Generalized regression operator estimation for continuous time functional data processes with missing at random response

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Abstract

In this paper, we are interested in nonparametric kernel estimation of a generalized regression function, including conditional cumulative distribution and conditional quantile functions, based on an incomplete sample \((X_t, Y_t, \zeta_t)_{t \in \mathbb{R}^+}\) copies of a continuous-time stationary ergodic process \((X, Y, \zeta)\). The predictor \(X\) is valued in some infinite-dimensional space, whereas the real-valued process \(Y\) is observed when \(\zeta = 1\) and missing whenever \(\zeta = 0\). Pointwise and uniform consistency (with rates) of these estimators as well as a central limit theorem are established. Conditional bias and asymptotic quadratic error are also provided. Asymptotic and bootstrap-based confidence intervals for the generalized regression function are also discussed. A first simulation study is performed to compare the discrete-time to the continuous-time estimations. A second simulation is also conducted to discuss the selection of the optimal sampling mesh in the continuous-time case. Finally, it is worth noting that our results are stated under ergodic assumption without assuming any classical mixing conditions.

Keywords: Asymptotic quadratic error, continuous time ergodic processes, confidence intervals, exchangeable bootstrap, functional data, generalized regression, missing at random.

Subject Classifications: 60F10, 62G07, 62F05, 62H15.

1 Introduction

Let \((E, d)\) be an infinite-dimensional space equipped with a semi-metric \(d(\cdot, \cdot)\). We consider some \(E \times \mathbb{R} \times \{0, 1\}\)-valued stationary and ergodic continuous time process \(\{Z_t = (X_t, Y_t, \zeta_t), \ t \in \mathbb{R}^+\}\) defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and observed at any time \(t \in [0, T]\), where \(\zeta\) is an indicator process taking values zero or one at any instant \(t\). For any \(y\) in a compact set \(S \subset \mathbb{R}\), let \(\psi_y(\cdot)\) be a real valued Borel function defined on \(S \times \mathbb{R}\). The generalized regression function of \(\psi_y(Y)\) given \(X = x\) is defined by \(m_\psi(x, y) := E(\psi_y(Y_t)|X_t = x) = E(\psi(y, Y_t)|X_t = x)\) which is supposed to exist for any \(x \in E\) and independent of \(t\).
Here, the process $Z$ has the same distribution as $(X, Y, \zeta)$, where $Y$ is a real integrable process and $X$ is a continuous time process valued in the functional space $\mathcal{E}$. This means that, for any fixed time $t = t_0$, $X_{t_0} \in \mathcal{E}$. Therefore, if, for instance, we take $\mathcal{E} := L^2[0, 1]$ the space of square integrable functions defined on $[0, 1]$, then for any fixed $t_0$, the process $X_t := \{X_{t_0}(s) : s \in [0, 1]\}$ describes a curve.

In this paper, we assume that the response variable $Y$ is subject to the **missing at random** mechanism (MAR). This means that for an available observed sample $(X_i, Y_i, \zeta_i)_{0 \leq i \leq T}$, $X_i$ is completely observed, whereas $\zeta_i = 1$ if $Y_i$ is observed at time $t$ and $\zeta_i = 0$ otherwise. The random variables $\zeta$ and $Y$ are supposed to be conditionally independent given $X$, that is $P(\zeta = 1|X = x, Y = y) = P(\zeta = 1|X = x) := p(x)$ almost surely (a.s.). The MAR phenomena of the response variable may occur in several situations. For instance, in survey sampling studies the non-response is an increasingly common problem, where the missing response reaches rates of 25% to 30% or even higher (see, e.g., Sikov (2018)). In such case the missing data become a real source of bias in survey sampling estimation. Another example where the response may be subject to the AMR phenomena is the household electricity consumption monitoring. Indeed, the real time collection of intraday electricity consumption is now possible after the deployment of smart meters at the household level. The transmission of the information from the smart meter towards the information system goes usually through WIFI or optical fiber networks which are significantly dependent on the weather conditions. Therefore, a response variable such as the daily total electricity consumption might be subject to missing at random mechanism due to bad weather conditions.

The statistical analysis involving missing data, based on an incomplete discrete sample $(X_i, Y_i, \zeta_i)_{1 \leq i \leq n}$ where the data are assumed independent and identically distributed (i.i.d), has been considered by several authors, see for instance Ling et al. (2015), Cheng (1994), Little and Rubin (2002), Nittner (2003), Tsiatis (2006), Liang et al. (2007), Efromovich (2011), Chaouch et al. (2017), and Chaouch et al. (2009). However, less attention has been given to the case when the covariate is infinite dimensional and the response is missing at random. One can cite, for instance, Ferraty et al. (2013), where the authors considered the estimation of the regression function based on an i.i.d. random sample, and Ling et al. (2015) extended their work to discrete time ergodic processes.

The literature on the estimation of the regression function based on a completely observed sample $(X_i, Y_i)_{i \geq 0}$ copies of a strongly mixing continuous time stationary process $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ is very extensive. One may refer to the monograph by Bosq (1998) and the references therein. Some of these results are extended by Didi and Louani (2014) and Bouzebda and Didi (2017) to the case where the underlying real-valued process is stationary and ergodic. Chaouch and Laïb (2019) have obtained an explicit upper bound of the asymptotic mean square error (AMSE) of kernel regression estimator when the data are sampled from a real-valued continuous time process with a missing at random response. Some of these results are extended to the case where $X$ is valued in an infinite-dimensional space and $Y$ is a real-valued completely observed. Under the $\alpha$-mixing condition Maillot (2008) established the convergence with rates of the regression operator, whereas Chesneau and Maillot (2014) obtained a superoptimal mean square convergence rate of the MSE of regression function for continuous functional time process with irregular paths.

This paper aims to extend Maillot (2008) and Chesneau and Maillot (2014) work at several levels. First, we suppose that the continuous time process satisfies an ergodic assumption rather than an $\alpha$-mixing one. Therefore, the dependence condition we consider is more general and involves several processes which do not satisfy the mixing property. On the other hand, this paper extends results established in Ling et al. (2015) in the context of discrete time functional data processes to the continuous time framework.
Indeed, given \((X_t, Y_t, \zeta_t)_{t \in \mathbb{R}^+}\) copies of a continuous-time stationary ergodic process \((X, Y, \zeta)\), we estimate the general operator \(m_\psi(x, y)\) which including conditional distribution function and conditional quantiles. It is worth noting that such extension is not obvious since it requires an appropriate definition of \(\sigma\)-fields adapted to continuous time context. Such adaptation is crucial when using martingale difference tools to establish asymptotic properties of the estimator. Second, the response variable considered here is affected by the MAR mechanism and therefore is not completely observed as in Maillot (2008). Moreover, in contrast to Maillot (2008) and Chesneau and Maillot (2014), we do not limit our study to the mean square convergence but we provide a more exhaustive inference on the regression operator estimator including pointwise and uniform almost sure convergence rate, identification of the limiting distribution of our estimator and provide two methods to build confidence intervals (central limit theorem and the bootstrap procedure).

The rest of this paper is organised as follows. In Section 2 we present the framework adapted to continuous time ergodic processes and introduce assumptions needed for establishing asymptotic results. The main asymptotic properties of the estimator are discussed in Section 3. An illustration of the performance of the proposed estimator is discussed through simulated data in Section 4. Section 5 discusses an application to conditional quantiles. Finally technical proofs are given in Section 6.

## 2 Framework and assumptions

To define the framework of our study, we need to introduce some definitions. Let \(X = (X_t)_{t \in [0, \infty)}\) be a continuous time process defined on a measurable probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and observed at any time \(t \in [0, T]\).

**Definition 2.1** Let \(\tau\) be a nonnegative real number, a measurable set \(A\) is \(\tau\)-invariant if \(T^\tau (A) = A\) for any \(\tau\)-shift transformation \(T^\tau\), i.e., \((T^\tau (x))_\lambda = x_{s+\lambda}\). The process \(X\) is said to be \(\tau\)-ergodic if for any \(\tau\)-invariant set \(A\), \(\mathbb{P}(A) = \mathbb{P}^2(A)\). It has the ergodic property if there exists an invariant distribution \(F(\cdot)\), such that for any measurable function \(h(\cdot)\) and a random variable \(\xi\) distributed as \(F(\cdot)\) satisfying \(\mathbb{E}(|h(\xi)|) < \infty\), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} h(X_t) dt = \mathbb{E}(h(\xi)) \quad \text{almost surely (a.s.)}. \tag{2.1}
\]

From now on, we will be working on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})\). For a positive real number \(\delta\) such that \(n = \frac{T}{\delta} \in \mathbb{N}\) and \(j \in \mathbb{N} \cap [1, n]\), consider the \(\delta\)- partition \(T_j = j \delta, 1 \leq j \leq n\) of the interval \([0, T]\). Furthermore, for \(t > 0\) and \(1 \leq j \leq n\), we define the following \(\sigma\)-fields:

\[
\mathcal{F}_{t-\delta} := \sigma((X_s, Y_s, \zeta_s) : 0 \leq s < t - \delta), \quad \mathcal{F}_j = \sigma((X_s, Y_s, \zeta_s), 0 \leq s < T_j),
\]

\[
\mathcal{S}_{t,\delta} := \sigma((X_s, Y_s, \zeta_s); (X_r, (1-\ell)Y_r) : 0 \leq s < t, \quad t \leq \tau \leq t + \delta, \quad \ell \in \{0, 1\}).
\]

Whenever \(s < 0\), \(\mathcal{F}_s\) stands for the trivial \(\sigma\)-field. Here, \(\zeta_s\) is a standard Bernoulli process (see Section below). Notice that, for any \(\delta > 0\) and \(t > 0\), \(\mathcal{S}_{t,\delta} \subset \mathcal{S}_{t,\delta}^1, \mathcal{F}_{t-\delta} \subset \mathcal{S}_{t,\delta}^1 \subset \mathcal{S}_{t,\delta}^2\), and for any \(j \geq 2\) such that \(T_{j-1} \leq t \leq T_j, \mathcal{F}_{j-2} \subset \mathcal{F}_{t-\delta} \subset \mathcal{S}_{t,\delta}^2\).

The estimator \(\hat{m}_{\psi, T}(x, y)\) adapted to MAR response may be defined, for any \(\zeta \neq 0\) and \(\int_0^T \zeta_t \Delta_t(x) dt \neq 0\), by

\[
\hat{m}_{\psi, T}(x, y) = \frac{\int_0^T \zeta_t \psi_t(Y_t) \Delta_t(x) dt}{\int_0^T \zeta_t \Delta_t(x) dt}, \tag{2.2}
\]
where $\Delta_t(x) = K \left( \frac{d(x, X_t)}{h_T} \right)$, $K(\cdot)$ is a kernel density function, $h_T$ is the smoothing parameter tending to zero as $T$ goes to infinity. Let $Z_1(x) := \int_0^T \Delta_t(x) dt$ and define the conditional bias as

$$B_T(x,y) := \frac{m_{\psi,T,2}(x,y)}{m_{\psi,T,1}(x,y)} - m_{\psi}(x,y) := C_T(x,y) - m_{\psi}(x,y).$$

(2.3)

where, for $j = 1, 2$,

$$m_{\psi,T,j}(x,y) := \frac{1}{nE(Z_1(x))} \int_0^T \mathbb{E} \left\{ \zeta_t(Y_t)^{j-1} \Delta_t(x) | \mathcal{F}_{t-\delta} \right\} dt.$$  \hspace{1cm} (2.4)

Let us now introduce the assumptions under which we establish our asymptotic results.

(A1) **Assumptions on the kernel function.** Let $K$ is a nonnegative bounded kernel of class $\mathcal{C}^1$ over its support $[0,1]$ such that $K(1) > 0$. The derivative $K'$ exists on $[0,1]$ and satisfies the condition $K'(v) < 0$, for all $v \in [0,1]$ and $\int_0^1 |K'(v)|dv < \infty$ for $j = 1, 2$.

(A2) **Assumptions related to the continuous time functional ergodic processes**

(i) $F_x(u) := \mathbb{P}(d(x, X_t) \leq u) = \phi(u)f(x) + o(\phi(u))$ as $u \to 0$.

(ii) For any $0 \leq s \leq t$, $F^+_x(u) := \mathbb{P}^+(d(x, X_t) \leq u) = \mathbb{P}(d(x, X_t) \leq u | \mathcal{F}_s) = \phi(u)f_{t,s}(x) + g_{t,s}(u)$ with $g_{t,s}(u) = o_{\alpha_s}(\phi(u))$ as $u \to 0$, $g_{t,s}(u)/\phi(u)$ a.s. bounded and $T^{-1} \int_0^T g_{t,s}(u) dt = o_{\alpha_s}(\phi(u))$ as $T \to \infty$ and $u \to 0$.

(iii) For any $x \in \mathcal{E}$: $\lim_{T \to \infty} \frac{1}{T} \int_0^T f_{t,t-\delta}(x) dt = f(x)$, a.s.

