Pfaffians, the $G$-Signature Theorem and Galois de Rham discriminants

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1 INTRODUCTION.

The principal objects of study in this article are the various bilinear forms on the de Rham cohomology of an arithmetic variety which arise from duality theory. S. Bloch provided a fundamental new insight into de Rham discriminants when he showed that, for a suitable arithmetic surface, the square root of the de Rham discriminant is equal to the conductor of the surface (see [Bl]). This striking result, which has been a fundamental influence on our work, may be seen as extending to surfaces the important fact that the discriminant of a ring of algebraic integers can be expressed in terms of Artin conductors. We remark that Bloch’s result has recently been extended to arithmetic varieties of higher dimension by K. Kato and T. Saito in [KS].

We consider a regular scheme $X$ which is projective and flat over $\text{Spec}(\mathbb{Z})$ of constant fibral dimension $d$, and which supports an action by a finite group $G$. We write $Y$ for the quotient scheme $X/G$. To each complex character $\theta$ of $G$ we can associate the Artin-Hasse-Weil $L$-function $L(Y, \theta, s)$ which conjecturally satisfies a functional equation

$$L(Y, \theta, s) = \varepsilon(Y, \theta) A(Y, \theta)^{-s} L(Y, \theta, d + 1 - s)$$

where $A(Y, \theta)$ denotes the conductor at $\theta$. The constant $\varepsilon(Y, \theta)$ is defined independently of the conjecture. By a theorem of Deligne and Langlands, after making certain choices which we suppress in our notation, there is a product formula

$$\varepsilon(Y, \theta) = \prod_v \varepsilon_v(Y, \theta)$$

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where the product extends over the places of the rational field \( \mathbb{Q} \) and almost all terms are 1. For details of the construction of the local constants \( \varepsilon_v(Y, \theta) \) see [D2]. A discussion of local constants which is well suited to the context of this paper is given in [CEPT2].

Suppose now that the group \( G \) acts tamely on \( X \) (for details see condition (T1) below). A character of \( G \) is said to be symplectic when it is the character of representation of \( G \) which supports a non-degenerate \( G \)-invariant alternating form. We shall show that for each symplectic character \( \theta \) of \( G \), the \( \theta \)-component of the de Rham discriminant has a natural square root called the Pfaffian. One of the main goals of this paper is to demonstrate an intimate relationship between the sign of such Pfaffians and the archimedean constant \( \varepsilon_\infty(Y, \theta) \). In this regard we are most grateful to Michael Atiyah who, early in our study, pointed out to us an analogous phenomenon for Selberg \( \zeta \)-functions. He showed in [Ms] that under suitable circumstances, the constant in the functional equation of such a \( \zeta \)-function is a signature invariant given by the regularised Pfaffian of the Laplacian. This observation provided us with valuable insight into the arithmetic situation which we consider here.

To describe a classical instance of our results, let \( N \subset \mathbb{C} \) be a number field which is tame and Galois over \( \mathbb{Q} \) with Galois group \( G \). The simplest non-trivial case concerns quadratic \( N \) over the rationals. Let \( \sigma \) denote the non-trivial automorphism of \( N \). Since 2 is non-ramified in \( N \), the ring of integers of \( N \) is equal to \( \mathbb{Z}[\alpha] \) where \( \alpha = \frac{1}{2}(1 + \sqrt{d_N}) \) and \( d_N \) is the discriminant of \( N \). Let \( \phi \) denote the non-trivial abelian character of \( G \); since \( \phi \) is real-valued the character \( 2\phi \) is symplectic. Using the fact that \( \alpha - \sigma(\alpha) \) is a Pfaffian for the trace form associated to \( \phi \) (see Theorem 18, page 200 in [F1]), it follows that the Pfaffian of the trace form associated to the symplectic character \( 2\phi \) is equal to \((\alpha - \sigma(\alpha))^2 = d_N \), and this is of course positive or negative according as \( N \) is real or imaginary quadratic. We then remark that this sign coincides exactly with the archimedean constant \( \varepsilon_\infty(Y, \theta) \). In this regard we are most grateful to Michael Atiyah who, early in our study, pointed out to us an analogous phenomenon for Selberg \( \zeta \)-functions. He showed in [Ms] that under suitable circumstances, the constant in the functional equation of such a \( \zeta \)-function is a signature invariant given by the regularised Pfaffian of the Laplacian. This observation provided us with valuable insight into the arithmetic situation which we consider here.

The results of this paper are best seen as a complement to the work in [CPT2]. There we considered certain Euler characteristics of bounded complexes of \( G \)-bundles which support metrics on the determinants of the isotypic components of their cohomology. We will refer to these Euler characteristics as equivariant Arakelov classes. In [CPT2] we considered the equivariant Arakelov classes associated to the de Rham complex, when the equivariant determinant of cohomology was endowed with Quillen metrics arising from a choice of Kähler metric. In this approach all signature information was lost. In [CPT1] we considered...
arithmetic surfaces in such a way as to retain the signature information. This was achieved by replacing the symplectic Arakelov group by the somewhat sharper hermitian class group of Fröhlich (see (5) in 3.1.2 for a comparison of these two groups). The main result of this article is to describe the de Rham discriminant of a suitable arithmetic variety, in the hermitian class group, and thereby describe in full the equivariant signature information of de Rham discriminant. In [CPT1] we were also able to determine the non-archimedean local epsilon constant \( \varepsilon_p(\mathcal{Y}, \theta) \) for a symplectic character \( \theta \) of \( G \) for an arithmetic surface \( \mathcal{Y} \), in terms of the hermitian structure of the local de Rham complex. Here we do not attempt to extend this result for non-archimedean local epsilon constants to higher dimensions; however, we do remark that the work of T. Saito in [Sa] appears to provide a good approach to this problem.

Prior to stating our main result, we first briefly comment on the methods and tools used in this paper. One elementary tool, which is nonetheless crucial, is the Pfaffian. Our reformulation of the Pfaffian extends the description of the discriminant of a non-degenerate bilinear form on a vector space as an element of the dual of the tensor square of the determinant of the vector space. This approach extends readily to complexes, and thus one may then view the discriminant as an element of the dual of the tensor square of the determinant of the cohomology of the complex. For an alternating form on a complex, the Pfaffian is a natural square root of the discriminant which is a functional on the determinant of cohomology. We then recast Fröhlich’s theory of hermitian classes in terms of the Pfaffian on determinants of cohomology, and this approach turns out to be extremely well suited to the calculation of the de Rham discriminant.

We also wish to highlight the crucial use of the Atiyah-Singer \( G \)-signature theorem. Recall that, when there is no group action, Hirzebruch gave a beautiful formula for the signature of the cup-product form on the middle dimensional cohomology group for a compact oriented real manifold \( Z \) whose dimension can be written \( 2d \) with \( d \) an even integer (see for instance Theorem 6.6 in [AS]). Using the Index formula in the case where a finite group \( G \) acts on \( Z \), Atiyah and Singer obtained a formula for the \( G \)-signature character of the virtual \( \mathbb{R}[G] \)-module

\[
\text{H}^d_B(Z, \mathbb{R})^+ - \text{H}^d_B(Z, \mathbb{R})^- 
\]

where \( \text{H}^d_B(Z, \mathbb{R})^\pm \) denotes a maximal \( \mathbb{R}[G] \)-module of the Betti cohomology group \( \text{H}^d_B(Z, \mathbb{R}) \) on which the cup-form product is positive definite resp. negative definite. In fact we shall need a modified version of such invariants: namely, we let \( \chi^\pm(Z) \) denote the dimension of \( \text{H}^\bullet_B(Z, \mathbb{R})^\pm \), the (virtual) maximal space of the total Betti cohomology, on which the cup product form is positive definite resp. negative definite (see 4.3 for details). Thus \( \chi^+(Z) + \chi^-(Z) \) is the Euler characteristic \( \chi(Z) \). Then, for an arbitrary compact oriented
real manifold $Z$ of even dimension $2d$, we put
\[
\delta(Z) = \begin{cases} 
\frac{\chi(Z)}{2}, & \text{if } d \text{ is odd}, \\
\chi^+(Z), & \text{if } d \equiv 2 \mod 4, \\
\chi^-(Z), & \text{if } d \equiv 0 \mod 4.
\end{cases}
\]

In order to present our main theorem we first need to introduce some notation. Throughout $\mathcal{X}$ is a flat projective scheme over $\text{Spec}(\mathbb{Z})$ which supports the action of a finite group $G$; we let $Y$ denote the quotient $\mathcal{X}/G$, and we further assume that the following two conditions are always satisfied:

(T1) the action of $G$ on $\mathcal{X}$ is “tame” (for every point $x$ of $\mathcal{X}$ the order of the inertia group $I_x \subset G$ is prime to the residual characteristic of $x$). Since $\mathcal{X}$ maps onto $\text{Spec}(\mathbb{Z})$, it follows that the locus of ramification locus of the action of $G$ is fibral, and so, writing $X$ for the generic fibre $\mathcal{X} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$, $G$ acts freely on $X$ and so the cover $X \to Y$ is etale;

(T2) both schemes $\mathcal{X}$ and $Y$ are regular and “tame” (i.e. they are regular and all their special fibres are divisors with normal crossings with multiplicities prime to the residue characteristic).

Now let $\Omega^1_{\mathcal{X}/\mathbb{Z}}$ denote the coherent sheaf of differentials of $\mathcal{X} \to \text{Spec}(\mathbb{Z})$. Since $\mathcal{X}$ is regular, we may choose a resolution of $\Omega^1_{\mathcal{X}/\mathbb{Z}}$ by a length two complex $K^•$ of $G$-equivariant locally free $\mathcal{O}_{\mathcal{X}}$-sheaves; for $i \geq 0$ we let $L^\wedge_i$ denote the $i$-th left derived exterior power functor of Dold-Puppe on perfect complexes of $G$-equivariant $\mathcal{O}_{\mathcal{X}}$-sheaves (that is to say, $\mathcal{O}_{\mathcal{X}}$-sheaves with a $G$-action which is compatible with the $G$-action on $\mathcal{O}_{\mathcal{X}}$). Thus $L^\wedge_i K^•$ denotes the complex arising from the application of $L^\wedge_i$ to $K^•$ and we define $L^\wedge • \Omega^1_{\mathcal{X}/\mathbb{Z}}$ to be the direct sum of the complexes $L^\wedge_i K^•[-i]$ for $0 \leq i \leq d$. For details of the Dold-Puppe exterior power functor the reader is referred to [DP], [I] and to 5.4-5.9 in [So]. We recall from [CEPT1] that, because $G$ acts tamely, $R\Gamma(\mathcal{X}, L^\wedge • \Omega^1_{\mathcal{X}/\mathbb{Z}})$ may represented by a perfect $\mathbb{Z}[G]$-complex. Note for future reference that on the generic fibre $X$ of $\mathcal{X}$ each $(L^\wedge i K^•) \otimes_\mathbb{Z} \mathbb{Q}$ is quasi-isomorphic to the sheaf of differentials $\Omega^1_{\mathcal{X}/\mathbb{Q}}$ viewed as a complex concentrated in degree zero.

In Sect. 4 we shall recall in detail from [CPT1] the symmetric $G$-invariant pairings on the cohomology groups
\[
\sigma^i_X : H^iR\Gamma(\mathcal{X}, L^\wedge • \Omega^1_{\mathcal{X}/\mathbb{Q}})[d] \times H^{-i}R\Gamma(\mathcal{X}, L^\wedge • \Omega^1_{\mathcal{X}/\mathbb{Q}})[d] \to \mathbb{Q}
\]
arising from Serre duality. In [F1] Fröhlich showed how to use the notion of Pfaffian to construct a refined discriminant, or hermitian class, for any locally free $\mathbb{Z}[G]$-module which supports a non-degenerate $G$-invariant symmetric form over $\mathbb{Q}$. In [CPT1] we extended this construction to perfect $\mathbb{Z}[G]$-complexes with non-degenerate $G$-invariant symmetric forms on the cohomology of the complex tensored by $\mathbb{Q}$. Thus, to the pair $(R\Gamma(\mathcal{X}, L^\wedge • \Omega^1_{\mathcal{X}/\mathbb{Z}})[d], \sigma^d_X)$,
we may associate a so-called hermitian Euler characteristic $\chi^s_H(R\Gamma(X, L \wedge^1 \Omega^1_{X/Z})[d], \sigma_X)$ which takes values in the hermitian class group $\text{HP}(\mathbb{Z}[G])$.

The hermitian Euler characteristic $\chi^s_H(R\Gamma(X, L \wedge^1 \Omega^1_{X/Z})[d], \sigma_X)$ was completely determined in [CPT1] when $X$ is an arithmetic surface. In this paper we shall essentially determine the hermitian Euler characteristic $\chi^s_H(R\Gamma(X, L \wedge^1 \Omega^1_{X/Z})[d], \sigma_X)$ for arbitrary fibral dimension $d$. To be a little more precise, we shall show that symplectic hermitian Euler characteristics decompose into the product of a metric invariant and a signature invariant. Writing $R^s_G$ for the group of virtual symplectic characters of $G$, we shall see that, under the above mentioned decomposition into metric and signature invariants (after tame extension of coefficients), the image of $\chi^s_H(R\Gamma(X, L \wedge^1 \Omega^1_{X/Z})[d], \sigma_X)$ lies in a group which is naturally isomorphic to

$$\text{Hom}_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(R^s_G, \mathbb{Q}^\times) \times \text{Hom}(R^s_G, \pm 1).$$

We let $\chi_1$ resp. $\chi_2$ denote the image of $\chi^s_H(R\Gamma(X, L \wedge^1 \Omega^1_{X/Z})[d], \sigma_X)$ in the first resp. second component. We shall see that $\chi_1$ coincides with the equivariant Arakelov class, which was fully described in [CPT2]; there we saw that this class is given by the $\varepsilon$-constant homomorphism which, for virtual characters $\theta$ of degree zero, maps $\theta$ to $\varepsilon(\mathcal{Y}, \theta)$. This product decomposition is of fundamental importance, for we note there are two natural sign invariants: as indicated above, the first sign invariant determines (conjecturally) the symmetry or skew-symmetry of the functional equation of the Artin-Hasse-Weil L-function; whereas the second sign invariant should be thought of as the archimedean signature. Such a double appearance of sign invariants was apparent in the work of Fröhlich (see for instance Corollary 3 page 192 in [F1]). The essential contribution of this article is the following evaluation of the signature class $\chi_2$.

**Theorem 1** For a symplectic character $\theta$ of $G$

$$\chi_2(\theta) = (-1)^{\delta(\mathcal{Y})\theta(1)/2} \varepsilon_\infty(\mathcal{Y}, \theta)$$

where $\varepsilon_\infty(\mathcal{Y}, \theta)$ is the archimedean constant described in the first part of the Introduction.

Note that it is a remarkable fact that the signatures of such equivariant de Rham discriminants, which come from de Rham cohomology, are determined by the archimedean $\varepsilon$-constants (at least for virtual characters of degree zero) which derive from the Hodge realisation of the real Artin motives $X_R \otimes_G V_\theta$ (see 5.3 in [D2], and see also Section 5 of [CEPT2]).

We conclude our introduction by providing a brief overview of the structure of this paper. The basic definitions and results on Pfaffians are all presented in Sect. 2. Then in Sect. 3 we introduce the symplectic hermitian classgroup $\text{HP}(\mathbb{Z}[G])$ where we define our hermitian Euler characteristics. It should be noted that this classgroup is in fact slightly
different from the hermitian class group defined by Fröhlich; we prefer to work with this
version, because it contains the symplectic equivariant Arakelov class group as a subgroup
in a natural way. In the Appendix we describe the natural map from Fröhlich’s class group
to our class group \( H^0(\mathbb{Z}|G|) \). In the Appendix, we shall also show how the hermitian Euler
characteristics defined in [CPT1], via lifts of pairings on cohomology to the whole perfect
complex, agree with the hermitian Euler characteristics that we use this paper, which are
defined via the Pfaffian on cohomology.

Finally in Sect. 4 we apply the foregoing theory to our arithmetic situation, and we
consider the de Rham discriminants: namely, the hermitian Euler characteristics of the de
Rham complex with forms on cohomology arising from the natural duality pairings. Here
we recall some basic results on archimedean \( \varepsilon \)-constants from [CEPT2] and we study the
signature properties of de Rham cohomology. This then provides us with all the tools we
need to complete the proof of the main theorem in the final sub-section.

2 PFAFFIANS.

In this section we work over an arbitrary field \( K \) of characteristic zero. All vector spaces are
assumed to be finite dimensional and all bilinear forms are assumed to be non-degenerate.

2.1 DISCRIMINANTS AND PFAFFIANS.

2.1.1 Determinants.

For a \( K \)-vector space \( V \) we let \( V^D \) denote the \( K \)-linear \( \text{Hom}_K(V, K) \), and if \( V \) has
dimension \( d \), we write \( \text{det}(V) = \wedge^d V \) and let \( \widehat{\text{det}}(V) \) denote the graded line \( (\text{det}(V), \dim V) \).
For a graded \( K \)-line \( (L, n) \), we put \( (L, n)^D = (L^D, -n) \). In the sequel we shall often write
\( L^{-1} \) for \( L^D \) and \( (L, n)^{-1} \) for \( (L, n)^D \).

Throughout this article we shall adopt the following convention in our use of exterior
products: we follow Deligne and we normalise the “twist” isomorphism between the tensor
product of the determinants of two vector spaces as follows: given two finite dimensional
vector spaces \( V, W \) over \( K \), the tensor product of the graded lines \( \widehat{\text{det}}(V) \) and \( \widehat{\text{det}}(W) \) is

\[
\widehat{\text{det}}(V) \otimes \widehat{\text{det}}(W) = \left( \text{det}(V) \otimes_K \text{det}(W), \dim(V) + \dim(W) \right)
\]

and we twist the standard isomorphism \( \text{det}(V) \otimes \text{det}(W) \cong \text{det}(W) \otimes \text{det}(V) \) according to
the Koszul rule of signs, i.e. by the factor \((-1)^{\dim(V)\dim(W)}\); thus, to be absolutely explicit,
under the new isomorphism

\[
(v_1 \wedge \cdots \wedge v_n) \otimes (w_1 \wedge \cdots \wedge w_m) \leftrightarrow (-1)^{\dim(V) \cdot \dim(W)} (w_1 \wedge \cdots \wedge w_m) \otimes (v_1 \wedge \cdots \wedge v_n).
\]
With this convention we then see that both the following diagram
\[
\begin{array}{ccc}
\hat{\det} (V) \otimes \hat{\det} (W) & \to & \hat{\det} (V \oplus W) \\
\downarrow & & \downarrow \\
\hat{\det} (W) \otimes \hat{\det} (V) & \to & \hat{\det} (W \oplus V)
\end{array}
\]
and the corresponding diagram where we then forget the grading commute. Here the horizontal maps are the maps induced by the isomorphisms \(\det (V) \otimes \det (W) \cong \det (V \oplus W)\) and the above description of \(\hat{\det} (V) \otimes \hat{\det} (W)\); the right-hand vertical arrow is induced by the natural isomorphism
\[V \oplus W \cong W \oplus V, \ v \oplus w \mapsto w \oplus v;\]
the left-hand vertical arrow is the above “twisted” isomorphism. This convention will help us avoid what Deligne calls the “nightmare of signs”.

