On Termination of Transactions over Semantic Document Models

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Abstract. We consider the framework of Document Modeling, which lays the formal basis for representing document lifecycle in Business Process Management systems. We address the question whether transactions given by a document model terminate on any input. We show that in general this problem is undecidable and formulate a number of sufficient conditions, which guarantee decidability and tractability of computing effects of transactions.

Keywords: Semantic Modeling, document model, transactions, chase

1 Introduction

In [10] a Document Modeling approach has been proposed as a fundamental basis for document processing in Business Process Management Systems (BPMS). Importantly, within this approach basic entities and primitives have been identified, which are common to BPMS such as Enterprise Resource Planning Systems, Customer Relationship Management Systems, etc. The approach rests on the natural idea that document lifecycle lies at the core of these systems. Typically, there is a static part, which describes the forms and statuses of documents (i.e., a schema), and a dynamic part, which describes changes in documents (i.e., transactions over them). In contrast to conventional approaches to BPMS, the approach of the Document Modeling shows that both parts can be given in a fully declarative fashion, thus making programming unnecessary. It suffices to describe the static part of a document model by giving a specification to document forms and fields, and to define the dynamic part via types of transactions, their conditions, and effects. Then, given an initial state of a document model (a collection of documents), the natural problem is to compute a state (an updated collection of documents), which results from the execution of a sequence of transactions. It is argued in the Document Modeling that this problem can be solved with the tools of formal logic such as automated deduction or model checking.

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In [11], the ideas of the Document Modeling have been implemented in a logical framework in terms of the language of the Semantic Programming (aka Semantic Modeling) [1]. It has been shown that the approach of the Document Modeling implemented this way goes beyond the common capabilities of today’s Business Process Management Systems. In particular, it allows to check document models for consistency and solve important problems like projection (i.e., what documents will be created after an accountant performs certain actions) and planning (what actions must be made in order to get an item on stock). The method follows the same line with some of the well-known approaches like Situation Calculus [13] and similar formalisms, but it addresses the topic of Business Process Management, which is a novel area of application for logic-based formalisms.

Obviously, an important question is how the above mentioned problems can be solved computationally. In this respect, the key problem is computing effects of transactions over a document model. Transactions can be fired due to an input of an oracle (a user or an algorithm, which provides some input to a document model), which in turn, can cause other transactions to fire, and so on. Thus potentially, this can result in an infinite chain of updates of a document model, under which a finite resulting state is never obtained. We consider this problem in the paper and formulate a number of complexity results, which demonstrate the expressiveness of document models.

The contributions of this work are as follows. We refine the formalization of the Document Modeling given in [11] and provide a more succinct formalization in an extension of the language of the Semantic Modeling with (non-standard) looping terms. We formulate the problem of transaction termination over document models and show that in general it is undecidable. Actually, with this result we demonstrate that the Document Modelling provides a computational model, which is Turing complete. Then we describe sufficient conditions, which guarantee decidability. For this we introduce a formal definition of a locally simple document theory (the notion previously discussed in [9]) and show that for any such theory the problem of transaction termination is decidable. Then we estimate the complexity of computing effects of transactions and identify a case when computation is tractable.

2 Preliminaries

Document Modeling follows the idea of declarative representation of documents and transactions over them. A document model consists of a description of fields, which can appear in documents (their cardinality and default values), a definition of document forms (given as collections of document fields), and a definition of so called daemons, which specify conditions and effects of document transactions and field triggers. Transactions can be fired on an input of a user or an external procedure (e.g., a Machine Learning algorithm like in [14]), or can be launched by other transactions. Field triggers can be viewed as a special case of transactions, but they can fire only on the event of changing a value of some document field.
The formalism of the Document Modeling includes at least three ingredients that can influence the complexity of computation. The first one is the set of operators over field values. In real-world applications of the Document Modeling, the language is restricted to basic arithmetic operations (like, summation, subtraction, etc.), which can be computed efficiently. For this reason, we do not consider the whole variety of operators over field values in our formalization. We describe only some basic operations, their implementation, and complexity for the purpose of illustration and in order to show that they make no contribution to the complexity of computing transactions. The second ingredient is the query language used in the Document Modeling to describe collections of documents having certain properties. Transactions can refer to document collections given by queries and hence, the complexity of the query language directly influences the complexity of computing transactions. We leave this effect out of the scope of this paper and focus on the complexity of transactions caused solely by their relationship to each other. For this, we adopt a simple query language implemented by predefined document filters, which can be used in the definition of transactions and are computationally simple.

In the remaining part of this section, we introduce basics of the Semantic Modeling and formulate conventions used in our formalization of the Document Modeling. We refer an interested reader to [1]-[5] for details on the Semantic Modeling.

2.1 Basics of the Semantic Modeling

The language of the Semantic Modeling is a first-order language with sorts “urelement” and “list”, in which only bounded quantification of the following form is allowed:

- a restriction onto the list elements \( \forall x \in t \) and \( \exists x \in t \);
- a restriction onto the initial segments of lists \( \forall x \trianglelefteq t \) and \( \exists x \trianglelefteq t \).

where \( t \) is a list term. A list term is defined inductively via constant lists, variables of sort “list”, and list functions given below. A constant list (which can be nested) is built over constants of sort “urelement” and a constant \( \langle \rangle \) of sort “list”, which represents the empty list. The list functions are:

1. **head** – the last element of a non-empty list and \( \langle \rangle \), otherwise;
2. **tail** – the list without the last element, for a non-empty list, and \( \langle \rangle \), otherwise;
3. **cons** – the list obtained by adding a new last element to a list;
4. **conc** – concatenation of two lists;

Terms of sort “urelement” are standard first-order terms. The predicates \( \in, \subseteq \) are allowed to appear in \( \Delta_0 \)-formulas without any restrictions, i.e., they can be used in bounded quantifiers and atomic formulas.

