The \( q \)-Onsager algebra and the positive part of \( U_q(\widehat{sl}_2) \)

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Abstract

The positive part \( U_q^+ \) of \( U_q(\widehat{sl}_2) \) has a presentation by two generators \( X, Y \) that satisfy the \( q \)-Serre relations. The \( q \)-Onsager algebra \( O_q \) has a presentation by two generators \( A, B \) that satisfy the \( q \)-Dolan/Grady relations. We give two results that describe how \( U_q^+ \) and \( O_q \) are related. First, we consider the filtration of \( O_q \) whose \( n \)th component is spanned by the products of at most \( n \) generators. We show that the associated graded algebra is isomorphic to \( U_q^+ \). Second, we introduce an algebra \( \Box_q \) and show how it is related to both \( U_q^+ \) and \( O_q \). The algebra \( \Box_q \) is defined by generators and relations. The generators are \( \{x_i \}_{i \in \mathbb{Z}_4} \) where \( \mathbb{Z}_4 \) is the cyclic group of order 4. For \( i \in \mathbb{Z}_4 \) the generators \( x_i, x_{i+1} \) satisfy a \( q \)-Weyl relation, and \( x_i, x_{i+2} \) satisfy the \( q \)-Serre relations. We show that \( \Box_q \) is related to \( U_q^+ \) in the following way. Let \( \Box_q^{\text{even}} \) (resp. \( \Box_q^{\text{odd}} \)) denote the subalgebra of \( \Box_q \) generated by \( x_0, x_2 \) (resp. \( x_1, x_3 \)). We show that (i) there exists an algebra isomorphism \( U_q^+ \to \Box_q^{\text{even}} \) that sends \( X \mapsto x_0 \) and \( Y \mapsto x_2 \); (ii) there exists an algebra isomorphism \( U_q^+ \to \Box_q^{\text{odd}} \) that sends \( X \mapsto x_1 \) and \( Y \mapsto x_3 \); (iii) the multiplication map \( \Box_q^{\text{even}} \otimes \Box_q^{\text{odd}} \to \Box_q, u \otimes v \mapsto uv \) is an isomorphism of vector spaces. We show that \( \Box_q \) is related to \( O_q \) in the following way. For nonzero scalars \( a, b \) there exists an injective algebra homomorphism \( O_q \to \Box_q \) that sends \( A \mapsto ax_0 + a^{-1}x_1 \) and \( B \mapsto bx_2 + b^{-1}x_3 \).

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1 Introduction

There is a family of algebras called tridiagonal algebras [33, Definition 3.9] that come up in the theory of \( Q \)-polynomial distance-regular graphs [31, Lemma 5.4] and tridiagonal pairs [21, Theorem 10.1], [33, Theorem 3.10]. A tridiagonal algebra has a presentation by two generators and two relations of a certain kind, called tridiagonal relations [33, Definition 3.9]. One example of a tridiagonal algebra is the positive part \( U_q^+ \) of the quantum affine algebra \( U_q(\widehat{sl}_2) \) [29, Corollary 3.2.6], [32, Lines (18), (19)]. The algebra \( U_q^+ \) has a presentation by generators \( X, Y \) and relations

\[
X^3Y - [3]_q X^2 YX + [3]_q XYX^2 - YX^3 = 0, \quad (1)
\]

\[
Y^3X - [3]_q Y^2 XY + [3]_q YXY^2 - XY^3 = 0, \quad (2)
\]
where \( [3]_q = (q^3 - q^{-3})/(q - q^{-1}) \). The relations (1), (2) are called the \( q \)-Serre relations [29]. Applications of \( U_q^+ \) to tridiagonal pairs can be found in [21, Example 1.7], [22], [24, Section 2], [33, Lemma 4.8]. The algebra \( U_q^+ \) plays a prominent role in the theory of \( U_q(\mathfrak{sl}_2) \) [12, 14, 19, 23, 25, 29]. Another example of a tridiagonal algebra is the \( q \)-Onsager algebra \( O_q \), [11, Section 2], [22, Section 1]. This algebra has a presentation by generators \( A, B \) and relations

\[
A^3 B - [3]_q A^2 BA + [3]_q ABA^2 - BA^3 = (q^2 - q^{-2})^2 (BA - AB),
\]

\[
B^3 A - [3]_q B^2 AB + [3]_q BAB^2 - AB^3 = (q^2 - q^{-2})^2 (AB - BA).
\]

The relations (3), (4) are called the \( q \)-Dolan/Grady relations [2, Line (5)]. Applications of \( O_q \) to tridiagonal pairs can be found in [21, 26, 27, 32, 33]. The algebra \( O_q \) has applications to quantum integrable models [1, 11], reflection equation algebras [9], and coideal subalgebras [13, 28]. There are algebra homomorphisms from \( O_q \) into \( U_q(\mathfrak{sl}_2) \) [10, line (3.15)], [10, line (3.18)], [28, Example 7.6], the \( q \)-deformed loop algebra \( U_q(L(\mathfrak{sl}_2)) \) [6, Prop. 2.2], [26, Prop. 8.5], [27, Props. 1.1, 1.13], and the universal Askey-Wilson algebra \( \Delta_q \) [34, Sections 9, 10].

Consider how \( U_q^+ \) and \( O_q \) are related. These algebras have at least a superficial resemblance, since for the \( q \)-Serre relations and \( q \)-Dolan/Grady relations their left-hand sides match. In this paper our goal is to describe how \( U_q^+ \) and \( O_q \) are related on an algebraic level. Our first main result is summarized as follows. For notational convenience abbreviate \( O = O_q \). We consider the filtration \( O_0 \subseteq O_1 \subseteq O_2 \subseteq \ldots \) of \( O \) such that for \( n \in \mathbb{N} \) the subspace \( O_n \) is spanned by the products \( g_1 g_2 \cdots g_r \) for which \( 0 \leq r \leq n \) and \( g_i \) is among \( A, B \) for \( 1 \leq i \leq r \). We consider the associated graded algebra \( \overline{O} \) in the sense of [17, p. 203]. We show that the algebras \( U_q^+ \) and \( \overline{O} \) are isomorphic. For our second main result, we introduce an algebra \( \Box_q \) and show how it is related to both \( U_q^+ \) and \( O_q \). Let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \) denote the cyclic group of order 4. The algebra \( \Box_q \) has a presentation by generators \( \{x_i\}_{i \in \mathbb{Z}_4} \) and relations

\[
\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = 1,
\]

\[
x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0.
\]

We show that \( \Box_q \) is related to \( U_q^+ \) in the following way. Let \( \Box_q^{\text{even}} \) (resp. \( \Box_q^{\text{odd}} \)) denote the subalgebra of \( \Box_q \) generated by \( x_0, x_2 \) (resp. \( x_1, x_3 \)). We show that (i) there exists an algebra isomorphism \( U_q^+ \to \Box_q^{\text{even}} \) that sends \( X \mapsto x_0 \) and \( Y \mapsto x_2 \); (ii) there exists an algebra isomorphism \( U_q^+ \to \Box_q^{\text{odd}} \) that sends \( X \mapsto x_1 \) and \( Y \mapsto x_3 \); (iii) the multiplication map \( \Box_q^{\text{even}} \otimes \Box_q^{\text{odd}} \to \Box_q \), \( u \otimes v \mapsto uv \) is an isomorphism of vector spaces. We show that \( \Box_q \) is related to \( O_q \) in the following way. For nonzero scalars \( a, b \) there exists an injective algebra homomorphism \( O_q \to \Box_q \) that sends

\[
A \mapsto ax_0 + a^{-1} x_1, \quad B \mapsto bx_2 + b^{-1} x_3.
\]

Our two main results are obtained under the following assumptions. The underlying field is arbitrary. The scalar \( q \) is nonzero and \( q^2 \neq 1 \). Our two main results are closely related, and will be proved more or less simultaneously. These proofs use only linear algebra, and do not
employ facts invoked from the literature. Our proof strategy is to introduce several algebras \( \hat{\square}_q, \square_q \) that are related to \( \square_q \) via surjective algebra homomorphisms \( \hat{\square}_q \to \hat{\square}_q \to \square_q \). The algebra \( \hat{\square}_q \) is very general, and an explicit basis will be given. Using this basis we will obtain some facts about \( \hat{\square}_q \), which give facts about \( \hat{\square}_q \) and \( \square_q \) via the homomorphisms \( \hat{\square}_q \to \hat{\square}_q \to \square_q \). These facts yield an algebra homomorphism \( O_q \to \hat{\square}_q \) such that the composition \( O_q \to \hat{\square}_q \to \square_q \) is injective. This composition is the homomorphism \( \mathbf{5} \).

Near the end of the paper we will discuss how \( \hat{\square}_q \) and \( \square_q \) are related to \( U_q(\mathfrak{sl}_2) \) and the \( q \)-tetrahedron algebra \( \mathfrak{X}_q \) from \([25]\). We will obtain a commuting diagram of algebra homomorphisms

\[
\begin{array}{c}
\hat{\square}_q \\
\downarrow \\
\square_q
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\longrightarrow
\end{array} \quad \begin{array}{c}
U_q(\mathfrak{sl}_2) \\
\mathfrak{X}_q
\end{array}
\]

In this diagram the homomorphism \( \square_q \to \mathfrak{X}_q \) is injective. Using the diagram we will explain the homomorphisms \( O_q \to U_q(\mathfrak{sl}_2) \) and \( O_q \to U_q(\mathfrak{L}(\mathfrak{sl}_2)) \) that we mentioned earlier in this section.

## 2 Preliminaries

We now begin our formal argument. Throughout this paper the following notation and assumptions are in effect. Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and integers \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \). We will be discussing algebras. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. Let \( \mathbb{F} \) denote a field and let \( \mathcal{A} \) denote an \( \mathbb{F} \)-algebra. Let \( H, K \) denote subspaces of the \( \mathbb{F} \)-vector space \( \mathcal{A} \). Then \( HK \) denotes the subspace of \( \mathcal{A} \) spanned by \( \{hk|h \in H, k \in K\} \). By an \( \mathbb{N} \)-grading of \( \mathcal{A} \) we mean a sequence \( \{A_n\}_{n \in \mathbb{N}} \) such that (i) each \( A_n \) is a subspace of the \( \mathbb{F} \)-vector space \( \mathcal{A} \); (ii) \( 1 \in A_0 \); (iii) the sum \( \mathcal{A} = \sum_{n \in \mathbb{N}} A_n \) is direct; (iv) \( A_r, A_s \subseteq A_{r+s} \) for \( r, s \in \mathbb{N} \). A \( \mathbb{Z} \)-grading of \( \mathcal{A} \) is similarly defined.

We will be discussing algebras defined by generators and relations. Let \( T \) denote the \( \mathbb{F} \)-algebra with generators \( x, y \) and no relations; \( T \) is often called a free algebra or tensor algebra. The generators \( x, y \) will be called standard. For \( n \in \mathbb{N} \), a word of length \( n \) in \( T \) is a product \( g_1 g_2 \cdots g_n \) such that \( g_i \) is a standard generator for \( 1 \leq i \leq n \). We interpret the word of length zero to be the multiplicative identity in \( T \). The words in \( T \) form a basis for the \( \mathbb{F} \)-vector space \( T \). For \( n \in \mathbb{N} \) the words of length \( n \) in \( T \) form a basis for a subspace \( T_n \) of \( T \). For example, 1 is a basis for \( T_0 \) and \( x, y \) is a basis for \( T_1 \). By construction the sum \( T = \sum_{n \in \mathbb{N}} T_n \) is direct. Also by construction \( T_r T_s = T_{r+s} \) for \( r, s \in \mathbb{N} \). By these comments the sequence \( \{T_n\}_{n \in \mathbb{N}} \) is an \( \mathbb{N} \)-grading of \( T \).

Fix \( 0 \neq q \in \mathbb{F} \) such that \( q^2 \neq 1 \). Define

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n \in \mathbb{Z}.
\]

All unadorned tensor products are meant to be over \( \mathbb{F} \).
3 The algebra $U_q^+$

Later in the paper we will discuss the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. In the meantime we consider a certain subalgebra of $U_q(\widehat{\mathfrak{sl}_2})$, denoted $U_q^+$ and called the positive part of $U_q(\widehat{\mathfrak{sl}_2})$.

**Definition 3.1.** (See [29, Corollary 3.2.6] ) Let $U_q^+$ denote the $\mathbb{F}$-algebra with generators $X,Y$ and relations

\begin{align*}
X^3Y - [3]_qX^2XY + [3]_qXYX^2 - YX^3 &= 0, \quad (6) \\
Y^3X - [3]_qY^2XY + [3]_qYXY^2 - XY^3 &= 0. \quad (7)
\end{align*}

The algebra $U_q^+$ is called the *positive part of $U_q(\widehat{\mathfrak{sl}_2})$*. The relations (6), (7) are called the *q-Serre relations*.

**Lemma 3.2.** Let $a,b$ denote nonzero scalars in $\mathbb{F}$. Then there exists an automorphism of $U_q^+$ that sends $X \mapsto aX$ and $Y \mapsto bY$.

*Proof.* Use Definition 3.1. \qed

Recall the free algebra $T$ with standard generators $x,y$. There exists an $\mathbb{F}$-algebra homomorphism $\mu : T \to U_q^+$ that sends $x \mapsto X$ and $y \mapsto Y$. The homomorphism $\mu$ is surjective. We now describe the kernel of $\mu$. Define elements $S_x, S_y$ in $T$ by

\begin{align*}
S_x &= x^3y - [3]_qx^2yx + [3]_qxy^2 - yx^3, \quad (8) \\
S_y &= y^3x - [3]_qy^2xy + [3]_qyx^2 - xy^3. \quad (9)
\end{align*}

Let $S = TS_xT + TS_yT$ denote the 2-sided ideal of $T$ generated by $S_x, S_y$. Then $S$ is the kernel of $\mu$. Recall the $\mathbb{N}$-grading $\{T_n\}_{n \in \mathbb{N}}$ of $T$. For $n \in \mathbb{N}$ let $U_q^+_n$ denote the image of $T_n$ under $\mu$. The $\mathbb{F}$-vector space $U_q^+_n$ is spanned by the products $g_1g_2\cdots g_n$ such that $g_i$ is among $X,Y$ for $1 \leq i \leq n$. Let $\mu_n$ denote the restriction of $\mu$ to $T_n$. We view $\mu_n : T_n \to U_q^+_n$. The kernel of $\mu_n$ is $S \cap T_n$. The subspace $S \cap T_n$ is described as follows. Note that $S_x, S_y \in T_4$. So for $r, s \in \mathbb{N}$, $T_rS_xT_s \subseteq T_{r+s+4}$ and $T_rS_yT_s \subseteq T_{r+s+4}$. Consequently for $n \in \mathbb{N}$,

\begin{equation}
S \cap T_n = \sum_{r,s \in \mathbb{N}} T_rS_xT_s + \sum_{r,s \in \mathbb{N}} T_rS_yT_s. \quad (10)
\end{equation}

Assume for the moment that $n \leq 3$. There does not exist $r, s \in \mathbb{N}$ such that $r+s = n-4$. Therefore $S \cap T_n = 0$, so $\mu_n : T_n \to U_q^+_n$ is an isomorphism. Taking $n = 0, 1$ we see that 1 is a basis for $U_q^+_0$ and $X,Y$ is a basis for $U_q^+_1$. We show that the sequence $\{U_q^+_n\}_{n \in \mathbb{N}}$ is an $\mathbb{N}$-grading of $U_q^+$. By (10) and $T = \sum_{r \in \mathbb{N}} T_r$ we obtain $S = \sum_{n \in \mathbb{N}} (S \cap T_n)$. Therefore the sum $U_q^+ = \sum_{n \in \mathbb{N}} U_q^+_n$ is direct. Recall that $T_rT_s = T_{r+s}$ for $r, s \in \mathbb{N}$. Applying $\mu$ we find that $U_q^+_rU_q^+_s = U_q^+_{r+s}$ for $r, s \in \mathbb{N}$. By these comments the sequence $\{U_q^+_n\}_{n \in \mathbb{N}}$ is an $\mathbb{N}$-grading of $U_q^+$. 

