On the Hopf Structure of $W_2$ - Algebra and N=1 Superconformal Algebra in the OPE Language

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Abstract

Hopf structure of the prototype realizations of the $W_2$-algebra and also $N = 1$ superconformal algebra are obtained using the bosonic and also fermionic Feigin-Fuchs type of free massless scalar fields in the operator product expansion (OPE) language.

1 Introduction

Conformal symmetry has played an important role in the developments of the physics contents of such models: i.e. (super)string theory \([1]\), statistical physics \([2]\), as well as in mathematics \([3]\). Its underlying symmetry algebra is Virasoro algebra which is a Lie algebra. It is well-known that a universal enveloping algebra of any simple Lie algebra is always a Hopf algebra \([4]\). Let $g$ a simple Lie algebra and $U(g)$ its universal enveloping algebra. Then define the comultiplication $\Delta$, the counit $\epsilon$ and the antipode $S$ for $U(g)$ as follows:

\[
\begin{align*}
\Delta (x) &= x \otimes 1 + 1 \otimes x, \\
\epsilon(x) &= 0, \\
S(x) &= -x.
\end{align*}
\]

\((1.1)\) \hspace{1cm} (1.2) \hspace{1cm} (1.3)

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In particular one can show that the coproduct rule define a Lie algebra homomorphism:
\[ \Delta ([x,y]) = \Delta(x) \Delta(y) - \Delta(y) \Delta(x) \]  
(1. 4)
\[ = [\Delta(x),\Delta(y)] \]  
(1. 5)

For example, if one take a Virasoro algebra (or also similar infinite dimensional algebras) and feed it into this machinery one obtains a Hopf algebra. Although nobody in general taken care of this structure, the purpose of this paper is to confirm this structure in these kind of algebras by using the operator product expansion (OPE) language. In this context we note that the universal enveloping algebra of the underlying symmetry algebra in the two dimensional (super)conformal field theory is essentially the same as that of the corresponding universal enveloping algebra of any simple Lie algebra. We must also emphasize here that one reason for the importance of the Hopf algebra is that the Hopf structure of an algebra facilitates the construction of representations of the sample algebra.

It is known that \( W_2 \)-algebra is equivalent to the Virasoro algebra in the W-algebra framework \[5\]. The \( \text{W}_2 \)-algebra, involving the modes of a spin-two field \( T(z) \equiv \sum_m L_m z^{-m-2} \), is described by the OPE
\[ T(z)T(w) = \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \]  
(1. 6)
where \( c \) is the central charge. Accordingly the Virasoro generators \( L_m \)'s, which are the Laurent coefficients of \( T(z) \), satisfy the Virasoro algebra
\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n,-m}, \]  
\[ [L_m, c] = 0 \]  
(1. 7)
One must emphasize here that the Virasoro algebra is a Lie algebra. Therefore the Hopf structure of this algebra is defined by:
\[ \Delta(L_m) = L_m \otimes 1 + 1 \otimes L_m, \]  
(1. 8)
\[ \Delta(c) = c \otimes 1 + 1 \otimes c, \]  
(1. 9)
\[ \epsilon(L_m) = 0, \]  
(1. 10)
\[ \epsilon(c) = 0, \]  
(1. 11)
\[ S(L_m) = -L_m, \]  
(1. 12)
\[ S(c) = -c, \]  
(1. 13)

In this work we will concentrate over conformal field concept, so we will only recapitulate the above Hopf structure for the energy-momentum tensor \( T(z) \) instead of \( L_m \) modes for that reason:
\[ \Delta(T(z)) = T(z) \otimes 1 + 1 \otimes T(z), \]  
(1. 14)
\[ \epsilon(T(z)) = T(z), \]  
(1. 15)
\[ S(T(z)) = -T(z) \]  
(1. 16)
It can be verified that this comultiplication rule is an algebraic homomorphism for $W_2$-algebra (1.6):

$$\Delta (T(z)) \Delta (T(w)) = \frac{\Delta(\frac{1}{2}c)}{(z-w)^4} + \frac{\Delta(2T(w))}{(z-w)^2} + \frac{\Delta(\partial T(w))}{z-w}$$  \hspace{1cm} (1. 17)

The organization of this paper is as follows. The next section contains the Hopf structure of the Feigin-Fuchs type of free massless scalar field quantization in the commutator and OPE language, respectively. In section 3, the Hopf structure of $W_2$-Algebra is realized. In section 4, both the Hopf structure of the Feigin-Fuchs type of free massless fermionic scalar field quantization and the Hopf structure of $N = 1$ superconformal algebra are realized, respectively.

