On complex-analytic $1\vert 3$-dimensional supermanifolds associated with $\mathbb{CP}^1$.

E.G. Vishnyakova

Abstract

We obtain a classification up to isomorphism of complex-analytic supermanifolds with underlying space $\mathbb{CP}^1$ of dimension $1\vert 3$ with retract $(k, k, k)$, where $k \in \mathbb{Z}$. More precisely, we prove that classes of isomorphic complex-analytic supermanifolds of dimension $1\vert 3$ with retract $(k, k, k)$ are in one-to-one correspondence with points of the following set:

$$\text{Gr}_{4k-4,3} \cup \text{Gr}_{4k-4,2} \cup \text{Gr}_{4k-4,1} \cup \text{Gr}_{4k-4,0}$$

for $k \geq 2$. For $k < 2$ all such supermanifolds are isomorphic to their retract $(k; k; k)$.

1 Introduction.

A classical result is that we can assign the holomorphic vector bundle, so called retract, to each complex-analytic supermanifold (see Section 2 for more details). Assume that the underlying space of a complex-analytic supermanifold is $\mathbb{CP}^1$. By the Birkhoff-Grothendieck Theorem any vector bundle of rank $m$ over $\mathbb{CP}^1$ is isomorphic to the direct sum of $m$ line bundles: $E \simeq \bigoplus_{i=1}^{m} L(k_i)$, where $k_i \in \mathbb{Z}$. We obtain a classification up to isomorphism of complex-analytic supermanifolds of dimension $1\vert 3$ with underlying space $\mathbb{CP}^1$ and with retract $L(k) \oplus L(k) \oplus L(k)$, where $k \in \mathbb{Z}$. In addition, we give a classification up to isomorphism of complex-analytic supermanifolds of dimension $1\vert 2$ with underlying space $\mathbb{CP}^1$.

The paper is structured as follows. In Section 2 we explain the idea of the classification. In Section 3 we do all necessary preparations. The classification up to isomorphism of complex-analytic supermanifolds of dimension $1\vert 3$ with underlying space $\mathbb{CP}^1$ and with retract $(k, k, k)$ is obtained in Section 4. The last section is devoted to the classification up to isomorphism of complex-analytic supermanifolds of dimension $1\vert 2$ with underlying space $\mathbb{CP}^1$.

1Supported by Fonds National de la Recherche Luxembourg

Mathematics Subject Classifications (2010): 51P05, 53Z05, 32M10.
The study of complex-analytic supermanifolds with underlying space $\mathbb{CP}^1$ was started in [BO]. There the classification of homogeneous complex-analytic supermanifolds of dimension $1|m$, $m \leq 3$, up to isomorphism was given. It was proven that in the case $m = 2$ there exists only one non-split homogeneous supermanifold constructed by P. Green [Gr] and V.P. Palamodov [B]. For $m = 3$ it was shown that there exists a series of non-split homogeneous supermanifolds, parameterized by $k = 0, 2, 3, \cdots$.

In [V] we studied even-homogeneous supermanifolds, i.e. supermanifolds which possess transitive actions of Lie groups. It was shown that there exists a series of non-split even-homogeneous supermanifolds, parameterized by elements in $\mathbb{Z} \times \mathbb{Z}$, three series of non-split even-homogeneous supermanifolds, parameterized by elements of $\mathbb{Z}$, and a finite set of exceptional supermanifolds.

2 Classification of supermanifolds, main definitions

We will use the word ”supermanifold” in the sense of Berezin – Leites [BL] [L], see also [BO]. All the time, we will be interested in the complex-analytic version of the theory. We begin with main definitions.

Recall that a complex superdomain of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space of the form $(U, \mathcal{F}_U \otimes \Lambda(m))$, where $\mathcal{F}_U$ is the sheaf of holomorphic functions on an open set $U \subset \mathbb{C}^n$ and $\Lambda(m)$ is the exterior (or Grassmann) algebra with $m$ generators.

