On the geodesic hypothesis in general relativity

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Abstract

In this paper, we give a rigorous derivation of Einstein’s geodesic hypothesis in general relativity. We use scaling stable solitons for nonlinear wave equations to approximate the test particle. Given a vacuum spacetime \([0, T] \times \mathbb{R}^3, h\), we consider the scalar field coupled Einstein equations. For all sufficiently small \(\epsilon\) and \(\delta \leq \epsilon^q, q > 1\), where \(\delta, \epsilon\) are the amplitude and size of the particle, we show the existence of solution \([0, T] \times \mathbb{R}^3, g, \phi^\epsilon\) to the coupled Einstein equations with the property that the energy of the particle \(\phi^\epsilon\) is concentrated along a timelike geodesic. Moreover, the gravitational field produced by \(\phi^\epsilon\) is negligibly small in \(C^1\), that is, the spacetime metric \(g\) is \(C^1\) close to \(h\). These results generalize those obtained by D. Stuart in [26], [27].

1 Introduction

In general relativity, Einstein’s geodesic hypothesis, which corresponds to Newton’s first law of motion in classical mechanics, states that a free massive test particle will follow a timelike geodesic in the spacetime, where by free we mean in the absence of all external forces except gravitation, which is ascribed to the spacetime curvature instead of a force. For the concept of test particle, one has to ignore its internal structure as well as the gravitational field produced by it. This paper is devoted to a rigorous mathematical derivation of geodesic hypothesis in the sense that we approximate the test particle by real particles which are scaled stable solitons for a class of nonlinear Klein-Gordon equations. Since the real particles will interact with the background spacetime, we consider the following scalar field coupled Einstein equations

\[
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu}(g, \phi^\epsilon; V_{\epsilon, \delta}), \\
\Box_g \phi^\epsilon + V'_{\epsilon, \delta}(\phi^\epsilon) &= 0.
\end{aligned}
\]

Here \(\Box_g\) is the covariant wave operator for the unknown spacetime metric \(g_{\mu\nu}\). \(\phi^\epsilon\) is the complex scalar field representing the real particles. \(R_{\mu\nu}\), \(R\) are the Ricci, scalar curvatures of the metric \(g\) respectively. \(V'_{\epsilon, \delta}\) is the first variation of the potential \(V_{\epsilon, \delta}\). \(T_{\mu\nu}\) is the energy momentum tensor for the scalar field \(\phi^\epsilon\) and is given as follows

\[
T_{\mu\nu}(g, \phi^\epsilon; V_{\epsilon, \delta}) = \langle \partial_\mu \phi^\epsilon, \partial_\nu \phi^\epsilon \rangle - \frac{1}{2}g_{\mu\nu} \langle \partial^\gamma \phi^\epsilon, \partial_\gamma \phi^\epsilon \rangle + 2V_{\epsilon, \delta}(\phi^\epsilon),
\]

where \(\langle, \rangle\) denotes the inner product of two complex numbers, namely, \(\langle a, b \rangle = \frac{1}{2}(ab + \bar{a}b)\).

The particle should have small size and small energy (or amplitude). We use the two small positive parameters \(\epsilon, \delta\) to denote its size and amplitude. In terms of local coordinate \((t, x)\), the normalized scalar field \(\phi\) takes the following form

\[
\phi(t, x) = \delta^{-1} \phi^\epsilon(\epsilon t, \epsilon x).
\]

Then the corresponding potential \(V\) for the normalized scalar field \(\phi\) is given by \(V_{\epsilon, \delta}\) via the following equation

\[
V(\phi) = \delta^{-2} \epsilon^2 V_{\epsilon, \delta}(\phi^\epsilon).
\]

In particular, one has \(V'_{\epsilon, \delta}(\phi^\epsilon) = \delta \epsilon^{-2} V'(\delta^{-1} \phi^\epsilon)\). For a solution \((M, g, \phi^\epsilon)\) of the above coupled Einstein equations, we find that the normalized scalar field \(\phi\) must satisfy the following nonlinear wave equation

\[
\Box g^\epsilon \phi + V'(\phi) = 0, \quad g^\epsilon(t, x) = g(\epsilon t, \epsilon x).
\]
Theorem 1. Let \( \bar{g} \) be the given vacuum metric. Our proof will show that the same result still holds if \( \epsilon \) is independent of \( \bar{g} \) is close to such solitons, then the real particle \( \phi^e \) is localized to a region of size \( \epsilon \). Hence solitons centered at the position of the particles can be used to approximate the particles.

We consider Einstein’s geodesic hypothesis on a given vacuum spacetime which is diffeomorphic to \( ([0, T] \times \mathbb{R}^3, h) \) with the vacuum Einstein metric \( h \). When the particle enters, it interacts with the given spacetime. To understand the motion of the particle, we consider the Cauchy problem for the scalar field coupled Einstein equations with initial data \( (\bar{g}, \bar{K}, \phi_0^e, \phi_1^e) \), where \( \bar{g} \) is a Riemannian metric on \( \mathbb{R}^3 \), \( \bar{K} \) is a symmetric two tensor, \( (\phi_0^e, \phi_1^e) \) are the initial data for the particle. The Cauchy problem is overdetermined which imposes compatibility conditions on the data: the constraint equations

\[
\bar{R}(\bar{g}) - |\bar{K}|^2 + (tr\bar{K})^2 = |\phi_1^e|^2 + |\nabla \phi_1^e|^2 + 2\nabla_c \delta(\phi_0^e), \quad \nabla_i \bar{K}_{ij} - \nabla_i tr\bar{K} = <\phi_1^e, \nabla_i \phi_0^e > .
\]

Here \( \bar{R}(\bar{g}) \) is the scalar curvature for the metric \( \bar{g} \) and \( \nabla \) is the covariant derivative with respect to \( \bar{g} \).

We have the following main result

**Theorem 1.** Let \( ([0, T] \times \mathbb{R}^3, h) \) be a given vacuum spacetime for arbitrary \( T > 0 \) with initial data \( (\bar{g}, \bar{K}, \phi_0^e, \phi_1^e) \). Consider the Cauchy problem for the Einstein equations (1) with initial data \( (\bar{g}, \bar{K}, \phi_0^e, \phi_1^e) \) satisfying the constraint equations. Assume \( (\bar{g}, \bar{K}) \) is sufficiently close to \( (h, \bar{K}) \) and the initial data \( (\phi_0^e, \phi_1^e) \) for the particle is close to some scaled stable soliton centered at the initial position \( P_0 \) of the particle. Then for sufficiently small \( \epsilon \), there exists a unique (up to diffeomorphism) solution \( (\bar{g}, \bar{K}, \phi_1^e) \) such that \( \phi_1^e \) is close some scaled soliton centered along a \( C^1 \) curve which is close to a timelike geodesic starting from \( P_0 \). Moreover, the spacetime metric \( g \) is close to the given vacuum metric \( h \).

**Remark 1.** Our proof will show that the same result still holds if \( \delta = \epsilon_0 \epsilon \) for sufficiently small \( \epsilon_0 \) which is independent of \( \epsilon \).

**Remark 2.** Existence of initial data satisfying conditions in the theorem will be shown in the last section.

**Remark 3.** The lower bound of \( p (\geq 2) \) is required for regularity purpose: the spacetime metric has to be in \( C^1 \). The upper bound \( p < \frac{3}{4} \) is needed to guarantee the existence of stable solitons.

**Remark 4.** We have the same conclusion for much more general potentials \( \mathcal{V} \), for example, potentials \( \mathcal{V} \) satisfying conditions given in [27].

A more precise statement of Theorem 1 as well as the definition of stable solitons can be found in the next two sections. We conclude from the theorem that in the limiting case when \( \epsilon \) goes to zero, the test particle moves along a timelike geodesic. This yields a rigorous derivation of Einstein’s geodesic hypothesis in general relativity.

Our theorem generalizes the result in [27] obtained by D. Stuart in two ways. First, our result describes the long time behavior of the test particle in the spacetime. In [27], it was shown that the solution \( (g, \phi^e) \) of the scalar field coupled Einstein equations exists only in a small portion \( [0, t^*) \times \mathbb{R}^3 \) of the given spacetime \( [0, T] \times \mathbb{R}^3 \) for some small positive constant \( t^* \). And the test particle travels along a timelike geodesic in the small time interval \( [0, t^*) \). Here our result shows that we can extend the solution to the whole spacetime \( [0, T] \times \mathbb{R}^3 \) such that the energy of the particle \( \phi^e \) is concentrated...
along a timelike geodesic for arbitrarily large given time $T$. In particular, our theorem implies that the test particle moves along a timelike geodesic up to any given time $T$. Second, our result holds under the assumption $\delta \leq \epsilon^q$, $q > 1$ which is weaker than $\delta \leq \epsilon^q$, $q \geq \frac{7}{4}$ imposed in [27]. Recall that $\delta$ denotes the amplitude or the energy of the particle. If one wants to show that the gravitational field produced by the particle is negligibly small in $C^1$, i.e., $\|g - h\|_{C^1} \leq C\epsilon$, one has to show that the energy momentum tensor $T_{\mu\nu}$ are bounded by $C\epsilon$ in $H^{s-1}$, $s > \frac{5}{6}$. This requires $\delta$ to be sufficiently small in terms of $\epsilon$. As the method for estimating the matter field $\phi' \in H^s$ requires $s$ to be an integer, D. Stuart proved the geodesic hypothesis under the strong condition $q \geq \frac{7}{4}$. On the other hand, notice that $\delta^{-1} \phi'$ is close to some soliton. In view of the equation (3), the potential $V_{\epsilon, \delta}$, in particular the energy momentum tensor $T_{\mu\nu}$, has size $\delta^2 \epsilon^{-2}$. Based on the heuristics that in the limiting case when $\epsilon$ goes to zero, the potential $V_{\epsilon, \delta}$ should be bounded, we see that the condition $\delta \leq \epsilon^q$, $q \geq 1$ is needed to allow a proof of geodesic hypothesis in this setting. Our theorem thus answers the question of D. Stuart on the optimal value of $q$ when $q > 1$. Furthermore, as was pointed out in Remark 1, the condition can be even weaker by merely assuming that $\delta = \epsilon_0 \epsilon$ for some small constant $\epsilon_0$ independent of $\epsilon$. This is robust and interesting as then the potential $V_{\epsilon, \delta}$ always has size $\epsilon_0^2$ even when $\epsilon$ goes to zero.

The first aspect that we can extend the solution to arbitrarily large time $T$ is based on a result on the orbital stability of solitons on small perturbations of Minkowski space. Note that the dynamics of the particle $\phi'$ are governed by the nonlinear wave equation (see the scalar field coupled Einstein equations (1)). In local coordinate $(t, x)$, the corresponding normalized scalar field $\phi$ solves the nonlinear wave equation (4) on the slowly varying background with metric $g^\prime(t, x) = g(\epsilon t, \epsilon x)$. The assumption on the initial data for the particle implies that the initial data for the normalized scalar field $\phi$ are close to some stable solitons. The property that the particle travels along a timelike geodesic can then be reduced to the orbital stability of stable solitons along a timelike geodesic on a slowly varying background. In other words, we need to show that the solution $\phi$ of the nonlinear wave (4) is close to some soliton for all $t$ and the center of the soliton propagates along a timelike geodesic.

The related problem of stability of solitons in Minkowski space ($g = g_0$, the Minkowski metric), including the existence of solitons of the form $e^{i\omega t} f_\omega(x)$ associated to the ground state of the corresponding elliptic equation

$$\Delta f_\omega - (m^2 - \omega^2) f_\omega + |f_\omega|^{p-1} f_\omega = 0, \quad |\omega| < m,$$

has been studied extensively. For the ground state $f_\omega(x)$, we summarize the relevant known results (see [1], [3], [19], [20], [24], [25] and references therein) in Theorem 2 in the next section. In particular, we have the $C^4$ regularity and exponential decay of the ground state $f_\omega(x)$. The stable solitons are those with phase $\omega$ such that the associated energy $\frac{1}{2} \|\nabla f_\omega\|_{L^2(\mathbb{R}^3)}^2$ is convex in $\omega$, see [21]. Stable solitons have been proven to be orbitally stable for all time in Minkowski space first by J. Shatah [21] with radial symmetry, and later by M. Grillakis, J. Shatah and W. Strauss in [15], [16] in a much more general framework. More precisely, it was shown that if the initial data are close to some stable soliton in $H^1$, then the solution to the nonlinear Klein-Gordon equation exists for all time and is close to some translated solitons(by the Lorentz transformation of Minkowski space). However, these works do not characterize the dynamics of the solitons. In particular, the centers of the solitons were not explicitly constructed.

The first step along this direction was taken by M. Weinstein. In [28], he proved the orbital stability of solitons to nonlinear Schrödinger equations and gave the additional information on the position and speed of the solitons by using modulation theory. For the modulation theory, one decomposes the solution $\phi$ into the soliton part and the remainder part $\phi = \phi_S + v$. The soliton part $\phi_S$ is unknown, depending on, e.g., its position and speed. Using the decomposition, the equations for $\phi$ then lead to a linearized equation for the remainder $v$. We remark here that the linearized equation also depends on the unknown soliton $\phi_S$. We choose the decomposition such that the remainder part $v$ is orthogonal to the generalized null space of the linearized equation at $\phi_S$. This orthogonality condition together with the equations for $\phi$ leads to a coupled system of nonlinear ODE’s (modulation equations), which governs the position and speed of the solitons.
Using this modulation approach, D. Stuart studied the stability of solitons to nonlinear wave equations. In [25], he proved the orbital stability of stable solitons and showed that the center of the soliton moves along a $C^1$ curve which is close to a straight line in Minkowski space. Later in [26], D. Stuart studied the stability of stable solitons on small perturbations of Minkowski space. More precisely, he considered the Cauchy problem for the nonlinear wave (4) with initial data which are close to stable solitons on a slowly varying background with metric $g^\epsilon(t, x) = g(et, cx)$. He showed that stable solitons are orbitally stable and move along timelike geodesics up to time $t^*/\epsilon$ for some small positive constant $t^*$. Due to the scaling, when applying this result to the problem of geodesic hypothesis, one can only show that the particle travels along a timelike geodesic in the short time interval $[0, t^*)$, see [27]. Although $t^*$ is independent of $\epsilon$, it was required to be sufficiently small.

A key improvement which allows us to obtain Theorem 1 is that we are able to extend D. Stuart’s stability result up to time $T/\epsilon$ for arbitrary large $T > 0$. We show that if the initial data are close to some stable solitons, then we can solve the nonlinear wave equation (4) up to time $T/\epsilon$ and demonstrate that the solution is close to stable solitons centered along a timelike geodesic. The time $T$ has to be fixed as we need to require $\epsilon$ to be sufficiently small depending on $T$. However, we no longer need the smallness of $T$ as it was in [26].

We will adapt the modulation approach to treat the orbital stability of stable solitons on a fixed slowly varying background. The new ingredient is that we can construct a coordinate system (Fermi coordinate system) such that the Christoffel symbols vanish along the trajectory of the center of the soliton, which is a timelike geodesic uniquely determined by the initial data. The motivation for choosing such a local coordinate system is to study the equation for the remainder part $v$ by using the modulation approach mentioned above. Recall that we decompose the solution $\phi$ of the equation (4) into soliton plus a remainder $\phi = \phi_S + v$. The remainder $v$ is supposed to be small. To control $v$, we need to estimate $\Box_{g^\epsilon}\phi_S$, in particular the term

$$\frac{1}{\sqrt{-\det g^\epsilon}} \partial_{\mu} \left( (g^\epsilon)^{\mu\nu} \sqrt{-\det g^\epsilon} \right) \partial_{\nu} \phi_S = -(g^\epsilon)^{\mu\nu} \Gamma^\nu_{\mu\gamma} \partial_{\nu} \phi_S,$$

where $\Gamma^\nu_{\mu\gamma}$ is the Christoffel symbols for the metric $g^\epsilon$. As the soliton $\phi_S$ is expected to be centered along a timelike geodesic, to control the above term, a natural way is to choose a good local coordinate system such that along that geodesic the Christoffel symbols vanish. Furthermore, under such a coordinate system, the geodesic equations are linear. In particular, the geodesic can be parameterized by $(t, u_0 t)$ for some constant vector $u_0 \in \mathbb{R}^3$. This parametrization is exactly the one for straight lines in Minkowski space. Now as the full derivative of the metric components vanishes along the geodesic and the metric is slowly varying, that is, $g^\epsilon(t, x) = g(\epsilon t, cx)$, we conclude that near the geodesic, the metric $g^\epsilon$ is higher order (at least $\epsilon^2$) perturbation of the flat metric $g(0, 0)$. Hence the errors contributed by the soliton $\phi_S$ in the equation for the remainder $v$ will have size $\epsilon^2$.

Nevertheless, for the orbital stability in the energy space $H^1$ on a slowly varying background, we are not going to study the equation for the remainder $v$ directly. Instead, as in [25], we decompose the almost conserved energies of the full solution $\phi$ around the soliton $\phi_S$. By using Gronwall’s inequality, we can prove the orbital stability up to time $T/\epsilon$. The key is that we have avoided arguments based on bootstrap. And by doing so we can remove the smallness assumption on $t^*$ which was used to close the bootstrap assumption as it was in [26]. For the higher order Sobolev estimates of the remainder $v$ which are needed to control the spacetime metric $g$, we turn to rely on the equation of $v$.

We now briefly discuss the proof for the main theorem as well as the second aspect of our work that we can improve the amplitude of the particle to be $\delta \leq \epsilon^q$, $q \geq 1$. We work under the Fermi coordinate system mentioned above. After scaling, it is equivalent to consider the scaled coupled Einstein equations for $(g^\epsilon, \phi)$. Starting with the bootstrap assumption on the unknown spacetime metric $g^\epsilon$

$$\|\partial_{\nu}(g^\epsilon - h^\epsilon)\|_{L^2(\mathbb{R}^3)}(t) \leq 2\epsilon^2, \quad 1 \leq s \leq 3, \quad t \leq T/\epsilon,$$  \hspace{1cm} (6)
we can show that the normalized scalar field $\phi$ is close to some solitons centered along the timelike geodesic $(t, u_0 t)$ in $H^3$. That is

$$\|\partial^s (\phi - \phi_S)\|_{L^2(\mathbb{R}^3)}(t) \leq C_\delta, \quad \forall s \leq 3, \quad t \leq T/\epsilon$$

for some constant $C$ independent of $\epsilon$, where $\phi_S$ are solitons centered along $(t, u_0 t)$. Here we must clarify that we have used bootstrap argument in order to estimate the spacetime metric $g$. But as discussed above, we avoid using bootstrap argument when estimating the scalar field $\phi$. The $H^1$ estimates are implied by the orbital stability of solitons. Then the higher Sobolev estimates follow by analyzing the nonlinear wave equation of the remainder $\phi - \phi_S$. These Sobolev estimates for the scalar field $\phi$ are used to control the the energy momentum tensor $\delta^2 T_{\mu\nu}(g^r, \phi; \mathcal{V}(\phi))$ (after scaling) so that we can estimate the spacetime metric $g^r$ in order to close the above bootstrap assumption. Choosing the relatively harmonic gauge condition $[5], [17]$, one can turn the Einstein equations into a hyperbolic system for the components of the metric $g^r$. Detailed reduction is carried out in Section 5. Combining with the vacuum Einstein equations for the given metric $h^r$, we can roughly obtain estimates for $g^r - h^r$ as follows

$$\sup_{t \leq T/\epsilon} \|\partial^s (g^r - h^r)\|_{L^2(\mathbb{R}^3)}(t) \leq C_\delta^2 \int_0^{T/\epsilon} \|T_{\mu\nu}\|_{H^s}(t) dt \leq C_\delta^2 \epsilon^{-1}$$

by using energy estimates for hyperbolic equations. This requires $\delta \leq \epsilon^q$, $q > \frac{1}{2}$ in order to close the bootstrap assumption. Such a lower bound on $q$ was suggested in [27] and the result there was proven under the even stronger condition $q \geq \frac{7}{4}$.

However, through another new observation, we are able to improve the condition on $\delta$ to be $\delta \leq \epsilon^q$, $q > 1$ or $\delta = \epsilon_0 \epsilon$ for sufficiently small $\epsilon_0$ which is independent of $\epsilon$. Notice that the tangent vector $X = \partial_t + u_0 \partial_x$ of the geodesic $(t, u_0 t)$ is timelike and can be extended to a uniformly timelike vector field on the whole spacetime. We have already shown that the scalar field $\phi$ decomposes into a soliton part $\phi_S$ and an error term. Hence the energy momentum tensor $\delta^2 T_{\mu\nu}(g^r, \phi; \mathcal{V}(\phi))$ splits into the soliton part $\delta^2 T_{\mu\nu}^S$, which moves along the timelike geodesic $(t, u_0 t)$ or quantitatively

$$\delta^2 |X \partial^s T_{\mu\nu}^S| \leq C_\delta^2 \epsilon, \quad \forall s \leq 3,$$

and the error part $\delta^2 T_{\mu\nu}^R$ which satisfies the estimates

$$\delta^2 \|\partial^s T_{\mu\nu}^R\|_{L^2(\mathbb{R}^3)}(t) \leq C_\delta^2 \epsilon, \quad s \leq 3, t \leq T/\epsilon.$$

To make use of the fact that the soliton part moves along the geodesic, when doing energy estimates, we multiply the hyperbolic equations by $X(g^r - h^r)$. Integrating by parts, we can pass the derivative $X$ to the soliton part $T_{\mu\nu}^S$ of the energy momentum tensor $T_{\mu\nu}$. As the soliton decays exponentially, using Hardy’s inequality, we can show that

$$\left| \int_{\mathbb{R}^3} T_{\mu\nu}^S \cdot X(g^r - h^r) dx \right| \leq \| (1 + |x|) X T_{\mu\nu}^S \|_{L^2} \| (1 + |x|)^{-1} (g^r - h^r) \|_{L^2} \leq C_\epsilon \|\partial (g^r - h^r)\|_{L^2}.$$

Since the error part is small, the above observation allows us to improve the condition on $\delta$. In fact, since the vector fields $X, \partial_t$ are uniformly timelike, the energy estimates for hyperbolic equations show that

$$\|\partial (g^r - h^r)\|_{L^2}^2(t) \leq C_\delta^4 + C_\epsilon \delta^2 \int_0^t \|\partial (g^r - h^r)\|_{L^2}^2(s) ds.$$

Here we have to require that initially $\|\partial (g^r - h^r)\|_{L^2}(0) \leq C_\delta^2$. Applying Gronwall’s inequality or using bootstrap argument, we obtain

$$\|\partial (g^r - h^r)\|_{L^2}(t) \leq C_\delta^2, \quad \forall t \leq T/\epsilon.$$
we are particularly interested in the ground state, that is, solution of (8) with lowest energy \( R \) on \( \omega \). Commuting the equations with \( \partial \), the bootstrap assumption (6) and hence to conclude our main theorem, it suffices to require that \( \delta \leq \epsilon^q \), \( q > 1 \) or \( \delta = \epsilon^3 \epsilon \) for sufficiently small \( \epsilon_0 \) which is independent of \( \epsilon \). Our main theorem then follows from the local existence result for Einstein equations in \( H^3 \), see [4], [9].

Finally, we discuss the existence of the initial data \((\Sigma_0, \bar{g}, \bar{K}, \phi_0, \phi_1)\). As we have mentioned previously, we require the data to satisfy the constraint equations and the estimates

\[
\|\partial^s (g^\epsilon - h^\epsilon)\|_{L^2(\Sigma_0)} \leq C \delta^2, \quad \forall 1 \leq s \leq 3.
\]

For given \((\phi_0^\epsilon, \phi_1^\epsilon)\), existence of initial data satisfying the constraint equations has been shown in [8], [5]. However, the above estimates do not follow directly from previous works. We will revisit the existence of initial data by using the implicit function theorem combined with the approach developed in [8], [6], [5]. To show that the data also satisfy the above estimates, we rely on a Hardy type inequality for a first order linear operator with trivial kernel in some weighted Sobolev space. We refer the reader to Lemma 16 in the last section for details.

In next section, we will address the basic setup, define solitons in Minkowski space and summarize the known results related to stability of solitons. In Section 3, we give the precise statement of our main theorem. In Section 4, we construct the good coordinate system such that the Christoffel symbols are vanishing along a given timelike geodesic. In Section 5, we reduce the Einstein equations to a hyperbolic system by choosing the relatively harmonic gauge condition. In Section 6, we prove the orbital stability of stable solitons along a timelike geodesic up to time \( T/\epsilon \) on a slowly varying background and show the higher Sobolev estimates for the solution which are used to control the energy momentum tensor \( T_{\mu\nu} \). In Section 7, we prove our main theorem. In the last section, we discuss the existence of initial data.

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## 2 Preliminaries and Stability Results in Minkowski Space

In Minkowski space, for the nonlinear Klein-Gordon equation

\[
\Box \phi - m^2 \phi + |\phi|^{p-1} \phi = 0, \quad \Box = -\partial_t^2 + \Delta = -\partial_t^2 + \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2, \quad m > 0, \quad p > 1
\]

of complex functions \( \phi(t, x) \), looking for the solitons or stationary waves, that is, \( \phi \) is of the form \( e^{i\omega t} f_\omega(x) \), \( \omega \in \mathbb{R} \), we are led to consider the elliptic equation

\[
\Delta f_\omega - (m^2 - \omega^2) f_\omega + |f_\omega|^{p-1} f_\omega = 0
\]

on \( \mathbb{R}^3 \). Such elliptic equation has been studied extensively and has infinite many solutions [2], of which we are particularly interested in the ground state, that is, solution of (8) with lowest energy

\[
E_\omega(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + (m^2 - \omega^2)|v|^2 - \frac{2}{p+1}|v|^{p+1} dx.
\]

It can be shown that the ground state has to be positive, radial symmetric. The existence of ground state as well as its properties are summarized in the following theorem, see [1], [3], [19], [20], [24], [25] and reference therein.

**Theorem 2.** For \( 1 < p < 5 \), \( \omega \in (-m,m) \), there exists a unique, positive, radial symmetric solution \( f_\omega(x) \in H^4(\mathbb{R}^3) \cap C^4(\mathbb{R}^3) \) of the equation (8). It is decreasing in \( |x| \) with the following properties:

1. Exponential decay up to fourth order derivatives

\[
|\nabla^\alpha f_\omega(x)| \leq C(\omega)e^{-c(\omega)|x|}, \quad \forall x \in \mathbb{R}^3, \quad |\alpha| \leq 4
\]

for some positive constant \( c(\omega) \);

2.
2. Asymptotical behavior
\[ \lim_{|x| \to \infty} \frac{f'(|x|)}{f(|x|)} = -\sqrt{m^2 - \omega^2}; \] (10)

3. Scaling of the solutions
\[ f_\omega(x) = (m^2 - \omega^2)^{1/2} f(\sqrt{m^2 - \omega^2}x), \] (11)

where \( f(x) \) is the solution for \( m^2 - \omega^2 = 1 \);

4. Identities of the energy
\[ \frac{3(p-1)(m^2 - \omega^2)}{2||\nabla f_\omega||_{L^2}^2} = \frac{5-p}{2||f_\omega||_{L^2}^2} = \frac{(p+1)(m^2 - \omega^2)}{||f_\omega||_{L^{p+1}}^{p+1}}. \] (12)

Existence of ground state has been obtained in [1], [3], [24]. K. McLeod [19] proved the uniqueness of the ground state. In [1], it was shown that \( f_\omega(x) \) is \( C^2 \) and decays exponentially up to second order derivatives, which was generalized to be \( C^4 \) in [25]. The asymptotic behavior of the solution (10) was first proven in [20]. Using integration by parts, identities (12) follow by multiplying the equation (8) with \( f_\omega, x \cdot \nabla f_\omega \) respectively.

Having the basic solitons \( e^{i\omega t}f_\omega(x) \) of (7) corresponding to the ground state \( f_\omega(x) \), one can study their stability. To start with, we must understand the symmetries of the equation (7), connecting to the symmetries of Minkowski space together with the structure of the equation (8), namely the scaling property of the ground state (11). These symmetries give an 8-parameter family of solitons. More precisely, denote \( \lambda = (\omega, \theta, \xi, u) \in \Lambda \) with
\[ \Lambda \equiv \{ (\omega, \theta, \xi, u) \in \mathbb{R}^8 : |u| < 1, \quad |\omega| < m \}. \]

A subset of \( \Lambda \) is of particular importance
\[ \Lambda_{stab} \equiv \{ (\omega, \theta, \xi, u) \in \Lambda, \quad \frac{p-1}{6-2p} < \frac{\omega^2}{m^2} < 1 \}, \quad 1 < p < \frac{7}{3}, \] (13)
corresponding to the stable solitons. Define
\[ z(x; \lambda) = A_u(x - \xi) = \rho P_u(x - \xi) + (I - P_u)(x - \xi), \quad \rho = (1 - |u|^2)^{-\frac{1}{2}}, \] (14)
\[ \Theta(x; \lambda) = \theta - \omega u \cdot z(x; \lambda), \]
where \( P_u : \mathbb{R}^3 \to \mathbb{R}^3 \) is the projection operator in the direction \( u \in \mathbb{R}^3, I \) is the identity map. Let
\[ \phi_S(x; \lambda) = e^{i\Theta(x; \lambda)}f_\omega(z(x; \lambda)), \]
\[ \psi_S(x; \lambda) = e^{i\Theta(x; \lambda)}(i\rho \omega f_\omega(z(x; \lambda)) - \rho u \cdot \nabla_z f_\omega(z(x; \lambda))). \]

Direct calculations show that \( \phi_S(x; \lambda) \) solves (7) if the curve \( \lambda(t) = (\omega(t), \theta(t), \xi(t), u(t)) \) obeys the evolution equations
\[ \dot{\omega} = 0, \quad \dot{\theta} = \frac{\omega}{\rho}, \quad \dot{\xi} = u, \quad \dot{u} = 0. \]

Here we use the dot to denote the derivative with respect to \( t \). We remark here that the centers \((t, ut)\) of the solitons forms a straight line (geodesic) in Minkowski space.

For \( \lambda \in \Lambda \), let
\[ V(\lambda) = (0, \frac{\omega}{\rho}, u, 0). \] (15)

We find that \( \psi_S(\lambda; x) = D_\lambda \phi_S(\lambda; x) \cdot V(\lambda) \)(inner product of vectors in \( \mathbb{R}^8 \)) and the important identity
\[ \Delta_x \phi_S - m^2 \phi_S + |\phi_S|^{p-1} \phi_S - D_\lambda \psi_S \cdot V(\lambda) = 0. \] (16)
We note that there exist a $C$ and $\eta$ defined in (14). For a $C^1$ curve $\lambda(t) = (\omega(t), \theta(t), \xi(t), u(t))$, let $\gamma(t) = (\omega(t), \pi(t), \eta(t), u(t))$ such that
\[
\dot{\lambda} = \dot{\gamma} + V(\lambda).
\] (17)

We note that
\[
\eta(t) = \xi(t) - \int_0^t u(s)ds, \quad \pi(t) = \theta(t) - \int_0^t \omega(s)\rho(s)ds.
\]

Consider the Cauchy problem for the equation (7) in Minkowski space with initial data $\phi_0(x) \in H^1(\mathbb{R}^3)$, $\phi_I(x) \in L^2(\mathbb{R}^3)$. The stability result for solitons is known. The following theorem is proven in [25].

**Theorem 3.** Let $1 < p < \frac{7}{4}$. For $\lambda_0 \in \Lambda_{\text{stab}}$, there exists a small constant $\epsilon(\lambda_0)$ such that if
\[
\epsilon = ||\phi(0,x) - \phi_S(x;\lambda_0)||_{H^1} + ||\partial_t \phi(0,x) - \psi_S(x;\lambda_0)||_{L^2} < \epsilon(\lambda_0),
\]
then there exist a $C^1$ curve $\lambda(t) \in \Lambda_{\text{stab}}$ and a solution $\phi(t,x) \in H^1$ of equation (7) satisfying
\[
||\phi(t,x) - \phi_S(x;\lambda(t))||_{H^1} + ||\partial_t \phi(t,x) - \psi_S(x;\lambda(t))||_{L^2} < C\epsilon
\]
and
\[
||\partial_t \lambda(t) - V(\lambda(t))|| < C\epsilon
\]
for some constant $C$ independent of $\epsilon$.

Stability of solitons with $\lambda_0 \in \Lambda_{\text{stab}}$ has first been shown by J. Shatah in [21] for radial symmetric initial data and was later put into a very general framework in [15], [16]. Their approach relies on the fact that the energy $E_\omega(f_\omega)$ is strictly convex in $\omega$ if initially $\lambda_0 \in \Lambda_{\text{stab}}$, see [21]. This condition on $\lambda_0$ is sharp in the sense that the solitons are unstable if the energy $E_\omega(f_\omega)$ is concave in $\omega$, see [22], [23]. Alternatively, the modulation approach, pioneered by M. Weinstein [28] for showing the stability of solitons to nonlinear Schrödinger equations, leads to Theorem 3 which additionally gives the behavior of the curve $\lambda(t)$, see the work of D. Stuart [25].

