Improved approximation algorithms for $k$-connected $m$-dominating set problems

Zeev Nutov

The Open University of Israel. nutov@openu.ac.il.

Abstract. A graph is $k$-connected if it has $k$ internally-disjoint paths between every pair of nodes. A subset $S$ of nodes in a graph $G$ is a $k$-connected set if the subgraph $G[S]$ induced by $S$ is $k$-connected; $S$ is an $m$-dominating set if every $v \in V \setminus S$ has at least $m$ neighbors in $S$. If $S$ is both $k$-connected and $m$-dominating then $S$ is a $k$-connected $m$-dominating set, or $(k,m)$-cds for short. In the $k$-Connected $m$-Dominating Set ($(k,m)$-CDS) problem the goal is to find a minimum weight $(k,m)$-cds in a node-weighted graph. We consider the case $m \geq k$ and obtain the following approximation ratios. For unit disc-graphs we obtain ratio $O(k \ln k)$, improving the ratio $O(k^2 \ln k)$ of [5,15]. For general graphs we obtain the first non-trivial approximation ratio $O(k^2 \ln n)$.

1 Introduction

A graph is $k$-connected if it has $k$ internally disjoint paths between every pair of its nodes. A subset $S$ of nodes in a graph $G$ is a $k$-connected set if the subgraph $G[S]$ induced by $S$ is $k$-connected; $S$ is an $m$-dominating set if every $v \in V \setminus S$ has at least $m$ neighbors in $S$. If $S$ is both $k$-connected and $m$-dominating set then $S$ is a $k$-connected $m$-dominating set, or $(k,m)$-cds for short. A graph is a unit-disk graph if its nodes can be located in in the Euclidean plane such that there is an edge between nodes $u$ and $v$ iff the Euclidean distance between $u$ and $v$ is at most 1. We consider the following problem for $m \geq k$ both in general graphs and in unit-disk graphs.

$k$-Connected $m$-Dominating Set ($(k,m)$-CDS)

Input: A graph $G = (V,E)$ with node weights $\{w_v : v \in V\}$ and integers $k,m$.

Output: A minimum weight $(k,m)$-cds $S \subseteq V$.

The case $k = 0$ is the $m$-DOMINATING SET problem. Let $\alpha_m$ denote the best known ratio for $m$-DOMINATING SET; currently $\alpha_m = O(1)$ in unit-disk graphs [5] and $\alpha_m = \ln(\Delta + m) + 1 < \ln \Delta + 1.7$ in general graphs [4], where $\Delta$ is the maximum degree of the input graph. The $(k,m)$-CDS problem with $m \geq k$ was studied extensively. In recent papers Zhang, Zhou, Mo, and Du [15] and Fukumaga [5] obtained ratio $O(k^2 \ln k)$ for the problem in unit-disc graphs. For unit-disc graphs and $k = 2$ Zhang et al. [15] also obtained an improved ratio $\alpha_m + 5$. In a related paper Zhang et al. [16] obtained ratio $O(k \ln \Delta)$ in
general graphs with unit weights, mentioning that no non-trivial approximation algorithm for arbitrary weights is known.

Let us say that a graph with a designated set $T$ of terminals and a root node $r$ is $k$-$(T, r)$-connected if it contains $k$ internally-disjoint $rt$-paths for every $t \in T$. Our ratios for $(k, m)$-CDS are expressed in terms of $\alpha_m$ and the best ratio for the following known problem:

**Rooted Subset $k$-Connectivity**

**Input:** A graph $G = (V, E)$ with edge-costs/node-weights, a set $T \subseteq V$ of terminals, a root node $r \in V \setminus T$, and an integer $k$.

**Output:** A minimum cost/weight $k$-$(T, r)$-connected subgraph of $G$.

Let $\beta_k$ and $\beta'_k$ denote the best known ratios for the Rooted Subset $k$-Connectivity problem with edge-costs and node-weights, respectively. Currently, $\beta_m = O(1)$ in unit-disc graphs [5], while in general graphs $\beta_2 = 2$ [3], $\beta_3 = 6 \frac{2}{3}$ [13], and $\beta_k = O(k \ln k)$ for $k \geq 4$ [11]. We also have $\beta'_k = O(k^2 \ln n)$ by [11] and the correction of Vakilian [14] to the algorithm and the analysis of [11]; see also [6].

Our main results are summarized in the following theorem.

**Theorem 1.** Suppose that the $m$-Dominating Set problem admits ratio $\alpha_m$ and that the Rooted Subset $k$-Connectivity problem admits ratios $\beta_k$ for edge-costs and $\beta'_k$ for node-weights. Then $(k, m)$-CDS with $m \geq k$ admits ratios $\alpha_m + \beta'_k + 2(k - 1) = O(k^2 \ln n)$ for general graphs and $\alpha_m + 5\beta_k + 2(k - 1) = O(k \ln k)$ for unit-disc graphs. Furthermore, $(3, m)$-CDS on unit-disc graphs admits ratio $\alpha_m + 5\beta_3 = \alpha_m + 33\frac{1}{3}$.

