The odd-even invariant
and
Hamiltonian circuits in tope graphs
by
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Abstract. In this paper we consider the question of the existence of Hamiltonian circuits in the tope graphs of central arrangements of hyperplanes. Some of the results describe connections between the existence of Hamiltonian circuits in the arrangement and the odd-even invariant of the arrangement. In conjunction with this, we present some results concerning bounds on the odd-even invariant. The results given here can be formulated more generally for oriented matroids and are still valid in that setting.

1. Introduction. An arrangement $\mathcal{A}$ of hyperplanes $H_i, i = 1, \ldots, n$, in $\mathbb{R}^d$ determines a partition of $\mathbb{R}^d$ into relatively open convex polyhedra. Following [2], we call the $d$-dimensional cells of this partition the topes of $\mathcal{A}$, and denote this set by $\mathcal{T}(\mathcal{A})$. In other words, the topes of the arrangement are the connected components of the complement of the union $\bigcup_i H_i$. The tope graph of $\mathcal{A}$ is the graph whose vertex set is $\mathcal{T}(\mathcal{A})$, with two topes (vertices) being adjacent if the intersection of their closures is a facet of each. The tope graph is always a connected bipartite graph. There is thus a coloring of the topes by two colors – say, burnt umber and chartreuse – such that no two adjacent topes receive the same color, and this coloring is unique up to reversing the colors of all the topes. The question of which pairs $(b, c)$ are possible for given $n$ and $d$, where $b$ and $c$ are, respectively, the number of burnt umber and chartreuse topes, has been studied since at least the 1970s. See, for example, [9, 10, 12, 13, 16, 17].

It is easy to see that if the tope graph possesses a Hamiltonian circuit, then $|b - c|$ (the odd-even invariant) is 0. This paper was largely motivated by a question of Vic Reiner [14], who asked if the converse holds. This question is studied here for central arrangements, that is, arrangements in which all the hyperplanes contain the origin. However the paper also includes results on bounds for the odd-even invariant concerning the broader
class of (not necessarily central) arrangements. In Section 2 we give the relevant definitions and background as well as several examples. In Section 3 we discuss the specific case of alternating arrangements and Hamiltonian circuits, and in Section 4, we establish the existence of arrangements having large odd-even invariants. In Section 5, Theorem 8 establishes the existence of arbitrarily large central arrangements in \( \mathbb{R}^d \) having odd-even invariant 0 and having no Hamiltonian circuit. Theorem 10 yields the fact that when \( d \) is odd, the limit superior as \( n \) goes to infinity of \( \frac{b}{c} \) for simple arrangements (arrangements in affine general position) of \( n \) hyperplanes in \( \mathbb{R}^d \) is 1. This settles a question mentioned by Grünbaum in [9] and studied extensively when \( d = 3 \) by Purdy and Wetzel in [13]. Theorem 11 gives a criterion to verify that certain tope graphs do not possess perfect matchings.

Although the paper technically answers Reiner’s question, it represents only a partial answer to the issue raised of the relationship between the odd-even invariant and the existence of Hamiltonian circuits in the tope graph. The paper concludes in Section 6 with several questions of interest.

The results are presented in the setting of hyperplane arrangements in \( \mathbb{R}^d \); however, the results obtain also for oriented matroids and the arguments can easily be adapted to the more general setting by appeal to the topological representation theorem for oriented matroids. See [2] for a statement of this theorem.

2. Background and examples. Let \( \mathcal{A} \) denote an arrangement of hyperplanes in \( \mathbb{R}^d \); that is, \( \mathcal{A} \) is a finite, indexed collection of \( (d - 1) \)-dimensional affine subspaces of \( \mathbb{R}^d \). If \( n \) denotes the number of hyperplanes, where \( n \geq 0 \), we may write \( \mathcal{A} = \{H_1, H_2, \ldots, H_n\} \).

If the hyperplanes \( H_i \) are linear subspaces the arrangement is said to be central. Further, an arrangement of hyperplanes in \( \mathbb{R}^d \) is simple if for \( 0 \leq k \leq d \) the intersection of each set of \( k \) of the hyperplanes of the arrangement has dimension \( d - k \), and the intersection of any \( d + 1 \) or more of the hyperplanes is empty. Equivalently, the hyperplanes of the arrangement are in affine general position. A central arrangement in \( \mathbb{R}^d \) having more than \( d \) hyperplanes cannot be simple. A central arrangement of hyperplanes in \( \mathbb{R}^d \) is centrally simple provided that for \( 0 \leq k \leq d \) the intersection of each \( k \) of the hyperplanes has dimension \( d - k \). Equivalently, the hyperplanes of the arrangement are in linear general position.

The rank of the central arrangement \( \mathcal{A} \) is the difference \( d - w \), where
\( w \) is the dimension of the linear space that is the intersection of the \( n \) hyperplanes. We will usually assume that \( w = 0 \) (that is, the intersection contains only the origin), so that the rank of \( \mathcal{A} \) is \( d \). If \( V \) is a linear subspace that is complementary to the linear subspace \( \bigcap_i H_i \), then the combinatorial structure of the arrangement is adequately reflected in the arrangement \( \{ H'_i = H_i \cap V \} \) of hyperplanes in \( V \), and for this arrangement, the dimension of \( V \) is the rank.

We will assume that positive and negative sides of each hyperplane in \( \mathcal{A} \) are specified. The open halfspace bounded by \( H \) and lying on its positive side will be denoted \( H^+ \), the other \( H^- \). A tope \( T \) may be specified by a function \( s_T : \mathcal{A} \rightarrow \{-1, 1\} \), where \( s_T(H) = 1 \) if \( T \subseteq H^+ \) and \( s_T(H) = -1 \) if \( T \subseteq H^- \). Then \( T = (\bigcap_{H \in \mathcal{A} : s_T(H) = 1} H^+) \cap (\bigcap_{H \in \mathcal{A} : s_T(H) = -1} H^-) \).

