UPPER SEMICONTINUITY OF THE DIMENSIONS OF AUTOMORPHISM GROUPS OF DOMAINS IN $\mathbb{C}^n$

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Abstract. Let $\mathcal{H}^n$ be the metric space of all bounded domains in $\mathbb{C}^n$ with the metric equal to the Hausdorff distance between boundaries of domains. We prove that the dimension of the group of automorphisms of domains is an upper semicontinuous function on $\mathcal{H}^n$. We also provide theorems and examples regarding the change in topological structure of these groups under small perturbation of a domain in $\mathcal{H}^n$.

0. Introduction

The automorphism group $\text{Aut}(D)$ (the group of biholomorphic self-maps of $D$) of a bounded domain $D$ in $\mathbb{C}^n$ is, in general, difficult to describe and little is known about it. However, it is known (see [SZ, BD]) that any compact Lie group can be realized as the group of automorphisms of a smooth strictly pseudoconvex domain, and (see [ShT]) that any linear Lie group can be realized as the group of automorphisms of a bounded domain. So, if we consider the group $\text{Aut}(D)$ as a function of $D$, the set of values is quite large.

If one considers this function on the metric space $\mathcal{H}^n$ of all bounded domains in $\mathbb{C}^n$ with the metric equal to the Hausdorff distance between boundaries of domains, one can expect that small perturbation of the boundary may only “decrease” the group, i.e., the function $\text{Aut}(D)$ is “upper semicontinuous”. Indeed, in [GK], [Ma] and [FP] the authors, using topologies different from $\mathcal{H}^n$, proved the upper semicontinuity of the function $\text{Aut}(D)$ in the sense that $\text{Aut}(\tilde{D})$ is isomorphic to a subgroup of $\text{Aut}(D)$ when $\tilde{D}$ is “close” to $D$. But, in general, this idea is not true according to the following theorem ([FP]).

Theorem 0.1. Let $M$ be a domain in $\mathbb{C}^n$. Then there exists an increasing sequence of bounded domains $M_k \subset M_{k+1} \subsetneq M$ such that $M = \bigcup M_k$ and $\text{Aut}(M_k)$ contains a subgroup isomorphic to $\mathbb{Z}_k$.

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This shows that domains in \( \mathbb{C}^n \) with an automorphism group containing \( \mathbb{Z}_k \) are everywhere dense in \( \mathcal{H}^n \), and it is well known that domains without non-trivial automorphisms are dense in \( \mathcal{H}^n \). So arbitrarily small perturbation of a domain in \( \mathcal{H}^n \) may create a domain with a larger automorphism group. But, for all known examples, this group is discrete, so it is of dimension zero. The natural question arises: can small perturbation in \( \mathcal{H}^n \) create domains with larger dimensions of automorphism groups?

In this paper we answer this question in the negative. Namely, we prove the following

**Theorem 0.2.** The function \( \dim \text{Aut}(D) \) is upper semicontinuous on \( \mathcal{H}^n \).

An immediate consequence is the following

**Corollary 0.3.** For each \( k > 0 \) the set of all domains in \( \mathcal{H}^n \) whose groups of automorphisms have dimensions greater than or equal to \( k \) is closed and, therefore, nowhere dense.

Thus a domain cannot be approximated by domains whose automorphism groups have strictly larger dimensions.

To prove Theorem 0.2 we consider a sequence of domains \( D_j \) converging in \( \mathcal{H}^n \) to a domain \( D \). The identity components \( \text{Aut}_0(D_j) \) of \( \text{Aut}(D_j) \) have the same dimensions as \( \text{Aut}(D_j) \). Also the dimensions of the Lie algebras of holomorphic vector fields generated by all one-parameter groups in \( \text{Aut}_0(D_j) \) coincide with \( \dim \text{Aut}_0(D_j) \). Lemma 2.4 states that the uniform norm of such fields on a compact set is bounded by its norm on an arbitrarily selected ball times a constant that, basically, depends on the size of the ball and the distances from the ball and the compact set to the boundary of a domain. This allows us to normalize bases in Lie algebras of \( \text{Aut}_0(D_j) \) and apply Theorem 2.5, which asserts the existence of non-trivial limits of those vector fields. The limits belong to the Lie algebra of \( \text{Aut}_0(D) \) and this gives us the proof.

