THE BRST SYMMETRY OF AFFINE LIE
SUPERALGEBRAS AND NON-CRITICAL STRINGS

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ABSTRACT

The topological field theories associated with affine Lie superalgebras are con-
structed. Their BRST symmetry is characterised by a Kazama algebra containing
spin 1, 2 and 3 operators and closes linearly. Under this symmetry all operators
are grouped into BRST doublets. The relation between the models constructed
and non-critical string theories is explored.
1. Introduction.

The study of topological quantum field theories [1,2] has aroused great interest recently. These theories, being endowed with a BRST symmetry, are models with no local degrees of freedom, so all local excitations can be eliminated once the topological symmetry has been fixed. In two dimensions there exist conformal field theories which, in addition, are topological field theories. These are the so-called Topological Conformal Field Theories (TCFT) [3].

A method to generate new TCFT’s consists in studying the BRST structure of different chiral algebras that extend the Virasoro symmetry. This method has been applied in ref. [4] to the case of an affine Lie algebra, whereas in ref. [5] this analysis was extended to the case of a superconformal current algebra. It is the purpose of this paper to explore, using the same procedure, the TCFT’s based on affine Lie superalgebras.

The topological symmetry of a TCFT is encoded in its topological algebra, which is the operator algebra closed by the chiral algebra of the TCFT and the BRST current. It was checked in refs. [4,5,6] that the topological algebra of a TCFT possessing a non-abelian current algebra must include operators of dimensions one, two and three. This algebra is the so-called Kazama algebra [7], which differs from the standard twisted $N = 2$ superconformal algebra [8]. The former includes two dimension-three operators and can be regarded as an extension of the latter. The extended nature of the Kazama algebra seems to be an unavoidable consequence of the underlying non-abelian current symmetry.

The representation of the BRST symmetry found in refs. [4,5,6] requires the level of the current algebra to be fixed to some critical value related to the dual Coxeter number of the Lie algebra. When the matter sector of the currents is realised by means of two decoupled currents, it is only the sum of the two levels that is constrained. In this two-current realization, the TCFT for a Lie algebra $G$ has a nice interpretation as a $G/G$ coset model [9]. After a suitable deformation these theories have been shown to be related to non-critical $W$-strings [10,11,12].
Lie superalgebras seem to play an important role in the construction and classification of extended superconformal algebras [13,14] and supersymmetric Toda field theories. Given a Lie superalgebra which admits a purely fermionic simple root system, one can construct an $N = 1$ supersymmetric Toda model [15,16,17]. Application of the method of hamiltonian reduction to Lie (super)algebras leads to (super) $\mathcal{W}$-algebras [18,19]. There is also a possibility that string theories may be classified in terms of superalgebras. For example, the twisted superconformal symmetry for strings with $N - 2$ supersymmetries has been constructed in [20] via the quantum hamiltonian reduction of $\text{osp}(N|2)$.

It therefore seems desirable to extend the formalism initiated in [4,5,6] to cover the general case of an arbitrary Lie superalgebra. We shall show below that the approach followed in refs. [4,5] can be easily generalised to TCFT’s possessing an affine Lie superalgebra. The BRST algebra for these theories is the same as in the bosonic case, which yields a new representation of the topological Kazama algebra. Moreover we provide arguments to support the idea that these theories are related to non-critical superstring theories.

This paper is organised as follows. In section 2, after a brief introduction to current superalgebras and the associated Sugawara constructions in two dimensions, the topological algebra is developed explicitly. A two-current construction is also possible, which paves the way for the interpretation of the theory as a gauged, supergroup-valued Wess-Zumino-Witten (WZW) model; this is done in section 3. In section 4 we turn our attention to the relation of the model constructed with the non-critical superstring theories. We shall find a suggestive connection between the $\text{osp}(1|2)$ and $\text{osp}(2|2)$ theories and non-critical superstring theories with one and two supersymmetries respectively. These results generalise the relation between non-critical $W_N$-strings and $\text{sl}(N)/\text{sl}(N)$ coset theories [10,11,12]. Finally, our work is summarised in section 5, together with some conclusions, comments and suggestions for future work.
2. Construction of the topological algebra.

Let \( \mathcal{G} \) be a finite-dimensional superalgebra over the complex field \([21,22,23]\). Call \( \mathcal{G}_B \) and \( \mathcal{G}_F \) the even (\( i.e. \), bosonic) and odd (\( i.e. \), fermionic) subspaces of \( \mathcal{G} \), spanned respectively by basis vectors \( T_a, a = 1, 2, \ldots d_B \), and \( T_\alpha, \alpha = 1, 2, \ldots d_F \), where \( d_B \) and \( d_F \) are the respective dimensions of \( \mathcal{G}_B \) and \( \mathcal{G}_F \). In the following, latin indices \( a, b, \ldots \) will run from 1 to \( d_B \), while greek indices \( \alpha, \beta, \ldots \) from 1 to \( d_F \). The superdimension of \( \mathcal{G} \) is \( d_B - d_F \). Denote the (generalised) Lie bracket by \([,]\). One has

\[
[T_a, T_b] = f^c_{ab} T_c \quad [T_a, T_\alpha] = f^\beta_{a\alpha} T_\beta \quad [T_\alpha, T_\beta] = f^c_{\alpha\beta} T_c. \tag{2.1}
\]

The generalised Lie bracket satisfies a graded Jacobi identity

\[
(-1)^g(x)g(z) [x, [y, z]] + (-1)^g(y)g(x) [y, [z, x]] + (-1)^g(z)g(y) [z, [x, y]] = 0 \tag{2.2}
\]

for any \( x, y \) and \( z \) in \( \mathcal{G} \), where \( g(x) = 0 \) if \( x \in \mathcal{G}_B \) and \( g(x) = 1 \) if \( x \in \mathcal{G}_F \). \( \mathcal{G} \) will be assumed to possess a real, non-degenerate, supersymmetric bilinear form \((,\)\) such that \( \mathcal{G}_B \) and \( \mathcal{G}_F \) are orthogonal. It will also be assumed to satisfy the invariance property

\[
([x, y], z) = (x, [y, z]) \tag{2.3}
\]

for all \( x, y \) and \( z \) in \( \mathcal{G} \). Call \( g_{ab} = (T_a, T_b), g_{\alpha\beta} = (T_\alpha, T_\beta) \); one has \( g_{ab} = g_{ba}, g_{\alpha\beta} = -g_{\beta\alpha} \). Indices are raised and lowered by contraction with the metric tensor and its inverse according to \( T^a = g^{ab} T_b, T_a = g_{ab} T^b, T^\alpha = T_\beta g^{\beta\alpha}, T_\alpha = T_\beta g_{\beta\alpha} \). Upon lowering of its upper index,

\[
f_{abc} = f^d_{ab} g_{dc}, \quad f_{a\beta\gamma} = f^\mu_{a\beta} g_{\mu\gamma}, \quad f_{\alpha\beta\gamma} = f_{a\beta\gamma} g_{dc}, \tag{2.4}
\]

the structure constants become superantisymmetric. The Jacobi identity (eq. (2.2)) imposes additional conditions on the structure constants. Some of these
conditions are:
\[ f_{\alpha \beta}^{\gamma} f_{\alpha}^{\delta} - f_{\gamma \alpha}^{\beta} f_{\delta}^{\alpha} = 0 \]
\[ f_{\alpha \beta}^{\delta} f_{\alpha}^{\gamma} + f_{\gamma \alpha}^{\beta} f_{\delta}^{\beta} = 0 \]
\[ f_{\alpha \beta}^{\delta} f_{\gamma}^{\delta} + f_{\beta \gamma}^{\delta} f_{\alpha}^{\gamma} + f_{\gamma \alpha}^{\delta} f_{\beta}^{\gamma} = 0 \]

Other relations satisfied by the structure constants can be obtained from the quadratic Casimir operator of the superalgebra. If \( C_A \) denotes the value of this operator in the adjoint representation, one has:
\[ g^{\beta c} f_{\alpha \beta}^{d} f_{\alpha}^{c} = C_A \delta_{\alpha}^{\delta} \]
\[ g^{\gamma c} f_{\alpha \beta}^{\mu} f_{\gamma}^{\mu} = \frac{C_A}{2} \delta_{\alpha}^{\mu} \]

