Solvability of the heat equation on a half-space with a dynamical boundary condition and unbounded initial data

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Abstract. We study the linear heat equation on a half-space with a linear dynamical boundary condition. We are interested in an appropriate choice of the function space of initial functions such that the problem possesses a solution. It was known before that bounded initial data guarantee solvability. Here, we extend that result by showing that data from a weighted Lebesgue space will also do so.

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1. Introduction

Let \( N \geq 2 \) and \( \mathbb{R}_+^N := \mathbb{R}_+^{N-1} \times \mathbb{R}_+ \). This paper is concerned with global solvability of the problem

\[
\begin{align*}
\partial_t u - \Delta u &= 0, \quad x \in \mathbb{R}_+^N, \quad t > 0, \\
\partial_t u + \partial_N u &= 0, \quad x \in \partial \mathbb{R}_+^N, \quad t > 0, \\
u(x, 0) &= \varphi(x), \quad x \in \mathbb{R}_+^N, \\
u(x, 0) &= 0, \quad x = (x', 0) \in \partial \mathbb{R}_+^N, \quad x' := (x_1, x_2, \ldots, x_{N-1}),
\end{align*}
\]  

(1.1)

where \( \partial_t := \partial/\partial t \), and \( \partial_N := -\partial/\partial x_N \). The boundary condition from (1.1) describes diffusion through the boundary in processes such as thermal contact with a perfect conductor or diffusion of solute from a well-stirred fluid or vapor (see, e.g., [4]). Various aspects of analysis of parabolic equations with dynamical boundary conditions have been treated by many authors, see, for example, [2,3,5–8,11–13,15,17,19–27].

In this paper, we focus on the simplest linear problem from a point of view which has not been considered yet (as far as we know). Namely, we are interested in an appropriate choice of the function space of initial functions \( \varphi \) such that problem (1.1) is solvable.

Throughout this paper, we often identify \( \mathbb{R}_+^{N-1} \) with \( \partial \mathbb{R}_+^N \). We introduce some notation. Let \( \Gamma_D = \Gamma_D(x, y, t) \) be the Dirichlet heat kernel on \( \mathbb{R}_+^N \), that is,

\[
\Gamma_D(x, y, t) := (4\pi t)^{-\frac{N}{2}} \left[ \exp \left( -\frac{|x-y|^2}{4t} \right) - \exp \left( -\frac{|x-y_*|^2}{4t} \right) \right]
\]  

(1.2)

for \( (x, y, t) \in \mathbb{R}_+^N \times \mathbb{R}_+^N \times (0, \infty) \), where \( y_* = (y', y_N) \) for \( y = (y', y_N) \in \mathbb{R}_+^N \). Set

\[
[S_1(t) \phi](x) := \int_{\mathbb{R}_+^N} \Gamma_D(x, y, t) \phi(y) \, dy, \quad (x, t) \in \mathbb{R}_+^N \times (0, \infty),
\]

(1.3)
for any measurable function $\phi$ in $\mathbb{R}^N_+$ if the right hand side of (1.3) is well defined. For $x = (x', x_N) \in \mathbb{R}^N_+$ and $t > 0$, set
\[
P(x', x_N, t) := C_N(x_N + t)^{1-N} \left( 1 + \frac{|x'|}{x_N + t} \right)^{-\frac{N}{2}},
\]
(1.4)
where $C_N$ is the constant chosen so that
\[\int_{\mathbb{R}^{N-1}} P(x', x_N, t)dx' = 1 \quad \text{for all } x_N \geq 0 \text{ and } t > 0.\]
Then, $P = P(x', x_N, t)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^N_+$ with the homogeneous dynamical boundary condition; that is, $P$ satisfies
\[
\begin{align*}
-\Delta P &= 0, & x \in \mathbb{R}^N_+, \ t > 0, \\
\partial_t P + \partial_{x_N} P &= 0, & x \in \partial \mathbb{R}^N_+, \ t > 0, \\
P(x, 0) &= \delta(x'), & x = (x', 0) \in \partial \mathbb{R}^N_+,
\end{align*}
\]
where $\delta = \delta(\cdot)$ is the Dirac delta function on $\partial \mathbb{R}^N_+ = \mathbb{R}^{N-1}$ (see, e.g., [1]). Set
\[
[S_2(t)\psi](x) := \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, t)\psi(y') dy', \quad (x, t) \in \mathbb{R}^N_+ \times (0, \infty),
\]
(1.5)
for any measurable function $\psi$ in $\mathbb{R}^{N-1}$ if the right hand side of (1.5) is well defined. Then, the function $\Psi(x, t) := [S_2(t)\psi](x)$ satisfies
\[
\begin{align*}
-\Delta \Psi &= 0, & x \in \mathbb{R}^N_+, \ t > 0, \\
\partial_t \Psi + \partial_{x_N} \Psi &= 0, & x \in \partial \mathbb{R}^N_+, \ t > 0, \\
\Psi(x, 0) &= \psi(x'), & x = (x', 0) \in \partial \mathbb{R}^N_+.
\end{align*}
\]
Consider
\[
\begin{align*}
\partial_t v &= \Delta v - F[v], & x \in \mathbb{R}^N_+, \ t > 0, \\
\Delta w &= 0, & x \in \mathbb{R}^N_+, \ t > 0, \\
v &= 0, & \partial_x w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial \mathbb{R}^N_+, \ t > 0, \\
v(x, 0) &= \varphi(x), & x \in \mathbb{R}^N_+, \\
w(x, 0) &= 0, & x = (x', 0) \in \partial \mathbb{R}^N_+,
\end{align*}
\]
(1.6)
where
\[
F[v](x, t) := \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, 0)\partial_{x_N} v(y', 0, t) dy'
+ \int_0^t \int_{\mathbb{R}^{N-1}} \partial_t P(x' - y', x_N, t - s)\partial_{x_N} v(y', 0, s) dy' ds.
\]
(1.7)
Following [12], we formulate the definition of a solution of (1.1).

