Local well-posedness for dispersion generalized Benjamin–Ono equations in Sobolev spaces

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We prove that the Cauchy problem for the dispersion generalized Benjamin–Ono equation

\[ \partial_t u + |\partial_x|^{1+\alpha} \partial_x u + uu_x = 0, \quad u(x, 0) = u_0(x), \]

is locally well-posed in the Sobolev spaces \( H^s \) for \( s > 1 - \alpha \) if \( 0 \leq \alpha \leq 1 \). The new ingredient is that we generalize the methods of Ionescu, Kenig and Tataru (2008) [13] to approach the problem in a less perturbative way, in spite of the ill-posedness results of Molinet, Saut and Tzvetkov (2001) [21]. Moreover, as a byproduct we prove that if \( 0 < \alpha \leq 1 \) the corresponding modified equation (with the nonlinearity \( \pm uu_x \)) is locally well-posed in \( H^s \) for \( s \geq 1/2 - \alpha/4 \).

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1. Introduction

In this paper, we consider the Cauchy problem for the dispersion generalized Benjamin–Ono equation

\[
\begin{aligned}
\partial_t u + |\partial_x|^{1+\alpha} \partial_x u + uu_x &= 0, \\
(u, x, 0) &= u_0(x),
\end{aligned}
\]

(11)

where \( 0 \leq \alpha \leq 1, \ u : \mathbb{R}^2 \to \mathbb{R} \) is a real-valued function and \( |\partial_x| \) is the Fourier multiplier operator with symbol \( |\xi| \). These equations arise as mathematical models for the weakly nonlinear propagation of long waves in shallow channels. Note that the case \( \alpha = 0 \) corresponds to the Benjamin–Ono equation and the case \( \alpha = 1 \) corresponds to the Korteweg–de Vries equation. During the past decades, both
of the two equations were extensively studied in a large number of literatures [23,12,2,14,15,3]. For example, see [25] for a thorough review.

In proving the well-posedness of the Cauchy problem (1.1) by direct contraction principle, the biggest enemy is the loss of derivative from the nonlinearity. It was proved by Molinet, Saut and Tzvetkov [21] that if \( 0 \leq \alpha < 1 \) then \( H^s \) assumption alone on the initial data is insufficient for a proof of local well-posedness of (1.1) via Picard iteration by showing the solution mapping fails to be \( C^2 \) smooth from \( H^s \) to \( C([0,T];H^s) \) at the origin for any \( s \). It is due to that the dispersive effect of the dispersive group of Eq. (1.1) when \( 0 \leq \alpha < 1 \) is too weak to spread the derivative in the nonlinearity and hence the high \times low interactions break down the \( C^2 \) smoothness. When \( \alpha = 0 \) a stronger ill-posedness was proved by Koch and Tzvetkov [22] that the solution mapping actually fails to be locally uniformly continuous in \( H^s \) for any \( s \). For the positive side, some weaker well-posedness results (only require the solution mapping to be continuous) were obtained. For the Benjamin–Ono equation \( (\alpha = 0) \), Tao [23] obtained the global well-posedness in \( H^s \) for \( s \geq 1 \) by performing a gauge transformation as for the derivative Schrödinger equation. This result was improved to \( s \geq 0 \) by Ionescu and Kenig [12].

For the KdV equation \( (\alpha = 1) \), the first well-posedness by contraction principle was due to Kenig, Ponce and Vega [15] who obtained LWP in \( H^s \) for \( s > 3/4 \). Bourgain [2] extended this result to GWP in \( L^2 \) by developing \( X^{1,0} \) space. Then Kenig, Ponce and Vega [14] obtained local well-posedness in \( H^s \) for \( s > -3/4 \) and Colliander, Keel, Staffilani, Takaoka and Tao [3] extended it to a global result where \( I \)-method was introduced. Local well-posedness in \( H^{-3/4} \) was obtained by Christ, Colliander, and Tao [4] using Miura transform and the \( H^{1/4} \) local well-posedness for the modified KdV equation. Recently, the author [7] obtained global well-posedness in \( H^{-3/4} \) by using directly the contraction principle to prove local well-posedness.

This paper is mainly concerned with Eq. (1.1) for \( 0 \leq \alpha < 1 \). If \( 0 < \alpha < 1 \), Kenig, Ponce and Vega [16] have shown that (1.1) is locally well-posed for data in \( H^s \) provided \( s \geq \frac{3}{2} (2 - \alpha) \) using the energy method enhanced with the smoothing effect. In [5] Colliander, Kenig and Staffilani obtained LWP for the data lying in some weighted Sobolev spaces by applying Picard iteration. S. Herr [9, 10] obtained LWP in \( H^s \cap \dot{H}^{1+rac{1}{3} - \frac{\alpha}{2}} \) for \( s > -\frac{3}{4} \alpha \) and global well-posedness for \( s \geq 0 \) by requiring the initial data has additional properties in low frequency to make the contraction principle work. Compared to the Benjamin–Ono equation, the dispersive group of (1.1) has stronger dispersive effect but it seems difficult to apply a gauge transform to (1.1) which can weaken the high–low interaction. Only very recently, Herr, Ionescu, Kenig and Koch [11] developed a para-differential gauge and proved \( L^2 \) well-posedness. When \( 0 < \alpha < 1 \), Eq. (1.1) is not completely integrable, but there are at least three conservation laws: if \( u \) is a smooth solution to (1.1) then

\[
\frac{d}{dt} \int_{\mathbb{R}} u(x,t) \, dx = 0, \tag{1.2}
\]

\[
\frac{d}{dt} \int_{\mathbb{R}} u(x,t)^2 \, dx = 0, \tag{1.3}
\]

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( |\partial_x|^{1+\alpha} u \right)^2 + \frac{1}{6} u(x,t)^2 \, dx = 0. \tag{1.4}
\]

These conservation laws provide a priori bounds on the solution. For example, we can easily get from (1.3) and (1.4) that for any smooth solution \( u \) of (1.1) on \([-T, T]\) then we have

\[
\| u(t) \|_{H^{1+\alpha}} \leq C (\| u_0 \|_{H^{1+\alpha}}), \quad \forall t \in [-T, T]. \tag{1.5}
\]

We will also need another symmetry. It is easy to see that Eq. (1.1) is invariant under the following scaling transform for any \( \lambda > 0 \)

\[
u(x, t) \rightarrow u_\lambda(x, t) = \lambda^{1+\alpha} u(\lambda x, \lambda^2 + \sigma t), \quad u_{0,\lambda} = \lambda^{1+\alpha} u_0(\lambda x). \tag{1.6}
\]


Then we see $H^{-\frac{1}{2}-\alpha}$ is the critical space in the sense of the scaling (1.6)

$$
\|u_{0,\lambda}\|_{H^{-\frac{1}{2}-\alpha}} = \|u_0\|_{H^{-\frac{1}{2}-\alpha}}.
$$

Now we state our main results:

**Theorem 1.1.**

(a) Let $0 \leq \alpha \leq 1$. Assume $s > 1 - \alpha$ and $u_0 \in H^\infty$. Then there exists $T = T(\|u_0\|_{H^s}) > 0$ such that there is a unique solution $u = S_t^\infty(u_0) \in C([-T, T]; H^\infty)$ of the Cauchy problem (1.1). In addition, for any $\sigma \geq s$

$$
\sup_{|t| \leq T} \|S_t^\infty(u_0)(t)\|_{H^{\sigma}} \leq C(T, \sigma, \|u_0\|_{H^s}). 
$$

(1.7)

(b) Moreover, the mapping $S_t^\infty : H^\infty \to C([-T, T]; H^\infty)$ extends uniquely to a continuous mapping

$$
S_t^s : H^s \to C([-T, T]; H^s).
$$

If $1/3 < \alpha \leq 1$, then $1 - \alpha < \frac{1+\alpha}{2}$. Thus from the a priori bound (1.5), and iterating Theorem 1.1, we obtain the following corollary.

**Corollary 1.2.** The Cauchy problem (1.1) is globally well-posed in $H^s$ for $s \geq \frac{1+\alpha}{2}$ if $1/3 < \alpha \leq 1$.

**Remark 1.3.** It is easy to see $1 - \alpha < \frac{1}{2}(2 - \alpha)$ for $0 \leq \alpha \leq 1$, thus our results improve the results in [16]. In view of the results for BO and KdV equation, the results are of course not optimal.

**Remark 1.4.** Although stronger results ($L^2$ GWP) were proved in [11] after the first version of this paper by using a para-differential gauge transform, our results have independent interests. Our proofs actually only rely on the order of dispersive effect and the energy structure, but not on the exact dispersion relation. Our methods also give a general strategy to deal with the exponential divergence in high–low interaction which may be useful in some other problems.

We discuss now some of the ingredients in the proof of Theorem 1.1. We will adapt the method of Ionescu, Kenig and Tataru [13] to approach the problem in a less perturbative way. It can be viewed as a combination of the energy methods and the perturbative methods. More precisely, we will define $F^s(T)$, $N^s(T)$ and energy space $E^s(T)$ and show that if $u$ is a smooth solution of (1.1) on $\mathbb{R} \times [-T, T]$ then

$$
\begin{align*}
\|u\|_{F^s(T)} &\lesssim \|u\|_{E^s(T)} + \|\partial_x(u^2)\|_{N^s(T)}; \\
\|\partial_x(u^2)\|_{N^s(T)} &\lesssim \|u\|_{F^s(T)}^2; \\
\|u\|_{E^s(T)}^2 &\lesssim \|\phi\|_{H^s}^2 + \|u\|_{F^s(T)}^2.
\end{align*}
$$

(1.8)

The inequalities (1.8) and a continuity argument still suffice to control $\|u\|_{F^s(T)}$, provided that $\|\phi\|_{H^s} \ll 1$ (which can by arranged by rescaling if $s \geq 0$). The first inequality in (1.8) is the analogue of the linear estimate. The second inequality in (1.8) is the analogue of the bilinear estimate. The last inequality in (1.8) is an energy-type estimate. To prove Theorem 1.1(b), we need to study the difference equation of Eq. (1.1). This difference equation has less symmetries, but some special symmetries for real-valued solutions in $L^2$. We then follow the methods in [13] to prove the continuity of the solution mapping in $H^s$ by adapting the Bona–Smith method [1].

We will develop the ideas in [13] to define the space structures. As was explained before, standard using of $X^{s,b}$ spaces does not work for (1.1). The failure is due to the high–low frequency interaction of
the type: $P_{\text{low}}u \partial_x P_{\text{hi}}u$. If measuring in $X^{s,b}$ space, we fail to control this interaction due to exponential divergence. Actually, we show (see Proposition 5.2 below)\[\| (\tau - \omega(\xi) + i)^{-1} \partial_x (uv) \|_{X_k} \lesssim 2^{k(1-\alpha)/2} \| \hat{u} \|_{X_0} \| \hat{v} \|_{X_k} \| \hat{v} \|_{X_k},\]

(1.9)

hold for all $u, v$, and (1.9) is optimal in the sense that there are $u, v$ such that
\[\| (\tau - \omega(\xi) + i)^{-1} \partial_x (uv) \|_{X_k} \gtrsim 2^{k(1-\alpha)/2} \| \hat{u} \|_{X_0} \| \hat{v} \|_{X_k} \| \hat{v} \|_{X_k}.\]

(1.10)

However, for the low frequency away from 0, one can overcome this by putting a weight on the $X^{s,b}$ structure. This was also observed in [11] and a para-differential gauge was used to remove $P \leq 0 u \partial_x P_{\text{hi}}u$. In this paper, we use a different strategy inspired by [13]. In order to control $P \leq 0 u \partial_x P_{\text{hi}}u$, we use $X^{s,b}$-type structures only on small, frequency dependent time intervals of length $2^{-[(1-\alpha)k]}$ in view of (1.10). The length of the time interval will be important. Generally, one needs to control the interaction in as large time interval as possible and leave the rest to be controlled in the energy estimates. Since we only control the interaction in short time then we need to define $\| u \|_{E^s(T)}$ sufficiently large to be able to still prove the linear estimate in (1.8). Finally, we use frequency-localized energy estimates and the symmetries of Eq. (1.1) to prove the energy estimates. Since the length of interval is chosen according to the dispersive effect, the price to pay is we need to deal with some technical difficulties caused by the fractional calculus and commutator estimates.

As a by-product, we use our estimates for the multiplier (we will study a very general class) to study the following modified equation
\[\partial_t u + |\partial_x|^{1+\alpha} \partial_x u \mp u^2 u_x = 0,\]

(1.11)

When $\alpha = 0$ and $\alpha = 1$, it corresponds to the modified Benjamin–Ono equation and modified Korteweg–de Vries equation. Both equations were also extensively studied [19,20,17,6]. The high–low interactions in the trilinear estimates are much weaker than that in the bilinear estimates. Indeed, it is known that for $\alpha = 1$ the high–low interactions are under control [14] and for $\alpha = 0$ the high–low interactions only cause logarithmic divergence which is removable [6]. So it is natural to conjecture that for $\alpha > 0$ the high–low interactions are also under control and a direct using of $X^{s,b}$ space would suffice for a well-posedness as in [14]. We proved the following

**Theorem 1.5.** Let $0 < \alpha \leq 1$ and $\phi \in H^s$ for $s \geq 1/2 - \alpha/4$. Then there exist $T = T (\| \phi \|_{H^{1/2-\alpha/4}}) > 0$ and a unique solution $u \in X^{s,1/2+}_T$ to (1.11) on $(-T, T)$. Moreover, the solution mapping $\phi \rightarrow u$ is locally Lipschitz continuous from $H^s$ to $C([-T, T] : H^s)$.

On the other hand, Eq. (1.11) has also several conservation laws: if $u$ is a smooth solution to (1.11) then
\[\frac{d}{dt} \int_R u(x, t) \, dx = 0,\]

(1.12)
\[\frac{d}{dt} \int_R u(x, t)^2 \, dx = 0,\]

(1.13)
\[\frac{d}{dt} \int_R |\partial_x|^{1+\alpha} u| \, dx = \frac{1}{12} u(x, t)^4 \, dx = 0.\]

(1.14)
It is easy to see that Eq. (1.11) is invariant under the following scaling transform: for any \( \lambda > 0 \)

\[
u(x, t) \rightarrow \nu(x, t) = \lambda^{1-\alpha} \nu(\lambda x, \lambda^{2+\alpha} t), \quad \nu_{0, \lambda} = \lambda^{1-\alpha} \nu_{0, \lambda}.
\] (1.15)

Then we see \( L^2 \) is subcritical space in the sense of the scaling and easily obtain the a priori bound: if \( \nu \) is a smooth solution to (1.11) (both focusing and defocusing) then for any \( t \in \mathbb{R} \)

\[
\|\nu\|_{H^{1/2}} \leq C(\|\phi\|_{H^{1/2}}).
\] (1.16)

From \( 1/2 - \alpha/4 < \frac{1+\alpha}{2} \) and the a priori bound (1.16), and iterating Theorem 1.1, we obtain the following corollary.

