Dynamically assisted Sauter-Schwinger effect in inhomogeneous electric fields

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Via the worldline instanton method, we study electron-positron pair creation by a strong electric field of the profile $E(\cosh^2(kx))$ superimposed by a weaker pulse $E'/\cosh^2(\omega t)$. If the temporal Keldysh parameter $\gamma_\omega = m\omega/(qE)$ exceeds a threshold value $\gamma_{\text{crit}}^0$ which depends on the spatial Keldysh parameter $\gamma_k = mk/(qE)$, we find a drastic enhancement of the pair creation probability − reporting on what we believe to be the first analytic non-perturbative result for the interplay between temporal and spatial field dependences $E(t,x)$ in the Sauter-Schwinger effect.

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I. INTRODUCTION

Despite the tremendous progress of quantum field theory as a fundamental description of nature, our understanding of its non-perturbative properties is still disappointingly incomplete. In quantum electrodynamics (QED), for example, a striking non-perturbative phenomenon is the Sauter-Schwinger effect predicting the creation of electron-positron pairs out of the vacuum $(QED)$, for example, a striking non-perturbative phenomenon as a fundamental description of nature, our understanding is not only unsatisfactory from a theoretical point of view, a deeper insight into the impact of understanding is not only unsatisfactory from a theoretical point of view, a deeper insight into the impact of understanding is not only unsatisfactory from a theoretical point of view, a deeper insight into the impact.

II. WORLDLINE INSTANTON METHOD

Let us start with a brief review of the worldline instanton method. Since the electron spin does not affect the exponent of the pair creation probability, we consider the vacuum persistence amplitude of scalar QED

$$\langle 0_{\text{out}} | 0_{\text{in}} \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{i \int d^4x (\partial_{\mu}\phi)^2 - m^2|\phi|^2}$$

with the covariant derivative $D_{\mu} = \partial_{\mu} + i q A_{\mu}$. After analytic continuation to Euclidean space, this functional path integral can be translated into the worldline representation where $\mathcal{D}\phi \mathcal{D}\phi^*$ is replaced by the sum over all closed loops $x_{\mu}(s)$ in Euclidean space. Then, via the saddle point method (with the electron mass $m$ playing the role of the large expansion parameter), the pair creation probability can be estimated as

$$P_{e^+e^-} = 1 - |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2 \sim e^{-S},$$

with the worldline instanton action

$$S = ma + i q \int_0^1 ds \dot{x}^{\mu} A_{\mu}(x^\nu).$$

Here $\dot{x}_\mu = dx_\mu/ds$ denotes the derivative of a closed loop $x_\mu(s) = x_\mu(s = 1)$ worldline loop $x_\mu(s)$ as a solution of the instanton equations

$$m\ddot{x}_\mu = iqF_{\mu\nu}\dot{x}^\nu a$$

with $\ddot{x}_\mu = d^2x_\mu/ds^2$ and $\dot{x}_\nu\dot{x}^\nu = a^2 = \text{const.}$

III. SUM OF SAUTER PULSES

Now let us apply the worldline instanton method to a space-time dependent electric field

$$E(t,x) = \left( \frac{E}{\cosh^2(kx)} + \frac{E'}{\cosh^2(\omega t)} \right) e_x$$

consisting of a strong spatial Sauter pulse $\propto E$ and a weaker temporal Sauter pulse $\propto E'$ where both field
strengths are sub-critical $E' < E < E_{\text{crit}} = m^2/q$. Furthermore, in order to be in the non-perturbative regime, we assume slowly varying pulses $\omega, k < m$. For convenience, we introduce the spatial and temporal Keldysh parameters via

$$\gamma_k = \frac{mk}{qE}, \gamma_0 = \frac{m\omega}{qE}. \quad (7)$$

The Euclidean vector potential reads

$$A_0(x_1) = \frac{E}{k} \tanh(kx_1), \quad A_1(x_0) = \frac{E'}{\omega} \tanh(\omega x_0). \quad (8)$$

with $x_0 = it$ and $x_1 = x$ as well as $A_2 = A_3 = 0$. As a result, the instanton equations [5] assume the form

$$\dot{x}_0 = + \frac{qEa}{m} \left( \frac{1}{\cosh^2(kx_1)} - \frac{E'}{E} \frac{1}{\cos^2(\omega x_0)} \right) \dot{x}_1, \quad (9)$$

and are analogous to the planar motion of a charged particle in a magnetic field $B(r) = B(x,y)\hat{e}_z$.

