Finite groups with $\mathbb{P}$-subnormal 2-maximal subgroups

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Abstract

A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$ if either $H = G$ or there is a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $|H_i : H_{i-1}|$ is prime for $1 \leq i \leq n$. In this paper we study the groups all of whose 2-maximal subgroups are $\mathbb{P}$-subnormal.

Keywords: finite group, $\mathbb{P}$-subnormal subgroup, 2-maximal subgroup.

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1 Introduction

We consider finite groups only. A subgroup $K$ of a group $G$ is called 2-maximal in $G$ if $K$ is a maximal subgroup of some maximal subgroup $M$ of $G$.

Let $H$ be a subgroup of a group $G$ and $n$ is a positive integer. If there is a chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = G,$$

such that $H_i$ is a maximal subgroup of $H_{i+1}$, $i = 0, 1, \ldots, n - 1$, then $H$ is called $n$-maximal in $G$.

For example, in the symmetric group $S_4$ the subgroup $I$ of order 2 from $S_3$ is 2-maximal in the chain of subgroups $I \subset S_3 \subset S_4$ and 3-maximal in the chain of subgroups $I \subset Z_4 \subset D_8 \subset S_4$. Here, $Z_4$ is the cyclic group of order 4 and $D_8$ is the dihedral group of order 8. For any $n \geq 3$, there exists a group in which some 2-maximal subgroup is $n$-maximal, see Example 1 below.

A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov in [1] introduced the following definition. Let $\mathbb{P}$ be the set of all prime numbers. A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$ if either $H = G$ or there is a chain

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

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of subgroups such that $|H_i : H_{i-1}|$ is prime for $1 \leq i \leq n$. In [1], [2] studied groups with $\mathbb{P}$-subnormal Sylow subgroups.

In [1] proposed the following problem:

Describe the groups in which all 2-maximal subgroups are $\mathbb{P}$-subnormal.

This problem is solved in the article. The following theorem is proved.

**Theorem.** Every 2-maximal subgroup of a group $G$ is $\mathbb{P}$-subnormal in $G$ if and only if $\Phi(G^u) = 1$ and every proper subgroup of $G$ is supersolvable.

Here, $G^u$ is the smallest normal subgroup of $G$ such that the corresponding quotient group is supersolvable, $\Phi(G^u)$ is the Frattini subgroup of $G^u$.

### 2 Preliminary results

We use the standart notation of [3]. The set of prime divisors of $|G|$ is denoted $\pi(G)$. We write $[A]B$ for a semidirect product with a normal subgroup $A$. If $H$ is a subgroup of a group $G$, then $\bigcap_{x \in G} x^{-1}Hx$ is called the core of $H$ in $G$, denoted $H_G$. If a group $G$ contains a maximal subgroup $M$ with trivial core, then $G$ is said to be primitive and $M$ is its primitivator. We will use the following notation: $S_n$ and $A_n$ are the symmetric and alternating groups of degree $n$, $E_{p^t}$ is the elementary abelian group of order $p^t$, $Z_m$ is the cyclic group of order $m$. Let $|G| = p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$, where $p_1 > p_2 > \ldots > p_k$. We say that $G$ has an ordered Sylow tower of supersolvable type if there exist normal subgroups $G_i$ with

$$1 = G_0 \leq G_1 \leq G_2 \leq \ldots \leq G_{k-1} \leq G_k = G,$$

and, where each factor $G_i/G_{i-1}$ is isomorphic to a Sylow $p_i$-subgroup of $G$ for all $i = 1, 2, \ldots, k$.

**Lemma 1.** [4, Theorem IX.8.3] Let $a$, $n$ be integers greater than 1. Then except in the cases $n = 2$, $a = 2^b - 1$ and $n = 6$, $a = 2$, there is a prime $q$ with the following properties:

1) $q$ divides $a^n - 1$;
2) $q$ does not divide $a^i - 1$ whenever $0 < i < n$;
3) $q$ does not divide $n$.

