Quantization of AdS $\times$ S particle in static gauge

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Abstract
We quantize the particle dynamics in AdS$^{N+1} \times$ S$^M$ spacetime in static gauge, which leads to the coordinate representation with wavefunctions depending only on spatial coordinates. The energy square operator is quadratic in canonical momenta and contains a scalar curvature term. We analyze the self-adjointness of this operator and calculate its spectrum. We then construct unitary representations of the isometry group SO($2, N$) $\times$ SO($M+1$) and calculate the quantum relation between the Casimir numbers.

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(Some figures may appear in colour only in the online journal)

Introduction

In this paper we quantize the AdS $\times$ S particle dynamics in static gauge. We use the quantization scheme of [1] based on a covariant construction of the energy square operator in the coordinate representation, where the wavefunctions depend only on spatial coordinates. Let us first outline the scheme, which we present here in a slightly modified form.

Particle dynamics in a spacetime with coordinates $x^\mu, \mu = (0, 1, \ldots, N)$ and a metric tensor $g_{\mu\nu}(x)$ can be described by the following action in the first-order formalism:

$$ S = \int dt \left( p_\mu \dot{x}^\mu - \frac{\lambda}{2} (g^{\mu\nu} p_\mu p_\nu + M^2) \right). $$

Here $M$ is the particle mass, $\lambda$ is a Lagrange multiplier and its variation provides the mass-shell condition

$$ g^{\mu\nu} p_\mu p_\nu + M^2 = 0. $$

Using the Faddeev–Jackiw reduction [2] in the gauge

$$ x^0 + p_0 \tau = 0, $$

we have...
from (1) we obtain an ordinary Hamiltonian system
\[
S = \int \text{d}\tau \left( p_n \dot{x}^n - \frac{1}{2} p_0^2 \right),
\]
(4)

where \( p_0 \), as a function of the spatial coordinates and momenta \((x^n, p_n) (n = 1, \ldots, N)\), is obtained from the constraint (2) and the gauge fixing condition (3).

Note that here we have modified the form of the standard static gauge \( x^0 = \tau \) used in [1]. However, this modification does not change the quantization scheme and it appears more convenient for a Hamiltonian treatment of (4) in a static spacetime, as well as for the generalization to string theory [3].

A static spacetime metric tensor can be represented in the form
\[
g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 \\ 0 & g_{mn} \end{pmatrix},
\]
(5)

where \( g_{00} \) and \( g_{mn} \) are functions only of the spatial coordinates \( x^n \). In this case the particle energy \( E(p, x) = -p_0 > 0 \) is conserved and from (2) it follows that
\[
E^2 = \Lambda(x) g_{mn}(x) p_m p_n + M^2 \Lambda(x),
\]
(6)

with \( \Lambda(x) := -g_{00}(x) > 0 \).

Thus, the Hamiltonian in (4) corresponds to a motion of a particle in the potential field \( V(x) = \frac{1}{2} M^2 \Lambda(x) \) and in a curved background with metric tensor
\[
h_{mn}(x) = \frac{1}{\Lambda(x)} g_{mn}(x).
\]
(7)

It is natural to quantize this system in the coordinate representation, where the wavefunctions \( \psi(x) \) form a Hilbert space with covariant scalar product
\[
\langle \psi_2 | \psi_1 \rangle = \int \text{d}^N x \sqrt{h(x)} \psi_2^*(x) \psi_1(x), \quad h(x) := \det h_{mn}(x).
\]
(8)

On the basis of DeWitt’s construction for quadratic in momenta operators [4], it was argued in [1] that the energy square operator is given by
\[
E^2 = -\Delta_h + \frac{N - 1}{4N} R_h(x) + M^2 \Lambda(x).
\]
(9)

Here \( \Delta_h \) is the covariant Laplace–Beltrami operator for the metric tensor \( h_{mn} \) and \( R_h(x) \) denotes the corresponding scalar curvature. The solution of the eigenvalue problem for (9) then provides the energy operator in diagonal form.

The coefficient in front of the scalar curvature term has been a subject of discussions during decades (see [5] and references therein). Therefore it is useful to comment on the value of this coefficient, \( \frac{N - 1}{4N} \), chosen in (9).

For the particle dynamics in AdS\(_{N+1}\) this coefficient was calculated in [1] from the commutation relations of the symmetry generators. In this case \( R_h(x) \) corresponds to the curvature of a semi-sphere and, therefore, it is constant. The obtained constant shift in the energy square operator provides its spectrum in the form \((E_0 + n)^2\), with fixed \( E_0 \) and a non-negative integer \( n \), that leads to the correct energy spectrum for the AdS particle.

