Spacetime and inner space of spinors in the theory of superalgebraic spinors

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Abstract. We constructed gamma operators which are superalgebraic analogs of the Dirac gamma matrices as well as two additional gamma operators which have no analogs in the Dirac theory. We found a new mechanism of the left-right symmetry breaking which is absent in the usual theory of Dirac spinors. We constructed Lorentz invariant gamma operators from operators of creation and annihilation of spinors. These gamma operators are also analogs of the Dirac gamma matrices, however they are not related to Lorentz transformations but generate vector fields as affine spinor connections. We have shown that the theory is equivalent to an extended version of the Pati-Salam theory.

1. Introduction
The question of the origin of the spacetime has long attracted the attention of physicists. At the same time, there are different approaches in attempts to substantiate the observed dimension and the spacetime signature.

One of the main directions is the theory of supergravity. It was shown in [1] that the maximum dimension of spacetime, at which supergravity can be built, is equal to 11. At the same time, multiplets of matter fields for supersymmetric Yang-Mills theories exist only when the dimension of spacetime is less than or equal to 10 [2].

Subsequently, the main attention was paid to the theory of superstrings and supermembranes. Various versions of these theories were combined into an 11-dimensional M-theory [3,4]. In [5], the most general properties of the theories of supersymmetry and supergravity in spaces of various dimensions and signatures were analyzed. Proceeding from the possibility of the existence of Majorana and pseudo-Majorana spinors in such spaces, it was shown that supersymmetry and supergravity of M-theory can exist in 11-dimensional and 10-dimensional spaces with arbitrary signatures, although depending on the signature the theory type will differ. Later, other possibilities were shown for constructing variants of M-theories in spaces of different signatures [6].

Other approaches are Kaluza-Klein theories. For example, in [7] it was shown that in the theories of Kaluza-Klein in some cases it is possible not to postulate, but to determine from the dynamics not only the dimension of the spacetime, but also its signature.

In [8-10], an attempt was made to find a signature based on the average value of the quantum fluctuating metric of spacetime.

An attempt was made in [11] to explain the dimension and signature of spacetime from the anthropic principle and the possibility of causality, in [12] from the existence of equations of
motion for fermions and bosons coinciding with four-dimensional ones, in [13] from the possibility of existence in spacetime classical electromagnetism.

In all the above approaches, the fermionic vacuum operator in the second quantization formalism is not constructed and the restrictions imposed by such a construction are not considered. Therefore, the possibility of the existence of a vacuum and fermions is not discussed. In particular, the vacuum should be a Lorentz scalar and have zero spin, but in the theory of algebraic spinors, which more generally describes spinors than the Dirac matrix theory, Clifford vacuum has the transformational properties of the spinor component, and not the scalar [14].

We develop an approach to the theory of spacetime, solving this problem. It is based on the theory of superalgebraic spinors – an extension of the theory of algebraic spinors, in which the generators of Clifford algebras (analogs of Dirac gamma matrices) are composite.

2. Analogs of Dirac gamma matrices

We showed in [15,16] that using Grassmann variables and derivatives with respect to them, one can construct an analog of matrix algebra, including analogs of matrix columns of 4-spinors and matrix rows of conjugate spinors. But at the same time, the spinors and their conjugates exist in the same space – in the same algebra.

In [17,18], this approach was developed – Grassmann densities \( \theta^a(p) \), \( a = 1, 2, 3, 4 \), and derivatives \( \frac{\partial}{\partial \theta^a(p)} \) with respect to them were introduced, with CAR-algebra

\[
\left\{ \frac{\partial}{\partial \theta^a(p)}, \theta^b(p') \right\} = \delta^a_b \delta(p - p').
\]  

Superalgebraic analogs \( \hat{\gamma}^\mu \) of Dirac gamma matrices \( \gamma^\mu \) are constructed of these densities, we call them gamma operators:

\[
\hat{\gamma}^0 = \int d^3p \left[ \frac{\partial}{\partial \theta^1(p)} \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \theta^2(p) + \frac{\partial}{\partial \theta^3(p)} \theta^3(p) + \frac{\partial}{\partial \theta^4(p)} \theta^4(p), \ast \right],
\]

\[
\hat{\gamma}^1 = \int d^3p \left[ \frac{\partial}{\partial \theta^1(p)} \frac{\partial}{\partial \theta^4(p)} - \theta^4(p) \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \frac{\partial}{\partial \theta^3(p)} - \theta^3(p) \theta^2(p), \ast \right],
\]

\[
\hat{\gamma}^2 = i \int d^3p \left[ -\frac{\partial}{\partial \theta^1(p)} \frac{\partial}{\partial \theta^4(p)} - \theta^4(p) \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \frac{\partial}{\partial \theta^3(p)} + \theta^3(p) \theta^2(p), \ast \right],
\]

\[
\hat{\gamma}^3 = \int d^3p \left[ \frac{\partial}{\partial \theta^1(p)} \frac{\partial}{\partial \theta^3(p)} - \theta^3(p) \theta^1(p) - \frac{\partial}{\partial \theta^2(p)} \frac{\partial}{\partial \theta^4(p)} + \theta^4(p) \theta^2(p), \ast \right],
\]

\[
\hat{\gamma}^4 = i \hat{\gamma}^5 = i \int d^3p \left[ \frac{\partial}{\partial \theta^1(p)} \frac{\partial}{\partial \theta^4(p)} + \theta^3(p) \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \frac{\partial}{\partial \theta^3(p)} + \theta^4(p) \theta^2(p), \ast \right].
\]

They convert \( \frac{\partial}{\partial \theta^a(p)} \), \( \frac{\partial}{\partial \theta^a(p)} \), \( \theta^a(p) \), \( \theta^a(p) \) and their linear combinations in the same way that Dirac matrices \( \gamma^\mu \) convert matrix columns and their linear combinations. The theory is automatically secondarily quantized and does not require normalization of operators.

