Verification of a Merkle Patricia Tree Library Using F★

SOTA SATO, Kyoto University, Japan
RYOTARO BANNO, Kyoto University, Japan
JUN FURUSE, Dailambda Inc., Japan
KOHEI SUENAGA, Kyoto University, Japan
ATSUSHI IGARASHI, Kyoto University, Japan

A Merkle tree is a data structure for representing a key-value store as a tree. Each node of a Merkle tree is equipped with a hash value computed from those of their descendants. A Merkle tree is often used for representing a state of a blockchain system such as Ethereum and Tezos since it can be used for efficiently auditing the state in a trustless manner. Due to the safety-critical nature of blockchains, ensuring the correctness of their implementation is paramount.

We show our formally verified implementation of the core part of Plebeia using F★, a programming language to implement a formally verified functional program. Plebeia, which is implemented in OCaml, is a library to manipulate an extension of Merkle trees (called Plebeia trees). It is being implemented as a part of the storage system of the Tezos blockchain system. To this end, we gradually ported Plebeia to F★; the OCaml code extracted from the modules ported to F★ is linked with the unverified part of Plebeia. By this gradual porting process, we can obtain a working code from our partially verified implementation of Plebeia; we confirmed that the binary passes all the unit tests of Plebeia.

More specifically, we verified the following properties on the implementation of Plebeia: (1) Each tree-manipulating function preserves the invariants on the data structure of a Plebeia tree and satisfies the functional requirements as a nested key-value store; (2) Each function for serializing/deserializing a Plebeia tree to/from the low-level storage is implemented correctly; and (3) The hash function for a Plebeia tree is relatively collision-resistant with respect to the cryptographic safety of the blake2b hash function. During porting Plebeia to F★, we found a bug in an old version of Plebeia, which was overlooked by the tests bundled with the original implementation. To the best of our knowledge, this is the first work that verifies a production-level implementation of a Merkle-tree library by F★.

Additional Key Words and Phrases: F★, Merkle Patricia Tree, Program Verification, Blockchain, Tezos

1 INTRODUCTION

Blockchain-based cryptocurrencies, such as Bitcoin [Nakamoto 2009] and Ethereum [Wood 2014], have become an important infrastructure to exchange virtual assets securely yet without a central authority. Any vulnerabilities [cve 2009; Lerner 2009] are potential attack vectors of malicious hackers because they are databases carrying a huge amount of financial values. Indeed, such attacks have caused significant financial losses in the past [Siegel et al. 2016]. Therefore, formally verifying the security of these frameworks is becoming a hot topic in PL research [Bhargavan et al. 2016; Coblenz 2017; Vyper Team [n.d.]].

In this context, the integrity of blockchain states, including transaction records and the (persistent) states of smart contracts, is another fundamental security property. A popular technique to ensure the integrity of blockchain states is to implement it using a data structure called Merkle trees [Merkle 1989], which are also called hash trees, and their extensions. Roughly, a Merkle tree is a tree whose leaves store data and whose internal nodes store the hash value computed from their children. Merkle trees enable efficient and trustless verification of the existence of a piece of information in a tree. Ethereum uses Merkle Patricia Trees (MPTs), which are a combination of Merkle trees and Patricia trees, to implement secure key-value stores.

Since time and space efficiency is important for the user experience of a blockchain system, various improvements of Merkle trees are being intensively developed. In Tezos [Goodman 2014],
one of the blockchain-based cryptocurrencies, this effort led to the storage system called **Plebeia**. Plebeia implements a variant of MPTs—called **Plebeia trees**—that is highly optimized to reduce the disk storage size\(^1\). Although each technique used in Plebeia is well known, the correctness of the entire implementation, which combines them in a sophisticated way, is nontrivial.

In this paper, we present our formally verified implementation of the core part of Plebeia. We verified the following properties of the current Plebeia implementation: (1) Each tree-manipulating function preserves the invariants on the data structure of a Plebeia tree and satisfies the functional requirements as a nested key–value store; (2) Each function for serializing/deserializing a Plebeia tree to/from the low-level storage is implemented correctly; (3) The hash function for a Plebeia tree is relatively collision-resistant with respect to the cryptographic safety of the blake2b hash function. To this end, we ported the core part of the Plebeia, which is implemented in OCaml, to F\(^\star\) [Swamy et al. 2016], which is a programming language for formal verification, and wrote the proofs of these properties. The F\(^\star\) type checker automatically verifies that our proofs are correct.

Since we ported only a part of Plebeia to F\(^\star\), our F\(^\star\) code cannot be built to working binary by itself. To address this problem, we use the code-extraction functionality of the F\(^\star\) compiler that extracts OCaml code from machine-checked proofs. Using this functionality, we extracted OCaml code from our proof and linked it with the unverified part of Plebeia. We confirmed that the resulting code runs and successfully passes all the tests bundled with Plebeia.

The rest of this paper is organized as follows. Section 2 reviews F\(^\star\); Section 3 introduces Plebeia trees and the structural and functional requirements of a Plebeia tree to be satisfied; Section 4 explains the extensions to Plebeia trees implemented in the actual code; Section 5 explains how we verified the core part of Plebeia with F\(^\star\). After the discussion on related work in Section 6, Section 7 concludes.

All the F\(^\star\) source code is available at https://gitlab.com/dailambda/plebeia/-/tree/banno@port_into_fstar/fstar.

## 2 F\(^\star\)

F\(^\star\) [Swamy et al. 2016] is a functional programming language for program verification. By using F\(^\star\), one can formalize the properties of a program and prove it. This section briefly introduces the features of F\(^\star\), which we used to verify the core part of Plebeia. For a detailed exposition, see http://www.fstar-lang.org/tutorial/.

### 2.1 Syntax

The syntax of F\(^\star\) is similar to that of OCaml, as the following example shows.

**Example 1.** The following is the definition of a function `sum` that takes an integer list and returns the sum of the integers in the list.

\[
\text{val sum : list int -> int}
\]

\[
\text{let rec sum = function}\ |\ [[]] -> 0 | \ x :: xs -> sum xs + x
\]

We remark the following differences of the F\(^\star\) syntax from the OCaml syntax:

- In F\(^\star\), a function definition can be preceded by a type declaration, whereas a type declaration has to be written in a separate signature definition in OCaml.
- Type parameters of a polymorphic type in F\(^\star\) are written like `t 'a 'b, . . . ,` whereas it is written (`'a, 'b, . . . ,`) `t` in OCaml.

\(^1\)https://www.dailambda.jp/blog/2019-08-08-plebeia/
2.2 Effects

The syntax of a function type in F* can be optionally accompanied by an effect; the syntax of a function type is $T_1 \rightarrow Eff \ T_2$, where $T_1$ is the type of the argument, $T_2$ is the type of the return value, and $Eff$ is an effect. An effect $Eff$ overapproximates the side effects that the function may cause when it is called. F* provides several primitive effects, including $Tot$, which indicates that functions do not cause any side effects and it terminates for any arguments. $ST$ indicates that functions may read from and write to memory.

The effect of a function is treated as $Tot$ if it is omitted. Hereafter, we say that a function is pure if the function has the effect $Tot$.

2.3 Refinement Types

The type system of F* incorporates refinement types [Freeman and Pfenning 1991]. A refinement type is a type constrained by a logical predicate. It can express a property of values that is more detailed than one expressed by a simple type.

A binding of a variable to a refinement type is written $x : T\{P\}$ where $T$ is a type and $P$ is a predicate; this refinement type is for values that have type $T$ and satisfy the predicate $P$. For example, $x : int\{x > 0\}$ is a binding of $x$ to the type for positive integers.

A programmer can use (1) comparison operators (e.g., $=\$, $>=\$); (2) logical operators (e.g., $\&\&\$, $\|\|\$, $\forall\$, $\exists\$); and (3) user-defined pure functions that return a value of type $bool$. For example, the following type declaration of $sum$ expresses that it only accepts a list of positive integers and it returns a positive integer.

$$val \ sum : list\ (p:\ int\{p > 0\}) \rightarrow r:\ int\{r > 0\}$$

One can use a dependent function type to express a relation between the arguments and the return value of a function. For example, the type of function $sum$ can be made more precise by the following (dependent) function type.

$$val \ sum : (v:\ list\ (p:\ int\{p > 0\})) \rightarrow (r:\ int\{r >= List\.length\ v\})$$

This type declaration asserts that the return value $r$ of the function $sum$ is greater than or equals to the length of the argument list $v$.

We use another notation that F* provides, which the following example illustrates, to express a type of a function with the effect $ST$.

$$val \ f : x:\ \tau_1 \rightarrow ST \ \tau_2 \ (requires\ (fun\ h \rightarrow P_1)) \ (ensures\ (fun\ h0 \ x\ h1 \rightarrow P_2))$$

The function of the above type takes $x$ of type $\tau_1$ and returns a value of type $\tau_2$. The effect $ST$ indicates that this function may read/write from/to memory. The clauses $requires$ and $ensures$ indicate the precondition and the postcondition of this function. The clause $requires\ (fun\ h \rightarrow P_1)$, where $h$ is of an abstract type representing heap and $P_1$ is a predicate, expresses that $f$ is supposed to be called where heap $h$ satisfies $P_1$. The clause $ensures\ (fun\ h0 \ x\ h1 \rightarrow P_2)$, where $P_2$ is a predicate, expresses that the call to $f$ modifies heap as described by $P_2$, where $h0$ is the heap before the call, $h1$ is the heap after the call, and $x$ is the value of type $\tau_2$ returned by this function.

By using the predicate $P_2$, one can express the relation among the returned value, the heap before the call, and one after the call.

