Elementary operations*

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Abstract

A Clifford algebra over the binary field $2 = \{0, 1\}$ is a second-order classical logic that is substantially richer than Boolean algebra. We use it as a bridge to a Clifford algebraic quantum logic that is richer than the usual Hilbert space quantum logic and admits iteration. This leads to a higher-order Clifford-algebraic logic. We formulate a toy Dirac equation with this logic. It is exactly Lorentz-invariant, yet it approximates the usual Dirac equation as closely as desired and all its variables have finite spectra. It is worth considering as a Lorentz-invariant improvement on lattice space-times.

1 Quantum hierarchy

The Hilbert-space lattice logic is obviously insufficient for quantum physics. It lacks hierarchy. It is quantum on one level, the next level, on which we talk about predicates, not quanta, is classical and distributive. We cannot describe a spin completely, but we assume that we can describe a predicate of the spin completely because the polarizers that control it are practically classical. The lattice logic is not yet a quantum logic but only a quantum first-order predicate algebra, with classical higher levels.

When we use elementary particles as probes, say of space-time structure, this one-tier-quantum logic is unsatisfactory. There are experimental indications of something like the set-theoretic hierarchy in physics. In the first place, space-time — which must ultimately be quantum — appears at least 4-dimensional to the physicist. This implies at least four hierarchic levels: point $\in$ line $\in$ plane $\in$ volume $\in$ space-time. Then particles and fields are are described by functions on time or space-time. This puts them still higher in the hierarchy; and by at least two levels, if a function is indeed a set of pairs. Finally quantum fields are built over single-quantum theories. This adds at least another level. Finally, Lagrangians are functions of quantum fields, adding at least another level. A simple physics (we assume) is quantum at every level, though the experimenter may may be described classically under low resolution. Therefore we seek a quantum correspondent to set theory.

Von Neumann understood this well. His Ph. D. thesis was already on a hierarchic logic. Von Neumann explicitly posed the problem of a quantum set theory that we may have solved here.

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2 Quantum cosmos

The standard quantum theory represents the operations on a given quantum system $S$ that can be carried out by an experimenter $T$ by operators on a Hilbert space $H(S)$ associated with $S$ up to isomorphism. To unify physics along this line one wants a space whose operators represent ideally possible quantum operations by any experimenter on any system. We see no alternative but this commonly made “$U$ assumption.”

All systems and experimenters are subsystems of one cosmic system $U$. (1)

We cannot determine the cosmos $U$ sharply. Strictly speaking its existence is metaphorical, not operational. We must imagine many possible approximate factorizations $U \approx S \otimes T(S)$ into system and exosystem, the exosystem $T(S)$ comprising everything not part of $S$, including an experimenter.

Sometimes the factorization into system and experimenter is called the Heisenberg cut, as though Heisenberg had discovered the distinction between map and territory. This is naive. We cannot trace the map-territory distinction back to its remote origins but it was clearly drawn by Plato, Boole, and Peirce well before quantum theory. It is only that physicists did not do much measurement theory during the golden age of naive classical physics. Instead they assumed, usually implicitly, that the system and its mathematical model were isomorphic, especially in their logics. In quantum theory logic is no longer scale-free. Microscopic spins (for example) have a different logic than macroscopic mathematical systems. It is no longer safe to identify map with territory, even for calculations.

The $U$ assumption renounces operationality. There is no actual experimenter to observe $U$ sharply and give operational meaning to its states. We take much the viewpoint in quantum physics that Laplace explicitly took in classical: that of a purely metaphorical Cosmic Experimenter or CE controlling the entire cosmos with maximal sharpness. The CE sees our little measurements on the system as reversible interactions between us and the system. Each experimenter partitions the cosmos $U$ into system $S$ and metasystem $T(S)$. We imagine that the metaphorical CE sees all these as subsystems of one quantum system $U$, the cosmos. Our CE differs from Laplace’s supreme intelligence, however, in that our CE is a quantum relativistic experimenter, while Laplace’s is a classical experimenter. Instead of Laplace’s cosmic phase space our CE has a cosmic Hilbert space $H(U)$ with cosmic initial states $|\alpha\rangle \in H(U)$ and dual final states $\langle\omega| \in \dual H(U)$, the dual Hilbert space. The cosmic Hilbert space factors into the operational Hilbert space $H(S)$ of the usual theory and a mostly inaccessible Hilbert space $H(T(S))$ for the exosystem.

