AN A.E. LOWER BOUND FOR HAUSDORFF DIMENSION 
UNDER VERTICAL PROJECTIONS IN THE HEISENBERG 
GROUP 

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Abstract. An improved a.e. lower bound is given for Hausdorff dimension 
under vertical projections in the first Heisenberg group. 

1. Introduction 

The aim of this work is to improve the known a.e. lower bounds for Hausdorff 
dimension under vertical projections in the Heisenberg group. The average 
behaviour of Hausdorff dimension under orthogonal projections in Euclidean space 
was first explored by Marstrand in 1954 [15]; many developments and generalisa-
tions have occurred since (see e.g. [16] [19] [10] [3] [17]). An effort began in [1] and 
[2] towards understanding the behaviour of Hausdorff dimension under projections 
in the Heisenberg group, which was further developed in [12] and [11] (see also [5]). One important open problem that remains is determining the a.e. behaviour 
of Hausdorff dimension under “vertical projections”.

All definitions relevant to this work will be restated here; further back-
ground is available in [1, 2]. Let $\mathbb{H}$ be the first Heisenberg group, which as a set will 
be identified with $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ (through exponential coordinates). The assumed 
convention for the group law on $\mathbb{H}$ is 

$$(z, t) \ast (\zeta, \tau) := (z + \zeta, t + \tau + 2 \text{Im}(z \zeta))$$ 

$$(z, t) \ast (\zeta, \tau) = (z + \zeta, t + \tau - 2z \wedge \zeta),$$

where $\wedge : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the standard wedge product on $\mathbb{R}^2$, given by 

$$(x_1, y_1) \wedge (x_2, y_2) = x_1 y_2 - x_2 y_1.$$ 

Define $\|(z, t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{1/4}$. The group $\mathbb{H}$ is a metric space when equipped 
with the left invariant metric $d_{\mathbb{H}}$, called the Korányi metric, defined by 

$$d_{\mathbb{H}}((z, t), (\zeta, \tau)) = \|(\zeta, \tau)^{-1} * (z, t)\|_{\mathbb{H}}$$ 

$$(1.1) = (|z - \zeta|^4 + |t - \tau - 2z \wedge \zeta|^2)^{1/4}.$$ 

see [8] for a proof of the triangle inequality. On any compact set, this metric is 
bih-Lipschitz equivalent to the usual Carnot-Carathéodory metric on $\mathbb{H}$ [1].
For a given metric space, Hausdorff dimension is defined through the underlying distance, which for the Heisenberg group will always be the Korányi metric. Hausdorff dimension is invariant under a bi-Lipschitz change of the metric, so the main results given here will hold for Carnot-Carathéodory metric too. The horizontal and vertical projections \( P_{\theta} : \mathbb{H} \to \mathbb{V}_{\theta} \) and \( P_{\theta}^\perp : \mathbb{H} \to \mathbb{V}_{\theta}^\perp \) are defined for each \( \theta \in [0, \pi) \) by

\[
P_{\theta}(z, t) = (\pi_{\theta}(z), 0), \quad P_{\theta}^\perp(z, t) = \left(\pi_{\theta}^\perp(z), t - 2\pi_{\theta}(z) \wedge \pi_{\theta}^\perp(z)\right),
\]

where \( \pi_{\theta} : \mathbb{C} \to \mathbb{C} \) denotes Euclidean projection onto the line

\[
\mathbb{V}_{\theta} := \{\lambda e^{i\theta} : \lambda \in \mathbb{R}\},
\]

\( \pi_{\theta}^\perp : \mathbb{C} \to \mathbb{C} \) denotes Euclidean projection onto

\[
\mathbb{V}_{\theta}^\perp = \{\lambda ie^{i\theta} : \lambda \in \mathbb{R}\},
\]

the horizontal subgroup \( \mathbb{V}_{\theta} \) is defined by

\[
\mathbb{V}_{\theta} := \{(\lambda e^{i\theta}, 0) \in \mathbb{C} \times \mathbb{R} : \lambda \in \mathbb{R}\},
\]

and the vertical subgroup \( \mathbb{V}_{\theta}^\perp \) is the Euclidean orthogonal complement of \( \mathbb{V}_{\theta} \) in \( \mathbb{R}^3 \):

\[
\mathbb{V}_{\theta}^\perp = \{(\lambda_1 ie^{i\theta}, \lambda_2) \in \mathbb{C} \times \mathbb{R} : \lambda_1, \lambda_2 \in \mathbb{R}\}.
\]

The term “projection” and the formulas for \( P_{\theta} \), \( P_{\theta}^\perp \) come from the unique way of writing an element

\[
(z, t) = P_{\theta}^\perp(z, t) * P_{\theta}(z, t),
\]

as a product of an element of \( \mathbb{V}_{\theta}^\perp \) on the left, with an element of \( \mathbb{V}_{\theta} \) on the right.

In [1] Theorem 1.4 it was shown that for any Borel (or analytic) set \( A \subseteq \mathbb{H} \),

\[
\dim P_{\theta} \perp A \geq \begin{cases} 
\dim A & \text{if } 0 \leq \dim A < 1 \\
1 & \text{if } 1 \leq \dim A < 3 \\
2 \dim A - 5 & \text{if } 3 \leq \dim A < 4,
\end{cases}
\]

for a.e. \( \theta \in [0, \pi) \), and it was conjectured that the lower bound \( \dim P_{\theta} \perp A \geq \dim A \) actually holds in the larger range \( 0 \leq \dim A \leq 3 \). The upper limit of 3 is necessary since the vertical subgroups \( \mathbb{V}_{\theta}^\perp \) have Hausdorff dimension 3 (the entire Heisenberg group \( \mathbb{H} \) has Hausdorff dimension 4). In [11], Fässler and Hovila proved

\[
\dim P_{\theta} \perp A \geq 1 + \frac{(s - 1)(s - 2)}{32 s^2}, \quad \text{for a.e. } \theta \in [0, \pi), \quad s = \dim A > 2,
\]

which improved [12] in the range \( 2 < \dim A < 3.00348 \) (approximately). The main result of this work is the following lower bound.

**Theorem 1.1.** If \( A \subseteq \mathbb{H} \) is an analytic set with \( \dim A > 1 \), then

\[
\dim P_{\theta} \perp A \geq \begin{cases} 
\frac{3 + \dim A}{4} & \text{if } \dim A \in (1, 3] \\
1 + \frac{\dim A}{6} & \text{if } \dim A \in (3, 4],
\end{cases}
\]

for a.e. \( \theta \in [0, \pi) \).

This improves [12] and [13] in the range \( 1 < \dim A < 3 + \frac{3}{11} \), and gives a positive answer to Question 4.2 from [11], which asks if the a.e. lower bound of 1 can be improved for \( 1 < \dim A \leq 2 \). The proof employs some of the techniques used by Orponen and Venieri in [18] for restricted families of projections in \( \mathbb{R}^4 \); the
main difficulty in adapting this to the Heisenberg setting lies in finding a substitute for Marstrand’s Three Circles Lemma (see [20, Lemma 3.2]).

1.1. Notation and preliminaries. Given two measure spaces $X$ and $Y$, a measure $\nu$ on $X$ and a measurable function $f : X \to Y$, the pushforward $f_\# \nu$ of $\nu$ under $f$ is a measure on $Y$, defined by $f_\# \nu(E) = \nu(f^{-1}(E))$ for each measurable set $E \subseteq Y$.

