Rational and iterated maps, degeneracy loci, and the generalized Riemann-Hurwitz formula

James F. Glazebrook and Alberto Verjovsky

Abstract

We consider a generalized Riemann-Hurwitz formula as it may be applied to rational maps between projective varieties having an indeterminacy set and fold-like singularities. The case of a holomorphic branched covering map is recalled. Then we see how the formula can be applied to iterated maps having branch-like singularities. Separately, we consider a further application involving the Chern classes of determinantal varieties when the latter are realized as the degeneracy loci of certain vector bundle morphisms.

Mathematics Subject Classification (2010): 57M12 32C10 57R19 32H50

Keywords: Riemann-Hurwitz formula, rational maps, iterated maps, degeneracy locus, determinantal variety.

1 Introduction

The two-fold aim of this paper is firstly to consider (generalized) higher dimensional versions of the classical Riemann-Hurwitz formula as initially applied to rational maps of complex projective varieties \( f : X \to Y \), where \( X \) and \( Y \) have the same complex dimension. The main results presented here (Theorem 2.1 and Theorem 2.2) are derived from the general setting of [12] formulated mainly in the category of CW-complexes, and then applying the basics of the Chern-Weil theory of characteristic classes. Other versions of a generalized Riemann-Hurwitz, such as in the differentiable category, had previously been obtained in [6, 28, 25]. In this first part we will be applying the main results to operations involving rational maps, as for instance, realized in various algebraic-geometric and complex-dynamical constructions (beyond, that is, the familiar case of holomorphic branched covering maps). The cases in question include:

1. Iterates of rational maps \( f : X \to Y \) [4] (cf. [10]).

2. Rational self-maps \( f : X \to X \), with respect to their fixed point sets [1], and degree lowering curves [10].

In the second part, we will apply those same main results and the allied constructions to the study of determinantal varieties when the latter are realized as the degeneracy loci of morphisms \( \psi : E \to F \), of complex vector bundles over \( X \). From a general formula established in §5.1, we pay attention to two particular cases: general symmetric bundle maps [19] and flagged bundles [11].

Throughout, the formulas are established in terms of Chern classes, as by now the traditional method for studying invariants in the algebraic-geometric category. Specifically, such formulas
regulate the necessary topological conditions for the existence of a given class of rational maps (or morphisms), just as the classical Riemann-Hurwitz formula applies when studying holomorphic maps of algebraic curves (viz. compact Riemann surfaces).

The construction and main results of [12] were adapted in [14] to cover the case of generalized monoidal transformations. A further work [15] is intended to further elaborate on this construction, as well as to bring into focus a number of results obtained by other authors concerning the blowing-up process of (singular) Chern classes following the original study undertaken by I. R. Porteous [26, 27] (see also [2, 13]).

2 The topological background

Given a topological group $G$, we start by recalling from [12] a general result valid in the characteristic ring of $G$-bundles when defined initially within the category of CW-complexes.

2.1 Adapted pairs

Following [12], a pair of CW-complexes $(M, M_1)$ will be called an $(n, \Lambda)$-adapted pair for any coefficient ring $\Lambda$ if it satisfies the following conditions: Let $M$ be an $n$-dimensional CW-complex and $M_1$ a subcomplex of codimension $r$, with $r \geq 2$, such that:

1) $H_q(M, \Lambda) = \begin{cases} 0 & \text{if } q > n \\ \Lambda & \text{if } q = n \end{cases}$

2) $H_q(M_1, \Lambda) = 0$ if $q \geq n - 1$;

3) there exists a neighborhood $N(M_1)$ of $M_1$, such that $M_1$ is a deformation retract of the interior $N^0(M_1)$ of $N(M_1)$, and the inclusion map $p : M \rightarrow (M, M - N^0(M_1))$ induces an isomorphism

$$p_* : H_n(M) \isom H_n(M, M - N^0(M_1)).$$

(2.1)

Next we form the subspace $K$ of $M \times I$, where

$$K = (M \times \partial I) \cup ((M, M - N^0(M_1)) \times I),$$

(2.2)

together with the double $S(M_1) \subset K$, given by

$$S(M_1) = (\partial N(M_1) \times I) \cup (N(M_1) \times \partial I).$$

(2.3)

Now let

$$K_1 = (M \times \{0\}) \cup ((M, M - N^0(M_1)) \times [0, \frac{3}{4}],$$

and

$$K_2 = (M \times \{1\}) \cup ((M, M - N^0(M_1)) \times [0, \frac{1}{4}],$$

and let $S(M_1)_i = S(M_1) \cap K_i$, for $i = 1, 2$.

By this construction, the spaces $K_i$ are homotopically equivalent to $M_1$ and the spaces $K/K_1$ and $S(M_1)/S(M_1)_1$, are both homotopically equivalent to the Thom space $M/(M - N^0(M_1))$. 

2
It follows from the cofibration
\[ S(M_1)_1 \longrightarrow S(M_1) \longrightarrow S(M_1)/S(M_1)_1 \] (2.4)
and 2) above that
\[ H_n(S(M_1)_1, \Lambda) \cong H_n(S(M_1)/S(M_1)_1) \cong \Lambda. \] (2.5)
A choice of generators for \( H_n(M, \Lambda) \) and \( H_n(S(M_1), \Lambda) \) will be called an orientation of \( M \) and \( S(M_1) \), respectively.

Observe that conditions 1) to 3) above are immediately satisfied when \( M \) is a orientable \( n \)-manifold, and \( M_1 \) is a codimension \( r \)-submanifold (with \( r \geq 2 \)), and \( \Lambda = \mathbb{Z} \). In the following, spaces such as \( M \) will initially be considered as orientable simple \( n \)-circuits (or ‘triangulated pseudomanifolds’ in the sense of [16]; see also [9, 23]), and \( M_1 \) an arbitrary subcomplex of codimension \( r \geq 2 \).

Let \( BG \) denote the classifying space of the topological group \( G \). If \( P \in H^q(BG, \Lambda) \) is a cohomology class, and \( E \longrightarrow X \) is a \( G \)-bundle over a space \( X \), then we shall denote by \( P(E) \in H^q(X, \Lambda) \) the class defined by the characteristic polynomial \( P(E) = \Phi^*_E(P) \) where \( \Phi_E : X \longrightarrow BG \) is the classifying map (for the basic details see e.g. [5, 7, 21]). Also, if \( \tau \in H_q(X, \Lambda) \), then we denote the Kronecker pairing by \( \langle P(E), \tau \rangle \), for which the degree of \( P(E) \) equals the dimension of the cycle \( \tau \).

We have then the general result from [12, Theorem 1.1]:

**Theorem 2.1.** Suppose \( (M, M_1) \) is an \( (n, \Lambda) \)-adapted pair and \( E, F \) are \( G \)-bundles over \( M \), such that on \( M - M_1 \) there exists a homotopy
\[ \theta : \Phi_E|_{M-M_1} \sim \Phi_F|_{M-M_1}. \] (2.6)
Then there exists a bundle \( \xi_\theta \longrightarrow S(M_1) \) and orientations \([M]\) and \([S(M_1)]\), such that for any class \( P \in H^n(BG, \Lambda) \), we have the following equality of Kronecker pairings as established in [12, Th. (1.1)]:
\[ \langle P(E) - P(F), [M] \rangle = \langle P(\xi_\theta), [S(M_1)] \rangle. \] (2.7)

**Remark 2.1.** An analogous result in the context of generalized monoidal transformations was given in [14].

### 2.2 The clutching construction

We now give a more explicit construction of the bundle \( \xi_\theta \longrightarrow S(M_1) \) which can be used for the applications to follow.

