LOCAL STABLE AND UNSTABLE SETS FOR POSITIVE ENTROPY $C^1$ DYNAMICAL SYSTEMS

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ABSTRACT. For any $C^1$ diffeomorphism on a smooth compact Riemannian manifold that admits an ergodic measure with positive entropy, a lower bound of the Hausdorff dimension for the local stable and unstable sets is given in terms of the measure-theoretic entropy and the maximal Lyapunov exponent. The mainline of our approach to this result is under the settings of topological dynamical systems, which is also applicable to infinite dimensional $C^1$ dynamical systems.

1. Introduction

Let $M$ be a compact manifold and let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism for some $\alpha > 0$. The stable manifold theory developed by Pesin among others [10, 17, 25, 26, 32] asserts that, roughly speaking, if $f$ is non-uniformly hyperbolic with respect to an $f$-invariant Borel probability measure $\mu$, then the stable and unstable sets for $\mu$-a.e. $x \in M$ are immersed submanifolds with complementary dimensions. To indicate that the $C^{1+\alpha}$ regularity hypothesis in Pesin’s stable manifold theory is essential, Pugh [29] gave an example of a $C^1$-diffeomorphism which admits an orbit with nonzero Lyapunov exponents but no invariant manifolds.

Pugh’s counter-example is about the non-existence of invariant manifolds for a single orbit of a concrete $C^1$ diffeomorphism. More recently, Bonatti, Crovisier and Shinohara [3] showed that non-existence of invariant manifolds is a generic phenomenon in the $C^1$ category. More precisely, for a diffeomorphism $f : M \to M$ on some Riemannian manifold $M$, let $d$ be the metric on $M$ induced by its Riemannian structure. Then for $x \in M$, the stable set of $x$ for $f$ is defined by

$$W^s(x, f) := \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}.$$ 

The unstable set $W^u(x, f)$ is simply defined as the stable set of $x$ for $f^{-1}$. In [3, Theorem 2] the authors proved the following statement.

**Theorem.** Let $M$ be a smooth compact manifold with $\dim M \geq 3$ and let $\text{Diff}^1(M)$ be endowed with the $C^1$-topology. Then there exists a non-empty open set $\mathcal{U} \subset \text{Diff}^1(M)$ and a dense $G_\delta$ subset $\mathcal{R} \subset \mathcal{U}$ satisfying the following property: each $f \in \mathcal{R}$ admits a hyperbolic and ergodic Borel probability measure $\mu$ such that

$$W^s(x, f) = W^u(x, f) = \{x\}$$

for every point $x$ in the support $\text{supp}(\mu)$ of $\mu$.

According to [3, Remark 3], for each $f \in \mathcal{R}$ in the above Theorem, the dynamics of $f$ on $\text{supp}(\mu)$ is a generalized adding machine (also called odometer or solenoid, see for example [5] or [22] for the definition); therefore, it is uniquely ergodic and the measure-theoretic entropy $h_\mu(f) = 0$. Then a
natural question arises: given \( f \in \text{Diff}^1(M) \) and \( f \)-invariant \( \mu \) with \( h_\mu(f) > 0 \), what can we say about the structure of \( W^s(x, f) \) and \( W^u(x, f) \) for \( \mu \)-a.e. \( x \in M \)?

As a partial answer of the question above, the purpose of this paper is to investigate the Hausdorff dimension of the local stable sets and local unstable sets for \( C^1 \) dynamical systems with positive measure-theoretic entropy (see also problem 6.2 in [8]).

Throughout this paper, by a topological dynamical system \((X, T)\) (TDS for short) we mean a compact metric space \((X, d)\) with a homeomorphism map \( T \) from \( X \) onto itself, where \( d \) refers to the metric on \( X \). For a TDS \((X, T)\), given \( x \in X \) and \( \delta > 0 \), the \( \delta \)-stable set of \( x \) is defined as

\[
W^s_\delta(x, T) = \{ y \in X : d(T^n x, T^n y) \leq \delta, \ \forall n \geq 0 \ \text{and} \ \lim_{n \to +\infty} d(T^n x, T^n y) = 0 \}.
\]

Similarly, the \( \delta \)-unstable set of \( x \) is defined as

\[
W^u_\delta(x, T) = \{ y \in X : d(T^{-n} x, T^{-n} y) \leq \delta, \ \forall n \geq 0 \ \text{and} \ \lim_{n \to +\infty} d(T^{-n} x, T^{-n} y) = 0 \}.
\]

Clearly, by definition, \( W^s_\delta(x, T) = W^u_\delta(x, T^{-1}) \). \( \delta \)-stable(resp. unstable) sets for some \( \delta > 0 \) are collectively called local stable(resp. unstable) sets.

Let \( M \) be a smooth compact Riemannian manifold. The Riemannian structure on \( M \) induces a complete metric \( d \) in the usual way. The Hausdorff dimension for subsets of \( M \) defined by this metric is denoted by \( \dim_H(\cdot) \). Given a \( C^1 \) diffeomorphism \( f \) on \( M \), let \( Df^n(x) \) denote the tangent map of \( f^n \) at \( x \) and \( \|Df^n(x)\| \) the operator norm of \( Df^n(x) \) induced by the Riemannian metric. For an \( f \)-invariant Borel probability measure \( \mu \) on \( M \), let us define the maximal Lyapunov exponent of \( f \) w.r.t. \( \mu \) by

\[
\chi_\mu(f) := \lim_{n \to +\infty} \frac{1}{n} \int_M \log^+ \|Df^n(x)\|d\mu(x),
\]

where \( \log^+ t = \max\{\log t, 0\} \) for \( t > 0 \). Let \( h_\mu(f) \) denote the measure-theoretic entropy of the measure preserving system \((M, f, \mu)\). The main result in this paper is as follows:

**Theorem 1.1.** Let \( M \) be a smooth compact Riemannian manifold and \( f : M \to M \) a \( C^1 \) diffeomorphism. Let \( \mu \) be an ergodic \( f \)-invariant Borel probability measure on \( M \) with positive entropy. Then the following holds for every \( \delta > 0 \):

\[
\dim_H(W^s_\delta(x, f)) \geq \frac{h_\mu(f)}{\chi_\mu(f)} \quad \text{and} \quad \dim_H(W^u_\delta(x, f)) \geq \frac{h_\mu(f)}{\chi_\mu(f^{-1})}, \quad \mu\text{-a.e. } x \in M.
\]  

**Remark.**

1. According to Margulis-Ruelle inequality, the assumption \( h_\mu(f) > 0 \) guarantees that both \( \chi_\mu(f) \) and \( \chi_\mu(f^{-1}) \) are strictly positive.
2. Without further assumption on the system \((M, f, \mu)\), the lower bound estimate of Hausdorff dimensions in (1.4) cannot be improved, even if \( f \) has distinct positive(or negative) Lyapunov exponents. For example, consider the product system \((M_i, f_i, \mu_i)\), \( i = 1, 2 \) as follows. \((M_1, f_1, \mu_1)\) satisfies that \( \mu_1 \) is an ergodic hyperbolic measure for \((M_1, f_1)\) such that

\[
W^u(p, f_1) = W^s(p, f_1) = \{p\}
\]

for each \( p \in \text{supp}(\mu_1) \), whose existence is guaranteed by Theorem 2 in [3] we cited before; in particular, \( h_{\mu_1}(f_1) = 0 \). \((M_2, f_2)\) is a two-dimensional hyperbolic torus automorphism induced by some matrix \( A \in \text{SL}(2, \mathbb{Z}) \) and \( \mu_2 \) is the Lebesgue measure on \( M_2 \). Let \( \lambda > \lambda^{-1} \) be the two eigenvalues of \( A \) and we further require that

\[
\lambda > \max_{p \in \mathbb{N}} \max \{\|Df_1(p)\|, \|Df_1^{-1}(p)\|\},
\]
say $A = \begin{pmatrix} n & n+1 \\ n-1 & n \end{pmatrix}$ for large $n$. Then

$$X_{\mu}(f^2) = X_{\mu_2}(f^2) = h_{\mu_2}(f) = h_{\mu_2}(f_2) = \log A,$$

and $\dim_H(W^{\mu}_n(x,f)) = \dim_H(W^{\mu_2}(x,f)) = 1$ for any $x \in \text{supp}(\mu_1) \times M_2$.

In fact, we shall prove a more general statement for TDS, which contains Theorem 1.1 as a special case. Let $(X, T)$ be a TDS with metric $d$ as before. Given $n \geq 1$ and $x \in X$, the pointwise Lipschitz constant of $T^n$ at $x$ is defined as follows:

$$\mathcal{L}_n(x) := \lim_{r \to 0} \left( \sup_{y \in B(x, r)} \left( \frac{d(T^n x, T^n y)}{d(x, y)} \right) \right) \in [0, +\infty], \quad (1.5)$$

where $B(x, r) := \{ y \in X : d(x, y) < r \}$ denotes the open ball of radius $r$ centered at $x$, and we adopt the convention that $\sup \emptyset = 0$ in (1.5). Then we have:

**Theorem 1.2.** Let $(X, T)$ be a TDS and let $\mu$ be an ergodic $T$-invariant Borel probability measure on $X$ with positive entropy. Suppose that there exists $r > 0$ such that

$$\int_X \log^+ \left( \sup_{y \in B(x, r)} \left( \frac{d(T x, T y)}{d(x, y)} \right) \right) d\mu < \infty. \quad (1.6)$$

Then

$$\chi_{\mu}(T) := \lim_{n \to +\infty} \frac{1}{n} \int_X \log^+ \mathcal{L}_n \, d\mu < \infty \quad (1.7)$$

is well-defined, and for each $\delta > 0$ we have:

$$\dim_H(W^{\mu}_{\delta}(x,T)) \geq h_{\mu}(T) \chi_{\mu}(T), \quad \mu\text{-a.e. } x \in X. \quad (1.8)$$

**Remark.**

1. We call $\chi_{\mu}(T)$ defined by (1.7) the maximal Lyapunov exponent of the measure-preserving system $(X, T, \mu)$, the pointwise version of which turns out to be equivalent to Kifer’s notion of maximal characteristic exponent introduced in [18]. Unlike the situation in Theorem 1.1, $h_{\mu}(T) > 0$ cannot exclude the possibility of $\chi_{\mu}(T) = 0$ (a concrete example is provided in § 4.3) in general, and once this happens, (1.8) simply means that $\dim_H(W^{\mu}_{\delta}(x,T)) = \infty$.