F* tries to automatically typecheck a definition against a refinement type by discharging the verification condition with an SMT solver. The SMT solver Z3 [de Moura and Bjørner 2008] is used as an SMT solver in our verification.

The type checker of F* has to prove the termination of a function if it is given the effect $Tot$, which is in general undecidable. The F* type checker verifies the termination according to the following strategy.
By default, F★ tries to prove that the argument of every recursive call in a function definition is structurally decreasing in a certain well-founded measure specified for the type of the argument. This heuristic is sufficient to prove the termination of the function \texttt{sum} in Example 1 automatically because the length of the list passed as the first argument of every recursive call to \texttt{sum} is shorter than that of the original list.

One can supply a hint on which argument is structurally decreasing; then F★ tries to prove that this argument is indeed decreasing. For example, the termination of the following \texttt{sum'}, which is a variant of \texttt{sum} with an accumulator argument, cannot be automatically proved by F★ because its first argument is not decreasing.

\begin{verbatim}
val sum' : int -> list int -> Tot int
let rec sum' c = function | [] -> c | x :: xs -> sum' (x + c) xs
\end{verbatim}

However, F★ can prove the termination after giving a hint by changing the type-declaration part to the following one whose \texttt{(decreases v)} annotation indicates that, in every recursive call, the length of the second argument is strictly less than that of \texttt{v}.

\begin{verbatim}
val sum' : int -> (v: list int) -> Tot int (decreases v)
\end{verbatim}

F★ generates the verification conditions for the length of the second argument is indeed decreasing and tries to discharge the condition by Z3. For this example of \texttt{sum'}, Z3 successfully discharges these conditions.

\section*{2.4 A Theorem as a Dependently-Typed Function}

A dependent function type is used to state a theorem in F★. For example, the following type declaration states and proves a theorem \texttt{n_plus_0} that asserts \( \forall n \in \mathbb{Z}. n + 0 = n \).

\begin{verbatim}
val n_plus_0 : (n: int) -> (_: unit {n + 0 = n})
let n_plus_0 n = ()
\end{verbatim}

A theorem is, as the above example shows, expressed as a dependent function type whose return type is a refinement of \texttt{unit}. The proof of a theorem (stated as a type declaration) is given by a function definition of this type. In fact, for the above \texttt{n_plus_0}, it suffices to give (\texttt{()}) as its body because Z3 SMT solver can automatically discharge \( \forall n \in \mathbb{Z}. n + 0 = n \).

Another way of stating a theorem in F★ is using a notation \texttt{Lemma (requires P) (ensures Q)}, where \( P \) and \( Q \) are predicates. For example, one can state the theorem \( \forall n \in \mathbb{Z}. n + 0 = n \) as follows.

\begin{verbatim}
val n_plus_0 : (n: int) -> Lemma (requires True) (ensures n + 0 = n)
\end{verbatim}

If \( P \) is \texttt{True}, then we can omit the \texttt{requires} clause and the \texttt{ensures} clause.

\begin{verbatim}
val n_plus_0 : (n: int) -> Lemma (n + 0 = n)
\end{verbatim}

In this paper, a theorem stated using the notation \texttt{Lemma (requires P) (ensures Q)} can be considered equivalent to the one mentioned above that stated using a refinement of \texttt{unit}. See Grimm et al. [2018] for detail.

We can prove a theorem that requires to use induction in F★. For example, consider the equality \( c + \text{sum} v = \text{sum'} v c \), where \text{sum} and \text{sum'} are those defined above. We can state that the above equality holds for any \( c \) and \( v \) in F★ as follows:

\begin{verbatim}
val sum_theorem : (v: list int) -> (c: int) -> Lemma (c + sum v = sum' c v)
\end{verbatim}

This theorem can be proved by mathematical induction on the length of the list \( v \), which is expressed as the following function definition.

\begin{verbatim}
let rec sum_theorem v c = match v with | [] -> () | x :: w -> sum_theorem w (x + c); ()
\end{verbatim}

The pattern-matching clause for \texttt{[]} corresponds to the base case of the induction, and the pattern-matching clause for \texttt{x :: w} corresponds to the step case. In the base case, since the list \( v \) is \texttt{[]}, the
equation is reduced to \( c + \emptyset = c \). Since the reduced equation is simple enough to be discharged by Z3 SMT solver, a programmer does not have to write anything other than () as a proof. In the step case, the list \( v \) can be decomposed to an integer \( x \) and the remaining list \( w \). The equation is reduced to \( c + x + \sum w = \sum (c + x) w \). To derive this equation, we use the induction hypothesis \( \sum \text{theorem } w (x + c) \), which is expressed by the call \( \sum \text{theorem } w (x+c) \) in the above example. F* adds this fact to the list of available assumptions at the point where \( \sum \text{theorem } w (x + c) \) is called.

3 PLEBEIA TREES

A Plebeia tree is an extension of a Merkle Patricia tree (MPT), which is a tree whose node keeps a hash value computed by those of its descendants. To explain a Plebeia tree and the invariants imposed on its structure, this section first explains an MPT. We will explain the most basic definition of Plebeia trees in this section. The implementation detail, which comes with various extensions, is explained in Section 4.

3.1 Merkle Patricia Tree

A Merkle Patricia tree (MPT) is a Patricia tree [Morrison 1968] (a binary radix tree) whose nodes are associated with hashes. An MPT is defined by the following F* type.

```fsharp
type side = L | R and key = list side and hash = string
(* The type for MPTs *)
type node = |
| Empty |
| Leaf of value * hash (* value here is the type of values *) |
| Internal of key * node * key * node * hash |
```

The type `key` represents non-empty lists of bits `L` and `R`. A key is associated with each edge in an MPT. Each node is accompanied by a hash represented by a string.

A node of MPT is either an empty tree (`Empty`), a leaf `Leaf(v, h)`, or an internal node `Internal(k1, n1, k2, n2, h)`. `Leaf(v, h)` has a value \( v \) and is associated with its hash \( h \). `Internal(k1, n1, k2, n2, h)` is the parent of the left child \( n1 \) and the right child \( n2 \), and is associated with its hash \( h \). The key \( k1 \) is on the edge from the internal node to \( n1 \); the key \( k2 \) is on the edge to \( n2 \).

A Merkle hash [Merkle 1989] is a hash of an MPT node that is computed in the following way:

- The Merkle hash of a leaf is the hash of its value.
- The Merkle hash of an internal node is computed by hashing the tuple of (1) the Merkle hashes of its subnodes and (2) the edge labels.

Practically assuming that there is no hash collision, the Merkle hash of the root node can identify an MPT: two trees are identical if and only if their Merkle hashes are the same under the no-hash-collision assumption. This property is suitable for blockchains; for example, participants in a blockchain network can efficiently verify that they share the same blockchain states by exchanging their Merkle hashes of the root node instead of communicating an (often large) MPT itself.

3.2 Plebeia Tree

3.2.1 Motivation and Definition. One of the major problems in a naïvely implemented MPT is the internal fragmentation of storage that is organized as a set of fixed-size records. To observe the problem, suppose that (1) the storage size required to keep a value or a pointer is 32 bits, (2) the size of a hash is 256 bits, and (3) the size of a key is \( k \) bits on average. Then, the size to store a leaf is \( 32 + 256 = 288 \) bits, whereas the size for an internal node is \( k + 32 + k + 32 + 256 = 2k + 320 \) bits. If \( k \) is close to 200, which is often the case in the current Tezos blockchain, the size for an internal node
is 2.5 times as large as that for a leaf. This imbalance in the sizes of a leaf and an internal node leads to wasted space in each fixed-size record in storage. To address this issue, Plebeia defines two kinds of internal nodes: a branch and an extender. Plebeia also has another kind of nodes called buds.

This extended MPT is called a Plebeia tree, whose type definition is as follows:

```haskell
type node = | Leaf of value * hash |
            | Branch of node * node * hash |
            | Extender of key * node |
```

A node `Branch(n1,n2,h)` has a Merkle hash `h` and two subnodes `n1` and `n2`, which are connected with edges with one-bit keys `L` and `R` (therefore omitted). A node `Extender(k,n)` has one subnode `n`; the edge to `n` has a non-empty key `k`. The hash field of a node `Extender(k,n)` is omitted; its hash value is defined as the concatenation of the hash value of its child node `n` and its key `k`, and hence its hash value can be calculated without hashing. Notice that an extender shoulders the role of bringing a key on the edges to its subnodes, which is the responsibility of an internal node in an MPT. Actually, this type cannot represent the empty tree; we use a bud node, introduced in Section 4.1.3, to express it.

Now the storage sizes of these three kinds of nodes are more balanced than the original definition: 288 bits for a leaf, $32 + 32 + 256 = 320$ bits for a branch, and $k + 32$ bits for an extender. This balanced-size design of nodes mitigates the problem of internal fragmentation.

3.2.2 Model of Plebeia Trees. A Plebeia tree `t` represents a key–value store `[[t]]`, which we call the model of `t`, defined as follows. For a key `k`, `[[t]](k)` is the value `v` if there is a path labeled by `k` from the root of `t` to a leaf `Leaf(v,h)`; if such a leaf does not exist, then `[[t]](k)` is undefined (`[[t]](k) = ⊥`). We write `Dom(t)` for the domain of `[[t]]`. Since only leaves can keep values, `Dom(t)` for an MPT `t` satisfies the following prefix-freedom property: for any distinct two keys in `Dom(t)`, neither is a prefix of the other.

3.2.3 Structural Invariants of Plebeia Trees. Plebeia does not allow (1) an extender to be a direct subnode of another extender and (2) an extender to have the empty key. These two restrictions are to ensure that there is a canonical representation of a key–value store by a Plebeia tree; if these restrictions were not met, then one could add arbitrarily many extenders below a branch without changing the key–value store it represents. We name these invariants on the structure of a Plebeia tree (called structural invariants) as follows.

