Scale to scale energy transfer rate in compressible two-fluid plasma turbulence

Supratik Banerjee
Department of Physics, Indian Institute of Technology Kanpur, Kalyanpur 208016, Uttar Pradesh, India

Nahuel Andrés
Institute of Astronomy and Space Physics, Ciudad Universitaria, Buenos Aires, Argentina and
Physics Department, University of Buenos Aires, Ciudad Universitaria, Buenos Aires, Argentina

(Dated: May 1, 2020)

We derive the exact relation for the energy transfer in three-dimensional compressible two-fluid plasma turbulence. In the long-time limit, we obtain an exact law which expresses the scale-to-scale average energy flux rate in terms of two point increments of the fluid variables of each species, electric and magnetic field and current density, and puts a strong constraint on the turbulent dynamics. The incompressible single fluid and two-fluid limits and the compressible single fluid limit are recovered under appropriate assumption. In the single fluid limits, analyses are done with and without neglecting the electron mass thereby making the exact relation suitable for a broader range of application. In the compressible two fluid regime, the total energy flux rate, unlike the single fluid case, is found to be unaltered by the presence of a background magnetic field. The exact relation provides a way to test whether a range of scales in a plasma is inertial or dissipative and is essential to understand the nonlinear nature of both space and dilute astrophysical plasmas.

I. INTRODUCTION

Turbulence is a highly nonlinear and complex phenomenon which is ubiquitous in nature. Starting from the tap water, turbulence is found in almost all natural fluids including astrophysical plasmas. In comparison with neutral fluids, plasma turbulence is more difficult to handle due to the presence of more than one characteristic scale (for instance, separately for ions and electrons), which involves different nonlinear dynamic regimes and probable dissipation scales e.g., ion and electron inertial length scales [1]. Beyond the hydrodynamic length and time scales, every species population (ions, electrons, neutral atoms) of a plasma can be modelled as a separate fluid whence a multi-fluid model can be appropriate to describe the entire plasma. If the plasma is highly ionized such that at every point the ionized and electronic charge densities are nearly equal, a single-fluid model, often called the Extended Magnetohydrodynamics (ExMHD), can be adopted. The ExMHD can be reduced to the popular Hall-MHD (HMHd) model when the electron mass is neglected with respect to the ion mass. Finally, if one is interested in the fluctuations of sufficiently large length scales (much greater than the ion inertial length or ion gyroradius) and time scales (much greater than the ion gyroperiod), the ordinary Magnetohydrodynamics (MHD) limit can be recovered. Due to its remarkable simplicity and interesting properties, this model has extensively been used to investigate plasma turbulence using analytical, numerical and observational studies [2, 3].

Whilst the initial understanding of energy transfer across different length scales of incompressible hydrodynamic (IHD) turbulence was principally phenomenological in nature, the pioneering works of von Kármán and Howarth [2] and Kolmogorov built the foundation of rigorous analytical exact relations for IHD turbulence. Assuming statistical stationarity, homogeneity and isotropy, in the limit of infinitely large Reynolds number, Kolmogorov [3, 4] (hereafter K41) derived an expression for the average energy flux rate $\varepsilon$ in terms of third-order velocity structure functions. This expression is only valid in the inertial range, i.e., the range of scales which are free from both the large-scale forcing and the small-scale dissipation dynamics. This first exact relation, also known as the $4/5$ law, represents one of the cornerstones of turbulence theories using two-point statistics [5] and can be written as,

$$\langle \delta u_x^3 \rangle = -\frac{4}{5} \varepsilon \ell, \quad (1)$$

where $u_x$ is the fluid velocity component along the increment direction $\ell$, $\delta \xi$ denotes the difference of a physical quantity $\xi$ between two arbitrary points $x$ and $x'$ ($\equiv x + \ell$) and $\langle \cdot \rangle$ stands for statistical average. Later, Monin and Yaglom [8] (hereafter MY75) proposed a vectorial form of the above relation without assuming isotropy as,

$$\nabla \cdot \langle \delta u^2 \delta u \rangle = -4 \varepsilon. \quad (2)$$

These exact results are crucial as they provide an accurate measure of the energy dissipation rate, and therefore, the heating rate of a system by the process of a turbulent cascade [9]. Following both K41 and MY75 formalism, Politano and Pouquet [4, 10] derived ex-
act relations for incompressible magnetohydrodynamics (MHD) turbulence both in terms of (i) fluid velocity and magnetic fields and (ii) the Elsässer variables, respectively. Other exact laws for incompressible turbulence were derived for MHD, HMHD and ExMHD (with and without electron inertia) turbulence describing the scale-to-scale transfer of total energy or other inviscid invariants (cross helicity, magnetic helicity, etc.) deep inside the inertial range. However, Galtier [12, 13] reported that unlike the case of ordinary IHD and IMHD, the average energy flux rate in incompressible HMHD (IHMHD) and electron MHD (ExMHD without electron inertia) turbulence cannot be expressed purely in terms of two-point increments. This issue was successfully dealt using an alternative formulation presented by Banerjee and Galtier [15] (BG17 hereafter), which expresses the energy cascade rate entirely in terms of two-point increments for IHD, IMHD, IHMHD and electron MHD turbulence without any assumption of isotropy. Very recently, this alternative form has been verified using direct numerical simulation [16, 17].

Seventy years after K41, a number of exact relations were derived for compressible hydrodynamic (CHD) turbulence using both isothermal and polytropic closures [18, 19]. Following the MY75 formalism, the general form of compressible exact relations may be written as,

\[-4\epsilon = \nabla \cdot \mathcal{F} + \mathcal{S}, \tag{3}\]

where only the so-called flux term \(\mathcal{F}\) survives for incompressible turbulence. The new source term \(\mathcal{S}\), being proportional to the local velocity divergence (i.e., \(\nabla \cdot \mathbf{u}\)), vanishes for the flow with constant density. Similar relations for compressible MHD (CMHD) and Hall-MHD (CHMHD) turbulence were also derived using this methodology [20, 21]. Recently, for isothermal CHD turbulence, Banerjee and Kritsuk [22] showed that unlike the kinetic energy correlation function, the thermodynamic energy correlator should be multiplied by a factor 1/2 to make it consistent with the spectral mode energy equipartition in the acoustic limit. This modification is found to rule out the role of the correlation between the velocity and the pressure dilatation in the process of energy cascade [24].

However, Andrés and Sahraoui [21] pointed out that flux-source formulation contains some terms which are neither a pure flux term nor as a source term. Again, there are certain terms which could be cast as both flux and source terms. It is only very recently [22] that an exact relation has been derived for isothermal and self gravitating CMHD turbulence, generalizing the BG17 findings and also modifying the thermodynamic energy correlator according to linear mode energy equipartition. The final relation is found to be remarkably compact with respect to the previous compressible exact relations and can be easily extended for rotational or HMHD turbulence.

Similar to incompressible turbulence, exact relations are found to be extremely useful for understanding various aspects of compressible turbulence in space and astrophysical fluids. They have been applied to in-situ observations in the fast and slow solar winds and also in the terrestrial magnetosheath to investigate the role of density fluctuations in the turbulence dynamics at the MHD scales and the efficient heating of the system by turbulent energy cascade [4, 24, 25]. Moreover, the exact laws for isothermal turbulence (both CHD and CMHD) have been validated using numerical simulations [see, e.g. 26, 27].

A considerable number of incompressible and compressible exact scaling relations have been derived in both neutral fluid and MHD turbulence. However, the MHD model constitutes only a very simplistic single fluid model of plasma turbulence, which is only applicable for length scales greater than the ion inertial scales. The Hall and electron MHD models, on the contrary, allow to probe partially into the sub-ion scales. Nevertheless, all those models, being the single fluid models of plasma, assume local charge neutrality (equal ion and electron number densities at every point of the flow field), which could not be the case for several natural plasma systems, especially where the plasma is weakly or partially ionized (e.g. cold molecular clouds, protoplanetary disks etc.). For those contexts, it is more appropriate to use the two-fluid (TF) description where the plasma is supposed to consist of two separate fluids of electrons and singly charged ions (for the sake of simplicity, we can neglect the neutral fluids for instance and can include without much problem). The local charge neutrality is no longer valid and each fluid has its individual characteristics. Following both the original von Kármán-Howarth and BG17 methods, the first exact relations for incompressible TF turbulence were derived recently by Andrés et al. [31, 32] for the total energy (ionic plus electronic kinetic energy and magnetic energy) and for the generalized ion and electron helicity. However, those exact relations were derived assuming quasi-neutrality and hence represent only a very limited subset of incompressible two-fluid turbulence.

