Convergence to Stable Limits for Ratios of Trimmed Lévy Processes and their Jumps

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Abstract

The distribution of an $r$-trimmed Lévy process conditional on its $r$ largest jumps up to a designated time $t$ depends only on the $r^{th}$ largest jump, $r = 2, 3, \ldots$. Assuming the underlying Lévy process is in the domain of attraction of a stable process as $t \downarrow 0$, this and similar identities we derive are applied to show joint convergence of the trimmed process divided by its large jumps to corresponding quantities constructed from a stable limiting process. This generalises related results in the 1-dimensional subordinator case developed in Kevei & Mason (2014) and produces new discrete distributions on the infinite simplex in the limit.

1 Introduction and Lévy Process Setup

Deleting the $r$ largest jumps up to a designated time $t$ from a Lévy process gives the “$r$-trimmed Lévy process”. We derive a useful identity for the conditional distribution of the process given its largest jumps, which shows that conditioning on the $r$ largest jumps up to a designated time $t$ is equivalent to conditioning on the $r^{th}$ largest jump alone. As corollaries, representations for the characteristic function of the trimmed process divided by its large jumps are found. Assuming $X$ is in the domain of attraction of a stable process as $t \downarrow 0$, the representations are applied to

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show joint convergence of those ratios to corresponding quantities constructed from the stable limiting process.

In the case of subordinators, Kevei & Mason (2014) considered one-dimensional convergence to stable subordinators and derived the limit distribution of the ratio of an \( r \)-trimmed subordinator to its \( r \)th largest jump. Our main result, Theorem 2.1, is a multivariate version of part of their Theorem 1.1, and, as a generalisation, we consider a trimmed Lévy process in the domain of attraction of a stable distribution with parameter \( \alpha \) in \((0, 2)\), taken as a ratio of one of its large jumps at time \( t \). We show the joint convergence of these ratios, as time \( t \) tends to 0, to corresponding quantities constructed from the stable limiting process. When \( 0 < \alpha < 1 \), the limit distribution in Theorem 2.1 is related to the generalised Poisson-Dirichlet distribution \( PD_{\alpha}^{(r)} \) in Ipsen & Maller (2017) derived from the trimmed stable subordinator, which includes as a special case the \( PD(\alpha, 0) \) distribution in Pitman & Yor (1997).

When \( \alpha > 1 \) the process is not a subordinator, and there is no direct connection with the Poisson Dirichlet distribution. In this case the process has to be centered appropriately to get the required convergence. We note that (since the Lévy measure has infinite mass) there are always infinitely many “large” jumps of \( X_t \), a.s., in any right neighbourhood of 0.

These considerations form the basis of further generalised versions of Poisson-Dirichlet distributions explored in Ipsen & Maller (2017). In the present paper we limit ourselves to proving Theorem 2.1 (in Section 2) and the foundational results needed for its proof (in Section 3). A second Theorem 2.2 proves a kind of “large trimming” result, showing that the trimmed process is of small order of the largest jump trimmed, uniformly in \( t \), as the order tends to infinity. Section 4 contains the proofs of the results in Section 2. For the remainder of this section we give a brief introduction to the Lévy process ideas we will need.

### 1.1 Lévy Process Setup

We consider a real valued Lévy process \((X_t)_{t \geq 0}\) on a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), with canonical triplet \((\gamma, \sigma^2, \Pi)\); thus, having characteristic function

\[
E e^{i\theta X_t} = e^{t\Psi(\theta)}, \quad t \geq 0, \quad \theta \in \mathbb{R},
\]

with exponent

\[
\Psi_X(\theta) := i\theta \gamma - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R} \setminus \{0\}} \left( e^{i\theta x} - 1 - i\theta x 1_{\{|x| \leq 1\}} \right) \Pi(dx).
\]

(1.1)

Here \( \gamma \in \mathbb{R}, \sigma^2 \geq 0 \) and \( \Pi \) is a Lévy measure on \( \mathbb{R} \), i.e., a Borel measure on \( \mathbb{R} \) with \( \int_{\mathbb{R} \setminus \{0\}} (x^2 \wedge 1) \Pi(dx) < \infty \). The positive, negative and two-sided tails of \( \Pi \) are defined for \( x > 0 \) by

\[
\Pi^+(x) := \Pi\{(x, \infty)\}, \quad \Pi^-(x) := \Pi\{(-\infty, -x)\}, \quad \text{and} \quad \Pi(x) := \Pi^+(x) + \Pi^-(x).
\]

(1.2)
Let $\Pi^\pm$ denote the inverse function of $\Pi$; thus,

$$\Pi^+(x) = \inf \{y > 0 : \Pi(y) \leq x\}, \quad x > 0,$$

(1.3)

and similarly for $\Pi^{\pm, \pm}$. Throughout, let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Write $\Delta X_t := X_t - X_{t-}$, with $\Delta X_0 = 0$, for the jump process of $X$, and $\Delta X_t^{(1)} \geq \Delta X_t^{(2)} \geq \cdots$ for the jumps ordered by their magnitudes at time $t > 0$. Assume throughout that $\Pi\{(0, \infty)\} = \infty$, so there are infinitely many positive jumps, a.s., in any right neighbourhood of 0. Thus the $\Delta X_t^{(i)}$ are positive a.s. for all $t > 0$ but $\lim_{t \downarrow 0} \Delta X_t^{(i)} = 0$ for all $i \in \mathbb{N}$. Our objective is to study the “one-sided trimmed process”, by which we mean $X_t$ minus its large positive jumps, at a given time $t$. Thus, the one-sided $r$-trimmed version of $X_t$ is

$$(r)X_t := X_t - \sum_{i=1}^{r} \Delta X_t^{(i)}, \quad r \in \mathbb{N}, \quad t > 0$$

(1.4)

(and we set $(0)X_t \equiv X_t$). Detailed definitions and properties of this kind of ordering and trimming are given in [Buchmann, Fan & Maller (2016)], where we identify the positive $\Delta X_t$ with the points of a Poisson point process on $[0, \infty)$.

Our main result, in Theorem 2.1, is to show that ratios formed by dividing $(r)X_t$, possibly after centering, by its ordered positive jumps, converge to corresponding stable ratios when $X$ is in the domain of attraction of a non-normal stable law.

## 2 Convergence of Lévy Ratios to Stable Limits

Throughout, $X$ will be assumed to be in the domain of attraction of a non-normal stable random variable at 0 (or at $\infty$). By this we mean that there are nonstochastic functions $a_t \in \mathbb{R}$ and $b_t > 0$ such that $(X_t - a_t)/b_t \xrightarrow{D} Y$, for an a.s. finite random variable $Y$, not degenerate at a constant, and not normally distributed, as $t \downarrow 0$. The Lévy tail $\Pi(x)$ is then regularly varying of index $-\alpha$ at 0, and the balance conditions

$$\lim_{x \downarrow 0} \frac{\Pi^\pm(x)}{\Pi(x)} = a_\pm,$$

(2.1)

where $a_+ + a_- = 1$, are satisfied. If this is the case then the limit random variable $Y$ must be a stable random variable with index in $\alpha \in (0, 2)$. We consider one-sided (positive) trimming, so we always assume $a_+ > 0$.

