A NEW BANDIT SETTING BALANCING INFORMATION FROM STATE EVOLUTION AND CORRUPTED CONTEXT

ABSTRACT

We propose a new sequential decision-making setting, motivated by mobile health applications, based on combining key aspects of two established online problems with bandit feedback. Both assume that the optimal action to play is contingent on an underlying changing state which is, however, not directly observable by the agent. They differ in what kind of side information can be used to estimate this state. The first one considers each state to be associated with a context, possibly corrupted, allowing the agent to learn the context-to-state mapping. Instead, the second one considers that the state itself evolves in a Markovian fashion, according to learnable principles, thus allowing the agent to estimate the current state based on history. We argue that it is realistic for the agent to have access to both these sources of information, i.e., an arbitrarily corrupted context obeying the Markov property. Thus, the agent is faced with a new challenge of balancing its belief about the reliability of information from learned state transitions versus context information. We present an algorithm that uses a referee to dynamically combine the policies of a contextual bandit and a multi-armed bandit. We capture the time-correlation of states through iteratively learning the action-reward transition model, allowing for efficient exploration of actions. Our setting is motivated by adaptive mobile Health interventions. Users transition through different unobserved, time-correlated but only partially observable internal states, which determine their current needs. The side-information about users might not always be reliable and standard approaches solely relying on the context risk incurring high regret. Similarly, some users might exhibit weaker correlations between subsequent states, leading to approaches that solely rely on state transitions risking the same. We evaluate our method on simulated medication adherence intervention data and on several real-world data sets, showing improved empirical performance compared to several popular algorithms.

Keywords Multi-Armed-Bandit, Contextual Bandit, Sequential Decision Making, Markov property, Nonstationarity

1 Introduction

In the area of sequential decision making, the multi-armed bandit problem has attracted significant attention due to its applicability in many real-world areas such as clinical trials Villar et al. [2015], Bastani and Bayati [2020], finance Shen et al. [2015], Huo and Fu [2017], routing networks Boldrini et al. [2018], Kerkouche et al. [2018], online
The earliest works have formalized this problem in the so-called Multi-Armed-Bandit (MAB) setting. Unlike in the full feedback setting, where the optimal action is revealed to the agent, the so-called bandit feedback means that only the reward of the action chosen by the agent is revealed, and the rewards of other actions remain unknown. This partial feedback mechanism requires a trade-off between exploitation and exploitation of actions under the assumption that rewards are drawn from some distribution. Thus, a single sample from each action is not enough to estimate their reward. The game between agent and the environment involves, in general, the following choice: the agent either exploits its current knowledge about the reward distributions by choosing the action that appears to yield the highest estimated reward; or it explores other actions to update own beliefs about their true reward, possibly incurring regret for not choosing the “best” action. The earliest works have formalized this problem in the so-called Multi-Armed-Bandit (MAB) setting. Significant effort has been done to design algorithms that provide optimal exploitation/exploration trade-off in this setting, depending on the nature of the reward distribution.

In the stochastic variant of the MAB problem Lai and Robbins [1985], the rewards the agent receives by playing an action are a sequence of i.i.d. random variables. Popular methods for the design of action selection strategies include using upper confidence bounds (UCB) on the mean rewards of actions, based on the principle of “optimism in the face of uncertainty” Lai and Robbins [1985], Auer et al. [2002]. Another approach relies on the Bayesian interpretation for optimal exploration and exploitation using Thompson sampling Thompson [1933], Chapelle and Li [2011]. The stochastic MAB problem assumes that the average rewards of actions do not change over time, that is, reward distributions are stationary. While a convenient and straightforward model of online decision-making, this formulation is quite limiting in many interesting practical settings, where the mean reward of actions may change over time. For example, a user’s taste in a particular movie genre might shift, or treatment might reduce effectiveness with repeated exposure. To overcome the limitation of the stationarity assumption in action rewards, a variety of non-stationary settings have been explored, such as switching bandits Garivier and Moulines [2011]. In this setting, the agent is faced with an environment where action rewards change over time by some random process. In the so-called adversarial bandits, the sequence of rewards are chosen by an adversary and may follow any arbitrary distribution Auer et al. [2003].

This paper is motivated by Mobile Health, particularly providing tailored digital interventions to promote behavior change. To model this setting, we assume that an underlying evolving state determines users’ needs and wants. Thus we expect non-stationarity in the rewards of interventions, so that different interventions will be beneficial depending on the state. For example, some patients are affected by stress and require a set of stress coping techniques. In contrast, others may experience little stress, but may require information and knowledge instead Abouserie [1994]. The underlying state, as it evolves, may induce a natural ordering of interventions as the patient transitions through different “stages”. For example, according to the transtheoretical model, patients often go through stages to enact a change in their habits. The ”Preparation” stage is characterized by the willingness to change behaviors, where small steps are taken towards the goal. In the “Maintenance” stage patients have changed their habits and are committed to prevent relapse in earlier stages Prochaska et al. [2015]. This assumption is partially modeled by several nonstationary bandit settings explored in the past, notably regime switching bandits Zhou et al. [2021]. In this setting, the agent is confronted with an environment, where a common hidden state determines the action-reward distributions. This hidden states evolves according to a Markov chain. The agent is thus required to learn an action selection strategy that performs well not only for a particular state, but also when states and action-reward distributions change.

The regime switching bandit setting assumes that the hidden state is not accessible to the agent at all, outside of action rewards. In practice, however, additional data can often be collected to estimate it. The state can often be appraised using questionnaires or passively collected data such as geo-location and physical activity metrics. Indeed, one particular bandit setting that exploits additional information has received significant attention. This extension to the MAB-problem is the Contextual Bandit (CB) problem, also known as the multi-armed-bandit problem with side information or associative reinforcement learning. Additional information at each time step effectively turns the nonstationary problem into a stationary one. This additional information is often referred to as context in the bandit literature, which the agent uses to select actions. The long-term goal of the agent, then, is to learn the relationship between context and the action rewards, thus maximizing cumulative reward. The CB-agent has a distinct advantage over MAB-agents in the CB setting: while the CB-agent can use the context to ”detect” changes in reward-distribution (or state), the MAB-agent has to discover changes through the rewards it receives. This may result in the need of the MAB-agent to discard prior knowledge about the reward distributions and relearn which comes with additional regret. For the contextual bandit setting with a linear context to reward mapping, authors in Chu et al. [2011] prove a regret lower bound of order $\sqrt{Td}$, with $d$ being the dimensionality of the context and $T$ being the number of time...
steps. A variety of algorithms have been developed that achieve regret of the lower bound. Some notable works for the contextual bandit problem include Linear Upper-Confidence-Bound (LinUCB) Chu et al. [2011], Li et al. [2010] and Linear Contextual Thompson Sampling (LinTS) Agrawal and Goyal [2013].

Naturally, the acquired context does not, in practice, give a full picture, and may be susceptible to noise corruption. We expect users not to necessarily know their state perfectly, or to pretend for various reasons, and thus provide us with context that is a noisy estimate of their underlying state. To equip our problem setting with this property, we consider a bandit problem introduced by Bouneffouf [2020], referred to as the contextual bandit with corrupted context, where the agent may not be able to use the provided contexts due to significant corruption. Thus, our problem setting exhibits a hidden state which evolves according to a Markov chain and determines context (potentially corrupted) and action-reward distribution.

Both the context and transition history can provide the agent with information about the underlying state. Thus, in order for an agent to perform well in our setting, it needs to balance its trust in the two sources of information: first, trust the context that can tell the agent directly in which state the environment is; or second, use the information learned about past state-correlations to determine the state.

The algorithmic approach we propose for the above problem setting has some similarities to several previous works. Combing algorithms that use different sources of information has been studied more extensively in recent years under the umbrella of online model selection. Noteworthy work in this area include the EXP3 and EXP4 algorithms Auer et al. [2003] that aim to combine multiple expert algorithms for decision making where each expert may use different sources of information to make decisions. Other major step in optimally combining policies is the seminal paper on corralling a band of bandits, where authors provide an algorithm, called CORRAL, providing a better balance between exploration and exploitation when combining learners instead of experts Agarwal et al. [2016]. We concentrate on a well proven strategy of “bandit over bandits approach”, where master algorithm (typically a bandit itself) chooses between the decisions of base algorithms.

We note that our setting also exhibits similarities to the broader literature on sequence-aware recommender systems where actions exhibit some underlying ordering structure that can be exploited to predict the following item (action) or set of items Natarajan et al. [2013], Jannach et al. [2015], Hariri et al. [2012], Bonnin and Jannach [2014], Mobasher et al. [2002], Nakagawa and Mobasher [2003]. For a comprehensive overview of sequence-aware recommender system literature, we refer to Quadrana et al. [2018].

Our main contributions are as follows: (1) we formulate a setting combining crucial aspects of two bandit problems. The correlation of actions rewards in time are determined by a hidden state that satisfies the Markov property and a potentially corrupted context is available that provides information about the underlying state; (2) we propose an algorithm for this setting and a learning mechanism for action to action correlation; (3) we evaluate our algorithm and several of it’s variants empirically on both simulated and real-world data. The performance is compared against a set of popular algorithms.

We organize the paper as follows: We discuss the background in Section 2, in particular, our problem setting, bandits and definitions used throughout the paper. We describe our simulation environment in Section 3. In section 4 we present the meta-algorithm COMBINE and its upper-confidence-bound (UCB) instantiation COMBINE-UCB. Additional calculation and theory of the algorithm are presented in Section 5. We outline, in Section 6, the setup of the empirical evaluation of our method on the simulation environment and real-world data, and in Section 7 we present and discuss the results. In section 8 we conclude the paper.

2 Background

In this section, we first informally introduce our problem setting. We are primarily concerned with an environment where observations of varying degrees of “usefulness” are available to the agent for decision making. A context (side information in the form of features) for example, allows the agent to learn the correct context to action reward mapping, improving its decision-making capabilities over time. Our particular setting is motivated by mobile health applications, where users provide feedback, which can be unreliable or conflicting. Users might also exhibit varying degrees of engagement and conscientiousness in reporting their state, thus timely side information for decision making may not be available to the agent at all times.

Besides information from contexts, users in a Mobile Health setting may also have specific goals, such as losing weight, requiring the progression through states to achieve the desired goal. Each of these states may require a specific interventions, thus knowing about the current state is essential. In the absence of any useful context to estimate the current state of the user, a reasonable approach would utilize transitions between states to guide action exploration more
effectively. We assume that the underlying evolving state of the user induces correlation or a sequence of actions to be played by the agent.

