Unbounded operators on Banach spaces over the quaternion field

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1 Introduction

The quaternion field is the algebra over \( \mathbb{R} \), but it is not the algebra over \( \mathbb{C} \), since each embedding of \( \mathbb{C} \) into \( \mathbb{H} \) is not central. Therefore, the investigation of operator algebras over \( \mathbb{H} \) can not be reduced to algebras of operators over \( \mathbb{C} \). On the other hand, the developed below theory of operator algebras over \( \mathbb{H} \) has many specific features in comparison with the general theory of operator algebras over \( \mathbb{R} \) due to the graded structure of \( \mathbb{H} \). Results of this work may be also used for the development of non-commutative geometry, superanalysis, quantum mechanics over \( \mathbb{H} \) and the representation theory of non locally compact groups such as groups of diffeomorphisms and loops of quaternion manifolds (see [1, 10, 3, 6, 7]). The vast majority of previous works on superanalysis was devoted to supercommutative superalgebras of the type of Grassman algebra, but for the non-commutative superalgebras it was almost undeveloped. The quaternion field serves as very important example of the superalgebra, which is not supercommutative. In this work the results of previous works on this theme of the author are used, in particular, non-commutative integral over \( \mathbb{H} \) [8, 9] that serves the analog of the

\[ \int \]
Cauchy-type integral known for \( \mathbb{C} \). Examples of quaternion unbounded operators are differential operators and among them in partial derivatives. They arise in the natural way, for example, the Klein-Gordon-Fock equation can be written in the form \((\partial^2/\partial z^2 + \partial^2/\partial \bar{z}^2)f = 0\) on the space of quaternion locally \((z, \bar{z})\)-analytical functions \(f\), where \(z\) is the quaternion variable, \(\bar{z}\) - is the adjoint variable, \(z\bar{z} = |z|^2\). The Dirac operator for spin systems over \(\mathbb{H}^2\) can be written in the form \((0 - \partial/\partial z)\), that is used in the theory of spin manifolds [5], but each spin manifold can be embedded into the quaternion manifold [9]. In this article main features of the quaternion case are given, since in this article it is impossible to give the same broad theory over \(\mathbb{H}\), as well-developed theory of operators over \(\mathbb{C}\) [2, 4].

2 Theory of unbounded operators

2.1. Definitions and Notes. Let \(X\) be a Banach space (BS) over the quaternion field \(\mathbb{H}\), that is, \(X\) is the additive group, multiplications of vectors \(v \in X\) on scalars \(a, b \in \mathbb{H}\) on the left and right satisfy axioms of associativity and distributivity, there exists the norm \(||v||\) on \(X\) relative to which, \(X\) is complete, where \(||av|| = |a|_\mathbb{H}||v||\), \(||vb|| = |b|_\mathbb{H}||v||\), \(||v + w|| \leq ||v|| + ||w||\) for each \(v, w \in X\), \(a, b \in \mathbb{H}\). Then \(X\) has also the structure \(X_\mathbb{R}\) of BS over \(\mathbb{R}\), since \(\mathbb{H}\) is the algebra over \(\mathbb{R}\) of dimension 4.

An operator \(T\) on a dense vector subspace \(\mathcal{D}(T)\) in \(X\) with values in BS \(Y\) over \(\mathbb{H}\) is called \((\mathbb{H})\)-right-linear (RLO), if \((Tv)a = av\), \((Tv)a + (Tw)b = (Tv)a + (Tv)b\) for each \(a, b \in \mathbb{H}\) and each \(v, w \in \mathcal{D}(T)\) and in addition \(T\) is \(\mathbb{R}\)-linear on \(\mathcal{D}(T)_\mathbb{R}\). FOR RLO we also write \(Tv\) instead of \(T(v)\). If \(T\) is \(\mathbb{R}\)-linear and \(b(Tav) = T(bav)\), \(T(av + bw) = a(Tv) + b(Tw)\) for each \(a, b \in \mathbb{H}\), then \(T\) is called \((\mathbb{H})\)-left-linear (LLO). For LLO we also write \(vT\) instead of \(T(v)\). An operator \(T\) is called \((\mathbb{H})\)-linear, if it is LLO and RLO simultaneously. An operator \(T : \mathcal{D}(T) \rightarrow \mathcal{R}(T)\) we call \((\mathbb{H})\)-quasi-linear (QLO), if it is additive \(T(v + w) = T(v) + T(w)\) and \(\mathbb{R}\)-homogeneous \(T(\alpha v) = \alpha T(v)\) for each \(\alpha \in \mathbb{R}, v\) and \(w \in X\), where \(\mathcal{R}(T) \subset Y\) denotes the range of values of this operator. For example, products of quaternion holomorphic functions are \(\mathbb{H}\)-quasi-linear (see [8]).

Let \(L_q(X, Y)\) \((L_r(X, Y); L_l(X, Y))\) denote BS of all bounded QLO \(T\) from \(X\) into \(Y\) (RLO and LLO respectively), \(||T|| := \sup_{v \neq 0} ||Tv||/||v||; L_q(X) := L_q(X, X), L_r(X) := L_r(X, X)\) and \(L_l(X) := L_l(X, X)\). The resolvent set
\[ \rho(T) \] of QLO \( T \) is defined as \( \rho(T) := \{ z \in \mathbb{H} : \text{there exists } R(z; T) \in L_q(X) \} \), where \( R(z; T) := R_z(T) := (zI - T)^{-1} \), \( I \) is the identity operator on \( X \), \( Iv = v \) for each \( v \in X \), analogously for RLO and LLO. A spectrum is defined as \( \sigma(T) := \mathbb{H} \setminus \rho(T) \).

2.2. Lemma. For each \( z_1 \) and \( z_2 \in \rho(T) \):

(i) \( R(z_2; T) - R(z_1; T) = R(z_1; T)(z_1 - z_2)R(z_2; T) \).

2.3. Lemma. If \( T \) is a closed QLO, then \( \rho(T) \) is open in \( \mathbb{H} \) and \( R(z; T) \) is quaternion holomorphic on \( \rho(T) \).

Proof. Let \( z_0 \in \rho(T) \), then the operator \( R(z_0; T) \) is closed and by the closed graph theorem it is bounded, since it is defined everywhere. If \( z \in \mathbb{H} \) and \( |z_0 - z| < \| (z_0I - T)^{-1} \|^{-1} \), then \( (zI - T) = (z_0I - T)(I - R(z_0; T)(z_0 - z)) \), such that

(i) \( R(z; T) = \{ \sum_{n=0}^{\infty} [R(z_0; T)(z_0 - z)]^n \} R(z_0; T) \in L_s(X) \), since this series \([R(z_0; T)(z_0 - z)]^n R(z_0; T)\) converges relative to the norm topology in \( L_s(X) \). In view of (i) \( R(z; T) \) is quaternion locally \( z \)-analytic, hence it is holomorphic on \( \rho(T) \) due to Theorem 2.16 [8].

2.4. Notes and Definitions. For BS \( X \) over \( \mathbb{H} \) its right-adjoint space \( X^*_r \) is defined as consisting of all functionals \( f : X \to \mathbb{H} \) such that \( f \) is \( \mathbb{R} \)-linear and \( \mathbb{H} \)-right-linear. Analogously we put \( X^*_l := L_l(X, \mathbb{H}) \) and \( X^*_l := L_l(X, \mathbb{H}) \), where \( X^*_l \) is the topologically quasi-adjoint space, \( X^*_l \) is the topologically left-adjoint space. Then \( X^*_l \) is BS over \( \mathbb{H} \) with the norm \( \| f \| := \sup_{x \neq 0} |f(x)|/\|x\| \). If \( X \) and \( Y \) are BS over \( \mathbb{H} \) and \( T : X \to Y \) belongs to \( L_s(X, Y) \), then \( T^\ast \) is defined by the equation: \( (T^\ast y^\ast)(x) := y^\ast \circ T(x) \) for each \( y^\ast \in Y^*_r \), \( x \in X \). Then \( T^\ast \in L_l(Y^*_r, X^*_r) \) for each \( T \in L_l(X, Y) \). If \( T \in L_l(X, Y) \), then \( T^\ast \in L_l(Y^*_r, X^*_r) \), since \( y^\ast \circ T(x) = xT^\ast y^\ast \) in the symmetrical notation, where \( x \in X \). If \( T \in L_q(X, Y) \), then \( T^\ast \in L_q(Y^*_q, X^*_q) \).

Let \( \hat{X} \) and \( \hat{Y} \) be images relative of the natural embedding \( X \) and \( Y \) into \( X^{**} \) and \( Y^{**} \) respectively. For each \( T \in L_s(X, Y) \) we define \( \hat{T} \in L_s(\hat{X}, \hat{Y}) \) by the equation \( \hat{T}(\hat{x}) = \hat{y}, \) where \( y = T(x) \). For each function \( S \) defined on the region \( X^{**} \supset dom(S) \supset \hat{X} \) and such that \( S(\hat{x}) = T(\hat{x}) \) for each \( \hat{x} \in \hat{X} \) is called the extension of \( T \).

Lemmas II.3.12; VI.2.2 - 4, 6, 7 and corollary II.3.13 from [2] are analogous in the case \( \mathbb{H} \) instead of \( \mathbb{C} \), taking in the proof of Lemma II.3.12 \( \| z \| = \| y + \alpha x \| = |\alpha| \| y + x \| \geq |\alpha|d \). Take by induction \( \mathcal{D}(T^n) := \{ x : x \in \mathcal{D}(T^{n-1}), T^{n-1}(x) \in \mathcal{D}(T) \} \), \( \mathcal{D}(T^\infty) := \bigcap_{n=1}^{\infty} \mathcal{D}(T^n) \), where \( T^n := I, T^0 := I, T^n(x) := T(T^{n-1}(x)) \).
2.5. Lemma. Let $T \in L_s(X)$, then $\sigma(T^*) = (\sigma(T))^\sim$ and $(R(\lambda, T))^* = R(\lambda^*, T^*)$, where $\lambda^* := (\lambda I)^*$, $s \in \{q, r, l\}$.

