Dynamic splitting of Gaussian pencil beams in heterogeneity-correction algorithms for radiotherapy with heavy charged particles

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Abstract
The pencil-beam algorithm is valid only when elementary Gaussian beams are small enough compared to the lateral heterogeneity of a medium, which is not always true in actual radiotherapy with protons and ions. This work addresses a solution for the problem. We found approximate self-similarity of Gaussian distributions, with which Gaussian beams can split into narrower and deflecting daughter beams when their sizes have overreached lateral heterogeneity in the beam-transport calculation. The effectiveness was assessed in a carbon-ion beam experiment in the presence of steep range compensation, where the splitting calculation reproduced a detour effect amounting to about 10% in dose or as large as the lateral particle disequilibrium effect. The efficiency was analyzed in calculations for carbon-ion and proton radiations with a heterogeneous phantom model, where the beam splitting increased computing times by factors of 4.7 and 3.2. The present method generally improves the accuracy of the pencil-beam algorithm without severe inefficiency. It will therefore be useful for treatment planning and potentially other demanding applications.

1. Introduction
In the treatment planning of radiotherapy with protons and ions, the pencil-beam (PB) algorithm is commonly used (Hong et al 1996, Kanematsu et al 1998, 2006, Schaffner et al 1999, Krämer et al 2000), where a radiation field is approximately decomposed into two-dimensionally arranged Gaussian beams that receive energy loss and multiple scattering in
matter. In the presence of heterogeneity, these beams grow differently to reproduce realistic fluctuation in the superposed dose distribution.

Comparisons with measurements and Monte Carlo (MC) simulations, however, revealed the difficulty of the PB algorithm at places with severe lateral heterogeneity such as steep areas of a range compensator and lateral interfaces among air, tissue and bone in a patient body (Goitein 1978, Petti 1992, Kohno et al 2004, Ciangaru et al 2005). One reason for the difficulty is that particles in a pencil beam are assumed to receive the same interactions, whereas they may be spatially overreaching beyond the density interface. The other reason is that only straight paths radiating from a point source are considered in beam transport, whereas actual particles may detour randomly by multiple scattering.

Schneider et al (1998) showed that a phase-space analysis based on the Fermi–Eyges theory (Eyges 1948) could address the overreach and detour effects for a simple lateral structure. Schaffner et al (1999) and Soukup et al (2005) subdivided a physical spot beam virtually into smaller beams to naturally reduce overreaches. However, their subdivision is arbitrary in granularity and will be insufficient or excessive, globally or locally in individual cases. Pfugfelder et al (2007) quantified lateral heterogeneity, with which subdivision could be optimized in principle. Unfortunately, the initial beam arrangement, even if it is optimized, is completely ineffective against beam-size growth during transport, which may be as large as millimeters.

For electrons, the overreach and detour effects are intrinsically much more severe. Shiu and Hogstrom (1991) developed a solution, the PB-redefinition algorithm, where minimal pencil beams are occasionally regenerated, considering electron flows in the Fermi–Eyges theory. The same idea was in fact partly applied to heavy particles for beam customization (Kanematsu et al 2008b), but the poly-energetic beam model to deal with heterogeneity could be severely inefficient in high-resolution calculations necessary for Bragg peaks.

In this study, we develop an alternative method to similarly address the overreach and detour effects. In the following sections, we incorporate our findings on the Gaussian distribution into the PB algorithm, test the new method in a carbon-ion beam experiment and discuss its efficiency for carbon-ion and proton radiations with a heterogeneous phantom model.

2. Materials and methods

2.1. Theory

2.1.1. Pencil-beam algorithm. The PB algorithm in this study basically follows our former works (Kanematsu et al 1998, 2006, 2008b). A pencil beam with index \( b \) is described by position \( \vec{r}_b \), direction \( \vec{v}_b \), number of particles \( n_b \), residual range \( R_b \), angular variance \( \theta^2_b \), angular-spatial covariance \( \theta \tau_b \) and spatial variance \( \tau^2_b \) of the involved particles. As described in appendix A, these parameters are initialized and modified with transport distance \( s \) in the Fermi–Eyges theory. The resultant beams with variance \( \sigma^2_b = \tau^2_b \) are superposed to form dose distribution

\[
D(\vec{r}) = \sum_b n_b D_{\Phi 0}(d_{br}) \exp \left( -\frac{|\vec{r}_{0b} + s_{br} \vec{v}_b - \vec{r}|^2}{2\sigma^2_b(s_{br})} \right),
\]

