LRS Bianchi type I universes exhibiting Noether symmetry in the scalar-tensor Brans-Dicke theory

Y. Kucukakca · U. Camci · İ. Semiz

Received: date / Accepted: date

Abstract Following up on hints of anisotropy in the cosmic microwave background radiation (CMB) data, we investigate locally rotational symmetric (LRS) Bianchi type I spacetimes with non-minimally coupled scalar fields. To single out potentially more interesting solutions, we search for Noether symmetry in this system. We then specialize to the Brans-Dicke (BD) field in such a way that the Lagrangian becomes degenerate (nontrivial) and solve the equations for Noether symmetry and the potential that allows it. Then we find the exact solutions of the equations of motion in terms of three parameters and an arbitrary function. We illustrate with families of examples designed to be generalizations of the well-known power-expansion, exponential expansion and Big Rip models in the Friedmann-Robertson-Walker (FRW) framework. The solutions display surprising variation, a large subset of which features late-time acceleration as is usually ascribed to dark energy (phantom or quintessence), and is consistent with observational data.

Keywords Bianchi type I spacetime; Noether symmetry; Brans-Dicke theory.

1 Introduction

The recent decade has witnessed the observational evidence for the cosmic acceleration of the universe, which has become the central theme of modern...
cosmology. This evidence is built up of observations of supernovae Type Ia (SNe Ia) [1], cosmic microwave background radiation (CMB) [2], and large-scale structure of the Universe [3]. In Einstein’s gravity, this acceleration cannot be explained by normal matter or fields on Friedmann-Robertson-Walker (FRW) metric background; therefore a mysterious cosmic fluid with negative pressure, the so called dark energy, is introduced. Presently, understanding the nature of the dark energy is one of the main problems in the research area of both theoretical physics and cosmology. A simple example of dark energy is the cosmological constant, equivalent to a fluid with the equation of state (EoS) parameter \( w = -1 \), where \( w = p/\rho \) in which \( p \) is the pressure of dark energy, and \( \rho \) its energy density. However, the cosmological constant model is subject to the so-called fine-tuning and coincidence problems [3].

Dark energy, if it is a perfect fluid, must have EoS parameter \( w < -1/3 \). If \(-1/3 > w > -1\), the dark energy is called the quintessence [5]; if \( w < -1 \), dark energy is dubbed the phantom fluid [6]. In spite of the fact that these models violate both the strong energy condition \( \rho + 3p > 0 \) and the dominant energy condition \( p + \rho > 0 \), and therefore may be physically considered undesirable, the phantom energy is found to be compatible with current data from SNe Ia observations, CMB anisotropy and the Sloan Digital Sky Survey (SDSS) [1,2,3].

Dark energy behavior can be exhibited by possible new fundamental fields acting on cosmological scales. The simplest such model is the single component scalar field. Other alternatives in the literature include the Cardassian expansion scenario [7], the tachyon [8], the quintom [9], the k-essence [10] models and more.

Alternatively, modifications of Einstein’s gravity have been proposed to explain the cosmic acceleration of the universe; among them \( f(R) \) theories [11]. For example, quintessential behavior of the parameter \( w \) can be achieved in a geometrical way in higher order theories of gravity [12]. Scalar fields with various couplings and potentials can be put in by hand, or follow from the model naturally.

One of the simplest modification of Einstein’s gravity is the Brans-Dicke (BD) gravity theory [13], a well known example of a scalar-tensor theory which represents the gravitational interaction using a scalar field in addition to the metric field. This theory is parametrized by one extra constant parameter, \( W \). In the limit \( W \to \infty \), the BD theory reduces to Einstein’s gravity [14]. The conditions, where the dynamics of a self-interacting BD field can account for the accelerated expansion, have been considered in Ref. [15], where it was concluded that accelerated expanding solutions can be obtained with a quadratic self-coupling of the BD field and a negative equation of state (EoS) parameter. Astrophysical data indicate that EoS parameter \( w \) lies in an interval of negative values roughly centered around \(-1\).

The majority of popular cosmological models, including all the ones referred to above, use the cosmological principle, that is, they assume that the universe is homogeneous and isotropic. On the other hand, there are hints in the CMB temperature anisotropy studies that suggest that the assumption of
statistical isotropy is broken on the largest angular scales, leading to some intriguing anomalies [17]. To provide predictions for the CMB anisotropies, one may consider the homogeneous but anisotropic cosmologies known as Bianchi type spacetimes, which include the isotropic and homogeneous FRW models [18]. In this study we consider the simplest of these, the locally rotationally symmetric (LRS) Bianchi type I spacetime as an anisotropic background universe model; note that this spacetime is a generalization of flat \((k = 0)\) FRW metric. Our aim is to investigate the solutions of the field equations of scalar-tensor, in particular, BD theory of gravity for the LRS Bianchi type I spacetime using the Noether symmetry approach.

The Noether symmetry approach was introduced by De Ritis et al. [19,20] and Capozziello et al. [21,22] to find preferred solutions of the field equations and the dynamical conserved quantity. The Noether theorem states that if the Lie derivative of a given Lagrangian
\[
\mathcal{L}_X \mathcal{L} = 0.
\]
then \(X\) is a symmetry for the dynamics, and it generates a conserved current. Recently some exact solutions have been presented in the scalar tensor theories following the Noether symmetry approach that allows the potential to be chosen dynamically, restricting the arbitrariness in a suitable way [19,20,21,22,23,24,25,26,27,28,29,30].

This paper is organized as follows. In the section 2, we present the field equations in scalar-tensor theory for the LRS Bianchi type I spacetime. In section 3, we search the Noether symmetry of the Lagrangian of scalar-tensor theory for the LRS Bianchi type I spacetime, and for the BD case, find new variables that include the cyclic one. In section 4 we derive the EoS of the BD field. In section 5 we give the solutions of the field equations by using new variables obtained in section 3. In section 6 we choose some examples corresponding to the well-known solutions in the isotropic cases such as power law, exponentially expanding and phantom/Big Rip models. We also make a conformal transform to the so-called Einstein frame in Section 7 and discuss the problem in that context. Finally, in section 8 we conclude with a brief summary.

2 The Lagrangian and the field equations

The general form of the action that involves gravity non-minimally coupled with a scalar field is given by [25], such that
\[
\mathcal{A} = \int d^4x \sqrt{-g} \left[ F(\Phi) R - \frac{W(\Phi)}{\Phi} \Phi, \Phi^c - U(\Phi) \right].
\]
Here \(R\) is the Ricci scalar, \(F(\Phi)\) and \(W(\Phi)\) are generic functions that describe the coupling, \(U(\Phi)\) is the potential for the scalar field \(\Phi\), and \(\Phi_a \equiv \Phi, A\) stand for the components of the gradient of the scalar field. Note that we use Planck
units. For $F(\Phi) = 1/2$ and $W(\Phi) = \Phi/2$, the action reduces to the form of Einstein-Hilbert action minimally coupled with a scalar field. The choice of $F(\Phi)$ and $W(\Phi)$ give us other gravity theories such as the BD theory, for which $F(\Phi) = \Phi$ and $W(\Phi) = \text{constant}$.

Variation of the general form of the action with respect to metric tensor yields the field equations

$$F(\Phi)G_{ab} = T^\phi_{ab}$$

where $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ is the Einstein tensor,

$$T^\phi_{ab} = \frac{W}{\Phi} \phi_a \phi_b - \frac{W}{2\Phi} g_{ab} \phi_c \phi^c - \frac{1}{2} g_{ab} U(\Phi) - g_{ab} \Box F(\Phi) + F(\Phi)_{,ab}$$

is the energy-momentum tensor of scalar field, and $\Box$ is the d’Alembert operator. It is clear that $T^\phi_{ab}$ includes the contributions from the non-minimal coupling and the scalar field parts of the action varying with respect to metric. In Ref.[31], it has been discussed that there are three possible and inequivalent ways of writing the field equations, corresponding to the ambiguity in the definition of the energy-momentum tensor of non-minimally coupled scalar field.

The variation with respect to $\Phi$ gives rise to the generalized Klein-Gordon equation governing the dynamics of the scalar field

$$2\frac{W}{\Phi} \Box \Phi + RF'(\Phi) + \left( \frac{W'}{\Phi} - \frac{W}{\Phi^2} \right) \phi_c \phi^c - U'(\Phi) = 0,$$

where the prime indicates the derivative with respect to $\Phi$. Note that this equation is equivalent to the contracted Bianchi identity. It follows from $T_{ab}^{\phi} = 0$ that using Eq.(3) together with Eq. (4), yields

$$\left[ \frac{1}{F} T^b_{a,b} \right] = 0 \Leftrightarrow T^b_{a,b} = - \frac{F_{,b}}{F} T^b_{a}$$

From (4), we compute $T^b_{a,b}$ to get

$$T^b_{a,b} = \Phi_a \left[ \left( \frac{W'}{2\Phi} \right) \phi_c \phi^c + \frac{W}{\Phi} \phi_c \phi^c - \frac{1}{2} U'(\Phi) + F_{,b} R^b_{a} \right]$$

Using Eq.(4), Eq.(7) and $F_{,b} = F'(\Phi)\phi_b$ in Eq.(8), we have

$$\Phi_a \left[ 2\frac{W}{\Phi} \Box \Phi + RF'(\Phi) + \left( \frac{W'}{\Phi} - \frac{W}{\Phi^2} \right) \phi_c \phi^c - U'(\Phi) \right] = 0.$$  

This is obviously the generalized Klein-Gordon equation for $\Phi_a \neq 0$.

