ONLINE REWEIGHTED LEAST SQUARES ALGORITHM FOR SPARSE RECOVERY AND APPLICATION TO SHORT-WAVE INFRARED IMAGING

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ABSTRACT

We address the problem of sparse recovery in an online setting, where random linear measurements of a sparse signal are revealed sequentially and the objective is to recover the underlying signal. We propose a reweighted least squares (RLS) algorithm to solve the problem of online sparse reconstruction, wherein a system of linear equations is solved using conjugate gradient with the arrival of every new measurement. The proposed online algorithm is useful in a setting where one seeks to design a progressive decoding strategy to reconstruct a sparse signal from linear measurements, so that one does not have to wait until all measurements are acquired. Moreover, the proposed algorithm is also useful in applications where it is infeasible to process all the measurements using a batch algorithm, owing to computational and storage constraints. It is not needed a priori to collect a fixed number of measurements; rather one can keep collecting measurements until the quality of reconstruction is satisfactory and stop taking further measurements once the reconstruction is sufficiently accurate. We provide a proof-of-concept by comparing the performance of our algorithm with the RLS-based batch reconstruction strategy, known as iteratively reweighted least squares (IRLS), on natural images. Experiments on a recently proposed focal plane array-based imaging setup shows up to 1 dB improvement in output reconstruction. Initial contributions to sequential CS were made by Malioutov et al. [8], who showed that a stopping rule can be designed for collecting measurements guaranteeing accurate reconstruction. Asif and Romberg [9] developed dynamic algorithms based on homotopy continuation for sequential CS assuming that the underlying signal changes slowly during measurement acquisition. Homotopy-based algorithms for solving online LASSO have also been developed in [10][11]. Angelosante et al. proposed time- and norm-weighted LASSO schemes [12] where the \( l_1 \)-norm weights are obtained from the recursive least-squares algorithm. Sequential reconstruction of sparse signals with slowly varying sparsity patterns has also been addressed by Vaswani et al. [13] and has been applied to dynamic magnetic resonance imaging. Their technique goes by name of Kalman-filter CS (KFCS).

The developments in the theory of CS have led to remarkable advances in imaging technologies in recent years [13][14]. CS asserts that using the sparsity prior, one can reconstruct a high-resolution (HR) image from its coded low-resolution (LR) measurements, thereby effectively increasing the sampling rate. Duarte et al. made pioneering contribution in compressed imaging by developing single pixel cameras (SPCs) [16], wherein one acquires coded measurements of a HR scene using a single photo detector and subsequently recovers the image by employing a CS reconstruction algorithm. Imaging in the short-wave infrared (SWIR) spectrum offers several benefits over visible imaging and is used in many applications. SWIR penetrates fog and smog, enables passive imaging in the dark and has a variety of applications in biomedical imaging [15]. Cameras for SWIR, however, are much more expensive than their counterparts in the visible spectrum. Motivated by the Single Pixel Camera (SPC) [16], Chen et al. [15] developed a proof-of-concept algorithms such as orthogonal matching pursuit [5], CoSaMP [6] etc. where the signal support is estimated in a greedy manner, followed by the estimation of amplitudes using least-squares and (ii) convex relaxation-based approaches such as LASSO [7], where a sparsity promoting convex regularization is incorporated to solve the otherwise ill-posed inverse problem.

The conventional CS deals with signal reconstruction in batch, wherein the entire set of measurements is available at one’s disposal. However, in most real applications, the measurements are acquired sequentially and it is important to develop online algorithms that reconstruct the underlying signal gradually as the measurements arrive. Moreover, in an online setting, the number of measurements does not have to be fixed in advance and one can stop as and when the quality of reconstruction is acceptable. Initial contribution to sequential CS was made by Malioutov et al. [8], who showed that a stopping rule can be designed for collecting measurements guaranteeing accurate reconstruction. Asif and Romberg [9] developed dynamic algorithms based on homotopy continuation for sequential CS assuming that the underlying signal changes slowly during measurement acquisition. Homotopy-based algorithms for solving online LASSO have also been developed in [10][11]. Angelosante et al. proposed time- and norm-weighted LASSO schemes [12] where the \( l_1 \)-norm weights are obtained from the recursive least-squares algorithm. Sequential reconstruction of sparse signals with slowly varying sparsity patterns has also been addressed by Vaswani et al. [13] and has been applied to dynamic magnetic resonance imaging. Their technique goes by name of Kalman-filter CS (KFCS).