For a generic \((N + 1)\)-dimensional static spacetime the same value of the coefficient \( \frac{N - 1}{4N} \) follows from the equivalence between the static gauge quantization and the covariant quantization based on the Klein–Gordon type equation.

The covariant quantization is a more conventional approach to the particle dynamics in AdS backgrounds [6, 7]. The general case with arbitrary dimensions and radii in this approach was analyzed in [8] from the perspective of scalar field propagators (see also [9]).
An interesting alternative quantization scheme based on the twistor and BRST formalisms was proposed in [10] for a massive bosonic particle in AdS$_5$. The obtained twistor construction was related to the oscillator construction of [11].

The quantization of a superparticle in the AdS$_5 \times S^5$ background was done in [12], using the lightcone gauge and the technique developed in [13] (see also [14], which is based on a different lightcone type gauge).

The dynamics of a massive bosonic particle in AdS$_5 \times S^5$ was considered in [15]. The authors used gauge invariant approach and Dirac bracket formalism. The obtained results were applied to the analysis of string energy spectrum at large coupling in the context of the AdS/CFT correspondence.

The main motivation of most of the papers on particle dynamics in AdS spaces is to develop useful methods for the quantization of strings in these backgrounds.

Our motivation is the same and in this paper we aim to apply the static gauge quantization to AdS$_{N+1} \times S^M$ particle. AdS$_{N+1}$ will be realized as a hyperbola $X^A X_A = -R^2$, with $A = (0', 0, 1, \ldots, N)$, embedded in $\mathbb{R}^{2,N}$ and $S^M$ as a $M$-dimensional sphere $Y_I Y_I = R_5^2$ in $\mathbb{R}^{M+1}$, with $I = (1, \ldots, M + 1)$.

In the next section we deal with the classical case and prepare the system for quantization.

**Geometry on AdS$_{N+1} \times S^M$**

AdS$_{N+1} \times S^M$ has $N = N + M$ spatial coordinates and we associate the first $N$ coordinates with the AdS part and the rest $M$ coordinates with $S^M$. The time coordinate $x^0$ is given by the polar angle $\theta$ in the $(X^0, X^0)$ plane.

We parameterize the embedding coordinates of $\mathbb{R}^{2,N}$ as follows:

$$X^{0'} = R \rho \sin \theta, \quad X^0 = R \rho \cos \theta, \quad X^a = R \rho x^a, \quad \rho := \sqrt{1 - x^b x^b},$$

(10)

where the coordinates $x^a$ are given on the $N$-dimensional unit disc.

The choice of coordinates $\phi^a$ on $S^M$ is not important for our calculations, since the quantization of the spherical part is trivial. We use the Greek letters $\alpha, \beta$ for tensorial indices on $S^M$ and they run from $N + 1$ to $N + M$.

The induced metric tensor on AdS$_{N+1} \times S^M$ has the structure (5) with

$$g_{00} = -\frac{R^2 \rho^2}{\rho^2}, \quad g_{aa} = \begin{pmatrix} g_{aa} & 0 \\ 0 & g_{a'b'} \end{pmatrix},$$

(11)

where $g_{aa}$ corresponds to the spatial part of AdS$_{N+1}$

$$g_{ab} = \frac{R^2}{\rho^2} \left( \delta_{ab} + \frac{x^a x^b}{\rho^2} \right)$$

(12)

and $g_{a'b'}$ is the metric tensor on $S^M$. Note that the scalar curvature for $g_{aa}$ is given by

$$R_g = -N(N-1) R^{-2} + M(M-1) R_S^{-2}.$$  

(13)

It is worth mentioning that the metric tensor (12) corresponds to the Euclidean AdS space (Lobachevski plane) and after the rescaling (7) with the Weyl factor

$$\frac{1}{\Lambda(x)} = \frac{\rho^2}{R^2},$$

(14)

it becomes the metric tensor on the unit semi-sphere (see figure 1) with

6 Other references on gauge invariant quantization of the AdS particle dynamics one can find in [1].
Figure 1. The unit disc here is tangent to the Lobachevski plane at the pole. The coordinates on the disc used in the parametrization (10) correspond to the stereographic projection onto the disc made from the origin of the 3D space. The unit half-sphere is also tangent to the disc at the pole. The same coordinates parameterize the unit half-sphere by the vertical projection.

\[ h_{ab}(x) = \delta_{ab} + \frac{x^a x^b}{\rho^2}, \quad \text{and} \quad \det h_{ab} = \frac{1}{\rho^2}, \]  
(15)

The total background metric tensor \( h_{mn} \) defined by (7) has a similar to (11) block structure and from (15) we find the integration measure in (8) to be

\[ \sqrt{h} = \kappa - M \]  
(16)

where \( \kappa = R/R_S \) and \( \mu_S \) is the SO\((M + 1)\) invariant measure on the unit sphere.