The line element of the LRS Bianchi type I spacetime has the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 (dy^2 + dz^2),$$
Noether-symmetric LRS Bianchi type I universes in scalar-tensor BD theory

describing an anisotropic universe with equal expansion rate in two of the three dimensions. The Ricci scalar of this spacetime is

\[ R = 2 \left[ \frac{\dot{A}}{A} + 2 \frac{\ddot{B}}{B^2} + 2 \frac{\dot{A} \dot{B}}{AB} \right] , \]  

(10)

where the dot represents differentiation with respect to \( t \). For the metric (9), the field and generalized Klein-Gordon equations can be obtained from Eqs.(3) and (5) respectively

\[ \frac{\ddot{B}}{B^2} + 2 \frac{\dot{A} \dot{B}}{AB} + \frac{F'}{F} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) \phi - \frac{1}{2F} \left[ W \phi^2 + U \right] = 0 , \]  

(11)

\[ 2 \frac{\ddot{B}}{B^2} + \frac{\ddot{A}}{A} + \frac{2F'}{F} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) \phi - \frac{1}{2F} \left[ \frac{W}{\phi} + 2F'' \right] \phi^2 - \frac{U}{2F} = 0 , \]  

(12)

\[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{A} \ddot{B}}{AB} + \frac{F'}{F} \left[ \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right] \phi - \frac{1}{2F} \left( \frac{W}{\phi} + 2F'' \right) \phi^2 - \frac{U}{2F} = 0 , \]  

(13)

\[ \frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B^2} + \frac{2\dot{A} \dot{B}}{AB} - \frac{W}{F' \phi} \left[ \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right] \phi - \frac{\dot{\phi}^2}{2F'} \left( \frac{W}{\phi} - \frac{W}{\phi^2} \right) - \frac{U'}{2F'} = 0 \]  

(14)

where \( F' \neq 0 \). The Lagrangian density of the LRS Bianchi type I spacetime is

\[ L = -2FAB^2 - 4FB\dot{A}B - 2F'B^2 \dot{A}B - 4F'AB\dot{B} \]  

\[ + AB^2 \left[ \frac{W}{\phi} \dot{\phi}^2 - U(\phi) \right] . \]  

(15)

Using such a Lagrangian, one may obtain the Euler-Lagrange equations as given in (12)-(14). The energy function, \( E_L \), associated with the Lagrangian (15) is found as

\[ E_L = \frac{\partial L}{\partial \dot{A}} \dot{A} + \frac{\partial L}{\partial \dot{B}} \dot{B} + \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} - L \]  

\[ = \frac{\ddot{B}}{B^2} + 2 \frac{\ddot{A} \ddot{B}}{AB} + \frac{F'}{F} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) \phi - \frac{1}{2F} \left[ W \phi^2 + U \right] . \]  

(16)

Therefore, it is obvious that the (0,0)-field equation given by (11) is equivalent to \( E_L = 0 \).
3 Noether symmetry approach

In this section we seek the Noether symmetry of the Lagrangian (15). The configuration space of this Lagrangian is \( Q = (A, B, \Phi, \dot{A}, \dot{B}, \dot{\Phi}) \), whose tangent space is \( TQ = (A, B, \Phi, \dot{A}, \dot{B}, \dot{\Phi}) \). The existence of Noether symmetry implies the existence of a vector field \( X \) such that

\[
X = \alpha \frac{\partial}{\partial A} + \beta \frac{\partial}{\partial B} + \gamma \frac{\partial}{\partial \Phi} + \dot{\alpha} \frac{\partial}{\partial \dot{A}} + \dot{\beta} \frac{\partial}{\partial \dot{B}} + \dot{\gamma} \frac{\partial}{\partial \dot{\Phi}} \tag{17}
\]

where \( \alpha, \beta \) and \( \gamma \) are depend on \( A, B \) and \( \Phi \). Hence the Noether equation given by (1) yields the following set of equations

\[
2 \frac{\partial \beta}{\partial A} + B \frac{F'}{F} \frac{\partial \gamma}{\partial A} = 0, \tag{18}
\]

\[
\frac{\alpha}{2} + B \frac{\partial \alpha}{\partial B} + A \frac{\partial \beta}{\partial B} + A \frac{F'}{F} \left( \frac{\gamma}{2} + B \frac{\partial \gamma}{\partial B} \right) = 0, \tag{19}
\]

\[
\frac{W}{\Phi} \left( \frac{\alpha}{2} + B \frac{\partial \gamma}{\partial \Phi} + A \frac{\partial \gamma}{\partial \Phi} \right) + A \left( \frac{W'}{\Phi} - \frac{W}{\Phi^2} \right) \gamma \]

\[= -F' \left( \frac{\partial \alpha}{\partial \Phi} + \frac{A \partial \beta}{B \partial \Phi} \right) = 0, \tag{20}
\]

\[
\beta + B \frac{\partial \beta}{\partial A} + A \frac{\partial \beta}{\partial A} + B \frac{\partial \gamma}{\partial \Phi} + B \frac{F'}{F} \left( \gamma + A \frac{\partial \gamma}{\partial A} + B \frac{\partial \gamma}{\partial B} \right) = 0, \tag{21}
\]

\[
2 \frac{\partial \beta}{\partial \Phi} + \frac{F'}{F} \left( 2 \beta + B \frac{\partial \alpha}{\partial A} + B \frac{\partial \gamma}{\partial \Phi} + 2 A \frac{\partial \beta}{\partial A} \right) \]

\[
+ \frac{F''}{F} B \gamma - \frac{W}{F \Phi} A B \frac{\partial \gamma}{\partial A} = 0, \tag{22}
\]

\[
\frac{\partial \alpha}{\partial \Phi} + A \frac{\partial \beta}{\partial \Phi} + \frac{F'}{F} \left( \frac{\alpha}{B} + A \frac{\beta}{B} + B \frac{\partial \alpha}{\partial B} + A \frac{\partial \beta}{\partial B} + A \frac{\partial \gamma}{\partial \Phi} \right) \]

\[
+ \frac{F''}{F} A \gamma - \frac{W AB}{2 F \Phi} \frac{\partial \gamma}{\partial B} = 0, \tag{23}
\]

\[
(B \alpha + 2 A \beta) U + A B \gamma U' = 0. \tag{24}
\]

Leaving these equations as reference for future work, we now specialize to the Brans-Dicke case, where \( F = \Phi \) and \( W \) is constant. We also require the Hessian determinant, \( D = \sum \left| \frac{\partial^2 L}{\partial Q_i \partial Q_j} \right| \), to vanish in order to get nontrivial solutions. This condition reads for our case as

\[
D = -\frac{16 A B^4 F}{\Phi} (3 \Phi F'^2 + 2 W F) = 0, \tag{25}
\]

which also determines a \( W \) value. So we will be working with the coupling functions

\[
F = \Phi, \quad W = -\frac{3}{2}. \tag{26}
\]
Thus the Lagrangian (15) becomes degenerate, and the BD action has the form

\[ A_{BD} = \int d^4x \sqrt{-g} \left[ \Phi R + \frac{3}{2\Phi} \Phi \varphi^2 - U(\Phi) \right] \] (27)

which can be easily related to the conformal relativity by defining new scalar field \( \varphi \) as \( \Phi = \varphi^2/12 \), transforming the action into

\[ A_{BD} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \frac{1}{6} \varphi^2 R + \varphi \varphi^2 - U(\varphi) \right]. \] (28)

This action is of course conformally invariant, since the application of the conformal transformation formulas together with the appropriate integration of the boundary term gives the same form of this action (see Refs. [14] and [27]).

Dabrowski et al. [27] have shown that the anisotropic non-zero spatial curvature models of Bianchi types I, III and Kantowski-Sachs type are admissible in \( W = -3/2 \) BD theory. They have given solutions of BD field equations for these spacetimes without the potential \( U(\Phi) \). In a previous work [28], we studied the case \( F = \frac{1}{12}(\Phi - \Phi_0)^2, W = \frac{\Phi}{2} \) for the Bianchi types I and III, and Kantowski-Sachs spacetimes, where \( \epsilon \) is a parameter depending on the signature of metric, but did not study the qualitative (acceleration etc.) behaviour of the universes in the solutions found.