Formulas in the language above are interpreted over hereditarily finite list superstructures \( HW(M) \), where \( M \) is a structure. Urelements are interpreted
as distinct elements of the domain of $\mathcal{M}$ and lists are interpreted as lists over urelements and the distinguished “empty list” $\langle \rangle$. In particular, the following equations hold in every $\text{HW}(\mathcal{M})$ (the free variables below are assumed to be universally quantified):

\[
\begin{align*}
\neg \exists x & \in \langle \rangle \quad \text{cons}(x, y) = \text{cons}(x', y') \rightarrow x = x' \land y = y' \\
\text{tail}(\text{cons}(x, y)) & = x, \quad \text{head}(\text{cons}(x, y)) = y \\
\text{tail}(\langle \rangle) & = \langle \rangle, \quad \text{head}(\langle \rangle) = \langle \rangle \\
\text{conc}(\langle \rangle, x) & = \text{conc}(x, \langle \rangle) = x \\
\text{cons}(\text{conc}(x, y), z) & = \text{conc}(x, \text{cons}(y, z)) \\
\text{conc}(\text{conc}(x, y), z) & = \text{conc}(x, \text{conc}(y, z))
\end{align*}
\]

It was shown in [12] that for any appropriate structure $\mathcal{M}$, there exists a representation of its superstructure of finite lists $\text{HW}(\mathcal{M})$, in which the value of any variable-free list term $t$ can be computed in time polynomial in the size of $t$ (given as as string). Throughout the text, we omit subtleties related to the representation of hereditarily finite structures and we assume w.l.o.g. that for any variable-free list term $t$ one can compute a constant list $t'$ in time polynomial in the size of $t$ such that $\text{HW}(\mathcal{M}) \models t = t'$, for any structure $\text{HW}(\mathcal{M})$. For list terms $t_1, \ldots, t_n$, $n \geq 1$, we will use $\langle t_1, \ldots, t_n \rangle$ as a shortcut for the term $\text{cons}(\text{cons}(\langle \rangle, t_1), t_2) \ldots, t_n \ldots)$. For a list $s$, the notation $|s|$ stands for the number of elements in $s$.

In [6,7,8], the language of the Semantic Modeling has been extended with non-standard list terms, which represent conditional operators (and correspond to the common “if-then-else’ or “switch” constructs of programming languages), bounded list search, and bounded recursion (similar to the restricted “while” operator). We refer to the obtained language as $\mathcal{L}$. The non-standard terms in $\mathcal{L}$ are called Cond-, bSearch- and Rec-terms, respectively, and are defined as follows. By default any standard term in the language of the Semantic Modeling is a $\mathcal{L}$-term and any formula of the language of the Semantic Modeling is a $\mathcal{L}$-formula.

If $t$ and $\theta(\overline{v}, x)$ is a $\mathcal{L}$-term of sort list and $\mathcal{L}$-formula, respectively, then the expression $\text{bSearch}^\mathcal{L}(\theta, t)(\overline{v})$ is a bSearch-term. It is equal to the last element $a$ of $t(\overline{v})$ such that $\theta(\overline{v}, a)$ holds and it is equal to $t(\overline{v})$, otherwise (i.e., if there is no such $a$).

If $\theta_0, \ldots, \theta_n$ are $\mathcal{L}$-formulas and $q_1, \ldots, q_{n+1}$ are $\mathcal{L}$-terms, where $n \geq 0$, then the term $\text{Cond}^\mathcal{L}[\theta_1, q_1][\theta_2, q_2] \ldots [\theta_n, q_n][q_{n+1}](\overline{v})$ is a Cond-term term with the following interpretation:
Finally, if \( f(\overline{v}), h(\overline{v}, y, z) \) and \( t(\overline{v}) \) are \( L \)-terms of sort list then the expression \( \text{Rec}[f, h, t](\overline{v}) \) is a \( \text{Rec-term} \) and its value is given by \( g(\overline{v}, t) \) with the following definition:

\[
\begin{aligned}
g(\overline{v}, \langle \rangle) &= f(\overline{v}) \\
g(\overline{v}, \text{cons}(\alpha, b)) &= h(\overline{v}, g(\alpha), b), \text{ for any lists } \alpha, b \text{ such that } \text{cons}(\alpha, b) \subseteq t
\end{aligned}
\]

In this paper, we refine the formalization of the Document Modeling from [11] in the language of the Semantic Modeling extended with the above mentioned non-standard list terms. In particular, we obtain a more succinct formalization in comparison with [11]. Further in Section 3 we introduce document theories, which formalize the key ingredients of the Document Modeling approach and in the following section we describe several conventions used in our formalization.

### 2.2 Conventions in Formalization of Document Theories

We use the following notions and informal conventions:

- There are pairwise disjoint sets \( \text{FieldNames}, \text{FormNames}, \text{FilterNames}, \) and \( \text{TransNames} \) of constants of sort \text{urelement}, which provide document field, form, and filter names, and transaction names, respectively, which can be used in the axioms of a document theory.

- Natural numbers are modelled in a straightforward way as lists consisting of \( n \) empty lists, for \( n \geq 0 \), and 0 is represented by the empty list \( \langle \rangle \) (we also show how to model real numbers in a decimal representation with a given precision).

- An \textit{instruction} is given as a list of the form \( \langle \text{formName}, \text{CreateDoc} \rangle \) (in which case it is called \text{CreateDoc-instruction}) or \( \langle \text{value}, \text{fieldName}, \text{docID}, \text{SetField} \rangle \) (a \text{SetField-instruction}), or \( \langle \text{params}, \text{docID}, \text{transName} \rangle \) (a \text{transaction}), where \( \text{formName} \in \text{FormNames}, \text{fieldName} \in \text{FieldNames}, \text{transName} \in \text{TransNames} \) and \( \text{docID} \) corresponds to a natural number, and \text{value, params} are some lists, which specify a field value and transaction parameters, respectively.

- A \textit{queue} is a list of instructions to be executed. A queue is updated by daemons, which implement triggers on the events such as changing a field value in a given document or executing a transaction. Creating a new document triggers no events.
A situation is a list of instructions, which represents the history of executed instructions. The last executed instruction appears first in a situation.

