MOND as the weak field limit of an extended metric theory of gravity with a matter-curvature coupling.

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In this article we construct an extended relativistic $f(R)$ theory of gravity with matter-curvature couplings $F(R, \mathcal{L}_{\text{mat}})$ for which its weak field limit of approximation recovers the simplest version of MOND. We do this by (a) performing an order of magnitude approach and (b) by perturbing the resulting field equations of the theory to the weakest field limit of approximation. We also compute the geodesic equation of the resulting theory and show that it has an extra force, a fact that commonly appears in general matter-curvature couplings.

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I. INTRODUCTION

The non-baryonic dark matter problem constitutes one of the most important unsolved problems in current research [cf. 1, 2]. Despite the huge research and its generally accepted success, the dark matter particle has never been detected. The gravitational anomaly that gives rise to the dark matter and/or energy hypothesis can also be understood as a modification of gravity at certain scales [cf. 3] as it was first discussed by the pioneer research of Milgrom [4, 5], with a MOdified Newtonian Dynamics (MOND) approach. A first coherent attempt to find a relativistic version was carried out by Bekenstein [6] with a TEnsor Scalar Vector (TEVES) theory, this idea has continued into finding a relativistic theory of gravity described by the action:

$$S = \frac{c^3}{16\pi G l_M^2} \int f(\chi) \sqrt{-g} \, d^4x + \frac{1}{c} \int \mathcal{L}_{\text{mat}} \sqrt{-g} \, d^4x,$$

where $\chi := L^2 R$, $R$ is the Ricci scalar, $L \propto r_g^{1/2} l^{1/2}$ with $r_g := GM/c^2$ the gravitational radius, $l := (GM/a_0)^{1/2}$ the “mass-length” scale of the system and $\mathcal{L}_{\text{mat}}$ is the standard matter Lagrangian, related to the energy-momentum tensor $T_{\alpha\beta}$ by:

$$T_{\alpha\beta} \sqrt{-g} \, \delta g^{\alpha\beta} = -2\delta \left( \sqrt{-g} \, \mathcal{L}_{\text{mat}} \right).$$

The constant $a_0 \approx 1.2 \times 10^{-10} \text{m/s}^2$ is Milgrom’s acceleration constant. This proposal is coherent with the results of gravitational lensing in individual, groups and clusters of galaxies [13] and at the same second perturbation order is coherent with a Parametrised Post-Newtonian (PPN) description where the parameter $\gamma = 1$ [14]. Another extension of gravity was performed by Barrientos and Mendoza [15], who analysed the action (1) but now using the Palatini approach, obtaining the same functional action $f(\chi) = \chi^{3/2}$ in order to recover the MONDian acceleration, with a mass dependence on the coupling length $L$.

The problem with action (1) is that it can only be applied in regions sufficiently far from the sources that produce the gravitational field, in order to approximate the system as a point mass source. There is however a cosmological attempt by Carranza, Mendoza, and Torres [16] in which the mass $M$ was thought of as the causal mass for a particular observer in the cosmic flow, yielding a good description of an accelerated expansion of the universe without the introduction of dark matter and/or energy.

Another recent exploration was carried out by Barrientos and Mendoza [17] who showed that the mass dependence in the coupling length $L$ can be avoided introducing derivatives of the matter Lagrangian in the action $f(\chi)$. In such proposal the coupling constant depends exclusively on the fundamental constants $c$, $a_0$, and $G$, but the price to pay is in the complexity of the field equations and the theoretical inconveniences that the introduction of the derivatives of the matter Lagrangian produce.

In this article we use an extension of a metric $f(R)$ theory of gravity with matter-curvature couplings $F(R, \mathcal{L}_{\text{mat}})$ following the approach by [18, 22] and show that with this generalised action a relativistic theory of MOND can be constructed. The article is presented in the following manner. In Section II an order of magnitude calculation is performed to show that a specific

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$F(R, \mathcal{L}_{\text{matt}})$ can reproduce MOND in its simplest form. Section III shows an exact solution for a point-mass source reproducing these results. In Section IV we use correct dimensional arguments to generalise an action for a $F(R, \mathcal{L}_{\text{matt}})$ and show that with this it is possible to recover either MOND or Newton’s gravity at the weakest field limit of the theory. Finally in Section V we discuss the results of the article and present our conclusions.