It is reasonable to ask whether \( \text{Aut}_0(D_j) \) are always isomorphic to a subgroup of \( \text{Aut}_0(D) \) when \( j \) is large. An example in Section 3 shows that the answer is negative.

If \( K_j \) is a maximal compact subgroup of \( \text{Aut}_0(D_j) \), then \( \text{Aut}_0(D_j) \) is diffeomorphic to \( K_j \times \mathbb{R}^{k_j} \) (see [MZ, p. 188]). The groups \( K_j \) may decrease or even disappear in the limit (see Example 3.2), while non-compact parts never vanish (see Theorem 3.3).
1. Some basic facts

Let $D$ be a bounded domain in $\mathbb{C}^n$. If the Lie group $\text{Aut}(D)$ has positive dimension, then it has one-parameter subgroups $g(\cdot, t)$, $-\infty < t < \infty$, i.e., $g(z, t + s) = g(g(z, t), s)$. Such subgroups generate vector fields

$$X(z) = \frac{\partial g}{\partial t}(z, 0)$$

that are holomorphic. Also, if $X$ is a holomorphic vector field on $D$ that is $\mathbb{R}$-complete, i.e., the initial value problem

$$\frac{\partial g}{\partial t}(z, t) = X(g(z, t)), \quad g(z, 0) = z$$

has a solution on $D \times \mathbb{R}$, then $g(z, t)$ is a one-parameter group.

The vector field $X$ has the following group property:

$$X(g(z, t)) = \frac{\partial g}{\partial z}(z, t)X(z). \quad (1)$$

For every two points $z$ and $w$ in $D$ among all holomorphic mappings of $D$ into the unit disk $\Delta$ we choose holomorphic functions $f$ such that $f(w) = 0$ and $f(z)$ is real and the maximal possible. Such functions $f$ exist and are called Carathéodory extremal functions for $z$ and $w$ on $D$. The quantity

$$\rho(0, f(z)) = \frac{1}{2} \ln \frac{1 + f(z)}{1 - f(z)} \quad (2)$$

is called the Carathéodory distance $c_D(z, w)$ on $D$. (Note that the formula for $\rho(0, a)$ gives the Poincaré distance between 0 and $a$ in the unit disc.) When $D$ is bounded this distance is non-degenerate and invariant, i.e., $c_D(g(z), g(w)) = c_D(z, w)$ for every $g \in \text{Aut}(D)$ (see [Sh, Ch. 5, §18]).

For a point $w \in D$ and a vector $Y$ in $\mathbb{C}^n$, among all holomorphic mappings of $D$ into the unit disk $\Delta$ we choose holomorphic functions $f$ such that $f(w) = 0$ and $(f'(w), Y)$ is real and the maximal possible. (Here $(Z, Y) = \sum_{j=1}^n z_j y_j$.) These functions are Carathéodory extremal functions for $Y$ at $w$ in $D$. The Carathéodory length of $Y$ at $w$ is defined to be $C_D(w, Y) = (f'(w), Y)$. It follows from [Sh, Ch. 5, §18] that if $w(t)$ is a smooth curve in $D$ with $w(0) = w$ and $Y = w'(0)$, then

$$c_D(w, w(t)) = C_D(w, Y)t + o(t). \quad (3)$$

Let $B(w, r)$ be the ball of radius $r$ centered at $w$ and let $|Y|$ be the Euclidean norm of $Y$. If $B(w, r) \subset D \subset B(w, R)$, then

$$\frac{|Y|}{R} \leq C_D(w, Y) \leq \frac{|Y|}{r}. \quad (4)$$
2. Proof of Theorem 0.2

Lemma 2.1. Let $D$ be a domain in $\mathbb{C}^n$ and let $d(z, w)$ be an invariant metric on $D$ satisfying the triangle inequality. If $g(z, t)$ is a group action on $D$, then for any $w, z \in D$

$$|d(g(w, t), z) - d(w, z)| \leq d(z, g(z, t)).$$

Proof. Apply the identity $d(w, z) = d(g(w, t), g(z, t))$ and the triangle inequality. \hfill \Box

Lemma 2.2. Let $w \in B(w, r) \subset D \subset B(w, R) \subset \mathbb{C}^n$. Then for any $Y \in \mathbb{C}^n$, $|Y| = 1$,

$$\text{Re} \left( \nabla f_s(w), Y \right) > \frac{1}{4R},$$

where $f_s(z)$ is a Carathéodory extremal function for $w$ and $w + sY$ in $D$, and $s$ is a real number such that

$$0 < s \leq \varepsilon = \frac{r^2}{16R}.$$ 

Proof. Let us fix $Y$ and introduce $D_Y = \{ \xi \in \mathbb{C} : w + \xi Y \in D \}$. Clearly, $\Delta(0, r) \subset D_Y \subset \Delta(0, R)$, where $\Delta(0, s)$ is the disk of radius $s$ centered at 0.

