UMBERLLAS AND POLYTOPAL APPROXIMATION OF THE EUCLIDEAN BALL

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ABSTRACT. There are two positive, absolute constants $c_1$ and $c_2$ so that the volume of the difference set of the $d$-dimensional Euclidean ball and an inscribed polytope with $n$ vertices is larger than

$$c_2 \, d \, n^{\frac{d-1}{d-1}} \, vol_d(B_d^2)$$

for $n \geq (c_1 \, d)^{\frac{d-1}{d-1}}$.

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We study here the approximation of a convex body in $\mathbb{R}^d$ by a polytope with at most $n$ vertices. There are many means to measure the approximation, the two most common are the Hausdorff distance or the symmetric difference metric. The Hausdorff distance between two convex bodies $K$ and $C$ is

$$d_H(K, C) = \max\left\{\max_{x \in C} \min_{y \in K} \|x - y\|_2, \max_{y \in K} \min_{x \in C} \|x - y\|_2\right\}$$

where $\|x\|_2$ is the Euclidean norm of $x$. The symmetric difference metric is the volume of the difference set.

$$d_S(K, C) = \text{vol}_d(K \triangle C).$$

Bronshtein and Ivanov [BI] and Dudley [D1, D2] showed that for every convex body there is a constant $c$ such that for every $n$ there is a polytope $P_n$ with at most $n$ vertices and

$$d_H(K, P_n) \leq cn^{-\frac{2}{d-1}}.$$

This can be used to show the same estimate for the symmetric difference metric. Gruber and Kenderov [GK] showed that the inverse inequality holds if $K$ has a $C^2$-boundary:

$$d_S(K, P_n) \geq cn^{-\frac{2}{d-1}}.$$

Macbeath [Mac] showed that the approximation of a convex body is always better than that of the Euclidean sphere. Gruber [Gr2] obtained an asymptotic formula. If a convex body $K$ in $\mathbb{R}^d$ has a $C^2$-boundary with everywhere positive curvature, then we have

$$\inf\{d_S(K, P_n) | P_n \subset K \text{ and } P_n \text{ has at most } n \text{ vertices}\} \sim \frac{1}{2} \text{del}_{d-1} \int_{\partial K} \kappa(x) \frac{1}{n^2} d\mu(x) \left(\frac{1}{n}\right) \frac{2}{d-1} \mu(dK)$$

where $\text{del}_{d-1}$ is a constant that is connected with triangulations. In [GMR1, GMR2] it was shown constructively that for all dimensions $d$, all convex bodies $K$, and all $n \geq 2$ there is a polytope $P_n$ with $n$ vertices that is contained in $K$ such that

$$\text{vol}_d(K) - \text{vol}_d(P_n) \leq c d \text{ vol}_d(K)n^{-\frac{2}{d-1}}$$

where $c$ is a numerical constant. This estimate can also be derived from [BI] and [D1, D2]. So the question was whether the factor $d$ was necessary, or, in other words, what is the order of magnitude of the constant $\text{del}_d$. The result in this paper shows that there are absolute constants $c_1$ and $c_2$ with

$$c_1 \leq \text{del}_d \leq c_2.$$

In fact, we have

$$\text{del}_{d-1} \leq \frac{32}{d} \left(\frac{\text{vol}_{d-1}(\partial B_d^d)}{\text{vol}_{d-1}(B_2^{d-1})}\right)^{\frac{2}{d-1}}.$$

This follows from estimate (1) below.
In this paper we want to show that the volume of the difference set of the $d$-dimensional Euclidean ball and an inscribed polytope with $n$ vertices is larger than

$$cd \ vol_d(B_2^d) n^{-\frac{d}{d-1}}$$

We want to reduce the computation of the volume of the difference set to that of the following set: The set between a $d-1$ dimensional face of the polytope and the boundary of the sphere. Intuitively it is clear that the faces should be simplices and that the polytope should have rather regular features. This leads us to the assumption that the volume of the set between a $d-1$ dimensional face of the polytope and the boundary of the sphere equals in average approximately the surface area of the face times the height of the cap of the Euclidean ball that is determined by that face.

There are two technical difficulties. The number of faces does not necessarily correspond to the number of vertices. In fact, a heuristic argument shows that the number of faces is of the order of the number of vertices times $d^\frac{d}{2}$. Secondly, although we may assume that the faces are simplices, we may not assume that they are regular or close to regular. This is expressed in the following way. If $F$ is a face and $H$ the hyperplane containing $F$ then the distance of the centers of gravity of $F$ and $H \cap B_2^d$ may be large.

