Subclasses of Baxter Permutations Based on Pattern Avoidance

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Abstract. Baxter permutations are a class of permutations which are in bijection with a class of floorplans that arise in chip design called mosaic floorplans. We study a subclass of mosaic floorplans called Hierarchical Floorplans of Order $k$ defined from mosaic floorplans by placing certain geometric restrictions. This naturally leads to studying a subclass of Baxter permutations. This subclass of Baxter permutations are characterized by pattern avoidance. We establish a bijection, between the subclass of floorplans we study and a subclass of Baxter permutations, based on the analogy between decomposition of a floorplan into smaller blocks and block decomposition of permutations. Apart from the characterization, we also answer combinatorial questions on these classes. We give an algebraic generating function (but without a closed form solution) for the number of permutations, an exponential lower bound on growth rate, and a linear time algorithm for deciding membership in each subclass. Based on the recurrence relation describing the class, we also give a polynomial time algorithm for enumeration. We finally prove that Baxter permutations are closed under inverse based on an argument inspired from the geometry of the corresponding mosaic floorplans. This proof also establishes that the subclass of Baxter permutations we study are also closed under inverse. Characterizing permutations instead of the corresponding floorplans can be helpful in reasoning about the solution space and in designing efficient algorithms for floorplanning.

Keywords: Floorplanning, Pattern Avoidance, Baxter Permutation

1 Introduction

Baxter permutations are a well studied class of pattern avoiding permutations having real world applications. One such application is to represent floorplans in chip design. A floorplan is a rectangular dissection of a given rectangle into a finite number of indivisible rectangles using axis parallel lines. These indivisible rectangles are locations in which modules of a chip can be placed. In the floorplanning phase of chip design, relative positions of modules are decided so as to optimize cost functions like wire length, routing, area etc. Given a set of modules and an associated cost function, the floorplanning problem is to find an optimal floorplan. The floorplanning problem for typical objective functions is
NP-hard [4, p. 94]. Hence combinatorial search algorithms like simulated annealing [10] are used to find an optimal floorplan. The optimality of the solution and performance of such algorithms depends on the class of floorplans comprising the search space and their representation. Wong and Liu [10] were the first to use combinatorial search for solving floorplanning problems. They worked with a class of floorplans called slicing floorplans which are obtained by recursively subdividing a given rectangle into two smaller rectangles either by a horizontal or a vertical cut. The slicing floorplans correspond to a class of permutations called separable permutations [1]. Later research in this direction focused on characterizing and representing bigger classes of floorplans so that search algorithms have bigger search spaces, potentially including the optimum. One such category of floorplans is mosaic floorplans which are a generalization of slicing floorplans. Ackerman et al. [1] proved a bijection between mosaic floorplans and Baxter permutations. We study a subclass of mosaic floorplans obtained by some natural restrictions on mosaic floorplans. We use the bijection of Ackerman et al. [1] as a tool to characterize and answer important combinatorial problems related to this class of floorplans. For the characterization of these classes we also use characterization of a class of permutations called simple permutations studied by Albert and Atkinson [2].

Given a floorplan and dimensions of its basic rectangles, the area minimization problem is to decide orientation of each cell which goes into basic rectangles so as to minimize the total area of the resulting placement. This problem is NP-hard for mosaic floorplans [9], but is polynomial time for both slicing floorplans [9] and Hierarchical Floorplans of Order 5 [3]. Hence Hierarchical Floorplans of Order $k$ is an interesting class of floorplans with provably better performance in area minimization [3] than mosaic floorplans. But the only representation of such floorplans is through a top-down representation known as hierarchical tree [3] and is known only for Hierarchical Floorplans of Order 5. Prior to this work it was not even known which floorplans with $k$ rooms are non-sliceable and is not constructible hierarchically from mosaic floorplans of $k-1$ rooms or less. Such a characterization is needed to extend the polynomial time area minimization algorithm based on non-dominance given in [3]. We give such a characterization and provide an efficient representation for such floorplans by generalizing generating trees to Skewed Generating Trees of Order $k$. We also give an exact characterization in terms of equivalent permutations.

Our main technical contributions are i) We establish a subclass of floorplans called Hierarchical Floorplans of Order $k$; ii) We characterize this subclass of floorplans using a subclass of Baxter permutations; iii) We show that the subclass is exponential in size; iv) We present an algorithm to check the membership status of a permutation in the subclass of Hierarchical Floorplans of Order $k$ and v) We present a simple proof of closure under inverse operation for Baxter permutations using the mapping between the permutations and floorplans, and the geometry of the rectangular dissection.

The remainder of the paper is organized as follows: in Section 2 we introduce the necessary background on floorplans and pattern avoiding permutations. In
Section 3 we motivate and characterize the subclasses of Baxter permutations studied in this paper. Section 4 is devoted to answering interesting combinatorial problems of growth, and giving generating function on these subclasses. Section 5 gives an algorithm for membership in each class as well as for deciding given a Baxter permutation the smallest \( k \) for which it is Hierarchical Floorplans of Order \( k \). Section 6 proves the closure of Baxter permutations under inverse. Section 7 lists some open problems. We also have a section Appendix (see A.1) which illustrates some floorplans which can be used to gain intuition about the floorplan classes we define.

2 Preliminaries

A floorplan is a dissection of a given rectangle by line segments which are axis parallel (see Figure 2). The rectangles in a floorplan which do not have any other rectangle inside are called basic rectangles or rooms. For the remainder of the paper we will refer to them as rooms. A floorplan captures the relative position of the rooms via four relations defined between rooms. Given a floorplan \( f \), the “left-of” relation denoted by \( L_f \) is defined as \((a, b) \in L_f \) if there is a vertical line segment of \( f \) going through the right edge of room \( a \) and left edge of room \( b \) or if there is a room \( c \) such that \((a, c) \in L_f \) and \((c, b) \in L_f \). When \((a, b) \in L_f \) we say that \( a \) is to the “left-of” \( b \) and is denoted by \( a <_l b \). For example in the floorplan given in Figure 1 the room labeled \( b \) is to the left of room labeled \( d \) because there is vertical segment through the right boundary of room \( b \) and left boundary of room \( d \). Similarly for a floorplan \( f \) the “above” relation denoted by \( A_f \) is defined as \((a, b) \in A_f \) if there is a horizontal line segment of \( f \) going through the bottom edge of room \( a \) and through the top edge of room \( b \) or if there is a room \( c \) such that \((a, c) \in A_f \) and \((c, b) \in A_f \). The other two relations are inverses of these relations: “right-of” is defined as \( R_f = \{(a, b) \mid (b, a) \in L_f \} \) and “below” is defined as \( B_f = \{(a, b) \mid (b, a) \in A_f \} \). A cross junction in a floorplan is an intersection of two line segments such that the intersection point is not an end point of either of the line segments. A mosaic floorplan is a floorplan where there are no cross junctions. This restriction is to ensure that, in a mosaic floorplan between any two rooms, exactly one of \( L_f, R_f, B_f, A_f \) holds [1, Observation 3.3]. We denote the set of all mosaic floorplans with \( k \) rooms by \( M_k \). The relations \( X \in \{L_f, A_f, R_f, B_f\} \) can be naturally extended to that between rooms and line segments, by defining \((a, l) \in X \) if room \( a \) is supported by line segment \( l \) from the respective direction \( X \) in \( f \). We call two mosaic floorplans \( f_1, f_2 \) equivalent if there is a bijective mapping \( \psi : f_1 \rightarrow f_2 \) such that \((a, b) \in X_{f_1} \) if and only if \((\psi(a), \psi(b)) \in X_{f_2} \) where \( X \in \{L, R, A, B\} \), i.e. \( \psi \) preserves the relative position of rooms and line segments. For example floorplans labeled \( a, b \) in Figure 3 are equivalent under this definition whereas \( a \) and \( c \) are not equivalent.

In this paper we study a subclass of mosaic floorplans called Hierarchical Floorplans of Order \( k \). The subclass Hierarchical Floorplans of Order \( k \) for \( k \geq 2, k \in \mathbb{N} \) (abbreviated as HFO\(_k\) in the remainder of the paper) is obtained
Fig. 1. ABLR relationships in a floorplan

\begin{align*}
& a \lessdot d \lessdot f \\
& a \lessdot d \lessdot f \Rightarrow a \lessdot f \\
& a \lessdot f \Rightarrow f \lessdot a \\
& c \lessdot e, c \lessdot d, e \lessdot f \\
& d \lessdot e, c \lessdot e, f
\end{align*}

Fig. 2. A floorplan with its rooms marked

Fig. 3. Equivalence of Floorplans - $a \equiv b$, but $a \not\equiv c$
by placing the following restriction on mosaic floorplans: a mosaic floorplan is 
HFO_k if it can be constructed using mosaic floorplans with at most k rooms by 
repeated application of an operation which we call \textit{insertion}.

