Robustly emulating vortex Majorana braiding in a finite time

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A ‘smoking gun’ experiment for the existence of Majorana zero modes in vortices of Fe-based superconductors is the demonstration of error-prone topological quantum computing by moving the vortices around each other infinitely slowly. This kind of braiding, however, poses yet unmet high experimental demands. We introduce a replacing emulation of braiding by only slightly moving vortices within a special spatial arrangement in a finite time. We present analytical results for the braiding gate and highlight their real-world robustness by a realistic tight-binding model for FeTe$_x$Se$_{1-x}$. The results are extended to the (magic) $\pi/8$ phase gate and the CNOT gate resulting in a scalable scheme for finite-time Majorana quantum computing without physically braiding vortices.

Introduction. Recent experiments on low-dimensional superconducting structures reveal localized electronic states at the Fermi level that are attributed to ‘half-fermionic’ exotic electronic states, the Majorana zero modes. Spatially isolated Majorana zero modes are not lifted from zero energy by proximity-coupling to ordinary quasi-particles and are a key ingredient for demonstrating nonuniversal topological quantum computing [1–4], despite some susceptibility to external noise [5–6]. Majorana modes have supposedly been detected at the ends of semiconducting wires [7,8], designed atomic chains with helical magnetism [9], and, in particular, at the center of Abrikosov vortices on superconductors with a superficial Dirac cone, where they result in spatially localized peaks in the density of states at zero bias voltage [10–14]. In the following, we call the latter vortex Majoranas.

Currently, the unambiguous discovery of Majorana zero modes remains debated. The next milestone is a ‘smoking gun experiment’ to realize quantum bit manipulation by Majorana braiding. I.e., moving vortex Majoranas around each other infinitely slowly. In principle, the set of Majorana quantum gates obtained with topological protection cannot be universal [4], but there are elaborate additional proposals for networks of superconducting wires how to realize universal quantum computing adiabatically, by projective measurements, or combinedly [2,13,21]. In particular, the braiding and the CNOT gate can be obtained by adiabatic braiding [2,22] and topologically non-protected measures can deliver the missing phase gate for universal quantum computing [21].

To naively demonstrate Majorana braiding in Fe-based superconductors, vortices need to be braided around each other, which poses major experimental problems despite the fundamental limitation that a true adiabatic evolution of perfectly degenerate levels cannot be achieved in principle. First, the length of the exchange path is on the order of micrometers and traversing it would take up to seconds in current setups, introducing high experimental demands on sample quality, temperature and experimental control for guaranteeing a coherent transport of the vortex without intermediate quasiparticle poisoning. Second, braiding the vortices results in twisted flux lines in the bulk of the Fe-based superconductor. This results in an energy instability [23], hindering braiding and eventually resulting in relaxation events that disturb the zero-energy subspace.

In this manuscript, we substantially simplify the direct approach of physically braiding vortex Majoranas, reduce braiding to a finite time, and extend the underlying idea to universal quantum computing. To this end, vortex Majoranas are spatially arranged such that changing the position of one of them on a short, well-defined path results in the desired quantum gates without actually braiding the vortices. For current experimental systems like FeTe$_x$Se$_{1-x}$ [13,24] the maximal gate frequency is on the order of a few GHz. Unwanted couplings that lift the degeneracy of the ground state are either excluded geometrically or exponentially suppressed by the Majorana superconducting coherence length $\xi$, the effective size of the vortex Majorana. We obtain the results for an analytically solvable time-dependent model and show that they remain valid for a vast range of realistic systems, where $\xi$ is comparable the Fermi wave length. This requirement is met in FeTe$_x$Se$_{1-x}$. If $\xi$ is much larger, controlled vortex manipulation becomes impractical, whereas if $\xi$ is much smaller, the energy scales fall below current experimental resolution. Since eventually the engineering of the
couplings between Majorana modes is the crucial mechanism here, the finite-time results are also applicable to Majoranas at the ends of topologically superconducting wires, edges states or other systems \([1, 3, 4, 5]\).

For our proposal, we presume a reliable mechanism for moving vortices. Tremendous progress in this regard has recently been made by moving vortices with the cantilever of a magnetic force microscope \([25–27]\). Additionally, the controlled nanoscale assembly of vortices has been achieved with a scanning tunneling microscope \([28]\) by letting vortices follow the locally heat-suppressed superconducting gap.

**System** In a low-energy, long-distance continuum model, the hybridization strength of two vortex Majoranas is \([29, 30]\)

\[
\epsilon(r) \propto \cos(k_F r + \pi/4)e^{-r/\xi}/\sqrt{r},
\]

where \(r\) is the distance between their centers, \(k_F\) the Fermi momentum and \(\xi\) as above. At distances \(r_i = (\pi i - 3/4))/k_F\), two vortex Majoranas decouple. In real systems and in the presence of multiple vortex Majoranas, the form of Eq. \((1)\) slightly changes, but its oscillatory behavior remains, resulting in only slightly shifted decoupling distances \(r_i\).

We next design a geometry for four vortex Majoranas, where most vortex Majoranas mutually decouple. To this end, we consider three vortex Majoranas \(\gamma_1, \gamma_2, \gamma_3\) at the corners of an equilateral triangle, see Fig. 1. Each vortex Majorana is a decoupling distance \(r_6\) away from the others, and hence does not hybridize with the others. A fourth vortex Majorana \(\gamma_m\) can be moved along a path close to the center. This path is defined by being exactly the decoupling distance \(r_4\) away from at least one exterior vortex. That most couplings vanish on this path is the key element of the setup. Especially, at the corners and on the edges of the path only two and three vortex Majoranas hybridize, respectively. Here, theoretically, all decoupling distances are equal but good can be chosen parameter dependent to reduce unwanted residual coupling or deformations of the geometry. For Fe\(_{1-x}\)Se\(_x\) with a superconducting coherence length of \(\approx 13.9\)nm, the choice of \(r_6\) and \(r_4\) results in a direct distance between exterior vortices of \(\approx 80\)nm, and a \(\gamma_m\) that is never further than \(\approx 6.5\)nm away from the center.

Since the Caroli-de Gennes-Matricon (CdGM) states are sufficiently gapped out in Fe-based superconductors \([12, 13]\), the low-energy Hamiltonian of the system is

\[
\mathcal{H}(t) = 2iJ(\lambda_1(t)\gamma_1 + \lambda_2(t)\gamma_2 + \lambda_3(t)\gamma_3)\gamma_m,
\]

with all \(\gamma\) obeying the Majorana algebra \([1]\) and \(J\) setting the energy scale, with \(J \approx 25\)μeV in Fe\(_{1-x}\)Se\(_x\). By having used decoupling distances inbetween most Majorana vortices, all additional Majorana couplings are forbidden. The setup has a \(C_3\) symmetry and a mirror symmetry, which we assume to hold in the following unless explicitly states otherwise.

