Magnetic field-induced anisotropic interaction in heavy quark bound states

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Abstract

In this article we have investigated how a strong magnetic field (B) could decipher the anisotropic interaction in heavy quark (Q) and antiquark (Q̄) bound states through the perturbative thermal QCD in real-time formalism. So we thermalize the Schwinger propagator for quarks in the lowest Landau level and the Feynman propagator for gluons to calculate the gluon self-energy up to one loop for massless flavours. For the quark-loop contribution to the self-energy, the medium does not have any temperature correction and the vacuum term gives rise an anisotropic term whereas the gluon-loop yields the temperature correction. This finding in quark-loop contribution corroborates the equivalence of a massless QED in (1+1)-dimension with the massless thermal QCD in strong magnetic field, which (quark sector) is reduced to (1+1)-dimension (longitudinal). This anisotropy in the self-energy is then being translated into the permittivity of the medium, which now behaves like as a tensor. Thus the permittivity of the medium makes the QQ̄ potential anisotropic in the presence of strong magnetic field in the coordinate space, which resembles with a contemporary results found in lattice studies [42, 43]. As a matter of fact, the potential for QQ̄-pairs aligned transverse to B is more attractive than the parallel alignment. However, the potential is always more attractive compared to the absence of B due to the softening of the electric screening mass. However, the (magnitude) imaginary-part of the potential becomes smaller compared to B = 0. We have next investigated the effects of strong B on the binding energies (B.E.) and thermal widths (Γ) of the ground states of cc and bb in a first-order time-independent perturbation theory, where the binding energies get increased and the widths get decreased compared to B = 0. The above medium modifications to the properties of QQ̄ bound states facilitated to study their quasi-free dissociation in the medium in a strong magnetic field. The dissociation temperatures estimated for J/ψ and Υ states quantitatively from an optimized criterion - B.E.=Γ/2 are obtained as 1.59Tc and 2.22Tc, respectively, which are higher than the estimate in the absence of strong magnetic field. Thus strong B impedes the early dissolution of QQ̄ bound states.

1 Introduction

Lattice Quantum Chromodynamics predicts that at extreme conditions of high temperatures and/or high densities, quarks confined inside the hadrons get deconfined and roam in an extended region of space (much bigger than the size of a hadron), known as quark-gluon plasma (QGP). This novel phenomenon can be seen as a generic property of nonabelian gauge theories at high energies, celebrated as Asymptotic Freedom. It is believed that such state of matter also existed in our present universe around one microsecond after the big bang, in the core of the dense stars, in the terrestrial laboratory of ultra-relativistic heavy ion collision (URHIC) experiments etc. As we know from the ongoing URHIC experiments at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC), a very strong magnetic field, perpendicular to the reaction plane, is produced in the very early stages of the collisions due to the very large relative velocity
of spectator quarks in non-central events [1, 2], ranging from $m_\text{Q}^2$ ($10^{18}$ Gauss) at RHIC [3] to $15m_\text{Q}^2$ at LHC [4]. Initially, it was believed that the magnetic field decays very fast just after it is produced, but the life time of the magnetic field is elongated if the medium have finite value of the electrical conductivity. So physicists have realized that in the presence of the background magnetic field various physical quantities associated with QGP will also get modified. In the recent years various research activities are going on in which the effects of the background magnetic field on the various properties of QGP have been incorporated, which in turn lead to many novel phenomena related to QCD, viz. (inverse) magnetic catalysis [5–7], chiral magnetic effect [3, 8], chiral vortical effect in rotating QGP [9, 10], axial magnetic effect [11, 12], the dilepton production rate [13–15], the conformal anomaly and production of soft photons [16, 17], dispersion relation in the magnetized thermal QED [18], refractive indices and decay constants [17, 19], thermodynamical [20, 21], magnetic [22] and transport properties [23, 24]. Out of many signatures of the QGP, the suppression of the heavy quarkonia is a very promising one.

The heavy quark and antiquark pairs are produced in URHICs on a very short time-scale \( \sim 1/2m_\text{Q} \) (\( m_\text{Q} \) is the mass of the heavy quark) and the pair develops into the physical resonances (heavy quarkonia) over a formation time and traverses the QGP and subsequently the hot hadronic matter before leaving the interacting system to decay into a dilepton to be detected. So by studying the properties of the heavy quarkonia, we can get some understanding about the medium and vice versa. Since the masses of the heavy quarks \( m_\text{Q} \) are much larger than the intrinsic QCD scale (\( \Lambda_{\text{QCD}} \)), the velocity of the heavy quarks (\( v \)) is very small in the bound states. The \( Q\bar{Q} \) pair could then be treated like a nonrelativistic system, possessing a hierarchy of energy scales: \( m_\text{Q} > m_\text{Q} > m_\text{Q}v^2 \) and integrating out the successive scales lead to a sequence of low-energy effective field theories (EFTs), viz. nonrelativistic QCD (NRQCD), potential nonrelativistic QCD (pNRQCD) etc. For example, pNRQCD (by integrating out the scale \( m_\text{Q}v \)) describes the \( QQ \) bound state by a two-point function satisfying the Schrödinger equation through the usual Cornell potential as the matching coefficients in the effective Lagrangian. For the quarkonia in a thermal medium, pNRQCD may be generalized to finite temperature [25], but the hierarchy in thermal scales (\( T > gT > g^2T \)) make the analysis complicated. For example, if the binding energies are larger than the temperature, although the \( Q\bar{Q} \) potential does not get modify but quarkonia develop a finite thermal width due to the medium induced singlet-octet transitions [25]. In the opposite limit (\( B.E < T < gT \)), the potential acquires an imaginary component [25]. However, the hierarchy in scales in EFT is not always evident and one needs lattice techniques to test the approach, where the modification of the quarkonium states can be studied in its spectral function in terms of the Euclidean meson correlation functions [26]. At finite temperature the construction becomes worst because the temporal-extent is decreasing, thus inadvertently supports the use of potential models at finite temperature to complement the lattice studies.

Thus the similarity in the time scales for the production of strong magnetic field and the formation of heavy quarkonia motivates us to study the effect of the strong magnetic field on \( QQ \) interaction. In the recent years the effect of the magnetic field have been studied on the production of the heavy quarkonia in [27, 28] and the evolution of \( J/\psi \) and the magnetic conversion of \( \eta_c \) to the \( J/\psi \) in the presence of the strong magnetic field in [29, 30]. Moreover, the static properties of quarkonia [31–35] as well as open heavy flavours [36–39] were studied in the presence of magnetic field. In the recent years, the properties of \( Q\bar{Q} \) bound states in a thermal QCD medium have been investigated by correcting both the perturbative and nonperturbative part of the \( Q\bar{Q} \) potential through the dielectric function in the real-time formalism [40] and later extended to the moving medium [41]. Two of us have also recently studied the effect of the strong magnetic field on the static properties of \( Q\bar{Q} \) bound states as well as their dissociation in a thermal QCD medium.

There have been lattice results on the heavy-quark potential and screening masses, both of which show novel anisotropic behaviors between transverse and longitudinal directions with respect to the magnetic field direction [42–44]. The anisotropic behaviours in the heavy quark
potential can be viewed in general as a manifestation of the breaking of rotational invariance in the presence of magnetic field. In nonrelativistic quantum mechanics, assuming the electron possessing the spin, the orientational term in the potential energy arises due to the interaction of spin magnetic moment with the external magnetic field. In relativistic quantum mechanics, the Dirac equation in the nonrelativistic limit manifests the aforesaid orientational term in Pauli-Schrodinger equation. However, in abovementioned potential studies at finite temperature [45–47] based on the pertubative thermal QCD, the magnetic field did not reveal any anisotropic nature in $Q\overline{Q}$ interaction, like in aforesaid lattice studies. Therefore, our aim is to uncover the tensorial (anisotropic) part in $Q\overline{Q}$ interaction by an an external magnetic potential. In fact, we have found and the potential becomes anisotropic and depends on the relative orientation of the quark pairs with respect to the direction of the magnetic field.

Initially the dissociation process of the heavy quarkonia was understood in terms of color screening. However, the broadening of the widths of the resonances is nowadays considered as the main reason behind the dissociation and arises either due to the inelastic parton scattering process mediated by the spacelike gluons known as Landau damping [48] or due to the gluodissociation process in which the color singlet state undergoes into a color octet state by a hard thermal gluon [49], photo-gluon dissociation. However, when the temperature of the medium is smaller than the binding energy of the particular resonance the later process become dominant. Thus, due to the broadening in the medium, the quarkonium resonances is dissociated at smaller temperatures with respect to the dissociation due to the color screening alone. We therefore first want to see the effects of strong magnetic field on the screening and Landau damping, which in turn gives the modified binding energies and widths of the resonance states, respectively. In the framework of potential model studies, the aforesaid studies are made possible by deriving the real- and imaginary-parts of potential in one-loop thermal QCD in a strong magnetic field.

This paper is organized as follows: In Section 2, we have revisited the heavy quarkonia in isotropic thermal QCD to make a baseline for our work in magnetic field in Section 3. As we know that the strong magnetic field generically causes a momentum anisotropy, so we have calculated the quark contribution to the gluon self-energy in strong $B$ at subsection 3.1 through a novel diagrammatic approach, by using Schwinger propagator. Thus results of subsection 3.1 facilitates to calculate the real-part of the complex permittivity from the static limit of resummed propagator, thus the real-part of an anisotropic heavy quark potential is obtained in strong magnetic field. Similarly we obtain the imaginary-part of the medium modified potential. In Section 4, we have studied how the properties of the charmonium and bottomonium ground states get affected by the strong magnetic field, which in turn explore the dissociation of the aforesaid states. We conclude in Section 5.

