MINIMAL ASYMPTOTIC TRANSLATION LENGTHS ON THE CURVE GRAPH OF TORELLI GROUPS OF PARTITIONED SURFACES

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Abstract. Putman introduced a notion of partitioned surface which is a surface with boundary with decoration restricting how they can be embedded into larger surfaces. Then he defined the Torelli group of a partitioned surface. In this paper, we show the minimal asymptotic translation length of Torelli groups of partitioned surfaces asymptotically behaves like the reciprocal of the Euler characteristic of the surface. This generalizes the previous result of the authors on Torelli groups for closed surfaces.

1. Introduction

Let $S$ be a connected orientable surface of finite type. When it is closed with genus $g$, we denote it by $S$. In this paper, we will also consider surfaces with punctures or boundary components but not both at the same time. For notation, we use $S_g^n$ to denote the surface of genus $g$ with $n$ boundary components, and $S_{g,n}$ to denote the surface of genus $g$ with $n$ punctures. In any case, we always assume $\chi(S) < 0$. The mapping class group of $S$, denoted by $\text{Mod}(S)$, is the group of homotopy classes of orientation-preserving homeomorphisms of $S$. For a surface $S^n_g$ with boundary components, $\text{Mod}(S^n_g)$ consists of orientation-preserving homeomorphisms up to homotopy fixing boundary pointwise.

For a closed surface $S_g$, one of the important proper normal subgroups of $\text{Mod}(S)$ is the Torelli group $I(S_g))$, which is the kernel of the following symplectic representation

$$\Psi : \text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z}).$$

This homomorphism is induced by the action on the first homology $H_1(S_g, \mathbb{Z})$ and the algebraic intersection number gives a symplectic structure. For more discussion on Torelli group, see [FM12] or [Joh83].

For a surface $S^n_g$ with boundary components, if $n > 1$, the algebraic intersection number becomes degenerate. Hence there is no such symplectic representation and the analogous definition of $I(S^n_g)$ does not work. In [Put07], Putman introduced the notion of a partitioned surface to develop “Torelli functor” on the category of partitioned surfaces and extend the usual definition of Torelli group of a closed surface. For the definition of partitioned surfaces, see Section 2 or [Put07].
In this paper, we will discuss the least asymptotic translation length of proper normal subgroups of Mod($S_g^n$) on the curve graph. This is motivated by the action of Torelli group of a closed surface $S_g$ on the curve graph $\mathcal{C}(S_g)$. Let $\mathcal{C}(S_g)$ be the curve graph of $S_g$ with path metric $d_{\mathcal{C}}(\cdot, \cdot)$, i.e., each edge in $\mathcal{C}(S_g)$ has length 1. Let $\ell_C(f)$ be the asymptotic translation length of $f \in \text{Mod}(S_g)$ defined by

$$\ell_C(f) = \liminf_{j \to \infty} \frac{d_{\mathcal{C}}(\alpha, f^j(\alpha))}{j},$$

where $\alpha$ is an element in $\mathcal{C}(S)$. Note that $\ell_C(f)$ is independent of the choice of $\alpha$. For any $H \subset \text{Mod}(S_g)$, define

$$L_C(H) = \min \{ \ell_C(f) : f \in H, \text{pseudo-Anosov} \}.$$

In [BS18], the first and second authors proved that

$$L_C(\mathcal{I}(S_g)) \propto \frac{1}{g},$$

that is, there are constants $C_1$ and $C_2$, independent of $g$, such that

$$\frac{C_1}{g} \leq L_C(\mathcal{I}(S_g)) \leq \frac{C_2}{g}.$$

On the contrary, Gradre–Tsai [GT11] showed that $L_C(\text{Mod}(S_g)) \asymp 1/g^2$. Hence the minimal asymptotic translation lengths from $\text{Mod}(S_g)$ and $\mathcal{I}(S_g)$ approach 0 at a different rate, $1/g^2$ and $1/g$, respectively.

