OPTIMAL CONTROL OF A DYNAMICAL SYSTEM WITH INTERMEDIATE PHASE CONSTRAINTS AND APPLICATIONS IN CASH MANAGEMENT

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Abstract. The aim of this work is to apply the results of R. Gabasov et al. [4, 14] to an extended class of optimal control problems in the Bolza form, with intermediate phase constraints and multivariate control. In this paper, the developed iterative numerical method avoids the discretization of the dynamical system. Indeed, by using a piecewise constant control, the problem is reduced for each iteration to a linear programming problem, this auxiliary task allows to improve the value of the quality criterion. The process is repeated until the optimal or the suboptimal control is obtained. As an application, we use this method to solve an extension of the deterministic optimal cash management model of S.P. Sethi [31, 32]. In this extension, we assume that the bank overdrafts and short selling of stock are allowed, but within the authorized time limit. The results of the numerical example show that the optimal decision for the firm depends closely on the intermediate moment, the optimal decision for the firm is to purchase until a certain date the stocks at their authorized maximum value in order to take advantage of the returns derived from stock. After that, it sales the stocks at their authorized maximum value in order to satisfy the constraint at the intermediate moment.

Introduction. Optimal control theory is an important area of applied mathematics, developed to find optimal way to control management systems, overcome the arduous tasks, predict and control future events and finally to optimize a certain criterion. This theory is applied in many fields of sciences, in particular engineering, physics, biology, economics and finance [24, 26, 30, 32]. Optimal control with intermediate phase constraints is often used to solve problems in which state constraints are imposed on intermediate times. On the other hand, these problems arise as auxiliary tasks for solving a nonlinear optimal control problem without phase constraints [21]. The optimal control problems in Bolza
form and intermediate phase constraints have important applications, in particular in financial economics.

Stochastic and deterministic optimal control models are the most important models in management systems, economics and finance [10, 24, 26, 31, 32]. The problem of cash management is one of the pioneering models on optimal control in finance and the central topic for research in microeconomics. This problem, in its simplest form, consists in formulating decision rules which control the level of a firm’s cash balance in order to satisfy its demands for cash at a minimum total cost, or equivalently to maximize the terminal value of its cash. We assume that the cash can be held in a bank account or invested in risky securities (stocks or bonds,...) [31, 32], we can assume another cash investment (certificate of deposits, Treasury bill, ... ). The first deterministic optimum cash management model has been developed by Baumol [11].

The concept of support, generalizing the basis in linear programming, has yielded the so-called support method developed by Gabasov et al. [19, 14]. The efficiency of this method has attracted attention of many researchers. In [1, 13] the authors have developed the necessary optimality conditions for optimal control problems with equality and inequality intermediate constraints. The paper [6] deals with a linear-quadratic optimal control problem with double terminal constraints on the trajectory and a scalar control. The set of controls in the solved problem [7] does not belong to a simple parallelepiped, but a polyhedron defined by equalities and box constraints. The authors in [16] consider the linear problem of optimization of the multidimensional nonstationary control systems with regard to the parallelepiped constraints. In [4], a numerical algorithm is constructed for solving a linear optimal control problem with phase constraints at intermediate times. In [21] the authors use the algorithm developed in [4] to construct open-loop and closed-loop solutions of linear-quadratic optimal control problems.

In this paper, we apply, in the first, the results of [3, 4, 6, 7, 14, 16, 21, 25] to an extended class of optimal control problems in Bolza form, with vector control and intermediate phase constraints. So we introduce the optimality and suboptimality criteria which allow to develop an algorithm for solving the considered problem. The suboptimality estimate permits to size up the difference between the current value and the optimum value of the cost functional. This iterative numerical method is based on the passage from an admissible control to another while improving the quality criterion.

Secondly, we present an extension of the deterministic optimal cash management model of S.P. Sethi [31, 32], in which we assume that the overdrafts on cash and short selling of stock are allowed, but within the authorized time limit. At the end, we give an illustrative example.

