hp-version time domain boundary elements for the wave equation on quasi-uniform meshes

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Abstract

Solutions to the wave equation in the exterior of a polyhedral domain or a screen in $\mathbb{R}^3$ exhibit singular behavior from the edges and corners. We present quasi-optimal $hp$-explicit estimates for the approximation of the Dirichlet and Neumann traces of these solutions on quasi-uniform meshes on the boundary. The results are applied to an $hp$-version of the time domain boundary element method, and an a posteriori error estimate is obtained towards adaptive mesh refinements for the Dirichlet problem. Numerical examples confirm the theoretical results for the Dirichlet problem both for screens and polyhedral domains.

Key words: boundary element method; approximation properties; hp methods; asymptotic expansion; wave equation.

1 Introduction

This article initiates the study of high-order boundary elements in the time domain. For elliptic problems, $p$- and $hp$-versions of the finite element method give rise to fast approximations of both smooth solutions and geometric singularities. These methods converge to the solution by increasing the polynomial degree $p$ of the elements, possibly in combination with reducing the mesh size $h$ of the quasi-uniform mesh. They were first investigated in the group of Babuska [2, 3, 14, 15]. See [41] for a comprehensive analysis for 2d problems.

The analogous $p$- and $hp$-versions of the boundary element method go back to [1, 44, 45]. More recent optimal convergence results for boundary elements on screens and polyhedral surfaces covering 3d problems have been obtained, for example, in [6, 7, 8, 9, 10].

Boundary element methods for time dependent problems have recently become of interest [40]. For stochastic heat equations $p$-version boundary elements in the time domain have been

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considered in [12]. In this paper we introduce a space-time \(hp\)-version of the time domain boundary element method for the wave equation. To be specific, in the exterior \(\Omega \subset \mathbb{R}^3\) of a polyhedral surface or screen \(\Gamma\) this article considers the initial-boundary value problem

\[
\begin{align*}
\frac{1}{c^2} & \partial_t^2 u(t,x) - \Delta u(t,x) = 0 \quad \text{in } \mathbb{R}_t^+ \times \Omega, \\
u(0,x) = \partial_t u(0,x) = 0 \quad \text{in } \Omega,
\end{align*}
\]

for given boundary data \(Bu\) on \(\Gamma = \partial \Omega\). We consider either Dirichlet \((Bu = u)\) or Neumann boundary conditions \((Bu = \partial_n u, n\text{ unit normal vector})\) and choose units such that \(c = 1\).

To solve (1) numerically, we reformulate it as a time dependent integral equation on \(\Gamma\) for the single layer or hypersingular operator. This integral equation is approximated using Galerkin \(hp\)-version boundary elements, based on tensor products of piecewise polynomial functions \(\tilde{V}_{p,q}^{\Delta t,h}\) on a quasi-uniform mesh in space and a uniform mesh in time of step size \(\Delta t\). The a posteriori error estimate in this article gives rise to adaptive \(hp\) mesh refinement procedures.

Similar to \(h\)-version boundary elements, the approximation rate is determined by the singularities of the solution \(u\) of (1) at non-smooth boundary points of the domain. Near an edge or a corner a singular decomposition of the solution into a leading part of explicit singular functions plus smoother terms has been obtained in work by Plamenevskii and collaborators [31, 32, 34, 39]. Their results imply that at a fixed time \(t\), the solution to the wave equation admits an explicit singular expansion with exactly the same behavior as for elliptic equations. (For the latter, see [13, 36, 37].)

This asymptotic expansion of the solution \(u\) and its normal derivative on \(\Gamma\) gives rise to quasi-optimal convergence rates for the \(p\)- and \(hp\)-versions in the time domain in polyhedral domains and on screens. For the screen \(\Gamma\), our main result for the approximation of the solutions to the boundary integral equations in space-time anisotropic Sobolev spaces is a consequence of:

**Theorem A.** Let \(\varepsilon > 0\) and \(\Gamma\) a flat open surface.

a) Let \(u\) be a strong solution to the homogeneous wave equation with inhomogeneous Neumann boundary conditions \(\partial_n u|\Gamma = g\), with \(g\) smooth. Further, let \(\phi_{h,\Delta t}\) be the best approximation in the norm of \(H^r_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{3}{2}-s}(\Gamma))\) to the Dirichlet trace \(u|\Gamma\) in \(\tilde{V}_{p,p}^{\Delta t,h}\) on a quasi-uniform spatial mesh with \(\Delta t \lesssim h\). Then

\[
\|u - \phi_{h,\Delta t}\|_{r,\frac{1}{2}-s,\Gamma}\lesssim \left(\frac{h}{p}\right)^{\frac{1}{2}+s-\varepsilon} + \left(\frac{h}{p}\right)^{-\frac{1}{2}+s+\eta} + \left(\frac{\Delta t}{p}\right)^{\mu+s-r-\frac{1}{2}},
\]

where \(r \in [0,p]\) and the regular part \(v_0 \in H^r_\sigma(\mathbb{R}^+, \tilde{H}^n(\Gamma))\) of the singular expansion of \(u\), with \(\eta, \mu\) sufficiently large.

b) Let \(u\) be a strong solution to the homogeneous wave equation with inhomogeneous Dirichlet boundary conditions \(u|\Gamma = g\), with \(g\) smooth. Further, let \(\psi_{h,\Delta t}\) be the best approximation in the norm of \(H^r_\sigma(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma))\) to the Neumann trace \(\partial_n u|\Gamma\) in \(\tilde{V}_{p,p}^{\Delta t,h}\) on a quasi-uniform spatial mesh.
with $\Delta t \lesssim h$. Then

$$
\| \partial_n u - \psi_{h, \Delta t} \|_{r, -\frac{1}{2}, \Gamma, \ast} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{2} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r},
$$

where $r \in [0, p + 1]$ and the regular part $\psi_0 \in H^{\mu + 1}_\sigma(\mathbb{R}^+, \tilde{H}^\eta(\Gamma))$ of the singular expansion of $\partial_n u$, with $\eta, \mu$ sufficiently large.

For the circular screen this result is the content of Theorem 15, while for the polygonal screen it is Theorem 19. It implies an approximation result for the solution to the relevant boundary integral formulations, see Corollary 16 for the circular screen, respectively Corollary 20 for the polygonal screen:

**Corollary B.** Let $\varepsilon > 0$ and $\Gamma$ a flat open surface.

a) Let $\phi$ be the solution to the hypersingular integral equation (7) and $\phi_{h, \Delta t}$ the best approximation in the norm of $H^r_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2} - \varepsilon}(\Gamma))$ to $\phi$ in $V^{p,p}_{\Delta t, h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$. Then

$$
\| \phi - \phi_{h, \Delta t} \|_{r, -\frac{1}{2} - s, \Gamma, \ast} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{2} + s - \varepsilon} + \left( \frac{h}{p} \right)^{-\frac{1}{2} + s + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}},
$$

where $r \in [0, p)$, $s \in [0, \frac{1}{2}]$ and the regular part $v_0 \in H^r_\sigma(\mathbb{R}^+, \tilde{H}^\eta(\Gamma))$ of the singular expansion of $\phi = u|_{\Gamma}$, with $\eta, \mu$ sufficiently large.

b) Let $\psi$ be the solution to the single layer integral equation (3) and $\psi_{h, \Delta t}$ the best approximation in the norm of $H^r_\sigma(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma))$ to $\psi$ in $V^{p,p}_{\Delta t, h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$. Then

$$
\| \psi - \psi_{h, \Delta t} \|_{r, -\frac{1}{2}, \Gamma, \ast} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{2} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r},
$$

where $r \in [0, p + 1]$ and the regular part $\psi_0 \in H^{\mu + 1}_\sigma(\mathbb{R}^+, \tilde{H}^\eta(\Gamma))$ of the singular expansion of $\psi$, with $\eta, \mu$ sufficiently large.

Indeed, on the flat screen the solutions to the integral equations are given by $\phi = |u|_{\Gamma}$ in terms of the solution $u$ which satisfies Neumann conditions $Bu = \partial_n u|_{\Gamma} = g$, respectively $\psi = [\partial_n u]|_{\Gamma}$ in terms of the solution $u$ which satisfies Dirichlet conditions $Bu = u|_{\Gamma} = f$.

We mention a generalization of these results to polyhedral domains in Section 4.5.

**Remark C.** Together with the a priori estimates for the time domain boundary element methods on screens [20, 22], Corollary B implies convergence rates for the $p$-version Galerkin approximations which are twice those observed for the quasi-uniform $h$-method in [19].

We first prove the approximation properties on the circular screen, without corners, and then discuss the approximation of the corner and corner-edge singularities on polygonal screens. On the square the convergence rate is determined by the singularities at the edges.
Our numerical experiments in Section 6.2 confirm the theoretical results and exhibit the predicted convergence rate for the Dirichlet problem for the time dependent wave equation outside a square screen. The convergence rate in the energy norm is doubled compared to the convergence rate of the $h$-version on a uniform mesh, as predicted. Our numerical experiments in Section 6.3 for the wave equation outside an icosahedron similarly confirm the predicted convergence of the $p$-method.

Related previous work for the time independent Laplace equation includes, in particular, the analysis of the $p$-version by Schwab and Suri [42] of the singularities of $u$ in polyhedral domains and their implications for the numerical approximation of the hypersingular integral equation by boundary elements. On geometrically graded meshes the $hp$-version was studied in [28], but the analysis does not yield a priori estimates on quasi-uniform meshes. Sharp estimates on piecewise flat open surfaces with quasi-uniform meshes are due to Bespalov and Heuer [8, 9, 10] for both the single layer and hypersingular integral equations. See [6, 7] for extensions to the Lamé equation.

The article is organized as follows: Section 2 recalls the boundary integral operators associated to the wave equation as well as their mapping properties between suitable space-time anisotropic Sobolev spaces. It concludes by reformulating the Dirichlet and Neumann problems for the wave equation (1) as boundary integral equations in the time domain. The following Section 3 introduces the space-time discretizations and a time domain boundary element method to solve the single layer and hypersingular integral equations. The asymptotic expansions of solutions to the wave equation and their $hp$ approximation are the content of Section 4, for circular and polygonal screens as well as for polyhedral surfaces. The a priori estimates are complemented by an a posteriori estimate and a resulting $hp$-adaptive method for the single layer equation in Section 5. The article presents numerical experiments both on screens and outside polyhedral domains in Section 6, before summarizing the conclusions in Section 7.

2 Boundary integral operators and Sobolev spaces

Let $\Gamma$ be the boundary of a polyhedral domain in $\mathbb{R}^3$, consisting of curved, polygonal boundary faces, or an open polyhedral surface (screen).

We make the following ansatz for the solution to (1) in terms of the single layer potential for the wave equation,

$$u(t, x) = \int_0^\infty \int_\Gamma G(t - \tau, x, y) \psi(\tau, y) \, ds_y \, d\tau .$$

Here $G$ is a fundamental solution to the wave equation and $\psi(\tau, y) = 0$ for $\tau < 0$. In 3 dimensions

$$u(t, x) = \frac{1}{4\pi} \int_\Gamma \frac{\psi(t - |x - y|, y)}{|x - y|} \, ds_y .$$

Taking Dirichlet boundary values on $\Gamma$ of the integral (2), we obtain the single layer operator $V$:

$$V \psi(t, x) = \int_0^\infty \int_\Gamma G(t - \tau, x, y) \psi(\tau, y) \, ds_y \, d\tau .$$
The wave equation (1) with Dirichlet boundary conditions, \( u = f \) on \( \Gamma \), is equivalent to the integral equation

\[
V \psi = u|_{\Gamma} = f .
\]  

In addition to \( V \), also the adjoint double layer operator \( K' \), the double layer operator \( K \) and the hypersingular operator \( W \) on \( \Gamma \) will be used:

\[
K \phi(t, x) = \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_y}(t - \tau, x, y) \phi(\tau, y) \, ds_y \, d\tau,
\]

\[
K' \phi(t, x) = \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_x}(t - \tau, x, y) \phi(\tau, y) \, ds_y \, d\tau,
\]

\[
W \phi(t, x) = \int_0^\infty \int_\Gamma \frac{\partial^2 G}{\partial n_x \partial n_y}(t - \tau, x, y) \phi(\tau, y) \, ds_y \, d\tau .
\]

**Remark 1.** On a flat screen \( \Gamma \subset \mathbb{R}^2 \times \{0\} \), \( \frac{\partial G}{\partial n} = 0 \) and therefore \( K \phi = K' \phi = 0 \).

These operators are studied in space-time anisotropic Sobolev spaces \( H^s_r(\mathbb{R}^+, \tilde{H}^s(\Gamma)) \), see [20] or [26]. To define the spaces for \( \partial \Gamma \neq \emptyset \), extend \( \Gamma \) to a closed, orientable Lipschitz manifold \( \tilde{\Gamma} \).

