Consistent Chiral Kinetic Theory in Weyl Materials: Chiral Magnetic Plasmons

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Introduction.— The study of the fundamental properties of magnetized relativistic matter attracted a lot of attention in recent years. The physical systems in question include the plasmas in the early Universe [1], relativistic heavy-ion collisions [2, 3], degenerate states of dense matter in compact stars [4], and a growing number of recently discovered three-dimensional Dirac and Weyl materials [5–7]. To large extent, the recent increased activity in the studies of magnetized relativistic matter is driven by the hope of detecting macroscopic implications of quantum anomalies. One of such implications is the celebrated chiral magnetic effect (CME) [8], which has been detected indirectly in the quark-gluon plasma created in heavy-ion collisions (for a review, see Ref. [3]), as well as in Dirac semimetals [9]. Note that the interpretation of the heavy-ion experiments is not without a controversy [10].

The search for macroscopic implications of quantum anomalies is greatly facilitated by the recent discovery of Dirac and Weyl materials, whose low-energy quasiparticle excitations are described by relativistic-like equations. Indeed, unlike most forms of truly relativistic matter, these novel condensed matter materials open the possibility for revealing and testing many anomalous effects in magnetized matter in tabletop experiments under controlled conditions. Moreover, they may even allow for modeling phenomena that are impossible in relativistic physics. A specific example is provided by a background pseudomagnetic (or, equivalently, axial magnetic) field \( B_5 \), which can be effectively produced by a mechanical strain in Dirac and Weyl materials [11–14]. In essence, the pseudomagnetic field \( B_5 \) resembles the ordinary magnetic field \( B \), but acts on opposite chirality quasiparticles so as if they had opposite charges. In the case of the Dirac semimetal Cd₃As₂, for example, the estimated strength of the strain-induced pseudomagnetic field could range from about \( B_5 \approx 0.3 \) T to \( B_5 \approx 15 \) T in twisted nanowires [14] and in bended thin films [5]. Similarly, a pseudoelectric field \( E_5 \) can be generated by time-dependent deformations.

Collective excitations are simple, but informative probes of plasma properties [16]. It is natural to ask, therefore, whether such modes in chiral plasmas can be affected by quantum anomalies. The authors of Ref. [17] proposed that the chiral anomaly implies the existence of a new type of collective excitation, i.e., the chiral magnetic wave (CMW), that originates from an interplay of chiral and electric charge density waves. In this study, we will investigate the collective modes in Weyl materials, using the framework of the chiral kinetic theory with a proper treatment of dynamical electromagnetism.

The central idea of this Letter is to use the correct definition of the electric current in the chiral kinetic theory for Weyl materials with strain-induced pseudoelectromagnetic fields. As we show, the current should necessarily include the Chern–Simons contribution, which is also known as the Bardeen-Zumino polynomial [18]. Such a correction restores the local conservation of the electric charge in the case of general electromagnetic and pseudoelectromagnetic fields. In addition, this topological term affects the properties of collective modes. For example, their plasma frequencies acquire a dependence on the chiral shift parameter, i.e., the momentum-space separation between the Weyl nodes.
Model.— The chiral kinetic theory is a semiclassical theory, which describes the time evolution of the one-particle distribution functions $f_\lambda$ for the right- ($\lambda = +$) and left-handed ($\lambda = -$) fermions. In the collisionless limit (assuming that the frequency of collective excitations $\omega$ is much larger that inverse relaxation time $1/\tau$), the kinetic equations are given by [13, 20]  

$$
\frac{\partial f_\lambda}{\partial t} + \left[ eE_\lambda + \frac{\epsilon_c}{c}(v \times B_\lambda) + \frac{\epsilon_c}{c} (\tilde{E}_\lambda \cdot B_\lambda)\Omega_\lambda \right] \cdot \nabla_p f_\lambda 
+ \left[ v + e(\tilde{E}_\lambda \times \Omega_\lambda) + \frac{\epsilon_c}{c} (v \cdot \Omega_\lambda)B_\lambda \right] \cdot \nabla_p f_\lambda = 0, \quad \text{(1)}
$$