Let us first adiabatically move \(\gamma_m\) along the path \(s_1\) (see Fig. 1). At the beginning, \(\gamma_{m1} = \gamma_{m2} = r_4\) so that \(\lambda_1\) and \(\lambda_2\) vanish and only \(\lambda_3\) survives. As \(\gamma_m\) moves towards \(\gamma_1\) with constant \(\gamma_m\), \(\lambda_1\) grows from zero and \(\lambda_3\) diminishes. When \(\gamma_m\) arrives at the position where \(\gamma_{m1} = \gamma_{m2} = r_4\), \(\lambda_1\) is the only non-zero coupling. Using the \(C_3\) symmetry of the setup determines the couplings of remaining path. The resulting adiabatic time evolution is

\[
\lim_{T \to \infty} U(T) \propto \frac{1}{\sqrt{2}} (1 - 2\gamma_1\gamma_2) = B_{1,2},
\]

where \(T\) is the time for traversing \(s_1\) and \(B_{1,2}\) is the braiding operator obtained when braiding \(\gamma_1\) around \(\gamma_2\) \([10]\). This can be seen by calculating the spherical angle enclosed by the path of \(X(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))\) in three-space \([31, 35]\). The result is independent of the concrete values of all \(X \neq 0\) as long as \(\lambda_i = 0\) whenever \(\gamma_i\) is furthest away from \(\gamma_m\) (which ensured by the definition of \(s_1\)). The braiding gate is hence realized without physically braiding vortex Majoranas around each other. By the same argument in conjunction with the \(C_3\) and the mirror symmetry of the setup, moving \(\gamma_m\) adiabatically along the path \(s_2\) of Fig. 1 results in the \(\pi/8\) phase gate. Although Majorana braiding and the \(\pi/8\) phase gate through adiabatic time evolution \([33, 34]\) or projective measurements \([20, 36]\) have been extensively studied in resembling setups for superconducting wires, a concrete proposal for real materials is missing in the literature. Here, we describe a protected finite-time realization and provide a realistic experimental recipe in \([37]\).

To resolve the non-adiabatic behavior, we first consider the following analytically solvable Hamiltonian and show further below that the exact form of the couplings is not relevant

\[
\lambda_1(t) = \sin \left(\frac{3\pi t}{2T}\right), \quad \lambda_2(t) = 0, \quad \lambda_3(t) = \cos \left(\frac{3\pi t}{2T}\right).
\]

This describes \(\gamma_m\) moving along the left edge of the path \(s_1\) (see Fig. 1), starting at \(t = 0\) where only \(\gamma_m\) and \(\gamma_3\) couple and ending at \(t = T/3\), where only \(\gamma_m\) and \(\gamma_1\) couple. In \([37]\), we verify the exact solution for the time evolution to be \(U_{3,1}\), where

\[
U_{i,j}(t) = e^{-i\chi_{ij}t} e^{i\gamma_{ij}Jt/h + i\gamma_{ij}3\pi t/2T}.\]

Notably, at discrete process times

\[
\mathcal{T}_n = 3\pi \sqrt{n^2 - 1/16} h/J,
\]

the finite-speed time evolution of Eq. \((5)\) is equivalent to adiabatic braiding \(U_{3,1}(\mathcal{T}_n) = B_{3,1}\). In Fe\(_{1-x}\)Se\(_x\) the corresponding frequencies are up to \(1/T_1 \approx 4.62\)GHz. This frequency is much smaller than the frequency corresponding to exciting the Majorana mode to the CdGM states \(\Delta^2/(\hbar E_F) \approx 300\)GHz, where \(\Delta\) is the size of the
superconducting gap and $E_F$ the Fermi energy. In contrast to Eq. (3), here $\gamma_m$ is only moved along one edge of $s_1$. The time evolution for moving $\gamma_m$ along all three edges is

$$U(T) = U_{1,2}(T)U_{2,3}(T)U_{3,1}(T). \quad (7)$$

such that $U(T_n) \propto B_{1,2}$. The model hence realizes braiding by moving $\gamma_m$ along $s_1$ in finite time $T_n$.

To realize the $\pi/8$ phase gate, we move $\gamma_m$ along the path $s_2$ of Fig. 1. For the edge of $s_2$ that crosses the center, we consider the couplings

$$\lambda_1 = \lambda_2 = \sin \left( \frac{3\pi t}{2T} \right) / \sqrt{2}, \quad \lambda_3 = \cos \left( \frac{3\pi t}{2T} \right). \quad (8)$$

By defining the auxiliary Majorana $\gamma_a = \frac{1}{\sqrt{2}} (\gamma_1 + \gamma_2)$, the Hamiltonian and hence the time evolution is equivalent to Eq. (5) when $\gamma_1$ is replaced by $\gamma_a$. Moving $\gamma_m$ along $s_2$ therefore realizes the $\pi/8$ phase gate in finite time (see 37 for details). Also, $s_2$ joined with its reflected and time-reversed partner forms $s_1$, which generates a braiding phase of $\pi/4$. This fixes the phase of $s_2$ to $\pi/8$.

**Deviations from the analytic model**  In real-world systems, deviations from Eq. (4) challenge the analytical findings. Yet, we can analytically show that the finite-time gates are protected and numerically verify this protection with a realistic tight-binding model for FeTe$_2$Se$_{1-x}$ [29]. We condense our findings to a step-by-step experimental recipe how to realize the braiding gate with vortex Majoranas in the Supplemental Material 37. For undisturbing deviations there will always be special process times for which the gates are perfectly realized. It is irrelevant to perform the motion of $\gamma_m$ at constant speed, as long as the $C_3$ and mirror symmetry are respected, because fluctuations in the speed are equivalent to fluctuations in the couplings, cf. Fig. 2.