Our algorithm uses the main ideas as well as partial results from the papers of Zhang et al. [15] and Fukunaga [5]. Let us say that a graph $G$ is $k$-T-connected if $G$ contains $k$ internally-disjoint paths between every pair of nodes in $T$. Both papers [15,5] consider unit-disc graphs and reduce the $(k, m)$-CDS problem with $m \geq k$ to the Subset $k$-Connectivity problem: given a graph with edge costs and a subset $T$ of terminals, find a minimum cost $k$-T-connected subgraph. The problem admits a trivial ratio $|T|^2$ for both edge-costs and node-weights, while for $|T| > k$ the best known ratios are $\frac{|T|}{|T|-k} O(k \ln k) = O(k^2 \ln k)$ for edge-costs and $\frac{|T|}{|T|-k} O(k^2 \ln n) = O(k^3 \ln n)$ for node-weights [12]; see also [8]. In fact, these ratios are derived by applying $O(k)$ times the algorithm for the Rooted Subset $k$-Connectivity problem. The main reason for our improvement over the ratios of [15,5] is a reduction to the easier Rooted Subset $k$-Connectivity problem. For small values of $k$ we present a refined reduction, but for unit disc graphs and $k = 2$ the performance of our algorithm and that of [15] coincide, since for $k = 2$ and edge-costs both Subset $k$-Connectivity and Rooted Subset $k$-Connectivity admit ratio 2 [3].
2 Proof of Theorem 1

For an arbitrary graph $H = (U, F)$ and $u, v \in U$ let $\kappa_H(u, v)$ denote the maximum number of internally disjoint $uv$-paths in $H$. We say that $H$ is $k$-in-connected to $r$ if $H$ is $k$-$(U \setminus \{r\}, r)$-connected, namely, if $\kappa_H(v, r) \geq k$ every $v \in U \setminus \{r\}$. For $A \subseteq U$ let $\Gamma_H(A)$ denote the set of neighbors of $A$ in $H$. The proof of the following known statement can be found in [7], see also [12]; part (i) of the lemma relies on the Mader’s Undirected Critical Cycle Theorem [9].

**Lemma 1.** Let $H_r$ be $k$-in-connected to $r$ and let $R = \Gamma_H(r)$.

(i) The graph $H = H_r \setminus \{r\}$ can be made $k$-connected by adding a set $J$ of new edges on $R$; furthermore, if $J$ is inclusionwise-minimal then $J$ is a forest.

(ii) Suppose that $|R| = k$. If $k = 2, 3$ then $H_r$ is $k$-connected.

Note that an inclusionwise-minimal edge set $J$ as in Lemma 1(i) can be computed in polynomial time, by starting with $J$ being a clique on $R$ and repeatedly removing from $J$ an edge $e$ if $H \cup (J \setminus e)$ remains $k$-connected.

A reason why the case $m \geq k$ is easier is given in the following lemma.

**Lemma 2.** If a graph $H = (V, E)$ has a $k$-dominating set $T$ such that $H$ is $k$-$T$-connected then $H$ is $k$-connected.

**Proof.** By a known characterization of $k$-connected graphs, it is sufficient to show that $|V \setminus (A \cup B)| \geq k$ holds for any subpartition $A, B$ of $V$ such that $E$ has no edge between $A$ and $B$. If both $A \cap T, B \cap T$ are non-empty, this is so since $H$ is $k$-$T$-connected. Otherwise, if say $A \cap T = \emptyset$, then since $T$ is a $k$-dominating set we have $|\Gamma_H(A)| \geq k$, and the result follows. $\square$

Finally, we will need the following known fact, c.f. [11].

**Lemma 3.** Given a pair $s, t$ of nodes in a node-weighted graph $G$, the problem of finding a minimum weight node set $P_{st}$ such that $G[P_{st}]$ has $k$ internally-disjoint $st$-paths admits a 2-approximation algorithm.

For arbitrary $k$, we will show that the following algorithm achieves the desired approximation ratio.

**Algorithm 1:** $(G = (V, E), w, m \geq k)$

1. compute an $\alpha_m$-approximate $m$-dominating set $T$
2. construct a graph $G_r$ by adding to $G$ a new node $r$ connected to a set $R \subseteq T$ of $k$ nodes by a set $F_r = \{rv : v \in R\}$ of new edges
3. compute a $\beta_k^G$-approximate node set $S \subseteq V \setminus T$ such that the subgraph $H_r$ of $G_r$ induced by $T \cup S \cup \{r\}$ is $k$-$(T, r)$-connected
4. let $H = H \setminus \{r\} = G[T \cup S]$ and let $J$ be a forest of new edges on $R$ as in Lemma 1(i) such that the graph $H \cup J$ is $k$-connected
5. for every $uv \in J$ find a 2-approximate node set $P_{uv}$ such that $G[T \cup S \cup P_{uv}]$ has $k$ internally-disjoint $uv$-paths; let $P = \bigcup_{uv \in J} P_{uv}$
6. return $T \cup S \cup P$
We now prove that the solution computed is feasible.

