UNIVERSAL COUNTING OF
LATTICE POINTS IN POLYTOPES

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Abstract. Given a lattice polytope $P$ (with underlying lattice $\mathbb{L}$), the universal counting function $U_P(\mathbb{L}') = |P \cap \mathbb{L}'|$ is defined on all lattices $\mathbb{L}'$ containing $\mathbb{L}$. Motivated by questions concerning lattice polytopes and the Ehrhart polynomial, we study the equation $U_P = U_Q$.

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1. The universal counting function

We will denote by $V$ a vector space of dimension $n$, by $\mathbb{L}$ a lattice in $V$, of rank $n$. Let

$$G_L = \mathbb{L} \rtimes GL(\mathbb{L})$$

be the group of affine maps of $V$ inducing isomorphism of $V$ and $\mathbb{L}$ into itself; in case

$$\mathbb{L} = \mathbb{Z}^n \subset V = \mathbb{Q}^n, G_n = \mathbb{Z}^n \rtimes GL(\mathbb{Z}^n)$$
corresponds to affine unimodular maps. An $\mathbb{L}$–polytope is the convex hull of finitely many points from $\mathbb{L}$; $\mathcal{P}_L$ denotes the set of all $\mathbb{L}$–polytopes. For a finite set $A$ denote by $|A|$ its cardinality. Finally, let $\mathcal{M}_L$ be the set of all lattices containing $\mathbb{L}$.

**Definition 1.** Given any $\mathbb{L}$–polytope $P$, the function $U_P : \mathcal{M}_L \to \mathbb{Z}$ defined by

$$U_P(\mathbb{L}') = |P \cap \mathbb{L}'|$$

is called the *universal counting function* of $P$.

This is just the restriction of another function $U : \mathcal{P}_L \times \mathcal{M}_L \to \mathbb{Z}$ to a fixed $P \in \mathcal{P}_L$, where $U$ is given by

$$U(P, \mathbb{L}') = |P \cap \mathbb{L}'|.$$ 

Note, further, that $U_P$ is invariant under the group, $G_{tr}$, generated by $\mathbb{L}$–translations and the reflection with respect to the origin, but, of course, not invariant under $G_L$.

**Example 1.** Take for $\mathbb{L}'$ the lattices $\mathbb{L}_k = \frac{1}{k} \mathbb{L}$ with $k \in \mathbb{N}$. Then

$$U_P(\mathbb{L}_k) = |P \cap \frac{1}{k} \mathbb{L}| = |kP \cap \mathbb{L}| = E_P(k)$$

where $E_P$ is the Ehrhart polynomial of $P$ (see [Ehr]). We will need some of its properties that are described in the following theorem (see for instance
[Ehr],[GW]). Just one more piece of notation: if $F$ is a facet of $P$ and $H$ is the affine hull of $F$, then the relative volume $\text{volume}$ of $F$ is defined as

$$\text{rvol}(F) = \frac{\text{Vol}_{n-1}(F)}{\text{Vol}_{n-1}(D)}$$

where $D$ is the fundamental parallelootope of the $(n - 1)$–dimensional sublattice of $H \cap \mathbb{L}$. For a face $F$ of $P$ that is at most $(n - 2)$–dimensional let $\text{rvol}(F) = 0$. Note that the relative volume is invariant under $\mathcal{G}_\mathbb{L}$ and can be computed, when $\mathbb{L} = \mathbb{Z}^n$, since then the denominator is the euclidean length of the (unique) primitive outer normal to $F$ (when $F$ is a facet).

**Theorem 1.** Assume $P$ is an $n$–dimensional $\mathbb{L}$–polytope. Then $E_P$ is a polynomial in $k$ of degree $n$. Its main coefficient is $\text{Vol}(P)$, and its second coefficient equals

$$\frac{1}{2} \sum_{F \text{ a facet of } P} \text{rvol}(F).$$

It is also known that $E_P$ is a $\mathcal{G}_\mathbb{L}$–invariant valuation, (for the definitions see [GW] or [McM]). The importance of $E_P$ is reflected in the following statement from [BK]. For a $\mathcal{G}_\mathbb{L}$–invariant valuation $\phi$ from $P_\mathbb{L}$ to an abelian group $G$, there exists a unique $\gamma = (\gamma_i)_{i=0,...,n}$ with $\gamma_i \in G$ such that

$$\phi(P) = \sum \gamma_i e_{P,i}$$

where $e_{P,i}$ is the coefficient of $k^i$ of the Ehrhart polynomial.

It is known that $E_P$ does not determine $P$, even within $\mathcal{G}_\mathbb{L}$ equivalence. [Ka] gives examples lattice–free $\mathbb{L}$–simplices with identical Ehrhart polynomial that are different under $\mathcal{G}_\mathbb{L}$. The aim of this paper is to investigate whether and to what extent the universal counting function determines $P$.