In our present study we find the solutions of BD field equations with potential \( U(\Phi) \) for the LRS Bianchi type I spacetime, although we derive the equations for a general function \( W(\Phi) \). In this context, the solutions of the above set of differential equations (18)-(24) for \( \alpha, \beta, \gamma \) and potential \( U(\Phi) \) are obtained as

\[ \alpha = \left( AB\Phi^{3/2} \right)^{-1}, \quad \beta = \frac{1}{2} \left( A^2\Phi^{3/2} \right)^{-1}, \quad \gamma = - \left( A^2B\Phi^{1/2} \right)^{-1}, \quad U(\Phi) = \lambda \Phi^2 \] (29)

where \( \lambda \) is a constant. The potential found indicates that for the Noether symmetry to be present, the scalar field must be a (massive) free field. The vector field \( X \) generating the Noether symmetry and determining the dynamics of LRS Bianchi type I metric is given by

\[ X = \frac{1}{AB\Phi^{3/2}} \left[ \frac{\partial}{\partial A} + \frac{B}{A} \frac{\partial}{\partial B} - \frac{\Phi}{A} \frac{\partial}{\partial \Phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{3\dot{\Phi}}{2\Phi} \right) \frac{\partial}{\partial A} \right. \\
- \left. \left( \frac{\dot{A}}{A} + \frac{3\dot{\Phi}}{4\Phi} \right) \frac{\partial}{\partial B} + \left( \frac{2\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{1\dot{\Phi}}{2\Phi} \right) \frac{\partial}{\partial \Phi} \right] \] (30)
4 Equation of state

Using $F(\Phi) = \Phi$ and $W = -3/2$ in Eq. (11) we can derive the energy density of the scalar field

$$\rho^\Phi = -\frac{3}{4} \frac{\dot{\Phi}^2}{\Phi} + \frac{U(\Phi)}{2} - 3H\dot{\Phi}$$

and the directional pressure due to the scalar field

$$p^\Phi_i = -\frac{3}{4} \frac{\dot{\Phi}^2}{\Phi} - \frac{U(\Phi)}{2} + \ddot{\Phi} + (3H - H_i) \dot{\Phi}, \quad i = x, y, z$$

for the $x, y$ and $z$ directions. Here $H_i$ represents the directional Hubble parameters in the directions of $x, y$ and $z$ respectively, and may be defined as

$$H_x = \frac{\dot{A}}{A}, \quad H_y = H_z = \frac{\dot{B}}{B}.$$

The mean Hubble parameter for LRS Bianchi type I metric is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{3} (H_x + 2H_y),$$

where we define $a = (AB^2)^{1/3}$ as the average scale factor of the Universe. Also of interest is the volumetric deceleration parameter (for the isotropic case, simply the “deceleration parameter”), defined by

$$q = -\frac{\ddot{a}a}{a^2} \equiv -1 - \frac{H}{H^2}$$

so that it is dimensionless.

The pressure is a vectorial quantity, as would be expected for anisotropic expansion, and thus the EoS parameter of the scalar field for the LRS Bianchi type I spacetime may be determined separately on each spatial axis:

$$w_i^\Phi(t) = \frac{p_i^\Phi}{\rho^\Phi}, \quad i = x, y, z$$

and the average EoS parameter of scalar field for the LRS Bianchi type I spacetime may be defined as

$$w^\Phi = \frac{p^\Phi}{\rho^\Phi} = \frac{1}{3} (w_x + 2w_y) = \frac{3 \frac{\dot{\Phi}^2}{\Phi} + \frac{U(\Phi)}{2} - \ddot{\Phi} - 2H\dot{\Phi}}{\frac{3 \frac{\dot{\Phi}^2}{\Phi} + \frac{U(\Phi)}{2} + 3H\dot{\Phi}}{\rho^\Phi}}$$

where the $p^\Phi$ is the isotropic pressure.

In the anisotropic case, $w$ does not directly determine the sign of $q$ as it does in the well-known (spatially) flat FRW cosmologies. To understand this, consider the Raychaudhuri Equation

$$\dot{\theta} + \frac{1}{3} \theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} + \frac{\kappa}{2} (\rho + 3p) - \Lambda = 0$$

(38)
where $\theta$ is the expansion scalar, $\sigma_{\mu\nu}$ is the shear tensor, $\omega_{\mu\nu}$ is the vorticity tensor (descriptions/definitions also given in [33]) and we take $\Lambda = 0$. If comoving particles in a metric move geodesically, the expansion scalar also gives the expansion rate of the space itself, when we apply this equation to a swarm of such particles. This is the case for the LRS Bianchi I metric, where $\theta$ turns out to be equal to $3H$, therefore $\dot{\theta} + \frac{1}{3} \theta^2$ is $-3H^2q$. The vorticity tensor vanishes for this case, and

$$\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{2B(t)^2}{3A(t)^2} \left[ \frac{d}{dt} \left( \frac{A(t)}{B(t)} \right) \right]^2.$$  \hfill (39)

Therefore in the isotropic case $\sigma_{\mu\nu}\sigma^{\mu\nu}$ also vanishes and since in that case $\rho \propto H^2$, we find that $q$ is proportional to $1 + 3w$. This correspondence is obviously broken when there is anisotropy, and therefore $w$ is not that useful a parameter.

5 The solutions from new coordinates and Lagrangian

The solutions of dynamical Eqs. (11)-(14) are not easy to evaluate in the present form. In order to simplify these equations we can search for the cyclic variable(s). In case of Noether symmetry, we should introduce new variables instead of old variables, i.e. a point transformation $\{A, B, \Phi\} \rightarrow \{\mu, \nu, u\}$ in which it is assumed that $\mu$ is the cyclic coordinate. A general discussion of this procedure has been given in Ref. [34]. The new variables $\{\mu, \nu, u\}$ satisfy the following equations

$$i_X d\mu = 1, \quad i_X d\nu = 0, \quad i_X du = 0$$  \hfill (40)

where $i_X$ is the interior product operator of $X$. Using Eq. (29), a solution of the above Eqs. (40) yield

$$\mu = A^2 B \Phi^{3/2}, \quad \nu = B \Phi^{1/2}, \quad u = \Phi.$$  \hfill (41)

The inverse transformation of these variables are

$$A = \frac{\mu}{\nu u}, \quad B = \frac{(\mu \nu)^{1/2}}{u}, \quad \Phi = \frac{\nu u^2}{\mu},$$  \hfill (42)

from which it follows that

$$a(t) = \frac{\mu^{2/3}}{u}.$$  \hfill (43)

for the average scale factor of the Universe defined by $a = (AB^2)^{1/3}$. Considering the transformation of variables, the coupling functions [20] and the potential $U(\Phi)$ given in [29], the Lagrangian becomes

$$L = -2u^{-1}\dot{\mu}\dot{\nu} - \lambda u^2$$  \hfill (44)
which does not depend on $\mu$ (i.e., $\frac{\partial L}{\partial \dot{\mu}} = 0$), as desired. This independence is preserved if we make further transformations in the $u - \nu$ plane; and also is inherent in the procedure described in Ref.[34]. We used this freedom to get as simple a Lagrangian as possible.

The Euler-Lagrange equations relative to this Lagrangian are

$$\dot{\nu} - l_0 u = 0,$$  \hspace{1em} (45)

$$\dot{\mu} - \frac{\lambda}{2} \nu^2 u^2 = 0,$$  \hspace{1em} (46)

$$\frac{\ddot{\mu}}{\mu} - \frac{\dot{\mu} \dot{\nu}}{\mu u} - \frac{\lambda \nu^2}{\mu} = 0,$$  \hspace{1em} (47)

where $l_0$ is a constant of motion associated with the coordinate $\mu$. Here $i_X\Theta_L = a_0 = -2l_0$ and $\Theta_L$ is the Cartan one-form, and Eq. (46) is equivalent to the vanishing of the energy functional, as noted before.

By substituting $u$ from Eq. (45) into Eq. (46) we get

$$\mu(\dot{\nu} \nu^3 + l_2),$$  \hspace{1em} (48)

where $l_1 = \lambda/(6l_0^2)$, $l_0 \neq 0$, and $l_2$ is an integration constant. The remaining equation (47) is identically satisfied now. Inserting (45) and (48) into (42), we obtain the metric functions and scalar field in terms of the arbitrarily specifiable $\nu$ (henceforth to be called the seed function), and $u = \dot{\nu}/l_0$ as follows:

$$A(t) = l_0 \frac{l_1 \nu(t)^3 + l_2}{\nu(t) \dot{\nu}(t)},$$  \hspace{1em} (49)

$$B(t) = \frac{l_0}{\dot{\nu}(t)} \sqrt{\nu(t) (l_1 \nu(t)^3 + l_2)},$$  \hspace{1em} (50)

$$\Phi(t) = \frac{\nu(t) \dot{\nu}(t)^2}{l_0^2 (l_1 \nu(t)^3 + l_2)^2}.$$  \hspace{1em} (51)

Thus, using (49) and (50), the LRS Bianchi type I metric (9) becomes

$$ds^2 = -dt^2 + l_0^2 (l_1 \nu(t)^3 + l_2) \frac{\nu(t)}{\dot{\nu}(t)^2} \left[ \left( l_1 + \frac{l_2}{\nu(t)^3} \right) dx^2 + dy^2 + dz^2 \right].$$  \hspace{1em} (52)

Then, the average scale factor of the Universe takes the form

$$a(t) = \frac{l_0}{\dot{\nu}(t)} (l_1 \nu(t)^3 + l_2)^{2/3}.$$  \hspace{1em} (53)

Also the case of isotropy can be seen, by requiring $A \propto B$, to correspond to $l_2 = 0$.