**Example 1.** For every $n \geq 3$ there exists a group in which some 2-maximal subgroup is $n$-maximal. Let $n = 3$. In the symmetric group $S_4$ the subgroup $I$ of order 2 from $S_3$ is 2-maximal in the chain of subgroups $I \subset S_3 \subset S_4$ and 3-maximal in the chain of subgroups $I \subset E_4 \subset D_8 \subset S_4$. Now let $n > 3$ and $a = 5$. By Lemma 1, there exists a prime $q$ such that $q$ divides $5^{n-1} - 1$ and $q$ does not divide $5^i - 1$ for all $i \in \{1, 2, \ldots, n - 2\}$. Hence
$GL(n - 1, 5)$ contains a subgroup $Z$ of order $q$ which acts irreducibly on the elementary abelian group $E_{5^{n-1}}$ of order $5^{n-1}$. In the group $X = [E_{5^{n-1}}]Z$ the identity subgroup 1 is 2-maximal in the chain of subgroups $1 \subset Z \subset X$ and $n$-maximal in the chain of subgroups $1 \subset E_5 \subset E_5^2 \subset \ldots \subset E_{5^{n-1}} \subset X$.

Recall that a Schmidt group is a finite non-nilpotent group in which every proper subgroup is nilpotent.

Example 2. Let $S = [P|Q$ be a Schmidt group of order $2^{11}11$, $A = \Phi(P)$, $|A| = 2$. Then $A \times Q$ is maximal in $S$, $A$ is 2-maximal in $S$, and $A$ is 10-maximal in $S$ because $A = A_0 \subset A_1 \subset \ldots \subset A_9 = P \subset S$, $|A_i : A_{i-1}| = 2$, $1 \leq i \leq 9$, $|S : P| = 11$.

Lemma 2. [11, Lemma 2.1] Let $N$ be a normal subgroup of a group $G$, $H$ an arbitrary subgroup of $G$. Then the following hold:

1) if $H$ is $P$-subnormal in $G$, then $(H \cap N)$ is $P$-subnormal in $N$, and $HN/N$ is $P$-subnormal in $G/N$;

2) if $N \subset H$ and $H/N$ is $P$-subnormal in $G/N$, then $H$ is $P$-subnormal in $G$;

3) if $H$ is $P$-subnormal in $K$, $K$ is $P$-subnormal in $G$, then $H$ $P$-subnormal in $G$;

4) if $H$ is $P$-subnormal in $G$, then $H^g$ is $P$-subnormal in $G$ for each element $g \in G$.

Example 3. In the alternating group $G = A_5$ the subgroup $H = A_4$ is $P$-subnormal. If $x \in G \setminus H$, then $H^x$ is $P$-subnormal in $G$. The subgroup $D = H \cap H^x$ is a Sylow 3-subgroup of the group $G$ and $D$ is not $P$-subnormal in $H$. Therefore an intersection of two $P$-subnormal subgroups is not $P$-subnormal. Moreover, if a subgroup $H$ is $P$-subnormal in a group $G$ and $K$ is an arbitrary subgroup of $G$, in general, their intersection $H \cap K$ is not $P$-subnormal in $K$.

Lemma 3. Let $H$ be a subgroup of a solvable group $G$, and assume that $|G : H|$ is a prime number. Then $G/H_G$ is supersolvable.

Proof. By hypothesis, $|G : H| = p$, where $p$ is a prime number. If $H = H_G$, then $G/H$ is cyclic of prime order $p$, and thus $G/H_G$ is supersolvable, as required. Assume now that $H \neq H_G$, i.e. $H$ is not normal in $G$. Then $G/H_G$ contains a maximal subgroup $H/H_G$ with trivial core. Therefore $G/H_G$ is primitive and its Fitting subgroup $F/H_G$ has prime order $p$. Since $F/H_G = C_{G/H_G}(F/H_G)$, it follows that $(G/H_G)/(F/H_G) \simeq H/H_G$ is isomorphic to a cyclic group of order dividing $p - 1$. Thus $G/H_G$ is supersolvable.

Lemma 4. Let $p$ be the largest prime divisor of $|G|$, and suppose that $P$ is a Sylow $p$-subgroup of $G$. Assume that $P$ is not normal in $G$, and that $H, K \subseteq G$ are subgroups with $N_G(P) \subseteq K \subseteq H$. Then $|H : K|$ is not prime.

Proof. It is clear that $N_G(P) = N_K(P) = N_H(P)$, and $P$ is a Sylow $p$-subgroup of $K$
and of $H$. By the lemma on indexes, we have

$$|H : N_H(P)| = |H : K| |K : N_K(P)|,$$

and, by the Sylow theorem,

$$|H : N_H(P)| = 1 + hp, \quad |K : N_K(P)| = 1 + kp, \quad h, k \in \mathbb{N} \cup \{0\}.$$  

Let $|H : K| = t$. Now,

$$1 + hp = t(1 + kp), \quad t = 1 + (h - tk)p.$$  

We see that $p$ divides $t - 1$, and thus $t > p$. If $t$ is prime, this contradicts the maximality of $p$.