This is analogous to the previous results of [Pen91] and [FLM08] regarding the stretch factor $\lambda$ of pseudo-Anosov mapping classes. For any $H \subset \text{Mod}(S_g)$, let us define

$$L(H) = \min \{ \log(\lambda(f)) : f \in H, \text{pseudo-Anosov} \}.$$

Notice that $L(\text{Mod}(S_g))$ can be thought of as the length spectrum of the moduli space of genus $g$ Riemann surfaces. Penner [Pen91] proved that $L(\text{Mod}(S_g)) \asymp 1/g$. On the contrary, Farb–Leininger–Margalit [FLM08] showed that $L(\mathcal{I}(S_g)) \asymp 1$, and that for level $m$ congruence subgroup $\text{Mod}(S_g)[m]$, $L(\text{Mod}(S_g)[m]) \asymp 1$ with $m \geq 3$. Later, Lanier–Margalit [LM] generalized this result and showed that for any proper normal subgroup $N$ of $\text{Mod}(S_g)$, $L(H) > \log(\sqrt{2})$ for $g > 3$.

Inspired by previous works, we generalize the result of [BS18] about Torelli groups to the case of surfaces with boundary components. The main theorem of the paper is as follows.

**Theorem 1.1.** For any partitioned surface $(S_g^n, P)$, the Torelli group of $(S_g^n, P)$ satisfies

$$L_C(\mathcal{I}(S_g^n, P)) \propto \frac{1}{|\chi(S_g^n)|}.$$
Note that Gradre–Tsai proved that
\[ L_C(\text{Mod}(S^n_g)) > \frac{1}{18(2g - 2 + n)^2 + 30(2g - 2 + n) - 10n}. \]
We will also prove the following to obtain an analogous result to Penner’s theorem. Hence it is natural to ask if
\[ L_C(\text{Mod}(S^n_g)) \lesssim \frac{1}{|\chi(S_g,n)|}. \]
We show that it is not the case.

**Theorem 1.2.** Suppose \( g < (1/4 - \epsilon)n \) for arbitrarily small \( \epsilon > 0 \). Then
\[ L_C(\text{Mod}(S^n_g)) \asymp \frac{1}{|\chi(S_g,n)|}. \]

It would be interesting to know if we have
\[ L_C(\text{Mod}(S^n_g)) \asymp \frac{1}{|\chi(S_g,n)|} \]
without any restriction on \((g,n)\).

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2. Quick review on Putman’s construction of Torelli groups for partitioned surfaces

2.1. Putman’s Construction of Torelli groups for surfaces with boundary. Let \( S^n_g \) be a surface of genus \( g \) with \( n \) boundary components, say with the labels \{\( b_1, \ldots, b_n \)\}. Let \( P \) be a partition of the set of boundary components of \( S^n_g \), then we call the pair \((S^n_g, P)\) a partitioned surface. For instance, one can think of an example where \( n = 7 \) and \( P = \{\{b_1, b_2, b_3\}, \{b_4, b_5\}, \{b_6\}, \{b_7\}\} \). (See Figure 1)

Let \((S^n_g, P)\) be a partitioned surface. As in [Put07], we define a capping of \((S^n_g, P)\) as an embedding \( S^n_g \hookrightarrow S_g \) where the set of the sets of boundary components of the connected components of \( S_g \setminus S^n_g \) is exactly the partition \( P \). The Torelli group \( \mathcal{I}(S^n_g, P) \) of the partitioned surface \( \mathcal{I}(S^n_g, P) \) is defined by
\[ \mathcal{I}(S^n_g, P) := \iota^{-1}(\mathcal{I}_g) < \text{Mod}(S^n_g). \]
for any capping \( \iota : S^n_g \hookrightarrow S_g \). Putman proved that this definition is independent of the choice of capping \( \iota \).
2.2. **Action on Homology groups.** Putman defined some version of homology group $H^P_1(S^a_g)$ depending only on the surface and the partition $P$. For completion, we describe the definition of $H^P_1(S^a_g)$ here (see [Put07] for detail).