Let $RV_0(\alpha)$ ($RV_{\infty}(\alpha)$) be the regularly varying functions of index $\alpha \in \mathbb{R}$ at 0 (or $\infty$). When $\Pi(\cdot) \in RV_0(-\alpha)$ with $0 < \alpha < \infty$ or, equivalently, the inverse function

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1The convergences in this section can be worked out as $t \downarrow 0$ or as $t \to \infty$. For definiteness and in keeping with modern trends in the area we supply the versions for $t \downarrow 0$, but little modification is needed for the case $t \to \infty$.  

3
\(\Pi^- (\cdot) \in RV_{\infty}(-1/\alpha)\) (e.g. Bingham, Goldie and Teugels (1987, Sect. 7, pp.28-29)), we have the easily verified convergence

\[
t \Pi(u \Pi^-(y/t)) \sim \frac{\Pi(u y^{-1/\alpha} \Pi^-(1/t))}{\Pi(\Pi)(1/t)} \rightarrow u^{-\alpha} y \text{ as } t \downarrow 0, \text{ for all } u, y > 0. \tag{2.2}
\]

For \(r > 0\) write

\[
P(\Gamma_r \in dx) = \frac{x^{r-1}e^{-x}dx}{\Gamma(r)} 1_{\{x > 0\}},
\]

for the density of \(\Gamma_r\), a Gamma(\(r, 1\)) random variable, which should not be confused with the Gamma function, \(\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx\). Denote the Beta random variable on (0, 1) with parameters \(a, b > 0\) by \(B_{a,b}\), having density function

\[
f_B(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1} = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, 0 < x < 1.
\]

Denote by \((S_t)_{t \geq 0}\) a stable process of index \(\alpha \in (0, 2)\) having Lévy measure

\[
\Lambda(dx) = \Lambda_S(dx) = -d(x^{-\alpha})1_{\{x > 0\}} + (a_-/a_+)d(-x)^{-\alpha}1_{\{x < 0\}}, \text{ } x \in \mathbb{R}, \tag{2.3}
\]

with characteristic exponent

\[
\Psi_S(\theta) := \int_{\mathbb{R}\setminus\{0\}} \left( e^{i\theta x} - 1 - i\theta x 1_{\{|x| \leq 1\}} \right) \Lambda(dx), \tag{2.4}
\]

and by \((\Delta S_t := S_t - S_{t-})_{t > 0}\) the jump process of \(S\). Let

\[
\Delta S_t^{(1)} \geq \Delta S_t^{(2)} \geq \cdots \geq \Delta S_t^{(n)} \geq \cdots
\]

be the ordered stable jumps at time \(t > 0\). These are uniquely defined a.s. (no tied values a.s.) since the Lévy measure of \(S\) has no atoms. The positive and negative tails of \(\Lambda\) are \(\overline{\Lambda}^+ (x) := \Lambda\{x, \infty\} = x^{-\alpha}\) and \(\overline{\Lambda}^- (x) := \Lambda\{-\infty, -x\} = (a_-/a_+)x^{-\alpha}\), for \(x > 0\). Since \(\overline{\Lambda}^+(0+) = \infty\), the \(\Delta S_t^{(i)}\) are positive a.s., \(i = 1, 2, \ldots\), but tend to 0 a.s. as \(t \downarrow 0\).

Define a centering function \(\rho_X(\cdot)\) for \(X\) by

\[
\rho_X(w) := \begin{cases} 
\gamma - \int_{[w,1]} x \Pi(dx), & 0 < w \leq 1, \\
\gamma + \int_{[-w,-1]\cup(1,w]} x \Pi(dx), & w > 1,
\end{cases} \tag{2.5}
\]

and similarly for \(\rho_S(w)\), but with \(\gamma\) taken as 0 and \(\Lambda\) replacing \(\Pi\) in that case.

To state Theorem 2.1 we need some further notation. For each \(n = 2, 3, \ldots\) and \(0 < u < 1\), suppose random variables \(J_{n-1}^{(1)}(u) \geq J_{n-1}^{(2)}(u) \geq \cdots \geq J_{n-1}^{(n-1)}(u)\) are
distributed like the decreasing order statistics of $n - 1$ independent and identically distributed (i.i.d.) random variables $(J_i(u))_{1 \leq i \leq n-1}$, each having the distribution
\[
P(J_1(u) \in dx) = \frac{\Lambda(dx)1_{\{1 \leq x \leq 1/u\}}}{1 - u^\alpha}, \quad x > 0. \tag{2.6}
\]
Also let $L_{n-1}^{(1)} \geq L_{n-1}^{(2)} \geq \cdots \geq L_{n-1}^{(n-1)}$ be distributed like the decreasing order statistics of $n - 1$ i.i.d. random variables $(L_i)_{1 \leq i \leq n-1}$, each having the distribution
\[
P(L_1 \in dx) = \Lambda(dx)1_{\{x > 1\}}. \tag{2.7}
\]
Define
\[
\psi(\theta) = \int_{(-\infty,1)} \left( e^{i\theta x} - 1 - i\theta x1_{\{|x| \leq 1\}} \right) \Lambda(dx), \quad \theta \in \mathbb{R}, \tag{2.8}
\]
and choose $\theta_0 > 0$ such that $|\psi(\theta)| < 1$ for $|\theta| \leq \theta_0$ (as is possible since $\psi(0) = 0$).
Also define $\phi(\theta, u) = \mathbb{E}e^{i\theta J_1(u)}$, $\theta \in \mathbb{R}$, with $J_1(u)$ having the distribution in (2.6):
\[
\phi(\theta, u) = (1 - u^\alpha)^{-1} \int_1^{1/u} e^{i\theta x} \Lambda(dx), \quad 0 < u < 1. \tag{2.9}
\]
Let $W = (W_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}$ with triplet $(0, 0, \Lambda(dx)1_{(-\infty,1)})$, and $\Gamma_{r+n}$ a Gamma $(r + n, 1)$ random variable independent of $W$.
When $x_k > 0$, $1 \leq k \leq n - 1$, $x_n = 1$, and $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$, write for shorthand
\[
x_{n+} = \sum_{k=1}^n x_k \quad \text{and} \quad \tilde{\theta}_{n+} = \tilde{\theta}_{n+}(x_1, \ldots, x_n) := \sum_{k=1}^n \frac{\theta_k}{x_k}, \tag{2.10}
\]
and let $\int_{x^1 \geq 1}$ denote integration over the region $\{x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq 1\} \subseteq \mathbb{R}^{n-1}$.
Recall that $(0) X \equiv X$.