The collection of all possible functions \( s : \mathcal{A} \rightarrow \{-1, 1\} \) forms the vertex set of the cube graph of order \( n \), with two such functions (vertices) being adjacent if their values differ on exactly one hyperplane \( H \in \mathcal{A} \). In the situation of most interest in this paper, the (indexed) hyperplanes of \( \mathcal{A} \) are distinct: if \( i \neq j \) then \( H_i \neq H_j \). In this case, the tope graph is a subgraph of the cube graph. The cube graph is bipartite, and we may 2-color its vertices with the colors chartreuse and burnt umber by assigning the color burnt umber to any vertex \( s \) (representing the map \( s : \mathcal{A} \rightarrow \{-1, 1\} \)) for which the set \( \{H \in \mathcal{A} : s(H) = 1\} \) has even cardinality, and otherwise assigning the color chartreuse to \( s \). Then, if \( s_1 \) and \( s_2 \) are adjacent, they have different colors. When the hyperplanes of \( \mathcal{A} \) are distinct, the tope graph is a subgraph of the cube graph, and the 2-coloring of the cube graph yields a 2-coloring of the tope graph.

The odd-even invariant (as defined in [11]) is

\[
\omega(\mathcal{A}) = \left| \sum_{T \in \mathcal{T}(\mathcal{A})} (-1)^{\sigma(T)} \right|
\]

where \( \sigma(T) \) denotes the number of hyperplanes \( H \) such that \( T \subseteq H^+ \). Therefore, \( \omega(\mathcal{A}) \) is the absolute value of the difference between the number of burnt umber topes and the number of chartreuse topes. We can further define the signed odd-even invariant as

\[
\text{s\omega}(\mathcal{A}) = \sum_{T \in \mathcal{T}(\mathcal{A})} (-1)^{\sigma(T)}.
\]
This sum yields the number of burnt umber topes minus the number of chartreuse topes.

Any central arrangement of \( n \) distinct hyperplanes, with \( n \) odd, has odd-even invariant zero. To see this, note that \( T = \bigcap_{H:s_{T}(H)=1} H^{+} \cap \bigcap_{H:s_{T}(H)=-1} H^{-} \) is a tope if and only if \( T^{*} \) is a tope, where \( s_{T^{*}}(H) = -s_{T}(H) \) for all \( H \in \mathcal{A} \). Then \( \sigma(T) = n - \sigma(T^{*}) \). When we have an odd number of hyperplanes, \((-1)^{\sigma(T)} \) and \((-1)^{\sigma(T^{*})} \) cancel out in the sum.

**Theorem 1.** For any arrangement \( \mathcal{A} \) whose tope graph admits a Hamiltonian circuit, \( \omega(\mathcal{A}) = 0 \).

**Proof.** Given a bipartite graph with a 2-coloring (chartreuse and burnt umber) of the vertices, the colors along any Hamiltonian circuit must alternate, as no similarly colored vertices are adjacent. Since every vertex is included, the number of chartreuse and burnt umber vertices must be equal. \( \square \)

The Hamiltonian circuits of certain special tope graphs have been well studied. We give a few examples here.

**Cube graphs.** The cube graphs themselves are the tope graphs of the arrangements consisting of the \( n \) coordinate hyperplanes in \( \mathbb{R}^{n} \). The odd-even invariant of such arrangements is zero, and there exist Hamiltonian circuits of their tope graphs. These circuits correspond to Gray codes (see [18]).

**Coxeter arrangements.** Another family of examples comes from the Coxeter arrangements of type \( A \). The \( \binom{n+1}{2} \) hyperplanes of this arrangement are the sets \( H_{i,j} = \{ (x_{0}, x_{1}, \ldots, x_{n}) \in V : x_{i} = x_{j} \} \), where \( V \) is the \( n \)-dimensional subspace of \( \mathbb{R}^{n+1} \) satisfying \( \sum x_{i} = 0 \). A point \( (x_{0}, x_{1}, \ldots, x_{n}) \in V \) that is not on any of the hyperplanes has no two coordinates equal, so the coordinates are ordered, \( x_{\pi(0)} < x_{\pi(1)} < \ldots < x_{\pi(n)} \), for some permutation \( \pi \). Thus, the topes of the arrangement correspond to the permutations of \( \{0, 1, \ldots, n\} \). Two topes are adjacent if and only if the corresponding permutations \( \pi_{1} \) and \( \pi_{2} \) differ by a transposition \( \tau \) that switches \( i \) and \( i-1 \) for some \( i \) with \( 1 \leq i \leq n \). That is, for \( k = 0, \ldots, n \), we have \( \pi_{1}(\tau(k)) = \pi_{2}(k) \), where

\[
\tau(k) = \begin{cases} 
  i & \text{if } k = i - 1, \\
  i - 1 & \text{if } k = i, \text{ and} \\
  k & \text{otherwise.}
\end{cases}
\]

These tope graphs are also well-known as the graphs of the permutahedra. For these arrangements, the odd-even invariant is zero: the permutations of
{0, \ldots, n} are divided into the two color classes by assigning the even permutations one of the colors and the odd permutations the other. Hamiltonian circuits for general \( n \) are given, for example, in [18].

As a specific example, in Figure 1 we take \( n = 2 \), with the hyperplanes \( \{H_{01}, H_{12}, H_{02}\} = \{x_0 = x_1, x_1 = x_2, x_0 = x_2\} \), shown projected on the subspace \( x_0 + x_1 + x_2 = 0 \). (In this case, the Hamiltonian circuit is the entire tope graph.)

The tope graphs of the other finite Coxeter arrangements also admit Hamiltonian circuits. See [3] for further details. The next class of arrangements warrants its own section.