Hyperplanes are usually denoted by $H$ and the closed halfspaces associated with $H$ by $H^+$ and $H^-$. $H(x, \xi)$ is the hyperplane that passes through $x$ and is orthogonal to $\xi$.

The $d-1$ dimensional faces of a polytope in $\mathbb{R}^d$ are denoted by $F_j$. The hyperplanes containing $F_j$ are denoted by $H_j$. $H_j^+$ denotes the halfspace containing $P$.

For a polytope $P$ that is contained in $B_2^d$ the height or width of $B_2^d \cap H_j^-$ is $h_j$ and the radius of $B_2^d \cap H_j$ is $r_j$.

cg$(M)$ is the center of gravity of the set $M$.

$[A, B]$ denotes the convex hull of the sets $A$ and $B$. The radial projection $rp(M)$ of a set $M$ in $B_2^d$ is

$$rp(M) = \{\xi \in \partial B_2^d | [0, \xi] \cap M \neq \emptyset\}.$$  

**Theorem 1.** There are two positive constants $c_1$ and $c_2$ so that we have for all $d, d \geq 2$, and all $n, n \geq (c_1 d)^\frac{d+1}{2}$, and all polytopes $P_n$ that are contained in the Euclidean unit ball $B_2^d$ and have $n$ vertices

$$vol_d(B_2^d) - vol_d(P_n) \geq c_2 \ d \ vol_d(B_2^d)n^{-\frac{d}{d-1}}.$$  

In particular we have by Theorem 1 that there are positive constants $c_3$ and $c_4$ such that

$$vol_d(B_2^d) - vol_d(P_n) \geq c_4 vol_d(B_2^d)$$  

if $n \leq (c_3 d)^\frac{d+1}{2}$.
Lemma 2. (i) For all $x, 0 < x$, there is a $\theta, 0 < \theta < 1$, such that
\[ \Gamma(x + 1) = \sqrt{2\pi x^{x+\frac{1}{2}}} \exp(-x + \frac{\theta}{12x}). \]

(ii) \[ \Vol_d(B_2^d) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)} \leq \frac{d \pi^{\frac{d-1}{2}} (2e)^{\frac{d}{2}}}{d^{\frac{d+1}{2}}}. \]

The following lemma is due to Bronshtein and Ivanov [BI] and Dudley [D1, D2].

Lemma 3. For all dimensions $d, d \geq 2$, and all natural numbers $n, n \geq 2d$, there is a polytope $Q_n$ that has $n$ vertices and is contained in the Euclidean ball $B_2^d$ such that
\[ d_H(Q_n, B_2^d) \leq \frac{16}{7} \left( \frac{\Vol_{d-1}(\partial B_{2}^d)}{\Vol_{d-1}(B_{2}^{d-1})} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}. \]

In particular, since a $Q_n$ which satisfies the hypothesis of Lemma 3 contains the Euclidean ball of radius $1 - d_H(Q_n, B_2^d)$, it follows that
\[ d_S(Q_n, B_2^d) \leq \Vol_d(B_2^d)(1 - d_H(Q_n, B_2^d))^d \leq \Vol_d(B_2^d)(1 - (1 - \frac{16}{7} \left( \frac{\Vol_{d-1}(\partial B_{2}^d)}{\Vol_{d-1}(B_{2}^{d-1})} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}})^d \]
and
\[ (1 - \frac{16}{7} \left( \frac{\Vol_{d-1}(\partial B_{2}^d)}{\Vol_{d-1}(B_{2}^{d-1})} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}})^{d-1} \Vol_{d-1}(\partial B_{2}^d) \leq \Vol_{d-1}(\partial Q_n). \]

We have that
\[ \Vol_{d-1}(\partial B_2^d) = d \Vol_d(B_2^d) = d \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)} = d \sqrt{\pi} \Gamma\left(\frac{d-1}{2} + 1\right) \Vol_{d-1}(B_{2}^{d-1}) \leq d \sqrt{\pi} \Vol_{d-1}(B_{2}^{d-1}). \]
Since $d^{\frac{2}{d-1}} \leq 4$ and $(1 - t)^d \geq 1 - dt$ we get from (1)
\[ d_S(Q_n, B_2^d) \leq (1 - (1 - \frac{64}{7} \pi n^{-\frac{2}{d-1}})^d) \Vol_d(B_2^d) \leq \frac{64}{7} \pi d n^{-\frac{2}{d-1}} \Vol_d(B_2^d). \]
Similarly we get from (2) that we have for $n \geq \left( \frac{128}{7} \pi d \right)^{\frac{d-1}{2}}$
\[ \Vol_{d-1}(\partial B_2^d) \leq 2 \Vol_{d-1}(\partial Q_n). \]
Proof. For every $n$ there is a $\theta_n > 0$ and a set $\{x_1, \ldots, x_n\} \subset \partial B^d_2$ so that for all $i \neq j$ we have

