DISCONTINUOUS VISCOSITY SOLUTIONS OF FIRST ORDER HAMILTON-JACOBI EQUATIONS

MICHEL BERTSCH, FLAVIA SMARAZZO, ANDREA TERRACINA, AND ALBERTO TESEI

Abstract. We consider the simplest example of a time-dependent first order Hamilton-Jacobi equation, in one space dimension and with a bounded and Lipschitz continuous Hamiltonian which only depends on the spatial derivative. We show that if the initial function has a finite number of jump discontinuities, the corresponding discontinuous viscosity solution of the corresponding Cauchy problem on the real line is unique. Uniqueness follows from a comparison theorem for semicontinuous viscosity sub- and supersolutions, using the barrier effect of spatial discontinuities of a solution. We also prove an existence theorem, as well as a comparison theorem for viscosity solutions with different initial data. In addition, we describe several properties of the evolution of the jump discontinuities.

As a byproduct of our analysis, we obtain new existence and uniqueness results for the initial-boundary value problem on an interval with (possibly singular) Neumann boundary conditions.

1. Introduction

After the introduction of continuous viscosity solutions of first order Hamilton-Jacobi (HJ) equations [15, 16], it was readily understood that the basic concepts and comparison results of the theory could be extended to the case of semicontinuous viscosity sub- and supersolutions. Addressing systematically discontinuities, both of the Hamiltonian itself and of solutions of HJ equations, was first undertaken in [22, 23] - an important issue, since discontinuous solutions are known to arise in many important applications (e.g., optimal control problems, differential game theory; see [3, 14] and references therein).

Although existence of possibly discontinuous viscosity solutions was proven in [23] by Perron’s method, uniqueness of such solutions remained unclear. In this connection, it should be noted that in general the comparison result for semicontinuous viscosity sub- and supersolutions does not imply uniqueness of viscosity solutions - apart from the trivial case of continuous data, in which case the unique viscosity solution is itself continuous (see [12]; see also Theorem 5.1 and Proposition 5.1 below). In fact, examples of nonuniqueness are known e.g. for the Cauchy problem of HJ equations, if the Hamiltonian is non-convex with explicit space and/or time dependence and the initial data function is discontinuous (see [3, 6, 21]).

Motivated by these difficulties, several different notions of discontinuous solutions of HJ equations have been proposed [2, 5, 7, 13, 21, 30], proving existence, comparison and uniqueness results under various assumptions (for instance, if the

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Hamiltonian is convex). Although interesting in their own and for their applications (e.g. to control problems), the mutual relationships between these different notions of solution have been elucidated only in some cases (see [14] [21]).

In the present paper we investigate discontinuous viscosity solutions, defined in the spirit of [23], in the simplest example of a time-dependent HJ equation, i.e. the one-dimensional equation

\[ U_t + H(U_x) = 0, \]

where the Hamiltonian \( H \) is bounded and uniformly Lipschitz continuous:

\[ H \in W^{1,\infty}(\mathbb{R}). \]

Observe that we do not assume convexity conditions on \( H \), nor the existence of the limits of \( H(p) \) as \( p \to \pm \infty \). Assuming boundedness of \( H \) is suggested by a mathematical model of ion etching [20] [28] [29].

The results of the present paper primarily concern the Cauchy problem

\[ (CP) \]

\[
\begin{aligned}
&U_t + H(U_x) = 0 \quad \text{in } \mathbb{R} \times (0,T) \\
&U = U_0 \quad \text{in } \mathbb{R} \times \{0\}, 
\end{aligned}
\]

where \( U_0 \) is a given locally bounded and piecewise continuous function on \( \mathbb{R} \) with a finite number of jump discontinuities (see \((H_2)\) below). The main results are the existence and uniqueness of a discontinuous viscosity solution (Theorems 3.3 and 3.6). We also prove some regularity results (see below), and a comparison result for discontinuous viscosity solutions with different initial data (Theorem 3.5).

Let us briefly describe the core of our approach. It is known (see [18]) that spatial discontinuities of a solution produce a barrier effect. For example, if a viscosity solution \( U \) is continuous in the set \((\mathbb{R} \setminus \{a\}) \times (0,\tau)\) and has spatial jump discontinuities at \( \{a\} \times (0,\tau) \), then the evolution of \( U \) in \((-\infty,a) \times (0,\tau)\) is totally independent of that in \((a,\infty) \times (0,\tau)\). More precisely, if the jump \( U(a^+,t) - U(a^-,t) \) \((t \in (0,\tau))\) is positive, then \( U \) satisfies on either side of \( a \) the singular Neumann problems

\[
\begin{aligned}
&U_t + H(U_x) = 0 \quad \text{in } (-\infty,a) \times (0,\tau) \\
&U_x(a,t) = \infty \quad \text{for } 0 < t < \tau \\
&U(x,0) = U_0(x) \quad \text{for } x < a,
\end{aligned}
\]

\[
\begin{aligned}
&U_t + H(U_x) = 0 \quad \text{in } (a,\infty) \times (0,\tau) \\
&U_x(a,t) = \infty \quad \text{for } 0 < t < \tau \\
&U(x,0) = U_0(x) \quad \text{for } x > a
\end{aligned}
\]

(see Lemma 5.3). Similarly if \( U(a^+,t) - U(a^-,t) \) is negative, with the condition \( U_x(a,t) = \infty \) replaced by \( U_x(a,t) = -\infty \).

For this reason we consider the Neumann problems

\[ (N) \]

\[
\begin{aligned}
&U_t + H(U_x) = 0 \quad \text{in } Q := \Omega \times (0,T) \\
&U_x = m_1 \quad \text{in } \{a\} \times (0,T) \\
&U_x = m_2 \quad \text{in } \{b\} \times (0,T),
\end{aligned}
\]

where \(-\infty \leq a < b \leq \infty\), \( \Omega := (a,b) \) and \( m_1, m_2 \in \mathbb{R} := [-\infty,\infty] \), with initial condition

\[ (1.2) \quad U = U_0 \quad \text{in } \Omega \times \{0\}. \]

Clearly, the conditions at \( x = a \) \((x = b)\) are dropped if \( a = -\infty \) \((b = \infty)\); in particular, problem \((N)-1.2\) coincides with the Cauchy problem \((CP)\) if \( a = -\infty \) and \( b = \infty \).

A major tool of our analysis is the comparison result for semicontinuous viscosity sub- and supersolutions of \((N)\) (Theorem 3.1), which seems to be new even in the case of general \( m_1, m_2 \in \mathbb{R} \) (see [27] for the case \( m_1 = m_2 = 0 \) in any space dimension).
Then, if $U_0$ is continuous in $\Omega$, existence and uniqueness of a viscosity solutions of $(N)-(1.2)$ can be proved in a rather standard way (see Propositions 5.1 and 6.4).

The above results for the Neumann problems are the basis for the proof of the main results. Assume that

$$\begin{cases} U_0 \in L^\infty(\Omega), \\ U_0 \text{ piecewise continuous in } \Omega \text{ with a finite number of discontinuities}, \end{cases}$$

and consider, to be specific, problem $(CP)$. It is proven that there is a positive time $\tau$ until which all discontinuities persist (see Lemma 5.2 and Remark 5.1), thus in a natural way the strip $R \times (0, \tau)$ is a finite disjoint union of subdomains. In view of the barrier effect, in each subdomain we solve problem $(N)$, the initial data being the restriction of $U_0$, with $m_1, m_2 = \pm \infty$ depending on the sign of the discontinuity jump as discussed above. The function determined by this procedure in $R \times (0, T)$ is proven to be the unique viscosity solution of $(CP)$ until the time $t = \tau$. If $\tau < T$ we iterate the procedure in $R \times (\tau, T)$ with a smaller number of discontinuities, thus well-posedness of $(CP)$ follows in a finite number of steps. In the above argument it is often enough to prove ”local” results, since by $(H_1)$ the “speed of propagation” is bounded by the Lipschitz constant $\|H'|_\infty$ (in this connection, see [16, 24]). As stated in Theorems 3.3 and 3.6 similar arguments and results hold for general Neumann problems.

As a by-product of our analysis, we prove that jump discontinuities of the solution cannot appear spontaneously nor disappear instantaneously in time (see Remark 3.4, claims (i)-(ii)). Moreover, the jumps are non-increasing in time and satisfy an explicit decay rate if $\limsup_{p \to \pm \infty} H(p) > \liminf_{p \to \pm \infty} H(p)$ (Proposition 3.4). Due to the boundedness of $H$, we also prove the Lipschitz continuity of solutions with respect to time (see (3.29)).

The paper is organized as follows. In the preliminary Section 2 we introduce the concept of semicontinuous envelopes of functions based on essential limits. In Section 3 we use the envelopes to define viscosity sub- and supersolutions, and we state the main results for the Neumann problem (which contain those for the Cauchy problem $(CP)$ if $(a, b) = \mathbb{R}$). In Section 4 we prove the basic comparison result for discontinuous viscosity sub- and supersolutions. In Section 5 we prove uniqueness of the viscosity solution, a comparison result for discontinuous viscosity solutions with different initial data, and some regularity results. Finally, in Section 6 we prove the existence of viscosity solutions.

Reassuring, the main novelty of the paper is the introduction of a procedure, based on the barrier effect of spatial discontinuities, which indicates how the comparison result for semicontinuous sub- and supersolutions can be used to prove uniqueness of viscosity solutions with discontinuous initial data. Although we have only done this for a particularly simple problem, preliminary calculations suggest that the procedure can be adapted to more general problems (to be addressed in future papers), namely the cases of initial data with infinitely many jump discontinuities and Hamiltonians with linear growth and explicit $x$ and $t$ dependance. The latter case is particularly interesting since it includes equations for which uniqueness of discontinuous viscosity solutions fails, as mentioned in the beginning of the Introduction. In particular our approach seems to suggest a mathematical uniqueness criterium.
Of course many open problems concerning discontinuous solutions of first order HJ equations remain to be solved, in particular the multidimensional case and that of Hamiltonians depending on $U$.

Finally let us observe that, setting $u := U_x$ and $u_0 := U'_0$, problem (CP) is formally related to the Cauchy problem for a scalar conservation law,

$$(CL) \begin{cases} u_t + [H(u)]_x = 0 & \text{in } \mathbb{R} \times (0, T) \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}, \end{cases}$$

although it is not trivial to make the correspondence rigorous ([11]; see also [4] for a statement in this direction). If $u_0 = U'_0$ is a Radon measure, it is possible to prove existence of suitably defined measure-valued entropy solutions of problem (CL) ([11]; see also [8, 9] for the case of positive initial measures). Moreover, if the singular part $u_0$ of $u_0$ (with respect to the Lebesgue measure) is a finite superposition of Dirac masses, uniqueness of such solutions can be proven, if additional compatibility conditions are satisfied near the support of $u_0$. Remarkably, the singularities of $u_0$ in (CL) have a barrier effect which corresponds to that produced by discontinuities of $U'_0$ in (CP), and well-posedness of (CL) can be proven using singular Dirichlet problems which are the counterpart of (N) for HJ equations (see [9, 11] for details).

2. Preliminaries

Let $\chi_E$ denote the characteristic function of $E \subset \mathbb{R}$. For every $u \in \mathbb{R}$ we set

$$[u]_x := \max\{ \pm u, 0 \}, \quad \sgn_x(u) := \pm \chi_{\mathbb{R}_x}(u), \quad \sgn(u) := \sgn_-(u) + \sgn_+(u).$$

Let $\Omega = (a, b)$ ($-\infty < a < b < \infty$). We say that a function $f : \Omega \to \mathbb{R}$, $f \in L^\infty(\Omega)$, is piecewise continuous if:

- $\Omega = \bigcup_{j=1}^{p+1} I_j$ ($p = 0, 1, \ldots$) with $I_1 := (a, x_1)$, $I_j := (x_{j-1}, x_j)$ for $j = 2, \ldots, p$,
- $I_{p+1} := (x_p, b)$;
- the restriction $f_j := f \upharpoonright I_j$ belongs to $C(\overline{I}_j)$ for every $j = 1, \ldots, p + 1$, and $f_j(x_j) \neq f_{j+1}(x_{j+1})$ for every $j = 1, \ldots, p$.

A function $f \in C(\overline{\Omega})$ is piecewise continuous (this corresponds to the case $p = 0$). If $\Omega$ is unbounded, a function $f \in L^\infty_{\text{loc}}(\overline{\Omega})$ is piecewise continuous in $\Omega$ if it is piecewise continuous in every bounded interval $(a_0, b_0) \subset \Omega$.

Let $D \subset \mathbb{R}^2$ be open and $(x_0, t_0) \in \partial D$. For every measurable function $z : D \to \mathbb{R}$ set

$$\begin{align*}
\operatorname{ess\,lim\,sup}_{D \ni (x, t) \to (x_0, t_0)} z(x, t) &= \inf_{\delta > 0} \left( \operatorname{ess\,sup}_{(x, t) \in D \cap B(x_0, t_0)} z(x, t) \right) = \lim_{\delta \to 0^+} \left( \operatorname{ess\,sup}_{(x, t) \in D \cap B(x_0, t_0)} z(x, t) \right), \\
\operatorname{ess\,lim\,inf}_{D \ni (x, t) \to (x_0, t_0)} z(x, t) &= \sup_{\delta > 0} \left( \operatorname{ess\,inf}_{(x, t) \in D \cap B(x_0, t_0)} z(x, t) \right) = \lim_{\delta \to 0^+} \left( \operatorname{ess\,inf}_{(x, t) \in D \cap B(x_0, t_0)} z(x, t) \right),
\end{align*}$$

where

$$B_r(x_0, t_0) := \{ (x, t) \in \mathbb{R}^2 \mid (x - x_0)^2 + (t - t_0)^2 < r^2 \} \quad (r > 0).$$

If $\operatorname{ess\,lim\,sup}_{D \ni (x, t) \to (x_0, t_0)} z(x, t) = \operatorname{ess\,lim\,inf}_{D \ni (x, t) \to (x_0, t_0)} z(x, t)$, we also set

$$\begin{align*}
\operatorname{ess\,lim\,inf}_{D \ni (x, t) \to (x_0, t_0)} z(x, t) &= \operatorname{ess\,lim\,sup}_{D \ni (x, t) \to (x_0, t_0)} z(x, t) = \operatorname{ess\,lim\,inf}_{D \ni (x, t) \to (x_0, t_0)} z(x, t), \\

\text{The quantities}
\end{align*}$$

$$\begin{align*}
\operatorname{ess\,lim\,sup}_{D \ni (x, t) \to (x_0, t_0)} z(x, t), \quad \operatorname{ess\,lim\,inf}_{D \ni (x, t) \to (x_0, t_0)} z(x, t)
\end{align*}$$
are defined replacing \( B_r(x_0, t_0) \) by \( B_r(x_0, t_0) \cap \{(x, t) \in \mathbb{R}^2 \mid t \geq t_0 \} \). Similarly,

\[
\text{ess lim sup}_{D(x,t) \rightarrow (x_0^r,t_0)} z(x,t), \quad \text{ess lim inf}_{D(x,t) \rightarrow (x_0^r,t_0)} z(x,t)
\]

are defined replacing \( B_r(x_0, t_0) \) by \( B_r(x_0, t_0) \cap \{(x, t) \in \mathbb{R}^2 \mid x \geq x_0 \} \).

Let \( z \in L^\infty(\overline{D}) \). By the essential upper semicontinuous envelope of \( z \) we mean the function \( z^* : \overline{D} \rightarrow \mathbb{R} \),

\[
(2.1) \quad z^*(x_0,t_0) := \text{ess lim sup}_{D(x,t) \rightarrow (x_0^r,t_0)} z(x,t) \quad \text{for any } (x_0,t_0) \in \overline{D}.
\]

Similarly, the essential lower semicontinuous envelope of \( z \) is the function \( z_* : \overline{D} \rightarrow \mathbb{R} \),

\[
(2.2) \quad z_*(x_0,t_0) := \text{ess lim inf}_{D(x,t) \rightarrow (x_0^r,t_0)} z(x,t) \quad \text{for any } (x_0,t_0) \in \overline{D}.
\]

We also set

\[
(2.3) \quad z^*(x_0,t_0^r) := \text{ess lim sup}_{D(x,t) \rightarrow (x_0,t_0^r)} z(x,t), \quad z_*(x_0,t_0^r) := \text{ess lim inf}_{D(x,t) \rightarrow (x_0,t_0^r)} z(x,t),
\]

\[
(2.4) \quad z^*(x_0^r,t_0) := \text{ess lim sup}_{D(x,t) \rightarrow (x_0^r,t_0)} z(x,t), \quad z_*(x_0^r,t_0) := \text{ess lim inf}_{D(x,t) \rightarrow (x_0^r,t_0)} z(x,t).
\]

Observe that

\[
(2.5) \quad z^*(x_0^r,t_0^r) \leq z^*(x_0,t_0), \quad z_*(x_0^r,t_0^r) \geq z_*(x_0,t_0),
\]

\[
(2.6) \quad z^*(x_0^r,t_0) \leq z^*(x_0,t_0), \quad z_*(x_0^r,t_0) \geq z_*(x_0,t_0).
\]

Similar definitions hold for any measurable function \( z : F \subseteq \mathbb{R} \rightarrow \mathbb{R} \). For shortness, we shall say “upper (respectively lower) envelope” instead of “essential upper (respectively lower) semicontinuous envelope”.

**Remark 2.1.** Since the definition of \( z^* \), \( z_* \) depends on the domain of definition of \( z \), different restrictions of \( z \) can have different upper and lower envelopes. In fact, let \( D_1 \subseteq D \) be an open set, and let \( z_1 := z \upharpoonright D_1 \) be the restriction of \( z \) to \( D_1 \). Then:

(i) if \( (x_0, t_0) \in D_1 \),

\[
(z_1)^*(x_0,t_0) := \text{ess lim sup}_{D_1(x,t) \rightarrow (x_0,t_0)} z(x,t) = \text{ess lim sup}_{D(x,t) \rightarrow (x_0,t_0)} z(x,t) = z^*(x_0,t_0),
\]

\[
(z_1)_*(x_0,t_0) := \text{ess lim inf}_{D_1(x,t) \rightarrow (x_0,t_0)} z(x,t) = \text{ess lim inf}_{D(x,t) \rightarrow (x_0,t_0)} z(x,t) = z_*(x_0,t_0);
\]

(ii) if \( (x_0, t_0) \in \partial D_1 \),

\[
(z_1)^*(x_0,t_0) := \text{ess lim sup}_{D_1(x,t) \rightarrow (x_0,t_0)} z(x,t) \leq \text{ess lim sup}_{D(x,t) \rightarrow (x_0,t_0)} z(x,t) = z^*(x_0,t_0),
\]

\[
(z_1)_*(x_0,t_0) := \text{ess lim inf}_{D_1(x,t) \rightarrow (x_0,t_0)} z(x,t) \geq \text{ess lim inf}_{D(x,t) \rightarrow (x_0,t_0)} z(x,t) = z_*(x_0,t_0).
\]

Observe that \( (2.3)-(2.6) \) are a particular case of the above inequalities. If \( D = \Omega \times (0,T) \) and \( t_0 = 0 \), inequalities \( (2.5)-(2.6) \) from above become equalities, namely

\[
(2.7) \quad z^*(x_0,0^+) = z^*(x_0,0), \quad z_*(x_0,0^+) = z_*(x_0,0) \quad \text{for any } x_0 \in \overline{\Omega}.
\]

More precisely, let \( D_1 \subseteq D \) be an open set, and let \( z_1 := z \upharpoonright D_1 \). Set

\[
(2.8) \quad D_1 := \{(x,t) \in \overline{D}_1 \mid \exists \delta_1 > 0 \text{ such that } B_{\delta_1}(x,t) \cap D \subseteq D_1 \}.
\]

Then for any \((x_0,t_0) \in \overline{D}_1\) there holds

\[
(2.9) \quad z^*(x_0,t_0) = (z_1)^*(x_0,t_0), \quad z_*(x_0,t_0) = (z_1)_*(x_0,t_0).
\]
In fact, if \((x_0, t_0) \in \tilde{D}_1\) then \(D \cap B_{\delta_1}(x_0, t_0) = D_1 \cap B_{\delta_1}(x_0, t_0)\) and, by (2.1),

\[
z^*(x_0, t_0) = \lim_{\delta \to 0^+} \left( \text{ess sup}_{(x,t) \in D \cap B_{\delta}(x_0, t_0)} z(x,t) \right) = \lim_{\delta \to 0^+} \left( \text{ess sup}_{(x,t) \in D_1 \cap \tilde{B}_{\delta}(x_0, t_0)} z_1(x,t) \right) = (z_1)^*(x_0, t_0).
\]

For further reference we consider some specific cases of \(D_1 \subseteq D = (a,b) \times (0,T)\), with \(-\infty < a < b < \infty\). The first two examples concern trapezoidal domains.