Due to $E'/E \ll 1$, the second term is negligible unless $\cos^2(\omega x_0)$ becomes very small near the poles of $E(x_0, x_1)$ at $\omega x_0 = \pm \pi/2$. Away from these poles, we may omit the second term and the above equations can be integrated

$$\dot{x}_0 = \frac{a}{\gamma_k} \tanh(kx_1) + ab,$$

$$\dot{x}_1 = \pm a \sqrt{1 - \left( \frac{\tanh(kx_1)}{\gamma_k} + b \right)^2}. \quad (10)$$

As mentioned after Eq. [5], the constant $a$ is given by $\dot{x}_0 \dot{x}' = a^2 = \text{const}$. The other integration constant $b$ determines the velocity $\dot{x}_0$ just before (or just after) crossing the $x_0$-axis, see Fig. 1.

Near the poles $\omega x_0 \approx \pm \pi/2$, on the other hand, the second term becomes important. Similar to the reflection of a charged particle at the region of a very strong magnetic field, the instanton trajectory is basically reflected by the “wall” at $\omega x_0 \approx \pm \pi/2$ if it reaches out far enough. Since this reflection occurs during a very short proper time $\Delta s$, we may neglect the regular terms in Eq. [9] and keep only the divergent contributions. Then, the equation for $x_1$ can be integrated approximately to

$$\dot{x}_1 \approx \frac{qEa}{m\omega} \tanh(\omega x_0) + \dot{x}_1^\text{in}, \quad (11)$$

and thus the equation for $x_0$ becomes

$$\dot{x}_0 \approx - \frac{(qEa)^2}{m^2\omega} \frac{\tanh(\omega x_0)}{\cos^2(\omega x_0)} \sim \frac{1}{(\omega x_0 \pm \pi/2)^3}. \quad (12)$$

As a result, the perpendicular velocity $\dot{x}_0$ is reversed by that reflection while the parallel velocity $\dot{x}_1$ has the same value $\dot{x}_1^\text{in}$ before and after the reflection.

IV. TUNNELLING PROBABILITY

Again due to $E \gg E'$, the instanton action reads

$$S \approx ma - \frac{qE}{k} \int_0^{1} ds \tanh(kx_1) \dot{x}_0. \quad (13)$$

In order to calculate the above integral, we split the closed loop into four quarters: from $x_1 = 0$ to the spatial turning point $x_1^*$, from $x_1^*$ to $x_1 = 0$, from $x_1 = 0$ to $-x_1^*$, and finally back to $x_1 = 0$, see Fig. 1. Since each quarter yields the same contribution, we get

$$S \approx ma - \frac{4m}{\gamma_k} \int_0^{x_1^*} dx_1 \tanh(kx_1) \left( \frac{\tanh(kx_1) + \gamma_k b}{\sqrt{\gamma_k^2 - (\tanh(kx_1) + \gamma_k b)^2}} \right). \quad (14)$$

where $x_1^*$ denotes the spatial turning point given by

$$\tanh(kx_1^*) + \gamma_k b = \gamma_k, \quad (15)$$

i.e., the zero of the square root in the integral in Eq. [14], where $dx_1/dx_0 = 0$. The constant $a$ is determined by

$$\dot{x}_0 \dot{x}' = a^2 \quad \text{and} \quad x_\mu(s = 0) = x_\mu(s = 1)$$

which gives

$$a = \frac{4}{\gamma_k} \int_0^{x_1^*} \frac{dx_1}{\sqrt{\gamma_k^2 - (\tanh(kx_1) + \gamma_k b)^2}}. \quad (16)$$

The remaining integration constant $b$ depends on the frequency $\omega$. If $\omega$ is too small and thus the poles at $\omega x_0 = \pm \pi/2$ are too far away, the instanton trajectory is not reflected at all and thus we have $b = 0$. In case of reflection, the integration constant $b$ is non-zero and determined by the implicit condition

$$\frac{4m}{\gamma_k} \int_0^{x_1^*} \frac{dx_1}{\sqrt{\gamma_k^2 - (\tanh(kx_1) + \gamma_k b)^2}} = \frac{\pi}{2\omega}. \quad (17)$$

Together with the above equations for $x_1^*$, $a$, and $b$, Eq. [14] is the main result of this paper.
The threshold condition \( b = 0 \) translates into

\[
\gamma_0 = \frac{\pi}{2} \frac{\gamma_k \sqrt{1 - \gamma_k^2}}{\arcsin(\gamma_k)} \equiv \gamma_{\omega}^{\text{crit}}. \tag{18}
\]