where $E_\lambda = E + \lambda E_5$ and $B_\lambda = B + \lambda B_5$ are effective electric and magnetic fields for fermions of chirality $\lambda$, $\Omega_\lambda = \lambda \hbar p/(2p^3)$ is the Berry curvature [21], $p \equiv |p|$, $E_\lambda = E_\lambda - (1/e) \nabla e p$, and the factor $1/(1+e(cB_\lambda \cdot \Omega_\lambda)/c)$ accounts for the correct phase-space density of chiral states in an effective magnetic field [22]. By making use of the fermion dispersion relation, valid up to the linear order in the background magnetic field $B_\lambda$ [20],

$$
\epsilon_p = v_F p \left[ 1 - (e/c)(B_\lambda \cdot \Omega_\lambda) \right], \quad \text{we derive the quasiparticle velocity } v = \nabla_p \epsilon_p, \text{ i.e.,}
$$

$$
v = v_F \frac{p}{p} \left[ 1 + 2 \frac{e}{c}(B_\lambda \cdot \Omega_\lambda) \right] - \frac{e v_F}{c p} B_\lambda(p \cdot \Omega_\lambda). \quad \text{(3)}
$$

Here $v_F$ is the Fermi velocity.

The equilibrium distribution functions for chiral fermions are given by the standard Fermi-Dirac distributions

$$
f_\lambda^{\text{eq}} = \left[ e^{(\epsilon_p - \mu_\lambda)/T} + 1 \right]^{-1}, \quad \text{(4)}
$$

where $T$ is the temperature (measured in energy units) and $\mu_\lambda = \mu + \lambda \mu_5$ are the effective chemical potentials for the right- and left-handed fermions. Note that $\mu$ and $\mu_5$ are the electric and chiral chemical potentials, respectively. The distribution functions for antiparticles $f_\lambda^{\text{eq}}$ are obtained by replacing $\mu_\lambda \rightarrow -\mu_\lambda$. In addition, for antiparticles, one should replace $e \rightarrow -e$ and $\Omega_\lambda \rightarrow -\Omega_\lambda$.

The charge and current densities are given by [20]

$$
\rho_\lambda = \sum_{p,a} e \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ 1 + \frac{\epsilon_p}{c} (B_\lambda \cdot \Omega_\lambda) \right] f_\lambda, \quad \text{(5)}
$$

$$
j_\lambda = \sum_{p,a} e \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ v + \frac{\epsilon_p}{c} (v \cdot \Omega_\lambda)B_\lambda + e(\tilde{E}_\lambda \times \Omega_\lambda) \right] f_\lambda 
+ \sum_{p,a} e v \times \int \frac{d^3 p}{(2\pi \hbar)^3} f_\lambda \epsilon_p \Omega_\lambda, \quad \text{(6)}
$$

where $\sum_{p,a}$ denotes the summations over particles and antiparticles and the last term describes a magnetization current.

Local charge nonconservation.— By using Eqs. (1), (5), and (6) together with the Maxwell’s equations, one can easily derive the following continuity equations for the chiral and electric currents:

$$
\frac{\partial \rho_\lambda}{\partial t} + \nabla \cdot j_\lambda = \frac{e^3}{2\pi^2 \hbar^2 c} \left[ (E \cdot B) + (E_5 \cdot B_5) \right], \quad \text{(7)}
$$

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot j = \frac{e^3}{2\pi^2 \hbar^2 c} \left[ (E \cdot B_5) + (E_5 \cdot B) \right]. \quad \text{(8)}
$$

The first equation is related to the celebrated chiral anomaly [22] and expresses the nonconservation of the chiral charge in the presence of electromagnetic or pseudoelectromagnetic fields. Physically, this nonconservation can be understood as pumping of the chiral charge between the Weyl nodes of opposite chiralities. The second equation describes the anomalous local nonconservation of the electric charge in electromagnetic and pseudoelectromagnetic fields.

