Finding an Integral vector in an Unknown Polyhedral Cone

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Abstract

We present an algorithm to find an integral vector in the polyhedral cone \( \Gamma = \{ X | AX \leq 0 \} \), without assuming the explicit knowledge of \( A \). About the polyhedral cone, \( \Gamma \), it is only given that, (i) the elements of \( A \) are in \( \{-d, -d + 1, \cdots, 0, \cdots, d - 1, d\} \), \( d \in \mathbb{N} \), and, (ii) \( Y = [y(1), y(2), \cdots, y(n)] \) is a non-zero integral solution to \( \Gamma \). The proposed algorithm finds a non-zero integral vector in \( \Gamma \) such that its maximum element is less than \( (2d)^{2n-1} - 1/2^{n-1} \).

Index Terms— Integer programming, Algorithm.

I. INTRODUCTION

Finding an integral vector\(^1\) in the polyhedral cone \( \Gamma = \{ X | AX \leq 0 \} \), for a given/known matrix \( A, A \in \mathbb{Z}^{n \times n} \), is a problem which has been considered in great detail \([1], [2], [3], [4]\).

In this paper, we consider the above-mentioned problem from another angle with a distinctly different assumption. Here, we assume there is no explicit knowledge about \( A \) (i.e., \( A \) is unknown), but a non-zero integer solution of \( \Gamma \) is given. Under these assumptions, we can show that not only does there exist another integer solution of \( \Gamma \) but also the maximum element of the obtained solution is less than \( (2d)^{2n-1} - 1/2^{n-1} \), when elements of \( A \) are in \( \{-d, -d + 1, \cdots, 0, \cdots, d - 1, d\} \), \( d \in \mathbb{N} \).

The rest of the paper is organized as follows. In Section \[\text{II}\] we present the main theorem of the paper. In Section \[\text{III}\] we consider an example. In Section \[\text{IV}\] we present some applications of the algorithm.

II. MAIN THEOREM

In this section, we present the main theorem of this paper. The proof of this theorem is constructive and along the proof we present the algorithm that archives the desired properties.

\(^1\)An integral vector is a vector with non-negative integer elements.

\(^2\)In other words, suppose that \( Y = [y(1), y(2), \cdots, y(n)] \) is a given non-zero integer solution of \( \Gamma \) such that \( \max_{1 \leq i \leq n} y(i) > (2d)^{2n-1} - 1/2^{n-1} \). Then, without knowing \( A \), we can find another non-zero integer solution of \( \Gamma \) such that its maximum element is less than \( (2d)^{2n-1} - 1/2^{n-1} \).
Theorem 1: Consider $\Gamma = \{X | AX \leq 0\}$, with this knowledge that the elements of $A$ are in $\{-d, -d+1, \ldots, 0, \ldots, d-1, d\}$. Assume that $Y = [y(1), y(2), \ldots, y(n)]$ is a given (arbitrary) non-zero integer solution of $\Gamma$. Then there exists an integer solution $X = [x(1), x(2), \ldots, x(n)]$ which satisfies $AX \leq 0$ and

$$\max_{1 \leq i \leq n} x(i) \leq \frac{(2d)^{2^n-1}}{2^{n-1}}. \quad (1)$$

Proof: Without loss of generality we assume that,$y(1) \leq y(2) \leq \ldots \leq y(n)$.

We consider the polyhedral cone $\Lambda_Y, \Lambda_Y \subset \Gamma$, which is defined by all inequalities of the form

$$\sum_{i=1}^{n} c_i x(i) \leq 0,$$

where $c_i \in \{-d, \ldots, d\}$ for all $i$ and $\sum_{i=1}^{n} c_i y(i) \leq 0$.

We will describe a procedure to construct a solution $X \in \Lambda_Y$ that satisfies

$$\max_{1 \leq i \leq n} x(i) \leq \frac{(2d)^{2^n-1}}{2^{n-1}}.$$

Definition 1: The sequence $\Upsilon_j$ is defined by the recurrence relation $\Upsilon_1 = d$, $\Upsilon_j = 2\Upsilon_{j-1}^2$. Thus

$$\Upsilon_j = \frac{1}{2} (2d)^{2^{j-1}}.$$

For $j \in \{1, \ldots, n-1\}$, let $\Lambda_Y^j$ denote the polyhedral cone in $n+1-j$ dimensions which is defined by all constraints of the form

$$\sum_{i=j}^{n} c_i x(i) \leq 0,$$

where for all $i$, $c_i$ is an integer with $|c_i| \leq \Upsilon_j$, and $\sum_{i=j}^{n} c_i y(i) \leq 0$.