**Definition 1.** A complex-analytic supermanifold of dimension $n|m$ is a $\mathbb{Z}_2$-graded locally ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_M)$ be a supermanifold and

$$\mathcal{J}_M = (\mathcal{O}_M)_1 + ((\mathcal{O}_M)_1)^2$$

be the subsheaf of ideals generated by odd elements in $\mathcal{O}_M$. We put $\mathcal{F}_M := \mathcal{O}_M/\mathcal{J}_M$. Then $(\mathcal{M}_0, \mathcal{F}_M)$ is a usual complex-analytic manifold, it is called the reduction or underlying space of $\mathcal{M}$. Usually we will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_M)$.

Denote by $\mathcal{T}_M$ the tangent sheaf or the sheaf of vector fields of $\mathcal{M}$. In other words, $\mathcal{T}_M$ is the sheaf of derivations of the structure sheaf $\mathcal{O}_M$. Since the sheaf $\mathcal{O}_M$ is $\mathbb{Z}_2$-graded, the tangent sheaf $\mathcal{T}_M$ also possesses the induced $\mathbb{Z}_2$-grading, i.e. there is the natural decomposition $\mathcal{T}_M = (\mathcal{T}_M)_0 \oplus (\mathcal{T}_M)_1$.

Let $\mathcal{M}_0$ be a complex-analytic manifold and let $\mathcal{E}$ be a holomorphic vector bundle over $\mathcal{M}_0$. Denote by $\mathcal{E}$ the sheaf of holomorphic sections of $\mathcal{E}$. Then the
ringed space \((\mathcal{M}_0, \wedge \mathcal{E})\) is a supermanifold. In this case \(\dim \mathcal{M} = n|m\), where \(n = \dim \mathcal{M}_0\) and \(m\) is the rank of \(\mathcal{E}\).

**Definition 2.** A supermanifold \((\mathcal{M}_0, \mathcal{O}_\mathcal{M})\) is called *split* if \(\mathcal{O}_\mathcal{M} \simeq \wedge \mathcal{E}\) (as supermanifolds) for a locally free sheaf \(\mathcal{E}\).

It is known that any real (smooth or real analytic) supermanifold is split. The structure sheaf \(\mathcal{O}_\mathcal{M}\) of a split supermanifold possesses a \(\mathbb{Z}\)-grading, since \(\mathcal{O}_\mathcal{M} \simeq \wedge \mathcal{E}\) and the sheaf \(\wedge \mathcal{E} = \bigoplus_p \wedge^p \mathcal{E}\) is naturally \(\mathbb{Z}\)-graded. In other words, we have the decomposition \(\mathcal{O}_\mathcal{M} = \bigoplus_p (\mathcal{O}_\mathcal{M})_p\). This \(\mathbb{Z}\)-grading induces the \(\mathbb{Z}\)-grading in \(T_\mathcal{M}\) in the following way:

\[
(\mathcal{T}_\mathcal{M})_p := \{v \in \mathcal{T}_\mathcal{M} \mid v((\mathcal{O}_\mathcal{M})_q) \subset (\mathcal{O}_\mathcal{M})_{p+q} \text{ for all } q \in \mathbb{Z}\}.
\]

In other words, we have the decomposition: \(\mathcal{T}_\mathcal{M} = \bigoplus_{p=-1}^m (\mathcal{T}_\mathcal{M})_p\). Therefore the superspace \(H^0(\mathcal{M}_0, \mathcal{T}_\mathcal{M})\) is also \(\mathbb{Z}\)-graded. Consider the subspace

\[
\text{End } \mathcal{E} \subset H^0(\mathcal{M}_0, (\mathcal{T}_\mathcal{M})_0).
\]

It consists of all endomorphisms of the vector bundle \(\mathcal{E}\) inducing the identity morphism on \(\mathcal{M}_0\). Denote by \(\text{Aut } \mathcal{E} \subset \text{End } \mathcal{E}\) the group of automorphisms of \(\mathcal{E}\), i.e. the group of all invertible endomorphisms of \(\mathcal{E}\). We have the action \(\text{Int}\) of \(\text{Aut } \mathcal{E}\) on \(\mathcal{T}_\mathcal{M}\) defined by

\[
\text{Int } A : v \mapsto AvA^{-1}.
\]

Clearly, the action \(\text{Int}\) preserves the \(\mathbb{Z}\)-grading \((\mathbb{I})\), therefore, we have the action of \(\text{Aut } \mathcal{E}\) on \(H^1(\mathcal{M}_0, (\mathcal{T}_\mathcal{M})_0)\).

