One-stage explicit extended RKN integrators
for quasilinear wave equations

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Abstract

In this paper, we study one-stage explicit extended Runge–Kutta–Nyström (ERKN) integrators for solving quasilinear wave equations. We introduce ERKN integrators as the semidiscretization in time. It is shown that one-stage explicit ERKN integrators in time have second-order convergence. By using a Fourier spectral method in space, full-discrete ERKN integrators are presented. Error bounds of this fully discrete scheme are also derived without requiring any CFL-type coupling of the discretization parameters. The error analysis given in this paper is based on energy techniques, which are widely applied in the numerical analysis of methods for partial differential equations.

Keywords: quasilinear wave equations, extended Runge–Kutta–Nyström integrators, second-order convergence.

MSC: 65M15, 65P10, 65L70, 65M20.

1 Introduction

It is well known that quasilinear wave equations play an important role in a variety of applications such as elastodynamics and general relativity (see, e.g. [6, 14, 20]). The quasilinear wave equations have long been used to approximately describe many examples from elasticity, fluid mechanics and general relativity (see, e.g. [15]). Compared with many publications about the analysis of these equations ([6, 14, 15, 20]), there is much less work devoted to the numerical solutions and numerical analysis for quasilinear wave equations. In this paper, we are devoted to the numerical analysis of a kind of trigo for quasilinear wave equations of the form (see [9])

\[
\partial_t^2 u = \partial_x^2 u - u + \kappa a(u) \partial_x^2 u + \kappa g(u, \partial_x u), \quad x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}), \quad t \in [0, T],
\]

where the functions \( g \) and \( a \) are smooth and real-valued and satisfy \( g(0, 0) = a(0) = 0 \). Here we use the real-valued parameter \( \kappa \) to emphasize the strength of the nonlinearities. In this paper, \( \kappa \) is
considered to be small $0 < \kappa \ll 1$ so that the nonlinearities are small. The system (1) is studied with $2\pi$-periodic boundary conditions in one space dimension and its solutions are assumed to be real-valued. The initial values at time $t = 0$ are given as

$$u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, 0) = \dot{u}_0.$$ (2)

These equations with small $\kappa$ have been extensively studied by [2, 3, 7, 10]. However, in the numerical discretization of (1), the quasilinear term $\kappa a(u)\partial_x^2 u$ is the principal difficulty, which needs to be dealt with carefully. In order to effectively solve (1), some implicit and semi-implicit methods of Runge-Kutta type for semi-discretization in time were proposed and researched recently in [13, 16] for a more general class of quasilinear evolution equations. More recently, the authors in [9] showed that a class of explicit exponential integrators given in [11, 12] can be used to numerically solve the quasilinear wave equation (1) with two regimes of $\kappa$ by using the energy technique with a modified discrete energy.

In this paper, we will rigorously study a class of one-stage explicit extended Runge–Kutta–Nyström (ERKN) integrators for effectively solving the quasilinear wave equation (1). We will prove second-order convergence for ERKN integrators in time. We will also show the convergence of a fully discrete ERKN integrator which is based on a combination with a spectral discretization in space. These ERKN integrators have been originally developed for highly oscillatory ordinary differential equations (ODEs) in [30]. Further researches of them are referred to [22, 25, 24, 26, 29]. Meanwhile, they were recognized to work well also for wave equations in the semilinear case (see, e.g. [17, 18, 27, 23, 28]).

The main contribution of this work is to show the error bounds of one-stage explicit ERKN integrators for quasilinear wave equations. In contrast to the analysis in [9], we do not use a modified discrete energy in this paper and just take the simpler and normal energy technique, which is widely used in the numerical analysis of partial differential equations (see, e.g. [1, 4, 5, 8, 19, 13, 16, 24]). The paper is displayed as follows. In Section 2 ERKN integrators for the discretization in time and full-discrete ERKN integrators of the quasilinear wave equation (1) are introduced. The main results of this paper are given in Section 3 and a numerical experiment is carried out to show the numerical behaviour and support the theoretical analysis. In Section 4 we prove the error bounds for ERKN integrators in time. Section 5 is devoted to the proof of error bounds for full-discrete ERKN integrators. For one of the ERKN integrators, a simple proof for the error bounds is presented in Section 6 by establishing a relationship between this integrator and a trigonometric integrator researched in [9]. Finally, in Section 7 we include the conclusions of this paper.

## 2 ERKN integrators

In this paper, we will use the following notations and properties, which have been used in [9].

- Denote by $H^s = H^s(\mathbb{T})$ with $s \geq 0$ the usual Sobolev space equipped with the norm $\| \cdot \|_s$ given by

$$\|v\|_s^2 = \sum_{j \in \mathbb{Z}} \langle j \rangle^{2s}|\hat{v}_j|^2 \quad \text{for} \quad v(x) = \sum_{j \in \mathbb{Z}} \hat{v}_j e^{ijx},$$ (3)

where the weights $\langle j \rangle$ for $j \in \mathbb{Z}$ are defined $\langle j \rangle = \sqrt{j^2 + 1}$. 


• The corresponding scalar product is defined by \( \langle \cdot, \cdot \rangle_s \):
\[
\langle u, w \rangle_s = \sum_{j \in \mathbb{Z}} (j)^{2s} \tilde{v}_j \tilde{w}_j \quad \text{for} \quad v(x) = \sum_{j \in \mathbb{Z}} \tilde{v}_j e^{ijx}, \quad w(x) = \sum_{j \in \mathbb{Z}} \tilde{w}_j e^{ijx}.
\]

• The solutions \((u(\cdot, t), \partial_t u(\cdot, t))\) of the quasilinear wave equation (1) are studied in the spaces \(H^{s+1} \times H^s\) with the norm
\[
\| (u, \dot{u}) \|_s = (\| u \|^2_{s+1} + \| \dot{u} \|^2_s)^{1/2}.
\]

• Another classical estimates for any smooth function \(G\) with \(G(0) = 0\) are (see Chapter 13 of [21])
\[
\| G(u) \|_s \leq \Lambda_s(\| u \|_s) \| u \|_s, \quad \| G(u) - G(v) \|_s \leq \Lambda_s(\| u \|_s + \| v \|_s) \| u - v \|_s,
\]
where \(\Lambda_s(\cdot)\) is a continuous nondecreasing function.

• It is noted that the norm and the scalar product have the following connection
\[
\| u \pm v \|^2_{1} = \| u \|^2_{1} + \| v \|^2_{1} \pm 2\langle u, v \rangle_{1}.
\]

2.1 ERKN integrators for the discretization in time

By letting
\[
f(u) = a(u)\partial_x^2 u + g(u, \partial_x u),
\]
and the linear operator
\[
\Omega = \sqrt{-\partial_x^2 + 1},
\]
the quasilinear wave equation (1) becomes
\[
\partial_t^2 u = -\Omega^2 u + \kappa f(u).
\]

In this paper, we use one-stage explicit ERKN integrators for the discretization in time of (8). The integrators were first formulated in [30] and we represent the scheme of them by the following definition.

**Definition 2.1 (See [30].)** A one-stage explicit ERKN integrator for solving (8) is defined by
\[
\begin{align*}
\begin{cases}
  u_{n+c_1} = \phi_0(c_1^2 V)u_n + h c_1 \phi_1(c_1^2 V)\dot{u}_n, \\
  u_{n+1} = \phi_0(V)u_n + h \phi_1(V)\dot{u}_n + kh^2 \bar{b}_1(V)f(u_{n+c_1}), \\
  \dot{u}_{n+1} = -h\Omega^2 \phi_1(V)u_n + \phi_0(V)\dot{u}_n + k h b_1(V)f(u_{n+c_1}),
\end{cases}
\end{align*}
\]
where \( h \) is a stepsize, \( c_1 \) is real constant, \( b_1(V) \) and \( \bar{b}_1(V) \) are operator-valued functions of \( V = h^2 \Omega^2 \), and
\[
\phi_j(V) := \sum_{k=0}^{\infty} \frac{(-1)^k V^k}{(2k+j)!}, \quad j = 0, 1, \ldots.
\]
It is noted that for $V = h^2\Omega^2$, one has
\begin{align*}
\phi_0(V) = \cos(h\Omega), \quad \phi_1(V) = \text{sinc}(h\Omega).
\end{align*}
According to the symmetry conditions of ERKN integrators given in [29], it is obtained that the one-stage explicit ERKN integrator (9) is symmetric if and only if
\begin{align}
c_1 = 1/2, \quad \text{sinc}(h\Omega)b_1(V) = (I + \cos(h\Omega))\bar{b}_1(V),
\end{align}
where $I$ is the identical operator. Under this condition, the ERKN integrator (9) can be rewritten as
\begin{align}
\begin{cases}
u_{n+1/2} = \cos(h\Omega)\nu_n + \frac{1}{2}h\text{sinc}(h\Omega)\dot{\nu}_n, \\
\dot{\nu}_n = \dot{\nu}_n + h\text{sinc}(h\Omega)b_1(V)f(\nu_{n+1/2}), \\
\Omega\nu_{n+1} = \left(\begin{array}{c}
\cos(h\Omega) \\
-\sin(h\Omega)
\end{array}\right) \left(\begin{array}{c}
\nu_n \\
\dot{\nu}_n
\end{array}\right), \\
\dot{\nu}_{n+1} = \dot{\nu}_{n+1} + h\text{sinc}(h\Omega)\bar{b}_1(V)f(\nu_{n+1/2}).
\end{cases}
\end{align}
We denote the numerical flow of this integrator by $\varphi_h$, i.e., $(\nu_{n+1}, \dot{\nu}_{n+1}) = \varphi_h(\nu_n, \dot{\nu}_n)$.

