Duality Invariant Non-Extreme Black Holes in Toroidally Compactified String Theory

Mirjam Cvetič and Ingo Gaida
Department of Physics and Astronomy
University of Pennsylvania
Philadelphia, PA 19104-6396, U.S.A.
(March 1997)

Abstract

We present duality invariant structure of the thermodynamic quantities of non-extreme black hole solutions of toroidally compactified Type II (M-theory) and heterotic string in five and four dimensions. These quantities are parameterized by duality invariant combinations of charges and the non-extremality parameter, which measures a deviation from the BPS-saturated limit. In particular, in $D = 5$ we find explicit S- and T-duality [U-duality] invariant expressions for solutions of toroidally compactified heterotic string [Type II string]. In $D = 4$, we consider general S-duality invariant expressions for non-extreme solutions of pure $N = 4$ supergravity and find to the leading order in non-extremality parameter the T- and S-duality invariant expressions of toroidally compactified heterotic string. General non-extreme solutions of toroidally compactified string in $D = 4$ are awaiting further investigation.

04.50.+h,04.20.Jb,04.70.Bw,11.25.Mj

Typeset using REVTeX
I. INTRODUCTION

There have been a number interesting developments in understanding the black hole physics in string theory (for a review, see e.g., [1,2,3]). In particular, string theory makes it possible to address microscopic properties of black holes, like the statistical origin of black hole entropy and their radiation rates.

The prerequisite for such investigations are the properties of the classical black hole solutions. A program to obtain the explicit form of the “generating solutions” for general rotating black hole solutions for toroidally compactified heterotic and Type II string theories in diverse dimensions has been explored in detail (for a review see [3]). (Note that this study excludes notable examples of black hole solutions for \( N = 2, D = 4,5 \) string vacua; see e.g., Ref. [4] and references therein.). Such string vacua possess enough (super)symmetries and well understood moduli spaces, thus allowing for a reliable treatment of at least BPS-saturated solutions.

While the generating solutions specify the \( D \)-dimensional space-time, the ultimate goal is to cast such solutions into manifestly duality invariant form. This goal has been accomplished for BPS-saturated rotating black hole solutions in \( D = 4 \) and \( D = 5 \) (in \( D \geq 6 \) these solutions have zero area of the horizon), i.e. the Bekenstein-Hawking entropy and the ADM mass of BPS-saturated solutions of toroidally compactified heterotic [Type II] string has been cast in S- and T-duality [U-duality] invariant form. On the other hand, for the non-extreme solutions, primarily the structure of the generating solution was given, and the duality invariant structure of non-extreme solutions awaits further investigation (These issues have also been addressed in Ref.[5,6,7]).

The explicit form of the generating solutions in \( D = 4,5 \) of toroidally compactified string was given in [10-13] (Particular examples of solutions have been obtained in a number of papers; for a review and references see [1] and references therein.). The main purpose of this paper is to fill in a gap by obtaining a duality invariant form of relevant thermodynamic quantities for non-extreme solutions. These quantities are expressed in terms of duality invariant combinations of (“dressed”) charges and the non-extremality parameter, which parameterizes a deviation from the BPS-saturated limit (This parameter can be in turn traded for the ADM mass and a duality invariant combinations of dressed charges.).

In this attempt we were successful for \( D = 5 \) non-extreme solutions (for \( D > 5 \) the proce-
II N = 4 AND N = 8 BLACK HOLES IN FIVE DIMENSIONS

The aim is to address general axi-symmetric solutions of the bosonic sector of the effective Lagrangian of toroidally compactified string theory. Since the bosonic sector includes the graviton, U(1) gauge fields, as well as massless scalar fields, the axi-symmetric solutions correspond to the dilatonic charged rotating black hole solutions. According to the “no-hair theorem” these black holes in D-dimensions are specified by the ADM mass $M$, $\left[\frac{D-1}{2}\right]$-components of angular momenta $J_{1,\ldots,\left[\frac{D-1}{2}\right]}$ and the number of allowed charge parameters associated with $U(1)$ gauge symmetry factors.

The most general black hole, compatible with the no-hair theorem, is obtained by acting on the generating solution with classical duality transformations. They do not change the D-dimensional Einstein-frame metric but do change the charges and scalar fields. One first
considers transformations, belonging to the maximal compact subgroup of duality transformations on the generating solution, which preserve the canonical asymptotic values of the scalar fields and show that all charges are generated in this way. Another duality transformation is used to change the canonical asymptotic values of the scalar fields to their arbitrary one. The aim is then to cast such a general solution in the manifestly duality invariant form.

We apply this method first to $D = 5$ non-extreme black holes of toroidally compactified string.

**A. The Generating Solution**

It has been shown in [13] that the generating solution for both the toroidally compactified Type II string and the heterotic string can be specified by the $U(1)$ charges in the NS-NS sector. Specifically in $D = 5$ the charge parameters of the generating solution are: $Q_1^{(1)}$, $Q_1^{(2)}$ and $P$. Here $Q_1^{(1)}$, $Q_1^{(2)}$ are the electric charges associated with $U(1)$ gauge fields $A_{\mu i}^{(1,2)}$ of the (momentum, winding) sector of NS-NS sector of the string theory, and $P$ is the electric charge of the gauge field, whose field strength is on-shell related to the field strength of the two-form field $B_{\mu \nu}$ by a duality transformation \[.\] In addition the generating solution is specified by two angular momenta $J_{\psi, \phi}$ and the non-extremality parameter $r_0$, which measures a deviation from the BPS-saturated limit. The explicit form of this generating solution has been given in [13]. The corresponding entropy reads

$$S = 4 \pi^2 \left\{ r_0^3 \left( \prod_{i=1}^{3} \cosh \delta_i + \prod_{i=1}^{3} \sinh \delta_i \right)^2 - \frac{1}{4} (J_{\phi} - J_{\psi})^2 \right\}^{1/2} +$$

$$\left\{ r_0^3 \left( \prod_{i=1}^{3} \cosh \delta_i - \prod_{i=1}^{3} \sinh \delta_i \right)^2 - \frac{1}{4} (J_{\phi} + J_{\psi})^2 \right\}^{1/2},$$

where the physical charges are

$$P = 2 \ r_0 \ \sinh \delta_p \ \cosh \delta_p ,$$

$$Q_1^{(1,2)} = Q_{1,2} = 2 \ r_0 \ \sinh \delta_{q1,2} \ \cosh \delta_{q1,2}.$$  \[II.2\]

\[In the following five dimensional case we take $G_N = \pi/8$ and $g^2 = e^{2\Phi_{\infty}}/3$.\]
and the angular momenta are defined as:

\[ J_{\phi,\psi} = 2 r_0 \left( a_{1,2} \prod_{i=1}^{3} \cosh \delta_i - a_{2,1} \prod_{i=1}^{3} \sinh \delta_i \right). \]  

