ENRICHT CATEGORIES AND TROPICAL MATHEMATICS

SOICHIRO FUJII

ABSTRACT. We point out a connection of enriched category theory over a quantale and tropical mathematics. Quantales or complete idempotent semirings, as well as matrices with coefficients in them, are fundamental objects in both fields. We first survey standard category-theoretic constructions on matrices, namely composition, right extension, right lifting and the Isbell hull. Along the way, we also review some known reformulations of such notions as tropical polytopes, directed tight spans and the Legendre–Fenchel transform by means of these constructions, illustrating their ubiquity in tropical mathematics and related fields. We then focus on complete semimodules over a quantale \( Q \), or equivalently skeletal and complete \( Q \)-categories, and show that they are precisely the injective objects with respect to fully faithful \( Q \)-functors among \( Q \)-categories. With the aim to bridge a gap between enriched category theory and tropical mathematics, we assume no knowledge in either field.

1. INTRODUCTION

Tropical mathematics (also called idempotent mathematics) is mathematics over an idempotent semiring, instead of over the fields of real or complex numbers as in many branches of ordinary mathematics. A prototypical example of idempotent semirings is the so-called min-plus semiring \( (\mathbb{R}_{\leq}, \max, 0, +) \), defined as the set \( \mathbb{R}_{\leq} \) of extended real numbers equipped with the minimum operation as (idempotent) addition and the ordinary addition as multiplication. Such a structure naturally arises in many fields, including operations research [CG91], formal language theory [Sim94, Pin98], mathematical physics [LMS01] and algebraic geometry [Vir01].

Enriched categories, on the other hand, are a generalisation of categories [ML98] defined relative to a base. We can choose arbitrary monoidal categories as the base, of which idempotent semirings are a special case. The theory works best when the base enjoys several completeness properties, and in the case of idempotent semiring, imposing these additional completeness assumptions results in the notion of (unital) quantale [Mul86]. Preordered sets, metric spaces and ultrametric spaces are all instances of enriched categories over a quantale [Law73].

In this paper we take a look at a branch of tropical mathematics, namely linear algebra over a quantale (cf. [CGQ04]), from the enriched categorical perspective.
We believe that enriched category theory can provide an abstract framework to tropical mathematics, which would help us to recognise various easy facts that hold for purely formal reasons, and focus on harder problems. On the other hand, tropical mathematics can offer novel interpretation to category theoretic concepts, which might eventually lead to discovery of useful categorical notions of tropical origin, along the lines of Lawvere’s notion of Cauchy completeness generalised from metric spaces to enriched categories [Law73]. The current paper attempts to demonstrate the possibility of the fruitful interplay between enriched categories and tropical mathematics, by working out the most basic case. Accordingly, we have also tried to make this paper mostly self-contained and accessible to both category theorists and tropical mathematician.

The outline of this paper is as follows. After reviewing the notion of quantale (Section 2), we move on to study matrices over a quantale $Q$ (called $Q$-matrices) in Section 3. Matrices over a quantale admit not only composition, but also the operations of right extension and right lifting, which may be computed by the formulas dual to the one for composition. Right extensions and right liftings of $Q$-matrices enables us to define the Isbell hull of a $Q$-matrix (Section 4), which is known to unify such constructions as directed tight spans of metric spaces [Wil13], lower semicontinuous convex functions on $\mathbb{R}^n$ [Wil15] and tropical polytopes [Ell17]; we also provide brief expositions of these results.

In Section 5 we study complete semimodules over a quantale (cf. [CGQ04, BK12]), a tropical analogue of vector spaces over a field. It turns out that the Isbell hull of a $Q$-matrix naturally admits a structure of a complete semimodule over $Q$, and conversely (as a consequence of our main theorem in Section 7), any complete semimodule over $Q$ can be realised as the Isbell hull of some $Q$-matrix. In Section 6 we introduce categories enriched over a quantale $Q$ (called $Q$-categories). Any complete semimodule over $Q$ naturally induces a $Q$-category, and those $Q$-categories arising from a complete semimodule over $Q$ are characterised by the intrinsic properties of skeletality and completeness. In the final section we shall prove a new characterisation of skeletal and complete $Q$-categories as the injective $Q$-categories (with respect to fully faithful $Q$-functors), generalising the classical characterisation of complete lattices as injective posets [BB67].

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2. Quantales

We first introduce quantales, also known as complete idempotent semirings. They are the enriching (or base) categories in the portion of enriched category theory we shall be concerned with; in tropical mathematics, elements of a quantale are often considered as scalars, and play the role similar to that of real or complex numbers in classical mathematics.
Definition 2.1. A (unital) quantale $Q$ is a complete lattice $(Q, \preceq)$ equipped with a monoid structure $(Q, I_Q, \circ_Q)$ such that the multiplication $\circ_Q$ preserves arbitrary sups in each argument: $(\bigvee_{i \in I} y_i) \circ_Q x = \bigvee_{i \in I} (y_i \circ_Q x)$ and $y \circ_Q (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \circ_Q x_i)$. We often omit the subscript $Q$ from the data of a quantale, writing it simply as $Q = (Q, \preceq, I)$.

Though we have introduced quantales in an order-theoretic manner, we could have introduced them in a more algebraic manner. In tropical mathematics, it is customary to start from the notion of idempotent semiring, namely a semiring $(Q, 0, I, \oplus, \circ)$ with an idempotent addition: $x \oplus x = x$. Given such an idempotent semiring, its additive part $(Q, 0, \oplus)$ is an idempotent commutative monoid, and thus induces a partial order $\preceq$ on $Q$ by $x \preceq y$ iff $x \oplus y = y$. The resulting poset $(Q, \preceq)$ is a sup-semilattice, with the least element $0$ and the binary sup operation $\oplus$. Therefore quantales may also be construed as idempotent semirings with an additional completeness property.

Notice that in the definition of quantale, we do not assume commutativity of the multiplication $\circ$ by default; quantales with commutative multiplication are said to be commutative. Although many examples of quantales we shall introduce below are in fact commutative, we work within the more general noncommutative setting in order to underline a certain parallelism between the structure of a quantale and that of matrices (of arbitrary dimensions) with coefficients in a quantale (Section 3). For matrices the multiplication (or composition) is not commutative; indeed, commutativity of multiplication does not even make sense.

2.1. Adjunctions. The notion of adjunction between posets is central to our approach (see also [CGQ04, BK12]). Recall that given posets $(L, \preceq)$ and $(L', \preceq')$, two functions $f : L \to L'$ and $u: L' \to L$ are said to form an adjunction (or Galois connection) iff for any $l \in L$ and $l' \in L'$,

$$f(l) \preceq' l' \iff l \preceq u(l')$$

holds. We call $f$ the left adjoint of $u$ and $u$ the right adjoint of $f$, and write them as $f \dashv u$. The adjointness relation (1) is powerful enough to determine each of the functions $f$ and $u$ from the other, and force both of them to be monotone functions [Str12].

We record the following well-known fact, to which we shall resort frequently in the sequel.

**Proposition 2.2.** Let $(L, \preceq)$ be a complete lattice and $(L', \preceq')$ be a poset. A function $f : L \to L'$ preserves arbitrary sups iff there exists a function $u: L' \to L$ such that $f \dashv u$.

**Proof.** Given such an $f$, define $u$ by

$$u(l') = \bigvee \{ l \in L \mid f(l) \preceq' l' \}$$

for each $l' \in L'$.

As the first application of Proposition 2.2, observe that in any quantale $Q = (Q, \preceq, I, \circ)$ there are two residuation operations: for any $x \in Q$, the function $(-) \circ x : Q \to Q$ preserves arbitrary sups, and hence has a right adjoint $(-) \vee$. 


$x : Q \rightarrow Q$ called the **right extension along** $x$; similarly, for any $y \in Q$ the function $y \circ (-)$ has a right adjoint $y \searrow (-)$, called the **right lifting along** $y$. Of course, in a commutative quantale the right extensions and right liftings coincide. The defining adjointness relations are:

$$y \preceq z \uparrow x \iff y \circ x \preceq z \iff x \preceq y \searrow z.$$  \hfill (3)

Focusing on the leftmost and rightmost formulas of $(3)$, we obtain

$$z \uparrow x \succeq y \iff x \preceq y \searrow z.$$  

That is, $z \uparrow (-) : Q \rightarrow Q$ (regarded as a function from the poset $Q = (Q, \preceq)$ to its dual $Q^{op} = (Q, \succeq)$) is the left adjoint of $(-) \searrow z : Q \rightarrow Q$ (from $Q^{op}$ to $Q$) for any $z \in Q$. The three types of adjunctions

$$Q \xleftarrow{(-) \circ x} \perp \xrightarrow{-} Q \quad Q \xleftarrow{y \circ (-)} \perp \xrightarrow{-} Q \quad Q \xleftarrow{z \uparrow (-)} \perp \xrightarrow{-} Q^{op}$$

are fundamental in theory of quantales. In Sections 3 and 5, we shall see that similar structures arise from matrices with coefficients in a quantale, and complete semimodules over a quantale, respectively.

We conclude this section with some examples of quantales.

**Example 2.3.** The **max-plus quantale** $\mathbb{R}_{\max,+} = ([-\infty, \infty], \leq, 0, +)$. Here, $([-\infty, \infty], \leq)$ is the poset $(\mathbb{R}, \leq)$ of all real numbers (with the usual order) extended with the least element $-\infty$ and the greatest element $\infty$. The multiplication $+$ is the unique sup-preserving extension to $[-\infty, \infty]$ of the ordinary addition of real numbers; the right extension/right lifting is then uniquely determined by the adjointness relation (3), and is an extension of the ordinary subtraction of real numbers; see Table 1. These extended addition and subtraction appear in [Law84]. The term “max-plus”, common in tropical mathematics, derives from the fact that as an idempotent semiring, the addition and the multiplication in $\mathbb{R}_{\max,+}$ are the max operation and (an extension of) the ordinary addition respectively.

Sometimes in the literature the **max-plus semifield** $([-\infty, \infty), -\infty, 0, \max, +)$, which is basically $\mathbb{R}_{\max,+}$ without the greatest element $\infty$, is used [CG91, LMS01]. Part of the reasons to prefer it seems to lie in the fact that it is a *semifield*, meaning that every non-zero (i.e., $\neq -\infty$) element has a multiplicative inverse (given by $x \mapsto -x$). However, it is our impression that quantales are much more convenient mathematical objects to work with than idempotent semifields. For instance, the underlying poset of an idempotent semifield cannot be a complete lattice except for trivial cases.