4
4 The q-Onsager algebra

In this section we recall the q-Onsager algebra and discuss its basic properties.

Definition 4.1. (See [1 Section 2], [33 Definition 3.9].) Let $O = O_q$ denote the $\mathbb{F}$-algebra with generators $A, B$ and relations

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = (q^2 - q^{-2})^2(BA - AB), \quad (11)$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = (q^2 - q^{-2})^2(AB - BA). \quad (12)$$

We call $O$ the q-Onsager algebra. We call (11), (12) the q-Dolan/Grady relations.

Consider the elements $\{A^iB^j\}_{r,s \in \mathbb{N}}$ in the $\mathbb{F}$-vector space $O$. We show that these elements are linearly independent. Let $A^r, B^s$ denote commuting indeterminates. Let $\mathbb{F}[A^r, B^s]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $A^r, B^s$ that have all coefficients in $\mathbb{F}$. The elements $\{(A^r)^*(B^s)^*\}_{r,s \in \mathbb{N}}$ form a basis for the $\mathbb{F}$-vector space $\mathbb{F}[A^r, B^s]$. By the nature of the relations (11), (12) there exists an $\mathbb{F}$-algebra homomorphism $O \rightarrow \mathbb{F}[A^r, B^s]$ that sends $A \mapsto A^r$ and $B \mapsto B^s$. By these comments the elements $\{A^iB^j\}_{r,s \in \mathbb{N}}$ in $O$ are linearly independent. In particular the elements 1, $A, B$ in $O$ are linearly independent. Recall the free algebra $T$ with standard generators $x, y$. There exists an $\mathbb{F}$-algebra homomorphism $\nu : T \rightarrow O$ that sends $x \mapsto A$ and $y \mapsto B$. The homomorphism $\nu$ is surjective. For $n \in \mathbb{N}$ let $O_n$ denote the image of $T_0 + T_1 + \cdots + T_n$ under $\nu$. The $\mathbb{F}$-vector space $O_n$ is spanned by the products $g_1g_2 \cdots g_i$ such that $0 \leq r \leq n$ and $g_i$ is among $A, B$ for $1 \leq i \leq r$. For example $O_0 = \mathbb{F}0$ and $O_1 = \mathbb{F}A + \mathbb{F}B$. For notational convenience define $O_{-1} = 0$. For $n \in \mathbb{N}$ we have $O_{n-1} + \nu(T_n) = O_n$ and in particular $O_{n-1} \subset O_n$. Since $\nu$ is surjective we have $O = \cup_{n \in \mathbb{N}}O_n$. By construction $O_0, O_s = O_{r+s}$ for $r, s \in \mathbb{N}$. The sequence $\{O_n\}_{n \in \mathbb{N}}$ is a filtration of $O$ in the sense of [17] p. 202]. For $n \in \mathbb{N}$ consider the quotient $\mathbb{F}$-vector space $O_n = O_n/O_{n-1}$. We view $O_0 = O_0$. The elements $\overline{A}, \overline{B}$ form a basis for $\overline{O}_1$, where $\overline{A} = A + O_0$ and $\overline{B} = B + O_0$. Now consider the formal direct sum $\overline{O} = \sum_{n \in \mathbb{N}} \overline{O}_n$. We emphasize that for all elements $u = \sum_{n \in \mathbb{N}} u_n$ in $\overline{O}$, the summand $u_n$ is nonzero for finitely many $n \in \mathbb{N}$. By construction $\overline{O}$ is an $\mathbb{F}$-vector space. We next define a product $\overline{O} \times \overline{O} \rightarrow \overline{O}$ that turns $\overline{O}$ into an $\mathbb{F}$-algebra. For $r, s \in \mathbb{N}$ the product sends $\overline{O}_r \times \overline{O}_s \rightarrow \overline{O}_{r+s}$ as follows. For $u \in \overline{O}_r$ and $v \in \overline{O}_s$ the product of $u + \overline{O}_{r-1}$ and $v + \overline{O}_{s-1}$ is $uv + \overline{O}_{r+s-1}$. We have turned $\overline{O}$ into an $\mathbb{F}$-algebra [17] p. 203]. By construction $\overline{O}_r, \overline{O}_s = \overline{O}_{r+s}$ for $r, s \in \mathbb{N}$. For $n \in \mathbb{N}$ the $\mathbb{F}$-vector space $\overline{O}_n$ is spanned by the products $g_1g_2 \cdots g_n$ such that $g_i$ is among $A, B$ for $1 \leq i \leq n$. The $\mathbb{F}$-algebra $\overline{O}$ is generated by $\overline{A}, \overline{B}$. The sequence $\{\overline{O}_n\}_{n \in \mathbb{N}}$ is an $\mathbb{F}$-grading of $\overline{O}$. The $\mathbb{F}$-algebra $\overline{O}$ is called the graded algebra associated with the filtration $\{O_n\}_{n \in \mathbb{N}}$ [17] p. 203]. We now construct an $\mathbb{F}$-algebra homomorphism $\overline{\nu} : T \rightarrow \overline{O}$. For $n \in \mathbb{N}$ let $\overline{\nu}_n$ denote the restriction of $\nu$ to $T_n$. We view $\nu_n : T_n \rightarrow O_n$. Consider the composition $\overline{\nu}_n : T_n \xrightarrow{\nu_n} O_n \xrightarrow{u \mapsto u + O_{n-1}} \overline{O}_n$.

The map $\overline{\nu}_n : T_n \rightarrow \overline{O}_n$ is $\mathbb{F}$-linear and surjective. Define an $\mathbb{F}$-linear map $\overline{\nu} : T \rightarrow \overline{O}$ that acts on $T_n$ as $\overline{\nu}_n$ for $n \in \mathbb{N}$. By construction $\overline{\nu}(T_n) = \overline{O}_n$ for $n \in \mathbb{N}$. The map $\overline{\nu}$ sends $x \mapsto \overline{A}$ and $y \mapsto \overline{B}$.

Lemma 4.2. The map $\overline{\nu} : T \rightarrow \overline{O}$ is an $\mathbb{F}$-algebra homomorphism.
Proof. Pick \( r, s \in \mathbb{N} \). It suffices to show that \( \nu(uv) = \nu(u)\nu(v) \) for \( u \in T_r \) and \( v \in T_s \). This equation is routinely verified using the definition of the map \( \nu \) and the algebra \( \overline{O} \).

The algebras \( U^+ \) and \( \overline{O} \) are related as follows. Using (11), (12) we obtain

\[
\begin{align*}
A^3B - [3]_q A^2B A + [3]_q ABA^2 - B A^3 &= 0, \\
B^3A - [3]_q B^2A B + [3]_q A AB^2 - A B^3 &= 0.
\end{align*}
\]

By (13), (14) there exists an \( \mathbb{F} \)-algebra homomorphism \( \psi : U^+ \rightarrow \overline{O} \) that sends \( X \mapsto A \) and \( Y \mapsto B \). The homomorphism \( \psi \) is surjective. By construction \( \psi(U^+_n) = \overline{O}_n \) for \( n \in \mathbb{N} \).

Lemma 4.3. The following diagram commutes:

\[
\begin{array}{ccc}
T & \xrightarrow{I} & T \\
\mu \downarrow & & \downarrow \nu \\
U^+ & \xrightarrow{\psi} & \overline{O}
\end{array}
\]

Proof. Each map in the diagram is an \( \mathbb{F} \)-algebra homomorphism. To verify that the diagram commutes, chase the standard generators \( x, y \) around the diagram.

Theorem 4.4. The map \( \psi : U^+ \rightarrow \overline{O} \) is an isomorphism of \( \mathbb{F} \)-algebras.

The proof of Theorem 4.4 will be completed in Proposition 10.32.

We mention one significance of Lemma 4.3 and Theorem 4.4. Consider the free algebra \( T \) with \( \mathbb{N} \)-grading \( \{T_n\}_{n \in \mathbb{N}} \). Pick \( u \in T \). For \( n \in \mathbb{N} \), we say that \( u \) is \( n \)-homogeneous whenever \( u \in T_n \). We say that \( u \) is homogeneous whenever there exists \( n \in \mathbb{N} \) such that \( u \) is \( n \)-homogeneous.

Definition 4.5. A subset \( \Omega \subseteq T \) is said to be homogeneous whenever each element in \( \Omega \) is homogeneous.

Proposition 4.6. Let \( \Omega \) denote a homogeneous subset of \( T \) such that the vectors \( \mu(z) \) (\( z \in \Omega \)) form a basis for the \( \mathbb{F} \)-vector space \( U^+ \). Then the vectors \( \nu(z) \) (\( z \in \Omega \)) form a basis for the \( \mathbb{F} \)-vector space \( \overline{O} \).

Proof. Recall the \( \mathbb{F} \)-algebra isomorphism \( \psi : U^+ \rightarrow \overline{O} \) from Theorem 4.4. For \( n \in \mathbb{N} \) let \( \Omega_n \) denote the set of \( n \)-homogeneous elements in \( \Omega \). We have \( \Omega_n \subseteq T_n \) and \( \mu(T_n) = U^+_n \), so \( \mu(z) \in U^+_n \) for \( z \in \Omega_n \). By assumption \( \Omega \) is homogeneous, so \( \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \). Since the vectors \( \mu(z) \) (\( z \in \Omega \)) form a basis for \( U^+ \) and the sum \( U^+ = \sum_{n \in \mathbb{N}} U^+_n \) is direct, we see that for \( n \in \mathbb{N} \) the vectors \( \mu(z) \) (\( z \in \Omega_n \)) form a basis for \( U^+_n \). We mentioned above Lemma 4.3 that \( \psi(U^+_n) = \overline{O}_n \), so the vectors \( \psi(\mu(z)) \) (\( z \in \Omega_n \)) form a basis for \( \overline{O}_n \). By Lemma 4.3 we have \( \psi(\mu(z)) = \nu(z) \) for \( z \in \Omega_n \), so the vectors \( \nu(z) \) (\( z \in \Omega_n \)) form a basis for \( \overline{O}_n \). Consequently the vectors \( \nu(z) \) (\( z \in \Omega \)) form a basis for a complement of \( \overline{O}_{n-1} \) in \( \overline{O}_n \). Therefore the vectors \( \nu(z) \) (\( z \in \Omega \)) form a basis for \( \overline{O} \).
Note 4.7. Assume that \( \mathbb{F} \) is algebraically closed with characteristic zero, and \( q \) is not a root of unity. In [22, Theorem 2.29], T. Ito and the present author display a homogeneous subset \( \Omega \) of \( T \) such that the vectors \( \mu(z) \) \((z \in \Omega)\) form a basis for the \( \mathbb{F} \)-vector space \( U^+ \). In [27, Theorem 2.1] the same authors show that the vectors \( \nu(z) \) \((z \in \Omega)\) form a basis for the \( \mathbb{F} \)-vector space \( \mathcal{O} \). We point out that [27, Theorem 2.1] follows from [22, Theorem 2.29] and Proposition 4.6.

5 The algebra \( \Box_q \)

We have been discussing the algebra \( U^+ = U^+_q \) which is the positive part of \( U_q(\hat{sl}_2) \), and the \( q \)-Onsager algebra \( \mathcal{O} = \mathcal{O}_q \). As we compare these algebras, it is useful to bring in another algebra \( \Box_q \). In this section we introduce \( \Box_q \), and describe how it is related to \( U^+ \) and \( \mathcal{O} \).

Let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \) denote the cyclic group of order 4.

Definition 5.1. Let \( \Box_q \) denote the \( \mathbb{F} \)-algebra with generators \( \{x_i\}_{i \in \mathbb{Z}_4} \) and relations

\[
qx_i x_{i+1} - q^{-1} x_{i+1} x_i = 1, \quad (15)
\]
\[
x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0. \quad (16)
\]

We have some comments.

Lemma 5.2. There exists an automorphism \( \rho \) of \( \Box_q \) that sends \( x_i \mapsto x_{i+1} \) for \( i \in \mathbb{Z}_4 \). Moreover \( \rho^4 = 1 \).

Lemma 5.3. For \( 0 \neq a \in \mathbb{F} \) there exists an automorphism of \( \Box_q \) that sends

\[
x_0 \mapsto ax_0, \quad x_1 \mapsto a^{-1} x_1, \quad x_2 \mapsto ax_2, \quad x_3 \mapsto a^{-1} x_3.
\]

Our next goal is to describe how \( \Box_q \) is related to \( U^+ \).

Definition 5.4. Define the subalgebras \( \Box_q^{\text{even}} \), \( \Box_q^{\text{odd}} \) of \( \Box_q \) such that

(i) \( \Box_q^{\text{even}} \) is generated by \( x_0, x_2 \);

(ii) \( \Box_q^{\text{odd}} \) is generated by \( x_1, x_3 \).

Proposition 5.5. The following (i)–(iii) hold:

(i) there exists an \( \mathbb{F} \)-algebra isomorphism \( U^+ \to \Box_q^{\text{even}} \) that sends \( X \mapsto x_0 \) and \( Y \mapsto x_2 \);

(ii) there exists an \( \mathbb{F} \)-algebra isomorphism \( U^+ \to \Box_q^{\text{odd}} \) that sends \( X \mapsto x_1 \) and \( Y \mapsto x_3 \);

(iii) the following is an isomorphism of \( \mathbb{F} \)-vector spaces:

\[
\Box_q^{\text{even}} \otimes \Box_q^{\text{odd}} \to \Box_q
\]

\[
u \otimes u \mapsto uv
\]
The proof of Proposition 5.5 will be completed in Section 10. Next we describe how $\bigotimes_q$ is related to $\mathcal{O}$.

**Proposition 5.6.** Pick nonzero $a, b \in \mathbb{F}$. Then there exists an $\mathbb{F}$-algebra homomorphism $\sharp: \mathcal{O} \to \bigotimes_q$ that sends $A \mapsto ax_0 + a^{-1}x_1, \quad B \mapsto bx_2 + b^{-1}x_3$. (17)

The proof of Proposition 5.6 will be completed in Section 8. In Theorem 10.33 we will show that the map $\sharp$ from Proposition 5.6 is injective.

# 6 The algebra $\widetilde{\bigotimes}_q$

In the previous section we introduced the algebra $\bigotimes_q$. As we investigate $\bigotimes_q$, it is useful to consider a certain homomorphic preimage denoted $\widetilde{\bigotimes}_q$. In this section we introduce $\widetilde{\bigotimes}_q$ and describe its basic properties.

**Definition 6.1.** Let $\widetilde{\bigotimes}_q$ denote the $\mathbb{F}$-algebra with generators $c_i^{\pm 1}, x_i$ ($i \in \mathbb{Z}_4$) and relations

$$c_i c_i^{-1} = c_i^{-1} c_i = 1, \quad c_i^{\pm 1} \text{ is central in } \widetilde{\bigotimes}_q, \quad \frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = c_i. \quad (18, 19, 20)$$

We have some comments.

**Lemma 6.2.** There exists an automorphism $\tilde{\rho}$ of $\widetilde{\bigotimes}_q$ that sends $c_i \mapsto c_{i+1}$ and $x_i \mapsto x_{i+1}$ for $i \in \mathbb{Z}_4$. Moreover $\tilde{\rho}^4 = 1$.

**Lemma 6.3.** There exists a unique $\mathbb{F}$-algebra homomorphism $\widetilde{\bigotimes}_q \to \bigotimes_q$ that sends $c_i^{\pm 1} \mapsto 1$ and $x_i \mapsto x_i$ for $i \in \mathbb{Z}_4$. This homomorphism is surjective.

**Definition 6.4.** The homomorphism $\widetilde{\bigotimes}_q \to \bigotimes_q$ from Lemma 6.3 will be called canonical.

