SEQUENTIAL AND EXACT SUBDIFFERENTIAL CALCULUS 
RULES FOR NONCONVEX INTEGRAL FUNCTIONS

RAFAEL CORREA†, ABDERRAHIM HANTOUTE‡, AND PEDRO PÉREZ-AROS§

Abstract. We are concerned in this work with the subdifferential of the integral functional 
\[ E_f(x) = \int_T f(t, x) d\mu(t), \]
for normal integrands \( f \) (possibly, not convex), defined in a \( \sigma \)-finite measure space and a separable Banach space whose dual is also separable. By means of techniques of variational analysis, we establish some sequential formulae to estimate the Fréchet subdifferential of \( E_f \). We also give upper-estimates for the limiting subdifferential of \( E_f \) for both Lipschitz and non-Lipschitz integrands. This last result is based on a Lipschitz-like condition.

Key words. Normal integrands, Integral functions and functionals, Subdifferential calculus.

AMS subject classifications. 49J52, 28B05, 28B20

1. Introduction. We are interested in this paper to the variations at a first-order of the integral function given in the form 
\[ E_f(x) = \int_T f(t, x) d\mu(t), \]
for a normal integrand \( f : T \times X \to \mathbb{R} \cup \{+\infty\} \), which is defined on a measurable space \((T, \mathcal{A}, \mu)\) and an infinite-dimensional Banach space \((X, \|\cdot\|)\). The normal integrand is possibly non-convex. Our aim is to provide sequential and exact formulae of the nonconvex subdifferentials of the integral function \( E_f \), including the Fréchet, the limiting, and the Clarke-Rockafellar subdifferentials. This will be achieved by means of estimates that make use of the data, namely the measurable selections of the subdifferential of the integrand \( f \). The function \( E_f \) can also be regarded as a functional operator acting on the subspace of constant functions on \( T \), as in [32] and [43] (see, also, [7, 11, 24, 28, 31, 33, 34, 42, 44]), but here in the current work \( E_f \) is considered as such; that is, as a continuous sum.

This work is inspired by the results of Ioffe [28] and Lopez-Thibault [34]. In [28], where the author deals with convex normal integrands, one can find characterizations of the Fenchel subdifferential of the integral function \( E_f \), given by means of limiting processes relying on the subdifferential of the data. Such characterizations do not involve any qualification condition. Namely, assuming that \( f \) is a convex normal integrand, satisfying a mere linear growth condition, given as

\[ f(t, x) \geq \langle a^*(t), x \rangle + \alpha(t), \quad \text{for all } t \in T, \ x \in X, \]

for integrable functions \( a^* : T \to X^* \) and \( \alpha : T \to \mathbb{R} \), the subdifferential of \( E_f \) at a given point \( x \in X \) can be characterized in terms of limits of the integral of measurable

---

*Submitted to the editors DATE.

Funding: This work is partially supported by CONICYT grants: Fondecyt 1151003, Fondecyt 1150909, Basal PFB -03 and Basal FB0003, CONICYT-PCHA/doctorado Nacional / 2014-21140621.
†Center for Mathematical Modeling (CMM), Universidad de Chile, Santiago, Chile (ahantoute@dim.uchile.cl).
‡Instituto de Ciencias de la Ingeniería, Universidad de O’Higgins, Rancagua, Chile (pe-dro.perez@uoh.cl).
we give sequential formulae for the Fréchet subdifferential of the integral functional
which allows applying Borwein-Preiss’ variational principle. Next, we adapt to our setting
the notion of robust local minima (see Definition 3.1), which we combine with some variational
principles, applied in the space X as well as in the functional space of p-integrable functions. Using this we extend and improve
the results of [28] and [34] (see also [14,15]).

Let us mention that all the results given in this paper have been developed in [41].
Related results can be found in [23, 24, 39]. Here, for the sake of simplifying the
presentation, we only give the aforementioned results for the Fréchet and the limiting
subdifferentials, both based on the notion of robust infima, instead of the common
presentation, we only give the aforementioned results for the Fréchet and the limiting
subdifferentials, both based on the notion of robust infima, instead of the common
approach using chain rules as developed in [34].

This work is organized as follows: Section 2 is dedicated to recall some notions
of variational analysis and the generalized subdifferentiation that are needed in the
sequel. In Section 3 we adapt to our setting the notion of robust local minima (see
Definition 3.1), which allows applying Borwein-Preiss’ variational principle. Next,
we give sequential formulae for the Fréchet subdifferential of the integral functional
(see Theorems 3.4 and 3.5 and Corollary 3.8). In Section 4 we introduce a Lipschitz-like
condition, which generalizes the classical Lipschitz continuity of integral functionals
(see, e.g., [12, Theorem 2.7.2]), and leads to upper-estimates for the limiting subdiffere-
tial, as well as for the Clarke-Rockafellar subdifferential of the integral functional.
Finally, for the sake of simplifying the presentation of the work, the technical results
and proofs are given in the Appendix.

2. Notation and preliminary results. In the following, \((X, \| \cdot \|)\) will be a
separable Asplund space and \(X^*\) its dual, which means that \(X^*\) is also separable
(see, e.g., [19] for more details). The norm in \(X^*\) will be denoted also by \(\| \cdot \|\). For a point \(x \in X\) and \(r > 0\), the closed ball of radius \(r\) and centered at \(x\) is denoted
by \(B_X(x, r)\), or simply \(B(x, r)\) when no confusion occurs, particularly, the unit closed
ball is simply denoted by \(B\). The bilinear form \(\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}\) is given by
\(\langle x^*, x \rangle := \langle x, x^* \rangle := x^*(x)\). The weak*-topology on \(X^*\) is denoted by \(w(X^*, X)\)
\((w^*, \text{for short})\). We write \(\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}\) with the conventions \(1/\infty = 0\),
\(0 \cdot \infty = 0 = 0 \cdot (-\infty)\) and \(\infty + (-\infty) = (-\infty) + \infty = +\infty\).

For a set \(A \subseteq X\) (or \(\subseteq X^*\)), we denote by \(\text{int}(A), \overline{A}, \text{co}(A),\) and \(\text{cc}(A)\) the interior,
the closure, the convex hull and the closed convex hull of \(A\), respectively. The linear
space spanned by \(A\) is denoted by \(\text{span}(A)\) and the negative polar cone of \(A\) is the set
\[ A^- := \{x^* \in X^* \mid \langle x^*, x \rangle \leq 0, \forall x \in A\}. \]
The indicator and the support functions of a set \(A \subseteq X, X^*\) are, respectively,
\[ \delta_A(x) := \begin{cases} 0 & x \in A, \\ +\infty & x \notin A, \end{cases} \text{ and } \sigma_A(x^*) := \sup\{\langle x^*, x \rangle : x \in A\}. \]

For a given function \(f : X \to \overline{\mathbb{R}}\), the (effective) domain of \(f\) is \(\text{dom} f := \{x \in X \mid f(x) < +\infty\}\). We say that \(f\) is proper if \(\text{dom} f \neq \emptyset\) and \(f > -\infty\), and sequentially
\(\tau\)-inf-compact (for some topology \(\tau\) on \(X\)) if for every \(\lambda \in \mathbb{R}\) and every sequence \((x_n) \subset [f \leq \lambda] := \{x \in X \mid f(x) \leq \lambda\}\) there exists a subsequence \(x_{n_k} \rightharpoonup x \in [f \leq \lambda]\). We write \(x \rightharpoonup x_0 \in X\) to say that \(x \to x_0\) with \(f(x) \to f(x_0)\).

Now, we consider a function \(f : X \to \mathbb{R}\), which is finite at \(x\). Then the Fréchet (or regular) subdifferential of \(f\) at \(x\) is defined by

\[
\partial f(x) := \left\{ x^* \in X^* \mid \liminf_{h \to 0} \frac{f(x + h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq 0 \right\}.
\]

Since the space \(X\) is Fréchet smooth, that is, it has an equivalent norm that is \(C^1\) for all \(x \neq 0\) (see, e.g., [19, Theorem 8.6]), we have that the Fréchet subdifferential coincides with the Viscosity Fréchet Subdifferential, that is to say, \(x^* \in \partial f(x)\) if and only if there are neighbourhood \(U\) of \(x\) and a \(C^1\) function \(\phi : U \to \mathbb{R}\) such that \(\nabla \phi(x) = x^*\) and \(f - \phi\) attains its local minimum at \(x\) (see, e.g., [9, 40]).

The limiting (or basic, or Mordukhovich subdifferential) and the singular subdifferentials are defined as (see, e.g., [35–37])

\[
\partial f(x) := \left\{ x^* - \lim x^*_n : x^*_n \in \partial f(x_n), \ x_n \rightharpoonup x \right\},
\]

\[
\partial^\infty f(x) := \left\{ x^* - \lim \lambda_n x^*_n : x^*_n \in \partial f(x_n), \ x_n \rightharpoonup x \text{ and } \lambda_n \to 0^+ \right\},
\]

respectively. Finally, the Clarke-Rockafellar subdifferential can be defined as (see, e.g., [35, Theorem 3.57])

\[
\partial_C f(x) := \overline{\partial} \{\partial f(x) + \partial^\infty f(x)\}.
\]

If \(|f(x)| = +\infty\), we set \(\partial f(x) := \partial f(x) := \partial^\infty f(x) := \partial_C f(x) := \emptyset\).

It is important to emphasize that when \(f\) is lower-semicontinuous (lsc), proper and convex, the Fréchet, the limiting and the Clarke-Rockafellar subdifferentials coincide with the convex (or Fenchel, Moreau-Rockafellar) subdifferential given for \(x \in \text{dom } f\) by

\[
\partial f(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \forall y \in X\}.
\]

Throughout the paper, \((T, \mathcal{A}, \mu)\) is a complete \(\sigma\)-finite measure space. A function \(x : T \to X (\text{or } X^*)\) is called simple if there are \(k \in \mathbb{N}\), sets \(T_i \in \mathcal{A}\) and elements \(x_i \in X, \ i = 0, ..., k\), such that \(x(\cdot) = \sum_{i=0}^k x_i \mathbb{1}_{T_i}(\cdot)\), where \(\mathbb{1}_{T_i}(t) = 1\) if \(t \in T_i\) and \(\mathbb{1}_{T_i}(t) = 0\) if \(t \notin T_i\). A function \(x : T \to X (\text{or } X^*)\) is called measurable if there exists a countable family \(x_n\) of simple functions such that \(\lim_{n \to \infty} \|x(t) - x_n(t)\| = 0\) \(\mu\)-almost everywhere (ae, for short). For \(p \in [1, \infty)\) we denote by \(L^p(T, X)\) and \(L^p(T, X^*)\) the sets of all (equivalence classes by the relation \(f = g \text{ ae}\)) measurable functions \(f\) such that \(\|f(\cdot)\|_p\) is integrable. As usual, the corresponding norm in these spaces is \(\|f\|_p := \left( \int_T \|f(t)\|^p d\mu(t) \right)^{1/p}\). For an integrable function \(x^*(\cdot)\) and a measurable set \(A\), the symbols \(\int_A x^*(t) d\mu(t)\) denote the Bochner integral of \(x^*\) over \(A\) (see [18, §II. Integration] and the details therein). The space \(L^\infty(T, X)\) consists of all (equivalent classes with respect to the the relation \(f = g \text{ ae}\)) measurable and essentially bounded functions \(x : T \to X\). The associated norm on \(L^\infty(T, X)\) is given by \(\|x\|_\infty := \text{ess sup}_{t \in T} \|x(t)\|\). In a similar way, the space \(L^\infty(T, X^*)\) is defined by the set of all measurable essentially bounded function \(x^* : T \to X^*\).
A sequence of functions \((\varphi_k)_{k \in \mathbb{N}} \subseteq L^1(T, \mathbb{R})\) is said to be uniformly integrable if
\[
\lim_{\alpha \to \infty} \sup_k \int_{\{|\varphi_k(t)| \geq \alpha\}} |\varphi_k(t)| d\mu(t) = 0.
\]

A function \(f : T \times X \to \overline{\mathbb{R}}\) will be called a normal integrand if it is \(A \otimes \mathcal{B}(X)\)-measurable (where \(\mathcal{B}(X)\) is the Borel \(\sigma\)-algebra, i.e., the \(\sigma\)-algebra generated by all open sets of \(X\)) and for every \(t \in T\), \(f_t := f(t, \cdot)\) is lsc. For \(\partial\) being any one of the subdifferentials above and any measurable function \(x : T \to X\), we denote \(\partial f(t, x(t)) := \partial f_t(x(t))\); i.e., the subdifferentials are taken with respect to the second variable.

An integral functional on \(L^p(T, X)\) (with \(p \in [1, +\infty]\)) is an extended-real-valued functional \(I_{f}^{\mu} : L^p(T, X) \to \overline{\mathbb{R}}\) of the form
\[
x \to I_{f}^{\mu}(x(\cdot)) := \int_T f(t, x(t)) d\mu(t) := \int_T f(t, x(t))^+ d\mu(t) + \int_T f(t, x(t))^− d\mu(t),
\]
where \(\alpha^+ := \max\{\alpha, 0\}\) and \(\alpha^− := \min\{\alpha, 0\}\); we simply write \(I_f\), when there is no ambiguity.

The main concern of this paper is the study of the subdifferential of the following particular class of integral functionals (also called continuous sum or integral sum), \(E_f : X \to \overline{\mathbb{R}}\), given by
\[
E_f^{\mu}(x) := \int_T f(t, x) d\mu(t);
\]
we simply write \(E_f\), when there is no confusion. It is worth mentioning that \(E_f\) can be understood as the integral functional \(I_f\) restricted to the constant measurable functions.

The subdifferential theory of functions \(I_f\) and \(E_f\) goes back to Ioffe-Tikhomirov [32] and Rockafellar [43] for convex normal integrands. Posteriorly, these functionals have been considered by several authors; for example, Rockafellar [42,44], Ioffe-Levin [31], Levin [33], Castaing-Valadier [11], Ioffe [28], Lopez-Thibault [34], Borwein-Yao [7], and Ginther-Penot [24], among others.

