A \(p\)-CONGRUENCE FOR A FINITE SUM OF PRODUCTS OF BINOMIAL COEFFICIENTS

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Abstract

For any prime \(p\) and any natural numbers \(\ell, n\) such that \(p\) does not divide \(n\), it holds that 
\[
\sum_{i \geq \ell+1} (-1)^i \binom{\left\lfloor \frac{n+1}{p} \right\rfloor p}{i} \binom{n-1+i(p-1)}{n-1+\ell(p-1)} \equiv 0 \mod p.
\]
Our proof involves Stirling numbers.

1. Introduction, notations and preliminaries

The purpose of the present paper is to establish that for any prime \(p\) and any natural numbers \(\ell, n\) such that \(p\) does not divide \(n\), we have
\[
\sum_{i \geq \ell+1} (-1)^i \binom{\left\lfloor \frac{n+1}{p} \right\rfloor p}{i} \binom{n-1+i(p-1)}{n-1+\ell(p-1)} \equiv 0 \mod p.
\]
In the following, \([x^n]f(x)\) denotes the coefficient of \(x^n\) in \(f(x)\), where \(f\) is a formal power series with the argument \(x\) and \(Df(x)\) is the derivative of \(f(x)\) with respect to \(x\). If \(x\) is a real number, we denote \(\lfloor x \rfloor\) the largest integer smaller or equal to \(x\). We also use the Iverson bracket notation: \([\mathcal{P}] = 1\) when proposition \(\mathcal{P}\) is true, and \([\mathcal{P}] = 0\) otherwise.

Lemma 1.1. Let \(f(w)\) be a formal power series, and \(\alpha\) a natural number, we have
\[
[[w^n]] f(w) = [[w^{n-1}]] \frac{Df(w)}{n}.
\]
Proof. This is clear, since 
\[
[[w^n]] f(w) = \frac{[[w^{n-1}]] Df(w)}{n}.
\]

We recall some basic properties of the binomial coefficients and Stirling numbers, which can be found for instance in [1]. The binomial coefficients \(\binom{n}{k}\), are defined by 
\[
\sum_k \binom{n}{k} x^k = (1 + x)^n,
\]
whatever the sign of integer \(n\). They obviously vanish when \(k < 0\). They are easily obtained by the basic recurrence relation \(\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}\). When \(n > 0\), we have \(\binom{-n}{k} = (-1)^k \binom{n+k-1}{n-1}\).
The cycle (or first kind) Stirling numbers \( \left[ \frac{n}{k} \right] \), \( n \geq 0 \) may be defined by the horizontal generating function
\[
\sum_k \left[ \frac{n}{k} \right] x^k = \prod_{j=0}^{n-1} (x + j), \tag{1.1}
\]
where an empty product is meant to be 1. Alternatively, they have the exponential generating function
\[
\sum_n \left[ \frac{n}{k} \right] \frac{x^n}{n!} = \frac{(-1)^k \left( \ln(1 - x) \right)^k}{k!}. \tag{1.2}
\]
They obviously vanish when \( k < 0 \) and \( k > n \). They are easily obtained by the basic recurrence
\[
\left[ \frac{n}{k} \right] = (n - 1) \left[ \frac{n-1}{k} \right] + \left[ \frac{n-1}{k-1} \right], \quad \text{valid for } n \geq 1, \quad \text{with } \left[ \frac{0}{k} \right] = \left[ k = 0 \right].
\]

We let \( \{ n \ k \} \), \( n \geq 0 \), be the partition (or second kind) Stirling number. They also vanish when \( k < 0 \) and \( k > n \). Their basic recurrence is \( \{ n \ k \} = k \{ n \ k-1 \} + \{ n-1 \ k \} \) for \( n \geq 1 \), with \( \{ 0 \ k \} = \left[ k = 0 \right] \). They have the following exponential generating function
\[
\sum_n \{ n \ k \} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!} \tag{1.3}
\]
and the following explicit expression
\[
\{ n \ k \} = \frac{(-1)^k}{k!} \sum_{j \geq 0} (-1)^j \binom{k}{j} j^n. \tag{1.4}
\]