We now briefly describe the modulation approach. Notice that equation (7) locally has a unique solution $(\phi(t,x), \partial_t \phi(t,x)) \in C^1([0,t^*); H^1 \times L^2)$. Decompose the solution $\phi$ as follows
\[
\phi(t,x) = \phi_S(x;\lambda(t)) + e^{i\theta(x;\lambda(t))}v(t,x),
\]
\[
\partial_t \phi(t,x) = \psi_S(x;\lambda(t)) + e^{i\theta(x;\lambda(t))}w(t,x)
\]
for a $C^1$ curve $\lambda(t) \in \Lambda_{\text{stab}}$ such that the following orthogonality condition hold
\[
< e^{-i\theta} D_\lambda \phi_S, w >_{dx} = < e^{-i\theta} D_\lambda \psi_S, v >_{dx}, \quad \forall t \in \mathbb{R}. \tag{18}
\]
Here in this paper for complex valued functions $a(x)$, $b(x)$, $< a(x), b(x) >_{dx}$ is short for
\[
\frac{1}{2} \int_{\mathbb{R}^3} \overline{a} \dot{b} + \dot{\overline{a}} b \quad dx
\]
on $\mathbb{R}^3$ with measure $dx$. Differentiating (18) in $t$ and using the equation (7), we can obtain a coupled system of ODE’s for $\lambda(t)$. To estimate the curve $\lambda(t)$, we must control the radiation term $(v, w)$. We first define two operators, $L_+$ and $L_-$, appeared in the linearization in the real and imaginary part of the solution to (7). These two operators act on functions of $z$ in $H^1(\mathbb{R}^3)$, defined as follows
\[
L_+ = -\Delta_z + (m^2 - \omega^2) - pf_\omega^{p-1}(z),
\]
\[
L_- = -\Delta_z + (m^2 - \omega^2) - f_\omega^{p-1}(z), \tag{19}
\]
which satisfy the following properties proven in [28].
Proposition 1. We have

(a) $L_-$ is a nonnegative self-adjoint operator in $L^2$ with null space $\ker L_- = \text{span}\{f_\omega\}$;
(b) $L_+$ is a self-adjoint operator in $L^2$ with null space $\ker L_+ = \text{span}\{\nabla_z f_\omega\}$. The strictly negative eigenspace of $L_-$ is one dimensional.

It can be shown from the nonlinear wave equation of $v(t,x)$ that the corresponding energy is

$$E_0(t) = \|w + \rho_u \cdot \nabla_z v - i\rho \omega \|_{L^2}^2 + <v_1, L_+ v_1 >_{dz} + <v_2, L_- v_2 >_{dz},$$

where $v, w$ are viewed as functions of $(t,z)$. Although the operators $L_+, L_-$ are not positive definite, one still can show that $E_0(t)$ is equivalent to $\|v\|_{H^1} + \|w\|_{L^2}$ under the orthogonality condition (18).

Proposition 2. Assume $\lambda \in \Lambda_{\text{stab}}$. Assume $v, w$ satisfy the orthogonality condition (18). Then there is a positive constant $C$ depending continuously on $\omega, u$ such that

$$C^{-1}(\omega, u)(\|w\|_{L^2}^2 + \|v\|_{H^1}^2) \leq E_0(t) \leq C(\omega, u)(\|w\|_{L^2}^2 + \|v\|_{H^1}^2).$$

We will use this proposition for our later argument. The proof could be found in [25], which is based on Proposition 1. The orthogonality condition $< e^{-i\theta} D g \phi, w >_{dz} = < e^{-i\theta} D g \psi, v >_{dz}$ which is equivalent to $< i\phi, w >_{dz} = < i\psi, v >_{dz}$ shows that the energy $E_0(t)$ is nonnegative. The proposition then follows by using a contradiction argument, see the detailed proof in [25]. Once we have control of $\|v\|_{H^1} + \|w\|_{L^2}$, by analyzing the ODEs for $\lambda(t)$, we can control the curve $\lambda(t)$ and obtain estimates for the solution $\phi(t,x)$.

3 Statement of the Main Theorem

Let $(\mathcal{M}, h)$ be a vacuum spacetime. Assume $\mathcal{M}$ is diffeomorphic to $[0,T] \times \mathbb{R}^3$ with coordinate system $(t,x)$. We assume the vacuum Einstein metric $h$ is $C^4$ and satisfies the following conditions

$$K_0^{-1} I_{3 \times 3} \leq \langle h^{kl} (t,x) \rangle \leq K_0 I_{3 \times 3}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^3,$$

$$K_0^{-1} - h^{00} \leq K_0, \quad \forall \mu, \nu = 0, 1, 2, 3, \quad ||h||_{C^4([0,T] \times \mathbb{R}^3)} + ||x^{1/2} \partial h||_{L^\infty} + ||x^{3/2} \partial h||_{L^\infty} \leq K_0, \quad |s| \leq 2$$

for some positive constant $K_0$, where $h = h^{\mu\nu}(t,x) dx^\mu dx^\nu$, $x^0 = t$, $h^{\mu\nu}(t,x) = (h^{-1})_{\mu\nu}(t,x)$, $\partial$ is short for $(\partial_t, \partial_x, \partial_x, \partial_x)$. The first two lines require that the metric $h$ is Lorentzian and the vector field $\partial_t$ is timelike. The decay assumption on $\partial h$ is used to estimate $g - h$ by using Hardy’s inequality as initially we do not have estimates for $\|g - h\|_{L^2(\Sigma_0)}$ (our construction of initial data implies that for general data $\|g - h\|_{L^2(\Sigma_0)}$ is not bounded).

The initial data $(\Sigma_0, g, \bar{K}, \phi_0, \phi_1)$ for the Cauchy problem of system (1) with $g$ a Riemannian metric and $\bar{K}$ a symmetric two tensor on $\Sigma_0$, have to satisfy the following constraint equations on the initial hypersurface $\Sigma_0$

$$\begin{align*}
\bar{R}(\bar{g}) - |\bar{K}|^2 + (tr \bar{K})^2 &= |\phi_1|^2 + |\nabla \phi_0|^2 + 2V_{\epsilon, \delta}(\phi_0), \\
\nabla^j \bar{K}_{ij} - \nabla_i tr \bar{K} &= <\phi_1, \nabla_i \phi_0>,
\end{align*}$$

where $\nabla$ is the covariant derivative with respect to the Riemannian metric $\bar{g}$ on $\Sigma_0$. The potential $V_{\epsilon, \delta}$ is defined as in (3), (5). We remark here that $\phi_1$ corresponds to $n \phi^2$ for $n$ the unit normal to $\Sigma_0$ embedded to the solution $(\mathcal{M}, g)$.

Since $\Sigma_0$ is diffeomorphic to $\mathbb{R}^3$, we work under the coordinate system $x$ on $\Sigma_0$. We assume the particle enters the spacetime at point $\xi_0 \in \Sigma_0$ and is approximated by a scaled and translated stable soliton centered at $\xi_0$. We hence have to define solitons on $(\mathcal{M}, h)$. The idea is that at any point $P \in \mathcal{M}$,
we simply define the solitons at $P$ to be those on the flat space with metric $h(P)$. More precisely, for any foliations $[0, T^*] \times \Sigma_\tau$ of $\mathcal{M}$ with coordinate system $(t, x)$, where $\Sigma_\tau \subseteq \mathbb{R}^3$, let

$$\Lambda_{\text{stab}}(t) = \{ (\omega, \theta, \xi, u_h) \in \mathbb{R}^8 | \frac{p - 1}{6 - 2p} < \frac{\omega^2}{m^2} < 1, \quad \partial_t + u_h \partial_x \text{ is timelike at } (t, \xi) \in [0, T^*) \times \Sigma_\tau \}.$$

We define

\textbf{Definition 1.} For a curve $\lambda(t) = (\omega(t), \theta(t), \xi(t), u_h(t)) \in \Lambda_{\text{stab}}(t)$, a soliton centered at $\xi(t)$ in the direction $u_h(t)$ on the space $([0, T^*) \times \Sigma_\tau, h)$ is defined as follows

$$\phi^\tau_S(x; \lambda(t)) = e^{ic\tau^{-1}(\theta(t) - \rho \omega(t)u_0 - \xi(t))Q} f_{\lambda(t)}(e^{-1}A_u Q^T (x - \xi(t))^T),$$

where

$$u = a^{-1}(\alpha + u_h Q), \quad \rho = (1 - |u|^2)^{-\frac{1}{2}}, \quad a > 0, \quad \alpha \in \mathbb{R}^3, \quad Q \text{ is } 3 \times 3 \text{ matrix such that}$$

$$h(t, \xi(t)) = \begin{pmatrix} -a^2 + \alpha \alpha^T & \alpha Q^T \\ Q \alpha^T & Q Q^T \end{pmatrix} = \begin{pmatrix} a & \alpha \\ 0 & Q \end{pmatrix} m_0 \begin{pmatrix} a & 0 \\ \alpha^T & Q^T \end{pmatrix}.$$  

We recall here that $A_u$ is $3 \times 3$ matrix defined in (14) and $m_0$ is the Minkowski metric. In particular, we can denote

$$\psi^\tau_S(x; \lambda(t)) = i e^{-1} \rho^{-1} \omega \phi^\tau_S(x; \lambda(t)) - u_h(t) \nabla_x \phi^\tau_S(x; \lambda(t))$$

corresponding to $\partial_t \phi^\tau_S$ in the direction $u_h$ and

$$n \phi^\tau_S(x; \lambda(t)) = i e^{-1} \rho^{-1} \phi^\tau_S - uQ^{-1} \cdot \nabla_x \phi^\tau_S(x; \lambda(t))$$

associated to $n \phi^\tau_S$, where $n$ is the unit normal to the hypersurface $\Sigma_0$ embedded to $(\mathcal{M}, h)$ at time $t$.

The above definition is well defined. We first show that $h(t, \xi(t))$ has a decomposition as in the definition, that is, $a, \alpha, Q$ exist. Notice that $h$ is Lorentzian metric. At point $(t, \xi(t))$, the $3 \times 3$ matrix $(h(t, \xi(t))_{kl} )$ is symmetric and positive definite. We hence can find a $3 \times 3$ matrix $Q$ such that

$$QQ^T = ((h(t, \xi(t)))_{kl}).$$

Then $a, \alpha$ are uniquely determined as follows

$$\alpha = (h_{01}, h_{02}, h_{03})(t, \xi(t))(Q^T)^{-1}, \quad a = \sqrt{-h(t, \xi(t))_{00} + \alpha \alpha^T}.$$  

Secondly, we prove that $|u| < 1$. In fact, notice that the vector $\partial_t + u_h \partial_x$ is timelike at $(t, \xi(t))$. We have

$$h(t, \xi(t))_{00} + 2h(t, \xi(t))_{0k} u_h^k + u_h^k h(t, \xi(t))_{kl} u_h^l < 0,$$

which implies that

$$|u|^2 = a^{-2}(\alpha + u_h Q)(\alpha + u_h Q)^T < 1.$$  

Finally, we demonstrate that the definition is independent of the choice of the $3 \times 3$ matrix $Q$. Let $\tilde{u}, \tilde{a}, \tilde{\alpha}, \tilde{Q}$ be another decomposition. Let $P = Q^{-1} \tilde{Q}$. We conclude that $PP^T = I$ and

$$\tilde{\alpha} = \alpha P, \quad \tilde{a} = a, \quad \tilde{u} = u P.$$  

Hence we can show that

$$\tilde{u} \cdot (x - \xi(t)) \tilde{Q} = u P \cdot (x - \xi(t))QP = u \cdot (x - \xi(t))Q,$$

$$A_u \tilde{Q}^T = A_u P P^T Q^T = P^T A_u Q^T.$$
Since \( f_\omega \) is spherical symmetric by Theorem 2, \( PP^T = I \), we thus have shown that \( \phi_S^\omega(x; \lambda(t)), \psi_S^\omega(x; \lambda(t)) \) are well defined.

For any function \( f \) on \( \mathbb{R}^3 \), we define the scaled Sobolev norm

\[
\|f\|_{H^s} = \|f(\epsilon)\|_{H^s} = \sum_{|\alpha| = 0}^s \epsilon^{|\alpha| - \frac{3}{2}} \|\nabla^\alpha f\|_{L^2}.
\]

Additional to the constraint equations (22), we assume \( \bar{g}, \bar{K}, \phi_0^\epsilon, \phi_1^\epsilon \) satisfy the estimates

\[
\|\delta^{-1}\phi_0^\epsilon(x) - \phi_S^\omega(x; \lambda_0)\|_{H^2} + \epsilon\|\delta^{-1}\phi_1^\epsilon(x) - n\phi_S^\omega(x; \lambda_0)\|_{H^2} \leq C_0 \epsilon,
\]

\[
\|\nabla(\bar{g} - h)\|_{H^2(\Sigma_0)} + \|\bar{K} - k\|_{H^2(\Sigma_0)} \leq C_0 \delta^2 \epsilon^{-1}
\]

for some constant \( C_0 \) and some point \( \lambda_0 = (\omega_0, \theta_0, \xi_0, u_h(0)) \in \Lambda_{stab}(0) \), where \( h \) is the Lorentzian metric \( h \) restricted to the initial hypersurface \( \Sigma_0 \) and \( k \) is the second fundamental form of \( \Sigma_0 \) embedded to \( (\mathcal{M}, h) \).

In particular, \( \lambda_0 \) gives a timelike geodesic \( (t, \gamma_0(t)) \) on \( (\mathcal{M}, h) \) such that \( \gamma_0(0) = \xi_0, \gamma_0'(0) = u_h(0) \). Existence of such initial data set \( (\Sigma_0, \bar{g}, \bar{K}, \phi_0^\epsilon, \phi_1^\epsilon) \) described above. We have the following main result.

**Theorem 4.** Let \( (\mathcal{M}, h) \) be a vacuum spacetime satisfying (21). Assume \( 2 \leq p < \frac{7}{3} \). Assume \( \delta \leq \epsilon^q, q > 1 \) or \( \delta = c_0 \epsilon \). Suppose the initial data \( (\Sigma_0, \bar{g}, \bar{K}, \phi_0^\epsilon, \phi_1^\epsilon) \) satisfy conditions (22), (24). Then there exists \( \epsilon^* > 0 \) such that for all \( \epsilon, c_0 \in [0, \epsilon^*] \) the Cauchy problem for system (1) admits a unique (up to diffeomorphism) solution \( (\mathcal{M}, g, \phi^\epsilon) \) with the following properties: there exists a foliation \( [0, T^*] \times \Sigma_\tau \) of \( \mathcal{M} \) with coordinates \( (s, y) \), where \( \Sigma_\tau \subseteq \mathbb{R}^3 \) and \( T^* \) depends only on \( T \) and \( \lambda_0 \), as well as a \( C^1 \) curve \( \lambda(s) = (\omega(s), \theta(s), \xi(s), u_h(s)) \in \Lambda_{stab}(s) \) such that

1. The spacetime \( (\mathcal{M}, g, \phi^\epsilon) \) is close to the given vacuum spacetime \( (\mathcal{M}, h) \)

\[
\|\partial(g - h)\|_{H^2(\Sigma_\tau)} \leq C\epsilon, \quad \forall s \in [0, T^*];
\]

2. \( \phi^\epsilon(t, x) \) is approximated by some translated soliton centered at \( \xi(s) \)

\[
\|\delta^{-1}\phi^\epsilon(s, y) - \phi_S(y; \lambda(s))\|_{H^2(\Sigma_\tau)} + \epsilon\|\delta^{-1}\phi^\epsilon(s, y) - \psi_S(y; \lambda(s))\|_{H^2(\Sigma_\tau)} \leq C\epsilon, \quad \forall s \in [0, T^*];
\]

3. The center of the particle \( \xi(s) \) is close to the given timelike geodesic \( (s, \gamma_0(s)) \)

\[
|\xi(s) - \gamma_0(s)| + |\omega(s) - \omega_0| + |u_h(s) - u_0(s)| \leq C\epsilon, \quad \forall s \in [0, T^*],
\]

where the constant \( C \) is independent of \( \epsilon \) and the timelike geodesic \( (s, \gamma_0(s)) \) is the one starting from \( \xi_0 \) with speed \( u_h(0) \).

4 Construction of Fermi Coordinate System

The geodesic hypothesis states that a test particle moves along a timelike geodesic. This timelike geodesic \( (t, \gamma_0(t)) \), using the notations in the previous section, is determined by the initial position \( \xi_0 \) and speed \( u_h(0) \). As having discussed in the introduction, our approach relies on the Fermi coordinate system such that the Christoffel symbols are vanishing along the timelike geodesic \( (t, \gamma_0(t)) \). In this section, we give an explicit construction of such local coordinate system.

**Lemma 1.** Let \( ([0, T] \times \mathbb{R}^3, h) \) be a smooth Lorentzian spacetime and \( (t, \gamma_0(t)) \) be a timelike geodesic. Assume that the metric \( h \) satisfies (21). Then for all \( \delta_1 \in (0, \frac{1}{2} T) \), there exists a subspace \( M \)

\[
[0, T - 2\delta_1] \times \mathbb{R}^3 \subset M \subset [0, T] \times \mathbb{R}^3
\]
with a new coordinate system \((s, y) \in [0, C'(0), h](T - \delta_1)] \times \mathbb{R}^3\) such that the given timelike geodesic is represented as \((s, u_0 s)\) for some constant vector \(u_0 \in \mathbb{R}^3\), \(|u_0| < 1\). Moreover, along this geodesic, we have
\[
\Gamma^\alpha_{\mu\nu}(s, u_0 s) = 0, \quad h_{\mu\nu}(s, u_0 s) = (m_0)_{\mu\nu}, \quad \forall s \in [0, C'(0), h](T - \delta_0)],
\]
where \(\Gamma^\alpha_{\mu\nu}\) are the associated Christoffel symbols and \(m_0\) is the Minkowski metric, that is \((m_0)_{00} = -1, (m_0)_{kk} = 1, (m_0)_{\mu\nu} = 0, \forall \mu \neq \nu\). Furthermore, under the new coordinate system \((s, y)\), there exists a positive constant \(K(\delta_1, \gamma_0(0))\) depending on \(\delta_1, \gamma_0(0), K_0\) such that
\[
h(X, X) \leq -K(\delta_1, \gamma_0(0)), \quad X = \partial_s + u_0^k \partial_{y_k}
\]
on the space \([0, C(\gamma_0(0), h)(T - \delta_1)] \times \mathbb{R}^3\). In particular, the vector field \(X = \partial_s + u_0^k \partial_{y_k}\) is timelike.

**Proof.** Let \((t, x)\) be the given coordinate system on \(([0, T] \times \mathbb{R}^3, h)\). Our first step is to choose a coordinate system such that the given geodesic is represented by \((t, 0)\). Let \((t, \tilde{x})\) be a new coordinate system defined as follows
\[
(t, \tilde{x}) = (t, x - \gamma_0(t)).
\]
With this translation, we can show that the geodesic is parameterized by \((t, 0)\). However, along this geodesic we only have \(\Gamma^\alpha_{00}(t, 0) = 0\). For simplicity, let’s still use \((t, x)\) to denote \((t, \tilde{x})\).

Next, we change the coordinates in a small neighborhood of the geodesic such that all the Christoffel symbols are vanishing along the geodesic. To achieve this, we reproduce the construction here, which is essentially the same as that in [13], [18]. Let \(\chi(x)\) be a cutoff function
\[
\begin{cases}
\chi(x) = 1, & |x| \leq r_0, \\
\chi(x) = 0, & |x| \geq 2r_0
\end{cases}
\]
for some positive number \(r_0\). Introduce new coordinates defined as follows
\[
\tilde{x}^\mu = x^\mu + a_k^\mu(t)x^k\chi(x) + \frac{1}{2}b_{kl}^\mu(t)x^kx^l\chi(x),
\]
where \((x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3)\). Here we recall that Greek letters \(k, l\) run from 1 to 3 while the Latin letters \(\mu, \nu\) run from 0 to 3. The geodesic in this new coordinate system is still parameterized by \((t, 0)\). The transformation law for Christoffel symbols
\[
\tilde{\Gamma}^\mu_{\nu\sigma} = \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\sigma} \left( \Gamma^\beta_{\gamma\delta} \frac{\partial x^\delta}{\partial x^\sigma} - \frac{\partial^2 x^\mu}{\partial x^\gamma \partial x^\sigma} \right)
\]
together with fact that \(\Gamma^\mu_{00}(t, 0) = 0\), implies that \(\tilde{\Gamma}^\mu_{\nu\sigma}\) vanishes along the geodesic \((t, 0)\) is equivalent to the following ODEs for \(a_k^\mu(t), b_{kl}^\mu(t)\)
\[
\begin{align*}
\frac{da_k^\mu}{dt}a_k^\mu &= \Gamma^\mu_{0k}(t, 0) + a_k^\mu \Gamma^\mu_{0k}(t, 0), \\
b_{kl}^\mu &= \Gamma^\mu_{kl}(t, 0) + a_k^\mu \Gamma^\mu_{kl}(t, 0).
\end{align*}
\]
Now take \(a_k^\mu(0) = b_{kl}^\mu(0) = 0\) initially. We conclude that the coefficients \(a_k^\mu, b_{kl}^\mu\) are uniquely determined up to any time \(T\). Moreover, we have the estimates
\[
|a_k^\mu| + |b_{kl}^\mu| \leq C(h, T, \gamma_0),
\]
where the constant \(C(h, T, \gamma_0)\) depends only on the metric \(h\), the time \(T\) and the geodesic \((t, \gamma_0(t))\).

Therefore for any \(\delta_1 > 0\), there exist \(r_0\) sufficiently small and the associated cutoff function \(\chi\) such that
\[
\tilde{x} : [0, T - \delta_1] \times \mathbb{R}^3 \to [0, T] \times \mathbb{R}^3, \quad \tilde{x}^{-1} : [0, T - 2\delta_1] \times \mathbb{R}^3 \to [0, T - \delta_1] \times \mathbb{R}^3
\]
are inclusions. Hence let \(M = \tilde{x}([0, T - \delta_1] \times \mathbb{R}^3)\) with the induced metric \(h\). We have
\[
[0, T - 2\delta_1] \times \mathbb{R}^3 \subset M \subset [0, T] \times \mathbb{R}^3
\]
with a coordinate system \( \hat{x} \) such that the Christoffel symbols \( \hat{\Gamma}_{\mu\nu}^\beta \) vanish along the timelike geodesic \((t, 0)\).

We still need to change the coordinate such that \( h = m_0 \) along the geodesic. We have shown that under the coordinate system \( \hat{x} = (t, x') \), \( \hat{\Gamma}_{\mu
u}^\beta(t, 0) = 0 \), which imply that \( \partial h(t, 0) = 0 \), \( h(t, 0) = h(0, 0) \). Since the geodesic is timelike, we have \( h_{00}(0, 0) < 0 \) and \((h(0, 0))_{kl} \) is positive definite. Assume

\[
h(0, 0) = \begin{pmatrix} -a & \alpha \\ \alpha^T & P \end{pmatrix}, \quad a > 0.
\]

Let \( Q_{3 \times 3} \) be such that \( QPQ^T = I_{3 \times 3} \). Consider the following new coordinate system

\[
(s, y) = (t, x') \begin{pmatrix} 1 & \alpha P^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \sqrt{a + \alpha P^{-1}a^T} & 0 \\ 0 & Q^{-1} \end{pmatrix} = (t\sqrt{a + \alpha P^{-1}a^T}, (\alpha P^{-1} + x')Q^{-1}).
\]

We can show that under this new coordinate system, the given timelike geodesic is parameterized by \((s, u_0s)\). Moreover, \( h(s, u_0s) = m_0 \) for some constant vector \( u_0 \in \mathbb{R}^3 \) given as follows

\[
u_0 = \frac{\alpha P^{-1}Q^{-1}}{\sqrt{a + \alpha P^{-1}a^T}} \in \mathbb{R}^3, \quad |u_0| < 1.
\]

By the previous construction, we have \((s, y) \in [0, (T - \delta_1)C(\gamma_0'(0), h)] \times \mathbb{R}^3 \), where we denote \( C(\gamma_0'(0), h) = \sqrt{a + \alpha P^{-1}a^T} \) depending only on \( \gamma_0'(0) \) and the metric \( h \).

Finally, notice that under the coordinate system \( \hat{x} = (t, x') \), the vector field \( \partial_\xi \) is timelike, that is, by (21) and the construction of \( \hat{x} \)

\[
h(\partial_\xi, \partial_\eta) = -K(\delta_1, \gamma_0'(0))
\]

for some constant depending on \( h, \delta_1, \gamma_0'(0) \). We remark here that the positive constant \( K(\delta_1, \gamma_0'(0)) \) also depends on the cutoff function \( \chi \). However, when \( \delta_1 \) is fixed, the cutoff function is also fixed. Therefore the corresponding vector field \( X = \partial_s + u_0^k \partial_{g_k} \) on the space \((M, h)\) with coordinate system \((s, y) \in [0, (T - \delta_1)\sqrt{a + \alpha P^{-1}a^T}] \times \mathbb{R}^3 \) is also timelike and satisfies

\[
h(X, X) \leq -K(\delta_1, \gamma_0'(0)).
\]

\[
5 \textbf{Reduced Einstein Equations}
\]

In this section, we reduce the scaled Einstein equations to a hyperbolic system for the components of the metric \( g_{\mu\nu} \) by fixing a relatively harmonic gauge condition, see [5], [17].

The Einstein equations are independent of the choice of local coordinate system. To solve the Einstein equations, we work under the Fermi coordinate system constructed in the previous section. More precisely, starting with the given vacuum spacetime \([0, T] \times \mathbb{R}^3, h\) with local coordinate system \((t, x)\), Lemma 1 shows that we can find a new coordinate system \((s, y) \in [0, T] \times \mathbb{R}^3 \) on \( M \), where \([0, T - 2\delta_1] \times \mathbb{R}^3 \subset M \subset [0, T] \times \mathbb{R}^3 \), such that the timelike geodesic is represented as \((s, u_0s)\) for some constant vector field \( u_0 \in \mathbb{R}^3, |u_0| < 1 \) and

\[
h(s, u_0s) = m_0, \quad \Gamma^\alpha_{\mu
u}(s, u_0s) = 0.
\]

Moreover, the vector field

\[
X = \partial_s + u_0^k \partial_{g_k}
\]

is uniformly timelike. To simplify the notations, we work on the space \([0, T] \times \mathbb{R}^3, h\) for arbitrary \( T > 0 \) with Fermi coordinate system \((s, y)\).

With this local coordinate system \((s, y)\), to understand the particle \( \phi^\nu \), which has small size \( \epsilon \) and small energy \( \delta \), we instead consider the scaled Einstein equations. More precisely, recall that the potential
$V_{\epsilon, \delta}$ satisfies the scaling $V_{\epsilon, \delta}(\phi) = \delta^2 \epsilon^{-2} V(\delta^{-1} \phi)$. We infer that if $(g(s, y), \phi'(s, y))$ solves system (1) on the space $[0, T] \times \mathbb{R}^3$, then

$$g'(s, y) = g(\epsilon s, \epsilon y), \quad \phi(s, y) = \delta^{-1} \phi'(\epsilon s, \epsilon y)$$

satisfy the rescaled Einstein equations

$$
\begin{align*}
R_{\mu\nu}(g') - \frac{1}{2} R(g')(g')_{\mu\nu} &= \delta^2 T_{\mu\nu}(g', \phi; V(\phi)), \\
\Box g' + \nu'(\phi) &= 0
\end{align*}
$$

(28)

on the rescaled space $[0, T/\epsilon] \times \mathbb{R}^3$. Conversely, a solution $(g', \phi)$ of (28) on the space $[0, T/\epsilon] \times \mathbb{R}^3$ gives a solution $(g, \phi')$ of (1) on the space $[0, T] \times \mathbb{R}^3$. It hence suffices to consider the scaled Einstein equations (28) on the space $[0, T/\epsilon] \times \mathbb{R}^3$ with initial data determined by $(\bar{g}, \bar{K}, \phi_0, \phi_1)$ as well as the coordinate change from $(t, x)$ to the Fermi coordinate system constructed in the previous section.

**Remark 5.** The scaling heavily relies on the Fermi local coordinate system. However, the system (28) itself is independent of choice of local coordinate system on the space $[0, T/\epsilon] \times \mathbb{R}^3$.

We must determine the initial data for the corresponding Cauchy problem for the rescaled Einstein equations (28). By the construction of the Fermi coordinate system in Lemma 1 and by Definition 1, we define

$$\phi_0(y) = \delta^{-1} \phi_0'(\epsilon y), \quad \phi_1(y) = \delta^{-1} \epsilon \phi_1'(\epsilon y), \quad \bar{g}'(y) = \bar{g}(\epsilon y), \quad \bar{K}'(y) = \epsilon \bar{K}(\epsilon y)$$

on the initial hypersurface $\mathbb{R}^3$ with coordinate system $\{y\}$. Denote

$$\lambda_0^0 = (\omega_0, \epsilon^{-1} \theta_0, 0, u_0), \quad n\phi_S(y; \lambda_0^0) = i \rho \phi_S(y; \lambda_0^0) - u_0 Q^{-1} \nabla_y \phi_S(y; \lambda_0^0),$$

where $\lambda_0 = (\omega_0, \theta_0, \xi_0, u_0(0))$ is given in Theorem 4 and $Q$ is the $3 \times 3$ matrix in Definition 1. By (22), we have

$$
\begin{align*}
\bar{R}(\bar{g}') - |\bar{K}'|^2 (tr \bar{K}')^2 &= \delta^2 (|\phi_1|^2 + |\nabla \phi_0|^2 + 2 \nu(\phi_0)), \\
\nabla^j K_j^i - \nabla_i tr \bar{K}' &= \delta^2 <\phi_1, \nabla_i \phi_0>,
\end{align*}
$$

(29)

where the covariant derivative is with respect to the metric $\bar{g}'$. Using this scaling, the condition (24) becomes

$$
\begin{align*}
\|\phi_0(y) - \phi_S(y; \lambda_0^0)\|_{H^3} + \|\phi_1(y) - n\phi_S(y; \lambda_0^0)\|_{H^3} &\leq C_0 \epsilon, \\
\|\nabla(\bar{g}' - \bar{h}')\|_{H^2} + \|\bar{K}' - \bar{k}'\|_{H^2} &\leq C_0 \delta^2,
\end{align*}
$$

(30)

where $\bar{h}'(y) = \bar{h}(\epsilon y)$, $\bar{k}'(y) = \bar{k}(\epsilon y)$, $\nabla$ is the covariant derivative for the metric $\bar{g}'$. We hence obtain the initial data $(\Sigma_0, \bar{g}', \bar{K}', \phi_0, \phi_1)$ for rescaled Einstein equations (28). We also remark here that the existence of the initial data $(\Sigma_0, \bar{g}', \bar{K}', \phi_0, \phi_1)$ in our main theorem 4 is equivalent to the existence of $(\Sigma_0, \bar{g}', \bar{K}', \phi_0, \phi_1)$ satisfying the conditions (29), (30) under the Fermi coordinate system. Existence of such initial data set under certain conditions is discussed in details in the last section.