A field is given as a list with two elements: ⟨value, fieldName⟩, where fieldName ∈ FieldNames and value is a list, a value for a field. Every field has a default value, which is assigned to it when a new document is created.

A document is a list of fields (the order of fields in the list is arbitrary).

A (document) model is a list consisting of tuples ⟨sit, form, doc, ID⟩, where ID corresponds to a natural number, doc is a document, form ∈ FormNames, and sit a situation. A model stores a version of each document in each situation which has ever taken place. The head of this list is a tuple, in which the situation is the current one, i.e., it consists of instructions (a history) that have given the model.

Situations represent contexts, in which documents are created or modified, and this information can be used in querying a document model. This feature is irrelevant for the results in this paper, but we prefer to keep situations to comply with the original formalization of document models from [11].

A document theory consists of axioms, which specify document fields, forms, filters (i.e., the static structure of documents and query templates), and axioms for the dynamic part. The latter is given by so called daemons (similar to the notion used in process programming), which specify the instructions that must be executed whenever certain event happens (i.e., whenever a value of a specific field in a document is changed or a certain transaction is fired).

3 Document Theories

We define a document theory $T$ as a theory in signature $\Sigma$, where $\Sigma$ consists of the list functions introduced in Section 2.1 and the predicate and function symbols introduced in the axioms below. In particular, $\Sigma$ contains pairwise disjoint finite subsets of constants FieldNames, FormNames, FilterNames, and TransNames, which specify field, form, filter, and transaction names, which can be used in the axioms of $T$. Besides, $\Sigma$ contains function Form, which gives a form name for a document, and the distinguished constants CreateDoc and SetField, ExecTrans, fault, which are used to represent instructions, and fault (similar to exception in programming languages).

We formulate the axioms of $T$ in the language of the Semantic Modeling with non-standard terms. Initially, this language contains only two sorts: urelement and list. For convenience, we will assume that there is also a subsort Real of the sort list, which corresponds to (non-negative) real numbers with a given precision. In the following subsection, we define the sort Real, together with the corresponding predicates and functions, and we show how basic arithmetic operations can be represented in terms of lists. In general, there can be many such implementations, so the next subsection is best viewed as a number of introductory examples to the language of the Semantic Modeling. The only important observation is that the proposed implementation is tractable, as
stated by Lemma 2 in Section 3.2. Throughout the text we assume that all the free variables are universally quantified.

### 3.1 Numeric Terms and Predicates

Let us define $Nat(x) \equiv \forall t \in x \ t = \langle \rangle$. We slightly abuse notation and assume that every natural number $n \in \omega$ is a shortcut for the list consisting of $n$ empty lists and 0 denotes the empty list $\langle \rangle$.

Given $prec \in \omega$, we define a subsort $Real$ of the sort $list$ by the following formula:

$$Real(x) \equiv len(x) = prec \land \forall t \in x \ Nat(t) \land len(t) \subseteq 9$$

where $len(y)$ is an abbreviation for the term $Rec[\langle \rangle, cons(g(\alpha), \langle \rangle), y]$, i.e., $len(y)$ gives the number of elements in a list $y$. In other words, a list of sort $Real$ corresponds to the decimal representation of a real number using $prec$-many digits, for a fixed $prec \in \omega$.

For a list $x$ and $i \in \omega$, let $x.i$ be a shortcut for the term

$$Cond[i = \langle \rangle \lor \neg (i \subseteq len(x)), fault][Rec[\langle \rangle, b, i]]$$

i.e., it gives the constant list $fault$, if $i = 0$ or $i$ is greater than the number of elements in $x$. Otherwise it gives the $i$-th element of $x$.

For lists $x, y$ of sort $Real$, let $x < y$ be a notation for the formula:

$$\exists i \subseteq prec \ (x.i \subseteq y.i \land x.i \neq y.i \land \forall j \subseteq prec \ (i \subseteq j \rightarrow x.j = y.j))$$

i.e., we assume that the first digit of a real number given by a list $x$ is $\text{head}(x)$. The corresponding formula $x \leq y$ is defined similarly.

For a non-empty list $t$, in which every element is of sort $Real$, let $\text{min}(t)$ be a notation for the term

$$Rec[\text{head}(t), Cond[b < g(\alpha), b][g(\alpha)], t]$$

The term $\text{max}(t)$ is defined similarly.

Finally, for lists of sort $Real$, let $x + y$ be a shortcut for the term

$$Cond[\text{tail}(sum) = 1, fault][\text{head}(\text{sum})]$$

where

$$sum \equiv Rec[\langle \langle \rangle, \langle \rangle \rangle, cons(\text{tail}(s), cons(\text{head}(g(\alpha)), \text{head}(s)))], prec],$$

$$s \equiv \text{sumnat}(\text{conc}(\text{conc}(\alpha, b), \text{tail}(g(\alpha))), \text{conc}(\alpha, b)), y.$$

$$\text{sumnat}(x, y) \equiv Cond[\text{conc}(x, y) \geq 10, 1, mod10(\text{conc}(x, y))][\text{conc}(0, \text{conc}(x, y))])$$

and

$$mod10(x) \equiv \text{head}(Rec[\langle \langle \rangle, \langle \rangle \rangle, Cond[\text{tail}(g(\alpha)) = 10,\text{conc}(\text{tail}(g(\alpha)), \text{conc}(\text{head}(g(\alpha)), b))], x])$$
We note that negative reals and other arithmetic operations, e.g., subtraction, multiplication, etc., can be defined in a similar fashion.

