A Beurling-Blecher-Labuschagne type theorem for Haagerup noncommutative $L^p$ spaces

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Abstract Let $M$ be a $\sigma$-finite von Neumann algebra, equipped with a normal faithful state $\varphi$, and let $A$ be maximal subdiagonal subalgebra of $M$ and $1 \leq p < \infty$. We prove a Beurling-Blecher-Labuschagne type theorem for $A$-invariant subspaces of Haagerup noncommutative $L^p(M)$ and give a characterization of outer operators in Haagerup noncommutative $H^p$-spaces associated with $A$.

Keywords subdiagonal algebras, Beurling’s theorem, invariant subspace, outer operator, Haagerup noncommutative $H^p$-space

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1 Introduction

Arveson introduced his notion of subdiagonal subalgebras of von Neumann algebras (see [1]), in effect, subdiagonal algebras are the noncommutative analogue of weak$^*$ Dirichlet algebras (for the definition of weak$^*$ Dirichlet algebras see [23]). For the finite and semi-finite case, most results on the classical Hardy spaces on the torus have been established in this noncommutative setting. We refer to [1, 3, 4, 5, 6, 10, 12, 15, 17, 20, 21, 23] (see also [9] for more historical references). It is natural to consider the case of $\sigma$-finite von Neumann algebras. But, the transition from finite or semifinite to $\sigma$-finite von Neumann algebras is not trivial, need some new techniques and some changes. For some results for this case, see [5, 12, 15, 16, 17, 20].

Let $M$ be a finite von Neumann and $A$ be its Arveson’s maximal subdiagonal subalgebras. In [6], Blecher and Labuschagne extended the classical Beurling’s theorem to describe closed $A$-invariant subspaces in noncommutative space $L^p(M)$ with $1 \leq p \leq \infty$. Sager [21] extended the work of Blecher and Labuschagne from a finite von Neumann algebra to semifinite von Neumann algebras, proved a Beurling-Blecher-Labuschagne theorem for $A$-invariant spaces of $L^p(M)$ when $0 < p \leq \infty$. The Beurling theorem has been generalized to the setting of unitarily invariant norms on finite and semifinite von Neumann algebras (see [1, 10, 22]).

When $A$ is a subdiagonal subalgebra of $\sigma$-finite von Neumann $M$, Labuschagne [17] showed that a Beurling type theory of invariant subspaces of noncommutative $H^2$-spaces holds true. A motivation for this paper is to extend the result in [17] to the setting of the Haagerup noncommutative $L^p$-spaces for $1 \leq p < \infty$.

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Blecher and Labuschagne [7] studied outer operators of the noncommutative $L^p$-spaces associated with Arveson’s subdiagonal subalgebras. They proved inner-outer factorization theorem and characterizations of outer operators for the case $1 \leq p < \infty$ (for the case $p < 1$, see [2]). In [8], they extended their generalized inner-outer factorization theorem in [7] and established characterizations of outer operators that are valid even in the case of operators with zero determinant. In this paper, we apply Labuschagne’s Beurling type theorem for $\mathcal{A}$-invariant subspaces of Haagerup noncommutative $L^2$-spaces to prove a Blecher-Labuschagne theorem for outer operators in Haagerup noncommutative $H^p$-spaces ($1 \leq p < \infty$).

The organization of the paper is as follows. In Section 2, we give some definitions and related results of Haagerup noncommutative $L^p$-spaces and $H^p$-spaces. A Blecher-Labuschagne-Beurling type theorem for Haagerup noncommutative $L^p$-spaces is presented in Section 3. In Section 4, we give characterizations of outer operators in Haagerup noncommutative $H^p$-spaces.

2 Preliminaries

Our references for modular theory are [19][24], for the Haagerup noncommutative $L^p$-spaces are [11][25] and for the Haagerup noncommutative $H^p$-spaces are [13][14]. Let us recall some basic facts about the Haagerup noncommutative $L^p$-spaces and the Haagerup noncommutative $H^p$-spaces, and fix the relevant notation used throughout this paper. Throughout this paper $\mathcal{M}$ will always denote a $\sigma$-finite von Neumann algebra on a complex Hilbert space $\mathcal{H}$, equipped with a distinguished normal faithful state $\varphi$. Let $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ be the one parameter modular automorphism group of $\mathcal{M}$ associated with $\varphi$. We denote by

$$\mathcal{N} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$$

the crossed product of $\mathcal{M}$ by $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$. It is well known that $\mathcal{N}$ is the semi-finite von Neumann algebra acting on the Hilbert space $L^2(\mathbb{R}, \mathcal{H})$, generated by

$$\{\pi(x) : x \in \mathcal{M}\} \cup \{\lambda(s) : s \in \mathbb{R}\},$$

where the operator $\pi(x)$ is defined by

$$(\pi(x)\xi)(t) = \sigma_t^\varphi(x)\xi(t), \quad \forall \xi \in L^2(\mathbb{R}, \mathcal{H}), \quad \forall t \in \mathbb{R},$$

and the operator $\lambda(s)$ is defined by

$$(\lambda(s)\xi)(t) = \xi(t - s), \quad \forall \xi \in L^2(\mathbb{R}, \mathcal{H}), \quad \forall t \in \mathbb{R}.$$ }

We will identify $\mathcal{M}$ and the subalgebra $\pi(\mathcal{M})$ of $\mathcal{N}$. The operators $\pi(x)$ and $\lambda(t)$ satisfy

$$\lambda(t)\pi(x)\lambda(t)^* = \pi(\sigma_t^\varphi(x)), \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathcal{M}.$$ }

Then

$$\sigma_t^\varphi(x) = \lambda(t)x\lambda(t)^*, \quad x \in \mathcal{M}, \quad t \in \mathbb{R}.$$ 

We denote by $\{\hat{\sigma}_t\}_{t \in \mathbb{R}}$ the dual action of $\mathbb{R}$ on $\mathcal{N}$, this is a one parameter automorphism group of $\mathcal{R}$ on $\mathcal{N}$, implemented by the unitary representation $\{W_t\}_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $L^2(\mathbb{R}, \mathcal{H})$:

$$\hat{\sigma}_t(x) = W(t)xW^*(t), \quad \forall x \in \mathcal{N}, \quad \forall t \in \mathbb{R},$$

(2.1)

where

$$W(t)(\xi)(s) = e^{-ist}\xi(s), \quad \forall \xi \in L^2(\mathbb{R}, \mathcal{H}), \quad \forall s, t \in \mathbb{R}.$$ 

Note that the dual action $\hat{\sigma}_t$ is uniquely determined by the following conditions: for any $x \in \mathcal{M}$ and $s \in \mathbb{R},$

$$\hat{\sigma}_t(x) = x \quad \text{and} \quad \hat{\sigma}_t(\lambda(s)) = e^{-ist}\lambda(s), \quad \forall t \in \mathbb{R}.$$ 

Hence

$$\mathcal{M} = \{x \in \mathcal{N} : \hat{\sigma}_t(x) = x, \forall t \in \mathbb{R}\}.$$ 

Let $\tau$ be the unique normal semi-finite faithful trace on $\mathcal{N}$ satisfying

$$\tau \circ \hat{\sigma}_t = e^{-t}\tau, \quad \forall t \in \mathbb{R}.$$
Also recall that the dual weight \( \hat{\varphi} \) of our distinguished state \( \varphi \) has the Radon-Nikodym derivative \( D \) with respect to \( \tau \), which is the unique invertible positive selfadjoint operator on \( L^2(\mathbb{R}, \mathcal{H}) \), affiliated with \( \mathcal{N} \) such that

\[
\hat{\varphi}(x) = \tau(Dx), \quad x \in \mathcal{N}_+.
\]

Recall that the regular representation of the above \( \lambda(t) \) is given by

\[
\lambda(t) = D^t, \quad \forall t \in \mathbb{R}.
\]

Now, we define Haagerup noncommutative \( L^p \)-spaces. Let \( L^0(\mathcal{N}, \tau) \) denote the topological \( * \)-algebra of all operators on \( L^2(\mathbb{R}, \mathcal{H}) \) measurable with respect to \( (\mathcal{N}, \tau) \). Then the Haagerup noncommutative \( L^p \)-spaces, \( 0 < p \leq \infty \), are defined by

\[
L^p(\mathcal{M}, \varphi) = \{ x \in L^0(\mathcal{N}, \tau) : \tilde{\sigma}_t(x) = e^{-t}x, \forall t \in \mathbb{R} \}.
\]

The spaces \( L^p(\mathcal{M}, \varphi) \) are closed selfadjoint linear subspaces of \( L^0(\mathcal{N}, \tau) \). It is not hard to show that

\[
L^\infty(\mathcal{M}, \varphi) = \mathcal{M}.
\]

Since for any \( \psi \in \mathcal{M}_\varphi^+ \), the dual weight \( \hat{\psi} \) has a Radon-Nikodym derivative with respect to \( \tau \), denoted by \( D_\psi : \)

\[
\hat{\psi}(x) = \tau(D_\psi x), \quad x \in \mathcal{N}_+.
\]

Then

\[
D_\psi \in L^0(\mathcal{N}, \tau)
\]

and

\[
\tilde{\sigma}_t(D_\psi) = e^{-t}D_\psi, \quad \forall t \in \mathbb{R}.
\]

So

\[
D_\psi \in L^1(\mathcal{M}, \varphi)_+.
\]

It is well known that the map \( \psi \mapsto D_\psi \) on \( \mathcal{M}_\varphi^+ \) extends to a linear homeomorphism from \( \mathcal{M}_\varphi^+ \) onto \( L^1(\mathcal{M}, \varphi) \) (equipped with the vector space topology inherited from \( L^0(\mathcal{N}, \tau) \)). This permits to transfer the norm on \( \mathcal{M}_\varphi^+ \) into a norm on \( L^1(\mathcal{M}, \varphi) \), denoted by \( \| \cdot \| \). Moreover, \( L^1(\mathcal{M}, \varphi) \) is equipped with a distinguished contractive positive linear functional \( \text{tr} \), defined by

\[
\text{tr}(D_\psi) = \psi(1), \quad \psi \in \mathcal{M}_\varphi^+.
\]

Therefore, \( \| x \|_1 = \text{tr}(\| x \|) \) for every \( x \in L^1(\mathcal{M}, \varphi) \).

