SKEW CLOSED STRUCTURE OF Gray-CATEGORIES

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Abstract. We define a skew-closed structure for Gray-categories extending the mapping space construction of Gohla [2014].

1. Definitions

Let \textbf{GrayCat} be the category \textbf{Gray}-categories and strict \textbf{Gray}-enriched functors. In [Gohla, 2014] we showed that there is for \textbf{Gray}-categories \( \mathcal{G}, \mathcal{H} \) a \textbf{Gray}-category \([\mathcal{G}, \mathcal{H}]\) consisting of pseudo-functors, pseudo-transformations, pseudo-modifications and perturbations. This construction can be restricted to a \textbf{Gray}-category \([\mathcal{G}, \mathcal{H}]^p\) of strict \textbf{Gray}-functors, pseudo-transformations, pseudo-modifications and perturbations.

We denote the composition operations in \([\mathcal{G}, \mathcal{H}]^p\) by \(*_i\) where \(i\) is the dimension of the intervening cell, i.e. we have \(\alpha' *_0 \alpha : H \Rightarrow \mathcal{H}''\) for pseudo-transformations \(\alpha : H \Rightarrow \mathcal{H}'\) and \(\alpha' : \mathcal{H}' \Rightarrow \mathcal{H}''\) where \(H, \mathcal{H}', \mathcal{H}''\) are \textbf{Gray}-functors \(\mathcal{G} \rightarrow \mathcal{H}\); \(\alpha\) and \(\alpha'\) are 1-cells in \([\mathcal{G}, \mathcal{H}]^p\) that coincide on \(\mathcal{H}'\). We will denote by \(*_{-1}\) the composition of functors, pseudo-transformations, pseudo-modifications and perturbations incident on a \textbf{Gray}-category, i.e. from the point of view of the mapping space \([\mathcal{G}, \mathcal{H}]^p\) the \textbf{Gray}-categories \(\mathcal{G}\) and \(\mathcal{H}\) have dimension \(-1\).

1.1. Claim. \([\mathcal{G}, \mathcal{H}]^p\) is functorial \(\mathcal{G}\) and \(\mathcal{H}\), meaning \([\_, \_]^p\) : \textbf{GrayCat}^{\text{op}} \times \textbf{GrayCat} \rightarrow \textbf{GrayCat} is a functor.

Proof. That \([\_, \mathcal{H}]^p\) is a functor each \(\mathcal{H}\) is seen easily, since it is defined in terms of pre-composition.

Let \(G : \mathcal{H} \rightarrow \mathcal{H}'\) be a \textbf{Gray}-functor, we define a mapping \([\mathcal{G}, G]^p : [\mathcal{G}, \mathcal{H}]^p \rightarrow [\mathcal{G}, \mathcal{H}']\) as follows:

- \((F : \mathcal{G} \rightarrow \mathcal{H}) \mapsto (GF : \mathcal{G} \rightarrow \mathcal{H}')\)
- \((\alpha : Q^1 \mathcal{G} \rightarrow \overline{\mathcal{H}}) \mapsto (Q^1 \alpha : Q^1 \mathcal{G} \rightarrow \overline{\mathcal{H}}')\)
- \((A : Q^1 \mathcal{G} \rightarrow \overline{\mathcal{H}}) \mapsto (Q^1 A : Q^1 \mathcal{G} \rightarrow \overline{\mathcal{H}}')\)
- \((\sigma : Q^1 \mathcal{G} \rightarrow \overline{\mathcal{H}}) \mapsto (Q^1 \sigma : Q^1 \mathcal{G} \rightarrow \overline{\mathcal{H}}')\)

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We check that \([\mathcal{G}, -]^p\) preserves unit and composition, i.e. that \([\mathcal{G}, \text{id}_{\mathcal{H}}]^p = \text{id}_{[\mathcal{G}, \mathcal{H}]^p}\) and \([\mathcal{G}, G'^p][\mathcal{G}, G]^p = [\mathcal{G}, G'G]^p\), by using the fact that \(\rightarrow\), \(\rightarrow\) and \(\rightarrow\) are endo-functors on \(\text{GrayCat}\) and composition of \(\text{Gray}\)-functors is associative.

Finally, we need to verify that \([\mathcal{G}, G]^p\) itself is a \(\text{Gray}\)-functor, meaning it needs to preserve the composition operations \(*_i\) of \([\mathcal{G}, \mathcal{H}]^p\) and the tensor.

Let \(\alpha: F \Rightarrow F'\) and \(\alpha': F' \Rightarrow F''\) be pseudo-transformations for \(\text{Gray}\)-functors \(F, F', F'': \mathcal{G} \rightarrow \mathcal{H}\). \(\alpha\) and \(\alpha'\) are given by \(\text{Gray}\)-functors \(\alpha, \alpha': Q^1\mathcal{G} \rightarrow \mathcal{H}\). Their composite along \(F'\) is defined as \(\alpha' \circ_0 \alpha = mQ^1(\alpha', \alpha)\delta\), hence \([\mathcal{G}, \mathcal{G}](\alpha' \circ_0 \alpha) = G mQ^1(\alpha', \alpha)\delta\). On the other hand \([\mathcal{G}, G][\alpha']\circ_0[\mathcal{G}, G](\alpha) = mQ^1(\mathcal{G} \alpha', \mathcal{G} \alpha)\delta = mQ^1(G \times_{d_0, d_1} G)Q^1(\alpha', \alpha)\delta\). Thus, to prove \([\mathcal{G}, G](\alpha' \circ_0 \alpha) = [\mathcal{G}, G](\alpha' \circ_0 [\mathcal{G}, G](\alpha)\), it remains we to be shown that

\[
\begin{align*}
\text{Q}^1(\mathcal{H}' \times_{d_0, d_1} \mathcal{H}) &\xrightarrow{m} \mathcal{H}' \\
\text{Q}^1(\mathcal{G} \times \mathcal{G}) &\xrightarrow{\mathcal{G}} \\
\text{Q}^1(\mathcal{H}' \times_{d_0, d_1} \mathcal{H'}) &\xrightarrow{m} \mathcal{H}'
\end{align*}
\]

commutes in \(\text{GrayCat}\). This can be seen when comparing the two paths as composites of \(\text{Q}^1\) graph maps \([\text{Gohla}, 2014]\). Their agreement as globular maps is obvious.

We proceed to check the equality of the 2-cocycle data: According to \([\text{Gohla}, 2014, \text{section 5}]\) the 2-cocyle on a 2-by-2 arrangement of squares is given by

\[
\begin{align*}
(\hat{f}'' \#_0 g'_2 \#_0 g_0) \\
\#_1(g'_2 \otimes g_2) \\
\#_1(\hat{g}'_1 \#_0 g_2 \#_0 f)
\end{align*}
\]

and because applying \(\mathcal{G}\) of a strict \(\text{Gray}\)-functor to cells of \(\mathcal{H}\) is given by applying \(G\) elementwise it is obvious that \(\square\) commutes.