The Laplace–Beltrami operator for the metric tensor \( h_{mn} \) then reads

\[ \Delta_h = h^{ab} \partial_2 a^b - (N + M) x^a \partial_a + \frac{k^2}{\rho^2} \Delta_S, \]  
(17)

where

\[ h^{ab} = \delta_{ab} - x^a x^b \]  
(18)

is the inverse to \( h_{ab} \) and \( \Delta_S \) is the Laplace–Beltrami operator on the unit sphere.

To calculate \( R_h \), one can use the transformation rule (A.1) of scalar curvatures under Weyl rescalings. With the help of (13) it leads to

\[ R_h(x) = (N + M) (N + M - 1) + \frac{M(M - 1)(k^2 - 1)}{\rho^2 (x)}. \]  
(19)

Now we consider the dynamical integrals related to the SO\((2, N) \times \) SO\((M + 1)\) isometry transformations

\[ J_{AB} = \gamma_{AB} p_a + \gamma_{AB}^0 p_0, \quad L_{IJ} = \gamma_{IJ}^0 \pi_a. \]  
(20)

Here \( p_a \) and \( \pi_a \) are the canonically conjugated variables to \( x^a \) and \( \phi^a \), respectively, \( p_0 \) is the negative square root of

\[ E^2 = \frac{R^2}{\rho^2} [g^{ab}(x)p_ap_b + g^{0\beta}(\phi)\pi_a\pi_\beta + \mathcal{M}^2] \]  
(21)
and the coefficients of the momentum variables are the components of the Killing vector fields
(see (A.3) in the appendix)
\[ \mathcal{V}_{ab}^0 = g^{00}(X_a \partial_b X_0 - X_0 \partial_b X_a), \quad \mathcal{V}_{ab}^a = g^{ab}(X_b \partial_a X_0 - X_0 \partial_a X_b), \]
(22)
\[ \mathcal{V}_{\alpha}^{ab} = g^{ab}(Y_b \partial_\alpha Y_a - Y_a \partial_\alpha Y_b). \]
(23)
Since we use dimensionless coordinates, the momentum variables \( p_\alpha, \pi_\alpha \) and the energy are also dimensionless. The minimal energy corresponds to the vanishing momenta and the maximal value of \( \rho \) (i.e. \( \rho = 1 \)) that yields \( E_{\text{min}}^2 = M^2 R^2 \).

The vector field components for the boost generators in (22) depend on the time coordinate \( \theta \) and to use them in (20), one has to make the replacement \( \theta \rightarrow E \tau \) corresponding to the gauge fixing (3). This calculation for the boost generators at \( \tau = 0 \) yields
\[ J_0 = E x^0, \quad J_0' = (p_\alpha x^0) x^\alpha - p_\alpha. \]
(24)
From the canonical Poisson brackets \( \{ p_\alpha, x^\beta \} = \delta_{\alpha\beta} \) follows that the energy square (21) and the boosts (24) satisfy the Poisson bracket relations
\[ \{ E^2, J_0 \} = -2EJ_0, \quad \{ E^2, J_0' \} = 2EJ_0'. \]
(25)
\[ \{ J_0, J_0' \} = -J_{ab} = [J_{ab}, J_{00}], \quad \{ J_0', J_0 \} = E \delta_{ab}, \]
(26)
where \( J_{ab} \) have the standard form of the rotation generators
\[ J_{ab} = p_a x^b - p_b x^a \]
and they correspond to the dynamical integrals (20) for the SO(N) rotations in AdS\(_{N+1}\). Equations (25) and (26) are then equivalent to a part of the commutation relations of the symmetry group Lie algebra. The rest part of the algebra is trivially fulfilled, since the rotation generators both in AdS\(_{N+1}\) and S\(^M\) commute with the energy square (21). Thus, the Poisson bracket algebra of the dynamical integrals (20) is not deformed by the Hamiltonian reduction, which is a consequence of their gauge invariance for the initial system (1).