Let \( 0 < p < \infty \) and \( x \in L^0(\mathcal{N}, \tau) \). If \( x = u|x| \) is the polar decomposition of \( x \), then \( x \in L^p(\mathcal{M}, \varphi) \) \( \Leftrightarrow \) \( u \in \mathcal{M} \) and \( |x| \in L^p(\mathcal{M}, \varphi) \) \( \Leftrightarrow \) \( u \in \mathcal{M} \) and \( |x|^p \in L^1(\mathcal{M}, \varphi) \). If we define

\[
\| x \|_p = \| |x|^p \|_1^{1/p}, \quad \forall x \in L^p(\mathcal{M}, \varphi),
\]

then for \( 1 \leq p < \infty \) (resp. \( 0 < p < 1 \)),

\[
(L^p(\mathcal{M}, \varphi), \| \cdot \|_p)
\]

is a Banach space (resp. a quasi-Banach space), and

\[
\| x \|_p = \| x^* \|_p = \| x^p \|, \quad \forall x \in L^p(\mathcal{M}, \varphi).
\]

It is proved in [11] and [25] that \( L^p(\mathcal{M}, \varphi) \) is independent of \( \varphi \) up to isometry. Hence, we denote \( L^p(\mathcal{M}, \varphi) \) by \( L^p(\mathcal{M}) \).

The usual Holder inequality also holds for the \( L^p(\mathcal{M}) \) spaces. It means that the product of \( L^0(\mathcal{N}, \tau), (x, y) \mapsto xy \), restricts to a contractive bilinear map

\[
L^p(\mathcal{M}) \times L^q(\mathcal{M}) \to L^{r}(\mathcal{M}),
\]

where
where \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \). In particular, if \( \frac{1}{p} + \frac{1}{q} = 1 \), then the bilinear form \((x, y) \mapsto \text{tr}(xy)\) defines a duality bracket between \(L^p(\mathcal{M})\) and \(L^q(\mathcal{M})\), for which \(L^{\infty}(\mathcal{M})\) coincides (isometrically) with the dual of \(L^p(\mathcal{M})\) (if \(p \neq \infty\)). Moreover, the \(\text{tr}\) have the following property:

\[
\text{tr}(xy) = \text{tr}(yx), \quad \forall x \in L^p(\mathcal{M}), \quad \forall y \in L^q(\mathcal{M}).
\]

Let \(0 < p \leq \infty\). For \(K \subset L^p(\mathcal{M})\), we denote the closed linear span of \(K\) in \(L^p(\mathcal{M})\) by \([K]_p\) (relative to the \(w^*\)-topology for \(p = \infty\)) and the set \(\{x^* : x \in K\}\) by \(J(K)\).

For \(0 < p < \infty\), \(0 \leq \eta \leq 1\), we have that

\[
L^p(\mathcal{M}) = [D^{\frac{1}{1-p}} \mathcal{M} D^{\frac{p}{p}}]^p.
\]

Let \(\mathcal{D}\) be a von Neumann subalgebra of \(\mathcal{M}\) and \(\mathcal{E}\) be a faithful normal conditional expectation from \(\mathcal{M}\) onto \(\mathcal{D}\).

**Definition 1** A \(w^*\)-closed subalgebra \(\mathcal{A}\) of \(\mathcal{M}\) is called a subdiagonal subalgebra of \(\mathcal{M}\) with respect to \(\mathcal{E}\) (or to \(\mathcal{D}\)) if

(i) \(\mathcal{A} + J(\mathcal{A})\) is \(w^*\)-dense in \(\mathcal{M}\),
(ii) \(\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y), \quad \forall x, y \in \mathcal{A}\),
(iii) \(\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}\).

The algebra \(\mathcal{D}\) is called the diagonal of \(\mathcal{A}\).

In [1], subdiagonal subalgebras are not assumed to be \(w^*\)-weakly closed. Since the \(w^*\) closure of an algebra that is subdiagonal with respect to \(\mathcal{E}\) will also be subdiagonal with respect to \(\mathcal{E}\) (see Remark 2.1.2 in [1]), we may assume that our subdiagonal subalgebras are always \(w^*\)-weakly closed (the definition as in [13,14,26]). Since \(\mathcal{M}\) is \(\sigma\)-finite, we may take a faithful normal state \(\phi\) on \(\mathcal{M}\) such that \(\phi \circ \mathcal{E} = \phi\). It is well known (cf. [24]) that the existence of a (unique) normal conditional expectation \(\mathcal{E} : \mathcal{M} \to \mathcal{D}\) such that \(\varphi \circ \mathcal{E} = \varphi\) is equivalent to \(\sigma_t^\mathcal{E}(\mathcal{D}) = \mathcal{D}\) for all \(t \in \mathbb{R}\). Hence, in the rest of this paper \(\mathcal{D}\) always denotes a normal faithful state satisfying \(\varphi \circ \mathcal{E} = \varphi\).

If \(\mathcal{A}\) is not properly contained in any other subalgebra of \(\mathcal{M}\) which is a subdiagonal with respect to \(\mathcal{E}\), We call \(\mathcal{A}\) is a maximal subdiagonal subalgebra of \(\mathcal{M}\) with respect to \(\mathcal{E}\) (or to \(\mathcal{D}\)). Let

\[
\mathcal{A}_0 = \{x \in \mathcal{A} : \mathcal{E}(x) = 0\}
\]

Then by [1] Theorem 2.2.1, \(\mathcal{A}\) is maximal if and only if

\[
\mathcal{A} = \{x \in \mathcal{M} : \mathcal{E}(yxz) = \mathcal{E}(yxz) = 0, \forall y \in \mathcal{A}, \forall z \in \mathcal{A}_0\}.
\]

It follows from [12] Theorem 2.4 and [20] Theorem 1.1 (also see [17] Theorem 1.1]) that a subdiagonal subalgebra \(\mathcal{A}\) of \(\mathcal{M}\) with respect to \(\mathcal{D}\) is maximal if and only if

\[
\sigma_t^\mathcal{E}(\mathcal{A}) = \mathcal{A}, \quad \forall t \in \mathbb{R}.
\]  

In this paper \(\mathcal{A}\) always denotes a maximal subdiagonal subalgebra in \(\mathcal{M}\) with respect to \(\mathcal{E}\).

**Definition 2** For \(0 < p < \infty\), we define the Haagerup noncommutative \(H^p\)-space that

\[
H^p(\mathcal{A}) = [AD^\frac{1}{p}]_p, \quad H^p_0(\mathcal{A}) = [A_0D^\frac{1}{p}]_p.
\]

If \(1 \leq p < \infty\), \(0 \leq \eta \leq 1\), then by [14] Proposition 2.1, we have that

\[
H^p(\mathcal{A}) = [D^{\frac{1}{1-p}} AD^\frac{1}{p}]_p, \quad H^p_0(\mathcal{A}) = [D^{\frac{1}{1-p}} A_0D^\frac{1}{p}]_p.
\]  

By [15] Proposition 2.7, we know that

\[
\mathcal{A} = \{x \in \mathcal{M} : \text{tr}(xa) = 0, \forall a \in H^1_0(\mathcal{A})\}.
\]

It is known that

\[
L^p(\mathcal{D}) = [D^{\frac{1}{p}} DD^\frac{1}{p}]_p, \quad \forall p \in [1, \infty), \quad \forall \eta \in [1, 0].
\]
Therefore, if $1 \leq p, q, r < \infty$ and $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$, then
\[ [H^q(A)D^\frac{1}{p}]_p = H^p(A) \quad \text{and} \quad [L^2(D)D^\frac{1}{p}]_p = L^p(D) \quad (2.6) \]

For $1 \leq p \leq \infty$, the conditional expectation $E$ extends to a contractive projection from $L^p(M)$ onto $L^p(D)$. The extension will be denoted still by $E$ (see [16, Proposition 2.3]). Let
\[ 1 \leq r, p, q < \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \]

Then
\[ E(xy) = E(x)E(y), \quad \forall x \in H^p(A), \quad \forall y \in H^q(A). \]

Let $M_a$ be the family of analytic vectors in $M$. Recall that $x \in M_a$ if only if the function $t \mapsto \sigma_t(x)$ extends to an analytic function from $\mathbb{C}$ to $M$. $M_a$ is a $w^*$-dense $*$-subalgebra of $M$ (cf. [19]).

The next result is known. For easy reference, we give its proof (see the proof of Theorem 2.5 in [19]).

**Lemma 1** Let $A_a$ and $D_a$ be respectively the families of analytic vectors in $A$ and $D$. If $1 \leq p < \infty$, then:

(i) $A_a$ is a $w^*$-dense in $A$, $(A_a)_0$ is a $w^*$-dense in $A_0$ and $D_a$ is a $w^*$-dense in $D$, where $(A_a)_0 = \{x \in A_a : E(x) = 0\}$;

(ii) $D^\frac{1}{p}A_a = A_aD^\frac{1}{p}$, \quad $D^\frac{1}{p}(A_a)_0 = (A_a)_0D^\frac{1}{p}$, \quad $D^\frac{1}{p}D_a = D_aD^\frac{1}{p}$;

(iii) $A_ad^\frac{1}{p}$ is dense in $H^p(A)$, $(A_a)_0d^\frac{1}{p}$ is dense in $H^p_0(A)$ and $D_ad^\frac{1}{p}$ is dense in $L^p(D)$.

