Derived bracket construction and anti-cyclic subcomplex of Leibniz (co)homology complex

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Abstract

An arbitrary Leibniz algebra can be embedded in a differential graded Lie algebra via the derived bracket construction. Such an embedding is called a derived bracket representation. We will construct the universal version of the derived bracket representation, prove that the principal part of the target dg Lie algebra defines a subcomplex of Leibniz (co)homology complex and that the existence of the subcomplex is a reflection of the anti-cyclicity of the Leibniz operad.

1 Introduction

Leibniz algebras are vector spaces equipped with binary bracket products satisfying the Leibniz identity. The notion of Leibniz algebra was introduced by Jean-Louis Loday, motivated by the study of algebraic K-theory. Hence the Leibniz algebras are sometimes called the Loday algebras. Today it is widely known that Leibniz algebras arise in many areas of mathematics not only K-theory. To study Leibniz algebras geometrically the derived bracket construction of Kosmann-Schwarzbach [10] is an effectual method. If $(\mathfrak{h}, D)$ is a differential graded Lie algebra (shortly, dg Lie algebra), then the derived bracket,

$$[x, y] := (Dx, y),$$

is an odd Leibniz bracket on $\mathfrak{h}$, where $(.,.)$ is the Lie bracket on $\mathfrak{h}$ and $x, y \in \mathfrak{h}$. The derived bracketing is a method of constructing a Leibniz algebra and one can think that the deriving differential $D$ has almost every information about the Leibniz bracket. It is known that the converse also holds. An arbitrary Leibniz algebra can be embedded in a dg Lie algebra via the derived bracket construction. Namely, given a Leibniz algebra $\mathfrak{g}$, there exists a dg Lie algebra $(\mathfrak{h}, D)$ which includes $\mathfrak{g}$ (precisely $\mathfrak{g}[1]$) and the Leibniz bracket on $\mathfrak{g}$ or $\mathfrak{g}[1]$ is expressed as the derived bracket.
on $\mathfrak{h}$. Such an embedding is considered to be a kind of representation of Leibniz algebra. So we call this type of representation a *derived bracket representation*. To find a useful representation is an interesting problem. For example, the symplectic realization for Courant algebroids is a derived bracket representation (for the details see Roytenberg [17]).

The first aim of this note is to construct the universal derived bracket representation. The derived bracket construction is regarded as a functor, which is denoted by $\mathcal{D}\mathcal{C}$, from the category of dg Lie algebras to the one of odd-Leibniz algebras. The universal representation is defined as the adjoint functor of the derived bracket construction.

$$\text{Hom}_{\text{dgLie}}(\mathcal{D}\mathcal{R}\mathfrak{g}[1], \mathfrak{h}) \cong \text{Hom}_{\text{Leib}}(\mathfrak{g}[1], \mathcal{D}\mathcal{C}\mathfrak{h}),$$

where $\mathcal{D}\mathcal{R}(\cdot)$ is the functor of the universal representation. The second aim is to show that the principal part of the dg Lie algebra $\mathcal{D}\mathcal{R}\mathfrak{g}[1]$ is a subcomplex of Loday’s complex over $\mathfrak{g}$ (the chain complex computing the Leibniz homology group). Since the Loday complex is a consequence of the bar construction, the relation between the derived bracket construction and the bar construction becomes clear. The existence of the subcomplex is closely related with the anti-cyclicity of the Leibniz operad. In [2] Chapoton proved that the Leibniz operad is anti-cyclic (See Section 2.2 below for the details). In general, if an operad is cyclic (not anti-cyclic), then the (co)homology complex of the operad-algebra can be reduced by its symmetry and the cyclic (co)homology group is defined (cf. Getzler-Kapranov [4]). Although the Leibniz operad is not cyclic, because it is still anti-cyclic, there exists a subcomplex or quotient complex. Our subcomplex is exactly that.

Our method of constructing the (sub)complex is not bar-construction, but the derived bracket construction or representation. An advantage of adopting the derived bracket theory, it is not necessary to use the Koszul duality theory. To compute the Koszul dual of Leibniz operad (Zinbiel operad) is not easy by comparison with the cases of the associative operad and the Lie operad. By using the derived bracket theory, one can avoid this problem.

The paper is organized as follows:

Section 2 is Preliminaries. We recall some basic properties of Leibniz algebras and derived bracket construction (of operadic).

In Section 3, we will construct the derived bracket representation of universal. To construct the universal representation an operad theory will be used. We will see that if $\mathfrak{g}$ is a Lie algebra as a commutative Leibniz algebra, then the second homology group of $\mathcal{D}\mathcal{R}\mathfrak{g}[1]$ is equal to the space of formal 0-forms, $\Omega^0\mathfrak{g}$, introduced by
Kontsevich [8]. According to Kontsevich, $\Omega^0 g$ is the target space of the universal invariant bilinear form. Invariant bilinear forms in the category of Lie algebras are symmetric pairings satisfying the well-known condition,

$$\langle x, [y, z] \rangle = \langle [x, y], z \rangle.$$ 

It is well-known that invariant bilinear forms are induced via the derived bracket construction (in the case of Drinfeld double by Kosmann-Schwarzbach [9] and in general case by Roytenberg [17]). Our result provides the universal version of the previous studies.

In Section 4.1, we will prove that the principal part of $\mathcal{DR} g[1]$ is a subcomplex of the Loday complex over $g$ and the deriving differential on $\mathcal{DR} g[1]$ is equal to the boundary map of Loday. In 4.2, we study the cohomology counter part of $\mathcal{DR} g[1]$. We will introduce the notion of anti-cyclic cochain for Leibniz algebras. The anti-cyclic cochains are defined as the linear functions satisfying a symmetry induced from the anti-cyclicity of Leibniz operad, like the cyclic cochains for associative algebras satisfy $\varphi(a_0, ..., a_n) = (-1)^n \varphi(a_n, a_0, ..., a_{n-1})$ (cf. Connes [3].) The symmetry that the anti-cyclic cochain satisfies is more complicated, for instance, $A(x_0, x_1, x_2)$ is an anti-cyclic 2-cochain if and only if

$$A(x_0, x_1, x_2) = A(x_0, x_2, x_1),$$

$$A(x_0, x_1, x_2) + A(x_2, x_0, x_1) + A(x_1, x_2, x_0) = 0.$$ 

We will prove that the set of anti-cyclic cochains is a subcomplex of Loday-Phirashviri complex over $g$ and that the coboundary map of Loday-Phirashviri is on the subcomplex the dual of the deriving differential on $\mathcal{DR} g[1]$.

In Section 5, we will study a tensor expression of the anti-cyclic cochains.