(a) Let \(D_1 = \tilde{A}\tau := \{(x,t) | x \in (a, b - \|H'\|_{\infty} t), t \in (0, \tau)\}\) for \(\tau \in (0, \tau_1]\), where

\[
\tau_1 := \min \left\{ \frac{b-a}{2\|H'\|_{\infty}}, T \right\}.
\]

Then \(\tilde{D}_1 = \{(x,t) | x \in [a,b - \|H'\|_{\infty} t), t \in [0, \tau)\}\) if \(\tau \in (0, \tau_1)\), and \(\tilde{D}_1 = \{(x,t) | x \in [a,b - \|H'\|_{\infty} t), t \in [0, T]\}\) if \(\tau = \tau_1 = T\);

(b) let \(D_1 = \tilde{B}\tau := \{(x,t) | x \in (a + \|H'\|_{\infty} t, b), t \in (0, \tau)\}\) for any \(\tau \in (0, \tau_1]\). Then \(\tilde{D}_1 = \{(x,t) | x \in (a, b - \|H'\|_{\infty} t), t \in [0, \tau)\}\) if \(\tau \in (0, \tau_1)\), and \(\tilde{D}_1 = \{(x,t) | x \in (a, b - \|H'\|_{\infty} t), t \in [0, T]\}\) if \(\tau = \tau_1 = T\);

(c) let \(D_1 = Q_{\tau, T} := \{(x,t) \in D | t \in (\tau, T)\}\) for any \(\tau \in (0, T)\). Then \(\tilde{D}_1 = \{(x,t) | x \in [a, b], t \in (\tau, T)\}\);

(d) let \(D_1 = \{(x,t) \in D | x \in (a, c), t \in (0, T)\}\) for any \(c \in (a, b)\). Then \(\tilde{D}_1 = \{(x,t) | x \in [a, c], t \in [0, T]\}\).

It is easily checked that the upper (lower) envelope \(z^* (z_*)\) is indeed upper (lower) semicontinuous in \(\tilde{D}\), namely for any \((x_0, t_0) \in \tilde{D}\)

\[
\limsup_{D \ni (x_0, t_0) \to (x_0, t_0)} z^*(x,t) \leq z^*(x_0, t_0), \quad \liminf_{D \ni (x_0, t_0) \to (x_0, t_0)} z_*(x,t) \geq z_*(x_0, t_0).
\]

Actually, the inequalities in (2.11) can be replaced by equalities:

**Lemma 2.1.** Let \(D \subseteq \mathbb{R}^2\) be open and \(z \in L^\infty_{\text{loc}}(\tilde{D})\). Then, for any \((x_0, t_0) \in \tilde{D}\),

\[
\begin{align}
(2.12a) \quad (z^*)^*(x_0, t_0) &= \text{ess sup}_{D \ni (x_0, t_0) \to (x_0, t_0)} z^*(x, t) = z^*(x_0, t_0) = \limsup_{D \ni (x_0, t_0) \to (x_0, t_0)} z^*(x, t), \\
(2.12b) \quad (z_*)^*(x_0, t_0) &= \text{ess lim inf}_{D \ni (x_0, t_0) \to (x_0, t_0)} z_*(x, t) = z_*(x_0, t_0) = \liminf_{D \ni (x_0, t_0) \to (x_0, t_0)} z_*(x, t).
\end{align}
\]

**Proof.** We only prove the result for \(z^*\). Since

\[
(z^*)^*(x_0, t_0) = \text{ess sup}_{D \ni (x_0, t_0) \to (x_0, t_0)} z^*(x, t) \leq \limsup_{D \ni (x_0, t_0) \to (x_0, t_0)} z^*(x, t) \leq z^*(x_0, t_0)
\]

(see (2.1) and (2.11)), it suffices to prove that

\[
(z^*)^*(x_0, t_0) \leq \text{ess sup}_{D \ni (x_0, t_0) \to (x_0, t_0)} z^*(x, t).
\]

Let \((x_0, t_0) \in \tilde{D}\). For every \(\varepsilon > 0\) there exists \(r_\varepsilon > 0\) such that

\[
\text{ess sup}_{(x,t) \in B_r(x_0, t_0) \cap D} z(x,t) > z^*(x_0, t_0) - \frac{\varepsilon}{2} \quad \text{for all} \quad r \leq r_\varepsilon.
\]

Therefore, for every such \(r\) there exists \(B_{r,\varepsilon} \subseteq B_r(x_0, t_0) \cap D, \varepsilon > 0\), such that

\[
z(x,t) \geq \text{ess sup}_{(x,t) \in B_{r,\varepsilon}(x_0, t_0) \cap D} z(x,t) - \frac{\varepsilon}{2} > z^*(x_0, t_0) - \varepsilon \quad \text{for a.e.} \quad (x,t) \in B_{r,\varepsilon}.
\]
Setting $B_\varepsilon := \bigcup_{r>\varepsilon} B_{r,\varepsilon}$, it follows that $|B_\varepsilon \cap B_r(x_0,t_0)| > 0$ for every $r > 0$ (hence $|B_\varepsilon| > 0$), and

$$z > z^*(x_0,t_0) - \varepsilon \quad \text{a.e. in } B_\varepsilon.$$ 

Let $(x,t) \in B_\varepsilon$ satisfy $|B_\delta(x,t) \cap B_\varepsilon| > 0$ for every $\delta > 0$; this choice is possible up to a null set $\mathcal{N}_\varepsilon \subset B_\varepsilon$, since $|B_\varepsilon| > 0$ and almost every $(x,t) \in B_\varepsilon$ is a Lebesgue point of $f(x,t) = \chi_{B_\varepsilon}(x,t)$ (e.g., see [19, Subsection 1.7.1]). Then we have

$$z > z^*(x_0,t_0) - \varepsilon \quad \text{a.e. in } B_\varepsilon \cap B_\delta(x,t), \quad (|B_\varepsilon \cap B_\delta(x,t)| > 0 \quad \text{for all } \delta > 0),$$

whence $z^*(x,t) \geq z^*(x_0,t_0) - \varepsilon$ for a.e. $(x,t) \in B_\varepsilon$. Since $|B_\varepsilon \cap B_r(x_0,t_0)| > 0$ for all $r > 0$, this implies that

$$\text{ess lim sup}_{B_\delta(x,t) \to (x_0,t_0)} z^*(x,t) \geq z^*(x_0,t_0) - \varepsilon,$$

and the conclusion follows from the arbitrariness of $\varepsilon$. \hfill \Box

An analogous result holds for $D \subseteq \mathbb{R}$ open and $z \in L^\infty_\text{loc}(\overline{D})$.

3. Results

3.1. Notation. We recall that $\Omega = (a,b)$ with $-\infty \leq a < b \leq \infty$ and $Q = \Omega \times (0,T)$. We set $Q := \Omega \times [0,T)$. We denote problem $(N)$ by $(\cdot)_R$ when $m_1, m_2 \in \mathbb{R}$, and by $(\cdot)_2$ when $m_1 = \pm \infty$ and/or $m_2 = \pm \infty$.

We shall also consider problem $(N)$ in trapezoidal domains of the following type:

(3.1) $A := \{(x,t) \mid x \in [a,d - \|H'\|_{\infty}t], \ t \in [0,T]\}$ with $d - a \geq \|H'\|_{\infty}T,$

(3.2) $B := \{(x,t) \mid x \in [c + \|H'\|_{\infty}t, b], \ t \in [0,T]\}$ with $\|H'\|_{\infty}T \leq b - c,$

(3.3) $C := \{(x,t) \mid x \in [c + \|H'\|_{\infty}t, d - \|H'\|_{\infty}t], \ t \in [0,T]\}$ with $d - c \geq 2\|H'\|_{\infty}T.$

Due to their slope, no boundary conditions on the oblique sides,

$$\{(x,t) \mid x = d - \|H'\|_{\infty}t, t \in (0,T)\}, \quad \{(x,t) \mid x = c + \|H'\|_{\infty}t, t \in (0,T)\},$$

will be needed for the well-posedness of $(N)$ in such domains. So if $(N)$ is stated in $(a,\infty) \times (0,T)$ or in the trapezoidal domain $A$ defined by (3.1) (respectively in $(-\infty,b) \times (0,T)$ or in $B$ defined by (3.2)), only the condition at $x = a$ (respectively at $x = b$) is assumed (see also Remark 3.2). Similarly, if $(N)$ is stated in $\mathbb{R} \times (0,T)$ (the Cauchy problem) or in $C$ defined by (3.3), all boundary conditions disappear.

3.2. Definitions. The following definitions are used throughout the paper.

Definition 3.1. Let $D \subseteq \mathbb{R}^2$ open and $U \in L^\infty_\text{loc}(\overline{D})$.

(i) $U$ is a viscosity subsolution of equation (1.1) in $D$ if for all $\varphi \in C^1(D)$ the following condition holds: if $(x,t) \in D$ is a local maximum point of $U^* - \varphi$ in $D$, then

$$\varphi_t(x,t) + H(\varphi_x(x,t)) \leq 0.$$  \hspace{1cm} (3.4)

(ii) $U$ is a viscosity supersolution of equation (1.1) in $D$ if for all $\varphi \in C^1(D)$ the following condition holds: if $(x,t) \in D$ is a local minimum point of $U_* - \varphi$ in $D$, then

$$\varphi_t(x,t) + H(\varphi_x(x,t)) \geq 0.$$  \hspace{1cm} (3.5)

Observe that if the open set $D \subseteq \mathbb{R}^2$ is bounded, viscosity sub and supersolutions of equation (1.1) in $D$ belong to the space $L^\infty(D)$. 

Definition 3.2. Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$, $\tilde{Q} = \overline{\Omega} \times (0, T]$. Let $m_1 \in \mathbb{R}$ and $m_2 \in \mathbb{R}$.

(i) By a viscosity subsolution of $(N_R)$ in $Q$ we mean any viscosity subsolution $U$ of (3.11) in $Q$, such that for all $\varphi \in C^1(\tilde{Q})$, if $(a, t)$ and/or $(b, t)$ are local maximum points of $U^* - \varphi$ in $Q$, then

$$
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0 & \text{if } \varphi_x(a^+, t) \leq m_1, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \leq 0 & \text{if } \varphi_x(b^-, t) \geq m_2.
\end{cases}
$$

(ii) By a viscosity supersolution of $(N_R)$ in $Q$ we mean any viscosity supersolution $U$ of (3.11) in $Q$, such that for all $\varphi \in C^1(\tilde{Q})$, if $(a, t)$ and/or $(b, t)$ are local minimum points of $U_* - \varphi$ in $Q$, then

$$
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \geq 0 & \text{if } \varphi_x(a^+, t) \geq m_1, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \geq 0 & \text{if } \varphi_x(b^-, t) \leq m_2.
\end{cases}
$$

Formally, conditions (3.6) for viscosity subsolutions of $(N_R)$ are void when $m_1 = -\infty$, $m_2 = \infty$; similarly, conditions (3.7) for viscosity supersolutions of $(N_R)$ are void when $m_1 = \infty$, $m_2 = -\infty$. This motivates the following definitions.

Definition 3.3. Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$. Let either $m_1 = \pm\infty$ and $m_2 \in \mathbb{R}$, or $m_1 \in \mathbb{R}$ and $m_2 = \pm\infty$. By a viscosity subsolution of $(N_S)$ in $Q$ we mean any viscosity subsolution $U$ of (3.11) in $Q$, such that for all $\varphi \in C^1(\tilde{Q})$:

(i) if $m_1 = \infty$ and $m_2 \in \mathbb{R}$, and $(a, t)$ and/or $(b, t)$ are local maximum points of $U^* - \varphi$ in $Q$, then

$$
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \leq 0 & \text{if } \varphi_x(b^-, t) \geq m_2;
\end{cases}
$$

(ii) if $m_1 = -\infty$ and $m_2 \in \mathbb{R}$, and $(b, t)$ is a local maximum point of $U^* - \varphi$ in $Q$, then

$$
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \leq 0 \quad \text{if } \varphi_x(b^-, t) \geq m_2;
$$

(iii) if $m_1 \in \mathbb{R}$ and $m_2 = \infty$, and $(a, t)$ is a local maximum point of $U^* - \varphi$ in $Q$, then

$$
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0 \quad \text{if } \varphi_x(a^+, t) \leq m_1;
$$

(iv) if $m_1 \in \mathbb{R}$ and $m_2 = -\infty$, and $(a, t)$ and/or $(b, t)$ are local maximum points of $U^* - \varphi$ in $Q$, then

$$
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0 & \text{if } \varphi_x(a^+, t) \leq m_1, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \leq 0.
\end{cases}
$$

Definition 3.4. Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$. Let either $m_1 = \pm\infty$ and $m_2 \in \mathbb{R}$, or $m_1 \in \mathbb{R}$ and $m_2 = \pm\infty$. By a viscosity supersolution of $(N_S)$ in $Q$ we mean any viscosity supersolution $U$ of (3.11) in $Q$, such that for all $\varphi \in C^1(\tilde{Q})$:

(i) if $m_1 = \infty$ and $m_2 \in \mathbb{R}$, and $(b, t)$ is a local minimum point of $U_* - \varphi$ in $Q$, then

$$
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \geq 0 \quad \text{if } \varphi_x(b^-, t) \leq m_2;
$$

(ii) if $m_1 = -\infty$ and $m_2 \in \mathbb{R}$, and $(a, t)$ and/or $(b, t)$ are local minimum points of $U_* - \varphi$ in $Q$, then

$$
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \geq 0, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \geq 0 \quad \text{if } \varphi_x(b^-, t) \leq m_2;
\end{cases}
$$

Similarly, conditions (3.8) for viscosity supersolutions of $(N_S)$ are void when $m_1 = -\infty$, $m_2 = \infty$; similarly, conditions (3.11) for viscosity subsolutions of $(N_S)$ are void when $m_1 = \infty$, $m_2 = -\infty$. This motivates the following definitions.
Let \( \Omega \) be an open trapezoid in \( (\mathbb{R}^d, d) \) with \( \partial \Omega = \partial A \). Let \( U \) be a viscosity supersolution of (1.1) in \( \Omega \).

Remark 3.1. If \( m_1 \in \mathbb{R} \) and \( m_2 = \infty \), and if \( (a, t) \) and/or \( (b, t) \) are local minimum points of \( U - \varphi \) in \( \hat{Q} \), then

\[
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \geq 0 & \text{if } \varphi_x(a^+, t) \geq m_1, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \geq 0;
\end{cases}
\]

(iv) if \( m_1 \in \mathbb{R} \) and \( m_2 = -\infty \), and if \( (a, t) \) is a local minimum point of \( U - \varphi \) in \( \hat{Q} \), then

\[
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \geq 0 & \text{if } \varphi_x(a^+, t) \geq m_1.
\]

**Definition 3.5.** Let \( \Omega = (a, b) \) with \( -\infty < a < b < \infty \). Let \( m_1 = \pm \infty \) and \( m_2 = \pm \infty \). By a **viscosity subsolution** of \((N_\varphi)\) in \( Q \) we mean any viscosity subsolution \( U \) of (1.1) in \( Q \), such that for all \( \varphi \in C^1(\hat{Q}) \):

(i) if \( m_1 = m_2 = \infty \) and \( (a, t) \) is a local maximum point of \( U^* - \varphi \) in \( \hat{Q} \), then

\[
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0;
\]

(ii) if \( m_1 = m_2 = -\infty \) and \( (b, t) \) is a local maximum point of \( U^* - \varphi \) in \( \hat{Q} \), then

\[
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \leq 0;
\]

(iii) if \( m_1 = \infty, m_2 = -\infty \), and \( (a, t) \) and/or \( (b, t) \) are local maximum points of \( U^* - \varphi \) in \( \hat{Q} \), then

\[
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \leq 0.
\end{cases}
\]

**Definition 3.6.** Let \( \Omega = (a, b) \) with \( -\infty < a < b < \infty \). Let \( m_1 = \pm \infty \) and \( m_2 = \pm \infty \). By a **viscosity supersolution** of \((N_\varphi)\) in \( Q \) we mean any viscosity supersolution \( U \) of (1.1) in \( Q \), such that for all \( \varphi \in C^1(\hat{Q}) \):

(i) if \( m_1 = m_2 = \infty \) and \( (b, t) \) is a local minimum point of \( U - \varphi \) in \( \hat{Q} \), then

\[
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \geq 0;
\]

(ii) if \( m_1 = m_2 = -\infty \) and \( (a, t) \) is a local minimum point of \( U - \varphi \) in \( \hat{Q} \), then

\[
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \geq 0;
\]

(iii) if \( m_1 = -\infty, m_2 = \infty \), and \( (a, t) \) and/or \( (b, t) \) are local minimum points of \( U - \varphi \) in \( \hat{Q} \), then

\[
\begin{cases}
\varphi_t(a, t) + H(\varphi_x(a^+, t)) \geq 0, \\
\varphi_t(b, t) + H(\varphi_x(b^-, t)) \geq 0.
\end{cases}
\]

**Remark 3.1.** If \( U \) is a viscosity subsolution of \((N)\) with \( m_1, m_2 \in \mathbb{R} \), it is also a viscosity subsolution of \((N)\) with \( m'_1, m'_2 \), for any \( m'_1 \in [-\infty, m_1] \) and \( m'_2 \in [m_2, \infty] \). Analogously, if \( U \) is a viscosity supersolution of \((N)\) with \( m_1, m_2 \in \mathbb{R} \), it is also a viscosity supersolution of \((N)\) with \( m'_1, m'_2 \), for any \( m'_1 \in [m_1, \infty] \) and \( m'_2 \in [-\infty, m_2] \).

**Remark 3.2.** Viscosity sub- and supersolutions of \((N)\) in \((a, \infty) \times (0, T)\) or in the open trapezoid \( \tilde{A} \), with \( \tilde{A} \) defined by (3.1) (respectively, in \((\mathbb{R}, t) \times (0, T)\) or in \( \tilde{B} \) with \( \tilde{B} \) defined by (3.2)), are defined as above, yet dropping conditions at \( x = b \) (respectively at \( x = a \)). For instance, set \( \tilde{A} := \{(x, t) \in A | t \in (0, T)\} \). If \( m_1 \in \mathbb{R} \), by a viscosity subsolution of \((N_R)\) in \( \tilde{A} \) we mean any viscosity subsolution \( U \) of (1.1).
in \( \hat{A} \), such that for all \( \varphi \in C^1(\hat{A}) \), if \((a, t)\) is a local maximum point of \( U^* - \varphi \) in \( \hat{A} \), then

\[
\varphi_t(a, t) + H(\varphi_x(a^*, t)) \leq 0 \quad \text{if} \quad \varphi_x(a^*, t) \leq m_1
\]

(see Definition 3.2).

Also observe that the above definitions make sense for any \( H \in C(\mathbb{R}) \).

**Remark 3.3.** Let \( Q = (a, b) \times (0, T) \) with \(-\infty < a < b < \infty\), and \( Q_1 = (a, b) \times (\tau_1, \tau_2) \), for \( 0 \leq \tau_1 < \tau_2 \leq T \). It is easily seen that if \( U \) is a viscosity subsolution of \((N)\) in \( Q \), its restriction \( U_1 := U \cap Q_1 \) is a viscosity subsolution of \((N)\) in \( Q_1 \); in fact, if \((a, t_0)\) is a local maximum point of \( (U_1)^* - \varphi \) in \( Q_1 \) and \( t_0 \in (\tau_1, \tau_2) \), by (3.3) it is also a local maximum point of \( U^* - \varphi \) in \( Q \), whereas in the case \( t_0 = \tau_1 \) it suffices to argue as in [17, Section 5.2]. Similar remarks hold for any \( D_1 \subseteq Q \) as in (a)-(b) of Remark 2.1 and for viscosity supersolutions of problem \((N)\) in \( Q \).

**Definition 3.7.** (i) A function \( U \) is called a **viscosity solution** of \((N)\) in \( Q \), if it is both a viscosity subsolution and a viscosity supersolution.

(ii) Let \( U_0 \in L^\infty_{loc}(\bar{\Omega}) \). A **viscosity solution** of \((N)\) in \( Q \) with initial condition \((1.2)\) is a viscosity solution of \((N)\) such that

\[
U^*(\cdot, 0) = (U_0)^*, \quad U_*(\cdot, 0) = (U_0)_* \quad \text{in} \quad \bar{\Omega}.
\]

### 3.3. Comparison, uniqueness and regularity.

The following comparison result will be proven (see Section 3).

**Theorem 3.1.** Let \( (H_1) \) hold. Let \( U, V \) be a viscosity sub- and supersolution of problem \((N)\) in \( Q \) with the same boundary conditions.

(i) Let \( Q = (a, b) \times (0, T) \) with \(-\infty < a < b < \infty\). Then

\[
\max_Q \left[ (U^*) - (V_*) \right]_* \leq \max_{\bar{\Omega}} \left[ (U^*) - (V_*) \right]_+.
\]

The same holds if \( Q \) is replaced by a trapezoidal domain of the form \((3.1)\)-(3.3), with \( \Omega = (a, d) \), or \( \Omega = (b, c) \), or \( \Omega = (c, d) \), respectively.

(ii) If \( Q = (a, \infty) \times (0, T) \), then for any trapezoidal domain \( A \subset Q \) as in \((3.1)\) there holds

\[
\max_A \left[ (U_A)^* - (V_A)_* \right]_* \leq \max_{[a, d]} \left[ (U^*) - (V_*) \right]_+,
\]

where \( U_A = U \cap A \) and \( V_A = V \cap A \).

(iii) If \( Q = (-\infty, b) \times (0, T) \), then for any trapezoidal domain \( B \subset Q \) as in \((3.2)\) there holds

\[
\max_B \left[ (U_B)^* - (V_B)_* \right]_* \leq \max_{[c, b]} \left[ (U^*) - (V_*) \right]_+,
\]

where \( U_B = U \cap B \) and \( V_B = V \cap B \).

(iv) If \( Q = \mathbb{R} \times (0, T) \), then for any trapezoidal domain \( C \subset Q \) as in \((3.3)\) there holds

\[
\max_C \left[ (U_C)^* - (V_C)_* \right]_* \leq \max_{[c, d]} \left[ (U^*) - (V_*) \right]_+,
\]

where \( U_C = U \cap C \) and \( V_C = V \cap C \).

