Anosov triangle reflection groups in
\( SL(3, \mathbb{R}) \)

Gye-Seon Lee, Jaejeong Lee & Florian Stecker

We identify all Anosov representations of compact hyperbolic triangle reflection groups into \( SL(3, \mathbb{R}) \). Specifically, we prove that such a representation is Anosov if and only if it lies in the Hitchin component of the representation space, or it lies in the Barbot component and the product of the three generators of the triangle group has distinct real eigenvalues.

Contents

1 Introduction .................................................. 1
2 Triangle reflection groups in \( SL(3, \mathbb{R}) \) .................. 6
3 Non–Anosov components ................................... 17
4 Constructing a boundary map .............................. 19
5Nested boxes ................................................. 23
6Duality ........................................................ 36
7Transversality .................................................. 37

1 Introduction

Given a finitely generated group \( \Gamma \) and a Lie group \( G \), it is a natural problem to find all discrete subgroups of \( G \) isomorphic to \( \Gamma \), or equivalently discrete and faithful representations of \( \Gamma \) into \( G \). When \( \Gamma \) is a fundamental group of a manifold or orbifold \( M \), this problem is closely linked to geometric structures on \( M \).

If \( \Gamma \) is the fundamental group of a closed surface \( S \) of genus \( g \geq 2 \) and \( G = \text{PGL}(2, \mathbb{R}) \), which is isomorphic to \( \text{Isom}(\mathbb{H}^2) \), the isometry group of the hyperbolic plane, then the discrete and faithful representations \( \Gamma \to G \) are fully understood: they form a union of

2020 Mathematics Subject Classification. 22E40, 51F15, 57S30

Key words and phrases. reflection groups, discrete subgroups of Lie groups, Anosov representations
two connected components of the representation space $\text{Hom}(\Gamma, G)$, and each representation up to conjugation corresponds to a hyperbolic structure on $S$. These components modulo conjugation are called the Teichmüller space of $S$.

In this paper, we are interested in the problem where $\Gamma$ is a compact hyperbolic triangle reflection group and $G = \text{PGL}(3, \mathbb{R}) \cong \text{SL}(3, \mathbb{R})$, the group of automorphisms of the projective plane $\mathbb{R}P^2$. Even for such “small” $\Gamma$ and $G$, it is still an open problem to understand all discrete and faithful representations of $\Gamma$ into $G$. Nevertheless, we can characterize all representations that are Anosov, a strengthening of discrete (see Theorem 1.2). This motivates us to conjecture that:

**Conjecture 1.1.** Let $\Gamma$ be a compact hyperbolic triangle reflection group. Then a representation $\rho: \Gamma \to \text{PGL}(3, \mathbb{R})$ is discrete and faithful if and only if $\rho$ lies in the closure of the space of Anosov representations in $\text{Hom}(\Gamma, \text{PGL}(3, \mathbb{R}))$.

### 1.1 Anosov representations

Anosov representations are discrete representations of a word–hyperbolic group $\Gamma$ into a Lie group $G$ with good dynamical properties. They have received a lot of attention and have been actively studied in recent years; see for example [Lab06; GW12; KLP17; GGKW17; BPS19]. Anosov representations have two key properties that set them apart from general discrete ones. The first is the existence of boundary maps (see Definition 2.8), which in fact characterizes Anosov representations with Zariski dense image. The second key property is openness: small deformations of Anosov representations are also Anosov. Hence if an Anosov representation is not isolated, it provides a family of new discrete representations.

Typical examples of Anosov representations of surface groups are Hitchin representations in a real split simple Lie group like $\text{PGL}(3, \mathbb{R})$, or maximal representations in a simple Lie group of Hermitian type. Similarly to surface group representations in $\text{PGL}(2, \mathbb{R})$, these representations form a closed and open subset of the representation space, hence a union of connected components. Such a component is called a higher Teichmüller space; see [GW18; Wie18].

In general, however, the space of Anosov representations is not closed in the representation space. An example is the component of surface group representations in $\text{PGL}(3, \mathbb{R})$ that contains a discrete and faithful representation preserving a disjoint pair of a point and a line in the projective plane; see [Bar10]. The shape of the space of Anosov representations, or even the number of its connected components, is not known in this case.
1.2 Results

The space of representations of a surface group into $\text{SL}(3, \mathbb{R})$ is very high–dimensional. We focus instead on the hyperbolic reflection group

$$\Gamma = \Gamma_{p_1, p_2, p_3} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_2 s_3)^{p_1} = (s_3 s_1)^{p_2} = (s_1 s_2)^{p_3} = 1 \rangle$$

where $2 \leq p_1 \leq p_2 \leq p_3$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. It is isomorphic to the group generated by reflections along the sides of the triangle with dihedral angles $\frac{\pi}{p_1}, \frac{\pi}{p_2}, \frac{\pi}{p_3}$ in the hyperbolic plane. Then the space of characters $\chi(\Gamma, \text{SL}(3, \mathbb{R}))$, which is the space of semisimple representations modulo conjugation, has dimension 0 or 1; see Lemmas 2.2 and 2.4.

As in the surface group case, there is a unique Hitchin component in $\chi(\Gamma, \text{SL}(3, \mathbb{R}))$, consisting of those representations which can be continuously deformed to a discrete and faithful representation into $\text{SO}(2, 1)$. If $p_1, p_2, p_3$ are all odd, there is also a unique component containing discrete and faithful representations which preserve a disjoint pair of a point and a line in the projective plane (see Section 2.3). We call it the Barbot component as it is analogous to the component studied in [Bar10].

Theorem 1.2. Let $\rho : \Gamma_{p_1, p_2, p_3} \to \text{SL}(3, \mathbb{R})$ be a representation.

(i) Assume $p_1, p_2, p_3$ are odd. Then $\rho$ is Anosov if and only if it is in the Hitchin component, or it is in the Barbot component and $\rho(s_1 s_2 s_3)$ has distinct real eigenvalues.

(ii) Otherwise, $\rho$ is Anosov if and only if it is in the Hitchin component.

In case (i), the set of Anosov characters in the Barbot component is the complement of a compact interval, as sketched in Figure 1.

Figure 1: top: sketch of the Barbot component showing the two open intervals of Anosov representations, each of them containing a single reducible representation. bottom: images of boundary maps into $\mathbb{R}P^2$ for three different representations of the $(3, 3, 5)$ triangle group. In case (B) the reducible representation shown is not semisimple, but line–irreducible (see Section 2.1).
While it has been known that every representation in the Hitchin component is Anosov [CG05; Lab06], our contribution in Theorem 1.2 is to show that \( \rho(s_1s_2s_3) \) having distinct eigenvalues is sufficient for Anosovness in the Barbot component, and that there are no Anosov examples in other components. As far as the authors know, our result is the first instance where a non–closed Anosov space in \( \text{Hom}(\Gamma, G) \) is completely identified for a nonelementary hyperbolic group \( \Gamma \) and a higher rank Lie group \( G \).

It is known that the representations on the boundary of the Anosov set are still discrete and faithful; see Remark 2.13. We show in Theorem 7.4 that they also admit continuous injective boundary maps. In contrast to Anosov representations, these boundary maps are not transverse; see Section 7.

An interesting consequence of Theorem 1.2, which demonstrates the explicitness of its criterion, is that the Anosov property can be checked with a few equalities and inequalities involving only the traces of group elements up to word length 3. Writing this out, we obtain the following.

**Corollary 1.3.** Let \( \rho: \Gamma_{p_1,p_2,p_3} \to SL(3, \mathbb{R}) \) be a representation. Assume that \( p_1 \geq 3 \) and define \( c_k = 2 \cos \frac{\pi}{p_k} \) for \( k \in \{1, 2, 3\} \), and

\[
\begin{align*}
t_1 &= \text{tr} \rho(s_2s_3), \quad t_2 = \text{tr} \rho(s_3s_1), \quad t_3 = \text{tr} \rho(s_1s_2), \\
x &= \text{tr} \rho(s_1s_2s_3), \quad y = \text{tr} \rho(s_3s_2s_1).
\end{align*}
\]

Then \( \rho \) is Anosov if and only if one of the following holds:

(i) \( t_k = c_k^2 - 1 \) for all \( k \) and \( x + t_1 + t_2 + t_3 < 0 \), or

(ii) \( p_1, p_2, p_3 \) are odd, \( t_k = 1 - c_k \) for all \( k \), and \( x^2y^2 - 4x^3 - 4y^3 + 18xy - 27 > 0 \).

**Remark 1.4.** Our result also gives some information about Anosov representations of surface groups: when \( p_1, p_2 \) and \( p_3 \) are odd, the fundamental group \( \pi_1(S_g) \) of the orientable surface \( S_g \) of genus \( g \) is a subgroup of \( \Gamma_{p_1,p_2,p_3} \) of finite index if and only if

\[
g = \frac{k}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} \right) \text{lcm}(p_1,p_2,p_3) + 1 \quad \text{for any } k \in \mathbb{N}
\]

where \( \text{lcm}(p_1, p_2, p_3) \) denotes the least common multiple of \( p_1, p_2 \) and \( p_3 \); see [EEK82]. In that case, the representations of \( \Gamma_{p_1,p_2,p_3} \) provide families of representations of the surface group \( \pi_1(S_g) \), and among such surface group representations we can characterize the Anosov ones.

**Remark 1.5.** Instead of compact hyperbolic triangle reflection groups, one may consider the “ideal hyperbolic triangle” reflection group \( W_3 = \mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2 \) along with certain unipotent conditions in order to obtain representations that preserve a circular limit curve. Such representations are not Anosov but may be regarded as “relatively Anosov” in a suitable sense.
When the target Lie group $G$ is a real form of $\text{SL}(3, \mathbb{C})$ the corresponding (relative) character space is also 1–dimensional and similar results have been obtained. More specifically, when $G = \text{Isom}(\mathbb{C}H^2) \cong \text{PU}(2, 1)$, the isometry group of the complex hyperbolic plane, Goldman and Parker [GP92] considered the representations $W_3 \to G$ that map the standard generators of $W_3$ to distinct, order two, complex reflections and satisfy the condition that any product of two distinct generators is parabolic. Among those representations, they conjectured exactly which ones are discrete and faithful. The conjecture was proved by Schwartz in [Sch01; Sch05]. Analogously, when $G = \text{SL}(3, \mathbb{R})$, Kim and Lee [KL] identified representations $W_3 \to G$ with an invariant circular limit curve in the flag manifold, among the representations that map the standard generators of $W_3$ to distinct involutions and satisfy the condition that any product of two distinct generators is “quasi-unipotent”.

1.3 Overview

In Section 2 we parametrize the space of characters, and review some properties of triangle reflection groups and Anosov representations. We also give a proof of Corollary 1.3 in Section 2.3, assuming Theorem 1.2. Then, in Section 3, we show that only the Hitchin and Barbot components can contain Anosov representations.

The remainder of the paper is devoted to proving that, if $\rho$ is in the Barbot component and $\rho(s_1s_2s_3)$ has distinct real eigenvalues, then $\rho$ is Anosov. We do this by approximating its boundary map with a collection of “boxes” in $\mathbb{R}P^2$. First, we show in Section 4 that if such boxes are mapped into each other by elements of $\Gamma$ in a certain way, they converge to a continuous boundary map. We then construct boxes with this property in Section 5, using the eigenvectors of $\rho(s_1s_2s_3)$. This yields a continuous boundary map into $\mathbb{R}P^2$. The next step, in Section 6, is to extend it to a map into the flag manifold. Finally, we show in Section 7 that the resulting map is transverse, and therefore $\rho$ is Anosov. The proof of Theorem 1.2 is given at the end of Section 7.

1.4 Acknowledgements

We are thankful for helpful conversations with Jean-Philippe Burelle, Jeffrey Danciger, and Richard Evan Schwartz.

G.-S. Lee was supported by the European Research Council under ERC-Consolidator Grant 614733 and by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No 2020R1C1C1A01013667). J. Lee is supported by the grant NRF-2019R1F1A1047703. F. Stecker received support from the Klaus Tschira Foundation, the RTG 2229 grant of the German Research Foundation and ERC grants 614733 and 715982.
2 Triangle reflection groups in $\text{SL}(3, \mathbb{R})$

2.1 Parametrizing Coxeter representations

Our goal is to find all Anosov representations of the hyperbolic triangle reflection group

$$\Gamma = \Gamma_{p_1, p_2, p_3} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_2 s_1)^{p_1} = (s_3 s_1)^{p_2} = (s_1 s_2)^{p_3} = 1 \rangle$$

into $\text{SL}(3, \mathbb{R})$, where $p_1, p_2, p_3 \geq 2$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. We call the product $s_1 s_2 s_3$ the Coxeter element.

In this section, we shall parametrize those representations $\rho : \Gamma \to \text{SL}(3, \mathbb{R})$ which send the generators $s_1, s_2, s_3$ to pairwise distinct non-trivial involutions in $\text{SL}(3, \mathbb{R})$. We call such a representation a Coxeter representation, and denote the space of them by $\text{Hom}^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R}))$. It is embedded into $\text{SL}(3, \mathbb{R})^3$ as the images of the generators $s_1, s_2, s_3$ and inherits its topology from this embedding.

The space of Coxeter characters is

$$\chi^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R})) = \text{Haus}(\text{Hom}^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R}))$$

where $\text{SL}(3, \mathbb{R})$ acts on the space of representations by conjugation, and Haus($X$) is the Hausdorff quotient of a topological space $X$. That is the quotient of $X$ by the equivalence relation $\sim$ defined by

$$x \sim y \iff x \approx y$$

for every equivalence relation $\approx$ such that $X/\approx$ is Hausdorff.

In particular, two points $x, y \in X$ with intersecting closures $\overline{\{x\}}$ and $\overline{\{y\}}$ represent the same element of Haus($X$). So two representations have the same character if their conjugacy classes have intersecting closures. The space of semisimple representations modulo conjugation is Hausdorff and it may be identified with the Hausdorff quotient of the representation space; see [Lun75; Lun76; RS90].

An involution $\sigma$ in $\text{SL}(3, \mathbb{R})$ can be written as

$$\sigma = b \otimes \alpha - 1, \quad \text{i.e.} \quad \sigma(v) = \alpha(v)b - v, \quad \forall v \in \mathbb{R}^3,$$

where $\alpha$ is a linear functional and $b$ is a vector of $\mathbb{R}^3$ such that $\alpha(b) = 2$. It uniquely determines the pair $(\alpha, b)$ up to the action of $\mathbb{R}^*$ by $\lambda \cdot (\alpha, b) = (\lambda \alpha, \lambda^{-1}b)$.

Lemma 2.1 ([Gol77, Chapter III]). Let $\sigma_1 = b_1 \otimes \alpha_1 - 1, \sigma_2 = b_2 \otimes \alpha_2 - 1$ be two distinct involutions in $\text{SL}(3, \mathbb{R})$ and $p \geq 2$ an integer. Then $(\sigma_1 \sigma_2)^p = 1$ if and only if

- $\alpha_1(b_2) \alpha_2(b_1) = 4 \cos^2\left(\frac{2\pi}{p}\right)$, where $1 \leq q \leq \frac{p}{2}$, and
- $\alpha_1(b_2)$ and $\alpha_2(b_1)$ are either both zero or both non-zero.
Proof. The subspace \( \ker \alpha_1 \cap \ker \alpha_2 \) of \( \mathbb{R}^3 \) is at least 1–dimensional, so 1 is an eigenvalue of \( \sigma_1 \sigma_2 \). A computation shows that

\[
\sigma_i \sigma_j = \alpha_i(b_j) b_i \otimes \alpha_j - b_i \otimes \alpha_i - b_j \otimes \alpha_j + 1 \quad \text{for} \ i, j = \{1, 2\},
\]

\[
\text{tr} \sigma_1 \sigma_2 = \alpha_1(b_2) \alpha_2(b_1) - 1.
\]

Now we assume \( (\sigma_1 \sigma_2)^p = 1 \). Then \( \sigma_1 \sigma_2 \) must be complex diagonalizable with eigenvalues \( 1, e^{2\pi iq/p}, e^{-2\pi iq/p} \), where \( 1 \leq q < p \). Possibly replacing \( q \) by \( p - q \), we can assume \( q \leq \frac{p}{2} \).

So \( \alpha_1(b_2) \alpha_2(b_1) = \text{tr} \sigma_1 \sigma_2 + 1 = 4 \cos^2\left(\frac{q}{p} \pi\right) \). If \( q = \frac{p}{2} \), we also have \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \). Since \( b_i \otimes \alpha_i \) and \( b_j \otimes \alpha_i \) are linearly independent in the space of \( 3 \times 3 \) matrices, this implies
\[
\alpha_1(b_2) = \alpha_2(b_1) = 0.
\]

Conversely, if \( \alpha_1(b_2) \alpha_2(b_1) = 4 \cos^2\left(\frac{q}{p} \pi\right) \) with \( 1 \leq q < \frac{p}{2} \), then \( \sigma_1 \sigma_2 \) has eigenvalues \( 1, e^{2\pi iq/p}, e^{-2\pi iq/p} \), so \( (\sigma_1 \sigma_2)^p = 1 \). If \( \alpha_1(b_2) = \alpha_2(b_1) = 0 \), then \( p \) is even and \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 = 1 \) by (1), so \( (\sigma_1 \sigma_2)^p = 1 \).

This motivates the following definition. A real matrix \( A = (a_{ij})_{1 \leq i, j \leq 3} \) is called a Cartan matrix if

(i) \( a_{ii} = 2 \) for all \( i = 1, 2, 3 \),

(ii) \( a_{ij}a_{ji} = 4 \cos^2\left(\frac{q_k}{p_k} \pi\right) \) for integers \( 1 \leq q_k \leq \frac{p_k}{2} \) and \( \{i, j, k\} = \{1, 2, 3\} \),

(iii) if \( a_{ij} = 0 \) then \( a_{ji} = 0 \) for all \( i, j = 1, 2, 3 \).

Two Cartan matrices are equivalent if they are conjugated by a diagonal matrix. We denote by \( \mathcal{C} \) the space of Cartan matrices modulo equivalence. It parametrizes the Coxeter characters as follows.