2.1.2 Pfaffians.

We begin by recalling the notion of discriminant for a non-degenerate bilinear form \(h\) on \(V\). Thus such a form \(h\) affords an isomorphism \(h : V \to V^D\), via the rule \(h (x) (y) = h (y, x)\). The discriminant \(d_h\) is then defined to be the linear isomorphism of one dimensional \(K\) vector spaces
\[d_h : \det (V) \otimes \det (V^D) \to K\]
given by using \(\det (V^D) \cong \det (V)^D\) and contraction.

Suppose now that \(h\) is an alternating form and let \(\dim (V) = 2n\); recall that \(V\) has a hyperbolic basis \(\{u_1, u'_1, u_2, u'_2, \ldots, u_n, u'_n\}\) where
\[h (u_i, u_j) = 0 = h (u'_i, u'_j) \text{ for all } i, j, \text{ and } h (u_i, u'_j) = \delta_{ij}.
\]
Thus in particular \(K u'_i\) identifies, via \(h\), as the dual line of \(K u_i\), and so we can define
\[\text{Pf}_h : \det (V) = \bigotimes_{i=1}^n \det (K u_i \oplus K u'_i) = \bigotimes_{i=1}^n \det (K u_i \oplus (K u_i)^D) \to \bigotimes_{i=1}^n K = K.
\]
Alternatively we see that \(\text{Pf}_h\) is the unique \(K\)-linear functional on \(\det (V)\) such that
\[\text{Pf}_h (u_1 \wedge u'_1 \wedge u_2 \wedge u'_2 \wedge \cdots \wedge u_n \wedge u'_n) = 1.
\]
(Note that had we defined \(h (x) (y) = h (x, y)\), then we would have \(1 = \text{Pf}_h (u'_1 \wedge u_1 \cdots)\).)

This notation suggests that in fact \(\text{Pf}_h\) does not depend on the choice of particular hyperbolic basis. That this is indeed the case follows from the first of the following two lemmas, both of whose proofs are routine.
Lemma 2 If \( \{v_1, v'_1, v_2, v'_2, \ldots, v_n, v'_n\} \) is a further hyperbolic basis of \( V \), with respect to \( h \), then, since a symplectic automorphism has determinant 1,
\[
Pf_h (v_1 \wedge v'_1 \wedge v_2 \wedge v'_2 \cdots \wedge v_n \wedge v'_n) = 1.
\]

Lemma 3 For \( i = 1, 2 \) let \( h_i \) be an alternating-form on the vector space \( V_i \). Let \( h_1 \oplus h_2 \) denote the orthogonal sum form on \( V_1 \oplus V_2 \). Then, with the above convention, \( \Pf_{h_1 \oplus h_2} = \Pf_{h_1} \otimes \Pf_{h_2} \) under the identification \( \det (V_1 \oplus V_2) = \det (V_1) \otimes \det (V_2) \).

The following lemmas describe the functorial properties of the Pfaffian which we shall require. The proofs are all completely routine and follow from the standard properties of determinants.

Lemma 4 For an alternating form \( h \) on \( V \) and for a given isomorphism of \( K \)-vector spaces \( \phi : V \to W \), let \( \phi^* h \) denote the form on \( W \) given by the rule
\[
\phi^* h (x, y) = h \left( \phi^{-1} x, \phi^{-1} y \right).
\]
Then the following diagram commutes
\[
\begin{array}{ccc}
\det (V) & \xrightarrow{\Pf_h} & K \\
\downarrow \det (\phi) & & \downarrow \\
\det (W) & \xrightarrow{\Pf_{\phi^* h}} & K.
\end{array}
\]
In particular if \( V = W \), then \( \Pf_{\phi^* h} = \det (\phi)^{-1} \Pf_h. \)

Proposition 5 For a given alternating form \( h \) on \( V \) and for an automorphism \( A \) of \( V \), let \( \hat{A} \) denote the adjoint of \( A \) with respect to \( h \); that is to say \( h(Ax, y) = h(x, \hat{A}y) \). Suppose \( A \) is self-adjoint, so that \( A = \hat{A} \), and define \( h' (x, y) = h(Ax, y) \). Then there is an automorphism \( B \) of \( V \) such that \( h = B^* h' \). This implies that \( A = \hat{B} B \) and by the above \( \Pf_{h'} = \det(B) \Pf_h \). The value \( \det(B) \) therefore depends only on \( A \) and we call it the Pfaffian of \( A \), denoted \( \pf(A) \), so that we have
\[
\Pf_{h'} = \pf(A) \Pf_h.
\]

Remark 6 In the sequel Pf will denote a functional on a \( K \)-line, whereas \( \pf \) will denote the Pfaffian of a matrix.
2.2 EXTENSION TO COMPLEXES.

Let $C^\bullet$ denote a bounded complex of vector spaces over a field $K$. We put

$$C^{ev} = C^0 \bigoplus_{i \geq 0} (C^{2i} \oplus C^{-2i}) \quad \text{and} \quad C^{odd} = \bigoplus_{i \geq 0} (C^{2i+1} \oplus C^{-2i-1}).$$

and we recall that $\det(C^\bullet) = \otimes \det(C^i)^{(-1)^i}$.

There is a natural map (given by reordering)

$$v_{C^\bullet} : \det(C^\bullet) \rightarrow \det(C^{ev}) \otimes \det(C^{odd})^{-1}$$

where in full the latter line is

$$\det(C^0) \otimes \det(C^2) \otimes \det(C^{-2}) \otimes \cdots \otimes \det(C^1)^{-1} \otimes \det(C^{-1})^{-1} \otimes \cdots$$

**Remark 7** (a) If $D^\bullet$ is a further $K$-complex and if all the terms of $C^\bullet$ and $D^\bullet$ have even dimension, then the map

$$\det(C^\bullet \oplus D^\bullet) \cong \det(C^\bullet) \otimes \det(D^\bullet)$$

given by using the Koszul-twist isomorphisms coincides with the naive map given by the reordering of terms.

(b) If again all the terms $C^i$ have even dimension, then the map

$$\det(C^\bullet) \rightarrow \det(C^{ev}) \otimes \det(C^{odd})^{-1}$$

given by using the Koszul-twist isomorphisms coincides with the naive map $v_{C^\bullet}$ given by the reordering of terms.

We shall write $H^\bullet(C^\bullet)$ for the complex $\{H^i(C^\bullet)\}_i$, with zero boundary maps. As above we write

$$H^{ev} = H^{ev}(C^\bullet) = H^0(C^\bullet) \bigoplus_{i \geq 0} (H^{2i}(C^\bullet) \oplus H^{-2i}(C^\bullet))$$

and

$$H^{odd} = H^{odd}(C^\bullet) = \bigoplus_{i \geq 0} (H^{2i+1}(C^\bullet) \oplus H^{-2i-1}(C^\bullet)).$$

From [KM] we recall that there is a canonical isomorphism of $K$-lines

$$\xi : \det(C^\bullet) \cong \det(H^\bullet(C^\bullet)).$$

**Definition 8** Suppose we are given alternating forms $h^{ev}$ on $H^{ev}$ and $h^{odd}$ on $H^{odd}$. Define $\text{Pf}_h$ to be the element of the dual of the line $\det(C^\bullet)$,

$$\text{Pf}_h : \det(C^\bullet) \rightarrow K,$$

given by composing

$$\text{Pf}_h = \text{Pf}^{ev}_h \otimes \text{Pf}^{-1}_{h^{odd}} : \det(H^{ev}(C^\bullet)) \otimes \det(H^{odd}(C^\bullet))^{-1} \rightarrow K$$

with the isomorphism $v_{H^\bullet(C^\bullet)} \circ \xi$. Note that in the sequel, for brevity, we shall usually write $h$ for the pair $\{h^{ev}, h^{odd}\}$.
2.3 EQUIVARIANT PFAFFIANS.

Suppose now that $G$ is a finite group, $K$ is a subfield of the real numbers $\mathbb{R}$, and that $W$ is a symplectic complex representation of $G$ with character $\theta$; thus, by definition, $W$ supports a non-degenerate $G$-invariant alternating-form $\kappa$.

**Lemma 9** If $\kappa'$ is a further such form on $W$, then, since every pair of non-degenerate alternating forms on $W$ are isomorphic, there is an automorphism $B$ of $W$ such that $\kappa' = B^* \kappa$. Since both forms are $G$-invariant, the self-adjoint automorphism $A = \hat{B}B$ is a $G$-automorphism of $W$.

**Definition 10** A symmetric $K[G]$-complex is a pair $(C^\bullet, \sigma)$ where $C^\bullet$ is a perfect $K[G]$-complex and where $\sigma^\text{ev}$ and $\sigma^\text{odd}$ are non-degenerate real-valued $G$-invariant symmetric forms on $H^\text{ev}(C^\bullet)$ and $H^\text{odd}(C^\bullet)$ respectively.

For a given symmetric complex $(C^\bullet, \sigma)$ and for $W$ and $\kappa$ as above, we define $\det(C^\bullet_W)$ to be the line $\det((C^\bullet \otimes_{\mathbb{R}} W)^G)$; thus we have the canonical isomorphism

$$\xi_W : \det(C^\bullet_W) \cong \det(H^\bullet(C^\bullet)_W).$$

By restricting $\sigma^\text{ev} \otimes \kappa$ to $(H^\text{ev} \otimes W)^G$ we obtain a non-degenerate alternating form which we denote by $(\sigma^\text{ev} \otimes \kappa)^G$; similarly we obtain a form $(\sigma^\text{odd} \otimes \kappa)^G$ on $(H^\text{odd} \otimes W)^G$. Thus we obtain the composite map

$$\det(C^\bullet_W) \cong \det(H^\bullet(C^\bullet)_W) \cong \det(H^\text{ev}(C^\bullet)_W) \otimes \det(H^\text{odd}(C^\bullet)_W)^{-1} \to K$$

where the right hand arrow is $\text{Pf}_{(\sigma \otimes \kappa)^G}^G$.

3 CLASS GROUPS.

3.1 HERMITIAN AND ARAKELOV CLASSGROUPS.

In this subsection we give the definition of the symplectic hermitian class group, and we also briefly recall the definition of the equivariant Arakelov classgroup - for full details on the latter see [CPT2].

3.1.1 Definition of classgroups.

Let $R_G$ denote the group of complex virtual characters of $G$, and let $R^\text{sv}_G$ be the subgroup of virtual symplectic characters. Let $\overline{Q}$ be an algebraic closure of $Q$ in $\mathbb{C}$, and define $\Omega = \text{Gal}(\overline{Q}/Q)$. Define $J_f$ (resp. $J_\infty$) to be the group of finite ideles (resp. the archimedean
ideles) of $\mathbb{Q}$. Thus $J_f$ is the direct limit of the finite idele groups of all algebraic number fields $E$ in $\mathbb{Q}$, and

$$J_\infty = \lim_{E \subset \mathbb{Q}} (E \otimes \mathbb{Q} \mathbb{R})^\times.$$ 

The idele group of $\mathbb{Q}$ is $J = J_f \times J_\infty$.

Let $\mathbb{Z} = \prod_p \mathbb{Z}_p$ denote the ring of integral finite ideles of $\mathbb{Z}$. For $x \in \mathbb{Z}[G]^\times$, the element $\text{Det}(x) \in \text{Hom}_\Omega(R_G, J_f)$ is defined by the rule that for a representation $T$ of $G$ with character $\psi$

$$\text{Det}(x)(\psi) = \det(T(x)),$$

the group of all such homomorphisms is denoted by

$$\text{Det}(\mathbb{Z}[G]^\times) \subseteq \text{Hom}_\Omega(R_G, J_f).$$

More generally, for $n > 1$ we can form the group $\text{Det}(GL_n(\mathbb{Z}[G]))$; as each group ring $\mathbb{Z}_p[G]$ is semi-local, we have the equality $\text{Det}(GL_n(\mathbb{Z}[G])) = \text{Det}(\mathbb{Z}[G]^\times)$ (see 1.2.6 in [T]).

Recall that by the Hasse-Schilling norm theorem

$$\text{Det}(\mathbb{Q}[G]^\times) = \text{Hom}_\Omega^+(R_G, \mathbb{Q}^\times),$$

where the right-hand expression denotes Galois equivariant homomorphisms whose values on $R_G^s$ are all totally positive. We then have a diagonal map

$$\Delta : \text{Hom}_\Omega^+(R_G, \mathbb{Q}^\times) \to \text{Hom}_\Omega(R_G, J_f) \times \text{Hom}(R_G, \mathbb{R}_{>0})$$

where $\Delta(f) = f \times |f|$. Given a homomorphism $f$ on $R_G$, we shall write $f^s$ for the restriction of $f$ to $R_G^s$; in particular we write

$$\Delta^s : \text{Hom}_\Omega^+(R_G^s, \mathbb{Q}^\times) \to \text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbb{R}_{>0})$$

for the restriction of $\Delta$ to $R_G^s$, so that

$$\Delta^s(f') = f' \times |f'| = f' \times f'.$$

**Definition 11** The group of symplectic hermitian classes $H^s(\mathbb{Z}[G])$ is defined to be the quotient group

$$H^s(\mathbb{Z}[G]) = \frac{\text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbb{R}^\times)}{\text{Im}(\Delta^s) \cdot (\text{Det}^s(\mathbb{Z}[G]^\times) \times 1)}$$

(2)

where $\text{Det}^s(\mathbb{Z}[G]^\times)$ denotes the restriction of $\text{Det}(\mathbb{Z}[G]^\times)$ to $R_G^s$. Note that this hermitian classgroup $H^s(\mathbb{Z}[G])$ is slightly different from the hermitian classgroup $HGL(\mathbb{Z}[G])$ used in [CPT1] and [F1]. There is a natural map between these two classgroups. For details see the Appendix.
We recall from Definition 3.2 in [CPT2] that the group of Arakelov classes is defined to be
\[
A(Z[G]) = \frac{\text{Hom}_Ω(R_G, J_f) \times \text{Hom}(R_G, R_{>0})}{\text{Im}(∆) \cdot (\text{Det}(Z[G]^×) \times 1)}
\] (3)
and that the group of symplectic Arakelov classes (see Definition 4.1 in [CPT2]) is defined to be
\[
A^s(Z[G]) = \frac{\text{Hom}_Ω(R^s_G, J_f) \times \text{Hom}(R^s_G, R_{>0})}{\text{Im}(∆^s) \cdot (\text{Det}^s(Z[G]^×) \times 1)}
\] (4)

**Remark 12** Firstly, from the above descriptions, we see that \(A^s(Z[G])\) is naturally a subgroup of \(H^s(Z[G])\). Secondly, from Lemma 2.1 on page 60 of [F2], we note that, since all symplectic characters are real-valued, there is a natural isomorphism induced by the inclusion \(\mathbb{Q} \subset \mathbb{C}\)
\[
\text{Hom}_Ω(R^s_G, J_∞) \cong \text{Hom}(R^s_G, R^×).
\]

### 3.1.2 Rational classes and signature classes.

Let \(-1_∞\) denote the idele which is 1 at all finite primes and which is \(-1\) at all infinite primes. We then consider the two subgroups of
\[
\text{Hom}_Ω(R^s_G, J) = \text{Hom}_Ω(R^s_G, J_f) \times \text{Hom}_Ω(R^s_G, J_∞)
\]
\[
\cong \text{Hom}_Ω(R^s_G, J_f) \times \text{Hom}(R^s_G, R^×)
\]
given by
\[
R(Z[G]) = \text{Hom}_Ω(R^s_G, Q^×) \times 1,
\]
\[
S_∞(Z[G]) = 1 \times \text{Hom}(R^s_G, ±1) = \text{Hom}(R^s_G, ±1_∞).
\]

**Theorem 13** The natural map from \(\text{Hom}_Ω(R^s_G, J)\) to \(H^s(Z[G])\) induces an injection on \(R(Z[G]) \times S_∞(Z[G])\); thus, in the sequel, we shall view \(R(Z[G]) \times S_∞(Z[G])\) as a subgroup of \(H^s(Z[G])\).

**Proof.** Let \(r \times s \in R(Z[G]) \times S_∞(Z[G])\). We must show that if \(r \times s \in \text{Im}(∆^s)\cdot(\text{Det}^s(Z[G]^×) \times 1)\), then \(r = 1 = s\). Now by the Hasse-Schilling theorem we see immediately that \(s\) is positive and hence 1. We therefore deduce that \(r \in R(Z[G]) \cap \text{Det}^s(Z[G]^×)\) which is known to be trivial by Proposition 6.1 in [CNT] (see also [F1] Theorem 17, p. 190).

The counterpart for Arakelov classes is the following result, which is shown in 4.D of [CPT2]:

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Theorem 14 The natural map from $\text{Hom}_G(R^*_G, J_f) \times \text{Hom}(R^*_G, R_{\geq 0})$ to $A^s(\mathbb{Z}[G])$ induces an injection on $R(\mathbb{Z}[G])$; thus in the sequel we may view $R(\mathbb{Z}[G])$ as a subgroup of $A^s(\mathbb{Z}[G])$.