The goal of the agent in our setting is to weight the usefulness of context against the usefulness of state correlations. The agent needs to learn their respective value through interacting with the environment. We note that “usefulness” is defined relative to the agent’s ability, that is, the information source is only as useful as the agent’s ability to utilize it for decision making. For example, a complex nonlinear context to action-reward mapping might provide no benefit to a linear CB; using state correlations instead of the context might prove more useful to the agent.

Thus, the exploration and exploitation trade-off not only involves deciding between using the context or correlations (or some combination). The agent also needs to take into account that the information sources’ usefulness may change as the agent learns to use them better. The time invested to better use one information source will lead to less experience, and thus less performance, with the other. It is not clear from the beginning which one of the information sources is better than the other; which one will ultimately have the lowest cumulative regret over some timeframe $T$. This requires a more subtle trade-off between exploration and exploitation: action selection policies that are optimal from the perspective of one information source are not necessarily optimal when using other sources, or a combination of them.

### 2.1 Contextual Bandit Problem and LinUCB

Our problem setting and algorithm combines two information sources, from contexts and state transitions. As mentioned before, the context provides the agent with a (potentially noisy) observation of the state. The contextual part exhibits similarities to the contextual bandit problem and the popular LinUCB algorithm, respectively. Therefore we find it instructive to start with introducing both. We follow the definition of the contextual bandit problem from Langford and Zhang [2008]. At each time step $t \in \{0, 1, \ldots, T\}$:

1. The environment reveals a d-dimensional feature vector $x_t \in \mathbb{R}^d$.
2. The agent chooses an action $a_t$ from a set $\mathcal{A}$ of $K$ alternatives according to its policy $\pi : \mathcal{X} \rightarrow \mathcal{A}$. After playing the action $a_t$, the action’s reward $r_{t,a_t}$ is revealed.
3. The agent updates its policy using the observations of context $x_t$, action $a_t$ and action-reward $r_{t,a_t}$ to improve action selection in future rounds.

This formulation of the CB problem does not make any assumptions about the specific relationship between contexts and action rewards. Indeed Langford and Zhang [2008] talk about regret compared to the best hypothesis in some hypothesis class. In our formulation we concentrate on linear hypotheses, that is, we make the linear realizability assumption. The expected reward of an action depends linearly on the context. more formally:

$$\mathbb{E}[r_a|x] = x^T \mu_a^*.$$  \hspace{1cm} (1)

where $\mu_a^*$ is some coefficient vector associated with action $a$.

The $CB$ problem formulation presented here does not explicitly include noisy or corrupted contexts. For our setting we adopt the notion of corrupted context from Bouneffouf, 2020. The agent receives a corrupted context that does not contain any information to learn the correct context to action-reward mapping. This setting is an extension of the $CB$ problem defined earlier. With probability $p$, the agent receives a corrupted context. How the context is corrupted is governed by the corruption function $\nu : \mathcal{X} \rightarrow \mathcal{X}$. This function is arbitrary, unknown to the agent and it is not possible to recover or otherwise compensate for it. The context the agent receives at every time step is defined as

$$\tilde{x}_t = \begin{cases} \nu(x_t) & \text{with probability } p \\ x_t & \text{with probability } 1 - p. \end{cases}$$

Contexts and action-rewards in this formulation are sampled i.i.d. from a joint distribution $D(x,r)$. This setting assumes that the context contains all information to estimate the context to action-reward mapping. In practice, some information important for the correct mapping is often not available due to, for example, privacy concerns. This may lead to a nonstationary context to action-reward mapping, that is, the learned mapping becomes inaccurate as time progresses. In our setting, context and rewards are sampled i.i.d. from a joint distribution $D(x,r,s)$, including the dependence on a hidden state $s$, following a more realistic approach for practical applications.

One popular algorithm is LinUCB Chu et al. [2011], Li et al. [2010], based on the idea of using UCB methods not only for the $MAB$ problem but also for the $CB$ problem. The key challenge for UCB-style methods in general is the construction of upper confidence bounds on the estimated action rewards and doing so in an efficient manner. This
is important since the agent chooses the action with the highest value UCB to manage the exploration/exploitation trade-off. For action-rewards that depend linearly on the context, authors in Li et al., 2010 provide an efficient and closed-form computation rule for constructing confidence intervals.

We focus on LinUCB with disjoint linear models described by Li et al., 2010. As in our setting, the linear realizability assumption holds. Coefficients $\mu_a$ in equation 1 are unknown to the agent and need to be estimated from collected data. The estimated coefficient vector $\hat{\mu}_a$ can be found via ridge regression:

$$\hat{\mu}_a = (D^\top a D_a + kI_d)^{-1} D^\top a b_a, \quad (2)$$

where $D_a$ is a design matrix with dimension $m \times d$ that contains $m$ contexts for action $a$ previously observed, $b_a$ is the reward vector for action $a$ and $I_d$ is the $d \times d$ identity matrix. $k > 0$ is a parameter, we set to 1.

At each time step $t$ LinUCB selects actions according to

$$a_t = \arg\max_{a \in A} \left( x_t^\top \hat{\mu}_a + \alpha \sqrt{x_t^\top A_a^{-1} x_t} \right), \quad (3)$$

where $A_a = D^\top a D_a + I_d$, $\log(2/\delta)$ being the natural logarithm and $\alpha = 1 + \sqrt{\log(2/\delta)/2}$, with $\delta > 0$, is a constant determining the level of exploration.

As a side note, the authors of LINUCB mention that the confidence interval (second term in equation 3), may also be derived from Bayesian principles using $\hat{\mu}_a$ and $A_a^{-1}$ as mean and covariance to the Gaussian posterior distribution of coefficient vector $\hat{\mu}$. This observation has been utilized in the LinTS algorithm Agrawal and Goyal [2013].

### 2.2 Regime Switching Bandits and a discounted UCB algorithm

The second ingredient to our problem setting includes nonstationarity of the action-rewards and context distribution modulated by an evolving hidden state. This mechanism on the action-rewards is similar to regime switching bandits Zhou et al. [2021]. We adopt part of their definition of the problem setting and start out by introducing the setting more formally. In the regime switching bandit problem, there exists a finite state Markov chain. This Markov chain evolves according to a transition kernel and each state is associated with a reward distribution depending on the state and the action. At each time step $t \in \{0, 1, \ldots, T\}$:

1. The agent plays action $a_t$ from a set $A$ of $K$ alternatives according to its policy $\pi : X \to A$.
2. The action’s reward $r_{s_t,a_t}$ according to the reward distribution $R(\cdot|s,a) := P(R_t \in \cdot|S_t = s,a_t = a)$ is revealed.
3. the agent updates its policy using reward $r_{s_t,a_t}$
4. the environment advances the Markov chain $\mathcal{M}$ by sampling the next state according to the transition kernel $\phi(s)$.

The agent has no a priori knowledge of either $\phi$, the state $s$ or reward distribution $R$ and needs learn about these quantities from collected data.

Previous settings with this special property are rested bandits Gittins [1979], where the reward of each action is coupled to a separate finite state Markov chain. Only the state of the played action evolves, the states of other actions remain “frozen”. A more general setting is the restless bandits Whittle [1988], Ortner et al. [2014], where states of other actions may also evolve at each timestep. What separates regime switching bandits from these previously investigated settings, is a common Markov chain that determines the reward distribution of all actions.

Agents in the regime switching bandits setting do not have access to side information that is commonly available in practical applications. Side information allows partial observation of the hidden state. This differentiates our setting from theirs, where we allow a noisy observation coupled to the hidden state to be used for decision making. More precisely, the distribution of this noisy observation is determined my the hidden state, forming a joint distribution $D(x,r,s)$. Thus allowing the agent to differentiate states by the context.

The authors in the regime switching bandits paper provided a relatively complicated algorithm based on spectral estimation. To aid interpretation and analysis of the experimental results, we opt for simpler algorithms that are augmented with the ability to utilize gained knowledge about state-transitions. As mentioned earlier in the introduction section, there are several state-of-the-art algorithms that deal with non-stationary action-rewards distributions. One of
the earliest and most straightforward approaches involves the use of algorithms developed for the stationary MAB problem and equipping them with the ability to discount prior experience about action-rewards. We introduce a discounted version of the UCB algorithm as an illustrative example, since we also use an adapted version in our algorithm. The idea behind this modified algorithm is not much different to the classical UCB algorithm Sutton and Barto [2018], that is, the agent chooses the action with the highest upper confidence bound (UCB) on the mean rewards collected so far:

$$a_t = \arg\max_{a \in A} UCB(a).$$  \hfill (4)

The UCB for each action is computed as

$$UCB(a) = \bar{R}(a) + \alpha B \sqrt{ \frac{\log(t)}{N_t(a)}},$$ \hfill (5)

where \( \log(t) \) is the natural logarithm, \( \bar{R}(a) \) is the mean reward of action \( a \) and \( N_t(a) \) is the number of times action \( a \) has been played so far. \( \alpha_b \in (0, \infty) \) is a scalar that modules the level of exploration. The discounted variant computes \( \bar{R}(a) \) as a discounted sum: \( \bar{R}_{t+1}(a) = \bar{R}_t(a) + \gamma (r_{at} - \bar{R}_t(a)) \), playing more weight on recent action-rewards. \( \gamma \in (0, 1] \) is again a scalar determining the weight of the most recent reward on the mean \( \bar{R}(a) \). This simple and modification allows the agent to adapt more quickly in the nonstationary setting when the distribution of action-rewards change.

While a straightforward and convenient method, prior knowledge about the action-rewards is inevitably discarded (or discounted). Referring back to the regime switching setting, if the environment revisits states, the agent needs to relearn the action-rewards, thus needlessly suffering regret. We introduce a modification later in section 4 that helps in mitigating this shortcoming. We equip the agent with the ability to focus on a reduced subset of actions that show correlations in rewards between states. This is achieved by learning an action-to-action adjacency matrix. Details of the exact procedure to learn the matrix are presented in section 5.2.