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2 Preliminaries

2.1 Leibniz algebras

(Left-)Leibniz algebras are by definition vector spaces $g$ equipped with binary brackets $[.,.]$ satisfying the (left-)Leibniz identity,

$$[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_2, [x_1, x_3]],$$

$$[x_1, x_2] + [x_2, x_1] = 0.$$
where \( x_1, x_2, x_3 \in g \). In the following, we usually suppose that the degree of the space \( g \) is homogeneously zero. Let \( g[1] \) be the shifted space of \( g \), where the degree of any element \( x \in g[1] \) is \(-1\). The space \( g[1] \) becomes an odd Leibniz algebra and the degree of the bracket is \(+1\).

We recall a basic property of Leibniz algebra, which will be used in the next section. Let \( h \) be a Leibniz algebra and let \( I = I_g \) be the space consisting of the symmetric brackets,
\[
I := \{ [x_1, x_2] + [x_2, x_1], x_1, x_2 \in g \}.
\]
Then \( I \) becomes an ideal of \( g \), in particular, \([I, g] = 0\). The quotient space \( g_{\text{Lie}} := g/I \) becomes a Lie algebra and this projection \( p : g \rightarrow g_{\text{Lie}} \) (so-called Liezation) is universal, i.e., for any Lie algebra \( h \), an arbitrary Leibniz algebra morphism \( f : g \rightarrow h \) factors through \( p, f = \psi \circ p \), where \( \psi : g_{\text{Lie}} \rightarrow h \) is the Lie homomorphism corresponding to \( f \).

### 2.2 Anti-invariant 2-forms for Leibniz algebras

In this section we suppose that \( g \) is a finite dimensional Leibniz algebra. In [2] Chapoton proved that the operad of Leibniz algebras is anti-cyclic. This means that invariant 2-forms, \( \omega(.,.) \), in the category of Leibniz algebras are anti-symmetric, i.e., \( \omega(x_1, x_2) = -\omega(x_2, x_1) \) and the invariant condition for \( \omega \) is defined by the following two formulas.

\[
\begin{align*}
\omega(x_1, [x_2, x_3]) &= -\omega([x_2, x_1], x_3), \\
\omega(x_1, [x_2, x_3]) &= \omega([x_1, x_3] + [x_3, x_1], x_2).
\end{align*}
\]

Chapoton’s original formula has been defined in the right-version and we will use the left-version above. If the 2-form is non-degenerate, then it is a symplectic structure. Hence we denote it by \( \omega \). Thanks to (1)-(2), one can immediately write-down the coadjoint action in the category of Leibniz algebras.

**Definition 2.1** (coadjoint action). Given a finite dimensional Leibniz algebra \( g \) and its dual space \( g^* \), the coadjoint action of \( g \) to \( g^* \) is by definition,
\[
\begin{align*}
\langle x_1, [x_2, a] \rangle &= -\langle [x_2, x_1], a \rangle, \\
\langle x_1, [a, x_2] \rangle &= \langle [x_1, x_2] + [x_2, x_1], a \rangle,
\end{align*}
\]
where \( x_1, x_2 \in g \), \( a \in g^* \) and \( \langle ., . \rangle \) the natural pairing between \( g \) and \( g^* \).
It is easy to see that the representation (3)-(4) is Leibniz, i.e.,
\[ [x_1, [x_2, a]] = [[x_1, x_2], a] + [x_2, [x_1, a]], \]
\[ [x_1, [a, x_2]] = [[x_1, a], x_2] + [a, [x_1, x_2]], \]
\[ [a, [x_1, x_2]] = [[a, x_1], x_2] + [x_1, [a, x_2]]. \]

Therefore, the semi-direct product, \( g \ltimes g^* = g \oplus g^* \), becomes a Leibniz algebra, whose bracket is
\[ [x_1 \oplus a_1, x_2 \oplus a_2] := [x_1, x_2] \oplus [x_1, a_2] + [a_1, x_2]. \]

The double space has a canonical symplectic structure,
\[ \omega(x_1 + a_1, x_2 + a_2) := \langle x_1, a_2 \rangle - \langle x_2, a_1 \rangle. \] (5)

The pair \( (g \ltimes g^*, \omega) \) satisfies (1)-(2).

### 2.3 Derived bracket construction of operadic

Let \( (\mathfrak{h}, (., .), d) \) be a dg Lie-algebra. We suppose that the degree of the Lie bracket is \( |(., .)| = 0 \) and the one of the differential is \( |d| := +1 \). Define an odd bracket by
\[ [x_1, x_2] := (dx_1, x_2), \] (6)
which is called a binary derived bracket or derived bracket for short ([10]). Then \( (\mathfrak{h}, [., .] = (d., .)) \) becomes an odd-Leibniz algebra.

**Remark 2.2 (sign).** Although in [10] the derived bracket has been defined as \((-1)^{|x_1|+1}(dx_1, x_2)\), because (6) has a good compatibility with operad theory, we will use (6) as a definition of the derived bracket.

**Definition 2.3 ([18]).** Lie-Leibniz algebras are by definition graded spaces with even-Lie brackets \( (., .) \) and odd-Leibniz brackets \([., .]\) satisfying two extra identities,
\[ [x_1, (x_2, x_3)] \] = \[ ([x_1, x_2], x_3) + (x_2, [x_1, x_3]) \] (7)
\[ [(x_1, x_2), x_3] \] = \[ ([x_1, x_2] - [x_2, x_1], x_3), \] (8)
where we put \( |x_1| = |x_2| = |x_3| := 0 \) simply.

The derived bracket Leibniz algebra \( (\mathfrak{h}, (., .), [., .] = (d., .)) \) is the model of the Lie-Leibniz algebra. When \( |x_1| = |x_2| = |x_3| := 1 \), since the Lie bracket is commutative, the second condition (8) has the following form
\[ [(x_1, x_2), x_3] = ([x_1, x_2] + [x_2, x_1], x_3). \]

In the following, we put \( |(., .)| := 0 \) and \([., .]| := 1. \)
Remark 2.4 (Jacobi identity and derived bracket construction). Many identities that the derived bracket satisfies are consequences of the Jacobi identity of the original Lie bracket. However (8) is completely independent from the Jacobi identity. It is a consequence of the derivation rule \( d(x_1, x_2) = (dx_1, x_2) + (x_1, dx_2) \).