Let $g_Y(z)$ be a Carathéodory extremal function for $Y$ at $w$ in $D$. For $\xi \in D_Y$ we introduce the functions $u(\xi) = \text{Re} F(\xi)$, where $F(\xi) = f_s(w + \xi Y)$, and $v(\xi) = \text{Re} G(\xi)$, where $G(\xi) = g_Y(w + \xi Y)$. All these functions are well-defined on $D_Y$ and $F(0) = G(0) = 0$, $G'(0) = C_D(w, Y)$ and $u(s) = F(s) \geq v(s)$. Let us prove that $v(t) \geq t/(2R)$ when $t \in [0, \varepsilon]$. Since

$$|v(t) - v'(0)t| \leq \frac{1}{2} \sup_{0 \leq x \leq \varepsilon} |v''(x)| \cdot t^2,$$

$\varepsilon \leq r^2/(16R) < r/2$ and by Cauchy estimate $|v''(x)| \leq 2/(r - \varepsilon)^2$ when $x < \varepsilon$, we see that

$$v(t) \geq v'(0)t - \frac{1}{(r - \varepsilon)^2}t^2 \geq v'(0)t - \frac{4}{r^2}t^2.$$

Since $t \leq r^2/(16R)$ and by (4)

$$v'(0) = G'(0) = C_D(w, Y) \geq \frac{1}{R},$$

$$v(t) \geq \frac{t}{R} - \frac{t}{4R} \geq \frac{t}{2R}.$$
for $0 \leq t \leq \varepsilon$. In particular,

$$v(s) = \text{Re} G(s) \geq \frac{s}{2R}.$$ 

Applying to the function $u(t)$ the same analysis as above we obtain

$$u(s) \leq u'(0)s + \frac{4}{r^2}s^2.$$ 

Hence

$$\frac{s}{2R} \leq v(s) \leq u(s) \leq u'(0)s + \frac{4}{r^2}s^2 \leq u'(0)s + \frac{s}{4R}.$$ 

Thus

$$\text{Re}(\nabla f_s(w), Y) = \text{Re} F'(0) = u'(0) \geq \frac{1}{4R}. \quad \square$$

**Lemma 2.3.** Let $B(0, r + a) \subset\subset D \subset\subset B(0, R)$, $r, a > 0$. Then there exists a positive $\delta = \delta(a, r, R) < a$ such that

$$\|X\|_{B(0, r+\delta)} \leq \frac{32R}{a}\|X\|_{B(0, r)}$$

for every holomorphic vector field $X$ generated by a one-parameter group action $g(z, t)$ on $D$.

**Proof.** Let $w$ belong to $B(0, r + a/2)$. Since

$$w \in B(w, a/2) \subset\subset D \subset\subset B(w, 2R),$$

by Lemma 2.2 there is an $\varepsilon = \varepsilon(a, R) > 0$ such that for every $w \in B(0, r + a/2)$, every $Y \in \mathbb{C}^n$, $|Y| = 1$, and every $s \in (0, \varepsilon]$

$$\text{Re}(\nabla f(w), Y) \geq \frac{1}{8R}, \quad (5)$$

where $f$ is a Carathéodory extremal function for $w$ and $w + sY$.

Let us take a positive number $\delta < a/2$ so small that for every $w \in B(0, r + \delta)$ and every unit vector $V$ there is a unit vector $Y$ such that $w + sY \in B(0, r)$ for some real $s$ with $|s| < \varepsilon$ and

$$|V - Y| < b = \frac{a}{32R}.$$ 

Clearly, the choice of this $\delta$ depends only on $a$, $r$ and $R$.

The lemma needs a proof only for non-trivial group actions when $X \neq 0$. Let $w \in \partial B(0, r + \delta)$, $X(w) \neq 0$ and let $V = X(w)/|X(w)|$. We choose a vector $Y$ and a real $s$ satisfying the above conditions.