Generalization of Dirac conjugation of a spinor \( \Psi \) is

\[
\bar{\Psi} = (\hat{M} \Psi)^+,
\]

where \( \hat{M} \) is an analog of the matrix constructed from gamma operators [18].

In the proposed theory, if we let \( \hat{M} = \hat{\gamma}^0 \), and equation (3) corresponds to usual Dirac conjugation, in addition to analogs of the Dirac matrices, there are two additional gamma operators \( \hat{\gamma}^6 \) and \( \hat{\gamma}^7 \) [18]:

\[
\hat{\gamma}^6 = i \int d^3p \left[ \frac{\partial}{\partial \theta^1(p)} \frac{\partial}{\partial \theta^3(p)} + \theta^2(p) \theta^1(p) - \frac{\partial}{\partial \theta^3(p)} \frac{\partial}{\partial \theta^1(p)} - \theta^4(p) \theta^3(p), \ast \right],
\]

\[
\hat{\gamma}^7 = \int d^3p \left[ \frac{\partial}{\partial \theta^1(p)} \frac{\partial}{\partial \theta^4(p)} - \theta^2(p) \theta^1(p) + \frac{\partial}{\partial \theta^4(p)} \frac{\partial}{\partial \theta^1(p)} - \theta^3(p) \theta^4(p), \ast \right].
\]
3. Possible transformations of superalgebraic spinors and breaking of the left-right symmetry

We showed in [18] that transformations of densities \( \theta^a(p) \) and \( \frac{\partial}{\partial p^i(p)} \), while maintaining their CAR-algebra of creation and annihilation operators, provide transformations of field operators \( \Psi \) of the form

\[
\Psi' = \left( 1 + i\gamma^a d\omega_a + \frac{1}{4} \gamma^{ab} d\omega_{ab} \right) \Psi, \tag{5}
\]

where \( \gamma^{ab} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a) \); \( a, b = 0, 1, 2, 3, 4, 6, 7 \) and \( d\omega_{ab} = -d\omega_{ba} \) - real infinitesimal transformation parameters. The multiplier \( 1/4 \) is added in equation (5) compared to [18] to correspond to the usual transformation formulas for spinors in the case of Lorentz transformations.

We can use other variants of conjugation. The most common form of the operator \( \hat{M} \) for the spacetime with signature \( (p, q) \) which is consistent with Lorentz transformation is [18]

\[
\hat{M} = c_+ \gamma^1 \gamma^2 \ldots \gamma^p + c_- \gamma^1 \gamma^2 \ldots \gamma^q, \tag{6}
\]

where \( c_+ \) and \( c_- \) are complex constants, \( \gamma^k_+ \) are gamma-operators with positive signature and \( \gamma^k_- \) are gamma-operators with negative one.

If the dimension \( n = p + q \) of the spacetime is odd \( n = 2m + 1 \) then the center of the Clifford algebra is nontrivial. In this case terms of equation (6) up to the element of this center are proportional one to another, and exists the only possible variant of the generalization of the Dirac conjugation.

If the dimension of the spacetime is even \( n = 2m \) then

\[
\hat{M} = \sum \gamma^a \gamma^b \ldots \gamma^p (c_+ + c_- \gamma_0), \tag{7}
\]

where

\[
\gamma_0 = \varepsilon \gamma^p_+ \ldots \gamma^2_+ \gamma^1_+ \gamma^1_- \ldots \gamma^q_-, \tag{8}
\]

and we choose \( \varepsilon = 1 \) or \( \varepsilon = i \) so that \( (\gamma_0)^2 = 1 \).

Variants when \( c_+ = 0 \) or \( c_- = 0 \) were investigated in [18]. In these cases signature of the spacetime and formula of generalized Dirac conjugation are one-to-one related. Variant with \( \hat{M} = c_+ \gamma^0_+ \) corresponds to the Euclidean space, variant with \( \hat{M} = c_- \gamma^0_- \) corresponds to the Lorentz signature of the spacetime, and so on. In these cases decomposition (5) performs, however corresponding number of gamma operators (2) and (4) get imaginary unit as multiplier.

In the 4-dimensional case we have \( \hat{\gamma} = \gamma^5 \). In [19] we proved that only one timelike axis \( \gamma^0 \) is possible, this will be discussed later. That is why we can rewrite condition (7) as

\[
\hat{M} = c_+ (\gamma^0 + \lambda \gamma^0 \gamma^5), \tag{9}
\]

where \( \lambda = c_- / c_+ = \lambda_1 + i\lambda_2 \).

Part of \( \hat{M} \) corresponding to the imaginary part of \( \lambda \) can be diagonalized by the unitary transformation

\[
\hat{M}' = T \hat{M} T^{-1}, \tag{10}
\]

where \( T = \exp(i\gamma^5 \varphi) \).

