Heavy quarks within the electroweak multiplet

J. Besprosvany and R. Romero

Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, Ciudad de México 01000, México

Abstract

Standard-model fields and their associated electroweak Lagrangian are equivalently expressed in a shared spin basis. The scalar-vector terms are written with scalar-operator components acting on quark-doublet elements, and shown to be parametrization-invariant. Such terms, and the t- and b-quark Yukawa terms are linked by the identification of the common mass-generating Higgs operating upon the other fields, after acquiring a vacuum expectation value $v$. Thus, the customary vector masses are related to the fermions', fixing the t-quark mass $m_t$ with the relation $m_t^2 + m_b^2 = v^2/2$ either for maximal hierarchy, or given the b-quark mass $m_b$, implying $m_t \simeq 173.9$ GeV, for $v = 246$ GeV. A sum rule is derived for all quark masses that generalizes this restriction. An interpretation follows that electroweak bosons and heavy quarks belong in a multiplet.

Keywords: Top quark, mass, multiplet, Lagrangian, spin, electroweak
1 Introduction

The standard model (SM) describes elementary-particle features and their interactions, which is praiseworthy, given its relatively limited required input, consisting of specific gauge and flavor symmetries, representations, and parameters, yet aspects remain within the model whose origin and connection to other tenets is absent, and that need to be addressed.

Thus, among its successes, the SM predicts mass values for the W and Z bosons that carry the short-range electroweak interaction, in terms of electroweak parameters, through the Higgs mechanism. However, one salient SM problem is that the fermion sector and its masses remain arbitrary, as they arise from Lagrangian terms, independent from the boson elements.

The electroweak sector hints it may provide this link, given that the W and Z vectors have universal couplings to SM fermions, and the Higgs field collectively gives mass to fermions and bosons. In addition, the similar order of magnitude of the measured masses of the W, Z, the recently discovered scalar excitation, associated with the Higgs, and the top quark (with the bottom quark’s the next highest), suggests connections among them, and thus, a common energy scale. Furthermore, fermions occupy the spin-1/2 and fundamental representations of the Lorentz and scalar groups, respectively, as vector bosons belong to the adjoint representation of each group, which implies bosons can be constructed in terms of fermions, suggesting composite structures and/or a common origin.

The above motivates looking for a formalism that takes account of discrete degrees of freedom in a single basis, including group representation properties, such as the fermion-boson fundamental-adjoint duality for the Lorentz-scalar representations, and  

---

1 As the Higgs occupies the SU_L(2) fundamental representation.
2 For the Abelian hypercharge group U(1)_Y, gauge invariance ensures boson-fermion quantum-number additivity.
that describes the combined action of operators on fields.

A previously proposed SM extension\[6\], based on a shared extended spin space, with a matrix formalism, satisfies these requirements, as it replicates SM fields with their features, and matrix multiplication accounts for operator action on fields. This space contains a (3+1)-dimensional \([d]\) subspace and one beyond 3+1, linked, respectively, to Lorentz and scalar degrees of freedom\[7\]. At each dimension, a finite number of Lorentz-invariant partitions are generated with specific symmetries and representations, reproducing particular SM features, where the cases with dimension 5+1\[8\], 7+1\[9\], and 9+1\[10\] were studied.

In this connection, it is worth recalling that a basis or representation choice can be useful, even essential, in the description of a system and its dynamics. It may reveal otherwise-hidden connections between its components, and provide a simpler framework to understand physical properties. Such a basis may describe effective degrees of freedom\[11\] accounting for collective interactions, allowing for a simpler near free-particle description, in a first approximation. For example, nucleon and associated boson interactive configurations give a tractable account of nuclear-motion modes\[12\]. Within condensed matter and low-temperature superconducting systems, a residual attractive interaction related to phonons couple electrons into Cooper pairs\[13\], which propagate freely, and lead to frictionless currents. In an application of this theory to quantum field theory and elementary particles\[14\], a four-fermion interaction produces fermion and composite-boson masses, linking their values. The quark model\[15\] conceives mesons and baryons in terms of constituent (dressed) quarks.

Leaving aside the more speculative nature of the spin SM extension, but complementarily to it, in this paper, we use it as a basis to derive SM connections, and the fields’ mass values in particular: SM heavy-fermion \(F\), vector \(V\), and scalar \(S\) fields are equivalently expressed in terms of the obtained common basis\[6\] for both Lorentz and electroweak degrees of freedom, in turn, recasting their Lagrangian com-
ponents $\mathcal{L} = \mathcal{L}_{FV} + \mathcal{L}_{SV} + \mathcal{L}_{SF}$; the identification of the scalar operator within the $\mathcal{L}_{SF}$ and $\mathcal{L}_{SV}$ vertices links univocally its defining (mass) parameters. Indeed, such universal electroweakly-invariant terms lead, under the Higgs mechanism, to a scalar whose lowest-energy condensate state pervades space, and generates particle masses through its vacuum expectation value $v$. Within the spin basis, this mechanism is similarly represented; as these fields shape elements on a matrix space, with a single associated scalar operator acting upon the others, their mass-generation property relates their coefficients.

Next, as we give the paper’s organization, we sketch the argument in more detail. Section 2 reviews the applied spin-extended space for symmetry generators and states. The paper focuses then on the (7+1)-d case that can describe the electroweak sector, and a quark doublet. For all sectors, $\mathcal{L}_{FV}, \mathcal{L}_{SV}, \mathcal{L}_{SF}$, the conventional and spin-space Lagrangian are equivalent, which is shown term-by-term in Appendices 1,2. Section 3 chooses one among two vector bases within $\mathcal{L}_{FV}$, where vectors with chiral properties are adequate. Section 4 writes $\mathcal{L}_{SV}$ equivalently with combinations of the scalars and their conjugates, with universal couplings to vectors, shown explicitly in Appendix 2; similarly for the spin-base representation, in which these two scalars induce a projection to flavor-doublet components (as $t,b$ quarks). Schematically, given the spin-space basis element $B_f$ for a field $f(x)$, we write $\mathcal{L}_{SV}$ in terms of $B_{S'}$ containing these two scalar components, obtaining the vector mass squared within $\mathcal{L}_{FV}$ as $[B_{S'},B_V]^\dagger[B_{S'},B_V]$. In Section 5, we show that the fermion masses within $\mathcal{L}_{SF}$ can be written $[B_S,B_F]$, where $B_F$ contains two terms with appropriate Yukawa coefficients. Within the spin-basis formalism, we derive that $B_{S'},B_S$ have the same operator structure; given their mass-giving nature, the identification of these operators and their coefficients translates a $v$-normalization restriction on $B_{S'}$ to $B_S$, implying a relation for the $t$ and $b$ quark masses. Section 6 shows a procedure exists that generalizes consistently this relation to all quarks in terms of a sum rule for their masses, taking advantage of the chiral projection properties of the scalar field in the
spin basis. In Section 7, we draw conclusions. We work in the classical framework afforded by the Lagrangian, and at tree-level, but also rely on a quantum-mechanical interpretation.

2 Symmetry generators and states in spin-extended space

In the following, we introduce the spin basis and its main features, where more information may be found in previous treatments [7]-[10]. Mainly, it describes SM discrete degrees of freedom in a single scheme, namely, for the Lorentz and scalar groups, and for both symmetry generators and state representations, using a common matrix space:

2.1 Matrix space

Such a space is rendered by a Clifford algebra $\mathcal{C}_N$, generated by a set of even-$N$ $2^{N/2} \times 2^{N/2}$ gamma matrices, obeying the defining property [16]

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta},$$

where $g_{\alpha\beta}$ is the metric tensor with signature $(+,-,...,-)$ and $\alpha, \beta = 0, 1, \ldots, 3, 5, \ldots, N$; whose combinations produce a complex matrix-space with dimension $2^N$.

The gamma matrices have Hermiticity properties

$$\begin{align*}
\gamma_0^\dagger &= \gamma_0, \\
\gamma_\delta^\dagger &= -\gamma_\delta \quad \delta = 1, \ldots, 3, 5, \ldots, N.
\end{align*}$$

\[3\] Following standard practice, the label 4 is omitted.
2.2 Operators and symmetry transformations

The Lorentz generators and transformations acting on spinors have standard expressions in the 4-d Clifford algebra $\mathcal{C}_4$, namely,

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \quad \text{with} \quad \mu, \nu = 0, \ldots, 3,$$

$$S(\Lambda) = e^{-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}},$$

with the $(3 + 1)$-d gamma matrices $\gamma_\mu$ transforming as vectors, while the remaining $N - 4$ gamma matrices $\gamma_a, a = 5, \ldots, N,$ and their products commuting with $\sigma_{\mu\nu}$, so they are indeed Lorentz scalars identified with generators of continuous symmetries, either gauge or global. Together with the 4-d pseudoscalar

$$\tilde{\gamma}_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3,$$

the scalars are accommodated in the unitary symmetry set

$$S_{N-4} = \frac{1}{2} (1 + \tilde{\gamma}_5) U \left( 2^{(N-4)/2} \right) \oplus \frac{1}{2} (1 - \tilde{\gamma}_5) U \left( 2^{(N-4)/2} \right),$$

where 1 stands for the $N$-d identity matrix.

A projector operator $\mathcal{P}$, obtained from elements of $S_{N-4}$, within a limited number partitions, is chosen to fit as closely the SM. The combined operator that acts on both the Lorentz generators $\mathcal{J}_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu}$ and the $S_{N-4}$ symmetry-operator space is likewise projected

$$\mathcal{J}'_{\mu\nu} = \mathcal{P} \mathcal{J}_{\mu\nu} = \mathcal{P} \left[ i (x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right],$$

$$S'_{N-4} = \mathcal{P} S_{N-4}.$$

Lorentz transformations are thus

$$S(\Lambda) = e^{-\frac{i}{4} \mathcal{P} \omega^{\mu\nu} \sigma_{\mu\nu}},$$

and scalar transformations have the form

$$U = \exp \left[ -i I_a \alpha_a(x) \right],$$
with $I_a \in S_{N-4}$. Symmetry generators within this space are described schematically in Fig. 1 in Ref. [9].

The inner product of two fields is defined according to a matrix space

$$\langle \phi | \Psi \rangle = \text{tr} \left( \phi^\dagger \Psi \right).$$

(10)

Under a unitary transformation, $\Psi \rightarrow U \Psi U^\dagger$, given the ket-bra matrix structure, with the bras interpreted as conjugate states. Thus, a Hermitian operator $O \rho$ within this space characterizes a state $\Psi$ with the eigenvalue rule

$$[O \rho, \Psi] = \lambda \Psi,$$

(11)

for real $\lambda$. This definition is consistent with the action of a derivative operator on a Hilbert space: $[\overrightarrow{\partial}, \Psi] = [-\overleftarrow{\partial}, \Psi] = [\Psi, \overleftarrow{\partial}].$ The direct product $\text{tr}\Psi_b^\dagger \Psi_a$ is also consistent associativity-wise with the operator rule, as $\text{tr}[O \rho, \Psi_b]^\dagger \Psi_a = \text{tr}\Psi_b^\dagger [O \rho^\dagger, \Psi_a]$.

2.3 Field Representation

Fields are usually assumed to exist on a Cartesian basis; for example, a vector field has components $A_\mu(x) = g_\mu^{\nu} A_\nu(x)$; alternatively, in the spin basis, it is expressed as $A_\mu(x)(\gamma_0 \gamma^\mu)_{\alpha\beta}$ (the $\alpha\beta$ indices now specify the vector character.)

More generally, a physical field with scalar quantum numbers is associated with elements of $C_N$, classified by operators from $C_4 \otimes S_{N-4}$, so it has the structure

$$\text{(elements of 3+1 space )} \times \text{( elements of } S_{N-4})$$

(12)

Fig. 2 in Ref. [9] shows the corresponding Lorentz states: scalars, vectors, fermions, and anti-symmetric tensors, arranged in the same matrix space. Next, we provide more details on the first three (physical) fields.
Fermion field

When $\Psi$ is a spin-1/2 particle, it may be seen schematically conformed as $\Psi \sim |\psi_1\rangle |a_1 F_1\rangle \langle F_2|$, with the ket carrying spin-1/2 and gauge-group fundamental representation $\psi_i$, $a_i$ quantum numbers, respectively, and both the bra and ket carrying flavor group $F_i$.

More specifically, a fermion can have the form

$$\psi_a^\alpha(x) L^\alpha P_F \Gamma^F_a,$$

where $\Gamma^F_a$ is an element of $S_{N-4}$, and $L^\alpha$ represents a spin polarization component, e.g., $L^1 = (\gamma_1 + i\gamma_2)$. The operator $P_F$ is a projection operator, e.g., $P_F = L_5$, where

$$R_5 = \frac{1}{2}(1 + \gamma_5), \quad L_5 = \frac{1}{2}(1 - \gamma_5),$$

implying

$$P_F \gamma^\mu = \gamma^\mu P_F,$$

with $P_F^c = 1 - P_F$, so that Lorentz and gauge generators act trivially on its rhs when evaluating commutators as in Eq. (11), since $P_F^c P_F = (1 - P_F)P_F = 0$.

Thus, for $U$ accounting for the Lorentz representation in Eq. 8 and the scalar transformation in Eq. 9, $\Psi$ transforms, unlike vector and scalar fields, as

$$\Psi \rightarrow U \Psi.$$  

This leads to fermions transforming as the fundamental representation of both the Lorentz and gauge groups.

Vector field

We may view vectors constructed as $\Psi \sim |\psi_1\rangle |a_1\rangle \langle a_2| \langle \psi_2|$, with the bra-ket configuration producing Lorentz vector and gauge group adjoint configurations, given the
vector and scalar $\gamma^\mu$, $\mu = 0, \ldots, 3$ and $\gamma^a$, $a = 5, \ldots, N$, respective transformation properties. Thus a vector field has form

$$A^a_\mu(x)\gamma_0\gamma^\mu I_a,$$

where $\gamma_0\gamma^\mu \in \mathcal{C}_4$ and $I_a \in \mathcal{S}_{N-4}'$ is a generator of a given unitary group.

**Scalar field**

$\Psi \sim |\psi_1\rangle |a_1\rangle \langle a_2| \langle \psi_2|$, with the bra-ket configuration producing Lorentz vector and gauge group fundamental configurations. In this case, a ket contains right-handed and a bra left-handed spin-1/2 components (or vice versa), reproducing the mass term and Higgs quantum numbers.

$$\phi^a(x)\gamma_0\Gamma^S_a,$$

with $\Gamma^S_a$ an element of $\mathcal{S}_{N-4}$.

