COMPLEX GENERALIZED KILLING SPINORS ON RIEMANNIAN SPIN\textsuperscript{c} MANIFOLDS

NADINE GROSSE AND ROGER NAKAD

ABSTRACT. In this paper, we extend the study of generalized Killing spinors on Riemannian Spin\textsuperscript{c} manifolds started by Moroianu and Herzlich to complex Killing functions. We prove that such spinor fields are always real Spin\textsuperscript{c} Killing spinors or imaginary generalized Spin\textsuperscript{c} Killing spinors, providing that the dimension of the manifold is greater or equal to 4. Moreover, we classify Riemannian Spin\textsuperscript{c} manifolds carrying imaginary and imaginary generalized Killing spinors.

1. INTRODUCTION

On a Riemannian Spin manifold \((M^n, g)\) of dimension \(n \geq 2\), a non-trivial spinor field \(\psi\) is called a complex generalized Killing spinor field with smooth Killing function \(K\) if
\[
\nabla_X \psi = K X \cdot \psi,
\]
for all vector fields \(X\) on \(M\), where \(\nabla\) denotes the spinorial Levi-Civita connection and \(\cdot\) the Clifford multiplication. Here \(K := a + ib\) denotes a complex function with real part function \(a\) and imaginary part function \(b\).

It is well known that the existence of such spinors imposes several restrictions on the geometry and the topology of the manifold. More precisely, on a Riemannian Spin manifold, a complex generalized Killing spinor is either a real generalized Killing spinor (i.e., \(b = 0\) and \(a \neq 0\)), an imaginary generalized Killing spinor (i.e., \(a = 0\) and \(b \neq 0\)) or a parallel spinor (i.e., \(b = a = 0\)) \([9, 11]\). Manifolds with parallel spinor fields are Ricci-flat and can be characterised by their holonomy group \([32, 16]\). Riemannian Spin manifolds carrying parallel spinors have been classified by M. Wang \([45]\).

When \(\psi\) is a real generalized Killing spinor, then \(a\) is already a nonzero constant, i.e., \(\psi\) is in fact a real Killing spinor. Those Killing spinors on simply connected Riemannian Spin manifolds were classified by C. Bär \([3]\). Real Killing spinors occur in physics, e.g. in supergravity theories, see \([12]\), but they are also of mathematical interest: The existence of real Killing spinor field implies that the manifold is a compact Einstein manifold of scalar curvature \(4n(n - 1)a^2\). In dimension 4, it has constant sectional curvature. Real Killing spinors are also special solutions of the twistor equation \([36, 37]\) and moreover, they are related to the spectrum of the Dirac operator. In fact, T. Friedrich \([14]\) proved a lower bound for the eigenvalues of the Dirac operator involving the infimum of the scalar curvature. The equality case is characterised by the existence of a real Killing spinor. More precisely, \(n^2 a^2\) is the smallest eigenvalue of the square of the Dirac operator \([14, 25]\). Other geometric and physics applications of the existence of real Killing spinors can be found in \([11, 13, 42, 44, 24, 25, 15, 17, 19, 20, 22, 21]\).

When \(\psi\) is an imaginary generalized Killing spinor, then \(M\) is a non-compact Einstein manifold. Moreover, two cases may occur: The function \(b\) could be constant (then \(\psi\) is called...
an imaginary Killing spinor) or it is a non-constant function (then we will continue to call \( \psi \) a imaginary generalized Killing spinor). H. Baum \([7, 6, 8]\) classified Riemannian Spin manifolds carrying imaginary Killing spinors. Shortly later, H-B. Rademacher extended this classification to imaginary generalized Killing spinors \([43]\) – here only so called type I Killing spinors can occur, see Section \([4]\).

Recently, Spin\(^c\) geometry became a field of active research with the advent of Seiberg-Witten theory. Applications of the Seiberg-Witten theory to 4-dimensional geometry and topology are already notorious. From an intrinsic point of view, Spin, almost complex, complex, Kähler, Sasaki and some classes of CR manifolds have a canonical Spin\(^c\) structure. Having a Spin\(^c\) structure is a weaker condition than having a Spin structure. Moreover, when shifting from the classical Spin geometry to Spin\(^c\) geometry, the situation is more general since the connection on the Spin\(^c\) bundle, its curvature, the Dirac operator and its spectrum will not only depend on the geometry of the manifold but also on the connection (and hence the curvature) of the auxiliary line bundle associated with the Spin\(^c\) structure.

A. Moroianu studied Equation (1) on Riemannian Spin\(^c\) manifolds when \( b = 0 \) and \( a \) is constant, i.e., when \( \psi \) is a parallel spinor or a real Killing spinor \([38]\). He proved that a simply connected complete Riemannian Spin\(^c\) manifold carrying a parallel spinor is isometric to the Riemannian product of a Kähler manifold (endowed with its canonical Spin\(^c\) structure) with a Spin manifold carrying a parallel spinor. Moreover, a simply connected complete Riemannian Spin\(^c\) manifold carrying a real Killing spinor is isometric to a Sasakian manifold endowed with its canonical Spin\(^c\) structure. In 1999, M. Herzlich and A. Moroianu considered Equation (1) for \( b = 0 \) on Riemannian Spin\(^c\) manifolds \([23]\). They proved that, if \( n \geq 4 \), real generalized Spin\(^c\) Killing spinor do not exist, i.e., they are already real Spin\(^c\) Killing spinor. In dimension 2 and 3, they constructed explicit examples of Spin\(^c\) manifolds carrying real generalized Killing spinor, i.e., where the real Killing function is not constant.

We recall also that the existence of parallel spinors, real Killing spinors and imaginary Killing spinors do not only give obstruction of the geometry and the topology of the Spin or Spin\(^c\) manifold \((M^n, g)\) itself, but also, the geometry and topology of hypersurfaces and submanifolds of \((M^n, g)\). In fact, The restriction of such Spin or Spin\(^c\) spinors is an effective tool to study the geometry and the topology of submanifolds \([1, 2, 4, 28, 27, 31, 30, 29, 41, 40]\).

In this paper, we extend the study of Equation (1) on Riemannian Spin\(^c\) manifolds. After giving some preliminaries of Spin\(^c\) structures in Section \([2]\) we consider general properties of complex generalized Killing spinors in Section \([3]\) and prove

**Theorem 1.1.** Let \((M^n, g)\) be a connected Riemannian Spin\(^c\) manifold of dimension \( n \geq 4 \), carrying a complex generalized Killing spinor \( \psi \) with Killing function \( K = a + ib \), \( a, b \in C^\infty(M, \mathbb{R}) \). Then, \( a \) or \( b \) vanishes identically on \( M \). In other words, \( \psi \) is already a real generalized Killing spinor with Killing function \( a \) (and hence \( a \) is constant, i.e, \( \psi \) is a real Killing spinor) or \( \psi \) is an imaginary generalized Killing spinor with Killing function \( ib \) (\( b \) is constant or a function).

Proof of Theorem 1.1 is based on the existence of differential forms which are naturally associated to the complex generalized Killing spinor field. For \( n = 2, 3 \) we still do not know whether there are complex generalized Killing spinors which are neither purely real or purely imaginary. But at least we know that \( a \) has to vanish in all points where \( b \) does not and vice versa, cf. Lemma \([5, 4]\). Thus, in case they exist they would be very artificial, cf. Remark \([3, 5]\). In dimension...
The bundle $L$ can be viewed as a real valued 2-form on $\Sigma$ denoted by $L$ compatible with the group covering $\pi$. The Killing function exist – but only in dimension $\geq 4$. In contrast to the $\text{Spin}^c$ case, see Theorem 4.1 and Proposition 4.3. But in contrast to the $\text{Spin}$ case, type II imaginary generalized Killing spinors with non-constant Killing function exist – but only in dimension 2, cf. Proposition 4.5 and Theorem 4.6.

2. Preliminaries

2.1. Conventions and general notations. Hermitian products $\langle .,\rangle$ are always anti-linear in the second component. If a vector is decorated with a hat, this vector is left out, e.g. $v_1, \ldots, \hat{v}_j, \ldots, v_n$ is meant to be $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$. The space of smooth sections of a bundle $E$ is denoted by $\Gamma(E)$.