Let $f$ be a Carathéodory extremal function for $w$ and $z = w + sY$. Since $B(w, a/2) \subset\subset D$, by Schwarz inequality,

$$|\text{Re}(\nabla f(w), Y - V)| \leq b|\nabla f(w)| \leq \frac{2b}{a}.$$
Hence by (5),

\[ \text{Re}(\nabla f(w), V) \geq \frac{2b}{a} \geq \frac{1}{8R} - \frac{1}{16R} = \frac{1}{16R}. \]

Let \( \zeta(t) = f(g(w, t)) \) and \( p = f(z) \). We introduce

\[ m(t) = \left| \frac{\zeta(t) - p}{1 - p \zeta(t(t))} \right|. \]

If \( \rho(\zeta, \xi) \) is the Poincare metric on \( U \), then

\[ \rho(0, p) = \frac{1}{2} \ln \frac{1 + m(t)}{1 - m(t)}. \]

A straightforward calculation shows that

\[ \frac{d}{dt} \rho(\zeta(0), p) = -\text{Re}(\nabla f(w), X(w)), \]

and, by using this calculation, we obtain

\[ \frac{d}{dt} \rho(\zeta(0), p) = -\text{Re}(\nabla f(w), X(w)). \]

Hence

\[ \rho(\zeta(-t), p) \geq \rho(0, p) + t\text{Re}(\nabla f(w), X(w)) \geq c_D(z, w) + \frac{t}{16R} |X(w)| \]

for small positive \( t \). Since the Carathéodory metric decreases under the holomorphic mapping \( f \),

\[ c_D(z, g(w, -t)) \geq \rho(\zeta(-t), p) \geq c_D(z, w) + \frac{t}{16R} |X(w)|. \tag{6} \]

By (6) and Lemma 2.1,

\[ \frac{t}{16R} |X(w)| \leq c_D(z, g(w, -t)) = c_D(z, w) \leq c_D(z, g(z, -t)). \]

By (3), \( c_D(z, g(z, -t)) = C_D(z, X(z))t + o(t) \). Note that \( B(z, a) \subset D \) and, therefore, \( C_D(z, X(z)) \leq 1/a \). Hence

\[ c_D(z, g(z, -t)) \leq 2C_D(z, X(z))t \leq \frac{2}{a} |X(z)|t \]

for small positive \( t \). Thus

\[ |X(w)| \leq \frac{32R}{a} |X(z)| \]

and

\[ \|X\|_{B(0,r+\delta)} \leq \frac{32R}{a} \|X\|_{B(0,r)}. \]

\[ \square \]
Lemma 2.4. Let $R > 2r > 2s > 0$. Let $K$ be a connected compact set containing $0$ in $\mathbb{C}^n$. Let $D$ be a domain in $\mathbb{C}^n$ such that $B(0, 2r) \subset D \subset B(0, R)$ and such that the $3s$-neighborhood of $K$ is contained in $D$. Then there exists a positive constant $C = C(K, R, s)$ such that $\|X\|_K \leq C\|X\|_{B(0, r)}$ for each holomorphic vector field $X$ generated by a one-parameter group action $g(z, t)$ on $D$.

Proof. Let $X$ be such a vector field on $D$. By the previous lemma there exist positive numbers $\delta = \delta(s, R) < s$ and $c = c(s, R)$ such that

$$\|X\|_{B(z, s + \delta)} \leq c\|X\|_{B(z, s)}$$

whenever $z \in D$ is at least $3s$ away from $\partial D$. There is a positive integer $N = N(K, \delta)$ such that for each $z \in K$ there is a set of $N$ points $\{z_1, \ldots, z_N\} \subset K$ with $z_1 = 0$, $z_N = z$, and $|z_{k+1} - z_k| < \delta$ for $k = 1, \ldots, N - 1$. Since $B(z_{k+1}, s) \subset B(z_k, s + \delta)$, we see that

$$\|X\|_{B(z_{k+1}, s)} \leq c\|X\|_{B(z_k, s)}$$

for $k = 1, \ldots, N - 1$. Thus,

$$\|X\|_{B(z, s)} \leq c^{N-1}\|X\|_{B(0, s)}.$$ 

In particular, $|X(z)| \leq c^{N-1}\|X\|_{B(0, s)} \leq c^{N-1}\|X\|_{B(0, r)}$. Therefore, $\|X\|_K \leq c^{N-1}\|X\|_{B(0, r)}$. \hfill $\Box$