Theorem 3.1 will be proven by a method of doubling variables adapted from [27], where only the case \( m_1 = m_2 = 0 \) was considered. We first prove the result when \( Q \) is a trapezoidal domain like \((3.1)\)-(3.2) (see Proposition 4.3), whence Theorem 3.1 easily follows. Observe that the cases \( m_1 \in \mathbb{R} \) and \( m_2 = \pm \infty \) or vice versa, as well as the cases \( m_1 = \pm \infty \) and \( m_2 = \pm \infty \), are covered by Theorem 3.1.
As for time regularity of viscosity solutions of (N) we have the following result, which might be guessed from the boundedness of the Hamiltonian $H$ in (1.1).

**Proposition 3.2.** Let $(H_1)$ hold. Let $U$ and $V$ be a viscosity sub- and supersolution of problem (N) in $Q$. Then for any $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$, and $\bar{x} \in \overline{\Omega}$

\[
U^*(\bar{x}, t_1) - U^*(\bar{x}, t_2) \quad \frac{t_1 - t_2}{\leq \sup_{u \in \mathbb{R}} (-H(u)),}
\]

(3.27a)

\[
V_*(\bar{x}, t_1) - V_*(\bar{x}, t_2) \quad \frac{t_1 - t_2}{\geq \inf_{u \in \mathbb{R}} (-H(u)).}
\]

(3.27b)

Theorem 3.3 will be used to prove uniqueness of discontinuous viscosity solutions of (N) with piecewise continuous (see Section 2) initial data. If $(H_2)$ holds in $\Omega = (a, b)$, with $-\infty \leq a < b \leq \infty$, we denote by $x_j$ the points where $U_0$ is discontinuous and by $J_0(x_j)$ the corresponding jumps:

\[
J_0(x_j) := U_0(x_j^+) - U_0(x_j^-) \quad (j = 1, \ldots, p).
\]

We also set $I_j := (x_{j-1}, x_j)$ for $j = 2, \ldots, p$, $I_1 := (a, x_1)$, $I_{p+1} := (x_p, b)$, $Q_j := I_j \times (0, T)$ ($j = 1, \ldots, p + 1$).

**Theorem 3.3.** Let $\Omega = (a, b)$ with $-\infty \leq a < b \leq \infty$ and let $(H_1)$-$(H_2)$ be satisfied.

(i) If $U$ and $V$ are viscosity solutions of problem (N) in $Q$ with initial condition (1.2), then $U = V$ a.e. in $Q$.

(ii) If $U$ is a solution of problem (N) in $Q$ with initial datum $U_0$, then:

(a) for every $j = 1, \ldots, p + 1$ the restriction $U_j := U \downarrow Q_j$ has a continuous representative $\tilde{U}_j$ in $\overline{Q}_j$;

(b) for every $j = 1, \ldots, p$ there exists a unique $\tau_j \in (0, T]$ such that

\[
\tilde{U}_j(x_j, t) \neq \tilde{U}_{j+1}(x_j, t) \quad \Leftrightarrow \quad t \in [0, \tau_j);
\]

(c) for every $j = 1, \ldots, p + 1$ the representative $\tilde{U}_j$ is Lipschitz continuous with respect to $t$ in $\overline{Q}_j$: for all $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$, and $x \in \overline{T}_j$

\[
\inf_{u \in \mathbb{R}} (-H(u)) \leq \tilde{U}_j(x, t_1) - \tilde{U}_j(x, t_2) \frac{t_1 - t_2}{\leq \sup_{u \in \mathbb{R}} (-H(u))}.
\]

(3.29)

**Remark 3.4.** Let assumptions $(H_1)$-$(H_2)$ be satisfied, and let $U$ be the viscosity solution of problem (N) in $Q$ with initial datum $U_0$ (existence of $U$ is ensured by Theorem 3.3 below). As a by-product of Theorem 3.3 we obtain several results:

(i) If $U_0(x^+) = U_0(x^-)$, then $U^*(x, t) = U_*(x, t)$ for all $t \in [0, T]$.

(ii) If $U_0$ has a jump discontinuity at $x_j$, then for every $t \in (0, T]$ there holds

\[
U^*(x_j, t) - U_*(x_j, t) = |\tilde{U}_j(x_j, t) - \tilde{U}_{j+1}(x_j, t)|,
\]

(3.30)

and by Theorem 3.3(b) there exists $\tau_j \in (0, T]$ such that

\[
\begin{cases}
U^*(x_j, t) - U_*(x_j, t) \neq 0 & \text{for } t \in [0, \tau_j) \\
U^*(x_j, t) - U_*(x_j, t) = 0 & \text{for } t \in [\tau_j, T] \quad \text{if } \tau_j < T.
\end{cases}
\]

(3.31)

(iii) If $U_0$ has a jump discontinuity at $x_j$, then for all $t \in [0, \tau_j)$ there holds

\[
U^*(x_j, t) - U_*(x_j, t) = \tilde{U}_j(x_j, t) - \tilde{U}_{j+1}(x_j, t) > 0 \quad \text{if } U_0(x_j^+) > U_0(x_j^-),
\]

(3.32)
Let important consequence of Theorem 3.1 is a comparison principle for such solutions: Proposition 3.4.

(Ω), then (see also Lemma 5.2 and Remark 5.1 below).

Theorem 3.6. Existence. Proposition 4.1.

If \( U(x_j, t) \) is Lipschitz continuous in time, since for every \( t_1, t_2 \in [0, \tau_j] \) we have

\[
|U^*(x_j, t_1) - U^*(x_j, t_2)| - |U^*(x_j, t_1) - U^*(x_j, t_2)| = \leq \left| \tilde{U}_{j+1}(x_j, t_1) - \tilde{U}_{j+1}(x_j, t_2) \right| + \left| \tilde{U}_{j}(x_j, t_1) - \tilde{U}_{j}(x_j, t_2) \right| \leq 2 \|H\|_{L^\infty(\Omega)} |t_1 - t_2|.
\]

Jumps turn out to be nonincreasing in time:

**Proposition 3.4.** Under the assumptions of Theorem 3.3 let \( U \) be a viscosity solution of problem (N) in \( Q \) with initial datum \( U_0 \). Assume that \( U_0(x_j^+) \neq U_0(x_j^-) \) for some \( x_j \in \Omega \), and let \( \tau_j \in (0, T) \) be as in Theorem 3.3 (b).

(i) If \( U_0(x_j^+) > U_0(x_j^-) \), then for every \( 0 \leq t_0 < t_1 < \tau_j \) there holds

\[
U^*(x_j, t_1) - U_*(x_j, t_1) \leq U^*(x_j, t_0) - U_*(x_j, t_0) - \limsup_{p \to -\infty} H(p) - \liminf_{p \to -\infty} H(p) (t_1 - t_0).
\]

(ii) If \( U_0(x_j^+) < U_0(x_j^-) \), then for every \( 0 \leq t_0 < t_1 < \tau_j \) there holds

\[
U^*(x_j, t_1) - U_*(x_j, t_1) \leq U^*(x_j, t_0) - U_*(x_j, t_0) - \limsup_{p \to -\infty} H(p) - \liminf_{p \to -\infty} H(p) (t_1 - t_0).
\]

In addition to the uniqueness of discontinuous viscosity solutions, another important consequence of Theorem 3.1 is a comparison principle for such solutions:

**Theorem 3.5.** Let \( \Omega = (a, b) \) with \( -\infty \leq a < b \leq \infty \), and let \((H_1)-(H_2)\) hold. If \( U \) and \( V \) are viscosity solutions of problem (N) in \( Q \) with initial data \( U_0 \leq V_0 \) a.e. in \( \Omega \), then \( U \leq V \) a.e. in \( Q \).

3.4. Existence.

**Theorem 3.6.** Let \( \Omega = (a, b) \) with \( -\infty \leq a < b \leq \infty \). Let \((H_1)-(H_2)\) hold. Then there exists a viscosity solution \( U \) of problem (N) in \( Q \) with initial condition \( (1.2) \).

4. Comparison: Proofs

To prove Theorem 3.1 we need two preliminary results of independent interest.

**Proposition 4.1.** Let \( H \in C(\mathbb{R}) \). Let \( Q_1 := (c, d) \times (t_1, t_2) \), \( \hat{Q}_1 := [c, d] \times (t_1, t_2) \) with \( -\infty < c < d < \infty \), \( 0 \leq t_1 < t_2 \leq T \).

(i) Let \( U \) be a viscosity subsolution of equation (1.1) in \( Q_1 \), and let \( \varphi \in C^1(\hat{Q}_1) \).

(a) Let \( (c, t_0) \), \( t_0 \in (t_1, t_2) \), be a local maximum point of \( U^* - \varphi \) in \( \hat{Q}_1 \). Then

\[
\varphi_t(c, t_0) + \inf \{ H(\xi_0) \xi \leq \varphi_x(c^+, t_0) \} \leq 0.
\]

(b) Let \( (d, t_0) \), \( t_0 \in (t_1, t_2) \), be a local maximum point of \( U^* - \varphi \) in \( \hat{Q}_1 \). Then

\[
\varphi_t(d, t_0) + \inf \{ H(\xi_0) |\xi \geq \varphi_x(d^-, t_0) \} \leq 0.
\]

(ii) Let \( U \) be a viscosity supersolution of equation (1.1) in \( Q_1 \), and let \( \varphi \in C^1(\hat{Q}_1) \).
Let \((c, t_0), t_0 \in (t_1, t_2]\), be a local minimum point of \(U_* - \varphi\) in \(\hat{Q}_1\). Then
\[
\varphi_t(c, t_0) + \sup\{H(\xi) | \xi \geq \varphi_x(c^+, t_0)\} \geq 0. \tag{4.3}
\]

Let \((d, t_0), t_0 \in (t_1, t_2]\), be a local minimum point of \(U_* - \varphi\) in \(\hat{Q}_1\). Then
\[
\varphi_t(d, t_0) + \sup\{H(\xi) | \xi \leq \varphi_x(d^-, t_0)\} \geq 0. \tag{4.4}
\]

The same results hold if \(\Omega = (a, \infty)\) for all \(a \leq c < d\), and if \(\Omega = (-\infty, b)\) for all \(c < d \leq b\).

**Proof.** Let \(-\infty < c < d < \infty\). We only prove \((4.3)\), the proofs of \((4.2)\) and \((4.4)\) being similar. Let \((c, t_0)\) be a local maximum point of \(U_* - \varphi\) in \(\hat{Q}_1\), then
\[
\limsup_{\overline{\Omega}_1(y, \tau) \rightarrow (c, t_0)} \frac{U^*(y, \tau) - U^*(c, t_0) - \varphi_x(c^+, t_0)(y - c) - \varphi_t(c, t_0)(\tau - t_0)}{[(y - c)^2 + (\tau - t_0)^2]^{1/2}} \leq 0. \tag{4.5}
\]

Clearly, by \((4.5)\) for every \(\xi \geq \varphi_x(c^+, t_0)\) there holds
\[
\limsup_{\overline{\Omega}_1(y, \tau) \rightarrow (c, t_0)} \frac{U^*(y, \tau) - U^*(c, t_0) - \xi(y - c) - \varphi_t(c, t_0)(\tau - t_0)}{[(y - c)^2 + (\tau - t_0)^2]^{1/2}} \leq 0,
\]

thus
\[
\xi := \inf\{\xi | U \in R | (4.0)\} \text{ holds} \leq \varphi_x(c^+, t_0).
\]

First suppose that \(\xi > -\infty\). In this case inequality \((4.0)\) holds with \(\xi = \xi\), hence there exists \(\psi \in C^1(\hat{Q}_1)\) such that \(\psi_x(c, t_0) = \xi, \psi_t(c, t_0) = \varphi_t(c, t_0)\), and \(U_* - \psi\) has a strict maximum at \((c, t_0)\) (e.g., see [25, Proposition 2.6]). Then for any \(\delta > 0\) sufficiently small the function \(U_* - \psi + \delta(x - c)\) has a maximum at some point \((x_\delta, t_\delta) \in (c, d) \times (t_1, t_2]\); observe that \(x_\delta > c\) by the minimality of \(\xi\), and \((x_\delta, t_\delta) \rightarrow (c, t_0)\) as \(\delta \rightarrow 0^+\). By \((4.3)\), for such values of \(\delta\) there holds \(\psi(x_\delta, t_\delta) + H(\psi_x(x_\delta, t_\delta) - \delta) \leq 0\). Letting \(\delta \rightarrow 0^+\) we obtain that \(\varphi_t(c, t_0) + H(\xi) \leq 0\). Since \(\xi \leq \varphi_x(c^+, t_0)\), it follows that \((4.1)\) holds if \(\xi > -\infty\).

Now let \(\xi = -\infty\). Then there exists a sequence \(\xi_n \rightarrow -\infty\) such that inequality \((4.0)\) holds for \(\xi = \xi_n\), thus for all \(\xi \geq \xi_n\) \((n \in \mathbb{N})\). Hence by the arbitrariness of \(n \in (-\infty, \varphi_x(c, t_0)]\) there exists \(\psi \in C^1(\hat{Q}_1)\) such that \(\psi_x(c, t_0) = \xi, \psi_t(c, t_0) = \varphi_t(c, t_0)\), and \(U_* - \psi\) has a strict maximum at \((c, t_0)\).

By Lemma \((4.2)\) there holds
\[
U^*(c, t_0) = \limsup_{\overline{\Omega}_1(y, \tau) \rightarrow (c, t_0)} U^*(y, \tau) = \text{ess lim sup}_{Q_1(y, \tau) \rightarrow (c, t_0)} U^*(y, \tau),
\]

thus there exists a sequence \(\{(y_n, \tau_n)\} \subseteq \overline{Q}_1\) such that
\[
y_n > c, \quad (y_n, \tau_n) \rightarrow (c, t_0), \quad U^*(y_n, \tau_n) \rightarrow U^*(c, t_0). \tag{4.7}
\]

In particular, for every \(\varepsilon > 0\) there exists \(n_\varepsilon \in \mathbb{N}\) such that
\[
U^*(c, t_0) - \varepsilon < U^*(y_n, \tau_n) < U^*(c, t_0) + \varepsilon \quad \text{for all} \quad n > n_\varepsilon.
\]

Let \(\varepsilon > 0\) be fixed, and for \(n > n_\varepsilon\) set
\[
\psi_{\varepsilon, n}(x, t) := \psi(x, t) - 2\varepsilon \left(1 - \left[1 - \frac{x - c}{y_n - c}\right]^2\right) \quad ((x, t) \in \hat{Q}_1).
\]

Then \(\psi_{\varepsilon, n} \in C^1(\hat{Q}_1)\), and
\[
(\psi_{\varepsilon, n})_t = \psi_t, \quad (\psi_{\varepsilon, n})_x = \psi_x - 4\varepsilon \left[\frac{y_n - x}{(y_n - c)^2}\right]_+ \quad \text{in} \quad \hat{Q}_1,
\]
(4.11) \[ \psi_{\varepsilon,n}(c,t) = \psi(c,t) \text{ for all } t \in (t_1, t_2), \]

(4.12) \[ \psi - 2\varepsilon \leq \psi_{\varepsilon,n} \leq \psi \text{ in } \hat{Q}_1. \]

Assuming without loss of generality that \( t_0 < t_2 \), we fix any \( \sigma \in (0, \min\{d - c, t_0 - t_1, t_2 - t_0\}) \), and set \( Q_2 := (c, c + \sigma) \times (t_0 - \sigma, t_0 + \sigma). \)

**Claim:** For any \( \varepsilon > 0 \) small enough and \( n > n_\varepsilon \), \( U^* - \psi_{\varepsilon,n} \) has a maximum in \( \hat{Q}_1 \), attained at a point \( (x_{n,\varepsilon}, t_{n,\varepsilon}) = (x_n, t_n) \in Q_2. \)

In fact, since \( U^* - \psi \) has a strict maximum at \( (c, t_0) \) and \( (4.12) \) holds, for all \( \varepsilon > 0 \) small enough and \( n > n_\varepsilon \) the function \( U^* - \psi_{\varepsilon,n} \) has a maximum in \( \hat{Q}_1 \), attained at a point \( (x_n, t_n) \in (c, c + \sigma) \times (t_0 - \sigma, t_0 + \sigma). \) Let us prove that \( x_n > c. \) Otherwise, were \( x_n = c \), it would follow that \( t_n = t_0 \), since \( U^* - \psi \) has a strict maximum at \( (c, t_0) \) and \( \psi_{\varepsilon,n}(c,t) = \psi(c,t) \) for every \( t \in (t_0 - \sigma, t_0 + \sigma) \) (see \( (4.11) \)). On the other hand, by \( (4.8) \) and the equality \( \psi_{\varepsilon,n}(y_n, \tau_n) = \psi(y_n, \tau_n) - 2\varepsilon \) (see \( (4.9) \)), we have that

\[ U^*(y_n, \tau_n) - \psi_{\varepsilon,n}(y_n, \tau_n) = U^*(y_n, \tau_n) - \psi(y_n, \tau_n) - 2\varepsilon > U^*(c, t_0) - \psi(y_n, \tau_n) + \varepsilon. \]

Since \( \psi(y_n, \tau_n) \to \psi(c, t_0) \) (see \( (4.7) \)), for every \( n \) sufficiently large we also get

\[ U^*(y_n, \tau_n) - \psi_{\varepsilon,n}(y_n, \tau_n) > U^*(c, t_0) - \psi(c, t_0) = U^*(c, t_0) - \psi_{\varepsilon,n}(c, t_0), \]

a contradiction since \( y_n > c \) by \( (4.7) \). Hence the Claim has been proved.

Since \( U \) is a viscosity subsolution of \( U_t + H(U) = 0 \) in \( Q_1 \), by the above Claim, inequality \( (3.4) \) and the first equality in \( (4.10) \) there holds

(4.13) \[ (\psi_{\varepsilon,n})_t(x_n, t_n) + H((\psi_{\varepsilon,n})_x(x_n, t_n)) = \psi_t(x_n, t_n) + H((\psi_{\varepsilon,n})_x(x_n, t_n)) \leq 0. \]

Since \( (\psi_{\varepsilon,n})_x(x_n, t_n) \leq \psi_x(x_n, t_n) \) (see \( (4.10) \)), from \( (4.13) \) we get

(4.14) \[ \psi_t(x_n, t_n) + \inf\{H(s) | s \leq \psi_x(x_n, t_n)\} \leq \psi_t(x_n, t_n) + H((\psi_{\varepsilon,n})_x(x_n, t_n)) \leq 0. \]

On the other hand, letting first \( n \to \infty \) and then \( \varepsilon \to 0^+ \) in \( (4.12) \) we find that \( (x_n, t_n) = (x_{n,\varepsilon}, t_{n,\varepsilon}) \to (c, t_0) \) (recall that \( c \leq x_{n,\varepsilon} \leq c + \sigma \) and \( \sigma > 0 \) is arbitrarily fixed). Since \( \psi_x(c, t_0) = \xi \), from \( (4.13) \) we obtain that

\[ \psi_t(c, t_0) + \inf\{H(s) | s \leq \xi\} \leq 0 \text{ for all } \xi \in (-\infty, \varphi_x(c, t_0)]. \]

Hence inequality \( (4.11) \) also holds if \( \xi = -\infty \). This completes the proof. \( \square \)

**Proposition 4.2.** Let \( H \in \text{Lip}(\mathbb{R}) \). Let the trapezoid \( A \) be defined by \( (3.1) \), let

\[ \hat{A} := \{(x, t) | x \in [a, d - \|H'\|_{\infty}], t \in (0, T)\}, \]

and let \( \varphi \in C^1(\hat{A}). \)

(i) Let \( U \) be a viscosity subsolution of equation \( (1.1) \) in \( \hat{A} \). Let \( (d - \|H'\|_{\infty})t_0, t_0) \), \( t_0 \in (0, T) \), be a local maximum point of \( U^* - \varphi \) in \( \hat{A} \). Then

(4.15) \[ \varphi_t(d - \|H'\|_{\infty}, t_0) + H(\varphi_x((d - \|H'\|_{\infty}^{-1}, t_0)) \leq 0. \]

(ii) Let \( V \) be a viscosity supersolution of equation \( (1.1) \) in \( \hat{A} \). Let \( (d - \|H'\|_{\infty})t_0, t_0) \), \( t_0 \in (0, T) \), be a local minimum point of \( V - \varphi \) in \( \hat{A} \). Then

(4.16) \[ \varphi_t(d - \|H'\|_{\infty}, t_0) + H(\varphi_x((d - \|H'\|_{\infty}^{-1}, t_0)) \geq 0. \]

Similar results hold for the trapezoids \( B \) and \( C \).
Proof. We only prove claim (i) for $A$. The remaining proofs are similar. Set
\[
W(y, t) := U(y - \|H\|_{\infty}, t) \quad \text{for } (y, t) \in \bar{A},
\]
where $A := [a + \|H\|_{\infty}T, d] \times [0, T]$ (observe that under the inverse transformation $(y, t) \mapsto (x, t) := (y - \|H\|_{\infty}, t)$ the set $A$ is mapped onto the proper subset \{(x, t) \mid x \in [a + \|H\|_{\infty}(T - t), d - \|H\|_{\infty}t], t \in [0, T]\} \subset \bar{A}

We set $H(p) := H(p) + \|H\|_{\infty}p$ for $p \in \mathbb{R}$. Then $H$ is Lipschitz continuous on $\mathbb{R}$ and nondecreasing. We claim that $W$ is a viscosity subsolution in $\bar{A}$ of the equation
\[
(4.17)
W_t + \tilde{H}(W_y) = 0 \quad \text{in } \bar{A}.
\]
In fact, fix any $\psi = \psi(y, t) \in C^1(\bar{A})$, where $\bar{A} := [a + \|H\|_{\infty}T, d] \times (0, T]$, and let $W^* - \psi$ have a local maximum at $(\bar{y}, \bar{t}) \in \bar{A}$; here by definition (see (2.1))
\[
W^*(y, t) = \text{ess lim sup}_{A \ni (y, t) \to (\bar{y}, \bar{t})} W(\eta, \tau) \quad \text{for any } (y, t) \in \bar{A}.
\]
It is easily seen that $W^*(y, t) = U^*(y - \|H\|_{\infty}t, t)$ for every $(y, t) \in \bar{A}$ such that $a + \|H\|_{\infty}T < y \leq d$. Therefore $U^* - \varphi$, where $\varphi = \varphi(x, t) := \psi(x + \|H\|_{\infty}t, t)$, has a local maximum at $(\bar{y} - \|H\|_{\infty}\bar{t}, \bar{t}) \in \bar{A}$. Since $U$ is a viscosity subsolution of equation (1.1) in $\bar{A}$, by (3.4), we obtain the claim:
\[
\varphi_t(\bar{y} - \|H\|_{\infty}\bar{t}, \bar{t}) + H(\varphi_y(\bar{y} - \|H\|_{\infty}\bar{t}, \bar{t})) =
\]
\[
\varphi_t(\bar{y}, \bar{t}) + \|H\|_{\infty}\psi_y(\bar{y}, \bar{t}) + H(\psi_y(\bar{y}, \bar{t})) = \varphi_t(\bar{y}, \bar{t}) + \tilde{H}(\psi_y(\bar{y}, \bar{t})) \leq 0.
\]
Now let $\varphi \in C^1(\bar{A})$, and let $(d - \|H\|_{\infty}t_0, t_0)$ $(t_0 \in (0, T)]$ be a local maximum point of $U^* - \varphi$ in $\bar{A}$. Then $(d, t_0)$ is a local maximum point of $W^* - \psi$ in $\bar{A}$, where $\psi \in C^1(\bar{A})$ is defined by $\psi = \psi(y, t) := \varphi(y - \|H\|_{\infty}t, t)$. Since $W$ is a viscosity subsolution in $\bar{A}$ of equation (4.17) and $\tilde{H}$ is nondecreasing, by (4.12) we have that
\[
0 \geq \varphi_t(d, t_0) + \inf\{\tilde{H}(\xi) \mid \xi \geq \psi_y(d, t_0)\} = \varphi_t(d, t_0) + \tilde{H}(\psi_y(d, t_0)) =
\]
\[
\varphi_t(d, t_0) + \|H\|_{\infty}\psi_y(d, t_0) + H(\psi_y(d, t_0)) =
\]
\[
\varphi_t(d, t_0) + \|H\|_{\infty}H(\psi_y(d, t_0)) =
\]
\[
\varphi_t(d, t_0) + \|H\|_{\infty}^2(\psi_y(d, t_0))^2 + H(\varphi_y((d - \|H\|_{\infty}t, t_0)).
\]
Hence inequality (1.15) follows.