2.2. Spin$^c$ structures on manifolds. We consider an oriented Riemannian manifold $(M^n, g)$ of dimension $n \geq 2$ without boundary and denote by $\text{SO}(M)$ the $\text{SO}_n$-principal bundle over $M$ of positively oriented orthonormal frames. A Spin$^c$ structure of $M$ is given by an $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1 M, \pi, M)$ of some Hermitian line bundle $L$ and a Spin$^c$-principal bundle $(\text{Spin}^c M, \pi, M)$ which is a 2-fold covering of the $\text{SO}_n \times \mathbb{S}^1$-principal bundle $\text{SO}(M) \times_M \mathbb{S}^1 M$ compatible with the group covering

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^c \longrightarrow \text{SO}_n \times \mathbb{S}^1 \longrightarrow 0.$$

The bundle $L$ is called the auxiliary line bundle associated with the Spin$^c$ structure. If $A : T(\mathbb{S}^1 M) \longrightarrow i\mathbb{R}$ is a connection 1-form on $\mathbb{S}^1 M$, its (imaginary-valued) curvature will be denoted by $F_A$, whereas we shall define a real 2-form $\Omega$ on $\mathbb{S}^1 M$ by $F_A = i\Omega$. We know that $\Omega$ can be viewed as a real valued 2-form on $M$ [18, 34]. In this case, $i\Omega$ is the curvature form of the auxiliary line bundle $L$ [18, 34].

Let $\Sigma := \text{Spin}^c M \times_{\rho_n} \Sigma_n$ be the associated spinor bundle where $\Sigma_n = \mathbb{C}^{2^n}$ and $\rho_n : \text{Spin}^c \longrightarrow \text{End}(\Sigma_n)$ the complex spinor representation [18, 35]. A section of $\Sigma$ will be called a spinor field. This complex vector bundle is naturally endowed with a Clifford multiplication, denoted by $\langle ., \cdot \rangle : C(\mathbb{T}M) \longrightarrow \text{End}(\Sigma M)$ which is a fiber preserving algebra morphism, and with a natural Hermitian scalar product $\langle ., . \rangle$ compatible with this Clifford multiplication [18, 26]. If such data are given, one can canonically define a covariant derivative $\nabla$ on $\Sigma M$ that is locally given by [18, 26, 39]:

$$\nabla_X \psi = X(\psi) + \frac{i}{4} \sum_{j=1}^n e_j \cdot \nabla_X e_j \cdot \psi + \frac{i}{2} A(s_+(X)) \psi,$$

where $X \in \Gamma(\mathbb{T}M)$, $\nabla_X$ is the Levi-Civita connection on $M$, $\psi = [b \times s, \sigma]$ is a locally defined spinor field, $b = (e_1, \ldots, e_n)$ is a local oriented orthonormal tangent frame over an open set $U \subset M$, $s : U \longrightarrow \mathbb{S}^1 M|_U$ is a local section of $\mathbb{S}^1 M$, $\tilde{b} \times s$ is the lift of the local section $b \times s : U \longrightarrow \text{SO}(M) \times_M \mathbb{S}^1 M|_U$ to the 2-fold covering and $X(\psi) = [\tilde{b} \times s, X(\sigma)]$.

The Dirac operator, acting on $\Gamma(\Sigma M)$, is a first order elliptic operator locally given by $D = \sum_{j=1}^n e_j \cdot \nabla e_j$, where $\{e_j\}_{j=1, \ldots, n}$ is any local orthonormal frame on $M$. An important tool when examining the Dirac operator on Spin$^c$ manifolds is the Schrödinger-Lichnerowicz formula.
Spin are not in general to defined on the trivial auxiliary line bundle is flat, then auxiliary bundle trivial auxiliary line bundle endowed with the trivial connection. Of course, Spin Furthermore, it is well known that a \[38, \text{Lemma 2}\] while the spinor bundle and product and \(e^k\) is the extension of the Clifford multiplication to differential forms. The Ricci identity is given, for all \(X \in \Gamma(TM)\), by

\[
\sum_{j=1}^{n} e_j \cdot \mathcal{R}(e_j, X)\psi = \frac{1}{2}\mathrm{Ric}(X) \cdot \psi - \frac{i}{2}(X \lrcorner \Omega) \cdot \psi,
\]

for any spinor field \(\psi\). Here, \(\mathrm{Ric}\) (resp. \(\mathcal{R}\)) denotes the Ricci tensor of \(M\) (resp. the Spin\(^c\) curvature associated with the connection \(\nabla\)), and \(\lrcorner\) is the interior product.

Let \(\omega_\mathbb{C} = i^{\frac{n(n+1)}{2}}e_1 \wedge \ldots \wedge e_n\) be the complexified volume element. The Clifford multiplication extends to differential forms so \(\omega_\mathbb{C}\) can act on spinors. If \(n\) is odd, the volume element \(\omega_\mathbb{C}\) acts as the identity on the spinor bundle. If \(n\) is even, \(\omega_\mathbb{C}^2 = 1\). Thus, by the action of the complex volume element on the spinor bundle decomposes into the eigenspaces \(\Sigma^\pm M\) corresponding to the \(\pm 1\) eigenspaces, the positive (resp. negative) spinors \([18, 26, 39]\). If \(\psi = \psi_+ + \psi_-\) for \(\psi_+ \in \Gamma(\Sigma^+ M)\), we set \(\bar{\psi} = \psi_+ - \psi_-\). Summarizing the action of the volume form we have

\[
\omega_\mathbb{C} \cdot \psi = \begin{cases} 
\bar{\psi} & \text{for } n \text{ even} \\
\psi & \text{for } n \text{ odd}.
\end{cases}
\]

Moreover, we recall that by direct calculation one sees immediately that

\[
\langle \delta \cdot \psi, \psi \rangle = (-1)^{\frac{k(k+1)}{2}} \langle \delta \cdot \bar{\psi}, \bar{\psi} \rangle
\]

for a \(k\)-form \(\delta\) and a spinor field \(\psi\).

Furthermore, it is well known that a Spin structure can be viewed as a Spin\(^c\) structure with a trivial auxiliary line bundle endowed with the trivial connection. Of course, Spin\(^c\) manifolds are not in general Spin manifolds – e.g. the complex projective space \(\mathbb{C}P^2\) is Spin\(^c\) but not Spin. However, a Spin\(^c\) structure on a simply connected Riemannian manifold \(M\) with trivial auxiliary bundle \(L\) is canonically identified with a Spin structure. Moreover, if the connection defined on the trivial auxiliary line bundle is flat, then \(\nabla\) on the Spin\(^c\) bundle \(\Sigma M\) corresponds to \(\nabla'\) on the Spin bundle \(\Sigma' M\), i.e., we have a global section on \(L\) which can be chosen parallel \([38\text{ Lemma 2.1]}\). In this case, (2) becomes

\[
\nabla_X' \psi = \nabla_X \psi = X(\psi) + \frac{1}{4} \sum_{j=1}^{n} e_j \cdot \nabla_X e_j \cdot \psi.
\]

Let \((M^n, g)\) be a Riemannian Spin\(^c\) manifold with Spin\(^c\) bundle \(\Sigma M\) and auxiliary bundle \(L\). Let now \((M^n, g)\) be equipped with another Spin\(^c\) structure, and let \(\Sigma' M\) (resp. \(L'\)) the corresponding Spin\(^c\) bundle (resp. auxiliary bundle). Then there is always a complex line bundle \(D\) such that \(\Sigma' M = \Sigma M \otimes D\) and \(L' = L \otimes D^2\). In particular, if \((M, g)\) is Spin and \(\Sigma' M\) denotes its spinor bundle and \(L'\) the trivial line bundle. Then, \(D^2 = L^{-1}\). Thus \(\Sigma M = \Sigma' M \otimes L^\perp\). Even if \(M\) is not Spin this is still true locally. This essentially means that, while the spinor bundle and \(L^\perp\) may not exist globally, their tensor product (the Spin\(^c\) bundle) can be defined globally.
2.3. Conformal Killing vector fields. We denote by \( \mathcal{L}_V g \) the Lie derivative of the metric \( g \) in direction of the vector field \( V \). A vector field \( V \) is a conformal Killing field if \( \mathcal{L}_V g = 2h g \) for a smooth real function \( h \). By taking traces one obtains \( \text{div} V = nh \). \( V \) is homothetic if \( h \) is a constant, and it is isometric if \( \mathcal{L}_V g = 0 \). Moreover, \( V \) is called closed if the corresponding 1-form \( w = g(V, .) \) is closed. H.-B. Rademacher proved that

**Theorem 2.1.** ([43] Theorem 2) Let \((M^n, g)\) be a complete Riemannian manifold with a non-isometric conformal closed Killing vector field \( V \), and let \( N \) be the number of zeros of \( V \). Then \( N \geq 4 \) and:

1. If \( N = 2 \), \( M \) is conformally diffeomorphic to the standard sphere \( S^n \).
2. If \( N = 1 \), \( M \) is conformally diffeomorphic to the Euclidean space \( \mathbb{R}^n \).
3. If \( N = 0 \), there exists a complete \((n-1)\)-dimensional Riemannian manifold \((F, g_F)\) and a smooth function \( h : \mathbb{R} \rightarrow \mathbb{R}^* \) such that the warped product \( F \times_h \mathbb{R} \) is a Riemannian covering of \( M \), and the lift of \( V \) is \( h \frac{\partial}{\partial t} \) where \( t \) denotes the coordinate of the \( \mathbb{R} \)-factor.