For the remaining operations one can argue similarly. \(\square\)

First, we need for every triple of \(\text{Gray}\)-categories \(\mathcal{G}, \mathcal{H}, \mathcal{K}\) a \(\text{Gray}\)-functor

\[
L^\mathcal{G}_{\mathcal{H}, \mathcal{K}}: [\mathcal{H}, \mathcal{K}]^p \rightarrow ([\mathcal{G}, \mathcal{H}]^p, [\mathcal{G}, \mathcal{K}]^p)^p
\]

natural in \(\mathcal{H}\) and \(\mathcal{K}\) and extranatural in \(\mathcal{G}\). For brevity we shall just write \(L\). Let \(H, \beta, B, \Delta\) be respectively a 0-, 1-, 2- and 3-cell from \([\mathcal{H}, \mathcal{K}]^p\), then we define \(L\) as giving the action by postcomposition:

\[
\begin{align*}
L(H) &= H \ast_{-1} (\_ ) \\
L(\beta) &= \beta \ast_{-1} (\_ ) \\
L(B) &= B \ast_{-1} (\_ ) \\
L(\Delta) &= \Delta \ast_{-1} (\_ ).
\end{align*}
\]
The composite \( *_{-1} \) is defined in section 4. Furthermore, let \( L(\beta)^2 \) be the family of perturbations indexed by composable pairs \( \alpha, \alpha' \) of pseudo-transformations in \([\mathcal{G}, \mathcal{H}]^p\) defined by

\[
(L(\beta)^2)^{\alpha, \alpha'}_x = \beta^2_{\alpha_x, \alpha_x}. \tag{4}
\]

1.2. Theorem. The map \( L \) given in (3) and (4) is well defined, that is,

- \( L(H) \) is a Gray-functor \((2.7)\) \( L(H) : [\mathcal{G}, \mathcal{H}]^p \rightarrow [\mathcal{G}, \mathcal{K}]^p \),
- \( L(\beta) \) is a pseudo-transformation \((2.3)\) \( L(\beta) : L(H) \rightarrow L(H') \),
- \( L(B) \) is a pseudo-modification \((2.4)\) \( L(B) : L(\beta) \rightarrow L(\beta') \),
- \( L(\Delta) \) is a perturbation \((2.5)\) \( L(\Delta) : L(B) \rightarrow L(B') \).

Proof. The claims are proved in lemmata 1.3, 1.4, 1.5, and 1.6. □

1.3. Lemma. \( L(H) : [\mathcal{G}, \mathcal{H}]^p \rightarrow [\mathcal{G}, \mathcal{K}]^p \) is a Gray-functor for \( H \) a Gray-functor, i.e. a 0-cell of \([[[\mathcal{G}, \mathcal{H}]^p, [\mathcal{G}, \mathcal{K}]^p]^p]^p\).

Proof. This is obvious from \( H \) being a Gray-functor. □

1.4. Lemma. Let \( \beta : H \rightarrow H' \) be a pseudo-transformation, i.e. a 1-cell in \([\mathcal{H}, \mathcal{K}]^p\), then \( L(\beta) : L(H) \rightarrow L(H') \), is a pseudo-transformation \((2.3)\) in a canonical way \( L(\beta) : L(H) \rightarrow L(H') \), i.e. a 1-cell in \([[[\mathcal{G}, \mathcal{H}]^p, [\mathcal{G}, \mathcal{K}]^p]^p]^p\).

Proof. First we give the details of definition of \( L(\beta) \) according to (3) and (4):

1. On 0-cells \( G \) of \([\mathcal{G}, \mathcal{H}]^p\) we get pseudo-transformations \( L(\beta)_G : L(H)(G) \rightarrow L(H')(G) \) by whiskering

\[
L(\beta)_G = \beta *_{-1} G
\]

by our definition (3) above. According to section 4 the components at cells of \( G \) are \( (L(\beta)_G)_x = \beta_{Gx} \), \((L(\beta)_G)(f) = \beta_{Gf} \), \((L(\beta)_G)_\varphi = \beta_{G\varphi} \) and 2-cocycle \( (L(\beta)_G)^2_{f, f'} = \beta^2_{Gf, Gf'} \). And by theorem 4.1 \( \beta *_{-1} G : HG \rightarrow H'G \) is a pseudo-modification, i.e. a 1-cell in \([\mathcal{G}, \mathcal{K}]^p\).

2. On 1-cells definition (3) \( \alpha : G \rightarrow G' \) of \([\mathcal{G}, \mathcal{H}]^p\) yields 2-cells in \([\mathcal{G}, \mathcal{K}]^p\) as follows:

\[
\begin{array}{ccc}
L(H)(G) & \xrightarrow{L(\beta)_G} & L(H')(G) \\
L(H)(\alpha) \downarrow & \xleftarrow{\varphi \beta \alpha} & \downarrow \varphi \beta \alpha \ \\
L(H)(G') & \xrightarrow{L(\beta)_{G'}} & L(H')(G')
\end{array}
\]

\[
\begin{array}{ccc}
HG & \xrightarrow{\beta *_{-1} G} & H'G \\
L(H)(\alpha) \downarrow & \xleftarrow{\varphi \beta \alpha} & \downarrow \varphi \beta \alpha \ \\
L(H)(G') & \xrightarrow{L(\beta)_{G'}} & L(H')(G')
\end{array}
\]

We note that by 4.1 \( \beta *_{-1} \alpha \) is a pseudo-modification, i.e. a 2-cell in \([\mathcal{G}, \mathcal{K}]^p\).
3. On 2-cells $A: \alpha \Rightarrow \alpha'$ of $[\mathcal{G}, \mathcal{H}]^p$ definition (3) gives us

\[
\begin{array}{c}
\xymatrix{
L(H)(G) \ar@{=>}[r]^{L(\beta)_G} & L(H')(G) \\
L(H)(G') \ar@{=>}[r]^{L(\beta)_{G'}} & L(H')(G') \\
L(H)(G'') \ar@{=>}[r]^{L(\beta)_{G''}} & L(H')(G'') \\
L(H)(A) \ar@{=>}[u]^{L(\beta)_A} & L(H)(A) \ar@{=>}[u]^{L(\beta)_{A}} & L(H)(A) \ar@{=>}[u]^{L(\beta)_{A}} & L(H)(A) \ar@{=>}[u]^{L(\beta)_{A}}
}
\end{array}
\]

(7)

4. By lemma 1.7 we get a family of perturbations indexed by composable pairs of 1-cells $G \xrightarrow{\alpha} G' \xrightarrow{\alpha'} G''$ in $[\mathcal{G}, \mathcal{H}]^p$:

\[
\begin{array}{c}
\xymatrix{
L(H)(G) \ar@{=>}[r]^{L(\beta)_G} & L(H')(G) \\
L(H)(G') \ar@{=>}[r]^{L(\beta)_{G'}} & L(H')(G') \\
L(H)(G'') \ar@{=>}[r]^{L(\beta)_{G''}} & L(H')(G'') \\
L(H)(A) \ar@{=>}[u]^{L(\beta)_A} & L(H)(A) \ar@{=>}[u]^{L(\beta)_{A}} & L(H)(A) \ar@{=>}[u]^{L(\beta)_{A}} & L(H)(A) \ar@{=>}[u]^{L(\beta)_{A}}
}
\end{array}
\]

(8)

each defined on 0-cells $x$ of $\mathcal{G}$ as $(L(\beta)_{\alpha,\alpha})_x = \beta^2_{\alpha_x,\alpha_x}$, i.e.

\[
\begin{array}{c}
\xymatrix{
HG_x \ar@{=>}[r]^{\beta_{G_x}} & H'G_x \\
HG' x \ar@{=>}[r]^{\beta_{G'x}} & H'G' x \\
HG'' x \ar@{=>}[r]^{\beta_{G''x}} & H'G'' x \\
H \alpha_x \ar@{=>}[u]^{\beta_{\alpha_x}} & H' \alpha_x \ar@{=>}[u]^{\beta_{\alpha_x}} & H' \alpha_x \ar@{=>}[u]^{\beta_{\alpha_x}} & H' \alpha_x \ar@{=>}[u]^{\beta_{\alpha_x}}
}
\end{array}
\]

(9)
We now check the axioms for a pseudo-transformation given in 2.3.

1. We get $L(\beta)_{id_G} = id_{L(\beta)G}$ because $\beta \ast_{-1} id_G = id_{\beta \ast_{-1} G}$ by 4.5.

2. For each 3-cell in $[G, H]^p$, i.e. a perturbation $\Gamma : A \Rightarrow A' : \alpha \Rightarrow \alpha'$ the following square commutes:

![Diagram](image)

By definition the components in (10) are given by

![Diagram](image)
Since this is an equation of perturbations it is sufficient to compare their values at 0-cells $x$ of $G$:

\[
\begin{array}{c}
HGx \xrightarrow{\beta_{Gx}} H'Gx \\
HG'x \xrightarrow{\beta_{G'x}} H'G'
\end{array}
\]

\[
\begin{array}{c}
HGx \xrightarrow{\beta_{Gx}} H'Gx \\
HG'x \xrightarrow{\beta_{G'x}} H'G'
\end{array}
\]

which holds by (63).

