Numerical study of Spherically Symmetric solutions on a Cosmological Dynamical Background using the BSSN Formalism

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We present a fully relativistic numerical method for the study of cosmological problems in spherical symmetry. This involves using the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism on a dynamical Friedmann-Lemaître-Robertson-Walker (FLRW) background. The regular and smooth numerical solution at the center of coordinates proceeds in a natural way by relying on the Partially Implicit Runge-Kutta (PIRK) algorithm described in Montero and Cordero-Carrión [arXiv:1211.5930]. We generalize the usual radiative outer boundary condition to the case of a dynamical background. We show the stability and convergence properties of the method in the study of pure gauge dynamics on a de Sitter background and present a simple application to cosmology by reproducing the Lemaître-Tolman-Bondi (LTB) solution for the collapse of pressure-less matter.

I. INTRODUCTION

Most cosmological models describe spatially isotropic Universe by using the FLRW metric [1]. There are interesting numerous situations, such as the studies of gravitational collapse or the dynamics of voids, in which spacetime is assumed to be spherically symmetric.

These studies are often performed either by assuming the Newtonian limit, in which the expansion of the Universe is considered as a small correction compared to the local gravitational fields and is adequately described by means of Newtonian dynamics, or the so called top-hat approximation, in which the local inhomogeneity assumes a density profile in the shape of a step function. It is then assumed that spacetime inside the spherical object is described by a separated FLRW solution [2]. The first procedure lacks the merit of providing a full general relativistic treatment of the problem. The second relies on debatable assumptions which would need sound reasons to be trusted throughout the whole spherical collapse process yet to be provided.

Some simple cases escape these considerations by resorting to using the LTB solution [3, 4]. This metric, in its original form, only accounts for the collapse of pressureless matter (dust). It is worth noting that recent works extended this solution to the case of a general fluid [5, 6]. In these, the Arnowitt-Deser-Misner (ADM) [7] formulation of General Relativity is used to write equations for the LTB-like degrees of freedom in the form of an initial value problem using a single coordinate chart. As the authors acknowledge, the resulting equations are difficult to solve and probably need a numerical treatment.

On the other hand, progresses in the field of Numerical Relativity over the last two decades have allowed to solve many problems on asymptotically flat spacetimes with great accuracy. In these studies, assuming spherical symmetry reduces the number of spatial dimensions along with the cost of numerical computation. When dealing with spherical coordinates one has to take into account the $1/r^p$ terms close to the center of coordinates $r = 0$. The PIRK methods presented in [8] are an easy way to solve the problem of instabilities without the need to include any further regularization technique. These methods were proved to be helpful in the study of Black Hole evolution with puncture gauge and spherical collapse of Tolman-Openheimer-Volkov solution solving Einstein’s equations in the BSSN formalism [9–11]. Another way of regularizing the solution described for the ADM formalism in [12] involves the inclusion of auxiliary variables and their corresponding evolution equations, which has been applied later successfully to the equations in BSSN formalism in [13]. This strategy is more complex and computer consuming than the PIRK methods.

In this paper, we study the gravitational collapse of a spherically distributed density of matter on a cosmological background. This study provides a fully relativistic treatment of the problem. It must be noted that the consideration of the BSSN formalism on an expanding background has already been studied by Shibata in [14], but for very different purpose.

The formalism used in this paper is developed in some details in Sect. 2. The specifications of the code are presented in Sect. 3, including description of the Numerics, evolution scheme and boundary conditions. The numerical analysis of the code used here is presented in some details in Sect. 4, where it is applied to the study of pure gauge dynamics. In Sect. 5 we show the application of the collapse of pressure-less matter and we compare it with the LTB solution. We use geometrical units in which $G = c = M_\odot = 1$. $\partial_i$ denotes partial derivative with respect to the corresponding variable.
II. FORMALISM

Under the assumption of spherical symmetry, the metric line element can be defined as

\[ ds^2 = -(\alpha - \beta^2)dt^2 + 2\beta dt dr + \psi^4 a^2(t)(\hat{\alpha} dr^2 + \hat{b} r^2 d\Omega^2), \]

where \( \alpha \) is the lapse function, \( \beta \) is the radial component of the shift vector (the only non zero component), \( \hat{\alpha} \) and \( \hat{b} \) are the non zero components of the conformal 3-metric, the conformal factor is written as \( \psi \sqrt{\alpha} \), and all the variables are functions of \( t \) and \( r \). The choice of the lapse function and the shift vector define the foliation of spacetime in spatial hypersurfaces. Following the strategy in [14], we factor out the cosmological scale factor \( a(t) \) from the spatial 3-metric, which only depends on the coordinate time \( t \).

