THE COEFFICIENT FIELD IN THE NILPOTENCE CONJECTURE FOR TORIC VARIETIES

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Abstract. The main result of the work “The nilpotence conjecture in $K$-theory of toric varieties” is extended to all coefficient fields of characteristic 0, thus covering the class of genuine toric varieties.

1. The statement

Let $R$ be a (commutative) regular ring, $M$ be arbitrary commutative, cancellative, torsion free monoid without nontrivial units, and $i$ be a nonnegative integral number. The nilpotence conjecture asserts that for any sequence $c = (c_1, c_2, \ldots)$ of natural numbers $\geq 2$ and any element $x \in K_i(R[M])$ there exists an index $j_x \in \mathbb{N}$ such that $(c_1 \cdot \cdot \cdot c_j)_*(x) \in K_i(R)$ for all $j > j_x$.

Here $R[M]$ is the monoid $R$-algebra of $M$ and for a natural number $c$ the endomorphism of $K_i(R[M])$, induced by the $R$-algebra endomorphism $R[M] \to R[M]$, $m \mapsto m^c, m \in M$, is denoted by $c_*$ (writing the monoid operation multiplicatively).

We call this action the multiplicative action of $\mathbb{N}$ on $K_i(R[M])$.

Theorem 1.2 in [G2] verifies the conjecture for the coefficient rings $R$ of type $S^{-1}k[T_1, \ldots, T_d]$ where $k$ is a number field and $S \subset k[T_1, \ldots, T_d]$ is arbitrary multiplicative subset of nonzero polynomials ($d \in \mathbb{Z}_+$). In particular, the nilpotence conjecture is valid for purely transcendental extensions of $\mathbb{Q}$. On the other hand $K$-groups commute with filtered colimits. Therefore, the following induction proposition, together with [G2, Theorem 2.1], proves the conjecture for all characteristic 0 fields.

Proposition 1. The validity of the nilpotence conjecture for a field $k_1$ of characteristic 0 transfers to any finite field extension $k_1 \subset k_2$.

This note should be viewed as an addendum to the main paper [G2]. In Sections 2 and 3 for the reader’s convenience we recall in a convenient form the needed results on Galois descent and Bloch-Stienstra operations in $K$-theory.

2. Galois descent for rational $K$-theory

We need the fact that rational $K$-theory of rings (essentially) satisfies Galois descent – a very special case of Thomason’s étale descent for localized versions of $K$-theory, first proved in [1] in the smooth case and then extended to the singular case as an application of the new local-to-global technique [14]. The argument
below follows closely [1, Lemma 2.13] and [TT, Proposition 11.10]. It is, of course, important that Thomason’s higher $K$-groups agree with those of Quillen in the affine (or, more generally, quasiprojective) case [TT, Theorem 7.6].

**Lemma 2.** Let $A \subset B$ be a finite Galois extension of noetherian rings with the Galois group $G$. Assume $[B] = n[A]$ in $K_0(A)$ where, naturally, $n = \#G$ (e. g. $B$ is a free $A$-module). Then $K_i(A) \otimes \mathbb{Q} = H^0(G, K_i(B) \otimes \mathbb{Q})$.

**Proof.** The push-out diagram of rings

\[ B \xrightarrow{\iota_1} B \otimes_A B \]

\[ A \xrightarrow{\iota} B \]

defines a pull-back diagram of the corresponding affine schemes in the category of schemes. The latter diagram satisfies all the conditions of Proposition 3.18 in [TT]. Therefore, by the mentioned proposition we have the equality of the endomorphisms

\[ \iota_* \circ \iota^* = (\iota_2)^* \circ (\iota_1)_* : K_i(B) \to K_i(B) \]

where $-^*$ refers to the functorial homomorphisms of the $K_i$-groups and $-^*$ refers to the corresponding transfer maps (contrary to the scheme-theoretical notation in [Q][I][TT]). Galois theory identifies the diagram (1) with the diagram

\[ B \xrightarrow{\Delta} B^n \]

\[ A \xrightarrow{\iota} B \]

