Bures Geometry of the Three-Level Quantum Systems. II

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For the eight-dimensional Riemannian manifold comprised by the three-level quantum systems endowed with the Bures metric, we numerically approximate the integrals over the manifold of several functions of the curvature and of its (anti-)self-dual parts. The motivation for pursuing this research is to elaborate upon the findings of Dittmann in his paper, “Yang-Mills equation and Bures metric” (Lett. Math. Phys. 46, 281-287 [1998]).

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I. INTRODUCTION

A metric of particular interest — the minimal member of the (nondenumerable) class of monotone metrics [1,2] that can be attached to the \((n^2-1)\)-dimensional convex sets of \(n\)-level quantum systems is the Bures metric [3,4] (cf. [5]). Dittmann [6] established that “the connection form (gauge field) related to the generalization of the Berry phase to mixed states proposed by Uhlmann satisfies the source-free Yang-Mills equation \(\ast D \ast D \omega\), where the Hodge operator is taken with respect to the Bures metric on the space of finite-dimensional nondegenerate density matrices” (cf. [7, p. 207]). Let us also note that Bilge et al [8,12], amongst others [9–11] (cf. [16, sec. 3.f.3], [17,18]), have studied the properties of Yang-Mills fields in eight dimensions (in contrast to the original, principally studied case of four [19]). Since the three-level \((n=3)\) quantum systems do, in fact, form an eight-dimensional convex set [20] — the \(n \times n\) density matrices, in general, forming \((n^2-1)\)-dimensional convex sets — it is of obvious interest to attempt to apply the analyses of Bilge et al in light of these findings of Dittmann concerning the relation between Yang-Mills fields and the Bures metric [8]. Such an effort constitutes, by and large, the substance of this communication. In the presentation of our findings, we will conform to the notation and conventions employed in [11], as detailed in the appendix to that paper, although we should indicate that in a related paper [20], Bilge replaces the notation used in [11] for the inner product \(\langle F, F \rangle\) of an \(SO(n)\)-valued curvature form \(F = (F_{ab})\) with \(\langle F, F \rangle\). We are compelled, however, to point out to the reader that in our computations we have not used the \(F_{ab}\)’s themselves, but rather as “proxies” for them, the \(R_{ab}\)’s, describing the curvature of the Riemannian manifold (cf. [21] eqs. (4), (5)] [22 eq. (8)]

Dittmann had noted [4] “that in affine coordinates (e. g. using the Pauli matrices for \(n > 2\)) the [Bures] metric becomes very complicated for \(n > 2\) and no good parameterization seems to be available for general \(n\). In the first part of this two-part paper [23], we attempted to fill this lacuna, by deriving the \(8 \times 8\) Bures metric tensor making use of a recently-developed Euler-angle parameterization of the \(3 \times 3\) density matrices [24,25]. Six Euler angles (denoted \(\alpha, \gamma, \alpha, \beta, b, \theta\), used in parameterizing an element of \(SU(3)\), are employed. Additionally, two (independent)
eigenvalues \((\lambda_1, \lambda_2)\) of the \(3 \times 3\) density matrices enter into this “Schur-Schatten”-type parameterization of the \(3 \times 3\) density matrices \([24, 28]\). (Of course, by the requirement of unit trace, \(\lambda_3 = 1 - \lambda_1 - \lambda_2\).) Several of the tensor elements were then, in fact, found to be identically zero. (The \(8 \times 8\) matrix tensor decomposes into a \(6 \times 6\) block and a \(2 \times 2\) one, in correspondence to the six Euler angles and the two independent eigenvalues.) However, we were not able to report concise symbolic expressions in \([24]\) for all the tensor elements.

II. ANALYSES

Subsequent to the issuance of Part I \([24]\), we have further pursued the analytical matters there, managing to derive symbolic expressions in relatively concise form for all the entries of the \(8 \times 8\) Bures metric tensor \((g_{ij})\), as well as its inverse \((g^{ij})\). Several of the entries, however, particularly two in the inverse, remain rather relatively cumbersome in nature, so far resisting further simplification. By way of illustration, making use of the LeafCount command in MATHEMATICA to measure the complexity of an expression, the largest leaf count (that is, the “total number of indivisible subexpressions”) for any single one of the \(g_{ij}\)’s is 352, while the largest for any of the \(g^{ij}\)’s is 233, aside from the two relatively complicated ones, having scores of 921 and 1,744. We note here that all these entries prove to be independent of the Euler angle \(\alpha\). Additionally, introducing the transformation \(\gamma = \tau - a\), all the entries also become independent of the Euler angle \(a\). (The entry with the largest leaf count is \(g^{\tau \tau}\), and the second largest, \(g^{\tau a}\).) Therefore, the two tensors are fully parameterizable with six, rather than eight variables. This reduction in dimensionality is exploited in the numerical integrations reported below.