(iv) There exists a nondecreasing bounded function $\tau_0$ such that, uniformly in $u \in [0,1]$,

$$\frac{\phi(hu)}{\phi(h)} = \tau_0(u) + o(1) \text{ as } h \downarrow 0 \quad \text{ and } \quad \int_0^1 (K(v))'\tau_0(v)dv < \infty.$$

(v) $T^{-1} \int_0^T b_{t,\alpha_0}(x) dt \to D_{\alpha_0}(x)$ as $T \to \infty$ with $0 < D_{\alpha_0}(x) < \infty$.

(A3) **Local smoothness and continuity conditions**

Suppose, for any $(y,t) \in S \times [0,T]$ and $r$ such that and $t \leq r \leq t + \delta$:

(i) $\mathbb{E} \left( \psi_{y}(Y_r)|\mathcal{S}_{t-\delta} \right) = \mathbb{E} (\psi_{y}(Y_r)|X_r) = m(X_r, y)$ a.s.

(ii) $\exists \beta > 0$ and a constant $c > 0$ such that, for any $(x', x'') \in \mathcal{E}^2$,

$$|m(x', y) - m_{y}(x'' , y)| \leq cd^\beta(x', x'').$$
In this section we investigate several asymptotic properties of the continuous time generalized regression estimator. Some particular cases, related to specific choices of the function $\psi_y(\cdot)$, including the conditional cumulative distribution function and the conditional quantiles will also be discussed.

### Comments on the assumptions

Condition (A1) is related to the choice of the kernel $K$, which is very usual in nonparametric functional estimation. Condition (A2)(i)-(ii) reflects the ergodicity property assumed on the continuous time functional process. It plays an important role in studying the asymptotic properties of the estimator. The functions $f_{t,s}$ and $f$ play the same role as the conditional and unconditional densities in finite dimensional case, whereas $\phi(u)$ characterizes the impact of the radius $u$ on the small ball probability as $u$ goes to 0. Several examples to satisfy these conditions are given in Laïb and Louani (2010) for discrete time functional data process.

Some examples are also given to satisfy this condition in Didi and Louani (2014) for the case where the observations $(X_t, Y_t)$ are sampled from an ergodic continuous time process taking values in $\mathbb{R}^d \times \mathbb{R}$ space.

Condition (A2)-(iii) involves the ergodic nature of the process where the random function $f_{t,t-\delta}$ belongs to the space of continuous functions $C^0$. Approaching the integral $\int_0^T f_{t,t-\delta}(x)dt$ by its Riemann’s sum: $T^{-1} \int_0^T f_{t,t-\delta}(x)dt \sim n^{-1} \sum_{j=1}^n f_{u_{j-1}, u_j}(x)$, it is easy to prove that the sequence $(f_{j\delta,(j-1)\delta}(x))_{j \geq 1}$ is stationary and ergodic (see, Didi and Louani (2014)). (A2)-(iv) is an usual condition when dealing with functional data. Assumption (A2)-(v) is a consequence of ergodic assumption. Condition (A3) is a Markov-type condition and characterizes the conditional moments of $\psi_y(Y)$. It is satisfied when considering, for instance, the model $\psi_y(Y_t) = m_\psi(X_t) + \epsilon_t$ where $(\epsilon_t)$ is a square integrable process independent of $(X_t)$ for any $t \geq 0$.

(A3)(iv) assumes the continuity of the conditional probability of observing a missing response. The moments $M_j$ are linked to the small probability function $\tau_0$. One can refer to Ferraty et al. (2007) for a discussion on the choice of $\tau_0$, the Kernel $K$ and the positivity of $M_j$.

### Main results

In this section we investigate several asymptotic properties of the continuous time generalized regression estimator. Some particular cases, related to specific choices of the function $\psi_y(\cdot)$, including the conditional cumulative distribution function and the conditional quantiles will also be discussed.
3.1 Almost sure consistency rates

3.1.1 Pointwise consistency

The following theorem establishes an almost sure pointwise consistency rate of \( \hat{m}_{\psi,T}(x,y) \).

**Theorem 3.1 (Pointwise consistency)** Assume that (A1)-(A3) hold true and the following conditions are satisfied

\[
\lim_{T \to \infty} T \phi(h_T) = \infty \quad \text{and} \quad \lim_{T \to \infty} \frac{\log T}{T \phi(h_T)} = 0. \quad (3.1)
\]

Then, we have for \( T \) sufficiently large that

\[
\hat{m}_{\psi,T}(x,y) - m_{\psi}(x,y) = O(h_T^\beta) + O\left( \sqrt{\frac{\log T}{T \phi(h_T)}} \right). \quad (3.2)
\]

**Remark 3.2** (i) Theorem 3.1 generalizes Theorem 1 of Laïb and Louani (2011) established in the context of discrete time stationary ergodic process and Theorem 3.4 of Ferraty et al. (2005) stated under a mixing assumption with completely observed response and the support of \( y \) is reduced to one point.

(iv) The function \( \phi(h_T) \) can decrease to zero at an exponential rate, whenever \( h_T \) goes to zero, therefore \( h_T \) should be chosen decreases to zero at a logarithmic rate.

3.1.2 Uniform consistency

To establish the uniform consistency with rate of the regression operator, we need additional assumptions that allow to express the uniform convergence rate as a function of the entropy number. Let \( C \) and \( S \) be compact sets in \( E \) and \( \mathbb{R} \), respectively. Consider, for any \( \epsilon > 0 \), let the \( \epsilon \)-covering number of the compact set \( C \), say, \( N_{\epsilon} = N(\epsilon, C, d) \), which measures how full is the class \( C \), and defined as:

\[
N_{\epsilon} := \min \{ n : \text{there exist } c_1, \ldots, c_n \in C \text{ such that } \forall x \in C \text{ we can find } 1 \leq i \leq n \text{ such that } d(x, c_i) < \epsilon \}. \]

The finite set of points \( c_1, c_2, \ldots, c_{N_{\epsilon}} \) is called an \( \epsilon \)-net of \( C \) if \( C \subset \bigcup_{k=1}^{N_{\epsilon}} B(c_k, \epsilon) \), where \( B(c_k, \epsilon) \) is the ball, with respect to the topology induced by the semi-metric \( d(\cdot, \cdot) \), centred at \( c_k \) and of radius \( \epsilon \). The quantity \( \varphi_C(\epsilon) = \log(N_{\epsilon}) \) is called the Kolmogorov’s \( \epsilon \)-entropy of the set \( C \) that may be seen as a tool to measure the complexity of the subset \( C \), in the sense that high entropy means that a large amount of information is needed to describe an element of \( C \) with an accuracy \( \epsilon \). Several examples of \( \varphi_C(\epsilon) \) covering special cases of functional process are given in Ferraty et al. (2010) and Laïb and Louani (2011).

(U0) Assume that (A2) holds uniformly, in the following sense:

(i) (A2)(i) and (A2)(ii) hold true with the remaining term \( o(\phi(u)) \) is uniform in \( x \),

(ii) For any \( x \in C \), \( \lim_{T \to \infty} \sup_{x \in C} \left| \frac{1}{T} \int_0^T f_{t,T} - f(x) \right| = 0 \) a.s.

(iii) \( T^{-1} \int_0^T b_{\alpha_0}(x)dt \to D_{\alpha_0}(x) \) as \( T \to \infty \) with \( 0 < \sup_{x \in C} D_{\alpha_0}(x) < \infty \).

(iv) \( b_0 < \inf_{x \in C} f(x) \leq \sup_{x \in C} f(x) < \infty \) for some nonegative real number \( b_0 \).

(v) \( \inf_{x \in C} p(x) > b_1 \) for some nonegative real number \( b_1 \).
The kernel function \( K \) satisfies the following conditions:

(i) \( K \) is a Hölder function of order one with a constant \( a_K \),

(ii) There exist two constants \( a_2 \) and \( a_3 \) such that \( 0 < a_2 \leq K(x) \leq a_3 < \infty \), for any \( x \in C \).

(U2) For \( 1 \leq \ell \leq 2 \), the sequence of random variables \( (\psi_\ell^y(Y_t))_t \) is ergodic and \( \mathbb{E} \left( |\psi_\ell^y(Y_0)| \right) < \infty \).

(U3) There exist \( c_\psi > 0 \) and nonnegative real number \( \gamma \) such that for any \( y \in S \)

\[
\sup_{y' \in [y-u,y+u] \cap S} |\psi_y(Y_t) - \psi_{y'}(Y_t)| \leq c_\psi u^\gamma.
\]

(U4) Let \( T_n \) be the integer part of \( T \) and suppose for \( T \) large enough that

\[
\frac{(\log T)^2}{T \phi(h_T)} < \varphi_C(\epsilon_n) < \frac{T \phi(h_T)}{\log T} \quad \text{with} \quad \epsilon_n = \frac{\log T_n}{T_n}.
\]

Conditions (U0) are standard in this context in order to get uniform consistency rate. Condition (U1) is usually used when we deal with nonparametric estimation for functional data, whereas (U2) requires to the existence of up to the order two moments. Hypothesis (U3) is a regularity condition upon the function \( \psi_y(\cdot) \) which is necessary to obtain a uniform result over the compact \( S \). Hypothesis (U4) allows to cover the subset \( C \) with a finite number of balls and to express the convergence rate in terms of the Kolmogorov’s entropy of this subset. Similar condition has been used in Ferraty et al. (2010), where the authors have pointed out that for a radius not too large, one requires the quantity \( \bar{\varphi}_C(\log T_n/T_n) \) is not too small and not too large. This condition seems to satisfy this exigence, since it implies that \( \frac{\bar{\varphi}_C(\log T_n/T_n)}{T \phi(h_T)} \) goes to 0 for sufficiently large \( T \), in addition some examples given in Ferraty et al. (2010) and Laïb and Louani (2011) satisfy (U4).

Theorem 3.3 states uniform consistency rate of the kernel regression estimator. It generalizes Theorems 2 in Laïb and Louani (2011) established in the context of discrete time stationary ergodic process with completely observed response.

**Theorem 3.3 (Uniform consistency).** Assume (A1), (U0)-(U4), (A3) hold true. Moreover, suppose Conditions (3.1) are satisfied and

\[
\sum_{n \geq 1} n^\gamma \exp\{(1-\eta)\varphi_C(\frac{\log n}{n})\} < \infty \quad \text{for some } \eta > 0 \text{ where } \gamma \text{ is as in (U3).} \tag{3.3}
\]

Then, we have

\[
\sup_{y \in S} \sup_{x \in C} |\hat{m}_{\psi,T}(x,y) - m_\psi(x,y)| = O_{a.s.} \left( h_T^2 \right) + O_{a.s.} \left( \frac{\varphi_C(\epsilon_T)}{T \phi(h_T)} \right) \quad \text{as } T \to \infty. \tag{3.4}
\]

### 3.2 Asymptotic conditional bias and risk evaluation

Before evaluating the conditional bias, let us introduce some additional notations. Consider for \( i = 1, 2 \), the following assumptions:

**BC1** Let \( d_i(x) = d(X_i,x) \) and suppose, for any \( t \geq 0 \), that

\[
\mathbb{E} \left[ m_\psi(X_t,y) - m_\psi(x,y) \mid d_i(x), \mathcal{F}_{t-\delta} \right] = \mathbb{E} \left[ m_\psi(X_t,y) - m_\psi(x,y) \mid d_i(x) \right] := \Psi_y(d_i(x)),
\]
the function $\Psi_y$ is differentiable at 0 and satisfies $\Psi_y(0) = 0$ and $\Psi'_y(0) \neq 0$ for any $y \in \mathbb{R}$. This condition was introduced in Ferraty et al. (2007) and used by Laïb and Louani (2010) to evaluate the conditional bias. The introduction of $\psi_0(\cdot)$ allows to make an integration with respect to the real random variable $d_t(x)$ rather than with respect to the couple of random variables $(d_t(x), X_t)$, where $X_t$ being functional continuous random variable.

The following Proposition gives an asymptotic expression of the conditional bias term, which generalizes Proposition 1 of Laïb and Louani (2010) for discrete time estimator to our setting. Its proof is similar and therefore is omitted.

**Proposition 3.4 (Conditional Bias)** Under assumptions (A1)-(A3), (BC1) and conditions (3.1), we have

$$B_T(x, y) = \frac{h_T \Psi'_y(0)}{M_1} \left[ K(1) - \int_0^1 (sK(s))' \tau_0(s)ds + o_{a.s.}(1) \right] + O_{a.s.} \left( h_T \sqrt{\log T \phi(h_T)} \right).$$

The next result gives an explicit expression of the asymptotic quadratic risk of the estimator $\hat{m}_{\psi,T}(x, y)$.