where $\Delta$ is the diagonal embedding and $(\Delta_G(b))_g = g(b)$. By the elementary properties of $K$-groups [Q, §2] and the equalities $(g^{-1})_* = g^*$, $g \in G$ the equality (2) and the diagram (3) imply $\iota_* \circ \iota^* = \sum_G (g^{-1})_* = \sum_G g_*$. The other composite $\iota^* \circ \iota_*$ is the multiplication by $n$ on $K_i(A)$. Therefore, the homomorphisms $\iota_* \otimes \mathbb{Q}$ and the corresponding restriction of $n^{-1}(\iota^* \otimes \mathbb{Q})$ establish the desired isomorphism. \( \square \)

### 3. Verschiebung and Frobenius

We also need the Bloch-Stienstra formula on the relationship between Verschiebung and Frobenius in the context of the action of big Witt vectors on Nil-$K$-theory.

For a (commutative) ring $R$ the additive group of Witt($R$) is the multiplicative group $1 + TR[[T]]$, $T$ a variable. The decreasing filtration of subgroups $I_m(R) \subset 1 + T^m R[[T]]$ makes Witt($R$) a topological group and any element $\omega(T) \in$ Witt($R$) has a unique convergent expansion $\omega(T) = \prod_{n \geq 1} (1 - r_n T^n)$, $r_n \in R$. When $\omega(T) \in I_m(R)$ then the expansion is of the type $\omega(T) = \prod_{n \geq m} (1 - r_n T^n)$. The multiplicative structure on Witt($R$) is the unique continuous extension to the whole Witt($R$) of the pairing

\[ (1 - r T^n) \star (1 - s T^n) = (1 - r^{n/d} s^{m/d} T^{mn/d})^d, \quad r, s \in R, \quad d = \gcd(m, n). \]
The assignment
\[ F_m : 1 - rT^n \mapsto (1 - r^{n/d}T^{n/d})^d, \quad d = \gcd(m, n), \quad r \in R \]
extends to a (unique) ring endomorphism \( F_m : \text{Witt}(R) \to \text{Witt}(R) \) which is called the \textit{Frobenius endomorphism}.

When \( \mathbb{Q} \subset R \) the \textit{ghost isomorphism} between the multiplicative and additive groups
\[ -T \cdot \frac{d(\log)}{dT} : 1 + TR[[T]] \to TR[[T]], \quad \alpha \mapsto -T \cdot \frac{d\alpha}{dt} \]
is actually a ring isomorphism \( \text{Witt}(R) \to \Pi_1^\infty R \) where the right hand side is viewed as \( TR[[T]] \) under the assignment \((r_1, r_2, \ldots) \mapsto r_1 \cdot T + r_2 T^2 + \cdots \). What we need here is the fact that the diagonal injection \( \lambda_T : R \to \Pi_1^\infty R = \text{Witt}(R) \) is invariant under \( F_m \) for all \( m \in \mathbb{N} \).

The \( R \)-algebra endomorphism \( R[T] \to R[T], \ T \mapsto T^m \) induces a group endomorphism \( V_m : NK_i(R) \to NK_i(R) \) – the \textit{Verschiebung} (see [8], Theorem 4.7) for such an identification of \( V_m \), in [8] this map is first defined in terms of the category \textbf{Nil}(R) of nilpotent endomorphisms).

Bloch [3] and then, in a systematic way, Stienstra [8] defined a Witt(\( R \))-module structure on \( NK_i(R) \). The action of \( 1 - rT^m \in \text{Witt}(R), \ r \in R \) on \( NK_i(R) \) is the effect of the composite functor
\[ \mathbb{P}(R[T]) \xrightarrow{t_m} \mathbb{P}(R[T]) \xrightarrow{\bar{r}} \mathbb{P}(R[T]) \xrightarrow{v_m} \mathbb{P}(R[T]) \]
where \( v_m \) corresponds to the base change through \( T \mapsto T^m \) (it gives rise to the \( m \)th Verschiebung), \( \bar{r} \) corresponds to the base change through \( T \mapsto rT \), and \( t_m \) corresponds to the scalar restriction through \( T \mapsto T^m \) (the corresponding endomorphism of \( NK_i(R) \) is the transfer). This determines the action of the whole Witt(\( R \)) because any element of \( NK_i(R) \) is annihilated by the ideal \( I_m(R) \) for some \( m \in \mathbb{N} \), i.e., \( NK_i(R) \) is a \textit{continuous} Witt(\( R \))-module.