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prototype hardware for compressive imaging in the SWIR spectrum. The setup, which is termed the Focal Plane Array-based Compressive Sensing (FPA-CS), captures several low-resolution measurements with its $64 \times 64$ SWIR sensor array, which are used for reconstructing an image of much higher resolution. We show that the proposed online approach fits naturally in the FPA-CS imaging framework and demonstrate results on a simulated setup.

Our main contribution is to propose an online algorithm for sparse reconstruction by exploiting the idea of iteratively reweighted least squares (IRLS), originally proposed by Daubechies et al. [17]. We refer to the proposed algorithm as online reweighted least squares (ORLS). Akin to IRLS, the key idea in ORLS is to approximate the sparsity inducing $\ell_1$ norm with a weighted $\ell_2$ penalty, where the weights are determined based on the current estimate of the signal. The underlying signal is assumed to be fixed during the course of measurement. With each new measurement, a system of linear equations is solved using the conjugate-gradient (CG) method, which requires less number of iterations to converge as more measurements are made available. We demonstrate that the ORLS algorithm is noise-robust and produces reconstructed signals on par with the batch-mode recovery using IRLS. Experiments on FPA-CS image reconstruction show that ORLS leads to an improvement over the total variation (TV) regularization-based batch recovery.

2. PROBLEM STATEMENT AND THE PROPOSED ORLS ALGORITHM

Our objective is to recover a sparse signal $x^* \in \mathbb{R}^n$ from its noisy linear projections of the form $y_t = a_t^T x^* + \xi_t$, revealed sequentially at time instants $t = 1, 2, \ldots$. Under this setting, we formulate the problem of sequential sparse recovery as an online convex optimization problem, where one predicts a vector $x_t$ in a convex set $S$, and suffers a convex loss $f_t(x_t)$, revealed at every time instant $t$. Naturally, the loss function is defined as $f_t(x) = (y_t - a_t^T x)^2$, so that it measures how well the current estimate explains the measurements. The set $S$ is defined as $S \triangleq \{x \in \mathbb{R}^n : \|x\|_1 \leq \tau\}$, the closed $\ell_1$-norm ball of radius $\tau$, to promote sparsity in the estimate. To perform online sparse estimation, it would be natural to solve

$$x_{t+1} = \arg \min_{x \in S} \sum_{j=1}^{t} \|y_j - a_j^T x\|^2,$$

where $x_{t+1}$ denotes the reconstructed sparse vector at time instant $t$. Solving (1) yields estimates $x_t$ that are consistent with the measurements received until time $t$, and are within the set $S$, thereby promoting sparsity.

2.1. Online reweighted least-squares (ORLS) algorithm

In order to develop the ORLS algorithm, we write (1) in its equivalent unconstrained form, given by

$$x_{t+1} = \arg \min_{x \in \mathbb{R}^n} \sum_{j=1}^{t} \|y_j - a_j^T x\|^2 + \lambda \|x\|_1,$$

where $\lambda > 0$ is an appropriately chosen regularization parameter which trades-off between sparsity and data fidelity. The problem in (2) is a convex program. However, because of the $\ell_1$ regularizer, it is not possible to find a closed-form solution to (2). To circumvent this problem, we propose to approximate the $\ell_1$ norm using a quadratic of the form $x^T W_t x$, where $W_t$ is a diagonal matrix, containing positive weights on its diagonal. The entries of $W_t$ are suitably updated based on the previous estimate $x_t$, using the formula $W_t(j) \leftarrow \frac{1}{|x_t(j)| + \delta}$ for every $j$, where $\delta > 0$ is a small positive constant that is introduced to avoid numerical instability. Therefore, at every $t$, one needs to solve the following optimization problem:

$$x_{t+1} = \arg \min_{x \in \mathbb{R}^n} \sum_{j=1}^{t} \|y_j - a_j^T x\|^2 + \lambda x^T W_t x. \quad (3)$$

The closed-form solution to (3) is given by

$$x_{t+1} = \left( \lambda W_t + \sum_{j=1}^{t} a_j a_j^T \right)^{-1} \left( \sum_{j=1}^{t} y_j a_j \right). \quad (4)$$