In the general anisotropic case, the directional Hubble parameter in $x$ direction defined in Eq. (53) is found as

$$H_x = \frac{\ddot{\nu}}{\nu} - \frac{\dot{\nu}}{\nu} - \frac{\ddot{u}}{u} = -\frac{\ddot{\nu}}{\nu} + \frac{\ddot{\nu}}{\nu} \left[ \frac{3l_1 \nu^3}{l_1 \nu^3 + l_2} - 1 \right],$$  \hspace{1em} (54)
the other directional Hubble parameters in the directions $y$ and $z$ are similarly found as
\[ H_y = H_z = \frac{1}{2} \left( \frac{\dot{\mu}}{\mu} + \frac{\dot{\nu}}{\nu} \right) - \frac{\dot{\nu}}{u} = -\frac{\dot{\nu}}{\nu} \left[ 3l_1 \nu^2 + \frac{1}{l_1 \nu^3 + l_2} \right], \tag{55} \]
and the mean Hubble parameter defined in Eq.(34) is found as
\[ H = \frac{2}{3} \frac{\dot{\mu}}{\mu} - \frac{\dot{\nu}}{u} = -\frac{\dot{\nu}}{\nu} + 2l_1 \nu^2 \frac{\dot{\nu}}{l_1 \nu^3 + l_2}. \tag{56} \]

At this point, we would like to emphasize that particular solutions can be found, that is, the scale factors $A(t)$, $B(t)$, the potential $\Phi(t)$ and the average scale factor $a(t)$ can be obtained by specifying the function $\nu(t)$. In the next section, we will find such examples.

6 Examples

To illustrate our family of solutions for the dynamics of LRS Bianchi type I cosmologies containing a scalar field, we choose examples corresponding to the well-known solutions in the isotropic case, that is, power-law models, i.e. $a_{\text{isotropic}} \propto t^n$, exponentially expanding models, i.e. $a_{\text{isotropic}} \propto e^{\chi t}$, and phantom/Big Rip models, i.e. $a_{\text{isotropic}} \propto (t_c - t)^{-\sigma}$, where $\eta$, $\chi$ and $\sigma$ are positive constants. We can find the form of each $\nu(t)$ by solving from Eq.(53) after setting $l_2 = 0$.

The most general form we find for the power-law models is $\nu(t) = \frac{C_1}{l_1 a^3_{1} t^{3n} + l_2}$, but for simplicity we use $\nu(t) = a_1 t^n$. For the exponential case, the most general form is $\nu(t) = \frac{C_3}{l_1 e^{-\chi t} + l_2}$, but we use $\nu(t) = a_2 e^{k t}$. Similarly, for the phantom/Big Rip models, the most general form is $\nu(t) = \frac{C_5}{l_1 a_{1} t^{3n} + l_2}$, but we use $\nu(t) = \frac{a_3}{(t_c - t)^{m}}$ ($n$, $k$ or $m$ are not necessarily integer).

6.1 Power-law seed

When we use $\nu(t) = a_1 t^n$, where $a_1$ and $n$ are nonzero constants, the scale factors read
\[ A(t) = \frac{l_0}{a_1^n} 1^{1-2n} (l_1 a_1^3 t^{3n} + l_2), \quad B(t) = \frac{l_0}{n \sqrt{a_1}} 1^{1-n/2} [l_1 a_1^3 t^{3n} + l_2]^{1/2}, \tag{57} \]
the average scale factor is found from Eq.(55) as
\[ a(t) = \frac{l_0}{a_1^n} \left[ l_1 a_1^3 t^{3n} + l_2 \right]^{2/3} t^{1-n}, \tag{58} \]
the scalar field from Eq.(51) as
\[ \Phi(t) = \frac{a_1^3 n^2 l^{3n-2}}{l_0^2 (l_1 a_1^3 t^{3n} + l_2)}. \tag{59} \]
and the mean Hubble parameter from Eq. (34) as
\[ H = \frac{2l_1a_1^3n^3a_3n^{-1}}{l_4a_4^3b_3} + \frac{1 - n}{t}. \] (60)

Inspection of the scale factors \( A(t), B(t) \) and \( a(t) \) shows that if \( l_1 \) or \( l_2 \) is zero, they will reduce to single powers of \( t \). But also in the general case where both \( l_1 \) and \( l_2 \) are nonzero, the scale factors will approach one power of \( t \) near zero and another near infinities. These powers depends on the sign of \( n \): For negative \( n \), near \( t = 0 \) we have \( A(t), B(t), a(t) \propto t^{1 + n} \), near infinities we have \( A(t) \propto t^{1 - 2n}, B(t) \propto t^{1 - n/2} \) and \( a(t) \propto t^{1 - n} \); for positive \( n \), the behaviors near zero and infinities are interchanged. Therefore, we need to treat the cases of vanishing \( l_1 \) or \( l_2 \) separately from the general case. These cases will be split into subcases, for example, according to the value of \( n \), etc. Each line in the Tables 1, 2 and 3 shows the timeline of an eventual subcase from \( t \rightarrow -\infty \) to \( t \rightarrow +\infty \), which usually describes more than one universe, since the timeline may be interrupted by singularities, shown by expressions in square brackets in the lines of the tables. Since solutions cannot be taken valid through singularities; each interval between singularities should be considered an independent solution/universe. The universes not extending to the right or left edge of a line have finite lifetime.

Obvious candidates for singularities are \( t = 0 \) and \( t \rightarrow \pm \infty \), in light of the above discussion. Moreover, the \((l_1a_1^3n^3a_3n^{-1} + l_2)\) terms in the scale factors can vanish at some finite \( t \) value \( t_1 \), or diverge at \( t = 0 \), if both \( l_1 \) and \( l_2 \) are nonzero. The meaning of a singularity not only depends on its mathematical behavior, but also on its relative time-relation to the observers: If observers see (or calculate) vanishing comoving volume in their finite past or finite future, they will call that singularity a Big Bang (BB) or a Big Crunch (BC); This is shown as “BC[0]BB” in the tables. Similarly, observers calculating diverging of the comoving volume in their finite future will call that singularity a Big Rip (BR), and the case of diverging comoving volume in the finite past we will call it an Inverse Big Rip (iBR).

The square brackets should contain an ordered triple, where the first entry shows the behavior of \( A(t) \), the second entry the behavior of \( B(t) \) and the third, the behavior of \( a(t) \). The behavior is indicated by the symbols 0, C or \( \infty \) to denote that the relevant scale factor vanishes/goes to a finite number/diverges at that point. To make the tables more compact, an ordered pair (describing the behavior of \( A(t) \) and \( B(t) \)); whenever these make the behavior of \( a(t) \) obvious) or a single entry (indicating that the behavior of all three scale factors is the same) may be used. Finally, one should also note that negative \( t \) can only be considered if \( n \) is a rational number with an odd denominator, not for general \( n \).

We discuss the three above-mentioned cases in order of increasing complexity: First, we discuss the isotropic \( l_2 = 0 \) case, then the simple anisotropic \( l_1 = 0 \) case, and finally the general anisotropic case where both \( l_1 \) and \( l_2 \) are nonzero.
6.1.1 The $l_2 = 0$ (isotropic) case.

As stated after Eq. (53), in this case the LRS Bianchi type I spacetime becomes isotropic, i.e. reduces to the well-known spatially flat FRW metric. The scale factors become proportional to each other, and

$$a(t) = a_0 t^{1+n}$$

(61)

where $a_0 = a_1 l_0^{1/3}$. This means that $a(t)$ can vanish or diverge at zero or infinity, and the critical $n$ values are -1 and 0, leading to the subcases shown in Table 1. The cases are

| $n$ | $t$ | $-\infty$ | 0 | $+\infty$ |
|-----|-----|-----------|---|-----------|
| $n < -1$ | 0 | Z | BR | I |
| $n = -1$ | (C) | static | (C) | static |
| $-1 < n < 0$ | $+\infty$ | I "decelerating" | BB | BB |
| $n \to 0$ | $+\infty$ | I linear | BB | BB |
| $n > 0$ | $+\infty$ | I "accelerating" | BB | BB |

Table 1 Universes derived for $l_2 = 0$ (isotropic universes) where $\nu(t)$ is a power of $t$. Each line shows a timeline from $-\infty$ to $\infty$, but may represent multiple universes delimited by singularities, shown by square brackets. The expressions in the brackets refer to the scale factor $a(t)$. Also, I: infinite scale factor at infinite future or past, BR: Big Rip, i: Inverse, BC: Big Crunch, BB: Big Bang, Z: vanishing scale factor at infinite future or past [Also see text and the note about negative time on page 12].

- $n < -1$: Negative times describe a universe that expands from a vanishingly small scale factor in the infinite past, and then the scale factor diverges within a finite time: one might call this a Zero-Big Rip (Z-BR) universe. Positive times describe another universe, whose behavior is the time-reverse of the first, an iBR-Z universe.
- $n = -1$: This universe, being static and flat, is equivalent to Minkowski space.
- $-1 < n < 0$: For positive times, this case represents decelerating (spatially) flat FRW universes, including the radiation-dominated ($n = -1/2$) and matter-dominated ($n = -1/3$) expansions, in particular. For negative times, however, the universes contract from infinite scale factor in the infinite past to a Big Crunch at $t = 0$; so the universes represented on the last three lines of Table 1 may be called Infinity-Big Crunch (I-BC) and Big Bang-infinity (BB-I) universes. The I-BC universes are decelerating, too: The negative $\dot{a}(t)$ [for positive $a(t)$] becomes more negative with time.
- $n \to 0$: This case can be analyzed with the help of an infinite scale transformation [note that Eqs. (45)-(47) are satisfied for constant $\nu(t)$], and describes universes linearly expanding (positive time) or contracting (negative time) with increasing time.

1 subject to the condition on $n$ mentioned on page 12.
• $n > 0$: Positive times describe a universe that starts with a Big Bang and expands forever with increasing speed; negative times$^1$ describe a universe with time-reversed behavior.