Let \( prec \in \omega \) be a given precision and let \( prec = k + m \), where \( k, m \) are some constants, which give the length of the integer/fractional part of real numbers, respectively. For a (non-negative) real number \( n \), let \( \text{dec}(n) \) be the decimal representation of \( n \) such that the number of digits in the integer and fractional part of \( \text{dec}(n) \) is exactly \( k \) and \( m \), respectively, which is achieved by using auxiliary zeros, e.g., for \( n = 3/2 \) and \( k = m = 2 \), we have \( \text{dec}(n) = 01.50 \). If \( \text{dec}(n) \) exists, let \( \text{List}(n) \) be the list representation of \( \text{dec}(n) \), i.e., the list such that \( \text{len}(\text{List}(n)) = prec \) and for all \( i \in \{1, \ldots, prec\} \) and \( j \in \omega \), it holds \( \text{List}(n).i = j \) iff \( j \) is the \((prec + 1 - i)\)-th digit in \( \text{dec}(n) \).

The following lemma sums up properties of the given formalization:

**Lemma 1 (Implementation of Arithmetic with Precision).** Let \( HW(M) \) be a list superstructure and \( prec \in \omega \) a precision. For any (non-negative) real numbers \( a_i \) such that \( \text{dec}(a_i) \) exists, for \( i = 1, \ldots, n \) and \( n \geq 3 \), it holds:

\[
\begin{align*}
- \text{dec}(a_1) \times \text{dec}(a_2) \text{ iff } & \text{HW}(M) \models \text{List}(a_1) \times \text{List}(a_2), \text{ for } \alpha \in \{<, =\} \\
- \text{dec}(a_1) + \text{dec}(a_2) = & \text{dec}(a_3) \text{ iff } \text{HW}(M) \models \text{List}(a_1) + \text{List}(a_2) = \text{List}(a_3) \\
- \text{dec}(\text{dec}(a_1) + \text{dec}(a_2)) \text{ does not exist } & \text{iff } \text{HW}(M) \models \text{List}(a_1) + \text{List}(a_2) = \text{fault}
\end{align*}
\]

For \( n \geq 1 \), the value of \( \min(\langle \text{List}(a_1), \ldots, \text{List}(a_n) \rangle) \) or \( \max(\langle \text{List}(a_1), \ldots, \text{List}(a_n) \rangle) \) in \( HW(M) \) is \( \text{List}(a) \) if \( a \) is minimal/maximal among \( \text{dec}(a_1), \ldots, \text{dec}(a_n) \), respectively.

Throughout the text, we will use the notation \( \bar{n} \) for \( \text{List}(n) \), where \( n \) is a real number such that \( \text{dec}(n) \) exists.

### 3.2 Document and Value Terms

In this section, we introduce notations for terms, which are used to access documents in a document model.

The following term gives the last used ID for a document in a model:

\[
\text{GetLastDocID(model)} \equiv \max(\text{cons}[\text{Rec}(\langle \rangle, \text{cons}(g(\alpha), \text{head}(b)), \text{model})), \emptyset)
\]

i.e., it implements a search for the greatest value occurring as the head of a tuple from \( \text{model} \) and outputs \( \emptyset \) if there are no documents in the model.

The next term gives an actual version of a document (from a model) by its ID. It implements search for the last tuple with a given ID (contained in a model) and outputs the found document. If no tuple with the given ID is present in the model, the term gives \( \text{fault} \).

\[
\text{GetDocByID(docID, model)} \equiv \text{Cond}[\text{doctuple} = \text{model}, \text{fault}[\text{head}(\text{tail}(\text{doctuple}))]]
\]

where \( \text{doctuple} = \text{bSearch}[\text{head}(x) = \text{docID}, \text{model}] \).
The next term provides a field value from the actual (the last) version of a document with a given ID:

\[
\text{GetFieldValue}(docID, fieldName, model) \equiv \\
\text{Cond}\{\text{document} = \text{fault}, \text{fault} \mid \text{bSearch}\{\text{head}(x) = \text{fieldName}, \text{document}\} \}
\]

where \( \text{document} = \text{GetDocByID}(docID, model) \).

Finally, we define the term \( \text{FindFieldPosition} \), which “splits” a given document into a partitioned one (denoted as \( \text{pdocument} \) below), which has the form \( \langle \text{list}_1, \text{list}_2 \rangle \) such that \( \text{conc}(\text{list}_1, \text{list}_2) = \text{document} \) and \( \text{head}(\text{list}_2) \) is a field with the required name (if there exists one in a document). This auxiliary term is employed in the axioms of a document theory to implement change of a field value in an existing document:

\[
\text{FindFieldPosition}(\text{document}, \text{fieldName}) \equiv \\
\text{Cond}\{\text{tail}(\text{pdocument}) = \langle \rangle, \text{fault} \mid \text{pdocument} \}
\]

where

\[
\text{pdocument} = \text{Rec}\{\langle \rangle, \\
\text{Cond}\{\text{head}(\text{tail}(g(\alpha))) = \text{fieldName}, \langle \text{tail}(g(\alpha)), \text{cons}(\text{head}(g(\alpha)), b) \rangle \} \}
\]

Now we define by induction the notion of value term, which generalizes the above introduced definitions.

**Definition 1 (Value-Term).** Any list term is a value term (in particular, any list of sort \( \text{Real} \) is a value-term). If \( s, t, u \) are value terms and \( i \in \omega \) then \( s.i, s + t, \min(s), \max(s), \text{GetLastDocID}(s), \text{GetDocByID}(s, t), \) and \( \text{GetFieldValue}(s, t, u) \) are value-terms. The definition of value-term is complete.

An important property of the above introduced terms is that they are computationally tractable as stated in the following lemma.

**Lemma 2 (Tractability of Value-Terms).** Let \( \text{HW}(M) \) be a list superstructure and \( \text{prec} \in \omega \). For any value-terms \( s(\overline{a}), t(\overline{b}) \) and vectors of constant lists \( \overline{a}, \overline{b} \):

- the value of \( s(\overline{a}) \) in \( \text{HW}(M) \) can be computed in time polynomial in the size of \( s(\overline{a}) \) and \( \text{prec} \);
- \( \text{HW}(M) \models s(\overline{a}) \propto t(\overline{b}) \), where \( \propto \in \{<, =\} \), can be decided in time polynomial in the size of \( s(\overline{a}), t(\overline{b}) \), and \( \text{prec} \).