That is why in the case of \( \lambda_1 = 0 \) and up to a non-essential phase multiplier we get \( \hat{M}' = \hat{\gamma}^0 \), and this case is equivalent to the case \( \hat{M} = \hat{\gamma}^0 \). Otherwise we have in the common case

\[
\hat{M}' = c'_+ \gamma^0 (1 + \lambda'_1 \gamma^5), \tag{11}
\]

where \( \lambda'_1 \) is real and \( c'_+ = \frac{1}{\sqrt{1 + (\lambda'_1)^2}} \) up to a non-essential phase multiplier.
Let
\[ \Psi_L = \frac{1 - \gamma^5}{2} \Psi, \quad \Psi_R = \frac{1 + \gamma^5}{2} \Psi. \] (12)

According equations (3) and (11)-(12) we have
\[ \Psi_L = c_4 (\gamma^0 + \lambda_1' \gamma^5) \frac{1 - \lambda_1'}{2} \Psi_L^+, \quad \Psi_R = \frac{1 + \lambda_1'}{\sqrt{1 + (\lambda_1')^2}} (\gamma^0 \Psi_R^+). \quad (13) \]

So, amplitudes of antispinors differ compared to usual Dirac theory by multiplier \( \frac{1 - \lambda_1'}{\sqrt{1 + (\lambda_1')^2}} \) for \( \Psi_L \) and by multiplier \( \frac{1 + \lambda_1'}{\sqrt{1 + (\lambda_1')^2}} \) for \( \Psi_R \). These multipliers break symmetry between left and write spinors and antispinors (conjugated spinors).

If \( \lambda_1' = -1 \) we have antispinor only for \( \Psi_L \) and have not antispinor for \( \Psi_R \). If \( \lambda_1' = 1 \) we have antispinor only for \( \Psi_R \) and have not antispinor for \( \Psi_L \).

If \( \lambda_1' \approx -1 \) or \( \lambda_1' \approx 1 \) both components exist, \( \Psi_L \) and \( \Psi_R \). However we have strong asymmetry between them in this case.

There are no any reasons why \( \lambda_1' \) must be equals to zero. That is why asymmetry between left and write spinors and antispinors must exists in a general case.

This is a new mechanism of the left-right symmetry breaking which is absent in the usual theory of Dirac spinors. The only question is why \( \lambda_1' \approx -1 \) for the neutrino.

4. Operators of creation and annihilation and operators of pseudo-orthogonal rotation

Operators \( \gamma^{ab} = \frac{1}{2} (\gamma^a \gamma^b - \gamma^b \gamma^a) \) are generators of pseudo-orthogonal rotations which have the form \( \exp(\gamma^{ab} \omega_{ab}/4) \). We call them gamma operators of rotations. They are generators of Lorentz rotations when \( a, b = 0, 1, 2, 3 \).

Operators of annihilation of spinors \( b_\alpha(p), \alpha = 1, 2 \) and of antispinors \( \bar{b}_\tau(p), \tau = 3, 4 \) are obtained by Lorentz rotations of \( \frac{\partial}{\partial \theta^a(0)} \) and \( \frac{\partial}{\partial \theta^a(0)} \), and the Dirac conjugated to them operators of creation \( \hat{b}_\alpha(p) \) and \( \hat{b}_\tau(p) \) – by Lorentz rotations from \( \theta^\alpha(0) \) and \( \theta^\tau(0) \) [17,18].

Anticommutation relations for \( b_\alpha(p) \) and \( \bar{b}_\tau(p) \) correspond to CAR-algebra
\[ \{ b_\alpha(p), \bar{b}_\tau(p') \} = \delta(p - p') \delta^3_k \] (14)

Examples of expressions for operators \( \gamma^{ab} \) are given in equations (15).
\[
\begin{align*}
\gamma^{01} & = \int d^3 p \left[ \frac{\partial}{\partial \theta^1} \frac{\partial}{\partial \theta^1} + \theta^1 \theta^1 + \frac{\partial}{\partial \theta^2} \frac{\partial}{\partial \theta^2} + \theta^2 \theta^2 \right], \\
\gamma^{12} & = -i \int d^3 p \left[ \frac{\partial}{\partial \theta^1} \theta^2 - \frac{\partial}{\partial \theta^2} \theta^1 \right] \right] , \\
\gamma^{67} & = -i \int d^3 p \left[ \frac{\partial}{\partial \theta^3} \theta^4 - \frac{\partial}{\partial \theta^4} \theta^3 \right] \right] .
\end{align*}
\] (15)

Expressions for all operators \( \gamma^{ab} \) are given in [19].

5. Discrete analogs of the creation and annihilation operators and fermionic vacuum

In [17-19] the author proposed a method for constructing a state vector of the vacuum. We divide the momentum space into infinitely small volumes and define operators
\[ B_k(p_j) = \frac{1}{\Delta^3 p_j} \int d^3 p \, b_k(p), \quad B_k(p_j) = \frac{1}{\Delta^3 p_j} \int d^3 p \, b_k(p). \] (16)
Using equation (14), we have
\[
\{ B_k(p_i), B_l(p_j) \} = \frac{1}{\Delta^3 p_1 \Delta^3 p_j} \int \frac{d^3 p}{\Delta^3 p_k} \int d^3 p' \{ b_k(p), b_l(p') \} = \frac{1}{\Delta^3 p_j} \delta_i^k \delta_l^j \tag{17}
\]
and
\[
(B_k(p_i))^2 = (B_k(p_i))^2 = 0. \tag{18}
\]

The expression \( \frac{1}{\Delta^3 p_j} \delta^i_j \) in equations (16)-(17) is a discrete analog of the delta function \( \delta(p-p') \).