3. Spin\(^c\) Complex Generalized Killing spinors

In this section, we want to establish general properties of complex generalized Killing spinors, i.e. a spinor satisfying (1). In particular, we will show that in dimension \( n \geq 4 \), the Killing function \( K \) is already purely real or purely imaginary. First, let us collect some general facts on Killing spinors.

**Lemma 3.1.** Let \( \psi \) be a complex generalized Killing spinor on a Riemannian Spin\(^c\) manifold \((M^n, g)\). Let \( K = a + ib \) be the corresponding Killing function. Then

1. \( n(n-1)K^2\psi - (n-1)dK \cdot \psi = \frac{1}{2}S\psi + \frac{1}{2}\Omega \cdot \psi \)
2. \( \langle da \cdot \psi, \psi \rangle = 2n ab |\psi|^2 \)
3. \( \frac{1}{2}(\text{Ric}(X) - iX \cdot \Omega) \cdot \psi = \nabla K \cdot X \cdot \psi + nX(K)\psi + 2(n-1)K^2 X \cdot \psi \)
4. \( \psi \) has no zeros.

**Proof.** All the calculations will be carried out at a point \( x \in M \) using a local orthonormal frame \( e_i \mid [e_i, e_j] = 0 \) and \( \nabla e_i e_j = 0 \) at \( x \). We calculate \( D\psi = \sum e_i \cdot \nabla e_i \psi = K \sum e_i \cdot e_i \cdot \psi = -nK\psi \).

Thus,

\[
D^2\psi = -ndK + n^2K^2\psi.
\]

Moreover, using \( \nabla^* \nabla = -\sum_{j=1}^n \nabla e_j \nabla e_j \), we deduce that

\[
\nabla^* \nabla \psi = -dK \cdot \psi + nK^2\psi.
\]

Using the last two equations and the Schrödinger Lichnerowicz formula (3), we obtain (1). Then, taking the imaginary part of the scalar product of (1) \( \psi \), we get that \( \langle da \cdot \psi, \psi \rangle = 2n ab |\psi|^2 \). For the last two claims the corresponding proofs for \( K \) real, i.e.,\([23] \) Lemma 2.2 for (iii) and\([36] \) Proposition 1 for (iv), carry over directly.

Let \( \omega_p \) be the \( p \)-form on \( M \) given by

\[
w_p(X_1, \ldots, X_p) := \langle (X_1 \wedge X_2 \wedge \cdots \wedge X_p) \cdot \psi, \psi \rangle,
\]

for any \( X_1, X_2, \ldots, X_p \in \Gamma(TM) \). These \( p \)-forms have been introduced in\([23] \) for real generalized Killing Spin\(^c\) spinors. In this case, the vector field \( V := i\omega_1^* \) is a Killing vector field. We point out that for complex generalized Killing Spin\(^c\) spinors, this is not the case, i.e., \( \xi \) is not necessary a Killing vector field, cp. Section 4.

**Lemma 3.2.** The forms \( \omega_{4k+1} \) and \( \omega_{4k+2} \) are imaginary-valued, but \( \omega_{4k+3} \) and \( \omega_{4k} \) are real-valued forms for all \( k \geq 0 \). Moreover, we have for all \( p \geq 0 \)

\[
d\omega_p = (K(-1)^p - K)\omega_{p+1}.
\]
In particular, for any \( k \geq 0 \),
\[
\begin{align*}
  db \wedge \omega_{2k+1} - 2ab\omega_{2k+2} &= 0, \\
  da \wedge \omega_{2k+2} + 2abi\omega_{2k+3} &= 0.
\end{align*}
\] (7) (8)

**Proof.** By (5), \( \omega_{4k+1} \) and \( \omega_{4k+2} \) are imaginary-valued; \( \omega_{4k+3} \) and \( \omega_{4k+4} \) are real-valued forms. We consider a local orthonormal frame \( \{ e_i, \ldots, e_n \} \) in a neighbourhood of \( x \in M \) such that \( [e_i, e_j] = 0 \) and \( \nabla_{e_i} e_j = 0 \) at \( x \). Then
\[
(p + 1)\omega_p(e_1, \ldots, e_{p+1})
\]
\[
= \sum_{i=1}^{p+1} (-1)^{i-1} e_i \left( \omega_p(e_1, \ldots, \hat{e}_i, \ldots, e_{p+1}) \right) + \sum_{i<j} (-1)^{i+j} \omega_p([e_i, e_j], e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{p+1})
\]
\[
= \sum_{i=1}^{p+1} (-1)^{i-1} \left[ \nabla_{e_i} e_1 \cdots \cdot \hat{e}_i \cdots \cdot e_{p+1} \cdot \psi, \psi \right] + \left( e_1 \cdots \cdot \hat{e}_i \cdots \cdot e_{p+1} \cdot \psi, \nabla_{e_i} \psi \right)
\]
\[
= \sum_{i=1}^{p+1} (-1)^{i-1} \left[ e_1 \cdots \cdot \hat{e}_i \cdots \cdot e_{p+1} \cdot \nabla_{e_i} \psi, \psi \right] + \left( e_1 \cdots \cdot \hat{e}_i \cdots \cdot e_{p+1} \cdot K e_i \cdot \psi \right)
\]
\[
= \sum_{i=1}^{p+1} (-1)^{i-1} \left[ e_1 \cdots \cdot \hat{e}_i \cdots \cdot e_{p+1} \cdot (K e_i \cdot \psi), \psi \right] - \nabla(e_1 \cdots \cdot \hat{e}_i \cdots \cdot e_{p+1} \cdot \psi, \psi)
\]
\[
= (p + 1)(K(-1)^p - K)\omega_{p+1}.
\]
Thus, we get
\[
\begin{align*}
  d\omega_2 &= 2ib\omega_{2k+1}, \\
  d\omega_{2k+1} &= -2abi\omega_{2k+2}.
\end{align*}
\]

After taking the differential of the last two equalities, we obtain \( db \wedge \omega_{2k+1} - 2ab\omega_{2k+2} = 0 \) and \( da \wedge \omega_{2k+2} + 2abi\omega_{2k+3} = 0 \). \( \square \)

If \( M \) is even dimensional, we can use the decomposition of the spinor bundle, see (4) and above, to define another sequence of \( p \)-forms on \( M \) by
\[
\overline{\omega}_p(X_1, \ldots, X_p) := \langle X_1 \cdot X_2 \cdots \cdot X_p \cdot \psi, \overline{\psi} \rangle
\]
for \( X_1, X_2, \ldots, X_p \in \Gamma(TM) \).