3. Alternating arrangements. The following construction yields an alternating arrangement of \( n \) hyperplanes of rank \( d \), and each combinatorial type of alternating arrangement arises in this way. For \( \alpha \in \mathbb{R} \), let \( H(\alpha) = \{(x_0, x_1, \ldots, x_{d-1}) : \sum_{k=0}^{d-1} x_k \alpha^k = 0\} \). Given real numbers \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \), let \( H_i = H(\alpha_i) \) for \( 1 \leq i \leq n \). The arrangement \( \mathcal{A}(n, d) = \{H_i : 1 \leq i \leq n\} \) is an alternating arrangement.

The topes of the arrangement are given in the following way. Each vector \( x = (x_0, x_1, \ldots, x_{d-1}) \in \mathbb{R}^d \) gives rise to a real polynomial \( p \) of degree \( d - 1 \) whose coefficients are the entries of \( x \):

\[
p(\tau) = x_0 + x_1 \tau + x_2 \tau^2 + \ldots + x_{d-1} \tau^{d-1}.
\]

We may identify \( \mathbb{R}^d \) with the vector space of such polynomials. Two polynomials \( p, q \) lie in the same tope of \( \mathcal{A} \) if \( p \) and \( q \) have the same nonzero sign when evaluated at each of the \( \alpha_i \)'s.
Alternatively, a function \( s : A \rightarrow \{-1, 1\} \) corresponds to some tope \( T \) if and only if the sequence \((s(H_1), s(H_2), \ldots, s(H_n))\) has at most \( d - 1 \) sign changes, for if this is the case then there is a polynomial of degree at most \( d - 1 \) having the required sign pattern. The combinatorial type therefore does not depend upon the particular choice of \( \alpha_i \)'s, but only upon their order. The odd-even invariants of the alternating arrangements were determined in [11]; we restate the relevant theorems here.

**Theorem 2.** The odd-even invariant of the alternating arrangement \( A(n, d) \), with \( n \) even and \( d \) odd, is
\[
2 \left( \frac{n}{2} - 1 \right) \left( \frac{d}{2} - 1 \right).
\]

For the remaining cases, that is, when \( n \) is odd and \( d \) is even, recall that for any central arrangement the odd-even invariant is always zero when \( n \) is odd. The same is true when \( d \) is even, provided the arrangement is centrally simple (as are the alternating arrangements).

**Theorem 3.** If \( A \) is a centrally simple arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \), then \( \omega(A) = 0 \) unless \( n \) is even and \( d \) is odd.

Considering the notion of a “mutation” lends some credence to Theorem 3, and this notion could be used to provide a proof that is valid in the current setting. (The proof in [11] does not use mutations, but is valid for uniform oriented matroids in general.)

Given an arrangement in \( \mathbb{R}^2 \) that contains the three lines pictured in Figure 2a, a *mutation* of the arrangement involving the triangular tope bounded by the three lines leads to an arrangement partly pictured in Figure 2b (we only mutate simplicial topes). Note that there may be more lines in the arrangement, but we may assume them to be unaffected by the mutation.

When positive sides of the lines are chosen, the parity, even or odd, of the number of negative sides that contain the tope changes, since the new
tope lies on different sides of the three lines from the old tope. Therefore, we increase the number of one color tope by one, and decrease the other by one (in this case, we lose a chartreuse tope, and gain a burnt umber tope, but all else remains the same).

In the analogous situation involving central arrangements, two opposite topes, each an open simplicial cone, are replaced by two other simplicial topes. When \( n \) is odd, a tope and its opposite have different colors to begin with, thus the odd-even invariant is unchanged. When \( n \) is even and \( d \) is odd, the odd-even invariant changes by \( \pm 4 \), as we change the signs of an odd number of hyperplanes for each tope (that is, the simplicial cone is bounded by an odd number of hyperplanes, and we change the sign of all the bounding hyperplanes in \( s_T \)). However, when \( d \) is even, the odd-even invariant does not change, as we switch the signs of an even number of hyperplanes for each antipodal tope.

For certain pairs \((n, d)\), we are able to describe Hamiltonian circuits in the tope graphs of the alternating arrangements \( A(n, d) \).

**Theorem 4.** The tope graph of the alternating arrangement \( A(n, 3) \) with \( n \) odd is Hamiltonian.

**Proof.** Consider the alternating arrangement as the set of sequences of \( \{+,-\} \)'s of length \( n \) and fewer than 3 changes of sign. Given a sequence \( s = s_1s_2\cdots s_n \in \{+,-\}^n \), we may represent the sequence as the set \( S = \{ i : s_i = - \} \). When we allow at most two sign changes, this means that the \( i \in S \) are consecutive modulo \( n \). We denote the possible sets \( S \) by \( S_0, S_n \), and, for \( 1 \leq j \leq n - 1 \) and \( 1 \leq k \leq n \), \( S_{j,k} \), described as follows. We put \( S_0 = \emptyset \), and \( S_n = \{1,2,\ldots,n\} \). For \( 1 \leq j \leq n - 1 \) and \( 1 \leq k \leq n \), let \( S_{j,k} \) be the set of \( j \) consecutive integers (modulo \( n \)) starting at position \( k \), for instance, \( S_{1,3} = \{3\} \) and \( S_{3,n} = \{n,1,2\} \). These \( 2 + n(n - 1) \) sets correspond to the topes, which are the vertices of the tope graph. We utilize the sets to describe the tope graph. The graph has two vertices of degree \( n \), \( n(n - 3) \) vertices of degree 4, and \( 2n \) vertices of degree 3. The vertex \( S_0 \) is adjacent to each vertex \( S_{1,k} \), and \( S_n \) is adjacent to \( S_{n-1,k} \), both for all \( k \), \( 1 \leq k \leq n \); \( S_0 \) and \( S_n \) are the vertices of degree \( n \). When \( 2 \leq j \leq n - 2 \) and \( 1 \leq k \leq n \), \( S_{j,k} \) is adjacent to \( S_{j-1,k}, S_{j-1,k+1}, S_{j+1,k}, S_{j+1,k-1} \); these are the vertices of degree 4. (We consider \( k \) modulo \( n \); e.g. \( S_{2,1} \) is adjacent to \( S_{1,1}, S_{1,2}, S_{3,1} \) and \( S_{3,0} = S_{3,n} \).) Also, \( S_{1,k} \) is adjacent to \( S_{2,k} \) and \( S_{2,k-1} \), in addition to \( S_0 \); and \( S_{n-1,k} \) is adjacent to \( S_{n-1,k} \) and \( S_{n-1,k+1} \), in addition to \( S_n \).
The next portion of the proof involves a great many indices, and the following is (hopefully) the clearest presentation. We populate an \( n \times (n-1) \) array with the sets \( S_{j,k} \) as follows. Express \( j \) \((1 \leq j \leq n - 1)\) as \( j = 4t + r \), where \( t \) and \( r \) are integers, and, as \( n \) is odd, \( r \in \{1, 3\} \). Then, \( S_{j,k} \) is placed in the \( j \)-th column and the \((k + t)\)-th row. It is harmless to assume (this will be justified later) that \( n \) is of the form \( 4s + 3 \), in which case the array looks something like:

\[
\begin{array}{cccccccc}
S_{1,1} & S_{2,1} & S_{3,1} & S_{4,1} & S_{5,1} & \cdots & S_{n-4,1} & S_{n-3,1} & S_{n-2,1} & S_{n-1,1} \\
S_{1,2} & S_{2,2} & S_{3,2} & S_{4,2} & S_{5,2} & \cdots & S_{n-4,2} & S_{n-3,2} & S_{n-2,2} & S_{n-1,2} \\
S_{1,3} & S_{2,3} & S_{3,3} & S_{4,3} & S_{5,3} & \cdots & S_{n-4,3} & S_{n-3,3} & S_{n-2,3} & S_{n-1,3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
S_{1,n-1} & S_{2,n-1} & S_{3,n-1} & S_{4,n-1} & S_{5,n-1} & \cdots & S_{n-4,n-1} & S_{n-3,n-1} & S_{n-2,n-1} & S_{n-1,n-1} \\
S_{1,n} & S_{2,n} & S_{3,n} & S_{4,n} & S_{5,n} & \cdots & S_{n-4,n} & S_{n-3,n} & S_{n-2,n} & S_{n-1,n} \\
\end{array}
\]

The Hamiltonian circuit we describe begins at \( S_0 \), which is adjacent to every vertex in the first column of our array. From there, we go to array element \((1, 1)\), and then zig-zag between pairs of columns, leaving one element in each column unvisited; these will be used to travel back from \( S_n \). In particular, for columns \( y \) with \( y \equiv 1 \mod 4 \) and rows \( x \), \( 1 \leq x \leq n - 2 \), we travel:

\[
\cdots \rightarrow (x, y) \rightarrow (x, y+1) \rightarrow (x+1, y) \rightarrow (x+1, y+1) \rightarrow \cdots.
\]

For \( x = n - 1 \), we continue to the next pair of columns, zig-zag our way back up to the first row, and on to the next pair of columns:

\[
\cdots \rightarrow (n-1, y) \rightarrow (n-1, y+1) \rightarrow (n-1, y+2) \rightarrow (n-1, y+3) \rightarrow (n-2, y+2) \rightarrow (n-2, y+3) \rightarrow \cdots
\]

\[
\cdots \rightarrow (1, y+2) \rightarrow (1, y+3) \rightarrow (1, y+4) \rightarrow (1, y+5) \rightarrow (2, y+4) \rightarrow (2, y+5) \rightarrow \cdots
\]

When we reach entry \((n-1, n-1)\) (in the case of \( n \equiv 3 \mod 4 \)) or entry \((1, n-1)\) (if \( n \equiv 1 \mod 4 \)), we continue to \( S_n \). (Every \( S_{n-1,k} \) is adjacent to \( S_n \), thus it does not matter if \( n \equiv 1 \mod 4 \) or if \( n \equiv 3 \mod 4 \) – i.e. whether we zig-zag up or down the last pair of columns.) Then, from \( S_n \), we travel back to \( S_0 \):

\[
S_n \rightarrow (n, n-1) \rightarrow (n, n-2) \rightarrow \cdots \rightarrow (n, 2) \rightarrow (n, 1) \rightarrow S_0,
\]
completing the Hamiltonian circuit. In diagram form (again, assuming $n = 4s + 3$), we have:

As an example, we give the described Hamiltonian circuit for $A(5, 3)$:

```
++ + + + +, − + + + +, − − + + +, + − + + +, + − − + +, + + − + +,
+ − − − −, + + + − −, + + + − −, − + + + −, − − + +, − − − − −,
− − − − −, − + + + −, + + + − −, − + + + −, + + + + +.
```

We may also view this circuit on the (spherical projection of the) arrangement itself; this is shown in Figure 3.

If $n = d$, the tope graph of $A(n, n)$ is the cube graph of dimension $n$, which, as stated above, is also Hamiltonian. We may use this fact to get a further class of $(n, d)$ pairs:

**Theorem 5.** The tope graph of the alternating arrangement $A(n, n − 1)$, where $n$ is odd, is Hamiltonian.

**Proof.** Set $N = 2^{n−2} − 1$ and suppose that $v_0, v_1, \ldots, v_N, v_0$ is a Hamiltonian circuit in $A(n − 2, n − 2)$, consecutive topes in this list being adjacent. Suppose further that $v_0 = +− +− \cdots +$ and $M$ is the integer, $0 < M < N$, and...
Figure 3: A Hamiltonian circuit on $A(5, 3)$, illustrating Theorem 4. The bounding edges of the front and back of the sphere are identified as indicated in the top left image. Dashed lines connecting squares indicate that the path goes from the front of the sphere to the back (or vice-versa) at this point.

Figure 4: A path on $A(3, 3)$ extended to $A(5, 4)$, illustrating Theorem 5.
for which \( v_M = - + - + \cdots + - \). Notice that \( M \) must be odd. Also note that neither \( v_0 + - \) nor \( v_M + - \) is a tope in \( A(n, n-1) \); these are the only two sequences of \( n + \)'s and \( - \)'s that are not.