(SI1) An extender must not have another extender as its direct child.

(SI2) An extender must not have an empty key.

3.3 Functions for Manipulating Plebeia Trees

Plebeia provides the following tree-manipulating functions.

- `get_value : t:node -> k:key -> option value` It returns `Some v` where `v` is the value on the leaf reached by following the key `k` from the root of the node `t`; if such a leaf does not exist, it returns `None`.

- `subtree : t:node -> k:key -> option node` It returns `Some t'` if `t'` is the tree reached by following the path according to key `k` from the root of `t`; if such `t'` does not exist, it returns `None`.

- `insert : t:node -> k:key -> v:value -> option node` It returns `Some t'` if `t'` is the tree obtained by inserting a leaf with value `v` to `t` at the position reached by following the path represented by the non-empty `k` from the root of `t`; if such insertion would violate the prefix-freedom condition (Section 3.2.2), it returns `None`. 


• update : t:node -> k:key -> v:value -> option node. It returns t' that is obtained by replacing the leaf at the position reached by following the path represented by the key k from the root of n with a leaf with the value v; if such a leaf does not exist at the position represented by k, it returns None.

• delete : t:node -> k:key -> option node. It returns Some t' if t' is obtained by deleting the leaf at the position reached by following the path represented by k from the root of t; if such a leaf does not exist, it returns None.

3.4 Functional Requirements for the Tree-Manipulating Functions

We state the functional requirements for the tree-manipulating functions. In the following, we describe those for get_value and insert.

The expected property of get_value can be stated as follows: for any Plebeia tree t and key k,

1. get_value t k = Some(v) \implies \llbracket t \rrbracket (k) = v
2. get_value t k = None \implies \llbracket t \rrbracket (k) = \bot,

where \llbracket t \rrbracket is the model of t defined in Section 3.2.2. The above requirements express that the function call get_value t k should look up the value of k in the mapping \llbracket t \rrbracket.

The expected property of the insert is as follows: For any Plebeia tree t, key k, and value v,

1. insert t k v = Some t' \implies \llbracket t' \rrbracket (k) = \bot \land \llbracket t' \rrbracket (k') = v \land \forall k'. k' \neq k \implies \llbracket t' \rrbracket (k') = \llbracket t \rrbracket (k')

2. insert t k v = None \implies \{k\} \cup \hbox{Dom}(t) \hbox{ is not prefix free } \lor \ hbox{ k \in Dom(t)}.

The first statement says that then the model of t' is the extension of \llbracket t \rrbracket with \{k \mapsto v\} if insert t k v evaluates to Some t'. The second statement says that, if insert t k v evaluates to None (hence fails), this means inserting k to Dom(n) violates the prefix-freedom condition or k already exists in Dom(t).

4 EXTENSIONS IN THE IMPLEMENTATION

In the actual implementation, Plebeia trees are implemented with several extensions for efficiency and persistence. We describe these extensions in this section.

4.1 Extension to Plebeia Trees

The type of Plebeia trees with these extensions are as follows.

\texttt{type node = | Leaf of value * index * hash | Branch of node * node * index * hash | Extender of key * node * index | Bud of option node * option index * option hash | Disk of index}

To explain this definition, we explain how it is obtained by gradually extending the original definition in Section 3.2.

4.1.1 Serialization of Trees. Since a Plebeia tree is often too large to be stored on memory, it is defined so that a part of a tree can be serialized to external storage; hence the definition of node is extended to the following, in which the changes from the original definition are marked.

\texttt{type node = | Leaf of value * index * hash | Branch of node * node * index * hash}
We (1) add a constructor \texttt{Disk} and (2) attach fields of type \texttt{index} to each constructor. The type \texttt{index} represents the addresses in the external storage. \texttt{Disk} \(i\) expresses the node that is stored at the address \(i\) of the storage. The \texttt{index} fields in the constructors \texttt{Leaf}, \texttt{Branch}, and \texttt{Extender} express to which address each node is supposed to be stored.

### 4.1.2 Lazy Computation of Hashes and Indices

The Merkle hash and the index of a node are not computed eagerly; they are computed only when the node is serialized to the storage. This design is to prevent the computation of the Merkle hash and the index of an ephemeral node that is temporarily constructed on memory and hence is not needed to be serialized to storage. To support such lazy computation of hashes and indices, we make the \texttt{index} and \texttt{hash} fields of \texttt{Leaf}, \texttt{Branch}, and \texttt{Extender} constructors to optional types:

\[
\text{type node} = \begin{cases} 
\text{Leaf of value * option index * option hash} \\
\text{Branch of node * node * option index * option hash} \\
\text{Extender of key * node * option index} \\
\text{Bud of option node * option index * option hash} \\
\text{Disk of index} 
\end{cases}
\]

The \texttt{option index} and \texttt{option hash} fields are \texttt{None} until they are computed. We say a node is \textit{indexed} if its field of type \texttt{index} \texttt{option} field is of the form \texttt{Some _}; a node is \textit{hashed} if (1) it has \texttt{hash} \texttt{option} field which is of the form \texttt{Some _} or (2) it is of the form \texttt{Extender(_*,n,_)} and \(n\) is hashed.

The computation of these fields happens according to the following rules:

- The hash and the index of a node are computed when the node is serialized to storage.
- All the descendants of a node \(n\) have to be indexed before \(n\) is indexed.
- All the descendants of a node \(n\) have to be hashed before \(n\) is hashed.

The second rule is to prevent a dangling pointer in the storage. The third rule is due to the definition of a Merkle hash (Section 3.1).

### 4.1.3 Bud Nodes

Plebeia is being developed to implement the storage system of Tezos, which is represented as a file system with a directory structure, which requires Plebeia to implement a mechanism to represent the directory structure. In order to represent a directory, a new kind of nodes called \texttt{bud} is introduced in the Plebeia tree implementation. The type \texttt{node} is extended as follows.

\[
\text{type node} = \begin{cases} 
\text{Leaf of value * option index * option hash} \\
\text{Branch of node * node * option index * option hash} \\
\text{Extender of key * node * option index} \\
\text{Bud of option node * option index * option hash} \\
\text{Disk of index} 
\end{cases}
\]

In the file system represented by this tree, a subtree whose root is a Bud node corresponds to a directory. A node \texttt{Bud(optn,i,h)} has a subnode \texttt{optn} of \texttt{option node} type; this field is \texttt{None} if this bud node represents an empty directory. Thus, the empty tree is represented by \texttt{Bud(None, i, h)}.

The model of a Plebeia tree with Bud nodes is extended from a key–value store to a \textit{nested} key–value store, corresponding to the fact that the Bud node represents a directory. Concretely, the model of a Plebeia tree \(t\) with bud nodes is defined as follows. For a key \(k\), \([|t|](k)\) is:

- The value \(v\) if there is a path labeled by \(k\) from the root of \(t\) to a leaf \texttt{Leaf}(\(v\), \(h\), \(i\));
Verifcation of a Merkle Patricia Tree Library Using F∗

- The nested key–value store \([t']\) if there is a path labeled by \(k\) from the root of \(t\) to a bud
  \(\text{Bud}(\text{Some}(t'),h,i);\)
- The empty key–value store \(\{}\) if there is a path labeled by \(k\) from the root of \(t\) to a bud
  \(\text{Bud}(\text{None},h,i);\) and
- Undefined (⊥) if such a leaf or bud does not exist.

\(\text{Dom}(t)\) is defined in the same manner as an MPT, and the prefix-free property is similarly satisfied on the domain of a Plebeia tree. For example, for the following tree (index and hash fields omitted for brevity):

\[
\begin{align*}
\text{Branch}(&\text{Branch}(\text{Leaf A}, \\
&\text{Bud}(\text{Some}(\text{Branch}(\text{Leaf B}, \text{Bud None})))), \\
&\text{Extender}([R;L],\text{Leaf D}))
\end{align*}
\]

its model is as follows. \(\llbracket t\rrbracket = \{\text{LL} \mapsto A, \text{LR} \mapsto \{\text{L} \mapsto B, \text{R} \mapsto \{\}, \text{RRL} \mapsto D\}\}.\) Notice how buds are used to express the nesting structure of the key–value store.

### 4.2 Zippers

One of the problems in the tree-manipulating functions in Section 3.4 is that each call to them has to traverse a tree from its root. This deteriorates the performance of these functions if the tree is large. The implementation of Plebeia prevents this problem by using zippers [Huet 1997].

A zipper is a purely functional data structure to express pointers to internal nodes. Intuitively, a zipper is a tree with a cursor that points to a node in the tree. The type for zippers is defined as follows.\(^2\)

\[
\begin{align*}
\text{type} & \quad \text{path} = \\
& \quad \mid \text{Top} \\
& \quad \mid \text{Left of} \quad \text{(* down to left from a Branch *)} \\
& \quad \quad \text{path} * \text{node} * \text{option index} * \text{option hash} \\
& \quad \mid \text{Right of} \quad \text{(* down to right from a Branch *)} \\
& \quad \quad \text{node} * \text{path} * \text{option index} * \text{option hash} \\
& \quad \mid \text{Extended of} \quad \text{path} * \text{key} * \text{option index} \quad \text{(* down from an Extender node *)} \\
& \quad \mid \text{Budded of} \quad \text{path} * \text{option index} * \text{option hash} \quad \text{(* down from a Bud node *)}
\end{align*}
\]

\[
\begin{align*}
\text{type} & \quad \text{zipper} = \text{path} * \text{node} \\
\text{type} & \quad \text{budzipper} = \text{path} * \text{n:node}\{\text{Bud? n}\}
\end{align*}
\]

A zipper is a pair of (1) a path, which describes how the node pointed to by the cursor can be reached from the root node, and (2) the subtree, whose root node is pointed to by the cursor. The intuition of the type path is as follows.