Lemma 2.2. The map

\[
\Psi : \chi^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R})) \to \mathcal{C}, \quad [\rho] \mapsto (\alpha_i(b_j))_{1 \leq i, j \leq 3},
\]

where \( \rho(s_i) = b_i \otimes \alpha_i - 1 \) and \( \alpha_i(b_i) = 2 \) for all \( i = 1, 2, 3 \), is a homeomorphism.

Proof. First note that since \( b_i \in \mathbb{R}^3 \) and \( \alpha_i \in (\mathbb{R}^3)^* \) are only determined up to the action of \( \mathbb{R}^* \), this gives us the matrix \((\alpha_i(b_j))_{1 \leq i, j \leq 3}\) up to equivalence. It is a Cartan matrix by Lemma 2.1. So \( \Psi \) is well–defined as a map from \( \text{Hom}^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R})) \), and it is continuous. Every continuous map from a topological space \( X \) to a Hausdorff space \( Y \) induces a unique continuous map from \( \text{Haus}(X) \) to \( Y \). So since \( \mathcal{C} \) is Hausdorff and \( \Psi \) is conjugation invariant, it descends to a map from \( \chi^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R})) \).

We construct a continuous map \( \Psi' \), which will be the inverse of \( \Psi \), as follows: For a Cartan matrix \( C \) we set \( \rho_C(s_i) = e_i \otimes \gamma_i - 1 \) where \( \{e_i\}_{i=1}^3 \) is the standard basis of \( \mathbb{R}^3 \), and \( \gamma_i \in (\mathbb{R}^3)^* \) is the \( i \)–th row of \( C \). Lemma 2.1 ensures that this indeed defines a representation of \( \Gamma \). If \( \Lambda \) is a diagonal matrix with entries \( \lambda_1, \lambda_2, \lambda_3 \) then the \( i \)–th row
of $\Lambda A^{-1}$ is $\lambda_i \Lambda^{-1}$, so $\rho_{\Lambda A^{-1}}(s_i) = \Lambda \rho_C(s_i) \Lambda^{-1}$. Hence we can define $\Psi'$ by setting $\Psi'([C]) = [\rho_C]$. It is easy to see that $\Psi \circ \Psi'$ is the identity map.

To see that $\Psi' \circ \Psi$ is also the identity, let $\rho$ be a representation defining $b_i$ and $\alpha_i$ as before, and let $A$ be the matrix with $i$–th row $\alpha_i$ and $B$ the matrix with $i$–th column $b_i$, for all $i$. Then the corresponding Cartan matrix is $C = AB$. We want to show that $[\rho_C] = [\rho]$. The matrix $B$ need not be invertible, but we can write $B = B\tilde{P}$ for an invertible matrix $\tilde{B}$ and a projection $\tilde{P}$ (that is $\tilde{P}^2 = \tilde{P}$). If we set $P_n = P + \frac{1}{n}(1 - P)$ then $P_n\rho = \tilde{P}P_n = P$, so $CP_n^{-1} = \tilde{B}\tilde{P}P_n^{-1} = C$ and $P_n^{-1}\tilde{B}^{-1}B = P_n^{-1}P = P$. Further $\tilde{A}\tilde{B}P_n$ converges to $C$, so we have

$\quad P_n\rho_C(s_i)P_n^{-1} = (P_n e_i) \otimes (\gamma_i P_n^{-1}) - 1 \rightarrow (P e_i) \otimes \gamma_i - 1,$

$\quad P_n^{-1}\tilde{B}^{-1}\rho(s_i)\tilde{B}P_n = (P_n^{-1}\tilde{B}^{-1}b_i) \otimes (\alpha_i\tilde{B}P_n) - 1 \rightarrow (P e_i) \otimes \gamma_i - 1.$

So the conjugacy classes of $\rho$ and $\rho_C$ have intersecting closures, which implies that they represent the same character in $\chi^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R}))$. □

We call a representation $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$

- **point–irreducible** if it does not preserve any one–dimensional subspace of $\mathbb{R}^3$,
- **line–irreducible** if it does not preserve any two–dimensional subspace of $\mathbb{R}^3$,
- **semisimple** if it is a product of irreducible representations.

Every Coxeter character $[\rho]$ has a point–irreducible, a line–irreducible, and a semisimple representative. An example for a line–irreducible one is $\rho_C$ as constructed in the proof of Lemma 2.2, and a point–irreducible representation can be obtained by a dual construction.

### 2.2 The space of Cartan matrices

The previous section showed that the Coxeter characters are parametrized by Cartan matrices. Luckily, the space of Cartan matrices $\mathcal{C}$ is quite simple. It consists of a number of connected components homeomorphic to $\mathbb{R}$ and possibly some isolated points.

**Definition 2.3.** Let $A = (a_{ij})_{i,j}$ be the Cartan matrix corresponding to a Coxeter representation $\rho$. We say $A$ is of type $(q_1, q_2, q_3)$ if $1 \leq q_k \leq \frac{2m_k}{\pi}$ and

$$a_{ij}a_{ji} = c_k^2, \quad c_k := 2 \cos \left( \frac{2m_k\pi}{p_k} \right)$$

for all $\{i, j, k\} = \{1, 2, 3\}$. These 2–cyclic products $a_{ij}a_{ji}$ are well–defined by the equivalence class $[A]$ of $A$. We also say a Coxeter representation $\rho: \Gamma_{p_1,p_2,p_3} \to \text{SL}(3, \mathbb{R})$ is of type $(q_1, q_2, q_3)$ if its Cartan matrix is.

Since $a_{ij}a_{ji}$ can take only a discrete set of values, the space $\mathcal{C}_{q_1,q_2,q_3} \subset \mathcal{C}$ of Cartan matrices of type $(q_1, q_2, q_3)$ is a union of connected components.
Lemma 2.4. If \( q_k = \frac{p_k}{2} \) for some \( k \in \{1,2,3\} \) then \( C_{q_1,q_2,q_3} \) is a single point. Otherwise it has two connected components, each homeomorphic to \( \mathbb{R} \).

Proof. If \( q_k = \frac{p_k}{2} \) for some \( k \), then \( a_{ij} = a_{ji} = 0 \) for \( \{i,j,k\} = \{1,2,3\} \). The Cartan matrix \( A \) is therefore equivalent to a symmetric matrix which is determined by \( (q_1,q_2,q_3) \) alone. For example, if \( q_1 = \frac{p_1}{2} \) then \( A \) is
\[
\begin{pmatrix}
 2 & a_{12} & a_{13} \\
 a_{21} & 2 & 0 \\
a_{31} & 0 & 2
\end{pmatrix}
\sim
\begin{pmatrix}
 2 & c_3 & c_2 \\
c_3 & 2 & 0 \\
c_2 & 0 & 2
\end{pmatrix}.
\]
So \( C_{q_1,q_2,q_3} \) is just a single point in this case.

If \( q_k < \frac{p_k}{2} \) for all \( k \in \{1,2,3\} \) then \( A \) is equivalent to a matrix of the form
\[
\begin{pmatrix}
 2 & -c_3 & -c_2 \\
-c_3 & 2 & -tc_1 \\
-c_2 & -t^{-1}c_1 & 2
\end{pmatrix}
\]
where \( t \in \mathbb{R} \setminus \{0\} \) (the minus signs are just a convention). This representative is unique since \( t = -a_{12}a_{23}a_{31}/c_1c_2c_3 \), which only depends on the equivalence class. So \( C_{q_1,q_2,q_3} \cong \mathbb{R} \setminus \{0\} \).

For a Coxeter representation \( \rho \colon \Gamma_{p_1,p_2,p_3} \to \text{SL}(3,\mathbb{R}) \) of type \( (q_1,q_2,q_3) \) with Cartan matrix \( (a_{ij})_{i,j} \) we therefore define its parameter
\[
t_\rho = -\frac{a_{12}a_{23}a_{31}}{c_1c_2c_3}.
\] (2)

If \( q_k < \frac{p_k}{2} \) for all \( k \in \{1,2,3\} \) then \( t_\rho \) can take any nonzero real value and parametrizes the Coxeter characters of type \( (q_1,q_2,q_3) \).

We can express the 2-cyclic products and 3-cyclic products by traces:
\[
\begin{align*}
\text{tr} \rho(s_1s_2) &= a_{12}a_{21} - 1, & \text{tr} \rho(s_2s_3) &= a_{23}a_{32} - 1, & \text{tr} \rho(s_3s_1) &= a_{31}a_{13} - 1, \\
\text{tr} \rho(s_1s_2s_3) &= a_{12}a_{23}a_{31} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13} + 3, \\
\text{tr} \rho(s_3s_2s_1) &= a_{21}a_{32}a_{13} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13} + 3.
\end{align*}
\] (3) (4) (5)

We immediately see from this that the determinant of the Cartan matrix is
\[
\det \begin{pmatrix}
 2 & a_{12} & a_{13} \\
 a_{21} & 2 & a_{23} \\
a_{31} & a_{32} & 2
\end{pmatrix} = \text{tr} \rho(s_1s_2s_3) + \text{tr} \rho(s_3s_2s_1) + 2.
\] (6)

Note the \( \rho \) is reducible if and only if this determinant is zero.
Lemma 2.5. Let $\rho : \Gamma \to \text{SL}(3, \mathbb{R})$ be a Coxeter representation. Assume that $p_1, p_2, p_3$ are odd and $\rho$ is of type $(\frac{p_1-1}{2}, \frac{p_2-1}{2}, \frac{p_3-1}{2})$. Then there is a real number $t_{\text{crit}} > 1$ such that $\rho(s_1s_2s_3)$

- has two non-real eigenvalues if $t_\rho < 0$ or $t_\rho \in (t_{\text{crit}}^{-1}, t_{\text{crit}})$,
- has a double eigenvalue and is not diagonalizable if $t_\rho \in \{t_{\text{crit}}^{-1}, t_{\text{crit}}\}$,
- has real eigenvalues with distinct absolute values if $t_\rho \in (0, t_{\text{crit}}^{-1}) \cup (t_{\text{crit}}, \infty)$.

Proof. By the proof of Lemma 2.4, the Cartan matrix of $\rho$ is uniquely equivalent to

$$A = \begin{pmatrix} 2 & -c_3 & -c_2 \\ -c_3 & 2 & -t_\rho c_1 \\ -c_2 & -t_\rho^{-1} c_1 & 2 \end{pmatrix},$$

where $t_\rho \neq 0$ and $c_i = 2 \cos\left(\frac{p_i-1}{2p_i} \pi\right)$. Then (4) and (5) above show that:

$$x = \text{tr} \rho(s_1s_2s_3) = -t_\rho c_1c_2c_3 - c_1^2 - c_2^2 - c_3^2 + 3$$
$$y = \text{tr} \rho(s_3s_2s_1) = -t_\rho^{-1} c_1c_2c_3 - c_1^2 - c_2^2 - c_3^2 + 3$$

Thus, the variables $x$ and $y$ satisfy

$$p(x, y) := (x - 3 + c_1^2 + c_2^2 + c_3^2)(y - 3 + c_1^2 + c_2^2 + c_3^2) = c_1^2 c_2^2 c_3^2.$$
From the discriminant of the characteristic polynomial (see [Gol90, Section 1.7]) we find that \(\rho(s_1s_2s_3)\) has three distinct real eigenvalues if and only if 
\[
\delta(x, y) := x^2y^2 - 4(x^3 + y^3) + 18xy - 27 > 0. 
\]  
(9) 

Using a change of variables \((u = \frac{x+y}{2}, v = \frac{x-y}{2})\), we obtain that:
\[
p(x, y) = c_1^2c_2^2c_3^2 \Leftrightarrow v^2 = (u - 3 + c_1^2 + c_2^2 + c_3^2)^2 - (c_1c_2c_3)^2 =: f(u)
\]
\[
\delta(x, y) = 0 \Leftrightarrow v^2 = u^2 + 12u + 9 \pm 2(2u + 3)^2 =: g_\pm(u)
\]

We set \(u_\pm := 3 - c_1^2 - c_2^2 - c_3^2 \pm c_1c_2c_3\), which are the two solutions of the equation \(f(u) = 0\). The points \((u, v) = (u_\pm, 0)\) correspond to the Coxeter characters with \(t_\rho = \mp 1\). Since \(p_i \geq 3\) we have \(0 < c_i \leq 1\), and \(c_i \leq 2\cos(\frac{2\pi}{3})\) for at least one \(i \in \{1, 2, 3\}\). Then a computation shows that \(0 \leq u_- < u_+ < 3\).

We now claim that the red curve in Figure 2, that is the connected component of the curve \(v^2 = f(u)\) containing the point \((u_+, 0)\), does not intersect with the black curve \(v^2 = g_\pm(u)\). Since
\[
\frac{df}{du} = 2(u - 3 + c_1^2 + c_2^2 + c_3^2) \quad \text{and} \quad \frac{dg_\pm}{du} = 2u + 12 \pm 6\sqrt{2u + 3},
\]
we have that for all \(u \geq -1\)
\[
\frac{dg_+}{du} - \frac{df}{du} = 6\sqrt{2u + 3} + 18 - 2(c_1^2 + c_2^2 + c_3^2) \geq 0
\]
and for all \(u \geq 3\)
\[
\frac{df}{du} - \frac{dg_-}{du} = 6\sqrt{2u + 3} - 18 + 2(c_1^2 + c_2^2 + c_3^2) \geq 0.
\]

Thus the red curve has no intersection with the black curve, as claimed.

As a result, if \(t_\rho < 0\), then \(\rho(s_1s_2s_3)\) has two non-real eigenvalues. Moreover, since \(\frac{df}{du} < 0\) for all \(u \leq u_-\), the blue curve, which is the connected component of the curve \(v^2 = f(u)\) containing the point \((u_-, 0)\), intersects \(v^2 = g_+(u)\) at two points \((u_\text{crit}, v_\text{crit})\) and \((u_\text{crit}, -v_\text{crit})\) with \(-1 < u_\text{crit} < u_-\), corresponding to \(t_\rho \in \{t_\text{crit}^{-1}, t_\text{crit}\}\). So \(\rho(s_1s_2s_3)\) has two non-real eigenvalues (resp. a double eigenvalue, resp. distinct real eigenvalues) if \(t_\rho \in (t_\text{crit}^{-1}, t_\text{crit})\) (resp. \(t_\rho \in \{t_\text{crit}^{-1}, t_\text{crit}\}\), resp. \(t_\rho \in (0, t_\text{crit}^{-1}) \cup (t_\text{crit}, \infty)\)). It is easy to see that if \(\rho(s_1s_2s_3)\) has eigenvalues of the form \(\lambda, -\lambda, -\lambda^{-2}\), then \(v^2 = u^2 - 1\) and \(u < 0\). But since \(u_- \geq 0\) this curve does not intersect \(v^2 = f(u)\).

In the case \(t_\rho \in \{t_\text{crit}^{-1}, t_\text{crit}\}, \rho(s_1s_2s_3)\) is not diagonalizable: assume it were, then it fixes a projective line pointwise. The intersection point \(p\) of this line with the reflection line of \(\rho(s_1)\) is fixed by \(\rho(s_2s_3)\). This element has a unique fixed point, which is fixed by \(\rho(s_2)\) and \(\rho(s_3)\) individually. So all of \(\rho(\Gamma)\) fixes \(p\), hence \(\rho\) is reducible. By (6), this implies \(u_\text{crit} = -1\), a contradiction. \(\Box\)
2.3 From $\text{PGL}(2, \mathbb{R})$ to $\text{SL}(3, \mathbb{R})$

The Hitchin and Barbot components in $\chi^{\text{Cox}}(\gamma, \text{SL}(3, \mathbb{R}))$ are distinguished by the fact that they contain certain representations factoring through $\text{PGL}(2, \mathbb{R})$ or $\text{SL}^{\pm}(2, \mathbb{R})$. We will describe these now. We start with a discrete and faithful representation $\rho_0: \Gamma \to \text{PGL}(2, \mathbb{R})$.

Let $\iota: \text{PGL}(2, \mathbb{R}) \to \text{PGL}(3, \mathbb{R}) \cong \text{SL}(3, \mathbb{R})$ be the irreducible embedding, which is unique up to conjugation. Concretely, it can be realized by the action of $\text{PGL}(2, \mathbb{R})$ on the projectivization of the symmetric square $\text{Sym}^2 \mathbb{R}^2 \cong \mathbb{R}^3$. The composition

$$\rho_F = \iota \circ \rho_0: \Gamma \to \text{SL}(3, \mathbb{R})$$

is called the Fuchsian representation and its component of $\chi^{\text{Cox}}(\Gamma, \text{SL}(3, \mathbb{R}))$ is the Hitchin component.

A second way to create special $\text{SL}(3, \mathbb{R})$ representations out of $\rho_0$ is by using the embedding $j: \text{SL}^{\pm}(2, \mathbb{R}) \to \text{SL}(3, \mathbb{R})$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \text{det}(A) \end{pmatrix}$.

Here $\text{SL}^{\pm}(2, \mathbb{R})$ is the group of $2 \times 2$ matrices with determinant $\pm 1$. This requires lifting $\rho_0$ to $\text{SL}^{\pm}(2, \mathbb{R})$, which is possible if and only if $p_1, p_2, p_3$ are all odd.

To see this, we first note that each $\rho_0(s_i)$ is a hyperbolic involution acting on $\mathbb{RP}^1$ with two distinct fixed points. In order to specify a lift of $\rho_0(s_i)$ in $\text{SL}^{\pm}(2, \mathbb{R})$ we put an arbitrary order on these fixed points and regard them as representing an oriented geodesic $s_i^-$ to $s_i^+$ in $\mathbb{H}^2$. The lift $\tilde{\rho}_0(s_i)$ corresponding to $s_i^- s_i^+$ is defined as the reflection having $s_i^+$ as the $(+1)$-eigenspace and $s_i^-$ as the $(-1)$-eigenspace.