$$\|x_i - x_j\| \geq \theta_n$$

and so that for every $x \in \partial B^d_2$ there is $i$ such that

$$\|x - x_i\| \leq \theta_n.$$ 

We choose $Q_n$ to be the convex hull of $\{x_1, \ldots, x_n\}$. We have

$$d_H(Q_n, B^d_2) \leq \frac{1}{2} \theta_n^2.$$ 

If not, then there is $x \in \partial B^d_2$ such that the Euclidean ball with radius $\frac{1}{2} \theta_n^2$ and center $x$ and $Q_n$ have an empty intersection. By the theorem of Hahn-Banach there is a hyperplane separating $Q_n$ and $B^d_2(x, \frac{1}{2} \theta_n^2)$. This hyperplane cuts off a cap of height greater than $\frac{1}{2} \theta_n^2$. The point at the top of this cap has a distance greater than $\theta_n$ from all $x_i$, $i = 1, \ldots, n$. This cannot be.

Now we estimate $\theta_n$ from above. The caps

$$\partial B^d_2 \cap H^-((1 - \frac{1}{8} \theta_n^2)x_i, x_i) \quad i = 1, \ldots, n$$

have disjoint interiors. Therefore we get

$$vol_{d-1}(\partial B^d_2) \geq \sum_{i=1}^n vol_{d-1}(\partial B^d_2 \cap H^-((1 - \frac{1}{8} \theta_n^2)x_i, x_i))$$

$$\geq n(\frac{1}{2} \theta_n \sqrt{1 - \frac{1}{16} \theta_n^2})^{d-1} vol_{d-1}(B^d_2).$$

We obtain

$$\frac{1}{2} \theta_n \sqrt{1 - \frac{1}{16} \theta_n^2} \leq \left(\frac{1}{n} \frac{vol_{d-1}(\partial B^d_2)}{vol_{d-1}(B^d_2)}\right)^{1/d-1}.$$ 

For $n = 2d$ we get that $\theta_n \leq \sqrt{2}$. Indeed, just consider the set $\{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$.

Thus it follows

$$\frac{\theta_n}{2} \sqrt{\frac{7}{8}} \leq \left(\frac{1}{n} \frac{vol_{d-1}(\partial B^d_2)}{vol_{d-1}(B^d_2)}\right)^{1/d-1}$$

and thus

$$\frac{1}{2} \theta_n^2 \leq \frac{16}{7} \left(\frac{1}{n} \frac{vol_{d-1}(\partial B^d_2)}{vol_{d-1}(B^d_2)}\right)^{2/d-1}.$$
Lemma 4. (i) For $k = 0, 1, 2, \ldots$ and $d = 1, 2, \ldots$ we have

$$\int_{\mathbb{R}^d_+} (\sum_{i=1}^d y_i)^k \exp(- (\sum_{i=1}^d y_i)^2) dy = \frac{\Gamma\left(\frac{k+d}{2}\right)}{2(d-1)!}.$$ 

(ii)

$$\int_{\mathbb{R}^d_+} (\sum_{i=1}^d y_i^2) \exp(- (\sum_{i=1}^d y_i)^2) dy = \frac{d^2}{2(d+1)!} \Gamma\left(\frac{d}{2}\right).$$

(iii) For $i \neq j$ we have

$$\int_{\mathbb{R}^d_+} y_i y_j \exp(- (\sum_{i=1}^d y_i)^2) dy = \frac{\Gamma\left(\frac{d}{2}\right)}{4(d+1)(d-1)!}.$$ 

Proof. (i) We denote $H_t = \{ y \in \mathbb{R}^d_+ | \sum_{i=1}^d y_i = t \}$. Let $d_{H_t}$ denote the $d - 1$-dimensional Lebesgue-measure on $H_t$. We have that $\text{vol}_{d-1}(H_t) = \frac{\sqrt{d}}{(d-1)!}$ and get

$$\int_{\mathbb{R}^d_+} (\sum_{i=1}^d y_i)^k \exp(- (\sum_{i=1}^d y_i)^2) dy = \int_{H_t} \frac{t^k}{\sqrt{d}} e^{-t^2} d_{H_t} dt$$

$$= \frac{1}{(d-1)!} \int_0^\infty t^{k+d-1} e^{-t^2} dt = \frac{1}{2(d-1)!} \int_0^\infty s^{k+d-1} e^{-s} ds = \frac{1}{2(d-1)!} \Gamma\left(\frac{k+d}{2}\right).$$