We give another description of $U_P$. Let $\pi: V \to V$ be any isomorphism satisfying $\pi(\mathbb{L}) \subset \mathbb{L}$. Define, with a slight abuse of notation,

$$U_P(\pi) = |\pi(P) \cap \mathbb{L}| = |P \cap \pi^{-1}(\mathbb{L})|. $$
Set $\mathbb{L}' = \pi^{-1}(\mathbb{L})$. Since $\mathbb{L}'$ is a lattice containing $\mathbb{L}$ we clearly have

$$U_P(\pi) = U_P(\mathbb{L}').$$

Conversely, given a lattice $\mathbb{L}' \in \mathcal{M}_L$, there is an isomorphism $\pi$ satisfying the last equality. (Any linear $\pi$ mapping a basis of $\mathbb{L}$ to a basis of $\mathbb{L}'$ suffices.) The two definitions of $U_P$ via lattices or isomorphisms with $\pi(\mathbb{L}) \subset \mathbb{L}$ are equivalent. We will use the common notation $U_P$.

Example 2. Anisotropic dilatations. Take $\pi : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by

$$\pi(x_1, \ldots, x_n) = (k_1 x_1, \ldots, k_n x_n),$$

where $k_1, \ldots, k_n \in \mathbb{N}$. The corresponding map $U_P$ extends the notion of Ehrhart polynomial and Example 1.

Simple examples show that $U_P$ is not a polynomial in the variables $k_i$.

2. A necessary condition

Given a nonzero $z \in \mathbb{L}^*$, the dual of $\mathbb{L}$, and an $\mathbb{L}$–polytope $P$, define $P(z)$ as the set of points in $P$ where the functional $z$ takes its maximal value. As is well known, $P(z)$ is a face of $P$. Denote by $H(z)$ the hyperplane $z \cdot x = 0$ (scalar product). $H(z)$ is clearly a lattice subspace.

Theorem 2. Assume $P, Q$ are $\mathbb{L}$–polytopes with identical universal counting function. Then, for every primitive $z \in \mathbb{L}^*$,

$$(* \quad \text{rvol } P(z) + \text{rvol } P(-z) = \text{rvol } Q(z) + \text{rvol } Q(-z).$$

The theorem shows, in particular, that if $P(z)$ or $P(-z)$ is a facet of $P$, then $Q(z)$ or $Q(-z)$ is a facet of $Q$. Further, given an $\mathbb{L}$–polytope $P$, there are only finitely many possibilities for the outer normals and volumes of the facets of another polytope $Q$ with $U_P = U_Q$. So a well–known theorem of Minkowski implies,
Corollary 1. Assume $P$ is an $\mathbb{L}$-polytope. Then, apart from lattice translates, there are only finitely many $\mathbb{L}$-polytopes with the same universal counting functions as $P$.

Proof of Theorem 2. We assume that $P, Q$ are full-dimensional polytopes. It is enough to prove the theorem in the special case when $\mathbb{L} = \mathbb{Z}^n$ and $z = (1, 0, \ldots, 0)$. There is nothing to prove when none of $P(z), P(-z), Q(z), Q(-z)$ is a facet since then both sides of (*) are equal to zero. So assume that, say, $P(z)$ is a facet, that is, $\text{rvol } P(z) > 0$.

For a positive integer $k$ define the linear map $\pi_k : V \to V$ by

$$\pi_k(x_1, \ldots, x_n) = (x_1, kx_2, \ldots, kx_n).$$

The condition implies that the lattice polytopes $\pi_k(P)$ and $\pi_k(Q)$ have the same Ehrhart polynomial. Comparing their second coefficients we get,

$$\sum_{F \text{ a facet of } P} \text{rvol } \pi_k(F) = \sum_{G \text{ a facet of } Q} \text{rvol } \pi_k(G),$$

since the facets of $\pi_k(P)$ are of the form $\pi_k(F)$ where $F$ is a facet of $P$.

Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{Z}^{n*}$ be the (unique) primitive outer normal to the facet $F$ of $P$. Then $\zeta' = (k\zeta_1, \zeta_2, \ldots, \zeta_n)$ is an outer normal to $\pi_k(F)$, and so it is a positive integral multiple of the unique primitive outer normal $\zeta''$, that is $\zeta' = m\zeta''$ with $m$ a positive integer. When $k$ is a large prime and $\zeta$ is different from $z$ and $\zeta_1 \neq 0$, then $m = 1$ and $\text{rvol } \pi_k(F) = O(k^{n-2})$. When $\zeta_1 = 0$, then $m = 1$, again, and the ordinary $(n - 1)$-volume of $\pi_k(F)$ is $O(k^{n-2})$. Finally, when $\zeta = \pm z$, $\text{Vol } \pi_k(F) = k^{n-1} \text{Vol } F$.