Suppose now that the \( n \)-adapted pair \( (M, M_1) \) consists of (oriented) combinatorial manifolds, with \( \text{codim}_RM_1 = r \). Also, without much loss of generality, we suppose that \( G \) is one of the Lie groups \( SU(\ell), \text{SO}(\ell) \) or \( \text{Sp}(\ell) \), for some \( \ell \). For ease of notation, we retain \( E \) and \( F \) to denote the associated vector bundles to those \( G \)-bundles as previously. We also consider a homomorphism \( \psi : E \longrightarrow F \), such that
i) \( \psi : E|_{M-M_1} \longrightarrow F|_{M-M_1} \), and
ii) \( \psi|_{M_1} \) has constant rank.
Here \( \psi \) is viewed as a ‘clutching function’ \(^3\) \(^2\) used to clutch \( E \) and \( F \) over \( S(M_1) \) (cf. \(^2\)). With these assumptions, we have then the following exact sequence of vector bundles on \( M_1 \):

\[
0 \rightarrow K_1 \rightarrow E|_{M_1} \xrightarrow{\psi} F|_{M_1} \rightarrow K_2 \rightarrow 0,
\]

(2.8)

where \( K_1 \cong \ker \psi \), and \( K_2 \cong \coker \psi \). Further, let

\[
L := \psi(E|_{M_1}) \subset F|_{M_1}.
\]

(2.9)

From this it is straightforward to deduce isomorphisms

\[
E|_{M_1} \cong K_1 \oplus L, \quad \text{and} \quad F|_{M_1} \cong K_2 \oplus L.
\]

(2.10)

In the development of ideas that follow, the vector bundle \( L \) in (2.9) along with its characteristic ring will be essential objects for producing formulas that will specialize the righthand side of (2.7).

2.3 The result in the characteristic ring

In proceeding, we recall the Gysin sequence (see e.g. \(^5\) \(^2\)) in the form of a lemma:

**Lemma 2.1.** Let \( q : V \rightarrow M_1 \) be an oriented sphere bundle with fibre \( S^r \). Then there exists a long exact sequence

\[
\ldots H^i(V) \xrightarrow{q^*} H^{i-r}(M_1) \xrightarrow{\cup e} H^{i+1}(M_1) \xrightarrow{q^*} H^{i+1}(V) \ldots
\]

(2.14)

for which the maps \( q^* \), \( \cup e \), and \( q^* \) are integration along the fibre, the product with the Euler class, and the natural pull-back respectively.

For a given vector bundle \( E \) and corresponding polynomial \( P_E \in \Lambda[y_1, \ldots, y_n] \), we shall consider the image \( \Phi^*_E(P) \) in \( H^*(M, \Lambda) \).

A differentiable version of the following result was established \(^2\) and stated explicitly in terms of the Chern forms of a principal \( U(q) \)-bundle (for some \( q \)).
Theorem 2.2. Let $K$ and $L$ be as in (2.13), with $q : S(M_1) \to M_1$ the $r$-sphere bundle as above. Then with respect to the clutched bundle $\xi = (E, \psi, F)$ in (2.13) and the classifying map $\Phi$, we have the following equality in characteristic numbers
\[
\langle \Phi^*_x(P), [S(M_1)] \rangle = \langle \Phi^*_x(P) \cup \Phi^*_q(L), [S(M_1)] \rangle = k\langle \Phi^*_y(P), [M_1] \rangle,
\]
for some constant $k$, and hence
\[
\langle P(\xi), [S(M_1)] \rangle = \langle P(E) - P(F), [M] \rangle = k\langle P(L), [M_1] \rangle.
\]

Proof. Let $\alpha \in \Phi^*_x(P)$, and $\beta \in \Phi^*_y(P)$. Then if $q_x$ and $q_y$ are the maps in Lemma 2.1 we have via fibre-integration along $S(M_1)|_{x \in M_1}$, the equality $q_x(q_y(\alpha) \cup \beta) = \alpha \cup q_x(\beta)$, which on integrating over $M_1$, yields
\[
\langle \Phi^*_x(P)\Phi^*_y(P), [S(M_1)_x] \cup [S(M_1)_y] \rangle = \langle \Phi^*_x(P), [S(M_1)_x] \rangle \langle \Phi^*_y(P), [S(M_1)_y] \rangle.
\]
Now $\langle \Phi^*_x(P), [S(M_1)_x] \rangle = \langle \Phi^*_x(P), [S(M_1)_x] \rangle$, where $K_x = (q_1^*K_1|_{M_1(x)}, \eta_x, q_2^*K_2|_{M_1(y)}$ is the vector bundle over $S(M_1)|_{x \in M_1}$ constructed via the transition function $\eta_x$ seen as the restriction of $\eta$ to $\partial B(M_1)_x$. That is, we have an isomorphism
\[
\eta_x : q_1^*K_1|_{\partial B(M_1)_x} \cong q_2^*K_2|_{\partial B(M_1)_y}.
\]
If $c(x, y)$ is a curve in $M_1$ joining two points $x, y \in M_1$, then the restrictions $K_1|_{c(x,y)}$, and $K_2|_{c(x,y)}$ are trivial. We have then the following diagram in which the vertical maps are isomorphisms
\[
\begin{array}{c}
q_1^*K_1|_{\partial B(M_1)_x} \\
\cong \downarrow \eta_x \uparrow \cong \\
q_2^*K_2|_{\partial B(M_1)_y}
\end{array}
\]
and modulo these isomorphisms, $\eta_x$, and $\eta_y$ are homotopic. Thus $K_x$ and $K_y$ regarded as bundles on $S^r \cong S(M_1)_x \cong S(M_1)_y$, are isomorphic. Since $M_1$ is connected, this implies $\langle \Phi^*_x(P), [S(M_1)_x] \rangle$ is a constant, $k$, say, independent of $x$ from which (2.15) follows. Then (2.16) follows by (2.7). \( \square \)

Observe that the purely topological nature of Theorem 2.2 is applicable for any characteristic polynomial $\Phi^*P(\ )$. In particular, if $\Phi^*P(\ ) = \varepsilon(\ )$ is the Euler polynomial, we obtain as in [25]:

Corollary 2.1. For $\xi$ as defined above, we have in terms of Euler classes
\[
\langle \varepsilon(\xi), [S(M_1)] \rangle = \langle \varepsilon(E) - \varepsilon(F), [M] \rangle = k\langle \varepsilon(L), [M_1] \rangle,
\]
where $k$ is a constant.

Proof. It is instructive to include the straightforward proof from [12] Lemma1.2 which incorporates Lemma 2.1 (cf. [25]). Starting from (2.13), we have
\[
\langle \varepsilon(\xi), [S(M_1)] \rangle = \langle \varepsilon(q^*L) \cup \varepsilon(K), [S(M_1)] \rangle \\
= \langle q^*e(L), e(K) \cap [S(M_1)] \rangle \\
= \langle e(L), q_*e(K) \cap [S(M_1)] \rangle \\
= k\langle e(L), [M_1] \rangle.
\]
\( \square \)
3 Rational maps of projective varieties

In the following, we shall be applying the general construction and results of §2 in the category of complex manifolds with morphisms the meromorphic maps. In this case there will be a slight adjustment in the roles played by $M$ (and $M_1$) as result of redefining certain terms. When the context is clear, it is assumed that complexified tangent bundles are taken in each case.

