BLACK HOLES: THEIR LARGE INTERIORS

Ingemar Bengtsson
Emma Jakobsson

Stockholms Universitet, AlbaNova
Fysikum
S-106 91 Stockholm, Sweden

Abstract:

Christodoulou and Rovelli have remarked on the large interiors possessed by static black holes. We amplify their remarks, and extend them to the spinning case.

The usual picture of a black hole is that of a compact object which eventually, after aeons of Hawking radiation, shrinks to a point and then disappears without a trace. But it is possible for a black hole to have a very large interior, in which case this picture is uncomfortably counterintuitive [1]. Recently Christodoulou and Rovelli (CR from now on) [2] pointed out that black holes always have very large interiors. More precisely they pointed out that a cross section of the event horizon of a spherically symmetric black hole taken at late times (much later than the disappearance of the collapsing matter) bounds a spatial volume which grows with time as

\[
Vol \sim 3\sqrt{3}\pi m^2 v ,
\]

where \( m \) is the mass of the black hole and \( v \) is the advanced time. A look at the Penrose diagram (Fig. 1) explains why. Consider a sphere \( \mathcal{S} \) in the event horizon at large advanced time. (We assume that the metric is known to the reader. If not, see below.) A spacelike sphere bounds many spacelike hypersurfaces [3]. Among them, choose one which is “close to null” just inside the sphere, then joins an \( r = \) constant hypersurface all the way down to the matter filled region, and is closed off there. In the Schwarzschild region the \( r = \) constant hypersurfaces are actually cylinders of constant radius. The Killing vector field \( \partial_v \) acts along them. They contribute to the volume through the integral
Figure 1: A sphere $\mathcal{S}$ on the event horizon bounds a spacelike hypersurface, a large portion of which coincides with an $r =$ constant hypersurface. We show this hypersurface with one dimension suppressed, and cut in the middle, omitting the long cylindrical part which gives the main contribution to its volume. We also illustrate the argument showing that most of the volume is contained in a region out of causal contact with matter that has advanced far into the black hole.

$$\text{Vol} = \int^v d\nu d\theta d\phi \sqrt{2m/r - 1} \nu^2 \sin \theta . \quad (2)$$

The lower integration limit is irrelevant since the integral will be dominated by its upper limit $v$. The smaller we choose $r$, the larger will be the first factor in the integrand—the cylinders are stretched. On the other hand their spherical cross sections will shrink. The coefficient in front of $v$ is maximized by choosing $r = 3m/2$, which yields eq. (1). The contribution from the part of the hypersurface in the matter region is small—at late advanced time the leading term is always contributed by the Schwarzschild region.

We must keep in mind that deviations from spherical symmetry are likely to have a large effect on the interior. For one thing, the singularity will no longer be Kasner-like (exhibiting stretching in one consistent direction, and
contraction in the two others). Will this affect the result? We think not. If
the mass of the black hole is large tidal forces will be small at the horizon,
but they will also be rather small at \( r = 3m/2 \). In fact we are rather far from
the singularity. Most of the volume is collected at late advanced time, from
a region which is out of causal contact with that part of the matter region
which has entered significantly into the black hole.

Let us make the last part of the argument a bit more quantitative. Recall
that the advanced time is defined as \( v = t + r_\ast \), and the retarded time as
\( u = t - r_\ast \), where

\[
 r_\ast(r) = \int^r \frac{dr}{1 - 2m/r} = r + 2m \ln (1 - r/2m) . \tag{3}
\]

Consider a radial null geodesic \( \gamma \) at constant \( u \) extending from \( r = r_0 \) to
\( r = r_1 \). It will cover an amount of advanced time equal to

\[
 \Delta v_0 = 2r_\ast(r_1) - 2r_\ast(r_0) . \tag{4}
\]

Let us assume that the event horizon has a radius of \( 10^6 \) km. If we set
\( r_1 = 3m/2 \) and \( \Delta v_0 = 10^3 \) years, we find that

\[
 \frac{c\Delta v_0}{2Gm} \approx 10^{10} \Rightarrow r_0 \approx 2 \left( 1 - e^{-10^{10}} \right) m . \tag{5}
\]

From Fig. 1 it is clear that we have identified a region extending only very
marginally indeed into the black hole, such that the main contribution to the
volume (the term proportional to \( \Delta v \)) is contained in its Cauchy develop-
ment.

If this argument is accepted it seems reasonable to repeat the calculation
for a black hole that settles down to the Kerr black hole at late advanced
time. If this were not the case the observation by CR would lose its bite,
since real black holes do rotate, more or less. We expect that at large (but
finite) advanced time the black hole will be close to Kerr not only outside
the event horizon, but also in a part of the interior which is close to it. Thus
we are interested in a region whose metric is closely approximated by \[4\]

\[
ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dv^2 + 2dvdr + \rho^2 d\theta^2 + \frac{A \sin^2 \theta}{\rho^2} d\phi^2 - \\
-2a \sin^2 \theta drd\phi - \frac{4amr}{\rho^2} \sin^2 \theta dv d\phi , \tag{6}
\]
where
\[ \Delta \equiv r^2 - 2mr + a^2 , \]  

(7)

\[ \rho^2 \equiv r^2 + a^2 \cos^2 \theta , \quad A \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta . \]  

(8)

The angular momentum is \( J = am \). The event horizon is at \( r = r_+ \), which is the largest root of \( \Delta = 0 \). Radially ingoing null geodesics have constant advanced time \( v \), and the coordinate \( r \) has a physical interpretation as an affine parameter along these geodesics.

