Coboson many-body formalism for atom-dimer scattering length

Shine-Yuan Shiau, Ching-Hang Chien, and Yia-Chung Chang
Research Center for Applied Sciences, Academia Sinica, Taipei, 115 Taiwan

Monique Combescot
Institut des NanoSciences de Paris, Université Pierre et Marie Curie, CNRS, Tour 22, 2 Place Jussieu, 75005 Paris, France
(Dated: October 31, 2017)

We use the composite boson (coboson) many-body formalism to tackle scattering lengths relevant to cold atoms. We show that bound dimers can be taken as elementary entities provided that fermion exchanges between them are treated exactly, as can be done through the coboson formalism. This alternative tool extended to cold atom many-body physics not only makes transparent the many-body processes at hand through Shiva diagrams specific to cobosons, but also greatly simplifies the numerical computation. Indeed, the integral equation we derive for the atom-dimer scattering length and easily solve by restricting the dimer relative motion to the ground state, gives the previously obtained values for all fermion mass ratios. This remarkable result is also true for the dimer-dimer scattering length, except for equal fermion masses where our restricted procedure gives a value slightly larger than the accepted one ($0.64a_d$ instead of $0.60a_d$).

I. INTRODUCTION

Interest in few-body systems has a long history that started long before the spectacular development of cold atom physics. In semiconductors, Lambert has predicted[1] in the 1960’s that semiconductor electrons and holes can form multi-body complexes such as excitons, trions, and biexcitons. In nuclear physics, the hadron-hadron scattering has also been an important subject of study for decades. Another peculiar phenomenon on its own right is the universal behavior that occurs in three-particle systems initiated by Efimov\cite{2, 3}, now known as Efimov physics.

The scattering of three fermions ($\alpha, \alpha, \beta$) interacting through a short-range attractive potential between different-species fermions only, is also known to exhibit universality. Indeed, the atom-dimer scattering length for equal fermion masses is equal to $1.18a_d$ where $a_d$ is the scattering length of the ($\alpha, \beta$) fermion pair, whatever the interaction strength\cite{4}. This problem has been taken up again in the 2000’s and extended to arbitrary mass ratios due to its interest in cold atom systems, by directly solving the three-body Schrödinger equation\cite{5, 6}, or through a finite-volume lattice method\cite{7}, or through a field theoretical many-body procedure\cite{8–10}. A precise analysis of limiting mass ratios can be found in Ref. \cite{11}.

In this work, we reconsider the three-fermion scattering problem from a different perspective using the coboson many-body formalism \cite{12, 13}. We have shown\cite{14} that the coboson formalism takes a much simpler form for particles like fermionic atoms which interact between different fermion species only. Besides its simplicity, the great advantage of this rather new many-body formalism is to provide a direct, visual understanding of the role of the Pauli exclusion principle in few-fermion scattering problems, through Shiva diagrams that are specific to cobosons. The important role of the Pauli exclusion principle in few-body scattering deserves some discussions, as we will do here.

The main result of this work is that the integral equation for the atom-dimer scattering length derived from the coboson many-body formalism leads to exact values for all fermion mass ratios, despite the fact that we solve it by restricting the dimer relative motion to the ground state — which greatly simplifies the numerical computation. In this work, we also reconsider dimer-dimer scattering. For this case too, the integral equation restricted to the relative motion ground state becomes analytically exact in the large mass ratio limit, so is the scattering length derived from it, while the numerically obtained scattering length for equal masses is equal to $0.64a_d$\cite{15}, which is slightly larger than the accepted value\cite{10, 16–18}, $0.60a_d$. Our work shows that, remarkably, cold atom dimers can be effectively treated as elementary entities in their many-body interactions with atoms or other dimers, provided the fermion exchanges induced by the Pauli exclusion principle are handled exactly.

This paper is organized as follows. In Sec. II, we consider a single dimer made of ($\alpha, \beta$) atoms, and we determine the threshold of the potential amplitude for the dimer to have a bound state. In Sec. III, we derive the atom-dimer scattering length. The coboson formalism allows us to link the scattering length to the energy induced by the atom-dimer interaction, while visualizing the fermion exchanges between atom and dimer in the scattering processes. We then analyze the integral equation in two atom mass ratio limits, and compute the scattering length for all mass ratios. In Sec. IV, we analyze in a similar way the dimer-dimer scattering in the large mass ratio limit. We then conclude.
II. DIMER SCATTERING LENGTH

We consider two species of fermionic atoms, α and β. The kinetic part of the Hamiltonian reads $H_0 = \sum_k \varepsilon_k a_k^\dagger a_k + \sum_k \varepsilon_k b_k^\dagger b_k$ with $\varepsilon_k = k^2/2(m_{\alpha,\beta})$, the operators $a_k^\dagger$ and $b_k^\dagger$ denoting momentum $k$ creation operators for free atoms $\alpha$ and $\beta$, respectively. Because of the low energy and temperature, effectively same-species atoms do not interact while different-species atoms interact via a short-ranged attractive potential that can be written as

$$V = -\sum_{K,p,p'} B_{K,p,p'}^\dagger v_{p-p'} B_{K,p}$$

with $B_{K,p,p'}^\dagger = a_{p+\gamma_\alpha K}^\dagger b_{p+\gamma_\beta K}^\dagger$ (see Fig. 1). For $\gamma_\alpha = 1 - \gamma_\beta = m_\beta/(m_\alpha + m_\beta)$, the operator $B_{K,p}^\dagger$ creates a free fermion pair with center-of-mass momentum $K$ that remains constant in the scattering, and relative momentum $p$ that changes from $p$ to $p'$ with a scattering amplitude $-v_{p-p'}$.

The boson many-body formalism treats the single-dimer eigenstates as entities while handling fermion exchanges between them in an exact way. This formalism reads in terms of dimer creation operators $B_i^\dagger$ defined as $|i\rangle = B_i^\dagger |v\rangle$ with $(H - E_i) |i\rangle = 0$ where $|v\rangle$ denotes the vacuum state, $H$ is equal to $H_0 + V$, and $E_i$ is the dimer $i$ energy. As the potential $V$ conserves the dimer center-of-mass momentum, it is convenient to have the center-of-mass momentum $K_i$ of dimer $i$ appearing explicitly in its creation operator, that is, to write $B_i^\dagger$ as

$$B_i^\dagger = B_{K_i,p_i}^\dagger = \sum_p B_{K_i,p_i}^\dagger \langle p|\nu_i\rangle.$$  

We then note that $V B_{K,p}^\dagger |v\rangle = -\sum_p v_{p-p'} B_{K,p'}^\dagger |v\rangle$, while $[H_0, B_{K,p}^\dagger] = E_{K,p} B_{K,p}^\dagger$ with $E_{K,p} = \varepsilon_{K}^{(d)} + \varepsilon_p$ where $\varepsilon_{K}^{(d)} = K^2/2(m_\alpha + m_\beta)$ and $\varepsilon_p = p^2/2\mu_\alpha$ for $\mu_\alpha = m_\alpha + m_\beta$. As a result, by projecting the dimer Schrödinger equation over a free pair state, we see that $\langle p|\nu_i\rangle$ corresponds to the dimer relative motion wave function in momentum space provided that

$$0 = (\varepsilon_p - \varepsilon_{\nu_i}) \langle p|\nu_i\rangle - \sum_{p'} v_{p-p'} \langle p'|\nu_i\rangle,$$

the energy of dimer $i$ being given by $E_i = \varepsilon_{K_i}^{(d)} + \varepsilon_{\nu_i}$.