3. A similar argument serves to verify equations (64) and (65).

4. We check that $L(\beta)^2$ as given by (8) satisfies the cocycle condition (66). But this is assured by the same condition for $\beta^2$: (13) holds for all 0-cells $x$ in $G$. Hence $L(\beta)^2$ is a 2-cocycle with components $L(\beta)^2_{\alpha',\alpha'}$: $(L(\beta)_{\alpha'} *_0 L(G)(\alpha)) *_1 (L(G')(\alpha') *_0 L(\beta)_{\alpha}) \Rightarrow L(\beta)_{\alpha'*0\alpha}$.

The cocycle $L(\beta)^2$ is also normalized, i.e. satisfies (67) because $\beta^2$ is so.

5. Finally we check compatibility of the cocycle $L(\beta)^2$ with whiskering of 2-cells, as shown in (68) and (69). Again, this suffices to check on 0-cells $x$ of $[G, H]^p$ and $\beta$ being a pseudo-transformation.

This completes the proof that $L(\beta)$ is a pseudo-transformation. \qed

1.5. Lemma. For a 2-cell in $[H, K]^p$, i.e. a pseudo-modification, $L(B)$ is a pseudo-modification (2.4) $L(B): L(\beta) \Rightarrow L(\beta')$.

Proof. We begin by stating the defining components of $L(B)$:

1. On 0-cells $G$ of $[G, H]^p$

\[
L(H)(G) \xrightarrow{L(\beta)G} L(H')(G) = H *_{-1} G \xrightarrow{L(\beta')G} L(H')(G) \quad (14)
\]
2. on 1-cells $\alpha$ of $[\mathcal{G}, \mathcal{H}]^p$

And we verify the axioms for this to constitute a pseudo-modification:

1. $L(B)\id_G = \id_{L(B)G}$ by $B *_{-1} \id_G = \id_{B*_{-1}G}$ from \ref{4.5}

2. Compatibility of $L(B)$ with the cocycles $L(\beta)^2$ and $L(\beta')^2$, i.e. \ref{13}, is assured by $B$ being a pseudo-modification.

3. For 2-cells $A: \alpha \Rightarrow \alpha': G \to G'$ in $[\mathcal{G}, \mathcal{H}]^p$ \ref{17} holds because it does so for $B$. $\square$

1.6. **Lemma.** For every 3-cell $\Delta: B \Rightarrow B': \beta \Rightarrow \beta'$ of $[\mathcal{H}, \mathcal{K}]^p$, $L(\Delta)$ is a perturbation $L(\Delta): L(B) \Rightarrow L(B')$.

**Proof.** The components of $L(\Delta)$ are given by $L(\Delta)_G = \Delta *_{-1} G$. But then $L(\Delta)$ satisfies the perturbation condition \ref{74} because $\Delta$ does so for every $G$ and for every $x$. $\square$

We define

\[
(L(\beta)'_\alpha)_x = G'Hx \quad \downarrow \quad G'H'x , \tag{16}
\]
(L(\beta)_{\alpha})_f = (\beta *_{-1} \alpha)_f

replace with ref to appendix
We pause here to note for later reference the composites of pseudo-transformations along the boundaries in (6), i.e. $L(G')(\alpha) L(\beta)_H = (G' *_{-1} \alpha) *_0 (\beta *_{-1} H) = \beta \triangleright \alpha$ and $L(\beta)_H L(G)(\alpha) = (\beta *_{-1} H') *_0 (G *_{-1} \alpha) = \beta \triangleleft \alpha$; these are given by their values on 0-, 1- and 2-cells and their 2-cocycle data. The data for $\beta \triangleright \alpha$ on 0- and 1-cells are:

\[
\begin{align*}
G H x & \xrightarrow{\beta_{Hx}} G' H x \xrightarrow{\alpha_{x'}} G' H' x \\
G H f & \xrightarrow{\beta_{Hf}} G' H f \xrightarrow{\alpha_{f'}} G' H' f \\
G H y & \xrightarrow{\beta_{Hy}} G' H y \xrightarrow{\alpha_{y'}} G' H' y
\end{align*}
\]

(18)

on 2-cells: see (19), 2-cocycle: see (20).

For $\beta \triangleleft \alpha$ on the other hand, the data on 0- and 1-cells are

\[
\begin{align*}
G H x & \xrightarrow{G \alpha x} G H x \xrightarrow{\beta_{Hx}} G' H x \\
G H f & \xrightarrow{G \alpha f} G H f \xrightarrow{\beta_{Hf}} G' H f \\
G H y & \xrightarrow{G \alpha y} G H y \xrightarrow{\beta_{Hy}} G' H y
\end{align*}
\]

(21)

on 2-cells finally, for the 2-cocycle: see (23).

In order to make sure that $L(\beta)_{\alpha}$ as shown in (6) is indeed a pseudo-modification we need to check the following conditions on the 3-cell components of $L(\beta)$, $\beta \triangleright \alpha$ and $\beta \triangleleft \alpha$, namely that

1. the $(L(\beta)_{\alpha})_f$ are compatible with the $(\beta \triangleright \alpha)^2_{f',f}$ and $(\beta \triangleleft \alpha)^2_{f',f}$,

2. for 2-cells $\varphi: f \rightarrow f': x \rightarrow y$ the 3-cells $(L(\beta)_{\alpha})_f$, $(L(\beta)_{\alpha})_{f'}$, $(\beta \triangleright \alpha)_{\varphi}$ and $(\beta \triangleleft \alpha)_{\varphi}$ are compatible.

On 3-cells . . . i.e. $L(\beta)_1$

1.7. LEMMA. For a pair of composable 1-cells $G \xrightarrow{\alpha} G' \xrightarrow{\alpha'} G''$ in $[G, H]^p\, L(\beta)^2_{\alpha', \alpha}$ is indeed a perturbation.
(\beta \triangleright \alpha)_\varphi

\[ (G'\alpha_y \#_0 \beta_{H_x}) \#_1 (G'\alpha_f \#_0 \beta_{H_f}) \]

\[ (G'\alpha_y \#_0 \beta_{H_x}) \#_1 (G'\alpha_f \#_0 \beta_{H_f}) \]

\[ (G'\alpha_y \#_0 \beta_{H_x}) \#_1 (G'\alpha_f \#_0 \beta_{H_f}) \]
PROOF. We need to verify that $L(\beta)^2_{\alpha', \alpha}$ with $(L(\beta)^2_{\alpha', \alpha})_x = \beta^2_{\alpha', \alpha}$ satisfies (76). That is, we need to verify that for every $x$ we have a commuting diagram:

This commutes by (25).
Verification of (24): along the top and right is \( ((L(\beta)_{\alpha'}) \circ_0 L(H)(\alpha)) \circ_1 (L(H')(\alpha') \circ_0 L(\beta)_{\alpha})_f \), the three arrows along the lower left make up \( (L(\beta)_{\alpha \circ \alpha})_f \).
1.8. **Lemma.** For pseudo-transformations and Gray-functors

\[
\begin{array}{c}
\xymatrix{ & H \ar[rd]_L \ar[dd]_H & \\
\mathcal{H} \ar[ru]_H \ar[rd]_{H'} & & \mathcal{K} \ar[ld]^{H''} \\
& H' \ar[lu]_{L(H)} & }
\end{array}
\]

we have \(L(\beta') \ast_0 L(\beta) = L(\beta' \ast_0 \beta),\) diagrammatically

\[
\begin{array}{c}
\xymatrix{ [G, \mathcal{H}]^p \ar[rr]_{L(H')} \ar[dd]_-{L(H'')} & & [G, \mathcal{K}]^p \ar[dd]^-{L(H'')} \ar[ll]^-{L(H)} \\
\downarrow_{L(\beta')} & & \downarrow_{L(\beta' \ast_0 \beta)} \\
[G, \mathcal{K}]^p & & [G, \mathcal{K}]^p \ar[ll]_-{L(\beta')} }
\end{array}
\]

\[
(26)
\]