2.3 Formal Problem Setting

In this section we describe the problem setting, introduced previously in an intuitive way, using formal notation. We consider an online learning problem with bandit feedback, where contexts can be corrupted and the context and action reward distribution is determined by an underlying hidden state evolving according to a Markov chain. We focus on Bernoulli rewards, since in practice, rewards are often collected in the form of "like/dislike" or "click/no click" on an ad (however, extension to arbitrary rewards is quite natural). Two sources of information are available to the agent: corrupted contexts and state correlations that result in time-correlation among actions. Learning these correlations can help in reducing the set of actions to consider, depending on the belief about the current state. The agent needs to decide which source of information to trust more for decision making. We model this problem via a repeated game between the agent and the environment. At every decision point \( t = 1, 2, \ldots, T \):

1. The environment generates context \( x_t \in \mathcal{X} \) from state \( s_t \in S \);
2. The context is corrupted \( \hat{x}_t = \nu(x_t), \nu: \mathcal{X} \to \mathcal{X} \) with probability \( p_t \);
3. The possibly corrupted context \( \hat{x}_t \in \mathcal{X} \) is revealed to the agent;
4. The agent chooses an action \( a_t = \pi(\hat{x}_t) \);
5. The environment reveals the reward \( r_{at, s_t} \in \{0, 1\} \);
6. The state \( s \) is updated: \( s_{t+1} = \phi(s_t) \).

where \( \phi: S \to S \) denotes a function determining the next underlying state. We describe this function in more detail in Section 3.1 for our simulation environment; we also demonstrate, through experiments on real-world datasets in Section 6.2, that it is a realistic assumption. In general, for finite state spaces, this function takes the form of a transition kernel. Thus, the state \( s_t \), is not drawn independently from an underlying stationary distribution but evolves according to a finite-state Markov chain. We do not impose a restriction on how the context \( x_t \) is generated from the state as long the linear realizability assumption holds, that is, the expected reward for each action depends linearly on the context. Each state is associated with a stationary context and action-reward distribution \( D_s(x, r) \). Contexts and rewards are sampled i.i.d. from \( D_s \).

The difference in cumulative reward between the policy that always chooses the optimal action on every time step, and the learned policy \( \pi \) of the agent, is commonly referred to as the dynamic regret. We compute the regret following the definition:
Dynamic regret  The dynamic regret after $T$ time steps is given as:

$$R(T) = \sum_{t=1}^{T} r_{t, a_t^*} - \sum_{t=1}^{T} r_{t, a_t},$$

where $a_t^*$ and $a_t := \pi(\hat{x}_t)$ denote the optimal action and the action chosen by the agents policy at time step $t$, respectively. When defining regret in this way, we note that our agent competes against the oracle that chooses the best action at every time step. Given that the hidden state evolution is Markov, an algorithm can perform much better by playing the best action in each state, compared to playing the action with the highest mean reward over all states and timesteps Ortner et al. [2014]. While achieving sublinear dynamic regret is impossible for settings where information about the hidden state is only indirectly available, this is not always true for our setting. The probability of getting an uncorrupted context is $1 - p$, that is the best-case scenario for contextual bandit algorithms. In our setting, for $p = 0$, the agent has access to the uncorrupted context, and thus, can use it to detect the state as well as changes in the state, before choosing an action. As a result, sublinear dynamic regret is achievable, even when the number of state changes are of order $T$.

3 Simulation Environment

In the following sections, we describe the simulation environment based on the medication adherence intervention scenario. This tool is used to generate artificial data under controlled settings, allowing us to gain in-depth understanding of the performance of the proposed algorithm, and compare it against state of the art algorithms. The code is freely available on github\(^1\) and follows the OpenAI gym api.

3.1 Context Corruption and Hidden State Update

According to our problem setting, the context the agent receives at each iteration has a chance of being arbitrarily corrupted such that it becomes useless for decision making. For our experiments, we corrupt the context $x$ by simply drawing a random state from a uniform distribution on the interval $[0, 1]$.

After the action is played by the agent, the hidden state is updated. The function $\phi$ maps the current state to a new state for time step $t + 1$. In our simulation $\phi$ is truncated random walk model limited to the domain $[0, 1]$.

The truncated random walk model simulates the natural evolution of the state. It can be interpreted as the natural change of the “health” state of the patient that may require specific interventions at particular “levels” for the hidden state $s_t$. This choice of state representation is based on computational convenience to directly generate contexts from the continuous state value. More inline with the problem setting definition, it is perfectly possible to alternatively define the following: a transitions kernel containing the transition probabilities between states, context distribution and action-reward distribution for each state.

At environment initialization, $T$ independent realizations are drawn from a Bernoulli distributed random variable $Z_t \sim B(p)$ constituting set $\mathbf{Z}$, where $p$ denotes the probability of success, i.e., $P(Z = 1) = p$. For our simulations, we consider $p = 0.5$. Function $\phi(s_t)$ transforms the state $s_t$ according to:

$$\phi(s_t) = \begin{cases} s_t + c & \text{if } Z_t = 1 \\ s_t - c & \text{otherwise} \end{cases},$$

for $\{Z_t \in \mathbf{Z} : t \in T\}$ and where $c \in [0, 1]$ is a constant. $c$ determines the dynamics of the state in each time step and may be different for different users. Once chosen for each user, it stays fixed for the whole simulation.

3.2 Context and Action Reward Correlation

We implemented the intuition of different “levels” or “stages” users might go through by discretizing the continuous space of state $s_t$ into intervals, with each interval having one best action. More formally, given a set of actions $A$ of cardinality $K$, we define a partition of the compact interval $Q = [0, 1]$ on the real line such that there exists a sequence of sub interval satisfying:

$$0 = q_0 < q_1 < \cdots < q_{K-1} = 1,$$

\(^1\)https://github.com/caisr-hh/CombineUserEnvironment
where, for every subinterval \([q_k, q_{k+1})\), there exists a corresponding best action \(a\). In other words, there exists a mapping between action \(a \in A\) and a sub-interval of partition \(q\). Each subinterval has a unique best actions, which can be the same or different between states. In our simulator, the number of actions equals the number of states, but this is not a required assumption for the problem setting. We do not put any constraints on what action corresponds to what interval, but for simplicity and interpretability, we assume that subinterval satisfies \([q_{k-1}, q_k] : a_k\), e.g., action \(a_1\) corresponds to subinterval \([q_0, q_1)\), and so forth. Note that this mapping is unknown to the agent and needs to be learned by interacting with the environment.

4 Meta-Algorithm: Competing Bandits with Corrupted Context and Action Correlations

Algorithm 1 Competing Bandits with Corrupted Context and Action Correlations

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1: procedure COMBINE
2: Input: Algorithm Parameters, Policies: \(CB, MAB\) and referee, action set, user set
3: Initialize: Book-keeping variables for \(CB, MAB\) and referee
4: for \(t = 1, 2, \ldots, T\) do
5:   for user \(i \in I\) do
6:     Observe context \(\hat{x}_{i,t}\)
7:     Sample Policy \(\pi(t)\) from referee
8:     if \(\pi_{i,t} = CB\) policy then
9:       choose action using \(CB\) policy
10:   else
11:     choose action using \(MAB\) policy from subset \(U_i\)
12:   Observe reward for chosen action
13:   if previous indicator action is not equal to the current chosen action then
14:     Update Adjacency matrix \(\Lambda_i\)
15:   if \(CB\) was chosen as a policy then
16:     Update \(CB\) policy
17:   else
18:     Update \(MAB\) policy
19:   Update referee
20:   if \(\text{reward} = 1\) then
21:     Update current indicator action
22:   else
23:     Update previous indicator action
24: choose action subset \(U_i\) to sample from next
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We present the high-level overview of the meta-algorithm COMPeting BandIts with corrupted coNtext and action corrElations (COMBINE) in algorithm 1, and its UCB variant, in algorithm 2. The key idea is that at each time step, after the agent observes the context \(\hat{x}_t\) (possibly corrupted), it decides to use either the \(CB\) or \(MAB\) policy (line 7). The \(MAB\) policy in particular chooses the actions to play from a reduced subset of actions that are determined by adjacency matrix \(\Lambda_i\). Therefore we call the \(MAB\) policy \(MAB_u\). The decision to play the \(CB\) or \(MAB_u\) is made based on the expected reward, estimated from past performance by another bandit, the so-called referee.

If the referee chooses the \(MAB_u\) policy, the \(MAB_u\) is presented with the actions from an action-subset \(U\) to choose from. \(U\) is dynamically computed and represents a candidate set of next promising actions. This candidate set is computed using the adjacency matrix, which counts the number of transitions observed between actions that provide a positive reward. If the \(CB\) is chosen, actions are selected according to the context from the full action set. The selected action action is played, and the agent observes the reward.

If the \(CB\) policy was chosen, then it is updated with the received reward (line 15). Otherwise, we update the \(MAB_u\) policy (line 18). Our reason here is that if the referee does not trust the context it receives, the \(CB\) may not gain anything by learning from the data point. We note that the referee indirectly determines the context’s usefulness through the performance of the \(CB\). We expect that discarding seemingly corrupted contexts will lead to more efficient learning of the context to action-reward mapping. Similarly, if the \(MAB_u\) has difficulty selecting the appropriate action, it may only harm its estimates of true action rewards when the state changes frequently. The referee is then updated using the reward and the choice of policy.
As mentioned before, the algorithm keeps track of state transitions through an action to action adjacency matrix that contains the number of observed transitions between the algorithms chosen actions. The matrix is updated after the reward is observed, provided that potential transition in state has occurred (line 13). To determine a change, we rely on the fact that most of the time, the algorithm will chose the action that provides the highest reward in the state. Thus if action selection changes towards an action that provides more reward over some period of time, the adjacency matrix updates its count of transitions between the chosen action to another.

The MAB\textsubscript{u} uses the adjacency matrix to select the next promising action. As part of the adjacency matrix update, we store and update two indicator actions, that is, actions for which the agent received a reward of 1 (Bernoulli distributed rewards). These two indicator actions are used to record what action leads into another. Finally, the action subset \( \mathcal{U} \) is updated, which is used in the next round (line 24).

