THE $q$-DEFORMED WIGNER OSCILLATOR IN QUANTUM MECHANICS

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Abstract

Using a super-realization of the Wigner-Heisenberg algebra a new realization of the $q$-deformed Wigner oscillator is implemented.

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Dedicated to the memory of Prof. Jambunatha Jayaraman, 28 January 1945-19 June 2003.

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1 Introduction

In 1989, independently, Biedenharn and Macfarlane [1], introduced the $q$-deformed harmonic oscillator and constructed a realization of the $SU_q(2)$ algebra, using a $q$-analogue of the harmonic oscillator and the Jordan-Schwinger mapping. The $q$-deformation of $SU(2)$, denoted by $SU_q(2)$, is one of the simplest examples of a quantum group.

The deformation of the conventional quantum mechanical laws has been implemented via different definitions and studied by several authors in the literature [2, 3, 4, 5, 6, 7, 8, 9]. Also, recently Palev et al. have investigated the 3D Wigner oscillator [9].

The main purpose of this work is to set up a realization of the $q$-deformed Wigner oscillator [2].

2 The $q$-deformed usual harmonic oscillator

In this section, we consider the $q$-deformed ladder operators of the harmonic oscillator, $a^-$ and its adjoint $a^+$, acting on the basis $|n\rangle$, $n = 0, 1, 2, \ldots$, as [1] $a^-|0\rangle = 0$, $|n\rangle = \frac{(a^+_q)^n}{(|n|)!}|0\rangle$ where $|n|! = |n|(|n-1|! \cdot \ldots \cdot |1|!$. The classical limit $q \rightarrow 1$ yields to the conventional ladder boson operators $a^\pm$, which satisfies $[a^-, a^+] = 1$, $a^-|n\rangle = \sqrt{n}|n-1\rangle$, $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$.

On the other hand, $su(1,1)$ algebra satisfies the following commutation relations $[K_0, K_{\pm}] = \pm K_{\pm}$, $[K_+, K_-] = -2K_0$ and the Casimir operator is given as $C = K_0(K_0 - 1) - K_+K_-$. The classical limit $q \rightarrow 1$ yields to the conventional ladder boson operators $a^\pm$, which satisfies $[a^-, a^+] = 1$, $a^-|n\rangle = \sqrt{n}|n-1\rangle$, $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$.

In Ref. [4] was found a realization of the $su_{q^2}(1,1)$ in terms of the generators of $su(1,1)$. The $q$-deformed ladder operators satisfy

\[ [\tilde{K}_0, \tilde{K}_{\pm}] = \pm \tilde{K}_{\pm}, \quad [\tilde{K}_+^, \tilde{K}_-] = -[2\tilde{K}_0]_{q^2}, \quad [x]_\mu = (\mu^x - \mu^{-x})/(\mu - \mu^{-1}). \]
\[ a_q^- a_q^+ = [N + 1], \quad a_q^+ a_q^- = [N], \quad (2) \]

where \( N \) is the number operator which is positive semi-definite. The \( q^- \)-analogue operators can be found in terms of the usual ladder boson operators \( a^- \) and \( a^+ \).

Note that we can write
\[
| n \rangle = \frac{a_q^+ (a_q^+)^{n-1}}{\sqrt{n!} (\langle n-1 \rangle)!} \quad | 0 \rangle = \frac{a_q^+}{\sqrt{n!}} \quad | n-1 \rangle \hspace{1cm} (3)
\]

Also, from (2) and \( a_q^- a_q^+ | n \rangle = [n + 1]^{\frac{1}{2}} a_q^- | n + 1 \rangle \), we get
\[
a_q^- | n \rangle = [n]^{\frac{1}{2}} | n-1 \rangle \quad (4)
\]

It's easy to verify that \([N, a_q^+] = a_q^+, \quad [N, a_q^-] = -a_q^-, \quad [N, q^N] = [a_q^- a_q^+, q^N] = 0, \quad a_q^- a_q^+ - qa_q^+ a_q^- = q^{-N}. We will show that a structure of this type exists for the Wigner oscillator.