We noted that the two expressions (both symmetric in \(\lambda_1\) and \(\lambda_2\))

\[
A = 3 - 7\lambda_1 + 4\lambda_1^2 + 7(-1 + \lambda_1)\lambda_2 + 4\lambda_2^2
\]  

(1)

and

\[
B = 4\lambda_1^3 + \lambda_1^2(-9 + 5\lambda_2) + \lambda_1(-1 + \lambda_2)(-7 + 5\lambda_2) + (-1 + \lambda_2)(2 + \lambda_2(-5 + 4\lambda_2))
\]  

(2)

frequently occurred in the tensor elements. These two expressions are displayed in Figs. 1 and 2, making use of the transformation employed in \([24]\): \(\lambda_1 = \cos^2 \zeta_1\), \(\lambda_2 = \sin^2 \zeta_1 \cos^2 \zeta_2\).

![FIG. 1. The subexpression A — that is (1), after conversion to spherical coordinates — appearing in several tensor elements](image-url)
FIG. 2. The subexpression $B$ — that is $\left[\begin{array}{ll}
\end{array}\right]$, after conversion to spherical coordinates — appearing in several tensor elements

For example,

$$g_{\theta\theta} = \frac{B + A(\lambda_1 - \lambda_2) \cos 2b}{2(-1 + \lambda_1)(-1 + \lambda_2)},$$

$$g^{\beta\beta} = -\frac{(B + A(\lambda_1 - \lambda_2) \cos 2b) \csc^2 \theta}{8(-1 + 2\lambda_1 + \lambda_2)^2(-1 + \lambda_1 + 2\lambda_2)^2},$$

$$g^{\alpha\alpha} = \frac{g^{\beta\beta} \csc^2 \beta \sec^2 \beta}{4}.$$  

In the two $6 \times 6$ (Euler angle) blocks of the Bures metric tensor and its inverse, the identically zero elements are $g_{\tau\beta}, g_{\tau\theta}, g_{\alpha\beta}, g_{\alpha\theta}, g^{\alpha\beta}, g^{\alpha\theta}$ and $g^{\alpha\theta}$.

The volume element of the Bures metric, in particular for the case $n = 3$,

$$v.e. = \sqrt{|g|} d\omega d\tau d\beta d\theta d\lambda_1 d\lambda_2,$$

where

$$\sqrt{|g|} = \frac{\sin 2b \sin 2\beta \sin^2 \theta \sin 2\theta}{8\sqrt{\lambda_1 \lambda_2 \lambda_3}} \frac{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)},$$

has been determined from certain general considerations \[29,30\]. We were then able, using this formula as a test, to numerically validate our calculations of the $g_{ij}$’s to a very high degree of accuracy. Also, we were able to reproduce — using our intermediate computation of the Ricci tensor — the results of Dittmann \[32\] (cf. \[33\]) for the scalar curvature of the Bures metric, as applied to the $n = 3$ case,

$$s.c. = \frac{2(28e_3 - 49e_2 - 9)}{e_3 - e_2},$$

where use is made of the elementary symmetric polynomials,

$$e_3 = \prod_{j=1}^3 \lambda_j, \quad e_2 = \lambda_1^2 + \lambda_2 + \lambda_1\lambda_2 - \lambda_2^2.$$  

While the convex set of $n \times n$ density matrices endowed with the Bures metric is of constant curvature for $n = 2$, it is not even a locally symmetric space for $n > 2$ \[31\] (cf. \[34\]). Dittmann \[33\] argues that the scalar curvature achieves its minimum for the fully mixed states, while others \[35\], using another monotone metric (the Kubo-Mori one), argue that the scalar curvature is maximum for the fully mixed states. This disparity is clearly a matter of convention,
depending upon what sign is designated for the Ricci tensor. (Somewhat relatedly, Grasselli and Streater have shown that “in finite dimensions, the only monotone metrics on the space of invertible density matrices for which the (+1) and (-1) affine connections are mutually dual are constant multiples of the Kubo-Mori metric” [36].)

