ASYMPTOTIC ESTIMATE OF COHOMOLOGY GROUPS VALUED IN PSEUDO-EFFECTIVE LINE BUNDLES

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Abstract. In this paper, we study questions of Demailly and Matsumura on the asymptotic behavior of dimensions of cohomology groups for high tensor powers of (nef) pseudo-effective line bundles over non-necessarily projective algebraic manifolds. By generalizing Siu’s $\partial\bar{\partial}$-formula and Berndtsson’s eigenvalue estimate of $\bar{\partial}$-Laplacian and combining Bonavero’s technique, we obtain the following result: given a holomorphic pseudo-effective line bundle $(L, h_L)$ on a compact Hermitian manifold $(X, \omega)$, if $h_L$ is a singular metric with algebraic singularities, then $\dim H^q(X, L^k \otimes E \otimes I(h_L^k)) \leq C k^n$ for $k$ large, with $E$ an arbitrary holomorphic vector bundle. As applications, we obtain partial solutions to the questions of Demailly and Matsumura.

1. Introduction

Numerical properties of cohomology groups valued in bundles play an important role to approach certain fundamental problems of complex algebraic geometry and complex analytic geometry. The concept of positivity is often involved in the study. If some “strong positivity” is satisfied, one can derive precise or asymptotic vanishing theorems of the cohomology groups, which can be used to study, say embedding problems, asymptotics of linear systems, extension problems of holomorphic sections, the minimal model program, for listing just a few (cf. [1, 2, 6, 9, 19, 21, 23, 27, 28, 33, 34, 45, 43, 51]).

When only some ”weaker positivity” can be assumed, some precise or asymptotic estimate of the cohomology groups are also expected, which is again used to study the algebraic and analytic geometric consequences.
about the manifolds. For instance, in this aspect, one has the Grauert-Riemenschneider conjecture (G-R conjecture for short) and the abundance conjecture.

The G-R conjecture says that given a hermitian holomorphic line bundle over a compact Hermitian manifolds, if the curvature form of the line bundle is semi-positive and positive on an open dense subset, then the base manifold is Moishezon, i.e. birational to a projective manifold.

Siu \cite{46} solved the G-R conjecture by giving an asymptotic estimate of the Dolbeault cohomology group. Shortly later, Demailly \cite{13, 14} gave another solution to the G-R conjecture by establishing the celebrated holomorphic Morse inequalities (giving asymptotic bounds on the cohomology of tensor bundles of holomorphic line bundles), which is an important development of the Riemann-Roch formula, and is used to study the Green-Griffiths-Lang conjecture by Demailly in \cite{18} recently. Bonavero \cite{5} considered the singular case and founded singular holomorphic Morse inequalities for line bundles admitting a singular metric with algebraic singularities, which was used to establish G-R type criterions by volumes of pseudo-effective line bundles by Boucksom and Popovici \cite{7, 40}. Berndtsson \cite{3} obtained an asymptotic eigenvalue estimate of $\bar{\partial}$-Laplace which also implies the G-R conjecture.

The abundance conjecture \cite{10, 24, 29, 30, 31, 48} asserts that $\kappa(X) = \nu(X)$ which is still an open question in algebraic geometry, where $\kappa(X)$ is the Kodaira dimension of the canonical line bundle $K_X$ on the projective manifold $X$ and $\nu(X)$ is the numerical dimension of $K_X$.

Let us recall some positivity concepts for holomorphic line bundles. Let $(X, \omega)$ be a compact Hermitian manifold of complex dimension $n$, $L \to X$ be a holomorphic line bundle over $X$.

- $L$ is said to be semi-positive (positive), if there is a smooth Hermitian metric $h$ of $L$, such that the curvature $i\Theta_h \geq 0$ ($> \varepsilon \omega$ for some $\varepsilon > 0$).
- $L$ is said to be pseudo-effective (big), if there is a singular Hermitian metric $h$ of $L$, such that the curvature current $i\Theta_h \geq 0$ ($> \varepsilon \omega$ for some $\varepsilon > 0$) in the sense of currents.
- $L$ is said to be nef (numerically effective or numerically eventually free), if for any $\varepsilon > 0$, there is a smooth Hermitian metric $h$ of $L$, such that the curvature $i\Theta_h \geq -\varepsilon \omega$.

An Hermitian metric $h$ of $L$ is said to be singular, if locally we can write $h = e^{-2\varphi}$, with $\varphi \in L^1_{loc}$. The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the ideal subsheaf of the germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in local coordinates near $x$. 


Let \( h_L = e^{-2\varphi} \) be a singular Hermitian metric on \( L \) where \( \varphi \in L^1_{\text{loc}}(X, \mathbb{R}) \). The multiplier ideal sheaf of \( h_L \) is defined by \( I(h_L) = I(\varphi) \), which is well-known to be coherent when \( \varphi \) is locally a plurisubharmonic function up to a bounded function.

In this paper, we are going to study the following two questions by Demailly and Matsumura on the asymptotic estimate of dimensions of cohomology groups valued in high tensor powers of (nef) pseudo-effective line bundles over a compact Hermitian manifold which is not necessarily a projective algebraic manifold.

**Question 1.1** (Demailly’s question [17]). For a holomorphic nef line bundle \( L \) and a holomorphic vector bundle \( E \) over a compact Hermitian \( n \)-fold \( X \), does the following estimate holds:

\[
\dim \mathbb{C} H^q(X, L^k \otimes E) \sim O(k^{n-q}) ?
\]

**Question 1.2** (Matsumura’s question [33 [35]). Let \( L \) be a line bundle on a compact Hermitian manifold with a singular metric \( h \) whose curvature is (semi)-positive. Then, for any holomorphic vector bundle \( M \) on \( X \) and any \( q \geq 0 \), one asks if the following estimate holds:

\[
\dim \mathbb{C} H^q(X, \mathcal{O}_X(M \otimes L^k) \otimes I(h^k)) \sim O(k^{n-q}) ?
\]

In Demailly’s book [17], the same estimate as in Question 1.1 was proved under the assumption that \( X \) is projective algebraic. The proof relies on the projective algebraic condition of \( X \), i.e. the existence of an ample line bundle on \( X \). However the existence of the ample line bundle is not guaranteed for general compact Hermitian (even Kähler) manifold. Demailly wrote in his book that ”Observe that the argument does not work any more if \( X \) is not algebraic. It seems to be unknown whether the \( O(k^{n-q}) \) bound still holds in that case”. We summarize the question as above Question 1.1. Note that the estimate of Question 1.1 in the projective case was used to study abundance conjecture (e.g. [29]).

In [33], Matsumura gave a positive answer to Question 1.2 when \( X \) is projective, and the existence of an ample line bundle on \( X \) is essentially needed in the proof. This type of estimate was used to prove Nadel type vanishing theorem via injectivity theorems, and thus called asymptotic cohomology vanishing theorems for high tensor powers of line bundles with singular metrics. Furthermore, Matsumura [33] wrote that if one can give a positive answer to Question 1.2 under the condition that \( X \) is a compact Kähler manifold, then the corresponding vanishing theorems can be generalized to the Kähler case. Question 1.2 for Kähler manifolds is also mentioned in Problem 3.4 in [35]. Here we summarize Question 1.2 for Hermitian manifolds.
Note that Berndtsson’s estimate answers both Question 1.1 and Question 1.2 under an extra assumption that \( L \) is semi-positive, i.e. \( L \) admits a smooth Hermitian metric with semi-positive curvature.

For the sake of convenience, we state Berndtsson’s result as follows: let \((X, \omega)\) be a compact Hermitian manifold with Hermitian metric \( \omega \), \( E \) and \( L \) holomorphic line bundles over \( X \). Assume that \( L \) is given a metric of semipositive curvature. Take \( q \geq 1 \). Then if \( 0 \leq \lambda \leq k \),

\[
h^{n,q}_{\leq k}(X, L^k \otimes E) \leq C(\lambda + 1)^q k^{n-q}.
\]

If \( 1 \leq k \leq \lambda \), then

\[
h^{n,q}_{\leq k}(X, L^k \otimes E) \leq C \lambda^n,
\]

where \( h^{n,q}_{\leq k}(X, L^k \otimes E) \) is the dimension of the linear span of \( \overline{\partial} \)-closed \( L^k \otimes E \)-valued \((n, q)\)-eigen-forms of the \( \overline{\partial} \)-Laplacian \( \Box \) with eigenvalue less than or equal to \( \lambda \). In particular, if \( \lambda = 0 \), \( h^{n,q}_{\leq 0}(X, L^k \otimes E) \) is just \( \dim_{\mathbb{C}} H^q(X, K_X \otimes E \otimes L^k) \). The proof of this result is a clever combination of localization technique, Siu’s \( \partial \overline{\partial} \)-formula [45] and a result of Skoda [50].