**Corollary 1.6.** The Cauchy problem (1.11) is globally well-posed in \( H^s \) for \( s \geq \frac{1+\alpha}{2} \) if \( 0 < \alpha \leq 1 \).

The rest of the paper is organized as follows: In Section 2 we present some notations and Banach function spaces. The estimates for the characterization multiplier will be given in Section 3. In Section 4 we prove Theorem 1.5. In Section 5 we prove some short-time bilinear estimates. We prove Theorem 1.1 in Section 6 using the energy estimates obtained in Section 7.

### 2. Notation and definitions

Throughout this paper, we fix \( 0 \leq \alpha < 1 \). For \( x, y \in \mathbb{R}^+ \), \( x \leq y \) means that there exists \( C > 0 \) such that \( x \leq Cy \). By \( x \sim y \) we mean \( x \leq y \) and \( y \leq x \). For \( x \in \mathbb{R} \), let \( [x] \) be the largest integer that is less than or equal to \( x \). Let \( a_1, a_2, a_3 \in \mathbb{R} \). It will be convenient to define the quantities \( a_{\text{max}} \geq a_{\text{med}} \geq a_{\text{min}} \) to be the maximum, median, and minimum of \( a_1, a_2, a_3 \) respectively. Usually we use \( k_1, k_2, k_3 \) and \( j_1, j_2, j_3 \) to denote integers, \( N_i = 2^{k_i} \) and \( L_i = 2^{j_i} \) for \( i = 1, 2, 3 \) to denote dyadic numbers.

For \( f \in \mathcal{S} \) we denote by \( \mathcal{F} \) or \( \mathcal{F}(f) \) the Fourier transform of \( f \) for both spatial and time variables,

\[
\mathcal{F}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-i\tau t} f(x, t) \, dx \, dt.
\]

Moreover, we use \( \mathcal{F}_x \) and \( \mathcal{F}_t \) to denote the Fourier transform with respect to space and time variable respectively. Let \( \mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty) \), and for \( k \in \mathbb{Z} \) let

\[
I_k = \{\xi: |\xi| \in [2^{k-1}, 2^{k+1}]\}, \quad I_{\leq k} = \{\xi: |\xi| \leq 2^{k+1}\}.
\]

Let \( \eta_0: \mathbb{R} \rightarrow [0, 1] \) be an even smooth function supported in \([-8/5, 8/5] \) and equal to 1 in \([-5/4, 5/4] \). For \( k \in \mathbb{Z} \) let \( \chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1}) \), and define \( \eta_k = \chi_k \) if \( k \geq 1 \) and \( \eta_k \equiv 0 \) if \( k \leq -1 \). \( \chi_{\leq k}(\xi) = \eta_0(\xi/2^k) \). For \( k \in \mathbb{Z} \) we define the operators on \( L^2(\mathbb{R}) \)

\[
\mathcal{P}_k u(\xi) = \chi_k(\xi) \hat{u}(\xi), \quad \mathcal{P}_{\leq k} u(\xi) = \chi_{\leq k}(\xi) \hat{u}(\xi).
\]

By a slight abuse of notations we also define the operators \( P_k \) on \( L^2(\mathbb{R} \times \mathbb{R}) \) by formulas \( \mathcal{F}(P_k u)(\xi, \tau) = \chi_k(\xi) \mathcal{F}(u)(\xi, \tau) \), and similar for \( P_{\leq k} \). For \( k, j \in \mathbb{Z}_+ \) let

\[
D_{k, j} = \{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega(\xi) \in I_j\}, \quad D_{k, \leq j} = \bigcup_{l \leq j} D_{k, l}.
\]
For $\xi \in \mathbb{R}$ let
\[
\omega(\xi) = -\xi |\xi|^{1+\alpha}
\] (2.1)
which is the dispersion relation of (1.1). For $\phi \in L^2(\mathbb{R})$ let $W(t)\phi \in C(\mathbb{R} : L^2)$ be the linear solution given by
\[
\mathcal{F}_x[W(t)\phi](\xi, t) = e^{it \omega(\xi)} \hat{\phi}(\xi).
\] (2.2)

We introduce the $X^{s,b}$ norm associated to Eq. (1.1) which is given by
\[
\|u\|_{X^{s,b}} = \left\|\langle \tau - \omega(\xi) \rangle^b \langle \xi \rangle^s \hat{u}(\xi, \tau)\right\|_{L^2(\mathbb{R}^2)},
\]
where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. The spaces $X^{s,b}$ turn out to be very useful in the study of low-regularity theory for the dispersive equations. These spaces were first used to systematically study nonlinear dispersive wave problems by Bourgain [2] and developed by Kenig, Ponce and Vega [14] and Tao [24]. Klainerman and Machedon [18] used similar ideas in their study of the nonlinear wave equation. We denote by $X^{s,b}_T$ the space that $X^{s,b}$ localized to the interval $[-T, T]$.

For $k \in \mathbb{Z}^+$ we define the dyadic $X^{s,b}$-type normed spaces $X_k(\mathbb{R}^2)$:
\[
X_k = \left\{ f \in L^2(\mathbb{R}^2) : \text{supp} f(\xi, \tau) \subset I_k \times \mathbb{R} \ (I_{\leq 0} \times \mathbb{R} \text{ if } k = 0) \text{ and } \|f\|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - W(\xi)) f(\xi, \tau)\|_{L^2(\mathbb{R}^2)} < \infty \right\}.
\]

These $l^1$-type $X^{s,b}$ structures were first introduced in [27] and used in [12,13,26,8]. The definition shows easily that if $k \in \mathbb{Z}^+$ and $f_k \in X_k$ then
\[
\left\| \int_{\mathbb{R}} |f_k(\xi, \tau')| \, d\tau' \right\|_{L^2_{\xi}} \lesssim \|f_k\|_{X_k}.
\] (2.3)

Moreover, it is easy to see (see [6]) that if $k \in \mathbb{Z}^+$, $l \in \mathbb{Z}^+$, and $f_k \in X_k$ then
\[
\sum_{j=l+1}^{\infty} 2^{j/2} \left\| X_j(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-d} \, d\tau' \right\|_{L^2}\]
\[
+ 2^{l/2} \left\| X_{\leq l}(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-d} \, d\tau' \right\|_{L^2} \lesssim \|f_k\|_{X_k}.
\] (2.4)

In particular, if $k \in \mathbb{Z}^+$, $l \in \mathbb{Z}^+$, $t_0 \in \mathbb{R}$, $f_k \in X_k$, and $\gamma \in \mathcal{S}(\mathbb{R})$, then
\[
\|\mathcal{F}[\gamma (2^l (t - t_0)) \cdot \mathcal{F}^{-1}(f_k)]\|_{X_k} \lesssim \|f_k\|_{X_k}.
\] (2.5)

At frequency $2^k$ we will use the $X^{s,b}$ structure given by the $X_k$ norm, uniformly on the $2^{-[(1-\alpha)k]}$ time scale. For $k \in \mathbb{Z}^+$ we define the normed spaces
\[
F_k = \left\{ \begin{array}{l}
 f \in L^2(\mathbb{R}^2) : \text{supp} \tilde{f}(\xi, \tau) \subset I_k \times \mathbb{R} \ (I \subseteq \mathbb{R} \text{ if } k = 0) \quad \text{and} \\
 \|f\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}\{f \cdot \eta_0(2^{(1-\alpha)k}(t-t_k))\}\|_{X_k} < \infty
\end{array} \right\}; \]

\[
N_k = \left\{ \begin{array}{l}
 f \in L^2(\mathbb{R}^2) : \text{supp} \tilde{f}(\xi, \tau) \subset I_k \times \mathbb{R} \ (I \subseteq \mathbb{R} \text{ if } k = 0) \quad \text{and} \\
 \|f\|_{N_k} = \sup_{t_k \in \mathbb{R}} \|\tau - \omega(\xi) + i2^{(1-\alpha)k} \|^{-1} \mathcal{F}\{f \cdot \eta_0(2^{(1-\alpha)k}(t-t_k))\}\|_{X_k} < \infty
\end{array} \right\};
\]

We see from the definitions that we still use \(X^{s,b}\) structure on the whole interval for the low frequency. Since the spaces \(F_k\) and \(N_k\) are defined on the whole line, we define then local versions of the spaces in standard ways. For \(T \in (0,1]\) we define the normed spaces

\[
F_k(T) = \left\{ f \in C([-T, T] : L^2) : \|f\|_{F_k(T)} = \inf_{\tilde{f} = f \in \mathbb{R} \times [-T, T]} \|\tilde{f}\|_{F_k}\right\};
\]

\[
N_k(T) = \left\{ f \in C([0, T] : L^2) : \|f\|_{N_k(T)} = \inf_{\tilde{f} = f \in \mathbb{R} \times [-T, T]} \|\tilde{f}\|_{N_k}\right\};
\]

We assemble these dyadic spaces in a Littlewood–Paley manner. For \(s \geq 0\) and \(T \in (0,1]\), we define the normed spaces

\[
F^s(T) = \left\{ u : \|u\|^2_{F^s(T)} = \sum_{k=1}^{\infty} 2^{sk} \|P_k(u)\|^2_{F_k(T)} + \|P_{\leq 0}(u)\|^2_{F_0(T)} < \infty \right\};
\]

\[
N^s(T) = \left\{ u : \|u\|^2_{N^s(T)} = \sum_{k=1}^{\infty} 2^{sk} \|P_k(u)\|^2_{N_k(T)} + \|P_{\leq 0}(u)\|^2_{N_0(T)} < \infty \right\};
\]

We define the dyadic energy space. For \(s \geq 0\) and \(u \in C([-T, T] : H^s)\) we define

\[\|u\|^2_{E(T)} = \|P_{\leq 0}(u(0))\|^2_{L^2} + \sum_{k=1}^{\infty} \sup_{t_k \in [-T, T]} 2^{2sk} \|P_k(u(t_k))\|^2_{L^2}.\]

As in [13], for any \(k \in \mathbb{Z}_+\) we define the set \(S_k\) of \(k\)-acceptable time multiplication factors

\[
S_k = \left\{ m_k : \mathbb{R} \rightarrow \mathbb{R} : \|m_k\|_{S_k} = \sum_{j=0}^{10} 2^{-j(1-\alpha)k} \|\partial^j m_k\|_{L^\infty} < \infty \right\}.
\]

For instance, \(\eta(2^{(1-\alpha)k}t) \in S_k\) for any \(\eta\) satisfies \(\|\partial^j \eta\|_{L^\infty} \leq C\) for \(j = 0, 1, 2, \ldots, 10\). Direct estimates using the definitions and (2.4) show that for any \(s \geq 0\) and \(T \in (0,1]\)

\[
\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \right\|_{F^s(T)} \lesssim \left( \sup_{k \in \mathbb{Z}_+} \|m_k\|_{S_k} \right) \cdot \|u\|_{F^s(T)};
\]

\[
\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \right\|_{N^s(T)} \lesssim \left( \sup_{k \in \mathbb{Z}_+} \|m_k\|_{S_k} \right) \cdot \|u\|_{N^s(T)};
\]

\[
\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \right\|_{E^s(T)} \lesssim \left( \sup_{k \in \mathbb{Z}_+} \|m_k\|_{S_k} \right) \cdot \|u\|_{E^s(T)}.
\]
3. A symmetric estimate

In this section we prove symmetric estimates which will be used to prove bilinear estimates. In order for later use, we work in an abstract setting.

**Definition 3.1.** Let $\beta > 0$ and $h : \mathbb{R} \to \mathbb{R}$ be smooth on $\mathbb{R} \setminus \{0\}$. $h$ is said to have $\beta$-order dispersive effect at high frequency and denoted by $h \in D_{hi}(\beta)$, if for $|\xi| \gtrsim 1$

$$|\partial^k_x h(\xi)| \sim |\xi|^{\beta-k}, \quad k = 1, 2; \quad |\partial^j_x h(\xi)| \lesssim |\xi|^{\beta-j}, \quad j \geq 3.$$

For example, the KdV dispersion $\xi^3 \in D_{hi}(3)$, and dispersion relation to (1.1) $\omega(\xi) = -|\xi|^{1+\alpha} \xi \in D_{hi}(2+\alpha)$. For $\xi_1, \xi_2 \in \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ let

$$\Omega(\xi_1, \xi_2) = h(\xi_1) + h(\xi_2) - h(\xi_1 + \xi_2). \quad (3.1)$$

This is the resonance function that plays a crucial role in the bilinear estimate of the $X^{s,b}$-type space. See [24] for a perspective discussion. For compactly supported nonnegative functions $f_1, f_2, f_3 \in L^2(\mathbb{R} \times \mathbb{R})$ let

$$J(f_1, f_2, f_3) = \int_{\mathbb{R}^4} f_1(\xi_1, \mu_1)f_2(\xi_2, \mu_2)f_3(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega(\xi_1, \xi_2)) \, d\xi_1 \, d\xi_2 \, d\mu_1 \, d\mu_2.$$

**Lemma 3.2.** Assume $\beta > 1$, $h \in D_{hi}(\beta)$ and $h$ is odd,$^1$ $k_i \in \mathbb{Z}$, $j_i \in \mathbb{Z}_+$, $N_i = 2^{k_i}$, $L_i = 2^{j_i}$, $i = 1, 2, 3$. Let $f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ be nonnegative functions supported in $I_{k_i} \times I_{j_i}$, $i = 1, 2, 3$. Then:

(a) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$,

$$J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \lesssim 2^{j_{\min}/2} 2^{k_{\min}/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2}. \quad (3.2)$$

(b) If $N_{\min} \ll N_{\med} \sim N_{\max}$, then for $q \in \{1, 2, 3\}$

$$J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \lesssim 2^{(j_1+j_2+j_3)/2} 2^{-(\beta-2)k_{\max}/2} 2^{-(j_q+k_q)/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2}. \quad (3.3)$$

(c) If $N_{\min} \sim N_{\med} \sim N_{\max} \gg 1$ then

$$J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \lesssim 2^{j_{\min}/4} 2^{j_{\med}/4} 2^{-(\beta-2)k_{\max}/4} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2}. \quad (3.4)$$

**Proof.** Simple changes of variables in the integration and the observation that the function $\omega$ is odd show that

$$|J(f_1, f_2, f_3)| = |J(f_2, f_1, f_3)| = |J(f_1, f_3, f_2)| = |J(\tilde{f}_1, f_2, f_3)|,$$

$^1$ If $h$ is not odd, the same methods work, although a little bit more complicated due to less symmetries.
where \( \widetilde{f}_1(\xi, \mu) = f_1(-\xi, -\mu) \). Let \( A_{k_i}(\xi) = \left[ \int_\mathbb{R} |f_{k_i,j_i}(\xi, \mu)|^2 \, d\mu \right]^{1/2}, \) \( i = 1, 2, 3 \). Using the Cauchy–Schwarz inequality we obtain

\[
J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \lesssim 2^{J_{\min}/2} \int_{\mathbb{R}^2} A_{k_1}(\xi_1) A_{k_2}(\xi_2) A_{k_3}(\xi_1 + \xi_2) \, d\xi_1 \, d\xi_2
\]

\[
\lesssim 2^{k_{\min}/2} 2^{J_{\min}/2} \prod_{i=1}^3 \| f_{k_i,j_i} \|_{L^2}.
\]

which is part (a), as desired.