Viewing $A^s(\mathbb{Z}[G])$ as a subgroup of $H^s(\mathbb{Z}[G])$, we obtain the natural decomposition

$$H^s(\mathbb{Z}[G]) = A^s(\mathbb{Z}[G]) \times S_\infty(\mathbb{Z}[G]).$$  \hfill (5)

3.2 Formation of Euler characteristics.

3.2.1 Definitions.

Sympelctic hermitian case. From now on we fix a set of symplectic $\mathbb{C}[G]$-representations $W_m$ whose characters $\theta_m$ form a $\mathbb{Z}$-basis of $R^*_G$. There is of course a natural $\mathbb{Z}$-basis for $R^*_G$ given by the irreducible symplectic characters and the sums of the irreducible non-symplectic characters and their contragredients; in the sequel we shall assume our basis to be of this form. We then fix a non-degenerate $G$-invariant alternating form $\kappa_m$ on $W_m$ and we let $\{w_{mn}\}$ denote a hyperbolic basis of $W_m$ with respect to $\kappa_m$.

Suppose now that we are given a perfect $\mathbb{Z}[[G]]$-complex $P^\bullet$ with $G$-invariant non-degenerate real-valued symmetric forms $\sigma^\text{ev}$ resp. $\sigma^\text{odd}$ on $H^\text{ev}(P^\bullet)$ resp. $H^\text{odd}(P^\bullet)$. For each prime $p$ of $\mathbb{Z}$ let $\{a_{ij}^p\}$ denote a $\mathbb{Z}_p[[G]]$ basis for $\mathbb{Z}_p \otimes \mathbb{Z} P^i$; similarly we choose a $\mathbb{Q}[[G]]$ basis $\{a_0^{ij}\}$ for $P^i = \mathbb{Q} \otimes \mathbb{Z} P^i$; then for each prime $p$ let $\lambda_p^i$ be the element of $GL(\mathbb{Q}_p[G])$ such that $\lambda_p^i a_{ij}^p = a_0^{ij}$.

The following lemma is now clear:

Lemma 15 For any free $\mathbb{C}[G]$-module $U$ with basis $\{u_i\}$, the map

$$r_G : U \otimes \mathbb{C} W_m \to (U \otimes \mathbb{C} W_m)^G$$

defined by $r_G(u \otimes w) = \sum g u \otimes gw$ is a surjection and $\{r_G(u_i \otimes w_{mn})\}_{i,n}$ is a basis of $(U \otimes \mathbb{C} W_m)^G$. \hfill \square

For each pair $i,m$ we put

$$b_{jm}^{im} = r_G(a_{0i}^{jm} \otimes w_{mn}).$$

Then by the above lemma, since $P^i$ is $\mathbb{Q}[G]$-free, $\{b_{jm}^{im}\}$ is a $\mathbb{C}$-basis of $(P^i \otimes \mathbb{Q} W_m)^G$. As in (2.3) we shall write $\xi_m$ for the canonical isomorphism

$$\det((P^i \otimes \mathbb{Q} W_m)^G) \cong \det((H^s(P^i) \otimes \mathbb{Q} W_m)^G).$$
Since all the terms in the complexes $P^*_Q$ and $H^*(P^*_Q)$ are $Q[G]$-modules, because the representation $W_m$ is symplectic, it follows that all the terms in the complexes $(P^*_Q \otimes Q W_m)^G$ and $(H^*(P^*_Q) \otimes Q W_m)^G$ are even dimensional. Indeed, let $M$ be an $R[G]$-module: if $W_m$ is an irreducible symplectic $R[G]$-module, then $W_m$ has real Schur index 2, and so $\dim(M \otimes_R W_m)^G$ is even; on the other hand if $W_m$ can be written as $V + V^*$ for some $C[G]$-module $V$ with $V^*$ denoting the contragredient of $V$, then $\dim(M \otimes_R V)^G = \dim(M \otimes_R V^*)^G$ and so again $\dim(M \otimes_R W_m)^G$ is even. In particular we note that this means that by Remark 7(b) we may treat the natural isomorphism $v_{H^*}$

$$\det((H^*(P^*_Q) \otimes Q W_m)^G) \rightarrow \det((H^*(P^*_Q) \otimes Q W_m)^G) \otimes \det((H^{\text{odd}}(P^*_Q) \otimes Q W_m)^G)^{-1}$$

(6)

as an identification with no sign changes; similarly, by Remark 7(a), given another perfect $Z[G]$-complex $Q^*$, we can and shall also identify

$$\det(((P^*_Q \oplus Q^*_Q) \otimes Q W_m)^G) = \det((P^*_Q \otimes Q W_m)^G) \otimes \det((Q^*_Q \otimes Q W_m)^G).$$

(7)

**Definition 16** We define $\chi^g_{H^*}(P^*_Q, \sigma) \in H^*(Z[G])$ to be the class represented by the character map which sends the character $\theta_m$ to

$$\Pi_{p<\infty} \text{Det}(\lambda^g_p)(\theta_m)^{(-1)^i} \times \text{Pf}(\sigma \otimes \kappa_m)^G \left( \xi_m(\otimes_i (\wedge_j b_{jm}^{im})(-1)^i) \right)$$

(8)

where the terms on the right are taken in lexicographic order. Thus in particular for fixed $i, m$ writing $b_{jm}$ for $b_{jm}^{im}$, then $\wedge_j b_{jm}$ is taken to mean the exterior product

$$b_{11} \wedge b_{12} \cdots \wedge b_m \wedge b_{21} \cdots \wedge b_{nm}.$$

We now wish to show that this class is independent of all choices.

It is clear that if we change basis from the given $Z_p[G]$-basis for $Z_p \otimes P^i$, $\{a^{ij}_p\}_j$, then we only change the representing character function by an element in $\text{Det}^g(Z_p[G]^*) \times 1$. Similarly, if we change the given $Q[G]$-basis for $Q \otimes P^i$, $\{a^{ij}_0\}_j$, then we only change the representing character function by an element in $\text{Im}(\Delta^g)$.

Next we consider the possible dependence on the alternating forms $\kappa_m$ and the chosen hyperbolic basis $\{w_{mn}\}$. Let $\eta_m$ be a further non-degenerate $G$-invariant alternating form on $W_m$, let $\{w'_{mn}\}$ denote a hyperbolic basis of $W_m$ with respect to $\eta_m$ and put

$$b_{jm}^{im} = r_G(a^{ij}_0 \otimes w'_{mn}).$$

In order to show that the value in (8) does not change, we must show that

$$\text{Pf}(\sigma \otimes \kappa_m)^G \left( \xi_m(\otimes_i (\wedge_j b_{jm}^{im})(-1)^i) \right)$$

$$= \text{Pf}(\sigma \otimes \eta_m)^G \left( \xi_m(\otimes_i (\wedge_j b_{jm}^{im})(-1)^i) \right).$$

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This will follow at once from Proposition 23 below and the fact that, by the corollary to Proposition 4.1 and 4.2 on page 35-37 of [F1], the right hand expression (12) in Proposition 23 is independent of the particular alternating form $\kappa_m$ used. \hfill $\square$

For future reference we record the following two results which follow at once from the above definition and Lemma 3.

**Lemma 17** (a) Suppose the complex $P^*$ is acyclic, and let $0$ denote the trivial form on the trivial vector space $\Gamma^0(P^*) = \{0\}$; then $\chi^s_H(P^*,0)$ is the trivial class.

(b) Given two perfect $\mathbf{Z}[G]$-complexes $P_i^*$ for $i = 1, 2$ with non-degenerate $G$-invariant forms $\sigma_i^{ev}, \sigma_i^{odd}$ on $H^0(P_i^*, \mathbf{Q})$, $H^{odd}(P_i^*, \mathbf{Q})$; we view $\sigma_i^{ev} \oplus \sigma_i^{odd}$ as a form on the even part of the cohomology of $P_1^* \oplus P_2^*$ via the identification $H^0(P_1^* \oplus P_2^*) = H^0(P_1^*) \oplus H^0(P_2^*)$; we then have the equality of classes

$$
\chi^s_H(P_1^* \oplus P_2^*, \sigma_1 \oplus \sigma_2) = \chi^s_H(P_1^*, \sigma_1) \chi^s_H(P_2^*, \sigma_2).
$$

**Arakelov case.** Here we briefly recall the construction of the Arakelov Euler characteristic given in [CPT2]. Let $\{V_i\}$ denote the distinct simple two sided ideals of the complex group algebra $\mathbf{C}[G]$, and let $\nu_r$ denote the hermitian form on $V_r$ given by the restriction of the standard non-degenerate $G$-invariant hermitian form $\nu_G : \mathbf{C}[G] \times \mathbf{C}[G] \to \mathbf{C}$

$$
\nu_G(\sum_{g \in G} l_g g, \sum_{h \in G} m_h h) = |G| \sum_{g \in G} \overline{l_g} m_g
$$

and we let $\{v_{rs}\}$ denote an orthonormal basis of $V_r$ with respect to $\nu_r$.

We next suppose that we are given a perfect $\mathbf{Z}[G]$-complex $P^*$ with metrics $h = \{h_r\}$ on the equivariant determinant of cohomology i.e. each $h_r$ is a metric on the line $\det((\Gamma^i(P^*) \otimes \mathbf{Q} V_r)^G)$. We again let $\{a_{ij}^i\}_j$ denote a $\mathbf{Z}_p[G]$-basis for $\mathbf{Z}_p \otimes P^i$ and let $\{a_{ij}^G\}_j$ denote a $\mathbf{Q}[G]$-basis for $\mathbf{Q} \otimes P^i$; as previously, we let $\lambda_p^i$ be the element of $GL(\mathbf{Q}_p[G])$ such that $\lambda_p^i a_{ij}^G = a_{ij}^G$. Then for each pair $i, r$ we put

$$
e^{jr}_{rs} = r_G(a_{ij}^G \otimes v_{rs})
$$

and again by Lemma 15 we know that $\{e^{jr}_{js}\}$ is a $\mathbf{C}$-basis of $(P^i \otimes \mathbf{Q} V_r)^G$.

**Definition 18** The equivariant Arakelov class $\chi_A(P^*, h) \in A(\mathbf{Z}[G])$ is defined to be the class represented by the following homomorphism on characters: if $V_r$ has character $\chi_r$, then the complex conjugate $\overline{\chi}$ is sent to the value

$$
\prod_{p < \infty} \text{Det}(\lambda_p^i((\overline{\chi}_r)^{-1})) \times h_r(\xi_r(\otimes_i (\lambda_{js} e^{jr}_{js})^{-1})))^{1/\chi_r(1)}. \tag{15}
$$

From 3.3 in [CPT2] we know that the class given by this character map is again independent of choices. The symplectic Arakelov class $\chi^s_A(P^*, h) \in A^s(\mathbf{Z}[G])$ is then given by restricting the above character map to symplectic characters.
3.2.2 The hermitian metrics associated to a symmetric bilinear form.

With the notation of the previous subsection, we suppose we are given non-degenerate $G$-invariant real-valued symmetric bilinear forms $\sigma^{ev}$, $\sigma^{odd}$ on $H^{ev}(P_{Q}^{i})$, $H^{odd}(P_{Q}^{i})$. We now briefly recall how this data naturally determines a system of metrics on the equivariant determinant of cohomology of $P^{i}$. We observe that $(\sigma^{ev} \otimes \nu_{r})^{G}$ is a non-degenerate hermitian form on the vector space $(H^{ev}(P_{Q}^{i}) \otimes Q V_{r})^{G}$; the determinant of this form affords a hermitian form $\det((\sigma^{ev} \otimes \nu_{r})^{G})$ on the complex line $\det((H^{ev}(P_{Q}^{i}) \otimes Q V_{r})^{G})$ which may be either positive or negative definite; multiplying by $-1$ if the form is negative definite, in all cases we then obtain a positive definite form which we denote by $\det((\sigma^{ev} \otimes \nu_{r})^{G})$. The positive definite form $\det((\sigma^{odd} \otimes \nu_{r})^{G})$ is defined similarly. We then write $h_{r}$ for the metric on the complex line $\det((H^{i}(P_{Q}^{i}) \otimes Q V_{r})^{G})$ corresponding via $\nu_{H^{i}(P^{i})}$ to the positive definite form $\det((\sigma^{ev} \otimes \nu_{r})^{G})$. Then $h = \{h_{r}\}$ is then the required system of metrics on the equivariant determinant of cohomology of $P^{i}$.

3.2.3 Independence under quasi-isomorphism.

We first recall the following result from Theorem 3.9 in [CPT2]: suppose $P_{1}^{i}$, resp. $P_{2}^{i}$ is a perfect $\mathbb{Z}[G]$-complex which supports metrics $h^{1} = \{h_{1}^{i}\}$, resp. $h^{2} = \{h_{2}^{i}\}$ on its equivariant determinant of cohomology. Suppose further that there is a quasi-isomorphism $\phi : P_{1}^{i} \rightarrow P_{2}^{i}$ in the derived category of bounded complexes of finitely generated $\mathbb{Z}[G]$-modules, which has the property that $\phi^{*}h^{1} = h^{2}$. Then we know that the formation of Arakelov classes is natural with respect to quasi-isomorphisms in the sense that

$$\chi_{A}(P_{1}^{i}, h^{1}) = \chi_{A}(P_{2}^{i}, h^{2}).$$

We now establish the corresponding result for hermitian classes. Prior to stating the result, we first need some more notation: for each $m$, for brevity we let $\det(P_{1}^{i})_{m}$ denote $\det((P_{1}^{i} \otimes Q W_{m})^{G})$; we write $\det(\phi)_{m}$ for the isomorphism $\det(P_{1}^{i})_{m} \cong \det(P_{2}^{i})_{m}$ induced by $\phi$, and we let $\xi_{m}$ denote the canonical isomorphism $\det(P_{1}^{i})_{m} \cong \det(H^{*}(P_{1}^{i}))_{m}$. Then we have the following result:
Theorem 19 Suppose $P_1^\bullet, P_2^\bullet$ are perfect $\mathbb{Z}[G]$-complexes which support non-degenerate $G$-invariant real-valued symmetric forms $\sigma_{1}^{ev}, \sigma_{1}^{odd}, \sigma_{2}^{ev}, \sigma_{2}^{odd}$, on their rational cohomology, and suppose, as previously, that there is a quasi-isomorphism $\phi : P_1^\bullet \to P_2^\bullet$ in the derived category of bounded complexes of finitely generated $\mathbb{Z}[G]$-modules, which has the property that
\[
Pf_{(\sigma_{1,C,\kappa_m})G} \circ \xi^1_m = Pf_{(\sigma_{2,C,\kappa_m})G} \circ \xi^2_m \circ \det (\phi_m). \tag{9}
\]
Then there is an equality of hermitian classes:
\[
\chi^s_H(P_1^\bullet, \sigma) = \chi^s_H(P_2^\bullet, \sigma).
\]

Proof. We may write the quasi-isomorphism $\phi : P_1^\bullet \to P_2^\bullet$ as
\[
P_1^\bullet \xleftarrow{\psi_1} Q^\bullet \xrightarrow{\psi_2} P_2^\bullet
\]
with $Q^\bullet$ a bounded complex of finitely generated $\mathbb{Z}[G]$-modules and with the $\psi_i$ quasi-isomorphic chain maps. As a first step, we observe that we can in fact choose $Q^\bullet$ to be a perfect $\mathbb{Z}[G]$-complex: this follows from a standard argument and the reader is referred to Lemma 5.1 in [CPT2] for the details.

Next we observe that, by adding on a sufficiently large acyclic free complex $A_1^\bullet$, we can ensure that there is chain map $\alpha$ such that
\[
\psi_1' = \psi_1 + \alpha : Q^\bullet \oplus A_1^\bullet \to P_1^\bullet
\]
is surjective. Then by VI.8.17 in [M] we can split $\psi_1'$ by a quasi-isomorphism $\beta_1 : P_1^\bullet \to Q^\bullet \oplus A_1^\bullet$, and so we obtain a direct decomposition
\[
Q^\bullet \oplus A_1^\bullet = \text{Im} (\beta_1) \oplus \ker (\psi_1').
\]
Let $\tau_1^{ev}, \tau_1^{odd}$ denote the transport of $\sigma_1^{ev}, \sigma_1^{odd}$ to $H^{ev}(\text{Im} (\beta_1)), H^{odd}(\text{Im} (\beta_1))$ and of course we endow the (zero) cohomology of the acyclic complex $\ker (\psi_1')$ with the zero form. We then repeat these constructions for $P_2^\bullet$. Putting this together and using Lemma 17 we get
\[
\chi^s_H(A_2^\bullet \oplus Q^\bullet \oplus A_1^\bullet, 0 \oplus \tau_1 \oplus 0) = \chi^s_H(A_2^\bullet, 0) \chi^s_H(Q^\bullet \oplus A_1^\bullet, \tau_1 \oplus 0)
\]
\[
= \chi^s_H(Q^\bullet \oplus A_1^\bullet, \tau_1 \oplus 0)
\]
while
\[
\chi^s_H(Q^\bullet \oplus A_1^\bullet, \tau_1 \oplus 0) = \chi^s_H(\text{Im} (\beta_1) \oplus \ker (\psi_1), \tau_1 \oplus 0)
\]
\[
= \chi^s_H(\text{Im} (\beta_1), \tau_1) \chi^s_H(\ker (\psi_1), 0)
\]
\[
= \chi^s_H(P_1^\bullet, \sigma_1) \chi^s_H(\ker (\psi_1), 0) = \chi^s_H(P_1^\bullet, \sigma_1).
\]
Thus we see that for $i = 1, 2$
\[
\chi^s_{\mathbb{H}}(A_2^* \oplus Q^* \oplus A_1^*, 0 \oplus \tau_i \oplus 0) = \chi^s_{\mathbb{H}}(P_i^*, \sigma_i)
\]
and so to prove the theorem it will suffice to show that
\[
\chi^s_{\mathbb{H}}(A_2^* \oplus Q^* \oplus A_1^*, 0 \oplus \tau_1 \oplus 0) = \chi^s_{\mathbb{H}}(A_2^* \oplus Q^* \oplus A_1^*, 0 \oplus \tau_2 \oplus 0).
\]
(Note here that the $0 \oplus \tau_i \oplus 0$ refer to different direct sum decompositions for $i = 1, 2$.) In order to show this it will suffice to show that for each $m$
\[
Pf_{((0 \oplus \tau_1 \oplus 0) \oplus \kappa_m)^G} = Pf_{((0 \oplus \tau_2 \oplus 0) \oplus \kappa_m)^G}.
\]
(10)
To establish this equality we consider the commutative diagram
\[
\begin{array}{ccc}
\text{det (Q*)}_m & \xrightarrow{\text{det} \psi_m} & \text{det (P1*)}_m \\
\downarrow \xi_{Q,m} & & \downarrow \xi_{1}^1 \\
\text{det (H* (Q*))}_m & \xrightarrow{\text{det} \text{H(\psi)*}_m} & \text{det (H* (P1*))}_m.
\end{array}
\]
Writing $\text{det (\phi)}_m = \text{det (\psi)}_2 \cdot \text{det (\psi)}_1^{-1}$, from (9) we get
\[
Pf_{(\sigma_1, C \otimes \kappa_m)^G} \circ \xi_{1}^1 \circ \text{det (\psi)}_m = Pf_{(\sigma_2, C \otimes \kappa_m)^G} \circ \xi_{2}^2 \circ \text{det (\psi)}_m
\]
and so from the commutative diagram we deduce that
\[
Pf_{(\sigma_1, C \otimes \kappa_m)^G} \circ \text{det (H (\psi1))}_m = Pf_{(\sigma_2, C \otimes \kappa_m)^G} \circ \text{det (H (\psi2))}_m.
\]
(11)
The equality (10) then follows since, by construction, under the isomorphism
\[
H^* (P1^*)_{H(\psi1)}^H (Q^*) = H^* (A_2^* \oplus Q^* \oplus A_1^*) = H^* (A_2^* \oplus \text{Im} (\beta_1) \oplus \ker (\psi_1) \oplus A_1^*)
\]
$\sigma_1$ transports to $0 \oplus \tau_1 \oplus 0$; thus identifying $H^* (Q^*)$ with $H^* (A_2^* \oplus Q^* \oplus A_1^*)$ we read (11) as (10), as required. □

3.2.4 Evaluation of Pfaffians.

The results we require in this subsection come from Appendix C in [CPT1]. Throughout this sub-section we again assume that all real and complex vector spaces are finite dimensional. We begin by considering the hermitian form associated to a $G$-invariant symmetric form.