Sobolev spaces of supported distributions in \( \Gamma \) are defined as:

\[
\tilde{H}^s(\Gamma) = \{ u \in H^s(\tilde{\Gamma}) : \text{supp} u \subset \tilde{\Gamma} \} , \quad s \in \mathbb{R} .
\]

To define an explicit scale of Sobolev norms, fix a partition of unity \( \alpha_i \) subordinate to a covering of \( \tilde{\Gamma} \) by open sets \( B_i \) and diffeomorphisms \( \phi_i \) mapping each \( B_i \) into the unit cube \( \subset \mathbb{R}^n \). They induce a family of norms from \( \mathbb{R}^d \):

\[
||u||_{s,\omega,\tilde{\Gamma}} = \left( \sum_{i=1}^p \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^s |\mathcal{F} \{ (\alpha_i u) \circ \phi_i^{-1} \} (\xi)|^2 \, d\xi \right)^{\frac{1}{2}} .
\]

Here, \( \mathcal{F} \) denotes the Fourier transform, and if \( \omega = 1 \) we often omit the index \( \omega \). The norms for different \( \omega \in \mathbb{C} \setminus \{0\} \) are equivalent. The above norms induce norms on \( H^s_r(\mathbb{R}^+) \), \( ||u||_{s,\omega,\mathbb{R}^+} = \inf_{v \in \tilde{H}^s(\tilde{\Gamma})} ||u + v||_{s,\omega,\tilde{\Gamma}} \), and on \( \tilde{H}^s(\Gamma) \), \( ||u||_{s,\omega,\Gamma,s} = ||e_+ u||_{s,\omega,\tilde{\Gamma}} \). Here, \( e_+ \) extends the distribution \( u \) by 0 from \( \Gamma \) to \( \tilde{\Gamma} \).

Weighted Sobolev spaces in time for \( r \in \mathbb{R} \) and \( \sigma > 0 \): are defined as

\[
H^r_{\sigma}(\mathbb{R}^+) = \{ u \in \mathcal{D}'_+ : e^{-\sigma t} u \in \mathcal{S}'_+ \text{ and } ||u||_{\sigma, r, \mathbb{R}^+} < \infty \} .
\]

Here, \( \mathcal{D}'_+ \) denotes the space of distributions on \( \mathbb{R} \) with support in \( [0, \infty) \), and \( \mathcal{S}'_+ \) the subspace of tempered distributions. The Sobolev spaces are Hilbert spaces endowed with the norm

\[
||u||_{\sigma, r, \mathbb{R}^+} = \left( \int_{-\infty}^{+\infty} |\omega|^{2r} |\hat{u}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} .
\]

The scale of space-time anisotropic Sobolev spaces combines the Sobolev norms in space and time:
Definition 2. For \( r, s \in \mathbb{R} \) and \( \sigma > 0 \) define
\[
H^r_\sigma(\mathbb{R}^+, H^s(\Gamma)) = \{ u \in \mathcal{D}'_+(H^s(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(H^s(\Gamma)) \text{ and } \| u \|_{r, s, \Gamma} < \infty \}, \\
H^r_\sigma(\mathbb{R}^+, \tilde{H}^s(\Gamma)) = \{ u \in \mathcal{D}'_+(\tilde{H}^s(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(\tilde{H}^s(\Gamma)) \text{ and } \| u \|_{r, s, \Gamma, \star} < \infty \}.
\]
\( \mathcal{D}'_+(E) \) denotes the space of distributions on \( \mathbb{R} \) with support in \( [0, \infty) \), taking values in \( E = H^s(\Gamma), \tilde{H}^s(\Gamma), \) and \( \mathcal{S}'_+(E) \) the subspace of tempered distributions. The Sobolev spaces are Hilbert spaces endowed with the norm
\[
\| u \|_{r, s, \Gamma} = \left( \int_{-\infty}^{\infty} |\omega|^{2r} \| \hat{u}(\omega) \|_{s, \omega, \Gamma}^2 \, d\omega \right)^{\frac{1}{2}}, \\
\| u \|_{r, s, \Gamma, \star} = \left( \int_{-\infty}^{\infty} |\omega|^{2r} \| \hat{u}(\omega) \|_{s, \omega, \Gamma, \star}^2 \, d\omega \right)^{\frac{1}{2}}.
\]
When \(| s | \leq 1\) one can show that the spaces are independent of the choice of \( \alpha_i \) and \( \phi_i \).

We state the mapping properties of the boundary integral operators:

Theorem 3 ([20]). The following operators are continuous for \( r \in \mathbb{R}, \sigma > 0 \):
\[
V : H^r_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)), \\
K' : H^r_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)), \\
K : H^r_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)), \\
W : H^r_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)).
\]

\( V \partial_t \) satisfies a coercivity estimate in the norm of \( H^0_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \): 
\[
\| \psi \|_{0, -\frac{1}{2}, \sigma, \star} \lesssim_\sigma \langle V \psi, \partial_t \psi \rangle.
\]
From the mapping properties of Theorem 3 one also has the continuity of the bilinear form associated to \( V \partial_t \) in a bigger norm: \( \langle V \psi, \partial_t \psi \rangle \lesssim_\sigma \| \psi \|_{1, -\frac{1}{2}, \sigma, \star}^2 \).

Similar estimates hold for \( W \partial_t ; \| \phi \|_{0, 1, \sigma, \star}^2 \lesssim_\sigma \langle W \phi, \partial_t \phi \rangle \lesssim_\sigma \| \phi \|_{1, -\frac{1}{2}, \sigma, \star}^2 \). Proofs and further information may be found in [20, 26].

The space-time Sobolev spaces allow a precise statement and analysis of the weak formulation for the Dirichlet problem (3): Find \( \psi \in H^1_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \) such that for all \( \Psi \in H^1_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \)
\[
\int_0^\infty \int_{\Gamma} (V \psi(t, x)) \partial_t \Psi(t, x) \, ds_x \, ds_t = \int_0^\infty \int_{\Gamma} f(t, x) \partial_t \Psi(t, x) \, ds_x \, ds_t,
\]
where \( ds_t = e^{-2\sigma t} dt \).

For the Neumann problem, a double layer potential ansatz for \( u \):
\[
u(t, x) = \int_{\mathbb{R}^+ \times \Gamma} \frac{\partial G}{\partial n_y}(t - \tau, x, y) \phi(\tau, y) \, d\tau \, ds_y,
\]
with \( \phi(s, y) = 0 \) for \( s \leq 0 \) leads to the hypersingular equation
\[
W \phi = \frac{\partial u}{\partial n} \bigg|_\Gamma = g.
\]
Lemma 8. Let \( \sigma > 0 \).

a) Assume that \( f \in H^2_\sigma(\mathbb{R}^+, H^\frac{1}{2} \sigma(\Gamma)) \). Then there exists a unique solution \( \psi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \) of (5) and
\[
\|\psi\|_{1, -\frac{1}{2}, \Gamma,*} \lesssim \|f\|_{2, \frac{1}{2}, \Gamma} .
\]

b) Assume that \( g \in H^2_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \). Then there exists a unique solution \( \phi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \) of (8) and
\[
\|\phi\|_{1, \frac{1}{2}, \Gamma,*} \lesssim \|g\|_{2, -\frac{1}{2}, \Gamma} .
\]

Practical computations use \( \sigma = 0 \), while the theoretical analysis requires \( \sigma > 0 \), [4, 17].

We finally mention some useful technical results: The first localizes estimates for fractional Sobolev norms [19]:

Lemma 5. Let \( \Gamma, \Gamma_j \) (\( j = 1, \ldots, N \)) be Lipschitz domains with \( \overline{\Gamma} = \bigcup_{j=1}^N \overline{\Gamma}_j, \tilde{u} \in H^r_\sigma(\mathbb{R}^+, \tilde{H}^s(\Gamma)), u \in H^r_\sigma(\mathbb{R}^+, H^s(\Gamma)) \), \( s \in \mathbb{R} \). Then for all \( s \in [-1,1], r \in \mathbb{R} \) and \( \sigma > 0 \)
\[
\sum_{j=1}^N \|u\|_{r,s,\Gamma_j}^2 \leq \|u\|_{r,s,\Gamma}^2 ;
\]
\[
\|\tilde{u}\|_{r,s,\Gamma,\ast}^2 \leq \sum_{j=1}^N \|\tilde{u}\|_{r,s,\Gamma_j,\ast}^2 .
\]

From Lemmas 8 and 9 in [19] we recall:

Lemma 6. Let \( r \geq 0, 0 \leq s_1, s_2 \leq 1, I_j = [0, h_j], u_2 \in \tilde{H}^{-s_2}(I_2), u_1 \in \tilde{H}^r_\sigma(\mathbb{R}^+, \tilde{H}^{-s_1}(I_1)) \). Then there holds
\[
\|u_1(t,x)u_2(y)\|_{r,-s_1-s_2, I_1 \times I_2,\ast} \lesssim \|u_1\|_{r,-s_1, I_1,\ast} \|u_2\|_{-s_2, I_2,\ast} .
\]

For positive Sobolev indices one has:

Lemma 7. Let \( r \geq 0, 0 \leq s \leq 1, I_j = [0, h_j], u_2 \in \tilde{H}^s(I_2), u_1 \in H^r_\sigma(\mathbb{R}^+, \tilde{H}^s(I_1)) \). Then there holds
\[
\|u_1(t,x)u_2(y)\|_{r,s, I_1 \times I_2,\ast} \lesssim \|u_1\|_{r,s, I_1,\ast} \|u_2\|_{s, I_2,\ast} .
\]

We also note the variants:

Lemma 8. Let \( r \geq 0, 0 \leq s \leq 1, u_2 \in \tilde{H}^{-s}(\Gamma), u_1 \in \tilde{H}^r_\sigma(\mathbb{R}^+) \). Then there holds
\[
\|u_1(t)u_2(x,y)\|_{r,-s, \Gamma,\ast} \lesssim \|u_1\|_{\sigma, r, \mathbb{R}^+} \|u_2\|_{-s, \Gamma,\ast} .
\]
Proof. This is a consequence of the estimate
\[
(\sigma^2 + |\omega|^2)^{r/2}(\sigma^2 + |\omega|^2 + \xi_1^2 + \xi_2^2)^{-s/2} \lesssim (\sigma^2 + |\omega|^2)^{r/2}(1 + \xi_1^2 + \xi_2^2)^{-s/2}
\]
in Fourier space.

We note a similar result for positive Sobolev indices:

**Lemma 9.** Let \( r \geq 0, 0 \leq s \leq 1, u_2 \in \tilde{H}^s(\Gamma), u_1 \in H^r_a(\mathbb{R}^+). \) Then there holds
\[
\|u_1(t)u_2(x,y)\|_{r,s,\Gamma} \lesssim \|u_1\|_{\sigma_r+s,\mathbb{R}^+}\|u_2\|_{s,\Gamma}.
\]

**Proof.** This is a consequence of the estimate
\[
(\sigma^2 + |\omega|^2)^{r/2}(\sigma^2 + |\omega|^2 + \xi_1^2 + \xi_2^2)^{s/2} \lesssim (\sigma^2 + |\omega|^2)^{(r+s)/2}(1 + \xi_1^2 + \xi_2^2)^{s/2}
\]
in Fourier space. \(\square\)

## 3 Discretization

For the time discretization we consider a uniform decomposition of the time interval \([0, \infty)\) into subintervals \([t_{n-1}, t_n)\) with time step \(\Delta t\), such that \(t_n = n\Delta t\) (\(n = 0, 1, \ldots\)).

In \(\mathbb{R}^3\), we may assume that \(\Gamma\) consists of closed triangular faces \(\Gamma_i\) such that \(\Gamma = \bigcup \Gamma_i\).

We choose a basis \(\{\xi_{h,i}^1, \ldots, \xi_{h,i}^N\}\) of the space \(V_h^q(\Gamma)\) of piecewise polynomial functions of degree \(q\) in space. Moreover we define \(\tilde{V}_h^q(\Gamma)\) as the space \(V_h^q(\Gamma)\), where the polynomials vanish on \(\partial \Gamma\) for \(q \geq 1\). For the time discretization we choose a basis \(\{\beta_{\Delta t}^1, \ldots, \beta_{\Delta t}^N\}\) of the space \(V_t^p\) of piecewise polynomial functions of degree \(p\) in time (continuous and vanishing at \(t = 0\) if \(p \geq 1\)).

Let \(T_S = \{\Delta_1, \ldots, \Delta_N\}\) be a quasi-uniform triangulation of \(\Gamma\) and \(T_T = \{[0, t_1), [t_1, t_2), \ldots, [t_{M-1}, T]\}\) the time mesh for a finite subinterval \([0, T]\).