For part (b), from symmetry we may assume \( k_1 \leq k_2 \leq k_3 \). If \( j_3 = j_{\max} \) then we will prove that if \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) functions supported in \( I_{k_i}, \) \( i = 1, 2, \) and \( g : \mathbb{R}^2 \to \mathbb{R}_+ \) is an \( L^2 \) function supported in \( I_{k_3} \times I_{\leq j_3} \), then

\[
\int_{\mathbb{R}^3} g_1(\xi_1) g_2(\xi_2) g(\xi_1 + \xi_2, \Omega(\xi_1, \xi_2)) \, d\xi_1 \, d\xi_2 \lesssim 2^{-2(\beta-1)k_{\max}/2} \| g_1 \|_{L^2} \| g_2 \|_{L^2} \| g \|_{L^2}.
\]

This suffices for (3.3) for \( q = 3 \) by Cauchy–Schwarz inequality.

To prove (3.5), we observe that since \( g_1 = \Omega(\xi_1, \xi_2 - \xi_1) \) we get that the left-hand side of (3.5) is bounded by

\[
\int_{|\xi_1| \sim N_1, |\xi_2| \sim N_2} g_1(\xi_1) g_2(\xi_2 - \xi_1) g(\xi_2, \Omega(\xi_1, \xi_2 - \xi_1)) \, d\xi_1 \, d\xi_2.
\]

Note that in the integration area we have

\[
|\partial_{\xi_1} \left[ \Omega(\xi_1, \xi_2 - \xi_1) \right]| = |h'(\xi_1) - h'(\xi_2 - \xi_1)| \sim N_2^{\beta-1},
\]

where we used \( h \in D_{hi}(\beta) \) and \( N_1 \ll N_2 \). By change of variable \( \mu_2 = \Omega(\xi_1, \xi_2 - \xi_1) \) we get that (3.6) is bounded by

\[
2^{-2(\beta-1)k_{\max}/2} \| g_1 \|_{L^2} \| g_2 \|_{L^2} \| g \|_{L^2}.
\]

If \( j_2 = j_{\max} \), then this case is identical to the case \( j_3 = j_{\max} \). If \( j_1 = j_{\max} \) it suffices to prove that if \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) functions supported in \( I_{k_i}, \) \( i = 2, 3, \) and \( g : \mathbb{R}^2 \to \mathbb{R}_+ \) is an \( L^2 \) function supported in \( I_{k_3} \times I_{\leq j_3} \), then

\[
\int_{\mathbb{R}^2} g_2(\xi_2) g_3(\xi_3) g(\xi_2 + \xi_3, \Omega(\xi_2, \xi_3)) \, d\xi_2 \, d\xi_3 \lesssim N_2^{-(\beta-2)/2} N_1^{-1/2} \| g_2 \|_{L^2} \| g_3 \|_{L^2} \| g \|_{L^2}.
\]

Indeed, by change of variables \( \xi'_2 = \xi_2, \xi'_3 = \xi_2 + \xi_3 \) and noting that in the integration area \( |\xi'_2| \sim 2^{k_2}, |\xi'_3| \sim 2^{k_3}, \)

\[
|\partial_{\xi_2} \left[ \Omega(\xi'_2, \xi'_3 - \xi'_2) \right]| = |h'(\xi'_2) - h'(\xi'_3 - \xi'_2)| \sim N_2^{\beta-2} N_1,
\]

then we get from Cauchy–Schwarz inequality that
\[
\begin{aligned}
&\int_{\mathbb{R}^2} g_2(\xi_2) g_3(\xi_3) g(\xi_2 + \xi_3, \Omega(\xi_2, \xi_3))
\lesssim \int_{|\xi_1| < 2^k_1, |\xi_2| < 2^j_2} g_2(\xi_2') g_3(\xi_2 - \xi_2') g(\xi_2', \Omega(\xi_2', \xi_2 - \xi_2'))
\lesssim N_2^{-(\beta-2)/2} N_1^{-1/2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2},
\end{aligned}
\]

which is (3.7) as desired.

We prove now part (c). For simplicity we denote \( f_i = f_{k_i,j_i}, i = 1, 2, 3 \). From symmetries we may assume \( \xi_j = \xi_{\max} \) and \(|\xi_1| \leq |\xi_2|\). Then writing \( d\sigma = d\xi_1 d\xi_2 d\mu_1 d\mu_2 \) we have

\[
\begin{aligned}
&\int_{\mathbb{R}^4} f_1(\xi_1, \mu_1) f_2(\xi_2, \mu_2) f_3(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega(\xi_1, \xi_2))
\lesssim \left( \int_{\xi_1, \xi_2 < 0} + \int_{\xi_1, \xi_2 > 0} \right) f_1(\xi_1, \mu_1) f_2(\xi_2, \mu_2) f_3(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega(\xi_1, \xi_2))
\lesssim I + II.
\end{aligned}
\]

For the contributions of \( I \), noting that if \( \xi_1 \cdot \xi_2 < 0 \) and \(|\xi_1| \leq |\xi_2|\) then

\[
|\partial_{\xi_1} \Omega(\xi_1, \xi_2 - \xi_1)| = |h'(\xi_1) - h'(\xi_2 - \xi_1)| \sim N_2^{\beta-1},
\]

thus by change of variable \( \xi_2 = \xi_2 - \xi_1 \) and as for part (b) we get

\[
I \lesssim 2^{(j_{\min} + j_{\med})/2} 2^{-(\beta-1)k_{\max}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}.
\]

Interpolating with part (a), we immediately get the bound (3.4) for this term.

For the contribution of \( II \), we break it into two parts

\[
II = \left( \int_{|\xi_1 - \xi_2| \leq R} + \int_{|\xi_1 - \xi_2| \geq R} \right) f_1(\xi_1, \mu_1) f_2(\xi_2, \mu_2) f_3(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega(\xi_1, \xi_2))
\lesssim II_1 + II_2,
\]

where \( R \) will be determined later. For \( II_1 \), we may assume \( j_1 = j_{\min} \) and as for part (a) we get

\[
II_1 = \int_{|\xi_1| \leq R} f_1(\xi_1 + \xi_2, \mu_1) f_2(\xi_2, \mu_2) f_3(\xi_1 + 2\xi_2, \mu_1 + \mu_2 + \Omega(\xi_1 + \xi_2, \xi_2))
\lesssim 2^{j_{\min}/2} R^{1/2} \prod_{i=1}^{3} \|f_i\|_{L^2},
\]

For \( II_2 \) we get from Cauchy–Schwarz inequality that
Corollary 3.3. Assume \( \beta > 1 \), \( h \in D_{hi}(\beta) \) and \( h \) is odd, \( k_i \in \mathbb{Z}, j_i \in \mathbb{Z}_+ \), \( N_i = 2^{k_i}, L_i = 2^{j_i}, i = 1, 2, 3 \). Let \( f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) be supported \( D_{k_i,j_i} \), \( i = 1, 2 \).

(a) For any \( k_1, k_2, k_3 \in \mathbb{Z} \) and \( j_1, j_2, j_3 \in \mathbb{Z}_+ \),

\[
\left\| \prod_{i=1}^{3} \left( f_{k_i,j_i} \right) \right\|_{L^2} \lesssim 2^{k_{\text{min}}/2} 2^{j_{\text{min}}/2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2}.
\]

(b) If \( N_{\text{min}} \ll N_{\text{med}} \sim N_{\text{max}} \), then for \( q \in \{1, 2, 3\} \)

\[
\left\| \prod_{i=1}^{3} \left( f_{k_i,j_i} \right) \right\|_{L^2} \lesssim 2^{(j_1+j_2+j_3)/2} N_{\text{max}}^{(\beta-2)/2} 2^{-(j_1+k_q)/2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2}.
\]

(c) If \( N_{\text{min}} \sim N_{\text{med}} \sim N_{\text{max}} \gg 1 \) then

\[
\left\| \prod_{i=1}^{3} \left( f_{k_i,j_i} \right) \right\|_{L^2} \lesssim 2^{j_{\text{med}}/4} 2^{(\beta-2)k_{\text{max}}/4} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2}.
\]

Remark 3.4. In Lemma 3.2, if assume \( f_{k_i,j_i} \) is supported in \( I_{k_i} \times I_{j_i} \) (\( I_{\leq 0} \times I_{\leq j_i} \) for \( k_i = 0 \)) for \( k_i, j_i \in \mathbb{Z}_+ \), then part (a) also holds, and part (b) holds for \( q \in \{1, 2, 3\} \) with \( k_q \neq 0 \). Similarly for Corollary 3.3.

4. Trilinear estimates

In this section we prove Theorem 1.5. The ingredients are the estimates for the characterization multiplier obtained in the last section and \( TT^* \) arguments as in [24]. The main issues reduce to prove the trilinear estimates and we refer the readers to [14] for the rest standard details.
Proposition 4.1. For all $u_1, u_2, u_3$ on $\mathbb{R} \times \mathbb{R}$ and $0 < \epsilon \ll 1$, we have
\[
\left\| (u_1 u_2 u_3) x \right\|_{X^{1/2-\alpha/4-1/2+\epsilon}} \lesssim \prod_{j=1}^{3} \left\| u_j \right\|_{X^{1/2-\alpha/4+1/2+\epsilon}} \tag{4.1}
\]
with the implicit constant depending on $\epsilon$.

This type of estimate was systematically studied in [24], see also [14] for an elementary method. We will follow the idea in [24] to prove Proposition 4.1. Let $Z$ be any abelian additive group with an invariant measure $d\xi$. In particular, $Z = \mathbb{R}^2$ in this paper. For any $k \geq 2$, let $\Gamma_k(Z)$ denote the hyperplane in $\mathbb{R}^k$

$$\Gamma_k(Z) := \{ (\xi_1, \ldots, \xi_k) \in \mathbb{Z}^k : \xi_1 + \cdots + \xi_k = 0 \}$$

endowed with the induced measure

$$\int_{\Gamma_k(Z)} f := \int_{\mathbb{Z}^{k-1}} f(\xi_1, \ldots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}) \, d\xi_1 \cdots d\xi_{k-1}.$$ 

Note that this measure is symmetric with respect to permutation of the co-ordinates.

A function $m : \Gamma_k(Z) \to \mathbb{C}$ is said to be a $[k; Z]$-multiplier, and we define the norm $\|m\|_{[k; Z]}$ to be the best constant such that the inequality
\[
\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_i(\xi_j) \right| \leq \|m\|_{[k; Z]} \prod_{j=1}^{k} \|f_i\|_{L^2} \tag{4.2}
\]
holds for all test functions $f_i$ on $Z$.

Proof of Proposition 4.1. By duality, Plancherel’s equality and the definition, it is easy to see that for (4.1), it suffices to prove
\[
\left\| \frac{(\xi_1 + \xi_2 + \xi_3)(\xi_4)^{1/2-\alpha}}{(\tau_4 - \omega(\xi_4))^{1/2-\epsilon} \prod_{j=1}^{3} (\xi_j)^{1/2-\alpha/4} (\tau_j - \omega(\xi_j))^{1/2+\epsilon}} \right\|_{[4; \mathbb{R} \times \mathbb{R}]} \lesssim 1. \tag{4.3}
\]

As in the proof of Corollary 6.3 in [24], we estimate $|\xi_1 + \xi_2 + \xi_3|$ by $\langle \xi_4 \rangle$. From the inequality

$$\langle \xi_4 \rangle^{2-\alpha/4} \lesssim \langle \xi_4 \rangle^{1/2} \sum_{j=1}^{3} \langle \xi_j \rangle^{1-\alpha/4}$$

and symmetry it reduces to show
\[
\left\| \frac{(\xi_4)^{1/2} (\xi_2)^{1/2}}{(\xi_1)^{2-\alpha/4} (\xi_3)^{2-\alpha/4} (\tau_4 - \omega(\xi_4))^{1/2-\epsilon} \prod_{j=1}^{3} (\tau_j - \omega(\xi_j))^{1/2+\epsilon}} \right\|_{[4; \mathbb{R} \times \mathbb{R}]} \lesssim 1.
\]

We may minorize $(\tau_2 - \omega(\xi_2))^{1/2+\epsilon}$ by $(\tau_2 - \omega(\xi_2))^{1/2-\epsilon}$. But then the estimate follows from $TT^*$ identity [24, Lemma 3.7] and the following proposition. □
Proposition 4.2. For all \( u, v \) on \( \mathbb{R} \times \mathbb{R} \) and \( 0 < \varepsilon \ll 1 \), we have
\[
\|uv\|_{L^2} \lesssim \|u\|_{X^{-1/2,1/2-\varepsilon}} \|v\|_{X^{1/2-\alpha/4,1/2+\varepsilon}}.
\]

Proof. By Plancherel's equality it suffices to show that
\[
\left\| \frac{\langle \xi_2 \rangle^{1/2}}{\langle \xi_1 \rangle^{1/2} (\tau_2 - \omega(\xi_2))^{1/2-\varepsilon} (\tau_1 - \omega(\xi_1))^{1/2+\varepsilon}} \right\|_{[3;\mathbb{R} \times \mathbb{R}]} \lesssim 1. \tag{4.4}
\]

By comparison principle (see [24]), it suffices to prove that
\[
\sum_{N_1,N_2,N_3} \sum_{L_1,L_2,L_3} \sum_H \frac{(N_2)^{1/2}}{(N_1)^{1/2} L_1^{1/2+\varepsilon} L_2^{1/2-\varepsilon}} \| X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \|_{[3;\mathbb{R}^2]} \lesssim 1,
\]
where \( N_i, L_i, H \) are dyadic numbers, \( h(\xi_1, \xi_2, \xi_3) = \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) \) and
\[
X_{N_1,N_2,N_3;H;L_1,L_2,L_3} = X_{\xi_1 \sim N_1, \xi_2 \sim N_2, \xi_3 \sim N_3} X_{h \sim H} X_{\tau_1 - \omega(\xi_1) \sim L_1, \tau_2 - \omega(\xi_2) \sim L_2, \tau_3 - \omega(\xi_3) \sim L_3}. \tag{4.5}
\]
The issues reduce to the estimates of
\[
\| X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \|_{[3;\mathbb{R}^2]} \tag{4.6}
\]
and dyadic summations.