For short-ranged atom-atom potential, $v_{p-p'}$ in Eq. (1) can be taken in a separable form,

$$v_{p-p'} = v w_p w_{p'} = v w_p w_{p'},$$

where $v$ is a constant and $w_p = 1$ up to a large cutoff $q_c$ and zero beyond. This cutoff, which mimics the natural decrease of the potential with momentum transfer, allows a proper handling of spurious divergences at large momenta. The above equation then gives the single-dimer relative motion eigenstates through

$$\langle p|\nu_i\rangle = \frac{v w_p}{\varepsilon_p - \varepsilon_{\nu_i}} \sum_{p'} w_{p'} \langle p'|\nu_i\rangle,$$

which readily leads to an integral equation for the relative motion energy

$$\frac{1}{v} = \sum_p \frac{w_p}{\varepsilon_p - \varepsilon_{\nu_i}},$$

since $w_p = w_p^2$. In three dimensions, this equation is known to allow one bound state only, $\varepsilon_{\nu_b} < 0$, provided that $v$ is larger than a $v^*$ threshold that corresponds to $\varepsilon_{\nu_b} = 0$, namely,

$$\frac{1}{v^*} = \sum_p \frac{w_p}{\varepsilon_p - \varepsilon_{\nu_b}^*},$$

As the dimer ground state corresponds to $g = (K_g = 0, \nu_g)$, Eq. (6) gives its energy, $E_g = E_g = \varepsilon_{\nu_b}$, through

$$\frac{1}{v^*} - \frac{1}{v} = \sum_p \frac{w_p}{\varepsilon_p - \varepsilon_{\nu_b}^*},$$

in which $w_p$ can be replaced by $1$ since the $p$ sum now converges for large momentum. A simple way to solve this equation and relate the dimer ground state energy $E_g^{(d)}$ to the potential amplitude $v$ is to use a dimensional argument to set

$$E_g^{(d)} = -\frac{1}{2\mu_\alpha a_d^2},$$

with $a_d$ real positive. The $p$ sum in Eq. (8) calculated with this $E_g^{(d)}$ then gives

$$\frac{1}{v^*} - \frac{1}{v} = \frac{1}{4\pi} \left( \frac{2\mu_\alpha L^3}{a_d} \right) > 0,$$

which relates $v$ to $E_g^{(d)}$ through $a_d$. As required, the obtained potential amplitude scales as the inverse of the sample volume, $L^3$.

The corresponding (normalized) dimer wave function, obtained from Eq. (5), reads

$$\langle p|\nu_g\rangle = \sqrt{\frac{\mu_\alpha}{2\pi}} \left( \frac{a_d}{L} \right)^{3/2} \frac{w_p}{1 + (p a_d)^2}.$$  

We see that $a_d$, which actually is the scattering length of the separable potential at hand — indeed positive in the case of bound state — is just the dimer spatial extension.
III. ATOM-DIMER SCATTERING

We now consider the scattering between a $\alpha$ atom and a $(\alpha, \beta)$ dimer. The difference between the $(\alpha, \alpha, \beta)$ ground state energy $E_{\alpha}^{(ad)}$ and the one of the $(\alpha, \beta)$ dimer then scales for large sample as the inverse of the sample volume. The scattering length $a_{ad}$ between a $\alpha$ atom and a $(\alpha, \beta)$ dimer follows from this difference as

$$\Delta^{(ad)} = \epsilon_{\alpha}^{(ad)} - \epsilon_{\alpha}^{(d)} = 4\pi \left( \frac{a_{ad}}{2\mu_{ad} L^3} \right),$$

where $\mu_{ad}^{-1} = m_\alpha^{-1} + (m_\alpha + m_\beta)^{-1}$ is the relative motion mass for the atom-dimer pair.

To get the scattering length $a_{ad}$, we thus have to determine the $(\alpha, \alpha, \beta)$ ground state energy, $(H - \epsilon_{\alpha}^{(ad)})|\Psi_{\alpha}^{(ad)}\rangle = 0$. Since free $(\alpha, \beta)$ fermion pairs can be written in terms of dimers according

$$B_{K,p} = \sum_i B_i^\dagger a_{K,i}^\dagger |\nu_i\rangle |p\rangle,$$

as easy to check from Eq. (2), we can expand the $(\alpha, \alpha, \beta)$ ground state on the (complete) atom-dimer basis as

$$|\Psi_{\alpha}^{(ad)}\rangle = \sum_{i,k} \Psi_{i,k}^{(ad)} B_i^\dagger a_{K,i}^\dagger |\nu_i\rangle |v_i\rangle.$$  

Note that

$$0 = \left[ B_i^\dagger, a_{K,i}^\dagger \right] = B_i^\dagger a_{K,i}^\dagger + \eta a_{K,i}^\dagger B_i^\dagger,$$

with $\eta = +$ when the operators $a_{K,i}^\dagger$ and $b_{K,i}^\dagger$ commute and $\eta = -$ when they anticommute,

$$0 = \left[ b_{K,i}^\dagger, a_{K,i}^\dagger \right]_{-\eta} = \eta a_{K,i}^\dagger B_i^\dagger,$$

the value of the scattering length $a_{ad}$ being independent of $\eta$, as can be ultimately checked.