\textbf{Definition 1 (Insertion).} Given a mosaic floorplan with k rooms \( f \in M_k \) and some fixed labeling of its rooms, \textit{insertion} of \( f \) by \( k \) mosaic floorplans \( f_1, f_2, f_3, \ldots, f_k \) denoted by \( f(f_1, \ldots, f_k) \) is the mosaic floorplan obtained by placing in \( f_i \) in \( i \)th room of \( f \).

Figure 4 illustrates \textit{insertion} of a floorplan with two rooms by two other floor-
plans. In \textit{insertion}, if two adjacent rooms in \( f \) (say \( a \) and \( b \)) have two segments 
coming from inserted floorplans \( f_a, f_b \) of same alignment (i.e., either both hori-
zontal or both vertical) touching each other making a cross junction, then to 
make the resulting floorplan mosaic, one of the line segments is moved by a small 
\( \delta > 0 \) as shown in Figure 5. Moving a line segment by a small \( \delta \) does not change 
the relative position of rooms. This ensures that \textit{insertion} produces floorplans 
which are mosaic.

We define a mosaic floorplan \( f \) to be \textit{decomposable} if there exists \( k > 1 \) for 
which there is a \( g \in M_k \) and \( k \) mosaic floorplans \( g_1, \ldots, g_k \) at least one of which 
is non trivial (i.e., has more than one room) and \( f = g(g_1, \ldots, g_k) \). A mosaic 
floorplan is called \textit{in-decomposable} if it is not decomposable.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig4}
\caption{Insertion operation on floorplans}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig5}
\caption{Avoiding cross junction}
\end{figure}
Ackerman et al. [1] established a representation for mosaic floorplans in terms of a class of pattern avoiding permutations called Baxter permutations. The bijection is established via two algorithms, one which produces a Baxter permutation given a mosaic floorplan and another which produces a mosaic floorplan given a Baxter permutation. For explaining the results in this paper we only need the algorithm which produces a Baxter permutation \( \pi_f \) given a mosaic floorplan \( f \). This algorithm has two phases, a labeling phase where every room in the mosaic floorplan \( f \) is given a unique number in \([n]\) and an extraction phase where the labels of the rooms are read off in a specific order to form a permutation \( \pi_f \in S_n \). The labeling is done by successively removing the top-left room of current floorplan by sliding it out of the boundary by pulling the edge which ends at a T junction (since no cross junctions are allowed in a mosaic floorplan, for any room every edge which is within the dissected rectangle is either a horizontal segment ending in a vertical segment forming a \( \shortmid \) or is a horizontal segment on which a vertical segment ends forming a \( \perp \)). The \( i \)th floorplan to be removed in the above process is labeled room \( i \) in the original floorplan. After the labeling phase we obtain a mosaic floorplan whose rooms are numbered from \([n]\). The permutation corresponding to the floorplan is obtained in the second phase called extraction where rooms from the bottom-left corner are successively removed by pulling the edge ending at a T junction. The \( i \)th entry of the permutation \( \pi_f \) is the label of the \( i \)th room removed in the extraction phase.

Figure 6 demonstrates the labeling phase and Figure 10 demonstrates the extraction phase. If room \( i \) is labeled before room \( j \) then room \( i \) is to the left or above of room \( j \), whereas if the room \( i \) is removed before room \( j \), i.e., \( \pi^{-1}[i] < \pi^{-1}[j] \) then room \( i \) is to the left of or below room \( j \) (see [1, Observation 3.4]). Since the permutation captures both the label and position of a room, it captures the above, below, left or right relations between rooms. Ackerman et al. (see [1, Observation 3.5]) also proved that two rooms share an edge in a mosaic floorplan \( f \) if and only if either their labels are consecutive or their positions in \( \pi_f \) are consecutive. For the rest of the paper we refer to this Algorithm of Ackerman et al. as FP2BP.

We now describe permutation classes which are used in this paper, including Baxter permutations mentioned earlier. For the convenience of defining pattern avoidance in permutations, we will assume that permutations are given in the one-line notation (for ex., \( \pi = 3142 \)). A permutation \( \pi \in S_n \) is said to contain a pattern \( \sigma \in S_k \) if there are \( k \) indices \( i_1, \ldots, i_k \) with \( 1 \leq i_1 < \cdots < i_k \leq n \) such that \( \pi[i_1], \pi[i_2], \pi[i_3], \ldots, \pi[i_k] \) called text has the same relative ordering as \( \sigma \), i.e., \( \pi[i_j] < \pi[i_l] \) if and only if \( \sigma_j < \sigma_l \). Note that the sub-sequence need not be formed by consecutive entries in \( \pi \). If \( \pi \) contains \( \sigma \) it is denoted by \( \sigma \leq \pi \). A
permutation $\pi$ avoids $\sigma$ if it does not contain $\sigma$. For example $\pi = 4321$ avoids $\sigma = 12$ because in 4321 every number to the right of a number is smaller than itself, but $\pi$ contains the pattern $\rho = 21$ because numbers at any two indices of $\pi$ are in decreasing order. A permutation $\pi$ is called separable if it avoids the pattern $\sigma_1 = 3142$ and its reverse $\sigma_2 = 2413$. Baxter permutations are a generalization of separable permutations in the following sense: they are allowed to contain 3142/2413 as long as any $\pi[i_1], \pi[i_2], \pi[i_3], \pi[i_4]$ which has the same relative ordering as 3142/2413 has $|\pi[i_1] - \pi[i_4]| > 1$. For example $\pi = 41532$ is not Baxter as 3 text 4153 in $\pi$ matches pattern 3142 and the absolute difference of entry matching 3 and entry matching 2 is 4 − 3 = 1. However $\pi = 41352$ is a Baxter permutation as the only text which matches 3142 is 4152 and the absolute difference of entries matching 3, 2 is 4 − 2 = 2 which is greater than 1.

Another class of permutations important to this study is the class of simple permutations. They are a class of block in-decomposable permutations. To define this in-decomposability we need the following definition: a block of a permutation is a set of consecutive positions such that the values from these positions form an interval $[i, j]$ of $\mathbb{N}$. Note that the values in the block need not be in ascending order as it is in the interval corresponding to the block $[i, j]$. The notion of block in-decomposability is defined by a decomposition operation called inflation. We recall the definition from Section 2 of [2].

**Definition 2 (Inflation).** Given a permutation $\sigma \in S_k$, inflation of $\sigma$ by $k$ permutations $\rho_1, \rho_2, \rho_3, \ldots, \rho_k$, denoted by $\sigma(\rho_1, \ldots, \rho_k)$ is the permutation $\pi$ where each element $\sigma_i$ of $\sigma$ is replaced with a block of length $|\rho_i|$ whose elements have the same relative ordering as $\rho_i$, and the blocks among themselves have the same relative ordering as $\sigma$.

For example inflation of 3124 by 21, 123, 1 and 12 results in $\pi = 65123478$ where 65 is the block corresponding to 21, 123 corresponds to 123, 4 corresponds to 1 and 78 corresponds to 12. If $\pi = \sigma(\rho_1, \ldots, \rho_k)$ then $\sigma(\rho_1, \ldots, \rho_k)$ is called a block-decomposition of $\pi$. A block-decomposition $\sigma(\rho_1, \rho_2, \ldots, \rho_k)$ is non-trivial if $\sigma \in S_k$ for $k > 1$ and at least one $\rho_i$ is a non-singleton permutation (i.e. of more than one element). A permutation is block-in-decomposable if it has no non-trivial block-decomposition. Note that inflation on permutations as defined above is analogous to insertion on mosaic floorplans defined earlier.

Block in-decomposable permutations can be thought of as building blocks of all other permutations by inflations. Albert and Atkinson [2] studied simple permutations which are permutations whose only blocks are the trivial blocks (which is either a single point $\pi[i]$ or the whole permutation $\pi[1 \ldots n]$). They also defined a sub class of simple permutations called exceptionally simple permutations which are defined based on an operation called one-point deletion. A one-point deletion on a permutation $\pi \in S_n$ is deletion of a single element at some index $i$ and getting a new permutation $\pi' \in S_{n-1}$ by rank ordering the remaining elements. For example one-point deletion at index 5 of 41352 gives 4135 which when rank ordered gives the permutation 3124. A permutation $\pi$ is exceptionally simple if it is simple and no one-point deletion of $\pi$ yields a simple permutation. Albert and Atkinson [2] characterized exceptionally simple
permutations and proved that for any permutation $\pi \in S_n$ which is *exceptionally simple* there exists two successive *one-point* deletions which yields a *simple* permutation $\pi' \in S_{n-2}$.

