Extremal functions of generalized critical Hardy inequalities

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Abstract

In this paper, we show the existence and non-existence of minimizers of the following minimization problems which include an open problem mentioned by Horiiuchi and Kumlin [20]:

\[ G_a := \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^N \, dx}{\left( \int_{\Omega} |u|^q f_{a,\beta}(x) \, dx \right)^{\frac{N}{q}}}, \quad \text{where} \quad f_{a,\beta}(x) := |x|^{-N} \left( \log \frac{aR}{|x|} \right)^{-\beta}. \]

First, we give an answer to the open problem when \( \Omega = B_R(0) \). Next, we investigate the minimization problems on general bounded domains. In this case, the results depend on the shape of the domain \( \Omega \). Finally, symmetry breaking property of the minimizers is proved for sufficiently large \( \beta \).

Keywords: Critical Hardy inequality, Optimal constant, Extremal function, Symmetry breaking

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1. Introduction

Let \( N \geq 2, \Omega \) be a bounded domain in \( \mathbb{R}^N \), \( 0 \in \Omega \), and \( 1 < p < N \). The classical Hardy inequality holds for all \( u \in W^{1,p}_0(\Omega) \) as follows:

\[ \left( \frac{N-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \, dx \leq \int_{\Omega} |\nabla u|^p \, dx, \quad (1) \]
where $W_{0}^{1,p}(\Omega)$ is a completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$. We refer the celebrated work by G. H. Hardy [17]. The inequality (1) has great applications to partial differential equations, for example stability, global existence, and instantaneous blow-up and so on. See e.g. [6], [3]. It is well-known that in $u \geq a$ which is called the critical Hardy inequality holds for all $u \in W_{0}^{1,p}(\Omega)$.

On the other hand, in the critical case where $p = N$, the following inequality which is called the critical Hardy inequality holds for all $u \in W_{0}^{1,N}(\Omega)$ and all $a \geq 1$, where $R = \sup_{x \in \Omega} |x|$:

$$\left(\frac{N-1}{N}\right)^N \int_{\Omega} \frac{|u|^N}{|x|^N(\log \frac{aR}{|x|})^N} dx \leq \int_{\Omega} |\nabla u|^N dx. \quad (2)$$

See e.g. [25], [24], [4], [5], [15] Corollary 9.1.2., [28], [34]. It is known that in $u$ for $q \in \mathbb{R}$, the following inequality (3) which are generalizations of (2):

$$G_a \left(\int_{\Omega} \frac{|u|^q}{|x|^N(\log \frac{aR}{|x|})^q} dx\right) \leq \int_{\Omega} |\nabla u|^N dx \quad (3)$$

for $u \in W_{0}^{1,N}(\Omega)$, $q, \beta > 1$, and $a \geq 1$. We define $G_a$ and $G_{a,rad}$ as the optimal constants of the inequalities (3) as follows:

$$G_a := \inf_{u \in W_{0}^{1,N}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^N dx}{\left(\int_{\Omega} \frac{|u|^q}{|x|^N(\log \frac{aR}{|x|})^q} dx\right)^{\frac{1}{N}}} , \quad G_{a,rad} := \inf_{u \in W_{0,rad}^{1,N}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^N dx}{\left(\int_{\Omega} \frac{|u|^q}{|x|^N(\log \frac{aR}{|x|})^q} dx\right)^{\frac{1}{N}}} \quad (4)$$

where $W_{0,rad}^{1,N}(\Omega) = \{ u \in W_{0}^{1,N}(\Omega) | u \text{ is radial} \}$. When $\Omega = B_R(0)$, $\beta = \frac{N-1}{N} q + 1$, and $q > N$, the exact optimal constant and the attainability of $G_{a,rad}$ are investigated by Horiuchi and Kumlin [20]. However we do not know the attainability of $G_a$ even if $\Omega = B_R(0)$. In fact, in their article [20] they mention that the attainability of $G_a$ is an open problem. See also [19]. Note that the continuous embedding $W_{0}^{1,N}(B_R(0)) \hookrightarrow L^{q}(B_R(0); |x|^{-N}(\log \frac{aR}{|x|})^{-\beta}dx)$ is not compact when $\beta = \frac{N-1}{N} q + 1$, $q \geq N$, and $a > 1$. In addition, the rearrangement technique does not work due to the lack of monotone decreasing property of the potential function $|x|^{-N}(\log \frac{aR}{|x|})^{-\beta}$ when $1 \leq a < e^\frac{1}{\beta}$. 