If $0 < \theta < \pi$ is the angle between the two intersecting oriented geodesics $s_1^- s_1^+$ and $s_2^- s_2^+$ then $\theta = \frac{\pi}{p_3}$ or $\theta = \pi - \frac{\pi}{p_3}$ depending on the chosen orientations. The product $\tilde{\rho}_0(s_1 s_2)$ is conjugate in $\text{SL}^{\pm}(2, \mathbb{R})$ to the rotation matrix $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. In order for $\rho_0$ to lift it is necessary that $R(\theta) p_3 = \text{id}$. But $R(\pi/p_3) p_3 = -\text{id}$ and $R(\pi - \pi/p_3) p_3 = (-1)^{p_3}(-\text{id})$, so $\rho_0$ can lift only if $p_3$ is odd.
If $p_1, p_2, p_3$ are odd, there are two possible lifts of $\rho_0$, corresponding to the choices of orientations with angles $\pi - \frac{\pi}{p_k}$ between $s_i^+$ and $s_j^+$, for all $\{i, j, k\} = \{1, 2, 3\}$; see the picture above. Let $\tilde{\rho}_0 : \Gamma \to \text{SL}^\pm(2, \mathbb{R})$ be the unique lift with $\text{tr} \tilde{\rho}_0(s_1 s_2 s_3) < 0$. Composing with $j$, we obtain the two representations

$$\rho_{\text{red}}, \tilde{\rho}_{\text{red}} : \Gamma \to \text{SL}(3, \mathbb{R}), \quad \rho_{\text{red}}(\gamma) = j(\tilde{\rho}_0(\gamma)), \quad \tilde{\rho}_{\text{red}}(\gamma) = j((-1)^{\ell(\gamma)} \tilde{\rho}_0(\gamma)) \quad \forall \gamma \in \Gamma,$$

where $\ell(\gamma)$ is the word length of $\gamma$.

**Lemma 2.6.** The representation $\rho_F$ is of type $(1, 1, 1)$ and has the parameter $t_{\rho_F} = 1$, and $\rho_{\text{red}}, \tilde{\rho}_{\text{red}}$ are of type $({\rho_{\text{red}}}_1, {\rho_{\text{red}}}_2, {\rho_{\text{red}}}_3)$ with $t_{\rho_{\text{red}}} = t_{\tilde{\rho}_{\text{red}}} = t_{\text{red}}^{-1}$, where $t_{\text{red}} > t_{\text{crit}} > 1$.

Furthermore, $\rho_{\text{red}}(s_1 s_2 s_3)$ has eigenvalues $-\lambda, -1, \lambda^{-1}$, for some $\lambda > 1$.

**Proof.** If $\rho$ is a Coxeter representation of type $(1, 1, 1)$ with $t_\rho = 1$, then it is the Fuchsian representation; see e.g. [Vin71, Proposition 24]. So, $\rho_F$ has type $(1, 1, 1)$ and $t_{\rho_F} = 1$.

In the case of $\rho_{\text{red}}$ we have $\text{tr}(A) = \text{tr}(A) + \det(A)$, so

$$\text{tr} \rho_{\text{red}}(s_i s_j) = \text{tr} j(\tilde{\rho}_0(s_i s_j)) = 2 \cos(\pi - \frac{\pi}{p_k}) + 1 = 4 \cos^2(\frac{p_k}{2 p_k} - \pi) - 1.$$

Hence $\rho_{\text{red}}$ is of type $({\rho_{\text{red}}}_1, {\rho_{\text{red}}}_2, {\rho_{\text{red}}}_3)$. To find $t_{\rho_{\text{red}}}$ we use that $\rho_{\text{red}}$ is reducible, so

$$-c_1 c_2 c_3 (t_{\rho_{\text{red}}} + t_{\rho_{\text{red}}}^{-1}) = \text{tr} \rho_{\text{red}}(s_1 s_2 s_3) + \text{tr} \rho_{\text{red}}(s_3 s_2 s_1) = -2.$$

This equation has exactly two solutions, which are positive and inverses of each other.

In $\text{SL}^\pm(2, \mathbb{R})$ we have $\text{tr} A^{-1} = \text{tr} A/\det A$, so our convention $\text{tr} \tilde{\rho}_0(s_1 s_2 s_3) < 0$ implies that $\text{tr} \tilde{\rho}_0(s_3 s_2 s_1) > 0$. Hence $\text{tr} \rho_{\text{red}}(s_3 s_2 s_1) < \text{tr} \rho_{\text{red}}(s_1 s_2 s_3)$. As $\rho_{\text{red}}(s_1 s_2 s_3)$ has a $-1$ eigenvalue, its eigenvalues must be of the form $-\lambda, -1, \lambda^{-1}$ for $\lambda > 1$. By (7) and (8) we also have $t_{\rho_{\text{red}}} > t_{\rho_{\text{red}}}^{-1}$. The inequality $t_{\rho_{\text{red}}} > t_{\text{crit}}$ then follows from Lemma 2.5 and the fact that $\rho_{\text{red}}(s_1 s_2 s_3)$ has three distinct real eigenvalues. \qed

**Definition 2.7.** A Coxeter representation $\rho : \Gamma_{p_1, p_2, p_3} \to \text{SL}(3, \mathbb{R})$ is in the

- **Hitchin component** if $\rho$ has type $(1, 1, 1)$ and $t_\rho > 0$,

- **Barbot component** if $\rho$ has type $(\frac{p_1 - 1}{2}, \frac{p_2 - 1}{2}, \frac{p_3 - 1}{2})$ and $t_\rho > 0$.

Now we can prove Corollary 1.3 from Theorem 1.2:

**Proof of Corollary 1.3.** Equations (3), (4) and (5) make it easy to identify the Hitchin and Barbot components using traces: by (2) the sign of $t_\rho$ is opposite to that of

$$\tau := a_{12} a_{23} a_{31} = \text{tr} \rho(s_1 s_2 s_3) + \text{tr} \rho(s_1 s_2) + \text{tr} \rho(s_2 s_3) + \text{tr} \rho(s_3 s_1)$$

If $\rho$ is not a Coxeter representation then $\text{tr} \rho(s_i s_j) \in \{-1, 3\}$ for at least one distinct pair $i, j \in \{1, 2, 3\}$. So $\rho$ is in the Hitchin component if and only if $\text{tr} \rho(s_i s_j) = 4 \cos^2(\frac{\pi}{p_k}) - 1$ for $\{i, j, k\} = \{1, 2, 3\}$ and $\tau < 0$, and in the Barbot component if and only if $\text{tr} \rho(s_i s_j) = 4 \cos^2(\frac{p_k - 1}{2 p_k} - \pi) = 1 = 2 \cos(\frac{\pi}{p_k})$ and $\tau < 0$. Together with (9), Theorem 1.2 therefore implies Corollary 1.3. \qed
2.4 Hyperbolic geometry of Coxeter axes

In this section, we describe some aspects of the geometry and combinatorics of Coxeter axes, which will be used in Section 5 and Section 7. Again, fix a discrete and faithful representation \( \rho_0 : \Gamma \to \text{PGL}(2, \mathbb{R}) \). It is unique up to conjugation and its generators \( \rho_0(s_1), \rho_0(s_2), \rho_0(s_3) \) are the reflections on the sides of a hyperbolic triangle \( T \) with angles \( \frac{\pi}{p_1}, \frac{\pi}{p_2}, \frac{\pi}{p_3} \). This triangle \( T \) is a fundamental domain for \( \Gamma \) and its translates tile the hyperbolic plane.

Figure 3: left: tiling of the hyperbolic plane in the case \( p_1 = p_2 = p_3 = 5 \), with the Coxeter axes in red. right: three triangles along the axis of \( s_3s_1s_2 \).

Adding the axes of all conjugates of the Coxeter element \( s_1s_2s_3 \) (shown in red in Figure 3) gives a finer tessellation. To understand its geometry, we consider the union \( s_2T \cup T \cup s_3T \cup s_3s_1T \) as in the right part of Figure 3. Let \( t_1, t_2, t_3 \) be the vertices of \( T \) and let \( A \) be the altitude triangle of \( T \), that is the vertices \( a_1, a_2, a_3 \) of \( A \) are the base points of the three altitudes of \( T \). Note that every geodesic in \( \mathbb{H}^2 \) intersects a \( \Gamma \)-translate of \( A \), since their complement is a disconnected union of compact polygons.

It is an elementary fact, true in hyperbolic as in Euclidean geometry, that the orthocenter of \( T \) is the incenter of \( A \); see [Fen89, Section VI.7]. In particular, \( \angle t_2a_2a_1 = \angle t_2a_2a_3 \) and hence the points \( s_2a_1, a_2, a_3 \) lie on a common geodesic. By the same argument, \( s_3a_1 \) and \( s_3s_1a_2 \) are also on that geodesic, which is therefore the axis of \( s_3s_1s_2 \).

Let \( F = s_2T \cup T \cup s_3T \) and consider the union \( N \) of its orbit under the glide reflection \( s_3s_1s_2 \). That is, \( s_3s_1s_2 \) acts on \( N \) with fundamental domain \( F \). We can also define \( N \) as
the union of all triangles that intersect the axis of $s_3s_1s_2$ (in the $\Gamma$–tessellation of $\mathbb{H}^2$ by $T$). In any case, $N$ is a neighborhood of the axis of $s_3s_1s_2$ with two piecewise geodesic boundaries. We claim that $N$ is convex if $p_1, p_2, p_3 \geq 3$. To see this, take a look at the vertices on the boundary of $N$. Every vertex belongs to three triangles in $N$ and the adjacent angles are all $\frac{\pi}{p_i}$ for some $k$. So their sum is at most $\pi = 3 \cdot \frac{\pi}{3}$. This means $N$ is a convex neighborhood of the axis. Hence this axis cannot intersect the reflection line of $s_1$. 

Now consider the $\langle s_1, s_2 \rangle$–orbit of the point $(s_1s_2s_3)_+$, the black dots in Figure 3. The sector bounded by the reflection lines of $s_1$ and $s_2$ contains exactly one orbit point. Since the axis of $s_3s_1s_2$ does not intersect $s_1$, and intersects $s_2$ before $s_3$, $(s_3s_1s_2)_+$ is this point. Then $(s_1s_2s_3)_+ = s_1s_2(s_3s_1s_2)_+$ is two sectors away. So we can label the orbit points $z_0, \ldots, z_{2p_3-1}$ in order along $S^1$, so that $z_0 = (s_1s_2s_3)_+$ and $z_2 = (s_3s_1s_2)_+$, and we have (with indices mod $2p_3$)

$$s_1z_i = z_{3-i}, \quad s_2z_i = z_{5-i} \quad \forall i \in \{0, \ldots, 2p_3 - 1\}.$$ 

### 2.5 Anosov representations

Finally, we will define Anosov representations and list their most important properties. Although they can be defined for any hyperbolic group $G$, we restrict to the case of triangle reflection groups $\Gamma$ and $G = \text{SL}(3, \mathbb{R})$. As before, we fix a discrete and faithful representation $\rho_0: \Gamma \to \text{PGL}(2, \mathbb{R})$. The corresponding action of $\Gamma$ on $\mathbb{H}^2$ extends to the visual boundary $S^1 = \partial \mathbb{H}^2$, which can be identified with the Gromov boundary $\partial \Gamma$ of $\Gamma$ as a word hyperbolic group. 

Let $\mathcal{F}$ be the flag manifold in $\mathbb{R}^3$, that is the space of all pairs $F = (F^{(1)}, F^{(2)})$ (called flags) where $F^{(i)}$ is an $i$–dimensional subspace of $\mathbb{R}^3$ and $F^{(1)} \subset F^{(2)}$. Alternatively $\mathcal{F}$ is the homogeneous space $\text{SL}(3, \mathbb{R})/B$ where $B$ is the subgroup of upper triangular matrices with determinant 1. It carries a natural action of $\text{SL}(3, \mathbb{R})$. Two flags $F, F'$ are transverse if $F^{(1)} \not\subset F'^{(2)}$ and $F'^{(1)} \not\subset F^{(2)}$.

**Definition 2.8.** A representation $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$ is Anosov if

1. there exists a map $\xi: S^1 \to \mathcal{F}$

   which is $\rho$–equivariant and continuous, maps the attracting fixed point $\gamma_+$ of every infinite order element $\gamma \in \Gamma$ to an attracting fixed point of $\rho(\gamma_+)$, and $\xi(x)$ and $\xi(y)$ are transverse whenever $x \neq y$, and

2. for every sequence $\gamma_n \to \infty$ in $\Gamma$ we have

   $${\sigma_1}(\rho(\gamma_n)) \to \infty, \quad {\sigma_2}(\rho(\gamma_n)) \to \infty,$$

   where $\sigma_1(A) \geq \sigma_2(A) \geq \sigma_3(A) \geq 0$ are the singular values of a matrix $A$. 


Such a map $\xi$ is unique and is called the \textit{limit curve} or \textit{boundary map} of $\rho$.

**Remark 2.9.** Definitions of Anosov representations into more general Lie groups usually have an additional qualifier, e.g. $P$–Anosov, $i$–Anosov, Borel Anosov, projective Anosov etc. In $\text{SL}(3, \mathbb{R})$ all these notions are equivalent, so we can just call them “Anosov”.

**Fact 2.10.** Anosov representations have a number of desirable properties, including

(i) The space of Anosov representations is open in $\text{Hom}(\Gamma, \text{SL}(3, \mathbb{R}))$.

(ii) If $\rho$ and $\rho'$ define the same point in $\chi(\Gamma, \text{SL}(3, \mathbb{R}))$, then $\rho$ is Anosov if and only if $\rho'$ is Anosov.

(iii) The image $\rho(\Gamma)$ of an Anosov representation $\rho$ is discrete in $\text{SL}(3, \mathbb{R})$.

(iv) If $\rho$ is Anosov and $\gamma \in \Gamma$ has infinite order then $\rho(\gamma)$ has distinct real eigenvalues.

(v) The boundary map varies continuously with the representation. More precisely, the map $\text{Hom}_{\text{Anosov}}(\Gamma, \text{SL}(3, \mathbb{R})) \to C^0(S^1, \mathcal{F})$ mapping a representation $\rho$ to its boundary map $\xi$ is continuous.

See [GW12; GGKW17] for proofs of these facts and more information on Anosov representations.

The representations $\rho_F, \rho_{\text{red}}$ and $\rho'_{\text{red}}$ from Section 2.3 are Anosov: since $\iota$ and $j$ map upper triangular matrices in $\text{PGL}(2, \mathbb{R})$ and $\text{SL}_\pm(2, \mathbb{R})$ into $B$, they induce maps $\mathbb{R}P^1 \to \mathcal{F}$, which are the boundary maps of $\rho_F$ respectively $\rho_{\text{red}}, \rho'_{\text{red}}$. Here $\mathbb{R}P^1$ is identified with $S^1 = \partial \mathbb{H}^2$ as the boundary of the upper half–plane model. It is easy to check that they satisfy all assumptions in Definition 2.8.

It is well–known that all representations in the same component as $\rho_F$ (the Hitchin component) are Anosov [CG05; Lab06]. More on Hitchin components of orbifold groups can be found in [ALS].

To prove that representations are Anosov, we will use another lemma from [GW12], which says that some parts of the definition are redundant for irreducible representations:

**Fact 2.11** ([GW12]). An irreducible representation $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$ is Anosov if and only if there exists a map $\xi: S^1 \to \mathcal{F}$ which is $\rho$–equivariant, continuous and transverse.

While Anosov representations of general hyperbolic groups can have a finite non–trivial kernel, they are always faithful for triangle groups.

**Lemma 2.12.** If a representation $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$ is Anosov, then it is faithful.
Proof. By [GW12, Theorem 1.7], the kernel of ρ is finite. But, since Γ is an irreducible infinite Coxeter group, any non-trivial normal subgroup of Γ is infinite (see e.g. Assertion 2 in the proof of [Par07, Proposition 4.3]). Here by irreducible, we mean that Γ cannot be written as the direct product of two nontrivial subgroups each of which is generated by a subset of the generating set \{s_1, s_2, s_3\}. So, the kernel of ρ is trivial, i.e. ρ is faithful. 

Remark 2.13. While the set of Anosov representations is open, the set of discrete and faithful representations is a closed subset of Hom(Γ, SL(3, R)). A proof is given in [GM87, Theorem 1.1] or [Kap01, Theorem 8.4].

3 Non–Anosov components

In this section we show that an Anosov representation ρ: Γ \rightarrow SL(3, R) of a hyperbolic triangle reflection group is either of type (1, 1, 1) or \(p_1, p_2, p_3\) are all odd and ρ is of type \((p_1 - 1, p_2 - 1, p_3 - 1)\). The basic topological reason is that a loop with winding number greater than 2 cannot be embedded in \(\mathbb{RP}^2\). We can restrict our attention to Coxeter representations as all others have a finite image (trivial, \(\mathbb{Z}/2\mathbb{Z}\) or a dihedral group), and thus cannot be Anosov.

Lemma 3.1. For integers \(p \geq 3\) and 1 \(\leq k \leq \frac{p}{2}\), let \(R \in SL(3, R)\) be a rotation by the angle \(\frac{2\pi}{p}\) (that is, \(R\) has eigenvalues 1 and \(e^{\pm 2\pi i/p}\) and \(γ: S^1 \rightarrow \mathbb{RP}^2\) an injective continuous curve satisfying \(γ(t + \frac{1}{p}) = R^k γ(t)\) for all \(t \in S^1 \cong \mathbb{R}/\mathbb{Z}\). Then

(i) either \(k = 1\) and \(γ\) is null–homotopic.

(ii) or \(p\) is odd, \(k = \frac{p-1}{2}\) and \(γ\) is not null–homotopic.

Proof. Observe that \(γ(\frac{1}{\gcd(k, p)}) = γ(0)\), so \(k\) and \(p\) must be coprime. In this case \(k\) is invertible modulo \(p\), i.e. there is an integer 1 \(\leq l < p\) with \(kl \equiv 1\) mod \(p\). If \(p = 3\) the statement is trivially true, so we can assume \(p \geq 4\).

We will use this simple consequence of the Jordan curve theorem: If \(x, y, z, w\) are distinct points on the boundary of a disk in this cyclic order, and \(x\) and \(z\) as well as \(y\) and \(w\) are connected by curves in the closed disk, then these curves intersect.