- \text{Top} represents the empty path.
- To explain the intuition of \text{Left}(p,n,i,h) (resp., \text{Right}(n,p,i,h)), let \(n'\) be the branch reached from the root by following the path \(p\); then, the path \text{Left}(p,n,i,h) (resp., \text{Right}(n,p,i,h)) represents the path from the root to the left (resp., right) child of \(n'\). The index \(i\) and the hash \(h\) are those for \(n'\). The tree \(n\) is the right (resp., left) subtree of \(n'\); we call this \(n\) an untracked tree at \text{Left}(p,n,i,h) (resp., \text{Right}(n,p,i,h)).
- Let \(n'\) be the extender reached from the root by following the path \(p\); then, the path \text{Extended}(p,k,i) represents the path from the root to the child of \(n'\) obtained by following the key \(k\). The index \(i\) is that for \(n'\).

\(^2\)This definition is slightly simplified from the actual formalization. See Remark 1.
Let \( n' \) be the bud reached from the root by following the path \( p \); then, the path \( \text{Budded}(p, i, h) \) represents the path from the root to the child of \( n' \). The index \( i \) and the hash \( h \) are those for \( n' \).

The type \( \text{budzipper} \) is for the zippers whose cursor points to a \( \text{Bud} \) node. The predicate \( \text{Bud}? n \), which is automatically generated by \( F^* \) from the definition of the type \( \text{node} \), holds if and only if \( n \) is of the shape \( \text{Bud} \_ \).

Using a zipper, one can replace the subtree pointed to by the cursor with another one in constant time in a purely functional manner. A cursor can be moved up and down in the tree (also in constant time).

For example, the following zipper:

\[
\text{let } \text{zipper}_\text{at}_b3 = \\
(\text{Left}(\text{Extended}(\text{Top}, [R;R], i_1), \\
\text{Extender}([R;L], \\
\text{Leaf}(C, i_7, h_7), i_4, i_2, h_2), \\
\text{Bud}(\text{Some}(\text{Branch}(\text{Leaf}(A, i_5, h_5), \\
\text{Leaf}(B, i_6, h_6), i_3, h_3)), i_8, h_8)) \\
\)
\]

represents the tree in Figure 1 wherein the cursor points to the node \( b3 \) and the path from the root to \( b3 \) is denoted by the dotted lines. The second element of \( \text{zipper}_\text{at}_b3 \) is the subtree rooted by \( b3 \). The first element expresses the path from the root of the entire tree to the node \( b3 \); \( b3 \) is reached from the extender at the root following its sole edge labeled with \([R;R]\) (reaching \( b1 \)) and then the left branch of \( b1 \).

### 4.3 Extensions to the Tree-Manipulating Functions

In the actual implementation, the tree-manipulating functions defined in Section 3.3 are defined so that they manipulate a tree via a zipper whose cursor points to a \( \text{Bud} \) node. For example, \( \text{insert} \) is defined as a function of type

\[
\text{z:budzipper} \to \text{k:key} \to \text{v:value} \to \text{option budzipper}
\]

instead of

\[
\text{z:node} \to \text{k:key} \to \text{v:value} \to \text{option node}
\]

The types of the other functions presented in Section 3.3 are changed in a similar way. This way, one can start a traversal of a tree from an internal node rather than the root, which reduces the cost of tree manipulations.

In addition to the functions in Section 3.3, Plebeia provides the following functions to serialize and deserialize a Plebeia tree.

- \( \text{commit_node} : \text{node} \to \text{node} * \text{index} \), which writes each node of the tree recursively from bottom to top and returns a tree in which the index of each node is computed. Plebeia uses a disk in an append-only style; therefore, \( \text{commit_node} \) has to store nodes at the indices which are disjoint from those already used.
- \( \text{load_node} : \text{index} \to \text{node} \), which reads data on the index from the disk and constructs a Plebeia tree recursively. It does not always deserialize the whole tree; several subtrees are not read and left as a \( \text{Disk} \_ \) node.
Remark 1. A tree-manipulation may lead to a recalculation of hashes and indexes. In the original implementation of Plebeia and our formalization, this recalculation is done lazily. To represent a hash and an index that need to be but have not been recalculated, the definition of path in the actual formalization is (an equivalent of) the following.

\[
\text{type path} = \ |
\text{Top} \\
\text{Left of} \ path \ast node \ast \text{option (option index \ast option hash)} \\
\text{Right of} \ node \ast path \ast \text{option (option index \ast option hash)} \\
\text{Extended of} \ path \ast key \ast \text{option (option index)} \\
\text{Budded of} \ node \ast \text{option (option index \ast option hash)}
\]

Notice that (1) the field of type option index \ast option hash in \text{Left}, \text{Right}, and \text{Budded} is replaced with option (option index \ast option hash) and (2) the field of type option index in \text{Extended} is option (option index). These fields are set to None if they need to be but yet to be recalculated. We omit this feature in this paper; see the source code for the actual formalization.

4.4 Structural Invariants for Extended Plebeia Trees

The extensions on Plebeia trees presented in Section 4.1, especially the lazy-computation feature in Section 4.1.2, requires the following structural invariants.

(SI3) If a node is hashed, its direct child must be hashed.
(SI4) If a node is indexed, it must be hashed.
(SI5) If a node is indexed, its direct child must be indexed.

There is an additional restriction on buds. There should not be consecutive occurrences of buds; this forces a canonical structure of a Plebeia tree to express a key–value store. A leaf should not be a direct child of a bud: this is to forbid the empty file name in the file system implemented with a Plebeia tree.

(SI6) A bud must not have another bud or leaf as its direct child.

4.5 Functional Requirements of Extended Tree-Manipulating Functions

The functional requirements described in Section 3.4 are restated so that each tree-manipulating function can receive and return a zipper instead of a tree. For this purpose, we need to define the model of a budzipper.

The model \([[[p,n]]]\) of a zipper \((p,n)\) is defined as a pair of the model of the path \([[p]]\) and that of the node \([[n]]\). \([[n]]\) is the same as that in Section 4.1.3. \([[p]]\) is a list of pairs of a nested key–value store and a key; in addition to the key from the root of the entire tree to \(n\), the model \([[p]]\) carries the information on the “untracked” trees. For example, let \(s\) be a nested key–value store \(s = \{\text{bird} \mapsto A, \text{mammal} \mapsto \{\text{dog} \mapsto \{\text{beagle} \mapsto B, \text{cat} \mapsto C\}, \text{fish} \mapsto D\}\). The zipper whose cursor points to the root of the tree corresponding to \(s\) is \([[],s\]). The model of the zipper whose cursor steps two levels down along the keys “mammal” and “dog” is \([[\{\text{dog},\{\text{cat} \mapsto C\}\},\{\text{mammal},\{\text{bird} \mapsto A,\text{fish} \mapsto D\}\}\},\{\text{beagle} \mapsto B\}\]). The first element of the model is the list of “mammal” and “dog”, each of which is associated with the nested key–value store that is not tracked.

On top of the above definition of a zipper, we can redefine the functional requirements. For example, the requirement for insert is as follows: For any budzipper \(z := (p,n)\), key \(k\), and value \(v\),

1. insert \((p,n)\ k\ v = \text{Some} \ (p',n') \implies \ [[p]] = [[p']\land [[n]](k) = \bot \land [[n']](k) = v \land \forall k'. k' \neq k \implies [[n'](k') = [[n]](k')

2. insert \((p,n)\ k\ v = \text{None} \implies \{k\} \cup \text{Dom}(n)\) is not prefix free \lor k \in \text{Dom}(n).
The condition $[[p]] = [[p']]$ states that the cursor stays at the same position after an insertion. Notice that `insert` is to insert values into the outermost MPT pointed to by the cursor; to insert into an inner MPT, one first has to move the zipper down to the corresponding bud.

5 VERIFICATION OF PLEBEIA WITH F★

This section explains how we verified a core part of Plebeia by F★. We remark that the F★ code we present in this section is sometimes simplified from the original version.

5.1 Overview

The properties we proved on the implementation of Plebeia are summarized as follows.

- The tree-manipulating functions in Section 3.3 preserve the structural invariants in Section 4.4;
- The tree-manipulating functions in Section 3.3 satisfy the functional requirements mentioned in Section 4.5;
- The functions that serialize and deserialize a Plebeia tree work correctly; and
- The Merkle hash defined on Plebeia trees is relatively collision-resistant (i.e., collision-free assuming the hash functions on the primitive data).

Currently, we have ported four core files of Plebeia into F★: (1) `node_type.ml` in which the structure of Plebeia tree is defined, (2) `cursor.ml` in which the tree-manipulating functions are defined, (3) `node_storage.ml` in which the functions for (de)serializing Plebeia trees, and (4) `hash.ml` in which the Merkle hash functions for Plebeia trees are defined. We started the verification of Plebeia from these four files because: (1) they implement the core functionality of Plebeia and therefore their implementation is already stable, and (2) the code in other files often uses the OCaml standard library that has not been ported to F★ yet.