**Example 2.4.** The **min-plus quantale** $\mathbb{R}_{\min,+} = ([-\infty, \infty], \geq, 0, +)$. The underlying poset $([-\infty, \infty], \geq)$ is the poset $(\mathbb{R}, \geq)$ of all real numbers ordered by the opposite of the usual order, extended with the greatest element $-\infty$ and the least element $\infty$. The extension of $+$ is again uniquely determined by the requirement that it should be part of a quantale structure.
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The operation tables for $y + x$ and $z - y$ in $\mathbb{R}_{\max,+}$ are as follows:

|       | $y + x$ | $x$ | $z - y$ | $y$ |
|-------|---------|-----|---------|-----|
| $\infty$ | $\infty$ | $s$ | $\infty$ | $t$ |
| $t$ | $\infty$ | $t + s$ | $-\infty$ | $u - t$ |
| $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ |

Table 1. The operation tables for $y + x (= y \circ x)$ and $z - y (= z \vee y = y \wedge z)$ in $\mathbb{R}_{\max,+}$. $s, t$ and $r$ denote real numbers.

In fact, the quantale $\mathbb{R}_{\min,+}$ is isomorphic to $\mathbb{R}_{\max,+}$ under the mapping $x \mapsto -x$. So they are just different ways of presenting the same structure. However, note that this does not mean that the operation tables are the same for $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\min,+}$; for instance, we have $-\infty + \infty = -\infty$ in $\mathbb{R}_{\max,+}$, but have $-\infty + \infty = \infty$ in $\mathbb{R}_{\min,+}$.

Example 2.5. The nonnegative min-plus quantale $\mathbb{R}_{\min,+}^{\geq} = ([0, \infty], \geq, 0, +)$. This is $\mathbb{R}_{\min,+}$ restricted to the nonnegative (i.e., $\geq 0$) part. The + operation is the restriction of $+$ for $\mathbb{R}_{\min,+}$ to $[0, \infty]$, which is anyway the only sensible extension of the ordinary addition. The right extension/right lifting is given by an extension of the truncated subtraction $\sim$, defined by $u - t = \max\{u - t, 0\}$ for nonnegative real numbers $t$ and $u$. This quantale is introduced in [Law73] for a category-theoretic approach to the theory of metric spaces.

Note that by restricting the isomorphism between $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\min,+}$, $\mathbb{R}_{\min,+}^{\geq}$ is isomorphic to the nonpositive max-plus quantale (in the obvious sense), but it is not isomorphic to “the nonnegative max-plus quantale”; indeed, there is no quantale that deserves this name. It is impossible to endow a quantale structure on the poset $([0, \infty], \leq)$ in such a way that its multiplication is given by an extension of the ordinary addition $+$, because the quantale axioms would force $0 + x = 0$ for all $x \in [0, \infty]$.

Example 2.6. One can also consider the “discrete” variants of the quantales $\mathbb{R}_{\max,+}^{\neq} (\cong \mathbb{R}_{\min,+}^{\neq})$ or $\mathbb{R}_{\min,+}^{\geq}$, replacing the real numbers by the integers. The discrete variant of $\mathbb{R}_{\min,+}$ is used in [Fuj14] in connection to discrete convex analysis [Mur03].

Example 2.7. The nonnegative min-max quantale $\mathbb{R}_{\min,\max}^{\geq} = ([0, \infty], \geq, 0, \max)$. Its underlying poset $([0, \infty], \geq)$ is the same as that of $\mathbb{R}_{\min,+}^{\geq}$. We take the binary max operation with respect to the usual ordering $\leq$, namely the binary meet operation with respect to $\geq$, as the multiplication. The right extension/right lifting is given by

$$\z \vee y = y \wedge z = \begin{cases} 0 & \text{if } y \geq z \\ z & \text{otherwise.} \end{cases}$$

This quantale is related to (a generalisation of) ultrametric spaces.

We remark that more generally, any locale, i.e., a complete lattice in which the binary meet operation $\wedge$ satisfies the infinitary distributive law $(\bigvee_{i \in I} y_i) \wedge x =$
\[ \bigvee_{i \in I} (y_i \land x), \] acquires a quantale structure with \( \land \) as the multiplication; indeed quantales were first introduced as a quantum theoretic generalisation of locales [Mul86]. The poset \([0, \infty], \geq\), or more generally any totally ordered complete lattice, is a locale.

**Example 2.8.** The truth value quantale \( 2 = (\{\bot, \top\}, \triangleright, \top, \land) \). The underlying poset of this quantale consists of \( \top \) for “truth” and \( \bot \) for “falsity”, ordered by the entailment relation \( \triangleright \), so that \( \bot \triangleright \top \). The monoid structure is given by conjunction \( \land \). This is also studied in [Law73].

**Example 2.9.** Let \( \mathcal{M} = (M, e, \cdot) \) be a monoid. The free quantale generated by \( \mathcal{M} \) is \( P\mathcal{M} = (P M, \subseteq, \{e\}, \cdot) \), where \( P M \) is the power set of \( M \) and the multiplication \( \cdot \) on \( P M \) is the unique sup-preserving extension of the original multiplication on \( M \), which is given by

\[ A \cdot B = \{ a \cdot b | a \in A, b \in B \} \]

for all \( A, B \in P M \). Unlike the previous examples, this quantale is not commutative unless \( \mathcal{M} \) is. It has been used in connection to formal language theory, both from the categorical side [Ros95] as well as from the tropical side [Pin98].

**Example 2.10.** Let \( A \) be a set. The poset \( (P(A \times A), \subseteq) \) of all binary relations on \( A \) admits a quantale structure \( (P(A \times A), \subseteq, I_A, \circ) \), where \( I_A \) denotes the diagonal relation on \( A \) and \( \circ \) denotes composition of relations. This quantale is not commutative in general. In fact, this is obtained from \( 2 \) via a general construction of matrix quantale; see the next section.

### 3. Matrices

In this section we study matrices over a quantale.

**Definition 3.1.** Let \( Q \) be a quantale, and \( A \) and \( B \) be (possibly infinite) sets. A \( Q \)-matrix (or simply a matrix) from \( A \) to \( B \) is a \((B \times A)\)-indexed family of elements of \( Q \). If \( X = (X_{b,a})_{b \in B, a \in A} \) is a \( Q \)-matrix from \( A \) to \( B \), we denote this by \( X : A \rightarrow B \).

In particular, if \( A = \{1, \ldots, m\} \) and \( B = \{1, \ldots, n\} \) for some natural numbers \( m \) and \( n \), then a \( Q \)-matrix \( X : A \rightarrow B \) is nothing but an \((n \times m)\)-matrix with coefficients in \( Q \), which we may write as

\[
X = \begin{bmatrix}
X_{1,1} & X_{1,2} & \cdots & X_{1,m} \\
X_{2,1} & X_{2,2} & \cdots & X_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
X_{n,1} & X_{n,2} & \cdots & X_{n,m}
\end{bmatrix}
\]

as usual.

For any set \( A \) we have the identity matrix \( I_A \)

\[
(I_A)_{a,a'} = \begin{cases} 
I_Q & \text{if } a = a', \\
0_Q & \text{otherwise},
\end{cases}
\]
where $0_Q$ denotes the least element in $(Q, \preceq_Q)$. Given matrices $X : A \to B$ and $Y : B \to C$, we may define their composition $Y \circ X : A \to C$ as follows:

$$(Y \circ X)_{c,a} = \bigvee_{b \in B} (Y_{c,b} \circ_Q X_{b,a}).$$

(5)

Since the sup $\bigvee$ corresponds to (possibly infinitary) addition in a quantale, (5) may be easily checked as the tropical analogue of the usual matrix multiplication formula. It may be easily checked that the composition operation satisfies the familiar laws, namely

$$X \circ I_A = X, \quad I_B \circ X = X$$

and

$$Z \circ (Y \circ X) = (Z \circ Y) \circ X$$

(6)

for all $X : A \to B$, $Y : B \to C$ and $Z : C \to D$.

We put a partial order $\preceq_{A,B}$ on the set $Q\text{-Mat}(A,B)$ of all $Q$-matrices from $A$ to $B$ in an entrywise manner, namely $X \preceq_{A,B} X'$ if for all $b \in B$ and $a \in A$, $X_{b,a} \preceq_Q X'_{b,a}$. The poset $(Q\text{-Mat}(A,B), \preceq_{A,B})$ is isomorphic to the $(B \times A)$-fold product of $(Q, \preceq_Q)$, and hence is again a complete lattice. We sometimes omit the subscripts $A, B$ in $\preceq_{A,B}$. One easily verifies that the composition operation preserves arbitrary sups in each argument:

$$(\bigvee_{i \in I} Y_i) \circ X = \bigvee_{i \in I} (Y_i \circ X) \quad \text{and} \quad Y \circ (\bigvee_{i \in I} X_i) = \bigvee_{i \in I} (Y \circ X_i)$$

(7)

for all $X, X_i : A \to B$ and $Y, Y_i : B \to C$.

Notice the evident parallelism of the structure of a quantale (Definition 2.1) and that of matrices. We see in particular that for any set $A$, the set of all matrices of type $A \to A$ admits a natural structure of a quantale, forming a matrix quantale (cf. Example 2.10). Indeed, taking into account the non-square matrices as well, we may summarise the equations (6) and (7) by saying that for any quantale $Q$, $Q$-matrices (of arbitrary dimensions) form a quantaloid $Q\text{-Mat}$. We will not need a formal definition of quantaloid in this paper, but informally it is a “many-object version” of quantale, just as groupoids are groups with many objects; see e.g., [Ros95, Stu05, SZ13].

The structural similarity of a quantale and matrices over it suggests the possibility of extending the constructions in the former, in particular that of residuation, to the latter. This is indeed the case, and to do so we need no more than Proposition 2.2. For any $Q$-matrix $X : A \to B$ and a set $C$, the first of the equations (7) says that the function

$$( - ) \circ X : Q\text{-Mat}(B,C) \to Q\text{-Mat}(A,C)$$

preserves arbitrary sups. Hence we obtain its right adjoint

$$(-) \bigvee X : Q\text{-Mat}(A,C) \to Q\text{-Mat}(B,C),$$

called the right extension along $X$. Similarly, for any $Q$-matrix $Y : B \to C$ and a set $A$, the function

$$Y \circ (-) : Q\text{-Mat}(A,B) \to Q\text{-Mat}(A,C)$$

has a right adjoint

$$Y \smallsetminus (-) : Q\text{-Mat}(A,C) \to Q\text{-Mat}(A,B),$$
called the right lifting along $Y$.