3 Characterizing Hierarchical Floorplans of Order $k$

In this section we characterize Hierarchical Floorplans of Order $k$ in terms of corresponding permutations using the notion of block decomposition defined earlier.

We note that this connection can be seen for a level of the hierarchy well studied in literature, namely HFO$_2$. HFO$_2$, the class of floorplans which can be built by repeated application of insertion of the two basic floorplans shown in Figure 6 are also called slicing floorplans. Slicing floorplans are known [1] to be in bijective correspondence with *separable permutations*. Separable permutations are also the class of permutations $\pi$ such that it can be obtained repeated *inflation* of 1 (the singleton permutation) by, 12 or 21. Note that both 12, 21 are *simple* permutations. Even though HFO$_2$ is well studied in literature and is known to be in bijective correspondence with separable permutations, the connection to block decomposition of permutations was not explicitly observed.

HFO$_5$ (shown in Figure 15 and Figure 16) floorplans are also studied in the literature, but the only characterization till date for these floorplans is based on a discrete structure called *generating trees*. We generalize this structure for an arbitrary $k$ in the following sense: a *generating tree* of order $k$ is a rooted tree, where each node is labeled by an in-decomposable mosaic floorplan, say $g$ of at most $k$ rooms, and the number of children of a node is equal to the number of rooms in the floorplan labeling the node. The children are arranged in the order $\pi^{-1}_g$ from left to right. That is the left most child corresponds to the first room to be removed in the extraction phase of FP2BP and second from left corresponds to second room to be removed and so on and so forth. The *generating tree* captures the top down application of *insertion’s* to yielding the given floorplan in the following sense: an internal node of a *generating tree* represents insertion of $f$ - the floorplan labeling the node - by the floorplans labeling its children, $f_1, \ldots, f_k$ (ordered from left to right). Figure 7 is a generating tree for an HFO$_5$ floorplan. There could be more than one generating tree for a floorplan owing to the fact there is ambiguity in consecutive vertical slices and in consecutive horizontal slices, as illustrated in Figure 8. But this can be removed (proved later) by introducing two disambiguation rules called “skew”. Skew rule insists that when there are multiple parallel vertical (respectively, horizontal) line segments touching the bounding box of the floorplan $f$, we consider only the insertion operation $f_1, f_2$ where $f_2$ is the floorplan contained to the right of (respectively, above) the first parallel line segment from left (respectively, bottom) and $f_1$ is the floorplan contained to the left of (respectively below) the first parallel line segment from left (respectively, bottom). Hence only the tree labeled $a$ satisfies “skew” rule among the generating trees in Figure 8. A generating tree satisfying “skew” rule is called *Skewed Generating Tree*. 
The connection between insertion and block decomposition and the fact the bijection of Ackerman et al.\cite{1} preserves this connection is the central idea of our paper. The following observation about the algorithm FP2BP, though not mentioned in the original paper, is not hard to see, but is useful for the characterization of HFO\(_k\).

**Lemma 1.** For a mosaic floorplan \( f \) let \( \pi_f \) denote the unique Baxter permutation obtained by algorithm FP2BP. If \( f = \operatorname{g}(g_1, \ldots, g_k) \) i.e., it is obtained by insertion of \( g \in M_k \) by \( g_1, \ldots, g_k \), then

\[
\pi_f = \pi_{\operatorname{g}}(\pi_{g_1}, \ldots, \pi_{g_k})
\]

where \( \pi_{\operatorname{g}}(\pi_{g_1}, \ldots, \pi_{g_k}) \) denotes the permutation obtained by inflating \( \pi_{\operatorname{g}} \) with \( \pi_{g_1}, \ldots, \pi_{g_k} \).

**Proof.** Since \( f \) is obtained by insertion of \( g \in M_k \) by \( g_1, \ldots, g_k \), each \( g_i \) is completely contained inside a rectangle, the \( i \)th room of \( g \). The theorem follows from the fact that FP2BP labeling labels all the rooms contained inside a rectangle before moving out, and it extracts all the rooms inside a rectangle before moving out of the rectangle.

We will first prove that the FP2BP labeling labels all the rooms contained inside a rectangle before moving out. To prove this assume to the contrary that there exists rooms \( a, b, c \) with \( a \) and \( b \) belonging to \( g_i \) and \( c \) belonging to \( g_j, j \neq i \).
such that they are labeled in the order $a, c, b$ without loss of generality. By the property (see [1, Observation 3.4]) of the labeling algorithm $a$ is to the left or above of $c$, and $c$ is to the left or above $b$ and since they are labeled consecutively there is a line segment shared by $a$ and $c$ as well as $c$ and $b$. They can only be oriented in one of the four ways shown in Figure 11 corresponding to whether $a <_l c$ or $a <_a c$ and $c <_l b$ or $c <_a b$. Among the four, except for $a <_l c <_a b$ and its symmetric counterpart $a <_a c <_l b$, it is clear that it cannot be the case that $a$ and $b$ are contained in one rectangle but $c$ in another. For the orientation $a <_l c <_a b$, the fact that there is a line segment shared by $b$ and $c$ removes the possibility of $a, b$ being in one rectangle and $c$ being in another.

A symmetric argument can be used to establish the same when $a <_a c <_l b$. A similar argument can be used to establish that the extraction algorithm moves to another rectangle only after exhausting all the rooms in the current rectangle.

We obtain the following useful corollary from Lemma 1 (see Appendix for a proof [3]):

**Corollary 1.** A mosaic floorplan $f$ is in-decomposable if and only if the Baxter permutation $\pi_f$ corresponding to it is block in-decomposable.

For the characterization we will also need the following connection between generating trees and block decomposition of permutations. Let $T_f$ be a generating tree corresponding to $f$, satisfying the “skew” rule, then $T_f$ captures the unique block decomposition of a permutation as defined in [2, Proposition 2]. Label every node of $T_f$ by Baxter permutation $\pi_{f_i}$ corresponding to the mosaic floorplan $f_i$ labeling it. Mosaic floorplan $g$ corresponding to the sub-tree rooted at $f_i$ is obtained by the insertion of $f_i$ by the floorplans labeling its children $f_{i_1}, \ldots, f_{i_k}$. Hence by applying Lemma 1 we get that $\pi_g = \pi_{f_i} \left( \pi_{f_{i_1}}, \ldots, \pi_{f_{i_k}} \right)$. So generating trees labeled by Baxter permutations $\pi_{f_i}$ captures the block decomposition of Baxter permutation $\pi_f$ corresponding to the floorplan $f$. Figure
12 illustrates the correspondence between inflation and insertion by showing the equivalence between inflating 3124 with 123, 21, 1 and 24, and inserting the floorplan corresponding to 3124 with floorplans corresponding to 123, 21, 1 and 24.

Fig. 12. Correspondence between inflation and insertion

**Theorem 1.** Skewed Generating Trees of Order $k$ are in bijective correspondence with HFO$_k$ floorplans. Moreover they capture the block decomposition of the Baxter permutation corresponding to the floorplan.

**Proof.** It follows from definition of HFO$_k$ that there is a generating tree of order $k$ capturing the successive applications of insertions resulting in the final floorplan. Since HFO$_k$ are a subclass of mosaic floorplans which are in bijective correspondence with Baxter permutations, there is unique Baxter permutation $\pi_f$ corresponding to the floorplan $f$. Lemma 1 can now be used to prove that a generating tree of order $k$ captures the block decomposition of $\pi_f$, by induction on the height of the tree. Consider the base case to be $h = 1$, i.e., the whole tree is one node labeled by an in-decomposable mosaic floorplan $f$ and by Corollary 1 $\pi_f$ is block in-decomposable. Assume that for any $h < l$, generating trees of order $k$ captures the block decomposition of $\pi_f$. Take a tree of height $h = l$ corresponding to a floorplan $f$, and let the root node be labeled by $g$ and children be labeled $g_1, \ldots, g_k$. By Lemma 1 $\pi_f = \pi_g (\pi_{g_1}, \ldots, \pi_{g_k})$. We can apply induction hypothesis on the children to get the decomposition of $\pi_{g_1}, \ldots, \pi_{g_k}$.