(ii)

$$\int_{\mathbb{R}^d_+} (\sum_{i=1}^d y_i^2) \exp(- (\sum_{i=1}^d y_i)^2) dy = \frac{1}{\sqrt{d}} \int_0^\infty \int_{H_t} \sum_{i=1}^d y_i^2 e^{-t^2} d_{H_t}(y) dt$$

$$= \frac{1}{\sqrt{d}} \int_0^\infty e^{-t^2} \left( \frac{d}{dt} \int_0^t \int_{H_s} \sum_{i=1}^d y_i^2 d_{H_s}(y) ds \right) dt$$

$$= \sqrt{d} \int_0^\infty e^{-t^2} \left( \frac{d}{dt} \int_0^t y_1^2 d_{H_s}(y) ds \right) dt$$

$$= d \int_0^\infty e^{-t^2} \frac{d}{dt} \left( \int_{0 \leq y_1 \leq t} y_1^2 d(y) \right) dt$$

$$= d \int_0^\infty e^{-t^2} \frac{d}{dt} \left( \int_{0 \leq y_1 \leq t} y_1^2 d(y) \right) dt$$

$$= \frac{d}{(d-1)!} \int_0^\infty e^{-t^2} \left( \int_{0 \leq y_1 \leq t} y_1^2 (t-y_1)^{d-1} dy_1 \right) dt.$$
\[ \frac{d}{(d-1)!} \int_0^\infty e^{-t^2} \frac{d}{dt} (t^{d+2} \int_0^1 s^2 (1-s)^{d-1} ds) dt \]

\[ = \frac{d}{(d-1)!} \int_0^\infty e^{-t^2} \frac{d}{dt} (t^{d+2} \Gamma(3) \Gamma(d) \Gamma(d+3)) dt \]

\[ = \frac{2d}{(d+1)!} \int_0^\infty e^{-t^2} t^{d+1} dt = \frac{d}{(d+1)!} \int_0^\infty s^2 e^{-s} ds = \frac{d \Gamma\left(\frac{d}{2} + 1\right)}{(d+1)!}. \]

(iii)

\[ \int_{\mathbb{R}^d_+} y_i y_j \exp(-\sum_{i=1}^d y_i^2) dy = \frac{1}{d^2 - d} \sum_{1 \leq k, l \leq d; k \neq l} \int_{\mathbb{R}^d_+} y_k y_l \exp(-\sum_{i=1}^d y_i^2) dy \]

\[ = \frac{1}{d^2 - d} \int_{\mathbb{R}^d_+} ((\sum_{i=1}^d y_i^2) - (\sum_{i=1}^d y_i^2)) \exp(-\sum_{i=1}^d y_i^2) dy \]

By (i) and (ii) we get for the above expression

\[ \frac{1}{d^2 - d} \left( \frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} - \frac{d^2}{2(d+1)!} \Gamma\left(\frac{d}{2}\right) \right) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{d^2}{2(d-1)!} - \frac{d^2}{2(d+1)!} = \Gamma\left(\frac{d}{2}\right) = \frac{d^2}{(d+1)(d-1)!}. \]

For the following lemma compare also [R].

**Lemma 5.** Let \( x_1, \ldots, x_d \) be points on the Euclidean sphere of radius 1, \( S \) the simplex \([x_1, \ldots, x_d]\), and \( \text{rp}(S) \) the radial projection of \( S \), i.e. the spherical simplex of the points \( x_1, \ldots, x_d \). Let \( X \) be the matrix whose columns are the vectors \( x_1, \ldots, x_d \). Then we have

\[ \text{vol}_{d-1}(\text{rp}(S)) = \frac{2}{\Gamma\left(\frac{d}{2}\right)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y) dy \]

and

\[ \text{vol}_d([0, \text{rp}(S)]) = \frac{2}{d \Gamma\left(\frac{d}{2}\right)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y) dy. \]