II. $F(R, \mathcal{L}_{\text{matt}})$ APPROACH

The lesson to learn from action $\mathcal{I}$ is that the matter Lagrangian $\mathcal{L}_{\text{matt}}$ needs to be inserted inside the gravitational action (see e.g. Mendoza [3]). The idea of a non-minimal coupling between the matter and the curvature has been already raised [23, 26]. To do so, we can use an extension of $f(R)$ of gravity introducing a $F(R, \mathcal{L}_{\text{matt}})$ described by Harko and Lobo [19]:

$$ S = \int F(R, \mathcal{L}_{\text{matt}}) \sqrt{-g} \, \mathrm{d}^4x, \quad (3) $$

with the following field equations:

$$ F_R R_{\alpha\beta} + (g_{\alpha\beta} \nabla^\mu \nabla_\mu - \nabla_\alpha \nabla_\beta) F_R - \frac{1}{2} (F - F_{\mathcal{L}_{\text{matt}}}) g_{\alpha\beta} = \frac{1}{2} F_{\mathcal{L}_{\text{matt}}} T_{\alpha\beta}, \quad (4) $$

where $F_R := \partial F/\partial R$ and $F_{\mathcal{L}_{\text{matt}}} := \partial F/\partial \mathcal{L}_{\text{matt}}$. Note that (a) $F(R, \mathcal{L}_{\text{matt}}) = c^4 R/16\pi G + \mathcal{L}_{\text{matt}}/c$ yields standard general relativity, (b) $F(R, \mathcal{L}_{\text{matt}}) = f(R)/2 + \mathcal{L}_{\text{matt}}/c$ is standard metric $f(R)$ gravity and (c):

$$ F(R, \mathcal{L}_{\text{matt}}) = \frac{c^3}{16\pi G} \frac{f(\chi)}{L^2} + \frac{1}{c} \mathcal{L}_{\text{matt}}, \quad (5) $$

is a correct generalisation of $\mathcal{I}$ in which the unknown length function $L = L(\mathcal{L}_{\text{matt}})$ is to be found; and together with the unknown function $f(\chi)$ must yield a correct MOND behaviour in the limit of low acceleration scales $a \lesssim a_0$.

III. MONDIAN LIMIT

Let us now show that with the assumptions made in section I it is possible to obtain the basic MOND relation based on the Tully-Fisher law. To do so, let us substitute equation (5) into the field equations (4) and take the trace of the resulting relation to yield:

$$ f_R(\chi) R - 2 f(\chi) + 3 L^2 \nabla^a \nabla_a \left( \frac{f_R(\chi)}{L^2} \right) = \frac{8\pi G L^2}{c^4} T_\alpha. \quad (6) $$

In order to find the correct MONDian limit equation, we follow the procedure by Bernal et al. [12] and so, let

$$ f(\chi) = \chi^b, \quad \text{and} \quad \mathcal{L}_{\text{matt}} = \rho c^2, \quad (7) $$

where we have assumed a point mass source generating the gravitational field, and so $\mathcal{L}_{\text{matt}}$ has a dust-like form. To order of magnitude, i.e. when $R \sim r_{\text{curv}}^{-2}$ - where $r_{\text{curv}}$ is the radius of curvature of space- and $\nabla \sim 1/r$, it follows that the first two terms on the left-hand side of equation (6) are smaller than the third when $r/r_{\text{curv}} \to 0$, i.e. when the equivalent acceleration $a$ is expected to be $\lesssim a_0$. Thus, the trace of the field equations that can be adapted to a MONDian regime of low acceleration scales is given by:

$$ 3 L^2 \nabla^\alpha \nabla_\alpha \left( \frac{f_R(\chi)}{L^2} \right) = \frac{8\pi G L^2}{c^4} T_\alpha. \quad (8) $$