The local nonconservation of the electric charge is a very serious problem. If taken at face value, it would imply that the electric charge is literally created out of nothing. It was suggested in Refs. [13, 14] that it may correspond to pumping of the charge between the bulk and the boundary of the system. However, it is unclear how such a spatially nonlocal process could resolve the problem.

As we argue below, the resolution of the problem is much simpler. It lies in the fact that Eqs. (7) and (8) are the so-called covariant anomaly relations that come from the fermionic sector of the theory, in which left- and right-handed fermions are treated in a symmetric way. Just like in quantum field theory, this is inconsistent with the gauge symmetry. The correct physical currents, satisfying the local conservation of the electric charge, are the consistent currents [24]. A very clear discussion of these concepts in the framework of a low-energy effective theory is given in Ref. [25]. Clearly, the same should apply to the chiral kinetic theory. This means that one should add the following topological contribution to the electric four-current density [18, 24, 25]:

$$
\delta j^\mu = \frac{e^3}{4\pi^2 \hbar^2 c} \epsilon^{\mu\nu\rho\lambda} A_\nu^5 F_{\rho\lambda}, \quad \text{(9)}
$$

where $A_\nu^5 = b_\nu + \tilde{A}_\nu^5$ is the axial vector potential, which is an observable quantity. Indeed, in Weyl materials, $b_0$ and $b$ correspond to the energy and momentum-space separations between the Weyl nodes. On the other hand, $\tilde{A}_\nu^5$ is expressed through the deformation tensor and describes strain-induced axial (pseudoelectromagnetic) fields. [Note that there is also a correction to the chiral current density, but it contains only pseudoelectromagnetic fields [22] and, thus, will not affect the plasmon properties, discussed later.] In components, Eq. (9) takes
the following form:

\[
\delta \rho = \frac{e^3}{2\pi^2 \hbar^2 c^2} (A^5 \cdot B),
\]

\[
\delta j = \frac{e^3}{2\pi^2 \hbar^2 c} A^5_0 B - \frac{e^3}{2\pi^2 \hbar^2 c} (A^5 \times E).
\]

For \( B_5 \) to be nonzero, the axial field \( \tilde{A}^5 \) should depend on coordinates. We will assume, however, that such a dependence is weak and \( \tilde{A}^5 \) in Eqs. (10) and (11) is negligible compared to the chiral shift \( \mathbf{b} \).

As is easy to check, the consistent current \( J^\mu = (c \rho + \delta \rho, j + \delta j) \) is nonanomalous, \( \partial_\mu J^\mu = 0 \), therefore, the electric charge is locally conserved. Note that the consistent current plays an important role even in the absence of strain-induced pseudoelectromagnetic fields. For example, in the equilibrium state with \( \mu_5 = -e b_0 \), the first term in Eq. (11) exactly cancels the corresponding CME term in Eq. (11). The distribution function is taken in the form \( f_\lambda = f_\lambda^{\text{eq}} + f_\lambda^{(1)} \), where \( f_\lambda^{\text{eq}} \) is the equilibrium distribution function (13) and \( f_\lambda^{(1)} = f_\lambda^{(1)} e^{-i\omega t} \) is a perturbation. To leading linear order in oscillating fields, the solution to the kinetic equation (14) reads

\[
f_\lambda^{(1)} \approx -\frac{e v_F}{\omega} \frac{\partial f_\lambda^{\text{eq}}}{\partial \epsilon_p} \left\{ (\mathbf{p} \cdot \mathbf{E}) \left[ 1 + \frac{\lambda e (B_{0,\lambda} \cdot \mathbf{p})}{2c^3 \rho^3} \right] - i (\mathbf{p} \cdot [B_{0,\lambda} \times \mathbf{E}]) \frac{e v_F}{2c^3 \rho^3} \right\}.
\]