We recall three known propositions, which will be used in the next section to prove the key-lemma of this note. Before that, we briefly recall algebraic operads. A collection \( \mathcal{P} = (\mathcal{P}(n)) \) consisting of modules \( \mathcal{P}(n) \) over the symmetric group \( S_n \) is called an \( S \)-module. Given an \( S \)-module \( \mathcal{P} \), a functor, so-called Schur functor, is defined by

\[
F_{\mathcal{P}} V := \bigoplus_{n \in \mathbb{N}} \mathcal{P}(n) \otimes_{S_n} V^\otimes n,
\]

where \( V \) is a vector space. Given two \( S \)-modules \( \mathcal{P} \) and \( \mathcal{Q} \), a tensor product \( \mathcal{P} \circ \mathcal{Q} \) is defined by

\[
(\mathcal{P} \circ \mathcal{Q})(n) := \bigoplus_{l_1 + \cdots + l_m = n} \mathcal{P}(m) \otimes_{S_m} (\mathcal{Q}(l_1), \ldots, \mathcal{Q}(l_m)) \otimes_{(S_{l_1}, \ldots, S_{l_m})} S_n.
\]

The \( S \)-module \( \mathcal{P} \) is called an operad, if \( F_{\mathcal{P}} \) is a triple (cf. MacLane [15]), namely, if there exists a morphism (natural transformation), \( F_{\mathcal{P}} F_{\mathcal{P}} \rightarrow F_{\mathcal{P} \circ \mathcal{Q}} \), more explicitly,

\[
(\mathcal{P} \circ \mathcal{Q})(n) := \bigoplus_{l_1 + \cdots + l_m = n} \mathcal{P}(m) \otimes_{S_m} (\mathcal{Q}(l_1), \ldots, \mathcal{Q}(l_m)) \otimes_{(S_{l_1}, \ldots, S_{l_m})} S_n.
\]

The \( S \)-module \( \mathcal{P} \) is called an operad, if \( F_{\mathcal{P}} \) is a triple (cf. MacLane [15]), namely, if there exists a morphism (natural transformation), \( F_{\mathcal{P}} F_{\mathcal{P}} \rightarrow F_{\mathcal{P} \circ \mathcal{Q}} \), and if by this operation \( F_{\mathcal{P}} \) becomes a unital associative monoid. When \( \mathcal{P} \) is an operad, the notion of \( \mathcal{P} \)-algebra is defined. If \( A \) is a \( \mathcal{P} \)-algebra, then the \( \mathcal{P} \)-algebra product on \( A \) is defined as a map of

\[
\gamma : F_{\mathcal{P}} A \rightarrow A.
\]

In particular, \( A = F_{\mathcal{P}} V \) is the free \( \mathcal{P} \)-algebra. The free operad over an \( S \)-module \( Q \), \( T Q \), is the free algebra in the category of operads and an algebraic operad \( \mathcal{P} \) is expressed as a quotient operad of the free operad \( \mathcal{P} := T Q/(R) \), where \( R \) is the relation of \( \mathcal{P} \) and \( (R) \) is the generated operadic ideal. If the free operad \( T Q \) is generated by \( Q(2) \) and if \( R \) is a sub \( S_3 \)-module of \( (T Q)(3) \), then the quotient operad \( \mathcal{P} := T Q/(R) \) is called a binary quadratic operad and \( R \) is called the quadratic relation. For example, the Lie operad (the operad of Lie algebras) is a binary quadratic operad over \( sgn_2 \),

\[
\mathcal{L}ie := T(sgn_2)/(R_{\mathcal{L}ie}),
\]

where \( sgn_2 \) is the sign representation of \( S_2 \) and \( R_{\mathcal{L}ie} \) is the Jacobi identity. As a result, \( \mathcal{L}ie(2) = sgn_2 \) and the base of \( \mathcal{L}ie(2) \) is identified with the universal Lie bracket, which is denoted by \((1, 2)\). Then \( R_{\mathcal{L}ie} \) is identified with the space generated by the Jacobiator,

\[
R_{\mathcal{L}ie} = < (1, (2, 3)) + (3, (1, 2)) + (2, (3, 1)) >.
\]
The Leibniz operad, $\text{Leib}$, is also defined by the same manner. The parity shift for operad, $\mathcal{P} \mapsto s\mathcal{P}$, is defined by $(s\mathcal{P})(n) := \mathcal{P}(n - 1) \otimes \text{sgn}_n$ for each $n$. If $|\mathcal{P}| = \text{even}$, then $s\mathcal{P}$-algebras are odd-$\mathcal{P}$-algebras.

Let us denote by $\mathcal{LL}$ the operad of Lie-Leibniz algebras, which is a binary quadratic operad

$$\mathcal{LL} := T(\text{Lie}(2) \oplus s\text{Leib}(2)) / (R_{\mathcal{LL}}),$$

where $R_{\mathcal{LL}}$ is the quadratic relation of $\mathcal{LL}$. It is obvious that $\mathcal{LL} = (\mathcal{LL}^i)$ is a graded operad, in particular, $\mathcal{LL}^0 = \text{Lie}$ and $\mathcal{LL}^\text{top} = s\text{Leib}$.

**Proposition 2.5 ([18]).** $\mathcal{LL} = \text{Lie} \otimes \mathcal{D} := (\text{Lie}(n) \otimes \mathcal{D}(n))$, where $\mathcal{D}$ is a graded operad defined as follows.

To introduce $\mathcal{D}$ we use the following expression of $\text{Com}$ (the operad of commutative associative algebras).

$$\text{Com}(2) = < 1 \otimes 1 >$$
$$\text{Com}(3) = < 1 \otimes 1 \otimes 1 >$$
$$\cdots = \cdots$$
$$\text{Com}(n) = < 1 \otimes \cdots \otimes 1 > .$$

Let $d$ be a formal 1-ary operator of degree +1 and $1 \otimes 1$ be the generator of $\text{Com}$. We consider a quadratic operad over $d$ and $1 \otimes 1$.

$$\mathcal{O} := T(d, 1 \otimes 1) / (R_{\mathcal{O}}),$$

where $R_{\mathcal{O}}$ is a quadratic relation generated by

$$(1 \otimes 1) \circ_1 (1 \otimes 1) = 1 \otimes 1 \otimes 1 = (1 \otimes 1) \circ_2 (1 \otimes 1),$$
$$d(1 \otimes 1) = d \otimes 1 + 1 \otimes d,$$
$$dd = 0.$$

Namely, $d$ is a differential in $\text{Com}$. Obviously, $\mathcal{O}$ is a graded operad, $\mathcal{O} = \{\mathcal{O}^i\}$, whose degree is the number of $d$. For each $n$,

$$\mathcal{O}(n) = \mathcal{O}^0(n) \oplus \mathcal{O}^1(n) \oplus \cdots \oplus \mathcal{O}^n(n).$$

and $\mathcal{O}^0 = \text{Com}$. There is no $\mathcal{O}^{n+1}(n)$ because $dd = 0$. The operad $\mathcal{D}$ is defined as a suboperad of $\mathcal{O}$:

**Definition 2.6 ([18]).** For each $n$,

$$\mathcal{D}(n) := \mathcal{O}^0(n) \oplus \mathcal{O}^1(n) \oplus \cdots \oplus \mathcal{O}^{n-1}(n).$$
This operad is expressed as follows,

\[\begin{align*}
\mathcal{D}^1(2) &= \langle d \otimes 1, 1 \otimes d \rangle, \\
\mathcal{D}^1(3) &= \langle d \otimes 1 \otimes 1, 1 \otimes d \otimes 1, 1 \otimes 1 \otimes d \rangle, \\
\mathcal{D}^2(3) &= \langle d \otimes d \otimes 1, d \otimes 1 \otimes d, 1 \otimes d \otimes d \rangle, \\
&\ldots = \ldots .
\end{align*}\]

The elements in \(\mathcal{LL} = \text{Lie} \otimes \mathcal{D}\) are identified with formal derived brackets, for instance,

\[(1, 2) \otimes (d \otimes 1) = (d1, 2),\]

\[(1, (2, 3)) \otimes (1 \otimes d \otimes 1) = (1, (d2, 3)),\]

where \((1, 2)\) is the Lie bracket in \(\text{Lie}(2)\). Hence the functor \((-) \otimes \mathcal{D}\) is considered to be a derived bracket construction of operadic.

**Proposition 2.7 ([18])**. For each \(n\), \((\mathcal{LL}(n), \delta)\) is complex.

\[\begin{align*}
\text{Lie}(n) &= \mathcal{LL}^0(n) \xrightarrow{\delta} \mathcal{LL}^1(n) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{LL}^{n-1}(n) = \text{sLeib}(n).
\end{align*}\]

The differential of the proposition is defined as follows. For each \(n\), \((\mathcal{D}(n), d)\) is clearly a complex, whose differential is defined by

\[d(x_1 \otimes \cdots \otimes x_n) := \sum_{1 \leq i \leq n} (-1)^{\left|x_1\right|+\cdots+\left|x_{i-1}\right|} (x_1 \otimes \cdots \otimes x_{i-1} \otimes dx_i \otimes \cdots \otimes x_n),\]

where \(x_j \in \{1, d\}\). Since \(\mathcal{LL} = \text{Lie} \otimes \mathcal{D}\), we obtain a differential on \(\mathcal{LL}\),

\[\delta := \text{Lie} \otimes d.\]

We should remark that \(\mathcal{LL}\) is not dg-operad.

**Proposition 2.8 ([18])**. The Lie-Leibniz identity (7)-(8) is a distributive law in the sense of Markl [16]. Therefore, the operad \(\mathcal{LL}\) is decomposed into \(\text{Lie}\) and \(\text{sLeib}\).

\[\mathcal{LL} = \text{Lie} \otimes \text{sLeib},\]

where \(\otimes\) is the tensor product in the category of \(S\)-modules, cf., (9).

As a corollary of this proposition, we have

**Corollary 2.9 ([18])**. Let \(\mathfrak{g}\) be a Leibniz algebra and let \(\mathfrak{g}[1]\) the shifted odd-Leibniz algebra. Then the free-Lie algebra over \(\mathfrak{g}[1]\), \(F_{\text{Lie}}\mathfrak{g}[1]\), is a Lie-Leibniz algebra. When \(\mathfrak{g}\) is free, the Lie-Leibniz algebra is also free.

We will study the Lie-Leibniz algebra \(F_{\text{Lie}}\mathfrak{g}[1]\) in the next section.
3 Derived bracket representation

It is known that an arbitrary Leibniz algebra can be embedded in a dg Lie algebra, via the derived bracket construction. There are some methods of proving this proposition. For instance, by extending the result in Grabowski et al [6], by the universal method that we will introduce in the following.

**Definition 3.1.** Let $\mathfrak{g}$ be a Leibniz algebra, let $\mathfrak{g}[1]$ be the shifted odd-Leibniz algebra and let $(\mathfrak{h}, (\cdot, \cdot))$ the derived bracket Leibniz algebra. A momorphism of Leibniz algebra (not necessarily embedding), $\mathfrak{g}[1] \to \mathfrak{h}$, is called a derived bracket representation.

The aim of this section is to construct the universal representation.

**Lemma 3.2.** If $\mathfrak{g}$ is a Leibniz algebra, then the graded space $F_{\text{Lie}}\mathfrak{g}[1]$ is a chain complex,

$$
\cdots \xrightarrow{d} F_{\text{Lie}}^3\mathfrak{g}[1] \xrightarrow{d} F_{\text{Lie}}^2\mathfrak{g}[1] \xrightarrow{d} \mathfrak{g}[1] \xrightarrow{0} 0.
$$

**Proof.** To define $d$ we use Propositions 2.7 and 2.8. The differential is defined as a composition map of $\delta$ and $\gamma$, where $\delta$ is the differential defined in (11) and $\gamma$ is the Leibniz product in (10).

$$
d : F_{\text{Lie}}^n\mathfrak{g}[1] = \text{Lie}(n) \otimes S_n \mathfrak{g}[1]^{\otimes n} \xrightarrow{\delta \otimes 1} \mathcal{L}\mathcal{L}^1(n) \otimes S_n \mathfrak{g}[1]^{\otimes n} = \text{Lie}(n-1) \otimes S_{\text{Leib}}(2) \otimes S_n \mathfrak{g}[1]^{\otimes n-1} = F_{\text{Lie}}^{n-1}\mathfrak{g}[1].
$$

In above sequence, the part of $\gamma$ is more precisely expressed as follows.

$$
\mathcal{L}\mathcal{L}^1(n) \otimes S_n \mathfrak{g}[1]^{\otimes n} = \bigoplus_{i+j+1=n-1} \text{Lie}(n-1) \otimes S_{n-1} (1^{\otimes i} \otimes S_{\text{Leib}}(2) \otimes 1^{\otimes j}) \otimes S_n \mathfrak{g}[1]^{\otimes n},
$$

and

$$
\text{Lie} \otimes \gamma = \bigoplus_{i+j+1=n-1} \text{Lie}(n-1) \otimes 1^{\otimes i} \otimes \gamma \otimes 1^{\otimes j}.
$$

From the odd Leibniz identity on $\mathfrak{g}[1]$, $\text{dd} = 0$ holds. $\square$
More explicitly $d$ is computed as follows. Denote the right normalized bracket $(x_1, (x_2, (x_3, ..., x_n)))$ by simply $\{x_1, ..., x_n\}$.

$$d\{x_1, ..., x_{n+1}\} = \sum_{\substack{i<j \\ i \leq n}} (-1)^{i-1}\{x_1, ..., x_i^\vee, ..., [x_i, x_j], x_{j+1}, ..., x_{n+1}\} + (-1)^{n-1}\{x_1, ..., x_{n-1}, [x_{n+1}, x_n]\}, \quad (12)$$

or equivalently,

$$= \sum_{\substack{i<j \\ i \leq n-1}} (-1)^{i-1}\{x_1, ..., x_i^\vee, ..., [x_i, x_j], x_{j+1}, ..., x_{n+1}\} + (-1)^{n-1}\{x_1, ..., x_{n-1}, [x_n, x_{n+1}] + [x_{n+1}, x_n]\}. \quad (13)$$

