\textit{k-Color Multi-Robot Motion Planning}\textsuperscript{*}

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\textbf{Abstract:} We present a simple and natural extension of the \textit{multi-robot motion planning} problem where the robots are partitioned into groups (colors), such that in each group the robots are interchangeable. Every robot is no longer required to move to a specific target, but rather to some target placement that is assigned to its group. We call this problem \textit{k-color multi-robot motion planning} and provide a sampling-based algorithm specifically designed for solving it. At the heart of the algorithm is a novel technique where the \textit{k}-color problem is reduced to several discrete multi-robot motion planning problems. These reductions amplify basic samples into massive collections of free placements and paths for the robots. We demonstrate the performance of the algorithm by an implementation for the case of disc robots in the plane and show that it successfully and efficiently copes with a variety of challenging scenarios, involving many robots, while a straightforward extension of prevalent sampling-based algorithms for the \textit{k}-color case, fails even on simple scenarios. Interestingly, our algorithm outperforms a state-of-the-art implementation for the standard multi-robot problem, in which each robot has a distinct color.

1 Introduction

\textit{Motion planning} is a fundamental problem in robotics and has applications in different fields such as the study of protein folding, computer graphics, computer-aided design and manufacturing (CAD/CAM) and computer games.

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The problem of motion planning, in its most basic form, is to find a collision-free path for a robot from start to goal placements while moving in an environment cluttered with obstacles.

An obvious extension of this problem is multi-robot motion planning, where several robots share a workspace and have to avoid collision with obstacles as well as with fellow robots. In many situations it is natural to assume that some robots are identical, in form and in functionality, and therefore are indistinguishable. In this setting every target position should be occupied by some robot of a kind (and not necessarily by a specific robot).

We consider the problem of $k$-color multi-robot motion planning—a simple and natural extension of the multi-robot problem where the robots are partitioned into $k$ groups (colors) such that within each group the robots are interchangeable. Every such group has a set of target positions, of size equal to the number of robots in that group. Every robot is no longer required to move to a specific target, but rather to some target position that is assigned to its group. However, we still require that all the target positions will be covered by the end of the motion of the robots. We term the special case where $k = 1$ the unlabeled multi-robot motion planning problem.

As an example consider a fleet of mobile robots operating in a factory that are given the task of cleaning a set of specific locations. The robots are indistinguishable from one another, and therefore any robot can be assigned to any location. Now assume that in addition to the mobile robots, another class of maintenance robots is employed by the factory; again, we consider all the maintenance robots to be of the same kind and interchangeable for the given task. This turns the unlabeled problem into a $k$-color problem, where $k = 2$ in this case.

From now on we will refer to the classic multi-robot motion planning problem as fully-colored, as it is a special case of the $k$-color problem where $k$ is equal to the number of robots and every group is of size one.

1.1 Previous Work

Throughout this work we will assume some familiarity with the basic terms in the area of motion planning. For more information on motion planning, see, e.g., [7, 15].
The first efforts in motion planning in general, and the multi-robot case in particular, were aimed towards the design of complete algorithms, guaranteed to find a solution when one exist or report that none exists otherwise. Schwartz and Sharir were the first to give [21] a complete algorithm for a multi-robot problem, specifically dealing with the case of coordinating disc robots in the plane. The running time of this algorithm is exponential in the number of robots. A work by Hopcroft et al. [10] presented soon after suggested that in some cases the exponential running time may be unavoidable, showing that even the relatively simple setting of rectangular robots bound in a rectangular region is PSPACE-hard in the number of robots.

The hardness of the multi-robot problem involving a large number of robots can be attributed to its high number of degrees of freedom (or dofs) — the sum of the dofs of the individual robots. Some efforts were made in the direction of reducing the effective number of dofs. Aronov et al. [2] showed that for the case of systems of two or three robots a path can be constructed, if one exists, where the robots are moved while maintaining contact, thus reducing the number of dofs by one or two, depending on the number of robots. Van den Berg et al. [5] proposed a general scheme for decomposing a multi-robot problem into a sequence of subproblems, each composed of a subset of robots, where every subproblem can be solved separately and the results can be combined into a solution for the original problem. This method reduces the number of dofs of the entire problem to the number of dofs of the largest subproblem.

An opposite approach to the complete planners is the decoupled approach, trading completeness with efficiency. Decoupled algorithms solve separate subproblems (usually for individual robots) and combine the individual solutions into a global solution. Although this approach can be efficient in some cases, it does not guarantee finding a solution if one exists and usually works only for a restricted set of problems. An example of such an algorithm can be found in the work of van den Berg and Overmars in [4] where every robot is given a priority and for each robot, the motion path is constructed to avoid collision with both static obstacles and lower-priority robots that are considered as moving obstacles. In other works, as in [16], individual paths are computed and velocity tuning is performed to avoid collision between robots.