Theorem 2.5. Suppose a sequence of domains $D_j$ converge in $\mathcal{H}^n$ to a domain $D$ and a ball $B(p, r + a)$, $r, a > 0$, belongs to all $D_j$. Also suppose that $g_j(z, t)$ are non-trivial one-parameter group actions on $D_j$ generating the holomorphic vector fields $X_j$. If $\|X_j\|_B = 1$, $B = B(p, r)$, then there is a subsequence of the group actions $g_{j_k}(z, t)$ that converges to a non-trivial group action $g(z, t)$ on $D$ uniformly on compacta in $D \times \mathbb{R}$ and

$$\lim_{k \to \infty} X_{j_k}(w) = X(w)$$

uniformly on compacta in $D$, where $X$ is the holomorphic vector field generated by $g$.

Proof. Let $K \subset D$. Choose $\delta > 0$ so that the $3\delta$-neighborhood of $K$ is contained in $D$ and in each $D_j$. Let $\hat{K}$ and $\tilde{K}$ denote the $2\delta$-neighborhood and the $\delta$-neighborhood of $K$ respectively. By Lemma 2.4 there exists $A > 0$ such that $\|X_j\|_{\hat{K}} \leq A$.

Let $\tau = \delta/(2A)$. Define the mapping $h_j : \hat{K} \times (-\tau, \tau) \to D_j$ as the solution of the initial value problem

$$\frac{\partial}{\partial t} h_j(z, t) = iX_j(h_j(z, t)), \quad h_j(z, 0) = z.$$
Since $\tau |X_j| < \delta$ in $\hat{K}$, it follows from the ODE’s theory that the mapping $h_j$ is well-defined.

For $M = \{ \zeta \in \mathbb{C} : |\text{Im}\zeta| < \tau \}$ we define $G_j : \hat{K} \times M \to D_j$ by $G_j(z, t + is) = g_j(h_j(z, s), t)$. Since $X_j$ is holomorphic, the mapping $G_j$ is holomorphic in $z$. We now prove that it is holomorphic in $\zeta = t + is$. It is clear that

$$\frac{\partial G_j}{\partial t}(z, t + is) = X_j(G_j(z, t + is)). \quad (7)$$

It follows immediately from the fact that the Poisson brackets $[X_j, iX_j] \equiv 0$, that

$$\frac{\partial G_j}{\partial s}(z, t + is) = iX_j(G_j((z, t + is)). \quad (8)$$

This fact also can be proved by a straightforward reasoning:

$$\frac{\partial G_j}{\partial s}(z, t + is) = \frac{\partial g_j}{\partial z}(h_j(z, s), t) \cdot iX(h_j(z, s)) = iX_j(g_j(h_j(z, s), t)) = iX_j(G_j(z, t + is));$$

the middle equality is by the infinitesimal group property (1). The equations (7) and (8) are the Cauchy-Riemann equations for $G_j$ in $\zeta$. So $G_j$ is holomorphic.

Passing to a subsequence, if necessary, we may assume that the mappings $G_j$ converge to a mapping $G$ uniformly on compacta in $\hat{K} \times M$. Consequently, the mappings $g_j(z, t)$ converge to $g(z, t)$ uniformly on compacta in $\hat{K} \times \mathbb{R}$, and the vector fields $X_j$ converge to

$$X(z) = \frac{\partial g}{\partial t}(z, 0)$$

uniformly on compacta in $\hat{K}$.

It follows that some subsequence of the sequence $\{g_j(z, t)\}$ converges to a mapping $g(z, t) = G(z, t)$ uniformly on compacta in $D \times \mathbb{R}$. Thus, $g(z, t)$ is a group action. Since $\|X\|_B = 1$, this group action is non-trivial. \qed

Proof of Theorem 0.2. Let $D_j$ be a sequence of domains converging in $\mathcal{H}^a$ to a domain $D$. Let us choose a ball $B(p, r + a)$, $r, a > 0$, belonging to all $D_j$ for sufficiently large $j$ and take $\delta > 0$ from Lemma 2.3. Let $\tilde{B} = B(p, r + \delta)$. We may assume that the dimensions of all groups $G_j = \text{Aut}_0(D_j)$ are the same and equal to $k$. Since the Lie algebra $A_j$ of all holomorphic vector fields on $D_j$ generated by one-parameter subgroups in $G_j$ has the same dimension as $G_j$, we can
choose $X^m_j \in A_j$, $1 \leq m \leq k$, such that
\[ \int_{\hat{B}} (X^m_j, X^l_j) \, dV = \delta_{ml}, \]
where $\delta_{ml}$ is Kronecker’s delta.