**Definition 6.5.** Define the subalgebras $\widetilde{\bigotimes}_q^{\text{even}}, \widetilde{\bigotimes}_q^{\text{odd}}, \widetilde{C}$ of $\widetilde{\bigotimes}_q$ such that

(i) $\widetilde{\bigotimes}_q^{\text{even}}$ is generated by $x_0, x_2$;

(ii) $\widetilde{\bigotimes}_q^{\text{odd}}$ is generated by $x_1, x_3$;

(iii) $\widetilde{C}$ is generated by $\{c_i^{\pm 1}\}_{i \in \mathbb{Z}_4}$.

**Lemma 6.6.** Let $\{\alpha_i\}_{i \in \mathbb{Z}_4}$ denote invertible elements in $\widetilde{C}$. Then there exists an $\mathbb{F}$-algebra homomorphism $\widetilde{\bigotimes}_q \to \widetilde{\bigotimes}_q$ that sends $x_i \mapsto \alpha_i x_i$ and $c_i \mapsto \alpha_i c_i^{\pm 1}$ for $i \in \mathbb{Z}_4$.

**Definition 6.7.** The homomorphism in Lemma 6.6 will be denoted by $\tilde{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$.
Lemma 6.8. Referring to Lemma 6.6 and Definition 6.7, assume that \(0 \neq \alpha_i \in \mathbb{F}\) for \(i \in \mathbb{Z}_4\). Then \(\widetilde{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) is an automorphism of \(\widehat{\square}_q\). Its inverse is \(\widetilde{g}^{-1}(\alpha_0^{-1}, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})\).

Proof. One checks that \(\widetilde{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) and \(\widetilde{g}^{-1}(\alpha_0^{-1}, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})\) are inverses. Therefore \(\widetilde{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) is invertible and hence an automorphism of \(\widehat{\square}_q\).

Our next goal is to obtain an analog of Proposition 5.5 that applies to \(\widehat{\square}_q\).

Lemma 6.9. In \(\widehat{\square}_q\),

\[
\begin{align*}
x_1 x_0 &= q^2 x_0 x_1 + (1 - q^2) c_0, \\
x_1 x_2 &= q^{-2} x_2 x_1 + (1 - q^{-2}) c_1, \\
x_3 x_2 &= q^2 x_2 x_3 + (1 - q^2) c_2, \\
x_3 x_0 &= q^{-2} x_0 x_3 + (1 - q^{-2}) c_3.
\end{align*}
\]

Proof. These are reformulations of (20).

Definition 6.10. The four relations in Lemma 6.9 will be called reduction rules for \(\widehat{\square}_q\).

We now express the reduction rules in a uniform way.

Lemma 6.11. Referring to the algebra \(\widehat{\square}_q\), pick \(u \in \{x_0, x_2\}\) and \(v \in \{x_1, x_3\}\). Then

\[
vu = u v q^{\langle u, v \rangle} + \gamma(u, v)(1 - q^{\langle u, v \rangle})
\]

where

\[
\begin{array}{c|cc}
\langle , \rangle & x_1 & x_3 \\
\hline
x_0 & 2 & -2 \\
x_2 & -2 & 2 \\
\end{array}
\quad
\begin{array}{c|cc}
\gamma(\ , \ ) & x_1 & x_3 \\
\hline
x_0 & c_0 & c_3 \\
x_2 & c_1 & c_2 \\
\end{array}
\]

Proof. Use Lemma 6.9.

Lemma 6.12. Fix \(r \in \mathbb{N}\). Referring to the algebra \(\widehat{\square}_q\), pick \(u_i \in \{x_0, x_2\}\) for \(1 \leq i \leq r\), and also \(v \in \{x_1, x_3\}\). Then

\[
v u_1 u_2 \cdots u_r = u_1 u_2 \cdots u_r v q^{\langle u_1, v \rangle} + \cdots + \langle u_r, v \rangle + \sum_{i=1}^{r} u_1 \cdots u_{i-1} u_{i+1} \cdots u_r \gamma(u_i, v) q^{\langle u_1, v \rangle} + \cdots + \langle u_{i-1}, v \rangle (1 - q^{\langle u, v \rangle}).
\]

The functions \(\langle \ , \ \rangle\) and \(\gamma(\ , \ )\) are from Lemma 6.11.

Proof. By Lemma 6.11 and induction on \(r\).

Lemma 6.13. In the algebra \(\widehat{\square}_q\),

\[
\begin{align*}
x_1 \widehat{\square}_q^{\text{even}} &\subseteq \widehat{\square}_q^{\text{even}} x_1 + \widehat{\square}_q^{\text{even}} c_0 + \widehat{\square}_q^{\text{even}} c_1, \\
x_3 \widehat{\square}_q^{\text{even}} &\subseteq \widehat{\square}_q^{\text{even}} x_3 + \widehat{\square}_q^{\text{even}} c_2 + \widehat{\square}_q^{\text{even}} c_3.
\end{align*}
\]
\textbf{Lemma 6.14.} We have
\[ \bar{\tilde{\gamma}}_q = \bar{\tilde{\gamma}}_q^{\text{even}} \bar{\tilde{\gamma}}_q^{\text{odd}} \cdot \tilde{\mathcal{C}}. \] \hfill (22)

\textbf{Proof.} By Definition \textbf{6.5} and Lemma \textbf{6.12}. \hfill \square

\textbf{Definition 6.15.} Let \{\lambda_i\}_{i \in \mathbb{Z}_4} denote mutually commuting indeterminates. Let \( \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}] \) denote the \( \mathbb{F} \)-algebra consisting of the Laurent polynomials in \( \{\lambda_i\}_{i \in \mathbb{Z}_4} \) that have all coefficients in \( \mathbb{F} \). We abbreviate
\[ L = \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}]. \] \hfill (23)

Recall the free algebra \( T \) with standard generators \( x, y \).

\textbf{Lemma 6.16.} The \( \mathbb{F} \)-vector space \( T \otimes T \otimes L \) has a unique \( \bar{\tilde{\gamma}}_q \)-module structure such that for all \( u, v \in T \) and \( w \in L \),
\begin{enumerate}
  \item \( c_i^{\pm 1} \) sends \( u \otimes v \otimes w \mapsto u \otimes v \otimes (\lambda_i^{\pm 1}w) \) for \( i \in \mathbb{Z}_4 \);
  \item \( x_0 \) sends \( u \otimes v \otimes w \mapsto (ux) \otimes v \otimes w \);
  \item \( x_1 \) sends \( 1 \otimes v \otimes w \mapsto 1 \otimes (xv) \otimes w \);
  \item \( x_2 \) sends \( u \otimes v \otimes w \mapsto (yu) \otimes v \otimes w \);
  \item \( x_3 \) sends \( 1 \otimes v \otimes w \mapsto 1 \otimes (yv) \otimes w \).
\end{enumerate}

\textbf{Proof.} We first show that the \( \bar{\tilde{\gamma}}_q \)-module structure exists. For notational convenience abbreviate \( V = T \otimes T \otimes L \). Let \( \text{End}(V) \) denote the \( \mathbb{F} \)-algebra consisting of the \( \mathbb{F} \)-linear maps from \( V \) to \( V \). We now define some elements in \( \text{End}(V) \). Momentarily abusing notation, we call these elements \( \{c_i^{\pm 1}\}_{i \in \mathbb{Z}_4} \) and \( \{x_i\}_{i \in \mathbb{Z}_4} \). For \( i \in \mathbb{Z}_4 \) there exist \( c_i^{\pm 1} \in \text{End}(V) \) that satisfy (i). There exists \( x_0 \in \text{End}(V) \) that satisfies (ii). There exists \( x_2 \in \text{End}(V) \) that satisfies (iv). We now define \( x_1 \in \text{End}(V) \) and \( x_3 \in \text{End}(V) \). To do this, we specify how \( x_1 \) and \( x_3 \) act on \( u \otimes v \otimes w \). Here we use Lemma \textbf{6.12} as a guide. Without loss of generality, we may assume that \( u \) is a word in \( T \). Write \( u \) as a product \( u_1u_2 \cdots u_r \) of standard generators. The element \( x_1 \) sends
\[ u \otimes v \otimes w \mapsto u \otimes (xv) \otimes wq^{(u_1,x)+\cdots+(u_r,x)} + \sum_{i=1}^{r} u_1 \cdots u_{i-1}u_{i+1} \cdots u_r \otimes v \otimes wq(\gamma(u_i,x), 1 - q^{(u_i,x)}) \],
and \( x_3 \) sends
\[ u \otimes v \otimes w \mapsto u \otimes (yu) \otimes wq^{(u_1,y)+\cdots+(u_r,y)} + \sum_{i=1}^{r} u_1 \cdots u_{i-1}u_{i+1} \cdots u_r \otimes v \otimes wq(\gamma(u_i,y), 1 - q^{(u_i,y)}) \],
where
where

\[
\begin{array}{c|cc}
\langle, \rangle & x & y \\
x & 2 & -2 \\
y & -2 & 2 \\
\end{array}
\quad \quad \quad
\begin{array}{c|cc}
\gamma(, ) & x & y \\
x & \lambda_0 & \lambda_3 \\
y & \lambda_1 & \lambda_2 \\
\end{array}
\]

We just specified how \(x_1\) and \(x_3\) act on \(u \otimes v \otimes w\). For \(u = 1\) these actions become (iii) and (v) in the lemma statement. We have defined the elements \(\{c_{±1}^i\}_{i \in \mathbb{Z}_4}\) and \(\{x_i\}_{i \in \mathbb{Z}_4}\) in \(\text{End}(V)\). One checks that these elements satisfy the defining relations (18)–(20) for \(\tilde{\Box}_q\). This turns \(V\) into a \(\tilde{\Box}_q\)-module, and by construction this \(\tilde{\Box}_q\)-module satisfies the requirements in the lemma statement. We have shown that the \(\tilde{\Box}_q\)-module structure exists. The \(\tilde{\Box}_q\)-module structure is unique in view of Lemma 6.12. \(\square\)

**Proposition 6.17.** The following (i)–(iv) hold:

(i) there exists an \(F\)-algebra isomorphism \(T \to \tilde{\Box}_q^{\text{even}}\) that sends \(x \mapsto x_0\) and \(y \mapsto x_2\);

(ii) there exists an \(F\)-algebra isomorphism \(T \to \tilde{\Box}_q^{\text{odd}}\) that sends \(x \mapsto x_1\) and \(y \mapsto x_3\);

(iii) there exists an \(F\)-algebra isomorphism \(L \to \tilde{C}\) that sends \(\lambda_i \mapsto c_{±1}^i\) for \(i \in \mathbb{Z}_4\);

(iv) the following is an isomorphism of \(F\)-vector spaces:

\[
\tilde{\Box}_q^{\text{even}} \otimes \tilde{\Box}_q^{\text{odd}} \otimes \tilde{C} \to \tilde{\Box}_q
\]

\[
u \otimes v \otimes c \mapsto uv c
\]

**Proof.** There exists a surjective \(F\)-algebra homomorphism \(f_1 : T \to \tilde{\Box}_q^{\text{even}}\) that sends \(x \mapsto x_0\) and \(y \mapsto x_2\). There exists a surjective \(F\)-algebra homomorphism \(f_2 : T \to \tilde{\Box}_q^{\text{odd}}\) that sends \(x \mapsto x_1\) and \(y \mapsto x_3\). There exists a surjective \(F\)-algebra homomorphism \(f_3 : L \to \tilde{C}\) that sends \(\lambda_i \mapsto c_{±1}^i\) for \(i \in \mathbb{Z}_4\). There exists an \(F\)-linear map \(f : T \otimes T \otimes L \to \tilde{\Box}_q\) that sends \(u \otimes v \otimes w \mapsto f_1(u)f_2(v)f_3(w)\) for all \(u, v \in T\) and \(w \in L\). It suffices to show that \(f\) is bijective. The map \(f\) is surjective by Lemma 6.14. To see that \(f\) is injective, view \(T \otimes T \otimes L\) as a \(\tilde{\Box}_q\)-module as in Lemma 6.16. The map \(f\) is injective because the composition

\[
T \otimes T \otimes L \xrightarrow{f} \tilde{\Box}_q \xrightarrow{z \mapsto z(1 \otimes 1 \otimes 1)} T \otimes T \otimes L
\]

is the identity map on \(T \otimes T \otimes L\). By these comments \(f\) is bijective. The result follows. \(\square\)

**Proposition 6.18.** For the algebra \(\tilde{\Box}_q\),

(i) the following is a basis for the \(F\)-vector space \(\tilde{\Box}_q^{\text{even}}\):

\[
u_1 \nu_2 \cdots \nu_r \quad r \in \mathbb{N}, \quad \nu_i \in \{x_0, x_2\}, \quad 1 \leq i \leq r; \quad (24)
\]

(ii) the following is a basis for the \(F\)-vector space \(\tilde{\Box}_q^{\text{odd}}\):

\[
u_1 \nu_2 \cdots \nu_s \quad s \in \mathbb{N}, \quad \nu_i \in \{x_1, x_3\}, \quad 1 \leq i \leq s; \quad (25)
\]
(iii) the following is a basis for the $\mathbb{F}$-vector space $\tilde{\mathbb{C}}$:

$$c_0^{n_0} c_1^{n_1} c_2^{n_2} c_3^{n_3}, \quad n_i \in \mathbb{Z}, \quad i \in \mathbb{Z}_4; \quad (26)$$

(iv) the $\mathbb{F}$-vector space $\tilde{\square}_q$ has a basis consisting of the elements

$$uvc \quad (27)$$

such that $u, v, c$ are contained in the bases (24), (25), (26), respectively.

Proof. By Proposition 6.17 \hfill \square

7 Some calculations in $\tilde{\square}_q$

We continue to investigate the algebra $\tilde{\square}_q$ from Definition 6.1. In this section, we obtain some results about $\tilde{\square}_q$ that will be used in the proof of Proposition 5.5.

Definition 7.1. For the algebra $\tilde{\square}_q$ define

$$S_i = x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 \quad i \in \mathbb{Z}_4. \quad (28)$$

We note that the elements $S_i$ from Definition 7.1 are in the kernel of the canonical homomorphism $\tilde{\square}_q \rightarrow \square_q$.

Lemma 7.2. We have $S_0, S_2 \in \tilde{\square}_q^{\text{even}}$ and $S_1, S_3 \in \tilde{\square}_q^{\text{odd}}$.

Proof. By Definition 6.5(i),(ii) and Definition 7.1 \hfill \square

Proposition 7.3. In the algebra $\tilde{\square}_q$ the following (i), (ii) hold for $i \in \mathbb{Z}_4$:

(i) $x_{i+1} S_i = q^4 S_i x_{i+1}$,

(ii) $x_{i-1} S_i = q^{-4} S_i x_{i-1}$.

Proof. (i) Without loss of generality we may assume that $i = 0$. Our strategy is to express $x_1 S_0 - q^4 S_0 x_1$ in the basis for $\tilde{\square}_q$ from Proposition 6.18(iv). By Definition 7.1

$$x_1 S_0 = x_1 x_0^3 x_2 - [3]_q x_1 x_0^2 x_2 x_0 + [3]_q x_1 x_0 x_2 x_0^2 - x_1 x_2 x_0^3, \quad (29)$$

$$S_0 x_1 = x_0^3 x_2 x_1 - [3]_q x_0^2 x_2 x_0 x_1 + [3]_q x_0 x_2 x_0^2 x_1 - x_2 x_0^3 x_1. \quad (30)$$

Consider the elements

$$x_1 x_0^3 x_2, \quad x_1 x_0^2 x_2 x_0, \quad x_1 x_0 x_2 x_0^2, \quad x_1 x_2 x_0^3, \quad S_0 x_1. \quad (31)$$

By Lemma 6.12 and (30), the elements (31) are weighted sums involving the following terms and coefficients:
We express this, expand and evaluate the result using the data in the above table. After a routine cancellation we obtain \( x_1S_0 - q^4S_0x_1 = 0 \). Therefore \( x_1S_0 = q^4S_0x_1 \).