We must not imagine that the CE can know $U$ “as it is.” We may imagine that the CE inputs the cosmos with a process $|\alpha\rangle$ of her choice before we begin our experiments and outtakes it with a bra $|\bra\omega\rangle$ of her choice after we finish. She works with probability amplitudes $\langle\omega|U|\alpha\rangle$ for $U$ transitions much as we do for atomic transition. Thus CE measurements cover and replace ours. Von Neumann already showed that such a higher-level experimenter can make observations consistent with ours.

One can then describe thermodynamically irreversible measurements by $T$ on $S$ as partly known invertible dynamical transformations of the meta-system $U = S \otimes T$. The $U$ assumption puts all actual experimenters, with all possible partitions, into one algebraic theory, extending relativity beyond what is possible for a truly operational theory. The quantum cosmos is not a new concept for quantum physics. Bohr at first rejected angrily the idea of a quantum universe, and later advocated such a higher relativity. Quantum field theorists already take the viewpoint of a CE implicitly. The person and apparatus of any actual physicist are just condensations in the same fields that the physicist is treating, and yet are lacking in the vacuum state of quantum field theory, which is Poincaré-invariant. That state must be the vacuum state of $U$ for the CE. In introducing the CE we simply make explicit what every field theorist has implicitly or explicitly done since Dirac’s quantum electrodynamics.

We distinguish therefore between operational theories, which have unitary groups limited to transforming the system but not the experimenter and the cosmos, and operation theories,
whose operations may radically change the entire cosmos but are not operational because they cannot be carried out by any experimenter in the cosmos. The same abstract algebraic structure can be interpreted either way. The metaphorical operations of the CE on the universe, including dynamical developments of \( U \), we call \( U \) operations. Those that we can actually carry out on the system \( S \), described from our viewpoint, we call \( S \) operations.

2.1 Group regularization

Segal pointed out that non-semisimple groups are unstable in an important sense and proposed to replace them with simple groups by making small changes in their commutation relations. A Lie group \( G \) with Lie algebra \( L_g \) is (Segal-) unstable if any neighborhood of its Lie product \( \otimes : L_g \otimes L_g \to L_g \) (in the topology of the manifold of tensors of type \( L_g \otimes L_g \to L_g \)) contains non-isomorphic products. That is, a group is unstable if the smallest change in its commutation relations suffices to change it to another group. And such a change also serves to stabilize it against further changes of that kind.

Indeed, many major discoveries of physics of the last century have had just this form. The group structure or algebraic structure of a quantum theory determines the quantum theory as follows. Given the unitary group \( U = U(H) \) of a Hilbert space \( H \), the Hilbert space \( H \) is uniquely determined up to isomorphism by the condition that its isometry group be \( U \).

We call this regularization process group flexing, since it introduces a curvature into the group manifold. The inverse process that returns to the singular unstable theory is group flattening. Group contraction and its inverse, group expansion, are special cases. The simple Lorentz group needs no regularization, but the Poincaré group and the Heisenberg group are still singular. Their most economical regularizations are orthogonal groups, whose representations are efficiently constructed using Clifford algebra.

If ultimately by such small changes in the commutation relations we arrive at a simple Lie group, it will be a finite quantum theory, with a discrete bounded spectrum for every observable. Its Hilbert space will be finite dimensional.

2.2 Clifford algebra concepts

We use the following Clifford algebra concepts and notation in what follows.