(II.3)

The ADM mass reads

\[ M = r_0 \sum_{i=1}^{3} (\cosh^2 \delta_i + \sinh^2 \delta_i). \]  

(II.4)

Here the non-extremality parameter \( r_0 \) and \( a_{1,2} \) parameterize respectively the ADM mass and the two components of the angular momentum of the \( D = 5 \) Kerr (neutral, rotating) solution (For \( r_0/2 \geq (|a_1| + |a_2|)^2 \) the solution has inner and outer horizon.). The three charges are determined in terms of \( r_0 \) and the three “boost” parameters \( \delta_i \), corresponding to the symmetry transformations of stationary solutions which generate charged solutions from neutral ones (for more details see [13]).

The BPS-saturated limit is obtained, by taking the non-extremality parameter \( r_0 \to 0 \) and the boost parameters \( \delta \to \infty \), while keeping the values of charges finite.

### B. Duality Invariant Black Holes of Toroidally Compactified Heterotic String

We start from the generating solution of [13] and a scalar non-linear \( \sigma \)-modell with coset space \( SO(1,1) \times \frac{SO(5,21)}{SO(5) \times SO(21)} \) of toroidally compactified heterotic string theory with the following bosonic field content in five dimensions: The graviton, 116 scalar fields (115 moduli in the matrix \( M \) and the dilaton), 26 U(1) gauge fields and one additional \( U(1) \) gauge field, which is the dual of the 3-form fieldstrength in five dimensions

We can derive a general T-duality invariant solution by acting with elements of \( SO(5) \times SO(21) \) (the maximal compact subgroup of \( SO(1,1) \times SO(5,21) \)) on the generating solution.

The electric charges are entries in a vector \( \vec{Q} \in O(5,21) \) with \( \vec{Q}^T = (Q_1^{(1)}, 0; Q_2^{(2)}, 0_20) \). Imposing a subset of T-duality transformations, lying in a coset \( SO(5) \times SO(21)/[SO(4) \times SO(20)] \) introduces 24 new charge parameters. Thus the theory is specified by 27 charge parameters [13]. The action of these transformations renders the three charges of the generating solution in duality invariant form. Namely

\[ Q_1^{(1)} \to X = \frac{1}{2} \sqrt{\vec{Q}^T M_+ \vec{Q}} + \frac{1}{2} \sqrt{\vec{Q}^T M_- \vec{Q}}, \]

\[ Q_2^{(2)} \to Y = \frac{1}{2} \sqrt{\vec{Q}^T M_+ \vec{Q}} - \frac{1}{2} \sqrt{\vec{Q}^T M_- \vec{Q}}, \]  

(II.5)
while $P$ remains intact, i.e. it is a singlet under $T$-duality transformations.

Using these three S- and T-duality invariant “coordinates” enables us to cast all parameters in the generating solution into S- and T-duality invariant form. For convenience we will discuss in the following only the entropy and the ADM mass. Introducing “non-extreme hatted” quantities we find the entropy

$$\hat{X}_i \equiv \sqrt{X_i^2 + r_0^2}, \quad X_i = (X, Y, P). \quad (II.6)$$

$$S = 2\pi \left\{ \left[ \hat{X} \hat{Y} \hat{P} + r_0^2 (\hat{X} + \hat{Y} + \hat{P}) + \sqrt{(\hat{X}^2 - r_0^2)(\hat{Y}^2 - r_0^2)(\hat{P}^2 - r_0^2) - (J_\phi - J_\psi)^2} \right]^{1/2} + \left[ \hat{X} \hat{Y} \hat{P} + r_0^2 (\hat{X} + \hat{Y} + \hat{P}) - \sqrt{(\hat{X}^2 - r_0^2)(\hat{Y}^2 - r_0^2)(\hat{P}^2 - r_0^2) - (J_\phi + J_\psi)^2} \right]^{1/2} \right\}. \quad (II.7)$$

The ADM mass reads

$$M = \hat{X} + \hat{Y} + \hat{P}. \quad (II.8)$$

This is a duality invariant form of the ADM mass in terms of the non-extremality parameter and the charges. In principle, the inversion of this expression yields the non-extremality parameter in terms of the ADM mass and duality invariant combinations of charges.

In the regular BPS limit ($r_0 \to 0, J_\phi = -J_\psi$) the entropy becomes independent of the moduli and is only a function of the bare quantized charges $\beta$ and $\vec{a}$, which are defined in terms of dressed charges as: $P = \beta/g^4$ and $\vec{Q} = g^2 M_\infty \vec{\alpha}$. This yields [13]

$$S_{BPS} = 2\pi \sqrt{(\vec{\alpha}^T \Lambda \vec{\alpha}) \beta} = 4J_\psi^2. \quad (II.9)$$

C. Duality Invariant Black Holes of Toroidally Compactified Type II String

The toroidally compactified Type II string theory has $N = 8$ supersymmetry and a scalar non-linear $\sigma$-modell with coset space $E_6(6)_{USp(8)}$. The Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of the theory yields the coset $SO(1, 1) \times \frac{SO(5, 5)}{SO(5) \times SO(5)}$ (which is also the coset space of the toroidal sector of the heterotic string). Since the toroidally compactified heterotic string and the NS-NS sector of the toroidally compactified Type II string theory have the same effective action they also have the same classical solutions. Moreover, it was shown in [15] that the generating solutions for black holes in toroidally compactified Type II string theory are the
same as the one of toroidally compactified heterotic string, i.e. the generating solution of toroidally compactified Type II string can be specified with the NS-NS charges, only. We will use this result here to present general U-duality invariant non-extreme solutions of Type II string theory in five dimensions.