**Theorem 2.1.** Assume $\Pi \in RV_0(-\alpha)$ for some $0 < \alpha < 2$ and (2.1).

(i) Then for each $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, as $t \downarrow 0$, we have the joint convergence
\[
\begin{pmatrix}
(r) X_t - t\rho_S(\Delta S^{(r+n)}_t) \\
\Delta S^{(r+1)}_t \\
\end{pmatrix} \quad \cdots \quad \begin{pmatrix}
(r) S_1 - \rho_S(\Delta S^{(r+n)}_1) \\
\Delta S^{(r+n)}_1 \\
\end{pmatrix} \xrightarrow{D} \begin{pmatrix}
(r) S_1 - \rho_S(\Delta S^{(r+n)}_1) \\
\Delta S^{(r+n)}_1 \\
\end{pmatrix}, \tag{2.11}
\]
(ii) When $r \in \mathbb{N}$, $n = 2, 3, \ldots$, the random vector on the RHS of (2.11) has characteristic function which can be represented, for $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$, as
\[
\mathbb{E} \exp \left( i \sum_{k=1}^n \frac{\theta_k (r) S_1 - \rho_S(\Delta S^{(r+n)}_1)}{\Delta S^{(r+n)}_1} \right) = \\
\int_{x^1 \geq 1} e^{i\tilde{\theta}_{n+} x_n + \mathbb{E}(e^{i\tilde{\theta}_{n+} W_{t+1}}) P(J_1^{(k)}(B_{r,n}) \in dx_k, 1 \leq k \leq n - 1)}, \tag{2.12}
\]
where $B_{r,n}$ is a Beta($r,n$) random variable independent of the $(J_i(u))$. Alternatively, recalling (2.8), when $\max_{1 \leq k \leq n} |\theta_k| \leq \theta_0$ the RHS of (2.12) can be written as

$$\int_{x^+ \geq 1} \frac{e^{i\theta_{n+1} + x_n}}{(1 - \psi(\theta_{n+1}))^{r+n}} P(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n - 1).$$

(2.13)

When $r = 0$, (2.12) and (2.13) remain true as stated if the res $J_{n-1}^{(k)}(B_{r,n}^{1/\alpha})$ are replaced respectively by $L_{n-1}^{(k)}$, being the order statistics associated with the distribution in (2.7).

(iii) When $r \in \mathbb{N}$, $n \in \mathbb{N}$ we have

$$\frac{\langle r \rangle X_t - t \rho_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+n)}} \xrightarrow{D} \frac{\langle r \rangle S_1 - \rho_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+n)}},$$

as $t \downarrow 0, (2.14)$

where, recalling (2.9), the random variable on the RHS of (2.14) has characteristic function

$$\frac{e^{i\theta}}{(1 - \psi(\theta))^{r+n}} E(\phi^{-1}(\theta, B_{r,n}^{1/\alpha})), \quad \theta \in \mathbb{R}, \ |\theta| \leq \theta_0.$$

(2.15)

When $r = 0$, (2.14) remains true as does (2.15), if $\phi(\theta, B_{r,n}^{1/\alpha})$ in (2.15) is replaced by $\phi(\theta, 0) := \int_1^\infty e^{i\theta x} \Lambda(dx)$.

Setting $n = 1$ in (2.14), and (since $\langle r \rangle X_t/\Delta X_t^{(r+1)} = 1 + \langle r+1 \rangle X_t/\Delta X_t^{(r+1)}$) replacing $r + 1$ by $r$ gives

**Corollary 2.1.** For each $r \in \mathbb{N}$, $\theta \in \mathbb{R}, \ |\theta| \leq \theta_0$,

$$\frac{\langle r \rangle X_t - t \rho_X(\Delta X_t^{(r)})}{\Delta X_t^{(r)}} \xrightarrow{D} \frac{\langle r \rangle S_1 - \rho_S(\Delta S_1^{(r)})}{\Delta S_1^{(r)}},$$

as $t \downarrow 0, (2.16)$

where

$$E(\psi^{\langle r \rangle}S_1 - \rho_S(\Delta S_1^{(r)}))/\Delta S_1^{(r)}) = E(\psi^{\langle r \rangle}W_{\Gamma}) = \frac{1}{(1 - \psi(\theta))}.$$

(2.17)

Further, $(\langle r \rangle S_1 - \rho_S(\Delta S_1^{(r)}))/\Delta S_1^{(r)} \overset{D}{=} W_{\Gamma},$ being a Gamma-subordinated Lévy process, is infinitely divisible for each $r \in \mathbb{N}$.

The unwieldy centering functions $\rho_X$ and $\rho_S$ in (2.11)–(2.17) can be simplified in many cases. Especially, when $X$ is a subordinator with drift $d_X$, $\rho_X$ can be replaced by $d_X$, and without loss of generality we can assume $d_X = 0$. The convergences in (2.11)–(2.16) can then be written in terms of Laplace transforms. This case recovers a result proved in Theorem 1.1 of Kevei & Mason (2014): assume $X$ is a driftless subordinator in the domain of attraction (at 0) of a stable random variable with index $\alpha \in (0,1)$. Then for $r \in \mathbb{N}$

$$\frac{\langle r \rangle X_t}{\Delta X_t^{(r)}} \xrightarrow{D} \langle r \rangle Y, \text{ as } t \downarrow 0,$$

(2.18)
where \( (r)Y \) is an a.s. finite non-degenerate random variable. From Theorem 2.1 we can identify \( (r)Y \) as having the distribution of \( (r)S_1/\Delta S_1^{(r)} \), in our notation. Kevei and Mason show, conversely, in this subordinator case, that when (2.18) holds with \( (r)Y \) a finite non-degenerate random variable, then \( X \) is in the domain of attraction (at 0) of a stable random variable with index \( \alpha \in (0, 1) \). They also give in their Theorem 1.1 a formula for the Laplace transform of \( (r)Y \). We can state an equivalent version as: suppose (2.18) holds. Then (2.17) becomes

\[
E(e^{-\lambda S_1/\Delta S_1}) = E(e^{-\lambda W_1}) = \frac{1}{(1 + \psi(\lambda))^r}, \quad r \in \mathbb{N},
\]

where now \( W = (W_v)_{v \geq 0} \) is a driftless subordinator with measure \( \Lambda(dx)1_{(0,1)} \), and

\[
\psi(\lambda) = \int_{(0,1)} (1 - e^{-\lambda x}) \Lambda(dx), \quad \lambda > 0.
\]

Remark 2.1 (Negative Binomial Point Process). The form of the Laplace transform in (2.19) suggests a connection with the negative binomial point process of Gregoire (1984), which connection is developed in Ipsen & Maller (2017), and also forms the basis for a general point measure treatment when \( 0 \leq \alpha \leq \infty \) in Ipsen, Maller & Resnick (2017), which contains a converse proof generalising that of Kevei & Mason (2014). Those results motivate further investigation of the “large trimming” properties of general Lévy processes in the spirit of Buchmann, Maller & Resnick (2016). But we do not explore this further here.