The matrix inversion in (3) can be performed using the Sherman-Morrison formula recursively, requiring $O(n^2 t)$ computations, provided that $a_i, i = 1 \cdots t$, are stored. Thus, for $m$ measurements, the overall run-time complexity of the ORLS algorithm is given by $T_{ORLS}(m) = \sum_{t=1}^{m} O(n^2 t) = O(m^3 n^2)$. If $a_i$s are not stored individually and one only keeps track of the accumulated value $\sum_{j=1}^{t} a_j a_j^T$, the inversion in (4) demands $O(n^3)$ complexity, which is unacceptably high for an online scheme. To circumvent this problem, we propose to find the estimate $x_{t+1}$ using the CG method. Computing $x_{t+1}$ using (4) is equivalent to solving the system of linear equations given by

$$A_t x = b_t,$$

where $A_t$ and $b_t$, for $t \geq 2$, are computed recursively:

$$A_t = \lambda W_t + Q_{t-1} + a_t a_t^T \quad \text{and} \quad b_t = b_{t-1} + y_t a_t. \quad (6)$$

by setting $Q_1 = a_1 a_1^T$ and $b_1 = y_1 a_1$. To evaluate $x_{t+1}$ using CG, we set the previous estimate $x_t$ as the initialization. Each iteration of CG requires $O(n^2)$ computation and one does not need to store the previous values of $y_t$ and $a_t$. An overall saving in computation and storage is realized if the number of iterations $L$ required in the CG algorithm is smaller than $m$. Typically, we observe that $L = \frac{n}{2}$ suffices to obtain equivalent performance as the ORLS algorithm.

3. EXPERIMENTAL RESULTS

To image a scene, random binary patterns of size $8 \times 8$ consisting of zeros and ones are used for scanning. Such patterns can be produced in practice by employing programmable digital micro-mirror devices (DMDs), wherein one can program the binary patterns and acquire spatially coded measurements. Using random patterns of zeros and ones amounts to acquiring the summation of intensities of a random subset of pixels over every $8 \times 8$ patch of the HR scene. We denote a generic HR patch by $z$, and assume that it admits a sparse representation of the form $z = D x$, where $D$ is the DCT dictionary. The ORLS algorithm recovers $z$ from sequentially obtained coded measurements of the form $y_t = c_t z = a_t^T x$, where $a_t = D^T c_t$, and $c_t$ denotes the random binary pattern. Each patch contains $n = 64$

$$^{1}$$Sherman-Morrison formula: $(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$. 

pixels and \( \frac{m}{n} \) denotes the fraction of measurements acquired until time \( t \). Experiments are conducted on color images and the reconstruction performance is shown for \( m \) varying from 1 to \( n \). Performance metrics for evaluating the quality of reconstruction are taken as the PSNR and the structural similarity index (SSIM), considering the ground-truth image as the reference.

The reconstructed images using ORLS corresponding to 25%, 75%, and 100% measurements are shown in Figures 1(b), 1(c), and 1(d), respectively. The ground-truth image is shown in Fig. 1(a) to facilitate visual comparison. We consider noiseless coded measurements in this experiment and the parameter \( \lambda \) is set to 1. The reconstruction in Fig. 1(d) corresponding to the full set of measurements is indistinguishable from the ground-truth with no color artifacts. This is also reflected in the PSNR and SSIM values plotted as a function of percentage measurements in Fig. 1(e). The values of PSNR and SSIM corresponding to 100% measurements are approximately 38 dB and 0.99, indicating a near-accurate recovery. We have also shown the number of iterations required by the CG algorithm (normalized by \( n \)) in order to achieve an accuracy of \( \epsilon = 10^{-5} \) with respect to the fraction of measurements. It is observed that the number of iterations required for convergence increases initially and attains a peak, thereafter falling almost monotonically as more measurements arrive. This trend indicates that a considerable saving in computation can be achieved over the direct inversion-based approach by employing CG to solve (5).