The central charge matrix of $N = 8$ supergravity in five dimensions can be brought to a skew-diagonal form by the use of an $USp(8)$ transformation $Z^0 = \Lambda Z \Lambda^T$. In this normal form the central charge matrix has four real eigenvalues $z_i$ ($i = 1, \ldots, 4$). General U-duality invariants can be expressed in terms of the central charges as:

$$J_{2n} \equiv \text{tr} \ (Z^+ Z)^{2n} = 2 \sum_{i=1}^{4} z_i^{2n}, \quad n = 1, 2, \ldots$$  \hspace{1cm} (II.10)

Using the $USp(8)$ symplectic matrix $\Omega$ one can introduce the cubic invariant $J_3$ which can be expressed in terms of central charge eigenvalues as:

$$J_3 \equiv \text{tr} \ (\Omega Z)^3 = 2 \sum_{i=1}^{4} z_i^3,$$  \hspace{1cm} (II.11)

and the constraint $J_1$:

$$J_1 \equiv \text{tr} \ (\Omega Z) = 2 \sum_{i=1}^{4} z_i = 0.$$  \hspace{1cm} (II.12)

Hence, for the generating solution the central charge matrix is parametrized by three independent real eigenvalues $z_{1,2,3,4}$:

$$z_{1,2} = P \pm Q_1 \pm Q_2,$$

$$z_{3,4} = -P \pm Q_1 \mp Q_2.$$  \hspace{1cm} (II.13)

Since the generating solution is given by three independent charges the general non-extreme U-duality invariant solution must be a function of three U-invariants. We choose these three invariants to be $J_2, J_3$ and $J_4$. Using the following identities, satisfied by the generating solution,

$$8J_4 - J_2^2 = 256 \left( P^2 Q_1^2 + P^2 Q_2^2 + Q_1^2 Q_2^2 \right),$$

$$J_3 = 48 \ P \ Q_1 \ Q_2,$$

$$J_2 = 8Q_1^2 + 8Q_2^2 + 8P^2,$$  \hspace{1cm} (II.14)

enables us to solve a cubic equation for all three charges of the generating solution in terms of $U$-invariants ($Q_{1,2}, P \rightarrow X_i$). We find three real solutions
\[ X_i = \sqrt{2\rho^{1/3} \cos\left(\frac{\varphi + 2n_i \pi}{3}\right) + \frac{1}{24} J_2}, \quad n_{1,2,3} = 0, 1, 2 \quad (\text{II.15}) \]

with
\[
\rho = \frac{1}{144} \sqrt{7J_2^2 - 24J_4}, \quad \rho \cos \varphi = \frac{1}{1024} J_2 \left( \frac{17}{108} J_2^2 - \frac{2}{3} J_4 \right) + \frac{1}{2} \left( \frac{J_3}{48} \right)^2. \quad (\text{II.16})
\]

The entropy and the ADM mass for general non-extreme U-duality invariant black holes in five dimensions is (II.7) and (II.8) in these coordinates. In the regular BPS limit the entropy is only a function of \( J_3 \) [14,15] and one angular momentum \( J_\psi = -J_\phi \equiv J \)
\[ S_{BPS} = \pi \sqrt{\frac{J_3^2}{12} - 8J^2}. \quad (\text{II.17}) \]

### III. N = 4 BLACK HOLES IN FOUR DIMENSIONS

In \( D = 4 \) the generating solution is parametrized by five charges [14,13]. Although we will comment on these general non-extreme black holes at the end of this chapter, we were not able to find the general non-extreme duality invariant thermodynamic quantities within our approach. However, starting with a truncated version of the generating solution, specified by four-charges, and further reducing it to two with “fixed values” of moduli, we find the duality invariant combinations of charges specifying non-extreme solutions of pure \( N = 4 \) supergravity.

In addition, in Section III, we shall address the duality invariant solutions of toroidally compactified string theory. Using the results for the non-extreme five (charge) parameter generating solution of [12] and the duality invariant structure of such solutions in the BPS limit [14], we were able to determine the duality invariant form of the near-extreme solution.

---

A special case of this general non-extreme solution is a Kerr-Newman type solution with \( Q_2 = Q_1 = P \). This solution can be completely determined by \( J_3 \). In this special case the non-extremality parameter is \( r_0^2 = \left( \frac{M}{\mathcal{A}} \right)^2 - \left( \frac{1}{18} J_3 \right)^{1/3} \) and the BPS mass reads \( M_{BPSS}^2 = \left( \frac{9}{16} J_3 \right)^{2/3} \). In the extremal limit \( r_0 = 2(a_1 + a_2)^2 \) with \( a_{1,2} \geq 0 \) we can eliminate one angular momentum parameter by keeping the other one and the non-extremality parameter. We find \( J_\psi + J_\phi = \frac{1}{2} (\hat{P}_+ - \hat{P}_-) \) with \( \hat{P}_\pm = (\hat{p} \pm r_0)^{3/2} \). The entropy in the extremal limit reads \( S_{EXT} = \pi \left\{ \sqrt{2(\hat{P}_+ + \hat{P}_-)^2 - (\hat{P}_+ - \hat{P}_- - 4J_\psi)^2} + \sqrt{\hat{P}_+ - \hat{P}_-} \right\} \). An analogous discussion holds for the toroidally compactified heterotic string in five dimensions (with \( N = 4 \) supersymmetry).
A. The Generating Solution

The starting point of the $D = 4$ black hole solutions is the Kerr-solution (neutral rotating black hole $^3$). The solution is specified by the non-extremality parameter $r_0$ and one rotational parameter $a$. Kerr metrics are all stationary and axisymmetric with Killing fields $\xi^a = (\frac{\partial}{\partial t})^a$ and $\psi^a = (\frac{\partial}{\partial \phi})^a$. The event horizons are located where the $r = \text{const}$ surface vanishes. Thus the inner and the outer horizon is given by

$$r_\pm = \frac{r_0}{2} \pm \sqrt{\frac{r_0^2}{4} - a^2}. \quad (\text{III.1})$$

provided $r_0/2 \geq a$. These horizons disappear for $a > r_0/2$ giving rise to a naked singularity. The limit $a \to r_0/2$ is known as the extremal limit. The physical singularity of the vacuum Kerr solution is time-like and concentrated on a ring at $r = 0$ and $\theta = \pi/2$.