Remark 2.2 (Modulus Trimming). Rather than removing large jumps from \( X \) as we do in (1.4), we can remove jumps large in modulus and obtain analogous formulae and results, with appropriate modifications. The centering function \( \rho_X \) in (2.5) should then be changed to \( \gamma - \int_{[-w, w]} x \Pi(dx) \) when \( 0 < w \leq 1 \), with a corresponding change when \( w > 1 \), and similarly for \( \rho_S \). The norming in Theorem 2.1 would then be by jumps large in modulus rather than by large (positive) jumps, and the convergence would be to the analogous modulus trimmed stable process. The identities in Section 3 required for the modified proofs can be obtained from analogous formulae for modulus trimming in Buchmann, Fan & Maller (2016).

Remark 2.3 (Connection with PD\(_\alpha^{(r)}\)). When \( X \) is a driftless subordinator, we obtain from (2.17) with \( n \in \mathbb{N} \) that

\[
\left( \Delta X_t^{(r+1)} / (r)X_t, \ldots, \Delta X_t^{(r+n)} / (r)X_t \right) \overset{D}{\to} \left( \Delta S_1^{(r+1)} / (r)S_1, \ldots, \Delta S_1^{(r+n)} / (r)S_1 \right), \quad \text{as } t \downarrow 0.
\]

When \( n \to \infty \), the \( n \)-vector on the RHS tends to a vector \( (V_1^{(r)}, V_2^{(r)}, \ldots) \) on the infinite simplex with the generalised Poisson-Kingman distribution PD\(_\alpha^{(r)} \) defined in
Ipsen & Maller (2017). When \( r = 0 \), this reduces to the Poisson-Kingman distribution generated from the stable subordinator, denoted by PD(\( \alpha, 0 \)) in Pitman & Yor (1997), which was first noted by Kingman (1975).

To complete this section we continue to consider the case when \( X \) is a driftless subordinator. Our final result in this section shows that ratios of the form

\[
\frac{(r+n)X_t}{\Delta X_t^{(r)}}
\]

have strong stability properties. In the next theorem the interesting aspect is the uniformity of convergence in neighbourhoods of 0; although \( \Delta X_t^{(r)} \downarrow 0 \) a.s. as \( t \downarrow 0 \), the remainder after removing an increasing number of jumps, \( r+n \), from \( X \) is shown to be small order \( \Delta X_t^{(r)} \), a.s., as \( n \to \infty \), uniformly on compacts.

**Theorem 2.2.** Suppose \( X \) is a driftless subordinator with \( \Pi \equiv \Pi^+ \in RV_0(-\alpha) \) for some \( 0 < \alpha < 1 \). Then for each \( r \in \mathbb{N} \)

\[
\frac{(r+n)X_t}{\Delta X_t^{(r)}} \to 0, \text{ a.s., as } n \to \infty,
\]

uniformly in \( t \in (0, t_0] \), for any \( t_0 > 0 \).

### 3 Representations for Trimmed Lévy Processes

In the present section we revert to considering an arbitrary real valued Lévy process \( (X_t)_{t \geq 0} \), set up as in Section 2 (see (1.1) and (1.2)), and derive the identities required for the proofs of the results in Section 2. Fundamental to these identities is a general representation for the joint distribution of \( (r)X_t \) and its large jumps, given in Buchmann et al. (2016), which allows for possible tied values in the jumps\(^2\). Our main theorem in this section, Theorem 3.1, derives some important properties from it. We expect these formulae will have useful applications in other areas too. Let \( \Pi^{r,+,-}(x) = \inf\{ y > 0 : \Pi^+(y) \leq x \}, x > 0 \), be the right-continuous inverse of \( \Pi^+ \).

**Theorem 3.1.** Take \( r \in \mathbb{N}, v_1 \geq v_2 \geq \cdots \geq v_r > 0, x \in \mathbb{R}, t > 0 \). Then the identity

\[
P((r)X_t \leq x \mid \Delta X_t^{(i)} = \Pi^+,(v_i), 1 \leq i \leq r) = P((r)X_t \leq x \mid \Delta X_t^{(r)} = \Pi^+,(v_r)),
\]

holds at points of increase \( v_i > 0, 1 \leq i \leq r \), of \( \Pi^+ \), and the identity

\[
P((r)X_t \leq x \mid \Delta X_t^{(r)} = \Pi^+,(v)) = P(X_t^{v} + G_t^{v} \leq x),\]

\(v \geq 1\)

\(\text{A different but equivalent distributional representation when } X \text{ is a subordinator is in Proposition 1 of Kevei & Mason (2013).}\)
holds at points of increase \( v > 0 \) of \( \Pi^+ \). In (3.2), \( (X^r_t)_{t \geq 0} \) is a \( \text{Lévy process, indexed by } v > 0 \), having canonical triplet

\[
\left( \gamma^r, \sigma^2, \Pi^r(dx) \right) := \\
\left( \gamma - 1 \{ \Pi^+ + v \leq x \} \right) \int_{\Pi^+ + v \leq x \leq 1} x \Pi(dx), \sigma^2, \Pi(dx) 1_{\{x < \Pi^+ + v \}} \right), \tag{3.3}
\]

while \( G^r_t = \Pi^+_{\Pi^+ + v} Y_{\kappa(v)} \) for \( v > 0, t > 0 \), with \( \kappa(v) := \Pi^+_{\Pi^+ + v} - v \) and \((Y_t)_{t \geq 0} a \text{ Poisson process with } EY = 1, \text{ independent of } X. \)

**Remark 3.1.** The RHS of (3.2) does not depend on \( r \), so it also equals \( P((1) X_t \leq x \mid \Delta X_t^{(1)} = \Pi^+_{\Pi^+ + v}) \). Similarly in (3.3) and (3.9) below.

**Proof of Theorem 3.1.** To state the Buchmann et al. (2016) formula for the distribution of the trimmed \( \text{Lévy process, let } v > 0 \) and introduce a \( \text{Lévy process (} X^r_t)_{t \geq 0} \) having the canonical triplet in (3.3), the Poisson process \( Y \), and the quantities \( G^r_t \) and \( \kappa \) as in the statement of Theorem 3.1. Let \( r \in \mathbb{N} \) and recall that \((\Gamma_i)\) are Gamma(i, 1) random variables, \( i \in \mathbb{N} \). Assume that \( X, (\Gamma_i) \) and \( Y \) are independent as random elements. Then Lemma 1, p.2333, of Buchmann et al. (2016) gives, for each \( t > 0, \)

\[
(r) X_t, \Delta X_t^{(1)}, \ldots, \Delta X_t^{(r)} \overset{D}{=} \left( X_t^{\Gamma, t} + G_t^r, \Pi^+_{\Pi^+ + v} (\Gamma_1/t, \ldots, \Pi^+_{\Pi^+ + v} (\Gamma_r/t) \right). \tag{3.4}
\]

From this we can compute, for \( t > 0, u > 0, \)

\[
P(r) X_t \leq x, \Delta X_t^{(r)} \leq u) = \int_0^\infty P(X_t^r + G_t^r \leq x) P(\Gamma \leq t dv)
\]

\[
= \int_{v \geq \Pi^+_{\Pi^+ + v}} P(X_t^r + G_t^r \leq x) P(\Pi^+_{\Pi^+ + v} (\Delta X_t^{(r)} \in dv). \tag{3.5}
\]

The LHS of (3.4) equals \( P((r) X_t \leq x, \Pi^+_{\Pi^+ + v} (\Delta X_t^{(r)} \geq \Pi^+_{\Pi^+ + v}) \), so we deduce

\[
P(r) X_t \leq x \mid \Pi^+_{\Pi^+ + v} (\Delta X_t^{(r)} = \Pi^+_{\Pi^+ + v} (v) = P(X_t^r + G_t^r \leq x), \tag{3.6}
\]

and the LHS of (3.5) equals \( P((r) X_t \leq x \mid \Delta X_t^{(r)} = \Pi^+_{\Pi^+ + v} (v)) \) at points of increase \( v \) of \( \Pi^+_{\Pi^+ + v} \). So we get (3.2), (3.1) follows similarly.