To apply Berndtsson’s technique in our case, we find some difficulties which we can not overcome by direct use of the technique. On one hand, one can not expect a smooth semi-positive representative in a nef class, we need to compensate the loss of arbitrarily small positivity. On the other hand, for a pseudo-effective line bundle, the singularities of the singular metric can be very complicated.

Fortunately, by generalizing and combining the techniques of Siu, Berndtsson and Bonavero, we can make some progress under the assumption that \( L \) admits a singular metric with algebraic singularities. Note that this type of assumption is often used to study some important problems in algebraic geometry [30, 31].

Now let us introduce the main results in this paper.

Firstly, we compute the vector bundle version of the so called Siu’s \( \partial \overline{\partial} \)-formula [45] in order to meet our needs. Given an \( E \otimes L \)-valued \((n, q)\)-form \( u \), we define an associated \((n-q, n-q)\)-form \( T_u \). Running a similar procedure like in [3], we get the following

**Proposition 1.1.** Let \((X, \omega)\) be a compact Hermitian manifold with Hermitian metric \( \omega \), \( E \) and \( L \) be holomorphic vector bundle of rank \( r \) and holomorphic line bundle respectively. Let \( u \) be an \( E \otimes L \)-valued \((n, q)\)-form. If \( u \)
is $\bar{\partial}$-close, the following inequality holds

$$i\bar{\partial}\bar{\partial}(T_u \wedge \omega_{q-1}) \geq (-2\text{Re}(\Box u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle - c|u|^2)\omega_n.$$ 

The constant $c$ is equal to zero if $\bar{\partial}\omega_{q-1} = \bar{\partial}\omega_q = 0$, hence in particular if $\omega$ is Kähler.

Actually, when $d\omega = 0$, we can get the following identity for smooth $E \otimes L$-valued $(n, q)$-form $u$.

$$i\partial\partial(T_u \wedge \omega_{q-1}) \geq (-2\text{Re}(\Box u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle - c|u|^2)\omega_n.$$ 

It is worth to mention that, by carefully checking the computations of the $\partial\partial$-formula, we are able to derive a similar formula in the case that the metric of $L$ is singular.

**Proposition 1.2.** Let $(X, \omega)$ be a compact Kähler manifold, $(E, h_E) \to X$ be a holomorphic Hermitian vector bundle over $X$, and $(L, h_L)$ be a holomorphic pseudo-effective line bundle with singular metric $h_L$ such that $i\Theta_{L, h_L} \geq \gamma$ with $\gamma$ a continuous real $(1, 1)$-form on $X$. Suppose that $u$ is an $E \otimes L$-valued $(n, q)$-form.

Then we have the following equality

$$i\partial\partial(T_u \wedge \omega_{q-1}) = (-2\text{Re}(\partial\partial u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle + |\partial\gamma|^2 + |\partial^\ast u|^2 - |\partial u|^2)\omega_n.$$ 

Moreover, if $u$ is $\partial\partial$-closed, then we have

$$i\partial\partial(T_u \wedge \omega_{q-1}) \geq (-2\text{Re}(\Box u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle)\omega_n.$$ 

The equality (3) can be used to prove vanishing theorems which is equivalent to solve $\partial$-equations on compact Kähler manifolds. By using Proposition 1.1, the main result of [3] is generalized to vector bundle version.

**Theorem 1.3.** Let $(X, \omega)$ be a compact Hermitian manifold with Hermitian metric $\omega$, $E$ be a holomorphic vector bundle and $L$ be a holomorphic line bundle over $X$. Assume that $L$ is given a metric of semipositive curvature. Take $q \geq 1$. Then if $0 \leq \lambda \leq k$,

$$h^{n,q}_{\leq \lambda}(X, L^k \otimes E) \leq C(\lambda + 1)^q k^{n-q}.$$ 

If $1 \leq k \leq \lambda$, then

$$h^{n,q}_{\leq \lambda}(X, L^k \otimes E) \leq C\lambda^n.$$ 

Then we consider the case of $L$ pseudo-effective with algebraic singularities, i.e. $L$ is equipped a singular metric $h_L$ with algebraic singularities whose curvature current is semi-positive in the sense of currents.

By combining Bonavero’s technique [5], Berndtsson’s technique [3], and Theorem 1.3, we are able to obtain the following
Theorem 1.4. Let $(X, \omega)$ be a compact Hermitian manifold of complex dimension $n$. $(L, h_L)$ is a pseudo-effective line bundle over $X$, such that $h_L$ is a singular metric with algebraic singularities. $E$ is a holomorphic vector bundle over $X$. Then we have the following estimate
\[ \dim H^q(X, L^k \otimes E \otimes I(h_L^k)) \leq Ck^{n-q} \]
for $k$ large, where $I(h_L^k)$ is the multiplier ideal sheaf associated to the metric $h_L^k$ of $L^k$.

Thus we give a positive answer to Question 1.2 under the assumption of algebraic singularities.

Combining with an argument related to an exact sequence, it follows from Theorem 1.4 that the following partial solution to Question 1.1 holds:

Theorem 1.5. Let $X$ be a compact complex manifold, $E \to X$ be a holomorphic vector bundle over $X$, and $L \to X$ be a holomorphic line bundle with a singular Hermitian metric $h_L$ with algebraic singularities such that the curvature current of $h_L$ is semi-positive. Assume that the dimension of the singular locus of $h_L$ is $m$. Then for $q > m$, we have that
\[ \dim H^q(X, \mathcal{O}_X(E \otimes L^k)) \leq Ck^{n-q}. \]

Furthermore, by a Diophantine approximation argument, we can weaken the assumption of algebraic singularities in Theorem 1.5.

Theorem 1.6. Let $X$ be a compact complex manifold, $E \to X$ be a holomorphic vector bundle over $X$, and $L \to X$ be a holomorphic line bundle with a singular Hermitian metric $h_L$ with analytic singularities such that the curvature current of $h_L$ is semi-positive. Let $h$ be an arbitrarily smooth Hermitian metric of $L$, and set $e^{-\psi} = h_L/h$. Suppose that there is a small $\varepsilon > 0$, such that $he^{-(1+\delta)^q}$ are singular metrics of $L$ with semi-positive curvature current for $|\delta| < \varepsilon$. Assume that the dimension of the singular locus of $h_L$ is $m$. Then for $q > m$, we have that
\[ \dim H^q(X, \mathcal{O}_X(E \otimes L^k)) \leq Ck^{n-q}. \]

Combining with injectivity theorem obtained in [33], we get the following two vanishing theorems.

Theorem 1.7. Let $X$ be a compact Kähler manifold, and $L$ be a holomorphic line bundle over $X$. Suppose that $L$ is pseudo-effective, and the singular metric $h_{\min}$ with minimal singularities of $L$ is with algebraic singularities. Then we have that
\[ H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes I(h_{\min})) = 0 \text{ for } q > n - \kappa(L). \]
Theorem 1.8. Let $X$ be a compact Kähler manifold, and $L$ be a holomorphic line bundle with non-negative Kodaira-Iitaka dimension over $X$. Suppose that $L$ is pseudo-effective, and the Siu’s metric $h_{\text{siu}}$ of $L$ is with algebraic singularities. Then we have that

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_{\text{siu}})) = 0 \text{ for } q > n - \kappa(L).$$

This paper is organized as follows: In Section 2, we recall some definitions and fundamental results which will be used. In Section 3, we derive the $\partial \bar{\partial}$-formula for forms valued in vector bundle on compact Hermitian manifolds including the case when the metric of the line bundle is singular with local quasi-psh potential on compact Kähler manifolds, which is a generalization of Siu’s $\partial \bar{\partial}$-formula on compact Kähler manifolds, and therefore prove the Proposition 1.1. In Section 4, we consider singular $\partial \bar{\partial}$-formula and prove Proposition 1.2. In Section 5, along the way of Berndtsson, we generalize Berndtsson’s eigenvalue estimate of $\bar{\partial}$-Laplacian to the case of powers of line bundle tensor with vector bundle and prove Theorem 1.3. In Section 6, we give the proof of our main Theorem 1.4. In Section 7, we give a proof of Theorem 1.5 which partially answers the question of Demailly. In Section 8, we give the proof of two vanishing theorems, i.e. Theorem 1.7 and Theorem 1.8.

2. Technical preliminaries

2.1. Algebraic singularities.

Definition 2.1. Let $L \to X$ be a holomorphic line bundle over a complex manifold $X$. Let $h = e^{-\varphi}$ be a singular metric of $L$.