**Proof.** In order for the two pseudo-transformations to be equal we need their respective defining data to coincide:

- by lemma \[1.9\] for all \(G : \mathcal{G} \longrightarrow \mathcal{H}\)

\[
(L(\beta') \ast_0 L(\beta))_G = L(\beta')_G \ast_0 L(\beta)_G = L(\beta' \ast_0 \beta)_G,
\]

\[
(27)
\]

- by lemma \[1.10\] for all \(\alpha : G \longrightarrow G'\)

\[
\begin{array}{c}
\xymatrix{ L(H)(G) \ar[rr]_-{L(H')(G)} \ar[dd]_-{L(H)(\alpha)} & & L(H')(G') \ar[dd]_-{L(H')(\alpha)} \\
L(\beta' \ast_0 \beta)_\alpha & & L(\beta' \ast_0 \beta)_\alpha \\
L(H)(G') \ar[rr]^-{L(H')(G')} & & L(H')(G') }
\end{array}
\]

\[
(28)
\]

fix notation

- by lemma \[??\] for all \(A : \alpha \Longrightarrow \alpha'\) the equation of 3-cells \[29\] holds,

- by lemma \[1.12\] finally, equation \[30\] holds for all composable pairs \(\alpha : H \Longrightarrow H', \alpha' : H' \Longrightarrow H''\).

In the sequel we prove the lemmas that constitute the substance of the above proof.

1.9. **Lemma.** \(L(\beta')_H \ast_0 L(\beta)_H = L(\beta' \ast_0 \beta)_H\) for all \(H : \mathcal{G} \longrightarrow \mathcal{H}\).
Proof. On 0-cells, which are Gray-functors $H : \mathcal{G} \rightarrow \mathcal{H}$: we have a pseudo-transformation $(L(\beta') \ast_0 L(\beta))_H : GH \rightarrow G''H$ given on 0-cells $x$ of $\mathcal{G}$ by

$$((L(\beta') \ast_0 L(\beta))_H)_x = ((\beta' \ast_{-1} H) \ast_0 (\beta \ast_{-1} H))_x = \beta'_{Hx} \#_0 \beta_{Hx} , \quad (31)$$

on 1-cells $f : x \rightarrow y$

$$((L(\beta') \ast_0 L(\beta))_H)_f = ((\beta' \ast_{-1} H) \ast_0 (\beta \ast_{-1} H))_f =$$

$$GHx \xrightarrow{\beta_{Hx}} G'Hx \xrightarrow{\beta'_{Hx}} G''Hx$$

$$\downarrow \beta_{Hy} \quad \downarrow \beta'_{Hy} \quad \downarrow \beta''_{Hy}$$

$$GHy \xrightarrow{\beta_{Hy}} G'Hy \xrightarrow{\beta'_{Hy}} G''Hy$$

(32)

on 2-cells by the diagram (33). On composable 1-cells $f', f$ from $\mathcal{G}$ we get the 2-cocycle shown in (34).

Now let us consider the right hand side of (27), that is, we need to give $L(\beta' \ast_0 \beta)_H$; this is simply substituting images under $H$ into the composite spelled out in section 3 which is given on 0-cells by

$$(L(\beta' \ast_0 \beta)_H)_x = ((\beta' \ast_0 \beta) \ast_{-1} H)_x = (\beta' \ast_0 \beta)_{Hx} = \beta'_{Hx} \#_0 \beta_{Hx} , \quad (35)$$

on 1-cells $f : x \rightarrow y$ by

$$L(\beta' \ast_0 \beta)_H)_f = ((\beta' \ast_0 \beta) \ast_{-1} H)_f$$

$$= (\beta' \ast_0 \beta)_Hf =$$

$$GHx \xrightarrow{\beta_{Hx}} G'Hx \xrightarrow{\beta'_{Hx}} G''Hx$$

$$\downarrow \beta_{Hy} \quad \downarrow \beta'_{Hy} \quad \downarrow \beta''_{Hy}$$

$$GHy \xrightarrow{\beta_{Hy}} G'Hy \xrightarrow{\beta'_{Hy}} G''Hy$$

(36)

on 2-cells $\varphi : f \Rightarrow f'$ by

$$(L(\beta' \ast_0 \beta)_H)_\varphi = ((\beta' \ast_0 \beta) \ast_{-1} H)_\varphi = (\beta' \ast_0 \beta)_H\varphi$$

with details given in (37).

For composable pairs of 1-cells $f', f$ we first consider the 2-cocycle associated to the pseudo-transformation $\beta' \ast_0 \beta : G \rightarrow G''$.

□

1.10. Lemma. For pseudo-transformations $\alpha : G \Rightarrow G'$ (28) holds.
\[(L(\beta') *_0 L(\beta))_\varphi = (\beta' *_0 \beta)_H\varphi\]
\[
(L(\beta') *_0 L(\beta))^2 \text{ tensor goes the wrong way? should be } ((L(\beta') *_0 L(\beta))_H)^2_{f,f}
\]
\[(L(\beta' \ast_0 \beta)_H)_\varphi = ((\beta' \ast_0 \beta) \ast_{-1} H)_\varphi = (\beta' \ast_0 \beta)_{H\varphi}\]
Proof. According to theorem 4.6, on 1-cells, which are pseudo-transformations $\alpha : G \Rightarrow G'$ we get a pseudo-modification

\[(L(\beta') *_0 L(\beta))_\alpha = (L(\beta') G *_{-1} L(\beta)) *_0 (L(\beta') *_{-1} L(\beta))_G\]

On 3-cells, i.e. pseudo-modifications $A : \alpha \Rightarrow \alpha' : H \Rightarrow H'$, we get a perturbation \((L(\beta) *_{-1} L(\beta))_A : (L(\beta) *_{-1} L(\beta))_\alpha \Rightarrow (L(\beta) *_{-1} L(\beta))_{\alpha'}\), which on 0-cells of $G$ is given by

\[((L(\beta') *_0 L(\beta))_A)_x\]

Finally, for the pseudo-transformation $L(\beta') *_0 L(\beta)$ we need to give the 2-cocycle \((L(\beta') *_0 L(\beta))^{2}_{\alpha',\alpha}\) for pairs of composable 1-cells from $[G, H]^p$, which means that for every compos-
able pair $\alpha, \alpha'$ of pseudo-transformations we need to give a perturbation

\begin{align*}
\text{(40)}
\end{align*}

This perturbation takes values on 0-cells of $\mathcal{G}$:

\begin{align*}
((\beta' \ast \beta_o \beta)_{\alpha', \alpha})_x
\end{align*}

\begin{align*}
\text{(41)}
\end{align*}

And on 1-cells the pseudo-modifications on the left and right hand sides of (40) are given
A natural transformation with components

\[ i_G : [1, \mathcal{G}]^p \to \mathcal{G} . \] (43)

An extranatural transformation with components

\[ j_G : 1 \to [\mathcal{G}, \mathcal{G}]^p . \] (44)

2. Functors and Transformations of \textit{Gray}-Categories

We collect some definitions from the literature.

2.1. Definition. [e.g. \textit{Gray}, 1974, \textit{Crans}, 1999] A \textbf{Gray-category} is an enriched category, enriched over the category of 2-categories with the \textit{Gray}-tensor product. A \textbf{Gray-functor} is correspondingly an enriched functor.
2.2. Remark. A Gray-functor $G : \mathcal{G} \to \mathcal{H}$ maps i-cells to i-cells and preserves units, compositions, whiskers and tensors.

In more detail, this means that the following equations hold.