**Theorem 3.5 (Quadratic risk)** Suppose that assumptions (A1)-(A3) and condition (2.1) hold true. Then, we have, for a fixed $(x, y) \in \mathcal{E} \times \mathbb{R}$, whenever $p(x) > 0$ and $f(x) > 0$, that

$$\text{MSE}(x, y) := \mathbb{E} \left[ (\hat{m}_{\psi,T}(x, y) - m_{\psi}(x, y))^2 \right] = A_1 h_T^2 \left[ K(1) - \int_0^1 (sK(s))' \tau_0(s)ds + o(1) \right] + A_2(x, y) + \frac{4 (W_2(x, y) + (m_{\psi}(x, y))^2)}{p(x) M_1^2 f(x)}.$$

**Remark 3.6** (i) Notice that for sufficiently large $T$, the expression of MSE becomes $A_2 h_T^2 + \frac{4 (x, y)}{T \phi(h_T)} h_T^2$. This result generalizes the one in Chaouch and Laïb (2019) established in the framework of real-valued continuous time processes where the bias term was of order $h_T^2$. Notice however that the bias term obtained in Chaouch and Laïb (2019) is of order $h_T^2$ which is smaller than $h_T$ given in Proposition 3.4. This increase in the bias term is because of the infinite dimensional characteristic of the functional space.

(ii) The mean squared error can be used as a theoretical guidance to select the “optimal” bandwidth by minimizing the quantity $A_1 h_T^2 + \frac{4 (x, y)}{T \phi(h_T)}$ with respect to $h_T$. The terms $A_1$ and $A_2(x, y)$ have unknown explicit from; and should be replaced by their empirical consistent estimators: $\Psi'_y(0)$, $(M_j, T)_{j=1,2}$, $\tau_0, T$, $p_T$, $W_2, T$, $f_T$. Notice that, $\Psi'_y(0)$ may be viewed as real regression function with response variable $m_{\psi}(X, y) - m_{\psi}(x, y)$ and covariable $d(X, x)$, it may be then estimated by a kernel regression estimate $\Psi'_y(0)$ by replacing $m_{\psi}(x, y)$ by its estimator $\hat{m}_{\psi,T}(X, x)$.

### 3.3 Asymptotic normality

**Theorem 3.7** Assume conditions (A1)-(A3) are fulfilled and

$$\lim_{T \to \infty} T \phi(h_T) = +\infty, \quad h_T^3 \sqrt{\phi(h_T)} = o(1) \quad \text{and} \quad h_T^2 \log T^{1/2} = o(1) \quad \text{as} \quad T \to \infty. \quad (3.5)$$

Then, we have, for any $(x, y) \in \mathcal{E} \times \mathcal{S}$ such that $f(x) > 0$

$$\sqrt{T \phi(h_T)} (\hat{m}_{\psi,T}(x, y) - m_{\psi}(x, y)) \overset{d}{\to} \mathcal{N}(0, \sigma^2(x, y)),$$
where
\[
\sigma^2(x, y) \leq \frac{1}{f(x)} \frac{M_2}{M_1^2 p(x)} \bar{W}_2(x, y) := \frac{1}{f(x)} \bar{V}(x, y) \quad \text{as} \quad T \to \infty, \tag{3.6}
\]
and
\[
\bar{W}_2(x, y) = \mathbb{E} \left[ (\psi_y(Y) - m_\psi(x, y))^2 \mid X = x \right].
\]

Note that, the statement (3.6) gives only an upper bound of the asymptotic variance \(\sigma^2(x, y)\). The following proposition gives an estimate of \(\tilde{V}(x, y)\) that will be needed to construct confidence bounds for the unknown operator \(m(x, y)\).

**Proposition 3.8** Suppose conditions of Theorem 3.7 hold and \(\sigma^2(x, y) > 0\), then
\[
\hat{\bar{V}}_T(x, y) := \frac{\sqrt{M_{1,T}}}{M_{2,T}} \sqrt{\frac{\bar{W}_{2,T}(x, y)}{TF_{x,T}(h_T)p_T(x)}},
\]

is an asymptotically consistent estimator for \(\bar{V}(x, y)\). The quantities \(M_{1,T}, M_{2,T}, \bar{W}_{2,T}, p_T(x)\) and \(F_{x,T}\) are empirical versions of \(M_2, M_1, \bar{W}_2, p(x)\) and \(F_x\) respectively.

The estimators \(M_{1,T}\) and \(M_{2,T}\) depend on the unknown quantity \(\tau_0\) given in (A2)(iv) that may be estimated by:
\[
\tau_{0,T}(u) = \frac{F_{x,T}(uh_T)}{F_{x,T}(h_T)} \quad \text{with} \quad F_{x,T}(u) = \frac{1}{T} \int_0^T \mathbb{1}_{\{d(x,T) \leq u\}} dt,
\]
whereas \(\bar{W}_{2,T}\) can be deduced from a nonparametric estimator for the conditional variance function \(\sigma_T^2\) (see Laïb and Louani (2010)).

### 3.4 Continuous time confidence intervals

#### 3.4.1 Asymptotic confidence intervals

Using the non-decreasing property of the cumulative standard Gaussian distribution function and the estimator \(\hat{\bar{V}}_T(x, y)\) we can then, with the help of Proposition 3.8 and Theorem 3.7, constructed confidence bands for the regression function \(m_\psi(x, y)\), which are similar to the those obtained in the discrete time case. This is the subject of the following corollary.

**Corollary 3.9** Assume conditions of Theorem (3.7) are fulfilled and conditions in (3.5) are replaced by
\[
\lim_{T \to \infty} TF_{x,T}(h_T) = +\infty \quad \text{and} \quad \lim_{T \to \infty} h_T^\beta \sqrt{TF_{x,T}(h_T)} = 0. \quad (3.7)
\]

Then, for any \(0 < \alpha < 1\), the \((1 - \alpha)100\%\) confidence bands for \(m_\psi(x, y)\) are given by
\[
\hat{m}_\psi,T(x, y) \pm c_{1-\alpha/2} \frac{M_{1,T}}{\sqrt{M_{2,T}}} \sqrt{\frac{\bar{W}_{2,T}(x, y)}{TF_{x,T}(h_T)p_T(x)}}, \quad \text{as} \quad T \to \infty, \tag{3.8}
\]
where \(c_{1-\alpha/2}\) is the quantile of standard Gaussian distribution.
3.4.2 Exchangeable bootstrap-based confidence intervals

In a variety of statistical problems, the bootstrap provides a simple method for circumventing technical difficulties due to the intractable distribution theory (this is the case in Theorem 3.7, where the limiting law depends crucially on the unknown variance). Resampling techniques become a powerful tool, e.g., for setting confidence bands, that we will illustrate in the sequel. The key idea behind the bootstrap is that if a sample is representative of the underlying population, then one can make inferences about the population characteristics by resampling from the current sample. To apply this approach for constructing confidence bands we have to define a bootstrap version that has the same asymptotic distribution as

$$\sqrt{T} \phi(h_T)(\hat{m}_{\psi, T}(x, y) - m_{\psi}(x, y)).$$

Inspection of the proof of Theorem 3.7 shows that the principal term leading to establish the asymptotic normality is given by

$$\sqrt{T} \phi(h_T) Q_T(x, y) \hat{m}_{\psi, T, 1}(x, y),$$

where $Q_T$ is defined in (6.1). Therefore, a bootstrap version for this quantity will be sufficient for estimating the true quantile $c_{1-\alpha/2}$ for a given $0 < \alpha < 1$. This may be done by combining Theorem 3.7, Theorem 2.1. of Pauly (2011) for martingale difference arrays sequences and Theorem 3 of Bouzebda et al. (2016). In this respect, define

$$S_T(x, y) := \sqrt{T} \phi(h_T) Q_T(x, y) = \sum_{i=1}^{n} \xi_{T,i}(x, y),$$

where

$$\xi_{T,i}(x, y) = \eta_{T,i}(x, y) - \mathbb{E} \left[ \eta_{T,i}(x, y) \bigg| \mathcal{F}_{i-1} \right],$$

and

$$\eta_{T,i}(x, y) = \frac{1}{\mathbb{E} \mathbb{Z}_T} \sqrt{\frac{\phi_T(h)}{n}} \int_{T_{i-1}}^{T_i} \xi \Delta_t(x) \psi_t(Y) - m_{\psi}(x, y) dt. \quad (3.9)$$

For every fixed $n \geq 1$, the sequence $(\xi_{T,i})_{1 \leq i \leq n}$ is a centred martingale difference array with respect $\mathcal{F}_{i-1}$ (see the proof of Lemma 6.7).

Define the resampling version of the quantity $S_T(x, y)$ as

$$S_T^*(x, y) = \sqrt{n} \sum_{i=1}^{n} W_{T,i} \xi_{T,i}(x, y), \quad (3.10)$$

where $W_{T,i}$ are weight functions satisfying conditions $W.1 - W.4$ in Bouzebda et al. (2016). The following proposition, which is a consequence of Theorem 3.7, Lemma 6.4 and the Slutsky’s lemma, states the asymptotic normality of the statistic $S_T^*$. Its proof may be obtained similarly to that of Theorem 3 of Bouzebda et al. (2016).

**Proposition 3.10** Assume that the above conditions related to bootstrap weights hold and the sequences of random variables $\{W_{T,i} : 1 \leq i \leq n\}$ and $\{\xi_{T,i} : 1 \leq i \leq n\}$ are independent. Then under the assumptions of Theorem 3.7 and for any $x \in \mathcal{E}$ such that $f(x) > 0$, we have

$$S_T^*(x, y) \xrightarrow{d} N(0, \sigma^2(x, y)) \quad as \quad T \to \infty,$$

where $\sigma^2(x, y)$ is as in Theorem 3.7. \( (3.11) \)

**Remark 3.11** Observe that $S_T^*$ depends on the unknown function $m_{\psi}(x, y)$. Substitute $m_{\psi}(x, y)$ by $\hat{m}_{\psi,T}(x, y)$ and $S_T^*$ by its estimate version $\hat{S}_T^*$, it follows from Lemma 6.1 and Theorem 3.1 that $\hat{S}_T^* - S_T^* = o(1)$ as $T$ goes to infinity. Therefore the statement (3.11) still holds if we replace $S_T^*$ by $\hat{S}_T^*$. 

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Proposition 3.10 turns out to be useful for obtaining confidence bands for the regression function \( m_\psi(x, y) \). Indeed, let \( 0 < \alpha < 1 \) be given, then, under conditions of Theorem 3.7, we have

\[
\lim_{T \to \infty} \mathbb{P} \left( \hat{m}_\psi, T(x, y) - c_{1-\alpha/2}(T \phi(h_T))^{-1/2} \leq m_\psi(x, y) \leq \hat{m}_\psi, T(x, y) + c_{1-\alpha/2}(T \phi(h_T))^{-1/2} \right) = \mathbb{P} \left( \mathcal{N}(0, \sigma^2(x, y)) \right) \leq c_{1-\alpha/2},
\]

(3.12)

where \( c_{1-\alpha/2} \) is the \( \alpha \)-quantile of \( \mathcal{N}(0, \sigma^2(x, y)) \). In practice the true quantile \( c_{1-\alpha/2} \) cannot be computed since the variance \( \sigma^2(x, y) \) is unknown. Nevertheless, from Proposition 3.10, we can approximate the true quantile \( c_{1-\alpha/2} \) by means of the quantiles of the bootstrapped distribution of \( \hat{S}_T^2 \) (see Subsection 3.5 for details).

### 3.5 Sampling schemes and computation of the confidence intervals

In the previous section, the process was supposed to be observable over \([0, T]\). However, in practice the data are often collected according to a sampling scheme since it is difficult to observe a path continuously at any time \( t \) over the interval \([0, T]\).