In the current paper, following the principles of the BG17 formulation, we derive an exact relation for three dimensional (3D), homogeneous and compressible two-fluid (CTF) plasma turbulence. For both ion and electron fluids, we use polytropic closures (with different polytropic indices), which are of interest for both laboratory and astrophysical plasmas. The paper is organized as follows, in Sec. II we present the basic equations for a polytropic CTF plasma. In Sec. III we demonstrate the conservation of the total energy in the CTF system, while in Sec. IV we present a detailed definition of the energy correlators and the derivation of the exact relation. Finally, in Sec. V we investigate different interesting limits and
summarize our main findings along with their potential implications in space and astrophysical plasma turbulence.

II. COMPRESSIBLE TWO-FLUID MODEL

A. Set of equations

In this paper, we consider a CTF plasma which is composed of a singly-charged ionic and an electronic species fluid both satisfying polytropic closures with different polytropic indices. Since in general a TF plasma is not locally quasi-neutral, we assume \( n_i \neq n_e \), where \( n_{i,e} \) denotes the number density for ions and electrons, respectively. In addition, we assume stationary large-scale forcings in the ion and electron momentum equations. Furthermore, for the sake of simplicity, we assume such a flow regime where the production and the loss rates of each species are roughly equal and each species fluid is undergoing polytropic equation of state with species specific polytropic index \( \gamma_s \). Therefore, the full set of dynamical equations are given by,

\[
\begin{align*}
\partial_t \rho_s + \nabla \cdot j_s &= 0, \tag{4} \\
\partial_t j_s + \nabla \cdot (j_s \otimes u_s) &= -\nabla p_s + \rho_{cs}(E + u_s \times B) + d_s + f_s, \tag{5} \\
\partial_t B &= -\nabla \times E, \tag{6} \\
\partial_t E &= \frac{1}{\mu_0 \epsilon_0} \nabla \times B - \frac{1}{\epsilon_0} J - \frac{1}{\epsilon_0} \sum_s \rho_{cs} u_s, \text{ and} \tag{7} \\
p_s &= K_s \rho_s^{\gamma_s}. \tag{8}
\end{align*}
\]

where the index \( s \) denotes the individual species of ions (i) and electrons (e). \( \rho_s, \rho_{cs}, u_s, j_s \equiv \rho_s u_s \) and \( p_s \) denote the mass density, the electric charge density, the fluid velocity, the mass current and the fluid pressure for each species \( s \), respectively. In the above equations, \( E \) is the electric field, \( B \) the magnetic field, \( J (\equiv \sum_s \rho_{cs} u_s) \) the electric current and \( \epsilon_0 \) and \( \mu_0 \) are the free space permittivity and permeability, respectively. Finally, \( f_s \) is the stationary large-scale forcings and \( d_s = -d_s = \nu p_s (u_s - \bar{u}_s) \) denotes the momentum exchange between the two species (with \( \rho_i = \rho_i \rho_e (\rho_i + \rho_e)^{-1} \) and \( K_s \) is a constant of proportionality. Note that, in the current study, for the sake of simplicity, forcing is added only in the momentum equation and not in the electromagnetic field evolution equations. It means that the energy is injected in the form of kinetic energy. However, electric and magnetic energy component will also be nourished through the interaction of the momentum field \( j \) and the velocity field \( u \) with the the electric field \( E \) and magnetic field \( B \).

B. Total energy conservation

In this section, we show that the total energy is conserved for a CTF plasma in the absence of any forcing and viscous term. Using the set of Eqs. (4)-(8) and setting \( f_s \) and \( d_s \) to zero, it is straightforward to show that,

\[
\begin{align*}
\partial_t \sum_s \rho_s u_s^2 &= 2 J \cdot E - \sum_s [\nabla \cdot (u_s^2 j_s) - u_s \cdot \nabla p_s - \gamma_s j_s \cdot \nabla e_s], \tag{9} \\
\partial_t \sum_s \rho_s e_s &= \sum_s [\nabla \cdot (j_s e_s) + p_s (\nabla \cdot u_s)], \tag{10} \\
\partial_t \left( \frac{B^2}{\mu_0} + \frac{\epsilon_0 E^2}{\epsilon_0} \right) &= -2 \left[ \nabla \cdot \left( \frac{E \times B}{\mu_0} \right) + J \cdot E \right], \tag{11}
\end{align*}
\]

where \( e_s = p_s / (\gamma_s - 1) \rho_s \) is the thermodynamic potential energy per unit mass for species \( s \). Noting that for a polytropic fluid \( \gamma_j \cdot \nabla e_s = u_s \cdot \nabla p_s \), we can show that for a CTF plasma the total energy is an inviscid...
invariant. The total energy density ($\mathcal{E}$), which is the sum of the densities of kinetic and thermodynamic potential energies of the individual species plus the density of the electromagnetic energy, can be written as,

$$\mathcal{E} = \frac{1}{2} \left( \sum_s \rho_s u_s^2 + \frac{B^2}{\mu_0} + \epsilon_0 E^2 \right) + \sum_s \frac{p_s}{\gamma_s - 1}. \quad (12)$$

In the next Sec. [III] we derive an exact relation related to the conservation of total energy in the absence of the forcing and the viscous terms.

### III. DERIVATION OF THE EXACT RELATION

In order to derive the exact relation for CTF turbulence, we define the two-point correlator for the total energy $\mathcal{E}$. For a neutral polytropic one-fluid, it can be shown that there is equipartition between the average kinetic energy and the average thermodynamic energy in the linear modes [34]. This type of equipartition can, in principle, be generalized for a charged fluid as well. Along with this fact, following Banerjee and Kritsuk [25], here we construct the thermodynamic energy correlator which is in accordance with spectral space energy equipartition between the kinetic and potential energies in the acoustic limit and also is equal to the thermodynamic energy density in the single point limit (where the primed and unprimed quantities coincide). To satisfy both the conditions, it turns out that the correlation function will include an appropriate combination of one-point and two-point contributions having the general form,

$$\frac{1}{n} \left( \rho \epsilon + \frac{\rho \epsilon'}{2n} \right), \quad \text{and search for the value of } n \text{ which gives spectral space equipartition of energy in the acoustic limit.}$$

For a polytropic fluid, the total energy density of the acoustic mode ($\mathcal{E}_a$) is given by [35],

$$\mathcal{E}_a = \frac{1}{2} \rho u^2 + \frac{C_s^2}{2 \rho_0^2}, \quad (14)$$

where $\rho_0$, $u$ and $\rho_1$ represent the mean density and the first-order perturbations in velocity and density, respectively. Here, $C = \sqrt{\gamma P_0/\rho_0}$ denotes the polytropic sound speed. The two terms on the right hand side of Eq. (14) correspond to the the acoustic kinetic and the thermodynamic energy densities, respectively. The polytropic two-point energy correlation function can therefore be written as,

$$\mathcal{R}_a(\ell) = \frac{1}{2} \rho_0 \langle u \cdot u' \rangle + \frac{C_s^2}{2 \rho_0^2} \langle \rho_1 \rho_1' \rangle$$

$$= \frac{1}{2} \rho_0 \langle u \cdot u' \rangle + \gamma \frac{P_0}{2 \rho_0^2} \langle \rho_1 \rho_1' \rangle. \quad (15)$$

In the formalism described in Sec. III of Ref. [25], it was shown, for an isothermal fluid, that the correlator could give the correct value for the thermodynamic energy density in the spectral space if $n = 2$ in equation (14). Generalising the same methodology for a polytropic fluid, we can expand the two-point contribution (it is only that part which can give us the energy density for the thermodynamic part in spectral space) in the general correlation function at acoustic limit (i.e. with only first-order perturbations), which gives

$$\langle \frac{\rho \epsilon' + \rho' \epsilon}{2n} \rangle = \frac{P_0}{2n(\gamma - 1)} \left( 1 + \frac{\rho_1}{\rho_0} \right) \left( 1 + \frac{\rho_1'}{\rho_0} \right)^{\gamma - 1} + \left( 1 + \frac{\rho_1}{\rho_0} \right) \left( 1 + \frac{\rho_1'}{\rho_0} \right)^{\gamma - 1}$$

$$\approx \frac{P_0}{2n(\gamma - 1)} \left[ 1 + \frac{\rho_1}{\rho_0} \left( 1 + (\gamma - 1) \frac{\rho_1'}{\rho_0} \right) + \left( 1 + \frac{\rho_1'}{\rho_0} \right) \left( \gamma - 1 \right) \frac{\rho_1}{\rho_0} \right] \quad (16)$$