Now we can prove Theorem 3.1 for trapezoidal domains.

**Proposition 4.3.** Let $(H_1)$ hold, and let $A$ be defined by (3.1). Let $U$ and $V$ be a viscosity sub- and supersolution of $(N)$ with the same boundary conditions. Then
\[
(4.18) \quad \max_A [U^* - V_*]_+ \leq \max_{[a,d]} [U^*(:,0) - V_*(:,0)]_+.
\]
Similar inequalities hold for $B$ and $C$ defined by (3.2) and (3.3).

**Proof.** We only deal with $A$. The proof is similar for $B$ and easier for $C$. Arguing by contradiction we suppose that for some $\sigma > 0$
\[
(4.19) \quad \max_A [U^* - V_*]_+ > \max_{[a,d]} [U^*(:,0) - V_*(:,0)]_+ + \sigma.
\]
Consider the function $F : A^2 \to \mathbb{R},$
\[
(4.20) \quad F = F(x, t, y, s) := U^*(x, t) - V_*(y, s) - \lambda(t + s) - \frac{|x - y + \alpha|^p + |t - s|^p}{\varepsilon^p},
\]
where $\lambda \in (0, \frac{\sigma}{4T})$ is fixed, $p \in (1, 2]$, $\varepsilon \in (0, \min\{\varepsilon_0, 1\})$ with

$$
\varepsilon_0 := \begin{cases}
\sqrt{\frac{d-n}{|m_1|}} & \text{if } m_1 \in \mathbb{R} \setminus \{0\}, \\
1 & \text{if } m_1 = 0, \\
d - a & \text{if } m_1 = \pm \infty.
\end{cases}
$$

This implies that $|\alpha| \varepsilon \leq \frac{d - a}{2}$, and so $\frac{a + d}{a} + \alpha \varepsilon \in [a, d]$.

Since $F$ is upper semicontinuous, it attains the maximum in $A^2$ at some point $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$. Observe that $F$ and $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ depend on $\varepsilon$ and $p$, but for notational simplicity we suppress this dependence. In view of (4.21), $F$ also depends on $m_1$.

Since $\frac{a + d}{a} + \alpha \varepsilon \in [a, d]$, there holds

$$
F\left(\left(\frac{a + d}{a} + \alpha \varepsilon, 0\right)\right) \leq F(\bar{x}, \bar{t}, \bar{y}, \bar{s}),
$$

which implies that

$$
\left|\bar{x} - \bar{y} + \alpha \varepsilon\right|^p + |\bar{t} - \bar{s}|^p \leq 4M, \text{ where } M := \max\left\{|\|U^*\|_{L^\infty(\mathcal{A})}, \|V_*\|_{L^\infty(\mathcal{A})}\right\}.
$$

Hence there holds $|\bar{t} - \bar{s}| \leq (4M)^{\frac{1}{p}} \varepsilon$ and

$$
|\bar{x} - \bar{y} + \alpha \varepsilon| \leq (4M)^{\frac{1}{p}} \varepsilon \quad \Rightarrow \quad |\bar{x} - \bar{y}| \leq \begin{cases}
\left((4M)^{\frac{1}{p}} + \frac{|m_1|}{2}\right) \varepsilon & \text{if } m_1 \in \mathbb{R}, \\
\left((4M)^{\frac{1}{p}} + \frac{1}{2}\right) \varepsilon & \text{if } m_1 = \pm \infty.
\end{cases}
$$

Observe that both estimates can be made independent of $p \in (1, 2]$.

Now consider the function $g : A \rightarrow \mathbb{R}$,

$$
g(x, t) := U^*(x, t) - V_*(x, t) - 2\lambda t = F(x, t, x, t) + |\alpha|^p.
$$

Observe that $g$ is upper semicontinuous, thus its maximum in $A$ exists.

Set $A_\tau := \{(x, t) \in A \mid t \in [0, \tau]\}$ for any $\tau \in (0, T]$, $A_T := A$. Below we prove the following claims.

**Claim 1:** There exists $t(\varepsilon, (0, T))$ such that

$$
\max_{A_\tau} g = \max_{A} g, \quad \text{and} \quad \max_{A_\tau} g > \max_{A} g.
$$

**Claim 2:** There exists $\varepsilon_1 \in (0, \varepsilon_0)$ which does not depend on $p \in (1, 2]$ such that for all $\varepsilon \in (0, \varepsilon_1)$: if either $m_1 \in \mathbb{R}$ and $p = 2$, or $m_1 = \pm \infty$ and $p \in (1, 2]$, there holds

$$
\max_{A^2} \left. F \right|_{(A_\tau, A_\tau)} = \max_{A^2} F \quad \text{with } \tau \in (0, T) \text{ given by Claim 1}.
$$

To prove Claim 1 set $G(t) := \max_{A} g \ (t \in (0, T])$. By (4.19) and since $\lambda < \frac{\sigma}{4T}$, we have

$$
G(T) \geq \max_{A} (U^*(\cdot, 0) - V_*(\cdot, 0) - 2\lambda T = \max_{A} [U^*(\cdot, 0) - V_*(\cdot, 0)]_+ - 2\lambda T
$$

$$
= \max_{[a, d]} [U^*(\cdot, 0) - V_*(\cdot, 0)]_+ + \sigma - 2\lambda T \geq \max_{[a, d]} [U^*(\cdot, 0) - V_*(\cdot, 0)]_+ + \frac{\sigma}{2}.
$$

Since $G$ is nondecreasing, there exists $\lim_{t \to 0^+} G(t) =: L_0$, and Claim 1 follows if we prove that

$$
L_0 \leq \max_{[a, d]} [U^*(\cdot, 0) - V_*(\cdot, 0)]_+.
$$

In fact, by (4.26) and (4.27) there exists $\tau \in (0, T)$ such that

$$
G(\tau) = \max_{A} g \leq \max_{[a, d]} [U^*(\cdot, 0) - V_*(\cdot, 0)]_+ + \frac{\sigma}{4} < G(T) = \max_{A} g.
$$
To prove (4.27), let \(\{t_n\}\) be a decreasing sequence such that \(t_n \to 0^+\), and let \((x_n, \tau_n) \in A_{\tau_n}\) be a maximum point - namely, \(G(t_n) = g(x_n, \tau_n)\). Clearly, there exists a converging subsequence (not relabelled) of \(\{(x_n, \tau_n)\} \subseteq A_t\), and a point \((\tilde{t}, \tilde{\tau})\), \(\tilde{\tau} \in [a, d]\), such that \((x_n, \tau_n) \to (\tilde{t}, \tilde{\tau})\) as \(n \to \infty\) (observe that \(\lim_{n \to \infty} \tau_n = \lim_{n \to \infty} t_n = 0\)). Then by the upper semicontinuity of \(g\) there holds
\[
L_0 = \lim_{n \to \infty} G(t_n) = \lim_{n \to \infty} \max_{A_{\tau}} g \leq g(\tilde{\tau}, 0^+) = U^*(\tilde{\tau}, 0^+) - V_*(\tilde{\tau}, 0^+) \leq \max_{[a, d]} [U^*(\cdot, 0) - V_*(\cdot, 0)]\,.
\]
This proves (4.27), hence Claim 1 follows.

To prove Claim 2, we preliminarily observe that, by (4.21) and (4.22), for every maximum point \((\bar{x}, \bar{t}, \bar{y}, \bar{s})\) of \(F\) there holds
\[
\max_A g - |\alpha|^p \leq F(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \leq U^*(\bar{x}, \bar{t}) - V_*(\bar{y}, \bar{s}) - \lambda(\bar{t} + \bar{s})
\]
In view of (4.21), this implies that
\[
(4.29) \quad \max_A g - \frac{m_1^2}{4} \varepsilon \leq U^*(\bar{x}, \bar{t}) - V_*(\bar{y}, \bar{s}) - \lambda(\bar{t} + \bar{s}) \quad \text{if} \quad m_1 \in \mathbb{R} \quad \text{and} \quad p = 2
\]
\[
(4.30) \quad \max_A g - \sqrt{\varepsilon} \leq U^*(\bar{x}, \bar{t}) - V_*(\bar{y}, \bar{s}) - \lambda(\bar{t} + \bar{s}) \quad \text{if} \quad m_1 = \pm \infty \quad \text{and} \quad p \in (1, 2)
\]
Now we argue by contradiction. Let \(m_1 \in \mathbb{R} \) and \(p = 2\). Were Claim 2 false, there would exist a sequence \(\{\varepsilon_n\} \subset \langle 0, \varepsilon_0 \rangle\) such that \(\varepsilon_n \to 0^+\), and a sequence of maximum points of \(F\)
\[
\{ (\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n) \} \equiv \{ (\bar{x}(\varepsilon_n, \bar{t}(\varepsilon_n), \bar{y}(\varepsilon_n), \bar{s}(\varepsilon_n)) \} \subset A_\tau\,.
\]
By the boundedness of \(\{(\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n)\}\) and the second inequality in (4.22), there would exist a converging subsequence (not relabelled) of \(\{(\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n)\}\) and a point \((\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t}) \in (A_\tau)^2\), such that \((\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n) \to (\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t})\) as \(n \to \infty\). By the upper semicontinuity of the function in the right-hand side of (4.29), rewriting (4.29) with \(\varepsilon = \varepsilon_n\), \((\bar{x}, \bar{t}, \bar{y}, \bar{s}) = (\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n)\) and letting \(n \to \infty\) we would obtain
\[
(4.31) \quad \max_A g \leq g(\tilde{x}, \tilde{t})
\]
which contradicts Claim 1 since \((\tilde{x}, \tilde{t}) \in A_\tau\). Hence Claim 2 follows in this case.

If \(m_1 = \pm \infty\) and \(p \in (1, 2]\) we argue similarly. Were the claim false, there would exist a sequence \(\{\varepsilon_n\} \subset \langle 0, \varepsilon_0 \rangle\) such that \(\varepsilon_n \to 0^+\), a sequence \(\{p_n\} \subset (1, 2]\) and a sequence of maximum points of \(F\)
\[
\{ (\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n) \} \equiv \{ (\bar{x}(\varepsilon_n, p_n), \bar{t}(\varepsilon_n, p_n), \bar{y}(\varepsilon_n, p_n), \bar{s}(\varepsilon_n, p_n)) \} \subset A_\tau\,.
\]
As before, there would exist a converging subsequence (not relabelled) and a point \((\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t}) \in (A_\tau)^2\), such that \((\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n) \to (\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t})\) as \(n \to \infty\). Letting \(n \to \infty\) in (4.20) rewritten with \(\varepsilon = \varepsilon_n\), \((\bar{x}, \bar{t}, \bar{y}, \bar{s}) = (\bar{x}_n, \bar{t}_n, \bar{y}_n, \bar{s}_n)\) we would obtain again inequality (4.31), a contradiction. Hence we have completed the proof of Claim 2.

Now we complete the proof. Henceforth we assume that \(\varepsilon \in (0, \varepsilon_1]\), so that we can use Claim 2. Then the function
\[
(x, t) \mapsto F(x, t, \bar{y}, \bar{s}) = U^*(x, t) - V_*(\bar{y}, \bar{s}) - \lambda(\bar{t} + \bar{s}) - \frac{|x - \bar{y} + \alpha \varepsilon|^p + |t - \bar{s}|^p}{\varepsilon^p}
\]

\[
=: U^*(x, t) - \phi(x, t)
\]
has a maximum at some point \((x, t) \in A \setminus A_T\) with \(\tau \in (0, T)\) given by Claim 1. Similarly, the function

\[
(y, s) \mapsto -F(x, t, y, s) = V_\epsilon(y, s) - U^*(x, t) + \lambda(x + s) + \frac{|x - y + \alpha \epsilon|}{\epsilon} + |t - s| =: \chi(y, s)
\]

has a minimum at some point \((\bar{y}, \bar{s}) \in A \setminus A_T\) with \(\tau \in (0, T)\) as above. Since \(U\) is a viscosity subsolution and \(V\) a viscosity supersolution of \((N)\), by Definition 3.1 if \(\bar{x} = a, \bar{y} = a\); (2) \(\bar{x} = a, \bar{y} > a\); (3) \(\bar{x} > a, \bar{y} > a\); (4) \(\bar{x} = \bar{y} = a\).

(1) \(\bar{x} > a, \bar{y} > a\): Since \(U\) is a viscosity subsolution and \((\bar{x}, \bar{t})\) is a maximum point of \(U^* - \phi\), there holds

\[
(4.32) \quad \phi_t(\bar{x}, \bar{t}) + H(\phi_x(\bar{x}, \bar{t})) = \lambda + \frac{2(\bar{t} - \bar{s})}{\epsilon^2} + H\left(\frac{2(\bar{x} - \bar{y})}{\epsilon^2} + m_1\right) \leq 0.
\]

This follows from Definition 3.1 if \(\bar{x} < d - \|H'\|_\infty \bar{t}\) (see 3.41), and from Proposition 4.2 (i) if \(\bar{x} = d - \|H'\|_\infty \bar{t}\) (see 4.15). On the other hand, since \(V_\epsilon\) is a viscosity supersolution and \((\bar{y}, \bar{s})\) is a minimum point of \(V_\epsilon - \chi\), there holds

\[
(4.33) \quad \chi_s(\bar{y}, \bar{s}) + H(\chi_y(\bar{y}, \bar{s})) = -\lambda + \frac{2(\bar{t} - \bar{s})}{\epsilon^2} + H\left(\frac{2(\bar{x} - \bar{y})}{\epsilon^2} + m_1\right) \geq 0.
\]

This follows from Definition 3.1 if \(\bar{y} < d - \|H'\|\infty \bar{s}\) (see 3.50), and from Proposition 4.2 (ii) if \(\bar{y} = d - \|H'\|_\infty \bar{s}\) (see 4.16). Subtracting inequality (4.33) from (4.32) we get \(2\lambda \leq 0\), a contradiction.

(2) \(\bar{x} = a, \bar{y} > a\): Since

\[
\phi_t(a, \bar{t}) + H(\phi_x(a, \bar{t})) = \lambda + \frac{2(\bar{t} - \bar{s})}{\epsilon^2} + H\left(\frac{2(a - \bar{y})}{\epsilon^2} + m_1\right) \leq 0,
\]

by Definition 3.2 (i) (see 3.6), we have that

\[
(4.34) \quad \phi_t(a, \bar{t}) + H(\phi_x(a, \bar{t})) = \lambda + \frac{2(\bar{t} - \bar{s})}{\epsilon^2} + H\left(\frac{2(a - \bar{y})}{\epsilon^2} + m_1\right) \leq 0,
\]

whereas now (4.33) reads

\[
\chi_s(\bar{y}, \bar{s}) + H(\chi_y(\bar{y}, \bar{s})) = -\lambda + \frac{2(\bar{t} - \bar{s})}{\epsilon^2} + H\left(\frac{2(a - \bar{y})}{\epsilon^2} + m_1\right) \geq 0.
\]

Subtracting from each other the above inequalities we get again \(2\lambda \leq 0\).

(3) \(\bar{x} > a, \bar{y} = a\): Since

\[
\chi_y(a^*, \bar{s}) = \frac{2(\bar{x} - a)}{\epsilon^2} + m_1 > m_1,
\]
by Definition 3.2 (ii) (see (3.7)) we have that
\[\chi_s(a, \bar{s}) + H(\chi_y(a^+, \bar{s})) = -\lambda + \frac{2(\bar{i} - \bar{s})}{\varepsilon^2} + H\left(\frac{2(\bar{x} - a)}{\varepsilon^2} + m_1\right) \geq 0.\]

On the other hand, now (4.32) reads
\[\phi_t(\bar{x}, \bar{t}) + H(\phi_x(\bar{x}, \bar{t})) = \lambda + \frac{2(\bar{i} - \bar{s})}{\varepsilon^2} + H\left(\frac{2(\bar{x} - a)}{\varepsilon^2} + m_1\right) \leq 0,\]

hence we get again $2\lambda \leq 0$.

(4) $\bar{x} = \bar{y} = a$: In this case there holds $\phi_x(a^+, \bar{t}) = \chi_y(a^+, \bar{s}) = m_1$, and inequalities (4.34), (4.35) (which hold by (3.6)-(3.7) of Definition 3.2) become
\[\phi_t(a, \bar{t}) + H(\phi_x(a^+, \bar{t})) = \lambda + \frac{2(\bar{i} - \bar{s})}{\varepsilon^2} + H(m_1) \leq 0,\]

whence again $2\lambda \leq 0$. This completes the proof if $m_1 \in \mathbb{R}$.

**Problem (N_S)**: $m_1 = \infty$. In this case we choose $\alpha = \sqrt{-p}$ (see (4.21)). We have two possibilities:

1. $\bar{x} = \bar{x}_p \geq a$, $\bar{y} = \bar{y}_p \geq a$ for some $\varepsilon \in (0, \varepsilon_1)$ and $p \in (1, 2]$: Arguing as for problem (N_R), and using Definitions 3.6-3.6 instead of Definition 3.2, we get
\[\phi_t(\bar{x}, \bar{t}) + H(\phi_x(\bar{x}, \bar{t})) = \lambda + \frac{p|\bar{i} - \bar{s}|p^{-1} \text{sgn}(\bar{i} - \bar{s})}{\varepsilon^p} + H\left(\frac{p|\bar{x} - a + \alpha \varepsilon p^{-1} \text{sgn}(\bar{x} - a + \alpha \varepsilon)}{\varepsilon^p}\right) \leq 0,\]

whence again $2\lambda \leq 0$.

