A complex-analytic approach to kinetic energy properties of irrotational traveling water waves

Olivia Constantin

Received: 15 September 2021 / Accepted: 22 March 2022 / Published online: 2 June 2022
© The Author(s) 2022, corrected publication 2022

Abstract
Relying on conformal mappings we prove the logarithmic convexity of certain flow quantities associated with irrotational periodic travelling waves that propagate at the surface of water over a flat bed. These results enable us to quantify the observation that the kinetic energy and the time-period of the particle paths are larger near the surface and reduce with increasing depth.

Keywords Travelling water waves · Kinetic energy · Conformal mapping

Mathematics Subject Classification 76B15 · 30C20

1 Introduction

Regular ocean waves propagating practically without change of form at constant speed in a fixed direction are commonly observed. Their sharp crests and flat troughs, departing from sinusoidal profiles, are indicative of the nonlinear processes at play. Considerable theoretical progress in the understanding of nonlinear flow properties was achieved in the last decades by relying on an interplay between harmonic and complex analysis for irrotational two-dimensional waves at the surface of a fluid of infinite depth (see [1, 15]), setting that is considered adequate to model deep-water waves. In recent years the flat-bed case received a lot of attention (see the discussions in [3, 6, 11, 14]), and complex-analytic considerations were central to these developments. The flat-bed hypothesis is physically realistic in many circumstances: other than arctic continental shelves with a layer of relatively shallow water that cover thousands of km², there are even vaster deep abyssal plains in some major sea and ocean basins, featuring depth variations of less than 0.01%.

Note that water waves are fluctuations of the free surface that signal tremendous energy transport (the associated mass transport being of a much lesser degree). For this reason, a better understanding of the properties of the kinetic energy associated to these wave phenomena is of great importance (see Sect. 5 and the discussions in [5, 9] for further details). It is

✉ Olivia Constantin
olivia.constantin@univie.ac.at

1 Faculty of Mathematics, University of Vienna, Oskar-Morgentern-Platz 1, 1090 Vienna, Austria
natural to start with the most regular wave patterns—two-dimensional periodic travelling waves in irrotational flow over a flat bed, for which complex analysis techniques prove to be very powerful in revealing aspects of the flow structure. We investigate the total kinetic energy of a particle and, in the process of doing this, we also obtain some by-products that are of independent interest. In particular, we clarify an issue relevant to the particle-path pattern. While a good theoretical understanding of the particle paths beneath such waves is already available, supported by numerical simulations and experimental results (see the discussions in [4], respectively [3, 10, 11, 16]), the uniformity in time of the particle paths, that is noticeable numerically and experimentally, has so far not been established theoretically. More precisely, each particle describes a repeated pattern. We show that the time it takes a particle to describe one of these repeating orbits is independent of the initial location of the particle, in the sense that it depends only on the corresponding streamline, thus confirming the experimental evidence (and showing the uniformity of the “elapsed time” concept introduced in [4] as the time needed for a particle initially located beneath the wave trough to regain a position beneath the wave trough). From now on we are going to refer to this specific time as to the streamline time-period. Thus, the maximal elevation of each particle is periodic in time with period equal to streamline time-period, while the horizontal position is shifted (the shift being dependent on the streamline) in the direction of wave propagation after each streamline time-period. An important step in our approach is to notice that the streamline time-period can be expressed as an integral over a horizontal segment of the modulus of a suitably defined analytic function, a feature which permits us to apply Hardy’s convexity theorem for an annulus. The link to such integrals of analytic functions helps us further in the study of the total kinetic energy of a particle over a streamline time-period.

The paper is organized as follows. In Sect. 2 we present the governing equations and a transformation to an equivalent form that better suits our purposes. Section 3 is devoted to the study of the streamline time-period of a particle and in Sect. 4 we derive some results about the total kinetic energy of a particle over a streamline time-period. Finally, some background material is provided in Sect. 5, putting the mathematical considerations made in the paper in a broader context of water-wave energy studies.

2 Preliminaries

For two-dimensional water waves it suffices to investigate the flow characteristics in a cross-section oriented towards the direction of wave propagation. We choose Cartesian coordinates \((X, Y)\) with the \(X\)-axis pointing in the direction of wave propagation and the \(Y\)-axis oriented upwards. Let \(Y = -d\) be the flat bed, \(Y = \eta(X, t)\) be the free surface and \((U(X, Y, t), V(X, Y, t))\) the velocity field. Under the physically reasonable assumption of a homogeneous inviscid flow, the governing equations for a flow determined by the balance between the restoring gravity force and the inertia of the system are the equation of mass conservation

\[
UX + VY = 0, \quad -d < Y < \eta(X, t),
\]

and Euler’s equations

\[
\begin{align*}
U_t + UUX + VUY &= -PX, \quad -d < Y < \eta(X, t), \\
V_t + UVX + VVY &= -PY - g.
\end{align*}
\]
where $P(X, Y, t)$ is the pressure and $g$ is the constant gravitational acceleration. The associated boundary conditions are the kinematic boundary conditions

$$V = \eta_t + U\eta_X \quad \text{on} \quad Y = \eta(X, t),$$

$$V = 0 \quad \text{on} \quad Y = -d,$$

expressing the fact that the water’s free surface and the flat bed are interfaces, and, the effect of surface tension being negligible, the dynamic boundary condition

$$P = P_{\text{atm}} \quad \text{on} \quad Y = \eta(X, t),$$

decoupling the motion of the water from that of the air above it, with $P_{\text{atm}}$ the constant atmospheric pressure at sea level. The absence of non-uniform currents is ensured by the irrotational character of the flow, requiring