A conformal current superalgebra is generated by a set of holomorphic bosonic \( J_a(z) \) and fermionic currents \( J_\alpha(z) \), satisfying the following operator product expansions (OPE's):
\[ J_a(z) J_b(w) = \frac{k g_{ab}}{(z-w)^2} + \frac{f_{ab}^{c}}{z-w} J_c(w) \]
\[ J_a(z) J_\beta(w) = \frac{f_{a \beta}^{\gamma}}{z-w} J_\gamma(w) \]
\[ J_\alpha(z) J_\beta(w) = \frac{k g_{\alpha \beta}}{(z-w)^2} + \frac{f_{\alpha \beta}^{c}}{z-w} J_c(w). \]

\( k \) is the level of the current superalgebra. The currents \( J_a \) and \( J_\alpha \) can be made into Virasoro primary fields by application of the Sugawara construction, whereby a bilinear in the \( J \)'s
\[ T^J = N (g^{ab} J_a J_b + g^{\alpha \beta} J_\alpha J_\beta) \]

is required to satisfy a Virasoro algebra, such that all the currents \( J \) have conformal dimension 1 with respect to \( T^J \); the normalisation constant \( N \) is fixed precisely by
this requirement. This analysis is standard and has been carried out, for the case of a Lie superalgebra, in [24,25,26,27,28]. The value of \( N \) is found to be

\[
N = \frac{1}{2k + C_A}, \tag{2.9}
\]

and the operator

\[
T^J = \frac{1}{2k + C_A}(g^{ab}J_aJ_b + g^{\alpha\beta}J_\alpha J_\beta) \tag{2.10}
\]

closes a Virasoro algebra with a central charge \( c_J \) given by

\[
c_J = \frac{2k(d_B - d_F)}{2k + C_A}. \tag{2.11}
\]

Let us now describe how one can construct a TCFT based on the algebra (2.7). The basic ingredient in our construction is the introduction of a ghost sector such that one can realise the BRST symmetry of the superalgebra (2.7). We shall use this BRST symmetry as the topological symmetry of the TCFT. In general, in order to represent the BRST symmetry of a given chiral algebra, one has to introduce a fermionic(bosonic) ghost system for every bosonic(fermionic) generator of the algebra. To these ghost systems one must assign ghost numbers in such a way that each antighost and its associated generator have the same spin. According to these general rules we must introduce in our case a spin-one ghost system for each current of the superalgebra (2.7). Moreover, the topological symmetry we are trying to implement is such that the BRST variation of the antighost equals the corresponding total current.

Let us denote by \((\rho^a, \gamma^a)\) to the fermionic ghosts for the currents \(J_a\) whereas the bosonic fields \((\lambda_\alpha, \eta^\alpha)\) will correspond to the currents \(J_\alpha\). Let us choose our conventions in such a way that the fermionic ghost fields \(\rho^a, \gamma^b\) satisfy the OPE

\[
\rho^a(z)\gamma^b(w) = \frac{-\delta^b_a}{z - w}. \tag{2.12}
\]

\(\rho^a\) and \(\gamma^b\) will be assumed to have conformal weights 1 and 0, and will be assigned ghost numbers \(-1\) and \(+1\), respectively. Their energy-momentum tensor \(T^{(\rho\gamma)}\) is
given by

\[ T^{(\rho \gamma)} = \rho_a \partial \gamma^a, \tag{2.13} \]

and has a central charge \( c_{(\rho \gamma)} = -2d_B \). Similarly the bosonic ghost fields \( \eta^\alpha, \lambda_\beta \) will satisfy

\[ \eta^\alpha(z)\lambda_\beta(w) = \frac{-\delta^\alpha_\beta}{z-w}. \tag{2.14} \]

\( \eta^\alpha \) and \( \lambda_\beta \) have conformal weights 0 and 1 and ghost numbers +1 and −1, respectively, and their energy-momentum tensor is:

\[ T(\eta \lambda) = \partial \eta^\alpha \lambda_\alpha. \tag{2.15} \]

The central charge of \( T(\eta \lambda) \) is \( c_{(\eta \lambda)} = 2d_F \). Altogether, the operator

\[ T = T^{J} + T^{(\rho \gamma)} + T^{(\eta \lambda)} \tag{2.16} \]

closes a Virasoro algebra with a central charge given by

\[ c_{(tot)} = \left[ \frac{2k}{2k+C_A} - 2 \right] (d_B - d_F). \tag{2.17} \]

A TCFT is expected to have a vanishing Virasoro anomaly. Notice that the condition \( c_{(tot)} = 0 \) is satisfied when \( k = -C_A \). This value of the level determines the topological point of the current + ghost system under consideration. Another way to see how a topological theory comes about is the following. Let us consider the combination

\[ J^h_a = f_{ab}^c \gamma^b \rho_c - \int_a^\gamma \gamma^b \lambda_a, \tag{2.18} \]

which is a bosonic, zero ghost-number field with conformal weight 1 with respect to \( T \) in eq. (2.16). Similarly, consider the object

\[ J^{gh}_a = f_{a\mu}^b \rho_b \eta^\mu + f_{a\gamma}^b \gamma^b \lambda_a, \tag{2.19} \]

with the same quantum numbers as above, but fermionic statistics. \( J^h_a \) and \( J^{gh}_a \) can be checked to verify the algebra (2.7), with a level \( k = C_A \). Actually one can
easily verify that \( J_a^{\text{gh}} \) and \( J_\alpha^{\text{gh}} \) represent the generators of the superalgebra in the space of ghost fields. Therefore the total bosonic and fermionic currents read

\[
\mathcal{J}_a = J_a + J_a^{\text{gh}}, \quad \mathcal{J}_\alpha = J_\alpha + J_\alpha^{\text{gh}},
\]

and satisfy the algebra (2.7) with a total level \( k_{(\text{tot})} = k + C_A \). According to the general arguments of refs. [4, 5], a topological current superalgebra should now appear at that particular value of \( k \) for which \( k_{(\text{tot})} \) vanishes, i.e., for \( k = -C_A \). That this is indeed correct is confirmed by the fact that this latter value, when substituted into eq. (2.17), gives \( c_{(\text{tot})} = 0 \).

We now work out the topological structure present in the theory. To begin with, a nilpotent BRST current \( Q \) having fermionic statistics, conformal weight 1 and ghost number +1 is needed. With the fields at hand, the combination

\[
Q = -\gamma^a J_a - \frac{1}{2} f^c_{ab} \gamma^a \gamma^b \rho_c + f^\mu_{a\beta} \gamma^a \eta^\beta \lambda_\mu - \eta^\alpha J_\alpha - \frac{1}{2} f^c_{\alpha\beta} \eta^\alpha \eta^\beta \rho_c \tag{2.21}
\]

satisfies the necessary requirements. Indeed \( Q \) is the canonical BRST charge for the Lie superalgebra (2.7). A tedious although straightforward calculation shows that

\[
Q(z)Q(w) = \frac{k + C_A}{z - w} (g_{ab} \partial \gamma^a \gamma^b + g_{\alpha\beta} \partial \eta^\alpha \eta^\beta), \tag{2.22}
\]

thus confirming again that only for the critical value \( k = -C_A \) is it possible to have a nilpotent topological symmetry. From now on we will always assume that we are working at the critical level.

One can now suspect that all the operators present in the theory appear in BRST doublets. To prove that this statement is true, let us begin with the total Kač–Moody currents as given in eq. (2.20). We have

\[
\begin{align*}
Q(z) \rho_a(w) &= \frac{1}{z - w} \mathcal{J}_a(w) \\
Q(z) \mathcal{J}_a(w) &= 0 \\
Q(z) \lambda_\alpha(w) &= \frac{1}{z - w} \mathcal{J}_\alpha(w) \\
Q(z) \mathcal{J}_\alpha(w) &= 0.
\end{align*}
\tag{2.23}
\]
so \( (\rho_a, J_a) \) and \( (\lambda_\alpha, J_\alpha) \) form weight 1 topological doublets. In eq. (2.23) one notices that the BRST variations of the antighosts \( \rho_a \) and \( \lambda_\alpha \) are the total currents \( J_a \) and \( J_\alpha \) respectively, which confirms the correctness of our choice for \( Q \). The operator algebra of \( \rho_a, \lambda_\alpha, J_b \) and \( J_\beta \) closes as follows:

\[
\begin{align*}
J_a(z)J_b(w) &= \frac{f^c_{ab}}{z-w}J_c(w) \\
J_a(z)J_\beta(w) &= \frac{f^\gamma_{\alpha\beta}}{z-w}J_\gamma(w) \\
J_\alpha(z)J_\beta(w) &= \frac{f^c_{\alpha\beta}}{z-w}J_c(w) \\
J_a(z)\rho_b(w) &= \frac{f^c_{ab}}{z-w}\rho_c(w) \\
J_a(z)\lambda_\alpha(w) &= \frac{f^\beta_{\alpha\alpha}}{z-w}\lambda_\beta(w) \\
J_\alpha(z)\rho_b(w) &= -\frac{f^\gamma_{\alpha\beta}}{z-w}\lambda_\gamma(w) \\
J_\alpha(z)\lambda_\beta(w) &= -\frac{f^c_{\alpha\beta}}{z-w}\rho_c(w).
\end{align*}
\]