Performing four boost transformations on the Kerr solution, i.e. symmetry transformations of stationary solutions, generate charged solutions (for more details see $^2$) specified by the four charges:

$$P^{(1,2)}_1 = 2 r_0 \sinh \delta_{p1,2} \cosh \delta_{p1,2},$$

$$Q^{(1,2)}_2 = 2 r_0 \sinh \delta_{q1,2} \cosh \delta_{q1,2}.$$ \quad (\text{III.2})

Here $(P^{(1)}_1, P^{(2)}_1)$ and $(Q^{(1)}_2, Q^{(2)}_2)$ are the magnetic [electric] charges associated with $U(1)$ gauge fields $A^{(1,2)}_\mu$ in the (momentum, winding) sector of the string theory, associated with the first [second] compactified direction. These charges are entries in a vector $\vec{Q} \in O(6, 22)$ and $\vec{P} \in O(6, 22)$ with $\vec{Q}^T = (0, Q^{(1)}_2, 0_4; 0, Q^{(2)}_2, 0_{20})$ and $\vec{P}^T = (P^{(1)}_1, 0_5; P^{(2)}_1, 0_{21})$.

The corresponding charged rotating solution has the angular momentum specified as $^1$

$$J = 4 r_0 a \left( \prod_{i=1}^4 \cosh \delta_i - \prod_{i=1}^4 \sinh \delta_i \right). \quad (\text{III.3})$$

This generating solution is thus parametrized by the non-extremality parameter $r_0$ and the four charges (\text{III.2}) and the angular momentum (\text{III.3}).

The four dimensional metric and the fields for this solution have been given in $^2$. The entropy and the ADM mass of this charged rotating solution is $^1$.

$^3$We take $G_N = \frac{1}{4} g^2$, $g^2 = e^{\Phi_{\infty}}$.

$^4$and the dilaton charge is of the form: $\Sigma = \hat{P}_1 + \hat{P}_2 - \hat{Q}_1 - \hat{Q}_2$. 

\[
S = \frac{\pi}{r_0} \left\{ r_+ \prod_{i=1}^{4} \sqrt{\hat{Q}_i + r_0} + r_- \prod_{i=1}^{4} \sqrt{\hat{Q}_i - r_0} \right\}, \quad (\text{III.4})
\]

\[
M = 2r_0 \sum_{i=1}^{4} (\cosh^2 \delta_i + \sinh^2 \delta_i) = \hat{Q}_1 + \hat{Q}_2 + \hat{P}_1 + \hat{P}_2. \quad (\text{III.5})
\]

Note that the above solution, specified by four-charges (III.2), corresponds to a special case of the generating solution for toroidally compactified heterotic and Type II string, which is specified by five charge parameters.

**B. Duality Invariant Black Holes in N = 4 Supergravity**

We start with this four-charge parameter generating solution of toroidally compactified heterotic (or Type II) string (described in the previous Section) in order to obtain the general S-duality invariant solutions of pure N = 4 supergravity theory (i.e., with “fixed moduli”).

The scalar sector of D = 4 toroidally compactified heterotic string corresponds to the non-linear \(\sigma\)-model with coset space \(SU(1,1)/U(1) \times O(6,22)/O(6) \times O(22)\). We can now derive a T-duality invariant solution by acting with elements of \(O(6) \times O(22)\) (maximal compact subgroup of \(O(6, 22)\)) on the four-charge parameter generating solution. This solution is thus specified by 54 independent charge parameters (i.e. 4 charge parameters of the generating solution and 50 parameters of the coset \([SO(6) \times SO(22)]/[SO(4) \times SO(20)] \subset O(6) \times O(22)\). Namely, the solution is specified by 28 electric and 28 magnetic charges subject to two charge constraints.

Thus, the action of the transformations renders the four charges of the generating solution to a T-duality invariant form:

\[
P_{1,2} \rightarrow \frac{1}{2} \sqrt{\bar{P}_t \mathcal{M}_+ \bar{P}} \pm \frac{1}{2} \sqrt{\bar{P}_t \mathcal{M}_- \bar{P}},
\]
\[
Q_{1,2} \rightarrow \frac{1}{2} \sqrt{\bar{Q}_t \mathcal{M}_+ \bar{Q}} \pm \frac{1}{2} \sqrt{\bar{Q}_t \mathcal{M}_- \bar{Q}}. \quad (\text{III.6})
\]

The corresponding solution obeys two \(O(6, 22)\) invariant constraints on the charges:

\[
\bar{P}_t \mathcal{M}_+ \bar{Q} = 0, \quad \bar{P}_t L \mathcal{M}_{\infty} L \bar{Q} = 0 \quad (\text{III.7})
\]

and is not invariant under \(SL(2, \mathbb{R})\) transformations. The dressed charges are related to bare quantized charges in the following way:

\[
\bar{P} = L \bar{\beta}, \quad \bar{Q} = e^{\Phi_{\infty}} \mathcal{M}_{\infty} (\bar{\alpha} + \Psi_{\infty} \bar{\beta}). \quad (\text{III.8})
\]
Setting, $\Phi_\infty = \Psi_\infty = 0$ and $M_\infty = I_{28}$, allows one to perform an $SO(2)$ (maximal compact subgroup of S-duality group $SL(2, \mathbb{R})$) transformation specified by angle $\theta$, which enables one to find a manifestly S-duality invariant form of the solution. This transformation removes one of the charge constraints. However, one constraint on charges remains and can be cast in the form:

$$
\tan \theta = -\frac{1}{2\bar{\alpha}^T \mu_+ \bar{\beta}} \left( g^2 \bar{\alpha}^T \mu_+ \bar{\alpha} - \frac{1}{g^2} \bar{\beta}^T \mu_+ \bar{\beta} - \gamma(\mu_+) \right) 
= -\frac{1}{2\bar{\alpha}^T L \bar{\beta}} \left( g^2 \bar{\alpha}^T L \bar{\alpha} - \frac{1}{g^2} \bar{\beta}^T L \bar{\beta} - \gamma(L) \right)
$$

