Running Waves in the Mixed State of Type II Superconductors

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Abstract
Nonlinear dynamics of the thermal and electromagnetic instabilities of the mixed state in type II superconductors has been analysed taking into account the effect of dissipation and dispersion. The existence of nonlinear running waves describing the final stage of evolution of the thermomagnetic instability in superconductors is demonstrated analytically.

The evolution of the thermal \( (T) \) and electromagnetic \( (\vec{E}, \vec{H}) \) perturbations is described by the nonlinear equation of the thermal conductivity [1]

\[
\nu \frac{dT}{dt} = \nabla \left[ \kappa \nabla T \right] + \vec{j} \vec{E},
\]

by the Maxwell equations

\[
\text{rot} \vec{E} = -\frac{1}{c} \frac{d\vec{H}}{dt},
\]

\[
\text{rot} \vec{H} = \frac{4\pi}{c} \vec{j}
\]

and by the equation of the resistive state

\[
\vec{j} = \vec{j}_c(T, \vec{H}) + \vec{j}_r(\vec{E}),
\]

where \( \nu = \nu(T) \) and \( \kappa = \kappa(T) \) are the heat capacity and the thermal conductivity respectively; \( \vec{j}_c \) is the critical current density and \( \vec{j}_r \) is the resistance current density.

The above system is essentially nonlinear because the right-hand part of Eq.(1) contains a term describing the Joule heat evolution in the region of the resistive phase. Such a set (1)-(4) of nonlinear parabolic differential equations in partial derivatives has no exact analytical solution.

Let us consider a planar semi-infinite sample \( (x > 0) \) placed in external magnetic field \( \vec{H} = (0, 0, H_e) \) growing at a constant rate \( \frac{d\vec{H}}{dt} = \text{const} \). According to the Maxwell equation (2), there is a vortex electric field \( \vec{E} = (0, E_e, 0) \) in the sample, directed parallel to the current density \( \vec{j} : \vec{E} \parallel \vec{j} \); where \( H_e \) is the amplitude of the external magnetic field and \( E_e \) is the amplitude of the external electric field.

The solution of the system of equations (1)-(4) may be presented as a function of new automodel variable \( \xi(x, t) \):
\[ T = T[\xi(x,t)], \]
\[ E = E[\xi(x,t)], \]  
\[ j = j[\xi(x,t)]. \]  

Substituting (5) into the system (1)-(4) gives, as a result of simple differentiation, the following system

\[ \frac{d \xi}{dt} = \kappa \left\{ \frac{d^2 \xi}{dx^2} \frac{dT}{d\xi} + \left( \frac{d\xi}{dx} \right)^2 \frac{d^2 T}{d\xi^2} \right\} + [j_c(T) + j_r(E)]E, \]  

(6)

\[ \frac{d^2 \xi}{dx^2} \frac{dE}{d\xi} + \left( \frac{d\xi}{dx} \right)^2 \frac{d^2 E}{d\xi^2} = \frac{4\pi}{c^2} \left[ \frac{dj_c}{dT} \frac{dT}{d\xi} + \frac{dj_r}{dE} \frac{dE}{d\xi} \right] \frac{d\xi}{dt}. \]  

(7)

In order the system (6),(7) was only function from \( \xi \) at the substation (5) it is required carrying out the following conditions:

\[ \frac{d\xi}{dt} = A(\xi), \]  

(8)

\[ \left( \frac{d\xi}{dx} \right)^2 = B(\xi) \frac{d\xi}{dt} = G(\xi), \]  

(9)

\[ \frac{d^2 \xi}{dx^2} = C(\xi) \frac{d\xi}{dt}, \]  

(10)

where \( A, C, G \) are functions from \( \xi \), the type of which will be determined below. Solving first two system of equations (8)-(10) we have a relation

\[ G(\xi) \frac{dA}{d\xi} = A(\xi) \frac{dG}{d\xi}. \]  

(11)

Whence just follows relationship between \( G \) and \( A \)

\[ G(\xi) = \frac{1}{u} A(\xi), \]  

(12)

where \( u \) is a free constant of integrating of the equation (11). From (8) and (9) follows that \( \xi(x,t) \) must satisfy single-line equation in private derivation

\[ \frac{d\xi}{dt} = u \frac{d\xi}{dx} \]  

(13)

the only solution of which is the function

\[ \xi(x,t) = F(x - ut). \]  

(14)

Using (14) we can immediately obtain
\( G(\xi) = 1, \quad A(\xi) = -F u, \quad C(\xi) = 0. \) \hspace{1cm} (15)

It is possible to ensure \( F = 1 \) by the transformation of coordinates and time. Thereby, we find final automodel substitution

\[ \xi = x - ut, \] \hspace{1cm} (16)

corresponding to solution of running type wave [2].