There is a functor denoting by \(\text{gr}\) from the category of supermanifolds to the category of split supermanifolds. Let us describe this construction. Let \(\mathcal{M}\) be a supermanifold and let as above \(\mathcal{J}_\mathcal{M} \subset \mathcal{O}_\mathcal{M}\) be the subsheaf of ideals generated by odd elements of \(\mathcal{O}_\mathcal{M}\). Then by definition we have \(\text{gr}(\mathcal{M}) = (\mathcal{M}_0, \text{gr } \mathcal{O}_\mathcal{M})\), where

\[
\text{gr } \mathcal{O}_\mathcal{M} = \bigoplus_{p \geq 0} (\text{gr } \mathcal{O}_\mathcal{M})_p, \quad (\text{gr } \mathcal{O}_\mathcal{M})_p = \mathcal{J}_\mathcal{M}^p/\mathcal{J}_\mathcal{M}^{p+1}, \quad \mathcal{J}_\mathcal{M}^0 := \mathcal{O}_\mathcal{M}.
\]

In this case \((\text{gr } \mathcal{O}_\mathcal{M})_1\) is a locally free sheaf and there is a natural isomorphism of \(\text{gr } \mathcal{O}_\mathcal{M}\) onto \(\wedge (\text{gr } \mathcal{O}_\mathcal{M})_1\). If \(\psi = (\psi_{\text{red}}, \psi^*) : (\mathcal{M}, \mathcal{O}_\mathcal{M}) \to (\mathcal{N}, \mathcal{O}_\mathcal{N})\) is a morphism of supermanifolds, then \(\text{gr}(\psi) = (\psi_{\text{red}}, \text{gr}(\psi^*))\), where \(\text{gr}(\psi^*) : \text{gr } \mathcal{O}_\mathcal{N} \to \text{gr } \mathcal{O}_\mathcal{M}\) is defined by

\[
\text{gr}(\psi^*)(f + \mathcal{J}_\mathcal{N}^p) := \psi^*(f) + \mathcal{J}_\mathcal{M}^p \text{ for } f \in (\mathcal{J}_\mathcal{N})^{p-1}.
\]

Recall that by definition every morphism \(\psi\) of supermanifolds is even and as consequence sends \(\mathcal{J}_\mathcal{N}^p\) into \(\mathcal{J}_\mathcal{M}^p\).
Definition 3. The supermanifold $\text{gr}(\mathcal{M})$ is called the \textit{retract} of $\mathcal{M}$.

For classification of supermanifolds we use the following corollary of the well-known Green Theorem (see [Gr], [BO] or [DW] for more details):

Theorem 2.1 [Green] Let $\mathcal{N} = (\mathcal{N}_0, \wedge \mathcal{E})$ be a split supermanifold of dimension $n|m$, where $m \leq 3$. The classes of isomorphic supermanifolds $\mathcal{M}$ such that $\text{gr}\mathcal{M} = \mathcal{N}$ are in bijection with orbits of the action $\text{Int}$ of the group $\text{Aut}\mathcal{E}$ on $H^1(\mathcal{M}_0, (\mathcal{T}\chi)^2)$.

Remark. This theorem allows to classify supermanifolds $\mathcal{M}$ such that $\text{gr}\mathcal{M} = \mathcal{N}$ is fixed up to isomorphisms that induce the identity morphism on $\text{gr}\mathcal{M}$.

3 Supermanifolds associated with $\mathbb{CP}^1$

In what follows we will consider supermanifolds with the underlying space $\mathbb{CP}^1$.

