Hamiltonian formalism of Minimal Massive Gravity

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Abstract

We study the three-dimensional Minimal Massive Gravity (MMG) in the Hamiltonian formalism. Canonical expressions for the asymptotic conserved charges are derived by defining the canonical gauge generators. Specifically, the construction of asymptotic structure of MMG requires to introduce suitable boundary conditions. For instance, the application of this procedure is done for the BTZ black hole as a solution to the MMG field equations. The related conserved charges give the energy and angular momentum of the BTZ black hole. We also show that the Poisson bracket algebra of the improved canonical gauge generators produces an asymptotic gauge group which includes two separable versions of Virasoro algebras. Finally, we calculate the entropy of black hole from Cardy formula using the parameters of the boundary conformal field theory and show the result is consistent with the value obtained from Smarr one.
\textbf{1 Introduction}

The non-renormalizability of general relativity (GR) in its simple Einstein-Hilbert (EH) form has changed the attentions to modify this theory nearly one half of a century. As if it seems that this dilemma is solved by adding higher order curvature terms to EH Lagrangian \cite{1}, but this work threatens unitarity of the theory. Another possibility is to study GR as a topological theory in three-dimensions (3D). But the 3D gravity, with or without a cosmological constant term \cite{2-5}, has no physical local degrees of freedom. Adding a gravitational Chern-Simons (CS) term to the EH action results in a physical degree of freedom, which is known as topologically massive gravity (TMG) \cite{6-8}.

Although TMG is a renormalizable theory of gravity in 3D, but it suffers from the bulk-boundary unitarity problem. In fact when one considers the spin-2 mode in 3D bulk gravity with an asymptotically anti-de Sitter (AdS) background, in correspondence there might be a dual conformal field theory (CFT) at the boundary. The unitarity of boundary CFT implies the positivity of central charges of asymptotic symmetry algebra. But in order to have positive energy BTZ black hole \cite{9} in TMG, the Newton’s constant must be positive which results in negative central charge in the boundary theory \cite{10}.

It seems that the problem is resolved if we consider the bulk theory at a critical point \cite{10,11}, known as chiral gravity at which the negative central charge vanishes. But specifically at this point the spin-2 mode is replaced by a logarithmic mode which results in a non-unitary logarithmic conformal field theory at the boundary \cite{11,12}. Other modified theories as new massive gravity (NMG) \cite{13,14} and its extensions \cite{15,16} are also introduced which suffer from similar problem in critical points \cite{17}.

Recently, a new version of 3D gravity proposed in \cite{18} that has one massive degrees of freedom as TMG which is called minimal massive gravity (MMG). Not only this theory is unitary in the bulk, but also results in a unitary CFT on the boundary, of course for some values in parameter space of the theory. A particular feature of this theory is that it has no Lagrangian in the metric formalism while is formulated in first order canonical structure. Some aspects of this theory as black hole solutions \cite{19,20} and chiral MMG \cite{21,22} have been considered.

In this paper we want to study the canonical Hamiltonian formalism of MMG based on \cite{23}. Initially, the gauge generators of asymptotic symmetries are constructed. Since the global properties of space-time affect symmetry of the asymptotic configurations which are related to the conservation laws, these gauge generators produce some conserved charges. Then we investigate the conserved charges as energy and angular momentum of the BTZ black hole in this formalism. The analysis for the case of torsion free TMG has been done in \cite{24,25}.

It has been shown that a suitably defined covariant Poisson bracket (PB) algebra of charge generators forms a centrally extended representation of the asymptotic symmetry algebra \cite{26}. The analysis of asymptotic symmetry algebra for the BTZ space-time in the Hamiltonian approach reveals the appearance of two Virasoro algebras with different conserved charges in the left-right sectors. Because of no metric
Lagrangian, we are unable to use the Wald formula \[27\] for computing the entropy for BTZ black hole, but one can derive it by application of the Smarr-like formula \[28\]. Finally, the consistency of calculations is done from holographic considerations by using the Cardy formula \[29\].

This paper is organized as follows: In section 2 we briefly review the MMG Lagrangian in canonical formalism and derive the field equations. In section 3 the canonical structure of the theory is discussed and the general form of gauge generators is calculated. In section 4 to clarify the application of this formalism, the BTZ black hole is studied and the asymptotic behavior of the theory is considered. In Section 5 we give a summary of discussions and concluding results.

Conventions: the metric in local Lorentz frame is mostly minus \(\eta_{ij} = (+, -, -)\); the Latin indices \((i, j, k, \ldots)\) and the Greek indices \((\mu, \nu, \lambda, \ldots)\) refer respectively to local Lorentz and coordinate frames and run over 0, 1, 2 while the letters \((\alpha, \beta, \gamma, \ldots)\) run over 1, 2; both Levi-Civita antisymmetric tensors \(\varepsilon^{ijk}\) and \(\varepsilon^{\mu\nu\rho}\) are normalized as \(\varepsilon^{012} = 1\). The space-time metric is a bilinear combination of the triad fields:

\[
g = g_{\mu\nu} \, dx^\mu \otimes dx^\nu = \eta_{ij} \, e^i \otimes e^j ,
\]

\[
g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu .
\]