Our procedure begins by setting $x(n) = 1$. For integers $j$ decreasing from $n-1$ down to 1, we describe a way to select and update the partial solutions, that is, in each iteration, say $n-j$, we derive a feasible solution with lower dimension called partial solution in that iteration and denote by $X^{(n-j)} = \{x(i) : i \geq j\} \in \Lambda_Y^j$.

For $j = n-1$ and $x(n) = 1$, any real partial solutions $X^{(n-1)} \in \Lambda_Y^{n-1}$ is calculated by constraints of the form

$$\gamma_{n-1}^u x(n-1) \leq \gamma_n^u \quad (2)$$

or

$$\gamma_{n-1}^l x(n-1) \geq \gamma_n^l \quad (3)$$
where $\gamma_{n-1}^u$, $\gamma_n^u$, $\gamma_{n-1}^l$, and $\gamma_n^l$ are integers satisfying
\begin{align}
0 &\leq \gamma_{n-1}^u \leq \Upsilon_{n-1} \\
1 &\leq \gamma_n^u \leq \Upsilon_{n-1} \\
1 &\leq \gamma_{n-1}^l \leq \Upsilon_{n-1} \\
0 &\leq \gamma_n^l \leq \Upsilon_{n-1} \\
\gamma_{n-1}^u y(n-1) &\leq \gamma_n^u y(n) \\
\gamma_{n-1}^l y(n-1) &\geq \gamma_n^l y(n)
\end{align}

We initially choose $x(n-1)$ to be a positive number that satisfies all constraints in $\Lambda_{n-1}$. Assuming that $x(n) = 1$. More specifically, we choose positive integers $\gamma_n^* \leq \Upsilon_{n-1}$ such that
\[
\frac{\gamma_n^*}{\gamma_{n-1}^*} = x(n-1) \leq x(n) = 1,
\]
and the constraints of $\Lambda_{n-1}$ are satisfied. We next multiply $x(n-1)$ and $x(n)$ by $\gamma_{n-1}^*$ to obtain an integral partial solution satisfying
\[
1 \leq x(n-1) \leq x(n) \leq \gamma_{n-1}^* \leq \Upsilon_{n-1},
\]
and this will be our initial partial solution when we begin to consider $x(n-2)$.

For $j$ decreasing from $n-2$ down to 1, suppose that we have an integral partial solution $X^{(n-(j+1))^*} = \{x^*(i) : i \geq j+1\} \in \Lambda_{j+1}^*$ with $x^*(n) \leq \prod_{i=j+1}^{n-1} \Upsilon_i$. We will use this partial solution to construct an integral partial solution $X^{(n-j)} \in \Lambda_j^*$ with $x(n) \leq \prod_{i=j}^{n-1} \Upsilon_i$. We begin by setting $x(i) = x^*(i)$, $j+1 \leq i \leq n$. Assuming that this is a legitimate assignment, in order for real partial solution $X^{(n-j)}$ to be an element of $\Lambda_j^*$, we must have that
\[
\sum_{i=j}^{n} c_i x(i) \leq 0,
\]
where for all $i$, $c_i$ is an integer with $|c_i| \leq \Upsilon_j$, and $\sum_{i=j}^{n} c_i y(i) \leq 0$. There are three cases to consider for $c_j$:

1) If $c_j = 0$, then since $X^{(n-(j+1))^*} \in \Lambda_{j+1}^*$ it follows that $\sum_{i=j+1}^{n} c_i x(i) \leq 0$ holds when $c_i$ is an integer with $|c_i| \leq \Upsilon_j$ for all $i \geq j+1$, and when $\sum_{i=j+1}^{n} c_i y(i) \leq 0$.