Additionally, it was interesting to observe that numerical evidence we adduced indicated that a certain necessary and sufficient condition for Riemannian connections to satisfy the Yang-Mills equation [27, 39],

\[(\nabla^g_Ric)(Y, Z) = (\nabla^g_Ric)(X, Z),\]

was not fulfilled in our particular situation. Here \(X, Y, Z\) denote arbitrary smooth vector fields on the manifold in question — that is, the eight-dimensional convex set of \(3 \times 3\) density matrices — which we will denote by \(M_8\), and \(\nabla^g\) the covariant derivative with respect to \(X\).

We utilized our determination of the \(g_{ij}\)’s and \(g^{ij}\)’s in performing the same type of calculations that have been shown in [1] to provide upper bounds on Pontryagin numbers for \(SO(n)\) Yang-Mills fields. These bounds are

\[\int_{M_8} (F, F)^2 \geq k \int_{M_8} p_1(E)^2,\]  \tag{11}

where

\[(F, F) = 2 \sum_{a=1}^{8} \sum_{b>a}^{8} (F_{ab}, F_{ab}),\]  \tag{12}

and

\[\int_{M_8} (F^2, F^2) \geq k' \int_{M_8} p_2(E).\]  \tag{13}

Here, the field strengths \(F_{ab}\)’s denote the skew-symmetric coordinate components of \(F\), an \(SO(n)\)-valued curvature two-form on the Euclidean space \(\mathbb{R}^8\). For the \(ij\)-entry of an \(8 \times 8\) matrix \(F_{ab}\) was computed by us as \(R_{ijab}\) — that is, an element of the fully covariant \((0,4)\)-Riemann curvature tensor [10, eq. (6.6.1)] (cf. [21] eqs. (4), (5)). Also, \(k\) and \(k'\) are certain constants, while \(p_1(E)\) and \(p_2(E)\) are the first and second Pontryagin classes for the bundle \(E\) in question. The principal \(U(n)\)-bundle \(E (n = 3)\) utilized by Dittmann [4] is \(Gl(n, \mathbb{C}) \to Gl(n, \mathbb{C})/U(n)\), where \(Gl(n, \mathbb{C})\) denotes the general linear group over the complex numbers \(\mathbb{C}\), that is the nonsingular \(n \times n\) matrices with complex entries [1], ex. 5.1. (The Pontryagin number for a four-dimensional compact manifold \(M_4\) is equal to the integral over \(M_4\) of a representative of the first, and unique Pontryagin class of the bundle, that is the Chern number \(\int_{M_4} \gamma_2\) of the complexified bundle [2].)

The first of the two bounds (11) was reported in [1] and the second (13) in [2]. The first is achieved by strongly [anti-] self-dual Yang-Mills fields, while the second is too restrictive in this regard [1]. An additional curvature invariant is \((\text{tr}F^2, \text{tr}F^2)\), so a “generic action density” can be taken to be of the form [1], eq. (17)

\[p(F, F)^2 + q(F^2, F^2) + r(\text{tr}F^2, \text{tr}F^2).\]  \tag{14}

(“The action density should be written in terms of the local curvature 2-form matrix in a way independent of the local trivialization of the bundle. Hence it should involve invariant polynomials of the local curvature matrix. We want to express the action as an inner product in the space of \(k\)-forms, which gives a quantity independent of the local coordinates” [1].)

A. Monte-Carlo numerical integrations

To evaluate the integrals over \(M_8\) of the three curvature invariants occurring in the generic action (14), we pursued a Monte-Carlo strategy [2, 4] in which we randomly selected points in the six-dimensional hyperrectangular domain determined by the six active parameters (four Euler angles — \(\tau, \beta, b, \theta\) — and two independent eigenvalues reexpressed using spherical coordinates, that is, \(\lambda_1 = \cos^2 \zeta_1, \lambda_2 = \sin^2 \zeta_1 \cos^2 \zeta_2\)) [25]. This six-dimensional domain has one side of length \(\pi\) (corresponding to the range of \(\tau\)), three sides (for \(\beta, b \) and \(\theta\)) of length \(\pi/2\), one side (for \(\zeta_1\)) of length \(\cos^{-1} \pi/3 \approx 0.555317\) and one (for \(\zeta_2\)) of length \(\pi/4\). The two remaining (seventh and eighth) sides — associated with the Euler angles \(\alpha\) and \(\delta\), which we have noted are absent from the simplified transformed expressions for the \(g_{ij}\)’s [24] — are of length \(\pi\). At each of the randomly selected points, we computed the products of the volume
element [6] with the quantities \((F, F)^2, (F^2, F^2), (\text{tr} F^2, \text{tr} F^2)\) — as well as \((F^3, F^3)^{2/3}\) and \((F^4, F^4)^{1/2}\). (In these MATHEMATICA computations, we utilized the relation \(\det(I + t F) = \sum_{k=0}^{n} \sigma_k t^k\), where \(\sigma_k\) are invariants of the local trivialization \([11, \text{eq. (12)}]\), \([16]\).) We multiplied the averages of each of the five products over the randomly chosen points by the full eight-dimensional Euclidean hyperrectangular volume, which is \(\frac{\pi^7}{32} \cos^{-1} \frac{1}{\sqrt{3}} \approx 90.1668\). For 21,238 randomly generated points, our set of five results (exactly as computed, without any “rounding”) is presented in the \textit{first} line of Table I.