Proof. If $S \in L_s(X, Y)$ and there exists $S^{-1} \in L_s(Y, X)$, then $S^* \in L_a(Y^*_s, X^*_s)$ has the inverse $(S^*)^{-1} \in L_a(X^*_s, Y^*_s)$ and $(S^{-1})^* = (S^*)^{-1}$, where $(s, u) \in \{(q, q); (r, l); (l, r)\}$. Then $(\lambda I - T)^* y^* = y^* \circ (\lambda I - T) = y^* \circ (\lambda I - y^* \circ T)$, consequently, $(\lambda^* I - T^*)[(\lambda I - T)^*]^{-1} = I$ and $(R(\lambda, T))^* = R(\lambda^*, T^*)$.

2.6. Definition and Note. Denote by $\mathcal{H}(T)$ the family of all quaternion holomorphic functions (QHF) $f$ on neighbourhoods $V_f$ for $\sigma(T)$, where $T \in L_s(X)$, $s \in \{q, r, l\}$, and for a QLO $T \in \mathcal{H}_\infty(T)$ is a set of all QHF on neighbourhoods $U_f$ of $\sigma(T)$ and $\infty$ in the one-point compactification $\hat{H}$ of the quaternion field. We choose a marked point $z_0 \in \sigma(T)$. For each $M = wJ + xK + yL \in \mathbf{H}_1$ with $|M| = 1$, where $w, x, y \in \mathbb{R}$, there exists a closed rectifiable path $\eta$ consisting of a finite union of arches $\eta(s) = z_0 + r_p \exp(2\pi sM)$ with $s \in [a_p, b_p] \subset [0, 1] \subset \mathbb{R}$ and segments of straight lines $\{z \in \mathbf{H} : z = z_0 + (r_p t + r_{p+1}(1 - t)) \exp(2\pi b_p M), t \in [0, 1]\}$ joining them, moreover $\eta \subset U \setminus \sigma(T)$, where $a_p < b_p$ and $0 < r_p < \infty$ for each $p = 1, ..., m, m \in \mathbb{N}, b_p = a_{p+1}$ for each $p = 1, ..., m - 1, a_1 = 0, b_m = 1$. Then there exists a rectifiable closed path $\psi$ homotopic to $\eta$ and a neighbourhood $U$ satisfying conditions of Theorem 3.9 [8] and such that $\psi \subset U \setminus \sigma(T)$. For $T \in L_q(X)$ we can define

(i) $f(T) := (2\pi)^{-1}(\int_\psi f(\zeta)R(\zeta; T)d\zeta)M^{-1}$,

where convergence is supposed in the weak operator topology. This integral depends on $f, T$ and it is independent from $U, \psi, \eta, \gamma, M$. For unbounded QLO $T$ let $A := -R(a; T)$ and $\Psi : \hat{H} \to \hat{H}$, $\Psi(z) := (z - a)^{-1}, \Psi(\infty) = 0, \Psi(a) = \infty$, where $a \in \rho(T)$. For $f \in \mathcal{H}_\infty(T)$ we define $f(T) := \phi(A)$, where $\phi \in \mathcal{H}_\infty(A)$ is given by the equation $\phi(z) := f(\Psi^{-1}(z))$.

2.7. Note. Consider BS $X$ over $\mathbf{H}$ as BS $X_\mathbf{C}$ over $\mathbf{C}$, then

(i) $X_\mathbf{C} = X_1 \oplus X_2j$,

where $X_1$ and $X_2$ are BS over $\mathbf{C}$, such that $X_1$ is isomorphic with $X_2$. The complex conjugation in $\mathbf{C}$ induces the complex conjugation of vectors in $X_m$, where $m = 1$ and $m = 2$. Each vector $x \in X$ can be written in the matrix form

(ii) $x = \begin{pmatrix} x_1 & x_2 \\ -\bar{x}_2 & \bar{x}_1 \end{pmatrix}$,

where $x_1 \in X_1$ and $x_2 \in X_2$. Then each QLO $T$ can be written in the form

(iii) $T = \begin{pmatrix} T_1 & T_2 \\ -\bar{T}_2 & \bar{T}_1 \end{pmatrix}$,

where $T_1 : X_1 \supset D(T_1) \to Y_1$, $T_2 : X_1 \supset D(T_2) \to Y_2$, $T(x) = Tx$ for
\(s \in \{g, r\}, T(x) = xT\) for \(s = l\).

(iv) \(T_m x := \overline{T_m x}\), where \(m = 1\) or \(m = 2\).

In particular, for the commutator [\(\zeta I, T\)] when \(\zeta = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \in \mathbf{H}\), where \(b \in \mathbf{C}\), is accomplished the formula

(v) \([\zeta I, T] = 2(-1)^{1/2}Im(b)\begin{pmatrix} 0 & T_2 \\ -T_2 & 0 \end{pmatrix}\), where \(Im(b)\) is the imaginary part of \(b\).

\[2(-1)^{1/2}Im(b) = (b - \bar{b}).\]

2.8. Theorem. If \(f \in \mathcal{H}_\infty(T)\), then \(f(T)\) does not depend from \(a \in \rho(T)\) and

(i) \(f(T) = f(\infty)I + (2\pi)^{-1}(\int \psi f(\lambda)R(\lambda; T)d\lambda)M^{-1}\).

Proof. If \(a \in \rho(T)\), then \(0 \neq b = (\lambda - a)^{-1}\) for \(\lambda \neq a\) and \((T - aI)(T - \lambda I)^{-1} = (bI - A)^{-1}b\), therefore, \(I + b^{-1}(T - \lambda I)^{-1} = b(bI - A)^{-1}b - bI\) for \(b \in \rho(A)\). If \(0 \neq b \in \rho(A)\), then \(A(bI - A)^{-1} = (T - \lambda I)^{-1}b^{-1}\) and \(\lambda \in \rho(T)\). The point \(b = 0 \in \sigma(A)\), since \(A^{-1} = T - aI\) is unbounded. Let \(a \notin V\), then \(U = \psi^{-1}(V) \supset \sigma(A)\) and \(U\) is open in \(\hat{\mathbf{H}}\), and \(\phi(z) = f(\psi^{-1}(z)) \in \mathcal{H}_\infty(U)\).

In view of Corollary 3.26 [8] we get (i).

2.9. Theorem. Let \(a, b, c, e \in \mathbf{H}\), \(f, g \in \mathcal{H}_\infty(T)\). Then

(i) \(af + bg \in \mathcal{H}_\infty(T)\) and \(af(T) + bg(T) = (af + bg)(T)\);

(ii) \(fg \in \mathcal{H}_\infty(T)\) and \(f(T)g(T) = (fg)(T)\);

(iii) if \(f\) is decomposed into a convergent series \(f(z) = \sum_k(b_k, z^k)\) in a neighbourhood \(\sigma(T)\), then \(f(T) = \sum_k(b_k, T^k)\) on \(\mathcal{D}(T^\infty)\), where \(b_k = (b_{k,1}, \ldots, b_{k,m(k)})\), \((b_k, z^k) := b_{k,1}z^{k_1} \ldots b_{k,m(k)}z^{k_{m(k)}}\), \(b_{k;j} \in \mathbf{H}\);

(iv) \(f \in \mathcal{H}_\infty(T^*)\) and \(f^*(T) = (f(T))^*\), where \(f^*(z) := (f(z^*))^*\).

Proof. (i). Take \(V_f \cap V_g =: V\) and for it we construct \(U, \eta\) and \(\psi\) as in \(\S 2.6\). Then the first statement follows from 2.6.(i).

(ii). In view of theorem 3.28 [8] the function \(\tilde{g}(\tilde{z}) =: \phi(\tilde{z})\) belongs to \(\mathcal{H}_\infty(T)\) and \(\tilde{\phi}(\tilde{z}) = g(z)\), where \(\tilde{z} = vI - wJ - xK - yL, z = vI + wJ + xK + yL, v, w, x, y \in \mathbf{R}, z \in V_g \subset \mathbf{H}\). Using \(\phi(A)\) we consider the case of bounded \(T\). In view of 2.6.(i): \((\tilde{g}^*\phi(\tilde{T})\tilde{h}) = (2\pi)^{-1}y^*M \int \tilde{\psi} R(\tilde{\zeta}, T)g(\tilde{\zeta})h,\) where \(\tilde{g}^*\tilde{T}h := (y^*Th)^*\) and \(\tilde{g}^*\tilde{h} := (y^*h)^*\) for each \(y^* \in X^*\) and \(h \in X\). Therefore, \((\tilde{g}^*\phi(\tilde{T})\tilde{h}) = (2\pi)^{-1}y^*M \int \tilde{\psi} R(\tilde{\zeta}, T)g(\tilde{\zeta})h,\) consequently, \((\phi(\tilde{T}))^* = (2\pi)^{-1}M \int \tilde{\psi} R(\tilde{\zeta}, T)g(\tilde{\zeta}) = g(T),\) since left and right integrals coincide in the space of quaternion holomorphic functions. The function \(fg\) is quaternion holomorphic on \(V\) (see \(\S 2.1\) and 2.12 [8]). There exist \(\psi_f\) and \(\psi_g\) as in \(\S 2.6\) and contained in \(U \setminus \sigma(T)\), where \(U \subset V\). In view of the Fubini theorem
there exists
\[ f(T)g(T) = (2\pi)^{-2} \int_{\psi_f} \int_{\psi_g} f(\zeta_1)R(\zeta_1; T)(d\zeta_1)(d\zeta_2)R(\zeta_2, T)g(\zeta_2), \]