The topological character of the theory is ensured if the energy-momentum tensor \( T \) in eq. (2.16) is \( Q \)-exact. In that case the BRST ancestor of \( T \), denoted by \( G \), would be a weight 2 fermionic field with ghost number \(-1\). A glance at eq. (2.23) can give some idea about its expression: a Sugawara-like bilinear of the form \( g^{ab}\rho_aJ_b \), \( g^{\alpha\beta}\lambda_\alpha J_\beta \), with some appropriate coefficients, will do. The precise combination

\[
G = \frac{-1}{C_A}(g^{ab}\rho_aJ_b + g^{\alpha\beta}\lambda_\alpha J_\beta),
\]

where the overall coefficient equals the one in eq. (2.10) for the critical level, satisfies

\[
Q(z)G(w) = \frac{d_B - d_F}{(z-w)^3} + \frac{1}{(z-w)^2}R(w) + \frac{1}{z-w}T(w).
\]

In eq. (2.26), \( R \) is a weight 1 bosonic field with ghost number zero given by

\[
R = \rho_a\gamma^a + \lambda_\alpha\eta^\alpha.
\]
Eq. (2.26) is characteristic of two-dimensional topological models. The residue at the simple pole is the energy-momentum tensor, which proves its BRST-exactness, while the operator appearing at the double pole is a $U(1)$ current. Indeed, one easily checks that

$$R(z)R(w) = \frac{d_B - d_F}{(z-w)^2}, \quad (2.28)$$

and $R$ can be understood as the ghost-number current. Indeed the $R$-charges of the different fields coincide with the ghost numbers we have assigned them. As for the triple pole in eq. (2.26), the coefficient is a c-number called the topological dimension $d$ of the model, which now equals the superdimension of the current superalgebra. The same arguments as those developed in [4] lead us to conclude that we are in fact describing a topological sigma model for the underlying supergroup manifold. A second BRST-doublet is thus given by $(G, T)$, and the OPE

$$Q(z)R(w) = \frac{-1}{z-w} Q(w) \quad (2.29)$$

shows that $(R, Q)$ too are BRST partners. One can now compute the remaining algebra between the generators above, with the result that

$$T(z)Q(w) = \frac{1}{(z-w)^2} Q(w) + \frac{1}{z-w} \partial Q(w)$$

$$T(z)R(w) = -\frac{(d_B - d_F)}{(z-w)^3} + \frac{1}{(z-w)^2} R(w) + \frac{1}{z-w} \partial R(w) \quad (2.30)$$

$$T(z)G(w) = \frac{2}{(z-w)^2} G(w) + \frac{1}{z-w} \partial G(w)$$

$$R(z)G(w) = \frac{-1}{z-w} G(w).$$

The above OPE’s are exactly those obtained upon twisting the $N = 2$ superconformal algebra [29,8] (the so-called topological algebra), so one might be led to believe that such an algebra is also present here. However, there is a fundamental difference now, because the BRST partner of $T$, $G$, is not nilpotent. Instead one
has

\[ G(z)G(w) = \frac{1}{z - w} W(w), \]

(2.31)

where \( W \) is a bosonic, dimension 3 operator with ghost number \(-2\) given by

\[
W = -\frac{1}{C_A} (\partial \rho^a \rho^a + \partial \lambda^a \lambda^a) \\
+ \frac{1}{(C_A)^2} \left( f_{abc} \rho^a \rho^b J_c + f_{\alpha \beta c} \lambda^\alpha \lambda^\beta J_c + 2 f_{\alpha \beta \gamma} \rho^\alpha \lambda^\beta J_\gamma \right).
\]

(2.32)

Since all operators in the theory so far have appeared as BRST doubles, one would expect this to hold for \( W \), too. And given that \( W \) is BRST-closed, i.e.,

\[
Q(z)W(w) = 0,
\]

(2.33)

we must look for a BRST ancestor for \( W \) with the following quantum numbers: fermionic statistics, conformal weight 3, and ghost number \(-3\). The combinations \( f_{abc} \rho^a \rho^b \rho^c \) and \( f_{\alpha \beta c} \lambda^\alpha \lambda^\beta \rho^c \) immediately come to mind. Defining

\[
V = \frac{1}{(C_A)^2} \left( \frac{1}{3} f_{abc} \rho^a \rho^b \rho^c + f_{\alpha \beta c} \lambda^\alpha \lambda^\beta \rho^c \right),
\]

(2.34)

one can check that

\[
Q(z)V(w) = \frac{1}{z - w} W(w),
\]

(2.35)

which proves our point: \((V, W)\) forms a new BRST doublet.

In trying to work out the topological structure present in the theory, we have found that the operator algebra is very similar to that of the twisted \( N = 2 \) models. But the appearance of \((V, W)\) forces us to compute their OPE’s with all other operators, and there is no guarantee that the resulting algebra will close on a finite number of fields. However, the algebra of \((G, T), (R, Q)\) and \((V, W)\) does
close, as some computation proves. The results are

\[
T(z)W(w) = \frac{3}{(z-w)^2} W(w) + \frac{1}{z-w} \partial W(w)
\]

\[
T(z)V(w) = \frac{3}{(z-w)^2} V(w) + \frac{1}{z-w} \partial V(w)
\]

\[
G(z)W(w) = \frac{3}{(z-w)^2} V(w) + \frac{1}{z-w} \partial V(w)
\]

\[
R(z)W(w) = \frac{-2}{z-w} W(w)
\]

\[
R(z)V(w) = \frac{-3}{z-w} V(w),
\]

while all other OPE's vanish identically. It is important to emphasise that, contrary to what happens with \( W \) algebras, the existence of higher-spin fields does not spoil the linearity of the algebra.

The above conclusions have also been obtained in [4,6], but our analysis here extends these results to the more general case of an arbitrary superalgebra. It should also be mentioned that the algebra exhibited in eqs. (2.26) to (2.36), which we shall call the Kazama algebra, first appeared in [7] as a consistent, non-trivial extension of the twisted \( N = 2 \) algebra. In [7] it was related to an \( N = 1 \) superconformal symmetry, but no explicit representation for the generators was given (see also ref. [30]).

To complete our analysis, it remains to study whether or not the currents \( J_a, J_\alpha \) and their BRST ancestors \( \rho_a, \lambda_\alpha \), on the one hand, and the generators \( T, G, R, Q, W \) and \( V \), on the other, are compatible. Some of the corresponding OPE's are trivial (for example, those expressing the Virasoro primary character of the currents); others have already been given (eq. (2.23)). Among those remaining, the only non-vanishing ones are
\[
G(z)\mathcal{J}_a(w) = \frac{1}{(z-w)^2} \rho_a(w) + \frac{1}{z-w} \partial \rho_a(w)
\]
\[
G(z)\mathcal{J}_\alpha(w) = \frac{1}{(z-w)^2} \lambda_\alpha(w) + \frac{1}{z-w} \partial \lambda_\alpha(w)
\]
\[
R(z)\rho_a(w) = -\frac{1}{z-w} \rho_a(w)
\]
\[
R(z)\lambda_\alpha(w) = -\frac{1}{z-w} \lambda_\alpha(w),
\]

which establishes that the topological and current superalgebra structures are indeed compatible.

