Double Ernst Solution in Einstein–Kalb–Ramond Theory

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Abstract

The Kähler formulation of 5–dimensional Einstein–Kalb–Ramond (EKR) theory admitting two commuting Killing vectors is presented. Three different Kramer–Neugebauer–like maps are established for the 2–dimensional case. A class of solutions constructed on the double Ernst one is obtained. It is shown that the double Kerr solution corresponds to a EKR dipole configuration with horizon.
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I. INTRODUCTION

In its low energy limit superstring theory leads to effective Lagrange systems in which the Einstein action is modified by terms depending on scalar, vector and tensor fields \([1]–[2]\). The Einstein–Kalb–Ramond (EKR) theory, being a system of this type, takes into account only the antisymmetric Kalb–Ramond field, which can be replaced on–shell by the pseudoscalar Pecci–Quinn axion in four dimensions. The situation becomes more complicated in the multidimensional case; the non–trivial properties of this theory were established in \([3]–[4]\) when dilaton and Abelian gauge fields are present.

In \([5]\) it was shown that the \((3+d)\)–dimensional EKR theory, being reduced to two dimensions, admits two different \(d \times d\)–matrix formulations: the real non–dualized (target space) and the Hermitian dualized (non–target space) ones; moreover, the Kramer–Neugebauer–like map between the matrix potentials was established.

In this paper we study the EKR system with \(d = 2\). It turns out that this system allows a Kähler representation which formally is defined by two vacuum Ernst potentials. A discrete transformation between the metric and Kalb–Ramond degrees of freedom is established; it gives rise to an alternative \(2 \times 2\)–matrix formulation of the model. It is shown that besides of the above mentioned Kramer–Neugebauer map there are two new transformations of this type in this case.

In Sec. III the 5–dimensional line element, explicitly depending on the Ernst potentials, is presented. After that, we take as starting solution the Kerr one in order to construct a class of 5–dimensional black hole solutions; in two of the three analyzed cases there is a Kalb–Ramond dipole inside of the horizon.

II. KRAMER–NEUGEBAUER MAPS

We start from the system with the action

\[
S = \int d^5x \sqrt{\mathcal{G}} \left\{ -\mathcal{R} + \frac{1}{12} \mathcal{H}^2 \right\},
\]

(1)
where $\mathcal{R}$ is the Ricci scalar constructed on the metric $\mathcal{G}_{MN}$, $(M = 0, 1, \ldots, 4)$ and

$$\mathcal{H}_{MNL} = \partial_M \mathcal{B}_{NL} + \text{cyc. perms.},$$

(2)

where $\mathcal{B}$ is the antisymmetric Kalb–Ramond field and $\mathcal{H}_{MNL}$ is the non–dualized axion one.

Such a system arises in the frames of the low energy limit of heterotic string theory. A complete investigation must include the dilaton and gauge vector fields, but we leave it to be studied in the near future. Thus, we consider the case when the vector fields are not present and suggest that the mixings of the metric (Kaluza–Klein vector fields) and Kalb–Ramond fields vanish:

$$\mathcal{G}_{\mu,n+2} = \mathcal{B}_{\mu,n+2} = 0,$$

(3)

where $\mu = 0, 1, 2$; $n = 1, 2$. It is evident that such a restriction does not provide any constraints on the remainder variables and can be considered as a non–trivial ansatz for the EKR theory.

After the Kaluza–Klein compactification of two dimensions on a torus, one obtains the $O(2, 2)$–symmetric $\sigma$–model constructed on the matrix fields $G_{mn} = \mathcal{G}_{m+2,n+2}$ and $B_{mn} = \mathcal{B}_{m+2,n+2}$ coupled to 3–gravity with the metric $g_{\mu\nu} = \mathcal{G}_{\mu\nu}$. The effective action of this system is \cite{1} and \cite{6}

$$3S = \int d^3 x \left| g \right|^{-\frac{1}{2}} \left\{ -\frac{3}{4} R + \frac{1}{4} Tr \left[ (J^G)^2 - (J^B)^2 \right] \right\},$$

(4)

where $J^G = \nabla G G^{-1}$ and $J^B = \nabla B G^{-1}$; $G$ and $B$ being symmetric and antisymmetric $2 \times 2$–matrices, correspondingly. If we parametrize them as follows:

$$G = p_1 \begin{pmatrix} p_2^{-1} & p_2^{-1} q_2 \\ p_2^{-1} q_2 & p_2 + p_2^{-1} q_2^2 \end{pmatrix}, \quad B = q_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

(5)

the “material part” of the action (4) takes the form

$$3S_m = \frac{1}{2} \int d^3 x \left| g \right|^{-\frac{1}{2}} \left\{ p_1^{-2} \left[ (\nabla p_1)^2 + (\nabla q_1)^2 \right] + p_2^{-2} \left[ (\nabla p_2)^2 + (\nabla q_2)^2 \right] \right\},$$