Suppose now that $U$ is a free $\mathbb{R}[G]$-module with basis $\{u_i : 1 \leq i \leq q\}$ which again supports a real-valued non-degenerate $G$-invariant symmetric form $\sigma$. We then write
\[
\bar{\sigma} : U \times U \to \mathbb{R}[G]
\]
for the associated group ring valued hermitian form (ref. page 25 in [F1]); thus for \( u, u' \in U \)
\[
\bar{\sigma} (u, u') = \sum_{g \in G} \sigma (gu, u') g^{-1}.
\]

Let \( r \mapsto \tau \) denote the \( \mathbb{R} \)-linear involution on \( \mathbb{R}[G] \) induced by group inversion. Note that
\[
\bar{\sigma} (u_i, u_j) = \bar{\sigma} (u_j, u_i).
\]

**Proposition 20** Let \( V \) be a \( \mathbb{C}[G] \)-module which supports a \( G \)-invariant non-degenerate form \( h \); then \( q \times q \) matrices with entries in \( \mathbb{C}[G] \) act on the direct sum of \( q \) copies of \( V \), and the matrix \( T = (\bar{\sigma} (u_i, u_j))_{i,j} \) is self adjoint with respect to \( h^{(q)} \), the orthogonal direct sum of \( h \) on \( q \)-copies of \( V \).

**Proof.** Indeed, writing \( v(i) \) for the vector in \( \oplus_{i=1}^q V \) which is \( v \) in the \( i \)-th position and zero elsewhere, we have
\[
\begin{align*}
\begin{split}
\quad & h^{(q)} (v(i), T v(j)) = h^{(q)} (v(i), \sum_k \bar{\sigma} (u_k, u_j) v(k)) = \\
\quad & = h^{(q)} (v(i), \bar{\sigma} (u_i, u_j) v(j)) = h^{(q)} (\bar{\sigma} (u_i, u_j) v(i), v(j)) = \\
\quad & = h^{(q)} (\bar{\sigma} (u_j, u_i) v(i), v(j)) = h(\bar{\sigma} (u_j, u_i) v, v').
\end{split}
\end{align*}
\]

while similarly
\[
\begin{align*}
\begin{split}
\quad & h^{(q)} (T v(i), v(j)) = h^{(q)} (\sum_i \bar{\sigma} (u_i, u_j) v(i), v(j)) = \\
\quad & = h^{(q)} (\bar{\sigma} (u_j, u_i) v(i), v(j)) = h(\bar{\sigma} (u_j, u_i) v, v').
\end{split}
\end{align*}
\]

\( \square \)

**Lemma 21** Suppose, as previously, that \( W_m \) is a symplectic \( \mathbb{C}[G] \)-module with non-degenerate \( G \)-invariant alternating form \( \kappa_m \). The map \( r_G : W_m \to (\mathbb{R}[G] \otimes \mathbb{R} W_m)^G \) (see Lemma 15) given by
\[
\begin{align*}
\quad & r_G (w) = \sum_{g \in G} gw
\end{align*}
\]

has the property that \( |G|^{-1} r_G \) is a \( G \)-isometry when \( (\mathbb{R}[G] \otimes \mathbb{R} W_m)^G \) is endowed with the form \( (\nu_R \otimes \kappa_m)^G \). (Recall that \( \nu_R \) was defined after Lemma 17; \( \nu_R \) is the restriction of \( \nu_C \) to the real group algebra \( \mathbb{R}[G] \).

**Proof.** First note that \( |G|^{-1} r_G \) has inverse \( \sum_a g \otimes w \mapsto \sum_a g^{-1} w \); we then observe that
\[
\begin{align*}
\begin{split}
\quad & (\nu \otimes \kappa_m)^G (|G|^{-1} r_G (w), |G|^{-1} r_G (w')) = |G|^{-2} (\nu \otimes \kappa_m) (\sum g \otimes gw, \sum h \otimes hw) \\
\quad & = |G|^{-1} \sum g \kappa_m (gw, gw') = \kappa_m (w, w').
\end{split}
\end{align*}
\]

\( \square \)

**Remark 22** Note for future reference that \( \text{Im}(r_G) \) is a left \( \mathbb{R}[G] \)-module by transport of structure; to be more precise, for \( h \in G \) we define \( h \cdot r_G (w) \) to be
\[
r_G (hw) = \sum_{g \in G} ghw = r_G (w)\left(h^{-1} \otimes 1\right).
\]
For a symplectic \( C[G] \)-module \( W_m \) we write \( T_{W_m}^{(q)} \) for the representation of \( G \) afforded by the direct sum of \( q \)-copies of \( W_m \); thus \( T_{W_m}^{(q)} \) provides an action of \( q \times q \) matrices with entries in \( C[G] \) on the direct sum of \( q \) copies of \( W_m \).

We shall henceforth identify \( U \) with \( \bigoplus_{i=1}^q R[G] \) and let \( \{ u_i \} \) be the \( R[G] \)-basis of \( U \) given by the canonical basis \( \{ 1_i \} \) of \( \bigoplus_{i=1}^q R[G] \). We define a form \( \nu \) on \( U \) by the rule

\[
\nu(\lambda u_i, \mu u_j) = \delta_{ij} \nu(\lambda, \mu).
\]

For \( w, w' \in W_m \) and given \( i, j \) we consider \( x = |G|^{-1} r_G(w_{(i)}) \), \( y = |G|^{-1} r_G(w'_{(j)}) \); note that by Lemma 21, if \( w \) ranges through a hyperbolic basis of \( W_m \) with respect to \( \kappa_m \) and if \( i \) ranges from 1 to \( q \), then \( x \) ranges through a hyperbolic basis of \( (U \otimes W_m)^G \) with respect to \( (\nu \otimes \kappa_m)^G \).

The following result will be fundamental in enabling us to calculate with Pfaffians.

**Proposition 23** With the above notation \( |G|^{-1} r_G \) defines an isometry

\[
\bigoplus_{i=1}^q W_m \cong (U \otimes W_m)^G
\]

where \( \bigoplus_{i=1}^q W_m \) is endowed with the form \( \kappa_m^{(q)} \) and \( (U \otimes W_m)^G \) is endowed with the form \( (\nu \otimes \kappa_m)^G \). By Proposition 20, \( T_{W_m}^{(q)}(\tilde{\sigma}(u_i, u_j)) \) is a self-adjoint automorphism with respect to \( \kappa_m^{(q)} \). There is an equality of Pfaffians

\[
\text{Pf}_{(\sigma \otimes \kappa_m)^G} \left( \wedge_{\text{in}} |G|^{-1} r_G (u_i \otimes m_{mn}) \right) = \text{pf}_{\kappa_m^{(q)}} \left( |G|^{-1} T_{W_m}^{(q)}(\tilde{\sigma}(u_i, u_j)) \right)
\]

(12)

where, as previously, \( \{ m_{mn} \} \) denotes a hyperbolic basis of \( W_m \) with respect to \( \kappa_m \) and the wedge product is of course taken in lexicographic order.

**Proof.** To prove the result, we claim that it will suffice to establish the equality

\[
(\sigma \otimes \kappa_m)(|G| x, |G| y) = \kappa_m^{(q)}(w_{(i)}, |G| T w'_{(j)}),
\]

(13)

where we put \( T = T_{W_m}^{(q)}(\tilde{\sigma}(u_i, u_j)) \). Indeed, assuming (13), we see that

\[
(\sigma \otimes \kappa_m)(x, y) = \kappa_m^{(q)}(w_{(i)}, |G|^{-1} T w'_{(j)})
\]

and so by Lemma 21

\[
\kappa_m^{(q)}(w_{(i)}, |G|^{-1} T w'_{(j)}) = (\nu \otimes \kappa_m)(|G|^{-1} r_G(w_{(i)}), |G|^{-1} r_G(|G|^{-1} T w'_{(j)}))
\]

\[
= (\nu \otimes \kappa_m)(x, |G|^{-1} T y).
\]

Hence by Proposition 6 it follows that

\[
\text{Pf}_{(\sigma \otimes \kappa_m)^G} \left( \wedge_{\text{in}} |G|^{-1} r_G (u_i \otimes m_{mn}) \right) = \text{pf}_{\kappa_m^{(q)}}(|G|^{-1} T) \text{Pf}_{(\nu \otimes \kappa_m^{(q)})^G} \left( \wedge_{\text{in}}(|G|^{-1} r_G(m_{mn,(i)})) \right)
\]

\[
= \text{pf}_{\kappa_m^{(q)}}(|G|^{-1} T)
\]

20
with the last equality holding since the \( \{|G|^{-1} r_G(w_{m,n,(i)})\} \) is a hyperbolic basis of the vector space \((U \otimes W_m)^G \) endowed with the form \((\nu \otimes \kappa_m^{(q)})^G \).

To show (13), we consider the left-hand side, which can be written as

\[
\sum_{g,h} \sigma(gu_i, hu_j) \kappa_m(gw, hw')
\]

which by \( G \)-invariance is

\[
|G| \sum_{h} \sigma(u_i, hu_j) \kappa_m(w, hw') = |G| \kappa_m(w, \sum_{h} \sigma(u_i, hu_j) hw').
\]

The result then follows since

\[
\kappa_m^{(q)}(w(i), |G| T w'(j)) = \kappa_m^{(q)}(w(i), |G| \sum_k T_k w'(k)) = \kappa_m^{(q)}(w(i), |G| T w'(i)) = |G| \kappa_m(w, \sum \sigma(u_i, u_j) w') = |G| \kappa_m(w, \sum \sigma(u_i, u_j) hw'). \quad \Box
\]

For future reference we also record the corresponding result for hermitian forms:

**Proposition 24** Suppose that \( V \) is a left ideal of \( \mathbb{C}[G] \) endowed with the non-degenerate \( G \)-invariant hermitian form \( \nu_V \) given by the restriction of \( \nu_\mathbb{C} \). As previously, we let \( T_V^{(q)} \) denote the \( G \)-representation afforded by the direct sum of \( q \) copies of \( V \). Let \( h_V \) denote the hermitian form on the complex vector space \((U \otimes_R V)^G \) given by restricting \( \sigma \otimes \nu_V \). Let \( \{v_{Vs}\} \) be an orthonormal basis of \( V \) with respect to \( \nu_V \). Then \( T_V^{(q)}(\tilde{\sigma}(u_i, u_j)) \) is a self-adjoint automorphism with respect to \( h_V \) and

\[
h_V(\wedge_is r_G(u_i \otimes v_{Vs})) = \left| \det(|G| T_V^{(q)}(\tilde{\sigma}(u_i, u_j))) \right|^{1/2}.
\]

**Proof.** First we note that \( \det(\sigma \otimes \nu_V)\left(\wedge_is r_G(u_i \otimes v_{Vs})\right) \) is equal to the determinant

\[
det((\sigma \otimes \nu_V) \left(\sum_{g,h \in G} gu_i \otimes gv_{Vs}, \sum_{h \in G} hu_i \otimes hv_{Vs'}\right)) = \det \left(\sum_{g,h \in G} (gu_i, hu_i) \nu_V((gv_{Vs}, hv_{Vs'}))\right),
\]

which by setting \( k = g^{-1}h \) and using the \( G \)-invariance of \( \sigma \) and \( \nu_V \) is

\[
= \det(|G| \sum_{k \in G} \sigma(u_i, ku_i) \nu_V(v_{Vs}, kv_{Vs'})).
\]

The matrix \( (\nu_V(v_{Vs}, kv_{Vs'}))_{s,s'} \) is exactly the matrix of \( T_V^{(q)}(k) \) relative to the orthonormal basis \( \{v_{Vs}\} \) of \( V \) with respect to \( \nu_V \). It therefore follows that the above determinant is equal to

\[
det(|G| T_V^{(q)}(\tilde{\sigma}(u_i, u_{i'})))\
\]

since \( \tilde{\sigma}(u_i, u_{i'}) = \sum_{k \in G} \sigma(u_i, ku_{i'}) k \). \quad \Box
3.2.5 Comparison of hermitian and Arakelov classes.

Recall that $\sigma^\text{ev}$ and $\sigma^\text{odd}$ denote $G$-invariant real-valued symmetric forms on $H^\text{ev}(P^\bullet)$ and $H^\text{odd}(P^\bullet)$ which, as per 3.2.2, induce metrics $h_\sigma = \{h_r\}$ on the equivariant determinant of cohomology of $P^\bullet$. In this subsection we shall compare the invariants $\chi^s(H(P^\bullet, \sigma))$ and $\chi^s(A(P^\bullet, h_\sigma))$. First we need the following algebraic result:

**Proposition 25** Given an $\mathbb{R}[G]$-module $M$ and a non-degenerate $G$-invariant symmetric form $\sigma$ on $M$, there exists a $G$-decomposition $M = M^+ \oplus M^-$ where $\sigma$ is positive definite on $M^+$ and negative definite on $M^-$. This decomposition is not necessarily unique, but the characters of the action of $G$ on $M^+$ and $M^-$ are independent of choices.

**Proof.** For full details see page 578 in [AS]; we briefly sketch a proof for the reader’s convenience. First we choose a $G$-invariant positive definite symmetric form $\tau$ on $M$; there is then a unique automorphism $A$ of $M$ such that for all $x, y \in M$

$$\sigma(x, y) = \tau(x, Ay).$$

As both $\sigma$ and $\tau$ are symmetric, $A$ is self adjoint with respect to $\tau$; furthermore, since both $\sigma$ and $\tau$ are $G$-invariant, $A$ commutes with the action of $G$; thus the different eigenspaces of $A$ are preserved by $G$; then, by considering the sums of eigenspaces for positive and negative eigenvalues, we obtain the required decomposition $M = M^+ \oplus M^-$. Clearly the above decomposition depends on the choice of $\tau$. To see that the characters of $M^+$ and $M^-$ are independent of the choice of $\tau$, we note that: the space of positive definite $G$-invariant forms on $M$ is connected; the maps $\tau \mapsto \text{char}(M^\pm)$ are continuous; $\text{char}(M^\pm)$ takes values in the discrete group $R_G$. □

A particularly simple, but nonetheless useful, instance of the above decomposition occurs when $(M, \sigma)$ is hyperbolic. To state this result we first need some notation. Recall that for an $\mathbb{R}[G]$-module $V$ the hyperbolic space is $\text{Hyp}(V) = V \oplus V^D$ endowed with the form $h$ such that

$$h(v \oplus f, v' \oplus f') = f(v') + f'(v) \quad \text{for } v, v' \in V, f, f' \in V^D.$$

**Lemma 26** There are $\mathbb{R}[G]$-isomorphisms $\text{Hyp}(V)^+ \cong V \cong \text{Hyp}(V)^-.$

**Proof.** First note that since $V$ is defined over $\mathbb{R}$, we know that $V \cong V^D$ as $\mathbb{R}[G]$-modules. As $\text{Hyp}(V \oplus W) \cong \text{Hyp}(V) \oplus \text{Hyp}(W)$, we see that it will suffice to prove the lemma when $V$ is irreducible over $\mathbb{R}$. The result then follows immediately, since $\text{Hyp}(V)^+$, $\text{Hyp}(V)^-$ are both $\mathbb{R}[G]$-submodules of $\text{Hyp}(V)$ and since hyperbolic spaces have zero signature. □

In the final section we shall need the following result on hyperbolic summands of quadratic modules:
Lemma 27 Let $K$ be an arbitrary field of characteristic zero, and let $\sigma$ be a non-degenerate $K$-valued $G$-invariant symmetric form on a finite dimensional $K[G]$-module $V$. Suppose that $W$ is an isotropic $K[G]$-submodule of $V$ and let $W^\perp$ denote the space of vectors orthogonal to $W$. Then there is an orthogonal decomposition of $K[G]$-modules

$$V \cong \text{Hyp}(W) \oplus \frac{W^\perp}{W}.$$  

Suppose further that $(V, \sigma)$ is a filtered quadratic $K[G]$-space in the following sense: we are given an increasing filtration $\{F_i\}$ of $K[G]$-submodules with $F_{-N} = (0)$ and $F_N = V$ for $N >> 0$, and with $F_i^\perp = F_{i-1}$; thus for all $i$, $\sigma$ induces isomorphisms

$$\text{Gr}_i \cong \text{Gr}_i^D$$

where $\text{Gr}_i$ denotes the $i$-th graded piece $F_i/F_{i-1}$. Then there is a (non-canonical) $K[G]$-decomposition of quadratic modules

$$V \cong \oplus_{i<0} \text{Hyp}(\text{Gr}_i) \oplus \text{Gr}_0.$$

Proof. To prove the first part for simplicity we may suppose without loss of generality that $W$ is irreducible. First choose an arbitrary decomposition of $K[G]$-modules $W^\perp = W \oplus U$; this is trivially an orthogonal decomposition. We then choose an arbitrary further decomposition of $K[G]$-modules $V = W^\perp \oplus W'$. Then the form $\sigma$ induces a map

$$W' \sigma' \rightarrow U^D \oplus W^D \rightarrow W^D$$  \hspace{1cm} (14)$$

and the composite is an isomorphism. We may then alter the initial decomposition $V = W^\perp \oplus W'$ by a homomorphism from $W'$ to $U$ to guarantee that the composition of $\sigma'$ with projection to $U^D$ is zero, as required.