This paper is organized as follows: In section 1 we formulate the linear optimal control problem in Bolza form with vector control and intermediate phase constraints, and we introduce the concept of support which is a basic element of the method. In section 2 the suboptimality estimate and optimality criterion are formulated. In section 3 we develop an iterative numerical method for solving the considered problem. In section 4 we describe an extension of the deterministic optimal cash management model of Sethi and we present a numerical example from which we have deduced the optimal decision for the firm.
1. Statement of the problem and definitions. In the class of piecewise constant controls, we consider the following optimal control problem:

\[
\begin{align*}
\max J(u) & = c_1^t x(t^*) + \int_0^{t^*} c_2(t)u(t)dt, \\
\dot{x} & = Ax + Bu + R, \quad x(0) = x_0, \\
g_s(x(t)) & \leq H_s x(t), t^* \in T = [0, t^*], s \in S, \\
d^- & \leq u(t) \leq d^+, t \in T = [0, t^*],
\end{align*}
\]

where \( \dot{x} = \frac{dx}{dt} \), \( x(t) \in \mathbb{R}^n \) is the state vector of the dynamical system at instant \( t \) and \( x_0 \) the initial state, \( u(t) \in \mathbb{R}^r \) is a vector control; \( A = A(K, K), B = B(K, J) \) and \( H_s = H_s(I(t_s), K) \) are respectively \( n \times n, n \times r \) and \( m_s \times n \)-matrices; \( g_s(x) = g_s(I(t_s)), g_s = g_s(I(t_s)) \) are \( m_s \)-vectors and \( d^- = d^-(J), d^+ = d^+(J) \) are \( r \)-vectors, with \( K = \{1, \ldots, n\}, J = \{1, \ldots, r\}, I(t_s) = \{1, \ldots, m_s\}, s \in S = \{1, \ldots, m\} \); \( c_1 \) and \( c_2 \) are vectors of corresponding dimension. The symbol \( (\cdot) \) represents the transposition operation.

Using the Cauchy formula

\[
x(t) = F(t)[x_0 + \int_0^t F^{-1}(\tau)Bu(\tau) + R(\tau)d\tau], t \in T, F(t) = \exp(At),
\]

we can write the dynamic optimization problem (1)-(4) in the following equivalent form:

\[
\begin{cases}
\max J(u) = c_1^t F(t^*)x_0 + \int_0^{t^*} c_1^t(t)u(t)dt + \int_0^{t^*} c_2^t(t)R(t)dt, \\
\notag \varphi(I(t_s), t)u(t)dt \leq \varphi(I(t_s), t)R(t)dt, \\
d^- \leq u(t) \leq d^+, t \in T = [0, t^*],
\end{cases}
\]

where \( c(t) = c_1^t F(t^*)F^{-1}(t)B + c_2, c_3 = c_1^t F(t^*)F^{-1}(t), \varphi(I(t_s), t)u(t)dt = \varphi(I(t_s), t)R(t)dt, \) \( \varphi(I(t_s), t) = \begin{cases} H_s F(t_s)F^{-1}(t)B, \quad \text{if } 0 \leq t \leq t_s, \\
0, \quad \text{if } t > t_s, \end{cases} \)

with

\[
\varphi(I(t_s), t) = \varphi(t_s, t) = (\varphi_{ij}(t_s, t), i \in I(t_s), j \in J) = (\varphi_{ij}(t_s, t), j \in J), s \in S,
\]

and we set

\[
\varphi(t) = \begin{pmatrix}
\varphi(I(t_1), t) \\
\varphi(I(t_2), t) \\
\vdots \\
\varphi(I(t_m), t)
\end{pmatrix}.
\]

In the adaptive method, the principal tool is the support which is directly connected to a nonsingular matrix. In order to define it, we get up a subset \( I_{\text{sup}}(t_s) \) in the set \( I(t_s) \); we form the set \( I_{\text{sup}} = \{ I_{\text{sup}}(t_s), s \in S \} \), with \( |I_{\text{sup}}| = p \leq \sum_{s \in S} m_s \). On the interval \( T \), let us choose a subset of isolated moments: \( T_{\text{sup}} = \{ t_k, k \in K^* \} \), with \( K^* = \{1, \ldots, k^*\}, k^* \leq p \).
Hence, the increment formula of the functional in (6) is expressed as follows:

\[
P_{\text{sup}} = P(Q_{\text{sup}}) = (P_{ij}(t_s, t_k), i \in I_{\text{sup}}(t_s), s \in S, j \in J_k, k \in K^*). \quad (9)
\]

**Definition 1.1.** A piecewise constant control \( u(t) \), \( t \in T \), is said to be an admissible control if it satisfies the constraints (3) and (4). An admissible control \( u^0(t) \), \( t \in T \), is called an optimal control if \( J(u^0) = \max J(u) \). The corresponding trajectory \( x^0(t) \) is said to be the optimal trajectory. Moreover, the admissible control \( u'(t) \), \( t \in T \), is called suboptimal (or \( \epsilon \)-optimal) if the inequality holds

\[
J(u^0) - J(u') \leq \epsilon,
\]

where \( u^0 \) is an optimal control of the problem and \( \epsilon \geq 0 \) is a given accuracy.