We pass to the universal cover \(S^2 \rightarrow \mathbb{RP}^2\) and write \(ι: S^2 \rightarrow S^2\) for its nontrivial deck transformation, the antipodal involution. Let \(\hat{γ}: [0, 1] \rightarrow S^2\) be one lift of \(γ\), the other one being \(ι \circ \hat{γ}\). The matrix \(R\) still acts as a rotation by \(\frac{2\pi}{p}\) on \(S^2\). Its two fixed points cannot be in the image of \(\hat{γ}\) or \(ι \circ \hat{γ}\). Choose one of them and let \(D\) be the smallest \(R\)-invariant closed disk around it which contains the images of \(\hat{γ}\) and \(ι \circ \hat{γ}\). We can assume that \(∂D\) intersects \(\hat{γ}\) in at least one point \(w\) (otherwise replace \(D\) by \(ιD\)). We may also assume that \(\hat{γ}(0) = w\).
The symmetry of $\gamma$ can lift in two ways: either $\hat{\gamma}(t + \frac{1}{p}) = R^k\hat{\gamma}(t)$ or $\hat{\gamma}(t + \frac{1}{p}) = \iota(R^k\hat{\gamma}(t))$. By continuity one of these relations holds for all $t \in S^1$. In the first case, consider the arcs $\hat{\gamma}|_{[0,1/p]}$ and $\hat{\gamma}|_{[1/p,(1+1)/p]}$. Their endpoints are

$$\hat{\gamma}(\{0, \frac{1}{p}\}) = \{w, R^k w\}, \quad \hat{\gamma}(\{\frac{1}{p}, \frac{1+1}{p}\}) = \{Rw, R^{k+1}w\}.$$ 

If $k \neq 1$ then these four points are distinct and their cyclic order along $\partial D$ is $w, Rw, R^kw, R^{k+1}w$. So the arcs have to intersect, which is a contradiction to the injectivity of $\gamma$. Furthermore, $\hat{\gamma}(1) = R^{kp}\hat{\gamma}(0) = \hat{\gamma}(0)$, so $\gamma$ is null–homotopic.

Now assume the second case, $\hat{\gamma}(t + \frac{1}{p}) = \iota(R^k\hat{\gamma}(t))$. Then we consider instead the arcs $\hat{\gamma}|_{[0,2/p]}$ and $\iota^l \circ \hat{\gamma}|_{[l/p,(l+2)/p]}$. Their endpoints are

$$\hat{\gamma}(\{0, \frac{2}{p}\}) = \{w, R^{2k}w\}, \quad \iota^l \hat{\gamma}(\{\frac{l}{p}, \frac{l+2}{p}\}) = \{Rw, R^{2k+1}w\}.$$ 

If these points are distinct their cyclic order is $w, Rw, R^{2k}w, R^{2k+1}w$, which again contradicts $\gamma$ being injective. So $2k$ must be congruent to $-1$, $0$, or $1$ modulo $p$. This only happens if $p$ is odd and $k = \frac{p-1}{2}$. In this case $\hat{\gamma}(1) = \iota^p R^{kp}\hat{\gamma}(0) = \iota \hat{\gamma}(0)$ since $p$ is odd, so $\gamma$ is not null–homotopic.

**Lemma 3.2.** Assume that one of $p_1, p_2, p_3$ equals 2 and $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$ is a Coxeter representation which is Anosov. Then $\rho$ is of type $(1,1,1)$.

**Proof.** We showed in the proof of Lemma 2.4 that if one of $p_1, p_2, p_3$ equals 2, then the Cartan matrix $A = (\alpha_i(b_j))_{1 \leq i,j \leq 3} = (a_{ij})_{1 \leq i,j \leq 3}$ is equivalent to a symmetric matrix. We may assume that $A$ is symmetric. Then there exists a scalar product $(\cdot, \cdot)$ in $\text{span}\{b_i\}_{1 \leq i \leq 3}$ such that $(b_i, b_j) = a_{ij} = a_{ji}$, and this scalar product is $\rho(\Gamma)$-invariant. The signature of $A$ should be $(3,0,0)$, $(2,1,0)$ or $(2,0,1)$. Here, a symmetric matrix has signature $(p,q,r)$ if the triple $(p,q,r)$ is the number of positive, negative and zero eigenvalues (counted with multiplicity).

In the case $(p,q,r) = (3,0,0)$, the image $\rho(\Gamma)$ lies in a compact subgroup, which is (a conjugate of) $O(3)$, hence $\rho$ cannot be discrete. In the case $(p,q,r) = (2,1,0)$, the image $\rho(\Gamma)$ lies in $SO(2,1)$, and acts properly discontinuously on a hyperbolic plane $\mathbb{H}^2$. The action is cocompact since the virtual cohomological dimension of $\Gamma$ is 2; see [GW12, Theorem 9.10]. After possibly negating the generators, we can apply [LM19, Lemma 5.4], to obtain that $\rho(\Gamma)$ is a hyperbolic reflection group, hence $\rho$ is of Hitchin type. In the case of $(p,q,r) = (2,0,1)$, the image $\rho(\Gamma)$ lies in $O(2) \rtimes \mathbb{R}^2$. This is a contradiction by Bieberbach’s Theorem.

**Proposition 3.3.** Let $\rho: \Gamma_{p_1, p_2, p_3} \to \text{SL}(3, \mathbb{R})$ be a representation of type $(q_1, q_2, q_3)$ which is Anosov. Then either $q_1 = q_2 = q_3 = 1$ or $p_1, p_2, p_3$ are all odd and $q_i = \frac{p_i - 1}{2}$ for all $i \in \{1, 2, 3\}$. Furthermore $t_\rho > 0$, so $\rho$ is in the Hitchin or Barbot component.
Proof. We can assume $p_1, p_2, p_3 \neq 2$, otherwise this follows from Lemma 3.2. As $\rho$ is Anosov, it comes with a continuous, injective, and equivariant boundary map $\xi: S^1 \to \mathbb{RP}^2$. Let $R_1 \in \text{SL}(3, \mathbb{R})$ be the rotation around the same axis as $\rho(s_1s_2)$, but only by the angle $\frac{2\pi}{p_3}$, so that $\rho(s_1s_2) = R^{p_3}$.

If we parametrize $\partial\Gamma_{p_1,p_2,p_3} = S^1$ by the unit interval so that the rotation $s_1s_2$ is a shift by $\frac{1}{p_3}$, then the assumptions of Lemma 3.1 are satisfied.

So if $\xi$ is null–homotopic then $q_3 = 1$, while if $\xi$ is not null–homotopic $p_3$ must be odd and $q_3 = \frac{p_3 - 1}{2}$. We can repeat the argument for the rotations $s_2s_3$ and $s_3s_1$ to obtain the analogous constraints for $q_1$ and $q_2$.

We claim that $t_{\rho} > 0$. In the case of $q_i = \frac{p_i - 1}{2}$ for all $i \in \{1, 2, 3\}$, it follows from Lemma 2.5 since $\rho(s_1s_2s_3)$ has distinct real eigenvalues. Now we assume that $q_i = 1$ for all $i \in \{1, 2, 3\}$. Consider two lines in the image of the dual boundary map, splitting $\mathbb{RP}^2$ into two bigons. As $\xi$ is transverse, it intersects each of them exactly once and consists of two arcs between them. Since it is null–homotopic, both arcs must lie in the same (closed) bigon. So $\xi(S^1)$ is contained in an affine chart. Then its convex hull $C$ is a properly convex set preserved by $\rho(\Gamma)$. By the same reason as in the proof of Lemma 3.2, $\rho(\Gamma)$ acts properly discontinuously and cocompactly on the interior $C^0$. Hence $\rho$ is of Hitchin type again by [LM19, Lemma 5.4]. That is, $t_{\rho} > 0$.

4 Constructing a boundary map

The setup in this section is more general than in the rest of the paper. Let $\Gamma$ be a cocompact discrete subgroup of the isometries of the hyperbolic plane $\mathbb{H}^2$, i.e. the fundamental group of a closed hyperbolic 2–orbifold. Its action extends to $\overline{\mathbb{H}^2} = \mathbb{H}^2 \sqcup S^1$.

Let $I \subset S^1$ be a proper closed interval and $T \subset \Gamma$ a finite subset which satisfies

(i) $\gamma I \subset I$ for all $\gamma \in T$,

(ii) if $\gamma \in T$ fixes an endpoint $x \in \partial I$, then $\gamma$ is hyperbolic and $x$ is its attracting fixed point,

(iii) and $\bigcup_{\gamma \in T} \gamma I = I$.

Now let $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$ be a representation. The goal of this section is to show

Proposition 4.1. Let $A \subset \mathbb{RP}^2$ be a closed set with non–empty interior which is contained in an affine chart. Assume that there is $N \in \mathbb{N}$ and a special element $\bar{t} \in T$ such that

(i) $\rho(t)A \subset A$ for all $t \in T$,

(ii) $\rho(t_1) \cdots \rho(t_N)A \subset A^0$ if $t_1, \ldots, t_N \in T$ and $t_N \neq \bar{t}$,

(iii) and the intersection of the sets $\rho(\bar{t})^iA$ for all $i \in \mathbb{N}$ is a point.
Then there exists a $\rho$-equivariant continuous map $\xi: S^1 \to \mathbb{RP}^2$ satisfying $\xi(I) \subset A$.

The proof needs some more setup. Fix a basepoint $o \in \mathbb{H}^2$ and a finite generating set $S \subset \Gamma$, and denote by $\ell: \Gamma \to \mathbb{N}_0$ the word length in $S$. We call a sequence $(g_n)_n \in \Gamma^\mathbb{N}$ a quasigeodesic ray to $x \in S^1$ if $g_n o \to x$ and $(g_n o)_n$ is a quasigeodesic ray in $\mathbb{H}^2$, i.e. $d(o, g_n o)$ is bounded by increasing affine linear functions of $n$ from below and above.

We call a sequence $(g_n)_n \in \Gamma^\mathbb{N}$ a code for $x \in S^1$ if $g_n^{-1} g_{n+1} \in T$ and $g_n^{-1} x \in I$ for all $n \geq 1$. Note that $g_1$ can be any element of $\Gamma$ satisfying the second condition. By assumption (iii) there is a code for every $x \in S^1$, although it is usually not unique.

To construct the boundary map $\xi$ for a representation $\rho$, we will take a code $(g_n)_n$ for $x \in S^1$, apply $\rho$ to it, and then take $\xi(x) \in \mathbb{RP}^2$ to be the limit of $(\rho(g_n))_n$ in a suitable sense. To make this work, we need to show that the limit exists and that it does not depend on the chosen code for $x$. The latter part is general and the content of Lemma 4.3 and Lemma 4.5. To show the existence of the limit in Lemma 4.7, we will use the set $A$ which gets mapped into itself by the elements of $\rho(T)$. The objective of Section 5 will then be to find such a set for the representations we are interested in.

**Remark 4.2.** The uniqueness part is inspired by [Sul85, Section 9] and the existence part follows the strategy of [Sch93]. A similar criterion for general Anosov representations is shown in [BPS19, Section 5].

What does it mean for a sequence $(g_n)_n \in \text{SL}(3, \mathbb{R})^\mathbb{N}$ to converge to $x \in \mathbb{RP}^2$? Let $\mu$ be the unique $\text{SO}(3)$-invariant Borel probability measure on $\mathbb{RP}^2$. Then we say $g_n \to x$ if $(g_n)_n \mu \to \delta_x$ in the weak topology of measures, where $\delta_x$ is the Dirac measure at $x$. Explicitly, this means that

$$\int_{\mathbb{RP}^2} f \circ g_n \, d\mu \to f(x)$$

for every continuous function $f$ on $\mathbb{RP}^2$. This mode of convergence is equivalent to the “flag convergence” defined in [KLP17]. If $g_n \to x$ and $g \in \text{SL}(3, \mathbb{R})$, it is clear from the definition that $g g_n \to g x$, while on the other hand $g_n g \to x$.

**Lemma 4.3.** Let $(g_n)_n \in \Gamma^\mathbb{N}$ be a code for $x \in S^1$. Then $(g_n o)_n$ is a quasigeodesic ray and $g_n o \to x$ in $\mathbb{H}^2$.

**Proof.** Let $I'$ be a slightly enlarged version of $I$ so that $\gamma I' \subset I^\circ$ for all $\gamma \in T$. The existence of $I'$ is guaranteed by properties (i) and (ii), but $I'$ will not satisfy property (iii). Let $\alpha$ be the hyperbolic geodesic connecting the two points of $\partial I'$. Then $\text{dist}(\gamma \alpha, \alpha) > 0$ for all $\gamma \in T$. Let $C$ be the minimum of these distances. Since the quasigeodesic property and the limit of $g_n o$ do not depend on the basepoint $o$, we can assume that $o \in \alpha$. So

$$d(g_n o, g_1 o) \geq \text{dist}(g_n \alpha, g_1 \alpha) \geq \sum_{i=1}^{n-1} \text{dist}(g_{i+1} \alpha, g_i \alpha) = \sum_{i=1}^{n-1} \text{dist}(g_i^{-1} g_{i+1} \alpha, \alpha) \geq C(n-1).$$

20
This shows that \((g_n o)_n\) is a quasigeodesic ray (the upper bound is clear). Its limit in \(S^1\) is in \(g_n I\) for all \(n\), so it must be \(x\).

**Lemma 4.4.** Let \((g_n)_n, (g'_n)_n \in \Gamma^N\) be quasigeodesic rays to \(x \in S^1\). There exists \(N \in \mathbb{N}\) such that for every \(n \in \mathbb{N}\) there is an \(m(n) \in \mathbb{N}\) with
\[
\ell(g_{m(n)}^{-1}g'_n) \leq N.
\]

**Proof.** Say \((z_n)_n = (g_n o)_n\) and \((z'_n)_n = (g'_n o)_n\) are both \((K, C)\)-quasigeodesic rays from \(o\) to \(x\), for some \(K\) and \(C\). The Morse lemma tells us that both are contained in the \(R\)-neighborhood of the geodesic \(ox\), for some \(R\). Denote by \(\pi\) the closest point projection of \(\mathbb{H}^2\) onto this geodesic. Since \(d(\pi(z_n), \pi(z_{n+1})) \leq d(z_n, z_{n+1}) \leq K + C\), every point on the ray \(ox\) is at most distance \(R' = \max\{(K + C)/2, d(o, \pi(z_1))\}\) from some \(\pi(z_m)\).

Now for every \(n \in \mathbb{N}\), choose \(m(n)\) such that \(d(\pi(z_{m(n)}), \pi(z'_n)) \leq R'\), and therefore
\[
d(z_{m(n)}, z'_n) \leq d(\pi(z_{m(n)}), \pi(z'_n)) + 2R \leq R' + 2R.
\]

The statement of the lemma follows since the orbit map of \(\Gamma\) is a quasiisometry from the word metric on \(\Gamma\) given by \(d_\Gamma(g, h) = \ell(g^{-1}h)\).

**Lemma 4.5.** Let \((g_n)_n, (g'_n)_n \in \Gamma^N\) be quasigeodesic rays to \(x \in S^1\). If \(\rho(g_n) \to y\) for some \(y \in \mathbb{RP}^2\), then also \(\rho(g'_n) \to y\).

**Proof.** By Lemma 4.4 there is a sequence \(m(n)\) such that \(g'_n = g_{m(n)} h_n\) with the \(h_n\) coming from a finite set. Clearly \(m(n) \to \infty\) as \(n \to \infty\), so \(\rho(g_{m(n)}) \to y\). For every subsequence along which \(h_n\) is constant we have \(\rho(g'_n) \to y\), so the same is true for the entire sequence.

**Lemma 4.6.** Let \(A \subset \mathbb{RP}^2\) be closed with non-empty interior and \((g_n)_n \in \text{SL}(3, \mathbb{R})^N\) a sequence satisfying \(g_{n+1} A \subset g_n A\) as well as \(\text{diam}(g_n A) \to 0\) (in any Riemannian metric on \(\mathbb{RP}^2\)). Then \(g_n \to x\) for some \(x \in \bigcap_{n \in \mathbb{N}} g_n A\).

**Proof.** By compactness of \(A\) the choice of Riemannian metric doesn’t matter. So we work with the spherical metric on \(\mathbb{RP}^2\). Let \(g_n = k_n a_n l_n\) be a singular value decomposition for \(g_n\), that is \(k_n, l_n \in \text{SO}(3)\) and \(a_n\) is a diagonal matrix with entries \(\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}\) sorted by absolute values, so that \(|\lambda_{1,n}| \geq |\lambda_{2,n}| \geq |\lambda_{3,n}|\). Passing to a subsequence, we can assume \(k_n \to k\) and \(l_n \to l\).

Then \(l A\) contains an open rectangle in homogeneous coordinates, which by an elementary computation is compressed to a point only if \(\lambda_{2,n}/\lambda_{1,n} \to 0\). This implies that \((a_n)_\mu\) converges to the Dirac measure at \([e_1] \in \mathbb{RP}^2\).

Now whenever \(n \geq m\) then \(g_n A \subset g_m A\) by assumption, so
\[
\delta_x(g_m A) \geq \limsup_{n \to \infty} (g_n)_\mu(g_m A) \geq \mu(A) > 0,
\]

21
hence \( x \in g_nA \) for all \( m \). This \( x \) is unique since \( \text{diam}(g_nA) \to 0 \), so the whole sequence converges.

\[ \tag*{\Box} \]

**Lemma 4.7.** Let \( A \subset \mathbb{RP}^2 \) be as in Proposition 4.1 and let \((g_i)_i \in \Gamma^\mathbb{N}\) be a code for \( x \in S^1 \).

Then \( \text{diam}(\rho(g_i)A) \to 0 \) and \( \rho(g_i) \) converges to the unique point \( y \in \bigcap_{i \in \mathbb{N}} \rho(g_i)A \).

**Proof.** We need to distinguish two cases: either \( g_i^{-1}g_i = \overline{1} \) for almost every \( i \), or not.

In the first case, the assumptions of Proposition 4.1 tell us that the intersection of the sets \( \rho(g_i)A \) is a single point, hence \( \text{diam}(\rho(g_i)A) \to 0 \). With Lemma 4.6 this proves the lemma. So we now assume that \( g_i^{-1}g_i \neq \overline{1} \) for infinitely many \( i \).

Fix an affine chart containing \( A \) and work with the Euclidean metric in this chart. Let \( h = t_1 \cdots t_N \) be any product of \( N \) elements of \( T \) with \( t_N \neq \overline{1} \), and let \( x, y \in \rho(h)A \) as well as \( a, b \in \mathbb{RP}^2 \setminus A^\circ \) be such that \( x, y, a, b \) lie on a projective line in that order. Let \( D = \text{diam}(\rho(h)A) \) and let \( d \) be the minimal distance between \( \rho(h)A \) and \( \mathbb{RP}^2 \setminus A^\circ \), which is positive since \( \rho(h)A \subset A^\circ \). Then the cross ratio satisfies

\[
[a : x : y : b] = \frac{|y - a||b - x|}{|x - a||b - y|} = \left(1 + \frac{|y - x|}{|x - a|}\right) \left(1 + \frac{|y - x|}{|b - y|}\right) \leq (1 + D/d)^2.
\]

Doing this for any \( h \) of this form gives a uniform upper bound on these cross ratios.