Now, let us recall the concept of a graph measurable multifunction. We recall that a Hausdorff topological space \(S\) is a Suslin space provided that there exists a Polish space \(P\) (complete, metrizable and separable) and a continuous surjection from \(P\) to \(S\) (see [10,11,46]). Typical examples of suslin spaces are a separable Banach space (with the norm topology) and its dual (with the weak*-topology).

Consider a Suslin space \(S\). A multifunction \(M : T \rightrightarrows S\) is said to be graph measurable (or simply measurable) if its graph, \(\text{gph} M := \{(t, s) \in T \times S : s \in M(t)\}\), belongs to \(A \otimes \mathcal{B}(S)\) (see, e.g., [2,11,27,29,50,51] for more details).

The next proposition corresponds to the Measurable Selection Theorem for graph measurable multifunction with values in Suslin spaces.

**Proposition 2.1.** [11, Theorem III.22] Consider a graph measurable multifunction \(M : T \rightrightarrows S\) with non-empty values. Then the exists a measurable function \(m : T \to S\) such that \(m(t) \in M(t)\) for all \(t \in T\).

For a multifunction \(M : T \rightrightarrows X^*\) and a measurable set \(A \in \mathcal{A}\), we define the Bochner integral of \(M\) over \(A\) by
\[
\int_A M(t) d\mu(t) := \left\{ \int_A x^*(t) d\mu(t) : x^* \in L^1(T, X^*) \text{ and } x^*(t) \in M(t) \text{ ae} \right\}.
\]
It is worth recalling that the original definition of integral of set-valued mappings is due to R. J. Aumann, given for multifunctions defined on closed intervals in \( \mathbb{R} \) (see, for example, [3]).

Unless stated otherwise, in the rest of this article we assume that \( f \) is an integrand from \( T \times X \) to \([0, +\infty] \), \( X \) is a separable Asplund space and its norm is \( C^1 \) away from the origin, we refer to [19] for more details about the theory of Asplund spaces. Although the assumption about the range of the values of the integrand appears less general, many of the results in the literature can be obtained in our setting by modifying appropriately the integrand. We will talk more in depth about these techniques in Theorems 3.4 and 3.5. It is important to recall that in our framework the integral functional

\[
I_p(f) := \inf_{v \in L^p(T, X)} \int_T f(t, v(t)) d\mu(t)
\]

is lsc (see, e.g., [24, Lemma 10]).

The next proposition corresponds to an extension of the well-known result of Rockafellar concerning the interchange between the infimum and integral in finite-dimensional spaces (see, e.g., [42–45]). Lately, this theorem was extended to separable infinite-dimensional spaces [11, Theorem VII-7]. We also refer to [15,22,24,26] for other versions of this result.

**Proposition 2.2.** Let \( h : T \times X \to \mathbb{R} \) be a normal integrand such that \( I_h(u_0) < \infty \) for some \( u_0 \in L^p(T, X) \). Then

\[
\inf_{v \in L^p(T, X)} \int_T h(t, v(t)) d\mu(t) = \int_T \inf_{x \in X} h(t, x) d\mu(t).
\]

**3. Sequential formulae for the subdifferential of integral functions.** In order to deal with an arbitrary complete \( \sigma \)-finite measure space \((T, \mathcal{A}, \mu)\), we adapt here the notion of **robust local minima** or **decoupled infima** (see, for example, [9, 30, 35, 40]) to the case of integral functionals.

**Definition 3.1.** Let \((T, \mathcal{A}, \mu)\) be a finite measure space. Consider a function \( f : T \times X \to \mathbb{R} \) and \( p \in [1, +\infty) \). We define the \( p \)-stabilized infimum of \( E_f \) on \( B \subseteq X \) by

\[
\land_{p,B} E_f := \sup_{\varepsilon > 0} \left\{ \int_T f(t, x(t)) d\mu(t) \mid \text{and} \int_T \|x(t) - y\|^p d\mu(t) \leq \varepsilon \right\}.
\]

The infimum of \( E_f \) on \( B \) is called \( p \)-robust if \( \land_{p,B} E_f = \inf_B E_f \) and these quantities are finite; then a minimizer of \( E_f \) on \( B \) will be called a \( p \)-robust minimizer on \( B \). A point \( x \) will be called a \( p \)-robust local minimizer of \( E_f \) provided the existence of some \( \eta > 0 \) such that \( x \) is a \( p \)-robust minimizer on \( B(x, \eta) \).

The above definition is given only for finite measures, due to the fact that whenever \( \mu(T) = +\infty \), we have that \( \int_T \|x(t) - y\|^p d\mu(t) = \infty \) for all \((x, y) \in L^p(T, X) \times X \setminus \{0\}\).

However, in many of the results, when we work with a general \( \sigma \)-finite measure, we can modify the measure space to work with an equivalent finite measure.

It is worth mentioning that one can easily prove (using Hölder’s inequality [5, Corollary 2.11.5]) that a \( p \)-robust minimizer of \( E_f \) is also an \( r \)-robust minimizer for every \( r \geq p \).

The following result gives sufficient conditions for \( p \)-robustness. Particularly, in the finite-dimensional setting, each local minimum is a \( p \)-robust minimum.
Proposition 3.2. Let \((T, \mathcal{A}, \mu)\) be a finite measure space. Consider \(p \in [1, +\infty), \)
\(q := p/(p-1),\) and \(B \subseteq X\) such that \(\text{dom} E_f \cap B \neq \emptyset.\) Suppose one of the following conditions is satisfied:

\((a)\) For almost every \(t \in T, \) \(f(t, \cdot)\) is \(\tau\)-lsc, \(B\) is \(\tau\)-closed and there exists \(A \in \mathcal{A}\) with \(\mu(A) > 0\) such that for all \(t \in A, \) \(f(t, \cdot)\) is sequentially \(\tau\)-inf-compact, with \(\tau\) being some topology, which is coarser than the norm topology (i.e. \(\tau \subseteq \tau(\|\cdot\|)\)).

\((b)\) For almost every \(t \in T, \) \(f(t, \cdot)\) is \(\tau\)-lsc, and \(B\) is sequentially \(\tau\)-inf-compact.

\((c)\) For almost every \(t \in T, \) \(f_t\) is Lipschitz on \(X\) with some \(q\)-integrable constant. Then

\[
\wedge_{p,B} E_f = \inf_B E_f.
\]

**Proof.** See Appendix C.1.

Now we present a fuzzy necessary condition for the existence of a \(p\)-robust minimum in terms of the subdifferential of the data function \(f.\) The proof of this result is based in the application of Borwein-Preiss variational principle to some appropriate function defined in \(X \times L^p(T, X).\)

Theorem 3.3. Let \(p, q \in (1, +\infty)\) with \(1/p + 1/q = 1.\) Assume that the measure \(\mu\) is finite and that \(x_0 \in X\) is a \(p\)-robust local minimizer of \(E_f.\) Then there are sequences \(y_n \in X, x_n \in L^p(T, X),\) and \(x_n^* \in L^q(T, X^*)\) such that:

\((a)\) \(x_n^*(t) \in \partial f(t, x_n(t))\) ae,

\((b)\) \(\|x_0 - y_n\| \to 0, \|x_0 - x_n(\cdot)\|_p \to 0,\)

\((c)\) \(\|x_n^*(\cdot)\|_q \|x_n(\cdot) - y_n\|_p \to 0,\)

\((d)\) \(\int_T x_n^*(t) d\mu(t) \to 0,\)

\((e)\) \(\int_T |f(t, x_n(t)) - f(t, x_0)| d\mu(t) \to 0.\)

**Proof.** See Appendix C.2.

Now we establish the two main results of this section. In order to show how to adapt some of the settings available in the literature to our framework, we consider in the following theorems two normal integrands \(f, g : T \times X \to \overline{\mathbb{R}},\) satisfying the following properties:

\((P)\)

For all \(t \in T\) and all \(x \in X, \) \(f(t, x) \geq g(t, x).\)

The functions \(g_t\) are \(C^1\) for all \(t \in T.\)

A typical example is \(g(t, x) = \langle a^*(t), x \rangle + \alpha(t)\) with \(a^* \in L^p(T, X^*)\) and \(\alpha \in L^1(T, \mathbb{R}),\) which corresponds to the convex case; i.e., when \(f(t, \cdot)\) is convex ae (see [15, 28, 34]). The first main result shows a fuzzy calculus rules for the Fréchet subdifferential. This calculus rule can be obtained using an appropriated modification of the measure space and the normal integrand function in order to satisfy the assumptions of Proposition 3.2, and then we transform a local minimum into a \(p\)-robust minimum. The proof of this result shows in particular the high potential of our general setting.

**Theorem 3.4.** Let \(f, g\) be two normal integrands satisfying \((P)\) and \(p, q \in (1, +\infty)\) with \(1/p + 1/q = 1.\) Assume that \(\mu\) is finite and the function \(\sup_{u \in X} \|\nabla g(\cdot, u)\|\) belongs to \(L^q(T, \mathbb{R}).\) Then for every \(x^* \in \partial E_f(x)\) and every \(w^*\)-continuous seminorm \(\rho\) in \(X^*,\) there exist sequences \(y_n \in X, x_n \in L^p(T, X), x_n^* \in L^q(T, X^*)\) such that:
(a) $x_n^*(t) \in \partial \hat{f}(t, x_n(t))$ ae.
(b) $\|x - y_n\| \to 0, \|x - x_n(\cdot)\|_p \to 0,$
(c) $\|x_n^*(\cdot)|_q\|x_n(\cdot) - y_n\|_p \to 0,$
(d) $\int_T \langle x_n^*(t), x_n(t) - x \rangle d\mu(t) \to 0,$
(e) $\rho \left( \int_T x_n^*(t) d\mu(t) - x^* \right) \to 0,$
(f) $\int_T |f(t, x_n(t)) - f(t, x)| d\mu(t) \to 0.$

Proof. First, assume that $g = 0$. Then consider $\varepsilon > 0$ and $\{e_i\}_{i=1}^{k}$ a finite family of points such that $\rho(\cdot) = \max\{\langle \cdot, e_i \rangle : i = 1, \ldots, k\}$, and denote by $L := \text{span}\{x, e_i\}_{i=1}^{k}$ and $K = L \cap \mathbb{B}(x, 1)$. Since $x^* \in \hat{\partial} E_f(x)$ there are a ball $\mathbb{B}(x, \eta)$ and $C^1$ function $\phi : \mathbb{B}(x, \eta) \to \mathbb{R}$ such that $\nabla \phi(x) = x^*$ and $E_f - \phi$ attains a local minimum at $x$.

Let us consider the measure space $(\bar{T}, \bar{\mathcal{A}}, \bar{\mu})$, where $\bar{T} = T \cup \{\omega_1, \omega_2\}$ (with $\omega_1, \omega_2 \notin T$), $\bar{\mathcal{A}} = \sigma(\mathcal{A}, \{\omega_1\}, \{\omega_2\})$ and $\bar{\mu}(A) = \mu(A \setminus \{\omega_1, \omega_2\}) + 1_A(\omega_1) + 1_A(\omega_2)$, together with the integrand function

\begin{equation}
\tilde{f}(t, x) = \begin{cases}
 f(t, x), & \text{if } x \in X \text{ and } t \in T,
 -\phi(x), & \text{if } x \in X \text{ and } t = \omega_1,
 \delta_K(x), & \text{if } x \in X \text{ and } t = \omega_2.
\end{cases}
\end{equation}

Now condition Item (a) of Proposition 3.2 holds for the integrand $\tilde{f}$ and the measure space $(\bar{T}, \bar{\mathcal{A}}, \bar{\mu})$. Furthermore, $E_{\tilde{f}}$ attains its minimum at $x$, and by Proposition 3.2 we have that $x$ is a $p$-robust minimizer of $E_{\tilde{f}}$. Whence, by Theorem 3.3 there exist sequences $\tilde{y}_n \in X, \tilde{x}_n \in L^p(\bar{T}, X), \tilde{x}_n^* \in L^q(\bar{T}, X^*)$ (with $1/p + 1/q = 1$) such that:

1. $\tilde{x}_n^*(t) \in \hat{\partial} \tilde{f}(t, \tilde{x}_n(t))$ ae,
2. $\|x - \tilde{y}_n\| \to 0,$ $\int_{\bar{T}} \|x - \tilde{x}_n(t)\|^p d\bar{\mu}(t) \to 0,$
3. $\|\tilde{x}_n^*(\cdot)|_q\|\tilde{x}_n(\cdot) - \tilde{y}_n\|_p \to 0,$
4. $\int_{\bar{T}} \|\tilde{x}_n^*(t) d\bar{\mu}(t)\| \to 0,$
5. $\int_{\bar{T}} |f(t, \tilde{x}_n(t)) - f(t, x)| d\bar{\mu}(t) \to 0.$

In particular, $\int_{\bar{T}} \tilde{x}_n^*(t) d\bar{\mu}(t)$ is bounded, and so $\langle \int_{\bar{T}} \tilde{x}_n^*(t) d\bar{\mu}(t), \tilde{y}_n - x \rangle \to 0$. Hence,

$$\left| \int_{\bar{T}} \langle \tilde{x}_n^*(t), \tilde{x}_n(t) - x \rangle d\bar{\mu}(t) \right| \leq \int_{\bar{T}} \langle \tilde{x}_n^*(t), \tilde{y}_n - x \rangle d\bar{\mu}(t)$$

$$+ \int_{\bar{T}} \|\tilde{x}_n^*(t)\| \|\tilde{x}_n(t) - \tilde{y}_n\| d\bar{\mu}(t) \to 0.$$

Now, define $x_n(t) := \tilde{x}_n(t), x_n^*(t) := \tilde{x}_n^*(t)$ with $t \in T$ and $y_n = \tilde{y}_n$. So, $x_n \in L^p(T, X), x_n^* \in L^q(T, X^*)$, $x_n^*(t) \in \hat{\partial} \hat{f}(t, x_n(t))$ ae, $\|x - y_n\| \to 0$, $\int_T \|x - x(t)\|^p d\mu(t) \to 0$, $\|x_n^*(\cdot)|_q\|x_n(\cdot) - y_n\|_p \to 0$, and $\int_T |f(t, x_n(t)) - f(t, x)| d\mu(t) \to 0$.