2 Minimal Massive Gravity

The MMG was introduced in \text{[18]} to resolve the deficiencies of TMG at the linearized level such that in the case of black hole objects leaves both mass and CFT central charge positive. The Einstein field equation of TMG \text{[7]} is given by

\[
\frac{1}{\mu} \, C_{\mu\nu} + \sigma \, G_{\mu\nu} + \Lambda_0 \, g_{\mu\nu} = 0 ,
\]

(2.1)

where \(G_{\mu\nu}\) and \(C_{\mu\nu}\) are the Einstein and symmetric traceless Cotton tensors. In \text{[18]} a new field equation is defined for TMG as a minimal construction:

\[
\frac{1}{\mu} \, C_{\mu\nu} + \bar{\sigma} \, G_{\mu\nu} + \bar{\Lambda}_0 \, g_{\mu\nu} = -\frac{\gamma}{\mu^2} \, J_{\mu\nu} ,
\]

(2.2)

with the symmetric tensor

\[
J_{\mu\nu} = \frac{1}{2|g|} \, \varepsilon^\rho_\mu \varepsilon^\tau_\nu \, S_{\rho\tau} S_{\lambda\eta} ,
\]

(2.3)

where \(S_{\mu\nu}\) is the Schouten tensor

\[
C_{\mu\nu} = \frac{1}{\sqrt{-|g|}} \, \varepsilon^\rho_\mu \varepsilon^\tau_\eta \, D_\tau S_{\rho\nu} , \quad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} \, g_{\mu\nu} \, R .
\]

(2.4)

\(\gamma\) is a non-zero dimensionless constant and \(\sigma\), \(\Lambda_0\) in the case of TMG \text{[21]} are replaced by \(\bar{\sigma}\), \(\bar{\Lambda}_0\) in \text{[22]} for MMG, since it is not obvious they should be equal to initial values. Because the linearization of \text{[22]} around a maximally symmetric background gives the linearized TMG with modified coefficients \text{[22]}, therefore MMG has the same degrees of freedom as TMG.
As seen this theory is constructed in terms of a field equation (2.2), which is not obtained from a metric action as TMG and NMG. So for further studies it would be better to study this theory in another formalism. The 3-form Lagrangian of MMG in the Chern-Simons-like formalism is given by

\[ L_{\text{MMG}} = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e \times e + h \cdot T + \frac{1}{2\mu} \left( \omega \cdot d\omega + \frac{1}{3} \omega \cdot \omega \times \omega \right) + \frac{\alpha}{2} e \cdot h \times h , \]  
(2.5)

where as before \( \sigma, \alpha \) are dimensionless parameters, \( \Lambda_0 \) is a cosmological constant term with mass-squared dimension and \( \mu \) is a parameter of dimension one. It has been shown that MMG resolves the problem of unitarity (no ghost/no tachyon) for some values of these parameters [18].

In order to have a torsion free condition we use a Lagrange multiplier field \( h \) which as spin connection \( \omega \) is an odd parity one dimensional field. For subsequent aims we rewrite the Lagrangian (2.5) in some useful form

\[ L_{\text{MMG}} = \frac{1}{2} \epsilon^{\mu\nu\rho} \left[ -\sigma e_i^\mu R_{i\nu\rho} + \frac{\Lambda_0}{3} \varepsilon_{ijk} e^{i\mu} e^{j\nu} e^{k\rho} + h^{i\mu} T_{i\nu\rho} \right. \\
+ \frac{1}{\mu} \left( \omega_i^\mu \partial_{i\nu} \omega_i^\rho + \frac{1}{3} \varepsilon_{ijk} \omega_i^\mu \omega_j^\nu \omega_k^\rho \right) + \alpha \varepsilon_{ijk} e_i^{\mu} h^j_\nu h^k_\rho \left. \right] , \]  
(2.6)

where the fields \( e_i^\mu, \omega_i^\mu, h^i_\mu \) are three Lorentz 1-form. The field strengths \( T^i(\omega) \) and \( R^i(\omega) \) are Lorentz covariant torsion and curvature 2-forms given by

\[ T^i = de^i + \varepsilon_{ijk} \omega^j e^k , \quad R^i = d\omega^i + \frac{1}{2} \varepsilon_{ijk} \omega^j \omega^k . \]  
(2.7)

In the case of \( \alpha = 0 \), the Lagrangian (2.5) gives rise to cosmological topological massive gravity in torsion free limit [4]. In addition for \( \mu \to \infty \) and \( \sigma = 1 \) it reduces to the custom 3D cosmological general relativity [5]. Varying the Lagrangian (2.6) with respect to 1-form fields \( e_i^\mu, \omega_i^\mu, h^i_\mu \) yields the following field equations, respectively,

\[ 0 = \epsilon^{\mu\nu\rho} \left( -\sigma R_{i\nu\rho} + \frac{\Lambda_0}{3} \varepsilon_{ijk} e^{j\nu} e^{k\rho} + D_\nu h_i^\rho + \alpha \varepsilon_{ijk} h^j_\nu h^k_\rho \right) , \]
\[ 0 = \epsilon^{\mu\nu\rho} \left( -\sigma T_{i\nu\rho} + \varepsilon_{ijk} e^{j\nu} h^k_\rho + \mu^{-1} R_{i\nu\rho} \right) , \]
\[ 0 = \epsilon^{\mu\nu\rho} \left( T_{i\nu\rho} + 2\alpha \varepsilon_{ijk} e^{j\nu} h^k_\rho \right) , \]  
(2.8)

in addition to some total derivative surface terms and that “\( D(\omega) = d + \omega \times \)" is the Lorentz covariant derivative. One can rewrite the equations of (2.8) as

\[ 0 = D(\Omega) h - \frac{\alpha}{2} h \times h + \sigma \mu (1 + \sigma \alpha) e \times h + \frac{\Lambda_0}{2} e \times e , \]
\[ 0 = R(\Omega) + \frac{\alpha \Lambda_0}{2} e \times e + \mu (1 + \sigma \alpha)^2 e \times h , \]
\[ 0 = T(\Omega) , \]  
(2.9)

where the last line represents the torsion free condition for connection field \( \Omega = \omega + \alpha h \) i.e.