We say $h$ is a singular metric with analytic singularities if $\varphi$ is a locally integrable function on $X$ which has locally the form

$$\varphi = \frac{c}{2} \log(\sum_{j=1}^{N} |f_j|^2) + \psi,$$

where $f_j$ are non-trivial holomorphic functions and $\psi$ is smooth and $c$ is a $\mathbb{R}_+$-valued, locally constant function on $X$.

We call $h$ a singular metric with algebraic singularities, if $c \in \mathbb{Q}_+$.

2.2. Multiplier ideal sheaves.

Definition 2.2 (cf. [17]). Let $L \to X$ be a holomorphic line bundle over a complex manifold $X$. Let $h = e^{-\varphi}$ be a singular metric of $L$ which has analytic singularities, i.e., $\varphi$ locally has the form (4). Then $\mathcal{I} = \mathcal{I}(\varphi/c)$ is defined to be the ideal of germs of holomorphic functions $h$ such that
\[ |h| \leq C e^{\varphi/c} \] for some constant \( C \). i.e. 
\[ |h| \leq C(|f_1| + \cdots + |f_N|). \]

This is a globally defined ideal sheaf on \( X \), locally equal to the integral closure \((f_1, \cdots, f_N)\), and \( \mathcal{I} \) is coherent on \( X \). If \((g_1, \cdots, g_{N'})\) are local generators of \( \mathcal{I} \), we still have
\[ \varphi = \frac{c}{2} \log \left( \sum_{j=1}^{N'} |g_j|^2 \right) + \psi', \]
where \( \psi' \) is smooth.

Usually, given a plurisubharmonic (psh for short) function \( \varphi \), it is not easy to compute the multiplier ideal sheaf \( \mathcal{I}(\varphi) \). When \( \varphi \) is with analytic singularities, the following facts are collected from [17].

(I) If \( \varphi \) has the form \( \varphi = \sum \lambda_j \log |g_j| \) where \( D_j = g_j^{-1}(0) \) are nonsingular irreducible divisors with normal crossings. Then \( \mathcal{I}(\varphi) \) is the sheaf of functions \( f \) on open sets \( U \subset X \) such that
\[ \int_U |f|^2 \prod |g_j|^{-2\lambda_j} dV < \infty. \]

Since locally the \( g_i \) can be taken to the coordinate functions from a local coordinate system \((z_1, \cdots, z_n)\), the condition is that \( f \) is divisible by \( \prod g_j^{m_j} \) where \( m_j - \lambda_j > -1 \) for each \( j \), i.e. \( m_j \geq |\lambda_j| \) (integral part). Hence
\[ \mathcal{I}(\varphi) = O(-[D]) = O(- \sum |\lambda_j| D_j). \]

(II) For the general case of analytic singularities, suppose that
\[ \varphi \sim \frac{c}{2} \log(|f_1|^2 + \cdots + |f_N|^2) \]
near the poles.

From Definition 2.2, one can assume that the \((f_j)\)'s are generators of the integrally closed ideal sheaf \( \mathcal{I} = \mathcal{I}(\varphi/c) \), defined as the sheaf of holomorphic functions \( f \) such that \(|f| \leq C \exp(\varphi/c)\).

There is a smooth modification \( \mu : \tilde{X} \to X \) of \( X \) such that \( \mu^* \mathcal{I} \) is an invertible sheaf \( O_{\tilde{X}}(-D) \) associated with a normal crossing divisor \( D = \sum \lambda_j D_j \), where \( (D_j) \) are the components of the exceptional divisor of \( \tilde{X} \), and \( \lambda_j \in \mathbb{N}_+ \).

Thus locally we have
\[ \varphi \circ \mu \sim c \sum \lambda_j \log |g_j| \]
where \( g_j \) are local generators of \( \mathcal{O}(-D_j) \). So
\[
I(\varphi \circ \mu) = \mathcal{O}(\sum [c\lambda_j]D_j),
\]
and
\[
I(\varphi) = \mu_\ast \mathcal{O}_\tilde{X}(\sum (\rho_j - [c\lambda_j])D_j),
\]
where \( R = \sum \rho_jD_j \) is the zero divisor of the Jacobian function \( J_\mu \) of the modification map.

2.3. Skoda’s Lemma.

**Definition 2.3.** For a psh function \( \varphi \) on an open set \( \Omega \subset \mathbb{C}^n \), the Lelong number of \( \varphi \) at \( x \) is defined to be
\[
\nu(\varphi, x) := \lim_{z \to x} \inf \frac{\varphi(z) - \varphi(x)}{\log |z - x|}.
\]

**Lemma 2.1** ([49]). Let \( \varphi \) be a psh function on an open set \( \Omega \subset \mathbb{C}^n \) and let \( x \in \Omega \).

(i) If \( \nu(\varphi, x) < 1 \), then \( e^{-2\varphi} \) is integrable in a neighborhood of \( x \), in particular, \( I(\varphi)_x = O_{\Omega,x} \).

(ii) If \( \nu(\varphi, x) \geq n + s \) for some integer \( s \geq 0 \), then \( e^{-2\varphi} \geq C|z - x|^{-2n-2s} \) in a neighborhood of \( x \) and \( I(\varphi)_x \subset m_{\Omega,x}^{s+1} \), where \( m_{\Omega,x}^{s+1} \) is the maximal ideal of \( \mathcal{O}_{\Omega,x} \).

2.4. An isomorphism theorem of cohomology groups.

**Lemma 2.2** (cf. [32]). Let \( \pi : \tilde{X} \to X \) be the blow-up of \( X \) with smooth center \( Y \subset X \). \( L \to X \) be a holomorphic line bundle with singular metric \( h_L \). Assume that in the neighborhood of any point of the exceptional divisor \( D \) of \( \pi \), a local weight \( \varphi \) of the metric \( h_L \) (\( h_L = e^{-2\varphi} \)) satisfies
\[
\varphi \circ \pi = c \log |f| + \psi,
\]
for some \( c > 0 \), \( f \) is a local definition function of \( D \) and \( \psi \) is quasi-psh. Then for any \( p > 1/c \) and \( q \geq 0 \), we have
\[
H^q(\tilde{X}, K_{\tilde{X}} \otimes \pi^\ast(L^p \otimes E) \otimes I(\pi^\ast h_{L^p})) \cong H^q(X, K_X \otimes L^p \otimes E \otimes I(h_{L^p})),
\]
i.e.
\[
H^{n,q}(\tilde{X}, \pi^\ast(L^p \otimes E) \otimes I(\pi^\ast h_{L^p})) \cong H^{n,q}(X, L^p \otimes E \otimes I(h_{L^p})),
\]

**Remark 2.1.** A real function is said to be quasi-psh, if it can be written as the sum of psh function and a smooth function locally. Lemma 2.2 is a consequence of Leray spectral theorem.
2.5. **Lebesgue decomposition of a current.** For a measure $\mu$ on a manifold $M$ we denote by $\mu_{ac}$ and $\mu_{sing}$ the uniquely determined absolute continuous and singular measures (with respect to the Lebesgue measure on $M$) such that

$$\mu = \mu_{ac} + \mu_{sing}$$

which is called the Lebesgue decomposition of the measure $\mu$.

If $T$ is a $(1, 1)$-current of order 0 on $X$, written locally $T = i \sum T_{ij} dz_i \wedge d\bar{z}_j$, we defines its absolute continuous and singular components by

$$T_{ac} = i \sum (T_{ij})_{ac} dz_i \wedge d\bar{z}_j,$$

$$T_{sing} = i \sum (T_{ij})_{sing} dz_i \wedge d\bar{z}_j.$$

The Lebesgue decomposition of $T$ is then

$$T = T_{ac} + T_{sing}.$$  

Note that a positive $(1, 1)$-current $T \geq 0$ is of order 0. If $T \geq 0$, it follows that $T_{ac} \geq 0$ and $T_{sing} \geq 0$. Moreover, if $T \geq \alpha$ for a continuous $(1, 1)$-form $\alpha$, then $T_{ac} \geq \alpha$, $T_{sing} \geq 0$.

It follows from the Radon-Nikodym theorem that $T_{ac}$ is (the current associated to) a $(1, 1)$-form with $L^1_{loc}$ coefficients. The form $T_{ac}(x)^n$ exists for almost all $x \in X$ and is denoted by $T_{ac}^n$.

Note that $T_{ac}$ in general is not closed, even when $T$ is, so that the decomposition doesn’t induce a significant decomposition at the cohomological level.

However, when $T$ is a closed positive $(1, 1)$-current with analytic singularities along a subscheme $V$, the residual part $R$ in Siu decomposition (cf. [44]) of $T$ is nothing but $T_{ac}$, and $\sum_k v(T, Y_k)[Y_k]$ is $T_{sing}$.