### 2.4 Lagrangian formulation

Interactive Lagrangians\[7\] can be given in terms of vector, scalar and fermion fields conforming to the general structure of operator action as in Eq. \[11\] and the inner product in Eq. \[10\] For example, a gauge-invariant fermion-vector Lagrangian is given by

$$\frac{1}{N_f}\text{tr}\Psi^\dagger\left\{[i\partial_\mu - gA^a_\mu(x)I_a] \gamma^0\gamma^\mu - M\gamma^0\right\}\Psi,$$

where $\Psi$ is a fermion field as in Eq. \[13\], $g$ is the coupling constant, $M$ is an appropriate mass operator, and $N_f$ contains the normalization. In the next subsection, we address the spin model in 7+1 d in connection with the SM, and whose basis states will allow to write $\mathcal{L}_{FV}$, $\mathcal{L}_{SV}$, $\mathcal{L}_{SF}$ in the next Sections.
2.5 (7+1)-dimensional model

We next make a brief description of resulting states in a (7+1)-dimensional spin space under a useful partition for the SM description, sketching the way to obtain it, and providing graphic description.

2.5.1 Operators

The Clifford algebra is generated by eight $16 \times 16$ matrices

$$\gamma_0, \gamma_1, \ldots, \gamma_8.$$  \hspace{1cm} (20)

The matrices $\gamma^0$, $\gamma^i$, $i = 1, 2, 3$ correspond to the Lorentz generators $\sigma_{\mu\nu}$, given in general in Eq. 3 and the remaining four matrices, together with all their different products, comprise the set $S_{N-4}$ of scalars, with a cardinality of 32. This set is, from Eq. 6, $S_4 = P_+ U(4) \oplus P_- U(4)$, with $P_{\pm} = \frac{1}{2} (1 \pm \tilde{\gamma}_5)$, 1 the $16 \times 16$ identity matrix and $\tilde{\gamma}_5$ the 4-d chirality matrix. The elements of U(4) consist of four matrices $\gamma_a$, $a = 5, \ldots, 8$, six pairs $\gamma_{ab} \equiv \gamma_a \gamma_b$, $a < b$, four triplets $\gamma_{abc} \equiv \gamma_a \gamma_b \gamma_c$, and one quadruplet $\gamma_5 \gamma_6 \gamma_7 \gamma_8$. The Cartan subalgebra $\mathfrak{h}$ of $S_{N-4}$ contains eight elements, and a suitable choice is given by

$$1, \tilde{\gamma}_5, \gamma_5 \gamma_6, \gamma_7 \gamma_8, \gamma_5 \gamma_6 \tilde{\gamma}_5, \gamma_7 \gamma_8 \tilde{\gamma}_5, \gamma_5 \gamma_6 \gamma_7 \gamma_8 \tilde{\gamma}_5.$$  \hspace{1cm} (21)

Since $\mathfrak{h}$ is conformed of all simultaneously diagonalizable operators, it is convenient to recast this basis in terms of the projection operators
\[ P_{R1} = \frac{1}{16}(1 + \tilde{\gamma}_5)(1 + i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8), \]
\[ P_{R2} = \frac{1}{16}(1 + \tilde{\gamma}_5)(1 + i\gamma_5\gamma_6)(1 - i\gamma_7\gamma_8), \]
\[ P_{R3} = \frac{1}{16}(1 + \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8), \]
\[ P_{R4} = \frac{1}{16}(1 + \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)(1 - i\gamma_7\gamma_8), \]
\[ P_{L1} = \frac{1}{16}(1 - \tilde{\gamma}_5)(1 + i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8), \]
\[ P_{L2} = \frac{1}{16}(1 - \tilde{\gamma}_5)(1 + i\gamma_5\gamma_6)(1 - i\gamma_7\gamma_8), \]
\[ P_{L3} = \frac{1}{16}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8), \]
\[ P_{L4} = \frac{1}{16}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)(1 - i\gamma_7\gamma_8), \]

which run along the diagonal in the matrix space (Fig. [1]).
Figure 1: (Color online) Matrix representation of the Cartan basis (cf. eq. (21)) in extended spin space in 7+1 dimensions. The eight-dimensional basis is represented here in terms of the projection operators $P_{R,L,i}$, $i = 1, \ldots, 4$. The subscripts $R, L$ refer to the chirality: $R$ for operators containing $1 + \tilde{\gamma}_5$ (right-handed), and $L$ for operators containing $1 - \tilde{\gamma}_5$ (left-handed).

The operators that classify the states, with examples in terms of the projectors, consist of the baryon-number operator

$$B = \frac{1}{6}(1 - i\gamma_5\gamma_6) = \frac{1}{3}(P_{R3} + P_{R4} + P_{L3} + P_{L4}),$$

(23)
the U(1) hypercharge generator

\[ Y_o = \frac{1}{3} (4P_{R3} - 2P_{R4} + P_{L3} + P_{LA}), \]

\[ = \frac{1}{6} (1 - i\gamma_5\gamma_6) \left(1 + i\frac{3}{2}(1 + \tilde{\gamma}_5)\gamma_7\gamma_8\right), \] (24)

and \( I_3 \) within the SU(2) weak isospin generators

\[ I_1 = \frac{i}{8}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)\gamma_7, \]

\[ I_2 = \frac{i}{8}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)\gamma_8, \]

\[ I_3 = \frac{1}{2}(P_{L3} - P_{LA}) = \frac{i}{8}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)\gamma_7\gamma_8. \] (25)

The charge operator is defined in the standard way by the Gell-Mann–Nishijima relation

\[ Q = I_3 + \frac{Y_o}{2}. \] (26)

There are also flavor operators, forming the groups SU(2)_f, SU(2)_{\hat{f}}, U(1)_f, and U(1)_{\hat{f}}, and given by

\[ f_1 = \frac{i}{8}(1 + \tilde{\gamma}_5)(1 + i\gamma^5\gamma^6)\gamma^7, \]

\[ f_2 = \frac{i}{8}(1 + \tilde{\gamma}_5)(1 + i\gamma^5\gamma^6)\gamma^8, \]

\[ f_3 = \frac{i}{8}(1 + \tilde{\gamma}_5)(1 + i\gamma^5\gamma^6)\gamma^7\gamma^8, \] (27)
\[
\hat{f}_1 = \frac{i}{8} (1 - \tilde{\gamma}_5) \left(1 + i\gamma^5\gamma^6\right) \gamma^7,
\]
\[
\hat{f}_2 = \frac{i}{8} (1 - \tilde{\gamma}_5) \left(1 + i\gamma^5\gamma^6\right) \gamma^8,
\]
\[
\hat{f}_3 = \frac{i}{8} (1 - \tilde{\gamma}_5) \left(1 + i\gamma^5\gamma^6\right) \gamma^7 \gamma^8,
\]

respectively for SU(2)\(_f\) and SU(2)\(_\hat{f}\), and

\[
f_0 = i\gamma^5\gamma^6\tilde{\gamma}_5,
\]

\[
\hat{f}_0 = i\gamma^5\gamma^6,
\]

for U(1)\(_f\), and U(1)\(_\hat{f}\). The operators \(f_3, \hat{f}_3, f_0\) and \(\hat{f}_0\) belong to \(\mathfrak{h}\). In Fig. 1 the matrix space is represented schematically. The diagonal operators classify the states (off-diagonal) acting from the left for states in the same row, and from the right for states in the same column, which is consistent with matrix multiplication.

We also define a combination of diagonal flavor operators that further classifies states, given by

\[
\hat{F} = -\frac{1}{4} \left(\hat{f}_0 + 4\hat{f}_3 - 8f_3\right).
\]
Figure 2: (Color online) Matrix representation of operators, massless quarks \((U^i_{L,R}, D^1_{L,R}, i = 1, \ldots, 4)\) and Higgs \((\phi^+_{1,2}, \phi^0_{1,2})\) degrees of freedom in \((7+1)\)-d spin space. The chiral projections of the diagonal operators \(B, I_3\) and \(Y_o\) are grouped together and represented by the sets \(Q_R = \frac{1}{2} (1 + \tilde{\gamma}_5) (B, I_3, Y_o)\) and \(Q_L = \frac{1}{2} (1 - \tilde{\gamma}_5) (B, Y_o)\). Following matrix multiplication rules, operators act from the left on states in the same row, and from the right on states in the same column.
Baryon number zero, Higgs-like scalars

| \( \phi_1 \) | \( I_3 \) | \( Y_0 \) | \( Q \) |
|----|----|----|----|
| \( \phi_1^+ \) | \( \phi_1^0 \) | \( \frac{1}{8} (1 - i \gamma^5 \gamma^6) (\gamma^7 + i \gamma^8) \gamma^0 \) | 1/2 | 1 |
| \( \phi_1^0 \) | \( \phi_1^+ \) | \( \frac{1}{8} (1 - i \gamma^5 \gamma^6) (1 + i \gamma^7 \gamma_5 \gamma_5) \gamma^0 \) | 1 | 1 |

| \( \phi_2 \) | \( \phi_2^+ \) | \( \phi_2^0 \) |
|----|----|----|
| \( \phi_2^+ \) | \( \phi_2^0 \) | \( \frac{i}{8} (1 - i \gamma^5 \gamma^6) (\gamma^7 + i \gamma^8) \gamma_5 \gamma_0 \) | 1/2 | 1 |
| \( \phi_2^0 \) | \( \phi_2^+ \) | \( \frac{i}{8} (1 - i \gamma^5 \gamma^6) (1 + i \gamma^7 \gamma_5 \gamma_5) \gamma^7 \gamma^8 \gamma^0 \) | 1 | 0 |

Table 1: Scalar Higgs-like doublets

2.5.2 States

States contain scalars, fermions and vectors. Only the first two are considered in this Section. The matrix space admits two Higgs doublets \( \phi_1 \) and \( \phi_2 \) (Table 1 and Fig. 2). They satisfy \( \phi_1 = \tilde{\gamma}_5 \phi_2 \). Their connection to Hermitian and SU(2) conjugates is clarified in Section 4.3.

Non-Higgs scalars can also be constructed that contribute to the diagonalization of massive states. Ref. [9] provides further information on their nature and their application to obtain fermion properties.

The massless-fermion states satisfy the general structure of Eq. [12] and have massless quark quantum numbers, when classified by baryon number, isospin, and hypercharge. The matrix space admits four generations of quarks of different flavor (Fig. 2), arranged in four SU(2)\(_L\) doublets and eight right-handed singlets, shown in Tables 2 3, respectively. After electroweak symmetry breaking, the Higgs generates a mass operator used in Section 5 to obtain fermion mass states.
Baryon number $1/3$, hypercharge $1/3$ and polarization $1/2$ (operator $\frac{3}{2}iB\gamma^1\gamma^2$), left-handed quark doublets

\[
\begin{array}{cccccc}
Q^1_{L1} &=& \begin{pmatrix} u^1_{L1} \\ d^1_{L1} \end{pmatrix} = & \begin{pmatrix} \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(\gamma^7 + i\gamma^8)(\gamma^0 + \gamma^3) \\ \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(1 - i\gamma^7\gamma^8)(\gamma^0 + \gamma^3) \end{pmatrix} & 1/2 & 2/3 & 1/2 & 0 & 3/2 \\
Q^1_{L2} &=& \begin{pmatrix} u^1_{L2} \\ d^1_{L2} \end{pmatrix} = & \begin{pmatrix} \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(1 + i\gamma^7\gamma^8)(\gamma^0 + \gamma^3) \\ \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(\gamma^7 - i\gamma^8)(\gamma^0 + \gamma^3) \end{pmatrix} & 1/2 & 2/3 & -1/2 & 0 & -1/2 \\
Q^1_{L3} &=& \begin{pmatrix} u^1_{L3} \\ d^1_{L3} \end{pmatrix} = & \begin{pmatrix} \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(\gamma^7 + i\gamma^8)\gamma^0(\gamma^0 - \gamma^3) \\ \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(1 - i\gamma^7\gamma^8)\gamma^0(\gamma^0 - \gamma^3) \end{pmatrix} & 1/2 & 2/3 & 0 & 1/2 & 1 \\
Q^1_{L4} &=& \begin{pmatrix} u^1_{L4} \\ d^1_{L4} \end{pmatrix} = & \begin{pmatrix} \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(1 + i\gamma^7\gamma^8)\gamma^0(\gamma^0 - \gamma^3) \\ \frac{1}{16} (1 - \tilde{\gamma}_5) (\gamma^5 - i\gamma^6)(\gamma^7 - i\gamma^8)\gamma^0(\gamma^0 - \gamma^3) \end{pmatrix} & 1/2 & 2/3 & 0 & -1/2 & 0 \\
\end{array}
\]

Table 2: Massless left-handed quark weak isospin doublets. Gauge and Lorentz operators act from the left and trivially from the right. To obtain the $-1/2$ polarization, the replacement must be made $(\gamma^0 + \gamma^3) \rightarrow (\gamma^1 - i\gamma^2)$, for $Q^1_{L1}, Q^1_{L2}$, and $(\gamma^0 - \gamma^3) \rightarrow (\gamma^1 - i\gamma^2)$, for $Q^1_{L3}, Q^1_{L4}$.
(operator $\frac{3}{2} iB\gamma^1\gamma^2$), right-handed quark singlets

| Baryon number 1/3 and polarization 1/2 | \(Y_\alpha\) | \(Q\) | \(f_3\) | \(\hat{f}_3\) | \(F\) |
|--------------------------------------|------------|-------|-------|--------|-------|
| \(U^1_{R1}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (\gamma^7 + i\gamma^8) \gamma^0 (\gamma^0 + \gamma^3)\) | \(4/3\) | \(2/3\) | \(1/2\) | \(0\) | \(3/2\) |
| \(D^1_{R1}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (1 - i\gamma^7\gamma^8) \gamma^0 (\gamma^0 + \gamma^3)\) | \(-2/3\) | \(-1/3\) | \(1/2\) | \(3/2\) |
| \(U^1_{R2}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (1 + i\gamma^7\gamma^8) \gamma^0 (\gamma^0 + \gamma^3)\) | \(4/3\) | \(2/3\) | \(-1/2\) | \(-1/2\) |
| \(D^1_{R2}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (\gamma^7 - i\gamma^8) \gamma^0 (\gamma^0 + \gamma^3)\) | \(-2/3\) | \(-1/3\) | \(-1/2\) | \(-1/2\) |
| \(U^1_{R3}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (\gamma^7 + i\gamma^8) (\gamma^0 - \gamma^3)\) | \(4/3\) | \(2/3\) | \(0\) | \(1/2\) | \(1\) |
| \(D^1_{R3}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (1 - i\gamma^7\gamma^8) (\gamma^0 - \gamma^3)\) | \(-2/3\) | \(-1/3\) | \(1/2\) | \(1\) |
| \(U^1_{R4}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (1 + i\gamma^7\gamma^8) (\gamma^0 - \gamma^3)\) | \(4/3\) | \(2/3\) | \(-1/2\) | \(0\) |
| \(D^1_{R4}\) = \(\frac{1}{16} (1 + \gamma_5) (\gamma^5 - i\gamma^6) (\gamma^7 - i\gamma^8) (\gamma^0 - \gamma^3)\) | \(-2/3\) | \(-1/3\) | \(-1/2\) | \(0\) |

Table 3: Massless right-handed quark weak isospin singlets. Gauge and Lorentz operators act from the left and trivially from the right. To obtain the $-1/2$ polarization, the replacement must be made $(\gamma^0 + \gamma^3) \rightarrow (\gamma^1 - i\gamma^2)$, for $U^1_{R1}, U^1_{R2}, D^1_{R1}, D^1_{R2}$, and $(\gamma^0 - \gamma^3) \rightarrow (\gamma^1 - i\gamma^2)$, for $U^1_{R3}, U^1_{R4}, D^1_{R3}, D^1_{R4}$.