**Definition 1.2.** The set \( Q_{\text{sup}} = \{ I_{\text{sup}}, J_{\text{sup}}, T_{\text{sup}} \} \) is called a support of the problem (6) if \( \det P_{\text{sup}} \neq 0 \). The pair \( \{ u, Q_{\text{sup}} \} \) formed by an admissible control \( u \) and a support \( Q_{\text{sup}} \) is called a support control.

**Definition 1.3.** The support control \( \{ u, Q_{\text{sup}} \} \) is said to be nondegenerate if

1. For each \( t_k \) of \( T_{\text{sup}} \) and for each index \( j \in J_k, k \in K^* \), one of these conditions is verified:
   - in the neighborhood of \( t_k \), the control \( u_j(t) \) is noncritical:
     \[ d_j^- < u_j(t) < d_j^+, \quad t \in [t_k - \delta, t_k + \delta], \quad \delta > 0, \]
   - or the control \( u_j(t) \), \( t \in T \), is discontinuous at \( t_k \);
2. Moreover, the following strict inequalities hold:
   \[ g_{s(i)}(s) < H_t(i, K)x(t_s) < g_{s(i)}^*, \quad \forall i \in I_{ns}(t_s) = I(t_s) \setminus I_{sup}(t_s), s \in S. \quad (10) \]

2. Optimality criterion.

2.1. **Increment formula of the quality criterion.** Let \( \{ u, Q_{\text{sup}} \} \) be a support control of the problem (1)-(4) and consider another arbitrary admissible control \( \pi(t) = u(t) + \Delta u(t) \) and its corresponding trajectory \( \pi(t) = x(t) + \Delta x(t), \quad t \in T \). Hence, the increment formula of the functional in (6) is expressed as follows:

\[
\Delta J(u) = J(\pi) - J(u) = c_j' F(t^*) x_0 + \int_0^{t^*} (c_j'(t) \pi(t) + c_j'(t) R(t)) dt - c_j' F(t^*) x_0 - \int_0^{t^*} (c_j'(t) u(t) + c_j'(t) R(t)) dt = \int_0^{t^*} c_j'(t) \Delta u(t) dt. \quad (11)
\]

We define

\[
c_{\text{sup}} = (c_j(t_k), j \in J_k, k \in K^*),
\]

where \( c_j(t), j \in J, \) is the \( j^{th} \) component of the vector \( c(t) \), and we construct the discrete potential function

\[ y = (y(s), s \in S), \quad y(s) = (y(s)i, i \in I(t_s)) = (y(s)(I_{sup}(t_s)), y(s)(I_{ns}(t_s))), \quad s \in S, \]

\[ J^* = \sum_{s \in S} J^*(s), \quad J^*(s) = J^*(s)(I_{ns}(t_s)), \quad s \in S, \]

\[ N_0 = N_{\text{sup}} + N_{\text{ns}}.
\]
with \( y = (y(I_{\text{sup}}), y(I_{\text{ns}})) \) and
\[
\begin{align*}
  y(I_{\text{sup}})' &= (y(s)_i, i \in I_{\text{sup}}(t_s), s \in S) = c_{\text{sup}}' \varphi_{\text{sup}}^{-1}, \\
  y(I_{\text{ns}})' &= (y(s)_i, i \in I_{\text{ns}}(t_s), s \in S) = 0.
\end{align*}
\]
(12)

Define the cocontrol
\[
E'(t) = \sum_{s \in S} y'_{(s)} \varphi(I(t_s), t) - c'(t), t \in T.
\]
(13)

Then we can write
\[
E'(t) = \sum_{s \in S} y'_{(s)} H(s) F(t_s) F^{-1}(t) B - (c_1' F(t^*) F^{-1}(t) B + c_2'(t)).
\]
(14)

Using the relations (13), the increment of the functional becomes:
\[
\Delta J(u) = J(\bar{u}) - J(u)
\]
\[
= \int_0^{t^*} c'(t) \Delta u(t) dt
\]
\[
= \int_0^{t^*} \left( \sum_{s \in S} y'_{(s)} \varphi(t_s, t) - E'(t) \right) \Delta u(t) dt
\]
\[
= \sum_{s \in S} y'_s \int_0^{t^*} \varphi(t_s, t) \Delta u(t) dt - \int_0^{t^*} E'(t) \Delta u(t) dt
\]
\[
= \sum_{s \in S} y'_{(s)} H(s) \Delta x(t_s) - \int_0^{t^*} E'(t) \Delta u(t) dt.
\]
(15)