| field          | \(\int_{M_8} (F, F)^2\) | \(\int_{M_8} (F^2, F^2)\) | \(\int_{M_8} (\text{tr} F^2, \text{tr} F^2)\) | \(\int_{M_8} (F^3, F^3)^{2/3}\) | \(\int_{M_8} (F^4, F^4)^{1/2}\) |
|----------------|--------------------------|--------------------------|---------------------------------|---------------------------------|---------------------------------|
| Bures metric   | 0.00174878               | 9.91872 \cdot 10^{-14}  | 0.00174878                      | 7.90173 \cdot 10^{-22}         | 5.07518 \cdot 10^{-26}         |
| anti-self-dual part | 2.0692                   | 1.5519                   | 3.62111                         | 1.41                            | 1.09736                         |
| self-dual part | 18.6228                  | 7.75951                  | 26.3823                         | 5.15961                         | 3.29208                         |

\textbf{TABLE I.} Curvature invariants approximated by evaluating the integrands at 21,238 points randomly selected in the six-dimensional hyperrectangle of active parameters
We also, following Baulieu and Shatashvili [47, eqs. (3.4)-(3.6)] (cf. [10, p. 207], [48, eq. (2.9)]), decomposed the curvature two-form for the $8 \times 8$ Bures metric tensor into two $Spin(7)$-irreducible components, according to $28 = 21 \otimes 7$, which can be called self-dual and anti-self-dual respectively. We then evaluated the same curvature norms. The corresponding Monte-Carlo results for the same set of 21,238 randomly selected points are presented in the second and third lines of Table I. It clearly appears, then, that the Yang-Mills field for the Bures metric on the three-level quantum systems is neither “self-dual” nor “anti-self-dual” (cf. [49–53]).

Since all the five integrands used in Table I are nonnegative in nature, it appears that for the Yang-Mills/Bures field (corresponding to the first line of the table), $(F_3^3, F_3^3)$ and $(F_4^4, F_4^4)$, at least, are themselves zero. Also, we see that in this same first line (but not the second and third), the integral of $(F, F)^2$ equals that of $(\text{tr} F^2, \text{tr} F^2)$. Here [11, eq. (21)]

$$\text{tr} F^2 = -2 \sum_{a=1}^{8} \sum_{b>a}^{8} F_{ab}^2.$$  

(15)

(For the octonionic $SO(7)$ instanton two-form $F$, the four-form $\text{Tr} F \wedge F$ is neither self-dual nor anti-self-dual [14].) The $8 \times 8$ skew-symmetric matrices $F_{ab}$ must have four pairs of imaginary eigenvalues of opposite sign. (“Strong self-duality or strong anti-self-duality can be characterized by requiring the equality of the absolute values of the [eight] eigenvalues” [11].) For the eight-dimensional Yang-Mills field $F$ over the three-level quantum systems, our analyses indicate that one of these four pairs is always $(0,0)$ — clearly indicative of some form of degeneracy. (Riemannian manifolds the skew-symmetric curvature operators of which have constant eigenvalues have been the subject of study [55], including the case of eight-dimensional manifolds [56].) This ensures that the determinant (the product of the eigenvalues), as well as the $(F_{ab}^3, F_{ab}^3)$’s themselves, and thus $(F_4^4, F_4^4)$, are zero. The apparent zero nature of $(F_2^2, F_2^2)$ and $(F_3^3, F_3^3)$ cannot be directly explained in this manner, that is, by the zero nature of one pair of eigenvalues. (Let us point out here the possible relevance for our study, yielding a “small action”, of [57].)