2. $\bar{x} = \bar{x}_p \geq a$, $\bar{y} = \bar{y}_p = a$ for every $p \in (1, 2]$ and $\varepsilon \in (0, \varepsilon_1)$: we fix $\varepsilon = \breve{\varepsilon} \in (0, \varepsilon_1)$ so small that
\[\sup_{\varepsilon \in (0, \varepsilon_1)} H(\varepsilon) \leq \limsup_{\varepsilon \to 0} H(\varepsilon) + \frac{1}{2} \lambda, \quad H(1/\breve{\varepsilon}) \geq \limsup_{\breve{\varepsilon} \to \infty} H(\breve{\varepsilon}) - \frac{1}{2} \lambda.\]

Hence $\bar{x} = \bar{x}_p$ and $\bar{y} = \bar{y}_p$ only depend on $p$, as we have chosen $\varepsilon = \breve{\varepsilon}$. Now inequality (4.36) reads
\[\phi_t(\bar{x}, \bar{t}) + H(\phi_x(\bar{x}, \bar{t})) = \lambda + \frac{p|\bar{i} - \bar{s}|p^{-1} \text{sgn}(\bar{i} - \bar{s})}{\varepsilon^p} + H\left(\frac{p|\bar{x} - a + \alpha \varepsilon p^{-1}}{\varepsilon^p}\right) \leq 0.\]

On the other hand, by Proposition 4.1 (ii) (see (4.3)) there holds
\[\chi_*(a, \bar{s}) \leq \sup\{H(\varepsilon) \mid \varepsilon \geq \chi_y(a^+, \bar{s})\} = -\lambda + \frac{p|\bar{i} - \bar{s}|p^{-1} \text{sgn}(\bar{i} - \bar{s})}{\varepsilon^p} + \sup\left\{H(\varepsilon) \mid \varepsilon \geq \frac{p|\bar{x} - a + \alpha \varepsilon p^{-1}}{\varepsilon^p}\right\} \geq 0\]

(since $\bar{x} - a + \alpha \varepsilon > 0$, $\text{sgn}(\bar{x} - a + \alpha \varepsilon) = 1$). Subtracting (4.40) from (4.39) gives
\[2\lambda + H\left(\frac{p|\bar{x} - a + \alpha \varepsilon p^{-1}}{\varepsilon^p}\right) \leq \sup\left\{H(\varepsilon) \mid \varepsilon \geq \frac{p|\bar{x} - a + \alpha \varepsilon p^{-1}}{\varepsilon^p}\right\}.\]
Letting \( p \to 1^+ \) in the above inequality gives
\[
2\lambda + H(1/\varepsilon) \leq \sup \left\{ H(\xi) \middle| \xi \geq 1/\varepsilon \right\},
\]
whence by \ref{eq:4.38} we get \( \lambda \leq 0 \), again a contradiction.

**Problem \((N_S)\), \( m_2 = -\infty \):** In this case we choose \( \alpha = -\frac{\sqrt{2}}{2} \) (see \ref{eq:4.21}). As before, we have two possibilities:

1. \( \bar{x} = \bar{x}_{\varepsilon,p} > a, \ \bar{y} = \bar{y}_{\varepsilon,p} \geq a \) for some \( \varepsilon \in (0,\varepsilon_1) \) and \( p \in (1,2] \); in this case we have again inequalities \ref{eq:4.36}-\ref{eq:4.37}, whence \( 2\lambda \leq 0 \).

2. \( \bar{x} = \bar{x}_p = a, \ \bar{y} = \bar{y}_p \geq a \) for every \( p \in (1,2] \) and \( \varepsilon \in (0,\varepsilon_1) \): we fix \( \varepsilon = \varepsilon \in (0,\varepsilon_1) \) so small that
\[
\inf_{\varepsilon \leq \varepsilon_1} H(\xi) \geq \lim \inf_{\varepsilon \to -\infty} H(\xi) - \frac{1}{2} \lambda, \quad H(-1/\varepsilon) \leq \lim \inf_{\xi \to -\infty} H(\xi) + \frac{1}{2} \lambda
\]
(observe that \( \bar{x} = \bar{x}_p \) and \( \bar{y} = \bar{y}_p \) only depend on \( p \)). By Proposition \ref{prop:1.1} (i) (see \ref{eq:4.1}) there holds
\[
\phi_\varepsilon(a,\bar{t}) + \inf \{ H(\xi) \middle| \xi \leq \phi_\varepsilon(a^+,\bar{t}) \} =
\]
\[
= \lambda + \frac{p|\bar{t} - \bar{s}|^{p-1} \text{sgn}(\bar{t} - \bar{s})}{\bar{p}} + \inf \left\{ H(\xi) \middle| \xi \leq -p|a - \bar{y} + \alpha \varepsilon|^{p-1} \right\} \leq 0
\]
\[
\text{(since} \ a - \bar{y} + \alpha \varepsilon < 0, \ \text{sgn}(a - \bar{y} + \alpha \varepsilon) = -1). \ \text{On the other hand, since} \ \bar{y} > a,
\]
\[
\chi_\varepsilon(a_\bar{y},\bar{s}) + H(\chi_\varepsilon(a_\bar{y},\bar{s})) =
\]
\[
= -\lambda + \frac{p|\bar{t} - \bar{s}|^{p-1} \text{sgn}(\bar{t} - \bar{s})}{\bar{p}} + H \left( \frac{p|a - \bar{y} + \alpha \varepsilon|^{p-1}}{\bar{p}} \right) \geq 0.
\]

Subtracting \ref{eq:4.44} from \ref{eq:4.43} gives
\[
2\lambda + \inf \left\{ H(\xi) \middle| \xi \leq -p|a - \bar{y} + \alpha \varepsilon|^{p-1} \right\} \leq H \left( \frac{-p|a - \bar{y} + \alpha \varepsilon|^{p-1}}{\bar{p}} \right).
\]

Letting \( p \to 1^+ \) in the above inequality gives
\[
2\lambda + \inf \left\{ H(\xi) \middle| \xi \leq -1/\varepsilon \right\} \leq H(-1/\varepsilon),
\]
whence by \ref{eq:4.42} we get \( \lambda \leq 0 \), again a contradiction. \( \square \)

Now we can prove Theorem \ref{thm:3.1}

**Proof of Theorem \ref{thm:3.1}**. (i) Let \( Q = (a,b) \times (0,T) \) with \( -\infty < a < b < \infty \), and \( Q_\tau := \{(x,t) \in Q \middle| t \in [0,\tau]\} \) for any \( \tau \in (0,T] \). Let \( U^*, V_* \) be defined in \( \overline{Q} \) by \ref{eq:2.1}-\ref{eq:2.2}. Set
\[
A_\tau := \{(x,t) \middle| x \in [a,b - \|H'\|_{\infty} t], t \in [0,\tau]\},
\]
\[
B_\tau := \{(x,t) \middle| x \in [a + \|H'\|_{\infty} t, b], t \in [0,\tau]\}
\]
for every \( \tau \in (0,\tau_1] \), with \( \tau_1 \) defined by \ref{eq:2.10}. By Remark \ref{rem:2.3} the restrictions \( U_{1,A} := U \cap A_\tau, V_{1,A} := V \cap A_\tau \) are viscosity sub- and supersolutions of problem \((N)\) in \( A_\tau \) (similarly for the restrictions \( U_{1,B} := U \cap B_\tau, V_{1,B} := V \cap B_\tau \)). Then by Proposition \ref{prop:1.3} there holds
\[
\max_{A_\tau} [(U_{1,A})^* - (V_{1,A})_*] \leq \max_{[a,b]} [(U_{1,A})^*(\cdot,0) - (V_{1,A})_*(\cdot,0)] \leq \max_{[a,b]} [U^*(\cdot,0) - V_*(\cdot,0)].
\]
(notice that \((U_{1,A})^*(x,t) \leq U^*(x,t)\) and \((V_{1,A})_*(x,t) \geq V_*(x,t)\), by the inclusion \(A_{t_1} \subseteq Q\),
\[
\max_{B_{t_1}} [(U_{1,B})^* - (V_{1,B})_*] \leq \max_{[a,b]} [(U_{1,A})^*(\cdot, 0) - (V_{1,A})_*(\cdot, 0)] \leq \max_{[a,b]} [U^*(\cdot, 0) - V_*(\cdot, 0)],
\]
whence, by Remark 2.1,
\[
\max_{[a,b]} [U^* - V_*] \leq \max_{[a,b]} [U^*(\cdot, 0) - V_*(\cdot, 0)]
\]
for every \(\delta \in (0, \tau_1)\). By the arbitrariness of \(\delta\) we get
\[(4.46) \quad \sup_{[a,b] \times (0, \tau_1]} [U^* - V_*] \leq \max_{[a,b]} [U^*(\cdot, 0) - V_*(\cdot, 0)] .
\]

Let \(\delta \in (0, \tau_1)\) be arbitrary and fixed. Arguing as before in the rectangle \(\overline{Q}_{\tau_2-\delta} \setminus \overline{Q}_{\tau_1-\delta}\), where \(\tau_2 := \min \{(b-a)/\|H^\prime\|_\infty, T\}\), we obtain
\[
[U^*(x,t) - V_*(x,t)]_* \leq \max_{[a,b]} [U^*(\cdot, (\tau_1 - \delta)^+) - V_*(\cdot, (\tau_1 - \delta)^)]_*
\]
for every \((x,t) \in [a,b] \times (\tau_1 - \delta, \tau_2 - \delta)\). Since
\[
U^*(\cdot, (\tau_1 - \delta)^+) - V_*(\cdot, (\tau_1 - \delta)^+) \leq U^*(\cdot, \tau_1 - \delta) - V_*(\cdot, \tau_1 - \delta)
\]
(see (2.5)), from the above inequality and (4.40) we obtain that
\[
\sup_{[a,b] \times (0, \tau_2-\delta]} [U^* - V_*] \leq \max_{[a,b]} [U^*(\cdot, 0) - V_*(\cdot, 0)] ,
\]
whence, by the arbitrariness of \(\delta\),
\[
\sup_{[a,b] \times (0, \tau_2]} [U^* - V_*] \leq \max_{[a,b]} [U^*(\cdot, 0) - V_*(\cdot, 0)] .
\]
It is now clear that in a finite number of steps the claim follows.

(ii) Let \(Q = (a, \infty) \times (0, T)\). Consider the family of trapezoids
\[
A := \{(x,t) \mid x \in [a, d - \|H^\prime\|_\infty], t \in [0, T]\}
\]
with \(d - a \geq \|H^\prime\|_\infty T\), and set 
\[
U_A := U \cup A, \quad V_A := V \cup A.
\]
Since \(U_A\) is a viscosity subsolution and \(V_A\) a viscosity supersolution of problem (N) in \(A\) (see Remark 5.3), by Proposition 3.1 we get
\[
\max_{A} [(U_A)^* - (V_A)_*] \leq \max_{[a,b]} [(U_A)^*(\cdot, 0) - (V_A)_*(\cdot, 0)] \leq \max_{[a,d]} [U^*(\cdot, 0) - V_*(\cdot, 0)],
\]
and the conclusion follows.

The proof for \(Q = (-\infty, b) \times (0, T)\) and \(Q = \mathbb{R} \times (0, T)\) is analogous, using trapezoids \(B\), respectively \(C\). \(\square\)

5. Uniqueness and regularity: Proofs

Proof of Proposition 3.3. We only prove when \(\bar{x} \in \Omega\), since the proof when \(\bar{x} \in \partial \Omega\) is similar. Set \(K = \sup_{u \in \mathbb{R}} (-H(u))\). By (2.5), (3.77) follows if we prove the stronger inequality
\[(5.1) \quad U^*(\bar{x}, t) \leq U^*(\bar{x}, t^*_1) + K(t - t_1) \quad \text{for all } t \in (t_1, T].
\]
Let us first consider the case \(\Omega = (a, b)\) with \(-\infty < a < b < \infty\). Let \(Q_{t_1, T} := \{(x,t) \in Q \mid t \in (t_1, T)\}\) and \(U_1 := U \cup Q_{t_1, T}\). By Remark 2.1 there holds
\[
(U_1)^*(x,t) = \begin{cases} 
U^*(x,t) & \text{if } t \in (t_1, T) \\
U^*(x,t^*_1) & \text{if } t = t_1,
\end{cases}
\]
thus in particular \( U_1 \) is a viscosity subsolution of problem \((N)\) in \( Q_{t_1,T} \) (see Remark 3.3). By Lemma 2.1 applied to \((U_1)^*\), for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
U^*(x,t_1^+) \leq U^*(x,t_1^+) + \varepsilon \quad \text{for all} \quad x \in (\bar{x} - \delta, \bar{x} + \delta).
\]

Define \( V : \overline{Q}_{t_1,T} \to \mathbb{R} \) by setting

\[
V(x,t) := \begin{cases} 
U^*(\bar{x}, t_1^+) + \varepsilon + K(t-t_1) & \text{if } (x,t) \in [\bar{x} - \delta, \bar{x} + \delta] \times [t_1,T], \\
\sup_{\bar{Q}_{t_1,T}}(\bar{x} - \delta, \bar{x} + \delta) \times [t_1,T]) \end{cases}
\]

with \( \varepsilon, \delta \) as in (5.2). Clearly \( V = V_* \). If \( \varphi \in C^1(\bar{Q}_{t_1,T}) \) and \((x_0,t_0)\) is a local minimum point of \( V_* - \varphi = V - \varphi \) in \( \hat{Q}_{t_1,T} \), then \( \varphi_t(x_0,t_0) \geq V_t(x_0,t_0) = K \). Therefore,

\[
\varphi_t(x_0,t_0) + H(\varphi_x(x_0,t_0)) \geq \sup_{\bar{Q}_{t_1,T}}(-H(u)) + H(\varphi_x(x_0,t_0)) \geq 0
\]

(with \( \varphi_x(x_0,t_0) \) replaced by \( \varphi_x(a^*, t_0) \) if \( x_0 = a \), and similarly for \( b \)). Therefore, \( V \) is a viscosity supersolution of problem \((N)\) in \( Q_{t_1,T} \). Applying Theorem 3.1 in \( Q_{t_1,T} \), and observing that, by (5.2)-(5.3), \((U_1)^*(\cdot,t_1) = U^*(\cdot,t_1^+) \leq V_*(\cdot,t_1) \) in \( \Omega \), we obtain that \( U_* \leq V \) in \( Q_{t_1,T} \). In particular,

\[
U^*(x,t) - U^*(\bar{x}, t_1^+) \leq \varepsilon + K(t-t_1) \quad \text{for any} \quad |x - \bar{x}| < \delta \quad \text{and} \quad t \in (t_1,T]
\]

and (5.1) follows from the arbitrariness of \( \varepsilon \).

The cases \( \Omega = (a,\infty) \), \( \Omega = (-\infty,b) \) and \( \Omega = \mathbb{R} \) will easily follow arguing as above, replacing the set \( Q_{t_1,T} \) by suitable trapezoidal domains

\[
A_{t_1,T} = \{(x,t) : x \in (a,d - \|H\|_\infty,t), t \in (t_1,T)\}, \quad \text{with} \quad a < \bar{x} < d - \|H\|_T,
\]

\[
B_{t_1,T} = \{(x,t) : x \in (c + \|H\|_\infty,t), t \in (t_1,T)\}, \quad \text{with} \quad c + \|H\|_T < \bar{x} < b,
\]

\[
C_{t_1,T} = \{(x,t) : x \in (c + \|H\|_\infty,t,d - \|H\|_\infty,t), t \in (t_1,T)\}
\]

with \( c + \|H\|_T < \bar{x} < d - \|H\|_\infty T \).

It is similarly seen that

\[
V_*(\bar{x}, t) \geq V_*(\bar{x}, t_1^+) + k(t-t_1) \quad \text{for all} \quad t \in (t_1,T],
\]

where \( k := \inf_{u \in \mathbb{R}} (-H(u)) \), where \( V \) is a supersolution of problem \((N)\) in \( Q \). From (5.4) and (5.5), we get (3.27), hence the result follows.

Next we prove Theorem 5.3 for continuous initial data, namely:

**Proposition 5.1.** Let \( \Omega = (a,b) \) with \( -\infty \leq a < b \leq \infty \). Let \((H_1)\) hold, and let \( U_0 \in C(\Omega) \). Let \( U \) and \( V \) be viscosity solutions of problem \((N)\) in \( Q \) with initial condition \( 1.2 \).

Then

(i) \( U = V \) a.e. in \( Q \);

(ii) \( U \) has a continuous representative \( \hat{U} \) in \( \overline{Q} \);

(iii) \( \hat{U} \) is Lipschitz continuous with respect to \( t \) in \( \overline{Q} \), and satisfies inequality \( \|\hat{U}\|_t \leq \|\hat{U}\|_{t_1} \) for every \( x \in \overline{\Omega} \) and \( t_1, t_2 \in [0,T] \), \( t_1 \neq t_2 \).

**Proof.** Let \( -\infty < a < b < \infty \). Let \( U \) and \( V \) be viscosity solutions of \((N)\) with the same initial data \( U_0 \). Since \( U_0 \in C(\overline{\Omega}) \), there holds \( (U_0)^* = (U_0)_* = U_0 \). Then by (5.2), (5.3)

\[
\max_{\overline{Q}} [U^* - V_*] = \max_{\overline{Q}} [U^*(\cdot,0) - V_* (\cdot,0)] = \max_{\overline{Q}} [(U_0)^* - (U_0)_*] = 0,
\]

\[
\max_{\overline{Q}} [V^* - U_*] = \max_{\overline{Q}} [V^*(\cdot,0) - U_* (\cdot,0)] = \max_{\overline{Q}} [(U_0)^* - (U_0)_*] = 0.
\]
It follows that \( U^* \leq V_* \leq V^* \leq U_* \), thus \( U^* = U_* = V^* = V_* \) in \( \overline{Q} \). Hence there holds \( U = V \) a.e. in \( Q \), and \( U \) has a continuous representative \( \tilde{U} \) in \( \overline{Q} \). By Proposition 5.2 \( \tilde{U} \) is Lipschitz continuous with respect to \( t \) in \( \overline{Q} \) and satisfies (3.29) for every \( x \in \overline{\Omega} \) and \( t_1, t_2 \in [0, T] \), \( t_1 \neq t_2 \). This proves the result if \( Q \) is bounded. If \( Q \) is unbounded we argue similarly, using (5.24) instead of (3.29) . \( \square \)

Using Proposition 5.2 we can prove the following lemma, which is needed to prove Theorem 3.3 for general piecewise continuous initial data.

**Lemma 5.2.** Let \( (H_1) \) hold and let \( U \) be a viscosity solution of problem (N) in \( Q = \Omega \times (0, T) \). Let \( t_0 \in [0, T) \) and \( G \in L^\infty_{\text{loc}}(\overline{\Omega}) \) satisfy the following conditions:

(a) for all \( x \in \overline{\Omega} \),

\[
U^*(x,t_0) = G^*(x), \quad U_*(x,t_0) = G_*(x) ;
\]

(b) for some \( x_0 \in \Omega \) there exist the essential limits \( \text{ess lim}_{x \to x_0}^+ G(x) := G(x_0^+) \), and

\[
G(x_0^+) \neq G(x_0^-) .
\]

Then there exists \( t \in (t_0, T] \) such that for all \( t \in (t_0, T] \):

(i) if \( G(x_0^+) > G(x_0^-) \), then

\[
U^*(x_0^+,t) = U^*(x_0^+,t) > U^*(x_0^-,t) ,
\]

\[
U_*(x_0^+,t) > U_*(x_0^-,t) = U_*(x_0,t) ;
\]

(ii) if \( G(x_0^+) < G(x_0^-) \), then

\[
U^*(x_0^+,t) < U^*(x_0^-,t) = U^*(x_0,t) ,
\]

\[
U_*(x_0^+,t) = U_*(x_0^+,t) < U_*(x_0^-,t) .
\]

**Remark 5.1.** Let \( U \) be a viscosity solution of problem (N) in \( Q = \Omega \times (0, T) \) with initial condition (1.2). It is easily seen that the conclusions of Lemma 5.2 hold true for \( t_0 = 0 \) and \( G = U_0 \) if the essential limits \( U_0(x_0^+) \) exist and \( U_0(x_0^+) \neq U_0(x_0^-) \). More precisely, under this assumption there exists \( t \in (0, T] \) such that for all \( t \in (0, T] \):

(i) inequalities (5.7) hold, if \( U_0(x_0^+) > U_0(x_0^-) \);

(ii) inequalities (5.8) hold, if \( U_0(x_0^+) < U_0(x_0^-) \).

**Proof.** We only address the case \( G(x_0^+) > G(x_0^-) \) and prove (5.7). The proof of (5.8) is similar. By assumption, for any \( \varepsilon > 0 \) there exists \( \delta \in (0, \varepsilon) \) such that

\[
|G(x) - G(x_0^-)| < \varepsilon \quad \text{for a.e. } x \in (x_0 - \delta, x_0) ,
\]

\[
|G(y) - G(x_0^+)| < \varepsilon \quad \text{for a.e. } y \in (x_0, x_0 + \delta) ,
\]

whence

\[
G(x_0^-) - \varepsilon \leq G_*(x) \leq G^*(x) \leq G(x_0^+) + \varepsilon
\]

for all \( x \in (x_0 - \delta, x_0) \), respectively

\[
G(x_0^+) - \varepsilon \leq G_*(y) \leq G^*(y) \leq G(x_0^+) + \varepsilon
\]

for all \( x \in (x_0, x_0 + \delta) \), respectively.
for all \( y \in (x_0, x_0 + \delta) \). Set as before \( K := \sup_{u \in \mathbb{R}} (-H(u)) \), and \( k := \inf_{u \in \mathbb{R}} (-H(u)) \).