The Einstein’s equations describe the dynamics of spacetime. In all formulations, these equations are split in two groups: the constraint equations and the evolution equations. The BSSN scheme has proved to be very stable and is one of the most used formulations in numerical simulations. The dynamical variables in spherical coordinates have been listed in [13]. We recall them here using slightly different notations to accommodate for the scale factor in the definition of the conformal factor. They consist in the lapse function and the shift vector (the (redefined) conformal metric functions \( \hat{\alpha} \) and \( \hat{b} \). One has to add to this list the components of the extrinsic curvature, \( \dot{K}_{ij} = -\frac{1}{2} \dot{\mathcal{L}}_n a_{ij} \), which can be decomposed in trace \( K \) and conformally-scaled trace-free part \( \hat{A}_{ij} \) as

\[ K_{ij} := \frac{1}{3} \pi_{ij} \pi + \psi^4 a^2 \hat{A}_{ij}. \]

In spherical symmetry, \( \hat{A}_{ij} \) has only two non zero components, \( A_a := \hat{A}_r \) and \( A_b := \hat{A}_\theta \). Since \( \hat{A}_{ij} \) must be traceless, one further has \( A_a + 2 A_b = 0 \).

The great stability of the BSSN scheme is due to the addition of the auxiliary 3-vector \( \hat{A}^i \). This vector has only one component in spherical symmetry which is defined as (see e.g. [13]):

\[ \hat{A}^i = \frac{1}{a} \left[ \frac{\partial_i \hat{\alpha}}{2 a} - \frac{\partial_i \hat{b}}{b} - 2 \frac{r}{a} \left( 1 - \frac{\hat{\alpha}}{b} \right) \right]. \]

This is the last of our dynamical variables.

In what follows, we limit ourselves to the case with zero shift, \( \beta = 0 \). There is no formal difficulty in choosing a different gauge, but for the present purpose this choice allows more straightforward comparison with other cosmological evolutions.

The Einstein’s equations are sourced by the energy content of the spacetime described by the energy-momentum tensor, \( T^{\mu\nu} \). From it, we define the matter source terms as seen by an Eulerian observer with 4-velocity \( n_\mu := (-\alpha, 0) \),

\[ E = n_\mu n_\nu T^{\mu\nu}, \quad j_i = -\gamma_{ij} n_\nu T^{\mu\nu}, \]

\[ S_{ij} = \gamma_{ij} \gamma_\nu T^{\mu\nu}, \]

where \( E, j_i \) and \( S_{ij} \) are the energy density, momentum density and stress energy tensor, respectively. Spherical symmetry reduces the number of such independent quantities to \( E, \gamma^r j_r, S_r := S^r_r \) and \( S_b := S^\theta_\theta \).

The evolution equations for the dynamical variables are [13]:

\[ \partial_t \hat{\alpha} = -2 \alpha \hat{\alpha} A_a, \]
\[ \partial_t \hat{b} = -2 \alpha \hat{b} A_b, \]
\[ \partial_t \psi = -\frac{1}{6} \alpha \psi K - \frac{1}{2} \frac{\dot{\alpha}}{a} \psi, \]
\[ \partial_t K = -\nabla^2 \alpha + \alpha (A_a^2 + 2 A_b^2 + \frac{1}{3} K^2) + 4 \pi \alpha (E + S_a + 2 S_b), \]
\[ \partial_t A_a = -\left( \nabla^2 \alpha - \frac{1}{3} \nabla^2 \alpha \right) + \alpha \left( R^r_r - \frac{1}{3} R \right) + \alpha K A_a - 16 \pi \alpha (S_a - S_b), \]
\[ \partial_t \hat{\Delta} = -\frac{2}{r} (A_a \partial_r \alpha + \alpha \partial_r A_a) + 2 \alpha \left( A_a \hat{\Delta} - \frac{2}{r} (A_a - A_b) \right) + \frac{\xi \alpha}{a} \left( \partial_r A_a - \frac{2}{3} \partial_r K + 6 A_a \partial_r \psi \right) + (A_a - A_b) \left( \frac{2}{r} + \frac{\partial_r \hat{b}}{b} \right) - 8 \pi j_r, \]

