Path Integral Quantization of Noncommutative Complex Scalar Field

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Abstract

Using noncommutative deformed canonical commutation relations, a model describing a noncommutative complex scalar field theory is considered. Using the path integral formalism, the noncommutative free and exact propagators are calculated to one-loop order and to the second order in the parameter of noncommutativity. Dimensional regularization was used to remove ultraviolet divergences that arise from loop graphs. It has been shown that these divergences may also be absorbed into a redefinition of the parameters of the theory.

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1. INTRODUCTION

Noncommutative geometry is presently one of the most important and extremely active area of research in theoretical physics, there is now a common belief that the usual picture of space-time as a smooth pseudo-Riemannian manifold should breakdown at very short distances of the order of the Planck length, due to the quantum gravity effects. The concept of noncommutative space-time was suggested very early on by the founding fathers of quantum mechanics and quantum field theory. This was motivated by the need to remove the divergences which had plagued quantum electrodynamics. However, this suggestion was ignored \[1\]. In recent years, the idea of noncommutative space-time has attracted considerable interest, and has penetrated into various fields in physics and mathematical physics, it leads to the investigation of some new and more fundamental physical and mathematical notions, the motivation for this kind of investigation is that the effects of noncommutativity of space may appear at very short distances of the order of the Planck length, or at very high energies, this may shed a light on the real microscopic geometry and structure of our universe \[1\]-\[14\]. One of the new features of noncommutative field theories is the UV/IR mixing phenomenon, in which the physics at high energies affects the physics at low energies, which does not occur in quantum field theories in which the coordinates commute \[1\]-\[12\]-\[13\]. Thus in the noncommutative space-time approach, the dynamical variables become operators, and, therefore, the formalism of the quantum field theory constructions must be modified. The discovery of noncommutative geometry has allowed the exploration of new directions in theoretical physics, in particular, several aspects of noncommutative quantum mechanics and two-dimensional noncommutative harmonic oscillators are an extremely active area of research and have been discussed extensively from different points of view \[15\]-\[34\].

Our paper is organized as follows: In Section 2, we consider a noncommutative action for a complex scalar field with self interaction, in section 3, we briefly recall basic results of path integral derivation of propagator and renormalization of $\varphi^4$ theory, in section 4, we consider the path integral derivation of the noncommutative free and exact propagators, in section 5 we consider the dimensional regularization of the noncommutative exact propagator, Finally in section 6, we draw our conclusions.
2. NONCOMMUTATIVE ACTION

In [35] a model describing a noncommutative complex scalar field theory, based on noncommutative deformed canonical commutation relations, was considered, the model has been quantized via the Peierls bracket [37]. Here we will use the path integral formalism to quantize the noncommutative complex scalar field theory.

Consider a complex scalar field \( \Phi(x) \) with Lagrangian density given by [36]-[40]

\[
\mathcal{L} = -\left(\partial_{\mu}\Phi\right)^{*}\left(\partial^{\mu}\Phi\right) - m^{2}\Phi^{*}\Phi - g\left(\Phi^{*}\Phi\right)^{2}
\]  

(1)

where \( m \) is the mass of the charged particles, and \( g \) is a positive parameter. The metric signature will be assumed to be \(-+++,\) in what follows, we take \( \hbar = c = 1. \)

The complex scalar field can be quantized using the canonical quantization rules, for this we express it in terms of its real and imaginary parts as \( \Phi = \frac{1}{\sqrt{2}}\left(\varphi_{1} + i\varphi_{2}\right) \), where \( \varphi_{1}, \varphi_{2} \) are real scalar fields; in terms of these real scalar fields the Lagrangian density reads

\[
\mathcal{L} = -\frac{1}{2}\left(\partial_{\mu}\varphi_{a}\right)^{2} - \frac{1}{2}m^{2}\left(\varphi_{a}\right)^{2} - \frac{1}{4}g\left(\varphi_{a}\varphi_{a}\right)^{2} = -\frac{1}{2}\left(\partial_{\mu}\varphi_{a}\right)^{2} - \frac{1}{2}\mu^{2}\left[\varphi\right]\left(\varphi_{a}\right)^{2}
\]  

(2)

where \( \mu^{2}\left[\varphi\right] = m^{2} + \frac{1}{2}g\left(\varphi_{a}\right)^{2} \).

Let \( \pi_{a} \) be the canonical conjugate to \( \varphi_{a} \)

\[
\pi_{a} = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_{a}} = \dot{\varphi}_{a}
\]  

(3)

The Hamiltonian density reads then

\[
\mathcal{H} = \pi_{a}\dot{\varphi}_{a} - \mathcal{L} = \frac{1}{2}\left(\pi_{a}\right)^{2} + \frac{1}{2}\left(\nabla\varphi_{a}\right)^{2} + \frac{1}{2}\mu^{2}\left[\varphi\right]\left(\varphi_{a}\right)^{2}
\]  

(4)

In the canonical quantization the canonical variables \( \varphi_{a} \) and the canonical conjugates \( \pi_{a} \) are assumed to be operators satisfying the canonical commutation relations

\[
\left[\varphi_{a}(t, \vec{x}), \pi_{b}(t, \vec{y})\right] = i\delta_{ab}\delta^{3}(\vec{x} - \vec{y})
\]

(5)

\[
\left[\varphi_{a}(t, \vec{x}), \varphi_{b}(t, \vec{y})\right] = 0
\]

\[
\left[\pi_{a}(t, \vec{x}), \pi_{b}(t, \vec{y})\right] = 0
\]

It is well known, since the birth of quantum field theory in the papers of Born, Dirac, Fermi, Heisenberg, Jordan, and Pauli, that the free field behaves like an infinite number
of coupled harmonic oscillators [36], using this analogy between free fields and an infinite number of coupled harmonic oscillators, one can impose noncommutativity on the configuration space of dynamical fields \( \varphi_a \), to do this we recall that the two-dimensional harmonic oscillator noncommutative configuration space can be realized as a space where the coordinates \( \hat{x}_a \), and the corresponding noncommutative momentum \( \hat{p}_a \), are operators satisfying the commutation relations

\[
[\hat{x}_a, \hat{x}_b] = i\theta^2 \varepsilon_{ab} \quad [\hat{p}_a, \hat{p}_b] = 0 \quad [\hat{x}_a, \hat{p}_b] = i\delta_{ab}
\]  

(6)

where \( \theta \) is a parameter with dimension of length, and \( \varepsilon_{ab} \) is an antisymmetric constant matrix.

It is well known that this noncommutative algebra can be mapped to the commutative Heisenberg-Weyl algebra [32]-[34]

\[
[x_a, x_b] = 0 \quad [p_a, p_b] = 0 \quad [x_a, p_b] = i\delta_{ab}
\]  

(7)

through the relations

\[
\hat{x}_a = x_a - \frac{1}{2} \theta^2 \varepsilon_{ab} p_b \quad \hat{p}_a = p_a
\]  

(8)

To impose noncommutativity on the configuration space of dynamical fields \( \varphi_a \), we assume that the noncommutative canonical variables \( \hat{\varphi}_a \) and the noncommutative canonical conjugates \( \hat{\pi}_a \) satisfy the noncommutative commutation relations

\[
[\hat{\varphi}_a (t, \vec{x}), \hat{\pi}_b (t, \vec{y})] = i\delta^3 (\vec{x} - \vec{y}) \delta_{ab}
\]  

\[
[\hat{\varphi}_a (t, \vec{x}), \hat{\varphi}_b (t, \vec{y})] = i\theta \varepsilon_{ab} \delta^3 (\vec{x} - \vec{y})
\]

\[
[\hat{\pi}_a (t, \vec{x}), \hat{\pi}_b (t, \vec{y})] = 0
\]

(9)

where \( \theta \) is the parameter of noncommutativity, which is assumed to be a constant, and \( \varepsilon_{ab} \) is a \( 2 \times 2 \) real antisymmetric matrix

\[
\varepsilon_{12} = -\varepsilon_{21} = 1
\]  

(10)

The noncommutative Hamiltonian density is assumed to have the form
\[ \hat{H} = \frac{1}{2} \hat{\pi}_a^2 + \frac{1}{2} \left( \hat{\nabla} \hat{\varphi}_a \right)^2 + \frac{1}{2} \mu^2 \left[ \hat{\varphi} \right] \left( \hat{\varphi}_a \right)^2 \]  

(11)

By generalizing the noncommutative harmonic oscillator construction an extension of quantum field theory based on the concept of noncommutative fields satisfying the noncommutative commutation relations \([9]\) has been proposed in \([27]-[29]\), where the properties and phenomenological implications of the noncommutative field has been studied and applied to different problem including scalar, gauge and fermionic fields \([27]-[31]\). Our approach is different, it is based on the relation between the noncommutative variables \(\hat{\varphi}_a\) and \(\hat{\pi}_a\) and the canonical variables \(\varphi_a\) and \(\pi_a\).

