POSITIVE AND INCREASING SOLUTIONS OF PERTURBED
HAMMERSTEIN INTEGRAL EQUATIONS WITH
DERIVATIVE DEPENDENCE

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Dedicated to Professor Juan J. Nieto on the occasion of his sixtieth birthday.

Abstract. We discuss the existence and non-existence of non-negative, non-decreasing solutions of certain perturbed Hammerstein integral equations with derivative dependence. We present some applications to nonlinear, second order boundary value problems subject to fairly general functional boundary conditions. The approach relies on classical fixed point index theory.

1. Introduction. The study of perturbed Hammerstein integral equations often arises in the study of real world phenomena. For example the equation

\[ u(t) = th(u(\beta)) + \int_0^1 k(t, s) f(s, u(s)) \, ds, \]

occurs when dealing with the solvability of the boundary value problem (BVP)

\[ u''(t) + f(t, u(t)) = 0, \quad u(0) = 0, \quad u'(1) = h(u(\beta)). \] (1)

The BVP (1) can be used as a model for the steady-states of heated bar of length 1, where the left end is kept at ambient temperature and a controller in the right end adds or removes heat according to the temperature registered by a sensor placed in a point \( \beta \) of the bar. The controller placed in the right end may act in a linear or in a nonlinear manner, depending on the nature of the function \( h \). There exists now a (relatively) wide literature on heat-flow problems of this kind, we refer the reader to the papers [7, 8, 10, 22, 23, 33, 39, 40, 41] for the cases of linear response and to [19, 21, 24, 25, 29, 35] for the nonlinear cases.

Note that the idea of using perturbed Hammerstein integral equations in order to deal with the existence of solutions of BVPs with nonlinear BCs has been used with success in a number of papers, see the manuscripts [1, 3, 4, 5, 9, 11, 12, 13, 14, 15, 16, 17, 19, 26, 36, 45, 46] and references therein. In particular, in the recent paper [17], by means of the classical Krasnosel’skii-Guo fixed point theorem of cone compression/expansion, Goodrich studied the existence of positive solutions of the equation

\[ u(t) = \gamma_1(t) h_1(\alpha_1[u]) + \gamma_2(t) h_2(\alpha_2[u]) + \lambda \int_0^1 k(t, s) f(s, u(s)) \, ds, \] (2)

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where $\lambda$ is parameter and $\alpha_1, \alpha_2$ are linear functionals on the space $C[0,1]$ realized as Stieltjes integrals with signed measures, namely

$$\alpha_i[u] := \int_0^1 u(s) \, dA_i(s),$$

with $A_i$ a function of bounded variation. The results of [17] complement the earlier ones by the author [19], where only positive measures were employed.

The functional formulation (3) has proven to be particularly useful in order to handle multi-point and integral BCs. For an introduction to nonlocal BCs, we refer the reader to the reviews [3, 6, 30, 34, 38, 44] and the papers [27, 28, 32, 37, 43].

On the other hand, in a recent paper [42], Webb gave, using fixed point index theory, a general set-up for the existence of positive solutions of second order BVPs where linear BCs of the type $\alpha[u'] = \int_0^1 u'(s) \, dA(s)$ occur, a particular example being the BVP

$$u''(t) + f(t, u(t)) = 0, \quad u(0) = 0, \quad u'(1) = \alpha[u'].$$

Also by means fixed point index theory, Zang and co-authors [47] discussed the existence of positive, increasing solutions of the BVP

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad u(0) = \alpha[u], \quad u'(1) = 0,$$

where $\alpha[u]$ is a linear, bounded functional on the space $C[0,1]$.

Nonlinear functional BCs were investigated by Mawhin et al. in [31], where the authors prove, by means of degree theory, the existence of a solution of a system of BVPs which, in the scalar case, reduces to

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad u(0) = a, \quad u'(1) = N[u'],$$

where $a$ is a fixed number and $N$ is a compact functional defined on the space $C[0,1]$.