The Cauchy problem for system (28) is underdetermined as any diffeomorphism of the spacetime $([0, T/\epsilon] \times \mathbb{R}^3, g', \phi)$ satisfying (28) leads to another solution. Such freedom can be removed by choosing a gauge condition such that system (28) is equivalent to a hyperbolic system for the components of the metric $g'$. Under the Fermi coordinate system on the space $([0, T/\epsilon] \times \mathbb{R}^3, h')$, we define

$$G^\lambda(g', h^\epsilon) = (g')^{\mu\nu}(\Gamma^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\mu\nu}), \quad \lambda = 0, 1, 2, 3,$$

where $\Gamma^\lambda_{\mu\nu}$, $\bar{\Gamma}^\lambda_{\mu\nu}$ are Christoffel symbols for the unknown metric $g'$ and the given vacuum metric $h'$ respectively. The gauge condition that we choose is

$$G^\lambda(g', h^\epsilon) = 0, \quad \forall \lambda = 0, 1, 2, 3,$$

(31)
which will be called the relatively harmonic gauge condition, see [5], [17]. Instead of considering the full Einstein equations (28), we consider the following reduced Einstein equations

$$R_{\mu\nu}(g^r) - \frac{1}{2}R(g^r)g_{\mu\nu}^r - \frac{1}{2}(g^r_{\mu\lambda}\nabla_\nu G^\lambda + g^r_{\nu\lambda}\nabla_\mu G^\lambda - g^r_{\mu\nu}\nabla_\lambda G^\lambda) = \delta^2 T_{\mu\nu}(g^r, \phi; \nu(\phi)),$$  

(32)

where the covariant derivative $\nabla$ is for the metric $g^r$ on the space $[0, T/e] \times \mathbb{R}^3$. Then a solution of the full Einstein equations (28) can be constructed as follows: we write down (32) under the Fermi coordinate free (due to the matter field equation of $\phi$ propagated, i.e., the relation (31) holds on $[0, t^*] \times \mathbb{R}^3, g^r, \phi$). We then construct initial data $(g^r(0), \partial_t g^r(0))$ from the given data $(g^r, \tilde{K}^r)$ such that the gauge condition (31) holds initially. Hence we can get a unique short time solution $([0, t^*) \times \mathbb{R}^3, g^r, \phi)$ of the reduced Einstein equations (32) coupled with the matter field equation for $\phi$ (second equation in (28)). We then argue that the gauge condition is propagated, i.e., the relation (31) holds on $[0, t^*) \times \mathbb{R}^3$. This implies that the solution $([0, t^*) \times \mathbb{R}^3, g^r, \phi)$ to the reduced Einstein equations is, in fact, a solution of the full Einstein equations (28).

Our first step is to check the hyperbolicity of the reduced Einstein equations (32). We notice that under the fixed Fermi coordinate system (can be viewed as a local coordinate system), the full Einstein equations can be written as

$$-(g^r)^{\alpha\beta}\partial_{\alpha\beta}g^r_{\mu\nu} + g^r_{\mu\lambda}\partial_\mu((g^r)^{\alpha\beta}\tilde{\Gamma}^\lambda_{\alpha\beta}) + g^r_{\nu\lambda}\partial_\nu((g^r)^{\alpha\beta}\tilde{\Gamma}^\lambda_{\alpha\beta}) + Q_{\mu\nu}(g^r) = \delta^2(2T_{\mu\nu} - trT \cdot g^r_{\mu\nu}),$$

where

$$Q_{\mu\nu}(g^r) = C^{\alpha\beta\gamma}_{\mu\lambda\nu\sigma}(g^r)\partial_\alpha g^r_\beta \partial_\beta g^r_\sigma,$$

and $C(g^r)$ denotes polynomials of $g^r_{\alpha\beta}$, $(g^r)^{\alpha\beta}$. Taking trace of the reduced Einstein equations (32), we get

$$-R(g^r) + \nabla_\mu G^\mu = trT.$$

Plug this into (32). We have the reduced Einstein equations

$$-(g^r)^{\alpha\beta}\partial_{\alpha\beta}g^r_{\mu\nu} + g^r_{\mu\lambda}\partial_\mu((g^r)^{\alpha\beta}\tilde{\Gamma}^\lambda_{\alpha\beta}) + g^r_{\nu\lambda}\partial_\nu((g^r)^{\alpha\beta}\tilde{\Gamma}^\lambda_{\alpha\beta}) + Q_{\mu\nu}(g^r) - G^\lambda \partial_\lambda g^r_{\mu\nu} = \delta^2(2T_{\mu\nu} - trT \cdot g^r_{\mu\nu}).$$

Similarly, we recall the vacuum Einstein equations for $h^r$ under the Fermi local coordinate system

$$-(h^r)^{\alpha\beta}\partial_{\alpha\beta}h^r_{\mu\nu} + h^r_{\mu\lambda}\partial_\mu((h^r)^{\alpha\beta}\tilde{\Gamma}^\lambda_{\alpha\beta}) + h^r_{\nu\lambda}\partial_\nu((h^r)^{\alpha\beta}\tilde{\Gamma}^\lambda_{\alpha\beta}) + Q_{\mu\nu}(h^r) = 0.$$

Subtract the above two equations. We can show that the reduced Einstein equations coupled with the matter field equation for $\phi$ are equivalent to the following hyperbolic system for the difference $\psi^r = g^r - h^r$

$$\begin{cases}
-(g^r)^{\alpha\beta}\partial_{\alpha\beta}\psi^r_{\mu\nu} + \delta P_{\mu\nu} + \delta Z_{\mu\nu} + \delta Q_{\mu\nu} = \delta^2(2T_{\mu\nu} - trT \cdot g^r_{\mu\nu}), \\
\square_{g^r} \phi + \nu(\phi) = 0,
\end{cases}$$

(33)

where

$$\begin{align*}
\delta Q_{\mu\nu} &= Q(g^r) - Q(h^r) - (g^r)^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta}(g^r_{\mu\nu}) + (h^r)^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta}(h^r_{\mu\nu}), \\
\delta Z_{\mu\nu} &= (g_{\mu\lambda}(g^r)^{\alpha\beta} - h^r_{\mu\lambda}(h^r)^{\alpha\beta}) \partial_\alpha \tilde{\Gamma}_{\beta\lambda} + (g_{\nu\lambda}(g^r)^{\alpha\beta} - h^r_{\nu\lambda}(h^r)^{\alpha\beta}) \partial_\alpha \tilde{\Gamma}_{\beta\lambda} - (\psi^r)^{\alpha\beta} \partial_{\alpha\beta} h^r_{\mu\nu}, \\
\delta P_{\mu\nu} &= \tilde{\Gamma}_{\alpha\beta}((g^r)^{\alpha\beta}\partial_\mu g^r_{\nu\lambda} + g^r_{\lambda\sigma}\partial_\mu(g^r)^{\alpha\beta} + g^r_{\nu\sigma}\partial_\mu(g^r)^{\alpha\beta} - (h^r)^{\alpha\beta}\partial_\lambda h^r_{\mu\nu} - h^r_{\nu\lambda}\partial_\mu(h^r)^{\alpha\beta} - h^r_{\nu\lambda}\partial_\mu(h^r)^{\alpha\beta})
\end{align*}$$

(34)

Hence we have shown that the reduced Einstein equations (32) are hyperbolic.

Secondly, we demonstrate that the gauge condition (31) propagates. Suppose $(g^r, \phi)$ satisfies system (33) on $[0, t^*) \times \mathbb{R}^3$ for some small positive time $t^*$. Take divergence of both sides of the reduced Einstein equations (32). Using Bianchi’s identity and the fact that the energy momentum tensor $T_{\mu\nu}$ is divergence free (due to the matter field equation of $\phi$), we obtain the evolution equations for $G^\lambda$

$$\nabla^\mu \nabla_\mu G^\nu + [\nabla_\mu, \nabla_\nu] G^\mu = 0, \quad \nu = 0, 1, 2, 3.$$
where the commutator $[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$. Hence $\mathcal{G}^\lambda$ satisfies the above wave equations on $[0, t^*) \times \mathbb{R}^3$. Thus $\mathcal{G}^\lambda$ vanishes on $[0, t^*) \times \mathbb{R}^3$ if $\mathcal{G}^\mu, \partial_t \mathcal{G}^\mu$ vanish initially. We can make $\mathcal{G}^\mu$ vanish on the initial hypersurface $\Sigma_0$ by choosing initial data for $\bar{g}_{0\mu}$, $\partial_t \bar{g}_{0\mu}$ (recall that only $\bar{g}^\lambda, \bar{K}^\lambda$ or $\bar{g}_{ij}, \partial_t \bar{g}_{ij}$ are given). Once we have $\mathcal{G}^\mu = 0$ on $\Sigma_0$, we can show that $\partial_t \mathcal{G}^\mu$ also vanishes on $\Sigma_0$ due to the constraint equations. In fact, since $(\bar{g}^\lambda, \phi)$ solves (32), the constraint equations (29) for $\bar{g}^\lambda, \bar{K}^\lambda$ together with the vacuum constraint equations for $\bar{h}^\lambda, \bar{k}^\lambda$ imply that

$$\nabla_0 \mathcal{G}_\nu + \nabla_\nu \mathcal{G}_0 - g_{0\nu} \nabla_\mu \mathcal{G}^\mu = 0, \quad \nu = 0, 1, 2, 3,$$

where the covariant derivative $\nabla$ is for the metric $\bar{g}^\lambda$. Hence $\partial_t \mathcal{G}^\mu = 0$ on $\Sigma_0$ if $\mathcal{G}^\mu = 0$ initially. Therefore, we have shown that as long as (31) holds on $\Sigma_0$, a solution $(\psi^\lambda, \phi)$ of the reduced Einstein equations (33) on $[0, t^*) \times \mathbb{R}^3$ leads to a solution $([0, t^*) \times \mathbb{R}^3, \psi^\lambda + h^\lambda, \phi)$ to the full Einstein equations (28). Since the solution of the Einstein equations (28) exists locally and is unique up to diffeomorphism [4], [9], it suffices to consider the reduced Einstein equations (33) with initial data $(\bar{g}^0(0, y), \partial_t \bar{g}^0(0, y))$ such that (31) holds on the initial hypersurface $\Sigma_0$.

It remains to construct initial data $(g^0(0, \partial_t g^0(0))$ from $(\bar{g}^\lambda, \bar{K}^\lambda)$ on $\Sigma_0$ such that the gauge condition (31) holds initially. Let

$$(g^0)^{i0}(0) = -\bar{N}^{-2}, \quad g_{0i}^0(0) = \beta_i, \quad i = 1, 2, 3,$$

where $\bar{N}$ is the lapse function and $\beta$ is the shift vector on $\Sigma_0$. Since the Riemannian metric $\bar{g}^\lambda$ is the Lorentzian metric $g^\lambda$ restricted to $\Sigma_0$ and $\bar{K}^\lambda$ is the second fundamental form, we can show that

$$g_{ij}^0(0) = \bar{g}_{ij}^0, \quad \partial_t g_{ij}^0(0) = 2\bar{N}\bar{K}_{ij}^0 + \nabla_i \beta_j + \nabla_j \beta_i,$$

where $\nabla$ is the covariant derivative for $\bar{g}^\lambda$ on $\Sigma_0$. That is $(g^0(0, \partial_t g^0(0)))$ is uniquely determined by $(\bar{g}^\lambda, \bar{K}^\lambda), \bar{N}, \beta_i$. We claim that $\partial_t g_{0\mu}(0)$ is given by the gauge condition (31). In fact, the gauge condition implies that

$$(2 - (m_0)_k^\nu) \partial_\nu \psi_\mu = 2\bar{N}^2(2(m_0)_k^\nu (g^0)^{i0} - (g^0)^{i0} (m_0)_k^\nu) \partial_t \psi_{ik} + \bar{N}^2(2(m_0)_0^\nu (g^0)^{k0} - (g^0)^{k0} (m_0)_0^\nu) \partial_t \psi_{k0} + \bar{N}^2 \bar{g}_{0\mu} (g^0)^{\nu\rho}(2\partial_\mu h^\nu_{\rho\beta} - \partial_\rho h^\nu_{\alpha\beta}), \quad \psi = g^\lambda - h^\lambda.$$

We choose $\partial_t g_{0\mu}$ on $\Sigma_0$ as above. Notice that different lapse functions $\bar{N}$ and shift vectors $\beta_i$ will lead to the same solution of the full Einstein equation (28) up to a change of local coordinate system. To better estimate the difference $\psi^\lambda$, we simply set

$$(g^0)^{00}(0, y) = (h^\lambda)^{00}(0, y), \quad g_{0i}^0(0, y) = h_{0i}^\lambda(0, y).$$

The above discussion shows that we have constructed the initial data $(g^0(0, \partial_t g^0(0))$ such that the gauge condition (31) holds initially.

Finally, we show that $g^\lambda$ is sufficiently close to $h^\lambda(0)$ and hence is Lorentzian initially. Notice that

$$\partial_t h_{ij}^\lambda = 2\bar{N}(\bar{K}_{ij}^0) + \nabla_i \beta_j + \nabla_j \beta_i,$$

where the covariant derivative is for the metric $\bar{h}^\lambda$. By (21), (30), we can show that

$$\|\nabla(g^\lambda - h^\lambda)\|_{H^2(\Sigma_\alpha)} + \|\partial_t (g^\lambda - h^\lambda)\|_{H^2(\Sigma_\alpha)} \leq C(h, C_0)\delta^2,$$  \hspace{1cm} (35)

where the constant $C(h, C_0)$ depends on $h, C_0$. Thus if $\epsilon$ is sufficiently small, $g^\lambda$ is a Lorentzian metric initially. We summarize what we have obtained in this section.

**Lemma 2.** Let $([0, T] \times \mathbb{R}^3, h)$ be a vacuum spacetime. The Cauchy problem for (1) with initial data $(\Sigma_0, \bar{g}, \bar{K}, \bar{\phi}_0, \bar{\phi}_0)$ is equivalent (up to a diffeomorphism and scaling) to the hyperbolic system (33), (34) with initial data $(\phi_0(0, y), \phi_1(0, y), g^0(0, y), \partial_t g^0(0, y))$ satisfying (30) and (35).
6 Stability of Stable Solitons on a Fixed Background

We have shown in the previous section that to solve the full Einstein equations (1), it suffices to consider the hyperbolic system (33) on the scaled space $[0, T/\epsilon] \times \mathbb{R}^3$ with initial data $(\phi_0(0, x), \phi_1(0, x), g^0(0, x), \partial_t g^0(0, x))$ satisfying (30) and (35). The standard local existence results imply that there is a unique short time solution $(g^\epsilon, \phi)$ on $[0, t^\epsilon] \times \mathbb{R}^3$. Using continuity argument, the solution can be extended to $T/\epsilon$ as long as $(g^\epsilon, \phi)$ satisfies condition (30), (35) for a constant $C$ independent of $\epsilon$. The estimate for $g^\epsilon - h^\epsilon$ follows from energy estimates for hyperbolic equations if the source term $T_{\mu\nu} - \frac{1}{2} tr T^\epsilon \cdot g^\epsilon_{\mu\nu} \in C([0, T/\epsilon]; H^3)$. Since it is expected that the unknown metric $g^\epsilon$ is close to the given vacuum metric $h^\epsilon$, the main difficulty for proving the main theorem is to show that $\phi$ exists and is close to some translated soliton up to time $T/\epsilon$, that is, orbital stability of stable solitons along a timelike geodesic on the slowly varying background $([0, T/\epsilon] \times \mathbb{R}^3, g^\epsilon)$.

In this section, we will let $([0, T] \times \mathbb{R}^3, h)$ be a Lorentzian spacetime with the Fermi coordinate system $(t, x)$ constructed in Lemma 1. More precisely, assume that $(t, u_0 t)$, $t \in [0, T]$ is a timelike geodesic, where the constant vector $u_0 \in \mathbb{R}^3$, $|u_0| < 1$. Along this geodesic, we have

$$h(t, u_0 t) = m_0, \quad \Gamma^\alpha_{\mu\nu}(t, u_0 t) = 0,$$

where $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbols for $h$. Moreover, the vector field

$$X = \partial_t + u_0^k \partial_k$$

is uniformly timelike on $([0, T] \times \mathbb{R}^3, h)$. We assume the metric $h \in C^4([0, T] \times \mathbb{R}^3)$ and satisfies (21) for some constant $K_0$. Let $g$ be another Lorentzian metric on $[0, T] \times \mathbb{R}^3$. Consider the Cauchy problem for the rescaled nonlinear wave equation

$$\begin{cases}
\Box_{g^\epsilon} \phi - m^2 \phi + |\phi|^{p-1} \phi = 0, \\
\phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x)
\end{cases} \quad (36)$$

on the rescaled space $[0, T/\epsilon] \times \mathbb{R}^3$, where the slowly varying metric $g^\epsilon(t, x) = g(\epsilon t, \epsilon x)$. We show the orbital stability of stable solitons to (36) if $g - h$ vanishes as $|x| \to \infty$ and satisfies

$$\|\partial^{s+1} \psi^\epsilon(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq 2 \epsilon^2, \quad |s| \leq 2, \quad \forall t \in [0, T/\epsilon], \quad (37)$$

where $\psi = g - h$, $\psi^\epsilon = g^\epsilon - h^\epsilon$. This condition can be viewed as a bootstrap assumption.

Our main result in this section is

**Theorem 5.** Let $g, h$ be Lorentzian metrics satisfying (21), (37) on the space $[0, T] \times \mathbb{R}^3$ with Fermi coordinate system $(t, x)$. Assume $(t, u_0 t)$ is a timelike geodesic for the metric $h$ for a vector $u_0 \in \mathbb{R}^3$, $|u_0| < 1$. Assume $2 \leq p < \frac{11}{3}$. Then for all $\lambda_0 = (\omega_0, \theta_0, 0, u_0) \in \Lambda_{\text{stab}}$, there exists a positive number $\epsilon^*$ depending on $h$, $\lambda_0$ such that for all positive $\epsilon < \epsilon^*$, if the initial data $\phi_0, \phi_1$ satisfy

$$\|\phi_0(x) - \phi_S(x; \lambda_0)\|_{H^1(\mathbb{R}^3)} + \|\phi_1(x) - \psi_S(x; \lambda_0)\|_{L^2(\mathbb{R}^3)} \leq \epsilon, \quad (38)$$

then there exists a unique solution $\phi(t, x) \in C([0, T/\epsilon]; H^1(\mathbb{R}^3))$ of the equation (36) with the following property: there is a $C^1$ curve $\lambda(t) = (\omega(t), \theta(t), \xi(t) + u_0 t, u(t) + u_0) \in \Lambda_{\text{stab}}$ such that

$$\|\phi(t, x) - \phi_S(x; \lambda(t))\|_{H^1(\mathbb{R}^3)} + \|\partial_t \phi(t, x) - \psi_S(x; \lambda(t))\|_{L^2(\mathbb{R}^3)} \leq C \epsilon, \quad \forall t \in [0, T/\epsilon]. \quad (39)$$

Moreover

$$|\lambda(0) - \lambda_0| \leq C \epsilon, \quad |\gamma| = |\lambda - V(\lambda)| \leq C \epsilon^2, \quad \forall t \in [0, T/\epsilon], \quad (40)$$

where $\gamma$, $V(\lambda)$ are defined in (15), (17) and the constant $C$ is independent of $\epsilon$.

**Remark 6.** When $g = h$, D. Stuart in [26] proved the orbital stability result up to time $t^* / \epsilon$ for sufficiently small $t^*$. 
To show that the metric $g^s$ is $C^1$, we need to control the energy momentum tensor $T_{\mu\nu}$ in $H^2$. We hence have to estimate the higher Sobolev norm of $\phi$. If initially $\phi_0 \in H^3$, $\phi_1 \in H^2$, then we have

**Proposition 3.** Assume

$$
\epsilon_1 = \|\phi_0(x) - \phi_S(x; \lambda_0)\|_{H^3} + \|\phi_1(x) - \psi_S(x; \lambda_0)\|_{H^2} < \infty.
$$

Let $\lambda(t)$ be the curve obtained in Theorem 5. Then

$$
\sum_{\vert s \vert \leq 3} \|\partial^s (\phi(t, x) - \phi_S(x; \lambda(t)))\|_{L^2(\mathbb{R}^3)} \leq C \max\{\epsilon, \epsilon_1\}, \quad \forall t \in [0, T/\epsilon]
$$

for a constant $C$ independent of $\epsilon, \epsilon_1$.

A direct corollary of the above proposition is the boundedness of the $H^3$ norm of the solution $\phi$.

**Corollary 1.** If the initially data $\phi_0 \in H^3$, $\phi_1 \in H^2$, then

$$
\sum_{\vert s \vert \leq 3} \|\partial^s \phi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C, \quad \forall t \in [0, T/\epsilon],
$$

where the constant $C$ is independent of $\epsilon$.

The above boundedness of $\phi$ in $H^3$ was used in [27]. In this paper, we have to use the fact that if initially the data $(\phi_0, \phi_1)$ are close to some soliton in $H^3$, then the solution $\phi$ stays close to some translated solitons for all $t \leq T/\epsilon$.

**Corollary 2.** If $\epsilon_1 \leq C_0 \epsilon$ for some constant $C_0$, then

$$
\sum_{\vert s \vert \leq 3} \|\partial^s (\phi - \phi_S(x; \lambda(t)))\|_{L^2(\mathbb{R}^3)} \leq C C_0 \epsilon, \quad \forall t \in [0, T/\epsilon].
$$

To avoid too many constants, we make a convention that $A \lesssim B$ means $A \leq CB$ for some universal constant $C$ depending on $h, T, \lambda_0, m, p$.

### 6.1 Decomposition of the Solution

Since solution $\phi$ of (36) exists locally, to begin with, we decompose the solution as follows

$$
\begin{align*}
\phi(t, x) &= \phi_S(x; \lambda(t)) + e^{i\theta(\lambda(t))} \frac{1}{q_0(t, x) \rho_0} v(t, x), \\
\partial_t \phi(t, x) &= \psi_S(x; \lambda(t)) + e^{i\theta(\lambda(t))} \frac{1}{\rho_0(t, x) \rho_0} w(t, x),
\end{align*}
$$

where $\lambda(t) \in \Lambda$. We also denote

$$
\tilde{p}_t(t, x) = \sqrt{-h^{00}(t, x)}, \quad \tilde{p}(t, x) = \sqrt{-h^{00}(t, x)}.
$$

We define three functions depending on $d_e, p_e, q_e$ as follows

$$
a_0 = q_e d_e, \quad a_1 = p_e d_e, \quad a = p_e^{-1} q_e = a_0 a_1^{-1}, \quad b = p_e^{-1} d_e.
$$

We now define the function $q_e(t, x)$ explicitly. Notice that $h(t, u_0 t) = m_0$ and the Christoffel symbols $\Gamma^\beta_{\mu\nu}$ for the metric $h$ are vanishing along $(t, u_0 t)$. In particular we have $\partial h(t, u_0 t) = 0$. Hence we can show that

$$
\begin{align*}
|h^{\mu\nu}(t, u_0 t + x)| &\leq |(m_0)^{\mu\nu}| + C_0 |x|^2, \quad \forall |x| \leq \delta_0, \quad t \in [0, T], \\
\sum_{k=1}^3 h^{k l}(t, u_0 t + x) \xi_k \xi_l &\geq (1 - C_0 |x|^2) \sum_{k=1}^3 |\xi_k|^2, \quad \forall |x| \leq \delta_0, \quad t \in [0, T]
\end{align*}
$$

(42)
for some positive constants $C_0, \delta_0$, where we recall $\mu, \nu \in \{0, 1, 2, 3\}$ and $k,l \in \{1,2,3\}$. Denote $\rho_0 = (1-|u_0|^2)^{-\frac{1}{2}}$. In particular, we have $\rho_0 \geq 1$. Without loss of generality, assume $\frac{1}{2}K_0^{-3}\rho_0^{-1} \leq 1 - 3\rho_0 C_0 \delta_0^2$, $K_0 \geq 10$ and $C_0 \delta_0^2 \rho_0 \leq \frac{1}{3}$. Then choose a function $q(x) \in C^3(\mathbb{R}^3)$ as follows

\[
q(x) = \begin{cases} 
1 - 3C_0 \rho_0 |x|^2, & |x| \leq \frac{1}{2} \delta_0, \\
\frac{4}{3}K_0^{-3}\rho_0^{-1} \delta_0^{-2} (|x|^2 - \frac{\delta_0^2}{4}) + (\frac{4}{3} \delta_0^{-2} - C_0 \rho_0) (\delta_0^2 - |x|^2), & |x| \in (\frac{1}{2} \delta_0, \delta_0), \\
\frac{1}{2}K_0^{-3}\rho_0^{-1}, & |x| \geq \delta_0.
\end{cases}
\]

We define $q_{\epsilon}(t, x)$ as

\[
q_{\epsilon}(t, x) = q(\epsilon(x - u_0t)).
\]  

(43)

We prove an inequality.

Lemma 3. Let $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then

\[
m^2 q_{\epsilon}^{-2} b^2 + q_{\epsilon}^{-2} (h^\epsilon)^{ki} y_k y_l - 2(q_{\epsilon} \tilde{p}_\epsilon)^{-1} (h^\epsilon)^{0k} y_k |y \cdot u| - 3\rho_0 m(q_{\epsilon} \tilde{p}_\epsilon)^{-1} (h^\epsilon)^{0k} y_k |b| \geq m^2 b^2 + |y|^2
\]

for all $b \in \mathbb{R}$, $(t, x) \in [0, T/\epsilon] \times \mathbb{R}^3$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$, $|u| \leq 1$.

Proof. Fix $t$. After scaling, at point $(t, u_0 t + x)$, it suffices to show that

\[
m^2 q_{\epsilon}^{-2} b^2 + q_{\epsilon}^{-2} h^{\epsilon} y_k y_l - 2(q_{\epsilon} \tilde{p}_\epsilon)^{-1} (h^\epsilon)^{0k} y_k |y \cdot u| - 3\rho_0 m(q_{\epsilon} \tilde{p}_\epsilon)^{-1} (h^\epsilon)^{0k} y_k |b| \geq m^2 b^2 + |y|^2, \forall t \in [0, T].
\]  

(44)

When $|x| \geq \delta_0$, we have $q(x) = \frac{1}{2} K_0^{-3} \rho_0^{-1}$. Hence the left hand side of (44)

\[
\geq m^2 q_{\epsilon}^{-2} b^2 + q_{\epsilon}^{-2} (1 - C_0 |x|^2) |y|^2 - 2q_{\epsilon}^{-1} (1 + C_0 |x|^2) C_0 |x|^2 |y|^2
\]

\[
- 3\sqrt{3} \rho_0 m q_{\epsilon}^{-1} |y| b |
\]

\[
\geq (q_{\epsilon} - 1) m^2 b^2 + q_{\epsilon}^{-2} (1 - q_{\epsilon}^2 - C_0 |x|^2) |y|^2 - 3q_{\epsilon} C_0 |x|^2 |y|^2
\]

\[
\geq 5C_0 \rho_0 |x|^2 m^2 b^2 + q_{\epsilon}^{-2} (1 - 2C_0 \rho_0 |x|^2) |y|^2 - 6\rho_0 |q_{\epsilon}^{-1} C_0 |x|^2 |y|^2 + m^2 b^2 + |y|^2
\]

\[
\geq m^2 b^2 + |y|^2.
\]

When $|x| \leq \delta_0$, we have $|h^{0k}(t, u_0 t + x)| \leq C_0 |x|^2$ by (42). Notice that $q(x) \leq 1 - 3C_0 \rho_0 |x|^2$ for $|x| \leq \delta_0$. By (21), we can show that the left hand side of (44)

\[
\geq m^2 q_{\epsilon}^{-2} b^2 + q_{\epsilon}^{-2} |y|^2 - 2q_{\epsilon}^{-1} (1 + C_0 |x|^2) C_0 |x|^2 |y|^2
\]

\[
- 3\sqrt{3} \rho_0 m q_{\epsilon}^{-1} |y| b |
\]

\[
\geq (q_{\epsilon} - 1) m^2 b^2 + q_{\epsilon}^{-2} (1 - q_{\epsilon}^2 - C_0 |x|^2) |y|^2 - 3q_{\epsilon} C_0 |x|^2 |y|^2
\]

\[
\geq 5C_0 \rho_0 |x|^2 m^2 b^2 + q_{\epsilon}^{-2} (1 - 2C_0 \rho_0 |x|^2) |y|^2 - 6\rho_0 |q_{\epsilon}^{-1} C_0 |x|^2 |y|^2 + m^2 b^2 + |y|^2
\]

\[
\geq m^2 b^2 + |y|^2.
\]

where we recall that $C_0 \rho_0 |x|^2 \leq C_0 \delta_0^2 \rho_0 \leq \frac{1}{3}$ and $\rho_0 = (1-|u_0|^2)^{-\frac{1}{2}} \geq 1$. Hence the lemma holds. \(\square\)

In application, we need a similar inequality for the metric $g^{\epsilon}$.

Corollary 3. Let $g^{\epsilon}(t, x) = g(\epsilon t, ex)$, $h^{\epsilon}(t, x) = h(\epsilon t, ex)$. Assume $\|g - h\|_{C^0} \leq \epsilon$. Then

\[
m^2 q_{\epsilon}^{-2} b^2 + q_{\epsilon}^{-2} (g^{\epsilon})^{ki} y_k y_l - 2(q_{\epsilon} \tilde{p}_\epsilon)^{-1} (g^{\epsilon})^{0k} y_k |y \cdot u| - 2m(q_{\epsilon} \tilde{p}_\epsilon)^{-1} ((g^{\epsilon})^{0k} y_k |b| \geq (1 - C\epsilon)(m^2 b^2 + |y|^2)
\]

for all $b \in \mathbb{R}$, $(t, x) \in [0, T/\epsilon] \times \mathbb{R}^3$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$, $|u| \leq 1$, where the constant $C$ depends only on $h$ and $m$.  

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6.2 Orthogonality Condition and Modulation Equations

The decomposition of the solution (41) relies on $\lambda(t)$, which we write as

$$
\lambda(t) = (\omega(t), \theta(t), \xi(t) + u_0 t, u(t) + u_0).
$$

Denote $\gamma(t) = (\omega(t), \pi(t), \eta(t), u(t))$ such that

$$
\dot{\lambda} = \dot{\gamma} + V(\lambda),
$$

where $V(\lambda) = (0, \frac{s}{p}, u + u_0, 0)$, $\rho = (1 - |u + u_0|^2)^{-\frac{1}{2}}$. Recall the notation defined in line (18). We choose $\lambda(t)$ such that the following orthogonality condition

$$
< D\lambda \phi_S, e^{i\theta} w >_{dx} = < D\lambda \phi_S, e^{i\theta} v >_{dx}
$$

(45)

holds. Differentiate it with respect to $t$. We can conclude that the orthogonality condition (45) holds if it holds initially and the curve $\lambda(t)$ satisfies

$$
< D^2 \dot{\lambda} \phi_S \cdot \dot{\lambda}, e^{i\theta} w >_{dx} + < D\lambda \phi_S, \partial_t (e^{i\theta} w) >_{dx} = < D^2 \dot{\lambda} \phi_S \cdot \dot{\lambda}, e^{i\theta} v >_{dx} + < D\lambda \phi_S, \partial_t (e^{i\theta} v) >_{dx}.
$$

Using the decomposition (41) and the relation $\dot{\lambda} = \dot{\gamma} + V(\lambda)$, we can show that the above equation is equivalent to

$$
\begin{align*}
& D\lambda(V(\lambda)D\lambda \psi_S), e^{i\theta} v >_{dx} < D\lambda(V(\lambda)D\lambda \psi_S), e^{i\theta} w >_{dx} - < D\lambda \psi_S, (a_1 - a_0)D\lambda \psi_S, \phi_t - \psi S >_{dx},
\end{align*}
$$

(46)

where we have replaced $< D\lambda V(\lambda) \cdot D\lambda \phi_S, e^{i\theta} w >$ with $< D\lambda V(\lambda) \cdot D\lambda \phi_S, e^{i\theta} v >$ by the orthogonality condition (45). Denote

$$
H(t, x) = b c \phi + a_1 \partial_t \phi - \Delta \phi = p^{-1} d e g \phi + p d_t \partial_t \phi - \Delta \phi
$$

$$
= \sum_{ij} \partial_{ij} \phi + 2 p^{-1} d e (g)^{ij} \partial_{ij} \phi + \frac{1}{pq} \partial_t \phi + a_{\mu k} \partial_{\mu k} \phi + b^\nu \partial_\nu \phi,
$$

where $a_{\mu k}$, $b^\nu$ are the corresponding coefficients. Recall the identity (16) for $\phi_S$. Using integration by parts and observing that $Re(e^{-i\theta} D\lambda \phi_S) = D\lambda |\phi_S|$, we can show that

$$
< D\lambda \phi_S, \Delta \phi >_{dx} = < D\lambda \phi_S, \phi - \psi S >_{dx}
$$

$$
= < D\lambda \phi_S, m^2 \phi - |\phi_S|^{-1} \phi + D\lambda \psi_S \cdot V(\lambda) >_{dx} + < D\lambda \phi_S, \phi - \psi S >_{dx}
$$

$$
= < D\lambda \phi_S, m^2 \phi - |\phi_S|^{-1} \phi + D\lambda \psi_S \cdot V(\lambda) >_{dx}.
$$

Now use the identity (47) to replace $p d_t \partial_t \phi$ in (46). The equation (36) of $\phi$ then implies that the right hand side of (46) can be written as

$$
F(t; \lambda(t)) = < a_0 D\lambda \psi_S, \phi - \psi S >_{dx} + < (a_0 - a_1) D\lambda \psi_S, \psi_t - \psi S >_{dx}
$$

$$
+ < D\lambda(V(\lambda)D\lambda \psi_S), \partial_t - \psi S >_{dx} + < D\lambda \phi_S, \psi_t - \psi S >_{dx}
$$

$$
< \partial_k (a_{\mu k} D\lambda \phi_S) - b^\nu D\lambda \phi_S, \partial_\nu >_{dx} - b D\lambda \phi_S, e^{i\theta} \mathcal{N}(\lambda) >_{dx}
$$

$$
+ < (b - 1) D\lambda \phi_S, m^2 \phi - |\phi_S|^{-1} \phi >_{dx} - < (b - 1) D\lambda |\phi_S|^{-1}, \phi - \psi S >_{dx}.
$$

(48)

where we denote

$$
\mathcal{N}(\lambda) = e^{-i\theta} \left( |\phi|^{-1} \phi - |\phi|^{-1} \phi - |\phi|^{-1} \phi - (p - 1)|\phi_S|^{-1} e^{i\theta} Re(e^{-i\theta}(\phi - \phi_S)) \right).
$$

(49)
as the nonlinearity depending also on $\lambda$. Here recall that $b = p_\varepsilon^{-1}d_\varepsilon$. In particular if $v = v_1 + iv_2$ for real functions $v_1, v_2$, then $Re(e^{-i\theta}(\phi - \phi_S)) = (q_1d_\varepsilon)^{-1}v_1$ by the decomposition (41).