It is easy to see that the noncommutative commutation relations \([9]\) can be mapped to the canonical commutation relations \([5]\) if the noncommutative variables \(\hat{\varphi}_a\) and \(\hat{\pi}_a\) are related to the canonical variables \(\varphi_a\) and \(\pi_a\) by the relations

\[ \hat{\varphi}_a = \varphi_a - \frac{1}{2} \theta \varepsilon_{ab} \pi_b \]  

(12)

\[ \hat{\pi}_a = \pi_a \]

Using these transformations, the noncommutative Hamiltonian density eq(11) can be rewritten, up to a total derivative term and up to second order in the parameter \(\theta\), as

\[ \hat{H} = \frac{1}{2} \pi^\sim \hat{M} \pi - \frac{1}{8} \theta^2 \pi^\sim \hat{D} \pi + \theta \pi^\sim N \varphi + \frac{1}{2} \varphi^\sim B \varphi + O \left( \theta^3 \right) \]  

(13)

where

\[ \hat{M} = \hat{\pi} + \frac{1}{4} \theta^2 \left( m^2 \hat{\pi} - g \varepsilon \hat{\varphi} \varepsilon \right) = M^\sim \]  

(14)

\[ \hat{\sigma}_{ab} = \frac{\delta^2}{\delta \varphi^a \delta \varphi^b} \left[ \frac{1}{4} \left( \varphi^\sim \varphi \right)^2 \right] = \varphi^\sim \varphi \delta_{ab} + 2 \varphi_a \varphi_b \]

\[ \hat{\sigma} = (\varphi^\sim \varphi) \hat{\pi} + 2 M \left[ \varphi \right] , \quad M_{ab} \left[ \varphi \right] = \varphi_a \varphi_b \]

\[ \hat{D} = \hat{\nabla}^2 \hat{\pi} = D^\sim \]

\[ \hat{N} = \frac{1}{2} \left( m^2 - \hat{\nabla}^2 + g \left( \varphi_a \right)^2 \right) \varepsilon = - N^\sim \]

\[ \hat{B} = \left( m^2 - \hat{\nabla}^2 + \frac{1}{2} g \left( \varphi_a \right)^2 \right) \hat{\pi} = B^\sim \]
with $\mathbb{I}$ denotes the $2 \times 2$ unit matrix, and $A^\sim$ denotes the transpose of the operator $A$.

From now on we keep only the modifications due to the noncommutativity up to second order in the parameter $\theta$.

The relation between $\pi_a$ and $\varphi_a$ is given by

$$\dot{\varphi}_a (x) = \frac{\delta \hat{H}}{\delta \pi_a (x)}$$

where $\hat{H} = \int d^3 x \hat{\mathcal{H}}$. Using the expression of $\hat{\mathcal{H}}$ and the symmetry properties of the operators $\mathbb{M} = \mathbb{M}^\sim$ and $\mathbb{D} = \mathbb{D}^\sim$, one gets

$$\dot{\varphi}_a (x) = \mathbb{M}_{ab} \pi_b (x) - \frac{1}{4} \theta^2 \mathbb{D}_{ab} \pi_b (x) + \theta N_{ab} \varphi_b (x)$$

From this relation we get the following iterative expression of $\pi_a$

$$\pi_a = \mathbb{K}_{ab} \left( \varphi_b - \theta N_{bc} \varphi_c \right) + \frac{1}{4} \theta^2 \mathbb{K}_{ab} \mathbb{D}_{bc} \pi_c$$

where $\mathbb{K}$ is the inverse of the matrix $\mathbb{M}$

$$\mathbb{K} = \mathbb{M}^{-1} = \mathbb{I} - \frac{1}{4} \theta^2 \left( m^2 \mathbb{I} - g \hat{\sigma} \hat{\varepsilon} \right)$$

Using the expression of the matrix $\mathbb{K}$ one gets, by iteration, the following expression of $\pi_a$

$$\pi_a = \left( \mathbb{I} + \frac{1}{4} \theta^2 \mathbb{D} \right)_{ab} \varphi_b - \theta N_{ab} \varphi_b$$

where $\mathbb{D} = \mathbb{D} - \left( m^2 \mathbb{I} - \varepsilon \hat{\sigma} \varepsilon \right) = \hat{\nabla}^2 \mathbb{I} - (m^2 \mathbb{I} - g \varepsilon \hat{\sigma} \varepsilon) = \mathbb{D}^\sim$.

We note that the noncommutative Hamiltonian density can be derived from the following noncommutative Lagrangian density

$$\hat{\mathcal{L}} = \frac{1}{2} \hat{\varphi}^\sim \left( \mathbb{I} + \frac{1}{4} \theta^2 \mathbb{D} \right) \hat{\varphi} + \theta \varphi^\sim \nabla \varphi - \frac{1}{2} \varphi^\sim \left( \mathbb{B} + \theta^2 \nabla^2 \right) \varphi$$

via the usual Legendre transformation $\hat{\mathcal{L}} = \pi_a \dot{\varphi}_a - \hat{\mathcal{H}}$. To get this expression we have used the symmetry properties of the operators $\mathbb{D}$, $N$ and $\mathbb{B}$. 
3. PROPAGATOR AND RENORMALIZATION

In this section we briefly recall basic results of path integral derivation of propagators and renormalization of $\varphi^4$– theory [40] [39] [36] [41]-[44].

Let us consider the theory of a real self-interacting scalar field specified by an action of the form

$$S[\varphi] = \int d^4x \left[ -\frac{1}{2} \left[ \partial_\mu \varphi (x) \right]^2 - \frac{1}{2} m^2 \left[ \varphi (x) \right]^2 - \frac{1}{4!} g \left[ \varphi (x) \right]^4 \right]$$

(21)

The action can be written as

$$S[\varphi] = S^{(0)}[\varphi] + S^{(1)}[\varphi]$$

(22)

where $S^{(0)}[\varphi]$ is the free action

$$S^{(0)}[\varphi] = -\frac{1}{2} \int d^4xd^4y \varphi (x) D(x,y) \varphi (y)$$

(23)

and $S^{(1)}[\varphi]$ is the interaction term

$$S^{(1)}[\varphi] = -\frac{1}{4!} g \int d^4x [\varphi (x)]^4$$

(24)

with

$$D(x,y) = (m^2 - \Box) \delta^4(x,y) = \int \frac{d^4q}{(2\pi)^4} \left( m^2 + q^2 \right) e^{iq(x-y)} = D(y,x)$$

(25)

The free propagator $\Delta(x,y)$ is defined as the vacuum expectation value [36] [41]-[44]

$$-i\Delta(x,y) = \langle 0 | T \{ \Phi (x) \Phi (y) \} | 0 \rangle$$

(26)

where $\langle 0 | T \{ \mathcal{A} [\Phi] \} | 0 \rangle$ denotes the vacuum expectation value of the chronological product $T \{ \mathcal{A} [\Phi] \}$, $\Phi (x)$ denotes the free quantum field operator corresponding to $\varphi (x)$, and $T$ is the time-ordering operator.

The vacuum expectation value eq(26) can be expressed as a Feynman functional integral [36] [41] [42]

$$-i\Delta(x,y) = \langle 0 | T \{ \Phi (x) \Phi (y) \} | 0 \rangle = \frac{\int \prod_x d\varphi (x) \left[ \varphi (x) \varphi (y) \right] e^{iS^{(0)}[\varphi]} }{\int \prod_x d\varphi (x) e^{iS^{(0)}[\varphi]}}$$

(27)
The functional integrals in eq(27) can now be performed with the help of the identity [36]

\[
\int \prod_x d\varphi (x) [\varphi (x_{l_1}) \varphi (x_{l_2}) ... \varphi (x_{2N})] e^{iS(0)[\varphi]} = \left[ \det \left( \frac{iD}{2\pi} \right) \right]^{-\frac{1}{2}} \sum_{\text{pairings of } (l_1,l_2,\ldots,l_{2N})} \prod_{\text{pairs } (l_j,l_k)} [-iD^{-1}(x_{l_j},x_{l_k})]
\]

(28)

where the sum is over all ways of pairing the indices \(l_1, l_2, \ldots, l_{2N}\), with two pairings being considered the same if they differ only by the order of the pairs, or by the order of indices within a pair, and \(D^{-1}(x,y)\) is the inverse of the matrix \(D(x,y)\).

Direct calculations give the following expression for the propagator

\[-i \Delta (x,y) = -iD^{-1}(x,y)\] (29)

to get the inverse of the matrix \(D(x,y)\) we write \(D(x,y)\) as a Fourier integral

\[D(x,y) = \int \frac{d^4q}{(2\pi)^4} D(q) e^{iq(x-y)}\] (30)

where

\[D(q) = m^2 + q^2 - i\epsilon\] (31)

Eq (29) can be rewritten as an integral equation

\[\int d^4y D(x,y) \Delta (y,z) = \delta (x,z)\] (32)

The solution of eq (32) is then

\[\Delta (x,y) = \int \frac{d^4q}{(2\pi)^4} \Delta (q) e^{iq(x-y)}\] (33)

where \(\Delta (q) = D^{-1}(q) = \frac{1}{m^2 + q^2 - i\epsilon}\) is the free-field propagator, and the \(i\epsilon\) terms have the effect of making the inverse well-defined for all real values of \(q\).