**Proof** (i) Let $x \in A$. We define
\[ x_n = \sqrt{\frac{n}{\pi}} \int_\mathbb{R} e^{-nt^2} \sigma_t(x) dt. \]

By (2.2), $x_n \in A$. Moreover by [19] p. 58, $x_n \in A_a$ and $x_n \rightharpoonup x$ $w^*$-weakly. Since
\[ \sigma_t^x(A_0) = A_0, \quad \sigma_t^x(D) = D, \quad \forall t \in \mathbb{R} \]
(see [12] p. 313]), a similar argument works for $A_0$ and $D$.

(ii) We prove only the first equivalence. The proofs of the two others are similar. Let $x \in A_a$. Then
\[ D^\frac{1}{p}x = [D^\frac{1}{p}x D^\frac{1}{p}]D^\frac{1}{p} = [\sigma_t^x(x)]D^\frac{1}{p} \in A_aD^\frac{1}{p}, \]
whence $D^\frac{1}{p}x \subseteq A_aD^\frac{1}{p}$. The inverse inclusion can be proved in a similar way.

(iii) Let $p'$ be the conjugate index of $p$. If $y \in L^{p'}(M)$ such that $tr(aD^\frac{1}{p}y) = 0$, $\forall a \in A_a$, then by (i),
\[ tr(aD^\frac{1}{p}y) = 0, \quad \forall a \in A, \]

since $D^\frac{1}{p}y \in L^1(M)$. Hence, by (2.3),
\[ tr(xy) = 0, \quad \forall x \in H^p(A) \]

By the Hahn-Banach theorem, $A_ad^\frac{1}{p}$ is dense in $H^p(A)$. Similarly, we can prove the two others.
3 $\mathcal{A}$-invariant subspaces of $L^p(\mathcal{M})$

We recall that a right (resp. left) $\mathcal{A}$-invariant subspace of $L^p(\mathcal{M})$, is a closed subspace $K$ of $L^p(\mathcal{M})$ such that $KA \subset K$ (resp. $AK \subset K$).

In the case when von Neumann algebra $\mathcal{M}$ is finite, for a right $\mathcal{A}$-invariant subspace $K$ of $L^2(\mathcal{M})$, Blecher and Labuschagne [2] defined the right wandering subspace of $K$ to be the space $W = K \oplus [KA_0]^2$; and they say that $K$ is type 1 if $W$ generates $K$ as an $\mathcal{A}$-module (that is, $K = [WA]^2$) and say that $K$ is type 2 if $W = \{0\}$ (also see [18], but the last notation conflicts with that of [15], where this class of subspaces is decomposed into two further subclasses which Nakazi and Watatani call type II and type III). If $p \neq 2$, Blecher and Labuschagne [6] defined the wandering quotient to be $K/[KA_0]^p$, and say that $K$ is type 2 if this is trivial. It turns out that the wandering quotient is an $L^p(\mathcal{D})$-module in the sense of Junge and Sheran (see [15]), and it is isometric to a canonically defined subspace of $K$ which can be called the right wandering subspace of $K$. They say that $K$ is type 1 if this subspace generates $K$ as an $\mathcal{A}$-module. For the case $1 \leq p < 2$ (resp. $p > 2$), they have shown that $K$ is type 1 iff $K \cap \mathcal{L}^2(\mathcal{M})$ (resp. $[K]^2$) is type 1 in the sense of the $\mathcal{L}^2$ case above.

Now, in the case that $\mathcal{M}$ is a $\sigma$-finite von Neumann algebra. Recall that if $K$ is a right $\mathcal{A}$-invariant subspace of $L^2(\mathcal{M})$, then

$$W = K \oplus [KA_0]^2$$

is often called the right wandering subspace of $K$. We say that $K$ is type 1 if $W$ generates $K$ as an $\mathcal{A}$-module (that is $K = [WA]^2$) and $K$ is type 2 if $W = \{0\}$ (see [14]).

**Proposition 1** Let $1 \leq p, q, r < \infty$, and $K$ be a closed subspace of $L^p(\mathcal{M})$. Suppose $\frac{1}{p} - \frac{1}{r} = \frac{1}{q}$, and $K_r = \{x \in K : xD^{-\frac{1}{q}} \in L^q(\mathcal{M})\}$. If $[K_r]^q = K$, then

$$[[K_r, D^{-\frac{1}{q}}]_q D^\frac{1}{r}]_p = K.$$

**Proof** (1) If $x \in [K_r, D^{-\frac{1}{q}}]_q$, then there is a sequence $(x_n) \subset K_r$ such that $x_n D^{-\frac{1}{q}} \to x$ in norm in $L^q(\mathcal{M})$. Hence, $x_n \to xD^\frac{1}{r}$ in norm in $L^p(\mathcal{M})$. It follows that $[K_r, D^{-\frac{1}{q}}]_q D^\frac{1}{r} \subset K$, and so $[[K_r, D^{-\frac{1}{q}}]_q D^\frac{1}{r}]_p \subset K$. On the other hand, since $K_r \subset [K_r, D^{-\frac{1}{q}}]_q D^\frac{1}{r}$, $K = [K_r]^q \subset [[K_r, D^{-\frac{1}{q}}]_q D^\frac{1}{r}]_p$. Therefore, we obtain the desired result.

**Lemma 2** Let $1 \leq p < \infty$, and let $K$ be an $\mathcal{A}$-invariant subspace of $L^p(\mathcal{M})$.

(i) If $1 \leq q, r < \infty$ and $\frac{1}{p} - \frac{1}{r} = \frac{1}{q}$, then $[K_r, D^{-\frac{1}{q}}]_q$ is a right $\mathcal{A}$-invariant subspace of $L^q(\mathcal{M})$, where $K_r = \{x \in K : xD^{-\frac{1}{q}} \in L^q(\mathcal{M})\}$.

(ii) If $1 \leq q, r < \infty$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$, then $[K_r, D^\frac{1}{r}]_q$ is a right $\mathcal{A}$-invariant subspace of $L^q(\mathcal{M})$.

**Proof** (i) It is clear that $[K_r, D^{-\frac{1}{q}}]_q \subset L^q(\mathcal{M})$. Using (ii) of Lemma [1] we get that

$$K_r D^{-\frac{1}{q}} A_a = K_r A_a D^{-\frac{1}{q}}.$$  

(3.1)

On the other hand, for any $a \in A_a$ and $x \in K_r$, we have that $xa \in K_r$. By (3.1), there is an element $a' \in A_a$ such that $xaD^{-\frac{1}{q}} = xD^{-\frac{1}{q}}a'$. It follows that $xaD^{-\frac{1}{q}} \in L^q(\mathcal{M})$, and so $xa \in K_r$. Hence, $K_r A_a \subset K_r$. From (3.1) follows that $K_r D^{-\frac{1}{q}} A_a \subset K_r D^{-\frac{1}{q}}$ and

$$[K_r, D^{-\frac{1}{q}}]_q A_a \subset [K_r, D^{-\frac{1}{q}}]_q.$$  

(3.2)

Now if $a \in A$, then by (i) in Lemma [1] we have a sequence $(a_n)$ in $A_a$ such that $a_n \to a$ w*-weakly. Hence, $tr(xD^{-\frac{1}{q}}a_n y) \to tr(xD^{-\frac{1}{q}}a y)$,  \forall x \in K_r,  \forall y \in L^q(\mathcal{M}),$

where $q'$ is the conjugate index of $q$. Since the weak closure of $K_r D^{-\frac{1}{q}} A_a$ is equal to $[K_r, D^{-\frac{1}{q}}]_q,$

$$xD^{-\frac{1}{q}} a \in [K_r, D^{-\frac{1}{q}}]_q.$$
Using (3.2), we get

\[ [K, D^{-\frac{1}{p}}]_q \subset [K, D^{-\frac{1}{q}}]_q. \]

Therefore,

\[ [K, D^{-\frac{1}{p}}]_q A \subset [K, D^{-\frac{1}{q}}]_q. \]

(ii) can be proved in a similar way.

Using same method as in the proof of Lemma 2, we get the following result.

**Lemma 3** Let \( 1 \leq p < \infty \), and let \( K \subset L^p(M) \). If \( 1 \leq q, r < \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), then

\[ [[K, A]_p D^{\frac{1}{q}}]_q = [K, D^{\frac{1}{q}}]_q, \quad [[K, A]_q D^{\frac{1}{q}}]_q = [K, D^{\frac{1}{q}}]_q. \]

and

\[ [[K, D]_p D^{\frac{1}{q}}]_q = [K, D^{\frac{1}{q}}]_q. \]

**Lemma 4** Let \( 1 \leq p < \infty \). If \( 1 < q, r < \infty \) and \( \frac{1}{p} - \frac{1}{q} = \frac{1}{r} \), then

\[ H^p(A)D^{-\frac{1}{q}} \cap L^q(M) = H^q(A) \quad \text{and} \quad D^{-\frac{1}{q}}H^p(A) \cap L^q(M) = H^q(A). \]

**Proof** Let \( x \in H^p(A)D^{-\frac{1}{q}} \cap L^q(M) \). Then there is an element \( y \in H^p(A) \) such that \( x = yD^{-\frac{1}{q}} \). If \( q' \) (resp. \( p' \)) is the conjugate index of \( q \) (resp. \( p \)), then \( \frac{1}{q'} = \frac{1}{p} + \frac{1}{r} \). Hence,

\[ \text{tr}(x D^{\frac{1}{r}} a) = \text{tr}(y D^{\frac{1}{r}} D^{\frac{1}{r}} a) = \text{tr}(y D^{\frac{1}{r}} a) = 0, \quad \forall a \in A_0. \]

Using (2.3), we get \( x \perp J(H^p_0(A)) \). By (14) Corollary 3.4] (or (5) (1.3)), \( x \in H^q(A), \) and so \( H^p(A)D^{-\frac{1}{q}} \cap L^q(M) \subset H^q(A). \) Conversely, from \( H^q(A)D^{-\frac{1}{q}} \subset H^p(A) \) it follows that \( H^q(A)D^{-\frac{1}{q}} \cap L^q(M) \supset H^q(A). \) Thus, we obtain the first result. The second result follows analogously.