Next, we get $\tilde{x}_n^*(\omega_1) = -\nabla \phi(\tilde{x}_n(\omega_1)) \rightharpoonup -\nabla \phi(x) = -x^*$. By using the convexity
of $K$, we have that for a large enough $n$, $\tilde{x}_n^*(\omega) \in N_K(\tilde{x}_n(\omega)) = L^\perp$. Therefore,

$$
\rho \left( \int_T x_n^*(t) d\mu(t) - x^* \right) \leq \rho \left( \int_T x_n^*(t) d\mu(t) + \tilde{x}_n^*(\omega_1) + \tilde{x}_n^*(\omega_2) \right) + \rho(-\tilde{x}_n^*(\omega_1) - x^*) + \rho(\tilde{x}_n^*(\omega_2))
$$

$$
\leq \| \int_T \tilde{x}_n^*(t) d\mu(t) \| + \rho(-\tilde{x}_n^*(\omega_1) - x^*) + \rho(\tilde{x}_n^*(\omega_2)) \to 0.
$$

On the one hand, since $\tilde{x}_n^*(\omega_1)$ is bounded and $\tilde{x}_n(\omega_1) \to x$, we have $\langle \tilde{x}_n^*(\omega_1), \tilde{x}_n(\omega_1) - x \rangle \to 0$. On the other hand, since for large enough $n$, $\langle \tilde{x}_n^*(\omega_2), \tilde{x}_n(\omega_2) - x \rangle = 0$, we get

$$
\left| \int_T \langle x_n^*(t), x_n(t) - x \rangle d\mu(t) \right| \leq \left| \int_T \langle \tilde{x}_n^*(t), \tilde{x}_n(t) - x \rangle d\mu(t) \right| + \left| \langle \tilde{x}_n^*(\omega_1), \tilde{x}_n(\omega_1) - x \rangle \right|
$$

$$
+ \left| \langle \tilde{x}_n^*(\omega_2), \tilde{x}_n(\omega_2) - x \rangle \right| \to 0.
$$

Finally, if $g$ is not zero, we know by Lemma A.2 that the gradient of $E_g$ is given by $\int_T \nabla g_t(x) d\mu(t)$. Then we apply the result to the integrand function $h := f - g$, with the gradient $g^* := x^* - \int_T \nabla g_t(x) d\mu(t) \in \partial E_h(x)$, and the result follows after some standard calculations.

The next theorem corresponds to the $p = +\infty$ version of Theorem 3.4. This theorem is obtained using Theorem 3.4, and modifying the measurable selection in a set of small measure. It is important to mention that this technique produces measurable selections in spaces of functions, which are not necessarily Asplund spaces (like $L^\infty(T,X)$ and $L^1(T,X^*)$). Consequently, it would not be possible to get this fuzzy calculus using simply the chain rule for the Fréchet subdifferential, as it was done in [34] for the convex subdifferential.

**THEOREM 3.5.** Let $f,g$ be two normal integrands satisfying (P). Assume that the function $\sup_{x \in X} \| \nabla g(x) \|$ belongs to $L^1(T,R)$. Then for every $x^* \in \partial E_f(x)$ and every $w^*$-continuous seminorm $\rho$ in $X^*$, there exist sequences $y_n \subset X$, $x_n \in L^\infty(T,X)$, $x_n^* \in L^1(T,X^*)$ such that

(a) $x_n^*(t) \in \partial f(t,x_n(t))$ ae.
(b) $\| x - y_n \| \to 0$, $\| x - x_n(\cdot) \|_\infty \to 0$.
(c) $\int_T \| x_n^*(t) \| \| x_n(t) - y_n \| d\mu(t) \to 0$.
(d) $\int_T \langle x_n^*(t), x_n(t) - x \rangle d\mu(t) \to 0$.
(e) $\rho(\int_T x_n^*(t) d\mu(t) - x^*) \to 0$.
(f) $\int_T |f(t,x_n(t)) - f(t,x)| d\mu(t) \to 0$.

**Proof.** Consider $\rho$ and $x^* \in \partial E_f(x)$ as in the statement. First we assume that $\mu$ is finite and $g = 0$, and so we have that $f(t,x) \geq 0$ for all $t \in T$ and all $x \in X$. Let $\varepsilon \in (0,1)$ and define $\tilde{f}(t,x') := f(t,x') + \delta_{B(x',\varepsilon)}(x')$. It follows that $x^* \in \partial E_{\tilde{f}}(x)$. Then by Theorem 3.4 there exist measurable functions $\tilde{x}_n \in L^2(T,X)$, $\tilde{x}_n^* \in L^2(T,X^*)$ such that:
(1) \( \tilde{x}_n^*(t) \in \partial f(t, \tilde{x}_n(t)) \) ae,
(2) \( \|x - \tilde{y}_n\| \to 0, \int_T \|x - \tilde{x}_n(t)\|^2 d\mu(t) \to 0, \)
(3) \( \|\tilde{x}_n^*()\|_2 \tilde{x}_n^*() - \tilde{y}_n\|_2, \)
(4) \( \rho(\int_T \tilde{x}_n^*(t) d\mu(t) - x^*) \to 0, \)
(5) \( \int_T |f(t, \tilde{x}_n(t)) - f(t, x)| d\mu(t) \to 0. \)

It is easy to see that if \( \|\tilde{x}_n(t) - x\| < \varepsilon, \) then \( \tilde{x}_n^*(t) \in \partial f(t, \tilde{x}_n(t)) \). Define the measurable sets \( \mathcal{A}_n := \{ t \in T : \|\tilde{x}_n(t) - x\| = \varepsilon \} \). The convergence in \( L^2(T, X) \) implies that \( \mu(\mathcal{A}_n) \to 0. \) We take \( n \in \mathbb{N} \) such that

\[
\begin{align*}
\text{(i) } & \|x - \tilde{y}_n\| \leq \varepsilon / 2, \rho(\int_T \tilde{x}_n^*(t) d\mu(t) - x^*) \leq \varepsilon / 3, \\
\text{(ii) } & \int_T \|\tilde{x}_n^*(t)\| d\mu(t) - \tilde{y}_n(t) d\mu(t) \leq \varepsilon^2 / 6, \int_T |f(t, \tilde{x}_n(t)) - f(t, x)| d\mu(t) \leq \varepsilon / 2, \\
\text{(iii) } & \int_T \langle x^*_n(t), x_n(t) - x \rangle d\mu(t) \leq \varepsilon / 3 \text{ and } \int_{\mathcal{A}_n} f(t, x) d\mu(t) \leq \varepsilon^2 / 24.
\end{align*}
\]

It follows from Item (ii) and the definition of \( \mathcal{A}_n \) that

\[
\frac{\varepsilon^2}{6} \geq \int_T \|\tilde{x}_n^*(t)\| d\mu(t) \geq \int_{\mathcal{A}_n} \|\tilde{x}_n^*(t)\| d\mu(t) \geq \int_{\mathcal{A}_n} \|\tilde{x}_n^*(t)\| (\|x - \tilde{x}_n(t)\| - \|x - \tilde{y}_n\|) d\mu(t) \geq \int_{\mathcal{A}_n} \|\tilde{x}_n^*(t)\| (\varepsilon - \varepsilon / 2) d\mu(t) \geq \frac{\varepsilon}{2} \int_{\mathcal{A}_n} \|\tilde{x}_n^*(t)\| d\mu(t).
\]

Therefore, \( \int_{\mathcal{A}_n} \|\tilde{x}_n^*(t)\| d\mu(t) \leq \frac{\varepsilon}{4}. \) We set \( \varepsilon(t) := f(t, x) \) and by the nonnegativity of the integrand, we have that \( x \) is a \( \varepsilon(t) \)-minimum of \( f(t, \cdot) \) for almost all \( t \in \mathcal{A}_n \). Then by Lemma B.3 there exist measurable functions \((y(t), y^*(t)) \in X \times X^* \) such that for almost all \( t \in \mathcal{A}_n \), \( y^*(t) \in \partial f(t, y(t)), \|y(t) - x\| \leq \varepsilon / 2, |f(t, y(t)) - f(t, x)| \leq \varepsilon(t) \) and \( \|y^*(t)\| \leq 8\varepsilon(t) / \varepsilon. \)

Let us define \( x(t) := \tilde{x}_n(t) 1_{\mathcal{A}_n}(t) + y(t) 1_{\mathcal{A}_n} \) and \( x^*(t) := \tilde{x}_n^*(t) 1_{\mathcal{A}_n}(t) + y^*(t) 1_{\mathcal{A}_n}. \)

Hence, \( x^*(t) \in \partial f(t, x(t)) \) ae, \( \|x - x(t)\|_\infty \leq \varepsilon, \)

\[
\int_T \|y^*(t)\| d\mu(t) = \int_{\mathcal{A}_n} \|\tilde{x}_n^*(t)\| d\mu(t) + \int_{\mathcal{A}_n} \|y^*(t)\| d\mu(t) \leq \mu(T)^{1/2} \|\tilde{x}_n^*(\cdot)\|_2 + \varepsilon / 3,
\]

\[
\int_T |f(t, x(t)) - f(t, x)| d\mu(t) = \int_{\mathcal{A}_n} |f(t, \tilde{x}_n(t)) - f(t, x)| d\mu(t) + \int_{\mathcal{A}_n} |f(t, y(t)) - f(t, x)| d\mu(t) \leq \varepsilon / 2 + \varepsilon / 2 = \varepsilon.
\]
Furthermore,

\[ \rho(\int_T x^*(t) d\mu(t) - x^*) \leq \rho(\int_T \tilde{x}^*_n(t) d\mu(t) - x^*) + \int_{A_n} \|\tilde{x}^*_n(t)\| d\mu(t) + \int_{A_n} \|y^*(t)\| d\mu(t) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon = \varepsilon, \]

so that

\[ \int_T \|x^*(t)\| \|\tilde{y}_n - x(t)\| d\mu(t) = \int_T \|\tilde{x}^*(t)\| \|\tilde{y}_n - \tilde{x}_n(t)\| d\mu(t) \]

\[ + \int_{A_n} \|y^*(t)\| (\|\tilde{y}_n - x\| + \|y(t) - x\|) d\mu(t) \leq \varepsilon^2/6 + \varepsilon^2/3 + \varepsilon^2/6 \leq \varepsilon. \]

Finally,

\[ \left| \int_T \langle x^*(t), x(t) - x \rangle d\mu(t) \right| \leq \left| \int_{A_n} \langle \tilde{x}^*_n(t), \tilde{x}_n(t) - x \rangle d\mu(t) + \int_{A_n} \langle y^*(t), y(t) - x \rangle d\mu(t) \right| \]

\[ + \int_{A_n} \|\tilde{x}^*_n(t)\| \cdot \|\tilde{x}_n(t) - x\| d\mu(t) + \int_{A_n} \|y^*(t)\| \cdot \|y(t) - x\| d\mu(t) \leq 2\varepsilon/3 + \varepsilon^2/6 \leq \varepsilon. \]

Now, if \( \mu \) is \( \sigma \)-finite, consider \( \nu(\cdot) = \int k(t) d\mu(t) \), where \( k > 0 \) is integrable and consider the integrand \( f(t, x) = f(t, x)/k(t) \). So, \( E_f^\nu = E_f^\mu \), and then by applying the previous part we easily get the result. The general case, when \( g \) is not zero, follows the same arguments given in the proof of Theorem 3.4.

**Remark 3.6.** It has not escaped to our notice that if one of the conditions of Proposition 3.2 holds, then the convergence of \( \int_T x_n^*(t) d\mu(t) \) to the subgradient \( x^* \) in Theorems 3.4 and 3.5, with respect to the seminorm \( \rho \), can be changed by the convergence in norm topology. Indeed, if one of the conditions of Proposition 3.2 holds, then we proceed similarly as in the proof of Theorem 3.4, by taking simply \( K = L = X \). So, the estimates follow similarly, but with the norm instead of the seminorm \( \rho \).

To illustrate our results we compute sequential formulae for series of lower semicontinuous functions using the measure space \( (N, \mathcal{P}(N)) \). This class of functions has been recently studied in the convex case (see, e.g., [14, 15, 47]), motivated by some applications to entropy minimization. Moreover, in this case we can apply techniques of separable reduction, and extend the results to an arbitrary Asplund space. The proof of the following result is written in Appendix C.3, for simplicity.

**Corollary 3.7.** The statement of Theorems 3.4 and 3.5 holds if we assume that \( X \) is a non-separable Asplund space and \( (T, A) = (N, \mathcal{P}(N)) \).

**Proof.** See Appendix C.3.
The final result corresponds to an extension of [34, Corollary 1.2.1] to the case $p = +\infty$, which characterizes the convex subdifferential of the integral functional $E_f$, when the data is a convex normal integrand.

**Corollary 3.8.** In the setting of Theorem 3.5, assume that $f$ is a convex normal integrand (i.e.: $f_t$ is convex for all $t \in T$). Then one has $x^* \in \partial E_f(x)$ if and only if there are nets $x_\nu, x_\nu^* \in L^\infty(T, X)$ and $x_\nu^* \in L^1(T, X^*)$ such that

(a) $x_\nu^*(t) \in \partial f(t, x_\nu(t))$ ae,

(b) $\|x - x_\nu(\cdot)\|_\infty \to 0$,

(c) $\int_T x_\nu^*(t) d\mu(t) \to x^*$,

(d) $\int_T (x_\nu^*(t), x_\nu(t) - x) d\mu(t) \to 0$,

(e) $\int_T |f(t, x_\nu(t)) - f(t, x)|d\mu(t) \to 0$.