The natural examples in this context include rational maps of (algebraic) projective varieties, and this fully enriched situation is the one to which we pay some attention. But we will point out now (as the astute minded reader can see) that the development of ideas, and constructions, etc. apply equally well if the spaces in question are just taken to be compact complex manifolds. But restricting to the algebraic case affords us some access to using significant numerical data, which otherwise might not necessarily be the case in the more general setting.

3.1 Application of the general result

Let $X$ and $Y$ be compact projective varieties, $\dim \mathbb{C} X = \dim \mathbb{C} Y = n$, and let $f : X \to Y$ be a rational map. In general, such a map will have an indeterminacy set $I_f$, namely the locus of points in $X$ for which $f$ fails to be holomorphic.

Let $X(s) := \{ x \in X : \text{rank}_\mathbb{C} f(x) \leq s < n \}$, (3.1)

be such that $f^{-1}(f(X(s))) = X(s)$. Letting $Z = I_f \cup X(s)$, we consider $(X, Z)$ as an $(n, \Lambda)$ adapted pair, with codim$_\mathbb{C} Z = r$. It is assumed that $Z$ is an ‘admissible space’ in the sense of [21, §4.2] with cycle $[Z]$, so that the Chern classes of complex vector bundles over $Z$ are defined. Thus we take $\Lambda = Z$.

Remark 3.1. We recall from e.g. [18] that such a rational map $f : X \to Y$ can be specified by a holomorphic map $\tilde{f} : X - I_f \to Y$, for which codim$_\mathbb{C} I_f \geq 2$. Thus for now, we are motivated to take $r \geq 2$, and view $Z$ as a ‘singular projective variety’. Also, in cases where $X = Y$, for instance, we might replace $X(s)$ above, by the fixed point set Fix($f$) of $f$, in the case of a self-map $f : X \to X$ (see e.g. §4.4).

3.2 A certain $2p$-cycle and application of Theorem [2.1]

Now take $M \subset X$ to be a compact oriented smooth submanifold, $\dim \mathbb{R} M = 2p$ (for $1 \leq p \leq n$) that intersects $Z$ transversally, and set

$N = M \cap Z = M \cap (X(s) \cup I_f)$, (3.2)

so that $\dim \mathbb{C} N = p - r$, for $p \geq r$. On applying our general result, we note that the isomorphisms in question are simply topological unless otherwise stated.

To this extent we take $(M, N)$ to be a $(2p, Z)$-adapted pair, and take $B(N)$ to be a tubular neighborhood of $N$ in $M$. Thus $N$ now plays the role of $M_1$ in [2.1]. With this slight modification in mind, we construct as in [2.2] the smooth double $S(N)$ producing the $S^{2r}$-fibration $q : S(N) \to N$, along with projections $q_i : B_i(N) \to N$ (for $i = 1, 2$) as in [2.11]. Note this produces a $2p$-cycle $[S(N)]$.

Here we will set

$E = TX|_{B_1(N)}$, and $F = f^*TY|_{B_2(N)}$, (3.3)
where, as before, \( \psi : TX \rightarrow f^*TY \) is an isomorphism when restricted to \( \partial B_1(N) = \partial B_2(N) \).

Following Theorem 2.1 and from the construction of 2.2 in the context of complex vector bundles with structure group \( \mathbb{U}(q) \), for some \( q \), we straightaway obtain

\[
\langle \Phi^*_E(P) - \Phi^*_F(P), [M] \rangle = \langle \Phi^*_\xi(P), [S(N)] \rangle,
\]

in terms of Chern polynomials \( \Phi_\phi(P) \).

On applying 2.2 together with 3.3, then 2.2 with \( \Phi^*_\phi(P) = c_p(\phi) \) (for \( 1 \leq p \leq n \)) reduces to the following form (cf. [6, 25]):

\[
\langle c_p(X) - f^*c_p(Y), [M] \rangle = \langle c_p(\xi), [S(N)] \rangle, \quad 1 \leq p \leq n,
\]

where \( \xi = (E, \psi, F) \) is given by 2.2.

### 3.3 The right-hand side of (3.3)

In order to deal with enumerating the right-hand side of (3.3), we return to the setting and conditions of 2.2. Here we take \( \psi \) to have constant rank \( n - r \) along \( N \), and following 2.2 we have the isomorphism

\[
TZ|_N \cong L = \psi(TZ)|_N,
\]

so that \( \operatorname{rank}_C L = n - r \). We also recall from 2.2 the relations

\[
E|_N \cong K_1 \oplus L, \quad F|_N \cong K_2 \oplus L,
\]

\[
E \cong q_1^* (K_1) \oplus q_1^* (L), \quad F \cong q_2^* (K_2) \oplus q_2^* (L),
\]

while noting that \( N \) is a deformation retract of \( B_i(N) \), for \( i = 1, 2 \). Hence on \( S(N) \) we have the isomorphism \( \xi \cong (q_1^* K_1, \eta, q_2^* K_2) \oplus q^* L = K \oplus q^* L \), as in 2.13, with \( \operatorname{rank}_C \xi = n \), from which we deduce \( \operatorname{rank}_C K = r \) (note that we have identified \( K_1 \) with the restriction to \( N \) of a complex rank \( r \) vector bundle normal to \( TZ \) in \( TX|_N \)).

As deduced from the total Chern classes of \( K \) and \( q^* (L) \), it is straightforward to show that

\[
c_p(K \oplus q^* L) = \sum_{\nu=1}^{r} c_p(K) \cup q^* c_{p-\nu} (L) + q^* c_p (L), \quad 1 \leq p \leq n - 1,
\]

\[
c_n(K \oplus q^* L) = \sum_{\nu=1}^{r} c_n(K) \cup q^* c_{n-\nu} (L).
\]

**Lemma 3.1.**

\[
\langle q^* c_p (L), [S(N)] \rangle = 0, \quad if 1 \leq p \leq n - 1.
\]

**Proof.** This follows from [6, §4](cf. [28] pp. 408–409) showing the existence on \( S(N) \) of \( \ell \) linearly independent trivializing sections of \( q^* (L) \), with \( \ell = (n - r) - (p - r) = n - p \). \( \square \)

**Theorem 3.1.** With regards to 3.2 and 3.6, we have for \( 1 \leq p \leq r \), and \( 1 \leq \nu \leq r \),

\[
\langle c_p(X) - f^* c_p(Y), [M] \rangle = \sum_{\nu=1}^{r} k_{p \nu} c_{p-\nu} (L), [N],
\]

where
even if $X$ is a constant independent of $x$ where we have applied Theorem 2.2, in particular (2.17), to show that map, such that there exists nowhere dense analytic subsets $X$: 

\[ f \]

The set $X \subseteq C$ with dim $\xi$ (3.8) and (3.9), we thus obtain for $\xi = (E, \psi, F)$ above,

\[ \langle c_p(\xi), [S(N)] \rangle = \left( \sum_{\nu=1}^{r} c_\nu(K) \cup q^* c_{p-\nu}(L), [S(N)] \right) \]

\[ = \left( \sum_{\nu=1}^{r} \left( c_\nu(K_x), [S(N)_x] \right) \cdot \langle c_{p-\nu}(L), [N] \rangle \right), \]

where we have applied Theorem 2.2 in particular (2.17), to show that $\sum_{\nu=1}^{r} \left( c_\nu(K_x), [S(N)_x] \right) = k_\nu$ is a constant independent of $x \in N$, in which $p_\nu \in \mathbb{N}$ satisfies $\nu p_\nu = p$ (for 1 $\leq \nu \leq r$).