The second part of the lemma then follows at once from the first part. \hfill $\Box$

Proposition 28 Let $U$ be a free $R[G]$-module with basis, $\{u_i\} i = 1, \ldots, q$, and suppose that $U$ supports a non-degenerate real-valued $G$-invariant form $\sigma$. Choose a decomposition $U = U^+ \oplus U^-$, as in Proposition 25, and for each $m$ define $n_m^\pm (\sigma) = \dim(U^\pm \otimes W_m)^G$. Then

$$\text{sign} \left( \text{pf}_{n_m^\pm} (T_{W_m}^q (\bar{\sigma}(u_i, u_j))) \right) = (\sqrt{-1})^{n_m^\pm (\sigma)}.$$

Note that the integers $n_m^\pm (\sigma)$ are all even, since they are the multiplicities of symplectic representations in real representations.

Remark 29 The authors are grateful to Boas Erez for stressing the importance of relating the sign of Fröhlich’s Pfaffian to signature invariants.
Proof. As previously, let \( \nu_{R} \) denote the standard \( G \)-invariant form on \( R[G] \) (see Lemma 21); we again define the form \( \nu \) on \( U \) by the rule
\[
\nu(\lambda u_i, \mu u_j) = \delta_{ij} \nu_{R}[G](\lambda, \mu).
\]
As in Proposition 25, there is a unique \( R[G] \)-automorphism \( A \) of \( U \), which is self-adjoint with respect to \( \nu \), such that for all \( x, y \in U \)
\[
\sigma(x, y) = \nu(x, Ay).
\]
Therefore the decomposition \( U = U^{+} \oplus U^{-} \) induces a decomposition
\[
(U \otimes W_{m})^{G} = (U^{+} \otimes W_{m})^{G} \oplus (U^{-} \otimes W_{m})^{G}
\]
and \( A \otimes 1 \) is diagonalisable on the subspaces \( (U^{\pm} \otimes W_{m})^{G} \) with positive resp. negative eigenvalues on \( (U^{+} \otimes W_{m})^{G} \) resp. \( (U^{-} \otimes W_{m})^{G} \). By construction \( (A \otimes 1)^{G} \) induces an isometry of alternating forms
\[
(\sigma \otimes \kappa_{m})^{G} \cong (\nu \otimes \kappa_{m})^{G}
\]
which we denote \( (A \otimes 1)^{G} \). Therefore by Proposition 5 we see that
\[
\text{Pf}_{(\sigma \otimes \kappa_{m})^{G}} = \text{pf}_{(\nu \otimes \kappa_{m})^{G}}((A \otimes 1)^{G} \cdot \text{Pf}_{(\nu \otimes \kappa_{m})^{G}}).
\]
We then evaluate both sides on \( \wedge_{\text{in}} |G|^{-1} r_{G}(u_{i} \otimes w_{mn}) \) and, noting that by Lemma 21 \( \{ |G|^{-1} r_{G}(u_{i} \otimes w_{mn}) \} \) is a hyperbolic basis with respect to \( (\nu \otimes \kappa_{m})^{G} \), we see that
\[
\text{Pf}_{(\sigma \otimes \kappa_{m})^{G}} \left( \wedge_{\text{in}} |G|^{-1} r_{G}(u_{i} \otimes w_{mn}) \right) = \text{pf}_{(\nu \otimes \kappa_{m})^{G}}((A \otimes 1)^{G}).
\]
On the other hand by Proposition 23
\[
\text{Pf}_{(\sigma \otimes \kappa_{m})^{G}} \left( \wedge_{\text{in}} |G|^{-1} r_{G}(u_{i} \otimes w_{mn}) \right) = \text{pf}_{\kappa_{m}^{(q)}} \left( |G|^{-1} T_{W_{m}}^{(q)} (\bar{\sigma}(u_{i}, u_{j})) \right).
\]
The result then follows by repeated use of the fact that (see page 40 in [F1])
\[
\text{pf} \left( \begin{array}{cc} d & 0 \\ 0 & d \end{array} \right) = d. \qed
\]
Recall that \( S_{\infty}(Z[G]) \) is the signature group defined in 3.1.2. We complete this section by stating the following result, which we shall prove in (5.3) of the Appendix.

Theorem 30 The class \( \chi_{A}^{a}(P^{\bullet}, \sigma)\chi_{A}^{s}(P^{\bullet}, h_{\sigma})^{-1} \) lies in \( S_{\infty}(Z[G]) \) and is represented by the character function which maps \( \theta_{m} \) to \( \bar{v}_{m}(\sigma) \) where
\[
n_{m}^{-}(\sigma) = n_{m}^{-}(\sigma^{\text{ev}}) - n_{m}^{-}(\sigma^{\text{odd}})
\]
denotes the virtual dimension of a maximal negative definite subspace of the \( W_{m} \)-isotypic component of the cohomology of \( P^{\bullet} \).


4 DE RHAM DISCRIMINANTS.

Throughout this section we again adopt the notation given in the Introduction. Thus the scheme $X$ is projective and flat over $\text{Spec}(\mathbb{Z})$ of relative dimension $d$ and $\pi : X \rightarrow Y$ is a $G$-cover which satisfies hypotheses (T1) and (T2) given in the Introduction. In most of this section we again let $X = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ resp. $Y = Y \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ denote the generic fibre of $X$ resp. $Y$.

4.1 De Rham pairings.

As in III Sect. 7 of [H] for $0 \leq i, j \leq d$ we have $G$-equivariant duality pairings

$$\sigma_{ij} : H^i(X, \Omega^j_X) \times H^{d-i}(X, \Omega^{d-j}_X) \rightarrow H^d(X, \Omega^d_X) |_{G^{-1}} \rightarrow \mathbb{Q}$$

where Tr is the trace map described after (17) below; note that here we divide the pairings used in [CPT1] by the group order; we used this normalisation in [CPT2], and the reason for choosing this normalisation will be explained after Lemma 32 below. For arbitrary coherent $X$-sheaves $F, G$ and for $x \in H^i(X, F)$, $y \in H^j(X, G)$, we know that $x \cup y = (-1)^{ij}y \cup x$ after identifying $F \otimes G$ and $G \otimes F$ by the “flip” isomorphism. Taking $F = \Omega^a_X$, $G = \Omega^b_X$ we see that $x \cup y = (-1)^{ij+ab}y \cup x$ in $H^{i+j}(X, \Omega^{a+b}_X)$; it therefore follows that

$$\sigma_{i,j}(x, y) = (-1)^{(d+1)(i+j)}\sigma_{d-i,d-j}(y, x). \quad (15)$$

We then symmetrise these pairings by the construction given in Sect. 3 of [CPT1]: namely, we define the twisted pairing $\sigma'_{i,j}$

$$\sigma'_{d-i,d-j}(y, x) = \sigma_{i,j}(x, y) = (-1)^{(d+1)(i+j)}\sigma_{d-i,d-j}(y, x). \quad (16)$$

We then define pairings $\sigma^t$ on the hypercohomology of $R\Gamma(X, L \wedge^1 \Omega^1_{X/Q})[d]$ as follows:

for $t < 0$ we put $\sigma^t = \oplus_{i+j=t+d} \sigma_{i,j}$

for $t > 0$ we put $\sigma^t = \oplus_{i+j=t+d} \sigma'_{i,j}$

and for $t = 0$ we set

$$\sigma^0 = \oplus_{i<d/2} \sigma_{i,d-i} \oplus \sigma_{d/2,d/2} \oplus \sigma_{d/2,d/2} \sigma'_{i,d-i}.$$ 

Here it is to be understood that the term $\sigma_{d/2,d/2}$ occurs only when $d$ is even. We note that in all cases $\sigma^t$ is symmetric by (15) and (16), and we then define

$$\sigma^{ev} = \oplus_{t \text{ even}} \sigma^t, \quad \sigma^{odd} = \oplus_{t \text{ odd}} \sigma^t.$$ 

Note that in all cases $\sigma^{odd}$ is a hyperbolic pairing; and $\sigma^{ev}$ is hyperbolic whenever $d$ is odd. To be more precise we have:
Proposition 31 There is a $\mathbb{Q}[G]$-isometry

$$
\left( H^i(X, \Omega_X^j) \oplus H^{d-i}(X, \Omega_X^{d-j}), \sigma_{ij} \oplus \sigma'_{d-i,d-j} \right) = \text{Hyp} \left( H^i(X, \Omega_X^j) \right)
$$

unless $d$ is even and $i = j = d/2$.

Since the signature of a hyperbolic form is always zero we know

Lemma 32 For any symplectic representation $W_m$ of $G$

$$
n_m^+(\sigma) - n_m^-(\sigma) = n_m^+(\sigma_{d/2,d/2}) - n_m^-(\sigma_{d/2,d/2})
$$

where the right hand side is to be interpreted as zero if $d$ is odd.

We conclude this subsection by considering the complex hermitian form associated to the de Rham pairing $\sigma$ by §3.2.2. We begin by considering the $L^2$-norms on cohomology.

Given a Kähler metric $h_Y$ on the complex tangent space of an arithmetic variety $Y$ which is invariant under complex conjugation, we denote by $h_X = h^{TX}$ the Kähler metric on $X(C)$ given by the pullback of $h_Y$; this is then also invariant under complex conjugation. Define $h^D_X$ to be the metric on the complex cotangent space of $X(C)$ which is dual to $h_X$.

Let $d_X$ denote the volume form given by the $d$-th exterior power of the $(1,1)$-form associated to $h^D_X$. Define the $L^2$-metric on $\Omega^{p,0}_{X(C)} \otimes \Omega^{0,q}_{X(C)}$ by

$$
\langle s, t \rangle_X = \frac{1}{d!} \int_{X(C)} |G|^{-1} \wedge^{p+q} h^D_X(s(x), t(x)) \left( \frac{i}{2\pi} \right)^d d_X
$$

where $\wedge^{p+q} h^D_X (-, -)$ denotes the inner product on $p+q$ forms given by the $p+q$-th exterior product of $h^D_X$ (see for instance page 131 in [So]). The reason for the normalisation factor $(i/2\pi)^d$ on the volume form will become apparent below: basically it will ensure that the $L^2$-norm is compatible with Serre duality. The reason for normalising by the factor $|G|^{-1}$ is that, since $X \to Y$ is etale, our metrics are then natural with respect to pullback in the sense that for $p$-forms $s', t'$ on $Y$, we then have $\langle \pi^* s', \pi^* t' \rangle_X = \langle s', t' \rangle_Y$ where

$$
\langle s', t' \rangle_Y = \frac{1}{d!} \int_{Y(C)} \wedge^{p+q} h^D_Y(s'(y), t'(y)) \left( \frac{i}{2\pi} \right)^d dy
$$

and $dy$ is the volume form given by the $d$-th exterior power of the $(1,1)$-form associated to $h^D_Y$. Let $\Delta^g = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial}$ be the Laplace operator on $\Omega^{p,0}_{X(C)} \otimes \Omega^{0,q}_{X(C)}$. The Hodge isomorphism

$$
H^q(X(C), \Omega^{p,0}_{X(C)}) = \ker (\Delta^g)
$$

then gives an $L^2$-metric on $H^q(X(C), \Omega^{p,0}_{X(C)})$. We will denote the resulting $L^2$-metric on the determinant of cohomology of $\Omega^{p,0}_{X(C)}$ by $|G|^{-1} \wedge^* \cdot |_{L^2}$ in order to emphasise the
appearance of the scaling factor $|G|^{-1}$. We then construct the associated Quillen metrics on the equivariant determinant of cohomology of $\Omega^{0,0}_{X(C)}$ by multiplying the above $L^2$-metrics by the inverse of the equivariant analytic torsion associated to $|G|^{-1} \wedge^p h_{\mathcal{Q}}^2$. This construction is described in more detail in the proof of Proposition 34. For a full discussion of this construction see Section 6 of [CPT2].

Identifying $H^d(X_C, \Omega^d_{\mathcal{Q}C})$ with the Dolbeault cohomology group $H^d(X_C, \Omega^d_{\mathcal{Q}C})$ and then integrating over $X$ affords a surjection $H^d(X_C, \Omega^d_{\mathcal{Q}C}) \otimes \mathbb{C} = H^d(X_C, \Omega^d_{\mathcal{Q}C}) \int_X \mathbb{C}$. From the above discussion we know that the following diagram commutes

\[
\begin{array}{ccc}
H^d(Y, \Omega^d_Y) & \xrightarrow{f_X} & \mathbb{C} \\
\pi^* \downarrow & & \downarrow \\
H^d(X, \Omega^d_X) & \xrightarrow{|G|^{-1} \int_X} & \mathbb{C}
\end{array}
\]

We then define the trace map

\[
H^d(X, \Omega^d_X) \xrightarrow{|G|^{-1} \text{Tr}} \mathbb{Q}
\]
to be induced by the map \(i^d_{(2\pi)^d! |G| \int_X} \). Recall that the following diagram commutes up to sign (see page 102 in [GH])

\[
\begin{array}{ccc}
H^i(X, \Omega^d_X) \times H^{d-i}(X, \Omega^{d-j}_X) & \xrightarrow{\cup} & H^d(X, \Omega^d_X) \\
\downarrow & & \downarrow \\
H^i_{\mathcal{Q}}(X) \times H^{d-i,j}_{\mathcal{Q}}(X) & \xrightarrow{\wedge} & H^d_{\mathcal{Q}}(X)
\end{array}
\]

where the upper horizontal map is cup product, the lower horizontal map is the exterior product of differential forms, and the vertical arrows are Dolbeault isomorphisms; in fact we shall compute this sign in 4.3.2. It now follows that the metrics $h_\sigma = \{h_r\}$ on the equivariant determinant of cohomology induced by $\sigma$ coincide with the metrics on the equivariant determinant of cohomology induced by $|G|^{-1} \wedge^p h^2_{\mathcal{Q}}$. (See 1.4 in [GSZ], and especially Theorem 7.8 in [CPT2] for a full account of the duality and metrics.) In summary we have now shown

**Lemma 33** The metrics $h_\sigma = \{h_r\}$, associated to the de Rham pairings $\sigma_{i,j}$ by 3.2.2, coincide with the $L^2$-metrics, and so we have the equality of Arakelov Euler characteristics

\[
\chi_\mathcal{A}(R\Gamma(X, L \wedge^\bullet \Omega^1_{X/Z})[d], h_\sigma) = \chi_\mathcal{A}(R\Gamma(X, L \wedge^\bullet \Omega^1_{X/Z})[d], |G|^{-1} \wedge^p h^2_{\mathcal{Q}}). \tag{18}
\]

Next we use a result of Ray-Singer (see Theorem 3.1 of [RS]) to show that:

**Proposition 34** The equivariant analytic torsion of the total de Rham complex vanishes (see below); thus we can write

\[
\chi_\mathcal{A}(R\Gamma(X, L \wedge^\bullet \Omega^1_{X/Z})[d], |G|^{-1} \wedge^p h^2_{\mathcal{Q}}) = \chi_\mathcal{A}(R\Gamma(X, L \wedge^\bullet \Omega^1_{X/Z})[d], |G|^{-1} \wedge^p h^2_{\mathcal{Q}}) \tag{19}
\]

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where \(|G|^{-1} \wedge^* h_{X,Q}^D\) denotes the equivariant Quillen metrics (see [B]) on the equivariant determinant of cohomology induced by \(|G|^{-1} \wedge^p h_X^D\).