Consider a partitioned surface $(S, P)$ and enumerate the partition $P$ as

$$P = \{\{b_1^1, \ldots, b_{k_1}^1\}, \ldots, \{b_1^m, \ldots, b_{k_m}^m\}\}.$$

Orient the boundary components $b_i^j$ so that $\sum_{i,j} [b^j_i] = 0$ in $H_1(S)$. Define

$$\partial H^P_1(S) = \langle [b_1^1] + \cdots + [b_{k_1}^1], \ldots, [b_1^m] + \cdots + [b_{k_m}^m]\rangle \subset H_1(S)$$

Let $Q$ be a set containing one point from each boundary component of $S$. Define $H^P_1(S)$ to be equal to the image of the following submodule of $H_1(S, Q)$ in $H_1(S, Q)/\partial H^P_1(S)$:

$$\langle\{[h] \in H_1(S, Q)\} \text{ either } h \text{ is a simple closed curve or } h \text{ is a properly embedded arc from } q_1 \text{ to } q_2 \text{ with } q_1, q_2 \in Q \text{ lying in boundary components } b_1 \text{ and } b_2 \text{ with } \{b_1, b_2\} \subset P \text{ for some } p \in P\}.$$ 

Putman proved that an element of $\text{Mod}(S^a_g)$ belongs to $I(S^a_g, P)$ if and only if it acts trivially on $H^P_1(S^a_g)$. In particular, this shows that $I(S^a_g, P)$ is well-defined, not depending on the choice of the capping $\iota$.

### 3. Lower Bound

Let $S$ be a surface of finite type with nonempty boundary and $\beta$ be one of its boundary components. Suppose $S'$ is the surface obtained from $S$ by capping the boundary component $\beta$ with a once-puncture disk. Let’s denote this puncture by $p_0$. Let $\text{Mod}(S, \{p_1, \ldots, p_k\})$ be the subgroup of $\text{Mod}(S)$ consisting of elements that fix the punctures $p_1, \ldots, p_k$ of $S$ (the set $\{p_i\}$ is possibly empty). Then we have a homomorphism

$$\varphi : \text{Mod}(S, \{p_1, \ldots, p_k\}) \to \text{Mod}(S', \{p_0, p_1, \ldots, p_k\})$$

and the following sequence is exact (see Proposition 3.19 in [FM12]):

$$1 \to (T_\beta) \to \text{Mod}(S, \{p_1, \ldots, p_k\}) \xrightarrow{\varphi} \text{Mod}(S', \{p_0, p_1, \ldots, p_k\}) \to 1.$$ 

As a consequence, we have the following proposition.

**Proposition 3.1.** There is a homomorphism

$$\Phi : \text{Mod}(S^a_g) \to \text{PMod}(S_{g,n}),$$

by capping each boundary component with a once-punctured disk.

**Proof.** By applying Proposition 3.19 in [FM12] repeatedly, we obtain a homomorphism $\Phi : \text{Mod}(S^a_g) \to \text{Mod}(S_{g,n})$. Since the capping homomorphism fix the puncture, the image of $\Phi$ fixes all punctures of $S_{g,n}$. Therefore, it is contained in $\text{PMod}(S_{g,n})$. $\square$
Proof. Let \( \Phi \equiv \text{equivariant} \). This is because the subgroup of \( \text{Mod}(n) \) acting trivially on \( C_{\text{equivariance}} \) for each \( g \) trivially on \( S \) of boundary components of \( S \) and this subgroup is contained in the kernel of \( \Phi \). Abusing the notation, we will use \( C_{g,n} \) for this curve complex (up to identification). By the \( \Phi \)-equivariance, for each \( f \in \text{Mod}(n) \), the actions of \( f \) and \( \Phi(f) \) on \( C_{g,n} \) are the same. From this, we immediately conclude that \( \ell_C(f) = \ell_C(\Phi(f)) \).

Before we obtain a lower bound on \( L_C(\mathcal{I}(n), P) \), we will prove a lemma which gives us a comparison between the groups \( \mathcal{I}(n), P \) for various partitions \( P \) of the set of boundary components. Let \( P, P' \) be partitions of the set of boundary components. We say \( P' \) is finer than \( P \) if each element of \( P \) is either an element of \( P' \) or a union of elements of \( P' \).

**Lemma 3.2.** Let \( P_1, P_2 \) be partitions of the set of boundary components of \( S_g^n \). Suppose \( P_2 \) is finer than \( P_1 \). Then we have \( \mathcal{I}(n), P_1 \subset \mathcal{I}(n), P_2 \).

**Proof.** Let \( j_i : (S_g^n, P_1) \rightarrow S_i \) be a capping for each \( i = 1, 2 \). We may assume that each connected component of \( S_i \) has no genus. Then \( \mathcal{I}(n), P_1 = (j_i)^{-1}(\mathcal{I}(S_i)) \) by definition (here \( S_i \) are closed surfaces so \( \mathcal{I}(S_i) \) are usual Torelli groups).