In this paper, the one-stage explicit ERKN integrator (9) is considered under the following assumption.

**Assumption 2.2** For the coefficient functions of the one-stage explicit ERKN integrator (9), we require the symmetry condition (10) and assume that there exists a constant $c$ such that
\begin{align}
|\xi \bar{b}_1(\xi^2)| \leq c, \quad |\xi^2\bar{b}_1(\xi^2)| \leq c, \quad |\bar{b}_1(\xi^2) - \cos(\frac{1}{2}\xi)| \leq c\xi^2, \\
|\xi\text{sinc}(\xi) - \bar{b}_1(\xi^2)| \leq c.
\end{align}
for all $\xi \geq 0$

### 2.2 Full-discrete ERKN integrators

For the full discretization of (8), we combine the ERKN integrators with a spectral Galerkin method for the discretization in space, which was considered in [29].

We denote the space of trigonometric polynomials of degree $K$ by
\begin{align*}
\mathcal{V}^K = \left\{ \sum_{j=-K}^{K} \hat{v}_je^{ijx} : \hat{v}_j \in \mathbb{C} \right\}
\end{align*}
and the $L^2$-orthogonal projection onto this ansatz space by
\begin{align}
P^K(v) = \sum_{j=-K}^{K} \hat{v}_je^{ijx} \quad \text{for} \quad v = \sum_{j=-\infty}^{\infty} \hat{v}_j e^{ijx} \in L^2.
\end{align}
Then we replace the nonlinearity $f(u)$ in the ERKN integrator in time [30] by the following new nonlinearity
\begin{align}
f^K(u) = P^K(f^K(u)),
\end{align}
where
\begin{align*}
f^K(u) = a^K(u)\partial_x^2u + g^K(u, \partial_xu) \quad \text{with} \quad a^K = \mathcal{T}^K \circ a, \quad g^K = \mathcal{T}^K \circ g.
\end{align*}
Here \( I^K \) denotes the trigonometric interpolation in the space \( \mathcal{V}^K \) of trigonometric polynomials of degree \( K \). This leads to the fully discrete ERKN integrator
\[
\begin{align*}
\frac{u^K_{n+1}}{2} &= \cos \left( \frac{1}{2} h \Omega \right) u^K_n + \frac{1}{2} h \text{sinc} \left( \frac{1}{2} h \Omega \right) \dot{u}_n^K, \\
u^K_{n+1} &= \cos (h \Omega) u^K_n + h \text{sinc} (h \Omega) \dot{u}_n^K + \kappa h^2 \bar{b}_1 (V) \dot{f}^K (u^K_{n+\frac{1}{2}}), \\
\dot{u}^K_{n+1} &= -\Omega \sin (h \Omega) u^K_n + \cos (h \Omega) \dot{u}_n^K + \kappa h b_1 (V) \dot{f}^K (u^K_{n+\frac{1}{2}}),
\end{align*}
\]
which computes approximations \( u^K_n \in \mathcal{V}^K \) and \( \dot{u}^K_n \in \mathcal{V}^K \) to \( u(\cdot, t_n) \) and \( \partial_t u(\cdot, t_n) \) respectively. In addition, the initial values \( u_0 \) and \( \dot{u}_0 \) of (2) are replaced by
\[
u^K_0 = \mathcal{P}^K (u_0), \quad \dot{u}^K_0 = \mathcal{P}^K (\dot{u}_0).
\]
We denote the fully discrete ERKN integrator \([15]\) as \((u^K_{n+1}, \dot{u}^K_{n+1}) = \varphi^K_h (u^K_n, \dot{u}^K_n)\).

**Remark 2.3** It is noted that the nonlinearity \( \dot{f}^K \) appearing in the fully discrete ERKN integrator \([15]\) can be computed efficiently by fast Fourier techniques (see [19]).

### 3 Main results and numerical experiment

In this section, we state the error bounds for the time-discrete ERKN integrator (9). The exact solution \( u(x, t) \) to (8) is required to satisfy the following assumption, which has been considered in [9].

**Assumption 3.1** (See [22].) The exact solution \((u(\cdot, t), \partial_t u(\cdot, t))\) to (8) is assumed to be in \( H^{5+s} \times H^{4+s} \) with
\[
\| (u(\cdot, t), \partial_t u(\cdot, t)) \|_{4+s} \leq M \quad \text{for} \quad 0 \leq t \leq T, \tag{16}
\]
where \( s \geq 0 \) and \( M > 0 \). Moreover, we assume that there are \( 0 < \delta < 1 \) and \( A_0 \geq 0 \) such that \( 1 + \kappa a (u(\cdot, t)) \leq \delta > 0 \) and \( \kappa a (u(\cdot, t)) \leq A_0 \) for \( 0 \leq t \leq T \).

**Remark 3.2** The regularity assumption (11) on the exact solution was considered in [22] and it holds locally in time for initial values in \( H^{5+s} \times H^{4+s} \) by local well-posedness theory (see [15] [20]).

#### 3.1 Main results

**Theorem 3.3** (Error bound for ERKN integrators in time.) Assume that Assumption 3.2 holds for the coefficient functions of ERKN integrators and Assumption 3.1 is true for the exact solution \((u(\cdot, t), \partial_t u(\cdot, t))\) with \( s = 0 \). Then there is a constant \( h_0 > 0 \) such that for \( 0 < h \ll 1 \) and for all \( \kappa \lesssim h < h_0 \), the global error bound for the time-discrete ERKN integrator \((u_n, \dot{u}_n)\) in \( H^2 \times H^1 \) is
\[
\| (u_n, \dot{u}_n) - (u(\cdot, t_n), \partial_t u(\cdot, t_n)) \|_1 \leq C h^2 \quad \text{for} \quad 0 \leq t_n = nh \leq T, \tag{17}
\]
where the constant \( C \) depends on the smooth functions \( a \) and \( g \) in (1), the constant \( c \) of Assumption 3.2, the constant \( M \) from Assumption 3.1 but is independent of the time step-size \( h \) and the final time \( T \).
Table 1: Three one-stage explicit ERKN integrators.

| Methods | $c_1$ | $\tilde{b}_1(V)$ | $b_1(V)$ |
|---------|-------|-----------------|--------|
| ERKN1   | $\frac{1}{2}$ | $\frac{1}{2} \phi_1^3(V/4)$ | $\phi_1^3(V/4) \phi_0(V/4)$ |
| ERKN2   | $\frac{1}{2}$ | $\frac{1}{2} \phi_1(V) \phi_1(V/4)$ | $\phi_1(V) \phi_0(V/4)$ |
| ERKN3   | $\frac{1}{2}$ | $\frac{1}{2} \phi_1(V) \phi_1^2(V/4)$ | $\phi_1(V) \phi_1(V/4) \phi_0(V/4)$ |

Theorem 3.4 (Error bound for full-discrete ERKN integrators.) Under the conditions in Theorem 3.3 but with a fixed $s \geq 0$ instead of $s = 0$, there is a constant $h_0 > 0$ such that for $0 < \kappa \ll 1$ and for all $\kappa \lesssim h < h_0$, the global error bound for the full-discrete ERKN integrator $(u_n^K, \dot{u}_n^K)$ of (15) in $H^2 \times H^1$ is

$$
\| (u_n^K, \dot{u}_n^K) - (u(\cdot, t_n), \partial_t u(\cdot, t_n)) \|_1 \leq C h^2 + C K^{-s-2} \quad \text{for} \quad 0 \leq t_n = nh \leq T. \tag{18}
$$

Theorem 3.5 (Error bounds for a special ERKN integrator.) For a special ERKN integrator ERKN2 which is presented in Table 1 and does not satisfy the last requirement in Assumption 2.2, it has the global error bound (17) for ERKN2 in time and the global error bound (18) for the full-discrete ERKN2.