We conducted verification by the following workflow. (The following explanation denotes parts in Figure 2 by the symbols in this figure.) Plebeia (a) is implemented in OCaml. It consists of several files, including `node_type.ml`, `cursor.ml`, `node_storage.ml`, and `hash.ml`. As denoted by the arrow (1), we manually translated the programs in these files in (a) to those of F★ files (b). These files

\[\text{Fig. 2. How we verified the Plebeia library.}\]
include \( F^* \) source files whose extensions are .fst (e.g., node_type_fstar.fst, cursor_fstar.fst, node_storage_fstar.fst, and hash_fstar.fst in (b)) and \( F^* \) interface files whose extensions are .fsti. The source files include the definition of Plebeia functions, theorems stating correctness, and their proofs. The interface files declare the types of the functions whose definitions have not been translated into \( F^* \) yet; these types are treated as assumptions in the proofs in node_type_fstar.fst, cursor_fstar.fst, node_storage_fstar.fst, and hash_fstar.fst. From these ported \( F^* \) files, we automatically extract OCaml source files (i.e., node_type.ml, cursor.ml, node_storage.ml, and hash.ml). We replace node_type.ml, cursor.ml, node_storage.ml, and hash.ml in Plebeia with the extracted files and build the entire project.

One crucial merit of our workflow is that it enables gradual verification of the entire implementation. One can run Plebeia before the verification of all the files completes. Another benefit of using \( F^* \) was that we were able to reuse most part of the implementation of Plebeia without rewriting them because, as we mentioned in Section 2, the \( F^* \) syntax is similar to that of OCaml.

5.2 Verification of the Structural Invariants

We explain how the verification of structural invariants of a Plebeia tree and a zipper proceeds in this section.

5.2.1 Formalizing for the Structural Invariants for Plebeia trees. We express the structural invariants of a Plebeia tree and a zipper as \( F^* \) functions whose return value is of type \( \text{bool} \) with effect \( \text{Tot} \). Concretely, the structural invariant for a Plebeia tree in Section 4.4 is expressed by the following function node_invariant.

\[
\text{let rec node_invariant (n:node) : Tot bool =}
\begin{align*}
&\text{node_hashed_invariant n }&& (* \text{SI3} *) \\
&\text{node_index_and_hash_invariant n }&& (* \text{SI4} *) \\
&\text{node_indexed_invariant n }&& (* \text{SI5} *) \\
&\text{node_shape_invariant n }&& (* \text{SI1, SI2, SI6} *) \\
&\text{match n with} \\
&\text{\mid Branch (l,r,i,h) -> node_invariant l }&& \text{node_invariant r} \\
&\end{align*}
\]

The invariant is expressed as the conjunction of node_hashed_invariant \( n \) that expresses the invariant on the hashedness of \( v \) (i.e., (SI3)); node_index_and_hash_invariant \( n \) that expresses the invariant on the dependency between indexedness and hashedness (i.e., (SI4)); node_indexed_invariant \( n \) that expresses the invariant on the indexedness of \( v \) (i.e., (SI5)); node_shape_invariant \( n \) that expresses the invariant on shape of \( v \) (i.e., (SI1), (SI2), and (SI6)); and the condition to recursively check these invariants on the descendants of \( v \). A predicate node_invariant \( (v:node) \) evaluates to \( \text{true} \) if the tree rooted by \( v \) satisfies the (SI1)–(SI6).

We only explain the implementation of node_shape_invariant here, which is defined by the following function.

\[
\begin{align*}
&\text{let node_shape_invariant (n:node) : Tot bool = match n with} \\
&\text{\mid Extender ([], _, _) -> false }&& (* \text{Checking SI2} *) \\
&\text{\mid Extender (_, (Extender _), _) -> false} \\
&\text{\mid Extender (_, _, _) -> true }&& (* \text{Checking SI6} *) \\
&\text{\mid Bud (Some (Bud _), _, _) }|| \text{Bud (Some (Leaf _), _, _) }&& \text{-> false}
\end{align*}
\]
The redefined type refines the original type for zippers so that the inhabiting values satisfy the invariants (SI1)–(SI6). The property (ZI-Connect) guarantees that the connection between the path \( p \) and the node of \( n \) does not violate (SI1)–(SI6); for example, this property violates that \( p \) is of the form \( \text{Extended} \). The property (ZI-Path) guarantees that the part of the tree expressed by the path \( p \) satisfies (SI1)–(SI6). The property (ZI-Node) guarantees that the node \( n \) satisfies (SI1)–(SI6). For a zipper \( (p, n) \), if \( p \) is of the form \( \text{Extended}(p', k, i) \), then (1-1) if \( p' \) is hashed, then \( p \) and \( n \) must be hashed; (1-2) if \( p' \) is indexed, then \( p \) and \( n \) must be indexed; (1-3) if \( p \) is indexed, then \( p \) must be hashed; and (1-4) \( n \) must satisfy (SI1)–(SI6). For a zipper \( (p, n) \), if \( p \) is of the form \( \text{Budded}(p', k, i) \), then (2-1) if \( p' \) is indexed, then \( p \) must be indexed; (2-2) \( p' \) must not be the form of \( \text{Extended} \); and (2-3) \( k \) must be non-empty. For a zipper \( (p, n) \), if \( p \) is of the form \( \text{Budded}(p', k, i) \), then (3-1) if \( p' \) is hashed, then \( p \) must be indexed; (3-2) if \( p' \) is indexed, then \( p \) must be indexed; (3-3) if \( p \) is indexed, then \( p \) must be hashed; and (3-4) \( p' \) must not be the form of \( \text{Budded} \).

**5.2.2 Structural Invariants for Zippers.** To prove that the tree-manipulating functions introduced in Section 4.3 preserve the structural invariants (SI1)–(SI6), we formulate the invariants of a zipper that are preserved by these functions and that are strong enough to imply (SI1)–(SI6) for the entire tree. For example, this property violates that \( p \) is of the form \( \text{Extended} \) and is indexed, then \( n \) must be indexed; and (3) if \( p \) is of the form \( \text{Extended} \), then \( n \) must not be of the form \( \text{Extended} \). Intuitively, for a zipper \( (p, n) \), the property (ZI-Node) forces that the node \( n \) satisfies (SI1)–(SI6). The property (ZI-Path) guarantees that the part of the tree expressed by the path \( p \) satisfies (SI1)–(SI6). The property (ZI-Connect) guarantees that the connection between the path \( p \) and the node of \( n \) does not violate (SI1)–(SI6); for example, this property violates that \( p \) is of the form \( \text{Extended} \).

Figure 3 explains these invariants schematically. The structural invariants of the part 2 in the figure is established by (ZI-Node); the part 1 is by (ZI-Path); the part 3 is by (ZI-Connect).

**5.2.3 Proving the Preservation of the Structural Invariants.** Based on the definitions so far, we prove that the tree-manipulating functions introduced in Section 3.3 preserve the structural invariants for a zipper. To state the preservation property compactly, we redefine the type of zipper as follows. The redefined type refines the original type for zippers so that the inhabiting values satisfy the structural invariants for zippers.

The redefinition is inserted before the definition of the tree-manipulating functions. Therefore, type-checking these functions proves the preservation of the structural invariants by the tree-manipulating functions. For example, the type of function \( \text{insert} \) is stated as follows. As mentioned
in Section 3.3, a zipper that is given to or returned from some of the tree-manipulation functions is restricted to the budzipper.

```markdown
type budzipper = z:zipper{let _,n = z in Bud? n}
val insert: (z:budzipper) -> (k:key{k <> []}) -> value -> option budzipper
```

Proving insert has the above type constitutes the proof of the preservation of the structural invariants by insert.

Quite surprisingly, this proof is done almost automatically: F⋆ backed by Z3 typechecks get_value, subtree, insert, create_subtree, update, and delete with only the following lemma on the length of a concatenated list:

```markdown
val list_app_length : (a:list 'a) -> (b:list 'a) ->
  Lemma (length (a @ b) = length a + length b)
```

This lemma is used in typechecking an auxiliary function that moves down the cursor of a zipper according to a key.

We also prove the following lemma that guarantees that the structural invariants for a zipper imply (SI1)–(SI6) for the tree to which the zipper is associated.

```markdown
val zipper_invariant_is_node_invariant : z:zipper ->
  Lemma (requires True)
  (ensures (let no = go_up_to_root z in node_invariant no))
  (decreases (zipper_path_len z))
```

Notice that the type zipper of z works as the assumption that z satisfies the structural invariants; therefore, the requires clause requires just True. The function zipper_path_len z computes the length of the path of z. The clause decreases (zipper_path_len z) is required to conduct the well-founded induction in the proof of this lemma (see below). The function go_up_to_root z is the function that computes the Plebeia tree to which z is associated.

```markdown
val go_up_to_root : z:zipper -> Tot node (decreases (zipper_path_len z))
let rec
  go_up_to_root (p,n) =
    match p with
    | Top -> n | _ -> let Some z' = go_up (p,n) in go_up_to_root z'
```

The type of go_up, which moves the cursor of a zipper up, is as follows:

```markdown
val go_up : (c:zipper) -> Tot r:(option zipper{zipper_path_len r << zipper_path_len c})
whose post-condition states that the length of the path of a returned zipper is strictly shorter than that of the passed zipper.

Then, zipper_invariant_is_node_invariant can be proved as follows.

```markdown
let rec
  zipper_invariant_is_node_invariant (p,n) = match p with
    | Top -> ()
    | _ -> let Some z' = go_up (p,n) in zipper_invariant_is_node_invariant z'
```

The proof is essentially the induction on the length of the path p. In the inductive step where p is not Top, go_up (p,n) generates the zipper z' in which its cursor is moved one-step up from that of (p,n). Since the type of go_up guarantees that the length of the path of z' is strictly shorter than that of (p,n), we can use the induction hypothesis zipper_invariant_is_node_invariant z' here.4

4Readers may be wondering why F⋆ can see that go_up (p,n) never returns None when the type of go_up does not say anything about when Some is returned; it seems that F⋆ combines the definition of go_up (not shown) and the fact that (p,n) is not Top to see it.
5.3 Verification of Functional Correctness

5.3.1 Definition of the Modeling Function. Since the functional correctness is stated using the function \([\cdot]\), we need to define a function to compute the model of a Plebeia tree. First, we define the types of models of zippers, paths, and nodes as follows.