**Proof.** For list terms, the first point follows from Lemma 2 in [12]. For arbitrary value-term \( t \), the claim is proved by induction on the form of \( t \). By analyzing the syntactic form of the terms \( .i, +, \min(), \max(), \text{GetLastDocID}(), \text{GetDocByID}(), \) and \( \text{GetFieldValue}() \), one can verify that each of these terms
can be computed in polynomial time. The second point of the lemma is shown by a careful analysis of the definition of $\leq$ ($<$, respectively): its form gives a polynomial time algorithm to verify whether there is a segment $i \subseteq prec$, for which the condition from the definition of $\leq$ ($<$, respectively) holds.

Let $s$ be a list term and $m$ a variable-free list term. A value term $t$ is called $(s, m)$-instantiated if $t$ is obtained from some value-term $p(\bar{v})$ by substituting every variable $z$ in a term $GetLastDocID(z), GetDocByID(y, z), GetFieldValue(x, y, z)$ from $p$ with $m$ and by substituting every other variable from $\bar{v}$ with $\text{head}(\text{head}(\ldots \text{head}(s) \ldots))$, for some $k \geq 1$.

### 3.3 Axioms of a Document Theory

A document theory has the form $\mathcal{T} = \mathcal{T}_f \cup \mathcal{T}_s \cup \mathcal{T}_d$, where the theory $\mathcal{T}_f$ gives predefined filters, which can be used to select collections of documents, $\mathcal{T}_s$ gives definitions to document fields and forms (i.e., it describes the data schema, hence, the subscript $s$), and $\mathcal{T}_d$ describes possible transactions and triggers, their execution rules, and instruction processing rules, which generate documents or update existing ones. Thus, $\mathcal{T}_d$ describes the dynamics of documents (hence, the subscript $d$). First, let us introduce auxiliary terms, which will be used in axioms of $\mathcal{T}$. The first one gives a form name of a document

$$Form(document) \equiv \text{head}tail(tail(document)))$$

while the second one gives a list, in which the order of elements is reversed:

$$\text{rev}(list) \equiv \text{Rec}([], \text{conc}(\langle b \rangle, g(\alpha)), list)$$

We begin with a definition of theory $\mathcal{T}_f$. For each $\text{name} \in \text{FilterNames}$, it contains a definition of a filter term of the form below. Every filter gives a list of IDs of (the last version of) those documents from a model, which satisfy conditions specified by the filter:

$$GetDocsByFilter_{\text{name}}(\text{formname, model, } \bar{v}) \equiv \text{head} \text{Rec}([], \text{selection, rev(model)})$$

where $\text{selection}$ is a term of the form

$$\text{Cond} \{ \text{head}(b) \in \text{tail}(g(\alpha)), g(\alpha) \}$$

$$[\text{filter}(\bar{v}, b), \langle \text{cons}(\text{tail}(g(\alpha)), \text{head}(b)), \text{cons}(\text{head}(g(\alpha)), b) \rangle][g(\alpha)]$$

$\text{filter}(\bar{v}, x)$ is a formula, which represents conditions on the documents to be selected:

$$\text{filter}(\bar{v}, document) \equiv Form(document) = \text{formname} \land \varphi$$

where $\varphi$ is a Boolean combination of formulas of the form $s \propto t$, where $\propto \in \{<, =\}$ and $s, t$ are value-terms over variables $\bar{v}$ and $document$ such that in every term $GetLastDocID(m), GetDocByID(x, m), \text{or GetFieldValue } (x, y, m)$ from $s$ or $t$, we have $m = model$. 
Next, we define the theory $T_s$. First of all, it contains axioms that describe fields, cardinalities for their values, and default values:

$$\text{Field}_{\text{name}}(x) \equiv \text{head}(x) = \text{name} \land \text{Card(tail}(x)) \land \text{tail}(x) = \text{defaultvalue} \quad (1)$$

for each $\text{name} \in \text{FieldNames}$, where $\text{Card}(y)$ is a cardinality predicate, which restricts the number of elements in a list $y$, and $\text{defaultvalue}$ is a constant list, which respects the cardinality restriction. We consider the following cardinalities:

- the list is empty
- i.e., the list contains zero or one element (we use notation "?" for such predicate)
- contains exactly one element (we use notation "!"
- contains one or more elements (notation "+")

For example, "?" is defined as

$$?(x) \equiv \forall t \in x \, \text{cons}(\langle \rangle, t) = x$$

The other predicates are defined similarly.

Further, $T_s$ introduces document forms by describing which fields (with their default values) are present in a blank document of a given form:

$$\text{Blank}(\text{name}) = \text{document} \equiv$$

$$\left( \bigwedge_{f \in \text{FormNames}} \text{name} \neq f \land \text{document} = \text{fault} \right) \lor \bigvee_{f \in \text{FormNames}} \varphi_f \quad (2)$$

where $\varphi_f$ is a conjunction of the form

$$\text{name} = f \land \left( \bigwedge_{i \in \text{N}_f} \exists x_i \in \text{document Field}_i(x_i) \right) \land \forall x \in \text{document} \left( \bigvee_{i \in \text{N}_f} x = x_i \right)$$

where $\text{N}_f \subseteq \text{FieldNames}$.

The definition of the theory $T_s$ is complete.

Now we are ready to define the theory $T_d$. It contains definitions of daemons and a definition of a recursive operator, which given a queue, updates a model and the queue to a new state, based on the definition of daemons. First, we define the recursive $\text{Update}$ operator. For the sake of readability, we split its definition into three formulas combined with disjunction and comment on them separately.