1. Units: for $x, f, \varphi, \Gamma$ 0-,1-,2-,3-cells respectively
   
   \begin{align*}
   G \text{id}_x &= \text{id}_{Gx} \\
   G \text{id}_f &= \text{id}_{Gf} \\
   G \text{id}_\varphi &= \text{id}_{G\varphi} \\
   G \text{id}_\Gamma &= \text{id}_{G\Gamma}.
   \end{align*}

2. Whiskers: for suitably incident cells $f, g, \varphi, \psi, \Gamma, \Delta$
   
   \begin{align*}
   G(g \#_0 \varphi) &= Gg \#_0 G\varphi \\
   G(\psi \#_0 f) &= G\psi \#_0 Gf \\
   G(g \#_0 \Gamma) &= Gg \#_0 G\Gamma \\
   G(\Delta \#_0 f) &= G\Delta \#_0 Gf \\
   G(\varphi \#_1 \Delta) &= G\varphi \#_0 G\Delta \\
   G(\Gamma \#_1 \psi) &= G\Gamma \#_1 G\psi.
   \end{align*}

3. Composites: for suitably incident cells $f, g, \varphi, \psi, \Gamma, \Delta$
   
   \begin{align*}
   G(g \#_0 f) &= Gg \#_0 Gf \\
   G(\psi \#_1 \varphi) &= G\psi \#_1 G\varphi \\
   G(\Delta \#_2 \Gamma) &= G\Delta \#_2 G\Gamma.
   \end{align*}

4. Tensors: for 2-cells incident on a 0-cell
   
   \begin{equation}
   G(\psi \otimes \varphi) = G\psi \otimes G\varphi.
   \end{equation}

The following notions were introduced in [Gohla, 2014] with some more generality; we give only the ones used in this paper.

2.3. Definition. For Gray-functors $F, G : \mathcal{G} \to \mathcal{H}$ a pseudo-transformation $\alpha : F \to G$ is given by the following data:

1. for each 0-cell $x$ of $\mathcal{G}$ a 1-cell $\alpha_x : Fx \to Gx$,

2. for each 1-cell $f : x \to y$ of $\mathcal{G}$ an invertible 2-cell
   
   \begin{equation}
   \begin{array}{ccc}
   FX & \xrightarrow{\alpha_x} & GX \\
   FF & \downarrow & \varphi_{\alpha_f} & \downarrow & GF \\
   FY & \xrightarrow{\alpha_y} & GY
   \end{array}
   \end{equation}
3. for each 2-cell $\varphi : f \to f'$ of $G$ an invertible 3-cell of $H$

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi} \ar[u]^-{F\varphi} \ar[d]_{Ff} & Ff' \ar[r]_-{\varphi_{f'}} \ar[u]^-{Ff'} & Gf' \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

(60)

4. for each pair of composable 1-cells $f : x \to y$, $f' : y \to z$ an invertible 3-cell

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff \ar[r]_-{\varphi_{f}} \ar[u]^-{Ff} & Gf \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f'}} \ar[u]^-{Ff'} & Gf' \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff \ar[r]_-{\varphi_{f}} \ar[u]^-{Ff} & Gf \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f'}} \ar[u]^-{Ff'} & Gf' \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

These must satisfy the following conditions:

1. On identities of 0-cells:

\[
\alpha_{id_x} = id_{\alpha_x}
\]  

(62)

2. for each 3-cell $\Gamma : \varphi \to \varphi'$ the square of 3-cells in $H$

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f}} \ar[u]^-{Ff} & Gf \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f'}} \ar[u]^-{Ff'} & Gf' \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

(63)

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f}} \ar[u]^-{Ff} & Gf \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f'}} \ar[u]^-{Ff'} & Gf' \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff \ar[r]_-{\varphi_{f}} \ar[u]^-{Ff} & Gf \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

\[
\begin{array}{c}
\xymatrix@C=10em{Fx \ar[r]^-{\alpha_x} & Gx \\
Ff' \ar[r]_-{\varphi_{f'}} \ar[u]^-{Ff'} & Gf' \ar[u]^-{GF} \ar[d]_{GF} \\
Fy \ar[r]_-{\alpha_y} & Gy}
\end{array}
\]  

commutes.
3. For every pair $\varphi: f \Rightarrow f', \varphi': f' \Rightarrow f''$:

\[
\begin{array}{c}
\text{\includegraphics{diagram.png}}
\end{array}
\]

and for identity 2-cells $\text{id}_f: f \Rightarrow f$ we have an identity 3-cell

\[
\alpha_{\text{id}_f} = \text{id}_{\alpha_f}.
\]  

(65)

4. The family of 3-cells has to satisfy a kind of cocycle condition: For a composable triple $f, f', f''$ of 1-cells $\alpha^2$ has to satisfy equation (66). Furthermore, $\alpha^2$ has to satisfy the normalization condition:

\[
\alpha^2_{f', f} = \begin{cases} 
\text{id}_{\alpha_f} & \text{if } f' = \text{id}_y \\
\text{id}_{\alpha_{f'}} & \text{if } f = \text{id}_x 
\end{cases}
\]  

(67)

5. The family of 3-cells $\alpha^2$ has to be compatible with left and right whiskering according to (68) and (69).

2.4. Definition. A \textbf{pseudo-modification} $A: \alpha \Rightarrow \alpha'$ between pseudo-modifications in the sense of definition 2.3 is given by the following data:

1. For every 0-cell $x$ in $G$ a 2-cell

\[
\begin{array}{c}
\text{\includegraphics{diagram2.png}}
\end{array}
\]

(70)

2. For every 1-cell $f: x \rightarrow y$ a 3-cell in $H$

\[
\begin{array}{c}
\text{\includegraphics{diagram3.png}}
\end{array}
\]

(71)
\[ F_x \xrightarrow{\alpha_x} G_x \]
\[ F_y \xrightarrow{\alpha_y} G_y \]
\[ F_z \xrightarrow{\alpha_z} G_z \]
\[ F_{f'} \xrightarrow{\alpha_{f'}} G_{f'} \]
\[ F_{f''} \xrightarrow{\alpha_{f''}} G_{f''} \]
\[ F_w \xrightarrow{\alpha_w} G_w \]

\[ (\alpha_{f''} \#_0 F)(f' \#_0 f) \]
\[ \#_1 (G(f'' \#_0 f') \#_0 \alpha_f) \]

\[ F_x \xrightarrow{\alpha_x} G_x \]
\[ F_y \xrightarrow{\alpha_y} G_y \]
\[ F_z \xrightarrow{\alpha_z} G_z \]
\[ F_{f'} \xrightarrow{\alpha_{f'}} G_{f'} \]
\[ F_{f''} \xrightarrow{\alpha_{f''}} G_{f''} \]
\[ F_w \xrightarrow{\alpha_w} G_w \]

\[ (\alpha_{f''} \#_0 f') \]
\[ \#_1 (G(f'' \#_0 f') \#_0 \alpha_f) \]

\[ (\alpha_{f'} \#_0 f) \]
\[ \#_1 (G(f' \#_0 f) \#_0 \alpha_f) \]
Compatibility of the cocycle $\alpha^2$ with left whiskers $\gamma\#_0 f$. 

\[(\alpha_2\#_0 F f) \#_1 (Gg'\#_0 \alpha_f) \]
\[F(x) \xrightarrow{\alpha_x} G(x) \]
\[F(f) \xrightarrow{\not\alpha_f} G(f) \]
\[F(y) \xrightarrow{\alpha_y} G(y) \]
\[F(z) \xrightarrow{\alpha_z} G(z) \]

\[(\alpha_{g'}\#_0 F f) \#_1 (G\gamma\#_0 f) \]
\[F(x) \xrightarrow{\alpha_x} G(x) \]
\[F(f) \xrightarrow{\not\alpha_f} G(f) \]
\[F(y) \xrightarrow{\alpha_y} G(y) \]
\[F(g') \xrightarrow{\not\alpha_{g'}} G(g') \]
\[F(z) \xrightarrow{\alpha_z} G(z) \]

\[\alpha^2_{g', f} \#_1 (G\gamma\#_0 f) \#_0 \alpha_z \]
\[F(x) \xrightarrow{\alpha_x} G(x) \]
\[F(f) \xrightarrow{\not\alpha_f} G(f) \]
\[F(y) \xrightarrow{\alpha_y} G(y) \]
\[F(g') \xrightarrow{\not\alpha_{g'}} G(g') \]
\[F(z) \xrightarrow{\alpha_z} G(z) \]

\[(\alpha_2\#_0 F (\gamma\#_0 f)) \#_1 \alpha^2_{g', f} \]
\[F(x) \xrightarrow{\alpha_x} G(x) \]
\[F(f) \xrightarrow{\not\alpha_f} G(f) \]
\[F(y) \xrightarrow{\alpha_y} G(y) \]
\[F(g') \xrightarrow{\not\alpha_{g'}} G(g') \]
\[F(z) \xrightarrow{\alpha_z} G(z) \]