Using Theorem 3.1 the conditional characteristic functions of \( (r) X_t \) can be written as in the next corollary.

**Corollary 3.1.** For \( v > 0, r = 2, 3, \ldots, t > 0, \theta \in \mathbb{R}, \)

\[
E(e^{i \theta (r) X_t} \mid \Delta X_t^{(1)}, \ldots, \Delta X_t^{(r-1)}, \Delta X_t^{(r)} = \Pi^+_{\Pi^+ + v}(v))
\]

\[
= E(e^{i \theta (r) X_t} \mid \Delta X_t^{(r)} = \Pi^+_{\Pi^+ + v}(v))
\]

\[
= \exp \left( i \theta r - t \sigma^2 \theta^2 / 2 + t \int_{-\infty}^{- \Pi^+_{\Pi^+ + v}(v)} \left( e^{i \theta x} - 1 - i \theta x 1_{\{|x| \leq 1\}} \right) \Pi(dx) \right)
\]

\[
\times \exp \left( t \kappa(v)(e^{i \theta \Pi^+_{\Pi^+ + v}(v)} - 1) \right), \tag{3.6}
\]

\[
\text{In general } \Pi^+_{\Pi^+ + v}(x) = y \text{ does not imply } x = \Pi^+_{\Pi^+ + v}(y), \text{ but this does hold when } x \text{ is a point of increase of } \Pi^+. \]
where $\gamma^v$ is the shift constant defined in (3.3).

Suppose $X$ is a subordinator (so $\sigma^2 = 0$) with drift $d_X := \gamma - \int_{0<x\leq 1} x \Pi(dx)$. Then (since $\Pi^{+,r} = \Pi^-$ in this case) the RHS of (3.6) can be replaced by

$$
\exp\left( i\theta t d_X + t \int_{(0,\Pi^{-}(v))} (e^{i\theta x} - 1) \Pi(dx) \right) \times \exp\left( t\kappa(v) (e^{i\theta} - 1) \right).
$$

(3.7)

The next corollary follows immediately from Corollary 3.1. Recall the definition of $\rho_X$ in (2.5).

**Corollary 3.2.** For $v > 0$, $\theta \in \mathbb{R}$, $t > 0$, $r \in \mathbb{N}$,

$$
E\left( \exp\left( i\theta (r) X_t - t\rho_X(\Delta X_t^{(r)}) \right) \left| \Delta X_t^{(1)}, \ldots, \Delta X_t^{(r-1)}, \Delta X_t^{(r)} = \Pi^{+,r}(v) \right. \right) \\
= E\left( \exp\left( i\theta (r) X_t - t\rho_X(\Delta X_t^{(r)}) \right) \left| \Delta X_t^{(r)} = \Pi^{+,r}(v) \right. \right) \\
= \exp\left( -t\sigma^2\theta^2/2(\Pi^{+,r}(v))^2 \right) \times \\
\times \exp\left( t \int_{(-\infty,1]} (e^{i\theta x} - 1) \Pi(\Pi^{+,r}(v) dx) \right) \times e^{t\kappa(v)(e^{i\theta} - 1)}.
$$

(3.8)

Suppose $X$ is a subordinator with drift $d_X$. Then (3.8) can be replaced by

$$
E\left( \exp\left( i\theta (r) X_t - t\Pi^{r}(X_t) \right) \left| \Delta X_t^{(1)}, \ldots, \Delta X_t^{(r-1)}, \Delta X_t^{(r)} = \Pi^{-}(v) \right. \right) \\
= E\left( \exp\left( i\theta (r) X_t - t\Pi^{r}(X_t) \right) \left| \Delta X_t^{(r)} = \Pi^{-}(v) \right. \right) \\
= \exp\left( t \int_{(0,1]} (e^{i\theta x} - 1) \Pi(\Pi^{-}(v) dx) \right) \times \exp\left( t\kappa(v)(e^{i\theta} - 1) \right).
$$

(3.9)

(Here we keep $r \geq 2$ in the first expectation of each identity.)

**Proof of Corollaries 3.1 and 3.2.** (3.6) follows from Theorem 3.1 using (3.3). Then (3.7) follows from (3.6) by rearranging the centering terms. (3.8) follows from (3.6) and (2.5), and (3.9) follows from (3.8). □

Another formula follows similarly from (3.1):

**Corollary 3.3.** Suppose $X$ is a subordinator with drift $d_X$. Then for $u \geq v > 0$ points of increase of $\Pi$, $\theta \in \mathbb{R}$, $t > 0$, $r \in \mathbb{N}$, $n \in \mathbb{N}$,

$$
E\left( \exp\left( i\theta (r+n) X_t - t\Pi^{r+n}(X_t) \right) \left| \Delta X_t^{(r)}, \Delta X_t^{(r+n)} = \Pi^{-}(v) \right. \right) \\
= \exp\left( t \int_{(0,\Pi^{-}(v))} (e^{i\theta x/\Pi^{-}(u)} - 1) \Pi(dx) + t\kappa(v)(e^{i\theta} - 1) \right).
$$

(3.10)
For the proofs in Section 4 we quote the following result, derived in Proposition 4.2 of Ipsen, Maller & Resnick (2017).

**Proposition 3.1.** Suppose \( \Pi(\cdot) \in RV_0(-\alpha) \) with \( \alpha \in (0, 2) \), and keep \( r \in \mathbb{N} \) and \( n = 2, 3, \ldots \). Take \( x_k \geq 1 \) for \( 1 \leq k \leq n - 1 \). Then

\[
\lim_{t \downarrow 0} P\left( \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} > x_k, \ 1 \leq k \leq n - 1 \ \bigg| \Delta X_t^{(r+n)} = \Pi^{-}(z/t) \right) = P\left( J_{n-1}^{(k)}(R_{r,n}) > x_k, 1 \leq k \leq n - 1 \right), \tag{3.11}
\]

where \( J_{n-1}^{(1)}(u) \geq J_{n-1}^{(2)}(u) \geq \cdots \geq J_{n-1}^{(n-1)}(u) \) are the order statistics associated with the distribution in (2.6), and \( R_{r,n} \) is a Beta\( (r, n) \) random variable independent of \( (J_i(u))_{1 \leq i \leq n-1} \).