**Lemma 5.** 1. A group is supersolvable if and only if the index of every of its maximal subgroup is prime.

2. Every subgroup of a supersolvable group is $\mathbb{P}$-subnormal.

3. A group is supersolvable if and only if the normalizers of all of its Sylow subgroups are $\mathbb{P}$-subnormal.

**Proof.** 1. This is Huppert’s classic theorem, see [3, Theorem VI.9.5].

2. The statement follows from (1) of the lemma.

3. If a group is supersolvable, then all of its subgroups are $\mathbb{P}$-subnormal, see (2).

Conversely, suppose that the normalizer of every Sylow subgroup of a group $G$ is $\mathbb{P}$-subnormal. By Lemma 4, for the largest $p \in \pi(G)$ a Sylow $p$-subgroup $P$ of $G$ is normal. It is easy to check that the conditions of the lemma are inherited by all quotient groups and so $G/P$ is supersolvable. In particular, $G$ has an ordered Sylow tower of supersolvable type. Since the class of all supersolvable groups is a saturated formation, we can assume, by the inductive hypothesis, that $G$ is primitive, in particular, $G = [P]M$, where $M$ is a maximal subgroup with trivial core. Since $M$ is supersolvable, it follows that $M = N_G(Q)$ for the largest $q \in \pi(M)$. It is obvious that $p \neq q$ and $M = N_G(Q)$ is $\mathbb{P}$-subnormal in $G$, by the condition of the lemma. Therefore $|P| = p$ and, by Lemma 3, $G$ is supersolvable.

**Lemma 6.** [5, Theorem 22], [6] Let $G$ be a minimal non-supersolvable group. We have:

1) $G$ is solvable and $|\pi(G)| \leq 3$.

2) If $G$ is not a Schmidt group, then $G$ has an ordered Sylow tower of supersolvable type.

3) $G$ has a unique normal Sylow subgroup $P$ and $P = G^{\text{aut}}$.

4) $|P/\Phi(P)| > p$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(G)$.

5) The Frattini subgroup $\Phi(P)$ of $P$ is supersolvable embedded in $G$, i.e., there exists a series

$$1 \subset N_0 \subset N_1 \ldots \subset N_n = \Phi(P)$$
such that $N_i$ is a normal subgroup of $G$ and $|N_i/N_{i-1}| \in \mathbb{P}$ for $1 \leq i \leq n$.

6) Let $Q$ be a complement to $P$ in $G$. Then $Q/Q \cap \Phi(G)$ is a minimal non-abelian group or a cyclic group of prime power order.

7) All maximal subgroups of non-prime index are conjugate in $G$, and moreover, they are conjugate to $\Phi(P)Q$.

3 Main results

Theorem. Every 2-maximal subgroup of a group $G$ is $\mathbb{P}$-subnormal in $G$ if and only if $\Phi(G^{ab}) = 1$ and every proper subgroup of $G$ is supersolvable.

Proof. Suppose that all 2-maximal subgroups of a group $G$ are $\mathbb{P}$-subnormal. We proceed by induction on $|G|$. Show first that $G$ has an ordered Sylow tower of supersolvable type. By Lemma 2, the conditions of the theorem are inherited by all quotient groups of $G$.

(1) $G$ has an ordered Sylow tower of supersolvable type.

Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the largest prime divisor of $|G|$. Suppose that $P$ is not normal in $G$. It follows that $N_G(P)$ is a proper subgroup of $G$. If $N_G(P)$ is not maximal in $G$, then there exists a 2-maximal subgroup $A$ containing $N_G(P)$. By the condition of the theorem, $A$ is $\mathbb{P}$-subnormal in $G$, and so $A$ is contained in a subgroup of prime index. This contradicts Lemma 4. Therefore $N_G(P)$ is maximal in $G$ and $|G : N_G(P)| \not\in \mathbb{P}$ by Lemma 4. If $N_G(P) = P$, then $G$ is solvable by Theorem IV.7.4 [3]. It follows that $N_G(P) \neq P$ and $N_G(P)$ has a maximal subgroup $B$ which contains $P$. We see that $B$ is 2-maximal in $G$ and, by the condition of the theorem, $B$ is $\mathbb{P}$-subnormal. Hence there exists a chain of subgroups