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In this paper, we study the existence, non-existence, and symmetry breaking property of the minimizers of $G_a$. First, we give an answer to the open problem except for $a = a_*$ which is a threshold number when $\Omega = B_R(0)$. More precisely, we show that there exists a minimizer of $G_a$ for $a \in (1, a_*)$ and there is no minimizer for $a > a_*$. Next, we extend the results to general bounded domains. Furthermore we investigate the positivity and the attainability of $G_1$ in general bounded domains. When $a = 1$, the positivity and the attainability of $G_1$ depend on geometry of the boundary of the domain since the potential function has singularities on the boundary. Finally, we show that when $\Omega = B_R(0)$, any minimizers of $G_a$ are non-radial for large $\beta$ and fixed $q > N$, and any minimizers are radial for any $\beta$ and any $q \leq N$.

Our problem is regarded as the critical case of one of Caffarelli-Kohn-Nirenberg type inequalities, see [20]. In the weighted subcritical Sobolev spaces $W_0^{1,p}(|x|^{\alpha}dx)$ where $p < N + \alpha$, the existence, nonexistence, and symmetry breaking property of the minimizers of Caffarelli-Kohn-Nirenberg type inequalities are well-studied especially for $p = 2$, see [35], [26], [12], [18], [8], [9], [10], [33], [14], [16], [11] and references therein.

Our minimization problem (4) is related to the following nonlinear elliptic equation with the singular potential:

\[
\begin{cases}
-\text{div} \left( |\nabla u|^{N-2} \nabla u \right) = b \frac{|u|^{q-2}u}{|x|^{N \log \frac{R}{|x|}}} & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } \partial \Omega.
\end{cases}
\]

(5)

The minimizer for $G_a$ is a ground state solution of the Euler-Lagrange equation (5) with a Lagrange multiplier $b$.

This paper is organized as follows: In section 2, necessary preliminary facts are presented. In section 3, we prove the (non-)attainability of $G_a$ when $\Omega = B_R(0)$ and $a > 1$. In section 4, we extend the results to several bounded domains, and we investigate the positivity and the attainability of $G_1$ in several bounded domains. In section 5, we show that symmetry breaking phenomena of the minimizers of $G_a$ occur for large $\beta$.

We fix several notations: $B_R(0)$ and $B_R^N(0)$ denote a $N$-dimensional ball centered 0 with radius $R$ and $\omega_{N-1}$ denotes an area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$. The Schwarz symmetrization $u^\# : \mathbb{R}^N \to [0, \infty]$ of $u$ is given by

\[u^\#(x) = u^\#(|x|) = \inf \left\{ \tau > 0 : \left| \{ y \in \mathbb{R}^N : |u(y)| > \tau \} \right| \leq |B_{|x|}(0)| \right\}.\]
2. Preliminaries

In this section, we give a necessary and sufficient condition of the positivity of \( G_a \) for \( a \in [1, \infty) \). Furthermore, we give the explicit value of \( G_a \) and the minimizers when \( \beta = \frac{N-1}{N}q + 1 \) and \( q > N \). First, we give a necessary and sufficient condition \([5]\) of the positivity of \( G_a \) when \( a > 1 \).

**Proposition 1.** Let \( a > 1, \Omega \subset \mathbb{R}^N \) be a bounded domain with \( 0 \in \Omega \), \( R = \sup_{x \in \Omega} |x|, N \geq 2 \) and \( q, \beta > 1 \). Then \( G_a > 0 \) if and only if \( \beta \) and \( q \) satisfy

\[
\begin{align*}
\text{either } & \beta > \frac{N-1}{N}q + 1 \text{ or } \\
& \beta = \frac{N-1}{N}q + 1, \quad q \geq N.
\end{align*}
\]

Essentially, Proposition \([1]\) is proved by the following theorem in \([27]\). The authors in \([27]\) show a necessary and sufficient condition of the positivity for more general inequalities in the critical Sobolev-Lorentz spaces \( H_{p,q}^s(\mathbb{R}^N) \). Note that the norm of \( H_{1,N}^1(\mathbb{R}^N) \) is equivalent to it of \( W^{1,N}(\mathbb{R}^N) \). We can obtain Proposition \([1]\) from Theorem A and simple calculations. We omit the proof here.

