HYPERBOLIC TRIANGLES OF THE MAXIMUM AREA
WITH TWO FIXED SIDES

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ABSTRACT. The aim of this paper is to consider the Lobachevskii geometry analog of a well-known Euclidian problem; namely: to find a triangle with two fixed sides and the maximum area.

1. Introduction

What is the triangle with two fixed sides and the maximum square? It is obvious that in Euclidian geometry this triangle is right-angled.

The aim of this paper is to describe the respective triangle (we shall call it the maximum square triangle) in Lobachevskii geometry. It appears that the maximum square triangle has a lot of properties corresponding to the properties of the Euclidian right-angled triangle (see Table 1 here $\alpha$, $\beta$ and $\gamma$ are the angles which lie opposite the sides $BC = a$, $AC = b$ and $AB = c$ correspondingly).

Table 1: Properties of maximum square triangles

| Euclidian Geometry | Lobachevskii Geometry |
|--------------------|-----------------------|
| 1) $\alpha = \beta + \gamma = \pi$; | 1) $\alpha = \beta + \gamma < \frac{\pi}{2}$; |
| 2) the center of the circumcircle coincides with the middle of the side $BC$; | 2) the center of the circumcircle coincides with the middle of the side $BC$; |
| 3) $\frac{S}{2} = \frac{b}{2} \cdot \frac{c}{2}$; | 3) $\sin \frac{S}{2} = \th \frac{b}{2} \cdot \th \frac{c}{2}$; |
| 4) $\cos \alpha = 0 = \text{const}$; | 4) $\cos \alpha = \th \frac{b}{2} \cdot \th \frac{c}{2} \neq \text{const}$; |
| 5) $a^2 = b^2 + c^2$. | 5) $\sh \frac{a}{2} = \sh \frac{b}{2} + \sh \frac{c}{2}$. |

So it is a maximum square triangle that should be considered analogous with a Euclidian right-angled triangle.

ACKNOWLEDGEMENTS. The author is grateful to P. V. Bibikov for useful discussions and for attention to this work.

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1This paper is prepared under the supervision of P. Bibikov and is submitted to a prize of the Moscow Mathematical Conference of High-school Students. Readers are invited to send their remarks and reports to mmks@mccme.ru
2. Poincare Disc Model

We will consider \textit{Poincare disc model of Lobachevskii geometry} (see [1, 2]). In this model \textit{Lobachevskii plane} is the interior of a unite disc; the boundary of this disc is called \textit{the absolute}. \textit{Points} are Euclidian points; \textit{lines} are either circle arcs that are orthogonal to the absolute, or diameters of the absolute (Fig. 1). \textit{The angle measure} in Poincare disc model is the same as in Euclidian geometry.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Poincare disk model}
\end{figure}

A \textit{triangle} consists of the circle arcs and the sum of its angles is less than \(\pi\). Let \(\delta\) be \textit{the defect of a triangle}, i.e., \(\delta = \pi - \alpha - \beta - \gamma\), where \(\alpha, \beta\) and \(\gamma\) are the angles of a triangle. It is clear that the defect of a triangle satisfies the following:

1) \(\delta > 0\);
2) if the triangles \(\triangle_1\) and \(\triangle_2\) are equal, then \(\delta_1 = \delta_2\);
3) if the triangle \(\triangle\) is decomposed into the triangles \(\triangle_1\) and \(\triangle_2\), then \(\delta = \delta_1 + \delta_2\).

So the defect of a triangle satisfies the axioms of \textit{the square}. It is proved (see [2]) that in Lobachevskii geometry

\[ S(\triangle) = \delta = \pi - \text{the sum of the angles}. \]

One can see that there is a huge difference between squares in Euclidian and Lobachevskii geometries. That is why a lot of Euclidian problems connected with the square become more difficult and interesting in Lobachevskii geometry.

3. \textbf{Equidistant of the Equal Squares}

\textbf{Definition 1.} Let \(p\) be a non-Euclidian line. An \textit{equidistant with the base} \(p\) is a set of points that are in a fixed half-plane and at a fixed distance from \(p\).

In other words an equidistant in Lobachevskii geometry is the analog of a Euclidian parallel line.

In Poincare disc model equidistants are either circle arcs, or chords of the absolute (see [1, 3]).

\textbf{Theorem 1} (O. V. Shvartsman, [5]). Suppose \(AB\) is a non-Euclidian segment and \(A\) coincides with the center of the absolute. Let \(B'\) be the point symmetric to \(B\) with respect to the absolute\(^2\) and let \(\lambda\) be the chord of the absolute, the continuation of which passes through \(B'\); then for any point \(C \in \lambda\) we have \(S(ABC) = 2\tau = \text{const}\), where \(\tau = \angle AB'C\).