We describe a Hamiltonian circuit in \( A(n, n-1) \). The path begins at \( v_0 + + \) and proceeds to \( v_0 + - \) and then \( v_0 - - \). From \( v_0 - - \) the circuit proceeds through \( v_1 - - , v_1 + - , \) and \( v_1 + + \) to \( v_1 -- \). For \( k = 2, 3, \ldots, M - 1 \), when \( k \) is even, the circuit proceeds from \( v_{k-1} -- \) through \( v_k - + , v_k ++ \), and \( v_k + - \) to \( v_k -- \); when \( k \) is odd, the circuit proceeds from \( v_{k-1} -- \) through \( v_k -- , v_k + - , \) and \( v_k ++ \) to \( v_k - + \). After \( v_{M-1} -- \), the circuit passes through \( v_M -- \) and \( v_M + - \) to \( v_M ++ \). From there it proceeds to \( v_{M+1} ++ , v_{M+1} + - , v_{M+1} -- , \) and \( v_{M+1} - + \). Then, for \( k = M + 2, M + 3, \ldots, N - 1 \), when \( k \) is odd, the circuit proceeds through \( v_k - + , v_k -- , v_k + - , \) and \( v_k ++ \); when \( k \) is even, it proceeds through \( v_k + + , v_k + - , v_k -- , \) and \( v_k - + \). From \( v_N ++ \) it returns to \( v_0 + + \).

We illustrate the case with \( A(3, 3) \) extending to \( A(5, 4) \) in Figure 4. These initial results and observations, as well as some small examples, give rise to a very natural question: do the tope graphs of all alternating oriented matroids with odd-even invariant equal to zero have Hamiltonian circuits?

For a bit of diversity, we give in Figure 5 an example of an arrangement \( A \) in \( \mathbb{R}^3 \) whose tope graph has no Hamiltonian circuit, even though the odd-even invariant is zero, together with a pictorial proof (Figure 6).

\[ \text{Figure 5: An arrangement with odd-even invariant zero but no Hamiltonian circuit. One side of the arrangement, with the tope graph superimposed, is given here. One tope is lightly shaded for use in a later proof.} \]

\[ A \] has nine hyperplanes (the eight drawn across the front, and the outer
border), thus $\sigma(A) = 0$. We have drawn a part of the tope graph superimposed on the sphere. If there exists a Hamiltonian circuit, we must be able to select a subgraph of the tope graph such that every vertex has degree two. This corresponds to selecting a subset $F$ of the bounding facets of the topes such that for every tope $T$, $|T \cap F| = 2$. We will show this is impossible, again pictorially, in Figure 6, by looking at the different possibilities for the upper five-sided, central tope (shaded in Figure 5), and the various associated subcases. In each case, precisely two facets of the tope must be included. Numbered, solid (blue) edges indicate the choices we make for a particular case; undecorated, solid (red) edges indicate bounding facets the inclusion of which is forced by those choices; solid, $\times$-ed (green) edges indicate the two bounding facets of a triangular tope the inclusion of which is not permitted. This is the contradiction, as it leaves just one bounding facet that may be selected (and thus, the path through the vertices is unable to continue). The offending tope in each case is shaded (orange). Note that there may be more than one “bad” tope for a particular set of choices.

4. Bounds on the odd-even invariant. In the literature there are several closely related but different settings in which to consider the problems of interest here. Until now we have considered only the setting of central arrangements in $\mathbb{R}^d$. By the usual process of projectivization (see [2]), we may consider instead arrangements in projective space of dimension $d - 1$. When the number of hyperplanes is even, the tope graph for the projective arrangement has odd-even invariant that is half the value of that of the corresponding central arrangement. When $n$ is odd, the tope graph of the projective arrangement is no longer bipartite and thus the odd-even invariant is not defined in the analogous way; however, in this case, we may define the odd-even invariant to be zero.

Likewise, by taking the intersections of the hyperplanes of the central arrangement with a hyperplane missing the origin, we may obtain a non-central arrangement in $\mathbb{R}^{d - 1}$. This latter process is called dehomogenization, the reverse process being homogenization, and this setting is of particular importance here, as it allows us to make use of previous research on non-central arrangements, especially non-central planar arrangements.

The question of the maximum ratio $\frac{b}{c}$ for an arrangement of lines in the plane, where $b$ and $c$ are the numbers of topes of each of the two colors, has been studied by several authors; see, e.g., [9, 12, 13, 17]. It is clear that this
Figure 6: An arrangement $\mathcal{A}$ with $\alpha(\mathcal{A}) = 0$ and non-Hamiltonian tope graph.
maximum ratio must be at least 1, since switching the colors on all the topes inverts the fraction. Many have noted that the ratio is less than two. More precisely, Simmons and Wetzel [17] showed that $b \leq 2c - 2 - \sum_p (\lambda(P) - 2)$, where the sum is over intersections of lines $P$, and $\lambda(P)$ is the number of lines containing $P$. For centrally simple arrangements in $\mathbb{R}^3$, it is easy to show that $b \leq \frac{2}{3}n(n - 1)$, with equality if and only if the triangular regions are precisely those regions that are colored burnt umber. Thus we see that in the 2-dimensional case, the problem of determining the maximum ratio is related to that of determining the maximum possible number of triangles in an arrangement of $n$ lines in the plane. This latter problem has been studied by many authors, motivated either by a question of Grünbaum in [8], by the 2-coloring problem, or both. See, for example, [6, 7, 13, 15]. The relatively recent paper of Bartholdi, Blanc, and Leisel [1] gives an account of the problem and adds infinitely many positive integers to the previous infinite list of Forge and Alfonsín [6] of integers $n$ for which it is known that there exists an arrangement of $n$ lines achieving the bound given by Simmons and Wetzel.

The following theorem shows that there exist arrangements with large odd-even invariant. This will be used in Section 5 to construct further arrangements with odd-even invariant zero and no Hamiltonian circuit.