If the space $X$ is reflexive we can take sequences instead of nets and the convergence of $\int_T x_\nu^*(t) d\mu(t)$ will be in the norm topology.

**Proof.** The construction of the net follows similar and classical arguments. Indeed, consider $x^* \in \partial E_f(x)$. Then take $N_0$ the neighborhood system of zero for the $w^*$-topology and consider the set $\mathcal{A} := \mathbb{N} \times N_0$, ordered by $(n_1, U_1) \leq (n_2, U_2)$ if and only if $n_1 \leq n_2$ and $U_2 \subseteq U_1$.

Then by Theorem 3.5 we have that for every $\nu = (n, U)$ there are $x_\nu, x_\nu^* \in L^\infty(T, X)$ and $x_\nu^* \in L^1(T, X^*)$ such that

1. $x_\nu^*(t) \in \partial f(t, x_\nu(t))$ ae,
2. $\|x - x_\nu(\cdot)\|_\infty \leq 1/n$,
3. $\int_T (x_\nu^*(t), x_\nu(t) - x) d\mu(t) \to 1/n$,
4. $\int_T x_\nu^*(t) d\mu(t) - x^* \in U,$
5. $\int_T |f(t, x_\nu(t)) - f(t, x)|d\mu(t) \to 0$.

Hence, the net $(x_\nu, x_\nu^*)$ satisfies the required properties. Conversely, assume that the net $(x_\nu, x_\nu^*)$ satisfies the above properties. Then for all $y \in X$

$$\langle x^*, y - x \rangle \leq \langle x^* - \int_T x_\nu^*(t) d\mu(t), y - x \rangle + \int_T \langle x_\nu^*(t), y - x_\nu(t) \rangle d\mu(t)$$

$$+ \int_T \langle x_\nu^*(t), x_\nu(t) - x \rangle d\mu(t)$$

$$\leq \langle x^* - \int_T x_\nu^*(t) d\mu(t), y - x \rangle + \int_T f(t, x_\nu(t)) - f(t, x) d\mu(t)$$

$$+ \int_T (x_\nu^*(t), x_\nu(t) - x) d\mu(t).$$

So, taking the limits we conclude $\langle x^*, y - x \rangle \leq E_f(y) - E_f(x)$, since $y$ is arbitrary we get the result.

Finally, when $X$ is reflexive, without loss of generality, we can assume that criterion Item (a) of Proposition 3.2 is satisfied; otherwise, we take $\tilde{f}_t := f_t + \delta_{\mathbb{B}(x, 1)}$. Then, by Remark 3.6, we can construct a sequence with the desired property using the norm instead of a family of seminorms.

**Remark 3.9.** It is worth comparing the results given in [34, Theorem 1.4.2] with Corollary 3.8: Let us recall that a functional $\lambda^* \in L^\infty(T, X)^*$ is called singular if there exists a sequence of measurable sets $T_n$ such that $T_{n+1} \subseteq T_n$, $\mu(T_n) \to 0$ as $n \to \infty$
and \( \lambda^*(g\mathbb{I}_T) = 0 \) for every \( g \in L^\infty(T, X) \). The set of all singular elements is denoted by \( L^{\text{sing}}(T, X) \). It is well-known that the dual of \( L^\infty(T, X) \) can be represented as the direct sum of \( L^1(T, X^*) \) and \( L^{\text{sing}}(T, X) \) (see, for example, [11,33]).

In [34, Theorem 1.4.2] the authors proved similar characterizations of the subdifferential of \( E_f \), where they established that \( x^* \in \partial E_f(x) \) if and only if there are nets \( x^*_\nu \in L^\infty(T, X) \), \( x^*_\nu \in L^1(T, X^*) \) and \( \lambda^*_\nu \in L^{\text{sing}}(T, X) \) such that

(a) \( x^*_\nu(t) \in \partial f(t, x^*_\nu(t)) \ ae \),
(b) \( \|x - x^*_\nu(\cdot)\|_\infty \to 0 \),
(c) \( \int_T x^*_\nu(t) d\mu(t) + A^*(\lambda^*_\nu) \xrightarrow{\nu^*} x^* \),
(d) \( \int_T \langle x^*_\nu(t), x^*_\nu(t) - x \rangle d\mu(t) + \lambda^*_\nu(x^*_\nu(\cdot) - A(x)) \to 0 \),
(e) \( \int_T |f(t, x^*_\nu(t)) - f(t, x)| d\mu(t) \to 0 \),

where \( A : X \to L^\infty(T, X) \) is the linear functional given by \( A(x) := x\mathbb{I}_T \) and \( A^* \) denotes its adjoint. Furthermore, the functionals \( \lambda^*_\nu \) belong to the normal cone of \( I^\infty_T \) at the constant function \( A(x) \). In other words, we have extended to the non-convex case this class of results by using the Fréchet subdifferential. Also, our characterizations of the subdifferential of \( E_f \) are tighter when the integrand is convex, since that we do not require the use of singular elements from the dual of \( L^\infty(T, X) \).

4. Limiting and Clarke-Rockafellar subdifferentials. The aim of this section is to establish upper-estimates for the limiting and Clarke-Rockafellar subdifferentials at a point \( x \in \text{dom } E_f \), in terms of the corresponding subdifferential of the data function \( f_t \) at the same point. We will focus on the case when \( X \) is a separable Asplund space.

In view of the results of the last section, we need to ensure the boundedness of the approximate sequences involved in the previous formulas of the subdifferential in order to establish upper-estimates, which are expressed at the exact point. So, the next part concerns criteria to guarantee this property. For this reason, we introduce the following definitions that allow us to extend the classical results, which generally consider some local Lipschitz continuity property of the integral functional (see for instance [12, Theorem 2.7.2] or [38]).

We introduce the concept of \( w^* \)-compact soles (see [13, Proposition 2.1]).

**Definition 4.1 (Integrable compact sole).** Consider a measurable multifunction \( C : T \rightrightarrows X^* \) with non-empty closed values.

(i) We say that \( C \) has an integrable compact sole if there exist \( e \in X \) and \( \gamma > 0 \) such that for every measurable selection \( c^* \) of \( C \)

\[ \gamma(c^*(t), e) \geq \|c^*(t)\| ae. \]

(ii) We denote

\[ UI(C) := \{ u \in X : \sigma_{C(t)}(u)^+ \in L^1(T, \mathbb{R}) \}, \]

where \( \sigma_{C(t)}(u)^+ := \max\{\sigma_{C(t)}(u), 0\} \).

Basically, the set \( UI(C) \) denotes the directions for which all the measurable selections are uniformly integrable.

In order to understand better this notion, we include in the Appendix a characterization of the integrable compact sole property in terms of the primal space (see Appendix C.5).
Theorem 4.2. Let \( x \in \text{dom} \, E_f \) and suppose there exist \( \varepsilon > 0 \), a measurable multfunction \( C : T \rightharpoonup X^* \) which has an integrable compact sole, and an integrable function \( K(\cdot) > 0 \) such that

\[
(4.1) \quad \hat{\partial} f(t, x') \subseteq K(t)\mathbb{B} + C(t), \forall x' \in \mathbb{B}(x, \varepsilon), \forall t \in T.
\]

Then

\[
(4.2) \quad \partial E_f(x) \subseteq \bigcap \left\{ \int_T \partial f(t, x) d\mu(t) + UI(C)^- + W^\perp \right\},
\]

\[
(4.3) \quad \partial^\infty E_f(x) \subseteq \bigcap \left\{ \int_T \partial^\infty f(t, x) d\mu(t) + UI(C)^- + W^\perp \right\},
\]

where the intersection is over all finite-dimensional subspaces \( W \subseteq X \). Consequently,

\[
(4.4) \quad \partial C E_f(x) \subseteq \overline{co}^\infty \left\{ \int_T \partial f(t, x) d\mu + \int_T \partial^\infty f(t, x) d\mu + UI(C)^- \right\}.
\]

Proof. Let \( x^* \in \partial E_f(x) \) and \( y^* \in \partial^\infty E_f(x) \). Consider a finite family of linearly independent points \( \{e_i\}_{i=1}^p \), \( W := \text{span}\{e_i\} \) and \( \rho(\cdot) := \max\{|\langle \cdot, e_i \rangle|\} \). Then by the auxiliary result Lemma C.2, proved in Appendix C.4, there exist sequences \( x_n, y_n \in L^\infty(T, X) \), \( x^*_n, y^*_n \in L^1(T, X^*) \) and \( \lambda_n \to 0^+ \) such that:

(i) \( x^*_n(t) \in \hat{\partial} f(t, x_n(t)) \) ae,

(ii) \( \|x - x_n(\cdot)\|_\infty \to 0 \),

(iii) \( \rho(\int_T x^*_n(t) d\mu(t) - x^*) \to 0 \),

(iv) \( \lim_{t \to \infty} \int_T |f(t, x_n(t)) - f(t, x)| d\mu(t) \to 0 \),

(i^\infty) \( y^*_n(t) \in \hat{\partial} f(t, y_n(t)) \) ae,

(ii^\infty) \( \|x - y_n(\cdot)\|_\infty \to 0 \),

(iii^\infty) \( \rho(\lambda_n \cdot \int_T y^*_n(t) d\mu(t) - y^*) \to 0 \),

(iv^\infty) \( \lim_{t \to \infty} \int_T |f(t, y_n(t)) - f(t, x)| d\mu(t) \to 0 \).

Hence, for large enough \( n \) relation (4.1) implies that \( x^*_n(t) \in K(t)\mathbb{B} + C(t) \) and \( y^*_n(t) \in K(t)\mathbb{B} + C(t) \) ae. Now, consider the multifunctions

\[
G_1(t) := \{(a^*, b^*) \in K(t)\mathbb{B} \times C(t) : x^*_n(t) = a^* + b^* \},
\]

\[
G_2(t) := \{(a^*, b^*) \in K(t)\mathbb{B} \times C(t) : y^*_n(t) = a^* + b^* \},
\]

which are graph measurable (see, e.g., [27, Proposition 1.41, Proposition 1.43 and Remark 1.44]). Therefore, by the measurable selection theorem (see Proposition 2.1) there are measurable selections \( h^1_n(t), h^2_n(t) \in \mathbb{B}(0, K(t)) \) and \( c^1_n(t), c^2_n(t) \in C(t) \) such that \( x^*_n(t) = h^1_n(t) + c^1_n(t) \) and \( y^*_n(t) = h^2_n(t) + c^2_n(t) \). From the fact that \( C \) has an integrable compact sole, there exist \( e \in X \) and \( \gamma > 0 \) such that \( \|c^i_n(t)\| \leq \gamma \|e\| \) for \( i = 1, 2 \) and almost all \( t \in T \). Then

\[
\int_T \|x^*_n(t)\| d\mu \leq \int_T K d\mu + \gamma \int_T \langle c^1_n(t), e \rangle d\mu(t)
\]

\[
= \int_T K(t) d\mu(t) + \gamma \left( \int_T \langle x^*_n(t), e \rangle d\mu(t) - \int_T \langle h^1_n(t), e \rangle d\mu(t) \right)
\]

\[
\leq (1 + \|e\|) \int_T K(t) d\mu(t) + \gamma \int_T \langle x^*_n(t), e \rangle d\mu(t).
\]
So, assuming that $e \in W$, the sequence $(x^*_n)$ is bounded in $L^1(T, X^*)$ and, obviously, the same holds for the sequence $(\lambda_n y^*_n)$. Then, by Lemma C.2, and observing that $C(x^*_n) \subseteq UI(C)^-$ and $C(\lambda_n y^*_n) \subseteq UI(C)^-$ (see the notation in Lemma C.2),

$$x^* \in \int_T \partial f(t, x) d\mu(t) + UI(C)^- + W^\perp$$

and

$$y^* \in \int_T \partial f^\infty(t, x) d\mu(t) + UI(C)^- + W^\perp.$$ 

Since $W$ was chosen arbitrary, (4.2) and (4.3) follow. Finally, (4.4) follows from (2.1).

**Remark 4.3.** When the measurable function $C$ has cone values, it is easy to see that (4.1) implies that for all $t \in T$, $\partial f(t, x) \subseteq C(t)$ and $UI(C) = \{u \in X : u \in C^-(t) \ a.e.\}$. In addition, if the values of $C$ are also $w^*$-closed and convex, then the integrable compact sole property can be understood in terms of the negative polar set $C^-(t)$ (see Lemma C.3). The most simple case is when $C$ is a fixed $w^*$-closed convex cone; in this case, Lemma C.3 characterizes the compact sole property as an interior non-emptiness condition of the polar cone $C^-(X)$. In particular, when the cone $C(t) = C = \{0\}$, we have that (4.1) implies that for almost all $t \in T$ the function $f_t$ is Lipschitz continuous on $B(x, \varepsilon)$ with constant $K(t)$ (see, e.g., [35, Theorem 3.52]), and consequently this recovers the classical framework of a normal integrand which is Lipschitz continuous (see, e.g., [12, Theorem 2.7.2]).

The next result corresponds to the explicit case when the measurable function $C$ in (4.1) is a fixed $w^*$-closed convex cone.

**Corollary 4.4.** In the setting of Theorem 4.2, we assume that the multifunction $C$ is a constant $w^*$-closed convex cone. Then

$$\partial E_f(x) \subseteq \bigcap \left\{ \int_T \partial f(t, x) d\mu(t) + C + W^\perp \right\}; \text{ and } \partial f^\infty E_f(x) \subseteq C,$$

where the intersection is over all finite-dimensional subspaces $W \subseteq X$. Consequently,

$$\partial C E_f(x) \subseteq \overline{w^*} \left\{ \int_T \partial f(t, x) d\mu(t) + C \right\}.$$

**Proof.** Let us check that $UI(C) = C^-$. Indeed, since that $\sigma_C = \delta_C^-$, we have that $\sigma_C(u)^+ \in L^1(T, \mathbb{R})$ if and only if $u \in C^-$, which means that $UI(C) = C^-$. Now, by the Bipolar theorem (see, e.g., [19, Theorem 3.38]), we have that $UI(C)^- = C$. Finally, using Theorem 4.2 we get the result.