For a real number $t$, let $\lceil t \rceil$ denote the least integer greater than or equal to $t$. Let $|x|$ denote the Euclidean norm of an element $x \in \mathbb{R}^n$. The Euclidean distance $|x - y|$ between $x$ and $y$ may also be denoted by $d_E(x,y)$. For $x \in \mathbb{R}^3$ and $r > 0$, let $B_E(x,r)$ and $B_H(x,r)$ be the Euclidean and Korányi balls around $x$ of radius $r$.

The following local Hölder condition from [4] will be useful later.

Lemma 1.2 ([4, Lemma 2.1]). For any $R > 0$, there exists a positive constant $c = c(R) > 0$ such that

$$\frac{|v - w|}{c} \leq d_H(v,w) \leq c|v - w|^{1/2},$$

for all $v, w \in \mathbb{R}^3$ with $|v|, |w| \leq R$.

In some situations, the following proposition gives a covering of a Euclidean ball by Korányi balls which is a more efficient than the single ball covering implied by the previous lemma.

Proposition 1.3 ([4, Proposition 3.4]). For any $R > 0$, there exists a positive integer $N = N(R) > 0$ such that any Euclidean ball $B_E(v,r)$ with $|v| \leq R$ and $r \in (0,1)$ can be covered by at most $\lceil N/r \rceil$ Korányi balls of radius $r$.

The following version of Frostman’s Lemma provides a characterisation of Hausdorff dimension for analytic sets (see [13, 16] for a proof). To state it, a subset $A$ of a complete separable metric space $X$ is called analytic if $A$ is the continuous image of a Borel set $B \subseteq Y$, for some complete separable metric space $Y$. In particular, every Borel set is analytic.

Lemma 1.4. Let $X$ be a complete separable metric space, let $A \subseteq X$ be an analytic subset of $X$ and let $s > 0$. If there exists a nonzero finite Borel measure $\nu$ on $A$ and a constant $C$, such that

$$\nu(B(x,r)) \leq r^s \quad \text{for all } r > 0 \text{ and } x \in A,$$

then $\dim A \geq s$. Conversely, if $\dim A > s$ then there exists a compactly supported, nonzero, finite Borel measure $\nu$ on $A$ satisfying \((1.4)\).

Measures satisfying \((1.4)\) are sometimes called fractal measures, or $s$-Frostman measures.

2. Proof of lemmas and the main theorem

Most of this section is devoted to proving the lemmas from which Theorem 1.1 will follow. The first lemma of this section is an abstract version of Lemma 2.5 from [18] (see also [14, Theorem 7.2]): the proof is not too different from the Euclidean case, but is included for completeness. In the statement of the lemma, $(\theta, x) \mapsto \pi_\theta(x)$ is an arbitrary continuous function, but all statements following the proof of the lemma will specialise to the case where $\pi_\theta = P_{\eta^\perp}$ is a vertical projection on $\mathbb{H}$. The lemma essentially says that, given a fractal measure on a set
A, if there is a quantitative restriction on how often the pushforward measure under the projection fails an $s$-Frostman condition, then a.e. the dimension of $\pi_\theta(A)$ is at least $s$ (where $s$ may be smaller than $\dim A$, but ideally as close to $\dim A$ as possible).

**Lemma 2.1.** Let $X, Y$ be metric spaces, with $X$ compact and $Y$ separable. Suppose that $\mu$ is a Borel probability measure on $X$, $\nu$ is a nonzero, finite, compactly supported Borel measure on $Y$, and $(\theta, y) \mapsto \pi_\theta(y)$ is a continuous function from $X \times Y$ into $Y$. Given $s > 0$, if there exist $\eta, \delta_0 > 0$ such that

\[
(2.1) \quad \nu \{ y \in Y : \mu \{ \theta \in X : \pi_\theta \# \nu(B(\pi_\theta(y), \delta)) \geq \delta^s \} \geq \delta^\eta \} \leq \delta^\eta,
\]

for all $\delta \in (0, \delta_0)$, then

\[
\dim \pi_\theta(\text{supp} \nu) \geq s \quad \text{for } \mu\text{-a.e. } \theta \in X.
\]

**Remark 2.2.** The proof of the lemma necessarily has a few measure-theoretic technicalities; the core part of the proof is the calculation following (2.6).

**Proof of Lemma 2.1.** Let $\mu, \nu, \eta, \delta_0, s$ be given. It is first shown that the sets occurring in (2.1) are measurable. For fixed $x \in X$, and any constant $c > 0$, the set

\[
S := \{ (\theta, y) \in X \times Y : d(\pi_\theta(x), \pi_\theta(y)) < c \},
\]

is open in $X \times Y$ by continuity. Since $Y$ is separable, the Borel sigma algebra on $X \times Y$ is equal to the one generated by the products of Borel sets \cite[Lemma 6.4.2]{6}, and is therefore contained in the class of $(\mu \times \nu)$-measurable sets, since $\mu$ and $\nu$ are Borel by assumption. Hence $S$ is $(\mu \times \nu)$-measurable. Therefore the function

\[
f(\theta, y) = \chi_{\pi_\theta^{-1}(B(\pi_\theta(x), c))}(y),
\]

is $(\mu \times \nu)$-measurable, and so the function

\[
(2.2) \quad \theta \mapsto \int f(\theta, y) \, d\nu(y) = \pi_\theta \# \nu(B(\pi_\theta(x), c))
\]

is $\mu$-measurable in $\theta$ by part (iv) of Fubini’s Theorem from \cite{9}. This proves $\mu$-measurability of the inner part of (2.1).

For the outer part, denoted by

\[
Z_\delta := \{ y \in Y : \mu \{ \theta \in X : \pi_\theta \# \nu(B(\pi_\theta(y), \delta)) \geq \delta^s \} \geq \delta^\eta \},
\]

a similar argument to that for (2.2) shows that for any $\delta > 0$ the function $(\theta, y) \mapsto \pi_\theta \# \nu(B(\pi_\theta(y), \delta))$ is $(\mu \times \nu)$-measurable, and hence the function

\[
y \mapsto \mu \{ \theta \in X : \pi_\theta \# \nu(B(\pi_\theta(y), \delta)) \geq \delta^s \}
\]

is a $\nu$-measurable function of $y$, by part (iii) of Fubini’s Theorem from \cite{9}. This shows that $Z_\delta$ is $\nu$-measurable.

Since $\nu$ is compactly supported and $\pi(\cdot)$ is continuous, to prove the lemma it may be assumed that $Y$ is compact. Let $\epsilon > 0$ and let $E \subseteq X$ be a compact set with

\[
(2.3) \quad \dim \pi_\theta(\text{supp} \nu) < s - \epsilon \quad \text{for every } \theta \in E.
\]

Any finite Borel measure on a compact metric space is inner regular, so it suffices to show $\mu(E) = 0$. Let $\epsilon' > 0$, and choose a positive $\delta_1 < \frac{\epsilon'}{4\dim Y}$ small enough to ensure $\delta^\eta \leq \epsilon'$. For each $\theta \in E$, using (2.3) let $\{ B(\pi_\theta(z_i(\theta)), \delta_i(\theta)) \}_{i=1}^\infty$ be a cover of $\pi_\theta \text{supp } \nu$ by balls in $Y$ of dyadic radii $\delta_i(\theta) < \delta_1$ such that $\sum_{i=1}^\infty \delta_i(\theta)^s < \epsilon'$. 