Concluding this section we comment on the Casimir numbers of the isometry groups. From (22) and (23) it follows (see (A.4) in appendix) that
\[ \frac{1}{2} J_{AB} J^{AB} = -R^2 (g^{00} E^2 + g^{ab} p_a p_b), \quad \frac{1}{2} L_{IJ} L^{IJ} = R^2 g^{0\rho} \pi_\alpha \pi_\rho \]
and due to the mass-shell condition (2) we find
\[ \frac{1}{2} J_{AB} J^{AB} - \kappa^2 \frac{1}{2} L_{IJ} L^{IJ} = M^2 R^2. \]
(28)
Note that this constant coincides with \( E_{\text{min}}^2 \).

Quantization
In this section we first investigate the eigenvalue problem for the energy square operator. Then we construct the isometry group generators and check the algebra of their commutators. Finally, we derive the quantum version of the relation between the Casimir numbers (29).

The energy square operator is defined by (9) and in our case its eigenfunctions can be written in the following factorized form
\[ \Psi = \psi(x) Y_L^{L^+ - L^-}, \]
(30)
where \( Y_L^{L^+ - L^-}(\phi) \) are the spherical harmonics on S\(^M\). These states are the eigenfunctions of the Laplace–Beltrami operator \( \Delta_S \) with eigenvalues \(-L(L + M - 1)\) for integer \( L \).

\[ \text{In fact, } E^2 \text{ corresponds to the energy square in units of } 1/R^2. \]
The operator $E^2$ on the AdS part then is given by

$$E^2 = -h^{ab} \partial_a \partial_b + (N + M) x^a \partial_a + \frac{C}{\rho^2} + C_1,$$

with constant parameters

$$C_1 = \frac{(N + M - 1)^2}{4} \quad (32)$$

and $C = \mathcal{M}^2 R^2 + C_2 + C_L$, where

$$C_2 = (\kappa^2 - 1) \frac{(N + M - 1)M(M - 1)}{4(M + N)} \quad (33)$$

corresponds to the scalar curvature term\(^8\) obtained by (13) and

$$C_L = \kappa^2 L(L + M - 1) \quad (34)$$

is the angular momentum contribution from rotations on $S^M$. The operator (31) has the same structure as in the AdS space, but with deformed parameters, which depend on the spherical part as well. Note that the pure AdS case corresponds to $M = 0 = L$.

Introducing the radial variable on the unit disc $r^2 = x^a x^a = 1 - \rho^2$, we find

$$x^a \partial_a = r \partial_r$$

and the second-order derivative operator in (31) takes the form

$$h^{ab} \partial_a \partial_b = \partial_{aa} - r^2 \partial_{rr}, \quad (35)$$

which is obviously invariant under the $SO(N)$ rotations in $\text{AdS}_{N+1}$.

Taking into account this symmetry in the operator (31), one can further factorize the eigenfunctions (30)

$$\psi(x) = F(r) Y_{\mu \nu \cdots}^l(\varphi). \quad (36)$$

Here $Y_{\mu \nu \cdots}^l(\varphi)$ are again the spherical harmonics, but now on the unit sphere $S^{N-1}$ ($N > 1$),\(^9\) which can be treated as the boundary of the unit disc. Using the parametrization of the radial variable $r = \cos \sigma$, with $\sigma \in (0, \pi/2]$, and the rescaling of the radial wavefunction

$$F(r) = (\sin \sigma)^{-\frac{3}{2}} (\cos \sigma)^{-\frac{3l}{2}} f(\sigma), \quad (37)$$

we find that $f(\sigma)$ satisfies the Schrödinger equation with the Pöschl–Teller potential [16]

$$-\frac{\partial^2}{\partial \sigma^2} + \frac{A}{\sin^2 \sigma} + \frac{B}{\cos^2 \sigma} f(\sigma) = E^2 f(\sigma), \quad (38)$$

where

$$A = C + \frac{1}{4} M(M - 2), \quad B = l(l + N - 2) + \frac{1}{4} (N - 1)(N - 3). \quad (39)$$

The integration measure for the scalar product of wavefunctions $f(\sigma)$ is just $d\sigma$ and the Schrödinger operator in (38), therefore, is Hermitian.

Note that $B \geq 3/4$, except for the two cases

$$l = 0, N = 3, \quad B = 0; \quad l = 0, N = 2, \quad B = -1/4. \quad (40)$$

To have the Schrödinger operator in (38) bounded from below (see [17]), we assume $A \geq -1/4$. This condition bounds the parameter $\mathcal{M}^2 R^2$.