**Theorem 6.**

1. If $d$ is an even positive integer then there is a constant $c_d > 0$ such that for arbitrarily large integer values of $n$ there exists an arrangement $\mathcal{A}$ of $n$ distinct hyperplanes in $\mathbb{R}^d$ for which $\omega(\mathcal{A}) > c_d n^d$.

2. If $d \geq 3$ is odd then there exists $c_d > 0$ such that for arbitrarily large integer values of $n$ there exists an arrangement $\mathcal{A}$ of distinct hyperplanes in $\mathbb{R}^d$ such that $\omega(\mathcal{A}) > c_d n^{d-1}$.

3. If $d \geq 3$ is odd then there exists $\bar{c}_d > 0$ such that for arbitrarily large integer values of $n$ there exists a central arrangement $\mathcal{A}$ of distinct hyperplanes in $\mathbb{R}^d$ for which $\omega(\mathcal{A}) > \bar{c}_d n^{d-1}$.

4. If $d \geq 4$ is an even positive integer then there exists $\tilde{c}_d > 0$ such that for arbitrarily large integer values of $n$ there exists a central arrangement $\mathcal{A}$ of distinct hyperplanes in $\mathbb{R}^d$ for which $\omega(\mathcal{A}) > \tilde{c}_d n^{d-2}$.
Proof. The result of Forge and Alfonsín [6] previously described implies the validity for \( d = 2 \), with \( c_2 = \frac{2}{3} \).

If \( d = 2m, m > 1 \), we show that \( c_d \) may be taken to be \((\frac{2\sqrt{c_2}}{d})^d \). Assume \( m | n \), and let \( s = \frac{n}{m} \). Let \( \mathcal{A}_0 = \{H_1, \ldots, H_s\} \) be an arrangement of \( \frac{n}{m} \) lines in the plane achieving \( sœ(\mathcal{A}_0) > c_2(\frac{n}{m})^2 \). Identify \( \mathbb{R}^d \) with \((\mathbb{R}^2)^m \).

For \( j = 1, \ldots, m \), let \( \pi_j \) be the linear function \( \pi_j : \mathbb{R}^d \to \mathbb{R}^2 \) that takes \((x_1, \ldots, x_{2j-1}, x_{2j}, \ldots, x_{2m}) \in \mathbb{R}^d \) to \((x_{2j-1}, x_{2j}) \in \mathbb{R}^2 \). For each \((i, j) \in [s] \times [m] \) let \( H_{i,j} = \pi_j^{-1}(H_i) \), a hyperplane in \( \mathbb{R}^d \), and let its positive side be the inverse image of the positive side of \( H_i \) under \( \pi_j \). Let \( \mathcal{A} = \{H_{i,j} : (i, j) \in [s] \times [m]\} \). The topes of the arrangement \( \mathcal{A} \) are the products \( T_{k_1} \times \ldots \times T_{k_m} \), where, for each \( r = k_1, \ldots, k_m \), \( T_r \) is a tope of \( \mathcal{A}_0 \). The signed odd-even invariant of \( \mathcal{A} \) is

\[
\text{soe}(\mathcal{A}) = \sum_{T_{k_1}, \ldots, T_{k_m} \in \mathcal{T}(\mathcal{A}_0)} (-1)^{\sigma(T_{k_1} \times \ldots \times T_{k_m})}
\]

\[
= \sum_{T_{k_1}, \ldots, T_{k_m} \in \mathcal{T}(\mathcal{A}_0)} (-1)^{\sigma(T_{k_1}) + \ldots + \sigma(T_{k_m})}
\]

\[
= \sum_{T_{k_1}, \ldots, T_{k_m} \in \mathcal{T}(\mathcal{A}_0)} \prod_{j=1}^{m} (-1)^{\sigma(T_{k_j})}
\]

\[
= \prod_{i=1}^{m} \sum_{T_r \in \mathcal{T}(\mathcal{A}_0)} (-1)^{\sigma(T_r)}
\]

\[
= (\text{soe}(\mathcal{A}_0))^m > c_2^m \left( \frac{n}{m} \right)^{2m},
\]

where, as before, \( \sigma(T) = |\{H : s_T(H) = 1\}| \). From this, (1) follows.

When \( d = 2m + 1 \) with \( m \geq 1 \), we may take \( c_d = c_{d-1} \), an appropriate arrangement being a set of \( n \) hyperplanes in \( \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R} \) that are the inverse images under the projection of \( \mathbb{R}^d \) to the first \( d-1 \) coordinates of the hyperplanes of an arrangement \( \mathcal{A}_0 \) in \( \mathbb{R}^{d-1} \) that satisfies \( \sigma(e(\mathcal{A}_0)) > c_{d-1} n^{d-1} \).

Statement (3) follows from (1) by homogenization, and similarly, statement (4) follows from (2).

We should note that when the requirement that the hyperplanes be distinct is dropped, statements (1) and (3) are trivial, and in (2) and (4) the order of \( n \) can be increased by one, trivially, by considering arrangements in which each hyperplane appears twice. In such an arrangement, the odd-even
5. Constructions forbidding Hamiltonian circuits. In this section, we give a theorem that will allow us to build arrangements with odd-even invariant zero and no Hamiltonian circuit. Before stating this theorem, we give a few necessary definitions. If $\mathcal{A}$ is an arrangement of $n \geq 1$ hyperplanes and $H \in \mathcal{A}$ then $\mathcal{A} \setminus \{H\}$ is also an arrangement of $n - 1$ hyperplanes in $\mathbb{R}^d$, called the deletion of $H$ from $\mathcal{A}$. Further, we may consider the collection of intersections $H' \cap H$, where $H' \in \mathcal{A} \setminus \{H\}$, to be an arrangement of hyperplanes in $\mathbb{R}^{d-1}$. We denote this arrangement by $\mathcal{A}/\{H\}$, and call it the restriction of $\mathcal{A}$ to $H$. (The arrangements $\mathcal{A} \setminus \{H\}$ and $\mathcal{A}/\{H\}$ are related to the deletion and contraction operations of oriented matroids; see [2].)