The total two-point contribution of second order of smallness is clearly given by $\frac{P_0}{2n(\gamma - 1)} (\rho_1 \rho_1')$. Equating this contribution to the thermodynamic energy contribution in Eqn. (14), we have $n = 2/\gamma$ (for the isothermal case, $\gamma = 1$ and one recovers $n = 2$ [25]). Therefore, we define the two-point symmetric correlator of the total energy for the TF plasma as, $\mathcal{R}(\ell) = (\mathcal{R}_\mathcal{E} + \mathcal{R}_\mathcal{E}^\prime) / 2$ with
where $B = B/\sqrt{\rho_s}$ and $E = \sqrt{\gamma_s} E$. Using Eqs. (4)-(8) (including the forcing and dissipation terms) and the two-point statistics for homogeneous turbulence, we obtain,

\begin{align}
R_c &= \frac{1}{2} \left[ \sum_s \left( j_s \cdot u'_s + \gamma_s \rho_s e'_s + (2 - \gamma_s) \rho_s e_s + B \cdot B' + E \cdot E' \right) \right], \\
R'_c &= \frac{1}{2} \left[ \sum_s \left( j' s \cdot u'_s + \gamma_s \rho'_s e'_s + (2 - \gamma_s) \rho'_s e'_s + B' \cdot B + E' \cdot E' \right) \right],
\end{align}

with these expressions, the evolution equation of the total correlation function $R$ can be written as,

\begin{align}
\partial_t R &= \frac{1}{\mu_0} \partial_t \left[ j_i \cdot u'_i + j_e \cdot u'_e + j'_i \cdot u_i + j'_e \cdot u'_e + 2B \cdot B' - 2E \cdot E' \right] \\
&\quad + \gamma_i \rho_i e'_i + (2 - \gamma_i) \rho_i e_i + \gamma_e \rho_e e'_e + (2 - \gamma_e) \rho_e e_e + \gamma_i \rho'_i e'_i + (2 - \gamma_i) \rho'_i e_i + \gamma_e \rho'_e e'_e + (2 - \gamma_e) \rho'_e e_e.
\end{align}

Using Eqs. (19)-(23), we obtain
\[ \partial_t R = \frac{1}{4} \left( -(u_i' \cdot u_i)(\nabla \cdot j_i) - \rho_i \nabla \left( \frac{u_i'^2}{2} \right) \cdot u_i' + u_i' \cdot (j_i \times \omega_i) - u_i' \cdot \nabla p_i + \rho_{ci} u_i' \cdot (E + u_i \times B) + u_i' \cdot (d_i' + f_i) \right. \\
- j_i \cdot \nabla \left( \gamma_i c_i + \frac{u_i'^2}{2} \right) + j_i \cdot (u_i' \times \omega_i) + \frac{c_i j_i}{m_i} \cdot (E' + u_i' \times B') + \frac{j_i}{\rho_i} \cdot (d_i' + f_i) \\
- (u_i' \cdot u_i)(\nabla \cdot j_i) - \rho_e \nabla \left( \frac{u_e'^2}{2} \right) \cdot u_e' + u_e' \cdot (j_e \times \omega_e) - u_e' \cdot \nabla p_e + \rho_{ce} u_e' \cdot (E + u_e \times B) + u_e' \cdot (d_e' + f_e) \\
- j_e \cdot \nabla \left( \gamma_e c_e + \frac{u_e'^2}{2} \right) + j_e \cdot (u_e' \times \omega_e) - \frac{c_j e_j}{m_e} \cdot (E' + u_e' \times B') + \frac{j_e}{\rho_e} \cdot (d_e' + f_e) \\
- (u_i \cdot u_i')(\nabla' \cdot j_i') - \rho_i' \nabla' \left( \frac{u_i'^2}{2} \right) \cdot u_i' + u_i' \cdot (j_i' \times \omega_i') - u_i' \cdot \nabla' p_i' + \rho_{ci} u_i' \cdot (E' + u_i' \times B') + u_i' \cdot (d_i' + f_i') \\
- j_i' \cdot \nabla \left( \gamma_i c_i + \frac{u_i'^2}{2} \right) + j_i' \cdot (u_i' \times \omega_i') - \frac{c_i j_i'}{m_i} \cdot (E + u_i \times B) + \frac{j_i'}{\rho_i} \cdot (d_i' + f_i) \\
- (u_e \cdot u_e')(\nabla' \cdot j_e') - \rho_e' \nabla' \left( \frac{u_e'^2}{2} \right) \cdot u_e' + u_e' \cdot (j_e' \times \omega_e') - u_e' \cdot \nabla' p_e' + \rho_{ce} u_e' \cdot (E' + u_e' \times B') + u_e' \cdot (d_e' + f_e') \\
- j_e' \cdot \nabla \left( \gamma_e c_e + \frac{u_e'^2}{2} \right) + j_e' \cdot (u_e' \times \omega_e') - \frac{c_j e_j}{m_e} \cdot (E + u_e \times B) + \frac{j_e'}{\rho_e} \cdot (d_e' + f_e) \\
- \frac{2}{\rho_0} \left[ (B_e \cdot (\nabla' \times E') + B_e' \cdot (\nabla \times E) - (\nabla \times B) \cdot E' - (\nabla' \times B') \cdot E - 2 (J \cdot E' + J' \cdot E) \\
- \gamma_e \left( \rho_e' \delta_{e' e} - \rho_e u_e' \cdot \nabla' e' \right) - \gamma_e \left( \rho_e' \delta_{e' e} - \rho_e u_e' \cdot \nabla' e' \right) \right. \\
- \gamma_e \left( \rho_e' \delta_{e' e} - \rho_e u_e' \cdot \nabla' e' \right) - \gamma_e \left( \rho_e' \delta_{e' e} - \rho_e u_e' \cdot \nabla' e' \right) \\
- (2 - \gamma_e) \rho_e \delta_{e' e} - (2 - \gamma_e) \rho_e \delta_{e' e} \right). \] 

(25)

Using statistical homogeneity, we can show,

\[ \langle [B \cdot (\nabla' \times E') + B_e' \cdot (\nabla \times E) - (\nabla \times B) \cdot E' - (\nabla' \times B') \cdot E - 2 (J \cdot E' + J' \cdot E)] \rangle = \] 

\[ \langle \nabla' \cdot (E' \times B) + \nabla \cdot (E \times B') - \nabla' \cdot (B \times E') - \nabla \cdot (B \times E) \rangle = \] 

\[ \nabla \cdot (E' \times B) - (E \times B') + (B \times E') - (B \times E) \rangle = 0. \]

(26)  

(27)  

(28)

Again Eq. (25) yields,

\[ 4 \partial_t R = \langle \delta j_i \cdot \delta ([u_i \cdot \nabla] u_i) + \delta u_i \cdot \delta [\nabla \cdot (j_i \otimes u_i)] + \delta j_e \cdot \delta ([u_e \cdot \nabla] u_e) + \delta u_e \cdot \delta [\nabla \cdot (j_e \otimes u_e)] \rangle + \] 

\[ + \gamma_e \langle \delta \rho_e \delta \left[ \frac{p_e \delta_{e e}}{p_e} + (u_e \cdot \nabla) e_e \right] + \delta u_e \cdot \delta [p_e \nabla e_e] \rangle + \gamma_e \langle \delta \rho_e \delta \left[ \frac{p_e \delta_{e e}}{p_e} + (u_e \cdot \nabla) e_e \right] + \delta u_e \cdot \delta [p_e \nabla e_e] \rangle \] 

\[ + \langle \delta (\rho_{ci} u_i \cdot \delta (u_i \times B)) + \delta u_i \cdot \delta (\rho_{ci} u_i \times B) + \delta (\rho_{ce} u_e \cdot \delta (u_e \times B)) + \delta u_e \cdot \delta (\rho_{ce} u_e \times B) \rangle + \] 

\[ + \langle \delta J \cdot \delta E - \delta u_i \cdot \delta (\rho_{ci} E) - \delta u_e \cdot \delta (\rho_{ce} E) \rangle + 4D + \mathcal{F}. \] 