Analogously to (3), the adjointness relation

$$Y \preceq_{B,C} Z \owns X \iff Y \circ X \preceq_{A,C} Z \iff X \preceq_{A,B} Y \downarrow Z$$

(8)

holds for any triple of matrices $X: A \rightarrow B$, $Y: B \rightarrow C$ and $Z: A \rightarrow C$, implying the third type of adjunction

$$Q\text{-Mat}(A, B) \cong Q\text{-Mat}(B, C)^\text{op} \downarrow Z$$

(9)

(sometimes called the Isbell adjunction [SZ13]). Note that, since elements of $Q$ can be identified with $Q$-matrices of type $1 \rightarrow 1$, where $1 = \{\ast\}$ is a singleton, (3) may be regarded as a special case of (8) where $A = B = C = 1$.

Let us now take a closer look at the operations of right extension and right lifting of matrices. Given matrices $X: A \rightarrow B$ and $Z: A \rightarrow C$, the right extension of $Z$ along $X$ is a matrix $Z \owns X: B \rightarrow C$. First observe that we have

$$(Z \owns X) \circ X \preceq_{A,C} Z,$$

(10)

since by (8) this is equivalent to $Z \owns X \preceq_{B,C} Z \owns X$, which holds trivially. In more detail, the adjointness relation (8) says that $Z \owns X$ is the largest matrix of type $B \rightarrow C$ with this property. Namely, it is characterised by (10) together with

for any $Y: B \rightarrow C$, $Y \circ X \preceq_{A,C} Z$ implies $Y \preceq_{B,C} Z \owns X$.

Explicitly, the matrix $Z \owns X$ is given by

$$(Z \owns X)_{c,b} = \bigwedge_{a \in A} (Z_{c,a} \owns X_{b,a}).$$

(11)

The formula (11) may either be deduced from (2) or be verified by showing the adjointness relation (8) directly. Similarly, the right lifting $Y \downarrow Z$ is abstractly characterised by the properties

$$Y \circ (Y \downarrow Z) \preceq_{A,C} Z$$

and

for any $X: A \rightarrow B$, $Y \circ X \preceq_{A,C} Z$ implies $X \preceq_{A,B} Y \downarrow Z$,

whereas concretely it is given by the formula

$$(Y \downarrow Z)_{b,a} = \bigwedge_{c \in C} (Y_{c,b} \downarrow Z_{c,a}).$$

(12)

We now introduce miscellaneous notions on matrices. For any matrix $X: A \rightarrow B$, its transpose $X^\top: B \rightarrow A$ is defined by $X^\top_{a,b} = X_{b,a}$. Recall that $1 = \{\ast\}$ is a singleton. As already mentioned, matrices of type $1 \rightarrow 1$ can be identified with elements of the base quantale $Q$, and thus are also called scalars. If $X: 1 \rightarrow 1$ is a scalar, then by abuse of notation we also denote by $X$ its unique entry $X_{\ast,\ast} \in Q$. For any set $A$, matrices of type $1 \rightarrow A$ are column vectors of size $A$, and dually those of type $A \rightarrow 1$ are row vectors of size $A$; cf. (4).
Note that a column vector \( X : 1 \rightarrow A \) and a row vector \( Y : A \rightarrow 1 \) of the same size compose, and give rise to a scalar

\[
Y \circ X = \bigvee_{a \in A} (Y_{*,a} \circ X_{a,*}),
\]

which may be thought of as the \emph{scalar product} of \( Y \) and \( X \). For any matrix \( X : A \rightarrow B \) and \( a \in A \), the \( a \)-th column vector of \( X \) is a column vector \( X_{-,a} : 1 \rightarrow B \) of size \( B \) defined by \((X_{-,a})_{b,*} = X_{b,a}\); dually, for any \( b \in B \) we have the \( b \)-th row vector of \( X \) \( X_{b,-} = ((X^\top)_{-,b})^\top \).

**Example 3.2.** Willerton [Wil15] observes that the Legendre–Fenchel transform (see e.g., [Roc70, Section 12]) is an instance of right extension/right lifting of matrices. Recall that the Legendre–Fenchel transform on a finite-dimensional real vector space \( \mathbb{R}^n \) maps any function \( f : \mathbb{R}^n \rightarrow [\mathbb{R}, \mathbb{R}] \) to its \textbf{conjugate}, which is a function \( f^* \) of type \( (\mathbb{R}^n)^* \rightarrow [-\mathbb{R}, \mathbb{R}] \), where \( (\mathbb{R}^n)^* = \mathbb{R}^n \) is the dual vector space of the original \( \mathbb{R}^n \). Using the \textbf{pairing function} \( \langle -,- \rangle \) defined by \( \langle p,v \rangle = \sum_{i=1}^n p_i v_i \) for each \( p = (p_1, \ldots, p_n) \in (\mathbb{R}^n)^* \) and \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), the formula for the conjugate \( f^* \) is given by

\[
f^*(p) = \sup_{v \in \mathbb{R}^n} \{ \langle p,v \rangle - f(v) \} \tag{13}
\]

for each \( p \in (\mathbb{R}^n)^* \).

To understand this construction via matrices, we first regard the pairing function \( \langle -,- \rangle \) as an \( \mathbb{F}_{\min,+} \)-matrix \( \langle -,- \rangle : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^* \). A function \( f \) of type \( \mathbb{R}^n \rightarrow [-\mathbb{R}, \mathbb{R}] \) may be identified with an \( \mathbb{F}_{\min,+} \)-matrix \( f : \mathbb{R}^n \rightarrow 1 \), and likewise a function \( g \) of type \( (\mathbb{R}^n)^* \rightarrow [-\mathbb{R}, \mathbb{R}] \) with an \( \mathbb{F}_{\min,+} \)-matrix \( g : 1 \rightarrow (\mathbb{R}^n)^* \). Then we have \( f^* = \langle -,- \rangle \vee f \); see (11). Note that infima in \( \mathbb{F}_{\min,+} \) (denoted by \( \wedge \) in (11)) amount to suprema with respect to the usual order (denoted by \( \sup \) in (13)).

The Legendre–Fenchel transform works in the converse direction as well, mapping a function \( g : (\mathbb{R}^n)^* \rightarrow [-\mathbb{R}, \mathbb{R}] \) to its conjugate \( g^* : \mathbb{R}^n \rightarrow [-\mathbb{R}, \mathbb{R}] \) given by essentially the same formula:

\[
g^*(v) = \sup_{p \in (\mathbb{R}^n)^*} \{ \langle p,v \rangle - g(p) \}.
\]

Of course, this can be expressed as \( g^* = g \vee \langle -,- \rangle \) using matrices.

The Legendre–Fenchel transform is an important tool in convex analysis. One reason for this importance is that for any function \( f : \mathbb{R}^n \rightarrow [-\mathbb{R}, \mathbb{R}] \), \( (f^*)^* \) is its \textbf{convex envelope}, meaning the least (with respect to the pointwise order \( \leq \)) lower-semicontinuous convex function above \( f \). In particular, lower-semicontinuous convex functions can be characterised as the “fixed points” of the Legendre–Fenchel transform; we shall revisit this point in the next section.

4. The Isbell Hull of a Matrix

In this section we introduce the notion of Isbell hull of a matrix. It incorporates many nontrivial constructions in tropical mathematics and related fields, such as the tropical polytope generated by a set of points [DS04], the directed
tight span of a (generalised) metric space \([HK12, KKO12]\), and the set of lower-semicontinuous convex functions on \(\mathbb{R}^n\).

**Definition 4.1.** Let \(Q\) be a quantale and \(Z: A \longrightarrow C\) be a \(Q\)-matrix. The **Isbell hull of** \(Z\), denoted by \(\text{Isb}(Z)\), is the set of all pairs \((X, Y)\) of \(Q\)-matrices \(X: A \longrightarrow 1\) and \(Y: 1 \longrightarrow C\) such that both \(Y = Z \vee X\) and \(X = Y \wedge Z\) hold.

In this section we treat the Isbell hull of a matrix only as a set of certain pairs of matrices, but in fact it has rich structure, as we shall see in Section 5.

The term "Isbell hull" is already used in \([KKO12]\] to refer to a special case of Definition 4.1. We find it appropriate to use this term in the more genernal setting, since Isbell has done pioneering work on this (or closely related) construction not only in the context of metric spaces \([Isb64]\] but also in the context of ordinary categories \([Isb60]\], contributing to deeper understanding of it.

The Isbell hull construction, like many important constructions, may be understood in several different manners. The first alternative view is provided by the following.

**Proposition 4.2.** Let \(Z: A \longrightarrow C\) be a \(Q\)-matrix. A pair \((X, Y)\) of \(Q\)-matrices \(X: A \longrightarrow 1\) and \(Y: 1 \longrightarrow C\) belongs to \(\text{Isb}(Z)\) iff it is a maximal pair under-approximating \(Z\), in the sense that \(i)\) \(Y \circ X \preceq Z\), and \(ii)\) if \((X', Y')\) satisfies \(Y' \circ X' \preceq Z\), \(X' \preceq X\) and \(Y \preceq Y'\), then \(X = X'\) and \(Y = Y'\).

**Proof.** Suppose \((X, Y) \in \text{Isb}(Z)\). Given \((X', Y')\) with \(Y' \circ X' \preceq Z\), \(X \preceq X'\) implies \(Y' \preceq Z \vee X' \preceq Z \vee X = Y\); similarly, \(Y \preceq Y'\) implies \(X' \preceq X\).

Conversely, suppose that \((X, Y)\) satisfies conditions \(i)\) and \(ii)\). Then the pair \((X, Z \vee X)\) satisfies \((Z \vee X) \circ X \preceq Z\), \(X \preceq X\) and, by \(i)\), \(Y \preceq Z \vee X\). So by \(ii)\) we conclude \(Y = Z \vee X\). Similarly, using the pair \((Y \wedge Z, Y)\) we see that \(X = Y \wedge Z\).

\[\square\]

4.1. **Fixed points of an adjunction.** The Isbell hull of a matrix can be seen as an instance of the set of fixed points of an adjunction. Suppose that

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{f} & \mathcal{L}' \\
\downarrow & & \downarrow \\
\mathbb{U} & \xrightarrow{u} & \mathbb{V}
\end{array}
\]

is an adjunction between posets \(\mathcal{L} = (L, \preceq)\) and \(\mathcal{L}' = (L', \preceq')\). By a **fixed point of the adjunction** \(f \dashv u\) we mean a pair \((l, l') \in L \times L'\) such that \(l' = f(l)\) and \(l = u(l')\). The set of all fixed points of \(f \dashv u\) is denoted by \(\text{Fix}(f \dashv u)\). For any matrix \(Z: A \longrightarrow C\), clearly \(\text{Isb}(Z)\) is \(\text{Fix}(Z \vee (-) \dashv (-) \wedge Z)\) for the adjunction \((9)\) with \(B = 1\).