From the identity
\[
\xi_1 + \xi_2 + \xi_3 = 0
\]
and
\[
\tau_1 - \omega(\xi_1) + \tau_2 - \omega(\xi_2) + \tau_3 - \omega(\xi_3) + h(\xi) = 0,
\]
then we must have for the multiplier in (4.6) to be nonvanishing
\[
N_{\text{max}} \sim N_{\text{med}},
\]
\[
L_{\text{max}} \sim \max(L_{\text{med}}, H), \tag{4.7}
\]
where we define \( N_{\text{max}} \geq N_{\text{med}} \geq N_{\text{min}} \) to be the maximum, median, and minimum of \( N_1, N_2, N_3 \) respectively. Similarly define \( L_{\text{max}} \geq L_{\text{med}} \geq L_{\text{min}} \). It's known (see [24, Section 4]) and from Lemma 5.1 that we may assume
\[
N_{\text{max}} \gtrsim 1, \quad L_1, L_2, L_3 \gtrsim 1, \quad H \sim N_{\text{max}}^{1+\alpha} N_{\text{min}}. \tag{4.8}
\]
Therefore, from Schur’s test [24, Lemma 3.11] it suffices to prove that
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N_{\text{min}}} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{(N_2)^{1/2}}{(N_1)^{1/2} L_1^{1/2+\varepsilon} L_2^{1/2-\varepsilon}} \| X_{N_1,N_2,N_3;L_{\text{max}};L_1,L_2,L_3} \|_{[3;\mathbb{R}^2]} \tag{4.9}
\]
and
are both uniformly bounded for all $N \gtrsim 1$.

Before we continue our proof, we restate Corollary 3.3 in the following lemma by taking $\beta = 2 + \alpha$, since $\omega \in D_h(2 + \alpha)$.

**Lemma 4.3.** Let $H$, $N_1$, $N_2$, $N_3$, $L_1$, $L_2$, $L_3 > 0$ obey (4.7) and (4.8). Then:

(i) If $N_{\max} \sim N_{\min}$ and $L_{\max} \sim N_{\max}^{1+\alpha} N_{\min}$, then we have

$$
(4.6) \lesssim t_{\min}^{1/2} \min^{1/4} N_{\max}^{1+\alpha} L_{\med}.
$$

(ii) If $N_2 \sim N_3 \gg N_1$ and $N_{\max}^2 N_{\min} \sim L_1 \gtrsim L_2, L_3$, then

$$
(4.6) \lesssim L_{\min}^{1/2} N_{\max}^{-(1+\alpha)/2} \min\left(N_{\max}^{1+\alpha} N_{\min}, N_{\max} L_{\med}\right)^{1/2}.
$$

Similarly for permutations.

(iii) In all other cases, we have

$$
(4.6) \lesssim t_{\min}^{1/2} N_{\max}^{-\alpha/2} \min\left(N_{\max}^{1+\alpha} N_{\min}, L_{\med}\right)^{1/2}.
$$

Fix $N$. We first prove (4.10). By (4.13) we reduce to

$$
\sum_{N_{\max} \sim N_{\med} \sim N} \sum_{L_{\max} \sim L_{\med} \sim N_{\max}^{1+\alpha} N_{\min}} \frac{\langle N_2 \rangle^{1/2}}{\langle N_1 \rangle^{1/2}} \frac{\langle N_2 \rangle^{1/2}}{\langle N_1 \rangle^{1/2}} L_{\min}^{1/2} N_{\max}^{1+\alpha} L_{\med}^{1/2} \lesssim 1.
$$

Using the estimate

$$
\frac{\langle N_2 \rangle^{1/2}}{\langle N_1 \rangle^{1/2}} \lesssim \frac{N^{1/2}}{\langle N_{\min} \rangle^{1/2}} L_{1}^{1/2+\varepsilon} L_{2}^{1/2-\varepsilon} \gtrsim L_{\min}^{1/2+\varepsilon} L_{\med}^{1/2-\varepsilon}
$$

and then performing the $L$ summations, we reduce to

$$
\sum_{N_{\max} \sim N_{\med} \sim N} \frac{N^{\alpha/2} \langle N_{\max}^{1+\alpha} N_{\min} \rangle^{\varepsilon}}{\langle N_{\min} \rangle^{1/2}} \lesssim 1
$$

which is certainly true since $0 < \alpha \leq 1$.

Now we show (4.9). We may assume $L_{\max} \sim N_{\max}^{1+\alpha} N_{\min}$.

We assume first $N_{\max} \sim N_{\min} \sim N$. In this case applying (4.11) we reduce to

$$
\sum_{L_{\max} \sim N_3} \frac{N^{1/2}}{N_{\min}^{1/2}} \frac{N^{1/2}}{N_{\max}^{1/2}} L_{\min}^{1/2} N_{\max}^{\alpha/2} L_{\med}^{1/2} \lesssim 1
$$

which is easily verified. Now we assume $N_{\max} \sim N_{\med} \gg N_{\min}$ where (4.12) applies. We have three cases.
\[ N \sim N_1 \sim N_2 \gg N_3; \quad H \sim L_3 \gtrsim L_1, L_2; \]
\[ N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3; \]
\[ N \sim N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gtrsim L_1, L_3. \]

In the first case we reduce to
\[
\sum_{N \ll N_1 \leq L_1, L_2 \leq N^{1+\alpha} N_3} N^{1/2} \left( \frac{N^2 \cdot \frac{q}{L_1^{1+2+\varepsilon}} L_2^{1/2-\varepsilon}}{N^{1/2} (N_1^{1+\alpha} N_1)^{1/2-\varepsilon} L_2^{1/2-\varepsilon}} \right)^{1/2} \left( \frac{N^{1+\alpha}}{N} \right)^{1/2} \left( \frac{N_{\text{max}}}{} N_{\text{med}} \right)^{1/2} \lesssim 1.
\]

Performing the \( N_3 \) summation we reduce to
\[
\sum_{N \ll N_1 \leq L_1, L_2 \leq N^{1+\alpha} N_1} N^{1/2} \left( \frac{N^2 \cdot \frac{q}{L_1^{1+2+\varepsilon}} L_2^{1/2-\varepsilon}}{N^{1/2} (N_1^{1+\alpha} N_1)^{1/2-\varepsilon} L_2^{1/2-\varepsilon}} \right)^{1/2} \left( \frac{N_{\text{max}}}{} N_{\text{med}} \right)^{1/2} \lesssim 1.
\]

which is easily verified.

To unify the second and third cases we replace \( L_1^{1+\varepsilon} \) by \( L_1^{1-\varepsilon} \). It suffices now to show the second case. We simplify using (4.12) to
\[
\sum_{N \ll N_1 \leq L_1, L_2 \leq N^2 N_1} N^{1/2} \left( \frac{N^2 \cdot \frac{q}{L_1^{1+2+\varepsilon}} L_2^{1/2-\varepsilon}}{N^{1/2} (N_1^{1+\alpha} N_1)^{1/2-\varepsilon} L_2^{1/2-\varepsilon}} \right)^{1/2} \left( \frac{N^{1+\alpha}}{N} \right)^{1/2} \left( \frac{N_{\text{max}}}{} N_{\text{med}} \right)^{1/2} \lesssim 1.
\]

We may assume \( N_1 \sim N^{-(1+\alpha)} \) since the inner sum vanishes otherwise. Performing the \( L \) summation we reduce to
\[
\sum_{N^{-(1+\alpha)} \ll N_1 \ll N} N^{1/2} \left( \frac{N^2 \cdot \frac{q}{L_1^{1+2+\varepsilon}} L_2^{1/2-\varepsilon}}{N^{1/2} (N_1^{1+\alpha} N_1)^{1/2-\varepsilon} L_2^{1/2-\varepsilon}} \right)^{1/2} \left( \frac{N^{1+\alpha}}{N} \right)^{1/2} \left( \frac{N_{\text{max}}}{} N_{\text{med}} \right)^{1/2} \lesssim 1.
\]

which is easily verified (with about \( N^{-\frac{q}{L}} \) to spare).

To finish the proof of (4.9) it remains to deal with the cases where (4.13) holds. This reduces to
\[
\sum_{N \ll N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim N^{1+\alpha} N_{\text{min}}} \left( \frac{N^2 \cdot \frac{q}{L_1^{1+2+\varepsilon}} L_2^{1/2-\varepsilon}}{N^{1/2} (N_1^{1+\alpha} N_1)^{1/2-\varepsilon} L_2^{1/2-\varepsilon}} \right)^{1/2} \left( \frac{N^{1+\alpha}}{N} \right)^{1/2} \left( \frac{N_{\text{max}}}{} N_{\text{med}} \right)^{1/2} \lesssim 1.
\]

Performing the \( L \) summations, we reduce to
\[
\sum_{N \ll N_{\text{med}} \sim N} \left( \frac{N^{1+\alpha} N_{\text{min}}^{1+\alpha}}{N_{\text{max}}^{1+\alpha} N_{\text{med}}^{1+\alpha}} \right)^{1/2} \left( \frac{N_{\text{max}}}{} N_{\text{med}} \right)^{1/2} \lesssim 1
\]

which is easily verified. \( \square \)

We see from the proof that \( \alpha > 0 \) plays crucial roles. The implicit constant in (4.1) depends on both \( \alpha \) and \( \varepsilon \). On the other hand, we do not need \( \alpha \lesssim 1 \) in the proof. Thus Theorem 1.5 actually holds for all \( \alpha > 0 \).
5. Short-time bilinear estimates

We prove some dyadic bilinear estimates. We will need an estimate on the resonance which follows from fundamental calculus.

Lemma 5.1. Assume \( \beta > 1 \), \( h \in D_{hi}(\beta) \) and \( h \) is odd. In the region \( |\xi|_{\text{max}} \gg 1 \)

\[
|\Omega(\xi_1, \xi_2)| \sim |\xi|_{\text{max}}^{-1}|\xi|_{\text{min}},
\]

where

\[
|\xi|_{\text{max}} = \max(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|), \quad |\xi|_{\text{min}} = \min(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|).
\]

First we give a counter-example which shows exponential divergence will occur in the standard using of \( X^{s,b} \) space. This is the main reason why we use the \( X^{s,b} \) structure in a short interval of such a length.

Proposition 5.2. Let \( 0 \leq \alpha \leq 1 \) and \( k \geq 100 \). Then for all \( u, v \)

\[
\| (\tau - \omega(\xi) + i)^{-1}\partial_x (uv) \|_{X_k} \lesssim 2^{k(1-\alpha)/2} \| \tilde{u} \|_{X_{\alpha}} \| \tilde{v} \|_{X_{\beta}}.
\]

Moreover, there exist \( u, v \) such that

\[
\| (\tau - \omega(\xi) + i)^{-1}\partial_x (uv) \|_{X_k} \gtrsim 2^{k(1-\alpha)/2} \| \tilde{u} \|_{X_{\alpha}} \| \tilde{v} \|_{X_{\beta}}.
\]

Proof. By the definition of \( X_k \) we get

\[
\| (\tau - \omega(\xi) + i)^{-1}\partial_x (uv) \|_{X_k} \lesssim \sum_{j_1, j_2, j_3 \geq 0} 2^{-j_3/2} 2^k \| 1_{D_{k,j_3}} u_{j_1} * v_{j_2} \|_{L^2},
\]

where \( u_{j_1} = \eta_{j_1}(\tau - \omega(\xi)) \tilde{u}, \ v_{j_2} = \eta_{j_2}(\tau - \omega(\xi)) \tilde{v} \). Then by Corollary 3.3(b) we get

\[
\sum_{j_1, j_2, j_3 \geq 0} 2^{-j_3/2} 2^k \| 1_{D_{k,j_3}} u_{j_1} * v_{j_2} \|_{L^2} \lesssim \sum_{j_1, j_2, j_3 \geq 0} 2^{-j_3/2} 2^k 2^{(j_1 + j_2)/2} 2^{-(\alpha+1)/2} \| u_{j_1} \|_2 \| v_{j_2} \|_2,
\]

which suffices to give the first inequality.