2) If $c_j = c_j^u > 0$, then we obtain an upper bound on $x(j)$:
\[
x(j) \leq -\frac{1}{c_j^u} \sum_{i=j+1}^{n} c_i x^*(i).
\]

Observe that
\[
y(j) \leq -\frac{1}{c_j^u} \sum_{i=j+1}^{n} c_i y(i).
\]
3) If \( c_j = c_i^j < 0 \), then we obtain a lower bound on \( x(j) \):

\[
x(j) \geq -\frac{1}{c_j^i} \sum_{i=j+1}^{n} c_i^j x^*(i).
\]

(12)

Observe that

\[
y(j) \geq -\frac{1}{c_j^i} \sum_{i=j+1}^{n} c_i^j y(i).
\]

(13)

In order for this approach to lead to a valid partial solution, we need to guarantee that all upper bounds on \( x(j) \) exceed all lower bounds on \( x(j) \). By (10)-(13) we want to establish that

\[
-\frac{1}{c_j^u} \sum_{i=j+1}^{n} c_i^u x^*(i) \geq -\frac{1}{c_j^i} \sum_{i=j+1}^{n} c_i^j x^*(i)
\]

(14)

when

\[
-\frac{1}{c_j^u} \sum_{i=j+1}^{n} c_i^u y(i) \geq -\frac{1}{c_j^i} \sum_{i=j+1}^{n} c_i^j y(i).
\]

(15)

Constraint (14) is equivalent to the condition

\[
\sum_{i=j+1}^{n} (c_i^j c_j^u - c_i^u c_j^i) x^*(i) \leq 0.
\]

(16)

The property (15) can be rewritten

\[
\sum_{i=j+1}^{n} (c_i^j c_j^u - c_i^u c_j^i) y(i) \leq 0.
\]

(17)

Notice that since \( |c_i^j| \leq \Upsilon_j \) and \( |c_i^u| \leq \Upsilon_j \) for all \( i \geq j \), it follows that \( |c_i^j c_j^u - c_i^u c_j^i| \leq 2\Upsilon_j^2 = \Upsilon_{j+1} \) for all \( i \geq j + 1 \). Since (17) holds and \( X^{(n-(j+1))}\star \in \Lambda_{Y+1} \) by assumption, it follows that (16) holds.

We choose \( x(j) \) to be the maximum value satisfying all constraints (10) and (12). Since \( x(j) \) may be of the form \( \sigma/\gamma_j^* \) for integer \( 1 \leq \gamma_j^* \leq \Upsilon_j \) and integer \( \sigma \), we multiply the partial solution by \( \gamma_j^* \) to obtain an integral partial solution satisfying

\[
x(n) \leq \gamma_j^* \prod_{i=j+1}^{n-1} \Upsilon_i \leq \prod_{i=j}^{n-1} \Upsilon_i.
\]

Observe that \( \Lambda_Y = \Lambda_{Y} \), so at the end of the procedure we have a solution vector \( X \) with

\[
0 \leq x(1) \leq \ldots \leq x(n) \leq \prod_{i=1}^{n-1} \Upsilon_i = \prod_{i=1}^{n-1} \frac{1}{2} (2d)^{2i-1} = \frac{(2d)^{2n-1}-1}{2^{n-1}}.
\]

Notice that, since \( 0 \leq y(1) \leq \ldots \leq y(n) \), we also have \( 0 \leq x(1) \leq \ldots \leq x(n) \), because of the definition of \( \Lambda_Y \).
In the following we present an example to illustrate the procedure.

III. AN EXAMPLE

Let \( \Gamma = \{X | AX \leq 0\} \), where \( A \) is a 4 \times 4 matrix and it is only given that its elements are in \( \{-1, 0, 1\} \). In addition, it is known that \( Y = [2, 3, 7, 29] \) is an integral solution of \( \Gamma \). Now, we find an integral vector in \( \Gamma \) such that its maximum element is 8, which satisfies the proposed bound, namely, it is less than \( \frac{(2d)^{n-1} - 1}{2^n} \) \( \big|_{d=1, n=4} = 16 \).

Recall that \( \Upsilon_1 = 1, \Upsilon_2 = 2 \) and \( \Upsilon_3 = 8 \). We construct a solution \([x(1), x(2), x(3), x(4)]\) for \( \Lambda_Y \) when \( y(1) = 2, y(2) = 3, y(3) = 7 \) and \( y(4) = 29 \).