Hereafter, we briefly discuss the effect of a sampling scheme on the construction of confidence intervals for the regression function \( m_\psi(\cdot, \cdot) \). Data sampling from a continuous time process at instants \((t_k)_{k=1, \ldots, n}\) can be made regularly, irregularly or even randomly. For a sake of simplicity, we consider here the case where the instants \((t_k)\) are irregularly spaced, that is \( \inf_{1 \leq k \leq n} |t_{k+1} - t_k| = \delta > 0 \). Now, for \( \ell \in \{0, 1\} \) and \( k \in \{1, \ldots n\} \), we define the following increasing families of \( \sigma \)-algebra:

\[
\mathcal{F}_k := \mathcal{F}_{t_k} = \sigma \left( (X_{t_1}, Y_{t_1}, \zeta_{t_1}), \ldots, (X_{t_k}, Y_{t_k}, \zeta_{t_k}) \right),
\]

and

\[
\mathcal{G}_k := \mathcal{G}_k = \sigma \left( (X_{t_1}, Y_{t_1}, \zeta_{t_1}), \ldots, (X_{t_k}, Y_{t_k}, \zeta_{t_k}) ; X_{t_{k+1}}, (1-\ell)\zeta_{t_{k+1}} \right).
\]

The purpose then consists in estimating \( m_\psi(\cdot, \cdot) \) given the discrete time ergodic stationary process \( (X_{t_k}, Y_{t_k}, \zeta_{t_k})_{k=1, \ldots, n} \) sampled from the underlying continuous time process \( \{(X_t, Y_t, \zeta_t) ; 0 \leq t \leq T\} \). In case of a regular sampling scheme, that is \( T = n\delta \), the estimator \( \hat{m}_\psi, T(x, y) \) defined in (2.2) becomes

\[
\hat{m}_\psi, n(x, y) = \frac{\sum_{k=1}^n \zeta_{t_k} \psi_y(Y_{t_k}) K \left( \frac{d(x, X_{t_k})}{h_n} \right)}{\sum_{k=1}^n \zeta_{t_k} K \left( \frac{d(x, X_{t_k})}{h_n} \right)}, \quad t_k = k\delta.
\]

(3.13)

a) Asymptotic confidence intervals

Notice that Theorem 3.5 still holds for the estimate \( \hat{m}_\psi, n(x, y) \) when replacing \( T \) by \( n\delta \). The limiting law is a Gaussian random variable with mean zero and variance function \( \sigma^2(x, y) = \int_{x} \frac{M_{2}}{M_{2} \phi(x, y)} \mathbb{W}_2(x, y) \). Making use of Corollary 3.9 and considering similar steps as in Laïb and Louani (2010), it follows that, for any \( 0 < \alpha < 1 \), the asymptotic confidence intervals of \( m_\psi(x, y) \) is given as:

\[
\hat{m}_\psi, n(x, y) \pm c_{1-\alpha/2} \sqrt{\frac{M_{n,1}}{M_{n,2}}} \sqrt{\frac{\mathbb{W}_{n,2}(x, y)}{nF_{x,n}(x)p_n(x)}}, \quad \text{as } n \to \infty,
\]

where \( c_{1-\alpha/2} \) is the quantile of standard Gaussian distribution.
b) Exchangeable bootstrap-based confidence intervals

The discrete-time resampling version of the quantity $\hat{S}^*_T(x,y)$ is defined for any fixed $x \in \mathcal{E}$ as

$$\hat{S}^*_n(x,y) = \sqrt{n} \left\{ \sum_{k=1}^{n} \left( W_{n,t_k} - \overline{W}_n \right) \hat{\xi}^*_{n,t_k}(x,y) \right\},$$

where $\overline{W}_n := n^{-1} \sum_{k=1}^{n} W_{n,t_k},$

$$\hat{\xi}^*_{n,t_k}(x,y) := \sqrt{\frac{F_{x,n}(h)}{n}} \xi_{t_k} \{ Y_{t_k} - \hat{m}_{\psi,n}(x,y) \} \frac{\Delta_{t_k}(x)}{\mathbb{E}(\Delta_{t_1}(x))} - \mathbb{E} \left\{ \sqrt{\frac{F_{x,n}(h)}{n}} \xi_{t_k} \{ Y_{t_k} - \hat{m}_{\psi,n}(x,y) \} \frac{\Delta_{t_k}(x)}{\mathbb{E}(\Delta_{t_1}(x))} \mid \mathcal{F}_{k-1} \right\},$$

and $\{ W_{n,t_k} : k = 1, \ldots, n \}$ the bootstrap weights generated at the instants $(t_k)_{k=1,\ldots,n}$, such that, for $n \geq 1$, the sequences of random variables $\{ \hat{\xi}^*_{n,t_k}(x,y) : k = 1, \ldots, n \}$ and $\{ W_{n,t_k} : k = 1, \ldots, n \}$ are independent.

In practice the bootstrap procedure can be summarized as follows:

- Suppose $\left( W^{(1)}_{n,t_1}, W^{(1)}_{n,t_2}, \ldots, W^{(1)}_{n,t_n} \right), \ldots, \left( W^{(B)}_{n,t_1}, W^{(B)}_{n,t_2}, \ldots, W^{(B)}_{n,t_n} \right)$ are $B$ independent vectors, sampled independently from the sample $\{ \hat{\xi}^*_{n,t_k}(x,y) : k = 1, \ldots, n \}$.

- Consider the random variables

$$\hat{S}^*_n(\ell,x,y) = \left| \sqrt{n} \left\{ \sum_{k=1}^{n} \left( W^{(\ell)}_{n,t_k} - \overline{W}_n \right) \hat{\xi}^*_{n,t_k}(x,y) \right\} \right|, \quad \ell = 1, \ldots, B.$$

- Using Proposition 3.10 and Remark 3.11, one can use the smallest $z > 0$ as an approximation of the unknown true quantile $c_{1-\alpha/2}$, such that

$$\frac{1}{B} \sum_{\ell=1}^{B} \mathbb{I}_{\{ \hat{S}^*_n(\ell,x,y) \leq z \}} \geq 1 - \alpha.$$

4 Simulation study

This section aims to discuss numerically some aspects related to continuous time processes that might affect the quality of estimation of the operator $m_{\psi}(\cdot,y)$. In this section we consider $\psi(Y_t) = Y_t$ and therefore $m_{\psi}(x,y)$ being the conditional expectation $m(x)$ of $Y_t$ given $X_t = x$. The first simulation aims to compare the quality of estimation of $m(x)$ based on a continuous time and discrete time processes. In simulation 2 we discuss the choice of the “optimal” sampling mesh $\delta$ in the case of continuous time processes and assess its sensitivity towards the missing at random mechanism.

4.1 Simulation 1: continuous-time versus discrete-time estimators

In this first simulation we try to compare the estimation of the regression operator when discrete and continuous time processes are considered. We want whether or not considering a continuous time processes may improve the quality of the predictions. We suppose that the functional space $\mathcal{E} = L^2([-1,1])$ endowed with its natural norm. The generation of continuous time processes $\{ X_t(s) : s \in [-1,1], Y_t \}_{t \in [0,T]}$ is obtained by considering the following steps:
1. First, we simulate an Ornstein-Uhlenbeck (OU) process \((Z_t)_{t \geq 0}\) solution of the following stochastic differential equation

\[
dZ_t = 2(5 - Z_t)dt + 7dW_t, \tag{4.1}
\]

where \(W_t\) denotes a Wiener process. Here, we take \(dt = 0.005\).

2. Let \(\Gamma(\cdot)\) be the operator mapping \(\mathbb{R}\) into \(L^2([-1,1])\) defined, for any \(z \in \mathbb{R}\), as follows:

\[
\Gamma(z) := (1 + [z] - z)P_{\text{num}(z)} + (z - [z])P_{\text{num}(z+1)},
\]

where \(P_j\) is the Legendre polynomials of degree \(j\) and \(\text{num}(z) := 1 + 2z \text{sign}(z) - \text{sign}(z)(1 + \text{sign}(z))/2\) and \([\cdot]\) denotes the floor function.

3. We consider that curves are sampled at 400 equispaced values in \([-1,1]\) and defined, for any \(t \in [0,T]\), as

\[
X_t(s) = \Gamma(Z_t)(s), \quad s \in [-1,1].
\]

4. To generate the real-valued process \((Y_t)_{t \in [0,T]}\), the following nonlinear functional regression model is considered:

\[
Y_t = m(X_t) + \epsilon_t, \tag{4.2}
\]

where \(m(x) := \int_{-1}^{1} x^2(t')dt'\) and \(\epsilon_t = U_t - U_{t-1}\) where \(U_t\) is a Wiener process independent of \(X_t\).

Observe that the OU process \(\{Z_t : t \in [0,T]\}\) is a real-valued continuous time process (since \(dt\) tends to zero). The operator \(\Gamma(\cdot)\) has a role to transform each observation in the process \(Z_t\) into a curve through the Legendre polynomials. In such way the functional variable \(X\) is generated continuously as is the process \((Z_t)\). Moreover, notice that steps 1, 2 and 3 are devoted to simulate the continuous time functional process \(\{X_t(s) : s \in [-1,1]\}_{t \in [0,T]}\) whereas in step 4 the real-valued continuous time process \((Y_t)_{t \in [0,T]}\) is generated. A sample of 20 simulated curves is displayed in Figure 1 and an example of the real-valued process \((Y_t)\) is given in Figure 2.

![Figure 1: A sample of 20 simulated curves \(\{X_t(s) : s \in [-1,1]\}\).](image-url)

Now, our purpose is to compare, in terms of estimation accuracy, the continuous-time estimator with the discrete-time one for different values of \(T = 50, 200, 1000\) and several missing at random rates. It is
worth noting that the continuous time process \((X_t, Y_t)\) is observed at every instant \(t = \delta, 2\delta, \ldots, n\delta\), where \(\delta = 0.005\) and \(n = T/\delta\). However, the discret-time process is observed only at the instants \(t = 1, 2, \ldots, n\).

As in Ferraty et al. (2013) and Ling et al. (2015) we consider that the missing at random mechanism is led by the following probability distribution:

\[
P(x) = P(\zeta_t = 1 | X_t = x) = \expit \left( \int_{-1}^{1} x^2(s)ds \right),
\]

where \(\expit(u) = e^u / (1 + e^u)\), for \(u \in \mathbb{R}\). Now, we specify the tuning parameters on which depends our estimation given in (2.2). We choose the quadratic kernel defined as \(K(u) = \frac{3}{4}(1 - u^2)(u)\) and because curves are smooth enough we choose as semi-metric the \(L_2\)-norm of the second derivatives of the curves, that is for \(t_1 \neq t_2\), \(d(X_{t_1}, X_{t_2}) = \left( \int_{-1}^{1} [X_{t_1}^{(2)}(s) - X_{t_2}^{(2)}(s)]^2ds \right)^{1/2}\). We used the local cross-validation method on the \(\kappa\)-nearest neighbours introduced in Ferraty and Vieu (2006) page 116 to select the optimal bandwidth for both discrete and continuous time regression estimators. The accuracy of the discrete and the continuous time regression estimators is evaluated on \(M = 500\) replications. The accuracy is measured, at each replication \(j = 1, \ldots, M\), by using the squared errors \(SE_T^j := \left( \hat{m}_T^j(x) - m(x) \right)^2\) and \(SE_n^j := \left( \hat{m}_n^j(x) - m(x) \right)^2\) for the continuous-time and the discrete-time estimators, respectively. Observe that the discrete time estimator of the regression operator is defined as:

\[
\hat{m}_n(x) := \frac{\sum_{t=1}^{n} \zeta_t Y_t \Delta_t(x)}{\sum_{t=1}^{n} \zeta_t \Delta_t(x)}.
\]

Table 1 shows that continuous time regression estimator is more accurate than the discrete time one. Moreover, when \(T\) increases the squared errors decrease much more quickly when working with the continuous time process.

### 4.2 Simulation 2: optimal sampling mesh selection

The purpose of this simulation is to investigate another aspect related to continuous time processes. The selection of the “optimal” sampling mesh is one of the most important topics in continuous time processes.
First of all we generate a continuous-time functional data process according to the following equation:

\[ X_t(s) = Z_t \left(1 - \sin(s - \pi/3)\right), \quad s \in [0, \pi/3] \quad \text{and} \quad t \in [0, T], \]

where \( Z_t \) is an OU process solution of the stochastic differential equation (4.1) and practically observed at the instants \( t = \delta, 2\delta, \ldots, n\delta \) with \( n = 200 \) fixed. Here, we take different values of sampling mesh \( \delta \), and we study the accuracy of the estimator to identify the optimal mesh, say \( \delta^* \), that minimises the Mean Integrated Square Error (MISE). Observe that each curve observed at the instant \( t \) is discretized in 100 equidistant points over the interval \([0, \pi/3]\). The response variable is obtained following the hereafter nonlinear functional regression model:

\[ Y_t = m(X_t) + \varepsilon_t, \]

where the operator \( m(\cdot) \) is defined as \( m(X_t) = \left(\int_0^{\pi/3} X_t'(s)ds\right)^2 \) and \( \varepsilon_t \sim N(0, 0.075) \).

Moreover, the missing at random mechanism in this simulation is also supposed to be the same as described in the first simulation as per equation (4.3). For the tuning parameters used to build the estimator, we considered the quadratic kernel and given the shape of the true regression operator, which depends on the first derivative of the functional covariate, the Euclidean distance between the first order derivatives of the curves is adopted as a semi-metric. Finally the bandwidth is selected according to the local cross-validation method based on the \( \kappa \)-nearest neighbours as detailed in Ferraty and Vieu (2006) page 116.

For each value of sampling mesh \( \delta \), the regression operator \( m(\cdot) \) is estimated over a grid of 50 different fixed curves \( x \) and the whole procedure is repeated over \( N = 500 \) replications. Finally the MISE is calculated, for each value of \( \delta \), according to the following equation

\[ MISE(\delta) := \frac{1}{N} \sum_{k=1}^{N} \frac{1}{50} \sum_{j=1}^{50} \left( m(X^{(k)}_j) - \hat{m}_\delta(X^{(k)}_j) \right)^2. \]

Observe that \( \hat{m}_\delta(\cdot) \) the estimator of \( m(\cdot) \) depends on the sampling mesh \( \delta \), so is the MISE. Figure 3 displays the values of MISE obtained for different values of sampling mesh \( \delta \) and a missing at random rate of 10%, 50% and 0% (complete data), respectively. One can observe that higher is the missing at random rate, higher will be errors in estimating the regression operator.