### 2.6 Fermion Yukawa elements

Bilinear fermion terms can be constructed that produce scalar elements transforming quarks into their different combinations. We use the (7+1)-d space represented in Fig. 2 with particular and general properties that can be distinguished.

There are two matrix configurations:

$$P^F_{\alpha\beta} = Q^\alpha_{R_\beta} \bar{Q}^\beta_{L_i} \quad i = 1, 2, 3$$ (32)
is contained in the Dirac projector with \((\alpha, \beta)\)-spin components and (positive or negative)-energy; the three \(P_{i}^{F^{\alpha\beta}}\) are the same up to a phase; \(Q\) are \(U\)- or \(D\)-type fermions obtained from Tables 2, 3 defining \(F\), the \(R\), \(L\) case taken as an example, and \(\bar{Q}_{L}^{\alpha} = Q_{L}^{\alpha} \gamma^{B}_{0}\),

\[\gamma^{B}_{0} = 2(\phi_{1}^{0} + \phi_{1}^{0\dagger}),\]  

(33)

\(\phi_{1}^{0}\) defined in Table 1. The \(i, j\) imply we choose a 3-generation (arbitrary) projection to reproduce the SM; we also note that \(Q_{R}^{\alpha} \bar{Q}_{L}^{\beta} = 0\) for \(i \neq j\).

On the other hand,

\[Y_{ij}^{F} = \bar{Q}_{R}^{\alpha} Q_{Lj}^{\alpha}\quad i, j = 1, 2, 3\]  

(34)

defines the Yukawa basis (full flavor transition matrix) to be used in Section 6, for the complete scalar-fermion SM Lagrangian component. One can check that \(Y_{ij}^{U}, \ Y_{ij}^{D}\) are the same (up to phases), so they are commonly labelled \(Y_{ij}^{F}\). The set \(R, L, \alpha\) is arbitrary and other choices will reproduce (up to phases) the nine \(Y_{ij}^{F}\) terms. Indeed, although the \((7+1)\)-d basis can accommodate four generations, the projection operator for, say, flavors 1,2,3

\[Y^{F 4} = Y^{F 11} + Y^{F 22} + Y^{F 33},\]  

(35)

induces the 3-generation subset with 9 elements, \(Y^{F 4 \dagger} Y_{ij}^{F} Y^{F 4}\). As the set is closed under matrix multiplication, the 4th generation is discarded (see Section 6.)

The resulting projection operators may be understood from the products of a fermion with matrix structure \(|\text{spin}\rangle \langle \text{flavor}|\) and an hermitian conjugate one, resulting in the form \(|\text{spin}\rangle \langle \text{spin}|\) for Eq. 32 and, inverting the order, \(|\text{flavor}\rangle \langle \text{flavor}|\) in Eq. 34.
3 Fermion-vector Lagrangian: chiral basis in spin space

Concentrating on the heaviest fermions, the SM two-quark electroweak interaction Lagrangian\cite{1} is\cite{2}

\begin{equation}
\mathcal{L}_{FV} = \bar{q}_L(x) \left[ i \partial_\mu + \frac{1}{2} g \tau^a W^a_\mu (x) + \frac{1}{6} g' B_\mu (x) \right] \gamma^\mu q_L(x) + \\
\bar{t}_R(x) \left[ i \partial_\mu + \frac{2}{3} g' B_\mu (x) \right] \gamma^\mu t_R(x) + \bar{b}_R(x) \left[ i \partial_\mu - \frac{1}{3} g' B_\mu (x) \right] \gamma^\mu b_R(x),
\end{equation}

where the spin-1/2 fields consist of $q_L(x) = \left( t_L(x), b_L(x) \right)$, a left-handed hypercharge $Y = \frac{1}{3}$ SU(2)$_L$-doublet, and $t_R(x), b_R(x)$, right-handed $Y = \frac{4}{3}, -\frac{2}{3}$ singlets, respectively; each term contains two polarizations as, e. g., $t_L(x) = \left( \psi_{1L}^T(x), \psi_{2L}^T(x) \right)$; $\psi_{qh}^\alpha(x)$ are wave functions\cite{3} for quarks $q = t, b$, with spin components $\alpha = 1, 2$, and chirality $h = L, R$; $W^a_\mu (x), a = 1, 2, 3$, and $B_\mu (x)$, are associated gauge-group weak and hypercharge vector bosons, with coupling constants, $g, g'$, respectively; $\tau^a$ are the Pauli matrices representing the SU(2)$_L$ generators.

An extended (7+1)-d Clifford algebra comprises a sufficiently large space to describe heavy SM particles\cite{4, 9}, with the 4-d Lorentz symmetry maintained, and spin-component generators $\frac{3}{2} B \sigma_{\mu \nu}$, where $\sigma_{\mu \nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu]$, and $\mu, \nu = 0, ..., 3$; additional scalar generators use $\gamma_5, ..., \gamma_8$, producing the baryon-number operator $B$ in Eq. \ref{23}, which conforms a spin-space projection partition, and gives quarks $1/3$ ($-1/3$ for antiparticles,) and bosons 0.

Other scalar-symmetry generators include the hypercharge $Y_o$ in Eq. \ref{24} with $\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3$, the weak SU(2)$_L$ terms in Eq. \ref{25} and flavor generators in Eqs. \ref{27}-\ref{30}; as

\footnote{A single generation is used, and CKM mixing is neglected; Eq. 36 describes the electroweak interaction for one quark color, and a sum is assumed over each such term.}

\footnote{We use units with $\hbar = c = 1$, and metric $g_{\mu \nu} = (1, -1, -1, -1)$ throughout.}

\footnote{For simplicity, spin and scalar representations are assumed that give the states’ form.}
required, \([I_i, I_j] = i\epsilon_{ijk} I_k, [I_i, Y_o] = [B, I_i] = [3B\sigma_{\mu\nu}, I_i] = [3B\sigma_{\mu\nu}, I_o] = 0\).

The \((7+1)\)-d space allows for a description of quark fields

\[
\begin{align*}
\Psi_{qL}(x) &= \sum_\alpha \psi_{\alpha L}^\alpha(x) T_L^\alpha + \psi_{bL}^\alpha(x) B_L^\alpha, \\
\Psi_{tR}(x) &= \sum_\alpha \psi_{\alpha R}^\alpha(x) T_R^\alpha, \\
\Psi_{bR}(x) &= \sum_\alpha \psi_{\alpha R}^\alpha(x) B_R^\alpha,
\end{align*}
\]

with hypercharges \(1/3, 4/3, -2/3\), respectively, and spinor components chosen in Table 4, given explicitly in Tables 2, 3; the quantum numbers \(\lambda\) are obtained from the operator structure \([Op, \Psi] = \lambda \Psi\) for the weak component \(I_3\), hypercharge \(Y_o\) (or charge \(Q = I_3 + 1/2 Y_o\)), and spin-polarization \(\frac{3}{2} B \gamma^1 \gamma^2\) operators.

The SM Lagrangian \(L_{FV}\) in Eq. 36 can be equivalently written in this basis: as derived in Ref. [7], and examined in Ref. [17]

\[
L_{FV} = \text{tr}\{\Psi_{qL}^\dagger(x) \left[i\partial_\mu + g I^a W_{\mu}^a(x) + \frac{1}{2} g' Y_o B_{\mu}(x)\right] \gamma^0 \gamma^\mu \Psi_{qL}(x) + \\
\Psi_{tR}^\dagger(x) \left[i\partial_\mu + \frac{1}{2} g' Y_o B_{\mu}(x)\right] \gamma^0 \gamma^\mu \Psi_{tR}(x) + \Psi_{bR}^\dagger(x) \left[i\partial_\mu + \frac{1}{2} g' Y_o B_{\mu}(x)\right] \gamma^0 \gamma^\mu \Psi_{bR}(x)\} P_f, (38)
\]

while gauge and Lorentz symmetries can be checked with the above transformation rule, or given the equivalence to the traditional formulation. A projection operator \(P_f\) that connects the two expressions[17] can be omitted by finding phases for \(\Psi\), which translates into finding an adequate \(\gamma_\mu\) basis. The trace coefficient is usually 1, as the field normalization factor accounts for reducible representations. A complete proof of the equivalence is given in Appendix 1.

The W-fermion vertex in \(L_{FV}\), Eq. 38, contains the matrix element \(\langle F'|W_{\alpha\mu}^i|F\rangle\), where the W contribution

\[
W_{\alpha\mu}^i = g\gamma_0 \gamma_\mu I^i
\]

describes the \(\text{SU}(2)_L\) inherently chiral action on fermion states \(|F\rangle, |F'\rangle\), as it carries the projection \(L_5 = \frac{1}{2} (1 - \tilde{\gamma}_5)\), predicted by the spin basis[9]; it is thus the natural

\footnote{The commutator is omitted as the operator acts trivially on one side.}

21
(a) hypercharge $1/3$ left-handed doublet

\[
\begin{pmatrix}
    T^1_L \\
    B^1_L
\end{pmatrix} = \begin{pmatrix}
    U^1_{L1} \\
    D^1_{L1}
\end{pmatrix}
\]

\[
\begin{array}{c|c|c|c}
I_3 & Q & \frac{3i}{2} B \gamma^1 \gamma^2 \\
\hline
1/2 & 2/3 & 1/2 \\
-1/2 & -1/3 & 1/2 \\
\end{array}
\]

(b) $I_3 = 0$ right-handed singlets

\[
\begin{array}{c|c|c|c}
Y & Q & \frac{3i}{2} B \gamma^1 \gamma^2 \\
\hline
T^1_R = U^1_{R1} & 4/3 & 2/3 & 1/2 \\
B^1_R = D^1_{R1} & -2/3 & -1/3 & 1/2 \\
\end{array}
\]

Table 4: (a) Quantum numbers of massless left-handed quark weak isospin doublet, and (b) right-handed singlets, with momentum along $\pm \hat{z}$, given explicitly in Tables 2, 3. The spin component along $\hat{z}$, $i \frac{3}{2} B \gamma^1 \gamma^2$, is used.

Choice. For example, this property is absent for $W^\prime_{\alpha\mu} = g\gamma_0 \gamma_\mu I'_i$, where $I'_i$ are the SU(2)$_L$ generators without $L_5$; although an equivalent interaction term results within this space, it requires the inclusion of $L_5$ within the vertex; worse, $[Y_o, I'_{1,2}] \neq 0$. 

22
4 Scalar-vector Lagrangian: extended charge-conjugate symmetry

4.1 Conventional $\mathcal{L}_{SV}$

In the SM, the Higgs particle is present[1] in the SU(2)$_L \times$U(1)$_Y$ gauge-invariant interacting Lagrangian-density component

$$\mathcal{L}_{SV} = H^\dagger(x)F^\mu\dagger(x)F_\mu(x)H(x), \quad (40)$$

with

$$F_\mu(x) = i\partial_\mu + \frac{1}{2}g\tau \cdot W_\mu(x) + \frac{1}{2}g'B_\mu(x), \quad (41)$$

$W_\mu^{-}(x) = (W_\mu^1(x), W_\mu^2(x), W_\mu^3(x))$, and the $Y = 1$ complex-doublet scalar $H(x) = \frac{1}{\sqrt{2}} \left( \eta_1(x) + i\eta_2(x) \right)$, composed of two charged (upper), and two neutral (lower) fields.

4.2 $\mathcal{L}_{SV}$ with Higgs and conjugate

$\mathcal{L}_{SV}$ can be equivalently written (with $B_\mu(x) \rightarrow -B_\mu(x)$) in terms of the orthogonal $Y = -1$ combination $\tilde{H}(x) = i\tau_2 H^*(x)$, which uses an antiunitary transformation $\mathcal{C}$ expressing charge-conjugation invariance (in addition to the CP symmetry in the electroweak sector, and approximate SU(2)$_L \times$SU(2)$_R$ symmetry[18]; a Hilbert space is assumed;) this is also a consequence of the SU(2) property that the conjugate representation is obtained from a similarity transformation, which ensures independence of the doublet choice. Appendix 2 shows that

$$\mathcal{L}_{SV} = \text{tr}[F'_\mu \tilde{H}_\chi \chi_\nu(x)]^\dagger F''^\mu \tilde{H}_\chi \chi_\nu(x), \quad (42)$$
where \( \bar{H}_{\chi_\ell \chi_b}(x) = (\chi_\ell \bar{H}(x), \chi_b \bar{H}(x)) \) is a 4 × 4 matrix, \( \chi_\ell, \chi_b \) are complex, and \( |\chi_\ell|^2 + |\chi_b|^2 = 1 \), with

\[
F'_\mu \bar{H}_{\chi_\ell \chi_b}(x) = (i\partial_\mu + \frac{1}{2} g \tau \cdot W_-(x)) \bar{H}_{\chi_\ell \chi_b}(x) + g' \bar{H}_{\chi_\ell \chi_b}(x) B_\mu(x) \tau_3, \tag{43}
\]

which is diagonal in \( \bar{H}(x), \bar{H}(x) \), and hence does not mix them. Moreover \( \mathcal{L}_{SV} \) is a sum of weighted positive-definite terms, meaning only the combination \( |\chi_\ell|^2 + |\chi_b|^2 \) results. This generalizes the expression\([19, 20]\) for \( \mathcal{L}_{SV} \) in terms of \( \bar{H}_{\frac{1}{\sqrt{2}} \sqrt{2}}(x) \). With the \( U(1) \) overall phase, a three-parameter subspace of the norm-conserving constraint \( |\chi_\ell|^2 + |\chi_b|^2 = 1 \) is generated. We associate this isometry with the \( \mathcal{L}_{SV} \) invariance under \( C \): \( -\tau_2 K F'_\mu \bar{H}_{\chi_\ell \chi_b}(x) \tau_2 K = F'_\mu \bar{H}_{\chi_b \chi_\ell}(x) \), with \( K \) the complex conjugate operator; \( \mathcal{L}_{SV} \) is also invariant under the \( \ll_3 \) transformation defined as \( \bar{H}_{\chi_\ell \chi_b}(x) \to \bar{H}_{\chi_\ell \chi_b}(x) \tau_3 \), together with the combination \( C \tau_3 \).