We have the following relationship between \( F_m \) and \( V_m \):
\[ V_m(F_m(\alpha) \star z) = \alpha \star z, \quad \alpha \in \text{Witt}(R), \quad z \in NK_i(R), \quad m \in \mathbb{N}, \]
where \( \star \) refers to the Witt(\( R \))-module structure. Actually, (4) is proved in [8, §6] for the elements \( \alpha \) in the image of the Almkvist embedding \( \text{ch} : K_0(\text{End}(R)) \to \text{Witt}(R) \)[A]. But the formula generalizes to the whole Witt(\( R \)) due to the fact that \( NK_i(R) \) is a continuous Witt(\( R \))-module and the elements of type \( 1 - rT^m \) are always in the mentioned image.

In particular, when \( \mathbb{Q} \subset R \) the equality (4) implies

**Lemma 3.** For any natural number \( m \) the Verschiebung \( V_m \) is an \( R \)-linear endomorphism of \( NK_i(R) \).

Weibel [W] has generalized the Witt(\( R \))-module structure to the graded situation so that for a graded ring \( A = A_0 \oplus A_1 \oplus \cdots \) and a subring \( R \subset A_0 \) there is a \textit{functorial} continuous Witt(\( R \))-module structure on \( K_i(A, A^+) \), where \( A^+ = 0 \oplus A_1 \oplus A_2 \oplus \cdots \). (Although here we restrict to the commutative case Weibel actually considers the general situation when \( R \) is in the center of \( A \); it is exactly the non-commutative case
that is used in [32]). In particular, the Witt($R$)-module structure on $NK_i(A, A^+)$ is the restriction of that on $NK_i(A)$ under the embedding $NK_i(A, A^+) \to NK_i(A)$ induced by the graded homomorphism

$$w : A_0 \oplus A_1 \oplus A_2 \oplus \cdots \to A[T] = A + TA + T^2A + \cdots,$$

$$a_0 \oplus a_1 \oplus a_2 \oplus \cdots \mapsto a_0 + a_1T + a_2T^2 + \cdots$$

and, at the same time, the latter module structure is just the scalar restriction of the Witt($R$)-module structure through Witt($R$) $\to$ Witt($A$). That $NK_i(A, A^+) \to NK_i(A)$ is in fact an embedding follows from the fact that $w$ splits the non-graded augmentation $A[T] \to A$, $T \mapsto 1$, in particular $K_i(A) \to K_i(A[T])$ is a split monomorphism.

Below we will make use of the fact, due to the split exact sequence $0 \to A^+ \to A \to A_0 \to 0$ ($A$ as above), that there is a natural isomorphism $K_i(A, A^+) = K_i(A)/K_i(A_0)$. Also, the latter group will be thought of as a direct summand of $K_i(A)$ in a natural way.

### 4. Proof of Proposition 1

For clarity we let $c \mapsto c_a/k$ denote the induced multiplicative action of $\mathbb{N}$ on $K_i(k[M])/K_i(k)$, $k$ a field.

Without loss of generality we can assume that $M$ is a finitely generated monoid. Fix arbitrary embedding of $M$ into a free monoid $\mathbb{Z}_+^*$ [BrG, 2.1]. This gives rise to graded structures on $k_1[M]$ and $k_2[M]$ so that the monoid elements are homogeneous of positive degree. In view of the previous section $K_i(k_2[M])/K_i(k_2)$ carries a $k_2$-linear structure, $K_i(k_1[M])/K_i(k_1)$ carries a $k_1$-linear structure, and the group homomorphism $K_i(k_1[M])/K_i(k_1) \to K_i(k_2[M])/K_i(k_2)$ is $k_1$-linear.