Remark 3.6 It is noted that this paper only considers one regime of $\kappa$ which is that $\kappa$ is small. The reason is that the bound (10) can be true only for this case. For the regime 1 of $\kappa$, we can only obtain that

$$
| \mathcal{R}(u_{n+1}, v_{n+1}, u_n, v_n) | \leq C_M h \| (u_n - v_n, \dot{u}_n - \dot{v}_n) \|_1,
$$

which is not sufficient for deriving the second-order convergence of the ERKN integrators. For the convergence of the ERKN integrators when applied to quasilinear wave equations with $\kappa = 1$, the possible way to work is to use a modified energy instead of the normal energy techniques and we will study it in future.

3.2 Numerical experiment

As an example, we present three practical one-stage explicit ERKN integrators and their coefficients are listed in Table 1. It can be checked easily that these three integrators except ERKN2 satisfy all the requirements in Assumption 2.2. For the convergence of ERKN2, we will give another proof in Section 6 which does not rely on Assumption 2.2. For comparison, we choose a trigonometric integrator (formula (15) with $c = 2$ of [9]) and denote it as NTI.

We consider the quasilinear wave equation (11) with $a(u) = u$ and $g(u, \partial_x u) = (\partial_x u)^2 + \kappa u^3$, which has been studied in [3, 9]. The initial values are chosen as

$$
u(x, 0) = \sum_{j \in \mathbb{Z}} \frac{1}{\sqrt{1 + |j|^{11+\sigma}}} e^{ijx}, \quad \partial_t u(x, 0) = \sum_{j \in \mathbb{Z}} \frac{1}{\sqrt{1 + |j|^{9+\sigma}}} e^{ijx}.
$$

For this choice, the initial values are in $H^5 \times H^4$, but not in $H^{5+\sigma} \times H^{4+\sigma}$ for $\sigma \geq 1/100$, so that the initial values just don’t fail to satisfy the regularity assumption (10) for $s = 0$. We solve
this problem in $[0,10]$ with $\kappa = 1/100$ and the setpsizes $h = \frac{j}{10}$ for $j = 1, 2, \cdots, 11$. The errors in $H^2 \times H^1$ of these three ERKN integrators are plotted in Figure 1. The observed convergence of these methods are two, which supports the results of Theorems 3.3-3.4. Moreover, it can be seen that the ERKN integrators behave better than the trigonometric integrator NTI.

4 Proof of error bounds for ERKN integrators in time

Theorem 3.3 will be proved in this section. Following [9] and in order to present this paper as a concise proof of concept, we limit ourselves to the exemplary case $g \equiv 0$ in (1), i.e.,

$$f(u) = a(u) \partial_x^2 u.$$  \hfill (19)

Since the most critical part of the nonlinearity in (1) is the quasilinear term $a(u) \partial_x^2 u$, it is straightforward to extend the proof to nonzero $g$, which will be noted after each step of the proof.

We remark that in the proof, denote by $C$ a generic constant that may depend on $a$, an upper bound $\max(1, |\kappa|)$ on $\kappa$ in (1) (but not on a lower bound), the order of the Sobolev space under consideration and on the constants in Assumptions 2.2 and 3.1. Denote by lower indices the additional dependencies of $C$, e.g., $C_M$ with $M$ from (16).

4.1 Bounds for a single time step

By the estimates (4)-(5) and the smoothness of $a$, some fundamental properties of the nonlinearity $f$ in (19) are obtained, which have been given in [9] and will be used in the proof.

**Lemma 4.1** (See [9]) For the nonlinearity $f$ in (19), it is true that

$$\|f(u)\|_s \leq \Lambda_s(\|u\|_\sigma) \|u\|_\sigma \|u\|_{s+2} \quad \text{with} \quad \sigma = \max(s,1),$$ \hfill (20)

and the Lipschitz property

$$\|f(u) - f(v)\|_s \leq \Lambda_s(\|u\|_{s+2} + \|v\|_{s+2})(\|u\|_{s+2} + \|v\|_{s+2}) \|u - v\|_{s+2}, \quad \hfill (21)$$

where $s \geq 0$, $u, v \in H^{s+2}$, and $\Lambda_s(\cdot)$ is a continuous non-decreasing function.

The following lemma shows that the time-discrete ERKN integrator $\varphi_h$ given by (11) maps $H^{s+1} \times H^s$ to itself for $s \geq 1$.

**Lemma 4.2** (Bounds for a single time step.) Let $s \geq 1$ and it is assumed that Assumption 2.2 holds. If a time-discrete ERKN integrator $(u_n, \dot{u}_n) \in H^{s+1} \times H^s$ satisfies $\|(u_n, \dot{u}_n)\|_s \leq M$, then it is true that

$$\left\|u_{n+\frac{1}{2}} \right\|_{s+1} \leq C_M$$

and $(u_{n+1}, \dot{u}_{n+1}) \in H^{s+1} \times H^s$ with

$$\|(u_{n+1}, \dot{u}_{n+1})\|_s \leq C_M.$$
Figure 1: The logarithm of the errors against the logarithm of stepsizes.
Proof From the definition of the ERKN integrator (9), it follows that

\[ \| u_{n+\frac{1}{2}} \|_{s+1} \leq \| \cos (\frac{1}{2}h\Omega) u_n \|_{s+1} + \frac{1}{2} \Omega^{-1} \sin (\frac{1}{2}h\Omega) \dot{u}_n \|_{s+1} \]

\[ = \| \cos (\frac{1}{2}h\Omega) u_n \|_{s+1} + \frac{1}{2} \| \sin (\frac{1}{2}h\Omega) \dot{u}_n \|_s \leq C_M. \]

Thus

\[ h^2 \| \tilde{b}_1(V) f(u_{n+\frac{1}{2}}) \|_{s+1} \leq \| \Omega^{-2} f(u_{n+\frac{1}{2}}) \|_{s+1} = \| f(u_{n+\frac{1}{2}}) \|_{s-1} \leq \Lambda_{s-1} \left( \| u_{n+\frac{1}{2}} \|_{s+1} \right) \| u_{n+\frac{1}{2}} \|_{s+1} \]

is seen from the second formula in (12) and (20). In a similar way, by the fourth formula in (12) it arrives that

\[ h \| b_1(h\Omega) f(u_{n+\frac{1}{2}}) \|_s \leq \| \Omega^{-1} f(u_{n+\frac{1}{2}}) \|_{s-1} \leq \Lambda_{s-1} \left( \| u_{n+\frac{1}{2}} \|_{s+1} \right) \| u_{n+\frac{1}{2}} \|_{s+1}^2. \]

Therefore, considering the scheme of ERKN integrator (9) again leads to

\[ \| u_{n+1} \|_{s+1} \leq \| \cos(h\Omega) u_n \|_{s+1} + \| \Omega^{-1} \sin(h\Omega) \dot{u}_n \|_{s+1} + h^2 \| \tilde{b}_1(h\Omega) f(u_{n+\frac{1}{2}}) \|_{s+1} \]

\[ \leq \| u_n \|_{s+1} + \| \dot{u}_n \|_s + \Lambda_{s-1} \left( \| u_{n+\frac{1}{2}} \|_{s+1} \right) \| u_{n+\frac{1}{2}} \|_{s+1}^2, \]

and

\[ \| \dot{u}_{n+1} \|_s \leq \| \Omega \sin(h\Omega) u_n \|_s + \| \cos(h\Omega) \dot{u}_n \|_s + h \| b_1(h\Omega) f(u_{n+\frac{1}{2}}) \|_s \]

\[ \leq \| u_n \|_{s+1} + \| \dot{u}_n \|_s + \Lambda_{s-1} \left( \| u_{n+\frac{1}{2}} \|_{s+1} \right) \| u_{n+\frac{1}{2}} \|_{s+1}^2. \]

\[ \square \]

Remark 4.3 It is noted that from the proof, it follows that this lemma is still true for a nonzero $g$ in (1).