The motivation for using the boundedness condition (4.1) comes from applications to stochastic programming; more precisely, applications to probability constraints (see [25, 48, 49]), where the authors impose boundedness conditions over the gradients of the involved functions to guarantee the interchange between the sign of the integral and the subdifferential.

The following examples show the importance of using the multifunction $C$ in Theorem 4.2 and Corollary 4.4.
Example 4.5. Consider the integrand $f : [0, 1] \times \mathbb{R} \to [0, +\infty)$ given by

$$f(t, x) = \begin{cases} x^{3/2}t^{-1+1} & \text{if } x > 0, \\ 0 & \text{if } \text{not}. \end{cases}$$

It is easy to check that $f$ is continuously differentiable with respect to $x$ and

$$E_f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0, \\ 0 & \text{if } \text{not}. \end{cases}$$

Then we easily get $\partial E_f(0) = [0, +\infty)$,

$$\hat{f}(t, x) = \begin{cases} \frac{3}{2}x^{1/2}t^{-1+1} + x^{3/2} \ln(t)t^{-1+1} & \text{if } x > 0, \\ 0 & \text{if } \text{not}, \end{cases}$$

and $\partial f(t, x) = \{0\}$. Then we can consider $C = [0, +\infty)$, so that

$$\partial E_f(0) = \int_{[0,1]} \partial f(t, 0) d\mu(t) + C = \{0\} + [0, +\infty).$$

The same example can be modified as

$$f(t, x) = \begin{cases} x^2t^{-1+1} & \text{if } x > 0, \\ 0 & \text{if } \text{not}. \end{cases}$$

Then one has

$$E_f(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } \text{not}. \end{cases}$$

So, the integral functional $E_f$ is Lipschitz continuous, but it is not true that $\partial E_f(0) = \{0, 1\}$ is included in $\int_{[0,1]} \partial f(t, 0) d\mu(t) = \{0\}$, as in classical results (see [36, Lemma 6.18] and also [38] for an extension of this result). However, Corollary 4.4 guarantees the inclusion $\partial E_f(0) \subseteq \int_{[0,1]} \partial f(t, 0) d\mu(t) + [0, +\infty)$.

Remark 4.6. As a final comment we recall that in the finite-dimensional setting two lsc functions $f_1, f_2$ satisfy the sum rule inclusion $\partial(f_1 + f_2)(x) \subseteq \partial f_1(x) + \partial f(x)$ at a point $x$ provided that the asymptotic qualification condition

$$x_1^* \in \partial^\infty f_1(x), \ x_2^* \in \partial^\infty f_2(x) \text{ and } x_1^* + x_2^* = 0 \Rightarrow x_1^* = x_2^* = 0.$$  

holds (see, e.g., [8, 9, 35, 37, 45]). However, the reader can notice that in the above example the integrand is continuously differentiable, then the singular subdifferential $\partial^\infty f_1(0) = \{0\}$ for all $t \in T$. In other words, it is not possible to recover similar criteria, as in the finite sum, in terms of the singular subdifferentials, to get an inclusion of the form $\partial E_f(x) \subseteq \int_T \partial f_t(x) d\mu(t)$.

The final result gives criteria for the Lipschitz continuity and differentiability of the function $E_f$.

Corollary 4.7. In the setting of Corollary 4.4, assume that the multifunction $C = \{0\}$. Then $E_f$ is locally Lipschitz around $x$. In addition, if $X$ is finite-dimensional and $\partial f(t, x')$ is single valued ae for all $x'$ in a neighborhood of $x$, then $E_f$ is continuous differentiable at $x$. 
Proof. By Corollary 4.4, the Clarke subdifferential $\partial C E_f$ is bounded by $M := \int K(t) d\mu(t)$ in a neighborhood of $x$. Then a straightforward application of Zagrodny’s Mean Value Theorem (see, e.g., [52, Theorem 4.3], or [35, Theorem 3.52]) shows that $E_f$ is Locally Lipschitz around $x$. Furthermore, if $X$ is finite-dimensional and $\partial f(t, x')$ is single valued ae for all $x'$ in a neighborhood of $x$, then $\partial C E_f$ is single-valued for all $x'$ in a neighborhood of $x$, and so [12, Proposition 2.2.4 and its Corollary] imply the result.

Concluding Remarks. In this paper, we gave new and explicit formulae for the subdifferential of non-necessarily convex integrals, that are defined on infinite-dimensional Banach spaces. The resulting formulae are given exclusively by means of the corresponding subdifferentials of the integrand functions. All this analysis is done without requiring any qualification conditions.

Acknowledgments. The authors are grateful to the anonymous referees for their valuable remarks and suggestions that have greatly helped to improve this manuscript.

Appendix. Next, in the last part of this paper we recall some results and we prove differentiability properties of integral functions. We also include here some technical lemmas relying on variational principles, and give necessary conditions for the existence of $p$-robust minima.

Appendix A. Continuity and differentiability of integral functionals. We shall need the following lemma, which shows that the convergence of the values of the integral functional implies a stronger convergence of the values of the data. This result has been proved in [24, Lemma 37] (see also [21]), but for the sake of completeness we present a proof.

**Lemma A.1.** Consider $x_n \in L^p(T, X)$ such that $x_n \overset{L^p}{\rightharpoonup} x$ and

$$\lim_{T} \int_T f(t, x_n(t)) d\mu(t) = \int_T f(t, x(t)) d\mu(t) \in \mathbb{R}.$$  

Then $\lim_{T} \int_T |f(t, x_n(t)) - f(t, x(t))| d\mu(t) = 0.$

**Proof.** Fix $\delta > 0$. Hence, by the lower semicontinuity of $I_f$ in $L^p(T, X)$ there exists $\varepsilon > 0$ such that $-\delta/4 \leq E_f(x) \leq I_f(y(\cdot))$ for every $y \in \mathbb{B}_{L^p(T, X)}(x, \varepsilon)$. Since $x_n \to x$, there exists $n_1 \in \mathbb{N}$ such that $x_n \in \mathbb{B}_{L^p(T, X)}(x, \varepsilon)$ for every $n \geq n_1$. In particular, for every $A \in \mathcal{A}$ and every $n \geq n_1$ the function $y := x_n \mathbb{1}_A + x \mathbb{1}_{A^c} \in \mathbb{B}_{L^p(T, X)}(x, \varepsilon)$, and then $-\delta/4 \leq \int_A f(t, x(t)) d\mu(t) \leq \int_A f(t, x_n(t)) d\mu(t)$ for every $A \in \mathcal{A}$. This yields, for all $A \in \mathcal{A}$, and all $n \geq n_1$,

$$-\delta/4 + \int_A f(t, x(t)) d\mu(t) \leq \int_A f(t, x_n(t)) d\mu(t)$$

$$= \int_T f(t, x_n(t)) d\mu(t) - \int_{A^c} f(t, x_n(t)) d\mu(t)$$

$$\leq \int_T f(t, x(t)) d\mu(t) - \int_{A^c} f(t, x(t)) d\mu(t) + \delta/4.$$
From the fact that \( \lim_{t \to 0^+} f(t, x_n(t)) d\mu(t) = \int_T f(t, x(t)) d\mu(t) \) there exist \( n_2 \geq n_1 \) such that \( \int_T f(t, x_n(t)) d\mu(t) \leq \int_T f(t, x(t)) d\mu(t) + \delta/4 \) for all \( n \geq n_2 \). Thus, for all \( A \in \mathcal{A} \) and all \( n \geq n_2 \)

\[
-\delta/4 + \int_A f(t, x(t)) d\mu(t) \leq \int_A f(t, x_n(t)) d\mu(t) \\
\leq \int_T f(t, x_n(t)) d\mu(t) - \int_A f(t, x(t)) + \delta/4 \\
\leq \int_T f(t, x(t)) d\mu(t) + \delta/4 - \int_A f(t, x(t)) + \delta/4 \\
= \int_A f(t, x(t)) d\mu(t) + \delta/2.
\]

Then, considering the measurable sets \( A^+_n := \{ t \in T : f(t, x_n(t)) - f(t, x(t)) > 0 \} \) and \( A^-_n := \{ t \in T : f(t, x_n(t)) - f(t, x(t)) < 0 \} \), we get

\[
\int_T |f(t, x_n(t)) - f(t, x(t))| d\mu(t) = \int_{A^+_n} f(t, x_n(t)) - f(t, x(t)) d\mu(t) \\
+ \int_{A^-_n} f(t, x(t)) - f(t, x_n(t)) d\mu(t) \\
\leq \delta/2 + \delta/4 < \delta;
\]

that is, \( \int_T |f(t, x_n(t)) - f(t, x(t))| d\mu(t) \to 0. \)

The following lemma is a simple application of classical rules concerning differentiation of integral functionals.

**Lemma A.2.** Let \( \mu \) be a finite measure and let \( f : T \times X \to \mathbb{R} \) be a normal integral Lipschitz on \( \mathbb{B}(x_0, \gamma) \) with some \( p \)-integrable constant, that is to say, there exists \( K \in L^p(T, \mathbb{R}) \) such that \( |f(t, x) - f(t, y)| \leq K(t)|x - y| \), for all \( x, y \in \mathbb{B}(x_0, \gamma) \) and all \( t \in T \). Assume that the functions \( f_t \) are Fréchet differentiable at \( x_0 \) ae. Then \( E_f \) is Fréchet differentiable at \( x_0 \), \( \nabla f(t, x_0) \) belongs to \( L^p(T, X^*) \) and \( \nabla E_f(x_0) = \int_T \nabla f_t(x_0) d\mu(t) \). Moreover, if \( f_t \) are \( C^1 \) on \( \text{int}(\mathbb{B}(x_0, \gamma)) \), then \( E_f \) is \( C^1 \) on \( \text{int}(\mathbb{B}(x_0, \gamma)) \).

**Proof.** First, the measurability and the integrability of the function \( t \to \nabla f_t(x_0) \) follows from the fact that for every \( h \in X \), \( \langle \nabla f_t(x_0), h \rangle = \lim_{s \to 0^+} f(t, x_0 + sh) - f(t, x_0) \) and \( \| \nabla f_t(x_0) \| \leq K(t) \) (see, e.g., [18, §2.1 Theorem 2 and §2.2 Theorem 2]). Now, take any sequence \( (0, \gamma) \supset s_n \to 0^+ \). Since \( \mathbb{B} \) is bounded we can assume that \( x_0 + s_nh \in \mathbb{B}(x_0, \gamma) \) for every \( n \in \mathbb{N} \) and \( h \in \mathbb{B} \), so that when the space \( X \) is separable, the measurability of

\[
t \to \sup_{h \in \mathbb{B}} \left| \frac{f_t(x_0 + s_nh) - f_t(x_0)}{s_n} - \langle \nabla f_t(x_0), h \rangle \right|
\]

follows from the Lipschitz continuity of the integrand and the separability of \( \mathbb{B} \). We notice that this function is bounded from above by \( K \); moreover, it converges to
zero (ae) as \( n \to \infty \). Then by Lebesgue’s dominated convergence theorem (see, e.g., [5, Theorem 2.8.1]) we get

\[
\lim_{n \to \infty} \sup_{h \in \mathcal{B}} \left| \frac{E_f(x_0 + s_nh) - E_f(x_0)}{s_n} - \int_T \langle \nabla f_t(x_0), h \rangle d\mu(t) \right| \xrightarrow{n \to \infty} 0,
\]

which concludes the first part.

To prove the continuity of the derivative \( \nabla E_f : \operatorname{int}(\mathcal{B}(x_0, \gamma)) \to (X^*, \|\cdot\|) \), consider \( x_n \to x \in \operatorname{int}(\mathcal{B}(x_0, \gamma)) \) with \( x_n \in \mathcal{B}(x_0, \gamma) \). Then for almost all \( t \in T \),

\[
\lim_{n \to \infty} \left| \nabla f_t(x) - \nabla f_t(x_n) \right| = 0,
\]

and

\[
g_n(t) := \sup_{h \in \mathcal{B}} |\langle \nabla f_t(x) - \nabla f_t(x_n), h \rangle| \leq 2K(t) \text{ ae.}
\]

Then, again by the Lebesgue dominated convergence theorem, we get

\[
|\nabla E_f(x) - \nabla E_f(x_n)| \xrightarrow{n \to \infty} 0.
\]

**Appendix B. Variational principles.** Now, we recall the Borwein-Preiss Variational Principle, for which we need to introduce the notion of type-gauge functions.

**Definition B.1.** [9, Definition 2.5.1] Consider \((\mathcal{X}, d)\) a metric space. We say that a continuous function \( \rho : \mathcal{X} \times \mathcal{X} \to [0, +\infty] \) is a gauge-type function provided that:

(i) \( \rho(x, x) = 0 \) for all \( x \in \mathcal{X} \),

(ii) for any \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that for all \( y, z \in \mathcal{X} \) we have \( \rho(y, z) \leq \eta \) implies that \( d(y, z) \leq \varepsilon \).

The next result corresponds to the Borwein-Preiss Variational Principle.

**Proposition B.2.** [9, Theorem 2.5.3] Let \((\mathcal{X}, d)\) be a complete metric space and let \( f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) be a lsc function bounded from below. Suppose that \( \rho \) is a gauge-type function and \( (\eta_i)_{i=0}^\infty \) is a sequence of positive numbers, and suppose that \( \varepsilon > 0 \) and \( z \in \mathcal{X} \) satisfy

\[
f(z) \leq \inf_{\mathcal{X}} f + \varepsilon.
\]

Then there exist \( y \) and a sequence \( (x_i) \) such that

(i) \( \rho(z, y) \leq \varepsilon/\eta_0, \rho(x_i, y) \leq \varepsilon/(2^i\eta_0) \) for all \( i = 1, 2, ... \),

(ii) \( f(y) + \sum_{i=0}^\infty \eta_i \rho(y, x_i) \leq f(z) \),

(iii) \( f(x) + \sum_{i=0}^\infty \eta_i \rho(x, x_i) > f(y) + \sum_{i=0}^\infty \eta_i \rho(y, x_i) \), for all \( x \in \mathcal{X} \setminus \{y\} \).