For concreteness, we focus on the braiding gate. Consider the real-world couplings to differ from Eq. (4) but to still obey $\lambda_3(t) = 0$. The general time evolution along the left edge of the path $s_1$ is then $U_{5,1}^s$ with

$$U_{ij}^s = b_1 + 2b_2\gamma_i\gamma_m + 2b_3\gamma_m\gamma_j + 2b_4\gamma_j\gamma_i, \quad (9)$$

with real coefficients $b_i$ that depend on the process time $T$. The time evolution along the remaining two edges ($U_{2,3}^s$ and $U_{1,2}^s$) follows by $C_3$ symmetry. We use two types of indicators for deviations from perfect braiding — unwanted quasiparticle excitations and the extension of the Berry phase to finite-time processes. The probability to excite unwanted quasiparticles after a full passage of $\gamma_m$ along $s_1$ is $|q|^2$ (see 37 with

$$q = \text{Tr} \left\{ (\gamma_1 + i\gamma_2) (\gamma_3 + i\gamma_m) U_{2,3}^s U_{1,2}^s U_{3,1}^s \right\} / 2 = \sqrt{2e^{\pi i}} (b_4 - b_1) \left( 1 - (\Sigma_i b_i^2) \right), \quad (10)$$

The amount of quasiparticle excitations is hence a product of real polynomials. For example, the analytical model of Eq. (4) results in

$$q = e^{i\pi \frac{\omega}{2} \sin(\frac{\theta}{2}) \left[ \cos(\theta) + 4\sqrt{4 + \omega^2} \sin(\theta) - 4 \right]} \sqrt{2e^{\pi i}} (b_4 - b_1) \left( 1 - (\sum_i b_i^2) \right), \quad (11)$$

where $\theta = \pi \sqrt{(4 + \omega^2)/(2\omega)}$ and $\omega = 6\pi h/(J T)$. At the process times $T_n$ of Eq. (4), as well as at additional transcendental process times, the unwanted quasiparticle excitations $|q|^2$ hence exactly vanish as in perfect adiabatic braiding. In real-world systems, $q$ generally has a different course than in the analytical solution, but remains a product of the real polynomials $b_1 - b_4$, and $1 - (\sum_i b_i^2)$, each of which is continuously connected to its counterpart in the analytical solution. Single zeros of $q$ therefore shift but cannot be lifted by small deviations. Only large deviations eventually result in the annihilation of two zeros simultaneously. Even away from
The analytical protocol, there are therefore special process times, where the quasiparticle poisoning $|q|^2$ vanishes. When no unwanted quasiparticle is excited, the system realizes a quantum gate acting on the degenerate ground state. Yet, the corresponding phase could deviate from the braiding gate. To resolve this, we determine the phase difference $\phi$ between quasiparticle states $|0\rangle$ and $(\gamma - i\gamma_m)|0\rangle$ that span the degenerate ground state. In the adiabatic limit, $\phi$ equals the Berry phase and reaches $\pi/2$ for perfect braiding. We analytically find that perfect braiding is realized at the zeros of $q$ that stem from $b_1 = b_1$ in Eq. (11). These zeros correspond to $T_n$ of Eq. (6) in the analytical model.

The numerical affirmation of the protection of the braiding gate is shown in Fig. 2 where the results for realistic couplings are compared to the analytical ones. To estimate such realistic couplings, we employ a tight-binding model for FeTe$_x$Se$_{1-x}$ and derive a low-energy model from the LDOS of the vortex Majoranas [37]. The extracted realistic couplings deviate significantly from the analytically solvable protocol of Fig. 2a. Yet, the zeros of $q$ persist and only shift slightly, see Fig. 2b. Additionally, as depicted in Fig. 2c, there are special process times where $\phi$ reaches $\pi/2$ exactly and the unwanted quasi-particle excitations vanish. Since a change in the speed of $\gamma_m$ is equivalent to a deformation of the time-dependent couplings, the speed of $\gamma_m$ during its motion does not need to be constant as well.

We furthermore find numerically [37] that the robustness of the zeros of the quasiparticle excitations survives the presence of additional unwanted couplings, such as $\lambda_2(t) \neq 0$ in Eq. (1) or couplings that lift the degeneracy of the ground state, as long as the $C_5$ symmetry is respected. This could be used to realize phase gates that deviate from $\pi/4$ and $\pi/8$.

**Multi qubit extension and readout.** Performing quantum computing requires more than two zero-energy Majorana vortices. To achieve this, we concatenate the elementary cells of Fig. 1, sketched in Fig. 3a. The most convenient assembly that respects $C_5$ and mirror symmetry is the hexagram depicted in Fig. 3b, accommodating six free Majoranas (encircled). The other zero modes are gapped out (connected by black lines). Each parity sector of a hexagram accommodates two qubits. Unwanted couplings between vortex Majoranas within a hexagram can not be easily omitted, which induces slight systematic errors in the computation scheme. Yet, unwanted couplings decrease exponentially with the superconducting gap (Eq. (1)). Readout can in principle be accomplished by removing an interior vortex Majorana (red) as in the rectangular area of Fig. 3c. The remaining free vortex Majoranas (blue) can be shifted in order to hybridize and reveal the occupation number of their mutual fermionic state [37, 38]. Majorana braiding and phase gates for each qubit are obtained in finite time as described above. Additionally, the CNOT gate can be realized [22] by braiding the vortex Majoranas from the left of a hexagram to the right, i.e.,

$$\text{CNOT} = (B_{6,1}B_{3,4})(B_{2,3}B_{3,4}B_{4,5}B_{3,4}B_{2,3}).$$

It is straightforward to achieve this in our setup by performing finite-time braiding gates on the respective triangular elementary cells, see Fig. 3d. On a triangular lattice of $n$ hexagrams, see Fig. 3c, the number of free Majoranas scales with $n$. The hexagram is the simplest possible unit cell with this scaling behavior. For simpler ones, the number of free Majoranas scales with $\sqrt{n}$. The qubits belonging to different hexagrams in the lattice can be shifted, such that the CNOT gate can be applied to arbitrary assemblies of free Majoranas. This is achieved by moving interior Majoranas only one edge along their paths $s_i$ such that a braiding gate is realized in finite time [37].
Conclusions. We show that braiding of vortex Majoranas in superconducting vortices can be achieved robustly and in finite time by only slightly moving vortex Majoranas in a special spatial arrangement. The procedure avoids an actual long-time, incoherent, physical braiding process and relies on the inter-Majorana coupling to vanish at material dependent distances. We analytically show that the introduced scheme can realize a universal set of gates for quantum computing. These are realized at specific process times for realistic systems as described by a tight-binding model for FeTe$_2$Se$_{2-x}$. The setup could be used not only as a ‘smoking gun’ experiment for the existence of Majorana zero modes, but eventually to realize universal, scalable quantum computing on the surface of Fe-based superconductors. To improve the control over the position vortices beyond Refs. [26–28], we propose to bring magnetic atoms nearby, which effectively attract or repel the flux line of the vortex. The coupling between vortex Majoranas could also be altered without an actual motion of the vortices, by bringing a nonmagnetic atom inbetween the respective vortex Majoranas. The Majorana coupling is then altered by manipulating the position of this adatom.