**Theorem A.** (\([27]\) Theorem 1.1.) Let \( N \in \mathbb{N}, 1 < p < \infty, 1 < r \leq \infty \) and \( 1 < \alpha, \beta < \infty \). Then there exists a constant \( C > 0 \) such that for all \( u \in H_{p,r}^s(\mathbb{R}^N) \), the inequality

\[
\left( \int_{B^1_2(0)} \frac{|u|^p}{|x|^N(\log \frac{1}{|x|})^\beta} \, dx \right)^\frac{1}{p} \leq C\|u\|_{H_{p,r}^s(\mathbb{R}^N)}
\]

holds true if and only if one of the following conditions (i)’(iii) is fulfilled

\[
\begin{align*}
(i) & \quad 1 + \alpha - \beta < 0, \\
(ii) & \quad 1 + \alpha - \beta \geq 0 \text{ and } r < \frac{\alpha}{1+\alpha-\beta}, \\
(iii) & \quad 1 + \alpha - \beta > 0, r = \frac{\alpha}{1+\alpha-\beta}, \text{ and } \alpha \geq \beta.
\end{align*}
\]

Next, we give a necessary and sufficient condition of the positivity of \( G_a \) when \( a = 1 \) and \( \Omega = B_R(0) \). Essentially, the following proposition follows from results in \([20]\).

**Proposition 2.** Let \( \Omega = B_R(0) \). Then \( G_1 > 0 \) if and only if \( \beta = q = N \).
In §4, we extend Proposition 2 to general bounded domains. Proposition 2 follows from Proposition 4 in §4. Thus we omit the proof of Proposition 2 here.

Finally, we give the explicit value of the optimal constant \( G_{a,\text{rad}} \) and the minimizers when \( \beta = \frac{N-1}{N}q + 1 \) and \( q > N \).

Logarithmic transformations related to \( G_{a,\text{rad}} \) are founded by [20], [21], [36], [30]. Especially, in the radial setting, the authors in [30] show an unexpected relation (9) that the critical Hardy inequality in dimension \( N \geq 2 \) is equivalent to the one of the subcritical Hardy inequalities in higher dimension \( m > N \) by using a transformation (10) as follows:

\[
\int_{\mathbb{R}^m} |\nabla u|^N \, dx - \left( \frac{m - N}{N} \right)^N \int_{\mathbb{R}^m} |u|^N \, dx = \frac{\omega_{m-1}}{\omega_{N-1}} \left( \frac{m - N}{N - 1} \right)^{N-1} \left( \int_{B^N_R(0)} |\nabla w|^N \, dy - \left( \frac{N - 1}{N} \right)^N \int_{B^N_R(0)} \frac{|w|^N}{|y|^N (\log \frac{R}{|y|})^N} \, dy \right), \tag{9}
\]

where \( u(|x|) = w(|y|) \) and \( \left( \log \frac{R}{|y|} \right)^{\frac{N-1}{N}} = |x|^{-\frac{N}{N-1}} \).

By using the transformation (10) and direct calculations, we can observe not only an equivalence between two Hardy inequalities but also the equivalence between Hardy-Sobolev type inequalities and generalized critical Hardy inequalities in the radial setting as follows:

\[
G_{1,\text{rad}} = \inf_{w \in W^{1,N}_{a,\text{rad}}(B^N_R(0)), \{0\}} \frac{\int_{B^N_R(0)} |\nabla w|^N \, dy}{\left( \int_{B^N_R(0)} \frac{|w|^q}{|y|^\alpha (\log \frac{R}{|y|})^\beta} \, dy \right)^{\frac{q}{q}}}
= \left( \frac{\omega_{N-1}}{\omega_{m-1}} \right)^{1 - \frac{N}{m}} \left( \frac{N - 1}{m - N} \right)^{N-1 + \frac{N}{m}} \inf_{u \in W^{1,N}_{a,\text{rad}}(\mathbb{R}^m), \{0\}} \frac{\int_{\mathbb{R}^m} |\nabla u|^N \, dx}{\left( \int_{\mathbb{R}^m} |x|^{\alpha} |u|^q \, dx \right)^{\frac{q}{q}}}, \tag{11}
\]

where \( \alpha = \frac{m-N}{N-1} (\beta - 1) - m \). The authors in [30] also give a transformation which is a modification of (10) when \( a > 1 \). Since the minimization problems on the right hand side of (11) are well-known (see e.g. [35], [26]), we can obtain the following proposition by using these transformations.

**Proposition 3.** Let \( \beta = \frac{N-1}{N} q + 1, q > N \), and \( \Omega = B_R(0) \). Then the followings hold.
(i) \(G_{a,\text{rad}}\) is independent of \(a \geq 1\). Furthermore, the exact value of the optimal constant is as follows:

\[
G_{a,\text{rad}} = G_{\text{rad}} := \omega_{N-1} (N-1) \left( \frac{N}{q} \right)^{\frac{2N}{q}} \left( 1 - \frac{N}{q} \right)^{-\frac{2N}{q}} \left( \frac{\Gamma \left( \frac{q(N-1)}{q-N} \right)}{\Gamma \left( \frac{qN}{q-N} \right)} \right)^{1-\frac{N}{q}},
\]

where \(\Gamma(\cdot)\) is the gamma function.