Further extension can be made for the scalars in the spin basis by attaching the \( \tilde{\gamma}_5 \) operator. Using the projection operators in Eq. \([14]\), \( \mathcal{L}_{SV} \) in Eq. \([42]\) is generalized with the substitutions

\[
F'_\mu \to (L_5)_{4 \times 4} F'_\mu \tag{44}
\]

\[
\bar{H} \to (L_5)_{4 \times 4}(\gamma_0)_{4 \times 4} \bar{H}, \tag{45}
\]

thus including spin degrees of freedom, leading to a combined spinor-electroweak description. An intermediate expression that connects to the spin basis, and ultimately to Yukawa components, is obtained

\[
\mathcal{L}_{SV} = \frac{1}{2} \text{tr}[L_5 F'_\mu L_5 \gamma_0 \bar{H}_{\chi_\ell \chi_b}(x)]^\dagger L_5 F'^\mu L_5 \gamma_0 \bar{H}_{\chi_\ell \chi_b}(x) \tag{46}
\]

\[
= \frac{1}{4} \text{tr}[(L_5 F'_\mu L_5 \gamma_0 \bar{H}_{\chi_\ell \chi_b}(x)]^\dagger + L_5 F'_\mu L_5 \gamma_0 \bar{H}_{\chi_\ell \chi_b}(x)) \tag{47}
\]

\[
(L_5 F'^\mu L_5 \gamma_0 \bar{H}_{\chi_\ell \chi_b}(x) + [L_5 F'^\mu L_5 \gamma_0 \bar{H}_{\chi_\ell \chi_b}(x)]^\dagger),
\]

with the trace also over spin degrees of freedom, the second equality using hermitian conjugates, \( R_5 L_5 = L_5 R_5 = 0 \), and trace properties which lead to only two identical
non-trivial terms. These forms will prove useful in comparing with Yukawa terms below.

4.3 $\mathcal{L}_{SV}$ in (7+1)-d spin space

In the spin basis, the four-scalar doublet structure above is reproduced. Indeed, it emerges naturally in the (7+1)-d spin basis, with the Higgs potential not altered under different definitions (chiral ones or not.) Table 1 presents two of these scalar elements (with two additional as their conjugates.) Together with coordinate dependence, they are

\begin{align*}
\phi_1(x) &= \frac{1}{\sqrt{2}} [\eta_1(x) + i\eta_2(x)] \phi_1^+ + \frac{1}{\sqrt{2}} [\eta_3(x) + i\eta_4(x)] \phi_0^0 \\
\phi_2(x) &= \frac{1}{\sqrt{2}} [\eta_1(x) + i\eta_2(x)] \phi_2^+ + \frac{1}{\sqrt{2}} [\eta_3(x) + i\eta_4(x)] \phi_0^0,
\end{align*}

(48)

and whose quantum numbers associate them to the Higgs doublet. These are unique within the (7+1)-d space[9]. Although new scalar fields are introduced in principle, here we concentrate on the SM-equivalent projections. Given the SM Higgs conjugate representation $\tilde{H}(x)$ the scalar components are interpreted through the assignments (see Table 1),

\begin{align*}
H(x) &\rightarrow \phi_1(x) - \phi_2(x) \\
\tilde{H}(x) &\rightarrow \phi_1(x) + \phi_2(x).
\end{align*}

(49)

This leads to the equivalent expressions

\begin{align*}
\mathcal{L}_{SV} &= \text{tr}\{[F''(x), H_{af}(x)]_{\pm} + [F''(x), H_{af}(x)]_{\pm}\}_{\text{sym}} \\
&= \frac{1}{2} \text{tr}\{[F''(x), H_{af}(x) + H_{af}^\dagger(x)]_{\pm} + [F''(x), H_{af}(x) + H_{af}^\dagger(x)]_{\pm}\}_{\text{sym}},
\end{align*}

(50)
where we introduced $H_{af}(x) = a\phi_1(x) + f\phi_2(x)$, and

$$F''(x) = [i\partial_{\mu} + gW_{\mu}^i(x)I_i + \frac{1}{2}g'B_{\mu}(x)Y_0]\gamma_0\gamma^\mu; \quad (52)$$

the subindex $sym$ means only symmetric $\gamma_\mu\gamma_\nu$ components are taken, to avoid the Pauli components; and the $\pm$ index means the commutator and the anticommutator should be used for the temporal and spatial $\gamma_\mu$ components, respectively. The equality for $L_{SV}$ implies that it accommodates SM parity-conserving scalar representations.

The complex parameters $a$, $f$, are constrained by the normalization rule $|a|^2 + |f|^2 = 1$.

These properties for $L_{SV}$ are shown explicitly in Appendix 2.

### 4.4 $L_{SV}$ mass components in conventional and (7+1)-d spin space

The spin representation can be connected with that of $H_{\chi_1\chi_2}(x)$ with the expression

$H_{af}(x) = \frac{1}{\sqrt{2}}(\chi_t H_t(x) + \chi_b H_b(x))$, where

$$H_t(x) = \phi_1(x) + \phi_2(x)$$
$$H_b(x) = \phi_1(x) - \phi_2(x), \quad (53)$$

with $\phi_i$ defined in Eq. 99 and this parameterization applies the unitary transformation $\chi_t = \frac{1}{\sqrt{2}}(a + f)$, $\chi_b = \frac{1}{\sqrt{2}}(a - f)$.

Under the Higgs mechanism, the SM scalars acquire [2, 3] a vacuum expectation value $v$, and only the neutral field $\eta_3(x)$ survives: $\langle \eta_3(x) \rangle = v$, $\langle H(x) \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while the charged and imaginary components are absorbed into vector bosons, as seen explicitly in the unitary gauge. Idem in the spin basis, as can be proved by the Lagrangian equivalence or directly; then,

$$\langle H_{af}(x) \rangle = H_n = \frac{v}{2}(\chi_t H_t^0 + \chi_b H_b^0), \quad (54)$$
where the normalized Higgs operator $H_n$ is defined, with the same 0, + component conventions as for the $\phi_i$, implying, as $\text{tr}H_i^0H_j^0 = 2\delta_{ij}$, \(i, j = t, b\),

$$\langle H_{af}^\dagger(x)H_{af}(x) \rangle = (|a|^2 + |f|^2)v^2/2 = (|\chi_t|^2 + |\chi_b|^2)v^2/2 = v^2/2. \tag{55}$$

The vector-Higgs vertex in $\mathcal{L}_{SV}$ determines the vector-boson masses, and within the spin basis, the trace is taken consistently with $H_n$. Thus, the mass component, extracted from Eq. 50, taking for $F''$ the $W, Z$ field terms, and for $H_{ab}$ its vacuum expectation value in Eq. 54.

$$\mathcal{L}_{SV_m} = \text{tr}[H_n, gW_0^m(x)I_m + \frac{g'}{2} B_0(x)]^\dagger[H_n, gW_0^l(x)I_l + \frac{g'}{2} B_0(x)] + \tag{56}$$

$$\{H_n, (gW^k(x)I_k + \frac{g'}{2} B_0(x))\gamma_\mu \gamma^\nu \}^\dagger \{H_n, (gW^l(x)I_l + \frac{g'}{2} B_0(x))\gamma_\mu \gamma^\nu \}$$

is produced. For the neutral massive vector boson, one derives the normalized $Z_\mu(x) = (-gW^3_\mu(x) + g'B_\mu(x))/\sqrt{g^2 + g'^2}$, and massless photon $A_\mu(x) = (g'W^3_\mu(x) + gB_\mu(x))/\sqrt{g^2 + g'^2}$, giving, e. g., the 0-component

$$\mathcal{L}_{SZ_{m0}} = \text{tr}[H_n, W_0^3(x)gI_3 + B_0(x)\frac{1}{2} g'Y_0] = [H_n, W_0^3(x)gI_3 + B_0(x)\frac{1}{2} g'Y_0] \tag{57}$$

$$= Z_0^2(x) \frac{1}{g^2 + g'^2} \text{tr}[H_n, g^2I_3 - \frac{1}{2} g'^2Y_0] = [H_n, g^2I_3 - \frac{1}{2} g'^2Y_0] = \frac{1}{2} Z_0^2(x)m_Z^2,$$

implying

$$\text{tr}\frac{1}{g^2 + g'^2}[\sqrt{2}H_n, g^2I_3 - \frac{1}{2} g'^2Y_0] = [\sqrt{2}H_n, g^2I_3 - \frac{1}{2} g'^2Y_0] = v^2(g^2 + g'^2)/4, \tag{58}$$

thus, \(m_Z = v\sqrt{g^2 + g'^2}/2, m_A = 0\).

Similarly, for $\mathcal{L}_{SW_m}$, the $W^3_{\mu}$ basis in Eq. 39 emerges, and defines the masses of the charged boson fields $W^\pm_\mu(x) = \frac{1}{\sqrt{2}}(W^1_\mu(x) \mp iW^2_\mu(x))$. Thus, the charged-vector boson component

$$\mathcal{L}_{SW_{m0}} = W_0^i(x)W_0^j(x)\text{tr}[H_n, W_0^i] = [H_n, W_0^j] = m_{W_i}^2 W_0^+(x)W_0^-(x), \tag{59}$$

\(i, j = 1, 2\) contains $m_{W_i}^2 = \text{tr}[H_n, W_0^+_{\mu}] = W_0^+_{\mu} = \frac{1}{\sqrt{2}} g\gamma_0 \gamma_\mu I^\pm, I^\pm = I_1 \pm iI_2$. This assignment is unique as this is the only way to maintain not
only the vertex condition (gauge invariance,) but also normalization (above.) When
written in terms of $H = H_n + H_n^\dagger$, interpreted as a fermion Hamiltonian, $m_W^2 = \text{tr}[H, W_{00}^+][H, W_{00}]$ and the other part is not affected, as $[H_n^\dagger, W_{00}] = 0$.

5 Scalar-fermion Lagrangian: heavy-quark doublet’s mass constraint

The Yukawa fermion-scalar interaction can be similarly parameterized in the Clifford basis

$$-\mathcal{L}_{SF} = \text{tr} \left[ \frac{\sqrt{2}}{v} \left( m_t \Psi_{tR}(x) H_t(x) \Psi_{qL}(x) + m_b \Psi_{qL}^\dagger(x) H_b(x) \Psi_{bR}(x) \right) + \{hc\} \right], \quad (60)$$

where $m_t$ and $m_b$ are the top and bottom masses, respectively, and the fermion fields $\Psi$ are defined in Eq. 37. We note that the Higgs scalar components have the correct chiral action over fermions: under the projection operators in Eq. 14 $L_5$, and $R_5$, e. g., $R_5 H_t(x)L_5 = H_t(x)$, $L_5 H_b(x)R_5 = H_b(x)$, $L_5 H_t(x)R_5 = 0$, $R_5 H_b(x)L_5 = 0$. For Eq. 60 the underlying mass operator is $H_m(x) = \frac{\sqrt{2}}{v}(m_t H_t(x) + m_b H_b(x))$, giving, under the Higgs mechanism,

$$\langle H_m(x) \rangle = H_m = m_t H_t^0 + m_b H_b^0. \quad (61)$$

Examples of quark massive basis states are summarized on Table 5 (see Tables 2-4), for both u and d-type quarks, with their quantum numbers. Only one polarization and one flavor are shown, as a more thorough treatment of the fermion-flavor states are given elsewhere [9].

This results in, e. g.,

$$H_m^h T_1^1_M = m_t T_1^1_M, \quad H_m^h T_1^c_M = -m_t T_1^c_M,$$

$$H_m^h B_1^1_M = m_b B_1^1_M, \quad H_m^h B_1^c_M = -m_b B_1^c_M. \quad (62)$$

28
where \( H_m^h = H_m + H_m^\dagger \), and \( T_M^c, B_M^c \) correspond to negative-energy solution states (and similarly for opposite spin components) and Eq. 62 justifies the \( m_t \) and \( m_b \) mass

| massive quarks | \( H_m^h \) | \( Q \) | \( \frac{3i}{2} B\gamma^1\gamma^2 \) |
|----------------|----------|------|------------------|
| \( T_M^1 = \frac{1}{\sqrt{2}} (T_L^1 + T_R^1) \) | \( m_t \) | 2/3 | 1/2 |
| \( B_M^1 = \frac{1}{\sqrt{2}} (B_L^1 - B_R^1) \) | \( m_b \) | -1/3 | 1/2 |
| \( T_M^c = \frac{1}{\sqrt{2}} (T_L^1 - T_R^1) \) | \( -m_t \) | 2/3 | 1/2 |
| \( B_M^c = \frac{1}{\sqrt{2}} (B_L^1 + B_R^1) \) | \( -m_b \) | -1/3 | 1/2 |

Table 5: Massive quark eigenstates of \( H_m^h \) given after Eq. 62.

interpretation.

Under the assumption of a single mass-producing field operator, we match a reparameterized \( H_n \) in Eq. 58 that gives the \( Z \) mass, to the fermion-mass term \( H_m \), in Eq. 62, resulting in \( \sqrt{2} H_n = H_m \): a multiplet structure is suggested. In other words, the operator identification derives from their mass eigenvalues, expressed schematically as \( | \langle Z|\sqrt{2}H_n|Z \rangle |^2 = m_Z^2 \) and \( \langle t|H_m + H_m^\dagger |t \rangle = m_t \), and the proportionality constant is derived accordingly. In this association, the simple real-field \( Z_\mu(x) \) nature justifies its use (similarly for each \( W_\mu^\pm(x) \)), as opposed to the complex \( W_\mu^\pm(x) \). Similarly, Eq. 50 is chosen over Eq. 51, as the latter adds the Higgs conjugate representation, unlike the SM. Thus, the vacuum expectation value reproduces the parameterization in Eq. 54, and identifies \( \chi_t, \chi_b \) as Yukawa parameters:

\[
\chi_t = m_t/\sqrt{2}, \quad \chi_b = m_b/\sqrt{2}.
\]  

The same argument can be made using the second scalar form in Eq. 51, as it also leads to Eq. 58. This results in

\[
-\mathcal{L}_{SF} = \text{tr} \sqrt{2} [\Psi_{tr}^\dagger(x) \mathbf{H}_{af}(x) \Psi_{qL}(x) + \Psi_{qL}^\dagger(x) \mathbf{H}_{af}(x) \Psi_{br}(x)] + \{hc\}.
\]  

(64)
Using Eq. 55, we obtain the relation for the $t$, $b$ quark masses

$$ (|a|^2 + |f|^2) v^2/2 = |m_t|^2 + |m_b|^2 = v^2/2. \quad (65) $$

The commutator arrangement in Eq. 57 is used in the above comparison; as it is set on the demand of a normalized scalar, the argument strengthens on the use of the same $Z$ operator acting on fermions in Eq. 38. The coefficient matching in $L_{SF}$ derives from the underlying freedom of choice in $L_{SV}$, and, in turn, from the underlying three-parameter $\tau_3$-$C$ symmetry that can be equally implemented in the spin basis. Looking at the matrix structure, the $\gamma_0$ operator within $H_m$ makes it a rank-2 reducible-representation operator, as expressed in Eq. 54 and can be read in Eq. 46: indeed, $H_m$ connects two fermion spin polarizations, but hits a single $W$ state’s components twice as $H_m$ duplicates the scalar representations, requiring the $\frac{1}{\sqrt{2}}$ normalization factor. In yet another interpretation, this relation is obtained from the normalization restriction in the Yukawa term in Eq. 55, dividing out the energy scale set by the vacuum expectation. To the extent that these arguments rely on a common metric vector space, they are geometric.