$$U_Y - V_X = 0,$$

A Stokes wave is a smooth travelling wave solution to the governing equations (1)-(6) for which there exists a period $\lambda > 0$ and a wave speed $c > 0$ such that the free surface profile $\eta$, the fluid velocity $(U, V)$ and the pressure $P$ have period $\lambda$ in the $X$ variable, $\eta$ depends only on $(X - cT)$, while $U$, $V$, and $P$ depend only on $(X - cT)$ and $Y$. Moreover, the wave profile is strictly monotonic between successive crests and troughs (in particular, there is a single crest and trough per period), and symmetric about the wave crest. Since we only consider smooth waves, the flow presents no stagnation points, that is,

$$U < c \quad \text{holds throughout} \quad -d < Y < \eta(X - ct).$$

It is known (see the discussion in [14]) that for irrotational travelling waves the breakdown of (7) can only occur at the wave crest, in which case the free surface is not a continuously differentiable curve, being real-analytic except at the crest where it is continuous with a corner containing an inner angle of $120^\circ$ (the setting for incipient wave breaking). In contrast to this, if (7) holds, then a global bifurcation approach ensures the existence of waves of small, moderate and large amplitude, whose symmetric profiles must be real-analytic and for which the velocity components have harmonic extensions across the free surface and the flat bed (see the discussion in [14]).

Taking advantage of the $(X, t)$-dependence of the form $(X - ct)$, passing to the moving frame

$$x = X - ct, \quad y = Y,$$  

and writing

$$U(X, Y, t) = u(X - ct, Y) = u(x, y), \quad V(X, Y, t) = v(X - ct, Y) = v(x, y),$$

$$P(X, Y, t) = p(X - ct, Y) = p(x, y),$$

we can reformulate the governing equations for Stokes waves in the form

$$(u - c)u_x + vu_y = -p_x, \quad -d < y < \eta(x),$$

$$(u - c)v_x + vv_y = -p_y - g, \quad -d < y < \eta(x),$$

$$u_x + v_y = 0, \quad -d < y < \eta(x),$$

$$u_y = v_x, \quad -d < y < \eta(x),$$

$$v = (u - c)\eta_x \quad \text{on} \quad y = \eta(x),$$

$$v = 0 \quad \text{on} \quad y = -d,$$
\[ p = P_{atm} \text{ on } y = \eta(x), \] (15)

with
\[ u - c < 0 \quad -d \leq y \leq \eta(x). \] (16)

Without loss of generality we can assume the wave crest to be located at \( x = 0 \), so that \( x = \pm \lambda/2 \) are trough lines and \( \eta(x) = \eta(-x) \) for all \( x \in \mathbb{R} \), with \( \eta \) increasing on \([-\lambda/2, 0]\) and decreasing on \([0, \lambda/2]\); see Fig. 1.

Structural properties of the system (9)–(15) can be used to reduce the number of unknowns without loss of information. First notice that the pressure can be eliminated by taking advantage of Bernoulli’s law: the expression
\[ \frac{(u-c)^2 + v^2}{2} + p + g(y + d) \]
is constant throughout the fluid. A further reduction can be achieved by introducing the stream function \( \psi(x, y) \), defined up to an additive constant by
\[ \psi_x = -v, \quad \psi_y = u - c. \] (17)

Note that the (11)–(12) ensure that \( \psi \) is harmonic throughout the fluid domain (in the moving frame)
\[ D = \{(x, y) : x \in \mathbb{R}, -d \leq y \leq \eta(x)\}, \]
while the kinematic boundary conditions (13)–(14) together with the non-existence of stagnation points, (16), ensure that \( \psi \) is constant and equal to its minimum value \( \psi_{\min} \) on the free surface \( y = \eta(x) \), and constant equal to its maximum value \( \psi_{\max} \) on the flat bed \( y = -d \).

For \( y \in [-d, \eta(\lambda)/2) \) we have
\[ \psi(\lambda/2, y) - \psi(-\lambda/2, y) = \int_{-\lambda/2}^{\lambda/2} \psi_x(s, y) \, ds = -\int_{-\lambda/2}^{\lambda/2} v(s, y) \, ds, \]
and this last term turns out to vanish since (11) and the periodicity of \( u \) yield
\[ \partial_y \left\{ -\int_{-\lambda/2}^{\lambda/2} v(s, y) \, ds \right\} = -\int_{-\lambda/2}^{\lambda/2} v_y(s, y) \, ds = \int_{-\lambda/2}^{\lambda/2} u_x(s, y) \, ds = u(\lambda/2, y) - u(-\lambda/2, y) = 0, \quad -d \leq y \leq \eta(\lambda)/2, \]
while (14) gives
\[ \int_{-\lambda/2}^{\lambda/2} v(s, -d) \, ds = 0. \]

Similarly one can show that
\[ \psi(3\lambda/2, y) = \psi(\lambda/2, y), \quad -d \leq y \leq \eta(\lambda/2). \]

The maximum principle for the harmonic function \( (x, y) \mapsto \psi(x + \lambda, y) - \psi(x, y) \) in the rectangular domain
\[ D_0 = \{(x, y) \in \mathbb{R}^2 : -\lambda/2 \leq x \leq \lambda/2, -d \leq y \leq \eta(x)\}, \]
now yields that \( \psi(x, y) \) is \( \lambda \)-periodic in the \( x \)-variable throughout the whole fluid domain \( D \). A straightforward argument using the maximum principle for the harmonic harmonic
function \((x, y) \mapsto \psi(x, y) - \psi(-x, y)\) in \(D_0\) then shows that, due to the symmetry of the wave profile, we must have

\[
\psi(x, y) = \psi(-x, y), \quad x \in \mathbb{R}, \quad y \in [-d, \eta(x)], \quad (18)
\]

and consequently we get

\[
\begin{aligned}
u(x, y) &= u(-x, y), \quad x \in \mathbb{R}, \quad y \in [-d, \eta(x)], \\
v(x, y) &= -v(-x, y), \quad x \in \mathbb{R}, \quad y \in [-d, \eta(x)].
\end{aligned} \quad (19)
\]

(Note that quite often the fact that \(u\) is even and \(v\) is an odd in the \(x\)-variable is assumed to be part of the symmetry of the flow pattern, while the above argumentation shows that these properties have to hold if the travelling wave profile is symmetric.) The relations (19) show that \(v(0, \cdot) = 0\) and, since \(v\) is periodic in the \(x\) variable, we also have \(v(\pm \lambda/2, \cdot) = 0\). Using these, the kinematic boundary conditions (13)–(14) together with the monotonicity of \(\eta\), we can now deduce that

\[
v(x, \cdot) < 0 \text{ for } x \in (-\lambda/2, 0) \text{ and } v(x, \cdot) > 0 \text{ for } x \in (0, \lambda/2),
\]

by the maximum principle applied to the harmonic function \(v\).