Then we get for every \( t \in (t_0, T] \) and \( x \in (x_0 - \delta, x_0) \):

\[
G(x_0) - \varepsilon + k(t - t_0) \leq G(x) + k(t - t_0) \leq U_*(x, t) \leq U^*(x, t) \leq G(x_0^*) + \varepsilon + K(t - t_0).
\]

Similarly, using (5.10) instead of (5.9) we get for all \( t \in (t_0, T] \) and \( y \in (x_0, x_0 + \delta) \):

\[
G(x_0^*) - \varepsilon + k(t - t_0) \leq G(y) + k(t - t_0) = U_*(y, t) \leq U^*(y, t) \leq G(x_0^*) + \varepsilon + K(t - t_0).
\]

In particular, from the above inequalities we get for all \( t \in (t_0, T] \):

\[
\begin{align*}
(5.11a) & \quad U_*(x, t) \leq U^*(x, t) \leq G(x_0^*) + \varepsilon + K(t - t_0) \quad \text{for any } x \in (x_0 - \delta, x_0), \\
(5.11b) & \quad G(x_0) - \varepsilon + k(t - t_0) \leq U_*(y, t) \leq U^*(y, t) \quad \text{for any } y \in (x_0, x_0 + \delta).
\end{align*}
\]

Now set

\[
\varepsilon_0 := \frac{G(x_0^*) - G(x_0)}{2}, \quad \mathcal{I}(\varepsilon) := \begin{cases} 
\{ t_0 + \frac{2(\varepsilon_0 - \varepsilon)}{K-k} \} & \text{if } K > k \; (\varepsilon \in (0, \varepsilon_0)), \\
T & \text{otherwise.}
\end{cases}
\]

Then for all \( \varepsilon \in (0, \varepsilon_0), \; t \in [t_0, \mathcal{I}(\varepsilon)] \) there holds

\[
G(x_0^*) - \varepsilon + k(t - t_0) \geq G(x_0) + \varepsilon + K(t - t_0).
\]

As a consequence, for all \( \varepsilon \in (0, \varepsilon_0), \; t \in (t_0, \mathcal{I}(\varepsilon)), \; x \in (x_0 - \delta, x_0) \) and \( y \in (x_0, x_0 + \delta) \), from inequalities (5.11) we get

\[
\begin{align*}
(5.14a) & \quad U^*(x, t) \leq G(x_0^*) + \varepsilon + K(t - t_0) \leq G(x_0^*) - \varepsilon + k(t - t_0) \leq U^*(y, t), \\
(5.14b) & \quad U_*(x, t) \leq G(x_0^*) + \varepsilon + K(t - t_0) \leq G(x_0^*) - \varepsilon + k(t - t_0) \leq U_*(y, t).
\end{align*}
\]

Plainly, from inequalities (5.14) we obtain for any \( t \in (t_0, \mathcal{I}(\varepsilon)) \)

\[
U^*(x_0^-, t) = \text{ess lim sup}_{Q(x, \tau) \rightarrow (x_0^-, t)} U^*(x, \tau) \leq G(x_0^-) + \varepsilon + K(t - t_0) < G(x_0) - \varepsilon + k(t - t_0) = U^*_0(t),
\]

\[
U_*(x_0^-, t) = \text{ess lim inf}_{Q(y, \tau) \rightarrow (x_0^-, t)} U_*(x, \tau) \leq G(x_0^-) + \varepsilon + K(t - t_0) < G(x_0^*) - \varepsilon + k(t - t_0) = U_*(x_0^-, t).
\]

(in these estimates the equalities follow by Lemma 2.4 applied to \( Q = (a, x_0) \times (0, T) \),

respectively \( Q = (x_0, b) \times (0, T) \)). Therefore, by Lemma 2.4 we get

\[
U^*(x_0, t) = \text{ess lim sup}_{Q(y, \tau) \rightarrow (x_0, t)} U^*(y, \tau) = U^*(x_0^+, t) > U^*(x_0^-, t),
\]

\[
U_*(x_0, t) = \text{ess lim inf}_{Q(y, \tau) \rightarrow (x_0, t)} U_*(x, \tau) = U_*(x_0^-, t) < U_*(x_0^+, t),
\]

and the conclusion follows.
Remark 5.2. By Lemma [5.2] if [5.6] is satisfied there holds
(5.15) \[ U^*(x_0, t) > U_*(x_0, t) \] for all \( t \in (t_0, t_1) \).

Also observe that [5.12] gives an expression of \( L \), e.g. by choosing \( \varepsilon = \frac{\alpha}{2} \).

The concept of barrier effect of a discontinuity, discussed in the Introduction, is
made precise by the following lemma.

Lemma 5.3. Let \(-\infty \leq a < c < b \leq \infty\), \(0 \leq t_1 < t_2 \leq T\), \(Q = (a,b) \times (0,T)\),
\(Q_1 = (a,c) \times (0,T)\) and \(Q_2 = (c,b) \times (0,T)\). Let \(U\) be a viscosity solution of \((N)\) in \(Q\),
and let \(U_i = U \cap Q^i (i = 1,2)\).

(i) If \(U^*(c^*, t) > U^*(c^-, t)\) for every \(t \in (t_1,t_2)\), then \(U_2\) is a viscosity solution of \((N)\) in \(Q_2\) with \(m_1 = \infty\).

(ii) If \(U_*(c^*, t) > U_*(c^-, t)\) for every \(t \in (t_1,t_2)\), then \(U_1\) is a viscosity solution of \((N)\) in \(Q_1\) with \(m_2 = \infty\).

(iii) If \(U^*(c^+, t) < U^*(c^-, t)\) for every \(t \in (t_1,t_2)\), then \(U_1\) is a viscosity solution of \((N)\) in \(Q_1\) with \(m_2 = -\infty\).

(iv) If \(U_*(c^+, t) < U_*(c^-, t)\) for every \(t \in (t_1,t_2)\), then \(U_2\) is a viscosity solution of \((N)\) in \(Q_2\) with \(m_1 = -\infty\).

Proof. We only prove (i), since the other proofs are similar. Since \((U_2)^* = U^*\) in \(Q_2\) (see Remark [21]) and \(U\) is a viscosity solution of \((N)\) in \(Q\), \(U_2\) is a viscosity solution of \((1.1)\) in \(Q_2\). By Definition [3.3] (i), it remains to prove that if \(\varphi \in C^1(\hat{Q}_2)\) and \((c,t_0)\) is a local maximum point of \((U_2)^* - \varphi\) in \(\hat{Q}_2\), then
(5.16) \[ \varphi(c, t_0) + H(\varphi_+(c^+, t_0)) \leq 0 \quad (t_0 \in (t_1, t_2)). \]

To prove (5.16), let \(t_0 \in (t_1, t_2)\) (if \(t_0 = t_2\) we can argue as in [17] Section 10.2) and observe first that, by assumption, there holds
(5.17) \[ U^*(c^-, t) < U^*(c, t) = U^*(c^+, t) = (U_2)^*(c, t) \quad \text{for all } t \in (t_1, t_2). \]

If \((c,t_0)\) is a strict local maximum point of \((U_2)^* - \varphi\) in \(\hat{Q}_2\), then
(5.18) \[ (U_2)^*(c, t_0) - \varphi(c, t_0) > (U_2)^*(y, \tau) - \varphi(y, \tau) \quad \text{for any } (y, \tau) \in B^*_r(c, t_0) \]
for some \(r > 0\), where \(B^*_r(c, t_0) := \{(y, \tau) \in B_r(c, t_0) \mid y > c\}\). Here \(r\) is chosen such that \(t_1 < t_0 - r < t_0 + r < t_2\). In view of (5.17) this implies that
(5.19) \[ U^*(y, \tau) = (U_2)^*(y, \tau) \quad \text{for all } (y, \tau) \in B^*_r(c, t_0). \]

From (5.18) and (5.19) we get
(5.20) \[ U^*(c, t_0) - \varphi(c, t_0) > U^*(y, \tau) - \varphi(y, \tau) \quad \text{for all } (y, \tau) \in B^*_r(c, t_0). \]

On the other hand, by [5.17] and the upper semicontinuity of \(U^*\) we also have that
\( \limsup_{(y, \tau) \to (c,t), y < c} U^*(y, \tau) \leq U^*(c^-, t) < U^*(c, t) \),
thus for some \(r > 0\) there holds
(5.21) \[ U^*(c, t) > U^*(y, \tau) \quad \text{for any } (y, \tau) \in B^*_r(c, t_0) = \{(y, \tau) \in B_r(c, t_0) \mid y < c\}. \]

Hence we can extend the definition of \(\varphi\) in \(\hat{Q}\) so that \(\varphi_x(c^-, t_0) = \varphi_x(c^+, t_0)\), and
(5.21) \[ U^*(c, t_0) - \varphi(c, t_0) > U^*(y, \tau) - \varphi(y, \tau) \quad \text{for any } (y, \tau) \in B^*_r(c, t_0). \]

By (5.20)-(5.21) \((c, t_0)\) is a local maximum point of \(U^* - \varphi\) in \(Q\), thus by [3.4] we obtain (5.16).
Now we are able to prove the uniqueness of discontinuous viscosity solutions. If $U_0 \in C(\Omega)$, uniqueness follows from Proposition 5.1. Lemma 5.3 will be used to handle possible discontinuities of the solutions if $U_0$ is piecewise continuous.

**Proof of Theorem 5.3** For simplicity we suppose that $U_0$ has a single jump discontinuity at $x_1 \in \Omega = (a, b)$, and that
\[
(5.22) \quad U_0(x_1^+) > U_0(x_1^-).
\]
If $U_0(x_1^+) < U_0(x_1^-)$ and if the number of jumps is finite, the proofs are similar.

Let $U$ and $V$ be two viscosity solutions of $(N)$ in $\Omega$ with initial datum $U_0$. By (5.22) and Remark 5.1 there exists $\delta_1 \in (0, T)$ such that for any $t \in [0, \delta_1)$
\[
(5.23) \quad U^*(x_1, t) = U^*(x_1^+, t) > U^*(x_1^-, t), \quad U_*(x_1, t) > U_*(x_1^+, t) = U_*(x_1, t),
\]
\[
(5.24) \quad V^*(x_1, t) > V^*(x_1^+, t) = V_*(x_1, t), \quad V_*(x_1, t) > V_*(x_1^+, t) = V_*(x_1, t).
\]
Therefore $\tau_1 := \sup \{ t \in (0, T) \mid (5.23) \text{ holds} \} > 0$ and, without loss of generality, we may assume that (5.24) is satisfied for all $t \in (0, \tau_1)$.

Set $Q_{1, \tau_1} := I_1 \times (0, \tau_1), \quad Q_{2, \tau_1} := I_2 \times (0, \tau_1), \quad Q_{\tau_1} := \Omega \times (0, \tau_1)$, where $I_1 \equiv (a, x_1)$, $I_2 \equiv (x_1, b)$. By Remark 5.1 and Lemma 5.3, the restrictions $U_j := U \cap Q_{j, \tau_1}$ and $V_j := V \cap Q_{j, \tau_1}$ ($j = 1, 2$) are viscosity solutions of $(N)$ in $Q_{j, \tau_1}$ (with infinite boundary conditions at least on one side of the lateral boundary of $Q_{j, \tau_1}$), with initial datum $U_{0, j} := U_0 \cap I_j$. Then by Proposition 5.1 there holds $U_1 = V_1$ a.e. in $Q_{1, \tau_1}$ and $U_2 = V_2$ a.e. in $Q_{2, \tau_1}$, thus $U = V$ a.e. in $Q_{\tau_1}$. Moreover, $U_j$ and $V_j$ admit a continuous representative $\hat{U}_j \in C(Q_{j, \tau_1})$, thus $U_j = V_j = \hat{U}_j$ a.e. in $Q_{j, \tau_1}$ ($j = 1, 2$), where $\hat{U}_j$ is Lipschitz continuous with respect to $t$ in $\overline{Q}_{j, \tau_1}$ and satisfies (5.29) (this proves Theorem 5.3(c)). Then it follows from (5.24) that
\[
(5.25) \quad \begin{cases} 
U^*(x_1, t) = U^*(x_1^+, t) = U_*(x_1, t) = \hat{U}_2(x_1, t) \\
U_*(x_1, t) = U_*(x_1^+, t) = U^*(x_1, t) = \hat{U}_1(x_1, t)
\end{cases}
\]
for all $t \in (0, \tau_1)$, and similarly for $V$. Hence by (5.23) there holds for all $t \in (0, \tau_1)$
\[
U^*(x_1, t) - U_*(x_1, t) = V^*(x_1, t) - V_*(x_1, t) = \hat{U}_2(x_1, t) - \hat{U}_1(x_1, t) > 0.
\]

We claim that $\hat{U}_2(x_1, \tau_1) = \hat{U}_1(x_1, \tau_1)$. Arguing by contradiction, it follows from the continuity of $\hat{U}_j$ in each rectangle $\overline{Q}_{j, \tau_1}$ ($j = 1, 2$) that there exists $\eta > 0$ such that $\hat{U}_2(x_1, \tau) - \hat{U}_1(x_1, \tau) \geq \eta$ for all $\tau \in (0, \tau_1)$ sufficiently close to $\tau_1$. Then, by Lemma 5.2 there exists $\delta > 0$, independent of $\tau$ (see (5.24) and Remark 5.2), such that (5.23) holds for every $t \in (\tau, \tau + \delta)$, a contradiction for $\tau > \tau_1 - \delta$ by the very definition of $\tau_1$.

Now observe that, since $U^* = U_* = V^* = V_* = \hat{U}_j$ in $Q_{j, \tau_1}$ and $\hat{U}_j \in C(\overline{Q}_{j, \tau_1})$, for any $t \in (0, \tau_1)$ there holds
\[
(5.26) \quad \sup_{\overline{Q}_{j, \tau_1}} [U^*(x, t) - V_*(x, t)]_* = \sup_{\overline{Q}_{j, \tau_1}} [V^*(x, t) - U_*(x, t)]_* = \hat{U}_2(x_1, t) - \hat{U}_1(x_1, t) = : J(t).
\]

Set $Q_{1, T} := \Omega \times (t, T)$ ($t \in (0, \tau_1)$). For the sake of brevity, we shall only consider the case of a bounded interval $\Omega = (a, b)$ with $-\infty < a < b < \infty$; otherwise, the conclusion will follow arguing in a similar way, considering suitable trapezoidal
By Lemma 2.1 applied to \((5.30)\) and \((5.31)\), for every 0
\[ \frac{U^*(x_j, t) - U^*(x_j, t_0)}{t_1 - t_0} \leq -\limsup_{p \to \infty} H(p), \]
\[ \frac{U_*(x_j, t_1) - U_*(x_j, t_0)}{t_1 - t_0} \geq -\liminf_{p \to \infty} H(p). \]

For every 0 ≤ t₀ < t₁ < τ_j, \((5.33)\) will follow by subtracting the inequalities
\[ \frac{U^*(x_j, t) - U^*(x_j, t_0)}{t_1 - t_0} \leq -\limsup_{p \to \infty} H(p), \]
\[ \frac{U_*(x_j, t_1) - U_*(x_j, t_0)}{t_1 - t_0} \geq -\liminf_{p \to \infty} H(p). \]

We only prove \((5.28a)\). We set \( P := (x, b) \times (t_0, \tau_j) \) and \( U_1 = U \cap P \). Hereafter, for the sake of brevity, we shall only consider the case \( b < \infty \). Otherwise, the conclusion will follow replacing the set \( P \) by some trapezoidal domain
\[ \overline{P} = \{(x, t) : x \in (x_j, d - \|H\|_\infty t), t \in (0, T)\}, \]
with \( d > \|H\|_\infty T + x_j \).

Observe that by Remark \(2.1\) and \((5.27)\) there holds
\[ \begin{cases} \quad (U_1)^*(x, t) = U^*(x, t) & \text{for } (x, t) \in P \\ \quad (U_1)^*(x, t) = U^*(x, t) & \text{for } t_0 < t < \tau_j. \end{cases} \]

By Lemma \(2.1\) applied to \((U_1)^*\), for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ (U_1)^*(x, t_0) \leq (U_1)^*(x, t_0) + \varepsilon \quad \text{for all } x \in [x_j, x_j + \delta]. \]

Let
\[ p_\varepsilon := \frac{\|U\|_{L^\infty(P)} - (U_1)^*(x, t_0) - \varepsilon}{\delta}, \]
and for every \( p > \max\{p_\varepsilon, 0\} \) define \( V : \overline{P} \mapsto \mathbb{R} \) by setting
\[ V(x, t; p) := (U_1)^*(x, t_0) + \varepsilon + p(x - x_j) - (t - t_0)H(p) \quad \text{for } (x, t) \in \overline{P}. \]

By \((5.30)-(5.31)\), for every \( p > \max\{p_\varepsilon, 0\} \) we get
\[ (U_1)^*(x, t_0) \leq V(x, t_0; p) = (U_1)^*(x, t_0) + \varepsilon + p(x - x_j) \quad \text{for all } x \in [x_j, b]. \]

Moreover, \( V \) is a viscosity supersolution of problem
\[ \begin{cases} V_t + H(V_x) = 0 & \text{in } P \\ V_x(x_j, t) = \infty & \text{for } t_0 < t < \tau_j \\ V_x(b, t) = p & \text{for } t_0 < t < \tau_j. \end{cases} \]

By \((5.27)\), Lemma \(3.3(i)\) and Remark \(3.1\) for every \( p > m_2 \) \( U_1 \) is a viscosity subsolution of \((5.33)\) (recall that \( m_2 \) is the boundary condition satisfied by \( U(c, t) \)).
at $b$). Hence from Theorem 3.1 and (5.32), for every $p > \max\{p_c, 0, m_2\}$ we get $(U_1)^* \leq V$ in $\overline{\Omega}$, thus

$$U^*(x_j, t_1) = (U_1)^*(x_j, t_1) \leq V(x_j, t_1) = (U_1)^*(x_j, t_0) + \varepsilon - (t_1 - t_0)H(p) \leq U^*(x_j, t_0) + 2\varepsilon - (t_1 - t_0)H(p)$$

for $t_0 < t_1 < \tau_j$ (here we have also used (5.34)). Choosing in the above inequality $p = p_n$, with $p_n \to +\infty$ such that $\lim_{n \to +\infty} H(p_n) = \limsup_{p \to +\infty} H(p)$, we obtain

$$U^*(x_j, t_1) \leq U^*(x_j, t_0) + 2\varepsilon - (t_1 - t_0)\limsup_{p \to +\infty} H(p),$$

and (5.28a) follows from the arbitrariness of $\varepsilon$. \hfill \square

We have already proved that discontinuous viscosity solutions are unique. Below we show that they also satisfy a comparison principle.

**Proof of Theorem 3.3** Let us first observe that, since by the assumption $U_0 \leq V_0$ a.e. in $\Omega$, there holds

$$\text{(5.34)} \quad (U_0)^* \leq (V_0)^* \quad \text{and} \quad (U_0)_* \leq (V_0)_* \quad \text{in} \quad \overline{\Omega}.$$  

Also observe that

$$S_0 := \{x \in \Omega \mid (U_0)^*(x) > (V_0)^*(x)\} \subseteq \mathcal{J}_0^U \cap \mathcal{J}_0^V,$$

where $\mathcal{J}_0^U$ and $\mathcal{J}_0^V$ denote the set of jump points of $U_0$ and $V_0$, respectively: in fact, at any point $\bar{x} \in (\overline{\Omega} \setminus \mathcal{J}_0^U) \cup (\overline{\Omega} \setminus \mathcal{J}_0^V)$ at least one between $U_0$ and $V_0$ is continuous, thus inequalities (5.34) give

$$\text{(5.35)} \quad (U_0)^*(\bar{x}) \leq (V_0)^*(\bar{x}) = (V_0)_*(\bar{x}) \quad \text{if} \quad V_0 \text{ is continuous in} \ \bar{x},$$

$$\text{(5.35)} \quad (U_0)^*(\bar{x}) = (U_0)_*(\bar{x}) \leq (V_0)_*(\bar{x}) \quad \text{if} \quad U_0 \text{ is continuous in} \ \bar{x}.$$  

In particular, $S_0$ consists of a finite number of points $x_1 < x_2 < \cdots < x_q$, and for every $x_k \in S_0$ the following holds:

(a) $(U_0)^*(x_k) > (U_0)_*(x_k)$ and $(V_0)^*(x_k) > (V_0)_*(x_k)$. In fact, were the claims false, from (5.34) we would get $(U_0)^*(x_k) = (U_0)_*(x_k) \leq (V_0)_*(x_k)$ or $(V_0)^*(x_k) = (V_0)_*(x_k)$, a contradiction since $x_k \in S_0$;

(b) if $U_0(x_k) > U_0(x_k^-)$, then $V_0(x_k^+) > V_0(x_k^-)$. Indeed, since $U_0 \leq V_0$ a.e. in $\Omega$, by assumption $(H_2)$ there holds $V_0(x_k^+) \geq U_0(x_k^-) = (U_0)^*(x_k) > (V_0)^*(x_k)$, thus $V_0(x_k^+) > V_0(x_k^-)$. Analogously, if $U_0(x_k) < U_0(x_k^-)$, then $V_0(x_k) < V_0(x_k^-)$;

(c) there exist $\tau_k', \tau_k'' \in (0, T]$ such that $U$ (respectively $V$) satisfies (5.31) at $x_k$ for all $t \in [0, \tau_k')$ (respectively for all $t \in [0, \tau_k'')$, see Theorem 3.3 (b) and Remark 3.3.

Set $\tau_k := \min\{\tau_k', \tau_k''\}$ ($k = 1, \cdots, q$), $T_1 := \min_{k=1,\cdots,q} \tau_k$. Set also $I_1 := (a, x_1)$, $I_k := (x_{k-1}, x_k)$ for $k = 2, \cdots, q$, $I_{q+1} := (x_q, b)$, $Q_1 := (a, x_1) \times (0, T_1)$, $Q_k := I_k \times (0, T_1)$ and $Q_{q+1} := (x_q, b) \times (0, T_1)$; moreover, let $U_{0,k} := U_0 \llcorner I_k$, $V_{0,k} := V_0 \llcorner I_k$, $U_k := U_0 \llcorner Q_k$ and $V_k := V_0 \llcorner Q_k$. For the sake of brevity, we shall consider below the case $\Omega = (a, b)$, with $-\infty < a < b < +\infty$; the case of an unbounded $\Omega$ can be addressed similarly replacing $Q_{q+1}$ and/or $Q_1$ by trapezoidal domains as in (3.1) and (3.2) (with $a = x_q$ and $b = x_1$, respectively).