Given a tope $T$ of $\mathcal{A}$, the restriction of the function $s_T$ to $\mathcal{A} \setminus \{H\}$ is a tope of $\mathcal{A} \setminus \{H\}$. If $T$ and $\tilde{T}$ are adjacent topes of $\mathcal{A}$ such that the value of $s_{\tilde{T}}$ differs from that of $s_T$ on $H$, then their common restriction to $\mathcal{A} \setminus \{H\}$ is a tope of $\mathcal{A}/\{H\}$.

**Theorem 7.** Suppose $\mathcal{A}$ is a central arrangement that includes a hyper-plane $H$ such that $\omega(\mathcal{A} \setminus \{H\}) > |\mathcal{T}(\mathcal{A}/\{H\})|$. Then the tope graph of $\mathcal{A}$ has no Hamiltonian circuit.

**Proof.** We will say that a tope $T$ of $\mathcal{A} \setminus \{H\}$ straddles $H$ if $T \cap H^+$ and $T \cap H^-$ are topes of $\mathcal{A}$. In this case, $T \cap H$ is a tope of $\mathcal{A}/\{H\}$.

Given a 2-coloring $\gamma_1 : \mathcal{A} \to \{\text{chartreuse, burnt umber}\}$ of the tope graph of $\mathcal{A}$, a 2-coloring $\gamma_2 : \mathcal{A} \setminus \{H\} \to \{\text{chartreuse, burnt umber}\}$ of $\mathcal{A} \setminus \{H\}$ can be obtained by giving each tope of $\mathcal{A} \setminus \{H\}$ lying in $H^+$ its color as a tope of $\mathcal{A}$, changing the color of each tope of $\mathcal{A} \setminus \{H\}$ lying in $H^-$, and giving each tope $T$ straddling $H$ the color of the tope $T \cap H^+$ of $\mathcal{A}$.

Suppose a Hamiltonian circuit of the tope graph of $\mathcal{A}$ exists. Remove the topes (vertices) of the circuit that border $H$ and lie in $H^-$. The number of such vertices is $|\mathcal{T}(\mathcal{A}/\{H\})|$, and their removal from the Hamiltonian circuit leaves a graph consisting of at most $|\mathcal{T}(\mathcal{A}/\{H\})|$ paths. Each of these paths lies entirely on one side of $H$, so the colors of its vertices alternate, and the topes along any of these paths contribute at most one to the odd-even invariant of $\mathcal{A} \setminus \{H\}$. It follows that $\omega(\mathcal{A} \setminus \{H\}) \leq |\mathcal{T}(\mathcal{A}/\{H\})|$. 

This theorem can be used to verify the nonexistence of Hamiltonian circuits in many tope graphs. First, recall from Theorem 6 that when $d$ is
odd there are central arrangements of \( n \) hyperplanes in \( \mathbb{R}^d \) having odd-even invariant \( \tilde{c}n^{d-1} \), for some positive constant \( \tilde{c} \). On the other hand, it is well-known that in \( \mathbb{R}^{d-1} \) any central arrangement has at most \( O(n^{d-2}) \) topes. By adding a hyperplane to an arrangement in \( \mathbb{R}^d \) that has a large odd-even invariant, and an even number of hyperplanes, we obtain, according to Theorem 7, an arrangement with no Hamiltonian circuit and odd-even invariant zero, as there is an odd number of hyperplanes. Thus as a corollary of Theorem 6 and this construction, we have the following theorem.

**Theorem 8.** If \( d \geq 3 \) is an odd integer there are arbitrarily large central arrangements for which the odd-even invariant is zero, but whose tope graph admits no Hamiltonian circuit.

Notice that the number of hyperplanes in an arrangement obtained by the construction is odd.

For use in the remainder of the paper we introduce a refinement of the notation that we have used. If \( T \) is any set of topes, we define

\[
\text{soe}(T) = \sum_{T \in T} (-1)^{\sigma(T)}.
\]

If \( \mathcal{A} \) is an arrangement of hyperplanes in \( \mathbb{R}^d \) then let \( T_b(\mathcal{A}) \) denote the set of bounded topes of \( \mathcal{A} \) and let \( T_\infty(\mathcal{A}) \) denote the set of unbounded topes of \( \mathcal{A} \). When \( \mathcal{A} \) is a simple arrangement, the topes in \( T_\infty(\mathcal{A}) \) correspond to the topes of the central arrangement obtained from \( \mathcal{A} \) by translating each hyperplane of \( \mathcal{A} \) so that it contains the origin. We will indicate this arrangement by \( \mathcal{A}_\infty \). If \( \mathcal{A} \) is simple then \( \mathcal{A}_\infty \) is a centrally simple arrangement.

**Theorem 9.** Suppose \( \mathcal{A} \) is a simple arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \), where \( d \) is odd. If \( n \) is odd, then \( \text{soe}(T(\mathcal{A})) = \text{soe}(T(\mathcal{A}_\infty)) = \text{soe}(T(\mathcal{A}_b)) = 0 \); otherwise, we have \( \text{soe}(T_b(\mathcal{A})) = -\frac{1}{2} \text{soe}(T(\mathcal{A}_\infty)) \).