(ii) \(G_{a,\text{rad}}\) is not attained for any \(a > 1\).

(iii) \(G_{1,\text{rad}}\) is attained by the family of the following functions \(U_\lambda:\)

\[
U_\lambda(y) = C \lambda^{-\frac{N-1}{N}} \left( 1 + \left( \lambda \log \frac{R}{|y|} \right)^{-\frac{q}{q-N}} \right)^{-\frac{N}{q}}, \quad \text{where } C \in \mathbb{R} \setminus \{0\} \text{ and } \lambda > 0.
\]

Here, we give a simple proof of Proposition 3(ii) by using a scaling argument.

**Proof of Proposition 3(ii).** Let \(\beta = \frac{N-1}{N} q + 1, q > N,\) and \(a > 1\). Assume that \(u \in W^{1,N}_{0,\text{rad}}(B_R(0))\) is a radial minimizer of \(G_{a,\text{rad}}\). We can assume that \(u\) is nonnegative without loss of generality. We shall derive a contradiction. For \(\lambda \in (0, 1),\) we consider a scaled function \(u_\lambda \in W^{1,N}_{0,\text{rad}}(B_R(0))\) which is given by

\[
u_{\lambda}(x) = \begin{cases} 
\lambda^{-\frac{N-1}{N}} u \left( \frac{|x|}{aR} \right)^{1-1} x 
& \text{if } x \in B_{\frac{a(1-\lambda^{-1})R}{R}}(0), \\
0 
& \text{if } x \in B_R(0) \setminus B_{\frac{a(1-\lambda^{-1})R}{R}}(0).
\end{cases}
\]

Then we have

\[
\frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} \frac{|u|^q}{|x|^q \log ( \frac{1}{|x|})^q} \, dx \right)^{\frac{N}{q}}} = \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} \frac{|u_\lambda|^q}{|x|^q \log ( \frac{1}{|x|})^q} \, dx \right)^{\frac{N}{q}}}
\]

which yields that \(u_\lambda\) is also a nonnegative minimizer of \(G_{a,\text{rad}}\). On the other hand, we can show that \(u_\lambda \in C^1(B_R(0) \setminus \{0\})\) and \(u_\lambda > 0\) in \(B_R(0) \setminus \{0\}\) by standard regularity argument and strong maximum principle to the Euler-Lagrange equation (5), see e.g. [13], [29]. However \(u_\lambda \equiv 0\) in \(B_R(0) \setminus B_{\frac{a(1-\lambda^{-1})R}{R}}(0)\). This is a contradiction. Therefore \(G_{a,\text{rad}}\) is not attained. □
3. Existence and non-existence of the minimizers

Let \( \Omega = B_R(0) \). In this section, we prove an existence and non-existence of the minimizers of \( G_a \). First result is as follows.

**Theorem 1.** Let \( a > 1 \) and \( q, \beta > 1 \) satisfy (6). Then the followings hold.

(i) If \( \beta > \frac{N-1}{N} q + 1 \), then \( G_a \) is attained.

(ii) If \( \beta = \frac{N-1}{N} q + 1 \) and \( q > N \), then there exists \( a_\ast \in (1, e^\frac{\beta}{N}] \) such that \( G_a \) is attained for \( a \in (1, a_\ast) \) and \( G_a \) is not attained for \( a > a_\ast \).

**Remark 1.** If \( a_\ast = e^\frac{\beta}{N} \), then we can show that \( G_a_\ast \) is not attained. In fact, if we assume that \( G_a_\ast \) is attained by \( u \), then \( u_\# \) is a radial minimizer of \( G_a_\ast \), which contradicts Proposition 3 (ii), see the proof of Theorem 1 (ii). However we do not know the value of \( a_\ast \).

In order to show Theorem 1, we need three lemmas. First we show the (non-)compactness of the embedding \( W_{0}^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); f_a,\beta(x)dx) \), where \( f_a,\beta(x) = |x|^{-N} (\log \frac{aR}{|x|})^{-\beta} \).

**Lemma 1.** Let \( a > 1 \) and \( q, \beta > 1 \) satisfy (6). Then the continuous embedding \( W_{0}^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); f_a,\beta(x)dx) \) is

(i) compact if \( \beta > \frac{N-1}{N} q + 1 \),

(ii) non-compact if \( \beta = \frac{N-1}{N} q + 1 \) and \( q \geq N \).