For example,

$$d(x_1, x_2) = [x_1, x_2] + [x_2, x_1], \quad (14)$$
$$d(x_1, (x_2, x_3)) = ([x_1, x_2], x_3) + (x_2, [x_1, x_3]) - (x_1, [x_2, x_3] + [x_3, x_2]). \quad (15)$$

We should remark that $(F_{\text{Lie}}g[1], d)$ is not dg Lie algebra. Although $d$ is not derivation, for any $\alpha_1, \alpha_2 \in F_{\text{Lie}}^2g_{\text{Lie}}[1]$, it still satisfies the rule of derivation,

$$d(\alpha_1, \alpha_2) = (d\alpha_1, \alpha_2) + (-1)^{|\alpha_1|}(\alpha_1, d\alpha_2) = [\alpha_1, \alpha_2] - (-1)^{|\alpha_1||\alpha_2|}[\alpha_2, \alpha_1],$$

where $[\alpha_1, \alpha_2]$ is the Lie-Leibniz bracket (recall Corollary 2.9). Hence one can think that $d$ is an almost derivation on the free Lie algebra. To define $d$ the Leibniz identity, or the third component of the Leibniz operad $\mathcal{Leib}(3)$, was not used. Therefore

**Corollary 3.3.** For any binary product on $g$, although in general $dd \neq 0$, a map $d$ is well-defined by the same manner as above.

**Proof.** Even if $g$ is not Leibniz algebra, if it has a binary product, then $s\mathcal{Leib}(2)$ acts on $g[1]^\otimes 2$. Hence by the same manner as above a map $d$ is well-defined. \qed

From (14), $H_1(F_{\text{Lie}}g[1], d) = g_{\text{Lie}}$. So we consider the chain complex

$$\mathcal{D}Rg[1] := \left( F_{\text{Lie}}^*g[1] \xrightarrow{d} g_{\text{Lie}} \xrightarrow{0} 0 \right),$$

where $g[1] \xrightarrow{d} g_{\text{Lie}}$ is the augmentation.
Theorem 3.4. The total space $DRg[1] = g_{Lie} \oplus F_{Lie}g[1]$ becomes a dg-Lie algebra and the Leibniz bracket on the Lie-Leibniz algebra $F_{Lie}g[1]$ is the derived bracket. Therefore the inclusion,

$$\iota : g[1] \to g_{Lie} \oplus F_{Lie}g[1],$$

is a derived bracket representation. This representation is universal, namely, an arbitrary derived bracket representation of $g$ is factors through $\iota$.

The derived bracket construction is the functor, $DC$, from the category of dg-Lie algebras to the one of odd Leibniz algebras. The theorem says that $DR$ is the adjoint functor of $DC$.

Proof. We denote by $\bar{x} := p(x)$, where $p : g \to g_{Lie}$ is the Liezation.

(A) The Lie algebra $g_{Lie}$ acts on $g[1]$ as follows.

$$(\bar{x}_1, x_2) := [x_1, x_2], \quad (16)$$

which is a representation of the Lie algebra. Since $(\bar{x}_1, -)$ is a linear map on $g[1]$, this operation can be extended on $F_{Lie}g[1]$ as a derivation on the free Lie algebra. Hence the semi-direct product plus the free Lie bracket,

$$(\bar{x}_1 \oplus \alpha_1, \bar{x}_2 \oplus \alpha_2) := (\bar{x}_1, \bar{x}_2) \oplus (\bar{x}_1, \alpha_2) - (\bar{x}_2, \alpha_1) + (\alpha_1, \alpha_2),$$

is a Lie bracket on $g_{Lie} \oplus F_{Lie}g[1]$. One can easily see that $g_{Lie} \oplus F_{Lie}g[1]$ becomes a dg Lie algebra. From (12) and (16), we notice that

Claim 3.5. The differential on $g_{Lie} \oplus F_{Lie}g[1]$ is generated from $d : g[1] \to g_{Lie}$.

(B) On the universality. Let $(\mathfrak{h}, (.,.), D)$ be a dg Lie algebra and let $f : g[1] \to (\mathfrak{h}, (.,.))$ a derived bracket representation of $g$. We should prove that an arbitrary dg Lie algebra mapping

$$\psi : g_{Lie} \oplus F_{Lie}g[1] \to (\mathfrak{h}, (.,.), D)$$

is factors through $\iota : g[1] \to g_{Lie} \oplus F_{Lie}$ and $f = \psi \circ \iota$.

(B1) Since $F_{Lie}g[1]$ is the free Lie algebra, by its universality, a Lie algebra morphism, $\psi_F : F_{Lie}g[1] \to \mathfrak{h}$, such that $\psi_F(x) = f(x)$ is uniquely well-defined, where $x \in g[1]$.

(B2) For any $\bar{x} \in g_{Lie}$, we define a map $\psi_{Lie} : g_{Lie} \to \mathfrak{h}$ as

$$\psi_{Lie}(\bar{x}) := Df(x).$$
We should check that $\psi_{\text{Lie}}$ is well-defined. It suffices to show that $Df(I[1]) = 0$, where $I[1] \subset g[1]$ is the ideal consisting of the symmetric brackets. Because $f$ is a Leibniz homomorphism, $f[x_1, x_2] = (Df(x_1), f(x_2))$. For any element of $I[1]$,

$$Df([x_1, x_2] + [x_2, x_1]) = Df([x_1, x_2]) = (Df(x_1), Df(x_2)) = (\psi\bar{x}_1, \psi\bar{x}_2).$$

Therefore, $\psi_{\text{Lie}}$ is well-defined.

(B3) We prove that $\psi := \psi_{\text{Lie}} \oplus \psi_F$ is a Lie algebra homomorphism. Firstly,

$$\psi(\bar{x}_1, \bar{x}_2) = \psi_{\text{Lie}}[x_1, x_2] = Df[x_1, x_2] = (Df(x_1), Df(x_2)) = (\psi\bar{x}_1, \psi\bar{x}_2).$$

Secondly,

$$\psi(\bar{x}_1, x_2) = \psi_F[x_1, x_2] = f[x_1, x_2] = (Df(x_1), f(x_2)) = (\psi\bar{x}_1, \psi(x_2))$$

and this implies that for any $\alpha \in F_{\text{Lie}}(g[1])$, $\psi(\bar{x}, \alpha) = (\psi(\bar{x}), \psi(\alpha))$.

(B4) Finally, we prove that $\psi$ is commutative with the differentials, i.e., $\psi d = D\psi$.