Let us now define the \emph{velocity potential} \(\varphi(x, y)\) as the harmonic conjugate of \(\psi\), defined uniquely up to an additive constant by

\[
\varphi_x = \psi_y, \quad \varphi_y = -\psi_x, \quad -d \leq y \leq \eta(x). \quad (21)
\]

From (21) we deduce that \(\varphi\) is constant and equal to its minimum value \(\varphi_{\min}\) on the vertical segment \(x = \lambda/2\), and constant equal to its maximum value \(\varphi_{\max}\) on \(x = -\lambda/2\). For simplicity, we shall from now on assume without loss of generality that \(\psi_{\min} = \varphi_{\min} = 0\), so that

\[
-\psi_{\max} = \int_{-d}^{\eta(x)} \psi_y(x, y) \, dy = \int_{-d}^{\eta(x)} [u(x, y) - c] \, dy, \quad x \in \mathbb{R},
\]

is the \emph{relative mass flux} (that is, the mass flux relative to the uniform horizontal flow at speed \(c\); see [2]). Since the function \((x, y) \mapsto \varphi(x + \lambda, y) - \varphi(x, \lambda)\) is the harmonic conjugate of the identically zero harmonic map \((x, y) \mapsto \psi(x + \lambda, y) - \psi(x, \lambda)\) throughout \(D\), we deduce that this map is constant. Thus

\[
\varphi(x + \lambda, y) - \varphi(x, y) = \varphi(\lambda/2, y) - \varphi(-\lambda/2, y) = -\varphi_{\max}, \quad (22)
\]

for any \((x, y)\) in the fluid domain \(D\). The fact that at any depth \(y\) below the wave trough level \(\eta(\lambda/2)\) we have

\[
\int_{-\lambda/2}^{\lambda/2} [u(x, y) - c] \, dx = \int_{-\lambda/2}^{\lambda/2} \varphi_x(x, y) \, dx = -\varphi_{\max},
\]

means that \(-\varphi_{\max}\) represents the strength of the \emph{uniform current} underlying the irrotational flow, currents being defined as means of the horizontal velocity component (see [2]).

In light of (11)–(12), the function \(F = \varphi + i\psi\), called the \emph{hodograph transform}, is analytic in the interior of \(D\) and we have \(F' = (u - c) + i(-v)\). In particular, \(F\) is a biholomorphic map between the interior of periodicity cell \([x, y) : x \in [-\lambda/2, \lambda/2], -d \leq y \leq \eta(x)]\) and the open rectangle \([q, p) : 0 < q < \psi_{\max}, 0 < p < \psi_{\max})\), and it extends to a
homeomorphism between the closures of these domains. The hodograph transform induces the following orientation-reversing conformal change of variables

\[
\begin{align*}
q &= \varphi(x, y), \\
p &= \psi(x, y),
\end{align*}
\] (23)

which is a global diffeomorphism from the fluid domain \( D \) to the closure of the horizontal strip

\[ \mathcal{R}_\psi = \{(q, p) : q \in \mathbb{R}, 0 < p < \psi_{\text{max}}\}. \]

Note that the conformal bijection \( q + ip \mapsto x + iy \) from the horizontal strip to the fluid domain in the moving frame, obtained as the inverse of the change of variables (23),

\[
\begin{align*}
x &= x(q, p), \\
y &= y(q, p),
\end{align*}
\] (24)

has the “periodicity” properties (see Fig. 1)

\[
\begin{align*}
x(q + \varphi_{\text{max}}, p) &= x(q, p) - \lambda \quad \text{for} \quad (q, p) \in \mathcal{R}_\psi, \\
y(q + \varphi_{\text{max}}, p) &= y(q, p) \quad \text{for} \quad (q, p) \in \mathcal{R}_\psi, \\
y(q, \psi_{\text{max}}) &= -d \quad \text{for all} \quad q \in \mathbb{R},
\end{align*}
\] (25)

where the first two relations above are mere reformulations of (22) and the periodicity of \( \psi \). The existence of a conformal mapping \( q + ip \mapsto x + iy \) from a horizontal strip \( \mathcal{R}_\psi \) to \( D \) that admits an extension as a homeomorphism between the closures of these domains and satisfies (25) is contingent upon the preservation of the scaling ratio \( \psi_{\text{max}}/\varphi_{\text{max}} \) (a geometric invariant). Indeed, the conformal diffeomorphism \( (q + ip) \mapsto e^{2\pi i(q + ip)/\psi_{\text{max}}} \) from the periodicity rectangle \( \{(q, p) : 0 \leq q < \varphi_{\text{max}}, 0 \leq p \leq \psi_{\text{max}}\} \) to the annulus \( \{z \in \mathbb{C} : e^{-2\pi \psi_{\text{max}}/\varphi_{\text{max}}} \leq |z| \leq 1\} \) shows that the claim is equivalent to a statement about the conformal equivalence of two such annuli, the necessary and sufficient condition being that the annuli have the same ratio of outer radius to inner radius (see [12]).
3 The streamline time-period

In this section we define the streamline time-period and establish some fundamental properties of this flow characteristic.