So the dominant term, when $k \to \infty$, is $k^{n-1}(\text{rvol } P(z) + \text{rvol } P(-z))$ since by our assumption $\text{rvol } P(z) > 0$. □

3. Dimension two

Let $P$ be an $\mathbb{L}$-polygon in $V$ of dimension two. Simple examples show again that $\mathcal{U}_P$ is not a polynomial in the coefficients of $\pi$. 5
In the planar case we abbreviate \( \text{rvol} P(z) \) as \( |P(z)| \). Extending (and specializing) Theorem 1 we prove

**Proposition 3.** Suppose \( P \) and \( Q \) are \( \mathbb{L} \)-polygons. Then \( U_P = U_Q \) if and only if the following two conditions are satisfied:

(i) \( \text{Area}(P) = \text{Area}(Q) \),

(ii) \( |P(z)| + |P(-z)| = |Q(z)| + |Q(-z)| \) for every primitive \( z \in \mathbb{L}^* \).

**Proof.** The conditions are sufficient: (i) and (ii) imply that, for any \( \pi \), \( \text{Area}(\pi(P)) = \text{Area}(\pi(Q)) \) and \( |\pi(P)(z)| + |\pi(P)(-z)| = |\pi(Q)(z)| + |\pi(Q)(-z)| \).

We use Pick’s formula for \( \pi(P) \), (see [GW], say):

\[
|\pi(P) \cup \mathbb{L}| = \text{Area} \pi(P) + \frac{1}{2} \sum_{z \text{ primitive}} |\pi(P)(z)| + 1.
\]

This shows that \( U_P = U_Q \), indeed.

The necessity of (i) follows from Theorem 1 immediately, (via the main coefficient of \( E_P \)), and the necessity of (ii) is the content of Theorem 2. \( \square \)

**Corollary 2.** Under the conditions of Proposition 3 the lattice width of \( P \) and \( Q \), in any direction \( z \in \mathbb{L}^* \) are equal.

**Proof.** The lattice width, \( w(z, P) \), of \( P \) in direction \( z \in \mathbb{L}^* \) is, by definition (see [KL],[Lo]),

\[
w(z, P) = \max \{ z \cdot (x - y) : x, y \in P \}.
\]

In the plane one can compute the width along the boundary of \( P \) as well which gives

\[
w(z, P) = \frac{1}{2} \sum_e |z \cdot e|
\]

where the sum is taken over all edges \( e \) of \( P \). This proves the corollary. \( \square \)
Theorem 3. Suppose $P$ and $Q$ are $\mathbb{L}$-polygons. Then $\mathcal{U}_P = \mathcal{U}_Q$ if and only if the following two conditions are satisfied:

(i) $\text{Area}(P) = \text{Area}(Q)$,

(ii) there exist $\mathbb{L}$-polygons $X$ and $Y$ such that $P$ resp. $Q$ is a lattice translate of $X + Y$ and $X - Y$ (Minkowski addition).

Remark. Here $X$ or $Y$ is allowed to be a segment or even a single point. In the proof we will ignore translates and simply write $P = X + Y$ and $Q = X - Y$.

Proof. Note that (ii) implies the second condition in Proposition 3. So we only have to show the necessity of (ii).

Assume the contrary and let $P, Q$ be a counterexample to the statement with the smallest possible number of edges. We show first that for every (primitive) $z \in \mathbb{L}^*$ at least one of the sets $P(z), P(-z), Q(z), Q(-z)$ is a point.

If this were not the case, all four segments would contain a translated copy of the shortest among them, which, when translated to the origin, is of the form $[0, t]$. But then $P = P' + [0, t]$ and $Q = Q' + [0, t]$ with $\mathbb{L}$-polygons $P', Q'$.

We claim that $P', Q'$ satisfy conditions (i) and (ii) of Proposition 3. This is obvious for (ii). For the areas we have that $\text{Area} P - \text{Area} P'$ equals the area of the parallelogram with base $[0, t]$ and height $w(z, P)$. The same applies to $\text{Area} Q - \text{Area} Q'$, but there the height is $w(z, Q)$. Then Corollary 2 implies the claim.