We consider the tensor product of the approximation spaces in space and time, \(V_h^q\) and \(V_t^p\), associated to the space-time mesh \(T_{S,T} = T_S \times T_T\), and we write
\[
V_{\Delta t,h}^{p,q} := V_{\Delta t}^p \otimes V_h^q. \tag{13}
\]

We analogously define
\[
\tilde{V}_{\Delta t,h}^{p,q} := V_{\Delta t}^p \otimes \tilde{V}_h^q. \tag{14}
\]

The Galerkin discretization of the Dirichlet problem (5) is then given by:

Find \(\psi_{\Delta t,h} \in V_{\Delta t,h}^{p,q}\) such that for all \(\Psi_{\Delta t,h} \in V_{\Delta t,h}^{p,q}\)
\[
\int_0^\infty \int_\Gamma (V\psi_{\Delta t,h}(t,x))\partial_t\Psi_{\Delta t,h}(t,x) \, ds_x \, ds_t = \int_0^\infty \int_\Gamma f(t,x)\partial_t\Psi_{\Delta t,h}(t,x) \, ds_x \, ds_t. \tag{15}
\]
For the Neumann problem (8), we have:

Find \( \phi_{\Delta t,h} \in \bar{V}^{p,q}_{t,h} \) such that for all \( \Phi_{\Delta t,h} \in \bar{V}^{p,q}_{t,h} \)

\[
\int_0^\infty \int_\Gamma (W\phi_{\Delta t,h}(t,x))\partial_t \Phi_{\Delta t,h}(t,x) \, ds_x \, d_\sigma t = \int_0^\infty \int_\Gamma g(t,x)\partial_t \Phi_{\Delta t,h}(t,x) \, ds_x \, d_\sigma t .
\] (16)

From the weak coercivity of \( V \), respectively \( W \), the discretized problems (15) and (16) admit unique solutions.

### 3.1 Approximation properties

While we use triangular meshes in our computations, for the ease of presentation we first discuss the approximation properties of meshes with rectangular elements. Reference [35] shows how to deduce approximation results on triangular meshes from the rectangular case.

Key ingredients in our analysis are projections from \( L^2(\Gamma) \) onto \( V^p_h \). We collect some key approximation properties used below, which are proven analogous to [24, Proposition 3.54 and 3.57], see also [20] for screens.

We recall the well-known results for \( V^p_h \) and \( V^q_{\Delta t} \), which we are going to need. See for example Theorem 4.1 in [11], respectively [4].

**Lemma 10.** Let \( \Pi_t^q \) the orthogonal projection from \( L^2(\mathbb{R}^+) \) to \( V^q_{\Delta t} \) and \( m \leq q \). Then for \( s \in [-\frac{1}{2}, \frac{1}{2}] \)

\[
||f - \Pi_t^q f||_{|s, \mathbb{R}^+|} \leq C_k \left( \frac{\Delta t}{q} \right)^{q+1-s} |f|_{|s,q+1, \mathbb{R}^+|}.
\]

**Lemma 11.** Let \( \Pi_t^p \) the orthogonal projection from \( L^2(\Gamma) \) to \( V^p_h \) and \( m \leq p \). Then for \( \varepsilon > 0 \) and \( s \in [-1, 0] \) we have in the norms of \( H^s(\Gamma) \) respectively \( \bar{H}^s(\Gamma) \):

\[
||f - \Pi_t^p f||_{|s, \Gamma|} \leq C \left( \frac{h}{p} \right)^{m+1-s} |f|_{m+1, \Gamma} ,
\]

\[
||f - \Pi_t^p f||_{|s, \Gamma|,*} \leq C \left( \frac{h}{p} \right)^{m+1-s} |f|_{m+1, \Gamma} ,
\]

for all \( f \in H^{m+1}(\Gamma) \cap \bar{H}^s(\Gamma) \).

The second estimate for \( \partial \Gamma \neq \emptyset \) follows by extending \( \Pi_t^p f \in V^p_h \) by zero outside \( \Gamma \) which allows to estimate the \( \bar{H}^{\pm 1} \) norm on the left hand side by standard Sobolev norms.

Combining \( \Pi_t^p \) and \( \Pi_t^q \) one obtains as in Proposition 3.54 of [24]:

**Lemma 12.** Let \( f \in H^m_\sigma(\mathbb{R}^+, H^m(\Gamma) \cap \bar{H}^r(\Gamma)) \), \( 0 < m \leq q + 1 \), \( 0 < s \leq p + 1 \), \( r \leq s \), \( |l| \leq \frac{1}{2} \) such that \( |l r| \geq 0 \). Then if \( l, r \leq 0 \) and \( \varepsilon > 0 \)

\[
||f - \Pi_t^p \circ \Pi_t^p f||_{r,l, \Gamma} \leq C \left( \frac{h}{p} \right)^{\alpha} + \left( \frac{\Delta t}{p} \right)^{\beta} ||f||_{s,m, \Gamma} ,
\] (17)

\[
||f - \Pi_t^p \circ \Pi_t^q f||_{r,l, \Gamma,*} \leq C \left( \frac{h}{p} \right)^{\alpha-\varepsilon} + \left( \frac{\Delta t}{p} \right)^{\beta} ||f||_{s,m, \Gamma} ,
\] (18)
where \(\alpha = \min\{m - l, m - \frac{m+l+r}{m+s}\}\), \(\beta = \min\{m + s - (l + r), m + s - \frac{m+s+l}{m}\}\). If \(l, r > 0\), \(\beta = m + s - (l + r)\).

Below the Lemma is mostly used for \(\Delta t \lesssim h\), where \(\Delta t\) may be replaced by \(h\).

The proof of the following result is given in [7, Theorem 3.1]:

**Lemma 13.** For \(\varepsilon > 0\), \(a < 1\) and \(s \in [-1, \min\{-a + \frac{1}{2}, 0\}]\) there holds with the piecewise polynomial interpolant of degree \(p\), \(\Pi_y^p y^{-a}\), of \(y^{-a}\)

\[
\|y^{-a} - \Pi_y^p y^{-a}\|_{s,[0,1],s} \lesssim \left( \frac{h}{y^2} \right)^{-a + \frac{1}{2} - s - \varepsilon}.
\]

For positive powers of \(y\) we recall [8, Theorem 3.1]:

**Lemma 14.** For \(\varepsilon > 0\), \(0 < a\) and \(s \in [0, a + \frac{1}{2}]\) there holds with the piecewise polynomial interpolant of degree \(p + 1\), \(\Pi_y^{p+1} y^a\), of \(y^a\)

\[
\|y^a - \Pi_y^{p+1} y^a\|_{s,[0,1],s} \lesssim \left( \frac{h}{y^2} \right)^{\min\{a + \frac{1}{2} - s, 2 - s\} - \varepsilon}.
\]

### 4 Approximation of singularities

Solutions of the wave equation (1) exhibit singularities at edges and corners of the domain. We here recall a decomposition of the solution near these non-smooth boundary points into a leading part given by explicit singular functions plus less singular terms.

Let \(0 \leq d \leq n - 2\) and \(K \subset \mathbb{R}^{n-d}\) an open cone with vertex at 0, which is smooth outside the vertex. Denote the wedge over \(K\) by \(\mathcal{K} = K \times \mathbb{R}^d\). We study the wave equation in \(\mathcal{K}\):

\[
\begin{align*}
\partial_t^2 u(t, x) - \Delta u(t, x) &= 0 \quad \text{in } \mathbb{R}_+^t \times \mathcal{K}, \quad (19a) \\
Bu &= g \quad \text{on } \Gamma = \partial \mathcal{K}, \quad (19b) \\
u(0, x) &= \partial_t u(0, x) = 0 \quad \text{in } \mathcal{K}, \quad (19c)
\end{align*}
\]

with either inhomogeneous Dirichlet boundary conditions \(Bu = \partial_n u|_\Gamma\) or Neumann boundary conditions \(Bu = \partial_n u|_\Gamma\) on \(\Gamma\). We aim to describe the asymptotic behavior of a solution in \(\mathcal{K}\) near \(\{0\} \times \mathbb{R}^d\). Locally, the edge of a screen in \(\mathbb{R}^3\) corresponds to \(d = 1\), a cone point to \(d = 0\).

After a separation of variables near the edge of \(\mathcal{K}\), we consider the operator \(\mathcal{A}_B(\nu) = \nu^2 + (n - d - 2)\nu - \Delta_S\) with \(B = D\) for Dirichlet and \(B = N\) for Neumann boundary conditions in the subset \(\Xi = K \cap S^{n-d-1}\) of the sphere. \(\Delta_S\) is the Laplace operator on \(S^{n-d-1}\), and its eigenvalues in \(\Xi\) are denoted by \(\{\mu_{k,B}\}_{k=0}^\infty\). The eigenvalues of \(\mathcal{A}_B(\nu)\) may then be expressed as \(-i\nu_{\pm k,B} = i\frac{(n-d-2)}{2} \mp i\lambda_{k,B}\) with \(\lambda_{k,B} = \frac{(\nu^2 + 4k\nu)^{1/2}}{2}\). We normalize the associated orthogonal eigenfunctions \(\Phi_{k,B}\) of the angular variables \(\theta\) as \(\|\Phi_{k,B}\|_{L^2(\Xi)}^2 = \lambda_{k,B}^{-1}\).

For \(d = 1, n = 3\), the nonzero eigenvalues \(-i\nu_{\pm k,B} = \pm \frac{k\pi}{\alpha}\) are simple provided \(\frac{k\pi}{\alpha} \not\in \mathbb{N}\), where \(\alpha\) denotes the opening angle of \(K \subset \mathbb{R}^2\). They have multiplicity 2 otherwise. For \(k > 0\) one has the explicit formulas \(\Phi_{k,N}(\theta) = (k\pi)^{-\frac{1}{2}} \cos(k\pi\theta/\alpha)\), \(\Phi_{k,D}(\theta) = (k\pi)^{-\frac{1}{2}} \sin(k\pi\theta/\alpha)\). In the case of Neumann boundary conditions, the eigenvalue \(-i\nu_{0,N} = 0\) has multiplicity 2.
The limit \( \alpha \) tends to \( 2\pi^- \) recovers a screen with flat boundary, and for circular edges one may adapt the discussion as in [38].

For \( d = 0, n = 3 \), the singular exponents in the corner need to be determined numerically. See [46] for a discussion of polyhedral domains.

The singular exponents determine the local asymptotic expansion of the solution to the inhomogeneous wave equation

\[
\begin{align*}
\partial_t^2 u(t, x) - \Delta u(t, x) &= f \quad \text{in } \mathbb{R}_t^+ \times \mathcal{K}_x, \\
Bu &= 0 \quad \text{on } \Gamma = \partial \mathcal{K}, \\
u(0, x) &= \partial_t u(0, x) = 0 \quad \text{in } \mathcal{K},
\end{align*}
\]

(20a) (20b) (20c)

near the singular points. For details, see [31, Theorem 7.4 and Remark 7.5] in the case of the Neumann problem in a wedge, and [32, Theorem 4.1] for the Dirichlet problem in a cone. The formulas for the asymptotic expansion involve special solutions of the Dirichlet or Neumann problem in \( K \), as in [32, (3.5)], respectively [31, (4.4)]:

\[
w_{-j,B}(y, \omega, \zeta) = \frac{2^{1-\lambda_{k,B}}}{\Gamma(\lambda_{k,B})} (i|y|\sqrt{-|\zeta|^2 + \omega^2})^{\lambda_{k,B}} K_{\lambda_{k,B}} (i|y|\sqrt{-|\zeta|^2 + \omega^2}) |y|^{\nu_{-k,B}} \Phi_{k,B}(y/|y|).
\]

Here \( K_{\lambda} \) denotes the modified Bessel function of the third kind.

Using \( w_{-j,B} \), the leading singularities near an edge or a cone point are given as

\[
|y|^{\nu_{j,B}} \Phi_{j,B}(\theta)(\partial_t^2 - \Delta_x)^m (i|y|)^{2m} F_{(\omega, \zeta) \rightarrow (t, z)}^{(j)} C_{j,B},
\]

(21)

with \( c_{j,B}(\omega, \zeta) = \langle \hat{f}(\cdot, \omega, \zeta), w_{-j,B}(\cdot, \omega, \zeta) \rangle_{L^2(K)} \), plus a remainder which is less singular [31, 32]. Here \( m \in \mathbb{N} \). Additional logarithmic terms in \( |y| \) appear if \( i\nu_{j,B} \in \mathbb{N} \). The regularity of \( c_{j,B}(\omega, \zeta) = \langle \hat{f}(\cdot, \omega, \zeta), w_{-j,B}(\cdot, \omega, \zeta) \rangle_{L^2(K)} \) is determined by the data \( f \) in the wave equation (20).

Further information can be obtained by combining the convolution representation

\[
F_{(\omega, \zeta) \rightarrow (t, z)}^{(j)} C_{j,B} = \int_{\mathbb{R}^d} dz_1 \int_{\mathbb{R}} dt_1 \int_{K} dy f(y, z_1, t_1) W_{-j,B}(y, t - t_1, z - z_1)
\]

with information about the singular functions \( W_{-j,B}(y, t, z) = F_{(\omega, \zeta) \rightarrow (t, z)}^{(-1)} w_{-j,B} \). The singular support of \( W_{-j,B} \) lies on a lightcone emanating from the edge, \( \{(y, t, z) \in \mathbb{R}^{n+1} : t = \sqrt{|y|^2 + |z|^2}\} \).