In recent years the sampling-based approach for motion-planning problems has become increasingly popular due to its efficiency, simplicity and the fact that it is applicable to a wide range of problems. Unlike the complete planners that explicitly build the configuration space (C-space) of a given problem, the state of all possible configurations of a robot, sampling-based algorithms construct an implicit representation of a robot C-space by sampling this space for valid robot placements and connecting nearby samples. The connections between samples form a roadmap whose vertices describe valid placements for the robot and the edges represent valid paths from one placement to the other. Due to the implicit representation of the C-space and their simplicity, sampling-based algorithms tend to be much faster than
complete planners in practice, and are applicable to problems with a large number of \textit{dofs} such as the multi-robot problem. Although these algorithms are not complete, most of them are \textit{probabilistically complete}, that is, they are guaranteed to find a solution, if one exists, given sufficient amount of time. Examples of such algorithms are the PRM algorithm \cite{Kavraki96} by Kavraki et al. and the RRT algorithm \cite{Kuffner00} by Kuffner and LaValle. Such algorithms can be easily extended to the multi-robot case by considering the fleet of robots as one large composite robot. Several tailor made sampling-based algorithms have been proposed for the multi-robot case \cite{Cadena11, Leng12}. For more information on sampling-based algorithms see, e.g., \cite{LaValle12}.

An abstract form of the multi-robot motion planning problem is the \textit{pebble motion on graphs problem} \cite{Kavraki97}. This is a general case of the famous \textit{15-puzzle} \cite{Knuth82} where pebbles occupying distinct vertices of a given graph are moved from one placement of vertices to another, where a move consists of transferring a pebble to an adjacent unoccupied vertex. In some variants it is assumed that the graph is a tree \cite{Lingas95}, while in others there is a single pebble and several movable obstacle-pebbles on the nodes of the graph \cite{Witkowski03}. In \cite{Kavraki96} an unlabeled version of the pebble problem is discussed, as well as other variants, such as a grid topology. We also mention the recent work \cite{Kavraki98} where an algorithm is given for the general pebble problem.

\textbf{1.2 Contribution}

In this paper we present a sampling-based algorithm for the \textit{k-color problem} (for any \textit{k}). This algorithm is aimed to solve the most general cases of this problem and does not make any assumptions regarding the workspace or the structure of the robots.

Our algorithm for the \textit{k-color problem}—the KPUMP algorithm—reduces the \textit{k-color problem} to several discrete pebble problems. Specifically, a sample generated by KPUMP represents a local \textit{k-color problem} that is embedded in a variant of the pebble motion problem. Those pebble problems are constructed in a manner that enables the algorithm to transform movements of pebbles into valid motions of the robots. This allows KPUMP to generate a wide range of motions and placements for the robots with minimal investigation of the configuration space, thus reducing the dependence of the algorithm on costly geometric tools such as the collision detector.

As reflected by the experiments reported below for the case of disc robots in the plane, KPUMP proves to be efficient, even on challenging scenes, and is able to solve problems involving a large number of robots using a modest number of samples. Interestingly it performs well even on inputs of the standard (fully-colored) multi-robot problem.

This algorithm is simple to implement and does not require special geometric components beyond single-robot local planners and single-robot collision detectors. We compare the performance of our algorithm with a variant of the PRM algorithm for the same problem that uses those components and show that the latter performs much slower than KPUMP and fails to solve
even problems that are considered to be simple for KPUMP. This implies that using the same components KPUMP provides a much more powerful alternative. Moreover, concentrating on the fully-colored case, KPUMP outperforms a state-of-the-art implementation of the PRM algorithm.

The organization of the paper is as follows. In Section 2 we give formal definitions of the unlabeled and $k$-color problems. In Section 3 we present a variant of the pebble problem and its properties which will be used in our algorithms. In Section 4 we present UPUMP—an algorithm for the unlabeled case—and describe the changes necessary to extend it to KPUMP. We present experimental results for the case of disc robots moving among polygonal obstacles in the plane in Section 5 and discuss the properties of KPUMP in Section 6. For lack of space we omit a full description of KPUMP and a subroutine used by UPUMP, as well as some other less crucial details. They are provided in the supplementary material [23].