2 Heavy Quarkonia in the absence of magnetic field

The inter-quark Cornell potential between $Q$ and $\overline{Q}$ in vacuum ($T = 0$), is

$$\hat{V}(r; T = 0) = -\frac{\alpha}{r} + \sigma r,$$

(1)

where $\alpha$ and $\sigma$ are the phenomenological constants, to be fitted to reproduce the ground state spectroscopy of heavy quarks bound states after including the spin-dependent term in the potential. The medium modification to the potential in the momentum space is obtained by the dielectric permittivity, $\epsilon (k)$ of the medium as

$$\hat{V}(k) = \frac{V(k)}{\epsilon(k)},$$

(2)
where $\epsilon(k)$ encodes the properties of the deconfined medium. $V(k)$ is the Fourier transform (FT) of the vacuum potential, where the FT of the linear term needs to regularize properly. Both terms are regulated by multiplying first by an exponentially damping factor and then switching off after the FT is evaluated. Thus the FT of $V(r; T = 0)$ becomes

$$V(k) = -\sqrt{\frac{2}{\pi}} \frac{\alpha}{k^2} - \frac{4\sigma}{\sqrt{2\pi}k^4}. \quad (3)$$

Finally, the medium modification to the potential in the co-ordinate space yields after taking the inverse FT

$$V(r; T) = \frac{1}{(2\pi)^{3/2}} \int d^3k \frac{V(k)}{\epsilon(k; T)} (e^{ikr} - 1), \quad (4)$$

Let us first revisit the dielectric permittivity, $\epsilon(k)$ in an isotropic hot QCD medium through the diagrammatic approach in real-time formalism, to make a baseline to compare our work in strong magnetic field.

### 2.1 Dielectric Permittivity in isotropic thermal QCD

In the real-time formalism of thermal field theory, the form of propagator generically develops a matrix structure

$$D^0 = \begin{pmatrix} D^0_{11} & D^0_{12} \\ D^0_{21} & D^0_{22} \end{pmatrix}, \quad (5)$$

whose $ij$-elements can be equivalently written in terms of retarded (R), advanced (A) and symmetric (S) propagators in Keldysh representation,

$$\Delta^0_R = D^0_{11} - D^0_{12} , \quad \Delta^0_A = D^0_{11} - D^0_{21} , \quad \Delta^0_S = D^0_{11} + D^0_{22}. \quad (6)$$

Similar representation holds good for the self-energy matrix, so the retarded, advanced and symmetric self-energy can be written from the components of the self-energy matrix $[\Pi_{ij}]$ can be written as

$$\Pi_R = \Pi_{11} + \Pi_{12} , \quad \Pi_A = \Pi_{11} + \Pi_{21} , \quad \Pi_S = \Pi_{11} + \Pi_{22}. \quad (7)$$

Then the full or resummed retarded (advanced) and symmetric propagators can be obtained by resumming the above respective propagators through the Dyson-Schwinger equation,

$$\Delta_{R,A} = \Delta^0_{R,A} + \Delta^0_{R,A} \Pi_{R,A} \Delta_{R,A} , \quad (8)$$

$$\Delta_S = \Delta^0_S + \Delta^0_R \Pi_R \Delta_S + \Delta^0_A \Pi_A \Delta_A + \Delta^0_R \Pi_S \Delta_A , \quad (9)$$

where the symmetric one (9) can be further expressed in terms of retarded and advanced ones

$$\Delta_S(K) = (1 + 2f_B) \text{sgn}(k_0) [\Delta_R(K) - \Delta_A(K)] + \Delta_R(K) [\Pi_S(K) - (1 + 2f_B) \text{sgn}(k_0) [\Pi_R(K) - \Pi_A(K)]] \Delta_A(K) . \quad (10)$$

However, for our problem on the static potential, only the longitudinal component of the resummed propagators will suffice our purpose, so the above resummed propagators (8) and (9) will be specifically

$$\Delta_{L,R,A} = \Delta_{L,R,A}^0 + \Delta_{L,R,A}^0 \Pi_{L,R,A} \Delta_{L,R,A}^0 , \quad (11)$$

$$\Delta^L_S = \Delta^L_S + \Delta^L_R \Pi^L_R \Delta^L_S + \Delta^L_A \Pi^L_A \Delta^L_A + \Delta^L_R \Pi^L_S \Delta^L_A , \quad (12)$$
which can be written for gluons ($\Delta = D^{\mu\nu}$, say) in Breit-Wigner form

\begin{align}
D^L_{R,A}(K) &= \frac{1}{k^2 - \text{Re}\Pi^L_{R}(K) \mp i\text{Im}\Pi^L_{R}(K)}, \\
D^L_S(K) &= \frac{2i \text{Im}\Pi^L_{R}(K)(1 + 2n_B(k_0))\text{sgn}(k_0)}{[k^2 - \text{Re}\Pi^L_{R}(K)]^2 + [\text{Im}\Pi^L_{R}(K)]^2},
\end{align}

wherein the relations between retarded and advanced self-energies for both real and imaginary parts have been used

\begin{align*}
\text{Re}\Pi^L_R(K) &= \text{Re}\Pi^A_R(K), \\
\text{Im}\Pi^L_R(K) &= -\text{Im}\Pi^A_R(K).
\end{align*}

The resummed retarded (or advanced) and symmetric propagators can be inverted to obtain the (real and imaginary parts) elements of full propagator matrix, namely the 11-element

\begin{align}
\text{Re} D^L_{11}(K) &= \text{Re} D^L_R(K), \\
\text{Im} D^L_{11}(K) &= \frac{\text{Im} D^L_S(K)}{2}.
\end{align}

The linear response theory gives the connection between the dielectric permittivity and the static limit of the 11-component of resummed gluon propagator by

\begin{equation}
\frac{1}{\epsilon(k)} = \lim_{k_0 \to 0} k^2 D^L_{11}(k_0, k).
\end{equation}

Thus the real and imaginary parts of 11-component give the respective components of the permittivity

\begin{align}
\frac{1}{\text{Re} \epsilon(k)} &= \lim_{k_0 \to 0} k^2 \text{Re} D^L_R(k_0, k), \\
\frac{1}{\text{Im} \epsilon(k)} &= \lim_{k_0 \to 0} k^2 \frac{\text{Im} D^L_S(k_0, k)}{2}.
\end{align}

We will now calculate the gluon self-energy to resum the propagators. Let us first begin with the form of gluon self-energy tensor ($\Pi^{\mu\nu}$) in vacuum, which could be written as a linear combination of the metric tensor, $g^{\mu\nu}$ and $K^\mu K^\nu$ (with the only four-vector available),

\begin{equation}
\Pi^{\mu\nu}(K) = \left(g^{\mu\nu} - \frac{K^\mu K^\nu}{K^2}\right)\Pi(K^2) \equiv P^{\mu\nu}\Pi(K^2),
\end{equation}

where the only (projection) tensorial basis, $P^{\mu\nu}$ satisfies the four-dimensional transversality condition

\begin{equation}
K_{\mu}P^{\mu\nu} = 0,
\end{equation}

with the additional relation

\begin{equation}
P^{\mu\rho}P_{\rho\nu} = P^{\mu\nu}.
\end{equation}

The above scalar function, $\Pi(K^2)$ is known as the structure factor (self-energy), which depends on the Lorentz invariant quantity $K^2$. 

5
Now bring the vacuum with the contact of a heat reservoir, which defines a local rest frame with four velocity, \( u^\mu = (1, 0, 0, 0) \) and hence breaks the Lorentz symmetry \( O(1, 3) \) of the vacuum into an \( O(3) \) rotational symmetry. Hence a larger basis is necessary and conveniently two orthogonal tensorial basis, \( P_{T}^{\mu \nu} \) and \( P_{L}^{\mu \nu} \) have been adopted compatible for the physical degrees of freedom to express the tensor \([50, 51]\) at finite temperature
\[
\Pi^{\mu \nu}(k_0, k) = \Pi_{T}(k_0, k)P_{T}^{\mu \nu} + \Pi_{L}(k_0, k)P_{L}^{\mu \nu}.
\] (22)

The forms of the basis are constructed as
\[
\begin{align*}
P_{T}^{\mu \nu} &= -g^{\mu \nu} + \frac{k^0}{k^2} (K^\mu u^\nu + u^\mu K^\nu) - \frac{1}{k^2} (K^\mu K^\nu + k^2 u^\mu u^\nu), \\
P_{L}^{\mu \nu} &= -\frac{k^0}{k^2} (K^\mu u^\nu + u^\mu K^\nu) + \frac{1}{k^2} \left( \frac{(k^0)^2}{k^2} K^\mu K^\nu + k^2 u^\mu u^\nu \right),
\end{align*}
\] (23) (24)

to satisfy the 4-dimensional transversality condition
\[
K_\mu P_{T}^{\mu \nu} = K_\mu P_{L}^{\mu \nu} = 0.
\]
In addition, they satisfy the following properties
\[
\begin{align*}
P_{T}^{\mu \rho} P_{T}^{\rho \nu} &= -P_{T}^{\mu \nu}, \\
P_{L}^{\mu \rho} P_{L}^{\rho \nu} &= -P_{L}^{\mu \nu}, \\
P_{T}^{\mu \rho} P_{L}^{\rho \nu} &= 0,
\end{align*}
\]

where the subscripts \( T \) and \( L \) label the transverse and longitudinal modes, respectively with respect to the three-momentum \( (k) \) and is justified by the dot products
\[
\begin{align*}
k_\iota P_{T}^{\iota \mu} &= 0, \\
k_\iota P_{L}^{\iota \mu} &= -\frac{(k^0)^2 k^\mu}{k^2}.
\end{align*}
\] (25) (26)

The structure factors, \( \Pi_{T} \) and \( \Pi_{L} \) are then called transverse and longitudinal components of self-energy tensor, respectively, which depend in the rest frame of the medium on both energy, \( k^0 (=K.u) \) and \( |k| (=k = ((K.u)^2 - K^2)^{1/2} \) separately due to the lack of Lorentz invariance at finite temperature. They are calculated in Hard Thermal Loop (HTL) approximation with the temperature as the hard scale for loop momentum, however, \( \Pi_{T} \) vanishes in the static limit.