(III.9)

with

$$
\gamma(y) = \sqrt{4 (\bar{\alpha}^T y \bar{\beta})^2 + (g^2 \bar{\alpha}^T y \bar{\alpha} - \frac{1}{g^2} \bar{\beta}^T y \bar{\beta})^2}.
$$

(III.10)

This constraint can be solved to give

$$
\mu_+ \bar{\alpha} \equiv (I + L) \bar{\alpha} = 2L \bar{\alpha}, \quad \mu_+ \bar{\beta} \equiv (I + L) \bar{\beta} = 2L \bar{\beta}.
$$

(III.11)

Thus we find $\mu_- \bar{\alpha} = \mu_- \bar{\beta} = 0$ and recover the result that a consistent $N = 4$ truncation of the heterotic string theory in four dimensions removes the “left-movers” by identifying the momentum with the winding modes and sets, in addition, the 16 Yang-Mills gauge fields of the heterotic string theory to zero. As the last step one can transform the solution back to the physical, dressed charges with arbitrary asymptotic values of the dilaton and axion fields.

Thus, in order to ensure that the remaining charge constraint (III.9) is (automatically) satisfied we ended up with an S-duality invariant solution of pure $N = 4$ supergravity (with “fixed moduli”) and the coset space $SU(1,1)/U(1)$. Namely, the corresponding matrix of the moduli is diagonal and constant and only the dilaton and the axion remain as dynamical scalar degrees of freedom. This truncated model contains now 6 vector fields and parametrizes 12 charges.

---

5Thus the notation is somehow redundant: $\bar{\alpha}^T L \bar{\alpha} = 2 \sum_{i=1}^{6} \alpha_i \alpha_i$. The analogous notation holds for the bare magnetic charges as well. The surviving symmetry group is only $SO(6)$, but for convenience we keep the “trivial” $SO(6,6)$ notation.
The same solution can be obtained directly by starting with the special case of the generating solution with \( P_1 = P_2 \) and \( Q_1 = Q_2 \). The corresponding solution is specified by two harmonic functions and fits into the pointlike supersymmetric Wilson-Israel-Perjés solutions of \([8]\). There the non-extreme entropy was also given in a duality invariant form and the corresponding thermodynamic quantities can be read off straightforward.

The charges of the generating solution \( P_1 \) and \( Q_1 \) emerge in the general solution of pure \( N = 4 \) supergravity to two (S-duality invariant) coordinates \( X \) and \( Y \) respectively.

\[
Q_1 = Q_2 \rightarrow X = \sqrt{\frac{1}{2} F(L, -\Gamma)}, \quad P_1 = P_2 \rightarrow Y = \sqrt{\frac{1}{2} F(L, \Gamma)}, \quad \text{(III.12)}
\]

with

\[
F(L, \pm \Gamma) = \frac{1}{2g^2} \left( \tilde{\mathbf{Q}}^T L \tilde{\mathbf{Q}} + \tilde{\mathbf{P}}^T L \tilde{\mathbf{P}} \pm \Gamma(L) \right), \quad \Gamma(L) = \sqrt{4 \left( \tilde{\mathbf{P}}^T L M_\infty L \tilde{\mathbf{Q}} \right)^2 + \left( \tilde{\mathbf{Q}}^T L \tilde{\mathbf{Q}} - \tilde{\mathbf{P}}^T L \tilde{\mathbf{P}} \right)^2}. \quad \text{(III.13)}
\]

These coordinates are - analogous to our five dimensional considerations - invariant under \( SO(6) \) and \( SL(2, \mathbb{R}) \) transformations. Thus they provide a basis to discuss T- and S-duality invariant quantities. In terms of the “non-extreme hatted” quantities

\[
\hat{X} \equiv \sqrt{X^2 + r_0^2}, \quad \hat{Y} \equiv \sqrt{Y^2 + r_0^2}, \quad \text{(III.14)}
\]

the ADM mass and the dilaton charge are of the form

\[
M = 2 \hat{X} + 2 \hat{Y}, \quad \Sigma = g^4 \left( 2 \hat{X} - 2 \hat{Y} \right). \quad \text{(III.15)}
\]

The non-extremality parameter \( r_0 \), which is invariant under T- and S-duality, can in turn be determined in terms of the duality invariant combinations of charges \( X \) and \( Y \) and the ADM mass as:

\[
4r_0^2 = \frac{M^2}{4} - F(L, \Gamma) - F(L, -\Gamma) + \left( \frac{F(L, \Gamma) - F(L, -\Gamma)}{M} \right)^2 \quad \text{(III.16)}
\]

This gives then the corresponding solution for the entropy of the Kerr solution with non-vanishing total angular momentum \( J = aM \).

\(^{6}\)These coordinates are also invariant under general \( O(6,22) \) transformations.
\[ S = \frac{\pi}{r_0} \left\{ r_+ (\hat{X} + r_0)(\hat{Y} + r_0) + r_- (\hat{X} - r_0)(\hat{Y} - r_0) \right\}. \]  

(III.17)

To calculate the surface gravity \( \kappa = \lim_{r \to r_+} \sqrt{g^{rr}} \partial_r \sqrt{-g_{tt}|_{\theta=0}} \) of the Kerr solution we need the metric components of the solution in duality invariant form. Using the results above they can be easily read off of the generating solution in [11]. We find the surface gravity to be

\[ \kappa = 2 r_0 \frac{r_+ - r_-}{r_+ (\hat{X} + r_0)(\hat{Y} + r_0) + r_- (\hat{X} - r_0)(\hat{Y} - r_0)}. \]  

(III.18)