(ii) We proceed as in part (i) above. Without loss of generality we may assume that \( i = 0 \). We express \( x_3S_0 - q^{-4}S_0x_3 \) in the basis for \( \tilde{\mathbb{K}}_q \) from Proposition (6.18(iv)). By Definition 7.1

\[
x_3S_0 = x_3x_0^3x_2 - [3]_q x_3x_0^2x_2x_0 + [3]_q x_3x_0x_2x_0^2 - x_3x_2x_0^3,
\]

(32)

\[
S_0x_3 = x_0^3x_2x_3 - [3]_q x_0^2x_2x_0x_3 + [3]_q x_0x_2x_0^2x_3 - x_0x_2x_0^3x_3.
\]

(33)

Consider the elements

\[
x_3x_0^2x_2, \quad x_3x_0^2x_2x_0, \quad x_3x_0^2x_2x_0^2, \quad x_3x_2x_0^3, \quad S_0x_3.
\]

(34)

By Lemma 6.12 and (33), the elements (34) are weighted sums involving the following terms and coefficients:

Now to write \( x_3S_0 - q^{-4}S_0x_3 \) in the basis for \( \tilde{\mathbb{K}}_q \) from Proposition (6.18(iv)), expand \( x_3S_0 - q^{-4}S_0x_3 \) using (32) and evaluate the result using the data in the above table. After a routine cancellation we obtain \( x_3S_0 - q^{-4}S_0x_3 = 0 \). Therefore \( x_3S_0 = q^{-4}S_0x_3 \).

\[
\text{Corollary 7.4. In the algebra } \tilde{\mathbb{K}}_q,\]

\[
S_0\tilde{\mathbb{K}}_{\text{odd}} = \tilde{\mathbb{K}}_q \text{odd } S_0, \quad S_1\tilde{\mathbb{K}}_{\text{even}} = \tilde{\mathbb{K}}_q \text{even } S_1, \quad S_2\tilde{\mathbb{K}}_{\text{odd}} = \tilde{\mathbb{K}}_q \text{odd } S_2, \quad S_3\tilde{\mathbb{K}}_{\text{even}} = \tilde{\mathbb{K}}_q \text{even } S_3.
\]

\[
\text{Proof. By Definition (6.5(i),(ii)) and Proposition 7.3}\]

\[
\square
\]
8 More calculations in $\tilde{\Box}_q$

We continue to investigate the algebra $\tilde{\Box}_q$ from Definition 6.1. In this section, we first obtain some results about $\tilde{\Box}_q$. We then use these results to prove Proposition 5.6.

For the algebra $\tilde{\Box}_q$ define

$$A = x_0 + x_1, \quad B = x_2 + x_3.$$  \hspace{1cm} (35)

Our next goal is to express

$$A^3B - [3]_qA^2BA + [3]_qABA^2 - BA^2 + (q^2 - q^{-2})^2c_0(AB - BA)$$  \hspace{1cm} (36)

in the basis for $\tilde{\Box}_q$ from Proposition 6.18(iv).

For the rest of this section, the notation (35) is in effect.

Lemma 8.1. In the algebra $\tilde{\Box}_q$,

(i) the element $A^2$ is a weighted sum involving the following terms and coefficients:

| term   | $x_0^2$ | $x_0x_1$ | $x_1^2$ | $c_0$ |
|--------|---------|----------|---------|-------|
| coefficient | $1$    | $1 + q^2$ | $1$     | $1 - q^2$ | 14

(ii) the element $AB$ is a weighted sum involving the following terms and coefficients:

| term   | $x_0x_2$ | $x_0x_3$ | $x_2x_1$ | $x_1x_3$ | $c_1$ |
|--------|----------|----------|----------|----------|-------|
| coefficient | $1$    | $1$     | $q^{-2}$ | $1$     | $1 - q^{-2}$ | 1

(iii) the element $BA$ is a weighted sum involving the following terms and coefficients:

| term   | $x_2x_0$ | $x_0x_3$ | $x_2x_1$ | $x_3x_1$ | $c_3$ |
|--------|----------|----------|----------|----------|-------|
| coefficient | $1$    | $q^{-2}$ | $1$     | $1$     | $1 - q^{-2}$ | 1

Proof. Use Lemma 6.9 and (35). \hfill $\blacksquare$

Next we use Lemma 8.1 to evaluate some terms in (36). Consider $A^3B = (A^2)(AB)$. In this equation, evaluate the right-hand side using Lemma 8.1(i),(ii) and expand the result; the details are in the table below. The table has two header columns that describe $A^2$, and two header rows that describe $AB$. The expressions inside parentheses are “out of order” and will be subject to further reduction shortly.

| $AB$ | $1$ | $1$ | $q^{-2}$ | $1$ | $1 - q^{-2}$ |
|------|-----|-----|----------|-----|-------------|
| $A^2$ | $x_0x_2$ | $x_0x_3$ | $x_2x_1$ | $x_1x_3$ | $c_1$ | 14
Similarly for $A^2BA = (A^2)(BA)$,

$$BA$$

$$
\begin{array}{c|cccccc}
A^2 & 1 & q^{-2} & 1 & 1 & 1 - q^{-2} \\
1 & x_0^2 & x_0x_0 & x_0x_3 & x_2x_1 & x_3x_1 & c_3 \\
1 + q^2 & x_0x_1 & x_0(x_1x_2) & x_0(x_1x_3) & x_0(x_1x_2)x_1 & x_0x_1x_3x_1 & x_0x_1x_3c_3 \\
1 & x_1^2 & (x_1^2x_2) & (x_1^2x_3) & (x_1^2x_2)x_1 & x_1^2x_3x_1 & x_1^2x_3c_3 \\
1 - q^2 & c_0 & x_2x_0c_0 & x_0x_3c_0 & x_2x_1c_0 & x_3x_1c_0 & c_0c_3 \\
\end{array}
$$

Similarly for $ABA^2 = (AB)(A^2)$,

$$A^2$$

$$
\begin{array}{c|cccc}
AB & 1 & 1 + q^2 & 1 & 1 - q^2 \\
1 & x_0x_2 & x_0x_2x_0 & x_0x_2x_1 & x_0x_2x_2 \\
1 & x_0x_3 & x_0(x_3x_0) & x_0(x_3x_1) & x_0(x_3x_2) \\
q^{-2} & x_2x_1 & x_2(x_1x_0) & x_2(x_1x_1) & x_2x_1c_0 \\
1 & x_1x_3 & (x_1x_3x_0) & (x_1x_3x_1) & x_1x_3c_0 \\
1 - q^{-2} & c_1 & x_2x_0c_1 & x_0x_1c_1 & x_1^2c_1 & c_0c_1 \\
\end{array}
$$

Similarly for $BA^3 = (BA)(A^2)$,

$$A^2$$

$$
\begin{array}{c|cccc}
BA & 1 & 1 + q^2 & 1 & 1 - q^2 \\
1 & x_2x_0 & x_2x_0x_0 & x_2x_0x_3 & x_2x_0x_2 \\
q^{-2} & x_0x_3 & x_0(x_3x_0) & x_0(x_3x_1) & x_0(x_3x_2) \\
1 & x_0x_1 & x_0(x_1x_0) & x_0(x_1x_1) & x_2x_1c_0 \\
1 & x_3x_1 & (x_3x_1x_0) & (x_3x_1x_1) & x_3x_1c_0 \\
1 - q^{-2} & c_3 & x_2x_0c_3 & x_0x_1c_3 & x_1^2c_3 & c_0c_3 \\
\end{array}
$$

The above four tables contain some parenthetical expressions. We will write these parenthetical expressions in the basis for $\Box_q$ from Proposition 6.18(iv). For the parenthetical expressions of length two, this is done using Lemma 6.9. For the remaining parenthetical expressions, this will be done over the next three lemmas.

**Lemma 8.2.** In the algebra $\Box_q$,

(i) the element $x_1x_0x_2$ is a weighted sum involving the following terms and coefficients:

| term | $x_0x_2x_1$ | $x_0c_1$ | $x_2c_0$ |
|------|-------------|----------|----------|
| coefficient | 1 | $q^*-1$ | $1 - q^2$ |

15
(ii) the element $x_1^2x_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_0x_1^2$ | $q^4$       |
| $x_1c_0$  | $1 - q^4$   |

(iii) the element $x_1^2x_2$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_2x_1^2$ | $q^{-4}$   |
| $x_1c_1$  | $1 - q^{-4}$|

(iv) the element $x_1x_2x_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_2x_0x_1$ | $1$       |
| $x_2c_0$  | $q^{-2} - 1$|
| $x_0c_1$  | $1 - q^{-2}$|

Proof. Apply Lemma 6.9 repeatedly.

Lemma 8.3. In the algebra $\tilde{\square}_q$,

(i) the element $x_1x_3x_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_0x_1x_3$ | $1$       |
| $x_1c_3$  | $1 - q^{-2}$|
| $x_3c_0$  | $q^{-2} - 1$|

(ii) the element $x_1^2x^2_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_0^2x_1$ | $q^4$       |
| $x_0c_0$  | $1 - q^4$   |

(iii) the element $x_3^2x^2_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_3^2x_0$ | $q^{-4}$   |
| $x_0c_3$  | $1 - q^{-4}$|

(iv) the element $x_3x_1x_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_0x_3x_1$ | $1$       |
| $x_1c_3$  | $q^{-2} - 1$|
| $x_3c_0$  | $1 - q^2$   |

Proof. Similar to the proof of Lemma 8.2.

Lemma 8.4. In the algebra $\tilde{\square}_q$,

(i) the element $x_1^2x_0x_2$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_0x_2x_1^2$ | $1$       |
| $x_0x_1c_1$  | $q^4 - 1$   |
| $x_2x_1c_0$  | $q^{-2} - q^2$|
| $c_0c_1$    | $(1 - q^2)(q^2 - q^{-2})$|

(ii) the element $x_1^2x_2x_0$ is a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $x_2x_0x_1^2$ | $1$       |
| $x_0x_1c_1$  | $q^{-2} - q^2$|
| $x_2x_1c_0$  | $q^{-4} - 1$|
| $c_0c_1$    | $(q^{-2} - 1)(q^2 - q^{-2})$|

\[16\]
(iii) the element $x_1 x_3 x_0^2$ is a weighted sum involving the following terms and coefficients:

| term  | $x_0^2 x_1 x_3$ | $x_0 x_1 c_3$ | $x_0 x_3 c_0$ | $c_0 c_3$ |
|-------|-----------------|----------------|----------------|------------|
| coefficient | $1$ | $q^2 - q^{-2}$ | $q^{-4} - 1$ | $(q^{-2} - 1)(q^2 - q^{-2})$ |

(iv) the element $x_3 x_1 x_0^2$ is a weighted sum involving the following terms and coefficients:

| term  | $x_0^2 x_3 x_1$ | $x_0 x_1 ^c_3$ | $x_0 x_3 c_0$ | $c_0 c_3$ |
|-------|-----------------|-----------------|----------------|------------|
| coefficient | $1$ | $q^4 - 1$ | $q^{-2} - q^2$ | $(1 - q^2)(q^2 - q^{-2})$ |

Proof. Similar to the proof of Lemma 8.2.

Referring to the algebra $\widetilde{\mathbb{C}}_q$, we now write the elements

$$A^3 B, \quad A^2 BA, \quad ABA^2, \quad BA^3$$

in the basis for $\widetilde{\mathbb{C}}_q$ from Proposition 6.18(iv).

**Lemma 8.5.** In the algebra $\widetilde{\mathbb{C}}_q$, the elements (37) are weighted sums involving the following terms and coefficients.

| term  | $A^3 B$ coeff. | $A^2 BA$ coeff. | $ABA^2$ coeff. | $BA^3$ coeff. |
|-------|-----------------|-----------------|----------------|----------------|
| $x_0^2 x_2$ | $1$ | $0$ | $0$ | $0$ |
| $x_0^2 x_2 x_0$ | $0$ | $1$ | $0$ | $0$ |
| $x_0 x_2 x_0^2$ | $0$ | $0$ | $1$ | $0$ |
| $x_2 x_0^3$ | $0$ | $0$ | $0$ | $1$ |
| $x_0 x_3 x_1$ | $1$ | $0$ | $0$ | $0$ |
| $x_0^2 x_3 x_1$ | $0$ | $1$ | $0$ | $0$ |
| $x_1 x_3 x_1^2$ | $0$ | $0$ | $1$ | $0$ |
| $x_3 x_1^3$ | $0$ | $0$ | $0$ | $1$ |
| $x_0^2 x_3 x_3$ | $1$ | $q^{-2}$ | $q^{-4}$ | $q^{-6}$ |
| $x_0 x_3 x_3$ | $q^{-6}$ | $q^{-4}$ | $q^{-2}$ | $1$ |
| $x_0 x_2 x_3 x_3$ | $q^{-3}[3]_q$ | $q^2$ | $0$ | $0$ |
| $x_0 x_1 x_3 x_1$ | $0$ | $q^2 + 1$ | $q^2 + 1$ | $0$ |
| $x_0 x_3 x_1^2$ | $0$ | $0$ | $1$ | $[3]_q$ |
| $x_0 x_2 x_1$ | $[3]_q$ | $1$ | $0$ | $0$ |
| $x_0 x_2 x_0 x_1$ | $0$ | $q^2 + 1$ | $q^2 + 1$ | $0$ |
| $x_2 x_0 x_1$ | $0$ | $0$ | $q^2$ | $q^2[3]_q$ |
| $x_0^2 x_1 x_3$ | $q^2[3]_q$ | $q^2 + 1$ | $1$ | $0$ |
| $x_0 x_3 x_1$ | $0$ | $1$ | $q^{-2} + 1$ | $q^{-2}[3]_q$ |
| $x_0 x_2 x_1^2$ | $q^{-2}[3]_q$ | $q^{-2} + 1$ | $1$ | $0$ |
| $x_2 x_0 x_1^2$ | $0$ | $1$ | $q^2 + 1$ | $q^2[3]_q$ |
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
term & $A^3B$ coef. & $A^2BA$ coef. & $ABA^2$ coef. & $BA^3$ coef. \\
\hline $x_0^2c_1$ & $(q^2 - 1)[3]_q$ & $q^2 - q^{-2}$ & $1 - q^{-2}$ & 0 \\
$x_0^2c_3$ & 0 & $1 - q^{-2}$ & $1 - q^{-4}$ & $q^{-2}(1 - q^{-2})[3]_q$ \\
$x_0x_2c_0$ & $(1 - q^2)(2 + q^2)$ & $q^{-2} - q^2$ & $1 - q^{-2}$ & 0 \\
x_2x_0c_0 & 0 & $1 - q^{-2}$ & $q^{-2} - q^2$ & $(1 - q^2)(2 + q^2)$ \\
\hline $x_0x_1c_1$ & $(q^2 - q^{-2})[3]_q$ & $2(q^2 - q^{-2})$ & $q^2 - q^{-2}$ & 0 \\
x_0x_1c_3 & 0 & $q^2 - q^{-2}$ & $2(q^2 - q^{-2})$ & $(q^2 - q^{-2})[3]_q$ \\
x_0x_3c_0 & $(1 - q^2)(2 + q^2)$ & $(q^2 - 1)(q^2 + 2)$ & $(q^2 - 1)[3]_q$ & $(q^2 - 1)(q^2 + 2)$ \\
x_2x_1c_0 & $(q^2 - 1)(2 + q^2)$ & $(q^2 - 1)[3]_q$ & $(q^2 - 1)(q^2 + 2)$ & $(1 - q^2)(2 + q^2)$ \\
\hline $x_1^2c_1$ & $q^{-2}(1 - q^{-2})[3]_q$ & $1 - q^{-4}$ & $1 - q^{-2}$ & 0 \\
x_1^2c_3 & 0 & $1 - q^{-2}$ & $q^2 - q^{-2}$ & $(q^2 - 1)[3]_q$ \\
x_1x_3c_0 & $(1 - q^2)(2 + q^2)$ & $q^{-2} - q^2$ & $1 - q^{-2}$ & 0 \\
x_3x_1c_0 & 0 & $1 - q^{-2}$ & $q^{-2} - q^2$ & $(1 - q^2)(2 + q^2)$ \\
\hline $c_0c_1$ & $-(q - q^{-1})^2(2 + q^2)$ & $(q^2 - 1)(q^2 - q^{-2})$ & $-(q - q^{-1})^2$ & 0 \\
c_0c_3 & 0 & $-(q - q^{-1})^2$ & $(q^2 - 1)(q^2 - q^{-2})$ & $-(q - q^{-1})^2(2 + q^2)$ \\
\hline
\end{tabular}
\end{center}

**Proof.** In the four tables below Lemma 8.1, evaluate the parenthetical expressions using Lemma 6.9 along with Lemmas 8.2, 8.3, 8.4. \qed

Recall the elements \{$(S_i)_{i \in \mathbb{Z}_4}$\} in $\tilde{\square}_q$, from Definition 7.1.