We recall that \((X, \|\cdot\|)\) is assumed to be a separable Asplund space and its norm is Fréchet differentiable away from the origin. The next result corresponds to a variational principle applied to integral functions.

**Lemma B.3.** Let \( z(\cdot) \) be a measurable function with values in \( X \), and let \( \varepsilon(\cdot) \) and \( \lambda(\cdot) \) be two strictly positive measurable functions. Suppose that \( z(t) \) is an \( \varepsilon(t) \)-minimum of \( f_t \). Then there are measurable functions \( y \) and \( y^* \) such that for almost all \( t \in T \), \( y^*(t) \in \partial f(t, y(t)), \|y(t) - z(t)\| \leq \lambda(t), |f(t, y(t)) - f(t, z(t))| \leq \varepsilon(t) \) and \( \|y^*(t)\| \leq 4\varepsilon(t)/\lambda(t) \).
Proof. Consider \( \eta_i > 0 \) with \( \eta_0 = 1 \) such that \( \sum_{i=0}^{\infty} \eta_i = 2 \). Then define \( \eta_i(t) := \eta_i \cdot \varepsilon(t)/\lambda^2(t) \), the space \( S = X \times \prod_{i=0}^{\infty} X \) with the product topology, and the function \( \varphi : T \times S \to \mathbb{R} \) given by

\[
\varphi(t, y, (x_i)) = \sum_{i=0}^{\infty} \eta_i(t) \|y - x_i\|^2.
\]

It is not hard to prove that \( S \) is a Polish space (i.e., metrizable, complete and separable) and that \( \varphi \) is measurable. Consider the set valued map \( A \) defined by

\[
y \in A(t, (x_i)) \text{ if and only if } y \in \operatorname{argmin}_X \left\{ f(t, \cdot) + \varphi(\cdot, (x_i)) \right\}:
\]

\[
hence, \text{ by } [2, \text{Theorem 8.2.11}] \text{ (see also } [11, 27, 45]), \text{ } A \text{ is a measurable set-valued map from } T \times \prod_{i=0}^{\infty} X \text{ to } X, \text{ while by } [11, \text{Proposition III.13}] \text{ we have that } gphA \subseteq A \otimes B(\prod_{i=0}^{\infty} X) \otimes B(X). \text{ Furthermore, by } [3, \text{Lemma 6.4.2}] \text{ (see also } [27, \text{Proposition 1.49}]\) we have that \( B(\prod_{i=0}^{\infty} X) \otimes B(X) = B(S) \), which implies that \( gphA \subseteq A \otimes B(S) \).

Now we consider the multifunction \( M \) so that \( (y, (x_i)) \in M(t) \) if and only if

(i) \( \|z(t) - y\| \leq \lambda(t), \|x_i - y\| \leq \lambda(t)/\sqrt{2} \) for all \( i = 1, 2, \ldots \),

(ii) \( f(y) + \varphi(t, (x_i)) \leq f(z(t)) \),

(iii) \( f(t, w) + \varphi(w, (x_i)) \geq f(t, y) + \varphi(y, (x_i)) \) for all \( w \in X \).

From the fact that every function involved in the definition of \( M \) is \( A \otimes B(S) \)-measurable and the fact that Item (iii) is equivalent to \( (t, y, (x_i)) \in gphA \), we have \( gphM \subseteq A \otimes B(S) \).

We claim that \( M(t) \) is non-empty for all \( t \in T \). Indeed, consider the type-gauge function \( \rho(a, b) := \|a - b\|^2 \). Then, applying the Borwein-Preiss Variational Principle (see Proposition B.2) to the \( \varepsilon(t) \)-minimum \( z(t) \) of \( f \), with \( \rho \) and the sequence \( \eta_i(t) \), there exists \( (y, (x_i)) \) that verifies Items (ii) and (iii). Furthermore, \( \rho(z(t), y) \leq \varepsilon(t)/\eta_0(t) = \lambda^2(t) \) and \( \rho(x_i, y) \leq \varepsilon(t)/(2^i \eta_0(t)) = \lambda^2(t)/2^i \), which clearly implies Item (i).

Now, by the Measurable Selection Theorem (see Proposition 2.1), there exist measurable functions \( (y(t), x_i(t)) \in M(t) \) ae. Hence, \( \|y(t) - z(t)\| \leq \lambda(t), \|x_i(t) - y(t)\| \leq \lambda(t)/\sqrt{2} \) for all \( i = 1, 2, \ldots \) and \( |f(t, y(t)) - f(t, z(t))| \leq \varepsilon(t) \) and \( f(t, w) + \varphi(t, w, (x_i(t))) \geq f(t, y(t)) + \varphi(t, y(t), (x_i(t))) \) for all \( w \in X \) ae.

Finally, it is easy to see that \( \phi(t, y) := \sum_{i=0}^{\infty} \eta_i(t) \|y - x_i(t)\|^2 \) is \( C^1 \) with respect to the second argument (see, e.g., Lemma A.2), \( \nabla \phi(t, y(t)) \) is measurable and

\[
\|\nabla \phi(t, y(t))\| \leq \sum_{i=0}^{\infty} 2 \eta_i(t) \|y(t) - x_i(t)\| \leq 4 \frac{\varepsilon(t)}{\lambda(t)}.
\]

Hence, \( f(t, \cdot) + \phi(t, \cdot) \) attains a minimum at \( y(t) \), and so \( y^*(t) := -\nabla \phi(t, y(t)) \) belongs to \( \partial f_i(y(t)) \).

The following result gives some preliminary consequences of the existence of a \( p \)-robust minimizer.

Lemma B.4. Let \( (T, A, \mu) \) be a finite measure space, \( p \in [1, +\infty) \) and \( x \in X \) be a \( p \)-robust minimizer of \( E_f \) on \( B \subseteq X \). Then for every sequence \( \varepsilon_n \to 0 \), and \( \varepsilon_n \)-minimizer \( (x_n(\cdot), y_n) \) of the function \( \varphi_n : L^p(T, X) \times X \to \mathbb{R} \), defined as

\[
\varphi_n(w, u) := \int_T f(t, w(t)) d\mu(t) + n \int_T \|w(t) - u\|^p d\mu(t) + \|x_0 - u\|^p + \delta_B(u),
\]

...
we have

(a) \( n\|x_n(t) - y_n\|_p^p, \|x_n(t) - x\|_p, \|y_n - x\| \to 0 \), and

(b) \( \int_T |f(t, x_n(t)) - f(t, x)|d\mu(t) \to 0 \).

In particular, we have

\[
\inf_{n \in N} \sup_{w \in L^p(T, X)} \inf_{u \in X} \varphi_n(w, u) = E_f(x) .
\]  

\textbf{Proof.} First, for \( n \geq 1 \) and \( \gamma > 0 \) define

\[
\nu_n := \inf \{ \varphi_n(w, u) \mid w \in L^p(T, X) \text{ and } u \in X \},
\]

\[
\kappa_n := \inf \{ \int_T f(t, w(t)) \mid \int_T \|w(t) - u\|^p d\mu(t) \leq \delta, \ w \in L^p(T, X) \text{ and } u \in B \}. 
\]

We have

\[
n(\|x_n(t) - y_n\|_p^p) \leq \int_T (f(t, x_n(t)) + n\|x_n(t) - y_n\|_p^p) d\mu(t) + \|y_n - x\|_p^p \leq E_f(x) + \varepsilon_n.
\]

The last inequality implies that \( \int_T \|x_n(t) - y_n\|^p d\mu(t) \to 0 \), and so, setting \( \gamma_n := \int_T \|x_n(t) - y_n\|^p d\mu(t) \),

\[
\kappa_n - \varepsilon_n \leq \int_T f(t, x_n(t))d\mu(t) - \varepsilon_n \leq \varphi_n(x_n, y_n) - \varepsilon_n \leq \nu_n \leq E_f(x_0).
\]

By taking the limits we conclude that \( \int_T f(t, x_n(t))d\mu(t) \to \int_T f(t, x_0)d\mu(t) \), and, consequently, Item (a) and (B.1) follow. Finally, Item (b) follows by using Lemma A.1.

\textbf{Appendix C. Some technical results.} In this part of the appendix we present the proofs of some technical results, which were used in the main sections of the article.

\textbf{C.1. Sufficient conditions for \( p \)-robust minima.} Here we present the proof of Proposition 3.2, which ensures conditions to have a \( p \)-robust infimum.

\textbf{Proof of Proposition 3.2.} (a) In the first case define

\[
\nu_n := \inf_{w \in L^p(T, X)} \left\{ \int_T f(t, w(t)) \mid \int_T \|w(t) - u\|^p \leq 1/n \right\},
\]

take \( \varepsilon_n \to 0^+ \) and \( (x_n, y_n) \in L^p(T, X) \times B \) such that

\[
-C_n + \int_T f(t, x_n(t)) \leq \nu_n,
\]

and \( \int_T \|x_n(t) - y_n\|^p \leq 1/n \). We can suppose that for every \( t_1 \in T \) and \( t_2 \in A \),

\[
\|x_n(t_1) - y_n\| \to 0 \text{ and } f(t_2, \cdot) \text{ is sequentially } \tau\inf\text{-compact. So, by Fatou’s lemma we have that, for every subsequence } x_{n_k} \text{ of } x_n,
\]

\[
\int_T \lim\inf f(t, x_{n_k}(t))d\mu(t) \leq \lim\inf \int_T f(t, x_{n_k}(t)) \leq \inf_{B} \int_T f(t, x_{n_k}(t)) \leq \inf_{B} E_f < +\infty.
\]
Then, in particular, for some \( t_0 \in A \), \( \liminf f(t_0, x_n(t_0)) < +\infty \), and there exist a subsequence \( x_{n_{k(t_0)}}(t_0) \) and a constant \( M_{t_0} \) such that \( f(t_0, x_{n_{k(t_0)}}(t_0)) \leq M_{t_0} \). Whence, by the inf-compactness of \( f(t_0, \cdot) \), there exists a subsequence \( z_n \) of \( x_{n_{k(t_0)}}(t_0) \) such that \( z_n \to w_0 \in X \). Because \( \|x_n(t_0) - y_n\| \to 0 \), we get the existence of a subsequence \( y_{\phi(n)} \) of \( y_n \) such that \( y_{\phi(n)} \to w_0 \in B \) (because \( B \) is \( \tau \)-closed). Then, from the fact that \( \|x_n(t) - y_n\| \to 0 \), we get \( x_{\phi(n)}(t) \to w_0 \) for all \( t \in T \). Finally, taking into account (C.1) and using the lsc of the integrand in (C.2) we obtain

\[
\inf_B E_f \leq E_f(w_0) \leq \int_T \liminf f(t, x_{\phi(n)}(t))d\mu(t) \\
\leq \liminf \int_T f(t, x_{\phi(n)}(t))d\mu(t) \leq \wedge_{p,B} E_f \leq \inf_B E_f.
\]

(b) The second case follows from the first part, by modifying the measure space and the integrand function as follows: For \( \omega_0 \not\in T \), define \((T, \tilde{A}, \tilde{\mu})\), where \( T = T \cup \{\omega_0\}, \tilde{A} = \sigma(A, \{\omega_0\}) \) (the \( \sigma \)-algebra generated by \( A \cup \{\omega_0\} \)) and \( \tilde{\mu}(A) = \mu(A \setminus \{\omega_0\}) + \mathbb{1}_A(\omega_0) \), and

\[
\tilde{f}(t, x) = \begin{cases} 
  f(t, x), & \text{if } t \in T, \\
  \delta_B(x), & \text{if } t = \omega_0.
\end{cases}
\]

Then, by the first part \( \wedge_{p,B} E_f^\mu = \inf_B E_f^{\tilde{\mu}} \) and, consequently,

\[
\inf_B E_f \geq \wedge_{p,B} E_f \geq \wedge_{p,B} E_f^{\tilde{\mu}} = \inf_B E_f^{\tilde{\mu}} = \inf_B E_f.
\]

(c) In the last case, let \( K \) be the \( q \)-integrable Lipschitz constant and consider \( w \in L^p(T, X) \) and \( y \in B \). Then

\[
\int_T f(t, w(t))d\mu(t) \geq -\int_T |f(t, w(t)) - f(t, y)|d\mu(t) + \int_T f(t, y)d\mu(t) \\
\geq -\int_T K(t)\|w(t) - y\|d\mu(t) + \inf_B E_f.
\]

So, the result follows taking the appropriate limits.

\[\square\]

**C.2. Proof of Theorem 3.3.** We prove Theorem 3.3 given in section 3.

**Proof of Theorem 3.3.** We recall that the norm in \( X \) is assumed to be \( C^1 \) away from the origin. Consider the function \( \ell(x) = \|x\|^p \). It is easy to see that \( \ell \) is \( C^1 \) everywhere. Moreover, we have that

\[
\|x\|^{p-1} \leq \|
abla \ell(x)\| \leq p\|x\|^{p-1} \text{ for all } x \in X.
\]