Thanks to the claim above, it suffices to check the two cases of $\psi d(x) = D\psi(x)$ and $\psi d(\bar{x}) = D\psi(\bar{x})$. The first case is

$$\psi d(x) = \psi_{\text{Lie}}(\bar{x}) = Df(x) = D\psi(x)$$

and the second case is obvious, because $d\bar{x} = 0$ and $D D = 0$. \hfill \Box

Finally of this section, we observe the second homology group, $H_2(F_{\text{Lie}}g[1], d)$. The invariant bilinear form in the category of Lie algebra is a symmetric pairing $\langle \cdot, \cdot \rangle$ satisfying the invariant condition

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle,$$  \hfill (17)

where $[\cdot, \cdot]$ is a Lie bracket. The universal invariant bilinear form on a Lie algebra $g$ is by definition the projection of $g \otimes g$ to $\Omega^0(g)$, where

$$\Omega^0(g) := g \otimes g / \{ x \otimes y - y \otimes x, [x, y] \otimes z - x \otimes [y, z] \}.$$

According to Kontsevich [8], $\Omega^0(g)$ is regarded as the space of formal functions (0-forms) over $g$ as a formal Lie-manifold. It is known that (17) (non-universal version) is a consequence of a derived bracket construction (cf. [9], [17]). The universal version also comes from the derived bracket construction, that is,

**Proposition 3.6.** If $g$ is a Lie algebra, then $H_2(F_{\text{Lie}}g[1], d) = \Omega^0(g)$. 

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Proof. Because $\mathfrak{g}[1]$ is an odd space, $F^2_{\text{Lie}}\mathfrak{g}[1]$ is the same as the symmetric tensor space, $S^2\mathfrak{g} := \mathfrak{g} \otimes \mathfrak{g}/\{x \otimes y - y \otimes x\}$, and because the bracket is Lie, $d = 0$ on $S^2\mathfrak{g}$. On the other hand,

$$d(y, (x, z)) = ([y, x], z) + (x, [y, z]) \sim -[x, y] \otimes z + x \otimes [y, z].$$

Therefore, the identity of the corollary holds. $\square$

Let us consider the Leibniz case. In this case, the universal symmetric bilinear form is not defined on $S^2\mathfrak{g}$, but on $\text{Ker}^2d$, and the target space of the bilinear form is $H^2(F_{\text{Lie}}\mathfrak{g}[1], d) = \text{Ker}^2d/\text{Im}^2d$. From (15), the invariant condition has the following form.

$$([x, y], z) + (y, [x, z]) = (x, [y, z] + [z, y]),$$

which is the same as the invariant condition for Courant algebroids (see [12] for the details). The meaning of (18) is clear. If $\mathcal{L}$ is a Lie subalgebra of the Leibniz algebra $\mathfrak{g}$, then $S^2\mathcal{L}$ is a subspace of $\text{Ker}^2d$ and then (18) reduces to the classical formula over $\mathcal{L}$. Namely, (18) is the relation for the Lie subalgebras of $\mathfrak{g}$.

## 4 Anti-cyclic subcomplex

The aim of this section is to describe how the complex $(F_{\text{Lie}}\mathfrak{g}[1], d)$ relates with the (co)homology complex of Leibniz algebra. In 4.1 we will prove that $(F_{\text{Lie}}\mathfrak{g}[1], d)$ is a subcomplex of Leibniz homology complex and $d$ is the same as the boundary map of Loday. In 4.2 we will introduce the notion of anti-cyclic cochain for Leibniz algebras by analogy with cyclic cochains for associative algebras and prove that the set of anti-cyclic cochains is a subcomplex of the cohomology complex of Loday-Phirashvili.

### 4.1 Homology side

Let $\mathfrak{g}$ be a Leibniz algebra. The complex over $\mathfrak{g}$ computing the Leibniz homology group is the tensor space $\bar{T}\mathfrak{g}[1] = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}[1] \otimes^n n$ with the boundary map,

$$\partial_L(x_1, \ldots, x_n) := \sum_{1 \leq i < j \leq n} (-1)^{i-1}(x_1, \ldots, x_i, x_j, x_{j+1}, \ldots, x_n).$$

The definition of $\partial_L$ is the left-version of Loday’s original formula. We call $(\bar{T}\mathfrak{g}[1], \partial_L)$ a Loday complex. The free Lie algebra $F_{\text{Lie}}\mathfrak{g}[1]$ is regarded as a subspace of the tensor space via the commutator,

$$\{x_1, \ldots, x_n\} = x_1 \otimes \{x_2, \ldots, x_n\} - (-1)^{n-1}\{x_2, \ldots, x_n\} \otimes x_1.$$
where \( \{x_1, \ldots, x_n\} \) is the right-normalized bracket used in (12).

**Theorem 4.1.** \( \partial_L = d \) on \( F_{\text{Lie}} g[1] \).

As a result, the free Lie algebra is a subcomplex of Loday complex.

**Proof.** Obviously \( \partial_L(x_1, x_2) = d(x_1, x_2) \).

**Lemma 4.2.** \( \partial_L(\{x_2, \ldots, x_n\} \otimes x_1) = (\partial_L \{x_2, \ldots, x_n\}) \otimes x_1 \)

**Proof.** From the defining equation of \( \partial_L \), we have

\[
\partial_L(\{x_2, \ldots, x_n\} \otimes x_1) = (\partial_L \{x_2, \ldots, x_n\}) \otimes x_1 + T
\]

where \( T \) is the term which has \([-, x_1]\),

\[
T = \sum (\ldots, x_i^\vee, \ldots) \otimes [x_i, x_1].
\]

We should prove \( T = 0 \). From (20),

\[
\{x_2, \ldots, x_n\} \otimes x_1 = x_2 \otimes \{x_3, \ldots, x_n\} \otimes x_1 - (-1)^{n-2} \{x_3, \ldots, x_n\} \otimes x_2 \otimes x_1. \tag{21}
\]

Therefore, \([x_2, x_1]\) appears in \( T \) in two ways. One is from the first term of (21)

\[
(-1)^{n-2}(-1)^{n-2} \{x_3, \ldots, x_n\} \otimes [x_2, x_1]
\]

and the other is from the second term

\[
-(-1)^{n-2}(-1)^{n-2} \{x_3, \ldots, x_n\} \otimes [x_2, x_1].
\]

Because the sign is reverse to each other, the terms with \([x_2, x_1]\) vanish. By repeating the same discussion, we obtain \( T = 0 \). \(\square\)

From (19), it is easy to see through

**Lemma 4.3.**

\[
\partial_L(x_1 \otimes \{x_2, \ldots, x_n\}) = \sum_{2 \leq i \leq n} \{x_2, \ldots, [x_1, x_i], \ldots, x_n\} - x_1 \otimes \partial_L \{x_2, \ldots, x_n\}.
\]