The exact propagator \(\Delta'(x,y)\) is given by [36] [41] [42]

\[-i \Delta'(x,y) = \langle 0 | T \{\Phi (x) \Phi (y)\} | 0 \rangle = \frac{\int \prod_x d\varphi (x) [\varphi (x) \varphi (y)] e^{iS[\varphi]} \cdot \int \prod_x d\varphi (x) e^{iS[\varphi]} }{\int \prod_x d\varphi (x) e^{iS[\varphi]} } \] (34)

In order to evaluate the functional integral eq(34), we make a Taylor expansion in powers of \(g\)
\[ e^{i\mathcal{S}[\varphi]} = e^{i\mathcal{S}^{(0)}[\varphi] + i\mathcal{S}^{(1)}[\varphi]} = e^{i\mathcal{S}^{(0)}[\varphi]} (1 + i\mathcal{S}^{(1)}[\varphi] + \ldots) \quad (35) \]

Using this Taylor expansion in eq(34) we obtain the following expressions

\[ \int \prod_x d\varphi(x) [\varphi(x) \varphi(y)] e^{i\mathcal{S}[\varphi]} = \int \prod_x d\varphi(x) [\varphi(x) \varphi(y)] e^{i\mathcal{S}^{(0)}[\varphi]} \]

\[ - \frac{i}{4!} g \int d^4 z \int \prod_x d\varphi(x) [\varphi(x) \varphi(y) [\varphi(z)]^4] e^{i\mathcal{S}^{(0)}[\varphi]} \]

and

\[ \int \prod_x d\varphi(x) e^{i\mathcal{S}[\varphi]} = \int \prod_x d\varphi(x) e^{i\mathcal{S}^{(0)}[\varphi]} - \frac{i}{4!} g \int d^4 z \int \prod_x d\varphi(x) [\varphi(z)]^4 e^{i\mathcal{S}^{(0)}[\varphi]} \quad (37) \]

The functional integrals in eq(36) and eq(37) can now be performed with the help of the identity eq(28), the exact propagator \( \Delta'(x, y) \) can be written as

\[ \Delta'(x, y) = \int \frac{d^4 q}{(2\pi)^4} \Delta(q) e^{iq(x-y)} \quad (38) \]

where

\[ \Delta'(q) = \Delta(q) + \Delta(q) [\Pi_{\text{Loop}}(q)] \Delta(q) \quad (39) \]

and \( \Pi_{\text{Loop}}(q) \) is given by the divergent integral

\[ \Pi_{\text{Loop}}(q) = \frac{i}{2} g \int \frac{d^4 p}{(2\pi)^4} \frac{1}{m^2 + p^2 - i\epsilon} \quad (40) \]

such ultraviolet divergences are typical of loop graphs. To deal with these divergent integrals that appear in quantum field theory, one can use some sort of regularization technique that makes these integrals finite. Dimensional regularization is the most convenient method for regulating divergent integrals, the idea is to treat the loop integrals as integrals over D-dimensional momenta, and then take the limit \( D \to 4 \), it turns out that the singularity of 1-loop graphs are simple poles in \( D = 4 \)

First we generalize the 4-dimensional action eq(28) to D-dimensions

\[ \mathcal{S}[\varphi_B] = \mathcal{S}^{(0)}_B [\varphi_B] + \mathcal{S}^{(1)}_B [\varphi_B] \quad (41) \]
with

\[ S^{(0)}_B [\varphi_B] = -\frac{1}{2} \int d^D x d^D y \varphi_B (x) \mathcal{D}^{(B)} (x, y) \varphi_B (y) \]  
(42)

\[ \mathcal{D}^{(B)} (x, y) = (m_B^2 - \Box) \delta^4 (x, y) = \int \frac{d^D q}{(2\pi)^D} \left( m_B^2 + q^2 \right) e^{iq(x-y)} \]  
(43)

and

\[ S^{(1)}_B [\varphi_B] = -\frac{1}{4!} \mu^{4-D} g_B \int d^D x [\varphi_B (x)]^4 \]  
(44)

where the scalar field $\varphi_B$ is a bare field, $m_B$ is the bare mass, and $\mu$ is an arbitrary mass parameter introduced to keep the bare coupling constant $g_B$ dimensionless. The bare quantities, such as $\varphi_B$, $m_B$ and $g_B$, are objects which are useful in the intermediate steps of the calculations, but they have no physical meaning. They are just chosen so that they cancel the divergences and leave us with the desired renormalized quantity. The physical or renormalized scalar field $\varphi$, mass $m$ and coupling constant $g$ are defined by

\[ \varphi = Z^{-\frac{1}{2}} \varphi_B , \quad m^2 = m_B^2 + \delta m^2 \]  
(45)

\[ g = Z_g^{-\frac{1}{4}} g_B , \quad Z_g = \frac{(1 + B)}{Z^2} \]  
(46)

with $Z$ and $\delta m^2$ to be chosen so that the propagators of the renormalized fields have poles in the same position and with the same residues as the propagators of the free fields in the absence of interactions.

The action may then be written in terms of renormalized quantities, as

\[ S [\varphi] = S^{(0)} [\varphi] + S^{(1)} [\varphi] + S^{(c)} [\varphi] \]  
(47)

where

\[ S^{(0)} [\varphi] = -\frac{1}{2} \int d^D x d^D y \varphi (x) \mathcal{D} (x, y) \varphi (y) \]  
(48)

\[ S^{(1)} [\varphi] = -\frac{1}{4!} \mu^{4-D} g \int d^D x [\varphi (x)]^4 \]  
(49)

\[ \mathcal{D} (x, y) = (m^2 - \Box) \delta^4 (x, y) = \int \frac{d^D q}{(2\pi)^D} \left( m^2 + q^2 \right) e^{iq(x-y)} \]  
(50)
and

\[ S^{(c)} [\varphi] = -\frac{1}{2} (Z - 1) \int d^D x d^D y \varphi (x) D (x, y) \varphi (y) + \frac{1}{2} Z \delta m^2 \int d^D x \varphi (x) \varphi (x) - \frac{B}{4!} \mu^{A-D} g \int d^D x [\varphi (x)]^4 \]  

(51)

The renormalized exact propagator \( \Delta^{(R)} (x, y) \) is given by

\[
- i \Delta^{(R)} (x, y) = \langle 0 | T \{ \Phi (x) \Phi (y) \} | 0 \rangle = \frac{\int \prod_x d\varphi (x) [\varphi (x) \varphi (y)] e^{iS[\varphi]} }{\int \prod_x d\varphi (x) e^{iS[\varphi]} } 
\]

(52)

We would like to evaluate the path integral for this theory, we now make a Taylor expansion in powers of \( g \)

\[ e^{iS[\varphi]} = e^{iS^{(0)}[\varphi] + iS^{(1)}[\varphi]} = e^{iS^{(0)}[\varphi]} \left( 1 + iS^{(1)} [\varphi] + iS^{(c)} [\varphi] + ... \right) \]  

(53)

Using this Taylor expansion in eq(52) we obtain the following expressions

\[
\int \prod_x d\varphi (x) [\varphi (x) \varphi (y)] e^{iS[\varphi]} = \int \prod_x d\varphi (x) [\varphi (x) \varphi (y)] e^{iS^{(0)}[\varphi]} 
\]

\[ - \frac{i}{4!} \mu^{A-D} g \int d^D z \int \prod_x d\varphi (x) [\varphi (x) \varphi (y) [\varphi (z)]^4] e^{iS^{(0)}[\varphi]} 
\]

\[ - \frac{i}{2} (Z - 1) \int d^D z d^D \tau D (z, \tau) \int \prod_x d\varphi (x) [\varphi (x) \varphi (y) \varphi (z) \varphi (\tau)] e^{iS^{(0)}[\varphi]} 
\]

\[ + \frac{i}{2} Z \delta m^2 \int d^D z \int \prod_x d\varphi (x) [\varphi (x) \varphi (y) [\varphi (z)]^2] e^{iS^{(0)}[\varphi]} 
\]

\[ - \frac{iB}{4!} \mu^{A-D} g \int d^D z \int \prod_x d\varphi (x) [\varphi (x) \varphi (y) [\varphi (z)]^4] e^{iS^{(0)}[\varphi]} 
\]

and

\[
\int \prod_x d\varphi (x) e^{iS[\varphi]} = \int \prod_x d\varphi (x) e^{iS^{(0)}[\varphi]} - \frac{i}{4!} \mu^{A-D} g \int d^D z \int \prod_x d\varphi (x) [\varphi (z)]^4 e^{iS^{(0)}[\varphi]} 
\]

\[ - \frac{i}{2} (Z - 1) \int d^D z d^D \tau D (z, \tau) \int \prod_x d\varphi (x) [\varphi (z) \varphi (\tau)] e^{iS^{(0)}[\varphi]} 
\]

\[ + \frac{1}{2} Z \delta m^2 \int d^D z \int \prod_x d\varphi (x) [\varphi (z)]^2 e^{iS^{(0)}[\varphi]} 
\]

\[ - \frac{B}{4!} \mu^{A-D} g \int d^D z \int \prod_x d\varphi (x) [\varphi (z)]^4 e^{iS^{(0)}[\varphi]} 
\]
The functional integrals in eq(52) can now be performed with the help of the identity eq(28), the renormalized exact propagator $\Delta^{(R)}(x,y)$ can be written as

$$\Delta^{(R)}(x,y) = \int \frac{d^D q}{(2\pi)^D} \Delta^{(R)}(q) e^{iq(x-y)}$$  \hspace{1cm} (54)$$

with

$$\Delta^{(R)}(q) = \Delta(q) + \Delta(q) \left[ \Pi^*(q^2) \right] \Delta(q) = \frac{1}{q^2 + m^2 - \Pi^*(q^2) - i\epsilon} + O(g^2)$$  \hspace{1cm} (55)$$

where $\Pi^*(q^2)$ is the self-energy function

$$\Pi^*(q^2) = - (Z - 1) (m^2 + q^2) + Z \delta m^2 + \Pi^*_{Loop}(q^2)$$  \hspace{1cm} (56)$$

and

$$\Pi^*_{Loop}(q^2) = i \frac{\mu^{4-D}}{2^D g} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2 - i\epsilon}$$  \hspace{1cm} (57)$$

The condition that $m^2$ is the true mass of the particle is that the pole of the propagator should be at $q^2 = -m^2$, so that [36]

$$\Pi^*(-m^2) = 0$$  \hspace{1cm} (58)$$

Also, the condition that the pole of the propagator at $q^2 = -m^2$ should have a unit residue (like the uncorrected propagator) is that

$$\left[ \frac{d}{dq^2} \Pi^*(q^2) \right]_{q^2=-m^2} = 0$$  \hspace{1cm} (59)$$

These conditions allow us to evaluate $Z$ and $\delta m^2$ [36]

$$Z\delta m^2 = -\Pi^*_{Loop}(-m^2)$$  \hspace{1cm} (60)$$

and

$$Z = 1 + \left[ \frac{d}{dq^2} \Pi^*_{Loop}(q^2) \right]_{q^2=-m^2}$$  \hspace{1cm} (61)$$

Direct calculations give the following expression for the loop integral [41]-[44] [45]
\[ \Pi^\ast_{\text{Loop}} (q^2) = \frac{i}{2} \mu^{4-D} g \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2 - i\epsilon} = -g \frac{m^2}{32\pi^2} \left( \frac{4\pi\mu^2}{m^2} \right)^{2-D} \Gamma \left( 1 - \frac{D}{2} \right) \] (62)

The divergence of this integral manifests itself in the pole of the Gamma function at the physical dimension \( D = 4 \).