\[ T^i = de^i + \varepsilon_{ijk} \Omega^j e^k = 0 . \]  
(2.10)
In order to have the same local degrees of freedom as TMG, in parameter space we must have 
\((1 + \sigma \alpha) \neq 0\). While for equality, the theory becomes a CS-like theory with no degrees of freedom [3].

The second equation of (2.9) yields
\[
h_{\mu \nu} = -\frac{1}{\mu (1 + \sigma \alpha)^2} \left[ S_{\mu \nu} + \frac{\alpha \Lambda_0}{2} g_{\mu \nu} \right], \tag{2.11}
\]
where by substituting (2.11) in the first equation of (2.9) and comparing the resultant equation by the metric field equation (2.2) gives
\[
\gamma = -\frac{\alpha}{(1 + \sigma \alpha)^2}, \quad \tilde{\Lambda}_0 = \Lambda_0 \left[ 1 + \sigma \alpha - \frac{\alpha^3 \Lambda_0}{4 \mu^2 (1 + \sigma \alpha)^2} \right], \quad \bar{\sigma} = \sigma + \alpha \left[ 1 + \frac{\alpha \Lambda_0}{2 \mu^2 (1 + \sigma \alpha)^2} \right], \tag{2.12}
\]
in other words, we have a canonical action (2.5) which yields the field equation (2.2).

3 Canonical structure

In this section we will consider the structure of the MMG as a gauge theory described by (2.6) in the canonical formalism [23]. The gauge symmetries determine the physical content of any gauge theory by means of some gauge generators which lead to a number of conserved charges. Often there are two classes of conserved charges: the exact ones associated to symmetries of the background solution and the asymptotic ones related to symmetries close to infinity or at the boundary. The calculation of these charges and their properties in the canonical formalism is the underlying idea of our work hereafter.

Here we will not discuss this procedure in details while enumerate the substantial constructions to reach the gauge generators of the MMG theory as [23, 24]:

1. The canonical momenta \((\pi_{i \mu}, \Pi_{i \mu}, p_{i \mu})\) are respectively defined for the Lagrangian variables
\[
\pi_{i \mu} \equiv \frac{\partial L}{\partial \dot{e}_{i \mu}}, \quad \Pi_{i \mu} \equiv \frac{\partial L}{\partial \dot{\omega}_{i \mu}}, \quad p_{i \mu} \equiv \frac{\partial L}{\partial \dot{h}_{i \mu}}, \tag{3.1}
\]

where by using the Lagrangian (2.6) we can define the following set of primary constraints
\[
\phi_i^0 \equiv \pi_i^0 \approx 0, \quad \phi_i^\alpha \equiv \pi_i^\alpha - \varepsilon^{0 \alpha \beta} h_{i \beta} \approx 0, \\
\Phi_i^0 \equiv \Pi_i^0 \approx 0, \quad \Phi_i^\alpha \equiv \Pi_i^\alpha + \varepsilon^{0 \alpha \beta} (\sigma e_{i \beta} - \mu^{-1} \omega_{i \beta}) \approx 0, \\
\psi_i^\mu \equiv p_i^\mu \approx 0. \tag{3.2}
\]

The canonical Hamiltonian in this construction is expressed as
\[
\mathcal{H}_c = \epsilon_{i 0} \mathcal{H}_i + \omega_{i 0} K_i + h_i^0 T_i + \partial_\alpha S^\alpha, \tag{3.3}
\]
\[
\mathcal{H}_i = -\varepsilon^{0 \alpha \beta} \left( -\sigma R_{i \alpha \beta} + \Lambda_0 \varepsilon_{i j k} e^j_\alpha h^k_\beta + \alpha \varepsilon_{i j k} h^j_\alpha h^k_\beta + D_\alpha h_{i \beta} \right), \\
K_i = -\varepsilon^{0 \alpha \beta} \left( -\sigma T_{i \alpha \beta} + \varepsilon_{i j k} e^j_\alpha h^k_\beta + \mu^{-1} R_{i \alpha \beta} \right), \\
T_i = -\varepsilon^{0 \alpha \beta} \left( T_{i \alpha \beta} + 2 \varepsilon_{i j k} e^j_\alpha h^k_\beta \right), \\
S^\alpha = \varepsilon^{0 \alpha \beta} \left( \omega_{i 0}^\beta - \sigma e_{i \beta} + \mu^{-1} \omega_{i \beta} \right) + \epsilon^i_0 h_{i \beta}. \]
where the last term in $\mathcal{H}_c$ is the surface term of (2.6). The total Hamiltonian constructed from the primary constraints given by