**Definition 1.1.** Let $\varphi$ be measurable function in $\mathbb{R}^N_+$. Let $0 < T \leq \infty$ and
\[
v, \ \partial_{x_N} v, \ w \in C(\mathbb{R}^N_+ \times (0, T)).
\]
We call $(v, w)$ a solution of (1.6) in $\mathbb{R}^N_+ \times (0, T)$ if
\[
[S_1(t)\varphi](x), \ \int_0^t [S_1(t - s)F[v](s)](x) \, ds, \ \int_0^t [S_2(t - s)\partial_{x_N} v(\cdot, 0, s)](x) \, ds
\]
are well defined and functions \( v \) and \( w \) satisfy

\[
v(x,t) = [S_1(t)\varphi](x) - \int_0^t [S_1(t-s)F[v](s)](x) \, ds, \]

\[
w(x,t) = \int_0^t [S_2(t-s)\partial x_N v(\cdot, 0, s)](x) \, ds,
\]

for \( x \in \mathbb{R}_+^N \) and \( t \in (0, T) \), respectively. Then, we say that \( u := v + w \) is a solution of (1.1) in \( \mathbb{R}_+^N \times (0, T) \).

In the case of \( T = \infty \), we call \((v, w)\) a global-in-time solution of (1.6) and \( u \) a global-in-time solution of (1.1).

We are ready to state the main results of this paper. For \( 1 \leq r \leq \infty \), we write \( | \cdot |_{L^r} := | \cdot |_{L^r(\partial \mathbb{R}_+^N)} \) and \( \| \cdot \|_{L^r} := \| \cdot \|_{L^r(\mathbb{R}_+^N)} \) for simplicity. Furthermore, for \( 1 \leq r \leq \infty \) and \( \alpha \geq 0 \), we define

\[
L^r_\alpha := \{ f \in L^r(\mathbb{R}_+^N) : \| f \|_{L^r_\alpha} < \infty \},
\]

where

\[
\| f \|_{L^r_\alpha} := \begin{cases} \left( \int_{\mathbb{R}_+^N} |f(x)|^r h(x_N)^{-\alpha r} \, dx \right)^{\frac{1}{r}} & \text{if } 1 \leq r < \infty, \\ \| f \|_{L^\infty} & \text{if } r = \infty, \end{cases}
\]

with \( h(x_N) := \frac{x_N}{x_N + 1} \). \hspace{1cm} (1.8)

Then, we can easily show that \( \| f \|_{L^r_\alpha} \leq \| f \|_{L^r_\beta} \) for \( r \in [1, \infty] \) and \( 0 \leq \alpha \leq \beta \).

**Theorem 1.1.** Let \( N \geq 2 \) and \( 1 \leq q \leq \infty \). Furthermore, let

\[
p \in \left( Nq/(N-1), \infty \right) \quad \text{if} \quad q < \infty \quad \text{and} \quad p = \infty \quad \text{if} \quad q = \infty.
\]

For \( r \in [q, \infty] \), put

\[
\alpha(r) = (N-1) \left( \frac{1}{q} - \frac{1}{r} \right) + \frac{1}{q},
\]

(1.9)

Assume \( \varphi \in L^q_{\alpha(p)} \). Then, problem (1.6) possesses a unique global-in-time solution \((v, w)\) with the following property: For any \( T > 0 \) there exists \( C_T > 0 \) such that

\[
\sup_{0 < t < T} \left[ t^N \left( \frac{q}{r} - \frac{1}{r} \right) \left( \| v(t) \|_{L^p} + t^\frac{1}{2} \| \partial x_N v(t) \|_{L^p} \right) + t^\frac{1}{2} |\partial x_N v(t)|_{L^r} \right] \leq C_T \| \varphi \|_{L^q_{\alpha(p)}},
\]

(1.10)

for \( r \in [q, p] \). Furthermore, \( v \) and \( w \) are bounded and smooth in \( \mathbb{R}_+^N \times I \) for any bounded interval \( I \subset (0, \infty) \).

Let us now explain the role of the space \( L^q_{\alpha(p)} \) in our study. Let \( 1 \leq q \leq \infty \) and take arbitrary functions \( \Phi \in L^q(\mathbb{R}^{N-1}), \vartheta \in L^q(1, \infty) \). Now set \( \varphi(x) := \Phi(x')\Psi(x_N) \) for \( x = (x', x_N) \in \mathbb{R}^N_+ \), where

\[
\Psi(x_N) := \begin{cases} x_N^\lambda & \text{if } 0 < x_N \leq 1, \quad \lambda \in \mathbb{R}, \\ \vartheta(x_N) & \text{if } x_N > 1. \end{cases}
\]

Choose \( p \) as in Theorem 1.1. Then, it is easy to check that \( \varphi \in L^q_{\alpha(p)} \) if and only if

\[
\lambda > (N-1) \left( \frac{1}{q} - \frac{1}{p} \right) > 0 \quad \text{if} \quad q < \infty.
\]
If $\lambda > 0$, then $\lim_{x, y \to 0} \psi(x) = 0$ which means that the condition $u(x', 0, 0) = 0$ in (1.1) is satisfied. This indicates that the choice of the space of initial functions is natural and also optimal in some sense since $\lambda$ can be arbitrarily close to 0 if $q$ is large enough.

We have not observed the importance of the behavior of $\psi$ near $\partial \mathbb{R}^N_+$ in the $L^\infty$-setting in [12]. The main novelty of this paper consists in working in an appropriate weighted $L^q$-space by which we extend a result from [12] significantly, as we explain below.

In [12], we studied the problem

\begin{equation}
\begin{aligned}
\partial_t u - \Delta u &= 0, \quad x \in \mathbb{R}^N_+, \quad t > 0, \\
\partial_t u + \partial_x u &= 0, \quad x \in \partial \mathbb{R}^N_+, \quad t > 0, \\
u(x, 0) &= \varphi(x), \quad x \in \mathbb{R}^N_+, \\
u(x, 0) &= \varphi_b(x'), \quad x = (x', 0) \in \partial \mathbb{R}^N_+,
\end{aligned}
\end{equation}

where $\varphi$ and $\varphi_b$ are bounded functions. A part of Theorem 1.1 in [12] reads as follows:

**Theorem 1.2.** Let $N \geq 2$, $\varphi \in L^\infty(\mathbb{R}^N_+)$ and $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$. Then problem (1.11) possesses a unique global-in-time solution $u$ which is bounded and smooth in $\mathbb{R}^N_+ \times I$ for any bounded interval $I \subset (0, \infty)$.

Hence, if $\varphi_b \equiv 0$, then Theorem 1.2 is a very special case of Theorem 1.1. If $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$ and $\varphi \in L^q(r)$ with $p, q$ as in Theorem 1.1, then we can combine Theorem 1.1 with Theorem 1.2 to obtain the existence of a solution of (1.11) easily, since the problem is linear.