When \( r = 0 \), (3.11) remains true if the RHS is replaced by

\[
P(L_{n-1}^{(k)} > x_k, 1 \leq k \leq n - 1) \tag{3.12}
\]

where \( L_{n-1}^{(k)} \) are the order statistics associated with the distribution in (2.7).

**Remark 3.2.** (3.11) and (3.12) can be stated in a unified fashion if we make the convention that \( B_{0,n} \equiv 0 \) a.s., put \( u = 0 \) in (2.6), and identify \( (J_i(0)) \) with a sequence \( (L_i) \) of independent and identically distributed random variables each having the distribution in (2.7). Similarly for the corresponding statements in Theorem 2.1.

4 **Proofs for Section 2**

Throughout this section \( X \) will be a Lévy process in the domain of attraction of a non-normal stable random variable. Thus the Lévy tail \( \Pi \) is regularly varying of index \(-\alpha, \alpha \in (0, 2), \) at 0, and the balance condition (2.1) holds at 0. Since \( a_+ > 0 \) in (2.4), also \( \Pi^+ \in RV_0(-\alpha) \) at 0.

**Proof of Theorem 2.1:** (i) Take \( r \in \mathbb{N}, n \in \mathbb{N}, \) and choose \( x_1 \geq \cdots \geq x_n \geq 1, \ x_n = 1, \ \theta_k \in \mathbb{R}, \ 1 \leq k \leq n, \) and \( v > 0. \) For shorthand, write \( M_t^{(r+n)} \) for \( \rho_X(\Delta X_t^{(r+n)}) \). We proceed by finding the limit as \( t \downarrow 0 \) of the conditional characteristic function

\[
\mathbb{E}\left( \exp\left( i \sum_{k=1}^{n} \frac{\theta_k^{(r)} X_t - t M_t^{(r+n)}}{\Delta X_t^{(r+k)}} \right) \left| \Delta X_t^{(r+k)} = x_k \Pi^{+,r}(v/t), 1 \leq k \leq n \right. \right) = \mathbb{E}\left( \exp\left( i \sum_{k=1}^{n} \frac{x_k \theta_k^{(r)} X_t - t M_t^{(r+n)}}{\Delta X_t^{(r+n)}} \right) \left| \Delta X_t^{(r+k)} = x_k \Pi^{+,r}(v/t), 1 \leq k \leq n \right. \right), \tag{4.1}
\]
Decompose \((r)X_t\) as follows:
\[
\frac{(r)X_t}{\Delta X_t^{(r+n)}} = \sum_{k=1}^{n} \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} + \frac{(r+n)X_t}{\Delta X_t^{(r+n)}},
\]
(4.2)
and recall the definitions of \(x_{n+}\) and \(\tilde{t}_{n+}\) in (2.10). Given the conditioning in (4.1), the first component on the RHS of (4.2) equals \(\sum_{k=1}^{n} x_k/x_n = x_{n+}\). For the second component, apply Corollary 3.2 with \(r\) replaced by \(r + n\) to replace the conditioning on \(\Delta X_t^{(r+k)}\), \(1 \leq k \leq n\), in (4.1), by conditioning on \(\Delta X_t^{(r+n)}\). Then the RHS of (4.1) can be written as
\[
e^{i\tilde{t}_{n+}x_{n+}} \times E\left( \exp \left( \frac{i\tilde{t}_{n+}^{(r+n)}X_t - tM^{(r+n)}_t}{\Delta X_t^{(r+n)}} \right) \right| \Delta X_t^{(r+n)} = \Pi^{+,\kappa}(v/t) \right).
\]
(4.3)
Then by (3.8) with \(r\) replaced by \(r + n\), \(\theta\) replaced by \(\tilde{t}_{n+}\), \(v\) replaced by \(v/t\), and \(\sigma^2 = 0\), the expression in (4.3) equals
\[
e^{i\tilde{t}_{n+}x_{n+}} \times \exp \left( \int_{(-\infty,1)} \left( e^{i\tilde{t}_{n+}x} - 1 - i\tilde{t}_{n+}x1_{\{|x| \leq 1\}} \right) t\Pi(\Pi^{+,\kappa}(v/t))dx \right)
\times \exp \left( t\kappa(v/t)(e^{i\tilde{t}_{n+}} - 1) \right).
\]
(4.4)
The term containing \(\kappa\) (defined in the statement of Theorem 3.1) is negligible here, as follows: \(\Pi^+\) in \(RV_0(-\alpha)\) implies \(\Delta \Pi^+(x) := \Pi^+(x-) - \Pi^+(x) = o(\Pi^+(x))\) as \(x \downarrow 0\). Hence
\[
t\kappa(v/t) &= \Pi^+(\Pi^{+,\kappa}(v/t)-) - v
\leq t \left( \Pi^+(\Pi^{+,\kappa}(v/t)-) - \Pi^+(\Pi^{+,\kappa}(v/t)) \right)
= t\Delta \Pi^+(\Pi^{+,\kappa}(v/t))
= o(t\Pi^+(\Pi^{+,\kappa}(v/t))) = o(1), \text{ as } t \downarrow 0.
\]
(4.5)
So we can ignore the \(\kappa\) term in (4.4) when \(t \downarrow 0\). The limit of the second factor in (4.4) can be found straightforwardly using integration by parts and applying (2.1) and (2.2). So the expression in (4.4) tends as \(t \downarrow 0\) to
\[
e^{i\tilde{t}_{n+}x_{n+}} \times \exp \left( v \int_{(-\infty,1)} \left( e^{i\tilde{t}_{n+}x} - 1 - i\tilde{t}_{n+}x1_{\{|x| \leq 1\}} \right) \Lambda(dx) \right).
\]
(4.6)
Thus, (4.1), to find the limit as \(t \downarrow 0\) of
\[
E \exp \left( i \sum_{k=1}^{n} \frac{\theta_k^{(r)}X_t - tM^{(r+n)}_t}{\Delta X_t^{(r+k)}} \right),
\]
we have to multiply (4.6) by the limit of
\[
d_x P \left( \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} \in dx_k, 1 \leq k \leq n - 1, \Delta X_t^{(r+n)} \leq \Pi^+(v/t) \right),
\]
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and integrate over $v$ and the $x_k$.\footnote{Use the result: $\int f_t(\omega)P_t(d\omega) \to \int f(\omega)P(d\omega)$ when $P_t \to P$ are probability measures and $f_t \to f$, $f$ cts, $|f| \leq 1$. In (4.7), the $f_t$ are characteristic functions and the limit distribution $P$ in (4.7) is continuous in all its variables.}

From (3.11) when $r \in \mathbb{N}$ and from (3.12) when $r = 0$ we see that this limit does not depend on $v$, and putting the RHS of (3.11) or (3.12) together with the expression in (4.6) we can write the limiting characteristic function of the $n$-vector on the LHS of (2.11) as

$$\int_{x^i \geq 1} e^{i\tilde{\theta}_{n+x}x} \int_0^\infty \exp \left( v \int_{-\infty}^1 \left( e^{i\tilde{\theta}_{n+x}x} - 1 - i\tilde{\theta}_{n+x}x 1_{\{x \leq 1\}} \right) \Lambda(dx) \right) P(\Gamma_{r+n} \in dv) \times P(J_n^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n - 1) \quad (4.7)$$

when $r \in \mathbb{N}$, and with each $J_n^{(k)}(B_{r,n}^{1/\alpha})$ replaced by $L_n^{(k)}$ when $r = 0$. Recall that $\int_{x^i \geq 1}$ denotes integration over the region $\{x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq 1\}$.