1) Initialize \( x(4) = 1 \).
2) \( \Lambda_Y^3 \) is the polyhedral cone with constraints \( c_3 x(3) + c_4 x(4) \leq 0 \), where \( c_3 \) and \( c_4 \) are integers with \( |c_3| \leq 8, |c_4| \leq 8 \), and \( c_3 y(3) + c_4 y(4) = 7c_3 + 29c_4 \leq 0 \). Observe that the defining inequalities for \( \Lambda_Y^3 \) for this example are \( 0 \leq x(3), 4x(3) \leq x(4) \) and \( 5x(3) \geq x(4) \); all the other inequalities that we consider are less restrictive.
3) Initialize \( x(3) = 1/4 \). Multiply the solution by 4 to obtain the integral partial solution \( x(3) = 1 \) and \( x(4) = 4 \).
4) \( \Lambda_Y^2 \) is the polyhedral cone with constraints \( c_2 x(2) + c_3 x(3) + c_4 x(4) \leq 0 \), where \( c_2, c_3 \) and \( c_4 \) are integers with \( |c_2| \leq 2, |c_3| \leq 2, |c_4| \leq 2 \), and \( c_2 y(2) + c_3 y(3) + c_4 y(4) = 3c_2 + 7c_3 + 29c_4 \leq 0 \). Observe that the defining inequalities for \( \Lambda_Y^2 \) for this example are \( 0 \leq x(2), 2x(2) \leq x(3) \) and \( 2x(2) + 2x(3) \leq x(4) \).
5) Initialize \( x(2) = 1/2 \). Multiply the solution by 2 to obtain the integral partial solution \( x(2) = 1, x(3) = 2 \) and \( x(4) = 8 \).
6) \( \Lambda_Y = \Lambda_Y^1 \) is the polyhedral cone with constraints \( c_1 x(1) + c_2 x(2) + c_3 x(3) + c_4 x(4) \leq 0 \), where \( c_1, c_2, c_3, c_4 \in \{-1, 0, 1\} \), and \( c_1 y(1) + c_2 y(2) + c_3 y(3) + c_4 y(4) = 2c_1 + 3c_2 + 7c_3 + 29c_4 \leq 0 \). Observe that the defining inequalities for \( \Lambda_Y^1 \) for this example are \( x(1) \geq 0, x(1) \leq x(2), x(1) + x(2) \leq x(3) \) and \( x(1) + x(2) + x(3) \leq x(4) \).
7) Choose \( x(1) = 1, x(2) = 1, x(3) = 2 \) and \( x(4) = 8 \).

The output of the algorithm is \( X = [1, 1, 2, 8] \). Thus, \( X \) is another integral vector in \( \Gamma \).

IV. APPLICATIONS

Some applications of the algorithm are as follows.

• An important feature of our algorithm is that we do not need to know matrix \( A \) to produce a bounded solution for the linear program. In many situations, we have a solution that satisfies the requirements of a linear system, but we want another solution with some upper bounds on its size. It might be impossible (or very time consuming) to measure entries of matrix \( A \). Using this method, we can have the bounded solution without knowing \( A \).

• In some streaming applications, we might have the unbounded solution in advance, but matrix \( A \) arrives later as a query, and we have to produce a bounded solution based on \( A \). In streaming problems, we have very limited time for processing the query. In our approach, we can solve the problem without the knowledge of query in advance. When the query arrives, we already have the answer.
Another interesting problem, that our bounds are useful for it, is the ellipsoid algorithm. Assume that we want to find a feasible solution for a linear program. We have to start with an ellipsoid that contains at least one feasible solution in the core. We then iteratively narrow down, and find a smaller ellipsoid. We have to start with an ellipsoid that contains some feasible non-zero integral solution of our linear program. In fact there has to be a non-zero integral point in the intersection of our polyhedral cone and the starting ellipsoid. If we start with a very small ellipsoid, we might not satisfy this property. If the starting ellipsoid is very large, we can not prove good bounds on the running time of the ellipsoid algorithm. In this case, we have to know some bounds on the size of the starting ellipsoid. Our algorithm can be used to get some bounds on the size of the starting ellipsoid. Our current bounds do not give polynomial bounds on the running time of ellipsoid algorithm, but we hope this approach can lead to such bounds by some polynomials.

Another potential application of our algorithm is for finding the routing capacity regions of networks. It is known that routing capacity regions of networks can be characterized using Farkas lemma as the solution set of infinite set of linear inequalities. But, our algorithm gives an upper bound (finite) on the set of inequalities needed to characterize the capacity regions, [5].

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