Equation 65 assumes the parity-conserving condition, constraining the quark masses.\footnote{8We neglect $t$-$b$ mixing as the CKM matrix is nearly diagonal\cite{4}, confirming this method can be applied here.} For maximal hierarchy\cite{7}, with $a$, $f$ dependence on one comparable large scale $O(a) \simeq O(f)$, ($m_b \ll m_t$) we get $\frac{1}{\sqrt{2}} v \simeq 173.95$, for $v = 246$ GeV, $m_b = 0$; alternatively, the quark-$b$ mass input predicts the top quark-mass as $m_t = \sqrt{v^2/2 - m_b^2} \simeq 173.90$ GeV, for $m_b = 4$ GeV (while renormalization effects give\cite{21} $m_b(m_t) \sim 2$ GeV.) These two calculations are consistent with the measured top pole mass\cite{4} $\tilde{m}_t = 173.21 \pm 0.51 \pm 0.71$ GeV, where systematic and statistic errors are quoted, respectively. Future precision improvements will test the limits of this tree-level calculation, with view of the bottom-quark influence.

\begin{itemize}
  \item \textbf{References:}
  \begin{itemize}
    \item \textbf{55} We neglect $t$-$b$ mixing as the CKM matrix is nearly diagonal\cite{4}, confirming this method can be applied here.
  \end{itemize}
\end{itemize}
6 Extended quark-mass relation

We place the heavy-quark mass relation in Eq. 65 in the larger SM context, and argue for a plausible generalization for all quarks, based on it. For these purposes, we first derive some SM field properties using the spin basis, assuming they can be also derived within the conventional SM basis, given their equivalent application. Needless to say, we demand consistency with the SM, and with experiment. At the Section’s end we identify some underlaying general assumptions.

Thus, we concentrate on the SM three-generation subset of the (7+1)-d model[9], as can be effected by the Yukawa operators in Eq. 34. Eq. 65 uses that the same single-scalar operator acts on the fermions and the vector bosons: such an operator is reproduced in the $SV$ and $SF$ terms, as the $SV$ term admits a basis expression that applies the associated $C$-symmetries in Section 4. This connection implies the equivalent expression that can be read from the Appendices,

$$L_{SV} = |\chi_t|^2 L_{SVu} + |\chi_b|^2 L_{SVd},$$

which shows separation of quark $i = u$- and $d$-type $L_{SVi}$ components, depending on scalars, and no mixing among them. We focus on the mass-generating scalar elements corresponding to the neutral $H^0_t$, $H^0_b$, from Eqs. 53, 54, and their hermitian conjugates. As mass relations are considered, we assume fields after the Higgs mechanism is applied.

In particular, a connection emerges between the normalized bilinear Higgs term that gives masses to the vector bosons, as the $Z$ mass in Eq. 58 and the fermions.

$$\frac{1}{2} \text{tr} H_m^\dagger H_m = 2\text{tr}[ (H_m T^\alpha_L T^\alpha_R \dagger)^\dagger H_m T^\alpha_L T^\alpha_R \dagger + (H_m B^\alpha_R B^\alpha_L )^\dagger H_m B^\alpha_R B^\alpha_L ]$$

$$= 2\text{tr}[ H_m^\dagger H_m T^\alpha_L T^\alpha_R \dagger T^\alpha_R T^\alpha_L \dagger + H_m^\dagger H_m B^\alpha_R B^\alpha_L B^\alpha_L B^\alpha_R ] = v^2 (\chi_t^2 + \chi_b^2),$$

where $H_m$ is defined in Eq. 61, $T^\alpha_R$, $T^\alpha_L$, $B^\alpha_R$, $B^\alpha_L$, are quarks at rest, defined in Table 4, $\chi_t$, $\chi_b$ are $H^0_t$, $H^0_b$ coefficients, as given in Eq. 54, in the second equality we use
the trace property, and the third expresses the $H_m$ normalization condition. Factor 2 comes as only one spin fermion component is used. Thus, $\mathcal{L}_{SV}$ elements can be written as a sum of inner products between Yukawa and SM scalar components. This relation derives from the projective nature of Higgs normal and dual terms, accompanied by a fermion chirality operator in $H_t(x), H_b(x)$ in Eq. 53.

Eq. 67 can also be understood from the substitutions in $trH_m^\dagger H_m$

$$H_t^0 \rightarrow H_t^0 T_L^\alpha T_R^\dagger = \frac{1}{\chi_t} y_{qt} H_t^0 T_L^\alpha Y^F T_R^\dagger$$

$$H_b^0 \rightarrow H_b^0 B_R^\alpha B_L^\dagger = \frac{1}{\chi_b} y_{bq} H_b^0 B_R^\alpha Y^F B_L^\dagger,$$

with terms extracted from $\mathcal{L}_{SF}$ in Eq. 60 using the trace permutation property. The identity in each substitution provides the link to the $t, b$ Yukawa constants for the $q = tb$ doublet, $t, b$ singlet cases. The arguments leading to the mass relation in Eq. 65 imply $y_{qt}^U = \chi_t, y_{bq}^D = \chi_b$, as given in Eq. 63, namely, a diagonal mass basis is assumed.

Since one can pick any fermion generation on Tables 2, 3, the interpretation of the $\chi_t, \chi_b$ coefficients as Yukawa constants within the $SV$ term leads to a generalization to other families and non-diagonal Yukawa elements. We now consider the extension of $\mathcal{L}_{SF}$ in Eqs. 60 and 64 with a fermion expansion that uses all Yukawa coefficients,

$$-\mathcal{L}_{SFT} = \text{tr} \left[ \sum_{iq} y_{iq}^U \Psi_{iR}(x)H_t(x)\Psi_{qL}(x)Y^F_{iq} + \sum_{jq} y_{jq}^D \Psi_{qR}(x)H_b(x)\Psi_{jL}(x)Y^F_{jq} \right] + \{hc\},$$

where the Yukawa operators $Y^F$ from Eq. 34 are necessary to connect the u- and d-type quark fields defined in Eq. 37 and $y_{iq}^U, y_{jq}^D$ are Yukawa coefficients, with the up, down, charm and strange quarks, also included, relabelling singlets $i = u, c, t$, $j = d, s, b$, and doublets $q = ud, cs, tb$.

The allowed Yukawa terms, diagonal and mixed, can be included using all combinations of a 3-generation set of normalized fermions on Table 2, where a projection operator as in Eq. 35 is applied. We evaluate the trace of bilinear $F_U = \frac{1}{\chi_t} y_{qt}^U U_{qL}^\alpha Y^F U_{tR}^\dagger$,
\[ F_D = \frac{1}{\chi_b} y_{jq}^D D^\alpha_{jR} Y^F_{jq} D^\alpha_{qL} \] terms with \( \mathcal{L}_{SV} \) components, extending Eq. 67, producing

\[
2 \text{tr}[(H_m F_U)^\dagger H_m F_U + (H_m F_D)^\dagger (H_m F_D)] = 1
\]

\[
= \frac{1}{2} v^2 (|y_{qi}|^2 \text{tr} H^0_t H^0_t + |y_{jq}|^2 \text{tr} H^0_b H^0_b)
\]

\[
= v^2 (|y_{qi}|^2 + |y_{jq}|^2),
\]

which may be also obtained by the substitution of the associated scalar coefficients in bilinear neutral Higgs terms \( \text{tr} H_m^\dagger H_m \)

\[
H_t(x) \rightarrow H^0_t F_U = H^0_t \frac{1}{\chi_b} y_{qi}^U U^\alpha_{qL} Y^F_{qi} U^\alpha_{iR} \]

\[
H_b(x) \rightarrow H^0_b F_D = H^0_b \frac{1}{\chi_b} y_{jq}^D D^\alpha_{jR} Y^F_{jq} D^\alpha_{qL} \].

The correspondence of \( \mathcal{L}_{SF} \) in Eqs. 60 and 64 to \( \mathcal{L}_{SFT} \) in Eq. 69 induces the sum of square mass-matrix elements in Eq. 70, which is equal (given the property \( \text{tr} M^\dagger M = \text{tr} M^\dagger M', M \) a matrix, \( M' \) its diagonal form) to the sum over the square masses,

\[
v^2 (\sum_{qi} |y_{qi}|^2 + \sum_{jq} |y_{jq}|^2) = 2(\sum_i m_i^2 + \sum_j m_j^2). \tag{72}
\]

A generalization with such a sum is induced, similar to relation Eq. 65 with the Higgs normalization condition, Eq. 55. Since Eq. 70 maintains the same structure as Eq. 67 following the generalization of \( \mathcal{L}_{SF} \) to \( \mathcal{L}_{SFT} \),

\[
m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2 = v^2 / 2. \tag{73}
\]

Implicitly, we used the \( SV \)-fermion symmetry, namely, no fermion preference. With today’s uncertainties in the quark-mass values, this relation is phenomenologically consistent with Eq. 63 as the same maximal hierarchy or quark b-mass input argument follows, and the rest of the quarks have comparably negligible masses. As this relation is independent of the mass diagonalization matrix, it is also of the CKM matrix[22].
The two quark-mass conditions in Eqs. 65 and 73 are interpreted. This paper shows SM features support a boson and fermion connection leading to the t,b quark mass condition in Eq. 65. If only such quarks belong in the same class as the other massive SM bosons, a different mass-generating mechanism is expected for the other fermions; one concludes that they are not affected by such dynamics, as their masses are comparably negligible. On the other hand, if there is a common dynamics, as suggested by the similar fermion-boson inner product, the all-quark condition Eq. 73 applies, given the fermion symmetry, and the structure similarity between Eq. 67 and Eq. 70.

Initial fermion states within the 3-generation set for $L_{SFT}$ in Eq. 69 remain within such a subspace, given the commuting property of the projection operator $Y^{F4}$ in Eq. 35 with baryon-number, Lorentz, gauge and mass operators ($B$, $B_{\sigma \mu \nu \ I}$, $Y_o$, $\phi_i$, $i = 1, 2$.) In other words, within the 3-generation subset of states, the substitution $Y_{ij}^F \rightarrow Y_{ij}^{F4}Y_{ij}^{F4}$ in $L_{SFT}$ is valid. This implies that no operator will connect the initial fermions outside the 3 generations. So is the case for the 3-generation extension of $L_{FV}$ in Eq. 38 requiring a sum over the (electroweak) flavors. We conclude the 3-generation spin-basis projection consistently describes the SM.

By construction, Eqs. 68, 71 imply masses represent $O(m_q/m_t)$ corrections. This is also the order of the Hamiltonian needed to obtain the other fermion masses. More assumptions are necessary to get further information on masses, and CKM matrix elements. For example, hierarchy arguments on the masses’ order of magnitude difference were derived[9] that explain how the associated W,Z,t,b, large scale mostly cancels for the other fermions at the vertical level (within a doublet) and horizontal level (between families). This leads to a consistent description in which such mechanisms coexist with the Higgs-generated one. While we produce above further consistency arguments for their parameters, more stringent constraints from the (7+1)-d will be tested elsewhere. Other arguments leading to hierarchy exist as textures[23].
We conclude Yukawa coefficients, contained in rest fermions as a device, connect to bilinear scalar combinations containing mass-generating Higgs terms in $\mathcal{L}_{SV}$, keeping the Lorentz or gauge structure of $SV$ unmodified, and ultimately consistently with the SM. We show above $\mathcal{L}_{SFT}$ in Eq. 69 induces a generalized sum rule for the square quark masses in Eq. 73. The latter is a plausible extension of Eq. 65 based on a subset of $\mathcal{L}_{SV}$ terms, after the Higgs mechanism. The same type of argument can be made for leptons, but given their smaller masses, their influence will be lesser, while similar conditions as in Eq. 67 will also lead to PNRS matrix\cite{24} independence.

7 Conclusions and Outlook

In summary, the formalism used places fields on a basis that simultaneously contains SM bosons and fermions. $SV$ and $SF$ terms are linked through the mass rendering of the scalar operator within them, using the electroweak $SV$ vertex independence of its components acting on different fermion-doublet elements, implicitly expected, but which we now expose. Supporting a SM prediction of a unique scalar, input from the normalized scalar-vector vertex, and the mass-parameter interpretation in the $SF$ vertex, relates $v$ and $m_t$, cf Eq. 65, the main result in the paper. The same relation can be argued by considering the scalar operator’s matrix rank, or assuming normalized Yukawa components. Based on chiral properties, the same Higgs-operator rule, and a correspondence between fermion-boson inner products and Yukawa terms, a plausible extended sum rule for the fermion square masses is proposed, given in Eq. 73. Both relations are consistent with the SM, given today’s particle-mass uncertainties. We conclude the spin basis is a useful platform to obtain, within the SM, the quark-mass electroweak relations.

The central argument input can be also read when $V$ terms in $\mathcal{L}_{FV}$, attached with the projector $L_5$ in Eq. 14 are carried into the intermediate $\mathcal{L}_{SV}$ chiral version in Eq.
and, after the 1/2 factor cancellation in its mass component, relate to $F$ terms in the Yukawa $\mathcal{L}_{SF}$. The spin-basis gives it further support as it classifies discrete degrees and produces SM features. Thus, the matrix space restricts representations, in turn, exhausting the space; electroweak $V$ fields belong to the adjoint, and $S, F$ fields to the fundamental representations. Additionally, the chiral property in the $FV$ electroweak term, associated to $V$, translates naturally to the $SV$ interaction components. Normalized fields define the Lagrangian terms, setting the trace coefficient, and the stage for the $\mathcal{L}_{SV}, \mathcal{L}_{SF}$ comparison. In the spin-basis context, the $S$ field’s chiral property is nominal, but consistent, as $\mathcal{L}_{SV}$ contains the $L_5$ projector from $V$, and within $\mathcal{L}_{SF}$, $S$ acts on chiral fermion components.

The scalar operator acting on vectors and fermions links their matrix elements, connecting parameters. The particles’ simultaneous participation in mass generation through the Higgs mechanism and related SM vertices, with assigned representations, implies a description with common dynamics, and at a given energy scale, already at the classical level, and suggests fields belong in a multiplet, supporting a common-origin unification assumption[7].