It is possible to choose the covers in such a way that the functions

\[(2.4) \quad \pi_{\theta \#} \nu(B(\pi_{\theta}(z_i(\theta)), c)), \quad \pi_{\theta \#} \nu \left( D_{\theta}^1 \right), \quad \nu \left( \pi_{\theta}^{-1} \left( D_{\theta}^j \right) \cap Z_\delta \right), \]

and

\[(2.5) \quad \nu \left( \pi_{\theta}^{-1} \left( D_{\theta,1}^j \right) \setminus Z_\delta \right), \quad \nu \left( \pi_{\theta}^{-1} \left( D_{\theta,2}^j \right) \setminus Z_\delta \right), \]

are \(\mu\)-measurable in \(\theta\) on \(E\), for any \(c, \delta > 0\), for each \(i\) and for any integer \(j\). Here

\[
D_{\theta}^j := \bigcup_{\delta_i(\theta) = 2^{-j}} B(\pi_{\theta}(z_i(\theta)), 2^{-j}),
\]

\(D_{\theta,1}^j\) is the subset of \(D_{\theta}^j\) defined as the union over those balls \(B(\pi_{\theta}(z_i(\theta)), 2^{-j})\) in \(D_{\theta}^j\) with \(\pi_{\theta \#} \nu \left( B(\pi_{\theta}(z_i(\theta)), 2^{-j}) \right) < 2^{-(j-1)s}\), and \(D_{\theta,2}^j\) is the union of the remaining balls in \(D_{\theta}^j\), equivalently \(D_{\theta,2}^j = D_{\theta}^j \setminus D_{\theta,1}^j\).

To verify the \(\mu\)-measurability of (2.4) and (2.5), the compactness of \(\pi_{\theta}(\text{supp } \nu)\) for each fixed \(\theta \in E\) ensures that there is a finite subcollection (not relabelled) of balls \(B(\pi_{\theta}(z_i(\theta)), \delta_i(\theta))\) which cover \(\pi_{\theta}(\text{supp } \nu)\). The union \(U_\theta\) of these balls is an open set, and therefore contains an open \(\delta'\)-neighbourhood \(N_{\delta'}(\pi_{\theta}(\text{supp } \nu))\) of \(\pi_{\theta}(\text{supp } \nu)\) for some \(\delta' > 0\). The compactness of \(Y\) (assumed without loss of generality) ensures that the map \((\theta, y) \mapsto \pi_{\theta}(y)\) is uniformly continuous on \(X \times Y\), which implies

\[
\pi_{\theta'}(\text{supp } \nu) \subseteq N_{\delta'}(\pi_{\theta}(\text{supp } \nu)) \subseteq U_\theta,
\]

for all \(\theta'\) in a sufficiently small ball \(B_\theta\) around \(\theta\). Therefore the balls \(B(\pi_{\theta}(z_i(\theta)), \delta_i(\theta))\) form a finite cover of \(\pi_{\theta'}(\text{supp } \nu)\) for \(\theta' \in B_\theta\). The sets \(B_\theta\) cover \(E\) as \(\theta\) ranges over \(E\), so by compactness of \(E\) there is a finite subcollection \(\{B_{\theta_1}, \ldots, B_{\theta_N}\}\) such that

\[
E = B_{\theta_1} \cup B_{\theta_2} \setminus B_{\theta_1} \cup \cdots \cup B_{\theta_N} \setminus \cup_{i=1}^{N-1} B_{\theta_i}.
\]

The functions \(z_i(\theta)\) and \(\delta_i(\theta)\) may then be taken to be constant on each part of this Borel partition of \(E\). By the piecewise constant property and the \(\mu\)-measurability of (2.2), the function \(\pi_{\theta \#} \nu(B(\pi_{\theta}(z_i(\theta)), c))\) is \(\mu\)-measurable for every \(i\) and any \(c > 0\). This proves the \(\mu\)-measurability of the first function in (2.4). Measurability of the other functions follows from a similar argument to the measurability of (2.2), using the piecewise constant property of the \(\delta_i(\theta)'s\) and the \(\nu\)-measurability of \(Z_\delta\). This shows that the covers \(\{B(\pi_{\theta}(z_i(\theta)), \delta_i(\theta))\}_{i=1}^{\infty}\) may be chosen to make the functions in (2.4) and (2.5) \(\mu\)-measurable over \(E\).

For each \(\theta \in E\),

\[(2.6) \quad \nu(Y) \leq \sum_{j > \log_2 \delta_1} \pi_{\theta \#} \nu \left( D_{\theta}^j \right),\]
by the definition of the cover and the sets $D_\theta^j$. Dividing both sides by $\nu(Y) \gtrsim 1$ and integrating over $E$ gives

$$\mu(E) \lesssim \int_E \sum_{j > |\log_2 \delta_1|} \pi_\theta \nu \left( D_\theta^j \right) d\mu(\theta)$$

(2.7)

$$\leq \sum_{j > |\log_2 \delta_1|} \int_E \nu \left( \pi_\theta^{-1} \left( D_\theta^j \right) \cap Z_{2^{-(j-1)}} \right) d\mu(\theta)$$

(2.8)

$$+ \int_E \sum_{j > |\log_2 \delta_1|} \nu \left( \pi_\theta^{-1} \left( D_{\theta,1}^j \right) \setminus Z_{2^{-(j-1)}} \right) d\mu(\theta)$$

(2.9)

$$+ \sum_{j > |\log_2 \delta_1|} \int_E \nu \left( \pi_\theta^{-1} \left( D_{\theta,2}^j \right) \setminus Z_{2^{-(j-1)}} \right) d\mu(\theta).$$

It remains to bound the integrals in (2.7), (2.8) and (2.9). Up to a constant the first sum, in (2.7), is bounded by $\delta_1^j \leq \epsilon'$ by the assumption on each $Z_\theta$ in the statement of the lemma, and the choice of $\delta_1$. The integral in (2.8) is $\lesssim \epsilon'$ by the condition $\sum_{i=1}^{\infty} \delta_i(\theta^j) \leq \epsilon'$ defining the cover and by the definition of $D_{\theta,1}^j$. It remains to bound (2.9). For each $j$ the set

$$\left\{ (\theta, y) \in E \times Y : \pi_\theta(y) \in D_{\theta,2}^j \right\}$$

is $(\mu \times \nu)$-measurable by the piecewise constant property of the defining cover, so an application of Fubini’s Theorem to each integral in (2.9) results in

$$\sum_{j > |\log_2 \delta_1|} \int_E \nu \left( \pi_\theta^{-1} \left( D_{\theta,2}^j \right) \setminus Z_{2^{-(j-1)}} \right) d\mu(\theta)$$

$$= \sum_{j > |\log_2 \delta_1|} \int_{E \setminus Z_{2^{-(j-1)}}} \mu \left\{ \theta \in E : \pi_\theta(y) \in D_{\theta,2}^j \right\} d\nu(y)$$

$$\leq \sum_{j > |\log_2 \delta_1|} \int_{E \setminus Z_{2^{-(j-1)}}} \mu \left\{ \theta \in E : \pi_\theta \# \nu \left( \left\{ \pi_\theta(y), 2^{-(j-1)} \right\} \right) \gtrsim 2^{-(j-1)s} \right\} d\nu(y)$$

$$\lesssim \sum_{j > |\log_2 \delta_1|} 2^{-j\eta} \text{ by definition of } Z_{2^{-(j-1)}},$$

$$\lesssim \epsilon',$$

by the condition $\delta_1^j \leq \epsilon'$ imposed in the choice of $\delta_1$. Therefore $\mu(E) \lesssim \epsilon'$ with $\epsilon'$ arbitrary, and thus $\mu(E) = 0$. This proves the lemma. \hfill \Box

The following lemma is a slightly refined version of Lemma 3.5 from [11], see also [2] Section 4. The lemma is a kind of transversality condition, which means that in a quantitative sense the paths of two fixed, distinct points under the family of vertical projections pass each other transversally. The proof has only minor adjustments to those in [2] [11] but is included for completeness.