Note that, with $\Lambda$ defined as in (2.3), $\tilde{\Lambda}(x) \in RV_0(-\alpha)$ and $\tilde{\Lambda}^+$ and $\tilde{\Lambda}^-$ satisfy (2.1). So exactly the same calculation\footnote{This easy correspondence is the reason for adopting the nonstandard centering in (2.4).} with $\Delta S_1^{(k)}$ replacing $\Delta X_1^{(k)}$, for $r \leq k \leq n$, and $\Lambda$ replacing $\Pi$ (and no limit on $t$ is necessary), shows that the characteristic function of the vector of stable ratios on the RHS of (2.11) equals (4.7) when $r \in \mathbb{N}$ or the corresponding version when $r = 0$.

(ii) To derive (2.12) and the corresponding version when $r = 0$, observe that the exponent inside the integral in (4.7) is the characteristic function of a Lévy process $(W_t)_{t \geq 0}$ having Lévy triplet $(0, 0, \Lambda(dx)1_{(-\infty, 1)})$, that is, of a Stable($\alpha$) process with jumps truncated below 1. So the integral with respect to $v$ in (4.7) is

$$\int_{v > 0} E(e^{i\tilde{\theta}_{n+W}v})P(\Gamma_{r+n} \in dv) = E(e^{i\tilde{\theta}_{n+W}v}),$$

and thus we obtain (2.12) when $r \in \mathbb{N}$. The corresponding version when $r = 0$ follows with $\phi(\theta, 0)$ defined as indicated after (2.13).

When $r \in \mathbb{N}$ and $n = 2, 3, \ldots$, the alternative representation in (2.13) is obtained by evaluating the $dv$ integral in (4.7), resulting in (recall $\psi(\cdot)$ defined in (2.8)):

$$E \exp \left( i \sum_{k=1}^n \frac{\theta_k ((\nu X_t - \nu \rho X(\Delta X_1^{(r+n)})))}{\Delta X_t^{(r+n)}} \right) \to E \exp \left( i \sum_{k=1}^n \frac{\theta_k ((\nu S_1 - \nu \rho S(\Delta S_1^{(r+n)})))}{\Delta S_1^{(r+n)}} \right) = \int_{x^i \geq 1} e^{i\tilde{\theta}_{n+x}x} \int_{v > 0} \frac{v^{r+n-1}e^{-v(1-\psi(\tilde{\theta}_{n+n}))}}{\Gamma(r+n)}dv \times P(J_n^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n - 1), \quad (4.8)$$
equal to the expression in (2.13). When \( r \in \mathbb{N}_0, n = 1 \), similar working shows that (4.7) can be replaced by

\[
\lim_{t \downarrow 0} E\left(e^{i\theta X_t - \theta X_t}\right) = e^{i\theta (\psi(\theta))^{r+1}} P(\Gamma_{r+1} \in dv)
\]

\[
\lim_{t \downarrow 0} E\left(e^{i\theta X_t - \theta X_t}\right) = e^{i\theta (\psi(\theta))^{r+1}}, \quad \theta \in \mathbb{R}. \tag{4.9}
\]

(iii) Finally, to prove (2.15) when \( r \in \mathbb{N} \), set \( \theta_1 = \cdots = \theta_{n-1} = 0, \theta_n = \theta \) (so \( \tilde{\theta}_n = \theta \)) and, recall, \( x_{n+} = x_1 + \cdots + x_{n-1} + 1 \) in (4.8) to get the characteristic function of the RHS of (2.14) equal to

\[
\int_{x^T \geq 1} \frac{\psi(\theta)^{r+n}}{(1 - \psi(\theta))^{r+n}} P(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1)
\]

\[
= \frac{\psi(\theta)}{(1 - \psi(\theta))^{r+n}} \int_0^{u < 1} E\left(e^{i\theta i^{(k)}} P(B_{r,n}^{1/\alpha} \in du)
\]

\[
= \frac{\psi(\theta)}{(1 - \psi(\theta))^{r+n}} \int_0^{u < 1} (\psi^{n-1}(\theta, B_{r,n}^{1/\alpha})),
\]

where \( \phi(\theta, u) = E\left(e^{i\theta i^{(k)}} \right) \), with \( |\psi(\theta)| < 1 \) when \( |\theta| \leq \theta_0 \). Similarly, (4.9) can alternatively be written as \( e^{i\theta} \) times the expression in (2.17). The \( r = 0 \) case follows as before. \( \square \)

**Proof of Theorem 2.2:** In this proof \( X \) is a driftless subordinator whose Lévy tail measure is in \( RV_0(-\alpha) \), \( 0 < \alpha < 1 \). From (3.10) we obtain the Laplace transform

\[
E\exp\left(-\lambda \frac{(r+n)X_t}{\Delta X_t^{(r)}}\right) = \int_{y > 0} \int_{w > y} e^{-t \left[1 - (1 - e^{-\lambda x/b})(1 - e^{-\lambda t/w})(1 - e^{-\lambda a/b})\right]} \times P(\Gamma_r \in dy, \Gamma_{r+n} \in dw), \tag{4.10}
\]

where \( \lambda > 0 \) and for brevity

\[
a = a(w, t) := \Pi^-(w/t) \leq b = b(y, t) := \Pi^+(y/t), \quad t > 0, \ w > y > 0
\]

(we can write \( \Pi^+ \) and \( \Pi^- \) for \( \Pi^+ \) and \( \Pi^+ \) in (3.10)). We derive an upper bound for the exponent in (4.10) as follows. Keep \( 0 < t \leq t_0 \) for a fixed \( t_0 > 0 \), throughout.