(68)
Compatibility of the cocycle $\alpha^2$ with right whiskers $g\#\delta$. 

\[(\alpha_g \otimes F\delta)^{-1} \#_1(Gg\#_0\alpha_f)\] 

\[(\alpha_g \#_0 Ff') \#_1(Gg\#_0\alpha_g)\] 

\[\alpha^2, f' \#_1(G\#_0\delta) \#_0 \alpha_z\] 

\[(\alpha_\#_0 f') \#_1(G\#_0) \#_0 \alpha_f\] 

\[Fz \quad \alpha_\#_0 \quad Gz\] 

(69)
such that the following conditions hold:

1. Units are preserved:
   \[ A_{\text{id}_x} = \text{id}_{A_x} \]  
   (72)

2. Compatibility with the cocycles of \( \alpha \) and \( \beta \) according to (73)

3. For 2-cells \( g : f \Rightarrow f' \) in \( \mathcal{G} \) the images under \( F \) and \( G \) as well the data of \( A \), \( \alpha \) and \( \beta \) are compatible as shown in (74)

2.5. Definition. A perturbation \( \Gamma : A \Ra A' \) between pseudo-modifications in the sense definition 2.4 is given by a family 3-cells indexed by 0-cells \( x \) of \( \mathcal{G} \):

\[
\begin{align*}
\begin{tikzpicture}
\node(xa)at(1,2){F x \xrightarrow{\alpha_x} G x};
\node(ya)at(1,0){F y \xleftarrow{\beta_y} G y};
\node(xb)at(3,2){F x \xrightarrow{\alpha_x} G x};
\node(yb)at(3,0){F y \xleftarrow{\beta_y} G y};
\node(xc)at(1,2){F x \xrightarrow{\alpha_x} G x};
\node(yc)at(3,2){F x \xrightarrow{\alpha_x} G x};
\node(xd)at(1,0){F y \xleftarrow{\beta_y} G y};
\node(yd)at(3,0){F y \xleftarrow{\beta_y} G y};
\end{tikzpicture}
\end{align*}
\]  
(75)

such that

\[
\begin{align*}
\begin{tikzpicture}
\node(xa)at(1,2){F x \xrightarrow{\alpha_x} G x};
\node(ya)at(1,0){F y \xleftarrow{\beta_y} G y};
\node(xb)at(3,2){F x \xrightarrow{\alpha_x} G x};
\node(yb)at(3,0){F y \xleftarrow{\beta_y} G y};
\node(xc)at(1,2){F x \xrightarrow{\alpha_x} G x};
\node(yc)at(3,2){F x \xrightarrow{\alpha_x} G x};
\node(xd)at(1,0){F y \xleftarrow{\beta_y} G y};
\node(yd)at(3,0){F y \xleftarrow{\beta_y} G y};
\end{tikzpicture}
\end{align*}
\]  
(76)

commutes.

3. Composites in \( [\mathcal{G}, \mathcal{H}]^p \)

3.1. Composites of Pseudo-Transformations. For reference we the data for the composite \( G \xrightarrow{\beta} G' \xrightarrow{\beta} G'' \) of pseudo-transformations. This also appears [Gohla, 2014, Appendix C], albeit for pseudo-functors.
Compatibility of the modification $A$ with the cocycles of $\alpha$ and $\beta$
Compatibility of 2-cells with $A$, $\alpha$ and $\beta$
1. On 0-cells:

\[(\beta' *_{0} \beta)_x = \beta_x' \#_{0} \beta_x .\]  

(77)

2. On 1-cells:

\[
\begin{align*}
&Gx \xrightarrow{\beta_x} G'x \xrightarrow{\beta'_x} G''x \\
&(\beta' *_{0} \beta)_f = Gf \\
&Gy \xrightarrow{\beta_y} G'y \xrightarrow{\beta'_y} G''y .
\end{align*}
\]  

(78)

3. On 2-cells: See (79).

4. On composable pairs: see (80).

3.2. Composites of Pseudo-Modifications. The composite \(A' *_{1} A\) of pseudo-modifications

\[
\begin{align*}
\begin{array}{c}
\alpha_x \\
\downarrow A_x \\
G \\
\downarrow G' \\
\downarrow A''_x \\
\end{array}
\end{align*}
\]  

is given by

\[(A' *_{1} A)_x = A'_x \#_{1} A_x \]  

(81)

\[
\begin{align*}
&Gx \xrightarrow{\alpha_x} (A' *_{1} A)_x G'x \\
&Gy \xrightarrow{\alpha'_x} (A' *_{1} A)_x G'y \\
&Gf \xrightarrow{\alpha'_f} (A' *_{1} A)_f G''f .
\end{align*}
\]  

(82)

3.3. Whiskers of Pseudo-Modifications. For a pseudo-transformation and a pseudo-modification

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\downarrow A \\
G \\
\downarrow G' \\
\downarrow \beta \\
\downarrow G'' \\
\end{array}
\end{align*}
\]  

we define the right whisker \(\beta *_{0} A\) as follows:

\[(\beta *_{0} A)_x = \beta_x' \#_{0} A_x \]  

(83)
(β' *₀ β)ϕ

\[
\begin{array}{c}
Gx \xrightarrow{β_x} G'x \xrightarrow{β'_x} G''x \\
G'y \xrightarrow{β_y} G''y \\
Gf \xrightarrow{β_f} G'f \\
Gf' \xrightarrow{β'_f} G''f
\end{array}
\]

(β'₀ #₀ β₀)

#₁(β'_₀ #₀ β₀)

(β'₀ #₀ β₀)

#₁(β'_₀ #₀ β₀)

(β'₀ #₀ β₀)

#₁(β'_₀ #₀ β₀)

(79)
\[(\beta' \circ \beta)^2_{f,f}\]
For a pseudo-transformation and a pseudo-modification

\[ G \xrightarrow{\alpha} G' \xrightarrow{\beta} G'' \]

we define the right whisker \( B \star_0 \alpha \) as follows:

\[ (B \star_0 \alpha)_x = B_x \#_0 \alpha_x \]  

(85)

and (86).

4. Horizontal Composition in \( \text{GrayCat}_Q \):  

We define the composition \( \_ \star_{-1} \_ \) of \( \text{Gray} \)-functors, pseudo-transformation, pseudo-modification, and perturbations along a \( \text{Gray} \)-category:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
*_{-1} & G & \alpha & A & \Gamma \\
\hline
H & H \star_{-1} G & H \star_{-1} \alpha & H \star_{-1} A & H \star_{-1} \Gamma \\
\beta & \beta \star_{-1} G & \beta \star_{-1} \alpha & \beta \star_{-1} A & \beta \star_{-1} \Gamma \\
B & B \star_{-1} G & B \star_{-1} \alpha & B \star_{-1} A & B \star_{-1} \Gamma \\
\Delta & \Delta \star_{-1} G & \Delta \star_{-1} \alpha & \Delta \star_{-1} A & \Delta \star_{-1} \Gamma \\
\hline
\end{array}
\]

On a 0-cell \( x \) each of these is defined as

\[
\begin{array}{|c|c|c|c|c|}
\hline
*_{-1} & G & \alpha: G \to G' & A & \Gamma \\
\hline
H & HGx & H\alpha_x & HA_x & H\Gamma_x \\
\beta: H \to H' & \beta_{Hx} &HGx & H'\alpha_x & H'\Gamma_x \\
B & B_{Gx} & B_{\alpha_x} & id & id \\
\Delta & \Delta_{Gx} & id & id & id \\
\hline
\end{array}
\]