We are looking for normalizable solutions of (38). To analyze them, it is convenient to introduce the parameters

$$\mu := \frac{1}{2} + \sqrt{\frac{A}{4} + \frac{1}{4}}, \quad \nu := l + \frac{N - 1}{2}, \quad (41)$$

which are the large roots of the equations $\mu (\mu - 1) = A$ and $\nu (\nu - 1) = B$, respectively.

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\(^8\) This term vanishes at equal radii $R = R_\kappa$ ($\kappa = 1$).

\(^9\) The case $N = 1$ will be treated separately.
The behavior of a solution of (38) at the boundaries is given by
\[ f(\sigma) \simeq c_1 \sigma^{\mu} + c_2 \sigma^{-1-\mu}, \sigma \to 0; \]
\[ f(\sigma) \simeq d_1 (\pi/2 - \sigma)^v + d_2 (\pi/2 - \sigma)^{-1-v}, \sigma \to \pi/2. \]  
(42)
The normalizability of \( f(\sigma) \) then requires that \( c_2 = 0 \) if \( \mu \geq 3/2 \) and \( d_2 = 0 \) if \( \nu \geq 3/2 \).

The case \( B \geq 3/4 \) is equivalent to \( \nu \geq 3/2 \) and the corresponding solution of (38) for \( d_1 = 1 \) and \( d_2 = 0 \) is given by
\[ f(\sigma) = (\sin \sigma)^{\mu} \cos \sigma^2 F_1(a, b, c; \cos^2 \sigma), \]  
with the following parameters of the hypergeometric function
\[ a = \frac{1}{2} (\mu + \nu - E), \quad b = \frac{1}{2} (\mu + \nu + E), \quad c = \nu + \frac{1}{2}. \]  
(44)
Since the wavefunction (37) has to be regular at \( \sigma = \pi/2 \), the solution (43) describes the two exceptional cases (40) as well. To analyze the behavior of this solution at the boundary \( \sigma = 0 \), let us consider the following two solutions of (38):
\[ f_1(\sigma) = (\sin \sigma)^{\mu} \cos \sigma^2 F_1(a, b, a + b - c + 1; \sin^2 \sigma), \]  
(45)
\[ f_2(\sigma) = (\sin \sigma)^{1-\mu} \cos \sigma^2 F_1(c - a, c - b, c - a - b + 1; \sin^2 \sigma), \]  
(46)
where \( f_1(\sigma) \) corresponds to \( c_1 = 1, \ c_2 = 0 \) in (42) and \( f_2(\sigma) \) to \( c_1 = 0, \ c_2 = 1 \).

The properties of hypergeometric functions then provide
\[ f(\sigma) = \frac{\Gamma(c - a - b) \Gamma(c)}{\Gamma(c - a) \Gamma(c - b)} f_1(\sigma) + \frac{\Gamma(a + b - c) \Gamma(c)}{\Gamma(a) \Gamma(b)} f_2(\sigma). \]  
(47)
If \( \mu \geq 3/2 \), then the normalizability condition \( c_2 = 0 \) requires \( a = -n \) with integer \( n \), and this condition leads to the energy spectrum
\[ E_{n,l,l} = \frac{N}{2} + \sqrt{\frac{1}{4} + l + 2n}. \]  
(48)
The dependence on \( L \) (and on the mass parameter and the radii) is contained in the parameter \( A \) (see (39) and (34)).

Let us now take \( \frac{1}{2} \leq \mu < \frac{3}{2} \), which corresponds to \( -\frac{1}{2} \leq A < \frac{3}{4} \). In this case the solution (43) is normalizable for any (even complex) values of \( E^2 \), which means that the Schrödinger operator in (38) is not essentially self-adjoint. However, the analysis of the deficiency indices [17] shows that this operator has self-adjoint extensions. There are two different acceptable self-adjoint extensions with wavefunctions \( f_1(\sigma) \) and \( f_2(\sigma) \), which are specified by the following boundary behavior at \( \sigma \to 0 \):
\[ f_1(\sigma) \simeq \sigma^\mu, \quad f_2(\sigma) \simeq \sigma^{-1-\mu}. \]  
(49)
The energy spectrum in the first case is obviously given again by (48) and in the second case one finds \( c - b = -n \), which is equivalent to
\[ E_{n,l,l}^{(-)} = \frac{N}{2} + \sqrt{\frac{1}{4} + l + 2n}. \]  
(50)
For \( N = 1 \) we use the parametrization \( x^1 = -\cos \sigma \), with \( \sigma \in (0, \pi) \), and the same rescaling as in (37). This leads again to the Schrödinger equation (38) with vanishing \( B \)
\[ \left( -\partial_{\sigma}^2 + \frac{A}{\sin^2 \sigma} \right) f(\sigma) = E^2 f(\sigma). \]  
(51)
If \( A \geq 3/4 \), then the Schrödinger operator in (51) is essentially self-adjoint with the spectrum
\[ E_n^2 = (\mu + n)^2, \]  
(52)
where \( n \) is a non-negative integer and \( \mu \) is given by (41).
If \(-1/4 < A < 3/4\), then the Schrödinger operator in (51) is not self-adjoint and one can consider two different self-adjoint extensions similarly to the higher-dimensional cases.