(5)

together with the evolution of the scale factor \( a(t) \), and the asymptotic value of the lapse function \( \alpha_{\text{bkg}} \) depending on the chosen slicing of spacetime (see next section for more details). Following [15], we specify \( \xi = 2 \). The above equations feature the radial component of the Ricci tensor, \( R^r_r \), its trace, \( R \), as well as the quantities \( \nabla^2 \alpha \) and \( \nabla^2 \alpha \). Their complete expressions in terms of the dynamical variables are given in the appendix.

The Hamiltonian and momentum constraint equations have to be fulfilled on each spatial hypersurface and are given by

\[ \mathcal{H} \equiv R - (A_a^2 + 2 A_b^2) + \frac{2}{3} K^2 - 16 \pi E = 0, \]
\[ \mathcal{M}^r \equiv \partial_r A_a - \frac{2}{3} \partial_r K + 6 A_a \partial_r \psi + (A_a - A_b) \left( \frac{2}{r} + \frac{\partial_r \hat{b}}{b} \right) - 8 \pi j_r = 0. \]

III. IMPLEMENTATION

A. Numerics

The radial dimension is approximated by a uniformly discretized cell-centred grid and radial derivatives are
computed with a fourth-order finite difference scheme. We use fourth order Kreiss-Oliger dissipation \cite{16}. The evolution equations are solved in time with the PIRK methods \cite{8}, and the applications to the evolution of BSSN variables has been described in \cite{15}. We only present here a short summary. The method involves a splitting of the evolution equations for the dynamical variables as follows:

\[
\begin{aligned}
\partial_t u &= L_1(u, t), \\
\partial_t v &= L_2(u) + L_3(u, v).
\end{aligned}
\] (8)

In a first step of the evolution, \( u \) is numerically evolved in an explicit way. The result is then used to evolve \( v \) partially implicitly, making use of updates values of \( u \) in the evaluation of the \( L_2 \) operator.\(^1\)

The cosmological variables \( a \) and the lapse of the background metric \( \alpha_{\text{bkg}} \) are first evolved explicitly along \( \dot{a}, \dot{b}, \psi \) and \( \alpha \). The updated values are then used to evolve \( \dot{a}, K \) and \( A_a \) partially implicitly. Finally, the update values are used to evolve \( \hat{\Delta}^r \) partially implicitly.

**B. Cosmological evolution and boundary conditions**

Regularity of the dynamical variables close to the origin is enforced, in part, by specifying their parity across the origin. To achieve this in time, a few virtual points of negative radius are added to the numerical grid.

The considered spacetimes are not asymptotically flat. Instead, they tend to the cosmological FLRW solution. It is important to note that this work takes the cosmological solution as a homogeneous background on which the local inhomogeneous fields has no influence.

The Friedmann and acceleration equations in the zero-shift gauge with arbitrary lapse are given by

\[
\begin{aligned}
\frac{1}{\alpha_{\text{bkg}}^2} \left( \frac{\dot{a}}{a} \right)^2 &= \frac{8\pi}{3} \rho_{\text{bkg}}, \\
\frac{1}{\alpha_{\text{bkg}}^2} \frac{\dot{a}}{a} &= -\frac{8\pi}{6} (\rho_{\text{bkg}} + 3p_{\text{bkg}}),
\end{aligned}
\] (9)

(10)

where \( \rho_{\text{bkg}} \) and \( p_{\text{bkg}} \) denote the homogeneous background energy density and pressure, respectively.