4.2 Stability

In this subsection, we will show the stability of ERKN integrators. Before presenting the result, the following two lemmas are needed.

Lemma 4.4 Assume that Assumption holds with constant $c$. For two time-discrete ERKN numerical solutions $(u_n, \dot{u}_n) \in H^{s+1} \times H^s$ and $(v_n, \dot{v}_n) \in H^{s+1} \times H^s$ with $s \geq 0$, one has that

\[ \| (u_{n+1} - v_{n+1}, \dot{u}_{n+1} - \dot{v}_{n+1}) \|_1^2 = \| (u_n - v_n, \dot{u}_n - \dot{v}_n) \|_1^2 + \kappa \mathcal{R}(u_{n+1}, v_{n+1}, u_n, v_n), \]

where the remainder is given by

\[ \mathcal{R}(u_{n+1}, v_{n+1}, u_n, v_n) = (2\text{sinc}(h\Omega)^{-1} b_1(V)(u_{n+1} - u_n - v_{n+1} + v_n), f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))_1. \]

(22)
Proof. In this proof, we will use the following results

\[ h \text{sinc}(h\Omega)\dot{u}_{n+1} = \cos(h\Omega)u_{n+1} + \kappa h^2 b_1(V) f(u_{n+\frac{1}{2}}) - u_n, \]
\[ h \text{sinc}(h\Omega)\dot{u}_n = -\cos(h\Omega)u_n - \kappa h^2 b_1(V) f(u_{n+\frac{1}{2}}) + u_{n+1}, \]  

(23)

which are obtained by considering the ERKN scheme (11) and its symmetry. The same relations hold for \( v \).

According to the third step of the integrator (11), it is obtained that

\[ \|\Omega(u_{n+1} - v_{n+1})\|_1^2 + \|\dot{u}_{n+1} - \dot{v}_{n+1}\|_1^2 = \|\Omega(u_n - v_n)\|_1^2 + \|\dot{u}_n - \dot{v}_n\|_1^2. \]  

(24)

By (4) and the fourth step of (11), we have

\[ \|\dot{u}_{n+1} - \dot{v}_{n+1}\|_1^2 = \left\|\dot{u}_{n+1} - \dot{v}_{n+1} - h\kappa \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\right\|_1^2 \]
\[ = \|\dot{u}_{n+1} - \dot{v}_{n+1}\|_1^2 + h^2 \kappa^2 \left\|\text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\right\|_1^2 \]
\[ -2h\kappa (\dot{u}_{n+1} - \dot{v}_{n+1}, \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})))_1. \]

Replacing the difference \( \dot{u}_{n+1} - \dot{v}_{n+1} \) with the help of the first relation of (23) yields

\[ h(\dot{u}_{n+1} - \dot{v}_{n+1}, \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})))_1 \]
\[ = \langle \text{sinc}(h\Omega)^{-1} \cos(h\Omega)(u_{n+1} - v_{n+1}) + \kappa h^2 \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})) \rangle_1 \]
\[ -\text{sinc}(h\Omega)^{-1}(u_n - v_n), \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})))_1 \]
\[ = \langle \text{sinc}(h\Omega)^{-2} \cos(h\Omega)b_1(V)(u_{n+1} - v_{n+1}), f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}) \rangle_1 \]
\[ +h\kappa^2 \left\|\text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\right\|_1^2 \]
\[ -\langle \text{sinc}(h\Omega)^{-2}b_1(V)(u_n - v_n), f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}) \rangle_1. \]

Here, we use the property \( \langle v, \Psi(h\Omega)w \rangle_1 = \langle \Psi(h\Omega)v, w \rangle_1 \), which is obtained by Parseval’s theorem.

Similarly, taking the second step of (11) and the second of (23) into account, one gets

\[ \|\dot{u}_n - \dot{v}_n\|_1^2 = \|\dot{u}_n - \dot{v}_n\|_1^2 + h^2 \kappa^2 \left\|\text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\right\|_1^2 \]
\[ +2h\kappa (\dot{u}_n - \dot{v}_n, \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})))_1 \]

and

\[ h(\dot{u}_n - \dot{v}_n, \text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})))_1 \]
\[ = \langle -\text{sinc}(h\Omega)^{-2} \cos(h\Omega)b_1(V)(u_n - v_n), f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}) \rangle_1 \]
\[ -\kappa h^2 \left\|\text{sinc}(h\Omega)^{-1}b_1(V)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\right\|_1^2 \]
\[ +\langle \text{sinc}(h\Omega)^{-2}b_1(V)(u_{n+1} - v_{n+1}), f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}) \rangle_1. \]
In light of the above analysis, the formula (24) can be expressed as

\[
\|(u_{n+1} - v_{n+1}, \dot{u}_{n+1} - \dot{v}_{n+1})\|_1^2 - \|(u_n - v_n, \dot{u}_n - \dot{v}_n)\|_1^2
= 2\kappa (\text{sinc}(h\Omega))^{-2}(I + \cos(h\Omega))b_1(V)(u_{n+1} - u_n - v_{n+1} + v_n), f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})\|_1.
\]

(25)

From the symmetry condition (10), it follows that

\[
\text{sinc}(h\Omega)^{-2}(I + \cos(h\Omega))b_1(V) = \text{sinc}(h\Omega)^{-1}b_1(V).
\]

Therefore, (26) yields the statement of this lemma with the remainder (22). □

The bound of the remainder \( R \) (22) is estimated by the following lemma.

**Lemma 4.5 (Bound of the remainder.)** Under the conditions given in Assumption 2.2, if time-discrete ERKN numerical solutions \((u_n, \dot{u}_n)\) and \((v_n, \dot{v}_n)\) belonging to \( H^3 \times H^2 \) satisfy

\[
\|(u_n, \dot{u}_n)\|_2 \leq M \quad \text{and} \quad \|(v_n, \dot{v}_n)\|_2 \leq M,
\]

we then obtain the bound for the remainder \( R \) as

\[
\|\kappa R(u_{n+1}, v_{n+1}, u_n, v_n)\| \leq C_M h\|(u_n - v_n, \dot{u}_n - \dot{v}_n)\|_1^2.
\]

(26)

**Proof.** It is obtained from the scheme of ERKN integrators (9) that

\[
\|(u_{n+1} - u_n) - (v_{n+1} - v_n)\|_1
\leq \|\cos(h\Omega) - I)(u_n - v_n)\|_1 + \|h\text{sinc}(h\Omega)(\dot{u}_n - \dot{v}_n)\|_1
+h^2 \|b_1(h\Omega)(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\|_1
\leq 2\|\sin(h\Omega/2)^2(u_n - v_n)\|_1 + h \|(u_n - \dot{v}_n)\|_1 + h \|\Omega^{-1}(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}))\|_1
\leq \|h\Omega(u_n - v_n)\|_1 + h \|(u_n - \dot{v}_n)\|_1 + h \|f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})\|_0
\leq h \|u_n - v_n\|_2 + h \|\dot{u}_n - \dot{v}_n\|_1 + h\lambda_0 \left(\|u_{n+\frac{1}{2}}\|_2 + \|v_{n+\frac{1}{2}}\|_2\right)
\left(\|u_{n+\frac{1}{2}}\|_2 + \|v_{n+\frac{1}{2}}\|_2\right) \|u_{n+\frac{1}{2}} - v_{n+\frac{1}{2}}\|_2,
\]

(27)

where the first formula in (12) and (21) were used here. By Lemma 4.2, we know that

\[
\|u_{n+\frac{1}{2}}\|_2 \leq C_M, \quad \|v_{n+\frac{1}{2}}\|_2 \leq C_M.
\]

Using the scheme of ERKN integrators (9) again leads

\[
\|u_{n+\frac{1}{2}} - v_{n+\frac{1}{2}}\|_2 \leq \|u_n - v_n\|_2 + \|\sin(h\Omega)\Omega^{-1} (\dot{u}_n - \dot{v}_n)\|_2
\leq \|u_n - v_n\|_2 + \|\dot{u}_n - \dot{v}_n\|_1.
\]

(28)