(29)

Finally, we assume a stationary state for which the left hand term of the above equation vanishes and we restrict ourselves to the length scales far away from the dissipation length scales whence we can neglect \( D \). As a result, we can identify \( \mathcal{F} = \epsilon \) and the final exact relation becomes,

\[ -4 \epsilon = \langle \delta j_i \cdot \delta ([u_i \cdot \nabla] u_i) + \delta u_i \cdot \delta [\nabla \cdot (j_i \otimes u_i)] + \delta j_e \cdot \delta ([u_e \cdot \nabla] u_e) + \delta u_e \cdot \delta [\nabla \cdot (j_e \otimes u_e)] \rangle + \] 

\[ + \gamma_e \langle \delta \rho_e \delta \left[ \frac{p_e \delta_{e e}}{p_e} + (u_e \cdot \nabla) e_e \right] + \delta u_e \cdot \delta [p_e \nabla e_e] \rangle + \gamma_e \langle \delta \rho_e \delta \left[ \frac{p_e \delta_{e e}}{p_e} + (u_e \cdot \nabla) e_e \right] + \delta u_e \cdot \delta [p_e \nabla e_e] \rangle \] 

\[ - \langle \delta (\rho_{ci} u_i \cdot \delta (u_i \times B)) + \delta u_i \cdot \delta (\rho_{ci} u_i \times B) + \delta (\rho_{ce} u_e \cdot \delta (u_e \times B)) + \delta u_e \cdot \delta (\rho_{ce} u_e \times B) \rangle + \] 

\[ + \langle \delta J \cdot \delta E - \delta u_i \cdot \delta (\rho_{ci} E) - \delta u_e \cdot \delta (\rho_{ce} E) \rangle. \] 

(30)
Equation (30) represents an exact law for energy transfer rate in homogeneous, polytropic TF plasma turbulence. It is worth mentioning that the final exact relation Eq. (30) is written only as a function of two-point increments and it is valid only in the inertial range (i.e., for length scales far away from the injection and dissipative scales). As discussed in [25], unlike incompressible turbulence, the net turbulent contribution in the energy flux rate cannot simply be expressed as a departure from the aligned states (generalized Beltrami flows). In the incompressible limit of two-fluid turbulence (see the next section), one can express $\varepsilon$ as a departure from different aligned states. In the next Sec. IV we discuss various important features of Eq. (30).

### IV. DISCUSSION

#### A. Incompressible TF limit

In the incompressible limit, both the ion and electron number densities, i.e., $n_i$ and $n_e$ (and hence the mass $\rho_{ci,i}$ and charge densities $\rho_{ce,ci}$), are constants in time and space. However, they are not necessarily equal. Using this assumption and statistical homogeneity, one can show for each species $s$,

$$
\langle \delta j_s \cdot \delta [(u_s \cdot \nabla) u_s] + \delta u_s \cdot \delta [\nabla \cdot (j_s \otimes u_s)] \rangle = 2\rho_s \langle \delta u_s \cdot \delta [(u_s \cdot \nabla) u_s] \rangle = -2\rho_s \langle \delta u_s \cdot \delta (u_s \times \omega_s) \rangle
$$

where $\nabla \cdot u_s = 0$. The thermodynamic energy terms also vanish for incompressible fluid. The, Eq. (30) is therefore simplified as,

$$
2\varepsilon = \langle \rho_{ci} \delta u_i \cdot \delta (u_i \times \omega_i) + \rho_{ci} \delta u_e \cdot \delta (u_e \times \omega_e) + \rho_{ce} \delta u_i \cdot \delta (u_i \times B) + \rho_{ce} \delta u_e \cdot \delta (u_e \times B) \rangle. \tag{31}
$$

where $\rho_{ci}$ and $\rho_{ce}$ are the mean mass densities and $\rho_{ce,ci}$ are the mean charge densities for electrons and ions, respectively. As expected for incompressible turbulence, the turbulent energy flux rate $\varepsilon$ is written in terms of the departure from aligned states which can be obtained as minimum energy states along with the constraints of the conservations of generalized helicities of both ions and electrons [36]. The two relaxed states in a barotropic or even in an incompressible two-fluid plasma are represented by the two following alignments: $(n_i, \omega_{i,e} + q_i,e u_{i,e}) \parallel n_{i,e} q_{i,e} u_{i,e}$. The above equation (31) evidently shows that $\varepsilon$ reduces to zero when both the alignments occur. In addition if the electron mass is neglected, then the corresponding alignment will simply be an alignment between electron fluid velocity and the magnetic field thereby leading to a magnetic force-free motion of the electrons. Interestingly, in the above equation, the terms associated directly with the electric field $E$, vanish under the incompressibility assumption. It is worth mentioning that Eq. (31) is valid only when the ion and electron density fluctuations are equal to zero. However, its mean values ($\rho_{ci}$ and $\rho_{ce}$) are not necessarily equal. For a TF plasma with very strong ionization, the charge quasi-neutrality assumption, i.e., $n_i = n_e = n_0$ can be satisfied and Eq. (31) then reduces to,

$$
2\varepsilon = n_0 \langle m_i \delta u_i \cdot \delta (u_i \times \omega_i) + m_e \delta u_e \cdot \delta (u_e \times \omega_e) + e\delta u_i \cdot \delta (u_i \times B) - e\delta u_e \cdot \delta (u_e \times B) \rangle \tag{32}
$$

$$
= n_0 M \left\langle \delta u_i \cdot \delta \left[ u_i \times \left( \frac{m_i}{M} \omega_i + \frac{e}{M} B \right) \right] + \delta u_e \cdot \delta \left[ u_e \times \left( \frac{m_e}{M} \omega_e - \frac{e}{M} B \right) \right] \right\rangle
$$

$$
= n_0 M \left\langle \delta u_i \cdot \delta \left[ u_i \times \left( \frac{e\mu_{m0}}{\sqrt{\mu_{m0}} M} B + \frac{m_i}{M} \omega_i \right) \right] - \delta u_e \cdot \delta \left[ u_e \times \left( \frac{e\mu_{m0}}{\sqrt{\mu_{m0}} M} B - \frac{m_e}{M} \omega_e \right) \right] \right\rangle
$$

$$
= n_0 M \left\langle \delta u_i \cdot \delta \left[ u_i \times \left( \frac{e\mu_{m0}}{\sqrt{\mu_{m0}} M} b + (1 - \mu) \omega_i \right) \right] - \delta u_e \cdot \delta \left[ u_e \times \left( \frac{e\mu_{m0}}{\sqrt{\mu_{m0}} M} b - \mu \omega_e \right) \right] \right\rangle
$$

$$
= \frac{n_0 M}{\lambda} \left\langle \delta u_i \cdot \delta \left[ u_i \times (b + \lambda (1 - \mu) \omega_i) \right] - \delta u_e \cdot \delta \left[ u_e \times (b - \lambda \mu \omega_e) \right] \right\rangle
$$

which has previously been derived by Andrés et al. [32]. In the above expressions, $M = m_i + m_e$, $\mu = m_e/M$,
and $\lambda = \sqrt{M/\epsilon}$. Similar to the IHD and IMHD turbulence, the turbulent energy flux in a two-fluid plasma can also be attributed to the IHD and IMHD turbulence, the turbulent energy balance, the contribution of the corresponding term to $\epsilon$ vanishes \cite{15}.