**Lemma 4.** The computed solution is feasible, namely, at the end of the algorithm $T \cup S \cup P$ is a $(k, m)$-cds.

**Proof.** Since $T$ is an $m$-dominating set, so is any superset of $T$. Thus the node set $T \cup S \cup P$ returned by the algorithm is an $m$-dominating set.

It remains to prove that $T \cup S \cup P$ is a $k$-connected set. We first prove that the graph $H_r$ computed at step 3 is $k$-in-connected to $r$. By Menger’s Theorem, $\kappa_{H_r}(v, r) \geq k$ if for all $A \subseteq T \cup S$ with $v \in A$

$$|\Gamma_{H_r \setminus R}(A)| + |A \cap R| \geq k. \tag{1}$$

Let $\emptyset \neq A \subseteq T \cup S$. If $A \cap T \neq \emptyset$ then (1) holds since $H_r$ is $k$-$(T, r)$-connected. If $A \cap S \neq \emptyset$ then $|\Gamma_{H_r \setminus R}(A)| \geq m \geq k$, since $T$ is an $m$-dominating set and thus every node in $A \cap S$ has at least $m$ neighbors in $T$. In both cases, (1) holds, hence $H_r$ is $k$-in-connected to $r$.

The graph $H \cup J$ is $k$-connected, which implies that the graph $G(T \cup S \cup P)$ is $(T \cup S)$-$k$-connected and thus $T$-$k$-connected. Furthermore, $T$ is a $k$-dominating set, since $m \geq k$. Applying Lemma 2 on the graph $G[T \cup S \cup P]$ we get that this graphs is $k$-connected, as required. \hfill $\Box$

**Lemma 5.** Algorithm $T$ has ratio $\alpha_m + \beta'_k + 2(k - 1)$.

**Proof.** Let $S^*$ be an optimal solution to $(k, m)$-CDS. Clearly, $w(T) \leq \alpha_m w(S^*) \leq \beta'_k w(S^*)$. We claim that $w(S) \leq \beta'_k w(S^* \setminus T)$. For this note that $S^* \setminus T$ is a feasible solution to the problem considered at step 3 of the algorithm, while $S$ is a $\beta'_k$-approximate solution. For the same reason, for each $w \in J$ the set $S^* \setminus (T \cup S)$ is a feasible solution to the problem considered at step 5, while the set $P_w$ computed is a 2-approximate solution; thus $w(P_w) \leq 2w(S^* \setminus (T \cup S))$. Finally, note that $|J| \leq k - 1$, and thus $w(P) \geq 2(k - 1)w(S^*)$. The lemma follows. \hfill $\Box$

This concludes the proof of the case of general $k$ and general graphs. Let us now consider unit disc graphs. Then we use the following result of [15].

**Theorem 2 (Zhang, Zhou, Mo, and Du [15]).** Any $k$-connected unit-disc graph has a $k$-connected spanning subgraph of maximum degree at most 5 if $k = 2$, and at most $5k$ if $k \geq 3$.

Note that any $k$-connected graph has minimum degree $k$. Thus Theorem 2 implies that when searching for a $k$-connected subgraph in a unit disc graph, one can convert node-weights to edge-costs while invoking in the ratio only a factor of 5/2 in the case $k = 2$ and 5 in the case $k \geq 3$. Specifically, given node weights $\{w_v : v \in V\}$ define edge-costs $c_{uv} = w_u + w_v$. Then for any subgraph $(S, F)$ of $G$ with maximum degree $\Delta$ and minimum degree $\delta$ we have:

$$\delta w(S) \leq c(F) \leq \Delta w(S)$$
since \( w_v \geq 0 \) for all \( v \in V \) and since
\[
c(F) = \sum_{uv \in E} (w_u + w_v) = \sum_{v \in V} d_F(v)w_v.
\]

We may use this conversion in some steps of our algorithm, and specifically in step 3, which concludes the proof of the case of general \( k \) and unit-disc graphs.

In the case \( k = 3 \) we use a result of Mader \cite{10} that any edge-minimal \( k \)-connected graph has at least \( \frac{(k-1)n+2}{2k-1} \) nodes of degree \( k \). At step 3 of the algorithm we “guess” such a node \( r \) and the 3 edges incident to \( r \) in some edge-minimal optimal solution, remove from \( G \) all other edges incident to \( r \), and run step 3 while omitting steps 4 and 5. By Lemma 1(ii) the graph \( G[S \cup T] \) will be already 3-connected.

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