To further simplify the above modulation equations, denote

$$
D = \langle D_\lambda \psi_S, D_\lambda \phi_S \rangle_{dx} - \langle D_\lambda \phi_S, D_\lambda \psi_S \rangle_{dx},
$$

$$
D_1 = \langle D_\lambda \psi_S, (a_0 - 1)D_\lambda \phi_S \rangle_{dx} - \langle D_\lambda \phi_S, (a_1 - 1)D_\lambda \psi_S \rangle_{dx},
$$

$$
D_2 = \langle D_\lambda^2 \phi_S, e^{i\theta}w \rangle_{dx} - \langle D_\lambda^2 \psi_S, e^{i\theta}v \rangle_{dx}.
$$

(50)

We point out here that $D, D_1, D_2$ are $8 \times 8$ matrices. Hence we choose the curve $\lambda(t)$ such that

$$
(D + D_1 + D_2)\dot{\gamma} = F(t; \lambda(t)), \quad \dot{\lambda} = \dot{\gamma} + V(\lambda), \quad \gamma(0) = \lambda(0)
$$

(51)

and we require $\lambda(0)$ satisfies the orthogonality condition (45).

Equations (51) are called the modulation equations for the curve $\lambda(t)$. If the orthogonality condition (45) holds initially and $\lambda(t)$ solves the ODE (51), then (45) holds as long as $\phi, \lambda(t)$ exist.

### 6.3 Initial Data

We have reduced the orthogonality condition (45) to a coupled system of ODE’s for $\lambda(t)$ if initially $\lambda(0)$ satisfies (45). We use the implicit function theorem to show the existence of the initial data $\lambda(0)$ satisfying the orthogonality condition.

**Lemma 4.** Denote

$$
\epsilon = \|\phi_0(x) - \phi_S(x; \lambda_0)\|_{H^1(\mathbb{R}^3)} + \|\phi_1(x) - \psi_S(x; \lambda_0)\|_{L^2(\mathbb{R}^3)}
$$

for some $\lambda_0 \in \Lambda_{\text{stab}}$. Then there exists a positive constant $\epsilon_1(\lambda_0)$, depending only on $\lambda_0$, such that if $\epsilon < \epsilon_1(\lambda_0)$, then there exists $\lambda(0) \in \Lambda_{\text{stab}}$ with the property that if

$$
\begin{align*}
\phi_0(x) &= \phi_S(x; \lambda(0)) + e^{i\theta(\lambda(0))} \frac{1}{q_0(x)d_\varepsilon} v(x; \lambda(0)), \\
\phi_1(x) &= \psi_S(x; \lambda(0)) + e^{i\theta(\lambda(0))} \frac{1}{p_0(x)d_\varepsilon} w(x; \lambda(0)),
\end{align*}
$$

then the orthogonality condition holds

$$
\langle D_\lambda \psi_S(x; \lambda(0)), e^{i\theta(\lambda(0))}v(x; \lambda(0)) \rangle_{dx} - \langle D_\lambda \phi_S(x; \lambda(0)), e^{i\theta(\lambda(0))}w(x; \lambda(0)) \rangle_{dx}.
$$

Moreover, we have

$$
|\lambda(0) - \lambda_0| \leq C(\lambda_0)\epsilon,
$$

$$
\|v(x; \lambda(0))\|_{H^1} + \|w(x; \lambda(0))\|_{L^2} \leq C(\lambda_0)\epsilon
$$

for some constant $C(\lambda_0)$ depending only on $\lambda_0$.

**Proof.** Define a functional $\mathcal{F} : H^1 \times L^2 \times \mathbb{R}^8 \to \mathbb{R}^8$ such that

$$
\mathcal{F}(v, w, \lambda) = \langle a_1D_\lambda \phi_S(\lambda; x), \phi_1(x) - \psi_S(\lambda; x) \rangle_{dx} - \langle a_0D_\lambda \psi_S(\lambda; x), \phi_0(x) - \phi_S(\lambda; x) \rangle_{dx}.
$$

In particular, we have $\mathcal{F}(0, 0, \lambda_0) = 0$. Notice that

$$
\mathcal{F}(0, 0, \lambda_0) = \langle a_0D_\lambda \psi_S, D_\lambda \phi_S \rangle_{dx} - \langle a_1D_\lambda \phi_S, D_\lambda \psi_S \rangle_{dx} = D + D_1.
$$

By Lemma 6 and Lemma 5 proven later, we can conclude that if $\epsilon$ is sufficiently small, then $D + D_1$ is nondegenerate initially. Since $\mathcal{F}$ is Lipschitz continuous in $(v, w)$, the implicit function theorem then implies that there exists $\lambda(0) \in \Lambda_{\text{stab}}$ satisfying the orthogonality condition (45) as well as the estimates in the lemma. \[\square\]
6.4 Bootstrap Argument

By Lemma 4, we can choose \(\lambda(0)\) close to \(\lambda_0\) such that the orthogonality condition (45) holds initially. The local existence result shows that there is a short time solution \(\phi(t, x)\) of (36). To prove the existence of solution \(\lambda(t)\) of the modulation equations (51), we have to demonstrate that the \(8 \times 8\) matrix \(D + D_1 + D_2\) is nondegenerate, which depends on \(\lambda(t)\) itself. We thus need to consider the modulation equations coupled to the nonlinear wave equation (36). Since initially the radiation term \((v, w)\) is small in \(H^1 \times L^2\), we show that for the coupled equations, \((v, w)\) stays small in \(H^1 \times L^2\) for all \(t \leq T/\epsilon\), which implies that the modulation equations are solvable and we can control the modulation curve \(\lambda(t)\).

**Proposition 4.** Let \((\phi(t, x), \lambda(t))\) be solutions of (36), (51) on \([0, T/\epsilon) \times \mathbb{R}^3\). Assume the complex functions \(v(t, x), w(t, x)\) satisfy the decomposition (41). Then for sufficiently small \(\epsilon\), we have

\[
\|v(t, x)\|_{H^1(\mathbb{R}^3)} + \|w(t, x)\|_{L^2(\mathbb{R}^3)} \leq C\epsilon, \quad \forall t \in [0, T/\epsilon],
\]

\[
|\dot{\gamma}(t)| \leq C\epsilon^2, \quad \forall t \in [0, T/\epsilon]
\]

for some constant \(C\) independent of \(\epsilon\).

We use bootstrap argument to prove this proposition. To estimate \(|\dot{\gamma}|\), we rely on the modulation equations (51) together with the fact that \(D + D_1 + D_2\), as an \(8 \times 8\) matrix, is nondegenerate. It turns out that \(D\) depends only on \(\omega\) and is nondegenerate for \(\omega = \omega_0\). Thus if \(\omega(t)\) is close to \(\omega_0\), we have the nondegeneracy of \(D\). From this point of view, we take a subset of \(\Lambda_{\text{stab}}\) defined as follows

\[
\Lambda_{\delta_0} = \{ (\omega, \theta, \xi) | \omega \in [\omega_0 - \delta_0, \omega_0 + \delta_0] \subset \{ \omega \mid \frac{p-1}{6-2p} < \frac{\omega^2}{m^2} < 1 \}, \quad |u - u_0| \leq \delta_0, \}
\]

(52)

where \(\delta_0\) is a positive constant, which will be determined in Lemma 9. Here we recall that \(\lambda_0 = (\omega_0, \theta_0, 0, u_0) \in \Lambda_{\text{stab}}\). Thus we can choose \(\delta_0\) sufficiently small such that \(\Lambda_{\delta_0}\) is nonempty. Our first bootstrap assumption is that \(\lambda(t) \in \Lambda_{\delta_0}\) for all \(t \in [0, T/\epsilon]\).

By the definition of \(D_1, D_2\) in line (50), we see that \(D_1\) has size \(\epsilon\) if the center \((t, \xi(t) + u_0t)\) of the soliton \(\phi_S(x; \lambda(t))\) does not diverge far away from the timelike geodesic \((t, u_0t)\). Hence we expect that \(|\xi(t)|\) is uniformly bounded, which is also suggested by Proposition 4. In fact since \(\xi = \dot{u}(t) + \ddot{\eta}_t\), if \(|\dot{\gamma}| = |(\dot{\omega}, \dot{\theta}, \dot{\xi}, \dot{\eta})| \leq C\epsilon^2\), then

\[
|\xi(t)| \leq |\xi(0)| + |u(0)|t + C\epsilon^2t^2 \leq |\xi(0)| + (C + 1)t^2, \quad \forall t \leq T/\epsilon.
\]

We remark here that this is compatible with Theorem 4 as if we scale it back to the space \([0, T] \times \mathbb{R}^3\), the center of the soliton becomes \((t, t\xi + u_0t)\) which is close to the geodesic \((t, u_0t)\). Similarly, \(D_2\) is an error term if \(\|v\|_{H^1(\mathbb{R}^3)} + \|v\|_{L^2(\mathbb{R}^3)}\) is small.

To prove Proposition 4, in addition to the assumption that \(\lambda(t) \in \Lambda_{\delta_0}\), we assume

\[
|\xi(t)| \leq 2C_2, \quad \forall t \in [0, T/\epsilon],
\]

(53)

\[
\|w(t, x)\|_{L^2(\mathbb{R}^3)} + \|v(t, x)\|_{H^1(\mathbb{R}^3)} \leq \delta_1, \quad \forall t \in [0, T/\epsilon]
\]

(54)

for some constants \(C_2, \delta_1\) which will be fixed later on. Without loss of generality, we assume \(C_2 > 1, \delta_1 < 1, C_3\epsilon < 1\). These are the bootstrap assumptions in this subsection.

As having mentioned previously, we do not have estimates for \(\|g^\epsilon - h^\epsilon\|_{L^2(\Sigma_0)}\) initially. In fact, our construction of initial data implies that \(g^\epsilon - h^\epsilon\) is not bounded in \(L^2\) for general data. To bound \(g^\epsilon - h^\epsilon\), we rely on Hardy’s inequality.

**Lemma 5.** Let \(f(x) \in C^1(\mathbb{R}^3)\). Assume \(f(x) \to 0\) as \(|x| \to \infty\). Then

\[
\|f(x)|x|^{-1}\|_{L^2(\mathbb{R}^3)} \leq 6\|\nabla f(x)\|_{L^2(\mathbb{R}^3)}.
\]

This inequality can be proven by using integration by parts under polar coordinate system. Detailed proof could be found in [12], [29]. In particular, using Sobolev embedding, the assumption (37) implies that

\[
\|\psi\|_{C^{1, \frac{1}{2}}(\mathbb{R}^3)} + \|\psi(1 + |x|)^{-1}\|_{L^2(\mathbb{R}^3)} + \|\partial^{s+1}\psi\|_{L^2(\mathbb{R}^3)} \lesssim \epsilon^2, \quad |s| \leq 2, \quad \forall t \in [0, T/\epsilon].
\]

(55)
To prove Proposition 4, we use bootstrap argument. We show that the matrix $D + D_1 + D_2$ are nondegenerate initially. Thus the modulation equations can be solved locally. If under the bootstrap assumptions (53), (54), we can show that Proposition 4 holds for some constant $C$ which is independent of $C_2$, then we can close the bootstrap assumptions if we take $C_2 = |\xi(0)| + (C + 1)T^2$, $\delta_1 = 2\epsilon$, $\delta_0 = 2CT\epsilon$. We thus can conclude Proposition 4 if $\epsilon$ is sufficiently small.

The strategy is as follows: we first show the nondegeneracy the modulation equations and obtain the estimates for $|\hat{\gamma}(t)|$. Under the orthogonality condition (45), we use energy estimate to demonstrate that the radiation $(v, w)$ is small in $H^1 \times L^2$.

6.4.1 Nondegeneracy of the Modulation Equations

To obtain estimates of $\lambda(t)$ and to show the local existence of the modulation equations (51), we demonstrate that, under the bootstrap assumption $\lambda(t) \in \Lambda_{\delta_0}$, the leading coefficient $D$ in (51) is nondegenerate.

**Lemma 6.** Let $D$ be the $8 \times 8$ matrix defined in (50). If $\lambda = (\omega, \theta, \xi, u) \in \Lambda_{\delta_0}$, then

$$|\det D| \geq C(\delta_0, \lambda_0) > 0$$

for some constant $C(\delta_0, \lambda_0)$ depending on $\delta_0, \lambda_0$. In particular, $D$ is nondegenerate.

**Proof.** Direct calculations show that

$$D_{\omega\theta} = D_{\omega}(\omega\|f_\omega\|_{L^2}^2), \quad D_{\omega\xi} = \rho uD_\omega B, \quad D_{\xi u} = -\rho B(1 + \rho^2 u \cdot u),$$

$$D_{\omega u} = D_{\theta \xi} = B_\xi = D_{\xi u} = D_{u u} = D_{\theta u} = D_{\theta \theta} = 0,$$

where we denote

$$B(t) = \omega^2\|f_\omega(x)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{3}\|\nabla_x f_\omega(x)\|_{L^2(\mathbb{R}^3)}^2.$$  \hspace{1cm} (57)

Notice that $D$ is antisymmetric. We can show that

$$\det D = |D_{\omega}(\omega\|f_\omega\|_{L^2}^2)|^2 \cdot |\det D_{\xi u}|^2 = |D_{\omega}(\omega\|f_\omega\|_{L^2}^2)|^2 B^6 \rho^{10}.$$  

Using the scaling property (11) of $f_\omega(x)$, when $\frac{p - 1}{6 - 2p} < \frac{\omega^2}{m^2} < 1$, we have

$$D_{\omega}(\omega\|f_\omega\|_{L^2}^2) = (m^2 - \frac{6 - 2p}{p - 1}\omega^2)(m^2 - \omega^2)\frac{\omega^2}{m^2 - \omega^2}\|f\|_{L^2(\mathbb{R}^3)}^2 < 0.$$  

Since $\rho \geq 1$, the lemma then follows.  

6.4.2 Estimates for the Modulation Curve

We have shown that $D$ is nondegenerate. To estimate $\gamma(t)$ by using the modulation equations, we have to show that $D_1, D_2, F(t; \lambda(t))$ are error terms. Estimates for $D_1, D_2$ can be obtained by using Cauchy-Schwartz inequality. The main difficulty for estimating $F(t; \lambda(t))$ lies in the nonlinear term $<D_3 \phi_S, e^{i\theta}N(\lambda)>_{dx}$. Before touching the equations, we prove two lemmas. The first lemma gives control of the nonlinearity $N(\lambda)$. Let

$$N(x) = \frac{1}{p + 1}|x|^{p + 1}$$

for complex number $x$. In particular, we have $D_{\bar{x}} N = \frac{1}{2}|x|^{p - 1}x$, where $\bar{x}$ is the complex conjugate of $x$.

By the definition (49) of the nonlinearity $N(\lambda)$, we can write

$$N(\lambda) = 2D_{\bar{x}} N(f_\omega + (q_\lambda d_e)^{-1}v) - 2D_{\bar{x}} N(f_\omega) - 2D_{\bar{x}} DN(f_\omega) \cdot (q_\lambda d_e)^{-1}v,$$

where $DN \cdot v = D_{\bar{x}} N v + D_{\bar{x}} N \bar{v}$.  

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Lemma 7. Let $\mathcal{N}(\lambda)$ be defined in line (49). Assume $\phi$ decomposes as (41). For all $p \geq 2$, we have

\begin{align*}
|\mathcal{N}(\lambda)| & \lesssim |v|^2 + |v|^p, \\
|N(f_\omega + v) - N(f_\omega) - DN(f_\omega)v - 1/2vD^2 N(f_\omega)v| & \lesssim |v|^3 + |v|^{p+1}.
\end{align*}

Proof. Let $\tilde{v} = (q, d_\omega)^{-1}v$. We can write $\mathcal{N}(\lambda)$ as an integral

$$
\mathcal{N}(\lambda) = 4 \int_0^1 \int_0^1 s\tilde{v}^2 D^2 D\tilde{x} N(f_\omega + ts\tilde{v}) dt ds.
$$

Since $p \geq 2$, we can show that

$$
|D^2 D\tilde{x} N(f_\omega + ts\tilde{v})| \lesssim 1 + |f_\omega + ts\tilde{v}|^{p-2} \lesssim 1 + |\tilde{v}|^{p-2}.
$$

Hence (58) holds. The second inequality (59) follows similarly.

This lemma implies that the nonlinearity in $F(t; \lambda(t))$ is in fact higher order error term. For the other terms in $F(t; \lambda(t))$, observe that if $g = h$, then $p_r - 1, d_r - 1, q_r - 1$ have size $\epsilon^2$ near the geodesic $(t, u_0 t)$ by (42) and the fact that $\phi_S$ decays exponentially. To pass $h^t$ to $g^t$ satisfying (55), we use the following lemma.

Lemma 8. Let $F_1(x), F_2(x)$ be two $C^1$ functions such that $F_1(0) = F_2(0) = 0$. Then we have

$$
\|F_1(\phi_S(\lambda; x)) F_2(g^t - F_2(m_0))\|_{L^r(\mathbb{R}^2)} \lesssim \|F_1\|_{C^1} \|F_2\|_{C^1} (1 + |\xi|^2)\epsilon^2, \quad \forall r \in [1, 2],
$$

where $\lambda = (\omega, \theta, \xi + u_0 t, u + u_0) \in \Lambda_{stab}$. Similarly

$$
\|F_1(\phi_S(\lambda)) F_2(\partial g^t)\|_{L^r(\mathbb{R}^2)} \lesssim \|F_1\|_{C^1} \|F_2\|_{C^1} (1 + |\xi|)\epsilon^2, \quad \forall r \in [1, \infty].
$$

Proof. Recall that $\phi_S(x; \lambda) = e^{i\theta} f_\omega(z), z = A_{u+u_0}(x - \xi - u_0 t)$. We conclude that $|z|^2 = |x - \xi - u_0 t|^2$.

Hence by Theorem 2, we have

$$
|F(\phi_S(\lambda))| \lesssim \|F\|_{C^1} e^{-c(\omega)|z|} \lesssim \|F\|_{C^1} e^{-c(\omega)|x - \xi - u_0 t|}.
$$

Since $\partial h(t, u_0 t) = 0, h(t, u_0 t) = m_0$, we can show that

$$
|F_2(g^t(t, u_0 t + x) - F_2(m_0)(t, u_0 t + x)| \leq \|F_2\|_{C^1} (\epsilon^2 |x|^2 \|h\|_{C^2} + |g^t - h^t|).
$$

Then (55) and Hölder’s inequality imply that

\begin{align*}
\|F_1(\phi_S(\lambda)) F_2(g^t - F_2(m_0))\|_{L^r(\mathbb{R}^2)} & \lesssim \|F_1(\phi_S(\lambda)) (1 + |z|)\|_{L_{\infty}} \|F_2\|_{C^1} \|\psi(1 + |z|)\|_{L^2} + \|F_1\|_{C^1} \|F_2\|_{C^2} (1 + |\xi|^2)\epsilon^2 \\
& \lesssim \|F_1\|_{C^1} \|F_2\|_{C^1} (1 + |\xi|^2)\epsilon^2, \quad r \in [1, 2].
\end{align*}

Similarly, by (55), we have

\begin{align*}
\|F_1(\phi_S(\lambda)) F_2(\partial g^t)\|_{L^r(\mathbb{R}^2)} & \lesssim \|F_1(\phi_S(\lambda)) F_2(\partial h^t)\|_{L^r(\mathbb{R}^2)} + \|F_2\|_{C^1} \|F_1(\phi_S) \partial \psi^t\|_{L^r} \\
& \lesssim \|F_1\|_{C^1} \|F_2\|_{C^1} (1 + |\xi|)\epsilon^2, \quad r \in [1, \infty].
\end{align*}

Having proven the above two lemmas, we are able to estimate $\lambda(t)$. It suffices to show that $D_1, D_2, F$ are error terms.
Proposition 5. Under the bootstrap assumptions (53), (54), we have

\[
\|D_1\| \lesssim (1 + |\xi|^2)\varepsilon^2 \lesssim (C_2\varepsilon)^2, \\
\|D_2\| \lesssim \|w\|_{L^2(\mathbb{R}^2)} + \|v\|_{H^1(\mathbb{R}^2)} \lesssim \delta_1, \\
\|F\| \lesssim (C_2\varepsilon)^2 + \|v\|_{H^1}^2.
\]

If \(\varepsilon, \delta_1\) is sufficiently small, by Lemma 6, estimates (60), (61) show that \(D\) dominates \(D_1 + D_2\). Hence \(D + D_1 + D_2\) is nondegenerate. Then using (62), we can estimate \(\dot{\gamma}\).

Proof. Inequality (60) follows from Lemma 8 and the bootstrap assumption (53). Estimate (61) for \(D_2\) can be obtained by using Cauchy-Schwartz’s inequality and the assumption (54).

For (62), we apply Lemma 7 to control the nonlinearity \(<bD_\lambda \phi_S, e^{i\Theta} N(\lambda) >_{dx}\) and use Lemma 8 to estimate other terms by observing that

\[
b - 1 = p^{-1}_e d_e - 1, \quad a_0 - 1 = q_e d_e - 1, \quad a_1 - 1 = p_e d_e - 1, \quad a^{k\mu}, \quad b^\mu
\]
can be written as the form \(F_1(g^e) - F_1(m_0)\). Hence we can show that

\[
\|F\| \lesssim \varepsilon^2(1 + |\xi|^2) + \|v^2 + |v|^p\|_{L^1} \lesssim (C_2\varepsilon)^2 + \|v\|_{H^1}^2.
\]

To make \(D + D_1 + D_2\) to be nondegenerate, we choose \(\delta_1\) in the following way. Fix \(\delta_0\). Then let \(\varepsilon, \delta_1\) be sufficiently small such that

\[
\|D_1\| + \|D_2\| \leq C(C_2\varepsilon)^2 + C\delta_1 \leq \frac{1}{10}C(\delta_0, \lambda_0)\delta_1,
\]

where \(C\) is the implicit constant in Proposition 5, which by our notations is independent of \(\varepsilon, C_2\). Here \(C(\delta_0, \lambda_0)\) is the constant in Lemma 6. So far, only \(\delta_0, C_2\) are unknown constants. However, all the implicit constants are independent of \(C_2, \varepsilon\). With this choice of \(\delta_1\), we can estimate \(\dot{\gamma}\).

Corollary 4. Let \(\delta_1\) be chosen as above. Suppose \(\lambda(t) \in \Lambda_{\delta_0}\) and \(\xi(t), w, v\) satisfy the bootstrap assumptions (53), (54). Then we have

\[
|\dot{\gamma}| \lesssim \|F\| \lesssim (C_2\varepsilon)^2 + \|v\|_{H^1}^2.
\]

This corollary shows that as long as the radiation term \(v\) in the decomposition (41) of the solution (41) is small, we can solve the modulation equations (51) and obtain estimates for the modulation curve \(\dot{\gamma}(t)\). The modulation equations are used to guarantee the orthogonality condition (45). Next, we show that under the orthogonality condition, the energy \(\|v\|_{H^1} + \|w\|_{L^2}\) of the radiation term \((v, w)\) is small.

6.4.3 Energy Decomposition

We use energy estimates to show that the radiation term \((v, w)\) is small in \(H^1 \times L^2\). We consider the almost conserved energies for the full solution \(\phi\) of the nonlinear wave equation (36). Using the decomposition (41), we decompose the energies for \(\phi\) around the solitons \(\phi_S\). The associated energy for the solitons \(\phi_S\) can be computed explicitly up to an error. Then combining with the orthogonality condition, we can obtain estimates for \(\|v\|_{H^1} + \|v\|_{L^2}\). We first define almost conserved energies for \(\phi\) and decompose them around the solitons \(\phi_S\).

We recall the energy momentum tensor with respect to the metric \(g^e\)

\[
T_{\mu\nu}[\phi] = \partial_{\mu}\phi, \partial_{\nu}\phi = -\frac{1}{2}g^e_{\mu\nu}(\partial^{\gamma}\phi, \partial_{\gamma}\phi) + 2\mathcal{V}(\phi),
\]

where

\[
\mathcal{V}(\phi) = \frac{m^2}{2}|\phi|^2 - \frac{1}{p+1}|\phi|^{p+1} = \frac{m^2}{2}|\phi|^2 - N(\phi).
\]
For a vector field $Y$ and a real function $\beta$, we have the identity
\[
D^\mu (\beta T_{\mu \nu} [\phi] Y^\nu) = \beta T^{\mu \nu} [\phi] \pi^Y_{\mu \nu} + \beta < \Box g^\phi - m^2 \phi + |\phi|^{p-1} \phi, Y(\phi) > + T(Y, D\beta),
\] (63)
where $\pi^Y_{\mu \nu} = \frac{1}{2} \mathcal{L}_Y g^\mu_{\nu}$ is the deformation tensor of $Y$ and $D\beta$ is the gradient of the function $\beta$. Consider the region $\mathbb{R}^3 \times [0, t]$. First we take $\beta \equiv 1$, $Y = \partial_\xi$. Using Stoke’s formula and equation (36), we have
\[
\mathcal{H}(t) = \mathcal{H}(0) - \int_0^t \int_{\mathbb{R}^3} T^{\mu \nu} [\phi] \pi^\beta_{\mu \nu} d\text{vol},
\] (64)
where
\[
\mathcal{H}(t) = - \int_{\mathbb{R}^3} - T_{\mu \nu} [\phi] (g^\tau)^{\mu \nu} d\sigma = \frac{1}{2} \int_{\mathbb{R}^3} < \partial^k \phi, \partial_k \phi > - < \partial^\xi \phi, \partial_\xi \phi > + 2V(\phi) d\sigma.
\] (65)
Here we recall that $d\sigma = \sqrt{-\det g^\tau} = d^2_\xi dx$, $d\text{vol} = d^2_\xi dtdx$.

Then let $\beta = a = p^{-1}_\xi q_\xi$, $Y = \partial_\xi$. We have
\[
\Pi_k(t) = \Pi_k(0) + \int_0^t \int_{\mathbb{R}^3} aT^{\mu \nu} [\phi] \pi^\beta_{\mu \nu} + \partial^\mu a T_{\mu k} [\phi] d\text{vol},
\] (66)
where
\[
\Pi_k(t) = \int_{\mathbb{R}^3} a T_{\mu k} [\phi] (g^\tau)^{\mu 0} d\sigma = \int_{\mathbb{R}^3} p^{-1}_\xi q_\xi < \partial^\xi \phi, \partial_\xi \phi > d\sigma.
\] (67)
We remark here that $\mathcal{H}$, $\Pi_k$ are conserved quantities of the equation (36) if the metric $g^\tau$ is flat. For general slowly varying metric $g^\tau$, we will show in the next section that these quantities are almost conserved, that is, the error is of higher order.

Since $\phi$ is complex valued, we define the charge of the solution $\phi$
\[
Q(t) = \int_{\mathbb{R}^3} a < i \partial^\xi \phi, \phi > d\sigma, \hspace{1em} d\sigma = d^2_\xi dx.
\] (68)

Using the equation (36), we can get an integral form of $Q(t)$
\[
Q(t) = Q(0) + \int_0^t \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (a d^2_\xi) < i \partial^\xi \phi, \phi > + ad^2_\xi ( < i \partial_\xi \partial^\xi \phi, \phi > + < i \partial^\xi \phi, \partial_\xi \phi > ) d\sigma.
\]
\[
= Q(0) + \int_0^t \int_{\mathbb{R}^3} d^2_\xi < i \partial_\mu a \partial^\mu \phi, \phi > + ad^2_\xi < i \partial^\mu \phi, \partial_\mu \phi > d\sigma
\] (69)
\[
= Q(0) + \int_0^t \int_{\mathbb{R}^3} < i \partial_\mu a \partial^\mu \phi, \phi > d^2_\xi d\sigma,
\]
where we have used the equation
\[
\partial_\mu \partial^\mu \phi + a^{-2} d^2_\xi \partial_\mu (a^2_\xi) \partial^\mu \phi - m^2 \phi + |\phi|^{p-1} \phi = 0
\]
together with the fact that the quadratic form $< i (g^\tau)_{\cdot \cdot}, \cdot >$ is antisymmetric.

Remark 7. The reason that we put some weight in the decomposition (41), also in the definition of the almost conserved quantities $\Pi_k$, $Q$, is to reduce the positivity of the energy of the radiation term $(v, w)$ to the case in Minkowski space, which has been proven in [25], see Proposition 2.

Next, we expand $\mathcal{H}$, $\Pi_k$, $Q$, as functionals of the full solution $\phi$, around the soliton $\phi_\xi$. The soliton part can be calculated explicitly. The crossing terms are close to the orthogonality condition (45) and hence are small. A combination of the quadratic terms in $v, w$ gives the energy $E_0(t)$ defined in (20), which is positive definite by Proposition 2. We thus end up with an estimate for $\|v\|_{L^2} + \|w\|_{H^1}$ if we can further show that $\mathcal{H}$, $\Pi_k$, $Q$ are almost conserved.
We first consider the angular momentum $\Pi_k(t)$. Using the decomposition (41), we can show that

$$\Pi_k(t) = \int_{\mathbb{R}^3} p^{-1}_k q_e < \partial' \phi, \partial_k \phi > \, d\sigma$$

$$= \int_{\mathbb{R}^3} p^{-1}_k q_e < (g')^{00} \partial_t \phi, \partial_k \phi > + p^{-1}_k q_e < (g')^{0i} < \partial_i \phi, \partial_k \phi > \, d\sigma$$

$$= \int_{\mathbb{R}^3} p^{-1}_k q_e (g')^{00} < \psi_S + e^{i\phi} (p_k dx)^{-1} w, \partial_k (\phi_S + e^{i\phi} (q_k dx)^{-1} v) >$$

$$+ p^{-1}_k q_e (g')^{0i} < \partial_i (\phi_S + e^{i\phi} (q_k dx)^{-1} v), \partial_k (\phi_S + e^{i\phi} (q_k dx)^{-1} v) > \, d\sigma.$$

Recall that $h \in C^4$ and $h = m_0$ along the geodesic $(t, u_0 t)$. Since $g'$ is close to $h'$ in terms of the condition (55), we compare the above integral with that associated to the metric $h$. We hence can write $\Pi_k(t)$ as a sum of main terms plus an error

$$\Pi_k(t) = - < \psi_S, \partial_k \phi_S >_{dx} - < \psi_S, \partial_k (e^{i\phi} v) >_{dx} - < e^{i\phi} w, \partial_k \phi_S >_{dx} - < e^{i\phi} w, \partial_k (e^{i\phi} v) >_{dx}$$

$$+ < (p_k q_e)^{-1} (g')^{00} \partial_i (e^{i\phi} v), \partial_k (e^{i\phi} v) >_{dx} + \text{Err}(\Pi_k),$$

in which, by using Lemma 8, we can show that the error term $\text{Err}(\Pi_k)$ can be bounded as follows

$$|\text{Err}(\Pi_k)| \lesssim c^2 (1 + |\xi|^2) (1 + \|v\|_{H^1}^2) + \|\partial g'\|_{L^\infty} (\|v\|_{L^2} \|v\|_{H^1} + \|v\|_{L^4}^2)$$

$$\lesssim (1 + |\xi|^2)e^2 + \|v\|_{L^2}^2 + \|v\|_{H^1}^2.$$  \hspace{1cm} (70)

We show the crossing terms are vanishing due to the orthogonality condition. In fact, recall the definition of $z$ in line (14). We find that

$$D_z \phi_S = D_z \phi_S \cdot \frac{\partial z}{\partial \xi} = -D_z \phi_S \cdot \frac{\partial z}{\partial x} = -\nabla_x \phi_S.$$

Since $\lambda(t)$ solves the modulation equations (51) and hence the orthogonality condition (45) holds, integration by parts implies that

$$< \psi_S, \nabla_x (e^{i\phi} v) >_{dx} + < e^{i\phi} w, \nabla_x \phi_S >_{dx} = - < e^{i\phi} w, D_z \phi_S >_{dx} + < D_z \psi_S, e^{i\phi} v >_{dx} = 0.$$

We now compute the soliton part. Since $dz = \rho dx$, we can compute

$$< \psi_S, \nabla_x \phi_S >_{dx} = < e^{i\phi} (i \rho \omega f_\omega - \rho (u + u_0) \cdot \nabla_x f_\omega), e^{i\phi} (-i \rho \omega f_\omega (u + u_0) + \nabla_x f_\omega \cdot \frac{\partial z}{\partial x}) >_{dx}$$

$$= - \int_{\mathbb{R}^3} \rho^2 \omega^2 f_\omega^2 (u + u_0) + \rho (u + u_0) \cdot \nabla_x f_\omega \nabla_z f_\omega \cdot \frac{\partial z}{\partial x} \, dx$$

$$= - \rho B(u + u_0),$$

where by Theorem 2, the ground state $f_\omega$ is spherical symmetric and $B$ is given in line (57). We hence can write

$$\Pi_k(t) = \rho B(u_k + u_0) - < e^{i\phi} w, \partial_k (e^{i\phi} v) >_{dx} + < (p_k q_e)^{-1} (g')^{00} \partial_i (e^{i\phi} v), \partial_k (e^{i\phi} v) >_{dx} + \text{Err}(\Pi_k),$$

where $u = (u^1, u^2, u^3), u_0 = (u_0^1, u_0^2, u_0^3)$ and the error term $\text{Err}(\Pi_k)$ satisfies (70).