The streamlines (i.e. the level sets of $\psi$) are smooth curves, since for every $p \in [0, \psi_{\text{max}}]$ and for every $x \in \mathbb{R}$ there exists a unique $f_p(x) \in [-d, \eta(x)]$ such that $\psi(x, f_p(x)) = p$ as a consequence of the implicit function theorem, with

$$f_p'(x) = -\frac{\psi_x(x, f_p(x))}{\psi_y(x, f_p(x))} = \frac{\varphi_y(x, f_p(x))}{\varphi_x(x, f_p(x))} = \frac{v(x, f_p(x))}{u(x, f_p(x)) - c}.$$  \hspace{1cm} (26)

From the symmetry (18) of $\psi$ it follows that $f_p$ is an even function of $x$. The periodicity of $\psi$ in the $x$ variable ensures that $f_p$ is periodic with period $\lambda$. Furthermore, (16) combined with (20) shows that $f_p$ is increasing on $[-\lambda/2, 0]$ and decreasing on $[0, \lambda/2]$.

Due to smoothness and the boundedness of the velocity field, the system describing the particle trajectories,

$$\begin{align*}
x'(t) &= u(x(t) - ct, y(t)), \\
y'(t) &= v(x(t) - ct, y(t)),
\end{align*}$$  \hspace{1cm} (27)

with initial data $x(t_0) = x_0$ and $y(t_0) = y_0$, has a unique global solution $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ that depends smoothly on $t$ and on the initial data $(t_0, x_0, y_0)$. It is easy to check that a solution viewed in the moving frame belongs to a level set of $\psi$, i.e. there is some $p \in [0, \psi_{\text{max}}]$ such that

$$\psi(x(t; t_0, x_0, y_0) - ct, y(t; t_0, x_0, y_0)) = p, \quad t \geq 0.$$  

To indicate that a solution corresponds to the level set $\psi = p$ we shall write $(x_p(t; t_0, x_0, y_0), y_p(t; t_0, x_0, y_0))$. Note that

$$y_p(t; t_0, x_0, y_0) = f_p(x_p(t; t_0, x_0, y_0) - ct), \quad t \geq 0.$$  \hspace{1cm} (28)

Obviously, each $(x_0, y_0)$ belongs at time $t_0$ to some streamline $\psi = p$, so that $y_0 = f_p(x_0 - ct_0)$. Thus, since $f_p$ is smooth, the dependence of the solution $(x_p(t; t_0, x_0, y_0), y_p(t; t_0, x_0, y_0))$ on $y_0$ is actually an implicit smooth dependence on $x_0$ and $t_0$. From now on we write $(x_p(t; t_0, x_0), y_p(t; t_0, x_0))$.

Our first result ensures that the particle trajectories viewed in the moving frame are periodic in time, with a period depending only on the streamline.

**Theorem 1** Let $p \in [0, \psi_{\text{max}}]$. Then for each $(t_0, x_0) \in \mathbb{R}^2$ the equation

$$x_p(T + t_0; t_0, x_0) - cT = x_0 - \lambda$$  \hspace{1cm} (29)

has a unique solution $T = T(p) > 0$ that is independent of $(t_0, x_0)$, and the following relation holds:

$$y_p(T + t_0; t_0, x_0) = y_p(t_0; t_0, x_0).$$  \hspace{1cm} (30)

Furthermore, the map $p \mapsto \log T(p)$ is a convex function and the map $p \mapsto T(p)$ is nonincreasing.

**Proof** We first show that equation (29) has a unique solution $T = T(p, t_0, x_0)$. To this end, let

$$h(t) = x_p(t + t_0; t_0, x_0) - ct - x_0 + \lambda, \quad t \in \mathbb{R}.$$  


We have
\[ h'(t) = u - c \leq -\delta, \quad t \in \mathbb{R}, \]
where
\[ \delta = \min_{(x,y) \in \Omega} \{c - u(x, y)\} > 0 \]
because there are no stagnation points in the fluid domain \( \mathcal{D} \). Since \( h(0) = \lambda > 0 \), this implies
\[ h(t) \leq -\delta t + \lambda, \quad t \geq 0, \]
thus ensuring the existence of \( T = T(p, t_0, x_0) > 0 \) with \( h(T) = 0 \). The uniqueness of \( T \) follows from the strict monotonicity of \( h \).

To see that \( T \) is independent of \( (t_0, x_0) \), consider the map
\[ \sigma(s) = x_p(s; t_0, x_0) - cs, \quad s \geq t_0. \]
Then
\[ \sigma'(s) = x'_p(s; t_0, x_0) - c = u(\sigma(s), f_p(\sigma(s))) - c, \]
and, by the definition of \( T \), we have \( \sigma(t_0) = x_0 - ct_0 \) and \( \sigma(T + t_0) = x_0 - ct_0 - \lambda \). Using these facts, we obtain the following formula for \( T \):
\[
T(p, t_0, x_0) = \int_{t_0}^{t_0 + T(p, t_0, x_0)} \frac{\sigma'(s)}{u(\sigma(s), f_p(\sigma(s))) - c} \, ds
= \int_{\sigma(t_0)}^{\sigma(t_0 + T(p, t_0, x_0))} \frac{d\sigma}{u(\sigma, f_p(\sigma)) - c}
= \int_{x_0 - ct_0}^{x_0 - ct_0 - \lambda} \frac{d\sigma}{c - u(\sigma, f_p(\sigma))} = \int_{-\frac{\lambda}{c}}^{\frac{\lambda}{c}} d\lambda.
\]
where the last step follows by periodicity. It is now clear that \( T \) depends only on \( p \). Furthermore, (30) follows easily from the periodicity of \( f_p \). Indeed,
\[
y_p(T + t_0; t_0, x_0) = f_p(x_p(T + t_0; t_0, x_0) - c(T + t_0)) = f_p(x_0 - ct_0 - \lambda) = f_p(x_0 - ct_0) = y_p(t_0; t_0, x_0).
\]

Now our strategy to show the convexity of \( p \mapsto \log T(p) \) is to use the hodograph transform to turn the integral expression for \( T \) into an integral of the modulus of an analytic function over a horizontal segment, which, using the periodicity, can be subsequently transformed into an integral over a circle, to which we may apply Hardy’s convexity theorem.