**Lemma 1.2.** For any natural number \( n \), \( n > 0 \), we have
\[
\left\{ \frac{n}{p-1} \right\} \equiv [p - 1 \text{ divides } n] \pmod{p}. \tag{1.5}
\]

**Proof.** We recall the Wilson theorem which states that \((p-1)! \equiv -1 \mod {p}\) for any prime \( p \), and some other well-known congruences, valid for any prime \( p \):
\[
\binom{p-1}{j} \equiv (-1)^j [0 \leq j \leq p-1] \pmod{p},
\]
\[
\sum_{p-1 \geq j \geq 1} j^k \equiv - [p - 1 \text{ divides } k] \pmod{p},
\]
so that, the claim readily follows from Equation (1.4). \( \square \)

### 2. \( p \)-congruences for Stirling numbers of first kind.

Let \( p \) be a prime number. We consider \( x^p := \prod_{j=0}^{p-1} (x + j) = \sum_k \left[ \frac{p}{k} \right] x^k \) as a polynomial in \( \mathbb{Z}/p\mathbb{Z} \). In that ring, we have \( \sum_k \left[ \frac{p}{k} \right] x^k = x^p - x \), since the polynomials
on both sides have have same degree $p$, same coefficient for $x^p$ and same roots: $0, -1, -2, \ldots, -(p-1)$. In particular, for $k$ such that $1 < k \leq p - 1$, we have $\binom{p}{k} \equiv 0 \mod p$.

Let $n$ be a non-negative integer and $r$, resp $q$ be the residue, resp. the quotient of the Euclidean division of $n$ by $p$, such that $n = qp + r$, with $0 \leq r \leq p - 1$. We may explicitly write and regroup the factors of $x^n$ so that

$$x^n = \prod_{t=0}^{q-1} ((x + tp)(x + tp + 1) \cdots (x + tp + (p - 1)) \prod_{u=0}^{r-1} (x + q + u),$$

with the convention that when $q = 0$ or $r = 0$ the empty products are meant to be equal to 1. Then, reducing modulo $p$, we have

$$\sum_k \binom{n}{k} x^k = \prod_{t=0}^{q-1} (x(x + 1) \cdots (x + (p - 1))) \prod_{u=0}^{r-1} (x + u) \mod p$$

$$\equiv (x(x + 1) \cdots (x + (p - 1)))^q \prod_{u=0}^{r-1} (x + u) \mod p$$

$$\equiv (x^p - x)^q \prod_{u=0}^{r-1} (x + u) \mod p$$

$$\equiv x^q (x^{p-1} - 1)^q \prod_{u=0}^{r-1} (x + u) \mod p$$

$$\equiv x^q \sum_{m=0}^{q} (-1)^{q-m} \binom{q}{m} x^{m(p-1)} \prod_{u=0}^{r-1} (x + u) \mod p$$

$$\sum_k \binom{n}{k} x^{k-q} = \left( \sum_{m=0}^{q} (-1)^{q-m} \binom{q}{m} x^{m(p-1)} \right) \left( \sum_{\ell=0}^{r} \binom{r}{\ell} x^{\ell} \right) \mod p$$

$$\equiv \sum_k \sum_{m,l} (-1)^{q-m} \binom{q}{m} \binom{r}{\ell} x^{k-q} \mod p.$$ 

Then

$$\binom{n}{k} \equiv \sum_{m,l} (-1)^{q-m} \binom{q}{m} \binom{r}{\ell} \mod p. \quad (2.1)$$

Now if $p$ divides $n$, we have $r = 0$, and there may exist only one possible solution in non-negative integers $\ell, m$ to the equation $m(p - 1) + \ell = k - q$ which may have
a non-zero contribution to the sum on the right hand side of the above congruence: this is when \( p - 1 \) divides \( k - \frac{n}{p} \) and we have \( \ell = 0 \) and \( m = \frac{k-q}{p-1} \). Otherwise, \( 1 \leq r < p \) and since \( \ell \leq r \) and \( r < p \), we have \( \ell < p \). But also \( \ell > 0 \), since \( \lfloor r \rfloor = 0 \), since \( r > 0 \). Then, there exists at most one solution in non-negative integers \( \ell, m \) to the equation \( m(p-1) + \ell = k - q \). Indeed, let \( \rho \) be the residue of the Euclidean division of \( k - q \) by \( p - 1 \). We have \( 0 \leq \rho < p - 1 \).