We set \( H(s) \Delta x(t_s) = v_{(s)} \) and by virtue of (12), the increment of the functional takes the following final form:
\[
\Delta J(u) = \sum_{s \in S} \sum_{i \in I_{\text{sup}}} y_{(s)}(i) v_{(s)i} - \int_0^{t^*} E'(t) \Delta u(t) dt.
\]
(16)

Therefore, it is clear that the maximum of this increment of the functional under the constraints:
\[
\begin{align*}
  &\left\{ g_{s}(i) - H(s)(i, K)x(t_s) \leq v_{(s)i} \leq g_{s}(i) - H(s)(i, K)x(t_s), i \in I_{\text{sup}}(t_s), s \in S; \\
  &d^- - u(t) \leq \Delta u(t) \leq d^+ - u(t), t \in T,
\end{align*}
\]
is equal to
\[
\beta(u, Q_{\text{sup}}) = \sum_{j=1}^{r} \left[ \int_{T_j^+} E_j(t)(u_j(t) - d_j^-) dt + \int_{T_j^-} E_j(t)(u_j(t) - d_j^+) dt \right]
\]
\[
+ \sum_{s \in S} \sum_{y_{(s)i} < 0, i \in I_{\text{sup}}(t_s)} y_{(s)i} v_{(s)i}^- + \sum_{y_{(s)i} > 0, i \in I_{\text{sup}}(t_s)} y_{(s)i} v_{(s)i}^+.
\]
(17)

where
\[
T_j^+ = \{ t \in T : E_j(t) > 0 \}, \ T_j^- = \{ t \in T : E_j(t) < 0 \}, \ j \in J,
\]
and
\[
\begin{align*}
  &v^-_{(I(t_s))} = (v^-_{(s)_i}, i \in I(t_s)) = g_{s}(s) - H(s)x(t_s), s \in S, \\
  &v^+_{(I(t_s))} = (v^+_{(s)_i}, i \in I(t_s)) = g_{s}(s) - H(s)x(t_s), s \in S.
\end{align*}
\]
The number \( \beta(u, Q_{sup}) \) is called the suboptimality estimate of the support control \( \{u, Q_{sup}\} \) and we have always the following inequality:

\[
\triangle J(u) = J(u^0) - J(u) \leq \beta(u, Q_{sup}).
\]

Therefore, if \( \beta(u, Q_{sup}) \leq \epsilon \), then the control \( u \) is an \( \epsilon \)-optimal one.

2.2. Optimality criterion.

**Theorem 2.1. Optimality criterion**\([3, 4, 5, 14]\)

Let \( \{u, Q_{sup}\} \) be a support control of the problem (1)-(4). The following relations

\[
\begin{align*}
E_j(t) &\geq 0, \text{ if } u_j(t) = d_j^-, \\
E_j(t) &\leq 0, \text{ if } u_j(t) = d_j^+, \\
E_j(t) &= 0, \text{ if } d_j^- < u_j(t) < d_j^+, \quad t \in T, \quad j \in J;
\end{align*}
\]

are sufficient, and in the case of nondegeneracy also necessary, for the optimality of the support control \( \{u, Q_{sup}\} \).

3. Construction of the algorithm. In this section we develop an iterative numerical method which avoids the discretization of the dynamical system. For this, let \( \{u, Q_{sup}\} \) be an initial support control with \( \beta(u, Q_{sup}) > \epsilon, \epsilon \geq 0 \). The aim of the algorithm is to construct a suboptimal control \( u^\prime \) or an optimal control \( u^0 \). An iteration of the algorithm consists in moving from \( \{u, Q_{sup}\} \) to another support control \( \{\overline{u}, \overline{Q}_{sup}\} \) such that \( J(\overline{u}) \geq J(u) \).

The developed algorithm has three procedures:

- The control transformation \( u \rightarrow \overline{u} \): by using a piecewise constant control, the problem is reduced for each iteration to a linear programming problem, this auxiliary task allows to construct a new support control such that \( J(\overline{u}) \geq J(u) \);
- The support transformation \( Q_{sup} \rightarrow \overline{Q}_{sup} \): this transformation is used to obtain via a dual control method a new support which yields a better suboptimality estimate, i.e, \( \beta(\overline{u}, \overline{Q}_{sup}) \leq \beta(u, Q_{sup}) \);
- The finishing procedure consists in determining an optimal support so that the corresponding quasi-control will be admissible and optimal.