We impose radiative boundary conditions at the outer boundary

\[
\partial_t f = \partial_t f_{\text{bkg}} - v \partial_r f - \frac{v}{r}(f - f_{\text{bkg}}),
\] (11)

where \( v \) is the speed of propagation of the variable \( f \) on the grid. This is inferred by considering the characteristic structure of the variables of the evolution system of equations. In the above \( f_{\text{bkg}} = f_{\text{bkg}}(t) \) denotes the spatially homogeneous asymptotic cosmological value of the variable \( f \) and \( \partial_t f_{\text{bkg}} \) its first time derivative. These expressions can be read from their asymptotic values

\[
\dot{a}(t, r), \dot{b}(t, r), \psi(t, r) \to 1, \\
\alpha(t, r) \to \alpha_{\text{bkg}}(t).
\] (12)

From the definition of the extrinsic curvature tensor, one can derive that

\[
A_a(t, r), A_\psi(t, r) \to 0, \\
K(t, r) \to -3 \frac{1}{\alpha_{\text{bkg}} a} \dot{a}.
\] (13)

and from the definition of the \( \hat{\Delta}^r \) variable one has that

\[
\hat{\Delta}^r \to 0.
\] (14)

**IV. CODE VALIDATION: PURE GAUGE DYNAMICS**

**A. Equations**

In order to validate our numerical code, we consider here pure gauge dynamics on a dynamical de Sitter background. This is the solution for a Universe filled with a constant homogeneous vacuum energy density with equation of state \( p_{\text{bkg}} = -\rho_{\text{bkg}} \). This is equivalent to adding a cosmological constant \( \Lambda \) to the Einstein’s equations such that \( \Lambda = 8\pi \rho_{\text{bkg}} \).

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Initially, we set

\[
\alpha(t = 0) = \alpha_{\text{bkg}}^0 + \frac{\alpha_0 r^2}{1 + r^2} \left[ e^{-(r - r_0)^2} + e^{-(r + r_0)^2} \right],
\] (15)

where \( \alpha_{\text{bkg}}^0 = \alpha_{\text{bkg}}(t = 0) \) and \( \alpha_0 \) is a constant which determines the amplitude of the gaussian perturbation. Setting \( \alpha_{\text{bkg}}^0 = 1 \) in (9) allows us to define the initial Hubble factor \( H_0 := \ddot{a}(t = 0) = \frac{\dot{a}_0}{a(t = 0)} \). Note that, since the energy density remains constant, one has

\[
\frac{1}{\alpha_{\text{bkg}} a} \dot{a} = H_0, \forall t.
\] (16)

The energy component is at all time equal to the constant homogeneous cosmological density:

\[
E = \rho_{\text{bkg}} = \frac{3}{8\pi} H_0^2.
\] (17)

---

\(^1\) The discrete evolution scheme used in the present paper is a second-order PIRK method, and involves a two-stage method described in details in \cite{8, 15}.
All the dynamical variables must fulfill both the Hamiltonian and the Momentum constraints. By comparison with the homogeneous cosmological case, we set

\[ \dot{a}(t = 0) = \dot{b}(t = 0) = \dot{\psi}(t = 0) = 1. \] (18)

These assumptions and the fact that \( A_b = -\frac{1}{2}A_a \) imply that the Hamiltonian and momentum constraints respectively reduce to

\[ \frac{3}{2} A_a^2 + \frac{2}{3} K^2 - 6H_0^2 = 0, \] (19)
\[ \partial_\tau A_a - \frac{2}{3} \partial_\tau K + 3 \frac{A_a}{r} = 0. \] (20)

Interestingly, upon setting \( x = 3A_a \), \( y = 2K \) these two equations can be rewritten as

\[ x^2 + y^2 = 36 H_0^2, \] (21)
\[ \partial_\tau x - \partial_\tau y + 3 \frac{x}{r} = 0, \] (22)

the former being the implicit equation of a circle of radius \( 6H_0 \). The general solution of these equations can be given in an implicit form in terms of a variable \( \theta \) by defining \( x = 6H_0 \cos \theta \), \( y = 6H_0 \sin \theta \). One finds:

\[ -e^\theta \cos \theta = Cr^3, \] (23)

with \( C \) an integration constant. The most trivial solution (and the only one in which the range of the coordinate radius \( r \) is \( [0, +\infty) \)) involves setting \( C = 0 \) and, therefore, \( \cos \theta = 0 \). This corresponds to

\[ K = \pm 3H_0, \quad A_a = 0. \] (24)

The minus sign is chosen in agreement with the cosmological expression for the background.