**Lemma 3.3.** If \( n \) is even, the \( p \)-form \( \overline{\omega}_p \) satisfies
\[
d\overline{\omega}_p = (K(-1)^p + K)\overline{\omega}_{p+1}.
\]

In particular, for any \( k \geq 0 \),
\[
\begin{align*}
  db \wedge \overline{\omega}_{2k+2} + 2abi\overline{\omega}_{2k+3} &= 0, \\
  da \wedge \overline{\omega}_{2k+1} - 2abi\overline{\omega}_{2k+2} &= 0.
\end{align*}
\] (9)

**Proof.** For \( X \in \Gamma(TM) \) the Clifford multiplication \( X \cdot \) is a map from \( \Gamma(\Sigma \pm M) \) to \( \Gamma(\Sigma \mp M) \). Thus, \( \nabla_X \psi = -KX \cdot \psi \). Now we can proceed as in Lemma 3.2 and obtain
\[
d\overline{\omega}_p(e_1, \ldots, e_{p+1}) = (K(-1)^p + K)\overline{\omega}_{p+1}.
\]

Thus, we get
\[
\begin{align*}
  d\overline{\omega}_2 &= 2abi\overline{\omega}_{2k+1}, \\
  d\overline{\omega}_{2k+1} &= -2abi\overline{\omega}_{2k+2}.
\end{align*}
\]

Taking the differential, we obtain \( db \wedge \overline{\omega}_{2k+2} + 2abi\overline{\omega}_{2k+3} = 0 \) and \( da \wedge \overline{\omega}_{2k+1} - 2abi\overline{\omega}_{2k+2} = 0 \). \( \square \)
Lemma 3.4. Let \((M^n, g)\) be a Riemannian \(\mathsf{Spin}^c\) manifold carrying a complex generalized Killing spinor with Killing function \(K = a + ib\). Then \(ab = 0\).

**Proof.** Let \(\psi\) denote the Killing spinor, and let \(e_1, e_2, \ldots, e_n\) be a local orthonormal frame of \(TM\). Firstly, assume that \(n\) is odd and set \(k = \frac{n-1}{2}\). Equality (8) for \(k\) implies that

\[
d\alpha \wedge \omega_{n-1} = -2abi\omega_n.
\]

We calculate each term of this equation separately. First, we have

\[
(da \wedge \omega_{n-1})(e_1, e_2, \ldots, e_n) = \sum_{j=1}^{n} (-1)^{j+1} \alpha(e_j) \omega_{n-1}(e_1, e_2, \ldots, \hat{e}_j, \ldots, e_n)
\]

\[
= \sum_{j=1}^{n} (-1)^{j+1} \alpha(e_j) \langle e_1 \cdot e_2 \cdots \cdot \hat{e}_j \cdots \cdot e_n \cdot \psi, \psi \rangle.
\]

Using (4), we get

\[
(da \wedge \omega_{n-1})(e_1, e_2, \ldots, e_n) = \sum_{j=1}^{n} (-1)^{j+1} (-1)^{-j} i^{-\frac{n+1}{2}} \alpha(e_j) \langle e_j \cdot \psi, \psi \rangle
\]

\[
= -i^{-\frac{n+1}{2}} \sum_{j=1}^{n} \alpha(e_j) \langle e_j \cdot \psi, \psi \rangle
\]

\[
= -i^{-\frac{n+1}{2}} \langle da \cdot \psi, \psi \rangle.
\]

On the other hand

\[
-2abi\omega_n(e_1, e_2, \ldots, e_n) = -2abi \langle e_1 \cdot e_2 \cdots \cdot e_n \cdot \psi, \psi \rangle = -2abii^{-\frac{n+1}{2}}|\psi|^2.
\]

Thus,

\[
-2abii^{-\frac{n+1}{2}}|\psi|^2 = -i^{-\frac{n+1}{2}} \langle da \cdot \psi, \psi \rangle.
\]

Together with Lemma 3.1(ii) and 3.1(iv), we obtain that \(2abi = 2nabi\). Hence, \(ab = 0\).

It remains the case that \(n\) is even. Then, (9) for \(k = \frac{n-2}{2}\) implies \(da \wedge \omega_{n-1} = 2abi\omega_n\), and an analogous calculation as in the first case gives again \(ab = 0\). \(\square\)

Now, we are able to prove Theorem 1.1.

**Proof of Theorem 1.1** We prove the claim by contradiction, i.e. let \(\psi\) be a Killing spinor to a Killing function \(a + ib\) where not both \(a\) and \(b\) vanish identically. Set \(\Omega := \{ x \in M \mid b(x) = 0 \}\). Then, \(\psi|_{\Omega}\) is a real generalized Killing spinor to the Killing function \(a|_{\Omega} \neq 0\). For \(n \geq 4\), this implies that \(a\) has to be constant on \(\Omega\) [23, Theorem 1.1]. But by Lemma 3.4, we know that \(ab = 0\). Thus, \(a|_{M \setminus \Omega} = 0\) which gives a contradiction to the smoothness of \(a\). \(\square\)

**Remark 3.5.** We conjecture that complex generalized Killing \(\mathsf{Spin}^c\) spinors also do not exist in dimension 2 and 3. Even, if this turns out to be wrong, these examples are very artificial: The manifold \(M\) consists of two closed subsets \(M_1\) and \(M_2\) where \(\psi|_{M_1}\) is a real generalized Killing spinor on \(M_1\) to the Killing function \(a\) and \(\psi|_{M_2}\) is a real generalized Killing spinor on \(M_1\) to the Killing function \(ib\). In particular, on \(M_1 \cap M_2\), we have \(a = b = 0\) and everything has to built such that it is smooth also over this “boundary” set. For the imaginary spinor part on \(M_2\), this is clearly possible when taking e.g. a warped product as in Theorem 4.1 by choosing \(k(t)\) carefully. But whether one can choose the real part such that the spinor has a good well-behaved zero set is still unclear.
4. Spin$^c$ Imaginary Generalized Killing spinors

On a Riemannian Spin$^c$ manifold $(M^n, g)$, we consider an imaginary generalized Killing spinor $\psi$ with Killing function $ib$, where $b$ is a smooth real function that is not identically zero on $M$. Let $f := |\psi|^2$. Moreover, define the vector field $V$ by

$$g(V, X) = i\langle X \cdot \psi, \psi \rangle \quad \text{for all} \quad X \in \Gamma(TM).$$

(10)

As in the Spin case we get by direct computation, [43] Section 3

$$\nabla f = 2bV, \quad \nabla_X V = 2bfX, \quad \mathcal{L}_V g = 4bfg,$$

(11)

for all $X \in \Gamma(TM)$. Hence, the vector field $V$ is a non-isometric conformal closed Killing vector field, [43] Section 2 and cf. Paragraph 2.3. Moreover, the function $q_\psi := f^2 - \|V\|^2$ is non-negative constant and

$$\frac{1}{f} V \cdot \psi = -i\psi \quad \text{for} \quad q_\psi = 0.$$

(12)

The proof of this follows exactly the one in the Spin case [6, Lemma 5 and below]. The spinor field $\psi$ is called of type I (resp. II) if $q_\psi = 0$ (resp. $q_\psi > 0$).

4.1. Imaginary Generalized Killing spinors of type I. We start with the type I imaginary generalized Killing spinors. It turns out that one only obtains the obvious generalization of the corresponding Spin result [43, Theorem 1a].

**Theorem 4.1.** Let $(M^n, g)$ be a complete connected Riemannian Spin$^c$ manifold admitting a imaginary generalized Killing spinor of type I with Killing function $ib$, $b \in C^\infty(M, \mathbb{R})$. Then, a Riemannian covering of $M$ is isometric to the warped product $F \times_k \mathbb{R} = (F^{n-1} \times \mathbb{R}, k(t)^2 \delta + dt^2)$, where $(F^{n-1}, h)$ is a complete Riemannian Spin$^c$ manifold admitting a non-zero parallel spinor field, and $k$ is a function on $t$. In particular, $f(t, x) = k(t)$ is also a function on $t$ alone and $b = \frac{f'}{f}$. Moreover, every manifold that fulfills these conditions admits a imaginary generalized Killing spinor of type I.