**Remark 2.2.** Suppose now $X$ is a compact complex manifold of complex dimension $n$. $(L, h_L)$ be a pseudo-effective line bundle over $X$, such that $h_L$ is a singular metric with analytic singularities. Then there is a smooth modification $\mu : \tilde{X} \to X$, such that $\mu^* \mathcal{F}$ is an invertible sheaf $\mathcal{O}_\tilde{X}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$, where $(D_j)$ are the components of the exceptional divisor of $\tilde{X}$. Now locally we can write

$$\mu^* \varphi = \varphi \circ \mu = c \log |s_D| + \psi,$$

where $s_D$ is the canonical section of $\mathcal{O}_\tilde{X}(-D)$, and $\psi$ is a smooth potential. This implies that we have the following Lebesgue decomposition

$$\frac{i}{\pi} \mu^* \Theta = \frac{i}{\pi} \partial \bar{\partial} (\mu^* \varphi) = c[D] + \beta$$

(5)
where \([D]\) is the current of integration over \(D\) and \(\beta\) is a smooth closed \((1, 1)\)-form. From the pseudo-effectiveness, i.e. \(\frac{1}{\pi} u^\alpha \Theta_{h_L} \geq 0\), we can conclude that \(\beta \geq 0\).

2.6. Regularity lemma.

**Lemma 2.3** ([52, Lemma 9.3]). Let \(\varphi\) be a function in \(L^1_{\text{loc}}(\Omega)\) such that \(\Delta \varphi\) is a measure, where \(\Omega\) is a domain in \(\mathbb{R}^m\) \((m \geq 2)\) and \(\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}\), then \(\frac{\partial}{\partial \nu} \varphi\) is a function in \(L^1_{\text{loc}}(\Omega)\).

**Remark 2.3.** If \(\varphi\) is a quasi-psh function, then \(\varphi\) is in \(L^1_{\text{loc}}\) and by Lemma 2.3 \(\nabla \varphi\) is also in \(L^1_{\text{loc}}\).

3. \(\bar{\partial} \bar{\partial}\)-formula for vector bundle valued forms

Let \((E, h_\alpha^\beta)\) be a Hermitian holomorphic vector bundle of rank \(r\) over an \(n\)-dimensional compact Hermitian manifold \((X, \omega)\). Let \((L, h_L = e^{-\psi} := e^{-2\varphi})\) be a Hermitian holomorphic line bundle over \(X\).

The curvature form \(\Theta_\alpha^\beta = -\sqrt{-1} \sum_{i,j} \Omega_{\alpha_i^j \beta_i^j} dz^i \wedge d\bar{z}^j\) of \(E\) is given by

\[
\Omega_{\alpha_i^j \beta_i^j} = \partial_i \partial_j h_\alpha^\beta - h_{\alpha \gamma} \partial_i h_{\beta \gamma} h_\gamma^\beta.
\]

Let \(D'\) be the \((1, 0)\)-part of the Chern connection associated to the vector bundle \(E \otimes L\) with respect to the Hermitian metrics of \(E\) and \(L\). Then one have the following formula

\[
D' \bar{\partial} + \bar{\partial} D' = \Theta_{E \otimes L},
\]

and

\[
\bar{\partial} = - * D' *.
\]

For the detailed computations, we refer to [26].

Let

\[
u^\alpha = \frac{1}{p! q!} \sum u^\alpha_{i_p j_q} dz^{i_p} \wedge d\bar{z}^{j_q}
\]

be an \(E\)-valued \((p, q)\)-form on \(X\).

Let \(u^\alpha\) be an \(E \otimes L\)-valued \((n, q)\)-form, we define an associated \((n-q, n-q)\)-form which in a local trivialization is written as

\[
T_u = c_{n-q} h_{\alpha \beta} \gamma^\alpha \wedge \bar{\gamma}^\beta e^{-\psi}
\]

where \(\gamma^\alpha = * u^\alpha, c_{n-q} = i^{(n-q)^2}\).
Here * denote the Hodge operator of the Hermitian manifold $X$, defined by the formula
\[ \xi \wedge \star \xi = |\xi|^2 \omega_n \]
where $\xi$ is a $(p, q)$-form on $X$.

The relation $*u^\alpha = \gamma^\alpha$ can be expressed as
\[ u^\alpha = c_{n-q} \gamma^\alpha \wedge \omega_q, \]
and moreover we have
\[ *\gamma^\alpha = (-1)^{n-q} c_{n-q} \gamma^\alpha \wedge \omega_q. \]

In the following we shall also use the relations $ic_q = (-1)^q c_{q-1}$ and $c_{q-1} = c_{q+1}$.

By direct computation, we get that
\[ h_{ab}(D' \gamma^a \wedge \gamma_b e^{-\psi}) = (1)^{n-q} c_{n-q} \gamma^a \wedge \omega_q, \]
and
\[ h_{ab}(D' \gamma^a \wedge \gamma_b e^{-\psi}) + (1)^{n-q} c_{n-q} \gamma^a \wedge \omega_q. \]

By using the commutator formula (6), we can get that
\[ \bar{\partial}(h_{ab} \gamma^a \wedge \gamma_b e^{-\psi}) = h_{ab}(\bar{\partial} \gamma^a \wedge \gamma_b + (-1)^{n-q} \gamma^a \wedge \bar{D} \gamma_b e^{-\psi}) \]
And then
\[ \bar{\partial}(h_{ab} \gamma^a \wedge \gamma_b e^{-\psi}) = h_{ab}(\bar{\partial} \gamma^a \wedge \gamma_b + (-1)^{n-q} \gamma^a \wedge \bar{D} \gamma_b e^{-\psi}) \]
\[ + (1)^{n-q} \gamma^a \wedge \gamma_b e^{-\psi}. \]

By using the formula (6), we can get that
\[ \bar{\partial} u^\alpha = - * D' \gamma^\alpha = (-1)^{n-q} c_{n-q-1} D' \gamma^\alpha \wedge \omega_{q-1}, \]
which implies that
\[ \bar{\partial} u^\alpha = (-1)^{n-q} c_{n-q-1} \bar{\partial} D' \gamma^\alpha \wedge \omega_{q-1} + (-1)^{n-q-1} D' \gamma^\alpha \wedge \bar{\partial} \omega_{q-1} \]
\[ = (-1)^{n-q} c_{n-q-1} \bar{\partial} D' \gamma^\alpha \wedge \omega_{q-1} + O(\bar{\partial} u^\alpha | \bar{\partial} \omega_{q-1}). \]

In complex geometry, $\partial \omega$ is called the torsion form of $\omega$ (the operator $\tau := [\Lambda, \partial \omega]$ is called the torsion operator), one can see that
\[ \bar{\partial} \omega_{q-1} = \bar{\partial} \omega \wedge \omega_{q-2}, \]
where the torsion comes into the game.

If the Hermitian metric $\omega$ is Kähler (i.e. $d \omega = 0$), this term disappears.

Multiply (9) by $ic_{n-q} \omega_{n-q}$, we have five terms. By (10), the second term in (9) equals
\[ -h_{ab} \bar{\partial} D' \gamma^a \wedge \gamma_b e^{-\psi} - h_{ab} \bar{\partial} \bar{\partial} u^\alpha \wedge \gamma_b e^{-\psi} = -(\bar{\partial} \bar{\partial} u, u) \omega_n \]
up to an error of size $O(\|\bar{\partial} u\|\|\bar{\partial}\omega_{q-1}\|u\|)$. Since the entire expression is real, the fifth term must be the conjugate of the second one, so these terms together give

$$-2\text{Re}\langle \bar{\partial}\partial u, u \rangle \omega_n.$$  

The first term is the curvature term

$$\langle \Theta_{E\otimes L} \wedge \Lambda u, u \rangle \omega_n = \langle (\Theta_E + \Theta_L \otimes 1_E) \wedge \Lambda u, u \rangle \omega_n.$$  

Checking signs, we see that the fourth term equals

$$|\bar{\partial}\partial u|^2 \omega_n,$$

so it only needs to analyze the third term.