First, the integral in the exponent of (4.10) is

\[
t \int_{(0,a)} \left(1 - e^{-\lambda x/b}\right) \Pi(dx) \leq t(\lambda/b) \int_0^a e^{-\lambda x/b} \Pi(x)dx \quad \text{(integrate by parts)}
\]

\[
= t \lambda \int_0^{a/b} e^{-\lambda x} \Pi(bx)dx. \tag{4.11}
\]
Now \( a(w,t) \rightarrow \Pi^+ (\infty) = 0 \) as \( w \rightarrow \infty \) or \( t \downarrow 0 \), and \( b(y,t) \rightarrow \Pi^- (\infty) = 0 \) as \( y \rightarrow \infty \) or \( t \downarrow 0 \). To compare the magnitudes of \( a \) and \( b \) we use the Potter bounds (Bingham, Goldie and Teugels (1987, p.25)). Since \( \Pi \in RV_0(-\alpha) \) with \( 0 < \alpha < 1 \), given \( \eta > 0 \), there are constants \( c > 0 \) and \( z_0 = z_0(\eta) > 0 \) such that

\[
\frac{\Pi(\mu z)}{\Pi(z)} \leq c\mu^{-\alpha-\eta} \text{ for all } \mu \in (0,1), \; z \in (0,z_0); \tag{4.12}
\]

and since \( \Pi^- \in RV_\infty(-1/\alpha) \) we also have

\[
\frac{\Pi^-(\mu z)}{\Pi^- (z)} \leq c\mu^{-1/\alpha+\eta} \text{ for all } \mu > 1, \; z > 1/z_0 \tag{4.13}
\]

(where \( c \) and \( z_0 \) may be chosen the same in both cases, and \( \eta < \alpha < 1/\alpha \)). Thus for \( 0 < x \leq a/b \leq 1 \) and \( 0 < b \leq z_0 \), using (4.12),

\[
t\Pi(bx) \leq ctx^{-\alpha-\eta}\Pi(b) = ctx^{-\alpha-\eta}\Pi^-(y/t) \leq cyx^{-\alpha-\eta},
\]

and we have \( b \leq z_0 \) if \( \Pi^-(y/t) \leq z_0 \), i.e., if \( y/t \geq \Pi(z_0) \). For \( w > y \) and \( y/t \geq 1/z_0 \), using (4.13),

\[
a/b = \frac{\Pi^+(w/t)}{\Pi^-(y/t)} = \frac{\Pi^+(w/t)}{\Pi^+(y/t)} \leq c \left( \frac{w}{y} \right)^{-1/\alpha+\eta} = c \left( \frac{y}{w} \right)^{1/\alpha-\eta}. \tag{4.14}
\]

Now keep \( y/t \geq z_1 := \Pi(z_0) \vee (1/z_0) \) and \( 0 < \eta < \alpha \). Then by (4.11)

\[
t \int_{(0,a)} (1 - e^{-\lambda x/b}) \Pi(dx) \leq t \lambda \int_0^{a/b} e^{-\lambda x} \Pi(bx) dx \\
\leq c \lambda y \int_0^{a/b} x^{-\alpha-\eta} dx = \frac{c \lambda y}{1-\alpha-\eta} \left( \frac{a}{b} \right)^{1-\alpha-\eta} \\
\leq c' \lambda y \left( \frac{y}{w} \right)^\beta =: \lambda g_1(w,y), \tag{4.15}
\]

where \( c' := c^2\alpha-\eta/(1-\alpha-\eta) > 0 \) and \( \beta := (1-\alpha-\eta)/(1/\alpha-\eta) > 0 \).

Alternatively, when \( y/t < z_1 \), we have \( b = \Pi^-(y/t) \geq \Pi^-(z_1) \), while \( t \leq t_0 \) implies \( a = \Pi^+(w/t) \leq \Pi^- (w/t_0) \). Then

\[
t \int_{(0,a)} (1 - e^{-\lambda x/b}) \Pi(dx) \leq t(\lambda/b) \int_{(0,a)} x \Pi(dx) \\
\leq t_0(\lambda/\Pi^+(z_1)) \int_{(0,\Pi^- (w/t_0))} x \Pi(dx) =: \lambda g_2(w). \tag{4.16}
\]

For the term containing \( \kappa \) in (4.10), we have, for all \( x \in (0,z_0) \),

\[
\Delta \Pi(x) = \Pi(x-) - \Pi(x) \leq \Pi(x/2) - \Pi(x) \leq 2^{\alpha+\eta} c \Pi(x)
\]

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by (4.13). Thus for all $t > 0$ and $w > y > 0$, using (4.5),

$$t \kappa(w/t) \leq t \Delta \Pi^{-}(w/t(t)) \leq 2^{\alpha + \eta} c \Pi^{-}(w/t(t)) \leq 2^{\alpha + \eta} c w,$$

because we kept $y/t > \Pi(z_0)$, as a consequence of which $\Pi^{-}(w/t(t)) \leq \Pi^{-}(y/t(t)) \leq z_0$. Then $t \kappa(w/t(t)) (1 - e^{-\lambda a/b}) \leq e^{2 \alpha + \eta} c w \lambda a/b$. When $w > y$ and $y/t \geq 1/z_0$, $t \kappa(w/t(t)) (1 - e^{-\lambda a/b}) \leq 2^{\alpha + \eta} c w \lambda \Pi^{-}(w/t_0(t))/\Pi^{-}(z_1(t))$. So an overall upper bound for the term containing $\kappa$ in (4.10) is

$$t \kappa(w/t(t)) (1 - e^{-\lambda a/b}) \leq \lambda g_3(w, y) := 2^{\alpha + \eta} \max \left( e^{2 \lambda w(y/w)^{1/\alpha - \eta}}, cw \lambda \Pi^{-}(w/t_0(t))/\Pi^{-}(z_1(t)) \right).$$

(4.17)

Combine (4.15)–(4.17) to get an upper bound for the negative of the exponent in (4.10) of the form

$$\lambda g(w, y) := \lambda \max (g_1(w, y) + g_2(w), g_3(w, y)).$$

So, for all $0 < t \leq t_0$, $n \in \mathbb{N}$,

$$E \exp \left( -\lambda \frac{(r+n)X_t}{\Delta X_t^{(r)}} \right) \geq \int_{y > 0} \int_{w > y} e^{-t_0 \lambda g(w, y)} P (\Gamma_r \in dy, \Gamma_{r+n} \in dw) = E(e^{-t_0 \lambda g(\Gamma_{r+n}, \Gamma_r)}).$$

(4.18)

Now when $w \to \infty$, $g_1(w, y) \to 0$ for each $y > 0$ (see (4.15)) and $g_2(w) \to 0$ as $w \to \infty$ because $\Pi^{-}(w) \to 0$ as $w \to \infty$ (see (4.16)); while $g_3(w, y) \to 0$ for each $y > 0$ because $\Pi^{-} \in RV_{\infty}(-1/\alpha)$ and $0 < \alpha < 1$ (see (4.17)).

Finally, since $\Gamma_{r+n} \overset{P}{\to} \infty$ as $n \to \infty$ for each $r \in \mathbb{N}$, we can let $n \to \infty$ and use Fatou’s lemma in (4.18) to see that

$$E \exp \left( -\lambda \frac{(r+n)X_t}{\Delta X_t^{(r)}} \right) \to 1, \text{ as } n \to \infty,$$

for each $r \in \mathbb{N}$, uniformly in $\lambda > 0$ and $t \in (0, t_0]$. We deduce convergence in probability in (2.21) uniformly in $t \in (0, t_0]$ from this, then since the LHS of (2.21) is monotone in $n$, we get the a.s. convergence.

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