**Lemma 2.3.** There exists a positive constant $C$ such that for any $v, w \in \mathbb{H}$ and any $\delta > 0$, the set

$$\left\{ \theta \in [0, \pi) : d_{\mathbb{H}} \left( P_{\psi_\theta}(v), P_{\psi_\theta}(w) \right) \leq \delta \right\},$$
is contained in a disjoint union of at most 41 intervals, each of length less than \( C \delta \). 

Proof. Fix \( v, w \in \mathbb{H} \) and write \( v = (z, t), w = (\zeta, \tau) \). If

\[
|z - \zeta| \geq |t - \tau - 2z \wedge \zeta|^{1/2},
\]

then

\[
\left\{ \theta \in [0, \pi) : d_{\mathbb{H}} \left( P_{V_\theta}^+(v), P_{V_\theta}^+(w) \right) \leq \delta \right\} \subseteq \left\{ \theta \in [0, \pi) : |\pi_{V_\theta}^+(z - \zeta)| \leq \delta \right\}.
\]

By writing \( z - \zeta = |z - \zeta|e^{i\phi} \) and rotating so that \( \phi = 0 \), the right hand side is contained two intervals of length \( \lesssim \frac{\delta}{|z - \zeta|} \lesssim \frac{\delta}{d_{\mathbb{H}}(v, w)} \). This proves the lemma in the case of (2.10).

If (2.10) fails, then

\[
|z - \zeta| < |t - \tau - 2z \wedge \zeta|^{1/2}.
\]

Suppose (2.11) holds and \( z = \pm \zeta \). Then

\[
d_{\mathbb{H}}(v, w) = \left( |z - \zeta|^4 + |t - \tau - 2z \wedge \zeta|^2 \right)^{1/4}
\]

\[
= \left( |z - \zeta|^4 + |t - \tau|^2 \right)^{1/4}
\]

\[
\leq 2^{1/4} |t - \tau|^{1/2}
\]

\[
= 2^{1/4} \left| t - \tau - 2\pi_{V_\theta}(z) \wedge \pi_{V_\theta}^+(z) + 2\pi_{V_\theta}(\zeta) \wedge \pi_{V_\theta}^+(\zeta) \right|^{1/2}
\]

\[
\leq 2^{1/4} d_{\mathbb{H}} \left( P_{V_\theta}^+(v), P_{V_\theta}^+(w) \right),
\]

and so

\[
\left\{ \theta \in [0, \pi) : d_{\mathbb{H}} \left( P_{V_\theta}^+(v), P_{V_\theta}^+(w) \right) \leq \delta \right\} \subseteq \left\{ \theta \in [0, \pi) : d_{\mathbb{H}}(v, w) \leq 2^{1/4} \delta \right\}.
\]

The right hand side is \( [0, \pi) \) if \( d_{\mathbb{H}}(v, w) \leq 2^{1/4} \delta \), which in this case is an interval of length \( \lesssim \frac{\delta}{d_{\mathbb{H}}(v, w)} \). Otherwise the right hand side is empty and there is nothing to show. This proves the lemma in the case of (2.11) with \( z = \pm \zeta \).

It remains to consider the case in which (2.11) holds but \( z \neq \pm \zeta \). Let

\[
a = \frac{t - \tau - 2z \wedge \zeta}{|z + \zeta||z - \zeta|}, \quad p = \frac{z - \zeta}{|z - \zeta|}, \quad q = \frac{z + \zeta}{|z + \zeta|}.
\]

Using \( z \wedge \zeta = \pi_{V_\theta}(z) \wedge \pi_{V_\theta}^+(\zeta) - \pi_{V_\theta}(\zeta) \wedge \pi_{V_\theta}^+(z) \),

\[
d_{\mathbb{H}} \left( P_{V_\theta}^+(v), P_{V_\theta}^+(w) \right) \geq \left| t - \tau - 2\pi_{V_\theta}(z) \wedge \pi_{V_\theta}^+(z) + 2\pi_{V_\theta}(\zeta) \wedge \pi_{V_\theta}^+(\zeta) \right|^{1/2}
\]

\[
= \left| t - \tau - 2z \wedge \zeta + 2\pi_{V_\theta}(z - \zeta) \wedge \pi_{V_\theta}(z + \zeta) \right|^{1/2}
\]

\[
= \left| t - \tau - 2z \wedge \zeta + 2(z - \zeta) \wedge \pi_{V_\theta}(z + \zeta) \right|^{1/2}
\]

\[
\geq |z + \zeta|^{1/2}|z - \zeta|^{1/2} |a + 2p \wedge \pi_{V_\theta}(q)|^{1/2}.
\]

If \( |a| \geq 4 \) then (2.11) implies that (2.13) \( \gtrsim d_{\mathbb{H}}(v, w) \), and the argument is similar to the case of (2.12). Hence it may be assumed that \( |a| < 4 \). With this assumption, (2.11) gives

\[
d_{\mathbb{H}}(v, w) \lesssim |t - \tau - 2z \wedge \zeta|^{1/2} \lesssim |z + \zeta|^{1/2}|z - \zeta|^{1/2},
\]
and putting this into \((2.13)\) yields
\[
d_{\mathbb{H}} \left( P_{\mathbb{H}}^\perp(v), P_{\mathbb{H}}^\perp(w) \right) \gtrsim d_{\mathbb{H}}(v, w) |a + 2p \wedge \pi_{V_v} q|^{1/2}.
\]
Therefore
\[
(2.14) \quad \left\{ \theta \in [0, \pi) : d_{\mathbb{H}} \left( P_{\mathbb{H}}^\perp(v), P_{\mathbb{H}}^\perp(w) \right) \leq \delta \right\}
\subseteq \left\{ \theta \in [0, \pi) : |a + 2p \wedge \pi_{V_v} q| \leq \left( \frac{K\delta}{d_{\mathbb{H}}(v, w)} \right)^{2} \right\},
\]
for a sufficiently large constant \(K\). Define \(F = F_{p,q}\) by
\[
F(\theta) = a + 2p \wedge \pi_{V_v} q,
\]
so that
\[
F'(\theta) = 2p \wedge \partial_{\theta} \pi_{V_v} (q), \quad F''(\theta) = 2p \wedge \partial_{\theta}^2 \pi_{V_v} (q).
\]
Using \(\pi_{V_v} (q) = (q, e^{i\theta}) e^{i\theta}\) gives
\[
\partial_{\theta} \pi_{V_v} (q) = (q, ie^{i\theta}) e^{i\theta} + (q, e^{i\theta}) ie^{i\theta}, \quad \partial_{\theta}^2 \pi_{V_v} (q) = 2 \left( (q, ie^{i\theta}) ie^{i\theta} - (q, e^{i\theta}) e^{i\theta} \right).
\]
Therefore \(\partial_{\theta} \pi_{V_v} (q)\) and \(\frac{1}{2} \partial_{\theta}^2 \pi_{V_v} (q)\) are orthonormal unit vectors in \(\mathbb{R}^2\), for each \(\theta \in [0, \pi)\), and so
\[
(2.15) \quad 1 = |p|^2 = \left| \frac{F'(\theta)}{2} \right|^2 + \left| \frac{F''(\theta)}{4} \right|^2 \quad \text{for all } \theta \in [0, \pi).
\]
It follows that for any \(b \in \mathbb{R}\), the equation \(F(\theta) = b\) has at most 2 solutions in any interval of length strictly less than 1/2. To see this, let \(I\) be an interval with \(|I| < 1/2\) and assume for a contradiction that \(F(\theta) = b\) for has three distinct solutions in \(I\). Then by Rolle’s Theorem \(F'\) has two distinct zeroes in \(I\), and by Rolle’s Theorem again \(F''\) has a zero \(\theta'\) in \(I\). Let \(\theta'\) be one of the zeroes of \(F'\). Then by \((2.15)\),
\[
2 = |F''(\theta')| = \left| \int_{\theta'} \frac{F''(\theta)}{4} \, d\theta \right| \leq 4|I| < 2,
\]
which is a contradiction.