A Clifford ring \( C \) consists of all polynomials in elements of a linear space \( C_1 \) called vectors, with a commutative unital ring of coefficients \( C_0 \) called scalars, obeying the Clifford law:

\[
\text{The square of any vector is a scalar.} \quad (3)
\]

For any vector \( v \) of \( C \) we define

\[
\|v\| =: v^\dagger v =: v^2 := \sum_m v^m v^m. \quad (4)
\]

This makes \( C_1 \) a quadratic space over the coefficient ring \( C_0 \). If \( C_0 \) is a field we call the Clifford ring a Clifford algebra.

A free Clifford algebra is defined by its quadratic space \( V = C_1 \) over the coefficient field \( C_0 \) and is written \( C = 2^V \). If \( C \) is isomorphic to the endomorphism algebra of a module \( S \), over some ring possibly different from \( C_0 \), \( C(V) \cong S \otimes \hat{S} \), then one calls \( S \) a spinor module for \( C(V) \), and we write this improperly as \( S = \Sigma C \).

The spinor space \( S \) supports a projective representation of \( \text{SO}(V) \). This representation of \( \text{SO}(V) \) is reducible. The proper irreducible subspaces of \( S \) are also called spinor spaces.

We write \( T : x \mapsto x^\dagger \), \( xy \mapsto y^\dagger x^\dagger \) for transposition, the natural anti-automorphism \( C \to C \) fixing every first-grade Clifford element \( e = e^\dagger \). We write \( C : x \mapsto x^c \) for Clifford conjugation, the automorphism of \( \text{Cliff} V \) that changes the sign of first grade elements: \( e^c = -e \). Then \( \mathcal{H} := TC \) is a natural anti-automorphism that changes the signs of the first-grade Clifford
elements: $e^{i\theta} = -e$. The *four-group* $G_4(C)$ of the Clifford algebra consists of the mappings $1, T, C, H : \text{Cliff } V \to \text{Cliff } V$, $1$ being the identity mapping.

Four natural quadratic forms on the Clifford algebra $C$ are the $\|z\| := \text{Re}(x^* x)$ where $x \in G_4(C)$ is any of the four-group elements. We use the one invariant under the largest group of inner automorphisms of $C$, $\|z\| := \|z\|$. The Clifford group $CG C$ of $C$ is the group of invertible elements of $C$.

For any algebra $A$ and any element $z \in A$, $Lz$ and $Rz$ designate the linear operators $A \to A$ of left-multiplication and right-multiplication by $z$: $(Lz)x = zx$, $(Rz)x = xz$. The associative law just states that $Lx$ and $Ry$ commute for arbitrary $x$ and $y$.

## 3 Clifford logic

### 3.1 Clifford classical logic

In classical physics one models a quantum class or predicate by an aggregate, now a quantum aggregate of quantum systems, not classical. We use this logic to describe physical processes within the physical system $S$. Its basic logical operation is to be a group operation. There are only two classical logical operations that are group operations: $\equiv$ (material equivalence, “neither or both”) and its negation $\neq$, usually called XOR (“either and not both”), designated here by $\sqcup$. Their group identities are FALSE and TRUE respectively. For the sake of familiarity we fix the logic group operation as $\sqcap$. Then we must represent FALSE by the multiplicative identity $1 \in L(S)$.

This corresponds to representing the vacuum by $1$ in Fock’s theory of fermions or bosons. Do not confuse this $1 \in L(S)$ representing FALSE, with the linear operator $I \in A(S)$ representing TRUE or GO in the lattice logic.

Every classical predicate can be expressed in classical (finite) logic as a XOR of disjoint atoms. We therefore represent the quantum atoms first.

We generate the classical XOR group with the relations

\[ \gamma \text{ XOR } \gamma = \gamma^2 = 1, \quad \gamma \gamma' = \gamma' \gamma \]

for different atom $\gamma \neq \gamma'$ of the predicate lattice. The XOR group is graded by the number of atoms in a predicate.

Over the binary field $2 = \{0, 1\}$, anticommuting is the same as commuting. The group (convolution) algebra of the XOR group is evidently a Clifford algebra $\text{Cliff}(S, 2) = \mathbb{2}^S$ over $\mathbb{2}$. We use it as a guide to the Clifford quantum logic. We call its cliffors *binors* for brevity.