In real-time formalism, the longitudinal component of retarded/advanced gluon self-energy tensor had been calculated \([52]\) in HTL perturbation theory\footnote{\( +i\epsilon \) \((-i\epsilon\) prescription is for the retarded (advanced) self-energy}
\[
\Pi_{R,A}^{L}(K) = m_{D}^{2}(T) \left( \frac{k_0}{2k} \ln \frac{k_0 + k \pm i\epsilon}{k_0 - k \pm i\epsilon} - 1 \right),
\] (27)

where \( m_{D}^{2}(T) \) is the leading-order result of the screening mass (also known as Debye mass) for an thermal QCD medium \([53]\) and is given by
\[
m_{D}^{2}(T) = \left( \frac{N_c}{3} + \frac{N_f}{6} \right) g^2 T^2.
\] (28)

Here \( g \) is the running strong coupling and its one-loop expression is given by \([54]\)
\[
\alpha(T) = \frac{g^2(T)}{4\pi} = \frac{6\pi}{(33 - 2N_f) \ln(\frac{Q}{\Lambda_{QCD}})},
\] (29)
with $N_f$ is the number of flavour (we take 3 massless flavours) and $\Lambda_{QCD}$ is scale ($\sim 0.200 \text{ GeV}$) of QCD. The scale, $Q$ is set at $2\pi T$.

The real- and imaginary parts of retarded self energy can thus be extracted as

$$\text{Re } \Pi_{R,A}^L(k_0,k) = m_D^2 \left( \frac{k_0}{2k} \ln \left| \frac{k_0 + k}{k_0 - k} \right| - 1 \right), \quad (30)$$

$$\text{Im } \Pi_{R,A}^L(k_0,k) = -\pi m_D^2 \frac{k_0}{2k}, \quad (31)$$

which could then help to evaluate the resummed retarded (or advanced) and symmetric propagator from the Breit-Wigner formulas (13) and (14), respectively and the real and imaginary-parts of the respective propagators are given by

$$\text{Re } D_{R,A}^L(k_0,k) = \frac{k^2 - m_D^2 \left( \frac{k_0}{2k} \ln \left| \frac{k_0 + k}{k_0 - k} \right| - 1 \right)}{\left( k^2 - m_D^2 \left( \frac{k_0}{2k} \ln \left| \frac{k_0 + k}{k_0 - k} \right| - 1 \right) \right)^2 + \left( \frac{\pi m_D^2 k_0}{2k} \right)^2}, \quad (32)$$

$$\text{Im } D_{S}^L(k_0,k) = -\frac{2T^2 m_D^2 \pi}{k} \left[ \left( k^2 - m_D^2 \left( \frac{k_0}{2k} \ln \left| \frac{k_0 + k}{k_0 - k} \right| - 1 \right) \right)^2 + \left( \frac{\pi m_D^2 k_0}{2k} \right)^2 \right], \quad (33)$$

where $k$ is $|k|$.

Using the equations (32) and (33), the above propagators in the static limit give the real- and imaginary-part of the (complex) dielectric permittivity in a thermal QCD medium

$$\frac{1}{\text{Re } \epsilon(k)} = \frac{k^2}{k^2 + m_D^2}, \quad (34)$$

$$\frac{1}{\text{Im } \epsilon(k)} = -\pi T m_D^2 \frac{k^2}{k(k^2 + m_D^2)^2}, \quad (35)$$

respectively.

### 2.2 Medium modification to $Q$-$\bar{Q}$ potential in a thermal QCD medium

By substituting the dielectric permittivities in (2), we could obtain the complex inter-quark potential in the isotropic hot QCD medium in the coordinate space, whose real-part is ($\hat{r} = r m_D$)

$$\text{Re } V(\hat{r}; T) = \left( \frac{2\sigma}{m_D(T)} - \alpha m_D(T) \right) e^{-\hat{r}} - \frac{2\sigma}{m_D(T)} + \frac{2\sigma}{m_D(T)} - \alpha m_D(T), \quad (36)$$

and the imaginary-part is

$$\text{Im } V(\hat{r}, T) = -\alpha T \phi_0(\hat{r}) - \frac{2\sigma T}{m_D^2} \Psi_0(\hat{r}), \quad (37)$$

where the following functions are,

$$\phi_0(\hat{r}) = \frac{-\hat{r}^2}{9} (-4 + 3\gamma_E + 3 \log \hat{r}),$$

$$\Psi_0(\hat{r}) = \frac{\hat{r}^2}{6} \left( -107 + 60\gamma_E + 60 \log \hat{r} \right) \hat{r}^4.$$
3 Heavy Quarkonia in the presence of strong magnetic field

In the presence of the magnetic field, only quarks, being electrically charged particles, are classically affected by the Lorentz force while gluons remain unaffected. To be precise, in strong magnetic field limit (|qfB| > T2 > m2f), the dominant scale for quarks becomes the magnetic field whereas for gluons the temperature remains the dominant scale even in the presence of strong magnetic field. As a result, in the presence of the strong magnetic field quarks and gluons are treated on different footing, hence the structure functions get decomposed into quark (q) and gluon (g) components. More specifically, the abovementioned structure functions, Π_L and Π_T in the absence of magnetic field will now be ascribed to gluons only and the two new structure functions, Π∥ and Π⊥ (later in (38)) are to be included in the gluon self-energy tensor for the quarks only (the notations, || and ⊥ denote the components along and transverse to the magnetic field, respectively).

3.1 Gluon self-energy tensor in the presence of strong magnetic field

In continuation with abovementioned discussion, the form of the gluon self-energy tensor can be written as [55, 56]

$$\Pi^{\mu\nu}(K) = \Pi^{g,T}_T(K) + \Pi^{g,L}_L(K) + \Pi^{q,||}_L(K) + \Pi^{q,\perp}_L(K),$$  \hspace{1cm} (38)

where $$\Pi^{g,L}$$ and $$\Pi^{g,T}$$ are the structure factors for gluons only and the new two structure factors, $$\Pi^{q,||}$$ and $$\Pi^{q,\perp}$$ appear for quarks only and their evaluation is to be done from the quark-loop. In the presence of the strong magnetic field (in the direction bμ), the rotational invariance of the thermal medium is broken and a much extended tensor basis is required and can be constructed with the help of vectors Kμ, uμ, bμ and the tensor gμν. So in addition to $$P^{\mu\nu}_T$$ and $$P^{\mu\nu}_L$$, two more projection tensors $$P^{\mu\nu}_{||}$$ and $$P^{\mu\nu}_{\perp}$$ have been constructed as [55, 56]

$$P^{\mu\nu}_{||} = -\frac{k_0^2}{k_{||}^2}(b^\mu u^\nu + u^\mu b^\nu) + \frac{1}{k_{||}^2}((k_0)^2b^\mu b^\nu + (k_z)^2u^\mu u^\nu),$$  \hspace{1cm} (39)

$$= -\left(g^{\mu\nu}_{||} - \frac{k_0^2k_z^2}{k_{||}^2}\right),$$  \hspace{1cm} (40)

$$P^{\mu\nu}_{\perp} = \frac{1}{k_{\perp}^2}[-k_\perp^2g^{\mu\nu} + k_0^2 (K^\mu u^\nu + u^\mu K^\nu) - k_z^2 (K^\mu b^\nu + b^\mu K^\nu) + k_0^2k_z^2(b^\mu u^\nu$$

$$+ u^\mu b^\nu) - K^\mu K^\nu + (k_0^2 - (k_z^2)^2)u^\mu u^\nu - K^2b^\mu b^\nu],$$  \hspace{1cm} (41)

$$= -\left(g^{\mu\nu}_{\perp} - \frac{k_0^2k_z^2}{k_{\perp}^2}\right),$$  \hspace{1cm} (42)

with the following notations:

$$u^\mu = (1, 0, 0, 0), \quad b^\mu = (0, 0, 0, -1),$$

$$g^{\mu\nu}_{||} = \text{diag}(1, 0, 0, -1), \quad g^{\mu\nu}_{\perp} = \text{diag}(0, -1, -1, 0),$$

$$K^2 = k_{||}^2 - k_{\perp}^2, \quad k_{||}^2 = (k_0)^2 - (k_z)^2,$$

$$k_{\perp}^2 = (k_x)^2 + (k_y)^2.$$
The tensorial basis satisfy the following properties [56]:

\[ P_{\parallel}^{\mu \rho} P_{\parallel}^{\rho \nu} = -P_{\parallel}^{\mu \nu}, \quad (43) \]
\[ P_{\perp}^{\mu \rho} P_{\perp}^{\rho \nu} = -P_{\perp}^{\mu \nu}, \quad (44) \]
\[ P_{\parallel}^{\mu \rho} P_{\perp}^{\rho \nu} = P_{\perp}^{\mu \rho} P_{\parallel}^{\rho \nu} = 0, \quad (45) \]
\[ P_{T}^{\mu \rho} P_{\perp}^{\rho \nu} = P_{\perp}^{\mu \rho} P_{T}^{\rho \nu} = -P_{\perp}^{\mu \nu}. \quad (46) \]

In the strong magnetic field, quarks are confined only in the lowest Landau level \((n = 0)\), resulting the transverse component of the quark momentum negligibly small \((P_{\perp} = 0)\). Consequently, \(\Pi_{q,\perp}^{0} \approx 0\) [56, 57], so the longitudinal component\(^2\) of the gluon self-energy tensor at finite T and strong magnetic field due to quark \((q)\) and gluon \((g)\) loops is given by

\[ \Pi_{q,L}^{L'}(K) = \Pi_{0,\parallel}^{0}(K) + \Pi_{g,L}^{0}(K), \quad (47) \]

because \(P_{T}^{00}, P_{\perp}^{00} = 0\).