**Proof.** The proof is analogous to the ordinary Spin case: For a type I imaginary Killing spinor $\psi$, $q_\psi = 0$ and hence, $\|V\| = f = |\psi|^2$. Then by Lemma 3.1(iv) $V$ has no zeros. By Theorem 2.1 a Riemannian covering of $M$ is the warped product of a complete Riemannian manifold $(F^{n-1}, h)$ and $(\mathbb{R}, dt^2)$, warped by a positive smooth function $k(t)$, i.e. $(F \times_k \mathbb{R}, k^2(t)h + dt^2)$. The lift of $V$ to this covering is given by $k \frac{\partial}{\partial t}$. Then, $k(t) = \|V\| = f$. Thus, using (11), we get

$$f' \frac{\partial}{\partial t} = \nabla f = 2bV = 2bf \frac{\partial}{\partial t}.$$

Hence, $b = \frac{f'}{2f}$. Moreover, the manifold $F_t := F \times_f \{t\} = (F, f(t)h)$ can be viewed as a hypersurface of $M$ whose mean curvature with respect to the unit normal vector field $\partial_t = \frac{1}{f}V$ is given by $-\frac{f'}{f}$ [5, Example 4.2]. Hence, $F := F_0$ carries an induced Spin$^c$ structure [39]. Using (12) and the Spin$^c$ Gauss formula [39] Proposition 3.3, we calculate for $\varphi = |\psi|_F$

$$\nabla^F_X \varphi = (\nabla_X \psi)|_F + \frac{f'}{2f} X \cdot \partial_t \cdot \psi|_F = ibX \cdot \psi|_F - ibX \cdot \psi|_F = 0,$$

where $\nabla^F$ is the Spin$^c$ connection on $F$. This gives a parallel spinor field $\varphi$ on $F$.

For the converse, let $\varphi$ be a nonzero parallel spinor on $(F^{n-1}, h)$. By parallel transport of $\varphi$ in $t$-direction we get $\varphi(t, x)$. Firstly assume that $n$ is odd, i.e., $n = 2m + 1$. Then, we can assume that w.l.o.g. that $\varphi$ is in one of the $S^F_k$ such that $\partial_t \cdot \varphi = (-1)^m i \varphi$ where $\varphi$ is now seen
as a spinor in $S_M$, cp. [6] Lemma 4. Set $\psi(t, x) = \eta(t)\varphi(t, x)$ with $\eta(t) = e^{-\int_0^t \frac{k'(s)}{2k} ds}$ and $b = (-1)^m \frac{k'}{2k}$. Then for $X \in \Gamma(TM)$ with $X \perp \partial_t$ we get

$$\nabla_X \psi = \eta \nabla_X \varphi - \frac{k'}{2k} X \cdot \partial_t \cdot \varphi = ibX \cdot \psi.$$  

Moreover, $\nabla_{\partial_t} \psi = \eta' \varphi = -i(-1)^m \eta' \partial_t \cdot \varphi = i(-1)^m \frac{k'}{2k} \eta \partial_t \cdot \varphi = ib \partial_t \cdot \psi$. Thus, $\psi$ is a Killing spinor to Killing function $b$. Moreover, $\|V\| = |g(V, \partial_t)| = |i(\partial_t \cdot \psi, \psi)| = |\psi|^2 = f$, thus, $\psi$ is of type I. Similar we obtain the Killing spinor when $n$ is even: As in [6] Lemma 4 $\tilde{\varphi} = \varphi \oplus \varphi$ can be seen as a spinor in $S_M$ with $\partial_t \cdot \tilde{\varphi} = (-1)^m i\tilde{\varphi} \neq 0$ and $n = 2m + 2$.

Set $b = (-1)^m \frac{k'}{2k}$. Then for $X \in \Gamma(TM)$ with $X \perp \partial_t$ we get $\nabla_X \psi = -\eta \frac{k'}{2k} X \cdot \partial_t \cdot \varphi = ibX \cdot \psi$ and $\nabla_{\partial_t} \psi = \eta' \varphi = ib \partial_t \cdot \psi$. Thus, $\psi$ is a Killing spinor to Killing function $b$. Moreover, $\|V\| = |g(V, \partial_t)| = |i(\partial_t \cdot \psi, \psi)| = |\psi|^2 = f$, thus, $\psi$ is of type I.

\begin{corollary}
Let $(M^n, g)$ be a complete connected Riemannian Spin$^c$ manifold admitting an imaginary Killing spinor of Killing number $i\mu$, $\mu \in \mathbb{R}$. If $\psi$ is of type I, a Riemannian covering of $M$ is isometric to the warped product $(F^{n-1} \times \mathbb{R}, e^{4\mu t}h \oplus dt^2)$, where $(F^{n-1}, h)$ is a complete Spin$^c$ manifold with a non-zero parallel spinor.
\end{corollary}

\begin{proof}
By Theorem [4] it only remains to determine $f$. As above we have, $f' = 2\mu f$. Thus, $f = ae^{2\mu t}$ for a positive constant $a$. By rescaling the metric $h$, we can assume that $a = 1$.
\end{proof}

4.2. Imaginary Generalized Killing spinor of type II. Next we study type II generalized imaginary Killing spinors to the Killing function $ib$. We will distinguish two cases:

4.2.1. $b$ is constant. Then it turns out that $M$ is already Spin$^c$.

\begin{proposition}
Let $(M^n, g)$ be a complete connected Riemannian Spin$^c$ manifold with an imaginary Killing spinor $\psi$ of Killing number $i\mu$, $\mu \in \mathbb{R} \setminus \{0\}$. If $\psi$ is of type II, $(M^n, g)$ is isometric to the hyperbolic space $\mathbb{H}^n(-4\mu^2)$ endowed with its trivial Spin$^c$ structure, i.e., its unique Spin$^c$ structure.
\end{proposition}

\begin{proof}
Let $\psi$ be of type II, i.e., $q_\psi > 0$. First we assume that $f = |\psi|^2$ has no critical points, then, by (11) the number of zeros of $V$ is 0. From Theorem [2] we obtain that a Riemannian covering of $M$ is isometric to the warped product $F \times_k \mathbb{R}$ where $k(t)$ is a function on $t$ alone, $F$ a complete Riemannian manifold and the lift of $V$ is $k \frac{\partial}{\partial t}$. Then again with (11) we obtain that $f$ also just depends on $t$ and $f' \frac{\partial}{\partial t} = \nabla f = 2\mu V$. Thus, $f' = 2\mu k$ and $f'' \frac{\partial}{\partial t} = 2\mu \nabla_{\partial_t} V = 4\mu^2 f \frac{\partial}{\partial t}$.

Hence, $f = Ae^{2\mu t} + Be^{-2\mu t}$ for constants $A, B$. Since $V$ and, hence, $f'$ has no zeros, $f' = 2\mu k$ and since $k$ and $f$ are everywhere positive, we obtain $f = Ae^{2\mu t}$, $A > 0$, and $k = f$. Hence, $k = \|V\| = f$ and $q_\psi = 0$ which gives a contradiction.

Hence, $f$ has critical points. Using (11) we obtain for $X, Y \in \Gamma(TM)$

$$\text{Hess} f(X, Y) = g(\nabla_X \nabla f, Y) = 2\mu g(\nabla_X Y, V) = 4\mu^2 g(X, Y)f.$$  

By [33] Theorem C), $M$ is isometric to the simply connected complete Riemannian manifold $(\mathbb{H}^n, (2|\mu|)^{-1}g_\mathbb{H})$ of constant curvature $-4\mu^2$. Since $\mathbb{H}^n$ is contractible, $\mathbb{H}^n$ admits only one Spin$^c$ structure – the canonical one coming from the Spin structure.

By Lemma [3,1][ii] we obtain

$$\text{Ric}(X) \cdot \psi - i(X \lrcorner \Omega) \cdot \psi = -4(n - 1)\mu^2 X \cdot \psi$$

for all $X \in \Gamma(TM)$. Since the Ricci tensor of $M$ is given by $\text{Ric} = -4(n - 1)\mu^2$, we obtain $(X \lrcorner \Omega) \cdot \psi = 0$ for all $X \in \Gamma(TM)$ and, hence, if $\Omega = 0$.  


Thus, the Spin\(^c\) structure is identified with the unique Spin structure on \(\mathbb{H}^n\). Here we recall that, on \(\mathbb{H}^n(-4\mu^2)\) endowed with its unique Spin structure, imaginary Killing spinors of Killing number \(i\mu\) and \(-i\mu\) form an orthogonal basis of \(\mathbb{H}^n\) with respect to the Hermitian scalar product defined on \(\Sigma M\) \(^6\). \(\square\)

4.2.2. \(b\) is not constant. On Spin manifolds, H.-B. Rademacher proved that there are no imaginary generalized Killing spinors of type II where \(b\) is non-constant. For dimension \(n \geq 3\), this will be still true for Spin\(^c\) manifolds. In contrast, in dimension 2 such spinors exist. In order to carry out the case of generalized Spin\(^c\) Killing spinors, we need the following auxiliary lemma.