Table 1: Summary statistics of \((\text{SE}_j)_{j=1,\ldots,500}\) for discrete and continuous time estimators of the regression function.

| MAR rate | Continuous \((\times 10^{-2})\) | Discrete \((\times 10^{-2})\) |
|----------|-------------------------------|-----------------------------|
|          | \( T \) \( Q_{25\%} \) Median Mean \( Q_{75\%} \) | \( 50 \) \( 200 \) \( 1000 \) \( 50 \) \( 200 \) \( 1000 \) |
| \( p = 20\% \) | | |
|          | 0.56 | 0.54 | 0.18 | 1.6 | 1.3 | 0.7 |
|          | 2.4 | 2.58 | 1.25 | 7.5 | 3.7 | 2.1 |
|          | 5.39 | 4.62 | 4 | 15.11 | 11.8 | 9.1 |
|          | 6.21 | 7.13 | 4.7 | 16.06 | 10.9 | 4.7 |
| \( p = 40\% \) | | |
|          | 0.49 | 0.6 | 0.3 | 1.7 | 1 | 0.5 |
|          | 4.77 | 2.7 | 2.2 | 4.9 | 6.1 | 2 |
|          | 6.93 | 8.8 | 4.6 | 12.4 | 9.9 | 8.3 |
|          | 9.39 | 10.6 | 4.5 | 16.05 | 11.3 | 11.1 |
Figure 3: The MISE(\delta) obtained for different values of sampling mesh \delta and several missing at random rates.

Table 2: The optimal sampling mesh, \delta^*, obtained for different MAR rates and some summary statistics of the MISE(\delta).

|                | \delta^* | MISE(\delta^*) | MISE(\delta) | Q_{25\%} | Q_{50\%} | Q_{75\%} |
|----------------|----------|----------------|--------------|-----------|-----------|-----------|
| Complete data  | 0.30     | 0.0476         | 0.0562       | 0.0510    | 0.0524    | 0.0550    |
| MAR=10%        | 0.36     | 0.0555         | 0.0643       | 0.0587    | 0.0612    | 0.0635    |
| MAR=50%        | 0.38     | 0.0635         | 0.0767       | 0.0714    | 0.0750    | 0.0770    |

Table 2 shows that higher is the missing at random rate longer we need to observe the underlying process to collect the \( n = 200 \) observations to be able to reasonably estimate the regression operator. Indeed, when the data is complete the optimal time interval \( T^* = n\delta^* = 200 \times 0.3 = 60 \). However, when \( MAR = 10\% \) (resp. \( 50\% \)) the optimal time interval is equal to \( T^* = 200 \times 0.36 = 72 \) (resp. \( T^* = 200 \times 0.38 = 76 \)). Consequently, it can be concluded that when the missing at random mechanism is heavily affecting the response variable, we need to collect data over a longer period of time. This allows to get sufficient information about the dynamic of the underlying continuous time process and therefore get a better estimate of the regression operator.

5 Application to conditional quantiles

Let \( x \in \mathcal{E} \) be fixed and \( y \in \mathbb{R} \), then if \( \psi_y(Y) = \mathbb{1}_{(-\infty, y]}(Y) \) the operator \( m_\psi(x, y) \) is no more but the conditional cumulative distribution function (df) of \( Y \) given \( X = x \), namely \( F(y|x) = \mathbb{P}(Y \leq y|X = x) \).
which may be estimated by \( \hat{F}_T(y|x) := \hat{m}_{\psi, T}(x, y) \). For a given \( \alpha \in (0, 1) \), the \( \alpha \)-th order conditional quantile of the distribution of \( Y \) given \( X = x \) is defined as \( q_\alpha(x) = \inf\{y \in \mathbb{R} : F(y|x) \geq \alpha\} \).

Notice that, whenever \( F(y|x) \) is strictly increasing and continuous in a neighbourhood of \( q_\alpha(x) \), the function \( F(\cdot|x) \) has a unique quantile of order \( \alpha \) at a point \( q_\alpha(x) \), that is \( F(q_\alpha(x)|x) = \alpha \). In such case

\[
q_\alpha(x) = F^{-1}(\alpha|x) = \inf\{y \in \mathbb{R} : F(y|x) \geq \alpha\},
\]

which may be estimated uniquely by \( \hat{q}_{T, \alpha}(x) = \hat{F}_T^{-1}(\alpha|x) \). Conditional quantiles have been widely studied in the literature when the predictor \( X \) is of finite dimension, see for instance, Gannoun et al. (2003) and Ferraty et al. (2005) for dependent functional data.

(a) **Almost sure pointwise and uniform convergence**

Under the same conditions of Theorem 3.1, the statement (3.2) still holds for the estimator of the cumulative conditional distribution function \( \hat{F}_T(y|x) \). That is \( \hat{F}_T(\alpha|x) \) converges, almost surely, towards \( F(y|x) \) with a rate \( O(h_T^\beta) + O(\sqrt{\log T/(T \phi(h_T))}) \).

Consequently, since \( F(q_\alpha(x)|x) = \alpha = \hat{F}_T(\hat{q}_{T, \alpha}(x)|x) \) and \( \hat{F}_T(\cdot|x) \) is continuous and strictly increasing, then we have \( \forall \epsilon > 0, \ \exists \eta(\epsilon) > 0, \ \forall y \), \( |\hat{F}_T(y|x) - \hat{F}_T(q_\alpha(x)|x)| \leq \eta(\epsilon) \Rightarrow |y - q_\alpha(x)| \leq \epsilon \) which implies that, \( \forall \epsilon > 0, \ \exists \eta(\epsilon) > 0, \)

\[
\mathbb{P} \left( |\hat{q}_{T, \alpha}(x) - q_\alpha(x)| \geq \eta(\epsilon) \right) \leq \mathbb{P} \left( \left| \hat{F}_T(\hat{q}_{T, \alpha}(x)|x) - \hat{F}_T(q_\alpha(x)|x) \right| \geq \eta(\epsilon) \right) = \mathbb{P} \left( |F(q_\alpha(x)|x) - \hat{F}_T(q_\alpha(x)|x)| \geq \eta(\epsilon) \right).
\]

Therefore, the statement (3.2) still holds for the conditional quantile estimator \( \hat{q}_{T, \alpha}(x) \) whenever conditions of Theorem 3.1 are satisfied. Ferraty et al. (2005) derived similar pointwise convergence rate by inverting the estimator of the conditional cumulative distribution function. Their result has been obtained under mixing condition and additional assumptions on the joint distribution, and the Lipschitz condition upon \( F(y|x) \) and its derivatives with respect \( y \).

Regarding the almost sure uniform convergence, observe that under conditions of Theorem 3.3, the statement (3.4) still holds true for the \( \sup_{y \in S} \sup_{x \in C} |\hat{F}_T(y|x) - F_T(y|x)| \), when \( \psi(y) \) is replaced by \( \mathbb{1}_{[\nu, \infty)}(Y) \). Moreover, assume that, for fixed \( x_0 \in C \), \( F(y|x_0) \) is differentiable at \( q_\alpha(x_0) \) with \( \frac{\partial}{\partial y} F(y|x_0)|_{y=q_\alpha(x_0)} := g(q_\alpha(x_0)|x_0) > \nu > 0 \), and \( g(\cdot|x) \) is uniformly continuous for all \( x \in C \). Knowing that \( \hat{F}_T(\hat{q}_{T, \alpha}(x)|x) = F(q_\alpha(x)|x) = \alpha \) and making use of a Taylor's expansion of the function \( F(\hat{q}_{T, \alpha}(x)|x) \) around \( q_\alpha(x) \), we can write

\[
F(\hat{q}_{T, \alpha}(x)|x) - F(q_\alpha(x)|x) = (\hat{q}_{T, \alpha}(x) - q_\alpha(x)) g(\hat{q}_{T, \alpha}(x)|x) \tag{5.2}
\]

where \( \hat{q}_{T, \alpha}(x) \) lies between \( q_\alpha(x) \) and \( \hat{q}_{T, \alpha}(x) \). It follows then from (5.2) that the inequality (5.1) still holds true uniformly in \( x \) and \( y \). Moreover, the fact that \( \hat{q}_{T, \alpha}(x) \) converges (a.s.) towards \( q_\alpha(x) \) as \( T \) goes to infinity, combined with the uniformly continuity of \( g(\cdot|x) \), allow to write that

\[
\sup_{x \in C} |\hat{q}_{T, \alpha}(x) - q_\alpha(x)| \sup_{x \in C} |g(q_\alpha(x)|x)| = O_{a.s.} \left( \sup_{y \in S} \sup_{x \in C} |\hat{F}_T(y|x) - F(y|x)| \right) \tag{5.3}.
\]

Since \( g(q_\alpha(x)|x) \) is uniformly bounded from below, we can then claim that the estimator \( \hat{q}_{T, \alpha}(x) \) converges uniformly towards \( q_\alpha(x) \) with the same convergence rate given in (3.4), as \( T \) goes to infinity.
(b) Confidence intervals

Confidence bounds for the conditional quantiles \( \hat{q}_\alpha(x) \) may be obtained according to the following steps. First, consider a Taylor’s expansion of \( \hat{F}_T(\cdot|x) \) around \( q_\alpha(x) \) and making use of the fact that \( \hat{q}_{T,\alpha}(x) \) converges (a.s.) towards \( q_\alpha(x) \) as \( T \) goes to infinity, one gets

\[
\hat{q}_{T,\alpha}(x) - q_\alpha(x) = -\frac{1}{\hat{q}_T(q_\alpha(x)|x)} \left( \hat{F}_T(q_\alpha(x)|x) - F(q_\alpha(x)|x) \right),
\]

where \( \hat{g}_T(\cdot|x) \) is a consistent estimator of \( g(\cdot|x) \). Then, replacing \( \psi(Y) \) by the indicator function, we get under conditions of Corollary 3.9, the following \((1-\alpha)100\%\) confidence bands for \( q_\alpha(x) \)

\[
\hat{q}_{T,\alpha}(x) \pm c_{1-\alpha/2} \frac{M_{T,1} \hat{g}_T(q_{T,\alpha}(x)|x)}{\sqrt{M_{T,2}}} \sqrt{\frac{\alpha(1-\alpha)}{TF_x T(h_T)p_T(x)}}, \quad \text{as } T \to \infty.
\]

6 Proofs

In this section and for sake of simplification the subscript \( \psi \) will be omitted. Consider the following quantities

\[
Q_T(x, y) := \left( \hat{m}_{T,2}(x, y) - \hat{m}_{T,2}(x, y) \right) - m(x, y)\left( \hat{m}_{T,1}(x, y) - \hat{m}_{T,1}(x, y) \right), \quad R_T(x, y) := -B_T(x, y)\left( \hat{m}_{T,1}(x, y) - \hat{m}_{T,1}(x, y) \right).
\]

We have then

\[
\hat{m}_{T}(x, y) - m(x, y) = \hat{m}_{T}(x, y) - C_T(x, y) + B_T(x, y) = B_T(x, y) + \frac{Q_T(x, y) + R_T(x, y)}{\hat{m}_{T,1}(x, y)},
\]

We start first by stating some technical lemmas that will be used later.

**Lemma 6.1** Assume that assumptions (A1)-(A2) are satisfied, then we have for any \( j \geq 1 \) and \( \ell \geq 1 \)

\[
(i) \quad \frac{1}{\phi(h_T)} \mathbb{E} \left( \Delta^j \psi(x) \mid F_{j-2} \right) = M_\ell f_\ell T_{j-2}(x) + O_{a.s.} \left( \frac{g_\ell T_{j-2}(x)}{\phi(h_T)} \right),
\]

\[
(ii) \quad \frac{1}{\phi(h_T)} \mathbb{E} \left( \Delta^1 \psi(x) \right) = M_\ell f_\ell(x) + o(1).
\]

**Proof.** The proof is similar to the proof of Lemma of Laïb and Louani (2010). □

**Lemma 6.2** Let \( (Z_n)_{n \geq 1} \) be a sequence of real martingale differences with respect to the sequence of \( \sigma \)-fields \( (\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n))_{n \geq 1} \) where \( \sigma(Z_1, \ldots, Z_n) \) is the sigma-field generated by the random variable \( Z_1, \ldots, Z_n \). Set \( S_n = \sum_{i=1}^n Z_i \). For any \( p \geq 2 \) and any \( n \geq 1 \), assume that there exist some nonnegative constants \( C \) and \( d_n \) such that \( \mathbb{E}(Z_i^p | \mathcal{F}_{n-1}) \leq C_{p} d_n^2 \) almost surely. Then, for any \( \epsilon > 0 \), we have

\[
\mathbb{P} \left( |S_n| > \epsilon \right) \leq 2 \exp \left\{ -\frac{\epsilon^2}{2(D_n + C\epsilon)} \right\},
\]

where \( D_n = \sum_{i=1}^n d_i^2 \).