3.1 Supermanifolds with underlying space $\mathbb{CP}^1$

Let $\mathcal{M}$ be a supermanifold of dimension $1|m$. Denote by $U_0$ and $U_1$ the standard charts on $\mathbb{CP}^1$ with coordinates $x$ and $y = \frac{1}{x}$, respectively. By the Birkhoff-Grothendieck Theorem we can cover $\text{gr}\mathcal{M}$ by two charts

$$(U_0, \mathcal{O}_{\text{gr}\mathcal{M}}|_{U_0}) \text{ and } (U_1, \mathcal{O}_{\text{gr}\mathcal{M}}|_{U_1})$$

with local coordinates $x, \xi_1, \ldots, \xi_m$ and $y, \eta_1, \ldots, \eta_m$, respectively, such that in $U_0 \cap U_1$ we have

$$y = x^{-1}, \quad \eta_i = x^{-k_i}\xi_i, \quad i = 1, \ldots, m,$$

where $k_i$ are integers. Note that a permutation of $k_i$ induces the automorphism of $\text{gr}\mathcal{M}$. We will identify $\text{gr}\mathcal{M}$ with the set $(k_1, \ldots, k_m)$, so we will say that a supermanifold has the retract $(k_1, \ldots, k_m)$. In this paper we study the case: $m = 3$ and $k_1 = k_2 = k_3 =: k$. From now on we use the notation $\mathcal{T} = \bigoplus \mathcal{T}_p$ for the tangent sheaf of $\text{gr}\mathcal{M}$.

3.2 A basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2)$.

Assume that $m = 3$ and that $\mathcal{M} = (k_1, k_2, k_3)$ is a split supermanifold with the underlying space $\mathcal{M}_0 = \mathbb{CP}^1$. Let $\mathcal{T}$ be its tangent sheaf. In [BO] the following decomposition

$$\mathcal{T}_2 = \sum_{i<j} \mathcal{T}^{ij}_2$$

(2)
was obtained. The sheaf $\mathcal{T}^{ij}_2$ is a locally free sheaf of rank 2; its basis sections over $(U_0, \mathcal{O}_M|_{U_0})$ are:

$$
\xi_i\xi_j \frac{\partial}{\partial x}, \, \xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l};
$$

(3)

where $l \neq i, j$. In $U_0 \cap U_1$ we have

$$
\xi_i\xi_j \frac{\partial}{\partial x} = -y^{2-k_i-k_j} \eta_i \eta_j \eta_l \frac{\partial}{\partial \eta_l},
$$

$$
\xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l} = y^{-k_i-k_j} \eta_i \eta_j \eta_l \frac{\partial}{\partial \eta_l}.
$$

(4)

The following theorem was proven in [V]. For completeness we reproduce it here.

**Theorem 3.1** Assume that $i < j$ and $l \neq i, j$.

1. For $k_i + k_j > 3$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}^{ij}_2)$ is:

$$
\begin{align*}
&x^{-n}\xi_i\xi_j \frac{\partial}{\partial x}, \, n = 1, \ldots, k_i + k_j - 3, \\
&x^{-n}\xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l}, \, n = 1, \ldots, k_i + k_j - 1;
\end{align*}
$$

(5)

2. for $k_i + k_j = 3$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}^{ij}_2)$ is:

$$
\begin{align*}
x^{-1}\xi_i\xi_j \xi_l \frac{\partial}{\partial \xi_l}, \quad x^{-2}\xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l};
\end{align*}
$$

3. for $k_i + k_j = 2$ and $k_l = 0$ the basis of $H^1(\mathbb{CP}^1, \mathcal{T}^{ij}_2)$ is:

$$
\begin{align*}
x^{-1}\xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l};
\end{align*}
$$

4. if $k_i + k_j = 2$ and $k_l \neq 0$ or $k_i + k_j < 2$, we have $H^1(\mathbb{CP}^1, \mathcal{T}^{ij}_2) = \{0\}$.

**Proof.** We use the Čech cochain complex of the cover $\mathcal{U} = \{U_0, U_1\}$ as in 3.1, hence, 1-cocycle with values in the sheaf $\mathcal{T}^{ij}_2$ is a section $v$ of $\mathcal{T}^{ij}_2$ over $U_0 \cap U_1$. We are looking for basis cocycles, i.e. cocycles such that their cohomology classes form a basis of $H^1(\mathcal{U}, \mathcal{T}^{ij}_2) \simeq H^1(\mathbb{CP}^1, \mathcal{T}^{ij}_2)$. Note that if $v \in Z^1(\mathcal{U}, \mathcal{T}^{ij}_2)$ is holomorphic in $U_0$ or $U_1$ then the cohomology class of $v$ is equal to 0. Obviously, any $v \in Z^1(\mathcal{U}, \mathcal{T}^{ij}_2)$ is a linear combination of vector fields (3) with holomorphic in $U_0 \cap U_1$ coefficients. Further, we expand these coefficients in a Laurent series in $x$ and drop
the summands $x^n$, $n \geq 0$, because they are holomorphic in $U_0$. We see that $v$ can be replaced by