$$\mathcal{H}_T = e^i_0 \mathcal{H}_i + \omega^i_0 \mathcal{K}_i + h^i_0 \mathcal{T}_i + u^i_\mu \phi^\mu_i + v^i_\mu \Phi^\mu_i + w^i_\mu \psi^\mu_i + \partial_\alpha S^\alpha,$$

the consistency conditions can be written as the approach introduced in [23], [24] which guarantee the primary constraints $\pi^0_i$, $\Pi^0_i$ and $p^0_i$ yield the secondary constraints:

$$\mathcal{H}_i \approx 0, \quad \mathcal{K}_i \approx 0, \quad \mathcal{T}_i \approx 0,$$

and due to the remaining primary constraints $\phi^{\alpha}_i$, $\Phi^{\alpha}_i$ and $p^{\alpha}_i$, the multipliers $u^i_\mu$, $v^i_\mu$ and $w^i_\mu$ are determined. Some of these consistency conditions are derived in appendix A.

2. The canonical structure of the asymptotic symmetry is described by the canonical gauge generators

$$G = -G_1 - G_2,$$

$$G_1 = \dot{\xi}^\rho (e^i_\rho \pi^0_i + \omega^i_\rho \Pi^0_i + h^i_\rho p^0_i)$$

$$+ \xi^\rho [e^i_\rho K_i + \omega^i_\rho K_i + h^i_\rho T_i + (\partial_\rho e^i_0)\pi^0_i + (\partial_\rho \omega^i_0)\Pi^0_i + (\partial_\rho h^i_0)p^0_i],$$

$$G_2 = \dot{\theta}^i \Pi^0_i + \theta^i [\bar{\mathcal{K}}_i - \varepsilon_{ijk} (e^0_j \pi^k + \omega^0_j \Pi^k + h^0_j p^k)].$$

where the dot sign is the time derivative and a factor of $\frac{1}{8\pi G} \int d^3 x$ is omitted for simplicity. The local Poincaré gauge transformations (PGT) are:

$$\delta_0 e^i_\mu = -\varepsilon^i_{jk} e^j_\mu \theta^k - (\partial_\mu \xi^0) e^i_\rho - \xi^0 \partial_\rho e^i_\mu,$$

$$\delta_0 \omega^i_\mu = -\nabla^i_\mu \theta^\rho - (\partial_\mu \xi^\rho) \omega^i_\rho - \xi^\rho \partial_\rho \omega^i_\mu,$$

$$\delta_0 h^i_\mu = -\varepsilon^i_{jk} h^j_\mu \theta^k - (\partial_\mu \xi^0) h^i_\rho - \xi^0 \partial_\rho h^i_\mu.$$

The symmetric parts of these equations give the asymptotic local translations $\xi^\mu$ and the antisymmetric ones give the local rotations $\theta^i$ of Poincaré transformations [24]. For example multiplying the first of (3.7) by $e^i_\nu$ and using (1.1) yields the general transformation of the metric

$$\delta_0 G_{\mu\nu} = -(\partial_\mu \xi^\rho) g_{\nu\rho} - (\partial_\nu \xi^\rho) g_{\mu\rho} - \xi^\rho \partial_\mu g_{\nu\rho}.$$

3. The variation of $G_2$ produces a total derivative term which by choosing the consistent boundary conditions in the next section has vanishing contribution after integration. Going over to variation of $G_1$ after some substitutions we have

$$\delta G_1 = \xi^\rho (e^i_\rho \delta \mathcal{H}_i + \omega^i_\rho \delta \mathcal{K}_i + h^i_\rho \delta \mathcal{T}_i) + \partial \mathcal{O} + R$$

$$= 2 \varepsilon^{\alpha \beta} \xi^\rho \partial_\alpha [e^i_\rho (\sigma \delta \omega^i_\beta - \frac{1}{2} \delta h^i_\beta) + \omega^i_\rho (\sigma \delta e^i_\beta - \frac{1}{2} \delta \omega^i_\beta) - h^i_\rho \delta e^i_\beta] + \partial \mathcal{O}_2 + R,$$

where the last term in $\mathcal{H}_c$ is the surface term of (2.6). The total Hamiltonian constructed from the primary constraints given by the canonical gauge generators of $G$ conditions in the next section has vanishing contribution after integration. Going over to variation the first of (3.7) by $e$ metric ones give the local rotations $\theta$ where the dot sign is the time derivative and a factor of $1$ are determined. Some of these consistency conditions are derived in appendix A.
where the term $\partial O_2$ is a boundary term that vanishes after integration and $R$ are some regular terms. Hereafter when we use $O_n$ it has the distance behavior as $\sim r^{-n}$, so using the Stokes theorem

$$\int_{M_2} d^2x \partial_\alpha v^\alpha = \int_{\partial M_2} v^\alpha df_\alpha = \int_0^{2\pi} v^1 d\varphi \quad (df_\alpha = \varepsilon_{\alpha\beta} dx^\beta),$$

the second term in (3.9) has no contribution in asymptotic conserved charges. Here the boundary of $M_2$ is a circle at infinity parametrized by the angular coordinate $\varphi$.