For the automodelling solution of the form (16), describing a running wave moving at a constant velocity \( v \) along the \( x \) axis, the system of equations (1)-(4) takes the following form

\[- v [N(T) - N(T_0)] = \kappa \frac{dT}{d\xi} - \frac{c^2}{4\pi v} E^2, \] \hspace{1cm} (17)

\[ \frac{dE}{d\xi} = - \frac{4\pi v}{c^2} j, \] \hspace{1cm} (18)

\[ E = \frac{v}{c} H. \] \hspace{1cm} (19)

The thermal and electrodynamic boundary conditions for equations (17)-(19) are as follows:

\[ T(\xi \to +\infty) = T_0, \quad \frac{dT}{d\xi}(\xi \to -\infty) = 0, \] \hspace{2cm} (20)

\[ E(\xi \to +\infty) = 0, \quad E(\xi \to -\infty) = E_c, \]

where \( T_0 \) is the temperature of the cooling medium.

Let us consider the Bean-London model of the critical state for the dependence \( j_c(T, H) \) [3]

\[ j_c(T) = j_0 [1 - a(T - T_0)] \] \hspace{1cm} (21)

where \( j_0 \) is the equilibrium current density, \( a \) is the thermal heat softening coefficient of the magnetic flux pinning force. The characteristic field dependence of \( j_c(E) \) in the region of sufficiently strong electric field \( (E > E_f) \) can be approximated by the piecewise linear function

\[ j_r \approx \sigma_f E, \] \hspace{2cm} (22)

where \( \sigma_f = \frac{\eta c^2}{H \Phi_0} \approx \sigma_n H_{c2}/H \) is the effective conductivity in the flux flow regime; \( \eta \) is the viscous coefficient, \( \Phi_0 = \frac{\pi h c}{2e} \) is the magnetic flux quantum, \( \sigma_n \) is the conductivity in the normal state; \( E_f \) is the boundary of the linear area in the voltage-current characteristics of the sample.

Excluding variables \( T(\xi) \) and \( H(\xi) \) from Eqs.(17) and (19), and taking into account the boundary conditions (20), we obtain an equation describing the electric field \( E(\xi) \) distribution (E-wave):

\[ \frac{d^2 E}{d\xi^2} + \left[ \frac{4\pi v}{c^2} \frac{dj_r}{d\xi} \frac{dE}{d\xi} + \frac{4\pi v^2 a}{c^2} \frac{N(T) - N(T_0)}{\kappa(T)} \right] - \frac{aE^2}{2\kappa(T)} = 0, \] \hspace{1cm} (22)
where the dependency $T = T \left( E, \frac{dE}{d\xi} \right)$ is defined by expression (2), (4) and have the form

$$T = T \left( E, \frac{dE}{d\xi} \right) = T_0 + \frac{1}{a} \left[ j_0 + j_r(E) + \frac{4\pi v}{c^2} \frac{dE}{d\xi} \right].$$

(23)

Here $N(T) = \int_0^T \nu(T) dT$.

The analysis of the phase plane $(E, \frac{dE}{d\xi})$ of Eq.(22) shows that there are two equilibrium points: $E_0 = 0, T = T_0$ is the stable node and $E = E_e, T = T* = T(E_e, 0)$ is the saddle. The solution may be represented in the form of the shock-wave-type with amplitude $E_e$, by joining these two equilibrium points. The velocity of $E$-wave is determined by the Eq. (22) with account of the boundary conditions (20):

$$v_E^2 = \frac{c^2}{8\pi} \frac{E_e^2}{N \left[ T_0 + \frac{1}{a} \left[ j_c(T) + j_r(E) \right] \right] - N(T_0)}.$$  

(24)

Using Eqs. (19) and (22) we find the expression for distribution of the magnetic field $H$ in the case of $H$-wave:

$$\frac{d^2 H}{d\xi^2} + \frac{4\pi v}{c^2} \left[ \frac{dj_r}{dE} \left|_{E = \frac{v}{c} H} \right. \frac{dH}{d\xi} + \frac{N(T) - N(T_0)}{\kappa(T)} \right] - \frac{av}{2c} \frac{H^2}{\kappa(T)} = 0.$$  

(25)

The velocity of $H$ wave is connected with its amplitude by the following expression

$$N(T) - N(T_0) = \frac{H_e^2}{8\pi}.$$  

(26)

The condition (26) reflects the adiabatical character of the wave propagation: the magnetic field energy transferred by the wave ensures local heating of the sample in the close vicinity of the wave front. Thereby, depending on the external conditions at the sample surface there may exist two kinds of thermomagnetic waves.

The system of Eqs. (1)-(4) is invariant with respect to an arbitrary translation. Therefore, the wave propagation conditions can be found for an arbitrary critical current density depending on $T$ and $H$. The results can also be obtained for an arbitrary temperature dependence of thermophysical parameters $\nu$ and $\kappa$ of superconducting material and for an arbitrary function $j_r(E)$.

Reference

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