In the neighborhood of this dimension we set

\[ D = 4 - 2s \] (63)

then

\[ \Pi^\ast_{\text{Loop}} (q^2) = -g \frac{m^2}{32\pi^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{-s} \Gamma (s - 1) \] (64)

using the relations

\[ \Gamma (s - 1) = -\left( \frac{1}{s} - \gamma + 1 \right) + O (s) \] (65)

and

\[ \left( \frac{m^2}{4\pi\mu^2} \right)^{-s} = 1 - s \ln \left( \frac{m^2}{4\pi\mu^2} \right) + O (s) \] (66)

we get

\[ \Pi^\ast_{\text{Loop}} (q^2) = g \frac{m^2}{32\pi^2} \left[ \frac{1}{s} + 1 - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] + O (s) \] (67)

hence

\[ Z = 1 + \left[ \frac{d}{dq^2} \Pi^\ast_{\text{Loop}} (q^2) \right]_{q^2 = -m^2} = 1 + O (g^2) \] (68)

and

\[ Z \delta m^2 = \delta m^2 = -\Pi^\ast_{\text{Loop}} (-m^2) = g \frac{m^2}{32\pi^2} \frac{1}{s} + g \frac{m^2}{32\pi^2} \left[ 1 - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] + O (g^2) \] (69)

to this order \( \Pi^\ast (q^2) \) is equal to zero

\[ \Pi^\ast (q^2) = 0 + O (g^2) \] (70)
4. PATH INTEGRAL DERIVATION OF NONCOMMUTATIVE PROPAGATOR

4.1. Noncommutative Free Propagator

The noncommutative action eq. (20) can be rewritten as

\[ \hat{S} [\varphi] = \int d^4x \hat{L} = \hat{S}^{(0)} [\varphi] + \hat{S}^{(1)} [\varphi] + \hat{S}^{(2)} [\varphi] + \hat{S}^{(3)} [\varphi] + \hat{S}^{(4)} [\varphi] + \hat{S}^{(5)} [\varphi] \]  

with

\[ \hat{S}^{(0)} [\varphi] = -\frac{1}{2} \int d^4x d^4y \varphi_a (x) \tilde{D}_{ab} (x, y) \varphi_b (y) \]  

\[ \hat{S}^{(1)} [\varphi] = -\frac{1}{4} g \int d^4x \left[ \varphi_a (x) \varphi_a (x) \varphi_b (x) \varphi_b (x) \right] \]  

\[ \hat{S}^{(2)} [\varphi] = \frac{1}{2} g \theta \int d^4x d^4y \hat{K}^{(1)}_{ab} (x, y) \left[ \varphi_a (x) \varphi_b (y) \varphi_c (x) \varphi_c (x) \right] \]  

\[ \hat{S}^{(3)} [\varphi] = \frac{1}{4} g \theta^2 \int d^4x d^4y d^4 \tau \hat{K}^{(2)} (x; y, \tau) \left[ \varphi_a (x) \varphi_a (y) \varphi_c (\tau) \varphi_c (\tau) \right] \]  

\[ \hat{S}^{(4)} [\varphi] = -\frac{1}{8} g \theta^2 \int d^4x d^4y d^4 \tau \hat{K}^{(3)} (x; y, \tau) \left[ \varphi_a (y) \varphi_a (\tau) \varphi_c (x) \varphi_c (x) \right] \]  

\[ \hat{S}^{(5)} [\varphi] = -\frac{1}{4} g \theta^2 \int d^4x d^4y d^4 \tau \hat{K}^{(3)} (x; y, \tau) \varepsilon_{ad} \varepsilon_{bc} \left[ \varphi_a (y) \varphi_d (x) \varphi_b (\tau) \varphi_c (x) \right] \]

where \( \tilde{D}_{ab} (x, y) \) is the symmetric matrix

\[ \tilde{D}_{ab} (x, y) = D_{ab} (x, y) - \frac{1}{4} \theta^2 M_{ab} (x, y) - \theta N_{ab} (x, y) = \tilde{D}_{ba} (y, x) \]  

\[ D_{ab} (x, y) = \delta_{ab} \left( m^2 - \Box \right) \delta^4 (x, y) = \delta_{ab} \int \frac{d^4q}{(2\pi)^4} \left( m^2 + q^2 \right) e^{iq(x-y)} = D_{ba} (y, x) \]  

\[ M_{ab} (x, y) = \delta_{ab} \left( m^2 - \Box \right) \left( m^2 - \vec{\nabla}^2 \right) \delta^4 (x, y) = \delta_{ab} \int \frac{d^4q}{(2\pi)^4} \left( m^2 + q^2 \right) \left( m^2 + \vec{q}^2 \right) e^{iq(x-y)} \]  

\[ M_{ab} (x, y) = M_{ba} (y, x) \]  

\[ N_{ab} (x, y) = \left( m^2 - \vec{\nabla}^2 \right) \varepsilon_{ab} \partial_i \delta^4 (x, y) = \varepsilon_{ab} \int \frac{d^4q}{(2\pi)^4} \left( m^2 + \vec{q}^2 \right) \left[ \varepsilon_{0} \right] e^{iq(x-y)} = N_{ba} (y, x) \]

while \( K^{(1)}_{ab} (x, y) \), \( K^{(2)} (x; y, \tau) \) and \( K^{(3)} (x; y, \tau) \) are given by
\[
\mathcal{K}^{(1)}_{ab} (x, y) = \varepsilon_{ab} \partial_t \delta^4 (x, y) = \int \frac{d^4q}{(2\pi)^4} \left[ \varepsilon_{ab} i q_0 \right] e^{iq(x-y)} = \mathcal{K}^{(1)}_{ba} (y, x) \tag{83}
\]
\[
\mathcal{K}^{(2)} (x; y, \tau) = \left[ \delta^4 (x, \tau) \left( m^2 - \nabla^2 \right) \delta^4 (x, y) - \left( \nabla \delta^4 (x, \tau) \right) \left( \nabla \delta^4 (x, y) \right) - \frac{1}{2} \left( \nabla^2 \delta^4 (x, \tau) \right) \delta^4 (x, y) \right] \tag{84}
\]
\[
\mathcal{K}^{(2)} (x; y, \tau) = \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \left[ m^2 + \frac{1}{2} (\vec{p} + \vec{q})^2 + \frac{1}{2} \vec{p}^2 \right] e^{ip(x-y)} e^{i(q-y) \cdot \partial_t} \tag{85}
\]
\[
\mathcal{K}^{(3)} (x; y, \tau) = \left[ \partial_t \delta^4 (x, y) \right] \left[ \partial_t \delta^4 (x, \tau) \right] = \mathcal{K}^{(3)} (x; \tau, y) \tag{86}
\]
\[
\mathcal{K}^{(3)} (x; y, \tau) = - \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} T q_0 p_0 e^{iq(x-y)} e^{i(p-y) \cdot \partial_t} = \mathcal{K}^{(3)} (x; \tau, y) \tag{87}
\]

The free noncommutative propagator \( \tilde{\Delta}_{ab} (x, y) \) is defined as the vacuum expectation value
\[
- i \tilde{\Delta}_{ab} (x, y) = \langle 0 | T \{ \Phi_a (x) \Phi_b (y) \} | 0 \rangle \tag{88}
\]

where \( \langle 0 | T \{ \mathcal{A} [\Phi] \} | 0 \rangle \) denotes the vacuum expectation value of the chronological product \( T \{ \mathcal{A} [\Phi] \} \), \( \Phi (x) \) denotes the free quantum field operator corresponding to \( \varphi (x) \), and \( T \) is the time-ordering operator.

