Borsuk’s partition problem in four-dimensional $\ell_p$ space

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Abstract

In 1933, Borsuk made a conjecture that every $n$-dimensional bounded set can be divided into $n + 1$ subsets of smaller diameter. Up to now, the problem is still open for $4 \leq n \leq 63$. In this paper, we firstly discuss the Banach-Mazur distance between the $n$-dimensional cube and the $\ell_p$ ball ($1 \leq p < 2$), then we study the generalized Borsuk’s partition problem in metric spaces and prove that all bounded sets $X$ in every four-dimensional $\ell_p$ space can be divided into $2^4$ subsets of smaller diameter.

Keywords: metric space, Banach-Mazur distance, covering functional, complete set, Borsuk’s partition problem.

1 Introduction

Let $\mathbb{E}^n$ be the $n$-dimensional Euclidean space, and in this paper an $n$-dimensional vector $x \in \mathbb{E}^n$ is always treated as a column vector. Let $K$ denote an $n$-dimensional convex body, a bounded compact convex set with non-empty interior $\text{int}(K)$ and boundary $\partial(K)$. By $\mathcal{K}^n$ we denote the set of convex bodies in $\mathbb{E}^n$.

Let $d(X)$ denote the diameter of a bounded set $X$ of $\mathbb{E}^n$ defined by

$$d(X) = \sup\{|x, y| : x, y \in X\},$$

where $|x, y|$ denotes the Euclidean distance between $x$ and $y$. Let $b(X)$ be the smallest number of subsets $X_1, X_2, \ldots, X_{b(X)}$ of $X$ such that

$$X = \bigcup_{i=1}^{b(X)} X_i$$

and $d(X_i) < d(X)$ holds for all $i \leq b(X)$. In 1933, K. Borsuk [1] proposed the following problem:

Borsuk’s partition problem. Is it true that

$$b(X) \leq n + 1$$

holds for every bounded set $X$ in $\mathbb{E}^n$?

Usually, the positive statement of this problem is referred as Borsuk’s conjecture. K. Borsuk [1] proved that the inequality $b(X) \leq 3$ holds for any bounded set $X \subseteq \mathbb{E}^2$. For $n = 3$, Borsuk’s conjecture was confirmed by H. G. Eggleston [4] in 1955. In 1945, H. Hadwiger [7] proved that the inequality $b(K) \leq n + 1$ holds for every $n$-dimensional convex body $K$ with smooth boundary. However, in 1993, J. Kahn and G. Kalai [9] discovered counterexamples to Borsuk’s conjecture in high dimensions. In 2021, C.

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Zong [18] gave a computer proof program to deal with this challenging problem. Up to now, the problem is still open for $4 \leq n \leq 63$.

Let $M^n$ denote the Minkowski space with respect to the metric $\| \cdot \|_C$ determined by a centrally symmetric convex body $C$. For a bounded set $X \subseteq M^n$, let $d_C(X)$ denote the diameter of $X$ and let $b_C(X)$ denote the smallest number such that $X$ can be divided into $b_C(X)$ subsets each of which has the diameter strictly smaller than $d_C(X)$.

In 1957, B. Grünbaum [6] firstly studied the problem in Minkowski planes $M^2$. It was mentioned in [3] that for every bounded set $X \subseteq M^2$, if the unit ball $C$ of $M^2$ is a not a parallelogram, then the inequality $b_C(X) \leq 3$ holds; otherwise, the inequality $b_C(X) \leq 4$ holds.

The covering number $\gamma(K)$ is the smallest number of translates of $\lambda K$ $(0 < \lambda < 1)$ such that their union contains $K$. In 1957, H. Hadwiger [8] raised the following conjecture, which has a close relation with the Borsuk’s partition problem.

**Hadwiger’s covering conjecture.** Every convex body $K$ in $E^n$ can be covered by $2^n$ translates of $\lambda K$ (or $\text{int}(K)$), where $\lambda$ is a suitable positive number satisfying $\lambda < 1$.

The two-dimensional case had been solved by F. W. Levi [10]. However, this conjecture is open for all $n \geq 3$ until now. In 2010, C. Zong [17] proposed a four-step program to attack this conjecture.

In 1965, V. G. Boltyanski and I. T. Gohberg [2] proved that

$$b_C(X) \leq \gamma(\hat{X}) \quad (1.1)$$

holds for all metrics and all bounded sets $X$ in $E^n$, where $\hat{X}$ denotes the closed convex hull of $X$. Based on this fact, they also proposed the following problem:

**Problem 1** Is it true that

$$b_C(X) \leq 2^n$$

holds for all $n$-dimensional metric spaces $M^n$ and all bounded sets $X \subseteq M^n$?