**Proof of Lemma 1.** (i) It is proved in [31]. However we give a proof here for the convenience of readers. Let \( (u_m)_{m=1}^\infty \subset W_{0}^{1,N}(B_R(0)) \) be a bounded sequence. Then there exists a subsequence \( (u_{m_k})_{k=1}^\infty \) such that

\[
\begin{align*}
&u_{m_k} \rightharpoonup u \quad \text{in} \quad W_{0}^{1,N}(B_R(0)), \\
&u_{m_k} \to u \quad \text{in} \quad L^r(B_R(0)) \quad \text{for any} \quad r \in [1, \infty).
\end{align*}
\]

(12)

Let \( \alpha \) satisfy \( \frac{N-1}{N} q + 1 < \alpha < \beta \). For all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\left( \log \frac{aR}{|x|} \right)^{\alpha-\beta} < \varepsilon \quad \text{for all} \quad x \in B_\delta(0).
\]

(13)

From (12) and (13), we have

\[
\int_{B_\delta(0)} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \leq \varepsilon \int_{B_\delta(0)} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx + C \delta^{-N} \left( \log \frac{aR}{\delta} \right)^{-\beta} ||u_{m_k} - u||_{L^q(B_\delta(0))}^q \leq C \varepsilon + C ||u_{m_k} - u||_{L^q(B_\delta(0))}^q \to 0 \quad \text{as} \quad \varepsilon \to 0, k \to \infty.
\]
Thus the continuous embedding \( W^{1,N}_0(B_R(0)) \hookrightarrow L^q(B_R(0); f_{a,\beta}(x)dx) \) is compact if \( \beta > \frac{N-1}{N}q + 1 \).

(ii) We can see a non-compact sequence \( (u_\lambda)_{\lambda=1}^\infty \) in \( W^{1,N}_0(B_R(0)) \), where for \( \lambda \in (0, 1] \) \( u_\lambda \) is defined in the proof of Proposition 3 (ii). Hence the continuous embedding \( W^{1,N}_0(B_R(0)) \hookrightarrow L^q(B_R(0); f_{a,\beta}(x)dx) \) is non-compact if \( \beta = \frac{N-1}{N}q + 1 \) and \( q \geq N \).

\[ \square \]

In [20], a continuity of \( G_a \) with respect to \( a \) is proved for \( a \in (1, \infty) \). However, in our argument, the continuity of \( G_a \) at \( a = 1 \) is needed.

**Lemma 2.** \( G_a \) is monotone increasing and continuous with respect to \( a \in [1, \infty) \).

**Proof of Lemma 2.** It is enough to show only the continuity of \( G_a \) at \( a = 1 \). From the definition of \( G_1 \), we can take \( (u_m)_{m=1}^\infty \subset C^\infty_c(B_R(0)) \) and \( R_m < R \) for any \( m \) such that \( \text{supp } u_m \subset B_{R_m}(0) \), \( R_m \not\rightarrow R \), and

\[
\frac{\int_{B_{R_m}(0)} |\nabla u_m|^N dx}{\left( \int_{B_{R_m}(0)} |u_m|^q f_{1,\beta}(x) dx \right)^\frac{N}{q}} = G_1 + o(1) \quad \text{as } m \to \infty.
\]

Set \( v(y) = u_m(x) \), where \( y = \frac{R}{R_m}x \). Then

\[
\frac{\int_{B_{R_m}(0)} |\nabla u_m|^N dx}{\left( \int_{B_{R_m}(0)} |u_m|^q f_{1,\beta}(x) dx \right)^\frac{N}{q}} \leq G_{a_m},
\]

where \( a_m = \frac{R}{R_m} \not\rightarrow 1 \) as \( m \to \infty \). Therefore we have \( G_{a_m} \leq G_1 + o(1) \). Since \( \int_{B_{R_m}(0)} f_{a_m,\beta}(x) dx \leq \int_{B_{R_m}(0)} f_{1,\beta}(x) dx \) for any \( x \in B_R(0) \), we have \( G_1 \leq G_{a_m} \). Hence we see that \( \lim_{a \not\rightarrow 1} G_a = G_1 \).

\[ \square \]

Third Lemma is concerned with the concentration level of minimizing sequences of \( G_a \).

**Lemma 3.** Let \( \beta = \frac{N-1}{N}q + 1, q > N, \) and \( a > 1 \). If \( G_a < G_{\text{rad}} \), then \( G_a \) is attained, where \( G_{\text{rad}} \) is given by Proposition 3 (i).