We decompose the charge $Q(t)$ in a similar way. Recall that $\beta = p^{-1}_k q_e$. We can show that

$$Q(t) = \int_{\mathbb{R}^3} p^{-1}_k q_e < i (g')^{00} (\psi_S + e^{i\phi} (p_k dx)^{-1} w), \phi_S + e^{i\phi} (q_k dx)^{-1} v >$$

$$+ p^{-1}_k q_e < i (g')^{0i} \partial_i (\phi_S + e^{i\phi} (q_k dx)^{-1} v), \phi_S + e^{i\phi} (q_k dx)^{-1} v > \, d\sigma$$

$$= - i \psi_S, \phi_S >_{dx} - < i w, v >_{dx} - i \psi_S, e^{i\phi} v >_{dx} - i e^{i\phi} w, \phi_S >_{dx}$$

$$+ < (p_k q_e)^{-1} (g')^{0i} i \partial_i (e^{i\phi} v), e^{i\phi} v >_{dx} + \text{Err}(Q).$$

(72)
where the error term $Err(Q)$ satisfies the estimate

$$|Err(Q)| \lesssim (1 + |\xi|^2)\epsilon^2 + \epsilon\|w\|_{L^2}^2 + \epsilon\|v\|_{H^1}^2. \quad (73)$$

Now observe that $D_\theta \phi_S = i\phi_S$, $D_\theta \psi_S = i\psi_S$. The orthogonality condition (45) together with integration by parts implies that

$$<i\psi_S, e^{i\Theta v}>_{dx} + <i\psi_S, e^{i\Theta v}>_{dx} = <D_\theta \psi_S, e^{i\Theta v}>_{dx} - <D_\theta \phi_S, e^{i\Theta v}>_{dx} = 0.$$

For the soliton part, we can compute

$$<i\psi_S, \phi_S>_{dx} = \int_{\mathbb{R}^3} -\rho \omega f^2 dx = -\omega \|f\|_{L^2}^2.$$

Therefore, we can write

$$Q(t) = \omega \|f\|_{L^2}^2 < iw, v>_{dx} + <(p,q_\epsilon)^{-1}(g')^{\frac{\nu}{2}} i\partial_k(e^{i\Theta v}), e^{i\Theta v} >_{dx} + Err(Q). \quad (74)$$

The error term $Err(Q)$ satisfies (73).

Finally, we consider the main energy $H(t)$, containing of quadratic terms and a higher order term corresponding to the nonlinearity. The quadratic part can be decomposed as follows

$$<\partial^k \phi, \partial_\nu \phi >_{dx} - <\partial^k \phi, \partial_\nu \phi >_{dx} + m^2 <\phi, \phi >_{dx}$$

$$= <(g')^k \phi, \partial_\nu \phi >_{dx} - <(g')^k \phi, \partial_\nu \phi >_{dx} + m^2 <\phi_S + e^{i\Theta(q, d_\epsilon)} v, \phi_S + e^{i\Theta(q, d_\epsilon)} v >_{dx}$$

$$= \int_{\mathbb{R}^3} |\nabla_x \phi_S|^2 + |\psi_S|^2 + m^2 |\phi_S|^2 dx + m^2 <q_\epsilon^{-2} v, v >_{dx} + <q_\epsilon^{-2} v, v >_{dx} + 2 <\nabla_x \phi_S, \nabla_x (e^{i\Theta v}) >_{dx} + <w, w >_{dx} + 2 <w, e^{i\Theta v}, \psi_S >_{dx} + 2 m^2 <\phi_S, e^{i\Theta v} >_{dx} + Err(Hq),$$

where

$$|Err(Hq)| \lesssim (1 + |\xi|^2)\epsilon^2 + \epsilon\|w\|_{L^1}^2 + \epsilon\|v\|_{H^1}^2. \quad (75)$$

We expand the nonlinear term up to second order

$$= \int_{\mathbb{R}^3} \frac{1}{p+1} f_\omega^{p+1} + f_\omega^p q_\epsilon^{-1} v_1 + \frac{1}{2} f_\omega^{p-1} q_\epsilon^{-2} (|v|^2 + (p-1)|v_1|^2) \ d\sigma$$

$$= \int_{\mathbb{R}^3} \frac{1}{p+1} f_\omega^{p+1} + f_\omega^p v_1 + \frac{1}{2} f_\omega^{p-1} (|v|^2 + (p-1)|v_1|^2) \ dx + Err(Hq),$$

where by using Lemma 7, we can control the error term

$$|Err(Hn)| \lesssim \|v\|_{H^1}^3 + \|v\|_{H^1}^{p+1} + (1 + |\xi|^2)\epsilon^2 \lesssim \|v\|_{H^1}^3 + (1 + |\xi|^2)\epsilon^2. \quad (76)$$

Group these together. We end up with

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi_S|^2 + |\psi_S|^2 + m^2 f_\omega^2 - \frac{2}{p+1} f_\omega^{p+1} + m^2 q_\epsilon^{-2} |v|^2 + <\nabla_x \phi_S, \nabla_x (e^{i\Theta v}) >_{dx}$$

$$+ <\nabla_x \phi_S, \nabla_x (e^{i\Theta v}) >_{dx} + \frac{1}{2} <w, w >_{dx} + <w, e^{i\Theta v}, \psi_S >_{dx} + m^2 <\phi_S, e^{i\Theta v} >_{dx} + Err(Hq)$$

$$- <f_\omega^p v_1 >_{dx} - \frac{1}{2} \int_{\mathbb{R}^3} f_\omega^{p-1} |v|^2 + (p-1)f_\omega^{p-1} v_1^2 dx + Err(Hn).$$
We can compute the soliton part
\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi_S|^2 + |\psi_S|^2 + m^2 f_\omega^2 - \frac{2}{p+1} f_\omega^{p+1} \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} \rho^2 \omega^2 |u|^2 f_\omega^2 + |\nabla_x \phi_S|^2 + \frac{\partial_z^2 \phi_S}{\partial x^2} + \rho^2 \omega^2 f_\omega^2 + \rho^2 \omega^2 f_\omega^2 + m^2 f_\omega^2 - \frac{2}{p+1} f_\omega^{p+1} \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} 2 \rho^2 \omega^2 f_\omega^2 + (m^2 - \omega^2) f_\omega^2 + \left( \frac{1}{3} + \frac{2}{3} \rho^2 \right) |\nabla_x f|^2 - \frac{2}{p+1} f_\omega^{p+1} \, dx
\]
\[
= \rho \left( \omega^2 \|f_\omega\|^2_{L^2} + \frac{1}{3} \|\nabla_x f\|^2_{L^2} \right) = \rho B.
\]

Using the identity (16) and the orthogonality condition (45), we can eliminate the crossing terms
\[
\langle \nabla_x \phi_S, \nabla_x (e^{i\theta} v) \rangle_{dx} + \langle \psi_S, e^{i\theta} w \rangle_{dx} + \langle m^2 \phi_S, e^{i\theta} v \rangle_{dx} - \langle f_\omega^p, v_1 \rangle_{dx}
\]
\[
= \langle \psi_S, e^{i\theta} w \rangle_{dx} + \langle -\Delta_x \phi_S + m^2 \phi_S - e^{i\theta} f_\omega^p, e^{i\theta} v \rangle
\]
\[
= \langle D_{\lambda} \phi_S \cdot V(\lambda), e^{i\theta} w \rangle - \langle D_{\lambda} \psi_S \cdot V(\lambda), e^{i\theta} v \rangle = 0.
\]

Combining all these together, we can write
\[
\mathcal{H}(t) = \rho B + \frac{1}{2} \int_{\mathbb{R}^3} \rho' q_{e}^{-2} |v|^2 + \langle q_{e}^{-2} (g^e)^{kl} \partial_k (e^{i\theta} v), \partial_l (e^{i\theta} v) \rangle + |w|^2 - f_\omega^{p-1} |v|^2
\]
\[
- (p - 1) f_\omega^{p-1} v_1^2 \, dx + \text{Err}(Hn) + \text{Err}(Hq), \tag{77}
\]

where the errors terms satisfy (75), (76).

We proceed by arguing that the energy of the radiation term \((v, w)\) is positive definite. Using the Fermi coordinate system, we have turned the stability of solitons on a slowly varying background into that on a small perturbation of Minkowski space. In Minkowski space, observe that the associated quantities \(\mathcal{H}, \Pi_k, Q\), as functionals of the solution \(\phi\) of (7), are conserved and that the soliton \(\phi_S\) is critical point of the Lagrange
\[
\mathcal{H} - u^k \Pi_k - \frac{\omega}{\rho} Q
\]
with Lagrange multipliers \(u^k, \frac{\omega}{\rho}\). To study the stability of stable solitons, we expand the above Lagrange around \(\phi_S\). It turns out that the soliton part is convex function of \(\omega, u\). The crossing term is vanishing since \(\phi_S\) is critical point. We remark here that this is also given by the orthogonality condition (18), which has been shown above. The quadratic part gives the energy of the remainder \(\phi - \phi_S\), which is proven to be positive definite under the orthogonality condition. One thus can obtain the orbital stability result of Theorem 3 in Minkowski space, see the work of D. Stuart [25]. We use a similar idea in our case by considering
\[
\mathcal{H}(t) - (u^k(0) + u^k_0) \cdot \Pi_k(t) - \frac{\omega}{\rho(0)} Q(t).
\]

Having the decomposition formulae (71), (74), (77), we can group the soliton part
\[
\rho B (1 - (u(0) + u_0) \cdot (u + u_0)) - \frac{\omega^2}{\rho(0)} \|f_\omega\|^2_{L^2(\mathbb{R}^3)}.
\]

We denote the quadratic part
\[
E(t) = \frac{1}{2} \int_{\mathbb{R}^3} \rho' q_{e}^{-2} |v|^2 + \langle q_{e}^{-2} (g^e)^{kl} \partial_k (e^{i\theta} v), \partial_l (e^{i\theta} v) \rangle + |w|^2 - f_\omega^{p-1} |v|^2
\]
\[
- (p - 1) f_\omega^{p-1} v_1^2 + 2 \frac{\omega}{\rho} < iw, v > - 2 \frac{\omega}{\rho(0)} < (p, q_e)^{-1} (g^e)^{0l} i \partial_l (e^{i\theta} v), e^{i\theta} v >
\]
\[
+ 2 < e^{i\theta} w, \partial_k (e^{i\theta} v) > (u^k + u^k_0) - 2(u^k(0) + u^k_0) < (p, q_e)^{-1} (g^e)^{0l} \partial_l (e^{i\theta} v), e^{i\theta} v > \, dx
\]

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as the energy of the radiation term \((v, w)\). Since \(|u(0) + u_0| < 1\), \(\rho(0) \geq 1\), \(|\omega| \leq m\), using Corollary 3 and condition (55), we can conclude that

\[
E(t) \geq \frac{1}{2} \int_{\mathbb{R}^3} m^2 |v|^2 + |\nabla_x (e^{i\Theta} v)|^2 + |w|^2 - f_\omega^{-1} |v|^2 - (p - 1) f_\omega^{-1} v^2 + \frac{2\omega}{\rho} \langle iw, v \rangle \tag{78}
\]

\[
+ 2 < e^{i\Theta} w, \partial_k (e^{i\Theta} v) \rangle \geq (u^k + u_0^k) dx - C_0 \varepsilon \|v\|_{H^1}^2
\]

\[
= \frac{1}{2\rho} (\|w + \rho (u + u_0) \nabla_z v - i\rho \omega |v|^2_{L^2(\mathbb{R}^3)} + <w, L_v v_i + v_1 + v_2, L_{-v_2} + d_z> - C_0 \varepsilon \|v\|_{H^1}^2)
\]

\[
= \frac{1}{2\rho} E_0(t) - C_0 \varepsilon \|v\|_{H^1}^2
\]

where the constant \(C_0\) depends only on \(h, m\). In the last line, we must view \(v, w\) as functions of \(z\) instead of \(x\). The operators \(L_+, L_-\) are defined in (19). We hence can write

\[
E(t) = (u^k(0) + u_0^k) \cdot \Pi_k(t) - \frac{\omega}{\rho(0)} Q(t) - \rho B(1 - (u(0) + u_0) \cdot (u + u_0)) - \frac{\omega^2}{\rho(0)} \|w\|_{L^2}^2
\]

\[
+ E(t) + \text{Err}_1 + \text{Err}_2,
\]

where \(\text{Err}_1\) consists of errors in \(E(t), \Pi_k(t), Q(t)\) satisfying the estimate

\[
|\text{Err}_1| = \left| \text{Err}(Hn) + \text{Err}(Hq) - (u^k(0) + u_0^k) \text{Err}(\Pi_k) - \frac{\omega}{\rho(0)} \text{Err}(Q) \right| \lesssim \|v\|_H^3 + (1 + |\xi|^2) e^2 + \varepsilon \|v\|_{H^1}^2 + \varepsilon \|w\|_{L^2}^2,
\]

\[
|\text{Err}_2| = \left| - (u^k - u^k(0)) < e^{i\Theta} w, \partial_k (e^{i\Theta} v) >_{dx} - \left( \frac{\omega^2}{\rho(0)} - \frac{\omega}{\rho(0)} \right) \langle iw, v \rangle_{dx} \right| \lesssim |u(t) - u(0)| \|w\|_{L^2} \|v\|_{H^1}.
\]

Now denote

\[
dE(t) = \rho B(1 - (u(0) + u_0) \cdot (u + u_0)) - \frac{\omega^2}{\rho(0)} \|w\|_{L^2}^2 - \frac{B(0)}{\rho(0)} + \frac{\omega(0)\omega}{\rho(0)} \|w\|_{L^2}^2(0)\tag{81}
\]

and

\[
\text{Err}_3 = H(0) - (u^k(0) + u_0^k) \Pi_k(0) - \frac{B(0)}{\rho(0)} - \frac{\omega}{\rho(0)} (Q(t) - Q(0)) \langle w, \Pi_k(0) \rangle.
\]

We obtain

\[
E(t) + dE(t) = H(t) - H(0) - (u^k(0) + u_0^k) \cdot (\Pi_k(t) - \Pi_k(0))
\]

\[
- \frac{\omega}{\rho(0)} (Q(t) - Q(0)) - \text{Err}_1 - \text{Err}_2 + \text{Err}_3.
\]

Since

\[
\rho(0) B(0) - |u(0) + u_0|^2 \rho(0) B(0) - \frac{B(0)}{\rho(0)} = 0,
\]

by Lemma 4, we can show that

\[
|\text{Err}_3| \lesssim \varepsilon^2.
\]

Since \((v, w)\) satisfies the orthogonality condition (45), Proposition 2 implies that \(E_0(t)\) is equivalent to the energy \(\|v\|_{H^1(\mathbb{R}^3)}^2 + \|w\|_{L^2(\mathbb{R}^3)}^2\) when \(\varepsilon\) is sufficiently small. Hence from (78), \(E(t)\) is equivalent to \(\|v\|_{H^1(\mathbb{R}^3)}^2 + \|w\|_{L^2(\mathbb{R}^3)}^2\) up to an error. The right hand side of (82) will be proven to be small in the next section. To obtain estimates for \(\|v\|_{H^1(\mathbb{R}^3)}^2 + \|w\|_{L^2(\mathbb{R}^3)}^2\), it suffices to show that \(dE\) is positive definite. In fact, we can show
Lemma 9. Let \( \lambda(t) \in \Lambda_{\delta_0} \). If \( \delta_0 \) is sufficiently small, depending only on \( \lambda(0) \), then
\[
d\mathcal{H}(t) \geq c(|\omega(t) - \omega(0)|^2 + |u(t) - u(0)|^2)
\]
for some positive constant \( c \) depending only on \( \lambda(0) \).

Proof. As functions of \( u \), we can compute
\[
D_u(\rho(1 - (u(0) + u_0) \cdot (u + u_0)))((0)) = D\rho(0)\rho(0)^{-2} - \rho(0)(u(0) + u_0) = 0,
\]
\[
D_u^2(\rho(1 - (u(0) + u_0) \cdot (u + u_0)))((0)) = D^2\rho(0)(1 - |u(0) + u_0|^2) - 2D\rho(0) \cdot (u(0) + u_0)
= \rho(0)I + \rho(0)^3(u(0) + u_0) \cdot (u(0) + u_0),
\]
which implies that the Hessian of \( \rho(1 - u(0) \cdot u) \) with respect to \( u \) is positive definite. Hence, if \( \delta_0 \) is sufficiently small, depending only on \( \lambda(0) \), then
\[
\rho(1 - (u(0) + u_0) \cdot (u + u_0)) \geq \rho(0)^{-1} + c_1|u(t) - u(0)|^2, \quad |u(t) - u(0)| \leq \delta_0
\]
for some constant \( c_1 \) depending only on \( u(0) + u_0 \).

For the other part on the right hand side of (81), we rely on the properties of the ground state \( f_\omega \). Recall the definition (57) of \( B(t) \) and the scaling property (11) as well as the energy identities, we can show that
\[
D_u \left( B(t) - \omega^2\|f_\omega\|_{L^2}^2 - B(0) + \omega(0)\omega\|f_\omega\|_{L^2}^2(0) \right)(0) = 0,
\]
\[
D_u^2 \left( B(t) - \omega^2\|f_\omega\|_{L^2}^2 - B(0) + \omega(0)\omega\|f_\omega\|_{L^2}^2(0) \right)(0)
= \left( \frac{6 - 2p}{p - 1} \omega(0)^2 - m^2 \right)(m^2 - \omega(0)^2)^{\frac{p-2}{p-1}} \|f\|_{L^2}^2 > 0.
\]
The positivity is due to the assumption \( \lambda(0) \in \Lambda_{\delta_0} \). Hence if \( \delta_0 \) is sufficiently small, we can conclude that
\[
B(t) - \omega^2\|f_\omega\|_{L^2}^2 - B(0) + \omega(0)\omega\|f_\omega\|_{L^2}^2(0) \geq c_2|\omega(t) - \omega(0)|^2, \forall \omega(t) \in (\omega_0 - \delta_0, \omega_0 + \delta_0)
\]
for some constant \( c_2 \) depending only on \( \lambda(0) \).

Therefore, we have shown that
\[
d\mathcal{H}(t) \geq \rho(0)^{-1}B(t) + c_1|u(t) - u(0)|^2B(t) - \rho(0)^{-1}(\omega^2\|f_\omega\|_{L^2}^2 + B(0) - \omega(0)\omega\|f_\omega\|_{L^2}^2(0))
\geq c_1|u(t) - u(0)|^2B(t) + c_2\rho(0)^{-1}|\omega(t) - \omega(0)|^2 \geq c(|\omega(t) - \omega(0)|^2 + |u(t) - u(0)|^2)
\]
for some constant \( c \) depending only on \( \lambda(0) \) if \( \delta_0 \) is sufficiently small.

We now choose \( \delta_0 \) such that Lemma 9 holds and the set \( \Lambda_{\delta_0} \) defined in line (52) is nonempty. By Proposition 2, we have shown the left hand side of (82) is bounded below as follows
\[
E(t) + d\mathcal{H} \geq c(\|v\|_{H^1}^2 + \|w\|_{L^2}^2(t) + |\omega(t) - \omega(0)|^2 + |u(t) - u(0)|^2)
\]
for some constant \( c \) depending only on \( \lambda(0) \) if \( \epsilon \) is sufficiently small.

6.4.4 Energy Estimates

We estimate the right hand side of (82) in this section by using the integral formulae (64), (66), (69). Denote the soliton part
\[
\mathcal{H}^S_\alpha(t) = \int_0^t \int_{\mathbb{R}^3} T_{\mu\nu}[\phi_S]^{\alpha} \pi_{\mu\nu}^\partial \psi_S dx dt, \quad \alpha \in \{0, 1, 2, 3\},
\]
where \( \partial_\alpha \phi_S \) should be replaced with \( \psi_S \).
Proposition 6. We have
\[ |\mathcal{H}(t) - \mathcal{H}(0) + \mathcal{H}_0^S(t)| \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + ||v||_{H^1}^2 + ||w||_{L^2}^2 ds, \]
\[ |\Pi(t) - \Pi(0) - \mathcal{H}_0^S(t)| \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + ||v||_{H^1}^2 + ||w||_{L^2}^2 ds, \]
\[ |Q(t) - Q(0)| \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + ||v||_{H^1}^2 + ||w||_{L^2}^2 ds \]
if \( \epsilon \) is sufficiently small.

Proof. We decompose the energy momentum tensor \( T_{\mu\nu}[\phi] \) as follows
\[ T_{\mu\nu}[\phi] = T_{\mu\nu}[\phi_S] + T_{\mu\nu}[\phi_S, \phi - \phi_S] + QT_{\mu\nu}[\phi - \phi_S], \]
where
\[ T_{\mu\nu}[\phi_S, \phi - \phi_S] = \langle \partial_{\mu}\phi_S, \partial_{\nu}((\phi - \phi_S)) > + \langle \partial_{\nu}\phi_S, \partial_{\mu}((\phi - \phi_S)) > \]
\[ - g^\alpha_{\mu\nu} \langle \partial^\alpha\phi_S, \partial_{\alpha}((\phi - \phi_S)) > + V(\phi_S, \phi - \phi_S). \]

Using Lemma 7 to estimate the nonlinear terms in \( QT_{\mu\nu}[\phi - \phi_S] \), we can show that
\[ \left| \int_0^t \int_{\mathbb{R}^3} T_{\mu\nu}[\phi - \phi_S](\pi^\alpha_{\mu\nu}) dv \right| \lesssim \epsilon \int_0^t ||v||_{H^1}^2 + ||w||_{L^2}^2 ds, \quad \forall \alpha \in \{0, 1, 2, 3\}. \]

For the linear term \( T_{\mu\nu}[\phi_S, \phi - \phi_S] \), we apply Lemma 8 to get
\[ \left| \int_0^t \int_{\mathbb{R}^3} T_{\mu\nu}[\phi_S, \phi - \phi_S](\pi^\alpha_{\mu\nu}) dv \right| \lesssim \epsilon^2 \int_0^t (||v||_{H^1} + ||w||_{L^2})(1 + |\xi(s)|) ds, \quad \forall \alpha \in \{0, 1, 2, 3\}. \]

For the soliton part, first we can drop the volume factor \( d^2 \) as
\[ \left| \int_0^t \int_{\mathbb{R}^3} T_{\mu\nu}[\phi_S](\pi^\alpha_{\mu\nu})(d^2 - 1) dx \right| \lesssim \epsilon^3 \int_0^t 1 + |\xi(s)|^2 ds. \]

The above estimates rely on the unknown upper bound of \( |\xi(t)| \). Although it is bounded by \( C_2 \) as a bootstrap assumption, we do not want \( C_2 \) to appear in the following estimates. Otherwise, we need smallness on \( t \) in order to close our bootstrap argument. We will instead use Gronwall’s inequality to control \( |\xi| \). First, we use the relation \( \xi = u + \dot{\eta} \) together with Corollary 4 to control \( \xi \) in terms of \( u, v, w \). We can control \( \dot{\eta} \) by Corollary 4 as follows
\[ |\dot{\eta}|^2 \lesssim |\gamma|^2 \lesssim (C_2 \epsilon)^4 + ||v||_{H^1}^2 \lesssim \epsilon^3 + ||v||_{H^1}^2. \]

If \( C_2 \epsilon < 1 \), then we can show that
\[ |\xi(t)|^2 \lesssim |\xi(0)|^2 + t \int_0^t |u|^2 + |\gamma|^2 ds \lesssim 1 + \epsilon^{-1} \int_0^t |u|^2 + ||v||_{H^1}^2 ds, \quad \forall t \leq T/\epsilon. \quad (84) \]

Plug this into above estimates. We conclude that
\[ |\mathcal{H}(t) - \mathcal{H}(0) + \mathcal{H}_0^S(t)| \lesssim \epsilon^3 \int_0^t 1 + |\xi|^2 ds + \epsilon \int_0^t ||v||_{H^1}^2 + ||w||_{L^2}^2 ds \]
\[ \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + ||v||_{H^1}^2 + ||w||_{L^2}^2 ds. \]
Similarly, we can show that
\[
\left| \Pi_k(t) - \Pi_k(0) - \mathcal{H}_S^k(t) - \int_0^t \int_{\mathbb{R}^3} (m_0)^{\mu\nu} \partial_\nu \beta T_{\mu k}^{m_0}[\phi_S] dx ds \right| \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds,
\]
\[
\left| Q(t) - Q(0) - \int_0^t \int_{\mathbb{R}^3} -i \partial_\mu \beta (m_0)^{\mu\nu} \partial_\nu \phi_S, \phi_S > dx ds \right| \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds,
\]
where we recall that \(m_0\) is the Minkowski metric, \(\beta = p^{-1} q_e\), and we denote \(T_{\mu\nu}^{m_0}[\phi_S]\) as the energy momentum tensor associated to the Minkowski metric \(m_0\). We thus can conclude the proposition if we can control the two integrals
\[
\int_0^t \int_{\mathbb{R}^3} (m_0)^{\mu\nu} \partial_\nu \beta T_{\mu k}^{m_0}[\phi_S] dx ds, \quad \int_0^t \int_{\mathbb{R}^3} < i \partial_\mu \beta (m_0)^{\mu\nu} \partial_\nu \phi_S, \phi_S > dx ds.
\]
The only smallness in the above integrals is contributed by \(\partial \beta\), which has size \(\epsilon\). To prove that they are higher order error terms, we have to exploit the properties of the solitons \(\phi_S\). The key observation is that the solitons travel in the direction \(\partial_t + u_0^1 \partial_k\). More precisely, we compute the second integral
\[
< i \partial_\mu \beta (m_0)^{\mu\nu} \partial_\nu \phi_S, \phi_S >= - \partial_\mu \beta < i \psi_S, \phi_S > + \partial_k \beta < i \partial_k \phi_S, \phi_S > = \rho_0 \mathcal{F}_F^2(z)(\partial_t + (u + u_0) \nabla)\beta.
\]
Observing that \(z = A_u(x - \xi - u_0 t)\), \(f_u(z)\) travels along the geodesic \((t, u_0 t)\), thus integration by parts may allow us to gain extra smallness. For any function \(F(z, \lambda(t))\) independent of \(\Theta\), we have the identity
\[
(\partial_t + (u + u_0) \nabla)\beta_0 \cdot F = \partial_t (\beta_0 F) - \beta_0 D_\lambda F \cdot (V(\lambda) + \dot{\gamma}) + (u + u_0) \nabla_\lambda \beta_0 \cdot F = \partial_\mu (\beta_0 F) - \beta_0 D_\lambda F \cdot \gamma - \beta_0 (u + u_0) D_\xi F + (u + u_0) \nabla_\lambda \beta_0 \cdot F \quad (85)
\]
where we have used \(D_\xi F = - D_x F\) as \(z = A_u(x - \xi - u_0 t)\). This key observation (85) is of particular importance in this paper. It allows us to prove Theorem 4 under the sharp condition \(q > 1\). Now let \(\beta_0 = \beta - 1, F = \rho_0 \mathcal{F}_F^2(z)\). The inequality (85) yields the estimate
\[
\left| \int_0^t \int_{\mathbb{R}^3} < i \partial_\mu \beta (m_0)^{\mu\nu} \partial_\nu \phi_S, \phi_S > dx ds \right| \lesssim \epsilon^2 (1 + |\xi(t)|^2) + \int_0^t |\gamma| \epsilon^2 (1 + |\xi(s)|^2) ds
\]
\[
\lesssim \epsilon^2 (1 + \epsilon^{-1} \int_0^t |u|^2 + \|v\|^2_{H^1} ds) + C_2^2 \epsilon^2 \int_0^t (C_2 \epsilon)^2 + \|v\|^2_{H^1} ds
\]
\[
\lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds
\]
for all \(t \leq T/\epsilon\) by Corollary 4 and the bootstrap assumption (53), \(C_2^2 \epsilon < 1\). We thus conclude the first and the third inequality of this proposition.

To show the second inequality of this proposition, it suffices to estimate the first integral above. Similarly, we can compute
\[
(m_0)^{\mu\nu} \partial_\nu \beta T_{\mu k}^{m_0}[\phi_S] = (m_0)^{\mu\nu} \partial_\nu \left( (\beta - 1) T_{\mu k}^{m_0}[\phi_S] \right) = (\beta - 1)(m_0)^{\mu\nu} \partial_\nu T_{\mu k}^{m_0}[\phi_S]
\]
\[
= (m_0)^{\mu\nu} \partial_\nu \left( (\beta - 1) T_{\mu k}^{m_0}[\phi_S] \right) - (\beta - 1) < - \partial_t \psi_S + \Delta_\lambda \phi_S - m^2 \phi_S + |\phi_S|^{p-1} \phi_S, \partial_k \phi_S >
\]
\[
= (m_0)^{\mu\nu} \partial_\nu \left( (\beta - 1) T_{\mu k}^{m_0}[\phi_S] \right) + (\beta - 1) < D_\lambda \psi_S \cdot \gamma, \partial_k \phi_S >
\]
by using the identity (16) and the relation \(\lambda = V(\lambda) + \dot{\gamma}\). Hence according to Lemma 8, we can show that
\[
\left| \int_0^t \int_{\mathbb{R}^3} (m_0)^{\mu\nu} \partial_\nu \beta T_{\mu k}^{m_0}[\phi_S] dx ds \right| \lesssim \epsilon^2 (1 + |\xi(t)|^2) + \int_0^t |\gamma| \epsilon^2 (1 + |\xi(s)|^2) ds
\]
\[
\lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds.
\]
We hence have proven the proposition. \(\Box\)
This proposition allows us to control the right hand side of (82).