Likewise, the quantities $b_\nu \in \mathbb{N}$ are determined at each stage by the rank value of $L$ as above, with dim $\mathbb{C} N = p - r$. Hence we obtain

\[ \langle c_p(\xi), [S(N)] \rangle = \left( \sum_{\nu=1}^{r} k_\nu c_{p-\nu}(L), [N] \right). \]

\[ (3.13) \]

\[ \Box \]

### 3.4 Interpreting Theorem §3.1

In view of applying the clutching construction of §2.2, Theorem §3.1 produces a significantly general formula that can be observed when regulating the topology of a rational map $f: X \rightarrow Y$ of compact projective varieties of equal (complex) dimension in terms of the cycles $[M]$ and $[N]$ as defined. Note that $[N]$ is a cycle which contains part of the (possibly large) singular variety $Z = X(s) \cup I_f$, once $Z$ is intersected by the 2p-cycle $[M]$ as in (3.2).

A working principle is to ‘resolve’ the indeterminacy set $I_f$, for instance by blowing up along $f(I_f)$, and then reduce matters to considering a holomorphic map $\hat{f}: X \rightarrow Y$. In which case we would set $N = M \cap X(s)$, and take $r \geq 1$. For instance, if $M = X$ (so $p = n$), then (3.10) in this case reduces to:

\[ \langle c_n(X) - \hat{f}^* c_n(Y), [X] \rangle = \sum_{\nu=1}^{r} k_\nu \langle c_{n-\nu}(X(s)), [X(s)] \rangle. \]

\[ (3.14) \]

More specifically, given $f: X \rightarrow Y$, one may pass to a proper modification $\hat{f}: \hat{X} \rightarrow Y$, and apply the formula in Theorem 3.1 for a holomorphic map, provided $\hat{X}$ is a compact complex manifold.

\[ \ast \]

We recall from [17] that a proper modification $\hat{f}: \hat{X} \rightarrow Y$ means that $\hat{f}$ is a proper surjective holomorphic map, such that there exists nowhere dense analytic subsets $X' \subset X$ and $Y' \subset Y$, such that: i) $\hat{f}(X') \subset Y'$, ii) $\hat{f}: \tilde{X} - X' \rightarrow Y - Y'$ is a biholomorphism, and iii) each fiber $\hat{f}^{-1}(y)$, for $y \in Y'$, consists of more than one point. The set $X' = \hat{f}^{-1}(y)$ is called the exceptional set. Note that $\hat{X}$ may not necessarily be an algebraic variety in general, even if $X$ has this property [20].
Theorem 3.1 holds with otherwise the basic construction leading to the various formulas remains the same. In particular, $\Delta_k$ and set $f$ denoted $\delta$.

Now we consider the case where $s = n - 1$ is constant, and let $f : X \rightarrow Y$ be a holomorphic branched covering map for which $X_1 := X(n-1) \subset X$ is the ramification divisor on which $\text{rank}_\mathbb{C}f|_{X_1} = n-1$, with $r = \text{codim}_\mathbb{C}X_1 = 1$. As before, let $M \subset X$ be any compact oriented smooth submanifold with $\text{dim}_\mathbb{R} M = 2p$ that meets $X_1$ transversally (with $1 \leq p \leq n$), with $N = M \cap X_1$. From (3.6), we have $L = TX_1|_N$. This leads to a version of the higher dimensional Riemann-Hurwitz formula as given in [3, Proposition 2] (see also [28, p. 409]):

$$\langle f^*c_p(Y) - c_p(X), [M] \rangle = (\mu - 1)\langle c_{p-1}(X_1), [N] \rangle,$$

(3.16)

where $\mu = \text{deg}(f|_{X_1}) \in \mathbb{Z}$ is the local topological degree of $f$ along $X_1$. Note that in general, the global degree $\text{deg}(f) := \delta \neq \mu$.

Together with (3.10), this case reveals the interest in enumerating the righthand side of (2.7) and (2.15) in general.

**Example 3.1.** Let $n = 2$, where $X$ is now a compact complex surface, and let $Y = \mathbb{C}P^2$. Consider a holomorphic map $f : X \rightarrow \mathbb{C}P^2$ which is branched over a curve $C$ of genus $g$ with normal crossings. Thus $f : X - X_1 \rightarrow \mathbb{C}P^2 - C$ is an unramified covering map of degree $\text{deg} f = \delta$ say, where the branch set $X_1 = f^{-1}(C)$.

As an example of showing consistency in the data, suppose in this case $X$ is a K3 surface. One can construct a map with $\delta = \mu = 4$, branched over a quartic curve $C$ (see e.g. [24]). In this top dimension, we have $\langle c_2(X), [X] \rangle = \chi(X) = 24$, and $\langle f^*c_2(\mathbb{C}P^2), [X] \rangle = 4\chi(\mathbb{C}P^2) = 4(3) = 12$. From (3.10), with $n = p = 2$, and the adjunction formula (e.g. [18, p.221]), it is straightforward to see that $\langle c_1(X_1), [X_1] \rangle = \chi(C) = -4$, and therefore $g = 3$.

4 **Rational self-maps**

4.1 **Iterated self-maps of projective varieties**

Now we consider the case where $X = Y$, and $f : X \rightarrow X$ is a rational map with $k$-th iterate denoted $f^k$. Let

$$X(k, s) := \{x \in X : \text{rank}_\mathbb{C}f^k(x) \leq s < n\},$$

(4.1)

and set $Z = X(k, s) \cup I_f$, with $\text{codim}_\mathbb{C}Z = r_k$. Let $\Delta_k = \text{deg}(f^k)$, and note that in general, $\Delta_k \neq \delta^k = \text{deg}(f)^k$.

Here we take $E = TX|_{B_i(N)}$, and $F = (f^k)^*TX|_{B_k(N)}$. This is only the essential difference, otherwise the basic construction leading to the various formulas remains the same. In particular, Theorem 3.1 holds with $X(s)$ replaced by $X(k, s)$ in (4.1).
4.2 Holomorphic and analytically stable maps

Let us deal first with a holomorphic map \( f : X \rightarrow X \), where the \( k \)-th iterate \( f^k \) is a holomorphic branched covering map with ramification divisor \( X_{1,k} := X(k,n - 1) \), with \( r_k = 1 \). Then (3.16) reads as

\[
\langle (f^k) c_p(X) - c_p(X), [M] \rangle = (\mu_k - 1) \langle c_{p-1}(X_{1,k}), [N] \rangle,
\]

where \( \mu_k \in \mathbb{Z} \) is the local topological degree of \( f^k \) along \( X_{1,k} \). We have \( \Delta_k \neq \mu_k \), in general.