2. Preliminaries

In this section, we prove several lemmata on $S_1(t)\phi$ and $F[v]$ and recall some properties of $S_2(t)v$. In what follows, by the letter $C$ we denote generic positive constants (independent of $x$ and $t$) and they may have different values also within the same line.

We first recall some properties of $S_1(t)\phi$ (see, e.g., [16] and [12, Lemma 2.1]).

\begin{enumerate}
\item[(G1)] For any $1 \leq q \leq r \leq \infty$,
\[ \|S_1(t)\phi\|_{L^r} \leq c_1 t^{-\frac{N}{2} \left(\frac{1}{q} \right) - \frac{1}{2}} \|\phi\|_{L^q}, \quad t > 0, \]
for all $\phi \in L^q(\mathbb{R}^N_+)$, where $c_1$ is a positive constant, independent of $q$ and $r$. In particular, if $q = r$, then
\[ \sup_{t > 0} \|S_1(t)\phi\|_{L^r} \leq \|\phi\|_{L^r}. \]

Furthermore, for any $1 \leq q \leq r \leq \infty$,
\[ \|\partial_x S_1(t)\phi\|_{L^r} \leq c_2 t^{-\frac{N}{2} \left(\frac{1}{q} \right) - \frac{1}{2}} \|\phi\|_{L^q}, \quad t > 0, \]
for all $\phi \in L^q(\mathbb{R}^{N-1}_+)$, where $c_2$ is a positive constant, independent of $q$ and $r$.
\item[(G2)] Let $\phi \in L^q(\mathbb{R}^N_+)$ with $1 \leq q \leq \infty$ and $T > 0$. Then, $S_1(t)\phi$ is bounded and smooth with respect to $x$ and $t$ in $\mathbb{R}^N_+ \times (T, \infty)$.
\end{enumerate}

**Lemma 2.1.** Let $1 \leq q \leq r \leq \infty$. Assume $\phi \in L^q_{\alpha(r)}$ with $\alpha(r)$ as in (1.9). Then, there exists $c_3 = c_3(N) > 0$ such that
\[ |\partial_x S_1(t)\phi|_{L^r} \leq c_3 t^{-\frac{1}{2}} \|\phi\|_{L^q_{\alpha(r)}}, \quad t > 0. \]
Proof. Let $\Gamma_d (d = 1, 2, \ldots)$ be the Gauss kernel in $\mathbb{R}^d$. It follows from (1.2) that

\[
\Gamma_D(x, y, t) = \Gamma_{N-1}(x' - y', t) \left( \Gamma_1(x_N - y_N, t) - \Gamma_1(x_N + y_N, t) \right), \tag{2.3}
\]

\[
K(x, y, t) := \partial_x \Gamma_D(x, y, t)
\]

\[
= \Gamma_{N-1}(x' - y', t) \left( -\frac{x_N - y_N}{2t} \Gamma_1(x_N - y_N, t) + \frac{x_N + y_N}{2t} \Gamma_1(x_N + y_N, t) \right), \tag{2.4}
\]

for $(x, y, t) \in \mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (0, \infty)$. Then, we have

\[
K(x', 0, y, t) = \frac{y_N}{t} \Gamma_{N-1}(x' - y', t) \Gamma_1(y_N, t) \tag{2.5}
\]

for $(x', y, t) \in \mathbb{R}^{N-1} \times \mathbb{R}^N_+ \times (0, \infty)$.

We first prove (2.2) for the case $r = \infty$. By (2.5) we can easily show that, for $q \in (1, \infty)$, it holds that

\[
\int_{\mathbb{R}^N_+} \left( |K(x', 0, y, t)| y_N^q \right) dy
\]

\[
= t^{-\frac{q'}{q}} \int_{\mathbb{R}^N_+} \left[ t^{\frac{N}{N-1}} \Gamma_{N-1}(y', t) \left( \frac{y_N}{t^{1/2}} \right)^{1+\frac{N}{q}} \Gamma_1(y_N, t) \right] \frac{y_N^q}{t} \leq C t^{-\frac{q'}{q}} \int_0^\infty \eta^q \exp \left( -C \eta^2 \right) d\eta
\]

\[
\leq C t^{-\frac{q'}{q}} \int_0^\infty \eta^q \exp \left( -C \eta^2 \right) d\eta \leq Ct^{-\frac{q'}{q}}
\]

for $t > 0$, where $1/q + 1/q' = 1$. Furthermore, for $q = 1$, it holds that

\[
|K(x', 0, y, t)| y_N^q = t^{\frac{N-1}{N}} \Gamma_{N-1}(x' - y', t) \left( \frac{y_N}{t^{1/2}} \right)^{1+N} \Gamma_1(y_N, t) \leq Ct^{-\frac{1}{2}},
\]

for $(x', y, t) \in \mathbb{R}^{N-1} \times \mathbb{R}^N_+ \times (0, \infty)$. For $q \in \{1, \infty\}$, by (1.3), (1.8), (2.6), and (2.7) we have

\[
|\partial_x [S_1(t)\phi](x', 0)|
\]

\[
\leq \int_{\mathbb{R}^N_+} |K(x', 0, y, t)| h(y_N)^{\frac{N}{q}} h(y_N)^{-\frac{N}{q}} |\phi(y)| dy
\]

\[
\leq \int_{\mathbb{R}^N_+} \left| K(x', 0, y, t) \right| y_N^q h(y_N)^{-\frac{N}{q}} |\phi(y)| dy \leq Ct^{-\frac{1}{2}q} \|\phi\|_{L^q_{R, \infty}} \tag{2.8}
\]

for $x' \in \mathbb{R}^{N-1}$ and $t > 0$. Furthermore, for $q \in (1, \infty)$, similarly to (2.8), applying Hölder’s inequality, we see that

\[
|\partial_x [S_1(t)\phi](x', 0)|
\]

\[
\leq C \left\{ \int_{\mathbb{R}^N_+} h(y_N)^{-\frac{N}{q}} |\phi(y)| dy \right\}^{\frac{1}{q}} \left\{ \int_{\mathbb{R}^N_+} \left| K(x', 0, y, t) h(y_N)^{\frac{N}{q}} \right|^q dy \right\}^{\frac{1}{q'}}
\]