We recall some well-known properties of \(\text{Fix}(f \dashv u)\). By definition, given an element \((l, l') \in \text{Fix}(f \dashv u)\), each of \(l\) and \(l'\) is determined from the other. This means that both projections \(\pi_1: \text{Fix}(f \dashv u) \longrightarrow L\) and \(\pi_2: \text{Fix}(f \dashv u) \longrightarrow L'\), mapping \((l, l')\) to \(l\) and \(l'\) respectively, are injective. Given elements \((l_1, l'_1)\) and \((l_2, l'_2)\) of \(\text{Fix}(f \dashv u)\), we have \(l_1 \preceq l_2\) iff \(l'_1 \preceq' l'_2\), because the former is equivalent to \(l_1 \preceq u(l'_2)\) and the latter to \(f(l_1) \preceq' l'_2\). Hence we obtain a natural partial
order on $\text{Fix}(f \dashv u)$ induced by either of the projections, turning it into a poset $\mathcal{F}ix(f \dashv u)$. The projections $\pi_1$ and $\pi_2$ are monotone by definition and moreover they have adjoints:

$$
\begin{array}{c}
\pi_1 \\
\downarrow \\
\mathcal{L} \\
\downarrow \\
\mathcal{L}' \\
\pi_2 \\
\downarrow \\
\mathcal{F}ix(f \dashv u).
\end{array}
$$

Here, the adjoints $g$ and $v$ are given by $g(l) = (uf(l), f(l))$ and $v(l') = (u(l'), fu(l'))$. Observe in particular that the image of $\pi_1$ (which may be identified with $Z$) is in fact a rich source of instances of naturally occurring Isbell hulls. In this case, a matrix $f$ is lower-semicontinuous convex iff $\pi_1$ coincides with the image of $u$. Infima and suprema in $\mathcal{F}ix(f \dashv u)$ are easy to describe. Given a family $((l_i, l'_i))_{i \in I}$ of elements of $\text{Fix}(f \dashv u)$, $\bigwedge_{i \in I} l_i \in \text{Fix}(f \dashv u)$ exists iff $\bigwedge_{i \in I} l_i \in L$ exists, and in this case we have $\bigwedge_{i \in I} l_i = (\bigwedge_{i \in I} l_i, f(\bigwedge_{i \in I} l_i))$; similarly, $\bigvee_{i \in I} l_i = (\bigvee_{i \in I} l_i, f(\bigvee_{i \in I} l_i))$ whenever either side of the equality exists. In particular, $\mathcal{F}ix(f \dashv u)$ is a complete lattice whenever either $L$ or $L'$ is.

**Corollary 4.3.** Let $Z: A \to C$ be a matrix.

1. The Isbell hull $\text{Isb}(Z)$ of $Z$ admits a partial order $\preceq$ defined by $(X, Y) \preceq (X', Y')$ iff $X \preceq_{A, 1} X'$, or equivalently iff $Y' \preceq_{1, C} Y$. The poset $(\text{Isb}(Z), \preceq)$ is a complete lattice.
2. A row vector $X: A \to 1$ is in $\text{Isb}(Z)$ (precisely: is in $\pi_1(\text{Isb}(Z))$) iff there exists some column vector $Y: 1 \to C$ such that $X = Y \setminus Z$.
3. For any pair $(X', Y') \in \mathcal{Q}\text{-Mat}(A, 1) \times \mathcal{Q}\text{-Mat}(1, C)$ under-approximating $Z$ (i.e., such that $Y' \circ X' \preceq Z$), there exists $(X, Y) \in \text{Isb}(Z)$ with $X' \preceq X$ and $Y' \preceq Y$.

*Proof.* We show only the third clause. Given any $X' \in \mathcal{Q}\text{-Mat}(A, 1)$, the pair $((Z \not\preceq X') \setminus Z, Z \not\preceq X')$ is in $\text{Isb}(Z)$ and we have $X' \preceq (Z \not\preceq X') \setminus Z$. If $Y' \in \mathcal{Q}\text{-Mat}(1, C)$ satisfies $Y' \circ X' \preceq Z$, then we also have $Y' \preceq Z \not\preceq X'$.

Let us see some examples of the Isbell hull construction. The case of tropical polytope requires some preparation and will be treated in Section 5.

**Example 4.4 ([Wil15]).** The set of all lower-semicontinuous convex functions of type $\mathbb{R}^n \to [-\infty, \infty]$ may be understood as the Isbell hull of the $\mathbb{R}_{\text{min, +}}$-matrix $\langle -, - \rangle: \mathbb{R}^n \to (\mathbb{R}^n)^*$ in Example 3.2. In light of that example, this is simply a restatement of the classical theorem that a function $f: \mathbb{R}^n \to [-\infty, \infty]$ is lower-semicontinuous convex iff $f = g^*$ for some $g: (\mathbb{R}^n)^* \to [-\infty, \infty]$ (iff $f = (f^*)^*$).

**Example 4.5.** The trivial-looking case of $\mathcal{Q} = 2$ is in fact a rich source of instances of naturally occurring Isbell hulls. In this case, a matrix $Z: A \to C$ may be identified with a relation $Z \subseteq C \times A$ consisting exactly of those pairs $(c, a) \in C \times A$ with $Z_{c,a} = \top$. Similarly, a row vector $X: A \to 1$ and a column
vector $Y : 1 \to C$ are identified with subsets $X \subseteq A$ and $Y \subseteq C$ respectively. Then we have an adjunction

\[
\begin{array}{ccl}
\mathcal{P}A & \xleftarrow{\perp} & (\mathcal{P}C)^{\text{op}} \\
\downarrow & & \downarrow \\
(-) \setminus Z & \xrightarrow{\perp} & Z (\cdot)
\end{array}
\]

given explicitly by

\[
\begin{align*}
Z \vee X &= \{ c \in C \mid \forall a \in X. (c, a) \in Z \} \\
Y \setminus Z &= \{ a \in A \mid \forall c \in Y. (c, a) \in Z \}
\end{align*}
\]

In fact, every adjunction between the powerset lattices $\mathcal{P}A$ and $(\mathcal{P}C)^{\text{op}}$ is of the form (14) for some relation $Z \subseteq A \times C$. (One can reconstruct a relation from such an adjunction by considering the images of singletons.)

Being a general form of adjunctions (Galois connections) between powerset lattices, this includes many classical dualities; see e.g., [Wil15, Section 1.1.1]. This is also the framework of formal concept analysis, in which one is given the data consisting of a set $A$ of attributes, a set $C$ of objects and a satisfaction relation $Z$ specifying which object satisfies which attribute, and aims to extract concepts, which are nothing but elements of $\text{Isb}(Z)$. Pavlovic [Pav12] proposes an enriched categorical generalisation of formal concept analysis via the nucleus of a profunctor, which is the same as the Isbell hull of a $\mathcal{Q}$-matrix for a general quantale $\mathcal{Q}$.

Another classical instance of this construction is where $A = C$ and $Z$ is a partial order on $A$. In this case, the poset $\text{Isb}(Z)$ (which is a complete lattice by Corollary 4.3) is known as the Dedekind–MacNeille completion of the poset $(A, Z)$. ■

Example 4.6. Willerton [Wil13] observes that the directed tight span of a generalised metric space [HK12, KKO12] is an instance of the Isbell hull construction with $\mathcal{Q} = \mathbb{R}_{\min,+}^\infty$. A generalised metric space $(A, d)$ consists of a set $A$ together with a function $d : A \times A \to [0, \infty]$ satisfying $d(a, a) = 0$ and $d(a, a') + d(a', a'') \geq d(a, a'')$ (the triangular inequality) for each $a, a', a'' \in A$. Given a generalised metric space $(A, d)$, its directed tight span is given by the set of all pairs of functions $(X, Y)$ where $X, Y : A \to [0, \infty]$, such that $X$ and $Y$ form a minimal pair satisfying $Y(a') + X(a) \geq d(a, a')$ for each $a, a' \in A$. In light of Proposition 4.2, this is clearly the Isbell hull of $d$ regarded as an $\mathbb{R}_{\min,+}^\infty$-matrix $d : A \to A$.

The directed tight span of a generalised metric space has a natural $l_\infty$-type metric on it, making it into a generalised metric space as well. (Indeed, the directed tight span of $(A, d)$ is characterised as the unique injective and tight extension of $(A, d)$; cf. Section 7.) As we shall see in Example 6.3, the notion of generalised metric space coincides with an enriched categorical notion of $\mathbb{R}_{\min,+}^\infty$-categories. We can also recover the $l_\infty$ metric on the directed tight span from the general theory, since as we shall see in Sections 5 and 6, the Isbell hull of a $\mathcal{Q}$-matrix always acquires a natural structure of a $\mathcal{Q}$-category. ■
5. Complete semimodules

In this section we study the notion of complete semimodule over a quantale \( Q \). It will turn out that the Isbell hull of any \( Q \)-matrix acquires such structure and conversely, any complete semimodule is isomorphic to the Isbell hull of itself (regarded as a \( Q \)-category; see Section 6).

**Definition 5.1** ([CGQ04, Section 2.2]). Let \( Q = (Q, \preceq_Q, I_Q, \circ_Q) \) be a quantale. A (left) complete semimodule over \( Q \) is a complete lattice \( M = (M, \preceq_M) \) equipped with a left action \( *_M : Q \times M \to M \) of the monoid \((Q, I_Q, \circ_Q)\) on the set \( M \), such that \( *_M \) preserves arbitrary sups in each argument: 
\[
\bigvee_{i \in I} x_i *_M m = \bigvee_{i \in I} (x_i *_M m) \quad \text{and} \quad x *_M \left( \bigvee_{i \in I} m_i \right) = \bigvee_{i \in I} (x *_M m_i).
\]

A right complete semimodule over \( Q \) is a left complete semimodule over the quantale \( Q^{\text{rev}} = (Q, \preceq_Q, I_Q, \circ_Q^{\text{rev}}) \), where \( x \circ_Q^{\text{rev}} y = y \circ_Q x \). ■

Analogously to the case of quantale, there is an algebraic variant of the above order-theoretic definition. In general, a semimodule over a semiring \( Q \) is a commutative monoid \( M = (M, \preceq_M, \ast_M) \) equipped with an action \( \ast_Q : Q \times M \to M \), which may be succinctly expressed as a semiring homomorphism from \( Q \) to the semiring \( \text{End}(M) \) of all endomorphisms on \( M \). If \( Q \) is an idempotent semiring then the existence of a semiring homomorphism \( Q \to \text{End}(M) \) forces \( \text{id}_M \in \text{End}(M) \) to be idempotent (with respect to addition). So a semimodule over an idempotent semiring can be equivalently given as a sup-semilattice with an appropriate action, and Definition 5.1 may be viewed as a complete version of it.