(1)

\[
s_{br} = (\vec{r} - \vec{r}_{0b}) \cdot \vec{v}_b, \quad d_{br} = R_0 - R_b(s_{br}),
\]

(2)

where \( \vec{r}_{0b} \) is the beam-\( b \) origin, \( s_{br} \) is the transport distance at the closest approach to point \( \vec{r} \), \( d_{br} \) is its equivalent water depth, \( R_0 \) is the beam range in water and \( D_{\Phi 0} \) is the in-water dose per in-air fluence or the tissue-phantom ratio.
2.1.2. Self-similarity of Gaussian distribution. Any normalized Gaussian distribution $G_{\mu,\sigma}(x)$ with mean $\mu$ and standard deviation $\sigma$ can be represented as

$$G_{\mu,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$  \tag{3}$$

with the standard normal distribution $N(x) = G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. \tag{4}

Incidentally, we have found that the binomial Gaussian function

$$N_2(x) = \frac{1}{2} \left[ G_{-\frac{1}{2}, \frac{\sqrt{3}}{2}}(x) + G_{\frac{1}{2}, \frac{\sqrt{3}}{2}}(x) \right]$$  \tag{5}$$

reasonably approximates $N(x)$ as shown in figure 1(a), where we first fixed symmetric displacement $\mu = \pm \frac{1}{2}$ for the binomial terms and determined their reduced standard deviation $\sigma = \sqrt{\frac{3}{2}}$ to conserve variance $\int_{-\infty}^{\infty} x^2 N_2(x) \, dx = 1$. Similarly, the daughter Gaussian terms in $N_2(x)$ splits into grand daughters to form the approximate function

$$N_3(x) = \frac{1}{4} \left[ G_{-\frac{3}{2}, \frac{\sqrt{3}}{2}}(x) + 2G_{0, \frac{\sqrt{3}}{2}}(x) + G_{\frac{3}{2}, \frac{\sqrt{3}}{2}}(x) \right]$$  \tag{6}$$

and then into grand grand daughters to form the approximate function

$$N_4(x) = \frac{1}{8} \left[ G_{-\frac{5}{2}, \frac{\sqrt{3}}{2}}(x) + 3G_{-\frac{1}{2}, \frac{\sqrt{3}}{2}}(x) + 3G_{\frac{1}{2}, \frac{\sqrt{3}}{2}}(x) + G_{\frac{5}{2}, \frac{\sqrt{3}}{2}}(x) \right],$$  \tag{7}$$

as shown in figures 1(b) and (c). Further splitting with the same displacement is not possible with valid ($\sigma > 0$) Gaussian terms.

With approximation $N(x) \to N_M(x)$ for $M \in \{2, 3, 4\}$, an overreaching Gaussian beam may split two-dimensionally into $M \times M$ smaller beams. Because beam multiplication will explosively increase computational amount, it must be applied only when and where necessary with $M$ optimum for size reduction/multiplication.

2.1.3. Lateral heterogeneity. For a patient with density distribution $\rho_0(\vec{r})$ in a grid-voxel model with interval $\delta_g$ and basis vector $\vec{e}_g$ for the axis $g \in \{1, 2, 3\}$ as shown in figure 2, we define the density gradient vector $\vec{\nabla} \rho_0$ as

$$\vec{\nabla} \rho_0 = \sum_{g=1}^{3} \max_{\delta_g} \left[ \rho_0(\vec{r} + \delta_g \vec{e}_g) - \rho_0(\vec{r}), \rho_0(\vec{r} - \delta_g \vec{e}_g) - \rho_0(\vec{r} - \delta_g \vec{e}_g) \right] \delta_g,$$  \tag{8}$$
where operation $\max[a,b] = a$ if $|a| \geq |b|$ or otherwise $b$. We quantify the lateral heterogeneity by effective lateral density gradient

$$\gamma_{xy}(\vec{r}) = \frac{1}{\sqrt{2}} \sqrt{|\nabla \rho_S|^2 - (\nabla \rho_S \cdot \vec{e}_z)^2},$$

with which we define the distance to an interface of density change $\kappa_{\rho}$ as