In this paper, we concern the $\ell_p$ space. For a real number $p \geq 1$, the $p$-norm of a $x \in E^n$ is defined by

$$\| x \|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

The maximum norm is the limit of the $p$-norm for $p \rightarrow \infty$, a.k.a.,

$$\| x \|_{\infty} = \max\{|x_1|, |x_2|, \cdots, |x_n|\}.$$

Let $M^n_p$ denote the $n$-dimensional $\ell_p$ space, the $n$-dimensional linear space with the $p$-norm. The $n$-dimensional $\ell_p$ ball is denoted by

$$C_{n,p} = \{ x \in R^n : \| x \|_p \leq 1 \}.$$

Denote by

$$C_n = \{ x \in R^n : \| x \|_{\infty} \leq 1 \} = [-1,1]^n,$$

the $n$-dimensional unit cube and denote the vertices of $C_n$ by $\{-1,1\}^n$.

In 2009, L. Yu and C. Zong [16] studied Problem 1 and obtained that $b_{C_3,p}(X) \leq 2^3$ holds for all bounded sets $X$ in every three-dimensional $\ell_p$ space. In 2021, Y. Lian and S. Wu [11] showed that each set $X$ having diameter 1 in three-dimensional $\ell_p$ space can be represented as the union of $2^3$ subsets of $X$ whose diameters are at most 0.925. In this paper, we continue studying the above problem in $M^4_p$. Our main result is:

**Theorem 1.1.** In every four-dimensional $\ell_p$ space

$$b_{C_4,p}(X) \leq 2^4$$

holds for all bounded sets $X$. 

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In order to prove this theorem, we rely on the Banach-Mazur distance. The Banach-Mazur distance between two $\sigma$-symmetric convex bodies $K$ and $L$ is defined as 

$$d_{BM}(K, L) = \min\{r > 0 : K \subset gL \subset rK, g \in GL(n, \mathbb{R})\},$$

where $GL(n, \mathbb{R})$ is the set of invertible linear operators.

In [15], F. Xue proved that

$$\alpha\sqrt{n} \leq d_{BM}(C_n, C_n, 1) \leq (\sqrt{2} + 1)\sqrt{n}$$

and $\alpha$ can actually be improved to $\frac{1}{\sqrt{2}}$. Here, we generalize the results to $C_n,p$ as follows:

**Theorem 1.2.**

$$2^{\frac{1}{p} - \frac{1}{q}}\sqrt{n} \leq d_{BM}(C_n, C_n,p) \leq (\sqrt{2} + 1)\sqrt{n}$$

for $1 \leq p < 2$.

This paper is organized as follows: In Section 2, we study the Banach-Mazur distance between $C_n$ and $C_n,p$ $(1 \leq p \leq 2)$; In Section 3, we prove Theorem 1.1.

## 2 On the Banach-Mazur distance between the cube and the $\ell_p$ ball

In order to find the Banach-Mazur distance between $C_n$ and $C_n,p$ $(1 \leq p \leq 2)$, one needs to find the minimum $r > 0$ such that there exists $g \in GL(n, \mathbb{R})$ with 

$$C_n,p \subset gC_n \subset rC_n,p.$$  

(2.1)

Assume that $g$ is the linear transformation with row vectors $c_1, c_2, ..., c_n$, i.e. $g = (c_i)_{n \times 1}$. Since $C_n = \text{conv}\{\sum_{i=1}^{n}\sigma_i e_i : \sigma_i \in \{-1, 1\}, i = 1, ..., n\} = \text{conv}\{v : v = (\pm 1, \pm 1, ..., \pm 1)^T\}$, where $e_i$ is the $i$-th unit vector, we have 

$$gC_n = \text{conv}\{gv, v \in \{-1, 1\}^n\}.$$ 

The maximum distance between any vertex of $gC_n$ and $o$ is 

$$\max_{v \in \{-1, 1\}^n} \|gv\|_p = \max_{v \in \{-1, 1\}^n} \|(c_1, v), (c_2, v), ..., (c_n, v)^T\|_p.$$ 

Here we have 

$$gC_n \subseteq \left(\max_{v \in \{-1, 1\}^n} \|gv\|_p\right) C_n,p.$$  

(2.2)

From the left part in (2.1), we have 

$$g^{-1}C_n, 1 = g^{-1}C_n = (gC_n)^* \subseteq C_n,p, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$ 

Then 

$$\max_i \|g^{-1}e_i\|_q \leq 1.$$  

(2.3)

Combining (2.3) with (2.2), we have 

$$d_{BM}(C_n, C_n,p) = r = \min_g \max_{v \in \{-1, 1\}^n} \|gv\|_p,$$  

(2.4)
where \( g \in GL(n, \mathbb{R}) \) and \( \max_i \|g^{-1}e_i\|_q \leq 1. \)

In order to show the upper bound of the distance, we use the Hadamard matrix. A Hadamard matrix is a square matrix whose entries are either +1 or −1, and whose rows are mutually orthogonal. Sylvester [14] provided one way to construct Hadamard matrices. Let

\[
H_1 = (1)
\]

\[
H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

\[
H_{2k} = \begin{pmatrix} H_{2k-1} & H_{2k-1} \\ H_{2k-1} & -H_{2k-1} \end{pmatrix}
\]

for \( k \geq 2 \), then \( H_{2k} \) are all Hadamard matrices.