**B. Numerical integrations using regular lattices**

As somewhat of a “cross-check” on the Monte-Carlo analyses reported above in Table I (motivated, in part, by some apparent instabilities in early, preliminary analyses), we also undertook numerical integrations by evaluating the various curvature quantities not at randomly selected points in the six-dimensional hyperrectangle, but at the nodes of regular lattices superimposed on this hyperrectangle. For a $2^6 = 64$-point lattice (dividing each parameter range into two equal segments and taking the midpoints of each segment as coordinates), we found

| field          | $\int_{M_8} (F, F)^2$ | $\int_{M_8} (F^2, F^2)$ | $\int_{M_8} (\text{tr} F^2, \text{tr} F^2)$ | $\int_{M_8} (F_3^3, F_3^3)^2/3$ | $\int_{M_8} (F_4^4, F_4^4)^{1/2}$ |
|---------------|-----------------------|------------------------|--------------------------------|----------------------------------|----------------------------------|
| Bures metric  | 0.0109779             | 1.399 $\cdot$ 10^{-26} | 0.0109779                        | 4.297 $\cdot$ 10^{-26}           | 7.231 $\cdot$ 10^{-26}           |
| anti-self-dual part | 1.77679               | 1.3326                 | 3.10939                          | 4.32074                          | 2.92568                         |
| self-dual part | 15.9911               | 6.66298                | 22.6541                          | 4.43048                          | 2.82686                         |
| SD part minus ASD part | 28.4287               | 21.3215                | 49.7502                          | 19.3719                          | 15.0766                         |

**TABLE II.** Curvature invariants approximated by evaluating the integrands at the nodes of a 64-point lattice imposed upon the six-dimensional hyperrectangle of active parameters
A fourth line is included in Table II for the difference \[17\], eq. (3.4)]

\[ F = F^+ - F^-, \]

of the self-dual \((F^+)\) and anti-self-dual \((F^-)\) fields, so

\[ F^\pm = \frac{1}{2}(F \pm \dagger F). \]

For a (finer) \(3^6 = 729\)-point lattice, we obtained

| field                   | \(\int_{M^8} (F, F)^2\) | \(\int_{M^8} (F^2, F^2)\) | \(\int_{M^8} (\text{tr}F^2, \text{tr}F^2)\) | \(\int_{M^8} (F^3, F^3)^2/3\) | \(\int_{M^8} (F^4, F^4)^{1/2}\) |
|-------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Bures metric           | 0.00029131               | 1.22054 \cdot 10^{-26}  | 0.00029131               | 1.15258 \cdot 10^{-35}  | 1.17924 \cdot 10^{-42}  |
| anti-self-dual part    | 2.11051                  | 1.58288                  | 3.69339                  | 1.43814                  | 1.11927                  |
| self-dual part         | 18.9946                  | 7.9144                   | 26.909                   | 5.2626                   | 3.3578                   |

**TABLE III.** Curvature invariants approximated by evaluating the integrands at the nodes of a 729-point lattice imposed upon the six-dimensional hyperrectangle of active parameters

Additionally, for a \(4^6 = 4096\)-point lattice, we computed

| field                   | \(\int_{M^8} (F, F)^2\) | \(\int_{M^8} (F^2, F^2)\) | \(\int_{M^8} (\text{tr}F^2, \text{tr}F^2)\) | \(\int_{M^8} (F^3, F^3)^2/3\) | \(\int_{M^8} (F^4, F^4)^{1/2}\) |
|-------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Bures metric           | 0.000824594              | 5.54578 \cdot 10^{-20}  | 0.000824594              | 3.95448 \cdot 10^{-30}  | 9.09798 \cdot 10^{-39}  |
| anti-self-dual part    | 2.08472                  | 1.56354                  | 3.64827                  | 1.42057                  | 1.10559                  |
| self-dual part         | 18.7625                  | 7.81771                  | 26.5802                  | 5.19831                  | 3.31677                  |

**TABLE IV.** Curvature invariants approximated by evaluating the integrands at the nodes of a 4096-point lattice imposed upon the six-dimensional hyperrectangle of active parameters
We see that the results here, in their overall aspects, are supportive of our much more extensive (and thus we are inclined to believe more accurate) Monte-Carlo analysis in Table I — to which we are in the process of adding still more randomly selected points. (It might be noted that none of our tabulated results — for the anti-self-dual and self-dual components — when divided by $\pi^n$, for $n = 4$ or other small integers, gives indications of being integral in nature [58, eq. (7)]. Also, there might seem to be a question of whether our tabulated results should be “corrected” by a factor of $3! = 6$, that is the number of ways of permuting the rows and columns of a $3 \times 3$ density matrix, but doing so would lead to density matrices that are simply permutations of one another being represented by distinct points in the manifold [23, 33, sec. III.B].)