Next we show the sharpness. Let \( N = 2^k \) and \( \beta = N^{-1+\alpha}/2 \). Take

\[
\tilde{u} = \beta^{-1/2} 1_{l_1}(\xi) \eta_1(\tau - \omega(\xi)), \quad \tilde{v} = \beta^{-1/2} 1_{l_2}(\xi) \eta_{\leq 10}(\tau - \omega(\xi)),
\]

where \( l_1, l_2 \) are intervals given by

\[
l_1 = [\beta/2, \beta], \quad l_2 = [N, N + \beta].
\]

Thus we see \( \| u \|_{X_0} \| v \|_{X_k} \sim 1 \). On the other hand

\[
\mathcal{F}[\partial_x (uv)](\xi, \tau) = i\xi \int_{\mathbb{R} \times \mathbb{R}} \tilde{u}(\xi_1, \tau_1) \tilde{v}(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1.
\]

If \( (\xi, \tau) \) satisfies \( \xi \in [N + \frac{3\beta}{2}, N + \frac{4\beta}{2}], \ |\tau - \omega(\xi)| \leq 1 \), then we get
Therefore, we get

$$
\| \partial_x (uv) \|_{X_k} \gtrsim N^{1/2} \gtrsim N^{(1-\alpha)/2},
$$

which proves the proposition. □

**Proposition 5.3 (High-low).** If \( k_3 \geq 20 \), \(|k_2 - k_3| \leq 5\), \(0 \leq k_1 \leq k_2 - 10\), then

$$
\left\| P_{k_3} \partial_x(u_k v_k) \right\|_{X_{k_3}} \lesssim \| u_{k_1} \|_{R_{k_1}} \| v_{k_2} \|_{R_{k_2}}.
$$

**(5.1)**

**Proof.** Using the definitions and (2.5), we obtain that the left-hand side of (5.1) is dominated by

$$
C \sup_{t_k \in \mathbb{R}} \left\| (\tau - \omega(\xi)) + i2^{[(1-\alpha)k_3]} \right\|^{-1} 2^{k_3} I_{k_3}(|\xi|) 
\cdot F[u_{k_1} \eta_0(2^{[(1-\alpha)k_3]} - (t - t_k))] \cdot F[v_{k_2} \eta_0(2^{[(1-\alpha)k_3]} - (t - t_k))] \|_{X_{k_3}}.
$$

**(5.2)**

To prove Proposition 5.3, it suffices to prove that if \( j_i \gtrsim [(1-\alpha)k_3] \) and \( f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}^+ \) are supported in \( D_{k_i,j_i} \) for \( i = 1, 2 \), then

$$
2^{k_3} \sum_{j_3 \gtrsim [(1-\alpha)k_3]} 2^{-j_3/2} \| 1_{D_{k_3,j_3}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2}) \|_{L^2} 
\lesssim 2^{(j_1+j_2)/2} \| f_{k_1,j_1} \|_{L^2} \| f_{k_2,j_2} \|_{L^2}.
$$

**(5.3)**

Indeed, let \( f_{k_1} = F[u_{k_1} \eta_0(2^{[(1-\alpha)k_3]} - (t - t_k))] \) and \( f_{k_2} = F[v_{k_2} \eta_0(2^{[(1-\alpha)k_3]} - (t - t_k))] \). Then from the definition of \( X_{k_3} \) we get that (5.2) is dominated by

$$
\sup_{t_k \in \mathbb{R}} 2^{k_3} \sum_{j_3 = 0}^{\infty} 2^{j_3/2} \sum_{j_1, j_2 \approx [(1-\alpha)k_3]} \| (2^{j_3} + i2^{[(1-\alpha)k_3]})^{-1} 1_{D_{k_3,j_3}} \cdot f_{k_1,j_1} \ast f_{k_2,j_2} \|_{L^2},
$$

**(5.4)**

where we set \( f_{k_i,j_i} = f_{k_i}(\xi, \tau) \eta_{j_i}(\tau - \omega(\xi)) \) for \( j_i \gtrsim [(1-\alpha)k_3] \) and the remaining part \( f_{k_i,j_i}^{\approx [(1-\alpha)k_3]} = f_{k_i}(\xi, \tau) \eta_{\approx [(1-\alpha)k_3]}(\tau - \omega(\xi)) \), \( i = 1, 2 \). For the summation on the terms \( j_3 < [(1-\alpha)k_3] \) in (5.4), we get from the fact \( 1_{D_{k_3,j_3}} \lesssim 1_{D_{k_3,j_3}} \) that

$$
\sup_{t_k \in \mathbb{R}} 2^{k_3} \sum_{j_3 < [(1-\alpha)k_3]} 2^{j_3/2} \sum_{j_1, j_2 \apпр [(1-\alpha)k_3]} \| (2^{j_3} + i2^{[(1-\alpha)k_3]})^{-1} 1_{D_{k_3,j_3}} \cdot f_{k_1,j_1} \ast f_{k_2,j_2} \|_{L^2} 
\lesssim \sup_{t_k \in \mathbb{R}} 2^{k_3} \sum_{j_1, j_2 \apпр [(1-\alpha)k_3]} 2^{-(1-\alpha)k_3}/2 \| 1_{D_{k_3,j_3}} \cdot f_{k_1,j_1} \ast f_{k_2,j_2} \|_{L^2}.
$$

**(5.5)**

From the fact that \( f_{k_i,j_i} \) is supported in \( D_{k_i,j_i} \) for \( i = 1, 2 \) and using (5.3), then we get that

$$
\sup_{t_k \in \mathbb{R}} 2^{k_3} \sum_{j_1, j_2 \apпр [(1-\alpha)k_3]} 2^{-(1-\alpha)k_3}/2 \| 1_{D_{k_3,j_3}} \cdot f_{k_1,j_1} \ast f_{k_2,j_2} \|_{L^2} 
\lesssim 2^{(\alpha-1/4)k_1} \sup_{t_k \in \mathbb{R}} \sum_{j_1, j_2 \apпр [(1-\alpha)k_3]} 2^{j_1/2} \| f_{k_1,j_1} \|_{L^2} 2^{j_2/2} \| f_{k_2,j_2} \|_{L^2}.
$$
Thus from the definition and using (2.4) and (2.5) we obtain (5.1), as desired. To prove (5.3), we apply Corollary 3.3(b) and Remark 3.4 that

$$2^{k_3} \sum_{j_3 \geq [(1-\alpha)k_3]} 2^{-j_3/2} \|1_{D_{k_3, \leq j_3}} \cdot (f_{k_1, j_1} \ast f_{k_2, j_2})\|_{L^2} \leq 2^{k_3} \sum_{j_3 \geq [(1-\alpha)k_3]} 2^{(j_1 + j_2 - j_3)/2} 2^{-(1+\alpha)k_3/2} \prod_{i=1}^2 \|f_{k_i, j_i}\|_{L^2} \leq 2^{(j_1 + j_2)/2} \prod_{i=1}^2 \|f_{k_i, j_i}\|_{L^2}.$$  

Therefore, we complete the proof of the proposition. \(\square\)

We see from the proof that if we only consider the interactions in short time, then the modulation has a bound below, thus we are able to control the high–low interactions in time interval of length \(2-[(1-\alpha)k]\).

**Proposition 5.4.** Assume \(k_3 \geq 20\). If \(|k_3 - k_2| \leq 5\) and \(|k_1 - k_2| \leq 5\) then we have

$$\|P_{k_3} \partial_x (u_{k_1} v_{k_2})\|_{N_{k_3}} \lesssim \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}}.$$  

**Proof.** As in the proof of Proposition 5.3, it suffices to prove that if \(j_1, j_2 \geq [(1-\alpha)k_3]\) and \(f_{k_i, j_i} : \mathbb{R}^2 \rightarrow \mathbb{R}^+\) are supported in \(D_{k_i, \leq j_i}, i = 1, 2\), then

$$2^{k_3} \sum_{j_3 \geq [(1-\alpha)k_3]} 2^{-j_3/2} \|1_{D_{k_3, \leq j_3}} (f_{k_1, j_1} \ast f_{k_2, j_2})\|_{L^2} \leq 2^{j_1/2} \|f_{k_1, j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2, j_2}\|_{L^2}. \quad (5.6)$$

Since by Lemma 5.1 we get in the area \(|\xi_i| \in I_{k_i}, i = 1, 2\) \(\cap \{\xi_1 + \xi_2 \in I_{k_3}\}\)

$$|\Omega(\xi_1, \xi_2)| \sim 2^{2(2+\alpha)k_3},$$

then by checking the support properties, we get \(1_{D_{k_3, \leq j_3}} \cdot (f_{k_1, j_1} \ast f_{k_2, j_2}) = 0\) unless \(j_{\text{max}} \geq (2+\alpha)k_3 - 30\). Then it follows from Corollary 3.3(a) that the left-hand side of (5.6) is bounded by

$$2^{k_3} \sum_{j_3 \geq [(1-\alpha)k_3]} 2^{-j_3/2} 2^{j_{\text{min}}/2} \sum_{j_3 \geq [(1-\alpha)k_3]} 2^{-j_3/2} 2^{j_{\text{min}}/2} \prod_{i=1}^2 \|f_{k_i, j_i}\|_{L^2}. \quad (5.7)$$

Then we get the bound (5.6) by considering either \(j_3 = j_{\text{max}}\) or \(j_3 \neq j_{\text{max}}\). \(\square\)

**Proposition 5.5.** If \(k_2 \geq 20\), \(|k_1 - k_2| \leq 5\) and \(0 \leq k_3 \leq k_1 - 10\), then we have

$$\|P_{k_3} \partial_x (u_{k_1} v_{k_2})\|_{N_{k_3}} \lesssim k_2^2 2^{-(1-\alpha)k_3} 2^{(1/2 - 2\alpha)k_2} \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}}.$$  

**Proof.** Let \(\beta : \mathbb{R} \rightarrow [0, 1]\) be a smooth function supported in \([-1, 1]\) with the property that

$$\sum_{n \in \mathbb{Z}} \beta^2(x - n) \equiv 1, \quad x \in \mathbb{R}.$$  

Using the definitions, the left-hand side of (5.8) is dominated by
Then it follows from Corollary 3.3(b) that the left-hand side of (5.9) is bounded by

\[ C \sup_{k \in \mathbb{R}} \sum_{k_j \leq k_3} \left\| (\tau - \omega(\xi) + i 2(1-\alpha)k_3^+) - 2k_j^3 \chi_{k_j^3}(\xi) \cdot \sum_{|m| \leq C2^1(1-\alpha)(k_j^2 - k_3^2)} F[u_{k_3} \eta_0(2(1-\alpha)k_j^3 + (t - t_k)) \beta(2(1-\alpha)k_j^2(t - t_k) - m)]^* \right\|_{X_{k_j}^3}.\]

We assume first \( k_3 = 0 \). In view of the definitions, (2.4) and (2.5), it suffices to prove that if \( j_1, j_2 \geq |(1-\alpha)k_2| \), and \( f_{k_1, k_2} : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \) are supported in \( D_{k_1, k_2}^{(2)} \), then

\[ \sum_{k_3 \leq 0} 2^{k_3^2} 2(1-\alpha)k_2 2^{-j_3/2} \left\| \chi_{k_3^3}(\xi) \eta_j \left( \tau - \omega(\xi) \right) \right\|_{L^2} \lesssim 2^{k_3^2} 2^{j_3/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| f_{k_2, j_2} \right\|_{L^2}. \]

To prove (5.9), we may assume \( k_3^3 \geq -10k_2 \), since otherwise we use Corollary 3.3(a). From Lemma 5.1 and the support properties as in Proposition 5.4, we get \( j_{\max} \geq (1 + \alpha)k_2 + k_3^3 - 30 \). Then it follows from Corollary 3.3(b) that the left-hand side of (5.9) is bounded by

\[ \sum_{k_3 \leq 0} 2^{k_3^2} 2(1-\alpha)k_2 2^{j_3/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| f_{k_2, j_2} \right\|_{L^2} \lesssim 2^{k_3^2} 2^{j_3/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| f_{k_2, j_2} \right\|_{L^2}. \]

We assume now \( k_3 \geq 1 \). It suffices to prove that if \( j_1, j_2 \geq |(1-\alpha)k_2| \), and \( f_{k_1, j_1} : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \) are supported in \( D_{k_1, j_1} \), then

\[ \sum_{k_3 \geq 1} 2^{k_3^2} 2(1-\alpha)k_2 2^{j_3/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| f_{k_2, j_2} \right\|_{L^2} \lesssim 2^{k_3^2} 2^{j_3/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| f_{k_2, j_2} \right\|_{L^2}. \]

which can be proved similarly as (5.9). \( \square \)

**Proposition 5.6.** If \( 0 \leq k_1, k_2, k_3 \leq 200 \), then

\[ \left\| P_{k_3} \partial_x(u_{k_1} v_{k_2}) \right\|_{N_{k_3}} \lesssim \left\| u_{k_1} \right\|_{F_{k_1}} \left\| v_{k_2} \right\|_{F_{k_2}}. \]

**Proof.** This follows immediately from the definitions, Corollary 3.3(a), Remark 3.4 and (2.4) and (2.5). \( \square \)

As a conclusion to this section we prove the bilinear estimates, using the dyadic bilinear estimates obtained above.