However, in this case where \( f \) is holomorphic and \( X = \mathbb{C}P^n \), we do have \( \Delta_k = \delta_k \) [10]. On the other hand, it is clear for \( M = \mathbb{C}P^n \), and \( p = n \), that such maps are thus regulated by the expression derived from (1.2):

\[
(n + 1)(\Delta_k - 1) = (\mu_k - 1) \langle c_{n-1}(X_{1,k}), [X_{1,k}] \rangle.
\]

Taking \( n = 2 \), a bimeromorphic map \( f : X \rightarrow X \) is said to be analytically stable if: i) for all \( k \geq 0 \), we have \( (f^k)^* = (f^*)^k \), and ii) for each curve \( C \) in \( X \), \( f^k(C) \notin I_f \) (see [10]).

There are several observations we can make in this case (\( n = 2 \)):

**Proposition 4.1.**

1. Let \( f : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2 \) be a holomorphic map. Suppose that the \( k \)-th iterate \( f^k : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2 \) is a holomorphic branched covering map over a curve of genus \( g \). Then necessarily the global degree \( \Delta_k \neq \mu_k \) (the local degree).

2. Let \( f : X \rightarrow X \) be a bimeromorphic map of (compact) algebraic surfaces. Then, through iterations \( f^k : X \rightarrow X \) (for all \( k \)), the characteristic numbers \( \langle c_p(\hat{X}), [\hat{X}] \rangle \) (for \( p = 1, 2 \)) are invariants of \( f \), given some proper modification \( \hat{f} : \hat{X} \rightarrow X \), where \( \hat{X} \) is a compact complex manifold.

**Proof.** For (1) we start by enumerating (1.3). This gives

\[
3(\Delta_k - 1) = 3(\delta_k - 1) = (\mu_k - 1)(2 - 2g),
\]

which is meaningless if \( \Delta_k = \mu_k \).

To go about proving (2), recall that the blow up of \( Y \) at \( y \), is the proper modification \( \hat{f} : \hat{X} \rightarrow Y \) which replaces \( y \) with the exceptional curve \( \pi^{-1}(y) \cong \mathbb{C}P^1 \), the set of holomorphic tangent directions at \( y \), and \( \hat{f} \) is a biholomorphism elsewhere. In fact, in this instance, any proper modification \( \hat{f} : \hat{X} \rightarrow Y \) arises as a composition of finitely many point blow ups (see e.g. [8 Th. 1.1]). Following [3, Th. 01], if \( f : X \rightarrow X \) is a bimeromorphic map, then there always exists a proper modification \( \hat{f} : \hat{X} \rightarrow X \) that lifts \( f \) to an analytically stable map. By this last condition, we therefore have on applying (3.16):

\[
\langle (\hat{f}^*)^k c_2(X), [\hat{X}] \rangle = \langle (\hat{f}^*)^k c_2(X), [\hat{X}] \rangle = \langle c_2(\hat{X}), [\hat{X}] \rangle.
\]

\[\Box\]

**Remark 4.1.** It is worth noting that in [3] it is shown, that for \( n = 2 \), there are countably many sequences \( \{d_\ell\} \subset \mathbb{N} \) for which a rational map \( f : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2 \) exists, satisfying \( \Delta_\ell = d_\ell \), for all \( \ell \in \mathbb{N} \).
4.3 Degree lowering curves

For a general rational map \( f : \mathbb{C}P^n \to \mathbb{C}P^n \), there may be any amount of peculiar behavior. For instance, it is possible that an iteration \( f^k \) may, for some \( k \), map an (irreducible) curve \( C \) into the indeterminacy set \( I_f \) (thus \( f \) cannot be analytically stable). In this case one sees that \( \Delta_k < \delta^k \) \(^{[10]}\), and so enumerating \((3.10)\), for \( p = n \), can easily be seen to give

\[
\Delta_k = 1 - \frac{1}{n+1} \sum_{\nu=1}^r k_{\nu} c_{n-\nu}(L), [N] < \delta^k,
\]

(4.6)

where \( N \) is given by \((3.2)\), \( L \) is as in \((3.6)\), and the constants \( k_{\nu} \) given by \((3.11)\).

4.4 Holomorphic maps with fixed point set

Next we will consider a holomorphic map \( f : X \to X \), with a (possibly singular) fixed point set \( S = \text{Fix}(f) \), with \( r = \text{codim}_C S \). In the general setting of \([2]\), we will regard \((X,S)\) as an adapted pair with \( S \) playing the role of \( M_1 \). Here we are partially motivated by certain constructions in \([1]\) (where \( X \) can be taken as a complex manifold) to which we will apply the general results of Theorem \(2.1\) and Theorem \(2.2\). This, on the other hand, will necessitate some re-defining of terms, and making a different assignment of data compared to that of the previous sections.

Specifically, let us start by letting \( Q_S \) denote the normal bundle to \( S \) in \( X \). Then let

\[
E = (Q_S)^{\otimes \nu_f}, \quad \text{and} \quad F = TX,
\]

(4.7)

where \( \nu_f \in \mathbb{N} \). We take up the hypotheses in \((2.2)\) in terms of a homomorphism \( \psi : (Q_S)^{\otimes \nu_f} \to TX \). In particular, as in \([1]\), the restricted morphism \( \psi|_S : (Q_S)^{\otimes \nu_f} \to TX|_S \), is identified with a holomorphic section of \( TX|_S \otimes (Q_S)|^{\otimes \nu_f} \). The bundles \( K \) and \( L \) are taken to be as in \((2.2)\) with

\[
L = \text{Im}(\psi|_S) \subset TX|_S.
\]

(4.8)

Let \( M \) be a \( 2p \)-dimensional submanifold \( M \subseteq X \) intersecting \( S \) transversally. As before, we set \( N = M \cap S \), with \( \dim_C N = p - r \) (for \( p \geq r \)).

Having made these adjustments for the bundles \( E \) and \( F \), etc., we apply Theorem \(2.4\) together with essentially the same proof as that used in proving Theorem \(3.1\) to obtain

\[
\langle c_p((Q_S)^{\otimes \nu_f}) - c_p(X), [M] \rangle = k\langle c_{p-r}(L), [N] \rangle.
\]

(4.9)

Example 4.1. Consider the case \( r = 1 \), and \( S \) is a (possibly singular) hypersurface regarded as an oriented \((n-1)\)-circuit. Then we have \( \text{rank}_C Q_S = 1 \), and \( \text{rank}_C L = n - 1 \), following which \((4.9)\) gives

\[
\nu_f \langle c_1^p(Q_S) - c_p(X), [M] \rangle = k\langle c_{p-1}(L), [N] \rangle.
\]

(4.10)

In particular, for \( M = X \) \((p = n)\), we obtain

\[
\nu_f \langle c_1^p(Q_S) - c_n(X), [X] \rangle = k\langle c_{n-1}(S), [S] \rangle.
\]

(4.11)

Remark 4.2. In the setting of \([1]\), the quantity \( \nu_f \) is considered as a measure of ‘order of contact’ between the map \( f \) and \( S \). For the case of such a hypersurface \( S \) there are the connected components \( \Lambda_\alpha \) of the union of singular sets \( \text{Sing}(X_f) \cup \text{Sing}(S) \), where \( \text{Sing}(X_f) \) is the set of zeros of a vector
field $X_f$ associated to $f$ that induces a (generally singular) holomorphic foliation. This leads to a residue formula \( \sum_a \text{Res}(X_f, S, \Lambda_a) = \langle c_{n-1}(S), [S] \rangle \) as in [11 Th. 01]. Our approach leads to somewhat different formulas as seen above. Though enumerating (4.11) for the case $n = p = 2$, shows that the right-hand (up to a constant) is such a residual quantity.