Applying Proposition 2.2, we obtain two residuation operations associated with a complete semimodule \( M = (M, \preceq_M, \ast_M) \): for each \( m \in M \), the right adjoint \( (\_ \ast_M)^{\to}_M m : M \to Q \) of \( (\_ \ast_M)^{\leftarrow}_M m : Q \to M \), and for each \( x \in Q \), the right adjoint \( x \ast_M (\_ \preceq_M)^{\leftarrow}_M : M \to M \) of \( x \ast_M (\_ \preceq_M)^{\to}_M : M \to M \). We again have the adjointness relation 
\[
x \preceq_Q n \ast_M m \iff x \ast_M m \preceq_M n \iff m \preceq_M x \ast_M n \quad (15)
\]
for each \( x \in Q \) and \( m, n \in M \). Adopting terminology from enriched category theory [Kel82], we call the function 
\[
\ast_M : Q \times M \to M
\]
the **tensor**, the function 
\[
\ast_M^{\to}_M : M \times M^{\text{op}} \to Q
\]
the **hom-functor**, and the function 
\[
\ast_M^{\leftarrow}_M : Q^{\text{op}} \times M \to M
\]
the **cotensor**.

**Proposition 5.2** (Cf. [CGQ04, Section 2.3] and [Stu05, Proposition 5.10]). Let \( Q = (Q, \preceq_Q, I_Q, \circ_Q) \) be a quantale and \( M = (M, \preceq_M, \ast_M) \) be a left complete semimodule over \( Q \). Then the triple \( M^{\text{op}} = (M, \preceq_M, \ast_M^{\leftarrow}_M) \) is a right complete semimodule over \( Q \) (called the **dual** of \( M \)). Moreover we have \((M^{\text{op}})^{\text{op}} = M\).
The resulting notion is (cotensor) as a primitive. This is indeed possible, by means of the hom-functor.

We regard a left (resp. right) complete subsemimodule itself as a left (resp. right) complete semimodule, by the induced order and operation.

Of course, a right complete subsemimodule of $\mathcal{M}$ is nothing but a left complete subsemimodule of the left complete semimodule $\mathcal{M}^{\text{op}}$ over $\mathcal{Q}^{\text{ev}}$.

The set of all left complete subsemimodules of a fixed left complete semimodule $\mathcal{M} = (M, \preceq_{\mathcal{M}}, \ast_{\mathcal{M}})$ is closed under arbitrary intersections. Hence for any subset $S \subseteq M$ there exists the least left complete subsemimodule containing $S$, the right complete subsemimodule generated by $S$. Similarly we have the right complete subsemimodule generated by $S$.

**Example 5.3.** Here are some immediate examples of (left or right) complete semimodules over a quantale $\mathcal{Q} = (Q, \preceq, I_{\mathcal{Q}}, \circ_{\mathcal{Q}})$; in the following, all complete semimodules are over $\mathcal{Q}$ unless otherwise stated. First, clearly the underlying complete lattice $(Q, \preceq)$ of $\mathcal{Q}$ acquires a left complete semimodule structure via $\circ_{\mathcal{Q}}$; we denote this left complete semimodule also by $\mathcal{Q}$. The dual right complete semimodule $\mathcal{Q}^{\text{op}}$ corresponding to it (Proposition 5.2) uses the right lifting $\setminus_{\mathcal{Q}}$ as action.

On the other hand, the complete lattice $(Q, \preceq)$ also underlies the quantale $\mathcal{Q}^{\text{rev}} = (Q, \preceq, I_{\mathcal{Q}}, \circ_{\mathcal{Q}}^{\text{rev}})$, hence via $\circ_{\mathcal{Q}}^{\text{rev}}$ it becomes a left complete semimodule over $\mathcal{Q}^{\text{rev}}$, i.e., a right complete semimodule (over $\mathcal{Q}$). This right complete semimodule is denoted by $\mathcal{Q}^{\text{ev}}$. The dual left complete semimodule $\mathcal{Q}^{\text{ev,op}} = (\mathcal{Q}^{\text{ev}})^{\text{op}}$ uses the right extension $\setminus$ as action.

Because left or right complete semimodules are closed under arbitrary products, for any pair of sets $A$ and $B$ we obtain a left complete semimodule $\mathcal{Q}^A \times (\mathcal{Q}^{\text{ev,op}})^B$ and a right complete semimodule $(\mathcal{Q}^{\text{op}})^A \times (\mathcal{Q}^{\text{ev}})^B$, which are easily seen to be the dual of each other.

The symmetry of the notions of left and right complete semimodule may leave one wonder whether it is possible to give the data of a complete semimodule in an unbiased manner, choosing neither the left action (tensor) nor the right action (cotensor) as a primitive. This is indeed possible, by means of the hom-functor. The resulting notion is skeletal and complete $\mathcal{Q}$-category; see Section 6.

**Definition 5.4.** Let $\mathcal{Q}$ be a quantale and $\mathcal{M} = (M, \preceq_{\mathcal{M}}, \ast_{\mathcal{M}})$ be a left complete semimodule over $\mathcal{Q}$. A **left complete subsemimodule of** $\mathcal{M}$ is a subset of $M$ closed under arbitrary sups in $\mathcal{M}$ and the tensor $x \ast_{\mathcal{M}} (-)$ by an arbitrary element $x \in Q$. A **right complete subsemimodule of** $\mathcal{M}$ is a subset of $M$ closed under arbitrary infs in $\mathcal{M}$ and the cotensor $\setminus_{\mathcal{M}} (-)$ by an arbitrary element $x \in Q$.

We regard a left (resp. right) complete subsemimodule itself as a left (resp. right) complete semimodule, by the induced order and operation.

Of course, a right complete subsemimodule of $\mathcal{M}$ is nothing but a left complete subsemimodule of the left complete semimodule $\mathcal{M}^{\text{op}}$ over $\mathcal{Q}^{\text{ev}}$.

The set of all left complete subsemimodules of a fixed left complete semimodule $\mathcal{M} = (M, \preceq_{\mathcal{M}}, \ast_{\mathcal{M}})$ is closed under arbitrary intersections. Hence for any subset $S \subseteq M$ there exists the least left complete subsemimodule containing $S$, the right complete subsemimodule generated by $S$. Similarly we have the right complete subsemimodule generated by $S$.

**Proposition 5.5** ([BK12, Theorem 3.1], cf. [Elli17, Theorem 3.3.9]). Let $Z: A \rightarrow C$ be a matrix. The Isbell hull $\text{Isb}(Z)$, considered as a subset of $\mathcal{Q}\text{-Mat}(A,1)$, is the right complete subsemimodule generated by the set $\{Z_{c,-} \mid c \in C\} \subseteq \mathcal{Q}\text{-Mat}(A,1)$ of all row vectors of $Z$, where $\mathcal{Q}\text{-Mat}(A,1)$ is regarded as the left
complete semimodule $\mathbb{Q}^A$. Dually, $\text{Isb}(Z)$ considered as a subset of $\mathbb{Q} \cdot \text{Mat}(1,C)$ is the left complete subsemimodule generated by $\{Z_{c,-} \mid c \in C\}$, where $\mathbb{Q} \cdot \text{Mat}(1,C)$ is regarded as the left complete semimodule $(\mathbb{Q} \text{revop})^C$.

Proof. Let us prove that $\text{Isb}(Z) \subseteq \mathbb{Q}^A$ is the right complete subsemimodule generated by $\{Z_{c,-} \mid c \in C\}$. Recall the second clause of Corollary 4.3, claiming that $X: A \rightarrow 1$ is in $\text{Isb}(Z)$ iff it is of the form $X = Y \setminus Z$ for some $Y: 1 \rightarrow C$. The key observation is that the equation $X = Y \setminus Z$ may be regarded as an expression of $X$ by a “tropical linear combination (in $(\mathbb{Q}^n)^A)$” of the row vectors $Z_{c,-}$. More precisely, we read from the formula (12) for right lifting that we have

$$X = Y \setminus Z = \bigwedge_{c \in C} (Y_{c,*} \setminus Y_{c,-}). \quad (16)$$

Observe that the column vector $Y$ plays the role of “coefficients” of the above tropical linear combination. Any row vector $X$ as in (16), expressible as the infimum of cotensors of $Z_{c,-}$, must lie in any right complete subsemimodule containing the row vectors $Z_{c,-}$. Also we see that for any $c \in C$, the row vector $Z_{c,-}$ itself may be expressible as a trivial linear combination, upon taking $Y$ to be

$$Y_{c,*} = \begin{cases} I_Q & \text{if } c = c', \\ 0_Q & \text{otherwise}. \end{cases}$$

That the set $\text{Isb}(Z)$ of all row vectors expressible as (16) is closed under infima and the cotensor $x \setminus X$ by an element $x$ of $\mathbb{Q}$ may be checked easily. Precisely, given a family $X_i = Y_i \setminus Z$, we may take new coefficient $\bigvee_{i \in I} Y_i$ to express their infimum: $\bigwedge_{i \in I} X_i = (\bigvee_{i \in I} Y_i) \setminus Z$. Given $X = Y \setminus Z$, we have $x \setminus X = (Y \circ x) \setminus Z$. $\square$

Example 5.6. By using Proposition 5.5, we can see how tropical polytopes [DS04] are (modulo some points at infinity) instances of the Isbell hull construction. Let $m$ and $n$ be natural numbers. In [DS04], the tropical polytope generated by $m$ points $Z_1, \ldots, Z_m \in \mathbb{R}^n$ is defined as the set of all points $X \in \mathbb{R}^n$ which can be expressed as a tropical linear combination

$$X = Y_1 \circ Z_1 \oplus \cdots \oplus Y_m \circ Z_m, \quad (17)$$

where $Y_i \in \mathbb{R}$ and $\oplus$ and $\circ$ denote the coordinate-wise minimum and the addition ($c \circ (a_1, \ldots, a_n) = (c + a_1, \ldots, c + a_n)$) respectively.\footnote{Strictly speaking, Develin and Sturmfels consider an additional step of projectivisation in the definition, hence our tropical polytopes are the unprojectivised version of theirs.} As observed in [Ell17], this is the finite part of the Isbell hull of the $\mathbb{R}_{\text{max},+}$-matrix $Z: n \rightarrow m$ (where $n$ (resp. $m$) denotes an $n$-element (resp. $m$-element) set) obtained from the coordinates of the points $Z_1, \ldots, Z_m$. Indeed, any expression of the form (17) may be expressed as $X = Y \setminus Z$ for some $\mathbb{R}_{\text{max},+}$-matrix $Y: 1 \rightarrow m$ with finite (i.e., $\neq \pm \infty$) entries and conversely.