2 Preliminaries and Definitions

Let $W$ be a workspace and $r$ a robot in this workspace. We denote the free space of a robot $r$—the collection of all collision-free single-robot configurations—by $F(r)$. Given $s,t \in F(r)$ a path for $r$ from $s$ to $t$ is a continuous function $\pi : [0,1] \to F(r)$ such that $\pi(0) = s, \pi(1) = t$.

Unlabeled Multi-Robot Motion Planning. We say that two robots $r,r'$ are geometrically identical if $F(r) = F(r')$. Let $R = \{r_1, \ldots, r_m\}$ be a set of $m$ geometrically identical robots, operating in a workspace $W$. As the robots are geometrically identical we use $F$ to denote $F(r_i)$ for any $1 \leq i \leq m$. Let $C = \{c_1, \ldots, c_m|c_i \in F\}$ be a set of $m$ single-robot configurations. $C$ is a configuration if for every $c,c' \in C, c \neq c'$ the robots $r,r' \in R$ placed in $c,c'$ do not collide. Notice that we reserve the unqualified term configuration to refer to a set of $m$ collision-free single-robot configurations. Other types of configurations will be qualified single-robot configurations, pumped configurations, etc.

Given two configurations $S = \{s_1, \ldots, s_m\}, T = \{t_1, \ldots, t_m\}$ named start and target configurations, respectively, we define $U = (R,S,T)$ as the unlabeled problem, which is shorthand for the unlabeled multi-robot motion planning problem. Our goal is to find an unlabeled path $\pi_U$, defined as follows. Firstly, $\pi_U$ is a collection of $m$ paths $\{\pi_1, \ldots, \pi_m\}$ such that for every $i$, $\pi_i$ is a collision-free path for the robot $r_i$ from $s_i$ to some $t \in T$. Secondly, the robots have to remain collision-free while moving on the respective paths, i.e., for every $\theta \in [0,1] \pi_U(\theta) = \{\pi_1(\theta), \ldots, \pi_m(\theta)\}$ is a configuration. Notice that this also implies that $\pi_U(1)$ is some permutation of $T$.

$k$-Color Multi-Robot Motion Planning. Let $U_1, \ldots, U_k$ be $k$ unlabeled problems where $U_i = (R_i, S_i, T_i)$. $C = \{C_1, \ldots, C_k\}$ is a composite configuration if for every $1 \leq i \leq k C_i$ is a configuration for $U_i$ and for any pair of robots $r \in R_i, r' \in R_j$ placed in some $c \in C_i, c' \in C_j$ do not collide.
The k-color problem \( \mathcal{L} \) is defined by the set \( \{U_1, \ldots, U_k\} \) and our goal is to find a \( k \)-color path \( \pi_L = \{\pi_{U_1}, \ldots, \pi_{U_k}\} \) where \( \pi_{U_i} \) is the unlabeled path of \( U_i \). In addition, we require that \( \pi_L(\theta) \) will be a composite configuration for every \( \theta \in [0, 1] \) (this ensures that robots will not collide with one another).

A special case of this problem, usually named simply multiple robots motion planning, is a \( k \)-color problem where every \( U_i \) is comprised of a single robot, namely \( |R_i| = 1 \). In our context we call this special case fully-colored.

### 3 The Pebble Motion Problem

In preparation for the algorithms presented in the next two sections we discuss a variant of the problem of pebble motion on graphs. This problem is a discretization of the unlabeled problem. Our variant is defined in a way that will allow us to transform local unlabeled problems into pebble problems such that a movement of the pebbles can be transformed back into valid robot motions.

**Formal Definition.** A pebble problem \([13]\) \( \mathcal{P}(G) \) is defined by an undirected graph \( G = (V, E) \), where \( |V| = n \). A pebble placement is an ordered set of distinct vertices of \( V \). Given two pebble placements \( S, T \) such that \( |S| = |T| = m \) for some \( m \) the pebbles \( \tau_1, \ldots, \tau_m \) are placed on the nodes \( S \). We wish to find a chain of placements (of size \( m \)) \( \pi^* = P_1, \ldots, P_\ell \) called a pebble path, which obeys the following set of rules. Firstly, we demand that \( P_1 = S \). Secondly, for every two consecutive placements \( P = \{p_1, \ldots, p_m\}, P' = \{p'_1, \ldots, p'_m\} \) and every \( i \in [m] \) it holds that \( (p_i, p'_i) \in E \) or \( p_i = p'_i \), i.e., the pebble \( \tau_i \) is allowed to stay in its current vertex or move to a neighboring vertex in the graph.