2. If the assumption on $T$ in (1.6) is replaced by $T^{-1}$, then applying Theorem 1.2 to $T^{-1}$ we conclude that

$$\dim_H(W^{\mu}_{\delta}(x,T)) \geq h_{\mu}(T^{-1}) \chi_{\mu}(T^{-1}) = \frac{h_{\mu}(T)}{\chi_{\mu}(T^{-1})}, \quad \mu\text{-a.e. } x \in X. \quad (1.9)$$

Our main result Theorem 1.1 follows from Theorem 1.2 immediately: for any $C^1$ diffeomorphism $f$ on a Riemannian manifold $M$, $f^n$ is automatically a Lipschitz map for each $n \geq 1$ and it is evident that as a continuous function, the pointwise Lipschitz constant $\mathcal{L}_n(x)$ coincides with $\|Df^n(x)\|$, and hence (1.7) coincides with (1.4) for $f = T$. Therefore, in the rest of this paper we only need to prove Theorem 1.2.

The key ingredient in our approach to Theorem 1.2 is to construct a measurable partition subordinate to local stable/unstable sets and to establish local entropy formula for disintegration over such a partition. This intermediate step can be stated as the following theorem, where the notions on measurable partition and disintegration will be specified in § 2.2 and the notations appearing in (1.10) will be explained in Definition 2.1.
Theorem 1.3. Let \((X,T)\) be a TDS and let \(\mu\) be an ergodic \(T\)-invariant Borel probability with \(h_\mu(T) > 0\). Then for any \(\delta > 0\), there exists a measurable partition \(\xi\) of \(X\) with the following properties.

1. \(\xi(x) \subseteq W^u_\delta(x)\) for each \(x \in X\), where \(\xi(x)\) is the atom of \(\xi\) containing \(x\).
2. Let \(\mu = \int_X \mu \cdot d\mu(x)\) be the measure disintegration of \(\mu\) over \(\xi\). Then for \(\mu\)-a.e. \(x \in X\),

\[
\bar{h}_\mu(T,y) = \bar{h}_\mu(T,y) = h_\mu(T), \quad \mu_x\text{-a.e. } y \in X. \tag{1.10}
\]

Remark. Let us say that the measurable partition appearing in the theorem above is subordinate to local unstable sets of \((X,T)\). Applying the theorem to \(T^{-1}\) instead of \(T\), we can obtain another measurable partition subordinate to local stable sets of \((X,T)\) with analogous properties.

Let us give a partial list for literatures closely related to the main theme of this paper. For results relating dimension with entropy and Lyapunov exponents under \(C^{1+\alpha}\) settings, see [37, 20, 21, 1]. For relevant results under \(C^1\) or even TDS settings, see [2, 33, 14, 9, 15, 16, 35, 36, 13, 8].

The paper is organized as follows. § 2 is a preliminary section in which we review some notions of ergodic theory. In § 3 we construct a measurable partition subordinate to local unstable sets on whose fibres the local entropy concentrates, and prove Theorem 1.3. Based on this construction, in § 4 we complete the proof of Theorem 1.2, and finally apply it to infinite dimensional \(C^1\) dynamical systems.

2. Preliminaries

In this section, we review some notions in ergodic theory that will be used in subsequent sections. We shall restrict ourselves to talking about compact metric space \((X,d)\) and TDS \((X,T)\). For a TDS \((X,T)\), let \(\mathcal{B}_X\) denote its Borel \(\sigma\)-algebra and let

\[\mathcal{E}_T := \{E \in \mathcal{B}_X : T^{-1}E = E\},\]

which is a sub-\(\sigma\)-algebra of \(\mathcal{B}_X\) and a \(\mathcal{B}_X\)-measurable function \(f\) on \(X\) is \(T\)-invariant if it is \(\mathcal{E}_T\)-measurable. Recall that by our assumption, \(T\) is a homeomorphism, so \(\mathcal{E}_T = \mathcal{E}_{T^{-1}}\).

Let \(M(X), M(X,T)\) and \(M^e(X,T)\) denote the collection of Borel measurable probability measures, \(T\)-invariant ones and \(T\)-ergodic ones respectively. Given \(\mu \in M(X)\), let \(\mathcal{B}_\mu\) be the completion of \(\mathcal{B}_X\) with respect to \(\mu\), so that \((X,\mathcal{B}_\mu)\) is a Lebesgue space (see, for example, [6, 7, 12, 23, 28, 30]). Denote the collection of \(\mu\)-integrable functions on \(X\) by \(L^1(\mu)\). For \(g \in L^1(\mu)\) and for any sub-\(\sigma\)-algebra \(\mathcal{C}\) of \(\mathcal{B}_\mu\), let \(\mathbb{E}_\mu(g|\mathcal{C})\) be (a representative of) the conditional expectation of \(g\) with respect to \(\mu\) and \(\mathcal{C}\).

2.1. Ergodic theorems. We shall make use of two well known generalizations of Birkhoff’s ergodic theorem listed below. The first one is a lemma named after Breiman as follows. See, for example [12, Lemma 14.34]), for a proof.

Lemma 2.1 (Breiman’s Lemma). Let \((X,T)\) be a TDS and let \(\mu \in M(X,T)\). Let \((g_n)\) be a sequence of measurable functions on \((X,\mathcal{B}_\mu)\) with the following properties:

- \(\sup_n |g_n| \in L^1(\mu)\);
- there exists a measurable \(g\) such that \(\lim_{n \to +\infty} g_n = g\) \(\mu\)-a.e. (and therefore in \(L^1(\mu)\)).

Then we have:

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_n \circ T^n = \mathbb{E}_\mu(g|\mathcal{E}_T) \quad \mu\text{-a.e. and in } L^1(\mu).
\]
Given a TDS \((X, T)\), a sequence of functions \(\phi_n : X \to [-\infty, +\infty)\) is called subadditive w.r.t. \(T\), if
\[
\phi_{m+n} \leq \phi_m + \phi_n \circ T^m, \quad \forall m, n \geq 1.
\]
The following subadditive ergodic theorem is first proved by Kingman in [19].

**Theorem 2.2** (Kingman’s subadditive ergodic theorem). Let \((X, T)\) be a TDS and let \(\mu \in \mathcal{M}(X, T)\). Let \(\phi_n\) be measurable and subadditive w.r.t. \(T\) and suppose \(\phi_1^+ \in L^1(\mu)\). Then \(\frac{1}{n} \phi_n\) converges \(\mu\)-a.e. to some \(T\)-invariant measurable function \(\psi : X \to [-\infty, +\infty)\), \(\psi^+ \in L^1(\mu)\) and
\[
\int \psi d\mu = \lim_{n \to \infty} \frac{1}{n} \int \phi_n d\mu = \inf_{n \geq 1} \frac{1}{n} \int \phi_n d\mu.
\]
Moreover, if \(\phi_n \geq 0\), then the convergence is also in \(L^1(\mu)\).

### 2.2. Measurable partition and disintegration.

For notions discussed in this subsection, please refer to, for example, [6, 7, 12, 23, 28, 30]. Let \(X\) be a compact metric space. By definition, a (Borel) partition of \(X\) is a collection of Borel sets in \(X\) such that \(X\) can be written as a disjoint union of them. All the partitions appearing in this paper consist of Borel sets, so we omit “Borel” and simply call them partitions. Elements in a partition are called its atoms. Given a partition \(\alpha\) of \(X\) and \(x \in X\), the unique atom in \(\alpha\) that contains \(x\) is denoted by \(\alpha(x)\). For two partitions \(\alpha\) and \(\beta\), we say that \(\beta\) is finer than \(\alpha\) or \(\alpha\) is coarser than \(\beta\), denoted by \(\alpha < \beta\) or \(\beta > \alpha\), if each element in \(\alpha\) is a union of a collection of elements in \(\beta\). This defines a partial order on partitions. For a finite or countable family of partitions \((\alpha_i)_{i \in I}\), denote their common refinement
\[
\bigvee_{i \in I} \alpha_i := \left\{ \bigcap_{i \in I} A_i : A_i \in \alpha_i, \forall i \in I \right\}.
\]
In other words, \(\bigvee_{i \in I} \alpha_i\) is the coarsest partition that is finer than every \(\alpha_i\). The meaning of \(\alpha \lor \beta\) and \(\bigvee_{i=1}^n \alpha_i\) are similar.