The Hadamard conjecture proposes that a Hadamard matrix of order \( 4k \) exists for every positive integer \( k \). So far this conjecture is still open.

**Lemma 2.1.** In dimension \( n = 2^k \), \( k \in \mathbb{Z}^+ \), we have \( d_{BM}(C_n, C_{n,p}) \leq \sqrt{n} \) for \( 1 \leq p \leq 2 \).

**Proof.** In dimension \( n = 2^k \), there exists a Hadamard matrix \( H_n \) with row vectors \( r_1, r_2, ..., r_n \). Choose the matrix \( g = \frac{1}{n^{\frac{1}{p}}} H_n \), and \( H_n^{-1} = \frac{1}{n} H_n^T \). Then we have

\[
\max_i \|g^{-1}e_i\|_q = \max_i \|n^{\frac{1}{p}} \cdot \frac{1}{n} H_n^T e_i\|_q = n^{\frac{1}{p}} \cdot \frac{1}{n} \cdot n^{\frac{1}{p}} = 1.
\]

Thus, we have

\[
r = \max_{v \in \{-1, 1\}^n} \|gv\|_p
\]

\[
= \frac{1}{n^{\frac{1}{p}}} \max_{v \in \{-1, 1\}^n} \|H_n v\|_p
\]

\[
= \frac{1}{n^{\frac{1}{p}}} \max_{v \in \{-1, 1\}^n} \|\langle r_1, v \rangle, \langle r_2, v \rangle, ..., \langle r_n, v \rangle\|_p
\]

\[
= \frac{1}{n^{\frac{1}{p}}} \max_{v \in \{-1, 1\}^n} \left( \|\langle r_1, v \rangle\|^p + \|\langle r_2, v \rangle\|^p + ... + \|\langle r_n, v \rangle\|^p \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{n^{\frac{1}{p}}} \max_{v \in \{-1, 1\}^n} \left( \|\langle r_1, v \rangle\|^2 + \|\langle r_2, v \rangle\|^2 + ... + \|\langle r_n, v \rangle\|^2 \right)^{\frac{p}{2}} \cdot n^{\frac{2-p}{2}}
\]

\[
= \frac{1}{n^{\frac{1}{p}}} \left( n^{\frac{2}{p}} \cdot n^{\frac{2-p}{2}} \right)^{\frac{1}{p}}
\]

\[
= \frac{1}{n^{\frac{1}{p}}} \cdot n^{\frac{2-p}{2} \cdot \frac{1}{p}}
\]

\[
= \sqrt{n}
\]

\( \square \)

**Remark 2.2.** By Lemma 2.1 let \( \sigma(p) \) denote the smallest number such that there exists a parallelootope \( P = gC_4 \) satisfying \( P \subseteq C_{4,p} \subseteq \sigma(p)P \). When we choose \( g \) to be \( \frac{1}{4^{\frac{1}{p}} \cdot p} H_4 \), where \( H_4 \) is a Hadamard matrix, then we have \( \sigma(p) \leq 2 \).

**Proof of the upper bound in Theorem 1.2.** By induction, assume that in dimension \( t \leq 2^k \) the upper bound of the distance is not bigger than \( (\sqrt{2} + 1)\sqrt{7} \) with cube determined by \( g_t \). That is to say, we have

\[
C_{t,p} \subseteq g_tC_t \subseteq rC_{t,p}
\]
and

\[ r = \max_{v \in \{-1,1\}^t} \|gtv\|_p \leq (\sqrt{2} + 1)\sqrt{t} \]

with

\[ \max_{1 \leq i \leq t} \|g_t^{-1}e_i\|_q \leq 1. \]

Then in dimension \( n = 2^k + t \) where \( t \leq 2^k \), let

\[ g_{2^k+t} = \begin{pmatrix} \frac{1}{(2^k)^{\frac{p}{2}}} H_{2^k} & 0 \\ 0 & g_t \end{pmatrix}. \]

Let \( e = \max_{1 \leq i \leq 2^k} \left\| \left( \frac{1}{(2^k)^{\frac{p}{2}}} H_{2^k} \right)^{-1} e_i \right\|_q \), \( f = \max_{1 \leq i \leq t} \|g_t^{-1}e_i\|_q \), obviously, we have \( f = e = 1 \) and

\[ \max_i \left\| (g_{2^k+t})^{-1} e_i \right\|_q = \max\{e, f\} = 1. \]

By Lemma 2.1, the distance is therefore

\[
\max_{v \in \{-1,1\}^n} \|g_{2^k+t}v\|_p \leq \max_{v \in \{-1,1\}^{2^k}} \left\| \frac{1}{(2^k)^{\frac{p}{2}}} H_{2^k} v \right\|_p + \max_{v \in \{-1,1\}^t} \|gtv\|_p \\
\leq \sqrt{(2^k) + \max_{v \in \{-1,1\}^t} \|gtv\|_p} \\
= \sqrt{(2^k) + (\sqrt{2} + 1)\sqrt{t}} \\
\leq (\sqrt{2} + 1)\sqrt{2^k + t} \\
= (\sqrt{2} + 1)\sqrt{n}
\]

The proof for the upper bound is finished. The Hadamard conjecture predicts the existence of a Hadamard matrix in dimension \( n = 4k \). When there indeed exists a Hadamard matrix \( H_n \), \( n = 4k \). The distance between \( C_{n,p} \) and the cube determined by \( H_n \) will not be larger than \( \sqrt{n} \) for \( 1 \leq p \leq 2 \).