Consider the bilinear form on $E \otimes L$ valued $(n-q,1)$-forms defined by

(11)

$$[\chi, \eta] \omega_n = ic_{n-q}(-1)^{n-q+1}h_{\alpha\bar{\beta}}\chi^{\alpha} \wedge \eta^{\beta} \wedge \omega_{q-1} = -c_{n-q+1}h_{\alpha\bar{\beta}}\chi^{\alpha} \wedge \eta^{\beta} \wedge \omega_{q-1}.$$  

Fix an arbitrary point $x \in X$, we can choose a good coordinate chart $(U, z)$ centered at $x$ and a good trivialization of $E$ and $L$ such that $\omega = \frac{1}{2}\bar{\partial}\partial|z|^2$, $h_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$, and $\psi = 0$ at $x$. Then the bilinear form (11) at $x$ reads

$$[\chi, \eta] \omega_n = -c_{n-q+1} \sum_{a=1}^{r} \chi^{\alpha} \wedge \eta^{\beta} \wedge \omega_{q-1}.$$  

For each $\alpha \in \{1, 2, \cdots, r\}$, we consider

(12)

$$[\chi, \eta] \omega_n = -c_{n-q+1} \chi^{\alpha} \wedge \eta^{\beta} \wedge \omega_{q-1}.$$  

It is proved in [3] that at $x$

- the form (12) is negative definite on the subspace $V_\alpha$ forms that can be written $\chi^{\alpha} = \chi_0^{\alpha} \wedge \omega$ (it then equals a negative multiple of the norm square of $\chi_0$),
- the annihilator $V_\alpha^\circ$ of $V_\alpha$ with respect to $[,]$ in (12), consists precisely of forms satisfying $\chi^{\alpha} \wedge \omega = 0$, and $V_\alpha \cap V_\alpha^\circ = \{0\}$,
- the form (12) is positive definite on $V_\alpha^\circ$,
- any $(n-q,1)$-form $\chi^{\alpha}$ can be decomposed uniquely

$$\chi^{\alpha} = \chi_1^{\alpha} + \chi_0^{\alpha} \wedge \omega$$

with $\chi_1^{\alpha} \in V_\alpha^\circ$.

Since the point $x \in X$ is arbitrarily chosen, it follows that any $E \otimes L$-valued $(n-q,1)$-form $\chi$ can be decomposed uniquely

$$\chi = \chi_1 + \chi_0 \wedge \omega.$$
with \( \chi_1 \in V^* \), where \( V^* \) is the annihilator of \( V \) with respect to \([,] \) in (11) and \( V \) is the subspace of \( E \otimes L \)-valued \((n - q, 1)\) forms on which the bilinear form \([,] \) in (11) is negative definite.

Now let \( \chi = \bar{\partial} \gamma \). Since \( u = c_n^{-q} \gamma \wedge \omega_q \) is \( \bar{\partial} \)-closed, we have that

\[
(13) \quad \bar{\partial} \gamma \wedge \omega_q = (-1)^{n-q-1} \gamma \wedge \bar{\partial} \omega_q.
\]

Decomposing \( \bar{\partial} \gamma = \chi_1 + \chi_0 \wedge \omega \) and plugging into (13), we have

\[
\chi_0 \wedge \omega \wedge \omega_q = (-1)^{n-q-1} \gamma \wedge \bar{\partial} \omega_q.
\]

Since \( \chi_0 \) is of bidegree \((n - q - 1, 0)\), this means that \( |\chi_0| = O(|\gamma||\bar{\partial} \omega_q|) = O(|u||\bar{\partial} \omega_q|) \). This means that the only possible negative contribution of \( [\bar{\partial} \gamma, \bar{\partial} \gamma] \) can be estimated by \( c|u|^2 \). If we also estimate the earlier error term

\[
O(|\bar{\partial} u||\bar{\partial} \omega_{q-1}| |u|) \leq C_\varepsilon |u|^2 + \varepsilon |\bar{\partial} u|^2,
\]

and collect all the terms and note that \( \bar{\partial} \bar{\partial} u = \Box u \) if \( u \) is \( \bar{\partial} \)-closed, we get that

\[
i\bar{\partial} \bar{\partial} u \wedge \omega_{q-1} \geq (-2 \text{Re}(\Box u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle + (1 - \varepsilon)|\bar{\partial} u|^2 - C_\varepsilon |u|^2) \omega_n.
\]

Thus we get the following

**Proposition 3.1.** Let \((X, \omega)\) be a compact Hermitian manifold with Hermitian metric \( \omega \), \( E \) and \( L \) be holomorphic vector bundle of rank \( r \) and holomorphic line bundle respectively. Let \( u \) be an \( E \otimes \omega \)-valued \((n, q)\)-form. If \( u \) is \( \bar{\partial} \)-close, the following inequality holds

\[
i\bar{\partial} \bar{\partial} u \wedge \omega_{q-1} \geq (-2 \text{Re}(\Box u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle - c|u|^2) \omega_n.
\]

The constant \( c \) is equal to zero if \( \bar{\partial} \omega_{q-1} = \bar{\partial} \omega_q = 0 \), hence in particular if \( \omega \) is Kähler.

**Remark 3.1.** Actually, when \( d\omega = 0 \), we can get the following identity for smooth \( E \otimes L \)-valued \((n, q)\)-form \( u \).

\[
i\bar{\partial} \bar{\partial} u \wedge \omega_{q-1} = (-2 \text{Re}(\bar{\partial} u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle + \bar{\partial} \gamma u|^2 + |\bar{\partial} u|^2 - |\bar{\partial} u|^2) \omega_n.
\]

**Proof.** To get the equality, we only need to analyse the term

\[
(-1)^{n-q+1} h_{\alpha\beta} \overline{\partial \gamma}^\alpha \wedge \overline{\bar{\partial} \gamma}^\beta e^{-\psi}.
\]

We need the following Lemma which is a variant of Lemma 4.2 in [4].

**Lemma 3.2.** Let \( \xi \) be a \( E \otimes L \) valued \((n - q, 1)\) form. Then

\[
ic_n^{-q}(1)^{n-q-1} h_{\alpha\beta} \xi^\alpha \wedge \xi^\beta \wedge \omega_{q-1} = (|\xi|^2 - |\xi \wedge \omega_q|^2) \omega_n.
\]
Proof. First we observe that the identity is pointwise. Fix arbitrary point \( x_0 \in X \), we can choose normal coordinates of \( X \) centered at \( x_0 \) and choose normal trivialization of \( E \) and any trivialization of \( L \), such that \( \omega(x_0) = \sum dz_i \wedge d\bar{z}_i \) and \( h_{\alpha\beta}(x_0) = \delta_{\alpha\beta} \).

Then the question in hand is reduced to the case considered in Lemma 4.2 in [4]. Thus we complete the proof of the Lemma. \( \square \)

From Lemma 3.2, Remark 3.1 follows easily along the line of the proof of Proposition 3.1. \( \square \)

It is worth to mention that from Remark 3.1, we can get the following estimate.

**Proposition 3.3.** Assume \((X, \omega)\) is a compact Kähler \( n \)-fold, \( E \) and \( L \) are holomorphic Hermitian vector bundles and line bundles over \( X \), and the curvature form \( \Theta_{E \otimes L} \) is strictly Nakano positive, i.e. \( \Theta_{E \otimes L} \geq c \omega \otimes 1_{E \otimes L} \) for some positive constant \( c \). Let \( u \) be an \( E \otimes L \) valued \((n, q)\)-form. Then we have

\[
qc \int_X |u|^2 \omega_n + \int_X |\partial u|^2 \omega_n \leq \int_X |D u|^2 \omega_n \leq \int_X |D^* u|^2 \omega_n.
\]

**Remark 3.2.** Remark 3.1 and Proposition 3.3 can be used to prove vanishing theorems and extension theorems for vector bundles, cf. [41, 42].

4. \( \partial \bar{\partial} \)-formula for the singular line bundle case

We are now concerned with the situation that the metric \( h_L \) of the line bundle \( L \) is singular. Suppose that the curvature \( \Theta_{h_L} \geq \gamma \) in the sense of current for some continuous \((1, 1)\)-form, i.e. the line bundle is quasi-pseu-effective. In this case, the local potentials \( \psi \) are quasi-psh functions.

From (10), one can see that if \( \partial \omega_{q-1} \neq 0 \), one can not get an estimate of the term \( O(|\partial u||\partial \omega_{q-1}|) \), since in this case our \( \psi \) is singular. For this reason, in this subsection, we work on compact Kähler manifold, i.e. the Hermitian metric \( \omega \) satisfies \( dw = 0 \) on \( X \).