By the above results, (27) becomes

\[
\|(u_{n+1} - u_n) - (v_{n+1} - v_n)\|_1 \leq C_M h\|(u_n - v_n, \dot{u}_n - \dot{v}_n)\|_1.
\]

(29)
Using the facts that $|\langle \cdot, \cdot \rangle| \leq C \| \cdot \|_1 \cdot \| \cdot \|_1$, the remainder \[22\] have the following bound
\[
|\kappa \mathcal{R}(u_{n+1}, v_{n+1}, u_n, v_n)| = \langle (u_{n+1} - u_n - v_{n+1} + v_n), 2\text{sinc}(h\Omega)^{-1}b_1(V)\kappa(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})) \rangle_1
\]
\[
\leq C \| \Upsilon \|_1 \| 2\text{sinc}(h\Omega)^{-1}b_1(V)\kappa(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})) \|_1
\]
\[
\leq C \| \Upsilon \|_1 \| h(h\Omega)^{-1}(f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}})) \|_1
\]
\[
\leq C \| \Upsilon \|_1 \| f(u_{n+\frac{1}{2}}) - f(v_{n+\frac{1}{2}}) \|_0,
\]
where $\Upsilon = u_{n+1} - u_n - v_{n+1} + v_n$. Based on this fact, the results \[21\], \[28\] and $\| \Upsilon \|_1 \leq C_M h\| (u_n - v_n, u_n - v_n) \|_1$, the bound \[26\] is obtained. \(\square\)

By the above two lemmas, we obtain the following estimate about the stability of ERKN integrators.

**Proposition 4.6 (Stability.)** Under the conditions given in Lemma 4.3, if the solution $(u, \partial_t u)$ to \[8\] in $H^3 \times H^2$ satisfies
\[
\| (u(\cdot, t_n), \partial_t u(\cdot, t_n)) \|_2 \leq M,
\]
then it holds that
\[
\| (u_{n+1}, \dot{u}_{n+1}) - \varphi_h(u(\cdot, t_n), \partial_t u(\cdot, t_n)) \|_1^2 \leq (1 + C_M h) \| (e_n, \dot{e}_n) \|_1^2,
\]
where the global error $(e_n, \dot{e}_n)$ is defined by
\[
(e_n, \dot{e}_n) = (u_n, \dot{u}_n) - (u(\cdot, t_n), \partial_t u(\cdot, t_n)).
\]

**Proof** This result is obtained immediately by letting
\[
(v_n, \dot{v}_n) = (u(\cdot, t_n), \partial_t u(\cdot, t_n)), \quad (\dot{v}_{n+1}, \ddot{v}_{n+1}) = \varphi_h(u(\cdot, t_n), \partial_t u(\cdot, t_n))
\]
in Lemmas 4.4 and 4.5. \(\square\)

**Remark 4.7** It is remarked that the statement of Lemma 4.4 remains valid for a nonzero $g$ in \[4\] with a new remainder $\mathcal{R}^*$ that contains additional terms with $g(u)$ instead of $a(u)\partial_x^2 u$. The remainder bound given in Lemma 4.4 can be extended to this case since $g(u)$ is more regular than $a(u)\partial_x^2 u$. Thus the result about the stability proposed in Proposition 4.6 is still true for the case that $g$ is nonzero.

### 4.3 Local error bound

Local error bound of time-discrete ERKN integrators is given by the following proposition.

**Proposition 4.8 (Local error in $H^2 \times H^4$.)** Under the conditions of Lemma 4.4, if the solution $(u(\cdot, t), \partial_t u(\cdot, t))$ to \[8\] is in $H^5 \times H^4$ with
\[
\| (u(\cdot, t), \partial_t u(\cdot, t)) \|_4 \leq M,
\]
then one gets
\[
\| (d_{n+1}, \dot{d}_{n+1}) \|_1 \leq C_M h^3,
\]
where the local error $(d_{n+1}, \dot{d}_{n+1})$ is defined by
\[
(d_{n+1}, \dot{d}_{n+1}) = \varphi_h(u(\cdot, t_n), \partial_t u(\cdot, t_n)) - (u(\cdot, t_{n+1}), \partial_t u(\cdot, t_{n+1})).
\]
Proof  Without loss of generality, the proof is given in the case $n = 0$, that is, we prove the result for

$$(d_1, \dot{d}_1) = (u_1, \dot{u}_1) - (u(\cdot, h), \partial_t u(\cdot, h)).$$

By the variation-of-constants formula, the exact solution of (8) at $t = h$ can be expressed by

$$
\begin{pmatrix}
    u(\cdot, h) \\
    \partial_t u(\cdot, h)
\end{pmatrix}
= R(h)
  \begin{pmatrix}
    u_0 \\
    \dot{u}_0
\end{pmatrix}
+ \kappa \int_0^h R(h-t)
  \begin{pmatrix}
    0 \\
    f(u(\cdot, t))
  \end{pmatrix}
dt,
$$

where

$$R(t) = \begin{pmatrix}
    \cos(t\Omega) & t\text{sinc}(t\Omega) \\
    -\Omega \sin(t\Omega) & \cos(t\Omega)
\end{pmatrix}.$$  

Taking this formula and the scheme of ERKN integrators (9) into account, it is arrived that

$$
\begin{align*}
\begin{pmatrix}
    d_1 \\
    \dot{d}_1
\end{pmatrix}
&= \kappa h
  \begin{pmatrix}
    \bar{b}_1(V)f(u_{\frac{1}{2}h}) \\
    b_1(V)f(u_{\frac{1}{2}h})
\end{pmatrix}
- \kappa \int_0^h R(h-t)
  \begin{pmatrix}
    0 \\
    f(u(\cdot, t))
  \end{pmatrix}
dt \\
  &- \kappa h R\left(\frac{h}{2}\right)
  \begin{pmatrix}
    0 \\
    f(u(\cdot, h))
  \end{pmatrix}
+ \kappa h R\left(\frac{h}{2}\right)
  \begin{pmatrix}
    0 \\
    f(u(\cdot, \frac{h}{2}))
  \end{pmatrix}
- \kappa \int_0^h R(h-t)
  \begin{pmatrix}
    0 \\
    f(u(\cdot, t))
  \end{pmatrix}
dt.
\end{align*}
$$

In the following proof, we will use the results (see [9])

$$
\left\| \frac{d^l}{dt^l} R(t) \begin{pmatrix}
    u \\
    \dot{u}
\end{pmatrix} \right\|_{1+l} = \left\| (u, \dot{u}) \right\|_{1+l},
\left\| \frac{d^{2-l}}{dt^{2-l}} f(u(\cdot, t)) \right\|_{1+l} \leq C_M,
$$

where $l = 0, 1, 2$.

- Bound of (33). According to (32), (33) is seen to be of the form

$$
\kappa h
  \begin{pmatrix}
    \bar{b}_1(V)f(u_{\frac{1}{2}h}) \\
    b_1(V)f(u_{\frac{1}{2}h})
\end{pmatrix}
- \kappa h R\left(\frac{h}{2}\right)
  \begin{pmatrix}
    0 \\
    f(u_{\frac{1}{2}h})
  \end{pmatrix}
= \kappa h
  \begin{pmatrix}
    \bar{b}_1(V) - \frac{1}{2}\text{sinc}\left(\frac{1}{2}h\Omega\right) f(u_{\frac{1}{2}h}) \\
    b_1(V) - \cos\left(\frac{1}{2}h\Omega\right) f(u_{\frac{1}{2}h})
  \end{pmatrix}.
$$
By the third and fifth formulae of (12), we obtain
\[
\| (b_1(V) - \frac{1}{2} \text{sinc}(\frac{1}{2}h\Omega)) f(u_{\frac{1}{2}}) \|_2 \leq C \| h\Omega f(u_{\frac{1}{2}}) \|_2 = Ch \| f(u_{\frac{1}{2}}) \|_3 \leq C h \Lambda_3(\|u_3\|_3) \|u_3\|_3 \leq Ch,
\]
\[
\| (b_1(V) - \cos(\frac{1}{2}h\Omega)) f(u_{\frac{1}{2}}) \|_1 \leq C \| h^2 \Omega^2 f(u_{\frac{1}{2}}) \|_1 = Ch^2 \| f(u_{\frac{1}{2}}) \|_3 \leq Ch^2 \Lambda_3(\|u_3\|_3) \|u_3\|_3 \|u_5\|_5 \leq Ch^2.
\]
Thus the term on right-hand side of (33) is bounded by \(Ch^3\).