A partition \(\xi\) of \(X\) is called measurable, if it is countably generated in the following sense: there exist a sequence of finite partitions \((\alpha_n)_{n \geq 1}\) such that \(\xi = \bigvee_{n=1}^\infty \alpha_n\). Given \(\mu \in \mathcal{M}(X)\), for any measurable partition \(\xi\), there exists an associated smallest \(\mu\)-complete sub-\(\sigma\)-algebra of \(\mathcal{B}_\mu\), denoted by \(\mathcal{C}_\mu\), which contains all the measurable sets that can be written as union of atoms in \(\xi\). In other words,
\[
\mathcal{C}_\mu = \{ A \in \mathcal{B}_\mu : A = \bigcup_{x \in A} \xi(x) \} \mu.
\]
where \(\mathcal{C}_\mu\) denotes the completion of \(\mathcal{C}\) with respect to \(\mu\) for any sub-\(\sigma\)-algebra \(\mathcal{C}\) of \(\mathcal{B}_\mu\). It is well known that for any sub-\(\sigma\)-algebra \(\mathcal{C}\) of \(\mathcal{B}_\mu\), there exists a measurable partition \(\xi\) of \(X\) such that \(\mathcal{C}_\mu = \mathcal{C}_\xi\). For a measurable partition \(\xi\), the conditional expectation \(\mathbb{E}_\mu(\cdot | \xi)\) is also denoted by \(\mathbb{E}_\mu(\cdot | \xi)\). Properties of the disintegration of \(\mu\) over \(\xi\) that will be used in this paper are summarized in the proposition below (see, for example, [7, Theorem 5.14] for a proof).

**Proposition 2.3.** Let \(\xi\) be a measurable partition of \((X, d)\) and let \(\mu \in \mathcal{M}(X)\). Then there exists a Borel set \(X'\) of full measure and a family \(\{\mu_x \in \mathcal{M}(X) : x \in X'\}\) which satisfy the following properties:

1. For every \(f \in L^1(X, \mathcal{B}_X, \mu)\), we have:
   \[
   \mathbb{E}_\mu(f|\xi)(x) := \mathbb{E}_\mu(f|\xi)(x) = \int_X f d\mu_x, \quad \mu\text{-a.e. } x \in X. \tag{2.1}
   \]
2. For every \(x \in X'\), \(\mu_x(X' \cap \xi(x)) = 1\), and for \(x, y \in X'\), \(\xi(x) = \xi(y)\) implies that \(\mu_x = \mu_y\).
2.3. Measure theoretic entropy. We summarize some basic concepts and useful properties related to measure-theoretic entropy here. See, for example, \cite{12, 23, 24, 28, 34} for reference.

Let $X$ be a compact metric space and let $\mathcal{P}_X$ denote the collection of all its finite (Borel) partitions. Given $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and a sub-$\sigma$-algebra $\mathcal{C} \subset \mathcal{B}_\mu$, the conditional information function and the conditional entropy of $\alpha$ with respect to $\mathcal{C}$ are defined by:

$$I_\mu(\alpha|\mathcal{C})(x) := - \sum_{A \in \alpha} 1_A(x) \log \mathbb{E}_\mu(1_A|\mathcal{C})(x)$$

and

$$H_\mu(\alpha|\mathcal{C}) := \int_X I_\mu(\alpha|\mathcal{C})d\mu = \sum_{A \in \alpha} \int_X -\mathbb{E}_\mu(1_A|\mathcal{C}) \log \mathbb{E}_\mu(1_A|\mathcal{C})d\mu,$$

where, as mentioned before, $\mathbb{E}_\mu(1_A|\mathcal{C})$ is the conditional expectation of $1_A$ (the indicator function of $A$) with respect to $\mathcal{C}$. One basic fact states that $H_\mu(\alpha|\mathcal{C})$ increases in $\alpha$ and decreases in $\mathcal{C}$.

Now let $(X, T)$ be a TDS. Given a partition $\alpha$ of $X$ and integers $m < n$, denote

$$\alpha_m^n := \bigvee_{k=m}^n T^{-k}\alpha, \quad \text{and} \quad \alpha_1^\infty := \bigvee_{k=1}^{+\infty} T^{-k}\alpha, \quad \alpha^{-1}_\infty := \bigvee_{k=1}^{-\infty} T^k\alpha.$$ When $\mu \in \mathcal{M}(X, T)$ and $\mathcal{C}$ is a $T$-invariant (i.e. $T^{-1}\mathcal{C} = \mathcal{C}$) sub-$\sigma$-algebra of $\mathcal{B}_\mu$, it is not hard to see that $H_\mu(\alpha_0^{n-1}|\mathcal{C})$ is a non-negative and subadditive sequence for any given $\alpha \in \mathcal{P}_X$, so

$$h_\mu(T, \alpha|\mathcal{C}) := \lim_{n \to +\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\mathcal{C}) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\mathcal{C})$$

is well defined.

For the trivial sub-$\sigma$-algebra $\mathcal{C} = \{\emptyset, X\}$, we denote $H_\mu(\alpha|\mathcal{C})$ and $h_\mu(T, \alpha|\mathcal{C})$ by $H_\mu(\alpha)$ and $h_\mu(T, \alpha)$ respectively. The measure-theoretic entropy of $\mu$ is defined by

$$h_\mu(T) = \sup_{\alpha \in \mathcal{P}_X} h_\mu(T, \alpha).$$

A basic and useful estimate for upper bound of entropy is:

$$h_\mu(T, \alpha|\mathcal{C}) \leq H_\mu(\alpha|\mathcal{C})(\text{or } h_\mu(T, \alpha)) \leq H_\mu(\alpha) \leq \log \#\alpha .$$

The following elementary properties of information functions are well known (see, for example \cite{12, 24}).

**Lemma 2.4.** Let $(X, T)$ be a TDS and let $\mu \in \mathcal{M}(X, T)$. Then we have:

1. (The information cocycle equation) For any $\alpha, \beta \in \mathcal{P}_X$ and any sub-$\sigma$-algebra $\mathcal{C} \subset \mathcal{B}_\mu$,

$$I_\mu(\alpha \lor \beta|\mathcal{C}) = I_\mu(\alpha|\mathcal{C}) + I_\mu(\beta|\mathcal{C} \lor \mathcal{C}).$$

2. For any $\alpha \in \mathcal{P}_X$ and any sub-$\sigma$-algebra $\mathcal{C} \subset \mathcal{B}_\mu$,

$$I_\mu(T^{-1}\alpha|T^{-1}\mathcal{C}) = I_\mu(\alpha|\mathcal{C}) \circ T.$$

Let $\mathcal{F}_n (n \geq 1)$ and $\mathcal{F}$ be sub-$\sigma$-algebras of $\mathcal{B}_\mu$. Denote

- $\mathcal{F}_n \nearrow \mathcal{F}$, if $\mathcal{F}_n$ is increasing in $n$ and $\mathcal{F}$ is the smallest $\sigma$-algebra containing each $\mathcal{F}_n$;
- $\mathcal{F}_n \searrow \mathcal{F}$, if $\mathcal{F}_n$ is decreasing in $n$ and $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$.

The following martingale-type properties of information functions are also well known(\cite{12, 24}).

**Proposition 2.5.** Let $(X, T)$ be a TDS and let $\mu \in \mathcal{M}(X, T)$. Let $\mathcal{F}$, $\mathcal{F}_n (n \geq 1)$ be sub-$\sigma$-algebras of $\mathcal{B}_\mu$ such that either $\mathcal{F}_n \nearrow \mathcal{F}$ or $\mathcal{F}_n \searrow \mathcal{F}$ holds. Then for any $\alpha \in \mathcal{P}_X$, we have:
(1) Chung’s Lemma:

\[ f := \sup_{n \geq 1} I_{\mu}(\alpha | F_n) \in L^1(\mu) \quad \text{and} \quad \int_X f \, d\mu \leq H_{\mu}(\alpha) + 1. \]

(2) Martingale Theorem:

\[ \lim_{n \to \infty} I_{\mu}(\alpha | F_n) = I_{\mu}(\alpha | F) \quad \mu \text{-a.e. and in } L^1(\mu). \]

The following is also a classical result (see for example [12, Lemma 18.2]).

Lemma 2.6. Let \((X, T)\) be a TDS and let \(\mu \in M(X, T)\). If \(\alpha, \tilde{\alpha},\beta \in \mathcal{P}^\infty \) with \(\alpha \prec \tilde{\alpha}\), then

\[ \lim_{n \to +\infty} H_{\mu}(\alpha | \tilde{\alpha} - \vee (T^n \beta)) = H_{\mu}(\alpha | \tilde{\alpha}). \]

2.4. Local entropy. Let us recall measure-theoretic local entropy for general Borel probability measures introduced by Feng and Huang in [11], which originates from the concept of Brin and Katok’s local entropy for invariant measures in [4].

Let \((X, T)\) be a TDS. For \(n \in \mathbb{N}, x \in X \) and \(r > 0\), let

\[ B_n(x, r, T) := \{ y \in X : \max_{0 \leq k \leq n-1} d(T^k x, T^k y) < r \} \]

be the open Bowen ball centered at \(x\) of step-length \(n\) and radius \(r\). We also denote it by \(B_n(x, r)\) for convenience when there is no risk of confusion for \(T\).