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Supplemental Material

In this Supplemental Material, we extract the Majorana couplings in the low-energy model from the data of the tight-binding model (Section 1), describe the employed realistic tight-binding model (Section 2), give a step-by-step experimental recipe how to realize finite-time Majorana braiding by following the correct path of the mediating Majorana vortex (Section 3), give details on the numerical time evolution obtained for the low-energy model (Section 3), show how an artificially broken $C_3$ symmetry results in unavoidable unwanted quasiparticle excitations (Section 4), derive and verify central formulas of the main text (Section 5, Section 6), show how the $\pi/8$ phase gate (magic gate) is obtained in finite times by moving the mediating Majorana $\gamma_m$ along the path $s_2$ of Fig. 4 of the main text (Section 8), and visualize the how free Majoranas can be reorganized in lattices of hexagrams in order to apply the CNOT gate to any combination of free Majoranas (Section 9).

Low-energy model from tight-binding model

We consider the case that the vortex Majoranas interact weakly. That is, the LDOS of an isolated vortex Majorana always exhibits particle-hole symmetry. Although the coupling of the two vortex Majoranas can destroy this symmetry, the distortion of the particle-hole symmetry for weak coupling can be neglected. Therefore, practically, we can estimate the LDOS within the vortex cores near zero energy. That this approach is valid, is also indicated by the simulated distortion of the particle-hole symmetry for weak coupling can be neglected. Therefore, practically, we can estimate the LDOS as discussed in the section below. In this section, we study if the LDOS, an observable, can reveal the hybridization of two vortex Majoranas described by the Hamiltonian

$$H_2 = iM_{ij}\gamma_i\gamma_j,$$

where $\gamma_i$ represents the Majorana mode at vortex core #i and $M_{ij} \geq 0$ by choosing the $Z_2$ gauge of $\gamma_i$. The energy peaks appear at $E = \pm M_{ij}$ so that the energy locations of the peaks determine the strength of the hybridization. The LDOS in the vortex cores #1 and #2 have an identical distribution.

Next, we consider a generic Hamiltonian with three hybridized Majoranas of the form

$$H_3 = i \sum_{1 \leq i < j \leq 3} M_{ij}\gamma_i\gamma_j.$$

Using $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, we have $H_3^2 = \sum_{1 \leq i < j \leq 3} M_{ij}^2$. Furthermore, due to the odd number of the Majoranas, one of them must have zero energy. Therefore, the peaks appear at $E = 0, \pm M$, where $M = \sqrt{\sum_{1 \leq i < j \leq 3} M_{ij}^2}$. By choosing a proper basis, the Hamiltonian can be written as

$$H_3 = i M\tilde{\gamma}_1\tilde{\gamma}_2,$$

where the transformed Majorana operators are $\tilde{\gamma}_1 = O_{11}\gamma_1$ and $O_{ij}$ is a real orthogonal matrix. Majoranas $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are coupled, whereas $\tilde{\gamma}_3$ has zero-energy. In the original basis, site $i$ (where $\gamma_i$ resides) has energy peaks at $E = \pm M$ with peak height $\propto (O_{11}^2 + O_{21}^2)/2$ and a zero-energy peak with height $\propto O_{31}^2$ appears. By comparing Eqs. \ref{eq:14} and \ref{eq:15}, we have

$$M_{ij} = M(O_{1i}O_{2j} - O_{2i}O_{1j}).$$

On the other hand, due orthogonality, we have

$$O_{3i} = \epsilon_{ijk}O_{1i}O_{2k}.$$

Hence, measuring the energy splitting $M$ and the zero-energy peak height $O_{3i}^2$ on each vortex can determine all Majorana coupling strengths $M_{ij} = M\epsilon_{ijk}O_{3k}$.

For the generic form of the couplings between four Majorana modes, the energy splitting and peaks height cannot fix all the values of the Majorana coupling strengths. The generic Hamiltonian is given by

$$H_4 = i \sum_{1 \leq i < j \leq 4} M_{ij}\gamma_i\gamma_j.$$

We want to use the LDOS as an observable to determine the six couplings $M_{ij}$ ($H_4$ is hermitian). By choosing a proper basis, the Hamiltonian can be written in the economic form

$$H_4 = i\epsilon_{1}\tilde{\gamma}_1\tilde{\gamma}_2 + i\epsilon_{2}\tilde{\gamma}_3\tilde{\gamma}_4,$$
FIG. SM-4. The vortex Majorana configuration in real scale for the braiding scheme in the continuous model (a) and the tight-binding model (b). The braiding path starts from the vertex between edge 1 and edge 2. The mediating Majorana $\gamma_m$ moves along edge 1 to edge 2 and then to edge 3. Finally, $\gamma_m$ is back to the starting position.

where $\tilde{\gamma}_i = O_{ij} \gamma_j$ and $O_{il} O_{jl} = \delta_{ij}$. The energy splittings are located at $E = \pm \epsilon_1, \pm \epsilon_2$. For vortex core #i, the peak heights are proportional to \( \frac{O_{21}^2 + O_{22}^2}{2} \) and \( \frac{O_{23}^2 + O_{24}^2}{2} \) for \( i = 1, 2, 3 \) are free observables of the peak heights. With two energy splittings, only five observables in total are unable to determine the values of the six coupling strengths. Hence, an STM (scanning tunneling microscope) measurement is unable to find generic coupling for more than three Majoranas.

Let us come back to the case of the three triangular vortex Majoranas with a mediating one as proposed in the main text. At the starting point of the braiding process as shown in Fig. SM-4, the mediating Majorana $\gamma_m$ is close to $\gamma_2$ near the center of the equilateral triangle with the three vertices ($\gamma_1, \gamma_2, \gamma_3$). With reflection symmetry the Hamiltonian describing the hybridization of the four vortex Majoranas is of the form

$$H = i (A \gamma_j + a \gamma_l + c \gamma_k) \gamma_m + ib \gamma_l \gamma_k + ic \gamma_j \gamma_l + ic \gamma_j \gamma_k.$$  \hfill (20)

The symmetry reduces the number of the unknown coupling strengths from 6 to 4. In this specific case, $j = 2, l = 1, k = 1$. Practically, the Majorana hybridization $A$ dominates and the remaining couplings as the residual hybridizations cannot be completely suppressed by fine-tuning. To estimate the values of the residual hybridizations we treat them as separate first order perturbations. First, let $a = c = 0$, then we have

$$H = i A \gamma_j \gamma_m + ib \gamma_l \gamma_k.$$  \hfill (21)