**Proof.** We have

\[ \text{vol}_d(B_2^d) = \frac{\pi^\frac{d}{2}}{\Gamma\left(\frac{d}{2} + 1\right)} \]

and

\[ \int_{\mathbb{R}^d_+} e^{-\|z\|^2} dz = \pi^\frac{d}{2}. \]
Therefore we get

\[
vol_d([0, rp(S)]) = \frac{1}{\Gamma\left(\frac{d}{2} + 1\right)} \int_{\{z = t\xi|\xi \in S \text{ and } t \in \mathbb{R}_+\}} e^{-\|z\|^2} dz.
\]

Using the substitution \(z = Xy\) we get that the latter expression equals

\[
\frac{1}{\Gamma\left(\frac{d}{2} + 1\right)} |\det(X)| \int_{y \geq 0} e^{-y^t X^t X y} dy.
\]

\[\square\]

**Lemma 6.** Let \(x_1, \ldots, x_d\) be points on the Euclidean sphere of radius 1, \(S\) the simplex \([x_1, \ldots, x_d]\), and let \(rp(S)\) be the radial projection of the simplex \(S\). Let \(H\) be the hyperplane containing the simplex \([x_1, \ldots, x_d]\) and \(r\) the radius of the \(d-1\)-dimensional Euclidean ball \(H \cap B^d_2\). Then we have

\[
vol_d([0, rp(S)]) - vol_d([0, S]) \geq \frac{d^2}{2(d+1)} \left(1 - \frac{1}{d} \sum_{i=1}^{d} x_i^2\right) vol_d([0, S])
\]

and

\[
vol_d([0, rp(S)]) - vol_d([0, S]) \geq \frac{d \sqrt{1 - r^2}}{2(d+1)} \left(1 - \frac{1}{d} \sum_{i=1}^{d} x_i^2\right) vol_{d-1}(S)
\]

**Proof.** By Lemma 5 we have

\[
vol_d([0, rp(S)]) - vol_d([0, S]) = \frac{2}{d \Gamma\left(\frac{d}{2}\right)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y) dy - \frac{|\det(X)|}{d!}
\]

By Lemma 5(i) with \(k = 0\) the last expression equals

\[
\frac{2}{d \Gamma\left(\frac{d}{2}\right)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y) - \exp\left(-\left(\sum_{i=1}^{d} y_i^2\right)\right) dy
\]

\[
= \frac{2}{d \Gamma\left(\frac{d}{2}\right)} |\det(X)| \int_{\mathbb{R}^d_+} \exp\left(\sum_{i=1}^{d} y_i^2 - y^t X^t X y\right) - 1 \exp\left(-\sum_{i=1}^{d} y_i^2\right) dy
\]

We use now the inequality \(1 + t \leq e^t\) and get that the above expression is greater than or equal to

\[
\frac{2}{d \Gamma\left(\frac{d}{2}\right)} |\det(X)| \int_{\mathbb{R}^d_+} \left(\sum_{i=1}^{d} y_i^2 - y^t X^t X y\right) \exp\left(-\sum_{i=1}^{d} y_i^2\right) dy
\]
\[\frac{2}{d!} |\det(X)| \sum_{i,j=1}^d (1 - x_i, x_j >) \int_\mathbb{R}_+ y_iy_j \exp(-(y^ty)^2)dy\]

Since we have \(1 = < x_i, x_j >\) for \(i = 1, \ldots, d\) we get by Lemma 5(iii) for the above expression
\[= \frac{2}{d!} |\det(X)| \sum_{i,j=1}^d (1 - x_i, x_j >) \frac{\Gamma\left(\frac{d}{2}\right)}{4(d+1)(d-1)!} = \frac{1}{2(d+1)!} (d^2 - \|\sum_{i=1}^d x_i\|^2) |\det(X)|.\]

\[\square\]

**Lemma 7.** Let \(A\) be a measurable subset of \(B_2^d\) such that the center of gravity of \(A\) is contained in a cap of height \(\Delta, \Delta \leq 1\). Then there is a cap \(C\) of height \(2\Delta\) so that
\[2 \ \text{vol}_d(C \cap A) \geq \text{vol}_d(A).\]

**Lemma 8.** Let \(P_n\) be a simplicial polytope with vertices \(x_1, \ldots, x_n\) that are elements of \(\partial B_2^d\). Let \(F_j, j = 1, \ldots, m\) be the \(d-1\)-dimensional faces of \(P_n\), \(H_j\) the hyperplane containing \(F_j\), \(h_j\) the height of the cap \(B_2^d \cap H_j^-\), and \(r_j\) the radius of \(B_2^d \cap H_j\). Let \(N\) be the set of integers \(j\) so that
\[h_j \leq \frac{1}{8} \left(\frac{\text{vol}_{d-1}(\partial P_n)}{\text{vol}_{d-1}(\partial B_2^d)} \frac{1}{4n}\right)^{\frac{1}{2r_j}}.\]