If \( \rho = 0 \), then the unique solution is \( \ell = p - 1 \) and \( m = \frac{k-q}{p-1} \).

If \( 0 < \rho \leq r \), then the unique solution is \( \ell = \rho \) and \( m = \frac{k-q-\rho}{p-1} \).

And finally if \( r < \rho < p - 1 \), there is no solution.

Putting everything together, we have the following theorem for the \( p \)-congruence of the Stirling numbers of the first kind

**Theorem 2.1.** Let \( p \) be a prime number and \( n, k \) non-negative integers such that \( 0 \leq k \leq n \) and let \( r \), (resp. \( q \)), be the residue, (resp. the quotient) of the Euclidean division of \( n \) by \( p \) and let \( \rho \) be the residue of the Euclidean division of \( k - q \) by \( p - 1 \). We have

\[
\binom{n}{k} \equiv (-1)^q \frac{k-q}{p-1} \binom{r}{j} \left( \frac{q}{k-q} \right) \pmod{p} \tag{2.2}
\]

with \( j = \rho + [\rho = 0] [r = p - 1] (p - 1) \).

**Remark.** Note that when \( \rho = 0 \), we have \( \binom{n}{k} \equiv 0 \mod p \) unless when \( r = p - 1 \) in which case, we have \( \binom{n}{k} \equiv (-1)^{q+1} \frac{k-q}{p-1} \binom{q}{p-1} \mod p \).

**Corollary 2.1.** Let \( p \) be a prime number, and \( i, m \) and \( s \) three natural numbers. We have

\[
\binom{m+s+m(p-1)}{m+s+i(p-1)} \equiv (-1)^{m-i} \binom{m+\lfloor \frac{s}{p} \rfloor}{i+\lfloor \frac{s}{p} \rfloor} \pmod{p}. \tag{2.3}
\]

**Proof.** We apply Theorem 2.2 with \( n = mp+s \) and \( k = m+s+i(p-1) \). We have \( q = m + \lfloor \frac{s}{p} \rfloor \) and \( k - q = s + i(p-1) - \lfloor \frac{s}{p} \rfloor = (i + \lfloor \frac{s}{p} \rfloor)(p-1) + r \). Then, when \( s - p\lfloor \frac{s}{p} \rfloor = r < p - 1 \) we have \( r = \rho < p - 1 \) and then \( j = r \) and then Congruence (3.1) reduces to Congruence (2.3). Otherwise \( s - p\lfloor \frac{s}{p} \rfloor = r = p - 1 \) and then \( \rho = 0 \) and then \( j = p - 1 \) and then Congruence (3.1) also reduces to Congruence (2.3).

### 3. An identity involving both kinds of Stirling numbers

Now, we present an identity involving binomial coefficients and the Stirling numbers of both kinds, which we believe is new.
Theorem 3.1. Let \( p \) be a positive integer and \( n, k \) non-negative integers. We have

\[
(-1)^{p-1} \binom{n-1}{p-1} \left( \binom{n-p+1}{k} \right) = \sum_{i} (-1)^{i} \binom{k-1+i}{i} \binom{n}{i+1} \binom{n}{i+k}.
\] (3.1)

Remark. It is interesting to compare this identity to Equation (6.28) in [1]. This is another three parameters identity but involving only the second kind of Stirling numbers, which is rather easily obtained from their exponential generating function. We reproduce it hereafter: under the condition that \( \ell, m, n \geq 0 \), we have

\[
\binom{\ell + m}{\ell} \binom{n}{\ell + m} = \sum_{k} \binom{k}{\ell} \binom{n-k}{m} \binom{n}{k}.
\] (3.2)

If we replace \( m \) by \(-m\) and \( n \) by \(-n\) in Equation (3.2), taking into account that \((-n) = (-1)^{k} \binom{n+k-1}{k}\) and the known duality \([-\alpha] = [\alpha]\) (see [1]), we obtain our Equation (3.1) with \( p-1 = \ell \). This shows that Equation (6.28) from [1] holds under the only condition \( \ell \geq 0 \), the conditions \( m, n \geq 0 \) given in [1] are not needed.