$$\delta_{\text{int}}(\vec{r}) = \min \left( \frac{\kappa_{\rho}}{\gamma_{xy}(\vec{r})}, 2\delta_{xy} \right),$$

where $\kappa_{\rho} = 0.1$ may be appropriate for interfaces among air ($\rho_S \approx 0$), soft tissues ($0.9 \lesssim \rho_S \lesssim 1.1$) and bones ($1.2 \lesssim \rho_S \lesssim 1.7$) (Kanematsu et al 2003). The effective lateral grid interval

$$\delta_{xy} = \frac{1}{\sqrt{2}} \sqrt{\sum_{g=1}^{3} \delta^2 - \sum_{g=1}^{3} (\vec{g} \cdot \vec{e}_z)^2}$$

multiplied by 2 is the effective distance to a second laterally adjacent grid, beyond which the distance to the interface cannot be estimated from the gradient.

2.1.4. Beam splitting. The pencil beams are examined at every transport step in a patient. Ones subject to splitting should not only be overreaching beyond a density interface but also sizable and influential in an absolute sense for computational efficiency. For mother beam $b$ to split, we thus require the conditions

$$\sigma_b > \delta_{\text{int}}, \quad \sigma_b > \frac{\delta_{xy}}{\sqrt{6}}, \quad n_b > \kappa_n n_{b0},$$

where $\sigma_b > \delta_{\text{int}}$ defines the state of overreaching, $\sigma_b > \delta_{xy}/\sqrt{6}$ suppresses splitting into beams narrower than effective resolution $\delta_{xy}/\sqrt{12}$ with $M = 2$ and $n_b > \kappa_n n_{b0}$ sets a limit on the number of particles $n_b$ with respect to that of the ancestral original beam, $n_{b0}$, with a cutoff parameter chosen as $\kappa_n = 0.1$. 

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**Figure 2.** (a) Definitions of the source coordinate system $(x, y, z)$, the beam-$b$ coordinate system $(s, t, u)$ and density grids at intervals $\delta_1, \delta_2$ and $\delta_3$ along the axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$, where parameters perpendicular to the viewing plane, $y, u, \delta_3$, $\vec{e}_y, \vec{e}_u, \vec{e}_3$, are not shown. (b) Schematic of splitting of a mother beam (gray) into daughter beams (black) radiating from a common focus.
Table 1. Factors of size reduction $\sigma_M$, displacement $\mu_M$, share fraction $f_M$ and the applicable beam-spread/interface-distance ratio $\sigma_b/\delta_{int}$ for splitting of multiplicity $M$.

| $M$ | $\sigma_M$ | $\mu_M$ | $f_M$ | $\sigma_b/\delta_{int}$ |
|-----|------------|---------|-------|-------------------------|
| 2   | $\frac{1}{\sqrt{2}}$ | $\{\frac{1}{2}, \frac{3}{2}\}$ | $\frac{1}{2}$ | $\{1, \frac{1}{2}\}$ |
| 3   | $\frac{1}{\sqrt{2}}$ | $\{1, 0, +1\}$ | $\frac{1}{2}$ | $\{\frac{1}{2}, \frac{1}{2}\}$ |
| 4   | $\frac{1}{\sqrt{2}}$ | $\{\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}\}$ | $\frac{1}{2}$ | $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ |

To avoid unnecessary multiplication, recursive splitting in particular, we choose a minimum multiplicity that will resolve overreaching state $\sigma_b > \delta_{int}$. Table 1 summarizes splitting modes and shows their applicable ranges for the ratio $\sigma_b/\delta_{int}$.

With beam-$b$ coordinate system $(s, t, u)$ shown in figure 2(a) and defined as

$$s = (\vec{r} - \vec{r}_b) \cdot \vec{e}_s,$$

$$t = (\vec{r} - \vec{r}_b) \cdot \vec{e}_t,$$

$$u = (\vec{r} - \vec{r}_b) \cdot \vec{e}_u,$$

(13)

daughter beams $b'_{\alpha\beta}$ ($\alpha, \beta \in [1, M]$) are initialized as

$$\vec{r}_{b'_{\alpha\beta}} = \vec{r}_b + \sigma_b (\mu_M \vec{e}_t + \mu_M \vec{e}_u),$$