4. We can write the relation (3.9) as

$$\delta G_1 = \partial_\alpha (\xi^0 \delta E^\alpha + \xi^2 \delta M^\alpha),$$

where

$$E^\alpha = 2 \varepsilon^{0\alpha\beta} \left[ e^i_0 (\sigma \delta \omega_{i\beta} - \frac{1}{2} \delta h_{i\beta}) + \omega^i_0 (\sigma \delta e_{i\beta} - \frac{1}{\mu} \delta \omega_{i\beta}) - h^i_0 \delta e_{i\beta} \right],$$

$$M^\alpha = 2 \varepsilon^{0\alpha\beta} \left[ e^i_2 (\sigma \delta \omega_{i\beta} - \frac{1}{2} \delta h_{i\beta}) + \omega^i_2 (\sigma \delta e_{i\beta} - \frac{1}{\mu} \delta \omega_{i\beta}) - h^i_2 \delta e_{i\beta} \right].$$

The conserved charges we are interested to calculate them are energy and angular momentum of the system which obtained by diffeomorphisms $\xi^0 = 1$ and $\xi^2 = 1$ as

$$E = \int_0^{2\pi} E^1 d\varphi, \quad J = \int_0^{2\pi} M^1 d\varphi.$$ (3.13)

4 Black Holes in MMG

Any solution of Enistein gravity with negative cosmological constant is automatically a solution of MMG. So BTZ solution constitutes a candidate for the black hole object of the theory as $AdS_3$ as vacuum solution. There is a parameter space for constant values in this theory which encourages us to exemplify the physical quantities for BTZ black holes.

4.1 Canonical BTZ solution

The most well-known maximally symmetric black hole solution for 3D gravity is BTZ black hole which its line element of space-time in the ADM [30] form is given by

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\varphi + N_\varphi dt)^2,$$ (4.1)

where the functions $N, N_\varphi$ are given by

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}, \quad N_\varphi = \frac{r_+ r_-}{l r^2}.$$ (4.2)

Note that we use the mostly minus signature for the metric of space-time by $(t, r, \varphi)$ coordinates and $r_+$ and $r_-$ are respectively the outer and inner horizons of BTZ black hole. This is a maximally symmetric solution such that

$$R^i = \frac{\Lambda}{2} \epsilon^i_{jk} e^j e^k.$$ (4.3)
and the relation between cosmological parameter and $l$ is $\Lambda = -1/l^2$. The components of triad 1-form $e^i$ in the first order formalism for (4.1) are

$$e^0 = N dt, \quad e^1 = N^{-1} dr, \quad e^2 = r (d \phi + N \phi dt),$$

(4.4)

and the components of spin connection $\Omega^i$ are computed from torsion-free condition (2.10) as

$$\Omega^0 = -N d \phi, \quad \Omega^1 = N^{-1} N \phi dr, \quad \Omega^2 = -\frac{r}{l^2} dt - r N \phi d \phi.$$  

(4.5)

Substituting the relation (4.3) in the second equation of (2.9) gives

$h^i = \mu C e^i,$

(4.6)

where the constant $C$ is defined in terms of parameter space

$$C = -\frac{(\Lambda + \alpha \Lambda_0)}{2 \mu^2 (1 + \sigma \alpha)^2}.$$  

(4.7)

### 4.2 Conserved charges

As mentioned in section 3 there are two types of conservation laws according to asymptotic symmetries. In fact those transformations that allow the field configurations under consideration to remain asymptotically invariant are asymptotic symmetries. They generate the known asymptotic symmetry group algebra [26].

On the other hand not only the asymptotic symmetries should be invariant under the isometry group of asymptotically $AdS_3$ backgrounds, $SL(2,R) \times SL(2,R)$, but also should have well-defined canonical generators. For the aforementioned reasons, there is no doubt that we must use suitable boundary conditions that respect them.

Thus in the case of BTZ solution the asymptotic behavior of triads are given by

$$e^i_\mu = e^i_\mu + E^i_\mu, \quad e^i_\mu = \begin{pmatrix} r/l & 0 & 0 \\ 0 & l/r & 0 \\ 0 & 0 & r \end{pmatrix}, \quad E^i_\mu \sim \begin{pmatrix} O_1 & O_4 & O_1 \\ O_2 & O_3 & O_2 \\ O_1 & O_4 & O_1 \end{pmatrix},$$

(4.8)

and for the connection field

$$\Omega^i_{\mu} = \Omega^i_{\mu} + \Upsilon^i_{\mu}, \quad \Omega^i_{\mu} \sim \begin{pmatrix} 0 & 0 & -r/l \\ 0 & -r_{+} r_{-}/r^3 & 0 \\ -r/l^2 & 0 & 0 \end{pmatrix}, \quad \Upsilon^i_{\mu} \sim \begin{pmatrix} O_1 & O_2 & O_1 \\ O_2 & O_3 & O_2 \\ O_1 & O_2 & O_1 \end{pmatrix}.$$  

(4.9)

The above asymptotic behavior of triad fields are derived from the Brown-Henneaux [26] ones for asymptotically $AdS_3$ space-times

$$g_{\mu \nu} = g_{\mu \nu} + G_{\mu \nu}, \quad g_{\mu \nu} = \begin{pmatrix} r^2/l^2 & 0 & 0 \\ 0 & -l^2/l^2 & 0 \\ 0 & 0 & -l^2 \end{pmatrix}, \quad G_{\mu \nu} \sim \begin{pmatrix} O_0 & O_3 & O_0 \\ O_4 & O_4 & O_4 \\ O_0 & O_3 & O_0 \end{pmatrix},$$