Here we study an integral equation related to (2), where we allow a dependence in the derivative of the nonlinearity $f$ and we allow the (not necessarily linear) functionals to act on the space $C^1[0,1]$, namely

$$u(t) = \eta_1 \gamma_1(t) h_1[u] + \eta_2 \gamma_2(t) h_2[u] + \lambda \int_0^1 k(t,s) f(s, u(s), u'(s)) \, ds,$$

where $h_1, h_2$ are suitable compact functionals on the space $C^1[0,1]$ and $\eta_1, \eta_2, \lambda$ are non-negative parameters. Multi-parameter problems of this kind have been studied recently by the author [21] in the context of systems of elliptic equations (without gradient dependence) subject to functional BCs. Here, in the spirit of the paper [21], we provide existence and non-existence results for the equation (4) that take into account the parameters $\eta_1, \eta_2, \lambda$. One advantage of considering the functionals in the space $C^1[0,1]$ is that it allows us to consider an interplay between function and derivative dependence in the BCs, this is illustrated in the examples of Section 3. Our methodology involves the classical fixed point index for the existence result and an elementary argument for the non-existence result.

As an application we discuss the solvability of the BVP

$$u''(t) + \lambda f(t, u(t), u'(t)) = 0, \quad u(0) = \eta_1 h_1[u], \quad u'(1) = \eta_2 h_2[u],$$

and illustrate, in two examples, how our methodology can be used in presence of nonlinear functionals that involve also nonlocal conditions.
2. Main results. In this Section we study the existence and non-existence of solutions of the perturbed Hammerstein equation of the type

\[ u(t) = \eta_1 \gamma_1(t) h_1[u] + \eta_2 \gamma_2(t) h_2[u] + \lambda \int_0^1 k(t, s) f(s, u(s), u'(s)) \, ds := Tu(t). \]  

Throughout the paper we make the following assumptions on the terms that occur in (5).

(C1) \( k : [0, 1] \times [0, 1] \to [0, +\infty) \) is measurable and continuous in \( t \) for almost every (a.e.) \( s \), that is, for every \( \tau \in [0, 1] \) we have

\[ \lim_{t \to \tau} |k(t, s) - k(\tau, s)| = 0 \text{ for a.e. } s \in [0, 1]; \]

furthermore there exist a function \( \Phi \in L^1([0, 1]) \) such that \( 0 \leq k(t, s) \leq \Phi(s) \) for \( t \in [0, 1] \) and a.e. \( s \in [0, 1] \).

(C2) For a.e. \( s \), the partial derivative \( \partial_t k(t, s) \) is non-negative and continuous in \( t \) except at the point \( t = s \) where there can be a jump discontinuity, that is, right and left limits both exist, and there exists \( \Psi \in L^1([0, 1]) \) such that

\[ 0 \leq \partial_t k(t, s) \leq \Psi(s) \text{ for } t \in [0, 1] \text{ and a.e. } s \in [0, 1]. \]

(C3) \( f : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is continuous.

(C4) We have \( \gamma_1, \gamma_2 \in C^1([0, 1]) \) and \( \gamma_1(t), \gamma_2(t), \gamma_1'(t), \gamma_2'(t) \geq 0 \) for every \( t \in [0, 1] \).

(C5) We have \( \eta_1, \eta_2, \lambda \in [0, +\infty) \).

Due to the hypotheses above, we use the space \( C^1[0, 1] \) endowed with the norm

\[ ||u|| := \max\{||u||_\infty, ||u'||_\infty\}, \]

where \( ||u||_\infty := \max_{t \in [0, 1]} |u(t)| \).

We recall that a cone \( K \) in a real Banach space \( X \) is a closed convex set such that \( \lambda x \in K \) for every \( x \in K \) and for all \( \lambda \geq 0 \) and satisfying \( K \cap (-K) = \{0\} \). Here, in order to discuss the solvability of (5), we work in the cone of non-negative, non-decreasing functions

\[ P := \{ u \in C^1[0, 1] : u(t), u'(t) \geq 0 \text{ for every } t \in [0, 1] \}. \]

and we require the nonlinear functionals \( h_1, h_2 \) to act positively on the cone \( P \) and to be compact, that is:

(C6) \( h_1, h_2 : P \to [0, +\infty) \) are continuous and map bounded sets into bounded sets.

We make use of the following basic properties of the fixed point index, we refer the reader to [2, 18] for more details.