where \( \zeta_1 \in \psi_f \) and \( \zeta_2 \in \psi_g \). There are accomplished the identities \( R(\zeta; T)d\zeta = d\zeta Ln(\zeta I - T) \) and \( (d\zeta)R(\zeta; T) = d\zeta Ln(\zeta I - T) \) for a chosen branch of \( Ln \) (see §3.7, 3.8 [8]), consequently, \( R(\zeta_1; T)(d\zeta_1)(d\zeta_2)R(\zeta_2; T) = d\zeta_1 d\zeta_2 Ln(\zeta_1 I - T)Ln(\zeta_2 I - T) = (d\zeta_1)R(\zeta_1; T)R(\zeta_2; T)d\zeta_2 \). In view of Lemma 2.2:
\[ (vi) \quad R(a; T)R(b; T) = [R(a; T) - R(b; T)](b - a)^{-1} \]
\[ +R(a; T)[R(b; T), (b - a)I](b - a)^{-1}, \]
\[ (vii) \quad [R(b; T), (b - a)I] = R(b; T)[T, (b - a)I]R(b; T), \]
since \([ (bI - T), (b - a)I ] = -T, (b - a)I \], where \( a, b \in \rho(T) \). Let in particular \( \psi_f \) and \( \psi_g \) are contained in the plane \( \mathbb{R} \oplus i\mathbb{R} \) in \( \mathbb{H} \), where \( i, j, k \) are generators of \( \mathbb{H} \) such that \( i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \). In view of 2.7.(v) and 2.8.(vii):
\[ (viii) \quad \int_{\psi_f} \int_{\psi_g} f(\zeta_1)R(\zeta_1; T)[R(\zeta_2; T), (\zeta_2 - \zeta_1)I](\zeta_2 - \zeta_1)^{-1}d\zeta_1 d\zeta_2 g(\zeta_2) = 0, \]
since the branch of \( Ln \) can be chosen the same along the axis \( j \) in \( \mathbb{H} \), in view of the argument principle 3.30 [8] it corresponds to the residue \( c(\zeta_1 - z)^{-1}(\zeta_2 - z)^{-1}b(\zeta_2 - \zeta_1)(\zeta_2 - z)^{-1}(\zeta_2 - \zeta_1)^{-1} \). Then from (v, vi, viii) it follows:
\[ (ix) \quad f(T)g(T) = (2\pi)^{-2} \int_{\psi_f} \int_{\psi_g} f(\zeta_1)[R(\zeta_1; T) - R(\zeta_2, T)](\zeta_2 - \zeta_1)^{-1}d\zeta_1 d\zeta_2 g(\zeta_2). \]
Choose \( \psi_g \) such that \( |\psi_g(s) - z_0| > |\psi_f(s) - z_0| \) for each \( s \in [0, 1] \). From the additivity of the integral along the path and the Fubini theorem:
\[ (x) \quad f(T)g(T) = (2\pi)^{-2} \int_{\psi_f} f(\zeta_1)R(\zeta_1; T)d\zeta_1(\int_{\psi_g} (\zeta_2 - \zeta_1)^{-1}d\zeta_2 g(\zeta_2)) \]
\[ -(2\pi)^{-2} \int_{\psi_g} f(\zeta_1)d\zeta_1 R(\zeta_2, T)(\zeta_2 - \zeta_1)^{-1}d\zeta_2 g(\zeta_2). \]
In view of Theorems 3.9, 3.24 [8] the second integral on the right of (x) is equal to zero, since \( \int_{\psi_f} f(\zeta_1)d\zeta_1 R(\zeta_2, T)(\zeta_2 - \zeta_1)^{-1} = 0 \), the first integral produces
\[ f(T)g(T) = (2\pi)^{-1}(\int_{\psi_f} f(\zeta_1)R(\zeta_1; T)d\zeta_1 g(\zeta_1))M^{-1} \]
where $\zeta \in \psi_f$.

(iii) follows from the application of (i,ii) by induction and convergence of the series in the strong operator topology.

(iv). In view of Lemma 2.5 $\sigma(T^*) = (\sigma(T))^*$, then $f \in \mathcal{H}_\infty(T^*)$. Since $(f(T))^*y^* = y^* \circ f(T)$ for each $y^* \in X^*$, then due to Lemma 2.5 $R(\zeta^*; T^*) = (R(\zeta; T))^*$, consequently, $(f(T))^*y^* = (2\pi)^{-1}(M^{-1})^* \int_\psi (d\zeta^*) R(\zeta^*; T^*)(f(\zeta))^*y^*$, where $(f(\zeta))^*y^* := y^* \circ f(\zeta)$. If $f(\zeta)$ is represented by the series converging in the ball: $f(\zeta) = \sum_n(a_n, \zeta^n)$, then $f(\zeta^*) = \sum_n a_n \zeta^{*n1} \ldots a_{n,m(n)} \zeta^{*m(n)}$, consequently, $[f(\zeta^*)]^* = \sum_n \zeta^{n(n)} a^*_n a^*_n$ and

$(f(T))^* = (2\pi)^{-1}(M^{-1})^* \int_\psi (d\zeta^*) R(\zeta^*; T^*) f^*(\zeta^*) = f^*(T^*)$.

2.10. Theorem. Let $QLO \ T$ be bounded, $f \in \mathcal{H}(T)$, then $f(\sigma(T)) = \sigma(f(T))$.

2.11. Theorem. Let $f \in \mathcal{H}_\infty(T)$, let also $f(U)$ be open for some open $U \subset \text{dom}(f) \subset \tilde{\mathcal{H}}$, $g \in \mathcal{H}_\infty(T)$ and $f(U) \supset \sigma(T)$, $\text{dom}(g) \supset f(U)$, then $F := g \circ f \in \mathcal{H}(T)$ and $F(T) = g(f(T))$.

Proof follows from §2.9 analogously to the case of the field $\mathbb{C}$ with the help of Theorems 2.16, 3.10, Corollary 2.13 [8], since $F \in \mathcal{H}_\infty(T)$.

2.12. Definition and Note. Let $\mathcal{A}$ be BS and an algebra over $\mathbb{H}$ with the unity $e$ having the properties: $|e| = 1$ and $|xy| \leq |x||y|$ for each $x$ and $y \in \mathcal{A}$, then $\mathcal{A}$ is called the Banach algebra (BA) or $C$-algebra (over $\mathbb{H}$). BA $\mathcal{A}$ is called quasi-commutative (QC), if there exists a commutative algebra $\mathcal{A}_{0,0}$ over $\mathbb{R}$ such that $\mathcal{A}$ is isomorphic with the algebra $\{T : T = \left( \begin{array}{cc} A & B \\ \bar{B} & \bar{A} \end{array} \right) : A, B \in \mathcal{A}_0\}$, where $\mathcal{A}_0 := \{A : A = A_0 + A_1 i; A_0 \in \mathcal{A}_{0,0}, A_1 \in \mathcal{A}_{0,0}\}$, $\bar{A} := A_0 - A_1 i$, $i := (-1)^{1/2}$.

Consider $X$ over $\mathbb{R}$: $X_\mathbb{R} = X_ee \oplus X_ii \oplus X_1j \oplus X_k k$, where $X_e$, $X_i$, $X_j$ and $X_k$ are pairwise isomorphic BS over $\mathbb{R}$. Then $\mathcal{A} = \mathcal{A}_0 e \oplus \mathcal{A}_i i \oplus \mathcal{A}_j j \oplus \mathcal{A}_k k$, where $\mathcal{A}_0$, $\mathcal{A}_i$, $\mathcal{A}_j$ and $\mathcal{A}_k$ are algebras over $\mathbb{R}$. Multiplying $\mathcal{A}$ on $S \in \{e, i, j, k\}$, we get automorphisms of $\mathcal{A}$, consequently, $\mathcal{A}_0$, $\mathcal{A}_i$, $\mathcal{A}_j$ and $\mathcal{A}_k$ are pairwise isomorphic.

2.13. Definitions and Notes. BA $\mathcal{A}$ is supplied with the involution, when there exists an operation $* : \mathcal{A} \ni T \mapsto T^* \in \mathcal{A}$ such that $(T^*)^* = T$, $(T + V)^* = T^* + V^*$, $(TV)^* = V^* T^*$, $(\alpha T)^* = T^* \bar{\alpha}$ for each $\alpha \in \mathbb{H}$.

If BA $\mathcal{A}$ (over $\mathbb{H}$) has a subalgebra $\mathcal{A}_{0,0}$ (over $\mathbb{R}$), then $T^* = \left( \begin{array}{cc} A^* & -B^* \\ -B^* & {\bar{A}}^* \end{array} \right)$. An element $x \in \mathcal{A}$ is called regular, if there exists $x^{-1} \in \mathcal{A}$. In the
contrary case it is called singular. Then the spectrum $\sigma(x)$ for $x$ is defined as the set of all $z \in H$, for which $ze - x$ is singular, his spectral radius is $|\sigma(x)| := \sup_{z \in \sigma(x)} |z|$. The resolvent set is defined as $\rho(x) := \{z \in H : ze - x$ is regular$\}$ and the resolvent is $R(z; x) := (ze - x)^{-1}$ for each $z \in \rho(x)$.

2.14. Lemma. A spectrum $\sigma(x)$ of an element $x \in A$ is a non-void compact subset in $H$. Its resolvent $x(z) := R(z; x)$ is quaternion holomorphic on $\rho(x)$, $x(z)$ converges to zero when $|z| \to \infty$ and

$x(z) - x(y) = x(z)(y - z)x(y)$ for each $y, z \in \rho(x)$.

Proof follows from $(ze - x)x(z)x(y) = x(y), x(z)x(y)(ye - x) = x(z), (ze - x)(x(z) - x(y))(ye - x) = (ye - x) - (ze - x) = (y - z)e$, consequently, $x(z) - x(y) = R(z; x)(y - z)R(y; x) = x(z)(y - z)x(y)$. Therefore, $x(z)$ is continuous by $z$ on $\rho(x)$ and there exists $(\partial [x(z + y)x^{-1}(y)]/\partial z).h = -x(y)h$ for each $h \in H$. For each marked point $y$ the term $x^{-1}(y)$ is constant on $A$, moreover, $(\partial [x(z + y)x^{-1}(y)]/\partial \bar{z}) = 0$, consequently, $x(z) \in H(\rho(x))$.