The Hawking temperature of the black hole is \( T = \frac{\kappa}{2\pi} \). Hence we recover the established relation [6, 5]:

\[ TS = r_+ - r_. \]

Moreover, the Killing field \( \chi^a = \xi^a + \Omega \psi^a \) is defined to vanish at the horizon \( (r = r_+) \). This yields the angular velocity

\[ \Omega = 4 r_0^2 \frac{a}{r_+ (\hat{X} + r_0)(\hat{Y} + r_0) + r_- (\hat{X} - r_0)(\hat{Y} - r_0)}. \]  

(III.19)

C. Extremal and BPS Limits

The extremal limit, in which the event horizon is about to disappear, is given by

\[ M^2 \to 2F(L, \Gamma) + 2F(L, -\Gamma) + 4\sqrt{F(L, -\Gamma)F(L, \Gamma) + 4J^2}. \]  

(III.20)

The extremal ADM mass (III.20) reduces to the BPS mass [10] in the limit \( J \to 0 \). This regular BPS limit, which is a special extremal limit, has no naked singularity. In the extremal limit we have \( r_0 \to 0 \) if and only if \( J \to 0 \). Hence the inequality \( r_0^2 \geq 4J^2 \) is saturated in the regular BPS limit. Thus the regular BPS limit is the extreme limit with vanishing angular momentum and the rotating non-extreme black holes lose all their angular momentum on their way to extremality. A possible dynamical process, that drives black holes to extremality, while they lose all their angular momentum, is the “Penrose Process” [17]. Once the BPS limit is reached due to this process, the remaining black hole is spherically symmetric. It is rather interesting that this classical process is compatible with the Bogomol’nyi-Gibbons-Hull bound, since the process stops if an irreducible mass is reached [18].
The Bogomol’nyi-Gibbons-Hull bound itself \[19\] determines the mass of particle states in extended supersymmetric theories in terms of the central charge matrix \(Z_{AB}\). (For a review see \[20\] and reference therein.)

\[
M \geq \max |Z_{AB}|. \tag{III.21}
\]

The central charge matrix \(Z_{AB}\) can be brought to a skew-diagonal form by a SU(4) transformation \[25\]: \(Z_{AB}^0 = \Lambda Z_{AB} \Lambda^T\). In terms of duality invariant quantities the absolute values of these eigenvalues are \[21\]

\[
|z_{1,2}| = \sqrt{2F(L, \Gamma)} \pm \sqrt{2F(L, -\Gamma)}. \tag{III.22}
\]

In the regular BPS limit we have

\[
\lim_{r_0 \to 0} M = M_{BPS} = |z_1|, \quad \lim_{r_0 \to 0} \Sigma = \Sigma_{BPS} = -|z_2|g^4. \tag{III.23}
\]

Moreover, the moduli fields, the dilaton and the axion are fixed values in the BPS limit at the horizon. The corresponding entropy only depends on the bare quantized charges \[26\]. Furthermore it has been shown in \[7\] that this result can be understood from a point of view of “supersymmetric attractors”. In \(N = 4\) supergravity we only have to find the fixed values of the dilaton and the axion at the horizon. Following \[7\] the dilaton charge has to vanish in the BPS limit \((z_2 = 0)\). This implies \(\Gamma(L) = 0\) and yields

\[
\Psi_{fix} = -\frac{\tilde{\alpha}^T L \tilde{\beta}}{\tilde{\beta}^T L \tilde{\beta}}, \quad g_{fix}^4 = \frac{(\tilde{\beta}^T L \tilde{\beta})^2}{(\tilde{\alpha}^T L \tilde{\alpha})(\tilde{\beta}^T L \tilde{\beta}) - (\tilde{\alpha}^T L \tilde{\beta})^2}. \tag{III.24}
\]

The corresponding area of the horizon is proportional to the largest eigenvalue of the central charge at the fixed points of the dilaton and the axion.

\[
|z_1|_{fix} = 2 \sqrt{2} \left[ (\tilde{\alpha}^T L \tilde{\alpha})(\tilde{\beta}^T L \tilde{\beta}) - (\tilde{\alpha}^T L \tilde{\beta})^2 \right]^{1/4}. \tag{III.25}
\]

The remaining theory in the BPS limit has \(N = 1\) supersymmetry and the Bekenstein-Hawking entropy is

\[
S_{BPS} = \frac{\pi}{16} |z_1|_{fix}^2. \tag{III.26}
\]
D. Duality Invariant Black Holes in N = 4 String Theory

Now we turn to a discussion of general static case of toroidally compactified heterotic string with the scalar coset space $\frac{SU(1,1)}{U(1)} \times \frac{O(6,22)}{O(6) \times O(22)}$. Although we were unable to present the complete duality invariant solution, however, we found the T- and S-duality invariant entropy of general near-extremal solutions.

Starting with the non-extreme five-parameter generating solution of [14] the general N = 4 spherically symmetric solution has the time component of the metric, which can be written in the form:

$$g_{tt} = \pi(r + r_0)(r - r_0)S^{-1}(r).$$

The function $S(r)$, which can be written in the form

$$S(r) = \pi \prod_{i=1}^{4} \sqrt{r + \lambda_i},$$

specifies at the horizon ($r = r_0$) the entropy of the non-extreme solution, i.e. $S \equiv S(r_0)$.

For the five-parameter generating solution $\lambda_i$ depend on five-charge parameters, and the non-extremality parameter $r_0$, which enters the expressions for $\lambda_i$ as a function of $r_0^2$, only.

The general solution, parametrized by 28 electric and 28 magnetic charges $Q_i$ and $P_i$ ($i = 1 \ldots 28$) respectively, is obtained by imposing on the generating solution $[SO(6) \times SO(22)]/[SO(4) \times SO(20)] \subset O(6,22)$ and $SO(2) \subset SL(2,R)$ transformations [12].