(15)

$$\vec{v}_{b'_{\alpha\beta}} = \left| \vec{v}_{b_{\alpha\beta}} - \frac{\vec{r}^2_{b_{\alpha\beta}}}{\frac{\partial r}{\partial t}} \vec{v}_b \right|^{-1} \left( \vec{r}_{b'_{\alpha\beta}} - \vec{r}_b + \frac{\vec{r}^2_{b_{\alpha\beta}}}{\frac{\partial r}{\partial t}} \vec{v}_b \right),$$

(16)

$$n_{b'_{\alpha\beta}} = f_{M,\beta} f_{M,\alpha} n_b,$$

$$R_{b'_{\alpha\beta}} = R_b,$$

$$\sigma^2_{b'_{\alpha\beta}} = \sigma^2_M t^2_{b'_{\alpha\beta}},$$

(17)

$$\frac{\partial}{\partial t} \sigma^2_{b'_{\alpha\beta}} = \sigma^2_M t^2_{b'_{\alpha\beta}},$$

$$\frac{\partial}{\partial \theta} \sigma^2_{b'_{\alpha\beta}} = \sigma^2_M \sigma^2_{b'_{\alpha\beta}} = \sigma^2_M t^2_{b'_{\alpha\beta}},$$

(18)

where $\vec{r}_{b_{\alpha\beta}}$ is the displaced position, $\vec{v}_{b_{\alpha\beta}}$ is the deflected direction, $n_{b_{\alpha\beta}}$ is the number of shared particles, $R_{b_{\alpha\beta}}$ is the conserved residual range, $\sigma^2_{b_{\alpha\beta}}$ is the reduced spatial variance, and variances $\sigma^2_{b_{\alpha\beta}}$ and $\sigma^2_{b_{\alpha\beta}}$ conserve the focal distance ($t^2/\theta^2$) and the local angular variance ($\theta^2/\sigma^2_{b_{\alpha\beta}}$) in splitting. The daughter beams are radiated from the focus or the virtual source (ICRU-35 1984) of the mother beam as shown in figure 2(b), which generally moves downstream in transport due to multiple scattering (Kanematsu 2009).

In this manner, a mother beam splits into smaller deflecting daughter beams to form different detouring paths. Sets of the initial parameters for the daughter beams are sequentially pushed on the stack of computer memory and the last set on the stack will be the first beam to be transported in the same manner, which will be repeated until the stack has been emptied before moving on to the next original beam.

2.2. Experiment

2.2.1. Apparatus. An experiment to assess the present method was carried out with accelerator facility HIMAC at the National Institute of Radiological Sciences. A $^{12}$C$^{6+}$ beam with nucleon kinetic energy $E/A = 290$ MeV was broadened to a uniform field of nominal
10 cm diameter by the spiral-wobbling method (Yonai et al 2008). The horizontal wobbler at $z = z_X = 527$ cm and the vertical wobbler at $z = z_Y = 470$ cm formed a spiral orbit of maximum 10 cm radius on the isocenter plane. A 0.8 mm thick Pb ($\rho_S = 5.77$, $X_0 = 0.561$ cm) foil was placed at $z = 425$ cm as a scatterer, which increased the instantaneous RMS beam size from pristine 8.3 mm to 25 mm at the isocenter. A large-diameter parallel-plate ionization chamber was placed at $z \approx 400$ cm for dose monitoring and beam-extraction control. An Al ($\rho_S = 2.12$, $X_0 = 8.90$ cm) ridge filter for semi-Gaussian range modulation of mean $\mu = 0.54$ cm and standard deviation $\sigma = 0.18$ cm in water (Schaffner et al 2000) was inserted at $z = 235$ cm to moderate the Bragg peak with a plain 2 mm thick Al base plate.

As shown in figure 3, a water ($\rho_S = 1$, $X_0 = 36.08$ cm) tank with a 1.9 cm thick PMMA ($\rho_S = 1.16$, $X_0 = 34.07$ cm) beam-entrance wall was placed at the irradiation site with the upstream face at $z = 16.9$ cm. The radiation field was defined by a 5 cm thick brass collimator of 8 cm square aperture whose downstream face was at $z = 65$ cm. Two identical 3 cm thick PMMA plates were inserted. The downstream plate was attached to the beam-entrance face of the tank covering only the $x > 0$ side to form a phantom system with a bump. The upstream plate was put with its downstream face at $z = 35$ cm covering only the $x < 0$ side to compensate the bump. Such arrangement is typical for range compensation and sensitive to the detour effects (Kohno et al 2004). These beam-customization elements were manually aligned to the nominal central axis at an uncertainty of 1 mm.