Clearly, $A$ and $b$ are determined by the observable energy splittings as explained above. Furthermore, consider $b = c = 0$, then two Majoranas with zero energy,

$$\frac{\gamma_l - \gamma_k}{\sqrt{2}}$$  \hfill (22)

commute with the Hamiltonian. The peak heights at zero energy are given by

$$G_j(0) = \frac{2a^2}{A^2}, \quad G_l(0) = G_k(0) = \frac{1}{2} + \frac{A^2}{4a^2 + 2A^2}, \quad G_m(0) = 0,$$  \hfill (23)

for the vortex core #j, l, k, m, respectively. The reflection symmetry leads to $G_l(0) = G_k(0)$. Since $A \gg a$, the value of $a$ can be estimated by $A \sqrt{G_j(0) / 2G_l(0)}$. Similarly, consider the simplified Hamiltonian

$$H = i \gamma_l (A \gamma_m + c \gamma_l + c \gamma_k).$$  \hfill (24)

The value of $c$ is approximately given by $A \sqrt{G_m(0) / 2G_l(0)}$. This approximation is applied in the main text to estimate the low-energy theory of the four Majoranas from the tight-binding model. In the special case, observing the LDOS of the vortex Majoranas can determine the strengths of the Majorana hybridizations.
Realistic tight-binding model for the braiding gate

In order to realize the braiding gate, the mediating vortex Majorana ideally moves along the path $s_1$, which is close to the center of the equilateral Majorana vortex triangle, cf. Fig. 1 of the main text. The main idea that makes it possible to braid Majorana modes without dramatically moving vortices is to steer the evolution of the Majorana qubit through a specific time-dependent manipulation of the Majorana couplings. The coupling strength between Majorana vortices is approximately described by Eq. (1) of the main text, stemming from a continuum model\cite{29, 30}. The distance-dependent oscillating behavior implies that the hybridization of the two Majorana vortices vanishes when the distance between both reaches $r_i = \frac{\pi (i - 1/2) - \alpha}{k_F}$, where $i$ is a positive integer. Hence, we choose the edge length of the equilateral triangle to be $r_0 = 2\pi/(4k_F)$, as shown in Fig. SM-4(a) so that the three unwanted couplings ($i\gamma_1\gamma_2, i\gamma_1\gamma_3, i\gamma_2\gamma_3$) vanish. Even if the edge length slightly deviates from the zero point, the resulting unwanted coupling will be strongly suppressed because of the general exponential decay of the Majorana couplings (Eq. (1) of the main text).

As the mediating Majorana $\gamma_m$ moves along the small triangular path $s_1$, each edge of the path keeps the same distance ($r_4 = 13\pi/(4k_F)$) between $\gamma_m$ and one of $\gamma_1, \gamma_2, \gamma_3$, since at least one of the three hybridizations $\gamma_1\gamma_m, \gamma_2\gamma_m, \gamma_3\gamma_m$ must be zero to emulate braiding. Thus, in principle two of the four vortex Majoranas can have zero energy during the entire braiding process as discussed in the main text.

This analytic approach, however, does not faithfully capture the realistic Majorana physics here. In particular, Eq. (1) of the main text is an approximation describing the Majorana hybridization for only two vortex Majoranas in the system. Since now there are four vortex Majoranas, any hybridization of two vortex Majoranas will be unavoidably affected by the remaining two. In addition, the presence of magnetic flux from the vortices can also alter the hybridization. Additionally, using a distance-independent phase $\alpha$ in the hybridization oscillation in Eq. (1) of the main text is only valid in the long-distance limit. Thus, we use a more realistic tight-binding model for FeTe$_{0.55}$Se$_{0.45}$\cite{9} with four vortices. This model captures the essence of the topological superconductivity in FeTe$_{0.55}$Se$_{0.45}$ by including a surface Dirac cone with real, material specific parameters — its chemical potential $\mu = 5$meV, Fermi velocity of the Dirac cone $v_F = 25$nm-meV, and Fermi momentum $k_F = \frac{\mu}{\nu} = 0.2$nm$^{-1}$. Furthermore, we introduce an s-wave superconducting gap ($\Delta = 1.8$meV)\cite{15} to have non-trivial superconductivity on the surface (where the Dirac cone appears) and add a vortex with one magnetic flux quantum going through (London penetration depth $\lambda = 500$nm\cite{30, 31}). In this setup, a vortex Majorana with zero energy arises at a vortex core on the surface of FeTe$_{0.55}$Se$_{0.45}$ (see more details of the simulation in Ref. \cite{9}). Its characteristic length scales are given by Majorana coherence length $\xi = v_F/\Delta = 13.9$nm and the oscillation length of the Majorana hybridization $\pi/k_F = 15.7$nm.

Now, we insert four vortices into FeTe$_{0.55}$Se$_{0.45}$ for the simulation of the braiding gate. To find their correct positions, we start with the vortex distribution from the continuum model as shown in Fig. SM-4(a), with ($r_4 = 13\pi/(4k_F) = 51.1$nm, $r_0 = 2\pi/(4k_F) = 82.5$nm). These distances are iteratively adjusted until the mediating Majorana is able to move on the ideal braiding path $s_1$. That is, only three types of the hybridizations are present — $r_1\gamma_m, r_2\gamma_m, r_3\gamma_m$. At the start of the time evolution, the mediating Majorana is closest to the Majorana $\gamma_2$ and further away from the other Majoranas with $r_1\gamma_m = \frac{\pi}{3}m$ as shown in Fig. SM-4(b). To have the ideal initial conditions, we need $iA\gamma_2\gamma_m$ as the only hybridization surviving in the Hamiltonian (20) and the residual hybridizations ($a, b, c$, see another section of the appendix) to vanish. Since many unobservable factors (e.g. magnetic flux, $\alpha$ phase correction) actually control the strength of the Majorana hybridization in the tight-binding model (as opposed to the analytical model), the residual hybridizations generally do not exactly vanish. Instead, we need to find the optimal distances inbetween the vortex Majoranas that minimize $a/A, b/A, c/A$ (these are the hybridization strengths compared with the dominant coupling) by solving the eigenvalue problem of the tight-binding model. By adjusting the edge lengths of the Majorana triangle to 77.25nm and the distance between Majorana $\gamma_2$ and the mediating Majorana $\gamma_m$ to 38nm, the minimal ratios of the residual hybridizations are reached and given by $a/A = 0.3\%$, $b/A = 0.9\%$, $c/A = 1.4\%$. After finding these optimal parameters, we start to move the mediating Majorana $\gamma_m$ towards Majorana $\gamma_3$. That is, the hybridization between $\gamma_2$ and $\gamma_m$ gradually decreases and the hybridization between $\gamma_3$ and $\gamma_m$ increases as $\gamma_m$ moves. To maintain the vanishing hybridization ($i\gamma_1\gamma_m$) between $\gamma_1$ and $\gamma_m$, we keep the distance $r_1\gamma_m$ constant during the movement. We thereby assume that the hybridization strength $i\gamma_1\gamma_m$ depends only on the absolute distance, as it is the case for the approximation based on the continuous model in Eq. (1) of the main text. When the mediating Majorana reaches the next corner of the triangular path, $r_1\gamma_m$ and $r_2\gamma_m$ have the same length and $r_3\gamma_m = 38$nm; the hybridization $i\gamma_3\gamma_m$ dominates. Following the same scheme, we move $\gamma_m$ towards $\gamma_1$ and finally return to $\gamma_2$. The evolution of the Hamiltonian is consistent with the description of the path $s_1$ in the main text.
Experimental recipe for braiding

Here, we provide this detailed step-by-step experimental recipe using FeTe$_{0.55}$Se$_{0.45}$ as our main platform for Majorana braiding. In the following, we discuss the four essential steps —

1. arranging the vortex locations
2. moving a Majorana vortex
3. probing the LDOS during braiding
4. reading out quantum states.