\[
\leq Ct^{-\frac{1}{2}q} \|\phi\|_{L^q_{R, \infty}}
\]

for $x' \in \mathbb{R}^{N-1}$ and $t > 0$, where $1/q + 1/q' = 1$. This together with (2.8) implies (2.2) for the case $r = \infty$. 
Next we consider the case \( r < \infty \). Let \( 1 \leq q \leq r < \infty \). Put
\[
\frac{1}{p} := 1 + \frac{1}{r} - \frac{1}{q}, \quad \beta = N \left( 1 - \frac{1}{p} \right). \tag{2.9}
\]
Then, for \( \theta \in (0, 1) \), it follows from Hölder’s inequality that
\[
\int_{\mathbb{R}^N} |K(x', 0, y, t)||\phi(y)| \, dy
\leq \left\{ \int_{\mathbb{R}^N} \left[ |K(x', 0, y, t)| h(y_N)^{\alpha(r)} \right]^{p} \left[ h(y_N)^{-\alpha(r)} |\phi(y)| \right] \frac{(1-\theta)p}{r} \, dy \right\}^{\frac{1}{p}}
\times \left\{ \int_{\mathbb{R}^N} \left[ h(y_N)^{-\alpha(r)} |\phi(y)| \right]^{q \theta'} \, dy \right\}^{\frac{1}{q'}}
\]
for \( x' \in \mathbb{R}^{N-1} \) and \( t > 0 \), where \( 1/p + 1/p' = 1 \). Put \( \theta = q/p' \). Then, since it follows from (2.9) that \( 1 - \theta = q/r \), we have
\[
\int_{\mathbb{R}^N} |K(x', 0, y, t)||\phi(y)| \, dy
\leq \left\{ \int_{\mathbb{R}^N} \left[ |K(x', 0, y, t)| h(y_N)^{\alpha(r)} \right]^{p} \left[ h(y_N)^{-\alpha(r)} |\phi(y)| \right] \frac{p\bar{q}}{r} \, dy \right\}^{\frac{1}{p}}
\times \left\{ \int_{\mathbb{R}^N} \left[ h(y_N)^{-\alpha(r)} |\phi(y)| \right]^{q \theta''} \, dy \right\}^{\frac{1}{q''}} \|\phi\|_{L^2_{\alpha(r)}}^{1 - \frac{q}{p}} \tag{2.10}
\]
for \( x' \in \mathbb{R}^{N-1} \) and \( t > 0 \). Furthermore, put \( \bar{\theta} = p/q' < 1 \). Then, since \( 1 - \bar{\theta} = p/r \) and \( \beta = N/p' \), applying Hölder’s inequality with (2.6), we see that
\[
\int_{\mathbb{R}^N} \left[ |K(x', 0, y, t)| y_N^{\alpha(r)} \right]^{p} \left[ h(y_N)^{-\alpha(r)} |\phi(y)| \right] \frac{p\bar{q}}{r} \, dy
\leq \left\{ \int_{\mathbb{R}^N} \left[ |K(x', 0, y, t)| y_N^{\bar{\theta} \bar{r}} \right]^{p} \, dy \right\}^{\bar{\theta}}
\times \left\{ \int_{\mathbb{R}^N} \left[ h(y_N)^{-\alpha(r)} |\phi(y)| \right]^{q \theta''} \, dy \right\}^{\frac{1}{q''}}
\]
for $x' \in \mathbb{R}^{N-1}$ and $t > 0$. This together with (2.10) yields that
\[
\int_{\mathbb{R}^{N}} |K(x', 0, y, t)||\phi(y)| \, dy 
\]
\[
\leq Ct^{-\frac{\beta}{2}} \left\{ \int_{\mathbb{R}^{N}} \left[ |K(x', 0, y, t)|y_N^{1-\gamma} \right]^p \left[ h(y_N)^{-\alpha(r)}|\phi(y)| \right]^q \, dy \right\}^{\frac{1}{q}} 
\]

(2.11)

for $x' \in \mathbb{R}^{N-1}$ and $t > 0$. Since
\[
\frac{\alpha(r) - \beta}{1 - \theta} = (N-1) \left( 1 - \frac{1}{p} \right) + 1,
\]
similarly to (2.6), it holds that
\[
\int_{\mathbb{R}^{N-1}} \left[ |K(x', 0, y, t)|y_N^{1-\gamma} \right]^p \, dx' 
\]
\[
= t^{-\frac{\beta}{2}} \int_{\mathbb{R}^{N-1}} \left[ \frac{N-1}{2} (1-\frac{1}{p}) + \frac{1}{2} \Gamma_{N-1}(y', t) \left( \frac{y_N}{t^{1/2}} \right)^{(N-1)(1-\frac{1}{p})+2} \right] \left( \frac{y_N}{t^{1/2}} \right)^{\alpha(r)^2} \, dx' 
\]
\[
\leq Ct^{-\frac{\beta}{2}} \left( \frac{y_N}{t^{1/2}} \right)^{(N-1)(p-1)+2p} \exp \left( - C'y_N^2 \right) \leq Ct^{-\frac{\beta}{2}}.
\]

This together with (2.11) implies
\[
|\partial_{x_N} [S_1(t)\phi]|_{L^r} 
\]
\[
\leq \int_{\mathbb{R}^{N-1}} \left\{ \int_{\mathbb{R}^{N}} |K(x', 0, y, t)||\phi(y)| \, dy \right\}^r \, dx' 
\]
\[
\leq Ct^{-\frac{\beta}{2}} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N}} \left[ |K(x', 0, y, t)|y_N^{1-\gamma} \right]^p \left[ h(y_N)^{-\alpha(r)}|\phi(y)| \right]^q \, dy \, dx' \|\phi\|_{L^p_{\alpha(r)}}^{r-q} 
\]
\[
\leq Ct^{-\frac{\beta}{2}} \|\phi\|_{L^p_{\alpha(r)}}^{r} 
\]

for $t > 0$, and we have (2.2). Thus, Lemma 2.1 follows. \hfill \square

Next we recall some properties of $S_2(t)\psi$.