5 Determinantal varieties

5.1 The degeneracy locus

In this section we commence the second part of the paper by turning to a related, but essentially more general setting. In the previous sections we considered applying the general result of [2] to rational maps of projective varieties. But now we tweak the setting of those sections somewhat with several terms redefined for the sake of replacing maps of projective varieties by vector bundle morphisms over a compact complex manifold $X$.

More specifically, consider a morphism $\psi : E \to F$ of complex (smooth) vector bundles over a projective variety $X$ (where dim$_\mathbb{C}$ $X = n$). For some given $s \in \mathbb{N}$, we have the degeneracy locus of $\psi$, as defined by

$$\Omega(s) := \{x \in X : \text{rank}_\mathbb{C}\psi(x) \leq s\}. \quad (5.1)$$

Once again we will apply the general setting of [2] where $(X, \Omega(s))$ is regarded as an $n$-adapted pair (so that $\Omega(s)$ plays the role of $M_1$ in [2]), with $r = \text{codim}_\mathbb{C}\Omega(s)$. Theorem 2.1 in particular, (2.7) immediately applies to give the general statement

$$\langle P(E) - P(F), [X] \rangle = \langle P(\xi_\theta), [S(\Omega(s))] \rangle.$$

(5.2)

**Remark 5.1.** The cohomology class $\{\Omega(s)\}$ of $\Omega(s)$ in $X$ can be determined by polynomials in the Chern classes of $E$ and $F$, and in certain cases the codimension of $\Omega(s)$ can be determined (see [11, 19] which also cover a historical background to the general problem in the algebraic geometric context). Note that [19] deals initially with results in the differentiable category, thus (5.2) applies in that case as well.

**Example 5.1.** If $E \to X$ is a complex vector bundle, rank$_\mathbb{C}$ $E = \ell$, $F = E^*$, and $\psi : E \to E^*$ is a general symmetric bundle map, then following [19 Th. 1], the cohomology class $\{\Omega(s)\}$ is given by a polynomial $P_s(c_1(E^*), \ldots, c_\ell(E^*))$. $\square$ This latter polynomial has an explicit expression given in terms of the determinant

$$2^{\ell-s} \begin{bmatrix} c_{\ell-s} & c_{\ell-s+1} & c_{\ell-s+2} \\ c_{\ell-s-2} & c_{\ell-s-1} & c_{\ell-s} \\ \vdots & \vdots & \ddots \end{bmatrix} = \{\Omega(s)\}, \quad (5.3)$$

where $c_i = c_i(E^*)$, and further, $\Omega(s)$ has codimension $r = \binom{\ell - s + 1}{2}$ for $s < \ell$.

---

1Let $V$ be a vector space and $V^*$ its dual vector space. A linear map $\psi : V \to V^*$ is said to be symmetric if $(\psi(x), y) = (\psi(y), x)$, for all $x, y \in V$, where $(\ ,\ )$ is the dual pairing between $V^*$ and $V$. Equivalently, $\psi$ is symmetric if $\psi = \psi^T$. The precise meaning of ‘general’ is explained in [19 Note 2, p.72]. Likewise, $\psi$ is skew-symmetric if $\psi = -\psi^T$. 

12
In keeping with the previous sections, we will be interested in finding expressions linking the Chern classes of some degree (p, say) of the spaces in question.

Let \( M \) be as in §3.2, transversally intersecting \( \Omega(s) \), and set \( N = M \cap \Omega(s) \), with \( \dim_{\mathbb{C}} N = p - r \), for \( p \geq r \) (again, the case \( r \geq p \) can be treated likewise). We recall the tubular neighborhoods \( B_i(N) \) of \( N \) (for \( i = 1, 2 \)), and consider \( E, F \) as restricted to \( B_1(N) \) and \( B_2(N) \) respectively, so that in accordance with §2.2, we have \( E|_{\partial B_1(N)} \cong F|_{\partial B_2(N)} \). Note that we do not yet assume that \( \psi \) has constant rank on \( \Omega(s) \), since we are still in the context of Theorem 2.1. We also recall from §2.2 the \( S^{2r} \)-fibration \( q : S(N) \to N \).

We shall be applying the same basic strategy as in §3 (and in §4). Thus (5.2) reduces to a statement that is more general than (3.5):

\[
\langle c_p(E) - c_p(F), [M] \rangle = \langle P(\xi_\theta)^{[2p]}, [S(N)] \rangle, \quad 1 \leq p \leq n,
\]

where \( P(\xi_\theta)^{[2p]} \) denotes the component of \( P(\xi_\theta) \) in dimension \( 2p \). Again, it is interesting to enumerate the right-hand side of (5.4) once the cohomology class \( \{\Omega(s)\} \) has been determined.

**Remark 5.2.** In view of this last comment, if the cohomology class \( \{N\} \) of \( N \) in \( M \) happens to be cohomologous to \( \{\Omega(s)\}^{[2p]} \) in \( H^*(\Omega(s), \mathbb{Z}) \), then the class \( P(\xi_\theta)^{[2p]} \) is expressible in the form

\[
P(\xi_\theta)^{[2p]} = q^* \{\Omega(s)\}^{[2p]} \cup \gamma,
\]

for some \( \gamma \in H^{2p}(S(N), \mathbb{Z}) \).

The following observations summarized as a proposition shows that, in the context of the symmetric bundle map of Example 5.1, there is indeed a restriction on components of the class \( P(\xi_\theta) \).

**Proposition 5.1.** With regards to the context of Example 5.1, we have for \( 1 \leq p \leq \ell \), the following relationships:

1. For \( p \) odd,

\[
\langle c_p(E) - c_p(F), [M] \rangle = \frac{1}{2} \langle P(\xi_\theta)^{[2p]}, [S(N)] \rangle.
\]

2. For \( p \) even, the class \( P(\xi_\theta)^{[2p]} \) is trivial in \( H^{2p}(S(N), \mathbb{Z}) \).

*Proof.* Noting that \( c_p(E^*) = (-1)^p c_p(E) \) (see e.g. [18, p.411]), we have from (5.4)

\[
(1 + (-1)^{p+1}) \langle c_p(E), [M] \rangle = \langle P(\xi_\theta)^{[2p]}, [S(N)] \rangle,
\]

for \( 1 \leq p \leq \ell \), from which the results follow.