Consider \( r \in (0, 1) \) such that \( x_0 \) is a \( p \)-robust minimizer of \( E_f \) on \( B := B(x_0, r) \), and fix a family of \( \eta_i > 0 \) such that \( \eta_0 = r \) and \( \sum_{i=0}^{+\infty} \eta_i = 1 \). Now define \( \varphi_n : L^p(T, X) \times X \rightarrow \mathbb{R} \) by

\[
\varphi_n(x, y) = \int_T f(t, x(t))d\mu(t) + n\int_T \ell(x(t) - y)d\mu(t) + \ell(y - x_0) + \delta_B(y).
\]
Then Lemma B.4 says that
\[
\sup_{n \in \mathbb{N}} \inf_{w \in L^p(T, X)} \varphi_n(w, u) = E_f(x_0),
\]
and so there exists \( \varepsilon_n \to 0^+ \) (with \( \varepsilon_n \in (0, \eta_0^2) \) for large enough \( n \)) such that \( (x_0, x_0) \) is an \( \varepsilon_n \)-minimum of \( \varphi_n \). Then, by applying the Borwein-Preiss Variational Principle (see Proposition B.2) with the type-gauge function \( \rho : (L^p(T, X) \times X)^2 \to \mathbb{R} \) given by
\[
\rho((w_1, u_1), (w_2, u_2)) := \int_T \ell(w_1(t) - w_2(t))d\mu(t) + \ell(u_1 - u_2),
\]
and the sequence \( (\eta_i)_{i \geq 0} \), to the function \( \varphi_n \) and the \( \varepsilon_n \)-minimum \( (x_0, x_0) \), we can find points \( (x_n^0, y_n^0) \in \mathbb{N}, (x_n^\infty, y_n^\infty) \in L^p(T, X) \times X \) such that:

- \( \ell(x_0 - x_n^\infty(t))d\mu(t) + \ell(x_0 - y_n^\infty) \leq \frac{\varepsilon_n}{\eta_0} \sum_{i=1}^{\infty} \eta_i \ell(w_i - y_n^0) \leq \frac{\varepsilon_n}{2\eta_0}, \) (BP.1)
- \( \varphi_n(x_n^\infty, y_n^\infty) + \phi_n(x_n^\infty, y_n^\infty) \leq \varphi_n(x_0, x_0), \) (BP.2)
- \( \varphi_n(w, u) + \phi_n(w, u) > \varphi_n(x_n^\infty, y_n^\infty) + \phi_n(x_n^\infty, y_n^\infty) \) for all \( (w, u) \in L^p(T, X) \times X \setminus \{(x_n^\infty, y_n^\infty)\}, \) (BP.3)

where
\[
\phi_n(w, u) = \sum_{i=1}^{\infty} \eta_i \left( \int_T \ell(w(t) - x_i^0(t))d\mu(t) \right) + \sum_{i=1}^{\infty} \eta_i \ell(u - y_i^0)
= \int_T \left( \sum_{i=1}^{\infty} \eta_i \ell(w(t) - x_i^0(t)) \right) d\mu(t) + \sum_{i=1}^{\infty} \eta_i \ell(u - y_i^0).
\]

On the one hand, by Item (BP.2) the quantity \( \int_T h_n(t, x_n^\infty(t))d\mu(t) \) is finite, where
\[
h_n(t, v) := f(t, v) + n\ell(v - y_n^\infty) + \sum_{i=1}^{\infty} \eta_i \ell(v - x_i^0(t))
\]
is a normal integrand functional, and by Item (BP.3) (taking \( u = y_n^\infty \))
\[
\int_T h_n(t, x_n^\infty(t))d\mu(t) = \inf_{w \in L^p(T, X)} \int_T h_n(t, w(t))d\mu(t)
= \int_T \inf_{u \in X} h_n(t, u)d\mu(t),
\]
where the last equality is given by Proposition 2.2. Then, by the sum rule (see, e.g., [9, Exercise 3.1.12]) we get
\[
0 \in \nabla f(x_n^\infty(t)) + nu_n^*(t) + v_n^*(t) ae,
\]
where \( u_n^*(t) := \nabla f(x_n^\infty(t) - y_n^\infty) \) and \( v_n^*(t) := \sum_{i=1}^{\infty} \eta_i \nabla f(x_n^\infty(t) - x_i^0(t)). \) The measurability and differentiability of these functions follow from Lemma A.2 (notice that this
infinite sum can also be seen as an integral functional). The estimate of the gradient of the function $\ell$ gives us $\|u_\infty^n(t)\|^q \leq p\|x_\infty^n(t) - y_\infty^n\|^p$ and $\int_T \|v_\infty^n(t)\|^q d\mu(t) \to 0$. On the other hand, by Item (BP.3) (taking $w = x_\infty^n$),

$$n \int_T \ell(x_\infty^n(t) - y_\infty^n) d\mu(t) + \ell(y_\infty^n - x_0) + \sum_{i=1}^\infty \eta_i \ell(y_\infty^n - y_i^n)$$

$$= \inf_{u \in X} \left( n \int_T \ell(x_\infty^n(t) - u) d\mu(t) + \ell(u - x_0) + \sum_{i=1}^\infty \eta_i \ell(y_\infty^n - y_i^n) \right).$$

Hence, again Lemma A.2 gives us the differentiability of these three functions, and after some calculus yields $0 = -n \int_T u_\infty^n(t) d\mu(t) + w_\infty^n$ with $\|w_\infty^n\| \to 0$. Thus, there exists $x_\infty^n := -nu_\infty^n(t) - v_\infty^n(t) \in L^q(T, X^*)$ such that $x_\infty^n(t) \in \hat{\partial} f(t, x_\infty^n(t))$ (see Equation (C.3)), and the previous computations give us

$$\left( \int_T \|x_\infty^n(t)\|^q \right)^{1/q} \leq n \left( \int_T \|u_\infty^n(t)\|^q \right)^{1/q} + \left( \int_T \|v_\infty^n(t)\|^q \right)^{1/q},$$

and $\|\int_T x_\infty^n(t) d\mu(t)\| \leq \|\int_T u_\infty^n(t) d\mu(t)\| + \|w_\infty^n\| \to 0$. By Item (BP.2) we have that $(x_\infty^n, y_\infty^n)$ is an $\varepsilon_n$-minimizer of $\phi_n$, and so by Lemma B.4 we conclude that $n(\|x_\infty^n(t) - y_\infty^n\|^p) \to 0$, $\int_T |f(t, x_\infty^n(t)) - f(t, x_0)| d\mu(t) \to 0$. Finally,

$$\|x_\infty^n\| \|x_\infty^n - y_\infty^n\|_p \leq n \left( \int_T \|u_\infty^n(t)\|^p d\mu(t) \right)^{1/q} + \left( \int_T \|v_\infty^n(t)\|^q d\mu(t) \right)^{1/q} \|x_\infty^n - y_\infty^n\|_p$$

$$\leq np \|x_\infty^n(-) - y_\infty^n\|^p + \|v_\infty^n\|_q \|x_\infty^n(-) - y_\infty^n\|_p$$

$$= np \|x_\infty^n(-) - y_\infty^n\|^p + \|v_\infty^n\|_q \|x_\infty^n(-) - y_\infty^n\|_p \to 0. \quad \square$$

C.3. Proof of Corollary 3.7. In this section, we give a proof of Corollary 3.7, by using some results on separable reductions for the Fréchet subdifferential.

Let us recall the concept of a rich family in a non-separable Banach space. The symbol $S(X \times X^*)$ denotes the family of sets $U \times Y$ where $U$ and $Y$ are (norm-) separable closed linear subspaces of $X$ and $X^*$. A set $R \subseteq S(X \times X^*)$ is called a rich family if for every $U \times Y \in S(X \times X^*)$, there exists $V \times Z \in R$ such that $U \subseteq V$, $Y \subseteq Z$ and $\bigcup_{n \in N} U_n \times \bigcup_{n \in N} Y_n \in R$ whenever the sequence $(U_n \times Y_n)_{n \in N} \subseteq R$ satisfy $U_n \subseteq U_{n+1}$ and $Y_n \subseteq Y_{n+1}$ (for more details, see [16, 17, 20] and references therein). In [16, Theorem 3.1] the authors showed that there exists a rich family in non-separable Asplund spaces as in the following proposition.

Proposition C.1. [16, Theorem 3.1] Let $(X, \| \cdot \|)$ be an Asplund space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any proper function. Then there exists a rich family $\mathcal{R} \subseteq S(X \times X^*)$ such that $Y_1 \subseteq Y_2$ whenever $V_1 \times Y_1, V_2 \times Y_2 \in \mathcal{R}$ and $V_1 \subseteq V_2$, such that for every $V \times Y \in \mathcal{R}$, the assignment $Y \ni x^* \mapsto x^*_V \in V^*$ is a surjective isometry from $Y$ to $V^*$, and for every $v \in V$ we have that

$$\left( \hat{\partial} f(v) \cap Y \right)_v = (\hat{\partial} f(v))_v = \hat{\partial} f_{|v}(v);$$

that is, in more details, if $v^* \in \hat{\partial} f_{|v}(v)$, then there exists a unique $x^* \in \hat{\partial} f(v) \cap Y$ such that $x^*_V = v^*$ and $\|x^*\| = \|v^*\|$. 


Besides, it has been proved that intersections of countably many rich families of a given space is (not only non-empty but even) rich (see [6, Proposition 1.1] or [20, Proposition 1.2]). Then for the case \((T, A) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))\) there must exist a rich family \(\mathcal{R}\) for the integrand function \((f_n)\), satisfying the properties of Proposition C.1 and with (C.4) uniformly for every \(n \in \mathbb{N}\), as well as for the integral functional \(E_f\). Using this family, we can extend all the previous statements to arbitrary Asplund spaces in the case when \((T, A) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))\).

Proof of Corollary 3.7. Assume that the assumptions on \(f, g\) in Theorem 3.4 hold in an Asplund space \(X\) (the assumptions in Theorem 3.5, respectively). Let \(x^* \in \partial E_f(x)\) and \(\rho\) be a \(w^*\)-continuous seminorm on \(X^*\); for instance, \(\rho = \max_{i=1,...,p} \langle \cdot, e_i \rangle\) with some \(e_i \in X\). Then consider \(V \times Y \in \mathcal{R}\) such that \(x, e_i \in V, \ i = 1,...,p\) and \(x^* \in Y\). Then \(x^*_V =: y^* \in \partial (E_f)_V(x)\). Then applying Theorem 3.4, there exist sequences \(y_n \in V, x_n \in L^p(T, V), z^*_n \in L^q(T, V^*)\) such that:

(a) \(z^*_n(t) \in \partial f|_V(t, x_n(t))\) ae,
(b) \(|x - y_n| \to 0, \int_T \|x(t) - x_n(t)\|^p d\mu(t) \to 0,\)
(c) \(\|z^*_n(\cdot)\|_q \|x_n(\cdot) - y_n\|_p \to 0, (\int_T \|x^*_n(\cdot)\|_q \|x_n(\cdot) - y_n\| d\mu(t) \to 0 \text{ resp.})\)
(d) \(\int_T \langle z^*_n(t), x_n(t) - x \rangle d\mu(t) \to 0, \rho \left(\int_T z^*_n(t) d\mu(t) - x^*\right) \to 0,\)
(e) \(\int_T |f(t, x_n(t)) - f(t, x)| d\mu(t) \to 0.\)

Then define \(x^*_n(\cdot)\) as the unique element in \(\partial f(t, x_n(t)) \cap Y\) such that \(\|x^*_n(\cdot)\| = \|z^*_n(\cdot)\|\) and \((x^*_n(\cdot))_V = z^*_n(\cdot)\). Now \(\|y_n(\cdot)\|_q = \|x^*_n(\cdot)\|_q\), which implies that

\[\|x^*_n(\cdot)\|_q \|x_n(\cdot) - y_n\|_p \to 0\]

\((\int_T \|x^*_n(\cdot)\|_q \|x_n(\cdot) - y_n\| d\mu(t) \to 0, \text{ respectively})\) and \(x^*_n \in L^q(T, X^*)\). From the fact that \(x_n(t), y_n, e_i, x \in V\) and \((x^*_n(\cdot))_V = z^*_n(\cdot)\) we conclude that \(\int_T \langle x^*_n(t), x_n(t) - x \rangle d\mu(t) \to 0\) and \(\rho \left(\int_T x^*_n(t) d\mu(t) - x^*\right) \to 0\). Then, the sequences \(y_n, x_n(\cdot)\) and \(x^*_n(\cdot)\) satisfy the required properties.

C.4. Fatou-type lemma. The next lemma gives the first sequential approximation of subgradients in the limiting subdifferential of the \(E_f\). This results uses the sequential representation of the Fréchet subdifferential found in Theorem 3.5, together with a Fatou’s lemma for sequences in Banach spaces. This will allow to get a relation between the limits of the integral of subgradients in the Fréchet subdifferential and the integral of the limiting subdifferential.

Lemma C.2. Consider \(x^* \in \partial E_f(x), y^* \in \partial^\infty E_f(x)\), a finite family of linearly independent points \(\{e_i\}_i^p, W := \text{span}\{e_i\}\) and \(\rho(\cdot) := \max\{|\langle \cdot, e_i \rangle|\}\). Then there exist sequences \(x_n, y_n \in L^\infty(T, X), x^*_n, y^*_n \in L^1(T, X^*)\) and \(\lambda_n \to 0^+\) such that:

(a) \(x^*_n(t) \in \partial f(t, x_n(t))\) ae,
(b) \(\|x(t) - x_n(\cdot)\|_\infty \to 0,\)
(c) \(\rho \left(\int_T x^*_n(t) d\mu(t) - x^*\right) \to 0,\)
(d) \(\lim_{T} f(t, x_n(t)) - f(t, x)| d\mu(t) = 0,\)

as well as

(e) \(y^*_n(t) \in \partial f(t, y_n(t))\) ae,
So, by Lemma A.1 we conclude that

$$(b^\infty) \|x - y_n(\cdot)\|_\infty \to 0,$$

$$(c^\infty) \rho(\lambda_n \cdot \int_T y_n^*(t)d\mu(t) - y^*) \to 0,$$

$$(d^\infty) \lim_T \int_T |f(t,y_n(t)) - f(t,x)|d\mu(t) = 0,$$

$$(e^\infty) y^* \in (B(y_n^*) \cap W)^-,$$

where

$$B(y_n^*) := \{u \in X : \liminf_T \int_T (y_n^*, u)^+ d\mu < +\infty\}.$$

Moreover, if there exists a bounded sequence $x_n^*$ (in $L^1(T, X^*)$) (or $\lambda_n y_n^*$, respectively) satisfying the above properties, then

$$x^* \in \int_T \partial f(t, x)d\mu(t) + C((x_n^*))^- + W^-, $$

$$y^* \in \int_T \partial^\infty f(t, x)d\mu(t) + C(\lambda_n y_n^*)^- + W^- (\text{respectively}),$$

where $C(x_n^*) := \{u \in X : ((x_n^*, u)^+)_{n \in \mathbb{N}} \text{ is uniformly integrable}\}.$

Proof. By the definition of $\partial E_f(x)$ and $\partial^\infty E_f(x)$, there exist sequences $z_n^* \in \hat{\partial} E_f(z_n)$, $s_n^* \in \hat{\partial} E_f(s_n)$ and $\lambda_n \to 0^+$ such that $z_n, s_n \to x, z_n^* \to x^*$ and $\lambda_n \cdot s_n^* \to y^*$. Whence, by Theorem 3.5 (and using a diagonal argument) there exist sequences $x_n, y_n \in L^\infty(T, X), x_n^*, y_n^* \in L^1(T, X^*)$ such that

(i) $x_n(t) \in \hat{\partial} f(t, x_n(t))$ ae,

(ii) $\|x - x_n(\cdot)\|_\infty \to 0$,

(iii) $\rho(\int_T x_n^*(t)d\mu(t) - x^*) \to 0$,

(iv) $\int_T f(t, x_n(t))d\mu(t) \to \int_T f(t, x)d\mu(t)$,

(iii) $y_n^*(t) \in \hat{\partial} f(t, y_n(t))$ ae,

(ii) $\|x - y_n(\cdot)\|_\infty \to 0$,

(iii) $\rho(\lambda_n \cdot \int_T y_n^*(t)d\mu(t) - y^*) \to 0$,

(iv) $\int_T f(t, y_n(t))d\mu(t) \to \int_T f(t, x)d\mu(t)$.