**Proof.** See See 8.2.2 of de la Peña and Giné (1999). □

**Proof of Theorem 3.1.** The proof of Theorem 3.1 is a consequence of decomposition (6.1) and Lemmas 6.3-6.4-6.5 established below. □
Lemma 6.3 Assume (A1)-(A3) tougher with conditions (3.1) hold true. Then, we have for $T$ sufficiently large

$$\hat{m}_{T,j}(x,y) - \bar{m}_{T,j}(x,y) = O_{a.s.}\left(\frac{\log T}{T\phi(h_T)}\right) \quad \text{with} \quad j \in \{1, 2\}.$$ 

Proof of Lemma 6.3. Consider the case where $j = 2$ and define the process

$$L_T := \hat{m}_{T,2}(x,y) - \bar{m}_{T,2}(x,y) = \frac{1}{nE(Z_1(x))} \sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} [\zeta_t \psi_y(Y_t) \Delta_t(x) - E(\zeta_t \psi_y(Y_t) \Delta_t(x)|\mathcal{F}_{t-\delta})] dt$$

Observe that, for any $j \geq 1$ and $t \in [T_{j-1}, T_j]$, $\mathcal{F}_{j-2} \subset \mathcal{F}_{t-\delta} \subset \mathcal{F}_{j-1}$, therefore $L_{T,j}(x,y)$ is $\mathcal{F}_{j-1}$-measurable, $E(|L_{T,j}(x,y)|) < \infty$ provided $E(\zeta_t^2) < \infty$ and $E(X_t^2) < \infty$ (in view of Cauchy-schwartz inequality). Moreover, letting $\eta_j := \int_{T_{j-1}}^{T_j} [\zeta_t \psi_y(Y_t) \Delta_t(x)] dt$, then $L_{T,j}(x,y) = \eta_j - E[\eta_j|\mathcal{F}_t-\delta]$ and

$$E\{L_{T,j}(x,y)|\mathcal{F}_{j-2}\} = E\{E[\eta_j|\mathcal{F}_{t-\delta}]|\mathcal{F}_{j-2}\} - E\{E[\eta_j|\mathcal{F}_{t-\delta}]|\mathcal{F}_{j-2}\} = 0.$$

Hence $(L_{T,j}(x,y))_{j \geq 1}$ is a sequence of martingale differences with respect to the family $(\mathcal{F}_{j-1})_{j \geq 1}$. To be able to apply Lemma 6.2 we need first to check its conditions. Applying Jensen and Minkowski inequalities, we get, for any $\kappa \geq 2$, that

$$E[|L_{T,j}(x,y)|^{\kappa}|\mathcal{F}_{j-2}] \leq 2^\kappa \int_{T_{j-1}}^{T_j} E[|\zeta_t \psi_y(Y_t) \Delta_t(x)|^{\kappa}|\mathcal{F}_{j-2}] dt.$$

Using successfully a double conditioning with respect to the $\sigma$-fields $\mathcal{S}_{t,\delta}^0$ and $\mathcal{S}_{t,\delta}^1$ combined with (A3)(i), (A3)(iii), (A3)(iv), (A3)(iv’)) and the fact that $\zeta$ and $Y$ are conditionally independent given $X$, one gets for any $t \in [T_{j-1}, T_j]$ and any $p \geq 2$, that

$$E[|\zeta_t \psi_y(Y_t) \Delta_t(x)|^p|\mathcal{F}_{j-2}] = E\left\{\Delta_t^p(x) U_p(X_t) W_p(X_t, y)|\mathcal{F}_{j-2}\right\} \leq C_{W,P} E(\Delta_t^p(x)|\mathcal{F}_{j-2}),$$

where $C_{W,P} = \max(\sup_x |U_p(x)|, \sup_{x,y} |W_p(x,y)|)$ which is a finite positive constant independent on $(x,y)$ in view of conditions (A3)(iii) and (A3)(iv’).

Since the kernel $K$ and the function $\tau_0$ are bounded from above by a positive constants $a_1$ and $c_0$ respectively, and the function $f_{t,T_{j-1}}(x)$ is bounded by the deterministic function $b_{t,2\delta}(x)$; using then assumption (A2)(ii) combined with Lemma 6.1 to get

$$E[|L_{T,j}(x,y)|^\kappa|\mathcal{F}_{j-2}] \leq \kappa!(2a_1)^{\kappa-2} \phi(\delta) \left[(2a_1)^2 C_{U,W} \int_{T_{j-1}}^{T_j} b_{t,2\delta}(x) dt + o(1)\right]. \quad (6.4)$$

Letting $d_{j-2} := \phi(\delta) \left[(2a_1)^2 C_{U,W} \int_{T_{j-1}}^{T_j} b_{t,2\delta}(x) dt + o(1)\right]$ and define

$$D_n := \frac{1}{n} \sum_{j=2}^{n} d_{j-2} = \phi(\delta) \left[(2a_1)^2 C_{U,W} \sum_{j=2}^{n} \int_{T_{j-1}}^{T_j} b_{t,2\delta}(x) dt + o(1)\right].$$

Since $T = n\delta$, it follows from (A2)(v) that $D_n = T\phi(\delta) \left[(2a_1)^2 C_{U,W} D_{2\delta}(x) + o(1)\right].$
Moreover, we have from the statement (ii) of Lemma 6.1 that \( n \mathbb{E}(Z_1(x)) = O(T\phi(h_T)) \). Lemma 6.2 combined with condition (3.1) allow to write, for any \( \epsilon_0 > 0 \), that

\[
\mathbb{P} \left( \left| \hat{m}_{T,2}(x, y) - m_{T,2}(x, y) \right| > \epsilon_0 \sqrt{\frac{\log T}{T\phi(h_T)}} \right) = \mathbb{P} \left( \sum_{j=1}^{n} L_{T,j}(x, y) > n\mathbb{E}(Z_1(x))\epsilon_0 \frac{\log T}{T\phi(h_T)} \right)^{1/2} \leq 2 \exp \left\{ -\epsilon_0^2 \log T \right\} = \frac{2}{T^{\epsilon_0}}
\]

where \( c \) is a positive constant. Let \( T_n := [T] \) be the integer part of \( T \) and choosing \( \epsilon_0 \) large enough, and using condition (3.1) with Borel-Cantelli Lemma to conclude that (3.2) is valid since \( \sum_{n \geq 1} \frac{1}{T_n} < \infty \). □

**Lemma 6.4** Under assumption (A1)-(A3), we have

\[
\hat{m}_{T,1}(x) \to p(x), \text{ a.s. as } T \to +\infty.
\]

**Proof.** Let us introduce the following decomposition

\[
\hat{m}_{T,1}(x) - p(x) = (\hat{m}_{T,1}(x) - m_{T,1}(x)) + (m_{T,1}(x) - p(x)) =: \mathcal{M}_{T,1}(x) + \mathcal{M}_{T,2}(x)
\]

We have from Lemma 6.3 that

\[
\mathcal{M}_{T,1} = O_{a.s.} \left( \sqrt{\frac{\log T}{T\phi(h_T)}} \right) = o_{a.s.}(1).
\]

Let us now focus on the second term \( \mathcal{M}_{T,2} \). Using a double conditioning with respect to the \( \sigma \)-field \( \mathcal{S}^{t-\delta} \), assumptions (A2)(ii)-(iii), (A3)(iv)-(iv') and Lemma 6.1 one gets

\[
\mathcal{M}_{T,2} = \frac{1}{n\mathbb{E}(Z_1(x))} \int_{0}^{T} \mathbb{E}[\Delta_t(x)p(X_t)|\mathcal{S}^{t-\delta}] dt = \frac{1}{n\mathbb{E}(Z_1(x))} \left( p(x) + o(1) \right) T\phi(h_T) \left\{ M_1 \frac{1}{T} \int_{0}^{T} f_{t,t-\delta}(x) dt + \frac{1}{T} \int_{0}^{T} O \left( \frac{g_{t,t-\delta,x}(h_T)}{\phi(h_T)} \right) dt \right\}
\]

\[
\mathcal{M}_{T,2} = (p(x) + o(1)) \frac{1}{n\mathbb{E}(Z_1(x))} O(T\phi(h_T)) = p(x) + o(1).
\]

Thus \( \mathcal{M}_{T,2} \) converges almost surely towards \( p(x) \) as \( T \to +\infty \), which achieve the proof of this lemma. □

**Lemma 6.5** Under hypotheses (A3)(i)-(ii), for a fixed \( x \in \mathcal{E} \), we have

\[
B_{T}(x, y) = O_{a.s.}(h_T^\beta),
\]

\[
R_{T}(x, y) = O_{a.s.} \left( h_T^\beta \left( \frac{\log T}{T\phi(h_T)} \right)^{1/2} \right) \text{ and } Q_{T}(x, y) = O_{a.s.} \left( \left( \frac{\log T}{T\phi(h_T)} \right)^{1/2} \right).
\]

**Proof.** Observe that

\[
B_{T}(x, y) = \frac{m_{T,2}(x, y) - m(x, y)m_{T,1}(x)}{m_{T,1}(x)} = \frac{\tilde{B}_{T}(x, y)}{m_{T,1}(x)}.
\]
Proof. Ignoring the product term as above, one may write
\[
\left| \bar{B}_T(x, y) \right| = \frac{1}{n\mathbb{E}(Z_1(x))} \int_0^T \mathbb{E}(\zeta_t \Delta_t(x) | m(X_t, y) - m(x, y) | \mathcal{F}_{t-\delta}) \, dt
\]
\[
\leq \sup_{u \in \mathcal{B}(x, h_T)} |m(u, y) - m(x, y)| \times \frac{1}{n\mathbb{E}(Z_1(x))} \int_0^T \mathbb{E}(\zeta_t \Delta_t(x) | \mathcal{F}_{t-\delta}) \, dt
\]
\[
= O(h_T^\beta) \times |\tilde{m}_{T,1}(x) |
\]
which implies that \( B_T(x, y) = O(h_T^\beta) \). The statements of (6.10) follow from (6.9) and Lemma 6.3. \( \blacksquare \)

**Proof of Theorem 3.5.** We have from (6.3) and Lemma (6.4) that
\[
\text{MSE}(x, y) = \mathbb{E}(\hat{m}_{T}(x, y) - m(x, y))^2 \simeq \mathbb{E}\left( B_T(x, y) + \frac{Q_T(x, y) + R_T(x, y)}{p(x)} \right)^2
\]
\[
\simeq \mathbb{E}(B_T^2(x, y)) + \frac{1}{p(x)} \left[ \mathbb{E}\left(Q_T^2(x, y)\right) + \mathbb{E}(R_T^2(x, y)) \right], \quad (6.12)
\]
where the products \( 2\mathbb{E}[B_T(x, y)(Q_T(x, y) + R_T(x, y))] \) and \( 2\mathbb{E}[Q_T(x, y) \times R_T(x, y)] \) have been ignored because by the Cauchy-Schwarz inequality
\[
\mathbb{E}[B_T(x, y)(Q_T(x) + R_T(x, y))] \leq \mathbb{E}(B_T^2(x, y))^{1/2} \times \mathbb{E}((Q_T(x, y) + R_T(x, y))^{1/2}
\]
\[
\leq \max\left\{ \mathbb{E}(B_T^2(x, y)), \mathbb{E}(Q_T^2(x, y) + R_T(x, y))^2\right\}.
\]
We have the same inequality for the second product. The proof of Theorem 3.5 results from Theorem 3.4 and Lemma 6.6 below, which gives an upper bound of the expectation of \( Q_T^2(x, y) \) and \( R_T^2(x, y) \), respectively.