Further enumeration of the righthand side of (5.7), can be carried out under the conditions of §2.2 which we will deal with next.
5.2 Constant rank case

Suppose now that $\text{rank}_\mathbb{C} \psi|_{\Omega(s)} = s < \ell$, is constant, and the bundle map $\psi$ is taken as a clutching map between $E$ and $F$, as in (5.2). In this case, there is the class of the $2p$-component given by $P(\xi_g)^{[2p]} = c_p(E, \psi, F)$. Also, Theorem 2.2 applies to give

$$\langle P(E) - P(F), [X] \rangle = k \langle P(L), \Omega(s) \rangle,$$

(5.8)

where, as before $L = \psi(E|_{\Omega(s)}) \subset F|_{\Omega(s)}$, with $\text{rank}_\mathbb{C} L = s$. In particular, $P(L) \in H^*(\Omega(s), \mathbb{Z})$; so knowing the cohomology of $\Omega(s)$ gives us a handle on the class $P(L)$. Applying Theorem 2.2 (cf. Theorem 3.1), we then have

$$\langle c_p(E) - c_p(F), [M] \rangle = \langle \sum_{\nu=0}^{s-1} k_\nu c_{s-\nu}^b(L), [N] \rangle,$$

(5.9)

where $b_\nu \in \mathbb{N}$ satisfies $b_\nu(s - \nu) = (p - r)$ (for $s > \nu$, $p > r$), and where the constants $k_\nu$ are given as in (5.11).

Example 5.2. In view of the above remarks, let us return to the context of Example 5.1. Here we have $L = \psi(E|_{\Omega(s)}) \subset E^*|_{\Omega(s)}$, and the cohomology class $P(L) = P(c_1(E^*), \ldots, c_s(E^*))$. On applying (5.9), we thus obtain for $p$ odd,

$$\langle c_p(E), [M] \rangle = \frac{1}{2} \sum_{\nu=0}^{s-1} k_\nu c_{s-\nu}^b(E^*), [N] \rangle = \kappa \langle P(c_1(E^*), \ldots, c_s(E^*)), [N] \rangle,$$

(5.10)

for some constant $\kappa$.

Example 5.3. There are analogous results in [19] for the cohomology class $\{\Omega(s)\}$ in the case of skew-symmetric maps (morphisms) $\psi : E \rightarrow E^*$. The cases $\psi : E \rightarrow E^* \otimes \mathcal{L}$, for $\mathcal{L}$ a complex line bundle, are also studied in the symmetric and skew-symmetric cases. For instance, when $\psi : E \rightarrow E^* \otimes \mathcal{L}$ is a general symmetric bundle map, then the cohomology class $\{\Omega(s)\}$ is given by (5.3), but now taking $c_i = c_i(E^* \otimes \sqrt{\mathcal{L}})$ [19, Th. 10]. The (general) skew-symmetric case can likewise be treated.

Example 5.4. (Application to variation of Hodge structure following [19]): Consider a family $\omega : \mathcal{C} \rightarrow X$ of curves of genus $g$. Let $H^{1,0}, H^{0,1}$ denote the corresponding Hodge bundles. We have then a period map $\Upsilon : X \rightarrow \text{Gr}(g, 2g)/\Gamma$, where $\text{Gr}(g, 2g)$ denotes a certain isotropic Grassmannian, and $\Gamma \subset \text{Aut}(\text{Gr}(g, 2g))$ is a discrete subgroup. Consequently, there is a bundle morphism $\psi : TX \rightarrow \text{Hom}(H^{1,0}, H^{0,1})$, that can be expressed alternatively as a symmetric bundle map $TX \rightarrow S^2(H^{1,0})^*$ [19].

When $X$ is an algebraic curve of genus $g_X$, and there are no singular fibres of $\omega$, then one can enumerate matters as follows. Setting $E = H^{1,0}$ (so $\text{rank}_\mathbb{C} E = g$), we have from [19, p.82] $c_1(\det(S^2 E^* \otimes \mathcal{O}(1)) \geq 0$. Observing that $c_1(S^2 E^*) = (g + 1)c_1(E^*)$, then this previous expression simplifies to $g(g_X - 1) \geq c_1(E)$.

It can be argued that if the variation of Hodge structure over $X$ is non-trivial, then by the local Torelli theorem, the period map $\Upsilon$ has maximal rank at some point of $X$, and by [19, Th. 10], the degeneracy locus $\Omega(g - 1)$ in this case, is not all of $X$. In the context of a general symmetric
bundle map, here given by $\psi : E \to E^* \otimes \mathcal{O}(-1)$, and from the remarks in Example 5.3 above, it follows that the cohomology class $\{\Omega(g - 1)\} = -2c_1(E \otimes \mathcal{O}(-\frac{2}{2})) = -2(c_1(E) - g(\chi - 1))$. This provides us with an enumeration of (5.9) with $M = X$, $N = \Omega(g - 1)$, and $F = \mathcal{O}(-\frac{1}{2})$, in the case of $p = 1$ and $s = g - 1$. In this case there is just a single constant $k = k_\nu = -\frac{1}{2}$.

5.3 Flagged bundles

Suppose now we consider, as in (11), the more general situation of (5.4 for a morphism $\psi : E \to F$, over $X$, for which

\[
E_1 \subset E_2 \subset \cdots \subset E_u = E
\]

\[
F = F_0 \to F_{u-1} \to \cdots \to F_1
\]  

(5.11)

are flags of subbundles and quotient bundles, respectively. Here we will take integers $s(\alpha, \beta) \in \mathbb{N}$ specified across the intervals $1 \leq \alpha \leq u$, and $1 \leq \beta \leq v$, and the degeneracy locus is then defined by

\[
\Omega(s) := \{x \in X : \text{rank}_\mathbb{C}(E_\alpha(x) \to F_\beta(x)) \leq s(\alpha, \beta), \forall (\alpha, \beta)\},
\]  

(5.12)

where $s$ is regarded as a certain rank function. As shown in (11), conditions on $s$ determine the irreducibility of $\Omega(s)$ as a projective variety, and further, the cohomology class $\{\Omega(s)\}$ can be determined in terms of the Chern classes of $E$ and $F$.

In the case $\psi|_\Omega(s)$ has constant rank $s(\alpha, \beta)$, and $\psi$ is a clutching map as before, we apply Theorem (2.2) to obtain $\langle P(E) - P(F), [X] \rangle = k(P(L), [\Omega(s)])$.

5.4 Complete flags

Following (11), we will exemplify matters in the case of ‘complete flags’ for the data $u = v = m$, and $E_i, F_i$ having (complex) rank $i$. In this case, $\Omega(s)$ is characterized by permutations in the symmetric group $S_m$. Given $w \in S_m$, let $\ell(w)$ be the length of $w$ (in other words, the number of inversions). Let $s_{\alpha}(\beta, \alpha) = \text{card}\{i \leq \beta : w(i) \leq \alpha\}$, and

\[
x_i = c_1(\text{Ker}(F_i \to F_{i-1})), \quad y_i = c_1(E_i/E_{i-1}), \text{ for } 1 \leq i \leq m.
\]  

(5.13)

Then one restricts attention to

\[
\Omega(w) = \Omega(s_w) := \{x \in X : \text{rank}_\mathbb{C}(E_\alpha(x) \to F_\beta(x)) \leq s_w(\beta, \alpha), \forall (\alpha, \beta)\}.
\]  

(5.14)

Here we make several observations from (11):

(i) The space $\Omega(w)$ has a natural structure of a scheme given by the vanishing of the induced maps from $\wedge^{s_w(\beta, \alpha)+1}(E_\alpha) \to \wedge^{s_w(\beta, \alpha)+1}(F_\beta)$.

(ii) The expected (maximum) value of $r = \text{codim}_\mathbb{P}\Omega(w)$, is $r = \ell(w)$.