Similarly to the situation in a magnetized nonrelativistic plasma (16), the leading-order perturbation \( f_\lambda^{(1)} \) is proportional to the magnitude of the oscillating electric field. By making use of this solution, we derive the following result for the polarization vector:

\[
P' = \frac{a_0}{4\pi} E' + \frac{a_1}{4\pi} (\mathbf{b} \times E') + \frac{a_2}{4\pi} (E' \times \mathbf{z}),
\]

where \( \mathbf{z} \) is the unit vector in the \( +z \) direction and

\[
a_0 = \frac{n_0^2 \Omega_e^2}{\omega^2}, \quad a_1 = -\frac{1}{2} \frac{e n_0^2 \alpha v_F}{\omega^2}, \quad a_2 = -\frac{i}{3\pi \omega^2} \sum_{\lambda = \pm} \left( B_{0,\lambda} \mu_5 \frac{h^2}{\omega^2} - \frac{B_{0,\lambda} \mu_5}{4T} F \left( \frac{\mu_5}{T} \right) \right)
\]

where \( \lambda \) and \( \nu_\lambda \) are the high- and low-temperature asymptotes of the function \( F(\nu_\lambda) \), respectively.

Note that the high- and low-temperature asymptotes of this function are given by \( F(\nu_\lambda) \approx 0.426 \nu_\lambda \) for \( \nu_\lambda \to 0 \) and \( F(\nu_\lambda) \approx 1/\nu_\lambda \) for \( \nu_\lambda \to \infty \), respectively.
By making use of Eqs. (12), (13), and (15), we obtain the spectral equation for the collective modes at $k = 0$

$$(n_0^2 + a_0) \left\{ (a_0^2 + a_0)^2 + a^2 b^2 + (a_2 - a_1 b_\parallel)^2 \right\} = 0,$$  

(20)

where we introduced the transverse $b_\perp = \sqrt{b_\perp^2 + b_\parallel^2}$ and longitudinal $b_\parallel = b_\parallel$, components of the chiral shift. Notice that the spectral equation is explicitly factorized. The corresponding approximate solutions are

$$\omega_i = \Omega_e, \quad \omega_{i\parallel} = \Omega_e \sqrt{1 \pm \delta \Omega_e / \Omega_e},$$  

(21)

where

$$\delta \Omega_e = \frac{2 e \alpha v_F}{3 \pi c h} \left\{ 9 \hbar^2 b_\perp^2 + \frac{2 v_F}{\Omega_e^2} (B_0 \mu + B_{0.5} \mu_5) \right\} - 3 \hbar b_{\parallel} - \frac{v_F \hbar^2}{4 \pi} \sum_{\lambda = \pm} B_{0,\lambda} F \left( \frac{\mu_\lambda}{T} \right)^2 \right\}^{1/2}.$$

(22)

In the absence of the chiral shift, the collective modes \cite{21} correspond to the longitudinal $(E' \parallel \hat{z})$ and transverse $(E' \perp \hat{z})$ waves. Moreover, Eq. (21) means that the effects of dynamical electromagnetism transform, as argued in Ref. \cite{17}, the CMW into a longitudinal plasmon, whose frequency coincides exactly with the Langmuir one at linear order in the (pseudo-)magnetic field. It is interesting to point out that the combined effect of the pseudomagnetic field $B_{0,5}$ and the chiral chemical potential $\mu_5$ on the collective modes is similar to that of the magnetic field $B_0$ and the electric chemical potential $\mu$. The qualitative dependence of the plasma frequencies \cite{21} on the magnetic field $B_0$ is presented graphically in Fig. 1 at fixed values of $b_\perp$ and $b_\parallel$.

According to the upper panel in Fig. 1 the plasma frequencies of all three collective modes are different when $b_\parallel \neq 0$. In this case, the smallest splitting occurs at $B_0 = 0$, where $\omega_{i\parallel} = \omega_{i\perp} = \delta \Omega_e = 2 e \alpha v_F b_\perp / (\pi c h)$.