### 4.1 Competing Bandits with Corrupted Context and Action Correlations UCB (COMBINE-UCB)

**Algorithm 2** Competing Bandits with Corrupted Context and Action Correlations UCB

```plaintext
1: procedure COMBINE-UCB
2: Input: \( \alpha \in (0, \infty) \), \( \alpha_B \in (0, 1) \), \( \gamma \in (0, \infty) \), \( \delta_R \in (0, \infty) \), action set \( A \), user set \( I \)
3: Initialize: preference values \( H = \{0\}^{1|I| \times 2} \), average action reward \( \bar{R} = \{0\}^{1|I| \times |A|} \), action play count \( n = \{0\}^{1|I| \times |A|} \), current indicator action \( a^+ = \{Na \}^{1|I| \times 1} \), previous indicator action \( a^- = \{Na \}^{1|I| \times 1} \), reach parameter \( \beta = \{1\}^{1|I| \times 1} \), action subset \( \mathcal{U}^{1|I| \times |A|} = A \), adjacency matrix per user \( \Lambda^i = \{1\}^{1|A| \times |A|} \), design matrix of LinUCB \( \forall a \in A : A_a = I_a \), reward vector \( \forall a \in A : b_a = \{0\}^{d \times 1} \)
4: for \( i = 1, 2, \ldots, |I| \) do
5: for user \( i \in I \) do
6: Observe context \( \hat{x}_i \)
7: \( pb = \sum_{a \in I} \theta_i(a) \) \# Compute probability of choosing either CB or MAB\textsubscript{u}
8: Sample policy \( \pi_i \sim B(pb_i) \)
9: if \( \pi_i = \pi_0 \) then \# Choose global CB
10: for action \( a = 1, \ldots, K \) do
11: \( \mu_a \leftarrow A_a^{-1}b_a \) \# Update weight vectors of global CB
12: \( a_i = \arg \max_{a \in 1, \ldots, K} \hat{x}_i^\top A_a^{-1} \hat{x}_i \)
13: else
14: \( a_i = \arg \max_{a \in 1, \ldots, K} \hat{x}_i^\top A_a^{-1} \hat{x}_i \) \# Choose MAB\textsubscript{u}
15: Observe reward \( r_{i,a} \) for action \( a_i \)
16: if \( a_i = a \) then
17: \( \Lambda_i^{a_i,a_i} = \Lambda_i^{a_i,a_i} + 1 \) \# Update adjacency matrix
18: if \( \pi_i = \pi_0 \) then \# Update global CB if it was chosen as a policy
19: \( A_a \leftarrow A_a + \hat{x}_i \hat{x}_i^\top \)
20: \( b_a \leftarrow b_a + r_{i,a} \hat{x}_i \)
21: else
22: \( \bar{R}_{i,a} \leftarrow \bar{R}_{i,a} + \gamma(r_{i,a} - \bar{R}_{i,a}) \) \# Update average action reward
23: \( n_{i,a} \leftarrow n_{i,a} + 1 \) \# Update action count for chosen action
24: \( \theta_{i,a} \leftarrow \bar{R}_{i,a} + \alpha_B \sqrt{\frac{2\ln(t)}{n_{i,a}}} \) \# Update action scores
25: \( H_{i,a} \leftarrow H_{i,a} + \delta_R(r_{i,a} - pb_i, \pi_i) \) \# Update gradient bandit for \( \pi_i \)
26: \( H_{i,a} \leftarrow H_{i,a} + \delta_R(1 - 2r_{i,a})(1 - pb_i, \pi_i) \) \# Update gradient bandit for \( \pi \neq \pi_i \)
27: if \( r_{i,a} = 1 \) then
28: \( \hat{a}^+ \leftarrow a_i \) \# Update current indicator action
29: else
30: \( \hat{a}^- \leftarrow \hat{a}^+ \) \# Update previous indicator action
31: \( \mathcal{U}_i \leftarrow \text{AdjSelect}(a^-, \Lambda^i, \beta_i, r_{i,a_i}) \) \# Choose action subset to sample from next
```

The UCB instantiation of the meta-algorithm presented in the previous section is shown in algorithm 2. At every time step, the agent observes a (possibly corrupted) version of the context: \( \hat{x}_i \). The decision to use the CB or MAB\textsubscript{u} policy to select an action is performed by a gradient bandit Sutton and Barto [2018]. The probabilities of choosing either CB
When the chosen action does not provide a positive reward (rewards are Bernoulli distributed), other actions are included worse compared to the $CB$. We discuss the implications for some of the design choices on the behavior of the algorithm.

Algorithm 3 is a simple strategy to dynamically adapt the subset of actions to play in each round. Using Algorithm 3 to ensure asymptotic optimality by enforcing optimistic behavior. Nonetheless, sufficient $pb$ after some time steps the agent may have for $MAB$ base algorithms receives, the less likely the gradient bandit is to choose the base algorithm. If the preference values. The preference values can be interpreted in the following manner: the less reward one of the action candidates for exploration is computed by Algorithm 3 using the individual action-to-action adjacency matrix for every user. After playing the action and observing reward $r_{i,a_i}$, we update the parameters of the policy chosen by the referee (line 18). Finally, we update the preference values $H_i$ (line 25) depending on the received reward. For example, if the $CB$ policy was chosen, that is, $\pi_i = \pi_0$ and a reward $r_{i,a_i} = 1$ was received, we increase the preference $H_{i,\pi_i}$ for choosing the $CB$ while decreasing the preference for the $MAB_u$ policy $H_{i,\neg\pi_i}$.

Algorithm 3 is a simple strategy to dynamically adapt the subset of actions to play in each round. Using reach $\beta$ the algorithm selects the top $\beta$ entries of the $a_i^-\ell$th row of individual adjacency matrix $\Lambda_i^\ell$, sorted in descending order. The reach-parameter can be thought of as the size of the set of action to play. The larger the reach, the more actions are included. action that show correlations in a first-order Markov chain are treated with higher priority and are included first. The algorithm greedily chooses the action subset (see line 6 and 7) depending on recent rewards. It can adapt quickly to changing reward distributions by increasing $\beta$ rapidly, thus including actions with less strong correlation to ensure sufficient exploration by the action selection algorithms. Note that algorithm 3 does not compute an explicit term (like UCB-type algorithms) to ensure asymptotic optimally by enforcing optimistic behavior. Nonetheless, sufficient exploration is ensured simply by the greediness of the algorithm when selecting the pool of possible candidate actions. When the chosen action does not provide a positive reward (rewards are Bernoulli distributed), other actions are included for potential exploration by the $MAB_u$ strategy.

5 Calculations and Theory

In this section, we go into more detail of the calculation steps for different parts of the presented algorithm. Furthermore, we discuss the implications for some of the design choices on the behavior of the algorithm.

5.1 Update Equations of the Gradient Bandit

The probability $pb_i$ of choosing the $CB$ or the $MAB_u$ is computed from a gradient bandit’s Sutton and Barto [2018] preference values. The preference values can be interpreted in the following manner: the less reward one of the base algorithms receives, the less likely the gradient bandit is to choose the base algorithm. If the $MAB_u$ performs worse compared to the $CB$, the preference values for the $MAB_u$ become smaller compared to the $CB$. For example, after some time steps the agent may have for $H_i = \{1, -2\}$. The probability of choosing the $CB$ (policy $\pi_0$) is then $pb_{i,\pi_0} = \frac{e^{1}}{e^{1} + e^{-2}} \sim 95\%$. The updates are:

$$H_{i,\pi_i} \leftarrow H_{i,\pi_i} + \delta R (r_{i,a_i} - pb_{i,\pi_i})$$
$$H_{i,\neg\pi_i} \leftarrow H_{i,\neg\pi_i} + \delta R (1 - 2r_{i,a_i}) (1 - pb_{i,\pi_i})$$
We modified the update equation of the gradient bandit presented in Sutton and Barto [2018], since the factor \((1 - \pi_i)\) in the original definition of \(H_{i, \pi_i}\) may favor algorithms with strong initial performance in the update, thus reducing adaptability in non-stationary environment due to vanishing gradients when probabilities are close to 1 Mei et al. [2020].

We also omitted comparing the received reward \(r_t\) and average reward \(\bar{R}_t\) as in the original definition. This comparison served as a baseline to modulate the magnitude and direction of the preference value update (second term on the RHS in equations above). Omitting this comparison allows for faster adaptability of the preference values since \(\bar{R}_t\) generally reduces the magnitude of the update. Our formulation punishes policies more strongly for mistakes if they were chosen more often. The gradient bandit can adapt much more quickly to changes in reward performance of the action selection policies, albeit with the caveat of being potentially more susceptible to reward noise. Note that the reward the gradient bandit receives depends on the performance of the chosen policy. Thus, referee and chosen bandit are updated in a joined manner via the combination of the referee’s choice of base algorithm and the base algorithms action selection.

### 5.1.1 Gradient Bandit Dynamics

The learning dynamics of the gradient bandit can be described by a solution the following nonlinear differential equation, see appendix A for the derivation:

\[
\frac{dp^*(t)}{dt} = \delta_t p^*(t)(\Delta_R - r^* + p^*(t)(2p^*(t) - 3)(\Delta_R p^*(t) - r^* + 1) + 1),
\]

where \(r^*\) is the average reward of the superior policy, \(p^*(t)\) is the probability of choosing the superior policy at time \(t\) and \(\Delta_R\) is the gap in average reward between the two policies. We do not solve for an explicit solution of \(p^*(t)\), but for a constant \(\Delta_R\) we can approximate it rather well by an exponential function. Thus, the convergences goes with approximately \(e^{-\Delta_R \rho c}\), where \(c > 0\) is some constant, reaching the following stationary probability of choosing the better performing policy:

\[
\lim_{t \to \infty} p^*(t) = (\Delta_R + 2r^* + \sqrt{9\Delta_R^2 - 4\Delta_R r^* + 4\Delta_R + 4r^*2 - 8r^* + 4 - 2})/(4\Delta_R) = C_\infty.
\]

Equation 9 follows readily from setting the LHS of equation 8 to 0 and solving for \(p^*(t)\). Thus, using equation 9, we let \(p^*(t)\) be approximately:

\[
p^*(t) \sim \left(C_\infty - p^*(0)\right)(1 - e^{-\Delta_R \rho c}) + p^*(0)
\]

with \(\Delta_R > 0\) and some problem-dependent constant \(c \neq 0\). We integrate over \(T\) to get the number of times the superior policy is chosen, that is

\[
\tau^* \sim \int_0^T \left(C_\infty - p^*(0)\right)(1 - e^{-\Delta_R \rho c}) + p^*(0)dt = \frac{1}{c\Delta_R} \left[e^{-T\Delta_R c}\left(C_\infty(1 - e^{T\Delta_R c}) + p^*(0)(e^{T\Delta_R c} - 1)\right)\right] + C_\infty T.
\]

Therefore, the number of time steps the inferior policy is chosen is sublinear in \(T\) when \(C_\infty\) converges to 1 eventually. From this analysis, we see that our gradient bandit does not commit as readily to the best policy, compared to the vanilla formulation by Sutton and Barto [2018]. Instead, it is preserving fast adaptability in a changing environment, with the caveat that additional regret may be incurred in scenarios where \(\Delta_R\) is small, and consequently \(C_\infty < 1\). This behavior of our gradient bandit can be readily observed in the results section (7) and leads to a dynamic equilibrium between choosing the \(CB\) and \(MAB_u\) policies according to the performance relative to each other.

### 5.2 Updating the Adjacency Matrix of COMBINE

We assume that we receive information from several users at each time step. For COMBINE-UCB and variants we assume that users in a state share the same context distribution, thus the learned action to reward mapping from each user could be generalized to other users. The meta-algorithm allows the possibility to include an individual model or global model for the \(CB\), \(MAB_u\) and referee. Thus we choose a global model for the \(CB\) and individual models for both \(MAB_u\) and referee. Both our \(MAB_u\) policy and the referee can adjust to each user. We also investigate learning a common adjacency matrix for all users. This common adjacency matrix is used by the individual \(MAB_u\) policies.
for decision making. We investigate four different ways of updating and utilizing the adjacency matrix for use in the multi-armed bandit part of COMBINE, described in the following subsections.