Next we depart from the problem definition in \([13]\). We demand that \( P_\ell \) is some permutation of the elements of \( T \). (The original formulation \([13]\) specified which pebble will reside on which specific vertex of \( T \).) We do, however, impose an additional requirement—the separation rule—which requires that the pebbles will move separately, i.e., for every two consecutive placements \( P, P' \), as defined above, exactly one pebble \( \tau_i \) makes a move on an edge, while the other pebbles remain stationary. More formally, there exists \( i \in [m] \) such that \( (p_i, p'_i) \in E \) and for every \( j \neq i \) it holds that \( p_j = p'_j \). The necessity of this restriction will become clear later on.

**Solvability.** The variant of the pebble problem used in this paper possesses the following property, which will come in handy in the following sections. Denote the number of connected components of \( G \) by \( \ell \) and let \( \{G_1, \ldots, G_\ell\} \) be the set of maximal connected subgraphs of \( G \), where \( G_i = (V_i, E_i) \). Given a placement \( V' \) for the pebble problem \( \mathcal{P}(G) \) we define the signature of \( V' \) as \( \text{sig}(G, V') = \{ |V' \cap V_i| \}_{i=1}^\ell \).

**Lemma 1.** For every pebble problem \( \mathcal{P}(G) \) and every two placements \( S, T \) such that \( \text{sig}(G, S) = \text{sig}(G, T) \) there exists a pebble path from \( S \) to \( T \).

This property is proved in \([6]\) where they refer to the problem defined above as moving chips in a graph. Even though an algorithm is provided as well we
decided to use a simpler algorithm which constructs spanning trees on the connected components and balances the pebbles within the trees. This algorithm is omitted here for lack of space; it can be found in the supplementary material [23].

4 Algorithm for the Unlabeled Case: Configuration Clusters

In this section we present our main contribution — a sampling-based algorithm for the unlabeled problem. The algorithm, UPUMP, samples sets of single-robot configurations of size larger than the actual number of robots, to seemingly accommodate an increased number of robots. We call these sets pumped configurations. Each sampled pumped configuration results in a structure we call configuration cluster (or simply cluster), which implicitly represent a collection of configurations and the connections between them. These connections are the result of an underlying pebble motion problem, defined on the elements of the pumped configuration, where movements of pebbles from one pebble placement to the other are mapped into motions of robots between configurations within the cluster.

This technique makes use of the fact that our problem does not involve one complex robot, but rather a collection of robots operating in the same configuration space. This is in stark contrast with a popular sampling-based technique that considers the group of robots as one composite robot. In our opinion, the latter suffers from an acute disadvantage compared to our technique. We will demonstrate this claim experimentally and discuss the benefits of UPUMP and KPUMP in depth later on.

After discussing the construction of clusters and exploring their various properties we show that clusters can be connected to generate paths for the robots not only moving from configuration to configuration within the same cluster but also between configurations belonging to different clusters. We conclude this section with a description of the full configuration-clusters sampling-based algorithm and state the changes required to extend UPUMP to KPUMP.

4.1 Construction of a Configuration Cluster

We define more formally some of the structures mentioned earlier.

**Definition 1.** [Pumped Configuration] Let $V = \{v_1, \ldots, v_{m'}\}$ for $m' \geq m$ be a set of single-robot configurations such that for every $v \in V$ it holds that $v \in \mathcal{F}$, where $\mathcal{F} = \mathcal{F}(r)$ for some $r \in \mathcal{R}$. $V$ is a pumped configuration if every $C \subseteq V$, such that $|C| = m$, is a configuration.

Given a pumped configuration $V$ we construct a graph $G = (V, E)$ whose vertices are the elements of $V$, and the edges represent paths in $\mathcal{F}$ for individual robots. We call this type of graph geometric. To generate the edges of $G$, and the respective paths, we utilize the following mechanism.

**Definition 2.** [Edge Planner] Given $v_i, v_j \in V$ such that $V$ is a pumped configuration and $v_i \neq v_j$ the edge planner generates a path $\pi_{v_i, v_j}$ or reports
failure. $\pi_{v_i,v_j}$ is a path for $r \in R$ from $v_i$ to $v_j$, such that for every $k \neq i,j$ the robot $r$, while moving on this path, does not collide with a (geometrically identical) robot placed in $v_k$.

The edge planner is applied on every pair $v_i \neq v_j$ in $V$. Upon a successful generation of a path $\pi_{v_i,v_j}$ the edge $(v_i,v_j)$ is added to $G$. An example of a pumped configuration, as well as its underlying graph, is given in Figure 2.