First, we will calculate \(\Pi_{q,\parallel}^{0}\) from the quark-loop up to one-loop. As we know, in strong magnetic field, only the lowest Landau level (LLL) are populated, so the quark propagator in vacuum in the momentum space is restricted to the LLL [58, 59]

\[ iS_{0}(P) = \frac{(1 + \gamma^{0} \gamma^{3} \gamma^{5})(\gamma^{0} p_{0} - \gamma^{3} p_{z} + m_{f})}{p_{\parallel}^{2} - m_{f}^{2} + i\epsilon} e^{-\frac{\rho_{f}^{2}}{m_{f}^{2}}}. \quad (48) \]

Now the above vacuum quark propagator at finite temperature in real-time formalism becomes a matrix

\[ S(P) = \begin{pmatrix} S_{0}(P) + n_{F}(p_{0})(S_{0}^{*}(P) - S_{0}(P)) & \sqrt{n_{F}(p_{0})(1 - n_{F}(p_{0}))(S_{0}^{*}(P) - S_{0}(P))} \\ -\sqrt{n_{F}(p_{0})(1 - n_{F}(p_{0}))(S_{0}^{*}(P) - S_{0}(P))} & -S_{0}^{*}(P) + n_{F}(p_{0})(S_{0}^{*}(P) - S_{0}(P)) \end{pmatrix}, \quad (49) \]

whose 11-element is

\[ iS_{11}(P) = \left[ \frac{1}{p_{\parallel}^{2} - m_{f}^{2} + i\epsilon} + 2\pi i n_{F}(p_{0})\delta(p_{\parallel}^{2} - m_{f}^{2}) \right] \left( 1 + \gamma^{0} \gamma^{3} \gamma^{5})(\gamma^{0} p_{0} - \gamma^{3} p_{z} + m_{f}) \right) \]
\[ \times e^{-\frac{\rho_{f}^{2}}{m_{f}^{2}}}. \quad (50) \]

Thus the 11-component of the quark-loop contribution can be written in strong magnetic field with the above quark propagator in LLL as

\[ \Pi_{q,1}^{\mu \nu}(K) = \frac{ig^{2}}{2} \sum_{f} \int \frac{dp_{\parallel}^{2} dp_{f}^{2}}{(2\pi)^{4}} \text{Tr}[\gamma^{\mu}(1 + \gamma^{0} \gamma^{3} \gamma^{5})(\gamma^{0} p_{0} - \gamma^{3} p_{z} + m_{f})\gamma^{\nu}(1 + \gamma^{0} \gamma^{3} \gamma^{5})] \]
\[ \times (\gamma^{0} q_{0} - \gamma^{3} q_{z} + m_{f}) \left[ \frac{1}{p_{\parallel}^{2} - m_{f}^{2} + i\epsilon} + 2\pi i n_{F}(p_{0})\delta(p_{\parallel}^{2} - m_{f}^{2}) \right] e^{-\frac{\rho_{f}^{2}}{m_{f}^{2}}} \]
\[ \times \left[ \frac{1}{q_{\parallel}^{2} - m_{f}^{2} + i\epsilon} + 2\pi i n_{F}(q_{0})\delta(q_{\parallel}^{2} - m_{f}^{2}) \right] e^{-\frac{\rho_{f}^{2}}{m_{f}^{2}}}. \quad (51) \]

\(^2\)new notation, \(L'\) is due to get rid of confusion from earlier notation of longitudinal component, \(L\) in the absence of magnetic field
Here $g'^2$ is the running strong coupling and runs with the magnetic field only because in strong magnetic limit the magnetic field is the hard scale [60]

$$\alpha'(eB) = \frac{g'^2(eB)}{4\pi} = \frac{1}{(\alpha^0(\mu_0))^{-1} + \frac{11N_c}{12\pi} \ln \left( \frac{k^2 + M_B^2}{\mu_0^2} \right) + \frac{1}{3\pi} \sum_f \frac{|q_f B|}{\sigma}},$$

where

$$\alpha^0(\mu_0) = \frac{12\pi}{11N_c \ln \left( \frac{\mu_0^2 + M_B^2}{\Lambda_V^2} \right)},$$

here $M_B$ is infrared mass. $\Lambda_V$ and $\mu_0$ are taken as 0.385 GeV and 1.1 GeV, respectively and $k_z = 0.1 \sqrt{eB}$.

The above self-energy tensor can be further expressed in terms of trace tensor, $L^{\mu\nu}$

$$\Pi_{11}^{\mu\nu}(K) = \sum_f \int \frac{dp^2_f dp^2_\parallel}{(2\pi)^4} L^{\mu\nu} \left[ \frac{1}{p^2_\parallel - m_f^2 + i\epsilon} + 2\pi i n_F(p_0)\delta(p^2_\parallel - m_f^2) \right] e^{-\frac{q^2_\parallel}{|q_f B|^2}},$$

with

$$L^{\mu\nu} = 8 \left[ p_\parallel^{\mu} q_\parallel^{\nu} + p_\parallel^{\nu} q_\parallel^{\mu} - g^{\mu\nu}(p_\parallel^{\mu} q_\parallel^{\nu} - m_f^2) \right].$$

The the (quark) loop momentum is factorizable into the longitudinal and the transverse component with respect to the direction of the magnetic field, which is consequently translated into the factorization in the external momentum of self-energy tensor as

$$\Pi_{11}^{q,\mu\nu}(K) = \Pi_{11}^{q,\mu\nu}(k_\parallel)B(k_\perp).$$

On integrating over the transverse component of the loop momentum, we get

$$B(k_\perp) = \frac{\pi |q_f B|}{2} e^{-\frac{q^2_\parallel}{|q_f B|^2}},$$

which, in the strong magnetic field limit ($k_\perp \approx 0$), yields into

$$B(k_\perp) = \frac{\pi |q_f B|}{2}.$$

### 3.1.1 Real-part of retarded self-energy

The real-part of the retarded (or advanced) gluon self-energy can be obtained from the real-part of the 11-component of the self energy matrix as

$$\text{Re} \, \Pi_{R,A}^{\mu\nu}(K) = \text{Re} \, \Pi_{11}^{\mu\nu}(K),$$

which can be obtained as the sum of the quarks(q) and gluons(g) loop diagrams. Since the gluon-loop are directly unaffected by the magnetic field, we are now going to calculate the quark-loop only, which is separated into the vacuum (vac) and medium ($n$, $n^2$) contributions

$$\Pi_{R,A}^{q,\mu\nu}(k_\parallel) = \Pi_{R,A}^{q,\mu\nu}(k_\parallel) + \Pi_{R,A}^{q,\mu\nu}(n)(k_\parallel) + \Pi_{R,A}^{q,\mu\nu}(n^2)(k_\parallel),$$

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where,

\[
\Pi_{R,A}^{q,\mu\nu}(k_{||}) = \frac{ig^2}{2(2\pi)^4} \int dp_0 dp_z L^{\mu\nu} \left[ \frac{1}{(q_{||}^2 - m_f^2 + i\epsilon)(q_{||}^2 - m_f^2 + i\epsilon)} \right],
\]

\[
\Pi_{R,A(n)}^{q,\mu\nu}(k_{||}) = -\frac{g^2}{2(2\pi)^3} \int dp_0 dp_z L^{\mu\nu} \left[ n_F(p_0) \frac{\delta(p_{||}^2 - m_f^2)}{(q_{||}^2 - m_f^2 + i\epsilon)} + n_F(q_0) \frac{\delta(q_{||}^2 - m_f^2)}{(p_{||}^2 - m_f^2 + i\epsilon)} \right],
\]

\[
\Pi_{R,A(n^2)}^{q,\mu\nu}(k_{||}) = -\frac{ig^2}{2(2\pi)^2} \int dp_0 dp_z L^{\mu\nu} \left[ n_F(p_0) \delta(p_{||}^2 - m_f^2)n_F(q_0)\delta(q_{||}^2 - m_f^2) \right],
\]

where the vacuum term is calculated as [45, 61–63]

\[
\text{Re} \Pi_{R,A(vac)}^{q,\mu\nu}(k_{||}) = \left( g_{\mu\nu}^{\mu\nu} - \frac{k_{||}^{\mu}k_{||}^{\nu}}{k_{||}^2} \right) \frac{g^2}{2\pi^3} \left[ \frac{2m_f^2}{k_{||}^2} \left( 1 - \frac{4m_f^2}{k_{||}^2} \right)^{-1/2} \left\{ \ln \left( \frac{1 - \frac{4m_f^2}{k_{||}^2}}{1 + \frac{4m_f^2}{k_{||}^2}} \right)^{1/2} - 1 \right\} + 1 \right].
\]

After multiplying the transverse momentum dependent factor (57), the real part of the longitudinal component (labeled as \(||\)) of the vacuum part for the massless quarks \((m_f=0)\) reduces to

\[
\text{Re} \Pi_{R,A(vac)}^{q,||}(k_0, k_z) = \frac{g^2}{4\pi^2} \sum_f |q_f| k_{||}^2.
\]

Next the real part of the longitudinal (\(||\)) component due to the medium contribution having single distribution (\(n\)) function (61) can be written as

\[
\text{Re} \Pi_{R,A(n)}^{q,||}(k_{||}) = \frac{g^2}{2(2\pi)^3} \int dp_0 dp_z L^{00} \left[ n_F(p_0) \left\{ \frac{\delta(p_0 - \omega_p) + \delta(p_0 + \omega_p)}{(q_0^2 - \omega_q^2)(2\omega_p)} \right\} \\
+ n_F(q_0) \left\{ \frac{\delta(q_0 - \omega_q) + \delta(q_0 + \omega_q)}{(p_0^2 - \omega_p^2)(2\omega_q)} \right\} \right],
\]

with the notations

\[
L^{00} = 8[p_0q_0 + p_zq_z + m_f^2],
\]

\[
\omega_p = \sqrt{p_z^2 + m_f^2},
\]

\[
\omega_q = \sqrt{(p_z - k_z)^2 + m_f^2}.
\]

In HTL approximation, the above medium contribution due to the quark-loop for massless flavours vanishes

\[
\text{Re} \ \Pi_{R,A(n)}^{q,||}(k_0, k_z) = 0,
\]

which ought to be because in strong magnetic field limit, QCD for massless flavours is equivalent to two dimensional massless QED (Schwinger model), which is not subject to any temperature/density corrections for the dimensional reason [38, 64].