By covering the interval \([0, \pi)\) with 7 intervals of length strictly less than 1/2, the equation \(F(\theta) = b\) has at most 14 solutions in \([0, \pi)\), for any \(b\), and therefore the second set in \((2.14)\) is the disjoint union of at most 15 subintervals of \([0, \pi)\). Equation \((2.15)\) implies that \(F'\) is 4-Lipschitz, so by using \(8\pi < 26\), these at most 15 intervals can be written as a union of at most 15 + 26 = 41 disjoint intervals \(I \subseteq [0, \pi)\), each of length at most 1/8, such that either \(|F'(\theta)| \geq 1/2\) for every \(\theta \in I\), or \(|F''(\theta)| > 3\) for every \(\theta \in I\). Lemma 3.3 from [7] asserts that each of these intervals has length \(\lesssim \frac{1}{\rho_{\mathbb{H}}(v,w)}\), assuming \(d_{\mathbb{H}}(v,w) \geq \delta\). If \(d_{\mathbb{H}}(v,w) \leq \delta\) the lemma holds trivially provided \(C > \pi\), so this finishes the proof.

**Lemma 2.4.** Fix \(s \in (1,4]\), let \(\nu\) be a nonzero finite compactly supported Borel measure on \(\mathbb{H}\) with \(\sup_{v \in \mathbb{H}} \frac{\nu(B_{\mathbb{H}}(v,r))}{r^s} < \infty\), and fix \(\kappa > \max \left\{ \frac{3(s-1)}{4}, \frac{5s}{6} - 1 \right\}\) with \(\kappa < s - 1\). Then there exist \(\delta_0, \eta > 0\) such that
\[
\nu \left\{ v \in \mathbb{H} : \mathcal{H}^1 \left( \left\{ \theta \in [0, \pi) : P_{\mathbb{H}} v \# \nu \left( B_{\mathbb{H}} \left( P_{\mathbb{H}} v \right) , \delta \right) \right\} \geq \delta^{s-\kappa} \right\} \geq \delta^{\eta} \right\} \leq \delta^\eta,
\]
whenever \(\delta \in (0, \delta_0)\).
Proof. Choose
\[ \eta = \frac{1}{10^4} \min \left\{ \kappa - \frac{3(s - 1)}{4}, \kappa - \left( \frac{5s}{6} - 1 \right) \right\} \]
which is strictly positive by the assumption on \( \kappa \). The choice of \( \delta_0 \) will be made implicitly to eliminate implicit constants and ensure that various trivial inequalities, such as \( |\log \delta| \leq \delta^{-\eta} \), hold for \( \delta < \delta_0 \). Fix \( \delta \in (0, \delta_0) \) and let
\[ Z = Z_\delta = \left\{ v \in \mathbb{H} : \mathcal{H}^1 \left( \theta \in [0, \pi) : P_{\mathbb{H}_v}^{-1} \left( B_{\mathbb{H}_v} \left( P_{\mathbb{H}_v}^{-1} (v), 2\delta \right) \right) \right) \geq \delta^{s-\kappa} \right\}. \]
By dyadic pigeonholing and the Frostman condition on \( \nu \), there exists \( t \geq \delta^{1 - 100\eta} \) with \( t \lesssim 1 \) and a \( \nu \)-measurable subset \( Z' \subseteq Z \) satisfying
\[ \nu(Z') \geq \delta^\eta \nu(Z), \]
and
\[ \mathcal{H}^1 (H'(v)) \geq \delta^{2\eta} \quad \text{for all} \ v \in Z', \]
where \( H'(v) \) is defined for any \( v \in \mathbb{H} \) by
\[ H'(v) = \left\{ \theta \in [0, \pi) : \nu \left( P_{\mathbb{H}_v}^{-1} \left( B_{\mathbb{H}_v} \left( P_{\mathbb{H}_v}^{-1} (v), 2\delta \right) \right) \right) \geq \delta^{s-\kappa+\eta} \right\}, \]
and \( A_{\mathbb{H}}(v, r, R) \) denotes the Korányi annulus in \( \mathbb{H} \) centred at \( v \) with inner radius \( r \) and outer radius \( R \). By inner regularity of \( \nu \), \( Z' \) can also be taken to be compact.

Fix \( v \in Z' \). Using (2.14), choose three subsets \( H'_j(v) \subseteq H'(v) \) separated \( (\text{mod} \ \pi) \) by a distance of at least \( \delta^{10^4} \) from each other, each contained in an interval of length \( \delta^{10^4} \), and each with 1-dimensional measure at least \( \delta^8 \). This can be done by partitioning \([0, \pi)\) into \( \lesssim \delta^{-10^4} \) intervals of length \( \delta^{10^4} \), choosing the 6 intervals with the largest intersection with \( H'(v) \) (in terms of \( \mathcal{H}^1 \)-measure), and then choosing 3 with gaps between them \( (\text{mod} \ \pi) \). By compactness of \( Z' \), this construction can be modified to ensure that for each \( j \), the sets \( H'_j(v) \) are piecewise constant in \( v \) over some disjoint Borel cover of \( Z' \).

For each \( v \in Z' \) and \( v_j \in \mathbb{H} \), define \( v \sim_j v_j \) if
\[ v_j \in P_{\mathbb{H}_v}^{-1} \left( B_{\mathbb{H}_v} \left( P_{\mathbb{H}_v}^{-1} (v), 2\delta \right) \right) \cap A_{\mathbb{H}}(v, t, 2t) \]
for some \( \theta \in H'_j(v) \). Set
\[ \alpha = \max \left\{ \frac{s - \kappa - 1 + 1000\eta}{s - 1}, \frac{\log t}{\log \delta} + 20\eta \right\} \in (0, 1). \]
The rest of the proof will consist of verifying the following inequality:
\[ \nu(Z) t^3 \delta^{1000\eta+3(s-\kappa-1)} \leq \nu^A \left\{ (v, v_1, v_2, v_3) \in Z' \times (\mathbb{H}^1)^3 : v \sim_j v_j \text{ for all } j \right\} \]
\[ \leq \max \left\{ \frac{t^{3s} \delta^{\frac{3s}{2}}}{t^{3s} \delta^{\frac{3s}{2}}} : \min \left\{ t^{2s} \delta^{\frac{3s}{2}} - 1000\eta, t^{s+1} \delta^{\frac{3s}{2}} \right\} \geq \delta^{\alpha} \right\} \]
where \( d_E \) refers to the Euclidean distance, \( \ell(a, b) \) means the infinite line through \( a \) and \( b \), \( \nu^A = \nu \times \nu \times \nu \times \nu \) and \( v = (\zeta, \tau) \in \mathbb{C} \times \mathbb{R} \). The lemma will essentially follow by comparing the two outer parts of (2.19). The piecewise constant property of the sets \( H'_j(v) \) ensures that the set in (2.19) is Borel measurable.
To prove the lower bound of (2.19), cover the interval \([0, \pi]\) with disjoint intervals of length \(\delta/t\), and fix \(v \in \mathbb{Z}'\), \(j \in \{1, 2, 3\}\). Since \(\mathcal{H}^1(H'_j(v)) \geq \delta^{8q}\), there are at least \(t\delta^{8q-1}\) intervals \(I_k = I_{k,j}\) intersecting \(H'_j(v)\), so pick some \(\theta_k = \theta_{k,j}\) in each intersection. Then

\[
\nu \left( P_{V_{\theta_k}}^{-1} \left( B_{\mathbb{H}} \left( P_{V_{\theta_k}}^{-1} (v), 2\delta \right) \right) \cap A_{\mathbb{H}}(v, t, 2t) \right) \geq \delta^{\alpha-\kappa+\eta},
\]

for each \(j\) and \(k\), which follows from \(H'_j(v) \subseteq H'(v)\) and the definition of \(H'(v)\).