It is easy to show Theorem 1 by these three lemmas. Therefore we give a proof of Theorem 1 before showing Lemma 3.
Proof of Theorem. (i) This is proved by Lemma (i). We omit the proof here.
(ii) Let $\beta = \frac{N-1}{N} q + 1$ and $q > N$. When $a \geq e^\beta$, the potential function $f_{a, \beta}$ is radially decreasing. Thus the Pólya-Szegő inequality and the Hardy-Littlewood inequality imply that
\[
\frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} |u|^q f_{a, \beta}(x) \, dx \right)^{\frac{N}{q}}} \geq \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} |u|^q f_{a, \beta}(x) \, dx \right)^{\frac{N}{q}}} \geq G_{a, \text{rad}}
\]
for any $u \in W_0^{1,N}(B_R(0))$ and $a \geq e^\beta$. Therefore $G_a = G_{a, \text{rad}} = G_{\text{rad}}$ for any $a \geq e^\beta$. Moreover we see $G_1 = 0$ by Proposition 2. Since $G_a$ is continuous and monotone increasing with respect to $a \in [1, \infty)$ by Lemma 2, there exists $a_* \in (1, e^\beta]$ such that $G_a < G_{\text{rad}}$ for $a \in [1, a_*)$ and $G_a = G_{\text{rad}}$ for $a \in [a_*, \infty)$. Hence $G_a$ is attained for $a \in (1, a_*)$ by Lemma 3. On the other hand, if we assume that there exists a nonnegative minimizer $u$ of $G_a$ for $a > a_*$, then we can show that $u \in C^1(B_R(0) \setminus \{0\})$ and $u > 0$ in $B_R(0) \setminus \{0\}$ by standard regularity argument and strong maximum principle to the Euler-Lagrange equation (5), see e.g. [13], [29]. Therefore we see that
\[
G_{\text{rad}} = G_a = \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} |u|^q f_{a, \beta}(x) \, dx \right)^{\frac{N}{q}}} \geq \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} |u|^q f_{a, \beta}(x) \, dx \right)^{\frac{N}{q}}} \geq G_{a, \text{rad}}.
\]
This is a contradiction. Therefore $G_a$ is not attained for $a > a_*$. \qed

Finally, we prove Lemma 3.

Proof of Lemma 3. Take a minimizing sequence $(u_m)_{m=1}^\infty \subset W_0^{1,N}(B_R(0))$ of $G_a$. Without loss of generality, we can assume that
\[
\int_{B_R(0)} |u_m|^q f_{a, \beta}(x) \, dx = 1, \quad \int_{B_R(0)} |\nabla u_m|^N \, dx = G_a + o(1) \text{ as } m \to \infty.
\]
Since $(u_m)$ is bounded in $W_0^{1,N}(B_R(0))$, passing to a subsequence if necessary, $u_m \rightharpoonup
\(u\) in \(W_{0}^{1,N}(B_{r}(0))\). Then by Brezis-Lieb lemma, we have

\[
G_{a} = \int_{B_{r}(0)} |\nabla u_{m}|^{N} \, dx + o(1)
\]

\[
= \int_{B_{r}(0)} |\nabla (u_{m} - u)|^{N} \, dx + \int_{B_{r}(0)} |\nabla u|^{N} \, dx + o(1)
\]

\[
\geq G_{a} \left( \int_{B_{r}(0)} |u_{m} - u|^{q} f_{a,\beta}(x) \, dx \right)^{\frac{N}{q}} + G_{a} \left( \int_{B_{r}(0)} |u|^{q} f_{a,\beta}(x) \, dx \right)^{\frac{N}{q}} + o(1)
\]

\[
\geq G_{a} \left( \int_{B_{r}(0)} (|u_{m} - u|^{q} + |u|^{q}) f_{a,\beta}(x) \, dx \right)^{\frac{N}{q}} + o(1)
\]

\[
= G_{a} \left( \int_{B_{r}(0)} |u_{m}|^{q} f_{a,\beta}(x) \, dx \right)^{\frac{N}{q}} + o(1) = G_{a}
\]

which implies that either \(u \equiv 0\) or \(u_{m} \to u \neq 0\) in \(L^{q}(B_{r}(0); f_{a,\beta}(x)dx)\) holds true from the equality condition of the last inequality. We shall show that \(u \neq 0\). Assume that \(u \equiv 0\). Then we claim that

\[
G_{rad} \leq \int_{B_{r}(0)} |\nabla u_{m}|^{N} \, dx + o(1). \tag{14}
\]

If the claim (14) is true, then we see that \(G_{rad} \leq G_{a}\) which contradicts the assumption. Therefore \(u \neq 0\) which implies that \(u_{m} \to u \neq 0\) in \(L^{q}(B_{r}(0); f_{a,\beta}(x)dx)\). Hence we have

\[
1 = \int_{B_{r}(0)} |u|^{q} f_{a,\beta}(x) \, dx, \quad \int_{B_{r}(0)} |\nabla u|^{N} \, dx \leq \liminf_{m \to \infty} \int_{B_{r}(0)} |\nabla u_{m}|^{N} \, dx = G_{a}.
\]