The evolution of the gauge dynamics is performed in the harmonic gauge slicing in which the evolution equation for the lapse is

\[ \partial_\tau a = -a^2 K. \] (25)

This choice of gauge for the entire domain also fixes the gauge of the cosmological background dynamics. In addition to equations (23) and (24), one thus also needs to solve

\[ \dot{a}_{\text{bkg}} = 3a_{\text{bkg}} \dot{a}. \] (26)

The only two independent variables for the background are \( a \) and \( a_{\text{bkg}} \). We choose to solve (10) and (26). Upon inserting (16), these equations become

\[ \dot{a}_{\text{bkg}} = 3a_{\text{bkg}} H_0, \] (27)
\[ \ddot{a} - \frac{\dot{a}}{a} = 4a_{\text{bkg}} H_0. \] (28)

We then use (16) to monitor the error on these equations in the manner of a constraint equation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Evolution of a pure gauge pulse in time on a de Sitter background with \( H_0 = 0.01 \). The asymptotic value of \( \alpha \) gets rescaled during the evolution.}
\end{figure}

\section*{B. Results}

We now come to discuss the stability of the scheme in the same terms as in [15]. This allows more straightforward comparisons.

For values of the initial expansion factor of the order \( H_0 \sim 10^{-3} \) or smaller, the exponential de Sitter expansion remains linear for time scales up to \( t \sim 10 \). In this case, the changes are small compared to the analysis carried out in [13, 15]. The code proceeds without difficulty yielding similar results for a value of the Courant-Friedrichs-Lewy (CFL) factor \( \Delta t/\Delta r = 0.5 \).

In what follows, we set \( H_0 = 0.01 \) and \( \alpha_0 = 0.01 \), that is well in the exponential regime of the cosmological expansion.\footnote{In comparison, the value of \( H_0 \) for our Universe expressed in the units of this paper is of the order of \( \sim 10^{-23} \).} To proceed with such large values of the expansion, the CFL factor must be reduced. The results performed in this section have been obtained with \( \Delta t/\Delta r = 0.25 \).

The dynamics of the lapse in the harmonic slicing (25) is that of a wave. As expected from the initial data, the initial gauge pulse splits in two parts travelling in opposite directions. One sees from Fig. 1, which shows the radial profile of \( \alpha \) for different values of the time, that the continuous background follows the evolution of \( \alpha_{\text{bkg}} \) imposed at the outer boundary condition and plotted in Fig. 2 as a function of time.

Fig. 3 shows the \( L^2 \)-norm (Root-Mean-Square) of the Hamiltonian and momentum constraints with a resolution of \( \Delta r = 0.05 \) as a function of time. The error hits a maximum when the left pulse hits the inner boundary. It goes down after it has bounced back and both pulses are travelling outward.
FIG. 2. Evolution of the scale factor $a(t)$ (upper panel) and background lapse function $\alpha_{\text{bkg}}$ (lower panel).

FIG. 3. $L^2$-norm of the Hamiltonian (upper panel) and the Momentum constraints (lower panel) in pure gauge dynamics ($\Delta r = 0.05$).

In Fig. 4 we have plotted the Hamiltonian constraint for three values of the resolution. The rescaling of the error when resolution is doubled proves the good agreement with the expected second-order convergence of the numerical method. The curves shown here display a striking resemblance with the similar quantity obtained in [15] on a flat background. This implies that the expansion of the background leads only to small changes in the dynamics of the error. Aside for the fact that the CFL factor used here is smaller than 0.5, another important difference with the simulations in [15] is that the convergence regime is attained for higher resolutions. This can be seen by looking at the Hamiltonian constraint profile shown in the inner plot of Fig. 4. The latter not being rescaled, its magnitude is of the correct order of magnitude but the profile is slightly different to those of the other curves.

V. APPLICATION: COSMOLOGICAL SPHERICAL COLLAPSE

A. Lemaître-Tolman-Bondi solution

We now apply the code to the study of the spherical collapse of pressure-less matter (dust). This case is of practical interest in cosmology. It is usually studied using geodesic slicing gauge condition, $\alpha = 1$. In this gauge, the most general solution to the Einstein’s equations is the so-called Lemaître-Tolman-Bondi (LTB) solution. It can be summed up in the form of the metric line element

$$\text{ds}^2 = -dt^2 + a_\parallel^2(\alpha_{\text{ltb}}(r), t, r) + a_\perp^2(\alpha_{\text{ltb}}(r)) \frac{r^2}{\Delta r^2},$$

(29)

with $a_\parallel = \partial_r(r a_\perp)$ and where $E_{\text{ltb}}(r)$ is a free function.