Let \( u \) be an \( E \otimes L \) valued \((n, q)\)-form, the associated \((n - q, n - q)\)-form \( T_u \) defined by (8). We have the following local data:

- Smooth metric \( h_{\alpha\beta} \) of \( E \) and singular metric \( h_L = e^{-\psi} \) locally with \( \psi \) a quasi-psh \((L^1_{\text{loc}})\) function.
- Singular Chern connection \( D = D' + \bar{\partial} \), with the \((1, 0)\)-connection matrix \( \Gamma_E \otimes 1_L - \partial \psi \otimes 1_E \), where \( \Gamma_E \) is the connection matrix of \( E \) and \( \partial \psi \) is a \((1, 0)\)-form with \( L^1_{\text{loc}} \) coefficients by Lemma 2.3.
- \( \Theta_{E \otimes L} = \Theta_E \otimes 1_L + \partial \bar{\partial} \psi \otimes 1_{E \otimes L} \), where \( \partial \bar{\partial} \psi \) is a closed positive current.
- The commutator formula (6) also holds: \( \partial D' + D' \bar{\partial} = \Theta_{E \otimes L} \).
- The $\overline{\partial}$-operator is also a first order differential operator with $L^1_{loc}$-coefficients.
- The operator $\partial\overline{\partial}$ also makes sense, hence the Laplace operator $\Box$ makes sense as well. In fact, we have already get the explicit formula for $\partial\overline{\partial}$ in (10), from which one can get the conclusion.

By the same computation in Section 3 under the extra assumption that $d\omega = 0$, we can get the following

**Proposition 4.1.** Let $(X, \omega)$ be a compact Kähler manifold, $(E, h^E) \to X$ be a holomorphic Hermitian vector bundle over $X$, and $(L, h_L)$ be a holomorphic pseudo-effective line bundle with singular metric $h_L$ such that $i\Theta_{E \otimes L} \geq \gamma$ with $\gamma$ a continuous real $(1, 1)$-form on $X$. Suppose that $u$ is an $E \otimes L$-valued $(n, q)$-form. Then we have the following equality

$$i\partial\overline{\partial}T_u \wedge \omega_{q-1} = (-2Re(\partial\overline{\partial} u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle) + |\partial u|^2 + |\overline{\partial} u|^2 - |\partial\overline{\partial} u|^2.)$$

Moreover, if $u$ is $\partial$-closed, then we have

$$i\partial\overline{\partial}T_u \wedge \omega_{q-1} \geq (-2Re(\Box u, u) + \langle \Theta_{E \otimes L} \wedge \Lambda u, u \rangle)\omega_n.$$

5. **Proof of Theorem 1.3**

Have the vector bundle version $\partial\overline{\partial}$-formula in hand, the proof of Theorem 1.3 can be copied word by word from [3].

Firstly, from the vector bundle version $\partial\overline{\partial}$-formula, we can obtain that

$$i\partial\overline{\partial}T_u \wedge \omega_{q-1} \geq (-2Re(\Box u, u) - c'|u|^2)\omega_n.$$

Following the same argument (a standard calculation) in [3], one can conclude that if $u \in \mathcal{H}^{n,q}_{\leq \lambda}(X, L^k \otimes E)$ with $\partial u = 0$, and the metric on $L$ as semi-positive curvature, then for $r < \lambda^{-1/2}$ and $r < c_0$,

$$\int_{|z| < r} |u|^2 \omega_n \leq Cr^q(\lambda + 1)^q \int_X |u|^2 \omega_n,$$

where the constants $c_0$ and $C$ are independent of $k, \lambda$ and the point $x$.

Secondly, one can get a pointwise norm estimate of $u$. In fact, fix an arbitrary point $x \in X$. Choose a local coordinates $z$, near $x$ such that $\overline{\partial}z(x) = 0$ and $\omega = \frac{i}{2}d\overline{\partial}|z|^2$ at $x$. Choose local trivializations of $E$ and $L$ such that the metrics of $E$ and $L$ take the following form

$$(14) \quad h_{\alpha\overline{\beta}}(z) = \delta_{\alpha\overline{\beta}} + O(|z|^2);$$

$$\psi(z) = \sum \mu_j |z_j|^2 + o(|z|^2).$$
We do the following rescaling trick. For any form $u$, we express $u$ in terms of the trivializations and local coordinates and put

$$u^{(k)}(z) = u(z/\sqrt{k}),$$

so that $u^{(k)}$ is defined for $|z| < 1$ if $k$ is large enough.

In the same time, the Laplacian is also scaled in the following form

$$k \Box^{(k)} u^{(k)} = (\Box u)^{(k)}.

As stated in [3], it is not hard to see that if $\Box$ is defined by the metric $k\psi$ on $L^k$ and $h_{\alpha\beta}$ on $E$, then $\Box^{(k)}$ is associated to the line bundle metric $(k\psi)(z/\sqrt{k})$ and the vector bundle metric $h_{\alpha\beta}(z/\sqrt{k})$.

From (14), we can see that $\Box^{(k)}$ is associated to

$$\sum \mu |z_j|^2 \otimes \delta_{\alpha\beta} + o(1),$$

hence converges to a $k$-independent elliptic operator. Hereafter, by almost the same argument as in [3], one can complete the proof of Theorem 1.3.

6. Proof of Theorem 1.4

Now let $\mu : \tilde{X} \to X$, $c$, $\lambda_j$ and $D_j$ be as in (II) of Section 2.2. For any Hermitian vector bundle $(F, h_F)$ on $X$, we set $(\tilde{F}, h_{\tilde{F}}) := (\mu^* F, \mu^* h_F)$.

From Remark 2.2, we have that

$$\tilde{\varphi} := \mu^* \varphi = \varphi \circ \mu = c \log |s_D| + \tilde{\psi},$$

where $s_D$ is the canonical section of $O_{\tilde{X}}(-D)$ with $D = \sum_j \lambda_j D_j$, and $\tilde{\psi}$ is a smooth potential.

From (I) in Section 2.2, we have

$$I(h_{\tilde{L}^p}) = O_{\tilde{X}}\left(-\sum_j \left\lfloor c \lambda_j p \right\rfloor D_j\right).$$

The main idea is to take advantage of the fact that $I(h_{\tilde{L}^p})$ is invertible and we write $\tilde{L}^p \otimes I(h_{\tilde{L}^p})$ as a tensor power of a fixed line bundle.

Fix $r \in \mathbb{N}$, $m \in \mathbb{N}^+$ such that $c = r/m$. Set $\tilde{D} = rD = mc \sum_j \lambda_j D_j$, $\tilde{L} = \tilde{L}^m \otimes O_{\tilde{X}}(-\tilde{D})$.

Let $h_{O_{\tilde{X}}(-\tilde{D})}$ be the singular Hermitian metric on $O_{\tilde{X}}(-\tilde{D})$ given by $|s_D|^{2r}$ locally. Then the local potential of $h_{O_{\tilde{X}}(-\tilde{D})}$ is $-r \log |s_D|$. Let $h_{\tilde{L}} = h_{\tilde{L}^m} \otimes h_{O_{\tilde{X}}(-\tilde{D})}$ be the metric on $\tilde{L}$ induced by $h_{\tilde{L}^m}$ and $h_{O_{\tilde{X}}(-\tilde{D})}$. 

It is easy to see that the metric $\hat{h}_{L}$ is smooth on $\tilde{X}$, and $i\Theta_{\hat{h}_{L}} \geq 0$. In fact, the local weight $\hat{\varphi}$ of $\hat{h}_{L}$ is
\[
\hat{\varphi} = m\varphi - r \log |s_D|
= m(\varphi - c \log |s_D|)
= m\hat{\psi}.
\]
By taking $\frac{i}{\pi} \partial \bar{\partial}$, we get that
\[
\frac{i}{\pi} \partial \bar{\partial} \hat{\varphi} = m \frac{i}{\pi} \partial \bar{\partial} \hat{\psi}
= m \frac{i}{\pi} \partial \bar{\partial} (\varphi - c \log |s_D|)
= m (i\Theta_{\hat{h}_{L}} - [D])
= m\beta \geq 0,
\]
where the last equality follows from Remark 2.2. That is to say, $\hat{L}$ can be equipped with a smooth Hermitian metric with semi-positive curvature.

We observe that for $p' \in \mathbb{N}$,
\[
I(h_{\tilde{L}^{m'}}) = O_X(-p'\tilde{D}), \quad \tilde{L}' = \tilde{L}^{m'} \otimes I(h_{\tilde{L}^{m'}}).
\]
Write $p = p'm + m'$ (where $c = r/m$ as above, $0 \leq m' < m, p', m' \in \mathbb{N}$; then $[c\lambda_j p'] = r\lambda_j p' + [c\lambda_j m']$.

Now we want to prove that for $p$ sufficiently large,
\[
\dim_{\mathbb{C}} H^{n,q}(\tilde{X}, \tilde{L}^p \otimes \tilde{E} \otimes I(h_{\tilde{L}^{m'}})) \leq C(p')^{n-q} \leq C p^{n-q},
\]
where the constant $C_0$ is independent of $p$.