Moving a vortex with a vortex Majorana is the key point braiding. Recently, there has been tremendous experimental progresses on manipulating Arbrikosov vortices. First, using magnetic force microscopy can control the location of the vortex in a thin film of superconducting Niobium$^{[25]}$. Very recently, a theoretical proposal$^{[26]}$ has extended this idea to move a vortex in FeTe$_{0.55}$Se$_{0.45}$. Second, experimentalists have recently successfully controlled the location of a single vortex by using a heated tip$^{[28]}$ of a scanning tunneling microscope (STM). The idea is using heat to locally suppress the superconductor gap, which attracts a vortex, so that the vortex intends to follow the motion of the heated tip. Manipulating the vortex location is a necessary preliminary step. Regarding our proposal, this STM manipulation potentially has the advantage of being able to move the vortex Majoranas and resolve their LDOS with the same instrument. Encouraged by the experimental developments, we believe that controlling vortex movement in FeTe$_{0.55}$Se$_{0.45}$ can be achieved in the near future.

Once the technique of controlling the position of a vortex Majorana is well developed, there is another step of preparation before we can perform the proposed braiding, namely the construction of the triangular structure shown in Fig. SM-4. Consider vortex Majoranas on the surface of FeTe$_{0.55}$Se$_{0.45}$. The strength of the magnetic field is the only experimentally controllable parameter for adjusting the average intervortex distance. In the low-field regime ($\leq 1T$), the arrangement of the vortices is close to a perfect triangular lattice distribution, which was observed by Hanaguri’s group$^{[16]}$. It is hence reasonable to assume that this material does not possess strong defects pinning vortices. For the Majorana triangle (see Fig. SM-4), we propose that its ideal edge length is around 77.25nm. The magnetic field strength should hence be less than 0.36T so that the intervortex distance is greater than 77.25nm.

We next move a vortex to the center of one Majorana triangle in the lattice and separate this formation of the four vortex Majoranas from others by moving other vortices away. Finally, we need to precisely adjust the locations of these four Majorana vortices such that the configuration is like the one of Fig. SM-4. In this geometry only $\gamma_m$ and $\gamma_2$ hybridize. The way to confirm that only hybridization remains in this setup is to observe that only two zero-bias tunneling peaks appear, namely at $\gamma_1$ and $\gamma_3$, and that only a small energy splitting of $\sim 25\mu eV$ is present at $\gamma_2$ and $\gamma_m$.

The preparation for braiding is finalized after the above steps. We are ready to move the mediating Majorana $\gamma_m$ to start the process. Before reading the quantum information in the Majorana state in order to confirm our proposed braiding operation, we suggest to check the evolution of the LDOS for each vortex Majorana during the braiding process by the STM. Sec. 1 shows that the strengths of the Majorana hybridization cannot be directly observed by any STM tip. However, we assume that only three types of the hybridizations in Eq. (3) are present in the entire braiding process. This is justified by the carefully constructed geometry. We neglect the residual couplings and then use the evolution of the three hybridization strengths to capture the changes of the LDOS for each vortex Majorana. At the beginning of the braiding, only $\gamma_2$ and $\gamma_m$ hybridize so that $\gamma_1$ and $\gamma_3$ exhibit zero-bias peaks as shown in panel (a) and (c) of Fig. SM-5. Furthermore, panel (a) shows that $\gamma_1$ always possesses a zero-bias peak during the first third of the movement. This supports the assumption of vanishing $i\gamma_1\gamma_m$ and the choice of the braiding path keeping $1\gamma_m1$ invariant in our tight-binding model. When $\gamma_m$ moves towards $\gamma_3$, the energy splitting starts to transit from $\gamma_2$ to $\gamma_3$. The zero-bias peaks appear at $\gamma_1$ and $\gamma_2$, as $\gamma_m$ is closest to $\gamma_3$. For the remaining braiding path, the idea for checking the correct experimental implementation is the same, namely that the change of the Majorana hybridizations leads to the transition of the zero-bias peak from one vortex to another. Throughout this process, two vortex Majorana zero modes are intact, having zero energy, so that two zero-bias peaks are present at each moment. Furthermore, the energy splitting away from zero energy transits from $\gamma_3$ to $\gamma_1$ and then back to $\gamma_2$. On the other hand, the LDOS of $\gamma_m$ never possesses any zero-bias peak as shown in Fig. SM-5d, since it always couples with another vortex Majorana. Observing the LDOS evolution for each vortex is the primary step to check the successful braiding. The information of the LDOS provides the energy splitting and the peak heights of the vortex Majoranas in the entire procedure so that the values of the three Majorana couplings can be determined at each moment; we can use these values to find the prefect time scale for braiding. On the other hand, the technical difficulty for the
expansion of the setup. The calculations conducted in the following section regard a course of the coupling where $\gamma$ is shown in the middle panel of Fig. 2b, and continue this course of coupling in order to respect the energy splitting. The LDOS for each vortex does not break particle-hole symmetry significantly, which supports the validity of a low-energy model as derived in another section of the appendix.

The low-energy model for $\gamma_1$ of the path $s_1$ (see Fig. SM-4 of the Supplemental Material) is

$$H_1(t) = 2iJ(\gamma_1 + \lambda_2(t)\gamma_2 + \lambda_3(t)\gamma_3)\gamma_m,$$

which is Eq. (2) of the main text. The explicit expressions for the $\lambda_i$ used here depend on the material system or on the used model. For the calculations of the main text, we take the data from our realistic tight-binding system that is shown in the middle panel of Fig. 2b and continue this course of coupling in order to respect the $C_3$ symmetry of the setup. The calculations conducted in the following section regard a course of the coupling where $C_3$ symmetry is broken. The full course of the couplings in this case is given by all three panels of Fig. 2b. Lastly, the explicit expressions for $\lambda_i$ in the analytical model are given in the main text.