Therefore \( F_{(\omega, \zeta) \rightarrow (t, z)}^{(-1)} C_{j,B} \) is smooth in

\[
\{(t, z) \in \mathbb{R}^{d+1} : t > \sup \{t_1 + \sqrt{|y|^2 + |z - z_1|^2} : (y, z_1, t_1) \in \text{singsupp } f \}\}.
\]

For smooth \( f \), \( \text{singsupp } f = \emptyset \) and therefore also \( F_{(\omega, \zeta) \rightarrow (t, z)}^{(j)} C_{j,B} \) is smooth everywhere.

The expansion for the inhomogeneous wave equation (20) implies an expansion for inhomogeneous boundary conditions (19). The argument is as for elliptic problems [37, Section 5]: If \( Bu = g \) on \( \mathbb{R}_t^+ \times \partial \mathcal{K} \), we choose an extension \( \tilde{g} \) in \( \mathbb{R}_t^+ \times \mathcal{K} \) with \( B\tilde{g} = g \) on \( \mathbb{R}_t^+ \times \partial \mathcal{K} \). Then \( U = u - \tilde{g} \)
satisfies the inhomogeneous wave equation \( \partial_t^2 U - \Delta U = f - \partial_t^2 \tilde{g} + \Delta \tilde{g} \) with homogeneous boundary condition \( BU = 0 \). The above discussion describes the asymptotic expansion of \( U \), and one concludes a corresponding expansion for \( u = U + \tilde{g} \). The resulting asymptotic expansions of the boundary values \( u|_\Gamma \) and \( \partial_n u|_\Gamma \) will be crucial for the analysis of the solutions to the boundary integral formulations.

In the case of a wedge, regularity results have also been obtained by Eskin [16] using Wiener-Hopf symbol factorizations.

### 4.1 Singularities for circular screens and approximation

We first illustrate the above expansion for the exterior of a circular wedge with exterior opening angle \( \alpha \). For \( \alpha \to 2\pi^- \), the wedge degenerates into the circular screen \( \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1\} \). Near the edge \( \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\} \) we use the coordinates \((y, z, \theta)\), where in polar coordinates in the \( x_1 - x_2\)-plane \( y = r - 1, \ z = \theta \). Using [38], an analogous expansion to (19) also holds in this curved geometry, with the same leading singular term \(|y|^{\nu}\), where \( \nu \to \frac{1}{2} \) as \( \alpha \to 2\pi^- \):

\[
\begin{align*}
    u(y, t, z)|_{\Gamma} &= a(t, z)|y|^\frac{1}{2} + v_0(y, z, t) , \\
    \partial_n u(y, t, z)|_{\Gamma} &= b(t, z)|y|^{-\frac{1}{2}} + \psi_0(y, z, t) .
\end{align*}
\]

Here \( a \) and \( b \) are smooth for smooth data.

From these decompositions we obtain optimal approximation properties for the \( hp\)-version.

**Theorem 15.** Let \( \varepsilon > 0 \). a) Let \( u \) be a strong solution to the homogeneous wave equation with inhomogeneous Neumann boundary conditions \( \partial_n u|_\Gamma = g \), with \( g \) smooth. Further, let \( \phi_{h, \Delta t} \) be the best approximation in the norm of \( H^s_p(\mathbb{R}^+, \tilde{H}^{\frac{1}{2} - s}(\Gamma)) \) to the Dirichlet trace \( u|_\Gamma \) in \( V^p_{\Delta t, h} \) on a quasi-uniform spatial mesh with \( \Delta t \lesssim h \). Then

\[
\| u - \phi_{h, \Delta t} \|_{r, \frac{1}{2} - s, \Gamma, *} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{p} + s - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{p} + s + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + s - \frac{1}{2}},
\]

where \( r \in [0, p] \) and the regular part \( v_0 \in H^s_p(\mathbb{R}^+, \tilde{H}^{\frac{1}{2} - s}(\Gamma)) \) of the singular expansion of \( u \), with \( \eta, \mu \) sufficiently large.

b) Let \( u \) be a strong solution to the homogeneous wave equation with inhomogeneous Dirichlet boundary conditions \( u|_\Gamma = g \), with \( g \) smooth. Further, let \( \psi_{h, \Delta t} \) be the best approximation in the norm of \( H^s_p(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma)) \) to the Neumann trace \( \partial_n u|_\Gamma \) in \( V^p_{\Delta t, h} \) on a quasi-uniform spatial mesh with \( \Delta t \lesssim h \). Then

\[
\| \partial_n u - \psi_{h, \Delta t} \|_{r, -\frac{1}{2}, \Gamma, *} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{p} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{p} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r},
\]

where \( r \in [0, p + 1] \) and the regular part \( \psi_0 \in H^{s+1}_p(\mathbb{R}^+, \tilde{H}^{\eta}(\Gamma)) \) of the singular expansion of \( \partial_n u \), with \( \eta, \mu \) sufficiently large.
Theorem 15 implies a corresponding result for the solutions of the single layer and hypersingular integral equations on the screen:

Corollary 16. Let $\varepsilon > 0$. a) Let $\phi$ be the solution to the hypersingular integral equation (7) and $\phi_{h,\Delta t}$ the best approximation in the norm of $H^s_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2} - s}(\Gamma))$ to $\phi$ in $V^{p,p}_{\Delta t,h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$. Then

$$\|\phi - \phi_{h,\Delta t}\|_{r,\frac{1}{2} - s, \Gamma,*} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} + \varepsilon} + \left( \frac{h}{p} \right)^{-\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu s - r - \frac{1}{2}},$$

where $r \in [0, p)$, $s \in [0, \frac{1}{2}]$ and the regular part $v_0 \in H^s_\sigma(\mathbb{R}^+, \tilde{H}^q(\Gamma))$ of the singular expansion of $\phi = u|_{\Gamma}$, with $\eta, \mu$ sufficiently large.

b) Let $\psi$ be the solution to the single layer integral equation (3) and $\psi_{h,\Delta t}$ the best approximation in the norm of $H^s_\sigma(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma))$ to $\psi$ in $V^{p,p}_{\Delta t,h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$. Then

$$\|\psi - \psi_{h,\Delta t}\|_{r,\frac{1}{2}, \Gamma,*} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r},$$

where $r \in [0, p + 1)$ and the regular part $\psi_0 \in H^{s+1}_\sigma(\mathbb{R}^+, \tilde{H}^q(\Gamma))$ of the singular expansion of $\psi = \partial_n u|_{\Gamma}$, with $\eta, \mu$ sufficiently large.

Indeed, on the flat screen the solutions to the integral equations are given by $\phi = [u]|_{\Gamma}$ in terms of the solution $u$ which satisfies Neumann conditions $Bu = \partial_n u|_{\Gamma} = g$, respectively $\psi = [\partial_n u]|_{\Gamma}$ in terms of the solution $u$ which satisfies Dirichlet conditions $Bu = u|_{\Gamma} = f$.

The theorem is shown in the following two subsections.

4.1.1 Approximation of the Neumann trace

Theorem 17. Under the assumptions of Theorem 15, for $\Delta t \lesssim h$ there holds

$$\|\partial_n u - \Pi_z \Pi_t \partial_n u\|_{r,\frac{1}{2}, \Gamma,*} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r}.$$

Proof. Using the decomposition (23) for $\partial_n u$, we can separate the singular and regular parts on the rectangular mesh:

$$\|\partial_n u - \Pi_z \Pi_t \partial_n u\|_{r,\frac{1}{2}, \Gamma,*} \leq \|b(t, z)|y|^{-\frac{1}{2}} - \Pi_z \Pi_t b(t, z)|y|^{-\frac{1}{2}}\|_{r,\frac{1}{2}, \Gamma,*} + \|\psi_0 - \Pi_z \Pi_t \psi_0\|_{r,\frac{1}{2}, \Gamma,*} \leq \|b(t, z)|y|^{-\frac{1}{2}} - \Pi_z \Pi_t b(t, z)|y|^{-\frac{1}{2}}\|_{r,\frac{1}{2}, \Gamma,*} + \|\Pi_z b(t, z)|y|^{-\frac{1}{2}} - \Pi_t \Pi_z b(t, z)|y|^{-\frac{1}{2}}\|_{r,\frac{1}{2}, \Gamma,*}$$

$$+ \|\psi_0 - \Pi_t \Pi_z \psi_0\|_{r,\frac{1}{2}, \Gamma,*} \leq \|b(t, z) - \Pi_t b(t, z)|y|^{-\frac{1}{2}} - \Pi_t b(t, z)|y|^{-\frac{1}{2}}\|_{r,\frac{1}{2}, \Gamma,*} + \|\Pi_z b(t, z)|y|^{-\frac{1}{2}} - \Pi_t \Pi_z b(t, z)|y|^{-\frac{1}{2}}\|_{r,\frac{1}{2}, \Gamma,*}$$

$$+ \|\Pi_t \Pi_z b(t, z)|y|^{-\frac{1}{2}} - \Pi_t \Pi_z b(t, z)|y|^{-\frac{1}{2}}\|_{r,\frac{1}{2}, \Gamma,*} + \|\psi_0 - \Pi_t \Pi_z \psi_0\|_{r,\frac{1}{2}, \Gamma,*}. $$

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Here, for the first term we have used Lemma 6, and for the second \( \Pi_x^p = \Pi_x^p \Pi_y^p \). The norm \( \| \cdot \|_{r, \varepsilon, -\frac{1}{2}} \) is the anisotropic space-time Sobolev norm in the \( t \) and \( z \) coordinates. We note that the first term is bounded by

\[
\| b(t, z) - \Pi_x^p b(t, z) \|_{r, \varepsilon, -\frac{1}{2}} \lesssim \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r} \| b(t, z) \|_{\mu + 1, \varepsilon, -\frac{1}{2}}.
\]

For the second and third terms we obtain with Lemma 6:

\[
\| \Pi_x^p b(t, z) \|_{\frac{1}{2} - \varepsilon} \lesssim \| \Pi_x^p \Pi_x^p b(t, z) \|_{\frac{1}{2} - \varepsilon} + \| \Pi_x^p \Pi_y^p b(t, z) \Pi_y^p y - \frac{1}{2} \|_{r, -\frac{1}{2}, \Gamma, *} + \| \Pi_x^p \Pi_y^p b(t, z) \Pi_y^p y - \frac{1}{2} \|_{r, -\frac{1}{2}, \Gamma, *}
\]

From Lemma 13 we have \( \| y^{-a} - \Pi_y^p y^{-a} \|_{\frac{1}{2}, I, *} \lesssim \left( \frac{h}{p} \right)^{-a + 1 - \varepsilon} \) and

\[
\| \Pi_x^p b(t, z) - \Pi_x^p \Pi_x^p b(t, z) \|_{r, \varepsilon, -\frac{1}{2}} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} + k - \varepsilon} \| b(t, z) \|_{r, k}.
\]

After possibly expanding finitely many terms, which may be treated as above, we assume that the regular part \( \psi_0 \) in (23) is \( H^n \) in space. Then using the approximation properties for \( \psi_0 \),

\[
\| \psi_0 - \Pi_x^p \Pi_x^p \psi_0 \|_{r, \frac{1}{2}, \Gamma, *} \lesssim \left( \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} \right) \| \psi_0 \|_{\mu + 1, \eta, \Gamma}.
\]

Combining the estimates for the different terms, we conclude that for \( \Delta t \lesssim h \) and sufficiently large \( k \)

\[
\| \partial_t u - \Pi_x^p \Pi_t \partial_t u \|_{r, \frac{1}{2}, \Gamma, *} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r}.
\]

\[
4.1.2 \quad \text{Approximation of the Dirichlet trace}
\]

We now consider the approximation of the solution \( u \) to the wave equation on the screen, with expansion (22), or equivalently the solution to the hypersingular integral equation. Apart from the energy norm, here the \( L^2 \)-norm is of interest, and we state the result for general Sobolev indices:

**Theorem 18.** For \( \Delta t \lesssim h, r \in [0, p) \) and \( s \in [0, \frac{1}{2}] \) there holds

\[
\| u - \Pi_x^p \Pi_t^p u \|_{r, \frac{1}{2} - s, \Gamma, *} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} + s - \varepsilon} + \left( \frac{h}{p} \right)^{s - \frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}}.
\]