C. Computation of Yang-Mills action functionals

For a $G$-bundle $E → X$, with connections $A$, the Yang-Mills functional can be expressed in the eight-dimensional case as [48, eq. (2.4)]

$$S_{YM}[A] = \frac{1}{2g^2} \int_X d^8 x \sqrt{g} \text{Tr} F_{\mu\nu} F^{\mu\nu} \equiv \frac{1}{2g^2} ||F||^2,$$

(18)

where $\frac{1}{g^2}$ represents the coupling constant of the gauge fields. For the Yang-Mills gauge field defining the Bures metric [3], based on 14,151 randomly chosen points in the six-dimensional hyperrectangle of active parameters, we obtained a value of .0145485 for $||F||^2$. For $||F^-||^2$, the comparable value was much larger, that is 5.33255 · $10^6$, and almost identical but slightly larger, that is 5.33268 · $10^6$, for $||F^+||^2$. “It can be shown that any self-dual 2-form defined by the above criteria [for self-duality in dimensions greater than 4] satisfies the Yang-Mills equations. However, the corresponding Yang-Mills action need not be extremal” [46, p. 4804]. “… it is no longer true that a minimum of the [Yang-Mills] action” [59, sec. 2.4].

Also Gao and Tian assert [48, eq. (2.5)]

$$S_{YM}[A] = \frac{1}{g^2(B + \sqrt{B^2 + 8A})} (2 \int_X \Omega \wedge \text{Tr}(F \wedge F) + \sqrt{B^2 + 8A} ||F^+||^2),$$

(19)

where [48, eq. (2.8)] $A = 6, B = -4$ and the four-form $\Omega$ is a “bispinor” [48, eq. (2.7)]. They state that “the Yang-Mills action can be written as a non-negative term proportional to $||F||^2$, plus a topological invariant. Clearly, such an action will reach its minimal values at $F^+ = 0$” [16, p. 3] (cf. [49, 33]).

In an exchange with Y. H. Gao on this matter, he has written: “Tentatively, the puzzle you mentioned (i.e. the Yang-Mills functional for the (anti) self-dual field is larger than for the full field) could stem from the problem of constructing a topological term using numerical methods. In the analytic world the difference between $||F||^2$ and $4||F_+||^2$ is ‘topological’ in the sense that its variation with respect to the connection vanishes. So, locally a minimum of $||F_+||^2$ is also a minimum of the original Yang-Mills action, and $||F||^2$ should be larger than $||F_-||^2$ provided $F$ and $F_-$ belong to the same ‘topological class’ (namely the topological term has the same value). In numerical computations, however, manifestation of topological invariance for an integral might not be as easy as in the analytic world. It could be the case that when $F$ is changed into $F_-$, the value of the “topological term” is also changed. In that case the argument in the analytic consideration would be no longer valid. Thus a naive suggestion is that during comparison between $||F||^2$ and $||F_-||^2$ numerically, one should also monitor possible changes in the topological term, to rule out the possibility that $F$ and $F_-$ are not really in the same topological class.”

So, it appears that it would be of interest in this matter to, in addition to our other computations, attempt to numerically approximate the term

$$\int_X \Omega \wedge \text{Tr}(F \wedge F),$$

(20)

in [19] by itself, as well as with $F$ replaced in it by $F_+$ as well as $F_-$.  

D. Monopole equations with $Spin(7)$-holonomy

Motivated by the Seiberg-Witten equations [60], Bilge et al formulated eight-dimensional monopole equations as follows [21, eq. (24)]:
Here (in the notation of (11)) \((\Psi\Psi^*)^+\) is the orthogonal projection of \(\Psi\Psi^*\) onto the spinor subbundle spanned by \(\rho^+(f_i), i = 2 \ldots 7\). We numerically solved the second set of equations (21) for the components of the spinor [16] eqs. (3a.57), (3f.102), (3f.103) (cf. [61, p. 30])

\[
\Psi = \frac{1}{2}(e_1 + e_4, e_2 + e_5, e_3 + e_6, e_0 + e_7, e_1 - e_4, e_2 - e_5, e_3 - e_6, e_0 - e_7),
\]

Only for what we have termed the “anti-self-dual” component \((F^-)\) did we obtain solutions, which always came in one of two forms. The first was \(e_2 = \frac{u\imath}{e_4}\), with \(u\) equal to a real constant, and the other six \(e\)'s all set to zero, while the second solution was \(e_0 = e_3 = e_6 = e_7 = 0\),

\[
e_1 = \frac{u\imath e_5}{e_4^2 + e_5^2}, \quad e_2 = \frac{u\imath e_4}{e_4^2 + e_5^2},
\]

where \(u\) is the same as in the first solution.