Proposition 1. [2, 18] Let \( K \) be a cone in a real Banach space \( X \) and let \( D \) be an open bounded set of \( X \) with \( 0 \in D_K \) and \( \overline{D}_K \neq K \), where \( D_K = D \cap K \). Assume that \( \hat{T} : \overline{D}_K \to K \) is a compact map such that \( x \neq \hat{T}x \) for \( x \in \partial D_K \). Then the fixed point index \( i_K(\hat{T}, D_K) \) has the following properties:

1. If there exists \( e \in K \setminus \{0\} \) such that \( x \neq \hat{T}x + \lambda e \) for all \( x \in \partial D_K \) and all \( \lambda > 0 \), then \( i_K(\hat{T}, D_K) = 0 \).

2. If \( \hat{T}x \neq \lambda x \) for all \( x \in \partial D_K \) and all \( \lambda > 1 \), then \( i_K(\hat{T}, D_K) = 1 \).

3. Let \( D^1 \) be open in \( X \) such that \( \overline{D}^1_K \subset D_K \). If \( i_K(\hat{T}, D_K) = 1 \) and \( i_K(\hat{T}, \overline{D}^1_K) = 0 \), then \( \hat{T} \) has a fixed point in \( D^1_K \setminus \overline{D}^1_K \). The same holds if \( i_K(\hat{T}, D_K) = 0 \) and \( i_K(\hat{T}, \overline{D}^1_K) = 1 \).

We define the set

\[ P_\rho := \{ u \in P : ||u|| < \rho \} \]
Theorem 2.1. Assume there exist $r, R \in (0, +\infty)$, with $r < R$ such that the following two inequalities are satisfied:

$$\max \left\{ \lambda \overline{f}_R K + \sum_{i=1}^{2} \eta_i \gamma_i(1) H_{i,R}, \lambda \underline{f}_R K^* + \sum_{i=1}^{2} \eta_i \| \gamma_i' \|_{\infty} H_{i,R} \right\} \leq R, \quad (6)$$

$$\lambda \underline{f}_R \min \{ K, K^* \} \geq r. \quad (7)$$

Then the equation $(5)$ has a solution $u \in P$ such that

$$r \leq \| u \| \leq R.$$

Proof. With a careful use of the Ascoli-Arzelà theorem, it is possible to prove that, under the assumptions $(C_1) - (C_6)$, the operator $T$ maps $P$ into $P$ and is compact.

If $T$ has a fixed point either on $\partial P_r$ or $\partial P_R$ we are done. Assume now that $T$ is fixed point free on $\partial P_r \cup \partial P_R$, we are going to prove that $T$ has a fixed point in $P_R \setminus P_r$.

We firstly prove that $\sigma u \neq Tu$ for every $u \in \partial P_R$ and every $\sigma > 1$. If this does not hold, then there exist $u \in \partial P_R$ and $\sigma > 1$ such that $\sigma u = Tu$. Note that if $\| u \| = R$ either $\| u \|_{\infty} = R$ or $\| u' \|_{\infty} = R$.

Assume that $\| u \|_{\infty} = R$. In this case we obtain, for $t \in [0, 1]$,

$$\sigma u(t) = \eta_1 \gamma_1(t) h_1[u] + \eta_2 \gamma_2(t) h_2[u] + \lambda \int_{0}^{1} k(t, s) f(s, u(s), u'(s)) ds \leq \eta_1 \gamma_1(1) H_{1,R} + \eta_2 \gamma_2(1) H_{2,R} + \lambda \overline{f}_R \int_{0}^{1} k(1, s) ds \leq R. \quad (8)$$

Taking the supremum for $t \in [0, 1]$ in $(8)$ gives $\sigma \leq 1$, a contradiction.

Assume that $\| u' \|_{\infty} = R$. In this case we obtain, for $t \in [0, 1]$,

$$\sigma u'(t) = \eta_1 \gamma_1'(t) h_1[u] + \eta_2 \gamma_2'(t) h_2[u] + \lambda \int_{0}^{1} \partial_t k(t, s) f(s, u(s), u'(s)) ds \leq \eta_1 \| \gamma_1' \|_{\infty} H_{1,R} + \eta_2 \| \gamma_2' \|_{\infty} H_{2,R} + \lambda \overline{f}_R \int_{0}^{1} \partial_t k(t, s) ds \leq R. \quad (9)$$

Taking the supremum for $t \in [0, 1]$ in $(9)$ yields $\sigma \leq 1$, a contradiction.