**Definition 3** Let \( 1 \leq p < \infty \), and let \( K \) be a right \( A \)-invariant subspace of \( L^p(M) \).

(i) If \( 1 \leq p \leq 2 \), \( \frac{1}{p} - \frac{1}{r} = \frac{1}{2} \) and \( W \) is the right wandering subspace of \( [K, D^{-\frac{1}{q}}]_2 \), we define the right wandering subspace of \( K \) to be the \( L^p \)-closure of \( W \).

(ii) If \( 2 < p < \infty \), \( \frac{1}{p} + \frac{1}{r} = \frac{1}{2} \) and \( W \) is the right wandering subspace of \( [K, D^{-\frac{1}{q}}]_2 \), we define the right wandering subspace of \( K \) to be the \( L^p \)-closure of \( W, D^{-\frac{1}{q}} \), where \( W_r = \{ x \in W : xD^{-\frac{1}{q}} \in L^p(M) \} \).

If \( K \) is a right \( A \)-invariant subspace of \( L^p(M) \), we say that \( K \) is type 1 if the right wandering subspace of \( K \) generates \( K \) as an \( A \)-module, and \( K \) is type 2 if \( 1 \leq p < 2 \) (resp. \( p > 2 \)) and \( K = [K, A]_p \) (resp. \( [K, D^{-\frac{1}{q}}]_2 = [K, D^{-\frac{1}{q}}]_2 = [K, D^{-\frac{1}{q}}]_2 = [K, D^{-\frac{1}{q}}]_2 \), where \( \frac{1}{p} + \frac{1}{r} = \frac{1}{2} \).

To extend the result in [17] to the setting of the Haagerup noncommutative \( L^p \)-spaces \( 1 \leq p < \infty \), we will use the column \( L^p \)-sum studied by Junge and Sherman [15] to investigate this. If \( X \) is a subspace of \( L^p(M) \), and \( \{ X_i : i \in I \} \) is a collection of subspaces of \( X \), which together densely span \( X \), with the property that \( X_i \cap X_j = \{ 0 \} \) if \( i \neq j \), then we say that \( X \) is the internal column \( L^p \)-sum \( \oplus_{i=1}^{\infty} X_i \).

**Theorem 1** Let \( 1 \leq p < 2 \) and \( K \) be a right \( A \)-invariant subspace of \( L^p(M) \). Suppose \( \frac{1}{p} - \frac{1}{q} = \frac{1}{2} \) and \( K_r = \{ x \in K : xD^{-\frac{1}{q}} \in L^p(M) \} \). If \( [K, A]_p = K \), then:

(i) \( K \) may be written uniquely as an \( L^p \)-column sum \( Z \oplus_{i=1}^{\infty} [Y, A]_p \), where \( Z \) is a type 2 right \( A \)-invariant subspace of \( L^p(M) \), \( Y \) is the right wandering subspace of \( K \) such that \( Y = [Y, D]_p \) and \( J(Y) \subset L^2(D) \).

(ii) If \( K \neq \{ 0 \} \) then \( K \) is type 1 if and only if \( K = \oplus_{i=1}^{\infty} u_i H^p(A) \), for \( u_i \) partial isometries with mutually orthogonal ranges and \( u_i^* u_i \in D \).

(iii) If \( K = K_1 \oplus_{i=1}^{\infty} K_2 \) where \( K_1 \) and \( K_2 \) are types 2 and 1 respectively, then the right wandering subspace of \( K \) equals the right wandering subspace for \( K_2 \).

(iv) The wandering quotient \( K/[K, A]_p \) is isometrically \( D \)-isomorphic to the right wandering subspace of \( K \).
(v) The wandering subspace $W$ of $K$ is an $L^p(D)$-module in the sense of Junge and Sherman.

Proof (i) By Lemma 2, $K' = [K_r D^\frac{1}{2}]_2$ is a right $A$-invariant subspace of $L^2(M)$. Using Theorem 2.3 and 2.8 in [17], we have that

$$K' = Z' \oplus \text{col} [Y', A],$$

where $Z'$ is a type 2 right $A$-invariant subspace of $L^2(M)$ and $Y'$ is the right wandering subspace of $K'$ with $Y' = [Y'D]_2$ and $J(Y')Y' \subset L^1(D)$. Let $Z = [Z'D^\frac{1}{2}]_p$ and $Y = [Y'D^\frac{1}{2}]_p$. By Lemma Z and Definition Y, $Z$ is a right $A$-invariant subspaces of $L^p(M)$ and $Y$ is the right wandering subspace of $K$. Using Lemma Z we know that $[Y'A]_2D^\frac{1}{2}]_p = [YA]_p$. For any $x \in Z', y \in [Y'A]_2$, we have that $x^*y = 0$, and so

$$D^\frac{1}{2} x^*y D^\frac{1}{2} = 0.$$

Hence, $J(Z)[Y.A]_p = \{0\}$. On the other hand, by Proposition 1 $K = [K'D^\frac{1}{2}]_p$. Therefore,

$$K = Z \oplus \text{col} [Y.A]_p.$$

Since $Z' = [Z'A]_2, Y' = [Y'D]_2$, by Lemma Z

$$Z = [Z'D^\frac{1}{2}]_p = [Z'A]_2D^\frac{1}{2}]_p = [Z'A_0D^\frac{1}{2}]_p = [Z^\prime D^\frac{1}{2}]_p$$

and

$$Y = [Y'D^\frac{1}{2}]_p = [Y'D]_2D^\frac{1}{2}]_p = [Y'D]_p.$$

Since

$$J(Y'D^\frac{1}{2})Y'D^\frac{1}{2} = D^\frac{1}{2} J(Y')Y'D^\frac{1}{2} \subset D^\frac{1}{2} L_1(D)D^\frac{1}{2} \subset L^1(D),$$

it follows that $J(Y')Y \subset L^1(D)$.

Now we prove the uniqueness. Suppose that $Z_1$ is a type 2 right $A$-invariant subspace of $L^p(M)$ and $Y_1$ is the right wandering subspace of $K$ such that

$$K = Z_1 \oplus \text{col} [Y_1.A] \quad \text{and} \quad Y_1 = [Y_1.D]_p.$$

Since $Y_1$ is the right wandering subspace of $K$, by Definition Z $Y_1 = [Y_1'D^\frac{1}{2}]_p$, where $Y_1'$ is the right wandering subspace of $[K, D^{-\frac{1}{2}} \cap L^2(M)]_2 = [K, D^{-\frac{1}{2}}]_2$. By the uniqueness assertion in Theorem 2.3 of [17], $Y' = Y_1$. It follows that $Y_1 = Y$. From $K = Z_1 \oplus \text{col} [Y.A]_p = Z \oplus \text{col} [Y.A]_p$, we obtain that $Z_1 = Z$.

(ii) Let $K \neq \{0\}$ and $K$ is type 1. From the proof of (1), we know that $[K_r D^{-\frac{1}{2}}]_2$ is type 1. So, by [17] (ii) of Theorem 2.8, there are partial isometries $u_i$ with mutually orthogonal ranges such that $u^*_i u_i \in D$,

$$[K_r D^{-\frac{1}{2}}]_2 = \oplus u^*_i u_i H_2(A).$$

Using Proposition 1 and 2, we get

$$K = \oplus u^*_i u_i H^p(A),$$

Conversely, let for $u_i$ as above,

$$K = \oplus u^*_i u_i H^p(A).$$

By Lemma Z $[H^p(A)D^{-\frac{1}{2}} \cap L^2(M)]_2 = H^2(A)$. Hence,

$$[K_r D^{-\frac{1}{2}}]_2 = \oplus u^*_i u_i [H^p(A)D^{-\frac{1}{2}} \cap L^2(M)]_2 = \oplus u^*_i u_i H^2(A).$$

So

$$[K_r D^{-\frac{1}{2}}]_2 = \oplus u^*_i u_i H_2^0(A).$$

Hence, the right wandering subspace $W$ of $[K_r D^{-\frac{1}{2}}]_2$ satisfies

$$W = \oplus u^*_i u_i L^2(D).$$

By Definition Z and 2, $\oplus u^*_i u_i L^p(D)$ is the right wandering subspace of $K$. Since

$$[\oplus u^*_i u_i L^p(D)A]_p = \oplus u^*_i u_i H^p(A) = K,$$
$K$ is type 1.