**Lemma 4.4.** Let \(\psi\) be a generalized Killing spinor to the Killing function \(ib, b \in C^\infty(M, \mathbb{R})\). Then, in all points of \(M\) where \(\nabla b \neq 0\)

\[
\langle X \cdot \psi, \psi \rangle = 0 \quad \text{for all } X \perp \nabla b.
\]

**Proof.** From \((7)\), we have that \(db \wedge \omega_1 = 0\). Let \(X \perp \nabla b\). Then

\[
0 = \langle db \wedge \omega_1, \nabla b, X \rangle = |\nabla b|^2 \omega_1(X) = |\nabla b|^2 \langle X \cdot \psi, \psi \rangle.
\]

Thus, \(\langle X \cdot \psi, \psi \rangle = 0\). \(\square\)

**Proposition 4.5.** Let \((M^n, g)\) be a complete connected Riemannian Spin\(^c\) manifold of dimension \(n \geq 3\). Then, every imaginary generalized Killing spinor of type II is already an imaginary Killing spinor.

**Proof.** We prove the claim by contradiction and assume that there is a Killing spinor \(\psi\) to a non-constant Killing function \(ib, b \in C^\infty(M, \mathbb{R})\). Then, there is a point \(x \in M\) where \(\nabla b\) is non-zero. In the following, we will always identify \(\nabla b\) and \(db\) using the metric \(g\). Then, \(db\) is non-zero in a neighbourhood \(U\) of \(x\), and one can find a local orthonormal frame \((e_1, \ldots, e_{n-1}, \frac{db}{|db|})\) of \(TU\). Then, by Lemma 4.4 \(\langle e_i \cdot \psi, \psi \rangle = 0\) for all \(1 \leq i \leq n - 1\) which will used in following without any further comment. In particular, this implies that the conformal Killing field \(V\) (cf. \((10)\)) is parallel to \(db\) and \(|V| = g(V, \frac{db}{|db|}) = -i \langle \frac{db}{|db|} \cdot \psi, \psi \rangle\). By Theorem 2.1 \(V\) has at most two zeros. Hence, there is an \(y \in U\) where \(\langle db \cdot \psi, \psi \rangle \neq 0\). The following calculations will be carried out at this point \(y\).

Take now \(e_i\), Clifford multiplied with the Lichnerowicz identity in Lemma 3.1(i) and its scalar product with \(\psi\):

\[
-n(n-1)b^2 \langle e_i \cdot \psi, \psi \rangle - i(n-1) \langle e_i \cdot db \cdot \psi, \psi \rangle = \frac{S}{4} \langle e_i \cdot \psi, \psi \rangle + \frac{i}{2} \langle e_i \cdot \Omega \cdot \psi, \psi \rangle.
\]

Taking the real part gives

\[
i(n-1) \langle e_i \cdot db \cdot \psi, \psi \rangle = \frac{1}{2} \Omega \left( e_i, \frac{db}{|db|} \right) \left\langle \frac{db}{|db|} \cdot \psi, \psi \right\rangle.
\]

On the other hand taking the scalar product of the Ricci identity in Lemma 3.1(iii) for \(X = e_i\) with \(\psi\) and using \(\langle e_j \cdot \psi, \psi \rangle = 0\) gives

\[
\frac{1}{2} \text{Ric} \left( e_i, \frac{db}{|db|} \right) \left\langle \frac{db}{|db|} \cdot \psi, \psi \right\rangle - \frac{i}{2} \Omega \left( e_i, \frac{db}{|db|} \right) \left\langle \frac{db}{|db|} \cdot \psi, \psi \right\rangle = i \langle db \cdot e_i \cdot \psi, \psi \rangle\).
\]

From the imaginary part of this equation we obtain

\[
\text{Ric} \left( e_i, \frac{db}{|db|} \right) \left\langle \frac{db}{|db|} \cdot \psi, \psi \right\rangle = 0 \quad \text{and, hence,} \quad \text{Ric} \left( e_i, \frac{db}{|db|} \right) = 0,
\]

\(14\).
Since \( n \geq 3 \), (13) and (15) imply
\[
\langle db \cdot e_i \cdot \psi, \psi \rangle = 0 \quad \text{and} \quad \Omega \left( e_i, \frac{db}{|db|} \right) = 0.
\] (16)

The Ricci identity in Lemma 3.1(iii) for \( X = \frac{db}{|db|} \) together with (16) and (14) gives
\[
\frac{1}{2} \text{Ric} \left( \frac{db}{|db|}, \frac{db}{|db|} \right) \cdot \psi = -i|db|\psi + ni|db|\psi - 2(n - 1)b^2 \frac{db}{|db|} \cdot \psi.
\]
In particular, \( \frac{db}{|db|} \cdot \psi \) is parallel to \( \psi \) and \( |V| \geq |g(V, \frac{db}{|db|})| = \left| \langle \frac{db}{|db|} \cdot \psi, \psi \rangle \right| = |\psi|^2 \). Hence, \( q_v \leq 0 \) which gives the contradiction. \( \square \)

We still have to carry out the 2-dimensional case.

**Theorem 4.6.** In dimension 2, there exists imaginary generalized Killing Spin\(^c\) spinors of type II with non-constant Killing function.

**Proof.** The proof is inspired by the construction of real generalized Killing Spin\(^c\) spinors in dimension 2, cf. [23, Theorem 2.5]. We consider the two-dimensional Euclidean space \( (\mathbb{R}^2, g_E = dx^2 + dy^2) \). Then \( \{\partial_x, \partial_y\} \) forms an orthonormal frame. We endow \( \mathbb{R}^2 \) with a conformal metric \( \tilde{g} \) on \( \mathbb{R}^2 \) by requiring the frame \( \{\tilde{\partial}_x := a\partial_x, \tilde{\partial}_y := a\partial_y\} \) be orthonormal. Let \( \nabla \) be the covariant derivative corresponding to \( \tilde{g} \). The function \( a \) will be specified later but depends only on \( x \). Then, \( [\tilde{\partial}_x, \tilde{\partial}_y] = a'\tilde{\partial}_y \). We denote by \( \tilde{\nabla} \) the Levi-Civita connection on \( (\mathbb{R}^2, \tilde{g}) \). Using the Koszul formula, one can check that
\[
\tilde{\nabla}_{\tilde{\partial}_x} \tilde{\partial}_x = 0 \quad \text{and} \quad \tilde{\nabla}_{\tilde{\partial}_x} \tilde{\partial}_y = -a' \tilde{\partial}_y.
\]
We denote by \( \Sigma \mathbb{R}^2 \) (resp. \( \tilde{\Sigma} \mathbb{R}^2 \)) the spinor bundle of \((\mathbb{R}^2, g_E)\) (resp. \((\mathbb{R}^2, \tilde{g})\)). By a slight abuse of notation, we denote the Clifford multiplication of \((\mathbb{R}^2, g)\) and \((\mathbb{R}^2, \tilde{g})\) by the same symbol \( \cdot \). Now, we consider the linear isomorphism of the tangent spaces of \( \mathbb{R}^2 \) w.r.t. the metrics \( g_E \) and \( \tilde{g} \) defined by \( \partial_x \mapsto \tilde{\partial}_x \) and \( \partial_y \mapsto \tilde{\partial}_y \). This map lifts to a fibrewise isometric isomorphism of the spinor bundles \( \Sigma \mathbb{R}^2 \to \tilde{\Sigma} \mathbb{R}^2 \), see [10]. Using this identification, let \( \tilde{\varphi}_+ \) denote the image of a positive parallel spinor \( \varphi_+ \) in \( \Sigma \mathbb{R}^2 \) with \( |\varphi_+| = 1 \). Note that \( \{\tilde{\varphi}_+, \tilde{\partial}_x \cdot \tilde{\varphi}_+\} \) forms an orthonormal basis of \( \Sigma \mathbb{R}^2 \). Let \( \varphi_- := \partial_y \cdot \varphi_+ \). Since \( i\partial_x \cdot \partial_y \cdot \varphi_+ = \varphi_+ \), see (4), we have \( \partial_y \cdot \varphi_\pm = i\varphi_\pm \). Using again the identification of the spinor bundles, we get \( \tilde{\varphi}_- = \tilde{\partial}_x \cdot \tilde{\varphi}_+ \) and \( \tilde{\partial}_y \cdot \tilde{\varphi}_\pm = i\tilde{\varphi}_\pm \). Together with (6), we then get
\[
\tilde{\nabla}_{\tilde{\partial}_x} \tilde{\varphi}_\pm = \frac{1}{2} \tilde{g}(\tilde{\nabla}_{\tilde{\partial}_x} \tilde{\partial}_x, \tilde{\partial}_y)\tilde{\partial}_x \cdot \tilde{\partial}_y \cdot \tilde{\varphi}_\pm = 0,
\]
and
\[
\tilde{\nabla}_{\tilde{\partial}_y} \tilde{\varphi}_\pm = \frac{1}{2} \tilde{g}(\tilde{\nabla}_{\tilde{\partial}_y} \tilde{\partial}_x, \tilde{\partial}_y)\tilde{\partial}_x \cdot \tilde{\partial}_y \cdot \tilde{\varphi}_\pm = -\frac{a'}{2} \tilde{\partial}_x \cdot \tilde{\partial}_y \cdot \tilde{\varphi}_\pm = \pm \frac{a'}{2} \tilde{\varphi}_\pm.
\]
For the Killing spinor on \( (\mathbb{R}^2, \tilde{g}) \) we make the following ansatz
\[
\varphi = -\cosh(c(x)) \tilde{\varphi}_+ + i \sinh(c(x)) \tilde{\varphi}_-.
\]
where \( c(x, y) = c(x) \), a real function depending only on \( x \), will again be specified later. We calculate