Recall that the hodograph transform \( F = \varphi + i \psi \) is analytic and we have \( F' = (u - c) + i(-v) = \varphi_x - i\varphi_y \). Inspired by this transform we perform a change of variables in the integral expression for \( T \) by setting
\[ q = \tilde{q}_p(x) = \varphi(x, f_p(x)). \]
Then, by (16), (17) and (21)
\[
\tilde{q}'_p(x) = \varphi_x(x, f_p(x)) + \varphi_y(x, f_p(x)) \cdot f'_p(x) = \frac{\varphi_x^2(x, f_p(x)) + \varphi_y^2(x, f_p(x))}{\varphi_x(x, f_p(x))}.
\]
\[ G(\frac{\lambda}{2}, f_p(\frac{\lambda}{2})) = \varphi_{\text{max}}, \quad \tilde{q}_p(\frac{\lambda}{2}) = \varphi(\frac{\lambda}{2}, f_p(\frac{\lambda}{2})) = 0. \]  

Hence

\[ T(p) = \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \frac{dx}{c - u(x, f_p(x))} = -\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \varphi(x, f_p(x)) \frac{dx}{G(x)} = \int_0^{\varphi_{\text{max}}} \frac{dq}{|F'(q, f_p(q))|^2}. \]

Notice that

\[ F(q^{-1}(q), f_p(q^{-1}(q))) = \left( \varphi(q^{-1}(q), f_p(q^{-1}(q))), \psi(q^{-1}(q), f_p(q^{-1}(q))) \right) = \left( \tilde{q}_p(q^{-1}(q), p) = (q, p), \right. \]

which implies

\[ T(p) = \int_0^{\varphi_{\text{max}}} \frac{1}{|F'(q, f_p(q))|^2} \frac{dq}{F'(q, f_p(q))} = \int_0^{\varphi_{\text{max}}} \left| (F^{-1})(q, p) \right|^2 dq. \]

The function \( G := (F^{-1})' \) is analytic in the rectangle \([0, \varphi_{\text{max}}] \times [0, \psi_{\text{max}}] \) and the periodicity of \( F' \) gives

\[ G(0, p) = \frac{1}{F'\left(F^{-1}(\varphi_{\text{max}}, f_p(\frac{\lambda}{2})), p\right)} = \frac{1}{F'(\varphi_{\text{max}}, f_p(\frac{\lambda}{2}))} = \frac{1}{F''(\varphi_{\text{max}}, f_p(\frac{\lambda}{2}))} = G(\varphi_{\text{max}}, p), \quad 0 \leq p \leq \psi_{\text{max}}. \]

By an appropriate transformation (see Fig. 2) we now pass from the periodicity rectangle \([0, \varphi_{\text{max}}] \times [0, \psi_{\text{max}}] \) to an annulus, where we may apply Hardy’s theorem. Denote

\[ \mathbb{A} = \{ z \in \mathbb{C} : 1 \leq |z| \leq e^{2\pi \varphi_{\text{max}} \psi_{\text{max}}} \} \]

and consider the principal branch of the logarithm,

\[ \log z = \log |z| + i \arg(z), \quad -\pi < \arg(z) \leq \pi. \]

Then (see Fig. 2)

\[ \beta(z) = \frac{\varphi_{\text{max}}}{2} \left[ \frac{i}{\pi} \log z + 1 \right] \]

maps \( \mathbb{A} \) bijectively onto \([0, \varphi_{\text{max}}] \times [0, \psi_{\text{max}}], \) and the map \( G \circ \beta \) is analytic in the interior of \( \mathbb{A} \) with the real-line segment \((-e^{2\pi \varphi_{\text{max}} \psi_{\text{max}}}, -1)\) excised. By property (35) the jump of \( \log \) gets smoothed out by the composition with \( G, \) and hence \( G \circ \beta \) extends to a continuous map on \( \mathbb{A}. \) By Morera’s theorem it now follows that \( G \circ \beta \) is analytic on \( \mathbb{A}. \)

Let us now fix \( p \in [0, \psi_{\text{max}}] \) and set \( r := e^{2\pi \psi_{\text{max}} p}. \) Then \( 1 \leq r \leq e^{2\pi \psi_{\text{max}} \varphi_{\text{max}}} \) and we have

\[ M_2(G \circ \beta, r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} |(G \circ \beta)(re^{i\theta})|^2 d\theta \]
where the last step above follows by the change of variables $q = \varphi_{\text{max}} \left( 1 - \frac{\theta}{\pi} \right)$. Hardy’s convexity theorem (see [13]) now ensures that the map $r \mapsto \log M_2(G \circ \beta, r)$ is a convex function of $\log r$ for $r \in [1, e^{2\pi \varphi_{\text{max}}}]$, i.e. for any $r_1, r_2 \in [1, e^{2\pi \varphi_{\text{max}}}]$ and for every $\alpha \in [0, 1]$ we have

$$M_2(G \circ \beta, r) \leq [M_2(G \circ \beta, r_1)]^\alpha \cdot [M_2(G \circ \beta, r_2)]^{1-\alpha}$$

where $\log r = \alpha \log r_1 + (1 - \alpha) \log r_2$. Since $\log r = \frac{2\pi}{\varphi_{\text{max}}} p$, we deduce from (36) that $p \mapsto \log T(p)$ is a convex function.