(6)

which allows us to introduce the independent each other Ernst–like potentials.
\[ \epsilon_1 = q_1 + ip_1, \quad \epsilon_2 = q_2 + ip_2. \]  

Using these definitions, the action of the model can be rewritten as

\[ ^3S = \int d^3x \left| g \right|^{\frac{1}{2}} \left\{ -3R + 2 \left( J^{\epsilon_1} J^{\epsilon_1} + J^{\epsilon_2} J^{\epsilon_2} \right) \right\}, \]  

where \( J^{\epsilon_1} = \nabla \epsilon_1 (\epsilon_1 - \tilde{\epsilon}_1)^{-1} \) and \( J^{\epsilon_2} = \nabla \epsilon_2 (\epsilon_2 - \tilde{\epsilon}_2)^{-1} \). Thus, the “material part” of the action for the theory under consideration corresponds to a double Ernst system.

Another \( 2 \times 2 \)–matrix representation arises from (5) using the replacement \( p_1 \to p_2, q_1 \to q_2 \), which, in view of (6), provides a discrete symmetry transformation for the action. This circumstance allows to combine the independent variables \( p_1, q_1, p_2 \) and \( q_2 \) in the matrices \( G' \) and \( B' \)

\[
G' = p_2 \begin{pmatrix} p_1^{-1} & p_1^{-1} q_1 \\ p_1^{-1} q_1 & p_1 + p_1^{-1} q_1^2 \end{pmatrix}, \quad B' = q_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};
\]

in terms of these magnitudes, the action of the system adopts the similar to (4) form

\[ ^3S = \int d^3x \left| g \right|^{\frac{1}{2}} \left\{ -3R + \frac{1}{4} Tr \left[ (J^G)^2 - (J^B)^2 \right] \right\}. \]  

In other words, the above mentioned substitution defines the discrete transformation

\[ G \to G', \quad B \to B', \]  

where the matrices \( G' \) and \( B' \) are considered as the new Kaluza–Klein and Kalb–Ramond ones; it mixes the gravitational and material degrees of freedom. It is an analogy of the Bonnor transformation for the stationary Einstein–Maxwell theory and can be used to generate new EKR solutions starting with the Kaluza–Klein ones.

In order to establish additional properties of the system under consideration, we reduce it to two dimensions. In this case, the 3–metric \( h_{ij} \) can be written in the Lewis–Papapetrou form

\[ ds^2 = h_{ij} dx^i dx^j = e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 d\tau^2, \]
where the function $\gamma$ as well as the “material” fields are $\tau$–independent. In the previous work \cite{5}, it was shown that the discussed system allows two different representations in two dimensions: the target space non–dualized and the non–target space dualized ones. The target space formulation is given in terms of $G$ and $B$ (or $G'$ and $B'$); in it the “material part” of the $\tau$–independent motion equations transforms to

$$\nabla(\rho J^B) - \rho J^G J^B = 0,$$

$$\nabla(\rho J^G) - \rho (J^B)^2 = 0,$$

where $\nabla = \{\partial_\rho, \partial_z\}$. They can be derived from the 2–dimensional action

$$2S = \frac{1}{4} \int d\rho dz \rho \text{Tr} \left[(J^G)^2 - (J^B)^2 \right].$$

In the same way we can obtain analogous 2–dimensional Euler–Lagrange equations and their corresponding action for $G'$ and $B'$. Eq. (13), being rewritten in the form

$$\nabla[\rho G^{-1}(\nabla B)G^{-1}] = 0,$$

constitutes the compatibility condition for the relation which defines the antisymmetric matrix $\Omega$ (by analogy we can define $\Omega'$):

$$\nabla \Omega = \rho G^{-1}(\nabla B)G^{-1}$$

where $\nabla_\rho = \nabla_z$ and $\nabla_z = -\nabla_\rho$ (see \cite{5}). The sets of matrices $(\Omega, G)$ and $(\Omega', G')$ provide alternative Lagrange descriptions of the 2–dimensional theory. In terms of $\Omega$ and $G$ the action of the system adopts the form

$$2S = \frac{1}{4} \int d\rho dz \rho \text{Tr} \left[\rho (J^G)^2 + \rho^{-1}(J^\Omega)^2 \right],$$

where $J^\Omega = G \nabla \Omega$. For the magnitudes $\Omega'$ and $G'$ we have the same relation. The Kramer–Neugebauer–like transformations which directly map the “material” field equations in terms of $(G, B)$ into the ones in terms of $(G, \Omega)$, and those given by $(G', B')$ into the ones given by $(G', \Omega')$ are

$$\begin{cases}
G \rightarrow \rho G^{-1} & G' \rightarrow \rho G'^{-1} \\
B \rightarrow i\Omega & B' \rightarrow i\Omega'
\end{cases}$$