If \( P_2 \) is finer than \( P_1 \) then \( S_1 \) has “more” simple closed curves than \( S_2 \). Since the primitive first homology classes of a surface can be realized by simple closed curves, this shows that \( (j_1)^{-1}(\mathcal{I}(S_1)) \subset (j_2)^{-1}(\mathcal{I}(S_2)) \). This proves the lemma.

**Theorem 3.3.** For any partitioned surface \( (S_g^n, P) \), we have

\[
L_C(\mathcal{I}(n), P) \gtrsim \frac{1}{|\chi(S_g^n)|}.
\]

**Proof.** Let \( f \in \mathcal{I}(n), P \), and let \( P_{\text{max}} \) be the finest partition of the set of boundary components of \( S_g^n \) (i.e., each element of \( P_{\text{max}} \) has only one boundary component). By Lemma 3.2, this implies that \( f \in \mathcal{I}(n), P_{\text{max}} \) (as pointed out in Put07, the idea of \( \mathcal{I}(n), P_{\text{max}} \) appears in Hain05). We have a capping \( i : (S_g^n, P_{\text{max}}) \rightarrow S_g \), hence \( f \in i^{-1}(\mathcal{I}(S_g)) \).

Let \( \Phi \) be the map in Proposition 3.1. Once we identify \( S_g,n \) with the surface obtained from \( S_g \) by removing one point from each connected component of \( S_g \). \( i \) can be seen as the composition \( S_g^n \rightarrow S_g,n \rightarrow S_g \). One can consider a map \( j_* : \text{Mod}(S_g^n) \rightarrow \text{Mod}(S_g,n) \) by extending the map as identity outside the image under \( j \), and then it would be the same map as \( \Phi \). Therefore the image under \( j_* \) is contained in \( \text{PMod}(S_g,n) \). Similarly one can define a map \( h_* : \text{PMod}(S_g,n) \rightarrow \text{Mod}(S_g) \).

Note that \( j_*(f) = \Phi(f) \in \text{PMod}(S_g,n) \) and \( h_*(\Phi(f)) = h_*(j_*(f)) = i_*(f) \). Since \( f \in i_*^{-1}(\mathcal{I}(S_g)) \), this implies that the Lefschetz number \( L(\Phi(f)) = L(h_*(\Phi(f))) = 2 - 2g < 0 \). By Tsai Tsa09, \( L(\Phi(f)) < 0 \) implies that there
exists a rectangle in the Markov partition for $\Phi(f)$ which intersects with its image under $\Phi(f)$.

Finally by [GT11] (see also [BS18]), $\ell_C(\Phi(f)) \gtrsim \frac{1}{|\chi(S_{n,g})|}$ and this implies $\ell_C(f) \gtrsim \frac{1}{|\chi(S_{n,g})|}$ (recall the discussion right before Lemma 3.2). This completes the proof. \[\square\]

4. Upper bound

In this section we will obtain the upper bound

$$L_C(\mathcal{I}(S_{n,g}^n, P)) \lesssim \frac{1}{|\chi(S_{n,g})|}.$$ To do this, we will use Penner’s construction to find a pseudo-Anosov element $f \in \mathcal{I}(S_{n,g}^n, P)$ such that $\ell_C(f) \lesssim 1/|\chi(S_{n,g})|.$

**Theorem 4.1.** For any partitioned surface $(S_{n,g}^n, P)$, we have

$$L_C(\mathcal{I}(S_{n,g}^n, P)) \lesssim \frac{1}{|\chi(S_{n,g})|}.$$
Proof. It suffices to show that there is a pseudo-Anosov mapping class \( f \in \mathcal{I}(S^m_g, P) \) such that \( \ell_C(f) \lesssim 1/|\chi(S^m_g)| \). The idea is as follows. We will find filling multicurves \( A \) and \( B \) consisting of separating curves. Then define \( f = T_AT_B^{-1} \), where \( T_A \) and \( T_B \) are multi-twists. Then \( f \) is pseudo-Anosov since it comes from Penner’s construction. Furthermore \( f \in \mathcal{I}(S^m_g, P) \) for any partitioned surface \( (S^m_g, P) \), since it acts trivially on \( H^P_1(S^m_g) \). We then find a simple closed curve \( \gamma \) such that \( f^N(\gamma) \) and \( \gamma \) don’t fill the surface together for some integer \( N \). This implies that \( d_C(f^N(\delta), \delta) \leq 2 \) and further that

\[
\ell_C(f) \leq \frac{2}{N}.
\]

We will consider when genus \( g \) is even and when \( g \) is odd, separately.