(P1) Let $\psi \in L^r(\mathbb{R}^{N-1})$ for some $r \in [1, \infty]$ and $t, t' > 0$. Then
\[
[S_2(t)\psi](x', x_N) = [S_2(t + x_N)\psi](x', 0),
\]
\[
[S_2(t + t')\psi](x) = [S_2(t)(S_2(t')\psi)](x),
\]

for $x = (x', x_N) \in \mathbb{R}^N$. Furthermore,
\[
\lim_{t \to 0} |S_2(t)\psi - \psi|_r = 0 \quad \text{if} \quad 1 \leq r < \infty.
\]
(P_2) For any 1 \leq r \leq q \leq \infty,
|S_2(t)\psi|_{L^q} \leq C t^{-(N-1)(\frac{1}{q} - \frac{1}{r})} |\psi|_{L^r}, \quad t > 0,
for all \psi \in L^r(\mathbb{R}^{N-1}). In particular, if q = r, then
\[
\sup_{t > 0} |S_2(t)\psi|_{L^q} \leq |\psi|_{L^q}.
\]
(2.12)

(P_3) Let 1 \leq r < \infty and Nr/(N-1) < q \leq \infty. Then
\[
\|S_2(t)\psi\|_{L^q} \leq C t^{-(N-1)(\frac{1}{q} - \frac{1}{r}) + \frac{1}{r}} |\psi|_{L^r}, \quad t > 0,
\]
for all \psi \in L^r(\mathbb{R}^{N-1}). Furthermore,
\[
\sup_{t > 0} \|S_2(t)\psi\|_{L^q} \leq C(\|\psi\|_{L^q} + |\psi|_{L^r})
\]
for all \psi \in L^q(\mathbb{R}^{N-1}) \cap L^r(\mathbb{R}^{N-1}). Properties (P_1), (P_2), and (P_3) easily follow from (1.5) (see, e.g., [10]) and imply that
\[
\sup_{t > 0} \|S_2(t)\psi\|_{L^\infty} \leq |\psi|_{L^\infty}
\]
for all \psi \in L^\infty(\mathbb{R}^{N-1}). Furthermore, by an argument similar to that in the proof of property (G_2) we have:

(P_4) Let \psi \in L^r(\mathbb{R}^{N-1}) with 1 \leq r \leq \infty. Then, for any T > 0, S_2(t)\psi is bounded and smooth in \(\mathbb{R}^N \times (T, \infty)\).

At the end of this section, we have the following (see also [11, Lemma 3.3]).

Lemma 2.2. Let 0 \leq a < 1 and 0 \leq b < 1 be such that 0 \leq a + b \leq 1. Let \gamma \geq 0 and T > 0. Then, for any \delta > 0, there exists a M_* \geq 1 such that
\[
\sup_{0 < t < T} e^{-Mt^\gamma} \int_0^t e^{M_*s^{-a}(t-s)^{-b}} ds \leq \delta \quad \text{for} \quad M \geq M_*.
\]

The proof of this lemma is almost the same as the proof of [11, Lemma 3.3], so we omit it here.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. By [12, Theorem 1.1] with \varepsilon = 1 and \varphi_b \equiv 0 we have Theorem 1.1 for the case p = q = \infty. So we focus on the case q < \infty.

Let T > 0, M \geq 1, 1 \leq q < \infty, and p \in (Nq/(N - 1), \infty]. Set
\[
X_{T,M} := \left\{ v : v, \partial_{x_N} v \in C(\mathbb{R}^N \times (0, T)), \|v\|_{X_{T,M}} < \infty, \quad \|v\|_{X_{T,M}} := \sup_{0 < t < T} e^{-Mt} E[v](t), \right\}
\]
where
\[
E[v](t) := \int_0^t (\frac{\partial v}{t}) \left( \|v(t)\|_{L^p} + t^\frac{1}{p} \|\partial_{x_N} v(t)\|_{L^p} \right) + \sup_{q \leq r \leq p} t^\frac{1}{2} |\partial_{x_N} v(t)|_{L^r}.
\]
Then, X_{T,M} is a Banach space equipped with the norm \| \cdot \|_{X_{T,M}}. We apply the Banach contraction mapping principle in X_{T,M} to find a fixed point of the functional
\[
Q[v](t) := S_1(t)\varphi - D[v](t)
\]
(3.1)
on $X_{T,M}$, where $D[v]$ is the function defined by
\[ D[v](t) := \int_0^t S_1(t-s)F[v](s) \, ds \tag{3.2} \]
and $F[v]$ is the function defined by (1.7).

**Lemma 3.1.** Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, and $p \in (Nq/(N-1), \infty]$. Assume that $v \in X_{T,M}$. Then, there exists $C > 0$, independent of $T$ and $M$, such that, for $p \in (Nq/(N-1), \infty)$, it holds that
\[ \|F[v](t)\|_{L^p} \leq C(1 + t^\frac{1}{q})t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}} \tag{3.3} \]
for $0 < t < T$. Furthermore, for any $r \in [q,p]$,
\[ \|F[v](\cdot, x_N, t)\|_{L^r(\mathbb{R}^{N-1})} \leq C\left(1 + \left(\frac{x_N^2}{r}t\right)^\frac{1}{2}\right)t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}} \tag{3.4} \]
for $x_N \in (0, \infty)$ and $0 < t < T$.

**Proof.** Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, $p \in (Nq/(N-1), \infty]$, and $v \in X_{T,M}$. It follows from (1.7) that
\[ F[v](x,t) = F_1[v](x,t) + F_2[v](x,t) \tag{3.5} \]
for $x \in \mathbb{R}^N_+$ and $0 < t < T$, where
\[ F_1[v](x,t) := \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, 0)\partial_{x_N} v(y', 0, t) \, dy', \]
\[ F_2[v](x,t) := \int_0^t \int_{\mathbb{R}^{N-1}} \partial_t P(x' - y', x_N, t-s)\partial_{x_N} v(y', 0, s) \, dy' \, ds. \]