If \(j = 2\), cover \([0, \pi]\) with disjoint intervals \(J_i\) of length \(\frac{\delta}{t}\). There are at least \(t\delta^{-\alpha+9\eta}\) intervals \(J_i\) which each intersect at least \(\delta^{\alpha-1+9\eta}\) different \(\theta_k\)’s, since otherwise the definition of \(\alpha\) would give the contradiction

\[
t\delta^{8q-1} \leq \left( t\delta^{-\alpha+9\eta} \right) \cdot \delta^{\alpha-1} + \left( \pi t\delta^{-\alpha} \right) \delta^{\alpha-1+9\eta} < t\delta^{8q-1}.
\]

After removing some of the sets \(J_i\), assume that the \(J_i\)’s are only those that intersect at least \(\delta^{\alpha-1+9\eta}\) different \(\theta_k\)’s. After removing some of the sets \(I_k\), assume that all the \(\theta_k\)’s intersect one of these sets \(J_i\), so there are at least \(t\delta^{20n-1}\) sets \(I_k\).

For fixed \(v \in \mathbb{Z}'\), \(v_3 \in \mathbb{H}\) with \(v \sim v, v \sim v_3\), and \(\frac{t}{2} \leq |\zeta - \zeta_1|, |\zeta - \zeta_3| \leq 2t\), it will be shown that for fixed \(l\),

\[
\nu \left\{ v_2 \in P_{V_{\theta_k}}^{-1} \left( \left( B_{\mathbb{H}} \left( P_{V_{\theta_k}}^{-1} (v), 2\delta \right) \right) \cap A_{\mathbb{H}}(v, t, 2t) \right) \cap A_{\mathbb{H}}(v, t, 2t) \text{ for some } \theta_k \in J_l : \right. \\
\left. \quad d_E (\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \right\} \geq \delta^{\alpha-1+11\eta} \delta^{\alpha-\kappa}.
\]

The sets in (2.20) are finitely overlapping over \(k\) by Lemma 2.3 and therefore summing (2.20) over those \(k\) with \(I_{k,2} \cap J_l \neq \emptyset\) gives

\[
\nu \left\{ v_2 \in P_{V_{\theta_k}}^{-1} \left( B_{\mathbb{H}} \left( P_{V_{\theta_k}}^{-1} (v), 2\delta \right) \right) \cap A_{\mathbb{H}}(v, t, 2t) \text{ for some } \theta_k \in J_l \right\} \geq \delta^{\alpha-1+10\eta} \delta^{\alpha-\kappa}.
\]

Hence (2.21) will follow from (2.22) combined with the observation that for

\[
\frac{t}{2} \leq |\zeta - \zeta_1|, |\zeta - \zeta_3| \leq 2t,
\]

the set

\[
E := \left\{ v_2 \in P_{V_{\theta_k}}^{-1} \left( B_{\mathbb{H}} \left( P_{V_{\theta_k}}^{-1} (v), 2\delta \right) \right) \cap A_{\mathbb{H}}(v, t, 2t) \text{ for some } \theta_k \in J_l : \\
\quad d_E (\zeta_2, \ell(\zeta_1, \zeta_3)) < \delta^\alpha \right\},
\]

is contained in a Korányi ball of radius \(\delta^{\alpha-100\eta}\), therefore contributing \(\leq \delta^{(\alpha-100\eta)s}\) to the measure in (2.20), which is smaller than \(\delta^{\alpha-1+s-\kappa+11\eta}\) by the definition of \(\alpha\). To prove that \(E\) is contained in the required Korányi ball, it will first be shown that the projected set

\[
F := \left\{ \zeta_2 \in \mathbb{R}^2 : (\zeta_2, \tau_2) \in A_{\mathbb{H}}(v, t, 2t) \text{ and } \left| \pi_{V_{\theta_k}} (\zeta_2 - \zeta) \right| < 2\delta \right. \\
\left. \quad \text{ for some } \tau_2 \in \mathbb{R} \text{ and } \theta_k \in J_l : d_E (\zeta_2, \ell(\zeta_1, \zeta_3)) < \delta^\alpha \right\},
\]


is contained in a Euclidean ball in $\mathbb{R}^2$ of radius $\delta^{\alpha-50\eta}$. To see this, fix $\zeta_2 \in F$ with $v_2 = (\zeta_2, \tau_2)$ and corresponding angle $\theta_k$ given. Define

$$\ell(\theta_k) = \{\zeta + \lambda e^{i\theta_k} : \lambda \in \mathbb{R}\},$$

so that by $\text{diam}\, J_l = \frac{\delta^\alpha}{t}$ and the definition of $F$,

$$F \subseteq N_{\delta^\alpha}(\ell(\zeta_1, \zeta_3)) \cap \bigcup_{\theta_m \in J_l} N_{2\delta}(\ell(\theta_m)) \subseteq N_{\delta^\alpha}(\ell(\zeta_1, \zeta_3)) \cap N_{2\delta}(\ell(\theta_k)).$$

It suffices to contain the right hand side in a ball of radius $\delta^{\alpha-50\eta}$. This will be done by establishing a lower bound on the angle $\theta$ between the two lines. Let $\phi_1 \in H^\perp_1(v)$ and $\phi_3 \in H^\perp_3(v)$ be such that $|\pi_{V_{\phi_1}}(\zeta - \zeta_j)| < 2\delta$ with $j \in \{1, 3\}$, so that $\phi_1$ and $\phi_3$ are $\delta^{4\eta}$ separated (mod $\pi$). Write

$$\zeta_2 = \lambda_1\zeta_1 + (1 - \lambda_1)\zeta_3 + \lambda_2\zeta_2, \quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \text{with} \quad |\lambda_2||\zeta_1 - \zeta_3| < \delta^\alpha,$$

so that

$$\left|(\zeta_1 - \zeta_3) \wedge \pi_{V_{\phi_1}}(\zeta_2 - \zeta)\right| \geq |(\zeta_1 - \zeta_3) \wedge (\zeta_2 - \zeta)| - 8t\delta$$

(2.23)

$$\geq |(\zeta_1 - \zeta) \wedge (\zeta_3 - \zeta)| - t(4\delta^\alpha + 8\delta)$$

$$\geq |\pi_{V_{\phi_1}}(\zeta_1 - \zeta) \wedge \pi_{V_{\phi_3}}(\zeta_3 - \zeta)| - t(4\delta^\alpha + 16\delta) - 4\delta^2$$

$$\geq \frac{t^2}{16} |\sin(\phi_3 - \phi_1)| - t(4\delta^\alpha + 16\delta) - 4\delta^2$$