2 The Hopf Structure of Free Massless Bosonic Scalar Fields

A Feigin-Fuchs type of free massless bosonic scalar field $\varphi(z)$ is a single-valued function on the complex plane and its mode expansion is given by

$$h(z) \equiv i \partial \varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}.$$  \hspace{1cm} (2. 1)

Canonical quantization gives the commutator relations

$$[a_m, a_n] = a_m a_n - a_n a_m = \kappa m \delta_{m+n,0},$$ \hspace{1cm} (2. 2)

where $\kappa$ is a central element commuting with all the modes $\{a_n\}$, $[a_n, \kappa] = 0$, and the aim of the central element $\kappa$ is to provide Hopf algebra structure for the free field mode algebra (2.2). This associative algebra is a Hopf algebra with

$$\Delta (a_m) = a_m \otimes 1 + 1 \otimes a_m,$$ \hspace{1cm} (2. 3)

$$\Delta(\kappa) = \kappa \otimes 1 + 1 \otimes \kappa,$$ \hspace{1cm} (2. 4)

$$\epsilon(a_m) = 0,$$ \hspace{1cm} (2. 5)

$$\epsilon(\kappa) = 0,$$ \hspace{1cm} (2. 6)

$$S(a_m) = -a_m,$$ \hspace{1cm} (2. 7)

$$S(\kappa) = -\kappa.$$ \hspace{1cm} (2. 8)

From the consistency of this Hopf structure with (2.2), i.e. the coproduct operation is given by

$$\Delta [a_m, a_n] = \Delta(a_m) \Delta (a_n) - \Delta(a_n) \Delta (a_m)$$ \hspace{1cm} (2. 9)

$$= 1 \otimes [a_m, a_n] + [a_m, a_n] \otimes 1$$ \hspace{1cm} (2. 10)

$$= \Delta(\kappa) m \delta_{m+n,0}.$$ \hspace{1cm} (2. 11)
On the other hand, the commutator relations (2.2) are equivalent to the contraction
\[ h(z)h(w) = \frac{\kappa}{(z-w)^2} + :h(z)h(w): \quad (2.12) \]

So, we shall now demonstrate that this contraction relation has a Hopf structure in the OPE language.

\[ \Delta(h(z)) = h(z) \otimes 1 + 1 \otimes h(z), \quad (2.13) \]
\[ \epsilon(h(z)) = 0, \quad (2.14) \]
\[ S(h(z)) = -h(z) \quad (2.15) \]

One can check that (2.13-15) satisfy the Hopf algebra axioms and that the defining relation (2.12) is consistent with them, i.e. the coproduct \( \Delta \)

\[ \Delta(h(z)) \Delta(h(w)) = 1 \otimes h(z)h(w) + h(z)h(w) \otimes 1 \\
\quad + h(z) \otimes h(w) + h(w) \otimes h(z) \quad (2.16) \]
\[ = \frac{\Delta(\kappa)}{(z-w)^2} + :\Delta(h(z)) \Delta(h(w)): \quad (2.17) \]

In the above, it is seen that the last two terms in the (2.16) does not contribute to the OPEs as the singular terms.