To prove the uniqueness of skewed generating trees we use the following theorem by Albert and Atkinson [2, Proposition 2] proving the uniqueness of the block-decomposition represented by skewed generating trees.

**Theorem 2.** For every non singleton permutation $\pi$ there exists a unique simple non singleton permutation $\sigma$ and permutations $\alpha_1, \ldots, \alpha_n$ such that

$$\pi = \sigma (\alpha_1, \ldots, \alpha_n)$$

Moreover if $\sigma \neq 12, 21$ then $\alpha_1, \ldots, \alpha_n$ are also uniquely determined. If $\sigma = 12$ (respectively, 21) then $\alpha_1$ and $\alpha_2$ are also uniquely determined subject to the additional condition that $\alpha_1$ cannot be written as $12 \beta, \gamma$ (respectively as $21 \beta, \gamma$)
The proof is completed by noting that the decomposition obtained by Skewed Generating Trees of Order $k$ satisfies the properties of the decomposition described in the above theorem. In a skewed generating tree if parent is $\sigma = 12$ (respectively, $21$), then its left child cannot be $12$ (respectively, $21$). Hence the block-decomposition corresponding to the left child, $\alpha$, cannot be $(12) [\beta, \gamma]$ (respectively, $(21) [\beta, \gamma]$). Since such a decomposition is unique, the skewed generating tree also must be unique. Hence the theorem.

To characterize $HFO_k$ in terms of pattern avoiding permutations the following insight is used: if a permutation $\pi$ is Baxter then it corresponds to a mosaic floorplan. Every mosaic floorplan is $HFO_k$ for some $k$. Hence for a Baxter permutation $\pi$ the corresponding floorplan $f_\pi$ is not $HFO_k$ for some specific $k$, it will be because of existence of a node in the unique skewed generating tree corresponding to $f_\pi$, which is labeled by an in-decomposable mosaic floorplan $g \in HFO_l$ for some $l > k$. Since $\pi$ is obtained by inflation of permutations including $\pi_g$ corresponding to $g$, $\pi$ will have some text which matches the pattern $\pi_g$ because of the Lemma 2. Thus if we can figure out all the patterns which correspond to in-decomposable mosaic floorplans which are $HFO_l$ for some $l > k$ then $HFO_k$ would be all Baxter permutations which avoid those patterns. We defer the proof of Lemma 2 to the Appendix (see \[7\]).

**Lemma 2.** If $\pi = \sigma (\rho_1, \ldots, \rho_k)$, then $\pi$ contains all patterns which any of $\sigma, \rho_1, \rho_2, \ldots, \rho_k$ contains.

We will use the following lemma which is proved in the Appendix (see \[8\]).

**Lemma 3.** If $\pi = \sigma (\rho_1, \ldots, \rho_k)$, then any block in-decomposable pattern in $\pi$ has a matching text which is completely contained in one of $\sigma, \rho_1, \rho_2, \ldots, \rho_k$.

Let $f$ be an in-decomposable mosaic floorplan which is $HFO_l$ for some fixed $l \in \mathbb{N}$. By Corollary 1, the permutation corresponding to $f$, $\pi_f$ would be block in-decomposable and hence it will be a simple permutation of length $l$. It is known (see [2, Theorem 5]) that a simple permutation of length $l$ has either a one-point deletion which yields another simple permutation or two one-point deletions giving a simple permutation. Hence by successive applications of one-point deletions we can reduce $\pi_f$ to a simple permutation of length $k$, or an exceptionally simple permutation of length $k + 1$ (at which point there is no further one-point deletion giving a simple permutation) for any $k < l$. Also if $\pi'$ is obtained from $\pi$ by a one-point deletion at index $i$, then $\pi [1, \ldots, i - 1, i + 1, \ldots, n]$ matches the pattern $\pi'$. That is $\pi$ contains all patterns $\pi'$ which are permutations obtained by one point deletion of $\pi$ at some index. Also since pattern containment is transitive by definition, if $\pi''$ is obtained by one-point deletion of $\pi'$ which in turn obtained from $\pi$ by a one-point deletion, then $\pi'' \leq \pi'$ and $\pi' \leq \pi$ implies that $\pi'' \leq \pi$. From these observations we get the following characterization of $HFO_k$.

**Theorem 3.** A mosaic floorplan $f$ is $HFO_k$ if and only if the permutation $\pi_f$ corresponding to $f$ (obtained by algorithm FP2BP) does not contain patterns from simple permutations of length $k + 1$ or exceptionally simple permutations of length $k + 2$. 


Proof. By Theorem 1, for any HFOₖ floorplan f there is a unique Skewed Generating Trees of Order k, Tₖ such that it captures the block-decomposition of πₖ. And in the block-decomposition of a generating tree of order k, permutations corresponding to the nodes are labeled by HFOₖ permutations of length at most k. Hence the block-decomposition of πₖ contains only block in-decomposable permutations of length at most k. By Lemma 2, πₖ cannot contain patterns which are block in-decomposable permutations of length strictly more than k. Thus πₖ cannot contain patterns from simple permutations of length k + 1 or from exceptionally simple permutations of length k + 2 as they are both classes of block in-decomposable permutations of length strictly greater than k.

For the reverse direction, we prove that any mosaic floorplan which is HFOₙ, l > k contains either a simple permutation of length k + 1 or an exceptionally simple permutation of length k + 2. From the fact that by definition any mosaic floorplan is HFOₖ for some j and the forward direction that no HFOₖ floorplan contains either a simple permutation of length k + 1 or an exceptionally simple permutation of length k + 2 proof is completed. Suppose if it is HFOₙ for l > 0 then πₖ would have a text matching a pattern σ ∈ Sₖ which is a simple permutation. Because the generating tree Tₖ will have σ and so would the block decomposition of the sub-tree rooted at node σ. And by Lemma 2 πₖ would also contain σ. From σ we can obtain by successive one-point deletions a permutation σ′ which is either a simple permutation of length k or is an exceptionally simple permutation of length k + 1. And σ′ would match a text in πₖ because πₖ had a text matching σ and σ contains this permutation, i.e., σ′ ≤ σ ≤ πₖ ⇒ σ′ ≤ πₖ.

From the above characterization it can be proved that the hierarchy HFOₖ (it is a hierarchy because by definition HFOₖ ⊆ HFOₖ₊₁) is strict for k ≥ 7, i.e., there is at least one floorplan which is HFOₖ but is not HFOₖ for any i < k. The natural candidates for such separation are in-decomposable mosaic floorplans on k rooms which corresponds to simple permutations of length k which are Baxter. It is easy to verify that for k = 5, π₅ = 41352 is such a permutation. Note that π₅ is of the form π[n − 1] = n and π[n] = 2. From π₅ we can obtain π₇ = 6413572 by inserting 7 between 5 and 2 and appending 6 at the beginning. It can be verified that π₇ is not HFO₅. It turns out that all permutations of length at most 4 which are Baxter are also HFO₂, making HFO₅ the first odd number from where one can prove the strictness of the hierarchy. Also every HFO₆ is HFO₅, hence for even numbers separation theorem can only start from 8. Hence we prove the separation theorem for k ≥ 7 generalizing the earlier stated idea. The generalization builds a πₖ₊₂ from a πₖ which is an in-decomposable HFOₖ having π[n − 1] = n and π[n] = 2, by setting πₖ₊₂[1] = n + 1, πₖ₊₂[i] = πₖ[i − 1], 2 ≤ i ≤ n, πₖ₊₂[n + 1] = n + 2 and πₖ₊₂[n + 2] = 2. The proof of the theorem is deferred to the Appendix (see 3).

**Theorem 4.** For any k ≥ 7, there exists a floorplan f which is in HFOₖ₊₂ but is not in HFOₖ for any l ≤ k + 1
4 Combinatorial study of HFO\textsubscript{k}

We will first prove for any fixed \(k\) the existence of a rational generating function for HFO\textsubscript{k}. Since we have proved that the number of distinct HFO\textsubscript{k} floorplans with \(n\) rooms is equal to the number of distinct Skewed Generating Trees of Order \(k\) with \(n\) leaves, it suffices to count such trees. Let \(t^5_n\) denote the number of distinct Skewed Generating Trees of Order \(k\) with \(n\) leaves and \(t^5_k\) represent a rectangle for any \(k\). Hence to provide a rational generating function for number of distinct HFO\textsubscript{k} floorplans with \(n\) rooms, it suffices to provide one for the count \(t^5_n\).