The vacuum expectation value eq\(26\) can be expressed as a Feynman functional integral
\[
- i \tilde{\Delta}_{ab} (x, y) = \langle 0 | T \{ \Phi_a (x) \Phi_b (y) \} | 0 \rangle = \frac{\int \prod d\varphi_a (x) [\varphi_a (x) \varphi_b (y)] e^{i\tilde{S}(0)[\varphi]} \prod_{a, x} d\varphi_a (x) e^{i\tilde{S}(0)[\varphi]} \prod_{a, x}}{\int \prod d\varphi_a (x) e^{i\tilde{S}(0)[\varphi]} \prod_{a, x}} \tag{89}
\]

The functional integrals in eq\(27\) can now be performed with the help of the identity
\[
\int \prod_x d\varphi (x) \left[ \varphi_{l_1} (x_1) \varphi_{l_2} (x_2) ... \varphi_{l_{2N}} (x_{2N}) \right] e^{i\tilde{S}(0)[\varphi]} = \left[ \det \left( i\hat{D} / 2\pi \right) \right]^{-\frac{1}{2}} \sum_{\text{pairings}} \prod_{\text{pairs}(x_j, x_k)} \left[ -i\hat{D}_{l_j l_k}^{-1} (x_j, x_k) \right] \tag{90}
\]

where the sum is over all ways of pairing the indices \( (x_1 l_1, x_2 l_2, ..., x_{2N} l_{2N}) \), with two pairings being considered the same if they differ only by the order of the pairs, or by the order of indices within a pair, and \( \hat{D}_{ab}^{-1} (x, y) \) is the inverse of the matrix \( \hat{D}_{ab} (x, y) \).
Direct calculations give the following expression for the propagator

\[ -i\tilde{\Delta}_{ab}(x, y) = -i\tilde{D}_{ab}^{-1}(x, y) \]  

(91)

to get the inverse of the matrix \(\tilde{D}_{ab}(x, y)\) we write \(\tilde{D}_{ab}(x, y)\) as a Fourier integral

\[ \tilde{D}_{ab}(x, y) = \int \frac{d^4q}{(2\pi)^4} \tilde{D}_{ab}(q) e^{iq(x-y)} \]  

(92)

where

\[ \tilde{D}_{ab}(q) = (m^2 + q^2)\delta_{ab} - \frac{1}{4}\theta^2\delta_{ab}(m^2 + q^2) - i\theta\varepsilon_{ab}(m^2 + \tilde{q}^2)q_0 \]  

(93)

Now using these relations and the fact that eq(91) can be rewritten as

\[ \int d^4y\tilde{\Delta}_{ab}(x, y)\tilde{D}_{bc}(y, z) = \delta_{ac}\delta_4(x, z) \]  

(94)

one can deduce that

\[ \tilde{\Delta}_{ab}(x, y) = \int \frac{d^4q}{(2\pi)^4} \tilde{\Delta}_{ab}(q) e^{iq(x-y)} \]  

(95)

with

\[ \tilde{\Delta}_{ab}(q) = \tilde{D}_{ab}^{-1}(q) = \frac{1}{m^2 + q^2}\delta_{ab} + \frac{1}{4}\theta^2\delta_{ab} \frac{m^2 + \tilde{q}^2}{m^2 + q^2} + \theta^2\delta_{ab} \frac{(m^2 + \tilde{q}^2)^2}{(m^2 + q^2)^3} (q_0)^2 + i\theta\varepsilon_{ab} \frac{m^2 + \tilde{q}^2}{(m^2 + q^2)^2}q_0 \]  

(96)

it is easy to show that the free noncommutative propagator has the following symmetry properties

\[ \tilde{\Delta}_{ab}(q) = \tilde{D}_{ab}^{-1}(q) = \delta_{ab}\tilde{\Delta}(q) + \varepsilon_{ab}\tilde{\eta}(q) = \delta_{ab}\tilde{\Delta}(q) + \theta\varepsilon_{ab}\tilde{\eta}(q) = \tilde{\Delta}_{ba}(-q) \]  

(97)

\[ \tilde{\Delta}(q) = \frac{1}{m^2 + q^2} + \frac{1}{4}\theta^2\frac{m^2 + \tilde{q}^2}{m^2 + q^2} + \theta^2\frac{(m^2 + \tilde{q}^2)^2}{(m^2 + q^2)^3} (q_0)^2 = \tilde{\Delta}(-q) \]  

(98)

\[ \tilde{\eta}(q) = \theta\tilde{\eta}(q) = i\theta\frac{m^2 + \tilde{q}^2}{(m^2 + q^2)^2}q_0 = -\tilde{\eta}(-q) \]  

(99)

4.2. Noncommutative Exact Propagator

The exact noncommutative propagator \(\tilde{\Delta}_{ab}'(x, y)\) is given by

\[ -i\tilde{\Delta}_{ab}'(x, y) = \langle 0\mid T\{\Phi_a(x)\Phi_b(y)\}\mid 0\rangle = \frac{\int \prod_a d\varphi_a(x) [\varphi_a(x)\varphi_b(y)] e^{i\mathcal{S}[\varphi]}}{\int \prod_a d\varphi_a(x) e^{i\mathcal{S}[\varphi]}} \]  

(100)
In order to evaluate the functional integral eq(100), we make a Taylor expansion in powers of $g$

$$e^{i\tilde{\mathcal{S}}[\varphi]} = \left[ 1 + i \sum_{k=1}^{5} \tilde{S}^{(k)}[\varphi] \right] e^{i\tilde{\mathcal{S}}^{(0)}[\varphi]}$$

(101)

Using this Taylor expansion in eq(100) and expanding the denominator by the binomial theorem, we obtain the following expression for the exact noncommutative propagator

$$-i\tilde{\Delta}_{ab}^{(1)}(x, y) = -i\Delta_{ab}(x, y) - \frac{i}{4} g \mathcal{I}_{ab}^{(1)}(x, y) + \frac{i}{2} \theta g \mathcal{I}_{ab}^{(2)}(x, y) + \frac{i}{4} g \theta^2 \left[ \mathcal{I}_{ab}^{(3)}(x, y) - \frac{1}{2} \mathcal{I}_{ab}^{(4)}(x, y) - \mathcal{I}_{ab}^{(5)}(x, y) \right]$$

(102)

with

$$\mathcal{I}_{ab}^{(1)}(x, y) = \int d^4 z \int_{(c)} \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] [\varphi_c(z) \varphi_c(z) \varphi_d(z) \varphi_d(z)] e^{i\tilde{\mathcal{S}}^{(0)}[\varphi]}$$

$$\mathcal{I}_{ab}^{(2)}(x, y) = \int d^4 \tau d^4 z \mathcal{K}_{cd}^{(1)}(\tau, z) \int_{(c)} \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] [\varphi_c(\tau) \varphi_d(z) \varphi_m(\tau) \varphi_m(\tau)] e^{i\tilde{\mathcal{S}}^{(0)}[\varphi]}$$

$$\mathcal{I}_{ab}^{(3)}(x, y) = \int d^4 z' d^4 z d^4 \tau \mathcal{K}_{cd}^{(2)}(z'; z, \tau) \int_{(c)} \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] [\varphi_c(z') \varphi_c(z) \varphi_d(\tau) \varphi_d(\tau)] e^{i\tilde{\mathcal{S}}^{(0)}[\varphi]}$$

$$\mathcal{I}_{ab}^{(4)}(x, y) = \int d^4 z' d^4 z d^4 \tau \mathcal{K}_{cd}^{(3)}(z'; z, \tau) \int_{(c)} \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] [\varphi_c(z) \varphi_c(\tau) \varphi_d(z') \varphi_d(z')] e^{i\tilde{\mathcal{S}}^{(0)}[\varphi]}$$

and

$$\mathcal{I}_{ab}^{(5)}(x, y) = \varepsilon_{mc} \varepsilon_{ld} \int d^4 z' d^4 z d^4 \tau \mathcal{K}_{cd}^{(3)}(z'; z, \tau) \times \int_{(c)} \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] [\varphi_m(\tau) \varphi_c(z') \varphi_l(z) \varphi_d(z')] e^{i\tilde{\mathcal{S}}^{(0)}[\varphi]}$$

where the subscript $(c)$ being added to the functional integral $\int_{(c)} \prod_{a,x} d\varphi_a(x)$ ... to remind us that $\mathcal{I}_{ab}^{(k)}(x, y), k = 1, 2, 3, 4, 5$, are connected Green’s functions (connected Green’s functions are obtained by disregarding all terms which factorize into two or more functions with no overlapping arguments).

The connected Green’s functions $\mathcal{I}_{ab}^{(k)}(x, y), k = 1, 2, 3, 4, 5$, can now be calculated with the help of the identity eq(90), direct but lengthy calculations lead to the following expressions for the connected Green’s functions
\[ I_{ab}^{(1)} (x, y) = 16i \int \frac{d^4 q}{(2\pi)^4} \Delta_{ac} (q) \Delta_{cb} (q) e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta} (p) \] (103)

\[ I_{ab}^{(2)} (x, y) = -8 \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \varepsilon_{cd} \Delta_{db} (q) \right] q_0 e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta} (p) + 8 \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} p_0 \tilde{\eta} (p) \] (104)

\[ I_{ab}^{(3)} (x, y) = 8i \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] (m^2 + \pi^2) e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta} (p) + 8i \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} (m^2 + \pi^2) \tilde{\Delta} (p) \] (105)

\[ I_{ab}^{(4)} (x, y) = 4i \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] (q_0)^2 e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta} (p) + 4i \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} (p_0)^2 \tilde{\Delta} (p) \] (106)

\[ I_{ab}^{(5)} (x, y) = 2i \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] (q_0)^2 e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta} (p) + 2i \int \frac{d^4 q}{(2\pi)^4} \left[ \Delta_{ac} (q) \Delta_{cb} (q) \right] e^{iq(x-y)} \int \frac{d^4 p}{(2\pi)^4} (p_0)^2 \tilde{\Delta} (p) \] (107)

Now, from eqs (103) – (107) one can easily show that

\[ \tilde{\Delta}'_{ab} (x, y) = \int \frac{d^4 q}{(2\pi)^4} \tilde{\Delta}'_{ab} (q) e^{iq(x-y)} \] (108)

with

\[ -i \tilde{\Delta}'_{ab} (q) = -i \tilde{\Delta}_{ab} (q) + \left[ -i \tilde{\Delta}_{ac} (q) \right] [i \Pi_{(Loop)cd} (q)] \left[ -i \tilde{\Delta}_{db} (q) \right] \] (109)

where

\[ \Pi_{(Loop)ab} (q) = 4ig\delta_{ab} \left[ \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta} (p) \right] + 4\theta g \int \frac{d^4 p}{(2\pi)^4} \left[ q_0 \varepsilon_{ab} \tilde{\Delta} (p) - \delta_{ab} p_0 \tilde{\eta} (p) \right] + i\theta^2 g\delta_{ab} \int \frac{d^4 p}{(2\pi)^4} \left[ -2 (m^2 + \pi^2) - 2 (m^2 + \pi^2) + (q_0)^2 + (p_0)^2 \right] \tilde{\Delta} (p) \] (110)
5. RENORMALIZED NONCOMMUTATIVE EXACT PROPAGATOR