The second statement follows from the consideration of the complexification $\mathbb{C} \otimes A$.

2.15. Theorem. Let $B$ be a closed ideal in QCBA $A$. The quotient algebra $A/B$ is isometrically isomorphic with $H$ if and only if $B$ is maximal.

The proof is analogous to the case of algebras over $C$ due to the definition of QCBA.

2.16. Definitions. A $C^*$-algebra $A$ over $H$ is a BA over $H$ with the involution $\ast$ such that $|x^{\ast}x| = |x|^2$ for each $x \in A$.

A scalar product on a linear space $X$ over $H$ (that is, linear relative to the right and left multiplications separately on scalars from $H$) is the biadditive $R$-bilinear mapping $< \ast, \ast > : X^2 \to H$ such that

1. $< x, x >= a_0 e$, where $a_0 \in R$;
2. $< x, x > > 0$ if and only if $x = 0$;
3. $< x, y >= < y, x >$ for each $x, y \in X$;
4. $< x + z, y >= < x, y > + < z, y >$;
5. $< xa, yb >= a < x, y > b$ for each $x, y, z \in X, a, b \in H$.

If $X$ is complete relative to the norm topology

6. $|x| := < x, x >^{1/2}$, then $X$ is called the quaternion Hilbert space (HS).

2.17. Lemma. $BA L_q(X)$ on $HS X$ with an involution:

1. $< Tx, y >= < x, Ty >$ for each $x, y \in X$

is a $C^*$-algebra.

2.18. Lemma. If $A$ is a QC $C^*$-algebra, then $|x^2| = |x|^2$, $|x| = |x^*|$ and $I^* = I$, where $I$ is the unity in $A$. 

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Proof. Each vector $x \in \mathcal{A}$ we represent in the form: $x = x_e e + x_i i + x_j j + x_k k$. Then $x^* = x_e^* e - x_i^* i - x_j^* j - x_k^* k$, since $(x_m S_m)^* = (-1)^\kappa S_m) x_m S_m$, where $S_m \in \{e, i, j, k\}$ for each $m \in \{e, i, j, k\}$, $\kappa(e) = 0$, $\kappa(i) = \kappa(j) = \kappa(k) = 1$. Therefore, $[x, x^*] = 0$ and $|x_m|^2 = |x_m|^2$. Then $|x|^2 = |x_i|^2 + |x_j|^2 + |x_k|^2$ and $|x|^2 = |(x^*)^*| = |(x^*)(xx^*)| = |x|^4$, consequently, $|x|^2 = |x|^2$. Since $I = I_e$, then $I^* = I_e^* = I_e = I$.

2.19. Definition. A homomorphism $h : \mathcal{A} \to \mathcal{B}$ of $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ preserving involutions: $h(x^*) = (h(x))^*$ is called a $*$-homomorphism. If $h$ is a bijective $*$-homomorphism of $\mathcal{A}$ on $\mathcal{B}$, then $h$ is called a $*$-isomorphism, $\mathcal{A}$ and $\mathcal{B}$ are called $*$-isomorphic. By $\sigma(\mathcal{A})$ is denoted a structural space for $\mathcal{A}$ and it is also called a spectrum of $\mathcal{A}$. A structural space is defined analogously to the complex case with the help of Theorem 2.15.

2.20. Proposition. For HS $X$ spaces $L_t(X, H)$ and $L_r(X, H)$ are isomorphic with $X$, for BS $X$ $L_q(X)$ is isomorphic with $L_t(X^2)$, $L_q(H) = H^4$, there exists a bijection between a family of all QLO $V$ on $D(T) \subset X$ a family of all LLO $V$ on $D(V) \subset X^2$.

Proof. Let $L_q(X) \ni \alpha = \alpha_e e + \alpha_i i + \alpha_j j + \alpha_k k$. Since $S \alpha S \in L_q(X)$ for each $S \in H$, there exist quaternion constants $S_{m,l,n}$ such that

\[
(1) \quad \alpha_m(x)m = \sum_n S_{m,1,n} \alpha(x) S_{m,2,n} \quad \text{for each} \quad m \in \{e, i, j, k\},
\]

where $S_{m,1,n} = \gamma_{m,n} S_{m,2,n}$ with $\gamma_{m,n} = (-1)^{\phi(m,n)}/4 \in \mathbb{R}$, $\phi(m,n) \in \{1, 2\}$, $S_{m,l,n} \in \mathbb{R}$ for each $n \in \{e, i, j, k\}$ (see §§3.7, 3.28 [8]). Applying for $x$ the decomposition from §2.12, due to (1) we get $4 \times 4$-block form of operators over $\mathbb{R}$ and the isomorphism of $L_q(X)$ with $L_t(X^2)$.

2.21. Note. For LLO (over $H$) the notions of point $\sigma_p(T)$, continuous $\sigma_c(T)$ and residual $\sigma_r(T)$ spectra are defined analogously to the case over the field $C$, due to 2.20 these notions spread on QLO.

2.22. Theorem. QC $C^*$-algebra is isometrically $*$-isomorphic with the algebra $C(\Lambda, H)$ of all continuous $H$-valued functions on its spectrum $\Lambda$.

The proof follows from the fact, that the mapping $x \mapsto x(\cdot)$ from $\mathcal{A}$ into $C(\Lambda, H)$ is the $*$-homomorphism, where $x(\mathcal{M})$ is defined by the equality $x + \mathcal{M} = x(\mathcal{M}) + \mathcal{M}$ for each maximal ideal $\mathcal{M}$. Let $x(\lambda) = \alpha_e e + \alpha_i i + \alpha_j j + \alpha_k k$, then $x^*(\lambda) = \beta_e e + \beta_i i + \beta_j j + \beta_k k$, where $\alpha_e, \ldots, \beta_k \in \mathbb{R}$. There exists the decomposition for $X : = C(\Lambda, H)$ from §2.12 with $X_e = C(\Lambda, H)$ and $x_m = \sum_n S_{m,1,n} z S_{m,2,n} n$ for each $z \in H$, where $z = x_e e + x_i i + x_j j + x_k k$, $x_m \in R$, $m, n \in \{e, i, j, k\}$ (see Proposition 2.20). Therefore, it can be applied the Stone-Weierstrass theorem for $H$-valued functions. If $\lambda_1 \neq \lambda_2$ are two maximal ideals in $\Lambda$, then $y(\lambda_1) \neq y(\lambda_2)$ for each $y \in \lambda_1 \setminus \lambda_2$. 

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Consequently, the algebra of functions $x(.)$ coincides with $C(\Lambda, H)$.

2.23. Definition. Let $X$ be HS over $H$ and $\mathcal{B}$ be a $\sigma$-algebra of Borel subsets of a Hausdorff topological space $\Lambda$. Consider a mapping $\hat{E}$ defined on $\mathcal{B} \times X^2$ and defining a unique $X$-projection-valued spectral measure $\hat{E}$ such that

(i) $< \hat{E}(\delta)x; y > = \hat{\mu}(\delta; x, y)$ is a regular (non-commutative) $H$-valued measure for each $x, y \in X$, where $\delta \in \mathcal{B}$. By our definition this means, that

\[ \hat{\mu}(\delta; x, y) = \hat{\mu}(x, y) \cdot \chi_{\delta} \]

(1) $\mu(\delta; x, y) = \mu(x, y).\chi_{\delta}$ and

(2) $\hat{\mu}(x, y).f := \sum_{m,l,n} \int_{\Lambda} S_{m,1,n} f(\lambda) S_{m,2,n} m\mu_{m,l}(d\lambda; x, y),$

where $\chi_{\delta}$ is the characteristic function for $\delta \in \mathcal{B}$, $S_{m,p,n} \in \mathbb{R}^n$ are the same as in §2.19, $\mu_{m,l}$ is a regular real-valued measure, $m, n, l \in \{e, i, j, k\}$, $p = 1$ or $p = 2$, $f$ is an arbitrary $H$-valued function on $\Lambda$, which is $\mu_{m,l}$-integrable for each $m, l$;

(3) $\hat{E}_S(\delta)^* = (-1)^{\kappa(S)}\hat{E}_S(\delta)$, where $\hat{E}_S(\delta).e := (\hat{E}_S(\delta))x = (\hat{E}(\delta).S)x,$

$S = cm$, $m \in \{e, i, j, k\}, c = const \in \mathbb{R}, x \in X$;

(4) $\hat{E}_{bS} = b\hat{E}_S \quad \text{for each} \quad b \in \mathbb{R} \quad \text{and each pure vector} \quad S = cm$;

(5) $\hat{E}_{S_1S_2}(\delta \cap \gamma) = \hat{E}_{S_1}(\delta)\hat{E}_{S_2}(\gamma)$ for each pure quaternion vectors $S_1$ and $S_2$ and each $\delta, \gamma \in \mathcal{B}$. Though from (3, 4) it follows, that $\hat{E}(\delta)\lambda = \lambda_e\hat{E}_e(\delta) + \lambda_i\hat{E}_i(\delta) + \lambda_j\hat{E}_j(\delta) + \lambda_k\hat{E}_k(\delta) =: \hat{E}_\lambda(\delta)$, but in general it may be $(\hat{E}(\delta)\lambda)x \neq (\hat{E}(\delta))\lambda x$, where $\lambda_e, ..., \lambda_k \in \mathbb{R}$.