On a 1-cell \( f \) each of these is defined as

\[
\begin{array}{|c|c|c|c|c|}
\hline
*_{-1} & G & \alpha: G \to G' & A & \Gamma \\
\hline
H & HGf & H\alpha_f & HA_f & id \\
\beta: H \to H' & \beta_{Gf} & (\beta \star_{-1} \alpha)_f & id & id \\
B & B_{Gf} & id & id & id \\
\Delta & id & id & id & id \\
\hline
\end{array}
\]
(\beta \ast_{0} A)_f =
$(B *_0 \alpha)_f =$

\[
\begin{array}{c}
\frac{Gx}{ \alpha_x} \xrightarrow{gf} \frac{G'x}{ \beta_x} \xrightarrow{B_x} \frac{G''x}{ \beta_x} \\
\frac{Gy}{ \alpha_y} \xrightarrow{\beta'_y} \frac{G'y}{ \beta'_y} \xrightarrow{B_y} \frac{G''y}{ \beta_y}
\end{array}
\]

\[
\left( \beta'_y \#_0 \alpha_f \right) \quad \left( B_f \#_0 \alpha_x \right)
\]

\[\#_1 \left( B_f \#_0 \alpha_x \right) \quad \#_1 \left( \beta'_y \#_0 \alpha_x \right) \]

(86)
\[(\beta \ast_{-1} \alpha)_f\]
On a 2-cell $\varphi$ each of these is defined as

\[
\begin{array}{c|cccc}
\ast_{-1} & G & \alpha & A & \Gamma \\
\hline
H & HG\varphi & H\alpha_{\varphi} & \text{id} & \text{id} \\
\beta & \beta_{G\varphi} & \text{id} & \text{id} & \text{id} \\
B & \text{id} & \text{id} & \text{id} & \text{id} \\
\Delta & \text{id} & \text{id} & \text{id} & \text{id}
\end{array}
\]

On a 3-cell $\Sigma$ each of these is defined as

\[
\begin{array}{c|cccc}
\ast_{-1} & G & \alpha & A & \Gamma \\
\hline
H & HG\Sigma & \text{id} & \text{id} & \text{id} \\
\beta & \text{id} & \text{id} & \text{id} & \text{id} \\
B & \text{id} & \text{id} & \text{id} & \text{id} \\
\Delta & \text{id} & \text{id} & \text{id} & \text{id}
\end{array}
\]

On a pair of composable 1-cells $f, f'$ each of these is defined as

\[
\begin{array}{c|cccc}
\ast_{-1} & G & \alpha: G \to G' & A & \Gamma \\
\hline
H & HG(f' \#_0 f) & H(\alpha^2_{f,f'}) & H A_{f'} & \text{id} \\
\beta: H \to H' & \beta^2_{Gf',Gf} & \text{id} & \text{id} & \text{id} \\
B & B_{Gf'} & \text{id} & \text{id} & \text{id} \\
\Delta & \text{id} & \text{id} & \text{id} & \text{id}
\end{array}
\]

We also need to define the non-dimension raising horizontal composites:

\[
\begin{array}{c|cccc}
\triangleleft_{-1} & G & \alpha: G \to G' & A: \alpha \Rightarrow \alpha' & \Gamma: A \Rightarrow A' \\
\hline
H & H \ast_{-1} G & H \ast_{-1} \alpha & H \ast_{-1} A & H \ast_{-1} \Gamma \\
\beta: H \to H' & \beta \ast_{-1} G & (\beta \ast_{-1} G') \ast_0 (H \ast_{-1} \alpha) & (\beta \ast_{-1} G') \ast_0 (H \ast_{-1} A) & (\beta \ast_{-1} G') \ast_0 (H \ast_{-1} \Gamma) \\
B: \beta \Rightarrow \beta' & B \ast_{-1} G & (B \ast_{-1} G') \ast_0 (H \ast_{-1} \alpha) & (B \ast_{-1} G') \ast_0 (H \ast_{-1} A) & (B \ast_{-1} G') \ast_0 (H \ast_{-1} \Gamma) \\
\Delta: B \Rightarrow B' & \Delta \ast_{-1} G & (\Delta \ast_{-1} G') \ast_0 (H \ast_{-1} \alpha) & (\Delta \ast_{-1} G') \ast_0 (H \ast_{-1} A) & (\Delta \ast_{-1} G') \ast_0 (H \ast_{-1} \Gamma)
\end{array}
\]

\[
\begin{array}{c|cccc}
\triangleright_{-1}, \triangleleft_{-1} & G & \alpha: G \to G' & A: \alpha \Rightarrow \alpha' & \Gamma: A \Rightarrow A' \\
\hline
H & H \ast_{-1} G & H \ast_{-1} \alpha & H \ast_{-1} A & H \ast_{-1} \Gamma \\
\beta: H \to H' & \beta \ast_{-1} G & (H' \ast_{-1} \alpha) \ast_0 (\beta \ast_{-1} G) & (H' \ast_{-1} A) \ast_0 (\beta \ast_{-1} G) & (H' \ast_{-1} \Gamma) \ast_0 (\beta \ast_{-1} G) \\
B: \beta \Rightarrow \beta' & B \ast_{-1} G & (H' \ast_{-1} \alpha) \ast_0 (B \ast_{-1} G) & (H' \ast_{-1} A) \ast_0 (B \ast_{-1} G) & (H' \ast_{-1} \Gamma) \ast_0 (B \ast_{-1} G) \\
\Delta: B \Rightarrow B' & \Delta \ast_{-1} G & (H' \ast_{-1} \alpha) \ast_0 (\Delta \ast_{-1} G) & (H' \ast_{-1} A) \ast_0 (\Delta \ast_{-1} G) & (H' \ast_{-1} \Gamma) \ast_0 (\Delta \ast_{-1} G)
\end{array}
\]

These cells have dimensions:

\[
\begin{array}{c|cccc}
\triangleright_{-1}, \triangleleft_{-1} & G & \alpha: G \to G' & A: \alpha \Rightarrow \alpha' & \Gamma: A \Rightarrow A' \\
\hline
H & 0 & 1 & 2 & 3 \\
\beta: H \to H' & 1 & 1 & 2 & 3 \\
B: \beta \Rightarrow \beta' & 2 & 2 & 3 & 3 \\
\Delta: B \Rightarrow B' & 3 & 3 & 3 & 3
\end{array}
\]
For reference we spell out the data for $\beta \triangleleft_{-1} \alpha$: are given in (88), (89) and (90).

\[
\begin{align*}
HGx \xrightarrow{H_\alpha x} HG'x & \xrightarrow{\beta_{G'}x} H'G'x \\
\downarrow HGF & \downarrow HGF' \quad \downarrow H'GF' \\
HGy \xrightarrow{H_\alpha y} HG'y & \xrightarrow{\beta_{G'y}} H'G'y
\end{align*}
\]

We also spell out the data for $\beta \triangleright_{-1} \alpha$: in dimensions 0 and 1 in (91), in dimension 2 in (92), and for composable 1-cells in (93).

\[
\begin{align*}
HGx \xrightarrow{\beta_{Gx}} H'Gx & \xrightarrow{H'\alpha x} H'G'x \\
\downarrow HGF & \downarrow HGF' \quad \downarrow H'GF' \\
HGy \xrightarrow{\beta_{Gy}} H'Gy & \xrightarrow{H'\alpha y} H'G'y
\end{align*}
\]

4.1. Theorem. The compositions defined above result in the following types of cells:

| dimension | type              | $H _{-1} G$          |
|-----------|------------------|----------------------|
| 0         | Gray-functor     | $H _{-1} G$          |
| 1         | pseudo-transformation | $H _{-1} \alpha, \beta _{-1} G$ |
| 2         | pseudo-modification      | $H _{-1} A, \beta _{-1} \alpha, B _{-1} G$ |
| 3         | perturbation        | $H _{-1} \Gamma, \beta _{-1} A, B _{-1} \alpha, \Delta _{-1} G$ |

Proof. In dimension 0 $HG$ is trivially a Gray-functor. Dimension 1 is equally trivial. In dimension 2 the only non-trivial case is $\beta _{-1} \alpha$ which is proved in lemma 4.2. In dimension 3 $H _{-1} \Gamma, \Delta _{-1} G$ are obviously perturbations. That $\beta _{-1} A, B _{-1} \alpha$ are such is shown in lemmata 4.3 and 4.4.