A weak-field limit coherent with bending of light in individual, groups and clusters of galaxies is obtained if the second perturbation order metric is given by [14]:

$$ \mathrm{d}s^2 = \left( 1 + \frac{2\phi}{c^2} \right) c^2 \mathrm{d}t^2 - \left( 1 - \frac{2\phi}{c^2} \right) \mathrm{d}x^2, \quad (9) $$

for a gravitational scalar potential $\phi$ and an isotropic space-time with a PPN parameter $\gamma \approx 1$ according to observations of such MONDian systems [13]. With this, the Ricci scalar takes the form: $R \approx -(2/c^2) \nabla^2 \phi$, which at order of magnitude yields: $R \sim a/rc$, for an acceleration $a = |\nabla \phi|$. Thus, to order of magnitude, equation (5) yields:

$$ a \sim G^{1/(b-1)} \rho^{1/(b-1)} r^{(b+1)/(b-1)} c^{2(b-4)/(b-1)} L^{-2}, \quad (10) $$

and so, in order to obtain MOND standard equation: $a = \sqrt{G a_0 M/r} \sim \sqrt{G a_0 \rho r}$, then $b = -3$ together with $L \propto (G\rho)^{-3/8} c^{5/4} a_0^{1/4}$, which yields:

$$ F(R, \mathcal{L}_{\text{matt}}) \propto R^{-3} \mathcal{L}_{\text{matt}}^3. \quad (11) $$

IV. A DIMENSIONALLY CORRECT GENERAL ACTION

Let us now consider an action motivated by equation $\mathcal{I}$ with the following form:

$$ S = \frac{c^3}{16\pi G a_0^3} \sqrt{-g} \int f(\chi, \xi) \, \mathrm{d}^4x + \frac{1}{c} \int \sqrt{-g} \mathcal{L}_{\text{matt}} \, \mathrm{d}^4x, \quad (12) $$

where $\chi$ and $\xi$ are dimensionless quantities given by:
\[ \xi := \frac{\xi_{\text{matt}}}{\lambda}, \quad \text{and} \quad \chi := \alpha R, \quad (13) \]

with \( \alpha \) and \( \lambda \) unknown “coupling” constants with dimensions of square length and energy density respectively.

The null variations with respect to the metric yields the following field equations:

\[ \alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(f - \xi f_\chi) = \left( \frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda} \right) T_{\mu\nu} - \alpha (g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) f_\chi, \quad (14) \]

with the standard definition of the energy-momentum tensor:

\[ T_{\mu\nu} = g_{\mu\nu} \xi_{\text{matt}} - \frac{2}{c^2} \frac{\partial \xi_{\text{matt}}}{\partial g_{\mu\nu}}, \quad (15) \]

in full agreement with equation (2).

The trace of equation (14) is given by:

\[ \chi f_\chi - 2(f - \xi f_\chi) + 3\alpha \Delta f_\chi = \left( \frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda} \right) T. \quad (16) \]

Since \( c, G \) and \( a_0 \) are independent fundamental constants, Buckingham’s II theorem of dimensional analysis implies that:

\[ \alpha = \kappa \frac{c^4}{a_0}, \quad \text{and} \quad \lambda = \kappa' \frac{a_0^2}{G}, \quad (17) \]

with \( \kappa \) and \( \kappa' \) pure dimensionless proportionality constants.

Following the previous approach, we can assume that:

\[ f(\chi, \xi) = \chi^\gamma \xi^\beta. \quad (18) \]

For the case of dust, the perturbation orders in the terms of the field equation are the following:

\[ \alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(f - \xi f_\chi) + \alpha (g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) f_\chi = \left( \frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda} \right) T_{\mu\nu}, \quad (19) \]

\[ \gamma = -3. \quad (24) \]

With this value, the Poisson-like equation (22) is:

\[ 3 \Delta f_\chi = \frac{3}{8\pi} \frac{(a_0 G)^2}{c^2} \nabla^2 \left( \left\{ \nabla^2 \phi \right\}^{-4} \rho \right) = \rho. \quad (25) \]

An analytic solution to the previous equation for the case of a point-mass source is given in the appendix A.