2.24. Theorem. Each QC $C^*$-algebra $A$ contained in $L_q(X)$ for HS $X$ over $H$ is isometrically $*$-equivalent with the algebra $C(\Lambda, H)$, where $\Lambda$ is its spectrum. Moreover, each isometric $*$-isomorphism $f \mapsto T(f)$ between $C(\Lambda, H)$ and $A$ defines a unique $X$-projection-valued spectral measure $\hat{E}$ on $\mathcal{B}(\Lambda)$ such that

(i) $< \hat{E}(\delta)x; y > = \hat{\mu}(\delta; x, y)$ is the regular $H$-valued measure for each $x, y \in X$, where $\delta \in \mathcal{B}$;

(ii) $\hat{E}_{S_1}(\delta).T(S_2f) = (-1)^{\kappa(S_1) + \kappa(S_2)}T(S_2\hat{E}_{S_1}(\delta).f)$ for each $f \in C(\Lambda, R), \delta \in \mathcal{B}$ and pure quaternion vectors $S_1, S_2$;

(iii) $T(f) = \int_{\Lambda} \hat{E}(d\lambda).f(\lambda)$ for each $f \in C(\Lambda, H)$, moreover, $\hat{E}$ is $\sigma$-additive in the strong operator topology.

Proof. We mention, that $\Lambda$ is compact. There exists a decomposition $C(\Lambda, H)$ as in §2.21. Each $\psi \in C^*_q(\Lambda, H)$ has a decomposition $\psi(f) = \psi_e(f)e + \psi_i(f)i + \psi_j(f)j + \psi_k(f)k$, where $f \in C(\Lambda, H)$ (see 2.20.(1)). Moreover, $\psi_l(f) = \psi_l(f_ee) + \psi_l(f_ii) + \psi_l(fjj) + \psi_l(fkk)$, where $f_m \in C(\Lambda, R), m, l \in \{e, i, j, k\}$. Then
(1) \[ \psi(f) = \sum_{m,n,l} \psi_l(S_{m,1,n} f S_{m,2,n}) I, \]
where \( m, n, l \in \{e, i, j, k\} \). In view of the Riesz representation theorem IV.6.3 [2]: \( \psi_l(gm) = \int_A g(\lambda) \mu_{m,l}(d\lambda) \) for each \( g \in C(\Lambda, \mathbb{R}) \), where \( \mu_{m,l} \) is a \( \sigma \)-additive real-valued measure. The accomplishment of the componentwise integration of matrix-valued functions gives

(2) \[ \psi(f) = \sum_{m,n,l} \int_A S_{m,1,n} f(\lambda) S_{m,2,n} \tilde{m} I \mu_{m,l}(d\lambda). \]
For \( \psi(f) := (T(f) x; y) > \) for each \( f \in C(\Lambda, \mathbb{H}) \) and marked \( x, y \in X \) from (2) it follows, that

(3) \[ \langle T(f) x; y \rangle = \sum_{m,n,l} \int_A S_{m,1,n} f(\lambda) S_{m,2,n} \tilde{m} I \mu_{m,l}(d\lambda; x, y), \]
since \( \big| \langle T(f) x; y \rangle \big| \leq \big| f ||x||y \big| \), consequently, \( \mu_{m,l}(\tilde{\delta}; x, y) \) for each \( a, b \in \mathbb{R} \), moreover,

(4) \[ \sup_{\delta \in B} \bigg( \sum_{\delta \in B} \big| \sum_{m,n} \zeta m \mu_{m,l}(\tilde{\delta}; x, y) \big|^2 \bigg)^{1/2} \leq |z| ||x||y| \text{ for each } z = z_e e + z_i i + z_j j + z_k k \in \mathbb{H}, \text{ since } ||l|| = 1. \]

From (3) it follows, that \( \mu_{m,l}(\tilde{\delta}; x, y) \) is \( \mathbb{R} \)-bihomogeneous and biadditive by \( x, y \). If \( f(\lambda) \in \mathbb{R} m \) for \( \mu \)-a.e. \( \lambda \in \Lambda \) for some \( m \in \{e, i, j, k\} \), then \( T(f) = T((-1)^{\kappa(m)} f) = (-1)^{\kappa(m)} T(f)^* \), consequently, \( \langle T(f) x; y \rangle = (-1)^{\kappa(m)} \langle T(f)^* x; y \rangle \)

Therefore, \( \mu_{m,l}(\tilde{\delta}; x, y) \) for each \( m, l \in \{e, i, j, k\} \), \( x, y \in X \).

2.25. Definition. An operator \( T \) on a quaternion \( HS X \) is called normal, if \( TT^* = T^* T \); \( T \) unitary, if \( TT^* = I \); \( T \) symmetrical, if \( < Tx; y > = < x; Ty > \) for each \( x, y \in D(T) \), \( T \) self-adjoint, if \( T^* = T \). Henceforth, for \( T^* \) it is supposed, that \( D(T) \) is dense in \( X \).

2.26. Lemma. An operator \( T \in L_q(X) \) is normal if and only if a minimal (\( \mathbb{H} \)-)subalgebra \( A \) in \( L_q(X) \) containing \( T \) and \( T^* \) is QC.

Proof. Let \( T \) be normal, then on \( X = X_e e \oplus X_i i \oplus X_j j \oplus X_k k \) it can be represented in the form \( T = T_e E + T_i i + T_j j + T_k k \), where \( \text{Range}(T_m) \subset X_m \) and \( T_m \in A \) for each \( m \in \{e, i, j, k\} \). In the block form \( T = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \) and \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{pmatrix} \), where \( x_1 = x_e e + x_i i \in X_e e \oplus X_i i, x_2 j = x_j j + x_k k \in X_j j \oplus X_k k, \)

\( \bar{x}_1 = x_e e - x_i i, \bar{A}x_1 := A \bar{x}_1 \). Then \( T^*(x_m m) = (-1)^{\kappa(m)} T(x_m m) \) for each \( m \in \{e, i, j, k\} \) and \( T^* = \begin{pmatrix} A^* & -B^* \\ B^* & A^* \end{pmatrix} \). Therefore, \( < mTmx; mTmy > = < mT^*mx; mT^*my > \) for each \( m \in \mathbb{H} \) with \( |m| = 1 \), consequently, \( (mTm)(mTm)^* = (mTm)^*(mTm) \). The space \( X \) is isomorphic with \( l_2(v, \mathbb{H}) \), in which \( < x; y > = \sum_{x \in v} b_x b_y \), x is a set, \( x = \{ l_x : l_x \in \mathbb{H}, l \in v \} \in l_2(v, \mathbb{H}) \). Then in \( L_q(l_2(v, \mathbb{H})) \) is accomplished \( T^* = \bar{T} \), moreover, \( A^* = A \) and \( B^* = B \). Therefore, \( TT^* = T^* T \) gives \( A^* A \) is normal, also an automorphism \( j : X \to X \) and the equality \( \mu_{m,l}(mTm)(mTm)^* = (mTm)^*(mTm) \) with \( m = \theta, m = (i + j)/2, m = (i + k)/2, m = (i + j)\theta/2, m = (i + k)\theta/2, \)
\[\theta = \exp(\pi i/4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\] leads to the pairwise commuting \(\{T_e, T_i, T_j, T_k\}\).

Vice versa, if \(A\) is quasi-commutative, then \(\{T_m : m = e, i, j, k\}\) are pairwise commuting, consequently, \(TT^* = T^*T\).

**2.27. Lemma.** Let \(T\) be a symmetrical operator and \(a \in H \setminus \mathbb{R}e\), then there exists \(R(a; T)\) and \(|x| \leq 2|R(a; T)x|/|a - \tilde{a}|\) for each \(x \in D(T)\). Let \(T\) be a closed operator, then the sets \(\rho(T), \sigma_p(T), \sigma_e(T)\) and \(\sigma_r(T)\) do not intersect and their union is the entire \(H\). For a self-adjoint QLO \(T\) \(\sigma(T) \subset \mathbb{R}e\), moreover, \(R(a; T)^* = R(a^*; T)\).

**2.28. Theorem.** For a self-adjoint QLO \(T\) there exists a uniquely defined regular countably-additive self-adjoint spectral measure \(\hat{E}\) on \(\mathcal{B}(H)\), \(\hat{E}|_{\rho(T)} = 0\) such that

(a) \(D(T) := \{x : x \in X; f_{\sigma(T)} < (\hat{E}(dz).z^2) x; x >> \infty\}\) and

(b) \(Tx = \lim_{n \to \infty} f_n^\prime(\hat{E}(dz).z)x, x \in D(T)\).

**Proof.** In view of Proposition 2.20 and Equality 2.16.(5) the space \(D(T)\) is \(H\)-linear. Let us use Lemma 2.27, the proof of which is analogous to the case over the field \(\mathbb{C}\), also take a marked element \(q \in \{i, j, k\}\), then there is \(h(z) := (q - z)^{-1}\) the homeomorphism of the sphere \(S^3 := \{z \in H : |z| = 1\}\) and for \(A := (q - z)R(z; T)(q - z) + (q - z)I\) for each \(z \in \rho(T) \setminus \{q\}\) is accomplished the equality \((hI - R(q; T))A = I\). If \(z = q\), then \(h = \infty\), consequently, \(h \notin \sigma(R(q; T))\). Let \(0 \neq h \in \rho(R(q; T))\), then there exists \(B := R(q; T)A\), where \(A := (hI - R(z; T))^{-1}\), consequently, \(B\) is bijective, \(R(B) = D(T)\) and \((zI - T)B = (z - q)I\), that is, \(z \in \rho(T)\). For \(h = 0 \in \rho(R(q; T))\) the operator \(R(q; T)^{-1} = (hI - T)\) is the bounded everywhere defined operator and this case is considered in Theorem 2.24. For each \(\delta \in \mathcal{B}(H)\) we put \(\hat{E}(\delta) := \hat{E}^1(h(\delta))\), where \(\hat{E}^1\) is the decomposition of the identity for the normal operator \(R(q; T)\), then the end of the proof is analogous to that of Theorem XII.2.3 [2].

**2.29. Note and Definition.** A unique spectral measure, related with a self-adjoint QLO \(T\) is called the decomposition of the identity for \(T\). For \(H\)-valued Borel function \(f\) defined \(\hat{E}\) almost everywhere \(f(T)\) is defined by the relations: \(D(f(T)) := \{x : \text{there exists lim}_n f_n(T)x, \text{where } f_n(z) := f(z) \text{ for } |f(z)| \leq n; f_n(z) := 0 \text{ for } |f(z)| > n; f(T)x := \lim_n f_n(T)x, x \in D(f(T))\), \(n \in \mathbb{N}\).