**Proof.** Following the approach in Section 4.1.1, we use the triangle inequality

\[
\| u - \Pi_x^p \Pi_t^p u \|_{r, \frac{1}{2} - s, \Gamma, *} \lesssim \| a(t, z) \|_{\frac{1}{2}} - \Pi_t^p a(t, z) \|_{r, \frac{1}{2} - s, \Gamma, *} + \| \Pi_t^p a(t, z) \|_{r, \frac{1}{2} - s, \Gamma, *} + \| v_0 - \Pi_x^p v_0 \|_{r, \frac{1}{2} - s, \Gamma, *}.
\]
We first estimate

\[ \|a(t, z)|y|^{\frac{1}{2}} - \Pi_2^p a(t, z)|y|^{\frac{1}{2}}\|_{r, \frac{1}{2} - s, \Gamma, s} \leq \|a(t, z) - \Pi_1^p a(t, z)\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \|y|^{\frac{1}{2}}\|_{\frac{1}{2} - s, s} \]

and note that

\[ \|a(t, z) - \Pi_1^p a(t, z)\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \lesssim \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}} \|a(t, z)\|_{\mu, \frac{1}{2} - s} . \]

For the second term we note with Lemma 7, respectively Lemma 6:

\[ \|\Pi_1^p a(t, z)|y|^{\frac{1}{2}} - \Pi_2^p \Pi_2^p a(t, z)|y|^{\frac{1}{2}}\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \]

\[ \leq \|\Pi_1^p a(t, z)|y|^{\frac{1}{2}} - \Pi_2^p \Pi_2^p a(t, z)|y|^{\frac{1}{2}} + \Pi_2^p \Pi_2^p a(t, z)|y|^{\frac{1}{2}} - \Pi_1^p \Pi_2^p a(t, z)|y|^{\frac{1}{2}}\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \]

\[ \leq \|\Pi_1^p a(t, z) - \Pi_1^p \Pi_2^p a(t, z)\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \|y|^{\frac{1}{2}}\|_{\frac{1}{2} - s, I, s} + \|\Pi_1^p \Pi_2^p a(t, z)\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \|y|^{\frac{1}{2}} - \Pi_2^p |y|^{\frac{1}{2}}\|_{\frac{1}{2} - s, I, s} . \]

Now note that

\[ \|\Pi_1^p a(t, z) - \Pi_1^p \Pi_2^p a(t, z)\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \lesssim \left( \frac{h}{p} \right)^{k - \frac{1}{2} + s} \|a(t, z)\|_{r, k} \]

and, from Lemma 14,

\[ \|y|^{\frac{1}{2}} - \Pi_2^p |y|^{\frac{1}{2}}\|_{\frac{1}{2} - s, I, s} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} + s - \epsilon} . \]

It remains to estimate the remainder

\[ \|v_0 - \Pi_2 \Pi_2^p v_0\|_{r, \frac{1}{2} - s, \mu, \frac{1}{2} - s} \lesssim \left( \frac{h}{p} \right)^{s - \frac{1}{2} + \eta} \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}} , \]

as in Section 4.1.1. Combining the estimates for the different terms, we conclude the proof of the theorem. \( \square \)

### 4.2 Singularities for polygonal screens and approximation

We consider the singular expansion of the solution to the wave equation (1) with Dirichlet or Neumann boundary conditions on a polygonal screen \( \Gamma \). Compared to (22), (23) additional singularities now arise from the corners of the screen. For simplicity, we restrict ourselves to the model case of a flat polygonal screen \( \Gamma \subset \mathbb{R}^3 \). In this geometry, for elliptic problems asymptotic expansions and their implications for the numerical approximation are discussed in [33, 36].

The following gives a decomposition of the solution and its normal derivative on \( \Gamma \) near the vertex \((0, 0)\), in terms of polar coordinates \((r, \theta)\) centered at this point [19]. Note that we have two boundary values, \(u_{\pm}\), from the upper and lower sides of the screen. Note that we use refined information about the edge-vertex singularity.
\[ u(t,x) = C(t)\chi(r)r^\lambda\Phi(\theta) + C_1(t)\tilde{\chi}(\theta)\beta_1(r)(\sin(\theta))^{\frac{s}{2}} \\
+ C_2(t)\tilde{\chi}(\frac{\pi}{2} - \theta)\beta_2(r)(\cos(\theta))^{\frac{s}{2}} + v_0(t,r,\theta) \]
\[ =: u^v + v_1^v + u_2^v + v_0, \]
\[ \partial_n u(t,x) = C'(t)\chi(r)r^{\lambda-\frac{1}{2}}\Phi'(\theta) + C'_1(t)\tilde{\chi}(\theta)\beta'_1(r)r^{\frac{s}{2}} + v'_0(t,r,\theta) \]
\[ =: \psi^v + \psi_1^v + \psi_2^v + \psi_0. \]

Here \( \beta_j(r) \) behaves like \( r^{\lambda-\frac{1}{2}} \) near \( r = 0 \), while \( \beta'_j(r) \) behaves like \( r^{\lambda} \), \( j = 1, 2, \ldots \). Compared to the local coordinates near the edge in the previous section, the polar angle \( \theta \) corresponds to the distance \( |y| \) to the edge and the radius \( r \) to the variable \( z \) along the edge. For \( \Gamma = (0,1) \times (0,1) \times \{0\} \), the corner exponent \( \lambda \approx 0.2966 \).

To control the remainder terms in these formal computations requires elliptic a priori weighted estimates near the singularities, as discussed in [34].

From the decomposition, similar to Theorem 15 we obtain the approximation properties of the \( hp \)-method. The error is dominated by the edge singularities, not the corners.

**Theorem 19.** Let \( \varepsilon > 0 \). a) Let \( u \) be a strong solution to the homogeneous wave equation with inhomogeneous Neumann boundary conditions \( \partial_n u|_\Gamma = g \), with \( g \) smooth. Further, let \( \phi_{h,\Delta t} \) be the best approximation in the norm of \( H^s_0(\mathbb{R}^+, \tilde{H}^\frac{3-s}{2}(\Gamma)) \) to the Dirichlet trace \( u|_\Gamma \) in \( V^{p,p}_{\Delta t,h} \) on a quasi-uniform spatial mesh with \( \Delta t \lesssim h \). Then
\[ \|u - \phi_{h,\Delta t}\|_{r,\frac{1}{2}-s,\Gamma} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{2}+\min\{\lambda,0\}+s-\varepsilon} + \left( \frac{h}{p} \right)^{-\frac{1}{2}+s+\eta} + \left( \frac{\Delta t}{p} \right)^{\mu+s-r-\frac{1}{2}}, \]
where \( r \in [0, p) \) and the regular part \( v_0 \in H^p_0(\mathbb{R}^+, \tilde{H}^\eta(\Gamma)) \) of the singular expansion of \( u \), with \( \eta, \mu \) sufficiently large.

b) Let \( u \) be a strong solution to the homogeneous wave equation with inhomogeneous Dirichlet boundary conditions \( u|_\Gamma = g \), with \( g \) smooth. Further, let \( \psi_{h,\Delta t} \) be the best approximation in the norm of \( H^s_0(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma)) \) to the Neumann trace \( \partial_n u|_\Gamma \) in \( V^{p,p}_{\Delta t,h} \) on a quasi-uniform spatial mesh with \( \Delta t \lesssim h \). Then
\[ \|\partial_n u - \psi_{h,\Delta t}\|_{r,-\frac{1}{2},\Gamma} \lesssim \left( \frac{h}{p^2} \right)^{\frac{1}{2}+\min\{\lambda,0\}-\varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2}+\eta} + \left( \frac{\Delta t}{p} \right)^{\mu+1-r}, \]
where \( r \in [0, p+1) \) and the regular part \( \psi_0 \in H^{\mu+1}_0(\mathbb{R}^+, \tilde{H}^\eta(\Gamma)) \) of the singular expansion of \( \partial_n u \), with \( \eta, \mu \) sufficiently large.

Theorem 19 follows from the results in Subsections 4.3 and 4.4 below, which approximate the leading vertex and edge-vertex singularities. The less singular remainders are approximated as in the previous section. The theorem implies a corresponding result for the solutions of the single layer and hypersingular integral equations on the screen.
Corollary 20. Let $\varepsilon > 0$. a) Let $\phi$ be the solution to the hypersingular integral equation (7) and $\phi_{h,\Delta t}$ the best approximation in the norm of $H^{p}_s(\mathbb{R}^+,H^{1/2-s}(\Gamma))$ to $\phi$ in $V^{p,p}_{\Delta t,h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$. Then

$$
\|\phi - \phi_{h,\Delta t}\|_{r,\frac{1}{2} - s,\Gamma,*} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} + \min\{\lambda,0\} - \varepsilon} + \left( \frac{h}{p} \right)^{-\frac{1}{2} + s + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}},
$$

where $r \in [0,p)$, $s \in [0,\frac{1}{2}]$ and the regular part $v_0 \in H^{p}_s(\mathbb{R}^+,H^{1/2-s}(\Gamma))$ of the singular expansion of $\phi = u|_{\Gamma}$, with $\eta, \mu$ sufficiently large.

b) Let $\psi$ be the solution to the single layer integral equation (3) and $\psi_{h,\Delta t}$ the best approximation in the norm of $H^{p}_s(\mathbb{R}^+,H^{-1/2}(\Gamma))$ to $\psi$ in $V^{p,p}_{\Delta t,h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$. Then

$$
\|\psi - \psi_{h,\Delta t}\|_{r,\frac{1}{2} - s,\Gamma,*} \lesssim \left( \frac{h}{p} \right)^{\frac{1}{2} + \min\{\lambda,0\} - \varepsilon} + \left( \frac{h}{p} \right)^{\frac{1}{2} + \eta} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r},
$$

where $r \in [0,p + 1)$ and the regular part $\psi_0 \in H^{p+1}_s(\mathbb{R}^+,H^{1/2}(\Gamma))$ of the singular expansion of $\psi = \partial_\nu u|_{\Gamma}$, with $\eta, \mu$ sufficiently large.

### 4.3 Vertex singularities

To prove Theorem 19 for the Dirichlet trace, we first consider the approximation of the vertex singularities. The regular part is estimated as on the circular screen, and the edge vertex singularities are the content of the following subsection.

From Theorem 3.6 in [9] and Theorem 3.6 in [8] and its extension to $hp$ in Theorem 6.1 of [10] we recall a key elliptic result for the vertex singularities:

**Theorem 21.** a) Assume $\lambda > 0$ and $p \geq \lambda$. There exists $U^v \in V^p_h$ such that for all $0 \leq s \leq 1$

$$
\|u^v - U^v\|_{s,\Gamma,*} \lesssim \left( \frac{h}{p^2} \right)^{\lambda + 1 - s - \varepsilon}.
$$

b) Assume $\lambda > -\frac{1}{2}$. There exists $\Psi^v \in V^p_h$ such that for all $-1 \leq s \leq \min\{0,\lambda\}$

$$
\|\psi^v - \Psi^v\|_{s,\Gamma,*} \lesssim \left( \frac{h}{p^2} \right)^{\lambda - s - \varepsilon}.
$$

We now estimate the error of approximating the vertex singularity (24) of the Dirichlet trace $u|_{\Gamma}$

$$
\|C(t)r^\lambda \Phi(\theta) - \Pi^p_{x,y} C(t)r^\lambda \Phi(\theta)\|_{r,\frac{1}{2} - s,\Gamma,*} \leq \|C(t)r^\lambda \Phi(\theta) - r^\lambda \Phi(\theta)\Pi^p_{x,y} C(t)\|_{r,\frac{1}{2} - s,\Gamma,*} + \|r^\lambda \Phi(\theta)\Pi^p_{x,y} C(t) - \Pi^p_{x,y} r^\lambda \Phi(\theta)\Pi^p_{x,y} C(t)\|_{r,\frac{1}{2} - s,\Gamma,*}.
$$

For the first term, with Lemma 9 ($\frac{1}{2} - s \geq 0$), respectively Lemma 8 ($\frac{1}{2} - s < 0$):

$$
\|C(t)r^\lambda \Phi(\theta) - r^\lambda \Phi(\theta)\Pi^p_{x,y} C(t)\|_{r,\frac{1}{2} - s,\Gamma,*} = \|r^\lambda \Phi(\theta)(1 - \Pi^p_{x,y}) C(t)\|_{r,\frac{1}{2} - s,\Gamma,*} \lesssim \|r^\lambda \Phi(\theta)\|_{s,\Gamma,*} \|(1 - \Pi^p_{\frac{1}{2}}) C(t)\|_{s,\frac{1}{2} - s + \frac{1}{2},\mathbb{R}^+} \lesssim \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}} \|C(t)\|_{s,\mu,\mathbb{R}^+}.
$$
For the second term we note
\[ \|r^\lambda \Phi(\theta) \Pi_t^p C(t) - \Pi_{x,y}^p r^\lambda \Phi(\theta) \Pi_t^p C(t)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \leq \|r^\lambda \Phi(\theta) \Pi_t^p C(t)\|_{r, -s, \Gamma, s, \ast} \]
\[ \sim \|r^\lambda \Phi(\theta)\|_{r, -s, \Gamma, s, \ast} \left\{ \Pi_t^p C(t)\right\}_{\sigma, r, s + \frac{1}{2}, \mathbb{R}^+}. \]