In [DS04] the following interesting duality result has been established: there is a bijective correspondence between (the combinatorial types of) tropical polytopes in $\mathbb{R}^n$ generated by $m$ points and tropical polytopes in $\mathbb{R}^m$ generated by $n$ points. From our point of view, this result can be understood abstractly from
the fact that for an \( \mathbb{R}_{\max,+} \)-matrix \( Z \), \( \text{Isb}(Z) \) and \( \text{Isb}(Z^\top) \) are canonically iso-
morphic.

\section{Q-categories}

In this section, we introduce Q-categories for a quantale \( Q \). They are instances of the well-studied notion of enriched category [Kel82].

\begin{definition}
Let \( Q = (Q, \preceq, I_Q, \circ_Q) \) be a quantale. A Q-category \( C \) consists of:

(CD1): a set \( \text{ob}(C) \) of objects;
(CD2): for each \( c, c' \in \text{ob}(C) \), an element \( C(c, c') \in Q \)

satisfying the following axioms:

(CA1): for each \( c \in \text{ob}(C) \), \( I_Q \preceq C(c, c) \);
(CA2): for each \( c, c', c'' \in \text{ob}(C) \), \( C(c', c'') \circ_Q C(c', c') \preceq Q C(c, c') \).

We also write \( c \in C \) for \( c \in \text{ob}(C) \).

Note that a Q-category \( C \) can be identified with a Q-matrix \( C : \text{ob}(C) \rightarrow \rightarrow \text{ob}(C) \) with \( C_{c, c} = C(c, c) \). In fact, a Q-category may be defined more succinctly as a set \( A \) (corresponding to (CD1)) together with a Q-matrix \( X : A \rightarrow \rightarrow A \) (CD2) on it such that \( I_A \preceq A X \) (CA1) and \( X \circ X \preceq A X \) (CA2) hold [BCSW83].

\begin{example}
In the case \( Q = 2 \), we may identify the data of a 2-category \( C = (\text{ob}(C), (C(c, c'))_{c, c' \in \text{ob}(C)}) \) with a set \( \text{ob}(C) \) equipped with a binary relation \( \preceq_C \) on it (defined as the set of all pairs \( (c, c') \in \text{ob}(C) \times \text{ob}(C) \) with \( C(c, c') = \top \)). The axioms (CA1) and (CA2) for a 2-category then translate to reflexivity and transitivity of \( \preceq_C \) respectively, hence a 2-category is nothing but a preordered set.

\end{example}

\begin{example}
In the case \( Q = \mathbb{R}_{\min,+}^\geq \), we may regard \( \mathbb{R}_{\min,+}^\geq \)-categories as generalised metric spaces [Law73] (cf. Example 4.6); objects of an \( \mathbb{R}_{\min,+}^\geq \)-category \( C \) are thought of as points and the element \( C(c, c') \in [0, \infty] \) as the distance from \( c \) to \( c' \). Notice that the axioms for \( \mathbb{R}_{\min,+}^\geq \)-category indeed translate to some of the axioms for metric spaces:

(CA1): for each \( c \in \text{ob}(C) \), \( 0 \geq C(c, c) \) (that is, \( C(c, c) = 0 \)); and
(CA2): for each \( c, c', c'' \in \text{ob}(C) \), \( C(c', c'') + C(c', c') \geq C(c, c'') \) (the triangular inequality).

Every metric space is an \( \mathbb{R}_{\min,+}^\geq \)-category, but not conversely. \( \mathbb{R}_{\min,+}^\geq \)-categories are more general than metric spaces in the following three aspects:

- distance may attain \( \infty \);
- distance is non-symmetric (or directed), i.e., \( C(c, c') \) may not be equal to \( C(c', c) \); and
- \( C(c, c') = C(c', c) = 0 \) does not necessarily imply \( c = c' \).

\end{example}

\begin{example}
Similarly, \( \mathbb{R}_{\min,max}^\geq \)-categories may be regarded as generalised ultrametric spaces; note that axiom (CA2) now reads:

\end{example}
(CA2): for each \( c, c', c'' \in \text{ob}(\mathcal{C}) \), \( \max\{ \mathcal{C}(c', c''), \mathcal{C}(c, c') \} \geq \mathcal{C}(c, c'') \).

As we have already mentioned, every left complete semimodule over \( \mathcal{Q} \) gives rise to a \( \mathcal{Q} \)-category. The precise construction is as follows.

**Definition 6.5.** Let \( \mathcal{M} = (M, \preceq_M, *,_\mathcal{M}) \) be a left complete semimodule over \( \mathcal{Q} \). The following data defines a \( \mathcal{Q} \)-category, again called \( \mathcal{M} \):

1. (CD1): the set of objects is \( \text{ob}(\mathcal{M}) = M \); and
2. (CD2): for each \( m, m' \in M \), the element \( \mathcal{M}(m, m') \in \mathcal{Q} \) is defined as \( m' \preceq_M m \) (note the order of the arguments \( m \) and \( m' \)).

The axioms can be checked easily by means of the adjointness relations (15). ■

The structure of a left complete semimodule \( \mathcal{M} \), namely the order relation \( \preceq_M \) and the action \( *_{\mathcal{M}} \), can be recovered from the induced \( \mathcal{Q} \)-category \( \mathcal{M} = (\mathcal{M}, (\mathcal{M}(m, m'))_{m, m' \in M}) \). To see this, first note that any \( \mathcal{Q} \)-category \( \mathcal{C} = (\text{ob}(\mathcal{C}), (\mathcal{C}(c, c'))_{c, c' \in \text{ob}(\mathcal{C})}) \) determines a preorder relation \( \preceq_{\mathcal{C}} \) on \( \text{ob}(\mathcal{C}) \) by

\[
c \preceq_{\mathcal{C}} c' \quad \text{iff} \quad I_{\mathcal{Q}} \preceq_{\mathcal{C}} \mathcal{C}(c, c').
\]

(18)

Two objects \( c, c' \in \mathcal{C} \) are said to be isomorphic iff both \( c \preceq_{\mathcal{C}} c' \) and \( c' \preceq_{\mathcal{C}} c \) hold. Isomorphic objects behave exactly in the same manner, namely if \( c, c' \in \mathcal{C} \) are isomorphic then for every object \( d \in \mathcal{C} \), we have \( \mathcal{C}(c, d) = \mathcal{C}(c', d) \) and \( \mathcal{C}(d, c) = \mathcal{C}(d, c') \). We call a \( \mathcal{Q} \)-category \( \mathcal{C} \) skeletal iff isomorphic objects in \( \mathcal{C} \) are equal, i.e., iff the induced preorder relation \( \preceq_{\mathcal{C}} \) on \( \text{ob}(\mathcal{C}) \) is antisymmetric and is a partial order relation. Now, if a \( \mathcal{Q} \)-category \( \mathcal{M} \) is derived from a left complete semimodule \( \mathcal{M} \) by the construction of Definition 6.5, then we actually recover the original order relation on the complete semimodule by (18) (see (15)). The action \( *_{\mathcal{M}} \) is then uniquely determined from the hom-functor \( \mathcal{M}(-, -) \) via the adjunction \((-) *_{\mathcal{M}} m \dashv \mathcal{M}(m, -)\).

In fact, it is also possible to define the counterpart of \( *_{\mathcal{M}} \) for general \( \mathcal{Q} \)-categories. Given a \( \mathcal{Q} \)-category \( \mathcal{C} \), an object \( c \in \mathcal{C} \) and an element \( x \in \mathcal{Q} \), an object \( c' \in \mathcal{C} \) is said to be a tensor product of \( x \) and \( c \) iff for any \( d \in \mathcal{C} \), the equation

\[
\mathcal{C}(c', d) = \mathcal{C}(c, d) \setminus x
\]

holds [Kel82, Section 3.7]. Tensor products of \( x \) and \( c \) may or may not exist in \( \mathcal{C} \), but when they exist they are unique up to isomorphism: if \( c' \) is a tensor product of \( x \) and \( c \), then an object \( c'' \in \mathcal{C} \) is also a tensor product of \( x \) and \( c \) iff \( c' \) and \( c'' \) are isomorphic. In particular, in a skeletal \( \mathcal{Q} \)-category tensor products are unique.

There is also a dual notion of cotensor product of \( x \in \mathcal{Q} \) and \( c \in \mathcal{C} \), which is defined as an object \( c' \in \mathcal{C} \) such that for any \( d \in \mathcal{C} \), the equation

\[
\mathcal{C}(d, c') = x \setminus \mathcal{C}(d, c)
\]

holds.

**Definition 6.6 ([Stu05, Stu06]).** A \( \mathcal{Q} \)-category \( \mathcal{C} \) is said to be:

- **tensored** iff for any \( x \in \mathcal{Q} \) and \( c \in \mathcal{C} \), a tensor product of \( x \) and \( c \) exists in \( \mathcal{C} \);
• **cotensored** iff for any \( x \in Q \) and \( c \in C \), a cotensor product of \( x \) and \( c \) exists in \( C \);
• **order-complete** iff the preordered set \((\text{ob}(C), \preceq_C)\) is complete (i.e., iff its poset reflection is a complete lattice);
• **complete** iff it is tensored and order-complete.

It is known that a \( Q \)-category is complete iff it is cotensored and order-complete [Stu05, Stu06].

It follows that the construction in Definition 6.5 provides a one-to-one correspondence between *left complete semimodules over* \( Q \) on the one hand, and *skeletal and complete* \( Q \)-categories on the other (cf. [Stu06, Section 4]; [Wil13, Section 5]).

Next we move on to the notion of morphism between \( Q \)-categories, called \( Q \)-functors.