(18)
these transformations are equivalent to the maps
\[
\begin{align*}
  p_1 &\rightarrow \rho p_1^{-1}, \\
  q_1 &\rightarrow i\omega, \\
  \epsilon_2 &\rightarrow -\epsilon_2^{-1}
\end{align*}
\begin{align*}
  p_2 &\rightarrow \rho p_2^{-1}, \\
  q_2 &\rightarrow i\omega', \\
  \epsilon_1 &\rightarrow -\epsilon_1^{-1}
\end{align*}
\] (19)

where \( \omega \) (\( \omega' \)) is the single non–trivial component of the matrix \( \Omega \) (\( \Omega' \)), i.e.,
\[
\Omega = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\] (20)

(a similar relation can be written for \( \Omega' \)).

The complex matrix transformations (18) have the same form as the well–known Kramer–Neugebauer one for the pure Einstein theory. These transformations algebraically map dualized matrix variables into the non–dualized ones.

It is easy to see that Eq. (16) is equivalent to
\[
\nabla \omega = \rho p_1^{-2} \nabla q_1;
\] (21)
a relation of the same kind connects \( \omega' \) with \( p_2 \) and \( q_2 \). Using such a double non–matrix dualization one can establish a new representation of the problem. Namely, the equations of motion on the language of \( p_1, \omega, p_2 \) and \( \omega' \) correspond to the action
\[
\frac{1}{2} S = \frac{1}{2} \int d\rho dz \left\{ \left[ \rho p_1^{-2}(\nabla p_1)^2 - \rho^{-1} p_1^2 (\nabla \omega)^2 \right] + \left[ \rho p_2^{-2}(\nabla p_2)^2 - \rho^{-1} p_2^2 (\nabla \omega')^2 \right] \right\}.
\] (22)
The map which transforms the motion equations of the action (22) into the ones of (6) in the two dimensional case, has exactly the Kramer–Neugebauer form for the sets of primed and non–primed variables:
\[
\begin{align*}
  p_1 &\rightarrow \rho p_1^{-1}, \\
  q_1 &\rightarrow i\omega, \\
  p_2 &\rightarrow \rho p_2^{-1}, \\
  q_2 &\rightarrow i\omega'.
\end{align*}
\] (23)
Thus, 5–dimensional EKR theory, being reduced to two dimensions, allows three different transformations of Kramer–Neugebauer type.
III. DOUBLE ERNST SOLUTION

In this section we construct solutions of the 5–dimensional EKR theory using its formal equivalence to the double Ernst system [14]. Namely, it is easy to see that the solution of the EKR problem, being reduced to two dimensions and parametrized by the functions $\epsilon_1$, $\epsilon_2$ and $\gamma$, can be constructed using the solutions of the double vacuum Einstein equations written in the Ernst form in terms of $\epsilon_k$ and $\gamma^\epsilon_k$ ($k = 1, 2$)

$$\nabla(\rho J^\epsilon_k) = \rho J^\epsilon_k (J^\epsilon_k - J^\bar{\epsilon}_k), \quad (24)$$

$$\partial_z \gamma^\epsilon_k = \rho \left[(J^\epsilon_k)_z(J^\epsilon_k)_\rho + (J^\bar{\epsilon}_k)_z(J^\bar{\epsilon}_k)_\rho\right], \quad (25)$$

$$\partial_\rho \gamma^\epsilon_k = \rho \left[(J^\epsilon_k)_\rho|^2 - |(J^\bar{\epsilon}_k)_z|^2\right], \quad (26)$$

if one identifies $\gamma \equiv \gamma^\epsilon_1 + \gamma^\bar{\epsilon}_2$.

Thus, the 5–dimensional line element for the metric ansatz defined by (5) and (12)

$$ds^2_5 = e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 d\tau^2 + G_{mn}dx^m dx^n, \quad (27)$$

can be expressed using the parametrization of the matrix $G_{mn}$ and the definition of the functions $\epsilon_1$ and $\epsilon_2$ (see Eq. (7)):

$$ds^2_5 = e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 d\tau^2 + \frac{\epsilon_1 - \bar{\epsilon}_1}{\epsilon_2 - \bar{\epsilon}_2}|du + \epsilon_2 dv|^2, \quad (28)$$

where $u = x^3$ and $v = x^4$. The Kalb–Ramond matrix is given by

$$B = \frac{\epsilon_1 - \bar{\epsilon}_1}{2i} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (29)$$

and the function $\gamma$ corresponds to the potentials $\epsilon_1$ and $\epsilon_2$ as it was pointed out below.