Given such graph $G$ for the pumped configuration $V$ we define the cluster of $G$ as $C_G = \{C|C \subseteq V, |C| = m\}$. Namely, $C_G$ is the collection of all configurations in the pumped configuration $V$. However, we do not require that all these configuration will be explicitly constructed within each cluster, and as we will see later on, an explicit representation of only some of the configurations will be required eventually. The following proposition discusses the existence of paths between configurations within the same cluster, (see Section 3 for the definition of $\text{sig}(G,V)$).

**Proposition 1.** Let $C, C' \in C_G$ and $\text{sig}(G,C) = \text{sig}(G,C')$. Then there exists a path $\pi_{U'}$ for $U' = (C,C')$.

**Proof.** By Lemma 1 there exists a pebble path $\pi^*$ for $P(G)$ from start to target pebble placements $V$ and $V'$, respectively. For a given pebble path $\pi^*$ we convert every movement of a pebble into a movement of the respective robot. Given a pebble $\tau_j$ that moves on the edge $(u,v)$ of $G$, from $u$ to $v$ we add an unlabeled path section where the robot $r_j$ moves from $u \in V$ to $v \in V$ along $\pi_{u,v}$, generated by the edge planner, while the rest of the robots in $R$ stay put throughout this motion. As $\pi^*$ obeys the separation rule only one robot can move at any given time. Moreover, moving robots cannot collide with obstacles or stationary robots as they are moving on paths generated by the edge planner. An example for such a path is given in Figure 2(b).

### 4.2 Connecting Configuration Clusters

Proposition 1 implies that certain unlabeled problems can be solved using a single cluster. However, this statement does not hold for many other instances.

![Fig. 2](image-url)
of the unlabeled problem. As an example consider $\mathcal{U} = (S, T)$ in which there exists at least one pair $s \in S, t \in T, s \neq t$ such that a robot $r \in R$ placed in $s$ overlaps with another robot $r' \in R$ placed in $t$. Thus, $s, t$ cannot be in the same pumped configuration.

Fortunately, we can combine several clusters in order to find paths for more general unlabeled problems. We first show that two clusters can be combined to generate paths for the robots moving from one cluster to the other.

**Lemma 2.** Given a pair of clusters $\mathcal{C}_G, \mathcal{C}_G'$ let $C, C'$ be two configurations such that $C \in \mathcal{C}_G, C' \in \mathcal{C}_G'$ and let $\pi_{C, C'}$ be a path for the unlabeled problem $\mathcal{U}' = (C, C')$. In addition, let $D, D'$ be two configurations such that $D \in \mathcal{C}_G, D' \in \mathcal{C}_G'$ and $\text{sig}(G, D) = \text{sig}(G, C)$ and $\text{sig}(G', D') = \text{sig}(G', C')$. Then there exists a path $\pi_{U''}$ for $\mathcal{U}'' = (D, D')$.

**Proof.** By Proposition 1 there is a path $\pi_{U_D}$ for $\mathcal{U}_D = (D, C)$ and, similarly, a path $\pi_{U_D'}$ for $\mathcal{U}_D' = (C', D')$. Thus, the concatenation of the paths $\pi_{U_D}, \pi_{C, C'}, \pi_{U_D'}$ yields a path for $\mathcal{U}'$.

Paths connecting configurations from different clusters are generated using the following component which generalizes the component local planner used in standard sampling-based algorithms.

**Definition 3.** [Connection Generator] Let $\mathcal{C}_G, \mathcal{C}_G'$ be two clusters. The connection generator returns a collection of paths such that every path $\pi_{C, C'}$ in the collection is a solution for the unlabeled problem $\mathcal{U}' = (C, C')$ where $C \in \mathcal{C}_G, C' \in \mathcal{C}_G'$.

In the implementation of the connection generator a collection of single-robot paths between single-robot configurations from the two clusters are generated. The interactions between these paths are modeled in a graph. Specifically, every vertex of the graph represents a single-robot path and two vertices are connected by an edge if robots moving on the respective paths collide. Consequently, an independent set of vertices in this graph represent a set of non-colliding paths. A full description of the connection generator is provided in [23].

### 4.3 Description of UPUMP

Next, we extend the result of Lemma 2 to generate still more complex paths. Algorithm UPUMP samples clusters (by generating random pumped configurations), then it generates paths connecting configurations form different clusters. Those paths will represent edges in a roadmap $\mathcal{H}$ whose vertices are configurations. Additional edges, that represent paths between configurations within the same cluster with identical signatures are added to $\mathcal{H}$ afterwards. We mention that even though the roadmap $\mathcal{H}$ does not contain clusters, every configuration in $\mathcal{H}$ is associated with a specific cluster. Throughout the text below we will employ the following notation. Let $\mathcal{C}_G$ be a cluster and let $C$ be a configuration such that $C \in \mathcal{C}_G$. To refer to the cluster of $C$ we use the notation $\mathcal{C}(C)$, i.e., $\mathcal{C}(C) = \mathcal{C}_G$. 