Denote local vacuum operator in the momentum space
\[
\Psi_{V,j} = (\Delta^3 p_j)^4 B_1(p_j) B_2(p_j) B_3(p_j) B_4(p_j), \tag{19}
\]
and define the fermionic vacuum \( \Psi_V \)
\[
\Psi_V = \prod_j \Psi_{V,j}, \tag{20}
\]
where the product goes over all physically possible values of \( j \). In this case, we will assume that all volumes \( \Delta^3 p_j \) are formed by Lorentz rotations from the volume \( \Delta^3 p_0 = 0 \) corresponding to \( p = 0 \), and the grid of angles \( \omega_{\mu
u} \) of these rotations is discrete.

Further, it will often be convenient to represent equation (20) in the form
\[
\Psi_V = \Psi_{V,j} \Psi_{V,j}^\dagger, \tag{21}
\]
where \( \Psi_{V,j} = \prod_{i \neq j} \Psi_{V,i} \) is the product of factors in equation (20), independent of \( p_j \).

We can replace in the formulas with participation of continuous operators \( b_k(p) \) and \( \bar{b}_k(p) \) to discrete \( B_k(p_j) \) and \( \bar{B}_k(p_j) \), and the integral \( \int d^3 p ... \) to the sum \( \sum_i \Delta^3 p_i ... \). In this case, all formulas using continuous operators \( b_k(p) \) and \( \bar{b}_k(p) \) are replaced by completely similar ones using discrete ones, with the replacement of the delta function \( \delta(p-p') \) by \( \delta^i_j \frac{1}{\Delta^3 p_i} \), where \( p_i \) corresponds to \( p \), and \( p_j \) corresponds to \( p' \). We will use for operators \( \gamma^n = \sum_i \Delta^3 p_i \gamma^n(p_i) \) and \( \gamma^{np} = \sum_i \Delta^3 p_i \gamma^{np}(p_i) \) after such a replacement the same notation as for the corresponding continuous ones, and we will call such \( \gamma^n \) as discrete gamma operators, and \( \gamma^{np} \) as discrete gamma operators of rotations.

6. Matrix representation of the gamma operators
Superalgebraic spinors for given momentum value \( p_j \) can be constructed as algebraic spinors \[19\]. Let \( e_{i,j} \) be basis Clifford vectors of the \( 2mN \)-dimensional complex Clifford algebra, where \( N \) is the number of discrete volumes of the momentum and index \( j \) corresponds to \( p_j \). We can set arbitrary signature of such basis for the complex Clifford algebra. Set it as \( (p,q) = (2mN,0) \).

We call it the large Clifford algebra \[19\].

We define discrete analogs of Grassmann densities \( \theta^a(p_j) \) and \( \frac{\partial}{\partial \theta^a(p_j)} \) as
\[
\theta^a(p_j) = \frac{1}{\sqrt{2 \Delta^3 p_j}} (e_{2\alpha-1,j} + ie_{2\alpha,j}), \quad \frac{\partial}{\partial \theta^a(p_j)} = \frac{1}{\sqrt{2 \Delta^3 p_j}} (e_{2\alpha-1,j} - ie_{2\alpha,j}), \tag{22}
\]
where \( \alpha = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, N \).
In accordance with equations (22) we have

$$\left\{ \theta^a(p_j), \frac{\partial}{\partial \theta^b(p_k)} \right\} = \frac{1}{2 \Delta^3_{p_j}} \{e_{2a-1,j} + ie_{2a,j}, e_{2\beta-1,k} - ie_{2\beta,k}\} = \frac{1}{\Delta^3_{p_j}} \delta^a_\beta \delta^b_\gamma. \quad (23)$$

Equation (23) is a discrete analog of equation (1).

All elements of the large Clifford algebra can be represented as square matrices of a size $2^{mN} \cdot 2^{mN}$. Generators of the large Clifford algebra are $\theta^a(p_j) + \frac{\partial}{\partial \theta^a(p_j)}$ and $\theta^a(p_j) - \frac{\partial}{\partial \theta^a(p_j)}$.

Lie algebra of the large Clifford algebra has generators $\hat{\gamma}^a$ and $\hat{\gamma}^{ab}$ for the case when $m = 2^\nu$ and $a, b = 1, 2, \ldots, 2\nu + 2$. Operators $\hat{\gamma}^a$ are generators of the Clifford algebra which we call small Clifford algebra [19]. Small Clifford algebra is more complicated compared to usual Clifford algebra. Its elements have properties of elements of the Clifford algebra only under one-particle state vectors or under one-particle field operators [19].

We can represent gamma operators as commutators of the square matrices of the size $2^{mN} \cdot 2^{mN}$. However there is much simpler matrix representation of the gamma operators if we use only one-particle state vectors or one-particle field operators for given momentum value $p_j$. We have $2^\nu$ elements $\theta^a(p_j)$ and $2^\nu$ elements $\frac{\partial}{\partial \theta^a(p_j)}$.

