Poincaré’s Observation and the Origin of Tsallis Generalized Canonical Distributions

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Abstract

In this paper, we present some geometric properties of the maximum entropy (MaxEnt) Tsallis-distributions under energy constraint. In the case \( q > 1 \), these distributions are proved to be marginals of uniform distributions on the sphere; in the case \( q < 1 \), they can be constructed as conditional distributions of a Cauchy law built from the same uniform distribution on the sphere using a gnomonic projection. As such, these distributions reveal the relevance of using Tsallis distributions in the microcanonical setup: an example of such application is given in the case of the ideal gas.

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I. INTRODUCTION

Nonextensive thermostatistics is a very active field nowadays, based on the concept of Tsallis’ information measure (or entropy) [1]. The concomitant maximum (Tsallis) entropy distributions play a role for the description of non-extensive systems akin of that of Gaussian ones for extensive systems. By Tsallis distributions, we mean the following $n$-variate probability densities:

- If $\frac{n}{n+2} < q < 1$, with $\beta = \frac{1}{2q-n(1-q)}$ and $A_q = \frac{\frac{1}{\Gamma(1-q)(\beta(1-q))^{n/2}}}{\pi^{n/2}}$, then
  $$ f_X(X) = A_q \left(1 - \beta (q - 1) x^T C^{-1} x\right)^\frac{1}{q-1} $$

- If $q > 1$ or $q < 0$, with $\beta = \frac{1}{2q-n(1-q)}$ and $A_q = \frac{\frac{1}{\Gamma(1-q)(\beta(q-1))^{n/2}}}{\pi^{n/2}}$, then
  $$ f_X(X) = A_q \left(1 - \beta (q - 1) x^T C^{-1} x\right)^\frac{1}{q-1} $$

where $x_+$ denotes $\max(x, 0)$.

These distributions are solutions of the following maximization problem:

$$ \max_f \frac{1}{1-q} \int_{\mathbb{R}^n} f_q \text{ such that } \int_{\mathbb{R}^n} x x^T f(x) \, dx = C. $$

In this sense, they are the counterparts of the Gaussian distribution with covariance matrix $C$ which is solution of

$$ \max_f \left(-\int_{\mathbb{R}^n} f \log f \right) \text{ such that } \int_{\mathbb{R}^n} x x^T f(x) \, dx = C $$

Gaussian distributions exhibit universal properties that allow to derive important statistical results such as the central limit theorem or the entropy power inequality. Some of these results have a "Tsallis-counterpart" [1]. A geometric approach like the one advanced here is interesting since it sheds new light on these properties.

II. GEOMETRIC APPROACH

A. stochastic properties

In this section, we propose a geometric characterization of Tsallis distributions based on their stochastic properties. Thus we begin by a detailed stochastic study of Tsallis random vectors. We denote as

$$ X \sim T_{q,C} $$
a Tsallis vector with nonextensivity parameter $q$ and covariance matrix $C$.
case $\frac{n}{n+2} < q < 1$

If $X \sim T_{q,C}$, then a stochastic representation of $X$ writes

$$X = \sqrt{m - 2C^{1/2}} \frac{N}{\sqrt{a}}$$

where $a$ is a chi-square distributed random variable with $m = \frac{2}{1-q} - n$ degrees of freedom, independent of the Gaussian random vector $N$ with unit covariance matrix. It is understood that $m$ may be non-integer.

duality result

If $X \sim T_{q,C}$ with $\frac{n}{n+2} < q < 1$ then the following random vector

$$Y = \frac{X}{\sqrt{1 + \frac{1}{m-2}X^TC^{-1}X}}$$

verifies

$$Y \sim T_{q^*,C^*}$$

with

$$C^* = C \frac{m - 2}{m + n}, \quad \frac{1}{q^* - 1} = \frac{1}{1 - q} - \frac{n}{2} - 1$$

case $q > 1$

From the duality result, we deduce a stochastic representation for $Y \sim T_{q^*,C}$ with $q^* > 1$ as follows

$$Y = \sqrt{m + n} C^{1/2} \frac{N}{\sqrt{a + N^TN}}$$

where $a$ is a chi-square random variable with $m = \frac{2q^*}{q^*-1}$ degrees of freedom, independent of the Gaussian random vector $N$ with unit covariance matrix.