We first obtain some estimates of $F_1[v]$. For $p \in (Nq/(N-1), \infty)$, by (1.5) and (2.13) we have
\[ \|F_1[v](t)\|_{L^p} \leq \liminf_{\varepsilon \to +0} \left(\int_{\mathbb{R}^{N-1}} \int \|S_2(x_N)\partial_{x_N} v(\cdot, 0, t)(x', 0)\|^p \, dx' \, dx_N\right)^{1/p} \]
\[ = \liminf_{\varepsilon \to +0} \left(\int_{\mathbb{R}^{N+1}} \|S_2(x_N)\partial_{x_N} v(\cdot, 0, t)(x', \varepsilon)\|^p \, dx\right)^{1/p} \tag{3.6} \]
\[ = \liminf_{\varepsilon \to +0} \left(\int_{\mathbb{R}^{N+1}} \|S_2(\varepsilon)\partial_{x_N} v(\cdot, 0, t)(x)\|^p \, dx\right)^{1/p} \]
\[ \leq C \left(\|\partial_{x_N} v(t)\|_{L^q} + \|\partial_{x_N} v(t)\|_{L^p}\right) \leq Ct^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}} \]
for $0 < t < T$. Similarly, for $r \in [q,p]$, by (P2) we obtain
\[ \|F_1[v](\cdot, x_N, t)\|_{L^r(\mathbb{R}^{N-1})} \leq \|S_2(x_N)\partial_{x_N} v(\cdot, 0, t)\|_{L^r} \]
\[ \leq \|\partial_{x_N} v(t)\|_{L^r} \leq t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}} \tag{3.7} \]
for $x_N \in (0, \infty)$ and $0 < t < T$. 
Next we obtain some estimates of $F_2[v]$. It follows from (1.4) that
\[
\partial_t P(x', x_N, t) = \frac{1}{x_N + t} \frac{|x'|^2 - (N - 1)(x_N + t)^2}{|x'|^2 + (x_N + t)^2} P(x', x_N, t),
\]
for $x = (x', x_N) \in \mathbb{R}_+^N$ and $t > 0$. This implies that
\[
|\partial_t P(x', x_N, t)| \leq C(x_N + t)^{-1} P(x', x_N, t) \leq CP(x', x_N, t)x_N^{-\frac{1}{2}} t^{-\frac{1}{2}}
\]
for $x = (x', x_N) \in \mathbb{R}_+^N$ and $t > 0$. Furthermore, for $p \in (Nq/(N - 1), \infty)$, by (2.12) we have
\[
\int_{\mathbb{R}^N} \left\{ (x_N + t - s)^{-1} [S_2(t - s)|\partial_{x_N} v(\cdot, 0, s)](x) \right\}^p \, dx
\]
\[
= \int_{0}^{\infty} (x_N + t - s)^{-p} |S_2(t - s + x_N)|\partial_{x_N} v(s)||_{L^p} dx_N
\]
\[
\leq C|\partial_{x_N} v(s)||_{L^p} \int_{0}^{\infty} (x_N + t - s)^{-p} dx_N \leq C(t - s)^{1-p} s^{-\frac{q}{2}} e^{Ms} \|v\|_{X_{T,M}}
\]
for $0 < s < t < T$. Then, by (3.8) and (3.9) we obtain
\[
\|F_2[v](t)\|_{L^p} \leq C \int_{0}^{t} \left( \int_{\mathbb{R}^N} \left\{ (x_N + t - s)^{-1} [S_2(t - s)|\partial_{x_N} v(\cdot, 0, s)](x) \right\}^p \, dx \right)^{\frac{1}{p}} \, ds
\]
\[
\leq C\|v\|_{X_{T,M}} \int_{0}^{t} (t - s)^{-1+\frac{1}{p}} s^{-\frac{q}{2}} e^{Ms} \, ds \leq Ct^{-\frac{1}{p}+\frac{1}{p}} e^{Mt} \|v\|_{X_{T,M}}
\]
for $0 < t < T$. In addition, for $r \in [q, p]$, by (P2) and (3.8) we see that
\[
\|F_2[v](\cdot, x_N, t)\|_{L^r(\mathbb{R}^N-1)}
\]
\[
\leq Cx_N^{-\frac{1}{2}} \int_{0}^{t} (t - s)^{-\frac{1}{2}} |S_2(x_N + t - s)\partial_{x_N} v(\cdot, 0, s)||_{L^r} \, ds
\]
\[
\leq Cx_N^{-\frac{1}{2}} \int_{0}^{t} (t - s)^{-\frac{1}{2}} |\partial_{x_N} v(s)||_{L^r} \, ds
\]
\[
\leq Cx_N^{-\frac{1}{2}} \|v\|_{X_{T,M}} \int_{0}^{t} (t - s)^{-\frac{1}{2}} s^{-\frac{q}{2}} e^{Ms} \, ds \leq Cx_N^{-\frac{1}{2}} e^{Mt} \|v\|_{X_{T,M}}
\]
for $x_N \in (0, \infty)$ and $0 < t < T$. Therefore, by (3.5), (3.6), (3.7), (3.10) and (3.11) we obtain (3.3) and (3.4). Thus, Lemma 3.1 follows.

\[\square\]

**Lemma 3.2.** Assume the same conditions as in Lemma 3.1. Let $D[v]$ be the function defined by (3.2). Then, there exists $M_* \geq 1$ such that
\[
\|D[v]\|_{X_{T,M}} \leq \frac{1}{2} \|v\|_{X_{T,M}}
\]
for $v \in X_{T,M}$ and $M \geq M_*$. Furthermore, $D[v]$ is bounded and smooth in $\mathbb{R}_+^N \times (\tau, T)$ for any $0 < \tau < T$. 

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Proof. We first prove (3.12). Let \( T > 0, M \geq 1, \) and \( 1 \leq q < \infty. \) For \( p \in (Nq/(N-1), \infty), \) by \((G_1)\) and \((3.3)\) we have

\[
\|D[v](t)\|_{L^p} \leq \int_0^t \|S_1(t-s)F[v](s)\|_{L^p} \, ds
\]

\[
\leq \int_0^t \|F[v](s)\|_{L^p} \, ds \tag{3.13}
\]

\[
\leq C\|v\|_{X_{T,M}} \int_0^t e^{Ms(1+s^{1/p})} s^{-\frac{1}{2}} \, ds
\]

\[
\leq C\|v\|_{X_{T,M}} \left(1 + T^{\frac{1}{p}}\right) \int_0^t e^{Ms} s^{-\frac{1}{2}} \, ds
\]

for \( v \in X_{T,M} \) and \( 0 < t < T. \) Furthermore, since it follows from \([12, \text{Lemma 2.3}]\) that

\[
\sup_{x > 0} \int_0^\infty \left(\frac{|x + y|}{t}\right)^k \Gamma_1(x \pm y, t) y^{-\frac{1}{2}} \, dy \leq Ct^{-\frac{k}{2}-\frac{j}{4}}, \quad k = 0, 1, \quad j = 0, 1, \tag{3.14}
\]