(1) $m' = 0$, i.e. $p = mp'$. Since the metric on $\tilde{L}$ is semipositive, then from Theorem 1.3 we have that for $P'$ sufficiently large
\[
\dim_{\mathbb{C}} H^{n,q}(\tilde{X}, \tilde{L}^p \otimes \tilde{E} \otimes I(h_{\tilde{L}^{m'}})) = \dim_{\mathbb{C}} H^{n,q}(\tilde{X}, \tilde{L}' \otimes \tilde{E})
\leq C_0(p')^{n-q} \leq C_0 p^{n-q},
\]
where $\tilde{E}_m = \tilde{L}^{m'} \otimes \tilde{E} \otimes O_X(-\sum_j [c\lambda_j m' D_j]).$

Then we get that
\[
\tilde{L}^{p'} \otimes \tilde{E} \otimes O_X(-\sum_j [c\lambda_j p' D_j]) = \tilde{L}' \otimes \tilde{E}_{m'}.
\]
Then since $\tilde{E}_{m'}$ is now a holomorphic line bundle, one can take a smooth Hermitian metric on $\tilde{E}_{m'}$, and by applying (15), we can get that for $p'$ sufficiently large and
\[
\dim_\mathbb{C} H^{n,q}(\tilde{X}, \tilde{L}^{p'} \otimes \tilde{E} \otimes I(h_{\tilde{L}^{p'}})) = \dim_\mathbb{C} H^{n,q}(\tilde{X}, \tilde{L}^{p'} \otimes \tilde{E}^{\prime} \otimes I(h_{\tilde{L}^{p'}})) \leq C_m(p')^{n-q} \leq C_m' p_m'^{n-q},
\]
where the constant $C_m'$ is independent of $p_m'$.

(3) To sum up, from the above $m$ cases, one can conclude that for $p'$ sufficiently large, the following estimate holds:
\[
\dim_\mathbb{C} H^{n,q}(\tilde{X}, \tilde{L}^{p} \otimes \tilde{E} \otimes I(h_{\tilde{L}^{p}})) \leq C p^{n-q},
\]
where the constant $C$ is independent of $p$.

Substituting $\tilde{E}$ by $\tilde{E} \otimes K_{\tilde{X}}^*$, we get that
\[
\dim_\mathbb{C} H^{n,q}(\tilde{X}, \tilde{L}^{p} \otimes \tilde{E} \otimes I(h_{\tilde{L}^{p}})) \leq C p^{n-q}.
\]

We now apply Lemma 2.2 to each step of the blowing-up of $\mathcal{I}$ performed in the modification $\mu$ in the assumption. In doing so, we replace $E$ by $E \otimes K_{\tilde{X}}^*$.

The hypothesis in Lemma 2.2 is satisfied, since our local weight $\varphi$ has analytic singularities and the centers of the blow-ups are included in the singular locus of the metric.

Thus we can apply Lemma 2.2 finitely many times and we get that for all $q \geq 0$, and $p$ large enough,

\[
H^q(X, L^p \otimes E \otimes I(h_L^p)) \simeq H^q(\tilde{X}, \tilde{L}^{p} \otimes \tilde{E} \otimes K_{\tilde{X}} \otimes K_{\tilde{X}}^* \otimes I(h_{\tilde{L}^{p}})).
\]

Then applying the above proof to the right-hand side of (16), we complete the proof of Theorem 1.4.

7. A partial solution to Question 1.1

Denote by $V_k$ the support of the multiplier ideal sheaf $\mathcal{O}_X/I(h_L^k)$. From the short exact sequence
\[
0 \to \mathcal{O}_X(E \otimes L^k) \otimes I(h_L^k) \to \mathcal{O}_X(E \otimes L^k) \to \mathcal{O}_{V_k}(E \otimes L^k) \to 0,
\]
we can get a long exact sequence
\[
\cdots \to H^q(X, \mathcal{O}_X(E \otimes L^k) \otimes I(h_L^k)) \to H^q(X, \mathcal{O}_X(E \otimes L^k)) \to H^q(V_k, \mathcal{O}_{V_k}(E \otimes L^k)) \to H^{q+1}(X, \mathcal{O}_X(E \otimes L^k) \otimes I(h_L^k)) \to \cdots.
\]

Since $h_L$ is a singular metric with analytic singularities, from Lemma 2.1 (or by the strong Noetherian property of the ideal sheaf $\mathcal{O}_X/I(h_L^k)$), we know that for sufficiently large $k$, $V_k$ is stationary, which is just the singular locus of the metric $h_L$. 
It follows from Theorem 1.4 that
\begin{equation}
\dim H^q(X, \mathcal{O}_X(E \otimes L^k) \otimes \mathcal{I}(h_L^k)) \leq C k^{n-q}.
\end{equation}

(18)

Suppose that the dimension of \( V_k \) for \( k \) large is \( m \). We have that
\begin{equation}
\dim H^q(V_k, \mathcal{O}_{V_k}(E \otimes L)) = 0, \text{ for } q > m.
\end{equation}

(19)

By combining (17), (18) and (19), we obtain that
\begin{equation}
\dim H^q(X, \mathcal{O}_X(E \otimes L^k)) \leq C k^{n-q}, \text{ for } q > m.
\end{equation}

(20)

In conclusion, we get the following

**Theorem 7.1.** Let \( X \) be a compact complex manifold, \( E \to X \) be a holomorphic vector bundle over \( X \), and \( L \to X \) be a holomorphic line bundle with a singular Hermitian metric \( h_L \) with algebraic singularities such that the curvature current of \( h_L \) is semi-positive. Assume that the dimension of the singular locus of \( h_L \) is \( m \). Then for \( q > m \), we have that
\begin{equation}
\dim H^q(X, \mathcal{O}_X(E \otimes L^k)) \leq C k^{n-q}.
\end{equation}

**Remark 7.1.** Theorem 7.1 is a partial answer to Question 1.1.

The assumption of algebraic singularities can be weakened. In fact, we have the following

**Theorem 7.2.** Let \( X \) be a compact complex manifold, \( E \to X \) be a holomorphic vector bundle over \( X \), and \( L \to X \) be a holomorphic line bundle with a singular Hermitian metric \( h_L \) with analytic singularities such that the curvature current of \( h_L \) is semi-positive. Let \( h \) be an arbitrarily smooth Hermitian metric of \( L \), and set \( e^{-\psi} = h_L/h \). Suppose that there is a small \( \varepsilon > 0 \), such that \( he^{-(1+\delta)\psi} \) are singular metrics of \( L \) with semi-positive curvature current for \( |\delta| < \varepsilon \). Assume that the dimension of the singular locus of \( h_L \) is \( m \). Then for \( q > m \), we have that
\begin{equation}
\dim H^q(X, \mathcal{O}_X(E \otimes L^k)) \leq C k^{n-q}.
\end{equation}

To prove Theorem 7.2, we need the following Diophantine approximation theorem due to Émile Borel.

**Lemma 7.3.** For every irrational number \( c \), there are infinitely many fractions \( \frac{p}{q} \), such that
\begin{equation}
\left| c - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.
\end{equation}

(20)
Proof of Theorem 7.2. Since $h_L$ is with analytic singularities, we have that, locally, $\psi$ can be written

$$\psi \equiv \frac{c}{2} \log \left( \sum_{j=1}^{N} |f_j|^2 \right) \mod C^\infty.$$ 

For any $(p, q)$ satisfies (20), we have

$$\left| 1 - \frac{1}{c \frac{p}{q}} \right| \leq \frac{1}{\sqrt{5}cq^2}.$$ 

Take sufficiently large $(p, q)$ such that $\frac{p}{q}$ satisfies (20) and $\frac{1}{\sqrt{5}cq^2} < \varepsilon$. Set $c_{p,q} = \frac{1}{c \frac{p}{q}}$, then $|c_{p,q} - 1| < \varepsilon$. From the assumption, we have that $h_{p,q} := he^{-c_{p,q}\psi}$ is a singular metric of $L$ with algebraic singularities such that the curvature current is semi-positive. In fact,

$$h_{p,q} = he^{-(1+(c_{p,q}-1))\psi} = he^{-(1+\delta)\psi}, |\delta| < \varepsilon,$$

$$c_{p,q}\psi \equiv \frac{p/q}{2} \log \left( \sum_{j=1}^{N} |f_j|^2 \right) \mod C^\infty.$$ 

But the singular locus of $h_{p,q}$ is exactly the same as the one of $h_L$. By applying Theorem 7.1, we can complete the proof of Theorem 7.2.

□

Remark 7.2. From the proof, we can see that the $\varepsilon$ in the assumption of Theorem 7.2 can be chosen to be arbitrarily small.