The monotonicity of $T$ now follows combining the above with the fact that $\upsilon$ vanishes on the flat bed. Indeed, the convexity of $p \mapsto \log T(p)$ implies the convexity of $p \mapsto T(p)$ which, at its turn, implies that $T'(p)$ is nondecreasing on $[0, \varphi_{\text{max}}]$. Therefore, in order to show that $T$ is nonincreasing it suffices to show that $T'(\varphi_{\text{max}}) = 0$. Due to (34), we have

$$T'(p) = \int_0^{\varphi_{\text{max}}} \frac{d}{dp} \left( \frac{1}{F'(F^{-1}(q, p))} \right)^2 dq = \int_0^{\varphi_{\text{max}}} \frac{d}{dp} \left( \frac{1}{((u - c)^2 + v^2)(F^{-1}(q, p))} \right)^2 dq.$$  

We claim that the evaluation of the integrand above at $p = \varphi_{\text{max}}$ vanishes. Denoting the components of $F^{-1}$ by $x$ and $y$, using the chain rule we obtain

$$\frac{\partial}{\partial p} \left( (u - c)^2 + v^2 \right) = 2(u - c) \left[ u_x \frac{\partial x}{\partial p} + u_y \frac{\partial y}{\partial p} \right] + 2v \left[ v_x \frac{\partial x}{\partial p} + v_y \frac{\partial y}{\partial p} \right].$$

We now evaluate the above expression at $p = \varphi_{\text{max}}$, so that all the terms containing $u$ and $v$ above are to be evaluated at

$$(x(q, \varphi_{\text{max}}), y(q, \varphi_{\text{max}})) = (x(q, \varphi_{\text{max}}), -d).$$
Since $v = 0$ on $y = -d$ and $u_y(-d, -d) = v_x(-d, -d)$ we infer that the last three of the four terms on the right side of (38) vanish. Furthermore, computing the inverse of the Jacobian matrix of $F$,
\[
\frac{\partial x}{\partial p}(q, \psi_{\text{max}}) = -\frac{v}{(u - c)^2 + v^2}(x(q, \psi_{\text{max}}), -d) = 0,
\]
so that the first term on the right side of (38) vanishes as well, and we can now conclude that $T'(\psi_{\text{max}}) = 0$. Thus $T$ is nonincreasing and, with this, the proof is complete. □ □

**Remark 1** If $(x_0, y_0)$ is the location of a particle at time $t = 0$, then there exists a unique $p \in [0, \psi_{\text{max}}]$ with $y_0 = f_p(x_0)$, and the location of the particle at time $t > 0$ is given by the solution $(x_p(t; x_0), y_p(t; x_0))$ to (27) with initial data $(x_p(0), y_p(0)) = (x_0, y_0)$. Theorem 1 ensures
\[
\begin{align*}
x_p(t + T(p); x_0) &= x_p(t; x_0) + c T(p) - \lambda, \\
y_p(t + T(p); x_0) &= y_p(t; x_0),
\end{align*}
\]
so that the elevation is $T(p)$-periodic while the horizontal position in shifted by $c T(p) - \lambda$ in the direction of wave propagation. Moreover, by Theorem 1 this shift depends monotonically on $p$, or, equivalently, on the minimal (or maximal) elevation of the particle, since $\psi_y = u - c < 0$ throughout the fluid domain. We infer that the particle trajectory consists of a repeating pattern, that is simply shifted horizontally after each streamline time-period $T(p)$, and this pattern depends solely on the streamline. For the precise pattern we refer to the theoretical study [4], the numerical simulations in [3] and the experiments reported in [16]. The fact that the pattern is repeated after a suitable shift was established in [4] by considering particles that are located initially beneath the wave crest and arguing that the relation $\psi_y = u - c < 0$ ensures that any particle will occupy this position at some time. Theorem 1 via relations (39) shows the uniformity in time of the particle trajectories, a result that is to be expected by glancing at the patterns depicted in [3, 4, 16], but which was not proven hitherto. □

### 4 The total kinetic energy of a particle

The main aim of this section is to investigate the total kinetic energy of an arbitrary particle over a streamline time-period. This quantity turns out to depend only on the corresponding streamline and we pursue a detailed investigation of this dependence.

**Theorem 2** The total kinetic energy of a particle located initially at $(x_0, y_0)$ with $y_0 = f_p(x_0)$, given by
\[
\mathcal{K}(p, x_0) = \frac{1}{2} \int_0^{T(p)} \left( [x'_p(t; x_0)]^2 + [y'_p(t; x_0)]^2 \right) dt,
\]
depends only on the corresponding streamline $\psi = p$ (i.e. is independent of $x_0$) and the map $p \mapsto \log \mathcal{K}(p)$ is a convex function, while the map $p \mapsto \mathcal{K}(p)$ is nonincreasing. Furthermore, the total kinetic energy of the particle relative to the moving frame, computed over a streamline time-period, given by
\[
\mathcal{K}(p, x_0) = \frac{1}{2} \int_0^{T(p)} \left( [x'_p(t; x_0) - c]^2 + [y'_p(t; x_0)]^2 \right) dt,
\]
is constant equal to $\varphi_{\text{max}}/2$. 

\[\text{Springer}\]
Remark 2  Theorem 2 ensures that the total kinetic energy of a particle increases with the elevation of the streamline over the flat bed (measured either below the crest or, equivalently, below the trough), i.e. the total kinetic energy of a particle above the flat bed is larger than that of a particle moving beneath it.

Proof  The proof is quite similar to the one of Theorem 1, so that we shall not provide all the details.

As already mentioned, the hodograph transform \( F = \varphi + i\psi \) is analytic and we have \( F' = (u - c) + i(-v) \). Using this notation we obtain

\[
2K(p, x_0) = \int_0^{T(p)} \left\{ [(x_p(t; x_0) - ct)' + c]^2 + [y_p'(t; x_0)]^2 \right\} \, dt \\
= \int_0^{T(p)} |F'(x_p(t; x_0) - ct, y_p(t; x_0)) + c|^2 \, dt \\
= \int_0^{T(p)} |F'(x_p(t; x_0) - ct, f_p(x_p(t; x_0) - ct)) + c|^2 \, dt ,
\]
due to (28). We first make the change of variables

\[
X = x_p(t; x_0) - ct \quad \text{with} \quad dX = \left( u(x_p(t; x_0) - ct, y_p(t; x_0)) - c \right) \, dt ,
\]
to get

\[
2K(p, x_0) = \int_{x_p(0)}^{x_p(T(p)) - c T(p)} |F'(X, f_p(X)) + c|^2 \frac{1}{u(X, f_p(X)) - c} \, dX \\
= \int_{x_0}^{x_0 - \lambda} |F'(X, f_p(X)) + c|^2 \frac{1}{\varphi_\lambda(X, f_p(X))} \, dX \\
= - \int_{-\lambda/2}^{\lambda/2} |F'(X, f_p(X)) + c|^2 \frac{1}{\varphi_\lambda(X, f_p(X))} \, dX ,
\]
where the last step follows by periodicity. This shows that \( K \) is independent of the particle, depending only on the corresponding streamline \( \psi = p \).