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Similarly the $n^{2}$-medium contribution (62) does not contribute to the real part
\[
\text{Re} \Pi_{R,A(n^{2})}^{\parallel}(k_{0}, k_{\parallel}) = 0,
\] (70)
which represents the inelastic processes.

Thus the real-part of the longitudinal component of retarded self-energy due to the quark loop simply becomes
\[
\text{Re} \Pi_{R,A}^{q,\parallel}(k_{0}, k_{\parallel}) = \frac{g'^{2}}{4\pi^{2}} \sum_{f} |q_{f}B| \frac{k_{\parallel}^{2}}{k_{0}^{2}},
\] (71)
which, in the static limit ($k_{z} = |k| \cos \beta_{n}$) takes the form
\[
\text{Re} \Pi_{R,A}^{q,\parallel}(k_{0} = 0, k \rightarrow 0) = -\frac{g'^{2}}{4\pi^{2}} \sum_{f} |q_{f}B| \cos^{2} \beta_{n},
\] (72)
where $\beta_{n}$ is the angle between the momentum $k$ and the direction of the anisotropy $n$ (the direction of the $B$).

Therefore, the real-part of the longitudinal component (denoted by $L'$) of the retarded gluon self-energy tensor (38) in strong magnetic field is the sum of quark (72) and gluon contribution (27) in the static limit
\[
\text{Re} \Pi_{R,A}^{L'}(k_{0} = 0, k \rightarrow 0) = \text{Re} \Pi_{R,A}^{q,\parallel}(k_{0} = 0, k \rightarrow 0) + \text{Re} \Pi_{R,A}^{g,\parallel}(k_{0} = 0, k \rightarrow 0),
\]
\[
= -N_{C} \frac{g^{2}T^{2}}{3} - \frac{g'^{2}}{4\pi^{2}} \sum_{f} |q_{f}B| \cos^{2} \beta_{n},
\]
\[
\equiv - \left[ m_{D}^{q,2}(T) + m_{D}^{q,2}(B) \cos^{2} \beta_{n} \right],
\] (73)
where
\[
m_{D}^{q,2}(T) = N_{C} \frac{g^{2}T^{2}}{3}
\] (74)
\[
m_{D}^{q,2}(B) = \frac{g'^{2}}{4\pi^{2}} \sum_{f} |q_{f}B|.
\] (75)

Thus in the presence of strong magnetic field, the Debye mass for the massless quarks in thermal QCD acquires angular dependence.

It is worth to mention here that an angular dependence in the Debye mass could arise from the momentum anisotropy inherited by a medium exhibited, where both quark- and gluon-loop contribute secularly to the anisotropy in the self-energy [40]. On the contrary, the anisotropy manifested in our case could be understood physically from the interaction of intrinsic spin (spin magnetic moment) with the external magnetic field. However, the magnetic field may also induce an anisotropy in the momentum distribution of quarks, $n_{F}(p_{0})$. As a consequence, the quark propagator in (50) becomes anisotropic, which, in turn, makes the self-energy in (51) anisotropic, which, for weak momentum anisotropy, is decomposable into isotropic and anisotropic components. We have found that the anisotropic component in the quark-loop is found to vanish for massless flavours (shown in Appendix A). The vanishing result can be understood by realizing the equivalence between the thermal QCD in strong $B$ in the limit of massless flavours and the massless QED in (1+1)-dimension, which does not allow to have any medium contribution because the momentum anisotropy discussed hereinafore is a medium description.
3.1.2 Imaginary-part of the retarded gluon self-energy

The imaginary part of the retarded gluon self-energy tensor in the real-time formalism is given by [65]

$$\text{Im}\Pi_{R}^{\mu\nu}(K) = \frac{\tanh\left(\frac{k_0}{2}\right)}{\varepsilon(k_0)}\text{Im}\Pi_{11}^{\mu\nu}(K),$$  \hspace{1cm} (76)

which can again be obtained from the quark(q) and gluon (g) loops. Now we calculate the imaginary-part of the quark-loop contribution wherein the vacuum contribution (60) in the massless $m_f = 0$ limit vanishes

$$\text{Im}\Pi_{R(\text{vac})}^{q,\parallel}(k_0, k_z) = 0,$$  \hspace{1cm} (77)

and the medium ($n$-dependent) contribution (61) also does not contribute

$$\text{Im}\Pi_{R(n)}^{q,\parallel}(k_0, k_z) = 0.$$  \hspace{1cm} (78)

The only nonvanishing contribution to the imaginary-part of the retarded self-energy comes out from the the $n^2$-term, which is calculated from the retarded current-current correlator in (1+1)-dimension because the transverse dynamics gets decoupled from the longitudinal dynamics of LLL states in Ref. [38] as

$$\text{Im}\Pi_{R}^{q,\parallel}(k_0, k_z) = -\frac{g'^2}{8\pi}\sum_f |q_f B| k_0 \left[\delta(k_0 + k_z) + \delta(k_0 - k_z)\right],$$  \hspace{1cm} (79)

where the factor $\frac{|q_f B|}{8\pi}$ appears as the transverse density of states for the LLL states.

While, the imaginary part due to the gluon-loop contribution can be obtained from the known result (27)

$$\text{Im}\Pi_{R}^{g,L}(k_0, k) = -\pi m_D^2 \frac{k_0}{2k}.$$  \hspace{1cm} (80)

Therefore, the longitudinal component of the imaginary part of gluon self energy in the presence of strong magnetic field

$$\text{Im}\Pi_{R}^{g,L}(k_0, k) = -\frac{g'^2}{8\pi}\sum_f |q_f B| k_0 \left[\delta(k_0 + k_z) + \delta(k_0 - k_z)\right] - \pi m_D^2 \frac{k_0}{2k},$$  \hspace{1cm} (81)

which in the static limit yields

$$\lim_{k_0 \to 0} \frac{\text{Im}\Pi_{R}^{g,L}(k)}{k_0} = -\frac{g'^2}{8\pi}\sum_f |q_f B| \left[\delta(k_z) + \delta(-k_z)\right] - \pi m_D^2 \frac{1}{2k}.$$  \hspace{1cm} (82)

3.2 Resummed gluon propagator and permittivity

Now, we are in a position to resum the retarded (or advanced) and symmetric propagators in a strong magnetic field. By substituting the real- and imaginary-part of the retarded (and advanced)
self-energy from (73) and (82), respectively, we have calculate the real-part of the resummed retarded propagator from the Briet-Wigner formula (13) in the static limit

$$\text{Re} \ D_{R,A}^L(k_0 = 0, k) = \frac{1}{k^2 + m_D^q \cos^2 \beta_n}.$$  \hspace{1cm} (83)

Similarly, the imaginary-part of the resummed symmetric propagator from Breit-Wigner formula (14) in the static limit can be written as the sum of the quark- and gluon-loop contributions

$$\text{Im} \ D_S^L(k) = \text{Im} \ D_S^{q\parallel}(k) + \text{Im} \ D_S^{g,L}(k),$$

$$= -\frac{Tg^2}{2\pi} \sum_f |q_f B| \left[ \delta(k_x) + \delta(-k_x) \right] \frac{1}{(k^2 + m_D^q \cos^2 \beta_n)^2} \frac{1}{k^2 + m_D^g \cos^2 \beta_n},$$

$$-2\pi T m_D^g \kappa \frac{1}{k^2 + m_D^g \cos^2 \beta_n}.$$  \hspace{1cm} (84)

Therefore, the real and the imaginary-part of the dielectric permittivity are obtained from the real-part of the resummed retarded and the imaginary-part of the symmetric propagators, respectively (15) and (16), where the real-part is

$$\frac{1}{\text{Re} \ \epsilon(k; T, B)} = \frac{k^2}{k^2 + m_D^q \cos^2 \beta_n},$$  \hspace{1cm} (85)

and the imaginary part is written as the sum of the quark and gluon contributions

$$\frac{1}{\text{Im} \ \epsilon(k; T, B)} = \frac{1}{\text{Im} \ \epsilon^q(k; T, B)} + \frac{1}{\text{Im} \ \epsilon^g(k; T, B)},$$  \hspace{1cm} (86)

with

$$\frac{1}{\text{Im} \ \epsilon^q(k; T, B)} = -\frac{Tg^2}{4\pi} \sum_f |q_f B| \left[ \delta(k_x) + \delta(-k_x) \right] \frac{k^2}{(k^2 + m_D^q \cos^2 \beta_n)^2},$$  \hspace{1cm} (87)

$$\frac{1}{\text{Im} \ \epsilon^g(k; T, B)} = -\pi T m_D^g \kappa \frac{k^2}{(k^2 + m_D^g \cos^2 \beta_n)^2}.$$  \hspace{1cm} (88)

### 3.3 Medium modification to $Q\bar{Q}$ potential in a strong magnetic field

We will use the real and imaginary parts of the dielectric permittivity to find the medium modification to the real and imaginary part of the $Q\bar{Q}$ potential, respectively.
3.3.1 Real-part