The situation is quite different in the case when $b_\perp = 0$, but $b_\parallel \neq 0$. This is demonstrated in the lower panel of Fig. 1. Now, while the three plasmons have generically different frequencies, one can make them degenerate by tuning the value of the magnetic field. The corresponding value of the magnetic field $B_0^*$, at which the frequency splitting vanishes, is given by

$$B_0^* = \frac{4 T}{v_F} \left[ \frac{2 v_F |B_0,5 \mu_5 - 3 \hbar \Omega_2 b_{\parallel}|}{8 T \mu - \hbar^2 \Omega_2 \sum_{\lambda = \pm} F \left( \frac{\mu_\lambda}{T} \right)} \right].$$  

(23)

Chiral magnetic plasmons.— It is worth discussing the chiral features of the collective excitations in more detail. It appears that these modes, including the longitudinal one, which describes the CMW with the effects of dynamical electromagnetism taken into account, are chiral plasmons, or rather chiral magnetic plasmons, when a background magnetic field is present. Their chiral nature is evident from the fact that they are accompanied by oscillations of not only the electric, but also the chiral current density. The result for the oscillating part of the electric current density is clear from the polarization vector if one uses Eqs. (12) and (15). As for the oscillating part of the chiral current density, it is given by the following expression:

$$J'_c = \sin (\omega t) \mathbf{E} \frac{2 e \alpha n_0^2 \mu_5}{3 \pi^2 \hbar^2 \omega} - \cos (\omega t) (\mathbf{E} \times \hat{z}) \frac{e \alpha n_0^2 v_F^2}{6 \pi^2 c} \times \sum_{\lambda = \pm} \frac{\lambda B_0,\lambda \mu_\lambda}{h^2 \omega^2} F \left( \frac{\mu_\lambda}{T} \right),$$

(24)

which is obtained using Eqs. (9) and (14). It is important to emphasize the topological origin of the first term in Eq. (24), which does not depend on temperature. In essence, it comes from a dynamical version of the chiral electric separation effect \cite{29}. The second term in Eq. (24) is related to a generalized Lorentz force.

We would like to note that the predicted frequencies and the splitting of plasmon frequencies as functions of an applied strain and/or magnetic field can be easily tested.
in experiment. As in the case of usual plasmons, this can be done by measuring the intensity and the phase shift of electromagnetic waves transmitted through a thin film of a Weyl material. The frequencies of transverse modes could be obtained from the peaks in the real part of optical conductivity, while the frequency of the longitudinal mode can be extracted from the energy loss function (e.g., see Ref. [20]).

Depending on the choice of a Weyl material, the estimated frequencies of the chiral magnetic plasmons could vary a lot. In Weyl semimetals such as NbP and TaAs, for example, the averaged Fermi velocity is about $v_F \approx 2 \times 10^7 \text{ cm/s}$ [31]. The corresponding Langmuir frequency may vary in a rather wide range between 1 THz to 100 THz depending on the actual values of the Fermi energy and temperature. The range of magnitude of the splitting between the transverse modes is more narrow, i.e., $\omega_{tr}^+ - \omega_{tr}^- \approx 0.3 b_\perp [\text{Å}^{-1}] \text{ THz}$, where the value of the chiral shift parameter $b_\perp$ varies from about $4 \times 10^{-3} \text{ Å}^{-1}$ (NbAs) to about $3 \times 10^{-2} \text{ Å}^{-1}$ (TaAs) [31].

**Conclusion.**—As we showed in this Letter, the consistent chiral kinetic theory in Weyl materials should necessarily include the topological Chern–Simons contribution that ensures the local conservation of the electric charge in electromagnetic and strain-induced pseudoelectromagnetic fields. Moreover, as we emphasized, such a term plays an important role even in the absence of pseudoelectromagnetic fields. It allows one to correctly describe the anomalous Hall effect in Weyl materials [27] and to reproduce the vanishing CME current in an equilibrium magnetic fields. Moreover, as we emphasized, such a term will be reported elsewhere.