So the universal counting functions of $P', Q'$ are identical. But the number of edges of $P'$ and $Q'$ is smaller than that of $P$ and $Q$. Consequently there are polygons $X', Y$ with $P' = X' + Y$, and $Q' = X' - Y$. But then, with $X = X' + [0, t], P = X + Y$ and $Q = X - Y$, a contradiction.
Next, we define the polygons $X, Y$ by specifying their edges. It is enough to specify the edges of $X$ and $Y$ that make up the edges $P(z), P(-z), Q(z), Q(-z)$ in $X + Y$ and $X - Y$. For this end we orient the edges of $P$ and $Q$ clockwise and set

$$P(z) = [a_1, a_2], P(-z) = [b_1, b_2], Q(z) = [c_1, c_2], Q(-z) = [d_1, d_2]$$

each of them in clockwise order. Then

$$a_2 - a_1 = \alpha t, b_2 - b_1 = \beta t, c_2 - c_1 = \gamma t, d_2 - d_1 = \delta t$$

where $t$ is orthogonal to $z$ and $\alpha, \gamma \geq 0$, $\beta, \delta \leq 0$ and one of them equals 0. Moreover, by condition (ii) of Proposition 3, $\alpha - \beta = \gamma - \delta$.

Here is the definition of the corresponding edges, $x, y$ of $X, Y$:

$$x = \alpha t, y = \beta t \text{ if } \delta = 0,$$

$$x = \beta t, y = \alpha t \text{ if } \gamma = 0,$$

$$x = \gamma t, y = -\delta t \text{ if } \beta = 0,$$

$$x = \delta t, y = -\gamma t \text{ if } \alpha = 0.$$  

With this definition, $X + Y$ and $X - Y$ will have exactly the edges needed. We have to check yet that the sum of the $X$ edges (and the $Y$ edges) is zero, otherwise they won’t make up a polygon. But $\sum(x + y) = 0$ since this is the sum of the edges of $P$, and $\sum(x - y) = 0$ since this is the sum of the edges of $Q$. Summing these two equations gives $\sum x = 0$, subtracting them yields $\sum y = 0$. \qed

4. An example and a question

Let $X$, resp. $Y$ be the triangle with vertices $(0, 0), (2, 0), (1, 1)$, and $(0, 0), (1, 1), (0, 3)$. As it turns out the areas of $P = X + Y$ and $Q = X - Y$ are equal. So Theorem 3 applies: $U_P = U_Q$. At the same time, $P$ and $Q$ are not congruent as $P$ has six vertices while $Q$ has only five.
However, it is still possible that polygons with the same universal counting function are equidecomposable. Precisely, $P_1, \ldots, P_m$ is said to be a subdivision of $P$ if the $P_i$ are $\mathbb{L}$-polygons with pairwise relative interior, their union is $P$, and the intersection of the closure of any two of them is a face of both. Recall from section 1 the group $\mathcal{G}_{tr}$ generated by $\mathbb{L}$-translations and the reflection with respect to the origin. Two $\mathbb{L}$-polygons $P, Q$ are called $\mathcal{G}_{tr}$-equidecomposable if there are subdivisions $P = P_1 \cup \cdots \cup P_m$ and $Q = Q_1 \cup \cdots \cup Q_m$ such that each $P_i$ is a translate, or the reflection of a translate of $Q_i$ with the extra condition that $P_i$ is contained in the boundary of $P$ if and only if $Q_i$ is contained in the boundary of $Q$.

We finish the paper with a question which has connections to a theorem of the late Peter Greenberg [Gr]. Assume $P$ and $Q$ have the same universal counting function. Is it true then that they are $\mathcal{G}_{tr}$-equidecomposable? In the example above, as in many other examples, they are.

References

[BK] U. Betke, M. Kneser, Zerlegungen und Bewertungen von Gitterpolytopen, J. Reine ang. Math. 358 (1985), 202–208.

[Eh] E. Ehrhart, Polinomes arithmétiques et méthode des polyédres en combinatoire, Birkhauser, 1977.

[Gr] P. Greenberg, Piecewise $SL_2$-geometry, Transactions of the AMS, 335 (1993), 705–720.

[GW] P. Gritzmann, J. Wills, Lattice points, in: Handbook of convex geometry, ed. P. M. Gruber, J. Wills, North Holland, Amsterdam, 1988.

[KL] R. Kannan, L. Lovász, Covering minima and lattice point free convex bodies, Annals of Math. 128 (1988), 577–602.

[Ka] J–M. Kantor, Triangulations of integral polytopes and Ehrhart polynomials, Beiträge zur Algebra und Geometrie, 39 (1998), 205–218.
[Lo] L. Lovász, *An algorithmic theory of numbers, graphs and convexity*, Regional Conference Series in Applied Mathematics 50, 1986.

[McM] P. McMullen, Valuations and dissections, in: *Handbook of convex geometry*, ed. P. M. Gruber, J. Wills, North Holland, Amsterdam, 1988.