**Definition 6.7.** Let \( Q \) be a quantale, and \( C \) and \( D \) be \( Q \)-categories.

1. A **\( Q \)-functor** \( F : C \rightarrow D \) is a function \( F : \text{ob}(C) \rightarrow \text{ob}(D) \) such that for each \( c, c' \in C \),

\[
C(c, c') \preceq_Q D(Fc, Fc')
\]

holds.
2. A \( Q \)-functor \( F \) is **fully faithful** iff for each \( c, c' \in C \), (19) is satisfied with equality. We call fully faithful \( Q \)-functors **embeddings** for short.
3. A \( Q \)-functor \( F \) is **bijective-on-objects** iff the corresponding function \( F : \text{ob}(C) \rightarrow \text{ob}(D) \) is bijective.
4. A \( Q \)-functor \( F \) is an **isomorphism** iff it is fully faithful and bijective-on-objects.

For any \( Q \)-category \( C \) we have the **identity \( Q \)-functor** \( \text{id}_C : C \rightarrow C \), and \( Q \)-functors are closed under composition. Note that an embedding \( F : C \rightarrow D \) of \( Q \)-categories need not be injective as a function \( F : \text{ob}(C) \rightarrow \text{ob}(D) \), though embeddings out of a skeletal \( C \) are injective.

Recall that a \( Q \)-category \( C \) may be identified with a \( Q \)-matrix \( C : \text{ob}(C) \rightarrow \text{ob}(C) \). We can then consider its Isbell hull \( \text{Isb}(C) \); by Proposition 5.5, it is a complete semimodule over \( Q \), hence by and Definition 6.5 we may view \( \text{Isb}(C) \) as a \( Q \)-category. A more explicit description of \( \text{Isb}(C) \) as a \( Q \)-category is the following.

**Definition 6.8.** Let \( C \) be a \( Q \)-category. The \( Q \)-category \( \text{Isb}(C) \) is defined as follows:

(\text{CA1}): its set of objects is \( \text{Isb}(C) \) as in Definition 4.1;
(\text{CA2}): given \((X, Y)\) and \((X', Y')\) in \( \text{Isb}(C) \), the element \( \text{Isb}(C)((X, Y), (X', Y')) \in Q \) is defined as

\[
\bigwedge_{c \in C} X'_{s,c} \searrow X_{s,c},
\]
or equivalently as
\[ \bigwedge_{c \in C} Y^c_{\ast \ast} \uparrow Y^c_{\ast \ast} \]
The \( \mathcal{Q} \)-functor \( J_C : \mathcal{C} \longrightarrow \text{Isb}(\mathcal{C}) \) is defined by mapping each \( c \in \mathcal{C} \) to \((\mathcal{C}(c, -), \mathcal{C}(-, c)) \in \text{Isb}(\mathcal{C})\), which turns out to be fully faithful.

We call the pair \((\text{Isb}(\mathcal{C}), J_C)\) the Isbell completion of \( \mathcal{C} \), and the \( \mathcal{Q} \)-functor \( J_C \) the Isbell embedding.

7. Complete semimodules are injective \( \mathcal{Q} \)-categories

We show that complete semimodules over \( \mathcal{Q} \) may be characterised by an abstract property of injectivity.

**Definition 7.1.** A \( \mathcal{Q} \)-category \( \mathcal{E} \) is said to be injective (with respect to the embeddings) iff for any \( \mathcal{Q} \)-categories \( \mathcal{C} \) and \( \mathcal{D} \), any \( \mathcal{Q} \)-functor \( F : \mathcal{C} \longrightarrow \mathcal{E} \), and any embedding \( G : \mathcal{C} \longrightarrow \mathcal{D} \), there exists a (not necessarily unique) \( \mathcal{Q} \)-functor \( H : \mathcal{D} \longrightarrow \mathcal{E} \) such that \( F = H \circ G \).

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{E} \\
\downarrow{G} & & \downarrow{H} \\
\mathcal{D} & & \\
\end{array}
\]

**Theorem 7.2.** A \( \mathcal{Q} \)-category is skeletal and complete iff it is injective.

The rest of this paper is devoted to a proof of this theorem. The case \( \mathcal{Q} = 2 \) of this theorem is essentially the same as the classical characterisation of complete lattices by injectivity due to Banaschewski and Bruns [BB67]. Indeed, our proof basically follows the corresponding proof in [BB67], adopting some techniques from [AHRT02] occasionally. The cases \( \mathcal{Q} = \mathbb{R}_{\min,+}^{\geq 0}, \mathbb{R}_{\min,max}^{\geq 0} \) are also equivalent to known theorems in the literature [KKO12, KO13]; see Remark 7.11.

First we show the easier direction.

**Lemma 7.3.** Skeletal and complete \( \mathcal{Q} \)-categories are injective.

**Proof.** Let \( \mathcal{E} \) be a skeletal and complete \( \mathcal{Q} \)-category. Given a diagram as in

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{E} \\
\downarrow{G} & & \downarrow{H} \\
\mathcal{D} & & \\
\end{array}
\]

we may define a \( \mathcal{Q} \)-functor \( H : \mathcal{D} \longrightarrow \mathcal{E} \) by

\[
Hd = \bigvee_{c \in \mathcal{C}} \mathcal{D}(Gc, d) \ast Fc \quad \text{or} \quad Hd = \bigwedge_{c \in \mathcal{C}} \mathcal{D}(d, Gc) \setminus_{\mathcal{E}} Fc. \hspace{1cm} ^2
\]

\(^2\text{That is, as the left or the right Kan extension of } F \text{ along } G.\)
Then it is easy to check that $H$ is indeed a $Q$-functor, and provided that $G$ is an embedding, $F = H \circ G$ holds.

To prove the converse direction, we need some preparation.

**Definition 7.4** (Cf. [AHRT02, Definitions 2.1]).

1. A $Q$-functor $F: \mathcal{C} \to \mathcal{D}$ is called an **essential embedding** iff (i) $F$ is an embedding, and (ii) for any $Q$-functor $G: \mathcal{D} \to \mathcal{E}$, if $G \circ F$ is an embedding then so is $G$.

2. An **injective hull** (or **injective envelope**) of a $Q$-category $\mathcal{C}$ is a pair $(\mathcal{D}, F)$ consisting of an injective $Q$-category $\mathcal{D}$ and an essential embedding $F: \mathcal{C} \to \mathcal{D}$.

**Remark 7.5.** For a $Q$-category $\mathcal{C}$, a pair $(\mathcal{D}, F)$ consisting of a (not necessarily injective) $Q$-category $\mathcal{D}$ and an essential embedding $F: \mathcal{C} \to \mathcal{D}$ is sometimes called a **tight extension of** $\mathcal{C}$, especially in the context where $Q = \mathbb{R}^{\geq 0}_{\text{min,+}}$ [Dre84, HK12].

Injective hulls of a $Q$-category are unique up to isomorphisms\(^3\).

**Lemma 7.6** ([AHRT02, Remarks 2.2 (2)]). Let $\mathcal{C}$ be a $Q$-category and $(\mathcal{D}, F), (\mathcal{D}', F')$ be injective hulls of $\mathcal{C}$. Then there exists an isomorphism $G: \mathcal{D} \to \mathcal{D'}$ such that $G \circ F = F'$.

**Proof.** By the injectivity of $\mathcal{D}'$, we obtain a $Q$-functor $G$ as in the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D}' \\
\downarrow{F} & & \downarrow{G} \\
\mathcal{D} & & \\
\end{array}
$$

We claim that any $Q$-functor $G$ between injective hulls as above (i.e., commuting with the essential embeddings) is an isomorphism. Since $F$ is an essential embedding and $F' = G \circ F$ an embedding, it follows that $G$ is also an embedding. Using the injectivity of $\mathcal{D}$, we get a $Q$-functor $H$ as below.

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\text{id}_\mathcal{D}} & \mathcal{D} \\
\downarrow{G} & & \downarrow{H} \\
\mathcal{D}' & & \\
\end{array}
$$

\(^3\)The usual definition of tight extension only requires that (i) $F$ is an embedding, and (ii') for any bijective-on-object $Q$-functor $G: \mathcal{D} \to \mathcal{E}$, if $G \circ F$ is an embedding, then so is $G$; i.e., $G$ is an isomorphism. Indeed this is sufficient to ensure that $F$ is an essential embedding, as is easily seen using the fact that every $Q$-functor may be (uniquely up to isomorphism) factorised as a bijective-on-object $Q$-functor followed by an embedding.

\(^4\)However, unlike many constructions in category theory, injective hulls are in general not unique up to canonical isomorphisms. As a consequence, the operation of taking the injective hulls of objects does not extend nicely to morphisms; see [AHRT02].
Note that $G$ is a section and hence one-to-one, whereas $H$ is a retraction and hence onto. Precomposing $F$ with the above diagram, we obtain the following.

\[
\begin{array}{c}
\mathcal{C} \\
F' \\
\mathcal{D}'
\end{array} \xymatrix{ 
\mathcal{C} \ar[r]^F \ar[d]_{F'} & \mathcal{D} \ar@{..>}[ld]^H \\
\mathcal{D}'}
\]

So $H$ is also a $Q$-functor between injective hulls. By iterating the same argument as above, we see that $H$ is a section and thus one-to-one; hence $H$ is a bijection and so is its section $G$. Since $G$ is an embedding, it is an isomorphism as required. □

We next give an intrinsic characterisation of essential embeddings; for this, we introduce the notions of presheaf and copresheaf over a $Q$-category.

**Definition 7.7.** Let $\mathcal{C}$ be a $Q$-category. The $Q$-category $\mathcal{P}\mathcal{C}$ of presheaves over $\mathcal{C}$ is defined as follows.

- An object is a presheaf over $\mathcal{C}$, that is a family $P = (P_c)_{c \in \mathcal{C}}$ of elements of $Q$ satisfying the inequality $P_c' \circ \mathcal{C}(c, c') \leq_Q P_c$ for each $c, c' \in \mathcal{C}$.
- The element $\mathcal{P}\mathcal{C}(P, P')$ of $Q$ is given by $\bigwedge_{c \in \mathcal{C}} P_{c'} \uparrow P_c$.

Dually, the $Q$-category $\mathcal{P}\mathcal{C}^!$ of copresheaves over $\mathcal{C}$ is defined as follows.

- An object is a copresheaf over $\mathcal{C}$, that is a family $R = (R_c)_{c \in \mathcal{C}}$ of elements of $Q$ satisfying the inequality $\mathcal{C}(c, c') \circ R_c \leq_Q R_{c'}$ for each $c, c' \in \mathcal{C}$.
- The element $\mathcal{P}\mathcal{C}^!(R, R')$ of $Q$ is given by $\bigwedge_{c \in \mathcal{C}} R_{c'} \downarrow R_c$.

