On \( k \)-Core Percolation in Four Dimensions

Giorgio Parisi\(^1\) and Tommaso Rizzo\(^2\)

\(^1\)Dipartimento di Fisica, Università di Roma “La Sapienza”, P.le Aldo Moro 2, 00185 Roma, Italy
\(^2\) “E. Fermi” Center, Via Panisperna 89 A, Complesso Viminale, 00184, Roma, Italy

The \( k \)-core percolation on the Bethe lattice has been proposed as a simple model of the jamming transition because of its hybrid first-order/second-order nature. We investigate numerically \( k \)-core percolation on the four-dimensional regular lattice. For \( k = 4 \) the presence of a discontinuous transition is clearly established but its nature is strictly first order. In particular, the \( k \)-core density displays no singular behavior before the jump and its correlation length remains finite. For \( k = 3 \) the transition is continuous.

After its introduction \([1]\), \( k \)-core percolation has been proposed to be relevant in a variety of different contexts, see \([2,3]\). The problem, also referred to as “Bootstrap percolation”, is defined as follows. The sites of a given lattice are populated with probability \( p \). Then each site with less than \( k \) neighbors is removed, the procedure being iterated until each site has at least \( k \) neighbors.

On regular lattices in dimension \( d \) the model exhibits two different behaviors depending on the value of \( k \). If \( k > d \) the cluster must be extended in order to survive the culling process but it is completely decimated for any \( p < 1 \) in the large size limit \([4]\). The behavior for large but finite system size has also been investigated \([2,3]\) but strong disagreement between the theoretical predictions and numerical simulations has been found. The highly non trivial origin of this discrepancy has been clarified only recently \([5]\). If \( k < d \) it is easy to realize that there are small “self-sustained” structures (e.g. \( d \)-dimensional hypercubes) that can survive culling irrespective of their environment. In this case the \( k \)-core always exists and the problem is rather if it percolates or not and what is the nature of the percolation transition \([10]\).

On the Bethe lattice the \( k \)-core percolation transition is known to be discontinuous \([11]\). Starting from high values of \( p \) the density of the \( k \)-core drops discontinuously to zero at \( p_c \). The transition however is not simply first order, the density near the transition is given by \( \rho(p) \sim p(p_c) + b(p-p_c)^{1/2} \). Furthermore it has been recently pointed out that the transition is also accompanied by diverging correlation lengths. This behavior has motivated the proposal of \( k \)-core percolation as a model of the jamming transition \([12]\). There are indeed evidences that this transition has a mixed character \([13]\).

This brought new interest into the question of whether the hybrid nature of the transition in the Bethe lattice survives in finite dimensions. This is certainly not the case for cubic lattices in \( d \leq 3 \). Indeed for \( k = 2 \) the transition is continuous and has the same critical point of ordinary percolation \([14]\). For \( k = 3 \) and \( d = 3 \) the transition is continuous \([11,15,16,17]\) with exponents consistent with those of ordinary \( d = 3 \) percolation \([16]\). These results are valid on cubic lattices and they not exclude the possibility of a mixed transition for \( d \leq 3 \) provided the structure of the lattice or the constraints are different. Indeed recently a 2-dimensional model with a mixed transition was exhibited \([18]\) and numerical evidences of a mixed transition in another 2-dimensional model were reported in \([19]\). As for regular lattices, an expansion in powers of \( 1/d \) has proven that turning on dimension perturbatively does not destroy the mixed nature of the transition \([10]\), thus suggesting that the hybrid transition may exist for some \((d > 3, 2 < k < d + 1)\). In this work we investigate numerically \( k \)-core percolation on the four-dimensional hypercubic lattice. For \( k = 3 \) we have found a continuous transition and we did not further investigate the critical behavior. In the case \((d = 4, k = 4)\) we find negative results concerning the hybrid transition: while the presence of a discontinuous transition is clearly established, it is strictly first order. More precisely for \( k = 4 \), at a critical value \( p_c = .6885(5) \) the system has a phase transition from a high \( p \) phase where there is a giant cluster with a finite density to a low-\( p \) phase where there is not. The density of the giant cluster is given by the \( k \)-core density minus the density of the small clusters, that is approximately \( \rho_{\text{giant}} \approx 0.04 \) near the transition, therefore the critical properties of the giant cluster can be safely extracted from the total density in the percolating phase. The \( k \)-core density exhibits a discontinuous transition, jumping from a \( \rho^{-} = 0.567(4) \) to \( \rho^{+} = 0.04(4) \). However the density displays no singular behavior at the transition, and the correlation length extracted from \( k \)-core percolation function \( G(i,j) = \langle v_i v_j \rangle - \langle v_i \rangle\langle v_j \rangle \) (where \( v_1 \) is 1 on the \( k \)-core and 0 otherwise \([10]\)) remains finite, \( \xi < 10 \).