As is well-known (and mentioned above), in FRW cosmology the behavior of the universe and the properties of the (possibly effective) fluid contained in the universe are related. In fact in our $l_2 = 0$ case the Hubble parameter, the deceleration and EoS parameters and the scalar field simplify to

$$H = \frac{n + 1}{t},$$  \hfill (62)

$$q = -\frac{n}{n + 1},$$  \hfill (63)

$$w = -\frac{3n + 1}{3(n + 1)},$$  \hfill (64)

$$\Phi(t) = \frac{6n^2}{\lambda}t^{-2}$$  \hfill (65)

and we can confirm that $q$ is proportional to $1 + 3w$; in particular, for $n = -1/2$ Eq.(64) gives $w = 1/3$ and for $n = -1/3$, it gives $w = 0$. For $n \to 0$, the deceleration parameter vanishes. We know that empty universes can have this property, and we see that the scalar field vanishes in this case. For $n > 0$, the effective fluid is equivalent to quintessence/quiescence ($-1 < w < -1/3$), approaching cosmological constant ($w = -1$) as $n \to \infty$. Only for $n < -1$ do we have phantom behavior ($w < -1$), and this is the case where a Big Rip appears (for negative times$^1$, at least).

The $n = -1$ case is particularly interesting. For this case, $q$ and $w$ seem to diverge, while in Table 1 this subcase does not seem to be more problematic than others. But, as $n$ approaches $-1$, the scale factor $a(t)$ approaches a constant function, therefore $\dot{a}(t)$ and $\ddot{a}(t)$ approach zero. Hence the deceleration parameter, which has division by $\dot{a}^2$ in its definition, is not a good parameter to use in this limit. Also, since this universe corresponds to Minkowski space, the stress-energy-momentum tensor should vanish. Let us calculate the energy density and isotropic pressure, using Eq.(65) and the potential expression in Eq.(29), in expressions (31) and (32):

$$\rho = \frac{18}{\lambda}n^2(n + 1)^2 t^{-4},$$  \hfill (66)

$$p = -\frac{18}{\lambda}n^2(n + 1)(n + \frac{1}{3}) t^{-4}.$$  \hfill (67)

These expressions verify that $n = 0$ corresponds to empty universes. It can also be seen that both the density and pressure vanish for $n = -1$, although the field does not vanish! Of course, this also makes $w$ meaningless in this subcase.
6.1.2 The $l_1 = 0$ case.

Since $l_1 = \lambda \left( \frac{6}{l_0^2} \right)$, in this case the potential $U(\Phi) = \lambda \Phi^2$ vanishes, therefore this case is equivalent considering the scalar field to be massless. Now $A(t) \propto t^{1-2n}$, $B(t) \propto t^{1-n/2}$ and $a(t) \propto t^{1-n}$, so that the critical $n$ values are $1/2$, $1$ and $2$; and we are led to Table 2 listing the subcases.

| $n$   | $t$                  | 0         | $+\infty$ |
|-------|----------------------|-----------|-----------|
| $n < 1/2$ | $[0, 0, 0]BB$     | $[\infty, \infty, \infty]$ |
| $n = 1/2$ | $[0, \infty, \infty]BB$ | $[0, 0, 0]BB$ |
| $n = 1/2$ | $[0, 0, 0]BB$     | $[0, \infty, \infty]BB$ |
| $n = 1/2$ | $[0, 0, 0]BB$     | $[0, 0, 0]BB$ |
| $n = 1$   | $[0, 0, 0]BB$      | $[0, 0, 0]BB$ |
| $n = 2$   | $[0, 0, 0]BB$      | $[0, 0, 0]BB$ |
| $n > 2$   | $[0, 0, 0]BB$      | $[0, 0, 0]BB$ |

Table 2 Universes derived for $l_1 = 0$, where $\nu(t)$ is a power of $t$. Each line shows a timeline from $-\infty$ to $\infty$, but may represent multiple universes delimited by singularities, shown by square brackets. The expressions in the brackets refer to $A(t)$, $B(t)$, and $a(t)$. Also, $I$: infinite scale factors at infinite future or past, BR: Big Rip, i: inverse, BC: Big Crunch, BB: Big Bang, 1 or 2: only one or two of the spatial dimensions, c: cigar-type, BD: Big Draw, IPa: "infinite pancake" of type a, etc., Z: vanishing scale factors at infinite future or past. (Also see text and the note about negative time on page 12.)

Let us note that

- For $n = 1/2$, we get universes evolving in the $y$ and $z$ dimensions only, either from infinite scale factors to a BC, or from a BB to infinite scale factors, in infinite time, possibly to be called 2I-2BC and 2BB-2I for two-dimensional
- For $1/2 < n < 1$, $A(t)$ diverges at $t = 0$, while the other scale factors vanish. This singularity is extremely anisotropic: As the universe contracts towards a BC in the $y$ and $z$ directions, it actually expands infinitely in the $x$ direction! But this expansion is not enough to prevent the vanishing of a comoving volume, and in this sense this event is a BB/BC. Singularities where one dimension diverges while two shrink to zero are called "cigar-type", so we might call this event a "cBC/cBB".

Again for $1/2 < n < 1$, $A(t)$ vanishes as $t \to \pm \infty$, while the other scale factors do not. This type of singularity is also extremely anisotropic, and is called a "pancake" singularity. In this particular subcase, the other scale factors diverge, so the singularity is called an infinite pancake. This type of singularity appears in the next two subcases too, but the behavior of the average scale factor $a(t)$ is different in each subcase. So we classified the "infinite pancakes" accordingly. (In [35], a classification is made according to the behavior of the three separate scale factors of a Bianchi I spacetime.)
- For $n = 1$, $A(t)$ diverges, $a(t)$ goes to a constant, $B(t)$ vanishes at $t = 0$. Although this $t = 0$ event is also a cigar-type singularity, since the comoving
volume neither vanishes nor diverges, it is not a BB/BC or BR/iBR. We suggest the names Big Draw (BD) and Inverse Big Draw (iBD).

- For \( n = 2 \), the universe evolves in the \( x \) dimension only, either from vanishing scale factor to a BR, or from an iBR to vanishing scale factor, in infinite time. For \( n > 2 \), this occurs in all three dimensions.

For \( l_1 = 0 \), the mean Hubble parameter, the deceleration parameter, the EoS parameter and the scalar field simplify respectively to

\[
H = \frac{1 - n}{t}, \tag{68}
\]

\[
q = \frac{n}{1 - n}, \tag{69}
\]

\[
w = \frac{-5n + 2}{3(n - 2)}, \tag{70}
\]

\[
\Phi(t) = \frac{a_1^3 n^2}{l_0^2} t^{3n-2}. \tag{71}
\]

There seem to be problems for \( n = 1 \) and \( n = 2 \), while in Table 2 these cases do not seem to be more problematic than others. The apparent divergence of \( q \) as \( n = 1 \) is understood as for the \( n = -1 \) line of Table 1: \( a(t) \) becomes constant. As discussed at the end of Section 4, the \( w \) parameter is not very useful when there is anisotropy, but one might still ask why it diverges for \( n = 2 \). Let us again calculate the energy density and isotropic pressure of the effective fluid for our subcase, recalling that the potential vanishes. Then, using Eq. (71), we get

\[
\rho^\Phi = \frac{3a_1^3}{4l_0^2 l_2^2} n^2(3n - 2)(3n - 2) t^{3n-4}, \tag{72}
\]

\[
p^\Phi = \frac{a_1^3}{4l_0^2 l_2^2} n^2(2 - 5n)(3n - 2) t^{3n-4}. \tag{73}
\]

explaining the divergence of \( w \) by the vanishing of \( \rho^\Phi \) for \( n = 2 \). We also see that the density and pressure both vanish (other than for the trivial, isotropic \( n = 0 \) case) for \( n = 2/3 \), due to the constancy of the field and vanishing of the potential.

6.1.3 The case with both \( l_1 \) and \( l_2 \) nonzero.

For this subcase, the limiting behaviors of the scale factors mentioned on page 12 show that they diverge near infinities for all \( n \). Moreover, at late time, the universe expands with acceleration, again for all \( n \), as can be seen from the asymptotic behavior mentioned after Eq. (60). But the behavior of the scale factors near zero determines the critical values of \( n \) as -1, 1/2, 1 and 2, leading to the subcases shown in Table 3. Outstanding features are

\footnote{since the volume of a metal drawn to produce a wire does not change.}

\footnote{As a specific example, consider \( n = 7/5 \), which gives \( w = 25/9 \) according to (70), which should give deceleration, but gives \( q = -7/2 \), that is, acceleration according to (69).}
The comoving volume diverges at $t = 0$ for $n < -1$ and for $n > 1$. Since it also diverges as $t \to \pm \infty$, this means that it must go through a minimum between two divergences — a Bounce (B). If $t_1$ does not exist, there will be a Bounce each in the $t < 0$ universe and the $t > 0$ universe; whereas if $t_1$ does exist, it will essentially bring one of the minima down to zero, inserting a BB, effectively splitting one of the universes into two. Hence the first line of Table[3] has an Infinity-Bounce-Big Rip (I-B-BR) universe and an iBR-B-I universe; the second line an I-BC universe, a finite-lifetime BB-BC universe and an iBR-B-I universe; and so on.