We want to show that \( \text{diam}(\rho(g_i)A) \to 0 \) and then employ Lemma 4.6. It is clear that this sequence is non-increasing. Assume it converges to \( c > 0 \). Then choose, for every \( i \), points \( x_i, y_i \in \rho(g_{i-N})A \) with \( |y_i - x_i| = \text{diam}(\rho(g_{i-N})A) \). Let \( a_i, b_i \) be the closest points of the boundary of \( \rho(g_i)A \) on the projective line through \( x_i \) and \( y_i \) in either direction, so that the points are ordered \( a_i, x_i, y_i, b_i \).

Then \( |y_i - x_i| \to c \) and \( |y_i - x_i| < |b_i - a_i| \leq \text{diam}(\rho(g_i)A) \), so also \( |b_i - a_i| \to c \). By the way the points are ordered, \( |y_i - a_i| \) and \( |b_i - x_i| \) must also converge to \( c \), while \( |x_i - a_i| \) and \( |b_i - y_i| \) go to \( 0 \). Hence the cross ratio \([a_i : x_i : y_i : b_i]\) goes to \( \infty \).

Now choose a subsequence \((g_{i_k})\) for which \( g_{i_k}^{-1}g_{i_k} \neq \overline{1} \), and also \( i_k \geq i_{k-1} + N \). Then since the cross ratio is a projective invariant and \( g_{i_k}^{-1}g_{i_k} \) is a product of \( N \) elements of \( T \), the last one different from \( \overline{1} \), the cross ratio \([a_{i_k} : x_{i_k} : y_{i_k} : b_{i_k}]\) equals one of the cross ratios we bounded above, for every \( k \). This is a contradiction, so \( \text{diam}(\rho(g_i)A) \to 0 \) and \( \rho(g_i) \) converges by Lemma 4.6.

\[ \tag*{\Box} \]

**Definition 4.8.** Under the assumptions of Proposition 4.1 we can define the map

\[ \xi : S^1 \to \mathbb{RP}^2 \]

by requiring that \( \rho(g_n) \to \xi(x) \) for every code \((g_n)_n \in \Gamma^\mathbb{N}\) for \( x \in S^1 \).
We know that such a code exists for every \( x \in S^1 \) and that \( \rho(g_n) \) converges by Lemma 4.7. The limit is independent of the choice of code by Lemma 4.3 and Lemma 4.5. More generally, \( \rho(g_n) \to \xi(x) \) for any quasigeodesic ray \( (g_n)_n \in \Gamma^N \) such that \( g_n o \to x \). So \( \xi \) is well–defined. Also note that every \( x \in I \) has a code \( (g_n)_n \) with \( g_1 = 1 \), so \( \xi(I) \subset A \).

**Lemma 4.9.** \( \xi \) is \( \rho \)–equivariant.

**Proof.** Let \( x \in S^1 \) and \( g \in \Gamma \). If \( (g_n)_n \in \Gamma^N \) is a quasigeodesic ray going to \( x \), then \( (gg_n)_n \) is a quasigeodesic ray going to \( gx \). So
\[
\xi(gx) \leftarrow \rho(gg_n) = \rho(g)\rho(g_n) \to \rho(g)\xi(x).
\]
This shows equivariance.

**Lemma 4.10.** \( \xi \) is continuous.

**Proof.** Let \( x \in S^1 \). We inductively construct a code \( (g_n)_n \in \Gamma^N \) for \( x \): first we choose \( g_1 \) so that \( g_1^{-1}x \in I^0 \). Then for every \( n \geq 1 \), since \( g_n^{-1}x \in I \), property (iii) ensures that \( g_n^{-1}x \in tI \) for some \( t \in T \). We set \( g_{n+1} = gn t \). If possible, we choose \( t \) so that \( g_n^{-1}x \in tI^0 \). Otherwise, if \( g_n^{-1}x \) is in \( I^0 \) but not in \( tI^0 \) for any \( t \in T \), then there are at least two different choices for \( t \), of which we choose one which results in \( x \) being the clockwise boundary point of \( g_{n+1}I \). Note that \( x \) is then also the clockwise boundary point of all \( g_m I \) for \( m > n \). The sequence constructed this way is a quasigeodesic ray going to \( x \) by Lemma 4.3.

Let \( \varepsilon > 0 \). Since \( \xi(x) \in \rho(g_n)A \) for all \( n \) and \( \text{diam}(\rho(g_n)A) \to 0 \) by Lemma 4.7 there is some \( n \) with
\[
\rho(g_n)\xi(I) \subset \rho(g_n)A \subset B_\varepsilon(\xi(x)).
\]
If \( g_n^{-1}x \in I^0 \), then \( g_n I \) is a neighborhood of \( x \), so this shows continuity at \( x \). On the other hand, if \( g_n^{-1}x \in \partial I \) then \( x \) is the clockwise boundary point of \( g_n I \) by the above construction. So in this case we only get semicontinuity of \( \xi \) at \( x \) in clockwise direction. But we can repeat the argument replacing “clockwise” by “counter–clockwise” to get full continuity.

This finishes the proof of Proposition 4.1.

5 Nested boxes

5.1 Intersecting conics

Assume \( p_1, p_2, p_3 \geq 3 \) are odd and not all equal to \( 3 \). Let \( \Gamma = \Gamma_{p_1, p_2, p_3} \) and \( \rho: \Gamma \to \text{SL}(3, \mathbb{R}) \) be a representation of type \( (p_1−1, p_2−1, p_3−1)/2 \) with parameter \( t_\rho \geq t_{\text{crit}} > 1 \). We
assume that \( \rho \) is line–irreducible, i.e. it does not preserve any line in \( \mathbb{RP}^2 \). As noted in Section 2.1 every Coxeter character has such a representative.

This section will use some long words in \( \rho(\Gamma) \). To simplify the notation, we will write
\[ a := \rho(s_1), \quad b := \rho(s_2) \quad \text{and} \quad c := \rho(s_3) \]
for the remainder of Section 5.

Recall from Section 2.4 that there are points \( z_0, \ldots, z_{2p_3-1} \in S^1 \), in order along \( S^1 \), so that \( z_0 \) is the attracting fixed point of \( s_1 s_2 s_3 \) and \( s_1 z_i = z_{3-i} \) as well as \( s_2 z_i = z_{5-i} \) for every \( i \). We treat these indices as elements of \( \mathbb{Z}/2p_3 \mathbb{Z} \) and will sometimes write e.g. \( z_{-1} \) instead of \( z_{2p_3-1} \).

To each of the points \( z_i \in S^1 \) we define a corresponding point \( w_i \) and a line \( \ell_i \) in \( \mathbb{RP}^2 \).

Let \( C \subset \mathbb{RP}^2 \) be the unique conic which passes through all the points \( w_i \). It is clearly invariant by \( a \) and \( b \). The complement of \( C \) has two connected components, a disk and a Möbius strip, which we call \( M \). Each of the lines \( \ell_i \) intersects \( C \) in two points, one of which is \( w_i \). The other point will be called \( u_i \).

We adopt the following convention: for any object \( O \) defined using the generators \( s_1, s_2, s_3 \), its “primed” version \( O' \) shall have the same definition, except that the generators are cyclically permuted, i.e. \( s_2, s_3, s_1 \) are used in place of \( s_1, s_2, s_3 \). For example, \( w_0 \) was defined as the attracting fixed point of \( abc \), so \( w_0' \) is the attracting fixed point of \( bca \), and therefore equal to \( w_3 = aw_0 \). Cyclically permuting another time, \( w_0'' \) is the attracting fixed point of \( cab \), hence equal to \( w_2 = baw_0 \).

![Figure 4: Coincidences of the points \( w_i, w_i', w_i'' \) and how the generators map them to each other. The same relations hold for \( \ell_i, \ell_i', \ell_i'' \) and \( z_i, z_i', z_i'' \) (with \( s_1, s_2, s_3 \) in place of \( a, b, c \)).](image)
It is easy to find more coincidences like this; they are shown in Figure 4. Analogous coincidences also hold for the $z_i$ and $\ell_i$, e.g. $z'_0 = (s_2 s_3 s_1)_+ = s_1 (s_1 s_2 s_3)_+ = z_3$. But this principle does not extend to the points $u_i$: for example, $u'_0$ is on the intersection of the line $\ell'_0 = \ell_3$ with the conic $C'$, while $u_3$ is on the intersection of the same line with $C$.

Every lemma we prove in this section also has a primed and a double–primed version: their statements are the same, just with every object $O$ replaced by $O'$ or $O''$ and $s_1, s_2, s_3, p_1, p_2, p_3$ replaced by $s_2, s_3, s_1, p_2, p_3, p_1$ or $s_3, s_1, s_2, p_3, p_1, p_2$, respectively. We only state and prove one version, but will make use of the others as needed.

Now we study how the conics $C$, $C'$ and $C''$ intersect. Figure 4 shows that

$$C \cap C' = \{w_0, w_2, w_3, w_5\}, \quad C \cap C'' = \{w_0, w_1, w_2, w_3\},$$

provided that the points $w_i$ are distinct (which is clear if $t_\rho = t_{\text{red}}$ and proved by Lemma 5.2 in general): if these conics intersected in five points, they would be equal, hence $C$ would be preserved by all of $\rho(\Gamma)$. But if $\rho(\Gamma)$ preserved a conic, it would be contained in a conjugate of $\text{SO}(2,1)$, which would imply $\text{tr} \rho(\gamma) = \text{tr} \rho(\gamma^{-1})$ for all $\gamma \in \Gamma$, and hence cannot happen for $t_\rho \neq 1$.

In fact, the configuration looks somewhat like Figure 5 left. We first show this in the case $t_\rho = t_{\text{red}}$ (i.e. if $\rho$ is reducible), and then deform to the general case $t_\rho > t_{\text{crit}}$.

![Figure 5](image-url)

**Figure 5:** left: the configuration of conics $C, C', C''$ and the order of the points $w_i, w_i', w_i''$ on them, in the case $(p_1, p_2, p_3) = (5, 5, 5)$. right: the order of the points $w_0, u_0, u'_2, cu_2, x$ on $\ell_0$ as in the proof of Lemma 5.1.

**Lemma 5.1.** Assume that $t_\rho = t_{\text{red}} > 1$, so $\rho$ is reducible, but line–irreducible. Then the $4p_3$ points $w_i$ and $u_i$ are in the cyclic order

$$w_0, u_1, u_2, w_3, w_4, u_5, u_6, w_7, \ldots$$
along C. Also, we have \( w_1 \in M' \).

**Proof.** The representation \( \rho \) fixes a point \( x \in \mathbb{R}P^2 \), and there is a continuous map from \( S^1 \) to the space of lines through \( x \), which maps the attracting fixed point of any infinite order \( \gamma \in \Gamma \) to the attracting line of \( \rho(\gamma) \), so in particular \( z_i \) to \( \ell_i \). Hence the lines \( \ell_i \) all pass through the point \( x \) and are ordered \( \ell_0, \ell_1, \ell_2, \ldots \). Along the conic \( C \), therefore, \( w_0 \) is followed by either \( w_1 \) or \( u_1 \). The next point along \( C \) after that is either \( w_2 \) or \( u_2 \). It must actually be \( w_2 \), as \( w_2 = (ba)w_0 \) is at an angle of \( \pi - \pi/p_3 \) from \( w_0 \). Applying the same argument to the odd indices, we find that the first four points along \( C \) are either

\[
(a) \ w_0, u_1, w_2, w_3 \quad \text{or} \quad (b) \ w_0, w_1, u_2, w_3
\]

After that the same pattern repeats with indices increased by 4, since \( w_{i+4} = (ba)^2w_i \) and \( (ba)^2 \) rotates \( C \) by an angle of \( 2\pi/p_3 \).

In case (a), the points \( w_0, u_2, w_3, u_0, w_2, w_5 \) are in this order along \( C \), while in case (b) their order is \( w_0, w_2, w_5, u_0, w_2, w_3 \). Here the points \( w_0, w_2, w_3, w_5 \) are exactly the intersection points of \( C \) and \( C' \). Since the conics intersect transversely, this means that in both cases exactly one of the points \( u_0 \) and \( u_2 \) is in \( M' \).

Now consider the line \( \ell_0 = \ell_2' \), which is fixed by the Coxeter element \( abc \). It contains the points \( w_0 = w_2' \) and \( x \), both of which are fixed by \( ab \), as well as \( w_0, cu_2 = cbau_0 \) and \( u_2' \). By Lemma 2.6 the eigenvalues of \( abc \) are \(-\lambda, -1, \lambda^{-1}\), with \( \lambda > 1 \), hence the action of \( abc \) on \( \ell_0 \) is orientation-preserving. So \( u_0 \) and \( cu_2 \) lie in the same component of \( \ell_0 \setminus \{w_0, x\} \). More precisely, the points are in the order \( u_0, w_0, u_2, w_2, x \) on \( \ell_0 \). As \( x \) is the unique fixed point of the rotations \( ab \) and \( bc \), it is not contained in \( M, M', \) or \( cM \).

So the intersection \( \ell_0 \cap M \) is the interval between \( w_0 \) and \( u_0 \) not containing \( x \), \( \ell_0 \cap cM \) is the interval from \( w_0 = cu_2 \) to \( cu_0 \), and \( \ell_0 \cap M' \) is the interval between \( w_0 = w_2' \) and \( w_2' \). As \( M' \) is invariant by \( c \), we have that exactly one of the points \( u_0 \) and \( cu_2 \) is in \( M' \).

So \( u_0 \) must lie between \( u_0 \) and \( cu_2 \). Therefore, we have \( u_0 \in M', cu_2 \notin M', u_0 \notin M, \) and \( u_0 \notin cM \).

The same reasoning applies to the “double–primed” situation, giving \( u_2 \notin M'' \) and \( u_2 \in bM'' \), which is equivalent to \( u_3 = bu_2 \in M'' \). But this is a contradiction to order (b): if the points \( u_2 \) and \( u_3 \) have none of the points \( w_0 \) between them, they must both be either in \( M'' \) or not. So we have order (a). In this order, the points \( u_0 \) and \( w_1 \) are neighbors, so \( u_0 \in M' \) implies \( w_1 \in M' \).

Now we consider a general representation \( \rho \) with \( t_\rho \geq t_{crit} \).

**Lemma 5.2.** The points \( w_1, \ldots, w_{2p_3-1} \) are pairwise distinct.

**Proof.** We have seen that in the reducible case, the \( w_i \) are distinct and in the order \( w_0, w_3, w_4, w_7, w_8, \ldots \) along \( C \). The same holds for the \( w_i' \) and \( w_i'' \). For contradiction, we take a path of representations starting from a reducible one and follow it until any
pair of the $w_i$ or $w_i'$ or $w_i''$ coincides for the first time. Without loss of generality, we can assume this collision happens among the $w_i$. Then by $(a, b)$–invariance we have either $w_0 = w_3$ or $w_3 = w_4$.

Note that $cb$ rotates $C'$ by the fixed angle $\frac{2\pi}{p_1} \pi$ and maps $w_i'$ to $w_i'_{i+2}$. So $w_i' \neq w_j'$ whenever $i - j$ is even and not a multiple of $2p_1$. In particular $w_0 = w_2' \neq w_0' = w_3$. So we have $w_3 = w_4$, and therefore $w_0'' = w_2' = bw_3 = bw_4 = w_1 = w_5'$. The four points $w_0'', w_4', w_5', w_6''$ have been in this cyclic order for every representation on the path, so they degenerate to either $w_0'' = w_4' = w_5''$ or $w_0'' = w_6'' = w_5''$. This is a contradiction if $p_2 \neq 3$ by the same argument as above, only using the rotation $ac$ instead of $cb$.

On the other hand, if $p_2 = 3$, then $w_0' = w_2'' = acw_0'' = acw_5'' = w_5'' = w_5'$, and we can repeat the same argument with $w_0'$ and $w_5'$ instead of $w_0''$ and $w_5''$. Again, this leads to a contradiction unless $p_1 = 3$ and $w_0 = w_2 = cbw_0' = cbw_5' = w_5' = w_5$. But $p_1, p_2, p_3$ cannot all be 3, so repeating the argument another time gives the contradiction we want.

Figure 6: The two possible orders of points along $C$. The shaded area represents the Möbius strip $M$, bounded by $C$, and the thin vertical lines are the $\ell_i$. The lines $\ell_i$ and $\ell_j$ can only intersect in $M$ if $\{i, j\} = \{2k - 1, 2k\}$ for some $k$.

**Lemma 5.3.** The (cyclic) order of the points $u_i$ and $w_i$ along $C$ is either

$$w_0, u_1, w_2, u_3, w_4, u_5, w_6, \ldots$$

as in the reducible case, or differs from it only by switching the order of $u_{2k-1}$ and $u_{2k}$ for all $k$ (or $u_{2k-1} = u_{2k}$ for all $k$). Furthermore, we have $w_1 \in M'$.

**Proof.** By Lemma 5.2 the order of the $w_i$ must be the same as in the reducible case. We only need to show that $w_2 \neq w_0$ and $w_2 \neq w_3$.

If $w_3 = w_2$, then the line $\ell_2$ contains both $w_2$ and $w_3 = bw_2$, so it is preserved by $b$. Since $cab$ also fixes $\ell_2$, so does $ca$. As $ca$ has finite order greater than 2, it has a unique fixed line, which is fixed by $a$ and $c$ individually. So we showed that $\ell_2$ is fixed by $a$, $b$ and $c$, contradicting our assumption that $\rho$ has no fixed line. The case $u_2 = w_0$ is similar.

We have $w_1 \in M'$ when $t_\rho = t_{red}$, and this cannot change under continuous deformation, since $w_1$ is never in $C \cap C'$.

27
Assuming $i \neq j$, we can read off from the order of $w_i, u_i, w_j, u_j$ on $C$ whether the intersection point of $\ell_i$ and $\ell_j$ is in the Möbius strip $M$ or in the disk bounded by $C$. If it is in $M$, we say $\ell_i$ and $\ell_j$ cross in $M$. Lemma 5.3 shows that $\ell_i$ and $\ell_j$ can only cross in $M$ if $\{i, j\} = \{2k - 1, 2k\}$ for some $k$; see Figure 6.