Following Feng and Huang [11], we introduce:

Definition 2.1. Given a TDS \((X, T)\) and \(\mu \in M(X)\), the measure-theoretic lower and upper entropy of \(\mu\) are defined respectively by

\[ h_{\mu}(T) := \int h_{\mu}(T, x)d\mu(x), \quad \bar{h}_{\mu}(T) := \int \bar{h}_{\mu}(T, x)d\mu(x), \]

where

\[ h_{\mu}(T, x) := \lim_{\epsilon \to 0} \lim_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon, T))}{n}, \]

\[ \bar{h}_{\mu}(T, x) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon, T))}{n}. \]

Recall that Brin and Katok [4] proved that for any \(T\)-invariant Borel probability measure \(\mu, h_{\mu}(T, x) = \bar{h}_{\mu}(T, x)\) for \(\mu\)-a.e. \(x \in X\) is a \(T\)-invariant function, and

\[ h_{\mu}(T) = \bar{h}_{\mu}(T) = h_{\mu}(T). \]

In particular, when \(\mu \in M^c(X, T)\), \(h_{\mu}(T, x) = \bar{h}_{\mu}(T, x) = h_{\mu}(T)\) for \(\mu\)-a.e. \(x \in X\).

3. Measurable partition subordinate to local stable/unstable sets

This section is denoted to the proof of Theorem 1.3 and we divide it into three parts. In § 3.1, we present a local version of Shannon-McMillan-Breiman theorem for later use; in § 3.2, we construct \(\xi\) explicitly and assertion (1) follows consequently; in § 3.3 we prove assertion (2).
3.1. A local version of Shannon-McMillan-Breiman theorem. The following statement can be seen as a local (conditional) version of the celebrated Shannon-McMillan-Breiman theorem.

**Proposition 3.1.** Let \((X, T)\) be a TDS and let \(\mu \in \mathcal{M}(X, T)\). Let \(\xi \subset \mathcal{B}_X\) be a measurable partition with \(T\xi \prec \xi\) and let \(\alpha \subset \mathcal{B}_X\) be a finite partition that satisfies \(\alpha^- \prec \xi\). Denote

\[
\gamma_n := \alpha^- \vee T^n \xi, \ \forall \ n \geq 0 \quad \text{and} \quad \mathcal{F} := \bigcap_{n=0}^{\infty} \gamma_n.
\]

Then we have:

\[
\lim_{N \to +\infty} \frac{1}{N} I_{\mu}(\alpha_0^{N-1}|\xi) = \mathbb{E}_\mu(I_{\mu}(\alpha|\mathcal{F})|\xi_T) \quad \mu\text{-a.e. and in } L^1(\mu).
\]

**Proof.** Define

\[
g_n = I_{\mu}(\alpha|\gamma_n), \ \forall \ n \geq 0 \quad \text{and} \quad g = I_{\mu}(\alpha|\mathcal{F}).
\]

From \(T\xi \prec \xi\) we know that \(\gamma_n \searrow \mathcal{F}\). Thus by Chung’s Lemma (see Proposition 2.5 (1)),

\[
\int_X \sup_{n \geq 0} g_n d\mu \leq H_{\mu}(\alpha) + 1 < \infty; \tag{3.1}
\]

and by decreasing Martingale Theorem (see Proposition 2.5 (2)),

\[
\lim_{n \to +\infty} g_n = g, \quad \mu\text{-a.e. and in } L^1(\mu). \tag{3.2}
\]

(3.1) and (3.2) enable us to apply Breiman’s Lemma (see Lemma 2.1) to \(g_n\) and \(g\), which implies that

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_n \circ T^n = \mathbb{E}_\mu(g|\xi_T), \quad \mu\text{-a.e. and in } L^1(\mu). \tag{3.3}
\]

On the other hand, applying Lemma 2.4 (1) repeatedly yields that

\[
I_{\mu}(\alpha_0^{N-1}|\xi) = I_{\mu}(\alpha|\xi) + I_{\mu}(\alpha_1^{N-1}|\alpha \vee \xi) = \cdots = I_{\mu}(\alpha|\xi) + \sum_{n=1}^{N-1} I_{\mu}(T^{-n}\alpha|\alpha_0^{n-1} \vee \xi).
\]

Then by Lemma 2.4 (2), and noting that \(\alpha^- \prec \xi\) implies \(\gamma_n = \alpha_{-n}^- \vee T^n \xi\) for every \(n \geq 1\), we obtain that

\[
I_{\mu}(\alpha_0^{N-1}|\xi) = I_{\mu}(\alpha|\xi) + \sum_{n=1}^{N-1} I_{\mu}(\alpha|\alpha_{-n}^- \vee T^n \xi) \circ T^n = \sum_{n=0}^{N-1} g_n \circ T^n.
\]

Combining this with (3.3), the conclusion follows. \(\square\)

**Remark.** In the statement of Proposition 3.1, if we assume that \(T\xi > \xi\) and \(\alpha^- > \xi\) instead, then by replacing \(\mathcal{F}\) with \(\alpha^-\), the same conclusion holds and the proof is analogous. In this case, we can recover the classical Shannon-McMillan-Breiman theorem simply by taking \(\xi = \{X\}\) to be the trivial partition.

As a direct corollary of Proposition 3.1, we have:

**Corollary 3.2.** Under the settings in Proposition 3.1, let us further assume that \(\mu\) is ergodic. Let \(\mu_x\) be the disintegration of \(\mu\) over \(\xi\) given in Proposition 2.3. Then there exists \(X_0 \in \mathcal{B}_X\) with \(\mu(X_0) = 1\) such that for

\[
\Xi_x := \xi(x) \cap X_0, \ \forall \ x \in X_0 \quad \text{and} \quad \alpha_N := \alpha_0^{N-1}, \ \forall \ N \geq 1,
\]

we have:
\[ \mu_x(\Xi_x) = 1 \quad \text{and} \quad \lim_{N \to +\infty} -\frac{\log \mu_x(\alpha_N(y))}{N} = H_\mu(\alpha|\mathcal{F}), \quad \forall \, x \in X_0, \forall \, y \in \Xi_x. \quad (3.4) \]

**Proof.** Note that by definition,

\[ -\log \mu_x(\alpha_N(x)) = I_\mu(\alpha_N|x)(x), \quad \mu\text{-a.e. } x \in X. \]

Also note that \( E_\mu(I_\mu(\alpha_N|\mathcal{F}|\mathcal{F}_T) = H_\mu(\alpha|\mathcal{F}) \) holds \( \mu\text{-a.e.} \) since \( \mu \) is assumed to be ergodic. Then it follows from Proposition 3.1 that there exists \( X_1 \in \mathcal{B}_X \) with \( \mu(X_1) = 1 \) such that

\[ \lim_{N \to +\infty} -\frac{\log \mu_x(\alpha_N(y))}{N} = H_\mu(\alpha|\mathcal{F}), \quad \forall \, y \in X_1. \]

On the other hand, by Proposition 2.3 (2), there exists \( X_2 \in \mathcal{B}_X \) with \( \mu(X_2) = 1 \) such that for any \( x \in X_2, \mu_x(\xi(x) \cap X_2) = 1 \) and for any \( x, y \in X_2, \xi(x) = \xi(y) \) implies that \( \mu_x = \mu_y \). As a result, the following holds for any \( N \geq 1 \):

\[ \mu_x(\alpha_N(y)) = \mu_y(\alpha_N(y)), \quad \forall \, x, y \in X_2 \quad \text{with} \quad \xi(x) = \xi(y). \]

Then for \( X_0 = X_1 \cap X_2 \), we have:

\[ \lim_{N \to +\infty} -\frac{\log \mu_x(\alpha_N(y))}{N} = H_\mu(\alpha|\mathcal{F}), \quad \forall \, x \in X_0, \forall \, y \in \xi(x) \cap X_0, \]

which completes the proof. \( \square \)

### 3.2. Construction of \( \xi \)

Let \((X, T), \mu \) and \( \delta > 0 \) be as given in the statement of Theorem 1.3. The following construction of \( \xi \) subordinate to the \( \delta \)-unstable sets is based on the construction in [31] or [23, § 6-2].

Recall that for a measurable partition \( \alpha \) of \( X, \alpha^- := \bigvee_{n=1}^{\infty} T^n\alpha \). Denote \( \partial \alpha := \bigcup_{A \in \alpha} \partial A \). Let \( \{\beta_i\}_{i=1}^{\infty} \) be a family of finite Borel partitions of \( X \) with the following properties:

- \( \beta_i \) is finer and finer, i.e. \( \beta_1 < \beta_2 < \beta_3 < \cdots \);
- \( \text{diam}(\beta_i) \leq \delta, \text{diam}(T\beta_i) \leq \delta \) and \( \lim_{i \to +\infty} \text{diam}(\beta_i) = 0 \);
- \( \mu(\partial \beta_i) = 0, \forall \, i \in \mathbb{N} \).

Starting from \( k_1 = 0 \), we can find a sequence of nonnegative integers \( k_1, k_2, \cdots \) such that for \( \alpha_q = \bigvee_{p=1}^{q} T^{k_p}\beta_p \), the following holds for every \( q \geq 2 \):

\[ H_\mu(\alpha_p|\alpha_{q-1}) - H_\mu(\alpha_p|\alpha_q) < \frac{1}{p \cdot 2^{q-p}}, \quad p = 1, 2, \cdots, q - 1. \quad (3.5) \]

The choice of such \( \alpha_q \) can be determined inductively on \( q \) as follows. Once \( k_p \) has been chosen for \( 1 \leq p < q \), applying Lemma 2.6 to \( \alpha = \alpha_p, \widetilde{\alpha} = \alpha_{q-1} \) and \( \beta = \beta_q \) for each \( 1 \leq p \leq q - 1 \), we can find \( k_q \) such that (3.5) holds.