It follows that the arguments provide a geometric approach to address problems as the electroweak-symmetry breaking origin. The formalism facilitates the fields’ composite description, as boson degrees of freedom may be written in terms of two fermions’. Expansions in such fields may be useful, independently of whether compositeness is physical or only a device.

Naturalness is hinted at in the $\phi_1, \phi_2$ associated single scale, which produces a hierarchy effect[7]. Thus, while this symmetry-breaking effect applies for heavy-quark masses, it could be valid also horizontally between generations in accordance with the fermions’ low masses. While here we considered the top-quark mass, the other fermions, besides the b-quark, may be included in this scheme, namely, considering bilinear fermion components for scalar particles, but they will have little influence on
this result, as their $SF$ interaction is proportional to their masses.

As the spin basis connects the vector and quark sectors, constraints may be derived for SM extensions as supersymmetry\cite{25}, composite models that require dynamical symmetry breaking\cite{14} as technicolor\cite{26} or, in an extension of such models, top and bottom quarks\cite{27} that conform condensate-producing massive particles.

Besides the fields’ spin representation connecting the scalar operator in two vertices, it highlights chiral components of particles and interactions that maintain their SM equivalence. Indeed, we showed two such valid chiral and non-chiral scalar bases for the $SV$ Lagrangian. This freedom could be clarified in other vertices, as with a SM extension with additional scalar degrees of freedom, whereas in this paper, we considered only their SM projection.

References

[1] S. L. Glashow, Nucl. Phys. 22 (1961) 579-588; S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264-1266; A. Salam, in Elementary Particle Theory, W. Svartholm (Ed.), Almquist and Wiskell, Stockholm, 1968, pp. 367-377.

[2] F. Englert and R. Brout, Phys. Rev. Lett. 13 (1964) 321-323.

[3] P. W. Higgs, Phys. Lett. 12 (1964) 132-133.

[4] K.A. Olive et al. (Particle Data Group), Chin. Phys. C 38 (2014) 090001 1-1675.

[5] G. Aad et al., (ATLAS Collaboration) 2012f, Phys. Rev. Lett. 108 (2012) 111803-19; S. Chatrchyan et al. (CMS Collaboration) Phys. Rev. Lett. 108 (2012) 111804.

[6] J. Besprosvany, Int. J. Theor. Phys. 39 (2000) 2797-2836; J. Besprosvany, Nuc. Phys. B (Proc. Suppl.) 101, 323-329 (2001); J. Besprosvany and R. Romero, in:
[7] J. Besprosvany and R. Romero, Int. J. Mod. Phys. A 29 (2014) 1450144-17.

[8] J. Besprosvany, Int. J. Mod. Phys. A 20 (2005) 77-93.

[9] R. Romero and J. Besprosvany, “Quark horizontal flavor symmetry and two-Higgs doublet in (7+1)-dimensional extended spin space”, arXiv:1611.07446v1 [hep-ph].

[10] J. Besprosvany, Phys. Lett. B 578 (2004) 181-186;

[11] L. B. Landau, and E. M. Lifshitz, Quantum Mechanics, Pergamon, London, 1965.

[12] F. Iachello and I. Talmi, Rev. Mod. Phys. 59 (1987) 339-361.

[13] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. 108 (1957) 1175-1204.

[14] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345-358.

[15] M. Gell-Mann and Y. Ne’eman, The Eightfold Way, Benjamin, New York, 1964.

[16] J. Snygg, Clifford algebra: a computational tool for physicists, Oxford University Press, 1997.

[17] J. Besprosvany and R. Romero, Nucl. and Part. Phys. Proc., Volumes 267-269 (2015) 199-206.

[18] M. Weinstein, Phys. Rev. D 8 (1973) 2511-2524

[19] P. Sikivie et al., Nuc. Phys. B 173 (1980) 189-207.

[20] R. S. Chivukula, Lectures presented at 1997 Les Houches Summer School, hep-ph/9803219v2
Acknowledgements The authors acknowledge support from DGAPA-UNAM through project IN112916, and discussions with A. Ayala.

Appendix 1: Fermion-vector $\mathcal{L}_{FV}$ fermion-scalar $-\mathcal{L}_{SF}$

Lagrangians

In this Appendix we show the Lagrangians’ equivalence in the conventional and spin bases by considering explicit expressions with accompanying wave functions (or fields). With hindsight, we use the same Lagrangian label in both bases.

First, we use an iterative procedure[16] to obtain a (7+1)-d $\gamma^\mu$ representation. Starting with the Pauli matrices $\sigma^1$, $\sigma^2$ and $\sigma^3$, we get the (3 + 1)-d representation.
\[ \alpha^0 = \sigma^1 \otimes \sigma^3 \quad \alpha^1 = -i\sigma^2 \otimes \sigma^3, \]
\[ \alpha^2 = I_2 \otimes i\sigma^1 \quad \alpha^3 = I_2 \otimes i\sigma^2; \]  
(74)

then, the \((5 + 1)\)-d representation

\[ \beta^0 = \alpha^0 \otimes \sigma^3 \quad \beta^1 = \alpha^1 \otimes \sigma^3, \]
\[ \beta^2 = \alpha^2 \otimes \sigma^3 \quad \beta^3 = \alpha^3 \otimes \sigma^3, \]
\[ \beta^5 = I_4 \otimes i\sigma^1 \quad \beta^6 = I_4 \otimes i\sigma^2, \]  
(75)

and finally, the \((7 + 1)\)-d representation

\[ \gamma^0 = \beta^0 \otimes \sigma^3 \quad \gamma^1 = \beta^1 \otimes \sigma^3, \]
\[ \gamma^2 = \beta^2 \otimes \sigma^3 \quad \gamma^3 = \beta^3 \otimes \sigma^3, \]
\[ \gamma^5 = \beta^5 \otimes \sigma^3 \quad \gamma^6 = \beta^6 \otimes \sigma^3, \]
\[ \gamma^7 = I_8 \otimes i\sigma^1 \quad \gamma^8 = I_8 \otimes i\sigma^2. \]  
(76)

The commuting property of the Lorentz and scalar symmetry operators implies that they can be represented as a tensor product. To compare with the spin-space basis, we write the conventional-basis generators as tensor products, choosing the \((7 + 1)\)-d space to represent them; thus the spin-1/2 and SU(2)\(_L\) terms, expressed by the \(4 \times 4\) Clifford basis, and Pauli matrices, respectively, generalize to, e.g., \(\tau_3 \otimes 1_{s_{2 \times 2}} \sim I^3\) and \(1_{w_{2 \times 2}} \otimes [\frac{i}{2} \gamma_L \gamma_2]_{2 \times 2} \sim \frac{i}{2} \gamma_L \gamma_2\), with \(\tau_3\) the 3-Pauli matrix, and corresponding spin and weak isospin unit operators \(1_{s_{2 \times 2}}, 1_{w_{2 \times 2}}\), respectively.

Similarly, states in the conventional basis can be obtained that are represented in \((7 + 1)\)-d space. For example, a left-handed (L), spin-1/2 polarization (1), top (T), state \(|L1T\rangle\) satisfies \(\frac{1}{2}(1 - \tilde{\gamma}_5)|L1T\rangle = -|L1T\rangle, \quad \frac{i}{2} \gamma_L \gamma_2 |L1T\rangle = \frac{1}{2}|L1T\rangle, \quad I^3|L1T\rangle = \frac{1}{2}|L1T\rangle\).

Eq. 36 in the paper implies spinors are labeled by the \(4 \times 4\) spin operator in the Dirac representation \(\frac{i}{2} \gamma_1 \gamma_2\), and the weak SU(L)\(_L\) \(\tau_3\) component. While most of the results in the paper are representation-independent, a unitary transformation may be applied to the \((7 + 1)\)-d matrices to show the conventional-basis description used in Eq.
36 in the paper. Indeed, $\frac{i}{2}P_L \gamma_1 \gamma_2$ has a Dirac form with the unitary transformation
\[
\gamma'_D = U_D^\dagger \gamma_D U_D \quad \text{with} \quad U_D = \frac{1}{\sqrt{2}}(1 - \gamma^1 \gamma^3)\gamma^0 \gamma^2 \gamma^3 \quad \text{(actually, it exchanges $\gamma^1$ and $\gamma^3$.)}
\]
and $|L1T\rangle$ is represented, after the unitary transformation $U_D^\dagger$, by
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, i, 0).
\]
Next, we write all the conventional-basis states in this basis, and their association to the spin-extended basis states, with corresponding quantum numbers (notation used in Table 4 and Ref. 7, written inbetween):
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, i, 0) \leftrightarrow T^1_L, U^1_{L1} \quad (77)
\]
\[
(0, 0, -i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \leftrightarrow T^2_L, U^2_{L1} \quad (78)
\]
\[
(0, 0, 0, 0, 0, 0, -i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \leftrightarrow T^1_R, U^1_{R1} \quad (79)
\]
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, i, 0) \leftrightarrow T^2_R, U^2_{R1} \quad (80)
\]
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -i) \leftrightarrow B^1_L, D^1_{L1} \quad (81)
\]
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, i) \leftrightarrow B^2_L, D^2_{L1} \quad (82)
\]
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \leftrightarrow B^1_R, D^1_{R1} \quad (83)
\]
\[
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -i, 0, 0, 0, 0, 0) \leftrightarrow B^2_R, D^2_{R1} \quad (84)
\]
where the spin-basis states are shown in extenso in Tables 2 and 3.

For the fermion wave functions $\psi^\alpha_{qh}(x)$, we use polar coordinates, where the conventional and spin terms contain, respectively, $\psi^\alpha_{qh}(x) \exp[i p^\alpha_{qh}(x)] \leftrightarrow \psi^\alpha_{qh}(x) \exp[i c^\alpha_{qh}(x)]$, for quarks $q = t, b$, with spin components $\alpha = 1, 2$, and chirality $h = L, R$. The magnitude part can be shown to be the same for both cases, as can be derived by comparing, e. g., the mass term. The vectors $W^\alpha_\mu(x), B_\mu(x)$ are real fields.
The phases appear in each term in both bases. For example, for the conventional basis and for the two polarizations $t_L(x) = \begin{pmatrix} \psi^1_{tL}(x) \\ \psi^2_{tL}(x) \end{pmatrix}$ within the left-handed hypercharge $Y = 1/3 \text{SU}(2)_L$-doublet, we use the association $t_L(x) \rightarrow \Psi^1_{tL}(x) \exp [i p^1_{tL}(x)] U_D + \Psi^2_{tL}(x) \exp [i p^2_{tL}(x)] U_D$, with $U_D$ applied to transform back from the Dirac representation, and we used the terms in Eqs. (4) and (5) in this Appendix; for the spin basis, $\Psi_{tL}(x) = \Psi^1_{tL}(x) \exp [ic^1_{tL}(x)] T^1_L + \Psi^2_{tL}(x) \exp [ic^2_{tL}(x)] T^2_L$.

The Lagrangians' identity is shown, by checking that the same terms are reproduced in both bases, and finding independent constant phases that connect the two
representations. In the following, we present the fermion-vector $\mathcal{L}_{FV}$ Lagrangian components: interactive (weak and hypercharge), kinetic; also the fermion-scalar (Yukawa) $\mathcal{L}_{SF}$ Lagrangian. The subtitle contains the two-basis Lagrangian expressions in a concise notation, and then one component is given in an expanded form; the equations that link the phases in the two representations are written as they derive from the terms.

### 1 Weak. $\bar{q}_L(x) \frac{1}{2} g \gamma^a W^a_\mu(x) \gamma^\mu q_L(x) \leftrightarrow \text{tr} \{ \Psi_{qL}(x) \dagger g I_a W^a_\mu(x) \gamma^0 \gamma^\mu \Psi_{qL}(x) \}$

\[
\begin{align*}
\frac{g}{2} \left( (W^3_3(x) - W^3_0(x)) \psi^1_{bL}(x)^2 - 2 \left( \cos \left( p^1_{bL}(x) - p^1_{tL}(x) \right) W^1_0(x) \right. \right. \\
- \cos \left( p^1_{bL}(x) - p^1_{tL}(x) \right) W^1_3(x) + \sin \left( p^1_{bL}(x) - p^1_{tL}(x) \right) \left( W^2_0(x) - W^2_3(x) \right) \right) \psi^1_{tL}(x) \\
+ \sin \left( p^1_{bL}(x) - p^2_{bL}(x) \right) W^3_2(x) \\
- \cos \left( p^1_{bL}(x) - p^2_{bL}(x) \right) W^3_1(x) \psi^2_{bL}(x) \\
- \left( \cos \left( p^1_{bL}(x) - p^2_{tL}(x) \right) W^1_1(x) - \sin \left( p^1_{bL}(x) - p^2_{tL}(x) \right) W^1_2(x) \right) \\
+ \sin \left( p^1_{bL}(x) - p^2_{tL}(x) \right) W^2_1(x) + \cos \left( p^1_{bL}(x) - p^2_{tL}(x) \right) \psi^2_{tL}(x) \right) \psi^1_{bL}(x) \\ \\
+ W^3_0(x) \psi^1_{tL}(x)^2 - W^3_3(x) \psi^1_{tL}(x)^2 - \left( W^3_0(x) + W^3_3(x) \right) \psi^2_{bL}(x)^2 + W^3_0(x) \psi^2_{tL}(x)^2 \\
+ W^3_3(x) \psi^2_{tL}(x)^2 - 2 \cos \left( p^1_{tL}(x) - p^2_{tL}(x) \right) W^3_1(x) \psi^1_{tL}(x) \psi^2_{tL}(x) \\
+ 2 \sin \left( p^1_{tL}(x) - p^2_{tL}(x) \right) W^3_2(x) \psi^1_{tL}(x) \psi^2_{tL}(x) \\
- 2 \psi^2_{bL}(x) \left( \cos \left( p^2_{bL}(x) - p^2_{tL}(x) \right) W^1_0(x) + \cos \left( p^2_{bL}(x) - p^2_{tL}(x) \right) W^1_3(x) \right) \\
+ \sin \left( p^2_{bL}(x) - p^2_{tL}(x) \right) \left( W^2_0(x) + W^2_3(x) \right) \right) \psi^2_{tL}(x) \\
- \left( \cos \left( p^1_{tL}(x) - p^2_{bL}(x) \right) W^1_1(x) - \sin \left( p^1_{tL}(x) - p^2_{bL}(x) \right) W^1_2(x) \right) \\
- \sin \left( p^1_{tL}(x) - p^2_{bL}(x) \right) W^2_1(x) - \cos \left( p^1_{tL}(x) - p^2_{bL}(x) \right) W^2_2(x) \right) \psi^1_{tL}(x) \right).
\end{align*}
\]

(85)

Comparing the corresponding expression in the spin basis, we derive the following phase relations (which retroactively provide such an expression).