**2.30. Theorem.** Let \(\hat{E}\) be a decomposition of the identity for a self-adjoint QLO \(T\) and \(f\) from §2.28. Then \(f(T)\) is a closed QLO defined on an everywhere dense domain, moreover:
(a) $\mathcal{D}(f(T)) = \{ x : \int_{-\infty}^{\infty} |f(z)|^2 < \hat{E}(dz); x > \infty \}$;
(b) $|f(T)x; y| = \int_{-\infty}^{\infty} < \hat{E}(dz), f(z)x; y >, x \in \mathcal{D}(f(T))$;
(c) $|f(T)x|^2 = \int_{-\infty}^{\infty} |f(z)|^2 < \hat{E}(dz); x >, x \in \mathcal{D}(f(T))$;
(d) $f(T)* = \tilde{f}(T)$; (e) $R(q; T) = \int_{-\infty}^{\infty} \hat{E}(dz)(q - z), q \in \rho(T)$.

**Proof.** Take $f_n$, with $\delta_n := \{ z : |f(z)| \leq n \}$. Then $|f(T)x|^2 = \lim_n |f_n(T)x|^2 = \int_{-\infty}^{\infty} |f(z)|^2 < \hat{E}(dz); x >$ for each $x \in \mathcal{D}(f(T))$, from this it follows (c), a closedness of $f(T)$ and (a) can be verified analogously to the complex case. A non-commutative measure $\hat{\mu}$ on the algebra $\mathcal{Y}$ of subsets of the set $\mathcal{S}$ corresponds to QLO with values in $\mathcal{H}$ and due to Proposition 2.20 it is characterized completely by $\mathcal{R}$-valued measures $\mu_{m,n}$ such that $\mu_{m,n}(f_m) = \hat{\mu}(f_m)n$ for each $\hat{\mu}$-integrable $\mathcal{H}$-valued function $f$ with components $f_m$, where $m, n \in \{ e, i, j, k \}$. Then it can be defined the variation $v(\hat{\mu}, U) := \sup_{W_i \subset U} \sum |\hat{\mu}(\chi_{W_i})|$ by all $\{ W_i \}$ finite disjunctive subsets $W_i \in \mathcal{Y}$ in $U$ with $\cup W_i = U$. If $\hat{\mu}$ is bounded, then it is QLO with bounded variation $v(\hat{\mu}, \mathcal{S}) \leq 16 \sup_{U \in \mathcal{Y}} |\hat{\mu}(\chi_U)|$, moreover, $v(\hat{\mu}, *)$ is additive on $\mathcal{Y}$. A function $f$ we call $\hat{\mu}$-measurable, if each $f_m$ is $\mu_{m,n}$-measurable for each $n$ and $m \in \{ e, i, j, k \}$. The space of all $\hat{\mu}$-measurable $\mathcal{H}$-valued functions $f$ with $v(\hat{\mu}, |f|^p)^{1/p} := |f|_p < \infty$ we denote by $L^p(\hat{\mu})$ for $0 < p < \infty$, $L^\infty(\hat{\mu})$ denotes the space of all $f$ for which there exist $|f|_\infty := ess_{v(\hat{\mu}, *)} - sup |f| < \infty$. In details we write $L^p(\mathcal{S}, \mathcal{Y}, \hat{\mu}, \mathcal{H})$ instead of $L^p(\hat{\mu})$. A subset $V$ in $\mathcal{S}$ we call $\mu$-zero-set, if $v^*(\hat{\mu}, V) = 0$, where $v^*$ is an extension of the total variation $v$ by the formula $v^*(\hat{\mu}, A) := \inf_{\mathcal{F} \subset \mathcal{A}} v(\hat{\mu}, F)$ for $A \subset \mathcal{S}$. A non-commutative measure $\hat{\lambda}$ on $\mathcal{S}$ we call absolutely continuous relative to $\hat{\mu}$, if $v^*(\hat{\lambda}, A) = 0$ for each subset $A \subset \mathcal{S}$ with $v^*(\hat{\mu}, A) = 0$. A measure $\hat{\mu}$ we call positive, if each $\mu_{m,n}$ is non-negative and $\sum_{m,n} \mu_{m,n}$ is positive. The usage of components $\mu_{m,n}$ and the classical Radon-Nikodym theorem (see Theorems III.10.2,10.7) lead to the following its non-commutative variants.

(i). If $(\mathcal{S}, \mathcal{Y}, \hat{\mu})$ is a space with a $\sigma$-finite positive non-commutative $\mathcal{H}$-valued measure $\hat{\mu}$, $\hat{\lambda}$ is absolutely continuous relative to $\hat{\mu}$ and it is a finite non-commutative measure defined on $\mathcal{Y}$, then there exists a unique $f \in L^p(\mathcal{S}, \mathcal{Y}, \hat{\mu}, \mathcal{H})$ such that $\hat{\lambda}(U) = \hat{\mu}(f_U)$ for each $U \in \mathcal{Y}$, moreover, $v(\hat{\mu}, \mathcal{S}) = |f|_1$.

(ii). If $(\mathcal{S}, \mathcal{Y}, \hat{\mu})$ is a space with a finite non-commutative $\mathcal{H}$-valued measure $\hat{\mu}$, $\hat{\lambda}$ is absolutely continuous relative to $\hat{\mu}$ and it is a non-commutative measure defined on $\mathcal{Y}$, then there exists a unique $f \in L^1(\hat{\mu})$ such that $\hat{\lambda}(U) = \hat{\mu}(f_U)$ for each $U \in \mathcal{Y}$. In view of (ii) there exists a Borel mea-
surable function $\phi$ such that $\hat{\nu}(\delta) := \hat{\mu}_{x,y}(\phi|\delta) = \langle \hat{E}(\phi|x)\delta; y \rangle > 0$ for each $\delta \in B(\mathcal{R})$. In view of (i) $|\phi(z)| = 1$ $\hat{\nu}$-almost every. Consider $f^1(z) := |f(z)|\phi(z)$, then in view of (a) $\mathcal{D}(f^1(T)) = \mathcal{D}(f(T))$ and $\langle f^1(T)x; y \rangle = \int_{-\infty}^{\infty}|f(z)|\hat{\nu}(dz)$. Therefore, $\langle f(T)x; y \rangle = \lim_{n} \int_{S_n} < \hat{E}(dz).f(z)x; y >$ and from this it follows (b).

(d). From $\hat{E}_S = (-1)^{s(E)}\hat{E}_S$ for each $S = cs$, $0 \neq c \in \mathcal{R}$, $s \in \{e, i, j, k\}$, it follows, that $\hat{E}.\hat{f} = \hat{E}.\hat{f}$. Take $x, y \in \mathcal{D}(\hat{f}(T)) = \mathcal{D}(f(T))$, then $\langle \hat{f}(T)x; y \rangle = \int_{-\infty}^{\infty} < \hat{E}(dz).\hat{f}(z)x; y > < x; f(T)y >$, consequently, $\hat{f}(T) \subset f(T)^*$. If $y \in \mathcal{D}(f(T)^*)$, then for each $x \in X$ and $m \in \mathcal{N}$: $\hat{f}_m(T)y = \hat{E}(\delta_m).f(T)y$ converges to $f(T)^*y$ while $m \to \infty$, consequently, $y \in \mathcal{D}(f(T))$.

In view of Theorem 2.24 Statement (e) follows from the fact, that $n\hat{E}(\delta) := \hat{E}(\delta_n \cap \delta)$ it is the decomposition of the identity for the restriction $T|_{X_n}$, where $X_n := \hat{E}(\delta_n)X$.

2.31. Theorem. A bounded normal operator $T$ on a quaternion HS is unitary, Hermitian or positive definite if and only if $\sigma(T)$ is contained in $S^3 := \{z \in \mathcal{H} : |z| = 1\}$, $\mathcal{R}$ or in $(0, \infty)$ respectively.

Proof. In view of Theorem 2.24 the equality $T^*T = TT^* = I$ is equivalent to $z\bar{z} = 1$ for each $z \in \sigma(T)$. If $\sigma(T) \subset [0, \infty)$, then $\langle Tx; x \rangle = \int_{\sigma(T)} < \hat{E}(dz).zx; x > \geq 0$ for each $x \in X$. The final part of the proof is analogous to the complex case, using the technique given above.

2.32. Definition. The family $\{T(t) : 0 \leq t \in \mathcal{R}\}$ of bounde QLO in $X$ is called a strongly continuous semigroup, if (i) $T(t+q) = T(t)T(q)$ for each $t, q \geq 0$; (ii) $T(0) = I$; (iii) $T(t)x$ is the continuous function by $t \in [0, \infty)$ for each $x \in X$.

2.33. Theorem. For each strongly continuous semigroup $\{U(t) : 0 \leq t \in \mathcal{R}\}$ of unitary QLO in HS $X$ over $\mathcal{H}$ there exists a unique self-adjoint QLO $B$ in $X$ such that $U(t) = \exp(itB)$, where $i = (-1)^{1/2}$.