We will first describe the method for HFO\textsubscript{5}. For simplicity of analysis let \(t_i = t^5_i\). Skewed Generating Trees of Order 5 are labeled by simple permutation of length at most 5 which are Baxter. There are only four of them - 12, 21, 25314 and 41352. Thus the root node of such a tree also must be labeled from one of these four permutations. We obtain a recurrence by partitioning the set of Skewed Generating Trees of Order 5 into four classes decided by the label of the root. Let \(a_n\) denote the number of Skewed Generating Trees of Order 5 with \(n\) leaves whose root is labeled 12, \(b_n\) denote the number of Skewed Generating Trees of Order 5 with \(n\) leaves whose root is labeled 21, \(c_n\) denote the number of Skewed Generating Trees of Order 5 with \(n\) leaves whose root is labeled 25314. Since these are the only in-decomposable HFO\textsubscript{k} permutations for \(k \leq 5\), the root (and also any internal node) has to be labeled by one of these permutations. Hence we get the following recurrence for \(t^5_n, t^5_n = a_n + b_n + c_n + d_n\).

In a skewed tree if the root is labeled 12, its left child cannot be 12 but it can be 21, 41352, 25314 or a leaf node. Hence the left child of the root of a tree in \(a_n\) has to be labeled from \(b, c\) or \(d\), but the right child has no such restriction. By definition of skewed generating trees if the root is labeled by a permutation of length \(l\), it will have \(l\) children, such that the number of leaves of the children sum to \(n\). Hence if root is labeled by 12, the two children will have leaves \(n - i\) and \(i\) for some \(i, 1 \leq i \leq n - 1\). This along with the skew rule dictates that \(a_n = \sum_{i=1}^{n-1} b_{n-i} + c_{n-i} + d_{n-i} t_i\). Similarly if the root is 21 then its left child cannot be 21 but it can be 12,41352, 25314 or a leaf node. But for trees whose roots are labeled 41352/25314, they can have any label for any of the five children. Hence we get, \(a_n = t^5_{n-1} + \sum_{i=2}^{n-1} (b_i + c_i + d_i) t^5_{n-i}, b_n = t^5_{n-1}.1 + \sum_{i=2}^{n-1} (a_i + c_i + d_i) t^5_{n-i}, c_n = \sum_{i,j,k,l,m \geq 0; i+j+k+l+m = n} t^5_i t^5_j t^5_k t^5_l t^5_m\) and \(d_n = \sum_{i,j,k,l,m \geq 0; i+j+k+l+m = n} t^5_i t^5_j t^5_k t^5_l t^5_m\). Note that \(c_n = d_n\). Since a node labeled 41352/25314 ought to have five children, \(c_n, d_n = 0\) for \(n < 5\). Summing up \(a_n\) and \(b_n\) and using the identity \(t^5_i = a_i + b_i + c_i + d_i\) we get the following recurrence for \(t^5_n\)

\[
t_n = t^5_{n-1} + \sum_{i=1}^{n-1} t^5_{n-i} t^5_i + 2 \sum_{i,j,k,l,m \geq 1; i+j+k+l+m = n} t^5_i t^5_j t^5_k t^5_l t^5_m + 2 \sum_{i,j,k,l,m \geq 1; i+j+k+l+m = n} t^5_i t^5_j t^5_k t^5_l t^5_m
\]
Define the ordinary generating function \( T(z) \) associated with the sequence \( t_n \) to be 
\[
T(z) = \sum_{n=1}^{\infty} t_n z^{n-1}.
\]
Multiplying the recurrence with \( \sum_{n=1}^{\infty} z^n = \frac{1}{1-z} \), we get 
\[
T(z) = z T(z) + z^2 T(z) + z^3 T(z) + z^4 T(z) + z^5 T(z) + t_1.
\]
Substituting \( t_1 = 1 \), gives the following polynomial equation in \( T(z) \), 
\[
z^5 T(z) + z^4 T(z) + z^3 T(z) + (z-1)T(z) + 1 = 0.
\]
Unfortunately this is a polynomial of sixth degree in \( T(z) \). Hence no general solution is available for its roots, which are needed to obtain the closed form expression for the above recurrence relation.

Note that in a similar way recurrence relation for any HFO \( k \) can be constructed. Again it will be a polynomial in \( T(z) \) with degree \( l \) where \( l \) is the smallest \( l \) such that HFO \( k \).

Even though the above recurrence fails to give a closed form solution it leads to a natural dynamic programming based algorithm for counting the number of HFO \( k \) floorplans with \( n \) rooms. For example the recurrence for HFO \( 5 \) is given by a sixth order recurrence relation given in Equation 1. Hence there is an \( O(n^6) \) tabular algorithm computing the value of \( t_n \) using dynamic programming which recursively computes all \( t_i \) for all \( i < n \) and then computes \( t_n \) from Equation 1. In general HFO \( k \) has a recurrence relation of order \( k \), and hence the algorithm for \( t_n^k \) would run in time \( O(n^{k+1}) \) using a similar strategy.

Using the argument which proved existence of an in-decomposable HFO \( k \) floorplan for any \( k \), we can get a simple lower bound on the number of HFO \( k \) floorplans with \( n \) rooms which are not HFO \( j \) for any \( j < k \). It is known [8] that the number of HFO \( 2 \) floorplans with \( n \) rooms is \( \theta(n^{(3+\sqrt{8})/2}) \). If in the generating tree corresponding to an HFO \( 2 \) floorplan an in-decomposable HFO \( k \) floorplan is inserted replacing one of the leaves (to be uniform, say the right most leaf), the resulting generating tree would be of order \( k \) and hence by Theorem 1 would correspond to an HFO \( k \) floorplan. Hence the number of HFO \( k \) floorplans with \( n \) rooms which are not HFO \( i \) is at least the number of generating trees of order 2 with \( n-k+1 \) leaves. And the number of generating trees of order 2 with \( n \) leaves equals the number of HFO \( 2 \) floorplans with \( n \) rooms thus giving the following exponential lower bound.

**Observation 1** For any \( k \geq 7 \), the number of HFO \( k \) floorplans with \( n \) rooms which are not HFO \( j \) for any \( j < k \) is at least
\[
\frac{(n-k)! \left(3 + \sqrt{8}\right)^{n-k}}{(n-k)^{1.5}}
\]

### 5 Algorithm for membership

For arriving at an algorithm for membership in HFO \( k \) we note that if a given permutation is Baxter then it is HFO \( k \) for some \( k \). And if it is HFO \( k \) by Theorem 1 there exists an order \( k \) generating tree corresponding to the permutation. By Theorem 1 the generating tree also captures the block decomposition of the
permutation. For sake of brevity we defer the formal description of the algorithm to the Appendix (see Algorithm 1). Our algorithm identifies the block-decomposition corresponding to the generating tree of order $k$, level by level. It can be thought of as a *deflating* algorithm, i.e., it finds the block decomposition which when *inflated* gives the input permutation. The algorithm first identifies the blocks length at most $k$ in the input permutation which corresponds to the leaves of the generating tree. Upon finding a block algorithm replaces the block with the interval $[i,j]$ where $[i,j]$ are the elements of the block. Hence after the first round the input permutation is changed to an ordered arrangement of entries which are intervals $[i,j]$ for some $i \leq j$. And in the subsequent round the algorithm tries to identify the blocks of at most $k$ such entries. The rounds continue until the permutation is reduced to a single entry $[1,n]$ or till a round fails to identify a block of length at most $k$. If the given permutation is reduced to a single permutation at the end, the algorithm guarantees that there is a block decomposition of the given permutation where the maximum in-decomposable block is of length $k$. Hence if the permutation, after running the algorithm is reduced to a single permutation, it is indeed HFO$_k$. And if the permutation is HFO$_k$ then there is a generating tree of order $k$ corresponding to it, and this guarantees that the algorithm would be able to reduce it to a single permutation by the level by level compression strategy.