**Lemma 5.4.** $u''_0 \in M$ and $cu_0 \in M$.

**Proof.** The points $w_0, w_2, w_3, u_0, w_1, w_2, w_3$ lie in this cyclic order on $C$. Since $w_1 \in M'$ by Lemma 5.3 and $C \cap C' = \{w_0, w_3, w_2, w_5\}$, we have that $u_0 \in M'$ and $u_2 \not\in M'$. Similarly, considering $w_0, w_3, w_1, w_2, w_3, w_5$ along $C'$ and using that $w_5 = w''_1 \in M$ by Lemma 5.3, we find that $u''_3 \in M$.

Now consider the line $\ell_2 = \ell'_3$ which contains the points $w_2 = w'_3, w'_3$, and $u_2$. Since $u'_3 \in M$ and $u_2 \not\in M'$ we see that $\ell_2 \cap M' \subset \ell_2 \cap M$. In particular $cu_0 \in \ell_2 \cap M' \subset M$. The other statement $u''_0 \in M$ is just a variant of $u_0 \in M'$.

**5.2 Definition of $I$, $T$, and the box**

In order to apply Proposition 4.1 and get a boundary map, we first choose an interval $I \subset S^1$ and a finite set $T \subset \Gamma$ satisfying the axioms in the beginning of Section 4. Then we construct a closed set $\square \subset \mathbb{RP}^2$ satisfying the assumptions of Proposition 4.1, in particular that $\rho(\gamma)\square \subset \square$ for all $\gamma \in \Gamma$.

Let $I = [z_3, z_0]$, that is the component of $S^1 \setminus \{z_0, z_3\}$ which does not contain the points $z_1, z_2$ (see Figure 7). Next, we define $T$ by

$$Q = \{s^j(s_1s_2)^\delta \mid \delta \in \{0, 1\}, 1 \leq j \leq \frac{p_3-1}{2}\}, \quad T = QQ'Q'.$$

Since $(s_1s_2)^j I'' = (s_1s_2)^j[z'_3, z'_0] = (s_1s_2)^j[0, 2] = [z_{-2j}, z_{-2j-2}]$ (see Figure 4), we have

$$\bigcup_{\gamma \in Q} \gamma I'' = \bigcup_{j=1}^{\frac{p_3-1}{2}} [z_{-2j}, z_{-2j-2}] \cup \bigcup_{j=1}^{\frac{p_3-1}{2}} [z_{1+2j}, z_{3+2j}] = [z_{p+1}, z_0] \cup [z_3, z_{p+2}] = I.$$

Together with the analogous versions $\bigcup_{\gamma' \in Q'} \gamma' I' = I'$ and $\bigcup_{\gamma'' \in Q''} \gamma'' I'' = I''$, this shows that $\bigcup_{\gamma \in T} \gamma I = I$, so $I$ satisfies assumptions (i) and (iii) of Section 4. The only $\gamma \in T$ which fixes an endpoint of $I$ is $\gamma = (s_1s_2)(s_3s_1)(s_2s_3) = (s_1s_2s_3)^2$, the attracting fixed point of which is the endpoint $z_0$. So assumption (ii) also holds.

It remains to find a closed subset $\square \subset \mathbb{RP}^2$ which satisfies

$$\rho(\gamma)\square'' \subset \square \quad \forall \gamma \in Q,$$

and therefore $\rho(\gamma')\square'' \subset \rho(\gamma)\square'' \subset \square$ for all $\gamma'' \gamma' \in T$.
The intervals \( I, I', I'' \) and the first level of subintervals. Here \( p_1 = p_2 = p_3 = 5 \), hence \( Q = \{s_2, s_1 s_2, s_2 s_1 s_1, s_1 s_2 s_1 s_2\} \), and \( Q' \) and \( Q'' \) are analogous. The points \( (s_1 s_2 s_3)_- \) and \( (s_2 s_3 s_1)_- \), used in Lemma 7.3, are also shown.

The set \( \Box \) is supposed to behave like the interval \( I \), so it should be bounded by \( \ell_0 \) and \( \ell_3 \) in the direction “along” the limit curve. In the transverse direction, there is no such obvious choice. A simple idea would be to “roughly” bound it by \( C \), e.g. to define \( \Box \) as convex hull of the points \( w_0, w_3, u_0, u_3 \). Unfortunately, this box not quite satisfy (10). We can fix this by cutting off two of its corners. This yields the convex hexagon we consider below.

Defining \( \Box \) as a convex hull requires some care, as the convex hull is only well-defined in a fixed affine chart. For a projective line \( \ell \) and points \( x_1, \ldots, x_n \in \mathbb{R}P^2 \) not on \( \ell \), we write

\[
\text{CH}_\ell(x_1, \ldots, x_n)
\]

for the convex hull of \( x_1, \ldots, x_n \) in the affine chart \( \mathbb{R}P^2 \setminus \ell \). Sometimes we have to switch from one affine chart to another. The following lemma will be useful for this.

**Lemma 5.5.** Consider a collection of points \( x_1, \ldots, x_n \in \overline{M} \) so that \( x_i \in \ell_{j_i} \) for some \( j_i \). Let \( \ell_k \) and \( \ell_m \) be two other lines which do not cross \( \ell_{j_i} \) in \( \overline{M} \) for any \( i \), and so that the points \( \{w_k, u_k\} \) lie in the same two components of \( C \setminus \{w_{j_1}, u_{j_1}, \ldots, w_{j_n}, u_{j_n}\} \) as the points \( \{w_m, u_m\} \). Then

\[
\text{CH}_{\ell_k}(x_1, \ldots, x_n) = \text{CH}_{\ell_m}(x_1, \ldots, x_n).
\]
Lemma 5.6. If \( \bigcup_i \{ w_{j_i}, u_{j_i} \} \), as are \( u_k \) and \( u_m \). We want to find a continuous path of lines \( \ell(t) \) from \( \ell_k \) to \( \ell_m \) which avoids \( x_1, \ldots, x_n \). Such lines \( \ell(t) \) are defined by their intersection points \( w(t) \) and \( u(t) \) with \( C \), which move from \( w_k \) to \( w_m \) and from \( u_k \) to \( u_m \), respectively.

By assumption we can find \( w(t) \) and \( u(t) \), and hence \( \ell(t) \), which avoid all points \( w_{j_i} \) and \( u_{j_i} \). In particular the cyclic order of \( w_{j_i}, w(t), u_{j_i}, u(t) \) is constant. This tells us that \( \ell_{j_i} \cap \ell(t) \not\in \mathcal{M} \) for all \( t \). So \( \ell(t) \) cannot contain the point \( x_i \).

Now we can define the box as (see Figure 8)

\[
\emptyset = \text{CH}_2\{w_0, w_3, w_5, bcw_0, abcw_0\}.
\]

Lemma 5.5 shows that \( \ell_1 \) can be used instead of \( \ell_2 \) in this definition: the 6 vertices defining \( \emptyset \) lie on the lines \( \ell_0, \ell_3, \ell_5, \ell_2 \) and in the order of the \( w_i \) and \( u_i \) on \( C \) both \( \{w_1, w_2\} \) and \( \{u_1, u_2\} \) are neighboring pairs. In particular, this shows \( a\emptyset = \emptyset \), as \( a\ell_2 = \ell_1 \) and the set of vertices is invariant.

5.3 The box inclusions

Our goal is now to show \( g\emptyset'' \subset \emptyset \) for all \( g \in \rho(Q) \), essentially by proving that all vertices of \( g\emptyset'' \) are in \( \emptyset \). For most \( g \) this is achieved by Lemma 5.10 together with Lemma 5.6, with a few special cases handled separately afterwards. Figure 8 shows the configuration of boxes in an example.

Many arguments in this section rely on the following simple fact: if \( x, y, z \) are three distinct points on a conic \( C \), splitting \( C \) into three arcs, and \( \ell \) is a line intersecting the conic in two of these arcs, then the third arc is completely contained in \( \text{CH}_\ell(x, y, z) \).

Lemma 5.6. \( w_i \in \emptyset \) for all \( i \notin \{1, 2\} \), \( w_1 \in \emptyset^o \) for all \( i \notin \{-2, 0, 1, 2, 3, 5\} \), and \( u_i \in \emptyset^o \) for all \( i \notin \{-1, 0, 1, 2, 3, 4\} \).

Proof. Splitting \( C \) into three arcs along \( w_0, w_3, \) and \( w_5 \), the arc from \( w_0 \) to \( w_3 \) contains \( u_2 \), and the arc from \( w_3 \) to \( w_5 \) contains \( w_2 \). Hence the arc from \( w_5 \) to \( w_0 \) is contained in \( \text{CH}_\ell(w_0, w_3, w_5) \subset \emptyset \). Since \( \emptyset \) is invariant by \( a \), it also contains the arc from \( w_3 \) to \( w_{-2} \) avoiding \( w_0 \). Together, these contain all points \( w_i \) with \( i \notin \{1, 2\} \). The interiors of these arcs are even contained in \( \emptyset^o \).

Lemma 5.7. \( u_0'' \) and \( cu_0 \) lie on the line \( \ell_2 \), and in \( M \) by Lemma 5.4. Since \( \ell_3 \) and \( \ell_2 \) do no intersect within \( M \), both points are in \( \text{CH}_{\ell_3}(w_0, w_2)^o \).

Lemma 5.8. \( w_{-2}' \in \text{CH}_{\ell_3}(w_2, w_0, cw_0)^o \).
Proof. Since the points $w_0, w_3, u_0, w_2, u_3$ lie in this order on the conic $C$, the arc from $u_0$ to $w_2$ avoiding $w_0$, and in particular the point $w_1$, is contained in $\text{CH}_\ell(w_0, w_2, u_0)$. So $w'''_2 = cu_1$ is contained in $c\text{CH}_\ell(w_0, w_2, u_0) = \text{CH}_\ell(w'_3, w'_2, cu_0)$.

Further $\text{CH}_\ell(w'_3, w'_2, cu_0) = \text{CH}_\ell(w'_3, w'_2, w'_5) = \text{CH}_\ell(w_2, w_0, cu_0)$, since $cu_0 \in \ell'_3$ and $w_0 \in M'$ (by Lemma 5.4) and the pairs $w'_0, w'_5$ as well as $u'_3, u'_0$ each lie on common arcs of $C' \setminus \{w'_2, w'_3, w'_2, u'_3\}$. \hfill \Box

Lemma 5.9. $cabu''_0 \in \text{CH}_\ell(w_2, cu_0)$. 

Proof. Since $u''_0 \in \text{CH}_\ell(w_2, u_2)$ by Lemma 5.7, we have (using Lemma 5.5 where necessary)

$$cabu''_0 \in c\text{CH}_\ell(w_0, u_0) = c\text{CH}_\ell(w_0, u_0) = \text{CH}_\ell(w_2, cu_0) = \text{CH}_\ell(w_2, cu_0).$$ \hfill \Box

Lemma 5.10. $\square'' \subset \text{CH}_\ell(w_3, w_2, u_0, u_2)$ for any $i \notin \{-1, 0, 1, 2\}$. In particular these $\ell_i$ do not intersect $\square''$, and $\ell_0$ and $\ell_2$ only intersect it in its boundary $\partial\square''$.

Proof. By definition,

$$\square'' = \text{CH}_\ell(w'_0, w'_3, w'_5, w''_2, abu''_0, cabu''_0) = \text{CH}_\ell(w_2, w_0, w'_1, w''_2, abu''_0, cabu''_0).$$

We need to show that the last four points are in the convex hull. For $w_1$ this directly follows from the order of the points $w_0, w_2, w_3, u_0, w_1, w_2, u_3$ along $C$. For $w''_2$ and $cabu''_0$ it comes from Lemma 5.8, Lemma 5.9 and Lemma 5.7. Finally, $abu''_0 \in \text{CH}_\ell(w_0, u_0) = \text{CH}_\ell(w_0, u_0)$ by Lemma 5.7 and Lemma 5.5. This proves the lemma for $i = 3$. We can then use Lemma 5.5 to change $i$ to what we want. \hfill \Box

If $g \in \rho(Q) \setminus \{b, ab\}$ then $g\square'' \subset \square$ follows from Lemma 5.10 (see Figure 8). For $g \in \{b, ab\}$ the lemmas above still show that most vertices of $g\square''$ are included in $\square$. An exception is the vertex $abu''_0$ of $ab\square''$. This one is a bit trickier, and in the case $p_2 = p_3 = 3$ it is not in $\square$ at all. We will work around this issue in Section 5.4.

Lemma 5.11. If $p_2 > 3$ or $p_3 > 3$ then $abu''_0 \in \square$.

Proof. By Lemma 5.7 and Lemma 5.5

$$abu''_0 \in \text{CH}_\ell(w_{-2}, u_{-2}) = \text{CH}_\ell(w_{-2}, u_{-2}).$$

If $p_3 > 3$ then $w_{-2} \in \square$ and $u_{-2} \in \square$ by Lemma 5.6, proving the lemma.

If however $p_3 = 3$, then $u_{-2} = u_4$, so Lemma 5.6 does not apply. In this case, consider the points $w''_3, w''_2, w''_5, w''_{-2}, w''_2$. They lie in this order on the conic $C''$. Therefore, the arc from $w''_3$ to $w''_{-2}$, which contains $u''_3$ if $p_2 > 3$, is contained in $\text{CH}_\ell(w''_3, w''_5, w''_{-2})$. Hence

$$abu''_0 = bu''_0 \in \text{CH}_\ell(w''_3, w''_5, w''_{-2}) = \text{CH}_\ell(w_5, w_4, bw''_{-2}).$$

The points $w_4$ and $w_5$ are in $\square$ by Lemma 5.6 and $bw''_{-2} \in \text{CH}_\ell(w_3, w_5, bcu_0) \subset \square$ by Lemma 5.8. So $abu''_0 \in \square$. \hfill \Box
Lemma 5.12. Assume that either $p_2 > 3$ or $p_3 > 3$ and let $g \in \rho(Q)$. Then $g\square'' \subset \square$.

We even have $g\{w''_{-2}, abu_0''\} \subset \square^o$, also $gw''_5 \in \square^o$ if $p_3 > 3$, and also $gw''_3 \in \square^o$ if in addition $g \notin \{b, ab\}$.

Proof. We can assume that $g = (ab)^j$ with $1 \leq j \leq \frac{p_3 - 1}{2}$. If $j > 1$ then $\square'' \subset CH_{\ell_2+2j}(w_0, w_2, u_0, u_2)$, so $g\square'' \subset CH_{\ell_2}(w_{-2j}, w_{-2j+2}, u_{-2j}, u_{-2j+2}) \subset \square$ by Lemma 5.10 and Lemma 5.6. Of the points $w_{-2j}, w_{-2j+2}, u_{-2j}, u_{-2j+2}$ the only one which can hit $\partial \square$ is $w_{-2j} = gw''_0$ if $j = 2$, so all other points in the convex hull are in $\square^o$.

Now suppose $j = 1$. Then $gw''_0 = w_0$, $gw''_3 = w_{-2}$, and $gw''_5 = w_{-1}$. These points are in $\square$ by Lemma 5.6, and even $gw''_5 \in \square^o$ if $p_3 > 3$. Further $gw''_{-2} = CH_{\ell_1}(w_0, w_{-2}, abc_0)^o \subset \square^o$ and $gabu_0'' \subset CH_{\ell_1}(w_0, abc_0) \subset \square$ by Lemma 5.8 and Lemma 5.9. And finally $gabu_0'' \in \square^o$ by Lemma 5.11.

This shows all six vertices of $g\square''$ are in $\square$. The only thing we still need is that $g\square'' \cap \ell_2 = \emptyset$, so that we can take the convex hull in the affine chart $\mathbb{R}P^2 \setminus \ell_2$. But by Lemma 5.10 $g\square'' \cap \ell_2 = \emptyset$ if $g\ell_i = \ell_2$ for some $i \notin \{-1, 0, 1, 2\}$. It is easy to see that this is true for all $g \in \rho(Q)$.

\[\square\]

Figure 8: The relevant points and conics to prove the inclusions $(ab)^j \square'' \subset \square$, in the case $p_1 = p_2 = p_3 = 5$. The inclusions $a(ab)^j \square'' \subset \square$ follow by symmetry.

32
5.4 The case $p_2 = p_3 = 3$

If $p_2 = p_3 = 3$ then $g \square''$ is generally not contained in $\square$ for $g \in \rho(Q)$. But we can skip one step and prove that $gg'' \square' \subset \square$ for all $g \in \rho(Q)$ and $g'' \in \rho(Q'')$. Again, we do this by showing that the vertices of these boxes are in $\square$ (see Figure 9).

![Figure 9: The relevant points for the proof of Lemma 5.13 and the conics defining them, in the case $(p_1, p_2, p_3) = (5, 3, 3)$. Note that $bca \square'$ and $ba \square'$ overlap.](image)

**Lemma 5.13.** Assume $p_2 = p_3 = 3$. The points $bcu_{-6}$, $bcu_0$, $bcu'_{-6}$ and $bcu'_{0}$ are all contained in $\square''$.