First of all, if the queue is not empty and the first instruction in the queue is not a valid one (i.e., it is neither $\text{CreateDoc}$, $\text{SetField}$ instruction, nor a transaction name $t \in \text{TransNames}$) the whole queue is skipped and the model given by the $\text{Update}$ operator is the initial model. If the queue is empty, then it
is assumed that all the instructions in the queue have been processed and thus, \( \text{Update} \) returns the value of \( \text{model} \):

\[
\text{Update}(\text{initialmodel}, \text{model}, \text{queue}) = \text{model}' \equiv \begin{cases} 
(\text{head}(\text{head}(\text{queue})) \notin \{\text{CreateDoc}, \text{SetField}, \text{tname}_1, \ldots, \text{tname}_k\} \land \\
\text{queue} \neq \langle \rangle \land \text{model}' = \text{initialmodel}) \lor \\
(\text{queue} = \langle \rangle \land \text{model}' = \text{model}) 
\end{cases} 
\] (3)

where \( \{\text{tname}_1, \ldots, \text{tname}_k\} = \text{TransNames} \).

Otherwise the queue contains an instruction to create a document of a specific form, change a field value in a document having a certain ID, or launch a specific transaction. In the first case, a blank document of a given form is created (which is implemented by using existential quantification) and added to the model, the instruction is removed from the queue, and the \( \text{Update} \) operator is evaluated recursively on the resulting input. If a blank document of a form with name \( \text{formName} \) can not be created (due to \( \text{formName} \notin \text{FormNames} \)) then the queue is skipped and \( \text{Update} \) returns the initial model:

\[
(\text{head}(\text{head}(\text{queue})) = \text{CreateDoc} \land \exists \text{document} \ \text{document} = \text{Blank}(\text{formName}) \land \\
(\text{document} = \text{fault} \land \text{model}' = \text{initialmodel}) \lor (\text{document} \neq \text{fault} \land \\
\text{model}' = \text{Update}(\text{initialmodel}, \text{cons}(\text{model}, \text{newdoc})), \text{tail}(\text{queue}))) \) \lor (4)
\]

where \( \text{formName} \) stands for \( \text{head}(\text{tail}(\text{head}(\text{queue}))) \), \( \text{newdoc} \) is a list term of the form \( \langle \text{newhistory}, \text{formName}, \text{document}, \text{GetLastDocID}(\text{model}) + 1 \rangle \), \( \text{newsituation} = \text{cons}(\text{Situation}(\text{model}), \langle \text{formName}, \text{CreateDoc} \rangle) \), and \( \text{Situation}(\text{model}) = \text{head}(\text{tail}(\text{tail}(\text{head}(\text{model}))))) \)

The case of \( \text{SetField} \) instruction in the queue is formulated similarly, but the formalization is technically more complex, because modifying an already existing document requires more steps than creating a fresh one:

\[
(\text{head}(\text{head}(\text{queue})) = \text{SetField} \land \\
(\text{pdocument} = \text{fault} \land \text{model}' = \text{initialmodel}) \lor \\
(\text{pdocument} \neq \text{fault} \land \text{model}' = \text{Update}(\text{initialmodel}, \\
\text{cons}(\text{model}, \langle \text{newsituation}, \text{updateddoc}, \text{docID} \rangle), \text{extendedQueue}))) \) \lor (5)
\]

where \( \text{pdocument} \) is an abbreviation for \( \text{FindFieldPosition}(\text{GetDocByID}(\text{docID}, \text{model}), \text{fieldName}) \)

and \( \text{updatedDoc} \) is a shortcut for

\[
\langle \text{tail}(\text{pdocument}), \langle \text{newFieldValue}, \text{fieldName} \rangle, \text{head}(\text{pdocument}) \rangle 
\]

where

\[
\text{docID} = \text{head}(\text{tail}(\text{head}(\text{queue}))) \\
\text{fieldName} = \text{head}(\text{tail}(\text{tail}(\text{head}(\text{queue}))))
\]
newFieldValue = head(tail(tail(head(queue))))
newsituation = ⟨Situation(model), ⟨newFieldValue, fieldName, docID, SetField⟩⟩
Situation(model) = head(tail(tail(head(model))))

(recall the instruction modeling conventions). Finally, \texttt{extendedQueue} is a shortcut for \texttt{SetFieldTrigger(docID, fieldName, newFieldValue, tail(queue), model)}

Thus, \texttt{updatedDoc} is a document with an updated field value and \texttt{extendedQueue} is a sequence of instructions provided by a trigger on a field value change. By the definition above, the whole queue is skipped whenever there is no field with the specified name in a given document. Note that in this case \texttt{tail(pdocument) = fault} holds by the definition of \texttt{FindFieldPosition} term.

Finally, if \texttt{head(head(queue))} is a transaction name, a call to the daemon is made, which defines the corresponding transaction:

\[
\bigvee_{tName \in \text{TransNames}} (\text{head(head(queue))} = tName \land model' = \\
\text{Update(initialmodel, model,} \\
\text{ExecTrans(tName, docID, params, tail(queue), model)))
\]

where \texttt{docID = head(tail(head(queue)))} is a document, for which the transaction is to be executed, and \texttt{params = head(tail(tail(head(queue))))} specifies parameters for the transaction.

Now we define functions, which implement daemons. Their purpose is to extend the queue with a sequence of instructions depending on whether a field value in an existing document is changed or a transaction is fired. Both functions have similar definitions:

\[
\text{SetFieldTrigger(docID, fName, fValue, queue, model) \equiv } \Phi \\
\text{ExecTrans(tName, docID, params, queue, model) \equiv } \Psi
\]

where

\[
\Phi = \text{Cond}[\theta_1, q_1], \ldots, [\theta_n, q_n]\]

and for all \( i \in \{1, \ldots, n\} \), \( \theta_i \) is a condition of the form

\[
\text{Form(GetDocByID(docID)) = formName \land fName = fieldName \land } \varphi
\]

(in this case \( \theta_i \) is called \textit{(formName, fieldName)-condition} and \( q_i = \text{conc(queue, instr}_i) \) such that