\textbf{Proof.} For an irreducible character \(\phi\) of \(G\), we let \(T_\phi(O_X^p, |G|^{-1} \wedge^p h_X^D)\) denote the analytic torsion associated to the hermitian sheaf \((\Omega_X^p, |G|^{-1} \wedge^p h_X^D)\). Thus, by definition,
\[
(\|G|^{-1} \wedge^* h_{X,Q}^D)_{Q,\phi} = T_\phi(O_X^p, |G|^{-1} \wedge^p h_X^D)^{-1}(\|G|^{-1} |.|_{L^2})_{\phi}.
\]

From Theorem 3.1 in [RS] we know that
\[
\prod_{p=0}^d T_\phi(O_X^p, |G|^{-1} \wedge^p h_X^D)^{(-1)^p} = 1.
\]

Moreover, it is standard (see for instance page 153 in [R]) that, if we scale the metrics \(\wedge^p h_X^D\) to \(c^2 \wedge^p h_X^D\) for a positive real number \(c\), then the total analytic torsion changes by a factor \(c\) to the power
\[
\sum_{p,q} (-1)^{p+q} q \zeta_{p,q}(0,\phi)
\]
where \(\zeta_{p,q}(s,\phi)\) denotes the \(\zeta\)-function for \(\phi\) associated to \(\Omega_X^p\) with the metric induced by \(\wedge^p \Omega_X^D\). However, from equation (3.2) in the proof of Theorem 3.1 in [RS] we know that for each \(q\)
\[
\sum_p (-1)^p \zeta_{p,q}(s,\phi) = 0
\]
and so we deduce that
\[
\sum_{p,q} (-1)^{p+q} q \zeta_{p,q}(0,\phi) = 0. \quad \square
\]

Recall that the principal goal of this paper is to describe the hermitian Euler characteristic
\[
\chi^s_H(R\Gamma(\mathcal{X}, L \wedge^* \Omega_{\mathcal{X}/\mathbb{Z}}, \sigma_X) \in H^s(\mathbb{Z}[G]));
\]
while from (5) of 3.1.2 we have the decomposition
\[
H^s(\mathbb{Z}[G]) = A^s(\mathbb{Z}[G]) \oplus S_\infty(\mathbb{Z}[G]).
\]

From Theorem 30 we know that the image of \(\chi^s_H(R\Gamma(\mathcal{X}, L \wedge^* \Omega_{\mathcal{X}/\mathbb{Z}}^1)[d], \sigma_X)\) in \(A^s(\mathbb{Z}[G])\) is the Arakelov class \(\chi^s_A(R\Gamma(\mathcal{X}, L \wedge^* \Omega_{\mathcal{X}/\mathbb{Z}}^1)[d], h_\sigma)\) where \(h_\sigma\) denotes the metrics on the equivariant determinant of cohomology afforded by the absolute values of the equivariant determinants of \(\sigma = \sigma_X\). With the above choices we have seen that \(h_\sigma\) coincides with the \(|G|^{-1} \wedge^* |.|_{L^2}\)-norm and so by (18) and (19)
\[
\chi^s_A(R\Gamma(\mathcal{X}, L \wedge^* \Omega_{\mathcal{X}/\mathbb{Z}}^1)[d], h_\sigma) = \chi^s_A(R\Gamma(\mathcal{X}, L \wedge^* \Omega_{\mathcal{X}/\mathbb{Z}}^1)[d], |G|^{-1} |.|_{L^2}) =
\]
\[
= \chi^s_A(R\Gamma(\mathcal{X}, L \wedge^* \Omega_{\mathcal{X}/\mathbb{Z}}^1)[d], |G|^{-1} \wedge^* h_{X,Q}^D).
\]
The crucial point here is that the latter Arakelov class was determined in Theorem 8.4 in [CPT2]: in particular, we show that on symplectic characters \( \theta \) of degree zero this class characterises the global constant \( \varepsilon(\mathcal{X}, \theta) \). Thus, to complete our description of the class \( \chi_1^h(R\Gamma(\mathcal{X}, L \wedge^\bullet \Omega^1_{\mathcal{X}/\mathbb{Z}})[d], \sigma) \), it remains to describe the image \( \chi_2^h(\mathcal{X}, \theta) \) of the class \( \chi_2^h(R\Gamma(\mathcal{X}, L \wedge^\bullet \Omega^1_{\mathcal{X}/\mathbb{Z}})[d], \sigma) \) in \( S_\infty(\mathbb{Z}[G]) \). From Theorem 30 we know that for \( \theta_m \) in \( R \mathcal{G} \)

\[ \chi_2(\theta_m) = i\bar{n}_m(\sigma). \]  

(20)

Thus, in order to prove Theorem 1, we are now required to show that

\[ i^{(Y)\theta_m(1)}\varepsilon(\mathcal{X}, \theta_m) = i\bar{n}_m(\sigma). \]  

(21)

4.2 Archimedean \( \varepsilon \)-constants.

Here we recall a number of results from Sect. 5 of [CEPT2]. Let \( F_\infty : \mathcal{X}(\mathbb{C}) \to \mathcal{X}(\mathbb{C}) \) denote the involution induced by complex conjugation on \( \mathcal{X}(\mathbb{C}) \), the space of complex points of \( \mathcal{X} \); then \( F_\infty \) acts on the Betti cohomology \( H^i_B(\mathcal{X}(\mathbb{C}), \mathbb{Q}) \) and, for a complex representation \( V \) of \( G \) with contragredient \( V^* \), we write \( H^i_B(\mathcal{X}(\mathbb{C}) \otimes \mathbb{Q} V^*)^G \) on which \( F_\infty \) acts by \( +1 \) resp. \( -1 \). (For a discussion of the motives \( \mathcal{X} \otimes \mathbb{Q} V \) see Section 2 of [CEPT2].) We then set

\[ \chi(\mathcal{X} \otimes \mathbb{Q} V) = \sum_{i=0}^{2d} (-1)^i \dim_C(H^i_B(\mathcal{X} \otimes \mathbb{Q} V)) \]

and we may extend \( \chi(\mathcal{X} \otimes \mathbb{Q} V) \) to virtual representations, since it is additive in \( V \).

The archimedean constant \( \varepsilon(\mathcal{X} \otimes \mathbb{Q} V) \) is constructed from the Hodge structure of the motive \( \mathcal{X} \otimes \mathbb{Q} V \); again it is additive in \( V \) and thus extends to virtual \( V \).

Lemma 35 Let \( W \) be a virtual symplectic complex representation of \( G \).

(a) Both \( \chi(\mathcal{X} \otimes \mathbb{Q} W) \) are even integers.
(b) If \( d \) is odd, then \( \varepsilon(\mathcal{X} \otimes \mathbb{Q} W) = 1 \).
(c) If \( d \) is even, then writing \( \pm \) for the sign of \( (-1)^{d/2+1} \) we have

\[ \varepsilon(\mathcal{X} \otimes \mathbb{Q} W) = i^{\chi(\mathcal{X} \otimes \mathbb{Q} W)} \]

and, moreover if \( \dim_C(W) = 0 \), then \( \varepsilon(\mathcal{X} \otimes \mathbb{Q} W) = i^{\chi+}(\mathcal{X} \otimes \mathbb{Q} W) = i^{\chi-}(\mathcal{X} \otimes \mathbb{Q} W) \).

Proof. Part (a) follows from the discussion in 3.2.1 after Lemma 15 which shows that each \( \dim_C(H^i_B(\mathcal{X} \otimes \mathbb{Q} W)) \) is even; (b) and (c) come from Lemma 5.1.1 in [CEPT2]. \( \square \)

4.3 Signature of cohomology.

Throughout all of this sub-section we shall suppose that the fibral dimension \( d \) is even.
4.3.1 Betti cohomology.

Since $d$ is even, $X(\mathbb{C})$ has real dimension divisible by 4; hence the cup product $c^d$ is a non-degenerate symmetric $G$-invariant form on $H_B^d(X(\mathbb{C}), \mathbb{R})$ via the map $H_B^{2d}(X(\mathbb{C}), \mathbb{R}) \to \mathbb{R}$. By Proposition 25 we know that $H_B^d(X(\mathbb{C}), \mathbb{R})$ admits a non-canonical decomposition of $G$-modules

$$H_B^d(X(\mathbb{C}), \mathbb{R}) = H_B^+ \oplus H_B^-$$

where $H_B^+$ is a maximal positive definite subspace and $H_B^-$ is a maximal negative definite subspace of $H_B^d(X(\mathbb{C}), \mathbb{R})$ with respect to $c^d$.

For $t < d$ we let $c^t$ denote the symmetrised $G$-invariant form on $H_B^t(X(\mathbb{C}), \mathbb{R}) \oplus H_B^{2d-t}(X(\mathbb{C}), \mathbb{R})$ induced by the cup product $c$ as per the construction of $\sigma^t$ in 4.1. Note that the symmetrisation here is the same as that used in 4.1: indeed, for $x \in H_B^t(X(\mathbb{C}), \mathbb{R})$, $y \in H_B^{2d-t}(X(\mathbb{C}), \mathbb{R})$, $t < d$

$$c(y, x) = (-1)^t c(x, y).$$

Whereas by (15) for $w \in H^i(X, \Omega^j_X)$, $z \in H^{d-i}(X, \Omega^{d-j}_X)$, if we set $t = i + j$, then, as $d$ is even, we have seen that

$$\sigma_{d-i, d-j}(z, w) = (-1)^{(i+j)} \sigma_{i, j}(w, z) = (-1)^t \sigma_{i, j}(w, z).$$

Thus for $t < d$, $c^t$ is hyperbolic and by Proposition 25 we have a decomposition of $\mathbb{R}[G]$-modules

$$H_B^{odd}(X(\mathbb{C}), \mathbb{R}) = H_B^{odd+} \oplus H_B^{odd-}$$

into positive and negative subspaces.

Applying Proposition 25 once again we obtain a decomposition

$$H_B^{ev}(X(\mathbb{C}), \mathbb{R}) = H_B^{ev+} \oplus H_B^{ev-}$$

where $H_B^{ev+} \subset H_B^{odd+}$, $H_B^{ev-} \subset H_B^{odd-}$.

Furthermore, by Lemma 26 and by hyperbolicity, we know that as $\mathbb{R}$-vector spaces

$$H_B^{ev+}/H_B^+ \cong H_B^{ev-}/H_B^- \cong \bigoplus_{\text{even}, t < d} H_B^t(X(\mathbb{C}), \mathbb{R}),$$

$$H_B^{odd+} \cong H_B^{odd-} \cong \bigoplus_{\text{odd}, t < d} H_B^t(X(\mathbb{C}), \mathbb{R}).$$

**Theorem 36** With the above notation and hypotheses, $H_B^{\bullet+}$ and $H_B^{\bullet-}$ are both free virtual $\mathbb{R}[G]$-modules.
Proof. Since $G$ acts freely on $X(C)$, by the Lefschetz Fixed Point theorem (see for instance [V]) for each $g \in G$, $g \neq 1$, the virtual character associated to $H^*_p(X(C), R)$ is zero when evaluated on such $g$; thus $H^*_B = H^*_B^+ + H^*_B^-$ is a free virtual $R[G]$-module.

Similarly we shall show that $H^*_B^+ - H^*_B^-$ is a free virtual $R[G]$-module; this will then establish the theorem. To see that $H^*_B^+ - H^*_B^-$ is free, we recall that by the G-Signature Theorem 6.12 in [AS] (see also V.18 in [S]), for each non-trivial element $g \in G$, the value of the virtual character of $H^*_B^+ - H^*_B^-$ evaluated on $g$ is presented in terms of data associated to the fixed point set $X(C)^g$. Since $g$ acts without fixed points, it then follows that this virtual character is zero on all such $g$. \hfill $\square$

4.3.2 De Rham cohomology and hypercohomology.

In this paragraph, to ease the notation, we use the symbol $X$ to denote either $X_{\mathbb{Q}} = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ or $X_{\mathbb{R}} = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{R})$ depending on the context. We follow the terminology of Grothendieck (see [G]); for a given integer $t$, we consider the (shifted) $t$-th Hodge cohomology group

$$H^t_{\text{Hod}}(X)[d] = H^t(X, \oplus_n \Omega^n_X[d - n]) = \oplus_n H^{t+d-n}(X, \Omega^n_X) = \oplus_m: m+n=t+d H^m(X, \Omega^n_X)$$

and similarly we put

$$H^t_{\text{Hod}}(X)[d] = \oplus_{t\text{ even}} H^t_{\text{Hod}}(X)[d], \quad H^t_{\text{Hod}}(X)[d] = \oplus_{t\text{ odd}} H^t_{\text{Hod}}(X)[d].$$

We then let $(H^t_{\text{Hod}}(X)[d], \sigma^{ev})$ denote $H^t_{\text{Hod}}(X)[d]$ endowed with the $G$-invariant symmetric form $\sigma^{ev}$ and similarly we have $(H^t_{\text{Hod}}(X)[d], \sigma^{odd})$. In what follows, we take $X = X_{\mathbb{R}}$. Applying Proposition 25 we have a decomposition of $R[G]$-modules into positive and negative spaces

$$H^t_{\text{Hod}}(X)[d] = H_{\text{Hod}}^{t, +}[d] \oplus H_{\text{Hod}}^{t, -}[d], \quad H^{\text{odd}}_{\text{Hod}}(X)[d] = H^{\text{odd}, +}[d] \oplus H^{\text{odd}, -}[d].$$

In order to obtain detailed information about these decompositions, we shall need to compare $(H^t_{\text{Hod}}(X)[d], \sigma^{ev})$ and $(H^{t, odd}_{\text{Hod}}(X)[d], \sigma^{odd})$ with the de Rham hypercohomology $H^{t, \bullet}_{\text{dR}}(X)[d] = H^{t, \bullet}(X, \Omega^{\bullet}_{X/R}[d])$ of $\Omega^{\bullet}_{X/R}[d]$ endowed with the $G$-invariant forms from duality theory: recall that duality for de Rham hypercohomology gives a perfect $R$-bilinear form

$$t^p : H^p_{\text{dR}}(X)[d] \times H^{-p}_{\text{dR}}(X)[d] \to H^0_{\text{dR}}(X)[2d] = H^0_{\text{dR}}(X) \to R,$$

where by the Wirtinger theorem (see page 31 in [GH]) the right hand map is given by real integration $\omega \mapsto |G|^{-1} \int_X \omega$ for a global real $2d$-form $\omega$. As $d$ is even, the map $t^p$ is symmetric (see below); note also that if $x \in H^p_{\text{dR}}(X)[d]$, $y \in H^{-p}_{\text{dR}}(X)[d]$, then $t^p(x, y) = (-1)^p t^{-p}(y, x)$ which again of course agrees with the commutation rule (15); hence, as per the construction in 4.1, we may then form the symmetrised duality maps $\tau^p$. 

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We write $\Omega_X^{\leq m}$ respectively $\Omega_X^{\geq m}$ for the complex

\[ O_X \xrightarrow{d} \Omega^1_{X/R} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{m-1}_{X/R} \]

\[ \Omega^m_{X/R} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d_{X/R} \]

where the term $O_X$ (respectively $\Omega^m_{X/R}$) is placed in degree zero (respectively degree $m$). We then consider the exact sequence of complexes

\[ 0 \to \Omega^m_{X/R} \to \Omega^m_{X/R} \to \cdots \to \Omega^d_{X/R} \to 0 \]

and we let $F^{-m+d/2}$ denote the image of $H^m_{\text{dR}}(X, \Omega^m_{X/R})$ in $H^m(X, \Omega^m_{X/R})$. Note that by the degeneration of the Hodge spectral sequence we know that in fact $H^m(X, \Omega^m_{X/R})$ injects into $H^m(X, \Omega^m_{X/R})$ under the natural map induced by $\Omega^m_{X/R} \xrightarrow{d} \Omega^m_{X/R}$.

**Theorem 37** The quadratic space $(H^m_{\text{dR}}(X)[d], \tau)$, when endowed with the filtration $\{F^i\}$, is a filtered quadratic space, as defined in Lemma 27. There is an isomorphism of $R[G]$-quadratic modules:

\[ (H^m_{\text{dR}}(X)[d], \tau) \cong (H^d/2(X, \Omega^{d/2}_X), \sigma_{d/2,d/2}) \oplus \text{Hyp}(\oplus_{i<d/2} H^i(X, \Omega^{d/2}_{X/R})) \oplus \text{Hyp}(H^m(X, \Omega^{d/2}_{X/R})). \]

**Proof.** Consider the shifted symmetrised duality maps

\[ H^p_{\text{dR}}(X) \times H^q_{\text{dR}}(X) \to H^{p+q}_{\text{dR}}(X) \to R. \]

It then follows immediately that the pairing

\[ H^*(X, \Omega^m_{X/R}) \times H^*(X, \Omega^{d-m}_{X/R}) \to R \]

factors through

\[ \tau'_m : \frac{H^*(X, \Omega^m_{X/R})}{H^*(X, \Omega^{m+1}_{X/R})} \times \frac{H^*(X, \Omega^{d-m}_{X/R})}{H^*(X, \Omega^{d-m+1}_{X/R})} \to R. \]

To see that this pairing is perfect, we again appeal to the decomposition of the Hodge spectral sequence to deduce that for all $m, n$

\[ \frac{H^n(X, \Omega^m_{X/R})}{H^n(X, \Omega^{m+1}_{X/R})} \cong H^n(X, \Omega^m_X). \]

Thus $\tau'_m$ induces forms

\[ \tau^n_m : H^n(X, \Omega^m_X) \times H^{d-n}(X, \Omega^{d-m}_X) \to R. \]

We now claim that under the above isomorphisms the forms $\tau^n_m$ agree with the pairings $\sigma_{n,m}$ up to the sign $(-1)^{(m+n)n}$, and in particular agree exactly when $m = d/2 = n$. To show this it is enough to apply $\otimes_R C$; identify the left-hand terms with the Dolbeault cohomology groups $H^{m,n}_{\partial}(X)$ and show
Proposition 38  The following diagram commutes:

\[
\begin{array}{ccc}
  H^0(X_C, \Omega^p_{\mathcal{O}_{X_C}}) & \times & H^{d-q}(X_C, \Omega^{d-p}_{\mathcal{O}_{X_C}}) \\
  \downarrow & & \downarrow (-1)^{(p+q)q} \\
  H^0_{d\sigma}(X_C) & \times & H^{d-p,d-q}_{d\sigma}(X_C) \\
\end{array}
\]

(22)

where the two left-hand vertical maps are Dolbeault isomorphisms and right-hand vertical map is the Dolbeault isomorphism multiplied by \((-1)^{(p+q)q}\). In particular if \(p = q\) then \((-1)^{(p+q)q} = 1\) and so in this case \(\cup\) and \(\wedge\) agree under the Dolbeault isomorphism.

Before proving the proposition, we first note that it will complete the proof of the theorem. Indeed, by Lemma 27, we know that

\[
(H^\bullet_{dR}(X)[d], \tau) \cong (\oplus_i H^i(\Omega^{d/2}_X), \tau_{d/2}) \oplus \text{Hyp}(H^\bullet(\Omega^{d/2}_X))
\]

\[
\cong (H^{d/2}(\Omega^{d/2}_X), \tau_{d/2}) \oplus \text{Hyp}(\oplus_i \tau_{<d/2} H^i(\Omega^{d/2}_X)) \oplus \text{Hyp}(H^\bullet(\Omega^{d/2}_X));
\]

however, by the above discussion together with the proposition, we know that \(\tau_{d/2}\) is equal to \(\sigma_{d/2,d/2}\) and the result will now follow.

**Proof of Proposition 38.** In unraveling the Dolbeault isomorphisms we shall follow the conventions given in Section 3 of Chapter 0 in [GH]. We first need some notation: let \(\mathcal{K}^{p,q}\) denote the sheaf of \(C^\infty\)-forms on \(X_C\) of type \((p, q)\) we let \(\mathcal{Z}^{p,q}\) denote the sheaf of \(\overline{\partial}\)-closed \(C^\infty\) forms of type \((p, q)\) and we write \(\Omega^p\) for the sheaf of holomorphic \(p\)-forms \(\mathcal{Z}^{p,0}\).