$$v = \sum_{n=1}^{\infty} a^n_{ij} x^{-n} \xi_i \xi_j \frac{\partial}{\partial x} + \sum_{n=1}^{\infty} b^n_{ij} x^{-n} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l},$$

where $a^n_{ij}, b^n_{ij} \in \mathbb{C}$. Using (4), we see that the summands corresponding to $n \geq k_i + k_j$ in the first sum of (6) and the summands corresponding to $n \geq k_i + k_j$ in the second sum of (6) are holomorphic in $U_1$. Further, it follows from (4) that

$$x^{2-k_i-k_j} \xi_i \xi_j \frac{\partial}{\partial x} \sim -k_i x^{1-k_i-k_j} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}.$$

Hence the cohomology classes of the following cocycles

$$x^{-n} \xi_i \xi_j \frac{\partial}{\partial x}, \quad n = 1, \ldots, k_i + k_j - 3,$$

$$x^{-n} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}, \quad n = 1, \ldots, k_i + k_j - 1,$$

generate $H^1(\mathbb{CP}^1, T_2^{ij})$. If we examine linear combination of these cocycles which are cohomological trivial, we get the result. □

Remark. Note that a similar method can be used for computation of a basis of $H^1(\mathbb{CP}^1, T_q)$ for any odd dimension $m$ and any $q$.

In the case $k_1 = k_2 = k_3 = k$, from Theorem 3.1 it follows:

**Corollary 3.2** Assume that $i < j$ and $l \neq i, j$.

1. For $k \geq 2$ the basis of $H^1(\mathbb{CP}^1, T_2^{ij})$ is

$$x^{-n} \xi_i \xi_j \frac{\partial}{\partial x}, \quad n = 1, \ldots, 2k - 3,$$

$$x^{-n} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}, \quad n = 1, \ldots, 2k - 1. \quad (7)$$

2. If $k < 2$, we have $H^1(\mathbb{CP}^1, T_2) = \{0\}$.

### 3.3 The group Aut E

This section is devoted to the calculation of the group of automorphisms Aut $E$ of the vector bundle $E$ in the case $(k, k, k)$. Here $E$ is the vector bundle corresponding to the split supermanifold $(k, k, k)$.

Let $(\xi_i)$ be a local basis of $E$ over $U_0$ and $A$ be an automorphism of $E$. Assume that $A(\xi_j) = \sum a_{ij}(x) \xi_i$. In $U_1$ we have

$$A(\eta_j) = A(y^{k_i} \xi_j) = \sum y^{k_j-k_i} a_{ij}(y^{-1}) \eta_l.$$
Therefore, $a_{ij}(x)$ is a polynomial in $x$ of degree $\leq k_j - k_i$, if $k_j - k_i \geq 0$ and is 0, if $k_j - k_i < 0$. We denote by $b_{ij}$ the entries of the matrix $B = A^{-1}$. The entries are also polynomials in $x$ of degree $\leq k_j - k_i$. We need the following formulas:

$$A(\xi_1 \xi_2 \xi_3 \frac{\partial}{\partial \xi_3})A^{-1} = \det(A) \sum_s b_{ks} \xi_1 \xi_2 \xi_3 \frac{\partial}{\partial \xi_3};$$

$$A(\xi_i \xi_j \frac{\partial}{\partial x})A^{-1} = \det(A) \sum_{k<s} (-1)^{l+r} b_{tl} \xi_k \xi_s \frac{\partial}{\partial x} + \det(A) \sum_s b'_{ls} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_s},$$

where $i < j$, $l \neq i, j$ and $r \neq k, s$. Here we use the notation $b'_{ls} = \frac{\partial}{\partial x}(b_{ls})$. By (8), in the case $k_1 = k_2 = k_3 = k$, we have:

**Proposition 1.** Assume that $k_1 = k_2 = k_3 = k$.

1. We have

$$\text{Aut } E \simeq \text{GL}_3(\mathbb{C}).$$

In other words

$$\text{Aut } E = \{ (a_{ij}) \mid a_{ij} \in \mathbb{C}, \det(a_{ij}) \neq 0 \}.$$

2. The action of $\text{Aut } E$ on $T^2$ is given in $U_0$ by the following formulas:

$$A(\xi_1 \xi_2 \xi_3 \frac{\partial}{\partial \xi_3})A^{-1} = \det(A) \sum_s b_{ks} \frac{\partial}{\partial \xi_3};$$

$$A(\xi_i \xi_j \frac{\partial}{\partial x})A^{-1} = \det(A) \sum_{k<s} (-1)^{l+r} b_{tl} \xi_k \xi_s \frac{\partial}{\partial x},$$

where $i < j$, $l \neq i, j$ and $r \neq k, s$. Here $B = (b_{ij}) = A^{-1}$

### 4 Classification of supermanifolds with retract $(k, k, k)$

In this section we give a classification up to isomorphism of complex-analytic supermanifolds with underlying space $\mathbb{C}P^1$ and with retract $(k, k, k)$ using Theorem 2.1. In previous section we have calculated the vector space $H^1(\mathbb{C}P^1, T_2)$, the group $\text{Aut } E$ and the action $\text{Int}$ of $\text{Aut } E$ on $H^1(\mathbb{C}P^1, T_2)$, see Theorem 3.2 and Proposition 1. Our objective in this section is to calculate the orbit space corresponding to the action $\text{Int}$:

$$H^1(\mathbb{C}P^1, T_2)/ \text{Aut } E.$$  

(10)

By Theorem 2.1 classes of isomorphic supermanifolds are in one-to-one correspondence with points of the set (10).
Let us fix the following basis of $H^1(\mathbb{CP}^1, T_2)$:

\begin{align*}
  v_{11} &= x^{-1}\xi_2\xi_3 \frac{\partial}{\partial x}, & v_{12} &= -x^{-1}\xi_1\xi_3 \frac{\partial}{\partial x}, & v_{13} &= x^{-1}\xi_1\xi_2 \frac{\partial}{\partial x}, \\
  \cdots \\
  v_{p1} &= x^{-p}\xi_2\xi_3 \frac{\partial}{\partial x}, & v_{p2} &= -x^{-p}\xi_1\xi_3 \frac{\partial}{\partial x}, & v_{p3} &= x^{-p}\xi_1\xi_2 \frac{\partial}{\partial x}, \\
  \cdots \\
  v_{p+1,1} &= x^{-1}\xi_2\xi_3 \frac{\partial}{\partial \xi_1}, & \cdots & v_{p+1,3} &= x^{-1}\xi_1\xi_3 \frac{\partial}{\partial \xi_3}, \\
  \cdots \\
  v_{q1} &= x^{-a}\xi_2\xi_3 \frac{\partial}{\partial \xi_1}, & \cdots & v_{q3} &= x^{-a}\xi_1\xi_3 \frac{\partial}{\partial \xi_3},
\end{align*}

(11)

where $p = 2k - 3$, $a = 2k - 1$ and $q = p + a = 4k - 4$. (Compare with Theorem 3.2.) Let us take $A \in \text{Aut } E \simeq \text{GL}_3(\mathbb{C})$, see Proposition 1. We get that in the basis (11) - (12) the automorphism Int $A$ is given by

\[ \text{Int } A(v_{is}) = \frac{1}{\det B} \sum_j b_{sj} v_{ij}. \]

Note that for any matrix $C \in \text{GL}_3(\mathbb{C})$ there exists a matrix $B$ such that

\[ C = \frac{1}{\sqrt{\det C}} B. \]

Indeed, we can put $B = \frac{1}{\sqrt{\det C}} C$. We summarize these observations in the following proposition:

**Proposition 2.** Assume that $k_1 = k_2 = k_3 = k$. Then

\[ H^1(\mathbb{CP}^1, T_2) \simeq \text{Mat}_{3 \times (4k-4)}(\mathbb{C}) \]

and the action Int of Aut $E$ on $H^1(\mathbb{CP}^1, T_2)$ is equivalent to the standard action of GL$_3(\mathbb{C})$ on Mat$_{3 \times (4k-4)}(\mathbb{C})$. More precisely, Int is equivalent to the following action:

\[ D \mapsto (W \mapsto DW), \]

(13)

where $D \in \text{GL}_3(\mathbb{C})$, $W \in \text{Mat}_{3 \times (4k-4)}(\mathbb{C})$ and $DW$ is the usual matrix multiplication.

Now we prove our main result.

**Theorem 4.1** Let $k \geq 2$. Complex-analytic supermanifolds with underlying space $\mathbb{CP}^1$ and retract $(k,k,k)$ are in one-to-one correspondence with points of the following set:

\[ \bigcup_{r=0}^{3} \text{Gr}_{4k-4,r}, \]

where $\text{Gr}_{4k-4,r}$ is the Grassmannian of type $(4k - 4, r)$, i.e. it is the set of all $r$-dimensional subspaces in $\mathbb{C}^{4k-4}$.

In the case $k < 2$ all supermanifolds with underlying space $\mathbb{CP}^1$ and retract $(k,k,k)$ are split and isomorphic to their retract $(k,k,k)$. 

8
Proof. Assume that $k \geq 2$. Clearly, the action on $\text{Mat}_{3 \times (4k-4)}(\mathbb{C})$ preserves the rank $r$ of matrices from $\text{Mat}_{3 \times (4k-4)}(\mathbb{C})$ and $r \leq 3$. Therefore, matrices with different rank belong to different orbits of this action. Furthermore, let us fix $r \in \{0, 1, 2, 3\}$. Denote by $\text{Mat}_r^{3 \times (4k-4)}(\mathbb{C})$ all matrices with rank $r$. Clearly, we have

$$\text{Mat}_{3 \times (4k-4)}(\mathbb{C}) = \bigcup_{r=0}^{3} \text{Mat}_r^{3 \times (4k-4)}(\mathbb{C}).$$

A matrix $W \in \text{Mat}_r^{3 \times (4k-4)}(\mathbb{C})$ defines the $r$-dimensional subspace $V_W$ in $\mathbb{C}^{4k-4}$. This subspace is the linear combination of lines of $W$. (We consider lines of a matrix $X \in \text{Mat}_{3 \times (4k-4)}(\mathbb{C})$ as vectors in $\mathbb{C}^{4k-4}$.) Therefore, we have defined the map $F_r$:

$$W \mapsto F_r(W) = V_W \in \text{Gr}_{4k-4,r}.$$

The map $F_r$ is surjective. Indeed, in any $r$-dimensional subspace $V \in \text{Gr}_{4k-4,r}$, where $r \leq 3$, we can take 3 vectors generating $V$ and form the matrix $W_V \in \text{Mat}_r^{3 \times (4k-4)}(\mathbb{C})$. In this case the matrix $W_V$ is of rank $r$ and $F_r(W_V) = V$. Clearly, $F_r(W) = F_r(DW)$, where $D \in \text{GL}_3(\mathbb{C})$. Conversely, if $W$ and $W' \in F_r^{-1}(V_W)$, then there exists a matrix $D \in \text{GL}_3(\mathbb{C})$ such that $DW = W'$. It follows that orbits of $\text{GL}_3(\mathbb{C})$ on $\text{Mat}_r^{3 \times (4k-4)}(\mathbb{C})$ are in one to one correspondence with points of $\text{Gr}_{4k-4,r}$. Therefore, orbits of $\text{GL}_3(\mathbb{C})$ on

$$\text{Mat}_{3 \times (4k-4)}(\mathbb{C}) = \bigcup_{r=0}^{3} \text{Mat}_r^{3 \times (4k-4)}(\mathbb{C})$$

are in one-to-one correspondence with points of $\bigcup_{r=0}^{3} \text{Gr}_{4k-4,r}$. The proof is complete.$\square$

5 Appendix. Classification of supermanifolds with underlying space $\mathbb{CP}^1$ of odd dimension 2.

In this Section we give a classification up to isomorphism of complex-analytic supermanifolds in the case $m = 2$ and $\text{gr} \mathcal{M} = (k_1, k_2)$, where $k_1$, $k_2$ are any integers. As far as we know the classification in this case does not appear in the literature, but it is known for specialists.