Note that checking if a set $S$ of $k$ elements form a range can be checked in constant time for a fixed value of $k$ by subtracting from each element $\min_{i \in S}(i)-1$ and then checking if the elements follow any of the $k!$ arrangements. We can also check if a set of $k$ elements form a Baxter permutation for a fixed $k$ in constant time by checking if their rank ordering is equivalent to any one of the Baxter permutations of length $k$(whose number is bounded by number of permutations, $k!$). After each round of algorithm at least one non-trivial block-decomposition is identified and deflated. Hence in each round the number of nodes in the corresponding generating tree reduces by at least one. Note that if the input permutation is not HFO$_k$, then algorithm progresses only till it can find a block-decomposition which can be deflated. Hence the number of rounds is linear in the number of nodes of the generating tree. And each round takes at most linear time. Since any tree with $n$ leaves where each internal node has degree at least 2 has, at most $n-1$ internal nodes, the total running time is $cn(2n-1)$. Hence the above algorithm runs in $O(n^2)$ time for a predetermined value of $k$.

For a fixed $k$ we can also achieve linear time for membership owing to a new fixed parameter algorithm of Marx and Guillemot [4] which given two permutations $\sigma \in S_k$ and $\pi \in S_n$ checks if $\sigma$ avoids $\pi$ in time $2^{O(k^2 \log k)} n$ and a linear time algorithm for recognizing Baxter permutations by Hart and Johnson [5]. Both results ([4],[5]) are highly non-trivial and deep. Theorem 3 guarantees that it is enough to ensure that $\pi$ is Baxter and $\pi$ and avoids simple permutations of length $k+1$ and exceptionally simple permutations of length $k+2$. Using the

\footnote{Figure 19 given in Appendix illustrates the decompositions identified at each round of the algorithm on input 13274685 checking whether it is HFO$_5$ or not.}
algorithm given by Hart and Johnson [5] we can check in linear time whether a given permutation is Baxter or not. Since there are at most \((k + 1)!\) simple permutations of length \(k\) and at most \((k + 2)!\) exceptionally simple permutations of length \((k + 2)\) using the algorithm given in [4] as a sub-routine we can do the latter in \(O((k + 2)!2^{(k+2)^2\log(k+2)}n)\) time. Since \(k\) is a fixed constant we get a linear time algorithm.

If the value \(k\) is unknown Algorithm 1 can be used to get an \(O(n^4)\) algorithm with a few modifications to find out the minimum \(k\) for which the input permutation is HFO\(_k\). The first modification is to make the algorithm check if the input permutation \(\pi\) is Baxter permutation. If it is not, it cannot be HFO\(_j\) for any \(j\) and hence is rejected. If it is a Baxter permutation then it is HFO\(_k\) for some \(k \leq n\). And in each round we check for the minimum \(j, 1 < j \leq |S|\) for which the top \(j\) elements form a range \([l, m]\) and is a Baxter permutation shifted by \(l\). Checking if a permutation is Baxter takes \(O(n^2)\) time. And as earlier there are at most \(2n\) rounds. In each round checking whether a set of elements forms a range takes \(O(n \log_2 n)\) time and checking if the resulting permutation is Baxter takes \(O(n^2)\) time. Since there are at most \(n\) elements in the stack at any time, the worst case cost of a round is \(O(n^3)\). Hence the running time of the algorithm is \(O(n^4)\).

We also note that algorithm for membership becomes much simpler if you want to check whether a permutation is HFO\(_k\) for a fixed \(k\). Because of Theorem 1 for any HFO\(_k\) permutation \(\pi\) there is a unique Skewed Generating Trees of Order \(k\), \(T_\pi\) such that the tree yields block decomposition of \(\pi\) when thought of as a parse tree. It is easy to see that the recurrence for generating trees or order \(k\) based on what root is labeled by gives a context free grammar for generating such tree. See Appendix A.4 for the details of the algorithm based on the context free grammar approach.

6 Closure properties of Baxter permutations

Only recently it has been proved that Baxter permutations are closed under inverse [6]. The proof in [6] uses an argument based on permutations and patterns. We give a simple alternate proof of this fact using the geometrical intuition derived from mosaic floorplans. We prove that the floorplan obtained by taking a mirror image of a floorplan along the horizontal axis is a floorplan whose permutation (under the bijection of Ackerman) is the inverse of the permutation corresponding to the starting floorplan.

The intuition is that when the floorplan’s mirror image about the horizontal axis is taken, it does not change the relationship between two rooms if one is to the left of the other. But if a room is below the other, it flips the relationship between the corresponding rooms. For any Baxter permutation \(\pi\) and two indices \(i, j\) where \(i < j\), if \(\pi[i] < \pi[j]\), since \(\pi[i]\) appears before \(\pi[j]\) by the property of the algorithm FP2BP \(\pi[i]\) is to the left of \(\pi[j]\) in \(\pi_f\). In the inverse of \(\pi\), \(\pi^{-1}\) indices \(\pi[i]\) and \(\pi[j]\) will be mapped to \(i\) and \(j\) respectively. Hence if \(\pi^{-1}\) is Baxter, or equivalently there is a mosaic floorplan corresponding to \(\pi^{-1}\), \(\pi_f^{-1}\),
the rooms labeled by $i$ and $j$ will be such that $i$ precedes $j$ in the top-left deletion ordering (as $i < j$) and also in bottom left deletion ordering (as $\pi[i] < \pi[j]$). Hence $i$ is to the left of $j$ in $\pi_f^{-1}$. If $\pi[i] > \pi[j]$, since $\pi[i]$ appears before $\pi[j]$ by the property of the algorithm FP2BP, $\pi[i]$ is below $\pi[j]$ in $\pi_f$. In the inverse of $\pi$, $\pi^{-1}$ indices $\pi[i]$ and $\pi[j]$ will again be mapped to $i$ and $j$ respectively. Hence if $\pi[i] > \pi[j]$, since $\pi[i]$ appears before $\pi[j]$ by the property of the algorithm FP2BP, $\pi[i]$ is below $\pi[j]$ in $\pi_f$. In the inverse of $\pi$, $\pi^{-1}$ indices $\pi[i]$ and $\pi[j]$ will again be mapped to $i$ and $j$ respectively. Hence if there is a mosaic floorplan corresponding to $\pi^{-1}_f$, $\pi^{-1}$, the rooms labeled by $i$ and $j$ will be such that $i$ precedes $j$ in the top-left deletion ordering (as $i < j$) but in bottom left deletion ordering $j$ precedes $i$ (as $\pi[i] < \pi[j]$). Hence $i$ is above $j$ in $\pi^{-1}_f$. Thus mirror image about horizontal axis satisfies all these constraints on the rooms.

For the formal proof of closure under inverse we use the following three lemmas. For sake of brevity we defer the proofs to the Appendix (see [10] and [9]).

**Lemma 4.** For any mosaic floorplan $f$, the floorplan obtained by deleting a room from the bottom-left corner of $f$ and then taking a mirror image about horizontal axis is equivalent to the floorplan obtained by taking a mirror image of $f$ about horizontal axis and then deleting a room from the top-left corner.

**Lemma 5.** For any mosaic floorplan $f$, let $g$ be the floorplan obtained from $f$ by taking a mirror image about the horizontal axis. Then the $i$th ($1 \leq i \leq n$) room deleted from $f$ during the extraction phase of algorithm FP2BP on $f$ is the $i$th room to be deleted in the labeling phase of algorithm FP2BP on $g$.

**Lemma 6.** For any mosaic floorplan $f$, let $g$ be the floorplan obtained from $f$ by taking a mirror image about the horizontal axis. Then the $i$th ($1 \leq i \leq n$) room deleted from $f$ during the labeling phase of algorithm FP2BP on $f$ is the $i$th room to be deleted in the extraction phase of algorithm FP2BP on $g$.

Now we can proceed to the proof of closure of Baxter permutations under inverse.

**Theorem 5.** Given a mosaic floorplan $f$, the floorplan $g$ obtained by taking the mirror image of $f$ about the horizontal axis is such that $\pi_f^{-1} = \pi_g$ where $\pi_f, \pi_g$ are the Baxter permutations corresponding to the mosaic floorplans $f$ and $g$ respectively.

**Proof.** Once again note that taking the mirror image of a mosaic floorplan results in a mosaic floorplan (as cross junctions do not appear through a rotation). From the definition algorithm FP2BP for any $i$, $\pi_f[i] = j$ is the $i$th room to deleted in the extraction phase of FP2BP on $f$. And $\pi_g^{-1}(i)$ is the $i$th room to be deleted from $g$ in the labeling phase of FP2BP. By Lemma [7] these rooms are one and the same. By Lemma [6] $j$th room to be labeled in $f$ is same as the $j$th room to be extracted in $g$. That is $\pi_g[j] = i$, which means that $\pi_g^{-1}(i) = j = \pi_f[i]$ for any $i$. Hence $\pi_g$ is the inverse of $\pi_f$.