From Theorem 21a) we conclude
\[ \|r^\lambda \Phi(\theta)\|_{r, -s, \Gamma, s, \ast} \sim \left( \frac{h}{p^2} \right)^{\lambda + \frac{1}{2} + s - \varepsilon}. \]

To estimate the approximation error for the Neumann trace \( \partial_n u|_\Gamma \), we note with the expansion from (25)
\[ \|C'(t) r^{\lambda - 1} \Phi'(\theta) - \Pi_t^p \Pi_{x,y}^p C'(t) r^{\lambda - 1} \Phi'(\theta)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \leq \|C'(t) r^{\lambda - 1} \Phi'(\theta) - r^{\lambda - 1} \Phi'(\theta) \Pi_t^p C'(t)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \]
\[ + \|r^{\lambda - 1} \Phi'(\theta) \Pi_t^p C'(t) - \Pi_{x,y}^p r^{\lambda - 1} \Phi'(\theta) \Pi_t^p C'(t)\|_{r, -\frac{1}{2}, \Gamma, s, \ast}. \]

For the first term, with Lemma 8,
\[ \|C'(t) r^{\lambda - 1} \Phi'(\theta) - r^{\lambda - 1} \Phi'(\theta) \Pi_t^p C'(t)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \]
\[ \sim \|r^{\lambda - 1} \Phi'(\theta)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \|\Pi_t^p C'(t)\|_{\sigma, r, \mathbb{R}^+} \]
\[ \lesssim \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r} \|C'(t)\|_{\sigma, \mu + 1, \mathbb{R}^+}. \]

For the second term we note
\[ \|r^{\lambda - 1} \Phi'(\theta) \Pi_t^p C'(t) - \Pi_{x,y}^p r^{\lambda - 1} \Phi'(\theta) \Pi_t^p C'(t)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \]
\[ \lesssim \|r^{\lambda - 1} \Phi'(\theta)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \|\Pi_t^p C'(t)\|_{\sigma, r, \mathbb{R}^+}. \]

From Theorem 21b) we conclude
\[ \|r^{\lambda - 1} \Phi'(\theta)\|_{r, -\frac{1}{2}, \Gamma, s, \ast} \lesssim \left( \frac{h}{p^2} \right)^{\lambda + \frac{1}{2} - \varepsilon}. \]

### 4.4 Edge-vertex singularities

To conclude the proof of Theorem 19, it remains to consider the approximation of the edge-vertex singularities.

From Theorem 3.4 in [9], Theorem 3.5 in [8] and its extension to \(hp\) in Theorem 5.1 of [10] we recall for the edge-vertex singularities:

**Theorem 22.** a) Assume \( \lambda > 0 \) and \( p \geq \min\{\lambda, 0\} \). There exists \( U^{ev} \in V_h^p \) such that for all \( 0 \leq s \leq \min\{\lambda + 1, 1\} \)
\[ \|u^{ev} - U^{ev}\|_{s, \Gamma, s} \lesssim \left( \frac{h}{p^2} \right)^{1 + \min\{\lambda, 0\} - s - \varepsilon}. \]

b) Assume \( \lambda > -\frac{1}{2} \) and \( p \geq 5 \). There exists \( \Psi^{ev} \in V_h^p \) such that for all \( -1 \leq s \leq \min\{0, \lambda\} \)
\[ \|\psi^{ev} - \Psi^{ev}\|_{s, \Gamma, s} \lesssim \left( \frac{h}{p^2} \right)^{\min\{\lambda, 0\} - s - \varepsilon}. \]
Concerning the edge-vertex singularities of the Dirichlet trace $u|_\Gamma$, we restrict ourselves to $u_1^{cv}$ in (24). We bound the corresponding approximation error by
\[
\|u_1^{cv} - \Pi_t^p u_1^{cv}\|_{r, \frac{1}{2} - s, \Gamma^*} \leq \|C(t) \tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}} - \Pi_t^p C(t) \tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}}\|_{r, \frac{1}{2} - s, \Gamma^*}
\leq \|(1 - \Pi_t^p) C(t) \tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}}\|_{r, \frac{1}{2} - s, \Gamma^*} + \|(1 - \Pi_t^p) C(t) \tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}}\|_{r, \frac{1}{2} - s, \Gamma^*}
\lesssim \|C(t) - \Pi_t^p C(t)\|_{\sigma, r - s + \frac{1}{2}, \mathbb{R}^+} \|\tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}}\|_{\frac{1}{2} - s, \Gamma^*} + \|(1 - \Pi_t^p) \tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}}\|_{\frac{1}{2} - s, \Gamma^*}.
\]

Here, for the first term we have used Lemma 9, respectively Lemma 8. We note that the first term is bounded by
\[
\|C(t) - \Pi_t^p C(t)\|_{\sigma, r - s + \frac{1}{2}, \mathbb{R}^+} \lesssim \left(\frac{\Delta t}{p}\right)^{\mu + s - r - \frac{1}{2}} \|C(t)\|_{\sigma, \mu, \mathbb{R}^+}.
\]

From Theorem 22a) we have
\[
\|(1 - \Pi_t^p) \tilde{\chi}(\theta) \beta_1(r)(\sin(\theta))^{\frac{1}{2}}\|_{\frac{1}{2} - s, \Gamma^*} \lesssim \left(\frac{h}{p^2}\right)^{\min(\lambda, 0) + \frac{1}{2} + s - \varepsilon}.
\]

For the edge-vertex singularities of the Neumann trace $\partial_n u|_\Gamma$, we consider $\psi_1^{cv}$ in (25) and estimate the approximation error as follows:
\[
\|

\psi_1^{cv} - \Pi_x^p\psi_1^{cv}\|_{r, -\frac{1}{2}, \Gamma^*} \leq \|C'(t) \tilde{\chi}(\theta) \beta_1'(r)(\sin(\theta))^{-\frac{1}{2}} - \Pi_x^p C(t) \tilde{\chi}(\theta) \beta_1'(r)(\sin(\theta))^{-\frac{1}{2}}\|_{r, -\frac{1}{2}, \Gamma^*}
\leq \|(1 - \Pi_x^p) C(t) \tilde{\chi}(\theta) \beta_1'(r)(\sin(\theta))^{-\frac{1}{2}}\|_{r, -\frac{1}{2}, \Gamma^*} + \|(1 - \Pi_x^p) C(t) \tilde{\chi}(\theta) \beta_1'(r)(\sin(\theta))^{-\frac{1}{2}}\|_{r, -\frac{1}{2}, \Gamma^*}
\lesssim \|C'(t) - \Pi_x^p C(t)\|_{\sigma, r, \mathbb{R}^+} \|\tilde{\chi}(\theta) \beta_1'(r)(\sin(\theta))^{-\frac{1}{2}}\|_{-\frac{1}{2}, \Gamma^*} \|C(t)\|_{\sigma, r, \mathbb{R}^+}.
\]

Here, for the first term we have used Lemma 8. We note that the first term is bounded by
\[
\|C'(t) - \Pi_x^p C(t)\|_{\sigma, r, \mathbb{R}^+} \lesssim \left(\frac{\Delta t}{p}\right)^{\mu + 1 - r} \|C(t)\|_{\sigma, \mu + 1, \mathbb{R}^+}.
\]

From Theorem 22b) we have
\[
\|(1 - \Pi_x^p) \tilde{\chi}(\theta) \beta_1'(r)(\sin(\theta))^{-\frac{1}{2}}\|_{-\frac{1}{2}, \Gamma^*} \lesssim \left(\frac{h}{p^2}\right)^{\min(\lambda, 0) + \frac{1}{2} - \varepsilon}.
\]

4.5 Singularities for polyhedral domains and approximation

The screen in the previous sections was the degenerate case of a polyhedral domain with opening angle $2\pi$ of the wedges, which leads to the strongest singularities. In general, for polyhedral domain with edge opening angles $< 2\pi$ the leading edge exponents of the solution $u$ in (21) with either Dirichlet or Neumann conditions are given by $\nu_{1,B} = \frac{\pi}{\alpha}$, where $\alpha$ is the opening angle of the wedge. Schwab and Suri [42] provide $p$-explicit approximation results for the Dirichlet case. We state the general approximation theorem for the elliptic case, which follows from the results of [42] and (for the Neumann trace) the stronger results of [9], see Theorems 21 and 22 above.
\textbf{Theorem 23.} a) There exists a function $u_{hp}$ such that for $s \in [0,1]$:  
\[ \|u - u_{hp}\|_{s,\Gamma,*} \lesssim \max \left\{ \frac{h^{k-s}}{p^{k-s}}, \frac{h^{\nu-s+\frac{1}{2}}}{p^{2\nu-2s+1}}, \frac{h^{\lambda-s+1-\varepsilon}}{p^{2\lambda-2s+2-2\varepsilon}} \right\}. \]

Here $v_0 \in H^k(\Gamma)$.

b) There exists a function $\psi_{hp}$ such that:
\[ \|\partial_n u - \psi_{hp}\|_{-\frac{1}{2},\Gamma,*} \lesssim \max \left\{ \frac{h^{k-s}}{p^{k-s}}, \frac{h^{\nu}}{p^{2\nu}}, \frac{h^{\lambda+\frac{1}{2}-\varepsilon}}{p^{2\lambda+1-2\varepsilon}} \right\}. \]

Here $\psi_0 \in H^k(\Gamma)$.

Here the second term in the maximum is the approximation error of the edge singular function, while the third is is the approximation error of the vertex singular function. The first term in the maximum is due to the approximation of the remainder of the asymptotic expansion.

Also in the time dependent case of the wave equation, the edge singularities dominate, except in domains with sharp reentrant corners [37]. For the Dirichlet and Neumann traces the exponents are the same as in the time independent case. Following the above analysis for the screen, by using the estimates for the approximation error of the time-independent singular functions at the vertices and edges from the proof of Theorem 23, one can show: If $u$ is the strong solution to the homogeneous wave equation with inhomogeneous Neumann boundary conditions $\partial_n u|_{\Gamma} = g$, with $g$ smooth, and $\phi_{h,\Delta t}$ the best approximation in the norm of $H^k_{\nu}(\mathbb{R}^+, \bar{H}^{\frac{1}{2}}(\Gamma))$ to the Dirichlet trace $u|_{\Gamma}$ in $V^{p,p}_{\Delta t,h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$, then for every $\varepsilon > 0$
\[ \|u - \phi_{h,\Delta t}\|_{\frac{1}{2} - s,\Gamma,*} \lesssim \max \left\{ \frac{h^{k-s}}{p^{k-s}}, \frac{h^{\nu-s+\frac{1}{2}}}{p^{2\nu-2s+1}}, \frac{h^{\lambda-s+1-\varepsilon}}{p^{2\lambda-2s+2-2\varepsilon}} \right\} + \left( \frac{\Delta t}{p} \right)^{\mu + s - r - \frac{1}{2}}. \]

Here $r \in [0,p)$ and the regular part $v_0 \in H^k_p(\mathbb{R}^+, \bar{H}^k(\Gamma))$ of the expansion of $u$.

This result generalizes Theorem A, part a), to polyhedral domains instead of flat screens, where $\lambda > 0$ and $\nu = \frac{1}{2}$.

Similarly, let $u$ be the strong solution to the homogeneous wave equation with inhomogeneous Dirichlet boundary conditions $u|_{\Gamma} = g$, with $g$ smooth. If $\psi_{h,\Delta t}$ is the best approximation in the norm of $H^k_{\nu}(\mathbb{R}^+, \bar{H}^{\frac{1}{2}}(\Gamma))$ to the Neumann trace $\partial_n u|_{\Gamma}$ in $V^{p,p}_{\Delta t,h}$ on a quasi-uniform spatial mesh with $\Delta t \lesssim h$, then for every $\varepsilon > 0$
\[ \|\partial_n u - \psi_{h,\Delta t}\|_{-\frac{1}{2},\Gamma,*} \lesssim \max \left\{ \frac{h^{k-s}}{p^{k-s}}, \frac{h^{\nu}}{p^{2\nu}}, \frac{h^{\lambda+\frac{1}{2}-\varepsilon}}{p^{2\lambda+1-2\varepsilon}} \right\} + \left( \frac{\Delta t}{p} \right)^{\mu + 1 - r}. \]

Here $r \in [0,p + 1)$ and the regular part $\psi_0 \in H^{\mu+1}_p(\mathbb{R}^+, \bar{H}^k(\Gamma))$ of the expansion of $\partial_n u$. 

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This estimate generalizes Theorem A, part b), to polyhedral domains.

Similar to Corollary B for a screen, also for a polyhedral domain the approximation rates for the Dirichlet and Neumann traces translate into approximation rates for appropriate boundary integral equations: $W \phi = (\frac{1}{2} - K')g$ for the Neumann problem, respectively $V \psi = (\frac{1}{2} - K)f$ for the Dirichlet problem.

5 hp a posteriori estimate for the single layer operator

This section extends the a posteriori error estimate of [21] for the $h$-version approximation of the Dirichlet Problem (5) to $hp$-version approximations on quasi-uniform meshes. For convenience, we denote the bilinear form associated to (5) by $B_D$.