As demonstrated here, the collective modes in Weyl materials are the chiral plasmons with interesting properties. Such modes are associated with the oscillations of both electric and chiral current densities. This is in contrast to the ordinary electromagnetic plasmons which are not connected with the oscillations of the chiral current density. It is worth mentioning that for the longitudinal mode, which corresponds to the CMW, these oscillations are of purely topological origin and are related to a dynamical version of the chiral electric separation effect.

While the plasma frequency of the longitudinal mode coincides with the Langmuir one, the frequencies of the transverse modes generically split up. The frequency splitting depends on both magnetic (pseudomagnetic) field and electric (chiral) chemical potential. As we showed, the qualitative features of this dependence on the magnetic field can be used to develop a protocol for experimentally extracting both the direction and magnitude of the chiral shift parameter in Weyl materials.

In this Letter, the study was restricted to the long-wavelength limit ($k = 0$) of the chiral magnetic plasmons and used an expansion to the linear order in background magnetic and pseudomagnetic fields. The generalization of this investigation to the case of nonzero wave vectors ($k \neq 0$) and higher orders in magnetic and pseudomagnetic fields will be reported elsewhere.

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[18] W. A. Bardeen, Phys. Rev. \textbf{184}, 1848 (1969); W. A. Bardeen and B. Zumino, Nucl. Phys. B \textbf{244}, 421 (1984).
[19] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. \textbf{109}, 162001 (2012).
[20] D. T. Son and N. Yamamoto, Phys. Rev. D \textbf{87}, 085016 (2013); D. T. Son and B. Z. Spivak, Phys. Rev. B \textbf{88}, 104412 (2013).
[21] M. V. Berry, Proc. R. Soc. A \textbf{392}, 45 (1984).
[22] D. Xiao, J. Shi, and Q. Niu, Phys. Rev. Lett. \textbf{95}, 137204 (2005) [Phys. Rev. Lett. \textbf{95}, 169903 (2005)]; C. Duval, Z. Horvath, P. A. Horvathy, L. Martina, and P. Stichel, Mod. Phys. Lett. B \textbf{20}, 373 (2006).
[23] S. L. Adler, Phys. Rev. \textbf{177}, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cim. A \textbf{60}, 47 (1969).
[24] K. Landsteiner, Phys. Rev. B \textbf{89}, 075124 (2014).
[25] K. Landsteiner, Acta Phys. Polon. B \textbf{47}, 2617 (2016).
[26] M. M. Vazifeh and M. Franz, Phys. Rev. Lett. \textbf{111}, 027201 (2013); G. Basar, D. E. Kharzeev, and H. U. Yee, Phys. Rev. B \textbf{89}, 035142 (2014).
[27] A. A. Burkov and L. Balents, Phys. Rev. Lett. \textbf{107}, 127205 (2011); A. G. Grushin, Phys. Rev. D \textbf{86}, 045001 (2012); P. Goswami and S. Tewari, Phys. Rev. B \textbf{88}, 245107 (2013).
[28] J. Buckeridge, D. Jevdokimovs, C. R. A. Catlow, and A. A. Sokol, Phys. Rev. B \textbf{93}, 125205 (2016).
[29] X. G. Huang and J. Liao, Phys. Rev. Lett. \textbf{110}, 232302 (2013); Y. Jiang, X. G. Huang, and J. Liao, Phys. Rev. D \textbf{91}, 045001 (2015).
[30] D. Pines, \textit{Elementary Excitations in Solids} (Benjamin, New York, 1964).
[31] C. C. Lee, S.-Y. Xu, S.-M. Huang, D. S. Sanchez, I. Belopolski, G. Chang, G. Bian, N. Alidoust, H. Zheng, M. Neupane, B. Wang, A. Bansil, M. Z. Hasan, and H. Lin, Phys. Rev. B \textbf{92}, 235104 (2015).