$, where $b^n := 1$. It has a completion $\mathcal{A}$ relative to the spectral norm $\| \ast \|_{sp}$ by Chapter 6 in [25]. This $\mathcal{A}$ in view of Theorem 6.15 and Corollaries 6.16, 6.17 in [25] is a Banach $\mathcal{C}(S\mathcal{P}(A),\mathcal{K})$, since $1 \in \mathcal{A}$ and $S\mathcal{P}(A)$ is compact such that $\mathcal{C}(E,\mathcal{K})$ is isomorphic with $\mathcal{C}_\infty(E,\mathcal{K})$ for compact $E$.

For $S = \{b\}$ each $\phi \in A^+$ is completely defined by $\phi(b)$, thus the map $\phi \mapsto \phi(b)$ is continuos and one-to-one, consequently, it is a homeomorphism from the compact space $S\mathcal{P}(A)$ onto the compact subset $S\mathcal{P}(b)$ of $\mathcal{K}$. Therefore, we identify $S\mathcal{P}(A)$ with $S\mathcal{P}(b)$. So the Gelfand transform of $b$ becomes the identity function on $S\mathcal{P}(b)$. Thus by Theorem 12 the identity representation of $A$ in $H$ gives rise to the $H$-projection-valued tight measure $P$ on $S\mathcal{P}(b)$ such that $\theta(b) = \int_D xP(dx)$ and $b = \lambda_bV \int_D xP(dx)$. Since the identity representation is faithful, then Proposition 16 shows that the closed support $D$ of $P$ is homeomorphic to $S\mathcal{P}(b)$ (up to the mapping $\chi|_{\mathcal{K}}$). Extending $P$ to $\mathcal{L}$ by setting $P(W) = P((W) \cap S\mathcal{P}(b))$, we thus obtain a tight projection-valued measure on $\mathcal{K}$ satisfying $(i - iii)$.

In the case of the family $S$ take $X = S\mathcal{P}(A)$ such that due to Theorem 2.3.20 in [8] about a diagonal mapping we can choose $a \leq \text{card } (\gamma) \leq \text{card } (A^+)$ while embedding $X \hookrightarrow B(\mathcal{K},0,1)$, where $a$ is the minimal cardinality of a family of subsets in $A^+$ separating points of $A$. In view of §6.2 in [25] $cl(X) = X \cup \{0\}$.

To show that $P$ is uniquely determined by $(i, ii)$, let $P'$ be another $H$-projection-valued tight measure on $\mathcal{K}$ satisfying $(i, ii)$ and let $E$ be a compact subset of $\mathcal{K}$ containing the supports of both $P$ and $P'$. Consider two presentations $T : f \mapsto \int_E f(x)P(dx)$ and $T' : f \mapsto \int_E f(x)P'(dx)$ of $C(E,\mathcal{K})$. If $w$ and $e$ are the identity function $w(x) = x$ and the constant function $e(x) = 1$ for each $x \in E$ respectively, then condition $(ii)$ satisfied by both $P$ and $P'$ shows that $T_w = T'_w = b$ and also $T_e = T'_e = 1$. By the Kaplansky theorem $w$ and $e$ generate $C(E,\mathcal{K})$ as a $C$-algebra [25, 27], hence $T' = T$ and by the uniqueness statement of Theorem 12 we have $P' = P$.

18. The $P$ of the above theorem is called the spectral measure of the operator $b$ or of a family $S$. In particular $S$ may be a commutative subalgebra of $\mathcal{L}(H)$. Evidently, each nilpotent operator $v$ in $\mathcal{L}(H)$ has $\|v\|_{sp} = 0$. If $v \in \mathcal{L}_0(H)$ has $e_j^*(ue_i) = 0$ for each $j \leq i$, then $v$ is nilpotent. Therefore, $\theta(\mathcal{L}_0(H))$ is isomorphic with the algebra of diagonal operators on $H$. Its
spectrum was found in §4.

19. Note. It is an interesting property of the non-Archimedean case that a condition of a normality of an operator $b$ is not necessary in Theorem 17.1 apart from the classical case. It is not so surprising if remind, that orthogonality in the non-Archimedean case is quite another than in the classical case. For example, two vectors $(1,0)$ and $(1,1)$ are orthonormal in $K^2$, but are not orthogonal in $C^2$.

This work was started by S. Ludkovsky. All matters of this paper were thoroughly discussed and investigated by both authors. Sections 2-4 were written by B. Diarra and section 5 was written by S. Ludkovsky.

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