When \( n = 4k + j \), \( j < 4 \), let the cube be determined by

\[ g_{4k+j} = \begin{pmatrix} I_j & 0 \\ 0 & \frac{1}{(4k)^{\frac{p}{2}}} H_{4k} \end{pmatrix}, \]

then

\[ \max_i \left\| (g_{4k+j})^{-1} e_i \right\|_q = 1, \]

so the distance is

\[
r = \max_{v \in \{-1,1\}^n} \|g_{4k+j}v\|_p \\
\leq \max_{v \in \{-1,1\}^{4k}} \left\| \frac{1}{(4k)^{\frac{p}{2}}} H_{4k} v \right\|_p + \max_{v \in \{-1,1\}^t} \|I_jv\|_p \\
\leq \sqrt{4k + j} \frac{p}{2} \\
< \sqrt{n} + 3.
\]
Therefore the upper bound will be $\sqrt{n} + 3$ for all $n$ if the Hadamard conjecture is true.

**Proof of the lower bound in Theorem 1.2**

Recall that $d_{BM}(C_n, C_{n,p}) = \min_g \max_{v \in \{-1,1\}^n} \|g v\|_p$, where $g \in GL(n, \mathbb{R})$ and $\max_i \|g^{-1} e_i\|_q \leq 1$. Without loss of generality, we consider $\det(g) > 0$ and write $g = \det(g)^{\frac{1}{n}} N$, where $N \in SL(n, \mathbb{R})$.

By using the Power Mean Inequality and $\|g^{-1} e_i\|_q \leq 1$ for $i = 1, \ldots, n$ and $q \geq 2$, we have

$$\det(g^{-1}) \leq \prod_{i=1}^n \|g^{-1} e_i\|_2 \leq n^{\frac{q}{2}} \prod_{i=1}^n \left( \frac{\|g^{-1} e_i\|_q^q}{n} \right)^{\frac{1}{q}} = n^{\frac{q}{2} - \frac{1}{q}}. \quad (2.5)$$

Let the row vectors of $N$ be $N_j$, i.e. $N = (N_j)_{n \times 1}$, then we have

$$\|N v\|_p = (|\langle N_1, v \rangle|^p + \ldots + |\langle N_n, v \rangle|^p)^{\frac{1}{p}}.$$

Also, since $\det(N) = 1$, by the definition of determinant,

$$\prod_{i=1}^n \|N_j\|_2 \geq 1,$$

and by the Arithmetic-geometric Inequality,

$$\left( \frac{\sum_{j=1}^n \|N_j\|_2^p}{n} \right)^{\frac{1}{p}} \geq 1. \quad (2.6)$$

By the Khintchine Inequality,

$$\left( \frac{1}{2^n} \sum_{v \in \{-1,1\}^n} |\langle u, v \rangle|^p \right)^{\frac{1}{p}} \geq 2^{\frac{1}{p} - \frac{1}{2}} \|u\|_2,$$

and

$$\frac{1}{2^n} \sum_{v \in \{-1,1\}^n} |\langle u, v \rangle|^p \geq 2^{\frac{1}{p} - 1} \|u\|_2^p, \quad (2.7)$$

Thus,

$$\max_{v \in \{-1,1\}^n} \|g v\|_p = \det(g)^{\frac{1}{p}} \max_{v \in \{-1,1\}^n} \|N v\|_p$$

$$= \det(g)^{\frac{1}{p}} \left( \max_{v \in \{-1,1\}^n} \sum_{j=1}^n |\langle N_j, v \rangle|^p \right)^{\frac{1}{p}}$$

$$\geq \det(g)^{\frac{1}{p}} \left( \frac{1}{2^n} \sum_{v \in \{-1,1\}^n} \sum_{j=1}^n |\langle N_j, v \rangle|^p \right)^{\frac{1}{p}}$$

$$= \det(g)^{\frac{1}{p}} \left( \frac{1}{2^n} \sum_{j=1}^n \sum_{v \in \{-1,1\}^n} |\langle N_j, v \rangle|^p \right)^{\frac{1}{p}}$$
\[ \geq \det(g)^{\frac{1}{n}} \left( 2^{\frac{p}{2} - 1} \sum_{j=1}^{n} \|N_j\|_2^p \right)^{\frac{1}{p}} \]  
(2.7)

\[ \geq \det(g)^{\frac{1}{n}} \left( 2^{\frac{p}{2} - 1} \cdot n \right)^{\frac{1}{p}} \]  
(2.6)

\[ \geq \frac{1}{n^{\frac{p}{2} - \frac{1}{p}}} \cdot n^{\frac{1}{p}} \cdot 2^{\frac{p}{2} - \frac{1}{p}} \]  
(2.5)

\[ = 2^{\frac{p}{2} - \frac{1}{p}} \cdot \sqrt{n}. \]

The proof for the lower bound is finished. \[ \square \]

3 Borsuk’s partition problem in four-dimensional \( \ell_p \) space

In order to prove Theorem 1.1, let us consider three main situations.