(iii) Set $K^{(r)}_1 = \{ x \in K_1 : xD^{-\frac{1}{r}} \in L^2(M) \}$ and $K^{(r)}_2 = \{ x \in K_2 : xD^{-\frac{1}{r}} \in L^2(M) \}$. If $x \in K_r$, then there exist $z \in K_1$ and $y \in K_2$ such that $x = z + y$ and $z^*y = 0$. It follows that $|xD^{-\frac{1}{r}}|^2 = |zD^{-\frac{1}{r}}|^2 + |yD^{-\frac{1}{r}}|^2$, and so $|xD^{-\frac{1}{r}}| \geq |zD^{-\frac{1}{r}}|$, $|xD^{-\frac{1}{r}}| \geq |yD^{-\frac{1}{r}}|$. Since $xD^{-\frac{1}{r}} \in L^2(M) \subset L^0(N)$, we get $yD^{-\frac{1}{r}} \in L^0(N)$. On the other hand,

$$\hat{\sigma}_t(D^{-\frac{1}{r}}) = e^{-\frac{t}{r}}D^{-\frac{1}{r}}, \quad \forall t \in \mathbb{R}.$$ 

Hence,

$$1 = \hat{\sigma}_t(D^{-\frac{1}{r}}D^{-\frac{1}{r}}) = e^{-\frac{t}{r}}D^{-\frac{1}{r}}\hat{\sigma}_t(D^{-\frac{1}{r}}), \quad \forall t \in \mathbb{R},$$

so that

$$\hat{\sigma}_t(D^{-\frac{1}{r}}) = e^{\frac{t}{r}}D^{\frac{1}{r}}, \quad \forall t \in \mathbb{R}.$$ 

Moreover,

$$\hat{\sigma}_t(zD^{-\frac{1}{r}}) = \hat{\sigma}_t(z)\hat{\sigma}_t(D^{-\frac{1}{r}}) = e^{-\frac{t}{r}+\frac{1}{r}}zD^{-\frac{1}{r}} = e^{-\frac{t}{r}}zD^{-\frac{1}{r}}$$

and

$$\hat{\sigma}_t(yD^{-\frac{1}{r}}) = \hat{\sigma}_t(y)\hat{\sigma}_t(D^{-\frac{1}{r}}) = e^{-\frac{t}{r}+\frac{1}{r}}yD^{-\frac{1}{r}} = e^{-\frac{t}{r}}yD^{-\frac{1}{r}}, \quad \forall t \in \mathbb{R}.$$ 

Thus $zD^{-\frac{1}{r}}$, $yD^{-\frac{1}{r}} \in L^2(M)$, i.e., $z \in K^{(r)}_1$ and $y \in K^{(r)}_2$.

Next, we prove that $[K^{(r)}_1]_{p} = K_1$. To this end let $P : K \to K_1$ be the projection operator. From the above, we know that $P(K_r) \subset K^{(r)}_1$. If $a \in K_1$, then $a \in K$. Since $[K_r]_{p} = K$, there exists a sequence $(a_n) \subset K$ such that $a_n \to a$. Hence $P(a_n) \to P(a) = a$. It follows that $a \in [K^{(r)}_1]_{p}$.

Therefore, $[K^{(r)}_1]_{p} = K_1$. Similarly, $[K^{(r)}_2]_{p} = K_2$.

$[K, D^{-\frac{1}{r}}]_2$ is a right $A$-invariant subspace of $L^2(M)$ and

$$[K, D^{-\frac{1}{r}}]_2 = [K^{(r)}_1 D^{-\frac{1}{r}}]_2 \oplus \text{col} [K^{(r)}_2 D^{-\frac{1}{r}}]_2$$

From the proof of (1), it follows that $[K^{(r)}_1 D^{-\frac{1}{r}}]_2$ and $[K^{(r)}_2 D^{-\frac{1}{r}}]_2$ are types 2 and 1 respectively. By [17] Proposition 2.7, the right wandering subspace for $[K, D^{-\frac{1}{r}}]_2$ equals the right wandering subspace for $[K^{(r)}_2 D^{-\frac{1}{r}}]_2$. By Definition 3, we obtain the desired result.

(iv) By (i), (ii) and (iii), we get that

$$K = Z \oplus \text{col} u_i H^p(A),$$

where $Z$ is a type 2, and $u_i$ are partial isometries with mutually orthogonal ranges such that $u^*u_i \in \mathcal{D}$ and $\text{col} u_i L^p(D)$ is the right wandering subspace of $K$. Using the properties of $E$, similar to the proof of (2) of Theorem 4.5 in [6], we prove the desired result. We omit the details.

(v) Since $J(W)W \subset L^2(D)$, $W$ is a right $L^p(D)$-module with inner product $\langle \xi, \eta \rangle = \xi^*\eta$ (see Definition 3.3).}

**Lemma 5** Let $2 < p < \infty$, $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ ($r > 2$) and $K$ be a right $A$-invariant subspace of $L^p(M)$. If $Y$ is the right wandering subspace of $[KD^\frac{1}{2}]_2$, then $[Y]_2 = Y$, where $Y_r = \{ x \in Y : xD^{-\frac{1}{r}} \in L^p(M) \}$.

**Proof** Let $K' = [KD^\frac{1}{2}]_2$. Then $K' = [K' A_0]_2 \oplus Y$. By [17] Theorem 2.3 and 2.8, $Y = \text{col} u_i L^2(D)$ where $u_i$ are partial isometries with mutually orthogonal ranges such that $u^*_i u_i \in \mathcal{D}$. Since $\text{col} u_i L^p(D) D^\frac{1}{2} \subset Y$, using (2.9), we get $[Y]_2 = Y$.

Similar to Theorem 1, we have the following result.

**Theorem 2** Let $2 < p < \infty$, $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ and $K$ be a right $A$-invariant subspace of $L^p(M)$. If $K = [[KD^\frac{1}{2}]_2 D^{-\frac{1}{r}} \cap L^p(M)]_{p}$, then:

(i) $K$ may be written uniquely as an $L^p$-column sum $Z \oplus \text{col} [YA]_p$, where $Z$ is a type 2 right $A$-invariant subspace of $L^p(M)$, $Y$ is the right wandering subspace of $K$ such that $Y = [YD]_p$ and $J(Y)Y \subset L^2(D)$. 

(ii) If \( K \neq \{0\} \) then \( K \) is type 1 if and only if \( K = \oplus_i^{\text{col}} u_iH^p(A) \), for \( u_i \) partial isometries with mutually orthogonal ranges and \( u_i^*u_i \in \mathcal{D} \).

(iii) If \( K = K_1 \oplus K_2 \) where \( K_1 \) and \( K_2 \) are types 2 and 1 respectively, then the right wandering subspace for \( K \) equals the right wandering subspace for \( K_2 \).

(iv) The wandering quotient \( K/[KA_0]_{p} \) is isometrically \( \mathcal{D} \)-isomorphic to the right wandering subspace of \( K \).

(v) The wandering subspace \( W \) of \( K \) is an \( L^p(\mathcal{D}) \)-module in the sense of Junge and Sherman.

\[ \text{Proof} \]

(i) By Lemma \([2] \) \( K' = [KD^{\frac{1}{2}}]_{2} \) is a right \( \mathcal{A} \)-invariant subspace of \( L^2(\mathcal{M}) \). Using Theorem 2.3 and 2.8 in \([17] \), we have that

\[ K' = Z' \oplus^{\text{col}} [Y',\mathcal{A}]_{2}, \]

where \( Z' \) is a type 2 right \( \mathcal{A} \)-invariant subspace of \( L^2(\mathcal{M}) \) and \( Y' \) is the right wandering subspace of \( K' \) with \( Y' = [Y'D]_{2} \) and \( J(Y')Y' \subset L^1(\mathcal{D}) \). For simplicity, we set

\[ K_r = \{ x \in K' : xD^{-\frac{1}{2}} \in L^p(\mathcal{M}) \}, \]
\[ Z_r = \{ x \in Z' : xD^{-\frac{1}{2}} \in L^p(\mathcal{M}) \}, \]
\[ Y_r = \{ x \in Y' : xD^{-\frac{1}{2}} \in L^p(\mathcal{M}) \}, \]
\[ X' = [Y',\mathcal{A}]_{2} \text{ and } X_r = \{ x \in X' : xD^{-\frac{1}{2}} \in L^p(\mathcal{M}) \}. \]

Let \( Z = [Z_r, D^{-\frac{1}{2}}]_{p} \) and \( Y = [Y_r, D^{-\frac{1}{2}}]_{p} \). By Lemma \([2] \) and Definition \([3] \) \( Z \) is a right \( \mathcal{A} \)-invariant subspaces of \( L^p(\mathcal{M}) \) and \( Y \) is the right wandering subspace of \( K \). We notice that \( K = [[KD^{\frac{1}{2}}]_{2}D^{-\frac{1}{2}} \cap L^p(\mathcal{M})]_{p} \) implies that \( K = [K_rD^{-\frac{1}{2}}]_{p} \).

Since \( KD^{\frac{1}{2}} \subset K_r \), we get \([K_r]_{2} = K' \). We use same method as in the proof of (iii) of Theorem \([1] \) to obtain that \( Z' = [Z_r]_{2} \), \( X' = [X_r]_{2} \) and

\[ K_r = Z_r \oplus^{\text{col}} X_r. \quad (3.3) \]

We have that

\[ [ZD^{\frac{1}{2}}]_{2} = [[Z_rD^{-\frac{1}{2}}]_{p}D^{\frac{1}{2}}]_{2} = [Z_rD^{-\frac{1}{2}}D^{\frac{1}{2}}]_{2} = [Z_r]_{2} = Z'. \]

Hence,

\[ [ZD^{\frac{1}{2}}A_0]_{2} = [[ZD^{\frac{1}{2}}A_0]_{2} = [Z',A_0]_{2} = Z' = [ZD^{\frac{1}{2}}]_{2}, \]

i.e., \( Z \) is a type 2 right \( \mathcal{A} \)-invariant subspace of \( L^p(\mathcal{M}) \). By Lemma \([1] \) we have that \( Y_rD_\alpha \subset Y_r \),

\[ Y_rD^{-\frac{1}{2}} \subset Y_rD^{-\frac{1}{2}}D_\alpha = Y_rD_\alpha D^{-\frac{1}{2}} \subset Y_rD^{-\frac{1}{2}} \]

and \([Y_rD^{-\frac{1}{2}}D_\alpha]_{p} = [Y_rD^{-\frac{1}{2}}D]_{p} \). Therefore, it follows that

\[ Y = [Y_rD^{-\frac{1}{2}}D]_{p} = [[Y_rD^{-\frac{1}{2}}D]_{p}D]_{p} = [YD]_{p}. \]

Since

\[ J(Y_rD^{-\frac{1}{2}})Y_rD^{-\frac{1}{2}} = D^{-\frac{1}{2}}J(Y_r)Y_rD^{-\frac{1}{2}} \subset D^{-\frac{1}{2}}L^1(\mathcal{D})D^{-\frac{1}{2}} \subset L^{\frac{1}{2}}(\mathcal{D}), \]

we deduce that \( J(Y)Y \subset L^{\frac{1}{2}}(\mathcal{D}) \).