FIG. SM-5. The LDOS evolution for each vortex core as the mediating Majorana $\gamma_m$ moves along in the braiding loop of Fig. SM-4. Since the energy splitting stemming from Majorana hybridization is around 25$\mu$eV, the energy resolution of the probe has to be better than this value to distinguish zero-bias peaks and energetically split ones. (a-d) show the LDOS of the corresponding vortex Majorana. Along this path, two zero-bias peaks are always simultaneously present at two of the three non-mediating vortices (1, 2, 3), while the mediating Majorana $\gamma_m$ always couples with another vortex Majorana resulting in energy splitting. The LDOS for each vortex does not break particle-hole symmetry significantly, which supports the validity of a low-energy model as derived in another section of the appendix.
The setup is $C_3$ symmetric such that the Hamiltonian for the remaining edges becomes

$$H_2(T/3 \leq t < 2T/3) = 2J\{\lambda_1(t - T/3)\gamma_2 + \lambda_2(t - T/3)\gamma_3 + \lambda_1(t - T/3)\gamma_1\} \gamma_m,$$

$$H_3(2T/3 \leq t \leq T) = 2J\{\lambda_1(t - 2T/3)\gamma_3 + \lambda_2(t - 2T/3)\gamma_1 + \lambda_1(t - 2T/3)\gamma_2\} \gamma_m,$$

respectively. The time evolution resulting in the data for Fig. 2b and Fig. 2c of the main text is obtained by taking this time-dependent Hamiltonian and simulating the Schrödinger equation for the unitary time evolution, i.e.,

$$\partial_t U_{i,j}(t) = -i\hbar H(t)U_{i,j}(t),$$

with an explicit Runge-Kutta method. Convergence of the method is sought for each data point individually, such that systematic errors do not add up by a successive evaluation of $U$ at different times. Additionally, the deviations of $U$ from unitarity is checked at each point, i.e., unitarity is not implicitly imposed. This serves as an additional convergence check. The deviations always remain not significantly larger than machine precision.

**Unwanted couplings and breaking of $C_3$ symmetry**

The proposed scheme is unexpectedly robust against fluctuations in the couplings as long as the $C_3$ symmetry of the setup, cf. Fig. 1 of the main text, is maintained. In this section, we give numerical evidence how unwanted quasiparticle excitations and deviations in the phase difference $\phi$ (see main text) emerge if these requirements are not met or if residual couplings are included that lift the zero energy subspace.

To this end, we employ the protocol shown in Fig. SM-6, which breaks $C_3$ symmetry, and incorporate the unwanted residual couplings determined at the end of another section of the appendix of the Supplemental Material.

We observe that the zeros of the quasiparticle excitation $|q|^2$ (defined in the main text) do not lift if the ground state degeneracy is lifted by unwanted residual couplings, see blue curve in Fig. SM-7a. The only effect of unwanted residual couplings caused on $|q|^2$ is that because the initial state is no longer a ground state of the Hamiltonian, the adiabatic limit still contains nonvanishing unwanted quasiparticle excitations. This is shown in the average increase of the blue curve for $T(J/\hbar) > 20$. The only mechanism we can find that lifts the protected zeros of $|q|^2$ is the artificial breaking of the $C_3$ symmetry of the couplings. Analytically, breaking the $C_3$ symmetry is allowed to result in the lifting of zeros because the constants $b$ in Eq. (10) of the main text in general become imaginary. The roots of $|q|^2$ can then be lifted, because roots of $q$ can evade into the complex plane instead of staying on the real axis.

Calculating the finite-time Berry phase $\phi$ (defined in the main text) for the energetically split ground state degeneracy, we observe a linear drift of this phase, indicated by a black-dotted line in Fig. SM-7b. This is expected because the ground state is no longer degenerate. The energy difference between the two former ground states results in an additional dynamical phase that accumulates over time $\frac{\pi}{4}$ $\frac{3\pi}{4}$.

In summary, our numerical results suggest that the robustness of the unwanted quasiparticle excitations $|q|^2$ is only weakened if the $C_3$ symmetry of the setup is broken. The finite-time Berry phase deviates from its perfect value of $\pi/2$ when the ground state degeneracy is lifted. However, this may actually turn out to be a merit and not a deficiency. In the case of an energetically split ground state degeneracy that still respects $C_3$ symmetry, each zero of the unwanted quasiparticle excitations corresponds to the realization of another phase gate. This has the advantage that no additional paths but only different speeds need to be realized in order to construct a complete set of gates for quantum computing.
Verification of the analytic time evolution

In Eq. [5] of the main text, we give the closed form of the time evolution corresponding to the Hamiltonian Eq. [2] with the time-dependent couplings of Eq. [4] of the main text. The time evolution operator is the solution of the time-dependent Schrödinger equation

$$\frac{\partial}{\partial t} U_{i,j}(t) = -i\hbar H(t)U_{i,j}(t),$$

(29)

which is obtained from the usual form of the Schrödinger equation $\frac{\partial}{\partial t} \Psi(t) = -i\hbar H(t)\Psi(t)$ by replacing $\Psi(t) = U(t)\Psi(0)$. Here

$$H(t) = 2iJ \left[ \cos \left( \frac{3\pi t}{2T} \right) \gamma_i + \sin \left( \frac{3\pi t}{2T} \right) \gamma_j \right] \gamma_m.$$

(30)

The solution to Eq. (29) can be found directly by a rotating-wave ansatz. Here, however, we constrain ourselves to verify that

$$U_{i,j}(t) = e^{-i\gamma_j \frac{3\pi t}{2T}} e^{2i\gamma_i \gamma_m J t/\hbar + i\gamma_j \frac{3\pi t}{2T}}$$

$$= \left[ \cos \left( \omega t \right) - 2\gamma_i \gamma_j \sin \left( \omega t \right) \right] \times \left[ \cos \left( \alpha t \right) + 2t \text{sinc} \left( \alpha t \right) \left( \gamma_i \gamma_m + \gamma_i \gamma_j \right) \right],$$

(31)

is the solution. This can be shown by using the algebra of the Majorana operators. However, for concreteness, we show it by using a specific (faithful) representation of the operators, namely

$$\gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma_4 = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