(2.24)

$$\geq \frac{t^2\delta^{4\eta}}{32} - t(4\delta^\alpha + 16\delta) - 4\delta^2$$

$$\geq t^2\delta^{5\eta},$$

since $t > \delta^{\alpha-20\eta}$ by (2.18). Hence the angle $\theta$ between the two lines satisfies

$$|\sin \theta| = \left|\frac{(\zeta_1 - \zeta_3) \wedge \pi_{V_{\phi_1}}(\zeta_2 - \zeta)}{|\zeta_1 - \zeta_3| \wedge \pi_{V_{\phi_1}}(\zeta_2 - \zeta)}\right| \geq \delta^{6\eta}.$$
The Euclidean projection of $E$ down to $\mathbb{R}^2$ is contained in $F$, and therefore in a ball of radius $\delta^{\alpha-50\eta}$. Combining this with (2.23) gives

$$d_H(v_2, v'_2) = (|\zeta_2 - \zeta_2'|^4 + \tau_2 - \tau_2' - 2\zeta_2 \wedge \zeta_2'|^2)^{1/4} \lesssim \delta^{\alpha-50\eta} + \delta^{1/2} \lesssim \delta^{\alpha-50\eta},$$

for any $v_2, v'_2 \in E$, by the definition of $\alpha$ in (2.18) and the choice of $\eta$ in (2.16). This shows that the Korányi diameter of $E$ is $\lesssim \delta^{\alpha-50\eta}$, and thus $E$ is contained in a Korányi ball of radius $\delta^{\alpha-100\eta}$. This implies (2.21) by (2.22), the Frostman condition on $\nu$ and the definition of $\alpha$.

For each $j$ and each $v \in Z'$, the sets in (2.20) are finitely overlapping over $k$ by Lemma 2.3, and therefore summing (2.20) over $k$ gives

$$\nu \{ v_j \in \mathbb{H} : v \sim_j v_j \} \gtrsim t\delta^{100\eta+s-\kappa-1}.$$  

Similarly, summing (2.21) over $l$ gives

$$\nu \{ v_2 \in \mathbb{H} : v \sim_2 v_2 : d_E(\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \text{ if } |\zeta - \zeta_1|, |\zeta - \zeta_3| \geq t/2 \} \gtrsim t\delta^{100\eta+s-\kappa-1}.$$  

Using these two inequalities and Fubini’s Theorem, gives

$$\nu^4 \{ (v, v_1, v_2, v_3) \in Z' \times (\mathbb{H}^1)^3 : v \sim_j v_j \text{ for all } j, d_E(\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \text{ if } |\zeta - \zeta_1|, |\zeta - \zeta_3| \geq t/2 \}$$

$$= \int_{Z'} \int \{ v_1 : v_1 \sim v_1 \} \int \{ v_3 : v_3 \sim v_3 \} \int \{ v_2 : v_2 \sim v_2 \text{ and } d_E(\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \text{ if } |\zeta - \zeta_1|, |\zeta - \zeta_3| \geq t/2 \} \nu(A_2, v_3, v_1, v)$$

$$\gtrsim \nu(Z') (t\delta^{100\eta+s-\kappa-1})^3 \gtrsim \nu(Z) t^{3s} \delta^{310\eta+3(s-\kappa-1)},$$

which implies the lower bound of (2.19).

For the upper bound, fix $v_j = (\zeta_j, \tau_j)$ for $j \in \{1, 2, 3\}$. Let

$$A = A(v_1, v_2, v_3) := \{ v \in Z' : v \sim_j v_j \text{ for all } j, d_E(\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \text{ if } |\zeta - \zeta_1|, |\zeta - \zeta_3| \geq t/2 \}.$$  

The upper bound of (2.19) will be obtained by bounding $\nu(A)$ and then integrating over $v_1, v_2, v_3$.

By the triangle inequality and Fubini,

$$\nu^4 \left\{ (v, v_1, v_2, v_3) \in Z' \times (\mathbb{H}^1)^3 : v \sim_j v_j \text{ for all } j \right\}$$

$$= \int_{\mathbb{H}} \int_{B_\delta(v_3, 4t)} \int_{B_\delta(v_3, 4t)} \nu \{ v \in Z' : v \sim_j v_j \text{ for all } j \} \nu(v_1) \nu(v_2) \nu(v_3)$$

$$\lesssim t^{3s}.$$  

This proves the second case of (2.19), so it may be assumed that $t > \delta^{\alpha-20\eta}$. Fix $v \in A$. For each $j \in \{1, 2, 3\}$, the inequality

$$d_H(P_{\zeta_j}(v), P_{\zeta_j}(v_j)) \leq 2\delta \text{ for some } \theta \in H_j(v),$$

for any $v_2, v'_2 \in E$, by the definition of $\alpha$ in (2.18) and the choice of $\eta$ in (2.16). This shows that the Korányi diameter of $E$ is $\lesssim \delta^{\alpha-50\eta}$, and thus $E$ is contained in a Korányi ball of radius $\delta^{\alpha-100\eta}$. This implies (2.21) by (2.22), the Frostman condition on $\nu$ and the definition of $\alpha$.
By combining this with the formula (2.26) therefore yields
\[ \nu\{v \in A : |\tau - \tau_j - 2\zeta \wedge \zeta_j|^{1/2} \geq |\zeta - \zeta_j|\text{ for some } j \in \{1, 2, 3\}\} \lesssim \delta^{n/2}. \]

It remains to bound \( \nu(A') \), where
\[ A' = \{v \in A : |\tau - \tau_j - 2\zeta \wedge \zeta_j|^{1/2} < |\zeta - \zeta_j|\text{ for all } j \in \{1, 2, 3\}\}. \]

Define \( G : \mathbb{H} \rightarrow \mathbb{R}^3 \) by
\[ G(\zeta, \tau) = \begin{pmatrix} \tau - \tau_1 - 2\zeta \wedge \zeta_1 \\ \tau - \tau_2 - 2\zeta \wedge \zeta_2 \\ \tau - \tau_3 - 2\zeta \wedge \zeta_3 \end{pmatrix}, \quad \text{so } DG(\zeta, \tau) = DG = \begin{pmatrix} -2y_1 & 2x_1 & 1 \\ -2y_2 & 2x_2 & 1 \\ -2y_3 & 2x_3 & 1 \end{pmatrix}, \]
where \( \zeta_j = x_j + iy_j \). Then \( A' \subseteq G^{-1}(B_E(0, C\delta)) \) for some constant \( C \), by (2.26). If \( v \in A' \), then \( \frac{1}{2} \leq |\zeta - \zeta_j| \leq 2t \) for each \( j \in \{1, 2, 3\} \) by the definition of the Korányi metric. Hence if \( A' \) is nonempty and there exists \( v_0 \in A' \), then by the condition \( d_E(\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \) in the definition of \( A \),
\[ \zeta_2 = \zeta_3 + \lambda_1(\zeta_1 - \zeta_3) + \lambda_2(\zeta_1 - \zeta_3), \quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \text{with } |\lambda_2| |\zeta_1 - \zeta_3| \geq \delta^\alpha. \]
The inequality \( |(\zeta_1 - \zeta) \wedge (\zeta_3 - \zeta)| \geq \delta^\alpha T \) follows similarly to the working from (2.28) to (2.24), and this gives
\[ |(\zeta_1 - \zeta) \wedge (\zeta_3 - \zeta)| \geq \delta^\alpha |\zeta_1 - \zeta| |\zeta_3 - \zeta|. \]

Combining this with the identity \( |z|^2|w|^2 = |(z, w)|^2 + |z \wedge w|^2 \) for \( z, w \in \mathbb{R}^2 \) and expanding out \( |(\zeta_1 - \zeta) \wedge (\zeta_3 - \zeta)|^2 \) gives \( |\zeta_1 - \zeta_3| \geq \frac{\delta^\alpha T}{2} \). Hence
\[ |\det DG| = 4|\zeta_1 \wedge \zeta_2 + \zeta_2 \wedge \zeta_3 + \zeta_3 \wedge \zeta_1| = 4|(\zeta_1 - \zeta_3) \wedge (\zeta_2 - \zeta_3)| = 4|\lambda_2| |\zeta_1 - \zeta_3|^2 \geq 2t \delta^\alpha T. \]