(4.10)

if so it is not a dynamical variable in the Hamiltonian formalism. Including the black hole geometries in the asymptotic limit demand that the asymptotics (4.8) and (4.9) under PGT behave as

$$\delta e^i \sim \begin{pmatrix} r/l - (r_{+}^2 + r_{-}^2)/2rl & 0 & 0 \\ 0 & l/r + l(r_{+}^2 + r_{-}^2)/2r^3 & 0 \\ -r_{+} r_{-}/rl & 0 & r \end{pmatrix},$$

(4.11)
and

\[ \delta \Omega^i \sim \begin{pmatrix} 0 & 0 & -r/l + (r^2 + r_+^2)/2rl \\ 0 & -r_+r_-/r^3 & 0 \\ -r/l^2 & 0 & -r_+r_-/rl \end{pmatrix}, \tag{4.12} \]

while for the vacuum configurations are \( \delta b^i = 0, \delta \Omega^i = 0 \). For the above boundary conditions the solution of transformations (3.7) are given by

\[ \xi^0 = l \left[ T + \frac{1}{2} \left( \frac{\partial^2 T}{\partial t^2} \right) \frac{l^4}{r^2} \right] + O_4, \]
\[ \xi^1 = -l \left( \frac{\partial T}{\partial \xi} \right) r + O_1, \tag{4.13} \]
\[ \xi^2 = S - \frac{1}{2} \left( \frac{\partial^2 S}{\partial \varphi^2} \right) \frac{l^2}{r^2} + O_4, \]

where the functions \( T(t, \varphi) \) and \( S(t, \varphi) \) satisfy the conditions

\[ \frac{\partial T}{\partial \varphi} = l \frac{\partial S}{\partial t}, \quad \frac{\partial S}{\partial \varphi} = l \frac{\partial T}{\partial t}. \tag{4.14} \]

These relations lead to the periodic conditions

\[ \partial (T \mp S) = 0, \tag{4.15} \]

where \( x^\pm = t/l \pm \varphi \) are light-cone coordinates. The commutation relations of the transformations (3.7) is closed and produce a Lie algebra

\[ [\delta_0, \delta''_0] = \delta'''_0 (T'''', S'''), \tag{4.16} \]

which to lowest order we obtain

\[ T''' = T' \partial_2 S'' + S' \partial_2 T'' - T'' \partial_2 S' - S'' \partial_2 T', \]
\[ S''' = S' \partial_2 S'' + T' \partial_2 T'' - S'' \partial_2 S' - T'' \partial_2 T'. \tag{4.17} \]

The improved form of the gauge generator (3.6) is \( \tilde{G} = G + K \) where the surface boundary term

\[ K = \oint df_\alpha (\xi^0 \mathcal{E}^\alpha + \xi^2 \mathcal{M}^\alpha) = \int_0^{2\pi} d\varphi (lT \mathcal{E}^1 + S \mathcal{M}^1), \tag{4.18} \]

depends only to leading terms in \( T \) and \( S \) not all the gauge in (4.13). The non-trivial conserved charges at the boundary are the energy and angular momentum of BTZ black hole. These quantities as mentioned are computed from (3.12) by choosing \( \xi^0 = 1 \) and \( \xi^2 = 1 \) and substituting the asymptotic values (4.18) - (4.12)

\[ \mathcal{E}^\alpha = 2 \varepsilon^{\alpha \beta \gamma} \left[ (\sigma + \alpha C) \Omega^\beta - \mu C \frac{3}{2} + \alpha(2\sigma + \alpha C) e^\gamma \frac{1}{2} \mu \Omega^2 \gamma + \frac{1}{l} (\sigma + \alpha C) e^\alpha \right] e^\beta, \]
\[ \mathcal{M}^\alpha = -2 \varepsilon^{\alpha \beta \gamma} \left[ (\sigma + \alpha C) \Omega^\beta - \mu C \frac{3}{2} + \alpha(2\sigma + \alpha C) e^\gamma \frac{1}{2} \mu \Omega^2 \gamma + \frac{1}{l} (\sigma + \alpha C) e^\alpha \right] e^\beta. \tag{4.19} \]
We applied the relations $\Omega = \omega + \alpha h$ and (4.6) with additionally the prefactor $\frac{1}{8\pi G}$ which yield

$$E = \frac{1}{2G} \left[ (\sigma + \alpha C) \frac{r_+^2 + r_-^2}{2l^2} + \frac{1}{\mu l} \frac{r_+ r_-}{l^2} \right],$$

$$J = \frac{1}{2G} \left[ (\sigma + \alpha C) \frac{r_+ + r_-}{l} + \frac{1}{\mu l} \frac{r_+^2 + r_-^2}{2l} \right],$$

(4.20)

where $G$ is the positive 3D Newton constant. These values are exactly consistent with the relations in (4.41) of [31] calculated for the BTZ black hole in TMG case ($\alpha = 0$) from holographic principles.

### 4.2.1 Asymptotic canonical algebra

The PB algebra of the gauge generators $\tilde{G}[\xi]$ is isomorphic to the Lie algebra of the asymptotic symmetries (4.16), but in general have a centrally extended term [26]. In fact for two sets of gauge generators $\tilde{G}' \equiv \tilde{G}[T', S']$ and $\tilde{G}'' \equiv \tilde{G}[T'', S'']$, the PB of the form $\{\tilde{G}'', \tilde{G}'\}$ is itself also a differentiable generator. Since each differentiable generator can defined up to a constant phase-space functional $k$, so this bracket leads to

$$\{\tilde{G}'', \tilde{G}'\} = \tilde{G}''' + k, \quad (4.21)$$

where this central extended term known as the central charge of the PB algebra.