### 3 The Hopf Structure of \( W_2 \)-Algebra

One can says that the \( W_2 \)-algebra is realized by the Feigin-Fuchs type of free massless scalar fields \( \{ h(z) \} \), of conformal spin-1. Let us define a conformal field \( T(z) \) having conformal spin-2 as follows:

\[ T(z) = \frac{1}{2} :h(z)h(z): \quad (3.1) \]

Using the contraction (2.12), we want to construct \( W_2 \)-algebra. So the non-trivial OPE of \( T(z) \) with itself takes the form

\[ T(z)T(w) = \frac{1}{4}\kappa^2 \frac{1}{(z-w)^4} + \frac{2\kappa T(w)}{(z-w)^2} \frac{\kappa \partial T(w)}{z-w} + :T(z)T(w): \quad (3.2) \]

For this construction, we used the following OPEs:

\[ T(z)h(w) = \frac{\kappa h(w)}{(z-w)^2} + \frac{\kappa \partial h(w)}{z-w} + :T(z)h(w):, \quad (3.3) \]
\[ h(z)T(w) = \frac{\kappa h(w)}{(z-w)^2} + :h(z)T(w): \quad (3.4) \]
The corresponding Hopf structure is given by

\[
\triangle (T(z)) = \frac{1}{2} : \triangle (h(z)) \triangle (h(z)) :
\]

\[
= 1 \otimes T(z) + T(z) \otimes 1 + h(z) \otimes h(z)
\]

(3. 5)

one can say here that this coproduct is not as in (1.6), but this additional term
\(h(z) \otimes h(z)\) will give us a relation between \(\triangle (e)\) and \(\triangle (\kappa^2)\) in the following calculations. Finally, it can be verified that this comultiplication rule is an algebraic homomorphism for the \(W_2\)-algebra (3.2).

\[
\triangle (T(z)) \triangle (T(w)) = 1 \otimes T(z)T(w) + T(z)T(w) \otimes 1 + h(w) \otimes T(z)h(w) + T(z)h(w) \otimes h(w) + h(z) \otimes h(z)T(w) + h(z)T(w) \otimes h(z) + h(z)h(w) \otimes h(z)h(w)
\]

(3. 7)

by using the explicit operator product expansions (3.2-4)

\[
= 1 \otimes \left\{ \frac{1}{(z-w)^4} + \frac{2\kappa T(w)}{(z-w)^2} + \frac{\kappa \partial T(w)}{z-w} + : T(z)T(w) : \right\}
\]

\[
+ \left\{ \frac{1}{2\kappa^2} \frac{2\kappa T(w)}{(z-w)^2} + \frac{\kappa \partial T(w)}{z-w} + : T(z)T(w) : \right\} \otimes 1
\]

\[
+ h(w) \otimes \left\{ \frac{\kappa h(w)}{(z-w)^2} + \frac{\kappa \partial h(w)}{z-w} + : T(z)h(w) : \right\}
\]

\[
+ \left\{ \frac{\kappa h(w)}{(z-w)^2} + \frac{\kappa \partial h(w)}{z-w} + : T(z)h(w) : \right\} \otimes h(w)
\]

\[
+ h(z) \otimes \left\{ \frac{\kappa h(w)}{(z-w)^2} + : h(z)T(w) : \right\} + \left\{ \frac{\kappa h(w)}{(z-w)^2} + : h(z)T(w) : \right\} \otimes h(z)
\]

\[
+ \frac{\kappa}{(z-w)^2} + : h(z)h(w) : \right\} \otimes \left\{ \frac{\kappa}{(z-w)^2} + : h(z)h(w) : \right\}
\]

(3. 8)

and also Taylor expansion for \(h(z)\) at \(w\)

\[
h(z) = h(w) + (z-w)\partial h(w) + \frac{1}{2}(z-w)^2 \partial^2 h(w) + \cdots
\]

(3. 9)

we obtain

\[
\triangle (T(z)) \triangle (T(w)) = \frac{\triangle (h(z)) \triangle (h(z))}{(z-w)^4} + \frac{\triangle (2\kappa T(w))}{(z-w)^2} + \frac{\triangle (\kappa \partial T(w))}{z-w}
\]

\[
+ : \triangle (T(z)) \triangle (T(w)) :
\]