One can form the binor algebra $\text{Cliff}(S, 2)$ of any finite set $S$ of distinct objects $s_0, s_1, \ldots, s_N$. We form unit sets $\gamma_n := is_n$ of grade 1 and provide them with the binary sum operation $+: i'S \times i'S \to i'S$ and product $\sqcup : i'S \times i'S \to i'S$. To keep the levels straight, it is important that the binor representing a set is the product of its unit subsets, not of its elements.

The general basis element of $\mathbb{2}^S$ is a monomial in the variables $\gamma_n \in i'S$. A variable whose square is 1 we call *uniquadratic*. Then the $\gamma_n$ make up a first-grade basis of $N$ (anti)commuting uniquadratic generators of $\mathbb{2}^S$ unique up to order.

#### 3.1.1 Interpretation

Every binor in $\mathbb{2}^S$ is a state of a variable subset of $S$.

The generator $\gamma_n$ toggles the $s_n$ into and out of existence. The uniquadratic property prevents multiple occupancy of any state. Like set theory, binor algebra incorporates the Pauli exclusion principle.

The binor product $a \sqcup b = ab$ is the logical XOR operation, with identity 1 (the empty set) and vanity 0 (meaningless). Writing “$x = 0$” makes it explicit that “$x$” is meaningless.
A monomial is a complete description of the variable set, giving its elements explicitly. The grade of a monomial is its degree, the number of factor generators, and the cardinality of that possibility.

The binor sum $+$ is addition (modulo 2) of polynomials in the (anti)commuting idempotents $\gamma_k$ with binary coefficients. A polynomial, a sum of monomials, lists possibilities and so gives an incomplete description unless it is a monomial.

Briefly put, monomials are sharp states, polynomials are crisp states, “1” means “nothing,” and “0” means nothing.

The usual complement of a set is multiplication by the top state, the product of all the basic generators.

For example let $a, b, c, \ldots$ be distinct generators. Then the binor product $abc$ represents the 3-element set with the three unit subsets $a, b, c$. The sum $a + b + c$ represents the class of the three possible unit sets $a, b, c$ with equal weights.

If $\gamma$ and $\gamma'$ have grade $g$ then so does $\gamma + \gamma'$. This sum is only a partial logical operation, since $\gamma + \gamma = 0$ is the undefined case. Nevertheless the sum is a group operation with 0 as its identity, and the product restricted to monomials is a group operation.

The binor sum is algebraically similar to Boolean addition modulo 2, but it is semantically different: The binor 0 means nothing while the Boolean 0 means something, namely “nothing.” The quantities combined by the binor sum are not idempotent but uniprincipal variables. Whenever $a = b$, the binor sum $a + b$ is 0, the undefined. None of the familiar classical logical operations is undefined in so many cases.

In the Boolean algebra of truth-valued functions $A, B, \ldots$ on a sample space, with the usual truth-values of 0 (false) and 1 (true), the following definitions apply, in which all arithmetic is modulo 2, and the default value in the definitions of POR and PAND is 0:

\[
\begin{align*}
A \text{ OR } B &:= \sup(A, B). \\
A \text{ AND } B &:= \inf(A, B). \\
A \text{ XOR } B &:= A + B. \\
A \text{ XAND } B &:= 1 + A B. \\
A \text{ POOR } B &:= A + B \text{ if } AB \equiv 0. \\
A \text{ PAND } B &:= 1 + A + B \text{ if } A + B \equiv 1.
\end{align*}
\]

They are listed in dual pairs under complementation, which replaces every predicate $A$ by its complement $1 - A$ ($= 1 + A$). POR and PAND are not truth-functional; that is, the truth values of $A$ POR $B$ and $A$ PAND $B$ at a point are not functions of the truth values of $A$ and $B$ at that point alone. Boole and Pierce needed them for probability theory.

The binor algebras are richer languages than Boolean algebra. If Boolean algebra represents mixtures with uniform weight, binor algebra represents aggregates and powers as well, and the undefined case (0) as well as the empty case (1).