The real-part of the medium modified potential is

\[ \text{Re } V(r; T, B) = \frac{1}{(2\pi)^{3/2}} \int d^3k \frac{V(k)}{\text{Re } \epsilon(k; T, B)} (e^{i k \cdot r} - 1), \]

\[ = -\frac{\alpha}{2\pi^2} \int d^3k \frac{1}{(k^2 + m_D^2 + m_B^2 \cos^2 \beta_n)} (e^{i k \cdot r} - 1), \]

\[ -\frac{4\sigma}{(2\pi)^2} \int d^3k \frac{1}{k^2(k^2 + m_D^2 + m_B^2 \cos^2 \beta_n)} (e^{i k \cdot r} - 1), \]

\[ \equiv \text{Re } V_C(r; T, B) + \text{Re } V_S(r; T, B), \tag{89} \]

where the Coulomb term is separated as

\[ \text{Re } V_C(r; T, B) = -\frac{\alpha}{2\pi^2} \int d^3k \frac{(e^{i k \cdot r} - 1)}{k^2 + \mu_D^2 + m_B^2 \cos^2 \beta_n}, \]

\[ = -\frac{\alpha}{2\pi^2} \int d^3k \frac{(e^{i k \cdot r} - 1)}{k^2 + \mu_D^2} + \frac{\alpha m_D^2}{4\pi^2} \int d^3k \frac{(e^{i k \cdot r} - 1) \cos 2\beta_n}{(k^2 + \mu_D^2)^2}, \]

\[ \equiv \text{Re } V_C^{(1)}(r; T, B) + \text{Re } V_C^{(2)}(r; T, B), \tag{90} \]

with \( \mu_D^2 = (m_D^2 + m_B^2). \)

Therefore the first term in the Coulomb potential \((\hat{r} = r \mu_D)\)

\[ \text{Re } V_C^{(1)}(r; T, B) = -\frac{\alpha}{2\pi^2} \int d^3k \frac{(e^{i k \cdot r} - 1)}{k^2 + \mu_D^2}, \]

\[ = -\alpha \mu_D e^{i \hat{r} \cdot \hat{r}} - \alpha \mu_D, \tag{91} \]

where the nonlocal term gives the correct limit of the \(V(r; T, B)\) as \(T, B \to 0\). Such term could arise naturally in thermal QCD from the real and imaginary-time correlators and from the basic computations of the real-time static potential in thermal QCD [48, 66].

For evaluating the second term, we first make the transformation with the purpose for converting the anisotropy in the momentum space to the coordinate space as

\[ \cos \beta_n = \cos \theta_r \cos \theta_{kr} + \sin \theta_r \sin \theta_{kr} \cos \phi_{kr}, \tag{92} \]

where \(\beta_n\) and \(\theta_r\) are the angle between \(k\) and \(n\) (in the momentum space), \(r\) and \(n\) (in the coordinate space), respectively. \(\theta_{kr}\) and \(\phi_{kr}\) are the angular variables for the vectors, \(k\) and \(r\), respectively, in the spherical polar coordinate system. Thus the second term in the Coulomb sector

\[ \text{Re } V_C^{(2)}(r, \theta_r; T, B) = \frac{\alpha m_D^2}{4\pi^2} \int d^3k \frac{(e^{i k \cdot r} - 1) \cos 2\beta_n}{(k^2 + \mu_D^2)^2}, \]

\[ = -\frac{\alpha m_D^2}{\mu_D} \left[ \frac{e^{-\hat{r} \cdot \hat{r}}}{\hat{r}^5} \left( \frac{1}{4} + \frac{1}{r^2} + \frac{1}{2} \right) - \frac{1}{r^3} \right] - \frac{3e^{-\hat{r} \cdot \hat{r}}}{\hat{r}^6} \left( \frac{1}{6} + \frac{1}{r^2} + \frac{1}{2} \right) \cos^2 \theta_r \right]. \tag{93} \]
Thus, the Coulomb potential in the presence of strong magnetic field is modified as

\[ \text{Re } V_C(r, \theta_r; T, B) = -\frac{\alpha m_D^2}{\mu_D} \left[ \frac{e^{-\hat{r}}}{\hat{r}} \left( \frac{1}{4} + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{2} + \mu_D^2 \frac{m_D^2}{m_D^2} \right) - \frac{1}{r^3} - \frac{1}{12} + \frac{\mu_D^2}{m_D^2} \right] \]
\[ - \left\{ \frac{3e^{-\hat{r}}}{\hat{r}} \left( \frac{5}{12} + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{2} \right) - \frac{1}{r^4} \right\} \cos^2 \theta_r. \]  

(94)

Similarly, the medium modification to the string part in (89) can be written as

\[ \text{Re } V_S(r; T, B) = -\frac{4\sigma}{(2\pi)^2} \int \frac{d^3k \left( e^{i\mathbf{k} \cdot \mathbf{r}} - 1 \right)}{k^2 (k^2 + \mu_D^2 + \frac{m_D^2}{2} \cos 2\beta_n)} \]
\[ = -\frac{4\sigma}{(2\pi)^2} \int \frac{d^3k \left( e^{i\mathbf{k} \cdot \mathbf{r}} - 1 \right)}{k^2(k^2 + \mu_D^2)} \frac{2\sigma m_D^2}{(2\pi)^2} \int \frac{d^3k (e^{i\mathbf{k} \cdot \mathbf{r}} - 1) \cos 2\beta_n}{k^2(k^2 + \mu_D^2)^2}, \]
\[ \equiv \text{Re } V_S^{(1)}(r, T, B) + \text{Re } V_S^{(2)}(r, T, B), \]  

(95)

where, the first term in the string part is

\[ \text{Re } V_S^{(1)}(r; T, B) = -\frac{4\sigma}{(2\pi)^2} \int \frac{d^3k \left( e^{i\mathbf{k} \cdot \mathbf{r}} - 1 \right)}{k^2(k^2 + \mu_D^2)}, \]
\[ = 2\frac{\sigma}{\mu_D} (\frac{e^{-\hat{r}}}{\hat{r}^2} - \frac{1}{\hat{r}^2}) + \frac{2\sigma}{\mu_D}. \]  

(96)

and using the same transformation (92), the second term is calculated as

\[ \text{Re } V_S^{(2)}(r, \theta_r; T, B) = \frac{2\sigma m_D^2}{(2\pi)^2} \int \frac{d^3k \left( e^{i\mathbf{k} \cdot \mathbf{r}} - 1 \right) (2 \cos^2 \beta_n - 1)}{k^2(k^2 + \mu_D^2)^2}, \]
\[ = \frac{4m_D^2}{\mu_D^3} \frac{\sigma}{\hat{r}} \left[ \frac{e^{-\hat{r}}}{\hat{r}} \left( \frac{1}{2\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{\hat{r}^3} + \frac{1}{8} \right) + \frac{1}{24\hat{r}^4} - \frac{1}{r^4} \right] \]
\[ - \left\{ \frac{3e^{-\hat{r}}}{\hat{r}} \left( \frac{5}{12\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{\hat{r}^3} + \frac{1}{12} \right) + \frac{1}{12\hat{r}^2} - \frac{1}{\hat{r}^4} \right\} \cos^2 \theta_r. \]  

(97)

Thus the medium modification to the string part in the presence of strong B becomes

\[ \text{Re } V_S(r, \theta_r; T, B) = \frac{4m_D^2}{\mu_D^3} \frac{\sigma}{\hat{r}} \left[ \frac{e^{-\hat{r}}}{\hat{r}} \left( \frac{1}{2\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{\hat{r}^3} + \frac{1}{8} + \frac{\mu_D^2}{2\hat{r}^2 m_D^2} \right) + \frac{1}{24\hat{r}^4} - \frac{1}{r^4} - \frac{\mu_D^2}{2\hat{r}^2 m_D^2} \right] \]
\[ + \frac{\mu_D^2}{2\hat{r}^2 m_D^2} \left\{ \frac{3e^{-\hat{r}}}{\hat{r}} \left( \frac{5}{12\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{\hat{r}^3} + \frac{1}{12} \right) + \frac{1}{12\hat{r}^2} - \frac{1}{\hat{r}^4} \right\} \cos^2 \theta_r. \]  

(98)

So the real-part of the medium modified potential consists of central and noncentral components

\[ \text{Re } V(r, \theta_r; T, B) = \text{Re } V_{\text{central}}(r; T, B) + \text{Re } V_{\text{noncentral}}(r, \theta_r; T, B), \]
where the central component is

\[
\text{Re } V_{\text{central}}(r; T, B) = -\frac{\alpha m_D^2}{\mu_D} \left[ \frac{e^{-\hat{r}}}{\hat{r}} \left( \frac{\hat{r}}{4} + \frac{1}{\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{2} + \frac{\mu_D^2}{m_D^2} \right) - \frac{1}{\hat{r}^3} - \frac{1}{12} + \frac{\mu_D^2}{m_D^2} \right]
+ 4\frac{m_D^2}{\mu_D^3} \sigma \hat{r} \left[ \frac{e^{-\hat{r}}}{\hat{r}} \left( \frac{1}{2\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{8} + \frac{\mu_D^2}{2\hat{r}m_D^2} \right) + \frac{1}{24\hat{r}} - \frac{1}{\hat{r}^4} \right]
- \frac{\mu_D^2}{2\hat{r}^2m_D^2} + \frac{\mu_D^2}{2\hat{r}^2m_D^2},
\]

(99)

and the noncentral component is

\[
\text{Re } V_{\text{noncentral}}(r, \theta; T, B) = \cos^2 \theta_r \left[ \frac{\alpha m_D^2}{\mu_D} \left\{ \frac{3e^{-\hat{r}}}{\hat{r}} \left( \frac{5}{12\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{12} \right) + \frac{1}{12\hat{r}^2} - \frac{1}{\hat{r}^4} \right\} \right]
- 4\frac{m_D^2}{\mu_D^3} \sigma \hat{r} \left\{ \frac{3e^{-\hat{r}}}{\hat{r}} \left( \frac{5}{12\hat{r}} + \frac{1}{\hat{r}^2} + \frac{1}{12} \right) + \frac{1}{12\hat{r}^2} - \frac{1}{\hat{r}^4} \right\}.
\]

(100)

It is thus inferred that the strong magnetic field introduces angular dependence into the $Q\bar{Q}$ interaction. To be specific, $QQ$ interaction is more attractive when the $QQ$ pair is aligned transverse to the magnetic field than when the pair is aligned (parallel alignment) along the magnetic field, which is reflected in Figure 1. We have observed that in the presence of the strong $B$, the $Q\bar{Q}$ potential gets less screened compared to its counterpart in the absence of magnetic field, which is due to softening of the screening/Debye masses.
To decipher the effects of strong magnetic field on the (real) potential (in Figure 1) minutely, we postmortem it by decomposing into the Coulomb and string terms in Figure 2. We have found that the (strong) magnetic field affects the string part more than the coulomb part and the effect is more pronounced in the perpendicular alignment. The sting part increases (decreases) in the perpendicular (parallel) alignment whereas the coulomb part increases very slightly in both cases.