5.2.1 Adjacency Matrix

Each user has their individual adjacency matrix, which the agent uses to select promising subsequent actions. The two bookkeeping variables, the previous indicator action denoted as $a_{-i}^t$ and the current indicator action denoted as $a_{+i}^t$, are used to record time-correlation of actions. If the previous indicator action is not equal to the current action, the algorithm did not receive a reward in the previous step (a transition occurred potentially) and may need to explore. We then update the adjacency matrix according to:

$$\Lambda^{i}_{a_{-i}^t,a_i^t} = \Lambda^{i}_{a_{-i}^t,a_i^t} + r_{i,a_i^t}. \quad (10)$$

If the agent received a reward $= 1$ for the currently played action $a_i^t$, we set $a_{+i}^t \leftarrow a_i^t$, otherwise we set $a_{-i}^t \leftarrow a_{+i}^t$.

While we cannot be sure using this mechanism actually recovers the true correlation among actions, it does so in expectation. As the agent learns what the best action is, it selects it more frequently and sooner. Errors, corresponding to lower rewards, give the algorithm an indication that something has changed. However, there is no mechanism to provide a guarantee, given that the agent may always make mistakes in estimating the current state. If something more is known about the setting, for example about the frequency of state changes, more elaborate solutions might be possible. Alternatively, there might be cases where users behave similarly over time, where we can learn the adjacency matrix jointly by combining the experience from multiple users. We investigate a variant where each user contributes to a common adjacency matrix. Referring back to the meta-algorithm, adjacency matrix $\Lambda$ (line 14 in algorithm 1) would constitute a global model of the action transitions. The common adjacency matrix $\Lambda_{com}$ is computed as the sum over all adjacency matrices at time $t$ for users in $I$:

$$\Lambda_{com,t} = \sum_{j=1}^{|I|} \Lambda_j^t. \quad (11)$$

5.2.2 Softmax Variation

We investigate a variant where we model transition probabilities for each action pair as the softmax distribution of a vector of action preferences. The adjacency matrix $\Lambda$ becomes an action preference matrix $\Lambda^*$ and is updated iteratively in the following manner:

$$\Lambda^*_{a_{-i}^t,a_i^t} = \Lambda^*_{a_{-i}^t,a_i^t} + \alpha S \left( r_{a_i^t} - \frac{\Lambda^*_{a_{-i}^t,a_i^t}}{\sum_{b=1}^{|A|} e^{\Lambda^*_{a_{-i}^t,b}}} \right), \quad (12)$$

where $\alpha S \in (0, \infty]$ is a step size parameter that modulates the rate of change of the preference matrix between time steps. The higher the value of this parameter, the larger the update towards the new preference matrix. Instead of simply adding the number of occurrences of some arbitrary transition $a_{i,t} \rightarrow a_{i,t+1}$ with $a_{i,t} \neq a_{i,t+1}$, particular transitions given the previous indicator action are more or less “preferred” by the agent. Subsequently, line 14 in algorithm 2 changes from the argmax over actions to sampling from a probability distribution over the action preferences. The probability of choosing action $a_i$ at time $t$ then becomes:

$$P(a_i | a_{-i}^t) = \frac{\Lambda^*_{a_{-i}^t,a_i} e^{\Lambda^*_{a_{-i}^t,a_i}}}{\sum_{b=1}^{|A|} e^{\Lambda^*_{a_{-i}^t,b}}},$$

where the column-indices (next possible actions) of $\Lambda^*$ are limited by reach $\beta$ and can vary from time step to time step, resulting in action preference vector $\Lambda^*_{a_{-i}^t}$. 

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Figure 1: State evolution for one user for each of the two groups. Users in group A exhibit a highly dynamic and rapidly changing state while having no corruption on the context (context corruption is not shown). The state of users in group B changes slowly, but their context is drawn from independent uniform noise.

As with the common adjacency matrix described above, we define a common preference matrix $\Lambda^{*}_{\text{com}}$, which is computed as a weighted sum of individual preference matrices using the hamming product:

$$\Lambda^{*}_{\text{com},t} = \left| \sum_{j=1}^{\vert I \vert} \Lambda^{*}_{t,j} \odot \Lambda_{t,j} \odot \Lambda^{*}_{\text{com},t} \right|,$$

(13)

where $\odot$ denotes the element-wise division. The computational rule can be explained intuitively: the more we observe a particular transition from a user, the more we are sure about said transition for all users, assuming that users behave similarly. We encode this fact by using the adjacency matrix $\Lambda_{t,j}$, which puts more weight on collected data from a particular user where the agent has observed a specific transition more often.

6 Experimentation

In this section, we describe the experimental setup for the simulation and real-world data sets.

6.1 Experimental Setup for the Simulation Environment

First, we investigate the performance of the proposed algorithm on a set of users that exhibit extreme behaviors in terms of context corruption and fast state changes. Users in group A have no context corruption but change their hidden state rapidly, and group B provides a useless context, but the hidden state (and best action) changes very slowly. An example of one user from each group is shown in figure 1. We expect the CB to perform quite well on group A, while the MAB would perform better on group B. The partition of the state to context as described in section 3.2 is the same for all users in the simulation. Given this set of users, we investigate the performance of the algorithms as the number of actions increases. We run the algorithm on both groups at the same time. The regret is measured over 2500 time steps since the rank order of performances did not change for longer simulations. The challenge is to perform well on both groups of users.

Distributing the actions space over discrete intervals of the context space, as defined in section 3.2, will lead to a nonlinear context to action-reward mapping. The action-reward distribution changes abruptly when a different state is visited resulting in a nonlinear “step” function from continuous state values to rewards. This mapping would result in sub-par performance of the CB algorithm, since it assumes linear realizability of rewards. To allow linear models to perform well such that deciding between CB and MAB becomes nontrivial, we construct simple linear features of the context, i.e., we represent the context as the one-hot encoding of the best action given the corrupted context. Our simulation environment “generates” context vector $x_{t,i}$ from $s_{t,i}$ by one-hot encoding the best action given the state
We use this simplification (unknown to the agent) only to increase the interpretability of results in the simulation environment, and neither the setting nor the algorithm depend on it, as shown in our real-world data experiments in Section 7.

For the simulation, we set the algorithm parameters to: $\delta_R = 0.5$, $\alpha = 1$, $\alpha_B = 1$, $\gamma = 0.1$ and $\alpha_S = 10$, if not otherwise mentioned. For LinTS, we set the algorithm specific parameter $v = 0.2$.

6.2 Experimental Setup for Real-World Datasets

We investigate the performance of the proposed algorithm on real-world data sets from the UCI Machine Learning Repository\(^2\). We primarily focused on time series or sequential data sets that exhibit ordering among classes (actions) in time. We evaluate the proposed algorithm on these data sets in particular: Activities of Daily Living Recognition Using Binary Sensors (ADL), Beijing Multi-Site Air-quality Data Set (BAQ) and Localization Data from Person Activity Data Set (Local).

For experimentation, we require the data to arrive in streams. We, therefore, sample each context (features) sequentially. At each round, the agent provides its choice from the pool of actions (classes). We then reveal, through the reward, whether the instance was classified correctly (1) or not (0). We evaluate the algorithm’s performance over a range of corruption levels using the same notion of regret defined in 2.3. As with the simulation, we set the algorithm parameters to: $\delta_R = 0.5$, $\alpha = 1$, $\alpha_B = 1$ and $\gamma = 0.1$, $\alpha_S = 10$ and the LinTS specific parameter $v = 0.2$.

All datasets underwent preprocessing. In the next paragraphs we explain what we did for each dataset in more detail.

Activities of Daily Living (ADLs) Data Set The ADLs dataset contains activities and corresponding sensor events for two people (A and B) in an intelligent home environment. In particular, the data recorded contains timestamps of several activities performed by the users throughout the day. The data constitute approximately 320 and 500 samples for persons A and B, respectively. We re-sampled the data to every minute and padded the gaps between events by forward imputation to increase the number of samples. The final number of instances is 19517 for person A and 30208 for person B. The context is represented as a one-hot encoding of the sensor information. There are 9 Sensors for person B and 11 for person A. We combined (summed) the representation of sensor events that happened simultaneously during activity and capped the resulting vectors between 0 and 1. For context corruption, we draw a sensor event (index of a sensor) uniformly at random and generate the one-hot representation. We study the performance on the data set of each individual and combine the performance for the summary results. We run experiments with five different random seeds and ten random starting locations of the data in the first 3000 time-steps. The regret is evaluated for the first 15000 time steps.

Beijing Multi-Site Air-quality Data Set (BAQ) This data set includes measurement from 12 nationally-controlled air-quality monitoring sites from 2013-2017. The data is recorded hourly, and all substations have the same amount of data. We run the algorithms on all 12 stations simultaneously (corresponding to “users”). We divided the particulate matter measurement (PM2.5) into five classes using equal frequency binning. We converted categorical features into their one-hot encoding. We used a random forest classifier to select the top 10 predictive features, all numeric except the South-West (SW) wind direction, which is binary. For the corrupted context, similar to the ADL dataset, we draw a random context from the uniform distribution with dimension $d = 9$ on the domain $[0, 1]$ and the SW wind direction from a binomial distribution with $p = 0.5$. The data set constitutes approximately 35000 instances. We run experiments with five different random seeds. The regret is evaluated over all time steps.

Localization Data from Person Activity Data Set (Local) The data set contains activities from five different people performing a sequence of 11 activities, each five times in a row. We combine the $x$, $y$, and $z$ coordinates of each of the four sensors into one set of features, constituting 12 features. We combine the data of all subjects (concatenation) for a total of approximately 35800 instances. As before, the corruption function constitutes random sampling of the context from a uniform distribution on the domain $[0, 1]$ with $d = 12$. We run experiments with five different random seeds and ten random starting locations of the data in the first 3000 time steps. The regret is measured over 30000 time steps.

6.3 Investigated Algorithms

We compare the performance of the proposed algorithm with a stationary (UCBBanditS) and nonstationary (UCBBanditNS) UCB bandit as well as two contextual bandits, LinUCB Li et al. [2010] and LinTS Agrawal and Goyal [2013]. UCBBanditS uses the UCB1 algorithm Auer et al. [2002]. UCBBanditNS uses UCB1 but computes the average action rewards as a discounted sum according to equation 5.