Then we have
\[\text{vol}_{d-1}(\bigcup_{j \in N} F_j) \leq \frac{1}{4} \text{vol}_{d-1}(\partial P_n).\]

**Proof.** We put
\[N_i = \{j \in N|x_i \in F_j\} \quad i = 1, \ldots, n\]
and
\[\rho = \frac{1}{8} \left(\frac{\text{vol}_{d-1}(\partial P_n)}{\text{vol}_{d-1}(\partial B_2^d)} \frac{1}{4n}\right)^{\frac{1}{2r_j}}.\]

Since \(h_j \leq \rho\) we have that \(\bigcup_{j \in N_i} F_j\) is contained in \(B_2^d(x_i, 2\sqrt{2}\rho)\). \(\bigcup_{j \in N_i} F_j\) is a subset of the boundary of the convex set \(P_n \cap B_2^d(x_i, 2\sqrt{2}\rho)\). Thus we get
\[\text{vol}_{d-1}(\bigcup_{j \in N_i} F_j) \leq \text{vol}_{d-1}(\partial(P_n \cap B_2^d(x_i, 2\sqrt{2}\rho))).\]

Since \(P_n \cap B_2^d(x_i, 2\sqrt{2}\rho)\) is a convex subset of the convex set \(B_2^d(x_i, 2\sqrt{2}\rho)\) we get
\[\text{vol}_{d-1}(\bigcup_{j \in N_i} F_j) \leq (8\rho)^{\frac{d-1}{2}} \text{vol}_{d-1}(\partial B_2^d) \leq \frac{1}{4n} \text{vol}_{d-1}(\partial P_n).\]

Therefore we get
\[\text{vol}_{d-1}(\bigcup_{j \in N} F_j) = \text{vol}_{d-1}(\bigcup_{i=1}^n \bigcup_{j \in N_i} F_j) \leq \sum_{i=1}^n \text{vol}_{d-1}(\bigcup_{j \in N_i} F_j) \leq \frac{1}{4} \text{vol}_{d-1}(\partial P_n).\]

\[\square\]
Lemma 9. Let $P_n$ be a simplicial polytope with vertices $x_1, \ldots, x_n$ that are elements of $\partial B_2^d$. Let $F_j$, $j = 1, \ldots, m$ be the $d-1$-dimensional faces of $P_n$, $H_j$ the hyperplane containing $F_j$, $h_j$ the height of the cap $B_2^d \cap H_j^-$, and $r_j$ the radius of $B_2^d \cap H_j$. Assume that we have for all $j, \ h_j \leq \frac{16}{7} \left( \frac{\text{vol}_{d-1}(\partial B_2^d)}{\text{vol}_{d-1}(B_2^{d-1})} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}$

and assume that

\[ \text{vol}_{d-1}(\partial B_2^d) \leq 2 \text{vol}_{d-1}(\partial P_n). \]

Let $\mathcal{M}$ be the set of integers $j$ so that

\[ \| cg(F_j) - cg(H_j \cap B_2^d) \|_2 \geq \frac{2^{22} - 1}{2^{22} r_j}. \]

Then we have

\[ \text{vol}_{d-1}(\bigcup_{j \in \mathcal{M}} F_j) \leq \frac{1}{4} \text{vol}_{d-1}(\partial P_n). \]

Proof. We put

\[ \theta = \frac{16}{7} \left( \frac{\text{vol}_{d-1}(\partial B_2^d)}{\text{vol}_{d-1}(B_2^{d-1})} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}} \leq \frac{16}{7} \left( 2d \sqrt{\pi} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}. \]

Since $h_j \leq \theta$ we have for all $j, \ h_j \leq \frac{\sqrt{2d\theta}}{2}$. We have that $cg(F_j)$ is contained in a cap of height $2^{-22}r_j$ of the $d-1$-dimensional Euclidean ball $H_j \cap B_2^d$. By Lemma 7 there is a subset $\tilde{F}_j$ of $F_j$ so that $\tilde{F}_j$ is contained in a cap of height $2^{-21}r_j$ and

\[ \text{vol}_{d-1}(F_j) \leq 2 \text{vol}_{d-1}(\tilde{F}_j). \]