Then we can specify our choice of \( \xi \):

\[ \mathcal{P} := \bigvee_{p=1}^{\infty} \alpha_p \quad \text{and} \quad \xi := \mathcal{P}^- . \]

Given \( p, m \in \mathbb{N} \), taking sum on both sides of (3.5) for \( q = p + 1, \cdots, p + m \) yields that

\[ H_\mu(\alpha_p|\alpha_{p+m}) - H_\mu(\alpha_p|\alpha_{p+m+1}) < \frac{1}{p}. \]
Let \( m \to +\infty \), one has
\[
H_\mu(\alpha_p|\xi) \geq H_\mu(\alpha_p|\alpha_p^-) - \frac{1}{p} = h_\mu(T, \alpha_p) - \frac{1}{p}.
\]
Define
\[
c_p = H_\mu(\alpha_p \bigcup_{\gamma_{p,n}}^{+\infty}), \quad \text{where} \quad \gamma_{p,n} := \alpha_p^- \vee T^n \xi.
\]
Since \( \alpha_p^- < \gamma_{p,n} < \xi \) for each \( n \), \( H_\mu(\alpha_p|\xi) \leq c_p \leq H_\mu(\alpha_p|\alpha_p^-) \), and hence
\[
h_\mu(T, \alpha_p) - \frac{1}{p} \leq c_p \leq h_\mu(T, \alpha_p).
\]
Then
\[
\sup_{p \geq 1} c_p = \sup_{p \geq 1} h_\mu(T, \alpha_p) = h_\mu(T).
\]

Next let us show that \( \bar{\xi}(x) \subset W_\delta^n(x, T) \) for each \( x \in X \). Given \( x \in X \), note that
\[
\bar{\xi}(x) \subset (T\beta_1)(x) \quad \text{and} \quad T^{-i}(\bar{\xi}(x)) = T^{-i}(\xi(x)) = (T^{-i}\xi(T^{-i}x)) \subset \beta_1(T^{-i}x), \quad \forall i \geq 1.
\]
Thus
\[
\text{diam}\left(T^{-i}(\bar{\xi}(x))\right) \leq \max\{\text{diam}(T\beta_1), \text{diam}(\beta_1)\} < \delta, \quad \forall i \geq 0.
\]
For any \( j \geq 1 \) and \( i \geq 0 \), one has
\[
T^{-(k_j+i)}(\bar{\xi}(x)) = T^{-(k_j+i)}(\xi(x)) = (T^{-(k_j+i)}\xi(T^{-(k_j+i)}x)) \subset \beta_j(T^{-(k_j+i)}x).
\]
Thus
\[
\text{diam}\left(T^{-(k_j+i)}(\bar{\xi}(x))\right) \leq \text{diam}(\beta_j), \quad \forall j \geq 1, \quad i \geq 0.
\]
Due to (3.7), (3.8) and \( \lim_{j \to +\infty} \text{diam}(\beta_j) = 0, \bar{\xi}(x) \subset W_\delta^n(x, T) \). The proof of assertion (1) is done.

3.3. **Proof of local entropy formula.** In this subsection we prove assertion (2) in Theorem 1.3. For preparation we need the following result.

**Lemma 3.3.** Let \((X, T)\) be a TDS and let \(\mu \in \mathcal{M}(X)\). Let \(\alpha \) and \(\beta_1 < \cdots < \beta_n < \cdots\) be finite partitions of \(X\). If there exists a constant \(c\) such that
\[
\lim_{n \to +\infty} \frac{-\log \mu((\alpha \vee \beta_n)(x))}{n} = c, \quad \mu\text{-a.e. } x \in X,
\]
then the following holds:
\[
\lim_{n \to +\infty} \frac{-\log \mu(\beta_n(x))}{n} = c, \quad \mu\text{-a.e. } x \in X.
\]
In particular, for any finite partition \(\alpha\) and any \(k \geq 1\), we have:
\[
\lim_{n \to +\infty} \frac{-\log \mu((\alpha \vee \beta_n)(x))}{n} = c, \quad \mu\text{-a.e. } x \in X \implies \lim_{n \to +\infty} \frac{-\log \mu(\beta_n(x))}{n} = c, \quad \mu\text{-a.e. } x \in X.
\]

**Proof.** Noting that
\[
\limsup_{n \to +\infty} \frac{-\log \mu(\beta_n(x))}{n} \leq \limsup_{n \to +\infty} \frac{-\log \mu((\alpha \vee \beta_n)(x))}{n},
\]
it suffices to show that for an arbitrary fixed constant \(\lambda < c\), we have
\[
\mu(E) = 0 \quad \text{for} \quad E := \{x \in X : \mu(\beta_n(x)) \geq e^{-\lambda n}\} \text{ holds for infinitely many } n.
\]
Letting \( m \) for each \( j \) holds for each \( E \) for each \( n \), we have \( \forall T \). We claim that \( E \subset \cup_{n=1}^{\infty} B(n,F) \). We can see this, pick an arbitrary \( x \in E \). Then there exists an infinite set \( S \subset \mathbb{N} \) such that \( \mu(\beta_n(x)) \geq \epsilon^{-bn} \) holds for \( n \in S \). Since \( \#\alpha \) is finite, by pigeonhole principle, there exists \( A \in \alpha \) and an infinite subset \( S' \) of \( S \) such that for each \( n \in S' \), \( A \) and \( B = \beta_n(x) \) satisfy property (a). Property (b) also follows for such a pair of \( A \) and \( B \). By the definition of \( E_A \), \( x \in E_A \), which verifies the claim.

To complete the proof, we only need to show that \( \mu(E_A) = 0 \) for each \( A \in \alpha \). Due to property (b) and the definition of \( E_A \),

\[
A \cap E_A \subset \{ x \in A : \mu((\alpha \cap \beta_n)(x)) \geq 1/\#\alpha \epsilon^{-bn} \text{ holds for infinitely many } n \}.
\]

Then by the assumption in the lemma, \( \mu(A \cap E_A) = 0 \). Note that for each \( m \geq 1 \), \( \cup_{n=1}^{\infty} F(n,A) \) can be written as a disjoint union \( \cup_{B \in C} B \), where \( C = C(m,A) \) is a subset of \( \cup_{n=1}^{\infty} B(n,A) \). On the other hand, for each \( B \in C \), \( \mu(B) \leq \#\alpha \cdot \mu(A \cap B) \). Taking sum over all \( B \in C \) on both sides yields that

\[
\mu(\cup_{n=1}^{\infty} F(n,A)) \leq \#\alpha \cdot \mu(A \cap \cup_{n=1}^{\infty} F(n,A)).
\]

Letting \( m \to \infty \), it follows that \( \mu(E_A) \leq \#\alpha \cdot \mu(A \cap E_A) = 0 \). The conclusion follows. \( \Box \)

Now we are ready to prove assertion (2) in Theorem 1.3. According to Corollary 3.2, the following holds for each \( j \geq 1 \):

\[
-\log \mu_s((\bigvee_{i=0}^{N-1} T^{-i}\alpha_j(y))) = c_j, \quad \mu_s\text{-a.e. } y \in X, \quad \mu\text{-a.e. } x \in X.
\]

Then by Lemma 3.3, it follows that

\[
-\log \mu_s((\bigvee_{i=0}^{N-1} (T^{-k_j}\alpha_j)(y))) = c_j, \quad \mu_s\text{-a.e. } y \in X, \quad \mu\text{-a.e. } x \in X.
\]

(3.9)

Given \( \epsilon > 0 \), we can find \( N = N(\epsilon) \in \mathbb{N} \) such that \( \text{diam}(\beta_j) \leq \epsilon \) for \( j \geq N \), as \( \lim_{i \to \infty} \text{diam}(\beta_i) = 0 \). Note that \( T^{-k_j}\alpha_j > \beta_j \). Hence when \( j \geq N \), for any \( n \geq 1 \) we have \( (\bigvee_{i=0}^{n-1} T^{-k_j}\alpha_j)(y) \subset B_\theta(y,\epsilon,T) \), and consequently

\[
-\log \mu_s(B_\theta(y,\epsilon,T)) \leq -\log \mu_s((\bigvee_{i=0}^{n-1} (T^{-k_j}\alpha_j)(y))), \quad \mu\text{-a.e. } x \in X, \forall y \in X.
\]

Thus for \( \mu\text{-a.e. } x \in X \) and any \( y \in X \), we have:

\[
\overline{h}_{\mu_s}(T,y) = \lim_{\epsilon \to 0} \sup_{n \to \infty} \frac{-\log \mu_s(B_\theta(y,\epsilon,T))}{n} \leq \sup_{j \geq 1} \lim_{n \to \infty} \frac{-\log \mu_s((\bigvee_{i=0}^{n-1} (T^{-k_j}\alpha_j)(y)))}{n}.
\]

(3.10)

Combining (3.9) and (3.10), we obtain that:

\[
\overline{h}_{\mu_s}(T,y) \leq \sup_{j \geq 1} c_j = h_\theta(T), \quad \mu_{\alpha}\text{-a.e. } y \in X, \quad \mu\text{-a.e. } x \in X.
\]
To complete the proof, it remains to show that for $\mu$-a.e. $x \in X$, $h_{\mu_x}(T, y) \geq h_y(T)$ holds for $\mu_x$-a.e. $y \in X$. Since $\sup_{j \geq 1} c_j = h_y(T)$, it suffices to prove that for any $j \in \mathbb{N}$ and $\mu$-a.e. $x \in X$, $h_{\mu_x}(T, y) \geq c_j$ holds for $\mu_x$-a.e. $y \in X$. Due to Corollary 3.2 and the definition of $c_j$ (recall that $c_j = H_y(\alpha_j | \mathcal{F})$ and $\mu(\partial \alpha_j) = 0$), the problem is reduced to verifying the lemma below.