The outstanding problem is to find the duality invariant form of the parameters $\lambda_i$ with $\lambda_i \equiv \lambda_i(r_0^2, Q_i, P_i)$. However, since $\lambda_i$ depend on $r_0^2$ only, we can determine the T- and S-duality invariant form of the entropy in the near-extremal limit. The entropy reads

$$S = \pi \prod_{i=1}^{4} \sqrt{\lambda_i^{[0]}} + \frac{\pi}{2} r_0 \sum_{i<j<k} \lambda_i^{[0]} \lambda_j^{[0]} \lambda_k^{[0]} + O(r_0^2)$$

(III.28)

Here the $\lambda_i^{[0]}$ are the eigenvalues in the BPS limit. Starting with the BPS five-charge (generating) solution of [14] and imposing T- and S-duality transformations one finds for the general 56-charge configuration the following T- and S-duality invariant quantities

$$S = \pi \prod_{i=1}^{4} \sqrt{\lambda_i^{[0]}} + \frac{\pi}{2} r_0 \sum_{i<j<k} \lambda_i^{[0]} \lambda_j^{[0]} \lambda_k^{[0]} + O(r_0^2)$$

(III.28)

Here the $\lambda_i^{[0]}$ are the eigenvalues in the BPS limit. Starting with the BPS five-charge (generating) solution of [14] and imposing T- and S-duality transformations one finds for the general 56-charge configuration the following T- and S-duality invariant quantities

$^7$U-duality invariant structure of non-extreme black holes have also been considered in [4]. However, in that case we were unable to reproduce the BPS limit of the five-charge parameter generating solution.
\[
\prod_{i=1}^{4} \lambda_i^{[0]} \equiv S_{BPS}^2 / \pi = \frac{1}{4} F(L, \Gamma) F(L, -\Gamma),
\]
\[
\sum_{i=1}^{4} \lambda_i^{[0]} \equiv M_{BPS} = \sqrt{F(M_+ , \Gamma)} + \sqrt{F(M_+ , -\Gamma)},
\]
\[
\sum_{i<j} \lambda_i^{[0]} \lambda_j^{[0]} = \frac{1}{2g^2} \left( \tilde{Q}^T L \tilde{Q} + \tilde{P}^T L \tilde{P} \right) + \sqrt{F(M_+ , \Gamma)} F(M_+ , -\Gamma),
\]
\[
\sum_{i<j<k} \lambda_i^{[0]} \lambda_j^{[0]} \lambda_k^{[0]} = \frac{1}{4g^2 M_{BPS}} \left\{ M_{BPS}^2 \left( \tilde{Q}^T L \tilde{Q} + \tilde{P}^T L \tilde{P} \right) \right.

\left. - \left( \tilde{Q}^T M_+ \tilde{Q} - \tilde{P}^T M_+ \tilde{P} \right) \left( \tilde{Q}^T L \tilde{Q} - \tilde{P}^T L \tilde{P} \right) \right.

\left. - 4 \left( \tilde{Q}^T L M_{\infty} L \tilde{P} \right) \left( \tilde{Q}^T M_+ \tilde{P} \right) \right\}. \tag{III.29}
\]

Here we used the following definitions
\[
F(M_+ , \pm \Gamma) = \frac{1}{2g^2} \left( \tilde{Q}^T M_+ \tilde{Q} + \tilde{P}^T M_+ \tilde{P} \pm \Gamma(M_+) \right),
\]
\[
\Gamma(M_+) = \sqrt{4 \left( \tilde{P}^T M_+ \tilde{Q} \right)^2 + \left( \tilde{Q}^T M_+ \tilde{Q} - \tilde{P}^T M_+ \tilde{P} \right)^2}. \tag{III.30}
\]

Since the five-charge parameter solution is the generating solution for \( D = 4 \) black holes of toroidally compactified Type II string, the U-duality invariant form of near-extremal solutions could be obtained along similar lines.

On the other hand the structure of T- and S-duality [U-duality] invariant solutions for non-extremal solutions of toroidally compactified heterotic [Type II] string theory in \( D = 4 \) remains an open problem.

Acknowledgement: We would like to thank F. Larsen for many helpful discussions and participation in the early stage of this work. The work is supported by U.S. DOE Grant Nos. DOE-EY-76-02-3071, the National Science Foundation Career Advancement Award No. PHY95-12732 and the NATO collaborative research grant CGR No. 940870 (M.C.).

IV. APPENDIX

In this appendix we summarize some basic facts concerning low-energy effective actions and duality symmetries of heterotic and Type II string theory. For a comprehensive review we refer to [16].
A. Heterotic Superstring on $T^6$ (in D = 4)

The moduli fields of the heterotic string theory in four dimensions can be combined to an $O(6,22)$ matrix valued scalar field $M$ with

$$M = M^T, \quad MLM^T = L. \quad (IV.1)$$

The gauge sector in four dimensions consists out of 28 U(1) gauge fields $A^{(a)}_\mu$ ($a = 1, \ldots, 28$). The antisymmetric tensor and the dilaton combine in four dimensions to a complex scalar $\lambda$ with the help of the Poincaré duality, which transforms on-shell the antisymmetric tensor $B_{\mu\nu}$ to an axion $\Psi$. Since we consider SL(2,$\mathbb{R}$) invariance, which is - first of all - only a symmetry of the equations of motion, we restrict ourselves to the on-shell case. This corresponding on-shell action is invariant under $O(6, 22)$ and $SL(2, \mathbb{R})$ transformations. The subgroup $O(6, 22; \mathbb{Z})$ (T-duality group) is an exact symmetry of the string theory, whereas the subgroup $SL(2, \mathbb{Z})$ (S-duality group) is a conjectured non-perturbative symmetry of string and M-theory.

1. $O(6, 22)$ Transformations

$\Omega \in O(6, 22)$ if $\Omega^T L \Omega = L$ with

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & I_{16} \end{pmatrix} \quad (IV.2)$$

The corresponding fields transform under $O(6, 22)$ transformations as follows

$$M \rightarrow \Omega M \Omega^T, \quad A^{(a)}_\mu \rightarrow \Omega_{ab} A^{(b)}_\mu, \quad (IV.3)$$

whereas $g_{\mu\nu}, B_{\mu\nu}$ and $\Phi$ are invariant under $O(6, 22)$ transformations. The projection operators we use are $\mu_\pm = M_\infty \pm L, M_\pm = (LML)_\infty \pm L$.