## 3 The q-deformed Wigner Oscillator

The one-dimensional Wigner super-oscillator Hamiltonian in terms of the Pauli’s matrices (\( \sigma_i \), i=1,2,3) is given by

\[
H(\lambda + 1) = \begin{pmatrix} H_-(\lambda) & 0 \\ 0 & H_+(\lambda) \end{pmatrix}, \quad H_-(\lambda) = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2} \lambda (\lambda + 1) \right\}, \quad (5)
\]

where \( H_+(\lambda) = H_-(\lambda + 1) \). The even sector \( H_-(\lambda) \) is the Hamiltonian of the oscillator with barrier or isotonic oscillator or Calogero interaction.

Thus, from the super-realized Wigner oscillator, its first order ladder operators given by \([2] a^\pm(\lambda + 1) = \frac{1}{\sqrt{2}} \left\{ \pm \frac{d}{dx} \pm \frac{(\lambda+1)}{x} \sigma_3 - x \right\} \sigma_1 \), the Wigner Hamiltonian and the Wigner-Heisenberg(WH) algebra ladder relations are readily obtained as

\[
H(\lambda+1) = \frac{1}{2} \left[ a^+(\lambda + 1), a^-(\lambda + 1) \right]_+, \quad \left[ H(\lambda + 1), a^\pm(\lambda + 1) \right]_- = \pm a^\pm(\lambda+1). \quad (6)
\]
Equations (6) and the commutation relation

\[
[a^-(\lambda + 1), a^+(\lambda + 1)]_\pm = 1 + 2(\lambda + 1) \sigma_3 \tag{7}
\]

constitutes the WH algebra [2] or deformed Heisenberg algebra [5, 7].

Let us consider an extension of the \(q\)-deformed harmonic oscillator commutation relation,

\[
a^- W a^+ W - qa^+ W a^- W = q^{-N}(1 + c \sigma_3), \quad c = 2(\lambda + 1) \tag{8}
\]
as a \(q\)-deformation of the Wigner oscillator commutation realization. These operators may be written in terms of the Wigner oscillator ladder operators, viz.,

\[
a^- W = \beta(N)a^- (\lambda + 1), \quad a^+ W = a^+(\lambda + 1)\beta(N), \quad N = a^+(\lambda + 1)a^-(\lambda + 1). \tag{9}
\]

Acting the ladder operators of the WH algebra in the Fock space, spanned by the vectors

\[
a^- (\lambda + 1)|2m >_c = \sqrt{2m}|2m - 1 >_c, \quad a^- (\lambda + 1)|2m + 1 >_c = \sqrt{2m + c + 1}|2m >_c, \quad a^+(\lambda + 1)|2m >_c = \sqrt{2m + c + 1}|2m + 1 >_c, \quad a^+(\lambda + 1)|2m + 1 >_c = \sqrt{2(m + 1)}|2m + 2 >_c,
\]
we obtain a recursion relation given by

\[
(2m + 2 - c)\beta^2(2m + 1) - q(2m + 1 + c)\beta^2(2m) = q^{-(2m+1)}(1 - c). \tag{10}
\]

This has, for the odd quanta and \(c = 0\), the following solution

\[
\beta(2m + 1) = \sqrt{\frac{1}{2m + 2} \frac{q^{2m+2} - q^{-(2m+2)}}{q - q^{-1}}} \Rightarrow \beta(N) = \sqrt{\frac{N + 1}{N + 1}}. \tag{11}
\]

Thus, the \(q\)-deformed Wigner Hamiltonian and the commutator \([a^- W, a^+ W]\), for \(c = 0\) become the \(q\)-deformed harmonic oscillator

\[
H_W = \frac{1}{2}[a^- W, a^+ W]_+ = H_b = \frac{1}{2}([N+1]+[N]), \quad [a^- W, a^+ W] = [N+1]-[N]. \tag{12}
\]

Also, from even quanta this same result is readily found for \(c = 0\).
4 Conclusion

In this work, we firstly presented a brief review on the $q$-deformation of the conventional quantum mechanical laws for the unidimensional harmonic oscillator. We have also implemented a new approach for the WH algebra. Indeed, the $q$-deformations of WH algebra are investigated via the super-realization introduced by Jayaraman-Rodrigues [2].

Also, we do not assume the relations of operators $a^+_W a^-_W$ and $a^-_W a^+_W$. They are derived from our defining set of relations $a^-_W = a^-_q = \sqrt{\frac{N+1}{N+1}} a^-$ and $a^+_W = a^+_q = a^+ \sqrt{\frac{N+1}{N+1}}$, for vanish Wigner parameter ($c = 0$) given by Eq. (2). The case with $c \neq 0$ will be presented in a forthcoming paper.

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