- **Bound of (14).** For (14), one has
\[
\| \text{term of (33)} \|_{1} = \kappa^2 h^2 \left| \Omega^{-1} \sin(\frac{1}{2}h\Omega)(f(u_{\frac{1}{2}}) - f(u(\cdot, \frac{1}{2}))) \right|_{2}^2
\]
\[
+ \kappa^2 h^2 \left| \cos(\frac{1}{2}h\Omega)(f(u_{\frac{1}{2}}) - f(u(\cdot, \frac{1}{2}))) \right|_{1}^2 \leq \kappa^2 h^2 \left| f(u_{\frac{1}{2}}) - f(u(\cdot, \frac{1}{2})) \right|_{1}^2 + \kappa^2 h^2 \left| f(u_{\frac{1}{2}}) - f(u(\cdot, \frac{1}{2})) \right|_{1}^2.
\]
\[
\left| u_{\frac{1}{2}} - u(\cdot, \frac{1}{2}) \right|_3 = |\kappa| \left| \int_{0}^{\frac{1}{2}} (\frac{1}{2} - t) \sin((\frac{1}{2} - t)\Omega) f(u(\cdot, t)) dt \right|_{3} \leq |\kappa| C_M \int_{0}^{\frac{1}{2}} (\frac{1}{2} - t) dt \leq C h^2 |\kappa|,
\]
we obtain
\[
\left| f(u_{\frac{1}{2}}) - f(u(\cdot, \frac{1}{2})) \right|_{1} \leq \Lambda_1 \left( \left| u_{\frac{1}{2}} \right|_3 + |u(\cdot, \frac{1}{2})| \right) \left( \left| u_{\frac{1}{2}} \right|_3 + |u(\cdot, \frac{1}{2})| \right) \| u_{\frac{1}{2}} - u(\cdot, \frac{1}{2}) \|_3 \leq Ch^2 |\kappa|,
\]
where (21) is used here. Therefore, it is arrived that \(\| \text{term of (33)} \|_{1} \leq Ch^3\).

- **Bound of (35).** The essential technology used here is the quadrature error of the mit-point rule. From its second-order Peano kernel \(K_2\), it follows that term of (35) = \(-h^3 \kappa \int_{0}^{1} K_2(\sigma) l''(\sigma h)d\sigma\) with \(l(t) = R(h - t) \left( \begin{array}{c} 0 \\ f(u(\cdot, t)) \end{array} \right) \). By (36), one arrives
\[
\| \text{term of (35)} \|_{1} \leq C_M h^3 |\kappa|.
\]

All these estimates together imply the result of this lemma. \(\square\)

**Remark 4.9** We remark that this lemma of the local error bound can be extended to a nonzero \(g\) in (1) since the proof is only based on the estimates (20) and (21).

### 4.4 Proof of Theorem 3.3

**Proof.** Denote by \(C_1\) and \(C_2\) the constants appearing in Propositions 4.6 and 4.8 respectively. Let \(h_0 = \sqrt{M/(C_2 T c^{C_1 T})}\) and it will be shown by induction on \(n\) that for \(h \leq h_0\)
\[
\| (u_n, \dot{u}_n) - (u(\cdot, t_n), \partial_t u(\cdot, t_n)) \|_{1} \leq C_2 e^{C_1 n h^2} \]
\[
\| (u_n, \dot{u}_n) - (u(\cdot, t_n), \partial_t u(\cdot, t_n)) \|_{1} \leq C_2 e^{C_1 n h^3} \] (37)
as long as $t_n = nh \leq T$.

Firstly, it is clear that (37) holds for $n = 0$. Assume that (37) holds for $n = 0, 1, \ldots, m - 1$. Choose $n = m - 1$ and then we have
\[
\| (u_{m-1}, \dot{u}_{m-1}) - (u(\cdot, t_{m-1}), \partial_t u(\cdot, t_{m-1})) \|_1 \leq C_2 |\kappa| e^{C_1|\kappa|(m-1)h} (m-1)h^3,
\]
which implies
\[
\| (u_{m-1}, \dot{u}_{m-1}) \|_1 \leq M + C_2 e^{C_1(m-1)h} (m-1)h^3 \leq 2M
\]
as long as $t_{m-1} - t_0 = (m-1)h \leq T$.

For the global error, one has
\[
\| (u_m, \dot{u}_m) - (u(\cdot, t_m), \partial_t u(\cdot, t_m)) \|_1
= \| \varphi_h(u_{m-1}, \dot{u}_{m-1}) - (u(\cdot, t_m), \partial_t u(\cdot, t_m)) \|_1
\leq \| \varphi_h(u_{m-1}, \dot{u}_{m-1}) - \varphi_h(u(\cdot, t_{m-1}), \partial_t u(\cdot, t_{m-1})) \|_1
+ \| \varphi_h(u(\cdot, t_{m-1}), \partial_t u(\cdot, t_{m-1})) - (u(\cdot, t_m), \partial_t u(\cdot, t_m)) \|_1.
\]
From Proposition 4.6, it follows that
\[
\| \varphi_h(u_{m-1}, \dot{u}_{m-1}) - \varphi_h(u(\cdot, t_{m-1}), \partial_t u(\cdot, t_{m-1})) \|_1
\leq (1 + C_1 h) \| (u_{m-1}, \dot{u}_{m-1}) - (u(\cdot, t_{m-1}), \partial_t u(\cdot, t_{m-1})) \|_1
\leq (1 + C_1 h) C_2 e^{C_1(m-1)h} (m-1)h^3.
\]
On the other hand, in the light of Proposition 4.8, one reaches
\[
\| \varphi_h(u(\cdot, t_{m-1}), \partial_t u(\cdot, t_{m-1})) - (u(\cdot, t_m), \partial_t u(\cdot, t_m)) \|_1 \leq C_2 h^3.
\]
Thus, it is obtained that
\[
\| (u_m, \dot{u}_m) - (u(\cdot, t_m), \partial_t u(\cdot, t_m)) \|_1 \leq (1 + C_1 h) C_2 e^{C_1(m-1)h} (m-1)h^3 + C_2 h^3.
\]
Expanding $e^{C_1(m-1)h}$ by Taylor series, the right-hand side of the above inequality becomes
\[
(1 + C_1 h) C_2 e^{C_1(m-1)h} (m-1)h^3 + C_2 h^3
= (1 + C_1 h) C_2 \sum_{k=0}^{\infty} \frac{(C_1(m-1)h)^k}{k!} (m-1)h^3 + C_2 h^3
= C_2 \sum_{k=0}^{\infty} \frac{1}{k!} (C_1)^k (m-1)^{k+1} h^{k+3} + C_2 \sum_{k=0}^{\infty} \frac{1}{k!} (C_1)^{k+1} (m-1)^{k+1} h^{k+4} + C_2 h^3
= C_2 m h^3 + C_2 \sum_{k=1}^{\infty} (m-1)^{k+1} + k(m-1)^k \frac{1}{k!} (C_1)^k h^{k+3}.
\]
According to the fact
\[
(m-1)^{k+1} + k(m-1)^k \leq (m-1 + 1)^{k+1} = m^{k+1} \quad \text{for} \quad m \geq 1, \ k \geq 1,
\]

we obtain
\[
(1 + C_1 h)C_2 e^{C_1(m-1)h}((m-1)h^3 + C_2 h^3) 
\leq C_2 mh^3 + C_2 \sum_{k=1}^{\infty} \frac{m^{k+1} h^k}{k!} = C_2 e^{C_1 mh} mh^3.
\]
Therefore, (37) holds for \( n = m \). By induction, it is true that
\[
\|(u_n, \dot{u}_n) - (u(\cdot, t_n), \partial_t u(\cdot, t_n))\|_1 \leq C_2 e^{C_1 T h^2} \leq C h^2,
\]
which proves the statement of Theorem 3.3.

\( \square \)

**Remark 4.10** The proof also holds for a nonzero \( g \) in (1) since it is based on Propositions 4.6 and 4.8, which are true for nonzero \( g \).