3.1 \( p > 2 \)

If \( p > 2 \), there exists a parallelotope \( P = C_4 \) satisfying

\[ \frac{1}{4} \cdot \frac{1}{n^{\frac{p}{2} - \frac{1}{p}}} \cdot n^{\frac{1}{p}} \cdot 2^{\frac{p}{2} - \frac{1}{p}} \]

(3.1)

For each bounded set \( X \), there is a point \( x \) and a minimal number \( \tau \) such that

\[ X + x \subseteq \tau P. \]

Since

\[ d_{C_4,p}(X + x) = d_{C_4,p}(X), \]

and \( X + x \) touches at least one pair of opposite facets of \( \tau P \), we have

\[ d_{C_4,p}(X) \leq 2\tau. \]  
(3.2)

On the other hand, \( \tau P \) can be covered by exactly 16 translates of \( \frac{1}{2} \cdot \tau P \). By (3.1) it follows that \( \tau P \) can be covered by 16 translates of \( \frac{1}{2} \cdot \tau C_4,C_4 \), and \( X \) can be divided into 16 corresponding subsets \( X_1, X_2, ..., X_{16} \) with

\[ d_{C_4,p}(X_i) \leq d_{C_4,p} \left( \frac{\tau C_4}{2} \right) = 4^{\frac{p}{2}} \cdot \tau < 2\tau \]  
(3.3)

for \( i = 1, ..., 16 \). Therefore, (3.2) and (3.3) together yield

\[ b_{C_4,p}(X) \leq 2^4. \]

Remark 3.1. Using the same method, we can prove \( b_{C_n,p}(X) \leq 2^n \) holds for all \( \log_2 n < p \leq +\infty \), \( n \geq 3 \) and all bounded set \( X \).

3.2 \( 1 < p \leq 2 \)

If \( 1 < p \leq 2 \), we recall from Lemma 2.1 and Remark 2.2 that

\[ \frac{1}{2} Q \subseteq C_4,p \subseteq Q, \]  
(3.4)
where $Q = gC_4$ and

\[ g = \frac{1}{4^{\frac{4}{d}}} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \]

Denote by $w(X, \mathbf{u})$ the Euclidean width of $X$ in the direction $\mathbf{u}$. Let $\mathbf{u}_i = ge_i$, i.e. $\mathbf{u}_1 = 4^{-\frac{4}{d}}(1, 1, 1, -1)^T$, then $w(C_{4,p}, \mathbf{u}_i) = 4^{-\frac{4}{d}}$ for $i = 1, \ldots, 4$.

For each bounded set $X$ with $d_{C_{4,p}}(X) = 2$ in $M^4_p$, we have $w(X, \mathbf{u}_i) \leq 4^{-\frac{4}{d}}$ for $i = 1, \ldots, 4$ and the equality holds if and only if there exists $\mathbf{a}_i, \mathbf{b}_i \in X$ such that

\[ \mathbf{a}_i - \mathbf{b}_i = 2\mathbf{u}_i. \tag{3.5} \]

Up to translation, we may assume that $X \subseteq \cap_{i \in [4]} \{ x : |(x, \mathbf{u}_i)| \leq w_i \} = Q_X$ with $w_i \leq 4^{-\frac{4}{d}}$. In fact, $Q = \cap_{i \in [4]} \{ x : |(x, \mathbf{u}_i)| \leq 4^{-\frac{4}{d}} \}$. Now we consider two cases:

Case 1. If there exists some $w_i < 4^{-\frac{4}{d}}$, then we have $X \subseteq Q_X \subseteq Q$ and $d_p(Q_X) < 4$. In this case, one can divided $Q_X$ into 16 smaller copies of $\frac{1}{4}Q_X$ with $d_{C_{4,p}}(\frac{1}{4}Q_X) < 2$. Then $X$ can also be divided into 16 corresponding parts with diameter strictly smaller than 2. Thus, $b_{C_{4,p}}(X) \leq 2^4$.