The effects of a background magnetic field on the screening of both electric and magnetic fields in the deconfined medium were much earlier studied by computing the electric and magnetic electric screening masses, respectively, by measuring the Polyakov loop correlators on the lattice [42–44]. They found that the magnetic field enhances an increase of both screening masses and in addition, induces an anisotropy in Polyakov loop correlators, which in turn is translated into an anisotropy in $Q\bar{Q}$ interaction. However, the lattice estimates for the electric screening masses are somehow much larger than our results, which may be due to the large nonperturbative effects, beyond the scope of our perturbative framework.

3.3.2 Imaginary-part

We had seen earlier in (86) that the imaginary-part of permittivity is separable into the quark- and gluon-loop contribution. So we first find out the quark-loop (labelled as $q$) contribution (87) to the imaginary-part of the potential, which, however, vanishes

$$\text{Im } V^q(r; T, B) = 0.$$  \hfill (101)

This happens due to the appearance of Dirac delta function, $\delta(k_z)$ in the $q$-contribution to the imaginary part of permittivity. The appearance of delta function can be understood from the
constraint on the motion of the quarks in LLL states, due to the strong magnetic field (z direction). As a subsequent consequence, the dispersion relation for the massless quarks in the LLL states will simply be, \( \omega = \pm k_z \). So, in the static limit, there will be no longitudinal energy-momentum transfer in the inelastic process involving massless quarks.

Next we will calculate the gluon-loop contribution of the permittivity (88), where the Coulomb term is given by

\[
\text{Im } V_C^g(r; T, B) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left( e^{ikr} - 1 \right) \left( -\sqrt{2} \frac{\alpha}{\pi k^2} \right) \left( -m_D^2 \frac{2\pi T k^2}{k(k^2 + m_D^2 + m_D^2 \cos^2 \beta_n)} \right).
\]

\[
= \frac{\alpha m_D^2 T}{2\pi} \int d^3k \left( e^{ikr} - 1 \right) \left( -m_D^2 \frac{m_D^2 T}{k(k^2 + m_D^2)^2} \right) \int d^3k \left( e^{ikr} - 1 \right) \cos 2\beta_n.
\]

\[
\equiv \Psi_1(\hat{r}) + \Psi_2(\hat{r}; \theta_r),
\]

(102)

where \( \Psi_1 \) is given by

\[
\Psi_1(\hat{r}) = \frac{-\alpha m_D^2 T}{\mu_D^2} \phi_0(\hat{r}).
\]

(103)

By substituting the transformation between the angular variables in momentum-space anisotropy and the coordinate-space anisotropy (92), \( \Psi_2 \) is obtained as

\[
\Psi_2(\hat{r}, \theta_r) = -\frac{2\alpha m_D^2 m_D^2 T}{\mu_D^4} \left[ \int_0^\infty \frac{zd\beta}{(z^2 + 1)^3} \left\{ \left( -\frac{\sin(\beta)}{\beta} - \frac{2\cos(\beta)}{\beta^3} + \frac{2\sin(\beta)}{\beta^3} \right) \right\} + \frac{1}{3} \int_0^\infty \frac{zd\beta}{(z^2 + 1)^3} \right].
\]

(104)

which, in addition to \( r \), also depends on the relative orientation of \( Q\bar{Q} \) pair with respect to the magnetic field.

Similarly the imaginary-part to the string term is obtained as

\[
\text{Im } V_s^g(r; T, B) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left( e^{ikr} - 1 \right) \left( -\frac{4\sigma}{\sqrt{2\pi k^4}} \right) \left( -m_D^2 \frac{2\pi T k^2}{k(k^2 + m_D^2 + m_D^2 \cos^2 \beta_n)} \right),
\]

\[
\equiv \Psi_3(\hat{r}) + \Psi_4(\hat{r}; \theta_r).
\]

(105)

where \( \Psi_3(\hat{r}) \) is calculated as

\[
\Psi_3(\hat{r}) = \frac{-2\sigma m_D^2 T}{\mu_D^4} \psi_0(\hat{r}),
\]

(106)

and \( \Psi_4 \) is obtained as

\[
\Psi_4(\hat{r}, \theta_r) = \frac{4\sigma m_D^2 m_D^2 T}{\mu_D^6} \left[ \int_0^\infty \frac{dz}{z(z^2 + 1)^3} \left\{ \left( -\frac{\sin(\beta)}{\beta} - \frac{2\cos(\beta)}{\beta^3} + \frac{2\sin(\beta)}{\beta^3} \right) \right\} + \frac{1}{3} \int_0^\infty \frac{dz}{z(z^2 + 1)^3} \right].
\]

(107)

Thus, like the real-part, the imaginary-part of the potential also consists of central and non-central components

\[
\text{Im } V(r; \theta_r; T, B) = \text{Im } V_{\text{central}}(r; T, B) + \text{Im } V_{\text{noncentral}}(r; \theta_r; T, B),
\]

19
where the central component is written as

\[
\text{Im } V_{\text{central}}(r; T, B) = \frac{\alpha m_{D}^{2} T^2}{9 \mu_{D}^{2}} (-4 + 3 \gamma_{E} + 3 \log r)
- \frac{2 \pi m_{D}^{2} T}{\mu_{D}^4} \left( \frac{r^2}{6} + \frac{(-107 + 60 \gamma_{E} + 60 \log \hat{r}) r^4}{3600} \right)
- \frac{2 \alpha m_{D}^{2} m_{q}^{2} T^2}{\mu_{D}^{4}} \int_{0}^{\infty} \frac{dz}{(z^2 + 1)^{3}} \left( \frac{-\sin (z \hat{r})}{(z \hat{r})} - \frac{2 \cos (z \hat{r})}{(z \hat{r})^2} + \frac{2 \sin (z \hat{r})}{(z \hat{r})^3} + \frac{1}{3} \right) 
- \frac{4 \alpha m_{D}^{2} m_{q}^{2} T^2}{\mu_{D}^{6}} \int_{0}^{\infty} \frac{dz}{z(z^2 + 1)^{3}} \left( \frac{2 \sin (z \hat{r})}{(z \hat{r})} - \frac{2 \cos (z \hat{r})}{(z \hat{r})^2} + \frac{2 \sin (z \hat{r})}{(z \hat{r})^3} + \frac{1}{3} \right) \cos^2 \theta_r. \tag{108}
\]

while the noncentral component is written as

\[
\text{Im } V_{\text{noncentral}}(r, \theta_r; T, B) = \left[ - \frac{2 \alpha m_{D}^{2} m_{q}^{2} T^2}{\mu_{D}^{4}} \left( \int_{0}^{\infty} \frac{dz}{(z^2 + 1)^{3}} \left( \frac{2 \sin (z \hat{r})}{(z \hat{r})} - \frac{6 \cos (z \hat{r})}{(z \hat{r})^2} - \frac{6 \sin (z \hat{r})}{(z \hat{r})^3} \right) \right) \right] \cos^2 \theta_r. \tag{109}
\]

We have now displayed the effect of strong magnetic field on the imaginary part of the heavy quark potential in Figure 3 as a function of interparticle separation \(r\) with respect to the direction of magnetic field for two orientations of \(r\) with respect to the direction of magnetic field (direction of anisotropy itself). It is found that the magnitude of imaginary-part in general gets reduced in strong \(B\) compared to \(B = 0\), which can again be attributed due to the softening of the Debye mass. However, the decrease (in magnitude) is lesser in the transverse direction than in the direction of magnetic field.
4 Properties of Quarkonia in strong $B$ and its dissociation

In order to study how the presence of an external strong magnetic field affects the in-medium properties of $Qar{Q}$ (nonrelativistic) bound states immersed in a hot QCD medium, we have solved the Schrödinger equation numerically with the potential thus obtained in (99) and (100). Since the potential is complex so the real- and imaginary-parts of the potential yield the binding energies and in-medium widths of the bound states in the presence of strong $B$, respectively. Since real-part has both spherical and nonspherical component and the nonspherical (angular) component is very small compared to the spherical component, so we have treated the nonspherical component as a perturbation and calculated the binding energies for the $J/\Psi$ and $\Upsilon$ states in a first-order perturbation theory. The binding energies thus obtained numerically decreases with the temperature (seen in Figure 4), however, its value gets enhanced in comparison to the absence of $B$ value.