B. Compressible single fluid limits

\textbf{a. Without neglecting electron mass:} In the single fluid limit, at every point $n_i = n_e = n$, however, $u$ is not necessarily constant. Now from the definitions of single fluid variables, we can write,

$$u = \frac{n_i m_i u_i + n_e m_e u_e}{n_i m_i + n_e m_e} = \frac{m_i u_i + m_e u_e}{m_i + m_e}$$  \hspace{1cm} (33)

$$J = e(n_i u_i - n_e u_e) = ne(u_i - u_e)$$  \hspace{1cm} (34)

Solving for $u_i$ and $u_e$, we get

$$u_i = u + \frac{\alpha}{1 + \alpha} \left( \frac{J}{ne} \right)$$ \hspace{1cm} (35)

$$u_e = u - \frac{1}{1 + \alpha} \left( \frac{J}{ne} \right)$$ \hspace{1cm} (36)

where $\alpha = m_e/m_i = \mu/(1 - \mu)$. Here we first derive the single fluid limit without neglecting electron mass and so that $\alpha < 1$ but $\alpha \neq 0$. This is important because the electron mass is responsible for magnetic reconnection to occur even in a collisionless plasma \cite{26} and turbulent reconnection is believed to affect the turbulent cascade \cite{27}. The value of $\alpha$ depends on the mass of the ion and is maximum $(1/1837 = 0.000544)$ for the lightest Hydrogen ions. So, for practical purpose, the effect of electron mass can be considered up to the first order of $\alpha$. This limit will lead to a regime of MHD which is more general than ordinary or Hall MHD and therefore can be called MHD with electron inertia. Note that this limit is nevertheless more restricted than the extended MHD model described in some previous works \cite{27,28} which did not neglect $\partial J/\partial t$, $\nabla \cdot (u \otimes J + J \otimes u)$ and the electron pressure gradient term in Generalized Ohm’s laws. These terms are certainly of theoretical interest. However, for most practical cases, these contributions are always neglected. Using the aforesaid expressions of ion and electron velocities, we can write

$$(u_i \cdot \nabla) u_i = (u \cdot \nabla) u + \frac{\alpha}{1 + \alpha} \left[ \left( \frac{J}{ne} \right) \cdot \nabla \right] u + \frac{\alpha^2}{(1 + \alpha)^2} \left[ \left( \frac{J}{ne} \right) \cdot \nabla \right] \left( \frac{J}{ne} \right)$$  \hspace{1cm} (37)

$$(u_e \cdot \nabla) u_e = (u \cdot \nabla) u - \frac{1}{1 + \alpha} \left[ \left( \frac{J}{ne} \right) \cdot \nabla \right] u + \frac{1}{(1 + \alpha)^2} \left[ \left( \frac{J}{ne} \right) \cdot \nabla \right] \left( \frac{J}{ne} \right)$$  \hspace{1cm} (38)

Using the above expressions, we obtain (keeping the terms up to first order of $\alpha$)

$$\delta j_i \cdot \delta [(u_i \cdot \nabla) u_i] + \delta j_e \cdot \delta [(u_e \cdot \nabla) u_e]$$

$$= \delta (\rho u) \cdot \delta [(u \cdot \nabla) u] + \frac{\alpha}{(1 + \alpha)^2} \delta \left( \frac{J}{ne} \right) \cdot \delta \left( \frac{J}{ne} \right) \left( \frac{J}{ne} \right)$$

$$+ \frac{\alpha}{1 + \alpha} \delta (\rho u) \cdot \delta \left( \frac{J}{ne} \right) \left( \frac{J}{ne} \right) - \frac{\alpha (1 - \alpha)}{(1 + \alpha)^3} \delta \left( \frac{J}{ne} \right) \cdot \delta \left( \frac{J}{ne} \right) \left( \frac{J}{ne} \right)$$

$$\approx \delta (\rho u) \cdot \delta [(u \cdot \nabla) u] + \alpha \delta \left( \rho \frac{J}{ne} \right) \cdot \delta \left( \frac{J}{ne} \right) \left( \frac{J}{ne} \right)$$

$$+ \alpha \delta \left( \rho \left( u - \frac{J}{ne} \right) \right) \cdot \delta \left( \frac{J}{ne} \right)$$  \hspace{1cm} (39)

Following the same methodology, we can also write

$$\rho_i u_i \otimes u_i = \frac{1}{1 + \alpha} \rho u \otimes u + \frac{\alpha}{(1 + \alpha)^2} \left( \rho \frac{J}{ne} \otimes u + \rho u \otimes \frac{J}{ne} \right) + \frac{\alpha^2}{(1 + \alpha)^4} \left( \rho \frac{J}{ne} \otimes \frac{J}{ne} \right)$$  \hspace{1cm} (40)

$$\rho_e u_e \otimes u_e = \frac{\alpha}{1 + \alpha} \rho u \otimes u - \frac{\alpha^2}{(1 + \alpha)^2} \left( \rho \frac{J}{ne} \otimes u + \rho u \otimes \frac{J}{ne} \right) + \frac{\alpha}{(1 + \alpha)^2} \left( \rho \frac{J}{ne} \otimes \frac{J}{ne} \right)$$  \hspace{1cm} (41)
where by definition, \( \rho_i = \frac{\rho_i}{1 + \alpha} \) and \( \rho_e = \frac{\rho_e}{1 + \alpha} \). After some steps of straightforward algebra, we get by retaining terms up to first order of \( \alpha \),

\[
\delta u_i \cdot \delta [\nabla \cdot (j_i \otimes u_i)] + \delta u_e \cdot \delta [\nabla \cdot (j_e \otimes u_e)] = \delta u \cdot \delta [\nabla \cdot (\rho u \otimes u)] + \frac{\alpha}{(1 + \alpha)^2} \delta \left( \frac{J_{ne}}{\gamma} \right) \cdot \delta \left[ \nabla \cdot \left( \rho \frac{J_{ne}}{\gamma} \otimes \frac{J_{ne}}{\gamma} \right) \right] + \frac{\alpha}{(1 + \alpha)^2} \delta u \cdot \delta \left[ \nabla \cdot \left( \rho \frac{J_{ne}}{\gamma} \otimes \frac{J_{ne}}{\gamma} \right) \right]
\]

\[
\approx \delta u \cdot \delta [\nabla \cdot (\rho u \otimes u)] + \alpha \delta \left( \frac{J_{ne}}{\gamma} \right) \cdot \delta \left[ \nabla \cdot \left( \rho \frac{J_{ne}}{\gamma} \otimes \frac{J_{ne}}{\gamma} \right) \right]
\]

For the contribution of the thermodynamic energy part, we can write for any species \( s \),

\[
(i) \quad \gamma_s \left( \delta n_{s} \delta \left[ \frac{\rho_s \theta_s}{\gamma_s} + (\mathbf{u}_s \cdot \nabla) \theta_s \right] \right) = \gamma_s m_{s}^{\gamma_s} \left[ \delta n \delta \left( K_{s} n^{\gamma_{s} - 1} \theta_{s} + (\mathbf{u}_s \cdot \nabla) K_{s} n^{\gamma_{s} - 1} \right) \right],
\]

\[
(ii) \quad \gamma_s \delta \mathbf{u}_s \cdot \delta [\rho_s \nabla \theta_s] = \gamma_s m_{s}^{\gamma_s} \delta \mathbf{u}_s \cdot \delta \left[ n \nabla \left( K_{s} n^{\gamma_{s} - 1} \right) \right].
\]

So, just by having an apparent view, one can think that the contribution from the electron fluid is negligibly small as it carries a ratio which is proportional to \( \alpha^{\gamma} (\gamma > 1) \) and also all other quantities \( (K_s, \theta_s, \gamma_s) \) are supposed to be approximately of the same order. At this point one has to be very careful. In practice, both the ion and electron pressure are of comparable magnitude and in fact with the same number density, electron fluid pressure is greater than ion fluid pressure due to having higher temperature. In terms of the polytropic closure equations, it simply says that \( K_i \) is considerably greater than \( K_e \) so that both pressures are significant. The total pressure can be written as,

\[
p = p_i + p_e = K_i \rho_i^{\gamma_i} + K_e \alpha \rho_e^{\gamma_e} = \frac{K_i}{(1 + \alpha)^{\gamma_i}} \rho^{\gamma_i} + \frac{K_e \alpha^{\gamma_e}}{(1 + \alpha)^{\gamma_e}} \rho^{\gamma_e}
\]
where \( \chi \equiv \nabla \cdot \left( \frac{\mathbf{B}}{\gamma} \right) \). In the Eqns. (48)–(50), one can notice that there are some terms containing electron fluid density, pressure and internal energy. In order to get rid of these terms, we need to introduce a new single fluid variable called the electrokinetic pressure \( P^E \) (as discussed in [33]) which can be written as

\[
P^E = \sum_s \frac{\rho_{cs} \delta}{\rho_s} = e \left( \frac{p_i - p_e}{m_i} \right).
\]