If the frequency is too low \( \gamma_\omega < \gamma_{\omega}^{\text{crit}} \), the instanton trajectory is basically not affected by the poles at \( \omega x_0 = \pm \pi/2 \) leading to \( b = 0 \) and thus the weak temporal pulse \( \propto E' \) has negligible impact. In this case \( b = 0 \), we get \( x_1 = \text{artanh}(\gamma_k)/k \) and all the integrals can be carried out analytically, yielding the same results as for a static Sauter pulse, which have already been obtained in [5]. If the frequency exceeds this threshold value \( \gamma_\omega > \gamma_{\omega}^{\text{crit}} \), on the other hand, the instanton trajectory is reflected at the poles (i.e., \( b > 0 \)) and thus the instanton action (14) is reduced by the weak temporal pulse \( \propto E' \), leading to a significant enhancement of the pair creation probability. In the homogeneous limit \( \gamma_k \downarrow 0 \), the threshold value (18) approaches \( \gamma_{\omega}^{\text{crit}} = \pi/2 \) consistent with the results of [5]. For \( \gamma_k \uparrow 1 \), the threshold \( \gamma_{\omega}^{\text{crit}} \) scales as \( \gamma_{\omega}^{\text{crit}} \propto \sqrt{1 - \gamma_k^2} \), i.e., very small frequencies \( \omega \) can have a significant impact in this case.

Unfortunately, due to the implicit nature of the condition for \( b \), we cannot provide a closed analytical expression for \( S \). However, near but above threshold, we can Taylor expand the involved quantities and obtain the following approximate formula for the instanton action

\[
S = \frac{m^2}{qE} \left( \frac{2\pi}{1 + \sqrt{1 - \gamma_k^2}} - \pi \frac{(1 - \gamma_k^2)^{3/2}}{\gamma_k^2 (\gamma_{\omega}^{\text{crit}})^4} \left[ \gamma_\omega - \gamma_{\omega}^{\text{crit}} \right]^2 \right) + O \left( \left[ \gamma_\omega - \gamma_{\omega}^{\text{crit}} \right]^3 \right). \tag{19}
\]

The zeroth order \( S_0 = \left( m^2/(qE) \right) 2\pi/\left(1 + \sqrt{1 - \gamma_k^2}\right) \) is just the result in the static case [5] and is valid below and at threshold \( \gamma_\omega \leq \gamma_{\omega}^{\text{crit}} \). For \( \gamma_k = 0 \), we recover Schwinger’s result [1] for a constant field [1]. Above threshold \( \gamma_\omega > \gamma_{\omega}^{\text{crit}} \) on the other hand, the action is reduced by the second-order term \( \propto (\gamma_\omega - \gamma_{\omega}^{\text{crit}})^2 \).

V. CONCLUSIONS

Via the worldline instanton technique, we derived an analytical estimate [14] for the electron-positron pair creation probability \( \omega \) induced by an electric field [6] which genuinely depends both on space and on time. Superimposing a strong spatial pulse by a weak temporal pulse [9], we found that the weak pulse is negligible for small frequencies \( \gamma_\omega \leq \gamma_{\omega}^{\text{crit}} \) but can enhance the pair creation probability significantly (dynamically assisted Sauter-Schwinger effect) for larger frequencies \( \gamma_\omega \geq \gamma_{\omega}^{\text{crit}} \) with the threshold (18) depending on the spatial Keldysh parameter \( \gamma_k \). In the homogeneous limit \( \gamma_k \downarrow 0 \), this threshold \( \gamma_{\omega}^{\text{crit}} \) converges to \( \pi/2 \) in accordance with [14]. If the spatial Keldysh parameter approaches unity \( \gamma_k \uparrow 1 \), on the other hand, the threshold \( \gamma_{\omega}^{\text{crit}} \) goes to zero. In this case \( \gamma_k \uparrow 1 \), the size of the spatial Sauter pulse is barely enough to produce electron-positron pairs (since the electrostatic potential difference is just above the gap of \( 2mc^2 \)) and the instanton loop becomes very large, cf. \( x_1 = \text{artanh}(\gamma_k)/k \) for \( b = 0 \). Quite intuitively, even comparably small frequencies (leading to poles at large distances to the origin) can have an impact in this limit.

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