\[
\nabla_{\partial_x} \varphi = - c'(x) \sinh(c(x)) \bar{\varphi}_+ + i c'(x) \cosh(c(x)) \bar{\varphi}_- \\
= i c'(x) (\cosh(c(x)) \bar{\partial}_x \cdot \varphi_+ + i \sinh(c(x)) \bar{\partial}_x \cdot \varphi_-) \\
= - i c'(x) \bar{\partial}_x \cdot \varphi.
\]

Moreover,

\[
\nabla_{\partial_y} \varphi = - i \frac{a'}{2} \cosh(c(x)) \bar{\varphi}_+ + \frac{a'}{2} \sinh(c(x)) \bar{\varphi}_-.
\]

We now consider the trivial line bundle \( L \) on \( \mathbb{R}^2 \) with connection form given by an imaginary 1-form \( i \tilde{\alpha} \) satisfying \( \tilde{\alpha}(\partial_x) = 0 \) and \( \tilde{\alpha}(\partial_y) = \alpha \). Here \( \alpha \) is a real function depending only on \( x \). We twist \( \tilde{\Sigma} \mathbb{R}^2 \) with \( L \) which yields a Spin\(^c\) structure on \( \mathbb{R}^2 \). Let \( \sigma \) be a non-zero constant section of \( L \) and consider \( \varphi \otimes \sigma \). W.l.o.g. let \( |\sigma| = 1 \). On \( \tilde{\Sigma} \mathbb{R}^2 \otimes L \), we consider the twisted connection \( \hat{\nabla} = \nabla \otimes \text{Id} + \text{Id} \otimes \nabla_L \), where \( \nabla_L \) is the covariant derivative on \( L \) given by \( \nabla_L \sigma = i \tilde{\alpha}(\sigma) \). Then

\[
\hat{\nabla}_{\partial_x} (\varphi \otimes \sigma) = - i c' \bar{\partial}_x \cdot (\varphi \otimes \sigma) \\
\hat{\nabla}_{\partial_y} (\varphi \otimes \sigma) = - i \left( \frac{a'}{2} + \alpha \right) \cosh(c(x)) \bar{\varphi}_+ \otimes \sigma + \left( \frac{a'}{2} - \alpha \right) \sinh(c(x)) \bar{\varphi}_- \otimes \sigma.
\]

In order to show that \( \varphi \otimes \sigma \) is a Killing spinor, we want the last term to be equal to

\[
- i c' \bar{\partial}_y \cdot (\varphi \otimes \sigma) = i c' \sinh(c(x)) \bar{\varphi}_+ \otimes \sigma - c' \cosh(c(x)) \bar{\varphi}_- \otimes \sigma.
\]

Thus, we should solve

\[
\begin{cases}
(\frac{a'}{2} + \alpha) \cosh(c(x)) = - c' \sinh(c(x)), \\
(\frac{a'}{2} - \alpha) \sinh(c(x)) = - c' \cosh(c(x)).
\end{cases}
\]

Any smooth function \( c(x) \) with no zeros together with

\[
\alpha(x) = \frac{1}{2} c' \left( \coth(c(x)) - \tanh(c(x)) \right), \\
a'(x) = - c' \left( \coth(c(x)) + \tanh(c(x)) \right),
\]

such that \( a' \) is bounded, gives a solution. E.g. take \( c(x) = 1 + \frac{1}{1 + e^x} \). Hence, such a \( c \) determines a Killing spinor to the Killing function \(- i c'\). Note that since \( a' \) is required to be bounded, the conformal factor \( a \) can be chosen such that it is everywhere positive as requested.

It remains to show that such spinors are of type II, i.e., that \( q_{\varphi \otimes \sigma} \) is positive. By definition \( q_{\varphi \otimes \sigma} = f^2 - \|V\|^2_g \). First, note that \( f = |\varphi \otimes \sigma|^2_g = \cosh^2(c(x)) + \sinh^2(c(x)) \). Moreover, we have

\[
\|V\|^2_g = \bar{g}(V, \bar{\partial}_x)^2 + \bar{g}(V, \bar{\partial}_y)^2 = \langle i \bar{\partial}_x \cdot (\varphi \otimes \sigma), \varphi \otimes \sigma \rangle^2 + \langle i \bar{\partial}_y \cdot (\varphi \otimes \sigma), \varphi \otimes \sigma \rangle^2.
\]

Together with

\[
\langle i \bar{\partial}_x \cdot (\varphi \otimes \sigma), \varphi \otimes \sigma \rangle \\
= (- i \cosh(c(x)) \bar{\psi}_- \otimes \sigma + \sinh(c(x)) \bar{\psi}_+ \otimes \sigma, - \cosh(c(x)) \bar{\psi}_+ \otimes \sigma + i \sinh(c(x)) \bar{\psi}_- \otimes \sigma) \\
= - \sinh(c(x)) \cosh(c(x)) - \sinh(c(x)) \cosh(c(x)) = - 2 \sinh(c(x)) \cosh(c(x))
\]

Thus, we have

\[
\|V\|^2_g = 2 \bar{g}(V, \bar{\partial}_x)^2 + 2 \bar{g}(V, \bar{\partial}_y)^2 = \langle i \bar{\partial}_x \cdot (\varphi \otimes \sigma), \varphi \otimes \sigma \rangle^2 + \langle i \bar{\partial}_y \cdot (\varphi \otimes \sigma), \varphi \otimes \sigma \rangle^2.
\]
and
\[
\langle \tilde{u} \partial_y \cdot (\varphi \otimes \sigma), \varphi \otimes \sigma \rangle \\
= \langle \cosh(c(x)) \tilde{\psi}_+ \otimes \sigma - i \sinh(c(x)) \tilde{\psi}_+ \otimes \sigma, - \cosh(c(x)) \tilde{\psi}_+ \otimes \sigma + i \sinh(c(x)) \tilde{\psi}_- \otimes \sigma \rangle \\
= -i \sinh(c(x)) \cosh(c(x)) + i \sinh(c(x)) \cosh(c(x)) = 0.
\]

we obtain
\[
f^2 - \|V\|^2 = (\cosh^2(c(x)) + \sinh^2(c(x)))^2 - 4 \cosh^2(c(x)) \sinh^2(c(x)) \\
= (\cosh^2(c(x)) - \sinh^2(c(x)))^2 = 1.
\]

\[ \square \]

Acknowledgment. We are indebted to the Institute of Mathematics of the University of Potsdam, especially to the group of Christian Bär, for their hospitality and support. The second author thanks also the Institute of Mathematics of the University of Leipzig and gratefully acknowledges the financial support of the Berlin Mathematical school.