5.1. Renormalized Noncommutative Action

In order to renormalize the noncommutative complex scalar field we proceed as in \( \varphi^4 \)-theory of a real self-interacting scalar field, that is we split the noncommutative action \( \hat{S}[\varphi_B] \), expressed in terms of bare couplings \( m_B, g_B \) and bare fields \( \varphi_B \), in a part depending on the renormalized parameters \( m, g \) and field \( \varphi \), and in a counter term part [41]-[44] [36] [45] [39]

\[
\hat{S}[\varphi_B] = \int d^4x \hat{L} = \hat{S}^{(0)}_B[\varphi_B] + \hat{S}^{(1)}_B[\varphi_B] + \hat{S}^{(2)}_B[\varphi_B] + \hat{S}^{(3)}_B[\varphi_B] + \hat{S}^{(4)}_B[\varphi_B] + \hat{S}^{(5)}_B[\varphi_B] \tag{111}
\]

The bare quantities, such as \( \varphi_B, m_B, \) and \( g_B \), are just chosen so that they cancel the divergences and leave us with the desired renormalized quantity. The physical or renormalized scalar field \( \varphi \), mass \( m \) and coupling constant \( g \) are defined by

\[
\varphi = Z^{-\frac{1}{2}} \varphi_B \quad , \quad m^2 = m_B^2 + \delta m^2 \tag{112}
\]

\[
g = Z_g^{-\frac{1}{2}} g_B \quad , \quad Z_g = \frac{(1 + B)}{Z^2} \tag{113}
\]

with \( Z \) and \( \delta m^2 \) to be chosen so that the propagators of the renormalized fields have poles in the same position and with the same residues as the propagators of the free fields in the absence of interactions. In this paper we are interested only to the propagator and its renormalization up to first order in the parameter \( g \), so we can assume that

\[
\varphi = \varphi_B \quad , \quad g = g_B \quad , \quad Z = Z_g = 1 \tag{114}
\]

\[
m^2 = m_B^2 + \delta m^2 \tag{115}
\]

The noncommutative action \( \hat{S}^{(0)}_B[\varphi_B] \) may then be written in terms of renormalized quantities, as

\[
\hat{S}^{(0)}_B[\varphi_B] = -\frac{1}{2} \int d^4x d^4y \varphi_a(x) \tilde{D}^{(B)}_{ab}(x,y) \varphi_b(y) \tag{116}
\]

where
\[ \mathcal{D}^{(B)}_{ab} (x, y) = \mathcal{D}^{(B)}_{ab} (x, y) - \frac{1}{4} \theta^2 \mathcal{M}^{(B)}_{ab} (x, y) - \theta \mathcal{N}^{(B)}_{ab} (x, y) \] (117)

using the expressions of \( \mathcal{D}^{(B)}_{ab} (x, y) \), \( \mathcal{M}^{(B)}_{ab} (x, y) \) and \( \mathcal{N}^{(B)}_{ab} (x, y) \)

\[ \mathcal{D}^{(B)}_{ab} (x, y) = \mathcal{D}^{(R)}_{ab} (x, y) - \delta m^2 \delta^4 (x, y) \delta_{ab} \] (118)
\[ \mathcal{M}^{(B)}_{ab} (x, y) = \mathcal{M}^{(R)}_{ab} (x, y) - \delta m^2 (m^2 - \Box) \delta^4 (x, y) \delta_{ab} - \delta m^2 \left( m^2 - \nabla^2 \right) \delta^4 (x, y) \delta_{ab} \] (119)
\[ \mathcal{N}^{(B)}_{ab} (x, y) = \mathcal{N}^{(R)}_{ab} (x, y) - \delta m^2 \partial_i \delta^4 (x, y) \varepsilon_{ab} \] (120)

one can write \( \mathcal{D}^{(B)}_{ab} (x, y) \) as

\[ \mathcal{D}^{(B)}_{ab} (x, y) = \mathcal{D}^{(R)}_{ab} (x, y) - \delta m^2 \delta^4 (x, y) \delta_{ab} + \frac{1}{4} \theta^2 \delta m^2 \left[ (m^2 - \Box) \delta^4 (x, y) + \left( m^2 - \nabla^2 \right) \delta^4 (x, y) \right] \delta_{ab} \]
\[ + \theta \delta m^2 \partial_i \delta^4 (x, y) \varepsilon_{ab} \]

where \( \mathcal{D}^{(R)}_{ab} (x, y) \) is given by eq(178)

\[ \mathcal{D}^{(R)}_{ab} (x, y) = \mathcal{D}^{(R)}_{ab} (x, y) - \frac{1}{4} \theta^2 \mathcal{M}^{(R)}_{ab} (x, y) - \theta \mathcal{N}^{(R)}_{ab} (x, y) \] (121)
\[ \mathcal{D}^{(R)}_{ab} (x, y) = \delta_{ab} (m^2 - \Box) \delta^4 (x, y) = \delta_{ab} \int \frac{d^4 q}{(2 \pi)^4} (m^2 + q^2) e^{iq(x-y)} \] (122)
\[ \mathcal{M}^{(R)}_{ab} (x, y) = \delta_{ab} (m^2 - \Box) \left( m^2 - \nabla^2 \right) \delta^4 (x, y) = \delta_{ab} \int \frac{d^4 q}{(2 \pi)^4} (m^2 + q^2) \left( m^2 + \nabla^2 \right) e^{iq(x-y)} \] (123)
\[ \mathcal{N}^{(R)}_{ab} (x, y) = (m^2 - \nabla^2) \varepsilon_{ab} \partial_i \delta^4 (x, y) = \varepsilon_{ab} \int \frac{d^4 q}{(2 \pi)^4} (m^2 + q^2) \left[ i q_0 \right] e^{iq(x-y)} = \mathcal{N}_{ba} (y, x) \] (124)

hence \( \tilde{S}^{(0)}_B \) can be rewritten as

\[ \tilde{S}^{(0)}_B \varphi = -\frac{1}{2} \int d^4 x d^4 y \varphi_a (x) \tilde{D}^{(R)}_{ab} (x, y) \varphi_b (y) + \Delta \tilde{S}^{(0)} \varphi \] (125)

where the counter term part \( \Delta \tilde{S}^{(0)} \varphi \) is given by
The interaction part of the noncommutative action $\hat{S}[\varphi]$ may then be written in terms of renormalized quantities as

$$\hat{S}_B[\varphi_B] = \hat{S}_R^{(k)}[\varphi] + O(g^2) \quad k = 1, 2, 3, 4, 5$$

(126)

where $\hat{S}_R^{(k)}[\varphi]$ are given by eq(13) – (19).

Now eq(125) and eq(126) lead to the following expression for the noncommutative action $\hat{S}[\varphi_B]$

$$\hat{S}[\varphi_B] = \hat{S}_R^{(0)}[\varphi] + \Delta \hat{S}^{(0)}[\varphi] + \hat{S}_R^{(1)}[\varphi] + \hat{S}_R^{(2)}[\varphi] + \hat{S}_R^{(3)}[\varphi] + \hat{S}_R^{(4)}[\varphi] + \hat{S}_R^{(5)}[\varphi]$$

(127)

The functional integrals in eq(129) can now be performed with the help of the identity eq(28), the renormalized noncommutative exact propagator $\tilde{\Delta}_{ab}^{(R)}(x, y)$ is given by

$$-i\tilde{\Delta}_{ab}^{(R)}(x, y) = \langle 0 | T \{ \Phi_a(x) \Phi_b(y) \} | 0 \rangle = \frac{\int \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] e^{i\hat{S}^{(0)}[\varphi]} \int \prod_{a,x} d\varphi_a(x) e^{i\hat{S}^{(0)}[\varphi]}}{\int \prod_{a,x} d\varphi_a(x) e^{i\hat{S}^{(0)}[\varphi]}}$$

(128)

making a Taylor expansion in powers of $g$, we get

$$-i\tilde{\Delta}_{ab}^{(R)}(x, y) = \frac{\int \prod_{a,x} d\varphi_a(x) [\varphi_a(x) \varphi_b(y)] \left[ 1 + i\Delta \hat{S}^{(0)}[\varphi] + i \sum_{k=1}^{5} \hat{S}_R^{(k)}[\varphi] [\varphi] \right] e^{i\hat{S}^{(0)}[\varphi]} \int \prod_{a,x} d\varphi_a(x) \left[ 1 + i\Delta \hat{S}^{(0)}[\varphi] + i \sum_{k=1}^{5} \hat{S}_R^{(k)}[\varphi] [\varphi] \right] e^{i\hat{S}^{(0)}[\varphi]}}{\int \prod_{a,x} d\varphi_a(x) \left[ 1 + i\Delta \hat{S}^{(0)}[\varphi] + i \sum_{k=1}^{5} \hat{S}_R^{(k)}[\varphi] [\varphi] \right] e^{i\hat{S}^{(0)}[\varphi]}}$$

(129)

The functional integrals in eq(129) can now be performed with the help of the identity eq(28), the renormalized noncommutative exact propagator $\tilde{\Delta}_{ab}^{(R)}(x, y)$ can be written as
\[-i \tilde{\Delta}_{ab}^{(R)}(x, y) = -i \tilde{\Delta}_{ab}'(x, y) - i \delta m^2 \int dz dz' \delta^4(z, z') \tilde{\Delta}_{ac}(x, z) \tilde{\Delta}_{cb}(z', y) + i \theta \delta m^2 \int dz dz' \tilde{\Delta}_{ac}(x, z) \varepsilon_{cd} \tilde{\Delta}_{db}(z', y) \partial_i \delta^4(z, z') + i \frac{\theta^2 \delta m^2}{4} \int dz dz' \tilde{\Delta}_{ac}(x, z) \tilde{\Delta}_{cb}(z', y) (m^2 - \Box) \delta^4(z, z') + i \frac{\theta^2 \delta m^2}{4} \int dz dz' \tilde{\Delta}_{cb}(z', y) (m^2 - \nabla^2) \tilde{\Delta}_{ac}(x, z)\]

where \(-i \tilde{\Delta}_{ab}'(x, y)\) is given by eqs(108) - (110).