for \( t > 0, \) by \((2.3)\) and \((3.4)\) we have

\[
|D[v](x, t)| \leq \int_0^t \int_{R^N} \Gamma_D(x, y, t-s) |F[v](y, s)| \, dy \, ds
\]

\[
\leq C \int_0^t \int_0^\infty \Gamma_1(x_N - y_N, t-s) \|F[v](\cdot, y_N, s)\|_{L^\infty(R^N)} \, dy_N \, ds \tag{3.15}
\]

\[
\leq C\|v\|_{X_{T,M}} \int_0^t \int_0^\infty \Gamma_1(x_N - y_N, t-s) \left(1 + (y_N^{-1} s)^{\frac{1}{2}}\right) s^{-\frac{1}{2}} e^{Ms} \, dy_N \, ds
\]

\[
\leq C\|v\|_{X_{T,M}} \int_0^t e^{Ms} \left(s^{-\frac{1}{2}} + (t-s)^{-\frac{1}{4}}\right) \, ds
\]

for \( v \in X_{T,M}, x \in \mathbb{R}_+^N, \) and \( 0 < t < T. \) Then, taking a sufficiently large \( M \geq 1 \) if necessary, we can apply Lemma 2.2 to (3.13) and (3.15), and for \( p \in (Nq/(N-1), \infty), \) it holds that

\[
\sup_{0 < t < T} e^{-Mt^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}} \|D[v](t)\|_{L^p} \leq \frac{1}{4} \|v\|_{X_{T,M}}. \tag{3.16}
\]
On the other hand, we observe from (2.4) and (3.4) that

\[
|\partial_{x_N} D[v](x, t)| \\
\leq \int_0^t \int_{\mathbb{R}^N_+} |K(x, y, t - s)||F[v](y, s)|\,dy\,ds \\
\leq C \int_0^t \int_{\mathbb{R}^N_+} \tilde{K}(x_N, y_N, t - s)||F[v](\cdot, y_N, s)||_{L^\infty(\mathbb{R}^{N-1})}\,dy_N\,ds \\
\leq C\|v\|_{X_{T,M}} \int_0^t \int_{\mathbb{R}^N} e^{Ms} \tilde{K}(x_N, y_N, t - s)\left(1 + (y_N^{-1}s)^{\frac{1}{2}}\right)s^{-\frac{1}{2}}\,dy_N\,ds \tag{3.17}
\]

for \(x \in \mathbb{R}^N_+\) and \(0 < t < T\), where

\[
\tilde{K}(x_N, y_N, t) = \frac{|x_N - y_N|}{t} \Gamma_1(x_N - y_N, t) + \frac{x_N + y_N}{t} \Gamma_1(x_N + y_N, t)
\]

for \(x_N \geq 0, y_N > 0\) and \(t > 0\). Then, by (3.14) and (3.17) we see that

\[
|\partial_{x_N} D[v](x, t)| \leq C\|v\|_{X_{T,M}} \int_0^t e^{Ms} \left(s^{-\frac{1}{2}}(t - s)^{-\frac{1}{2}} + (t - s)^{-\frac{3}{4}}\right)\,ds \tag{3.18}
\]

for \(x \in \mathbb{R}^N_+\) and \(0 < t < T\). Furthermore, similarly to (3.17) and (3.18), we see that

\[
|\partial_{x_N} D[v](t)|_{L^r} \leq C\|v\|_{X_{T,M}} e^{-Mt} \int_0^t e^{Ms} \left(s^{-\frac{1}{2}}(t - s)^{-\frac{1}{2}} + (t - s)^{-\frac{3}{4}}\right)\,ds, \quad t > 0.
\]

This together with (3.18) yields

\[
\begin{align*}
e^{-Mt^\frac{1}{2}} |\partial_{x_N} D[v](t)|_{L^r} \\
&\leq C\|v\|_{X_{T,M}} e^{-Mt^\frac{1}{2}} \int_0^t e^{Ms} \left(s^{-\frac{1}{2}}(t - s)^{-\frac{1}{2}} + (t - s)^{-\frac{3}{4}}\right)\,ds, \quad r \in [q, p], \tag{3.19}
\end{align*}
\]
for $0 < t < T$. Moreover, for $p \in (Nq/(N-1), \infty)$, by (2.1) and (3.3) we have

$$\|\partial_{x_N} D[v](t)\|_{L^p} \leq \int_0^t \|\partial_{x_N} S_1(t-s)F[v](s)\|_{L^p} \, ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|F[v](s)\|_{L^p} \, ds$$

$$\leq C \|v\|_{X_{T,M}} \int_0^t (t-s)^{-\frac{1}{2}} (1 + s^{\frac{1}{p}}) s^{-\frac{1}{2}} e^{Ms} \, ds$$

$$\leq C \|v\|_{X_{T,M}} \left(1 + T^{\frac{1}{p}}\right) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} e^{Ms} \, ds$$

(3.20)

for $0 < t < T$. Then, by Lemma 2.2 with (3.18), (3.19), and (3.20), taking a sufficiently large $M \geq 1$ if necessary, for $p \in (Nq/(N-1), \infty]$ and $r \in [q,p]$, we see that

$$\sup_{0 < t < T} e^{-Mt^{\frac{N}{p}/2}} \|\partial_{x_N} D[v](t)\|_{L^p} \leq \frac{1}{8} \|v\|_{X_{T,M}},$$

$$\sup_{0 < t < T} e^{-Mt^{\frac{1}{2}}} \|D[v](t)\|_{L^r} \leq \frac{1}{8} \|v\|_{X_{T,M}}.$$ 

This together with (3.16) implies (3.12).

Next we prove the boundedness and smoothness of $D[v]$. It follows from the semigroup property of $S_1(t)$ that

$$D[v](x,t) = \int_0^t [S_1(t-s)F[v](s)](x) \, ds$$

$$= S_1(t-\tau)D[v](x,\tau) + \int_\tau^t [S_1(t-s)F[v](s)](x) \, ds$$

for $x \in \mathbb{R}^N_+$ and $0 < \tau < t < T$. Then, by (3.2) and (G2) we see that

$$S_1((t-\tau))D[v](x,\tau)$$

is bounded and smooth in $\mathbb{R}^N_+ \times (\tau,T)$. Furthermore, by (3.4) we apply the same argument as in [14, Section 3, Chapter 1] to see that

$$\int_\tau^t [S_1(t-s)F[v](s)](x) \, ds$$

is bounded and smooth in $\mathbb{R}^N_+ \times (\tau,T)$. (See also [9, Proposition 5.2] and [18, Lemma 2.1].) Then we deduce that $D[v]$ and $\partial_{x_N} D[v]$ are bounded and smooth in $\mathbb{R}^N_+ \times (\tau,T)$ for $0 < \tau < T$. Thus, Lemma 3.2 follows.