An interesting feature of the above construction is the fact that it can also be performed with two independent sets of currents \(J^1, J^2\). Suppose \(J^1\) and \(J^2\) satisfy the algebra (2.7) with levels \(k_1\) and \(k_2\), respectively. Then the Sugawara energy-momentum tensor \(T^J\) is given by

\[
T^J = \frac{1}{2k_1 + C_A} (g^{ab} J^1_a J^1_b + g^{\alpha\beta} J^1_\alpha J^1_\beta) + \frac{1}{2k_2 + C_A} (g^{ab} J^2_a J^2_b + g^{\alpha\beta} J^2_\alpha J^2_\beta),
\]

and the corresponding central charge is

\[
c_J = \left[ \frac{2k_1}{2k_1 + C_A} + \frac{2k_2}{2k_2 + C_A} \right] (d_B - d_F).
\]

Imposing \(k_1 + k_2 = -C_A\) and setting \(k_1 = k\) for simplicity, eqs. (2.38) and (2.39) reduce to

\[
T^J = \frac{1}{2k + C_A} [g^{ab} (J^1_a J^1_b - J^2_a J^2_b) + g^{\alpha\beta} (J^1_\alpha J^1_\beta - J^2_\alpha J^2_\beta)]
\]

and

\[
c_J = 2(d_B - d_F),
\]

which exactly cancels the ghost central charge \(c_{(\rho\gamma)} + c_{(\eta\lambda)}\). That this is indeed a new topological point is again confirmed by the following arguments. The new
total currents are
\[ \mathcal{J}_a = J^1_a + J^2_a + J^g_a \]
\[ \mathcal{J}_\alpha = J^1_\alpha + J^2_\alpha + J^g_\alpha, \]
(2.42)
with \( J^g_a \) as in eqs. (2.18), (2.19), and their algebra is
\[ \mathcal{J}_a(z)\mathcal{J}_b(w) = \frac{(k_1 + k_2 + C_A)g_{ab}}{(z - w)^2} + \frac{f_{\alpha\beta}^c}{z - w}\mathcal{J}_c(w) \]
\[ \mathcal{J}_a(z)\mathcal{J}_\beta = \frac{f_{a\beta}^\gamma}{z - w}\mathcal{J}_\gamma(w) \]
\[ \mathcal{J}_\alpha(z)\mathcal{J}_\beta = \frac{(k_1 + k_2 + C_A)g_{\alpha\beta}}{(z - w)^2} + \frac{f_{\alpha\beta}^c}{z - w}\mathcal{J}_c(w). \]
(2.43)
The new BRST current making them BRST-exact is
\[ Q = -\gamma^a(J^1_a + J^2_a) - \frac{1}{2}f_{ab}^c\gamma^a\gamma^b\rho_c + f_{a\beta}^\mu\gamma^a\eta^\beta\lambda_\mu - \eta^a(J^1_\alpha + J^2_\alpha) - \frac{1}{2}f_{\alpha\beta}^c\eta^\alpha\eta^\beta\rho_c, \]
(2.44)
where again \( \rho_a \) and \( \lambda_\alpha \) are their BRST ancestors. Nilpotency of \( Q \) occurs only at the topological point, since
\[ Q(z)Q(w) = \frac{k_1 + k_2 + C_A}{z - w}(g_{ab}\partial\gamma^a\gamma^b + g_{\alpha\beta}\partial\eta^\alpha\eta^\beta). \]
(2.45)
One can now repeat the above analysis and work out the expressions for all the operators, with the result that the topological algebra is satisfied without changes.
The ghost number current \( R \) and the topological dimension \( d \) remain the same, but the other generators have to be modified as follows:
\[ G = \frac{1}{2k + C_A}\left[ g^{ab}\rho_a(J^1_b - J^2_b) + g^{\alpha\beta}\lambda_\alpha(J^1_\beta - J^2_\beta) \right] \]
\[ W = \frac{1}{(2k + C_A)^2}\left[ -C_A(\partial\rho_a\rho^a + \partial\lambda_\alpha\lambda^\alpha) + 2f_{a\beta}^\gamma\rho^\alpha\lambda^\beta(J^1_\gamma + J^2_\gamma) + f_{ab}^\rho\rho^a(J^1_b + J^2_b) + f_{\alpha\beta}^c\lambda^\alpha\lambda^\beta(J^1_c + J^2_c) \right] \]
\[ V = \frac{1}{(2k + C_A)^2}\left[ \frac{1}{3}f_{abcd}\rho^a\rho^b\rho^c + f_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\rho^\gamma \right]. \]
(2.46)
Although the topological algebra is the same as in the one-current case, this two-current construction is interesting because it allows for a lagrangian interpretation.
of the theory as a $G/G$ coset. This point is examined in the next section. Before finishing this one, let us point out that the topological algebra we have studied admits deformations both in its one and two current realizations. Indeed, if $\alpha^a$ are c-number constants, one can redefine $T$, $G$ and $R$ as follows:

\[
T \rightarrow T + \sum_a \alpha^a \partial J_a \\
G \rightarrow G + \sum_a \alpha^a \partial \rho_a \\
R \rightarrow R + \sum_a \alpha^a J_a,
\]

The operators $Q$, $V$ and $W$ are left unaffected by the deformation. One easily checks that the transformed generators satisfy the extended topological algebra for any value of the $\alpha^a$ constants. Of course, after the deformation, the currents are no longer primary dimension-one operators. Transformations of the type displayed in eq.(2.47) will play an important role in section 4, where we shall relate our results with the non-critical string theories.

3. A gauged, supergroup-valued WZW model.

The topological algebra described in the previous section has a lagrangian interpretation that we now develop. We shall show below that it is possible to give a lagrangian description of the two-current construction of section 2. The main result of this section is the interpretation of the gauged, $G$-valued WZW model as a theory in which the extended topological algebra closed by $(T, G)$, $(Q, R)$ and $(W, V)$ is realised. A similar conclusion has also been reported in [4] for the bosonic case (i.e., when $d_F = 0$), but our presentation here is totally general. An earlier reference on gauged WZW models is [31]. For arbitrary supergroups, a lagrangian construction of $G/G$ has already been put forward in [32].
Our starting point is the $G$-gauged WZW functional

$$\Gamma(g, A) = \Gamma(g) - \frac{1}{\pi} \int_{\Sigma} d^2 z \text{str}(g^{-1} A \bar{z} g A z - A \bar{z} g^{-1} + g^{-1} \partial \bar{z} g A z - A z A \bar{z}), \quad (3.1)$$

with $\Gamma(g)$ given by

$$\Gamma(g) = \frac{1}{2\pi} \int_{\Sigma} d^2 z \text{str}(g^{-1} \partial \bar{z} g^{-1} \partial z g) + \frac{i}{12\pi} \int_{M} \epsilon^{\mu \nu \rho} \text{str}(g^{-1} \partial \mu g g^{-1} \partial \nu g g^{-1} \partial \rho g). \quad (3.2)$$

g is a function taking values in the supergroup whose Lie superalgebra is $G$, and the 3-manifold $M$ is such that $\partial M = \Sigma$. The na"ive partition function is

$$Z = \int Dg DA \bar{z} DA z \exp \left[-k \Gamma(g, A)\right]. \quad (3.3)$$

The gauge invariance of $\Gamma(g, A)$ is well known. Also useful is the Polyakov-Wiegmann identity satisfied by $\Gamma(g)$,

$$\Gamma(gh) = \Gamma(g) + \Gamma(h) + \langle g, h \rangle, \quad (3.4)$$

where

$$\langle g, h \rangle = \frac{1}{\pi} \int_{\Sigma} d^2 z \text{str}(g^{-1} \partial \bar{z} g \partial z h h^{-1}). \quad (3.5)$$

Parametrise the gauge fields as

$$A_{\bar{z}} = h^{-1} \partial \bar{z} h \quad A_{z} = \bar{h}^{-1} \partial z \bar{h} \quad (3.6)$$

with $h$ and $\bar{h}$ taking values in the supergroup, and change variables in the functional integral (3.3). One has

$$DA_{z} DA_{\bar{z}} = J[h, \bar{h}] Dh D\bar{h}, \quad (3.7)$$

where $J[h, \bar{h}]$ is the Jacobian for the change of variables $A_{z}, A_{\bar{z}} \rightarrow h, \bar{h}$. This Jacobian can be represented as a functional integral over ghost fields that take
values in the adjoint representation of $\mathcal{G}$. Denoting, as in the previous section, these ghost fields by $(\rho_a, \gamma^a)$ and $(\lambda_\alpha, \eta^\alpha)$, we have:

$$J[h, \bar{h}] = \exp [C_A \Gamma(h^{-1} \bar{h})] \int D\rho D\gamma D\lambda D\eta \exp \left[\frac{-1}{\pi} \int_{\Sigma} d^2 z (\rho_a \partial_z \gamma^a + \lambda_\alpha \partial_z \eta^\alpha + c.c.) \right].$$

(3.8)

Taking into account that

$$\Gamma(g, A) = \Gamma(h^{-1} g \bar{h}) - \Gamma(h^{-1} \bar{h}),$$

(3.9)

the partition function becomes

$$Z = \int Dg Dh \bar{D}h D\rho D\gamma D\lambda D\eta \exp \left[-k \Gamma(h^{-1} g \bar{h}) + (k + C_A) \Gamma(h^{-1} \bar{h})\right] \exp \left[\frac{-1}{\pi} \int_{\Sigma} d^2 z (\rho_a \partial_z \gamma^a + \lambda_\alpha \partial_z \eta^\alpha + c.c.) \right].$$

(3.10)

Changing variables as $h^{-1} g \bar{h} \to g$ and choosing the gauge $h = 1$ (which does not introduce any new Faddeev-Popov ghosts) we obtain:

$$Z = \int Dg D\bar{h} D\rho D\gamma D\lambda D\eta \exp \left[-k \Gamma(g) + (k + C_A) \Gamma(\bar{h})\right] \exp \left[\frac{-1}{\pi} \int_{\Sigma} d^2 z (\rho_a \partial_z \gamma^a + \lambda_\alpha \partial_z \eta^\alpha + c.c.) \right].$$

(3.11)

This gauge-fixed form for the partition function clearly exhibits the necessary elements to construct the $\mathcal{G}/\mathcal{G}$ theory, as realised with two currents: one needs two independent, ungauged WZW models such that their levels add up to $-C_A$, plus a compensating ghost sector in order to set the total level to zero. Let us finally point out that the lagrangian interpretation we have discussed in this section allows to interpret the BRST symmetry of the superalgebra $\mathcal{G}$ as the basic symmetry of a topological sigma model having a Lie supergroup as target space.
4. Relation with non-critical superstrings.