```ocaml
let is_model_of_node l =  
  let keys = List.map fst l in  
  is_prefix_free keys && is_sorted keys  

type nestedvalue = Value of value | Map of model_of_node  
and model_of_node = l:list (key * nestedvalue){ is_model_of_node l }  

let model_of_path = list (key * model_of_node)  
let model = model_of_path * model_of_node
```

These definitions follow the definition of the model of a zipper described in Section 4.5. We encode a nested key–value store as an association list refined by a predicate `is_model_of_node`. This predicate `is_model_of_node l` forces (1) the keys in `l` to satisfy the prefix-freedom and (2) the association list to be sorted in the lexicographic order over keys. The former property is mentioned in Section 3.2.2; the latter is for the convenience of the proof.

A function `modelize` that computes the model of a zipper is defined as follows.

```ocaml
val modelize_node_aux : node -> model_of_node  
let rec modelize_node_aux n =  
  match n with  
  | Leaf(v,_,_) -> [([],Value v)]  
  | Branch(l,r,_,_) ->  
    let tl,tr = modelize_node_aux l, modelize_node_aux r in  
    (prepend_all [L] tl) @ (prepend_all [R] tr)  
  | Extender(seg,n',_,_) -> prepend_all seg (modelize_node_aux n')  
  | Bud(None,_,_) -> [([],Map [])]  
  | Bud(Some n',_,_) -> [([],Map (modelize_node_aux n'))]
```

```ocaml
val modelize_node : (n:node{match n with Bud _ -> true | _ -> false}) -> model_of_node  
let modelize_node n =  
  let [([],Map res)] = modelize_node_aux n in  
val modelize_path : path -> model_of_path  
let rec modelize_path p = match p with  
  | Top -> []  
  | _ ->  
    let p',n = go_up_to_bud_or_top (p,(Leaf(dummy,None,None))) in  
    let n' = match n with Bud _ -> n | _ -> Bud(Some(n)) in  
    let l = modelize_node n' in  
    let k = key_to_nearest_bud p in  
    let k, List.remove_assoc l k :: modelize_path p'  
    val modelize : z:budzipper -> model  
    let modelize (p,n) = (modelize_path p,modelize_node n)
```

The function `modelize` computes the model of a zipper `(p,n)` from the model of `p` and the model of `n`. The function `modelize_node_aux` recursively constructs the model following the definition of the model of a Plebeia tree discussed in Section 4.1.3. The function `prepend_all : k:key -> m:model_of_node -> model_of_node` prepends `k` to every key of the model `m`. The call to a lemma `model_branch_lemma tl tr` states that the constructed model.
(prepend_all [L] tl) @ (prepend_all [R] tr) satisfies is_model_of_node. For the definition and the proof of model_branch_lemma, see the source code. The function modelize_path p returns the model of path p as an association list following the definition described in Section 4.5. Given a path p, it creates a zipper whose path is p and whose cursor points to a dummy leaf node Leaf(dummy, None, None). Then, it moves up the zipper (p, Leaf(dummy, None, None)) to (p', n') where n' is the nearest bud node; if there is no such bud node, then n' points to the root. Using this zipper, it creates the model l of node n' (modelize_node n') and computes the path from the root of n' to that of n (key_to_nearest_bud p). Finally, it recursively computes modelize_path p' and combines with the entry for key k. The value of k has to be the association list obtained by removing the entry for k from l, which is the dummy node created at the beginning of this function.

5.3.2 Formalization of Functional Correctness. Using the above modeling functions, we formalize the functional correctness of get_value, subtree, insert, create_subtree, update, and delete. For example, the functional correctness of insert is formalized by the following declaration.

```
val insert_functionality :
  z:budzipper -> k:key -> v:value -> Lemma
  (let x = insert z k v in
   let p, l = modelize z in
   if is_prefix_free (List.Tot.map fst l) k then
     match x with
     | Some z' -> let p', l' = modelize z' in
                   p = p' ∧ is_extended_key_value_store l' l k v
     | None -> False
   )
```

The above code states the functional correctness of insert discussed in Section 4.5. This code uses is_extended_key_value_store l' l k v, which holds if (1) the mapping represented by l' is an extension of the mapping represented by l with \{k \mapsto v\} and (2) k is not in the domain of the mapping represented by l.

5.3.3 Proof of Functional Correctness. We describe the strategy to prove the functional properties of the tree-manipulating functions with insert_functionality as an example. The correctness of the other functions is proved in the source code.

To prove insert_functionality, we designate

\[ m_{\text{insert}} : (p,n) : \text{model} -> k : \text{key} \{k \neq []\} -> v : \text{value} -> \text{model} \]

which expresses the desired behavior of insert on models of zippers. Then, we prove this theorem by reducing it to the functional correctness of m_{\text{insert}} as follows.

(A) We prove the following equation on insert and m_{\text{insert}}:

\[ \text{modelize} \circ \text{insert} = m_{\text{insert}} \circ \text{modelize}. \]

This proves that insert correctly reflects the behavior of m_{\text{insert}} with respect to the function modelize.

(B) Then, we prove that m_{\text{insert}} is functionally correct in that it satisfies the following lemma model_insert_functionality:

```
val model_insert_functionality :
  (p,n) : model -> k : key \{k \neq []\} -> v : value -> Lemma
  (let y = m_insert (p,n) k v in
   if is_prefix_free (List.Tot.map fst n) k then
     match y with
```

17
model_insert_functionality states that \( m_{\text{insert}} \) satisfies the functional correctness of \( \text{insert} \) in Section 3.4 stated in terms of models of zippers. \( \text{model}_{\text{insert}}\text{functionality} \) is easily proved by the induction on the structure of the association list \( ml \).

By using (A) and (B) in combination, we proved \( \text{insert}_{\text{functionality}} \) as follows.

\[
\begin{align*}
\text{let} & \quad \text{insert}_{\text{functionality}} z \ k \ v = \\
& \quad \text{insert}_{\text{is homomorphism}} z \ k \ v; \\
& \quad \text{model}_{\text{insert}}\text{functionality} \ (\text{modelize} \ z) \ k \ v
\end{align*}
\]
Verification of a Merkle Patricia Tree Library Using F

```f
let f (storage:storage) (i:index) (ch:char) : ST unit
= (* Write two bytes to index i *)
  let buf = make_buf storage i in (* Create a buffer mapped to index i. *)
  set_char buf 10 ch;
  set_char buf 11 ch

(* The modelized version of f. *)
let spec_f
  (h:mem) (* Buffer modeled with LowStar.Buffer.buffer *)
  (storage:storage_model) (* Storage modeled with LowStar.Buffer.buffer *)
  (i:index) (ch:char)
  : Tot mem
= (* Relay heaps via arguments of functions *)
  let buf = spec_make_buf h storage i in (* Modelized make_buf *)
  let h1 = spec_set_char h buf 10 ch in (* Modelized set_char *)
  let h2 = spec_set_char h1 buf 11 ch in
  h2 (* Explicitly return updated heap *)

Then, in the type of f, we can state that f and spec_f are equivalent to each other.

(* Type declaration of function f *)
val f : storage:storage -> i:index -> ch:char -> ST unit
  (requires (fun _ -> true))
  (ensures (fun h _ h' ->
    (* In the postcondition of a function with ST effect,
      we can take h and h' that represent
      the state of the heap before and after calling this function. *)
      h' == spec_f h (model_of_storage storage) i ch
  ))

To encode the assumption on the termination of the disk-accessing functions, we augmented
their parameters with a natural number fuel and modified their implementation so that they
decrement fuel by one at every recursive function call. If the computation of a disk-accessing
function terminates before fuel reaches 0, then it returns Some r where r is the result; otherwise,
it returns None. The argument fuel is marked as a ghost variable that does not appear in the
extracted code. We separately assumed an oracle get_enough_fuel that can compute a sufficiently
large value for fuel from the state of a disk and a buffer.

5.5 Verification of Relative Collision Resistance of the Merkle Hash of Plebeia tree

We also prove that the Merkle hash defined on the Plebeia tree is relatively collision-resistant:
the Merkle hash is collision-resistant under the assumption that the blake2b hash function is
cryptographically secure. To describe the verification of the relative collision-resistance property,
we first explain the definition of the collision-free property of hash functions.

The collision-free property [Damgård 1987] is an indispensable property for the hash functions
used in the authentication. A collision of a hash function f is a pair (a,b) such that a ≠ b ∧ f(a) = f(b).
If a collision of the hash function of Plebeia tree is found in a reasonable amount of time, it
can be abused to forge invalid account data in the Tezos blockchain. Therefore, it is important to
verify the collision-free property on the hash Merkle of Plebeia tree.
The collision-free property is usually formalized on a parameterized hash functions to formally state the computational infeasibility for finding a collision [Damgård 1987]. In this formalization, the probability of finding a collision is bounded from the above by the inverse of any polynomial function of the parameter. This formalization would be suitable for the theorem provers specialized for cryptographic verification. However, we do not use this formalization because this definition requires probabilistic evaluation, which F★ is not good at now.

Instead, we use the human ignorance approach introduced by Rogaway [Rogaway 2006], which is a proof technique for algorithms that use cryptographic primitives and is used in the formalization of the collision-free property of the Merkle hash implemented in HACL*, a verified library for cryptographic primitives [Zinzindohoué et al. 2017]. In this approach, one can prove some cryptographic security of such an algorithm—the algorithm to compute the Merkle hash of Plebeia in our case—with respect to the cryptographic safety of the cryptographic primitives—blake2b [Aumasson et al. 2013] in our case—on which the algorithm is based. The proof is formalized by defining a reduction function that receives a collision of the Merkle hash and generates a collision or a preimage of a certain value of the blake2b, either of which destroys the cryptographic security of the blake2b.