**Corollary 5.** We have
\[
\left| \mathcal{H}(t) - \mathcal{H}(0) - (u^k(0) + u_0^k) \cdot (\Pi_k(t) - \Pi_k(0)) - \frac{\omega}{\rho(0)} (Q(t) - Q(0)) \right| \\
\lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds.
\]

**Proof.** Notice that $T^{\mu
u}[\phi_S]$ is a function of $(z, \lambda(t))$ and is independent of $\Theta$. By applying (85), we can show that
\[
\left| \int_0^t \int_{\mathbb{R}^3} T^{\mu
u}[\phi_S](\partial_t + (u + u_0)\nabla_x)(g^\tau)_{\mu\nu} dxds \right| \lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds.
\]
Since $\pi^{\alpha_\mu} = \frac{1}{2} \partial_\alpha (g^\tau)_{\mu\nu}$, we have
\[
\left| \mathcal{H}_S^0 (t) + (u(0) + u_0)^k \mathcal{H}_S^k (t) \right| \lesssim \left| \int_0^t \int_{\mathbb{R}^3} T^{\mu
u}[\phi_S](\partial_t + (u + u_0)\nabla_x)(g^\tau)_{\mu\nu} dxds \right| \\
+ \left| \int_0^t \int_{\mathbb{R}^3} T^{\mu
u}[\phi_S](u(s) - u(0))^k \partial_k (g^\tau)_{\mu\nu} dxds \right| \\
\lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds + \epsilon^2 \int_0^t (1 + |\xi|)(|u| + \epsilon) ds \\\n\lesssim \epsilon^2 + \epsilon \int_0^t |u|^2 + \|v\|^2_{H^1} + \|w\|^2_{L^2} ds,
\]
where we have used $|u(s) - u(0)| \lesssim |u(s)| + \epsilon$ by Lemma 4. Then the corollary follows from Proposition 6. \hfill \Box

### 6.4.5 Proof of Proposition 4 and Theorem 5

We are now able to improve the bootstrap assumptions and to conclude Proposition 4 and Theorem 5. Denote
\[
\mathcal{E}(t) := |u|^2 + |\omega(t) - \omega(0)|^2 + \|w\|^2_{L^2} + \|v\|^2_{H^1}.
\]
According to Proposition 2 and Lemma 9, we have
\[
\mathcal{E}(t) + d\mathcal{H}(t) \geq c(1 - C_0 \epsilon) \mathcal{E}(t)
\]
for some positive constants $c$, $C_0$ independent of $\epsilon$, $C_2$. Hence by (82), (79), (80), (83) and Corollary 5, for sufficiently small $\epsilon$, we can show that
\[
\mathcal{E}(t) \lesssim \epsilon^2 (1 + |\xi|^2) + \epsilon \int_0^t \mathcal{E}(s) ds + \mathcal{E}(t)^{\frac{3}{2}} + \epsilon \mathcal{E}(t) \\\n\lesssim \epsilon^2 + \epsilon \int_0^t \mathcal{E}(s) ds + \mathcal{E}(t)^{\frac{3}{2}} + \epsilon \mathcal{E}(t),
\]
where we have used inequality (84) to control $|\xi|^2$. Since the implicit constant is independent of $\epsilon$ and initially $\mathcal{E}(0) \lesssim \epsilon^2$, using Gronwall’s inequality, we have
\[
\mathcal{E}(t) \lesssim \epsilon^2, \quad \forall t \leq T/\epsilon.
\]
Let $C_3$ be the universal implicit constant appeared before. By our notation, $C_3$ depends on $h$, $m$, $\lambda_0$, $T$ and is independent of $\epsilon$, $C_2$. In particular, we have
\[
\mathcal{E}(t) = |u(t)|^2 + |\omega(t) - \omega(0)|^2 + \|v\|^2_{H^1}(t) + \|w\|^2_{L^2}(t) \leq C_3 \epsilon^2, \quad \forall t \leq T/\epsilon.
\]

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By Corollary 4, this implies that

\[
|\xi(t)| \leq |\xi(0)| + \int_0^t |u| + |\dot{\gamma}| ds
\]

\[
\leq C_3 \epsilon + C_3 \int_0^t |C_3^2 \epsilon + C_2^2 \epsilon^2 + \|v\|^2_{H_1} ds
\]

\[
\leq C_3 \epsilon + C_3^2 T + C_3 T + C_3^2 \epsilon T,
\]

where we let \( \epsilon \) to be small such that \( C_4^2 \epsilon < 1 \). Take

\[
C_2 = C_3 + C_3^2 T + C_3 T + 2C_3^2 T.
\]

If

\[
\epsilon \leq \min\{C_4^{-4}, \frac{1}{10} C_2^{-1} \delta_1, 1\},
\]

then for all \( t \leq T/\epsilon \), we have

\[
\|v\|_{H^1} + \|w\|_{L^2} \leq \mathcal{E}(t)^{1/2} \leq C_2 \epsilon \leq \frac{1}{2} \delta_1,
\]

\[
|\xi| \leq C_0 \epsilon + C_0^2 T + C_0 T + C_0^3 \epsilon T \leq C_2.
\]

This improves the bootstrap assumptions (54), (53). Therefore we can conclude that Proposition 4 follows from Corollary 4.

For Theorem 4, notice that there is a unique short time solution \( \phi(t, x) \) on \([0, t^*) \times \mathbb{R}^3 \). Then Lemma 6 implies that the modulation equations (51) admits a local solution \( \lambda(t) \), \( t \in [0, t^*] \subset [0, t^*) \). Proposition 4 then shows that \( \phi(t, x), \lambda(t) \) satisfy (39), (40) with a constant \( C \) independent of \( \epsilon \). Since Proposition 4 holds for any \( t^* \leq T/\epsilon \), we can conclude that the solution \( \phi(t, x), \lambda(t) \) can be uniquely extended to \([0, T/\epsilon] \times \mathbb{R}^3 \) and satisfy estimates (39), (40).

### 6.5 Estimates of Higher Sobolev Norms

In the previous section, we have proven the orbital stability of stable solitons of equation (36) on a slowly varying background in the energy space \( H^1 \times L^2 \). To solve the full Einstein equations (1) and to obtain a \( C^1 \) spacetime \(([0, t^*] \times \mathbb{R}^3, g) \), we need higher Sobolev estimates for the matter field \( \phi \). In this section, we prove Proposition 3 and Corollary 1.

Since we already have a solution \( \phi(t, x) \) and a curve \( \lambda(t) \) satisfying estimates (39), (40), we use energy estimates to obtain the higher Sobolev estimates by considering the equation of the remainder \( v \). However, to simply the argument in the sequel and to avoid taking fourth order derivative of \( \gamma(t) \), we choose a modified curve \( \tilde{\lambda}(t) \in \Lambda_{\text{stab}} \), defined as the integral curve of \( V(\lambda) \), that is

\[
\partial_t \tilde{\lambda} = V(\lambda), \quad \tilde{\lambda}(0) = (\omega_0, \theta_0, 0, u_0).
\]

Using this modified curve \( \tilde{\lambda}(t) \), we decompose the solution \( \phi \) as follows

\[
\phi(t, x) = \phi_S(\tilde{\lambda}; x) + e^{i\Theta}(\tilde{\lambda}) \tilde{v}.
\]

Then we claim that Proposition 3 is reduced to the following estimates.

**Proposition 7.** Assume the initial data \( \phi_0 \in H^2, \phi_1 \in H^2 \). Let \( \epsilon_1 \) be defined in Proposition 3. Then

\[
\sum_{|\alpha| \leq 3} \|\partial^\alpha \tilde{v}(t, \cdot)\|_{L^2} \lesssim \max\{\epsilon, \epsilon_1\}, \quad \forall t \leq T/\epsilon.
\]
In fact, if Proposition 7 holds, observing that
\[ |\lambda - \hat{\lambda}| \lesssim \int_0^t |\gamma| ds + |\lambda(0) - \hat{\lambda}(0)| \lesssim \epsilon^2 t + \epsilon, \quad \forall t \leq \frac{T}{\epsilon}, \]

we have
\[ \partial_t \Theta(\check{\lambda}) = \frac{\omega}{\rho} + \rho_0 \omega_0 u_0 (u_0 + u), \quad \nabla_x \Theta(\check{\lambda}) = -\rho_0 \omega_0 u_0, \]

then we have
\[ \|\phi - \phi_S(x; \lambda)\|_{H^3} = \|\phi_S(x; \check{\lambda}) + e^{i \Theta(\check{\lambda})} \tilde{v} - \phi_S(x; \lambda)\|_{H^3} \lesssim |\lambda(t) - \check{\lambda}(t)| + \|\tilde{v}\|_{H^3} \lesssim \max\{\epsilon, \epsilon_1\}. \]

This partially explains Proposition 3 as \( \|\partial_t^4 (\phi - \phi_S(x; \lambda))\|_{L^2} \) requires taking higher order derivatives of the modulation curve \( \lambda(t) \). We will estimate the higher derivatives of \( \lambda(t) \) in next section.

Although we have modified the curve \( \lambda(t) \), the remainder \( \tilde{v} \) is still small in \( H^1 \).

**Lemma 10.** We have
\[ \|\tilde{v}\|_{H^1} + \|\partial_t \tilde{v}\|_{L^2} \lesssim \epsilon, \quad \forall t \in [0, T/\epsilon]. \]

**Proof.** Since \( \|v\|_{H^1} + \|w\|_{L^2} \lesssim \epsilon \), by using the decomposition (41), we can show that
\[ \|e^{i\Theta(\check{\lambda})} \tilde{v}\|_{H^1} = \|\phi_S(\lambda; x) + e^{i\Theta(\check{\lambda})}(p, d_e)^{-1} v - \phi_S(\check{\lambda}; x)\|_{H^1} \lesssim \epsilon, \]
\[ \|e^{i\Theta(\check{\lambda})} \partial_t \tilde{v}\|_{L^2} = \|\partial_t (\phi - \phi_S(\check{\lambda}; x)) - i\partial_t \Theta(\check{\lambda}) e^{i\Theta(\check{\lambda})} \tilde{v}\|_{L^2} \]
\[ = \|D_\lambda \phi_S(\lambda; x) \cdot V(\lambda) + e^{i\theta} (p, d_e)^{-1} w - D_\lambda \phi_S(\check{\lambda}; x) \cdot V(\lambda) - i\partial_t \Theta(\check{\lambda}) e^{i\Theta(\check{\lambda})} \tilde{v}\|_{L^2} \lesssim \epsilon. \]

\[ \blacksquare \]

**Remark 8.** The reason that we consider the modified curve \( \check{\lambda} \) is that we must avoid taking the fourth derivative of \( \lambda \). Otherwise, we have to take third derivative of the nonlinearity \( |\phi|^{p-1} \phi \), which is impossible since \( p \) is assumed to be less than 3. Using the modified curve guarantees that when we differentiate the equation of the radiation term \( \tilde{v} \) twice, we only need third derivative of \( \lambda \). We remark here that \( \check{\lambda} \) still depends on \( \lambda \).

### 6.5.1 Estimates for Higher Derivatives of \( \lambda(t) \)

Since \( \lambda(t) \) satisfies the modulation equations (51), by differentiating the equations, we are able to derive estimates for derivatives of \( \lambda(t) \). It turns out that we first have to estimate the derivatives of the nonlinearity \( N(\lambda) \).

**Lemma 11.** Let \( N(\lambda) \) be defined in line (49). Assume \( p \geq 2 \). Then for any vector field \( Y \) on \( [0, T/\epsilon] \times \mathbb{R}^3 \), we have
\[ |Y \nabla(\lambda)| \lesssim (|v| + |v|^{p-1})(|Y f_\omega| + |Y v| + |v|), \]
\[ |Y^2 \nabla(\lambda)| \lesssim (1 + \|v\|_{L^\infty} + |Y \ln f_\omega|)(|Y^2 f_\omega| + |Y v|^2 + |v|^2) + (|v| + |v|^{p-1})(|Y^2 f_\omega| + |Y^2 v|) \]
\[ + |Y^2 d_e|(|v|^2 + |v|^p). \]

**Proof.** It follows by direct calculations and the properties of \( f_\omega \) summarized in Theorem 2. \( \blacksquare \)

Using this lemma together with the modulation equations (51), we are able to estimate the higher order derivatives of the modulation curve \( \lambda(t) \).

**Proposition 8.** Let \( \gamma(t) \) be defined in line (17). Assume \( \gamma(t) \) satisfies the modulation equations (51). Then we have
\[ |\dot{\gamma}| \lesssim \epsilon^2, \]
\[ |\partial_t^3 \gamma| \lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \epsilon \|D_\lambda \phi_S X^2 v\|_{L^2}. \]
**Proof.** Differentiate the modulation equations (51), we get the ODE for \( \dot{\gamma} \)

\[
(D + D_1 + D_2)\dot{\gamma} + \partial_t(D + D_1 + D_2)\dot{\gamma} = \partial_t F(t; \lambda(t)).
\]

Since \( |\dot{\lambda}(t)| \lesssim 1, |\partial g| \lesssim 1 \), we can show that

\[
|\partial_t D| + |\partial_t D_1| \lesssim |\dot{\lambda}| + |\partial g^t| \lesssim 1
\]

according to the definition of \( D, D_1 \) given in line (50). For \( \partial_t D_2 \), we have to use the equation of \( \phi \). In fact, by (36), (47), we can show that

\[
|\partial_t D_2| \lesssim |\dot{\lambda}| \cdot \|w\|_{L^2} + |\partial_t \phi S, \partial_t (e^{\phi} w) > dx| + |\dot{\lambda}| \cdot \|v\|_{L^2} + |\partial_t \phi S, \partial_t (a_0 (\phi - \phi S)) > dx |
\]

\[
\lesssim 1 + \|D_2^2 \phi S, (\partial_t - \partial_t \phi S) a_1 > dx| + \|D_2^2 \phi S, a_0 (\partial_t \phi - D_\lambda \phi S(V(\lambda) + \dot{\gamma})) > dx |
\]

\[
\lesssim 1 + \|D_2^2 \phi S, H(t, x) + \Delta \phi - b(m^2 \phi - |\phi|^{p-1} \phi) > dx| + \|D_2^2 \phi S, a_0 (\partial_t \phi - D_\lambda \phi S) > dx |
\]

\[
\lesssim 1 + \|\partial_t (a_0 D_2^2 \phi S), \partial_t \phi > dx| + \|\nabla_x D_2^2 \phi S, \nabla_x \phi > dx|
\]

\[
\lesssim 1 + \|v\|_{H^1} + \|w\|_{L^2} \lesssim 1.
\]

We hence have shown

\[
|\partial_t (D + D_1 + D_2)| \lesssim 1.
\]

Then Lemma 6 implies that estimates |\( \dot{\gamma} \)| \( \lesssim \epsilon^2 \) follow if we can show that

\[
|\partial_t F(t; \lambda(t))| \lesssim \epsilon^2. \tag{87}
\]

By the definition (48) of \( F(t; \lambda(t)) \), it suffices to estimate

\[
< bD_2 \phi S, e^{\phi} \partial_t N(\lambda) > dx, \quad < \dot{a}_1 D_2 \phi S, \partial_t > dx, \quad < a_0 D_2 \phi S, \phi - \phi S > dx.
\]

All the other terms can be estimated similarly to \( F(t; \lambda(t)) \) in Lemma 5. We first consider the nonlinear term, which is in fact the main term in \( \partial_t F(t; \lambda(t)) \) as other terms are errors from the slowly varying metric \( g^t \). The key observation is that for vector field \( X = \partial_t + u_0 \nabla_x \) and any \( C^1 \) function \( F_1 \), we have

\[
\|X F_1(f_\omega(z))\| \lesssim \|\partial_t + u_0 \nabla_x\| f_\omega(z) = \|D_\omega f_\omega + \nabla_x f_\omega \| + u_0 \nabla_x f_\omega A_u
\]

\[
\lesssim \|\dot{\gamma}\| + \|\nabla_x f_\omega (A_u u_0 + A_u (-\dot{\xi} - u_0))\|
\]

\[
\lesssim \epsilon^2 + \|u\| \lesssim \epsilon \tag{88}
\]

by using Theorem 5 as well as the relation \( \dot{\xi} = u + \dot{\gamma} \). Hence by Lemma 11 and Lemma 7, we can show

\[
|< bD_2 \phi S, \partial_t N(\lambda) > dx| \lesssim |< bD_2 \phi S, XN(\lambda) > dx| + |< bD_2 \phi S, D_\lambda \nabla_x N(\lambda) > dx|
\]

\[
\lesssim \|D_2 \phi S (|v| + |v|^{p-1}) \| Xf_{\omega} + \|Xv| + |v| \| \|_{L^1} + |< D_\lambda \nabla_x (bD_2 \phi S), N(\lambda) > dx |
\]

\[
\lesssim \|D_2 \phi S \cdot X \|_{L^2} + \epsilon (\|v\|_{L^2} + \|v\|_{H^1}^2) + \|v\|_{H^1} + \|v\|_{L^2} \]
\]

\[
\lesssim \epsilon^2 + \epsilon \|D_2 \phi S \|_{L^2} \lesssim \epsilon^2 + \|D_2 \phi S \|_{L^2} + \|D_2 \phi S \|_{L^2} \tag{89}
\]

We must remark here that using the decomposition (41) we have

\[
|\partial_{v} v| \lesssim |v| + |\dot{\theta} v|.
\]

Although \( \dot{\theta} \) depends on \( z \), \( |\dot{\theta} D_2 \phi S| \) decays exponentially in \( z \) by Theorem 2.

For the second term \( < \dot{a}_1 D_2 \phi S, \partial_t > dx \), we use equations (36), (47) and then use integration by parts. We can bound

\[
|< \dot{a}_1 D_2 \phi S, \partial_t > dx| = |< \partial_t \ln a_1 D_2 \phi S, a_0 \partial_{t} \partial_{t} \partial_{t} \phi + b^t \partial_{t} \partial_{t} \partial_{t} \phi + \Delta \phi - b(m^2 \phi - |\phi|^{p-1} \phi) > dx |
\]

\[
\lesssim \epsilon^2 + |< \partial_{t} (a_0 \partial_{t} \partial_{t} \partial_{t} \phi) + b^t \partial_{t} \partial_{t} \partial_{t} \phi > dx| + \|< \nabla_x (\partial_{t} \ln a_1 D_2 \phi S), \nabla_x \phi > dx |
\]

\[
\lesssim \epsilon^2 + \|\partial_{t}^{2} g^t D_2 \phi S \|_{L^2} \lesssim \epsilon^2 + \|\partial_{t}^{2} (g^t + h^t) \|_{L^2} + \|\partial_{t}^{2} h^t D_2 \phi S \|_{L^2} \lesssim \epsilon^2
\]

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by using the assumption (37) and the fact that $h^c(t, x) = h(\epsilon t, \epsilon x) \in C^2$.

The third term $< d_0 D\lambda \psi_S, \phi - \phi_S >_{dx}$ can be estimated similarly. We can show that
\[
< d_0 D\lambda \psi_S, \phi - \phi_S >_{dx} \lesssim \epsilon^2 + \|\partial^2 g' D\lambda \phi_S\|_{L^2} \lesssim \epsilon^2.
\]
Therefore, we have shown (87). Hence $|\tilde{\eta}| \lesssim \epsilon^2$.

Having proven $|\tilde{\eta}| \lesssim \epsilon^2$, to estimate the third order derivative of $\gamma$, differentiate the modulation equations (51) twice
\[
(D + D_1 + D_2) \partial^3 t \gamma + 2 \partial_t (D + D_1 + D_2) \gamma + \partial_{tt} (D + D_1 + D_2) \gamma = \partial_{tt} F(t; \lambda(t)).
\]
Similarly to estimating $\partial_t F(t; \lambda(t))$ carried out above, we can show that
\[
|\partial_{tt} (D + D_1 + D_2)| \lesssim |\lambda| + |\lambda| + \|\partial^2 (g' - h')\|_{L^2} \lesssim 1.
\]
It hence suffices to estimate $\partial_t F(t; \lambda(t))$. The strategy is similar to that of $\partial_t F(t; \lambda(t))$. The main term is that with derivative hitting on the nonlinearity $< e^{-i\Theta} b D\lambda \phi_S, \partial_t \tilde{\eta} N(\lambda) >$, which by Lemma 11 and by inequalities (88), (89), can be estimated as follows
\[
< e^{-i\Theta} b D\lambda \phi_S, \partial_t \tilde{\eta} N(\lambda) >_{dx} \lesssim < e^{-i\Theta} b D\lambda \phi_S, \partial_t \tilde{\eta} N(\lambda) >_{dx} \\
\lesssim \epsilon \|D\lambda \phi_S X^2 \nu\|_{L^2} + \|D\lambda \phi_S \partial^2 g' (|v|^2 + |v|^p)\|_{L^1} \\
+ |< u_0 \nu_x (e^{-i\Theta} b D\lambda \phi_S), N(\lambda) >_{dx} | + |< u_0 \nu_x (e^{-i\Theta} b D\lambda \phi_S), N(\lambda) >_{dx} |
\lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \|D\lambda \phi_S X^2 \nu\|_{L^2} + \|\partial^2 (g' - h')\|_{L^2} \|v\|_{H^1}^2 \\
\lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \|D\lambda \phi_S X^2 \nu\|_{L^2}.
\]
For those terms when the derivative hits on $\partial_t \phi$, we rely on the equation (36) together with the identity (47) and then use integration by parts to pass the derivative to the metric $g'$. We hence can show that
\[
|\partial_t F(t; \lambda(t))| \lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \|D\lambda \phi_S X^2 \nu\|_{L^2} + \|\partial^3 (g') (|D\lambda \phi_S| + |D\lambda \psi_S|)\|_{L^1} \\
\lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \|D\lambda \phi_S X^2 \nu\|_{L^2}.
\]
Then Lemma 6 yields the estimate
\[
|\partial^3 t \gamma| \lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \|D\lambda \phi_S X^2 \nu\|_{L^2}.
\]

\[ \square \]

6.5.2 Linearized Equation for $\tilde{v}$ and Energy Estimates

Using the modified curve $\bar{\lambda}(t)$ and the corresponding decomposition (86), we can find the equation for $\tilde{v}$
\[
L_\epsilon \tilde{v} + N(\bar{\lambda}) + \tilde{F} = 0, 
\]
where
\[
\tilde{F} = e^{-i\Theta} (\Box g' \phi_S \bar{\lambda}, x) - m^2 \phi_S + |\phi_S|^{p-1} \phi_S + i \Box g' \Theta \cdot \bar{v}.
\]
For any complex function $v(t, x) = v_1(t, x) + iv_2(t, x)$, the linear operator $L_\epsilon$ is defined as follows
\[
L_\epsilon v = \Box g' v + A(\bar{\lambda}) v + 2i \partial_\mu (\partial_\nu v - m^2 v + f_{\nu v}^{p-1}(z)v + (p-1)f_{\nu v}^{p-1}(z)v_1
\]
with
\[
A(\bar{\lambda}) = -(g')^{\mu\nu} \partial_\mu \Theta (\bar{\lambda}) \partial_\nu \Theta (\bar{\lambda}) = e^{-i\Theta} \Box g' e^{i\Theta} - i \Box g' \Theta.
\]
We have the following energy estimates for the linear operator $L_\epsilon$. 

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Lemma 12. For all $t \leq T/\epsilon$, we have
\[
\|\partial v\|_{L^2(t)}^2 \lesssim \|\partial v(0, x)\|_{L^2}^2 + \|v(0, x)\|_{L^2}^2 + \epsilon^{-1} \int_0^t \|L_v v\|_{L^2(s)}^2 ds + \sup_{0 \leq s \leq t} \|v\|_{L^2}^2.
\]

Proof. Recall the energy momentum tensor $\tilde{T}_{\mu\nu}[v]$ for the operator $\Box_{\gamma^r}$
\[
\tilde{T}_{\mu\nu}[v] = \langle \partial_\mu v, \partial_\nu v \rangle - \frac{1}{2} g_{\mu\nu} < \partial^r v, \partial_r v > .
\]
For any vector field $Y$, we have the identity
\[
D^\mu(\tilde{T}_{\mu\nu}[v]Y^\nu) = \tilde{T}^\mu_{\mu\nu}[v]Y^\nu + < \Box g_{r}, Y(\phi) > .
\]
Take $Y = X = \partial_t + u_0^k \partial_k$. Then integrate on the region $[0, t] \times \mathbb{R}^3$. We obtain
\[
\int_{\mathbb{R}^3} \tilde{T}_{\mu\nu}[v]X^\mu n^\nu d\sigma(t) = \int_{\mathbb{R}^3} \tilde{T}_{\mu\nu}[v]X^\mu n^\nu d\sigma(0) + \int_0^t \int_{\mathbb{R}^3} \tilde{T}_{\mu\nu}[v]X^\mu n^\nu + < \Box g_{r}, X(v) > d\text{vol}.
\]
By replacing $\Box g_{r} v$ with $L_v v$, we must estimate the other terms respectively. First, consider the term $< 2i\partial_\mu \Theta \cdot \partial^\mu v, X(v) > d\sigma$. We use integration by parts. We can write
\[
< 2i\partial_\mu \Theta \cdot \partial^\mu v, X(v) > d\sigma = d_2^2 \partial^\mu \Theta < 2i\partial_\mu v, X(v) > + d_2^2 \partial^\mu \Theta (\partial_\mu < iv, X(v) > - X < iv, \partial_\mu v >)
\]
\[
= \partial_\mu (d_2^2 \partial^\mu \Theta < iv, X(v) >) - X (d_2^2 \partial^\mu \Theta < iv, \partial_\mu v >) - \partial_\mu (d_2^2 \partial^\mu \Theta) < iv, X(v) > + X(d_2^2 \partial^\mu \Theta) < iv, \partial_\mu v >.
\]
Recall the definition of $\lambda$. We can compute
\[
\partial_t \Theta = \omega - \rho_0 \omega_0 (|u_0|^2 + u_0 u) = \rho_0 \omega_0 + \rho_0 \omega_0 u_0 u + \frac{\omega}{\rho} - \frac{\omega_0}{\rho_0},
\]
\[
\nabla_\nu \Theta = -\rho_0 \omega_0 u_0.
\]
Therefore
\[
\left| \int_0^t \int_{\mathbb{R}^3} < 2i\partial_\mu \Theta \cdot \partial^\mu v, X(v) > d\text{vol} \right| = \left| \int_0^t \int_{\mathbb{R}^3} < 2i\partial_\mu \Theta \cdot \partial^\mu v, X(v) > d_2^2 dx dt \right|
\]
\[
\lesssim \left| \int_{\mathbb{R}^3} < iv, \partial^\mu \Theta X(v) - \partial^\mu \Theta \partial_\mu v > d\sigma \right| + \int_0^t \int_{\mathbb{R}^3} \|\partial(d_2^2 \partial^\mu \Theta)v \partial v\| d\text{vol}
\]
\[
\lesssim \|v\|_{L^2(t)} \|\partial v\|_{L^2(t)} + \|v\|_{L^2(0)} \|\partial v\|_{L^2(0)} + \epsilon \int_0^t \|v\|_{L^2}^2 + \|\partial v\|_{L^2}^2 ds.
\]
For the other terms, notice that for any real function $F_1$, we have
\[
2 \int_0^t \int_{\mathbb{R}^3} < F_1 v, X(v) > d\text{vol} = \int_0^t \int_{\mathbb{R}^3} F_1 X(|v|^2) d_2^2 dx dt
\]
\[
= \int_{\mathbb{R}^3} F_1 |v|^2 |d\sigma| \left|_0^t \right. - \int_0^t \int_{\mathbb{R}^3} X(F_1 d_2^2) |v|^2 dx dt.
\]
When $F_1 = f^{\nu \theta}_{\omega_0}^{-1}$, since $|XF_1| \lesssim \epsilon$ by (88), we have
\[
\|X(f^{\nu \theta}_{\omega_0}^{-1} d_2^2) |v|^2 \|_{L^1(\mathbb{R}^3)} \lesssim \epsilon \|v\|^2_{L^2} \lesssim \epsilon \|v\|^2_{H^1}.
\]
When $F_1 = A(\gamma)$, since $|\partial \Theta| \lesssim 1$, $|\partial^2 \Theta| \lesssim |\gamma| \lesssim \epsilon^2$, we have
\[
\|X(A(\gamma) d_2^2) |v|^2 \|_{L^1} = \|X((g^r)^{\nu \theta} \partial_\nu \Theta \partial_r \Theta d_2^2) |v|^2 \|_{L^1} \lesssim \epsilon \|v\|^2_{L^1} \lesssim \epsilon \|v\|^2_{L^2}.
\]
Combine all these together. We have shown that

$$\left| \int_0^t \int_{\mathbb{R}^3} < A(\gamma)v - m^2 v + f^{p-1}_{\omega_0}(z)v + (p - 1)f^{p-1}_{\omega_0}(z)v_1, X(v) > \text{dvol} \right|$$

$$\lesssim \|v\|_{L^2(t)}\|v\|_{H^1(t)} + \|v\|_{L^2(0)}\|v\|_{H^1(0)} + \epsilon \int_0^t \|v\|^2_{H^1} \text{d}s.$$

Now, since the unit normal vector field \((-h^*)^{0\mu} - \bar{\tau}(h^*)^{0\mu}\partial_\mu\) to the hypersurface \(\mathbb{R}^3\) as well as the vector field \(X = \partial_t + u_0^0\partial_k\) are timelike with respect to the metric \(h^*\), we can conclude that the unit normal vector field \(n = (-g^*)^{0\mu} - \bar{\tau}(g^*)^{0\mu}\partial_\mu\) together with the vector field \(X = \partial_t + u_0^0\partial_k\) are also timelike for the metric \(g^*\) if \(\epsilon\) is sufficiently small. Therefore, there exists a positive constant \(c\), depending only \(h, u_0\), such that

$$T_{\mu\nu}[v]X^\mu n^\nu \geq c|\partial v|^2.$$ 

Since \(|\pi^X_{\mu\nu}| \lesssim \epsilon\), the energy identity then implies that

$$\|\partial v\|^2_{L^2(t)} \lesssim \|\partial v(0, x)\|^2_{L^2} + \|v(0, x)\|^2_{L^2} + \epsilon \int_0^t \|\partial v\|^2_{L^2}(s) \text{d}s + \epsilon^{-1} \int_0^t \|L_\epsilon v\|^2_{L^2} \text{d}s$$

$$+ \|v\|^2_{L^2} + \epsilon \int_0^t \|v\|^2_{H^1}(s) \text{d}s, \ \forall t \leq T/\epsilon.$$ 

The Lemma then follows by using Gronwall’s inequality. \(\square\)

### 6.5.3 \(H^2\) Estimates

Commuting the equation (90) with the vector field \(X = \partial_t + u_0^0\partial_k\), we obtain

$$L_\epsilon X\tilde{v} + [X, L_\epsilon]\tilde{v} + XN(\tilde{v}) + X\tilde{F} = 0. \quad (92)$$

Since we have computed

$$|\partial \Theta| \lesssim 1, \ |\partial^2 \Theta| \lesssim |\tilde{\gamma}| \lesssim \epsilon^2, \ |\partial A(\tilde{\gamma})| \lesssim \epsilon, \ |Xf_\omega| \lesssim \epsilon,$$

using Lemma 10, we can estimate the commutator

$$\|X, L_\epsilon\tilde{v}\|_{L^2} \lesssim \|(X, \partial g')\tilde{v}\|_{L^2} + \|X\partial^\mu \Theta \cdot \partial_\mu \tilde{v}\|_{L^2} + \|X f^{p-1}_{\omega_0}(\tilde{v} + (p - 1)\tilde{v}_1)\|_{L^2} + \|X A(\gamma)\tilde{v}\|_{L^2}$$

$$\lesssim \epsilon \|\partial^2 \tilde{v}\|_{L^2} + \|\partial^2 (g')\partial \tilde{v}\|_{L^2} + \epsilon(\|\partial \tilde{v}\|_{L^2} + \|\tilde{v}\|_{L^2})$$

$$\lesssim \epsilon \|\partial^2 \tilde{v}\|_{L^2} + \|\partial^2 (g' - h^*)\|_{L^2}\|\partial \tilde{v}\|_{L^2} + \epsilon^2$$

$$\lesssim \epsilon \|\partial^2 \tilde{v}\|_{L^2} + \|\partial^2 \tilde{v}\|_{H^1}\|\partial \tilde{v}\|_{H^1} + \epsilon^2$$

$$\lesssim \epsilon \|\partial^2 \tilde{v}\|_{L^2} + \epsilon^2 \|\partial \tilde{v}\|_{H^1}^2 + \epsilon^2 \lesssim \epsilon \|\partial^2 \tilde{v}\|_{L^2} + \epsilon^2.$$

For the nonlinearity \(XN(\tilde{\lambda})\), Lemma 11 yields the estimates

$$\|XN(\tilde{\lambda})\|_{L^2} \lesssim \|\tilde{v} + |\tilde{v}|^{p-1}(|Xf_\omega| + |X\tilde{v}|)\|_{L^2}$$

$$\lesssim \|\tilde{v} + |\tilde{v}|^{p-1}\|_{L^2}(\epsilon + \|X\tilde{v}\|_{L^2})$$

$$\lesssim \|\tilde{v}\|_{H^1} + |\tilde{v}|^{p-1}(\epsilon + \|\partial^2 \tilde{v}\|_{L^2}) \lesssim \epsilon^2 + \epsilon \|\partial^2 \tilde{v}\|_{L^2}.$$

As for \(X\tilde{F}\), first using the identity (16) we have

$$\Box \phi_S(x; \tilde{\lambda}) - m^2 \phi_S(x; \tilde{\lambda}) + |\phi_S|^{p-1} \phi_S = V(\tilde{\lambda})D^2_\lambda \phi_S V(\tilde{\lambda}) - V(\lambda)D^2_\lambda \phi_S V(\lambda) - D_\lambda \phi_S \cdot \partial_1 V(\lambda).$$

By Theorem 5, we can show that

$$|V(\lambda) - V(\tilde{\lambda})| = |(0, \frac{\omega}{\rho}, u + u_0, 0) - (0, \frac{\omega_0}{\rho_0}, u_0, 0)| \lesssim \epsilon.$$
The key observation is that for all $k \leq 4$, we have
\[ \|X \nabla^k f_\omega\|_{L^2} \lesssim \epsilon, \]
which can be proven similarly to (88). In particular, we have
\[ \|X (e^{-i \theta} D^2_{\lambda} \phi_S)\|_{L^2} \lesssim \epsilon. \]

Therefore, by Proposition 8, we can estimate $X \tilde{F}$ as follows
\[ \|X \tilde{F}\|_{L^2} \lesssim \|X (V(\tilde{\lambda}) e^{-i \theta} D^2_{\lambda} \phi_S V(\tilde{\lambda}) - V(\lambda) e^{-i \theta} D^2_{\lambda} \phi_S V(\lambda) - e^{-i \theta} D_{\lambda} \phi_S \cdot \partial_t V(\lambda))\|_{L^2} \]
\[ + \|X (e^{-i \theta} (\Theta - \bar{\Theta}) \phi_S)\|_{L^2} + \|X (\Theta \cdot \tilde{v})\|_{L^2} \]
\[ \lesssim \gamma |\gamma| + \epsilon \|X (e^{-i \theta} D^2_{\lambda} \phi_S)\|_{L^2} + \|\partial^2 (g^t)(\nabla^2 f_{\omega_0} + |\nabla f_{\omega_0}| + f_{\omega_0})\|_{L^2} \]
\[ + \|(g^t - m_0)(|\nabla^3 f_{\omega_0}| + |\nabla^2 f_{\omega_0}| + |\nabla f_{\omega_0}| + f_{\omega_0})\|_{L^2} + \|\partial^2 (g^t) \tilde{v}\|_{L^2} + \|\partial (g^t) \tilde{v}\|_{L^2} \]
\[ \lesssim \epsilon^2 + \|\partial^2 \tilde{v}\|_{H^1} + \epsilon \|\tilde{v}\|_{L^2} \lesssim \epsilon^2. \]

The energy estimate Lemma 12 then implies that
\[ \|\partial X \tilde{v}\|_{L^2}^2 \lesssim \max\{\epsilon^2, \epsilon_1^2\} + \epsilon^{-1} \int_0^t \|[[L_x, X] \tilde{v}]\|_{L^2}^2 + \|X N(\tilde{\lambda})\|_{L^2}^2 + \|X \tilde{F}\|_{L^2}^2 ds \quad (93) \]
\[ \leq \max\{\epsilon^2, \epsilon_1^2\} + \epsilon \int_0^t \|\partial^2 \tilde{v}\|_{L^2}^2 ds \]
for all $t \leq T/\epsilon$, where $\epsilon_1$ is the size of the initial data given in Proposition 3.