Figure [13] illustrates the above mentioned link between inverse and the geometry.
From Theorem 5 we get the following corollary,

**Corollary 2.** For any $k \in \mathbb{N}$, if $\pi \in \text{HFO}_k$ then so is $\pi^{-1}$.

**Proof.** Since $\pi \in \text{HFO}_k$, $\pi$ is also a Baxter permutation. And according to Theorem 5 $\pi^{-1}$ is also a Baxter permutation whose corresponding floorplan (under the bijection of Ackerman et. al) is obtained by taking the mirror image of $f_\pi$ about the horizontal axis. Theorem 1 guarantees that there is a generating tree $T_\pi$ of order $k$ corresponding to $f_\pi$. The nodes of the tree are labeled by Baxter permutations of length at most $k$. Now obtain a new tree $T'$ by relabeling each node, starting from root, by inverse of the permutation labeling the node and moving the children of the node accordingly. For a node $u \in T_\pi$ the corresponding node $u' \in T'$ gets labeled by inverse of the Baxter permutation labeling $u$. Note that this still is Baxter permutation of length at most $k$. Hence the generating tree $T'$ represents an HFO$_k$ permutation because of Theorem 1. It is not hard to verify that the floorplan represented by $T'$ is the mirror image of $f_\pi$ taken about the horizontal axis.

We also observe that there is a geometric interpretation for reverse of a Baxter permutation. Note that it is easy to see that Baxter permutations are closed under reverse because the patterns they avoid are reverses of each other (3142/2413). We observe, without giving a proof, that for a Baxter permutation $\pi$ its reverse $\pi^r$ corresponds to the mosaic floorplan that is obtained by first rotating by $90^\circ$ clockwise and then by taking a mirror image along the horizontal axis. See Figure 20 in the Appendix for an illustration of this link.

### 7 Open Problems

One natural open problem arising from this work is that of exact formulae for the number of HFO$_k$ floorplans. The only $k$ for which exact count is known is $k = 2$. Our proof of closure under inverse for Baxter permutation gives rise to the following open problem. For a class of permutations characterized by pattern avoidance, like Baxter permutations, to be closed under inverse is it enough that the forbidden set of permutations defining the class is closed under inverse.
A Appendix

A.1 Example Floorplans

We provide example floorplans for various HFO\(_k\). Figure 14 shows an HFO\(_2\) with more than 2 rooms. Figure 15 shows the smallest non-slicing (non HFO\(_2\)) floorplan (and it contains five rooms). The structure is called a “pin-wheel” and there are only two of them, one right rotating and another left rotating as shown in Figure 15. Figure 16 shows an HFO\(_5\) which is not HFO\(_2\) by slicing a wheel. Figure 17 shows an HFO\(_8\) which is not an HFO\(_7\) floorplan. Figure 18 shows another HFO\(_8\) with 10 rooms.

Fig. 14. A slicing(HFO\(_2\)) floorplan with 4 rooms
Fig. 15. Only non HFO\textsubscript{2} floorplans with at most five rooms

Fig. 16. An HFO\textsubscript{3} which is not HFO\textsubscript{2}

Fig. 17. An HFO\textsubscript{8} which is not HFO\textsubscript{7}

Fig. 18. An HFO\textsubscript{8} with 10 rooms, with the blocks identified
A.2 Proof’s omitted from the main paper

Corollary 3. A mosaic floorplan $f$ is in-decomposable if and only if the Baxter permutation $\pi_f$ corresponding to it is block in-decomposable.

Proof. If $f$ is decomposable then Lemma 1 guarantees that $\pi_f$ is also decomposable. Suppose $\pi_f$ is decomposable say $\pi_f = \rho(\sigma_1, \ldots, \sigma_k)$, then Lemma 1 guarantees that the floorplan $f'$ obtained by insertion of $f_p$ by $f_{\sigma_1}, \ldots, f_{\sigma_k}$ (the floorplans corresponding to permutations $\rho, \sigma_1, \ldots, \sigma_k$) also corresponds to $\pi_f$. Since mosaic floorplans are in bijective correspondence with Baxter permutations it must be the case that $f \equiv f'$.

Lemma 7. If $\pi = \sigma(\rho_1, \ldots, \rho_k)$, then $\pi$ contains all patterns which any of $\sigma, \rho_1, \rho_2, \ldots, \rho_k$ contains.

Proof. Let $\pi = \sigma(\rho_1, \ldots, \rho_k)$. If $\sigma$ has a text indexed by $i_1, \ldots, i_m$ matching a pattern $\alpha \in S_m$, then by taking an arbitrary element from each block corresponding to $\rho_1, \ldots, \rho_m$ in $\pi$, a text matching $\alpha$ is obtained. This is because inflation orders blocks corresponding to $\rho_1, \rho_2, \ldots, \rho_k$ by $\sigma$. Similarly if some $\rho_i$ has a text matching some pattern $\alpha$ then $\pi$ having the block corresponding to $\rho_i$ contains the same text as this block preserves the relative ordering of elements inside the block according to $\rho_i$.

Lemma 8. If $\pi = \sigma(\rho_1, \ldots, \rho_k)$, then any block in-decomposable pattern in $\pi$ has a matching text which is completely contained in one of $\sigma, \rho_1, \rho_2, \ldots, \rho_k$.

Proof. Recall that a pattern is block in-decomposable if it does not have any non-trivial blocks. Suppose if the pattern $\gamma$ which is block in-decomposable has at least two matching text elements from any $\rho_i$ and also contains a matching text element from $\rho_j$ for $j \neq i$, then the matching text elements from $\rho_i$ forms a non-trivial block of $\gamma$ as $\rho_i$’s form blocks in $\pi$ by definition of inflation. Hence $\gamma$ can have either at most one matching text element from each $\rho_i$ in which case $\sigma$ also contains the text by definition of inflation, or it is completely contained in one of $\rho_i$’s. Hence the theorem.

Lemma 9. For any mosaic floorplan $f$, let $g$ be the floorplan obtained from $f$ by taking a mirror image about the horizontal axis. Then the $i$th ($1 \leq i \leq n$) room deleted from $f$ during the labeling phase of algorithm $FP2BP$ on $f$ is the $i$th room to be deleted in the extraction phase of algorithm $FP2BP$ on $g$.

The proof is similar to the proof of earlier lemma.

Theorem 6. For any $k \geq 7$, there exists a floorplan $f$ which is in $\text{HFO}_{k+2}$ but is not in $\text{HFO}_l$ for any $l \leq k + 1$.

Proof. We generalize the above procedure to one which obtains $\pi_{k+2}$ from a $\pi_k$ which is an in-decomposable $\text{HFO}_k$ having $\pi[n-1] = n$ and $\pi[n] = 2$, as follows $\pi_{k+2}[1] = n+1$, $\pi_{k+2}[i] = \pi_k[i-1]$, $2 \leq i \leq n$, $\pi_{k+2}[n+1] = n + 2$ and $\pi_{k+2}[n+2] = 2$. If $\pi_k$ is a Baxter permutation which is also simple then
\( \pi_{k+2} \) is also Baxter and simple. Before discussing the proof we can see why this help prove the theorem. For \( k = 5, \pi_5 = 41352 \) satisfies the conditions, and the procedure makes sure that successive \( \pi_k \)'s obtained also have \( \pi [n - 1] = n \) and \( \pi [n] = 2 \) allowing us to separate any odd \( k \) and \( k + 2 \). For even \( k \) we observe that \( \pi_8 = 75146382 \) satisfies the necessary conditions hence proving a similar theorem for even number \( k, k \geq 8 \).

We prove that \( \pi_{k+2} \) obtained as above is both simple and Baxter. Since \( \pi_k \) is simple, if there is a nontrivial block in \( \pi_{k+2} \) it must include one of the newly inserted elements (i.e., \( n + 1 \) or \( n + 2 \)) or the moved element (i.e., \( 2 \)). But if the non-trivial block includes \( n + 2 \), it must involve \( n + 1 \) as there are no elements greater than \( n + 2 \), but in this case the block will have to include 1 as it is between \( n + 2 \) and \( n + 1 \), but if it includes 1 it becomes a trivial block which is the entire permutation. Similarly if the block involves \( n + 1 \) it must include \( n \), or \( n + 2 \) but in both cases 1 must be included, thus making the block trivial. If it involves \( 2 \) it must also include \( n + 2 \) as it is the only number adjacent to it, but we have already proved that the non-trivial block cannot contain \( n + 2 \). Hence \( \pi_{k+2} \) is simple.