By combining this with the formula \((DG)^{-1} = (\det DG)^{-1} \text{adj} DG\) for the inverse, where \text{adj} refers to the adjugate, the operator norm satisfies \( \|(DG)^{-1}\| \lesssim t^{-1} \delta^{-\alpha - 5\alpha}. \) Hence
\[ A' \subseteq G^{-1}(B_E(0, C\delta)) \subseteq B_E(v_0, C^t t^{-1} \delta^{1-\alpha - 5\alpha}). \]

The radius of this ball is less than 1 by the definitions of \( \eta, \alpha \) in (2.10), (2.18) and the assumption \( t > \delta^{-20n} \). Proposition 1.4 therefore implies that \( A' \) can be covered by \( \lesssim t \delta^{-(1-\alpha - 5\alpha)} \) Korányi balls of radius \( t^{-1} \delta^{1-\alpha - 5\alpha} \). Also, by Lemma 1.7 \( A' \) is contained in a single Korányi ball of radius \( \lesssim t^{1/2} \delta^{1-\alpha - 5\alpha} \). Hence by the Frostman condition on \( \nu \),
\[ \nu(A') \lesssim \min \left\{ \frac{t^{1-s} \delta^{(1-\alpha - 5\alpha)(s-1)}}{t^{-s} \delta^{(1-\alpha - 5\alpha)s}}, \frac{2t}{\delta} \right\}. \]
Combining (2.28) with (2.27) therefore yields
\[ \nu(A) \lesssim \max \left\{ \frac{1}{\delta^{1/2}}, \min \left\{ \frac{1-s}{t^{1-s} \delta^{(1-\alpha - 5\alpha)(s-1)}}, \frac{t^{1/2}}{\delta} \right\} \right\}. \]
By the triangle inequality and Fubini,
\[
\nu^s \left\{ (v, v_1, v_2, v_3) \in \mathbb{H}' \times \left( \mathbb{H}^1 \right)^3 : v_j \sim v_j \text{ for all } j \right\}
\]
\[
d_E(\zeta_2, \ell(\zeta_1, \zeta_3)) \geq \delta^\alpha \text{ if } |\zeta - \zeta_1|, |\zeta - \zeta_2| \geq t/2
\]
\[
= \int \mathbb{H} \int \mathbb{B}_3(v_3, 4t) \int \mathbb{B}_3(v_3, 4t) \nu(A(v_1, v_2, v_3)) \, dv(v_1) \, dv(v_2) \, dv(v_3)
\]
\[
\lesssim \max \left\{ t^{2s} \delta^s / 2, \min \left\{ t^{\frac{2s}{7}} \delta^{(1-\alpha)(s-1)} / 2, t^{s+1} \delta^{(1-\alpha)(s-1)} \right\} \right\}.
\]
This implies the upper bound in the first part of (2.19), which finishes the proof of (2.19).

If \( t \leq \delta^{\alpha-20}\), then by (2.19),
\[

\nu(Z) \leq \delta^{\alpha(s-1)-(s-\kappa-1)-1001\eta} \leq \delta^\eta,
\]
by the definition of \( \alpha \) in (2.18).

Therefore it may be assumed that \( t \geq \delta^{\alpha-20}\). By comparing the lower and upper bounds from (2.19),
\[

(2.29) \quad \nu(Z)t^3 \delta^{1000\eta+3(s-\kappa-1)} \\
\lesssim \max \left\{ t^{2s} \delta^s / 2, \min \left\{ t^{\frac{2s}{7}} \delta^{(1-\alpha)(s-1)} / 2, t^{s+1} \delta^{(1-\alpha)(s-1)-1000\eta} \right\} \right\}.
\]
The definition of \( \alpha \) and \( \kappa \) imply the inequality \( t^{2s} \delta^s / 2 \leq t^{\frac{2s}{7}} \delta^{(1-\alpha)(s-1)-1000\eta} \). If \( 1 < s \leq 2 \), then taking the first term in the minimum gives
\[
\nu(Z) \leq t^{\frac{2s}{7}} \delta^{\alpha(s-1)-3(s-\kappa-1)-3000\eta}
\]
\[
\leq \delta^{\frac{s}{7} + \alpha(s-3)-3(s-\kappa-1)-4000\eta}
\]
\[
\leq \delta^{\frac{s}{7} + \left( \frac{s-\kappa-1}{s-1} \right)(s-3)-3(s-\kappa-1)-10^4\eta}
\]
\[
\leq \delta^{\eta \delta \left( \frac{s-\kappa-1}{s-1} \right)(s-3)-3(s-\kappa-1)-10^4\eta}
\]
\[
\leq \delta^\eta,
\]
by the definition of \( \eta \) in (2.10). This proves the lemma in the case \( 1 < s \leq 2 \).

In the remaining case \( s \geq 2 \), using \( t \lesssim 1 \) and taking the second term in the minimum yields
\[
\nu(Z) \leq \max \left\{ \delta^{\frac{s}{7} - 3(s-\kappa-1)-3000\eta}, \delta^{(1-\alpha)(s-1)-3(s-\kappa-1)-3000\eta} \right\}
\]
\[
\leq \delta^\eta \max \left\{ \delta^{3(s-\kappa-1)-2000\eta}, \delta^{(1-\alpha)(s-1)-3(s-\kappa-1)-3000\eta} \right\}
\]
\[
\leq \delta^\eta
\]
by the definition of \( \eta \) in (2.10). This covers the final case and finishes the proof of the lemma. \( \square \)

Finally, the proof of the main theorem can be given by combining Lemma 2.1 and Lemma 2.4 with Lemma 1.4 (Frostman).

**Proof of Theorem 1.1.** Let \( A \subseteq \mathbb{H} \) be an analytic set with \( s := \dim A > 1 \) and let \( \epsilon \in (0, s-1) \). By Lemma 1.4 (Frostman), there is a nonzero, finite, compactly
supported Borel measure $\nu$ on $A$ with $\nu(B_{\mathbb{H}}(v, r)) \leq r^{s-\epsilon}$ for every $v \in \mathbb{H}$ and $r > 0$. By Lemma 2.4 with

$$\kappa = \max \left\{ \frac{3(s-1)}{4}, \frac{5s}{6} - 1 \right\} > \max \left\{ \frac{3(s-\epsilon-1)}{4}, \frac{5(s-\epsilon)}{6} - 1 \right\}$$

there exist $\delta_0, \eta > 0$ such that

$$\nu \left\{ v \in \mathbb{H} : H^1 \left\{ \theta \in [0, \pi) : P_{V^\perp} \# \nu \left( B_{\mathbb{H}} \left( P_{V^\perp}(v), \delta \right) \right) \geq \delta^{s-\epsilon-\kappa} \right\} \geq \delta^\eta \right\} \leq \delta^\eta,$$

whenever $\delta \in (0, \delta_0)$. By Lemma 2.1

$$\dim P_{V^\perp} A \geq \dim P_{V^\perp} (\text{supp} \nu) \geq s - \epsilon - \kappa$$

for a.e. $\theta \in [0, \pi)$. Letting $\epsilon \to 0$ results in

$$\dim P_{V^\perp} A \geq s - \kappa = \begin{cases} \frac{3+s}{4} & \text{if } s \in (1, 3] \\ 1 + \frac{s}{6} & \text{if } s \in (3, 4] \end{cases}$$

for a.e. $\theta \in [0, \pi)$.

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