**Proposition 8.6.** For the algebra $\tilde{\square}_q$ define

$$A = x_0 + x_1, \quad B = x_2 + x_3.$$ 

Then both

$$A^3B - [3]_qA^2BA + [3]_qABA^2 - BA^3 + (q^2 - q^{-2})^2c_0(AB - BA) = S_0 + S_1, \quad (38)$$

$$B^3A - [3]_qB^2AB + [3]_qBAB^2 - AB^3 + (q^2 - q^{-2})^2c_2(AB - BA) = S_2 + S_3. \quad (39)$$

**Proof.** To obtain (38), write (36) in the basis for $\tilde{\square}_q$ from Proposition 6.18 iv). To do this, evaluate the terms (37) using Lemma 8.5 and the term $c_0(AB - BA)$ using Lemma 8.1 ii), iii). Assertion (38) follows after a routine computation. Assertion (39) is obtained from (38) by applying the square of the automorphism $\tilde{\rho}$ from Lemma 6.2 \qed

**Proof of Proposition 5.6** The composition

$$\tilde{\square}_q \xrightarrow{\tilde{g}((a,a^{-1}),b,b^{-1})} \tilde{\square}_q \xrightarrow{\text{can}} \square_q$$

is an $\mathbb{F}$-algebra homomorphism. Apply this homomorphism to everything in Proposition 8.6 \qed
9 The algebra  \( \hat{q} \)

We have been discussing the algebras \( q \) and \( \tilde{q} \). As we compare these algebras, it is useful to bring in an “intermediate” algebra \( \hat{q} \) such that the canonical homomorphism \( \tilde{q} \to q \) has a factorization \( \tilde{q} \to \hat{q} \to q \).

**Definition 9.1.** Let \( \hat{q} \) denote the \( \mathbb{F} \)-algebra with generators \( c_i^{\pm 1}, x_i \ (i \in \mathbb{Z}_4) \) and relations

\[
\begin{align*}
 c_i c_i^{-1} &= c_i^{-1} c_i = 1, \\
 c_i^{\pm 1} &\text{ are central in } \hat{q}, \\
 q x_i x_{i+1} - q^{-1} x_{i+1} x_i &= c_i, \\
 x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} + [3]_q x_i x_{i+2} x_{i+1} - x_{i+2} x_i^3 &= 0.
\end{align*}
\]

We have some comments.

**Lemma 9.2.** There exists an automorphism \( \hat{\rho} \) of \( \hat{q} \) that sends \( c_i \mapsto c_{i+1} \) and \( x_i \mapsto x_{i+1} \) for \( i \in \mathbb{Z}_4 \). Moreover \( \hat{\rho}^4 = 1 \).

**Lemma 9.3.** There exists an \( \mathbb{F} \)-algebra homomorphism \( \hat{q} \to q \) that sends \( c_i^{\pm 1} \mapsto c_i^{\pm 1} \) and \( x_i \mapsto x_i \) for \( i \in \mathbb{Z}_4 \). This homomorphism is surjective.

**Lemma 9.4.** There exists an \( \mathbb{F} \)-algebra homomorphism \( \hat{q} \to q \) that sends \( c_i^{\pm 1} \mapsto 1 \) and \( x_i \mapsto x_i \) for \( i \in \mathbb{Z}_4 \). This homomorphism is surjective.

**Definition 9.5.** The homomorphisms \( \tilde{q} \to \hat{q} \) from Lemma 9.3 and \( \hat{q} \to q \) from Lemma 9.4 will be called *canonical*.

**Note 9.6.** In Definition 6.4 we defined the canonical homomorphism \( \tilde{q} \to q \), and in Definition 9.5 we defined the canonical homomorphisms \( \tilde{q} \to \hat{q} \) and \( \hat{q} \to q \). When we speak of the canonical homomorphism, it should be clear from the context which version we refer to.

**Lemma 9.7.** The following diagram commutes:

\[
\begin{array}{ccc}
\tilde{q} & \xrightarrow{\text{can}} & \hat{q} \\
\downarrow & & \downarrow \text{can} \\
\hat{q} & \xrightarrow{\text{can}} & q
\end{array}
\]

**Proof.** By Definitions 6.4, 9.5.

**Definition 9.8.** Define the subalgebras \( \hat{q}^{\text{even}} \), \( \hat{q}^{\text{odd}} \), \( \hat{C} \) of \( \hat{q} \) such that

1. \( \hat{q}^{\text{even}} \) is generated by \( x_0, x_2 \);
2. \( \hat{q}^{\text{odd}} \) is generated by \( x_1, x_3 \);
(iii) \( \hat{C} \) is generated by \( \{c_i^{\pm 1}\}_{i \in \mathbb{Z}_4} \).

**Lemma 9.9.** Let \( \{\alpha_i\}_{i \in \mathbb{Z}_4} \) denote invertible elements in \( \hat{C} \). Then there exists an \( F \)-algebra homomorphism \( \hat{\square}_q \to \hat{\square}_q \) that sends \( x_i \mapsto \alpha_i x_i \) and \( c_i \mapsto \alpha_i \alpha_{i+1} c_i \) for \( i \in \mathbb{Z}_4 \).

**Definition 9.10.** The homomorphism in Lemma 9.9 will be denoted by \( \hat{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \).

**Lemma 9.11.** Referring to Lemma 9.9 and Definition 9.10, assume that \( 0 \neq \alpha_i \in F \) for \( i \in \mathbb{Z}_4 \). Then \( \hat{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \) is an automorphism of \( \hat{\square}_q \). Its inverse is \( \hat{g}(\alpha_0^{-1}, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}) \).

**Proof.** Similar to the proof of Lemma 6.8.

Our next goal is to obtain an analog of Propositions 5.5, 6.17 that applies to \( \hat{\square}_q \).

**Definition 9.12.** Let \( J \) denote the 2-sided ideal of \( \tilde{\square}_q \) generated by \( \{S_i\}_{i \in \mathbb{Z}_4} \). Thus

\[
J = \sum_{i \in \mathbb{Z}_4} \tilde{\square}_q S_i \tilde{\square}_q.
\]  

(44)

**Lemma 9.13.** The canonical homomorphism \( \tilde{\square}_q \to \hat{\square}_q \) has kernel \( J \).

**Proof.** Compare Definitions 6.1, 9.1.

**Definition 9.14.** Let \( J^{\text{even}} \) (resp. \( J^{\text{odd}} \)) denote the 2-sided ideal of \( \tilde{\square}_q^{\text{even}} \) (resp. \( \tilde{\square}_q^{\text{odd}} \)) generated by \( S_0, S_2 \) (resp. \( S_1, S_3 \)). Thus

\[
J^{\text{even}} = \tilde{\square}_q^{\text{even}} S_0 \tilde{\square}_q^{\text{even}} + \tilde{\square}_q^{\text{even}} S_2 \tilde{\square}_q^{\text{even}},
\]  

(45)

\[
J^{\text{odd}} = \tilde{\square}_q^{\text{odd}} S_1 \tilde{\square}_q^{\text{odd}} + \tilde{\square}_q^{\text{odd}} S_3 \tilde{\square}_q^{\text{odd}}.
\]  

(46)

Recall the free algebra \( T \) with standard generators \( x, y \). Below (9) we defined the 2-sided ideal \( S \) of \( T \).

**Lemma 9.15.** The following (i), (ii) hold:

(i) \( J^{\text{even}} \) is the image of \( S \) under the isomorphism \( T \to \tilde{\square}_q^{\text{even}} \) from Proposition 6.17(i);  
(ii) \( J^{\text{odd}} \) is the image of \( S \) under the isomorphism \( T \to \tilde{\square}_q^{\text{odd}} \) from Proposition 6.17(ii).

**Proof.** Compare Definition 9.14 with the definition of \( S \) below (9).

**Lemma 9.16.** Referring to the vector space isomorphism from Proposition 6.17(iv), the preimage of \( J \) is

\[
J^{\text{even}} \otimes \tilde{\square}_q^{\text{odd}} \otimes \tilde{C} + \tilde{\square}_q^{\text{even}} \otimes J^{\text{odd}} \otimes \tilde{C}.
\]  

(47)
Proof. Let $\tilde{m}$ denote the isomorphism in question. Let $J'$ denote the image of $\tilde{m}$ under $\tilde{m}$. We show that $J = J'$. We have $J \supseteq J'$ by construction and since each of $J_{\text{even}}$, $J_{\text{odd}}$ is contained in $J$. To obtain $J \subseteq J'$, by (44) it suffices to show that $\tilde{m}_q S_q \tilde{m}_q \subseteq J'$ for $i \in \mathbb{Z}_4$.

Let $i$ be given, and first assume that $i$ is even. By Lemma 6.14, Corollary 7.4, and since $\tilde{C}$ is central in $\tilde{\Box}_q$,

$$\tilde{\Box}_q S_q \tilde{\Box}_q = \tilde{\Box}_q^\text{even} \tilde{\Box}_q^\text{odd} \tilde{\Box}_q^\text{even} \tilde{\Box}_q^\text{odd} \tilde{\Box}_q^\text{even} \tilde{\Box}_q^\text{odd} \tilde{\Box}_q^\text{even} S_q \tilde{\Box}_q^\text{odd} \tilde{\Box}_q \subseteq \tilde{\Box}_q^\text{even} S_q \tilde{\Box}_q^\text{odd} \tilde{\Box}_q^\text{even} S_q \tilde{\Box}_q^\text{odd} \tilde{\Box}_q.$$

By Lemma 6.14, line (45), and the definition of $\tilde{m}$,

$$\tilde{\Box}_q^\text{even} S_q \tilde{\Box}_q = \tilde{\Box}_q^\text{even} S_q \tilde{\Box}_q^\text{even} \tilde{\Box}_q^\text{odd} \tilde{\Box}_q^\text{even} \tilde{\Box}_q^\text{odd} \tilde{\Box}_q = \tilde{\Box}(\tilde{J}_{\text{even}} \otimes \tilde{\Box}_q^\text{odd} \otimes \tilde{\Box}_q) \subseteq J'.$$

We have shown that $\tilde{\Box}_q S_q \tilde{\Box}_q \subseteq J'$ for $i$ even. We similarly show that $\tilde{\Box}_q S_q \tilde{\Box}_q \subseteq J'$ for $i$ odd. Therefore $J \subseteq J'$. We have shown that $J = J'$, and the result follows.

Lemma 9.17. In the algebra $\tilde{\Box}_q$,

(i) $J \cap \tilde{\Box}_q^\text{even} = J_{\text{even}}$,

(ii) $J \cap \tilde{\Box}_q^\text{odd} = J_{\text{odd}}$,

(iii) $J \cap \tilde{C} = 0$.

Proof. By Lemma 9.16 and since neither of $J_{\text{even}}$, $J_{\text{odd}}$ contains 1.

Recall from Definition 9.8 the subalgebras $\tilde{\Box}_q^\text{even}$, $\tilde{\Box}_q^\text{odd}$, $\tilde{\Box}$ of $\tilde{\Box}_q$.

Lemma 9.18. For the canonical homomorphism $\tilde{\Box}_q \to \tilde{\Box}_q$, the images of $\tilde{\Box}_q^\text{even}$, $\tilde{\Box}_q^\text{odd}$, $\tilde{\Box}$ are $\tilde{\Box}_q^\text{even}$, $\tilde{\Box}_q^\text{odd}$, $\tilde{\Box}$, respectively.

Proof. By Definitions 6.5, 9.5, 9.8 and Lemma 9.3

Definition 9.19. For the canonical homomorphism $\tilde{\Box}_q \to \tilde{\Box}_q$, the restrictions to $\tilde{\Box}_q^\text{even}$, $\tilde{\Box}_q^\text{odd}$, $\tilde{\Box}$ induce surjective $F$-algebra homomorphisms

$$\tilde{\Box}_q^\text{even} \to \tilde{\Box}_q^\text{even}, \quad \tilde{\Box}_q^\text{odd} \to \tilde{\Box}_q^\text{odd}, \quad \tilde{\Box} \to \tilde{\Box}.$$

Each of the homomorphisms (48) will be called restricted canonical.

Lemma 9.20. The following (i)–(iii) hold:

(i) the restricted canonical homomorphism $\tilde{\Box}_q^\text{even} \to \tilde{\Box}_q^\text{even}$ has kernel $J_{\text{even}}$;

(ii) the restricted canonical homomorphism $\tilde{\Box}_q^\text{odd} \to \tilde{\Box}_q^\text{odd}$ has kernel $J_{\text{odd}}$;

(iii) the restricted canonical homomorphism $\tilde{\Box} \to \tilde{\Box}$ is a bijection.

Proof. By Lemmas 9.13, 9.17.
Proposition 9.21. The following (i)–(iv) hold:

(i) there exists an \( F \)-algebra isomorphism \( U^+ \to \hat{q}^{\text{even}} \) that sends \( X \mapsto x_0 \) and \( Y \mapsto x_2 \);

(ii) there exists an \( F \)-algebra isomorphism \( U^+ \to \hat{q}^{\text{odd}} \) that sends \( X \mapsto x_1 \) and \( Y \mapsto x_3 \);

(iii) there exists an \( F \)-algebra isomorphism \( L \to \hat{C} \) that sends \( \lambda_i^{\pm 1} \mapsto c_i^{\pm 1} \) for \( i \in \mathbb{Z}_4 \);

(iv) the following is an isomorphism of \( F \)-vector spaces:

\[
\hat{q}^{\text{even}} \otimes \hat{q}^{\text{odd}} \otimes \hat{C} \to \hat{q},
\]

\[
u \otimes v \otimes c \mapsto uv c
\]

Proof. (i) Recall the free algebra \( T \) with standard generators \( x, y \). By Proposition 6.17(i) there exists an \( F \)-algebra isomorphism \( T \to \hat{q}^{\text{even}} \) that sends \( x \mapsto x_0 \) and \( y \mapsto x_2 \). The inverse of this isomorphism will be denoted by \( \hat{\theta} \). Consider the \( F \)-algebra homomorphism \( \mu : T \to U^+ \) that sends \( x \mapsto X \) and \( y \mapsto Y \). By Lemma 9.15(i), the composition

\[
\hat{q}^{\text{even}} \xrightarrow{\hat{\theta}} T \xrightarrow{\mu} U^+
\]

is surjective with kernel \( J^{\text{even}} \). Therefore there exists an \( F \)-algebra isomorphism \( \theta : \hat{q}^{\text{even}} / J^{\text{even}} \to U^+ \) that sends \( x_0 + J^{\text{even}} \mapsto X \) and \( x_2 + J^{\text{even}} \mapsto Y \). The restricted canonical homomorphism \( \hat{\theta} \big|_{\hat{q}^{\text{even}}} \to \hat{\theta} \big|_{\hat{q}^{\text{even}}} \) sends \( x_0 \mapsto x_0 \) and \( x_2 \mapsto x_2 \). This map is surjective by construction, and has kernel \( J^{\text{even}} \) by Lemma 9.20(i). Therefore there exists an \( F \)-algebra isomorphism \( \hat{\theta} : \hat{q}^{\text{even}} / J^{\text{even}} \to \hat{q}^{\text{even}} \) that sends \( x_0 + J^{\text{even}} \mapsto x_0 \) and \( x_2 + J^{\text{even}} \mapsto x_2 \). The composition

\[
U^+ \xrightarrow{\hat{\theta}^{-1}} \hat{q}^{\text{even}} / J^{\text{even}} \xrightarrow{\theta} \hat{q}^{\text{even}}
\]

is the desired \( F \)-algebra isomorphism.