Now we are ready to complete the proof of Theorem 1.1.
Proof of Theorem 1.1. Let \( T > 0, M \geq 1, 1 \leq q < \infty, \) and \( p \in (Nq/(N - 1), \infty) \). Then, since \( \| \varphi \|_{L^q_r} \leq \| \varphi \|_{L^q_{t_r}} \) for \( r \in [1, \infty] \) and \( 0 \leq \alpha \leq \beta \), by (G1) and (2.2) we have

\[
e^{-Mt} E[S_1(t)\varphi](t) \leq (c_1 + c_2 + c_3)\| \varphi \|_{L^q_{(p)}} \tag{3.21}
\]

for \( t > 0 \), where \( c_1, c_2, \) and \( c_3 \) are positive constants given in (G1) and Lemma 2.1, respectively, and \( \alpha(p) \) is given in (1.9). Furthermore, by Lemma 3.2, taking a sufficiently large \( M \geq 1 \) if necessary, we see that

\[
\| D[v] \|_{X_{T,M}} \leq \frac{1}{2} \| v \|_{X_{T,M}}, \quad v \in X_{T,M}, \tag{3.22}
\]

for \( 0 < t < T \). Set

\[
m := 2(c_1 + c_2 + c_3)\| \varphi \|_{L^q_{(p)}}. \tag{3.23}
\]

We deduce from (3.1), (3.21), (3.22), and (3.23) that

\[
\| Q[v] \|_{X_{T,M}} \leq \sup_{0 < t < T} e^{-Mt} E[S_1(t)\varphi](t) + \| D[v] \|_{X_{T,M}}
\]

\[
\leq (c_1 + c_2 + c_3)\| \varphi \|_{L^q_{(p)}} + \frac{1}{2} \| v \|_{X_{T,M}} \leq m \tag{3.24}
\]

for \( v \in X_{T,M} \) with \( \| v \|_{X_{T,M}} \leq m \). Similarly, it follows from (3.22) that

\[
\| Q[v_1] - Q[v_2] \|_{X_{T,M}} = \| D[v_1] - v_2 \|_{X_{T,M}} \leq \frac{1}{2} \| v_1 - v_2 \|_{X_{T,M}} \tag{3.25}
\]

for \( v_i \in X_{T,M} \) (\( i = 1, 2 \)). Then, by (3.24) and (3.25), applying the contraction mapping theorem, we find a unique solution \( v \in X_{T,M} \) with \( \| v \|_{X_{T,M}} \leq m \) such that

\[
v = Q[v] = S_1(t)\varphi - D[v](t) \quad \text{in} \quad X_{T,M}.
\]

In particular, we see that

\[
\| v \|_{X_{T,M}} \leq C\| \varphi \|_{L^q_{(p)}}.
\]

Moreover, by (G2) and Lemma 3.2, we see that \( v \) is bounded and smooth in \( \overline{\mathbb{R}^N}_+ \times (T_1, T) \) for any \( 0 < T_1 < T \). Set

\[
w(x, t) = \int_0^t [S_2(t - s)\partial_{x_N} v(\cdot, 0, s)](x) \, ds
\]

for \( x \in \overline{\mathbb{R}^N}_+ \) and \( t \in (0, T) \). By (2.13) and (3.23) we obtain

\[
\| w(t) \|_{L^p} \leq \int_0^t \| S_2(t - s)\partial_{x_N} v(\cdot, 0, s) \|_{L^p} \, ds
\]

\[
\leq C \int_0^t \left( |\partial_{x_N} v(s)|_{L^p} + |\partial_{x_N} v(s)|_{L^p} \right) \, ds
\]

\[
\leq C \int_0^t e^{Ms} s^{-\frac{1}{2}} \| v \|_{X_{T,M}} \, ds \leq C e^{MT} T^{\frac{1}{2}} \| \varphi \|_{L^q_{(p)}} < \infty,
\]
and

\[ |w(t)|_{L^r} \leq \int_0^t |S_2(t - s)\partial_{x_N} v(\cdot, 0, s)|_{L^r} \, ds \]

\[ \leq C \int_0^t |\partial_{x_N} v(s)|_{L^r} \, ds \]

\[ \leq C \int_0^t e^{M_s s^{-\frac{1}{2}}} \|v\|_{X_{T,M}} \, ds \leq Ce^{MT}T^{\frac{1}{2}} \|\varphi\|_{L^q(\nu(p))} < \infty, \]

for \( 0 < t < T \). Furthermore, by \((P_3)\) we apply an argument similar to that in the proof of Lemma 3.2 and see that \( w \) is bounded and smooth in \( \mathbb{R}_+^N \times (T_1, T) \) for any \( 0 < T_1 < T \). Therefore, we deduce that \((v, w)\) is a solution of (1.6) in \( \mathbb{R}_+^N \times (0, T) \) satisfying (1.10).

Let \((\tilde{v}, \tilde{w})\) be a solution of (1.6) in \( \mathbb{R}_+^N \times (0, T_*) \) for any \( T_* > T \) and such that \( \tilde{v} \in X_{T_*,M_*} \) with some \( M_* > 0 \). Then, \( \tilde{v} \in X_{T,M} \) and since

\[ v - \tilde{v} = Q[v] - Q[\tilde{v}] = D[v - \tilde{v}] \quad \text{in} \quad X_{T,M}, \]

by (3.12) we have

\[ \|v - \tilde{v}\|_{X_{T,M}} \leq \frac{1}{2} \|v - \tilde{v}\|_{X_{T,M}}. \]

This implies that \( v = \tilde{v} \) in \( X_{T,M} \). Therefore, we deduce that \((v, w)\) is a unique global-in-time solution of (1.6) satisfying (1.10). Thus, Theorem 1.1 holds for the case \( q < \infty \). Furthermore, by [12, Theorem 1.1] with \( \varepsilon = 1 \) and \( \varphi_b \equiv 0 \) we have Theorem 1.1 for the case \( p = q = \infty \), and the proof of Theorem 1.1 is complete. \( \square \)

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