| $n, t_1$ | $t$ | $-\infty$ | $-|t_1|$ | $|t_1|$ | $+\infty$ |
|----------|------|------------|-------------|-------------|----------|
| $n < -1$ | $t_0 > 0$ | $\infty$ | $B$ | $BR(\infty)$ | $B$ | $I(\infty)$ |
| $t_0 \leq 0$ | $\infty$ | $|t_1|$ | $B$ | $BR|t_1|$ | $B$ | $I(\infty)$ |
| $n = -1$ | $t_0 > 0$ | $\infty$ | $B$ | $BC(\infty)$ | $B$ | $I(\infty)$ |
| $t_0 \leq 0$ | | | | | | |
| $n > 0$ | $t_0 > 0$ | $\infty$ | $B$ | $BC(\infty)$ | $B$ | $I(\infty)$ |
| $t_0 \leq 0$ | | | | | | |

Table 3: Universes derived for $t_1 \neq 0$ and $t_2 \neq 0$, where $\nu(t)$ is a power of $t$. Each line shows a timeline from $-\infty$ to $\infty$, but may represent multiple universes delimited by singularities, shown by square brackets. A single expression in the brackets refers to all scale factors, two expressions refer to $A(t)$ and $B(t)$, three expressions refer to $A(t), B(t)$ and $a(t)$. Also, I: infinite scale factors at infinite future or past, B: Bounce, BR: Big Rip, i: Inverse, BC: Big Crunch, BB: Big Bang, 1 or 2: only one or two of the spatial dimensions, c: cigar-type, BD: Big Draw [See text, and also note about negative time on page 12. *Sign of bounce time or $t_1$ may be positive or negative].
• The third line features an iBR-BC universe, i.e. a finite-lifetime universe which starts with infinitely large scale factors that immediately contract to finite values, and keep on contracting to a BC.
• For \( n = -1 \), the scale factors become constant at \( t = 0 \). Therefore the sub-case without \( t_1 \) features the only universe of the table without beginning or end, a bouncing universe.
• The \( n \to 0 \) case is the same as the corresponding case of Table I. In fact, it is isotropic.
• For all cases with \( n > 0 \), the singularity at \( t = 0 \) is qualitatively the same as in the corresponding \( n \) value Table II.

Interestingly, a wide range of cosmological possibilities, including late-time acceleration, can be reproduced by using a power of \( t \) as the seed function. This even includes the isotropic case, where the time-dependence of the source scalar field does not change with \( n \): \( \Phi(t) \propto t^{-2} \). This surprising richness, including the possibility of vanishing energy-momentum tensor while the field is nonzero, seems to be due to the coupling coefficient selected by the Hessian determinant condition, and the potential selected by the Noether symmetry approach.

6.2 Exponential seed

When we use \( \nu(t) = a_1 e^{kt} \) where \( a_2 \) and \( k \) are non-zero constants, in Eqs. (49)-(51), (53), (56) and (35); the scale factors, the scalar field, the mean Hubble parameter and the deceleration parameter become

\[
A(t) = \frac{l_0}{ka_2^2} \left( l_1 a_2^3 e^{3kt} + l_2 \right) e^{-2kt},
\]
\[
B(t) = \frac{l_0}{k \sqrt{a_2}} \left( l_1 a_2^3 e^{3kt} + l_2 \right)^{1/2} e^{-kt/2},
\]
\[
a(t) = \frac{l_0}{ka_2^2} \left( l_1 a_2^3 e^{3kt} + l_2 \right)^{2/3} e^{-kt},
\]
\[
\Phi(t) = \frac{k^2 a_2^3}{l_0^2} \frac{e^{3kt}}{l_1 a_2^3 e^{3kt} + l_2},
\]
\[
H = k \left[ l_1 a_2^3 e^{3kt} - l_2 \right] \left[ l_1 a_2^3 e^{3kt} + l_2 \right],
\]
\[
q = -\frac{l_2^2 a_2^6 e^{6kt} + 4l_1 l_2 a_2^3 e^{3kt} + l_2^2}{(l_1 a_2^3 e^{3kt} - l_2)^2}.
\]

Unlike in Section 6.1, these scale factors are never singular at \( t = 0 \), however the \( (l_1 a_2^3 e^{3kt} + l_2) \) terms in the scale factors can vanish at some finite \( t \) value \( t_2 \), if \( l_1 \) and \( l_2 \) are both nonzero. Since they do not diverge at finite time, there is no Big Rip.

In that case, the scale factors diverge exponentially at both infinities, regardless of the signs of \( l_1, l_2 \) and \( k \). Therefore, if \( t_2 \) does not exist, the universe
is a bouncing universe, infinite in both time directions. This solution is qualitatively similar to the one in the fourth line of Table 3. If $t_2$ does exist, then the solution represents two universes; one collapsing from infinite scale factors to a Big Crunch, one expanding from a Big Bang to infinite scale factors. This solution is qualitatively similar to the one in the fifth line of Table 3. For all solutions (with nonzero $l_1$ and $l_2$), at late time, $H \to |k|$ and $q \to -1$, confirming the de Sitter-like asymptotic behavior of the universe(s). For positive $k$, the scalar field $\Phi$ approaches a constant.

If either one of $l_1$ and $l_2$ is zero, the universe either expands or contracts exponentially, depending on the sign of $k$ and on which constant vanishes. In these cases, $q = -1$ at all times, and for $l_2 = 0$ (isotropic), $\Phi$ is constant. A constant scalar field is the standard (simplest) way in the literature for causing exponential expansion, and in the isotropic case can be interpreted as creating a cosmological constant. For positive $k$, the solution approaches isotropy at late times.

According to recent astrophysical data, the deceleration parameter of the universe lies in the interval $-1.72 < q < -0.58$. In all models of this subsection at late time $q$ converges to $-1$, so in this sense these models are consistent with the observational results.

### 6.3 Shifted-reversed power seed

We may also use $\nu(t) = \frac{a_3}{(t_c - t)^m}$, where $a_3$, $t_c$ and $m$ are non-zero constants, since this function is similar to the FRW Big Rip models. Then Eqs. (49)-(51) become

$$A(t) = \frac{l_0}{a_3^3 m}(t_c - t)^{1+2m} \left[ l_1 a_3^3(t_c - t)^{-3m} + l_2 \right], \quad (80)$$

$$B(t) = \frac{-l_0}{m a_3^3}(t_c - t)^{1+m/2} \left[ l_1 a_3^3(t_c - t)^{-3m} + l_2 \right]^{1/2}, \quad (81)$$

$$\Phi(t) = \frac{a_3^2 m^2 (t_c - t)^{-3m-2}}{l_0^2 \left[ l_1 a_3^3(t_c - t)^{-3m} + l_2 \right]}. \quad (82)$$

The substitutions $m \to -n$ and $t \to t_c - t$ bring these expressions into the same form as Eqs. (57) and (59). Therefore all solutions from Section 6.1 apply, we just have to read the lines of the tables in that section from right to left, and switch the sign of $n$ (and of course, shift any “i” designation to the other side of the singularity). So this seed does not only give Big Rip models (see Ref. [37] for detailed discussion).

### 6.4 Other seeds and parameter choices

Of course, $\nu(t)$ being a free function, an infinite number of choices is possible, not to mention the freedom in the choices of $l_1$ and $l_2$. Among these, we are
naturally interested in those choices that give simple metric functions $A(t)$ and $B(t)$.

One way of achieving this would be to set $l_1$ or $l_2$ equal to zero. But $l_2 = 0$ takes us to isotropic solutions, which are not in line with the emphasis of the present work. The choice $l_1 = 0$, although it will give simple-looking solutions, limits us to the case of massless field, and there seems to be no motivation for choosing a particular solution above others, except possibly $\nu(t) = t$ (since the comoving volume stays constant), but this is already covered in subsection 6.1.2.

One can also choose combinations to make both $\nu(t)$ and $(l_1 \nu + l_2)$ simple. For example, choosing $\nu(t) = \cos^{2/3}(t)$, $l_0 = -2/3$, $l_1 = -1$ and $l_2 = 1$ yields

$$ds^2 = -dt^2 + \cos^{4/3}(t) \left[ \tan^2(t) dx^2 + dy^2 + dz^2 \right]$$

(83)

a metric with scale factors

$$A(t) = \frac{\sin(t)}{\cos^{1/3}(t)}$$

(84)

$$B(t) = \cos^{2/3}(t)$$

(85)

$$a(t) = [\sin(t) \cos(t)]^{1/3}$$

(86)

i.e. a set of universes that start with a 1-D Big Bang and end with a cigar-like Big Crunch (for $0 \leq t \leq \pi/2$, in appropriate units of time) or start with a cigar-like Big Bang and end with a 1-D Big Crunch (for $\pi/2 \leq t \leq \pi$).

Another solution,

$$ds^2 = -dt^2 + \sinh^{4/3} t \left( \coth^2 t dx^2 + dy^2 + dz^2 \right)$$

(87)

is mathematically similar, but describes universes with infinite lifetime, for example the $t > 0$ universe starts with a cigar-like Big Bang and approaches de Sitter-like isotropically expanding universe at late time.