We have also calculated the in-medium widths ($\Gamma$) of quarkonia with the imaginary part of the $Q\bar{Q}$ potential in first-order perturbation theory. Assuming the ground states of $c\bar{c}$ and $b\bar{b}$ states (which are $J/\psi$ and $\Upsilon$, respectively) as the Coulombic bound states, the widths are calculated numerically from the relation:

$$\Gamma = -\int d^3r \, |\Psi(r)|^2 \, \text{Im} V(r, \theta_r; T, B),$$

where $\Psi(r)$ is the ground state wave function and is given by

$$\Psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}$$

with $a_0$ as the radius of the first Bohr orbit for $Q\bar{Q}$ bound state. The widths, $\Gamma$'s are found to increase with the temperature (Figure 4), but its magnitude gets reduced compared to $B = 0$ result.

Finally we have studied the quasi-free dissociation of quarkonium states by the competition of binding energies and medium widths, which originate from the real- and imaginary-parts of the potential in a medium, respectively. The quantitative study of the dissociation is made by the above competition between screening and Landau damping, in particular, when the binding energy of a particular quarkonium state ($i$) is equal to the half of its width, i.e. $\text{B.E.}_i = \frac{\Gamma_i}{2}$. Since both quantities (B.E. and $\Gamma$) depend on the temperature and the strength of the strong magnetic field, so the above relation gives the temperature of the hot medium ($T_d$) in a strong $B$ at which the $Q\bar{Q}$ state gets excited and moves to the continuum. We have obtained the $T_d$'s for $J/\Psi$ and $\Upsilon$ as 1.59 $T_C$ and 2.22 $T_C$, respectively.

5 Conclusions and future outlook

In this work, we have have delved into the effects of strong magnetic field on the ground state properties of $J/\psi$ and $\Upsilon$ in a hot QCD medium through the color screening and the Landau damping phenomena. For that purpose, we first thermalize the Schwinger propagator in the lowest Landau level (LLL) and the Feynman propagator for quarks and gluons, respectively to obtain the gluon self-energy for a thermal QCD medium with massless flavours in a strong magnetic field. We found that the medium does not contribute to the quark-loop contribution rather its vacuum contribution yields an angular dependence to the self-energy. This finding can be envisaged by the
equivalence between the massless QED in (1+1) dimension (Schwinger model) with the massless thermal QCD in strong magnetic field, which forbids any medium (finite temperature) correction to the self-energy. Thus the self-energy introduces the angular dependence in the resummed propagators and, hence the permittivity of the medium becomes anisotropic, i.e. behaves like a tensor, which in fact inserts the nonspherical (anisotropic) term in $Q\bar{Q}$ potential. Overall the real part of the potential in the strong $B$ is found more attractive as compared to the thermal medium in the absence of magnetic field ($B = 0$), due to the softening of the Debye masses. Moreover, the $Q\bar{Q}$ potential is more attractive in the transverse alignment as compared to the parallel alignment of the $Q\bar{Q}$ pair with respect to the magnetic field, which is also seen in the lattice studies [42, 43]. On the other hand, the magnitude of the imaginary-part decreases, compared to $B = 0$ case. However, this decrease is less pronounced in the transverse direction than the parallel alignment.

Finally we have solved the Schrodinger equation with the spherical part of the (real) potential numerically to obtain the eigen function, which in turn is used to calculate the first-order correction due to the nonspherical part of (real) of the potential in the time-independent perturbation theory. The binding energies for $J/\Psi$ and $\Upsilon$ thus obtained are found larger in comparison to the $B = 0$ case. Similarly we have also studied numerically the effect of the magnetic field on the medium induced width of quarkonia from the imaginary part of the potential in first-order perturbation theory, which, on the contrary is smaller than in the absence of $B$. Finally, with these inputs on the properties of quarkonia in strong $B$, we have studied the quasi-free dissociation of $J/\psi$ and $\Upsilon$ in the magnetized thermal QCD medium with an optimized criterion on the binding energy and width of a particular resonance - B.E.= Width $(\Gamma)/2$. The dissociation temperatures ($T_d$) are thus found as $1.59T_c$ and $2.22T_c$, respectively, which are larger than the $T_d$’s in the absence of $B$. Thus the presence of strong magnetic field does not favour the early dissolution of quarkonia in the medium.

The nonspherical (anisotropic) interaction in the potential could have consequences on the meson spectrum in heavy-ion phenomenology in the meson spectrum [27, 36, 67–69] because the
perturbation to the energy levels due to the nonspherical interaction may modify their production as well as decay rates etc. As we have noticed that the strong magnetic field affects the string part more than the Coulomb part, especially for the perpendicular alignment of $Q\bar{Q}$ pairs. One of the possible consequence is that the particle production, mainly mesons through the strong breaking could be affected. One of the corollary of the anisotropic interaction may affect the thermalization process, which could be verified through the measurement of the elliptic flow.

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Appendices

A Anisotropic contribution in gluon self-energy

In a magnetic field, the energy levels of a quark ($f$) in vacuum get discretized into Landau ($n = 0, 1, 2, \cdots$) levels as

$$\omega_{f,n}(p_L) = \sqrt{p_L^2 + m_f^2 + 2n|q_f B|}.$$  \hfill (A.112)

However, if the magnetic field is strong enough ($|q_f B| \gg T^2$), quarks are confined to be in the LLL ($n = 0$, due to the large energy gap ($\sim O(\sqrt{eB})$) between LLL and higher Landau levels ($n = 1, 2, \cdots$) and results an anisotropy ($p_L \ll p_T$) in the momentum distribution of quarks with a negative anisotropic parameter $\xi = \frac{p_T^2}{2p_L^2} - 1)$. Therefore, the distribution functions for the quarks $n_F(p_0)$ in (50) can be approximated by the isotropic one, at least, for weak anisotropy ($\xi \ll 1$)

$$n^{\text{aniso}}_F(p) = \frac{1}{e^\beta \sqrt{p^2 - \xi(p.n)^2 + m_f^2}} + \text{...}$$  \hfill (A.113)

where, $p = (0, 0, p_z)$ and $n$ is the direction of the anisotropy, i.e. the direction of magnetic field. For weak anisotropy, we may expand the distribution function (A.113) in the powers of $\xi$ and retain the term linear in $\xi$ only,

$$n^{\text{aniso}}_F(p_0) = n_F(p_0) + \frac{(p.n)^2}{2p_0^2e^\beta} \xi n_F^2(p_0) + \cdots$$  \hfill (A.114)

Therefore, the quark-loop contribution (51) in a strong magnetic field deviates from the same in isotropic medium (38) and is decomposed into isotropic and anisotropic components,

$$\Pi_{11}^{\mu\nu}(k_\parallel) = \Pi_{11,\text{(iso)}}^{\mu\nu}(k_\parallel) + \Pi_{11,\text{(aniso)}}^{\mu\nu}(k_\parallel).$$  \hfill (A.115)

We are now going to calculate the anisotropic contribution up to the order $\xi$

$$\Pi_{11,\text{(aniso)}}^{\mu\nu}(k_\parallel) = \xi \left(I_1^{\mu\nu}(k_\parallel) + I_2^{\mu\nu}(k_\parallel) + I_3^{\mu\nu}(k_\parallel) + I_4^{\mu\nu}(k_\parallel)\right),$$  \hfill (A.116)
where $I_{\mu \nu}^1, I_{\mu \nu}^2, I_{\mu \nu}^3$ and $I_{\mu \nu}^4$ are given by

$$I_{\mu \nu}^1(k_\parallel) = -\frac{g^2}{2} \int \frac{d p_\parallel^2}{(2\pi)^4} L^{\mu \nu} \left( \frac{2\pi n_\parallel^2(p_0) \exp(\frac{|p_0|}{T}) (p.n)^2}{2p_0 T} \right) \delta(p_\parallel^2 - m_\parallel^2), \quad \text{(A.117)}$$

$$I_{\mu \nu}^2(k_\parallel) = -\frac{g^2}{2} \int \frac{d p_\parallel^2}{(2\pi)^4} L^{\mu \nu} \left( \frac{2\pi n_\parallel^2(p_0) \exp(\frac{|p_0|}{T}) (p.n)^2 2\pi i n_F(q_0)}{2p_0 T} \right) \delta(p_\parallel^2 - m_\parallel^2) \times \delta(q_\parallel^2 - m_\parallel^2), \quad \text{(A.118)}$$

$$I_{\mu \nu}^3(k_\parallel) = -\frac{g^2}{2} \int \frac{d p_\parallel^2}{(2\pi)^4} L^{\mu \nu} \left( \frac{2\pi n_\parallel^2(q_0) \exp(\frac{|q_0|}{T}) (q.n)^2}{2q_0 T} \right) \delta(q_\parallel^2 - m_\parallel^2), \quad \text{(A.119)}$$

$$I_{\mu \nu}^4(k_\parallel) = -\frac{g^2}{2} \int \frac{d p_\parallel^2}{(2\pi)^4} L^{\mu \nu} \left( \frac{2\pi n_\parallel^2(q_0) \exp(\frac{|q_0|}{T}) (q.n)^2 2\pi i n_F(p_0)}{2q_0 T} \right) \delta(q_\parallel^2 - m_\parallel^2) \times \delta(p_\parallel^2 - m_\parallel^2). \quad \text{(A.120)}$$

The longitudinal component of the real-part of the anisotropic component (A.115) comes out to be zero

$$\text{Re} \Pi_{11(\text{aniso})}^{q_\parallel}(k_\parallel) = 0, \quad \text{(A.121)}$$

because

$$\text{Re} I_{\mu \nu}^1(k_\parallel) + \text{Re} I_{\mu \nu}^3(k_\parallel) = 0, \quad \text{Re} I_{\mu \nu}^2(k_\parallel) = 0, \quad \text{Re} I_{\mu \nu}^4(k_\parallel) = 0.$$

Similarly the imaginary-part of the anisotropic contribution also vanishes

$$\text{Im} \Pi_{11(\text{aniso})}^{q_\parallel}(k_\parallel) = 0. \quad \text{(A.122)}$$

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