Thus the diameter of $\tilde{F}_j$ is less than $2^{-9}r_j \leq \frac{\sqrt{2d\theta}}{512}$. The set of all integers $j$ such that $x_i \in \tilde{F}_j$ is denoted by $\mathcal{M}_i$. We have that $\bigcup_{j \in \mathcal{M}_i} \tilde{F}_j$ is a subset of the boundary of the convex set $P_n \cap B_{2^d}(x_i, 2^{-9} \sqrt{2d\theta})$ and has a smaller surface area than $B_{2^d}(x_i, 2^{-9} \sqrt{2d\theta})$.

\[ \text{vol}_{d-1}(\bigcup_{j \in \mathcal{M}_i} \tilde{F}_j) \leq \left( \frac{\sqrt{2d\theta}}{512} \right)^{d-1} \text{vol}_{d-1}(\partial B_2^d) \leq \frac{4d \sqrt{\pi} \left( \sqrt{\frac{32}{512}} \right)^{d-1} \text{vol}_{d-1}(\partial P_n)}{n}. \]

Since $d \leq 2^{d-1}$ we get that the latter expression is smaller than

\[ \frac{4 \sqrt{\pi}}{n} \left( \frac{\sqrt{2}}{128} \right)^{d-1} \text{vol}_{d-1}(\partial P_n) \leq \frac{\sqrt{2\pi}}{32n} \text{vol}_{d-1}(\partial P_n) \leq \frac{1}{8n} \text{vol}_{d-1}(\partial P_n). \]

Therefore we get

\[ \text{vol}_{d-1}(\bigcup_{j \in \mathcal{M}_i} \tilde{F}_j) \leq \frac{1}{8n} \text{vol}_{d-1}(\partial P_n). \]
\[
\leq 2 \sum_{i=1}^{n} \text{vol}_{d-1} \left( \bigcup_{j \in M_i} \tilde{F}_j \right) \leq 2 \sum_{i=1}^{n} \text{vol}_{d-1} \left( \bigcup_{j \in M_i} \tilde{F}_j \right) \leq \frac{1}{4} \text{vol}_{d-1}(\partial P_n).
\]

\[\square\]

**Proof of Theorem 1.** We consider numbers of vertices \(n\) such that \(n \geq \left( \frac{512}{7 \pi d} \right)^{\frac{d-1}{4}}\).

Let \(P_n\) be a polytope with \(n\) vertices so that \(\text{vol}_d(B^d_2) - \text{vol}_d(P_n)\) is minimal. Let \(Q_n\) be a polytope with \(n\) vertices so that \(d_H(B^d_2, Q_n)\) is minimal. By Lemma 3 we have that for all \(j\)

\[d_H(B^d_2, Q_n) \leq \frac{16}{7} \left( \frac{\text{vol}_{d-1}(\partial B^d_2)}{\text{vol}_{d-1}(B^d_2)} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}.\]

We consider now the convex hull of \(P_n\) and \(Q_n\).

\[P = [P_n, Q_n].\]

\(P\) has at most \(2n\) vertices. Its \(d-1\)-dimensional faces are denoted by \(F_j, j = 1, \ldots, m\). \(H_j\) is the hyperplane containing \(F_j\), \(h_j\) the height of the cap \(B^d_2 \cap H_j\), and \(r_j\) the radius of \(B^d_2 \cap H_j\). We may assume that \(P\) is simplicial. We have that

\[h_j \leq d_H(B^d_2, Q_n) \leq \frac{16}{7} \left( \frac{\text{vol}_{d-1}(\partial B^d_2)}{\text{vol}_{d-1}(B^d_2)} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}.\]

By the assumption on \(n\) we have that

\[(5)\quad h_j \leq \frac{1}{8} \quad \text{and} \quad r_j = \sqrt{2h_j - h_j^2} \leq \frac{1}{2}.\]

Also we have by (4) that

\[\text{vol}_{d-1}(\partial B^d_2) \leq 2\text{vol}_{d-1}(\partial Q_n) \leq 2\text{vol}_{d-1}(\partial P).
\]

We apply Lemma 8 and 9 to \(P\) that has at most \(2n\) vertices. Thus a factor 2 enters the estimates. Let \(\mathcal{L}\) be the set of integers \(j\) so that

\[(6)\quad \frac{1}{8} \left( \frac{\text{vol}_{d-1}(\partial P_n)}{\text{vol}_{d-1}(B^d_2)} \right)^{\frac{2}{d-1}} \leq h_j \leq \frac{16}{7} \left( \frac{\text{vol}_{d-1}(\partial B^d_2)}{\text{vol}_{d-1}(B^d_2)} \right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}
\]

and

\[(7)\quad \|cg(F_j) - cg(H_j \cap B^d_2)\|_2 < \frac{222 - 1}{222} r_j.
\]