7 Solutions in Einstein frame

It is known that a scalar-tensor theory can be transformed to the so-called Einstein frame, where the gravitational scalar field becomes minimally coupled to curvature. The price to pay for this simplification is the equivalence principle: Massive point particles do not follow geodesics any more, in contrast to the Jordan frame, which we used so far in this work [14][15]. For the BD theory, the case $W = -3/2$ is special: We derived it from the vanishing of the Hessian [26], it represents a fixed-point of the conformal transformation [15], and marks the boundary beyond which ghosts appear [27]. In the Einstein frame, this specialness is reflected in the gravitational scalar field becoming non-dynamical. Hence, it would be interesting to look at our solutions also in the Einstein frame.
The conformal transformation
\[ g_{ab} = \Omega^{-2} \tilde{g}_{ab}, \quad g^{ab} = \Omega^{2} \tilde{g}^{ab}, \]  
(88)
takes us from the Jordan frame to the Einstein frame. Defining \( \Phi = e^{-\sigma} \), the \( W = -3/2 \) BD action given in Eq. (27) takes the form
\[ A_{BD} = \int \! d^4x \sqrt{-g} e^{-\sigma} \left[ R + \frac{3}{2} \sigma \dot{\sigma} - U_{1}(\sigma) \right] \]  
(89)
where \( U_{1}(\sigma) = U(\Phi)/\Phi \). It is easy to show that under a choice of a conformal factor
\[ \Omega = e^{-\sigma/2}, \]  
(90)
the above action (89) transforms into
\[ A_{E} = \int \! d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - U_{2}(\sigma) \right], \]  
(91)
This is exactly the Einstein-Hilbert action with a potential \( U_{2}(\sigma) = e^{\sigma} U_{1}(\sigma) = U(\Phi)/\Phi^{2} \), and using the potential obtained in (29) it takes the form \( U_{2}(\sigma) = \lambda \), where \( \lambda \) is a constant and could be interpreted as the cosmological constant in Einstein frame. In this case, the (transformed) scalar field \( \sigma \) does not appear in the action, hence the scalar field is non-dynamical as stated in the beginning of this section; and the field equations take the form
\[ \tilde{R}_{ab} = \frac{\lambda}{2} \tilde{g}_{ab}. \]  
(92)
where the Ricci tensor \( \tilde{R}_{ab} \) refers to the transformed metric \( \tilde{g}_{ab} \).

The LRS Bianchi I spacetime can be brought back to its original form
\[ ds^2 = -dt^2 + \tilde{A}^2 dx^2 + \tilde{B}^2 (dy^2 + dz^2), \]  
(93)
by a simple coordinate transformation after the conformal transformation. The transformations of time coordinate and scale factors from the Jordan frame to the Einstein frame are given by
\[ \tilde{t} = \int \sqrt{\Phi} \! dt, \quad \tilde{A} = \sqrt{\Phi} A, \quad \tilde{B} = \sqrt{\Phi} B. \]  
(94)
For this spacetime in Einstein frame the field equations (92) give
\[ 2 \frac{\tilde{A} \dot{\tilde{B}}'}{AB} + \left( \frac{\dot{\tilde{B}}'}{B} \right)^2 = \frac{\lambda}{2}, \]  
(95)
\[ 2 \frac{\tilde{B}''}{B} + \left( \frac{\dot{\tilde{B}}'}{B} \right)^2 = \frac{\lambda}{2}, \]  
(96)
\[ \frac{\tilde{A}''}{A} + \frac{\tilde{B}''}{B} + \frac{\ddot{\tilde{A}} \dot{\tilde{B}}'}{AB} = \frac{\lambda}{2} \]  
(97)
where the prime represents derivative with respect to tilted time coordinate $\tilde{t}$. These equations can be solved exactly, giving

\[ \tilde{A} = c_3 (\tilde{t} + c_1)^{-1/3}, \quad \tilde{B} = c_2 (\tilde{t} + c_1)^{2/3}, \]  

(98)

for $\lambda = 0$;

\[ \tilde{A} = c_3 \sinh(k\tilde{t} + c_1) \cosh^{-1/3}(k\tilde{t} + c_1), \quad \tilde{B} = c_2 \cosh^{2/3}(k\tilde{t} + c_1), \]  

(99)

for $\lambda = 8k^2/3 > 0$;

\[ \tilde{A} = c_3 \sin(k\tilde{t} + c_1) \cos^{-1/3}(k\tilde{t} + c_1), \quad \tilde{B} = c_2 \cos^{2/3}(k\tilde{t} + c_1), \]  

(100)

for $\lambda = -8k^2/3 < 0$. We note here that the obtained scale factors given in (98) represent the well known Kasner solution. We would like to check the consistency between these solutions and those in the Jordan frame, (49)-(51).

For given $\lambda$, the Jordan frame solutions contain the arbitrary function $\nu(t)$ and two arbitrary constants $l_0$ and $l_2$ ($l_1$ is not independent). The Einstein frame solutions contain one arbitrary function $\Phi(\tilde{t})$ (which does not appear in the metric, however) and three arbitrary constants, $c_1$, $c_2$ and $c_3$. To show consistency, we need to transform the Jordan frame solutions to the Einstein frame and show that they agree with solution found in that frame.

Applying the transformations in (94) to (49) and (50), using (51), we get

\[ \tilde{A} = \sqrt{l_1 \nu(t)^3 + l_2 \nu(t)}, \]  

(101)

\[ \tilde{B} = \nu(t), \]  

(102)

Using (102) in (51), we can write for $\Phi$ in the transformed coordinates

\[ \Phi(t) = \frac{\tilde{B}^2 \dot{\tilde{B}}}{l_0^2 (l_1 \tilde{B}^3 + l_2)}, \]  

(103)

but using (94),

\[ \dot{\tilde{B}} = \frac{d\tilde{B}}{dt} = \frac{d\tilde{B}}{dt} \frac{dt}{d\tilde{t}} = \dot{\tilde{B}}^* \sqrt{\Phi} \implies \Phi = \frac{\tilde{B}^2 \dot{\tilde{B}}^2}{l_0^2 (l_1 \tilde{B}^3 + l_2)}, \]  

(104)

so $\Phi$ disappears from the equation: it cannot be determined in the transformed coordinates. This was to be expected, since the scalar field is absent from the action (91). The arbitrariness (or information) in $\nu(t)$ in the Jordan frame has shifted to $\Phi(\tilde{t})$ in the Einstein frame.

Returning to (104) and using the solution found for $\tilde{B}(\tilde{t})$, e.g. for positive $\lambda$,

\[ l_0^2 (l_1 c_2^2 \cosh^2(k\tilde{t} + c_1) + l_2) = c_3^4 \frac{4}{9} k^2 \sinh^2(k\tilde{t} + c_1), \]  

(105)
where it should be noted that \( k^2 = 3\lambda/8 = 9l_1l_2^2/4 \) (see after eq. (18)). This will hold, if \( c_2 \) is chosen such that \( l_2 \) is equal to \(-l_1c_3^2\). This choice can be made, since obviously the set \( \{c_1, c_2, c_3\} \) cannot be independent of the set \( \{l_0, l_2\} \).

If two solutions describe the same physical reality, the parameter \( s \) (arbitrary constants) of one should be expressible in terms of the parameter \( s \) of the other, although information could hide in the arbitrary function(s) in this case.

Similarly, (101) will agree with (99), if \( c_3 \) is properly chosen; but \( c_1 \) is not related to \( l_0 \) or \( l_2 \), it comes from the integration in the first term in (94).

The same calculations can be made for the cases of negative and vanishing \( \lambda \), establishing consistency of Einstein frame solutions with the Jordan frame solutions.

For example, we can transform the solutions of subsection 6.1.1 \((l_2 = 0, \text{ isotropic})\) as

\[
\sqrt{\Phi} = \frac{3n}{2kl} \implies t = t_1 e^{\frac{2k}{3}i}, \quad \tilde{A} = \frac{2k\alpha_1}{3l_0} t_0^{\frac{1}{2}} e^{\frac{2k}{3}i}, \quad \tilde{B} = a_1 t_0^{\frac{1}{2}} e^{\frac{2k}{3}i}\tag{106}
\]

and those of subsection 6.1.2 \((l_1 = 0)\) as

\[
\sqrt{\Phi} = \left[ \frac{n^2 \alpha_3^2}{l_0^3} \right]^{1/2} t^{(3/2)n-1} \implies \quad t = t_0 (\tilde{t} + \tilde{c}_1)^{2/(3n)} , \quad \tilde{A} = A_0 (\tilde{t} + \tilde{c}_1)^{-1/3}, \quad \tilde{B} = B_0 (\tilde{t} + \tilde{c}_1)^{2/3}\tag{107}
\]

where

\[
t_0 = \left[ \frac{9\alpha_3 l_2}{4\alpha_1^2} \right]^{1/(3n)}, \quad A_0 = \left[ \frac{2l_2}{3l_0} \right]^{1/3} \quad \text{and} \quad B_0 = \left[ \frac{9\alpha_3 l_2}{4} \right]^{1/3}\tag{108}
\]

Both solutions (106) and (107) identically satisfy the Einstein frame equations (95)-(97), if one recalls that \( l_1 = 0 \) implies vanishing of \( \lambda \). The constants \( t_1 \) in (106) and \( \tilde{c}_1 \) in (107) correspond to the constants \( c_1 \) that appear in (98)-(100).

To summarize, the solutions we found in the Einstein frame, although containing only three parameters, are the transformed versions of the solutions in the Jordan frame, which contained an arbitrary function. This correspondence between a solution for the metric in the Einstein frame and an infinite number of metrics in the Jordan frame is possible, since the scalar field, which determines the transformation between the two frames, is arbitrary in the Einstein frame.