Note that, for comparable ion and electron fluid pressures, \( P^E \) is effectively negative. Using simple algebra, one can show that

\[
p_e \approx \alpha \left( \frac{p - m_i p^E}{\gamma - 1} \right), \quad e_e \approx \frac{(1 + \alpha)}{\gamma} \left( \frac{p - m_i p^E}{e - \rho} \right)
\]

So in principle, one can write the Eqn. (50) in terms of single fluid quantities assuming quasi-neutrality and a polytropic closure for the resultant single fluid. However, for the sake of comparison between different contributions, we keep the terms with \( p_e, p_e \) and \( e_e \) as they are. In the next step, we investigate the reduced form of the total electromagnetic contribution of the exact relation in the single fluid limit. For doing so, we need to use the specific form of Generalized Ohm’s law which reduces to the Ohm’s law for Hall-MHD in the limit \( \alpha \to 0 \). We therefore use the following form [33]:

\[
ne \left( \frac{1}{m_i} + \frac{1}{m_e} \right) \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{\mathbf{J}}{\sigma_0} \right) = \left( \frac{1}{m_e} - \frac{1}{m_i} \right) \left( \mathbf{J} \times \mathbf{B} \right)
\]

\[
\Rightarrow \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{\mathbf{J}}{\sigma_0} \right) = \frac{1 - \alpha}{1 + \alpha} \left( \mathbf{J} \times \mathbf{B} \right) \quad \text{(54)}
\]

where, \( \sigma_0 = \frac{m_e^2}{m_i m_e} \) is the conductivity of the single fluid. In the ideal limit, where the single fluid has infinite conductivity (\( \sigma_0 \to \infty \)), the resulting Ohm’s law can be expressed as

\[
\Rightarrow (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \left( \frac{1 - \alpha}{1 + \alpha} \right) \left( \frac{\mathbf{J} \times \mathbf{B}}{ne} \right) \quad \text{(55)}
\]

\[
\Rightarrow (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = \left( \frac{1}{1 + \alpha} \right) \left( \frac{\mathbf{J} \times \mathbf{B}}{ne} \right) \quad \text{(56)}
\]

\[
\Rightarrow (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{\mathbf{J} \times \mathbf{B}}{ne} \right) \quad \text{(57)}
\]
Now we obtain the total electromagnetic contribution for compressible single fluid limit. For the sake of easier calculation, we start the analysis from the correlation can be substantially reduced if we completely neglect the effect of electron mass i.e. $\alpha = 0$. In this limit, the exact relation is simply expressed as

\[ -4 \varepsilon = \left( \delta (\rho u) \cdot \delta [u \cdot \nabla] u + \delta u \cdot \delta [\nabla \cdot (\rho u \otimes u)] + \gamma \delta \rho \delta \left( \frac{p\chi}{\rho_c} \right) + \delta \rho \delta [u \cdot \nabla] e + \delta u \cdot \delta (\rho \nabla e) \right) \]

\[ + \gamma \delta \rho \delta [u \cdot \nabla] e + \alpha \gamma \delta \rho \delta \left( \frac{J_{ne}}{ne} \right) \cdot \delta \left( \frac{J_{ne}}{ne} \right) \cdot \delta (\rho \nabla e) \]

\[ + \alpha \gamma \delta \left( \frac{J_{ne}}{ne} \right) \cdot \delta (\rho \nabla e) - \gamma \delta \left( \frac{J_{ne}}{ne} \right) \cdot \delta (\rho \nabla e) \]

\[ + \delta u \cdot \delta (B \times J) + \delta j \cdot \delta \left( \frac{B \times J}{\rho} \right) + 2 \delta J \cdot \delta \left[ B \times \left( u - \frac{J}{ne} \right) \right] + 4 \alpha \delta J \cdot \delta \left( \frac{B \times J}{ne} \right) \]

This exact relation is a generalised version of Hall MHD turbulence where the electron mass is neglected.
but the electron pressure is not. This is much more realistic model than usual Hall MHD or electron MHD model to probe into the scales comparable or smaller than the electron inertial length. This is the exact relation for the total energy transfer rate in homogeneous and polytropic CHMHD turbulence where electron fluid pressure is not neglected. From this expression, one can easily obtain the exact relations for compressible ordinary MHD turbulence obtained in Banerjee and Kritsuk [25] except the thermodynamic energy term which was calculated from an isothermal closure in [25] and the self-gravity term which is absent here. Now, a simplified estimate gives,

\[
\frac{|u|}{J/ne} \approx ne\mu_0 \frac{|u|}{\nabla \times B} \approx \frac{ne\mu_0 u_\ell}{\sqrt{\mu_0 m_e} b_\ell} = \left( \frac{\ell}{\lambda_i} \right) \left( \frac{u_\ell}{b_\ell} \right),
\]  

(62)

where \( \nabla \sim 1/\ell \) with \( \ell \) being the length scale which we are interested in, \( u_\ell \) and \( b_\ell \) are the velocity and magnetic field fluctuations corresponding to the scale \( \ell \). For typical systems where \( u_\ell \sim b_\ell \), the ratio \( |u|/|J/ne| \) can be given by the factor \( \ell/\lambda_i \) and for ordinary MHD, one is particularly interested in the fluctuation during a very large scale (\( \ell >> \lambda_i \)). Hence, at that limit, \( |u|/|J/ne| >> 1 \) and consequently \( |u|/|\alpha J/ne| >> 1 \). Under the above assumptions, one can finally approximate \( u_\ell \approx u_e \approx u \). Interestingly, this does not imply a zero current density \( J \) but just indicates a negligible value of the term \( J/ne \) with respect to the fluid velocity \( u \) at length scales much larger than \( \lambda_i \). Due to the same reason, all the terms of Eqn. (61) containing electron fluid contribution (also containing \( J/ne \)) can approximately be neglected with respect to similar terms containing single fluid variables (also containing \( u \)). Using the above simplifications, we can easily show that in the limit of CMHD turbulence, the resulting exact relation becomes,

\[
-4\epsilon = \left\langle \delta j \cdot \delta \left[ (u \cdot \nabla) u \right] + \delta u \cdot \delta \left[ \nabla \cdot (J \otimes u) \right] + \gamma \delta \rho \delta \left[ \frac{\mu_0}{\rho} (u \cdot \nabla) e \right] + \gamma \delta u \cdot \delta \left[ \rho \nabla e \right] + \delta u \cdot \delta (B \times J) + 2 \delta j \cdot \delta \left( \frac{B}{\rho} \right) \right\rangle.
\]  

(63)

\[C. \text{ Incompressible single fluid limits}\]

In the incompressible single fluid limit, both ion and electron number densities are equal and constant in space and time. In this limit, we also study two different cases:

a. **Without neglecting electron mass**: As in the previous section, here we also keep the contribution of the electron mass up to first order. The corresponding incompressible single fluid limit can be achieved either by (i) taking the single fluid limit of Eqn. (32) or (ii) re-writing the equation (60) in the incompressible limit. In the following, we shall only derive the single fluid limit of Eqn. (60). For that, we use pre-defined variables:

\[
u_i = u + \frac{\alpha}{1 + \alpha} \left( \frac{J}{ne} \right) \approx u + \alpha \left( \frac{J}{ne} \right) \quad \text{and} \quad \nu_e = u - \frac{1}{1 + \alpha} \left( \frac{J}{ne} \right) \approx u - (1 - \alpha) \left( \frac{J}{ne} \right),
\]

and hence we obtain \( \omega_i \approx \omega + \alpha \Omega \) and \( \omega_e \approx \omega - (1 - \alpha) \Omega \), where \( \Omega = \nabla \times \left( \frac{J}{ne} \right) = \frac{\Sigma \times J}{ne} \) (using incompressibility). Now a few steps of straightforward algebra give
\[ \text{where } Z = \left( \frac{\delta}{\eta \varepsilon} \right) \times \omega + u \times \Omega. \] Adding the above two equations, we get
\[ \rho \left( \delta \mathbf{u} \cdot \delta \mathbf{u} + \alpha \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \cdot \left( \frac{\delta}{\eta \varepsilon} \times \Omega \right) + \alpha \delta \left( \frac{\delta}{\eta \varepsilon} \right) \cdot \delta \mathbf{Z} \right). \] (64)