The inhomogeneous counterparts of the Friedmann and acceleration equations are

$$\frac{\dot{a}_\parallel^2}{a_\parallel^2} = \frac{M(r)}{a_\parallel^3} + \frac{2E_{\text{ltb}}(r)}{a_\parallel^2},$$

(30)

$$\frac{\ddot{a}_\perp}{a_\perp} = -\frac{M(r)}{2a_\perp^2},$$

(31)

in which $M$ is another free function related to the energy density through

$$8\pi \rho = \frac{\partial_r(M r^3)}{a_0^3 a_\perp^3 r^2}.$$ 

One of the main interests of using the geodesic slicing resides in the simplicity of the solution of the evolution
equation for the dust density. Indeed, conservation of energy implies

\[ \partial_t \rho + \frac{1}{2} (\gamma^{rr} \partial_r \gamma_{rr} + 2 \gamma^{\theta\theta} \partial_t \gamma_{\theta\theta}) \rho = 0. \]  

(33)

In terms of the LTB metric components, the solution of previous equation reads

\[ \rho = \rho_0 \left( \frac{a}{a_\perp} \right)^2, \]

(34)

where \( \rho_0 \) is the initial density profile. It can be shown that this is equivalent to Eq. (32).

\[ \rho = \rho_0 \left( \frac{a}{a_\perp} \right)^2, \]

B. Initial Data

Building the initial data for the evolution of the LTB spacetime involves specifying an initial profile for three functions amongst \( a_\perp, a_\parallel, E_{\text{ltb}}, \rho \) and \( M \). The remaining variables can then be inferred from Eqs. (30) and (32). We wish to compare the evolution in the LTB and BSSN variables. We choose then to build the initial data from the constraints in the BSSN formulation and compute their equivalent in terms of the LTB variables.

To allow direct comparison, we choose as gauge variable \( \alpha = 1 \). The initial values of the variables defined in Sect. II are

\[ \hat{a}(t = 0) = \hat{b}(t = 0) = 1, \quad K(t = 0) = -3H_0, \]
\[ A_\parallel(t = 0) = A_\parallel(t = 0) = 0, \]
\[ E(r, t = 0) = [1 + \delta_m(r)] \rho_0^{0\text{bkg}}, \]

(35)

where \( \rho_{0\text{bkg}} = \rho_{\text{bkg}}(t = 0) \). In our work, we choose a density contrast profile in the form of a bump function,

\[ \delta_m(r) = \delta_m^0 \exp \left( -\frac{r^2}{r_0^2 - r^2} \right), \]

(36)

where \( \delta_m^0 \) and \( r_0 > 0 \) are constants. This profile has the property of being smooth and to have a compact support spanning the region \([0, r_0] \). Other profiles have been used though not documented here.

In particular, the choice of \( K \) and \( A_\parallel \) imposes that

\[ H_0 = \hat{a}_0 \frac{\hat{a}}{a_0} = \gamma^{rr} \partial_r \gamma_{rr} |_{r = 0} = \gamma^{\theta\theta} \partial_t \gamma_{\theta\theta} |_{t = 0}. \]

(37)

The equation for the initial value of the conformal factor is found by plugging initial conditions from Eqs. (35) in the Hamiltonian constraint, which reduces to

\[ a^{-2} \psi^{-5} \left( \partial_r^2 \psi + \frac{2}{r} \partial_r \psi \right) + 6H_0^2 = 16\pi \rho_{0\text{bkg}}^0 (1 + \delta_m^0(r)). \]

(38)

Using Friedmann equation, previous expression becomes

\[ \partial_r^2 \psi + \frac{2}{r} \partial_r \psi = 16\pi \rho_{0\text{bkg}}^0 \delta_m^0(r) a_\perp^2 \psi^5. \]

(39)

FIG. 5. Initial conformal factor in the case of a dust matter over-density of central value \( \delta_m^0 = 0.1 \) (plain line). The solution agrees well with the asymptotic value imposed as boundary condition (dotted line).