An interesting case here is \(A = 0\) (i.e. \(\mu = 1\)), which corresponds to a free particle in a box. The operator \(-\partial^2_{x^i}\) is then characterized by two different self-adjoint extensions. The corresponding eigenstates are given by the trigonometric functions
\[
\begin{align*}
  f_{1,n}(\sigma) &= \sin((n+1)\sigma), & f_{2,n}(\sigma) &= \cos n\sigma, & (n \geq 0),
\end{align*}
\]
where \(f_{1,n}\) and \(f_{2,n}\) satisfy the Dirichlet and Neumann boundary conditions, respectively.

We discuss the isometry group generators. The quantization of the rotation generators both on \(S^M\) and \(\text{AdS}_{n+1}\) is trivial. Up to the factor \((-i)\), these operators are given by the corresponding vector fields obtained from (22)–(23). Therefore, it suffices to discuss the construction of the boost generators only.

We introduce the boost operators, which correspond to the functions (24), similarly to the AdS case [1]
\[
J_{a\alpha'} = \sqrt{E}x^\alpha \sqrt{E}, \quad J_{a0} = i\sqrt{E}\left(V_a - \frac{N + M - 1}{2} x^a\right) \frac{1}{\sqrt{E}},
\]
with
\[
V_a := h^{ab} \partial_b = \partial_a - x^b \partial_b x_b.
\]
The calculation of the commutation relations of the operators \(x^a\) and \(V_a\) with the energy square operator (31) is straightforward (see (A.7) in the appendix) and we obtain the quantum version of the Poisson bracket relations (25)
\[
[E^2, J_{a0}] = 2iJ_{a0} E + J_{a\alpha'}, \quad [E^2, J_{a\alpha'}] = -2iJ_{a0} E + J_{a0}.
\]
Note that the value (32) for the constant \(C_1\) is important to verify the second relation in (56). This confirms the value of the coefficient of the scalar curvature term in (9).

From the commutation relations (56) follow the commutators
\[
[E, Z_a] = -Z_a, \quad [E, Z^*_a] = Z^*_a,
\]
where \(Z_a = J_{a\alpha'} - iJ_{a0}\) and \(Z^*_a = J_{a\alpha'} + iJ_{a0}\) are the lowering and raising operators in the \(so(2, N)\) algebra. Then it remains to calculate the commutators between the boosts only.

In particular, by (54) one obtains
\[
[J_{a\alpha'}, J_{b\beta}] = \sqrt{E}(x^a E x^b \sqrt{E} - x^b E x^a \sqrt{E}) \frac{1}{\sqrt{E}}.
\]
To simplify the right-hand side here, we apply the same trick as in [1]. Using the commutation relation \([E, J_{a0}] = iJ_{a0}\), which follows from (57), one finds the operator equality
\[
E(x^a E - x^a E) = \frac{N + M - 1}{2} x^a - V_a.
\]
The operator in the parentheses of (58), therefore, corresponds to the generator of \(\text{SO}(N)\) rotations (27) given by \(iJ_{\alpha\beta} = x^b \partial_b - x^b \partial_b x_b\). Since \([E, J_{a\beta}] = 0\), one can neglect the \(\sqrt{E}\) terms in (58) and obtain
\[
[J_{a\alpha'}, J_{b\beta}] = iJ_{\alpha\beta}.
\]
The other commutators between the boost operators are computed in a similar way and they realize the algebra (26) on the quantum level.