\(^2\)https://archive.ics.uci.edu/ml/datasets.html
Table 1: Regret performance on the simulated data. The regret is averaged over number of actions for both group A and group B (mean ± std).

| Method               | Group A    | Group B    | Total       |
|----------------------|------------|------------|-------------|
| UCBBanditNS          | 1825 ± 584.3 | 220.4 ± 148.2 | 1023 ± 913.0 |
| UCBBanditS           | 1714 ± 574.4 | 850.3 ± 462.9  | 1282 ± 676.3 |
| LinUCB               | 33.31 ± 77.84 | 1918 ± 357.73 | 975.8 ± 984.5 |
| LinTS                | 14.73 ± 4.447  | 1944 ± 324.46 | 979.8 ± 999.4 |
| LinUCB+UCBBanditNS    | 111.5 ± 103.4 | 266.8 ± 184.2  | 189.1 ± 122.7 |
| LinTS+UCBBanditNS     | 69.32 ± 28.12  | 273.0 ± 189.4  | 171.2 ± 169.0 |
| COMBINE-UCB          | 25.05 ± 6.452  | 325.7 ± 237.9  | 175.4 ± 225.4 |
| COMBINE-UCB common    | 25.01 ± 6.253  | 323.6 ± 237.7  | 174.3 ± 224.7 |
| COMBINE-softmax       | 24.89 ± 6.0    | 213.0 ± 161.0  | 118.9 ± 147.5 |
| COMBINE-softmax common| 25.01 ± 6.589  | 212.0 ± 155.5  | 118.7 ± 144.3 |

We also compare different combinations of CB and MABs: LinUCB and Non-stationary UCB (LinUCB + UCBBanditNS), LinTS and Nonstationary UCB (LinTS + UCBBanditNS), both without reach parameter $\beta$. We test these against four versions of the meta-algorithm: COMBINE-UCB, COMBINE-UCB common (COMBINE-UCB with common adjacency matrix for all users), COMBINE-softmax (using a preference matrix for each user) and COMBINE-softmax common (a common preference matrix for all users). All algorithms which are not named COMBINE do not use the adjacency matrix for action selection.

7 Results

In this section, we show the results from the experiments described above. We start with the simulated data, followed by the results on the real-world data sets.

7.1 Simulation analysis

Table 1 shows the regret over 2500 time steps of the investigated algorithms averaged over actions for both groups of users. We investigated 2, 5, 7, 9, 12 and 15 actions. For each combination of the parameters, we run five different random seeds and ten different initializations of the state per group, i.e., corresponding to ten users in each group.

All algorithms utilizing the context performed well on group A, which is somewhat expected given that these users exhibit no context corruption, and therefore the context to action mapping is quickly learned. The regret on users B is significantly worse for the competing algorithms since the context is highly corrupted, prohibiting the learning of expected reward for each action effectively. The combination approach works best on both groups of users as we dynamically adapt the chosen strategy based on how the MAB and CB perform during learning. The softmax variant performs the best overall since the next promising action is selected using the adjacency matrix compared to UCB that might continue to explore and incur additional regret.

Figure 2 shows the regret of the algorithms for our simulation study over a different number of actions. In general, increasing the number of actions leads to an expected increase of regret on both groups A and B. For group A, notable exceptions from this rule are the single CB and combination approaches. We see that an increase in the number of actions does not necessarily lead to strictly worse average regret. This is to some extent expected since the regret lower bound for contextual bandit problems scale with the dimensionality of the context space Chu et al. [2011] not with the number of actions. This property still seems to hold when using the CB as base algorithms in combination approaches.

We note that combination approaches mostly outperform single algorithms on Group B, exhibiting high context corruption but low state change. On group A, on the other hand, we observe that combining might not always lead to the lowest incurred regret. We observe that LinTS outperforms COMBINE. We attribute this performance deficit to the referee, who must choose between the two base algorithms. This shows the potential limitations of COMBINE when in environments with low to little context noise, a concern we expect to play a less important role in practical applications, where some form of context noise is ubiquitous. Furthermore, high state fluctuations effectively result in i.i.d.-like sampling of the context, turning the problem into the standard contextual bandit setting. In such a setting, COMBINE may incur more regret due to the initial exploration of the referee, especially if there is little or no context noise.
Figure 2: Regret for all investigated algorithms over actions and state for both groups A and B. Shaded areas show the standard deviation. (Top) An increase in the number of actions results in an expected increase in incurred regret for all algorithms not using the context and both groups A and B. (Bottom) The same experimental setup but COMBINE trains on all data without ignoring noisy data points. We observe that COMBINE is now more sensitive to the number of actions, particularly for group B.

Figure 3: Regret over corruption levels and state instability for five actions. Notice the deteriorating performance of the nonstationary bandit as instability levels increase, providing a pseudo upper bound on the regret for the MAB components of the combination algorithms. A combination of high state instability (rapid change of actions within a time period) and high corruption levels leads to sub-par performance due to the dynamics of the referee. Shaded areas show standard deviation.

To illustrate the referee’s influence on regret, we ran additional experiments over a range of state instabilities and corruption levels. In settings where a high level of corruption on contexts or state instability exists, choosing between $MAB_u$ or $CB$ might lead to higher overall regret, as both exhibit periods of better and worse performance; the referee biases action selection to one or the other intermittently. This "indecision" or switching by the referee comes with an additional cost where the pure $CB$ achieves lower regret in total compared to the combination. Figure 3 illustrates this fact, showing the average regret for five actions.

To investigate this point further, we explore the switching dynamics of the referee. We primarily focus on the simulation where all users exhibit the same levels of corruption and state instability, i.e., all users are the same. We observe that, in the setting of high instability and low corruption, the bandit is overtaken by the $CB$ as expected. However, even
slight corruption or inaccuracies in learning the mapping between contexts and action rewards will lead to the $MAB_u$ exhibiting better performance.

Figure 4 shows the dynamics of the referee over time, corruption levels and state stability for ten actions. The referee responds to changes in corruption levels or state instability quickly, adjusting its choice of base algorithm according to the performance of either $CB$ or $MAB_u$ in the most recent time steps. We observe that the probability of choosing either the $CB$ or the $MAB_u$ falls into a dynamic equilibrium. This equilibrium exhibits higher variance in areas where either the $CB$ or $MAB_u$ show higher short-term performance. In case of high state instability and corruption, neither $CB$ or $MAB_u$ exhibits strictly higher performance on average. As a result, a combination of both does not yield superior performance compared to using a $CB$ or $MAB_u$ individually in the extreme case.

Overall, at appropriate levels of corruption and state instability, the agents can decide between either contextual bandit or MAB as one performs better than the other over time. For lower state instability, the referee settles for a dynamic equilibrium, which shows better regret performance than single approaches for a broad range of context corruption, particularly for higher levels of corruption.

### 7.2 Real life data sets analysis

Table 2 illustrates the average results for the three investigated data sets. As can be seen, our algorithm with the softmax bandit variant performs the best overall on the data sets when averaged over corruption levels investigated. On the BAQ data set, we can also observe that the performance difference between the algorithms is the least significant compared to the ADL and Local data sets. This can be explained by high state permanence in both data sets (ADL and Local), allowing the agent to learn the state transition matrix effectively. The BAQ data set exhibits strong fluctuations between actions in time and nonlinear correlations between context and pollution level, violating the linear realizability assumption. This makes it quite tricky for the $CB$ as well as $MAB_u$ to perform well. We observe that both UCBBanditS and UCBBanditNS perform relatively poorly on their own, and the $CB$s have a better chance of incurring less regret. We see that our algorithm with the softmax bandit variant performs the best for high corruption levels. We attribute the better performance to using the action transition probabilities instead of the upper confidence bound of action rewards, resulting in less exploration of actions that do not follow sequentially.

We applied a slight smoothing to the features and target variable (second-order Savitzky-Golay-filter, windows size 151). This improved the performance of all algorithms slightly without making the problem significantly easier for the
We observe a significant change in behavior of the referee for all algorithms on the BAQ data set. We attribute this to the fact that the non-softmax variable and less pure action-to-action transition probabilities. Smoothing helps sparsify the adjacency matrix and, therefore, more predictable transitions, leading to significantly improved regret performance.

Figure 6 shows the learned adjacency matrix of the softmax bandit for all data sets. In most cases, using the softmax version leads to more sparse matrices and, therefore, less exploration when choosing the subsequent promising actions. This works well for the ADL and Local data sets. The BQA data set exhibits significant fluctuations in the target variable and less pure action-to-action transition probabilities. Smoothing helps sparsify the adjacency matrix and, therefore, more predictable transitions, leading to significantly improved regret performance.

Figure 7 shows the dynamics of the referee on the real-world data sets. The ADL data set exhibits strong ordering as well as low action change frequency such that the MAB variants of the algorithms can achieve low regret without knowing the context. This is further exacerbated when adding corruption to the context, further diminishing the performance of the CB.

Similarly to the ADL dataset, the MAB outperforms the CB early on the Local data set. We note that our algorithm’s non-softmax version performs comparatively worse on this data set due to the MAB_a component being noncompetitive. We attribute this to the fact that the MAB_a component still uses UCB to select actions. Given that we do not update the MAB_a if it was not chosen by the referee for actions selection, UCB might explore more often, incurring additional regret and increasing the reach parameter β rapidly. This may effectively lead to (i) the whole actions space being available for exploration and (ii) the "over-exploitation" of actions that, due to β, rarely have been tried before. These actions receive a large exploration bonus due to the high upper-confidence-bound estimate (line 24 in algorithm 2), and it may require a significant number of trails to reduce said bonus.

We observe a significant change in behavior of the referee for all algorithms on the BAQ data set. Contexts (and therefore actions) can frequently change, coupled with the nonlinear context that does not allow a good mapping from contexts to actions, the performance of all algorithms suffer. Interestingly, the softmax version does seem to provide enough advantage over the CB to be chosen more frequently, further amplified by corrupting the context. Smoothing the data set allows the MAB_a to overtake the CB rapidly, reducing regret, particularly for the softmax variation of our algorithm.

### 7.2.1 Tuning $\alpha_B$ and $\alpha_S$

As we have mentioned in earlier sections, the non-softmax MAB_a components explore actions in a manner that does not allow competitive regret performance to the softmax variant. All combination algorithms have a UCB MAB in their core component, and as such, we need to tune the $\alpha_B$ parameter for optimal exploration and exploitation. We briefly investigate the effect of the exploration parameter $\alpha_B$ to see if we can improve upon regret.
Figure 5: The softmax bandit version of COMBINE consistently outperforms other algorithms over all time steps. 5a Combined regret curves of both users in ADL data set. The softmax bandit version of our algorithm performs best mainly due to the good performance of the MAB component. 5b Similar to the ADL data set, the softmax bandit variant of our algorithm performs the best. The algorithm quickly converges to using the softmax bandit for action selection, incurring the least regret. 5c BAQ data set. For the non-smoothed version, the level of pollution can change rapidly between time steps resulting in sub-par performance. Only for high corruption levels, we observe the MAB\textsubscript{u} having an advantage. 5d Slight smoothing applied significantly improves performance. Shaded areas show standard deviation.
Figure 6: Adjacency Matrix for all data sets of COMBINE-UCB (top), COMBINE using the softmax bandit variation (middle) and the respective ground truth (GT) (bottom). The adjacency matrix has been split for the ADL data set into user A and user B. For most data sets, we observe that the algorithm learns the adjacency matrix, leading to more targeted action selection and less regret incurred. A notable exception is the BAQ data set without smoothing, where the agent did not approximate the true adjacency matrix particularly well.