As before, inspired by the hodograph transform, we perform a further change of variables

\[
q = \tilde{q}_p(x) = \varphi(x, f_p(x)) .
\]

Hence, by (31)–(32)–(33), we obtain

\[
2K(p) = \int_0^{\varphi_{\max}} \frac{|F' + c|^2}{|F'|^2} (\tilde{q}^{-1}(q), f_p(\tilde{q}^{-1}(q))) \, dq \\
= \int_0^{\varphi_{\max}} \frac{|F' + c|^2}{|F'|^2} (F^{-1}(q, p)) \, dq \\
= \int_0^{\varphi_{\max}} \left| 1 + \frac{c}{F'(F^{-1}(q, p))} \right|^2 \, dq \\
= \int_0^{\varphi_{\max}} \left| 1 + c (F^{-1})'(q, p) \right|^2 \, dq . \tag{40}
\]
The function $H = 1 + c(F^{-1})'$ is analytic in the interior of the rectangle $[0, \varphi_{\text{max}}] \times [0, \psi_{\text{max}}]$ and the periodicity of $F'$ gives

$$H(0, p) = 1 + \frac{c}{F'(F^{-1}(\varphi(\frac{c}{2}, f_p(\frac{c}{2})), p)} = 1 + \frac{c}{F'(\frac{c}{2}, f_p(\frac{c}{2}))} = 1 + \frac{c}{F'(-\frac{c}{2}, f_p(-\frac{c}{2}))}$$

$$= H(\varphi_{\text{max}}, p), \quad 0 \leq p \leq \psi_{\text{max}}.$$

The last expression for $K(p)$ is given by an integral on a horizontal segment of the modulus of an analytic function. The exact same argument as the one used in the proof of Theorem 1 now shows that $p \mapsto \log K(p)$ is a convex function.

The second assertion follows by taking advantage of the first one. Indeed, the convexity of $p \mapsto \log K(p)$ implies the convexity of $p \mapsto K(p)$ which, at its turn, implies that $K'(p)$ is nondecreasing on $[0, \varphi_{\text{max}}]$. Therefore, in order to show that $K$ is nonincreasing it suffices to show that $K'(\psi_{\text{max}}) = 0$. We have

$$2K'(p) = \int_{0}^{\psi_{\text{max}}} \frac{d}{dp} \left[ \frac{F'(F^{-1}(q, p)) + c}{F'(F^{-1}(q, p))} \right] dq$$

$$= \int_{0}^{\psi_{\text{max}}} \frac{d}{dp} \left( \frac{u^2 + v^2}{(u - c)^2 + v^2} (F^{-1}(q, p)) \right)^2 dq.$$

We claim that the evaluation of the integrand above at $p = \psi_{\text{max}}$ vanishes. From (38) we have

$$\frac{\partial}{\partial p} \left[ (u - c)^2 + v^2 \right] F^{-1}(q, \psi_{\text{max}}) = 0,$$  \hspace{1cm} (41)

and since this formula is actually valid also for $c = 0$, we can now conclude that $K'(\psi_{\text{max}}) = 0$. Thus $K$ is nonincreasing.

The last assertion in our statement follows immediately by letting $c = 0$ in the above argument. More precisely, for $c = 0$, relation (40) becomes $K(p, x_0) = \varphi_{\text{max}}/2$. \hfill \square

**Remark 3** Using exactly the same method as above, one can prove that the conclusion of Theorem 2 remains true for the $L^s$-norm of the kinetic energy, i.e. for expressions of the form

$$K_s(p, x_0) = \frac{1}{2} \left( \int_{0}^{T(p)} \left[ x'_p(t; x_0) \right]^2 + [y'_p(t; x_0)]^2 \right)^{s} dt \right)^{1/s}$$

for the physical frame, respectively

$$K_s(p, x_0) = \frac{1}{2} \left( \int_{0}^{T(p)} \left[ x'_p(t; x_0) - c \right]^2 + [y'_p(t; x_0)]^2 \right)^{s} dt \right)^{1/s}$$

for the moving frame; here $s > 0$ is a rational number. More precisely, one can show that $K_s(p, x_0)$ and $K_s(p, x_0)$ are independent of $x_0$, the functions $p \mapsto \log K_s(p)$ and $p \mapsto \log K_s(p)$ are convex and the maps $p \mapsto K_s(p)$ and $p \mapsto K_s(p)$ are nonincreasing.

Indeed, if $s = m/n$ with $m, n \in \mathbb{N}$, $n \neq 0$, the analogue of (40) becomes

$$K_s^*(p) = \frac{1}{2s} \int_{0}^{\psi_{\text{max}}} \frac{|F' + c|^{2s}}{|F'|^2} (F^{-1}(q, p)) dq$$
\[
\phi(x) = \frac{1}{2} \int_0^{\phi_{\text{max}}} \left| \frac{(F' + c)^n}{(F')^n} (F^{-1}(q, p)) \right|^{2/n} dq ,
\]

the analytic function to which we apply Hardy’s convexity theorem being in this case \( \frac{(F' + c)^n}{(F')^n} (F^{-1}(q, p)) \). Analogously, we have

\[
2 \mathcal{K}_s(p) = \left( \int_0^{\phi_{\text{max}}} |F'(F^{-1}(q, p))|^{2(s-1)} dq \right)^{1/s} ,
\]

(42)

and we apply Hardy’s convexity theorem to the analytic function \( F'(F^{-1}(q, p)) \). Notice that \( \mathcal{K}_1(p) = \varphi_{\text{max}}/2 \), as already shown in Theorem 2.