A. Coboson formalism

To go further, we use two commutation relations constructed along the coboson many-body formalism. The first one is a commutator (see Appendix IA)

$$\left[ H, B_i^\dagger \right] = E_i B_i^\dagger + V_i^\dagger.$$  

It defines the creation potential $V_i^\dagger$ that describes the interaction of the $(\alpha, \beta)$ dimer in state $i$ with the rest of the system. This operator gives zero when acting on vacuum, $V_i^\dagger |v\rangle = 0$, while its interaction with a $\alpha$ atom follows from a second commutation relation (see Appendix IA), which can be a commutator or an anticommutator depending on if the $(\alpha, \beta)$ fermion operators commute or anticommute, namely

$$\left[ V_i^\dagger, a_{K,i}^\dagger \right] = \sum_{i', k'} \xi_{i', k'}^{(k', k)} B_{i'}^\dagger a_{K,i'}^\dagger.$$  

The energy-like scattering $\xi_{i', i}^{(k', k)}$ corresponds to the direct interaction of the $\alpha$ atom in state $k$ and the $\beta$ atom of the $(\alpha, \beta)$ dimer in state $i$, since $\alpha$ atoms interact with $\beta$ atoms only. It is precisely given by

$$\xi_{i', i}^{(k, k')} = -\langle \nu_i | k_{\alpha} + K_{\alpha} | \nu_i \rangle \langle \nu_i | k_{\beta} + K_{\beta} | \nu_i \rangle.$$  

as can be directly read from its Shiva diagram shown in Fig. 2(b) (see Appendix IB). These two commutators, used in the Schrödinger equation $(H - \epsilon_{\alpha}^{(ad)})|\Psi_{\alpha}^{(ad)}\rangle = 0$, give

$$0 = \sum_{i, k} \left\{ (E_i - \epsilon_{\alpha}^{(ad)}) |\Psi_{\alpha}^{(ad)}\rangle + \sum_{i', k'} \xi_{i', i}^{(k', k)} |\Psi_{\alpha}^{(ad)}\rangle \right\} B_{i'}^\dagger a_{K,i'}^\dagger |v\rangle,$$

with $E_i = E_i + \epsilon_{\alpha}^{(ad)}$.

The next step is to project the above equation on the state $|v\rangle a_{K,i}^\dagger B_{i'}^\dagger$ and use the scalar product

$$|v\rangle a_{k}^\dagger B_{i'}^\dagger a_{K,i}^\dagger |v\rangle = \delta_{ij'} \delta_{k'k} - \lambda_{i'}^{(k', k)}$$  

that follows from two other commutators whatever $\eta$ (see Appendix IA)

$$\left[ B_{i'}^\dagger, B_i^\dagger \right] = \delta_{ij'} - D_{ij'},$$

$$\left[ D_{i'}^\dagger, a_{k}^\dagger \right] = \sum_{k'} \lambda_{i'}^{(k', k)} a_{k'}^\dagger.$$  

The dimensionless Pauli scattering $\lambda_{i'}^{(k', k)}$ corresponds to fermion exchange between $\alpha$ atom and $(\alpha, \beta)$ dimer. It
Next, in the sum of Eq. (25), we separate the lowest term,
\[ \lambda^{k',k}_{i'\ i} = \delta_{k' + K, \ k + K} \langle \nu_i | k - \gamma_0 K | \nu_i \rangle \langle k' - \gamma_0 K | \nu_i \rangle, \]
which is precisely given by
\[ \lambda^{k',k}_{i'\ i} = \delta_{k' + K, \ k + K} \langle \nu_i | k - \gamma_0 K | \nu_i \rangle \langle k' - \gamma_0 K | \nu_i \rangle, \]
as can be directly read from its Shiva diagram shown in
Fig. 2(a) (see Appendix I B).

Equation (20) then leads to
\[ 0 = \left( E_{i'k} - \varepsilon^{(ad)}_g \right) \psi^{(ad)}_{i'k} + \sum_{i',k'} \zeta^{k'k}_{i'\ i} \psi^{(ad)}_{i',k'}, \]
where \( \zeta^{k'k}_{i'\ i} \) is an effective scattering that contains
direct and exchange processes (see Appendix I C)
\[ \zeta^{k'k}_{i'\ i} = \frac{\varepsilon^{\text{int}}_{i'\ i} (k'k)}{2} + \frac{\varepsilon^{\text{exch}}_{i'\ i} (k'k)}{2}. \]

The “in” exchange interaction scattering, shown by
the Shiva diagram of Fig. 2(c), is defined as \( \varepsilon^{\text{exch}}_{i'\ i} (k'k) = \sum_{j,p} \lambda^{k'p}_{i'\ i} \). With the help of Eq. (3), it can be
written in a form such that the potential amplitude \( v \) does not appear explicitly, namely (see Appendix I B)
\[ \varepsilon^{\text{exch}}_{i'\ i} (k'k) = \frac{\varepsilon^{\text{in}}_{i'\ i} (k'k)}{2} - \frac{\varepsilon^{\text{int}}_{i'\ i} (k'k)}{2} \]
with \( \varepsilon^{\text{in}}_{i'\ i} (k'k) \) replaced by \( \varepsilon^{\text{int}}_{i'\ i} (k'k) \).

The \( \zeta^{k'k}_{i'\ i} \) effective scattering also has a less obvious part,
namely the dimensionless Pauli scattering multiplied
by an energy to make it energy-like,
\[ \zeta^{k'k}_{i'\ i} = \frac{\varepsilon^{\text{exch}}_{i'\ i} (k'k)}{2} \frac{E_{i'k} + E_{\nu_i k} - \varepsilon^{(ad)}_g}{2}. \]

It is possible to show (see Appendix I C) that the effective scattering which rules the \( (\alpha, \alpha, \beta) \) ground state fulfills
\[ \zeta^{k'k}_{i'\ i} = \zeta^{k'k}_{i'\ i}, \]
as physically required by time reversal symmetry.

To obtain \( \Delta^{(ad)} \) in Eq. (12), we replace \( \varepsilon^{(ad)}_g \) by \( \varepsilon^{(ad)}_g + \Delta^{(ad)} \) in Eq. (25), but by \( \varepsilon^{(ad)}_g \) only in Eq. (29)
because \( \Delta^{(ad)} \) scales as \( 1/L^3 \), so it gives a contribution vanishingly small compared to the other terms of \( \zeta^{k'k}_{i'\ i} \).

Next, in the sum of Eq. (25), we separate the lowest energy term \( (g, 0) \) from the other \( \{i, k\} \neq (g, 0) \) terms.

We can replace \( \varepsilon^{(ad)}_g = E_{g, 0} + \Delta^{(ad)} \) in the above integral equation by \( E_{g, 0} \) because \( \Delta^{(ad)} \propto 1/L^3 \) is negligible in front of any energy difference for \( (i', k') \neq (g, 0) \).

As \( \Psi^{(ad)}_{g, 0} \neq 0 \) by construction, Eq. (30) readily gives
\[ \Delta^{(ad)} = \zeta^{00}_{g, g}. \]

To go further, we note that the center-of-mass momentum of the \( (g, 0) \) pair is equal to zero, so are the ones of the \( (i', k') \) pairs coupled to it in Eq. (31) because all scattering processes conserve momentum. As a result, the \( \zeta^{k'k}_{i'\ i} \) scatterings of interest are \( \zeta^{k'k}_{i'\ i} \equiv \zeta^{k'k}_{i'\ i}. \)

The relevant “in” exchange interaction scattering precisely reads
\[ \zeta^{in}_{i'\ i} (k, k') = \frac{\varepsilon^{\alpha}_{i'\ i} (k, k')}{2} \frac{E_{i'k} + E_{\nu_i k} - \varepsilon^{(ad)}_g}{2} \]
as obtained from Eqs. (7) and (10). So, the potential amplitude \( v \), and hence the direct interaction scattering \( \xi \) (see Eq. (19)), go to zero when the cutoff \( q_c \) gets large. By contrast, although the exchange interaction scattering also has a \( v \) prefactor, the sum it contains makes it finite for large \( q_c \), as seen from Eq. (28). So, the direct interaction scattering can be neglected in front of the other parts of the \( \zeta \) effective scattering (26).