For each $Q$-functor $F: \mathcal{C} \rightarrow \mathcal{D}$, we have $Q$-functors $F^*: \mathcal{D} \rightarrow \mathcal{P}\mathcal{C}$ and $F_*: \mathcal{D} \rightarrow \mathcal{P}\mathcal{C}^!$, defined as $F^*d = (\mathcal{D}(Fc, d))_{c \in \mathcal{C}}$ and $F_*d = (\mathcal{D}(d, Fc))_{c \in \mathcal{C}}$ respectively. We call $F$ dense iff $F^*$ is an embedding, and codense iff $F_*$ is an embedding.

Note that for each $Q$-category $\mathcal{C}$ we have a functor $(\text{id}_\mathcal{C})^*: \mathcal{C} \rightarrow \mathcal{P}\mathcal{C}$, which maps $c \in \mathcal{C}$ to $(\mathcal{C}(c', c))_{c' \in \mathcal{C}}$. $(\text{id}_\mathcal{C})^*$ turns out to be an embedding, is called the Yoneda embedding, and is denoted by $Y_\mathcal{C}$. In particular, $\text{id}_\mathcal{C}$ is both dense and codense. Moreover it turns out that $(Y_\mathcal{C})^*: \mathcal{P}\mathcal{C} \rightarrow \mathcal{P}\mathcal{C}$ is the identity $Q$-functor $\text{id}_{\mathcal{P}\mathcal{C}}$ on $\mathcal{P}\mathcal{C}$ (the Yoneda lemma); in particular, $Y_\mathcal{C}$ is dense (but in general not codense).

The embedding $J_\mathcal{C}: \mathcal{C} \rightarrow \text{Isb}(\mathcal{C})$ of $\mathcal{C}$ into its Isbell completion is both dense and codense, because $(J_\mathcal{C})^*: \text{Isb}(\mathcal{C}) \rightarrow \mathcal{P}\mathcal{C}$ and $(J_\mathcal{C})_*: \text{Isb}(\mathcal{C}) \rightarrow \mathcal{P}\mathcal{C}^!$ coincide with the second and the first projections respectively.

**Lemma 7.8** (Cf. [BB67, Lemma 3]). A $Q$-functor is an essential embedding iff it is a dense and codense embedding.

**Proof.** Suppose that $F: \mathcal{C} \rightarrow \mathcal{D}$ is an essential embedding. Then the composite $F^* \circ F: \mathcal{C} \rightarrow \mathcal{P}\mathcal{C}$ maps each $c \in \mathcal{C}$ to $(\mathcal{D}(Fc',Fc))_{c' \in \mathcal{C}} = (\mathcal{C}(c', c))_{c' \in \mathcal{C}}$, i.e., it is
the Yoneda embedding $Y_C$. In particular, $F^* \circ F$ is an embedding and hence so is $F^*$, showing that $F$ is dense. A similar argument shows that $F$ is codense.

Conversely, suppose that $F : C \to D$ is a dense and codense embedding. Take any $Q$-functor $G : D \to E$ such that $G \circ F$ is an embedding. We aim to show that $G$ is also an embedding, namely that for each $d, d' \in D$ we have $D(d, d') = E(Gd, Gd')$. Since $G$ is a $Q$-functor, it suffices to show the inequality

$$E(Gd, Gd') \preceq Q D(d, d').$$

(20)

Since $F$ is dense, we have

$$D(d, d') = PC(F^*d, F^*d') = \bigwedge_{c \in C} D(Fc, d').$$

(21)

Since $F$ is codense, we have

$$D(Fc, d') = \bigwedge_{c' \in C} D(Fc, d').$$

(22)

Substituting (22) into (21), we obtain

$$D(d, d') = \bigwedge_{c \in C} \left( \bigwedge_{c' \in C} D(d', Fc') \downarrow D(Fc, Fc') \right) \uparrow D(Fc, d)$$

(23)

which can be checked easily as follows:

$$D(d', Fc') \circ E(Gd, Gd') \circ D(Fc, d) \preceq E(GFc, GFc') \circ E(Gd, Gd') \circ E(GFc, Gd) \preceq E(GFc, GFc')$$

$$= C(c, c') \preceq D(Fc, Fc').$$

□

It follows from this lemma that for any $Q$-category $C$, its injective hull is given by $J_C : C \to \text{Isb}(C)$.

Now we can show that any injective $Q$-category is skeletal and complete, finishing the proof of Theorem 7.2. If $C$ is an injective $Q$-category, then (in addition to $J_C$) $\text{id}_C : C \to C$ is also its injective hull, so by the uniqueness of injective hulls (Lemma 7.6) we see that $J_C$ is an isomorphism. Since $\text{Isb}(C)$ is skeletal and complete, so is $C$. 
Corollary 7.9. A \( \mathcal{Q} \)-category \( \mathcal{C} \) is injective (or equivalently, is skeletal and complete) iff the embedding \( J_\mathcal{C} : \mathcal{C} \rightarrow \text{Isb}(\mathcal{C}) \) is an isomorphism. In particular, any skeletal and complete \( \mathcal{Q} \)-category, hence any complete semimodule over \( \mathcal{Q} \), is isomorphic to one of the form \( \text{Isb}(\mathcal{Z}) \) for some matrix \( \mathcal{Z} \).

The embedding \( J_\mathcal{C} : \mathcal{C} \rightarrow \text{Isb}(\mathcal{C}) \) is characterised by the essentialness of \( J_\mathcal{C} \) and the injectivity of \( \text{Isb}(\mathcal{C}) \). In fact it is also “extremal” with respect to these two properties, as is immediately seen from their abstract definitions.

Corollary 7.10 (Cf. [BB67, Proposition 2]). Let \( \mathcal{C} \) be a \( \mathcal{Q} \)-category.

1. For any essential embedding \( F : \mathcal{C} \rightarrow \mathcal{D} \), there exists a (not necessarily unique) embedding \( G : \mathcal{D} \rightarrow \text{Isb}(\mathcal{C}) \) with \( G \circ F = J_\mathcal{C} \).
2. For any embedding \( H : \mathcal{C} \rightarrow \mathcal{E} \) into an injective \( \mathcal{E} \), there exists a (not necessarily unique) embedding \( K : \text{Isb}(\mathcal{C}) \rightarrow \mathcal{E} \) with \( K \circ J_\mathcal{C} = H \).

\[ \begin{array}{ccc}
\mathcal{C} & \xrightarrow{J_\mathcal{C}} & \text{Isb}(\mathcal{C}) \\
F & \downarrow & H \\
\mathcal{D} & \xrightarrow{G} & \mathcal{E}
\end{array} \]

Remark 7.11. Kemajou, Künzi and Otafudu characterise injective \( \mathbb{R}_{\text{min,+}}^0 \)-categories as (skeletal) Isbell convex \( \mathbb{R}_{\text{min,+}}^0 \)-categories [KKO12, Theorem 1], extending the classical characterisation (due to Aronszajn and Panitchpakdi [AP56]) of injective metric spaces as hyperconvex metric spaces to the non-symmetric (or directed) setting. We comment on the relationship of their work and the \( \mathcal{Q} = \mathbb{R}_{\text{min,+}}^0 \) case of our theorem.

Modulo minor modifications, an \( \mathbb{R}_{\text{min,+}}^0 \)-category \( \mathcal{C} \) is called Isbell convex iff for any family \( \{(c_i, x_i, y_i)\}_{i \in I} \) where \( c_i \in \mathcal{C} \) and \( x_i, y_i \in [0, \infty) \), if \( y_i + x_j \geq C(c_i, c_j) \) holds for each \( i, j \in I \), then the intersection of “balls” \( \bigcap_{i \in I} B(c_i; x_i) \cap B'(c_i; y_i) \subseteq \text{ob}(\mathcal{C}) \) is nonempty, where the “balls” \( B(c_i; x_i) \) and \( B'(c_i; y_i) \) are defined as

\[ B(c_i; x_i) = \{ c \in \mathcal{C} \mid x_i \geq C(c, c_i) \} \quad \text{and} \quad B'(c_i; y_i) = \{ c \in \mathcal{C} \mid y_i \geq C(c_i, c) \}, \]

respectively. In terms of matrices, the same condition can be phrased as follows: for any pair of \( \mathbb{R}_{\text{min,+}}^0 \)-matrices \( X : \text{ob}(\mathcal{C}) \rightarrow 1 \) and \( Y : 1 \rightarrow \text{ob}(\mathcal{C}) \) with \( Y \circ X \geq \mathcal{C} \), there exists an object \( c \in \mathcal{C} \) such that both \( X \geq C(c, -) \) and \( Y \geq C(-, c) \) hold. Thus in light of Proposition 4.2 and the third clause of Corollary 4.3, this

---

\(^5\) The difference between our definition of Isbell convexity and the corresponding definition in [KKO12, Definition 2] is that in Kemajou et al.’s definition, \( x_i \) and \( y_i \) are not allowed to take the value \( \infty \) (perhaps because they only consider \( [0, \infty) \)-valued distance by default). According to their definition, any nonempty “discrete” \( \mathbb{R}_{\text{min,+}}^0 \)-category (in which every two different objects have the distance \( \infty \)) would trivially be Isbell convex and skeletal, but of course it is not injective (unless it has only one object).
is equivalent to requiring that the embedding $J_C: C \rightarrow \text{Isb}(C)$ be surjective on objects; and for skeletal $C$, equivalent to requiring that $J_C$ be an isomorphism.\footnote{Indeed, (even in the absence of the skeletality condition) Isbell convexity is equivalent to (categorical) completeness. Thus Isbell convexity gives a purely geometric characterisation of completeness for $\mathbb{R}^{\geq 0}_{\text{min},+}$-categories.}

Hence on the one hand, our Corollary 7.9 yields Kemajou et al.’s characterisation, and on the other hand, (since it is easy to see that Isbell hulls are skeletal and complete) their characterisation essentially proves the harder direction of our Theorem 7.2. The proof in [Kko12], however, uses Zorn’s lemma and is quite different from our choice-free proof.

A completely parallel comment applies to the relationship of our theorem when $Q = \mathbb{R}^{\geq 0}_{\text{min},\text{max}}$ and Künzi and Otafudu’s characterisation of injective $\mathbb{R}^{\geq 0}_{\text{min},\text{max}}$-categories by $q$-spherical completeness in [ko13, Theorem 2].

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: s.fujii.math@gmail.com