The lowering and raising operators \(Z_a\) and \(Z^*_a\) provide an alternative way for the calculation of the energy spectrum. One can start with the ground state \(\psi_0\) of the operator (31). This state is a \(\text{SO}(N)\) scalar and it is annihilated by the lowering operators
\[
Z_a = \sqrt{E} \left(V_a + x^a \left(E - \frac{N + M - 1}{2}\right)\right) \frac{1}{\sqrt{E}}.
\]
The ground state wavefunction $\psi_0(\rho)$ then satisfies the equations

$$V_0 \psi_0(\rho) = \xi^a \left( \frac{N + M - 1}{2} - E_0 \right) \psi_0(\rho), \quad a = 1,$$

where $E_0$ denotes the energy of the ground state. Using (A.6), we find the solution

$$\psi_0(\rho) = c_0 \rho^{E_0 - \frac{N + M - 1}{2}},$$

up to a normalization constant $c_0$. The normalizability condition with integration measure (16) restricts the minimal energy by the unitarity bound in the AdS space [6]

$$E_0 > \frac{N}{2} - 1.$$

The wavefunction (63) should also be an eigenfunction of the energy square operator (31) with the eigenvalue $E_0^2$. This condition relates $E_0$ to other parameters of the theory in the form

$$\left( E_0 - \frac{N}{2} \right)^2 = A + \frac{1}{4},$$

where $A$ is given by (39). If $1/4 \lesssim A < 3/4$, then one obtains two solutions of $E_0$ as above. However, since $A$ is unbounded for increasing $L$, one has to take only the large root $E_0^+$ of (65). The action of the raising operators on the ground state shifts the energy levels by 1 and we obtain the same spectrum as above and the following representation of the isometry group

$$\sum_{L \geq 0} D_{E_0^+}^{\pm}(L) \otimes D_L(O(M + 1)),$$

where $D_{E_0^+}(L)$ and $D_L(O(M + 1))$ are the corresponding standard representations.

Taking the case of equal radii ($\kappa = 1$) and $M = N + 1$, we obtain

$$E_0^+(L) = \frac{N}{2} + \sqrt{\lambda^2 R^2 + L(L + N) + \frac{N^2}{4}}.$$

Finally, we consider the quantum version of the relation (29) between the Casimir numbers. In the calculation of the Casimir number operator of the SO(2, N) group one can use the identity (59) for the term $-J_{0\theta} J_{0\theta}$ and the factors $\sqrt{\lambda}$ in the term $-(J_{0\theta} J_{0\theta} + J_{0\theta} J_{0\theta})$ can be removed by the same trick as in (58). The computation of the remaining part is straightforward, and taking into account that $\frac{1}{2} L_{IJ} L_{IJ} = -\Delta_{SJ}$, we find

$$\frac{1}{2} J_{\theta\theta} J^{AB} - \kappa^2 \frac{1}{2} L_{IJ} L_{IJ} = \lambda^2 R^2 + C_2 - \frac{N^2 - (M - 1)^2}{4}.$$

Note that the quantum correction here vanishes for $\kappa = 1$ (i.e. $R = R_s$) and $N + 1 = M$.

**Discussion**

A consistent quantization of a scalar particle dynamics in $\text{AdS}_{N+1}$ should provide an unitary irreducible representation $D_{E_0}(O(2, N))$ of the isometry group O(2, N). This representation is characterized by the minimal value of energy $E_0$, with $E_0 > \frac{N}{2} - 1$. The Casimir number $J^2 = \frac{1}{2} J_{AB} J^{AB}$ is related to the minimal energy by $J^2 = E_0(E_0 - N)$ and there are two different representations for a given $J^2$, if

$$-\frac{N^2}{4} < J^2 < -\frac{N^2}{4} + 1.$$

These are well-known results and the quantization schemes mentioned in the introduction provide various realizations of Hilbert spaces for the dynamics in AdS backgrounds.

The static gauge quantization leads to position-dependent wavefunctions, which seems to be more natural. However, the relativistic principles require a non-local character for the
energy and boost operators that creates complicated ordering ambiguities. Due to this ordering problem the static gauge quantization was usually avoided in the literature. The construction of the energy square operator, proposed in [1], simplifies the ordering ambiguity problem and allows to solve it in AdS_{N+1}.

The main result of this paper is the generalization of the static gauge quantization to the AdS $\times$ S spaces. This result is summarized in equations (48), (65)–(68).

It is remarkable that the Hamiltonian of the system in the static gauge coincides with the energy square, which is characterized by the classical parameters $M$, $R$ and $R_S$. Upon quantization the quantum minimal energy and the quantum relation between the Casimir numbers are expressed through these classical parameters. These expressions exhibit the quantum deformations of the corresponding classical relations

$$E_{\text{min}} = MR = \sqrt{J^2 - \kappa^2 L^2}.$$  

Note that the connection to the classical mass parameter sometimes is lost in various quantization schemes.