\[
c^1_{bL}(x) = p_{W2} + p^1_{bL}(x) + \frac{\pi}{2} \tag{86}
\]

\[
c^1_{tL}(x) = p_{W2} + p^1_{tL}(x) + \pi \tag{87}
\]

43
for arbitrary real constants $p_{W1}$, $p_{W2}$, $p_{W3}$, $p_{W4}$, $p_{Z1}$, $p_{Z2}$, requiring the identities

\[ p_{W1} = p_{W3} = p_{W4} = p_{W2}. \]

.2 Hypercharge. $\bar{q}_L(x)\gamma^\mu q_L(x) + \bar{t}_R(x)\gamma^\mu t_R(x) + \bar{b}_R(x)\gamma^\mu b_R(x) \leftrightarrow \text{tr}\{\Psi q_L(x)\gamma^\mu q_L(x) + \Psi q_L(x)\gamma^\mu q_L(x)\}

\[
\bar{q}_L(x)\frac{1}{6}g' B_\mu(x) \gamma^\mu q_L(x) + \bar{t}_R(x)\frac{2}{3}g' B_\mu(x) \gamma^\mu t_R(x) + \\
\bar{b}_R(x)[-\frac{1}{3}g' B_\mu(x) \gamma^\mu b_R(x) \leftrightarrow \text{tr}\{\Psi q_L(x)\gamma^\mu q_L(x) + \Psi q_L(x)\gamma^\mu q_L(x)\}
\]

\[
g' \left( (B_0(x) - B_3(x)) \Psi^1_{\mu L}(x)^2 - 2 \left( \cos (p^1_{\mu L}(x) - p^2_{\mu L}(x)) B_1(x) \right) - \sin (p^1_{\mu L}(x) - p^2_{\mu L}(x)) B_2(x) \right) \Psi^2_{\mu L}(x) - 2B_0(x) \Psi^1_{\mu R}(x)^2 - 2B_3(x) \Psi^1_{\mu L}(x)^2 + 4B_0(x) \Psi^2_{\mu R}(x)^2 + 4B_3(x) \Psi^2_{\mu L}(x)^2 + 2B_3(x) \Psi^2_{\mu R}(x)^2 + B_0(x) \Psi^2_{\mu L}(x)^2 + 4B_0(x) \Psi^2_{\mu R}(x)^2 - 4B_3(x) \Psi^2_{\mu L}(x)^2 + 4\cos (p^1_{\mu R}(x) - p^2_{\mu R}(x)) B_1(x) \Psi^1_{\mu R}(x) \Psi^2_{\mu R}(x) - 4\sin (p^1_{\mu R}(x) - p^2_{\mu R}(x)) B_2(x) \Psi^1_{\mu R}(x) \Psi^2_{\mu R}(x) - 2\cos (p^1_{\mu L}(x) - p^2_{\mu L}(x)) B_1(x) \Psi^1_{\mu L}(x) \Psi^2_{\mu L}(x) + 2\sin (p^1_{\mu L}(x) - p^2_{\mu L}(x)) B_2(x) \Psi^1_{\mu L}(x) \Psi^2_{\mu L}(x) - 8\cos (p^1_{\mu R}(x) - p^2_{\mu R}(x)) B_1(x) \Psi^1_{\mu R}(x) \Psi^2_{\mu R}(x) + 8\sin (p^1_{\mu R}(x) - p^2_{\mu R}(x)) B_2(x) \Psi^1_{\mu R}(x) \Psi^2_{\mu R}(x) \right)
\]

(94)
.3 Kinetic. \( \bar{q}_L(x)i_\frac{1}{2} \rightarrow \partial_\mu \gamma^\mu q_L(x) + \bar{f}_R(x)i_\frac{1}{2} \rightarrow \partial_\mu \gamma^\mu f_R(x) + \bar{b}_R(x)i_\frac{1}{2} \rightarrow \partial_\mu \gamma^\mu b_R(x) \)

\[
\gamma^\mu b_R(x) \leftrightarrow \text{tr}_{\frac{1}{2}} \{ \Psi_{q_L}(x)i_\partial_\mu \gamma^0 \gamma^\mu \Psi_{q_L}(x) + \Psi_{t_R}(x)i_\partial_\mu \gamma^0 \gamma^\mu \Psi_{t_R}(x) + \\
\Psi_{b_R}(x)i_\partial_\mu \gamma^0 \gamma^\mu \Psi_{b_R}(x) \}
\]

Using the fields’ integrability property (belonging to Hilbert space), integration by parts has been applied to make the derivative substitution \( i\partial_\mu \rightarrow i_\frac{1}{2} \rightarrow \partial_\mu \).
\[-\psi^2_{1L}(x)^2 \left( \partial_x p^2_{1L}(x) \right) + \psi^2_{1R}(x)^2 \left( \partial_x p^2_{1R}(x) \right) - 2\sin \left[ p^1_{bR}(x) \right] - p^2_{bR}(x) \right] \psi^1_{bR}(x) \psi^2_{bR}(x) \left( \partial_y p^1_{bR}(x) \right) - 2\sin \left[ p^1_{bR}(x) \right] - p^2_{bR}(x) \right] \psi^1_{bR}(x) \psi^2_{bR}(x) \left( \partial_y p^2_{bR}(x) \right) + 2\cos \left[ p^1_{bR}(x) - p^2_{bR}(x) \right] \psi^2_{bR}(x) \left( \partial_y \psi^1_{bR}(x) \right) + \cos \left[ p^1_{1L}(x) - p^2_{1L}(x) \right] \psi^2_{1L}(x) \left( \partial_y \psi^1_{1L}(x) \right) + \cos \left[ p^1_{1R}(x) - p^2_{1R}(x) \right] \psi^2_{1R}(x) \left( \partial_y \psi^1_{1R}(x) \right) - 2\cos \left[ p^1_{bR}(x) - p^2_{bR}(x) \right] \psi^1_{bR}(x) \left( \partial_y \psi^2_{bR}(x) \right) + 2\cos \left[ p^1_{bR}(x) \right] - p^2_{bR}(x) \right] \psi^1_{bR}(x) \psi^2_{bR}(x) \left( \partial_y p^1_{bR}(x) \right) + \psi^2_{1L}(x) \left( \cos \left[ p^1_{1L}(x) - p^2_{1L}(x) \right] \left( \partial_y \psi^2_{1L}(x) \right) + \psi^2_{1L}(x) \left( \sin \left[ p^1_{1L}(x) - p^2_{1L}(x) \right] \left( \partial_y p^1_{1L}(x) \right) - \cos \left[ p^1_{1L}(x) - p^2_{1L}(x) \right] \left( \partial_x \psi^1_{1L}(x) \right) + \sin \left[ p^1_{1L}(x) - p^2_{1L}(x) \right] \left( \partial_x \psi^2_{1L}(x) \right) \right) - \psi^1_{1R}(x) \left( \cos \left[ p^1_{1R}(x) - p^2_{1R}(x) \right] \left( \partial_y \psi^1_{1R}(x) \right) + \psi^2_{1R}(x) \left( \sin \left[ p^1_{1R}(x) - p^2_{1R}(x) \right] \left( \partial_y p^1_{1R}(x) \right) + \sin \left[ p^1_{1R}(x) - p^2_{1R}(x) \right] \left( \partial_x \psi^2_{1R}(x) \right) \right) - 2\psi^1_{bR}(x)^2 \left( \partial_y p^1_{bR}(x) \right) - \psi^1_{1L}(x)^2 - \left( \partial_x p^1_{1L}(x) \right) + \partial_x p^1_{1L}(x) - \psi^1_{1R}(x)^2 \left( \partial_x p^1_{1R}(x) + \partial_y p^1_{1R}(x) \right) - 2\psi^2_{bR}(x)^2 \left( \partial_y p^2_{bR}(x) \right) - \psi^2_{1L}(x)^2 \left( \partial_y p^2_{1L}(x) \right) - \psi^2_{2R}(x)^2 \left( \partial_y p^2_{2R}(x) \right) \right) \right).
.4 Yukawa. \[ \frac{\sqrt{2}}{v} [m_t \bar{t}_R(x) \tilde{H}^\dagger(x) q_L(x) + m_b \bar{q}_L(x) H(x) b_R(x)] + \{hc\} \leftrightarrow \text{tr} \frac{\sqrt{2}}{v} [m_t \bar{t}_{tR}(x) H_t(x) \Psi_{qL}(x) + m_b \bar{q}_{bL}(x) H_b(x) \Psi_{bR}(x)] + \{hc\} \]

The representation of scalars in the conventional and spin bases uses the association, e.g., \( H_\gamma^0_{4 \times 4} \rightarrow H_t \); the conventional phases, written explicitly in Appendix 2, are set to fit the spin basis, as both operators act equally on fermions, and we applied the gamma-matrix representation freedom of choice.

\[
\frac{1}{v} \left[ \cos \left( p_{W2} - p_{Z1} - p_{\eta 1}(x) - p_{1bR}(x) + p_{1tL}(x) \right) m_b \eta^r_1(x) \psi^1_{bR}(x) \psi^1_{tL}(x) \right. \\
+ \cos \left( p_{W2} - p_{Z2} - p_{\eta 0}(x) + p_{1tL}(x) - p_{1tR}(x) \right) m_t \eta^r_0(x) \psi^1_{tR}(x) \psi^1_{tL}(x) \\
+ \psi^1_{bL}(x) \left( \sin \left( p_{W2} - p_{Z2} + p_{\eta 1}(x) + p_{1bL}(x) - p_{1tR}(x) \right) m_t \eta^r_1(x) \psi^1_{tR}(x) \\
- \sin \left( p_{W2} - p_{Z1} + p_{\eta 0}(x) + p_{1bL}(x) - p_{1bR}(x) \right) m_b \eta^r_0(x) \psi^1_{bR}(x) \right) \\
- \cos \left( p_{W2} - p_{Z1} - p_{\eta 1}(x) - p_{2bR}(x) + p_{2tL}(x) \right) m_b \eta^r_1(x) \psi^2_{bR}(x) \psi^2_{tL}(x) \\
- \cos \left( p_{W2} - p_{Z2} - p_{\eta 0}(x) + p_{2tL}(x) - p_{2tR}(x) \right) m_t \eta^r_0(x) \psi^2_{tL}(x) \psi^2_{tR}(x) \\
+ \psi^2_{bL}(x) \left( \sin \left( p_{W2} - p_{Z1} + p_{\eta 0}(x) + p_{2bL}(x) - p_{2bR}(x) \right) m_b \eta^r_0(x) \psi^2_{bR}(x) \\
- \sin \left( p_{W2} - p_{Z2} + p_{\eta 1}(x) + p_{2bL}(x) - p_{2tR}(x) \right) m_t \eta^r_1(x) \psi^2_{tR}(x) \right],
\]

requiring the identities \( p_{Z1} - \pi = p_{Z2} - \frac{\pi}{2} = p_{W1} \).

Appendix 2: Scalar-vector Lagrangian \( \mathcal{L}_{SV} \); conjugate-Higgs invariance

For the scalar components, we also use expressions in polar coordinates, and in which the phase is written explicitly, to see its workings. Thus, for the conventional basis,

\[
\mathbf{H}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta^r_1(x) e^{ip_{1t} + ip_{\eta 1}(x)} \\ \eta^r_0(x) e^{ip_{1b} + ip_{\eta 0}(x)} \end{pmatrix} \]  \hspace{1cm} \text{(97)}
\[
\mathbf{\tilde{H}}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\eta^r_0(x) e^{ip_{b0} - ip_{\eta 0}(x)} \\ i\eta^r_1(x) e^{ip_{b1} - ip_{\eta 1}(x)} \end{pmatrix} \]  \hspace{1cm} \text{(98)}

where \( p^1_b, p^1_t, p^0_b, p^0_t \) are charged and neutral phases, respectively, and \( \bar{\mathbf{H}}_{\chi_t, \chi_b}(x) = (\chi_t \mathbf{H}(x), \chi_b \bar{\mathbf{H}}(x)) \) is a \( 4 \times 4 \) matrix, \( \chi_t, \chi_b \), can be assumed real and their dependence in all terms is through the factor \( \chi_t^2 + \chi_b^2 \), so their explicit form constitutes a likewise demonstration for \( \mathcal{L}_{SV} \).

For the spin basis, we use a generalized expression for the scalar term with conjugated terms weighted by a multiplicative parameter \( \lambda \), to keep track of terms, and with a normalization that makes \( \mathcal{L}_{SV} \) \( \lambda \)-independent:

\[
\mathbf{H}_{ab}^{tot\lambda}(x) = \frac{1}{\sqrt{1 + \lambda^2}} \left[ \chi_t \left[ \eta_1^r(x) e^{i\phi_1^+ + ip_0(x)} (\phi_1^+ + \phi_2^+) + \eta_0^r(x) e^{i\phi_0^+ + ip_0(x)} (\phi_0^+ + \phi_2^+) \right] \\
+ \chi_b \left[ \eta_1^r(x) e^{i\phi_1^- - ip_0(x)} (\phi_1^- - \phi_2^-)^\dagger + \eta_0^r(x) e^{i\phi_0^- - ip_0(x)} (\phi_0^- - \phi_2^-)^\dagger \right] \\
+ \lambda \chi_t \left[ \eta_1^r(x) e^{i\phi_1^+ - ip_0(x)} (\phi_1^+ - \phi_2^+) \right] \\
+ \lambda \chi_b \left[ \eta_1^r(x) e^{i\phi_1^- + ip_0(x)} (\phi_1^- + \phi_2^-)^\dagger \right] \right],
\]

(99)

where \( \phi_{1,2}^+, \phi_{1,2}^0 \) are defined in Table 1, and \( \phi_1^+, \phi_1^-, \phi_0^+, \phi_0^- \) are charged and neutral phases, respectively, and those with \( \lambda \) correspond to the hermitian-conjugate function (see Eqs. (50), (51) in the paper.) Given the chiral nature of the scalar components, they do not mix with their hermitian-conjugate components.