Again the electromagnetic contribution can be written as (upto first order of \( \alpha \))
\[ \rho \left( \delta \mathbf{u} \cdot \delta \mathbf{u} + \alpha \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \cdot \left( \frac{\delta}{\eta \varepsilon} \times \Omega \right) + \alpha \delta \left( \frac{\delta}{\eta \varepsilon} \right) \cdot \delta \mathbf{Z} \right) \]
\[ \approx \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \cdot \delta (\mathbf{J} \times \mathbf{B}) + \delta \mathbf{J} \cdot \delta \mathbf{u} + 2\alpha \delta \left( \frac{\delta}{\eta \varepsilon} \right) \cdot \delta (\mathbf{J} \times \mathbf{B}) \] (65)

Putting all the contributions together and using Eqn. 62, finally the exact relation for incompressible single fluid plasma turbulence is given by
\[ 2\varepsilon = \rho \left( \delta \mathbf{u} \cdot \delta \mathbf{u} + \alpha \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \cdot \left( \frac{\delta}{\eta \varepsilon} \times \Omega \right) + \alpha \delta \left( \frac{\delta}{\eta \varepsilon} \right) \cdot \delta \mathbf{Z} \right) \]
\[ + \left( \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \cdot \delta (\mathbf{J} \times \mathbf{B}) + \delta \mathbf{J} \cdot \delta \mathbf{u} + 2\alpha \delta \left( \frac{\delta}{\eta \varepsilon} \right) \cdot \delta (\mathbf{J} \times \mathbf{B}) \right) \] (66)

\[ 2\varepsilon = \rho \left( \delta \mathbf{u} \cdot \delta \mathbf{u} + \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \cdot \delta (\mathbf{J} \times \mathbf{B}) + \delta \mathbf{J} \cdot \delta \mathbf{u} \right), \] (67)
\[ = \rho \left( \delta \mathbf{u} \cdot \delta \mathbf{u} + \delta \mathbf{u} \cdot \delta (\mathbf{J} \times \mathbf{B}) + \delta \mathbf{J} \cdot \delta \left( \mathbf{u} - \frac{\delta}{\eta \varepsilon} \right) \times \mathbf{B} \right) \] (68)

which is the exact relation in the incompressible Hall MHD limit. For non-relativistic IMHD and IGMHD limits, one can neglect the displacement current term \((\times \partial \mathbf{E}/\partial t)\) and then write \(\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\). Using normalized (to velocity) magnetic field \(b \equiv \mathbf{B}/\sqrt{\mu_0 \rho_0}\) and \(\mathbf{j}_b = \nabla \times \mathbf{b}\) and also assuming incompressibility \((n \text{ is constant})\), one can show that \(J/\eta \varepsilon = \lambda_i \mathbf{j}_b\), where \(\lambda_i\) is the ion inertial length scale which is defined as
\[ \lambda_i = \frac{\text{speed of light (c)}}{\text{ion plasma frequency} (\omega_{pi})} = \frac{1}{e} \sqrt{\frac{m_i}{n \mu_0}} \] (69)
The equation (65) can therefore be expressed as
\[ 2\varepsilon = nM \left( \delta \mathbf{u} \cdot \delta \mathbf{u} + [\delta \mathbf{j}_b \cdot \delta \{ (\mathbf{u} - \lambda_i \mathbf{j}_b) \times \mathbf{b} \} + \delta \mathbf{u} \cdot \delta (\mathbf{j}_b \times \mathbf{b}) \right), \] (70)

which is similar to what is obtained (where the constant fluid density is normalized to unity) previously.
by Banerjee and Galtier \[15\] for IHMHD turbulence. For the IMHD limit, we are interested in the length

\[2\varepsilon = \rho \left( \delta \mathbf{u} \cdot \delta \left( \mathbf{u} \times \mathbf{\omega} \right) + \delta j_b \cdot \delta \left( \mathbf{u} \times \mathbf{b} \right) + \delta \mathbf{u} \cdot \delta \left( j_b \times \mathbf{b} \right) \right),\]

(71)

which is also equal to the exact relation derived by \[15\] for IMHD turbulence, when the uniform density is assumed to be unity.

D. Effect of mean magnetic field

The total magnetic field \( \mathbf{B} \) at each point can be written as a sum of a mean or uniform

\[\langle \delta (\rho c_i \mathbf{u}_i) \cdot \delta \mathbf{u}_i \times \mathbf{B}_0 + \delta \mathbf{u}_i \cdot \delta (\rho c_i \mathbf{u}_i) \times \mathbf{B}_0 + \delta (\rho c_e \mathbf{u}_e) \cdot \delta \mathbf{u}_e \times \mathbf{B}_0 + \delta \mathbf{u}_e \cdot \delta (\rho c_e \mathbf{u}_e) \times \mathbf{B}_0 \rangle = 0\]

and hence the equation \[30\] remains unaffected by the application of a uniform background field. In the single fluid HMHD limit, the electric field can be expressed in terms of the magnetic field as \( \mathbf{E} = \mathbf{u} \times \mathbf{B} \) and hence the non-zero contribution comes only from the terms which were originally containing electric field in eqn. \[30\] and the corresponding residual contribution due to \( \mathbf{B}_0 \) comes to be

\[\langle -\delta \mathbf{J} \cdot \delta \mathbf{u}_e \times \mathbf{B}_0 + \delta \mathbf{u} \cdot \delta (\rho c_e \mathbf{u}_e) \times \mathbf{B}_0 - \delta \left( \mathbf{u} - \frac{\mathbf{J}}{\rho c_e} \right) \cdot \delta (\rho c_e \mathbf{u}_e) \times \mathbf{B}_0 \rangle = \langle \mathbf{0} \rangle \]

(72)

In the limit of ordinary MHD, \(|\mathbf{u}|/|\mathbf{J}/\rho c_e| >> 1\), and hence the net contribution due to \( \mathbf{B}_0 \) comes to be equal to \( \langle -\delta \mathbf{J} \cdot \mathbf{B}_0 \times \delta \mathbf{u} + \delta \left( \frac{\mathbf{J}}{\rho c_e} \right) \cdot \delta \mathbf{J} \times \mathbf{B}_0 \rangle \), which is identical to the contribution of \( \mathbf{B}_0 \) to the turbulent energy flux rate in compressible MHD turbulence as obtained previously \[25\].

V. CONCLUSIONS

For fully developed turbulence, we derive the exact relation for a 3D compressible two-fluid plasma model. Equation \[30\] can be used to compute the total energy dissipation or transfer rate in a weakly ionized plasma which can mostly be seen in the dilute astrophysical plasmas like cold molecular clouds or protoplanetary disks. Till now, astrophysical turbulence has been mostly studied using hydrodynamic simulation and very few times using MHD single fluid model (for a detailed review, see \[40\] which is a simplistic assumption for an astrophysical medium. This current work will facilitate the study of the astrophysical plasma turbulence to a large extent. In particular, two-fluid compressible plasma turbulence model can be expected to give a reliable estimate of the star formation efficiency in the cold molecular clouds which can then be studied using appropriate numerical simulation. In addition, owing to this law, the effect of background magnetic field in the turbulent star forming regions can also be studied using CTF plasma turbulence simulation. An interesting study will be to understand at which scale the electric field contribution gets converted to the background magnetic field contribution as obtained in the Eqn. \[72\]. Furthermore, this exact relation will help estimate the energy dissipation or heating rate of the ionosphere due to
Although these instabilities can steepen the magnetic and kinetic energy spectra, the form of the exact relation does not a priori assume any isotropy. It is known that turbulence eddies in the form of current sheets and vorticity layers can become more and more anisotropic and finally unstable to plasma instabilities when a large inertial range is considered. Although these instabilities can steepen the magnetic and kinetic energy spectra, the form of the exact relation is expected to be unaltered provided the dissipation rate $\varepsilon$ at the smallest scales. Therefore, it provides a way to test whether a range of scales in a plasma is inertial or dissipative. Over the last years, the sustained increase in the spatial and temporal resolution of space missions such as Cluster (ESA) or the new NASA MMS (Magnetospheric MultiScale) mission has opened the possibility to investigate small-scale plasma phenomena as never before. The exact laws derived here allow investigation of the nature of turbulent magnetic field fluctuations at a broad range of scales in space plasmas, and will be essential to understand the nonlinear nature of turbulence at the electron scales in the solar wind.