3.1. Control transformation. Let \( \epsilon > 0 \) be a given number and \( \{u, Q_{sup}\} \) a support control verifying \( \beta(u, Q_{sup}) > \epsilon \). We construct another admissible control \( \overline{u}(t) = u(t) + \theta \triangle u(t), t \in T \), such that \( J(\overline{u}) \geq J(u) \), where \( \triangle u(t) \) is an ascent direction and \( \theta \geq 0 \) is the step along this direction. For this, let \( \alpha > 0 \) and \( h > 0 \), be the parameters of the algorithm and we construct the sets:

\[
T_\alpha = \{t \in T : \eta(t) \leq \alpha\}, \quad T_\alpha = T \setminus T_\alpha, \quad \text{with } \eta(t) = \min_{j \in J} |E_j(t)|, \quad t \in T.
\]

We subdivide \( T_\alpha \) into intervals \([\tau_k, \tau^k], k = \frac{1}{N}, \tau_k < \tau^k \leq \tau_{k+1}, T_\alpha = \bigcup_{k=1}^N [\tau_k, \tau^k], \) so that \( \tau^k - \tau_k \leq h; u_j(t) = u_{jk} = \text{const}, t \in [\tau_k, \tau^k], k = \frac{1}{N}, \tau_k \in J. \)
Then we compute the following quantities
\[
\beta_{jk} = -\int_{\tau_k}^{\tau_{k+1}} E_j(t) dt, \quad q(s)_{jk} = \int_{\tau_k}^{\tau_{k+1}} \varphi_j(t), t) dt, \quad k = 1, N, j \in J, s \in S; \tag{20}
\]
\[
\beta_{N+1} = -\sum_{j=1}^{r} \int_{T_s}^{T_{s+1}} E_j(t) \Delta u_j(t) dt + \sum_{i \in I_{sup}(t_s), s \in S} y(s_i) v(s_i); \tag{21}
\]
\[
q(s)_{i(N+1)} = \sum_{j=1}^{r} \int_{T_s}^{T_{s+1}} \varphi_j(t, s) t) \Delta u_j(t) - v(s_i), \quad i \in I_{sup}(t_s), s \in S; \tag{22}
\]
\[
q(s)_{i(N+1)} = \sum_{j=1}^{r} \int_{T_s}^{T_{s+1}} \varphi_j(t, s) t) \Delta u_j(t) dt, \quad i \in I_{ns}(t_s), s \in S, \tag{23}
\]
where
\[
\overline{v}_{(s_i)} = \begin{cases} \frac{g_s(i) - H_s(i, K)x(t_s)}{v}, & \text{if } y(s_i) > 0, \quad i \in I_{sup}(t_s), \quad s \in S, \\
\frac{g_s(i) - H_s(i, K)x(t_s)}{v}, & \text{if } y(s_i) < 0, \quad i \in I_{sup}(t_s), \quad s \in S,
\end{cases} \tag{24}
\]
and
\[
\Delta u_j(t) = \begin{cases} d^+_j - u_j(t), & \text{if } E_j(t) < -\alpha, \\
d^-_j - u_j(t), & \text{if } E_j(t) > \alpha, \quad t \in T_s, \quad j = 1, r. \tag{25}
\end{cases}
\]
We set
\[
f_s(s) = (f_s(I_{sup}), f_s(I_{ns})), \quad f^*_s = (f^*_s(I_{sup}), f^*_s(I_{ns})),
\]
with
\[
f_s(I_{ns}(t_s)) = g_s(I_{ns}(t_s)) - H_s(I_{ns}(t_s), K)x(t_s), \quad f^*(I_{ns}(t_s)) = g^*(I_{ns}(t_s)) - H_s(I_{ns}(t_s), K)x(t_s), \quad s \in S,
\]
\[
f_s(I_{sup}) = f^*_s(I_{sup}) = 0;
\]
\[
l = (l_1, ..., l_{1N}, ..., l_{r1}, ..., l_{rN}, l_{N+1})', \tag{26}
\]
\[
\beta = (\beta_1, ..., \beta_{1N}, ..., \beta_{r1}, ..., \beta_{rN}, \beta_{N+1})', \tag{27}
\]
where \(l\) and \(\beta\) are \((Nr + 1)\)-vectors. Using these quantities, we formulate the following support linear problem:
\[
\max \beta' l, \tag{28a}
\]
\[
f_s(s) \leq \sum_{j=1}^{r} \sum_{k=1}^{N} q(s)_{jk} l_{jk} + q(s)_{N+1} l_{N+1} \leq f^*_s(s), \quad s \in S, \tag{28b}
\]
\[
d^-_j - u_j(t) \leq l_{jk} \leq d^+_j - u_j(t), \quad j = 1, r, \quad k = 1, N, \quad 0 \leq l_{N+1} \leq 1. \tag{28c}
\]
We solve the linear programming problem (28) with the adaptive method, presented in [8, 18].