Case 2. If $w_i = 4^{-\frac{4}{d}}$ for all $i = 1, \ldots, 4$, let $F_i = \{ x : \langle x, \mathbf{u}_i \rangle = 4^{-\frac{4}{d}} \}$ and $F_{-i} = \{ x : \langle x, -\mathbf{u}_i \rangle = 4^{-\frac{4}{d}} \}$, then $X$ touches each pair of opposite facets of $Q$. Assuming that $X$ touches $Q \cap F_i$ at one point $\mathbf{a}_i$ and touches $Q \cap F_i$ at point $\mathbf{b}_i$ satisfying (3.5). In addition, since $C_{4,p}$ is strictly convex when $1 < p \leq 2$, then $X$ cannot touch $F_i$ (as well as $F_{-i}$) at more than one point. Also, all $\mathbf{a}_i, \mathbf{b}_i$ must be in the interior of each facet of $Q$. If not, suppose $\mathbf{a}_i$ is on the boundary of one facet of $Q$. Without of loss generality, let $\mathbf{a}_1 \in Q \cap F_1 \cap F_2$, by $\mathbf{a}_1 - \mathbf{b}_1 = 2\mathbf{u}_1$, then $\mathbf{b}_1 \in (Q \cap F_1 \cap F_2)$. Since $d_{C_{4,p}}(X) = 2$ and $\mathbf{a}_1, \mathbf{b}_1 \in (Q \cap F_2)$, there is no point of $X$ on the opposite facet $(Q \cap F_2)$, which contradicts to the assumption that $X$ intersects all facets of $Q$.

For $1 < p \leq 2$, by the strictly convexity of $C_{4,p}$ and (4.4), the diameter of $\frac{1}{4}Q$ in $M^4_p$ is only determined by its eight pairs of symmetric vertices:

\[ d_{C_{4,p}} \left( \frac{1}{4}Q \right) = 2 = d_{C_{4,p}} \left( \frac{1}{2}g\mathbf{v}_0, \frac{1}{2}g(-\mathbf{v}) \right), \quad \mathbf{v} \in \{ 1, -1 \}^4 = \Sigma_{i=1}^4 \delta_i e_i, \quad \delta_i \in \{ 1, -1 \}. \]

Now we still divided $Q$ into 16 smaller copies of $\frac{1}{4}Q$, that is, $Q = \bigcup_{i=1}^{16} (\frac{1}{4}Q + \mathbf{y}_i)$ with $\mathbf{y}_i \in \{ 1, -1 \}^4$. Then we also get 16 corresponding subsets $X_i = X \cap (\frac{1}{4}Q + \mathbf{y}_i), i = 1, \ldots, 16$. For every translating point pair $(\frac{1}{2}g\mathbf{v}_0, \frac{1}{2}g(-\mathbf{v}_0))$ with $\mathbf{v}_0 = \Sigma_{i=1}^4 \sigma_i e_i, \sigma_i \in \{ 1, -1 \}$. Then

\[ \frac{1}{2}g\mathbf{v}_0 + \mathbf{y}_i = \frac{1}{2}g(\Sigma_{i=1}^4 (\sigma_i + \delta_i) e_i), \tag{3.6} \]

\[ \frac{1}{2}g(-\mathbf{v}_0) + \mathbf{y}_i = \frac{1}{2}g(\Sigma_{i=1}^4 (\delta_i - \sigma_i) e_i). \tag{3.7} \]

If all $\delta_i = \sigma_i, i = 1, \ldots, 4$, then the point (3.6) is contained in $\cap_{i=1}^{16} F_{\delta,(i)}$; if $\delta_i = \sigma_i, i = 1, \ldots, 3$ and $\delta_4 \neq \sigma_4$, then the point (3.6) is contained in $\cap_{i=1}^{3} F_{\delta,(i)} \cap Q$; if $\delta_i = \sigma_i, i = 1, 2$ and $\delta_3 \neq \sigma_3, j = 3, 4$, then the point (3.6) is contained in $\cap_{i=1}^{2} F_{\delta,(i)} \cap Q$; if $\delta_1 = \sigma_1, \delta_i \neq \sigma_i, i = 2, \ldots, 4$, then the point (3.7) is contained in $\cap_{i=2}^{4} F_{\delta,(i)} \cap Q$; if all $\delta_i \neq \sigma_i, i = 1, \ldots, 4$, then the point (3.7) is contained in $\cap_{i=1}^{4} F_{\delta,(i)}$.

By above discussions and $X$ touches each facet of $Q$ at exactly one interior point, we have $d_{C_{4,p}}(X_i) < 2$ for all $i = 1, \ldots, 16$. Therefore, $b_{C_{4,p}}(X) \leq 2^4$. 

8
3.3 \( p=1 \)

By \([17]\), determining the covering number of a convex body is useful for solving the Borsuk’s partition problem. Let \( m \) be a positive integer and let \( \gamma_n(K) \) be the smallest positive number \( r \) such that \( K \) can be covered by \( m \) translates of \( rK \). Clearly, \( \gamma_n(K) < 1 \) is equivalent to \( \gamma(K) \leq m \). Firstly, we estimate the values of \( \gamma_{2n}(C_{n,p}) \).