In this section we shall explore the relation between the topological theories constructed in the previous sections and the non-critical superstring theories. In particular, we will argue that, for the Lie superalgebras osp(1|2) and osp(2|2), the corresponding topological coset models are related to the $N = 1$ and $N = 2$ superstring theories, respectively. In order to find this correspondence one must first conveniently deform the two-current model, as was explained at the end of section 2. In the deformed theory one can implement a quantum Drinfeld-Sokolov hamiltonian reduction in such a way that the reduced model can be identified with the corresponding string theory.

The connection of the topological current system with non-critical strings is more transparent if a free field realisation of the two currents involved in the $G/G$ coset is used. Roughly speaking, one can associate one of the two currents with the matter sector of the string whereas the other current is related to the Liouville degrees of freedom. Moreover, some of the ghosts of the deformed $G/G$ coset can be identified with those of string theory. The remaining fields coming from the ghost and current sectors can be organised into topological quartets and one can invoke the standard Kugo-Ojima confinement mechanism to eliminate these quartets from the physical Hilbert space.

Before studying the osp(1|2) and osp(2|2) cases, let us for completeness recall [10,11] the relation between the $sl(N)/sl(N)$ cosets and the non-critical $W_N$-strings. First of all let us briefly describe the root system of the $sl(N)$ Lie algebra. If $\vec{e}_i$ is a unitary vector in $I\mathbb{R}^N$ along the $i$th axis and $\vec{e}_i = \vec{e}_i - \vec{e}_j$, then the positive roots of $sl(N)$ are the elements of the set $\Delta_+ = \{ \vec{e}_{ij}, \ j > i \}$, while the simple roots are given by $\vec{\alpha}_i = \vec{e}_{i,i+1}$ ($i = 1, \cdots, N - 1$). Any positive root $\vec{\alpha} \in \Delta_+$ can be written as $\vec{\alpha} = \sum_{i=1}^{N-1} n^i_\alpha \vec{\alpha}_i$ where the $n^i_\alpha$'s are non-negative integers. The height of $\vec{\alpha}$ is given by:

$$h_\alpha = \sum_{i=1}^{N-1} n^i_\alpha. \quad (4.1)$$
In particular, for \( j > i \), \( h_{\epsilon_{ij}} = j - i \). Denoting by \( \delta \) half the sum of positive roots (i.e. \( \delta = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \)), the height of any \( \alpha \in \Delta_+ \) is simply given by \( h_\alpha = \delta \cdot \alpha \).

We adopt the free field realisation of the \( \mathfrak{sl}(N) \) current algebra of ref. [33]. In this realisation, a spin-one bosonic \( \beta \gamma \) system is introduced for every positive root \( \alpha \in \Delta_+ \). If we call these bosonic fields \( (w_\alpha, \chi_\alpha) \), then the expression for the currents associated with the positive roots is of the form:

\[
J_\alpha = w_\alpha + \cdots, \tag{4.2}
\]

where the dots denote terms which are non-linear in the fields. Notice that the conformal weights of \( (w_\alpha, \chi_\alpha) \) are \( \Delta(w_\alpha) = 1 \) and \( \Delta(\chi_\alpha) = 0 \). One also needs to introduce a set of \( N - 1 \) scalar fields \( \tilde{\phi} = (\phi_1, \cdots, \phi_{N-1}) \). Then the expression for the Cartan currents can be given in general. In fact if we represent by \( \tilde{\mu} \cdot \tilde{H} \) the current along the direction of an arbitrary Cartan vector \( \tilde{\mu} \), we have:

\[
\tilde{H} = i\sqrt{k + N} \partial \tilde{\phi} - \sum_{\alpha \in \Delta_+} \tilde{\alpha} w_\alpha \chi_\alpha, \tag{4.3}
\]

where \( k \) is the level of the \( \mathfrak{sl}(N) \) algebra. It is also possible to give a simple expression for the Sugawara energy-momentum tensor in terms of these free fields:

\[
T^J = \frac{1}{2(k + N)} g^{ab} J_a J_b = \sum_{\alpha \in \Delta_+} w_\alpha \partial \chi_\alpha - \frac{1}{2} (\partial \tilde{\phi})^2 - \frac{i}{\sqrt{k + N}} \tilde{\delta} \cdot \partial^2 \tilde{\phi}. \tag{4.4}
\]

A simple calculation using the Freudenthal-de Vries “strange” formula \( (12 \tilde{\delta}^2 = N(N^2 - 1)) \) shows that the energy-momentum tensor in eq. (4.4) indeed has the correct central charge for an affine algebra at level \( k \).

The topological \( \mathfrak{sl}(N)/\mathfrak{sl}(N) \) coset is obtained by combining two \( \mathfrak{sl}(N) \) currents with levels \( k_1 = k \) and \( k_2 = -k - 2N \). Let the corresponding free fields carry the
labels 1 and 2. We must also add a pair of fermionic ghosts for each independent current direction. In what follows we shall denote the ghost along the Cartan direction by \((\rho^i, \gamma^i)\) \((i = 1, \cdots, N - 1)\) and those associated with the positive (negative) roots of the algebra by \((\rho_\alpha, \gamma_\alpha)\) \((\rho_{-\alpha}, \gamma^{-\alpha})\) respectively.

In order to make contact with non-critical string theory we must first deform the theory. It turns out that the appropriate deformation of the total energy-momentum tensor \(T\) is:

\[
T_{\text{improved}} = T + \vec{\delta} \cdot \partial \vec{H}.
\]

(4.5)

In eq. (4.5) \(\vec{H}\) is the total Cartan current of the \(\text{sl}(N)/\text{sl}(N)\) coset (see eq. (2.42)). Let us separate in \(T_{\text{improved}}\) the contribution of the currents from those of the ghosts:

\[
T_{\text{improved}} = T^J_{\text{improved}} + T^\text{gh}_{\text{improved}}.
\]

(4.6)

Using the free-field representation of \(T^J\) and \(\vec{H}\) given in eqs. (4.4) and (4.3) we can write the explicit expression of \(T^J_{\text{improved}}\):

\[
T^J_{\text{improved}} = -\frac{1}{2} (\partial \tilde{\phi}_1)^2 - \frac{1}{2} (\partial \tilde{\phi}_2)^2 + i \frac{t+1}{\sqrt{t}} \tilde{\delta} \cdot \partial^2 \tilde{\phi}_1 - \frac{t+1}{\sqrt{t}} \tilde{\delta} \cdot \partial^2 \tilde{\phi}_2 +
\]

\[
+ \sum_{i=1,2} \sum_{\alpha \in \Delta^+} [ (1 - h_\alpha) w_\alpha^i \partial \chi_\alpha^i - h_\alpha \partial w_\alpha \chi_\alpha^i ],
\]

(4.7)

where \(t = k + N\). Notice that, in the deformed theory, the fields \((w_\alpha, \chi_\alpha)\) acquire a conformal weight that depends on the height of the root \(\tilde{\alpha}\) \((\Delta(w_\alpha) = 1 - h_\alpha, \Delta(\chi_\alpha) = h_\alpha)\). In order to compute the ghost contribution to the improved energy-momentum tensor, we need to know the part of \(\vec{H}\) that depends on the ghost fields. From the commutation relations of \(\text{sl}(N)\) one easily gets:

\[
\vec{H}^\text{gh} = \sum_{\alpha \in \Delta^+} \tilde{\alpha} [ \gamma^\alpha \rho_\alpha - \gamma^{-\alpha} \rho_{-\alpha} ].
\]