We have

\[(8)\quad \text{vol}_{d-1}(\bigcup F_j) \geq \frac{1}{2} \text{vol}_{d-1}(\partial P).\]
We apply Lemma 6

\[ \text{vol}_d(B_2^d) - \text{vol}_d(P_n) \geq \text{vol}_d(B_2^d) - \text{vol}_d(P) \geq \sum_{j \in \mathcal{L}} (\text{vol}_d([0, rp(F_j)]) - \text{vol}_d([0, F_j])) \]

\[ \geq \sum_{j \in \mathcal{L}} \sqrt{\frac{1 - r_j^2}{4}} (1 - \|cg(F_j)\|^2_2) \text{vol}_{d-1}(F_j). \]

By (5) we have \( r_j \leq \frac{1}{2} \) and get that the latter expression is greater than

\[ \sum_{j \in \mathcal{L}} \frac{1}{8} (1 - \|cg(F_j)\|^2_2) \text{vol}_{d-1}(F_j). \]

We have

\[ \|cg(F_j)\|^2_2 = (1 - h_j)^2 + \|cg(F_j) - cg(H_j \cap B_2^d)\|^2_2. \]

By (7) we get for \( j \in \mathcal{L} \)

\[ 1 - \|cg(F_j)\|^2_2 \geq 1 - (1 - h_j)^2 - \left(\frac{2^{22} - 1}{2^{22}} r_j\right)^2 \]

\[ = 1 - (1 - h_j)^2 - \left(\frac{2^{22} - 1}{2^{22}}\right)^2 (2h_j - h_j^2) = (2^{-21} - 2^{-44})(2h_j - h_j^2) \geq 2^{-21} h_j. \]

Therefore

\[ \text{vol}_d(B_2^d) - \text{vol}_d(P) \geq \frac{1}{2^{24}} \sum_{j \in \mathcal{L}} h_j \text{vol}_{d-1}(F_j). \]

By (6) we get that this expression is greater than

\[ \frac{1}{2^{27}} \left(\frac{\text{vol}_{d-1}(\partial P)}{\text{vol}_{d-1}(\partial B_2^d)} \right) \frac{1}{8n} \sum_{j \in \mathcal{L}} \text{vol}_{d-1}(F_j). \]

By (8) this expression is greater than

\[ \frac{1}{2^{29}} \left(\frac{\text{vol}_{d-1}(\partial P)}{\text{vol}_{d-1}(\partial B_2^d)} \right) \frac{1}{8n} \text{vol}_{d-1}(\partial P) \]

\[ \geq \frac{1}{2^{36}} \text{vol}_{d-1}(\partial B_2^d) n^{-\frac{2}{d+1}}. \]

\( \square \)

References

[BI] E.M. Bronshtein and L.D. Ivanov, The approximation of convex sets by polyhedra, Siberian Mathematical Journal 16 (1975), 1110–1112.

[D1] R. Dudley, Metric entropy of some classes of sets with differentiable boundaries, Journal of Approximation Theory 10 (1974), 227–236.

[D2] R. Dudley, Correction to "Metric entropy of some classes of sets with differentiable boundaries"., Journal of Approximation Theory 26 (1978), 192-193.
[F–T] L. FejesToth, Über zwei Maximumsaufgaben bei Polyedern, Tohoku Mathematical Journal 46 (1940), 79–83.

[GMR1] Y. Gordon, M. Meyer, and S. Reisner, Volume approximation of convex bodies by polytopes—a constructive method, Studia Mathematica 111 (1994), 81–95.

[GMR2] Y. Gordon, M. Meyer and S. Reisner, Constructing a polytope to approximate a convex body, Geometriae Dedicata 57 (1995), 217-222.

[Gr1] P.M. Gruber, Volume approximation of convex bodies by inscribed polytopes, Mathematische Annalen 281 (1988), 292–245.

[Gr2] P.M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies II, Forum Mathematicum 5 (1993), 521–538.

[GK] P.M. Gruber and P. Kenderov, Approximation of convex bodies by polytopes, Rend. Circolo Mat. Palermo 31 (1982), 195–225.

[Mac] A.M. Macbeath, An extremal property of the hypersphere, Proceedings of the Cambridge Philosophical Society 47 (1951), 245–247.

[Mü] J.S. Müller, Approximation of the ball by random polytopes, Journal of Approximation Theory 63 (1990), 198–209.

[R] C.A. Rogers, Packing and Covering, Cambridge University Press, 1964.

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