## 5 Proof of error bounds for full-discrete ERKN integrators

In this section, we prove the error bounds for full-discrete ERKN integrators. Throughout the proof, we use the following approximation property of the \( L^2 \)-orthogonal projection \( P^K \):
\[
\| P^K(v) \|_s \leq \| v \|_s \quad \text{for} \quad v \in H^s
\]
and
\[
\| v - P^K(v) \|_{s'} \leq K^{-(s-s')} \| v \|_s \quad \text{for} \quad v \in H^s,
\]
where \( s \geq s' \geq 0 \). In addition, for the trigonometric interpolation \( I^K \), we use the approximation property for \( s \geq s' \geq 0 \) with \( s - s' > \frac{1}{2} \)
\[
\| v - I^K(v) \|_{s'} \leq C_{s,s'} K^{-(s-s')} \| v \|_s \quad \text{for} \quad v \in H^s
\]
and its stability
\[
\| I^K(v) \|_s \leq C_s \| v \|_s \quad \text{for} \quad v \in H^s.
\]

It is noted that all estimates in the following are uniform in the spatial discretization parameter \( K \).

### 5.1 Stability

The result of Lemma 4.4 can be extended to the full-discrete ERKN integrator (15) directly.

**Lemma 5.1** Under the conditions given in Lemma 4.4, it follows that
\[
\|(u_{n+1}^{K} - v_{n+1}^{K}, \dot{u}_{n+1}^{K} - \dot{v}_{n+1}^{K})\|_1^2 = \|(u_n^{K} - v_n^{K}, \dot{u}_n^{K} - \dot{v}_n^{K})\|_1^2 + \kappa R^K(u_n^{K}, v_n^{K}, u_n^{K}, v_n^{K})
\]
with the remainder
\[
R^K(u_n^{K}, v_n^{K}, u_n^{K}, v_n^{K}) = \langle 2 \text{sinc}(h\Omega)^{-1}\hat{b}_1(V)(u_n^{K} - v_n^{K} + \dot{v}_n^{K}), \dot{f}_K(u_n^{K}) - \dot{f}_K(v_n^{K}) \rangle_1.
\]

For this remainder \( R^K \), we have the following bound.
Lemma 5.2 (Bound of the remainder.) Under the conditions given in Lemma 4.5, the remainder $R^K$ is bounded by

$$|κR^K(u_{n+1}^K, v_{n+1}^K, u_n^K, v_n^K)| \leq C_M h \| (u_n^K - v_n^K, u_n^K - v_n^K) \|_1^2.$$ \hspace{2cm} (43)

Proof. This lemma is proved in a similar way to that of Lemma 4.5 by using in addition the bounds (38) and (41) on $P^K$ and $I^K$ and the property

$$⟨v^K, P^K(w)⟩_s = ⟨v^K, w⟩_s$$ \hspace{2cm} (44)

for $v^K ∈ V^K, w ∈ H^s$ with $s = 1$. □

The stability of the full-discrete ERKN integrator (15) is obtained immediately by these two lemmas.

Proposition 5.3 (Stability.) Under the conditions given in Proposition 5.3, we have

$$\| (u_{n+1}^K, \dot{u}_{n+1}^K) - φ^K_h (u^K(·, t_n), \partial_t u^K(·, t_n)) \|^2 \leq (1 + C_M h) \| (e_n^K, \dot{e}_n^K) \|_1^2,$$

where the global error $(e_n^K, \dot{e}_n^K)$ is defined by

$$(e_n^K, \dot{e}_n^K) = (u_n^K, \dot{u}_n^K) - (u^K(·, t_n), \partial_t u^K(·, t_n)).$$

5.2 Local error bound

For full-discrete ERKN integrator (15), the local error bound is presented as follows.

Proposition 5.4 (Local error in $H^2 \times H^1$.) Under the conditions of Proposition 5.3 one has

$$\| (d_{n+1}^K, \dot{d}_{n+1}^K) \|_1 \leq C_M h^3 + C_M hK^{-s-2},$$

where the local error $(d_{n+1}^K, \dot{d}_{n+1}^K)$ is defined by

$$(d_{n+1}^K, \dot{d}_{n+1}^K) = φ^K_h (u^K(·, t_n), \partial_t u^K(·, t_n)) - (u^K(·, t_{n+1}), \partial_t u^K(·, t_{n+1})).$$
Proof Similar to the proof of Proposition 4.8, we only consider the case \( n = 0 \). Using this formula and letting \( \tilde{f}^K(u) = \mathcal{P}^K \circ f \), the local error can be rewritten in the form

\[
\begin{pmatrix}
\frac{d_1^K}{d_1'} \\
\frac{\hat{d}_1^K}{d_1'}
\end{pmatrix} = \kappa h \begin{pmatrix}
\hat{b}_1(V) \hat{f}^K(u^K) \\
b_1(V) \hat{f}^K(u^K)
\end{pmatrix} - \kappa \int_0^h R(h - t) \begin{pmatrix}
0 \\
\hat{f}^K(u(\cdot, t))
\end{pmatrix} dt
\]

\[
= \kappa h \begin{pmatrix}
\hat{b}_1(V) \hat{f}^K(u^K) \\
b_1(V) \hat{f}^K(u^K)
\end{pmatrix} - \kappa h R(\frac{h}{2}) \begin{pmatrix}
0 \\
\hat{f}^K(u^K)
\end{pmatrix} 
\]

\[
+ \kappa h R(\frac{h}{2}) \begin{pmatrix}
0 \\
\hat{f}^K(u^K)
\end{pmatrix} - \kappa h R(\frac{h}{2}) \begin{pmatrix}
0 \\
\hat{f}^K(u(\cdot, \frac{h}{2}))
\end{pmatrix} 
\]

\[
+ \kappa h R(\frac{h}{2}) \begin{pmatrix}
0 \\
\hat{f}^K(u(\cdot, \frac{h}{2}))
\end{pmatrix} - \kappa \int_0^h R(h - t) \begin{pmatrix}
0 \\
\hat{f}^K(u(\cdot, t))
\end{pmatrix} dt.
\]

These terms can be estimated similarly as in the proof of Proposition 4.8 using in addition the properties (39)-(41) of \( \mathcal{P}^K \) and \( \mathcal{I}^K \) and the assumed regularity of the exact solution.

Bounds of (46), (48) and (49) are similar to those in the proof of Proposition 4.8. For the bound of (47), we have

\[ \hat{f}^K - \tilde{f}^K \leq \mathcal{P}^K \circ (f^K - f). \]

By the arguments of the proof of Proposition 4.8 as well as (48) and (49), the estimate \( C_M h K^{-s-4} |\kappa| \) in \( H^2 \times H^1 \) and \( C_M h K^{-s-3} |\kappa| \) in \( H^3 \times H^2 \) for (47) can be obtained. \( \square \)

5.3 Proof of Theorem 3.4

Proof Based on the above analysis given in this section, the proof of Theorem 3.4 is similar to that of Theorem 3.3 with some obvious adjustments. \( \square \)

Remark 5.5 It is noted that the proof of error bounds for full-discrete ERKN integrators does not require any CFL-type coupling of the discretization parameters.
6 Proof of Theorem 3.5

We consider the following Strang splitting method

1. \((q^n_+, p^n_+) = \Phi_{h/2,L}(q^n, p^n) : \)
   \[
   \begin{pmatrix}
   q^n_+ \\
   p^n_+
   \end{pmatrix} = \begin{pmatrix}
   \cos\left(\frac{h\Omega}{2}\right) & \Omega^{-1}\sin\left(\frac{h\Omega}{2}\right) \\
   -\Omega\sin\left(\frac{h\Omega}{2}\right) & \cos\left(\frac{h\Omega}{2}\right)
   \end{pmatrix}
   \begin{pmatrix}
   q^n \\
   p^n
   \end{pmatrix},
   \]

2. \((q^n_-, p^n_-) = \Phi_{h,NL}(q^n_+, p^n_+) : \)
   \[
   \begin{pmatrix}
   q^n_- \\
   p^n_-
   \end{pmatrix} = \begin{pmatrix}
   q^n_+ \\
   p^n_+ + h\Upsilon(h\Omega)g(q^n_+)
   \end{pmatrix},
   \]

3. \((q^{n+1}, p^{n+1}) = \Phi_{h/2,L}(q^n_+, p^n_+) : \)
   \[
   \begin{pmatrix}
   q^{n+1} \\
   p^{n+1}
   \end{pmatrix} = \begin{pmatrix}
   \cos\left(\frac{h\Omega}{2}\right) & \Omega^{-1}\sin\left(\frac{h\Omega}{2}\right) \\
   -\Omega\sin\left(\frac{h\Omega}{2}\right) & \cos\left(\frac{h\Omega}{2}\right)
   \end{pmatrix}
   \begin{pmatrix}
   q^n \\
   p^n
   \end{pmatrix}.
   \]