**Theorem 24.** Let $\psi \in H^1_0(\mathbb{R}^+, \mathbb{H}^{-\frac{1}{2}}(\Gamma))$, $\psi_{h,\Delta t} \in V$ be the solutions to (5), respectively its discretization (15). Assume that $\mathcal{R} = \partial_t f - V \partial_t \psi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma))$. Then

$$\|\psi - \psi_{h,\Delta t}\|_{0, -\frac{1}{2}, \Gamma,*} \lesssim \max\{\Delta t, h\} \left( \|\partial_\Gamma \mathcal{R}\|^2_{0,0,\Gamma} + \|\nabla \mathcal{R}\|^2_{0,0,\Gamma} \right).$$

**Proof.** We first note that for all $\phi_{h,\Delta t} \in V_{h,\Delta t}^p$

$$\|\psi - \psi_{h,\Delta t}\|^2_{0, -\frac{1}{2}, \Gamma,*} \lesssim B_D(\psi - \psi_{h,\Delta t}, \psi - \psi_{h,\Delta t})$$

$$= \int_0^\infty \int_\Gamma \partial_t f(\psi - \psi_{h,\Delta t}) \, ds_x \, d_\sigma t - B_D(\phi_{h,\Delta t}, \psi - \psi_{h,\Delta t})$$

$$= \int_0^\infty \int_\Gamma \partial_t f(\psi - \phi_{h,\Delta t}) \, ds_x \, d_\sigma t - B_D(\psi_{h,\Delta t}, \psi - \phi_{h,\Delta t})$$

$$= \int_0^\infty \int_\Gamma (\partial_t f - V \partial_t \psi_{h,\Delta t})(\psi - \phi_{h,\Delta t}) \, ds_x \, d_\sigma t.$$  

The last term may be estimated by:

$$\int_0^\infty \int_\Gamma (\partial_t f - V \partial_t \psi_{h,\Delta t})(\psi - \phi_{h,\Delta t}) \, ds_x \, d_\sigma t$$

$$\leq \|\mathcal{R}\|_{0,1,\Gamma} \|\psi - \phi_{h,\Delta t}\|_{0, -\frac{1}{2}, \Gamma,*}.$$  

We use $\phi_{h,\Delta t} = \psi_{h,\Delta t}$ together with the interpolation inequality

$$\|\mathcal{R}\|^2_{0,\frac{1}{2},\Gamma} \leq \|\mathcal{R}\|_{0,0,\Gamma} \|\mathcal{R}\|_{0,1,\Gamma}.$$  

As the residual is perpendicular to $V_{h,\Delta t}^p$,

$$\|\mathcal{R}\|^2_{0,0,\Gamma} = \langle \mathcal{R}, \mathcal{R} \rangle = \langle \mathcal{R}, \mathcal{R} - \tilde{\psi}_{h,\Delta t} \rangle$$

$$\leq \|\mathcal{R}\|_{0,0,\Gamma} \|\mathcal{R} - \tilde{\phi}_{h,\Delta t}\|_{0,0,\Gamma}$$

for all $\tilde{\phi}_{h,\Delta t} \in V_{h,\Delta t}^p$, we obtain

$$\|\mathcal{R}\|_{0,0,\Gamma} \leq \inf\{\|\mathcal{R} - \tilde{\phi}_{h,\Delta t}\|_{0,0,\Gamma} : \tilde{\psi}_{h,\Delta t} \in V_{h,\Delta t}^p \}.$$
Choosing $\bar{\phi}_{h,\Delta t} = \bar{\Pi}_{h,\Delta t} R$, based on the interpolation operator defined earlier, we obtain

$$\|R\|_{0,0,\Gamma} \lesssim \frac{\Delta t}{p} \|\partial_t R\|_{0,0,\Gamma} + \left\| \frac{h}{p} \cdot \nabla R \right\|_{0,0,\Gamma}.$$  

The theorem follows. \(\square\)

### 5.1 Space-time $h_p$-adaptive procedure

The a posteriori error estimates lead to adaptive mesh refinement procedure, based on the four steps:

**SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE.**

For this purpose, we define the local error indicators

$$\eta^2_j(\Delta_i) = \frac{\max\left\{\Delta t, h\right\}}{p} \left( \|\partial_t R\|^2_{0,0,[t_j,t_{j+1}]} \times \Delta_i + \|\nabla R\|^2_{0,0,[t_j,t_{j+1}]} \times \Delta_i \right)$$

in a space-time element $[t_j,t_{j+1}] \times \Delta_i$. The precise algorithm is given as follows:

**Space–time Adaptive Algorithm:**

Input: Mesh $T = (T_S \times T_T)_0$, refinement parameter $\theta \in (0,1)$, tolerance $\epsilon > 0$, data $f$.

1. Solve $V \dot{\psi}_{h,\Delta t} = \dot{f}$ on $T$.
2. Compute the error indicators $\eta_j(\Delta_i)$.
3. Stop if $\sum_{i,j} \eta^2_j(\Delta_i) < \epsilon^2$.
4. Mark all $[t_j,t_{j+1}] \times \Delta_i \in T$ with $\eta_j(\Delta_i) > \theta \max \eta_j(\Delta_i)$.
5. Decide $p$ or $h$ refinement based on local smoothness indicators [30].
6. Refine to obtain new mesh:
   - $h$-marked triangles: divide $\Delta_i$ into 4 new triangles, replace $\Delta t$ by $(\Delta t)/2$.
   - $p$-marked triangles: increase polynomial degree by 1.
7. Go to 1.

Output: Approximation of $\dot{\psi}$.

See [21, 24] for first studies of $h$-adaptive time domain boundary elements.
6 Numerical experiments

6.1 Implementation of single layer operator

On the left hand side of (15), we use ansatz, respectively test functions

$$\psi_{\Delta t, h}(t, x) = \sum_{m=1}^{N_t} \sum_{l=1}^{N_s} c_{m}^{l} \gamma_{\Delta t}^{m} (t) \psi_{h}^{l} (x) \in V_{h, \Delta t}^{p}, \quad \Psi_{n,t}^{l} (t, x) = \gamma_{\Delta t}^{n} (t) \psi_{h}^{l} (x) \in V_{h, \Delta t}^{p}$$

to obtain for the single layer potential:

$$\int_{0}^{\infty} \int_{\Gamma} (V \psi_{\Delta t, h}) \gamma_{\Delta t}^{n} \psi_{h}^{l} ds \, dt = \sum_{m, i} c_{m}^{i} \frac{1}{4 \pi} \int_{0}^{\infty} \int_{\Gamma} \frac{1}{|x-y|} \gamma_{\Delta t}^{m} (t - |x-y|) \psi_{h}^{i} (y) \gamma_{\Delta t}^{n} (t) \psi_{h}^{l} (x) ds \, ds \, dt$$

$$= \sum_{m, i} c_{m}^{i} \frac{1}{4 \pi} \int_{\Gamma} \psi_{h}^{i} (y) \psi_{h}^{l} (x) \int_{0}^{\infty} \gamma_{\Delta t}^{m} (t - |x-y|) \gamma_{\Delta t}^{n} (t) \, dt \, ds \, ds$$

for all $n = 1, ..., N_t$ and $l = 1, ..., N_s$. Here, we use a dot to denote the time derivative.

For example, for piecewise linear basis functions, $p = 1$ in space and time, a calculation of the time integral shows:

$$\int_{0}^{\infty} \gamma_{\Delta t}^{m} (t - |x-y|) \gamma_{\Delta t}^{n} (t) \, dt$$

$$= - \frac{1}{2 (\Delta t)^2} (t_{n-m+2} - |x-y|)^2 \chi_{E_{n-m+1}} (x, y)$$

$$+ \left( \frac{1}{(\Delta t)^2} (t_{n-m+1} - |x-y|)^2 + \frac{1}{2 (\Delta t)^2} (t_{n-m} - |x-y|)^2 - 1 \right) \chi_{E_{n-m}} (x, y)$$

$$- \left( \frac{1}{(\Delta t)^2} (t_{n-m-1} - |x-y|)^2 + \frac{1}{2 (\Delta t)^2} (t_{n-m} - |x-y|)^2 - 1 \right) \chi_{E_{n-m-1}} (x, y)$$

$$+ \frac{1}{2 (\Delta t)^2} (t_{n-m-2} - |x-y|)^2 \chi_{E_{n-m-2}} (x, y) ,$$

with

$$E_{l} = \{(x, y) \in \Gamma \times \Gamma : t_{l} \leq |x-y| \leq t_{l+1}\} .$$

Formulas for higher polynomial degree may be found in [43]. After the time integral is evaluated analytically, the spatial integrals are approximated using a composite $hp$-graded quadrature [17].

The Galerkin discretization leads to a block–lower–Hessenberg system of equations, see Figure 1. Here the blocks $V^t$ correspond to the matrix with entries

$$V_{il}^{m-n} = \int_{0}^{\infty} \int_{\Gamma} (V \gamma_{\Delta t}^{m} \psi_{h}^{i}) \gamma_{\Delta t}^{n} \psi_{h}^{l} ds \, dt .$$

The system can be solved with an approximate time stepping scheme, respectively a space-time preconditioned GMRES method [23].

Note that the common, but non-conforming MOT time stepping schemes are based on piecewise constant test functions in time. Then $E_{l+1}$ does not contribute to the matrix entries of $V$, so the block $V^{-1} = 0$, and one obtains a block–lower–triangular system of equations.
6.2 Wave equation outside a screen

Example 1. Using the discretization by piecewise polynomials of degree $p$ described above, we compute the solution to the integral equation $V \psi = f$ on $\mathbb{R}^+ \times \Gamma$, with the square screen $\Gamma = \{(x, y, 0) : -\frac{1}{2} \leq x, y \leq -\frac{1}{2}\}$ depicted in Figure 2. We use a discretization with 8 triangles and 9 nodes in space, a time step $\Delta t = 0.5$, respectively 1.0, and study the convergence of the numerical solution as the polynomial degree is increased. Several right hand sides $f$ are considered. We compute the solution up to times $T = 5$ and compare to the extrapolated energy norm $\langle V \psi, \partial_t \psi \rangle^{1/2}$.

From [19], the convergence rate in energy norm of the uniform $h$-method on the screen is 0.5 as $h$ tends to 0. A cross section at $y = 0$ of the solution for the right hand side

$$f_1(t, x) = \sin^5(t)x^2$$

is shown in Figure 3, for a uniform triangulation of $\Gamma$ with 1250 triangles at times $t = 1.0$ and 1.4. The cross section shows the edge singularities of the solution, as well as unphysical oscillations as numerical errors near the boundary. It indicates the difficulty of approximating the singularities numerically.

For this right hand side $f_1$, Figure 4 depicts the convergence in energy norm of a $p$-method up to polynomial degree $p = 6$ in space and time. The empirical convergence rate for $\Delta t = 0.5$
Figure 3: Density $\psi$ computed by $h$-method on a uniform mesh with 1250 triangles for $f_1$ (left), cross section $y = 0$ at $t = 1.0, 1.4$ (right).

Figure 4: Relative error in energy norm for single-layer equation on square screen, Example 1.

Figure 5: Energy as function of time for time-singular $f_4$, Example 1.
For $\Delta t = 1.0$ the convergence rate is 1.18 (yellow crosses). The results reflect the expected doubling of the convergence rate for the $p$-method, compared to the $h$-method.

The results are confirmed for plane-wave right hand sides at low frequencies. For the right hand side

$$f_2(t, x) = \exp(-2/t^2) \cos(\omega t - kx),$$

with $k = (2, 0.5, 0.1)$ and $\omega = |k|$, Figure 4 (red squares) shows the convergence in energy norm of the $p$-version with rate 1.02 up to $p = 7$, for $\Delta t = 0.5$. For the higher-frequency wave

$$f_3(t, x) = \exp(-2/t^2) \cos(\omega t - kx),$$

with $k = (6, 0.5, 0.1)$ and $\omega = |k|$, piecewise linear or quadratic polynomials provide a poor approximation, as shown in Figure 4 (black diamonds) when $\Delta t = 0.5$. At higher $p$ the convergence rate becomes approximately 1.01, in agreement with the results for $f_1$ and $f_2$.

Finally, a source which is nonsmooth in time is considered,

$$f_4(t, x) = \sin^5(t)|1 - t|^\alpha \cos(k \cdot x),$$

with $\alpha = \frac{1}{2}$ and $k = (6, 0.5, 0.1)$. Note the square-root singularity in time in this right hand side. Figure 5 shows the “energy” $E(t) = \frac{1}{2} \langle V\psi, \partial_t \psi \rangle_{0,t \times \Gamma} - \langle f, \partial_t \psi \rangle_{0,t \times \Gamma}$ as a function of time at multiples of the time step $\Delta t = 0.5$, for $p = 1, 3, 5, 7$. While the solutions for different $p$ closely agree for short times, after the kink of the right hand side at $t = 1$ only higher polynomial degrees $p$ provide similar approximations. The convergence rate in energy norm here is 0.78, see Figure 4 (green stars), less than for $f_1$, $f_2$ and $f_3$.