(4.8)
Using eq. (4.8) one obtains

\[
T_{\text{improved}}^{\text{gh}} = \sum_{i=1}^{N-1} \rho_i \partial \gamma^i + \sum_{\alpha \in \Delta_+} [(1 - h_\alpha) \rho_\alpha \partial \gamma^\alpha + h_\alpha \gamma^\alpha \partial \rho_\alpha] + 
\]

\[
+ \sum_{\alpha \in \Delta_+} [(1 + h_\alpha) \rho_{-\alpha} \partial \gamma^{-\alpha} - h_\alpha \gamma^{-\alpha} \partial \rho_{-\alpha}].
\]

The central charge of the field $\bar{\phi}_1$ can be computed from the background charge displayed in eq. (4.7). A simple calculation shows that

\[
c_{\bar{\phi}_1} = (N - 1) [1 - N(N + 1) \frac{(t - 1)^2}{t}].
\]

When $t = \frac{q}{p}$ with $q, p \in \mathbb{Z}$ (i.e. when $k + N = \frac{q}{p}$), the central charge in eq. (4.10) is precisely that of the minimal $(p, q)$ model of $W_N$ matter. It is also easy to check from eq. (4.10) that $\bar{\phi}_2$ has the correct background charge to be considered as the $W_N$-Liouville field. Moreover it can be seen that one can always combine in a Kugo-Ojima topological quartet the $(w, \chi)$ systems with ghost fields having the same conformal weights. We can pair, for example, the $(w_1^1, \chi_1^1)$ fields with the ghosts $(\rho_\alpha, \gamma^\alpha)$ corresponding to the positive roots. Also the Cartan ghosts $(\rho_i, \gamma^i)$ can be paired with the fields $(w_\alpha^2, \chi_\alpha^2)$ when $\tilde{\alpha}$ is a simple root (i.e. when $h_\alpha = 1$), since in this case both systems have dimensions $(1, 0)$ and there are equal number of them. The $(w_\alpha^2, \chi_\alpha^2)$ fields with $h_\alpha \geq 2$ can be paired with some of the $(\rho_{-\alpha}, \gamma^{-\alpha})$ ghosts. By looking at the conformal weights of these last two systems one concludes that, in order to combine $(w_\alpha^2, \chi_\alpha^2)$ with $(\rho_{-\alpha'}, \gamma^{-\alpha'})$, the heights of $\tilde{\alpha}$ and $\tilde{\alpha}'$ must satisfy $h_\alpha - h_{\alpha'} = 1$. A simple calculation tells one how many fields can be paired in this way. Since the number of roots of $\mathfrak{sl}(N)$ with height $h$ is $N - h$, the difference between the number of $(\rho_{-\alpha'}, \gamma^{-\alpha'})$ and $(w_\alpha^2, \chi_\alpha^2)$ systems is $N - h_{\alpha'} - (N - h_\alpha) = h_\alpha - h_{\alpha'} = 1$. It follows that, for a given height $h$, there always remains one unpaired $(\rho, \gamma)$ system with conformal weights $(1 + h, -h)$. We are thus left with a set of $N - 1$ anticommuting ghost fields with conformal weights $(2, -1), \ldots, (N, 1 - N)$. Let us denote these fields by $(b_j, c_j)$ where the
conformal weight of $b_j$ is $j + 1$ for $j = 1, \cdots, N - 1$. Notice that they correspond to the ghost system of the $W_N$ string. Therefore, writing $\vec{\phi}_M$ ($\vec{\phi}_L$) instead of $\vec{\phi}_1$ ($\vec{\phi}_2$ respectively), the reduced energy-momentum tensor is given by:

$$T_{\text{reduced}} = -\frac{1}{2} (\partial \vec{\phi}_M)^2 - \frac{1}{2} (\partial \vec{\phi}_L)^2 + i \frac{t - 1}{\sqrt{t}} \vec{\delta} \cdot \partial^2 \vec{\phi}_M - \frac{t + 1}{\sqrt{t}} \vec{\delta} \cdot \partial^2 \vec{\phi}_L +$$

$$+ \sum_{j=1}^{N-1} [(j + 1)b_j \partial c_j - j c_j \partial b_j],$$

which, as stated above, corresponds to that of matter coupled to $W_N$-gravity.

Next let us consider the osp(1|2) Lie superalgebra. This algebra contains three bosonic currents $J_\pm$ and $H$ that close an sl(2) algebra at level $k$. In addition there are two fermionic currents that we shall denote by $j_\pm$. The affine osp(1|2) superalgebra can be realised [18] in terms of one scalar field $\phi$, one bosonic $\beta\gamma$ system (denoted by $(w, \chi)$), and one fermionic $bc$ system (denoted by $(\bar{\psi}, \psi)$), with dimensions $\Delta(w) = \Delta(\bar{\psi}) = 1$ and $\Delta(\chi) = \Delta(\psi) = 0$, satisfying the following basic OPE’s:

$$w(z) \chi(w) = \psi(z) \bar{\psi}(w) = \frac{1}{z - w} \quad \phi(z) \phi(w) = -\log(z - w). \quad (4.12)$$

In terms of these fields the explicit form of the osp(1|2) currents is:

$$J_+ = w$$

$$J_- = -w \chi^2 + i\sqrt{2k + 3} \chi \partial \phi - \chi \psi \bar{\psi} + k \partial \chi + (k + 1) \psi \partial \psi$$

$$H = -w \chi + i\sqrt{2k + 3} \partial \phi - \frac{1}{2} \psi \bar{\psi}$$

$$j_+ = \bar{\psi} + w \psi$$

$$j_- = -\chi(\bar{\psi} + w \psi) + i\sqrt{2k + 3} \psi \partial \phi + (2k + 1) \partial \psi. \quad (4.13)$$

Notice that for osp(1|2), with our conventions, $C_A = 3$. Using this value and the metric tensor extracted from OPE’s of the currents in eq. (4.13), we can write the
Sugawara energy-momentum tensor:

\[ T^J = \frac{1}{2k+3} [J_+ J_- + J_- J_+ + 2H^2 - \frac{1}{2} j_+ j_- + \frac{1}{2} j_- j_+]. \quad (4.14) \]

Substituting the representation given in eq. (4.13) into eq. (4.14) one gets:

\[ T^J = w \partial \chi - \bar{\psi} \partial \psi - \frac{1}{2} (\partial \phi)^2 - \frac{i}{2 \sqrt{2k+3}} \partial^2 \phi. \quad (4.15) \]

In order to realise the topological osp(1|2)/osp(1|2) coset model we must combine two current systems with levels \( k_1 = k \) and \( k_2 = -k - 3 \). Let us denote the anticommuting ghost systems for the currents \( H \) and \( J_\pm \) by \( (\rho_0, \gamma^0) \) and \( (\rho_\pm, \gamma^\pm) \) respectively. The commuting ghosts associated to the \( j_\pm \) currents will be similarly denoted by \( (\lambda_\pm, \eta^\pm) \). In complete parallel with the \( \text{sl}(N) \) case, let us deform the energy-momentum tensor by adding a derivative along the total Cartan current \( \mathcal{H} \):

\[ T^J_{\text{improved}} = T^J_{\text{improved}} + T^\text{gh}_{\text{improved}} = T + \partial \mathcal{H}. \quad (4.16) \]

In eq. (4.16) we have separated the contributions of the currents from those of the ghosts. Let us consider \( T^J_{\text{improved}} \) first. Using eq. (4.15) one immediately arrives at:

\[ T^J_{\text{improved}} = -\frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 + \frac{i}{2} \frac{t - 1}{\sqrt{t}} \partial^2 \phi_1 - \frac{1}{2} \frac{t + 1}{\sqrt{t}} \partial^2 \phi_2 - \sum_{i=1,2} \left[ \partial w^i \chi^i + \frac{1}{2} \bar{\psi}^i \partial \psi^i + \frac{1}{2} \psi^i \partial \bar{\psi}^i \right], \quad (4.17) \]

where now \( t = 2k + 3 \) and, as in the \( \text{sl}(N) \) case, the indices 1 and 2 label the two currents. On the other hand, from the basic OPE’s of osp(1|2), the contribution
of the ghost fields to $\mathcal{H}$ is easily obtained. One gets:

$$
\mathcal{H}^{gh} = \gamma^+ \rho_+ - \gamma^- \rho_- - \frac{1}{2} \eta^+ \lambda_+ + \frac{1}{2} \eta^- \lambda_-. 
$$