We can now proceed just as in Schwarzschild: We start with a sphere on the event horizon at late advanced time, connect it to a spacelike hypersurface at constant \( r \) in the interior, and close it once we reach values of \( v \) where the geometry starts to deviate from the above. The main contribution to the volume of this hypersurface will be

\[
\text{Vol} = \int^v dv d\theta d\phi \sqrt{-\Delta} \rho \sin \theta =
\]

(9)

\[
= 2\pi v \sqrt{-\Delta} \left( \sqrt{r^2 + a^2} + \frac{r^2}{2a} \ln \frac{\sqrt{r^2 + a^2} + a}{\sqrt{r^2 + a^2} - a} \right).
\]

In the extreme limit \( a/m = 1 \), the region where \( \Delta < 0 \) disappears, and there is no such term.

It would make no sense to maximize this expression analytically, instead we give the results as Fig. 2. What we see is that the value of the numerical coefficient in front of \( v \) at maximal volume decreases with \( a/m \), but the effect is not dramatic as long as \( a/m < 0.99 \). Thus we conclude that large volumes are present for realistic values of \( a/m \). Of course the Kerr metric will fail to give a good approximation of the situation throughout much of the interior of the black hole. But if its exterior becomes indistinguishable from the Kerr black hole at \( v = v_0 \), then we only have to assume that this is so also for a very thin shell inside the event horizon. If we wait for another \( 10^3 \) years the region where we perform the calculation will be out of causal contact with the interior of that shell. We convince ourselves of this by looking at radial null geodesics, just as we did for Schwarzschild. With “radial” we now mean that they belong to the outgoing Kerr congruence. Then eq. (4) still holds, but with the modification that
The volume of a hypersurface of constant $r$ in the Kerr geometry is given by

$$\text{Vol} = 2\pi m^2 v f\left(\frac{r}{m}, \frac{a}{m}\right) \quad \text{(eq. (9))},$$

where $f$ depends on our choice of $r$, for values of $a/m$ ranging from 0.1 to 0.9 in even steps, and for $a/m = 0.99$ on the innermost curve.

The figure shows how the numerical factor $f$ depends on our choice of $r$, for values of $a/m$ ranging from 0.1 to 0.9 in even steps, and for $a/m = 0.99$ on the innermost curve.

Figure 2: The volume of a hypersurface of constant $r$ in the Kerr geometry is given by $\text{Vol} = 2\pi m^2 v f\left(\frac{r}{m}, \frac{a}{m}\right)$ (eq. (9)). The figure shows how the numerical factor $f$ depends on our choice of $r$, for values of $a/m$ ranging from 0.1 to 0.9 in even steps, and for $a/m = 0.99$ on the innermost curve.

Then, if we start at a value of $r$ which maximizes the volume, and go back a thousand years in time, we find that eq. (5) generalizes to

$$r_0 \approx \left(1 - e^{-\frac{r_+ - r_-}{r_+}10^{10}}\right) r_+ . \quad \text{(11)}$$

where $r_\pm$ are the roots of $\Delta = 0$ (radii of the outer and inner horizons). Then, if we start at a value of $r$ which maximizes the volume, and go back a thousand years in time, we find that eq. (5) generalizes to

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It remains true that the main contribution to the volume comes from the Cauchy development of a hypersurface that extends only marginally into the interior of the black hole. And we are all the time avoiding the neighbourhood of the inner horizon, where instabilities are likely to pile up. We conclude that the reasoning by CR survives the generalization to realistic spinning black holes.

In their paper CR estimate that the black hole at the centre of the Milky Way—whose area radius they assume to be not much larger than the distance to the Moon—now contains enough space to fit a million solar systems. A decent estimate for its spin appears to be $a/m \approx 0.9$ \[5\]. It follows that CR overestimate the volume, but only by a factor of 10 or somewhat less.
There have been a number of recent attempts to define the volume of a black hole [6, 7, 8, 9]. There is considerable freedom here, but we think that the observation made by CR is a striking one. In their paper they actually solve a kind of isoperimetric problem: Given a round sphere on the event horizon of a spherically symmetric black hole created by a collapsing null shell, what is the volume of the largest spherically symmetric hypersurface bounded by the sphere? The answer is as given above [2]. In fact it has to be, because \( r = \frac{3m}{2} \) defines a maximal hypersurface in Schwarzschild [10, 11]. In the Kerr case hypersurfaces of constant \( r \) are never maximal, so we do not expect our result to be optimal, but it may be nearly so.

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