By the \(\overline{\partial}\)-Poincaré Lemma we have exact sequences of sheaves

\[
0 \rightarrow \mathcal{Z}^{p,q} \rightarrow \mathcal{K}^{p,q} \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,q+1} \rightarrow 0. \tag{23}
\]

For each \(n\), \(0 \leq n < q\), we then consider the exact sequence

\[
0 \rightarrow \mathcal{Z}^{p,q} \otimes \mathcal{Z}^{d-p,d-q-n-1} \rightarrow \mathcal{Z}^{p,q} \otimes \mathcal{K}^{d-p,d-q-n-1} \xrightarrow{1 \otimes \overline{\partial}} \mathcal{Z}^{p,q} \otimes \mathcal{Z}^{d-p,d-q-n} \rightarrow 0. \tag{24}
\]

For brevity we write \(H^a(\mathcal{Z}^{p,q})\) for \(H^a(X_C, \mathcal{Z}^{p,q})\) etc. and we let \(\delta^a_1\) resp. \(\delta^a_2\) denote the \(n\)-th coboundary map associated to the cohomology of the exact sequence (23) resp. (24). Then for \(x \in H^0(\mathcal{Z}^{p,q})\), \(y \in H^n(\mathcal{Z}^{d-p,d-q-n})\), by using the cocycle description of the boundary maps, we obtain \(x \wedge \delta^a_1(y) = \delta^a_2(x \wedge y)\). Hence for each such \(n\), \(0 \leq n < d - q\), we get a commutative diagram

\[
\begin{array}{ccc}
  H^0(\mathcal{Z}^{p,q}) & \times & H^n(\mathcal{Z}^{d-p,d-q-n}) \\
  \downarrow 1 & & \downarrow \delta^a_1 \\
  H^0(\mathcal{K}^{p,q}) & \times & H^{n+1}(\mathcal{Z}^{d-p,d-q-n-1}) \\
\end{array}
\]

\[
\xrightarrow{\cup} \begin{array}{ccc}
  H^n(\mathcal{Z}^{p,q} \wedge \mathcal{Z}^{d-p,d-q-n}) \\
  \downarrow \delta^a_2 \\
  H^{n+1}(\mathcal{Z}^{p,q} \wedge \mathcal{Z}^{d-p,d-q-n-1}) \\
\end{array}
\]

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and we claim that $\delta_2^n$ agrees with $(-1)^{p+q}$ times the Dolbeault map $D^n : H^n(\mathbb{Z};d,d-n) \to H^{n+1}(\mathbb{Z};d-d-n-1)$. Recall that $D^n$ is an isomorphism for positive $n$ and that $D^0$ is a surjection which induces the isomorphism

$$H^0_{\mathcal{D}}(X) := H^0(\mathbb{Z};d) / \Omega(H^0(\mathbb{Z};d-q-1)) \to H^1(\mathbb{Z};d-q-1).$$

In proving the claim it will be useful to have the following cocycle formula for the Dolbeault maps $D^n : H^n(\mathbb{Z};a,b) \to H^{n+1}(\mathbb{Z};a,b-1)$. Once and for all we fix a sufficiently fine cover $U = \{U_i \}_{i \in I}$ of $X$. For a $\mathbb{Z}$-valued $n$-cochain $\omega^n$, we write $\omega(i_0, \ldots, i_n)$ for $\omega(U_{i_0} \cap \cdots \cap U_{i_n})$. Then

$$(\delta^n(i_0, \ldots, i_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \delta^{n+1}_{\mathcal{D}}(\omega^n(i_k))$$

where $\omega^n(i_k)$ means $\omega^n(i_0, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_{n+1})$. Now let $\gamma \in H^0(\mathbb{Z};d) \ , \ \omega^n \in H^n(\mathbb{Z};d-p,d-q-n)$. Then

$$(\delta^n(\gamma \wedge \omega^n)(i_0, \ldots, i_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \delta^{n+1}_{\mathcal{D}}(\gamma \wedge \omega^n(i_k))$$

while

$$\delta_2^n(\gamma \wedge \omega^n)(i_0, \ldots, i_{n+1}) = \sum_{k=0}^{n+1} (-1)^k ((1 \otimes \mathcal{D})^{-1} \gamma \wedge \omega^n(i_k)).$$

The claim then follows by the Leibniz rule for $\delta$ and using the fact the $\delta_{\mathcal{D}}(\gamma(i_0)) = 0$: indeed, for $k > 0$

$$\delta^{n+1}_{\mathcal{D}}(\gamma(i_0) \wedge \omega^n(i_k)) = (-1)^{p+q} \gamma(i_0) \wedge \delta^{n+1}_{\mathcal{D}}(\omega^n(i_k)) = (-1)^{p+q} (1 \otimes \delta)_{\mathcal{D}}^{-1}(\gamma(i_0) \wedge \omega^n(i_k))$$

and similarly when $k = 0$

$$\delta^{n+1}_{\mathcal{D}}(\gamma(i_1) \wedge \omega^n(i_0)) = (-1)^{p+q} (1 \otimes \delta)_{\mathcal{D}}^{-1}(\gamma(i_1) \wedge \omega^n(i_0)).$$

Next, for $m$, $0 \leq m < q$, we consider the exact sequences

$$0 \to \mathbb{Z}^{p,m-1} \otimes \Omega^{d-p} \to \mathbb{Z}^{p,m-1} \otimes \Omega^{d-p} \to \mathbb{Z}^{p,m} \otimes \Omega^{d-p} \to 0 \tag{25}$$

and we write $\delta_2^m$ for the $m$-th coboundary map associated to the cohomology of this exact sequence. For $x \in H^m(\mathbb{Z};d) \ , \ y \in H^q(\Omega^{d-p})$, we know that $\delta_1^m(x) \wedge y = \delta_2^m(x \wedge y)$ ([Go] Ch. 6.5, p. 255). Therefore for each such $m$ we obtain a commutative square

$$\begin{array}{ccc}
H^m(\mathbb{Z};d) & \to & H^{d-q}(\Omega^{d-p}) \\
\downarrow \delta_1^m & & \downarrow 1 \\
H^{m+1}(\mathbb{Z};d) & \to & H^{d-q+1}(\mathbb{Z};d) \\
\end{array}$$

\[34\]
and we now claim that $\delta_m^3$ agrees with the Dolbeault map $D^{d-q+m}$. For the sake of brevity we put $p' = d - p$, $q' = d - q$ and we let $\omega^m \in H^m(\mathcal{Z}^{p,m})$, $\nu^{q'} \in H^{q'}(\Omega^{q'})$. Then

$$D^{m+q'}(\omega^m \wedge \nu^{q'})(i_0, \ldots, i_{m+q'+1}) = \sum_{k=0}^{m+q'+1} (-1)^k \partial^{-1}(\omega^m \wedge \nu^{q'})(\hat{i}_k)$$

while

$$\delta_m^3(\omega^m \wedge \nu^{q'})(i_0, \ldots, i_{m+q'+1}) = \sum_{k=0}^{m+q'+1} (-1)^k (\partial \circ 1)^{-1}(\omega^m \wedge \nu^{q'})(\hat{i}_k).$$

Since $\partial \circ 1 = \partial$ as we have seen in (25) we obtain that $\delta_m^3$ agrees with $D^{d-q+m}$. Recall now that the Dolbeault isomorphisms ([GH] p. 45) are obtained as a composition of a succession of Dolbeault maps (there are $q$ of these maps)

$$H^{p,q}_d(X) \cong H^{0}(\mathcal{Z}^{p,q})/\overline{\partial}(H^{0}(\mathcal{Z}^{p,q}+1)) \xrightarrow{D} \cdots \xrightarrow{D} H^{q-1}(\mathcal{Z}^{p,1}) \xrightarrow{D} H^q(\Omega^p).$$

By combining the above results with an inductive argument, we can now see that the sign discrepancy between the two pairings is equal to the product of $q$ copies of $(-1)^{p+q}$, therefore equal to $(-1)^{p+q}$. □

**Corollary 39** There is a non-canonical $\mathbb{R}[G]$-isometry

$$(H^\bullet_{dR}(X)[d], \tau) \otimes_{\mathbb{Q}} \mathbb{R} \cong (H^\bullet_{Hod}(X)[d], \sigma) \otimes_{\mathbb{Q}} \mathbb{R}. $$

**Proof.** By the above proposition we know that each of the above quadratic spaces is isometric to the orthogonal sum of $(H^{d/2}(X, \Omega^{d/2}_X/\mathbb{R}), \sigma_{d/2,d/2})$ and a hyperbolic space. On the other hand by the degeneration of the Hodge spectral sequence we know that $H^\bullet_{dR}(X) \otimes \mathbb{C}$ and $H^\bullet_{Hod}(X) \otimes \mathbb{C}$ are isomorphic $\mathbb{C}[G]$-modules; hence $H^\bullet_{dR}(X)$ and $H^\bullet_{Hod}(X)$ are isomorphic $\mathbb{R}[G]$-modules; therefore we may conclude that the two hyperbolic spaces are isometric, as required. □

Next we recall that from Proposition 1.4 on p 319 in [D]:

**Proposition 40** The comparison isomorphism $H^\bullet_{dR}(X) \otimes \mathbb{C} \cong H^\bullet_{B}(X) \otimes \mathbb{C}$ identifies $H^\bullet_{dR}(X) \otimes \mathbb{Q} \mathbb{R}$ with $H^\bullet_{B+} \oplus (H^\bullet_{B-} \otimes \mathbb{R} i\mathbb{R})$.

Writing

$$H^{ev,+}_{B} = H^{ev}_{B+} \cap H^{ev,+}_{B}, \quad H^{ev,-}_{dR} = H^{ev}_{dR}(X),$$

and similarly with odd in place of even, we conclude from the above proposition that:

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Corollary 41

\[ \dim(H_{dR}^{ev,+}) = \dim(H_{B+}^{ev}) + \dim(H_{B-}^{ev}), \quad \dim(H_{dR}^{ev,-}) = \dim(H_{B+}^{ev}) + \dim(H_{B-}^{ev}), \]
\[ \dim(H_{dR}^{odd,+}) = \dim(H_{B+}^{odd}) + \dim(H_{B-}^{odd}), \quad \dim(H_{dR}^{odd,-}) = \dim(H_{B+}^{odd}) + \dim(H_{B-}^{odd}). \]

Moreover by Corollary 39 the same statements hold with \( H_{Hod}^{ev} \) in place of \( H_{dR} \) etc. on the left-hand side.

4.4 Proof of Theorem 1.

Before embarking on the proof of the theorem, we first need:

Proposition 42 \( H_{Hod}^{ev}(X) - H_{Hod}^{odd}(X) \) is a (virtually) free \( \mathbb{Q}[G] \)-module.

Proof. This follows from Theorem 36 and the comparison isomorphism \( H_*^{\bullet}(X) \otimes_{\mathbb{Q}} C \cong H_*^{\bullet}(X) \otimes_{\mathbb{R}} C. \)

Throughout this sub-section we assume \( W \) to be a virtual complex symplectic representation of \( G \). Recall from (21) that in order to prove the theorem it is sufficient to show

\[ i^{\delta(Y)} \dim(W) e_{\infty}(\mathcal{X}, W) = i^{n_W(\sigma)}. \] (26)

Initially we shall suppose that \( \dim(W) = 0 \); then, by linearity in both sides, we shall conclude the proof by dealing with the case where \( W \) is two copies of the trivial representation.

By the above and Lemma 35, in order to prove Theorem 1 when \( \dim(W) = 0 \), we are required to establish the congruence modulo 4

\[ n_W(\sigma) \equiv \begin{cases} 0, & \text{if } d \text{ is odd}, \\ \chi_-(X \otimes G W), & \text{if } d \text{ is even}. \end{cases} \]

Case 1. \( d \) is odd. On the one hand we know that \( \sigma \) is hyperbolic; on the other hand by the above proposition we know that \( H_{Hod}^{ev}(X)_{\mathbb{R}} \) is a free \( \mathbb{R}[G] \)-module; thus by Lemma 26 we know that \( H_{Hod}^{ev}(X)_{\mathbb{R}} - H_{Hod}^{odd}(X)_{\mathbb{R}} \) is also a free \( \mathbb{R}[G] \)-module; therefore, because \( W \) has dimension zero, it follows at once that \( n_W(\sigma) = 0 \), as required.

Case 2. \( d \) is even. Since \( d \) is even we may freely use the notation of 4.3. Then by Corollary 41

\[ n_W(\sigma) = \dim(H_{B+}^{\bullet} \otimes W)^G + \dim(H_{B-}^{\bullet} \otimes W)^G. \] (27)

By Theorem 36 \( H_{B+}^{\bullet} = H_{B+}^{\bullet} + H_{B-}^{\bullet} \) is a free \( G \)-module and again since \( \dim(W) = 0 \), we see that

\[ 0 = \dim(H_{B+}^{\bullet} \otimes W)^G = \dim(H_{B+}^{\bullet} \otimes W)^G + \dim(H_{B-}^{\bullet} \otimes W)^G \]
and therefore by (27)

\[ n_W(\sigma) = -\dim(H^*_{B^-} \otimes W)^G + \dim(H^*_{B^+} \otimes W)^G. \]

By Proposition 25 we know that each \( H^*_{B^\pm}(X(C), R) \) is an \( R[G] \)-module, and so, reasoning as after the proof of Lemma 15, we see that \( \dim(H^*_{B^-} \otimes W)^G \) is even, and it therefore follows that we have the congruence mod 4

\[ n_W(\sigma) \equiv \dim(H^*_{B^-} \otimes W)^G + \dim(H^*_{B^+} \otimes W)^G = \chi_-(X \otimes_G W) \]

as required.

To conclude we now suppose that \( W \) is two copies of the trivial representation; so that \( H^*_{B^\pm}(X) \otimes_G W \) is now just two copies of \( H^*_{B^\pm}(Y) \). We are now required to establish the congruence mod 4

\[ n_W(\sigma) \equiv \begin{cases} 
\chi(Y), & \text{if } d \text{ is odd,} \\
2\chi^+(Y) + 2\chi^-(Y), & \text{if } d \text{ is even and } \pm \text{ denotes the sign of } (-1)^{d/2+1}.
\end{cases} \tag{28} \]

Suppose first that \( d \) is odd. Since \( \sigma \) is hyperbolic, by Lemma 26 \( n_W(\sigma) = \chi(Y) \), as required.

Suppose next that \( d \) is even, so that again we may use the notation of 4.3. Recall that \( (H^*_{B^\pm}(X) \otimes W)^G \) is two copies of \( H^*_{B^\pm}(Y) \), and, as previously, put \( \chi^\pm(Y) = \dim(H^*_{B^\pm}(Y)) \) etc.; more generally, we shall write \( \chi^\pm(Y) \) for \( \dim(H^*_{B^\pm}(Y)) - \dim(H^*_{B^\pm}(Y)) \). Observe that by Corollary 41 we again have congruences mod 4

\[ n_W(\sigma) = 2\chi^-(Y) + 2\chi^+(Y) = 2\chi^-(Y) - 2\chi^-(Y) + 2\chi^+(Y) \equiv 2\chi^-(Y) - 2\chi^-(Y) - 2\chi^+(Y) \equiv 2\chi^-(Y) - 2\chi^-(Y). \]

Case 1. \( d \equiv 2 \text{ mod } 4 \). In this case by (28) we have to show the congruence mod 4

\[ 2\chi^-(Y) - 2\chi^-(Y) \equiv 2\chi^+(Y) + 2\chi^-(Y) \]

which is clear since

\[ \chi^+(Y) + \chi^-(Y) = \chi(Y) = \chi^+(Y) + \chi^-(Y). \]

Case 2. \( d \equiv 0 \text{ mod } 4 \). This follows at once since we have to show the congruence mod 4

\[ 2\chi^-(Y) + 2\chi^-(Y) \equiv 2\chi^-(Y) - 2\chi^-(Y) \]

which is immediate.
5 Appendix: Comparison of definitions.

The symplectic hermitian class group that we have used, namely \( H^s(\mathbb{Z}[G]) \), is very well suited to comparison with Arakelov invariants; indeed, from (5) we see that it is the natural vehicle for carrying discriminantal signs associated to Arakelov discriminants. In this Appendix we briefly indicate how the class group \( H^s(\mathbb{Z}[G]) \), and hermitian classes formed in this group, relate to the previous hermitian classes and classgroups, such as those used for instance in [F1] and [CPT1].

5.1 Hermitian class groups.

Recall that in Definition 11 we defined the symplectic hermitian class group \( H^s(\mathbb{Z}[G]) \) to be
\[
H^s(\mathbb{Z}[G]) = \frac{\text{Hom}_{\mathbb{Q}}(R^s_G, J_f) \times \text{Hom}(R^s_G, \mathbb{R}^\times)}{\text{Im}(\Delta^s) \cdot (\text{Det}(\mathbb{Z}[G]^\times) \times 1)}.
\] (29)
By contrast in [F1] and [CPT1] the hermitian class group \( \text{HCl}(\mathbb{Z}[G]) \) is used, which is described in terms of character functions as
\[
\text{HCl}(\mathbb{Z}[G]) = \frac{\text{Hom}_{\mathbb{Q}}(R_G, J_f) \times \text{Det}(\mathbb{R}[G]^\times) \times \text{Hom}_{\mathbb{Q}}(R^s_G, \mathbb{Q}^\times)}{\text{Im}(\tilde{\Delta}) \cdot (\text{Det}(\mathbb{Z}[G]^\times \times \mathbb{R}[G]^\times) \times 1)}
\] (30)
where \( \tilde{\Delta} \) is the twisted diagonal map
\[
\tilde{\Delta} : \text{Det}(\mathbb{Q}[G]^\times) \to \text{Hom}_{\mathbb{Q}}(R_G, J_f) \times \text{Det}(\mathbb{R}[G]^\times) \times \text{Hom}_{\mathbb{Q}}(R^s_G, \mathbb{Q}^\times)
\]
given by \( \tilde{\Delta}(\text{Det}(a)) = \text{Det}(a) \times \text{Det}(a) \times \text{Det}^s(a)^{-1} \). We therefore have a natural map from
\[
\phi : \text{HCl}(\mathbb{Z}[G]) \to H^s(\mathbb{Z}[G])
\] (31)
induced by the map
\[
\text{Hom}_{\mathbb{Q}}(R_G, J_f) \times \text{Det}(\mathbb{R}[G]^\times) \times \text{Hom}_{\mathbb{Q}}(R^s_G, \mathbb{Q}^\times) \to \text{Hom}_{\mathbb{Q}}(R^s_G, J_f) \times \text{Hom}(R^s_G, \mathbb{R}^\times)
\]
which: takes the first left-hand factor into the first right-hand factor by restriction from \( R_G \) to \( R^s_G \); which is trivial on the second left-hand factor; and which maps the third left-hand factor to the second right-hand factor by inverting the natural map induced by the inclusion \( \mathbb{Q} \hookrightarrow \mathbb{C} \).