(ii), (iii). Similar to the proof of (i) above.

(iv). The multiplication map \( \hat{m} : \hat{q}^{\text{even}} \otimes \hat{q}^{\text{odd}} \otimes \hat{C} \to \hat{q} \), \( u \otimes v \otimes c \mapsto uv c \) is \( F \)-linear. We show that \( \hat{m} \) is a bijection. By Proposition 6.17(iv), the multiplication map \( \hat{m} : \hat{q}^{\text{even}} \otimes \hat{q}^{\text{odd}} \otimes \hat{C} \to \hat{q} \), \( u \otimes v \otimes c \mapsto uv c \) is an isomorphism of \( F \)-vector spaces. Recall the canonical homomorphism \( \hat{\eta} : \hat{q} \to \hat{q} \) from Definition 9.3. By construction the following diagram commutes:

\[
\begin{array}{ccc}
\hat{q}^{\text{even}} \otimes \hat{q}^{\text{odd}} \otimes \hat{C} & \xrightarrow{u \otimes v \otimes c \mapsto \text{can}(u) \otimes \text{can}(v) \otimes \text{can}(c)} & \hat{q}^{\text{even}} \otimes \hat{q}^{\text{odd}} \otimes \hat{C} \\
\hat{\eta} & \downarrow & \hat{\eta} \\
\hat{q} & \xrightarrow{\text{can}} & \hat{q}
\end{array}
\]

The map \( \hat{m} \) is surjective by these comments and the last assertion of Lemma 9.3. The map \( \hat{m} \) is injective in view of Lemma 9.16 along with Lemmas 9.13, 9.20. Therefore \( \hat{m} \) is a bijection. \( \square \)

For the rest of this section, we describe how \( \hat{q} \) is related to the \( q \)-Onsager algebra \( \mathcal{O} \).
Proposition 9.22. For the algebra \( \hat{\Box}_q \), define

\[ A = x_0 + x_1, \quad B = x_2 + x_3. \]

Then

\[
A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = (q^2 - q^{-2})^2c_0(BA - AB),
\]

\[
B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = (q^2 - q^{-2})^2c_2(AB - BA).
\]

**Proof.** Apply the canonical homomorphism \( \tilde{\Box}_q \to \hat{\Box}_q \) to each side of (38), (39). 

The following more general version of Proposition 9.22 is obtained by applying Lemma 9.9.

Corollary 9.23. Let \( \{\alpha_i\}_{i \in \mathbb{Z}_4} \) denote invertible elements in \( \hat{\mathcal{C}} \). For the algebra \( \hat{\Box}_q \), define

\[ A = \alpha_0x_0 + \alpha_1x_1, \quad B = \alpha_2x_2 + \alpha_3x_3. \]  

Then

\[
A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = (q^2 - q^{-2})^2\alpha_0\alpha_1c_0(BA - AB),
\]

(50)

\[
B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = (q^2 - q^{-2})^2\alpha_2\alpha_3c_2(AB - BA).
\]

(51)

**Proof.** Apply the homomorphism \( \hat{g}(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \) from Definition 9.10 to everything in Proposition 9.22.

Proposition 9.24. Let \( \{\alpha_i\}_{i \in \mathbb{Z}_4} \) denote elements in \( \hat{\mathcal{C}} \) such that \( \alpha_0\alpha_1c_0 = 1 \) and \( \alpha_2\alpha_3c_2 = 1 \). Then there exists an \( \mathbb{F} \)-algebra homomorphism \( \sharp : \mathcal{O} \to \hat{\Box}_q \) that sends

\[ A \mapsto \alpha_0x_0 + \alpha_1x_1, \quad B \mapsto \alpha_2x_2 + \alpha_3x_3. \]  

(52)

**Proof.** By Definition 4.1 and Corollary 9.23.

In Theorem 10.34 we will show that the map \( \sharp \) from Proposition 9.24 is injective.

Let the elements \( \{\alpha_i\}_{i \in \mathbb{Z}_4} \) and the map \( \sharp \) be as in Proposition 9.24. By construction there exist nonzero \( a, b \in \mathbb{F} \) such that the canonical homomorphism \( \hat{\Box}_q \to \Box_q \) sends

\[ \alpha_0 \mapsto a, \quad \alpha_1 \mapsto a^{-1}, \quad \alpha_2 \mapsto b, \quad \alpha_3 \mapsto b^{-1}. \]  

(53)

Using \( a, b \) we obtain the \( \mathbb{F} \)-algebra homomorphism \( \sharp : \mathcal{O} \to \Box_q \) as in Proposition 5.6.

Lemma 9.25. With the above notation, the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{k} & \hat{\mathcal{C}}_q \\
\downarrow & & \downarrow \text{can} \\
\mathcal{O} & \xrightarrow{\sharp} & \Box_q
\end{array}
\]

**Proof.** Each map in the diagram is an \( \mathbb{F} \)-algebra homomorphism. The \( \mathbb{F} \)-algebra \( \mathcal{O} \) is generated by \( A, B \). To verify that the diagram commutes, chase \( A, B \) around the diagram using (17) and (52), (53).
10 The algebra $\square_q$, revisited

In Section 5 we defined the algebra $\square_q$, and in Sections 6–9 we investigated its homomorphic preimages $\hat{\square}_q$, $\hat{\square}_q$. In this section we return our attention to $\square_q$. We will first prove Proposition 5.5. We will then prove Theorem 4.4, and establish the injectivity of the maps $\sharp$ and $\natural$ from Propositions 5.6 and 9.24, respectively.

Recall the $\mathbb{F}$-algebra $L$ from (23). Let $L_0$ denote the ideal of $L$ generated by $\{\lambda_i - 1\}_{i \in \mathbb{Z}_4}$. Thus

$$L_0 = \sum_{i \in \mathbb{Z}_4} L(\lambda_i - 1).$$

(54)

The ideal $L_0$ is the kernel of the $\mathbb{F}$-algebra homomorphism $L \to \mathbb{F}$ that sends $\lambda_i \mapsto 1$ for $i \in \mathbb{Z}_4$. The sum $L = L_0 + \mathbb{F}$ is direct.

**Definition 10.1.** Let $K$ denote the 2-sided ideal of $\hat{\square}_q$ generated by $\{c_i - 1\}_{i \in \mathbb{Z}_4}$. Since the $\{c_i\}_{i \in \mathbb{Z}_4}$ are central,

$$K = \sum_{i \in \mathbb{Z}_4} \hat{\square}_q(c_i - 1).$$

**Lemma 10.2.** The canonical homomorphism $\hat{\square}_q \to \square_q$ has kernel $K$.

**Proof.** Compare Definition 5.1 and Definition 9.1.

Recall from Definition 9.8 the subalgebras $\hat{\square}_q^{\text{even}}$, $\hat{\square}_q^{\text{odd}}$, $\hat{C}$ of $\hat{\square}_q$.

**Definition 10.3.** Let $K_0$ denote the ideal of $\hat{C}$ generated by $\{c_i - 1\}_{i \in \mathbb{Z}_4}$. Thus

$$K_0 = \sum_{i \in \mathbb{Z}_4} \hat{C}(c_i - 1).$$

(55)

**Lemma 10.4.** The ideal $K_0$ is the image of $L_0$ under the isomorphism $L \to \hat{C}$ from Proposition 9.21(iii).

**Proof.** Compare (54), (55).

**Lemma 10.5.** Referring to the vector space isomorphism from Proposition 9.21(iv), the preimage of $K$ is

$$\hat{\square}_q^{\text{even}} \otimes \hat{\square}_q^{\text{odd}} \otimes K_0.$$  

(56)

**Proof.** By Definitions 10.1, 10.3

**Lemma 10.6.** In the algebra $\hat{\square}_q$,

(i) $K \cap \hat{\square}_q^{\text{even}} = 0$;

(ii) $K \cap \hat{\square}_q^{\text{odd}} = 0$;
(iii) \( K \cap \hat{C} = K_0 \).

**Proof.** Use Lemma 10.5. \( \square \)

Recall from Definition 5.4 the subalgebras \( \Box_q^{\text{even}}, \Box_q^{\text{odd}} \) of \( \Box_q \).

**Lemma 10.7.** For the canonical homomorphism \( \hat{\Box}_q \to \Box_q \), the images of \( \hat{\Box}_q^{\text{even}}, \hat{\Box}_q^{\text{odd}}, \hat{C} \) are \( \Box_q^{\text{even}}, \Box_q^{\text{odd}}, \mathbb{F} \), respectively.

**Proof.** By Definitions 5.4, 9.8 and Lemma 9.4. \( \square \)

**Definition 10.8.** For the canonical homomorphism \( \hat{\Box}_q \to \Box_q \), the restrictions to \( \hat{\Box}_q^{\text{even}}, \hat{\Box}_q^{\text{odd}}, \hat{C} \) induce surjective \( \mathbb{F} \)-algebra homomorphisms

\[
\hat{\Box}_q^{\text{even}} \to \Box_q^{\text{even}}, \quad \hat{\Box}_q^{\text{odd}} \to \Box_q^{\text{odd}}, \quad \hat{C} \to \mathbb{F}.
\]

Each of the homomorphisms (57) will be called restricted canonical.

**Lemma 10.9.** The following (i)–(iii) hold:

(i) the restricted canonical homomorphism \( \hat{\Box}_q^{\text{even}} \to \Box_q^{\text{even}} \) is an isomorphism;

(ii) the restricted canonical homomorphism \( \hat{\Box}_q^{\text{odd}} \to \Box_q^{\text{odd}} \) is an isomorphism;

(iii) the restricted canonical homomorphism \( \hat{C} \to \mathbb{F} \) has kernel \( K_0 \).

**Proof.** By Lemmas 10.2, 10.6. \( \square \)

**Proof of Proposition 5.5** (i) The desired isomorphism is the composition of the isomorphism \( U^+ \to \hat{\Box}_q^{\text{even}} \) from Proposition 9.21(i) and the isomorphism \( \hat{\Box}_q^{\text{even}} \to \Box_q^{\text{even}} \) from Lemma 10.9 (i).

(ii) The desired isomorphism is the composition of the isomorphism \( U^+ \to \hat{\Box}_q^{\text{odd}} \) from Proposition 9.21(ii) and the isomorphism \( \hat{\Box}_q^{\text{odd}} \to \Box_q^{\text{odd}} \) from Lemma 10.9 (ii).

(iii) By Lemma 10.5 and since \( 1 \notin K_0 \) we see that, for the vector space isomorphism in Proposition 9.21(iv), the preimage of \( K \) has zero intersection with \( \hat{\Box}_q^{\text{even}} \otimes \hat{\Box}_q^{\text{odd}} \). The result follows. \( \square \)

We now turn our attention to Theorem 4.4 and the maps \( \sharp, \natural \). We comment on the notation. In earlier sections we discussed the algebras \( \Box_q, \Box_q^{\text{even}}, \Box_q^{\text{odd}} \). In our discussion going forward, in order to simplify the notation we will drop the reference to \( q \), and write \( \Box = \Box_q, \Box^{\text{even}} = \Box_q^{\text{even}}, \Box^{\text{odd}} = \Box_q^{\text{odd}} \).

Recall the \( \mathbb{F} \)-algebra \( U^+ = U_q^+ \) and its \( \mathbb{N} \)-grading \( \{ U_n^+ \}_{n \in \mathbb{N}} \).

**Definition 10.10.** Recall from Proposition 5.5(i) the \( \mathbb{F} \)-algebra isomorphism \( U^+ \to \Box_q^{\text{even}} \) that sends \( X \mapsto x_0 \) and \( Y \mapsto x_2 \). Under this isomorphism, for \( n \in \mathbb{N} \) the image of \( U_n^+ \) will be denoted by \( \Box_n^{\text{even}} \). Recall from Proposition 5.5(ii) the \( \mathbb{F} \)-algebra isomorphism \( U^+ \to \Box_q^{\text{odd}} \) that sends \( X \mapsto x_1 \) and \( Y \mapsto x_3 \). Under this isomorphism, for \( n \in \mathbb{N} \) the image of \( U_n^+ \) will be denoted by \( \Box_n^{\text{odd}} \).
Lemma 10.11. The sequence \( \{\square_n^{\text{even}}\}_{n \in \mathbb{N}} \) is an \( \mathbb{N} \)-grading of \( \square^{\text{even}} \). The sequence \( \{\square_n^{\text{odd}}\}_{n \in \mathbb{N}} \) is an \( \mathbb{N} \)-grading of \( \square^{\text{odd}} \).

Proof. By Definition 10.10 and since \( \{U_n^{+}\}_{n \in \mathbb{N}} \) is an \( \mathbb{N} \)-grading of \( U^{+} \).

Definition 10.12. Recall from Proposition 5.5(iii) the isomorphism of \( \mathbb{F} \)-vector spaces \( \square^{\text{even}} \otimes \square^{\text{odd}} \to \square \). Under this isomorphism, for \( r, s \in \mathbb{N} \) the image of \( \square_r^{\text{even}} \otimes \square_s^{\text{odd}} \) will be denoted by \( \square_{r,s} \).

Lemma 10.13. The following (i), (ii) hold for \( r, s, t \in \mathbb{N} \):

(i) \( \square_t^{\text{even}} \square_{r,s} \subseteq \square_{r+t,s} \);

(ii) \( \square_{r,s} \square_t^{\text{odd}} \subseteq \square_{r,s+t} \).

Proof. (i) By Definition 10.12 and since \( \square_t^{\text{even}} \square_t^{\text{even}} \subseteq \square_{r+t}^{\text{even}} \).

(ii) By Definition 10.12 and since \( \square_t^{\text{odd}} \square_t^{\text{odd}} \subseteq \square_{s+t}^{\text{odd}} \).

Definition 10.14. For \( n \in \mathbb{Z} \) define

\[
\square_n = \sum_{r,s \in \mathbb{N}} \square_{r,s}.
\]

Referring to Definition 10.14 our next goal is to show that the sequence \( \{\square_n\}_{n \in \mathbb{Z}} \) is a \( \mathbb{Z} \)-grading of \( \square \).

Lemma 10.15. The sum \( \square = \sum_{n \in \mathbb{Z}} \square_n \) is direct. Moreover \( 1 \in \square_0 \) and \( x_0, x_2 \in \square_1 \) and \( x_1, x_3 \in \square_{-1} \).

Proof. By Definition 10.14 and the comments above Lemma 10.13.

Lemma 10.16. The following (i), (ii) hold for \( n \in \mathbb{Z} \) and \( t \in \mathbb{N} \):

(i) \( \square_t^{\text{even}} \square_n \subseteq \square_{n+t} \);

(ii) \( \square_n \square_t^{\text{odd}} \subseteq \square_{n-t} \).

Proof. Use Lemma 10.13 and Definition 10.14.

Lemma 10.17. For \( r, s \in \mathbb{N} \),

\[
\square_s^{\text{odd}} \square_r^{\text{even}} \subseteq \sum_{\ell=0}^{\min(r,s)} \square_{r-\ell,s-\ell}.
\]

Moreover \( \square_s^{\text{odd}} \square_r^{\text{even}} \subseteq \square_{r-s} \).

Proof. To obtain (58) use Lemma 6.12 and induction on \( s \). The last assertion follows from (58) and Definition 10.14.
Lemma 10.18. We have $\square_r \square_s \subseteq \square_{r+s}$ for $r, s \in \mathbb{Z}$.

Proof. By Definition [10.14], Lemma [10.16] and the last assertion of Lemma [10.17].

Proposition 10.19. The sequence $\{\square_n\}_{n \in \mathbb{Z}}$ is a $\mathbb{Z}$-grading of $\square$.

Proof. By Lemmas [10.15], [10.18].