It can be checked that the ERKN integrator \(\phi_h\) can be expressed by this Strang splitting as

\[\phi_h = \Phi_{h/2,L} \circ \Phi_{h,NL} \circ \Phi_{h/2,L}\] (50)

if and only if

\[\Upsilon(h\Omega) = b_1(h\Omega)\cos^{-1}\left(\frac{1}{2}h\Omega\right) = 2b_1(h\Omega)\text{sinc}^{-1}\left(\frac{1}{2}h\Omega\right).\] (51)

On the other hand, for the Strang splitting

\[\hat{\phi}_h = \Phi_{h/2,NL} \circ \Phi_{h,L} \circ \Phi_{h/2,NL},\]

it is identical to a trigonometric integrator ((XIII.2.7)–(XIII.2.8) given on p.481 of [11])

\[q^{n+1} = \cos(h\Omega)q^n + hsinc(h\Omega)p^n + \frac{1}{2}h^2\text{sinc}(h\Omega)\Upsilon(h\Omega)g(q^n),\]
\[p^{n+1} = -\Omega\sin(h\Omega)q^n + \cos(h\Omega)p^n + \frac{1}{2}h\left(\cos(h\Omega)\Upsilon(h\Omega)g(q^n) + \Upsilon(h\Omega)g(q^{n+1})\right).\] (52)

This trigonometric integrator with the choice

\[\Upsilon(h\Omega) = \text{sinc}(h\Omega)\] (53)

has been discussed in [9] for quasilinear wave equations. Thus based on the following important connection

\[\phi_h \circ \cdots \circ \phi_h = \Phi_{h/2,L} \circ \Phi_{h/2,NL} \circ \cdots \circ \Phi_{h/2,NL} \circ \Phi_{h/2,L}.\] (54)

it is arrived that ERKN2 and the trigonometric integrator (52)-(53) have similar error bounds when they are used to solving quasilinear wave equations. Therefore, the error bounds of ERKN2 are obtained immediately by considering the results given in [9] and the proof of Theorem 3.5 is complete.
7 Concluding remarks

This paper studied error bounds of one-stage explicit extended ERKN integrators for solving quasi-linear wave equations. Second-order convergence for the semidiscretization in time was proved and the error bounds of fully discrete scheme were also presented without requiring any CFL-type coupling of the discretization parameters.

Last but not least, the analysis of ERKN integrators in this paper can be extended to quasilinear wave equations (1) without Klein-Gordon term $-u$ and also works for higher spatial dimensions. The application and analysis of ERKN integrators for quasilinear wave equations with $\kappa = 1$ and for more general quasilinear wave equations or other kinds of PDEs will be our future work.

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References

[1] B. Cano, Conservation of invariants by symmetric multistep cosine methods for second-order partial differential equations, BIT 53 (2013) 29-56.

[2] M. Chirilus-Bruckner, W.-P. Düll, G. Schneider, NLS approximation of time oscillatory long waves for equations with quasilinear quadratic terms, Math. Nachr. 288 (2015) 158-166.

[3] C. Chong, G. Schneider, Numerical evidence for the validity of the NLS approximation in systems with a quasilinear quadratic nonlinearity, ZAMM Z. Angew. Math. Mech. 93 (2013) 688-696.

[4] D. Cohen, E. Hairer, C. Lubich, Conservation of energy, momentum and actions in numerical discretizations of non-linear wave equations, Numer. Math. 110 (2008) 113-143.

[5] X. Dong, Stability and convergence of trigonometric integrator pseudospectral discretization for N-coupled nonlinear Klein-Gordon equations, Appl. Math. Comput. 232 (2014) 752-765.

[6] W. Dörfler, H. Gerner, R. Schnaubelt, Local well-posedness of a quasilinear wave equation, Appl. Anal. 95 (2016) 2110-2123.

[7] W.-P. Düll, Justification of the nonlinear Schrödinger approximation for a quasilinear Klein–Gordon equation, Comm. Math. Phys. 355 (2017) 1189-1207.

[8] L. Gauckler, Error analysis of trigonometric integrators for semilinear wave equations, SIAM J. Numer. Anal. 53 (2015) 1082-1106.

[9] L. Gauckler, J. Lu, J. L. Marzuola, F. Rousset, K. Schratz, Trigonometric integrators for quasilinear wave equations, Math. Comp. (2017/2018). https://doi.org/10.1090/mcom/3339
[10] M. D. Groves, G. Schneider, Modulating pulse solutions for quasilinear wave equations, J. Diff. Equa. 219 (2005) 221-258.

[11] E. Hairer, C. Lubich, G. Wanner, Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations, (Second Edition), Springer-Verlag, Berlin, Heidelberg, 2006.

[12] M. Hochbruck, A. Ostermann, Exponential integrators, Acta Numer. 19 (2010) 209-286.

[13] M. Hochbruck, T. Pažur, Error analysis of implicit Euler methods for quasilinear hyperbolic evolution equations, Numer. Math. 135 (2017) 547-569

[14] L. Hörmander, Lectures on nonlinear hyperbolic differential equations, vol. 26 of Mathématiques Applications, Springer-Verlag, Berlin, 1997.

[15] T. J. R. Hughes, T. Kato, J. E. Marsden, Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity, Arch. Rational Mech. Anal. 63 (1977) 273–294.

[16] B. Kovács, C. Lubich, Stability and convergence of time discretizations of quasi-linear evolution equations of Kato type, Numer. Math. (2017) doi:10.1007/s00211-017-0909-3.

[17] C. Liu, A. Iserles, X. Wu, Symmetric and arbitrarily high-order Birkhoff–Hermite time integrators and their long-time behaviour for solving nonlinear Klein–Gordon equations, J. Comput. Phys. 356 (2018) 1-30.

[18] C. Liu, X. Wu, Arbitrarily high-order time-stepping schemes based on the operator spectrum theory for high-dimensional nonlinear Klein–Gordon equations, J. Comput. Phys. 340 (2017) 243-275.

[19] C. Lubich, A. Ostermann, Runge-Kutta approximation of quasi-linear parabolic equations, Math. Comp. 64 (1995) 601-627.

[20] M. E. Taylor, Pseudodifferential operators and nonlinear PDE, vol. 100 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1991.

[21] M. E. Taylor, Partial differential equations III. Nonlinear equations., vol. 117 of Applied Mathematical Sciences, Springer, New York, 2011.

[22] B. Wang, A. Iserles, X. Wu, Arbitrary-order trigonometric Fourier collocation methods for multi-frequency oscillatory systems, Found. Comput. Math. 16 (2016) 151-181.

[23] B. Wang, X. Wu, The formulation and analysis of energy-preserving schemes for solving high-dimensional nonlinear Klein-Gordon equations, IMA J. Numer. Anal. (2018) DOI: 10.1093/imanum/dry047.

[24] B. Wang, X. Wu, Error analysis of one-stage explicit extended RKN integrators for semilinear wave equations, Numer. Algo. (2018) https://doi.org/10.1007/s11075-018-0585-0.

[25] B. Wang, X. Wu, J. Xia, Error bounds for explicit ERKN integrators for systems of multi-frequency oscillatory second-order differential equations, Appl. Numer. Math. 74 (2013) 17-34.

[26] B. Wang, H. Yang, F. Meng, Sixth order symplectic and symmetric explicit ERKN schemes for solving multi-frequency oscillatory nonlinear Hamiltonian equations, Calcolo 54 (2017) 117-140.
[27] X. Wu, C. Liu, L. Mei, A new framework for solving partial differential equations using semi-analytical explicit RK(N)-type integrators, J. Comput. Appl. Math. 301 (2016) 74-90.

[28] X. Wu, K. Liu, W. Shi, Structure-preserving algorithms for oscillatory differential equations II, Springer-Verlag, Heidelberg, 2015.

[29] X. Wu, X. You, B. Wang, Structure-Preserving Algorithms for Oscillatory Differential Equations, Springer-Verlag, Berlin, Heidelberg, 2013.

[30] X. Wu, X. You, W. Shi, B. Wang. ERKN integrators for systems of oscillatory second-order differential equations. Comput. Phys. Comm. 181 (2010) 1873-1887.