(4.18)

Taking eq. (4.18) into account it is straightforward to compute $T_{\text{improved}}^{gh}$:

$$
T_{\text{improved}}^{gh} = \rho_0 \partial \gamma^0 + \gamma^+ \partial \rho_+ + 2 \rho_- \partial \gamma^- - \gamma^- \partial \rho_- \\
+ \frac{1}{2} \partial \eta^+ \lambda_+ - \frac{1}{2} \eta^+ \partial \lambda_+ + \frac{3}{2} \partial \eta^- \lambda_- + \frac{1}{2} \eta^- \partial \lambda_-.
$$

(4.19)

Notice that in the deformed theory the conformal weights of the antighost fields are:

$$
\Delta(\rho_0) = 1 \quad \Delta(\rho_+) = 0 \quad \Delta(\rho_-) = 2 \quad \Delta(\lambda_+) = \frac{1}{2} \quad \Delta(\lambda_-) = \frac{3}{2}.
$$

(4.20)

Therefore $(\rho_-, \gamma^-)$ and $(\lambda_-, \eta^-)$ have acquired the right dimensions to become the superdiffeomorphism ghosts of the $N = 1$ string. A glance at eqs. (4.19) and (4.17) reveals that the other ghosts can be accommodated in quartets with fields coming from the current sector. Indeed one can pair $(\rho_0, \gamma^0)$ and $(\rho_+, \gamma^+)$ with $(w^1, \chi^1)$ and $(w^2, \chi^2)$. The commuting ghosts $(\lambda_+, \eta^+)$ can be paired with the $(\frac{1}{2}, \frac{1}{2})$ fermionic system obtained from, say, the fields $\frac{1}{\sqrt{2}}(\psi^1 + \bar{\psi}^1)$ and $\frac{1}{\sqrt{2}}(\psi^2 + \bar{\psi}^2)$. After this process there remain two Majorana fields $\psi_M = \frac{i}{\sqrt{2}}(\psi^1 - \bar{\psi}^1)$ and $\psi_L = \frac{i}{\sqrt{2}}(\psi^2 - \bar{\psi}^2)$. Calling $\phi_M, \phi_L, b, c, \beta$ and $\gamma$ to $\phi_1, \phi_2, \rho_-, \gamma^-, \lambda_-$ and $\eta^-$ respectively, we can write the reduced energy-momentum tensor as:

$$
T_{\text{reduced}} = -\frac{1}{2} (\partial \phi_M)^2 - \frac{1}{2} (\partial \phi_L)^2 + \frac{i}{2} \frac{t-1}{\sqrt{t}} \partial^2 \phi_M - \frac{1}{2} \frac{t+1}{\sqrt{t}} \partial^2 \phi_L \\
- \frac{1}{2} \psi_M \partial \psi_M - \frac{1}{2} \psi_L \partial \psi_L + 2 b \partial c - c \partial b + \frac{3}{2} \partial \gamma \beta + \frac{1}{2} \gamma \partial \beta,
$$

(4.21)

which is indeed the one corresponding to the $N = 1$ RNS superstring. Furthermore the matter central charge in $T_{\text{reduced}}$ is:

$$
c_M = \frac{3}{2} \left( 1 - 2 \frac{(t-1)^2}{t} \right).
$$

(4.22)

When $t = 2k + 3 = \frac{2}{p}$ with $p, q \in \mathbb{Z}$, eq. (4.22) gives the central charge of the minimal models of the $N = 1$ superconformal symmetry.
To finish this section let us now analyse the osp(2|2) current algebra. This algebra contains four bosonic currents \((J_\pm, H, J)\) along with other four fermionic ones \((j_{\pm\pm})\). The currents \(J_\pm\) and \(H\) close an \(sl(2)\) algebra, while \(J\) is a \(U(1)\) current. One can represent this algebra by means of two scalar fields \(\phi\) and \(\varphi\), one \( (1,0) \) commuting \( \beta\gamma \) system (denoted by \((w, \chi)\)) and two \((1,0)\) anticommuting \(bc\) systems \(((\bar{\psi}_+, \psi_-)\) and \((\bar{\psi}_-, \psi_+)\)). We shall use the conventions of eq. (4.12) for the OPE’s of the fields \((w, \chi)\), \(\phi\) and \(\varphi\). For the fermionic fields, the basic OPE’s are:

\[
\psi_+(z)\bar{\psi}_-(w) = \psi_-(z)\bar{\psi}_+(w) = \frac{1}{z-w},
\]

(4.23)

Then the explicit representation\[18\] of the osp(2|2) currents is:

\[
\begin{align*}
J_+ &= w \\
J_- &= -w\chi^2 + i\sqrt{2k+2}\ \chi\partial\phi - \chi(\psi_+\bar{\psi}_- + \psi_-\bar{\psi}_+) \\
&\quad - \sqrt{2k+2}\ \psi_+\psi_-\partial\phi + k\partial\chi + (k+1)[\psi_-\partial\psi_+ + \psi_+\partial\psi_-] \\
H &= -w\chi + \frac{i}{\sqrt{2k+2}}\ \partial\phi - \frac{1}{2}[\psi_-\bar{\psi}_+ + \psi_+\bar{\psi}_-] \\
J &= -\frac{1}{2}[\psi_-\bar{\psi}_+ - \psi_+\bar{\psi}_-] + \frac{\sqrt{2k+2}}{2}\ \partial\phi \\
j_{\pm\pm} &= \bar{\psi}_\pm + w\psi_\pm \\
j_- &= -\chi(\bar{\psi}_\pm + w\psi_\pm) + \sqrt{2k+2}\ \psi_\pm\partial(\pm\varphi) + (2k+1)\partial\psi_\pm + \psi_\pm\psi_\mp\bar{\psi}_\pm.
\end{align*}
\]

(4.24)

A direct computation using the operator algebra closed by the currents of eq. (4.24) shows that \(C_A = 2\) for osp(2|2). This same calculation yields the values of the metric tensor. Using these values we can write down the Sugawara tensor:

\[
T^J = \frac{1}{2k+2} [J\bar{J} + J\bar{J} + 2H^2 - 2J^2 - \frac{1}{2}(j++ j-- j-- j++ + j++ j-- - j-- j++)].
\]

(4.25)

Taking eq.(4.24) into account, one can obtain the expression of \(T^J\) in terms of the
free fields. After some calculation one gets:

\[ T^J = w \partial \chi - \bar{\psi}_+ \partial \psi_- - \bar{\psi}_- \partial \psi_+ - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \varphi)^2. \] (4.26)

Notice that the fields \( \phi \) and \( \varphi \) have vanishing background charges. It is also evident by inspecting eq. (4.26) that the central charge of the osp(2|2) WZW model is zero. This also follows from the fact that \( d_B = d_F = 4 \) for osp(2|2) (see eq. (2.11)).

As in the osp(1|2) case, we shall denote the ghosts for the currents \( H \) and \( J_{\pm} \) by \( (\rho_0, \gamma^0) \) and \( (\rho_{\pm}, \gamma^\pm) \), whereas \( (\rho_J, \gamma^J) \) and \( (\lambda_{\pm \pm}, \eta_{\pm \pm}) \) will correspond to the currents \( J \) and \( j_{\pm \pm} \) respectively. The topological osp(2|2) current system will be realised by combining these ghosts with two currents whose levels are \( k_1 = k \) and \( k_2 = -k - 2 \). In order to make contact with the non-critical \( N = 2 \) superstring we must first improve the energy-momentum tensor. Let us assume that we deform the total operator \( T \) as \( T \to T + \partial H \), where \( H \) is the total Cartan current along the \( H \)-direction. The contributions to \( H \) of the currents with levels \( k \) and \( -k - 2 \) can be read from the third equation in (4.24). Moreover, using the structure constants of the osp(2|2) algebra, it is easy to compute the ghost contribution to \( H \). One gets:

\[ H_{gh} = \gamma^+ \rho_+ - \gamma^- \rho_- - \frac{1}{2} \eta^{++} \lambda_{++} - \frac{1}{2} \eta^{+-} \lambda_{+-} + \]

\[ + \frac{1}{2} \eta^{-+} \lambda_{-+} + \frac{1}{2} \eta^{--} \lambda_{--}. \] (4.27)