Note that equation (24) represents a non-linear generalisation of the standard Poisson equation \( \nabla^2 \phi \propto \rho \).

A family of these non-linear generalisations was discussed by \[27\], with Poisson-like equations of the form \( \nabla \cdot \left( \mu \left( \left\{ \nabla \phi \right\} \nabla \phi \right) \right) \propto \rho \) satisfying conformal invariance in all cases studied. Equation (25) does not fall into that category and as such, it differs from the standard AQUAL proposal [28]. This is due to the fact that the nonlinearity of equation (25) does not only apply to the scalar potential \( \phi \) but also to the mass density \( \rho \), since this last one appears inside the Laplacian operator on the left-hand side of relation (25).

B. Poisson’s equation for Newtonian gravity.

Another possible choice for equation (19) is \( \gamma + \beta = 1 \) which yields:

\[ \alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(f - \xi f_\chi) = \left( \frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda} \right) T_{\mu\nu}. \quad (26) \]

This lowest perturbation order choice means that:
\[
(g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) f_\chi = 0. \tag{27}
\]

Taking the trace of equation (26) for dust, a relation between the Ricci scalar and the matter density is obtained:

\[
R = \left( - \frac{16\pi}{\gamma + 1} (\kappa \kappa')^{1-\gamma} \right)^{1/\gamma} \frac{G}{c^2} \rho. \tag{28}
\]

At the lowest perturbation order, when \( R = -(2/c^2) \nabla^2 \phi \), this previous equation can be constructed -with the appropriate coupling constants- to yield Newtonian gravity (Poisson’s equation) for any value of \( \gamma \neq -1 \).

V. DISCUSSION

In this article we have shown that it is possible to show, exactly and by an order of magnitude approach, that a \( F(R, \mathcal{L}_{\text{matt}}) \) theory of gravity described by:

\[
f(\chi, \xi) = \chi^{-3} \xi^3, \quad \chi := \alpha R, \quad \xi := \mathcal{L}_{\text{matt}}/\lambda, \tag{29}
\]

is a good candidate for a full relativistic extension of MOND, in regions where the acceleration of test particles \( \lesssim a_0 \). In the weak-field limit of approximation it converges to standard MOND for a point mass source \( M \), with \( \rho = M \delta(r) \) and \( \mathcal{L}_{\text{matt}} = \rho c^2 \). It is our intention to explore this interpretation with applications to lensing and dynamics of individual, groups and clusters of galaxies as well as with cosmology. The advantage of this approach is that it is a full metric formalism and does not involve interpretations of gravity using Palatini formalism or torsion as we have previously explored \cite{24, 30}. Furthermore, it is a correct generalisation to the first attempts made by Bernal et al. \cite{12}.

At first sight, the action given by the Lagrangian density: \( R^{-3} \mathcal{L}_{\text{matt}} \) from which we have proved the MONDian behaviour is obtained, seems to diverge in the Minkowskian regime, namely when \( R \to 0 \). In order to show that this is not so, we proceed in the following way. Using relations \( \{17\}, \{18\}, \{29\} \), and the fact that \( \gamma = -\beta \), expression \( \{21\} \) turns into:

\[
- \frac{9}{8\pi k^4 k'^3} \left( \frac{a_0 G}{c^6} \right)^2 \Delta (R^{-4} \mathcal{L}_{\text{matt}}^3) = T, \tag{30}
\]

which in the weak-field limit for a point-mass source is:

\[
- \frac{9}{8\pi k^4 k'^3} \left( \frac{a_0 G}{c^6} \right)^2 \nabla^2 (R^{-4} \mathcal{L}_{\text{matt}}^3) = M \delta(r). \tag{31}
\]

Using the well known result:

\[
\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(r), \tag{32}
\]

the following relation is satisfied:

\[
R^{-4} \mathcal{L}_{\text{matt}}^3 = \frac{2\pi k^4 k'^3}{9} \left( \frac{c^5}{a_0 G} \right)^2 \frac{M}{r}. \tag{33}
\]

Therefore, in the weak field limit, this proposal has the following relation: \( \mathcal{L}_{\text{matt}}^3 \propto R^4/r \). This implies that the Lagrangian density for the action that we are interested in converges to \( R^{-3} \mathcal{L}_{\text{matt}}^3 \propto R/r \to 0 \) as \( r \) increases.