We compare the performance of ORLS with its batch counterpart, namely IRLS, with the full set of measurements (\( m = 64 \) per patch). The HR scene to be recovered is corrupted with additive Gaussian noise with a PSNR of 22.10 dB. The reconstructed images obtained using the batch IRLS and ORLS algorithms corresponding to \( \lambda = 40 \) are shown in Figures 2(a) and 2(b), respectively. The difference image is shown in Figure 2(c). The values lie in the range \([0, 76]\) and are scaled in the figure for better visualization. The variations of PSNR and SSIM of the output of ORLS are plotted in Fig. 2(d) as a function of measurements. We observe that ORLS leads to a reconstruction with PSNR and SSIM matching that of batch IRLS (shown using dotted horizontal lines), indicating that ORLS performs on par with its batch counterpart.

### 4. Application of ORLS to FPA-CS

Focal-plane-array-based CS (FPA-CS) is the architecture proposed in [15] that leverages CS for short-wave infrared (SWIR) imaging. It is optically identical to several SPCs arranged in an array and provides increased measurement rates, thereby resulting in higher spatio-temporal resolution. A schematic of the FPA-CS imaging setup is shown in Fig. 3. The object to be imaged is focused onto a DMD array, which acts as a random binary mask on the image. The light from the DMD is refocused by the relay lens onto a DMD array, which acts as a random binary mask on the image.
reveals that the image reconstruction task in FPA-CS fits naturally
input image is taken as based batch mode reconstruction as in [15]. The patch size on the
performance of ORLS is compared with the TV regularization-
gions to form the input image. For our experiment, we have simu-
average of the neighboring reconstructed patches on overlapping re-
the input image might have overlaps and one should compute the
Since the acquisition for each patch is independent of

\[ y_t(p) = c(P)^T z(P) + \xi_t(P), \]

where \( c(P) \) represents the combined effect of blurring and the DMD pattern. The binary mask can in fact be chosen to be periodic without loss of generality, so that it is same for every patch \( P \), thus requiring less storage. Substituting \( z(P) = D x(P) \), where \( D \) is the DCT dictionary and \( x(P) \) is sparse, we have that

\[ y_t(p) = a(P)^T x(P) + \xi_t(P), \quad t = 1, 2, \cdots \quad (7) \]

where \( a(P) = D^T c(P) \). The modified measurement equation (7) reveals that the image reconstruction task in FPA-CS fits naturally in the framework considered in ORLS. Notably, the patches \( z(P) \) of the input image might have overlaps and one should compute the average of the neighboring reconstructed patches on overlapping regions to form the input image. For our experiment, we have simulated the FPA-CS imaging setup such that there is no overlap of the patches in \( z \). Since the acquisition for each patch is independent of the others, it is possible to parallelize the patch-wise computations.

Experiments are conducted on a simulated FPA-CS setup and the performance of ORLS is compared with the TV regularization-based batch mode reconstruction as in [15]. The patch size on the input image is taken as \( 8 \times 8 \). The results are reported in Fig. 4 which shows a comparison of the ORLS reconstruction (Fig. 4(c)) using 64 measurements (which corresponds to 100% of the measurements), with the TV-based batch reconstruction (Fig. 4(b)). We observe that

Fig. 3. A schematic of the FPA-CS imaging setup.

ORLS leads to a reconstruction with better visual quality, which is reflected in higher PSNR (28.99 dB versus 27.98 dB) and SSIM (0.89 versus 0.88) values. The difference between the ground-truth and the ORLS-based reconstruction is shown in Fig. 4(d) with intensity values scaled to fall in the range \([0, 113]\) to ensure better visual assessment. We observe that although some textures in the target image are not recovered accurately, most of the edge information is reliably reconstructed.

5. CONCLUSIONS AND OUTLOOK

We have developed an online regularized least squares algorithm for sparse reconstruction. The ORLS algorithm solves a system of linear equation using CG with the arrival of each new measurement. We have demonstrated that the CG algorithm takes considerably less number of iterations than the signal dimension if the current estimate of the signal is used as the initialization. This leads to progressively less amount of computation as more measurements are acquired. We did not consider measurement adaptation in our formulation and assumed that the signal remains fixed during the measurement process. Numerical experiments on natural images show that the ORLS scheme is robust to noise and reconstructs images that are on par with the batch IRLS technique in terms of PSNR and SSIM. The ORLS reconstruction is also free from any color artifacts. We have demonstrated that an improvement of 1 dB in output PSNR can be achieved using ORLS as compared with the batch-mode reconstruction that uses the TV regularizer.

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