So, by Lemma A.1 we conclude that

$$\lim_T \int_T |f(t, x_n(t)) - f(t, x)|d\mu(t) = 0$$

and

$$\lim_T \int_T |f(t, y_n(t)) - f(t, x)|d\mu(t) = 0.$$
where $Ls^w\{x^*_n(t)\}$ represents the sequential upper limit of the sequence $(x^*_n(t))$. Moreover, if $\sup_n \int_T \|x^*_n(t)\|d\mu(t)$ is finite, then (up to subsequences)

$$w^*_n := \int_T x^*_n(t)d\mu(t) \to w^*_0.$$ 

Moreover, $w^*_n = P^*_W(w^*_0) + w^*_0 - P^*_W(w^*_0)$ and by Item (iii) we get that $P^*_W(w^*_0) \to P^*_W(x^*)$. Therefore, $w^*_0 - P^*_W(x^*) \in W^\perp$, and then we conclude that $x^* = w^*_0 + P^*_W(x^*) - w^*_0 + x^* - P^*_W(x^*) \in \int T Ls^w\{x^*_n(t)\}d\mu(t) + C(x^*)^- + W^\perp$.

Finally, we have to prove that $Ls^w\{x^*_n(t)\} \subseteq \partial f(t,x)$ for $ae t$. Indeed, from the previous convergence (and taking a subsequence if necessary, see, e.g., [1, Theorem 13.6]) we can take a measurable set $T$ such that $\mu(T \setminus \tilde{T}) = 0$ and for every $t \in \tilde{T}$, $x_n(t) \to x_0$ and $f(t,x_n(t)) \to f(t,x)$. Then take an integrable selection $a^*(t) \in Ls^w\{x^*_n(t)\}$ and fix $t_0 \in \tilde{T}$. So, there exists a subsequence $x_{nk(t_0)}(t_0) \to a^*(t_0)$, and then $x_{nk(t_0)}(t_0) \to x_0$ and $f(t_0,x_{nk(t_0)}) \to f(t_0,x)$. Hence, $a^*(t_0) \in \partial f(t_0,x)$. The case for the point $y^*$ is similar, and so we omit the proof.

C.5. Characterization of Integrable compact sole property. The final lemma allows us to understand the definition of integrable compact soles in terms of the primal space instead of the dual space, using the polar cone.

**Lemma C.3.** Let $C : T \to X$ be a measurable multifunction with non-empty $w$-closed convex values, let $e \in X$ and let $\delta > 0$. Then the following statements are equivalent:

(a) For every measurable selection $c^*$ of $C^-(t) := (C(t))^-$ one has

$$\delta^{-1}\langle c^*(t), -e \rangle \geq \|c^*(t)\| \quad ae.$$ 

(b) The ball in $L^\infty(T,X)$ around $e$ with radius $\delta$ is contained in

$$\{x(\cdot) \in L^\infty(T,X) : x(t) \in C(t) \quad ae\}.$$ 

**Proof.** First, by Castaing’s representation there exist sequences of measurable selections $c_n$ and $c^*_n$ of $C$ and $C^*$ respectively such that $C(t) = \{c_n(t)\}_{n \in \mathbb{N}}$ and $C^-(t) = \{c^*_n(t)\}_{n \in \mathbb{N}}$ (the measurability of $C^-$ follows from the fact that $C^-(t) = \bigcap_{n \in \mathbb{N}}\{x^* \in X^* : \langle c_n(t), x^* \rangle \leq 0\}$).

Suppose that Item (a) holds and consider $h \in L^\infty(T,X)$ and $\|h\|_{\infty} \leq 1$. Then

$$\|e + \delta h(t), c^*_n(t)\| \leq \langle e, c^*_n(t) \rangle + \delta \|c^*_n(t)\| \leq 0 \quad ae.$$ 

Since the last inequality holds for all $n$ and the measurable selection $c^*_n(t)$ is dense in $C^-(t)$, we conclude that $e + \delta h(t) \in (C^-(t))^-$ ae. Then, using the Bipolar theorem (see, e.g., [19, Theorem 3.38]), we obtain that $e + \delta h(t) \in C(t) \quad ae$.

Now suppose that Item (b) holds and consider a measurable selection $c^*$ of $C^-$ and $(0,1) \ni \varepsilon_n \to 0$. Take a measurable selection $h_n(t) \in B(0,1)$ such that $\langle h_n(t), c^*(t) \rangle \geq \|c^*(t)\| - \varepsilon_n \quad ae$. So, $e + \delta h(t) \in C(t) \quad ae$, and hence $\|c^*(t), e + \delta h(t)\| \leq 0 \quad ae$. Therefore, $\delta^{-1}\langle c^*(t), -e \rangle \geq \|c^*(t)\| - \varepsilon_n \quad ae$. From the stability of null sets under countable intersections we get $\delta^{-1}\langle c^*(t), -e \rangle \geq \|c^*(t)\|$ for $ae t$.

**References**

[1] C. D. Aliprantis and K. C. Border, *Infinite-dimensional analysis*, vol. 4 of Studies in Economic Theory, Springer-Verlag, Berlin, 1994. A hitchhiker’s guide.
[2] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition [MR1048347].

[3] R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl., 12 (1965), pp. 1–12.

[4] E. J. Balder and A. R. Sambucini, *Fatou’s lemma for multifunctions with unbounded values in a dual space*, J. Convex Anal., 12 (2005), pp. 383–395.

[5] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007.

[6] J. Borwein and W. B. Moors, *Separable determination of integrability and minimality of the Clarke subdifferential mapping*, Proc. Amer. Math. Soc., 128 (2000), pp. 215–221.

[7] J. M. Borwein and L. Yao, *Legendre-type integrands and convex integral functions*, J. Convex Anal., 21 (2014), pp. 261–288.

[8] J. M. Borwein and Q. J. Zhu, *A survey of subdifferential calculus with applications*, Nonlinear Anal., 39 (1999), pp. 687–773.

[9] J. M. Borwein and Q. J. Zhu, *Techniques of variational analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 20, Springer-Verlag, New York, 2005.

[10] N. Bourbaki, *Eléments de mathématique. VIII. Première partie: Les structures fondamentales de l’analyse. Livre III: Topologie générale. Chapitre IX: Utilisation des nombres réels en topologie générale*. Actualités Scientifiques et Industrielles, 1045, Hermann et Cie., Paris, 1948.

[11] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin-New York, 1977.

[12] F. H. Clarke, *Optimization and nonsmooth analysis*, vol. 5 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second ed., 1990.

[13] B. Cornet, V.-F. Martins-da Rocha, et al., *Fatou’s lemma for unbounded Gelfand integrable mappings*, preprint, 109 (2002).

[14] R. Correa, A. Hantoute, and P. Pérez-Aros, *Characterizations of the subdifferential of convex integral functions under qualification conditions*, (2018), submitted.

[15] R. Correa, A. Hantoute, and P. Pérez-Aros, *Qualification conditions-free characterizations of the ε-subdifferential of convex integral functions*, (2018), submitted.

[16] M. Cúth and M. Fabian, *Asplund spaces characterized by rich families and separable reduction of Fréchet subdifferentiability*, J. Funct. Anal., 270 (2016), pp. 1361–1378.

[17] M. Cúth and M. Fabian, *Rich families and projectional skeletons in Asplund WCG spaces*, J. Math. Anal. Appl., 448 (2017), pp. 1618–1632.

[18] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[19] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011. The basis for linear and nonlinear analysis.

[20] M. Fabian and A. Ioffe, *Separable reductions and rich families in the theory of Fréchet subdifferentials*, J. Convex Anal., 23 (2016), pp. 631–648.

[21] E. Giner, *On the Clarke subdifferential of an integral functional on Lp, 1 ≤ p < ∞*, Canad. Math. Bull., 41 (1998), pp. 41–48.

[22] E. Giner, *Necessary and sufficient conditions for the interchange between infimum and the symbol of integration*, Set-Valued Var. Anal., 17 (2009), pp. 321–357.

[23] E. Giner, *Clarke and limiting subdifferentials of integral functionals*, J. Convex Anal., 24 (2017), pp. 661–678.

[24] E. Giner and J.-P. Penot, *Subdifferentiation of integral functionals*, Math. Program., 168 (2018), pp. 401–431.

[25] A. Hantoute, R. Henrion, and P. Pérez-Aros, *Subdifferential characterization of probability functions under gaussian distribution*, Mathematical Programming, (2018).

[26] F. Hiai and H. Umegaki, *Integrals, conditional expectations, and martingales of multivalued functions*, J. Multivariate Anal., 7 (1977), pp. 149–182.

[27] S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis. Vol. I, Vol. II*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1997. Theory.

[28] A. Ioffe, *Three theorems on subdifferentiation of convex integral functionals*, J. Convex Anal., 13 (2006), pp. 759–772.

[29] A. D. Ioffe, *Survey of measurable selection theorems: Russian literature supplement*, SIAM J. Control Optim., 16 (1978), pp. 728–732.

[30] A. D. Ioffe, *On the theory of subdifferentials*, Adv. Nonlinear Anal., 1 (2012), pp. 47–120.

[31] A. D. Ioffe and V. L. Levin, *Subdifferentials of convex functions*, Trudy Moskov. Mat. Obšč., 26 (1972), pp. 3–73.

[32] A. D. Ioffe and V. M. Tikhomirov, *Duality of convex functions and extremum problems*, Russian Mathematical Surveys, 23 (1968), p. 53.
[33] V. L. Levin, Convex integral functionals and the theory of lifting, Russian Mathematical Surveys, 30 (1975), p. 119.
[34] O. Lopez and L. Thibault, Sequential formula for subdifferential of integral sum of convex functions, J. Nonlinear Convex Anal., 9 (2008), pp. 295–308.
[35] B. S. Mordukhovich, Variational analysis and generalized differentiation. I, vol. 330 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2006. Basic theory.
[36] B. S. Mordukhovich, Variational analysis and generalized differentiation. II, vol. 331 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2006. Applications.
[37] B. S. Mordukhovich, Variational Analysis and Applications, vol. 8, Springer, 2018.
[38] B. S. Mordukhovich and N. Sagara, Subdifferentials of nonconvex integral functionals in Banach spaces with applications to stochastic dynamic programming, J. Convex Anal., 25 (2018), pp. 643–673.
[39] J.-P. Penot, Image space approach and subdifferentials of integral functionals, Optimization, 60 (2011), pp. 69–87.
[40] J.-P. Penot, Calculus without derivatives, vol. 266 of Graduate Texts in Mathematics, Springer, New York, 2013.
[41] P. Pérez Aros, Subdifferential calculus in the framework of Epi-pointed variational analysis, integral functions, and applications, PhD thesis.
[42] R. T. Rockafellar, Integrals which are convex functionals, Pacific J. Math., 24 (1968), pp. 525–539.
[43] R. T. Rockafellar, Convex integral functionals and duality, in Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), Academic Press, New York, 1971, pp. 215–236.
[44] R. T. Rockafellar, Integrals which are convex functionals. II, Pacific J. Math., 39 (1971), pp. 439–469.
[45] R. T. Rockafellar and R. J.-B. Wets, Variational analysis, vol. 317 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1998.
[46] L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures, Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. Tata Institute of Fundamental Research Studies in Mathematics, No. 6.
[47] C. Vallée and C. Zălinescu, Series of convex functions: subdifferential, conjugate and applications to entropy minimization, J. Convex Anal., 23 (2016), pp. 1137–1160.
[48] W. van Ackooij and R. Henrion, Gradient formulae for nonlinear probabilistic constraints with Gaussian and Gaussian-like distributions, SIAM J. Optim., 24 (2014), pp. 1864–1889.
[49] W. van Ackooij and R. Henrion, (Sub-)gradient formulae for probability functions of random inequality systems under Gaussian distribution, SIAM/ASA J. Uncertain. Quantif., 5 (2017), pp. 63–87.
[50] D. H. Wagner, Survey of measurable selection theorems, SIAM J. Control Optimization, 15 (1977), pp. 859–903.
[51] D. H. Wagner, Survey of measurable selection theorems: an update, in Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), vol. 794 of Lecture Notes in Math., Springer, Berlin-New York, 1980, pp. 176–219.
[52] D. Zagrodny, Approximate mean value theorem for upper subderivatives, Nonlinear Anal., 12 (1988), pp. 1413–1428.