Clearly, $\|X^m_j\|_{\hat{B}} \geq \text{Vol}(\hat{B})^{-1}$. On the other hand, by Cauchy estimates and Lemma 2.4, for some constants we have
\[ 1 \geq C_1 \|X^m_j\|_{\hat{B}} \geq C_2 \|X^m_j\|_{\hat{B}}. \]

Let $g^m_j$ be the one-parameter groups generated by $X^m_j$. By Theorem 2.5 one can choose a subsequence $\{j_k\}$ such that $g^m_{j_k}$ converge, uniformly on compacta in $D \times \mathbb{R}$, to a one-parameter group $g^m(z, t)$ on $D$, and $X^m_{j_k}$ converge to a vector field $X^m$ uniformly on compacta in $D$. Since
\[ \int_{\hat{B}} (X^m, X^l) \, dV = \delta_{ml}, \]
the dimension of $\text{Aut}_0(D)$ is at least $k$.

3. Structural theorems

By Iwasawa’s theorem (see [MZ, p. 188]) the group $\text{Aut}_0(D)$ is diffeomorphic to $K \times \mathbb{R}^k$, where $K$ is a maximal compact subgroup and $k$ is the characteristic number of $\text{Aut}(D)$. It is interesting to find out what happens with $K$ and $\mathbb{R}^k$ under small perturbations of domains. Let us look at maximal compact subgroups first. The argument of Corollary 4.1 in [FP] provides the following theorem.

**Theorem 3.1.** Let $D$ be a bounded domain in $\mathbb{C}^n$, let $z_0$ be a point in $D$, and let $W$ be a compact set in $D$. If $\hat{D}$ is sufficiently close to $D$ in $\mathcal{H}^n$ and for some maximal compact subgroup $\hat{K}$ in $\text{Aut}_0(\hat{D})$ the orbit $\hat{K}(z_0) \subset W$, then $\hat{K}$ is isomorphic to a subgroup of $\text{Aut}_0(D)$.

Next example shows that without the condition in the above theorem of orbits being contained in a fixed compact set, it is possible that $\text{Aut}(D)$ does not contain a compact subgroup while close domains have $\text{Aut}_0(\hat{D})$ isomorphic to $S^1$. Let $\Delta$ denote the unit disc in $\mathbb{C}$.

**Example 3.2.** There is a sequence $\{D_j\}$ of bounded pseudoconvex domains in $\mathbb{C}^2$ converging to a domain $D$ such that $\text{Aut}(D_j) \cong S^1$ for each $j$, and $\text{Aut}(D) \cong \mathbb{R}$.

**Construction.** Let $Q_j = \{z \in \Delta : |z - 2^{-1} + 2^{-j}| > 1/2\}$, $Q = \{z \in \Delta : |z - 2^{-1}| > 1/2\}$, $D_j = \{(z, w) : z \in Q_j, w \in \Delta, w \neq z\}$, $D = \{(z, w) : z \in Q, w \in \Delta, w \neq z\}$.
1. One can see that $D_j \rightarrow D$.
2. The domains $D_j$ and $D$ are bounded and pseudoconvex.
3. We now prove that $\text{Aut}(D) \cong \mathbb{R}$. Let $F \in \text{Aut}(D)$. On each fiber $(z, \cdot)$, $F$ is bounded and has an isolated singularity, so $F$ extends to be an automorphism of $Q \times \Delta$. Thus, $F$ has the form $F(z, w) = (f(z), g(w))$, or $F(z, w) = (g(w), f(z))$. For both cases, one has, by the definition of $D$, that

$$f(z) = g(z), \quad z \in Q.$$  \hfill (9)