Following [14], this equation is solved numerically as a boundary value problem with conditions

\[ \partial_r \psi \to 0, \quad \text{for } r \to 0; \]
\[ \psi \to 1 + \frac{C_\psi}{2r}, \quad \text{for } r \to \infty. \]

The parameter \( C_\psi \) is adjusted by specifying an additional outer boundary condition:

\[ \partial_r \psi \to -\frac{C_\psi}{2r^2}. \]

(42)

The solution to the initial boundary value problem is shown in Fig. 5 for \( \delta_m^0 = 0.1 \) and \( r_0 = 5 \) (plain line). Its behavior agrees well with the imposed asymptotic solution (dashed line).

In terms of the BSSN variables, taking into account that \( \hat{a} \hat{b}^2 = 1 \) always holds, the solution of Eq. (33) is:

\[ \rho = \rho_0 \left( \frac{a_\parallel^0 \psi_0^6}{a_\parallel^0 \psi_0^6} \right), \]

(43)

where \( \psi_0 = \psi(t = 0) \). This expression generalizes the rescaling equation of dust in cosmology to the case of a non-homogeneous spacetime.

We now come to set the initial data in terms of the LTB variables. Since the analysis is performed in the zero-shift gauge, it can be assumed that the radius coordinates of both ansatz of the metric should only differ up to a constant factor throughout the integration. Fixing this factor equal to one in the initial data allows to compare the metric components themselves between both methods.

From the decomposition introduced in Eq. (1) and using the fact that \( a_\parallel = \partial_r (r a_\perp) \), one obtains the initial
values of $a_\perp$ and $a_\parallel$ by differentiation:

$$a_\perp^0 = \psi_0^3 a_0,$$

$$a_\parallel^0 = \psi_0^2 a_0 + 2\psi_0 \frac{d\psi_0}{dr} a_0 r. \quad (45)$$

By comparing the radial part of the spatial metric, one then finds the form of the energy function $E_{\text{ltb}}(r)$ of (29). In agreement with (37), the initial time derivatives are

$$a_\perp^0 = a_0^0 H_0, \quad a_\parallel^0 = a_0^0 H_0. \quad (46)$$

Using Eq. (30), $M(r)$ is deduced and can then be used for the evolution of $\dot{a}$ using Eq. (31).

**C. Evolution and Results**

The background evolution proceeds in the same way as for the case of gauge dynamics. We solve the acceleration equation, which in the geodesic slicing and in presence of dust only reduces to

$$\frac{\ddot{a}}{a} = -\frac{8\pi}{6} \rho_{\text{bkg}}. \quad (47)$$

The homogeneous part of the dust energy density is evolved simply as $\rho_{\text{bkg}} = \rho_{\text{bkg}}^0 a_0^3 / a^3$.

Fig. 6 shows the result of the evolution of the $\gamma_{rr}$ and $\gamma_{\theta\theta}$ 3-metric components using the LTB variables (lines) and the BSSN equations (crosses and circles) for different values of the coordinate time $t$, up to $t = 15$. The shape of the curves remains basically unchanged for subsequent values of $t$. The simulation has been performed using $H_0 = 0.1$ and $\Delta r = 0.1$. The maximum of the relative difference between the curves shown in Fig. 6 is of the order $\sim 10^{-5}$ and is lower with higher resolution.

The long term stability analysis of the code is better analyzed by looking at the evolution of the $L_2$-norm of the Hamiltonian constraint, displayed in Fig. 7 for different resolutions. We obtain similar shapes in all the curves. The difference in magnitude despite the rescaling indicate an order of convergence above second-order. We except this to be a result of the fact that the evolution of matter is simple enough to make the dominant error come partially from the finite difference scheme used to compute spatial derivatives (fourth-order) rather than the time evolution integration.

The dust density contrast profile, $\delta(t, r)$, is defined through the expression $\rho(t, r) = \rho_{\text{bkg}}(t)(1 + \delta(t, r))$. It is plotted for three different values of $t$ in Fig. 8. The profile grows exponentially and its shape changes in time departing from the initial bump profile given in Eq. (36). We see no effect of the central coordinate singularity on the profile. This shows the good reliability of the PIRK algorithm. One useful tool in cosmology is the value of the central density contrast as a function of time, plotted in Fig. 9. The numerical simulation can be used to investigate the non-linear regime of growth of dust matter density. The background scale factor is shown in Fig. 10, along with the local conformal scale factor defined as the product $a^2(t) \psi(t, r = 0)$. In ordinary studies, it is assumed that virialisation should occur when the local scale factor decreases to half its maximum value [2].