Proof of Theorem 3.1. The following proof, using the coefficient extractor method and the generating functions (1.2) and (1.3) is due to Marko Riedel [2]. Let \( S \) be the right hand side of (3.1). We have

\[
S = \sum_{i=p-1}^{n-k} (-1)^{i} \binom{k-1+i}{i} \binom{n}{i+k}.
\]

Then by Lemma 1.1, we have

\[
S = \frac{(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-1}] \right] \frac{1}{1-w} \sum_{i=p-1}^{n-k} (-1)^{i}[i] ([\varepsilon - 1]^{i})^{p-1} \left( \log \frac{1}{1-w} \right)^{i+k-1}
\]

\[
= \frac{(-1)^{k-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-1}] \right] \frac{1}{1-w} \sum_{i=p+k-2}^{n-1} (-1)^{i}[i] z^{k-1} ([\varepsilon - 1]^{i})^{p-1} \left( \log \frac{1}{1-w} \right)^{i}
\]

\[
= \frac{(-1)^{k-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-1}] \right] \frac{1}{1-w} \sum_{i\geq p+k-2} \left( -\log \frac{1}{1-w} \right)^{i} ([\varepsilon - 1]^{i})^{p-1}.
\]

since \( \left( \log \frac{1}{1-w} \right)^{i} = w^{i} + \ldots \).
Now, we have \((e^z - 1)^{p-1} = z^{p-1} + \cdots\), then the lowest power of \(z\) in the power series of \(z^{k-1}(e^z - 1)^{p-1}\) is \(z^{p+k-2}\) and then

\[
S = \frac{(-1)^{k-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-1}] \frac{1}{1-w} \sum_{i \geq 0} \left(-\log \frac{1}{1-w}\right)^i[[z^i]]z^{k-1}(e^z - 1)^{p-1}\right.
\]

\[
= \frac{(-1)^{k-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-1}] \frac{1}{1-w} \left(-\log \frac{1}{1-w}\right)^{k-1}(e^{\log \frac{1}{1-w}}) - 1)^{p-1}\right.
\]

\[
= \frac{(-1)^{k-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-1}] \frac{1}{1-w} \left(-\log \frac{1}{1-w}\right)^{k-1}(-w)^{p-1}\right.
\]

\[
= \frac{(-1)^{p-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-p}] \frac{1}{1-w} \left(\log \frac{1}{1-w}\right)^{k-1}\right.
\]

\[
= \frac{(-1)^{p-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-p}] \frac{1}{1-w} \left(\log \frac{1}{1-w}\right)^{k-1}(n-p+1)\right.
\]

\[
= \frac{(-1)^{p-1}(n-1)!}{(p-1)!(k-1)!} \left[ [w^{n-p+1}] \frac{1}{1-w} \left(\log \frac{1}{1-w}\right)^{k-1}(n-p+1)\right.
\]

\[
= (-1)^{p-1}(n-1)! \left[ [w^{n-p+1}] \left(\log \frac{1}{1-w}\right)^{k-1}(n-p+1)\right.
\]

\[
= (-1)^{p-1}\left[ \binom{n-1}{p-1} \left[ n-p+1 \right] \right].
\]

\[
\square
\]

As a corollary, we have another \(p\)-congruence involving the Stirling numbers of first kind.