Thus we can show that \(u\) is a minimizer of \(G_{a}\). We shall show the claim (14). Since \(u_{m} \to 0\) in \(L^{r}(B_{r}(0))\) for any \(r \in [1, \infty)\) and the potential function \(f_{a,\beta}\) is bounded away from the origin, for any small \(\varepsilon > 0\) we have

\[
1 = \int_{B_{r}(0)} |u_{m}|^{q} f_{a,\beta}(x) \, dx = \int_{B_{r}(0)} |u_{m}|^{q} f_{a,\beta}(x) \, dx + o(1).
\]

Let \(\phi_{\varepsilon}\) be a smooth cut-off function which satisfies the followings:

\[
0 \leq \phi_{\varepsilon} \leq 1, \quad \phi_{\varepsilon} \equiv 1 \text{ on } B_{\frac{\varepsilon}{2}}(0), \quad \text{supp } \phi_{\varepsilon} \subset B_{\varepsilon R}(0), \quad |\nabla \phi_{\varepsilon}| \leq C \varepsilon^{-1}.
\]
Set \( \tilde{u}_m(y) = u_m(x) \) and \( \tilde{\phi}_\varepsilon(y) = \phi_\varepsilon(x) \), where \( y = \frac{x}{\varepsilon} \). Then we have

\[
1 = \left( \int_{B_{\varepsilon R}(0)} |u_m|^q f_{u, \beta}(x) \, dx \right)^{\frac{1}{q}} + o(1)
\]

\[
\leq \left( \int_{B_{\varepsilon R}(0)} |u_m\phi_\varepsilon(x)|^q f_{u, \beta}(x) \, dx \right)^{\frac{1}{q}} + o(1)
\]

\[
= \left( \int_{B_{\varepsilon R}(0)} |\tilde{u}_m\tilde{\phi}_\varepsilon(x)|^q f_{\varepsilon^{-1}, \beta}(x) \, dx \right)^{\frac{1}{q}} + o(1) \leq G_{ae^{-1}}^{-1} \int_{B_{\varepsilon R}(0)} |\nabla(\tilde{u}_m\tilde{\phi}_\varepsilon)|^N \, dx + o(1).
\]

We see that \( ae^{-1} \geq e^{\frac{\beta}{q}} \) for small \( \varepsilon \). Since \( G_{ae^{-1}} = G_{a, \text{rad}} = G_{\text{rad}} \) from the proof of Theorem 1(ii), we have

\[
1 \leq G_{\text{rad}}^{-1} \int_{B_{\varepsilon R}(0)} |\nabla(\tilde{u}_m\tilde{\phi}_\varepsilon)|^N \, dx + o(1)
\]

\[
\leq G_{\text{rad}}^{-1} \left( \int_{B_{\varepsilon R}(0)} |\nabla u_m|^N \, dx + N \int_{B_{\varepsilon R}(0) \setminus B_{\varepsilon R}(0)} |\nabla u_m|^{N-2} \nabla u_m \cdot \nabla \phi_\varepsilon u_m \phi_\varepsilon^{N-1} \, dx \right) + o(1)
\]

\[
\leq G_{\text{rad}}^{-1} \left( \int_{B_{\varepsilon R}(0)} |\nabla u_m|^N \, dx + NC e^{-1} ||\nabla u_m||_{L^N}^{N-1} ||u_m||_{L^N} \right) + o(1)
\]

\[
\leq G_{\text{rad}}^{-1} \int_{B_{\varepsilon R}(0)} |\nabla u_m|^N \, dx + o(1) \leq G_{\text{rad}}^{-1} \int_{B_{\varepsilon R}(0)} |\nabla u_m|^N \, dx + o(1).
\]

Therefore we obtain the claim (14). The proof of Lemma 3 is now complete. \( \square \)

4. In the case of general bounded domain

We extend Theorem 1 and Proposition 2 to bounded domains. Throughout this section, we assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain, \( 0 \in \Omega \), and \( \beta \) and \( q \) satisfy (6). Set \( R = \sup_{x \in \Omega} |x| \).

First we extend Proposition 2 to general bounded domains. If there exists \( \Gamma \subset \partial \Omega \cap \partial B_R(0) \) such that \( \Gamma \) is open in \( \partial B_R(0) \), then we can obtain the same result as Proposition 2 as follows.