In the canonical algebra when the constraints do not change under gauge transformations $\delta_0$, we can approximate the PB as $\{\tilde{G}'', \tilde{G}'\} = \delta_0' \tilde{G}'' \approx \delta_0'' K''$. So getting together $\tilde{G}''' \approx K'''$ and (4.21) have

$$\delta_0'' K'' \approx K''' + k. \quad (4.22)$$

According to periodicity conditions (4.15) we can define $f^\pm = T \pm S$ and $U_\pm = (lE^1 \pm M^1)/2$, so the asymptotic transformation under $\delta_0'' K''$ are given by

$$\delta U_\pm = -f^\pm \partial_\pm U_\pm - 2(\partial_\pm f^\pm)U_\pm + \left((\sigma + \alpha C)\frac{1}{\mu l}\right) \partial_\pm^3 f^\pm, \quad (4.23)$$

the last term is related to the second term of (4.22) as

$$k = k_- + k_+ = -\left((\sigma + \alpha C)\frac{1}{\mu l}\right) \int_0^{2\pi} d\varphi (\partial_\pm^3 f^+) f''- \left((\sigma + \alpha C)-\frac{1}{\mu l}\right) \int_0^{2\pi} d\varphi (\partial_\pm^3 f^+) f''+, \quad (4.24)$$

and $\partial_\pm = (l\partial_t \pm \partial_\varphi)/2$. We can define Fourier modes as

$$L_n^\pm = -\tilde{G}[f^\pm = e^{inx^\pm}], \quad (4.25)$$

likewise the asymptotic generator is a linear composition as

$$\tilde{G} = -\sum_{-\infty}^{+\infty} (a_n L_n^+ + \bar{a}_n L_n^-), \quad (4.26)$$

where the periodic behavior of functions $f^\pm$ is defined in accordance with the condition (4.16).
The canonical algebra constructed from these Fourier modes takes the form of two Virasoro algebras as
\[
\{L^+_n, L^-_m\} = -i(n-m)L^+_{n+m} - \frac{c_L}{12}n^3\delta_{n+m,0}, \\
\{L^-_n, L^-_m\} = -i(n-m)L^-_{n+m} - \frac{c_R}{12}n^3\delta_{n+m,0}, \\
\{L^+_n, L^-_m\} = 0,
\] (4.27)
where \(c_L\) and \(c_R\) are left-right central charges
\[
c_L = \frac{3l}{2G}\left(\sigma + \alpha C - \frac{1}{\mu l}\right), \quad c_R = \frac{3l}{2G}\left(\sigma + \alpha C + \frac{1}{\mu l}\right),
\] (4.28)
which are exactly the values obtained in [18].

### 4.3 Thermodynamics

The thermodynamical variables for the BTZ black hole as Hawking temperature and angular velocity of horizon \(r_+\) are given by
\[
T_H = \frac{1}{2\pi}\kappa|_{r=r_+} = \frac{r_+}{2\pi} \left(1 - \frac{r_+^2}{r_-^2}\right), \quad \Omega_h = \frac{1}{l}N_\phi|_{r=r_+} = \frac{r_-}{l r_+},
\] (4.29)
where \(\kappa\) is the surface gravity in the ADM form
\[
\kappa = \frac{1}{l}\sqrt{g^{rr}N'|_{r=r_+}}.
\] (4.30)

The values of energy and angular momentum from (4.20) and thermodynamical parameters in (4.29) satisfy in the Smarr-like formula
\[
E = T_H S_{BTZ} + \Omega_h J,
\] (4.31)
therefore the entropy of BTZ black hole for the MMG should be
\[
S_{BTZ} = \frac{A_H}{4G} \left[\sigma + \alpha C + \frac{1}{\mu l} r_-\right],
\] (4.32)
where \(A_H = 2\pi r_+\) is the area of event horizon. As seen the result is consistent with [20] in the case of TMG when we set \(\alpha = 0\) in (4.32).

This value is also consistent with the Cardy formula from the holographic considerations. As shown in the previous section the central charges of dual conformal field theory are given by (4.28) where from the Cardy formula
\[
S = \frac{\pi^2}{3}(c_L T_L + c_R T_R),
\] (4.33)
and that the left and right temperatures for the BTZ black hole are
\[
T_L \equiv \frac{r_+ - r_-}{2\pi l}, \quad T_R \equiv \frac{r_+ + r_-}{2\pi l},
\] (4.34)
we obtain again the entropy (4.32).
For this solution the physical parameters given by relations (4.20), (4.29) and (4.32) satisfy a modified form of differential first law of black hole thermodynamics [33]:

\[ dE = 2T_H dS + \Omega_h dJ. \]  

(4.35)

In the calculation of the entropy in MMG a same result is obtained from another approach in [34].

5 Conclusions

In this paper we have considered the asymptotic structure of the MMG in the canonical first order formalism. Although this theory has an additive multiplier field \( h \) but has the same degrees of freedom as TMG for \( 1 + \sigma \alpha \neq 0 \). From the field equations, it has been shown that in order to have a torsion free gravity one can define new 1-form field \( \Omega = \omega + \alpha h \). Then we construct the canonical Hamiltonian of the MMG Lagrangian (2.6) by defining the canonical momenta for different dynamical fields.