(3. 10)
we must emphasize here that the aims of the element $\kappa$ is to provide to Hopf algebra structure for the free field algebra (2.12), then the coproduct of central element $c$ for the abstract $W_2$-algebra (1.6) must be $\Delta(c) = c \otimes 1 + 1 \otimes c$ as in equation (1.9), but in the present realization $\tilde{c} = 1$ since there is one free field and one can says that the contribution of the one free field to the central term is only one, and then it is seen that the central term and all structure constants of the $W_2$-Algebra depend only generator $\kappa$, so there is a relation between the central element $\Delta(c)$ and $\Delta(\kappa^2)$ in general, but this appears in the present work as

$$\Delta(c) = \Delta(\tilde{c} \kappa^2) = \Delta(\tilde{c}) \Delta(\kappa^2) = \Delta(\tilde{c}) \Delta(\kappa) \Delta(\kappa)$$  (3.11)

$$= \kappa^2 \otimes 1 + 1 \otimes \kappa^2 + 2 \kappa \otimes \kappa$$  (3.12)

where $\Delta(\tilde{c} = 1) = 1 \otimes 1$. Therefore, this relations prevent a doubling of the central charge $c$ and $\kappa^2$. This points of view is also valid for $N=1$ superconformal algebra.

4 The Hopf Structure of Superconformal Algebra

$N = 1$ superconformal algebra \cite{5} is generated by a fermionic spin-3/2 chiral field $G(z)$ and stress-energy tensor $T(z)$, which are satisfy the following OPEs:

$$G(z)G(w) = \frac{\frac{2c}{(z-w)^3} + \frac{2T(w)}{(z-w)}}{(z-w)}$$  (4.1)

$$T(z)G(w) = \frac{\frac{4G(w)}{(z-w)^2} + \partial G(w)}{z-w}$$  (4.2)

$$T(z)T(w) = \frac{\frac{1}{4}c}{(z-w)^4} + \frac{\frac{2T(w)}{(z-w)^2} + \partial T(w)}{z-w}$$  (4.3)

This algebra can be realized a Feigin-Fuchs Type of free massless scalar field $h(z) \equiv i \partial \varphi(z)$ (2.1) and a real fermion field

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-\frac{3}{2}}$$  (4.4)

In addition to the Hopf structure of the Feigin-Fuchs Type of free massless scalar boson field quantization as in section 2, we have to give the Hopf structure of the Feigin-Fuchs Type of free massless scalar fermion field quantization. So the canonical quantization gives the following anti-commutator statement,

$$\{\psi_m, \psi_n\} \equiv \psi_m \psi_n + \psi_n \psi_m = \kappa \delta_{m+n,0}$$  (4.5)
and \([\psi_n, \kappa] = 0\). This associative algebra is a Hopf algebra with

\[
\begin{align*}
\Delta(\psi_m) &= \psi_m \otimes 1 + 1 \otimes \psi_m, \quad (4.6) \\
e(\psi_m) &= 0, \quad (4.7) \\
S(\psi_m) &= -\psi_m \quad (4.8)
\end{align*}
\]

From the consistency of this Hopf structure with eqn. (4.5), i.e. the coproduct operation is given by

\[
\begin{align*}
\Delta \{ \psi_m, \psi_n \} &= \Delta(\psi_m) \Delta(\psi_n) + \Delta(\psi_n) \Delta(\psi_m) \quad (4.9) \\
&= 1 \otimes \{ \psi_m, \psi_n \} + \{ \psi_n, \psi_m \} \otimes 1 \quad (4.10) \\
&= \Delta(\kappa) \delta_{m+n,0} \quad (4.11)
\end{align*}
\]

where we used a parity condition \((a_1 \otimes b_1)(a_2 \otimes b_2) = -a_1 a_2 \otimes b_1 b_2\) (if \(b_1\) and \(a_2\) are odd). On the other hand, the anti-commutator relations (4.5) are equivalent to the contraction statement

\[
\psi(z) \psi(w) = \frac{\kappa}{z - w} + : \psi(z) \psi(w) : \quad (4.12)
\]

So, we shall now demonstrate that this contraction relation has a Hopf structure in the OPE language.

\[
\begin{align*}
\Delta(\psi(z)) &= \psi(z) \otimes 1 + 1 \otimes \psi(z) \quad (4.13) \\
e(\psi(z)) &= 0 \quad (4.14) \\
S(\psi(z)) &= -\psi(z) \quad (4.15)
\end{align*}
\]