Sampling: UPUMP samples a collection of pumped configurations and generates the respective respective geometric graphs and clusters.

Connecting configurations in different clusters: The connection generator is applied on every pair of clusters \( C_G, C'_G \). For every returned path \( \pi_{C,C'} \) UPUMP adds \( C, C' \) as vertices to \( H \). In addition it adds an edge between those two vertices with the attached information \( \pi_{C,C'} \). As \( S \) and \( T \) can be viewed as clusters, containing a single configuration, we try to connect them to different clusters using the connection generator as well.

Connecting configurations within the same cluster: By Proposition 1 for every two nodes \( C, C' \) of \( H \), such that \( \mathcal{L}(C) = \mathcal{L}(C') \) and \( \text{sig}(\mathcal{L}(C), C) = \text{sig}(\mathcal{L}(C'), C') \), there exists a path for \( U' = (C, C') \). Therefore, we may add an edge in \( H \) between \( C, C' \). We do not generate the respective paths at this point as only some of them will eventually participate in the final path connecting \( S \) to \( T \).

The following corollary allows us to transform vertex paths in \( H \) into a path for the unlabeled problem \( U \). It is a direct result of Lemma 2.

**Corollary 1.** Suppose that \( H \) contains a vertex path \( C_0, \ldots, C_\ell \) such that \( C_0 = S^*, C_\ell = T^* \), for every \( 1 \leq i \leq \ell \) it holds that \( (C_{i-1}, C_i) \) is an edge in \( H \). Then there exists an unlabeled path for \( U \).

As the paths between different clusters have already been constructed by the connection generator, we are only required to generate paths induced by pebble motions between \( C_{i-1}, C_i \) such that \( \mathcal{L}(C_{i-1}) = \mathcal{L}(C_i), \text{sig}(\mathcal{L}(C_{i-1}), C_{i-1}) = \text{sig}(\mathcal{L}(C_i), C_i) \).

### 4.4 Extension to KPUMP

We briefly describe some of the changes necessary to transform UPUMP to KPUMP—an algorithm for the \( k \)-color problem. Although it may seem natural to allow robots from different unlabeled problems to share common geometric graphs we chose to take a different approach. KPUMP simultaneously samples several pumped configuration—each corresponds to a distinct color (namely, a different unlabeled problem). The resulting geometric graphs are constructed in a manner which prevents collision between robots from different unlabeled problems. This calls for a redefinition of the edge planner as well as other components. Similarly to UPUMP, KPUMP construct a roadmap, with the exception that its vertices are no longer configurations but composite configurations (see Section 2). The full description of KPUMP is provided in [23].

### 5 Experimental Results

We describe experimental results for the case of disc robots in the plane moving amidst polygonal obstacles. We show results for five challenging scenarios and compare the performance of KPUMP with two other sampling-based algorithms. Specifically we compare KPUMP with the PRM implementation of the OOPSMP package [20] on inputs of the fully-colored problem. For other
inputs we use a basic sampling-based algorithm for the $k$-color problem called KBASIC, described later on.

KPUMP was implemented in C++ using CGAL Arrangements [8] and the Boost Graph Library (BGL) [22]. The code was tested on a PC with Intel i7-2600 3.40GHz processor with 8GB of memory, running a Windows 7 64-bit OS. For the implementation of the edge generator and single-robot local planner (used by the connection generator) a straight-line connection strategy [1] was used. This strategy attempts to move the robot along a straight line drawn between the start and target positions.

**Parameters of KPUMP.** The algorithm has three parameters that affect its performance: $n$ describes the number of sampled clusters; $c$ is the number of paths that the connection generator attempts to produce between every two clusters; $\mu$ is the maximal number of single-robot configurations that one sample comprises. For instance, in the case of an unlabeled problem the size of every sampled pumped configuration will be at most $\mu$. The value of the latter parameter depends on the input problem. For unlabeled problems increasing $\mu$ results in increased connectivity of the resulting geometric graphs. Thus, it will be beneficial that the pumped configurations will be as large

![Fig. 3](image_url)