It remains for us to attempt to solve the first of the two sets of equations in (21), that is the Dirac equation, together with the Coulomb gauge condition.

### III. Discussion

What we have considered to be the “anti-self-dual” part \((F^+)\) of the Yang-Mills field over the eight-dimensional convex set of three-level quantum systems satisfies the “set \(b\)” of twenty-one equations in [12],

\[
F_{12} - F_{34} = 0; \quad F_{12} - F_{56} = 0; \quad F_{12} - F_{78} = 0; \quad F_{13} + F_{24} = 0; \quad F_{13} - F_{57} = 0; \quad F_{13} + F_{68} = 0
\]

\[
F_{14} - F_{23} = 0; \quad F_{14} + F_{67} = 0; \quad F_{14} + F_{58} = 0; \quad F_{15} + F_{26} = 0; \quad F_{15} + F_{37} = 0; \quad F_{15} - F_{48} = 0
\]

\[
F_{16} - F_{25} = 0; \quad F_{16} - F_{38} = 0; \quad F_{16} - F_{47} = 0; \quad F_{17} + F_{28} = 0; \quad F_{17} - F_{35} = 0; \quad F_{17} + F_{46} = 0
\]

\[
F_{18} - F_{27} = 0; \quad F_{18} + F_{36} = 0; \quad F_{18} + F_{45} = 0
\]

that is, to the negative eigenvalue, \(-3\), of a fourth rank tensor invariant under \(SO(7)\). (A skew-symmetric matrix satisfying the set (24) must have all its [imaginary] eigenvalues equal in absolute value [11].)

The “self-dual” part \((F^+)\) of the Yang-Mills/Bures field satisfies the “set \(a\)” of seven equations [12],

\[
F_{12} + F_{34} + F_{56} + F_{78} = 0; \quad F_{13} - F_{24} + F_{57} - F_{68} = 0; \quad F_{14} + F_{23} - F_{67} - F_{58} = 0; \quad F_{15} - F_{26} - F_{37} + F_{48} = 0
\]

\[
F_{16} + F_{25} + F_{38} + F_{47} = 0; \quad F_{17} - F_{28} + F_{35} - F_{46} = 0; \quad F_{18} + F_{27} - F_{36} - F_{45} = 0
\]

that is to the positive eigenvalue 1.

“The solutions of set \(a\) and set \(b\) can be viewed as analogues of self-dual 2-forms in four dimensions from different aspects. The strongly self-dual 2-forms, hence the solutions of set \(b\) saturate various topological lower bounds . . . but they form an overdetermined system . . . we show that the solutions of set \(b\) for an \(N\)-dimensional gauge group, depend exactly on \(N\) arbitrary constants, provided that the system is consistent. Thus the set \(b\) lacks the rich structure of the self-duality equations in four dimensions. On the other hand, the solutions of set \(a\) do not saturate the topological bounds . . . but these equations form an elliptic system under the Coulomb gauge condition” [12].

“In fact, one often finds in the literature that [our] equations (25) and (24) are referred to as the self-duality and anti-self-duality equations respectively. This nomenclature suggests a symmetry between these equations which is not present in the octonionic case since, for example, the spaces have different dimension. In our opinion, self-duality and anti-self-duality correspond to which way the division algebra \& acts: if on the left or on the right, and are hence related by a change of orientation on the manifold. Although there has been some work in the literature concerning equation (24), we believe this equation is not as fundamental as (25). This can also be seen not just in the results
for D strengths in the self-dual Yang-Mills equation are necessarily Ricci-flat. The holonomy group is then 21-dimensional. Acharya and O’Loughlin replaced the field matrices studied in [64] endowed with the Bures metric have sought to calculate the Yang-Mills functionals, as in sec. II C, for the four-dimensional convex set of 3 × 3 density matrices studied in [5] endowed with the Bures metric

\[ \rho = \frac{1}{2} \begin{pmatrix} v + z & 0 & x - iy \\ 0 & 2 - 2v & 0 \\ x + iy & 0 & v - z \end{pmatrix} \]  

(26)

The Yang-Mills functionals for the original and self-dual and anti-self-dual fields turn out to be infinite in this case, though.