**VI. CONCLUSION AND PERSPECTIVES**

We have presented a full relativistic numerical method suited for cosmological studies of problems with spherical symmetry. The stability of the algorithm at the center of coordinate is ensured by the use of the PIRK meth-
The present paper generalizes the results obtained in [15], in which the same methods were applied to asymptotically flat spacetimes. We have given a generalization of the treatment of a radiative boundary condition to the case of a dynamical background and we have provided proofs of the stability and convergence of the code by solving for the dynamics of a pure gauge pulse on an expanding de Sitter background. One of the key steps in the process of building a numerical scheme on a flat background involves testing it on the most basic spherically symmetric vacuum solution, namely the Schwarzschild Black Hole. We have generalized this study by applying our code to study the numerical spherical collapse of dust which is adequately described by the LTB solution. We have shown how our code reproduces the same solution in presence of identical initial data and by comparing the metric components.

Further directions include the application of the method to the practical study of the cosmological spherical collapse for different cosmological models including new scalar degrees of freedom (quintessence). This would provide an invaluable tool to discriminate between various Dark Energy candidates. The approach presented here can also be straightforwardly adapted to the case of a fluid with pressure.

**Appendix: Algebraic Expressions of Used Quantities**

The following expressions are extracted from [13]. They are repeated here for the convenience of the reader with the slight modifications regarding the change of notations, mainly the introduction of the scale factor in the 3-metric.

\[
\begin{align*}
R^r_r &= - \frac{1}{a \dot{a} \psi} \left[ \frac{\partial^2 \dot{a}}{2 \dot{a}} - \dot{a} \partial_r \hat{\Delta} \tau - \frac{3}{4} \left( \frac{\partial_r \dot{\psi}}{\dot{a}} \right)^2 + \frac{1}{2} \left( \frac{\partial_r \dot{b}}{b} \right)^2 ight. \\
&\quad - \frac{1}{2} \Delta \tau \partial_r \dot{a} + \frac{\partial_r \ddot{b}}{rb} + \frac{2}{r^2} \left( 1 - \frac{\dot{a}}{b} \right) \left( 1 + r \partial_r \dot{b} \right) \\
&\quad + 4 \frac{\partial^2 \dot{\psi}}{\psi} - 4 \left( \frac{\partial_r \dot{\psi}}{\psi} \right)^2 - 2 \left( \frac{\partial_r \dot{\psi}}{\psi} \right) \left( \frac{\partial_r \dot{a}}{a} - \frac{\partial_r \dot{b}}{b} - \frac{2}{r} \right) \left. \right]. \\
\end{align*}
\]
\[ R = -\frac{1}{a\dot{\psi}} \left[ \frac{\partial^2 \dot{\psi}}{2a^2} + \frac{\partial^2 \dot{b}}{b} - \dot{a} \ddot{\psi} - \left( \frac{\partial_r \hat{a}}{\hat{a}} \right)^2 \right] \]

\[ + \frac{1}{2} \left( \frac{\partial_r \dot{b}}{b} \right)^2 + 2 \frac{\partial_r \dot{b}}{rb} \left( 3 - \frac{\dot{a}}{b} \right) + 4 \left( 1 - \frac{\dot{a}}{b} \right) \]

\[ + 8 \frac{\partial_r^2 \psi}{\psi} - 8 \left( \frac{\partial_r \dot{\psi}}{\psi} \right) \left( \frac{\partial_r \hat{a}}{2a} - \frac{\partial_r \dot{b}}{b} - \frac{2}{r} \right) \]. (A.2)

\[ \nabla^2 \alpha = \frac{1}{a \dot{\psi}} \left[ \partial_r^2 \alpha - \partial_r \alpha \left( \frac{\partial_r \hat{a}}{2a} - \frac{\partial_r \dot{b}}{b} - 2 \frac{\partial_r \dot{\psi}}{\psi} - \frac{2}{r} \right) \right]. \] (A.3)

\[ \nabla' \nabla' \alpha = \frac{1}{a \dot{\psi}} \left[ \partial_r^2 \alpha - \partial_r \alpha \left( \frac{\partial_r \hat{a}}{2a} + 2 \frac{\partial_r \dot{\psi}}{\psi} \right) \right]. \] (A.4)

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