A multichannel ionization chamber (MCIC) with 96 vented sense volumes aligned at intervals of 2 mm along the $x$-axis was installed in the water tank at $y = 0$. The MCIC system was electromechanically movable along the $z$-axis and the upstream limit at $z_{ref} = 14.77$ cm was chosen for the reference height with reference depth $d_{ref} = 2.44$ cm of equivalent water from the tank surface.

2.2.2. Measurement. With a reference open field without the PMMA plates or the collimator, we measured reference dose/MU reading $M_{ref,i}$ at reference height $z_{ref}$ for every channel $i$ for a calibration purpose. Every dose/MU reading $M_i(z)$ of channel $i$ at height $z$ for any field is divided by corresponding reference reading $M_{ref,i}$ to measure dose $D$ at position $(x_i, z)$ as

$$D(x_i, z) = \frac{M_i(z)}{M_{ref,i}} \frac{z_X}{z_X - z_{ref}} \frac{z_Y}{z_Y - z_{ref}},$$

where the divergence-correction factor $z_X/(z_X - z_{ref}) \cdot z_Y/(z_Y - z_{ref}) = 1.054$ is to measure the doses in dose unit $U$ that would be the isocenter dose for the reference depth of the reference field.
Gaussian beam splitting for heterogeneity corrections

Figure 4. (a) Tissue-phantom ratio $D_0(d)$ with indications for the measurement (c) and the reference and 80% dose depths ($d_{\text{ref}}$ and $d_{80}$) and (b) effective density $\rho_S(x, z)$ and (c) effective lateral density gradient $\gamma_{xy}(x, z)$ distributions in gray scale at $y = 0$ in the calculation model.

Table 2. Estimated contributions of beam-line elements at height $z$ to the beam range $R$, scattering angle $\theta$, and source sizes $\sigma_X$ and $\sigma_Y$.

| Element      | $z$ | $-\Delta R$ | $\sqrt{\Delta \theta^2}$ | $\sqrt{\Delta \sigma_X^2}$ | $\sqrt{\Delta \sigma_Y^2}$ |
|--------------|-----|-------------|---------------------------|-----------------------------|-----------------------------|
| Pristine     | 1   | 0.68 cm     | 1                         | 8.3 mm                      | 8.3 mm                      |
| Scatterer (Pb) | 425 cm | 0.46 cm | 5.5 mrad | 5.6 mm | 2.5 mm |
| Ridge filter (Al) | 235 cm | 0.96 cm | 3.2 mrad | 9.3 mm | 7.5 mm |
| Total        | 1   | 2.10 cm     | 1                         | 13.7 mm                     | 11.5 mm                     |

We then measured reference-field doses $D_0(z)$ in the phantom at varied z positions, from which we get the tissue-phantom ratio

$$D_{00}(d) = D_0(z) \frac{z_X - z}{z_X} - \frac{z_Y - z}{z_Y}, \quad d = d_{\text{ref}} + z_{\text{ref}} - z. \quad (20)$$

Beam range $R_0$ with Gaussian modulation was equated to distal 80% dose depth $d_{80} = 14.14$ cm as shown in figure 4(a) (Koehler et al 1975).

With the collimator and the PMMA plates in place, lateral dose profiles were measured in the same manner with particular interest around $z = 3.3$ cm, 6.8 cm and 10.3 cm, where the Bragg peaks were expected for the primary ions passing through none, either and both of the PMMA plates.

2.2.3. Calculation. Table 2 shows range loss $-\Delta R$ and scattering $\Delta \theta^2$ for the beam-line elements, and the resultant contributions to source sizes $\sigma_X$ and $\sigma_Y$ estimated by back projection to the sources. The ridge filter with the base plate was modeled as plain aluminum of average thickness. The scattering for the scatterer was estimated from measured beam size 25 mm quadratically subtracted by pristine size in a distance of 425 cm. The total range loss 2.10 cm was deduced from the range 16.24 cm expected for $E/A = 290$ MeV carbon ions (Kanematsu 2008c), where unaccounted range deficit 0.68 cm may be attributed to minor materials in the beam line.