Proof. If $\{T(t) : 0 \leq t\}$ is a semigroup continuous in the uniform topology, then due to Theorem VIII.1.2 [2] and Proposition 2.20 there exists a bounded operator $A$ in $X$ such that $T(t) = \exp(tA)$ for each $t \geq 0$. If $Re(z) := (z + \bar{z})/2 > |A|$, then $|\exp(-t(zI - A))| \leq \exp(t(|A| - Re(z))) \to 0$ while $t \to \infty$. For such $z \in \mathcal{H}$ due to the Lebesgue theorem: $(zI - A)\int_{0}^{\infty}\exp(-t(zI - A))dt = I$ and by Lemma 2.3 there exists $R(z; A) = \int_{0}^{\infty}\exp(-t(zI - A))dt$. For each $\epsilon > 0$ let $A_{\epsilon}x := (T(\epsilon)x - x)/\epsilon$, where $x \in X$, for which there exists $\lim_{\epsilon \to 0} A_{\epsilon}x$, a set of such $x$ we denote $\mathcal{D}(A)$. Evidently, that $\mathcal{D}(A)$ is the $\mathcal{H}$-vector space in $X$. We take on it the infinites-
imal QLO $Ax := \lim_{\epsilon \to 0} A \epsilon x$. Considering $H$ as BS over $R$ we get analogs of Lemmas 3,4,7, Corollaries 5, 9 and Theorem 10 from §VIII.1 [2], moreover, $\mathcal{D}(A)$ is dense in $X$, $A$ is closed QLO on $\mathcal{D}(A)$. Let $w_0 := \lim_{t \to \infty} \ln(|T(t)|)/t$ and $z \in H$ with $Re(z) > w_0$. For each $w_0 < \delta < Re(z)$ due to Corollary VIII.1.5 [2] there exists a constant $M > 0$ such that $|T(t)| \leq M \exp(\delta t)$ for each $t \geq 0$. Then there exists $R(z)x := \int_0^\infty \exp(-t(zI - A))x dt$ for each $x \in X$ and $Re(z) > w_0$, consequently, $R(z)x \in \mathcal{D}(A)$. Let $T_z$ be QLO corresponding to $z^{-1}A$ instead of $T$ for $A$, where $0 \neq z \in H$, moreover, $\mathcal{D}(A) = \mathcal{D}(z^{-1}A)$. Then $z^{-1}A \int_0^\infty \exp(-t(I - z^{-1}A))x dt = \int_0^\infty \exp(-t(I - z^{-1}A)z^{-1}A)x dt$, consequently, $R(z)(zI - A)x = x$ for each $x \in \mathcal{D}(A)$ and $R(z) = R(z; A)$. Therefore, $R(z; A)x = \int_0^\infty \exp(-t(zI - A))x dt$ for each $z \in \rho(A)$ and $x \in X$. 

With the help of 2.35. Lemma. For QLO $A$ there exists QLO $B$ such that $A = iB$, where $B = \begin{pmatrix} B_1 & B_2 \\ B_3^* & -B_1 \end{pmatrix}$. Since $U(t)U(t)^* = U(t)^*U(t) = I$, then $A$ commutes with $A^*$ and $\exp(t(A + A^*)) = I$. From $R(z; B)^* = R(\bar{z}, B)$ it follows, that $B = B^*$. If $\hat{E}$ it is the decomposition of the identity for $B$ and $V(t) := \exp(itB)$, by Theorem 2.30 $< V(t)x,y > = \int_0^\infty < \hat{E}(dz).\exp(itz)x,y > dt = \int_0^\infty \int_{-\infty}^\infty < \hat{E}(dz).\exp(-b - iz)t)x,y > dt = \int_0^\infty \exp(-bt)x,y > dt = \int_0^\infty < U(t).\exp(-bt)x,y > dt$ while $Re(b) > 0$. In view of Lemma VIII.1.15 $< V(t).\exp(-et)x,y >= < U(t).\exp(-et)x,y >$ for each $t \geq 0$ and $Re(b) > 0$, consequently, $U(t) = V(t)$. 

2.34. Notations. Let $X$ be a $H$-linear locally convex space. Consider left, right and two-sided $H$-linear spans of a family of vectors $\{v^a : a \in A\}$, where $\text{span}_H^l\{v^a : a \in A\} := \{z \in X : z = \sum_{a \in A} q_a v^a\}$; $\text{span}_H^r\{v^a : a \in A\} := \{z \in X : z = \sum_{a \in A} q_a v^a\}$; $\text{span}_H\{v^a : a \in A\} := \{z \in X : z = \sum_{a \in A} q_a v^a r_a\}$. 

2.35. Lemma. In the notation of §2.34 $\text{span}_H^l\{v^a : a \in A\} = \text{span}_H^r\{v^a : a \in A\} = \text{span}_H\{v^a : a \in A\}$. 

Proof. In view of continuity of the additivity and multiplication of vectors in $X$ and using convergence of a net of vectors it is sufficient to prove the statement of the lemma for a finite set $A$. Then the space $Y := \text{span}_H\{v^a : a \in A\}$ is finite-dimensional over $H$ and evidently, that left and right $H$-linear spans are contained in it. Then in $Y$ it can be chosen a basis over $H$ and each vector can be written in the form $v^a = \{v^a_1, ..., v^a_n\}$, where $n \in N$, $v^a_n \in H$. Each quaternion $q \in H$ can be written in the form of $4 \times 4$ real matrix, there-
fore, for each vector \( y \in Y \) there exist matrices \( A \) and \( B \), elements of which belong to \( H \) such that \( AV = y \) and \( WB = y \), where \( W = \{ v^a : a \in A \} \), \( V = W^t \) is the transposed matrix, since \( A, B, W \) and \( V \) can be written in the block form over \( R \) with blocks \( 4 \times 4 \). Therefore, \( \text{span}_H^t \{ v^a : a \in A \} \cap \text{span}_H \{ v^a : a \in A \} \supset \text{span}_H \{ v^a : a \in A \} \), that together with the inclusion \( \text{span}_H^t \{ v^a : a \in A \} \cup \text{span}_H \{ v^a : a \in A \} \subset \text{span}_H \{ v^a : a \in A \} \) proved above leads to the statement of this lemma.

2.36. Lemma. Let \( X \) be HS over \( H, X_R \) be the same space considered over the field \( R \). A vector \( x \in X \) is orthogonal to a \( H \)-linear subspace \( Y \) in \( X \) relative to the \( H \)-valued scalar product in \( X \) if and only if \( x \) is orthogonal to \( Y_R \) relative to the scalar product in \( X_R \). The space \( X \) is isomorphic to the standard HS \( l_2(\alpha, H) \) over \( H \) of converging by the norm sequences \( v = \{ v^a : a \in \alpha \} \) with the scalar product

\[
\langle v, w \rangle := \sum a \overline{v^a} w_a, \text{ moreover, } \text{card}(\alpha)\mathbb{R}_0 = \|w(X)\|, \text{ where } \text{card}(\alpha) \text{ is the cardinality of the set } \alpha, \mathbb{R}_0 = \text{card}(\mathbb{N}).
\]

Proof. In view of Lemma 2.35 and transfinite induction in \( Y \) there exists a \( H \)-linearly independent system of vectors \( \{ v^a : a \in A \} \) such that \( \text{span}_H^t \{ v^a : a \in A \} \) is everywhere dense in \( Y \). In another words in \( Y \) there exists a Hamel basis over \( H \). A vector \( x \) is by definition orthogonal to \( Y \) if and only if \( \langle v; x \rangle = 0 \) for each \( v \in Y \), that is equivalent to \( \langle v^a; x \rangle = 0 \) for each \( a \in A \). The space \( X_R \) is isomorphic with the direct sum \( X_e \oplus X_i \oplus X_j \oplus X_k \), where \( X_e, X_i, X_j \) and \( X_k \) are pairwise isomorphic HS over \( R \). The scalar product \( \langle x; y \rangle \) in \( X \) can be written in the form

\[
(i) \quad \langle x; y \rangle = \sum_{m,n \in \{e,i,j,k\}} \langle x_m; y_n \rangle m \bar{n},
\]

where \( \langle x_m; y_n \rangle \in R \) due to 2.16.(3,5). Then the scalar product \( \langle x; y \rangle \) in \( X \) induces the scalar product

\[
(ii) \quad \langle x; y \rangle_R = \sum_{m \in \{e,i,j,k\}} \langle x_m; y_m \rangle
\]

in \( X_R \). Therefore, from the orthogonality of \( x \) to the subspace \( Y \) relative to \( \langle x; y \rangle \) it follows orthogonality of \( x \) to the subspace \( Y_R \) relative to \( \langle x; y \rangle_R \). In view of Lemma 2.35 from \( y \in Y \) it follows, that \( m y_m \in Y \) for each \( m \in \{e,i,j,k\} \). Then from \( \langle x; y_m \rangle_R = 0 \) for each \( y \in Y \) and \( m \) due to 2.16.(5) it follows \( \langle x; y \rangle = 0 \) for each \( y \in Y \). Then by theorem about transfinite induction \([11]\) in \( X \) there exists an orthonormal basis over \( H \), in which each vector can be represented in the form of a converging series of left (or right) \( H \)-linear combinations of basis vectors. For each \( x \in X \) in view of normability of \( X \) the base of neighbourhoods is countable and for the topological density we have the equality \( d(X) = \text{card}(\alpha)\mathbb{R}_0 \), since \( H \) is separable, hence \( w(X) = d(X) \). From this the last statement of this lemma
follows.

**2.37. Lemma.** For each QLO $T$ in $H^{S}X$ over $H$ an adjoint operator $T^{*}$ in $X$ relative to a $H$-scalar product coincides with an adjoint operator $T^{*}_{R}$ in $X_R$ relative to a $R$-valued scalar product in $X_R$.

**Proof.** Let $D(T)$ be a domain of operator $T$, which is dense in $X$. In view of Formulas 2.36.(i, ii) and the existence of the automorphisms $z \mapsto zm$ in $H$ for each $m \in \{e, i, j, k\}$ it follows that the continuities of $<Tx;y>$ and $<Tx;y>_R$ by $x \in D(T)$ are equivalent, therefore, in view of Lemma 2.35 the family of all $y \in X$, for which $<Tx;y>$ is continuous by $x \in D(T)$ forms a $H$-linear subspace in $X$ and it is the same relative to $<Tx;y>_R$, that is the domain $D(T^{*})$ of the operator $T^{*}$. Then the adjoint operator $T^{*}$ is defined by the equality $<Tx;y>=<x;T^{*}y>$, while $T^{*}_{R}$ is given by $<Tx;y>_R=<x;T^{*}_{R}y>_R$, where $x \in D(T)$ and $y \in D(T^{*})$. In view of Formulas 2.36.(i, ii) $<x_{m};(T^{*})_{m}>=<x_{m};(T_{m}^{*})>_R$ for each $x \in D(T)$, $y \in D(T^{*})$ and $m \in \{e, i, j, k\}$. Since due to Proposition 2.20 and Lemma 2.35 $D(T)$ and $D(T^{*})$ are $H$-linear, then automorphisms of the field $H$ given above lead to $T^{*}=T^{*}_{R}$.