One can check that equations (4.13-15) satisfy the Hopf algebra axioms and that the defining relation equation (4.12) is consistent with them. I.e. the coproduct \(\Delta\)

\[
\begin{align*}
\Delta(\psi(z)) \Delta(\psi(w)) &= 1 \otimes \psi(z) \psi(w) + \psi(z) \psi(w) \otimes 1 \\
&\quad + \psi(z) \otimes \psi(w) + \psi(w) \otimes \psi(z) \quad (4.16) \\
&= \Delta(\kappa) \frac{z - w}{z - w} + : \Delta(\psi(z)) \Delta(\psi(w)) : \quad (4.17)
\end{align*}
\]

In order to realize the Hopf structure of \(N = 1\) Superconformal algebra which are given as in eqn.(4.1-3). Let us define a conformal field \(G(z)\) having conformal spin-3\(\frac{3}{2}\), as follows:

\[
G(z) = : \psi(z) h(z) : \quad (4.18)
\]

By using the statements (2.12) and (4.12), the OPE of \(G(z)\) with itself takes the form,

\[
G(z) G(w) = \frac{\kappa^2}{(z - w)^2} + \frac{2 \kappa T(w)}{(z - w)} + : G(z) G(w) : \quad (4.19)
\]
where $T(z)$ is stress-energy tensor, which is in the form of:

$$T(z) = \frac{1}{2} : h(z)h(z) : + \frac{1}{2} : \psi(z)\partial\psi(z) : \quad (4.20)$$

and the OPEs with itself and also with $G(z)$ are

$$T(z)T(w) = \frac{1}{2}\kappa^2 (z-w)^2 + \frac{2\kappa T(w)}{(z-w)^2} + \frac{\kappa \partial T(w)}{z-w} + : T(z)T(w) : \quad (4.21)$$

$$T(z)G(w) = \frac{\kappa G(w)}{(z-w)^2} + \frac{\kappa \partial G(w)}{z-w} + : T(z)G(w) : \quad (4.22)$$

respectively. For this realization, we emphasize here that we used the OPEs in the equations (3.3-4), and also the following OPEs:

$$T(z)\psi(w) = \frac{1}{2}\kappa h(w) + \frac{\kappa \partial \psi(w)}{z-w} + : T(z)\psi(w) : \quad (4.23)$$

$$G(z)h(w) = \frac{\kappa \psi(w)}{(z-w)^2} + \frac{\kappa \partial \psi(w)}{z-w} + : G(z)h(w) : \quad (4.24)$$

$$G(z)\psi(w) = \frac{\kappa h(w)}{z-w} + : G(z)\psi(w) : \quad (4.25)$$

The Hopf structure of $N=1$ superconformal algebra is given by

$$\Delta (G(z)) = : \Delta (\psi(z)) \Delta (h(z)) : \quad (4.26)$$
$$= 1 \otimes G(z) + G(z) \otimes 1 + h(z) \otimes \psi(z) + \psi(z) \otimes h(z) \quad (4.27)$$

and

$$\Delta (T(z)) = \frac{1}{2} : \Delta (h(z)) \Delta (h(z)) : + \frac{1}{2} : \Delta (\psi(z)) \Delta (\partial \psi(z)) : \quad (4.28)$$
$$= 1 \otimes T(z) + T(z) \otimes 1 + h(z) \otimes h(z)$$
$$+ \psi(z) \otimes \partial \psi(z) + \partial \psi(z) \otimes \psi(z) \quad (4.29)$$

Finally, one can easily verify that these comultiplication rules are an algebraic homomorphism for the $N=1$ superconformal algebra (4.1-3).
5 Conclusions

In this latter we have presented that the universal enveloping algebra of the underlying symmetry algebra in the two dimensional (super)conformal field theory is essentially the same as that of the corresponding universal enveloping algebra of any simple Lie algebra, with examples only for the $W_2$-algebra and also $N = 1$ superconformal algebra. We will try to extend these studies beyond the $W_2$-algebra and also at least $N = 2$ superconformal algebra. The investigations in this directions are under study. At this point this paper does not have a composed system, but, besides some previous articles a detailed calculations will be also given for the connection between the two products in the subsequent works which will be the complementary to this one.

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