\(^2\)I.e., \(B'\) is the image of \(B\) under the inversion with respect to the absolute.
Proof. Let $BC$ be the Euclidian segment and let $a$ be the circle that passes through $B$, $B'$ and $C$. Then $a$ is orthogonal to the absolute (see [6]) so the intersection of $a$ with the Poincare disc defines the non-Euclidian line (Fig. 2).

It is obvious that the angle between the segment $BC$ and the arc $\widehat{BC}$ of circle $a$ is equal to $\tau$. Therefore, the sum of the Euclidian angles in Euclidian triangle $ABC$ equals $\alpha + \beta + \gamma + 2\tau = \pi$. Finally, we get

$$S(ABC) = \pi - (\alpha + \beta + \gamma) = 2\tau = \text{const}.$$

Definition 2. Let us remember that a chord of the absolute is an equidistant in Lobachevskii geometry. The chord that satisfies the conditions of the Shvartsman theorem will be called the equidistant of the equal squares for segment $AB$.

One can see that the Shvartsman theorem is the analog of a well-known fact of Euclidian geometry: the set of points $C$ such that $S(ABC) = \text{const}$ is a line parallel to $AB$.

4. Maximum square triangles and their properties

Now we can solve the main problem: find a non-Euclidian triangle $ABC$ with two fixed sides $AB$ and $AC$, and the maximum square.

Without loss of generality it can be assumed that vertex $A$ coincides with the center of the absolute. Let us fix the side $AB$; then the vertex $C$ lies on circle $\omega$ with center $A$ and fixed radius. Consider point $B'$ symmetrical to $B$ with respect to the absolute and equidistant of the equal squares $\lambda$ for fixed segment $AB$. By $S/2$ denote the angle between $AB'$ and $\lambda$. By the Shvartsman theorem triangle $ABC$ with sides $AB$ and $AC$ such that $S(ABC) = S$ exists if and only if the chord $\lambda$ intersects the circle $\omega$ (Fig. 3). So we should find the equidistant $\lambda_{\text{max}}$ that intersects $\omega$ and forms the maximum angle with $AB'$. It is obvious that $\lambda_{\text{max}}$ is a tangent line to $\omega$. So to build the maximum square triangle $ABC$ we must build the tangent line $B'C$ to circle $\omega$.

Since a maximum square triangle is an analog of a Euclidian right-angled triangle, it will be interesting to describe this analogy using corresponding properties of these triangles (see Table 1). We shall state these properties in the following theorem.
Theorem 2. Suppose $ABC$ is triangle with fixed sides $AC = b$ and $AB = c$; then the following conditions are equivalent:

0. $ABC$ has the maximum square;
1. $\alpha = \beta + \gamma < \frac{\pi}{2}$;
2. the center of the circumcircle coincides with the middle of the side $BC$;
3. $\sin \frac{S}{2} = \frac{b}{2} \cdot \frac{c}{2}$;
4. $\cos \alpha = \frac{b}{2} \cdot \frac{c}{2} \neq \text{const}$;
5. $\text{sh}^2 \frac{a}{2} = \text{sh}^2 \frac{b}{2} + \text{sh}^2 \frac{c}{2}$.

Proof. We will use the construction described above.

0 $\iff$ 1 First suppose that triangle $ABC$ has the maximum square. Then $\angle ACB' = \frac{\pi}{2}$ and we get $\alpha = \beta + \gamma = \frac{\pi}{2}$.

Secondly suppose that the condition $\alpha = \beta + \gamma$ holds. Then the Euclidian angle $\angle ACB'$ in the triangle $AB'C$ is equal to $\pi - \alpha - \frac{S}{2} = \frac{\pi}{2}$. So the line $B'C$ is tangent to $\omega$ (Fig. 3) and the triangle $ABC$ has the maximum square.

1 $\iff$ 2 This statement is well known and belongs to the absolute geometry.

0 $\iff$ 3 If the triangle $ABC$ has the maximum square, then $\sin \frac{S}{2} = \frac{AC_E}{AB_E}$, where by $AC_E$ and $AB_E$ we denote the Euclidian lengths of the Euclidian segments $AC$ and $AB'$ respectively. As $B'$ is symmetric to $B$ with respect to the absolute, then $\frac{1}{AB_E} = AB_E$. It is proved in [3] that $AB_E = \frac{b}{2}$ and $AC_E = \frac{c}{2}$. Finally, we get $\sin \frac{S}{2} = AC_E \cdot AB_E = \frac{b}{2} \cdot \frac{c}{2}$.

The converse statement follows from the sinus theorem: $\frac{AB'_E}{\sin \angle ACB'} = \frac{AC_E}{\sin \frac{S}{2}}$. If we combine this with (3), we get $\sin \angle ACB' = 1$. So $\angle ACB' = \frac{\pi}{2}$ and the line $B'C$ is tangent to $\omega$.