For the rest of this section, fix nonzero $a, b \in \mathbb{F}$. Using $a, b$ we obtain the $\mathbb{F}$-algebra homomorphism $\sharp : \mathcal{O} \to \square$ as in Proposition 5.6. Define

$$A^+ = ax_0, \quad A^- = a^{-1}x_1, \quad B^+ = bx_2, \quad B^- = b^{-1}x_3$$ (59)

so that

$$\sharp(A) = A^+ + A^-, \quad \sharp(B) = B^+ + B^-.$$ (60)

By Lemma [10.15] and (59),

$$A^+ \in \square_1, \quad A^- \in \square_{-1}, \quad B^+ \in \square_1, \quad B^- \in \square_{-1}.$$ (61)

By (60), (61),

$$\sharp(A) \in \square_1 + \square_{-1}, \quad \sharp(B) \in \square_1 + \square_{-1}.$$ (62)

Lemma 10.20. Let $n \in \mathbb{N}$. For $1 \leq i \leq n$ pick $g_i \in \{A, B\}$. Then

$$\sharp(g_1 g_2 \cdots g_n) = \sum g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_n^{\epsilon_n},$$ (63)

where the sum is over all sequences $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that $\epsilon_i \in \{+, -\}$ for $1 \leq i \leq n$.

Proof. To verify (63), expand the left-hand side using (60) and the fact that $\sharp$ is an algebra homomorphism.

Lemma 10.21. Refer to Lemma [10.20]. In the sum on the right in (63), consider any summand $g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_n^{\epsilon_n}$. This summand is contained in $\square_\ell$, where

$$\ell = \left| \{i | 1 \leq i \leq n, \, \epsilon_i = +\} \right| - \left| \{i | 1 \leq i \leq n, \, \epsilon_i = -\} \right|.$$ (64)

Proof. By Lemma [10.18] and (61).

Lemma 10.22. For $n \in \mathbb{N}$,

$$\sharp(\mathcal{O}_n) \subseteq \sum_{\ell = -n}^n \square_\ell.$$ (64)

Proof. We mentioned below Definition [1.1] that $\mathcal{O}_n$ is spanned by the products $g_1 g_2 \cdots g_r$ such that $0 \leq r \leq n$ and $g_i$ is among $A, B$ for $1 \leq i \leq r$. The result follows from this and Lemmas [10.20], [10.21].
Definition 10.23. For \( r \in \mathbb{Z} \) define an \( \mathbb{F} \)-linear map \( \pi_r : \square \to \square \) such that \((\pi_r - I)\square_r = 0\) and \(\pi_r\square_s = 0\) for \( s \in \mathbb{Z}, s \neq r \). Thus \(\pi_r\) is the projection from \(\square\) onto \(\square_r\).

Lemma 10.24. For \( n \in \mathbb{N} \) and \( r \in \mathbb{Z} \), the composition

\[
\mathcal{O}_n \xrightarrow{\pi_n} \square \xrightarrow{\pi_r} \square
\]

is zero unless \(-n \leq r \leq n\).

Proof. By Lemma 10.22 and Definition 10.23.

Definition 10.25. Pick \( n \in \mathbb{N} \) and recall the \( \mathbb{F} \)-vector space \( \overline{\mathcal{O}}_n = \mathcal{O}_n/\mathcal{O}_{n-1} \) from Section 4. Define the \( \mathbb{F} \)-linear map \( \varphi_n : \mathcal{O}_n \to \square \) to be the composition

\[
\varphi_n : \mathcal{O}_n \xrightarrow{\pi_n} \square \xrightarrow{\pi_r} \square.
\] (65)

By Lemma 10.24 we have \( \varphi_n(\mathcal{O}_{n-1}) = 0 \). Therefore \( \varphi_n \) induces an \( \mathbb{F} \)-linear map \( \overline{\varphi}_n : \overline{\mathcal{O}}_n \to \square \) that sends \( u + \mathcal{O}_{n-1} \mapsto \varphi_n(u) \) for all \( u \in \mathcal{O}_n \).

Lemma 10.26. For \( n \in \mathbb{N} \) the map \( \overline{\varphi}_n : \overline{\mathcal{O}}_n \to \square \) is described as follows. For \( 1 \leq i \leq n \) pick \( g_i \in \{ A, B \} \). Then \( \overline{\varphi}_n \) sends \( g_1 g_2 \cdots g_n \mapsto g_1^n g_2^+ \cdots g_n^+ \), where we recall \( A^+ = ax_0 \) and \( B^+ = bx_2 \).

Proof. By construction \( g_1 g_2 \cdots g_n = g_1 g_2 \cdots g_n + \mathcal{O}_{n-1} \in \overline{\mathcal{O}}_n \). By this and Definition 10.25 the map \( \overline{\varphi}_n \) sends \( g_1 g_2 \cdots g_n \) to \( \varphi_n(g_1 g_2 \cdots g_n) \), which is equal to \( \pi_n(\varphi(g_1 g_2 \cdots g_n)) \). To compute this last quantity, apply \( \pi_n \) to each side of \( (63) \), and evaluate the result using Lemma 10.21 and Definition 10.23. For the sum on the right in \( (63) \), \( \pi_n \) fixes the summand \( g_1^n g_2^+ \cdots g_n^+ \) and sends the remaining summands to zero. The result follows.

Lemma 10.27. Let \( r, s \in \mathbb{N} \). Then \( \overline{\varphi}_r(u)\overline{\varphi}_s(v) = \overline{\varphi}_{r+s}(uv) \) for all \( u \in \overline{\mathcal{O}}_r \) and \( v \in \overline{\mathcal{O}}_s \).

Proof. Use Lemma 10.26.

Definition 10.28. Define an \( \mathbb{F} \)-linear map \( \overline{\varphi} : \overline{\mathcal{O}} \to \square \) that acts on \( \overline{\mathcal{O}} \) as \( \overline{\varphi}_n \) for \( n \in \mathbb{N} \).

Lemma 10.29. The map \( \overline{\varphi} : \overline{\mathcal{O}} \to \square \) from Definition 10.28 is an \( \mathbb{F} \)-algebra homomorphism.

Proof. It suffices to check that \( \overline{\varphi}(u)\overline{\varphi}(v) = \overline{\varphi}(uv) \) for all \( u, v \in \overline{\mathcal{O}} \). This checking is routine using Lemma 10.27, Definition 10.28, and the definition of the \( \mathbb{F} \)-algebra \( \overline{\mathcal{O}} \) in Section 4.

Lemma 10.30. The map \( \overline{\varphi} : \overline{\mathcal{O}} \to \square \) from Definition 10.28 sends \( \overline{A} \mapsto ax_0 \) and \( \overline{B} \mapsto bx_2 \).

Proof. We have \( A, B \in \overline{\mathcal{O}}_1 \). By Lemma 10.26 and Definition 10.28

\[
\overline{\varphi}(A) = \overline{\varphi}_1(A) = A^+ = ax_0, \quad \overline{\varphi}(B) = \overline{\varphi}_1(B) = B^+ = bx_2.
\]

By Lemma 10.23 there exists an automorphism of \( U^+ \) that sends \( X \mapsto aX \) and \( Y \mapsto bY \). Denote this automorphism by \( \phi \). Recall the surjective \( \mathbb{F} \)-algebra homomorphism \( \psi : U^+ \to \overline{\mathcal{O}} \) from above Lemma 10.3. By Proposition 5.5(i) there exists an injective \( \mathbb{F} \)-algebra homomorphism \( U^+ \to \square \) that sends \( X \mapsto x_0 \) and \( Y \mapsto x_2 \). Call this homomorphism \( \overline{b} \).
Lemma 10.31. With the above notation, the following diagram commutes:

\[
\begin{array}{c}
U^+ \xrightarrow{\psi} O \\
\downarrow \phi \quad \downarrow \varphi \\
U^+ \xrightarrow{b} \square
\end{array}
\]

Proof. Each map in the diagram is an $\mathbb{F}$-algebra homomorphism. The $\mathbb{F}$-algebra $U^+$ is generated by $X,Y$. To verify that the diagram commutes, chase $X,Y$ around the diagram using Lemma 10.30. \hfill \square

Proposition 10.32. Theorem 4.4 holds. Moreover the map $\varphi$ is injective.

Proof. Consider the commuting diagram in Lemma 10.31. By construction $\phi$ is an isomorphism, $b$ is injective, and $\psi$ is surjective. Now since the diagram commutes, $\psi$ and $\varphi$ are injective. The map $\psi$ is an isomorphism since it is both injective and surjective. Therefore Theorem 4.4 holds. \hfill \square

Theorem 10.33. The map $\sharp$ from Proposition 5.6 is injective.

Proof. Pick a nonzero $u \in O$. We show that $\sharp(u) \neq 0$. There exists a unique $n \in \mathbb{N}$ such that $u \in O_n$ and $u \notin O_{n-1}$. The map $\varphi_n$ is injective, by Definition 10.28 and since $\varphi$ is injective by Proposition 10.32. By this and Definition 10.25, the kernel of $\varphi_n$ is equal to $O_{n-1}$. This kernel does not contain $u$, so $\varphi_n(u) \neq 0$. Now $\sharp(u) \neq 0$ in view of (65). The result follows. \hfill \square

Theorem 10.34. The map $\natural$ from Proposition 9.24 is injective.

Proof. By Lemma 9.25 and Theorem 10.33. \hfill \square

11 The $q$-tetrahedron algebra $\boxtimes_q$

The $q$-tetrahedron algebra $\boxtimes_q$ was introduced in [25] and investigated further in [20], [30]. In this section, we consider how $O$ and $\square_q$ are related to $\boxtimes_q$. We first display an injective $\mathbb{F}$-algebra homomorphism $\square_q \to \boxtimes_q$. Next, we compose this map with the map $\natural : O \to \square_q$ from Proposition 5.6 to get an injective $\mathbb{F}$-algebra homomorphism $O \to \boxtimes_q$.

Definition 11.1. (See [25, Definition 6.1].) Let $\boxtimes_q$ denote the $\mathbb{F}$-algebra defined by generators

\[
\{x_{ij} \mid i,j \in \mathbb{Z}_4, \ j - i = 1 \text{ or } j - i = 2\}
\]

and the following relations:

(i) For $i,j \in \mathbb{Z}_4$ such that $j - i = 2$,

\[
x_{ij}x_{ji} = 1.
\]
(ii) For \( i, j, k \in \mathbb{Z}_4 \) such that \((j - i, k - j)\) is one of \((1, 1), (1, 2), (2, 1),\)
\[
\frac{qx_{ij}x_{jk} - q^{-1}x_{jk}x_{ij}}{q - q^{-1}} = 1.
\] (68)

(iii) For \( i, j, k, \ell \in \mathbb{Z}_4 \) such that \( j - i = k - j = \ell - k = 1,\)
\[
x_{ij}^3x_{kl} - [3]_q x_{ij}^2x_{kl}x_{ij} + [3]_q x_{ij}x_{kl}x_{ij}^2 - x_{kl}x_{ij}^3 = 0.
\] (69)

We call \( \boxtimes_q \) the \( q \)-tetrahedron algebra.

Lemma 11.2. There exists an automorphism \( \varrho \) of \( \boxtimes_q \) that sends each generator \( x_{ij} \mapsto x_{i+1,j+1} \). Moreover \( \varrho^4 = 1. \)

Lemma 11.3. There exists an \( \mathbb{F} \)-algebra homomorphism \( \square_q \to \boxtimes_q \) that sends \( x_i \mapsto x_{i-1,i} \) for \( i \in \mathbb{Z}_4 \).

Proof. Compare Definitions [5.1][11.1].

Definition 11.4. The homomorphism \( \square_q \to \boxtimes_q \) from Lemma [11.3] will be called canonical.

Our next goal is to show that the canonical homomorphism \( \square_q \to \boxtimes_q \) is injective.

Definition 11.5. (See [30] Section 4.1.) Define the subalgebras \( \boxtimes_q^{\text{even}}, \boxtimes_q^{\text{odd}}, \boxtimes_q^{\times} \) of \( \boxtimes_q \) such that

(i) \( \boxtimes_q^{\text{even}} \) is generated by \( x_{30}, x_{12} \);

(ii) \( \boxtimes_q^{\text{odd}} \) is generated by \( x_{01}, x_{23} \);

(iii) \( \boxtimes_q^{\times} \) is generated by \( x_{02}, x_{20}, x_{13}, x_{31} \).

Proposition 11.6. (See Miki [30] Prop. 4.1.) The following (i)–(iv) hold:

(i) there exists an \( \mathbb{F} \)-algebra isomorphism \( U^+ \to \boxtimes_q^{\text{even}} \) that sends \( X \mapsto x_{30} \) and \( Y \mapsto x_{12} \);

(ii) there exists an \( \mathbb{F} \)-algebra isomorphism \( U^+ \to \boxtimes_q^{\text{odd}} \) that sends \( X \mapsto x_{01} \) and \( Y \mapsto x_{23} \);

(iii) the \( \mathbb{F} \)-algebra \( \boxtimes_q^{\times} \) has a presentation by generators \( x_{02}, x_{20}, x_{13}, x_{31} \) and relations
\[
x_{02}x_{20} = x_{20}x_{02} = 1, \quad x_{13}x_{31} = x_{31}x_{13} = 1;
\]

(iv) the following is an isomorphism of \( \mathbb{F} \)-vector spaces:
\[
\boxtimes_q^{\text{even}} \otimes \boxtimes_q^{\times} \otimes \boxtimes_q^{\text{odd}} \to \boxtimes_q
\]
\[
u \otimes v \otimes w \mapsto uvw
\]

Note 11.7. In [30] Prop. 4.1] Miki assumes that \( \mathbb{F} = \mathbb{C} \) and \( q \) is not a root of unity. We emphasize that Proposition [11.6] holds without this assumption. There is a proof of Proposition [11.6] that is analogous to our proof of Proposition 5.5.
Proposition 11.8. The canonical homomorphism $\square_q \to \square_q$ is injective.

Proof. Compare Proposition 5.5 and Proposition 11.6. \hfill \square

Proposition 11.9. Pick nonzero $a, b \in \mathbb{F}$. Then there exists an $\mathbb{F}$-algebra homomorphism $\mathcal{O} \to \square_q$ that sends

$$A \mapsto ax_{30} + a^{-1}x_{01}, \quad B \mapsto bx_{12} + b^{-1}x_{23}.$$ 

This homomorphism is injective.

Proof. The desired homomorphism is the composition of the homomorphism $\# : \mathcal{O} \to \square_q$ from Proposition 5.6 and the canonical homomorphism $\square_q \to \square_q$. The last assertion follows from Theorem 10.33 and Proposition 11.8. \hfill \square

12 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$

In the literature on the $q$-Onsager algebra $\mathcal{O}$, there are algebra homomorphisms from $\mathcal{O}$ into the quantum algebra $U_q(\widehat{\mathfrak{sl}_2})$ due to P. Baseilhac and S. Belliard [10, line (3.15)], [10, line (3.18)] and S. Kolb [28, Example 7.6]. Also, there are algebra homomorphisms from $\mathcal{O}$ into the $q$-deformed loop algebra $U_q(L(\mathfrak{sl}_2))$ due to P. Baseilhac [6, Prop. 2.2], and also T. Ito and the present author [26, Prop. 8.5], [27, Props. 1.1, 1.13]. In the case of [28, Example 7.6] and [27, Props. 1.1, 1.13] the homomorphism was shown to be injective. In fact, all of the above homomorphisms are injective, and this can be established using the results from the previous sections of the present paper. In each case the proof is similar. In this section we illustrate what is going on with a single example [28, Example 7.6]. This example involves $U_q(\widehat{\mathfrak{sl}_2})$, which we now define.