The relevant “in” exchange interaction scattering precisely reads
\[ \zeta^{in}_{i'\ i} (k, k') = -\frac{\varepsilon^{\alpha}_{i'\ i} (k, k')}{2} \frac{E_{i'k} + E_{\nu_i k} - \varepsilon^{(ad)}_g}{2} \]
and
\[ -8\pi \frac{a_d}{2\mu d L^3} \sum_{i',k'} \frac{\delta_{i'k'}}{1+\frac{\Delta^{(ad)}}{\varepsilon^{(ad)}_g}} I_{k,k'-I_{k',k}} (34) \]
with \( \bar{k} = k_0 a_d \), the cutoff part being possibly replaced by 1 since \( q_c a_d \gg 1 \). The “out” exchange interaction scattering reads as \( \zeta^{exch}_{i'\ i} (k, k') \) with \( I_{k,k'} \) replaced by \( I_{k',k} \).

Turning to the Pauli part of the effective scattering, we find that it reads
\[ \zeta^{exch}_{i'\ i} (k, k') = 4\pi \frac{a_d}{2\mu d L^3} \frac{\delta_{i'k'}}{1+\frac{\Delta^{(ad)}}{\varepsilon^{(ad)}_g}} I_{k,k'-I_{k',k}} (35) \]
and
\[ = 4\pi \frac{a_d}{2\mu d L^3} \frac{\delta_{i'k'}}{1+\frac{\Delta^{(ad)}}{\varepsilon^{(ad)}_g}} I_{k,k'-I_{k',k}}, \]
as \( \mu_{ad}(1 - \gamma_\alpha^2) = \mu_d \). All this leads to
\[
\zeta(k, k') = 4\pi \left( \frac{a_d}{2\mu_dL^3} \right) \frac{2k^2}{k^2 - k'^2} I_{k,k'} + (k \leftrightarrow k') .
\]

Next, we perform the angular integration
\[
\tilde{I}(k, k') = \frac{1}{2} \int_0^\pi \sin \theta_{kk'} \, d\theta_{kk'} I_{k,k'} = \frac{1}{4\gamma_\alpha kk'} \ln \left( 1 + \frac{(\bar{k} + \gamma_\alpha k')^2}{1 + (k - \gamma_\alpha k')^2} \right) .
\]

Inserting this result in Eq. (36) gives the effective scattering \( \zeta \) as
\[
\tilde{\zeta}(k, k') = 8\pi \left( \frac{a_d}{2m_\alpha L^3} \right) \frac{1}{k} \frac{\bar{k}^2}{k^2 - k'^2} \ln \left( 1 + \frac{(\bar{k} + \gamma_\alpha k')^2}{1 + (k - \gamma_\alpha k')^2} \right) + (k \leftrightarrow k') .
\]

Using the above expression, it becomes easy to compute \( \zeta_0^{(0)} \), solution of Eq. (31), and obtain the atom-dimer scattering length. Before going further, we wish to stress that the integral equation (31) is different from the one obtained using a field theoretical approach\(^{11}\) — which in fact is normal because they do not give the same quantity. Yet, the value of the scattering length \( a_{ad} \) we obtain is the same for all mass ratios. To better grasp this result, let us first consider very heavy and very light \( \beta \) atoms.

**B. \( m_\beta/m_\alpha \to \infty \)**

When the \( \beta \) atom is very heavy, \( \mu_d \simeq m_\alpha \simeq \mu_{ad} \) and \( \gamma_\alpha \simeq 0 \); so, the \( m_\beta \) mass disappears from the problem. As a result, the scattering length does not depend on the mass ratio \( \gamma = m_\beta/m_\alpha \). In this limit, Eq. (38) reduces to
\[
\tilde{\zeta}(k, k') \simeq 8\pi \left( \frac{a_d}{2m_\alpha L^3} \right) \frac{1}{k} \frac{1}{(1 + k^2)(1 + k')^2} .
\]

It is worth noting that \( \zeta(k, k') \) in Eq. (36) is equal to this quantity without angular averaging. This physically shows that only s-wave scattering survives in this limit.

The integral equation (31) then appears as
\[
\tilde{\zeta}(k, 0) \simeq 8\pi \left( \frac{a_d}{2m_\alpha L^3} \right) \frac{1}{1 + k^2} \frac{d\tilde{\zeta}(k', 0)}{1 + k'^2} + \int_{-\infty}^{\infty} \frac{d\bar{k}'}{k'} \ln \left( 1 + (\bar{k'} - k')^2 \right) \tilde{\zeta}(k', 0) ,
\]

in agreement with the result using a field theoretical approach\(^{11}\). This integral equation can be solved to give
\[
a_{ad} \simeq a_d \sqrt{\frac{m_\alpha}{m_\beta}} .
\]

It is of interest to note that the mass-dependent prefactor of the Pauli part (35) in the effective scattering \( \zeta \) is \( 1/\mu_{ad} \), since the \( (I_{k,k'} - I_{k',k}) \) difference scales as \( 1 - \gamma_\alpha^2 \). By contrast, the mass-dependent prefactor of the exchange interaction part (34) is \( 1/\mu_d \). So, in the large \( m_\beta \) limit, these two parts of the effective scattering play an equally important role, while the scattering length in the small \( m_\beta \) limit is controlled by exchange interaction scatterings only.

---

**FIG. 3: Atom-dimer scattering length obtained using Eq. (38).** The value for \( m_\alpha = m_\beta \) is \( a_{ad} \simeq 1.18a_d \).