Therefore,

\[
\partial_L \{x_1, \ldots, x_n\} = \sum_{2 \leq i \leq n} \{x_2, \ldots, [x_1, x_i], \ldots, x_n\} - x_1 \otimes \partial_L \{x_2, \ldots, x_n\} - (-1)^{n-1}(\partial_L \{x_2, \ldots, x_n\}) \otimes x_1.
\]

By assumption of induction, \( \partial_L \{x_2, \ldots, x_n\} = d\{x_2, \ldots, x_n\} \). Hence

\[
\partial_L \{x_1, \ldots, x_n\} = \sum_{2 \leq i \leq n} \{x_2, \ldots, [x_1, x_i], \ldots, x_n\} - \{x_1, d\{x_2, \ldots, x_n\}\}
\]

\[
= d\{x_1, \ldots, x_n\}
\]

The proof is completed. \(\square\)
In the following we denote by $HA_{n-1}(g) := H_\bullet(F_{Lie}g[1], d)$. Hence $HA_0(g) = g_{Lie}$ and if $g$ is Lie, then $HA_1(g) = \Omega^0(g)$.

4.2 Cohomology side

We recall the cohomology complex for Leibniz algebra [14]. Let $g$ be a Leibniz algebra and $M$ a $g$-module or representation of $g$. The cochain complex which computes the cohomology group of $g$ with coefficients in $M$ is

$$LP(g, M) := \text{Hom}_K(g[1], M[1])$$

equipped with a differential defined by

$$(d_{LP}f)(x_1, ..., x_{n+1}) := [f(x_1, ..., x_{n-1}, x_n)] + \sum_{i=1}^{n} (-1)^{i+n}[x_i, f(x_1, ..., x_i^\vee, ..., x_{n+1})] - \sum_{i<j \leq n+1} (-1)^{i+n}f(x_1, ..., x_i^\vee, ..., [x_i, x_j], x_{j+1}, ..., x_{n+1}),$$

where $f \in LP^n(g, M)$ and $|f| := n - 1$. This definition of the derivation is the left-version of the original formula introduced in [14].

Let $LP^n(g)$ be the space of $n+1$-linear functions on the tensor space $Tg[1]$,

$$LP^n(g) := \text{Hom}(g[1] \otimes^{n+1}, K).$$

The differential $d_{LP}$ can be extended on $LP^\bullet(g)$ by the following manner,

$$(b_{LP}\tilde{f})(x_1, ..., x_{n+1}, x_{n+2}) := (-1)^n\tilde{f}(x_1, ..., x_n, [x_{n+1}, x_{n+2}] + [x_{n+2}, x_{n+1}]) + \sum_{i<j \leq n} (-1)^{i-1}\tilde{f}(x_1, ..., x_i^\vee, ..., [x_i, x_j], x_{j+1}, ..., x_{n+2}),$$

where $\tilde{f} \in LP^n(g)$. When $g$ is finite dimensional and $f \in LP^n(g, g^*)$, if we put

$$\tilde{f}(x_1, ..., x_{n+1}) := \omega(f(x_1, ..., x_n), x_{n+1}),$$

then $b_{LP}\tilde{f} = (-1)^n\tilde{d_{LP}}f$, where $\omega$ is the canonical structure in (5).

Now we define the notion of anti-cyclic cochain. Before giving a general definition, let us observe the elementary case. Let $g$ be a finite dimensional Leibniz algebra. Consider an Abelian extension of $g$ by $g^*$,

$$0 \longrightarrow g^* \longrightarrow g \oplus g^* \longrightarrow g \longrightarrow 0.$$
Proof. If linear function, $A$ is an ac-cochain.

Lemma 4.5 (Implicit definition). $A$ is in $ALP^n(g)$ if and only if there exists a linear function, $A'$, on $F_{Lie}^n g[1]$ and

$$A(x_1, ..., x_{n+1}) = A'(x_1, ..., x_{n+1}).$$

Proof. If $A$ is an ac-cochain, $A' := \frac{1}{n+1} A\epsilon$. The converse is also easy (See Appendix).

Theorem 4.6. $ALP^*(g)$ is a subcomplex of $(LP^*(g), b_LP)$.
Proof. Suppose that $A(x_1, ..., x_{n+1})$ is an ac n-cochain. From (23) and the assumption,

$$(b_{LPA}(x_1, ..., x_{n+1}, x_{n+2}) = (-1)^n A(x_1, ..., x_n, [x_{n+1}, x_{n+2}] + [x_{n+2}, x_{n+1}]) + \sum_{i<j} (-1)^{i-j} A(x_1, ..., x_i, ..., [x_i, x_j], x_{j+1}, ..., x_{n+2}) =$$

$$= (-1)^n A'(x_1, ..., x_n, [x_{n+1}, x_{n+2}] + [x_{n+2}, x_{n+1}]) + \sum_{i<j} (-1)^{i-j} A'(x_1, ..., x_i, ..., [x_i, x_j], x_{j+1}, ..., x_{n+2}),$$

on the other hand, from (13), the right-hand side is equal to $(A'd)(x_1, ..., x_{n+2})$.

Hence we obtain

$$(b_{LPA}(x_1, ..., x_{n+2}) = (A'd)(x_1, ..., x_{n+2}),$$

which yields the theorem, i.e., $(b_{LPA})' = (A'd)$.

Denote by $HA^\bullet(g) := H^\bullet(ALP(g), b_{LPA})$ the cohomology group of anti-cyclic cochains. The space of ac 0-cochains is equal to the dual space $g^* := \text{Hom}(g, \mathbb{K})$.

When $A$ is an ac 0-cochain, then $b_{LPA}A = 0$ if and only if $A = 0$ on the ideal $I$.

Hence $HA^0(g) = I^\perp \cong g^*_{\text{Lie}} = (HA_0g)^*$. When $g$ is Lie, if $A$ is an ac 1-cocycle, then

$$A([x, y], z) + A(y, [x, z]) = A(z, [x, y]) - A([z, x], y) = 0.$$ 

Hence $HA^1(g) = (HA_1g)^* = (\Omega^1 g)^*$. We here prove a classical theorem. Let $g$ be a finite dimensional Leibniz algebra. We consider a subclass of Abelian extensions of $g$ by $g^*$ such that

(i) the Leibniz algebra of the middle position, $g \oplus g^*$, satisfies (1)-(2) with respect to $\omega$,

(ii) the isomorphisms between extensions preserve $\omega$.

Theorem 4.7. Such extensions are classified into $HA^2(g)$.

Proof. In general, an isomorphism between Abelian extensions is given by $e^\tau := 1 + \tau$, where $\tau : g \to g^*$. This preserves $\omega$ if and only if $\tilde{\tau}$ is a symmetric tensor or ac 1-cochain.