2. $SL(2, \mathbb{R})$ Transformations

In general the axion-dilaton and the gauge fields transform under $SL(2, \mathbb{R})$ transformations, while the metric $g_{\mu\nu}$ and the moduli $M$ are invariant. The corresponding $SL(2, \mathbb{R})$ transformations of the dressed physical charges and the string coupling constant are:
\[ \vec{Q} \rightarrow (c \Psi_\infty + d) \vec{Q} + c e^{-\Phi_\infty} M_\infty L \vec{P}, \]
\[ \vec{P} \rightarrow (c \Psi_\infty + d) \vec{P} - c e^{-\Phi_\infty} M_\infty L \vec{Q}, \]
\[ g^2 \rightarrow [(c\Psi_\infty + d)^2 + c^2 e^{-2\Phi_\infty}]^2 g^2. \] (IV.4)

The corresponding SL(2,\mathbb{R}) transformations of the bare charges are
\[ \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \rightarrow \mathcal{L} \omega \mathcal{L}^T \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \] (IV.5)
with
\[ \mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad ad - bc = 1 \] (IV.6)

Note that an SO(2) \subset SL(2, R) transformation preserves the special background with \( \Psi = \Phi = 0 \). This SO(2) transformation can be parametrised by one angle \( \theta \):
\[ \omega = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \] (IV.7)

Using this SO(2) transformations one finds
\[ g^2 \bar{\alpha}^T y \bar{\alpha} \rightarrow g^2 \bar{\alpha}^T y \bar{\alpha} + \tan \theta \bar{\alpha}^T y \bar{\beta}, \]
\[ \frac{1}{g^2} \bar{\beta}^T y \bar{\beta} \rightarrow \frac{1}{g^2} \bar{\beta}^T y \bar{\beta} - \tan \theta \bar{\alpha}^T y \bar{\beta}. \] (IV.8)

**B. The Heterotic/Type II String on** \( T^5 \) (in D = 5)

For the low energy effective heterotic string compactified on \( T^5 \) [13] the T-duality group is \( SO(5, 21, \mathbb{Z}) \) with analogous transformations of the fields with respect to the four dimensional case. On the other hand, the S-duality group becomes simply \( SO(1, 1, \mathbb{Z}) \).

The low energy effective Type II string compactified on \( T^5 \) [14, 23] has T-duality group \( SO(5, 5, \mathbb{Z}) \) with analogous transformations of the fields with respect to the four dimensional case. Moreover, the U-duality group in five dimensions is \( E_{6(6)}(\mathbb{Z}) \) [24]. The maximal compact subgroup is the unitary symplectic group \( USp(8) \). The 27 abelian gauge fields of \( N = 8 \) supergravity and their corresponding charges transform as a 27 vector of \( E_{6(6)} \).
References

1. G. Horowitz, UCSBTH-96-07, [gr-qc/9604051].

2. J. Maldacena, Ph.D. thesis, [hep-th/9607233].

3. M. Cvetić, UPR-714-T, [hep-th/9701152].

4. K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W.K. Wong, Phys. Rev. D54 (1996) 6293;
   G. Lopes-Cardoso, D. Lüst and T. Mohaupt, Phys. Lett. B388 (1996) 266;
   K. Behrndt, G. Lopes-Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, [hep-th/9610105];
   W. Sabra, [hep-th/9703101].

5. A. Sen, Nucl. Phys. B440 (1995) 421;
   A. Sen, Mod. Phys. Lett. A10 (1995) 2081.

6. R. Kallosh, A. Linde, T. Ortin, A. Peet and A. Van Proeyen, Phys. Rev. D46 (1992) 5278.

7. S. Ferrara and R. Kallosh, Phys. Rev. D54 (1996) 1514, Phys. Rev. D54 (1996) 1525.

8. T. Ortin, Phys. Rev. D47 (1993) 3136.
   R. Kallosh and T. Ortin, Phys. Rev. D48 (1993) 742.
   E. Bergshoeff, R. Kallosh and T. Ortin, Nucl. Phys. B478 (1996) 156.

9. R. Kallosh and A. Rajaraman, Phys. Rev. D54 (1996) 6381.

10. M. Cvetič and D. Youm, Phys. Rev. D53 (1996) 584.

11. M. Cvetič and D. Youm, Phys. Rev. D54 (1996) 2612.

12. M. Cvetič and D. Youm, Nucl. Phys. B 472 (1996) 249.

13. M. Cvetič and D. Youm, Nucl. Phys. B 476 (1996) 118.

14. M. Cvetič and A.A. Tseytlin, Phys. Rev. D53 (1996) 5619.
15. M. Cvetič and C.M. Hull, Nucl. Phys. **B 480** (1996) 296.

16. A. Sen, Int. J. Mod. Phys **A9** (1994) 3702.

17. R. Penrose, Riv. Nuovo Cimento **1** (1969) 252.

18. D. Christodoulou, Phys. Rev. Lett. **25** (1970) 1596.

19. G.W. Gibbons and C.M. Hull, Phys. Lett. **B 109** (1982) 190.

20. L. Andrianopoli, R. D’Auria and S. Ferrara, [hep-th 9612105](http://arxiv.org/abs/hep-th/9612105).

21. M.J. Duff, J.T. Liu and J. Rahmfeld, Nucl. Phys. **B 459** (1996) 125.

22. C.M. Hull and P.K. Townsend, Nucl. Phys. **B 438** (1995) 109.

23. E. Cremmer, J. Scherk and J.H. Schwarz, Phys. Lett. **B 84** (1979) 83.

24. E. Cremmer, Proceedings of the Gravity Workshop, Cambridge, June (1980) page 267, and Proceedings of the ICTP Spring School on Supergravity, Trieste, April (1981) page 313.

25. B. Zumino, J. Math. Phys. **3** (1962) 1055.  
S. Ferrara, C. A. Savoy and B. Zumino, Phys. Lett. **100B** (1981) 393.

26. F. Larsen and F. Wilczek, Phys. Lett. **B375** (1996) 37.