The complementation duality of binor algebra is top-multiplication $A \rightarrow \top A$. We extend the above definitions to binor logic thus, with 1 now meaning false (absent, the vacuum) and $\top$ true (present, the plenum):

\[
\begin{align*}
A \text{ OR } B &:= \sup(A, B). \\
A \text{ AND } B &:= \inf(A, B). \\
A \text{ XOR } B &:= A B. \\
A \text{ XAND } B &:= \top A B. \\
A + B &:= A + B.
\end{align*}
\]

Here the infimum of two binors consists of the monomial terms present in both, and the supremum consists of the monomial terms present in either or both. The sum $A + B$ is self-dual. To contrast these operations we note that

\[
\begin{align*}
1 \text{ XOR } 1 = 1 &\text{ OR } 1 = 1, \quad 1 + 1 = 0. \\
0 \text{ XOR } A = 0 &\text{ OR } A = A; \quad 0 + A = A.
\end{align*}
\]
We represent a binor algebra $2^S$ by the formal sum of all its monomials, which we designate by the same symbol. Then

$$2^S = \prod_s (1 + \iota s) \quad (10)$$

If $S = \sum_s s$ has $N$ terms then $2^S$ has $2^N$ terms.

### 3.2 Clifford quantum logic

We interpret a real or complex Clifford algebra as a quantum logic in close parallel to our interpretation of the binary Clifford algebra. We use the Clifford algebra as a Hilbert space of state vectors. Generators $\gamma$ are again sharp specifications of an individual quantum system $S$. The Clifford algebra defines a quadratic form for vectors. It follows from the Clifford property as usual that mutually orthogonal vectors anticommute. This projectively represents the commutative law of the classical XOR algebra.

Obviously $C$ contains a representative of every predicate in the lattice logic of $S$. These are just the monomial cliffors that Grassmann called “real.”

The predicates of $C$ do not represent filters, which often stop the system. They represent phase plates, which flip phase rather than stop. This is a purely quantum concept. The classical irreversible logic arises as a degenerate form when one uses most of the system as a heat bath and information dump, making some operations on the residual system effectively irreversible.

By the statistics of the system we mean the rule for constructing the algebraic structure of the aggregate system from that of its constituent. This is a special case of the process in classical logic that the Scottish logician William Hamilton called quantification in 1842. This is obviously a more correct term than “second quantization” and we use it here.

Here we take the ket space $H(S)$ of the system $S$ to be a Clifford algebra:

$$H(S) = 2^V(S). \quad (11)$$

### 3.3 Hierarchic quantum logic

Unlike lattice logic, Clifford quantum logic iterates neatly to make a hierarchic logic that is quantum at every level, and so is at least a candidate for a higher-order quantum logic. And unlike Grassmann logic, which we explored in earlier work [11], and bosonic logic, which we advocated in one unfortunate paper, Clifford logic is stable in the Segal sense [20] and finite at every finite order, and brings in naturally the indefinite metrics and spinors that one needs relativistic gauge theories like the standard model. Clifford logic is the most promising single candidate logic we have found for physics so far.

The Clifford exponential functor $\text{Cliff} : V \mapsto C = 2^V$ produces an algebra from a quadratic space. We make every Clifford algebra $C$ a quadratic space with the natural norm $\|z\| = \Re z^2$ for $z \in C$. Then Cliff can be iterated just like the power set functor $\mathcal{P} : S \to 2^S$.

We designate the vector image in $\text{Cliff}$ of a cliffor $\gamma \in C$ by $\iota \gamma$. $\iota$ is isometric:

$$\|\iota a\| = \Re(\iota a)^2 = \Re a^2 = \|a\|. \quad (12)$$

We define the infinite-dimensional hierarchic Clifford algebra $\text{Cliff}^\infty(\iota)$ as the least real Clifford algebra including $\mathbb{R}$ and closed under $\iota$. $\text{Cliff}^\infty(\iota)$ is the limit (union) of the nested sequence of Clifford algebras $C_n$ that starts from the null set for $C_0$ and proceeds by the iteration