Therefore we have $i_{P}(T, P_R) = 1$.

We now consider the function $g(t) := t$ in $[0, 1]$, note that $g \in P \setminus \{0\}$. We show that

$$u \neq Tu + \sigma g \text{ for every } u \in \partial P_r \text{ and every } \sigma > 0.$$ 

If not, there exists $u \in \partial P_r$ and $\sigma > 0$ such that $u = Tu + \sigma g$. 

and the quantities

$$\overline{f}_R := \max_{[0,1] \times [0, \rho]} f(t, u, v), \quad \underline{f}_R := \min_{[0,1] \times [0, \rho]} f(t, u, v), \quad H_{i,R} := \sup_{u \in \partial P_r} h_i[u],$$

$$K := \int_{0}^{1} k(1, s) ds, \quad K^* := \sup_{t \in [0, 1]} \int_{0}^{1} \partial_t k(t, s) ds.$$
Assume that $\|u\|_\infty = r$. In this case we obtain, for $t \in [0, 1]$,

$$
    u(t) = \eta_1 \gamma_1(t) h_1[u] + \eta_2 \gamma_2(t) h_2[u] + \lambda \int_0^1 k(t, s) f(s, u(s), u'(s)) \, ds + \sigma t
$$

(10)

$$
\geq \lambda \int_0^1 k(t, s) \, ds + \sigma t.
$$

From (10) and the fact that $u$ which proves the result.

Taking the supremum for $t \in [0, 1]$, we obtain

$$
r = \max_{t \in [0, 1]} u(t) = u(1) \geq \lambda \int_0^1 k(1, s) \, ds + \sigma \geq r + \sigma,
$$

a contradiction.

Assume that $\|u'\|_\infty = r$. In this case we obtain, for $t \in [0, 1]$,

$$
    u'(t) = \eta_1 \gamma'_1(t) h_1[u] + \eta_2 \gamma'_2(t) h_2[u] + \lambda \int_0^1 \partial_k k(t, s) f(s, u(s), u'(s)) \, ds + \sigma
$$

(11)

$$
\geq \lambda \int_0^1 \partial_k k(t, s) \, ds + \sigma.
$$

Taking the supremum for $t \in [0, 1]$ in (11) yields $r \geq r + \sigma$, a contradiction.

Thus we obtain $i_P(T, P_\tau) = 0$.

Therefore we have

$$
i_P(T, P_\tau \setminus P_\tau) = i_P(T, P_\tau) - i_P(T, P_\tau) = 1,
$$

which proves the result. \qed

We now prove, by an elementary argument, a non-existence result.

**Theorem 2.2.** Assume that there exist $\tau, \xi_1, \xi_2 \in (0, +\infty)$ such that

$$
    0 \leq f(t, u, v) \leq \tau u, \text{ for every } (t, u, v) \in [0, 1] \times [0, \infty)^2,
$$

$$
    h_i[u] \leq \xi_i \|u\|_\infty, \text{ for every } u \in P \text{ and } i = 1, 2,
$$

$$
    \lambda \tau K + \sum_{i=1}^2 \eta_i \xi_i \gamma_i(1) < 1.
$$

(12)

Then the equation (5) has at most the zero solution in $P$.

**Proof.** Assume that there exist $u \in P \setminus \{0\}$ such that $u$ is a fixed point for $T$. Then $\|u\|_\infty = \rho$, for some $\rho > 0$. Then we have

$$
    u(t) = \eta_1 \gamma_1(t) h_1[u] + \eta_2 \gamma_2(t) h_2[u] + \lambda \int_0^1 k(t, s) f(s, u(s), u'(s)) \, ds
$$

$$
\leq \eta_1 \gamma_1(1) h_1[u] + \eta_2 \gamma_2(1) h_2[u] + \lambda \tau \int_0^1 k(1, s) u(s) \, ds
$$

(13)

$$
\leq \eta_1 \gamma_1(1) \xi_1 \|u\|_\infty + \eta_2 \gamma_2(1) \xi_2 \|u\|_\infty + \lambda \tau \|u\|_\infty \int_0^1 k(1, s) \, ds.
$$