**Proof.** Applying $b$ to Lemma 5.9 gives $bcu'_{0} \in \mathrm{CH}_{\ell_1''}(bcw'_{3}, bcu_{0})^o$. The “double primed” version of this is $abcau'_{0} \in \mathrm{CH}_{\ell_2''}(w'_{3}, abu'_{0})^o$, but by Lemma 5.5 we could also take $\ell_1''$ instead of $\ell_2''$. If we apply $bc$ to that we get

$$bcu'_{0} \in \mathrm{CH}_{bcu_{1}''}(bcw'_{3}, bcu_{0})^o = \mathrm{CH}_{\ell_2}(w_{3}, bcu_{0})^o \subset \mathrm{CH}_{\ell_2}(w_{3}, bcu_{0})^o \subset \square^o.$$  

Applying $a$ to the double primed version of Lemma 5.8 gives $au'_{-2} \in \mathrm{CH}_{w_{1}''}(w'_{3}, w'_{5}, abu'_{0})^o$. Again, Lemma 5.5 allows to replace $\ell_2''$ by $\ell_1''$, and therefore

$$bcu_{-6} \in \mathrm{CH}_{bcu_{1}''}(bcw'_{3}, bcw'_{5}, bcu_{0})^o = \mathrm{CH}_{\ell_2}(w_{3}, bw'_{-2}, bcu_{0})^o \subset \square^o,$$

where $bw'_{-2} \in \square$ follows from Lemma 5.8.
Next, in the proof of Lemma 5.11 we showed that \( abab'' \in CH_{\ell_2}(w_5, w_4, bw_{-2}''\sigma) \) if \( p_3 = 3 \) and \( p_2 > 3 \). In the present case we have \( p_2 = 3 \) and \( p_1 > 3 \), so \( caca'u_0 \in CH_{\ell_2}(w_5', w_4', aw_{-2}'\sigma) \). Applying \( b \) to this shows

\[
bcac'u_0 \in CH_{\ell_2}(bw_5', bw_{-2}'\sigma, bw_{-2}'\sigma) = CH_{\ell_2}(w_{-2}, bw_{-2}'\sigma, abw_{-2}'\sigma) \subset c^0.
\]

Finally, consider the points \( w_4 = bcau_0', bcau_{-2} = bcau_3 \) and \( bw_{-2}' = bcau_3' \) and the line \( \ell_{-2}' = bcau_3' \). We showed above that these points are in \( c \). The points \( w_0', w_3', w_4', w_5', w_6, w_7 \) are in this order along \( C' \), so \( u_0' \in CH_{\ell_2}(w_0', w_4', w_7) \) and therefore

\[
bcac'u_0 \in CH_{\ell_2}(bcau_0', bcau_3, bcau_3') = CH_{\ell_2}(w_4, bcau_{-2}', bw_{-2}'\sigma) \subset c^0.
\]

For the last inclusion we used that \( \ell_{-2}' \cap c = \emptyset \) by Lemma 5.10.

**Lemma 5.14.** Assume \( p_2 = p_3 = 3 \) and let \( g \in \rho(Q) \) and \( g'' \in \rho(Q'') \). Then \( gg''c' \subset c \). Furthermore, \( gg''\{w_{-2}', w_5\} \subset c^0 \cup \{w_{-2}, w_5\} \).

**Proof.** Recall that \( \rho(Q)\rho(Q'') = \{ba, aba, bca, abca\} \). Since \( a\emptyset = \emptyset \) we can assume \( g'' \in \{ba, bca\} \). First, we show that the vertices of \( ba\emptyset' \) are in \( \emptyset \): \( bw_0' = w_5 \in \emptyset \) by Lemma 5.6, \( bcau_0' \in c^0 \) by Lemma 5.13, and \( bw'_5 = bw_{-2}' \in CH_{\ell_2}(w_3, w_5, bcu_0) \subset c^0 \) by Lemma 5.8. The remaining three vertices follow by symmetry: for every vertex \( x \) of \( \emptyset' \) such that \( bx \in \emptyset \), the point \( bx \) is another vertex of \( \emptyset' \) and \( ba\emptyset = abax \in a\emptyset = \emptyset \).

The vertices of \( bca\emptyset' \) are \( bcau_{-2}', bcau_0', bcau_3', \) which are in \( \emptyset' \) by Lemma 5.13, \( bcau_5' = bw_{-2}' \) which was shown to be in \( c^0 \) in the previous paragraph, and \( bcau_0' = w_3 \) and \( bcau_3' = w_{-2} \), also in \( \emptyset \). By Lemma 5.10 \( \emptyset' \cap \emptyset'' = \emptyset \), and \( bca\ell_4' = \ell_2 \), so \( bca\emptyset' \) also avoids \( \ell_2 \), hence \( bca\emptyset' \subset \emptyset \).

### 5.5 Iteration

If \( p_1 > 3 \) and \( h = gg''g' \in \rho(QQ'Q') = \rho(T) \) then either \( h\emptyset \subset gg''c' \subset g\emptyset'' \subset c \) by Lemma 5.12, or, if \( p_2 = p_3 = 3 \), then \( h\emptyset \subset gg''\emptyset' \subset \emptyset \) by Lemma 5.14. To apply Proposition 4.1 we need a bit more, that \( h\emptyset \subset c^0 \) for most \( h \in \rho(T) \). We get this by carefully examining which vertices of \( h\emptyset \) can end up on the boundary of \( c \).

**Lemma 5.15.** Assume \( p_1 > 3 \) and \( h \in \rho(T) \). Then \( h\{w_{-2}, w_5, bcu_0\} \subset c^0 \) and \( hw_3 \in c^0 \cup \{w_{-2}, w_5\} \).

**Proof.** We write \( h = gg''g' \in \rho(Q)\rho(Q'') \rho(Q') \). If \( x \in \{w_{-2}, w_5, bcu_0\} \), then \( g'x \in c^0 \) by Lemma 5.12, so \( hx \in c^0 \).

Lemma 5.12 also tells us that \( g'w_3 \in c^0 \) unless \( g' \in \{c, bc\} \), in which case \( g'w_3 \in \{w_{-2}', w_5\} \). If \( p_2 > 3 \) then \( g''\{w_{-2}', w_5\} \subset c^0 \), so \( hw_3 \in \emptyset \). If \( p_2 = 3 \) and \( p_3 > 3 \), then still \( g''w_{-2}' \in c^0 \), but \( g''w_5' \in \{w_{-2}', w_5\} \), and \( g\{w_{-2}', w_5\} \subset c^0 \), so \( hw_3 \in \emptyset \). Finally, in the case \( p_2 = p_3 = 3 \) we have \( hw_3 \in gg''\{w_{-2}', w_5\} \subset \{w_{-2}, w_5\} \) by Lemma 5.14. □
Lemma 5.16. Assume \( p_2 > 3 \) or \( p_3 > 3 \) and let \( g \in \rho(Q) \). Then \( g\Box'' \cap \ell_0 = \emptyset \) unless \( g = ab \). Similarly, if \( p_2 = p_3 = 3 \) and \( g \in \rho(Q) \), \( g'' \in \rho(Q'') \), then \( gg''\Box' \cap \ell_0 = \emptyset \) unless \( g = ab \) and \( g'' = ca \).

Proof. By Lemma 5.10 \( g\Box'' \cap \ell_0 = \emptyset \) if \( g\ell_i = \ell_0 \) for some \( i \not\in \{-1, 0, 1, 2\} \). This is true for all \( g \in \rho(Q) \) except if \( g = ab \) or \( g = b \) and \( p_3 = 3 \). So assume \( p_3 = 3 \) and \( b\Box'' \cap \ell_0 \neq \emptyset \). As \( b\Box'' \subset \Box \) and \( \ell_0 \) intersects \( \Box \) only on its boundary by Lemma 5.10, \( b\Box'' \cap \ell_0 \) must contain a vertex of \( b\Box'' \). These vertices are \( bw_0'' = w_3 \), \( bw_1'' = w_5 \), and \( bw_3'' = w_4 \), which are not on \( \ell_0 \), \( babu''_0 \) and \( bw''_{-2} \), which are in \( \Box'' \) by Lemma 5.12, and \( bcabu''_0 \). But \( bcabu''_0 \in CH_{\ell_2}(w_3, bcu_0) \subset \ell_3 \cap M \) by Lemma 5.9, in particular it is not in \( \ell_0 \).

Using the same argument in the case \( p_2 = p_3 = 3 \), if \( gg''\Box' \cap \ell_0 \) is non–empty it contains a vertex of \( gg''\Box' \). Now \( gg'' \in \{ba, aba, bca\} \), but we can ignore the case \( aba\Box' = ba\Box' \). If we just list the twelve vertices of \( ba\Box' \) and \( bca\Box' \) we see they are only ten distinct points. We showed in the proof of Lemma 5.14 already that seven of them are in \( \Box'' \). The remaining points are \( beaw_0' = w_3 \), \( baw_1' = w_5 \), and \( beaw_3' = baw_3' = w_2 \); see Figure 9. None of them are on the line \( \ell_0 \).

\( \Box \)

Lemma 5.17. Assume \( p_1 > 3 \) and let \( h \in \rho(T) \). Then \( h\Box \cap \ell_0 = \emptyset \) unless \( h = abcabc \).

Proof. We write \( h = gg''g' \in \rho(QQ''Q') \). If \( h\Box \cap \ell_0 \neq \emptyset \) and \( p_2 > 3 \) or \( p_3 > 3 \), then \( g\Box'' \cap \ell_0 \neq \emptyset \), so \( g = ab \) by Lemma 5.16. Hence \( g(g''\Box' \cap \ell_0') = gg''\Box' \cap \ell_0 \neq \emptyset \), so \( g'' = ca \) by the same lemma. And finally \( gg''(g''\Box \cap \ell_0') = h\Box \cap \ell_0 \neq \emptyset \), so \( g' = bc \). In summary we get \( h = gg''g' = abcabc \). The case \( p_2 = p_3 = 3 \) is similar, we just use the second part of Lemma 5.16 in place of the first two steps.

\( \Box \)

Proposition 5.18. Assume \( p_1 > 3 \) and let \( h_1, h_2, h_3 \in Z \). Then either \( h_3 = abcabc \) or \( h_1h_2h_3\Box \subset \Box'' \).

Proof. We already know that \( h_1h_2h_3\Box \subset \Box \). Assume \( h_1h_2h_3\Box \cap \partial \Box \) is non–empty. Then \( h_1h_2\Box \cap \Box \) is also non–empty, and by convexity it must be a union of closed edges and vertices of \( h_1h_2\Box \). But Lemma 5.15 shows that none of \( h_1h_2\{w_2, w_5, bcu_0, w_3\} \) can be in \( \partial \Box \), so

\[ h_1h_2\Box \cap \partial \Box \subset h_1h_2CH_{\ell_2}(w_0, abcu_0) \subset h_1h_2\ell_0. \]

So \( h_1h_2h_3\Box \cap \partial \Box \subset h_1h_2(h_3\Box \cap \ell_0) \), which is empty by Lemma 5.17 unless \( h_3 = abcabc \).

\( \Box \)
6 Duality

The results from the previous two sections allow us to construct boundary maps into $\mathbb{RP}^2$ for representations $\rho$ of type $(\frac{p_1-1}{2}, \frac{p_2-1}{2}, \frac{p_3-1}{2})$ with parameter $t_\rho \in [t_{\text{crit}}, \infty)$. Now we can leverage two forms of duality: first to extend this to the case $t_\rho \in (0, t_{\text{crit}}^{-1}]$ and then to also construct a boundary map into the dual projective plane $(\mathbb{RP}^2)^*$. Note that in this section we write $\partial \Gamma$ instead of $S^1$ for the group boundary, to be more precise when two different groups are involved.

**Lemma 6.1.** Let $\rho: \partial \Gamma_{p_1,p_2,p_3} \rightarrow \text{SL}(3, \mathbb{R})$ have type $(\frac{p_1-1}{2}, \frac{p_2-1}{2}, \frac{p_3-1}{2})$ and parameter $t_\rho \in (0, t_{\text{crit}}^{-1}] \cup [t_{\text{crit}}, \infty)$. Then there exists a continuous $\rho$–equivariant map

$$\xi^{(1)}: \partial \Gamma_{p_1,p_2,p_3} \rightarrow \mathbb{RP}^2.$$

**Proof.** As we discussed in Section 2.5, every reducible representation in this component is Anosov. So we can assume $\rho$ is irreducible. If $t_\rho \geq t_{\text{crit}}$ then the set $\square$ from Proposition 5.18 satisfies the assumptions of Proposition 4.1, hence there exists a continuous $\rho$–equivariant map $\xi^{(1)}: \partial \Gamma_{p_1,p_2,p_3} \rightarrow \mathbb{RP}^2$.

Now assume $t_\rho \in (0, t_{\text{crit}}^{-1}]$ and let $\psi: \Gamma_{p_1,p_2,p_3} \rightarrow \Gamma_{p_1,p_2,p_3}$ be the group isomorphism which fixes $s_1$ and interchanges $s_2$ with $s_3$. It is an isometry of Cayley graphs and hence induces a homeomorphism $\partial \psi: \partial \Gamma_{p_1,p_2,p_3} \rightarrow \partial \Gamma_{p_1,p_2,p_3}$ of the group boundaries, which is $\psi$–equivariant.

Using the notation $c_i = 2 \cos(\frac{p_i-1}{2p_i} \pi)$ as in the proof of Lemma 2.4, the Cartan matrix of $\rho \circ \psi$ is (equivalent to)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -c_3 & -c_2 \\ -c_3 & 2 & -t_\rho c_1 \\ -c_2 & -t_\rho^{-1} c_1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -c_2 & -c_3 \\ -c_2 & 2 & -t_\rho^{-1} c_1 \\ -c_3 & -t_\rho c_1 & 2 \end{pmatrix}.$$

Hence $\rho \circ \psi$ is a representation of $\Gamma_{p_1,p_2,p_3}$ of type $(\frac{p_1-1}{2}, \frac{p_2-1}{2}, \frac{p_3-1}{2})$ with parameter $t_{\rho \circ \psi} = t_\rho^{-1} \in [t_{\text{crit}}, \infty)$. So there exists a continuous $(\rho \circ \psi)$–equivariant boundary map $\xi^{(1)}: \partial \Gamma_{p_1,p_3,p_2} \rightarrow \mathbb{RP}^2$ by the above. But then

$$\xi^{(1)} = \xi^{(1)} \circ \partial \psi^{-1}: \partial \Gamma_{p_1,p_2,p_3} \rightarrow \mathbb{RP}^2$$

is $\rho$–equivariant. □

**Proposition 6.2.** Let $\rho: \partial \Gamma_{p_1,p_2,p_3} \rightarrow \text{SL}(3, \mathbb{R})$ be as in Lemma 6.1. Then there exists a continuous $\rho$–equivariant map

$$\xi: \partial \Gamma_{p_1,p_2,p_3} \rightarrow F.$$

into the flag manifold $F$. It maps the attracting (resp. repelling) fixed point $\gamma_{\pm}$ of the Coxeter element $\gamma = s_1 s_2 s_3$ to the attracting (repelling) flag of $\rho(\gamma)$. 36
Proof. We can again assume that \( \rho \) is irreducible. By Lemma 6.1 there is \( \rho \)-equivariant boundary map \( \xi^{(1)}: \partial \Gamma \to \mathbb{R}P^2 \).

To obtain a dual boundary map, we consider the inverse transposed representation \( \rho^{-T} \). Since its Cartan matrix is just the transpose of that of \( \rho \), we have

\[
(\rho(s_i)^{-T}) = (b_i \otimes \alpha_i - 1)^T = (\alpha_i^T \otimes b_i^T - 1)
\]

its Cartan matrix is just the transpose of that of \( \rho \). Hence the type of \( \rho^{-T} \) is also \((\frac{p_{11} - 1}{2}, \frac{p_{22} - 1}{2}, \frac{p_{33} - 1}{2})\), but \( t_{\rho^{-T}} = t_{\rho}^{-1} \). By Lemma 6.1 there is a \( \rho^{-T} \)-equivariant continuous boundary map \( \xi': \partial \Gamma \to \mathbb{R}P^2 \). If we write \( D: \mathbb{R}P^2 \to (\mathbb{R}P^2)^* \) for the duality induced by the standard scalar product on \( \mathbb{R}^3 \), then

\[
\xi^{(2)} = D \circ \xi': \partial \Gamma \to (\mathbb{R}P^2)^*
\]

is \( \rho \)-equivariant.

For the remaining part, note that the Coxeter element \( \rho(s_1s_2s_3) \) does not have three distinct eigenvalues if \( t_\rho \in \{t_{\text{crit}}^{-1}, t_{\text{crit}}\} \) (Lemma 2.5). But we can choose \( \gamma \in \{s_1s_2s_3, s_3s_2s_1\} \) so that \( \rho(\gamma) \) is proximal, i.e. has a unique eigenvalue of maximal modulus. We write \( \rho(\gamma)_+ \) for its attracting point and \( \rho(\gamma)^- \) for its repelling line in the projective plane.

By irreducibility \( \xi^{(1)}(\partial \Gamma) \) cannot be contained in a line, so there is some \( z \in \partial \Gamma \) with \( \xi^{(1)}(z) \notin \rho(\gamma)^- \). Hence \( \xi^{(1)}(\gamma^nz) = \rho(\gamma)^n \xi^{(1)}(z) \to \rho(\gamma)_+ \). We can assume \( z \neq \gamma_- \). But then \( \gamma^nz \to \gamma_+ \) in \( \partial \Gamma \). So continuity implies \( \xi^{(1)}(\gamma_+) = \rho(\gamma)_+ \). An analogous argument shows that \( \xi^{(2)}(\gamma_-) = \rho(\gamma)^- \).

If \( t_\rho \notin \{t_{\text{crit}}^{-1}, t_{\text{crit}}\} \) then \( \rho(\gamma)^{-1} \) is also proximal, so we can apply the same to it and get \( \xi^{(1)}(\gamma_-) = \rho(\gamma)^- \) and \( \xi^{(2)}(\gamma_+) = \rho(\gamma)^+ \). Otherwise, \( \rho(\gamma) \) is not diagonalizable by Lemma 2.5. Hence it has exactly two fixed points \( \rho(\gamma)_+ \) and \( \rho(\gamma)_- \) and two fixed lines \( \rho(\gamma)^\pm \) with \( \rho(\gamma)_\pm \in \rho(\gamma)^\pm \). Since \( \rho(\gamma)_+ \notin \rho(\gamma)^- \), we necessarily have \( \xi^{(1)}(\gamma_-) = \rho(\gamma)_- \) and \( \xi^{(2)}(\gamma_+) = \rho(\gamma)^+ \).