- \texttt{formName} \in \texttt{FormNames} and \texttt{fieldName} \in \texttt{FieldNames}
- \( \varphi \) is a Boolean combination of formulas of the form \( p \gg r \), where \( \gg \in \{<, \leq, =\} \) and \( p, r \) are \((fValue), model\)-instantiated value-terms

and \texttt{instr}_i \ (called \texttt{queue extension}) is either \texttt{()} or the list term

\[
\text{cons}(...) \text{cons}(...(\text{cons}(\texttt{()}, s_1), s_2 \ldots), \ldots), s_k)
\]

where
- each $s_i$ (called **instruction**), for $i = 1, \ldots, k$, is a list term $\langle \text{val}, \text{fldName}, \text{docID}, \text{SetField} \rangle$ or $\text{Rec}[\langle \rangle, h, \text{DocFilter}]$ with a definition $g(\langle \rangle) = \langle \rangle$, $g(\text{cons}(\alpha, \text{id})) = h(\text{id})$
- $\text{fldName} \in \text{FieldNames}$ and $\text{val}$ is $\langle \text{fValue}, \text{model} \rangle$-instantiated value-term (then $s_i$ is called **(formName, fldName)-instruction**)
- $\text{DocFilter} = \text{GetDocsByFilter}_\text{name}(\text{frmName}, \text{model}, \mathcal{T})$
- $\mathcal{T}$ is a vector of $\langle \text{fValue}, \text{model} \rangle$-instantiated value-terms
- $h = \text{conc}(\langle \langle \text{params}', \text{id}, \text{transName} \rangle \rangle, g(\alpha))$
- $\text{name} \in \text{FilterNames}$, $\text{frmName} \in \text{FormNames}$, and $\text{transName} \in \text{TransNames}$ (then $s_i$ is called **(formName, transName)-instruction**)
- $\text{params}'$ is a list of $\langle \text{params}, \text{model} \rangle$-instantiated value-terms

The definition of the document theory $\mathcal{T}$ is complete.
We let the size of $\mathcal{T}$ be the total size of its axioms (given as strings).

### 4 Termination of Transactions

The recursive definition of $\text{Update}$ predicate yields a natural notion of chase operator, which given lists $\text{model}, \text{queue}$ computes their update $\text{model}', \text{queue}'$ obtained after processing the first instruction from $\text{queue}$ (i.e., $\text{head}(\text{queue})$). In other words,

$$\text{Update}(\text{model}, \text{model}, \text{queue}) = \text{Update}(\text{model}, \text{model}', \text{queue}')$$

and $\text{model}', \text{queue}'$ are obtained in one step of recursion by the definition of $\text{Update}$ in a document theory $\mathcal{T}$. We denote this fact as $\langle \text{model}, \text{queue} \rangle \mapsto \langle \text{model}', \text{queue}' \rangle$. A chase sequence for a list $\langle m_0, q_0 \rangle$ of the form above is a sequence of lists $\langle m_0, q_0 \rangle, \langle m_1, q_1 \rangle, \ldots$, where $(m_i, q_i) \mapsto (m_{i+1}, q_{i+1})$, for all $i \geq 0$. A chase sequence is terminating if it is of the form $s_0, \ldots, s_k$, for some $k \geq 0$, and there is no $s_{k+1}$ such that $s_k \mapsto s_{k+1}$. In this case $s_k = \langle m_k, q_k \rangle$, where $\text{head}(\text{head}(q_k)) \notin \langle \text{SetField}, \text{CreateDoc}, \text{ExecTrans} \rangle$ (in particular, $q_k$
may be ( ) and \( m_k \) represents a collection of documents, which cannot be further modified (by processing the instruction queue).

First, we note that there may not exist a terminating chase sequence for a given list \( \langle m, q \rangle \) and a theory \( T \). Then we formulate sufficient conditions on the form of \( T \), which guarantee chase termination, and finally we estimate the complexity of computing the chase. Due to space constraints we provide here only proof sketches. Full proofs can be found in the extended version of the paper.

**Theorem 1 (Termination of Transactions is Undecidable).** Given a document theory \( T \) and a list \( s = \langle \text{model}, \text{queue} \rangle \) it is undecidable whether there is a terminating chase sequence for \( s \).

*Proof.* The theorem is proved by a reduction of the halting problem for Turing machines to chase termination. Given a Turing machine \( M \), we define a document theory \( T \), which encodes \( M \). The theory \( T \) contains axioms, which specify a single document form and a field used to represent the content of the tape cells of \( M \), and axioms for daemons, which encode transitions of \( M \). Then we define a list \( \text{initqueue} \) of instructions, which encodes the first two symbols of the initial configuration of \( M \) and enforces execution of a transaction over a document, whose single field contains the initial state symbol \( q_0 \). Then there is a terminating chase sequence for \( \langle \langle \rangle, \text{initqueue} \rangle \) if and only if \( M \) halts.

In fact, the construction of the theory \( T \) in the proof of Theorem 1 shows that non-termination may be caused by the ability to change a field value of the same document or execute the same transaction infinitely many times. Thus, in general the definition of \( \text{SetFieldTrigger} \) and \( \text{ExecTrans} \) functions of \( T \) allows for cyclic references between instructions and transactions. In the following, we observe that if one forbids cycles then chase termination is guaranteed.

**Definition 2 (Dependency Graph).** A dependency graph over a document theory \( T \) is a directed graph with the set of vertices \( V \) equal to \( \text{FormNames} \times (\text{FieldNames} \cup \text{TransNames}) \) and the set of edges \( E \) defined as follows.

For any vertices \((\text{form}, \text{name}), (\text{form}', \text{name}') \in V\), there is an edge from \((\text{form}, \text{name})\) to \((\text{form}', \text{name}')\) if there is a pair \([\theta, q]\) in the definition of \( \text{SetFieldTrigger} \) or \( \text{ExecTrans} \) function in \( T \), where \( \theta \) is a \((\text{form}, \text{name})\)-condition and \( q = \text{conc}(\text{queue}, \text{instr}) \), for a list queue and queue extension instr such that a \((\text{form}', \text{name}')\)-instruction occurs in instr.

**Definition 3 (Locally Simple Document Theory).** A document theory \( T \) is called locally simple if the dependency graph over \( T \) is acyclic.