5 Tensor expression

In this section we study a tensor expression of anti-cyclic cochains. In the following suppose that $g$ is a finite dimensional Leibniz algebra. Let $e_1, ..., e_{\dim g}$ be a base
of $\mathfrak{g}$. The degree of $e_i$ is $|e_i| = -1$ for each $i$. If $A$ is an ac 2-cochain on $\mathfrak{g}$, then $A(e_i, e_j, e_k) = A_{ijk}$. Hence the cochain is expressed as $A = A_{ijk}e^i \otimes e^j \otimes e^k$, where $e^i$ is the dual base of $e_i$. The coefficient part, $A_{ijk}$, satisfies

$$A_{ijk} = A_{ikj},$$

$$A_{ijk} + \text{cyclic} = 0,$$

which is the symmetry that the normalized Lie bracket $(e_i, (e_j, e_k))$ satisfies. Hence the symmetry of the tensor part, $e^i \otimes e^j \otimes e^k$, should be the dual of the one of $(e_i, (e_j, e_k))$. We denote such a tensor by $\{e^i, e^j, e^k\}$ and call the bracket $\{\ldots, e^k\}$ a dual Lie bracket. In general, the dual Lie bracket is defined as follows

**Definition 5.1** (dual Lie brackets).

$$\{x^1, \ldots, x^n\}_* := x^1 \otimes \{x^2, \ldots, x^n\}_* - (-1)^{n-1} x^n \otimes \{x^1, \ldots, x^{n-1}\}_*,$$

(28)

where we put $|x^i| := \text{odd or } +1$. In particular, $\{x^1\}_* = x^1$.

For example, $\{x^1, x^2\}_*$ is equal to the symmetric tensor $\{x^1, x^2\}_* = x^1 \otimes x^2 + x^2 \otimes x^1$, the 3-ary bracket is

$$\{x^1, x^2, x^3\}_* = x^1 \otimes x^2 \otimes x^3 + x^1 \otimes x^3 \otimes x^2 - x^3 \otimes x^1 \otimes x^2 - x^3 \otimes x^2 \otimes x^1.$$

From (28) the total cyclic summation of the dual-Lie bracket is zero.

$$\oint \{x^1, \ldots, x^n\}_* = 0,$$

where $\oint$ is the cyclic permutation for all variables. An ac $n$-cochain on $\mathfrak{g}$ is expressed by using the dual Lie bracket as follows.

$$A = \frac{1}{n+1} \sum A_{i_1 \ldots i_{n+1}} \{e^{i_1}, \ldots, e^{i_{n+1}}\}_*.$$

**Definition 5.2** (contraction). If $f$ is a linear function on $\mathfrak{g}$, then

$$i_f \{x^1, \ldots, x^n\}_* := f(x^1)\{x^2, \ldots, x^n\}_* - (-1)^{n-1} f(x^n)\{x^1, \ldots, x^{n-1}\}_*.$$

Our interesting space is not $\mathfrak{g}$ but the double space $\mathfrak{g} \oplus \mathfrak{g}^\ast$. By analogy with the Lie algebra case, the notion of Cartan 3-form is defined by $C(x, y, z) := \omega([x, y], z)$ on $\mathfrak{g} \oplus \mathfrak{g}^\ast$, where $[\ldots]$ is the Leibniz bracket of $\mathfrak{g} \times \mathfrak{g}^\ast$. The structure constant of the Leibniz bracket is defined by using the Cartan 3-form

$$C^k_{ij} := C(e_i, e_j, e_k) = \omega([e_i, e_j], e_k).$$
Denote $\mu_{\text{Leib}} := C_{ij}^k \{e^i, e^j, e_k\}_*$. Then for any linear functions $f_1, f_2$ on $g \oplus g^*$, the Leibniz bracket $[f_1, f_2]$ is computed by

$$[f_1, f_2] = i_{f_2} f_1 \mu_{\text{Leib}}.$$ 

For $\widetilde{H}$ in (24)-(25), denote $H_{ijk} := \widetilde{H}(e_i, e_j, e_k)$. Then $\widetilde{H} = \frac{1}{3} H_{ijk} \{e^i, e^j, e^k\}_*$. Therefore, the total structure with the twisting term $\widetilde{H}$ is expressed as follows.

$$\theta_{\text{Leib}} := C_{ij}^k \{e^i, e^j, e_k\}_* + \frac{1}{3} H_{ijk} \{e^i, e^j, e^k\}_*.$$ 

Classical structures for Lie algebras are expressed by using the wedge product,

$$\theta_{\text{Lie}} = \frac{1}{2} C_{ij}^k e^i \wedge e^j \wedge e_k + \frac{1}{6} H_{ijk} e^i \wedge e^j \wedge e^k.$$ 

On the other hand in the Leibniz world, the structure tensors are expressed by the dual Lie bracket instead of the wedge product.

**Appendix –Proof of Lemma 4.5–**

Denote by $\{,\}$ the map of higher bracketing

$$\{,\} : x_1 \otimes \cdots \otimes x_n \mapsto \{x_1, ..., x_n\}.$$ 

Then the following identity holds.

$$\{,\} \epsilon \{,\} = n \{,\},$$ 

where $n$ is the length of word. The lemma is a consequence of this identity, hence we prove (29).

When $n = 1, 2$ the identity obviously holds. By the Jacobi identity, in general, the normalized bracket satisfies

$$\{, \ldots, \epsilon \{, ..., \}, ..., x_f \} = \{, \ldots, \{, ..., \}, ..., x_f \},$$ 

where $x_f$ is the fixed variable which lies the most right position. For example,

$$\epsilon \{x_1, x_2\} = x_1 \otimes x_2 + x_2 \otimes x_1 \text{ and }$$

$$\{\epsilon \{x_1, x_2\}, x_3\} = \{x_1, x_2, x_3\} + \{x_2, x_1, x_3\} = \{\{x_1, x_2\}, x_3\}.$$ 

From the definition,

$$\epsilon \{x_1, ..., x_{n+1}\} = x_1 \otimes \epsilon \{x_2, ..., x_{n+1}\} - (-1)^n \epsilon \{x_2, ..., x_{n+1}\} \otimes x_1.$$  

Applying $\{,\}$ to (31), we obtain

$$\{,\} \epsilon \{x_1, ..., x_{n+1}\} = \{x_1, \{\epsilon \{x_2, ..., x_{n+1}\}\} - (-1)^n \{\epsilon \{x_2, ..., x_{n+1}\}, x_1\}.$$ 

By the assumption of induction, the first term is equal to $n \{x_1, ..., x_n\}$ and by (30) the second term is equal to $\{x_1, ..., x_{n+1}\}$. Hence

$$\{,\} \epsilon \{x_1, ..., x_{n+1}\} = (n+1) \{x_1, ..., x_{n+1}\}.$$ 

The proof is completed.
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