$$P \subseteq B = B_0 \subset B_1 \subset \ldots \subset B_{t-1} = V \subset B_t = G,$$

$$|B_i : B_{i-1}| \in \mathbb{P}, 1 \leq i \leq t.$$

The subgroup $V$ is maximal in $G$ and $V$ different from $N_G(P)$, because $|G : N_G(P)|$ is not a prime number, whereas $|G : V|$ is prime. Besides, $t \geq 3$. Thus $V \cap N_G(P) = B$ and $N_V(P) = V \cap N_G(P) = B = N_{B_1}(P)$. We have $|B_1 : N_{B_1}(P)| \in \mathbb{P}$, this contradicts Lemma 4. Therefore the assumption is false and $P$ is normal in $G$. By induction on $|G|$, every proper subgroup of $G/P$ is supersolvable, and by Lemma 6, $G/P$ has an ordered Sylow tower of supersolvable type. Thus $G$ has an ordered Sylow tower of supersolvable type, in particular, $G$ is solvable.

(2) Every proper subgroup of $G$ is supersolvable.
Suppose that \( G \) contains a non-supersolvable maximal subgroup \( H \). Then, by Lemma 5, \( H \) contains a maximal subgroup \( K \) of non-prime index. Since \( K \) is 2-maximal in \( G \), there exists a chain of subgroups

\[
K = K_0 \subset K_1 \subset \ldots \subset K_{n-1} = T \subset K_n = G
\]
such that \( |K_i : K_{i-1}| \in \mathbb{P} \) for all \( i = 1, 2, \ldots, n \). It is clear that \( H \neq T \) and \( H \cap T = K \).

Assume that \( G = HT \). In this case,

\[
|G : T| = |H : H \cap T| = |H : K| \in \mathbb{P},
\]
this is a contradiction. Hence \( G \neq HT \). Since \( H \) and \( T \) are distinct maximal subgroups of \( G \), and \( G \) is solvable, by Theorem II.3.9 \([3]\), we have \( T = H^g \) for some \( g \in G \). Since \( H \neq T \), we see that \( H \) is a non-normal maximal subgroup of prime index in \( G \). By Lemma 3, the quotient group \( G/H \) is supersolvable. Since

\[
H_G \subseteq H \cap H^g = H \cap T = K,
\]
we have \( K/H_G \) is maximal in \( H/H_G \). By Lemma 5,

\[
|H : K| = |H/H_G : K/H_G| \in \mathbb{P},
\]
this is a contradiction. Therefore the assumption is false and every proper subgroup of \( G \) is supersolvable.

(3) \( \Phi(G^\alpha) = 1 \)

If \( G \) is supersolvable, then \( G^\alpha = 1 \), it follows that \( \Phi(G^\alpha) = 1 \). Assume now that \( G \) is non-supersolvable. Then \( G \) has the properties listed in Lemma 6. We keep the notation of that lemma. Now \( G^\alpha = P \) and \( \Phi(P)Q \) is maximal in \( G \).

Suppose that \( \Phi(P) \neq 1 \). Assume that \( A = N_{m-1} \) is a maximal subgroup of \( \Phi(P) \), and that \( A \) is normal in \( G \). Then \( [A]Q \) is a 2-maximal subgroup of \( G \). By the condition of the theorem, \( [A]Q \) is \( \mathbb{P} \)-subnormal in \( G \). Hence, there exists a chain of subgroups \( [A]Q \subseteq B \subseteq G \) such that \( |G : B| \in \mathbb{P} \). Since \( G = [P]Q \) and \( Q \subseteq B \), by the Dedekind identity, we have \( B = (B \cap P)Q \), and \( B \cap P \) is maximal in \( P \). Therefore \( \Phi(P) \subseteq B \cap P \) and \( \Phi(P)Q \) is conained in \( B \), where \( \Phi(P)Q \) is maximal in \( G \). Thus \( B = \Phi(P)Q \) and \( p = |G : B| = |P : \Phi(P)| \), this contradicts Lemma 6. Therefore our assumption is false and \( \Phi(P) = 1 \). The necessity is proved.