**Lemma 6.6** *Assume that assumptions (A1)-(A3) hold true, then*
\[
\mathbb{E}(Q_T^2(x, y)) \simeq \frac{4p(x)(W_2(x, y) + (m(x, y))^2)M_2}{T \hat{\sigma}(h_T)M_T f(x)}, \quad (6.13)
\]

**Proof.** Ignoring the product term as above, one may write
\[
\mathbb{E}[Q_T^2(x, y)] \simeq \mathbb{E}[\hat{m}_{T,2}(x, y) - \overline{m}_{T,2}(x, y)]^2 + m^2(x, y) \mathbb{E}[\hat{m}_{T,1}(x) - \overline{m}_{T,1}(x)]^2 := I_{T1} + m^2(x, y) I_{T2}.
\]
The terms \( I_{T1} \) and \( I_{T2} \) can be handled similarly. We will just evaluate the first one. Since \((T_j = j\delta)_{0 \leq j \leq n}\) is a \( \delta \)-partition of \([0, T]\), we have
\[
\hat{m}_{T,2}(x, y) - \overline{m}_{T,2}(x, y) = \frac{1}{n\mathbb{E}(Z_1(x))} \sum_{j=1}^n \int_{T_{j-1}}^{T_j} \left[ \zeta_t \psi_y(Y_t) \Delta_t(x) - \mathbb{E}\{\zeta_t \psi_y(Y_t) \Delta_t(x)|\mathcal{F}_{t-\delta}\} \right] \, dt
\]
\[
= \frac{1}{n\mathbb{E}(Z_1(x))} \sum_{j=1}^n L_{T,j}(x, y), \quad (6.14)
\]
Since \((L_{T,j}(x, y))_{j \geq 1}\) is a sequence of martingale differences with respect to the family \((\mathcal{F}_{j-1})_{j \geq 1}\), then \( \mathbb{E}(L_{T,j}(x, y)L_{T,k}(x, y)) = 0 \) for every \( j, k \in \{1, \ldots, n\} \) such that \( j \neq k \). Therefore (by ignoring the product term), we have
\[
I_{T1} = \mathbb{E}(\hat{m}_{T,2}(x, y) - \overline{m}_{T,2}(x, y))^2 \simeq \frac{1}{n^2(\mathbb{E}(Z_1(x))^2} \sum_{j=1}^n \mathbb{E}(L_{T,j}(x, y))^2. \quad (6.15)
\]
Lemma (6.4) implies, under assumption (A1)-(A3), that \( \hat{\psi}(A3)(iii)-(iv) \), \( I_{T1} \) may bounded as

\[
I_{T1} \leq \frac{4}{n^2(E(Z_1(x))^2} \sum_{j=1}^{n} \int_{T_j}^{T_i} E [\Delta^2_t(x)p(x)W_2(x_t, y)] \, dt.
\]

\[
= \frac{4(p(x) + o(1))(W_2(x_t, y) + o(1))[M_2f(x) + o(1)]}{T\phi(h_T)[M_1f(x) + o(1)]^2}.
\]

On the other side, we can easily show that

\[
I_{T2} \leq \frac{4(p(x) + o(1))[M_2f(x) + o(1)]}{T\phi(h_T)[M_1f(x) + o(1)]^2}.
\]

Therefore,

\[
E(Q^2_I(x, y)) \simeq I_{T1} + m^2(x, y)I_{T2} = \frac{4p(x)(W_2(x_t, y) + o(1))^2M_2}{T\phi(h_T)M_1f(x) + o(1)}.
\]

Moreover, using the decomposition (6.2) and Theorem 3.4 and Lemma 6.3 one can see that \( E(Q^2_I(x, y)) \) is negligible with respect to \( E(Q^2_I(x, y)) \). This completes the proof.

**Proof of Theorem 3.7.** The proof of Theorem 3.7 is based essentially on Lemma 6.7 established below, which gives the normality asymptotic of the principal term in the decomposition (6.3). Indeed, using decomposition (6.3), one may write

\[
\sqrt{T\phi(h_T)}(\hat{m}_T(x,y) - m(x, y)) = \sqrt{T\phi(h_T)}B_T(x, y) + \sqrt{T\phi(h_T)}Q_T(x, y) + \sqrt{T\phi(h_T)}R_T(x, y).
\]

(6.16)

Lemma (6.4) implies, under assumption (A1)-(A3), that \( \hat{m}_{T,1}(x) \rightarrow p(x) \) a.s. as \( T \rightarrow \infty \). Moreover, using Lemma (6.5), we get under (A3)(i)-(ii) combined with conditions (3.5) that \( \sqrt{T\phi(h_T)}B_T(x, y) = O_{a.s.}(h_T^{1/2}\log T) \). Therefore,

\[
\sqrt{T\phi(h_T)}R_T(x, y) = O_{a.s.} \left( \sqrt{T\phi(h_T)}h_T^{1/2}(\frac{\log T}{T\phi(h_T)})^{1/2} \right) = O_{a.s.}(1)
\]

The proof may be then achieved by Lemma 6.7 and Slutsky’s Theorem.

**Lemma 6.7** Under conditions (A1)-(A3), we have

\[
\sqrt{T\phi(h_T)}(\hat{m}_T(x,y) - m(x, y)) \overset{d}{\rightarrow} N(0, \hat{\sigma}^2(x,y)) \quad \text{where}
\]

\[
\hat{\sigma}^2(x, y) \leq \frac{M_2}{M_1f(x)}p(x)W_2(x, y) \quad \text{as} \quad T \rightarrow \infty.
\]

**Proof.** We have

\[
\sqrt{T\phi(h_T)}Q_T(x, y) = \sum_{i=1}^{n} \xi_{T,i}(x, y), \quad \text{with} \quad \xi_{T,i}(x, y) = \eta_{T,i}(x, y) - E[\eta_{T,i}(x, y) \mid F_{t_{i-1}}]
\]

and

\[
\eta_{T,i}(x, y) = \frac{1}{E(\phi_T(h_T))} \int_{T_{i-1}}^{T_i} \zeta \Delta(x) [\psi(y_t - m(x, y))] \, dt.
\]

(6.17)
Now observe that for any \( i \geq 1 \) and \( t \in [T_{i-1}, T_i] \), \( F_{i-2} \subset F_t \subset F_{i-1} \). Therefore, \((\xi_{T,i}(x,y))_{i \geq 1}\) is \( F_{i-1}\)-measurable, and \( \mathbb{E}(|\xi_{T,i}|) < \infty \) provided \( \mathbb{E}(\xi_i^2) < \infty \) and \( \mathbb{E}(X_i^2) < \infty \). Moreover, we have for any \( 1 \leq i \leq n \) that \( \mathbb{E} (\xi_{T,i} \mid F_{i-2}) = \mathbb{E} \{ \mathbb{E} [\eta_i \mid F_{i-2}] \mid F_{i-2} \} - \mathbb{E} \{ \mathbb{E} [\eta_i \mid F_{i-2}] \mid F_{i-2} \} = 0 \).

Hence \((\xi_{T,i}(x,y))_{i \geq 1}\) is a sequence of martingale differences with respect to the \( \sigma \)-fields \((F_{i-1})_{i \geq 1} \). To prove the asymptotic normality, it suffices to check the two following conditions (see, Corollary 3.1, p. 56, Hallqnd Heyde (1980)):

(a) \( \sum_{i=1}^{n} \mathbb{E} [\xi_{T,i}^2(x,y) \mid F_{i-2}] \overset{p}{\to} \sigma^2(x,y) \)

and (b) \( n \mathbb{E} [\xi_{T,i}^2(x,y) \mathbb{1}_{[\{\xi_{T,i}(x,y)\} > 1]}] = o(1) \) holds, for any \( \epsilon > 0 \).

**Proof of (a).** Observe now that

\[
\left| \mathbb{E} \left[ \xi_{T,i}^2(x,y) \mid F_{i-2} \right] - \mathbb{E} \left[ \xi_{T,i}^2(x,y) \mid F_{i-2} \right] \right|^2 \leq \sum_{i=1}^{n} \left( \mathbb{E} \left[ \eta_{T,i}(x,y) \mid F_{i-2} \right] \right)^2
\]

Using (A1), (A3)(i)-(iv) with Lemma 6.1, and conditioning two times with respect to the \( \sigma \)-field \( \mathcal{S}_{i-1}^1 \) and the fact that \( n \mathbb{E} Z_1(t) = O(T \phi(h)) \), we have

\[
\left| \mathbb{E} (\eta_{T,i} \mid F_{i-2}) \right| = \frac{1}{n \mathbb{E} Z_1} \sqrt{\frac{\phi_T(h)}{n}} \left| \int_{T_{i-1}}^{T_i} \mathbb{E} \left( p(X_t) \Delta_t \mathbb{1}_{[m(X_t, y) - m(x, y)]} dt \mid F_{i-2} \right) \right| \leq \sqrt{\frac{\phi_T(h)}{n \mathbb{E} Z_1}} \sup_{u \in \mathcal{B}(x,h)} |m(u, y) - m(x, y)| \sup_{u \in \mathcal{B}(x,h)} |p(u) - p(x)| \left| \int_{T_{i-1}}^{T_i} \mathbb{E} (\Delta_t dt \mid F_{i-2}) \right| \leq O \left( \sqrt{n \phi_T(h)} h^{\delta} \right) O \left( \phi(h_T) \int_{T_{i-1}}^{T_i} f_i(x, y) dt + o(1) \right) \frac{1}{n \mathbb{E} Z_1} \]

It follows by (A2)-(iii) and the Cauchy-Schwarz inequality that

\[
\sum_{i=1}^{n} \left( \mathbb{E} \left[ \eta_{T,i}(x,y) \mid F_{i-2} \right] \right)^2 = O(h^{2\delta} \phi(h)) = o(1).
\]

Thus we have only to show that: \( \sum_{i=1}^{n} \mathbb{E} [\eta_{T,i}^2(x,y) \mid F_{i-2}] \overset{p}{\to} \sigma^2(x,y) \). Using again the Cauchy-Schwarz inequality, one may write

\[
\sum_{i=1}^{n} \mathbb{E} [\eta_{T,i}^2(x,y) \mid F_{i-2}] = \frac{1}{(\mathbb{E} Z_1)^2} \phi_T(h) n \sum_{i=1}^{n} \mathbb{E} \left[ \left( \int_{T_{i-1}}^{T_i} \zeta_t \Delta_t(x) [\psi(y) - m(x, y)] dt \right)^2 \mid F_{i-2} \right] \leq \frac{\delta}{(\mathbb{E} Z_1)^2} \phi_T(h) n \sum_{i=1}^{n} \mathbb{E} \left[ \left( \int_{T_{i-1}}^{T_i} \zeta_t^2 \Delta_t^2(x) [\psi(y) - m(x, y)]^2 dt \right)^2 \mid F_{i-2} \right] + \frac{\delta}{(\mathbb{E} Z_1)^2} \phi_T(h) n \sum_{i=1}^{n} \mathbb{E} \left[ \left( \int_{T_{i-1}}^{T_i} \zeta_t \Delta_t^2(x) [m(X_t, y) - m(x, y)]^2 dt \right)^2 \mid F_{i-2} \right]
\]

\[
=: A_n + C_n
\]

(6.18)
Conditioning three times with respects to $\mathcal{F}_{t-\delta}$ and $\mathcal{S}_{t-\delta}^i$, and using Conditions (A3)(ii)-(iv)-(iv') and the fact that $T = n\delta$, to get from Lemma 6.1 that

$$A_n = \frac{\delta}{(\mathbb{E}Z)^2} \frac{\phi_T(h)}{n} \sum_{i=1}^n \mathbb{E} \left[ \int_{T_{i-1}}^{T_i} p(X_i) \Delta_i^2(x) \mathcal{W}_2(X_i,y)dt \bigg| \mathcal{F}_{i-2} \right]$$

$$\leq \frac{\delta}{(\mathbb{E}Z)^2} \frac{\phi_T(h)}{n} (p(x) + o(1)) \frac{1}{\mathbb{E}Z} \sum_{i=1}^n \mathbb{E} \left[ \frac{1}{\phi_T(h)} \Delta_i^2(x) \bigg| \mathcal{F}_{i-2} \right] dt$$

$$\leq (\delta + o(1)) p(x) \mathcal{W}_2(x,y) \phi_T^2(h) \frac{1}{\mathbb{E}Z} \sum_{i=1}^n \int_{T_{i-1}}^{T_i} f_{i,T_2-2}(x)dt + O_{a.s.} \left[ \frac{1}{T} \sum_{i=1}^n \int_{T_{i-1}}^{T_i} \frac{g_{i,T_2-2}(h_T)}{\phi_T(h)} dt \right]$$

Using the Riemann’s sum combined with condition (A2)(iii), it follows that

$$\frac{1}{T} \sum_{i=1}^n \int_{T_{i-1}}^{T_i} f_{i,T_2-2}(x)dt \leq \frac{1}{T} \int_0^T f_{i,T_2-2}(x)dt \longrightarrow f(x) \text{ a.s. as } T \longrightarrow \infty.$$ 

and by (A2)(ii), which states that $\frac{g_{i,T_2-2}(h_T)}{\phi_T(h)} = o(1)$ as $T \longrightarrow \infty$. Thus,

$$A_n \leq (\delta + o(1)) p(x) \mathcal{W}_2(x,y) \phi_T^2(h) \frac{1}{\mathbb{E}Z} \sum_{i=1}^n \int_{T_{i-1}}^{T_i} f_{i,T_2-2}(x)dt + O_{a.s.} \left[ \frac{1}{T} \sum_{i=1}^n \int_{T_{i-1}}^{T_i} \frac{g_{i,T_2-2}(h_T)}{\phi_T(h)} dt \right]$$

(6.19)

Making use of the same arguments as above combined with the fact that $\sup_{u \in B(x,h)} |m(x) - m(u)| = h^{2\beta}$, we get $C_n = o_{a.s.}(1)$.

**Proof of part (b).** Using successively Hölder, Markov, Jensen and Minkowski inequalities combined with conditions (A3)(iii), (A3)(iv') and Lemma (6.1), we get, for any $\epsilon > 0$, any $p$ and $q$ such that $1/p + 1/q = 1$, that

$$n\mathbb{E} [\xi_T^2(x,y) I_{\{T \leq \eta_{T,i}(x,y) > \epsilon\}}] \leq 4n(\epsilon/2)^{-2q/p} \mathbb{E} [\eta_{T,i}^{2q}] = O \left( T \phi(h_T)^{-\gamma/2} \right) = o_{a.s.}(1)$$

by taking $2q = 2 + \gamma$ ($0 < \gamma < 1$), since $T \phi(h_T)$ towards to infinity as $T$ goes to infinity.