Now we prove that

\[ K = Z \oplus^{\text{col}} [Y,\mathcal{A}]_{p}. \]

By \([17] \) Theorem 2.8, there are partial isometries \( u_i \) with mutually orthogonal ranges such that \( [u_i] \in \mathcal{D} \),

\[ X' = \oplus_i^{\text{col}} u_iH_2(A) \quad \text{and} \quad Y' = \oplus_i^{\text{col}} u_iL_2(D). \]

Using Lemma \([4] \) we get that

\[ X_rD^{-\frac{1}{2}} = \oplus_i^{\text{col}} u_i(H^2(A)D^{-\frac{1}{2}} \cap L^p(\mathcal{M})) = \oplus_i^{\text{col}} u_iH^p(A). \]

and

\[ Y_rD^{-\frac{1}{2}} = \oplus_i^{\text{col}} u_i(L^2(D)D^{-\frac{1}{2}} \cap L^p(\mathcal{M})) = \oplus_i^{\text{col}} u_iL^p(D). \]

So, it follows that \([X_rD^{-\frac{1}{2}}]_{p} = [Y,\mathcal{A}]_{p}. \)
We claim that $K_r D^{-\frac{1}{2}}$ is closed. Indeed, if $x \in [K_r D^{-\frac{1}{2}}]_p$, then there is a sequence $(y_n)$ in $K_r$ such that $y_n D^{-\frac{1}{2}} \to x$ in norm in $L^p(M)$. It follows that $y_n \to x D^{\frac{1}{2}}$ in norm in $L^2(M)$. Set $y = x D^{\frac{1}{2}}$. It is clear that $y \in K_r$. Hence, $x = y D^{-\frac{1}{2}} \in K_r D^{-\frac{1}{2}}$, i.e., $K_r D^{-\frac{1}{2}}$ is closed. Similarly, we can prove that $Z_r D^{-\frac{1}{2}}$ and $X_r D^{-\frac{1}{2}}$ are closed. Thus

$$K = K_r D^{-\frac{1}{2}}, \quad Z = Z_r D^{-\frac{1}{2}} \quad \text{and} \quad [Y,A]_p = X_r D^{-\frac{1}{2}}.$$ 

Applying (3.3), we obtain that $K = Z \otimes^c [Y,A]_p$. The remainder of the proof can be done the same way as in the proof of Theorem 1.

Remark 1 Let $1 \leq p < \infty$ and $K$ be a right $\mathcal{A}$-invariant subspace of $L^p(M)$. In general, if $1 \leq p < 2$ and $\frac{1}{2} - \frac{1}{p} = \frac{1}{2}$, then $[K_r]_p \subset K$; if $2 < p < \infty$ and $\frac{1}{2} + \frac{1}{p} = \frac{1}{2}$, then $K \subset [K D^{\ast}]_2 D^{-\ast} \cap L^p(M)_p$. It is unknown at the time of this writing whether for the general case, the results in Theorem 1 and 2 are hold.

We use same method as in the proof of [17] Proposition 2.4 to obtain the following result, we give its proof.

**Proposition 2** Let $K$ is a right $\mathcal{A}$-invariant subspace of $L^2(M)$, and let $W$ be the right wandering subspace of $K$. If $W$ has a cyclic and separating vector for the $\mathcal{D}$-action, then there is an isometry $u \in \mathcal{M}$ such that $W = uL^2(\mathcal{D})$.

**Proof** By an adaption of an argument from [15] (see p.13) there exists an isometric $\mathcal{D}$-module isomorphism $\psi : L^2(\mathcal{D}) \to W$. Let $h = \psi(D^{\frac{1}{2}}) \in W$. Then

$$tr(d^* h^* hd) = ||\psi(D^{\frac{1}{2}} d)||_2^2 = tr(d^* Dd), \quad \forall d \in \mathcal{D}.$$ 

By [17] (5) of Theorem 2.3, $h^* h \in L^1(\mathcal{D})$, and so $h^* h = D$. Hence there exists an isometry $u$ with initial projection 1 such that $h = u D^{\frac{1}{2}}$. Since $\psi$ is $\mathcal{D}$-module map, we have that

$$\psi(D^{\frac{1}{2}} d) = \psi(D^{\frac{1}{2}}) d = u D^{\frac{1}{2}} d, \quad \forall d \in \mathcal{D}.$$ 

Since $L^2(\mathcal{D}) = [D^{\frac{1}{2}} \mathcal{D}]$, it follows that $\psi(L^2(\mathcal{D})) = uL^2(\mathcal{D})$. Thus $W = uL^2(\mathcal{D})$ and $u^* u = 1$.

Similar to Proposition 2 we have the following result.

**Proposition 3** Let $K$ is a left $\mathcal{A}$-invariant subspace of $L^2(M)$, and let $W$ be the left wandering subspace of $K$. If $W$ has a cyclic and separating vector for the $\mathcal{D}$-action, then there is a partial isometry $v \in \mathcal{M}$ such that $vv^* = 1$ and $W = L^2(\mathcal{D})v$.

## 4 Outer operators of $H^p(\mathcal{A})$

In the case when von Neumann algebra $\mathcal{M}$ is finite, from the Beurling-Blecher-Labuschagne theorem follows a generalized ‘inner-outer’ factorization. Let $x \in L^p(\mathcal{M})$ $(1 \leq p \leq \infty)$ and $K = [x]_p$. If the right-wandering subspace of $K$ (respectively right-wandering quotient of $K$) has a nonzero separating and cyclic vector for the right action of $\mathcal{D}$, then $x$ is of the form $x = uh$ for some some outer operator $h \in H^p(\mathcal{A})$ and a unitary $u \in \mathcal{M}$ (see the lines before the Closing remark of [6]). For more details on outer operators we refer to [2][7][8].

In this section, we consider outer operators in the case that $\mathcal{M}$ is a $\sigma$-finite von Neumann algebra. Similar to the finite case, we define the outer operators as following.

**Definition 4** Let $0 < p \leq \infty$. An operator $h \in H^p(\mathcal{A})$ is called a left outer operator, a right outer operator or a bilaterally outer operator according to $[h\mathcal{A}]_p = H^p(\mathcal{A})$, $[Ah]_p = H^p(\mathcal{A})$ or $[Ah\mathcal{A}]_p = H^p(\mathcal{A})$.

**Proposition 4** Let $1 \leq p < \infty$, and let $h \in H^p(\mathcal{A})$. The following are equivalent:

(i) $h$ is a bilaterally outer operator;
(ii) $\mathcal{E}(h)$ is a bilaterally outer operator in $L^p(\mathcal{D})$ and $[Ah\mathcal{A}]_p = [Ah\mathcal{A}]_p = H^p(\mathcal{A})$;
(iii) $\mathcal{E}(h)$ is a bilaterally outer operator in $L^p(\mathcal{D})$ and $\mathcal{E}(h) - h \in [Ah\mathcal{A}]_p = [Ah\mathcal{A}]_p$. 
Proof (i) ⇒ (ii). If $h$ is a bilaterally outer operator, then for $D^{\frac{1}{2}}$ there exist two sequences $(a_n)$, $(b_n) \subset A$ such that

$$\|a_n h b_n - D^{\frac{1}{2}}\|_p \to 0 \quad \text{as} \quad n \to \infty.$$  \hspace{1cm} (4.1)

By continuity of $E$, we get

$$\|E(a_n)E(h)b_n - D^{\frac{1}{2}}\|_p \to 0 \quad \text{as} \quad n \to \infty.$$  

Hence, by (2.5), we have that

$$L^p(D) = [D^{\frac{1}{2}}D]_p \subset [DE(h)D]_p \subset L^p(D).$$

So, $E(h)$ is a bilaterally outer operator in $L^p(D)$. Using (2.3) and (1.1), we deduce that

$$[AhA_0]_p = [A_0 h A]_p = H^p_0(A).$$

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i). It is clear that

$$D^{\frac{1}{2}} \in [DE(h)D]_p \subset [AE(h)A]_p$$

and $h \in [AhA]_p$. Hence, $E(h) = (E(h) - h) + h \in [AhA]_p$. It follows that $D^{\frac{1}{2}} \in [AhA]_p$. By (2.3), we obtain that $H^p(A) = [AhA]_p$.

Similar to Proposition 4, we have the following result.

Proposition 5 Let $1 \leq p < \infty$, and let $h \in H^p(A)$. The following are equivalent:

(i) $h$ is a left outer operator (resp. a right outer operator);

(ii) $E(h)$ is a left outer operator (resp. a right outer operator) in $L^p(D)$ and $[hA_0]_p = H^p_0(A)$ (resp. $[Ah_0]_p = H^p_0(A)$);

(iii) $E(h)$ is a left outer operator (resp., a right outer operator) in $L^p(D)$ and $E(h) - h \in [hA_0]_p$ (resp. $E(h) - h \in [Ah_0]_p$).

Proposition 6 Let $1 \leq p < \infty$. If $h \in H^p(A)$ is a left outer operator (resp. a right outer operator), then $E(h)$ and $h$ are left outer operator (resp. a right outer operator) in $L^p(M)$.

Proof Let $h \in H^p(A)$ be a left outer operator. From Proposition 5, it follows that $[E(h)D]_p = L^p(D)$. Since $D^{\frac{1}{2}} \in L^p(D) = [E(h)D]_p$, there is a sequence $(d_n) \subset D$ such that $E(h)d_n \to D^{\frac{1}{2}}$ in norm in $L^p(M)$. Therefore, $[E(h)M]_p = L^p(M)$.

Notice that $E(h) \in H^p(A) = [hA]_p$. It follows that there is a sequence $(a_n) \subset A$ such that $ha_n \to E(h)$, and so $[hA]_p = L^p(M)$. The alternative claim follows analogously.