B. Geometric representation

The preceding stochastic representations allow to derive easily a geometric construction of Tsallis random vectors as follows.
An $n-$variate random vector $U$ is uniformly distributed on the ellipsoid $E_{C,n} = \{ Z \in \mathbb{R}^n | Z^T C Z = 1 \}$ if and only if it writes
\[
U = C^{-1/2} N \sqrt{N^T N}
\]
where $N$ is a Gaussian vector with unit covariance matrix. Comparing this stochastic representation with the Tsallis vector’s one \[1\], the following result can be proved:

**Theorem 1** If $Y \sim T_{q^*,C}$ with $q^* > 1$ or $q^* < 0$ then $Y$ is the $n-$variate marginal vector of an $(m+n)-$variate random vector $U$ uniformly distributed on the ellipsoid $E_{C^{-1},m+n}$ with $m = \frac{2q^*}{q^*-1}$.

We remark that, according to this characterization, the $(m+n-1)-$variate marginal random vector is the only one distributed according to a Tsallis law with negative non-extensivity index (namely $q^* = -1$); moreover, the $(m+n-2)-$variate marginal is uniform inside a $(m+n-2)-$dimensional ellipsoid and corresponds to a non-extensivity index $q^* = +\infty$. All other lower dimension marginals have finite and positive index $q^* > 1$.

**case $q < 1$**

For the sake of simplicity, we address here the uncorrelated case $C = I_n$. We note that this is not a loss of generality, since
\[
X \sim T_{q,C} \Rightarrow C^{-1/2} X \sim T_{q,1}.
\]

**Theorem 2** Assume point $P$ is uniformly distributed on the sphere $E_{I_{n,n}}$; consider the intersection $M$ of vector $OP$ with the hyperplane $H = \{ Z \in \mathbb{R}^n | Z_n = 1 \}$.\[1] Then point $M$ follows a $(n-1)-$variate Cauchy distribution in $H$:
\[
f_M(y_1, \ldots, y_{n-1}) \propto (1 + \sum_{i=1}^{n-1} y_i^2)^{-\frac{n}{2}}.
\]

Moreover, any distribution of point $M$ conditioned on variables $y_{k+1}, \ldots, y_{n-1}$ follows a $k-$variate Tsallis distribution with $m = n - k$ degrees of freedom \[10\]:
\[
f_M(y_1, \ldots, y_k | y_{k+1}, \ldots y_{n-1}) \propto (1 + \lambda \sum_{i=1}^{k} y_i^2)^{-\frac{k}{q^*}}
\]

\[1\] point $M$ is called the gnomonic projection of point $P$. 

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In Fig. 1 below, the $n = 3$ dimensional sphere is depicted, with the Cauchy distribution of point $M$: the white curve represents, up to a constant, the one dimensional conditional density $f_M(y_1|y_2)$ which is Tsallis distributed with $m = 2$ degrees of freedom ($q = \frac{1}{3}$).

III. COMMENTS ON SOME PARTICULAR CASES

In this section, we comment on the geometric results obtained.

A. the case $q > 1$ and Poincaré’s observation

The characterization of Tsallis random vectors - with $q > 1$ and identity covariance matrix - as marginals of vectors uniformly distributed on a sphere can be related to a result attributed to Poincaré called ”Poincaré’s Observation” [2] (for a discussion on this paternity, see [8]).

**Theorem 3** If $X = (x_1, \ldots, x_p)$ is uniformly distributed on the sphere $S_p$ with radius $\sqrt{p}$, then
for fixed \( n < +\infty \),

\[
\lim_{p \to \infty} \Pr ( \cap_{i=1}^{n} (a_i \leq x_i \leq b_i)) = \int_{a_1}^{b_1} \frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}} dx \ldots \int_{a_n}^{b_n} \frac{e^{-\frac{x_n^2}{2}}}{\sqrt{2\pi}} dx
\]

This result shows that a Gaussian (Tsallis \( q = 1 \) maximizer) random vector with fixed dimension \( n \) can be considered as the marginal vector of an infinite-dimensional vector uniformly distributed on the sphere \( S_{\infty} \). The results obtained above demonstrate that, in contrast, a Tsallis \( q > 1 \) random vector of dimension \( n \) can be viewed as the marginal of a finite-dimensional vector uniformly distributed on the sphere \( S_p \) with

\[
p = \frac{2q}{q - 1} + n.
\]

As \( q \to 1^+ \), \( p \to +\infty \) and we recover Poincaré’s observation.