Let \(l^0 = 0\) be an initial feasible solution of the support problem (28). After a number of iterations, we obtain an \(\epsilon\)-optimal solution \(l^\epsilon\).

Thus, the new admissible control \(\overline{u}\) of the problem (1)-(4) is expressed as follows:
\[
\overline{u}_j(t) = \begin{cases} u_j(t) + \overline{v}_{jk}, & t \in [\tau_k, \tau^k], \quad k = 1, N, \\
u_j(t) + \overline{v}_{N+1} \Delta u_j(t), & t \in T_s, \quad j = 1, r. \tag{29}
\end{cases}
\]

So the new control (29) verifies the inequality \(J(\overline{u}) \geq J(u)\). Let us calculate the new suboptimality estimate \(\beta(\overline{u}, Q_{sup})\). If \(\beta(\overline{u}, Q_{sup}) \leq \epsilon\), then \(u\) is an \(\epsilon\)-optimal
control for the problem (1)-(4). Otherwise we perform either a new iteration with a support control \(\{\pi, Q_{sup}\}\) and parameters \(\alpha < \alpha, \bar{h} < h\), or we do the change of the support.

3.2. Change of support. Let \(\{\pi, Q_{sup}\}\) be the support control found after solving the problem (28). Using the formula (12)-(13) we calculate the cocontrol \(E(t)\) corresponding to \(\{\pi, Q_{sup}\}\). After that, we construct the quasi-control:

\[
\begin{aligned}
w_j(t) &= \begin{cases}
\frac{d^-_j}{d^-_j} & \text{if } E_j(t) > 0, \\
\frac{d^+_j}{d^+_j} & \text{if } E_j(t) < 0, \\
\in [d^-_j, d^+_j] & \text{if } E_j(t) = 0, j = 1, \ldots, r, t \in T,
\end{cases}
\end{aligned}
\]

and the corresponding quasi-trajectory \(\chi(t)\), \(t \in T\):

\[
\chi(t) = A\chi(t) + Bw(t) + R(t); \chi(0) = x_0.
\]

Then compute:

\[
\gamma(J_{\pi}, T_{\pi}) = \varphi_{\pi}^{-1}(g^*_{(s)}(I_{sup}(t_s))) - H_{(s)}(I_{sup}(t_s), K)\chi(t_s), s \in S),
\]

\[
\gamma^*_{(s)}(I_{ns}(t_s)) = (\gamma^*_{(s)i}, i \in I_{ns}(t_s) = I(t_s)\setminus I_{sup}(t_s)),
\]

where

\[
g^*_{(s)}(s) = \begin{cases}
g_{s}^{i}, & \text{if } y_i(s) < 0, \\
g_{s}^{i}, & \text{if } y_i(s) \geq 0,
\end{cases}
\]

and

\[
\begin{aligned}
\gamma^*_{(s)i} &= \sum_{j \in J, k \in K^*} \varphi_{ij}(t_s, t_k) \gamma(j, t_k) + H_{(s)}(i, K)\chi(t_s) - g^*_{(s)i}, \\
\gamma_{(s)i} &= \sum_{j \in J, k \in K^*} \varphi_{ij}(t_s, t_k) \gamma(j, t_k) + H_{(s)}(i, K)\chi(t_s) - g_{(s)i}.
\end{aligned}
\]

We introduce an enough small parameter \(\mu > 0\). Thus two cases can occur:

- if the following relations

\[
\|\gamma(J_{\pi}, T_{\pi})\| \leq \mu,
\]

\[
\gamma^*_{(s)}(I_{ns}(t_s)) \geq 0, \gamma_{(s)}(I_{ns}(t_s)) \leq 0, s \in S,
\]

are verified, then we perform the finishing procedure.