We will keep all previous notations throughout this section. If $h$ is a left outer operator and it is also a right outer operator, then we call $h$ an outer operator.

Lemma 6 Let $0 < p < \infty$.

(i) If $h \in H^p(A)$ is an outer operator in $H^p(A)$ and $h = u|h|$ is the polar decomposition of $h$, then $u$ is a unitary.

(ii) If $d \in L^p(D)$ is an outer operator in $L^p(D)$ and $d = v|d|$ is the polar decomposition of $d$, then $v$ is a unitary in $D$.

Proof (i) Since $h$ is a left outer operator, there exists a sequence $(a_n) \subset A$ such that $ha_n \to D^{\frac{1}{2}}$ in norm in $L^p(M)$. Let $l(h)$ be the left support projection of $h$. Then $l(h) = l(h)^{-1} = l(h)D^{\frac{1}{2}}$ in norm in $L^p(M)$. On the other hand, $l(h)h = 0$ for all $n$, and so $l(h) = l(h)^{-1}D^{\frac{1}{2}} = 0$. Since $D^{\frac{1}{2}}$ is invertible, $l(h) = 0$. Hence, $h$ must have dense range, i.e., $uu^* = l(h) = 1$. Similarly, from the fact that $h$ is a right outer operator, we obtain that $u^*u = r(h) = 1$, where $r(h)$ is the right support projection of $h$. Thus $u$ is a unitary.

(ii) The proof is similar to the proof of (i).
Theorem 3 Let $1 \leq p < \infty$, and let $d \in L^p(\mathcal{D})$. The following are equivalent:

(i) $d$ is an outer operator in $L^p(\mathcal{D})$;
(ii) $d$ is an outer operator in $H^p(\mathcal{A})$;
(iii) The left and right support projections of $d$ are 1;
(iv) $d$ is an outer operator in $L^p(\mathcal{M})$.

Proof (i) $\Rightarrow$ (ii) Since $D^\frac{d}{p} \in L^p(\mathcal{D}) = [d\mathcal{D}]_p = [\mathcal{D}d]_p$, there are sequences $(a_n)$ and $(b_n)$ in $\mathcal{D} \subset A$ such that $da_n \to D^\frac{d}{p}$ and $b_n d \to D^\frac{d}{p}$. Hence, $[d\mathcal{A}]_p = [\mathcal{A}d]_p = H^p(\mathcal{A})$.

(ii) $\Rightarrow$ (iii) is follows from the proof of Lemma 3.

(iii) $\Rightarrow$ (iv). First we prove $d$ is a left outer operator in $L^p(\mathcal{M})$. Let $p'$ be the conjugate index of $p$. If $x \in L^{p'}(\mathcal{M})$ such that $tr(xd) = 0$ for all $z \in \mathcal{M}$, then $xd = 0$. Hence, $x = xdd^{-1} = 0$, and so $[d\mathcal{M}]_p = L^p(\mathcal{M})$. Using the same method, we can prove that $d$ is a right outer operator in $L^p(\mathcal{M})$.

(iv) $\Rightarrow$ (i). Since $D^\frac{d}{p} \in [d\mathcal{M}]_p = [\mathcal{M}d]_p$, there are sequences $(a_n)$ and $(b_n)$ in $\mathcal{M}$ such that $da_n \to D^\frac{d}{p}$ and $b_n d \to D^\frac{d}{p}$ in norm in $L^p(\mathcal{M})$. Using the continuity of $E$, we obtain that $dE(a_n) \to D^\frac{d}{p}$ and $E(b_n)d \to D^\frac{d}{p}$ in norm in $L^p(\mathcal{D})$. Hence, we get the desired result.

Corollary 1 Let $1 \leq p < \infty$ and $0 < r < \infty$. If $d \in L^p(\mathcal{D})$ is an outer operator and $rp \geq 1$, then $[d]^r \in L^{rp}(\mathcal{D})$ is an outer operator.

Proof It is clear that $[d] \in L^p(\mathcal{D})$ is an outer operator. Hence, by Theorem 3 $[d]^r$ is an outer operator.

Corollary 2 Let $1 \leq p < \infty$ and $d \in L^1(\mathcal{D})^+$ be an outer operator. If $0 \leq \eta < 1$, then

$$H^p(\mathcal{A}) = [d\frac{1-\eta}{p} A d\frac{1-\eta}{p}]_p, \quad H^p_0(\mathcal{A}) = [d\frac{1-\eta}{p} A_0 d\frac{1-\eta}{p}]_p, \quad L^p(\mathcal{D}) = [d\frac{1-\eta}{p} D d\frac{1-\eta}{p}]_p$$

and $L^p(\mathcal{M}) = [d\frac{1-\eta}{p} M d\frac{1-\eta}{p}]_p$.

Lemma 7 Let $1 \leq p < \infty$, $1 \leq q, r < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{4}$. If $d \in L^p(\mathcal{D})$ is outer and $dD^\frac{d}{p}, D^\frac{d-1}{p}d \in L^q(\mathcal{M})$, then $dD^\frac{d-1}{p}, D^\frac{d-1}{p}d \in L^q(\mathcal{D})$ are outer operators.

Proof Since $dD^\frac{d-1}{p} \in H^p(\mathcal{A})D^\frac{d-1}{p} \cap L^q(\mathcal{M})$ and $dD^\frac{d-1}{p} \in J(H^p(\mathcal{A}))D^\frac{d-1}{p} \cap L^q(\mathcal{M})$, by Lemma 3 we get $dD^\frac{d-1}{p} \in H^q(\mathcal{A}) \cap J(H^q(\mathcal{M}) = L^q(\mathcal{D})$. Similarly, $dD^\frac{d-1}{p} \in L^q(\mathcal{D})$. Using Theorem 3 we obtain the desired result.

Lemma 8 Let $1 \leq p < \infty$, $1 \leq q, r < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{q}$.

(i) If $h \in H^p(\mathcal{A})$ is an outer operator, then $hD^\frac{d}{p}$ and $D^\frac{d-1}{p}h \in H^q(\mathcal{A})$ are outer operators.

(ii) If $d \in L^p(\mathcal{D})$ is an outer operator, then $dD^\frac{d}{p}, D^\frac{d-1}{p}d \in L^q(\mathcal{D})$ are outer operators.

Proof (i) We only prove $hD^\frac{d}{p}$ is an outer operator. A similar argument works for $D^\frac{d}{p}h$. By (4), $[H^p(\mathcal{A})D^\frac{d}{p}h]_q = H^q(\mathcal{A})$. We use same method as in the proof of (3) of Lemma 3 to obtain that $[hA]_p = [A_h]_p = H^p(\mathcal{A})$. Hence, $[hA_0]_pD^\frac{d}{p}]_q = H^q(\mathcal{A})$. Using Lemma 3 we get

$$H^q(\mathcal{A}) = [hA_0]_pD^\frac{d}{p}]_q = [hA_0D^\frac{d}{p}]_q = [hD^\frac{d}{p}A]_q \subset [hD^\frac{d}{p}A]_q \subset H^q(\mathcal{A}).$$

Thus $hD^\frac{d}{p}$ is a left outer operator. Similarly we can show $hD^\frac{d}{p}$ is a right outer operator.

(ii) follows analogously.

Proposition 7 Let $1 \leq p < \infty$ and $h \in H^p(\mathcal{A})$. Suppose that $E(h)$ is an outer operator in $L^p(\mathcal{D})$ and one of the the following conditions holds.

(i) $\frac{1}{p} - \frac{1}{r} = \frac{1}{2}$ (r > 2) and $hD^\frac{d}{p}, D^\frac{d-1}{p}h \in L^2(\mathcal{M})$;

(ii) $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ (r > 2).

Then there is a left outer operator $g \in H^p(\mathcal{A})$ and an isometry $u \in \mathcal{A}$ such that $h = ug$ (resp. there is a right outer operator $g' \in H^p(\mathcal{A})$ and $v \in \mathcal{A}$ such that $vu^* = 1$ and $h = g'v$).
Proof} First assume that condition (i) holds. By Lemma [4], we get $hD^{-\frac{1}{2}} \in H^2(A)$. Let $p'$ be the conjugate index of $p$. Then for any $d \in D$, we have that

$$tr(\mathcal{E}(h)D^{-\frac{1}{2}}D^{\frac{1}{2}}d) = tr(\mathcal{E}(h)D^{\frac{1}{2}}d) = tr(\mathcal{E}(h)D^{-\frac{1}{2}}D^{\frac{1}{2}}d) = tr(\mathcal{E}(h)D^{-\frac{1}{2}}D^{\frac{1}{2}}d).$$

By (2.3), we get

$$tr(\mathcal{E}(h)D^{-\frac{1}{2}}f) = tr(\mathcal{E}(h)D^{-\frac{1}{2}}f), \quad \forall f \in L^2(D).$$

Hence, $\mathcal{E}(h)D^{-\frac{1}{2}} = \mathcal{E}(h)D^{-\frac{1}{2}}$. On the other hand, by Lemma [4], $\mathcal{E}(h)D^{-\frac{1}{2}}$ is an outer operator in $L^2(D)$.

We consider the orthogonal projection

$$P : [hD^{-\frac{1}{2}}A]_2 \to [\mathcal{E}(h)D^{-\frac{1}{2}}]D]_2.$$ 

Then $P = \mathcal{E}|_{[hD^{-\frac{1}{2}}A]_2}$ and $[\mathcal{E}(h)D^{-\frac{1}{2}}]D]_2 = [hD^{-\frac{1}{2}}A]_2 \odot [hD^{-\frac{1}{2}}A_0]_2$. It follows that $\mathcal{E}(h)D^{-\frac{1}{2}}$ is a cyclic separating vector for the wandering subspace $[\mathcal{E}(h)D^{-\frac{1}{2}}]D]_2$ of $[hD^{-\frac{1}{2}}A]_2$. By Proposition [2], there exists an isometry $u \in M$ such that $[hD^{-\frac{1}{2}}A]_2 = uH^2(A)$.