First consider the case with four independent Grassmann variables $\theta^a(p_j)$ and four Grassmann variables $\frac{\partial}{\partial \theta^a(p_j)}$ for given $p_j$. In this case $\nu = 2$, $m = 2^\nu = 4$, and we have 6-dimensional Clifford algebra. We construct it explicitly. We define basis vectors

$$\frac{\partial}{\partial \theta^1(p_j)} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \quad \frac{\partial}{\partial \theta^2(p_j)} = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \quad \theta^3(p_j) = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right), \quad \theta^4(p_j) = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right). \quad (24)$$

So spinors and Dirac conjugated spinors can be represented as

$$\Psi(p_j) = \psi^\alpha \frac{\partial}{\partial \theta^\alpha(p_j)} + \psi_\tau \theta^\tau(p_j), \quad (25)$$
$$\Psi^\dagger(p_j) = (\psi^\alpha)^* \theta^\alpha(p_j) - (\psi_\tau)^* \frac{\partial}{\partial \theta^\tau(p_j)}. \quad (26)$$

where $\alpha = 1, 2; \tau = 3, 4$, $\psi^\alpha$ and $\psi_\tau$ are complex coefficients and * is complex conjugation.

Superalgebraic spinor $\Psi_1(p_j)$ and Dirac conjugated superalgebraic spinor $\Psi_2(p_j)$ exist in the same space

$$\Phi(p_j) = \Psi_1(p_j) + \Psi_2(p_j) = \left( \begin{array}{c} \Psi_1(p_j) \\ \Psi_2(p_j) \end{array} \right). \quad (27)$$

We define block Pauli matrices and unity matrices

$$\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad e = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \quad (28)$$
$$\Sigma_1 = \left( \begin{array}{cc} 0 & e \\ e & 0 \end{array} \right), \quad \Sigma_2 = \left( \begin{array}{cc} 0 & -ie \\ ie & 0 \end{array} \right), \quad \Sigma_3 = \left( \begin{array}{cc} e & 0 \\ 0 & -e \end{array} \right), \quad E = \left( \begin{array}{cc} e & 0 \\ 0 & e \end{array} \right) \quad (29)$$
$$\Sigma_1' = \left( \begin{array}{cc} 0 & E \\ E & 0 \end{array} \right), \quad \Sigma_2' = \left( \begin{array}{cc} 0 & -iE \\ iE & 0 \end{array} \right), \quad \Sigma_3' = \left( \begin{array}{cc} E & 0 \\ 0 & -E \end{array} \right), \quad E' = \left( \begin{array}{cc} E & 0 \\ 0 & E \end{array} \right) \quad (30)$$

Representation of the gamma operators acting on spinors are usual Dirac matrices:

$$\gamma^0 = \Sigma_3, \quad \gamma^1 = i \sigma_1 \Sigma_2, \quad \gamma^2 = i \sigma_2 \Sigma_2, \quad \gamma^3 = i \sigma_3 \Sigma_2, \quad \gamma^5 = \Sigma_1. \quad (31)$$
It is easy to prove that according to equations (2), (3) and (24)-(31) matrix representation of gamma operators $\hat{\gamma}^\mu$, $\mu = 0, 1, 2, 3, 5$, which act on the spinor $\Phi(p_j)$ are

$$\gamma^{0'} = \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix} = \Sigma_3 \Sigma_3', \quad \gamma^{1'} = \begin{pmatrix} \gamma^1 & 0 \\ 0 & -\gamma^1 \end{pmatrix} = i\sigma_1 \Sigma_2 \Sigma_3', \quad \gamma^{2'} = \begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix} = i\sigma_2 \Sigma_2 E', $$

$$\gamma^{3'} = \begin{pmatrix} \gamma^3 & 0 \\ 0 & -\gamma^3 \end{pmatrix} = i\sigma_3 \Sigma_2 \Sigma_3', \quad \gamma^{5'} = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix} = \Sigma_1 E', \quad \gamma^{4'} = i\gamma^{5'}. \quad (32)$$

Additionally to Dirac theory occur matrices corresponding to the gamma operators $\hat{\gamma}^6$ and $\hat{\gamma}^7$:

$$\gamma^{6'} = \begin{pmatrix} 0 & -\sigma_2 \Sigma_3 \\ \sigma_2 \Sigma_3 & 0 \end{pmatrix} = -i\sigma_2 \Sigma_3 \Sigma_2', \quad \gamma^{7'} = \begin{pmatrix} 0 & i\sigma_2 \Sigma_3 \\ i\sigma_2 \Sigma_3 & 0 \end{pmatrix} = i\sigma_2 \Sigma_3 \Sigma_1'. \quad (33)$$

Matrix representation of the gamma operators $\hat{\gamma}^{ab}$ can be calculated as

$$\gamma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a); \quad a, b = 0, 1, 2, 3, 4, 6, 7. \quad (34)$$

Operators $\hat{\gamma}^6$, $\hat{\gamma}^7$, $\gamma^{ab}$, $\gamma^{a7}$ as well as matrices $\gamma^6$, $\gamma^7$, $\gamma^{ab}$, $\gamma^{a7}$ correspond to inner degrees of freedom of superalgebraic spinors. We can continue definitions similar to equations (29) and (30) of the block matrices when we increase twice value of $m$, that is when we increase twice the number of columns and rows in the matrices. Formula for gamma matrices $\gamma^a_{2m}$ with $2m$ columns and $2m$ rows on this step is simpler than equations (32)-(33):

$$\gamma_{2m}^a = \begin{pmatrix} \gamma^a_m & 0 \\ 0 & -\gamma^a_m \end{pmatrix}, \quad \gamma^{2m-2} = i \begin{pmatrix} 0 & E' \cdots' \\ E' \cdots' & 0 \end{pmatrix}, \quad \gamma^{2m-1} = \begin{pmatrix} 0 & -E' \cdots' \\ E' \cdots' & 0 \end{pmatrix}, \quad (35)$$

where $E' \cdots'$ is the unity matrix $m \cdot m$.