**Lemma 3.4.** Under the settings in Corollary 3.2, assume that $\mu(\partial \alpha) = 0$ additionally. Then for $\mu$-a.e. $x \in X$, $h_{\mu_x}(T, y) \geq H_y(\alpha | \mathcal{F})$ holds for $\mu_x$-a.e. $y \in X$.

The lemma above follows from the argument of Brin and Katok [4] in bounding their lower local entropy from below. For completion, we give a self-contained proof of it in the appendix. This completes the proof of Theorem 1.3.

4. Lower bound of Hausdorff dimension

In this section, we shall complete the proof Theorem 1.2 in § 4.2. For preparation, we need an alternative definition of maximal Lyapunov exponent introduced by Kifer [18] and show the equivalence between two definitions in § 4.1. Finally, we apply Theorem 1.2 to infinitely dimensional $C^1$ systems in § 4.3.

4.1. On maximal Lyapunov exponent. Following [18], define (here we still set $\sup \varnothing = 0$)

$$L^\mu_n(x) := \sup_{y \in B_n(x, r)} \frac{d(T^n x, T^n y)}{d(x, y)} \in [0, +\infty].$$

By continuity of $T$ and openness of $B_n(x, r)$, it is easy to see that $L^\mu_n$ is lower-semicontinuous. Recall $L_n$ defined in (1.5) and note that by continuity, for each $n \geq 1$, $L^\mu_n$ monotonically converges to $L_n$ as $r \downarrow 0$. It follows that $L_n$ is a Borel function, and therefore $\int_X \log^+ L_n d\mu$ in (1.7) is well-defined. Moreover, as observed by Kifer [18], for each $r > 0$, $\log^+ L_n$ is a subadditive sequence of functions in $n$ w.r.t. $T$. To proceed, let us make the following simple observation:

**Lemma 4.1.** Let $(X, T)$ be a TDS and let $\mu \in \mathcal{M}(X, T)$. Let $\phi^m_n \geq 0$ be measurable functions on $X$ with $\phi^1_n \in L^1(\mu)$. Suppose that for each $m$, $\phi^m_n$ is subadditive in $n$ w.r.t. $T$, and for each $n$, $\phi^m_n$ is decreasing in $m$. Then the following two iterated limits converge in both $\mu$-a.e. and $L^1(\mu)$ sense, and the order is exchangeable:

$$\lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{n} \phi^m_n = \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{n} \phi^m_n .$$

**Proof.** Since $0 \leq \phi^m_n \leq \phi^1_n \in L^1(\mu)$, by subadditive ergodic theorem, $\psi^m := \lim_{n \to \infty} \frac{1}{n} \phi^m_n$ converges $\mu$-a.e. and in $L^1(\mu)$; moreover, for each $m$ we have

$$0 \leq \int \psi^m d\mu = \inf_{n \to \infty} \frac{1}{n} \int \phi^m_n d\mu \leq \int \phi^1_n d\mu < \infty .$$

Noting that $0 \leq \psi^m \leq \psi^1 \in L^1(\mu)$ is decreasing in $m$, $\psi := \lim_{m \to \infty} \psi^m$ also converges $\mu$-a.e. and in $L^1(\mu)$. On the other hand, for the same reason, $\varphi_n := \lim_{m \to \infty} \phi^m_n$ converges $\mu$-a.e. and in $L^1(\mu)$. As pointwise limit of subadditive sequences, $\varphi_n \geq 0$ is still subadditive and $\varphi^1 \in L^1(\mu)$, so that $\Psi := \lim_{n \to \infty} \frac{1}{n} \varphi_n$ converges $\mu$-a.e. and in $L^1(\mu)$; moreover, $0 \leq \int \Psi d\mu = \inf_{n \to \infty} \frac{1}{n} \int \varphi_n d\mu < \infty .$

By definition, $\frac{1}{n} \varphi_n \leq \frac{1}{m} \phi^m_n \mu$-a.e. for any $m, n$. First letting $n \to \infty$ and then letting $m \to \infty$ yields that $\Psi \leq \psi \mu$-a.e.. To complete the proof, it suffices to show that $\int \Psi d\mu \geq \int \psi d\mu$. To this end, observe
that
\[ \int \psi d\mu \leq \int \psi^n d\mu \leq \frac{1}{n} \int \phi^m_n d\mu , \quad \forall m, n. \]

First letting \( m \to \infty \) and then letting \( n \to \infty \), the conclusion follows. \( \square \)

As a direct corollary of Lemma 4.1 together with subadditivity of \( \log^+ \mathcal{L}_n \), we have:

**Proposition 4.2.** Let \((X, T)\) be a TDS and let \( \mu \in \mathcal{M}(X, T) \). Assume that \( \log^+ \mathcal{L}_n^r \in L^1(\mu) \) for some \( r > 0 \). Then all the limits below converge in both \( \mu \)-a.e. and \( L^1(\mu) \) sense:

\[ \chi(x) := \lim_{n \to \infty} \frac{1}{n} \log^+ \mathcal{L}_n(x), \quad \Lambda'(x) := \lim_{n \to \infty} \frac{1}{n} \log^+ \mathcal{L}_n'(x), \quad r \in (0, r_\ast]. \]

Moreover, for any sequence \( r_m \searrow 0 \), the following equality also holds \( \mu \)-a.e. and in \( L^1(\mu) \):

\[ \chi(x) = \lim_{m \to \infty} \Lambda'^m(x). \]

**Proof.** Given an arbitrary sequence \( r_m \searrow 0 \) starting with \( r_1 \leq r_\ast \), the family \( \phi^m_n := \log^+ \mathcal{L}_n^m \) satisfy all the assumptions in Lemma 4.1. All the assertions follow. \( \square \)

**Remark.** We call \( \chi(x) \) the maximal Lyapunov exponent at \( x \) provided the limit exists. The proposition guarantees that under assumption (1.6), \( \chi_\mu(T) \) appearing in Theorem 1.2 satisfies \( \chi_\mu(T) = \int_X \chi(x)d\mu < \infty \); in particular, in the ergodic case, \( \chi(x) = \chi_\mu(T) \) for \( \mu \)-a.e. \( x \in X \). Besides, the proposition says that \( \chi(x) \) coincides with the notion of characteristic maximal exponent introduced by Kifer [18].

### 4.2. Proof of Theorem 1.2.

Let us begin with another simple observation:

**Lemma 4.3.** Following the settings and notations in Proposition 4.2, the statement below holds for \( \mu \)-a.e. \( x \in X \) with \( \chi(x) < \infty \). Given \( \lambda > \chi(x) \) and \( \epsilon > 0 \), there exist \( \eta > 0 \) and \( N \geq 1 \) such that \( B(x, \frac{\eta}{\epsilon^n}) \subset B_n(x, \epsilon) \) for any \( n \geq N \).

**Proof.** According to Proposition 4.2, the following holds for \( \mu \)-a.e. \( x \in X \):

\[ \chi(x) = \lim_{m \to \infty} \Lambda'^m(x) < \infty; \]

so given \( \lambda > \chi(x) \) and \( \epsilon > 0 \), there exists \( r \in (0, \epsilon] \) such that \( \Lambda'(x) < \lambda - c \), where \( c := \frac{1}{2} \lambda(\lambda - \chi(x)) > 0 \). It suffices to prove the statement for such an \( x \) and for \( r \) instead of \( \epsilon \). Let \( N_0 \geq 1 \) with \( e^{-cN_0} < r \). By definition of \( \Lambda'(x) \), there exists \( N \geq N_0 \) such that \( \mathcal{L}_n^r(x) < e^{(\lambda - c)n} \) when \( n \geq N \). By continuity, we can find \( \eta \in (0, 1) \) such that \( B(x, \frac{\eta}{e^n}) \subset B_n(x, r) \), so the conclusion holds for \( n = N \). By induction, we may suppose that \( B(x, \frac{\eta}{e^n}) \subset B_n(x, r) \) for some \( n \geq N \). It follows that

\[ y \in B(x, \frac{\eta}{e^{n+1}}) \subset B(x, \frac{\eta}{e^n}) \subset B_n(x, r) \quad \implies \quad d(T^nx, T^n y) \leq \mathcal{L}_n^r(x) \cdot d(x, y) < \frac{e^{(\lambda - c)n}}{\eta} \cdot \frac{\eta}{e^n} < e^{-cN_0} < r. \]

Thus we obtain that \( B(x, \frac{\eta}{e^{n+1}}) \subset B_{n+1}(x, r) \), which completes the induction. \( \square \)

Now we are ready to prove Theorem 1.2. \( \chi_\mu(T) \) has been shown well-defined and finite-valued by Proposition 4.2. For the rest, fixing \( \delta > 0 \), let \( \xi \) be the measurable partition subordinate to \( \delta \)-local unstable sets asserted in Theorem 1.3 (1). Let \( \{\mu_x \in \mathcal{M}(X) : x \in X'\} \) be the disintegration of \( \mu \) over \( \xi \) as stated in Proposition 2.3. Firstly, observe that for each \( x \in X' \), the lower local dimension of \( \mu_x \) below satisfies that

\[ d_{\mu_x}(y) := \liminf_{r \searrow 0} \frac{\log \mu_x(B(y, r))}{\log r} = \liminf_{n \to \infty} \frac{-\log \mu_x(B(y, e^{-n\lambda}))}{n\lambda}, \quad \forall y \in X, \lambda > 0. \]  