The metric (28) algebraically depends on the Ernst potentials, so that the situation in the EKR theory differs from the one that takes place in the pure Einstein theory. In latter case, in order to construct the metric, one must use the function $\omega$, which is related with $q_1$ by the dualization procedure (21), instead of the imaginary part $q_1$ of the (single) Ernst potential $\epsilon_1$. 

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As an example of the developed technique which allows to construct EKR solutions, we present the solution which arises from the double Kerr one. The Ernst potentials corresponding to two Kerr solutions with sources in different points of the symmetry axis are:

\[ \mathcal{E}_k = 1 - \frac{2m_k}{r_k + ia_k \cos \theta_k}, \]  

(30)

where \( m_k \) and \( a_k \) are constant parameters which define masses and rotations of the sources of the Kerr field configurations. Our potentials are \( \epsilon_k = i\mathcal{E}_k \). The two entered coordinate sets are connected with the polar system as

\[ \rho = [(r_k - m_k)^2 + a_k^2]^{1/2} \sin \theta_k, \quad z = z_k + (r_k - m_k) \cos \theta_k, \]  

(31)

where \( z_k \) denote the locations of the sources and \( a_k^2 = m_k^2 - a_k^2 \). In this case, for the function \( \gamma_k \) one has:

\[ e^{2\gamma_k} = \frac{P_k}{Q_k}, \]  

(32)

where \( P_k = \Delta_k - a_k^2 \sin^2 \theta_k, \quad Q_k = \Delta_k + \sigma_k^2 \sin^2 \theta_k \) and \( \Delta_k = r_k^2 - 2m_k r_k + a_k^2 \).

Then, the Kalb–Ramond matrix \( B_{mn} \) will be defined by the function

\[ q_1 = \frac{2m_1 a_1 \cos \theta_1}{\Sigma_1}, \]  

(33)

where \( \Sigma_k = r_k^2 + a_k^2 \cos^2 \theta_k \). This kind of field configuration corresponds to a Kalb–Ramond dipole with momentum \( m_1 a_1 \); it is located in the point \( z_1 \).

In the particular case when \( \epsilon_1 = i\mathcal{E} \) and \( \epsilon_2 = i \), the line element adopts the form (here indeces are dropped)

\[ ds^2 = P \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \Delta \sin^2 \theta d\tau^2 + \left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) \left( du^2 + dv^2 \right). \]  

(34)

This metric describes the gravitational field originated by the massive (with mass \( m \)) Kalb–Ramond dipole which is hidden inside of the horizon

\[ r_h = m + \sqrt{m^2 - a^2}. \]  

(35)
The discrete transformation (11), which can be written as $\epsilon_1 \to \epsilon_2$, $\epsilon_2 \to \epsilon_1$, maps this solution into the one defined by the potentials $\epsilon_1 = i$ and $\epsilon_2 = i\bar{E}$. The corresponding metric is:

$$
\begin{align*}
\frac{ds^2}{5} &= P \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \Delta \sin^2 \theta d\tau^2 + \left( \frac{\sum}{\Delta - a^2 \sin^2 \theta} \right) \left| du + i \left( 1 - \frac{2m}{r - i\cos \theta} \right) dv \right|^2. 
\end{align*}
$$

(36)

In this case the Kalb–Ramond field is absent, i.e., this solution is a pure Kaluza–Klein one. Thus, the Kalb–Ramond dipole vanishes, but the metric (36) acquires an additional degree of freedom (the non–trivial value of $G_{uv}$) with respect to (34). This configuration corresponds to the field of a source with mass $m$ which is located inside of the horizon (35).

The double Kerr solution (28)–(29) can be understood as the “superposition” of the two considered cases. The simplest “superposition” of this kind takes place when the two Ernst potentials coincide: $\epsilon_1 = \epsilon_2 = i\bar{E}$. Therefore, the 5–dimensional metric reads

$$
\begin{align*}
\frac{ds^2}{5} &= \frac{P^2}{Q} \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \Delta \sin^2 \theta d\tau^2 + \left| du + i \left( 1 - \frac{2m}{r - i\cos \theta} \right) dv \right|^2. 
\end{align*}
$$

(37)

This metric has non–diagonal form and, simultaneously, the Kalb–Ramond field describes a massive (with mass $m$) dipole; it is also located inside of the horizon defined by the formula (35).

IV. CONCLUSION

The Kähler formulation of 5–dimensional EKR theory, admitting two commuting Killing vectors, is presented. In two dimensions there are three pairs of descriptions of the theory; each pair contains a target space (non–dualized) representation and a non–target space (dualized) one. The corresponding Kramer–Neugebauer–like maps between the dualized and non–dualized variables are established.

A class of solutions constructed on the double Ernst one is obtained. It turns out that the 5–dimensional line element explicitly depends on the Ernst potentials. The double Kerr solution corresponds to an asymptotically flat EKR dipole configuration.
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