**C. \( m_\beta/m_\alpha \to 0 \)**

When the \( \alpha \) atom is very heavy, \( \mu_d \simeq m_\beta \), \( \gamma_\alpha \simeq 1 \), and \( \mu_{ad} \simeq m_\alpha/2 \). So, in this limit, the two masses remain in the problem and the scattering length must depend on the mass ratio \( \gamma = m_\beta/m_\alpha \). To study the \( \gamma \) dependence of \( a_{ad} \), we expand the logarithmic function in Eq. (38) up to first order in \( \gamma \),
\[
\ln \left( \frac{1 + (\bar{k} + \gamma_\alpha k')^2}{1 + (k - \gamma_\alpha k')^2} \right) \simeq \ln \left( \frac{1 + (\bar{k} + k')^2}{1 + (k - k')^2} \right) + 2\gamma \frac{1 + \bar{k}(k + k')}{1 + (k + k')^2} - \frac{1 + \bar{k}(k - k')}{1 + (k - k')^2} .
\]

The integral equation (31) then appears, since \( \tilde{\zeta}(k, 0) = \tilde{\zeta}(-k, 0) \), as
\[
2\pi \gamma \tilde{\zeta}(k, 0) \simeq 8\pi^2 a_d \frac{\bar{k}}{m_\alpha L^3} \frac{1}{1 + k^2} + \int_{-\infty}^{\infty} \frac{d\bar{k}'}{k'} \ln \left( 1 + (\bar{k'} - k')^2 \right) \tilde{\zeta}(k', 0) ,
\]
in agreement with the result using a field theoretical approach\(^{11}\). This integral equation can be solved to give
\[
a_{ad} \simeq a_d \ln \frac{m_\alpha}{m_\beta} .
\]
D. Numerical results

The above results show that the integral equation we have derived gives analytically exact atom-dimer scattering length in the large and small \( m_\beta \) limits, despite the fact that we have obtained these limits with the dimer relative motion restricted to the ground state. For finite mass ratios, we numerically solved Eq. (31) with the effective scattering also given in Eq. (38). The results are shown in Fig. 3. Even for equal masses, which is expected to be the worst case, the scattering length value is equal to the accepted value

\[
a_{ad} \simeq 1.18 a_d .
\]

This leads us to conclude that our coboson many-body approach allows us to recover the previously obtained exact results for all mass ratios.

IV. DIMER-DIMER SCATTERING

In a previous work[15], we have studied the dimer-dimer scattering length \( a_{dd} \). It follows in a similar fashion from the ground state energy \( E_g^{(dd)} \) of \((\alpha, \alpha, \beta, \beta)\) fermions through

\[
\Delta^{(dd)} = E_g^{(dd)} - 2E_g = 4\pi \left( \frac{g_{dd}}{2m_{dd}L^3} \right),
\]

with \( \mu_{dd}^{-1} = 2(m_\alpha + m_\beta)^{-1} \) being the relative motion mass of the two dimers. As for atom-dimer scattering, the energy difference \( \Delta^{(dd)} \) can be identified with \( \tilde{\zeta}^{(gg)} \) obtained from the integral equation

\[
\tilde{\zeta}^{(gg)}(i_g) = \tilde{\zeta}^{(gg)}(i_g) + \sum_{mn} \frac{1}{E_{gg} - E_{nm}} \tilde{\zeta}^{(mg)}(ng)
\]

with \( E_{ij} = E_i + E_j \). Here also, the kernel scattering

\[
\zeta^{(mj)}(mi) = \xi^{(mj)} - \frac{\xi^{in}(mj)}{2} + \frac{\xi^{out}(nj)}{2} - \frac{E_{mm} + E_{ij} - \xi^{(dd)}}{2} \lambda^{(nj)}(mi)
\]

\[
= \zeta^{(mj)} - \xi^{exch}(mj) - \xi^{Pauli}(mj)\]

contains the fundamental dimer-dimer scatterings represented by the Shiva diagrams of Fig. 4. Their precise expressions are given in Appendix II.

Here also, to obtain the dimer-dimer scattering length as a function of mass ratio, we have numerically computed Eq. (47) with the dimer relative motion restricted to the ground state \( n_\gamma \) [15]. This amounts to neglecting excitations to relative motion states other than the ground state. For the worst case, that is, for equal masses, we obtained 0.64\( a_d \) instead of 0.60\( a_d \). Although we have reported[15] the numerical value of this scattering length \( a_{dd} \), it is of interest to analyze the limiting case when one of the atoms is very heavy, e.g., \( \gamma = m_\beta/m_\alpha \to 0 \), as we have done for atom-dimer scattering, in order to investigate the consequences of using a dimer basis restricted to the relative motion ground state.

When the \( \alpha \) atom mass is large, \( (\gamma_\beta, \gamma_\alpha) \to (0,1) \). The “in” exchange interaction scattering given in Eq. (C.11) and the Pauli scattering given in Eq. (C.12) reduce, for \( \xi^{(K',\nu)} = \xi^{(K',\nu)} = \xi^{(K',\nu)} \), to

\[
\xi^{in}(K',K) \simeq -\frac{8\pi a_d}{\mu_d L^3} \frac{1}{Q^2 + 4},
\]

\[
\lambda(K',K) \simeq \left( \frac{a_d}{L} \right)^3 \frac{16\pi}{Q^2 + 4},
\]

with \( Q = (K - K') a_d \). As for atom-dimer scattering, the direct interaction scattering \( \xi \) given in Eq. (C.9) contains a r factor, and so can be neglected when the cutoff \( q_c \) is large. Moreover, for small \( \gamma \), the mass-dependent prefactor of the Pauli part, \( \xi^{exch} \) in the above \( \xi \) scattering is \( \mu_d/\mu_{dd} \), smaller than the exchange interaction part, \( \xi^{exch} \). So, in this limit, the dimer-dimer scattering is controlled by the exchange interaction part only. Equation (48) then reduces to

\[
\zeta^{(K',K)} \simeq \frac{8\pi a_d}{\mu_d L^3} \frac{1}{Q^2 + 4}.
\]

After angular averaging, we get

\[
\zeta^{(K',K)} = \frac{1}{2} \int_0^\pi \sin \theta_{KK'} d\theta_{KK'} \zeta^{(K',K)} \simeq \frac{2\pi a_d}{\mu_d L^3 K K'} \ln \left( \frac{4 + (K + K')^2}{4 + (K - K')^2} \right).
\]
Inserting this result into Eq. (47) leads to
\[
\pi \gamma K \bar{\zeta} \left(K, 0\right) \simeq \frac{8\pi^2 a_d}{m_\alpha L^3} \frac{\bar{K}}{4 + K^2}
\int_0^\infty \frac{d\bar{K}' \ln \left(4 + (\bar{K} + \bar{K}')^2\right)}{4 + (\bar{K} - \bar{K}')^2} \bar{\zeta}(K', 0).
\]
This equation is identical to the one obtained from a field theoretical approach (see Eq. (56) in Ref. [18]). Its analytical resolution gives the dimer-dimer scattering length as
\[
a_{dd} \simeq \frac{a_d}{2} \ln \frac{m_\alpha}{m_\beta} \simeq \frac{a_{dd}}{2}.
\]
We conclude that our coboson many-body procedure again gives the exact value for the dimer-dimer scattering length, except for a slight deviation near equal atom masses.

V. CONCLUSION

We extend the coboson many-body formalism to cold atoms and derive the atom-dimer and dimer-dimer scattering lengths for arbitrary mass ratios. For atom-dimer, we recover the previously obtained exact values for all masses. This remarkable result also holds true for dimer-dimer except near equal atom masses where its value slightly deviates. These results show that restricting the dimer relative motion basis to the ground state, as we have done to numerically solve the integral equation for effective scattering in an easy way, is an excellent approximation. Our work also shows that atom-dimer scattering and dimer-dimer scattering can be physically seen as two-body scattering, provided fermion exchange with dimers is properly handled.