**Lemma 5.7.**

(a) If \( s \geq 1 - \alpha, T \in (0, 1) \), and \( u, v \in F^s(T) \) then

\[ \left\| \partial_x(uv) \right\|_{N^s(T)} \lesssim \left\| u \right\|_{F^1(T)} \left\| v \right\|_{F^{1-(\alpha)}(T)} \left\| v \right\|_{F^1(T)}. \]
(b) If \( T \in (0, 1] \), \( u \in F^0(T) \) and \( v \in F^{\alpha}(T) \) then
\[
\| \partial_x (uv) \|_{N^0(T)} \lesssim \| u \|_{F^0(T)} \| v \|_{F^{1-\alpha}(T)}.
\]

**Proof.** In view of definition, we get
\[
\| \partial_x (uv) \|_{N^0(T)}^2 \lesssim \sum_{k_3=0}^{\infty} 2^{2sk_3} \| P_{k_3} (\partial_x (uv)) \|_{N_{k_3}(T)}^2.
\]
For \( k \in \mathbb{Z}_+ \), we fix extensions \( u_k, v_k \) of \( P_k u, P_k v \) such that \( \| u_k \|_{F_k} \leq 2 \| P_k (u) \|_{F_k(T)} \) and \( \| v_k \|_{F_k} \leq 2 \| P_k (v) \|_{F_k(T)} \). Then we get
\[
\| P_{k_3} (\partial_x (uv)) \|_{N_{k_3}(T)} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}_+} \| P_{k_3} (\partial_x (P_k u P_k v)) \|_{N_{k_3}(T)}
\lesssim \sum_{k_1, k_2 \in \mathbb{Z}_+} \| P_{k_3} (\partial_x (u_k v_k)) \|_{N_{k_3}}.
\]
From symmetry we may assume \( k_1 \leq k_2 \). Dividing the summation on the right-hand side into several parts, we get
\[
\sum_{k_1, k_2 \in \mathbb{Z}_+} \| P_{k_3} (\partial_x (u_k v_k)) \|_{N_{k_3}} \leq \sum_{i=1}^{4} \sum_{A_i} \| P_{k_3} (\partial_x (u_k v_k)) \|_{N_{k_3}},
\]
where we denote
\[
A_1 = \{ k_1 \leq k_2 : |k_2 - k_3| \leq 5, k_1 \leq k_2 - 10, \text{ and } k_2 \geq 20 \};
A_2 = \{ k_1 \leq k_2 : |k_2 - k_3| \leq 5, |k_1 - k_2| \leq 10, \text{ and } k_2 \geq 20 \};
A_3 = \{ k_1 \leq k_2 : k_3 \leq k_2 - 10, |k_1 - k_2| \leq 5, \text{ and } k_1 \geq 20 \};
A_4 = \{ k_1 \leq k_2 : k_1, k_2, k_3 \leq 200 \}.
\]
For part (a), it suffices to prove that for \( i = 1, 2, 3, 4 \) then
\[
\| 2^{sk_3} \sum_{A_i} P_{k_3} (\partial_x (u_k v_k)) \|_{N_{k_3}} \lesssim \| 2^{sk_3} v_k F_k \|_{F_k} \| 2^{(1-\alpha)} u_k F_k \|_{F_k}^\alpha,
\]
which follows from Propositions 5.3–5.6. For part (b), it suffices to prove
\[
\| \sum_{k_1, k_2 \in \mathbb{Z}_+} P_{k_3} (\partial_x (u_k v_k)) \|_{N_{k_3}} \lesssim \| v_k F_k \|_{F_k} \| 2^{(1-\alpha)} u_k F_k \|_{F_k}^\alpha
\tag{5.11}
\]
Similarly we divide the summation on the left-hand side of (5.11) into many pieces, but now we do not have symmetries. We denote for \( i = 1, 2, 3, 4 \)
\[
\bar{A}_i = \{ (k_1, k_2) : (k_2, k_1) \in A_i \}.
\]
For the summation in $\hat{A}_1 \cup A_1$ we can get easily control it using Proposition 5.3. The contributions of the summation in $\hat{A}_2 \cup A_2$ and $\hat{A}_4 \cup A_4$ are acceptable due to Propositions 5.4 and 5.6. For the summation in $\hat{A}_3 \cup A_3$ we use Proposition 5.5 since for $0 \leq \alpha \leq 1$ we have $1 - \alpha > 1/2 - 2\alpha$. \hfill $\Box$

6. Proof of Theorem 1.1

In this section we devote to prove Theorem 1.1. The main ingredients are energy estimates which are proved in the next section and short-time bilinear estimates obtained in the last section. The idea is due to Ionescu, Kenig and Tataru [13].

**Proposition 6.1.** Let $s \geq 0$, $T \in (0, 1)$, and $u \in F^s(T)$, then

$$\sup_{t \in [-T, T]} \| u(t) \|_{H^s} \lesssim \| u \|_{F^s(T)}.$$  \hfill (6.1)

**Proof.** In view of the definitions, it suffices to prove that if $k \in \mathbb{Z}_+$, $t_k \in [-1, 1]$, and $\tilde{u}_k \in F_k$ then

$$\| F[\tilde{u}_k(t_k)] \|_{L^2} \lesssim \| \mathcal{F}[\tilde{u}_k \cdot \eta_0(2^{(1-\alpha)k})(t - t_k)] \|_{X_k}.$$  \hfill (6.2)

Let $f_k = \mathcal{F}[\tilde{u}_k \cdot \eta_0(2^{(1-\alpha)k})(t - t_k)]$, so

$$\mathcal{F}[\tilde{u}_k(t_k)](\xi) = c \int_{\mathbb{R}} f_k(\xi, \tau) e^{it_k \tau} \, d\tau.$$  \hfill (6.3)

From the definition of $X_k$, we get that

$$\| \mathcal{F}[\tilde{u}_k(t_k)] \|_{L^2} \lesssim \| \int_{\mathbb{R}} f_k(\xi, \tau) |d\tau| \|_{L^2} \lesssim \| f_k \|_{X_k},$$

which completes the proof of the proposition. \hfill $\Box$

**Proposition 6.2.** Assume $T \in (0, 1)$, $u, v \in C([-T, T] : H^\infty)$ and

$$u_t + |\partial_x|^{1+\alpha} \partial_x u = v \quad \text{on } \mathbb{R} \times (-T, T).$$  \hfill (6.4)

Then for any $s \geq 0$,

$$\| u \|_{F^s(T)} \lesssim \| u \|_{E^s(T)} + \| v \|_{N^s(T)}.$$  \hfill (6.5)

**Proof.** In view of the definitions, we see that the square of the right-hand side of (6.4) is equivalent to

$$\| P_{\leq 0}(u(0)) \|_{L^2}^2 + \| P_{\leq 0}(v) \|_{N_k(T)}^2 + \sum_{k \geq 1} \left( \sup_{t_k \in [-T, T]} 2^{2sk} \| P_k(u(t_k)) \|_{L^2}^2 + 2^{2sk} \| P_k(v) \|_{N_k(T)}^2 \right).$$

Thus, from definitions, it suffices to prove that if $k \in \mathbb{Z}_+$ and $u, v \in C([-T, T] : H^\infty)$ solve (6.3), then

$$\begin{cases} \| P_{\leq 0}(u) \|_{F_0(T)} \lesssim \| P_{\leq 0}(u(0)) \|_{L^2} + \| P_{\leq 0}(v) \|_{N_0(T)}; \\
\| P_k(u) \|_{F_k(T)} \lesssim \sup_{t_k \in [-T, T]} \| P_k(u(t_k)) \|_{L^2} + \| P_k(v) \|_{N_k(T)} \text{ if } k \geq 1.
\end{cases}$$  \hfill (6.6)
We only prove the second inequality in (6.5), since the first one can be treated in the same way. Fix $k \geq 1$ and let $\tilde{v}$ denote an extension of $P_k(v)$ such that $\|\tilde{v}\|_{N_k} \leq C\|v\|_{N_k(T)}$. In view of (2.7), we may assume that $\tilde{v}$ is supported in $[-T - 2^{-(1-\alpha)k-10}, T + 2^{-(1-\alpha)k-10}]$. Indeed, let $\theta(t)$ be a smooth function such that

$$\theta(t) = 1, \quad \text{if } t \geq 1; \quad \theta(t) = 0, \quad \text{if } t \leq 0.$$ 

Thus $\theta(2^{(1-\alpha)k+10}(t + T + 2^{-(1-\alpha)k-10}))$, $\theta(2^{(1-\alpha)k+10}(t - T - 2^{-(1-\alpha)k-10}))$ is supported in $S_k$. Then we see that $\theta(2^{(1-\alpha)k+10}(t + T + 2^{-(1-\alpha)k-10}))\theta(2^{(1-\alpha)k+10}(t - T - 2^{-(1-\alpha)k-10}))$ is supported in $[-T - 2^{-(1-\alpha)k-10}, T + 2^{-(1-\alpha)k-10}]$ and equal to 1 in $[-T, T]$. From (2.7) we consider $\tilde{v}[\theta(2^{k+10}(t + T + 2^{-(1-\alpha)k-10}))\theta(2^{k+10}(t - T - 2^{-(1-\alpha)k-10}))]$ instead. For $t \geq T$ we define

$$\tilde{u}(t) = \eta_0(2^{(1-\alpha)k+5}(t - T)) \left[ W(t - T)P_k(u(T)) + \int_{T}^{t} W(t - s)(P_k(\tilde{v}(s))) \, ds \right].$$

For $t \leq -T$ we define

$$\tilde{u}(t) = \eta_0(2^{(1-\alpha)k+5}(t + T)) \left[ W(t + T)P_k(u(-T)) + \int_{-T}^{t} W(t - s)(P_k(\tilde{v}(s))) \, ds \right].$$

For $t \in [-T, T]$ we define $\tilde{u}(t) = u(t)$. It is clear that $\tilde{u}$ is an extension of $u$ and we get from (2.7) that

$$\|u\|_{F_k(T)} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{(1-\alpha)k}(t - t_k))]\|_{X_k}. \quad \text{(6.6)}$$

Indeed, to prove (6.6), it suffices to prove that

$$\sup_{t_k \in \mathbb{R}} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{(1-\alpha)k}(t - t_k))]\|_{X_k} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{(1-\alpha)k}(t - t_k))]\|_{X_k}. \quad \text{(6.7)}$$

For $t_k > T$, since $\tilde{u}$ is supported in $[-T - 2^{-(1-\alpha)k-5}, T + 2^{-(1-\alpha)k-5}]$, it is easy to see that

$$\tilde{u}\eta_0(2^{(1-\alpha)k}(t - t_k)) = \tilde{u}\eta_0(2^{(1-\alpha)k}(t - T))\eta_0(2^{(1-\alpha)k}(t - t_k)).$$

Therefore, we get from (2.5) that

$$\sup_{t_k > T} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k}(t - t_k))]\|_{X_k} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k}(t - t_k))]\|_{X_k}. \quad \text{(6.8)}$$

Using the same method for $t_k < -T$, we obtain (6.6) as desired.

Now we prove the second inequality in (6.5). In view of the definitions, (6.6) and (2.5), it suffices to prove that if $\phi_k \in L^2$ with $\hat{\phi}_k$ supported in $I_k$, and $v_k \in N_k$ then

$$\|\mathcal{F}[u_k \cdot \eta_0(2^{(1-\alpha)k}t)]\|_{X_k} \lesssim \|\phi_k\|_{L^2} + \left\| (T - \omega(\xi) + i2^{(1-\alpha)k})^{-1} \cdot \mathcal{F}(v_k) \right\|_{X_k}. \quad \text{(6.9)}$$

where

$$u_k(t) = W(t)(\phi_k) + \int_{0}^{t} W(t - s)(v_k(s)) \, ds.$$
Straightforward computations show that
\[
\mathcal{F}[u_k \cdot \eta_0(2^{[(1-\alpha)k]t})](\xi, \tau)
= \widehat{\phi_k}(\xi) \cdot 2^{-[(1-\alpha)k]} \hat{\eta}_0(2^{-(1-\alpha)k}(\tau - \omega(\xi)))
\]
\[+ C \int_{\mathbb{R}} \mathcal{F}(v_k)(\xi, \tau') \cdot \frac{\hat{\eta}_0(2^{-(1-\alpha)k})(\tau - \tau') - \hat{\eta}_0(2^{-(1-\alpha)k})(\tau - \omega(\xi))}{2^{[(1-\alpha)k]}(\tau' - \omega(\xi))} d\tau'.
\]
We observe now that
\[
\left| \frac{\hat{\eta}_0(2^{-(1-\alpha)k})(\tau - \tau') - \hat{\eta}_0(2^{-(1-\alpha)k})(\tau - \omega(\xi))}{2^{[(1-\alpha)k]}(\tau' - \omega(\xi))} \cdot (\tau' - \omega(\xi) + i2^{[(1-\alpha)k]}) \right|
\leq 2^{-(1-\alpha)k}(1 + 2^{-(1-\alpha)k}|\tau - \tau'|^{-4} + 2^{-(1-\alpha)k}(1 + 2^{-(1-\alpha)k}|\tau - \omega(\xi)|^{-4}.
\]
Using (2.3) and (2.4), we complete the proof of the proposition. \(\square\)

Now we turn to prove Theorem 1.1. To prove Theorem 1.1(a), by the scaling (1.6) we may assume that
\[
\|u_0\|_{H^s} \ll \epsilon \ll 1. \quad (6.10)
\]
The uniqueness follows from the classical energy methods. We only need to construct the solution on the time interval \([-1, 1]\). In view of the classical results, it suffices to prove that if \(T \in (0, 1]\) and \(u \in C([-T, T]; H^\infty)\) is a solution of (1.1) with \(\|u_0\|_{H^s} \ll \epsilon \ll 1\) then
\[
\sup_{t \in [-T, T]} \|u(t)\|_{H^2} \lesssim \|u_0\|_{H^2}. \quad (6.11)
\]

It follows from Proposition 6.2, Lemma 5.7 and the energy estimate Proposition 7.2 that for any \(T' \in [0, T]\) we have
\[
\begin{cases}
\|u\|_{F^s(T')} \lesssim \|u\|_{E^s(T')} + \|\partial_x(u^2)\|_{N^s(T')}; \\
\|\partial_x(u^2)\|_{N^s(T')} \lesssim \|u\|^2_{F^s(T')}; \\
\|u\|^2_{E^s(T')} \lesssim \|\phi\|^2_{H^s} + \|u\|^3_{F^s(T')}.
\end{cases} \quad (6.12)
\]

We denote \(X(T') = \|u\|_{E^s(T')} + \|\partial_x(u^2)\|_{N^s(T')}\). Then by a similar argument as in the proof of Lemma 4.2 in [13], we know \(X(T')\) is continuous and satisfies
\[
\lim_{T' \to 0} X(T') \lesssim \|u_0\|_{H^s}.
\]
On the other hand, we get from (6.12) that
\[
X(T')^2 \lesssim \|u_0\|^2_{H^s} + X(T')^3 + X(T')^4.
\]
If \(\epsilon_0\) is sufficiently small, then we can get from (6.10), the continuity and the standard bootstrap that \(X(T') \lesssim \|u_0\|_{H^s}^2\) and therefore we obtain
\[
\|u\|_{F^s(T')} \lesssim \|u_0\|_{H^s}. \quad (6.13)
\]
For \( \sigma \geq s \) we obtain from Proposition 6.2, Lemma 5.7(a) and the energy estimate Proposition 7.2 that for any \( T' \in [0, T] \) we have

\[
\begin{align*}
\|u\|_{F^s(T')} & \lesssim \|u\|_{E^s(T')} + \|\partial_x(u^2)\|_{N^s(T')}, \\
\|\partial_x(u^2)\|_{N^s(T')} & \lesssim \|u\|_{E^s(T')} \|u\|_{F^s(T')}, \\
\|u\|_{E^s(T')}^2 & \lesssim \|\phi\|_{H^s}^2 + \|u\|_{F^s(T')}^2.
\end{align*}
\]  

(6.14)

Then from (6.13) we get \( \|u\|_{F^s(T')} \ll 1 \) and hence

\[
\|u\|_{E^s(T')} \lesssim \|u_0\|_{H^s},
\]

(6.15)

which in particular implies (6.11) as desired. We complete the proof of part (a).