8 Concluding remarks

In this paper we have examined the scalar-tensor Brans-Dicke theory of gravity for LRS Bianchi type I spacetime admitting Noether symmetry. This symmetry approach is important because it provides us with a theoretical motivation to select a region of the solution space (for other motivations, leading to different regions of the solution space for the Bianchi type I spacetimes, see
e.g. Refs. [38]–[42]). The Lagrangian density (15) of LRS Bianchi type I becomes degenerate for $W(\Phi) = -3/2$, when the Brans-Dicke coupling function $F(\Phi) = \Phi$ is used. This degeneracy is required for nontrivial solutions, hence we use this value as the BD parameter. The existence of Noether symmetry also restricts the form of potential $U(\Phi)$, and allows us to find a transformation given by (42) in which the metric potentials and the scalar field are stated in terms of new dynamical variables $(\mu, \nu, u)$, where the variable $\mu$ is cyclic. Under the transformation (42) the Lagrangian (15) reduces to a new, simpler one (44).

We have obtained the new set of field equations (45)–(47) for the LRS Bianchi type I spacetime by using these transformations. We have found the general class of solutions of BD field equations with potential $U(\Phi) = \lambda \Phi^2$ in the background of LRS Bianchi type I spacetime exhibiting Noether symmetry. This solution family contains an arbitrary function, called $\nu(t)$ in this work, and two arbitrary constants.

In Section 6, we gave examples using some simple forms of the seed function $\nu(t)$; first as powers, then exponentials, powers of $(t_c - t)$, and some others. The first three are chosen to be similar to popular models in FRW cosmology. The solutions are shown concisely in tables, clearly showing the relation between the behavior of the model universes and the parameters of the seed function.

Because of the specialness of the value $W = -3/2$ for the BD-theory when one considers conformal transformations, we considered the problem also in the Einstein frame in Section 7. For this value, the scalar field becomes non-dynamical, taking on the arbitrary nature of the function $\nu(t)$ in the Jordan frame. The solutions found in the Einstein frame are consistent with those found in the Jordan frame; they can be transformed into each other.

The models found in Section 6 show a wide range of behaviors, featuring Big Bangs, Big Crunches, Big Rips, Bounces, various singularities of the cigar or pancake types, etc. While some of these solutions are of theoretical interest only, there are many expanding-universe solutions with acceleration, consistent with observational data. For example, all the solutions in Table 3 and all solutions in subsection 6.2 except the few strictly contracting solutions, feature late-time acceleration.

Even the isotropic special case with power-seed function displays surprising richness, despite the time-dependence of the scalar field being the same for all powers. One solution is particularly interesting: It is Minkowski space, containing a nontrivial scalar field. The particular nonminimal coupling and the potential selected by the Hessian Determinant condition and the Noether symmetry make this possible. Given the important guiding role of the concept of symmetry in modern theoretical physics, we believe that the family of solutions found and analyzed in this work constitute a potentially more relevant set among all possible solutions for LRS Bianchi type-I cosmological models containing a Brans-Dicke field.
Acknowledgements

This work was supported by Akdeniz University, Scientific Research Projects Unit.

References

1. Riess, A. G., et al.: Astron J. 116 1009 (1998)
   Perlmutter, S. et al.: Astrophys. J. 517 565 (1999)
2. Spergel, D. N., et al.: Astrophys. J. Suppl. 148, 213 (2003)
3. Bennett, C. N., et al.: Astrophys. J. Suppl. 148 1 (2003)
4. Copeland, E. J., Sami, M. and Tsujikawa, S.: Int. J. Mod. Phys. D. 15 1753 (2006)
5. Caldwell, R.R., Dave, R., Steinhardt, P.J.: Phys. Rev. Lett. 80 1582 (1998)
   Armendariz-Picon, C., Mukhanov, V.F., Steinhardt, P.J.: Phys. Rev. Lett. 85 4438 (2000)
   Sahni, V.: Class. Quant. Grav. 19 3435 (2002)
   Gao, C.J., Shen, Y.G.: Chin. Phys. Lett. 19 1396 (2002)
6. Torres, D.F.: Phys. Rev. D 66 043522 (2002)
   Alimohammadi, M., Mohseni Sadjadi, H.: Phys. Rev. D73 083527 (2006)
   Caldwell, R.R.: Phys. Lett. B 545 23 (2002)
   Caldwell, R.R., Kamionkowski, M., Weinberg, N.N.: Phys. Rev. Lett. 91 071301 (2003)
   Elizalde, E., Nojiri, S., Odintsov, S.D.: Phys. Rev. D70 043539 (2004)
   Faraoni, V.: Class. Quantum Grav. 22 3235 (2005)
   Setare, M.R.: Eur. Phys. J. C 50 991 (2007)
7. Freese, K., Lewis, M.: Phys. Lett. B 540 1 (2002)
8. Sen, A.: J. High Energy Phys. 207 65 (2002)
9. Li, M.Z., Feng, B., Zhang, X.M.: J. Cosmol. Astropart. Phys. 512 002 (2005)
10. Armendariz-Picon, C., Mukhanov, V., Steinhardt, P.J.: Phys. Rev. Lett. 85 4438 (2000)
11. Vollick Dan, N.: Phys. Rev. D68 063510 (2003)
   Capozziello, S., Nojiri, S., Odintsov, S. D., Troisi, A.: Phys. Lett. B 639 1135 (2006)
   Nojiri, S., Odintsov, S. D.: Phys. Rev. D74 086005 (2006)
   Multamaki, T., Vilja, I.: Phys. Rev. D74 064022 (2006)
   Nojiri, S., Odintsov, S.D.: Int. J. Meth. Mod. Phys. A 115 (2007)
   Hu, W., Sawicki, I.: Phys. Rev. D76 064004 (2007)
   Capozziello, S., Francaviglia, M.: Gen. Relativ. Gravit. 40 357 (2008)
   Sotiriou, T. F., Faraoni, V.: Rev. Mod. Phys. 82 451 (2010)
12. Capozziello, S.: Int. J. Mod. Phys. D11 483 (2002)
13. Brans, C., Dicke, R.H.: Phys. Rev. 124 925 (1961)
14. Fujii, Y., Maeda, K.: The scalar-Tensor Theory of Gravitation, p.42, Cambridge Univ.
   Press, Cambridge (2004)
15. Fraonil, V.: Phys. Lett. B 665 135 (2008)
16. Bertolami, O., Martins, P.J.: Phys. Rev. 61 064007 (1999)
17. Copi, C. J., Huterer, D., Schwarz, E. J., Starkman, G. D.: Phys. Rev. D75 023507 (2007)
18. Barrow, J. D., Maartens, R.: Phys. Rev. D 59 043502 (1999)
19. de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P., Stornaiolo, C.: Phys. Rev. D42 1091 (1990)
20. Demianski, M., de Ritis, R., Rubano, C., Scudellaro, P.: Phys. Rev. D 46 1391 (1992)
21. Capozziello, S., de Ritis, R.: Phys. Lett. A 177 1 (1993)
22. Capozziello, S., Lambiase, G.: Gen. Relativ. Gravit. 32 673 (2000)
23. Sanyal, A.K., Modak, B.: Class. Quantum Grav. 18 3767 (2001)
24. Sanyal, A.K.: Phys. Lett. B 524 177 (2002)
25. Sanyal, A.K., Rubano, C., Piedipalumbo, E.: Gen. Relativ. Gravit. 35 1617 (2003)
26. Kamila, S., Modak, B., Biswas, S.: Gen. Relativ. Gravit. 36 661 (2004)
27. Dabrowski, M.P., Denkiewicz, T., Blaschke, D.B.: Ann. Phys. (Leipzig) 16, 237 (2007)
28. Camci, U., Kucukakca, Y.: Phys. Rev. D 76, 084023 (2007)
29. Kucukakca, Y., Camci, U.: Astrophys. Space. Sci. 338, 211 (2012)
30. Wei, H., Guo, X.J., Wang, L.F.: Phys. Lett. B 707, 298 (2012)
31. Faraoni, V.: Phys. Rev. D 62, 023504 (2000)
32. Calogero, S., Heinze, J.M.: Ann. Inst. Henri Poincare 10, 225 (2009)
33. Gron, O., Hervik, S.: Einstein’s General Theory of Relativity, Springer (2007)
34. Capozziello, S., de Ritis, R., Rubano, C., Scudellaro, P.: Riv. Nuovo Cimento 19, 1 (1996)
35. Roy, S.R., Narain, S., Singh, J.P.: Aust. J. Phys. 38, 239 (1985)
36. Melchiorri, A., Mersini, L., Odman, C.J., Trodden, M.: Phys. Rev. D 68, 043509 (2003)
37. Nojiri, S., Odintsov, S.D., Tsujikawa, S.: Phys. Rev. D 71, 063004 (2005)
38. Rodrigues, D.C.: Phys. Rev. D 77, 023534 (2008)
39. Koivisto, T., Mota, D.F.: Astrophys. J. 679, 1 (2008)
40. Koivisto, T., Mota, D.F.: J. Cosmol. Astropart. Phys. 6, 018 (2008)
41. Akarsu, O., Kilinc, C.B.: Gen. Relativ. Gravit. 42, 119 (2010)
42. Calogero, S., Heinze, J.M.: Gen. Relativ. Gravit. 42, 1491 (2010)
43. D’Inverno, R.: Introducing Einstein’s Relativity, Clarendon Press, Oxford, p.314, (2003).