(32)

and plug Eq. (31) into Eq. (30). The second part of Eq. (31) can also be verified by hand by this approach, using the fact that the parity sectors decouple, and $e^{i\alpha n \sigma} = \cos(\alpha) + i \sigma \cdot n \sin(\alpha)$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector containing all Pauli matrices and $n$ is a vector of length 1. We obtain the matrix representations of both sides of Eq. (31) ($\hbar = 1$) as

$$(-i\hbar H(t))_{i,j} = (\partial_t U(t))_{i,j} = A_{i,j}(t)$$
with

\[
A_{1,1}(t) = i \left( -\omega \sin(t\omega) \sin(t\sqrt{\omega^2 + 1}) + \cos(t\omega) \left( \sqrt{\omega^2 + 1} \cos(t\sqrt{\omega^2 + 1}) + i \sin(t\sqrt{\omega^2 + 1}) \right) \right),
\]

\[
A_{2,2}(t) = \frac{i\omega \sin(t\omega) \sin(t\sqrt{\omega^2 + 1}) + \cos(t\omega) \left( -\sin(t\sqrt{\omega^2 + 1}) - i\sqrt{\omega^2 + 1} \cos(t\sqrt{\omega^2 + 1}) \right)}{\sqrt{\omega^2 + 1}},
\]

\[
A_{3,3}(t) = \frac{i \left( -\omega \sin(t\omega) \sin(t\sqrt{\omega^2 + 1}) + \cos(t\omega) \left( \sqrt{\omega^2 + 1} \cos(t\sqrt{\omega^2 + 1}) + i \sin(t\sqrt{\omega^2 + 1}) \right) \right)}{\sqrt{\omega^2 + 1}},
\]

\[
A_{4,4}(t) = \frac{i\omega \sin(t\omega) \sin(t\sqrt{\omega^2 + 1}) + \cos(t\omega) \left( -\sin(t\sqrt{\omega^2 + 1}) - i\sqrt{\omega^2 + 1} \cos(t\sqrt{\omega^2 + 1}) \right)}{\sqrt{\omega^2 + 1}}.
\]

\[
A_{1,4}(t) = i \sin(t\omega) \cos(t\sqrt{\omega^2 + 1}) + \frac{\sin(t\sqrt{\omega^2 + 1}) (\sin(t\omega) + i\omega \cos(t\omega))}{\sqrt{\omega^2 + 1}},
\]

\[
A_{2,3}(t) = \frac{\sin(t\sqrt{\omega^2 + 1}) (\sin(t\omega) - i\omega \cos(t\omega))}{\sqrt{\omega^2 + 1}} - i \sin(t\omega) \cos(t\sqrt{\omega^2 + 1}),
\]

\[
A_{3,2}(t) = -i \left( \sin(t\omega) \cos(t\sqrt{\omega^2 + 1}) + \frac{\sin(t\sqrt{\omega^2 + 1}) (\omega \cos(t\omega) - i \sin(t\omega))}{\sqrt{\omega^2 + 1}} \right),
\]

\[
A_{4,1}(t) = i \left( \sin(t\omega) \cos(t\sqrt{\omega^2 + 1}) + \frac{\sin(t\sqrt{\omega^2 + 1}) (\omega \cos(t\omega) + i \sin(t\omega))}{\sqrt{\omega^2 + 1}} \right).
\]

The solution is hence verified.

**Derivation of the formula for the unwanted quasiparticle excitations \( q \)**

For four Majoranas, we use the Fock basis \( |0,0\rangle, |1,1\rangle, |0,1\rangle, |1,0\rangle \) that are eigenvectors of the quasi-particle operators \( d_1 = (\gamma_1 + i\gamma_2) / \sqrt{2} \) and \( d_2 = (\gamma_3 + i\gamma_1) / \sqrt{2} \). From \( H|0\rangle \) we see that \( d_1 \) is a zero mode, i.e., a quasiparticle with energy zero. The ground state space is hence spanned by \( |0,0\rangle \) with parity 0 and \( |1,0\rangle \) with parity 1. A time evolution \( U \) that starts and ends at the same Hamiltonian is in principle able to leave the ground space by exciting quasiparticles of type \( d_2 \). Because \( U \) preserves parity, however, a quasiparticle of type \( d_1 \) is simultaneously excited. The probability of this to happen, when starting in \( |0,0\rangle \), is \( |q|^2 \) with

\[
q = (0,0|d_1 d_2 U|0,0).
\]

Here \( |q|^2 = 1 \) when both a \( d_1 \) and a \( d_2 \) particle are excited and \( |q|^2 = 0 \) if no unwanted quasiparticle is excited. We rewrite Eq. (33) as a basis independent formula, which becomes Eq. (10) of the main text by setting \( \gamma_4 = \gamma_m \) and inserting the full time evolution \( U = U_{2,2}^R U_{2,1}^R U_{3,1}^R \). Hence,

\[
q = (0,0|d_1 d_2 U|0,0) = \text{Tr} \{ d_1 d_2 U \} = \text{Tr} \{ (\gamma_1 + i\gamma_2) (\gamma_3 + i\gamma_4) U \} / 2
\]

**Finite-time magic gate (\( \pi/8 \) phase gate)**

In this section, we show that the \( \pi/8 \) phase gate is realized in finite time if the mediating Majorana \( \gamma_m \) is moved along the path \( s_2 \) of Fig. 1 of the main text. This gate is also known as “magic” gate and, together with the braiding gate and the CNOT gate discussed in the main text, forms a set of complete gates for universal quantum computing. We proceed as in the main text and regard the probability \( |q|^2 \) for unwanted quasiparticle excitations and the non-adiabatic extension of the Berry phase difference \( \phi \) on the degenerate ground states for moving \( \gamma_m \) along \( s_2 \). The results are shown in Fig. SM-8. The analytical function \( q \) for the unwanted quasiparticle excitations of the magic gate, has zeros at specific process times, see Fig. SM-8a, similar to the behavior of the finite-time braiding gate (see main text). At every fifth of these zeros, the Berry phase difference \( \phi \) simultaneously vanishes, as shown in Fig. SM-8b.
FIG. SM-8. The $\pi/8$ phase gate (magic gate), which is obtained by moving the mediating Majorana $\gamma_m$ along the path $s_2$ of Fig. 1 of the main text. a) Unwanted quasiparticle excitations $|q|^2$. b) Berry phase difference $\phi$. For each fifth zero of $q$, $\phi$ vanishes as well.

FIG. SM-9. The qubits from concatenated hexagrams can be reordered by moving the interior Majoranas along a single edge of $s_1$ (arrows). The CNOT gate can therefore be applied on any pair of qubits.

Reorganizing free Majoranas in hexagram lattices

This section visualizes the possibility of shifting the free Majoranas within concatenated hexagrams to arbitrary places. By this, the CNOT gate can be applied to any combination of free Majoranas. The Majoranas are shifted by moving mediating vortex Majoranas along a single edge of their path $s_1$ as described in the main text. The reorganization of free Majoranas from two hexagrams is shown in Fig. SM-9.