**2.38. Definition.** A bounded QLO $P$ in $H^{S}X$ over $H$ is called a partial $R$- (or $H$-) isometry if there exists a closed $R$- (or $H$-) linear subspace $Y$ such that $\|Px\|=\|x\|$ for each $x \in Y$ and $P(Y_{R}^{\perp})=\{0\}$ (or $P(Y_{H}^{\perp})=\{0\}$) respectively, where $Y_{R}^{\perp}:=\{z \in X: <z;y>=0 \forall y \in Y\}$, $Y_{H}^{\perp}:=\{z \in X_H: <z;y>_R=0 \forall y \in Y\}$.

**2.39. Theorem.** If $T$ is closed QLO in $H^{S}X$ over $H$, then $T=PA$, where $P$ is a partial $R$-isometry on $X_R$ with a domain $cl(Range(T^{*}))$ and $A$ is a self-adjoint QLO such that $cl(Range(A))=cl(Range(T^{*})))$. If $T$ is $H$-linear, then $P$ is a partial $H$-isometry.

**Proof.** In view of the spectral Theorem 2.28 a self-adjoint QLO $T$ is positive if and only if its spectrum $\sigma(T) \subset [0, \infty)$ (see also Lemma XII.7.2 [2]). In the field $H$ each polynomial has a root (see Theorem 3.17 [8]). Therefore, $T$ is a positive self-adjoint QLO, then there exists a unique positive QLO $A$ such that $A^{2}=T$ (see also Lemma XII.7.2 [2]). Then there exists the positive square root $A$ of the operator $T^{*}T$. Moreover, $A$ is $H$-linear, if $T$ is $H$-linear. Put $SAx=Tx$ for each $x \in D(T^{*}T)$ and $V$ be an isometrical extension of $S$ on $cl(Range(A))$. The space $cl(Range(A))$ is $R$-linear. If $A$ is in addition left- (or right-) $H$-linear, then $cl(Range(A))$ is a $H$-linear subspace due to Lemma 2.35. In view of Lemma 2.36 there exists the perpendicular projector $E$ from $X$ on $cl(Range(A))$, moreover, $E$ is $H$-linear, if $cl(Range(A))$ is the
isolated subset \( \lambda \) there exists a neighbourhood \( \eta \) where 

\[
p_3 \leq 3.22 \quad (8),
\]

denote (closed and open simultaneously) in \( \sigma \). A closed rectifiable path in \( U \) is \( H \)-linear subspace. Then put \( P = VE \). From \( \langle Ax; Ax \rangle = \langle Tx; Tx \rangle \) for each \( x \in D(T^*T) \) it follows, that \( PAx = Tx \) for each \( x \in D(T^*T) \). The rest of the proof can be done analogously to the proof of Theorem XII.7.7 [2] with the help of Lemmas 2.35-37.

2.40. Note and Definition. Apart from the case of \( C \) nontrivial polynomials of quaternion variables one has roots, which are not points, but closed submanifolds in \( H \) with dimensions over \( R \) from 0 up to 3 (see [8]).

A closed subset \( \lambda \subset \sigma(T) \) is called an isolated subset of the spectrum, if there exists a neighbourhood \( U \) of a subset \( \lambda \) such that \( \sigma(T) \cap U = \lambda \). An isolated subset \( \lambda \) of a spectrum \( \sigma(T) \) is called a pole of a spectrum (of order \( p \)), if \( R(z; T) \) has zero on \( \lambda \) (of order \( p \)), that is, each \( z \in \lambda \) is zero of order \( 0 < p(z) \leq p \) for \( R(z; T) \) and \( \max_{z \in \lambda} \| p(z) = p \| \). A subset \( \lambda \) which is clopen (closed and open simultaneously) in \( \sigma(T) \) is called a spectral set. Let \( \eta_1 \) be a closed rectifiable path in \( U \) encompassing \( \lambda \) and not intersecting with \( \lambda \), characterized by a vector \( M_1 \in H \), \( |M_1| = 1 \), \( M_1 + M_1 = 0 \) (see Theorem 3.22 [8]), denote

\[
(\phi_n(z, T) := (2\pi)^{-1}\left\{ \int_{\eta_1} R(\zeta; T)((\zeta - a)^{-1}(z - a))^{n}(\zeta - a)^{-1}d\zeta \right\} M_1^{-1},
\]

where \( \eta_1 \subset B(H, a, R) \setminus B(H, a, r), B(H, a, R) \subset U \), \( \lambda \subset B(H, a, r) \), \( 0 < r < R < \infty \). We say, that an index of \( \lambda \) is equal to \( p \) if and only if there exists a vector \( x \in X \) such that

\[
(\phi_n(z, T) : \phi_{n+1}(z, T)v_1 \cdots (zI - T)^{s_n}v_{2m-1} \cdots (zI - T)^{s_m} v_{2m-1} \cdots (zI - T)^{s_1} x = 0
\]

for each \( z \in \lambda \) and each \( 0 \leq s_n \in \mathbb{Z} \) with \( s_1 + \cdots + s_m = p \) and each \( v_1, \ldots, v_{2m-1}, \)

where \( v_1 = v_1(\delta, T) \in H, \ldots, v_{2m-1} = v_{2m-1}(\delta, T) \in H \), \( m \in \mathbb{N} \), \( \delta := \delta(z) \in \mathcal{B}(\lambda) \), while expression in (ii) is not equal to zero for some \( z \in \lambda \) and \( s_1, \ldots, s_m \) with \( s_1 + \cdots + s_m = p - 1 \).

2.41. Theorem. A subset \( \lambda \) is a pole of order \( p \) of QLO \( T \in L_q(X) \) for \( U = B(H, a, R) \), \( 0 < R < R' < \infty \) in Definition 2.40, where \( 0 < r < \infty \) if and only if \( \lambda \) has an index \( p \).

Proof. Choose with the help of a homotopy relative to \( U \setminus \lambda \) closed paths \( \eta_1 \) and \( \eta_2 \) homotopic to \( \gamma_1 \) and \( \gamma_2 \), moreover, \( \inf_{\theta} \| \eta_1(\theta) \| > \sup_{\theta} \| \eta_2(\theta) \| \), where \( \gamma_1 \) and \( \gamma_2 \) are chosen as in Theorem 3.22 [8] (see also Theorem 3.9 there), \( \theta \in [0, 1] \). In view of Theorem 3.22 [8] the quaternion Loran decomposition of
\( R(z; T) \) in the neighbourhood \( B(H, a, R) \setminus B(H, a, r) \) has the form \( R(z; T) = \sum_{n=0}^{\infty} (\phi_n(z, T) + \psi_n(z, T)) \), where \( \phi_n \) is given by Formula 2.40.(i) and

\[
(i) \quad \psi_n(z, T) := (2\pi)^{-1} \left\{ \int_{\eta_2} R(\zeta; T)(z-a)^{-1}((\zeta-a)(z-a)^{-1})^{n}d\zeta \right\} M_2^{-1}.
\]

If \( \lambda \) is a pole of order \( p \), then \( \phi_p = 0 \) and \( \phi_{p-1} \neq 0 \), then there exists \( x \in X \) such that

\[
(ii) \quad \phi_p(z, T)x = 0 \quad \text{for each} \quad z \in \lambda, \quad \text{and}
\]

\[
(iii) \quad \phi_{p-1}(z, T)x \neq 0 \quad \text{for some} \quad z \in \lambda.
\]

An analogous decompositions with the corresponding \( \phi_n \) are true for the products \( f(T)R(z; T)g(T) \), where \( f \) and \( g \) are quaternion holomorphic functions on a neighbourhood \( \sigma(T) \) not equal to zero everywhere on \( \lambda \). Functions \( \phi_n \) for \( R(z; T) \) can be approximated with any precision in the strong operator topology in the form of left \( H \)-linear combinations of functions taking part in 2.40.(ii) due to Lemma 2.35 and the definition of the quaternion line integral along a rectifiable path, since while \( |\xi| > \sup |\chi| \) the series for \( R(\xi; T_\chi) \) converges in the uniform operator topology for each spectral set \( \chi \) of the spectrum \( \sigma(T) \), where \( T_\chi = T|_{X_\chi}, \quad X_\chi := \hat{E}_\chi(\chi; T)X \). The variation of \( f \) and \( g \) implies, that the index of \( \lambda \) is not less than \( p \). Vice versa, let Conditions (ii) be satisfied for some \( n \). The resolvent \( R(z; T)x \) is regular on \( H \setminus B(H, a, r) \) and

\[
x = (2\pi)^{-1} \left\{ \int_{\eta} R(\zeta; T)x d\zeta \right\} M_1^{-1} = (2\pi)^{-1} \left\{ \int_{\eta_2} R(\zeta; T)x d\zeta \right\} M_2^{-1} = \omega(T)x,
\]

where \( \omega(T) \) is the function equal to 1 on a neighbourhood of \( \lambda \) and equal to zero on \( H \setminus U \), \( \eta \) is the corresponding closed rectifiable path encompassing \( \sigma(T) \) and characterized by \( M \in H, \quad |M| = 1, \quad M + \bar{M} = 0 \). Then due to 2.40.(ii) \( \phi_p(z, T)x = 0 \) for each \( z \in \lambda \).

2.42. Note. An isolated point \( \lambda \) of a spectrum \( \sigma(T) \) for normal QLO \( T \in L_q(X) \) on HS \( X \) over \( H \) may not have eigenvectors because of non-commutativity of a projection-valued measure \( \hat{E} \) apart from the case of linear operators on HS over \( C \).

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