To derive the full estimates for $\|\partial^2 \tilde{v}\|_{L^2}$, merely commuting the equation with the vector field $X$ is not sufficient. The key point to obtain estimates (93) is that the soliton $\phi_S$ travels along the timelike geodesic $(t, u_0 t)$ or quantitatively the vector field $X = \partial_t + u_0 \nabla_x$ acting on $f_\omega$ leads to the estimates $|X f_\omega| \lesssim \epsilon$. For general vector field, estimates (93) may not hold. To retrieve the full estimates $\|\partial^2 \tilde{v}\|_{L^2}$, we rely on the following elliptic estimates.

**Lemma 13.** Let $A^{ij}(x) \in C^\alpha(\mathbb{R}^3)$ for some positive constant $0 < \alpha < 1$. Assume that $A^{ij}$ is uniformly elliptic. That is $\exists K$ such that
\[ K^{-1}|y|^{2} \leq A^{ij}(x)y_i y_j \leq K|y|^2, \quad \forall x, y \in \mathbb{R}^3. \]
Then there exists a constant $C$ such that
\[ \|\phi\|_{H^2} \leq C\|A\|_{C^\alpha}^2 (\|A^{ij} \partial_{ij} \phi\|_{L^2} + \|\phi\|_{L^2}) \]
for any $\phi \in H^2(\mathbb{R}^3)$. Here $\|A\|_{C^\alpha} = \sup_{i,j} \|A^{ij}\|_{C^\alpha}$.

**Proof.** Let $\chi$ be a cut-off function supported in the ball $B_2$ with radius 2 and equal to 1 in the unit ball $B_1$. Then the elliptic estimates show that
\[ \|\phi\|_{H^2(B_1)} \leq C\|A\|_{C^\alpha} (\|A^{ij} \partial_{ij} (\chi \phi)\|_{L^2} + \|\phi\|_{L^2(B_2)}) \leq C\|A\|_{C^\alpha} (\|A^{ij} \partial_{ij} \phi\|_{L^2(B_2)} + \|\phi\|_{H^1(B_2)}) \]
for some constant $C$ independent of $\phi, \chi$. The above estimate holds for any ball $B_2$. We cover the whole space $\mathbb{R}^3$ with radius 1 balls such that every point is covered for at most 10 times. Add all the estimates, we conclude that
\[ \|\phi\|_{H^2} \leq C\|A\|_{C^\alpha} (\|A^{ij} \partial_{ij} \phi\|_{L^2} + \|\phi\|_{H^1}). \]
Interpolating between $H^2$ and $L^2$, we have
\[ \|\phi\|_{H^1} \leq C\|\phi\|_{H^2}^{1/2} \|\phi\|_{L^2}^{1/2} \leq \frac{1}{2} C^{-1} \|A\|_{C^\alpha} \|\phi\|_{H^2} + 2C\|A\|_{C^\alpha} \|\phi\|_{L^2}. \]

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Plug this into the above inequality. We get
\[ \|\phi\|_{H^2} \leq C\|A\|^2_{C^\infty} (\|A^{ij}\partial_{ij}\phi\|_{L^2} + \|\phi\|_{L^2}). \]

Using this lemma and estimates (93), we are able obtain the $H^2$ estimates of $\tilde{v}$. First, write the wave operator $\Box_{g^r}$ as follows
\[ \Box_{g^r} \tilde{v} = ((g^r)^{kl} + (g^r)^{00} u_0^k u_0^l - 2(g^r)^{0k} u_0^l)\partial_{kl} \tilde{v} + ((g^r)^{00}(\partial_t - u_0 \nabla) + 2(g^r)^{0k} \partial_k)X \tilde{v} + d_r^{-2}\partial_{\mu}(d_r^2(g^r)^{\mu
u})\partial_{\nu} \tilde{v}. \]
Since $X = \partial_t + u_0^k \partial_k$ is timelike with respect to the metric $g^r$, we conclude that the $3 \times 3$ matrix
\[ (g^r)^{kl} + (g^r)^{00} u_0^k u_0^l - 2(g^r)^{0k} u_0^l \]
is positive definite. That is there exists a constant $K$, depending only on $h, u_0$ such that
\[ K^{-1}|y|^2 \leq \sum_{1 \leq k,l \leq 3} y_k y_l ((g^r)^{kl} + (g^r)^{00} u_0^k u_0^l - 2(g^r)^{0k} u_0^l) \leq K|y|^2, \quad \forall t \leq T/\epsilon, \quad x, y \in \mathbb{R}^3. \]
Hence by Lemma 13 and equation (90), we can show that
\[ \|\tilde{v}\|_{H^2} \lesssim \epsilon + \|\partial X \tilde{v}\|_{L^2}. \]
Then inequality (93) implies that
\[ \|\partial X \tilde{v}\|_{L^2}^2 \lesssim \max\{\epsilon^2, \epsilon_1^2\} + \epsilon \int_0^t \epsilon^2 + \|\partial X \tilde{v}\|_{L^2}^2 ds \lesssim \max\{\epsilon^2, \epsilon_1^2\} + \epsilon \int_0^t \|\partial X \tilde{v}\|_{L^2}^2 ds, \quad \forall t \leq T/\epsilon. \]
By using Gronwall’s inequality, we conclude that
\[ \|\partial^2 \tilde{v}\|_{L^2}^2 \lesssim \max\{\epsilon^2, \epsilon_1^2\} + \|\partial X \tilde{v}\|_{L^2}^2 \lesssim \max\{\epsilon^2, \epsilon_1^2\}, \quad \forall t \leq T/\epsilon. \]

### 6.5.4 $H^3$ Estimates

Having the $H^2$ estimates, we first can improve the estimate of $\partial_t^3 \gamma$ obtained in Proposition 8. According to the decomposition (41) corresponding to the curve $\lambda(t)$, we can show that
\[ \|v\|_{L^\infty} \lesssim \|v\|_{H^2} \lesssim \|a_0 e^{-i\Theta(\lambda)}(\phi_S(\lambda; x) - \phi_S(\tilde{\lambda}; x)) - e^{i\Theta(\tilde{\lambda})} \tilde{v}\|_{H^2} \lesssim \|\lambda(t) - \tilde{\lambda}(t)\|_{H^2} + \|\partial^2 a_0 \tilde{v}\|_{L^2} \lesssim \max\{\epsilon, \epsilon_1\} + \|\tilde{v}\|_{H^2} \|\partial^2 a_0\|_{H^1} \lesssim \max\{\epsilon, \epsilon_1\}. \]
We also need to estimate $X^2 \tilde{v}$, which, as having pointed out previously, does not follow directly from the estimates of $\tilde{v}$. However, notice that
\[ \|D_\lambda \phi_S X^2(a_0 e^{-i\Theta(\lambda)})\|_{L^2} \lesssim 1 + \|X^2 a_0\|_{L^2} \lesssim 1, \]
\[ \|\phi_S(\lambda; x) - \phi_S(\tilde{\lambda}; x) - e^{i\Theta(\tilde{\lambda})} \tilde{v}\|_{L^\infty} \lesssim \|\tilde{v}\|_{L^\infty} + |\lambda(t) - \tilde{\lambda}(t)| \lesssim \max\{\epsilon, \epsilon_1\}. \]
By the decompositions (41), (86), we can show that
\[ \|D_\lambda \phi_S X^2 \|_{L^2} \lesssim \|D_\lambda \phi_S X^2 \|_{L^2} \lesssim \|D_\lambda \phi_S X^2(a_0 e^{-i\Theta(\lambda)}(\phi_S(\lambda; x) - \phi_S(\tilde{\lambda}; x) - e^{i\Theta(\tilde{\lambda})} \tilde{v}))\|_{L^2} \lesssim \max\{\epsilon, \epsilon_1\} + |\lambda(t) - \tilde{\lambda}(t)| + |\gamma| + |\tilde{\gamma}| \lesssim \max\{\epsilon, \epsilon_1\}. \]
Thus Proposition 8 implies that
\[ |\partial_t^3 \gamma| \lesssim \epsilon^2 (1 + \|v\|_{L^\infty}) + \epsilon \|D_\lambda \phi_S X^2 v\|_{L^2} \lesssim \epsilon \max\{\epsilon, \epsilon_1\}. \]
We proceed to estimate the $H^3$ norm of $\bar{v}$. Commute the equation (92) with the vector field $X = \partial_t + u^0_0 \partial_k$ again. We have the equation for $X^2\bar{v}$

$$L_\gamma X^2 \bar{v} + [X^2, \Box_{\gamma}] \bar{v} + X^2\mathcal{N}(\bar{\lambda}) + X^2\bar{F} = 0.$$  

(96)

By Lemma 11, we can estimate the nonlinearity

$$||X^2\mathcal{N}(\bar{\lambda})||_{L^2} \lesssim \epsilon^2 + ||X\bar{v}||^2 + ||\bar{v}||^2 + (||\bar{v}|| + ||\bar{v}||^{p-1})(||\bar{v}|| + ||X^2\bar{v}||) + ||X^2a_0||(|\bar{v}||^2 + |\bar{v}|^p)||_{L^2}$$

$$\lesssim \epsilon^2 + ||X\bar{v}||_{L^\infty}||X\bar{v}||_{L^2} + ||\bar{v}||^2 + \epsilon ||X^2\bar{v}||_{L^6} + ||X^2a_0||_{H^1}||\bar{v}||^2_{H^1}$$

$$\lesssim \epsilon^2 + \epsilon ||X\bar{v}||_{H^2} + ||\bar{v}||^2 + \epsilon ||X^2\bar{v}||_{H^2}$$

$$\lesssim \epsilon \max\{\epsilon, \epsilon_1\} + \epsilon ||\partial^2 X\bar{v}||_{L^2}.$$

For the commutator, using (88), we can show that

$$||X^2, L_\gamma \bar{v}||_{L^2} \lesssim ||X^2, \Box_{\gamma} \bar{v}||_{L^2} + ||X^2, A(\gamma)\Box \bar{v}||_{L^2} + ||X^2, \partial^2 \Phi \Box \bar{v}||_{L^2} + ||X^2, f_{\gamma}^{-1}||_{L^2}$$

$$\lesssim ||\partial(g')\partial^2 X\bar{v}||_{L^2} + ||\partial^2 (g')\partial^2 \bar{v}||_{L^2} + ||\partial^3 (g')\partial \bar{v}||_{L^2} + ||\partial^2 (g')\bar{v}||_{L^2}$$

$$+ ||\partial^2 (g')\bar{v}||_{L^2} + ||X^2 f_{\gamma}^{-1}||_{L^2} + ||X f_{\gamma}^{-1}||_{L^2}$$

$$\lesssim \epsilon ||\partial^2 X\bar{v}||_{L^2} + ||\partial^2 (g' - h')||L^\infty||\partial^2 \bar{v}|| + ||\partial \bar{v}|| + ||\bar{v}||_{L^2} + ||\partial^3 (g' - h')||L^2||\partial \bar{v}||_{L^\infty} + \epsilon^2$$

$$\lesssim \epsilon ||\partial^2 X\bar{v}||_{L^2} + ||\partial^2 (\bar{v} + \epsilon ||\partial^2 \bar{v}||_{H^1})||L^2 + ||\partial^3 \bar{v}||L^2||\partial \bar{v}||_{H^2} + \epsilon^2$$

$$\lesssim \epsilon ||\partial^2 X\bar{v}||_{L^2} + \epsilon ||\partial^3 \bar{v}||_{L^2} + \epsilon^2.$$

Using the improved estimate (95), similarly to $X\bar{F}$, we can estimate $X^2\bar{F}$ as follows

$$||X^2\bar{F}||_{L^2} \lesssim \epsilon^2 + ||\partial^2 \gamma||_{L^2} + ||\partial^2 g'\partial^2 \phi||_{L^2} + ||\partial^3 g'\partial \phi s||_{L^2} + ||\partial^2 g'\partial \bar{v}||_{L^2} + \epsilon \max\{\epsilon, \epsilon_1\} + ||\partial^3 (g' - h')||_{L^2} + ||\partial^2 (g' - h')||_{H^1}$$

Thus apply Lemma 12 to equation (96). We get

$$||\partial X^2\bar{v}||_{L^2} \lesssim \epsilon^2 + \epsilon ||\partial^2 X\bar{v}||_{L^2} + \epsilon ||\partial^2 \bar{v}||_{H^2} + \epsilon^2 ||\partial^3 \bar{v}||_{L^2} + \epsilon \max\{\epsilon, \epsilon_1\} + \epsilon \int_0^t ||\partial^3 \bar{v}||_{L^2} ds.$$

To retrieve the full estimates $||X^2\bar{v}||_{L^2}$, we apply Lemma 13 to the equation

$$L_\gamma \partial \bar{v} + [\partial, L_\gamma] \bar{v} + \partial \mathcal{N}(\bar{\lambda}) + \partial \bar{F} = 0.$$

We can show that

$$||\partial \bar{v}||_{H^2} \lesssim \max\{\epsilon, \epsilon_1\} + ||\partial X^2\bar{v}||_{L^2} + ||[\partial, L_\gamma] \bar{v} + \partial \mathcal{N}(\bar{\lambda}) + \partial \bar{F}||_{L^2}$$

$$\lesssim \max\{\epsilon, \epsilon_1\} + ||\partial^2 X^2\bar{v}||_{L^2} + ||[\partial, L_\gamma] \bar{v} + \partial \mathcal{N}(\bar{\lambda})||_{L^2}$$

$$\lesssim \max\{\epsilon, \epsilon_1\} + ||\partial^2 X^2\bar{v}||_{L^2}.$$

In particular, we have

$$||X \bar{v}||_{H^2} \lesssim \max\{\epsilon, \epsilon_1\} + ||\partial X^2\bar{v}||_{L^2}.$$

Therefore from the estimates for $\partial X^2\bar{v}$ we have obtained above, we can show that

$$||\partial X^2\bar{v}||_{L^2}^2 \lesssim \max\{\epsilon^2, \epsilon_1^2\} + \epsilon \int_0^t ||\partial^3 \bar{v}||_{L^2} ds \lesssim \max\{\epsilon^2, \epsilon_1^2\} + \epsilon \int_0^t ||\partial^3 X^2\bar{v}||_{L^2} ds$$

$$\lesssim \max\{\epsilon^2, \epsilon_1^2\} + \epsilon \int_0^t \partial X^2\bar{v}||_{L^2} ds, \quad \forall t \leq T/\epsilon.$$
Then Gronwall’s inequality implies that
\[ \|\partial^3 v\|_{L^2} \lesssim \max\{\epsilon, \epsilon_1\} \quad \forall t \in [0, T/\epsilon]. \]
This together with estimates (94) proves Proposition 7.

Finally, the estimates (95) imply that
\[ \|\partial^s (\phi - \phi_S(x; \lambda(t)))\|_{L^2} \lesssim \|\partial^s (\phi_S(x; \tilde{\lambda}(t)) + e^{i\Theta(\tilde{\lambda})} - \phi_S(x; \lambda(t)))\|_{L^2} \lesssim \max\{\epsilon, \epsilon_1\} + |\partial^2 \gamma| \lesssim \max\{\epsilon, \epsilon_1\} \]
for all \(|s| \leq 3\). Hence we have finished proving Proposition 3.

\section{Proof of the Main Theorem}

We use bootstrap argument to prove the main Theorem 4. Using the Fermi coordinate system, we have shown the existence of solution \( \phi \) of equation (36) as well as its properties in Theorem 5 and Proposition 3 under the assumption (37), which could be viewed as a bootstrap assumption for the full reduced Einstein equations (33). We consider the equations of \( \psi^s = g^s - h^s \) to improve this bootstrap assumption and thus to conclude the main Theorem 4.

\subsection{Estimates of the Metric \( g^s \)}

Let \((g^s, \phi)\) be a solution of the system (33) with initial data satisfying conditions (30), (35) on the space \([0, T/\epsilon] \times \mathbb{R}^3, h^s\). We have shown in Lemma 2 that the difference \( \psi^s = g^s - h^s \) satisfies the following hyperbolic system

\[ -(g^s)_{\alpha\beta} \partial_\alpha \psi^s_{\mu\nu} + \delta P_{\mu\nu} + \delta Z_{\mu\nu} + \delta Q_{\mu\nu} = 2\delta^2 (T_{\mu\nu} - \frac{1}{2} tr T \cdot g^s_{\mu\nu}), \]

where \( \delta Q_{\mu\nu}, \delta Z_{\mu\nu}, \delta P_{\mu\nu} \) are given in (34). We show in this subsection that

\textbf{Proposition 9.} If \( \psi^s = g^s - h^s \) satisfies condition (37), then

\[ \|\partial^{s+1} \psi^s\|_{L^2}(t) \lesssim \delta^2, \quad \forall t \leq T/\epsilon, |s| \leq 2. \]

The key observation that allows \( \delta \leq \epsilon^q, q > 1 \) is based on the fact that the energy momentum tensor \( T_{\mu\nu}[\phi] \) splits into soliton part, which travels along the timelike geodesic \((t, u_0 t)\), and the error term which is small by Proposition 3. When doing energy estimate, we multiply the equations by \( X \psi^s = (\partial_t + u_0 \nabla_x) \psi^s \). By using integration by parts, we can pass the derivative \( X \) to the soliton part of \( T_{\mu\nu}[\phi] \). This, according to (88), allows us to prove Proposition 9 for all \( \delta \leq \epsilon^q, q > 1 \).

\textbf{Proof.} Since the initial data \((\phi_0, \phi_1)\) satisfy condition (30), we conclude according to Theorem 5 and Proposition 3 that \( \phi \) decomposes as (86) associated to the modified curve \( \tilde{\lambda}(t) \) such that the remainder \( \tilde{v} \) satisfies the estimates

\[ \|\partial^s \tilde{v}\|_{L^2}(t) \lesssim \epsilon, \quad \forall |s| \leq 3, \quad t \leq T/\epsilon. \]

Using the modified decomposition (86), we can write

\[ T_{\mu\nu} - \frac{1}{2} tr T \cdot g^s_{\mu\nu} = < \partial_\mu \phi, \partial_\nu \phi > + \mathcal{V}(\phi) g^s_{\mu\nu} = T^S_{\mu\nu} + T^R_{\mu\nu} \]

with the soliton part given by

\[ T^S_{\mu\nu} = < \partial_\mu \phi_S(x; \tilde{\lambda}(t)), \partial_\nu \phi_S(x; \tilde{\lambda}(t)) > + \mathcal{V}(\phi_S(x; \tilde{\lambda}(t))) g^s_{\mu\nu} \]

\[ = \partial_\mu f_{\omega_0} \partial_\nu f_{\omega_0} + \partial_\mu \Theta(\tilde{\lambda}) \partial_\nu \Theta(\tilde{\lambda}) f_{\omega_0}^2 + \mathcal{V}(f_{\omega_0}) g^s_{\mu\nu}. \]
The error term $T_{\mu\nu}^R$ is small by Proposition 7. In fact, we can show that
\[
\sum_{|s| \leq 2} \| \partial^s T_{\mu\nu}^R \|_{L^2} (t) \lesssim \epsilon + \sum_{|s| \leq 2} \| \partial^{s+1} \psi \|_{L^2} + \| \partial^s (V(\phi) - V(\phi_S)) \|_{L^2} + \| \partial^2 g'(V(\phi) - V(\phi_S)) \|_{L^2} \\
\lesssim \epsilon + \| \partial^2 (g' - h') \|_{L^2} + \| V''(\phi) - V''(\phi_S) \|_{L^2} + \| \partial' \phi - \partial' \phi_S \|_{L^2} \lesssim \epsilon, \quad \forall t \leq T/\epsilon.
\]

Here we recall that $V(\phi)$ is given in line (5) and $p \geq 2$.

Since $X = \partial_t + u_0^h \partial_x$ is timelike, apply formula (63) with $Y = X, \beta = 1$ to the above hyperbolic system for $\psi_{\mu\nu}^s$ commuting with the vector field $\partial^s$. We obtain the energy estimates
\[
\| \partial^s \psi^s \|_{L^2}^2 (t) \lesssim \| \partial^s \psi^s \|_{L^2}^2 (0) + \int_0^t \left( \| \partial^2 g' \cdot \partial^2 \psi^s \|_{L^2} + \| \partial^s \left( \delta P_{\mu\nu} + \delta Z_{\mu\nu} + \delta Q_{\mu\nu} \right) \|_{L^2} \right) \| X \partial^s \psi^s \|_{L^2} \, ds \\
+ \epsilon \int_0^t \| \partial^s \psi^s \|_{L^2}^2 \, ds + \delta^2 \int_0^t \| \partial^s T_{\mu\nu}^R \|_{L^2} \| X \partial^s \psi^s \|_{L^2} \, ds + \delta^2 \int_0^t \int_{\mathbb{R}^3} \partial^s T_{\mu\nu}^S \cdot X \partial^s \psi^s_{\mu\nu} \, d\nuol, \]
where $|s_1| \leq |s|$. Recall that $h'(t, x) = h(t, \epsilon x)$. By assumptions (21), we conclude that
\[
\| |x| \partial^{s+2} h' \|_{L^\infty} + \| |x| (\partial h' \cdot \partial h') \|_{L^\infty} + \| \partial^{s+1} h' \|_{L^\infty} \lesssim \epsilon, \quad |\psi|^2 + |\partial \psi|^2 \lesssim \epsilon.
\]

By the definitions of $\delta Q_{\mu\nu}, \delta Z_{\mu\nu}, \delta P_{\mu\nu}$ given in (34), for $|s| \leq 2$, we can estimate
\[
\| \partial^s \delta Z_{\mu\nu} \|_{L^2} \lesssim \epsilon \sum_{s_1 \leq s-1} \| \partial^{s_1+1} \psi^s \|_{L^2} + \sum_{s_2 \leq s} \| |x| \partial^{s_2+2} h' \cdot |x|^{-1} \psi^s \|_{L^2} \lesssim \epsilon \sum_{s_1 \leq s-1} \| \partial^{s_1+1} \psi^s \|_{L^2},
\]
\[
\| \partial^s \delta P_{\mu\nu} \|_{L^2} \lesssim \epsilon \sum_{s_1 \leq s} \| \partial^{s_1+1} \psi^s \|_{L^2} + \| |x| \partial^s (\partial h' \cdot \partial h') \cdot |x|^{-1} \psi^s \|_{L^2} \lesssim \epsilon \sum_{s_1 \leq s} \| \partial^{s_1+1} \psi^s \|_{L^2},
\]
\[
\| \partial^s \delta Q_{\mu\nu} \|_{L^2} \lesssim \epsilon \sum_{s_1 \leq s} \| \partial^{s_1+1} \psi^s \|_{L^2} + \| |x| \partial^s (\partial h' \cdot \partial h') \cdot |x|^{-1} \psi^s \|_{L^2} + \| \partial^2 \psi^s \cdot \partial^2 \psi^s \|_{L^2} \lesssim \epsilon \sum_{s_1 \leq s} \| \partial^{s_1+1} \psi^s \|_{L^2} + \| \partial^2 \psi^s \|_{H^1}^2.
\]

Here we use Lemma 5 to bound $\| |x|^{-1} \psi^s \|_{L^2}$.

For the soliton part, first notice that with the modified curve $\tilde{\lambda}$, we can compute
\[
X f_{\omega_0} (z; \tilde{\lambda}) = -\nabla_z f_{\omega_0} A_{\omega_0} u, \quad |u(t)| \lesssim \int_0^t |\gamma| \, ds \lesssim \epsilon.
\]

Therefore by Proposition 8 and inequality (95), we have
\[
\| X \partial^s f_{\omega_0} (1 + |x|) \|_{L^2} (t) \lesssim |u| + |\gamma| + |\gamma| + |\partial^2 \gamma| \lesssim \epsilon, \quad \forall t \leq T/\epsilon, \quad |s| \leq 4.
\]

Using integration by parts and Lemma 5, we can show that
\[
\left| \int_0^t \int_{\mathbb{R}^3} \partial^s T_{\mu\nu}^S \cdot X \partial^s \psi^s_{\mu\nu} \, d\nuol \right| \lesssim \| \partial^s T_{\mu\nu}^S \cdot \partial^s \psi^s_{\mu\nu} \|_{L^1} (t) + \| \partial^s T_{\mu\nu}^S \cdot \partial^s \psi^s_{\mu\nu} \|_{L^1} (0) + \int_0^t \| X \partial^s T_{\mu\nu}^S \cdot \partial^s \psi^s_{\mu\nu} \|_{L^1} \, ds \\
\lesssim \| \partial^s T_{\mu\nu}^S (1 + |x|) \|_{L^2} \| \partial^s \psi^s_{\mu\nu} (1 + |x|) \|_{L^2} + \| \partial^s T_{\mu\nu}^S (1 + |x|) \|_{L^2} \| \partial^s \psi^s_{\mu\nu} (1 + |x|) \|_{L^2} \| d\nuol \\
+ \int_0^t \| X \partial^s T_{\mu\nu}^S (1 + |x|) \|_{L^2} \| \partial^s \psi^s_{\mu\nu} (1 + |x|) \|_{L^2} \| d\nuol \\
\lesssim \| \partial^{s+1} \psi^s \|_{L^2} (t) + \| \partial^{s+1} \psi^s \|_{L^2} (0) + \epsilon \int_0^t \| \partial^{s+1} \psi^s \|_{L^2} \, ds.
\]

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For \( \| \partial^2 g' \cdot \partial^2 \psi' \|_{L^2} \), note that
\[
\| \partial^2 g' \cdot \partial^2 \psi' \|_{L^2} \lesssim \epsilon^2 \| \partial^2 \psi' \|_{L^2} + \| \partial^2 \psi' \cdot \partial^2 \psi' \|_{L^2} \lesssim \epsilon^2 \| \partial^2 \psi' \|_{L^2} + \| \partial^2 \psi' \|_{H^1}^2.
\]
Hence by the conditions (35), we can show that
\[
\sum_{|s| \leq 2} \| \partial^s \psi' \|_{L^2}^2(t) \lesssim \delta^4 + \epsilon \sum_{|s| \leq 2} \int_0^t \| \partial^{s+1} \psi' \|_{L^2}^2 ds + \int_0^t \| \partial^{s+1} \psi' \|_{L^2}^2 ds + \delta^2 \int_0^t \| \partial^{s+1} \psi' \|_{L^2} ds
\]
\[
+ \delta^2 \| \partial^{s+1} \psi' \|_{L^2}(t).
\]
Since we have assumed
\[
\| \partial^{s+1} \psi' \|_{L^2} \leq 2\epsilon, \quad \forall t \leq T/\epsilon,
\]
Gronwall's inequality then implies that
\[
\| \partial^{s+1} \psi' \|_{L^2}(t) \lesssim \delta^2, \quad \forall t \leq T/\epsilon, \quad |s| \leq 2.
\]
Thus the proposition follows.