**Proof.** Suppose \( n \) is odd. We obtain a centrally simple arrangement of hyperplanes \( \mathcal{A}' \) in \( \mathbb{R}^{d+1} \) by viewing the hyperplanes of \( \mathcal{A} \) as lying in \( \mathbb{R}^d \times \{1\} \subseteq \mathbb{R}^{d+1} \). We take the hyperplanes of \( \mathcal{A}' \) to be those hyperplanes in \( \mathbb{R}^{d+1} \) generated by the hyperplanes (in \( \mathbb{R}^d \times \{1\} \)) of \( \mathcal{A} \), together with the one additional hyperplane, \( H_0 = \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^{d+1} \), whose positive side \( H_0^+ \) is the open halfspace containing \( \mathbb{R}^d \times \{1\} \). Since \( d + 1 \) is even and \( \mathcal{A}' \) is centrally simple, \( \text{soe}(T(\mathcal{A}')) = 0 \), by Theorem 3. The topes of \( \mathcal{A}' \) lying in \( H_0^+ \) are the cones generated by the topes of \( \mathcal{A} \), and the topes of \( \mathcal{A}' \)
lying in $H_0^+$ are the reflections of these through the origin. Since $n+1$ is even, it is clear that $\sigma(T(A')) = -2\sigma(T(A))$. Also, clearly, $\sigma(T(A)) = \sigma(T_b(A)) + \sigma(T_{\infty}(A))$. We have $\sigma(T_{\infty}(A)) = \sigma(T(A_{\infty}))$, and, since $n$ is odd, $\sigma(T(A_{\infty})) = 0$. It follows that $\sigma(T_b(A)) = 0$ as well, as claimed.

Suppose $n$ is even. Again we obtain an arrangement $A'$ in $\mathbb{R}^{d+1}$, this time omitting the additional hyperplane $H_0$. Then we have $\sigma(T(A')) = 2\sigma(T_b(A)) + \sigma(T(A_{\infty}))$. Since $d+1$ is even and $A'$ is centrally simple, $\sigma(T(A')) = 0$.

As a corollary we obtain an asymptotic bound on $\frac{b}{c}$, as $n \to \infty$.

**Theorem 10.** Suppose $\mathcal{A}$ is a simple arrangement of $n$ hyperplanes in $\mathbb{R}^d$, where $d$ is a fixed odd positive integer, and suppose $\epsilon > 0$. If the tope graph of $\mathcal{A}$ is 2-colored, $b$ topes being colored burnt umber and $c$, chartreuse, then $\frac{b}{c} \leq 1 + \epsilon$ provided that the number of hyperplanes is sufficiently large.

**Proof.** We have that $b + c$ is the total number of topes, which is \( \sum_{k=0}^{d} \binom{n}{k} \) as the arrangement of the $n$ hyperplanes in $\mathbb{R}^d$ is simple. This sum is a polynomial of degree $d$ in $n$. The odd-even invariant is the difference, $b - c$, and by Theorem 9, this is certainly less than the number of topes of the arrangement $\mathcal{A}_{\infty}$, which is $2 \sum_{k=0}^{d-1} \binom{n}{d-1-2k}$, since it is a centrally simple arrangement. This is a polynomial of degree $d - 1$ in $n$, and the statement follows. \( \square \)

Of course, when $n$ is odd (and assuming $d$ is odd as in the theorem), $\frac{b}{c} = 1$ by Theorem 9.

The following theorem is analogous to Theorem 7, for perfect matchings instead of Hamiltonian circuits. It includes the additional hypothesis that the arrangement be centrally simple and requires a more restrictive bound.

**Theorem 11.** Suppose $\mathcal{A}$ is a simple central arrangement of $n$ hyperplanes in $\mathbb{R}^d$ that includes a hyperplane $H$ such that $\sigma(\mathcal{A} \setminus \{H\}) > 2|\mathcal{T}(\mathcal{A}/\{H\})|$. Then the tope graph of $\mathcal{A}$ has no perfect matching.

**Proof.** Since $\sigma(\mathcal{A} \setminus \{H\}) \neq 0$, $n - 1 = |\mathcal{A} \setminus \{H\}|$ is even, and, by Theorem 3, $d$ is odd. Let $b$ be the number of burnt umber topes and $c$ be the number of chartreuse topes of $\mathcal{A} \setminus \{H\}$ that lie in $H^+$. We may assume $b \geq c$. Since $|\mathcal{A} \setminus \{H\}|$ is even, the reflections of topes through the origin are of the same color. The remaining topes $T$ of $\mathcal{A} \setminus \{H\}$ correspond to the topes of $\mathcal{A}/\{H\}$, and induce a subgraph of the tope graph of $\mathcal{A} \setminus \{H\}$ that is isomorphic to
the tope graph of $\mathcal{A}/\{H\}$. By Theorem 3, $\omega(\mathcal{A}/\{H\}) = 0$. It follows that the topes having a facet in $H$ make no net contribution to the odd-even invariant of $\mathcal{A} \setminus \{H\}$. We therefore have $\omega(\mathcal{A} \setminus \{H\}) = 2(b - c)$.

In any perfect matching of the tope graph of $\mathcal{A}$, at least $b - c$ of the topes lying in $H^+$ and not having $H$ as a facet must be matched to topes in $H^+$ that have $H$ as a facet. But $b - c$ is larger than $|\mathcal{T}(\mathcal{A}/\{H\})|$, which is the number of such topes.

When $d = 3$, Theorem 11 and the existence of centrally simple arrangements with large odd-even invariant establishes the existence of arbitrarily large centrally simple arrangements in $\mathbb{R}^3$ whose tope graphs have no perfect matchings.

6. Questions. The foregoing leaves many unanswered questions, of which we mention only a few.

1. We have seen that there exist many examples of central arrangements $\mathcal{A}$ with $n$ hyperplanes in $\mathbb{R}^d$ whose tope graphs have no Hamiltonian circuit, even though $\omega(\mathcal{A}) = 0$, with $d$ odd and $n$ odd. Is the same true for even $d$, or for even $n$?

2. The construction described at the end of Section 5 does not yield centrally simple arrangements, in general. Are there such examples, when the requirement of central simplicity is added?

3. Can the exponent $d - 1$ in (4) of Theorem 6 be increased to $d$? This would imply a positive answer to Question 1.

4. Is it true that the tope graph of an alternating arrangement $\mathcal{A}$ of $n$ hyperplanes in $\mathbb{R}^d$ has a Hamiltonian circuit if and only if $\omega(\mathcal{A}) = 0$?

There are analogous questions for non-central arrangements.

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