**Lemma 3.2 \([17]\).** Let \( K \) be an \( n \)-dimensional convex body, \( \lambda \) be a real number satisfying \( 0 < \lambda < 1 \), \( R \) be a closed region on \( \partial(K) \) with boundary \( \Gamma \) and a relatively interior point \( p \). If \( \Gamma \cup \{p\} \subset \lambda K + y \) holds for some point \( y \), then we have \( R \subset \lambda K + y \).

**Lemma 3.3.** \( \gamma_{2n}(C_{n,p}) \leq \left( \frac{n-1}{n} \right)^{\frac{1}{n}} \) holds for all \( 1 \leq p \leq 2 \) and \( n \geq 2 \).

**Proof.**\( \text{Let } m = \left( \frac{1}{n} \right)^{\frac{1}{n}} \text{ and } \lambda = \left( \frac{n-1}{n} \right)^{\frac{1}{n}}. \text{ Let } y_i = me_i, y_{n+i} = -me_i, i = 1, \ldots, n. \text{ Here we show that} \)

\[
C_{n,p} \subseteq \bigcup_{i=1}^{n} ((\lambda C_{n,p} + y_i) \cup (\lambda C_{n,p} + y_{n+i})). \tag{3.8}
\]

Let \( \Pi_i = \{x = (x_1, x_2, \ldots, x_n) : x_i = m, x \in \partial(C_{n,p})\} \). Let \( R_i \) be a closed region on \( \partial(C_{n,p}) \) with boundary \( \Pi_i \) and a relatively interior point \( e_i, i = 1, \ldots, n. \) For every point \( x \in \Pi_i \), we have

\[
|x_1|^p + |x_2|^p + \ldots + |x_{i-1}|^p + |x_{i+1}|^p + \ldots + |x_n|^p = 1 - \frac{1}{n} = \frac{n-1}{n},
\]

so \( x \in \lambda C_{n,p} + y_i \). And \( e_i \in \lambda C_{n,p} + y_i \), since

\[
|1 - m|^p = \left| 1 - \left( \frac{1}{n} \right)^{\frac{1}{n}} \right|^p \leq \frac{n-1}{n}.
\]

By Lemma 3.2 we have \( R_i \subseteq \lambda C_{n,p} + y_i, i = 1, \ldots, n. \) In other words, \( \{x : x_i \geq m\} \cap C_{n,p} \subseteq \lambda C_{n,p} + y_i, i = 1, \ldots, n. \) Similarly, \( \{x : x_i \leq -m\} \cap C_{n,p} \subseteq \lambda C_{n,p} + y_{n+i}, i = 1, \ldots, n. \)

Since \( C_{n,p} \setminus \bigcup_{i=1}^{n} (\{x : x_i \geq m\} \cap C_{n,p}) \) is an \( n \)-dimensional parallelotope with vertices \( (\pm m, \pm m, \ldots, \pm m) \) and

\[
\text{conv}\{0\} \cup \{x : x_i = \pm m, \ |x_j| \leq m, j \neq i\} \subseteq \text{conv}\{0\} \cup \{\{x : x_i = \pm m\} \cap C_{n,p}\}, i = 1, \ldots, n,
\]

it is sufficient to verify that \( 0 \in \lambda C_{n,p} + y_i, i = 1, \ldots, 2n. \)

As

\[
\left( \frac{1}{n} \right)^{\frac{1}{n}} = \frac{n}{n} \leq \frac{n-1}{n}, n \geq 2,
\]

we know \( 0 \in \lambda C_{n,p} + y_i, i = 1, \ldots, 2n. \)

Therefore, (3.8) holds. Consequently,

\[
\gamma_{2n}(C_{n,p}) \leq \left( \frac{n-1}{n} \right)^{\frac{1}{n}}
\]

holds for all \( 1 \leq p \leq 2 \) and \( n \geq 2 \). This completes the proof of the lemma. \( \square \)

In order to show the case of \( p = 1 \), we use the concept of completeness. A bounded set is called complete if it is not properly contained in a set of the same diameter. Clearly, a complete set is convex and compact. In \([2]\), H. G. Eggleston showed that any bounded set \( X \subseteq M^n \) can be embedded in a complete set \( A \) of the same diameter, the complete set \( A \) is called the completion of \( X \). Generally, \( A \) is not unique. For
every bounded set $X \subseteq \mathbb{M}^n$, we have $b_C(X) \leq b_C(A)$, since $X \subseteq A \cap X \subseteq \bigcup_{i=1}^{h_c(A)} (A_i \cap X) = \bigcup_{i=1}^{h_c(A)} (X_i)$ and $d_C(X_i) = d_C(A_i \cap X) \leq d_C(A_i) < d_C(A) = d_C(X)$.