Fig. 3: [Best viewed in color] Start positions of the robots are indicated by discs while target positions are illustrated as circles in respective colors (unless otherwise indicated) (a) Unlabeled scene with twenty five robots. (b) 2-Color scene; the two groups are required to switch positions. (c) Fully-colored scene with eight robots. (d) Fully-colored scene with five robots. (e) 4-Color scene; every group has to move in a clockwise manner to the next room, e.g., blue group should move to the bottom room.
as possible (limited by the topology of the scenario). On the other hand, in $k$-colored problems where $k > 1$ the value $\mu$ has to be set more carefully as an excessively high value of $\mu$ will reduce the connectivity of the geometric graphs. This stems from the fact that the a single-robot path produced by the edge planner has to avoid collision with robots from other groups. Consequently, as the value of $\mu$ grows it becomes harder to connect single-robot configurations using an edge planner.

**Test Scenarios.** The scenarios are illustrated in Figure 3 and represent a variety of challenging problems. The unlabeled problem in (a) involves the motion of a large collection of robots. Scenarios (b) and (e) describe 2-color and 4-color problems comprising a large number of robots as well. Although scenarios (c),(d) do not involve as many robots, they are nevertheless extremely challenging. This range of problems demonstrate the work of the various components of the KPUMP algorithm. In the first three scenarios the resulting geometric graphs have a low number of connected components due to the low value of $k$ (as in scenario (a)) or high clearance from obstacles (as in (e)). Therefore, large portions of the resulting paths involve the motions of the robots on paths induced by pebble problems. While the generated graphs in scenarios (c) and (d) have low connectivity, KPUMP still performs well—due to the use of the connection-generator component.

The results of running KPUMP for specific parameters are given in Table 1. In addition to the parameters mentioned above, the table contains the values $k$ for the number of colors, $m_i$ the number of robots in every group and $M$ the total number of robots. The running times are given in seconds. The parameters used by KPUMP, as well as the parameters of other algorithms mentioned later on, were manually optimized over a concrete set.

**Comparison with Other Algorithms.** The first part of the comparison involves solely inputs of the fully-colored problem. We compare KPUMP with the implementation of PRM provided by OOPSMP, which, by our experience, is very efficient. This algorithm is designed for solving single-robot and fully-colored multi-robot motion planning problems. While OOPSMP required 100 seconds to solve scenario (d), KPUMP managed to solve it in 1.9 seconds. Scenario (c) proved to be even more challenging for OOPSMP, which failed to solve it, even when was given 5000 seconds of preprocessing time, whereas KPUMP solved in 213.7 seconds.

In order to provide a more informative comparison we ran both algorithms on scenarios (c),(d), only that now we increased the difficulty of these scenarios gradually—incrementally introducing the robots to the scenarios, i.e.,
starting with a single robot and adding the others one by one, as long as
OOPSMP succeeded solving the new inputs in reasonable time. In this case
OOPSMP was able to solve scenario (c) with five robots, while six robots
was out of its reach (when given 5000 seconds of preprocessing time). The
speedup of KPUMP compared to OOPSMP for this new range of scenarios is
depicted in Figure 4 along with an additional test case (“decoupled-simple”),
which is a simpler variant of scenario (c) with some of the obstacles removed
and the radius of the robots is decreased. The latter was designed to test the
performance of OOPSMP on problems involving a higher number of robots.

As we are not aware of any other algorithms for the $k$-color problem ($k > 1$)
we designed a basic algorithm to compare KPUMP with. This algorithm,
which we call KBASIC, is an extension of the PRM algorithm for the
$k$-color case. It can be described as a special case of KPUMP that samples
configurations, instead of pumped configurations.

The entire set of scenarios (a)-(e)(in their original form) proved to be
too challenging for KBASIC that spent at times more than tens of minutes.
Similarly to the previous comparison we designed a set of simple test scena-
rios. Specifically, scenario (e) was converted into five $k$-color problems for
$1 \leq k \leq 5$ by partitioning the robots into $k$ groups such that a robot number
$i$ was assigned to the group $i \mod k$. Then, as in the previous comparison
the robots were introduced incrementally to the five scenarios. Figure 4 de-
picts the speedup of KPUMP compared with KBASIC for each of the $k$-color
problems. This shows that KPUMP outperforms KBASIC in every possible
setting, be it $k$-color, unlabeled or fully-colored problem.

6 Discussion

In this section we discuss the various properties of the KPUMP algorithm
and novelties it encompasses.