4.2. Lemma. $\beta _{-1} \alpha$ as defined above is a pseudo-modification

\[
\beta _{-1} \alpha; \beta \triangleleft_{-1} \alpha \implies \beta \triangleright_{-1} \alpha \\
= (\beta _{-1} G') *_0 (H _{-1} \alpha) \implies (H' _{-1} \alpha) *_0 (\beta _{-1} G).
\]

Proof. We need to verify that $\beta _{-1} \alpha$ satisfies the conditions 2.4. The conditions of definition (72) is obvious from (87).

Next, we check that our definition satisfies (73), that is, we need to verify that (95) commutes, which we do in (96).

Finally, we check that (74) holds for $\beta _{-1} \alpha$, i.e. that (97) holds, this is carried out in (98).

4.3. Lemma. Our definition of $-_{-1}$ makes $\beta _{-1} A$, a perturbation.

Proof. Complete the proof
Details of \((\beta <_1 \alpha)_\phi\)
Details of \((\beta \triangleleft \alpha)^2_{f,f}\)

\[
\begin{align*}
\beta'^g &\triangleleft 0 \Delta \alpha, \\
\beta'^t &\triangleleft 0 \Delta \alpha, \\
\beta'^t &\triangleright H'^t \triangleright \alpha,
\end{align*}
\]
Details of \((\beta \triangleright -1 \alpha)_\varphi\)

\[
\begin{align*}
HGx & \xrightarrow{\beta_{Gx}} H'Gx \xrightarrow{H'\alpha_x} H'G'x \\
H'Gx & \xrightarrow{\beta_{Gx}} H'G'x \\
HGy & \xrightarrow{\beta_{Gy}} H'Gy \xrightarrow{H'\alpha_y} H'G'y \\
H'Gy & \xrightarrow{\beta_{Gy}} H'G'y \\
HGf & \xrightarrow{\beta_{Gf}} H'Gf \\
H'Gf & \xrightarrow{H'\alpha_f} H'G'f \\
HG'f & \xrightarrow{H'\alpha_f} H'G'_f \\
\end{align*}
\]
Details of $(\beta \triangleright \alpha)^2_{f, f}$

\[ (H'\alpha x \#_0 \beta_f \#_0 H G f) \]
\[ \#_1 (H'\alpha x' \#_0 \beta_f) \]
\[ \#_1 (H'G'f' \#_0 H'\alpha f' \#_0 \beta G x) \]

\[ (H'\alpha x \#_0 \beta_{G'f} \#_0 H G f) \]
\[ \#_1 (H'\alpha x' \#_0 \beta_{G'f}) \]
\[ \#_1 (H'G'f' \#_0 H'\alpha f' \#_0 \beta G x) \]

\[ (H'\alpha x \#_0 \beta_{G'f} \#_0 H G f) \]
\[ \#_1 (H'\alpha x' \#_0 \beta_{G'f}) \]
\[ \#_1 (H'G'f' \#_0 H'\alpha f' \#_0 \beta G x) \]
Verification of (95). Unlabeled subdiagrams commute by naturality. All 3-cells shown are canonical and invertible.
Verification of (97). Unlabeled subdiagrams commute by naturality. All 3-cells shown are canonical and invertible.
4.4. **Lemma.** Our definition of $\_ *_{-1} \_ \text{ makes } B *_{-1} \alpha$, a perturbation.

**Proof.** We complete the proof the derived horizontal compositions $\triangleright, \triangleleft, \ldots$

4.5. **Theorem.** The $*_{-1}$-composition preserves units, i.e. $\beta *_{-1} \text{id}_G = \text{id}_{\beta *_{-1} G}$.

**Proof.** We see that $(\beta *_{-1} \text{id}_G)_x = \text{id}_{\beta *_{-1} G}$ by (62) and $(\beta *_{-1} \text{id}_G)_f = \beta *_{-1} \text{id}_G$ using (57) and (67).

4.6. **Theorem.** For Gray-functors and pseudo-transformations

we have the following compatibility of $*_{-1}$ and $*_{0}$:

\[
\begin{align*}
\beta *_{-1} G & \quad H'' G & \beta *_{-1} G & \quad H'' G \\
\beta *_{-1} \alpha & \quad H'' G & \beta *_{-1} \alpha & \quad H'' G \\
G & \quad H & \quad G & \quad H \\
\beta *_{-1} \alpha & \quad H'' G' & \beta *_{-1} \alpha & \quad H'' G' \\
\beta *_{-1} G & \quad H G & \beta *_{-1} G & \quad H G \\
\beta *_{-1} \alpha & \quad G' & \beta *_{-1} \alpha & \quad G' \\
\end{align*}
\]

\[
\begin{align*}
\beta *_{-1} G & \quad H'' G & \beta *_{-1} G & \quad H'' G \\
\beta *_{-1} \alpha & \quad H'' G & \beta *_{-1} \alpha & \quad H'' G \\
\beta *_{-1} G & \quad H G & \beta *_{-1} G & \quad H G \\
\beta *_{-1} \alpha & \quad G' & \beta *_{-1} \alpha & \quad G' \\
\end{align*}
\]

\[
(99)
\]

**Proof.** We need to show that the two pseudo-modifications defined in (99) are equal.

The left-hand and the right-hand side of (99) can be evaluated according the definitions of $*_{0}$ and $*_{-1}$ given in this section and in 3. On 0-cells we get

\[
(((\beta' *_{-1} G') *_{0} (\beta *_{-1} \alpha)) *_{1} ((\beta' *_{-1} \alpha) *_{0} (\beta *_{-1} G')))_x = (\beta' *_{0} \beta)_{\alpha_x} = ((\beta' *_{0} \beta) *_{-1} \alpha)_x.
\]

On 1-cells the left hand side is given by (100); the right-hand side of (99) is given by (101). We verify their equality in (105).
The left-hand side of (99), i.e. $(((\beta' \ast \neg 1 G') \ast 0 (\beta \ast \neg 1 \alpha)) \ast \neg 1 ((\beta' \ast \neg 1 \alpha) \ast 0 (\beta \ast \neg 1 G)))_f$, according to (82), (84) and (84).
$((\beta' \ast_0 \beta) *_{-1} \alpha)_f$. See (102), (103), and (104) for a breakdown of the 3-cells.
Details of $((\beta' \star_0 \beta)^2\#_0 G_f)^{-1}$

\[
\begin{align*}
&\beta_{G_x}' \\
&\beta_{G_x} \\
&H G_x \\
&H'G_f \\
&H'G_y \\
&H''G_y \\
&H''G_f \\
&H''G_x
\end{align*}
\]
Details of $(\beta' \ast_0 \beta)_{\alpha_f}$

(104)
Verification of the equality of (100) (top row) and (101) (remaining outline). Unlabeled subdiagrams commute by naturality.
References

Sjoerd E. Crans. A tensor product for \textbf{Gray}-categories. \textit{Theory Appl. Categ.}, 5:No. 2, 12–69 (electronic), 1999. ISSN 1201-561X.

Björn Gohla. Mapping spaces of Gray-categories. \textit{Theory Appl. Categ.}, 29:100–187, 2014. ISSN 1201-561X/e.

John W. Gray. Formal category theory: Adjointness for 2-categories. Lecture Notes in Mathematics. 391. Berlin-Heidelberg-New York: Springer-Verlag. XII, 282 p., 1974.

Ross Street. Skew-closed categories. \textit{J. Pure Appl. Algebra}, 217(6):973–988, 2013. ISSN 0022-4049. doi: 10.1016/j.jpaa.2012.09.020.