$$C_{n+1} = \text{Cliff} C_n = \text{Endo} S_{n+1}. \quad (13)$$

The first terms in this sequence are

- $C_1 = \text{Cliff}(0, 0) = \mathbb{R}$,
- $C_2 = \text{Cliff}(1, 0) = \mathbb{C}$,
- $C_3 = \text{Cliff}(2, 0)$,
\[
\begin{align*}
C_4 &= \text{Cliff}(3, 1) = \mathbb{S}, \\
C_5 &= \text{Cliff}(10, 6), \\
C_6 &= \text{Cliff}(32832, 32704), \\
\vdots
\end{align*}
\] (14)

Here \( \mathbb{S} \) is the 16-dimensional algebra of Dirac spin operators that Eddington called the sedenion algebra.

The higher algebras \( C_7, C_8, \ldots \) have astronomically large dimensionality \( d_n \). They have the signature
\[
s_n = \sqrt{d_n}.
\] (15)

This is as though the individual generators had random signatures \( \pm 1 \) and the law of large numbers held exactly for numbers beyond 2, instead of on the average. We provisionally take this mathematical hierarchy of quantum logics for the physical one and use these algebras as basic ingredients in our constructions. For example, we use \( C_4 \) as the Dirac algebra of fermion theory.

The hierarchy of sets has a well-known fractal-like structure. All the set-making operations that can be applied to the null set can be applied to any other set. Thus growing from every vertex of the tree of sets is an image of the entire tree, and many other sets as well. \( \text{Cliff}^\infty(\iota) \) has the same fractal-like property. All the Clifford algebras that occur in it repeat everywhere in the hierarchy.

## 4 Elementary operations

When the state vector space has an algebraic product expressing statistics we call it the state algebra. We use the \( \iota \) algebra \( \text{Cliff}^\infty(\iota) \) as state algebra to describe operations of the cosmic quantum computer \( U \). This provides us with a discrete lexicon of elementary (first-grade) unipotent operations \( o, o', \ldots, (o^2 = 1) \). The most general state vector \( |\alpha\rangle \) for \( U \) is a polynomial in the elementary operations.

Elementary operations that have been considered (sometimes implicitly) for the cosmic state algebra include

- Dirac spins with Maxwell-Boltzmann statistics (Feynman),
- Weyl spins with various parastatistics (Weizsäcker et al.),
- Pauli spins with Bose statistics (Penrose),
- Weyl spins with Bose statistics (Finkelstein), and
- iterated Fermi-Dirac statistics (Finkelstein).

None of these efforts reached the level of a dynamical theory with interactions. We have also considered the 2-valued real Clifford statistics recently. Two-valued complex representations of the permutation group were formulated by Wiman 1898, completely catalogued by Schur 1911, and extensively studied in works such as Hoffman and Humphreys 1992. Extended to the orthogonal group, they begin to be considered for physics in Wilczek and Zee 1982, Nayak and Wilczek 1994, and Wilczek 1998. We do not use them here because they are not seen in the vacuum. The single-valued Clifford statistics we use here reverts to the underlying Clifford logic of Finkelstein 1982. We use the Clifford algebra as the state space, not as the algebra of observables. This Clifford statistics is a slight generalization of Fermi-Dirac statistics, to which it readily specializes. Particles obeying Clifford statistics we call cliffordons.

## 5 Finite Dirac equation

We construct an illustrative toy example of a finite quantum theory of a spin-1/2 particle serving as a probe to define the space-time structure. We regularize the theory by replacing compound groups by nearby simple groups.
We make the following regularization of differential geometry within Clifford algebra. Let \( \mu = 1, 2, 3, 4 \) be a Lorentz index. Let \( \alpha = 1, \ldots, 8 \) be an octad index extending \( \mu \). Let \( n = 1, \ldots, N \) enumerate \( N \) octads of Clifford generators \( \gamma_\alpha(n) \). For the space-time differentials we set

\[
idx^\mu \leftarrow \tau \gamma^{\mu 5}.
\]

(16)