**Case 1.** Assume \( g = 2 + 2m \) for some \( m \). Let \( A \) consist of red curves and \( B \) consist of blue curves as in Figure 2(a). In this figure, if the number of boundary components \( n \) is odd, then we place \( n - 1 \) boundary components in the middle and put the last boundary component, say \( \beta \), on the left bottom corner. If \( n \) is even, we place all boundary components in the middle and \( \beta \) doesn’t represent the boundary component. Then the numbers \( k \) and \( l \) are defined by

\[
k = m + \frac{1 + (-1)^{m+1}}{2} \quad \text{and} \quad l = n - 1 - \frac{1 + (-1)^{n+1}}{2}.
\]

Note that \( k \) is always even and \( l \) is always odd. Define \( f = T_AT_B^{-1} \).

Now we follow the same notation as in the proof of Theorem 6.1 in [GT11]. For a finite collection of curves \( \{c_j\}_{j=1}^k \subset A \cup B \) such that \( c_1 \cup \cdots \cup c_k \) is connected, let \( N(c_1 \cdots c_k) \) be the regular neighborhood of \( c_1 \cup \cdots \cup c_k \). Let \( \delta \) be the curve \( b_{\lfloor \frac{1}{2} \rfloor} \) and let \( \gamma \) be the curve as in Figure 3. Recall that

\[
\ell_C(f) \leq \frac{2}{N}.
\]
$f = T_A T_B^{-1}$ and consider the images of $f$. One can see that

$$f(\delta) \subset \begin{cases} \mathcal{N}(b_{\lfloor \frac{n}{4} \rfloor - 1} b_{\lfloor \frac{n}{4} \rfloor} b_{\lfloor \frac{n}{4} \rfloor + 1}), & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is even,} \\ \mathcal{N}(b_{\lfloor \frac{n}{4} \rfloor - 2} b_{\lfloor \frac{n}{4} \rfloor - 1} b_{\lfloor \frac{n}{4} \rfloor} b_{\lfloor \frac{n}{4} \rfloor + 1} b_{\lfloor \frac{n}{4} \rfloor + 2}), & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is odd,} \end{cases}$$

and consider the images of $f$. One can see that

$$f^2(\delta) \subset \begin{cases} \mathcal{N}(b_{\lfloor \frac{n}{4} \rfloor - 3} \cdots b_{\lfloor \frac{n}{4} \rfloor + 3}), & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is even,} \\ \mathcal{N}(b_{\lfloor \frac{n}{4} \rfloor - 4} \cdots b_{\lfloor \frac{n}{4} \rfloor + 4}), & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is odd,} \end{cases}$$

and consider the images of $f$. One can see that

$$f^\frac{3}{2}(\delta) \subset \mathcal{N}(b_1 \cdots b_t)$$

and consider the images of $f$. One can see that

$$f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta) \subset \mathcal{N}(b_1 \cdots b_t)$$

and consider the images of $f$. One can see that

$$f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta) \subset \mathcal{N}(a_t \cdots a_{k+t}), \quad \text{where } t = 2 \left\lceil \frac{k}{4} \right\rceil - 1.$$

Note that $\gamma$ can be realized as not included $\mathcal{N}(a_t \cdots a_{k+t})$ and hence $\gamma$ doesn’t intersect $f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta)$. This implies that $f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta)$ and $\delta$ doesn’t fill the surface, i.e., $d_C(f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta), \delta) \leq 2$. Hence we have

$$\ell_C(f) \leq \frac{2}{\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor} \leq \frac{2}{\frac{k}{4} + \frac{n}{4} - 1}$$

$$\leq \frac{8}{k + n - 4} = \frac{8}{\frac{g}{2} + n + 4}$$

$$\leq \frac{16}{g + 2n - 10} < \frac{16}{g + \frac{n}{2} - 10}$$

$$= \frac{32}{|\chi(S_g^n)| - 18}.$$ 

Therefore, $\ell_C(f) \lesssim \frac{1}{|\chi(S_g^n)|}$.