We prove now Theorem 1.1(b), following the ideas in [13]. Fixing \( u_0 \in H^s \), then we choose \( \{\phi_n\} \subset H^{2s} \) such that \( \lim_{n \to \infty} \phi_n = u_0 \) in \( H^s \). It suffices to prove the sequence \( S^\infty_T(\phi_n) \) is a Cauchy sequence in \( C([-T, T] : H^s(\mathbb{R})) \). From the definition it suffices to prove that for any \( \delta > 0 \) there is \( M_\delta \) such that

\[
\sup_{t \in [-T, T]} \|S^\infty_T(\phi_m) - S^\infty_T(\phi_n)\|_{H^s} \leq \delta, \quad \forall m, n \geq M_\delta.
\]

For \( K \in \mathbb{Z}_+ \) let \( \phi^K_n = P_{\leq K} \phi_n \). Since \( \phi^K_n \to u^K_0 \) in \( H^2 \), then we see for any fixed \( K \) there is \( M_{\delta, K} \) such that

\[
\sup_{t \in [-T, T]} \|S^\infty_T(\phi^K_m) - S^\infty_T(\phi^K_n)\|_{H^s} \leq \delta/2, \quad \forall m, n \geq M_{\delta, K}.
\]

On the other hand, we get from Proposition 7.3 and Proposition 6.1 that

\[
\sup_{t \in [-T, T]} \|S^\infty_T(\phi_n) - S^\infty_T(\phi^K_n)\|_{H^s} \lesssim \|S^\infty_T(\phi_n) - S^\infty_T(u^K_0)\|_{F^s(T)} \\
\lesssim \|\phi_n - \phi^K_n\|_{H^s} + \|\phi^K_n\|_{H^{2s}} \|\phi_n - \phi^K_n\|_{L^2} \\
\lesssim \|\phi - \phi_n\|_{H^s} + \|\phi - \phi^K_n\|_{H^s}.
\]

Thus we obtain that for any \( \delta > 0 \) there are \( K \) and \( M_\delta \) such that

\[
\sup_{t \in [-T, T]} \|S^\infty_T(\phi_n) - S^\infty_T(\phi^K_n)\|_{H^s} \leq \delta/2, \quad \forall n \geq M_\delta.
\]

Therefore, we complete the proof of part (b) of Theorem 1.1.

7. Energy estimates

In this section we prove the energy estimates, following the ideas in [13]. The difficulty is we need to handle fractional derivative and the commutator estimates. Assume that \( u, v \in C([-T, T]; L^2) \) and

\[
\begin{align*}
u_t + |\partial_x|^{1+\alpha} \partial_x u &= v, \quad (x, t) \in \mathbb{R} \times (-T, T); \\
u(x, 0) &= \phi(x).
\end{align*}
\]

(7.1)

Then we multiply by \( u \) and integrate to conclude that
\[ \sup_{|k| \leq T} \left\| u(t_k) \right\|_{L^2}^2 \leq \left\| \phi \right\|_{L^2}^2 + \sup_{|k| \leq T} \left| \int_{\mathbb{R} \times [0,T]} u \cdot \nu \, dx \, dt \right|. \] \quad (7.2)

**Lemma 7.1.**

(a) Assume \( T \in (0, 1], k_1, k_2, k_3 \in \mathbb{Z}_+ \) with \( \max(k_1, k_2, k_3) \geq 1 \), and \( u_i \in F_{k_i}(T) \), \( i = 1, 2, 3 \). Then if \( k_{\min} \leq k_{\max} - 5 \), we have

\[ \left| \int_{\mathbb{R} \times [0,T]} u_1 u_2 u_3 \, dx \, dt \right| \leq 2^{-\alpha k_{\max}} \prod_{i=1}^3 \| u_i \|_{F_{k_i}(T)}. \] \quad (7.3)

(b) Assume \( T \in (0, 1], k \in \mathbb{Z}_+, 0 \leq k_1 \leq k - 10, u \in F_k(T) \), and \( v \in F_{k_1}(T) \). Then

\[ \left| \int_{\mathbb{R} \times [0,T]} P_k(u) P_k (\partial_t u \cdot P_{k_1}(v)) \, dx \, dt \right| \leq 2^{k_1 - \alpha k_{\max}} \| v \|_{F_{k_1}(T)} \sum_{|k' - k| \leq 10} \| P_{k'}(u) \|_{F_{k'}(T)}^2. \] \quad (7.4)

**Proof.** For part (a), from symmetry we may assume \( k_1 \leq k_2 \leq k_3 \). In order for the integral to be nontrivial we must also have \( |k_2 - k_3| \leq 4 \). We fix extension \( \tilde{u}_i \in F_{k_i} \) such that \( \| \tilde{u}_i \|_{F_{k_i}} \leq \| u_i \|_{F_{k_i}(T)} \), \( i = 1, 2, 3 \). Let \( \gamma : \mathbb{R} \to [0, 1] \) denote a smooth function supported in \([-1, 1]\) with the property that

\[ \sum_{n \in \mathbb{Z}} \gamma^3(x - n) \equiv 1, \quad x \in \mathbb{R}. \]

The left-hand side of (7.3) is dominated by

\[ C \sum_{|n| \leq 2^{1-(\alpha)k_1}} \left| \int_{\mathbb{R} \times \mathbb{R}} (\gamma (2^{(1-\alpha)k_1} t - n) 1_{[0,T]}(t) \tilde{u}_1) \right. \]

\[ \cdot \left( \gamma (2^{(1-\alpha)k_1} t - n) \tilde{u}_2 \right) \cdot \left( \gamma (2^{(1-\alpha)k_1} t - n) \tilde{u}_3 \right) \, dx \, dt \right|. \] \quad (7.5)

We observe first that

\[ |A| = \left| \left\{ n : \gamma (2^{(1-\alpha)k_1} t - n) 1_{[0,T]}(t) \text{ nonzero and } \neq \gamma (2^{(1-\alpha)k_1} t - n) \right\} \right| \leq 4. \]

We assume first that \( k_1 \leq k_3 - 5 \). For the summation of \( n \in A^c \) on the left-hand side of (7.5), as was explained in the proof of Proposition 5.3, for (7.3) it suffices to prove that if \( f_{k_i, j_i} \) are \( L^2 \) functions supported in \( D_{k_i, j_i} \) for \( i = 1, 2, 3 \) then

\[ 2^{(1-\alpha)k_3} \sum_{j_1, j_2, j_3 \geq 1-(\alpha)k_3} \left| J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \right| \leq 2^{-\alpha k_{\max}} \sum_{j_i \geq 0} \prod_{i=1}^3 2^{j_i/2} \| f_{k_i, j_i} \|_2. \] \quad (7.6)

Clearly we may assume \( \max(k_1, k_2, k_3) \geq 10 \), otherwise we can get (7.6) by using Lemma 3.2(a). We get from Lemma 3.2(b) that the left-hand side of (7.6) is bounded by
\[ 2^{(1-\alpha)k_3/2} \sum_{j_1,j_2,j_3 \geq (1-\alpha)k_1} 2^{(j_1+j_2+j_3)/2} 2^{-(1+\alpha)k_3/2} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_2 \]
\[ \lesssim 2^{-\alpha k_{\max}} \sum_{j_i \geq 0} 2^{j_i/2} \| f_{k_i,j_i} \|_2, \quad (7.7) \]

which is (7.6) as desired.

For the summation of \( n \in A \), we observe that if \( I \subset \mathbb{R} \) is an interval, \( k \in \mathbb{Z}_+ \), \( f_k \in X_k \), and \( f_k^I = \mathcal{F}(1_I(t) \cdot \mathcal{F}^{-1}(f_k)) \) then

\[ \sup_{j \in Z_+} 2^{j/2} \| \eta_j(\tau - \omega(\xi)) \cdot f_k^I \|_{L^2} \lesssim \| f_k \|_{X_k}. \quad (7.8) \]

Indeed, to prove (7.8) it suffices to prove for any \( j_1 \geq 0 \) and \( f_{k,j_1} = f_k(\xi,\tau) \eta_{j_1}(\tau - \omega(\xi)) \) then

\[ \sup_{j \in Z_+} 2^{j/2} \| \eta_j(\tau - \omega(\xi)) \cdot f_{k,j_1} \|_{L^2} \lesssim 2^{j/2} \| f_{k,j_1} \|_{L^2}. \quad (7.9) \]

If \( j \leq j_1 + 20 \), then (7.9) follows from Plancherel’s equality. If \( j \geq j_1 + 20 \) then from

\[ 2^{j/2} \eta_j(\tau - \omega(\xi)) \| f_{k,j_1} \|_{L^2} \lesssim 2^{j/2} \eta_j(\tau - \omega(\xi)) \int \| f_{k,j_1}(\xi,\tau) \|_{\tau - \tau'}^{-1} d\tau' \]

we get (7.9) from (2.4) since \( |\tau - \tau'| \sim 2^j \). For the summation of \( n \in A \) on the left-hand side of (7.5), clearly we may assume \( j_1 \leq 10k_3 \). Then as before we can get (7.3) due to \( \alpha < 1 \).

For part (b), we denote the commutator of \( T_1, T_2 \) by \( [T_1, T_2] = T_1T_2 - T_2T_1 \). Then the left-hand side of (7.4) is dominated by

\[ \left| \int_{\mathbb{R} \times [0,T]} P_k(u)P_k(\partial_\xi u)P_{k_1}(v) \, dx \, dt \right| + \left| \int_{\mathbb{R} \times [0,T]} P_k(u)\left[ P_k, P_{k_1}(v) \right](\partial_\xi u) \, dx \, dt \right|. \quad (7.10) \]

For the first term in (7.10) we integrate by parts and then use (7.3). For the second term it follows from (7.3) and the similar argument in the proof of Lemma 6.1 in [13]. We omit the details. \( \square \)

**Proposition 7.2.** Assume that \( T \in (0,1] \) and \( u \in C([-T,T];H^\infty) \) is a solution to Eq. (1.1) on \( \mathbb{R} \times (-T,T) \). Then for \( s \geq 1 - \alpha \) we have

\[ \| u \|_{E^s(T)}^2 \lesssim \| u_0 \|_{H^s}^2 + \| u \|_{F^{1-\alpha}(T)} \| u \|_{F^s(T)}. \quad (7.11) \]

**Proof.** From definition we have

\[ \| u \|_{E^s(T)}^2 - \| u_0 \|_{H^s}^2 \lesssim \sum_{k \geq 1} \sup_{t_k \in [-T,T]} 2^{2sk} \| P_k(u(t_k)) \|_{L^2}^2. \]

Then we can get from (7.2) that

\[ 2^{2sk} \| P_k(u(t_k)) \|_{L^2}^2 \leq 2^{2sk} \| P_k(u_0) \|_{L^2}^2 \lesssim 2^{2sk} \int_{\mathbb{R} \times [0,t_k]} P_k(u)P_k(u_0) \cdot \partial_\xi u \, dx \, dt. \quad (7.12) \]
It is easy to see that the right-hand side of (7.12) is dominated by
\[
C2^{2sk} \sum_{k_1 \leq k-10} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(u) P_k(P_{k_1}u \cdot \partial_x u) \, dx \, dt \right| \\
+ C2^{2sk} \sum_{k_1 \geq k-9, k_2 \in \mathbb{Z}_+} \left| \int_{\mathbb{R} \times [0,t_k]} P_k^2(u) P_{k_1}(u) \cdot \partial_x P_{k_2}(u) \, dx \, dt \right|. 
\tag{7.13}
\]

For the first term in (7.13), using (7.4) then we get that it is bounded by
\[
C2^{2sk} \sum_{k_1 \leq k-10} 2^{k_1-a} \|u\|_{F_k(T)} \sum_{|k'-k| \leq 10} \|P_k(u)\|_{F_k(T)}^2 \lesssim \|u\|_{F^{1-a}(T)} \sum_{|k'-k| \leq 10} \|u\|_{F_k(T)}^2 
\leq 2^{2sk} \|u\|_{F^{1-a}(T)} \sum_{|k'-k| \leq 10} \|u\|_{F_k(T)}^2.
\]
which implies that the summation of the first term is bounded by \(\|u\|_{F^{1-a}(T)}\|u\|_{F(T)}^2\) as desired.

For the second term in (7.13), using (7.3) we get that it is bounded by
\[
C2^{2sk} \sum_{k_1 \leq k-10, k_2 \leq k+10} 2^{k_2-a} \|P_k(u)\|_{F_k(T)} \|P_{k_1}(u)\|_{F_k(T)} \|P_{k_2}(u)\|_{F_k(T)} \lesssim \|u\|_{F^{1-a}(T)} \sum_{|k'-k| \leq 10} \|u\|_{F_{k'}(T)}^2 
\leq 2^{2sk} \|u\|_{F^{1-a}(T)} \sum_{|k'-k| \leq 10} \|u\|_{F_{k'}(T)}^2.
\]
Therefore, we complete the proof of the proposition. \(\square\)

**Proposition 7.3.** Let \(0 \leq \alpha < 1\). Assume \(\sigma > 1 - \alpha\). Let \(u_1, u_2 \in F^\sigma(1)\) be solutions to (1.1) with initial data \(\phi_1, \phi_2 \in H^\infty\) satisfying
\[
\|\phi_1\|_{H^\sigma} + \|\phi_2\|_{H^\sigma} \leq \epsilon_0 \ll 1.
\]
Then we have
\[
\|u_1 - u_2\|_{F^\sigma(1)} \lesssim \|\phi_1 - \phi_2\|_{L^2}, \tag{7.14}
\]
and
\[
\|u_1 - u_2\|_{F^\sigma(1)} \lesssim \|\phi_1 - \phi_2\|_{H^\sigma} + \|\phi_1\|_{H^{2\sigma}} \|\phi_1\|_{H^\sigma}. \tag{7.15}
\]