Moreover, through Shiva diagrams, the coboson many-body formalism provides a visual understanding of the role of the Pauli exclusion principle for composite bosons (for scatterings relevant to cold atom physics). Just like Feynman diagrams for elementary particles, Shiva diagrams make this coboson formalism an alternative powerful tool for understanding and calculating few-body as well as many-body effects involving composite quantum particles, that is, nearly all particles of physical interest.

Appendix I. ATOM-DIMER INTERACTION

A. Commutators or anticommutators

The coboson formalism we have developed for fermion pairs interacting with fermion pairs makes use of commutators only. In its extension to dimer interacting with fermions, commutators are mixed with anticommutators when the \((\alpha, \beta)\) fermions commute,
\[
0 = \left[b^+_k, a^\dagger_k\right]_\eta = b^+_k a^\dagger_k + \eta a^\dagger_k b^+_k,
\]
with \(\eta = (-)\) for a commutator and \(\eta = (+)\) for an anticommutator. Let us here show why this has to be so. The simplest way to understand when to use commutator or anticommutator for \((A, B)\) operators is to start with \(AB\) and push \(A\) to the right.

(i) The commutation relation between \((H, B^\dagger_i)\) must be a commutator whatever \(\eta\)

Let us consider \(HB^\dagger_i\). We find, for \(\{a^\dagger_k, a^\dagger_k\} = \delta_{k', k}\),
\[
H_0^{(a)} B^\dagger_i = \sum K \frac{\alpha K}{a^\dagger_k a^\dagger_k a^\dagger_{\bar{K}} K_{p+\gamma_k K}} b^\dagger_{p+\gamma_k K} (p) |\nu_i\rangle
\]
\[
= \sum K \frac{\alpha K}{a^\dagger_k a^\dagger_{\bar{K}} K_{p+\gamma_k K}} b^\dagger_{p+\gamma_k K} (p) |\nu_i\rangle
\]
\[
- \sum \gamma K \frac{\alpha K}{a^\dagger_k a^\dagger_{\bar{K}} K_{p+\gamma_k K}} a_k b^\dagger_{p+\gamma_k K} (p) |\nu_i\rangle. \tag{A.2}
\]

The second term on the right-hand side of the above equation is equal to \(+B^\dagger_i H_0^{(a)}\), with a similar result for \(H_0^{(b)} B^\dagger_i\) and \(V B^\dagger_i\). So, by combining these terms with the corresponding left-hand sides, we end with a commutator \([H, B^\dagger_i]\) whatever \(\eta\).

(ii) The commutation relation between \((V^\dagger_i, a^\dagger_k)\) depends on \(\eta\)

From the definition of \(V^\dagger_i\) given in Eq. (17), we get
\[
V^\dagger_i a^\dagger_k = \left(-E_i B^\dagger_i + [H, B^\dagger_i]\right) a^\dagger_k
\]
\[
= -E_i B^\dagger_i a^\dagger_k + H B^\dagger_i a^\dagger_k - B^\dagger_i H a^\dagger_k. \tag{A.3}
\]
To go further, we note that
\[
H_0 a^\dagger_k = \frac{\alpha K}{a^\dagger_k} + a^\dagger_k H_0, \tag{A.4}
\]
which leads us to consider \([H_0, a^\dagger_k]\) whatever \(\eta\); and similarly for \(V a^\dagger_k\). Equation (A.3) then reads, with the help of Eq. (15),
\[
V^\dagger_i a^\dagger_k = \eta E_i a^\dagger_k B^\dagger_i - \gamma \left([H, a^\dagger_k]_\eta + a^\dagger_k H\right) B^\dagger_i
\]
\[
- B^\dagger_i \left([H, a^\dagger_k]_\eta + a^\dagger_k H\right), \tag{A.5}
\]
which can be written in a compact form as
\[
V^\dagger_i a^\dagger_k = -\eta a^\dagger_k V^\dagger_i - \left[B^\dagger_i, [H, a^\dagger_k]_\eta\right]. \tag{A.6}
\]
This leads us to consider \([V^\dagger_i, a^\dagger_k]_\eta\), that is, whether to use a commutator or an anticommutator depends on \(\eta\).

(iii) The commutation relation between \((B^\dagger_i, B^\dagger_i)\) must be a commutator whatever \(\eta\)

From the definition of \(B^\dagger_i\) given in Eq. (2), we find
\[
B^\dagger_i B^\dagger_i = \sum_{p p'} \langle \nu_i | p' \rangle \langle p | \nu_i \rangle \times b_{-p' + \gamma_k K} a_{p + \gamma_k K} a^\dagger_{p + \gamma_k K} b^\dagger_{-p + \gamma_k K}, \tag{A.7}
\]
which also reads
\[ B_i, B_j^\dagger = \sum_{pp'} \langle \nu_i | p' \rangle \langle p | \nu_i \rangle \times \left\{ b_{-p'+\gamma \beta K, i} \left[ a_{p'+\gamma \alpha K, j} a_{p+\gamma \alpha K, j}^\dagger \right] + b_{p+\gamma \beta K, i} \right\} \]
(6.8)

By writing the second term of the above curly bracket as
\[ a_{p+\gamma \alpha K, j} a_{p'+\gamma \alpha K, j}^\dagger, \]
and using
\[ b_{-p'+\gamma \beta K, i} b_{p+\gamma \beta K, i} = \left[ b_{-p'+\gamma \beta K, i} b_{p+\gamma \beta K, i}^\dagger \right] - b_{p+\gamma \beta K, i} b_{-p'+\gamma \beta K, i}^\dagger, \]
the second term of Eq. (6.8) leads to \(+ B_i, B_j^\dagger \). So, here again, we must consider a commutator \([ B_i, B_j^\dagger \] whatever \( \eta \).

(iv) The commutation relation between \((D_{\nu_i}, a_k^\dagger)\) must be a commutator whatever \( \eta \).

From the definition of \( D_{\nu_i} \) given in Eq. (6.3), we get
\[ D_{\nu_i} a_k^\dagger = \left( \delta_{\nu_i} - \left[ B_{\nu_i}, B_j^\dagger \right] \right) a_k^\dagger \]
(6.10)

To go further, we note that
\[ B_{\nu_i} a_k^\dagger = \sum_{pp'} \langle \nu_i | p' \rangle \langle p | \nu_i \rangle \left\{ a_{p'+\gamma \beta K, j} a_{p+\gamma \beta K, j}^\dagger \right\} + a_{p+\gamma \alpha K, j} a_{p'+\gamma \alpha K, j}^\dagger \]
(6.11)

As the second term on the right-hand side of the above equation is equal to \(- \eta a_k^\dagger B_{\nu_i} \), we are led to consider \([ B_{\nu_i}, a_k^\dagger \] whatever \( \eta \).