**Corollary 3.1.1.** Let \(p\) be a prime number, and \(n, k\), non-negative integers, the following congruence holds

\[
\sum_{i \geq 0} \binom{k-1+i}{k-1} \binom{n}{i+k} \equiv [p \text{ divides } n \left[\frac{n-p+1}{k}\right]] \pmod p. \tag{3.3}
\]

**Proof.** Consider Equation [3.1] when \(p\) is prime. When \(i = 0\), we have \(\binom{i}{p-1} = 0\), and when \(i > 0\), by Lemma 1.2 we have \(\binom{i}{p-1} \equiv [p - 1 \text{ divides } i] \pmod p\). Moreover \((-1)^{p-1} \equiv 1 \pmod p\), then, modulo \(p\), the rhs of Equation (3.1) is

\[
\sum_{i > 0} \binom{k-1+i}{k-1} \left[ n \right].
\]
Now, for the lhs of Equation (3.1), when \( p \) does not divide \( n \), by Kummer theorem, whenever \( n - p \equiv 0 \mod p \) we have \( \binom{n-1}{p-1} \equiv 0 \mod p \), since in this case the addition \( (n - p) + (p - 1) \) in base \( p \) has at least one carry (at the lowest digit). And eventually, when \( p \) divides \( n \), we have \( \binom{n-1}{p-1} \equiv (-1)^{p-1} \equiv 1 \mod p \).

4. Back to binomial coefficients

We can now prove our claim from the introduction.

**Theorem 4.1.** Let \( p \) be a prime number, and \( m, \ell \) and \( s \) three natural numbers such that \( m > \ell \). We have

\[
\sum_{i \geq \ell + 1} (-1)^{m-i} \binom{m + \lceil \frac{p}{s} \rceil}{i + \lceil \frac{p}{s} \rceil} \left( m + s - 1 + i(p - 1) \right) \left( m + s - 1 + \ell(p - 1) \right) \equiv \begin{cases} p \divides s & \left( -1 \right)^{m-1-\ell} \binom{m-1}{\ell + \frac{p}{s}} \pmod{p} \end{cases}. \tag{4.1}
\]

In particular, for any natural number \( n \) such that \( p \) does not divides \( n \), we have

\[
\sum_{i \geq \ell + 1} (-1)^i \binom{n}{i} \left( -1 \right)^m \left( \ell + \frac{p}{s} \right) = 0 \pmod{p}. \tag{4.2}
\]

**Proof.** If we substitute \( m + s + \ell(p - 1) \) for \( k \) and \( mp + s \) for \( n \), Congruence (3.3) becomes

\[
\sum_{0 \leq i \leq \ell+1} \left( m + s - 1 + \ell(p - 1) + i \right) \binom{mp + s}{i + m + s + \ell(p - 1)} \equiv \begin{cases} p \divides s & \left( m-1 \right)p + s + 1 \pmod{mp + s} \end{cases} \left( m + s + \ell(p - 1) \right) \pmod{p}.
\]

which, by the appropriate index change, is

\[
\sum_{i \geq \ell + 1} \left( m + s - 1 + i(p - 1) \right) \binom{mp + s}{m + s + i(p - 1)} \equiv \begin{cases} p \divides s & \left( m-1 \right)p + s + 1 \pmod{s + 1} \end{cases} \left( m + s + \ell(p - 1) \right) \pmod{p}.
\]

Finally, we make use of Corollary (2.1.1) and we have

\[
\sum_{i \geq \ell + 1} \left( m + s - 1 + i(p - 1) \right) (-1)^{m-i} \binom{m + \lceil \frac{p}{s} \rceil}{i + \lceil \frac{p}{s} \rceil} \equiv \begin{cases} p \divides s & \left( -1 \right)^{m-1-\ell} \binom{m-1}{\ell + \lceil \frac{p}{s} \rceil} \pmod{p} \end{cases}.
\]
which is the claim, since when $p$ divides $s$, we have $\left\lfloor \frac{s+1}{p} \right\rfloor = \frac{s}{p}$. Eventually, Congruence \((4.2)\) is obtained from \((4.1)\) by letting $s$ be the residue of the Euclidean division of $n$ by $p$ and substituting $\left\lfloor \frac{n}{p} \right\rfloor p$ for $m$. \qed

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References

[1] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Adison-Wesley Publishing Company, 2nd Edition (1994).

[2] M. Riedel, A three-parameters identity involving Stirling numbers of both kinds, URL (version: 2020-03-24): https://math.stackexchange.com/q/3592354.