If we write \(\tilde{\Delta}_{ab}^{(R)}(x, y)\) as a Fourier integral

\[\tilde{\Delta}_{ab}^{(R)}(x, y) = \int \frac{d^4 q}{(2\pi)^4} \tilde{\Delta}_{ab}^{(R)}(q) e^{iq(x-y)}\]  

then

\[
\tilde{\Delta}_{ab}^{(R)}(q) = \tilde{\Delta}_{ab}'(q) + \delta m^2 \tilde{\Delta}_{ac}(q) \tilde{\Delta}_{cb}(q) - i \theta \delta m^2 q_0 \tilde{\Delta}_{ac}(q) \varepsilon_{cd} \tilde{\Delta}_{db}(q) - \frac{1}{4} \theta^2 \delta m^2 (m^2 + q^2) \tilde{\Delta}_{ac}(q) \tilde{\Delta}_{cb}(q) - \frac{1}{4} \theta^2 \delta m^2 (m^2 + \vec{q}^2) \tilde{\Delta}_{ac}(q) \tilde{\Delta}_{cb}(q)\]

where

\[\tilde{\Delta}_{ab}'(q) = \tilde{\Delta}_{ab}(q) + \Delta_{ac}(q) \Pi_{(Loop)cd}^{*}(q) \tilde{\Delta}_{db}(q) = \]

and

\[\Pi_{(Loop)ab}^{*}(q) = 4ig\delta_{ab} \left[ \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta}(p) \right] + 4g\varepsilon_{ab} \left[ \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta}(p) \right] q_0 - 4\theta^2 g\delta_{ab} \left[ \int \frac{d^4 p}{(2\pi)^4} \tilde{\Delta}(p) \right] p_0 \eta(p)\]

\[+ i \theta^2 \delta m^2 \int \frac{d^4 p}{(2\pi)^4} \left[ -2 (m^2 + \vec{q}^2) - 2 (m^2 + \vec{p}^2) + (q_0)^2 + (p_0)^2 \right] \tilde{\Delta}(p)\]

from eq(131) and eq(132) we get

\[\tilde{\Delta}_{ab}^{(R)}(q) = \tilde{\Delta}_{ab}(q) + \Delta_{ac}(q) \Pi_{(Loop)cd}^{*}(q) \tilde{\Delta}_{db}(q) = \left[ \frac{1}{\Delta^{-1}(q) - \Pi^{*}(q^2) - ie} \right]_{ab} + O(g^2)\]

where \(\Pi_{ab}^{*}(q)\) is the self-energy part (of the propagator)
\[ \Pi_{ab}(q) = \delta^2 \delta_{ab} - i\theta \delta m^2 q_0 \varepsilon_{ab} - \frac{1}{4} \theta^2 \delta m^2 (m^2 + q^2) \delta_{ab} - \frac{1}{4} \theta^2 \delta m^2 (m^2 + q^2) \delta_{ab} + \Pi_{(\text{Loop})ab}^*(q) \]  

(135)

Using the expression eq(133) of \( \Pi_{(\text{Loop})ab}^*(q) \), the self-energy part \( \Pi_{ab}^*(q) \) can be rewritten as

\[ \Pi_{ab}^*(q) = \left[ \left( 1 - \frac{1}{2} \theta^2 m^2 \right) \delta^2 \delta_{ab} + \pi_{ab}^{(1)} \right] - \left[ i\theta \delta m^2 \varepsilon_{ab} - \pi_{ab}^{(2)} \right] q_0 \]  

- \left[ \frac{1}{2} \theta^2 \delta m^2 \delta_{ab} - \pi_{ab}^{(3)} \right] \overrightarrow{q}^2 + \left[ \frac{1}{4} \theta^2 \delta m^2 \delta_{ab} - \pi_{ab}^{(4)} \right] (q_0)^2  

(136)

where

\[ \pi_{ab}^{(1)} = 4g_i \int \frac{d^4p}{(2\pi)^4} \left[ 1 - \frac{1}{2} \theta^2 (2m^2 + \overrightarrow{p}^2) + \frac{1}{4} \theta^2 (p_0)^2 \right] \tilde{\Delta} (p) \delta_{ab} - 4g\theta^2 \int \frac{d^4p}{(2\pi)^4} p_0 \overline{\eta} (p) \delta_{ab} \]  

(137)

\[ \pi_{ab}^{(2)} = 4g\theta \left[ \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right] \varepsilon_{ab} , \quad \pi_{ab}^{(3)} = -2g_i \theta^2 \left[ \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right] , \quad \pi_{ab}^{(4)} = g_i \theta^2 \left[ \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right] \]  

(138)

the last three terms of eq(136) can be written as

\[ \left[ i\theta \delta m^2 \varepsilon_{ab} - \pi_{ab}^{(2)} \right] q_0 = i\theta \left[ \delta^2 + 4g_i \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right] \varepsilon_{ab} q_0 \]  

(139)

\[ \left[ \frac{1}{2} \theta^2 \delta m^2 \delta_{ab} - \pi_{ab}^{(3)} \right] \overrightarrow{q}^2 = \frac{1}{2} \theta^2 \left[ \delta^2 + 4g_i \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right] \delta_{ab} \overrightarrow{q}^2 \]  

(140)

\[ \left[ \frac{1}{4} \theta^2 \delta m^2 \delta_{ab} - \pi_{ab}^{(4)} \right] (q_0)^2 = \frac{1}{4} \theta^2 \left[ \delta^2 + 4g_i \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right] (q_0)^2 \]  

(141)

while the first term of eq(136) reads

\[ \left[ \left( 1 - \frac{1}{2} \theta^2 m^2 \right) \delta^2 \delta_{ab} + \pi_{ab}^{(1)} \right] = \left( \delta^2 + 4g_i \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta} (p) \right) \left[ \delta_{ab} - \frac{1}{2} \theta^2 m^2 \right] \]  

\[ + 2g_i \theta^2 \left( \int \frac{d^4p}{(2\pi)^4} \left[ -m^2 - \overrightarrow{p}^2 + \frac{1}{2} (p_0)^2 \right] \tilde{\Delta} (p) + 2g_i \int \frac{d^4p}{(2\pi)^4} p_0 \overline{\eta} (p) \right) \delta_{ab} \]

Putting \( \delta m^2 = \delta m_0^2 + \frac{1}{2} \theta^2 \delta \mu_0^2 \) in eq(136), we get
\[ \Pi^{*}_{ab}(q) = \left( \delta m_0^2 + 4gi \int \frac{d^4p}{(2\pi)^4} \Delta(p) \right) \pi^{*}_{ab}(q) + \frac{1}{2} \theta^2 \left[ \delta \mu^2 \delta_{ab} + \varpi^{*}_{ab} \right] \]  

(142)

where

\[ \Delta(p) = \frac{1}{m^2 + p^2} \]  

(143)

is the free-field propagator, while \( \pi^{*}_{ab}(q) \) and \( \varpi^{*}_{ab} \) are given by

\[ \pi^{*}_{ab}(q) = \left[ \delta_{ab} - i\theta \varepsilon_{ab}q_0 - \frac{1}{2} \theta^2 \delta_{ab} (m^2 + \overrightarrow{q}^2) + \frac{1}{4} \theta^2 \delta_{ab} (q_0)^2 \right] \]  

(144)

\[ \varpi^{*}_{ab} = 8gi\delta_{ab} \int \frac{d^4p}{(2\pi)^4} \left[ -\frac{1}{4} + \frac{m^2 + \overrightarrow{p}^2}{(m^2 + p^2)^3} (p_0)^4 \right] \]  

(145)

5.2. Dimensional Regularization

To deal with the divergent integrals in eq(142) and eq(145), one can use some sort of regularization technique that makes these integrals finite. Dimensional regularization is the most powerful and popular method of regulating divergent integrals in perturbation theory. The idea is to treat the loop integrals as integrals over D-dimensional momenta, then we could analytically continue the integrals back to \( D = 4 \). The ultraviolet divergences will then appear as singularities (poles) in the deviation, \( \epsilon = 4 - D \), from four dimensions \[41]-[44], [36], [40], [45].