Using the primary first class constraints and appropriate Poincaré gauge transformations, we obtained the gauge generators (3.6) in according to the procedure in [23]. The asymptotic variation of generators under PGT generate some total derivatives. The contribution of \( G_2 \) vanishes while the one for \( G_1 \) has a finite term after using the Stokes theorem. After all we have found expressions for the conserved charges (3.12) and (3.13) in the asymptotic region.

We calculated the energy and angular momentum of the BTZ black hole in this formalism which their values are given by (4.20). These conserved charges are achieved by suitable asymptotic boundary conditions (4.8)–(4.12) for the canonical fields \( e^i_\mu \) and \( \Omega^i_\mu \). The PB of the improved generators \( \tilde{G} = G + K \) produced two version of Virasoro algebra with the different central terms (4.28) which is consistent with the asymptotic symmetry group of locally AdS\(_3\) BTZ solution in [26]. One can easily see that the insertion of \( \alpha = 0 \) in (4.28) gives the values of TMG case.

Typically, the entropy of the black holes in gravitational theories are computed from the Wald formula. But since MMG has no metric Lagrangian, we have found it by using the Smarr formula (4.31). We also consider that the resultant entropy (4.32) accompanied by the energy and angular momentum satisfy in a modified first law of black hole thermodynamics (4.35). This entropy is also consistent with the expression obtained from the Cardy formula in dual CFT. In brief, the effect of MMG on all calculated physical quantities is summarized in the product of \( \alpha \) and \( C \) parameters relative to TMG ones.

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A The algebra of constraints

The necessary and sufficient conditions for $G$ as a gauge generator are

$$G = \text{primary}, \quad \{G, H\} = \text{primary}, \quad \{G, \text{any constraint}\} = \text{constraints}. \quad (A.1)$$

The Hamiltonian equations yields the following constraints as (3.2) and (3.5)

$$G|_{\phi_{\rho} = 0} = 0, \quad \{G, H_c\}|_{\phi_{\rho} = 0} = 0. \quad (A.2)$$

where $\phi_{\rho}$ are primary constraints and $H$ is the canonical Hamiltonian of system in the gauge theory. The PB algebra of these primary and secondary constraints ($\phi_i^\alpha, \Phi_i^\alpha, \psi_i^\alpha, H_i, K_i, T_i$) are given by

$$\{\phi_i^\alpha, \Phi_j^\beta\} = \sigma \epsilon^{0\alpha\beta} \eta_{ij} \delta, \quad \{\phi_i^\alpha, \psi_j^\beta\} = -\epsilon^{0\alpha\beta} \eta_{ij} \delta, \quad \{\Phi_i^\alpha, \Phi_j^\beta\} = -2 \mu^{-1} \epsilon^{0\alpha\beta} \eta_{ij} \delta,$$

$$\{\psi_i^\alpha, H_j\} = \epsilon^{0\alpha\beta} (-\eta_{ij} \partial_{\beta} + 2 \epsilon_{ij k} h_{k\beta}) \delta, \quad \{\psi_i^\alpha, K_j\} = -2 \epsilon^{0\alpha\beta} \epsilon_{ijk} e_{k\beta} \delta, \quad \{\psi_i^\alpha, T_j\} = -4 \epsilon^{0\alpha\beta} \epsilon_{ijk} e_{k\beta} \delta,$$

$$\{\phi_i^\alpha, H_j\} = -2 \Lambda_0 \epsilon^{0\alpha\beta} \epsilon_{ijk} e_{k\beta} \delta, \quad \{\phi_i^\alpha, K_j\} = 2 \sigma \epsilon^{0\alpha\beta} \eta_{ij} \partial_{\beta} \delta - 2 \epsilon^{0\alpha\beta} \epsilon_{ijk} (\sigma \omega_{k\beta} + \Lambda_0 e_{k\beta}) \delta,$$

$$\{\phi_i^\alpha, T_j\} = -2 \epsilon^{0\alpha\beta} \eta_{ij} \partial_{\beta} \delta + 2 \epsilon^{0\alpha\beta} \epsilon_{ijk} (\omega_{k\beta} - 2 h_{k\beta}) \delta,$$

$$\{\Phi_i^\alpha, H_j\} = 2 \sigma \epsilon^{0\alpha\beta} \eta_{ij} \partial_{\beta} \delta - 2 \epsilon^{0\alpha\beta} \epsilon_{ijk} (\omega_{k\beta} - h_{k\beta}) \delta,$$

$$\{\Phi_i^\alpha, K_j\} = 2 \mu^{-1} \epsilon^{0\alpha\beta} \eta_{ij} \partial_{\beta} \delta + 2 \epsilon^{0\alpha\beta} \epsilon_{ijk} (\sigma - \mu^{-1}) \omega_{k\beta} \delta,$$

$$\{\Phi_i^\alpha, T_j\} = 2 \epsilon^{0\alpha\beta} \epsilon_{ijk} e_{k\beta} \delta,$$

that $\partial$ is the partial derivative and $\delta$ refers to $\{e_i^\mu(x), \pi_j^\nu(x')\} = \delta_{ij} \delta^\nu_{\mu} \delta(x - x').$

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