REFERENCES

[1] AMMANN, B. The Willmore conjecture for immersed tori with small curvature integral. *Manuscripta Math.* 101, 1 (2000), 1–22.
[2] AMMANN, B. Ambient Dirac eigenvalue estimates and the Willmore functional. In *Dirac operators: yesterday and today*. Int. Press, Somerville, MA, 2005, pp. 221–228.
[3] BÄR, C. Real Killing spinors and holonomy. *Comm. Math. Phys.* 154, 3 (1993), 509–521.
[4] BÄR, C. Extrinsic bounds for eigenvalues of the Dirac operator. *Ann. Global Anal. Geom.* 16, 6 (1998), 573–596.
[5] BÄR, C., GAUDUCHON, P., AND MOROIANU, A. Generalized cylinders in semi-Riemannian and Spin geometry. *Math. Z.* 249, 3 (2005), 545–580.
[6] BAUM, H. Complete Riemannian manifolds with imaginary Killing spinors. *Ann. Global Anal. Geom.* 7, 3 (1989), 205–226.
[7] BAUM, H. Odd-dimensional Riemannian manifolds with imaginary Killing spinors. *Ann. Global Anal. Geom.* 7, 2 (1989), 141–153.
[8] BAUM, H. Variétés riemanniennes admettant des spineurs de Killing imaginaires. *C. R. Acad. Sci. Paris Sér. I Math.* 309, 1 (1989), 47–49.
[9] BAUM, H., FRIEDRICH, T., GRUNEWALD, R., AND KATH, I. Twistor and Killing spinors on Riemannian manifolds, vol. 108 of *Seminarberichte [Seminar Reports]*. Humboldt Universität Sektion Mathematik, Berlin, 1990.
[10] BOURGUIGNON, J.-P., AND GAUDUCHON, P. Spineurs, opérateurs de Dirac et variations de métriques. *Comm. Math. Phys.* 144, 3 (1992), 581–599.
[11] CAHEN, M., GUTT, S., LEMAIRE, L., AND SPINDEL, P. Killing spinors. *Bull. Soc. Math. Belg. Sér. A* 38 (1986), 75–102 (1987).
[12] DUFF, M. J., NILSSON, B. E. W., AND POPE, C. N. Kaluza-Klein supergravity. *Phys. Rep.* 130, 1-2 (1986), 1–142.
[13] FRANC, A. Spin structures and Killing spinors on lens spaces. *J. Geom. Phys.* 4, 3 (1987), 277–287.
[14] FRIEDRICH, T. Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung. *Math. Nachr.* 97 (1980), 117–146.
[15] FRIEDRICH, T. A remark on the first eigenvalue of the Dirac operator on 4-dimensional manifolds. *Math. Nachr.* 102 (1981), 53–56.
[16] FRIEDRICH, T. Zur Existenz paralleler Spinorfelder über Riemannschen Mannigfaltigkeiten. *Colloq. Math.* 44, 2 (1981), 277–290 (1982).
[17] FRIEDRICH, T. On the conformal relation between twistors and Killing spinors. In *Proceedings of the Winter School on Geometry and Physics* (Srní, 1989) (1990), no. 22, pp. 59–75.
[18] FRIEDRICH, T. Dirac operators in Riemannian geometry, vol. 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.
[19] FRIEDRICH, T., AND GRUNEWALD, R. On the first eigenvalue of the Dirac operator on 6-dimensional manifolds. *Ann. Global Anal. Geom.* 3, 3 (1985), 265–273.
[20] FRIEDRICH, T., AND KATH, I. Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator. *J. Differential Geom.* 29, 2 (1989), 263–279.
[21] FRIEDRICH, T., AND KATH, I. 7-dimensional compact Riemannian manifolds with Killing spinors. *Comm. Math. Phys.* 133, 3 (1990), 543–561.
[22] FRIEDRICH, T., AND KATH, I. Compact 5-dimensional Riemannian manifolds with parallel spinors. *Math. Nachr.* 147 (1990), 161–165.
[23] HERZLICH, M., AND MOROIANU, A. Generalized Killing spinors and conformal eigenvalue estimates for Spin^c manifolds. *Ann. Global Anal. Geom.* 17, 4 (1999), 341–370.
[24] HUJAZI, O. Caractérisation de la sphère par les premières valeurs propres de l’opérateur de Dirac en dimensions 3, 4, 7 et 8. *C. R. Acad. Sci. Paris Sér. I Math.* 303, 9 (1986), 417–419.
[25] HUJAZI, O. A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors. *Comm. Math. Phys.* 104, 1 (1986), 151–162.
[26] HUJAZI, O. Spectral properties of the Dirac operator and geometrical structures. In *Geometric methods for quantum field theory (Villa de Leyva, 1999)*. World Sci. Publ., River Edge, NJ, 2001, pp. 116–169.
[27] HUJAZI, O., MONTIEL, S., AND ROLDÁN, A. Eigenvalue boundary problems for the Dirac operator. *Comm. Math. Phys.* 231, 3 (2002), 375–390.
[28] HUJAZI, O., MONTIEL, S., AND URBANO, F. Spin^c geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds. *Math. Z.* 253, 4 (2006), 821–853.
[29] HUJAZI, O., MONTIEL, S., AND ZHANG, X. Dirac operator on embedded hypersurfaces. *Math. Res. Lett.* 8, 1–2 (2001), 195–208.
[30] HUJAZI, O., MONTIEL, S., AND ZHANG, X. Eigenvalues of the Dirac operator on manifolds with boundary. *Comm. Math. Phys.* 221, 2 (2001), 255–265.
[31] HUJAZI, O., MONTIEL, S., AND ZHANG, X. Conformal lower bounds for the Dirac operator of embedded hypersurfaces. *Asian J. Math.* 6, 1 (2002), 23–36.
[32] HITCHIN, N. Harmonic spinors. *Advances in Math.* 14 (1974), 1–55.
[33] KANAI, M. On a differential equation characterizing a Riemannian structure of a manifold. *Tokyo J. Math.* 6, 1 (1983), 143–151.
[34] KOBAYASHI, S., AND NOMIZU, K. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.
[35] LAWSON, JR., H. B., AND MICHELSON, M.-L. *Spin geometry*, vol. 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
[36] LICHTNEROWICZ, A. Spin manifolds, Killing spinors and universality of the Hijazi inequality. *Lett. Math. Phys.* 13, 4 (1987), 331–344.
[37] LICHTNEROWICZ, A. Les spineurs-twisteurs sur une variété spinorielle compacte. *C. R. Acad. Sci. Paris Sér. I Math.* 306, 8 (1988), 381–385.
[38] MOROIANU, A. Parallel and Killing spinors on Spin^c manifolds. *Comm. Math. Phys.* 187, 2 (1997), 417–427.
[39] NAKAD, R. The energy-momentum tensor on Spin^c manifolds. *Int. J. Geom. Methods Mod. Phys.* 8, 2 (2011), 345–365.
[40] NAKAD, R., AND ROTH, J. Hypersurfaces of Spin^c manifolds and Lawson type correspondence. *Ann. Global Anal. Geom.* 42, 3 (2012), 421–442.
[41] NAKAD, R., AND ROTH, J. The Spin^c Dirac operator on hypersurfaces and applications. *Differential Geom. Appl.* 31, 1 (2013), 93–103.
[42] NILSSON, B. E. W., AND POPE, C. N. Scalar and Dirac eigenfunctions on the squashed seven-sphere. *Phys. Lett. B* 135, 1–2 (1983), 67–71.
[43] RADEMACHER, H.-B. Generalized Killing spinors with imaginary Killing function and conformal Killing fields. In *Global differential geometry and global analysis (Berlin, 1990)*, vol. 1481 of Lecture Notes in *Math.* Springer, Berlin, 1991, pp. 192–198.
[44] SULANKE, S. Der erste Eigenwert des Dirac-Operators auf S^5/T. *Math. Nachr.* 99 (1980), 259–271.
[45] WANG, M. Y. Parallel spinors and parallel forms. *Ann. Global Anal. Geom.* 7, 1 (1989), 59–68.
MATHEMATISCHES INSTITUT, UNIVERSITÄT LEIPZIG, 04009 LEIPZIG, GERMANY
E-mail address: grosse@math.uni-leipzig.de

DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF NATURAL AND APPLIED SCIENCES, NOTRE DAME UNIVERSITY-LOUAIZE, P.O. BOX 72, ZOUK MIKAEL, ZOUK MOSBEH, LEBANON.
E-mail address: rnakad@ndu.edu.lb