Remark 7.3. If $(L, h_L) \rightarrow X$ is a singular metric with analytic singularities such that the curvature current is semi-positive in the sense of current and the singular locus of $h_L$ are isolated points, then we can see that $L$ admits a smooth Hermitian metric with semi-positive curvature. From this, we can conclude that if $L$ satisfies the assumption in Theorem 7.1 and furthermore $L$ is not semi-positive, then the dimension of the singular locus is positive.

Remark 7.4. A pseudo-effective line bundle $L$ is nef if there is a singular metric on $L$ with semi-positive curvature current such that the Lelong number of the local potential is zero everywhere. More precisely, the necessary and sufficient condition for a pseudo-effective line bundle to be nef is characterized in [38, 39] by Păun.

We want to mention that one may not hope that for every nef line bundle $L$, there exists a singular metric $h$ on $L$ with semi-positive curvature current, such that the Lelong number of the local potential of $h$ is everywhere zero. Actually, it is closely related to the so called non-Kähler locus or non-nef locus which was systematically studied in [8] and [11].
To finish this section, we mention an example of Demailly-Peternelle-Schneider in [20] as a supplement of Remark 7.4.

Let $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\text{Im}\tau > 0$, be an elliptic curve and let $E$ be the rank 2 vector bundle over $\Gamma$ defined by

$$E = \mathbb{C} \times \mathbb{C}^2 / (\mathbb{Z} + \mathbb{Z}\tau)$$

where the action of $\mathbb{Z} + \mathbb{Z}\tau$ is given by the two automorphisms

$$g_1(x, z_1, z_2) = (x + 1, z_1, z_2);$$
$$g_\tau(x, z_1, z_2) = (x + \tau, z_1 + z_2, z_2),$$

where the projection $E \rightarrow \Gamma$ is induced by the first projection $(x, z_1, z_2) \mapsto x$. Then $\mathbb{C} \times \mathbb{C} \times \{0\} / (\mathbb{Z} + \mathbb{Z}\tau)$ is a trivial line subbundle $\mathcal{O} \hookrightarrow E$, and the quotient $E / \mathcal{O} \cong \Gamma \times \{0\} \times \mathbb{C}$ is also trivial.

Let $L$ be the line bundle $L = \mathcal{O}_E(1)$ over the ruled surface $X = \mathbb{P}(E)$. From the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{O} \rightarrow 0,$$

it is shown in [20] that $L$ is nef over $X$.

Moreover, the only possible metric $h$ of $L$ with semi-positive curvature is shown to be a singular metric with analytic singularities and moreover $\frac{k}{\pi} \Theta_{L, h} = [C]$, where $[C]$ is the current of integration over a curve $C$. For detailed computations, the reader is referred to see [20, Example 1.7].

8. TWO VANISHING THEOREMS

**Definition 8.1** (Kodaira-Iitaka dimension of a line bundle). For a holomorphic line bundle over a compact complex manifold $X$, the Kodaira-Iitaka dimension of $L$ is defined to be

$$\kappa(L) := \limsup_{k \to +\infty} \frac{\log \dim \mathcal{H}^0(X, L^k)}{\log k}.$$  

It is worth to mention that for any compact complex manifold $X$ and a holomorphic line bundle $L$ over $X$, if the Kodaira-Iitaka dimension of $L$ is non-negative, then there is a singular metric $h_L$ with analytic singularities on $L$ such that the curvature current is semi-positive in the sense of current.

Moreover, by using sections of tensor powers $L^k$ of $L$, one can define Siu’s metric as follows: for a basis $\{s_j^k\}_{j=1}^{N_k}$ of $\mathcal{H}^0(X, L^k)$, we define a metric $\varphi_k$ by

$$\varphi_k := \frac{1}{2k} \log \sum_{j=1}^{N_k} |s_j^k|^2.$$  

(21)
Taking a convergent series \( \{\varepsilon_k\}_{k=1}^{\infty} \), one can define a metric \( h_{\text{suz}} \) on \( L \) whose local weight is equal to \( \log \sum_{k=1}^{\infty} \varepsilon_k e^{\phi_k} \). This type of metric is called Siu’s metric which was first introduced by Siu and plays important role in [47].

Siu’s metric \( h_{\text{suz}} \) and the associated multiplier ideal sheaf \( I(h_{\text{suz}}) \) depend on the choice of \( \{\varepsilon_k\}_{k=1}^{\infty} \), but \( h_{\text{suz}} \) always admits an analytic Zariski decomposition, i.e. \( H^0(X, L^k) = H^0(X, L^k \otimes I(h_{\text{suz}}^k)) \).

**Theorem 8.1.** Let \( X \) be a compact Kähler manifold and \( L \) be a holomorphic line bundle over \( X \). Suppose that \( L \) is pseudo-effective, and the singular metric \( h_{\text{min}} \) with minimal singularities of \( L \) is with algebraic singularities. Then we have that

\[
H^q(X, O_X(K_X \otimes L) \otimes I(h_{\text{min}})) = 0 \quad \text{for} \quad q > n - \kappa(L).
\]

To prove the above Theorem, we need the following Theorem which is a consequence of injectivity theorem.

**Theorem 8.2** ([33, Corollary 3.3]). Let \( (L, h_L) \) and \( (M, h_M) \) be line bundles with singular metrics on a compact Kähler manifold \( X \). Assume the following conditions:

- There exists a subvariety \( Z \) on \( X \) such that \( h_L \) and \( h_M \) are smooth on \( X \setminus Z \).
- \( \sqrt{-1} \Theta_{h_L}(L) \geq \gamma \) and \( \sqrt{-1} \Theta_{h_M}(M) \geq \gamma \) for some smooth \((1, 1)\)-form \( \gamma \) on \( X \).
- \( \sqrt{-1} \Theta_{h_L}(L) \geq 0 \) on \( X \setminus Z \).
- \( \sqrt{-1} \Theta_{h_L}(L) \geq \varepsilon \sqrt{-1} \Theta_{h_M}(M) \) on \( X \setminus Z \) for some positive number \( \varepsilon > 0 \).

Assume that \( h^q(X, O_X(K_X \otimes L) \otimes I(h_L)) \) is not zero. Then we have

\[
\dim H^0_{\text{bdd}, h_M}(X, M) \leq h^q(X, O_X(K_X \otimes L \otimes M) \otimes I(h_L h_M)),
\]

where \( H^0_{\text{bdd}, h_M}(X, M) \) is the space of sections of \( M \) with bounded norm

\[
H^0_{\text{bdd}, h_M}(X, M) := \{ s \in H^0(X, M) | \sup_X |s|_{h_M} < +\infty \}.
\]

**Proof of Theorem 8.1.** Suppose to the contrary, we assume that \( h^q(X, O_X(K_X \otimes L) \otimes I(h_{\text{min}})) \) for \( q > n - \kappa(L) \) is not zero.

Since \( h_{\text{min}} \) is of minimal singularities, it admits an analytic Zariski decomposition, which means that

\[
h^0(X, L^{k-1}) = h^0_{\text{bdd}, h_{\text{min}}^{k-1}}(X, O_X(L^{k-1})) \leq h^q(X, O_X(K_X \otimes L^k) \otimes I(h_{\text{min}}^k)),
\]

where the equality follows from the property that \( h_{\text{min}} \) is a singular metric with minimal singularities and the inequality follows from Theorem 8.2.
By the definition of Kodaira-Iitaka dimension $\kappa(L)$, we have that
\[
\limsup_{k \to +\infty} \frac{h^0(X, L^k)}{(k-1)^{\kappa(L)}} > 0.
\]

On the other hand, by Theorem 1.4, we have $h^q(X, \mathcal{O}_X(K_X \otimes L^k \otimes \mathcal{I}(h_{\min}^k)) = O(k^{n-q})$ as letting $k$ go to infinity. It is a contradiction to the inequality $> n - \kappa(L)$.

**Remark 8.1.** The injectivity theorem used in the proof of Theorem 8.2 has been already proved for arbitrary singular metrics in [36].

**Remark 8.2.** Theorem 8.1 is a generalization of Theorem 1.4 (1) in [33] from the case of smooth projective manifold to compact Kähler manifold under the assumption that the singular metric with minimal singularities on $L$ is with algebraic singularities.

**Remark 8.3.** Metrics with minimal singularities do not always have algebraic singularities (see [33] and reference therein).

By the same argument as in the proof of Theorem 8.1 we can obtain the following

**Theorem 8.3.** Let $X$ be a compact Kähler manifold and $L$ be a holomorphic line bundle with non-negative Kodaira-Iitaka dimension over $X$. Suppose that $L$ is pseudo-effective and the Siu’s metric $h_{\text{siu}}$ of $L$ is with algebraic singularities. Then we have that
\[
H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_{\text{siu}})) = 0 \quad \text{for} \quad q > n - \kappa(L).
\]

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