5.2 Hermitian classes.

Next we recall the construction of hermitian Euler characteristics used in [F1] and [CPT1]; we compare this definition with that given in 3.2.1, and then see how they match under the comparison map \( \phi \) above.
So here we consider a perfect $\mathbb{Z}[G]$-complex $P^\bullet$

$$\ldots \rightarrow P^i \xrightarrow{\partial^i} P^{i+1} \rightarrow \ldots$$

which supports non-degenerate $G$-invariant $Q$-valued forms

$$\sigma^i : H^i(P^\bullet)_Q \times H^{-i}(P^\bullet)_Q \rightarrow Q$$

which are symmetric in the sense that $\sigma^i(x, y) = \sigma^{-i}(y, x)$. In Proposition 2.7 of [CPT1] we show that, after adding an acyclic complex to $P^\bullet$ if necessary, each $P^i_Q = Q \otimes P^i$ admits a $G$-decomposition $P^i_Q = B^i \oplus H^i \oplus U^i$ with $B^i = \text{Im}(\partial^i)$ and with $U^i$ mapped isomorphically onto $B^{i+1}$ by $\partial^i$, and there exist non-degenerate $G$-invariant pairings

$$\overline{\varphi}_H : H^i \times H^{-i} \rightarrow Q, \quad \overline{\varphi}_B : B^i \times U^{-i} \rightarrow Q, \quad \overline{\varphi}_U : U^i \times B^{-i} \rightarrow Q$$

which lift the $\sigma^i$ in the following sense: under the identification $H^i = H^i(P^\bullet_Q)$

$$\overline{\varphi}_H = \sigma^i : H^i \times H^{-i} \rightarrow Q;$$

and for $b \in B^i, u \in U^{-i}$

$$\overline{\varphi}_B(b, u) = \overline{\varphi}_U^{-i}(u, b);$$

furthermore these pairings have the crucial property that for all $i$

$$\overline{\varphi}_U^{-i}(u^{-i}, \partial^{-i}u^{-i}) = \overline{\varphi}_B(\partial^{-i}(u^{-i}), u^{-i}).$$

The above $\overline{\varphi}_H, \overline{\varphi}_B, \overline{\varphi}_U$ then induce pairings

$$p^0 : P^0_Q \times P^0_Q \rightarrow Q$$

$$p^i : (P^i_Q \oplus P^{-i}_Q) \times (P^i_Q \oplus P^{-i}_Q) \rightarrow Q, \text{ for } i > 0$$

which are non-degenerate, symmetric and $G$-invariant. By construction, we see that each $p^i$ is an orthogonal sum of forms $p |_{H^i \oplus H^{-i}}$ on $H^i \oplus H^{-i}, p |_{B^i \oplus U^{-i}}$ on $B^i \oplus U^{-i}, p |_{U^i \oplus B^{-i}}$ on $U^i \oplus B^{-i}$ when $i > 0$; and when $i = 0$ we have $p |_{H^0} = \overline{\varphi}_H^0, p |_{B^0 \oplus U^0} = \overline{\varphi}_B^0$.

With the same notation as used in 3.2.1, the hermitian class in [CPT1] associated to the pair $(P^\bullet, \sigma)$ is denoted $d(P^\bullet, \sigma)$; then $\phi(d(P^\bullet, \sigma))^{-1}$ is that class in $H^s(\mathbb{Z}[G])$ represented by the character map which takes $\theta_m \in R^*_G$ to the value

$$\prod_{p < \infty} \text{Det}(\lambda^i_p)(\theta_m)^{(-1)^i} \times \prod_{i \geq 0} \text{pf} \left( T_m(\overline{\varphi}^i(a^0_{ij}, a^{-i}_{ij}))_{j,j'} \right)^{(-1)^i}$$

where $T_m$ is a symplectic representation with character $\theta_m$ and where for $i > 0$ the above Pfaffian term is formed with respect to the $\{a^0_{ij}, a^{-i}_{ij}\}_j$ and $\{a^0_{ij}\}_j$ again denotes a chosen $Q[G]$-basis of $P^i \otimes \mathbb{Q}$.  

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(To see why we need to invert the class $\phi(d(P^\bullet, \sigma))$ to get the above representative, note: firstly, the map $\phi$ involves inversion of the archimedean coordinate (see (31)); secondly, the formula for the finite coordinate in Definition 4.5 in [CPT1] is the inverse of the finite coordinate that we use.)

In Theorem 2.9 of [CPT1] it is shown that the class $\phi(d(P^\bullet, \sigma))$ is independent of choices, and so only depends on the complex $P^\bullet$ and the cohomology pairings $\sigma_i$. Recall that for brevity we put

\[ b^{im}_{jn} = r_G(a_0^{ij} \otimes w_{mn}). \]

With the notation of the previous sub-section we shall show

**Proposition 43**

(a) For each integer $m$

\[ \sum_{i \geq 0} (-1)^i n_m^-(p^i) = \sum_{i \geq 0} (-1)^i n_m^-(\sigma^i). \]

(b) For each integer $m$

\[ \text{Pf}_{(\sigma \otimes \kappa_m)G} \left( \xi_m(\otimes_i \wedge_j b^{im}_{jn}(-1)^i) \right) = |G|_{\chi(P^\bullet|G)}^{(\theta_m(1)/2) \prod_{i} \text{Pf} \left( \tilde{p}^{i}(a_0^{ij}, a_0^{-i}j')_{j,j'} \right)}^{(-1)^i} \]

where

\[ \chi(P^\bullet) = |G|^{-1} \sum_i (-1)^i \dim_Q(P^i_Q) = |G|^{-1} \sum_i (-1)^i \dim_Q(H^i(P^\bullet_Q)). \]

(c) Let $h^i_m$ denote the metric on $\det(P^0_m)$ resp. on $\det(P^i_m \oplus P^{-i}_m)$ for $i = 0$ resp. $i > 0$ afforded by $p^0$ resp. $p^i$, and let $x_i \in \det(P^i_m)$ all be non-zero. Then

\[ h^i_m(x_0) \prod_{i > 0} h^i_m(x_i \otimes x_{-i})^{(-1)^i} = h_{\sigma,m}(\xi_m(\otimes_i x^i_{(-1)^i})). \]

Before proving this proposition, we first note that part (b) together with (36) has the following important implication

**Theorem 44**

\[ \chi^I_H(P^\bullet, \sigma) = \phi_d(P^\bullet, |G| \sigma)^{-1}. \]

**Remark 45** To understand conceptually why we are obliged to renormalise the forms $\sigma^{ev}, \sigma^{odd}$ by a factor $|G|$, it is helpful to consider the special case of a tame Galois extension $N/Q$ with Galois group $G$. In [F1] and [CPT1] one works with the hermitian pair $(O_N, Tr_{N/Q})$; however, in [CPT2] we are obliged to work with $(O_N, |G|^{-1} Tr_{N/Q})$ for the
following reason: the metric associated to the trace map is of course the pullback of the trivial (or standard) metric on $Q$; however, as explained in 4.1 (after Lemma 32), we need to normalise the pullback metric by a factor $|G|^{-1}$ to ensure that it agrees with the original metric on $G$-fixed sections.

**Example 46** Let $HP = \text{Hyp}(\mathbb{Z}[G])$ denote the free hyperbolic plane, that is to say the module $\mathbb{Z}[G] \oplus \text{Hom}_\mathbb{Z}(\mathbb{Z}[G], \mathbb{Z})$ with the evaluation pairing $\sigma$ (see 3.2.5). Then from [F1] pages 42-43, we know that under the decomposition (26) $d(HP, \sigma)$ is represented by the character function which sends the symplectic character $\theta_m$ to the value

$$1 \times 1 \times (-1)^{\theta_m(-1)/2}. $$

Thus we see that by the above theorem the class $\chi^*_H(HP, |G|^{-1} \sigma)$ respectively $\chi^*_H(HP, \sigma)$ in $H^*(\mathbb{Z}[G])$ is represented by the character function which maps $\theta_m$ to

$$1 \times (-1)^{\theta_m(-1)/2}, \text{ resp. } 1 \times (-|G|)^{\theta_m(-1)/2}. $$

**Proof of proposition.** To prove (a) we note that for $i > 0$, $p \mid_{H^i \oplus H^{-i}}, p \mid_{B^i \oplus U^{-i}},$ and $p \mid_{B^{-i} \oplus U^i}$ are all hyperbolic; hence

$$n_m^{-}(p^i) = n_m^{-}(\sigma^i) + \dim(B^i) + \dim(U^i)$$

and the result follows at once since

$$\dim(U^0) + \sum_{i>0} (-1)^i (\dim(B^i) + \dim(U^i)) =$$

$$\dim(U^0) + \sum_{i>0} (-1)^i (\dim(U^{-i-1}) + \dim(U^i)) = 0.$$ 

In proving (b) we shall use Deligne’s “Koszul rule of signs” in reordering wedge products and tensor products; however, we shall see that all terms used have even grade since they arise as determinants of symplectic isotypic parts of real representations. Note also that for non-zero elements $l, l'$ of a complex line $L$, we shall write $l' l^{-1}$ for the complex number such that $l' = (l' l^{-1}) l$.

For fixed $m$, for brevity we shall put

$$r_i = \wedge_{jn} b_{jn}^{im} \in \det(P_m^i).$$

We then use the decompositions

$$P_m^i = B_m^i \oplus H_m^i \oplus U_m^i$$

$$r_i = x_{B^i} \otimes x_{H^i} \otimes x_{U^i}$$

where for a vector space $V$, $x_V$ denotes an element of $\det V$. As indicated previously, we note that each of the above $x$-terms is a wedge product with even grade, and so in the
and again, after multiplying the terms $x_{H^i}$ by suitable scalars we may assume that each $x_{U^i}$ is mapped to $x_{B^i+1}$ by det$(\partial^i)$. Thus

$$\xi_m(\otimes_i r_i^{(-1)i}) = \otimes_i x_{H^i}^{(-1)i} = x_{H^0} \otimes (\otimes_{i>0} (x_{H^2i} \otimes x_{H^{-2i}})) \otimes (\otimes_{i \geq 0} (x_{H^{2i+1}} \otimes x_{H^{-2i+1}})^{-1}). \quad (37)$$

Next we recall the orthogonal decompositions of the $p^i$ given prior to the statement of the proposition and we note that by (34) we know that

$$\text{Pf}_{p|B^{-1} \otimes U^i} (x_{B^i} \otimes x_{U^i}) = \text{Pf}_{p|B^{-1} \otimes U^i} (\det(\partial) x_{U^{-i-1}} \otimes x_{U^i})$$

$$= \text{Pf}_{p|U^{-i-1} \otimes B^i+1} (x_{U^{-i-1}} \otimes \det(\partial) x_{U^i})$$

$$= \text{Pf}_{p|U^{-i-1} \otimes B^i+1} (x_{U^{-i-1}} \otimes x_{B^i+1})$$

$$= \text{Pf}_{p|B^i+1 \otimes U^{-i-1}} (x_{B^i+1} \otimes x_{U^{-i-1}}).$$

We may therefore use Lemma 3 to deduce that the product

$$\text{Pf}_{p^0 \otimes \kappa_m} (r_0) \prod_{i>0} \text{Pf}_{p^0 \otimes \kappa_m} (r_2i) \prod_{i \geq 0} \text{Pf}_{p^{2i+1} \otimes \kappa_m} (r_{2i+1})^{-1}$$

telescopes to

$$\text{Pf}_{p^0 \otimes \kappa_m} (x_{H^0}) \prod_{i>0} \text{Pf}_{p^{2i} \otimes \kappa_m} (x_{H^{2i}} \otimes x_{H^{-2i}}) \prod_{i \geq 0} \text{Pf}_{p^{2i+1} \otimes \kappa_m} (x_{H^{2i+1}} \otimes x_{H^{-2i+1}})^{-1}$$

and by (38) this is equal to

$$\text{Pf}_{p^0 \otimes \kappa_m} (\xi_m(\otimes_i r_i^{(-1)i})). \quad (38)$$

On the other hand, by Proposition 23, for each $i$

$$\text{pf} (\{G\mid T_{W_m}^q (p^i a_{ij}^{ij}, a_{0j}^{ij})\}) = \text{Pf}_{p^i \otimes \kappa_m} (r_i) \quad (39)$$

and so comparing with (39) above (b) is proved.

The proof of (c) is very similar to that of (b); we therefore only sketch the proof. Again we decompose each $x_i$ as

$$x_i = x_{B^i} \otimes x_{H^i} \otimes x_{U^i}$$

and again, after multiplying the $x_{H^i}$ by suitable scalars, we may suppose that each $x_{U^i}$ is mapped to $x_{B^i+1}$ by det$(\partial^i)$; hence, as previously,

$$\xi_m(\otimes_i (x_{B^i} \otimes x_{H^i} \otimes x_{U^i})^{(-1)i}) =$$

$$= x_0 \otimes (\otimes_{i>0} (x_{H^{2i}} \otimes x_{H^{-2i}})) \otimes (\otimes_{i \geq 0} (x_{H^{2i+1}} \otimes x_{H^{-2i+1}})^{-1}). \quad (40)$$
Furthermore by \((34)\) we know that
\[
 h(B^{-i} \oplus U^i)(x_{B^{-i}} \otimes x_{U^i}) = h(B^{-i} \oplus U^i)(\det(\partial)x_{U^{-i-1}} \otimes x_{U^i})
\]
\[
 = h(B^{i+1} \oplus U^{-i-1})(x_{U^{-i-1}} \otimes \det(\partial)x_{U^i})
\]
\[
 = h(B^{i+1} \oplus U^{-i-1})(x_{U^{-i-1}} \otimes x_{B^{i+1}}).
\]

Hence the product \(h_m^0(x_0) \prod_{i>0} h_m^i(x_i \otimes x_{-i})^{(-1)^i}\) telescopes to
\[
 h_{H^0}(x_{H^0}) \prod_{i>0} h_{H^{2i} \oplus H^{-2i}}(x_{H^{2i}} \otimes x_{H^{-2i}}) \prod_{i>0} h_{H^{2i+1} \oplus H^{-2i-1}}(x_{H^{2i+1}} \otimes x_{H^{-2i-1}})^{-1}
\]
and by \((41)\) this is equal to \(h\sigma,m(\xi_m(\otimes)x_i^{(-1)^i})\), as required. \(\Box\)

### 5.3 Proof of Theorem 30.

We again adopt the notation of the theorem, so that \(P^*\) is a perfect \(\mathbb{Z}[G]\)-complex and \(\sigma^{\text{ev}}, \sigma^{\text{odd}}\) are \(G\)-invariant non-degenerate symmetric forms on the even and odd parts of the cohomology of \(P^*\). As in \((5.2)\), after extending \(P^*\) by an acyclic perfect complex if necessary (which leaves \(\chi_H^*(P^*, \sigma)\) unchanged by Theorem 19), we obtain “lifts” \(p^i\) as in \((32)-(35)\), and hence \(\mathbb{Q}[G]\)-valued pairings
\[
 p^0 : P^0_Q \times P^0_Q \to \mathbb{Q}[G]
\]
\[
 p^i : (P^i_Q \oplus P^{-i}_Q) \times (P^i_Q \oplus P^{-i}_Q) \to \mathbb{Q}[G], \text{ for } i > 0.
\]

Using \((5)\) and Theorem 44 we obtain a decomposition
\[
 \chi_H^*(P^*, \sigma) = \phi(d(P^*, \{[G|p^i]\}))^{-1} = x_1 \times x_2 \in A^s(\mathbb{Z}[G]) \times S_\infty(\mathbb{Z}[G]).
\]

Here, with the notation of \((36)\), the class \(x_1 \times x_2\) is represented by the character map which takes the symplectic character to the value:
\[
 \prod_{p<\infty} \text{Det}(\chi_p^*(\theta_m)^{(1)^i}) \times \prod_{i \geq 0} \left| \text{pf}(T_m([G|\tilde{p}^i(a_0^{ij}, a_0^{-ij'}))]_{jj'})^{(-1)^i} \right|
\]
respectively
\[
 \text{sign} \left( \prod_{i \geq 0} \text{pf}(T_m([G|\tilde{p}^i(a_0^{ij}, a_0^{-ij'})])_{jj'})^{(-1)^i} \right).
\]

First we consider the Arakelov class \(x_1\). By linearity we may suppose that \(W_m\) is a left ideal of \(\mathbb{C}[G]\); then we denote by \(\{v_{m^s}\}\) an orthonormal basis of \(W_m\) with respect to the hermitian form \(\nu_{W_m}\), and we put
\[
 c_{js}^{im} = r_G(a_0^{ij} \otimes v_{m^s}).
\]
Then by Proposition 24 we have

\[ \prod_{i \geq 0} \left| \text{pf}(T_m(|G|\bar{p}(a_{ij}, a_{ij}'))_{j,j'})^{(-1)i} \right| = \prod_{i \geq 0} \left| \det(T_m(|G|\bar{p}'(a_{ij}, a_{ij}'))_{j,j'})^{(-1)i} \right|^{1/2} \]

\[ = \prod_{i \geq 0} h_{p^i,m} \left( \Lambda_{js} c_{js}^{i,m} \right)^{(-1)i} \]

and by Proposition 43(c) this latter real number is equal to

\[ h_{\sigma,m}(\xi_m(\otimes_i(\Lambda_{js} c_{js}^{i,m})^{(-1)i})). \]

We now consider the signature class \( x_2 \). By Proposition 28 we know that

\[ \text{sign} \left( \prod_{i \geq 0} \text{pf}(|G|T_m(\bar{p}'(a_{ij}, a_{ij}'))_{j,j'})^{(-1)i} \right) = \prod_{i \geq 0} (\sqrt{-1})^{n_{\tilde{m}}(p^i)} \]

and by Proposition 43(a) the right-hand sign is equal to \( (\sqrt{-1})^{n_{\tilde{m}}(\sigma)} \), as required. On considering the definition of \( \chi_{H}^\sigma(P^*, h_\sigma) \) we see that this then implies Theorem 30. \( \square \)

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