In the context of D-dimensional Euclidean gravity, Acharya and O’Loughlin have defined a generalization of the self-dual Yang-Mills equations as duality conditions on the curvature two-form \( R \) of a Riemannian manifold [21]. For \( D = 8 \), solutions to these self-duality equations are provided by manifolds of \( Spin(7) \) holonomy [65–67], which are necessarily Ricci-flat. The holonomy group is then 21-dimensional. Acharya and O’Loughlin replaced the field strengths in the self-dual Yang-Mills equation

\[ F_{\mu\nu} = \frac{1}{2} \phi_{\mu\nu\lambda\rho} F_{\lambda\rho} \]  

(27)

by the components \( R_{ab} \) of the curvature two-form (as we have above).

Using the technique of embedding the spin connection in the gauge connection (cf. [68, sec. 10.5]), they construct a self-dual gauge field directly from the self-dual metric. They let “\( G_{ab} \) be the generators of one of \( SU(2), SU(3), G_2 \) or \( Spin(7) \). The ansatz for the gauge field is \( A = \gamma G_{ab} \omega_{ab} \). The form index of \( A \) comes from the form index of \( \omega \) while the Lie algebra structure of \( A \) comes from that of \( G \). One sees then that

\[ F = dA + A \wedge A \]  

(28)

\[ = \gamma G_{ab} d\omega_{ab} + \kappa \gamma^2 G_{ab} \omega_{ac} \wedge \omega_{cb} \]  

(29)

where \( \kappa \) is a constant that depends upon the group generated by \( G_{ab} \). In each case it is trivial to solve for \( \gamma \) giving \( F = \gamma G_{ab} R_{ab} \). Duality of \( F \) follows from that of \( R \) and the symmetry of \( R_{\mu\nu\lambda\rho} \) between the first pair and second pair of indices” [21]. We shall attempt to follow this prescription in our continuing research on the topic of this paper.

The duality operator, \( \phi_{abcd} \) for \( D = 8 \), is the unique \( Spin(7) \)-invariant four-index antisymmetric tensor which is Hodge self-dual. Proposition 10.6.7 of [67] relates the first Pontryagin class of a manifold \( M \) with a \( Spin(7) \) structure \( (\Omega, g) \) to the integral over \( M \) of \( |R|^2 \), where \( R \) is the Riemann curvature of \( g \). Also, if \( (M, \Omega, g) \) is a compact \( Spin(7) \)-manifold, then \( g \) has holonomy \( Spin(7) \) if and only if \( M \) is simply connected and the Betti numbers of \( M \) satisfy \( b^3 + b^4 = b^2 + 2b^3 + 25 \) [7]. Thm. 10.6.8).

It has been known for some time that the Bures metric on the two-level quantum systems is isometric to the standard metric on the three-sphere [4,69]. “The Bures metric for a two-dimensional system corresponds to the surface of a unit four-ball, i.e., to the maximally symmetric three-dimensional space of positive curvature (and may be recognized as the spatial part of the Robertson-Walker metric in general relativity). This space is homogeneous and isotropic, and hence the Bures metric does not distinguish a preferred location or direction in the space of density operators. Indeed, as well as rotational symmetry in Bloch co-ordinates (corresponding to unitary invariance), the metric has a further set of symmetries generated by the infinitesimal transformations

\[ r \rightarrow r + \epsilon (1 - r^2)^{1/2} \]  

(30)

(where \( r \) is an arbitrary three-vector)” [24, p. 128], \( r \) being the radial distance — the length of \( r \) — in the Bloch sphere of two-level systems (from the fully mixed state, \( r = 0 \)). Petz and Súdár observe that “in the case of the Bures metric, the tangential component is independent of \( r \)” [1, p. 2667].

Contrastingly, Dittmann [31] established that the Bures metric on the three-level quantum systems is not a space of constant curvature nor even locally symmetric. (Neither, is the Einstein-Yang-Mills equation fulfilled with a certain cosmological constant [4], as it is for the two-level quantum systems [7].) Nevertheless, we have obtained in Tables
strong indications that the curvature (known to satisfy the Yang-Mills equation \[9\]) of the Bures metric on the three-level quantum systems is quite flat in character. ("quantum information manifolds are equipped with two natural flat connections: the mixture connection, obtained from the linear structure of trace class operators themselves, and the exponential connection, obtained when combinations of states are performed by adding their logarithms...the Bogoliubov-Kubo-Mori metric is, up to a factor, the unique monotone Riemannian metric with respect to which the exponential and mixture connections are dual \[36\].)

Due to the relative complexity (high leaf count) of our current formulas for the entries of the \(8 \times 8\) Bures metric tensor and its inverse, we have had to have recourse here, by and large, to numerical calculations in order to gain insights into the nature of the associated Yang-Mills field. If further progress in simplifying these entries can be achieved, it may be possible to proceed further with more highly desirable exact symbolic computations.

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