0 $\iff$ 4 This equivalence is proved in the same way as the previous one.

4 $\iff$ 5 To prove this statement, we use the cosine theorem (see [3]): $\text{ch} a = \text{ch} b \text{ch} c - \text{sh} b \text{sh} c \cos \alpha$. If we replace $\cos \alpha$ by $\frac{b}{2} \cdot \frac{c}{2}$ in the cosine theorem, we obtain (5). The converse statement is proved in the same way.

Thus most of the right-angled triangle properties correspond to those of the maximum square triangle (and not the right-angled one, as might have been expected) in Lobachevskii geometry. So it is the maximum square triangle that should be considered analogous with the Euclidian right-angled triangle.

Remarks. 1. Using the proof of the (0) and (3) properties equivalence, we get the formula to find out the square of an arbitrary non-Euclidian triangle through its two sides and the angle...
between them, i.e., the analog of the Euclidian formula \( S = \frac{bc \sin \alpha}{2} \):

\[
\text{ctg} \frac{S}{2} = \frac{\text{cth} \frac{b}{2} \cdot \text{cth} \frac{c}{2} - \cos \alpha}{\sin \alpha}.
\]

Besides, the Shvartsman theorem makes it easy to prove many other non-Euclidian formulae connected with the triangle square. It also explains why it is the half of the square and the halves of the sides that these formulae contain.

2. Properties (1)–(5) from Theorem 2 transform into corresponding Euclidian properties (see Table 1) as \( b, c \to 0 \). This again shows that a maximum square triangle is an analog of a Euclidian right-angled triangle in Lobachevski geometry.

3. However there are some differences between these two types of triangles. The most extraordinary one is property (4): \( \cos \alpha = \text{th} \frac{b}{2} \cdot \text{th} \frac{c}{2} \). First of all, the angle \( \alpha \) depends on the sides \( b \) and \( c \), whereas in a Euclidian right-angled triangle \( \alpha = \frac{\pi}{2} = \text{const} \). Second, as \( b, c \to \infty \), the angle \( \alpha \) tends to 0! Moreover, the longer sides \( b \) and \( c \) are, the smaller the angle \( \alpha \) is (Fig. 4).

4. It is natural to call property (5) the non-Euclidian pythagorean theorem, because this relation looks more like the Euclidian pythagorean theorem \( a^2 = b^2 + c^2 \) than the relation \( \text{ch} a = \text{ch} b \text{ch} c \) in the non-Euclidian right-angled triangle.

5. Application: the isoperimetric problem

One of the most famous and interesting extremum problems is the so-called isoperimetric problem: find a fixed length curve that bounds a figure of the maximum square. The answer in Euclidian geometry is well known: this curve is the circle (see [4]). Using the properties of the maximum square triangle, we shall now solve this problem in Lobachevski geometry.

**Theorem 3.** Any fixed length curve that bounds a figure of a maximum square is a circle.

**Proof.** Let \( F \) be a figure of maximum square \( S \) bounded by the curve \( f \) of length \( l \) (the existence of \( F \) is proved in the same way as in Euclidian geometry; see [4]). It is obvious that \( F \) is convex. Let \( BC \) be the diameter of \( F \), i.e., \( BC \) halves the perimeter of \( F \). Then \( BC \) also halves the square of \( F \).

Let us show that all points of \( f \) are at a fixed distance from the middle \( O \) of segment \( BC \). Let \( A \in f \) be an arbitrary point. We claim that triangle \( ABC \) has the maximum square. Indeed, assume the converse. Consider the half of \( F \) bounded by \( BC \) and \( f \), and containing \( A \). Let us fix sides \( AB \) and \( AC \) and change the angle \( \angle BAC \) so that the square of triangle \( ABC \)
be maximum. Then we get the figure with the perimeter $l/2$ and the square larger than $S/2$. If we reflect this figure with respect to line $BC$, we get figure $F'$ with the perimeter $l$ and the square larger than $S$. This contradiction proves that triangle $ABC$ has the maximum square. Using property (2) of Theorem 2 we get $OA = OB = OC$. So, the curve $f$ is a circle. □

References

[1] N. V. Efimov, Higher Geometry, M.: Fismatlit, 2003.
[2] A. P. Norden, Elementary Introduction into Lobachevskii Geometry, M.: GIITL, 1953.
[3] V. V. Prasolov, Lobachevskii Geometry, 3-d ed. M.: MCCME, 2004.
[4] V. Yu. Protasov, Maximums and Minimums in Geometry, M.: MCCME, 2005.
[5] O. V. Shvartsman, Commentary on the P. Bibikov and I. Tkachenko Article "Triangle Trisection and Bisection on Lobachevskii Plane", Mat. Prosveshenie, 3:11 (2007), 127–130.
[6] A. A. Zaslavsky, Geometric Transformations, M.: MCCME, 2003.

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