Observe that by the very definition of the set $S_0$, and since $U_0 \leq V_0$ a.e. in $\Omega$, for any $k = 1, \cdots, q + 1$ there holds

$$\text{(5.36)} \quad (U_{0,k})^* \leq (V_{0,k})_* \quad \text{in} \quad T_k.$$
Also observe that, by claims (b)-(c) and Lemma 3.3 for every $k = 2, \ldots, q$ the restrictions $U_k$ and $V_k$ are viscosity solutions of $(N)$ in $Q_k$ with initial data $U_{0,k}$ and $V_{0,k}$ respectively, and the same boundary conditions $m_1$ (at $x_{k-1}$) and $m_2$ (at $x_k$):

\begin{equation}
(5.37) \quad m_1 = \begin{cases} 
\infty & \text{if } U_0(x_{k-1}) > U_0(x_{k-1}) \\
-\infty & \text{if } U_0(x_{k-1}) < U_0(x_{k-1}).
\end{cases} \quad m_2 = \begin{cases} 
\infty & \text{if } U_0(x_k) > U_0(x_k) \\
-\infty & \text{if } U_0(x_k) < U_0(x_k).
\end{cases}
\end{equation}

Similar remarks hold in $Q_1$ and $Q_{q+1}$. Then by inequality (5.36), applying Theorem 3.1 to each rectangle $Q_k$, and using inequality (5.39) we get

\begin{equation}
(5.38) \quad (U_k)^* \leq (V_k)^* \quad \text{in } \Omega_k.
\end{equation}

Since both $U_k$ and $V_k$ are piecewise continuous in $\overline{Q}_k$ (see Theorem 3.3 (ii), claim (a)), the above inequality plainly gives $U_k \leq V_k$ a.e. in $Q_k$ for every $k = 1, \ldots, q+1$, whence

\begin{equation}
(5.39) \quad U \leq V \quad \text{a.e. in } \Omega \times (0, T_1).
\end{equation}

If $T_1 = T$ the proof is complete. If not, we assume that there exist a unique $\tilde{k} \in \{1, \ldots, q\}$ such that $T_1 = \tau_{k} < T$ (the argument is similar if more than one $\tilde{k}$ has this property). Set $S_1 := S_0 \setminus \{\tilde{x}_k\}$, $T_2 := \min_{k=1, \ldots, q, k \neq \tilde{k}} \tau_k > T_1$, $\Omega_{k} := I_k \cup I_{k+1}$. Plainly, in $(\Omega \setminus \Omega_{k}) \times (0, T_2)$ we can argue as before, and we obtain that

\begin{equation}
(5.40) \quad U \leq V \quad \text{a.e. in } (\Omega \setminus \Omega_{k}) \times (0, T_2).
\end{equation}

Further, set $\bar{Q}_k := \Omega_k \times (0, T_2)$, $\bar{U}_k := U \cup \bar{Q}_k$ and $\bar{V}_k := V \cup \bar{Q}_k$. Observe that $\bar{U}_k$ and $\bar{V}_k$ are viscosity solutions of problem $(N)$ in $\Omega_{k} \times (\tau, T_2)$ for every $\tau \in (0, T_1)$, with boundary conditions as in (5.37) with: $k = k$ for $m_1$ and $k = k+1$ for $m_2$. Then by Theorem 3.1 for every $(x, t) \in \Omega_{k} \times (\tau, T_2)$ we get

\begin{equation}
(5.41) \quad [(\bar{U}_k)^*(x, t) - (\bar{V}_k)^*(x, t)]_+ \leq \max_{x \in \Omega_{k}} [(\bar{U}_k)^*(x, \tau) - (\bar{V}_k)^*(x, \tau)]_+ = [U^*(x_k, \tau) - V^*(x_k, \tau)]_+.
\end{equation}

Now observe that, if $T_1 = \tau_{k'}$, by Theorem 3.3 (c) and (3.31)-(3.32) there holds

\begin{equation}
(5.42) \quad \lim_{\tau \to T_1} [U^*(x_k, \tau) - V^*(x_k, \tau)]_+ \leq \lim_{\tau \to T_1} [U_*(x_k, \tau) - V_*(x_k, \tau)]_+ \quad + \quad \lim_{\tau \to T_1} [U^*(x_k, \tau) - U_*(x_k, \tau)] = U^*(x_k, T_1) - U_*(x_k, T_1) = 0.
\end{equation}

Similarly, if $T_1 = \tau_{k''}$, then

\begin{equation}
(5.43) \quad \lim_{\tau \to T_1} [U^*(x_k, \tau) - V^*(x_k, \tau)]_+ \leq \lim_{\tau \to T_1} [U^*(x_k, \tau) - V^*(x_k, \tau)]_+ \quad + \quad \lim_{\tau \to T_1} [V^*(x_k, \tau) - V_*(x_k, \tau)] = V^*(x_k, T_1) - V_*(x_k, T_1) = 0.
\end{equation}

Then letting $\tau \to T_1$ in (5.41) we get

\begin{equation}
(\bar{U}_k)^*(x, t) - (\bar{V}_k)^*(x, t) \leq 0 \quad \text{for all } (x, t) \in \Omega_k \times [T_1, T_2].
\end{equation}

Since both $\bar{U}_k$ and $\bar{V}_k$ are piecewise continuous in $\overline{\Omega}_k \times [T_1, T_2]$, from the above inequality we get

\begin{equation}
(5.44) \quad U \leq V \quad \text{a.e. in } \Omega_k \times (T_1, T_2).
\end{equation}
By \textbf{(5.39)}, \textbf{(5.40)} and \textbf{(5.43)} we have that $U \leq V$ a.e. in $\Omega_k \times (0, T_2)$, whence the result follows if $T_2 = T$. Otherwise, we obtain the result by iterating the above argument finitely many times. \hfill $\square$

6. Existence: Proofs

Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$. Let $f_{1, \varepsilon}, f_{2, \varepsilon}, f_{3, \varepsilon} \in C^\infty(\mathbb{R})$ ($\varepsilon \in (0, 1)$) be a partition of unity:

\[
\begin{cases}
0 \leq f_{i, \varepsilon} \leq 1, & \sum_{i=1}^{3} f_{i, \varepsilon} = 1 \quad \text{in } \mathbb{R}, \\
f_{1, \varepsilon} = 1 \quad \text{in } (-\infty, a + 2\sqrt{\varepsilon}], & \text{supp } f_{1, \varepsilon} \subseteq (-\infty, a + 3\sqrt{\varepsilon}], \\
f_{2, \varepsilon} = 1 \quad \text{in } [a + 3\sqrt{\varepsilon}, b - 3\sqrt{\varepsilon}], & \text{supp } f_{2, \varepsilon} \subseteq [a + 2\sqrt{\varepsilon}, b - 2\sqrt{\varepsilon}], \\
f_{3, \varepsilon} = 1 \quad \text{in } [b - 2\sqrt{\varepsilon}, \infty), & \text{supp } f_{3, \varepsilon} \subseteq [b - 3\sqrt{\varepsilon}, \infty),
\end{cases}
\]

such that for $i = 1, 2, 3$

\[
\sup_{\varepsilon \in (0, 1)} \|f'_{i, \varepsilon}\|_{L^1(\mathbb{R})} < \infty, \quad \sup_{\varepsilon \in (0, 1)} \sqrt{\varepsilon} \|f''_{i, \varepsilon}\|_{L^1(\mathbb{R})} < \infty, \quad \sup_{\varepsilon \in (0, 1)} \sqrt{\varepsilon} \|f'''_{i, \varepsilon}\|_{L^1(\mathbb{R})} < \infty.
\]

Let $U_0 \in C^3(\overline{\Omega})$. For every $x \in \overline{\Omega}$, set

\[
(6.1) \quad U_{0,\varepsilon}(x) := U_0(a) + \int_a^x u_{0,\varepsilon}(s)ds,
\]

where

\[
(6.2) \quad u_{0,\varepsilon} := m_1 f_{1, \varepsilon} + m_2 f_{2, \varepsilon} \left[ \eta_{\varepsilon} * U_0' \right] + m_3 f_{3, \varepsilon};
\]

here $m_1, m_2 \in \mathbb{R}$, $\{\eta_{\varepsilon}\}$ is a sequence of standard mollifiers, with supp $\eta_{\varepsilon} \subseteq [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$ (observe that $u_{0,\varepsilon}$ also depends on $m_1, m_2$; we disregard this dependence to make notations simpler). Then $U_{0,\varepsilon} \in C^\infty(\overline{\Omega})$, and $U_{0,\varepsilon}' = u_{0,\varepsilon}$. Moreover, $u_{0,\varepsilon} = m_1$ in $[a, a + \sqrt{\varepsilon}]$ and $u_{0,\varepsilon} = m_2$ in $[b - \sqrt{\varepsilon}, b]$.

\[
\|u_{0,\varepsilon}\|_{L^\infty(\Omega)} \leq \max\{m_1, m_2\}, \quad \|U_0'\|_{L^\infty(\Omega)} \quad \text{for any } \varepsilon \in (0, 1),
\]

\[
(6.3) \quad \sup_{\varepsilon \in (0, 1)} \|u'_{0,\varepsilon}\|_{L^1(\Omega)} < \infty, \quad \sup_{\varepsilon \in (0, 1)} \sqrt{\varepsilon} \|u''_{0,\varepsilon}\|_{L^1(\Omega)} < \infty,
\]

\[
(6.4) \quad u_{0,\varepsilon}(x) \rightarrow U_0'(x) \quad \text{for all } x \in \Omega, \quad U_{0,\varepsilon} \rightarrow U_0' \quad \text{in } C(\overline{\Omega}),
\]

\[
\text{and } u_{0,\varepsilon}' \rightharpoonup U_0' \quad \text{in } L^1(\Omega), \quad u_{0,\varepsilon} \rightarrow U_0' \quad \text{in } L^p(\Omega) \quad \text{for every } 1 \leq p < \infty.
\]

Let $H$ satisfy $(H_1)$. Set

\[
H_{\varepsilon}(u) := g_{\varepsilon}(u) \left[ \eta_{\varepsilon} * H \right](u) - [\eta_{\varepsilon} * H](0) \quad (u \in \mathbb{R}),
\]

where the family $\{g_{\varepsilon}\} \in C^\infty_c(\mathbb{R})$ satisfies $g_{\varepsilon} = 1$ in $(-1/\varepsilon, 1/\varepsilon)$, $0 \leq g_{\varepsilon}(x) \leq 1$ and $|g'(x)| \leq 1$ for any $x \in \mathbb{R}$, supp $g_{\varepsilon} \subseteq (-2/\varepsilon, 2/\varepsilon)$. It is easily seen that

\[
H_{\varepsilon} \rightarrow H \quad \text{uniformly on the compact subsets of } \mathbb{R}.
\]

Let $u_{\varepsilon} \in C^{2,1}(\overline{Q})$ be the unique classical solution of the parabolic problem

\[
(D_{\varepsilon}) \quad \begin{cases}
\sup_{\varepsilon \in (0, 1)} \|\nabla H_{\varepsilon}(u_{\varepsilon})\|_{L^1(\mathbb{R})} < \infty, \\
H_{\varepsilon} \rightarrow H \quad \text{uniformly on the compact subsets of } \mathbb{R}.
\end{cases}
\]
with $m_1, m_2 \in \mathbb{R}$, $u_{0,\varepsilon}$ and $H_\varepsilon$ as above (e.g., see [25]). By the maximum principle and (6.2) there holds
\[(6.6) \quad \|u_\varepsilon\|_{L^\infty(Q)} \leq \max \left\{ |m_1|, |m_2|, \|U_0'\|_{L^\infty(\Omega)} \right\} \quad \text{for any } \varepsilon \in (0,1).
\]
Moreover, there exists $c > 0$ (depending on $m_1$, $m_2$, $\|U_0''\|_{L^1(\Omega)}$, and $\|H\|_{W^{1,\infty}(\mathbb{R})}$) such that for any $\varepsilon \in (0,1)$
\[(6.7) \quad \|u_{xx}\|_{L^\infty(0,T;L^1(\Omega))} \leq c,
\]
\[(6.8) \quad \|u_{xt}\|_{L^\infty(0,T;L^1(\Omega))} \leq c,
\]
\[(6.9) \quad \varepsilon \|u_{xx}\|_{L^\infty(Q)} \leq c.
\]
In fact, arguing as in the proof of [31] Proposition 3.1 (see also [11]) and using (6.3) gives (6.6), whence (6.9) easily follows (see [10] Lemma 5.2 for details).

By estimates (6.6)–(6.8) the family $\{u_\varepsilon\}$ is bounded in $L^\infty(Q)$, and there exists $M > 0$ such that $\sup_{\varepsilon(0,1)} \|u_\varepsilon\|_{W^{1,1}(Q)} \leq M$. Then by embedding theorems there exist a sequence $\{u_{\varepsilon_k}\} \subseteq \{u_\varepsilon\}$ and a function $u \in L^1(Q)$ such that
\[(6.10) \quad u_{\varepsilon_k} \rightarrow u \quad \text{in } L^1(Q) \quad \text{as } k \rightarrow \infty.
\]
It should be mentioned (see [11]) that $u$ is the unique entropy solution of the problem
\[(D_\varepsilon) \quad \begin{cases}
\quad u_t + [H(u)]_x = 0 & \text{in } Q \\
\quad u = m_1 & \text{in } \{a\} \times (0,T) \\
\quad u = m_2 & \text{in } \{b\} \times (0,T) \\
\quad u = U_0' & \text{in } \Omega \times \{0\}.
\end{cases}
\]

The following result will be used (see [10] Lemma 5.4).

**Lemma 6.1.** Let $u$ be given by (6.10). Then for every $t \in (0,T)$
\[(6.11) \quad \|u(\cdot,t)\|_{L^1(\Omega)} \leq \|U_0'\|_{L^1(\Omega)} + 2 \|H\|_\infty t.
\]

It is easily seen that the function
\[(6.12) \quad u_\varepsilon(x,t) := -\int_0^t \{H_\varepsilon(u_\varepsilon(x,s)) - \varepsilon u_{xx}(x,s)\} \, ds + U_{0,\varepsilon}(x) \quad ((x,t) \in \overline{Q})
\]
satisfies $u_{xx} = u_\varepsilon$ in $\overline{Q}$ and is the unique classical solution of the initial-boundary value problem
\[(N_\varepsilon) \quad \begin{cases}
\quad U_{ct} + H_\varepsilon(U_{xx}) = \varepsilon U_{xx} & \text{in } Q \\
\quad U_{xx} = m_1 & \text{in } \{a\} \times (0,T) \\
\quad U_{xx} = m_2 & \text{in } \{b\} \times (0,T) \\
\quad U_\varepsilon = U_{0,\varepsilon} & \text{in } \Omega \times \{0\}.
\end{cases}
\]

Then, by (6.6)–(6.9), with the same constant $c > 0$ as above for any $\varepsilon \in (0,1)$ there holds
\[(6.13a) \quad \|U_{xx}\|_{L^\infty(Q)} \leq \max \left\{ |m_1|, |m_2|, \|U_0'\|_{L^\infty(\Omega)} \right\},
\]
\[(6.13b) \quad \|U_{xx}\|_{L^\infty(0,T;L^1(\Omega))} \leq c,
\]
\[(6.13c) \quad \|U_{xx}\|_{L^\infty(0,T;L^1(\Omega))} \leq c,
\]
\[(6.13d) \quad \varepsilon \|U_{xx}\|_{L^\infty(Q)} \leq c.
\]
\begin{equation}
\|U_{ct}\|_{L^\infty(Q)} \leq c + \|H\|_{\infty},
\end{equation}
where follows from and the equality $U_{ct} = \varepsilon U_{xxx} - H_x(U_{xx})$.

Let us first prove Theorem \ref{thm:viscosity}. When $\Omega$ is bounded, $U_0$ is smooth and $m_1, m_2 \in \mathbb{R}$.

**Proposition 6.2.** Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$, and let $(H_1)$ hold. Then for every $U_0 \in C^\infty(\overline{\Omega})$ there exists a viscosity solution of problem $(N_R)$ with initial condition \ref{initial_condition}. Moreover, $U \in W^{1,\infty}(Q)$ and there holds
\begin{equation}
\|U_x\|_{L^\infty(Q)} \leq \max\{m_1, m_2\}, \|U'_0\|_{L^\infty(\Omega)} ,
\end{equation}
\begin{equation}
\|U_t\|_{L^\infty(Q)} \leq \|H\|_{\infty}.
\end{equation}

**Proof.** By estimates \ref{estimate_1}, \ref{estimate_2} the family $\{U_x\}$ is bounded in $W^{1,\infty}(Q)$. Hence there exist a sequence $\{U_{e_k}\} \subseteq \{U_x\}$ and a function $U \in C(\overline{Q})$, with $U_t, U_x \in L^\infty(Q)$, such that $U_{e_k} \rightarrow U$ in $C(\overline{Q})$ (in particular, $U_{e_k}(0) = U_{0,e_k} \rightarrow U_0$ in $C(\overline{Q})$; see \ref{Lemma_3.5}). Inequality \ref{inequality_4} is a direct consequence of \ref{estimate_3}, since by \ref{estimate_1}, \ref{estimate_2} (possibly extracting a subsequence, not relabelled) $U_{e_k} \rightarrow U$ a.e. in $Q$. As for \ref{Lemma_3.5}, it follows from \ref{Lemma_3.5} as soon as we prove that $U$ is a viscosity solution of problem $(N_R)$.

To this purpose, we shall only check conditions \ref{Lemma_3.5} and \ref{Lemma_3.7}, the proof being analogous for \ref{Lemma_3.5} and \ref{Lemma_3.7}. Without loss of generality we may assume that $\varphi \in C^2(\Omega)$ and $U - \varphi$ assumes a strict local maximum.

(a) Let $U - \varphi$ assume a strict local maximum at $(x, t) \in \Omega \times (0, T)$. Since $U_{e_k} \rightarrow U$ in $C(\overline{Q})$, there exists a sequence $\{(x_k, t_k)\} \subseteq \Omega \times (0, T)$ such that $(x_k, t_k) \rightarrow (x, t)$ as $k \rightarrow \infty$, and the function $U_{e_k} - \varphi$ assumes a local maximum at $(x_k, t_k) \in \Omega \times (0, T)$. The latter property and the regularity of $U_{e_k}$ imply that

$$U_{e_k}(x_k, t_k) = \varphi_x(x_k, t_k), \quad U_{e_k}(x_k, t_k) \leq \varphi_t(x_k, t_k),$$

whence

\begin{equation}
\varphi_t(x_k, t_k) + H_{e_k}(\varphi_x(x_k, t_k)) \leq U_{e_k}(x_k, t_k) + H_{e_k}(U_{e_k}(x_k, t_k)) =
\end{equation}

\begin{equation}
\varepsilon_k U_{e_k}(x_k, t_k) \leq \varepsilon_k \varphi_{xx}(x_k, t_k).
\end{equation}

As $k \rightarrow \infty$, by \ref{Lemma_3.5} we get inequality \ref{inequality_3}. 

(b) Let $U - \varphi$ assume a strict local maximum at $(a, t) \in (0, T)$, and let $\varphi_x(a^+, t) \leq m_1$. Suppose first that $\varphi_x(a^+, t) < m_1$. Arguing as in (a), there exists a sequence $\{(x_k, t_k)\} \subseteq [a, b] \times (0, T)$ such that $(x_k, t_k) \rightarrow (a, t)$ as $k \rightarrow \infty$, and the function $U_{e_k} - \varphi$ assumes a local maximum in $(x_k, t_k)$. Observe that $x_k > a$ for every $k$, for otherwise we would have $m_1 = U_{e_k}(a, t_k) \leq \varphi_x(a, t_k) < m_1$. Hence also in this case \ref{Lemma_3.5} holds, whence as $k \rightarrow \infty$ we get $\varphi_t(a, t) + H(\varphi_x(a^+, t)) \leq 0$, the second inequality in \ref{inequality_3}.

Next, let $\varphi_x(a^+, t) = m_1$. Set
\begin{equation}
\varphi_{\delta}(x, t) := \varphi(x, t) - \delta(x - a) \quad ((x, t) \in \hat{Q}, \, \delta > 0);
\end{equation}

notice that $\varphi_{\delta t} = \varphi_t$, $\varphi_{\delta x} = \varphi_x - \delta$, and $\varphi_{\delta} \rightarrow \varphi$ in $C(\overline{\hat{Q}})$ as $\delta \rightarrow 0$. Then, since $U - \varphi$ has a strict local maximum at $(a, t)$, there exists a sequence $\{(x_{\delta_k}, t_{\delta_k})\} \subset [a, b] \times (0, T)$ such that
\begin{equation}
(x_{\delta_k}, t_{\delta_k}) \rightarrow (a, t) \text{ as } k \rightarrow \infty, \text{ and } U - \varphi_{\delta_k} \text{ has a local maximum at } (x_{\delta_k}, t_{\delta_k}).
\end{equation}

If $x_{\delta_k} \in (a, b)$, as in (a) we obtain that
\begin{equation}
\varphi_t(x_{\delta_k}, t_{\delta_k}) + H(\varphi_x(x_{\delta_k}, t_{\delta_k}) - \delta_k) \leq 0.
\end{equation}
On the other hand, if \( x_{\delta_k} = a \), for every \( k \) sufficiently large we get \( t_{\delta_k} = t \) (recall that \( U - \varphi \) achieves a strict local maximum at the point \( (a, t) \)), hence \( U - \varphi_{\delta_k} \) admits a local maximum at the point \( (a, t) \). Since \( \varphi_{\delta_k}(a, t) = \varphi_x(a^*, t) - \delta_k < m_1 \), by the first part of case (\( \beta \)), we get inequality (6.18) in \( (a, t) \), namely
\[
\varphi_t(a, t) + H(\varphi_x(a^*, t) - \delta_k) \leq 0,
\]
and the conclusion follows by the continuity of \( H \), taking the limit in (6.18)-(6.19) as \( k \to \infty \).