Taking the supremum for $t \in [0, 1]$ in (13) gives $\rho < \rho$, a contradiction. \qed
3. **Two examples.** We now illustrate the applicability of the results of Section 2. In particular we focus on the BVP

\[ u''(t) + \lambda f(t, u(t), u'(t)) = 0, \quad u(0) = \eta_1 h_1[u], \quad u'(1) = \eta_2 h_2[u]. \]  

(14)

It is routine to show (for some details, see for example [20]) that the solutions of (14) can be written in the form

\[ u(t) = \eta_1 \gamma_1(t) h_1[u] + \eta_2 \gamma_2(t) h_2[u] + \lambda \int_0^1 k(t, s) f(s, u(s), u'(s)) \, ds, \]

where the kernel \( k \) is the Green’s function associated to the right focal BCs

\[ u(0) = u'(1) = 0, \]

namely

\[ k(t, s) = \begin{cases} s & \text{if } s \leq t, \\ t & \text{if } s > t, \end{cases} \]

and \( \gamma_1(t) = 1 \) and \( \gamma_2(t) = t \) are solutions of the BVPs

\[ \gamma_1''(t) = 0, \quad \gamma_1(0) = 1, \quad \gamma_1'(1) = 0, \]

\[ \gamma_2''(t) = 0, \quad \gamma_2(0) = 0, \quad \gamma_2'(1) = 1. \]

In this case we have

\[ \gamma_1'(t) = 0, \quad \gamma_2'(t) = 1 \quad \text{and} \quad \partial_t k(t, s) = \begin{cases} 0 & \text{if } s < t, \\ 1 & \text{if } s > t. \end{cases} \]

Therefore the assumptions \((C_1), (C_2)\) and \((C_4)\) are satisfied with \( \Phi(s) = s \) and \( \Psi(s) = 1 \). By direct calculation we have \( K = \frac{1}{2} \) and \( K^* = 1 \).

**Example 1.** Let us consider the BVP

\[ u''(t) + \lambda e^{u(t) + u'(t)} = 0, \quad u(0) = \eta_1 h_1[u], \quad u'(1) = \eta_2 h_2[u], \]  

(15)

where

\[ h_1[u] = u(1/4) + (u'(3/4))^2, \quad h_2[u] = \int_0^1 u^3(s) + u'(s) \, ds. \]

Let us fix \( r = 1/20 \) and \( R = 1 \), then we have

\[ f_1 = e^2, \quad f_1 = 1, \quad H_{1,1}, H_{2,1} \leq 2. \]

Therefore the condition (6) is satisfied if

\[ \max \left\{ \frac{\lambda e^2}{2} + 2\eta_1 + 2\eta_2, \lambda e^2 + 2\eta_2 \right\} \leq 1, \]  

(16)

and the condition (7) reads

\[ \lambda \geq \frac{1}{10}. \]  

(17)

For the range of parameters that satisfy the inequalities (16)-(17), Theorem 2.1 provides the existence of at least a nondecreasing, nonnegative solution \( u \) of the BVP (15) with \( 1/20 \leq \| u \| \leq 1 \); this occurs, for example, for \( \lambda = 1/10, \eta_1 = 1/11, \eta_2 = 1/12 \).
Example 2. Let us now consider the BVP
\[ u''(t) + \lambda u(t)(2 - t \sin(u(t))u'(t)) = 0, \quad u(0) = \eta_1 h_1[u], \quad u'(1) = \eta_2 h_2[u], \]  
(18)
where
\[ h_1[u] = u(1/4) \cos^2(u'(3/4)), \quad h_2[u] = u(3/4) \sin^2(u'(1/4)). \]
In this case we may take \( \tau = 3, \xi_1 = \xi_2 = 1 \). Then the condition (12) required by Theorem 2.2 reads
\[ \frac{3}{2} \lambda + \eta_1 + \eta_2 < 1. \]
(19)

For the range of parameters that satisfy the inequality (19), Theorem 12 guarantees that the only possible solution in \( P \) of the BVP (18) is the trivial one; this occurs, for example, for \( \lambda = 1/3, \eta_1 = 1/4, \eta_2 = 1/5 \).

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