(4.1)
Secondly, note that \( \chi(y) = \chi_\mu(T) \) holds for \( \mu \)-a.e. \( y \in X \) because \( \mu \) is assumed to be ergodic. Then Lemma 4.3 implies that there exists \( Y \subset X \) of full measure with the following properties: for each \( y \in Y, \chi(y) = \chi_\mu(T) \); moreover, given \( \lambda > \chi_\mu(T) \) and \( \epsilon > 0 \), there exists \( \eta > 0 \) such that

\[
\liminf_{n \to \infty} \frac{-\log \mu(B(y, \epsilon^{\eta}))}{n} \geq \liminf_{n \to \infty} \frac{-\log \mu(B_{\mu}(y, \epsilon, T))}{n}, \quad \forall x \in X', y \in Y.
\]

Since the left hand side of the above inequality is independent of \( \eta \), taking \( \eta = 1 \) and letting \( \epsilon \to 0^+ \) yields that

\[
\liminf_{n \to \infty} \frac{-\log \mu(B(y, e^{-n\lambda}))}{n} \geq \lim_{\epsilon \to 0^+} \liminf_{n \to \infty} \frac{-\log \mu(B_{\mu}(y, \epsilon, T))}{n} = \mu_{\mu}(T, y), \quad \forall x \in X', y \in Y. \quad (4.2)
\]

Thirdly, combining (4.1) and (4.2) with Theorem 1.3 (2), we conclude that for any \( \lambda > \chi_\mu(T) \), the following holds for \( \mu \)-a.e. \( x \in X \):

\[
d_{\mu}(y) \geq \frac{\mu_{\mu}(T, y)}{\lambda} = \frac{\mu(T)}{\lambda}, \quad \mu_{\mu}\text{-a.e. } y \in \xi(x) \cap Y.
\]

Since \( \mu_{\mu}(\xi(x) \cap Y) = 1 \) for \( \mu \)-a.e. \( x \in X \), from the inequality above and the mass distribution principle (see, for example, [27, 28]), we obtain that

\[
\dim_H(W^u_{\mu}(x)) \geq \dim_H(\xi(x) \cap Y) \geq \frac{\mu(T)}{\lambda}, \quad \mu\text{-a.e. } x \in X.
\]

Since \( \lambda > \chi_\mu(T) \) is arbitrary, the proof of Theorem 1.2 is completed.

4.3. On infinitely dimensional \( C^1 \) systems. As \( M \) is of finite dimension is inessential in our proof of Theorem 1.1 for \( \delta \)-unstable set, an analogous conclusion in infinite dimensional case also holds for \( \delta \)-unstable set. More precisely, let \( B \) be a Banach space, let \( U \subset B \) be an open subset and let \( f : U \to B \) be a \( C^1 \) map. Suppose that \( X \subset U \) is compact and \( f \) maps \( X \) homeomorphically onto itself. Then for \( T := f|_X, (X, T) \) is a TDS, where the metric \( d \) on \( X \) is induced by the norm \( \| \cdot \| \) on \( B \). Given \( x \in X \) and \( r > 0 \), denote open ball centered at \( x \) of radius \( r \) in \( B \) (respectively \( X \)) by \( O(x, r) = \{ y \in B : \| x - y \| < r \}\) (respectively \( B(x, r) = O(x, r) \cap X \)). When \( O(x, r) \subset U \), by mean-value theorem and convexity of \( O(x, r) \),

\[
\mathcal{L}_1^r(x) = \sup_{y \in B(x, r) \cap X} \frac{d(Tx, Ty)}{d(x, y)} \leq \sup_{y \in O(x, r)} \| Df(y) \|.
\]

This fact together with continuity of \( \| Df \| \) on \( U \) and compactness of \( X \) implies that \( \mathcal{L}_1^r \) is bounded from above on \( X \) for some \( r > 0 \), so the integrability condition (1.6) is automatically satisfied for this \((X, T)\). Therefore, we have:

**Theorem 4.4.** Let \( B, f, X \) and \( T \) be as above and let \( \mu \in \mathcal{M}^e(X, T) \) be of positive entropy. Then the following holds for every \( \delta > 0 \):

\[
\dim_H(W^\mu_\delta(x, T)) \geq \frac{\mu(T)}{\chi_\mu(T)}, \quad \mu\text{-a.e. } x \in X.
\]

**Remark.**

1. Evidently, for either finite dimensional or infinite dimensional case, once \( f^{-1} \) exists and is \( C^1 \)-map on an open neighborhood of \( X \), similar lower bound estimate for \( \text{Hausdorff} \) dimension of local stable sets holds.

2. It might be mentioned that for infinite dimensional systems, it could happen that \( h_{\mu}(T) > 0 \) while \( \chi_{\mu}(T) = 0 \). The following is such an example.
Example 4.5. There exist a Hilbert space $\mathbb{H}$, a bounded linear operator $T$ on $\mathbb{H}$ of spectrum radius 1, a compact $T$-invariant $X \subset \mathbb{H}$ and $\mu \in \mathcal{M}(X, T)$ with positive entropy.

Proof. Given a sequence $a = (a_k)_{k \geq 0}$ of positive reals, denote
\[
\mathbb{H}_a := \{(x_n)_{n \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z} \mid \sum_{n \in \mathbb{Z}} a_n |x_n|^2 < +\infty\}.
\]

Then $\mathbb{H}_a$ is a Hilbert space over $\mathbb{R}$ with respect to the inner product below:
\[
\langle x, y \rangle_a := \sum_{n \in \mathbb{Z}} a_n |x_n| y_n \quad \text{for} \quad x = (x_n)_{n \in \mathbb{Z}}, \; y = (y_n)_{n \in \mathbb{Z}} \in \mathbb{H}_a.
\]

Let $\| \cdot \|_a$ denote the norm on $\mathbb{H}_a$ induced by its inner product. Let us say that the sequence $a = (a_k)_{k \geq 0}$ above sub-exponentially decreases to 0, if it satisfies the following two properties:

- $a_k$ is strictly decreasing to 0 as $k \to \infty$;
- there exist $C > 0$ and a sequence $b = (b_k)_{k \geq 0}$ of positive reals with $\lim_{k \to \infty} \frac{1}{k} |\log b_k| = 0$ such that
  \[
  \frac{a_k}{a_l} \leq C b_{k-l}, \quad \forall \; k, l \geq 0.
  \]

From now on suppose that $a = (a_k)_{k \geq 0}$ sub-exponentially decreases to 0. We claim that the left-shift map $T$ acting on $\mathbb{H}_a$ is a bounded linear operator with spectrum radius 1. The linearity of $T$ and the fact that $\|T^k\|_a > 1$ for each $k \geq 1$ are evident. On the other hand, for $x = (x_n)_{n \in \mathbb{Z}} \in \mathbb{H}_a$,
\[
\|T^k x\|^2_a = \sum_{n \in \mathbb{Z}} a_{|n-k|} |x_n|^2 \leq C \cdot \max\{b_0, \cdots, b_k\} \cdot \|x\|^2_a,
\]
which implies that the spectrum radius of $T$ is bounded by 1 from above.

Now further assume that $\sum_{k=0}^{\infty} a_k < \infty$; for example, let $a_k = \frac{1}{k^{1+\varepsilon}}$. Then $X = [0, 1]^{\mathbb{Z}}$ can be identified as a subset of $\mathbb{H}_a$ in the natural way. The norm on $\mathbb{H}_a$ induces a metric on $X$ that makes $X$ a compact metric space, which is compatible with the standard product topology on $X$. Let $\mu = \left(\frac{1}{2} \delta_0 + \frac{1}{2} \delta_1\right)^\mathbb{Z}$. Then $\mu \in \mathcal{M}(X, T)$ with $h_\mu(T) = \log 2$. Since the spectrum radius of $T : \mathbb{H}_a \to \mathbb{H}_a$ is 1, the maximal Lyapunov exponent $\chi_\mu(T)$ of $T : X \to X$ is 0. \qed

Appendix A. Proof of Lemma 3.4

Following the argument of Brin and Katok [4], it suffices to prove the technical statement below, where we denote $c = H_\mu(a|\mathcal{F})$.

Claim. For any $\varepsilon > 0$ sufficiently small, there exist $\tau > 0$ and a measurable subset $I$ of $X$ with $\mu(I) > 1 - \varepsilon^\frac{1}{4}$ for which the following statement holds. For any $x \in I$, we can find a measurable subset $D$ of $X$ such that
\[
\mu_x(D) > 1 - 4\varepsilon^\frac{1}{4}, \quad \text{and} \quad \liminf_{n \to +\infty} \frac{-\log \mu_x(B_n(y, \tau, T))}{n} \geq c - 3(\Delta + \varepsilon), \quad \forall y \in D,
\]
where
\[
\Delta := 2 \sqrt{\varepsilon} \log(1 - \varepsilon) - 2 \sqrt{\varepsilon} \log 2 \sqrt{\varepsilon} - (1 - 2 \sqrt{\varepsilon}) \log(1 - 2 \sqrt{\varepsilon}).
\]

Since the proof is a little bit long, let us provide an outline here briefly. Firstly, we construct the set $I$ explicitly. Secondly, given $x \in I$, the associated set $D$ will be taken with the form $E \setminus \bigcup_{n=0}^{\infty} E_n$; to define $E_n$, we need to introduce a pseudo-metric $\rho_n$ on $a_n$ in advance. Finally we show that $D$ satisfies the desired properties.
Proof of Claim. Let $\epsilon > 0$ be a small number whose upper bound will be specified in (A.6). We may suppose $\#x \geq 2$ because otherwise there is nothing to prove.