As described in the appendix, pencil beams were defined to cover the collimated field at intervals of $\delta_1 = 1$ mm on the isocenter plane, where the open field was assumed to have
uniform unit fluence $\Phi_0 = n_b/\delta^2 = 1$. Exact collimator modeling was omitted because we were interested in the density interface in the middle of the field. The upstream PMMA plate was modeled as a range compensator with range loss $S = 3.48$ cm for $x < 0$ or $S = 0$ for $x \geq 0$, where the original beams were generated, followed by the range loss and scattering. The phantom system comprised of the downstream PMMA plate and the water tank was modeled as density voxels at grid intervals of $\delta_1 = \delta_2 = \delta_3 = 1$ mm for a 2 L volume of $|x| \leq 5$ cm, $|y| \leq 5$ cm and $0 \leq z \leq 20$ cm. Figures 4(b) and (c) show the density and lateral heterogeneity distributions. For the modeled experimental system, we carried out dose calculations with beam splitting enabled (splitting calculation) and disabled (non-splitting calculation).

2.3. Applications

To examine the effect and efficiency of this method in the presence of realistic heterogeneity, a 3 cm diameter cylindrical air cavity at $(x, z) = (-3 \text{ cm}, 13 \text{ cm})$ and two 1 cm diameter bone rods with density $\rho_S = 2$ at (2 cm, 13 cm) and (4 cm, 13 cm) were added to the phantom in the calculation model.
We carried out splitting and non-splitting dose calculations of the same carbon-ion radiation to monitor changes in frequencies of splitting modes, number of stopped beams, total path length \( \sum_b \int ds \), total effective volume \( \sum_b \int 12\sigma_b^2 ds \) in the heterogeneous phantom and computing time with a 2.4 GHz Intel Core 2 Duo processor on Apple MacBook computer.

Protons generally suffer larger scattering, thus, naturally with larger overreach and detour effects. We verified this hypothesis with equivalent dose calculations for protons with enhanced scattering angle by a factor of 3.61 from equation (A.7) in otherwise the same configuration as for the carbon ions including the tissue-phantom ratios.

3. Results

3.1. Experiment

Figure 5 shows the two-dimensional dose distributions measured in the carbon-ion beam experiment and the corresponding splitting and non-splitting calculations. Figure 6 shows their lateral profiles in the plateau and at depths for sub peak, main peak and potential sub peak expected for particles that penetrated both, either and none of the PMMA plates. A dip/bump structure was commonly formed along the \( x = 0 \) line due to lateral particle disequilibrium (Goitein 1978). There was actually a sub peak in the measurement and in the splitting calculation, while it was naturally absent in the non-splitting calculation. The observed loss of the main-peak component was also reproduced by the splitting calculation. The potential sub peak was only barely noticeable in the splitting calculation.

3.2. Applications

Figure 7 shows details of the heterogeneous phantom and the dose distributions by the splitting calculation for the carbon-ion and proton radiations. The larger scattering for protons naturally led to the larger dose blurring. Figure 8 shows the dose profiles at the main peak and where the heterogeneity effects were large in the splitting and non-splitting calculations. In addition to the loss of the main-peak component at \( x \approx 0 \), the finer split beams resulted in slightly sharper dose profiles especially for the carbon ions although we do not have reference data to compare...
Figure 8. Lateral dose profiles in the heterogeneous phantom by splitting (solid) and non-splitting (dashed) calculations for carbon-ion radiation at heights (a) 7.3 cm, (b) 6.8 cm and (c) 5.8 cm and for proton radiation at heights (d) 7.3 cm, (e) 6.8 cm and (f) 5.8 cm.

Table 3. Statistics per original beam in non-splitting and splitting calculations for carbon-ion and proton radiations in the heterogeneous phantom.

| Projectile | Carbon ion | Proton |
|------------|------------|--------|
| Beam splitting | | |
| Frequency of $M = 2$ | No | Yes | No | Yes |
| Frequency of $M = 3$ | 0 | 0.243 | 0 | 3.857 |
| Frequency of $M = 4$ | 0 | 0.132 | 0 | 0.714 |
| Number of stopped beams | 1 | 26.8 | 1 | 25.1 |
| Mean path length (cm) | 20.0 | 394.6 | 20.0 | 500.8 |
| Mean effective volume (cm$^3$) | 3.52 | 23.1 | 30.8 | 380.9 |
| Mean computing time (ms) | 0.35 | 1.64 | 0.65 | 2.11 |

these results with. Table 3 shows the statistical quantities averaged for 8281 original beams in each field of carbon-ion and proton radiations incident into the heterogeneous phantom, where the increases due to splitting were factors of 27 and 25 in number, 20 and 25 in path length, 6.6 and 12 in volume and 4.7 and 3.2 in computing time.

