Quadratic Residues and Non-residues for Infinitely Many Piatetski-Shapiro Primes

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Abstract In this paper, we prove a quantitative version of the statement that every nonempty finite subset of \( \mathbb{N}^+ \) is a set of quadratic residues for infinitely many primes of the form \([n^c]\) with \(1 \leq c \leq 243/205\). Correspondingly, we can obtain a similar result for the case of quadratic non-residues under reasonable assumptions. These results generalize the previous ones obtained by Wright in certain aspects.

Keywords Quadratic residue, quadratic non-residue, Piatetski-Shapiro prime

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1 Introduction

Distributions of quadratic residues and non-residues have received great attention over the recent decades. Wright [1] has proved by combinatorial and reasonably elementary method that

1) Every nonempty finite subset of \( \mathbb{N}^+ \) is a set of quadratic residues for infinitely many primes.

2) For every nonempty finite subset \( \mathcal{A} \subseteq \mathbb{N}^+ \), which contains no squares, we write \( \Pi_{\text{odd}}(a) = \{ p \text{ prime} : p^a || a, \alpha \text{ odd} \} \), \( \Pi = \bigcup_{a \in \mathcal{A}} \Pi_{\text{odd}}(a) \) and \( \mathfrak{P} = \{ \Pi_{\text{odd}}(a) : a \in \mathcal{A} \} \).

If \( \mathfrak{P} \) contains at most 4 elements, then \( \mathcal{A} \) is a set of quadratic non-residues for infinitely many primes if and only if \( \mathfrak{P} \) does not contain a 3-cycle included in \( \{ \mathcal{C} : \mathcal{C} \neq \emptyset, \mathcal{C} \subseteq \Pi \} \).

We intend to give a quantitative version of such results in the present paper. To be precise, for an arbitrarily given set \( \mathcal{S} \) of finite cardinality \( S = \# \mathcal{S} \), consider the following quantity

\[ S(x) = \# \left\{ p \in \mathcal{P} \cap (x, 2x] : \left( \frac{s}{p} \right) = 1 \text{ for each } s \in \mathcal{S} \right\}, \]

and correspondingly

\[ \overline{S}(x) = \# \left\{ p \in \mathcal{P} \cap (x, 2x] : \left( \frac{s}{p} \right) = -1 \text{ for each } s \in \mathcal{S} \right\}, \]

where \( \mathcal{P} \) is the set consisting of all the primes, \( \left( \frac{s}{p} \right) \) is the Legendre symbol mod \( p \). Clearly, we aim to show that \( S(x) \gg x^{\theta_1} \) and \( \overline{S}(x) \gg x^{\theta_2} \) for certain \( \theta_1, \theta_2 > 0 \) under reasonable assumptions.

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On the other hand, a classical result due to Piatetski-Shapiro states that there exists $c > 1$ such that there are infinitely many primes of the form $[n^c]$ with $n \in \mathbb{N}^+$, where $[x]$ denotes the integral part of $x$. More precisely, Piatetski-Shapiro [2] proved that

$$\sum_{n \leq x \atop [n^c] \in \mathcal{P}} 1 = \left( \frac{1}{c} + o(1) \right) \frac{x}{\log x}, \quad \text{as } x \to +\infty$$

for $1 < c < 12/11 = 1.09$. There are many works devoted to enlarging the reasonable range of $c$, see [3–9] for instance. The most recent result is due to Rivat and Wu [9], who succeeded in showing that

$$\sum_{n \leq x \atop [n^c] \in \mathcal{P}} 1 \gg \frac{x}{c \log x}, \quad \text{as } x \to +\infty$$

for the wider range $1 \leq c \leq 243/205 = 1.18536 \cdots$. Moreover, there are many other problems concerning primes, as well as some related to the analytic theory of automorphic forms (see [10] for instance), which also restrict the primes to be of the form $[n^c]$.

Motivated by these works on the distribution of Piatetski-Shapiro primes, we consider the following quantity

$$S_c(x) = \# \left\{ x < n \leq 2x : [n^c] \in \mathcal{P}, \quad \left( \frac{s}{[n^c]} \right) = 1 \text{ for each } s \in S \right\},$$

and correspondingly

$$\mathcal{S}_c(x) = \# \left\{ x < n \leq 2x : [n^c] \in \mathcal{P}, \quad \left( \frac{s}{[n^c]} \right) = -1 \text{ for each } s \in S \right\}.$$

Clearly, $S_1(x) = S(x)$, $\mathcal{S}_1(x) = \mathcal{S}(x)$.

We shall prove by analytic methods that

**Theorem 1.1** Every nonempty finite set $S \subseteq \mathbb{N}^+$ is a set of quadratic residues of infinitely many primes of the form $[n^c]$ with $1 \leq c \leq 243/205$.

To be precise, let $c$ be a number with $1 \leq c \leq 243/205$. Then for sufficiently large $x$, we have

$$S_c(x) = \left( \# \mathcal{H} + 1 \right) \left( \frac{1}{c \cdot 2^{\#S}} + o(1) \right) \frac{x}{\log x},$$

where $\mathcal{H}$ denotes the collection of all the nonempty subsets of $S$ satisfying that the product of all the elements of each subset is a square integer.

**Theorem 1.2** Considering all of the nonempty subsets of the nonempty finite set $S \subseteq \mathbb{N}^+$, the product of the elements of each one is a square. If such subsets of even cardinalities are not less than those of odd cardinalities, then $S$ is a set of quadratic non-residues for infinitely many primes of the form $[n^c]$ with $1 \leq c \leq 243/205$.

More precisely, we have

$$\mathcal{S}_c(x) = \frac{1}{c \cdot 2^{\#S}} \left[ 1 + \sum_{T \in \mathcal{H}} (-1)^{\#T} + o(1) \right] \frac{x}{\log x},$$

where $\mathcal{H}$ is the same as that in Theorem 1.1.

**Corollary** For any given finitely many primes, all of them are the quadratic residues for certain infinitely many primes of the form $[n^c]$ with $1 \leq c \leq 243/205$, and in the meanwhile, are also the quadratic non-residues for infinitely many primes of the form $[n^c]$ with $1 \leq c \leq 243/205$. 