When used into Eq. (6.11), this gives
\[ D_{\nu_i} a_k^\dagger = \delta_{\nu_i} a_k^\dagger + B_j^\dagger \left( \left[ B_{\nu_i}, a_k^\dagger \right] \right) \eta a_k^\dagger B_{\nu_i} \]
(6.12)

which again leads us to consider the commutator \([ D_{\nu_i}, a_k^\dagger \] whatever \( \eta \).

B. Relevant atom-dimer scatterings

In this appendix, we derive the various \((\xi_{\nu_i}, \zeta_{\xi_{\nu_i}}, \lambda)\) scatterings that enter the effective scattering \( \zeta \) in the case of cold atoms interacting through different-species fermions only. We will do it by reading their expressions from the Shiva diagrams that represent them. The algebraic procedure to recover these expressions starting from their definitions in terms of commutation relations, is similar to the one used for excitons (see Ref. [19]).

(i) Pauli scattering for fermion exchange

![Diagram](image)

**FIG. 5:** Diagrammatic representation of the Pauli scattering \( \lambda_{\nu_i}^{k'k} \).

The Pauli scattering \( \lambda_{\nu_i}^{k'k} \) is shown by the Shiva diagram of Fig. 5. The dimer \( i \) expands on \( B_{k_i}^\dagger p \) pairs through a vertex \( \langle p | \nu_i \rangle \), denoted by a dashed box. After exchanging with the \( \alpha \) atom in state \( k \), the \( \alpha \) atom of the dimer \( i \), initially in state \( p + \gamma \alpha K_i \), ends in state \( k' \); so, \( k' = p + \gamma \alpha K_i \). Similarly, the \( \alpha \) atom in state \( k \) ends in state \( p + \gamma \alpha K_i \) to form the dimer \( i' \) through a vertex \( \langle \nu_i | p' \rangle \); so, \( k = p' + \gamma \alpha K_i \). In this exchange, the \( \beta \) atom stays unchanged; so, \(-p' + \gamma \beta K_i = -p + \gamma \beta K_i \). All this leads to

\[ \lambda_{\nu_i}^{k'k} = \sum_{pp'} \langle \nu_i | p' \rangle \langle p | \nu_i \rangle \times \delta_{k, p' + \gamma \alpha K_i} \delta_{k, p + \gamma \alpha K_i} \delta_{p' - \gamma \beta K_i, -p + \gamma \beta K_i}, \]
(6.1)

which readily gives the expression of \( \lambda_{\nu_i}^{k'k} \) in Eq. (24).

(ii) Direct interaction scattering

![Diagram](image)

**FIG. 6:** Diagrammatic representation of the direct interaction scattering \( \xi_{\nu_i}^{k'k} \).

The direct interaction scattering \( \xi_{\nu_i}^{k'k} \) is shown by the Shiva diagram of Fig. 6. As for Pauli scattering, this scattering contains the two vertices \( \langle \nu_i | p' \rangle \) and \( \langle p | \nu_i \rangle \).
Since $\alpha$ atoms do not interact between themselves, the $\alpha$ atom making the dimer $i$ and $i'$ stays unchanged; so, $k_\alpha = p' + \gamma_\alpha K_\nu = p + \gamma_\alpha K_i$. By contrast, the $V$ potential transforms the pair made of the free $\alpha$ atom and the $\beta$ atom of the dimer $(k, -p + \gamma_\beta K_i)$ into $(k', -p' + \gamma_\beta K_{i'})$ with a scattering amplitude $-\nu w_{q'q} w_{q'i'}$ where $q'$ and $q$ are the final and initial relative motion momenta of the pair, its center-of-mass momentum $K$ staying unchanged; so, $K = k - p + \gamma_\beta K_i = k' - p' + \gamma_\beta K_{i'}$, while $k = q + \gamma_\alpha K$ and $k' = q' + \gamma_\alpha K$. All this leads to

$$\xi^{i'i'}_{k'k} = -\nu \sum_{pp'} \langle \nu'|p'|\langle p|\nu\rangle \delta_{k,p+\gamma_\alpha K_{i'}} \delta_{k',p+\gamma_\alpha K_i} \sum_{qq'} w_{q'q} \delta_{k,q+\gamma_\alpha K_{i'}} \delta_{k',q+\gamma_\alpha K_i},$$

which readily gives the expression of $\xi^{i'i'}_{k'k}$ in Eq. (19).

(iii) “In” exchange interaction scattering

![Diagram of “In” exchange interaction scattering](image)

FIG. 7: Diagrammatic representation of the “in” exchange interaction scattering $\xi^{i'i'}_{k'k}$.

The “in” exchange interaction scattering $\xi^{i'i'}_{k'k}$ is shown by the Shiva diagram of Fig. 7. As for the direct interaction and Pauli scatterings, the exchange interaction scattering $\xi^{i'i'}_{k'k}$ contains the two vertices $\langle p|\nu\rangle$ and $\langle \nu'|p'\rangle$. Due to fermion exchange, the $\alpha$ atom from the dimer $i$ goes from state $p + \gamma_\alpha K_i$ to state $k'$; so, $k' = p + \gamma_\alpha K_i$. The potential $V$ transforms the pair of $(\alpha, \beta)$ atoms $(k, -p + \gamma_\beta K_i)$ into the pair $(p' + \gamma_\alpha K_{i'}, -p' + \gamma_\beta K_{i'})$ with the $i'$ dimer, with a scattering amplitude $-\nu w_{q'q} w_{q'i'}$ where $q'$ and $q$ are the final and initial relative motion momenta of the pair, the center-of-mass momentum $K$ of this pair staying unchanged. So, $K = k - p + \gamma_\beta K_i = K_{i'}$, while $k = q + \gamma_\alpha K$ and $p' + \gamma_\beta K_{i'} = q' + \gamma_\alpha K$. All this leads to

$$\xi^{i'i'}_{k'k} = -\nu \sum_{pp'} \langle \nu'|p'|\langle p|\nu\rangle \delta_{k,p+\gamma_\alpha K_i} \delta_{k',p+\gamma_\alpha K_i} \sum_{qq'} w_{q'q} \delta_{k,q+\gamma_\alpha K_{i'}} \delta_{k',q+\gamma_\alpha K_i},$$

which readily gives

$$\xi^{i'i'}_{k'k} = -\delta_{k',K_{i'}} \delta_{k,K_i} \langle k' - \gamma_\alpha K_i|\nu\rangle \sum_{p'} \left\{ w_{k'\gamma_\alpha K_{i'}} \sum_{q'} w_{q'\gamma_\alpha K_{i'}} \langle k' - \gamma_\alpha K_i|\nu\rangle \right\}. \quad \text{(B.4)}$$

Actually, it is possible to make the potential amplitude $v$ formally disappear from $\xi^{i'i'}_{k'k}$ by noting that, through Eq. (3), the curly bracket of the above equation is equal to $(\varepsilon_k - \varepsilon_{K_{i'}})(\varepsilon_k - \varepsilon_{K_{i'}})$. So, we end with the expression of $\xi^{i'i'}_{k'k}$ given in Eq. (28).