For the space-time coordinates we set

\[
ix^\mu \leftarrow \tau \sum_n \gamma^{\mu 5}(n).
\]

(17)

For the conjugate momenta we set

\[
ipp_\mu \leftarrow \frac{n\hbar}{N\tau} \sum_n \gamma^{\mu 6}.
\]

(18)

The eigenvalues of the time coordinate all have the form \( n\tau \) for integer \( n \) and fall between \( \pm N\tau \). So \( \tau \) is a time quantum or “chronon.” There is an analogous energy quantum or “ergon”

\[
\epsilon := \frac{\hbar}{N\tau}.
\]

(19)

For the imaginary unit of finite quantum mechanics, we infer from the commutation relations that

\[
i \leftarrow \sum_n \gamma^{56}(n)/N =: \eta
\]

(20)

This \( \eta \) is an operator in the algebra and hence a quantum variable. In quaternionic quantum mechanics (Finkelstein Jauch and Speiser 1959) the variable \( \eta = \eta(x) \) is a natural (Stückelberg-) Higgs field; indeed, this is still the only Higgs field since gravity that was not introduced ad hoc. Clifford algebra is a generalized quaternion algebra, and it is natural optimism to expect the variable \( \eta \in 2^{8N\mathbb{R}} \) to serve as Higgs field once again. The \( i \) given above, however, has a rich spectrum of values in \([0, 1]\) for its absolute value. To account for the observed near-constancy of \( i \) (as the transformation from anti-symmetric symmetry generators to symmetric observables) we must appeal to a long range order among all \( N \) of the \( \gamma^{56}(n) \) in the vacuum ground state; much as if the usual \( i \) is the effective vacuum value of a Higgs field.

Using these correspondences it is easy to reconstruct the Dirac equation for a neutral particle as a correspondence limit of a finite Clifford algebraic quantum theory. We suppose that the rest mass \( m \) is the correspondence limit of a conjugate variable to a proper time variable \( \tau \)

\[
m = \imath \hbar \frac{d}{d\tau}.
\]

(21)

We assign the proper-time coordinate \( \tau \) to \( T \), not \( S \). From the viewpoint of the CE, they are all \( U \) variables.

The Dirac equation that we have to regularize is then the operator equation of motion for any one-particle operator \( X \), not explicitly depending on \( \tau \), based on a proper-time generator \( -iM \):

\[
\frac{d}{d\tau} X = -i[M, X]
\]

(22)

with mass operator

\[
M = \gamma^{\mu 5} \partial_\mu.
\]

(23)

The flexed algebra is a large Clifford algebra \( C = 2^V \) over a quadratic space \( V = 8N\mathbb{R} \). \( 2^V \) is a Clifford product of \( N \) octadic Clifford algebras \( 2^8 \). This is isomorphic by a Jordan-Wigner transformation to a tensor product of \( N \) octadic Clifford algebras:

\[
2^{N8} \cong \bigotimes_{n=1}^{N} 2^8 \cong \bigotimes_{n=1}^{N} 2^8
\]

(24)
which represents a Maxwell-Boltzmann aggregate of octads. This is how we account for the fact that space-time points have Maxwell-Boltzmann statistics in the standard physics.

This calculation ultimately has to be made self-consistent, as whenever one postulates a spontaneous symmetry breaking. We must show that the dynamics can lead to a ground mode with the spontaneous symmetry breaking by $\eta$ that we have assumed.

This project further requires flexing the gauge group of present differential-geometric physics to arrive at a Clifford algebraic theory with interaction. This is one of several problems now being studied.

Since this meeting a finite quantum linear harmonic oscillator has been worked out [?]/

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Contents

1 Quantum hierarchy  
2 Quantum cosmos  
  2.1 Group regularization  
  2.2 Clifford algebra concepts  
3 Clifford logic  
  3.1 Clifford classical logic  
    3.1.1 Interpretation  
  3.2 Clifford quantum logic  
  3.3 Hierarchic quantum logic  
4 Elementary operations  
5 Finite Dirac equation  
6 Acknowledgments