For $r > 0$ and $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$ and

$$V_r(x) = \{x \in X : B(x, r) \subseteq x\}.$$

Since $\bigcap_{r > 0} V_r(x) = \partial x$, we have $\lim_{r \searrow 0} \mu(V_r(x)) = \mu(\partial x) = 0$. Then we can choose $\tau > 0$ such that $\mu(V_\tau(x)) < \epsilon$ for any $0 < r \leq \tau$.

For any subset $F$ of $X$, let $1_F$ be its indicator function. For each $n \geq 1$ denote

$$A_n \coloneqq \{y \in X : \frac{1}{k} \sum_{i=0}^{k-1} 1_{V_\tau(y)}(T^iy) < 2\sqrt{\epsilon} \text{ for any } k \geq n\}.$$

Since $\mu(V_\tau(x)) < \epsilon$, there exists $\ell_0 \geq 1$ such that

$$\mu(A_n) > 1 - 2\sqrt{\epsilon}, \quad \forall n \geq \ell_0, \tag{A.3}$$

and here the ergodicity of $\mu$ is not required. To see this, note that by Birkhoff’s ergodic theorem and Egorov’s theorem, $\frac{1}{n} \sum_{i=0}^{n-1} 1_{V_\tau(y)} \circ T^i$ converges to $\mathbb{E}(1_{V_\tau(y)}|\mathcal{F}_T)$ almost uniformly. On the other hand, by Chebyshev’s inequality,

$$\mu(\{y \in X : \mathbb{E}(1_{V_\tau(y)}|\mathcal{F}_T)(y) \geq \sqrt{\epsilon}\}) < \sqrt{\epsilon}.$$

Then (A.3) follows easily.

Define $Q_n \coloneqq \{x \in X : \mu_x(A_n) \geq 1 - 2\epsilon^{\frac{1}{4}}\}$. Then

$$\mu(Q_n) \cdot 2\epsilon^{\frac{1}{4}} \leq \int X \mu_x(A_n^c)d\mu(x) = \mu(A_n^c) < 2\sqrt{\epsilon}, \quad \forall n \geq \ell_0.$$

Thus $\mu(Q_n) > 1 - \epsilon^{\frac{1}{4}}$ for $n \geq \ell_0$.

Recall that we denote $c = H_\mu(\alpha|\mathcal{F})$. By (3.4) in Corollary 3.2 there exists $X_0 \subset X$ with $\mu(X_0) = 1$ such that for $x \in X_0$, $\Xi_x \coloneqq \xi(x) \cap X_0$ and $\alpha_n \coloneqq \bigvee_{i=0}^{n-1} T^{-i}\alpha$, we have:

$$\mu_x(\Xi_x) = 1, \quad \text{and} \quad \lim_{n \to +\infty} \frac{-\log \mu_x(\alpha_n(y))}{n} = c, \quad \forall y \in \Xi_x. \tag{A.4}$$

Let $I = X_0 \cap Q_{\ell_0}$. Then $\mu(I) = \mu(Q_{\ell_0}) > 1 - \epsilon^{\frac{1}{4}}$.

Now fix an arbitrary $x \in I$ and let

$$B_n \coloneqq \{y \in \Xi_x : \frac{-\log \mu_x(\alpha_n(y))}{n} \geq c - \epsilon \text{ for any } k \geq n\}.$$

Then by (A.4) we can find $\ell_1 \geq \ell_0$ such that $\mu_x(B_{\ell_1}) \geq 1 - \epsilon^{\frac{1}{4}}$. Let $E := A_{\ell_1} \cap B_{\ell_1}$. From (A.3) and the definition of $E$ we know that $\mu_x(E) > 1 - 3\epsilon^{\frac{1}{4}}$.

For nonempty $V \subset X$, let $\alpha(V)$ denote the unique atom in $\alpha$ that contains $V$ once it makes sense. For any $n \geq 1$ and any $V \in \alpha_n$, $\alpha(T^kV)$ is well defined for each $0 \leq k < n$. Let $\rho_n$ be a pseudo-metric\(^1\) on $\alpha_n$ defined by

$$\rho_n(V, W) \coloneqq \frac{1}{n} \cdot \#\{0 \leq i < n : \alpha(T^iV) \neq \alpha(T^iW)\}.$$

\(^1\)More precisely, $\rho_n(V, W) = 0$ may not imply $V = W$, and except for this, $\rho_n$ satisfies all the other axioms of a metric.
Observe that if $z \in B_n(y, \tau, T)$, then for any $0 \leq i < n$, either $T^iy$ and $T^iz$ belong to the same element of $\alpha$ or $T^iy \in V_0(\alpha)$. Hence when $y \in E$ and $n \geq \ell_1$, 
$$z \in B_n(y, \tau, T) \implies \rho_n(\alpha_n(y), \alpha_n(z)) < 2\sqrt{\epsilon}.$$ 
In other words, if we denote the $2\sqrt{\epsilon}$-“open ball” of $W$ under $\rho_n$ by $\mathcal{V}_n(W)$, i.e.
$$\mathcal{V}_n(W) := \{V \in \alpha_n \mid \rho_n(V, W) < 2\sqrt{\epsilon}\}$$
for each $W \in \alpha_n$, then
$$B_n(y, \tau, T) \subset \bigcup_{V \in \mathcal{V}_n(\alpha_n(y))} V, \quad \forall y \in E, \ n \geq \ell_1.$$ 
On the other hand, by the definition of $\mathcal{V}_n(W)$, we have:
$$\#\mathcal{V}_n(W) \leq \sum_{i=0}^{m} \binom{n}{i}(\delta/1 - 1)^i < m \cdot \bigg(\frac{n}{m}\bigg)^m, \quad \text{for} \ m = \lfloor 2n \sqrt{\epsilon} \rfloor.$$ 
Then Stirling’s formula implies that (see for example [17, Page 144]) for $\epsilon > 0$ small enough, there exists $\ell_2 \geq 1$ such that the following holds:
$$\#\mathcal{V}_n(W) \leq \exp((\Delta + \epsilon)n), \quad \forall n \geq \ell_2,$$ 
where $\Delta$ is defined in (A.2).

We shall choose $D$ appearing (A.1) as a subset of $E$ by removing some “bad parts” $E_n$ for large $n$ that will be specified below. To this end, let
$$\mathcal{F}_n := \{V \in \alpha_n : \mu_\delta(V) > \exp((-c + 2(\Delta + \epsilon)n)\},$$
and let
$$E_n := \bigcup_{V \in \mathcal{F}_n} \{y \in E : \rho_n(\alpha_n(y), V) < 2\sqrt{\epsilon}\}.$$ 
By the definition of $\mathcal{F}_n$ and noting that $\mu_\delta(X) = 1$, we have:
$$\#\mathcal{F}_n \leq \exp(c - 2(\Delta + \epsilon)n), \quad \forall n \geq 1.$$ 
To estimate the size of $E_n$, for each $n \geq 1$, let
$$\mathcal{G}_n := \bigcup_{W \in \mathcal{F}_n} \{V \in \mathcal{V}_n(W) : \mu_\delta(V) < \exp((-c + \epsilon)n)\} \subset \bigcup_{W \in \mathcal{F}_n} \mathcal{V}_n(W).$$ 
It follows that
$$\#\mathcal{G}_n \leq \exp((\Delta + \epsilon)n) \cdot \#\mathcal{F}_n \leq \exp((c - (\Delta + \epsilon)n), \quad \forall n \geq \ell_2.$$ 
By the definition of $E_n$ and noting that for any $y \in E$, $\mu_\delta(\alpha_n(y)) < \exp((-c + \epsilon)n)$ holds for $n \geq \ell_1$, we obtain that
$$E_n \subset \bigcup_{W \in \mathcal{G}_n} W \implies \mu_\delta(E_n) \leq \exp((-c + \epsilon)n) \cdot \#\mathcal{G}_n \leq \exp(-\Delta n), \quad \forall n \geq \ell_3,$$ 
where $\ell_3 = \max(\ell_1, \ell_2)$. Then there exists $\ell \geq \ell_3$ such that $\sum_{n=\ell}^{\infty} \mu_\delta(E_n) < \epsilon^\frac{1}{2}$. Let $D = E \setminus \bigcup_{n=\ell}^{\infty} E_n$. Then $\mu_\delta(D) > 1 - 4\epsilon^\frac{1}{2}$. Given $y \in D$ and $n \geq \ell$, since $y \in E \setminus E_n$, it is clear that for each $V \in \alpha_n$ with $\rho_n(V, \alpha_n(y)) < 2\sqrt{\epsilon}$, one has
$$\mu_\delta(V) \leq \exp((-c + 2(\Delta + \epsilon)n).$$ 
Moreover combining this with (A.5) and (A.6) we have
$$\mu_\delta(B_n(y, \tau, T)) \leq \exp((\Delta + \epsilon)n) \cdot \exp((-c + 2(\Delta + \epsilon)n)$$
$$= \exp((-c + 3(\Delta + \epsilon)n).$$
Thus for any $y \in D$ and $n \geq \ell$,
\[
- \log \mu(B_n(y, \tau, T)) \geq c - 3(\Delta + \epsilon)
\]
which completes the proof. \qed

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