Expanding the Sampling-Based Scheme. Typically, sampling-based al-
gorithm, such as PRM and RRT, decompose the input problem into subprob-
lems that are assumed to be easier than the original problem. Specifically,
the task of path finding between two remote configurations is transformed
into several (often many) problems where nearby pairs of configurations are
connected. Consider a more general view of this approach where one is given
a powerful “black-box” component that is able to solve specific instances of
the motion-planning problem. To harness the power of this component one

Fig. 4: [Best viewed in color] Comparing KPUMP with OOPSMP/PRM and KBASIC
would decompose the input problem into subproblems that can be solved with the component and combine the solutions to the subproblem into a solution to the input problem. As a deterministic decomposition is not always possible, one would try to sample subproblems with the hope that some of them contain information that is crucial for solving the full problem.

KPUMP is an example of such an algorithm. Specifically, KPUMP decomposes the $k$-color problem into local $k$-color problems—instances that can transformed into pebble problems. Subproblem sampling is achieved by sampling configuration clusters while the connections of the subproblems is achieved with the connection generator component. The pebble problem allows us to transform problems in the continuous configuration space into a discrete problem on graphs. While a transformation to a discrete representation occurs also in other sampling-based algorithm it is only used for the purpose of modeling paths that were generated in the configuration space. In our case, however, the pebble problem produces new paths that were not previously generated in the configuration space. The variant of the pebble motion on graphs used in this paper allows us to efficiently generate such paths and considerably reduces the use of heavy-duty geometric operations such as collision detection.

**Shortcomings of the Composite Robot Approach.** The traditional composite robot approach to the multi-robot problem treats the group of robots as one composite robot whose configuration space is the Cartesian product of the configuration spaces of the individual robots. With this approach single-robot tools, such as sampling-based algorithms, can be used to solve multi-robot problems. For instance, this technique is used in the software packages OOPSMP and OMPL [12, 20] where PRM is applied to the fully-colored problem, and in the KBASIC algorithm discussed above. Paths generated by this approach usually force the robots to move simultaneously from one placement to the other, where none of the robots remains in the same position while the others are moving.

We believe that such paths are unnatural in the multi-robot setting and are more difficult to produce than paths that involve motion of only few robots at a time. Given collision-free placements for all the robots it is usually possible to move some of the robots to different placements without altering the placements of the rest of the robots, i.e., those robots remain still. For instance, consider a configuration $C = \{c_1, \ldots, c_m\}$ for some unlabeled problem $\mathcal{U}$. Unless the workspace is extremely tight, another configuration $C'$ can be derived from $C$ where $c_1$ is changed to $c'_1$. Moreover, connecting two such configurations by a path requires only a single-robot collision-free path for which the moving robot does not collide with the other robots placed in $c_2, \ldots, c_m$. In contrast, the connection of two “unrelated” configurations by a path imposes much harder constraints—$m$ single-robot collision-free paths have to be created and in addition robots moving along those paths must not collide with each other.
KPUMP utilizes this observation by restricting the movements of the robots along certain path sections—induced by pebble problems—to motions of individual robots. We mention that KPUMP does not preclude simultaneous movements of robots when necessary, specifically on path sections where the robots move from one cluster to the other along paths generated by the connection generator.

**Amplification of Samples.** The pumped configurations sampled by KPUMP, and the resulting geometric graphs are fairly simple structures that require only little effort to generate. Yet, using pebble problems, such samples can be amplified to describe not only placements and paths for single robots, but also to represent an incredible amount of paths and positions all the robots in a given problem. Configuration clusters that result from such pumped configurations, represent a collection of \( \prod_{i=1}^{k} \binom{m_i}{m'} \) configurations and a large number of connections between them. However, this information is not explicitly represented and only little storage space is required to represent a single cluster. In particular, one is only required to store the pumped configurations and the geometric graphs that define the cluster as well as a modest number of individual configurations from this cluster. Specifically, those are configurations through which the cluster connects to other clusters. Such configurations are selected by the connection generator. Similarly, this component does not require an explicit representation of the clusters and can deduce the structure of the cluster from the pumped configurations that define it. Furthermore, continuing the theme presented here that one action leads to a large number of outcomes, namely, a sample of a pumped configuration results in many configurations, a path generated by a connection generator not only connects two configurations from different clusters but also a large number of configurations from those clusters, which are not necessarily directly connected. Thus, these properties enable KPUMP to generate a variety of configurations and motions of the robots, using only few samples. To reproduce this variety by KBASIC one must generate far more samples.

An additional advantage of the use of clusters lies in the fact that they can be connected more easily than two configurations, when a powerful component as the connection generator is at hand. Using this component KPUMP succeeds in solving difficult scenarios even in situations where the generated geometric graphs suffer from low connectivity, as in scenarios (c) and (d).

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