Figure 8 shows the regret achieved over a range of values of the parameter for the investigated data sets. The performance of the algorithms slightly improves for an optimal $\alpha_B$ unique to each data set. Even with tuning, the softmax bandit variation of our algorithm outperforms all other investigated methods with the added benefit of not needing any tuning. Even when tuning the exploration rate from very high to very low exploration, we do not significantly improve performance such that we are on par with the softmax $MAB_u$ variant.

Figure 9 shows the regret for various values of $\alpha_S$ for the algorithms that use the adjacency matrix for action selection. The most significant effect on regret can be observed for low values between $0 − 20$. The softmax-bandit variant of COMBINE shows the highest sensitivity to this parameter, which is not surprising since updates to the preference matrix can significantly affect the resulting probability distribution over actions. Biasing action selection towards promising next actions with fewer iterations, effectively results in more “greedy” action selection and less exploration. The effect on COMBINE-UCB version of the algorithm is less significant given that the ranking of the actions is not influenced as strongly by $\alpha_S$.

8 Conclusions

We introduced a sequential decision-making problem setting where the observed corrupted contexts are determined by an underlying state that obeys the Markov property. This state evolution allows the incorporation of knowledge about the order of actions that is used to guide the exploration and exploitation of actions. We combine, in a novel way, two previously investigated settings, namely the Contextual Bandit with Corrupted Contexts and Regime Switching Bandits. In our setting, the agent needs to balance exploration and exploitation of actions based on its belief about the reliability of information from learned state transitions and context side information. We propose a meta-algorithm, called COMBINE, and an instantiation of it based on upper-confidence bound combined with one-step transition probabilities.
Figure 7: Probability of choosing the CB over the MAB for the different real-world data sets. 7a The ADL data almost universally favor the MA, with the stationary MAB experiencing periods where the CB is preferred, mostly in regions of action transition. 7b The MAB approach outperforms the CB quickly on the Local data set. As a failure case, the MAB component of COMBINE-UCB does not perform well, resulting in additional regret. 7c Neither CB nor MAB performs exceptionally well, resulting in fluctuating behavior of the referee. 7d Smoothing the BAQ data set reduces sporadic action changes and allows the MAB to outperform the CB. Shaded areas show standard deviation.

Figure 8: Regret over a range of $\alpha_B$ values of the MAB component of each algorithm. Each data set has a unique optimal value. The softmax bandit version performs best even after tuning the other MAB agents. Shaded areas show standard deviation.
Figure 9: Regret over a range of $\alpha_S$ values for algorithms using the adjacency matrix. Best results are obtained for sufficiently large $\alpha_S$, e.g., in $5 - 10$ range.

We show improved performance compared to contemporary methods through empirical evaluation of the algorithm on simulated and real-world data, mainly when using the softmax variant of COMBINE. On simulated data, we observe that for extreme cases of high state fluctuations and low context corruption, or, alternatively, low state fluctuations and high context corruption, using single approaches, i.e., either a $CB$ or $MAB$, might minimize incurred regret. In more realistic settings, however, when there is enough exploitable information for the agent to make decisions, the COMBINE approach significantly outperforms single methods.

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References

S. S. Villar, J. Bowden, and J. Wason. Multi-armed Bandit Models for the Optimal Design of Clinical Trials: Benefits and Challenges. Stat Sci, 30(2):199–215, 2015. URL https://doi.org/10.1214/14-STSS504.

Hamsha Bastani and Mohsen Bayati. Online decision making with high-dimensional covariates. Operations Research, 68(1):276–294, 2020. doi: 10.1287/opre.2019.1902. URL https://doi.org/10.1287/opre.2019.1902.

Weiwei Shen, Jun Wang, Yu-Gang Jiang, and Hongyuan Zha. Portfolio choices with orthogonal bandit learning. In International Conference on Artificial Intelligence, IJCAI’15, page 974–980. AAAI Press, 2015. ISBN 9781577357384.

Xiaoguang Huo and Feng Fu. Risk-aware multi-armed bandit problem with application to portfolio selection. Royal Society Open Science, 4(11):171377, 2017. doi: 10.1098/rsos.171377. URL https://royalsocietypublishing.org/doi/abs/10.1098/rsos.171377.

Stefano Boldrini, Luca De Nardis, Giuseppe Caso, Mai Le, Jocelyn Fiorina, and Maria-Gabriella Di Benedetto. mumab: A multi-armed bandit model for wireless network selection. Algorithms, 11(2):13, Jan 2018. ISSN 1999-4893. doi: 10.3390/a11020013. URL http://dx.doi.org/10.3390/a11020013.

R. Kerkouche, R. Alami, R. Féraud, N. Varsier, and P. Maillé. Node-based optimization of lora transmissions with multi-armed bandit algorithms. In 2018 25th International Conference on Telecommunications (ICT), pages 974–980. DOI 10.1109/ICT.2018.8464949.

Zheng Wen, Branislav Kveton, Michal Valko, and Sharan Vaswani. Online influence maximization under independent cascade model with semi-bandit feedback. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems, volume 30, pages 3022–3032, Curran Associates, Inc., 2017. URL https://proceedings.neurips.cc/paper/2017/file/7137debd45ae4d0ab9aa953017268b20-Paper.pdf.

Eric Schwartz, Eric Bradlow, and Peter Fader. Customer acquisition via display advertising using multi-armed bandit experiments. Marketing Science, 36, 04 2017. doi: 10.1287/mksc.2016.1023.

Q. Wang, C. Zeng, W. Zhou, T. Li, S. S. Iyengar, L. Shwartz, and G. Y. Grabarnik. Online interactive collaborative filtering using multi-armed bandit with dependent arms. IEEE Transactions on Knowledge and Data Engineering, 31 (8):1569–1580, 2019. doi: 10.1109/TKDE.2018.2866041.

Linas Baltrunas, Karen Church, Alexandros Karatzoglou, and Nuria Oliver. Frappe: Understanding the usage and perception of mobile app recommendations in-the-wild. CoRR, abs/1505.03014, 2015. URL http://arxiv.org/abs/1505.03014.
T.L. Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. *Advances in Applied Mathematics*, 6(1):4 – 22, 1985. ISSN 0196-8858. doi: 10.1016/0196-8858(85)90002-8. URL http://www.sciencedirect.com/science/article/pii/0196885885900028.

Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47:235–256, 05 2002. doi: 10.1023/A:1013689704352.

William R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294, 1933. ISSN 00063444. URL http://www.jstor.org/stable/2332286.

Olivier Chapelle and Lihong Li. An Empirical Evaluation of Thompson Sampling. In J. Shawe-Taylor, R. S. Zemel, P. L. Bartlett, F. Pereira, and K. Q. Weinberger, editors, *Neural Information Processing*, pages 2249–2257. Curran Associates, Inc., 2011. URL http://papers.nips.cc/paper/4321-an-empirical-evaluation-of-thompson-sampling.pdf.

Aurélien Garivier and Eric Moulines. On upper-confidence bound policies for switching bandit problems. In Jyrki Kivinen, Csaba Szepesvári, Esko Ukkonen, and Thomas Zeugmann, editors, *Algorithmic Learning Theory*, pages 174–188, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg. ISBN 978-3-642-24412-4.

Peter Auer, Nicolò Cesa-Bianchi, Yoav Freund, and Robert E. Schapire. The nonstochastic multiarmed bandit problem. *SIAM J. Comput.*, 32(1):48–77, January 2003. ISSN 0097-5397. doi: 10.1137/S0097539701398375. URL https://doi.org/10.1137/S0097539701398375.

Reda Abouserie. Sources and levels of stress in relation to locus of control and self esteem in university students. *Educational Psychology*, 14(3):323–330, 1994. doi: 10.1080/0144341940140306. URL https://doi.org/10.1080/0144341940140306.

James O Prochaska, Colleen A Redding, Kerry E Evers, et al. The transtheoretical model and stages of change. *Health behavior: Theory, research, and practice*, 97, 2015.

Xiang Zhou, Yi Xiong, Ningyuan Chen, and Xuefeng Gao. Regime switching bandits, 2021. URL https://arxiv.org/abs/2001.09390.

Wei Chu, Lihong Li, Lev Reyzin, and Robert Schapire. Contextual bandits with linear payoff functions. volume 15 of *Machine Learning Research*, pages 208–214, Fort Lauderdale, FL, USA, 11–13 Apr 2011. JMLR Workshop and Conference Proceedings. URL http://proceedings.mlr.press/v15/chu11a.html.

Lihong Li, Wei Chu, John Langford, and Robert E. Schapire. A contextual-bandit approach to personalized news article recommendation. *International conference on World wide web - WWW '10*, 2010. doi: 10.1145/1772690.1772758. URL http://dx.doi.org/10.1145/1772690.1772758.

Shipra Agrawal and Navin Goyal. Thompson sampling for contextual bandits with linear payoffs. volume 28 of *Machine Learning Research*, pages 127–135, Atlanta, Georgia, USA, 17–19 Jun 2013. PMLR. URL http://proceedings.mlr.press/v28/agrawal13.html.

Djallel Bouneffouf. Online learning with Corrupted context: Corrupted Contextual Bandits. Number NeurIPS, 2020. URL http://arxiv.org/abs/2006.15194.

Alek Agarwal, Haipeng Luo, Behnam Neyshabur, and Robert E. Schapire. Corralling a band of bandit algorithms. *CoRR*, abs/1612.06246, 2016. URL http://arxiv.org/abs/1612.06246.

Nagarajan Natarajan, Donghyuk Shin, and Inderjit S. Dhillon. Which app will you use next? collaborative filtering with interactional context. In *ACM Conference on Recommender Systems*, RecSys ’13, page 201–208, New York, NY, USA, 2013. Association for Computing Machinery. ISBN 9781450324090. doi: 10.1145/2507157.2507186. URL https://doi.org/10.1145/2507157.2507186.

Dietmar Jannach, Lukas Lerche, and Iman Kamekhkosh. Beyond "hitting the hits": Generating coherent music playlist continuations with the right tracks. In *ACM Conference on Recommender Systems*, RecSys ’15, page 187–194, New York, NY, USA, 2015. Association for Computing Machinery. ISBN 9781450336925. doi: 10.1145/2792838.2800182. URL https://doi.org/10.1145/2792838.2800182.