**Case 2.** Assume $g = 3 + 2m$ for some $m$.

In this case, the computation is almost identical to Case 1. Let $A$ consist of red curves and $B$ consist of blue curves as in Figure 2(b). Here, the numbers $k$ and $l$ are defined by

$$k = m + \frac{1 + (-1)^{m+1}}{2} \quad \text{and} \quad l = n - 1 - \frac{1 + (-1)^{n+1}}{2}.$$ 

Define $f = T_A T_B^{-1}$ and choose $\gamma$ and $\delta$ as in Figure 3. Then by the same argument as in Case 1, we conclude that $\gamma$ doesn’t intersect $f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta)$. Then $f^\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor (\delta)$ and $\delta$ doesn’t fill the surface, and hence we have again that

$$\ell_C(f) \leq \frac{2}{\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor} \leq \frac{32}{|\chi(S_g^n)| - 18}.$$
Therefore, \( \ell_C(f) \lesssim \frac{1}{|\chi(S^n_g)|} \).

\[ \square \]

5. Further discussions

As we pointed out in the introduction, for closed surfaces we have \( L_C(\text{Mod}(S_g)) \asymp \frac{1}{g^2} \) \((\text{GT11})\) while \( L_C(\text{Mod}(S_g)) \asymp \frac{1}{g} \) \((\text{BS18})\).

One might wonder if an analogous statement would be true for either \( S^n_g \) or \( S_{g,n} \). First of all, Valdivia showed in \text{Val14} \((\text{for (1), see also BS18})\) that

1. When \( g \) is fixed, \( L_C(\text{PMod}(S_{g,n})) \asymp \frac{1}{|\chi(S_{g,n})|} \).

2. When \( g = rn \) for some \( r \in \mathbb{Q} \), \( L_C(\text{Mod}(S_{g,n})) \asymp \frac{1}{|\chi(S_{g,n})|^2} \).

As we pointed out before, \( L_C(\text{Mod}(S_n^g)) = L_C(\text{PMod}(S_{g,n})) \), since we have a map \( \Phi : \text{Mod}(S_n^g) \rightarrow \text{PMod}(S_{g,n}) \) so that their actions on the curve complexes is \( \Phi \)-equivariant. Hence by (1) above, it is not possible to have \( L_C(\text{Mod}(S_n^g)) \asymp \frac{1}{|\chi(S_n^g)|} \).

We conjecture that both phenomena are true in general.

Conjecture 5.1. The followings hold.

1. \( L_C(\text{PMod}(S_{g,n})) \asymp \frac{1}{|\chi(S_{g,n})|} \).
2. \( L_C(\text{Mod}(S_{g,n})) \asymp \frac{1}{|\chi(S_{g,n})|^2} \).

Valdivia’s result is a partial evidence to both parts of Conjecture 5.1. As an additional partial evidence to Conjecture 5.1 (1), we show the following.

Theorem 5.2. Suppose \( g < (1/4 - \epsilon)n \) for arbitrarily small \( \epsilon > 0 \). Then \( L_C(\text{PMod}(S_{g,n})) \asymp \frac{1}{|\chi(S_{g,n})|} \).

Proof. We first remark that it is sufficient to show that \( L_C(\text{PMod}(S_{g,n})) \gtrsim \frac{1}{|\chi(S_{g,n})|} \) due to the construction given in Section 4 of \text{Val14}. For this, we roughly follow the proof of Theorem 7.1 \((\text{see also the proof of Theorem 5.1})\) in \text{BS18}.

Let \( \phi \in \text{PMod}(S_{g,n}) \) be a pseudo-Anosov element and \( \tau \) be its invariant train track obtained from Bestvina-Handel algorithm \text{BH95}. What we need to know about the train track obtained from Bestvina-Handel algorithm is summarized in Section 2 of \text{BS18}. Here is the list of facts we need.