We started the numerical investigations considering hypercubic lattices with periodic boundary conditions (PBC). In fig. \([20]\) we plot the density of the \( k \)-core for a sample of size \( L = 320 \), corresponding to \( O(10^{10}) \) sites. The density of the \( k \)-core has a discontinuous transition at a \( p = .6869 \), where it jumps from a high-density percolating phase to a low-density non-percolating phase \( \rho \approx .05 \). The behavior appears to be consistent with a singular behavior at the transition but a careful study of the data in order to extract the critical \( p \) and the exponent \( \beta \) shows some inconsistencies. Indeed the curve seems to be fit at best with the exponent \( \beta = 1/2 \) (the mean-field Bethe-lattice value) but with a value of the critical probability \( p \approx .6862 \) definitively lower than that at which the transition is actually observed. In order to assess if this behavior can be considered a finite-size
In figure (2) we plot the inverse correlation length $\xi$ vs. probability for the same sample of fig. 1. The arrow marks the point of the actual transition. Although the behavior is consistent with a divergence at $p \approx .6862$, we never observed a correlation length at the actual transition exceeding $\xi \approx 8 - 10$, i.e. large but much smaller than the sample size $L = 320$. The vertical line marks the value of the true critical probability $p_c$, see text.

In order to assess the validity of this interpretation and to determine whether there is a true percolation transition at higher values of $p$ we considered systems with completely empty boundaries. These boundary conditions guarantee that the system is completely isolated from the outside, as a consequence the density at finite size is a lower bound to the density in the thermodynamic limit. Furthermore it turned out that in this case numerical methods are rather safe for extrapolating the behavior of the infinite system at variance with the more delicate $k > d$ case where finite-size effects are extremely large as mentioned above. The possibility of choosing these boundary condition is a special feature of the case $k < d$ because otherwise no site can resist culling if the boundaries of the hypercube are empty.

In order to estimate the density at a given value of $p$ we generated lattices of increasing size. In figure (3) we plot the results for the total density $\rho_L$ at various sizes $L$. The total density $\rho_L$ is affected by the presence of the empty boundaries and is an increasing function of $L$. The monotonicity property allows to safely conclude from fig. (3) that there is indeed a discontinuous transition in the large $L$ limit. We also measured the bulk density, i.e. the density of a smaller hypercube inside the sample whose boundaries are far enough from the surfaces. The bulk density provides a direct estimate of the density in thermodynamic limit and strengthens the conclusion that there is a discontinuous transition, see fig. (4). The transition probability decreases with the sample size and tends to a critical value $p_c = .6885(5)$ that was estimated through extrapolation. We expect that the in-

FIG. 1: $k$-core density vs probability with $k = 4$ for a four-dimensional sample of size $L = 320$ and periodic boundary conditions. The arrow marks a discontinuous transition in the density that jumps to $p \approx .05$ at lower values of $p$. The data are fit in the region near the transition with the function $\rho(p) \approx \rho(p_c) + b(p-p_c)^{1/2}$ but the actual transition probability (marked by the arrow) is above the $p_c$ estimated from the fit. Inset: rescaled plot of the data and the fit near the transition.