We can also assume that $k_1 \subset k_3$ is a Galois extension. In fact, if $k_1 \subset k_3$ is a finite Galois extension such that the conjecture is true for the monoid $k_3$-algebras and $k_2 \subset k_3$ then the commutative squares

\begin{equation}
\begin{array}{ccc}
K_i(k_3[M])/K_i(k_3) & \xrightarrow{c_a/k_3} & K_i(k_3[M])/K_i(k_3), \\
\uparrow & & \uparrow \\
K_i(k_2[M])/K_i(k_2) & \xrightarrow{c_a/k_2} & K_i(k_2[M])/K_i(k_2)
\end{array}
\end{equation}

imply the validity of the nilpotence conjecture for $k_2$ too because the vertical homomorphisms in (5) are monomorphisms. To see this observe that $k_3[M]$ is a free module over $k_2[M]$ of rank $[k_3 : k_2]$ and, hence, the composite of the functorial map $K_i(k_3[M]) \to K_i(k_3[M])$ with the corresponding transfer map $K_i(k_3[M]) \to K_i(k_2[M])$ is the multiplication by $[k_3 : k_2]$, which restricts to an automorphism of the rational vector subspace $K_i(k_2[M])/K_i(k_2) \subset K_i(k_2[M])$.

Let $G$ be the Galois group of the extension $k_1 \subset k_2$. Then $k_1[M] \subset k_2[M]$ is a Galois extension of rings with the same Galois group. The action of $G$ on Witt($k_2$) shows that the $k_1$-vector space $K_i(k_2[M])/K_i(k_2)$ is a Galois $k_1$-module. Then
Lemma 2 implies

\[ K_i(k_2[M]) / K_i(k_1) = K_i(k_1[M]) / K_i(k_1) \otimes k_1. \]

Since the nilpotence conjecture is valid for \( k_1 \) every element of \( K_i(k_1[M]) / K_i(k_1) \) is annihilated by high iterations of the multiplicative action of \( \mathbb{N} \). It follows from (6) and the commutative squares of type (6) for the extension \( k_1[M] \subset k_2[M] \) that the same is true for the elements of certain generating set of the \( k_2 \)-vector space \( K_i(k_2[M]) / K_i(k_2) \). So we are done once it is shown that \( c_* / k_2 \) is a \( k_2 \)-linear map for every natural number \( c \).

Consider the commutative square of \( k_2 \)-algebras (\( c \in \mathbb{N} \))

\[
\begin{array}{ccc}
\kappa_2[M][T] & \xrightarrow{T \mapsto T c} & \kappa_2[M][T] \\
\downarrow{T^{\deg(-)(-)^c}} & & \uparrow{T^{\deg(-)(-)}} \\
\kappa_2[M] & \xrightarrow{(-)^c} & \kappa_2[M]
\end{array}
\]

where every element \( m \in M \) is fixed by the upper horizontal homomorphism and

- \( (\cdot)^c(m) = m^c \),
- \( (T^{\deg(-)(-)^c})(m) = T^{\deg(m)} m^c \),
- \( (T^{\deg(-)})(m) = T^{\deg(m)} m \).

Here \( \deg(-) \) is the degree with respect to the fixed graded structure on \( k_2[M] \). Thinking of \( k_2[M][T] \) as a graded ring with respect to the powers of \( T \) the vertical maps in (7) become graded homomorphisms. We arrive at the commutative diagrams of groups

\[
\begin{array}{ccc}
NK_i(k_2[M]) & \xrightarrow{V} & NK_i(k_2[M]) \\
\downarrow & & \uparrow \\
K_i(k_2[M]) / K_i(k_2) & \xrightarrow{c_* / k_2} & K_i(k_2[M]) / K_i(k_2)
\end{array}
\]

whose vertical maps are \( k_2 \)-linear by functoriality and the upper horizontal map is \( k_2 \)-linear by Lemma 2. The left vertical homomorphism is actually a monomorphism because the map \( T^{\deg(-)(-)} \) in diagram (7) is of the type \( w \), discussed at the end of Section 3. These conditions altogether imply the desired linearity. \( \square \)

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