We now introduce several auxiliary notions, which will be used in the remaining part of the paper.

Given a document theory \( T \), a list \( i \) is called CreateDoc-instruction if \( i = \langle \text{formName}, \text{CreateDoc} \rangle \), where \( \text{formName} \in \text{FormNames} \), \( i \) is called SetField-instruction if \( i = \langle \text{val}, \text{fieldName}, \text{docID}, \text{SetField} \rangle \), where \( \text{fieldName} \in \text{FieldNames}, \text{val} \) is some list, \( \text{docID} \in \omega \). Finally, \( i \) is called transaction if
Let $G$ be a dependency graph over $T$. Given a list model, an a instruction or transaction $i$ of the form above is said to have rank $k$, if

- $i$ is a CreateDoc-instruction and $k = 0$;
- $i = \{ \text{val, fieldName, docID, SetField} \}$, there is a list $d \in \text{model}$ such that $\text{head}(d) = \text{docID}$, $\text{tail}(\text{tail}(d)) = \text{formName}$, where $\text{formName} \in \text{FormNames}$, and the longest path outgoing from the vertex $(\text{formName, fieldName})$ in $G$ has $k$ vertices;
- $i = \{ \text{params, docID, transName} \}$, there is a list $d \in \text{model}$ such that $\text{head}(d) = \text{docID}$, $\text{head}(\text{tail}(\text{tail}(d))) = \text{formName}$, where $\text{formName} \in \text{FormNames}$, and the longest path outgoing from the vertex $(\text{formName, transName})$ in $G$ has $k$ vertices.

Theorem 2 (Local Simplicity Implies Termination of Transactions). For any locally simple document theory $T$, a list model, and a list of instructions $\text{queue}$, there is a terminating chase $c = s_0, \ldots, s_k$, $k \geq 0$, for $s_0 = \{ \text{model}, \text{queue} \}$.

Proof. We show that there is a finite chase sequence $c = s_0, s_1, \ldots, s_m$, where $m \geq 1$, such that $s_0 = \{ \text{model}, \text{queue} \}$, and $s_n = \{ \text{model}', \text{tail}(\text{queue}) \}$, for some list $\text{model}'$ such that $|\text{model}'| = |\text{model}| + p$, for some $p \geq 0$. This yields that any instruction from $\text{queue}$ can be processed in a finite number of steps, from which the claim follows.

Although the theorem above shows that local simplicity guarantees effective computation of chase, it does not give any insight on how complex the computation can be. The next result indicates that the complexity is enormous, which is due to the recursive terms allowed in the definition of transactions.

For a given $n \geq 0$, let $1^{\text{exp}}(n)$ be the notation for $2^n$ and for $k \geq 1$, let $(k+1)^{\text{exp}}(n) = 2^{k^{\text{exp}}(n)}$.

Theorem 3 (Computing Effects of Transactions is Hard). For any $k, n \geq 1$ there exists a locally simple document theory $T$ and a list of instructions $\text{queue}$, both of size polynomial in $k, n$, such that the terminating chase sequence for $s_0 = \{ \text{queue} \}$ has the form $c = s_0, s_1, \ldots, s_m$, where $m \geq \text{1exp}(n)$ and $s_m = \{ \text{model}, \text{queue}' \}$, for a list $\text{queue}'$ such that $|\text{model}| \geq \text{kexp}(n)$.

Finally, we formulate a sufficient condition, which guarantees tractable computation of chase.

Theorem 4 (Computing Effects of Transactions in Polytime). Let $T$ be a locally simple document theory, in which the queue extensions from the definitions of SetFieldTrigger and ExecTrans functions satisfy the following properties:

a. a transaction may appear at most once in a queue extension;
b. if an instruction of the form \texttt{Rec}[^⟨⟩, h, \texttt{DocFilter}] appears in a queue extension, then \( h = \text{conc}[^⟨t⟩, g(α)] \), where \( t \) is a \texttt{SetField}-instruction.

Then for any list \texttt{model} and a list of instructions \texttt{queue}, a terminating chase sequence \( c = s_0, s_1, \ldots, s_n \) for \( s_0 = [\texttt{model}, \texttt{queue}] \) can be computed in time polynomial in the size of \( T \) and \( s \).

\textbf{Proof.} Let \( m \) be the maximal number of instructions in a queue extension from the definition of \texttt{SetFieldTrigger} or \texttt{ExecTrans} functions of \( T \). We show that there is a chase sequence \( c = s_0, s_1, \ldots, s_n \), which can be computed in time polynomial in the size of \( T \) and \( s \), where \( s_n = [\texttt{model'}, \texttt{tail(queue)}] \), \( |\texttt{model'}| = |\texttt{model}| + p \), and both \( p \) and \( n \) are linearly bounded by the size of \( T \) and \( s_0 \).

5 Conclusions

We have shown that document theories (and thus, the Document Modeling approach) provide a Turing-complete computation model even if the language of arithmetic operations (over document field values) and queries (to select collections of documents) is tractable. This confirms that one of the main sources of complexity is the definition of daemons, which specify transactions and relationships between them. If the definition allows us to execute the same transaction or change the value of a document field infinitely many times, then it is possible to implement computations of any Turing machine. We have shown that disallowing cyclic relationships between transactions guarantees decidability of transaction termination (importantly, cycles can be easily detected by the syntactic form of the axioms of a document theory). But the complexity of computing effects of transactions even in the acyclic case is very high, whenever creating documents is allowed in cycles. In fact, using cycles in transactions is natural, since they allow to perform modifications over collections of documents. If documents can be only modified in cycles, but not created then the complexity of computing effects of transactions is decreased and we have shown a case, when it is tractable. In further research, we plan to make a more detailed complexity analysis for various (practical) restrictions onto the definition of transactions. In this paper, we did not consider the contribution of the query language to the complexity of computing transactions. Since transactions use document queries to perform modifications over collections of documents, it would be important to study the interplay between these two sources of complexity.

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