FIG. 2: Inverse of the correlation length $\xi$ vs. probability for the same sample of fig. 1. The arrow marks the point of the actual transition. Although the behavior is consistent with a divergence at $p \approx .6862$, in all sample studied we never observed a correlation length at the actual transition exceeding $\xi \approx 8 - 10$, i.e. large but much smaller than the sample size $L = 320$. The vertical line marks the value of the true critical probability $p_c$, see text.
and the high-density phase in the bulk penetrates more and more in the sample for $p \to p_c$, as a consequence in smaller systems the transition will occur at higher values of $p$. The percolation value $p_c = 0.6885(5)$ is larger than the actual value of the transition in the case of periodic boundary condition, e.g. $p = 0.6869$ for $L = 320$, in figure 4 we plot the bulk density and compare it with the density of the system in the case of periodic boundary conditions. For $p \geq p_c$ (marked by a vertical line in the plot) the two densities are equal and we clearly see that the density in the case of periodic boundary conditions is the analytic continuation of the percolating-phase density in the unstable region. We also verified that the correlation length of the bulk is the same of the systems with periodic boundary conditions at the corresponding values of $p$, see fig. 2. By looking at figures 4 and 2 we immediately see that the density and correlation length are regular at the estimated value of the real transition probability $p_c = 0.6885$. In particular using this value of $p_c$ we estimate $\xi_c \approx 2.5$, while in the unstable phase we can observe $\xi$ up to ten lattice spacings. We note that due to time and memory constraints, few lattices of the largest size ($L = 320$) can be studied, however the sample-to-sample fluctuations are small enough to ensure a good estimate the density, for instance the bulk density estimates computed from hypercubes of different size $L = 100, 150, 200$ do not show significant deviations, see inset of fig. 2.

The qualitative features of the finite size effect in the case of empty boundaries can be understood in the same way as in thermodynamical first-order transitions, e.g. a ferromagnetic Ising model in a small positive field with the spins on the boundaries forced to be negative. As we noted above, the critical transition probability is shifted to higher values of $p$ and the true $p_c$ can be estimated through extrapolation (for instance for $L = 320$ the transition is at $p = 0.689$, see 2). In general the finite size corrections to the density should scale as $1/L$, i.e.

$$\rho_L(p) = \rho(p) - \frac{1}{L} c_1(p) + O\left(\frac{1}{L^2}\right)$$

and figure 3 suggests that the factor $c_1(p)$ diverges at $p_c$. This factor is determined by the density profiles $\rho(p, z)$ at distance $z$ from an empty surface:

$$c_1(p) = \int_0^{\infty} (\rho(p) - \rho(p, z)) dz .$$

Direct inspection of the profiles $\rho(p, z)$ shows that they are consistent with a divergence at the transition, consistently with the expectation that the transition is determined by the penetration deep inside the sample of the interface between the low-density and the percolating phases. The precise nature of the divergence of $c_1(p)$ would require a more detailed analysis which goes beyond the scope of this work. In figure 3 we plot the behavior of the inverse of $c_1$ and of the transition probability for different sample sizes from which $p_c$ was estimated.

In conclusion the density of the $k$-core in four dimension with $k = 4$ exhibits a discontinuous transition at $p_c = 0.6885(5)$ from a high-density percolating phase to a low-density non-percolating phase. The transition loses its hybrid character with respect to the Bethe lattice case: the density is regular at the transition and
the correlation length is finite. These results are most clearly seen numerically considering samples with empty boundary conditions, instead in finite size systems with periodic boundary conditions it is possible to follow the percolating phase in the unstable region $p < .6885(5)$. The behaviour of the unstable phase is consistent with a pseudo-transition at a lower probability $p \approx .686$. This pseudo-transition appears to have a hybrid character with pseudo-exponent $\beta = 1/2$ and diverging correlation length but it cannot be observed because the unstable phase decays at $p \approx .6869$ through nucleation of low-density droplets determined by spatial fluctuations of the density.

It would be very interesting to confirm these results through analytical methods. Although the $1/d$ expansion [10] gives no hint about the observed disappearance of the mixed transition, maybe different approaches to perturbation theory around the Bethe solution [19, 20] could be able to see it, notwithstanding the possibility that non-perturbative effects should be taken into account.

FIG. 5: probability vs. inverse of the $1/L$ correction $c_1$ for different sample sizes, the data are consistent with a divergence of the prefactors at the transition. Inset: probability at which the total density is $\rho = .3 < \rho_c$ as a function of the inverse of the size of the sample, it tends to $p_c$ as the size of the system increases.

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