Let us compute the 1-cohomology with values in the tangent sheaf $\mathcal{T}_2$. The sheaf $\mathcal{T}_2$ is a locally free sheaf of rank 1. Its basis section over $(U_0, \mathcal{O}_\mathcal{M}|_{U_0})$ is $\xi_1 \xi_2 \frac{\partial}{\partial x}$. The transition functions in $U_0 \cap U_1$ are given by the following formula:

$$\xi_1 \xi_2 \frac{\partial}{\partial x} = -y^{2-k_1-k_2} \eta_1 \eta_2 \frac{\partial}{\partial y}.$$
Therefore, a basis of $H^1(\mathbb{C}P^1, \mathcal{T}_2)$ is

$$x^{-n}\xi_1\xi_2 \frac{\partial}{\partial x}, \ n = 1, \cdots, k_1 + k_2 - 3.$$ 

Let $(\xi_i)$ be a local basis of $E$ over $U_0$ and $A$ be an automorphism of $E$. As in the case $m = 3$, we have that $a_{ij}(x)$ is a polynomial in $x$ of degree $\leq k_j - k_i$, if $k_j - k_i \geq 0$ and is 0, if $k_j - k_i < 0$. We need the following formulas:

$$A(x^{-n}\xi_1\xi_2 \frac{\partial}{\partial x})A^{-1} = (\det A)x^{-n}\xi_1\xi_2 \frac{\partial}{\partial x}.$$ 

Denote

$$v_n = x^{-n}\xi_1\xi_2 \frac{\partial}{\partial x}, \ n = 1, \cdots, k_1 + k_2 - 3.$$ 

We see that the action $\text{Int}$ is equivalent to the action of $\mathbb{C}^*$ on $\mathbb{C}^{k_1+k_2-3}$, therefore, the quotient space is $\mathbb{C}P^{k_1+k_2-4} \cup \{\text{pt}\}$ for $k_1 + k_2 \geq 4$ and $\{\text{pt}\}$ for $k_1 + k_2 < 4$. We have proven the following theorem:

**Theorem 5.1** Assume that $k_1 + k_2 \geq 4$. Complex-analytic supermanifolds with underlying space $\mathbb{C}P^1$ and retract $(k_1, k_2)$ are in one-to-one correspondence with points of $\mathbb{C}P^{k_1+k_2-4} \cup \{\text{pt}\}$.

In the case $k_1 + k_2 < 4$ all supermanifolds with underlying space $\mathbb{C}P^1$ and retract $(k_1, k_2)$ are split and isomorphic to their retract $(k_1, k_2)$.

**Acknowledgment**

The author is grateful to A. L. Onishchik for useful discussions.

**References**

[B] Berezin F.A. Introduction to superanalysis. Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Mathematical Physics and Applied Mathematics, 9. D. Reidel Publishing Co., Dordrecht, 1987.

[BL] Berezin F.A., Leites D.A. Supermanifolds. Soviet Math. Dokl. 16, 1975, 1218-1222.

[BO] Bunegina V.A. and Onishchik A.L. Homogeneous supermanifolds associated with the complex projective line. Algebraic geometry, 1. J. Math. Sci. 82 (1996), no. 4, 3503-3527.
[DW] Donagi R. and Witten E. Supermoduli space is not projected. arXiv:1304.7798, 2013.

[Gr] Green P. On holomorphic graded manifolds. Proc. Amer. Math. Soc. 85 (1982), no. 4, 587-590.

[L] Leites D.A. Introduction to the theory of supermanifolds. Russian Math. Surveys 35 (1980), 1-64.

[V] Vishnyakova E. G. Even-homogeneous supermanifolds on the complex projective line. Differential Geometry and its Applications 31 (2013) 698-706.

Elizaveta Vishnyakova
University of Luxembourg
E-mail address: VishnyakovaE@googlemail.com