4. Discussion

Subdivision of a radiation field into virtual pencil beams is an arbitrary process in the PB algorithm although the beam sizes and intervals should be limited by lateral heterogeneity of a given system. In the PB-redefinition algorithm (Siu and Hogstrom 1991), beams are defined in uniform rectilinear grids and hence regeneration in areas with little heterogeneity may be wasteful. In the beam-splitting method, beams are automatically arranged in accordance with local heterogeneity. In other words, the field will be covered by optimally arranged beams in a density-modulated manner as a result of independent self-similar splitting.
While every individual integral dose is guaranteed to conserve in beam splitting by
\[ \int_{-\infty}^{\infty} N_M(x) \, dx = 1 \quad (M \in \{2, 3, 4\}) \]
relative approximation errors \((N_M - N)/N\) are the worst at \(x = 0\) amounting to \(-2.3\%, -3.3\%\) and \(-8.5\%\). Fortunately, the consequent dose errors will be normally an order of magnitude smaller than those due to contributions from many other overlapping beams.

The effectiveness of the splitting method was demonstrated in the experiment. The most prominent detour effect was the loss of range-compensated main-peak component in figure 6(c), which amounted to about 10% in dose and approximately as large as the bump/dip due to lateral particle disequilibrium. The splitting calculation and the measurement generally agreed well, considering that the experimental errors in device alignment could have been 1 mm or more. The potential sub peak for particles detouring around both PMMA plates was not detected, which may be natural because detouring itself requires scattering. The dose resolution of the MCIC system of about 1% of the maximum should have also limited the detectability. In comparison between carbon ions and protons, contrary to our expectation, the overreach and detour effect by beam splitting was not specifically different because other scattering effects were also enhanced for protons.

Computing time is a practical concern. In principle, the total path length determines the computational amount for path integrals \((A.6)-(A.9)\) and the total effective volume determines that for dose convolution equation \((2)\) (Kanematsu et al. 2008a). Accuracy and speed also depend strongly on the cutoff parameters and logical conditions in algorithmic implementation, size and heterogeneity of a patient model, and spatial resolution clinically required. In fact, the slowing factors for beam splitting were 3–5 in our examples. Common computational overheads should have superficially reduced these factors and implemented algorithmic techniques, to be reported elsewhere, could have also contributed. The present performance, \(1/4–1/3\) min for 2 L volume in 1 mm grids, may be already affordable for clinical applications.

The automatic multiplication of tracking elements resembles the shower process of physical particle interactions that are normally handled in MC simulations. In fact, the MC method has many things in common. Transport and stacking of the elements are essentially the same and the probability for scattering may be equivalent to our Gaussian approximation. The essential factors that differentiate the efficiency between the MC and PB methods are the number of tracking elements and reliance on stochastic behavior of random numbers. With regard to accuracy, it is difficult for beam splitting or any beam model in general to deal with interactions that spoil the uniformity of involved particles, such as nuclear fragmentation processes (Matsufuji et al. 2005). In our model, their influences are only implicitly involved in the tissue-phantom ratios.

The beam-splitting method is intuitively based on the self-similarity principle and can be applied to any Gaussian beam model to fill the gap between MC particle simulations and beam-model calculations in accuracy and efficiency. In our case, the algorithm only requires rationally determined control parameters \(\kappa_r\) and \(\kappa_n\) other than conventional experimental data such as beam range, source sizes and tissue-phantom ratios. Simplicity and rationality will ease implementation and the use of the algorithm in clinical practice.

5. Conclusions

In this work we applied our finding of the self-similar nature of Gaussian distributions to the dose calculation of heavy charged particle radiotherapy. The self-similarity enables dynamic, individual and independent splitting of Gaussian beams that have grown to overreach the local
lateral heterogeneity. As a result, the detour effects can be addressed by small deflecting beams optimally arranged with modulated areal density.