B. the case \( m = 1, q < 1 \)

The case \( m = 1 \) and \( q < 1 \) corresponds to Cauchy-Lorentz distributions, which are not \textit{stricto sensu} Tsallis distributions since they have no finite covariance matrix. However, they appear in many relevant physical situations \[4\]. They are defined as

\[
f_X (X) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{\frac{n+1}{2}}} (1 + X^T X)^{-\frac{n+1}{2}}
\]

and have for stochastic representation

\[
X = \frac{N}{M},
\]

where \( N \) is an \( n \)-variate Gaussian vector and \( M \) is a scalar Gaussian variable, independent of \( N \). Applying the above results, we deduce that they can be obtained as gnomonic projections of uniformly distributed points on the \( n \)-sphere, as illustrated in the following figure in the case \( n = 3 \): as point \( P \) describes uniformly the surface of \( S_3 \), point \( M_2 \) in the plane \( X_3 = 1 \) is distributed with Tsallis law \( T_{1/3, t_2} \).

IV. APPLICATION: THE IDEAL GAS

We adopt here notations of Minlos \[5\]. Denote as \( q_i \) the velocity of the i-th particle among \( N \) of an ideal gas in volume \( \Lambda \). Let us define the set

\[
\Omega_{\Lambda,N,E} = \{(q_1, \ldots, q_N) \mid H(q_1, \ldots, q_N) = E\}
\]
FIG. 2: point $M_2$ distributed with Tsallis law $T_{1/3, I_2}$ while point $M_1$ is distributed with Tsallis law $T_{-1, I_2}$

where the Hamiltonian writes

$$H(q_1, \ldots, q_N) = \sum_{i=1}^{N} \frac{1}{2} m q_i^2.$$  

In the microcanonical setup, one assumes that the velocities have a distribution $\nu$ uniform on surface $\Omega_{\Lambda, N, E}$:

$$\nu(q_1, \ldots, q_N) = \frac{2 |\Lambda|^N \pi^{3N/2}}{m \Gamma(\frac{3N}{2})} \left( \frac{2E}{N} \right)^{\frac{3N}{2} - 1}.$$  

Let us now consider a subsystem of the gas, consisting of $N_0 < N$ particles in a volume $\Lambda_0 \subset \Lambda$: their distribution can be easily computed as

$$\nu(q_1, \ldots, q_{N_0}) \propto \left( E - \sum_{i=1}^{N_0} \frac{1}{2} m q_i^2 \right)^\frac{3}{2} (N-N_0) - 1.$$  

A usual approach consists then in taking the thermodynamic limit: assuming the total volume $\Lambda$ converges to $\mathbb{R}^3$, the number of particle per volume $\frac{N}{|\Lambda|}$ converges to density $\rho$ and the energy per volume $\frac{E}{|\Lambda|}$ converges to energy density $e$, we deduce

$$\nu(q_1, \ldots, q_{N_0}) \propto \exp \left( -\frac{3}{2e} \sum_{i=1}^{N_0} \frac{1}{2} m q_i^2 \right)$$

and recover the celebrated Boltzmann distribution of velocities.
If, however, we do not take the thermodynamic limit but remain under the finite-dimensional assumption, then we remark that distribution (2) is an $N_0$-variate Tsallis distribution with nonextensivity index

$$q = \frac{N - N_0}{N - N_0 - \frac{N}{3}} > 1.$$  \hspace{1cm} (3)

In other terms, the distribution of the finite dimensional ideal gas in the microcanonical ensemble maximizes Tsallis entropy with parameter $q$ as defined by (3). As $N - N_0$ is exactly the dimension of the heat bath seen by the $N_0$ particles, we may conclude that, in this example, Tsallis entropy appears as the finite-dimensional counterpart of Shannon entropy. This idea was already derived by one of the authors [6] years ago, but without the details we provide here. Moreover, Adib et al. [7] have characterized Tsallis $q > 1$ distributions as marginals of uniform distributions, in the case of more general Hamiltonians verifying an homogeneity property, but they assume the (stronger) ergodic hypothesis to derive this result.

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