The second case is impossible, since implies that $f(Q) = \Delta$, $g(\Delta) = Q$, and $f(Q) = g(Q)$, which leads to a contradiction that $\Delta$ coincides with a subset of $Q$. Therefore, $F$ has the form $F(z, w) = (f(z), g(w))$, where $f \in \text{Aut}(Q)$, $g \in \text{Aut}(\Delta)$. By (9), $f = g|_Q$. Let $\phi(w) = -i(w+1)/(w-1)$. Then $\phi$ is a biholomorphic map from $\Delta$ to the upper half-plane $\Pi = \{ \zeta \in \mathbb{C} : \text{Im } \zeta > 0 \}$, and $\phi(Q) = \Lambda \equiv \{ \zeta \in \mathbb{C} : 0 < \text{Im } \zeta < 1 \}$. Now $\phi \circ g \circ \phi^{-1}$ is an automorphism of $\Pi$, and its restriction to $\Lambda$ is an automorphism of $\Lambda$. Thus $\phi \circ g \circ \phi^{-1}(\zeta) = \zeta + t$ for some $t \in \mathbb{R}$. It follows that $\text{Aut}(D) = \{ F_t : t \in \mathbb{R} \} \cong \mathbb{R}$, where $F_t(z, w) = (g_t(z), g_t(w))$, and

$$g_t(w) = \phi^{-1}(\phi(w) + t) = \frac{2w + i(w-1)t}{2 + i(w-1)t}.$$

4. In a way very similar to the above argument, one can prove that $\text{Aut}(D_j) \cong S^1$ for each $j$. \hfill \Box

By Theorem 0.2 the creation of compact subgroups with larger dimensions by small perturbations must be compensated by an elimination of some non-compact subgroups so that the total dimension will not go up. It seems to us that the other way around is impossible: characteristic numbers are upper semicontinuous on $\mathcal{H}^n$. While we cannot prove this statement, the following theorem certifies that non-compact parts cannot be created from nothing.

**Theorem 3.3.** Let $D \subset \mathbb{C}^n$ be a bounded domain such that $\text{Aut}_0(D)$ is compact. Then for all $\tilde{D}$ sufficiently close in $\mathcal{H}^n$ to $D$ the group $\text{Aut}_0(\tilde{D})$ is also compact.

**Proof.** If the statement is not true, then there is a sequence $\{D_j\}$ of domains converging to $D$ such that for each $j$ the identity component $G_j = \text{Aut}_0(D_j)$ is noncompact. Write $G = \text{Aut}_0(D)$. Fix a $z_0 \in D$. The orbit $G(z_0)$ is compact. We may assume that $G(z_0) \subset D_j$ for each $j$. For each connected component $F$ of $\text{Aut}(D)$, either the set $F(z_0)$ coincides with $G(z_0)$ or $G(z_0) \cap F(z_0) = \emptyset$. Indeed, if $h \in F$ and $h(z_0) \in G(z_0)$, then $G(z_0) = Gh(z_0) = G(z_0)$, since $H = Gh$. Now we claim that there exists a positive number $a$ such that
a < d(H(z_0), G(z_0)) for each component H of Aut(D) with H(z_0) ≠ G(z_0), where d is the euclidean distance. Otherwise, there is a sequence \{H_k\} of distinct components of Aut(D) with H_k(z_0) ≠ G(z_0) such that d(H_k(z_0), G(z_0)) \to 0. Passing to a subsequence if necessary, we may assume that there are h_k ∈ H_k such that h_k(z_0) tends to a point in G(z_0). It follows that some subsequence of \{h_k\} converges in the compact-open topology to a (g) \in Aut(D); but this is impossible because h_k belong to different components of the Lie group Aut(D). Therefore, such an a exists. Decreasing a if necessary, we see that the open set

\[ V = \{ z \in D : d(z, G(z_0)) < a \} \]

is relatively compact in D and in each D_j, and satisfies \( \overline{V} \cap \text{Aut}(D)(z_0) = G(z_0) \). This implies that \( \partial V \cap \text{Aut}(D)(z_0) = \emptyset \). Since \( G_j \) is noncompact, \( G_j(z_0) \) is noncompact, hence \( G_j(z_0) \cap \partial V \neq \emptyset \). It follows that for each j there is a \( g_j \in G_j \) with \( g_j(z_0) \in \partial V \). Some subsequence of the sequence \{g_j\} converges uniformly on compacta to a \( g \in \text{Aut}(D) \). It is clear that \( g(z_0) \in \partial V \), contradicting \( \partial V \cap \text{Aut}(D)(z_0) = \emptyset \).

\[ \square \]

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