5.5.1 The statement of the relative collision-resistance. As mentioned above, the Merkle hash of Plebeia is constructed from the blake2b hash function, which we assume to be secure [Guo et al. 2014; Luykx et al. 2016]. More precisely, we assume the following two cryptographic-security properties of function \(B'\), which takes the first 222 bits of blake2b:

Assumption 1. \(B'\) is collision-free: It is hard to find a pair \(a, b \in \{0, 1\}^*\) such that \(a \neq b \land B'(a) = B'(b)\).

Assumption 2. It is hard to find a string \(z\) that satisfies \(0 \cdots 0^{222}\); i.e., a preimage of the all-zero-bit string.

As we will see soon, the second assumption is essential to prevent a collision on a Bud(\(\text{None}, \_, \_\)) node and the first assumption is for the other kinds of nodes. Mathematically, there is a collision of \(B'\), as is easily shown by the pigeonhole principle. However, a collision or a preimage of the all-zero-bit string has not been found yet, and it is unlikely to be found within a reasonable amount of time.

Based on these assumptions, what we would like to prove is the following theorem about the Merkle hash \(MH\):

**Theorem 1.** If the collision of the Merkle hash (i.e., a pair of Merkle tree \((n_1, n_2)\) such that \(n_1 \neq n_2 \land MH(n_1) = MH(n_2)\)) is given, then either of the following can be efficiently constructed from the collision:

(C1) a pair of strings \((x_1, x_2)\) that satisfies \(x_1 \neq x_2 \land B'(x_1) = B'(x_2)\); or

(C2) a string \(y\) that satisfies \(B'(y) = 0^{222}\).

Our proof is not fully formal. We formally show that a certain function can construct (C1) or (C2) from a collision of the Merkle hash. Since the time complexity of the function is linear in the size of the input, we know (C1) or (C2) can be efficiently constructed but we do not formalize the argument about complexity.

---

6As we see, the length of a Merkle hash can vary (depending on the kind of a node) but it is often 28 bytes, which is equal to 224 bits, in the current implementation. We will explain the other two bits below.
5.5.2 Definition of the Merkle Hash of Plebeia Tree. We now describe the Merkle hash of Plebeia in more detail. The Merkle hash of Plebeia is defined as follows.\footnote{The actual implementation of the Merkle hash uses some stateful operations, such as updating a bytestring or loading a hash value from the disk for better performance. In what follows, we ignore Disk for simplicity and present a purely functional definition but the actual proof deals with stateful functions, using a device similar to the one used for verifying persistence.}

\[
\text{let hash222_2 b v = update b 27 ((get (blake2B_28 b) 27 & 0xfc) | (int_to_byte v))}
\]

\[
\text{let (^^) = Bytes.append}
\]

\[
\text{let hash = function}
\]

\[
\text{| Leaf (v,_,_) -> hash222_2 v 0b10}
\]

\[
\text{| Branch (l,r,_,_) ->}
\]

\[
\text{let hl,hr = hash l,hash r in}
\]

\[
\text{hash222_2 (hl ^^ hr ^^ int_to_byte (Bytes.length hr))) 0b00}
\]

\[
\text{| Extender (n',k,_,_) -> hash n' ^^ (key_to_bytes k)}
\]

\[
\text{| Bud(None,_,_) -> Bytes.make 28 '\0'}
\]

\[
\text{| Bud(Some(n'),_,_) -> hash222_2 (hash n') 0b11}
\]

Here, the function blake2B_28 is the blake2b hash function with the hash size parameter is set to 28 (bytes); key_to_bytes and int_to_bytes are (injective) functions to compute some byte sequence from a list of keys and integers, respectively.

The size of a Merkle hash depends on the constructor of a tree. The length of the Merkle hash of Leaf, Bud, and Branch nodes is 28 bytes, but the length of the Merkle hash of Extender node is greater than 28 bytes. We use the last 2 bits of a 28-byte hash to identify the kind of a node. The function hash222_2 is used to compute blake2b of the first argument and overwrite the last two bits of the 28 bytes according to the second argument. (An Extender does not use this function because the length of its Merkle hash tells that it is an Extender.)

5.5.3 Formalization of the collision-free statement in F∗. We formalized Theorem 1 as the following function merkle_hash_collision2hash_collision, which returns either Collision (corresponding to C1) or Preimage (corresponding to C2) from a collision of the Merkle Hash.

\[
\text{let first_222_bit b = update b 27 (get b 27 & 0xfc)}
\]

\[
\text{let hash b = first_222_bit (blake2B_28 b)}
\]

\[
\text{type instance_of_attack =}
\]

\[
\text{| Collision of (s1s2:(Seq.seq UInt8.t * Seq.seq UInt8.t){}
\text{let s1,s2 = s1s2 in s1 <> s2 ∧ hash s1 = hash s2 }}
\]

\[
\text{| Preimage of (s:Seq.seq UInt8.t{ hash s = Seq.create 28 0uy })}
\]

\[
\text{val merkle_hash_collision2hash_collision :}
\]

\[
\text{n1:node(node_invariant n1) -> n2:node(node_invariant n2) -> Pure instance_of_attack}
\]

\[
\text{(requires (n1 <> n2 ∧ merkle_hash n1 = merkle_hash n2))}
\]

\[
\text{(ensures (fun _ -> True))}
\]

\[
\text{let rec merkle_hash_collision2hash_collision n1 n2 = (* ... *)}
\]
The computational complexity of merkle_hash_collision2hash_collision is $O(N_1 + N_2)$, where $N_i$ is the size of the tree $n_i$. Currently, there is no feature implemented in F* to describe the computational complexity of a function. However, the implementation of the function is apparent from the source code, which is simple enough to examine its complexity. We place the source code of its implementation in Appendix A.

5.6 Result of the Verification

Our proof on the structural invariants of Plebeia trees in Section 5.2 consists of 341 lines of F* code, whereas the functional correctness in Section 5.3 consists of 4416 lines of F* code, the correctness of the data-persistence process in Section 5.4 consists of 3394 lines of F* code, and the relative collision-resistant in Section 5.5 consists of 224 lines of F* code. Our proof is included in the source code.

We executed the F* compiler in an environment with 1.1GHz CPU and 32GB RAM. the version of the F* compiler was 0.9.7.0-alpha1; the version of the backend SMT solver Z3 was 4.8.5. The time spent for type checking was 1722 seconds. The memory space used during the type checking was 4400 megabytes.8

We built Plebeia by linking the OCaml code extracted from our proof with the unverified OCaml code. We confirmed that the resulting executable passed all the tests bundled in the original Plebeia implementation. The time spent to execute the Plebeia benchmark with the extracted code was 275 seconds, whereas the original Plebeia ran in 261 seconds.

It is worth mentioning that we found a bug that existed in an old version of Plebeia through our verification: The structural invariant (SI4) was not preserved in delete. This bug remains in Plebeia from the commit 518f00db on April 9th, 2019 to the commit 12045fb1 on July 23rd, 2019. Fortunately, this bug had been, in fact, fixed before we discovered although we discovered this bug independently of the developers.

6 RELATED WORK

Project Everest [Bhargavan et al. 2017] is a project for implementing TLS 1.3 [Rescorla 2018]—one of the essential components in HTTPS protocol—in F* to obtain a verified TLS library written in OCaml and C. HACL* [Zinzindohoué et al. 2017] and EverCrypt [Protzenko et al. 2020] are the libraries developed in Project Everest. HACL* is a verified library for cryptographic primitives, which are used in Mozilla Firefox9 and Tezos. HACL* is implemented with Low* [Protzenko et al. 2017], which is a part of F* that supports extraction to the C language. The properties verified in HACL* includes memory safety, freedom from several side-channel attacks, and functional correctness of each cryptographic primitive. EverCrypt is a cryptographic library, which uses HACL* and assembly code verified with Vale [Bond et al. 2017], which is an intermediate language that can express an assembly. EverCrypt provides a verified implementation of Merkle trees; the cryptographic primitives implemented in EverCrypt are used in these implementations.

Lochbihler and Maric [Lochbihler and Maric 2020] formalized the authenticated data structures, which generalizes the Merkle hash for various types of trees, as the datatype in Isabelle/HOL framework. They formalized the Merkle tree construction step as the Merkle functor, which generates a Merkle tree from a tree. They verified the correctness of abstracted algorithms of Merkle hash and Merkle proofs, whereas we directly verified a production-level implementation of the Plebeia tree.

8F* compiler can cache verification results for future verification. We turned off the caching in measuring efficiency.
9https://blog.mozilla.org/security/2017/09/13/verified-cryptography-firefox-57/
Verification of complex data structures has been conducted by several authors. For example, Xi and Pfenning [1999] verified red-black trees by a dependently-typed ML-based language DML. Zee et al. [2008] verified various linked data structures by the Jahob verification system. Malecha et al. [2010] verified B+ trees with Coq and used it to implement a verified relational database management system (RDBMS). Lammich and Lochbihler [2010] implemented a fast and verified data-structure library for Isabelle. The GitHub repository of F*\textsuperscript{10} also includes verified implementation of various data structures (e.g., Merkle trees and red-black trees).

Bhargavan et al. [2016] propose a framework to use F* for verifying Ethereum smart contracts. They propose (1) a certified translation from Solidity, a high-level programming language to write Ethereum smart contracts, to a subset of F* called Solidity* and (2) a certified translation from Ethereum bytecode to a subset of F* called EVM*. The translated code can further be verified using F*. Their framework is for the correctness verification of a smart contract, whereas we verified the correctness of a core part of Tezos, which is an infrastructure on which a smart contract can be executed.