**Proof of Corollary 3.9.** We have

$$\sqrt{\frac{T F_{x,T}(h_T)}{V_T^2(x,y)}} (\hat{m}_T(x,y) - m(x,y)) = \sqrt{\frac{F_{x,T}(h_T)}{\phi(h_T)f(x)}} \sqrt{\frac{\sigma^2(x,y)f(x)}{V_T^2(x,y)}} \sqrt{\frac{T \phi(h_T)}{\sigma^2(x,y)}} (\hat{m}_T(x,y) - m(x,y))$$

(6.21)

We have from the consistency of $F_{x,T}(h_T)$ and A2(i) that $\frac{F_{x,T}(h_T)}{\phi(h_T)f(x)}$ goes to 1 a.s. as $T$ goes $+\infty$. By Theorem 3.7 that quantity $\sqrt{\frac{F_{x,T}(h_T)}{\phi(h_T)f(x)}} (\hat{m}_T(x,y) - m(x,y))$ converges to $N(0,1)$ as $T \rightarrow \infty$. Using then the non-decreasing property of the cumulative standard Gaussian distribution function $\Psi$, we get, for a given risk $0 < \alpha < 1$, the $(1 - \alpha)$- pseudo-confidence bands:

$$\sqrt{\frac{T \phi(h_T)}{\sigma^2(x,y)}} |\hat{m}_T(x,y) - m(x,y)| \leq \Psi^{-1}(1 - \frac{\alpha}{2}).$$

(6.22)
Considering now the statement (3.6) combined with Proposition 3.8, it holds that
\[
\lim_{T \to \infty} \frac{\sigma^2(x,y)f(x)}{V_n^2(x,y)} \leq \lim_{T \to \infty} \frac{f(x)V^2(x,y)}{V_n^2(x,y)} = \lim_{T \to \infty} \frac{\hat{V}^2(x,y)}{V_n^2(x,y)} = 1 \quad \text{a.s.,}
\]  
(6.23)
because \( \hat{V}_n^2(x,y) \) is a consistent estimator of \( V^2(x,y) \). The proofs follows then from the statements (6.21), (6.22) and (6.23).

\[\square\]

**Proof of Theorem 3.3**

Letting \( \lambda_T = \sqrt{\frac{2\epsilon_T}{T \phi(\eta_T)}} \) with \( \epsilon_T = \frac{\log T}{n} \) and consider \( n \delta = T, n = [T] (T \geq 1) \), consequently \( 1 \leq \delta < 2 \). Observe that
\[
\sup_{y \in S} \sup_{x \in C} |\tilde{m}(x,y) - m(x,y)| \leq \sup_{y} \sup_{x \in C} |B(x,y)| + \sup_{y} \frac{\sup_{x \in C} |\tilde{m}(x,y)|}{\sup_{x \in C} |\hat{m}(x,y)|}.
\]  
(6.24)

Since \( \inf_{x \in C} |\tilde{m}_{T,1}(x)| > \inf_{x \in C} |p(x)| - \sup_{x \in C} |\tilde{m}_{T,1}(x) - p(x)| \), using the same steps of the proof of Lemma 6.4, one can show, under conditions (A1), (U0), (A3)(iv)-(i), (A3)(i)-(ii), the second term in the above inequality equals zero. Thus, making use (U0)(v), we get for sufficiently large \( T \) that \( \inf_{x \in C} |\tilde{m}_{T,1}(x)| > b_1 \) a.s.

Concerning the conditional bias, inspection of the proof of the statement (6.12) shows that the term \( \tilde{B}_T(x,y) \) is bounded above by a constant which is independent of \( x \) and \( y \), therefore, we have under (A3)(i)-(ii) that
\[
\sup_{y} \sup_{x \in C} |B_T(x,y)| \leq \sup_{y} \sup_{x \in C} |\tilde{B}_T(x,y)| \leq \frac{1}{b_1} \sup_{y} \sup_{x \in C} |\tilde{B}_T(x,y)| = O_{a.s.}(h_T^\beta).
\]  
(6.25)

Moreover, under the assumption of Lemma 6.8, making use of the decomposition (6.1), it follows from (U0)(iv) that \( \sup_{y \in S} \sup_{x \in C} |Q_T(x,y)| = O_{a.s.}(\lambda_T) \). In the other hand conditions (6.2) and (6.25) allow to conclude \( \sup_{y \in S} \sup_{x \in C} |R_T(x,y)| = O_{a.s.}(\lambda_T h_T^\beta) \). The mean task is to prove the following Lemma that allows with the statement (6.3) to achieve the proof of the Theorem 3.3

**Lemma 6.8** Assume that (A1), (U0)(i)-(iii), (A3)(i), (A3)(iii-iv), (A3)(iv'-i'), (U1)-(U4) together with conditions (3.1) and (3.3) are satisfied. Then we have
\[
\sup_{y \in S} \sup_{x \in C} |\tilde{m}_{T,2}(x,y) - m_{T,2}(x,y)| = O_{a.s.}(\lambda_T).
\]

**Proof of Lemma 6.8**

Let \( \epsilon > 0 \) be given and consider a covering of the class of functions \( C \) by closed balls
\[
B(c_k,\epsilon) = \{ x \in C : d(x,c_k) < \epsilon \}, 1 \leq k \leq \mathcal{N}(\epsilon,C,d) := N_\epsilon, \text{ that is } C \subset \bigcup_{k=1}^{N_\epsilon} B(c_k,\epsilon).
\]
Then we have
\[
\sup_{y \in S} \sup_{x \in C} |\tilde{m}_{T,2}(x,y) - m_{T,2}(x,y)| \leq \sup_{y \in S} \max_{1 \leq k \leq N_\epsilon} \sup_{x \in B(c_k,\epsilon)} |\tilde{m}_{T,2}(x,y) - \tilde{m}_{T,2}(c_k,y)| \\
+ \sup_{y \in S} \max_{1 \leq k \leq N_\epsilon} |\tilde{m}_{T,2}(c_k,y) - m_{T,2}(c_k,y)| \\
+ \sup_{y \in S} \max_{1 \leq k \leq N_\epsilon} \sup_{x \in B(c_k,\epsilon)} |m_{T,2}(x,y) - m_{T,2}(c_k,y)| \\
=: H_{1,T} + H_{2,T} + H_{3,T}.
\]  
(6.26)
Let us now focus on the first term $\mathcal{H}_{T,1}$. We have for any $x \in B(c_k, \epsilon)$ and $y \in S$

$$\hat{m}_{T,2}(x, y) - \hat{m}_{T,2}(c_k, y) = \frac{1}{nE(Z_1(x))} \int_0^T \zeta \psi_y(Y_t) [\Delta_t(x) - \Delta_t(c_k)] dt$$

$$+ \frac{1}{nE(Z_1(x))E(Z_1(c_k))} \int_0^T \zeta \psi_y(Y_t) \Delta_t(c_k) [E(Z_1(c_k)) - E(Z_1(x))] dt$$

$$:= \mathcal{I}_{T,1}(c_k, y) + \mathcal{I}_{T,2}(c_k, y).$$

(6.27)

Making use of the property of ergodicity, conditions (U1) and (U2) and the boundedness of $\zeta$, we get for $T$ sufficiently large and any $(x, y) \in B(c_k, \epsilon) \times S$ that

$$|\mathcal{I}_{T,1}(c_k, y)| \leq a_3 \zeta \frac{\epsilon}{h_T} \frac{1}{T} \int_0^T |\psi_y(Y_t)| dt$$

$$\leq a_3 \zeta \frac{\epsilon}{h_T} \mathcal{O}_{a.s. \sup_{y \in S}} |\psi_y(Y_0)| = \mathcal{O}_{a.s.} \left( \frac{\epsilon}{h_T} \right).$$

(6.28)

On the other hand, we have under the above conditions

$$|\mathcal{I}_{T,2}(c_k, y)| \leq \frac{c_\zeta a_3^2}{a_2} \frac{1}{h_T} \int_0^T |\psi_y(Y_t)| dt = \frac{c_\zeta a_3^2}{a_2} \frac{1}{h_T} \sup_{y \in S} |\psi_y(Y_0)| = \mathcal{O}_{a.s.} \left( \frac{\epsilon}{h_T} \right).$$

(6.29)

The constants in the right hand side of (6.28) and (6.29) are independent of $x$ and $y$, thus

$$\mathcal{H}_{T,1} = \mathcal{O}_{a.s.} \left( \frac{\epsilon_T}{h_T} \right)$$

Similarly we get, under the above conditions, the same bound of the term $\mathcal{H}_{T,3}$, using condition (U4) we conclude, for $T$ sufficiently large enough, that

$$\mathcal{H}_{T,1} = \mathcal{H}_{T,3} = \mathcal{O}_{a.s.} \left( \lambda_T \right)$$

with $\epsilon_T = \frac{\log T}{T}$

(6.31)

Consider now, the intermediate term $\mathcal{H}_{T,2}$. For this purpose, cover $S$ with $\nu_T = [T^\gamma] + 1$ intervals $I_k = [\ell_n - y_k, \ell_n + y_k]$ of centre $y_k \in S$ and length $2\ell_T \leq c/\nu_T$, for some $\gamma > 0$, such that $S \subset \bigcup_{k=1}^{\nu_T} I_k$. Then we have

$$\mathcal{H}_{T,2} = \sup_{y \in S} \max_{1 \leq k \leq N} |\hat{m}_{T,2}(c_k, y) - \hat{m}_{T,2}(c_k, y)|$$

$$\leq \max_{1 \leq k \leq \nu_T} \max_{1 \leq k \leq N} \sup_{y_k \in I_k} |\hat{m}_{T,2}(c_k, y) - \hat{m}_{T,2}(c_k, y)|$$

$$+ \max_{1 \leq k \leq \nu_T} \max_{1 \leq k \leq N} |\hat{m}_{T,2}(c_k, y_k) - \hat{m}_{T,2}(c_k, y_k)|$$

$$+ \max_{1 \leq k \leq \nu_T} \max_{1 \leq k \leq N} \sup_{y_k \in I_k} |\hat{m}_{T,2}(c_k, y_k) - \hat{m}_{T,2}(c_k, y)|$$

$$:= \mathcal{R}_{1,T} + \mathcal{R}_{2,T} + \mathcal{R}_{3,T}$$

(6.32)

Using (U1)(ii), (U2) and (U3) combined with the ergodic property and because $\zeta$ is bounded, one may write

$$|\hat{m}_{T,2}(c_k, y) - \hat{m}_{T,2}(c_k, y_k)| = \left|TE(\Delta_0(c_k))\right|^{-1} \int_0^T |\zeta| |\Delta_t(c_k)| \psi_y(Y_t) - \psi_{y_k}(Y_t)| dt$$

$$\leq c_{\zeta} a_3 \frac{1}{a_2} \frac{1}{\nu_T} \mathcal{E}(|\zeta_0|) = \mathcal{O}_{a.s.}(T^{-\gamma}).$$

(6.33)
The same bound may be obtained for the quantity $m_{T,2}(c_k, y_{k'}) - m_{T,2}(c_k, y)$. Since the constants in the above terms are independent of $c_k$ and $y_{k'}$, we conclude that

$$R_{1,T} = R_{3,T} = O_{a.s.}(\lambda_T) \quad (6.34)$$

because $\lim_{T \to +\infty} \frac{1}{\lambda_T \nu^2} = 0$ in view of condition (U4).

We turn now to the intermediate term $R_{2,T}$. Using the fact that $\varphi_C(\epsilon_n) = \log N_\epsilon$, $\nu_T = [T^\gamma] + 1$, the statement (6.5) which still true under conditions (A1), (U0)(i)-(iii), (A3)(i), (A3)(iii-iv) and (A3)(iv'), we get for any $\epsilon_0 > 0$, whenever condition (3.1) and the assumption (U4) are satisfied, that

$$P(R_{2,T} > \lambda_T) = P \left( \max_{1 \leq k' \leq \nu_T} \max_{1 \leq k \leq N_\epsilon} |\hat{m}_{T,2}(c_k, y_{k'}) - m_{T,2}(c_k, y_{k'})| > \lambda_T \right)$$

$$\leq \sum_{k'=1}^{\nu_T} \sum_{k=1}^{N_\epsilon} P(|\hat{m}_{T,2}(c_k) - m_{T,2}(c_k)| > \lambda_T)$$

$$\leq 2\nu_T N_\epsilon \exp \{ -c\epsilon_0^2 \varphi_C(\epsilon_n) \} = 2\nu_T N_\epsilon^{1-\epsilon_0^2}$$

$$\leq 2n^\gamma N_\epsilon^{1-\epsilon_0^2} \quad (6.35)$$

because $T = n\delta$ and $1 \leq \delta < 2$. Choosing $\epsilon_0^2 = \eta$ and considering condition (3.3) which is equivalent to

$$\sum_{n \geq 1} n^\gamma N_\epsilon^{1-\eta} < \infty \quad \text{for some } \eta > 0$$

to conclude by Borel-Cantelli Lemma the end of the proof of Lemma 6.8. \[\square\]

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