In any case \( \xi^{(1)}(\gamma_+) \in \xi^{(2)}(\gamma_+) \), and since the orbit of \( \gamma_+ \) is dense in \( \partial \Gamma \), we have \( \xi^{(1)}(x) \in \xi^{(2)}(x) \) for all \( x \in \partial \Gamma \), so \( \xi^{(1)} \) and \( \xi^{(2)} \) combine to a map \( \xi \) into \( \mathcal{F} \).

\[\square\]

7 Transversality

In this section we assume that \( t_\rho \in (0, t_{\text{crit}}^{-1}) \cup (t_{\text{crit}}, \infty) \). We will show that in this case the boundary map \( \xi \) from Proposition 6.2 is transverse, that is \( \xi^{(1)}(x) \notin \xi^{(2)}(y) \) whenever \( x \neq y \). This is not true if \( t_\rho \in \{t_{\text{crit}}, t_{\text{crit}}^{-1}\} \), as the Coxeter element is not diagonalizable by Lemma 2.5 and its attracting and repelling flags are therefore not transverse. We are first going to prove transversality for pairs in \( S^1 = \partial \Gamma \) of which one element is a fixed point of the Coxeter element, then extend this to a certain open subset of pairs, and finally show that the \( \Gamma \)-orbit of this subset comprises all distinct pairs.
Lemma 7.1. Let $\gamma = s_1s_2s_3$ be the Coxeter element and $z \in S^1 \setminus \{\gamma_+ , \gamma_-\}$. Then $\xi(z)$ is transverse to $\xi(\gamma_+)$ and $\xi(\gamma_-)$.

Proof. If $\xi^{(1)}(z) \in \xi^{(2)}(\gamma_-)$, then we would have, by Proposition 6.2,

$$\rho(\gamma)_+ = \xi^{(1)}(\gamma_+) = \lim_{n \to \infty} \xi^{(1)}(\gamma^nz) = \lim_{n \to \infty} \rho(\gamma)^n \xi^{(1)}(z) = \lim_{n \to \infty} \rho(\gamma)^n \xi^{(2)}(\gamma_-) = \rho(\gamma)^-.$$  

This is clearly false if $\gamma$ has distinct real eigenvalues, so $\xi^{(1)}(z) \notin \xi^{(2)}(\gamma_-)$. An analogous argument shows $\xi^{(2)}(z) \notin \xi^{(1)}(\gamma_-)$ and applying the same to $\gamma^{-1}$ instead of $\gamma$ also shows transversality to $\xi(\gamma_+)$.

Recall from Section 5 the definition of the points $z_0, \ldots, z_{2p_1-1} \in S^1$ such that

$$z_0 = (s_1s_2s_3)_+, \quad z_3 = (s_2s_3s_1)_+, \quad s_2s_1z_i = z_{i+2}$$

and the corresponding points $w_i, u_i \in \mathbb{RP}^2$, and lines $\ell_i \subset \mathbb{RP}^2$. It follows from Proposition 6.2 that $\xi(z_i) = (w_i, \ell_i)$ for all $i$. We write $[z_i, z_j]$ for the closed interval in $S^1$ containing the points $z_i, z_{i+1}, \ldots, z_j$. As before, there are also “primed” and “double–primed” versions of these points, obtained by cyclically permuting $s_1, s_2, s_3$ in the definition.

Lemma 7.2. Assume $t_p > t_{\text{crit}}$. If $z \in [z_j, z_{j+1}]$ and $z' \in [z_k, z_{k+1}]$ such that $\xi^{(1)}(z) \in \xi^{(2)}(z')$, then either $|j - k| \leq 1$ or $j$ and $k$ are even and $|j - k| = 2$.

Proof. First assume $t_p > t_{\text{crit}}$. Let $P(C)$ be the set of unordered pairs in the conic $C$, i.e. the quotient of $C^2$ by flipping the order. Define a map $f : S^1 \to P(C)$ by mapping $x \in S^1$ to the two intersection points of the line $\xi^{(2)}(x)$ with $C$. A priori $\xi^{(2)}(x)$ and $C$ might intersect in only one or no points, but we will see below that this does not happen. We have $f(z_i) = \{w_i, u_i\}$ and Lemma 7.1 also shows that $w_i \notin f(x)$ for all $x \neq z_i$.

Figure 10: The proof of Lemma 7.2 in the case $k = 0$ and $j = 3$. The line $\xi^{(2)}(z')$ (in green) can only intersect points in $M$ between $\ell_{-2}$ and $\ell_3$, but $\xi^{(1)}(z)$ is in the red region between $\ell_3$ and $\ell_4$.

For a fixed $k$, let $\tilde{f} : [z_k, z_{k+1}] \to C^2$ be the lift of $f$ with $f(z_k) = (w_k, u_k)$. If $k$ is odd, it follows from the order of the points along $C$ (described in Lemma 5.3) that $\tilde{f}(z_{k+1}) = (w_{k+1}, u_{k+1})$. Hence $\tilde{f}([z_k, z_{k+1}]) \subset [w_k, w_{k+1}] \times (w_{k-1}, w_{k+2})$, which means that $\xi^{(2)}(z')$ cannot intersect $\ell_{k-1}$ or $\ell_{k+2}$ within the Möbius strip $M$. So if $\xi^{(1)}(z) \notin \xi^{(2)}(z')$ then $\xi^{(1)}(z)$ and $\xi^{(1)}(z')$ have to be in the same component of $\mathbb{RP}^2 \setminus (\ell_{k-1} \cup \ell_{k+2})$. Since $\xi^{(1)}$ can
only cross $\ell_{k-1}$ and $\ell_{k+2}$ at $z_{k-1}$ resp. $z_{k+2}$ by Lemma 7.1, this shows $z \in [z_{k-1}, z_{k+2}]$, i.e. $|j-k| \leq 1$.

If $k$ is even, we have $f(z_{k+1}) = (u_{k+1}, w_{k+1})$, so $f([z_k, z_{k+1}]) \subset [w_k, w_{k+3}) \times (w_{k-2}, w_{k+1})$ (see Figure 10). By the same argument as above, $z$ must be in the interval $[z_{k-2}, z_{k+3}]$, so $|j-k| \leq 2$.

Recall that $I = [z_3, z_0] = [(s_2s_3s_1)_+, (s_1s_2s_3)_+]$, and let $J = [(s_2s_3s_1)_-, (s_1s_2s_3)_-]$, and $K = [z_1, z_2]$. We can see in Figure 7 that $K \subset J$.

**Lemma 7.3.** Let $A \subset S^1 \times S^1$ be a $\Gamma$-invariant subset which contains $I \times K$, $I' \times K'$, $I'' \times K''$, as well as $K \times I$, $K' \times I'$, and $K'' \times I''$. Then $A$ contains every pair $(x, y) \in S^1 \times S^1$ of distinct points which are not the two fixed points of a conjugate of $s_1s_2s_3$.

**Proof.** A pair $(x, y)$ of distinct points in $S^1$ defines an oriented geodesic $xy$ in the hyperbolic plane. As we noted in Section 2.4, it intersects one of the “altitude triangles” bounded by Coxeter axes (axes of elements conjugate to $s_1s_2s_3$). Hence some $\Gamma$–translate of either $(x, y)$ or $(y, x)$ is contained in $I \times J$, or $I' \times J'$, or $I'' \times J''$; see Figure 7. Due to the symmetry of the assumptions of the lemma, we can assume $(x, y) \in I \times J$, and in fact $(x, y) \in I^0 \times J^0$ if $xy$ is not a Coxeter axis.

It remains to show $I^0 \times J^0 \subset A$. To do this, we will decompose $J^0$ into a union of translates of $K$, $K'$ and $K''$. Let $\eta = (s_1s_3s_2)^2$, and note using Figure 4 that $s_1s_3z_2' = z_1$ and analogously $s_2s_1z_2'' = z_1'$. So the intervals $K$ and $s_1s_3K'$ share an endpoint, as do $K'$ and $s_2s_1K''$. Therefore,

$$[\eta z_2, z_1] = s_1s_3s_2s_1K'' \cup s_1s_3K' \cup K$$

Now $I' = [z_2, z_3] \subset [z_2, z_4] = s_2s_1[z_0, z_2] = s_2s_1I''$, so

$$I \subset s_1s_3I' \subset s_1s_3s_2s_1I'' \subset \eta I.$$

So not only $I \times K \subset A$ but also $I \times s_1s_3K' \subset s_1s_3(I' \times K') \subset A$ and $I \times s_1s_3s_2s_1K'' \subset s_1s_3s_2s_1(I'' \times K'') \subset A$. Together, this gives $I \times [\eta z_2, z_1] \subset A$. Similarly, we find that $I \times \eta^{k+1}z_2, \eta^{k}z_2] \subset \eta^{k}(I \times [\eta z_2, z_1]) \subset A$, for all $k \in \mathbb{N}$.

Let $L = (\eta_+, z_2)$ be the union of the sequence $[\eta^{k+1}z_2, \eta^{k}z_2]$ of adjacent intervals. Then $L \cup s_1L = (\eta_+, s_1\eta_+) = J^0$, so using $s_1I = I$ we obtain $I \times J^0 \subset A$. □

**Theorem 7.4.** Let $\rho: \Gamma \to \text{SL}(3, \mathbb{R})$ be a representation of type $\left(\frac{p_1-1}{2}, \frac{p_2-1}{2}, \frac{p_3-1}{2}\right)$ with parameter $t_\rho \in (0, t_{\text{crit}}^{-1}] \cup [t_{\text{crit}}, \infty)$, and $\xi: S^1 \to \mathcal{F}$ the $\rho$–equivariant continuous boundary map from Proposition 6.2.

Then $\xi^{(1)}$ and $\xi^{(2)}$ are injective. If $t_\rho \notin \{t_{\text{crit}}^{-1}, t_{\text{crit}}\}$, then $\xi(x)$ and $\xi(y)$ are even transverse for every distinct pair $x, y \in S^1$, so $\rho$ is an Anosov representation.
Proof. We can assume that \( \rho \) is irreducible, as we already know it is Anosov otherwise (see Section 2.5, in particular Fact 2.10(ii)). We can also assume that \( t_\rho > 1 \), otherwise we consider the representation \( \rho \circ \psi \) instead, as in the proof of Lemma 6.1.

We first consider the case \( t_\rho > t_{\text{crit}} \). If \( x \in I = [z_3, z_0] \) and \( y \in K = [z_1, z_2] \) then \( \xi(x) \) and \( \xi(y) \) are transverse by Lemma 7.2. The same is true if \( (x, y) \) is in \( I' \times K' \) or \( I'' \times K'' \).

Transversality is symmetric, so we can apply Lemma 7.3 to extend this to all pairs \((x, y)\) with \( x \neq y \), unless \( x \) and \( y \) are the fixed points of a conjugate \( \gamma \) of the Coxeter element. But if \( x = \gamma_+ \) and \( y = \gamma_- \), then \( \xi(x) \) and \( \xi(y) \) are transverse by Proposition 6.2 since \( \gamma \) has distinct real eigenvalues by Lemma 2.5. Fact 2.11 then shows that \( \rho \) is Anosov.

Now assume \( t_\rho = t_{\text{crit}} \). We know that \( \xi^{(1)}(\gamma_+) \neq \xi^{(1)}(\gamma_-) \) for every conjugate \( \gamma \) of the Coxeter element. So by Lemma 7.3, to prove injectivity of \( \xi^{(1)} \) it suffices to show \( \xi^{(1)}(I) \) does not intersect \( \xi^{(1)}(K) \). By Proposition 4.1 \( \xi^{(1)}(I) \subset \square \), and \( K = [z_0, z_2] \cap [z_1, z_3] = I'' \cap s_1I'' \), so \( \xi^{(1)}(K) \subset \rho(s_1)\square'' \cap \square'' \). But by Lemma 5.10 \( \square'' \) does not intersect \( \ell_3 \) and \( \rho(s_1)\square'' \) does not intersect \( \ell_0 \), so \( \xi^{(1)}(K) \) is in one component of \( \mathbb{R}P^2 \setminus (\ell_0 \cup \ell_3) \). Also by Lemma 5.10, \( \ell_0 \) and \( \ell_3 \) intersect \( \square \) only on its boundary, so \( \xi^{(1)}(I) \) is in the closure of a component of \( \mathbb{R}P^2 \setminus (\ell_0 \cup \ell_3) \). These components cannot be the same (see e.g. Figure 10), hence \( \xi^{(1)} \) is injective. Using the constructions in Section 6, the result extends to \( \xi^{(2)} \) and to \( t_\rho = t_{\text{crit}}^{-1} \). \( \square \)

Proof of Theorem 1.2. If \( \rho: \Gamma \to \text{SL}(3, \mathbb{R}) \) is not a Coxeter representation, it has a finite image, so it cannot be Anosov. If \( \rho \) is Anosov, it is in the Hitchin or Barbot component by Proposition 3.3, and by Fact 2.10(iv) \( \rho(\gamma) \) has distinct real eigenvalues for every \( \gamma \in \Gamma \) of infinite order, in particular for \( \gamma = s_1s_2s_3 \).

Conversely, if \( \rho \) is in the Hitchin component it is Anosov by [CG05; Lab06]. If \( \rho \) is in the Barbot component and \( \rho(s_1s_2s_3) \) has distinct real eigenvalues, then it has type \( (\frac{p_1-1}{2}, \frac{p_2-1}{2}, \frac{p_3-1}{2}) \) and \( t_\rho \in (0, t_{\text{crit}}^{-1}) \cup (t_{\text{crit}}, \infty) \) by Lemma 2.5. So \( \rho \) is Anosov by Theorem 7.4. \( \square \)

References

[ALS] D. Alessandrini, G.-S. Lee, and F. Schaffhauser. Hitchin components for orbifolds. J. Eur. Math. Soc. (JEMS), to appear. arXiv: 1811.05366 († 16).

[Bar10] T. Barbot. Three-dimensional Anosov flag manifolds. In: Geom. Topol. 14.1 (2010), pp. 153–191 († 2, 3).

[BPS19] J. Bochi, R. Potrie, and A. Sambarino. Anosov representations and dominated splittings. In: J. Eur. Math. Soc. (JEMS) 21.11 (2019), pp. 3343–3414 († 2, 20).

[CG05] S. Choi and W. M. Goldman. The deformation spaces of convex \( \mathbb{R}P^2 \)-structures on 2-orbifolds. In: Amer. J. Math. 127.5 (2005), pp. 1019–1102 († 4, 16, 40).

[EEK82] A. L. Edmonds, J. H. Ewing, and R. S. Kulkarni. Torsion free subgroups of Fuchsian groups and tessellations of surfaces. In: Invent. Math. 69.3 (1982), pp. 331–346 († 4).
W. Fenchel. *Elementary geometry in hyperbolic space*. Vol. 11. De Gruyter Studies in Mathematics. With an editorial by Heinz Bauer. Walter de Gruyter & Co., Berlin, 1989, pp. xii+225 († 14).

W. M. Goldman and J. J. Millson. *Local rigidity of discrete groups acting on complex hyperbolic space*. In: Invent. Math. 88.3 (1987), pp. 495–520 († 17).

W. M. Goldman. *Convex real projective structures on compact surfaces*. In: J. Differential Geom. 31.3 (1990), pp. 791–845 († 11).

W. M. Goldman and J. R. Parker. *Complex hyperbolic ideal triangle groups*. In: J. Reine Angew. Math. 425 (1992), pp. 71–86 († 5).

W. M. Goldman. *Affine manifolds and projective geometry on surfaces*. Senior thesis. Princeton University, 1977 († 6).

F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard. *Anosov representations and proper actions*. In:Geom. Topol. 21.1 (2017), pp. 485–584 († 2, 16).

O. Guichard and A. Wienhard. *Anosov representations: domains of discontinuity and applications*. In: Invent. Math. 190.2 (2012), pp. 357–438 († 2, 16, 17, 18).

O. Guichard and A. Wienhard. *Positivity and higher Teichmüller theory*. In: European Congress of Mathematics. Eur. Math. Soc., Zürich, 2018, pp. 289–310 († 2).

M. Kapovich. *Hyperbolic manifolds and discrete groups*. Vol. 183. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2001, pp. xxvi+467 († 17).

M. Kapovich, B. Leeb, and J. Porti. *Anosov subgroups: dynamical and geometric characterizations*. In: Eur. J. Math. 3.4 (2017), pp. 808–898 († 2, 20).

S. Kim and J. Lee. *Three-punctured sphere groups in SL(3, R)*. (in preparation) († 5).

F. Labourie. *Anosov flows, surface groups and curves in projective space*. In: Invent. Math. 165.1 (2006), pp. 51–114 († 2, 4, 16, 40).

G.-S. Lee and L. Marquis. *Anti–de Sitter strictly GHC-regular groups which are not lattices*. In: Trans. Amer. Math. Soc. 372.1 (2019), pp. 153–186 († 18, 19).

D. Luna. *Sur certaines opérations différentiables des groupes de Lie*. In: Amer. J. Math. 97 (1975), pp. 172–181 († 6).

D. Luna. *Fonctions différentiables invariantes sous l’opération d’un groupe réductif*. In: Ann. Inst. Fourier (Grenoble) 26.1 (1976), pp. ix, 33–49 († 6).

L. Paris. *Irreducible Coxeter groups*. In: Internat. J. Algebra Comput. 17.3 (2007), pp. 427–447 († 17).

R. W. Richardson and P. J. Slodowy. *Minimum vectors for real reductive algebraic groups*. In: J. London Math. Soc. (2) 42.3 (1990), pp. 409–429 († 6).

R. E. Schwartz. *Pappus’ theorem and the modular group*. In: Inst. Hautes Études Sci. Publ. Math. 78 (1993), 187–206 (1994) († 20).

R. E. Schwartz. *Ideal triangle groups, dented tori, and numerical analysis*. In: Ann. of Math. (2) 153.3 (2001), pp. 533–598 († 5).

R. E. Schwartz. *A better proof of the Goldman-Parker conjecture*. In: Geom. Topol. 9 (2005), pp. 1539–1601 († 5).
[Sul85] D. Sullivan. Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups. In: Acta Math. 155.3-4 (1985), pp. 243–260 (↑ 20).

[Vin71] È. B. Vinberg. Discrete linear groups that are generated by reflections. In: Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), pp. 1072–1112 (↑ 13).

[Wie18] A. Wienhard. “An invitation to higher Teichmüller theory”. In: Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures. World Sci. Publ., Hackensack, NJ, 2018, pp. 1013–1039 (↑ 2).

GYE-SEON LEE
Department of Mathematics, Sungkyunkwan University, 2066 Seobu-ro, Jangan-gu, Suwon, 16419, South Korea
Email: gyeseonlee@skku.edu

JAEJEONG LEE
Institute of Basic Science, Sungkyunkwan University, 2066 Seobu-ro, Jangan-gu, Suwon, 16419, South Korea
Email: j.lee@skku.edu

FLORIAN STECKER
Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712, USA
Email: math@florianstecker.net

42