This scheme is similar to the usual Cartan’s scheme of constructing matrix representations of the Clifford algebras of increasing dimensions [20]. However we keep diagonal matrices generated by the diagonal matrices of the previous steps.

7. Lorentz invariant gamma operators

It is easy to construct Lorentz invariant analogs $\hat{\Gamma}^a$ and $\hat{\Gamma}^{ab}$ of gamma operators $\hat{\gamma}^a$ and $\hat{\gamma}^{ab}$ [19]. To do this, it is enough in equations (2),(4),(15) and so on replace all operators $\frac{\partial}{\partial \theta^a(p)}$ by $b_k(p)$, and operators $\theta^k(p)$ by $\bar{b}_k(p)$. For example,

$$\hat{\Gamma}^0 = \int d^3p \left[ b_1(p)\bar{b}_1(p) + b_2(p)\bar{b}_2(p) + b_3(p)\bar{b}_3(p) + b_4(p)\bar{b}_4(p), * \right], \quad (36)$$

$$\hat{\Gamma}^1 = \int d^3p \left[ b_1(p)b_4(p) - \bar{b}_1(p)\bar{b}_4(p) + b_2(p)b_3(p) - \bar{b}_2(p)\bar{b}_3(p), * \right], \quad (37)$$

$$\hat{\Gamma}^{67} = -i \int d^3p \left[ b_1(p)\bar{b}_1(p) + b_2(p)\bar{b}_2(p) - b_3(p)\bar{b}_3(p) - b_4(p)\bar{b}_4(p), * \right]. \quad (38)$$

Formulas of all $\hat{\Gamma}^a$ and $\hat{\Gamma}^{ab}$ are done in [19]. In the discrete version of the theory, continuous operators $b_k(p)$ and $\bar{b}_k(p)$ are replaced by discrete $B_k(p)$ and $\bar{B}_k(p)$ in the operators $\hat{\Gamma}^a$ and $\hat{\Gamma}^{ab}$, and integrals $\int d^3p \ldots$ by sums $\sum_i \Delta^3 p_i \ldots$. In contrast to $\hat{\gamma}^a$ and $\hat{\gamma}^{ab}$, operators $\Gamma^a$ and $\Gamma^{ab}$ do not change either by the Lorentz transformations [19].

Since the commutation relations (14) for $b_k(p)$ and $\bar{b}_k(p)$ are the same as equation (1) for $\frac{\partial}{\partial \theta^a(p)}$ and $\theta^k(p)$, the commutation relations for $\Gamma^a$ and $\Gamma^{ab}$ are the same as for $\hat{\gamma}^a$ and $\hat{\gamma}^{ab}$.
That is, $\hat{\Gamma}^\mu$ are also analogs of Dirac matrices [19].

We introduce the superalgebraic analogs [17] of the operators of the number of particles $\hat{N}_1(p)$, $\hat{N}_2(p)$ and antiparticles $\hat{N}_3(p)$, $\hat{N}_4(p)$ and the charge operator $\hat{Q}$ in the method of second quantization:

$$\hat{N}_k(p) = [\bar{b}_{\leq k>}(p)b_{\leq k>}(p), \ast] = -[b_{\leq k>}(p)\bar{b}_{\leq k>}(p), \ast],$$

$$\hat{Q} = \int d^3p \left( \hat{N}_1(p) + \hat{N}_2(p) - \hat{N}_3(p) - \hat{N}_4(p) \right).$$ (39)

where there is no summing by indexes in the triangle brackets. Equations (36) and (38) can be rewritten in the form:

$$\hat{\Gamma}^0 = -\int d^3p \left( \hat{N}_1(p) + \hat{N}_2(p) + \hat{N}_3(p) + \hat{N}_4(p) \right),$$

$$\hat{\Gamma}^{67} = i \int d^3p \left( \hat{N}_1(p) + \hat{N}_2(p) - \hat{N}_3(p) - \hat{N}_4(p) \right) = i\hat{Q}.$$ (40)

That is, $-\hat{\Gamma}^0$ is the operator of the total number of spinors and antispinors, and $\hat{\Gamma}^{67}$ is related to the charge operator $\hat{Q}$ by the ratio $\hat{\Gamma}^{67} = i\hat{Q}$.