The improved energy-momentum tensor of the ghosts can be easily computed from eq.(4.27):

\[ T_{\text{improved}}^{gh} = \rho_0 \partial \gamma^0 + \rho_J \partial \gamma^J + \gamma^+ \partial \rho_+ + 2 \rho_- \partial \gamma^- - \gamma^- \partial \rho_- + \]

\[ + \sum_{\alpha = \pm} \left[ \frac{1}{2} \partial \eta^{+\alpha} \lambda_{+\alpha} - \frac{1}{2} \eta^{+\alpha} \partial \lambda_{+\alpha} + \frac{3}{2} \partial \eta^{-\alpha} \lambda_{-\alpha} + \frac{1}{2} \eta^{-\alpha} \partial \lambda_{-\alpha} \right]. \] (4.28)

Therefore the conformal weights that the different antighosts acquire after the
deformation are:

\[ \Delta(\rho_0) = \Delta(\rho_J) = 1 \quad \Delta(\rho_+) = 0 \quad \Delta(\rho_-) = 2 \]

\[ \Delta(\lambda_{\pm}) = \frac{1}{2} \quad \Delta(\lambda_{\mp}) = \frac{3}{2}. \]

Moreover, from eqs. (4.26) and (4.24), one can obtain the improved energy-momentum tensor in the current sector. If we label with the indices 1 and 2 the free fields coming from the two currents, it is straightforward to arrive at the result:

\[ T_{\text{improved}}^I = \sum_{i=1,2} \left[ -\frac{1}{2} (\partial \phi_i)^2 - \frac{1}{2} (\partial \bar{\phi}_i)^2 \right] + \frac{i}{2} \sqrt{2k+2} \partial^2 \phi_1 - \frac{1}{2} \sqrt{2k+2} \partial^2 \phi_2 - \sum_{i=1,2} \left[ \partial w^i \chi^i + \frac{1}{2} \bar{\psi}_-^i \partial \psi_-^i + \frac{1}{2} \bar{\psi}_+^i \partial \psi_+^i + \frac{1}{2} \psi_+^i \partial \bar{\psi}_-^i + \frac{1}{2} \psi_-^i \partial \bar{\psi}_+^i \right]. \]

(4.30)

Let us now see how one can pair ghost fields from eq.(4.28) with fields in eq. (4.30) in such a way that, after the reduction, we are left with the field content of the non-critical \( N = 2 \) superstring. Indeed, as in the osp(1|2) case, we can pair the systems \((\rho_0, \gamma^0)\) and \((\rho_+, \gamma^+)\) with \((w^1, \chi^1)\) and \((w^2, \chi^2)\). Moreover we can form a quartet with the ghosts \((\lambda_{\pm \pm}, \eta_{\pm \pm})\) and the \((\frac{1}{2}, \frac{1}{2})\) anticommuting system formed from the Majorana fermions \(\frac{1}{2}(\psi^1_\pm + \bar{\psi}^1_\mp)\) and \(\frac{1}{2}(\psi^2_\pm + \bar{\psi}^2_\mp)\). The remaining fields can be assigned to matter and Liouville degrees of freedom. Let us first consider the bosonic fields. If the labels M and L refer matter and Liouville fields, we can define:

\[ \phi_M = \frac{1}{\sqrt{2}} (\phi_1 + i\varphi_1) \quad \bar{\phi}_M = \frac{1}{\sqrt{2}} (\phi_1 - i\varphi_1) \]

\[ \phi_L = \frac{1}{\sqrt{2}} (\phi_2 + i\varphi_2) \quad \bar{\phi}_L = \frac{1}{\sqrt{2}} (\phi_2 - i\varphi_2). \]

(4.31)
Similarly we can define the Dirac fermionic fields:

$$\psi = \frac{i}{2} \left[ \psi^1_+ - \psi^1_- + i(\psi^1_+ - \psi^1_-) \right] \quad \bar{\psi} = \frac{i}{2} \left[ \psi^1_+ - \psi^1_- - i(\psi^1_+ - \psi^1_-) \right]$$

$$\psi_L = \frac{i}{2} \left[ \psi^2_+ - \psi^2_- + i(\psi^2_+ - \psi^2_-) \right] \quad \bar{\psi}_L = \frac{i}{2} \left[ \psi^2_+ - \psi^2_- - i(\psi^2_+ - \psi^2_-) \right]. \quad (4.32)$$

It should be noticed that $\psi_M$ and $\bar{\psi}_M$ ($\psi_L$ and $\bar{\psi}_L$) are the components of the fermionic fields used to represent the osp(2|2) current at level $k_1$ (respectively $k_2$) that are not affected by the reduction described above. Splitting the reduced energy-momentum tensor as:

$$T_{\text{reduced}} = T_{\text{M+L}} + T_{\text{gh}} \quad (4.33)$$

we can easily write down the matter + Liouville contributions. Using the definitions (4.31) and (4.32), one arrives at:

$$T_{\text{M+L}} = - \partial \phi_M \partial \bar{\phi}_M - \partial \phi_L \partial \bar{\phi}_L + \frac{\sqrt{k+1}}{2} \left( i \partial^2 (\phi_M + \bar{\phi}_M) - \partial^2 (\phi_L + \bar{\phi}_L) \right) -$$

$$- \frac{1}{2} \bar{\psi}_M \partial \psi_M - \frac{1}{2} \psi_M \partial \bar{\psi}_M - \frac{1}{2} \bar{\psi}_L \partial \psi_L - \frac{1}{2} \psi_L \partial \bar{\psi}_L. \quad (4.34)$$

The ghost part of $T_{\text{reduced}}$ can be read form eq. (4.28). Relabelling the ghost fields unaffected by the reduction as $\rho_\pm \to b$, $\gamma^- \to c$, $\rho_J \to \tilde{b}$, $\gamma^J \to \tilde{c}$, $\lambda_{\pm} \to \beta_\pm$ and $\eta^- \to \gamma^\pm$, one can finally write $T_{\text{gh}}$ in the form:

$$T_{\text{gh}} = 2 b \partial c - c \partial b + \frac{3}{2} \partial \gamma^+ \beta_+ + \frac{1}{2} \gamma^+ \partial \beta_+ + \frac{3}{2} \partial \gamma^- \beta_- + \frac{1}{2} \gamma^- \partial \beta_- + \tilde{b} \partial \tilde{c}, \quad (4.35)$$

which indeed coincides with the energy-momentum tensor for the ghosts of the $N = 2$ string. Notice that the contribution of this $N = 2$ ghost system to the
Virasoro central charge is $-6$. Moreover the matter central charge $c_M$ and the osp(2|2) level $k$ are now related as:

$$c_M = -6k - 3,$$  \hspace{1cm} (4.36)

in complete agreement with ref. [18]

5. Summary and conclusions.

In this paper we have analysed the topological structure associated with the Kač–Moody symmetry based on an arbitrary Lie superalgebra $G$. The resulting topological algebra turns out to be an extension of the twisted $N = 2$ superconformal algebra [29,8] by a BRST doublet of spin 3 operators, upon whose introduction the algebra closes linearly. Our method consists in introducing a ghost sector that allows to define a BRST cohomology, in such a way that all operators appear as BRST doublets. The resulting theory has been interpreted as a topological sigma model for supergroup manifolds, i.e., as a $G/G$ coset.

By means of free field representations we have related the osp(1|2) and osp(2|2) models with the $N = 1$ and $N = 2$ non-critical superstrings, respectively. We have established that, after performing a suitable deformation of the coset model, one can identify the matter, Liouville and ghost sectors of the coset with those of the corresponding string theory. Moreover the remaining degrees of freedom can be accommodated in topological doublets.

In this paper we have restricted ourselves to comparing the field content of the deformed coset models and non-critical superstrings. It remains to see whether or not the BRST cohomologies of both models are related. For bosonic $W_N$-strings this relation has been shown to exist [10,11,12] and thus one expects a similar result for the supersymmetric string models (see ref. [32] for an analysis of the osp(1|2) case). There are many other questions that remain open in the relation between the $G/G$ cosets and string theories. For example, one would expect to have a
current algebra prescription to compute correlation functions in non-critical string theories. Another problem that, in our opinion, deserves future investigation, is the relation between the $\mathcal{G}/\mathcal{G}$ BRST symmetry and the topological symmetry of non-critical strings discovered in ref. [34] (see also [35]).

The $\mathcal{G}/\mathcal{G}$ coset models can also be regarded as models of topological matter. The coupling of this $\mathcal{G}/\mathcal{G}$ matter to topological gravity gives rise to a topological string model. A prescription for this coupling was given in ref. [30] at the level of the operator algebra. The extended character of the BRST algebra plays an important role in this analysis. It would be interesting to understand this coupling within a lagrangian formalism. This could possibly shed light on the nature and implications of the BRST symmetry of current algebras.

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