We may write $hD^{-\frac{1}{2}} = uf$, for $f \in H^2(A)$. Then

$$[fA]_2 = u^*u[fA]_2 = u^*[hD^{-\frac{1}{2}}A]_2 = u^*uH^2(A) = H^2(A),$$

i.e., $f$ is a left outer operator. On the other hand,

$$0 = tr(hD^{-\frac{1}{2}}aD^{\frac{1}{2}}b) = tr(u(uaD^{\frac{1}{2}}b)), \quad \forall a \in A_0, \quad \forall b \in A.$$

Since $f$ is a left outer operator, by Proposition [5], $[fA_0]_2 = H^2(A)$. Hence, using (2.3), we obtain that $[fA_0D^{\frac{1}{2}}A]_1 = H^2_0(A)$. It follows that $0 = tr(ua)$ for all $a \in H^2_0(A)$. By (2.4), $u \in A$. Let $g = fD^{\frac{1}{2}}$. From the proof of Lemma [8], we know that $g$ is a left outer operator. This gives the desired result. Similarly, we prove the alternative claim.

If condition (ii) holds. The proof is similar to the above.

Lemma 9 If $x \in L^2(M)$ and $u \in M$ is a contraction such that $\|ux\|_2 = \|x\|_2$, then $x = u^*ux$.

Proof} We have that $x^*u^*ux \leq x^*x$ and $tr(x^*u^*ux) = \|ux\|_2^2 = \|x\|_2^2 = tr(x^*x)$. Hence,

$$\|x^*x - x^*u^*ux\|_1 = tr(x^*x - x^*u^*ux) = 0,$$

so that $x^*x = x^*u^*ux$. Thus $\|(1 - u^*)u^*x\|_2^2 = \|x^*-(1 - u^*)x\|_1 = 0$, therefore $(1 - u^*)x = (1 - u^*)u^*[1 - u^*u]x = 0$, and $x = u^*ux$.

In the finite case, $h \in H^2(A)$ is a right outer operator if and only if there is a cyclic separating vector for the right action $\mathcal{D}$ on the wandering subspace of $[hA]_2$ and $\|\mathcal{E}(h)\|_2 = \|P(h)\|_2$, where $P$ is the orthogonal projection from $[hA]_2$ to $[hA_0]_2 \odot [hA_0]_2$ (see [7] Proposition 4.8 or [8] Theorem 4.4). This result was extend to the case $1 \leq p < \infty$ (see [8] Theorem 4.4).

The following result extends [8] Theorem 4.4 to the Haagerup noncommutative $H^p$-space case.

Theorem 4 Let $1 \leq p, r < \infty$, and let $h \in H^p(A)$.

1. If $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$, then $h$ is an outer operator if and only if $\mathcal{E}(h)$ is an outer operator in $L^p(D)$ and $\|\mathcal{E}(h)D^{\frac{1}{2}}\|_2 = \|P(hD^{\frac{1}{2}})\| = \|P^*(hD^{\frac{1}{2}})\|$, where $P$ is the orthogonal projection from $[hD^{\frac{1}{2}}A]_2$ to $[hD^{\frac{1}{2}}A_0]_2 \odot [hD^{\frac{1}{2}}A_0]_2$ and $P^*$ is the orthogonal projection from $[AhD^{\frac{1}{2}}]_2$ to $[AhD^{\frac{1}{2}}]_2 \odot [AhD^{\frac{1}{2}}]_2$.

2. Suppose that $\frac{1}{p} - \frac{1}{r} = \frac{1}{2}$ and $hD^{-\frac{1}{2}} \in L^2(M)$. If $\mathcal{E}(h)$ is an outer operator in $L^p(D)$ and $\|\mathcal{E}(h)D^{-\frac{1}{2}}\|_2 = \|P(hD^{-\frac{1}{2}})\| = \|P^*(hD^{-\frac{1}{2}})\|$, where $P$ is the orthogonal projection from $[hD^{-\frac{1}{2}}A]_2$ to $[hD^{-\frac{1}{2}}A_0]_2 \odot [hD^{-\frac{1}{2}}A_0]_2$ and $P^*$ is the orthogonal projection from $[AhD^{-\frac{1}{2}}]_2$ to $[AhD^{-\frac{1}{2}}]_2 \odot [AhD^{-\frac{1}{2}}]_2$, then $h$ is an outer operator.
Proof (i) "⇒". Using Proposition 5 we obtain that $\mathcal{E}(h)$ is an outer operator in $L^p(\mathcal{D})$. Since $\mathcal{E}$ is a contractive projection from $H^2(\mathcal{A})$ onto $L^2(\mathcal{D})$ with kernel $H^2(\mathcal{A})$, we deduce that

$$\|\mathcal{E}(hD^\frac{1}{2})\|_2 = \inf_{h_0 \in H^2(\mathcal{A})} \|hD^\frac{1}{2} + h_0\|_2.$$ 

On the other hand, by Lemma 8, $hD^\frac{1}{2}$ is an outer operator in $H^2(\mathcal{A})$. Using Proposition 5 we obtain that

$$\|\mathcal{E}(hD^\frac{1}{2})\|_2 = \inf_{h_0 \in H^2(\mathcal{A})} \|hD^\frac{1}{2} + h_0\|_2 = \|P(hD^\frac{1}{2})\|.$$ 

Similarly, we can prove $\|\mathcal{E}(hD^\frac{1}{2})\|_2 = \|P^*(hD^\frac{1}{2})\|.$

"⇐". By Proposition 7, $h = u g$, where $g \in H^p(\mathcal{A})$ is a left outer operator and $u \in \mathcal{A}$ is an isometry. On the other hand, it is clear that $gD^\frac{1}{2}$ is a left outer operator in $H^2(\mathcal{A})$, Hence,

$$\|\mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2})\|_2 = \|\mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2})\|_2 \leq \|\mathcal{E}(gD^\frac{1}{2})\|_2 = \inf_{a_0 \in A_0} \|gD^\frac{1}{2} + gD^\frac{1}{2} a_0\|_2 = \inf_{a_0 \in A_0} \|u^*(hD^\frac{1}{2} + hD^\frac{1}{2} a_0)\|_2 \leq \inf_{a_0 \in A_0} \|hD^\frac{1}{2} + hD^\frac{1}{2} a_0\|_2 = \|P(hD^\frac{1}{2})\| = \|\mathcal{E}(hD^\frac{1}{2})\|_2.$$ 

This gives $\|\mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2})\|_2 = \|\mathcal{E}(gD^\frac{1}{2})\|_2$. Using Proposition 5 we get $\mathcal{E}(gD^\frac{1}{2})$ is a left outer operator in $L^2(\mathcal{D})$, and so the left support of $\mathcal{E}(gD^\frac{1}{2})$ is 1. Applying Lemma 9 we obtain that $\mathcal{E}(u)$ is an isometry. On the other hand, we have that $\mathcal{D}\mathcal{E}(hD^\frac{1}{2}) = \mathcal{D}\mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2}) \subset \mathcal{D}\mathcal{E}(gD^\frac{1}{2})$. Hence,

$$L^2(\mathcal{D}) = [\mathcal{D}\mathcal{E}(hD^\frac{1}{2})]_2 = [\mathcal{D}\mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2})]_2 \subset [\mathcal{D}\mathcal{E}(gD^\frac{1}{2})]_2 \subset L^2(\mathcal{D}),$$

i.e., $\mathcal{E}(gD^\frac{1}{2})$ is a right outer operator. So, $\mathcal{E}(gD^\frac{1}{2})$ is an outer operator. From $\mathcal{E}(hD^\frac{1}{2}) = \mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2})$ follows that

$$\mathcal{E}(u)\mathcal{E}(u^*)\mathcal{E}(hD^\frac{1}{2}) = \mathcal{E}(u)\mathcal{E}(gD^\frac{1}{2}) = \mathcal{E}(hD^\frac{1}{2}).$$

Hence, $\mathcal{E}(u)\mathcal{E}(u^*) = 1$, and so $\mathcal{E}(u)$ is a unitary. Therefore, $\mathcal{E}((u - \mathcal{E}(u))^*)(u - \mathcal{E}(u)) = 0$. So $u = \mathcal{E}(u) \in \mathcal{D}$ and $h$ is a right outer operator.

(ii) From the proof of Proposition 7 we know that $\mathcal{E}(hD^{-\frac{1}{2}})$ is an outer operator. Using same method as in the proof of (i), we obtain that $hD^{-\frac{1}{2}}$ is an outer operator in $H^2(\mathcal{A})$. Hence, $h$ is an outer operator in $H^p(\mathcal{A})$.

Let $d$ be a positive outer operator in $L^1(\mathcal{D})$ with $\|d\|_1 = 1$. By Theorem 3 $d$ is an invertible positive selfadjoint operator. Set

$$\phi(x) = tr(xd), \quad \forall x \in \mathcal{M}.$$ 

It is clear that $\phi$ is a normal faithful state on $\mathcal{M}$. Since $tr(\mathcal{E}(x)) = tr(x)$ for $x \in L^1(\mathcal{M})$ (see [16 (2.4)]), we get that

$$\phi(\mathcal{E}(x)) = tr(\mathcal{E}(x)d) = tr(\mathcal{E}(x))d = tr(xd) = \phi(x), \quad \forall x \in \mathcal{M}.$$ 

We denote the dual weight of $\phi$ by $\hat{\phi}$. Then $d$ is the Radon-Nikodym derivative of $\hat{\phi}$ with respect to $\tau$ and

$$\hat{\phi}(x) = \tau(xd), \quad x \in \mathcal{N}_+.$$ 

Hence, the role of $d$ is similar to that of $D$. It follows that if we replace $D$ by $d$ in Section 3 and 4, then the related results still hold.

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