Negar Hariri, Bamshad Mobasher, and Robin Burke. Context-aware music recommendation based on latenttopic sequential patterns. In *ACM Conference on Recommender Systems*, RecSys ’12, page 131–138, New York, NY, USA, 2012. Association for Computing Machinery. ISBN 9781450312707. doi: 10.1145/2365952.2365979. URL https://doi.org/10.1145/2365952.2365979.

Geoffray Bonnin and Dietmar Jannach. Automated generation of music playlists: Survey and experiments. *ACM Comput. Surv.*, 47(2), November 2014. ISSN 0360-0300. doi: 10.1145/2652481. URL https://doi.org/10.1145/2652481.
B. Mobasher, Honghua Dai, Tao Luo, and M. Nakagawa. Using sequential and non-sequential patterns in predictive web usage mining tasks. In *IEEE International Conference on Data Mining*, pages 669–672, 2002. URL [https://ieeexplore.ieee.org/document/1184025](https://ieeexplore.ieee.org/document/1184025).

Miki Nakagawa and Bamshad Mobasher. Impact of site characteristics on recommendation models based on association rules and sequential patterns. 05 2003. URL [https://www.researchgate.net/publication/228479790_Impact_of_site_characteristics_on_recommendation_models_based_on_association_rules_and_sequential_patterns](https://www.researchgate.net/publication/228479790_Impact_of_site_characteristics_on_recommendation_models_based_on_association_rules_and_sequential_patterns).

Massimo Quadrana, Paolo Cremonesi, and Dietmar Jannach. Sequence-aware recommender systems. *ACM Comput. Surv.*, 51(4), July 2018. ISSN 0360-0300. doi: 10.1145/3190616. URL [https://doi.org/10.1145/3190616](https://doi.org/10.1145/3190616).

John Langford and Tong Zhang. The epoch-greedy algorithm for multi-armed bandits with side information. In J. Platt, D. Koller, Y. Singer, and S. Roweis, editors, *Neural Information Processing*, volume 20, pages 817–824. Curran Associates, Inc., 2008. URL [https://proceedings.neurips.cc/paper/2007/file/4b04a686b0ad13dce35fa99fa4161c65-Paper.pdf](https://proceedings.neurips.cc/paper/2007/file/4b04a686b0ad13dce35fa99fa4161c65-Paper.pdf).

J. Gittins. Bandit processes and dynamic allocation indices. *Journal of the royal statistical society series b-methodological*, 41:148–164, 1979.

P. Whittle. Restless bandits: Activity allocation in a changing world. *Journal of Applied Probability*, 25:287–298, 1988. ISSN 0021-9002. doi: 10.2307/3214163. URL [http://www.jstor.org/stable/3214163](http://www.jstor.org/stable/3214163).

Ronald Ortner, Daniil Ryabko, Peter Auer, and Rémi Munos. Regret bounds for restless markov bandits. *Theoretical Computer Science*, 558:62–76, 2014. ISSN 0304-3975. doi: 10.1016/j.tcs.2014.09.026. URL [https://www.sciencedirect.com/science/article/pii/S030439751400704X](https://www.sciencedirect.com/science/article/pii/S030439751400704X). Algorithmic Learning Theory.

Richard S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction*. A Bradford Book, Cambridge, MA, USA, 2018. ISBN 0262039249.

Jincheng Mei, Chenjun Xiao, Csaba Szepesvari, and Dale Schuurmans. On the global convergence rates of softmax policy gradient methods, 2020.
A Derivation of the Gradient Bandit Learning Dynamics

We iterate the update equations for our gradient bandit formulation:

\[ H_{i,\pi} \leftarrow H_{i,\pi} + \delta R (r_{i,a} - pb_{i,\pi}) \]  
\[ H_{i,\neg\pi} \leftarrow H_{i,\neg\pi} + \delta R (1 - 2r_{i,a}) (1 - pb_{i,\pi}). \]  

(14)  

(15)

First, we are looking for the change in probability \( pb \) of choosing a policy depending on the change in weights of the softmax policy and the change of the weights \( H_{\pi} \) over time, that is:

\[ \frac{dpb_{i,\pi}}{dt} = \frac{dpb_{i,\pi}}{dH_{i,\pi}} \times \frac{dH_{i,\pi}}{dt}. \]  

(16)

The derivative of the softmax policy with respect to weights \( H_{i,\pi} \) is simply Sutton and Barto [2018]:

\[ \frac{dpb_{i,k}}{dH_{i,j}} = pb_{i} (I_{k,j} - pb_{j}), \]  

(17)

where \( I \) is the indicator function being 1 if \( k = j \) and 0 otherwise. We now derive the expression \( \frac{dH_{i,\pi}}{dt} \). For the derivation, we assume that the gradient bandit needs to decide between two policies and assume Bernoulli distributed rewards. Furthermore, we omit the subscript \( i \) for ease of notation. Thus, given the update equations above, we have to consider the following four cases:

Case 1: \( H_{\pi_t} = \pi_0 \land r_{t,a_t} = 1 \):

\[ H_{\pi_0} \leftarrow H_{\pi_0} + \delta R (1 - pb_{\pi_0}) \]  
\[ H_{\pi_1} \leftarrow H_{\pi_1} - \delta R (1 - pb_{\pi_0}) \]

Case 2: \( H_{\pi_t} = \pi_0 \land r_{t,a_t} = 0 \):

\[ H_{\pi_0} \leftarrow H_{\pi_0} - \delta R pb_{\pi_0} \]  
\[ H_{\pi_1} \leftarrow H_{\pi_1} + \delta R (1 - pb_{\pi_0}) \]

Case 3: \( H_{\pi_t} = \pi_1 \land r_{t,a_t} = 1 \):

\[ H_{\pi_0} \leftarrow H_{\pi_0} - \delta R pb_{\pi_0} \]  
\[ H_{\pi_1} \leftarrow H_{\pi_1} + \delta R pb_{\pi_0} \]

Case 4: \( H_{\pi_t} = \pi_1 \land r_{t,a_t} = 0 \):

\[ H_{\pi_0} \leftarrow H_{\pi_0} + \delta R pb_{\pi_0} \]  
\[ H_{\pi_1} \leftarrow H_{\pi_1} - \delta R (1 - pb_{\pi_0}) \]

Note the second term on the RHS of each of these update equations can be interpreted as a case dependent derivative of the respective weights. Note that we have expressed all probabilities in terms of one policy \( \pi_0 \), which we can do since \( pb_{1} := 1 - pb_{0} \).

First note that for each case, given average reward \( r_{\pi_0} \) for policy \( \pi_0 \) and average reward \( r_{\pi_1} \) for policy \( \pi_1 \), we construct an expectation over derivatives by an weighted average for each of the cases above. We start with derivative \( \frac{dH_{\pi_t}}{dt} \). If \( \pi_t = \pi_0 \), as a first step, consider the weighted sum of all the update equations for \( H_{\pi_0} \) where \( r_{t,a_t} = 1 \) or \( r_{t,a_t} = 0 \). The weights in the sum are determined by the average reward \( r_{\pi_0} \). For \( r_{t,a_t} = 1 \) we weight the update with \( r_{\pi_0} \) and for \( r_{t,a_t} = 0 \) with \((1 - r_{\pi_0})\) leading to the partial term:

\[ r_{\pi_0} \delta R (1 - pb_{\pi_0}) - (1 - r_{\pi_0}) \delta R pb_{\pi_0}. \]  

(18)
Now $H_{\pi_0}$ is also updated when $\pi_t = \pi_1$. The weights are now determined by the average reward $r_{\pi_1}$, thus we get for the second partial term:

$$-r_{\pi_1}\delta_R p_{\pi_0} + (1 - r_{\pi_1})\delta_R p_{\pi_0}. \tag{19}$$

For the final result, we need to consider the probability of these two events happening. This is determined by probabilities $p_{\pi_0}$ and $p_{\pi_1}$. We multiply equations 18 and 19 with the respective probabilities getting:

$$p_{\pi_0}\left(r_{\pi_0}\delta_R(1 - p_{\pi_0}) - (1 - r_{\pi_0})\delta_R p_{\pi_0}\right), (1 - p_{\pi_0})\left(-r_{\pi_1}\delta_R p_{\pi_0} + (1 - r_{\pi_1})\delta_R p_{\pi_0}\right), \tag{20}$$

where we substituted $p_{\pi_1} = (1 - p_{\pi_0})$. $\frac{dH_{\pi_1}}{dt}$ can be derived in a similar fashion. We then have for the derivatives of the weights $H_{\pi}$:

$$\frac{dH_{\pi_0}}{dt} = p_{\pi_0}\left(r_{\pi_0}\delta_R(1 - p_{\pi_0}) - (1 - r_{\pi_0})\delta_R p_{\pi_0}\right) + (1 - p_{\pi_0})\left(-r_{\pi_1}\delta_R p_{\pi_0} + (1 - r_{\pi_1})\delta_R p_{\pi_0}\right), \tag{21}$$

$$\frac{dH_{\pi_1}}{dt} = p_{\pi_0}\left(-r_{\pi_0}\delta_R(1 - p_{\pi_0}) + (1 - r_{\pi_0})\delta_R(1 - p_{\pi_0})\right) + (1 - p_{\pi_0})\left(r_{\pi_1}\delta_R p_{\pi_0} - (1 - r_{\pi_1})\delta_R(1 - p_{\pi_0})\right). \tag{22}$$

We now derive the final expression in equation 16, we focus on $\frac{dp_{\pi_0}}{dt}$, since the change in probability of $\pi_1$ follows directly by the fact $\frac{dp_{\pi_1}}{dt} = -\frac{dp_{\pi_0}}{dt}$. In order to describe $\frac{dp_{\pi_0}}{dt}$ we need to consider the total differential:

$$\frac{dp_{\pi_0}}{dt} = \frac{dp_{\pi_0}}{dH_{\pi_0}} \times \frac{dH_{\pi_0}}{dt} + \frac{dp_{\pi_0}}{dH_{\pi_1}} \times \frac{dH_{\pi_1}}{dt}. \tag{23}$$

Putting equations 17, 21 and 22 in 23, we get after some simplification:

$$\frac{dp^*(t)}{dt} = \delta_R p^*(t)(\Delta_R - r^* + p^*(t)(2p^*(t) - 3)(\Delta_R p^*(t) - r^* + 1) + 1), \tag{24}$$

where $\Delta_R = r_{\pi_0} - r_{\pi_1} > 0$, $r^* = r_{\pi_0}$ and $p^*(t) = p_{\pi_0}(t)$.