To see why \( \pi_{k+2} \) is Baxter, assume to the contrary there exists a text at indices \( i_1, i_2, i_3, i_4 \) which matches 3142/2413 with \( |\pi [i_1] - \pi [i_4]| = 1 \), then at least one of \( \pi [i_1], \pi [i_2], \pi [i_3], \pi [i_4] \) must be from \( \{ n + 1, n + 2 \} \). Because otherwise the same text would appear in \( \pi_k \) also as we have not changed the relative positions of elements in \( \pi_k \) while constructing \( \pi_{k+2} \). If it involves \( n + 2 \) it must be the case that it matches 4 as it is the largest element in \( \pi_{k+2} \). Since there are no two elements after \( n + 2 \) in one-line form of \( \pi_{k+2} \) it cannot match 2413, and if it matches 3142 it must be the case that 2 in \( \pi_{k+2} \) matches the 2 in the pattern as it is the only number to the right of \( n + 2 \) in the one-line form of \( \pi_{k+2} \). But if 2 matches 2 in the pattern, 1 of \( \pi_{k+2} \) must match 3 of the pattern hence 3 of \( \pi_{k+2} \) must match 3 in the pattern as we need absolute difference of numbers matching 3 and 2 to be 1. But in this case we can replace \( n + 2 \) in the text with \( n \) to get a new text which is also present in \( \pi_k \) matching 3142, contradicting our assumption that \( \pi_k \) is Baxter. Hence it cannot involve \( n + 2 \), but if it involves \( n + 1 \) by virtue of it having only element greater than itself in \( [n + 2] \), it cannot match 1 or 2 in the pattern. Also it cannot match 4 in 3142/2413 as there is no number to the left of \( n + 1 \) in one-line form of \( \pi_{k+2} \). Similarly it cannot match 3 in 2413 as there is nothing to the left of \( n + 1 \). Thus it can only match 3 in 3142. But if \( n + 1 \) matches 3 then \( n + 2 \) must match 4 as it is the only number greater than \( n + 1 \) and thus forcing \( n \) to match 2 as the absolute value of difference is 1. But \( n + 2 \) matching 4 lies to the right of \( n \) matching 2, rendering such a case impossible.

Thus the permutation obtained by above construction is both simple and Baxter. Hence it contains a pattern which is a simple permutation of length \( k + 2 \) (the whole permutation). This fact along with Theorem 8 implies that the permutation thus obtained cannot be HFO\( \ell \) for \( \ell < k + 2 \).

**Lemma 10.** For any mosaic floorplan \( f \), the floorplan obtained by deleting a room from the bottom-left corner of \( f \) and then taking a mirror image about
**horizontal axis is equivalent to the floorplan obtained by taking a mirror image of f about horizontal axis and then deleting a room from the top-left corner.**

*Proof.* Ackerman et al. proved that deletion operation does not change the relation between any two blocks (see proof of, [1] Observation 3.4) of a mosaic floorplan. And the top-left room of the floorplan obtained by taking a mirror image of f about horizontal axis is the bottom-left room of f. This combined with the fact that image about the horizontal axis does not change the relationship between two rooms if one is to the left of other, but flips relationship between two rooms if one is below the other proves the theorem.

**Lemma 11.** For any mosaic floorplan f, let g be the floorplan obtained from f by taking a mirror image about the horizontal axis. Then the ith (1 ≤ i ≤ n) room deleted from f during the extraction phase of algorithm FP2BP on f is the ith room to be deleted in the labelling phase of algorithm FP2BP on g.

*Proof.* Note that g is also a mosaic floorplan as reflections cannot introduce cross junctions. The proof is an induction on i. When i = 1 the first room to be deleted from f in the extraction phase is the bottom-left room of f. Clearly it is the top-left room in the floorplan g. In the labelling phase of FP2BP on g the first room to be labelled is the top-left room. Hence they are one and the same.

Assume that the hypothesis is true for i, that is the ith room to be deleted from f is the ith room labelled in g. Let f' be the floorplan obtained from f by deleting i rooms from the bottom-left corner. Similarly g' is obtained from g by deleting i rooms from the top-left corner. Repeated application of Lemma [4] implies that mirror image f' about horizontal axis is equivalent to g'. The bottom-left room of f' is the i+1th room to be deleted from in f in the extraction phase of FP2BP. The bottom-left room of f' is the top-left room of g' as mirror image of f' about horizontal axis is equivalent to g'. Hence the it is the room to be deleted from the top-left corner of g'. But by definition of g' it is the i + 1th room to be labeled in the labeling phase of FP2BP on g. Hence the theorem.
A.3 Pseudo-code for the Algorithm for membership

**Input:** A permutation \( \pi \) of length \( n \)

1. Stack \( S \) ← φ;
2. Stack \( SS \) ← φ;
3. Boolean Deflated ← true;
4. for \( i = 1 \) to \( n \) do
5. \( S \).push(\([\pi[i], \pi[i]]\));
6. end
7. while Deflated \( \text{AND} \) \( S \).size() ≠ 1 do
8. Deflated = false;
9. while There exists a \( l,j \), such that \( j \) is the maximum such number less than \( k \) for which top \( j + 1 \) elements of \( S \) is \( [l,l+j] \) do
10. if \( S \) [top...(top−j)] is a Baxter permutation shifted by \( l−1 \) then
11. \( R = [l,l+j] \);
12. if \( l \neq l+j \) then
13. Deflated = true;
14. end
15. for \( m = j \) downto 0 do
16. \( S \).pop();
17. end
18. \( SS \).push(\( R \));
19. end
20. end
21. while \( SS \).size() ≠ 0 do
22. \( S \).push(\( SS \).pop());
23. end
24. end
25. if \( S \).size() = 1 then
26. Accept;
27. else
28. Reject;
29. end

**Algorithm 1:** Algorithm for checking if a permutation is \( \text{HFO}_k \)

The following figure demonstrates the identification of blocks as employed by the above algorithm on an \( \text{HFO}_5 \) permutation.

| Stage 1       | 1 2 3 4 5 6 7 |
|---------------|---------------|
| Stage 2       | 1 2 3 4 7     |
| Stage 3       | 1 2 7         |

**Fig. 19.** Running of algorithm for \( \text{HFO}_5 \) recognition
A.4 Context Free Grammar Approach for checking membership

The context free grammar corresponding to Skewed Generating Trees of Order 5, which yields block-decomposition of all HFO₅ permutations is given below.

\[
\langle \text{tree-node} \rangle ::= \langle \text{vertical-slice} \rangle \\
| \langle \text{horizontal-slice} \rangle \\
| \langle \text{right-rotating-wheel} \rangle \\
| \langle \text{left-rotating-wheel} \rangle \\
| '1'
\]

\[
\langle \text{vertical-slice} \rangle ::= '12 [ \langle \text{left-skew} \rangle \langle \text{tree-node} \rangle '1']
\]

\[
\langle \text{left-skew} \rangle ::= \langle \text{horizontal-slice} \rangle \\
| \langle \text{right-rotating-wheel} \rangle \\
| \langle \text{left-rotating-wheel} \rangle \\
| '1'
\]

\[
\langle \text{horizontal-slice} \rangle ::= '21 [ \langle \text{right-skew} \rangle \langle \text{tree-node} \rangle '1']
\]

\[
\langle \text{right-skew} \rangle ::= \langle \text{vertical-slice} \rangle \\
| \langle \text{right-rotating-wheel} \rangle \\
| \langle \text{left-rotating-wheel} \rangle \\
| '1'
\]

\[
\langle \text{left-rotating-wheel} \rangle ::= '25314 [ \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle '1']
\]

\[
\langle \text{right-rotating-wheel} \rangle ::= '41352 [ \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle \langle \text{tree-node} \rangle '1']
\]

Now by using CYK-algorithm ¹¹ one can check whether a block-decomposition is generated by a Skewed Generating Trees of Order 5, in time $O(n^3)$. And to produce the block-decomposition of the given permutation a similar stack based algorithm can be used. The algorithm will run in time order of number of blocks in the decomposition, which is at most $2n$, as the number of nodes in the generating tree and the number of blocks in the corresponding Baxter permutation are the same. Hence for a fixed $k$ this approach of finding the block-decomposition and using CYK-algorithm to see if the corresponding block decomposition is generated, takes $O(n^3)$ time.

A.5 The equivalence between reverse of a Baxter permutation and rotation on a mosaic floorplan
Fig. 20. Obtaining a mosaic floorplan corresponding to the reverse of a Baxter permutation