8. Spacetime signature in the presence of the spinor vacuum

Under the action on the vacuum (20) $\hat{\Gamma}^0\Psi_V = \hat{\Gamma}^{mn}\Psi_V = 0$, $m, n = 1, 2, 3, 4, 6, 7$, and $\hat{\Gamma}^m\Psi_V \neq 0$, $\hat{\Gamma}^{0m}\Psi_V \neq 0$. Therefore, we can measure only the eigenvalues of the operators $\hat{\Gamma}^0$ and $\hat{\Gamma}^{mn}$ with $m, n = 1, 2, 3, 4, 6, 7$ [19]. The reason for the difference between the action on the vacuum and the state vectors of the operators $\hat{\Gamma}^0$ and $\hat{\Gamma}^{mn}$, on the one hand, and $\hat{\Gamma}^m$ and $\hat{\Gamma}^{0m}$ on the other, is related to the structure of these operators in equations (2), (4), (8), (15), (36)-(38). Since the vacuum state vector has a multiplier $B_{\leq a>}(0)\overline{B}_{\leq a>}(0)$, the action on the vacuum of operators consisting only of terms of the form $[\overline{B}_{\leq t>}(0)B_{\leq k>}(0), \ast]$ will always give zero, since, by virtue of equations (17) and (18)

$$[\overline{B}_{\leq t>}(0)B_{\leq k>}(0)B_{\leq k>}(0)B_{\leq t>}(0)\overline{B}_{\leq t>}(0)] = 0.$$ (41)

But the terms of the form $[B_{\leq k>}(0)\overline{B}_{\leq t>}(0), \ast]$ and $[\overline{B}_{\leq k>}(0)\overline{B}_{\leq t>}(0), \ast]$ will give a non-zero result. Summing the results of Lorentz rotations leads to similar conclusion for $\hat{\Gamma}^0$, $\hat{\Gamma}^{mn}$, on the one hand, and $\hat{\Gamma}^m$, $\hat{\Gamma}^{0m}$, on the other.

Thus, the operator $\hat{\Gamma}^0$ is fundamentally different from operators $\hat{\Gamma}^m$. It annihilates vacuum, but $\hat{\Gamma}^m$ do not. Multiplying $\hat{\Gamma}^m$ by imaginary unit changes signature of $\hat{\Gamma}^m$ (and, correspondingly, of $\gamma^m$) from $-1$ to $+1$. However the space of Clifford vectors with the same signature must be isotropic. Therefore, other than $\hat{\Gamma}^0$ Clifford vectors could not have the same signature as $\hat{\Gamma}^0$. That is why it is possible only one timelike axis in the small Clifford algebra [19].

9. Decomposition of the spinor field operator and internal degrees of freedom

We can replace Grassmann densities $\theta^k(p)$ and $\frac{d}{[\gamma^a(p)]}$ by the creation and annihilation operators $\bar{b}_k(p)$ and $b_k(p)$ in the proof of the decomposition (5) for given momentum $p$. So we replace gamma operators $\gamma^a$ and $\gamma^{ab}$ in the decomposition by the gamma operators $\Gamma^a$, $\Gamma^{ab}$ [19]

$$\Psi' = \left( 1 + i\Gamma^a d\omega_a + \frac{1}{4} \Gamma^{ab} d\omega_{ab} \right) \Psi,$$ (42)

where $a, b = 0, 1, 2, 3, 4, 6, 7$ and $d\omega_a$, $d\omega_{ab}$ – arbitrary infinitesimal real constants.
Consider the decomposition (42) in the case of an infinitely small change in coordinates $dx^\mu$ [19]. Values $d\omega_a$ and $d\omega_{ab}$ must be proportional to $dx^\mu$

$$dw_a = F_{a\mu}dx^\mu, \quad dw_{ab} = F_{ab\mu}dx^\mu.$$  

This is meaning that

$$d\Psi = \Psi' - \Psi = \left(i\hat{\Gamma}^aF_{a\mu} + \frac{1}{4}\hat{\Gamma}^{ab}F_{ab\mu}\right)dx^\mu\Psi,$$  

where $a, b = 0, 1, 2, 3, 4, 6, 7$.

Energy-momentum operator $\hat{P}_\mu$ in the superalgebraic formalism [19] is given by the formula

$$\hat{P}_\mu = \int d^3p\, p_\mu \left[\bar{b}_1b_1 + \bar{b}_2b_2 + \bar{b}_3b_3 + \bar{b}_4b_4, *\right]$$  

If we set $F_{0\mu} = -p_\mu$, we receive $-\hat{\Gamma}^0F_{0\mu} = \hat{P}_\mu$. Operator $\hat{Q} = -i\hat{\Gamma}^{67}$ is charge operator in the formalism of second quantization [19].

Denote $F_{67\mu} = -F_{6\mu}$ as $gA_\mu$, where $g$ is any constant of interaction. So we can write the covariant derivative of spinor field in the form [19]

$$D_\mu = -i\hat{P}_\mu + i\frac{g}{2}\hat{Q}A_\mu + \hat{\Gamma}^kF_{k\mu} + \frac{1}{4}\hat{\Gamma}^{cd}F_{cd\mu},$$  

where $k = 1, 2, 3, 4, 6, 7$; $c, d = 0, 1, 2, 3, 4, 6, 7$ but $cd \neq 67$, $cd \neq 76$.

Physical sense of the field $A_\mu$ is obvious, it is vector potential of the electromagnetic field. Physical sense of affine spinor connections $F_{k\mu}$ and $F_{cd\mu}$ is not clear.

It was written in [19] that we can replace $-i\hat{P}_\mu$ with $\partial_\mu$, however it is proper only if $F_{k\mu} = 0$ and $F_{0k\mu} = 0$, since $\partial_\mu$ commute with $\hat{\Gamma}^k$ and $\hat{\Gamma}^{0k}$ but $-i\hat{P}_\mu$ anticommute. Moreover, such replacement is correct only if the fields $A_\mu$, $F_{k\mu}$ and $F_{cd\mu}$ do not change with the coordinate. That is why equation (46) which is corresponding to the second quantization formalism is more common then usual classical (not quantum) approaches of the General Relativity. In addition, this equation opens a new way in quantum fields theory, because it turns out to be possible to construct a quantum theory of spacetime and quantum fields without using partial derivatives.