Finally, we discuss the geodesic equation of the theory. Following a similar procedure as the one shown in \cite{18, 31}, the geodesic equation is given by:

\[
\frac{dx^\mu}{ds} + \Gamma^\mu_{\nu\alpha} \frac{dx^\nu}{ds} \frac{dx^\alpha}{ds} = f^\mu, \tag{34}
\]

where

\[
f^\mu = (g_{\mu\nu} - u_\mu u_\nu) \nabla^\nu \ln \left[ (16\pi k \kappa' + f_\xi) \frac{d\mathcal{L}_{\text{matt}}^3}{d\rho} \right]. \tag{35}
\]

As expected, the usual relation \( u_\mu f^\mu = 0 \) is obtained. This means that the extra force is perpendicular to the four-velocity. For dust, the extra-force takes the following form:

\[
f_\nu = (g_{\mu\nu} - u_\mu u_\nu) \nabla^\mu \ln \left[ 16\pi \kappa \kappa' + f_\xi \right]. \tag{36}
\]

This type of extra force has been studied and interpreted in the literature \cite{32} and in a very different context to the one discussed in this article to yield MOND-like accelerations by \cite{33}. Investigations into its nature and its astrophysical consequences requires further research.

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Appendix A: Poisson-like equation

Let us begin by rewriting equation \( \{26\} \) as:

\[
K \nabla^2 \left( \nabla^2 \phi \right)^{-4} \rho^3 = \rho, \tag{A1}
\]

where for simplicity we have defined:
\[ K := \frac{3}{8\pi} \left( \frac{a_0 G}{2\kappa} \right)^2. \]

The matter density for a point-mass source is given by:

\[ \rho = \frac{M}{4\pi r^2} \delta(r), \]

and since the Laplacian for a spherically symmetric problem is:

\[ \nabla^2 \psi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right), \]

then, equation (A1) turns into:

\[ 4\pi K \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right)^4 \frac{1}{\rho^3} = M \delta(r). \]

Integration of the previous equation yields:

\[ 4\pi K \frac{d}{dr} \left( \frac{\nabla^2 \phi}{\rho^3} \right)^4 = \frac{M}{r^2}, \]

which after another integration gives:

\[ 4\pi K \left( \nabla^2 \phi \right)^4 \rho^3 = -\frac{M}{r}. \]

Using again eqs. (A3) and (A4) and after some algebraic steps, we obtain:

\[ (-K)^{1/4} \left( \frac{M}{4\pi} \right)^{1/2} \left( \frac{r^3}{\delta(r)} \right)^{1/4} \delta(r) = \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right). \]

which after another integration is written as:

\[ \left( \frac{r^3}{\delta(r)} \right)^{1/4} \left. \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) \right|_0 = r^2 \frac{d\phi}{dr}. \]

Using the fact that the acceleration \( a = |\mathbf{a}| = |\nabla \phi| \) and the Dirac’s delta function is given by:

\[ \delta(r = 0) = \lim_{r \to 0} \frac{1}{2\pi r}, \]

then the relation for the accelerations is given by:

\[ \left( -K \right)^{1/4} \left( \frac{M}{2\pi} \right)^{1/2} \frac{1}{r} = a. \]

Substitution of the value of \( K \) given in equation (A2), yields to:

\[ \frac{3}{4\kappa^3 \pi^2} \left( \frac{a_0 GM}{\kappa} \right)^{1/2} \frac{1}{r} = a. \]

Thus, the choice \( \kappa^3 = -3/4\pi^2 \kappa^4 \) yields a MONDian acceleration \( a = \sqrt{GMa_0/r} \).

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