(iii) The cohomology class $\{\Omega(w)\} = \mathcal{G}_w(x, y)$, where $\mathcal{G}_w(x, y) = \mathcal{G}(x_1, \ldots, x_m, y_1, \ldots, y_m)$, is the double Schubert polynomial for $w$, this being a homogeneous polynomial in the $2m$ variables of degree $\ell(w)$ (see (11) for details of the latter).
Theorem 2.1 applies directly to give
\[ \langle P(E) - P(F), [X] \rangle = \langle P(\xi_\theta), [S(\Omega(s))] \rangle, \]  
(5.15)

In the case \( \psi|_{\Omega(w)} \) has constant rank (less than maximal), we deduce from Theorem 2.2 that
\[ \langle P(E) - P(F), [X] \rangle = \langle \widehat{S}_w(x,y), [\Omega(w)] \rangle, \]  
(5.16)

where \( \widehat{S}_w(x,y) \) is a double Schubert polynomial in the class \( \{\Omega(w)\} \). Thus, with respect to the cycles \([M]\) and \([N]\) as previously defined, we have
\[ \langle c_p(E) - c_p(F), [M] \rangle = \langle (\widehat{S}_w(x,y))^{[2p]}, [N] \rangle. \]  
(5.17)

5.5 Final remark and a further example

We have already mentioned, in the Introduction, the modification of the main result of [12] to the topology of generalized monoidal transformations [14, 15]. In closing, we should add that there are likely to be further situations to which Theorem 2.1 can be applied. As an example of such a situation, in a similar vein to the development of 5.1, consider the following.

Example 5.5. This follows from [27]. Let \( \mathcal{L} \rightarrow X \) be complex line bundle (\( X \) here can be a complex manifold), and let \( h : \mathcal{L} \rightarrow \mathbb{C}^{\ell+1} \) be a transversal linear system on \( X \) in the sense of 27. Let \( E = Q\mathcal{L} \) be the vector bundle on \( X \) whose sections consist of the \( \mathbb{C} \)-invariant vector fields on \( \mathcal{L} \), and let \( F = \text{Hom}(Q\mathcal{L}, \mathbb{C}^{\ell+1}) \). From \( h \), one can define a complex vector bundle morphism \( \psi : E \rightarrow F \), whose singular set, called the Jacobian set \( J(h) \), can be formulated in a similar way to (5.12) (and plays a similar role to \( \Omega(s) \)). The main results of 2 likewise apply to the adapted pair \((X, J(h))\) (cf. (5.2)), and further enumeration in the constant rank case produces a formula similar to (5.9) for Chern classes of appropriate order (cf. [27]).

References

[1] M. Abate, F. Bracci, and F. Tovena: Index theorems for holomorphic self-maps. Ann. of Math. 159(2) (2004), 819–864.

[2] P. Aluffi: Chern classes of blow-ups. Math. Proc. Cambridge Philos. Soc. 148 no. 2 (2010), 227–242.

[3] M. F. Atiyah: K-Theory. Benjamin, New York, 1967.

[4] A. M. Bonifant and J. E. Fornæss: Growth or degree for iterates of rational maps in several variables. Indiana Univ. Math. J. 49(2) (2000), 751–778.

[5] R. Bott and L. W. Tu: Differential Forms in Algebraic Topology. Grad. Texts in Math. 82. Springer-Verlag, New York Heidelberg Berlin, 1982.

[6] J.-P. Brasselet: Sur une formule de M. H. Schwartz relative aux revêtements ramifiés. C. R. Acad. Sci. Paris Sér. A-B 283 (2) (1976), A41–A44.
[7] S. S. Chern: *Complex Manifolds without Potential Theory*. Springer-Verlag, Berlin, New-York, 1979.

[8] J. Diller and C. Favre: Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.* 123 (2001), 1135–1169.

[9] S. Eilenberg and N. Steenrod: *Foundations of Algebraic Topology*. Princeton Mathematical Series 15, Princeton Univ. Press, Princeton, NJ, 1952.

[10] J. E. Fornæss and N. Sibony: Complex dynamics in higher dimension II. *Ann. Math. Studies* 137, Princeton Univ. Press, Princeton, NJ, 1995.

[11] W. Fulton: Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. *Duke Math. J.* 65 (1992), no. 3, 381-420.

[12] S. Gitler, J. F. Glazebrook, and A. Verjovsky: On the generalized Riemann-Hurwitz formula. *Boletin de la Sociedad Matemática Mexicana* 30(1) (1985), 1–11.

[13] S. Gitler: The cohomology of blow ups. *Papers in honor of José Adem (Spanish)*. *Bol. Soc. Mat. Mexicana* (2) 37 (1992), no. 1–2, 167-175.

[14] J. F. Glazebrook and A. Verjovsky: Residual circuits in generalized monoidal transformations. *Boletín de la Sociedad Matemática Mexicana* 33(1) (1988), 19–25.

[15] J. F. Glazebrook and A. Verjovsky: On the topology of generalized monoidal transformations. In preparation.

[16] M. Goresky and R. MacPherson: Intersection homology theory. *Topology* 19 (1980), 135–162.

[17] H. Grauert and K. Fritzsche: *Several Complex Variables*. Graduate Texts in Mathematics 38. Springer-Verlag, New York-Heidelberg, 1976.

[18] P. A. Griffiths and J. Harris: *Principles of Algebraic Geometry*. Wiley, New York, 1978.

[19] J. Harris and L. W. Tu: On symmetric and skew-symmetric determinantal varieties. *Topology* 23 (1984), no. 1, 71-84.

[20] H. Hironaka: Flattening theorem in complex algebraic geometry. *Amer. J. Math.* 97 (1975), 503–547.

[21] F. Hirzebruch: *Topological Methods in Algebraic Geometry*. Grundlehren 131, 3rd Ed., Springer Verlag 1966.

[22] M. Karoubi: *K-Theory, an introduction*. Grundlehren der matematiche Wissenschaften 226, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[23] S. Lefschetz: *Algebraic Topology*. American Math. Soc., Providence RI, 1942.

[24] D. Morrison: The geometry of K3 surfaces. Lectures delivered at the Scuola Matematica Internuniversitaria Cortona, Italy, July 31-August 27, 1988.
[25] Ngô Van Quê: Generalisation de la formula de Riemann-Hurwitz. *Canadian J. Math.* **24**(5) (1972), 761–767.

[26] I. R. Porteous: Blowing up Chern classes. *Proc. Camb. Phil. Soc.* **56** (1960), 118–124.

[27] I. R. Porteous: Todd’s canonical classes. Proceedings of Liverpool Singularities Symposium I, (1969/70), pp. 308–312. *Lecture Notes in Math.* **192**, Springer, Berlin, 1971.

[28] M.-H. Schwartz: Champs de repères tangents à une variété presque complexe. *Bull. Soc. Math. Belg.* **19** (1967), 389-420.

James F. Glazebrook.
Department of Mathematics and Computer Science
Eastern Illinois University
600 Lincoln Ave., Charleston, IL 61920–3099 USA
jfglazebrook@eiu.edu
(Adjunct Faculty)
Department of Mathematics
University of Illinois at Urbana–Champaign
Urbana, IL 61801, USA

Alberto Verjovsky
Instituto de Matemáticas
Universidad Autónoma de México
Av. Universidad s/n, Lomas de Chamilpa
Cuernavaca CP 62210, Morelos, Mexico
alberto@matcuer.unam.mx