Prove the sufficiency. Assume that every proper subgroup of \( G \) is supersolvable and \( \Phi(G^\alpha) = 1 \). If a group is supersolvable, then every its maximal subgroup has a prime index, it follows that every 2-maximal subgroup of a supersolvable group is \( \mathbb{P} \)-subnormal. Let \( G \) be non-supersolvable. Then \( G \) is minimal non-supersolvable and the structure of \( G \) is described in Lemma 6. We keep for \( G \) the notation of that lemma, in particular, we have:
$P = G^{ul}$, $Φ(P) = 1$ and $Q$ is a maximal subgroup of $G$. Let $H$ be an arbitrary 2-maximal subgroup of the group $G$. If $H ⊆ M$, where $M$ is a maximal subgroup of $G$ and $|G : M| ∈ \mathbb{P}$, then $H$ is $\mathbb{P}$-subnormal in $G$, because $M$ is supersolvable. If $H ⊆ K$, where $K$ is a maximal subgroup of the group $G$ and $|G : K| ∉ \mathbb{P}$, then, by Lemma 6, the subgroup $H$ contained in $Q^g$ for some $g ∈ G$. Therefore $PH$ is a proper subgroup of $G$, thus $PH$ is supersolvable, and $H$ is $\mathbb{P}$-subnormal in $PH$. Let $T$ be a maximal subgroup of $G$ containing $PH$. Since $T$ is supersolvable and $|G : T| ∉ \mathbb{P}$, we see that $PH$ is $\mathbb{P}$-subnormal in $G$. Using Lemma 2, we deduce that $H$ is $\mathbb{P}$-subnormal in $G$. The theorem is proved.

**Corollary.** Suppose that every 2-maximal subgroup of a group $G$ is $\mathbb{P}$-subnormal. If $|π(G)| ≥ 4$, then $G$ is supersolvable.

**Proof.** Let every 2-maximal subgroup of a group $G$ be $\mathbb{P}$-subnormal. Suppose that $G$ is not supersolvable. By the previous theorem, $G$ is a minimal non-supersolvable group. By Lemma 6, the order of $G$ has at most three prime divisors, i.e. $|π(G)| ≤ 3$, which is a contradiction. Therefore, our assumption is false and $G$ is supersolvable.

The following examples show that for $|π(G)| = 2$ and for $|π(G)| = 3$ there exist non-supersolvable groups in which every 2-maximal subgroup is $\mathbb{P}$-subnormal.

**Example 4.** There are three non-isomorphic minimal non-supersolvable groups of order 400:

$$[E_{52}](<a><b>),\ |a| = |b| = 4.$$ 

Numbers of these groups in the library of SmallGroups are [400,129], [400,130], [400,134]. The Sylow 2-subgroups of these groups are non-abelian and have the form: $[Z_4 × Z_2]Z_2$ and $[Z_4]Z_4$. Suppose that $G$ is one of these groups. Then $G^{ul} = [E_{52}]$ and $Φ(G^{ul}) = 1$. All subgroups of the group $G$ are $\mathbb{P}$-subnormal, except the maximal subgroup $<a><b>$.

**Example 5.** The general linear group $GL(2, 7)$ contains the symmetric group $S_3$ which acts irreducibly on the elementary abelian group $E_{72}$ of order 49. The semidirect product $[E_{72}]S_3$ is a minimal non-supersolvable group, it has subgroups of orders 14 and 21. Therefore, in the group $[E_{72}]S_3$, every 2-maximal subgroup is $\mathbb{P}$-subnormal.

**References**

[1] Vasilyev A. F., Vasilyeva T. I., Tyutyanov V. N. On finite groups similar to supersoluble groups // Problems of physics, mathematics and technics. 2010. No. 2 (3). P. 21–27.

[2] Vasilyev A. F., Vasilyeva T. I., Tyutyanov V. N. On the finite groups of supersoluble type // Siberian Mathem. J. 2010. Vol. 51, No. 6. P. 1004–1012.
[3] Huppert B. Endliche Gruppen I. Berlin, Heidelberg, New York. 1967. 792 p.

[4] Huppert B., Blackburn N. Finite groups, II. Berlin–Heidelberg–New York: Springer. 1982.

[5] Huppert B. Normalteiler und maximale Untergruppen endlicher Gruppen // Mathematische Zeitschrift. 1954. Vol. 60. P. 409–434.

[6] Doerk K. Minimal nicht überauflösbare, endliche Gruppen // Mathematische Zeitschrift. 1966. Vol. 91. P. 198–205.

[7] GAP (2009) Groups, Algorithms, and Programming, Version 4.4.12. www.gap-system.org.

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