Definition 12.1. (See [18, p. 262].) Let $U_q(\widehat{\mathfrak{sl}_2})$ denote the $\mathbb{F}$-algebra with generators $e^\pm_i, k^\pm_i, i \in \{0, 1\}$ and the following relations:

$$k_i k^{-1}_i = k^{-1}_i k_i = 1, \\
k_0 k_1 = k_1 k_0, \\
k_i e^\pm_i k^{-1}_i = q^\pm 2 e^\pm_i, \\
k_i e^\pm_i k^{-1}_i = q^{\mp 2} e^\mp_j, \quad i \neq j, \\
[e^+_i, e^-_j] = \frac{k_i - k^{-1}_i}{q - q^{-1}}, \\
[e^\pm_0, e^\pm_1] = 0, \\
(e^\pm_i)^2 e^\pm_j - [3]_q (e^\pm_i)^2 e^\pm_j + [3]_q e^\pm_i e^\pm_j (e^\pm_i)^2 - e^\pm_j (e^\pm_i)^3 = 0, \quad i \neq j.$$

We call $e^\pm_i, k^\pm_i, i \in \{0, 1\}$ the Chevalley generators for $U_q(\widehat{\mathfrak{sl}_2})$.

In the following three lemmas we describe some automorphisms of $U_q(\widehat{\mathfrak{sl}_2})$; the proofs are routine and omitted.
Lemma 12.2. There exists an automorphism of $U_q(\hat{\mathfrak{sl}}_2)$ that sends
\[ e_i^+ \mapsto e_i^-, \quad e_i^- \mapsto e_i^+, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \quad i \in \{0, 1\}. \]

Lemma 12.3. There exists an automorphism of $U_q(\hat{\mathfrak{sl}}_2)$ that sends
\[ e_i^+ \mapsto e_i^+ k_i, \quad e_i^- \mapsto k_i^{-1} e_i^-, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \quad i \in \{0, 1\}. \]

Lemma 12.4. Let $\varepsilon_0, \varepsilon_1$ denote nonzero scalars in $\mathbb{F}$. Then there exists an automorphism of $U_q(\hat{\mathfrak{sl}}_2)$ that sends
\[ e_i^+ \mapsto \varepsilon_i e_i^+, \quad e_i^- \mapsto \varepsilon_i^{-1} e_i^-, \quad k_i^{\pm 1} \mapsto k_i^{\mp 1}, \quad i \in \{0, 1\}. \]

Definition 12.5. The automorphism of $U_q(\hat{\mathfrak{sl}}_2)$ from Lemma 12.4 will be denoted by $\xi(\varepsilon_0, \varepsilon_1)$.

We now recall the equitable presentation of $U_q(\hat{\mathfrak{sl}}_2)$.

Lemma 12.6. (See [23, Theorem 2.1].) The $\mathbb{F}$-algebra $U_q(\hat{\mathfrak{sl}}_2)$ has a presentation with generators $y_i^+, k_i^{\pm 1}, i \in \{0, 1\}$ and relations
\[
\begin{align*}
    k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\
    k_0 k_1 &\text{ is central,} \\
    q y_i^+ k_i - q^{-1} k_i y_i^+ &= 1, \\
    q k_i y_i^- - q^{-1} y_i^- k_i &= 1, \\
    q y_i^- y_i^+ - q^{-1} y_i^+ y_i^- &= 1, \\
    q y_i^+ y_j^- - q^{-1} y_j^- y_i^+ &= k_0^{-1} k_1^{-1}, \quad i \neq j, \\
    (y_i^\pm)^3 y_j^\mp - [3]_q (y_i^\pm)^2 y_j^\mp y_i^\pm + [3]_q y_i^\pm y_j^\mp (y_i^\pm)^2 - y_j^\pm (y_i^\pm)^3 &= 0, \quad i \neq j.
\end{align*}
\]

An isomorphism with the presentation in Definition 12.1 is given by:
\[
\begin{align*}
    k_i^{\pm} &\mapsto k_i^{\pm}, \quad (70) \\
    y_i^- &\mapsto k_i^{-1} + e_i^- (q - q^{-1}), \quad (71) \\
    y_i^+ &\mapsto k_i^{-1} - k_i^{-1} e_i^+ q (q - q^{-1}). \quad (72)
\end{align*}
\]

The inverse of this isomorphism is given by:
\[
\begin{align*}
    k_i^{\pm} &\mapsto k_i^{\pm}, \\
    e_i^- &\mapsto (y_i^- - k_i^{-1}) (q - q^{-1})^{-1}, \\
    e_i^+ &\mapsto (1 - k_i y_i^+) q^{-1} (q - q^{-1})^{-1}.
\end{align*}
\]

Definition 12.7. Referring to Lemma 12.6, we call $y_i^\pm, k_i^{\pm 1}, i \in \{0, 1\}$ the equitable generators for $U_q(\hat{\mathfrak{sl}}_2)$. The isomorphism described in (70)–(72) will be denoted by $\tau$. 

32
We now describe an $\mathbb{F}$-algebra homomorphism $\hat{\square}_q \to U_q(\hat{\mathfrak{sl}}_2)$ and an $\mathbb{F}$-algebra homomorphism $U_q(\hat{\mathfrak{sl}}_2) \to \mathbb{B}_q$. In this description we use the equitable presentation of $U_q(\hat{\mathfrak{sl}}_2)$.

**Lemma 12.8.** There exists an $\mathbb{F}$-algebra homomorphism $\hat{\sigma} : \hat{\square}_q \to U_q(\hat{\mathfrak{sl}}_2)$ such that

\[
\begin{array}{c|cccc|ccc}
\sigma(u) & x_0 & x_1 & x_2 & x_3 & c_0 & c_1 & c_2 & c_3 \\
\hline
u & y_0 & y_1 & y_2 & y_3 & 1 & k_0^{-1}k_1^{-1} & 1 & k_0^{-1}k_1^{-1}
\end{array}
\]

**Proof.** Compare the defining relations (40)–(43) for $\hat{\square}_q$, with the defining relations for $U_q(\hat{\mathfrak{sl}}_2)$ given in Lemma 12.6

**Lemma 12.9.** There exists an $\mathbb{F}$-algebra homomorphism $\sigma : U_q(\hat{\mathfrak{sl}}_2) \to \mathbb{B}_q$ such that

\[
\begin{array}{c|cccc|ccc}
\sigma(u) & x_0 & x_1 & x_2 & x_3 & k_0 & k_0^{-1} & k_1 & k_1^{-1} \\
\hline
u & y_0 & y_1 & y_2 & y_3 & x_{30} & x_{10} & x_{23} & x_{31} & x_{31} & x_{13}
\end{array}
\]

**Proof.** Compare the defining relations for $U_q(\hat{\mathfrak{sl}}_2)$ given in Lemma 12.6 with the defining relations for $\mathbb{B}_q$ given in Definition 11.1

**Lemma 12.10.** The following diagram commutes:

\[
\begin{array}{ccc}
\hat{\square}_q & \xrightarrow{\hat{\sigma}} & U_q(\hat{\mathfrak{sl}}_2) \\
\downarrow & & \downarrow \\
\square_q & \xrightarrow{\text{can}} & \mathbb{B}_q
\end{array}
\]

**Proof.** Each map in the diagram is an $\mathbb{F}$-algebra homomorphism. To verify that the diagram commutes, chase the $\hat{\square}_q$ generators $x_i, c_i^{\pm 1}$ $(i \in \mathbb{Z}_4)$ around the diagram using Definitions 9.5, 11.4 and Lemmas 12.8, 12.9

We now obtain the homomorphism $\mathcal{O} \to U_q(\hat{\mathfrak{sl}}_2)$ due to Kolb [28, Example 7.6].

**Proposition 12.11.** (See [28, Example 7.6].) For $i \in \{0, 1\}$ pick $s_i \in \mathbb{F}$ and $0 \neq \gamma_i \in \mathbb{F}$. Then for $U_q(\hat{\mathfrak{sl}}_2)$ the elements

\[
B_i = e_i - \gamma_i e_i^+ k_i^{-1} + s_i k_i^{-1}, \quad i \in \{0, 1\}
\]

satisfy

\[
\begin{align*}
B_0^3 B_1 - [3]_q B_0^2 B_1 B_0 + [3]_q B_0 B_1 B_0^2 - B_1 B_0^3 &= q(q + q^{-1})^2 \gamma_0 (B_1 B_0 - B_0 B_1), \\
B_1^3 B_0 - [3]_q B_1^2 B_0 B_1 + [3]_q B_1 B_0 B_1^2 - B_0 B_1^3 &= q(q + q^{-1})^2 \gamma_1 (B_0 B_1 - B_1 B_0).
\end{align*}
\]

**Proof.** Let $\lambda$ denote an indeterminate. Replacing $\mathbb{F}$ by its algebraic closure if necessary, we may assume without loss that $\mathbb{F}$ is algebraically closed. There exist scalars $\{\alpha_i\}_{i \in \mathbb{Z}_4}$ in $\mathbb{F}$ such that $\alpha_0, \alpha_1$ are the roots of the polynomial

\[
\lambda^2 - s_0 \lambda + \gamma_0 q(q - q^{-1})^{-2}
\]

33
and \( \alpha_2, \alpha_3 \) are the roots of the polynomial
\[
\lambda^2 - s_1 \lambda + \gamma_1 q(q - q^{-1})^{-2}.
\]

By construction
\[
\begin{align*}
\alpha_0 + \alpha_1 &= s_0, & \alpha_0 \alpha_1 &= \gamma_0 q(q - q^{-1})^{-2}, \\
\alpha_2 + \alpha_3 &= s_1, & \alpha_2 \alpha_3 &= \gamma_1 q(q - q^{-1})^{-2}.
\end{align*}
\]

Note that \( \alpha_i \neq 0 \) for \( i \in \mathbb{Z}_4 \). Define
\[
\varepsilon_0 = \alpha_0 (q - q^{-1}), \quad \varepsilon_1 = \alpha_2 (q - q^{-1}).
\]

Note that \( \varepsilon_0 \neq 0 \) and \( \varepsilon_1 \neq 0 \). In the algebra \( \mathfrak{I}_q \) define
\[
A = \alpha_0 x_0 + \alpha_1 x_1, \quad B = \alpha_2 x_2 + \alpha_3 x_3.
\]

Then \( A, B \) satisfy (50), (51) by Corollary 9.23. Consider the composition
\[
\zeta : \mathfrak{I}_q \xrightarrow{\hat{\vartheta}} U_q(s\mathfrak{l}_2) \xrightarrow{\tau} U_q(s\mathfrak{l}_2) \xrightarrow{\xi(\varepsilon_0, \varepsilon_1)} U_q(\hat{s}\mathfrak{l}_2),
\]
where \( \hat{\vartheta} \) is from Lemma 12.8, \( \tau \) is from Definition 12.7 and \( \xi(\varepsilon_0, \varepsilon_1) \) is from Definition 12.5. By construction \( \zeta : \mathfrak{I}_q \rightarrow U_q(\hat{s}\mathfrak{l}_2) \) is an \( \mathbb{F} \)-algebra homomorphism. Using (76) and \( k_0 \varepsilon_0^+ = q^2 \varepsilon_0^+ k_0 \) we find that \( \zeta \) sends \( A \mapsto B_0 \). Similarly \( \zeta \) sends \( B \mapsto B_1 \). By Lemma 12.8, \( \hat{\vartheta} \) sends \( \varepsilon_0 \mapsto 1 \) and \( \varepsilon_1 \mapsto 1 \). Therefore \( \zeta \) sends \( \varepsilon_0 \mapsto 1 \) and \( \varepsilon_1 \mapsto 1 \). Applying \( \zeta \) to each side of (50), (51) we obtain (74), (75).

**Proposition 12.12.** (See [28, Example 7.6].) Referring to Proposition 12.11, assume that
\[
\gamma_0 = q^{-1}(q - q^{-1})^2, \quad \gamma_1 = q^{-1}(q - q^{-1})^2.
\]

Then there exists an \( \mathbb{F} \)-algebra homomorphism \( \mathcal{O} \rightarrow U_q(\hat{s}\mathfrak{l}_2) \) that sends \( A \mapsto B_0 \) and \( B \mapsto B_1 \). This homomorphism is injective.

**Proof.** The desired \( \mathbb{F} \)-algebra homomorphism \( \mathcal{O} \rightarrow U_q(\hat{s}\mathfrak{l}_2) \) exists, since under the assumption (77) the relations (74), (75) become the \( \mathbb{F} \)-Dolan/Grady relations. Call the above homomorphism \( \partial \). We show that \( \partial \) is injective. Consider the composition
\[
\eta : \mathcal{O} \xrightarrow{\partial} U_q(\hat{s}\mathfrak{l}_2) \xrightarrow{\xi(\varepsilon_0^{-1}, \varepsilon_1^{-1})} U_q(\hat{s}\mathfrak{l}_2) \xrightarrow{\tau^{-1}} U_q(\hat{s}\mathfrak{l}_2) \xrightarrow{\sigma} \mathfrak{I}_q,
\]
where \( \xi(\varepsilon_0^{-1}, \varepsilon_1^{-1}) \) is from Definition 12.5. \( \tau \) is from Definition 12.7 and \( \sigma \) is from Lemma 12.9. By construction \( \eta : \mathcal{O} \rightarrow \mathfrak{I}_q \) is an \( \mathbb{F} \)-algebra homomorphism. One checks that \( \eta \) coincides with the \( \mathbb{F} \)-algebra homomorphism \( \mathcal{O} \rightarrow \mathfrak{I}_q \) from Proposition 11.9 where \( a = \alpha_0 \) and \( b = \alpha_2 \). The map from Proposition 11.9 is injective, so \( \partial \) is injective.

**Note 12.13.** In [28, Example 7.6] Kolb assumes that \( \mathbb{F} \) has characteristic zero and \( q \) is not a root of unity. We emphasize that Propositions 12.11 and 12.12 hold without this assumption.

34
13 Directions for future research

In this section we give some suggestions for future research.

Problem 13.1. For the $\mathbb{F}$-algebras $\hat{\square}_q$, $\hat{\hat{\square}}_q$, $\hat{\square}_q$ find their automorphism group.

Problem 13.2. The Lusztig automorphisms of $U_q(\hat{\mathfrak{sl}}_2)$ are described in [19, p. 294]. Find analogous automorphisms for $\hat{\square}_q$, $\hat{\hat{\square}}_q$, $\hat{\square}_q$.

Problem 13.3. Referring to the algebra $\square_q$, for $i \in \mathbb{Z}_4$ define

$$N_i = \frac{q(1 - x_i x_{i+1})}{q - q^{-1}} = \frac{q^{-1}(1 - x_{i+1} x_i)}{q - q^{-1}}.$$  

Note that $N_i x_i = q^2 x_i N_i$ and $N_i x_{i+1} = q^{-2} x_{i+1} N_i$. Determine how $N_i$ is related to $x_{i+2}$ and $x_{i+3}$.

Problem 13.4. Referring to Problem 13.3 determine how $N_i$, $N_j$ are related for $i, j \in \mathbb{Z}_4$.

Problem 13.5. Referring to Problem 13.3 consider the $q$-exponential $E_i = \exp_q(N_i)$. For $u \in \square_q$ compute $E_i u E_i^{-1}$ and determine if the result is contained in $\square_q$. If it always is, then conjugation by $E_i$ gives an automorphism of $\square_q$. In this case, describe the subgroup of $\text{Aut}(\square_q)$ generated by $\{E_i^{\pm 1}\}_{i \in \mathbb{Z}_4}$.

Conjecture 13.6. Let $V$ denote a finite-dimensional irreducible $\hat{\square}_q$-module. Then $V$ becomes a $\hat{\mathbb{F}}_q$-module such that for $i \in \mathbb{Z}_4$ the action of $x_i$ on $V$ is a scalar multiple of the action of $x_{i-1,i}$ on $V$. The $\hat{\mathbb{F}}_q$-module $V$ is irreducible.

Conjecture 13.7. Let $A, B$ denote a tridiagonal pair over $\mathbb{F}$ that has $q$-Racah type in the sense of [16, p. 259]. Then the underlying vector space $V$ becomes a $\square_q$-module on which $A$ (resp. $B$) is a linear combination of $x_0, x_1$ (resp. $x_2, x_3$). The $\square_q$-module $V$ is irreducible.

Problem 13.8. For the $\mathbb{F}$-algebras $\hat{\square}_q$, $\hat{\hat{\square}}_q$, $\hat{\square}_q$ find their center.

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