In our experiment the splitting calculation was prominently effective in the region with steep range adjustment by an upstream range compensator. The maximum detour effect of about 10% in dose, which was comparable to the lateral particle disequilibrium effect, was reproduced at an expense of slowing down by a factor of 4.7 for carbon ions and 3.2 for protons. Between carbon ions and protons, the significance of splitting may not be essentially different because other scattering effects should also be larger for protons.

The present method of Gaussian-beam splitting is intuitive in principle, rigorous in formulation and easy to use with no additional experimental or free parameters. The moderate increase of computing time would be justified by improvement in accuracy and continuing advancement in computing technology.

Appendix. Pencil beam generation and transport

On generation of pencil beam $b$ on a plane at height $z_0$ as shown in figure 2, beam position $\vec{r}_b$, residual range $R_b$, and variances $\theta^2_b, \theta^2_{tb}$, and $t^2_b$ are initialized as

$$\vec{r}_b(0) = \vec{r}_b = \vec{r}_b + \frac{z_0}{v_{bc}}, \quad R_b(0) = R_0, \quad (A.1)$$

$$\frac{\delta^2}{\theta^2_{tb}}(0) = \frac{1}{2} \left( \frac{\sigma_X}{z_X - z_0} \right)^2 + \frac{1}{2} \left( \frac{\sigma_Y}{z_Y - z_0} \right)^2, \quad (A.2)$$

$$t^2_b(0) = \frac{z_X - z_0}{z_X} \frac{z_Y - z_0}{z_Y} \frac{\delta^2}{\theta^2_{tb}}, \quad (A.3)$$

$$\frac{\delta}{\theta^2_{tb}}(0) = \frac{\vec{r}_b(0)}{\sqrt{z_X - z_0} \sqrt{z_Y - z_0}}, \quad (A.4)$$

where $\vec{r}_b = (x_b, y_b, z_0)$ is the beam-$b$ origin, $\vec{r}_b = (x_{ib}, y_{ib}, 0)$ is the beam position on the isocenter plane, $\sigma_X$ and $\sigma_Y$ are the source sizes at virtual source heights $z_X$ and $z_Y$, $R_0$ is the initial residual range and $v_b = (v_{bx}, v_{by}, v_{bc})$ is the beam direction radiating from the virtual sources with

$$v_{bx} = -\frac{x_b}{z_X}, \quad v_{by} = -\frac{y_b}{z_Y}, \quad v_{bc} = -\left( \frac{x_b^2}{z_X} + \frac{y_b^2}{z_Y} + 1 \right)^{-\frac{1}{2}}. \quad (A.5)$$

Because nuclear interactions are effectively handled in the tissue-phantom ratio $D_{aq}(d)$ in the dose calculation, number of particles $n_b$ is modeled as invariant.

The modified Fermi–Eyges theory (Kanematsu 2008c, 2009) gives increments of the PB parameters in step $\Delta s$ within a voxel of tissue modeled as water with variable effective density $\rho_s$ (Kanematsu et al 2003) by

$$\Delta \vec{r}_b = \vec{v}_b \Delta s, \quad \Delta R_b = -\rho_s \Delta s, \quad (A.6)$$

$$\Delta \theta^2_b = 1.00 \times 10^{-3} z^{-0.16} \left( \frac{m}{m_p} \right)^{-0.92} \ln \frac{R_b}{R_b + \Delta R_b}, \quad (A.7)$$

$$\Delta \theta^2_{tb} = \left( \theta^2_{tb} + \frac{\Delta \theta^2_b}{2} \right) \Delta s, \quad (A.8)$$
\[ \Delta T_{b} = \left[ 2 \theta_{b} + \frac{2 \Delta \theta_{b}}{3} \right] \Delta s, \]  

where \( z \) and \( m/m_p \) are the particle charge and mass in units of those of a proton. For the last physical step with \( R_0 \to 0 \) and diverging \( \Delta \theta_{b} \), the growth is directly given by \( \Delta T_{b} = 0.0224 z^{-0.16} (m/m_p)^{-0.92} (R_0/\rho_0)^2 \) and then disabled by \( \Delta T_{b} = 0 \) in the unphysical \( R_0 \leq 0 \) region.

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