VI. ACKNOWLEDGEMENTS

N.A. thanks Daniel O. Gómez for useful discussions. S.B. acknowledges research grant from DST INSPIRE fellowship (DST/PHY/2017514). N.A. acknowledges financial support from CNRS/CONICET Laboratoire International Associé (LIA) MAGNETO.

[1] R. Bruno and V. Carbone, “The Solar Wind as a Turbulence Laboratory,” Living Reviews in Solar Physics 2, 4 (2005).
[2] W. H. Matthaeus and M. L. Goldstein, “Measurement of the rugged invariants of magnetohydrodynamic turbulence in the solar wind,” Journal of Geophysical Research 87, 6011 (1982).
[3] U. Frisch, Turbulence: The Legacy of A. N. Kolmogorov (Cambridge University Press, 1995).
[4] H. Politano and A. Pouquet, “von Kármán–Howarth equations for hall magnetohydrodynamics and its consequences on third-order longitudinal structure and correlation functions,” Physical Review E 57, R21  (1998).
[5] T. von Kármán and L. Howarth, “On the statistical theory of isotropic turbulence,” Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 164, 6011 (1938).
[6] A. N. Kolmogorov, “The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds’ Numbers,” Akademiia Nauk SSSR Doklady 30, 301 (1941a).
[7] A. N. Kolmogorov, “Energy dissipation in locally isotropic turbulence,” Comptes Rendus de l’Académie des Sciences de l’URSS 2, 19 (1941b).
[8] A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 2 (Cambridge, MA: MIT Press., 1975).
[9] S. Banerjee, L. Z. Hadid, F. Sahraoui, and S. Galtier, “Scaling of compressible magnetohydrodynamic turbulence in the fast solar wind,” The Astrophysical Journal Letters 829, L27 (2016).
[10] H. Politano and A. Pouquet, “Dynamical length scales for turbulent magnetized flows,” Geophysical Research Letters 25, 273–276 (1998).
[11] H. Politano, T. Gomez, and A. Pouquet, “von Kármán–Howarth relationship for helical magnetohydrodynamic flows,” Physical Review E 68, 026315 (2003).
[12] S. Galtier, “Exact scaling laws for 3D electron MHD turbulence,” Journal of Geophysical Research: Space Physics 113, A1 (2008).
[13] S. Galtier, “von kármán–howarth equations for hall magnetohydrodynamic flows,” Physical Review E 77, 015302 (2008).
[14] R. Meyrand and S. Galtier, “A universal law for solar-wind turbulence at electron scales,” The Astrophysical Journal 721, 1421-1424 (2010).
[15] S. Banerjee and S. Galtier, “An alternative formulation for exact scaling relations in hydrodynamic and magnetohydrodynamic turbulence,” Journal of Fluid Mechanics 742, 230-242 (2014).
[16] N. Andres and S. Banerjee, “Statistics of incompressible hydrodynamic turbulence: An alternative approach,” Physical Review Fluids 4, 024603 (2019).
[17] R. Ferrand, S. Galtier, F. Sahraoui, R. Meyrand, N. Andres and S. Banerjee, “On exact laws in incompressible Hall MHD turbulence,” The Astrophysical Journal 851, 50 (2019).
[18] S. Galtier and S. Banerjee, “Exact relation for correlation functions in compressible isothermal turbulence,” Physical Review Letters 107, 134501 (2011).
[19] S. Banerjee and S. Galtier, “A kolmogorov-like exact relation for compressible polytropic turbulence,” Journal of Fluid Mechanics 742, 230-242 (2014).
[20] S. Banerjee and S. Galtier, “Exact relation with two-point correlation functions and phenomenological approach for compressible magnetohydrodynamic turbu-
lence,” Physical Review E 87, 013019 (2013).
[21] N. Andrés and F. Sahraoui, “Alternative derivation of exact law for compressible and isothermal magnetohydrodynamics turbulence,” Physical Review E 96, 053205 (2017).
[22] N. Andrés, S. Galtier, and F. Sahraoui, “Exact law for homogeneous compressible Hall magnetohydrodynamics turbulence,” Physical Review E 97, 013204 (2018).
[23] S. Banerjee and A. G. Kritsuk, “Exact relations for energy transfer in self-gravitating isothermal turbulence,” Physical Review E 96, 053116 (2018).
[24] H. Aluie, “Compressible turbulence: the cascade and its locality,” Physical Review Letters 106, 174502 (2011).
[25] S. Banerjee and A. G. Kritsuk, “Energy transfer in compressible magnetohydrodynamic turbulence for isothermal self-gravitating fluids,” Physical Review E 97, 023107 (2018).
[26] S. Galtier, S. Hadid, F. Sahraoui, and S. Galtier, “Energy cascade rate in compressible fast and slow solar wind turbulence,” The Astrophysical Journal 838, 9 (2017).
[27] L. Z. Hadid, F. Sahraoui, S. Galtier, and S. Huang, “Compressible magnetohydrodynamic turbulence in the earth’s magnetosheath: estimation of the energy cascade rate using in situ spacecraft data,” Physical Review Letters 120, 055102 (2018).
[28] N. Andrés, F. Sahraoui, S. Galtier, L. Z. Hadid, R. Ferrand, and S. Y. Huang, “Energy Cascade Rate Measured in a Collisionless Space Plasma with MMS Data and Compressible Hall Magnetohydrodynamic Turbulence Theory,” Physical Review Letters 123, 245101 (2019).
[29] A. G. Kritsuk, R. Wagner, and M. L. Norman, “Energy cascade and scaling in supersonic isothermal turbulence,” Journal of Fluid Mechanics 729, R1 (2013).
[30] N. Andrés, F. Sahraoui, S. Galtier, L. Z. Hadid, P. Dmitruk, and P. D. Mininni, “Energy cascade rate in isothermal compressible magnetohydrodynamic turbulence,” Journal of Plasma Physics 84, 905840404 (2018).
[31] N. Andrés, P. D. Mininni, P. Dmitruk, and D. O. Gomez, “von Kármán–Howarth equation for three-dimensional two-fluid plasmas,” Physical Review E 93, 063202 (2016).
[32] N. Andrés, S. Galtier, and F. Sahraoui, “Exact scaling laws for helical three-dimensional two-fluid turbulent plasmas,” Physical Review E 94, 063206 (2016).
[33] J. A. Bittencourt, Fundamentals of Plasma Physics (Springer, 2004).
[34] E. G. Zweibel and C. F. McKee, “Equipartition of energy for turbulent astrophysical fluids: Accounting for the unseen energy in molecular clouds,” The Astrophysical Journal 439, 779–792 (1995).
[35] L. D. Landau and E. M. Lifshitz, “Fluid Dynamics,” Elsevier Science (2013).
[36] L. C. Steinhauer and A. Ishida, “Relaxation of a Two-Specie Magnetofluid,” Physical Review Letters 79, 3423–3426 (1997).
[37] N. Andrés, L. Martin, P. Dmitruk, and D. Gómez, “Effects of electron inertia in collisionless magnetic reconnection,” Physics of Plasmas 21, 072904 (2014).
[38] L. Comisso, Y. M. Huang, M. Lingam, E. Hirvijoki, and A. Bhattacharjee, “Magnetohydrodynamic Turbulence in the Plasmoid-mediated Regime,” The Astrophysical Journal 854, 103 (2018).
[39] H. M. Abdelhamid Y Kawazura and Z. Yoshida, “Hamiltonian formalism of extended magnetohydrodynamics,” Journal of Physics A: Mathematical and Theoretical 48, 235502 (2015).
[40] W. Schmidt, Numerical Modelling of astrophysical turbulence (Springer Briefs in astronomy, 2014).
[41] A. Miura, “Nonlinear Evolution of the Magnetohydrodynamic Kelvin-Helmholtz Instability,” Physical Review Letters 49, 779–782 (1982).
[42] L. Comisso, M. Lingam, Y. M. Huang, and A. Bhattacharjee, “General theory of the plasmoid instability,” Physics of Plasmas 23, 100702 (2016).