1. \( \tau \) has two types of branches, real and infinitesimal.
2. The number of real branches is smaller or equals to \( 9|\chi(S_{g,n})| \).
3. Let \( M_{\tau}^q \) be the transition matrix for \( \phi \) on the real branches of \( \tau \). If \( q \) is a positive integer so that \( M_{\tau}^q \) has a positive diagonal entry, then the \( \ell_C(\phi) \gtrsim \frac{1}{q|\chi(S_{g,n})|} \) \((\text{this follows from Proposition 2.2 of BS18})\) which is based on the work of \text{MM99}, \text{GT11}, \text{GHKL13}.

Hence, it is enough to show that the number \( q \) above is uniformly bounded by a constant.

As explained in the proof of Theorem 5.1 in \text{BS18}, each puncture is contained in a distinct ideal polygons obtained as connected components
of the complement of $\tau$. Let $k_1$ be the number of punctures contained in monogons and $k_2$ be the number of punctures contained in bigons. Let $\mathcal{S}$ be the set of singularities of the invariant foliation for $\phi$ whose index is greater than equals to 3.

For each singularity $s \in \mathcal{S}$, let $P_s$ denote its index. By the Euler-Poincaré formula \cite{FLP79} and the fact that $P_s \geq 3$, we have

$$k_1 \geq \frac{n - k_2 + 4 - 4g}{2} \geq \frac{4e\epsilon n - k_2 + 4}{2}.$$ 

The second inequality follows from our hypothesis $g < (1/4 - \epsilon)n$.

Now we are going to divide the situation into two cases. First, let us assume that $k_2 < 2\epsilon n$. Then $k_1 > \epsilon n + 2$.

Suppose there are at least $N$ real branches attached at the monogons containing punctures. Then there are in total at least $k_1N/2$ many real branches in $\tau$. Since $k_1 > \epsilon n + 2$, we have

$$k_1N/2 \geq (N/2)(\epsilon n + 2),$$

which gives us a lower bound on the number of real branches.

On the other hand, by Fact (2) above, the number of real branches is bounded above by $9|\chi (S_{g,n})| = 9(3n/2 - 2)$. Hence, if $N$ is sufficiently large, we get a contradiction.

This shows that the there exists a uniform number $N$ such that there exists at least one monogon where the number of attached real branches is bounded above by $N$. Since these real branches are permuted by $\phi$, this gives a uniform upper bound on $q$ in Fact (3).

For the second case, let us assume $k_2 \geq 2\epsilon n$. Again, suppose there are at least $N$ real branches at each of the bigons containing punctures. Then there are at least $Nk_2/2 \geq N\epsilon n$ many real branches in $\tau$. Since the number of real branches is bounded above by $9(3n/2 - 2)$, for sufficiently large $N$, we get a contradiction as before. This implies there exists a uniform number $N$ such that $\tau$ has at least one bigon containing a puncture where less than $N$ real branches are attached. This again gives a uniform upper bounded on $q$.

In either case, since $q$ is uniformly bounded by a constant, we get the desired result.

□

The main result of this paper is that $L_C(\mathcal{I}(S_{g,n}^p, P)) \asymp 1/|\chi (S_{g,n}^p)|$ for any partition $P$ of $\partial S_{g,n}^p$. Hence, from the perspective of Conjecture \ref{conj} (1), Mod($S_{g,n}^p$) and their proper normal subgroups seem to be not distinguishable by the least asymptotic translation length on the curve complex unlike the case of closed surfaces.

On the other hand, in the case of punctured surface, it is much more hopeful. If one can show Conjecture \ref{conj} then it would be a strong partial evidence toward that Mod($S_{g,n}$) and their proper normal subgroups
are distinguishable by the least asymptotic translation length on the curve complex.

Furthermore, one way to define the Torelli subgroup \( I_{g,n} \) of \( \text{Mod}(S_{g,n}) \) is to define it as \( \Phi^*_\ast (I(S^n_n, P)) \) where \( P \) is the maximal partition of the boundary components. Then \( I_{g,n} \). In this case, our main result implies that \( L_C(I_{g,n}) \approx 1/|\chi(S_{g,n})| \), hence it is an additional partial evidence.

In summary, we propose the following

**Conjecture 5.3.** There exists a uniform constant \( C > 0 \) such that for any proper normal subgroup \( H \) of \( \text{Mod}(S_{g,n}) \), \( L_C(H) \geq C/|\chi(S_{g,n})| \).

This is an extended version of Question 1.2 in [BS18].

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