Let us consider the divergent integral

\[ I_1 = 4gi \int \frac{d^4p}{(2\pi)^4} \Delta(p) \]  

(146)

The first step in doing this divergent integral is to analytically continue the Feynman integrals to a continuous space-time dimension in the neighborhood of the physical dimension \( D = 4 \)

\[ I_1 = 4gi\mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{m^2 + p^2} \]  

(147)

where \( \mu \) is an arbitrary mass parameter introduced to keep the bare coupling constant \( g \) dimensionless.
The next step is called a Wick rotation \[36\] \[42\] \[44\], instead of integrating \(p^0\) on the real axis from \(-\infty\) to \(+\infty\), we integrate it on the imaginary axis from \(-i\infty\) to \(+i\infty\). That is, we can write \(p^0 = iq_4, p^2 = \vec{p}^2 - (p_0)^2 = \vec{q}^2 + (q_4)^2 = q^2\), with \(q_4\) integrated over real values from \(-\infty\) to \(+\infty\).

\[I_1 = 4g\mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{m^2 + p^2} = -4g\mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{m^2 + q^2} \tag{148}\]

Using the identity \[46\] \[38\] \[42\] \[44\]

\[a^{-z} = \frac{1}{
\Gamma(z) \int_0^\infty dt t^{z-1} e^{-at} \tag{149}\]

which is valid for \(\text{Re} \, z > 0\) and \(\text{Re} \, a > 0\), where \(\Gamma(z)\) is the Gamma function

\[\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} \tag{150}\]

defined for \(\text{Re}(z) > 0\), we obtain the following expression

\[I_1 = -4g\mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{m^2 + q^2} = -4g\mu^{4-D} \int_0^\infty dt \int d^Dqe^{-(m^2+q^2)t} \tag{151}\]

integration over the D-dimensional momentum integral can be performed by means of the relation

\[\int d^Dq e^{-(m^2+q^2)t} = \left(\frac{\pi}{t}\right)^{\frac{D}{2}} e^{-m^2t} \tag{152}\]

Then we can perform the integral over \(t\) using the formula

\[\int_0^{+\infty} dt t^{2s-1} e^{-at} = \frac{a^{-s}}{2} \Gamma(s) \tag{153}\]

yielding to

\[I_1 = -4g\mu^{4-D}\pi^{\frac{D}{2}} (m^2)^{\frac{D}{2}-1} \int_0^\infty dt t^{\frac{D}{2}-1} e^{-t} = -4g\frac{m^2}{16\pi^2} \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D}{2}-2} \Gamma\left(1 - \frac{D}{2}\right) \tag{154}\]

The divergence of this integral manifests itself in the pole of the Gamma function at the physical dimension \(D = 4\). In the neighborhood of this dimension we set

\[D = 4 - 2s \tag{155}\]

then
\[ I_1 = -\frac{g m^2}{4\pi^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{-s} \Gamma (s - 1) \] (156)

The singularity at small \( s \to 0 \) may be isolated by using the expansions

\[ \Gamma (s - 1) = -\left( \frac{1}{s} - \gamma + 1 \right) + O(s) \] (157)

and

\[ \left( \frac{m^2}{4\pi\mu^2} \right)^{-s} = 1 - s \ln \left( \frac{m^2}{4\pi\mu^2} \right) + O(s) \] (158)

we get then

\[ I_1 = 4g \mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{m^2 + p^2} = \frac{g m^2}{4\pi^2} \frac{1}{s} + \frac{g m^2}{4\pi^2} \left[ 1 - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \] (159)

Let us now evaluate the divergent integral eq(145)

\[ I_2 = 8g \int \frac{d^Dp}{(2\pi)^D} \left[ -\frac{1}{4} + \frac{m^2 + p^2}{(m^2 + p^2)^3} (p_0)^4 \right] - \frac{1}{4} + \frac{m^2 + q^2}{(m^2 + q^2)^3} (q_0)^4 \] (160)

dimensional regularization gives

\[ I_2 = 8g \mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \left[ -\frac{1}{4} + \frac{m^2 + q^2}{(m^2 + q^2)^3} (q_0)^4 \right] = -8g \mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \left[ -\frac{1}{4} + \frac{m^2 + q^2}{(m^2 + q^2)^3} (q_0)^4 \right] \] (161)

where we have performed the usual Wick rotation.

Next we use the relation eq(139)

\[ \frac{1}{(m^2 + q^2)^3} = \frac{1}{\Gamma(3)} \int_0^\infty dt t^2 e^{-(m^2 + q^2)t} \] (162)

and the fact that [44]

\[ \int \frac{d^Dq}{(2\pi)^D} = 0 \] (163)

to get

\[ I_2 = -8g \mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \frac{m^2 + q^2}{(m^2 + q^2)^3} (q_0)^4 = -8g \mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \frac{m^2 + q^2}{(m^2 + q^2)^3} (q_0)^4 \int_0^\infty dt t^2 e^{-(m^2 + q^2)t} \] (164)
$I_2$ can be rewritten as

$$I_2 = -4g\frac{\mu^{4-D}}{(2\pi)^{D}} \int_0^\infty dt t^2 e^{-m^2t} \int d^{D-1}q (m^2 + q^2) e^{-\vec{q}^2t} \int_{-\infty}^{+\infty} dq_4 (q_4)^4 e^{-q_4^2t} \quad (165)$$

the integration over the D-dimensional momentum integral on the right-hand side can be performed with the help of the relations [44] [46] [38] [42]

$$\int d^n q f(q^2) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{+\infty} dk k^{n-1} f(k) \quad (166)$$

and

$$\int_0^{+\infty} dt t^{2s-1} e^{-at} = \frac{a^{-s}}{2} \Gamma (s) \quad (167)$$

hence

$$I_2 = -4g\frac{\mu^{4-D}}{(2\pi)^{D}} \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \Gamma\left(\frac{5}{2}\right) \int_0^\infty dt t^{-\frac{D}{2}} e^{-m^2t} \int_0^{+\infty} dk k^{D-2} (m^2 + k^2) e^{-k^2t} \quad (168)$$

using eq(167), we get

$$I_2 = -g\frac{3\mu^{4-D}}{(2\pi)^{D}} \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} m^2 \int_0^\infty dt \left[ t^{-\frac{D}{2}} + \frac{D-1}{2} t^{-\frac{D+2}{2}} \right] e^{-t} \quad (169)$$

using again eq(167) to perform the integration over $t$, leads to

$$I_2 = -g\frac{3m^4}{16\pi^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{\frac{D}{2}} \left[ \Gamma \left( \frac{D}{2} + 1 \right) + \frac{D-1}{2} \Gamma \left( \frac{D}{2} \right) \right] \quad (170)$$

The divergence of this integral manifests itself in the pole of the Gamma function at the physical dimension $D = 4$.

In the neighborhood of this dimension we set

$$D = 4 - 2s$$

then

$$I_2 = -g\frac{3m^4}{16\pi^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{-s} \left[ \Gamma (s-1) + \left( \frac{3}{2} - s \right) \Gamma (s-2) \right] \quad (171)$$

using the expansions
\[
\Gamma (s - 1) = - \left( \frac{1}{s} - \gamma + 1 \right) + O(s) \quad (172)
\]
\[
\Gamma (s - 2) = \frac{1}{2} \left( \frac{1}{s} - \gamma + \frac{3}{2} \right) + O(s) \quad (173)
\]
and
\[
\left( \frac{m^2}{4\pi\mu^2} \right)^{-s} = 1 - s \ln \left( \frac{m^2}{4\pi\mu^2} \right) + O(s) \quad (174)
\]
we get the following expression
\[
I_2 = \frac{3gm^4}{64\pi^2} \left[ \frac{1}{s} - \gamma + \frac{3}{2} - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \quad (175)
\]
Hence
\[
\pi^*_a b = 8gi\mu^{4-D}\delta_{ab} \int \frac{d^D p}{(2\pi)^D} \left[ -\frac{1}{4} + \frac{m^2 + \vec{p}^2}{(m^2 + p^2)^3} (p_0)^4 \right] = \frac{3gm^4}{64\pi^2} \delta_{ab} \left[ \frac{1}{s} - \gamma + \frac{3}{2} - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \quad (176)
\]
and
\[
\Pi^*_a b (q) = \left( \delta m^2_0 + \frac{gm^2}{4\pi^2} \frac{1}{s} + \frac{gm^2}{4\pi^2} \left[ 1 - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \right) \pi^*_a b (q)
+ \frac{1}{2} g^2 \left( \delta \mu^2_0 + \frac{3gm^4}{64\pi^2} \frac{1}{s} + \frac{3gm^4}{64\pi^2} \left[ \frac{3}{2} - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \right) \delta_{ab} \quad (177)
\]
where
\[
\pi^*_a b (q) = \left[ \delta_{ab} - i\theta\varepsilon_{ab} q_0 - \frac{1}{2} \theta^2 \delta_{ab} (m^2 + \vec{q}^2) + \frac{1}{4} \theta^2 \delta_{ab} (q_0)^2 \right] \quad (178)
\]
The condition that \( m^2 \) is the true mass of the particle is that the pole of the propagator should (like the uncorrected propagator) be at \( q^2 = -m^2 \), so that
\[
[\Pi^*_a b (q)]_{q^2=-m^2} = \Pi^*_a b \left( \overrightarrow{0}, m \right) = 0 \quad (179)
\]
This condition allows us to evaluate \( \delta m^2_0 \) and \( \delta \mu^2_0 \)
\[
\delta m^2_0 = - \frac{gm^2}{4\pi^2} \frac{1}{s} - \frac{gm^2}{4\pi^2} \left[ 1 - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \quad (180)
\]
\[
\delta \mu^2_0 = - \frac{3gm^4}{64\pi^2} \frac{1}{s} - \frac{3gm^4}{64\pi^2} \left[ \frac{3}{2} - \gamma - \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] \quad (181)
\]
to this order \( \Pi^* (q) \) is equal to zero
\[
\Pi^* (q) = 0 + O \left( g^2 \right) \quad (182)
\]
6. CONCLUSION

Thought this work we have considered a noncommutative complex scalar field theory with self interaction, by imposing non commutativity to the canonical commutation relations. The action and all relevant quantities are expanded up to second order in the parameter of noncommutativity $\theta$. Using the path integral formalism, the noncommutative free and exact propagators are calculated to one-loop order and to the second order in the parameter of noncommutativity $\theta$. Dimensional regularization was used to remove ultraviolet divergences that arise from loop graphs. It has been shown that these divergences may also be absorbed into a redefinition of the parameters of the theory.

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