We can find terms in equation (46) which are simultaneously measurable with energy-momentum operators $\hat{P}_\mu$. That is, which of them commute with $\hat{P}_\mu$. We already mentioned that $\hat{\Gamma}^k$ and $\hat{\Gamma}^{0k}$ anticommute with $\hat{P}_\mu$. This is meaning that fields $F_{k\mu}$ with charge operators $\hat{\Gamma}^k$ and fields $F_{0k\mu}$ with charge operators $\hat{\Gamma}^{0k}$ are not measurable simultaneously with energy-momentum operators. However operators $\hat{\Gamma}^{kl}$, where $k, l = 1, 2, 3, 4, 6, 7$, commute with $\hat{P}_\mu$. Therefore, fields $F_{kl\mu}$ with charge operators $\hat{\Gamma}^{kl}$ are measurable simultaneously with energy-momentum operators, and $\hat{Q} = -i\hat{\Gamma}^{67}$ is one of these operators. It is interesting that we can consider $\hat{\Gamma}^0$ as a charge operator for vector field $p_\mu$, that is for the momentum of the spinor.

Now consider cases with doubling the number of independent Grassmann variables $\theta^k(p)$ for the given momentum $p$. Doubling is necessary for existence of the small Clifford algebra. We have two additional gamma operators for each doubling. We assume that spacetime is four-dimensional, and all additional axes of the space of Clifford vectors correspond to the internal degrees of freedom. Therefore, the momentum space is also four-dimensional, and doubling the number of creation and annihilation operators corresponds to doubling the number of fermion types.

Let

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$  

(47)
where $\Psi_1$ is first fermion, that is first half of the spinor $\Psi$ components, and $\Psi_2$ is second one.

In accordance with equation (35), fermion $\Psi_2$ will be transformed in the same way as $\Psi_1$ by the gamma operators, if in the matrix column first two components of $\Psi_2$ correspond to Grassmann variables $\theta^k(p)$ and last two components correspond to $\frac{\partial}{\partial \theta^k(p)}$. That is, the order of positive-frequency and negative-frequency components is reversed compared with $\Psi_1$. We have two additional Lorentz invariant gamma operators $\hat{\Gamma}^8$, $\hat{\Gamma}^9$ and corresponding additional operators of rotation $\hat{\Gamma}^{89}$, $\hat{\Gamma}^{a9}$, $a = 0, 1, 2, 3, 4, 6, 7$. In this case equations (42)-(46) remain valid, only the maximum index increases to 9 and number of operators of creation and annihilation in equation (45) doubles.

Operators
\[
\hat{\tau}_1 = \frac{i}{2}\left(\hat{\Gamma}^4\hat{\Gamma}^9 - \hat{\Gamma}^9\hat{\Gamma}^4\right) = i\hat{\Gamma}^{49} = -i\hat{\Gamma}^{94},
\]
\[
\hat{\tau}_2 = \frac{i}{2}\left(\hat{\Gamma}^4\hat{\Gamma}^8 - \hat{\Gamma}^8\hat{\Gamma}^4\right) = i\hat{\Gamma}^{48},
\]
\[
\hat{\tau}_3 = -i\hat{\Gamma}^{89}
\]
play role of generators of electroweak interaction, with the exception of chirality factor $\frac{1-\hat{\gamma}_5}{2}$. Thus, we have version of the Pati-Salam theory [21] with “upper” fermion $\Psi_1$ and “bottom” fermion $\Psi_2$, electroweak fields $W^\mu_k$, $k = 1, 2, 3$, and the same as in equation (46) interaction constant $g$:
\[
F^\mu_{48} = gW^\mu_k, \quad F^\mu_{94} = gW^\mu_2, \quad F^\mu_{89} = gW^\mu_3.
\]

In this version of the theory fermion $\Psi_1$ has electrical charge 0, fermion $\Psi_2$ has electrical charge $-1$, and $\hat{Q} = -\frac{\hat{\Gamma}^{67}}{2}$ is hypercharge operator. It has eigenvalues $+\frac{1}{2}$ for fermions and $-\frac{1}{2}$ for antifermions. However we have no fermions corresponding to quarks. And we have additional fields $\hat{\Gamma}^k F^\mu_k$ and $\hat{\Gamma}^a F^\mu_{l\alpha}$, $k = 1, 2, 3, 4, 6, 7$; $l = 0, 1, 2, 3$; $a = 0, 1, 2, 3, 4, 6, 7$, which are absent in the Pati-Salam theory.

A further increase in the numbers of Grassmann variables allows us to construct a complete version of the Pati-Salam theory. However, with the increase in the numbers of additional compared to this theory fields.

10. Conclusion
The article develops an approach to the theory of spacetime which is based on the theory of superalgebraic spinors.

We constructed gamma operators which are superalgebraic analogs of the Dirac gamma matrices as well as two additional gamma operators which have no analogs in the Dirac theory.

We found a new mechanism of the left-right symmetry breaking which is absent in the usual theory of Dirac spinors.

We constructed Lorentz invariant gamma operators from operators of creation and annihilation of spinors. These operators are also analogs of the Dirac gamma matrices, however they are not related to Lorentz transformations but generate vector fields as affine spinor connections. We have shown that this theory is equivalent to an extended version of the Pati-Salam theory.

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