- Otherwise, we will change the support \((Q_{sup} \rightarrow Q_{sup}')\) with an iteration of the dual method [4, 14, 25]. Then we perform a new iteration with \(\{u, Q_{sup}\} := \{u, Q_{sup}'\}\).
3.3. Finishing procedure. Assume that the relations (36) and (37) are verified for a quasi-control \( w(t) \), \( t \in T \), and its quasi-trajectory \( \chi(t) \), \( t \in T \), constructed by the support \( Q^\sup \). The finishing procedure consists in finding an optimal support \( Q^\sup_* = \{I^\sup_*, J^\sup_*, T^\sup_*\} \) and the potential vector \( y^* \) in order to have \( g_{s(s)}(i, K) \leq H(s) \chi^*(t_s) \leq g_{s(\phi)} \), \( s \in S \), where \( \chi^*(t) \) is the corresponding quasi-trajectory. Thus, The support \( Q^\sup_* \) is determined by solving the following equations:

\[
\begin{align*}
&\sum_{j \in J_k} \sum_{k \in K^*} (d_j^+ - d_j^-) \text{sign} \frac{d}{dt} (t_k) \int_{t_k}^{V_k(T^\sup_*)} \varphi_{ij}(t_s, t) dt - g_{s(s)}^* \chi^*(t_s) + \\
&H(s)(i, K) \chi^*(t_s) = 0, \ i \in I^\sup_*(t_s), s \in S, \\
&E_j(V_k(T^\sup_*), T^\sup_*) = 0, V_k(T^\sup_*) = t_k, \ j \in J_k, \ k \in K^*; \\
&E(t, T^\sup_*) = \sum_{s \in S} \chi^*(s) \varphi(t_s, t) - c(t).
\end{align*}
\]

We solve the equations (38) for the non-degenerate case:

\[
\frac{dE_j(t_k)}{dt}(t_k) \neq 0, \ j \in J_k, \ k \in K^*.
\]

Assume \( Q^\sup_* \) is a \( n \)th approximation, and \( Q^\sup_{n+1} \) an initial approximation, with \( I^\sup_n = I^\sup_0, \ j^\sup_0 = J^\sup_0 \) and \( T^\sup_n = T^\sup_0, \ J^\sup_0 = J^\sup_k = K^* \). Then the \((n+1)^{th}\) approximation is constructed as follows:

\[
T^\sup_{n+1} = T^\sup_n + \left\{ \frac{1}{d_j^+ - d_j^-} \text{sign} \frac{d}{dt} (t_k) \gamma_{ij}(t_k), j \in J_k, k \in K^* \right\}. \quad (39)
\]

Moreover, the \((n+1)^{th}\) approximation will be constructed in such a way that the relations (37) will be satisfied. Thus, for each approximation, if the conditions (37) are not verified, then we change the support via the dual method [4, 14, 25] in order to get the relation (37).

We perform a new iteration until the successive approximations are equal. Let \( Q^\sup_* = \{I^\sup_*, J^\sup_*, T^\sup_*\} \) be a solution of the system (38). Then the quasi-control \( w^*(t) \), \( t \in T \), calculated with the support \( Q^\sup_* \) and (30) is an admissible and optimal control of the problem (1)-(4).

4. Optimal cash management model. In this section, we develop a deterministic cash management model which is close to Sethi model [31, 33], assuming as an extension that the overdrafts on cash and short selling of stock are allowed, but within the authorized time limit.

Consider a firm which has a known demand \( d(t) \) for cash at time \( t \). This demand can be positive or negative. We assume that the amount invested in the bank account at time \( t \) is \( x_1(t) \) and the amount invested in stock is \( x_2(t) \), \( t \in [0, t^*] \), where \( t^* \) defines the time horizon. The interest rate earned on the bank account is \( r_1(t) \) and the returns derived from stock at time \( t \) takes two forms: capital gains rate \( r_2(t) \) and cash dividend rate \( r_3(t) \). For each unit of stock that is bought or sold, the firm pays the broker’s commission for a cost \( \alpha \) per an unit worth of stock. The control variables at time \( t \) are \( u_1(t) \) and \( u_2(t) \), representing respectively the sale and the purchase, where \( 0 \leq u_1(t) \leq M_1 \) and \( 0 \leq u_2(t) \leq M_2 \).
The state equations are
\[
\begin{aligned}
\dot{x}_1 &= r_1 x_1 + r_3 x_2 + u_1 - u_2 - \alpha (u_1 + u_2) - d, \quad x_1(0) = x_1^0, \\
\dot{x}_2 &= r_2 x_2 - u_1 + u_2, \quad x_2(0) = x_2^0,
\end{aligned}
\] (40)
and the objective function is
\[
\max V = x_1(t^*) + x_2(t^*).
\] (41)

In this paper, we assume that the overdrafts on cash and short selling of stock are allowed, but within the authorized time limit. Some considerations are required to simplify this assumption. These conditions will be checked if we impose the following constraints:
\[
x_1(t_s) \geq 0, \ s \in S = \{1, \ldots, m\}, \ x_2(t'_s) \geq 0, \ s \in S' = \{1, \ldots, m'\},
\] (42)
where \( t_s - t_{s-1} = h_1, s \in S, \ t'_s - t'_{s-1} = h_2, s \in S' \) and \( t_m = t'_m = t^* \).