C. Derivation of the effective scattering $\xi^{i'i'}_{k'k}$

In this appendix, we derive the effective scattering $\xi^{i'i'}_{k'k}$ in terms of the direct scattering $\xi$, the two exchange interaction scatterings ($\xi^{i'i'}_{k'k}$, $\xi^{i'i'}_{k'k}$), and the Pauli scattering $\lambda$.

We start with Eq. (20) that we project over the atom-dimer state $|\nu\rangle |\nu\rangle_B$. With the help of Eq. (21), we find

$$0 = \left( E_{i',k'} - \epsilon_g^{(ad)} \right) |\psi_{i',k'}^{(ad)}\rangle + \sum_{i',i''} \lambda^{i'i''}_{i',k'} \left( E_{i,k} - \epsilon_g^{(ad)} \right) |\psi_{i',k'}^{(ad)}\rangle$$

$$- \sum_{i,k} \lambda^{i'i'}_{i,k} \left( E_{i,k} - \epsilon_g^{(ad)} \right) |\psi_{i',k'}^{(ad)}\rangle,$$

the scattering in the last term actually being the “in” exchange interaction scattering

$$\xi^{i'i'}_{k'k} = \sum_{i,k} \lambda^{i'i'}_{i,k} \xi^{i'i'}_{k'k}.$$

As a result, Eq. (C.1) is an integral equation for $|\psi_{i',k'}^{(ad)}\rangle$.

$$0 = \left( E_{i,k} - \epsilon_g^{(ad)} \right) |\psi_{i',k'}^{(ad)}\rangle + \sum_{i',i''} \lambda^{i'i''}_{i',k'} |\psi_{i',k'}^{(ad)}\rangle,$$

with an effective scattering $\lambda$ that appears as

$$\xi^{i'i'}_{k'k} = \xi^{i'i'}_{k'k} - \lambda^{i'i'}_{i,k} \left( E_{i,k} - \epsilon_g^{(ad)} \right). \quad \text{(C.4)}$$

We can write this effective scattering in a symmetric form with respect to initial and final atom-dimer states by noting that

$$\xi^{i'i'}_{k'k} = \lambda^{i'i'}_{i,k} |\psi_{i',k'}^{(ad)}\rangle + \chi^{i'i'}_{k'k} \left( E_{i,k} - \epsilon_g^{(ad)} \right) |\psi_{i',k'}^{(ad)}\rangle,$$

where the “out” exchange interaction scattering is defined as $\xi^{i'i'}_{k'k} = \sum_{i',k'} \chi^{i'i'}_{k'k} \lambda^{i'i'}_{i,k}$. 

A simple way to show the relation (C.5) is to calculate \( \langle v | a_k B_i H B_i^d | v \rangle \) using the commutator (17). We get, for \( H \) acting on the right,

\[
\langle v | a_k B_i H B_i^d | a_k^\dagger v \rangle = E_{i,k'} \langle v | a_k B_i B_i^d a_k^\dagger | v \rangle + \sum_{i',k''} \langle v | a_k B_i B_i^d a_{k''}^d | v \rangle \langle k'' | v \rangle ,
\]

or with the help of Eq. (21),

\[
E_{i,k} \delta_{i'i'} \delta_{kk'} + \xi_{i'i'}^{(k,k')} - \xi_{i'i'}^{(k',k')} \lambda_{i'i'}^{(k,k')} E_{i,k'}.
\]

We can also calculate this quantity through the above equation. Equation (C.5) then follows from \( \lambda_{i'i'}^{(k,k')} = \xi_{i'i'}^{(k,k')}(\nu,\bar{v}) \) and \( \lambda_{i'i'}^{(k,k')} = \lambda_{i'i'}^{(k',k')} \), but

\[
\xi_{i'i'}^{(k,k')} = \xi_{i'i'}^{(k',k')}.
\]

Appendix II. RELEVANT DIMER-DIMER SCATTERINGS

Here, we recall the \( \xi, \xi_{out} \), and \( \lambda \) scatterings in the case of cold atoms. Detailed derivations in the case of excitons can be found in Ref. 20. The direct interaction scattering between \((\alpha_1, \beta_1)\) dimer and \((\alpha_2, \beta_2)\) dimer with the total center-of-mass momentum equal to zero reads as

\[
\xi^{(\nu_m,-K') (\nu_n,-K)} = -v_Q \sum_{k,p} \langle \nu_m | k + \beta Q | \nu_n \rangle \langle p + \alpha Q | p \rangle + \langle \nu_m | k - \alpha Q | \nu_n \rangle \langle p - \beta Q | p \rangle.
\]

with \( Q = K - K' \), while the “in” exchange interaction scattering is given by

\[
\xi_{in}^{(\nu_m,-K') (\nu_n,-K)} = -\sum_{k,p} v_P \left( \langle \nu_m | k + \frac{P - \alpha}{2} | \nu_n \rangle \langle k - \frac{P + \beta}{2} | \nu_n \rangle \right) \langle k + \frac{P - \alpha}{2} | \nu_n \rangle \langle k - \frac{P + \beta}{2} | \nu_n \rangle.
\]

As for the atom-dimer scattering, we can eliminate the potential amplitude \( v \) in this exchange interaction scattering by using Eq. (3). We then find

\[
\xi_{in}^{(\nu_m,-K') (\nu_n,-K)} = \sum_k \left( \xi_{in}^{(\nu_m,-K') (\nu_n,-K)} \right) \langle \nu_m | k + \frac{P}{2} | \nu_n \rangle \langle k - \frac{P}{2} | \nu_n \rangle \langle k + \frac{P}{2} | \nu_n \rangle \langle k - \frac{P}{2} | \nu_n \rangle.
\]

Finally, the Pauli scattering for fermion exchange between two dimers is given by

\[
\lambda^{(\nu_m,-K') (\nu_n,-K)} = \sum_k \langle \nu_m | k + \frac{P}{2} | \nu_n \rangle \langle k - \frac{P}{2} | \nu_n \rangle \langle k + \frac{P}{2} | \nu_n \rangle \langle k - \frac{P}{2} | \nu_n \rangle.
\]
[18] F. Alzetto, R. Combescot, and X. Leyronas, Phys. Rev. A 87, 022704 (2013).
[19] S.-Y. Shiau, M. Combescot, and Y.-C. Chang, Phys. Rev. B 86, 115210 (2012).
[20] S.-Y. Shiau, M. Combescot, and Y.-C. Chang, arXiv:1301.7266; Annals of Physics 336, 309 (2013).