**7.2 Proof of Theorem 4**

The foliation of the spacetime is not preserved under the change of coordinate system constructed in Lemma 1. Since the argument of Theorem 5 relies on the new Fermi coordinate system, we first extend the given vacuum spacetime \((0, T] \times \mathbb{R}^3, h)\) to \((0, T + \delta_1] \times \mathbb{R}^3, h)\) for some small positive constant \(\delta_1\). This can be obtained due to the assumptions (21) together with the local existence result for Einstein equations [4]. The metric \(h\) still satisfies condition (21) but with some new constant \(K_0\). Therefore Lemma 1 implies that one can choose a Fermi coordinate system \((s, y) \in [0, c_0(T + \delta_1)] \times \mathbb{R}^3\) on a subspace \(M\) such that
\[
M = [0, T] \times \mathbb{R}^3 \subset M \subset [0, T + \delta_0] \times \mathbb{R}^3,
\]
where \(c_0 = C(u_h(0), h)\), depending only on the initial data \(\lambda_0 \in \Lambda_{\text{stab}}(0)\) (see the definition in the proof of Lemma 1). We now can identify \(M\) with the space \([0, c_0(T + \delta_1)] \times \mathbb{R}^3\) with the Fermi coordinate system \((s, y)\). Then on the rescaled space \([0, c_0(T + \delta_1)/\epsilon] \times \mathbb{R}^3\), the hyperbolic system (33) with initial data described in Lemma 2 admits a unique solution \((g', \phi)\) on \([0, t^*] \times \mathbb{R}^3\) for some small time \(t^*\). Moreover, Proposition 9 implies that under the assumption
\[
\sup_{0 \leq \lambda \leq t^*} \sum_{|\alpha| \leq 2} \| \partial^\alpha \psi' \|_{L^2}(s) \leq 2\epsilon^2,
\]
we in fact can show that
\[
\sup_{0 \leq \lambda \leq t^*} \sum_{|\alpha| \leq 2} \| \partial^\alpha \psi' \|_{L^2}(s) \leq C_3 \delta^2 \leq C_3 \epsilon^{2q}(or C_3 c_0^2 \epsilon^2 if \delta = c_0 \epsilon)
\]
for some constant \(C_3\) independent of \(\epsilon\). Note that \(q > 1\). Additional to the requirement on \(\epsilon\) in Theorem 5, if we choose \(\epsilon (or c_0)\) such that
\[
C_3 \epsilon^{2q-2}(or C_3 c_0^2) \leq 1,
\]
then we can improve the bootstrap assumption (37). This also implies that the solution \((g', \phi)\) of (33) can be extended to the whole space \([0, c_0(T + \delta_1)/\epsilon] \times \mathbb{R}^3\) such that
\[
\sum_{1 \leq |\beta| \leq 3} \| \partial^\beta (g' - h') \|_{L^2}(s) \lesssim \delta^2, \quad \forall s \leq c_0(T + \delta_1)/\epsilon,
\]
\[
\sum_{|\beta| \leq 3} \| \partial^\beta (\phi - \phi_S(y; \lambda'(s))) \|_{L^2}(s) \lesssim \epsilon, \quad \forall s \leq c_0(T + \delta_1)/\epsilon,
\]
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where we define the modulation curve $\lambda'(s)$ as follows

$$\lambda'(s) = (\omega(\epsilon s), \epsilon^{-1} \theta(\epsilon s), \epsilon^{-1} \zeta(\epsilon s) + u_0 s, u_0 + u(\epsilon s)) \in A_{\text{stab}}.$$

Moreover, this curve is close to the given time like geodesic $(s, u_0 s)$ in the sense that

$$|\zeta(s)| + |\omega(s) - \omega(0)| + |u(s)| \lesssim \epsilon, \quad \forall s \leq c_0(T + \delta_1).$$

We now use these results to construct solutions of the Einstein equations (1). Under the Fermi coordinate system, rescale the spacetime $([0, c_0(T + \delta_1)/c] \times \mathbb{R}^3, g^\epsilon, \phi)$ to $([0, c_0(T + \delta_1)] \times \mathbb{R}^3, g, \phi^\epsilon)$ in the following way

$$g(s, y) = g^\epsilon(s/\epsilon, y/\epsilon), \quad \phi^\epsilon(s, y) = \delta \phi(s/\epsilon, y/\epsilon).$$

Making use of Lemma 2, we conclude that $([0, c_0(T + \delta_1)] \times \mathbb{R}^3, g, \phi^\epsilon) = (M, g, \phi^\epsilon)$ solves the Einstein equations (1) and satisfies the initial data $(\Sigma, g, \phi^\epsilon)$ given in Theorem 4. The Fermi coordinate system $(s, y)$ on $M$ leads to a foliation $\Sigma_\tau$ of $M$ in a natural way $\Sigma_\tau := \{ s = \tau \}$ such that for such foliation we have the following estimates for the solution $(g, \phi^\epsilon)$

$$\sum_{1 \leq |\beta| \leq 3} \epsilon^{3-\frac{3}{2}} \| \partial^\beta (g - h) \|_{L^2}(s) \lesssim \delta^2, \quad \forall s \leq c_0(T + \delta_1),$$

$$\sum_{|\alpha| \leq 3} \epsilon^{\alpha - \frac{3}{2}} \| \partial^\alpha (\delta^{-1} \phi^\epsilon - \phi_S(y/\epsilon; \lambda'(s/\epsilon))) \|_{L^2}(s) \lesssim \epsilon, \quad \forall s \leq c_0(T + \delta_1).$$

Let the $C^1$ curve

$$\lambda(s) = (\omega(s), \theta(s), \zeta(s) + u_0 s, u_0 + u(s)) \in A_{\text{stab}}$$

be defined from $\lambda'(s)$, which has been given above. Recall Definition 1 for $\phi_S(y; \lambda(s))$. We note that

$$\phi_S(y/\epsilon; \lambda'(s/\epsilon)) = \phi_S(y; \lambda(s))$$

if $h(s, \zeta(s) + u_0 s) = m_0$. However, since

$$|\zeta(s)| \lesssim \epsilon, \quad h(s, u_0 s) = m_0,$$

we conclude from (42) that

$$\sum_{|\alpha| \leq 3} \epsilon^{\alpha - \frac{3}{2}} \| \partial^\alpha (\phi_S(y; \lambda(s)) - \phi_S(y/\epsilon; \lambda'(s/\epsilon))) \|_{L^2}(s) \lesssim |h(s, \zeta(s) + u_0 s) - h(s, u_0 s)| \lesssim \epsilon^2.$$

In particular, we have shown that

$$\| \partial(g - h) \|_{H^2} \lesssim \epsilon^{-1} \delta^2 \lesssim \epsilon,$$

$$\| \delta^{-1} \phi^\epsilon(s, y) - \phi_S(y; \lambda(s)) \|_{H^2} + \epsilon \| \delta^{-1} \phi^\epsilon(s, y) - \phi_S(y; \lambda(s)) \|_{H^2} \lesssim \epsilon, \quad \forall s \leq c_0(T + \delta_1).$$

This proves estimates (25), (26).

Finally, since the space $M$ (diffeomorphic to $[0, c_0(T + \delta_1)]$) can be viewed as an extension of $\mathcal{M}$ by (97), restricting the solution $(M, g, \phi^\epsilon)$ to $\mathcal{M}$, we obtain a solution $(\mathcal{M}, g, \phi^\epsilon)$ of (1) as well as a $C^1$ curve $\lambda(s) = (\omega(s), \theta(s), \zeta(s) + u_0 s, u_0 + u(s))$ such that (27) holds. This solution is unique up to diffeomorphism by a result in [9]. This completes the proof of the main Theorem 4.
8 Existence of Initial Data

In this section, we discuss the existence of the initial data \((\mathbb{R}^3, \bar{g}, \bar{K}, \phi_0, \phi_1')\) satisfying the conditions in Theorem 4.

Let \((\mathbb{R}^3, \bar{h}, \bar{k})\) be the given initial data for the vacuum spacetime \((\mathcal{M}, h)\), satisfying the vacuum constraint equations

\[
R(\bar{h}) - |\bar{k}|^2 + (\text{tr}\bar{k})^2 = 0, \quad \nabla^j \bar{k}_{ij} - \nabla_i \text{tr}\bar{k} = 0, \tag{98}
\]

where \(R\) denotes the scalar curvature on \((\mathbb{R}^3, \bar{h})\), \(\nabla\) is the covariant derivative with respect to \(\bar{h}\). Let \(\{x(x_1, x_2, x_3)\}\) be a coordinate system on \(\mathbb{R}^3\). Assume \(\phi_0', \phi_1'\) are given functions on \(\mathbb{R}^3\) of the form

\[
\phi_0'(x) = \delta \phi_0(x/\epsilon) = \delta \phi_0, \quad \phi_1'(x) = \delta \epsilon^{-1} \phi_1(x/\epsilon) = \delta \epsilon^{-1} \phi_1, \tag{99}
\]

where \(\delta = \epsilon^q, q > 1\) or \(\delta = \epsilon_0\epsilon\). We want to show that there exists a Riemannian metric \(\bar{g}\) and a symmetric two tensor \(\bar{K}\) on \(\mathbb{R}^3\) satisfying the constraint equations

\[
\begin{cases}
\bar{R}(\bar{g}) - |\bar{K}|^2 + (\text{tr}\bar{K})^2 = \delta^2 \epsilon^{-2} (|\phi_1|^2 + |\nabla \phi_0|^2 + 2\nu(\phi_0))(1/\epsilon), \\
\nabla^j \bar{K}_{ij} - \nabla_i \text{tr}\bar{K} = \delta^2 \epsilon^{-2} < \phi_1, \nabla_i \phi_0 > (1/\epsilon),
\end{cases}
\]

as well as the estimates

\[
\|\nabla (\bar{g} - \bar{h})\|_{H^2_0(\mathbb{R}^3)} + \|\bar{K} - \bar{k}\|_{H^2_0(\mathbb{R}^3)} \leq C(\phi_0, \phi_1, \bar{h}, \bar{k}) \delta^2 \epsilon^{-1}. \tag{100}
\]

Here \(\nabla\) is the covariant derivative for the unknown metric \(\bar{g}\) and the function \(\nu\) is defined in (5).

We define the weighted Sobolev space \(H^{s,w}\) on \(\mathbb{R}^3\)

\[
\|\phi\|_{H^{s,w}} := \left( \sum_{0 \leq m \leq s} \int_{\mathbb{R}^3} |\partial^m \phi|^2 (1 + |x|^2)^{w+m} dx \right)^{\frac{1}{2}},
\]

and weighted Hölder space \(C^{0,w}\)

\[
\|\phi\|_{C^{0,w}} := \sup_x \{(1 + |x|)^w|\phi(x)|\}.
\]

We define the metric space \(M^{s,w}\) on \(\mathbb{R}^3\) as follows:

\[
M^{s,w} := \{\text{Riemannian metric } g; g_{ij} - (m_0)_{ij} \in H^{s,w}\},
\]

where \(m_0\) is the Euclidean metric on \(\mathbb{R}^3\), that is, \((m_0)_{ii} = 1, (m_0)_{ij} = 0\) if \(i \neq j\). We have the following existence result of the initial data \((\mathbb{R}^3, \bar{g}, \bar{K}, \phi_0, \phi_1')\).

**Theorem 6.** Let \((\Sigma_0, \bar{h}, \bar{k})\) be the initial data for the vacuum spacetime \((\mathcal{M}, h)\) such that \(\bar{h} \in M^{4,-1}, \bar{k} \in H^{3,0}, \text{tr}\bar{k} = 0\). Assume the matter field \(\phi_0 \in H^{3,-1}, \phi_1 \in H^{2,0}\). Then there exists \(\epsilon_0 > 0\) such that for all \(\epsilon < \epsilon_0\), there exists a Riemannian metric \(\bar{g}\) and a symmetric two tensor \(\bar{K}\) satisfying the constraint equations (99) and the estimates (100).

**Remark 9.** The method here also applies to the case \(\text{tr}\bar{k} = \text{constant}\) which has been studied in \([8]\). However, the assumption \(\bar{k} \in H^{3,0}\) together with \(\text{tr}\bar{k} = \text{constant}\) imply that \(\text{tr}\bar{k} = 0\). For the general case \(\text{tr}\bar{k} \neq 0\), see the work of Corvino and Schoen [11].

The existence of \(\bar{g}, \bar{K}\) has been shown in \([5, 8]\) by using implicit function theorem. For consistence, we repeat the proof. However, the difficulty here is to show that \(\bar{g}, \bar{K}\) obey the estimates (100) for all \(\delta = \epsilon^q, q > 1\) or \(\delta = \epsilon_0\epsilon\), in particular the estimate

\[
\|\nabla (\bar{g} - \bar{h})\|_{L^2} + \|\bar{K} - \bar{k}\|_{L^2} \leq C \delta^2 \epsilon^\frac{q}{2}.
\]
The approach of previous works can only imply
\[ \| \nabla (\bar{g} - \bar{h}) \|_{L^2} + \| \bar{K} - \bar{k} \|_{L^2} \lesssim C \delta^2 \epsilon^{-\frac{3}{2}}. \]
We improve this estimate by relying on a Hardy’s inequality (see Lemma 16) for the first order linear operator \( L_V \) (defined below), which does not have any nontrivial kernel in the class \( H^{2,-1} \). Before proving this theorem, we make a convention that \( A \lesssim B \) means \( A \leq CB \) for some constant \( C \) depending on \( h, \phi_0, \phi_1 \).

**Remark 10.** Suggested by our argument for the main theorem, the previous results in [5], [8] may imply the above existence theorem if we consider the rescaled constraint equations. However, the estimates depend on the \( H^{4,-1} \) norm of the given metric \( h \). After scaling, the \( H^{4,-1} \) norm of the scaled metric \( \tilde{h}(x) = \frac{h(\epsilon x)}{\epsilon} \) depends on \( \epsilon \). In fact, it can be shown that the scaled norm has size \( \epsilon^{-\frac{3}{2}+w} = \epsilon^{-\frac{3}{2}} \). Hence considering the scaled constraint equations can not lead to the above theorem directly.

We denote the **Hamiltonian constraint**
\[
\mathcal{H}((\tilde{g}, \tilde{K}), (\phi_0, \phi_1)) = R(\tilde{g}) - |\tilde{K}|^2 + (\text{tr}\tilde{K})^2 - \delta^2 \epsilon^{-2}(2H(\phi_1) + |\nabla\phi_0|^2 - 2V(\phi_0))(\cdot/\epsilon),
\]
and the **momentum constraint**
\[
\mathcal{M}((\tilde{g}, \tilde{K}), (\phi_0, \phi_1)) = \nabla \tilde{K} - \nabla_{\text{tr}} \tilde{K} - \delta^2 \epsilon^{-2} < \phi_1, \nabla \phi_0 > (\cdot/\epsilon),
\]
where \( \nabla \) is the covariant derivative with respect to \( \tilde{g} \). Define the spaces
\[
X := \{(\tilde{g} - m_0, \tilde{K}) | \tilde{g} \in M^{3,-1}, \tilde{K} \in H^{2,0}\},
\]
\[
Y := \{((\phi_0, \phi_1)) | \phi_0 \in H^{3,-1}, \phi_1 \in H^{2,0}\},
\]
\[
Z := \{((\rho, J)) | \rho \in H^{1,1}, J \in H^{1,1}\},
\]
where \( \rho \) is scalar function, \( J \) is vector valued function on \( \mathbb{R}^3 \). We define the constraint map
\[
\Phi : X \times Y \to Z,
\]
\[
(\bar{g} - m_0, \bar{K}) \times (\bar{\phi}_0, \bar{\phi}_1) \mapsto (\mathcal{H}((\bar{g}, \bar{K}), (\bar{\phi}_0, \bar{\phi}_1)), \mathcal{M}((\bar{g}, \bar{K}), (\bar{\phi}_0, \bar{\phi}_1))).
\]
The fact that \( \Phi \) is a map from \( X \times Y \) to \( Z \) follows from the multiplication and embedding properties of the weighted Sobolev spaces. We state Lemma 2.4 and Lemma 2.5 in [6] here.

**Lemma 14.** We have
\[
\| f g \|_{H^{s,w}} \leq C \| f \|_{H^{s_1,w_1}} \| g \|_{H^{s_2,w_2}}, \quad s_1 + s_2 > s + \frac{3}{2}, \quad w_1 + w_2 > w - \frac{3}{2},
\]
\[
\| f \|_{C^{0,w'}} \leq C \| f \|_{H^{s,w}}, \quad s > \frac{3}{2}, \quad w' < w + \frac{3}{2},
\]
where the constant \( C \) depends only on \( s, w, s_1, w_1, s_2, w_2, w' \).

Let
\[
x_0 = (\bar{h}, \bar{K}), \quad y_0 = (0,0), \quad x = (\bar{g}, \bar{K}), \quad y = \delta^2 \epsilon^{-2}(\phi_0, \phi_1)(\cdot/\epsilon).
\]
Then the vacuum constraint equations (98) become \( \Phi(x_0, y_0) = 0 \). We define the linear map
\[
D\Phi(x_0, y_0) : X \to Z,
\]
\[
(g, K) \mapsto (DH, DM)
\]
as the linearization of \( \Phi(x, y) \) at the point \( (x_0, y_0) \), which can be computed as follows
\[
DH = -\Delta_h (tr_h g) + div_h(div_h g) - g \cdot Ric(h) - 2K \cdot K + 2(tr_h K)(tr_h K),
\]
\[
DM = \nabla_j K_i - \nabla_i (tr_h K) + \frac{1}{2} K_i \nabla_j tr_h g - \frac{1}{2} k H_j \nabla_i g_i.
\]
Here the covariant derivative $\nabla$ is for the metric $\tilde{h}$. The above formulae could be found in [5] or can be obtained by straightforward computations, noticing that the linearization of the connection is given by

$$\delta \Gamma^i_{ij} = \frac{1}{2} \left( \nabla_i g^i_j + \nabla_j g^i_j - \nabla^i g_{ij} \right), \quad g^i_i = g_i \tilde{h}^{ij},$$

for symmetric two tensor and $g_{ij}$. Moreover, using Lemma 14, we can show that

$$\|\Phi((g + \tilde{h}, K + k), y_0) - \Phi((\tilde{h}, k), y_0) - (DH, DM)\|_Z \lesssim \|(g, K)\|_X.$$

To apply the implicit function theorem, we must show that the linear map $D\Phi(x_0, y_0)$ is surjective from $X$ to $Z$, that is, for any $z = (\rho, J) \in Z$, the equations

$$DH = \rho, \quad DM = J \tag{101}$$

have at least one solution $(g, K) \in X$. Notice that the above equations are underdetermined. The linear map $D\Phi(x_0, y_0)$ has nontrivial kernel. We are instead looking for a solution of the form

$$g = \frac{1}{3} \lambda \tilde{h}, \quad K = L_V \tilde{h} - \text{div}(V) \tilde{h} - \frac{1}{2} \lambda \tilde{k} \tag{102}$$

for some real function $\lambda$ and vector fields $V$ on $\mathbb{R}^3$. Here $L_V \tilde{h}$ is the deformation tensor of the vector fields $V$ on $(\mathbb{R}^3, \tilde{h})$ defined as follows

$$(L_V \tilde{h})_{ij} = \nabla_i V_j + \nabla_j V_i.$$ 

Note that $tr \tilde{k} = 0$ on $\mathbb{R}^3$. Using (98), the equations (101) are reduced to the following elliptic systems

$$-\Delta \tilde{k} + \tilde{k}^2 = 3 \tilde{k} - 2 \lambda \tilde{h}, \quad \text{div}(L_V \tilde{h}) = J. \tag{103}$$

We must show that the above elliptic systems have a unique solution $(\lambda, V) \in H^{s,w}$ for any $z = (\rho, J) \in Z$. Since the systems are splitting. We first consider the second equation (104) which is independent of $\lambda$. The following lemma indicates that the operator $\text{div}(L_{(\cdot)} \tilde{h})$ is injective from $H^{2,-1}$ to $H^{0,1}$.

**Lemma 15.** Let $\tilde{h} \in M^{4,-1}$. If $\text{div}(L_V \tilde{h}) = 0$, $V \in H^{2,-1}$, then $V = 0$.

This result has been proven in [7]. From a geometric point of view, a Killing vector field $V$ is uniquely determined by $V|_p$, $\nabla V|_p$. The condition $\text{div}(L_V \tilde{h}) = 0, V \in H^{2,-1}$ implies that $V$ is Killing and $V$, $\nabla V$ vanish at infinity. Hence $V$ vanishes everywhere. However, we give another proof inspired by the method in [10], see Theorem 3.3 there.

**Proof.** Since $V \in H^{2,-1}$, we have

$$0 = \int_{\mathbb{R}^3} \tilde{h}(\text{div}(L_V \tilde{h}), V) d\sigma = -\frac{1}{2} \int_{\mathbb{R}^3} |L_V \tilde{h}|^2 d\sigma$$

by approximating $V$ with vectorfields $V_n \in C_0^\infty$. Hence $L_V \tilde{h} = 0$, that is, $V$ is Killing. In local coordinates, we have $\nabla_{\partial_i} V_j + \nabla_{\partial_j} V_i = 0$. In particular, we can compute

$$\nabla_{\partial_i} \nabla_{\partial_j} V_k = -R(\partial_i, \partial_j, \partial_k), \tag{105}$$

where $\partial_i$ is the vector field $\partial_{x_i}$, $R$ is the Riemann curvature tensor defined as follows

$$R(\partial_i, \partial_j, \partial_j, \partial_k) = \tilde{h}(\nabla_{\partial_i} \nabla_{\partial_j} \partial_j - \nabla_{\partial_i} \nabla_{\partial_j} \partial_k, \partial_k).$$

Using Lemma 14 by taking $w' = \frac{9}{4} < -1 + 2 + \frac{3}{2}$, we have

$$|R(\partial_i, \partial_j, \partial_j, \partial_k)| \lesssim |\nabla^2 \tilde{h}| \lesssim (1 + |x|)^{-\frac{r}{2}}, \quad |V| \lesssim (1 + |x|)^{-\frac{r}{4}}.$$
Hence by Lemma 5 (or Poincaré inequality), we can show that
\[\|(1 + |x|)\nabla^2 V\|_{L^2(\mathbb{R}^3/B_{R_0})} \lesssim (1 + R)^{-\frac{3}{4}}\|(1 + |x|)^{-1}V\|_{L^2(\mathbb{R}^3/B_{R_0})} \lesssim (1 + R)^{-\frac{3}{4}}\|(1 + |x|)\nabla^2 V\|_{L^2(\mathbb{R}^3/B_{R_0})}\]
for any ball \(B_{R_0}\) with radius \(R_0\). Choose \(R_0\) large enough. We can conclude that \(V\) is vanishing outside the ball \(B_{R_0}\).

Now consider the set
\[S := \{x \in \mathbb{R}^3, \quad V|_x = 0, \quad \nabla V|_x = 0\}.
\]
We show that the set \(S\) is open. In fact, let \(x \in S\). Notice that \(V \in C^0\). We have
\[
\int_0^r |V|^2(x + s\omega)ds \lesssim r^2 \int_0^r |\nabla V|^2ds \lesssim r^4 \int_0^r |\nabla^2 V|(x + s\omega)ds \lesssim r^4 \int_0^r |V|^2(x + s\omega)ds, \quad \forall \omega \in S^2,
\]
where we have used the equation (105). Choosing \(r\) small enough, we can show that the ball \(B_r(x) \subset S\). Hence \(S\) is open. Notice that \(S\) is closed and nonempty. We conclude that \(V \equiv 0\) on \(\mathbb{R}^3\).

This lemma also implies that the first order linear operator \(L(\cdot)\bar{h}\) is injective. We prove a Hardy’s inequality for this operator, which will be used to improve estimates for \(\|\nabla \lambda, \nabla V\|_{L^2}\).

**Lemma 16.** Assume \(\bar{h} \in M^{1, -1}, \quad V \in H^{2, -1}\). Then
\[\|\nabla^{-1} V\|_{L^2} \lesssim \|L_V \bar{h}\|_{L^2}.
\]

**Proof.** Choose \(R_0\) such that
\[|\text{Ric}| \leq \frac{1}{10}(1 + |x|)^{-2}, \quad |x| \geq R_0.
\]
We claim that
\[\|V\|_{L^2(B_{R_0})} \lesssim \|L_V \bar{h}\|_{L^2} \quad (106)
\]
In fact, if the above inequality does not hold, assume \(V_n \in H^{2, -1}\) such that
\[1 = \|V_n\|_{L^2(B_{R_0})} \geq n\|L_{V_n} \bar{h}\|_{L^2}.
\]
Integration by parts, we have
\[
\frac{1}{2}\|L_V \bar{h}\|_{L^2}^2 = \int_{\mathbb{R}^3} \nabla^i V^j \nabla_i V_j + \nabla^i V^j \nabla_i V_j d\sigma
= \int_{\mathbb{R}^3} \nabla^i V^j \nabla_i V_j - V^i \nabla^i \nabla_j V_j + V^i \nabla V^j \nabla_i V_j - V^i \nabla_i \nabla_j V_j d\sigma
= \int_{\mathbb{R}^3} |\nabla V|^2 + |\text{div}(V)|^2 - \text{Ric}(V, V) d\sigma.
\]
Hence we conclude from the assumptions on \(V_n\) that
\[\|\nabla V_n\|_{L^2} \leq 1 + \|\text{Ric}\|_{C^0}[V_n]_{L^2(B_{R_0})} + \frac{1}{10}(1 + |x|)^{-1}V_n\|_{L^2(\mathbb{R}^3/B_{R_0})},\]
which, by using Lemma 5, implies that
\[\|\nabla V_n\|_{L^2} \leq 2 + 2\|\text{Ric}\|_{C^0}.
\]
Therefore, there exists vectorfields \(V\) such that (up to a subsequence)
\[
\nabla V_n \rightharpoonup \nabla V \text{ weakly, \quad } V_n \rightarrow V \text{ strongly in } L^2(B_0).
\]

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In particular, we have

$$\|L_V \vec{h}\|_{L^2} \leq \lim_{n \to \infty} \|L_{V_n} \vec{h}\|_{L^2} = 0, \quad \|V\|_{L^2(B_{R_0})} = 1.$$  

That is $V$ is killing vector field. Moreover, by (105), we have $V \in H^{2,-1}$. Then Lemma 15 implies that $V \equiv 0$, which contradicts to $\|V\|_{L^2(B_{R_0})} = 1$. Therefore the desired inequality (106) holds. Thus, for $V \in H^{2,-1}$, we can show that

$$\|x^{-1}V\|_{L^2}^2 \lesssim \|\nabla V\|_{L^2}^2 = 2\|L_V \vec{h}\|_{L^2}^2 + \int_{\mathbb{R}^3} \text{Ric}(V, V) - |\text{div}(V)|^2 d\sigma$$

$$\leq 2\|L_V \vec{h}\|_{L^2}^2 + \|\text{Ric}\|_{C^0(\mathbb{R}^3)} \|V\|_{L^2(\mathbb{R}^3)} + \frac{1}{10}\|(1+|x|)^{-1}V\|_{L^2(\mathbb{R}^3)},$$

which implies that

$$\|x^{-1}V\|_{L^2}^2 \lesssim 2\|L_V \vec{h}\|_{L^2}^2 + \|\text{Ric}\|_{C^0(\mathbb{R}^3)} \|V\|_{L^2(\mathbb{R}^3)} \lesssim \|L_V \vec{h}\|_{L^2}^2.$$ 

**Remark 11.** An alternative approach for Lemma 15 and Lemma 16 is to use continuity argument. We sketch the prove here. Let

$$L_t(V) = L_V(t\vec{h} + (1-t)m_0), \quad t \in [0, 1].$$

If for some $t_0 \in [0, 1]$ such that

$$\|x^{-1}V\|_{L^2} \leq C_0\|L_{t_0}(V)\|_{L^2}, \quad \forall V \in H^{1,-1},$$

then we can show that

$$\|L_{t_0}(V)\|_{L^2} \leq C(\vec{h})\|L_t(V)\|_{L^2}$$

for $t$ close to $t_0$. Since Lemma 16 follows from Lemma 5 if $\vec{h} = m_0$, we thus conclude that Lemma 16 holds for all $\vec{h} \in M^{2,-1}$, $V \in H^{2,-1}$. In particular, Lemma 15 follows from Lemma 16.

We now proceed to prove Theorem 6. By Lemma 15, the operator $L_{t_0}(t\vec{h} + (1-t)m_0)$ is injective from $H^{s+2,-1}$ to $H^{s,1}$, for $s = 0$ or 1. Now, for $t = 0$, the operator $L_t(t) m_0 = 2\Delta$ is a diffeomorphism from $H^{s+2,-1}$ to $H^{s,1}$, see Theorem 5.1 in [6]. Hence the method of continuity [14] implies that the operator $L_t(\vec{h})$ is a diffeomorphism from $H^{s+2,-1}$ to $H^{s,1}$. In particular, there exists a unique solution $V \in H^{s+2,-1}$ of (104) such that

$$\|V\|_{H^{s+2,-1}} \lesssim \|J\|_{H^{s,1}}.$$ 

Moreover, by Lemma 16, we can show that

$$\|L_V \vec{h}\|_{L^2}^2 = -2\int_{\mathbb{R}^3} \vec{h}(V, \text{div}(L_V \vec{h})) d\sigma \lesssim \|x^{-1}V\|_{L^2} \cdot \|\text{div}(J)\|_{L^2} \lesssim \|L_V \vec{h}\|_{L^2} \cdot \|x|J|\|_{L^2},$$

which implies that

$$\|L_V \vec{h}\|_{L^2} \lesssim \|x|J|\|_{L^2}. \quad (107)$$

Having obtained $V$, for equation (103), by Theorem 6.6 in [6], the operator $-\nabla \vec{h} + |k|^2$ is a diffeomorphism from $H^{s+2,-1}$ to $H^{s,1}$, for $s = 0$ or 1. By Lemma 14, $k \cdot L_V \vec{h} \in H^{1,1}$. Therefore, there exists a unique solution $\lambda$ of (103) such that

$$\|\lambda\|_{H^{s+2,-1}} \lesssim \|k \cdot L_V \vec{h}\|_{H^{s,1}} + \|\rho\|_{H^{s,1}} \lesssim \|\rho\|_{H^{s,1}} + \|J\|_{H^{s,1}}, \quad s = 0, 1.$$ 

Moreover, multiply equation (103) by $\lambda$ and integration by parts, we can show that

$$\|\nabla \lambda\|_{L^2}^2 + \|k \lambda\|_{L^2}^2 = -\int_{\mathbb{R}^3} \lambda \left(3k \cdot L_V \vec{h} + \frac{3}{2} \rho\right) d\sigma \lesssim \|x^{-1}\lambda\|_{L^2} \left(\|x|\rho|_{L^2} + \|L_V \vec{h}\|_{L^2}\right),$$

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Now we have shown that the linear map $X$ from $H^{3,0}$ to $Z$ can be decomposed as

Now we have shown that the linear map $D\Phi(x_0, y_0)$ is surjective from $X$ to $Z$. Hence the Banach space $X$ can be decomposed as $X = X_1 + X_2$ such that $D\Phi(x_0, y_0)(X_2) = 0$, $D\Phi(x_0, y_0)$ is a diffeomorphism from $X_1$ to $Z$. In particular, for $y = \delta^2 \epsilon^{-2} (\phi_0, \phi_1)(\cdot/\epsilon)$, the implicit function theorem shows that there is a solution $(\bar{g}, \bar{K}) = (\bar{h} + g, \bar{k} + K) \in X$ of (99) if $\epsilon$ is sufficiently small, depending only on $\bar{h}, \phi_0, \phi_1$. Moreover, we can require that $(g, K)$ is of the form (102) for $(\lambda, V) \in Z$. Therefore

which implies that

We must remark here that in local coordinate $\{x\}$ we have $\phi_0(\cdot/\epsilon) = \phi_0(x/\epsilon)$. Similarly

which shows that

It remains to estimate $\|\nabla (\bar{g} - \bar{h})\|_{L^2} + \|\bar{K} - \bar{k}\|_{L^2}$, which we rely on (107), (108). Note that

Making use of the multiplication properties of $H^{a,w}$ in Lemma 14, we can show that

where $N(g, K)$ is the nonlinear term in equation (99) for $\bar{g} = \bar{h} + g, \bar{K} = \bar{k} + K$. Hence

That is the solution $(\bar{g}, \bar{K})$ satisfies the constraint equations (99) as well as the estimate (100).

References

[1] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.*, 82(4):313–345, 1983.

[2] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.*, 82(4):347–375, 1983.

[3] H. Berestycki, P.-L. Lions, and L. A. Peletier. An ODE approach to the existence of positive solutions for semilinear problems in $R^N$. *Indiana Univ. Math. J.*, 30(1):141–157, 1981.

[4] Y. Choquet-Bruhat. Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires. *Acta Math.*, 88:141–225, 1952.
[5] Y. Choquet-Bruhat. *General relativity and the Einstein equations.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2009.

[6] Y. Choquet-Bruhat and D. Christodoulou. Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity. *Acta Math.*, 146(1-2):129–150, 1981.

[7] Y. Choquet-Bruhat, A. E. Fischer, and J. E. Marsden. *Maximal Hypersurfaces and Positivity of Mass.* Isolated gravitating systems in general relativity. Proceedings of the International School of Physics Enrico Fermi (67), North-Holland, New York, 1979.

[8] Y. Choquet-Bruhat, A. Fisher, and J. Marsden. Équations des contraintes sur une variété non compacte. *C. R. Acad. Sci. Paris Sér. A-B*, 284(16):A975–A978, 1977.

[9] Y. Choquet-Bruhat and R. GerochRobert. Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.*, 14:329–335, 1969.

[10] D. Christodoulou and N. Ó Murchadha. The boost problem in general relativity. *Comm. Math. Phys.*, 80(2):271–300, 1981.

[11] J. Corvino and R. Schoen. On the asymptotics for the vacuum Einstein constraint equations. *J. Differential Geom.*, 73(2):185–217, 2006.

[12] M. Dafermos and I. Rodnianski. The redshift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math.*, 62(7):859–919, 2009.

[13] A. D. Dolgov and I. B. Khriplovich. Normal coordinates along a geodesic. *General Relativity and Gravitation*, 15(11), 1983.

[14] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, reprint of the 1998 edition edition, 2001.

[15] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987.

[16] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal.*, 94(2):308–348, 1990.

[17] S. W. Hawking and G. R. Ellis. *The large scale structure of space-time.* Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.

[18] F. K. Manasse and C. W. Misner. Fermi normal coordinates and some basic concepts in differential geometry. *Journal of Mathematical Physics*, 4(6):735–745, 1963.

[19] K. McLeod. Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$. II. *Trans. Amer. Math. Soc.*, 339(2):495–505, 1993.

[20] L. A. Peletier and J. Serrin. Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.*, 81(2):181–197, 1983.

[21] J. Shatah. Stable standing waves of nonlinear Klein-Gordon equations. *Comm. Math. Phys.*, 91(3):313–327, 1983.

[22] J. Shatah. Unstable ground state of nonlinear Klein-Gordon equations. *Trans. Amer. Math. Soc.*, 290(2):701–710, 1985.

[23] J. Shatah and W. Strauss. Instability of nonlinear bound states. *Comm. Math. Phys.*, 100(2):173–190, 1985.

[24] W. A. Strauss. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.*, 55(2):149–162, 1977.
[25] D.M.A Stuart. Modulational approach to stability of non-topological solitons in semilinear wave equations. *J. Math. Pures Appl. (9)*, 80(1):51–83, 2001.

[26] D.M.A Stuart. The geodesic hypothesis and non-topological solitons on pseudo-Riemannian manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(2):312–362, 2004.

[27] D.M.A Stuart. Geodesics and the Einstein nonlinear wave system. *J. Math. Pures Appl. (9)*, 83(5):541–587, 2004.

[28] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

[29] S. Yang. Global solutions to nonlinear wave equations in time dependent inhomogeneous media. 2010. arXiv:math.AP/1010.4341.

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