Thus, the optimal cash management model is reduced to the final following optimal control problem
\[
\max V = x_1(t^*) + x_2(t^*),
\] (43)
\[
\begin{aligned}
\dot{x}_1 &= r_1 x_1 + r_3 x_2 + u_1 - u_2 - \alpha (u_1 + u_2) - d, \quad x_1(0) = x_1^0, \\
\dot{x}_2 &= r_2 x_2 - u_1 + u_2, \quad x_2(0) = x_2^0, \\
x_1(t_1) \geq 0, \ s \in S = \{1, \ldots, m\}, \\
x_2(t'_1) \geq 0, \ s \in S' = \{1, \ldots, m'\}, \\
0 \leq u_1(t) \leq M_1, \ 0 \leq u_2(t) \leq M_2, \ t \in [0, t^*].
\end{aligned}
\] (44)

4.1. Example (optimal cash management). Consider the model of optimal cash management with the following numerical values: \( x_1(0) = 500, \ x_2(0) = 0; \ r_1(t) = 0.04, \ r_2(t) = 0.15, \ r_3(t) = 0, \) and \( d(t) = 50, \ \forall t \in [0, t^*], \ t_1 = t'_1 = 6, t_2 = t'_2 = t^* = 12, \alpha = 0 \) and \( M_1 = M_2 = 100 \). Thus, the optimal cash management model takes the form:
\[
\max V = x_1(t^*) + x_2(t^*),
\]
\[
\begin{aligned}
\dot{x}_1 &= 0.04 x_1 + u_1 - u_2 - 50, \quad x_1(0) = 500, \\
\dot{x}_2 &= 0.15 x_2 - u_1 + u_2, \quad x_2(0) = 0, \\
x_1(t_1) = 6 \geq 0, \ x_2(t'_1) = 6 \geq 0, \\
x_1(t_2) = 12 \geq 0, \ x_2(t'_2) = 12 \geq 0, \\
0 \leq u_1(t) \leq 100, \ 0 \leq u_2(t) \leq 100, \ t \in [0, 12].
\end{aligned}
\] (45)

We apply the algorithm with the following parameters:
\[
A = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.15 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R(t) = \begin{pmatrix} -50 & 0 \\ 0 & 0 \end{pmatrix}, \quad H(1) = H(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{s(1)} = g_{s(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
and we consider the initial control \( u^0(t) = (u^0_1(t), u^0_2(t)), \ t \in [0, 12] \), where \( u^0_1(t) = 0, \ for \ t \in [0, 12] \) and
\[
u^0_2(t) = \begin{cases} 0, & \text{for } t \in [0, 11.8[ \\ 100, & \text{for } t \in [11.8, 12]. \end{cases}
\] (46)
The control \( u^0(t) \) is admissible and \( V(u^0) = 38.1670 \).

So, with an accuracy of \( \epsilon = 10^{-4} \), we obtain the following \( \epsilon \)-optimal control:
\[
u^*_1(t) = \begin{cases} 0, & \text{for } t \in [0, 4.1614[ \cup [6, 7.3704[ \\ 100, & \text{for } t \in [4.1614, 6] \cup [7.3704, 12], \end{cases}
\] (47)
\[
u^*_2(t) = \begin{cases} 
100, & \text{for } t \in [0, 4.1614] \cup [6, 7.3704], \\
0, & \text{for } t \in [4.1614, 6] \cup [7.3704, 12], 
\end{cases}
\] (48)

with the maximum value \( V(u^*) = 988.1833. \)

**Figure 1.** Optimal control \( u^*_1(t). \)

**Figure 2.** Optimal control \( u^*_2(t). \)

From the results, the optimal decision for the firm is to purchase until a certain
date the stocks at their authorized maximum value in order to take advantage of
the returns derived from stock. After that, it sales the stocks at their authorized
maximum value in order to satisfy the constraint (bank account will be positive) at
time \( t_s. \)

**Conclusion.** In this work, we deal in the first with the problem of optimal control
of a linear dynamical system. By using the method developed in [3, 4, 6, 14, 21, 25]
we have extended this method to the class of optimal control problems in Bolza form,
with vector control and intermediate phase constraints. Secondly, we have presented
the results of the extension for an optimal cash management. These results show
that the optimal decision for the firm depends closely on the intermediate moment,
because the firm must satisfy the constraint (bank account will be positive) at time \( t_s. \) In a future work, we will propose an extension of the model for an optimal cash
management on considering uncertain capital gains and cash demands.
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