Thus, \( \mathcal{L}_{SV} = \mathbf{H}^\dagger(x) \mathbf{F}_\mu(x) \mathbf{F}_\mu(x) \mathbf{H}(x) \), with \( \mathbf{F}_\mu(x) = i\partial_\mu + \frac{1}{2} g \mathbf{g} \cdot \mathbf{W}_\mu(x) + \frac{1}{2} g' B_\mu(x) \), \( \mathbf{W}_\mu(x) = (W_\mu^1(x), W_\mu^2(x), W_\mu^3(x)) \) (cf. Eqs. (42) and (43)) is compared with

\[
\frac{1}{2} \text{tr} \left\{ [\mathbf{F}''(x), \mathbf{H}_{ab}^{tot\lambda}(x)]_\pm [\mathbf{F}''(x), \mathbf{H}_{ab}^{tot\lambda}(x)]_\pm \right\}_{\text{sym}},
\]

where \( \mathbf{F}''(x) = [i\partial_\mu + g W_\mu^i(x) I_i + \frac{1}{2} g' B_\mu(x) Y_\alpha] \gamma_\alpha \gamma^\mu \), the subindex \( \text{sym} \) means only symmetric \( \gamma_\mu \gamma_\nu \) components are taken.
\[ \text{.5 Square W.} \quad \text{tr} \bar{H}^\dagger(x) \frac{1}{2} g \tau \cdot W_\mu(x) \frac{1}{2} g \tau \cdot W^\mu(x) \bar{H}(x) \leftrightarrow \]
\[ \frac{1}{2} \text{tr} \{ [g W_0^n(x) I_n \gamma^0, H_{ab}^{\text{tot} \lambda}(x)]^\dagger [g W_0^m(x) I_m \gamma^0, H_{ab}^{\text{tot} \lambda}(x)] + \]
\[ \{ g W_j^n(x) I_n \gamma^j, H_{ab}^{\text{tot} \lambda}(x) \}^\dagger \{ g W_k^m(x) I_m \gamma^k, H_{ab}^{\text{tot} \lambda}(x) \} \}
\]
\[ \frac{1}{8} g^2 (\chi_i^2 + \chi_0^2) (\eta_0(x)^2 + \eta_1(x)^2) W_\mu^n(x) W^\mu(x) \quad (100) \]

\[ \text{.6 Square B.} \quad \text{tr} \bar{H}^\dagger(x) \frac{1}{2} g' B_\mu(x) \frac{1}{2} g' B^\mu(x) \bar{H}(x) \leftrightarrow \]
\[ \frac{1}{2} \text{tr} \{ [\frac{1}{2} g' B_0(x) Y_0 \gamma^0, H_{ab}^{\text{tot} \lambda}(x)]^\dagger [\frac{1}{2} g' B_0(x) Y_0 \gamma^0, H_{ab}^{\text{tot} \lambda}(x)] + \]
\[ \{ \frac{1}{2} g' B_j(x) Y_0 \gamma^j, H_{ab}^{\text{tot} \lambda}(x) \}^\dagger \{ \frac{1}{2} g' B_k(x) Y_0 \gamma^k, H_{ab}^{\text{tot} \lambda}(x) \} \}
\]
\[ \frac{1}{8} g'^2 (\chi_i^2 + \chi_0^2) (\eta_0(x)^2 + \eta_1(x)^2) B_\mu(x) B^\mu(x) \quad (101) \]
\[ \frac{1}{4} g g' \left( \chi^2 + \chi_b^2 \right) \left( -B_0(x) W^3_0(x) \eta^0_0(x)^2 + B_1(x) W^3_1(x) \eta^0_1(x)^2 \right. \\
+ B_2(x) W^3_2(x) \eta^0_1(x)^2 + B_3(x) W^3_3(x) \eta^0_0(x)^2 \right) \\
+ 2 \cos \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_0(x) W^1_0(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \sin \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_0(x) W^2_0(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \cos \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_1(x) W^1_1(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \sin \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_1(x) W^2_1(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \cos \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_2(x) W^1_2(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \sin \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_2(x) W^2_2(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \cos \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_3(x) W^1_3(x) \eta^0_0(x) \eta^1_1(x) \\
+ 2 \sin \left[ p^0_t - p^1_b - p_{q0}(x) + p_{q1}(x) \right] B_3(x) W^2_3(x) \eta^0_0(x) \eta^1_1(x) \\
+ B_0(x) W^3_0(x) \eta^1_1(x)^2 - B_1(x) W^3_1(x) \eta^1_1(x)^2 \\
- B_2(x) W^3_2(x) \eta^1_1(x)^2 - B_3(x) W^3_3(x) \eta^1_1(x)^2 \right). \]
.8 Cross W-derivative. \( \text{tr} \{ \bar{\mathbf{H}}(x) \frac{1}{2} g \tau \cdot \mathbf{W}_\mu(x) i \bar{\partial}^\mu \bar{\mathbf{H}}(x) - \)

\[
\bar{\mathbf{H}}(x) i \bar{\partial}^\mu \frac{1}{2} g \tau \cdot \mathbf{W}_\mu(x) \bar{\mathbf{H}}(x) \leftrightarrow \frac{1}{2} \text{tr} \left\{ \left[ -i \bar{\partial}_0 \gamma_0 \gamma^0, \mathbf{H}_{ab}^{\text{tot} \lambda}(x) \right] \hat{\dagger} [g W^n_0(x) I_n \gamma_0 \gamma^0, \mathbf{H}_{ab}^{\text{tot} \lambda}(x)] \right. \\
+ [g W^n_0(x) I_n \gamma_0 \gamma^0, \mathbf{H}_{ab}^{\text{tot} \lambda}(x)] \hat{\dagger} [i \partial_0 \gamma_0 \gamma^0, \mathbf{H}_{ab}^{\text{tot} \lambda}(x)] + \\
\left\{ -i \bar{\partial}_j \gamma_0 \gamma^j, \mathbf{H}_{ab}^{\text{tot} \lambda}(x) \right\} \hat{\dagger} \left\{ g W^n_k(x) I_n \gamma_0 \gamma^k, \mathbf{H}_{ab}^{\text{tot} \lambda}(x) \right\} + \\
\left\{ g W^n_j(x) I_n \gamma_0 \gamma^j, \mathbf{H}_{ab}^{\text{tot} \lambda}(x) \right\} \hat{\dagger} \left\{ i \partial_k \gamma_0 \gamma^k, \mathbf{H}_{ab}^{\text{tot} \lambda}(x) \right\} \right. 
\]

As for the kinetic term in \( \mathcal{L}_{FV} \), the fields’ integrability property leads to derivatives in the form \( i \frac{1}{2} \bar{\partial}_\mu \); similarly for the cross B-derivative and d’Alembert terms next.
\[
\frac{1}{2} g \left( \chi^2 + \chi_b^2 \right) \left( \eta''_1(x)^2 \right) \left( W^3_3(x) (\partial_x p_{q_1}(x)) + W^3_2(x) (\partial_y p_{q_1}(x)) \right) \\
+ W^3_1(x) (\partial_x p_{q_1}(x)) - W^3_0(x) (\partial_t p_{q_1}(x)) \\
- \eta''_1(x) \left( W^2_3(x) \left( \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] \eta''_0(x) (\partial_x p_{q_0}(x) + \partial_z p_{q_1}(x)) \right) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] \left( \partial_z \eta''_0(x) \right) \right) \\
+ W^1_3(x) \left( -\cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] \eta''_0(x) (\partial_z p_{q_0}(x) + \partial_z p_{q_1}(x)) \right) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] \left( \partial_z \eta''_0(x) \right) \right) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_2(x) \eta''_0(x) (\partial_y p_{q_0}(x)) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_2(x) \eta''_0(x) (\partial_y p_{q_1}(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_2(x) \eta''_0(x) (\partial_y p_{q_1}(x)) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_2(x) \eta''_0(x) (\partial_y p_{q_1}(x)) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_2(x) \left( \partial_y \eta''_0(x) \right) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_2(x) \left( \partial_y \eta''_0(x) \right) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_1(x) \eta''_0(x) (\partial_x p_{q_0}(x)) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_1(x) \eta''_0(x) (\partial_x p_{q_0}(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_1(x) \eta''_0(x) (\partial_x p_{q_1}(x)) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_1(x) \eta''_0(x) (\partial_x p_{q_1}(x)) \\
+ \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_1(x) \left( \partial_y \eta''_0(x) \right) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_1(x) \left( \partial_y \eta''_0(x) \right) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_0(x) \eta''_0(x) (\partial_x p_{q_0}(x)) \\
- \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) \eta''_0(x) (\partial_x p_{q_0}(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_0(x) \eta''_0(x) (\partial_x p_{q_1}(x)) \\
- \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) \left( \partial_x \eta''_0(x) \right) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) \left( \partial_x \eta''_0(x) \right) \\
+ \eta''_0(x) \left( -W^3_3(x) \eta''_0(x) (\partial_x p_{q_0}(x)) + \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^3_3(x) (\partial_x \eta''_0(x)) \right) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_3(x) (\partial_x \eta''_1(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_2(x) (\partial_x \eta''_1(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_2(x) (\partial_y \eta''_1(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_1(x) (\partial_y \eta''_1(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_1(x) (\partial_x \eta''_0(x)) \\
- \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_x \eta''_0(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_0(x) (\partial_y \eta''_0(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_y \eta''_0(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_0(x) (\partial_x \eta''_0(x)) \\
- \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_x \eta''_0(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_x \eta''_0(x)) \\
+ \eta''_0(x) \left( -W^3_3(x) \eta''_0(x) (\partial_x p_{q_0}(x)) + \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^3_3(x) (\partial_x \eta''_0(x)) \right) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_3(x) (\partial_x \eta''_1(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_2(x) (\partial_x \eta''_1(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_2(x) (\partial_y \eta''_1(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_1(x) (\partial_y \eta''_1(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_1(x) (\partial_x \eta''_0(x)) \\
- \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_x \eta''_0(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_0(x) (\partial_y \eta''_0(x)) \\
- \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_y \eta''_0(x)) \\
+ \cos \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^1_0(x) (\partial_x \eta''_0(x)) \\
- \sin \left[ p^0_b - p^1_b - p_{q_0}(x) + p_{q_1}(x) \right] W^2_0(x) (\partial_x \eta''_0(x)) \right) ,
\]

from which one derives the phase connections.
\[
\phi^1_\lambda = p^0_t - p^1_t + \phi^0_\lambda - \frac{\pi}{2}
\]  
(104)

\[
\phi^1_b = p^0_t - p^1_t + \phi^0_b - \frac{\pi}{2}
\]  
(105)

\[
\phi^1_t = p^0_b - p^1_b + \phi^0_t + \frac{\pi}{2}
\]  
(106)

\[
\phi^1_\lambda = p^0_b - p^1_b + \phi^0_\lambda + \frac{\pi}{2}.
\]  
(107)

.9 Cross B-derivative. \[\text{tr}\{\tilde{\mathbf{H}}^\dagger(x)\frac{1}{2}g' B_\mu(x) i \partial^\mu \tilde{\mathbf{H}}(x) - \tilde{\mathbf{H}}^\dagger(x) \frac{\chi^2}{2} g' B_\mu(x) \tilde{\mathbf{H}}(x) \} \leftrightarrow \frac{1}{2} \text{tr}\{[-i \partial_0 \gamma^0_0, \mathbf{H}^\text{tot}_a(x)]^\dagger[\frac{1}{2}g' B_0(x) Y_0 \gamma^0_0, \mathbf{H}^\text{tot}_a(x)]
\]

\[
+ \frac{1}{2}g' B_0(x) Y_0 \gamma^0_0, \mathbf{H}^\text{tot}_a(x)]^\dagger[i \partial_0 \gamma^0_0, \mathbf{H}^\text{tot}_a(x)] + \}
\]

\[
+ \left\{i \partial_0 \gamma^0_0, \mathbf{H}^\text{tot}_a(x)\right\}^\dagger\left\{\frac{1}{2}g' B_k(x) Y_0 \gamma^k, \mathbf{H}^\text{tot}_a(x)\right\} + \}
\]

\[
+ \n\}
\]

\[
\frac{1}{2} \left(\chi^2_b + \chi^2_t \right) g' \left(\left(B_3(x) \left(\partial_2 p_{\eta 0}(x)\right) + B_2(x) \left(\partial_3 p_{\eta 0}(x)\right)\right) \right)\eta^0_0(x)^2 + \left(B_3(x) \left(\partial_2 p_{\eta 0}(x)\right) + B_2(x) \left(\partial_3 p_{\eta 0}(x)\right)\right) \eta^0_0(x)^2 + \}
\]

\[
+ B_1(x) \left(\partial_2 p_{\eta 0}(x)\right) - B_0(x) \left(\partial_2 p_{\eta 0}(x)\right)) \eta^0_0(x)^2 + (B_3(x) \left(\partial_2 p_{\eta 1}(x)\right) + B_2(x) \left(\partial_3 p_{\eta 1}(x)\right) - B_0(x) \left(\partial_2 p_{\eta 1}(x)\right)) \eta^0_1(x)^2\right]\).  
\]  
(108)

In addition to the above equations, we derive

\[
p^0_t = -p^0_b + p^1_b + p^1_t.
\]  
(109)

As the similarity transformation phases in e.g. \(\mathbf{H}, \tilde{\mathbf{H}}\), this relation accounts for the sign change for complex conjugate components.
d’Alembert. \( \text{tr} \overleftrightarrow{\bar{H}}(x) \overleftrightarrow{\partial^\mu} \partial_\mu \bar{H}(x) \leftrightarrow \)

\[
- \frac{1}{2} \text{tr} \left\{ \left[ \overleftrightarrow{\partial_0} \gamma_0 \gamma^0, H_{ab}^{\text{tot}\lambda}(x) \right] \right\} \left[ \partial_0 \gamma_0 \gamma^0, H_{ab}^{\text{tot}\lambda}(x) \right] - \\
\left\{ \partial_j \gamma_0 \gamma^j, H_{ab}^{\text{tot}\lambda}(x) \right\} \left\{ \partial_k \gamma_0 \gamma^k, H_{ab}^{\text{tot}\lambda}(x) \right\}
\]

\[
- \frac{1}{2} \left( \chi_b^2 + \chi_i^2 \right) \left( (\partial_z \eta^r_0(x))^2 + \eta^r_0(x)^2 (\partial_z p_{\eta 0}(x))^2 + \eta^r_i(x)^2 (\partial_z p_{\eta 1}(x))^2 \\ + (\partial_z \eta^r_0(x))^2 + (\partial_y \eta^r_i(x))^2 + \eta^r_0(x)^2 (\partial_y p_{\eta 0}(x))^2 + \eta^r_i(x)^2 (\partial_y p_{\eta 1}(x))^2 \\ + \eta^r_0(x)^2 (\partial_x p_{\eta 0}(x))^2 + \eta^r_i(x)^2 (\partial_x p_{\eta 1}(x))^2 + (\partial_x \eta^r_0(x))^2 - (\partial_x \eta^r_i(x))^2 \\ - \eta^r_0(x)^2 (\partial_t p_{\eta 0}(x))^2 - \eta^r_i(x)^2 (\partial_t p_{\eta 1}(x))^2 - (\partial_t \eta^r_0(x))^2 \right).
\]

Each of the \( L_{SV} \) terms is indeed proportional to the combination \( \chi_b^2 + \chi_i^2 \), which manifests the t-b symmetry of this component, as the phases that connect the two representations were obtained.

We thus completed the demonstration of the SM Lagrangian terms’ equivalence in two bases; we conclude the spin-space representation reproduces the same properties of SM generators.