**Proof.** We prove first (7.14). Since \(\|\phi_1\|_{H^\sigma} + \|\phi_2\|_{H^\sigma} \leq \epsilon_0 \ll 1\), then from the proof of Theorem 1.1(a) in the last section we know
\[
\|u_1\|_{F^\sigma(1)} \ll 1, \quad \|u_2\|_{F^\sigma(1)} \ll 1. \tag{7.16}
\]
Let \(v = u_2 - u_1\), then \(v\) solves the equation
\[
\begin{cases}
\partial_t v + |\partial_x|^{1+\alpha} \partial_x v = -\partial_x \left[ v(u_1 + u_2)/2 \right]; \\
v(0) = \phi = \phi_2 - \phi_1.
\end{cases} \tag{7.17}
\]
Then from Proposition 6.2 and Lemma 5.7(b) we obtain

\[ \left\{ \begin{array}{l}
\|v\|_{F^0(1)} \lesssim \|v\|_{E^0(1)} + \|\partial_x[v(u_1 + u_2)/2]\|_{N^0(1)}; \\
\|\partial_x[v(u_1 + u_2)/2]\|_{N^0(1)} \lesssim \|v\|_{F^0(1)} \left( \|u_1\|_{F^{1-\alpha}(1)} + \|u_2\|_{F^{1-\alpha}(1)} \right).
\end{array} \right. \tag{7.18} \]

We now devote to derive an estimate on \(\|v\|_{E^0(1)}\). As in the proof of Proposition 7.2, we get from (7.2) that

\[ \|v\|_{E^0(1)}^2 \lesssim \sum_{k \geq 1} \int_{\mathbb{R} \times [0,t_k]} P_k(v)P_k(\partial_x \cdot (u_1 + u_2)) \, dx \, dt \]

\[ + \sum_{k \geq 1} \int_{\mathbb{R} \times [0,t_k]} P_k(v)P_k(v \cdot \partial_x(u_1 + u_2)) \, dx \, dt. \tag{7.19} \]

For the first term on right-hand side of (7.19), using Lemma 7.1 we can bound it by

\[ C \sum_{k \geq 1} \sum_{k_1 \leq k-10} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(v)P_k(\partial_x v \cdot p_{k_1}(u_1 + u_2)) \, dx \, dt \right| \]

\[ + C \sum_{k \geq 1} \sum_{k_1 \geq k-9, k_2 \in \mathbb{Z}_+} \left| \int_{\mathbb{R} \times [0,t_k]} P_k^2(v)\partial_x p_{k_2}(v) \cdot p_{k_1}(u_1 + u_2) \, dx \, dt \right| \]

\[ \lesssim \|v\|_{F^0(1)}^2 \left( \|u_1\|_{F^\sigma(1)} + \|u_2\|_{F^\sigma(1)} \right). \]

The second term on right-hand side of (7.19) is dominated by

\[ \sum_{k \geq 1} \sum_{k_1, k_2 \in \mathbb{Z}_+} \left| \int_{\mathbb{R} \times [0,t_k]} P_k^2(v)p_{k_1}(v) \cdot \partial_x p_{k_2}(u_1 + u_2) \, dx \, dt \right| \lesssim \|v\|_{F^0(1)}^2 \left( \|u_1\|_{F^\sigma(1)} + \|u_2\|_{F^\sigma(1)} \right). \]

Therefore, we obtain the following estimate

\[ \|v\|_{E^0(1)}^2 \lesssim \|\phi\|_{L^2}^2 + \|v\|_{F^0(1)}^2 \left( \|u_1\|_{F^\sigma(1)} + \|u_2\|_{F^\sigma(1)} \right). \tag{7.20} \]

which combined with (7.18) implies (7.14) in view of (7.16).

We prove now (7.15). From Proposition 6.2 and Lemma 5.7 we obtain

\[ \left\{ \begin{array}{l}
\|v\|_{F^\sigma(1)} \lesssim \|v\|_{E^\sigma(1)} + \|\partial_x[v(u_1 + u_2)/2]\|_{N^\sigma(1)}; \\
\|\partial_x[v(u_1 + u_2)/2]\|_{N^\sigma(1)} \lesssim \|v\|_{F^\sigma(1)} \left( \|u_1\|_{F^\sigma(1)} + \|u_2\|_{F^\sigma(1)} \right).
\end{array} \right. \tag{7.21} \]

Since \(\|P_{\leq 0}(v)\|_{E^\sigma(1)} = \|P_{\leq 0}(\phi)\|_{L^2}\), it follows from (7.16) that

\[ \|v\|_{F^\sigma(1)} \lesssim \|P_{\leq 1}(v)\|_{E^\sigma(1)} + \|\phi\|_{H^\sigma}. \tag{7.22} \]

To bound \(\|P_{\geq 1}(v)\|_{E^\sigma(1)}\), we observe that

\[ \|P_{\geq 1}(v)\|_{E^\sigma(1)} = \|P_{\geq 1}(\Lambda^\sigma v)\|_{E^0(1)}. \]
where $\Lambda^\sigma$ is the Fourier multiplier operator with the symbol $|\xi|^\sigma$. Thus we apply the operator $\Lambda^\sigma$ on both sides of Eq. (7.17) and get

$$
\partial_t \Lambda^\sigma v + |\partial_x|^{1+\sigma} \partial_x \Lambda^\sigma v = -\Lambda^\sigma \partial_x \left[ v(u_1 + u_2)/2 \right].
$$

We rewrite the nonlinearity in the following way

$$
\Lambda^\sigma \partial_x \left[ v(u_1 + u_2)/2 \right] = \Lambda^\sigma \left[ \partial_x v(u_1 + u_2)/2 + v \partial_x (u_1 + u_2) \right]
$$

$$
= \frac{1}{2} \left[ \Lambda^\sigma, (u_1 + u_2) \right] \partial_x v + \frac{1}{2} \left[ \Lambda^\sigma, v \right] \partial_x (u_1 + u_2)
$$

$$
+ \frac{1}{2} (u_1 + u_2) \Lambda^\sigma \partial_x v + \frac{1}{2} v \Lambda^\sigma \partial_x (u_1 + u_2).
$$

(7.23)

The right-hand side of (7.23) can be rewritten as

$$
\frac{1}{2} \left[ \Lambda^\sigma, (u_1 + u_2) \right] \partial_x v + \frac{1}{2} \left[ \Lambda^\sigma, v \right] \partial_x (u_1 + u_2) + u_2 \Lambda^\sigma \partial_x v + v \Lambda^\sigma \partial_x u_1.
$$

We write the equation for $U = P_{\geq -10}(\Lambda^\sigma v)$ in the form

$$
\begin{cases}
\partial_t U + |\partial_x|^{1+\sigma} \partial_x U = P_{\geq -10}(-u_2 \cdot \partial_x U) + P_{\geq -10}(G);

U(0) = P_{\geq -10}(\Lambda^\sigma \phi),
\end{cases}
$$

(7.24)

where

$$
G = -P_{\geq -10}(u_2) \cdot \Lambda^\sigma \partial_x P_{\leq -11}(v) - P_{\leq -11}(u_2) \cdot \Lambda^\sigma \partial_x P_{\leq -11}(v)
$$

$$
- \frac{1}{2} \left[ \Lambda^\sigma, (u_1 + u_2) \right] \partial_x v - \frac{1}{2} \left[ \Lambda^\sigma, v \right] \partial_x (u_1 + u_2) - v \cdot \Lambda^\sigma \partial_x u_1.
$$

It follows from (7.2) and (7.24) that

$$
\|U\|^2_{L^2(\Omega(1))} - \|\phi\|^2_{H^\sigma} \lesssim \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) P_k(u_2 \cdot \partial_x U) \, dx \, dt \right|
$$

$$
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0,t_k]} P^2_k(U) P_{\geq -10}(u_2) \cdot \Lambda^\sigma \partial_x P_{\leq -11}(v) \, dx \, dt \right|
$$

$$
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) \left[ \Lambda^\sigma, (u_1 + u_2) \right] \partial_x v \, dx \, dt \right|
$$

$$
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) \left[ \Lambda^\sigma, v \right] \partial_x (u_1 + u_2) \, dx \, dt \right|
$$

$$
+ \sum_{k \geq 1} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) v \cdot \Lambda^\sigma \partial_x u_1 \, dx \, dt \right|
$$

$$
:= I + II + III + IV + V.
$$
For the contribution of $I$ we can bound it as in (7.19) and then get that
\[
I \lesssim \|U\|_{F^0(1)}^2 \|u_2\|_{F^\sigma(1)}.
\]

For the contribution of $II$, since the derivatives fall on the low frequency, then we can easily get
\[
II \lesssim \|U\|_{F^0(1)}^2 \|u_2\|_{F^\sigma(1)}.
\]

We consider now the contribution of $III$:
\begin{align*}
V & \leq \sum_{k \geq 1} \sum_{k_1, k_2 \in \mathbb{Z}^+} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) \cdot P_k_1(v) \cdot A^\sigma \partial_x P_k_2(u_1) \, dx \, dt \right| \\
& \leq \sum_{k \geq 1} \sum_{|k-k_2| \leq 5, k_1 \leq k-10} 2^{k(\sigma+1-\alpha)} \|P_k(U)\|_{F_k(1)} \|P_k_1(v)\|_{F_k_1(1)} \|P_k_2(u_1)\|_{F_k_2(1)} \\
& \quad + \sum_{k \geq 1} \sum_{k_1 \geq k-10} 2^{k(\sigma+1)-2\alpha \max(k_1,k_2)} \|P_k(U)\|_{F_k(1)} \|P_k_1(v)\|_{F_k_1(1)} \|P_k_2(u_1)\|_{F_k_2(1)} \\
& \lesssim \|U\|_{F^0(1)}^2 \|v\|_{F^0(1)} \|u_1\|_{F^{2\alpha}(1)} + \|U\|_{F^0(1)}^2 \|u_1\|_{F^\sigma(1)}.
\end{align*}

For the contribution of $III$, we obtain
\begin{align*}
III & \lesssim \sum_{k \geq 1} \sum_{k_1 \leq k-10} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) \left[ A^\sigma, P_k_1(u_1 + u_2) \right] \partial_x P_k_2(v) \, dx \, dt \right| \\
& \quad + \sum_{k \geq 1} \sum_{k_1 \geq k-9} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) \left[ A^\sigma, P_k_1(u_1 + u_2) \right] \partial_x P_k_2(v) \, dx \, dt \right| \\
& \quad := III_1 + III_2.
\end{align*}

We note that in the term $III_2$, the component $\left( u_1 + u_2 \right)$ can spare derivative, and thus we get
\begin{align*}
III_2 & \lesssim \sum_{k \geq 1} \sum_{k_1 \geq k-9} \left| \int_{\mathbb{R} \times [0,t_k]} A^\sigma P_k(U) P_k_1(u_1 + u_2) \partial_x P_k_2(v) \, dx \, dt \right| \\
& \quad + \sum_{k \geq 1} \sum_{k_1 \geq k-9} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(U) P_k_1(u_1 + u_2) A^\sigma \partial_x P_k_2(v) \, dx \, dt \right| \\
& \lesssim \sum_{k \geq 1} \sum_{k_1 \geq k-9} 2^{-2\alpha k_1} 2^{k \sigma} 2^{k_2} \|P_k(U)\|_{F_k(1)} \|P_k_1(u_1 + u_2)\|_{F_k_1(1)} \|P_k_2(v)\|_{F_k_2(1)} \\
& \quad + \sum_{k \geq 1} \sum_{k_1 \geq k-9} 2^{-2\alpha k_1} 2^{k_2(\sigma+1)} \|P_k(U)\|_{F_k(1)} \|P_k_1(u_1 + u_2)\|_{F_k_1(1)} \|P_k_2(v)\|_{F_k_2(1)} \\
& \lesssim \|U\|_{F^0(1)}^2 \left( \|u_1\|_{F^{\sigma}(1)} + \|u_2\|_{F^{\sigma}(1)} \right).
\end{align*}

For the contribution of $III_1$ we need to exploit the cancellation of the commutator. By taking $\gamma$ and extending $U, u_1, u_2, v$ as in the proof of Lemma 7.1, then we get
\[ III_I \lesssim \sum_{k \geq 1} \sum_{|k| \leq 10} \sum_{|n| \leq C^{1/1-\alpha} 2^{k-10}} \int \left( \gamma \left( 2^{[(1-\alpha)k_1]} t - n \right) I_{[0, t_k]}(t) P_k(U) \right) \cdot [\Lambda^\sigma, \gamma \left( 2^{[(1-\alpha)k_1]} t - n \right) P_k(u_1 + u_2)] \hat{\partial}_x P_{k_2} \left( \gamma \left( 2^{[(1-\alpha)k_3]} t - n \right) v \right) dx \, dt. \]

Let \( f_k = \gamma \left( 2^{[(1-\alpha)k_1]} t - n \right) P_k(U) \), \( g_k = \gamma \left( 2^{[(1-\alpha)k_1]} t - n \right) P_k(u_1 + u_2) \) and \( h_k = P_{k_2} \left( \gamma \left( 2^{[(1-\alpha)k_1]} t - n \right) v \right) \). It is easy to see from \(|k_2 - k| \leq 3\) that
\[
|\mathcal{F}([\Lambda^\sigma, g_k] \hat{\partial}_x h_k)(\xi, \tau)| \lesssim \int_{\mathbb{R}^2} |\widehat{g_k}(\xi - \xi', \tau - \tau')| 2^{k_2} 2^{\sigma_k} \hat{h}_k(\xi', \tau') d\xi' d\tau'.
\]

Then using a similar argument in the proof of Lemma 7.1 we can get that
\[
III_I \lesssim \sum_{k \geq 1} \sum_{|k| \leq 10} 2^{k_1} 2^{\sigma_k} 2^{\alpha_k} \|P_k(U)\|_{F^\sigma(1)} \|P_k(u_1 + u_2)\|_{F^\sigma(1)} \|P_{k_2}(v)\|_{F^{\sigma_k}(1)}.
\]

The contribution of IV is identical to the one of III from symmetry. Therefore, we have proved that
\[
\|U\|_{E^\sigma(1)}^2 \lesssim \|\phi\|_{H^\sigma}^2 + \|U\|_{F^\sigma(1)}^2 (\|u_1\|_{F^\sigma(1)} + \|u_2\|_{F^\sigma(1)}) + \|U\|_{F^{\sigma_k}(1)}^2 \|v\|_{F^{\sigma_k}(1)} \|u_1\|_{F^{2\sigma_k}(1)}.
\]

By (7.16), Theorem 1.1(a), (7.14) and (7.22) we get
\[
\|U\|_{E^\sigma(1)} \lesssim \|\phi_1 - \phi_2\|_{H^\sigma} + \|\phi_1 - \phi_2\|_{L^2} \|\phi_1\|_{H^{2\sigma}},
\]
which combined with (7.22) completes the proof of the proposition. \(\square\)

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