An interesting new observation is the existence of two different self-adjoint extensions of the Hamilton operator for those values of the parameter $MR$ which correspond to (69).

This gives a new interpretation to two non-equivalent representations of the O(2, N) isometry group for a given Casimir number $J^2$.

In the pure AdS space, which corresponds to $M = 0 = L$, we reproduce the results of [1] for arbitrary AdS_{N+1}. Note that the energy spectrum of the AdS$_5$ particle obtained in [10] with an additional requirement of the CPT symmetry, corresponds the Breitenlohner–Freedman bound [6] with $E_0 = 2$ and $J^2 = -4$.

The most interesting case is AdS$_5$ $\times$ S$^5$ with equal radii, where the minimal energy is given by (67). At large $MR$ one obtains the following expansion:

$$E_0 = MR + 2 + \frac{L(L + 4) + 4}{2MR} + O((MR)^{-2}).$$  

This result was obtained in [15] on the basis of Dirac bracket quantization and it has been used for the analysis of the string energy spectrum in perturbation theory at large coupling.

Finally we would like to comment on the static gauge quantization of string dynamics. The ordering problem for the Poincaré group generators in the Minkowski space has been solved in [3] similarly to the particle case and the critical dimension $D = 26$ has been reproduced by the closure of the Poincaré algebra. The application of the same scheme to the AdS background provides a convenient framework for the calculation of the string spectrum in perturbation theory at strong coupling and work in this direction is in progress.

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Appendix

In this appendix we collect some useful formulae used in the main text.
Weyl transformation of a scalar curvature

If a metric tensor \( h_{mn}(x) \) is a Weyl transformed \( g_{mn}(x) \),

\[
h_{mn}(x) = \frac{1}{\Lambda(x)} g_{mn}(x),
\]

then its scalar curvature is given by

\[
R_h = \frac{1}{\Lambda(x)} \left( R_g + (N-1)\Delta \log \Lambda + \frac{(N-1)(N-2)}{4} g^{mn} \partial_m (\log \Lambda) \partial_n (\log \Lambda) \right). \tag{A.1}
\]

Killing vector fields on AdS\(_{N+1}\) and S\(_M\)

The Killing vector fields on AdS\(_{N+1}\) and S\(_M\) are treated similarly and it is sufficient to consider the spherical part only.

Let us introduce vector fields \( \mathcal{V}_I \) on S\(_M\) with components \( \mathcal{V}_\alpha^I = g^{\alpha\beta} \partial_\beta Y^I \). These fields are obtained from the equations \( \mathcal{V}_I \partial g = dY^I \) and they satisfy the relations

\[
\mathcal{V}_I (Y_K) = \delta_{IK} - Y_I Y_K R^2. \tag{A.2}
\]

The Killing vector fields \( \mathcal{V}_{IJ} \) with components (23) then can be written as \( \mathcal{V}_{IJ} = Y_J \mathcal{V}_I - Y_I \mathcal{V}_J \), and due to (A.2) they provide the infinitesimal form of the isometry transformations

\[
\mathcal{V}_{IJ} (Y_K) = \delta_{IK} Y_J - \delta_{JK} Y_I. \tag{A.3}
\]

The metric on S\(_M\) can be written as \( dY_I dY^I = \frac{1}{2} \theta_{IJ} \theta^{IJ} \), with \( \theta_{IJ} = Y_I dY^J - Y_J dY^I \), and in local coordinates one finds

\[
\frac{1}{2} \mathcal{V}^\alpha_{IJ} \mathcal{V}_\beta^{IJ} = g^{\alpha\beta}. \tag{A.4}
\]

Algebra of the boost operators

The energy square operator (31) is linear in \( H = -h^{ab} \partial_a \partial_b \) and \( D = x^a \partial_a \), whereas the boost generators (54) are linear in \( V_a \) and in the multiplication operator \( x^a \). The commutators of these operators form the following algebra:

\[
[D, x^a] = x^a, \quad [H, x^a] = -2V_a
\]

\[
[D, V_a] = -2x^a D - V_a, \quad [H, V_a] = 2V_a - 2x^a H,
\]

and, in addition, one has

\[
[V_a, \rho] = V_a (\rho) = -x^a \rho. \tag{A.5}
\]

From these commutators follow

\[
[E^2, x^a] = -2V_a + (M + N)x^a, \quad [E^2, V_a] = -2x^a (E^2 - C_1) - (M + N - 2)V_a, \tag{A.6}
\]

which turn the equations in (56) to identities.

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