7 CONCLUSION
We verified the correctness of Plebeia, an implementation of a key–value store using an MPT-like data structure. To this end, we ported the core part of Plebeia to F*. The properties we verified are: (1) the structural invariants of a Plebeia tree are preserved by the tree-manipulating functions, (2) the functional correctness of the tree-manipulating functions, (3) the data-persistence process of Plebeia works correctly, and (4) the relative collision resistance of the functions that compute the Merkle hash of Plebeia tree. To the best of our knowledge, our work is the first one that formally verified a product-level implementation of an MPT-like data structure, which is heavily used in blockchain systems.

ACKNOWLEDGMENTS
This research is partially supported by JST CREST Grant Number JPMJCR2012 and MEXT/JSPS KAKENHI Grant Number JP19H04084.

REFERENCES
2009. [Full Disclosure] CVE-2012-2459 (block merkle calculation exploit). https://bitcointalk.org/index.php?topic=102395.0 (Accessed on February 1st, 2021.).
Jean-Philippe Aumasson, Samuel Neves, Zooko Wilcox-O’Hearn, and Christian Winnerlein. 2013. BLAKE2: Simpler, Smaller, Fast as MD5. In Applied Cryptography and Network Security - 11th International Conference, ACNS 2013, Banff, AB, Canada, June 25-28, 2013. Proceedings (Lecture Notes in Computer Science, Vol. 7954), Michael J. Jacobson Jr., Michael E. Locasto, Payman Mohassel, and Rei Sahni Safavi-Naini (Eds.). Springer, 119–135. https://doi.org/10.1007/978-3-642-38980-1_8
Karthikeyan Bhargavan, Barry Bond, Antoine Delignat-Lavaud, Cédric Fournet, Chris Hawblitzel, Catalin Hritcu, Samin Ishtiaq, Markulf Kohlweiss, Rustan Leino, Jay R. Lorch, Kenji Maillard, Jianyang Pan, Bryan Parno, Jonathan Protzenko, Tahina Ramananandro, Ashay Rane, Aseem Rastogi, Nikhil Swamy, Laure Thompson, Peng Wang, Santiago Zanella Béguelin, and Jean Karim Zinzindohoue. 2017. Everest: Towards a Verified, Drop-in Replacement of HTTPS. In SNAPL 2017 (LIPIcs, Vol. 71), Benjamin S. Lerner, Rastislav Bodik, and Shriram Krishnamurthi (Eds.). Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 1:1–1:12. https://doi.org/10.4230/LIPIcs.SNAPL.2017.1
Karthikeyan Bhargavan, Antoine Delignat-Lavaud, Cédric Fournet, Anitha Gollamudi, Georges Gonthier, Nadim Kobeissi, Natalia Katkova, Aseem Rastogi, Thomas Sibut-Pinote, Nikhil Swamy, and Santiago Zanella Béguelin. 2016. Formal Verification of Smart Contracts: Short Paper. In PLAS@CCS 2016, Toby C. Murray and Deian Stefan (Eds.). ACM, 91–96. https://doi.org/10.1145/2993600.2993611
Barry Bond, Chris Hawblitzel, Manos Kapritisos, K. Rustan M. Leino, Jacob R. Lorch, Bryan Parno, Ashay Rane, Srinath Setty, and Laure Thompson. 2017. Vale: Verifying High-Performance Cryptographic Assembly Code. In Proceedings of the 26th USENIX Conference on Security Symposium (Vancouver, BC, Canada) (SEC’17). USENIX Association, USA, 917–934.

\textsuperscript{10}https://github.com/FStarLang/FStar/tree/master/examples
A SOURCE CODE OF THE COLLISION REDUCTION FUNCTION

This source code is simplified from the original version by removing the fields for an index and a hash from each tree constructor.

This function recursively destructs a collision of the Merkle hash to generate an instance of the_instance_of_attack, which is either a collision or a preimage of the all-zero-bit string of blake2b. The function merkle_hash_collision2hash_collision n1 n2 is recursively called at most min(h1, h2) times, where h1 and h2 are the heights of the trees n1 and n2, respectively.

```ocaml
let first_222_bit b = update b 27 (get b 27 \& 0xfc)
let hash b = first_222_bit (blake2B_28 b)
(* before_hash_seq satisfies the following equation
  hash (before_hash_seq n) ==
  first_222_bit (model_of_merkle_hash n) *)
val before_hash_seq : (n:node{node_invariant n} -> True)
let before_hash_seq = (* ... *)

type instance_of_attack =
  | Collision of (s1s2:(Seq.seq UInt8.t \* Seq.seq UInt8.t){
      let s1, s2 = s1s2 in s1 <> s2 \&\& hash s1 = hash s2 })
  | Preimage of (s:Seq.seq UInt8.t{ hash s = Seq.create 28 \0uy })

val merkle_hash_collision2hash_collision :
  n1:node{node_invariant n1} -> n2:node{node_invariant n2} ->
  Pure instance_of_attack
  (requires (n1 <> n2 \&\&
     model_of_merkle_hash n1 = model_of_merkle_hash n2))
  (ensures (fun _ -> True))

let rec merkle_hash_collision2hash_collision n1 n2 =
```

Verification of a Merkle Patricia Tree Library Using F*

Dirk Siegel, Jens Hermann Paulsen, Arnaud Michelet, Peter Wiedmann, and Leo Tacke. 2016. https://www2.deloitte.com/content/dam/Deloitte/de/Documents/Innovation/Deloitte_Blockchain_Institute_Whitepaper_The.DAO.pdf

Nikhil Swamy, Catalin Hrițcu, Chantal Keller, Aseem Rastogi, Antoine Delignat-Lavaud, Simon Forest, Karthikeyan Bhargavan, Cédric Fournet, Pierre-Yves Strub, Markulf Kohlweiss, Jean-Karim Zinzindohoué, and Santiago Zanella-Béguelin. 2016. Dependent Types and Multi-Monadic Effects in F*. In POPL 2016. ACM, 256–270. https://doi.org/10.1145/2837614.2837655

Vyper Team. [n.d.]. Vyper documentation. https://vyper.readthedocs.io/en/latest/ (Accessed on February 1st, 2021.).

Gavin Wood. 2014. Ethereum: A secure decentralised generalised transaction ledger. https://ethereum.github.io/yellowpaper/paper.pdf (Accessed on February 1st, 2021.).

Hongwei Xi and Frank Pfenning. 1999. Dependent Types in Practical Programming. In Proceedings of the 26th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (San Antonio, Texas, USA) (POPL ’99). Association for Computing Machinery, New York, NY, USA, 214–227. https://doi.org/10.1145/292540.292560

Karen Zee, Viktor Kuncak, and Martin Rinard. 2008. Full Functional Verification of Linked Data Structures. In Proceedings of the 29th ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI’08) (Tucson, AZ, USA). Association for Computing Machinery, New York, NY, USA, 349–361. https://doi.org/10.1145/1375581.1375624

Jean Karim Zinzindohoué, Karthikeyan Bhargavan, Jonathan Protzenko, and Benjamin Beurdouche. 2017. HACL*: A Verified Modern Cryptographic Library. In CCS 2017, Bhavani M. Thuraisingham, David Evans, Tal Malkin, and Dongyan Xu (Eds.). ACM, 1789–1806. https://doi.org/10.1145/3133956.3134043
(* exclude impossible patterns. *)
begin match n1,n2 with
  | Leaf _, Leaf _ | Bud(Some _), Bud(Some _) | Bud None, _ | _, Bud None
  | Branch _, Branch _ | Extender _, Extender _ -> ()
  | Extender _ _, _ | _, Extender _ ->
    (* a pair of Extender and another type of node
       should not be a collision of the Merkle hash
       because the length of the Merkle hash differs *)
    assert (Seq.length (model_of_merkle_hash n1) <>
            Seq.length (model_of_merkle_hash n2))
  | Bud(Some _), Branch _ | Branch _, Bud(Some _)
  | Leaf _, Bud(Some _) | Bud(Some _), Leaf _
  | Leaf _, Branch _ | Branch _, Leaf _ ->
    (* a pair of Extender and Bud(Some _) 
       should not be a collision of the Merkle hash
       because the last 2 bit of the Merkle hash differs *)
    assert (last_2_bit_of_hash (model_of_merkle_hash n1) <>
            last_2_bit_of_hash (model_of_merkle_hash n2))
end;
match n1,n2 with
  | Bud(Some n1'), Bud(Some n2') when
    (model_of_merkle_hash n1' = model_of_merkle_hash n2') ->
    (* (n1',n2') is a collision of the Merkle hash *)
    merkle_hash_collision2hash_collision n1' n2'
  | Extender (k1,n1'), Extender (k2,n2') ->
    assert (k1 = k2);
    (* (n1',n2') is a collision of the Merkle hash *)
    merkle_hash_collision2hash_collision n1' n2'
  | Branch(l1,r1), Branch(l2,r2) when
    (model_of_merkle_hash l1 = model_of_merkle_hash l2 &&
     model_of_merkle_hash r1 = model_of_merkle_hash r2) ->
    if l1 <> l2 then
      (* (l1,l2) is a collision of the Merkle hash *)
      merkle_hash_collision2hash_collision l1 l2
    else (* r1 <> r2 *)
      (* (r1,r2) is a collision of the Merkle hash *)
      merkle_hash_collision2hash_collision r1 r2
  | Bud None, n | n, Bud None ->
    let s = before_hash_seq n in
    (* s is a preimage of zeros *)
    Preimage_of_Zeros s
  | _ ->
    let s1 = before_hash_seq n1 in
    let s2 = before_hash_seq n2 in
    (* (s1,s2) is a collision of B' *)
    Collision (s1,s2)