**Remark 4** The total kinetic energy (in the moving frame) on a streamline \( \gamma_p \), with

\[
\gamma_p(x) = (x, f_p(x)), \quad \lambda/2 \leq x \leq \lambda/2,
\]

is given by

\[
\mathcal{E}(p) = \frac{1}{2} \int_{\gamma_p} \left( |u(x, y) - c|^2 + v^2(x, y) \right) ds ,
\]

where \( s \) represents the arclength parameter. We can show that the map \( p \mapsto \log \mathcal{E}(p) \) is convex, while \( p \mapsto \mathcal{E}(p) \) is increasing.

The method of the proof is the same as for Theorems 1–2, so we only sketch some of the main steps. Using the derivative of the hodograph transform \( F' = (u - c) + i(-v) \) together with relation (26), we infer

\[
\mathcal{E}(p) = \frac{1}{2} \int_{-\lambda/2}^{\lambda/2} \left( |u(x, f_p(x)) - c|^2 + v^2(x, f_p(x)) \right) \sqrt{1 + \left| f_p'(x) \right|^2} dx
\]

\[
= \frac{1}{2} \int_{-\lambda/2}^{\lambda/2} \left( |u(x, f_p(x)) - c|^2 + v^2(x, f_p(x)) \right) \sqrt{1 + \frac{v^2}{(u - c)^2}} dx
\]

\[
= \frac{1}{2} \int_{-\lambda/2}^{\lambda/2} |F'(x, f_p(x))|^3 \cdot \frac{1}{-\varphi_x(x, f_p(x))} dx ,
\]

since \( \varphi_x = u - c < 0 \) by (16), (17) and (21). Performing the change of variables \( q = q_p(x) = \varphi(x, f_p(x)) \) and using the relations (31)–(33), one finds

\[
\mathcal{E}(p) = \frac{1}{2} \int_0^{\varphi_{\text{max}}} |F'(F^{-1}(q, p))| dq = \sqrt{2} (\mathcal{K}_{3/2}(p))^{3/2} .
\]

where the last equality follows from (42).

\[\square\]

### 5 Physical motivation

In order to explain the broader relevance of our considerations, let us note that the energy of a water wave comprises the kinetic energy of the orbital motion of the particles beneath the surface and the potential energy associated with the vertical rise and fall of the water surface from its still-water, undisturbed state. Ocean wave energy is becoming an important renewable source due to its vast but yet somewhat untapped potential. In particular, with wave power density averages up to 50 kW per meter of wave front near the coast, and up to 100 kW/m offshore (see [7]), ocean wave energy compares favourably with typically less
energy-dense sources such as wind and solar, with the added benefit that the lower profile of the energy converters reduces their visual impact. The fact that ocean wave harvesting devices convert the kinetic energy into electricity (see [17]) or make it available directly for other purposes, like taking advantage of the associated pressure to run desalinization processes (see [8]), provides a practical motivation for a better understanding of the structure of the kinetic energy of water waves.

Acknowledgements The author is grateful for helpful comments from the referee, which brought considerable improvement.

Funding Open access funding provided by University of Vienna.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Amick, C.J., Toland, J.F.: On periodic water-waves and their convergence to solitary waves in the long-wave limit. Philos. Trans. R. Soc. A 303, 633–669 (1981)
2. Bühler, O.: Waves and Mean Flows. Cambridge University Press, Cambridge (2014)
3. Clamond, D.: Note on the velocity and related fields of steady irrotational two-dimensional surface gravity waves. Philos. Trans. R. Soc. Lond. A 370, 1572–1586 (2012)
4. Constantin, A., Strauss, W.: Pressure beneath a Stokes wave. Comm. Pure Appl. Math. 53, 533–557 (2010)
5. Constantin, O., Persson, A.-M.: A complex-analytic approach to kinetic energy properties of irrotational flows. Proc. Am. Math. Soc. 150, 2647–2653 (2022)
6. Craig, W., Sulem, C.: Numerical simulation of gravity waves. J. Comput. Phys. 108, 73–83 (1993)
7. EPRI: Mapping and assessment of the United States ocean wave energy resource (2011)
8. Leijon, J., Boström, C.: Freshwater production from the motion of ocean waves—a review. Desalination 435, 161–171 (2018)
9. Longuet-Higgins, M.S.: On the decrease of velocity with depth in an irrotational water wave. Math. Proc. Camb. Philos. Soc. 49, 552–560 (1953)
10. Nachbin, A., Ribeiro-Junior, R.: A boundary integral formulation for particle trajectories in Stokes waves. Discrete Contin. Dyn. Syst. 34, 3135–3153 (2014)
11. Nachbin, A., Ribeiro-Junior, R.: Capturing the flow beneath water waves. Philos. Trans. R. Soc. A 376, 17 (2018). (Art. 20170098)
12. Rudin, W.: Real and Complex Analysis. McGraw-Hill Book Co., New York (1987)
13. Sarason, D.: The $H^p$ spaces of an annulus. Mem. Am. Math. Soc. 56, 78 (1965)
14. Strauss, W.: Steady water waves. Bull. Am. Math. Soc. 47, 671–694 (2010)
15. Toland, J.F.: Stokes waves. Topol. Methods Nonlinear Anal. 7, 1–48 (1996)
16. Umebayashi, M.: Eulerian-Lagrangian analysis for particle velocities and trajectories in a pure wave motion using particle image velocimetry. Philos. Trans. R. Soc. Lond. A 370, 1687–1702 (2012)
17. Xie, J., Zuo, L.: Dynamics and control of ocean wave energy converters. Int. J. Dyn. Control 1, 262–276 (2013)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.