### 6.3 Wave equation outside an icosahedron

**Example 2.** Using the discretization by piecewise polynomials of degree $p$ described above, we compute the solution to the integral equation $V\psi = f$ on $\mathbb{R}^+_t \times \Gamma$, for the icosahedron $\Gamma$ depicted in Figure 6. We use the discretization given by the 20 triangular faces of the icosahedron with 12 vertices and a time step $\Delta t = 0.5$. The convergence of the numerical solution is studied as the polynomial degree is increased. Different right hand sides $f$ are considered. We compute the solution for long times up to $T = 11$ and compare to an extrapolated benchmark energy as in Example 1. From the analysis in Section 4.5 for the direct integral equation $V\psi = (\frac{1}{2} - K)f$ one expects a convergence rate for the $p$-version of 1.62, dominated by the edge singularities.

A picture of the smooth solution at time $t = 0.5$ for the right hand side

$$f_1(t, x) = \sin^5(t)x^2$$

is shown in Figure 7, computed using an $h$-method on a uniform triangulation of $\Gamma$ with 1280 triangles and time step $\Delta t = 0.1$.

Figure 8 shows the convergence of the $p$-method in the energy norm for the right hand side $f_1$ from above (blue circles). The empirical convergence rate is 1.46 as the polynomial degree
Figure 6: Icosahedron with 20 triangles and 12 vertices.

Figure 7: Density $\psi$ computed by $h$-method on a uniform mesh with 1280 triangles for $f_1$.

Figure 8: Relative error in energy norm of $p$-method for single-layer equation on icosahedron, Example 2.
Figure 9: Energy as function of time up to $t = 11$ for the right hand side $f_1$, Example 2.

$p$ is increased. Figure 9 shows the possibility of long-time simulations and plots the energy of the numerical solution with $p = 6$ as a function up to time $t = 11$ at multiples of the time step $\Delta t = 0.5$. Figure 10 depicts the difference $|E_6(t) - E_p(t)|$ between the energy of the $p$-method solution for $p = 6$ and the numerical solutions for $p = 1, 2, \ldots, 5$. The error remains stable over the time interval, reflecting the space-time variational discretization used [22].

A second right hand side investigates a plane-wave

$$f_2(t, x) = \exp(-2/t^2)\cos(\omega t - kx) ,$$

with $k = (3, 0.5, 0.1)$ and $\omega = |k|$. The convergence rate in this case is approximately 1.61, see Figure 8, in agreement with the analysis and slightly higher than for $f_1$.

Finally, a right hand side with a singularity in space is considered,

$$f_3(t, x) = \sin^5(t) |\sin(kx)|^\alpha ,$$

$\alpha = \frac{1}{2}$ and $k = (2, 0.5, 0.1)$. The convergence rate here is lower, 1.22. Note that the solution $\psi$ has a singularity in space on the lines $kx = k\pi$, $k \in \mathbb{Z}$, similar to the edge singularities in Example 1. The convergence rate in Figure 8 is therefore reduced to values closer to those seen for screen problems in Example 1.

7 Conclusions

In this work we initiate the study of $p$- and $hp$-version boundary elements for the wave equation. The analysis and numerical experiments show the efficient approximation of both smooth solutions and geometric singularities in polyhedral domains, with the same convergence rates as known for $p$- and $hp$-approximations of time independent problems [11, 42].

For singular solutions the quasi-optimal $hp$-explicit estimates in this article complement the recent analysis of low-order approximations on algebraically graded meshes, for both finite and
boundary element methods [19, 27]. In both cases the convergence is determined by the singularities of the solution at non-smooth boundary points of the domain. The analysis combines the time independent approximation results [11] with the work by Plamenevskii and co-authors on the leading singular terms in the time dependent problem [39]. For screen problems the energy error $O(p^{-1})$ of the $p$-version has the same convergence rate as for an $h$-version on a 2-graded mesh. For open polyhedral domains the solutions are less singular, and accordingly higher convergence rates are obtained. Numerical experiments illustrate these on the icosahedron.

The a posteriori error estimate in this article provides a basis for $hp$-adaptive mesh refinement procedures for time domain boundary elements, building on first works for adaptive $h$-version boundary elements [21, 24]. Given the success of time independent $hp$-adaptive methods for time independent problems [25, 41], their extension to boundary element methods in the time domain provides a goal for future work.

References

[1] E. Alarcon and A. Reverter, $p$-adaptive boundary elements, Internat. J. Numer. Methods Engrg. 23 (1986), 801-829.

[2] I. Babuska, M. Suri, The optimal convergence rate of the $p$-version of the finite element method, SIAM J. Numer. Anal. 24 (1987), 750-776.

[3] I. Babuska, B. A. Szabo, I. N. Katz, The $p$-version of the finite element method, SIAM J. Numer. Anal. 18 (1981), 515-545.

[4] A. Bamberger, T. Ha Duong, Formulation variationnelle espace-temps pour le calcul par potentiel retardé d’une onde acoustique, Math. Meth. Appl. Sci. 8 (1986), 405-435 and 598-608.

Figure 10: Energy difference between $p = 6$ and lower $p$, as function of time, Example 2.
[5] L. Banz, H. Gimperlein, Z. Nezhi, E. P. Stephan, *Time domain BEM for sound radiation of tires*, Computational Mechanics 58 (2016), 45-57.

[6] A. Bespalov, *The hp-Version of the BEM with quasi-uniform meshes for a three-dimensional crack problem: the case of a smooth crack having smooth boundary curve*, Numer. Methods Partial Differential Eq. 24 (2008) 1159-1180.

[7] A. Bespalov, N. Heuer, *The p-version of the boundary element method for a three-dimensional crack problem*, J. Integral Eq. Appl. 17 (2005), 243-258.

[8] A. Bespalov, N. Heuer, *The p-version of the boundary element method for hypersingular operators on piecewise plane open surfaces*, Numer. Math. 100 (2005), 185-209.

[9] A. Bespalov, N. Heuer, *The p-version of the boundary element method for weakly singular operators on piecewise plane open surfaces*, Numer. Math. 106 (2007), 69-97.

[10] A. Bespalov, N. Heuer, *The hp-version of the boundary element method with quasi-uniform meshes in three dimensions*, ESAIM: M2AN 42 (2008), 821-849.

[11] A. Bespalov, N. Heuer, *The hp-version of the boundary element method with quasi-uniform meshes for weakly singular operators on surfaces*, IMA Journal of Numerical Analysis 30 (2010), 377-400.

[12] A. Chernov and C. Schwab, *Sparse p-version BEM for first kind boundary integral equations with random loading*, Appl. Numer. Math. 59 (2009), 2698-2712.

[13] M. Dauge, *Elliptic boundary value problems in corner domains*, Lecture Notes in Mathematics 1341, Springer-Verlag, 1988.

[14] M. R. Dorr, *The approximation theory for the p-version of the finite element method*, SIAM J. Numer. Anal. 21 (1984), 1180-1207.

[15] M. R. Dorr, *The approximation of solutions of elliptic boundary-value problems via the p-version of the finite element method*, SIAM J. Numer. Anal. 23 (1986), 58-77.

[16] G. Eskin, *The wave equation in a wedge with general boundary conditions*, Comm. Partial Differential Equations 17 (1992), 99-160.

[17] H. Gimperlein, M. Maischak, E. P. Stephan, *Adaptive time domain boundary element methods and engineering applications*, Journal of Integral Equations and Applications 29 (2017), 75-105.

[18] H. Gimperlein, F. Meyer, C. Özdemir, E. P. Stephan, *Time domain boundary elements for dynamic contact problems*, Computer Methods in Applied Mechanics and Engineering 333 (2018), 147-175.

[19] H. Gimperlein, F. Meyer, C. Özdemir, D. Stark, E. P. Stephan, *Boundary elements with mesh refinements for the wave equation*, Numerische Mathematik 139 (2018), 867-912.
[20] H. Gimperlein, Z. Nezhi, E. P. Stephan, *A priori error estimates for a time-dependent boundary element method for the acoustic wave equation in a half-space*, Mathematical Methods in the Applied Sciences 40 (2017), 448-462.

[21] H. Gimperlein, C. Özdemir, D. Stark, E. P. Stephan, *A residual a posteriori estimate for the time-domain boundary element method*, preprint.

[22] H. Gimperlein, C. Özdemir, E. P. Stephan, *Time domain boundary element methods for the Neumann problem and sound radiation of tires: Error estimates and acoustic problems*, J. Comp. Mathematics, 36 (2018), 70-89.

[23] H. Gimperlein, D. Stark, *On a preconditioner for time domain boundary element methods*, Engineering Analysis with Boundary Elements 96 (2018), 109-114.

[24] M. Gläfke, *Adaptive Methods for Time Domain Boundary Integral Equations*, Ph.D. thesis, Brunel University London (2012).

[25] J. Gwinner, E. P. Stephan, *Advanced Boundary Element Methods – Treatment of Boundary Value, Transmission and Contact Problems*, draft of book (2017).

[26] T. Ha-Duong, *On retarded potential boundary integral equations and their discretizations*, Topics in computational wave propagation, Lect. Notes Comput. Sci. Eng. 31 (2003), 301-336.

[27] F. Müller, C. Schwab, *Finite Elements with mesh refinement for wave equations in polygons*, J. Comp. Appl. Math. 283 (2015), 163-181.

[28] N. Heuer, M. Maischak, and E. P. Stephan, *Exponential convergence of the hp-version for the boundary element method on open surfaces*, Numer. Math. 83 (1999), 641-666.

[29] H. Holm, M. Maischak, E. P. Stephan, *The hp-version of the boundary element method for Helmholtz screen problems*, Computing 57 (1996), 105-134.

[30] P. Houston, E. Süli, *A note on the design of hp-adaptive finite element methods for elliptic partial differential equations*, Computer Methods in Applied Mechanics and Engineering 194 (2005), 229-243.

[31] A. Y. Kokotov, P. Neittaanmäki, B. A. Plamenevskii, *The Neumann problem for the wave equation in a cone*, J. Math. Sci. 102 (2000), 4400-4428.

[32] A. Y. Kokotov, P. Neittaanmäki, B. A. Plamenevskii, *Diffraction on a cone: The asymptotics of solutions near the vertex*, J. Math. Sci. 109 (2002), 1894-1910.

[33] M. Maischak, E. P. Stephan, *The h-p-version of the BEM with geometric meshes in 3D*, Boundary Element Topics (1997), 351-362.

[34] S. I. Matyukevich, B. A. Plamenevskii, *On dynamic problems in the theory of elasticity in domains with edges*, Algebra i Analiz 18 (2006), 158-233.
[35] T. von Petersdorff, *Randwertprobleme der Elastizitätstheorie für Polyeder-Singularitäten und Approximation mit Randelementmethoden*, Ph.D. thesis, Technische Universität Darmstadt (1989).

[36] T. von Petersdorff, E. P. Stephan, *Regularity of mixed boundary value problems in $\mathbb{R}^3$ and boundary element methods on graded meshes*, Math. Methods Appl. Sci. 12 (1990), 229-249.

[37] T. von Petersdorff, E. P. Stephan, *Decompositions in edge and corner singularities for the solution of the Dirichlet problem of the Laplacian in a polyhedron*, Math. Nachr. 149 (1990), 71-103.

[38] T. von Petersdorff, E. P. Stephan, *Singularities of the solution of the Laplacian in domains with circular edges*, Appl. Analysis 45 (1992), 281-294.

[39] B. A. Plamenevskii, *On the Dirichlet problem for the wave equation in a cylinder with edges*, Algebra i Analiz 10 (1998), 197-228.

[40] F.-J. Sayas, *Retarded Potentials and Time Domain Boundary Integral Equations: A Road Map*, Springer Series in Computational Mathematics 50 (2016).

[41] C. Schwab, *p- and hp- finite element methods: theory and applications in solid and fluid mechanics*, Oxford University Press, 1998.

[42] C. Schwab and M. Suri, *The optimal p-version approximation of singularities on polyhedra in the boundary element method*, SIAM J. Numer. Anal. 33 (1996), 729-759.

[43] E. P. Stephan, M. Maischak, E. Ostermann, *Transient boundary element method and numerical evaluation of retarded potentials*, Computational Science–ICCS 2008, 2008.

[44] E. P. Stephan and M. Suri, *On the convergence of the p-version of the boundary element Galerkin method*, Math. Comp. 52 (1989), 31-48.

[45] E. P. Stephan and M. Suri, *The h-p version of the boundary element method on polygonal domains with quasi-uniform meshes*, RAIRO Model. Math. Anal. Numer. 25 (1991), 783-807.

[46] E. P. Stephan and J. R. Whiteman, *Singularities of the Laplacian at corners and edges of threedimensional domains and their treatment with finite element methods*, Math. Meth. Appl. Sci. 10 (1988), 339-350.