(\( \gamma \)) If \( U - \varphi \) achieves a local maximum at \( (b, t) \), with \( t \in (0, T] \) and \( \varphi_x(b^-, t) > m_2 \), arguing as in step (\( \beta \)) the conclusion follows by considering first the case \( \varphi_x(b^-, t) > m_2 \) and then the case \( \varphi_x(b^-, t) = m_2 \) (we omit the details). Hence the result follows.

\[ \square \]

Next we prove Theorem 3.6 when \( \Omega \) is bounded, \( U_0 \) is smooth and \( m_1, m_2 \in \mathbb{R} \).

**Proposition 6.3.** Let \( \Omega = (a, b) \) with \( -\infty < a < b < \infty \), and let \((H_1)\) hold. Then for every \( U_0 \in C^\infty(\Omega) \) there exists a viscosity solution \( U \) of problem \((N_3)\) with initial condition \((1.2)\). Moreover, \( u \in W^{1,1}(Q) \) and \((6.14)\) holds true.

**Proof.** Let at least one of \( m_1, m_2 \) be infinite. For every \( n \in \mathbb{N} \) consider the problem

\[
(N_n) \quad \begin{cases}
U_t + H(U_x) = 0 & \text{in } Q \\
U_x = m_{1,n} & \text{in } \{a\} \times (0, T) \\
U_x = m_{2,n} & \text{in } \{b\} \times (0, T) \\
U = U_0 & \text{in } \Omega \times \{0\},
\end{cases}
\]

where
\[
m_{i,n} := \begin{cases}
m_i & \text{if } m_i \in \mathbb{R}, \\
\pm n & \text{if } m_i = \pm \infty
\end{cases} \quad (i = 1, 2).
\]

Observe that
\[
m_n := \max\{|m_{1,n}|, |m_{2,n}|\} = n \quad \text{for } n \text{ large enough}.
\]

Let \( U_n \in W^{1,\infty}(Q) \) be the viscosity solution of problem \((N_n)\), which exists by Proposition 6.2. By the proof of Proposition 6.2 \( U_n \) is the uniform limit in \( Q \) of a sequence \( \{U_{n\varepsilon_k}\} \) of solutions of problem \((N_{\varepsilon_k})\) with \( \varepsilon = \varepsilon_k \) and \( m_1 = m_{1,n}, m_2 = m_{2,n} \):

\[
(N_{\varepsilon_k,n}) \quad \begin{cases}
U_{n\varepsilon_k} + H_{\varepsilon_k}(U_{n\varepsilon_k}) + \varepsilon_k U_{n\varepsilon_k xx} & \text{in } Q \\
U_{n\varepsilon_k} = m_{1,n} & \text{in } \{a\} \times (0, T) \\
U_{n\varepsilon_k} = m_{2,n} & \text{in } \{b\} \times (0, T) \\
U_{n\varepsilon_k} = U_{0,\varepsilon_k} & \text{in } \Omega \times \{0\}.
\end{cases}
\]

It is also known that in \( Q \) there holds \( U_{n\varepsilon_k} = u_{n\varepsilon_k} \), where \( u_{n\varepsilon_k} \) is the unique classical solution of problem \((D_{\varepsilon_k})\) with \( \varepsilon = \varepsilon_k \) and \( m_1 = m_{1,n}, m_2 = m_{2,n} \):

\[
(D_{\varepsilon_k,n}) \quad \begin{cases}
\left[ u_{n\varepsilon_k} + H_{\varepsilon_k}(u_{n\varepsilon_k}) \right]_x = \varepsilon_k U_{n\varepsilon_k xx} & \text{in } Q \\
u_{n\varepsilon_k} = m_{1,n} & \text{in } \{a\} \times (0, T) \\
u_{n\varepsilon_k} = m_{2,n} & \text{in } \{b\} \times (0, T) \\
u_{n\varepsilon_k} = U_{0,\varepsilon_k} & \text{in } \Omega \times \{0\}.
\end{cases}
\]

Now let \( w_{n\varepsilon_k} \) be the solution of problem \((D_{\varepsilon_k,n})\) with \( m_1 = m_2 = m_n \) and initial data
\[
w_{n\varepsilon_k}(x, 0) = m_n f_{1,\varepsilon_k}(x) + f_{2,\varepsilon_k}(x) [\eta_{\varepsilon_k} * |U_0'|](x) + m_n f_{3,\varepsilon_k}(x),
\]
where \( \eta_{x_k} \) and \( f_{i,x_k} \) (\( i = 1,2,3 \)) are the functions considered at the beginning of this section (observe that \( w_{\eta_{x_k}}(\cdot,0) \to [U'_0] \) in \( L^1((a,b)) \)). Similarly, let \( w_{(n)\epsilon_k} \) be the solution of problem \((D_{\epsilon_k,n})\) with \( m_1 = m_2 = -m \) and initial data \( w_{(n)\epsilon_k}(\cdot,0) = -w_{\eta_{x_k}}(\cdot,0) \). Then by (6.24) and standard comparison results there holds
\[
(6.21) \quad w_{(n-q)\epsilon_k} \leq w_{(n)\epsilon_k} \leq U_{n\epsilon_k} \leq w_{\epsilon_k} \leq w_{(n+q)\epsilon_k} \quad \text{in } Q
\]
for \( n \in \mathbb{N} \) large enough and \( q \in \mathbb{N} \). By estimates \((6.6) - (6.8)\) and embedding results, there exist subsequences (not relabelled) \( \{w_{(n)\epsilon_k}\}, \{U_{n\epsilon_k}\}, \{w_{\epsilon_k}\}, \) and \( w_{-n}, w_n, z \in L^1(Q) \) such that
\[
(6.22) \quad w_{(n)\epsilon_k} \to w_{-n}, \ U_{n\epsilon_k} \to z, \ w_{\epsilon_k} \to w_n \quad \text{in } L^1(Q) \quad \text{as } k \to \infty.
\]
It is easily seen that \( z = U_{n\epsilon_k} \), thus from \((6.21) - (6.22)\) we get
\[
(6.23) \quad w_{-n} \leq w_{-n} \leq U_{n\epsilon_k} \leq w_n \leq w_{n+q} \quad \text{a.e. in } Q
\]
for \( n \in \mathbb{N} \) large enough and \( q \in \mathbb{N} \). In addition, since by \((6.11)\) for \( n \in \mathbb{N} \) there holds
\[
|w_{\epsilon_k}(\cdot,t)|_{L^1(\Omega)} \leq |U'_0|_{L^1(\Omega)} + 2\|H\|_{\infty} t \quad (t \in (0,T)),
\]
arguing by monotonicity shows that there exist \( w_1, w_2 \in L^1(Q) \) such that
\[
(6.24) \quad w_{-n} \to w_1, \ w_n \to w_2 \quad \text{in } L^1(Q) \quad \text{as } n \to \infty.
\]
Then letting \( q \to \infty \) in \((6.23)\) we obtain
\[
(6.25) \quad w_1 \leq w_{-n} \leq (U_{n\epsilon_k})_t \leq w_n \leq w_2 \quad \text{a.e. in } Q
\]
for all \( n \in \mathbb{N} \) large enough.

It follows that for all \( x_1, x_2 \in \Omega, t \in (0,T) \) and \( n \) as above,
\[
(6.26) \quad |U_n(x_2,t) - U_n(x_1,t)| \leq \left| \int_{x_1}^{x_2} w(x,t) \, dx \right|
\]
where \( w(x,t) := \max\{|w_1(x,t)|, |w_2(x,t)|\} \). On the other hand, by \((6.14b)\),
\[
(6.27) \quad |U_n(x,t_2) - U_n(x,t_1)| \leq \|H\|_{\infty} |t_2 - t_1| \quad (x \in \Omega; t_1, t_2 \in (0,T)).
\]
By inequalities \((6.24) - (6.25)\), the sequence \( \{U_n\} \) is uniformly equicontinuous. Hence, possibly up to a subsequence (not relabelled) there holds
\[
(6.28) \quad U_n \to U \quad \text{in } C(\overline{Q}),
\]
for some \( U \in W^{1,1}(Q) \) with \( U_t \in L^\infty(Q) \). Moreover, by \((6.25)\), \( U_t \) satisfies \((6.14b)\).

Let us prove that \( U \) is a viscosity solution of problem \((N_S)\). Again we only check conditions \((3.3)\) and \((3.6)\), since the proof is analogous for \((3.5)\) and \((3.7)\).

(a) Let \( U - \varphi \) assume a strict local maximum at \((x,t) \in \Omega \times (0,T) \). Then, by \((6.26)\) there exists a sequence \( \{(x_n,t_n)\} \subseteq \Omega \times (0,T) \) such that \((x_n,t_n) \to (x,t) \) as \( n \to \infty \) and the function \( U_n - \varphi \) assumes a local maximum in \((x_n,t_n) \in \Omega \times (0,T) \). Since \( U_n \) is the viscosity solution of problem \((N_n)\), it follows that
\[
(6.29) \quad \varphi_t(x_n,t_n) + H(\varphi(x_n,t_n)) \leq 0.
\]
Letting \( n \to \infty \) in \((6.27)\) gives the conclusion in this case.

(b) We only consider the case \( m_1 = \infty \), since otherwise the proof is analogous to that of Proposition \((6.27)\). Let \( \varphi \in C^1(\overline{\Omega} \times (0,T)) \), and let \( U - \varphi \) assume a strict local maximum in \((a,t), t \in (0,T) \). Fix any \( \delta > 0 \) sufficiently small. Then there exists a sequence \( \{(x_n,t_n)\} \subseteq [a,b] \times (0,T) \) such that: (i) \((x_n,t_n) \to (a,t) \) as \( n \to \infty \), \( 0 < t - \delta \leq t_n \leq t + \delta < T \) for every \( n \) large enough; (ii) the function \( U_n - \varphi \) achieves a local maximum at \((x_n,t_n) \); (iii) there holds \( \varphi_t(x,t) < 0 \) for all \((x,t) \in \overline{\Omega} \times [t - \delta, t + \delta]\).
Since $U_n$ is the viscosity solution of $(N_n)$ and $\varphi_{x}(x_n, t_n) < n$, we get again inequality (6.24), whence the conclusion follows in this case, too. The case when $U - \varphi$ achieves a local maximum in $(b, t)$, with $t \in (0, T)$, can be similarly settled. Hence the result follows also in this case. This completes the proof. □

Now we can prove Theorem 3.3 for general $\Omega$ and $m_1, m_2 \in \mathbb{R}$, provided that $U_0$ is continuous in $\Omega$.

**Proposition 6.4.** Let $\Omega = (a, b)$ with $-\infty \leq a < b \leq \infty$, and let $(H_1)$ hold. Then for every $U_0 \in C(\Omega)$ there exists a viscosity solution of problem $(N)$ with initial condition (1.2).

**Proof.** First suppose $-\infty < a < b < \infty$. Let $U_0 \in C(\Omega)$, and let $\{U_{0,n}\} \subseteq C^\infty(\Omega)$, $U_{0,n} \to U_0$ in $C(\Omega)$. Let $U_n$ be the viscosity solution of problem $(N)$ with initial condition $U_n(\cdot, 0) = U_{0,n}$, which exists by Propositions 6.2-6.3. By Theorem 3.1-(i) (see (6.23)) there holds

(6.28) $\max_{\Omega} |U_m - U_n| \leq \max_{\Omega} |U_{m,n} - U_{0,n}|$ for all $m, n \in \mathbb{N}$.

By (6.28) $\{U_{0,n}\}$ is a Cauchy sequence in $C(\Omega)$, hence there exists $U \in C(\Omega)$ such that $U_n \to U$ in $C(\Omega)$. Arguing as in Proposition 6.3 shows that $U$ is a viscosity solution of problem $(N)$ with initial condition (1.2). Hence the result follows.

Now let $\Omega = (a, \infty)$ (the argument is similar for $\Omega = (-\infty, b)$ or $\Omega = \mathbb{R}$). Let $\Omega_j := (a, b_j)$, $b_j \leq b_{j+1}$ for every $j \in \mathbb{N}$, $b_j \to \infty$ as $j \to \infty$. Let $U_0 \in C(\Omega)$, $U_{0,j} \in C(\Omega_j)$, supp $U_{0,j} \subseteq \Omega_j$, and let $U_{0,j} \to U_0$ uniformly on compact subsets of $[a, \infty)$. Let $U_j$ be the viscosity solution of $(N)$ in $Q_j := \Omega_j \times (0, T)$ with initial condition $U_j(\cdot, 0) = U_{0,j}$ in $\Omega_j$, with the given boundary condition $U_{j,\varepsilon} = m_1$ at $\{a\} \times (0, T)$ and arbitrary boundary condition $U_{j,\varepsilon} = m_2$ at $\{b_j\} \times (0, T)$. For every $b > a$ set $K := [a, b] \times [0, T]$, and let $j_0 \in \mathbb{N}$ be fixed such that $b_j > b + \|H\|_{\infty}T$ for all $j \geq j_0$. Applying inequality (4.18) to the trapezoid $\tilde{Q}$ with basis $[a, b + \|H\|_{\infty}T] \times \{0\}$ we obtain in particular for every $j, k \geq j_0$

$$\max_{K} |U_j - U_k| \leq \max_{[a, b + \|H\|_{\infty}T]} |U_{0,j} - U_{0,k}|.$$ 

By the above inequality $\{U_j\}$ is a Cauchy sequence, thus a converging sequence in $C(K)$. Then by the arbitrariness of $K$ and a diagonal argument there exists a subsequence of $\{U_j\}$ (not relabelled) and $U \in C(\Omega)$ such that $U_j \to U$ uniformly on the compact subsets of $\Omega$. Arguing as before it is shown that $U$ is a viscosity solution, thus the conclusion follows. □

Finally we prove Theorem 3.6 in the general case.

**Proposition 6.5.** Let $\Omega = (a, b)$ with $-\infty \leq a < b \leq \infty$, and let $(H_1), (H_2)$ hold. Then there exists a viscosity solution of problem $(N)$ with initial condition (1.2).

**Proof.** Set $U_{0,j} := U_0 \cap I_j$ ($j = 1, \ldots, p + 1$). For every $j = 2, \ldots, p$ let $U_j$ be the viscosity solution of $(N_j)$ in $Q_j := I_j \times (0, T)$ with initial condition $U_j(\cdot, 0) = U_{0,j}$ in $I_j$, and

(i) $m_1 = m_2 = \infty$, if min $\{J_0(x_{j-1}), J_0(x_j)\} > 0$;
(ii) $m_1 = \infty$, $m_2 = -\infty$, if $J_0(x_{j-1}) > 0 > J_0(x_j)$;
(iii) $m_1 = -\infty$, $m_2 = \infty$, if $J_0(x_{j-1}) < 0 < J_0(x_j)$;
(iv) $m_1 = m_2 = -\infty$, if max $\{J_0(x_{j-1}), J_0(x_j)\} < 0$.
(here \(J_0(x_j)\) is defined by (3.28)). Moreover, let \(U_1\) be the viscosity solution of (N) in \(Q_1\) with initial condition \(U_1(\cdot, 0) = U_{0,1}\) in \(I_1\), with \(m_1\) given at \(\{a\} \times (0, T)\) if \(a > -\infty\) (or without boundary condition if \(a = -\infty\)), and with \(m_2 = \pm\infty\) if \(J_0(x_1) \nless 0\). Similarly, let \(U_{p+1}\) be the viscosity solution of (N) in \(Q_{p+1}\) with initial condition \(U_{p+1}(\cdot, 0) = U_{0,p+1}\) in \(I_{p+1}\), with \(m_2\) given at \(\{b\} \times (0, T)\) if \(b < \infty\) (or without boundary condition if \(b = \infty\)), and with \(m_1 = \pm\infty\) if \(J_0(x_{p+1}) \nless 0\). Observe that such viscosity solutions \(U_j\) exist by Proposition 6.1; moreover, \(U_j \in C(\overline{Q_j})\) by Proposition 6.1 (\(j = 1, \ldots, p + 1\)).

Set

\[
\hat{t}_j := \sup \{t \in (0, T) | U_j(x_j^-, t) + U_j(x_j^+, t)\} \quad (j = 1, \ldots, p); \tag{6.29}
\]

observe that, by the continuity of \(U_j\) and \(U_{j+1}\), the definition of \(\hat{t}_j\) is well-posed. Also set \(\tau_1 := \min \{\hat{t}_1, \ldots, \hat{t}_p\}\), and

\[
U := U_j \text{ in } I_j \times [0, \tau_1] \quad (j = 1, \ldots, p + 1). \tag{6.30}
\]

Let us prove that \(U\) is a viscosity solution of (N) in \(Q_{\tau_1} := \Omega \times (0, \tau_1)\) with initial condition 1.2. By Definition 5.1 it suffices to prove that for any \(\varphi \in C(\overline{Q}_{\tau_1})\):

- if \((x, t) \in Q_{\tau_1}\) is a local maximum point of \(U^* - \varphi\) in \(Q_{\tau_1}\), then
  \[
  \varphi_t(x, t) + H(\varphi_x(x, t)) \leq 0;
  \]

- if \((x, t) \in Q_{\tau_1}\) is a local minimum point of \(U_* - \varphi\) in \(Q_{\tau_1}\), then
  \[
  \varphi_t(x, t) + H(\varphi_x(x, t)) \geq 0
  \]

(see (5.4)–(5.5)). Clearly, since \(U_j\) is a viscosity solution of (N) in \(Q_j\), we only need to prove that the above conditions are satisfied for \(x = x_j\) \((j = 1, \ldots, p)\).

To this purpose, observe that for any \(t \in (0, \tau_1]\) and \(j = 1, \ldots, p\)

\[
J_0(x_j) > 0 \Rightarrow \begin{cases} U^*(x_j, t) = U_{j+1}(x_j^+, t) = (U_{j+1})^*(x_j, t), \\ U_*(x_j, t) = U_j(x_j^-, t) = (U_j)^*(x_j, t) \end{cases} \tag{6.31}
\]

\[
J_0(x_j) < 0 \Rightarrow \begin{cases} U^*(x_j, t) = U_j(x_j^-, t) = (U_j)^*(x_j, t), \\ U_*(x_j, t) = U_{j+1}(x_j^+, t) = (U_{j+1})^*(x_j, t) \end{cases} \tag{6.32}
\]

Let \(\varphi \in C(\overline{Q}_{\tau_1})\), and denote by \(\varphi_j\) its restriction to \(\hat{Q}_{j,\tau_1}\). Let \(x_j\) be fixed, and suppose that \(J_0(x_j) > 0\). If \((x_j, t)\) is a local maximum point of \(U^* - \varphi\) in \(Q_{\tau_1}\), by (6.31) it is also a local maximum point of \((U_{j+1})^* - \varphi_{j+1}\) in \(\hat{Q}_{j+1,\tau_1}\), thus

\[
\varphi_t(x_j, t) + H(\varphi_x(x_j, t)) = (\varphi_{j+1})_t(x_j, t) + H((\varphi_{j+1})_x(x_j^+, t)) \leq 0.
\]

On the other hand, if \((x_j, t)\) is a local minimum point of \(U_* - \varphi\) in \(Q_{\tau_1}\), by (6.31) it is also a local minimum point of \((U_j)_* - \varphi_j\) in \(\hat{Q}_{j,\tau_1}\), thus

\[
\varphi_t(x_j, t) + H(\varphi_x(x_j, t)) = (\varphi_j)_t(x_j, t) + H((\varphi_j)_x(x_j^-, t)) \geq 0.
\]

Now suppose that \(J_0(x_j) < 0\). If \((x_j, t)\) is a local maximum point of \(U^* - \varphi\) in \(Q_{\tau_1}\), by (6.32) it is also a local maximum point of \((U_j)^* - \varphi_j\) in \(\hat{Q}_{j,\tau_1}\), thus

\[
\varphi_t(x_j, t) + H(\varphi_x(x_j, t)) = (\varphi_j)_t(x_j, t) + H((\varphi_j)_x(x_j^-, t)) \leq 0.
\]

On the other hand, if \((x_j, t)\) is a local minimum point of \(U_* - \varphi\) in \(Q_{\tau_1}\), by (6.32) it is also a local minimum point of \((U_{j+1})_* - \varphi_{j+1}\) in \(\hat{Q}_{j+1,\tau_1}\), thus

\[
\varphi_t(x_j, t) + H(\varphi_x(x_j, t)) = (\varphi_{j+1})_t(x_j, t) + H((\varphi_{j+1})_x(x_j^+, t)) \geq 0.
\]
If \( \tau_1 = T \), from the above considerations the result follows. Instead, if \( \tau_1 \in (0, T) \), for some \( k = 1, \ldots, p \) there exists \((x_k, \hat{t}_k)\) such that \( U_k(x_k, t) \neq U_{k+1}(x_k^+, \hat{t}_k) \) for \( 0 \leq t < \tau_1 = \hat{t}_k \), yet

\[
U(x_k^-, \hat{t}_k) = U_k(x_k^-, \hat{t}_k) = U_{k+1}(x_k^+, \hat{t}_k) = U(x_k^+, \hat{t}_k)
\]

(otherwise, by the continuity of \( U \)) there would exist \( \hat{t}_k \in (\hat{t}_k, T) \) such that \( U_k(x_k^-, t) \neq U_{k+1}(x_k^+, t) \) for \( t \in [\hat{t}_k, \hat{t}_k] \), which contradicts the definition of \( \hat{t}_k \). By (6.33) we can repeat the above arguments with a lesser number of discontinuities of \( U \), hence in a finite number of steps the conclusion follows. \( \square \)

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