The complete sets have a useful characterization in terms of supporting slabs. A supporting slab of the convex body $K \in \mathcal{K}^n$ is any closed set $\Sigma \supseteq K$ that is bounded by two parallel supporting hyperplanes $H, H'$ of $K$. The distance between $H$ and $H'$ is called the width of $\Sigma$. For any other convex body $M$, we say that the supporting slab $\Sigma$ of $K$ is $M$-regular if the supporting slab of $M$ that is parallel to $\Sigma$ has the property that at least one of its bounding hyperplanes contains a smooth boundary point of $M$ (a boundary point through which passes only one supporting hyperplane of $M$). In \cite{12} and \cite{13}, J. P. Moreno and R. Schneider gave a new characterization of the complete sets in $\mathbb{M}^n$. They also states that in the case of a polyhedral norm, the space of translation classes of complete sets of given diameter is a finite polytopal complex.

**Lemma 3.4** (\cite{12}). Let $d > 0$. The $n$-dimensional convex body $K \in \mathcal{K}^n$ is a complete set of diameter $d$ if and only if the following properties hold:

(a) Every $C$-regular supporting slab of $K$ has width $\leq d$, $C$ is the unit ball of $\mathbb{M}^n$.

(b) Every $K$-regular supporting slab of $K$ has width $d$.

**Lemma 3.5** (\cite{13}). Let $\Sigma_1, \ldots, \Sigma_k$ be the $C$-regular supporting slabs of the polytopal unit ball $C$. Each complete set $K$ with diameter $2$ is of the form

$$K = \bigcap_{i=1}^{k} (\Sigma_i + t_i)$$

with $t_i \in \mathbb{R}^n$, $i = 1, \ldots, k$.

For the polytopal unit ball $C_{4,1}$ of $M_4^1$, its supporting slabs are $\Sigma_1$ with outer normal vectors $\pm u_1 = \pm(1, 1, 1, 1)$, $\Sigma_2$ with outer normal vectors $\pm u_2 = \pm(-1, -1, 1, 1)$, $\Sigma_3$ with outer normal vectors $\pm u_3 = \pm(1, -1, -1, 1)$, $\Sigma_4$ with outer normal vectors $\pm u_4 = \pm(-1, -1, -1, 1)$, $\Sigma_5$ with outer normal vectors $\pm u_5 = \pm(1, 1, 1, -1)$, $\Sigma_6$ with outer normal vectors $\pm u_6 = \pm(-1, 1, 1, 1)$, $\Sigma_7$ with outer normal vectors $\pm u_7 = \pm(1, -1, -1, -1)$ and $\Sigma_8$ with outer normal vectors $\pm u_8 = \pm(1, 1, -1, 1)$. Each slab $\Sigma_i$ is bounded by two parallel hyperplanes $\Phi_i = \{x : \langle x, u_i \rangle = 1\}$ and $\Phi_{-i} = \{x : \langle x, -u_i \rangle = 1\}$, $i = 1, \ldots, 8$.

For every bounded set $X \subseteq M_4^1$ with $d_{C_{4,1}}(X) = 2$, there always exists a completion $D$ of $X$. Up to some translation and by Lemma 3.5, we may assume that

$$X \subseteq D = D(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$= \cap_{i=5}^{8} (\Sigma_i + \alpha_i u_i)$$

$$= (C_{4,1} \cup (\bigcup_{i=1}^{4} \Sigma_{\pm i})) \cap \bigcap_{i=1}^{4} (\Sigma_i + \alpha_i u_i)$$

$$= \bigcup_{i=5}^{8} D_i,$$

where $D_1 = C_{4,1} \cap \bigcap_{i=1}^{4} (\Sigma_i + \alpha_i u_i)$, $D_j = S_{j-1} \cap \bigcap_{i=1}^{4} (\Sigma_i + \alpha_i u_i)$ or $D_j = S_{-(j-1)} \cap \bigcap_{i=1}^{4} (\Sigma_i + \alpha_i u_i)$ with $|\alpha_i| \leq \frac{1}{2}$, $j = 2, \ldots, 5$, and

$$S_i = \text{conv} \left( (C_{4,1} \cap \Phi_i) \cup \frac{1}{2} u_i \right), S_{-i} = -S_i, \quad i = 1, \ldots, 4.$$
For $i = 1, \ldots, 4$, we obtain that $d_{C_{4,1}}(S_i) = d_{C_{4,1}}(S_{-i}) = 2$ and that there are exactly five vertices of $S_i$ or $S_{-i}$ such that the distance between each pair is 2. Neither $S_i$ nor $S_{-i}$ is a complete set by Lemma 3.4 since there exist a $S_i (S_{-i})$-regular supporting slab of $S_i (S_{-i})$ with width 1. By Lemma 3.3, $C_{4,1}$ can be covered by 8 smaller copies of $C_{4,1}$. In fact, some vertices with neighbour also have been covered from the covering of $C_{4,1}$, so the remaining part of $S_i$ or $S_{-i}$ has diameter strictly smaller than 2. Therefore, $X$ can be divided into at most 12 parts, each of which has diameter strictly smaller than 2. Consequently, $b_{C_{4,1}}(X) \leq 2^4$.

In conclusion, $b_{C_{4,p}}(X) \leq 2^4$ holds for all four-dimensional $\ell_p$ space and all bounded set $X$. This completes the proof of the theorem.

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