**Proposition 4.** Assume that there exists \( \Gamma \subset \partial \Omega \cap \partial B_R(0) \) such that \( \Gamma \) is open in \( \partial B_R(0) \). Then \( G_1 > 0 \) if and only if \( \beta = q = N \).
Proof of Proposition 4. First we show that $G_1 = 0$ if $\beta > \frac{N-1}{N}q + 1$. Set $x = r\omega (r = |x|, \omega \in S^{N-1})$ for $x \in \mathbb{R}^N$. From the assumption, we can take $\delta > 0$ and $\Gamma \subset \Gamma$ such that $\bar{\Gamma}$ is open in $\partial B_R(0)$ and

$$\left\{ (r, \omega) \in [0, R) \times S^{N-1} \left| R - 2\delta \leq r \leq R, \omega \in \frac{1}{R} \bar{\Gamma} \right. \right\} \subset \Omega.$$  

Let $0 \neq \psi \in C_c^\infty (\frac{1}{R} \bar{\Gamma})$ and $\phi \in C_c^\infty ([0, \infty))$ satisfy $\phi \equiv 1$ on $[R - \delta, R]$ and $\phi \equiv 0$ on $[0, R - 2\delta]$. Set $u_s(x) = (\log \frac{R}{r})^s \psi(\omega)\phi(r)$. Then we have

\[
\int_{\Omega} |\nabla u_s|^N dx = \int_{S^{N-1}} \int_0^R \left| \frac{\partial u_s}{\partial r} + \frac{1}{r} \nabla_{S^{N-1}} u_s \right|^N r^{N-1} dr dS_{\omega} 
\leq 2^{N-1} \int_{S^{N-1}} \int_0^R \left| \frac{\partial u_s}{\partial r} \right|^N r^{N-1} + |\nabla_{S^{N-1}} u_s|^N r^{-1} dr dS_{\omega} 
\leq s^N C \int_{R-\delta}^R \left( \log \frac{R}{r} \right) \frac{R^{(s-1)N}}{r} dr + C \int_{R-\delta}^R \left( \log \frac{R}{r} \right) s^{N} \frac{dr}{r} + C 
\leq s^N C \int_{0}^{\log \frac{R}{\bar{r}}} t^{(s-1)N} dt + C < \infty \quad \text{if } s > \frac{N-1}{N}.
\]

Thus $u_s \in W^{1,N}_0 (\Omega)$ for all $s > \frac{N-1}{N}$. However, direct calculation shows that

\[
\int_{\Omega} \frac{|u_s|^q}{|x|^N (\log \frac{R}{|x|})^\beta} dx \geq C \int_{R-\delta}^R \left( \log \frac{R}{r} \right)^{-s \beta} \frac{dr}{r} = C \int_0^{\log \frac{R}{\bar{r}}} t^{s \beta} dt
\]

which implies that

\[
\int_{\Omega} \frac{|u_s|^q}{|x|^N (\log \frac{R}{|x|})^\beta} dx = \infty
\]

for $s$ close to $\frac{N-1}{N}$ since $\beta > \frac{N-1}{N}q + 1$. Therefore we see that

\[G_1 = 0 \quad \text{if } \beta > \frac{N-1}{N}q + 1. \quad (15)\]

Next we show that $G_1 = 0$ if $\beta > N$. Set $x_\epsilon = (R - 2\epsilon) \frac{y}{R}$ for $y \in \partial B_R(0)$. Note that $B_{\epsilon}(x_\epsilon) \subset \Omega$ for small $\epsilon > 0$ and some $y \in \Gamma$. Then we define $u_\epsilon$ as follows:

\[
u_\epsilon (x) = \begin{cases} v \left( \frac{|x-x_\epsilon|}{\epsilon} \right) & \text{if } x \in B_{\epsilon}(x_\epsilon), \\ 0 & \text{if } x \in \Omega \setminus B_{\epsilon}(x_\epsilon), \end{cases} \quad \text{where } v(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2(1-t) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}
\]
Since \( \log t \leq t - 1 \) for \( t \geq 1 \), we obtain

\[
\int_{\Omega} |\nabla u_\varepsilon(x)|^N \, dx = \int_{B_1(0)} |\nabla v(|z|)|^N \, dz < \infty,
\]

\[
\int_{\Omega} \left| \frac{u_\varepsilon(x)}{|x|^N} \left( \log \frac{|x|}{\varepsilon} \right) \varepsilon \right| \, dx \geq C \int_{B_\varepsilon(x_\varepsilon)} \frac{|u_\varepsilon(x)|^q}{(R - |x|)^\rho} \, dx \geq \frac{C}{(3\varepsilon)^\rho} \int_{B_\varepsilon(x_\varepsilon)} \, dx = C \varepsilon^{N-\beta} \to \infty
\]

as \( \varepsilon \to 0 \) when \( \beta > N \). Hence we see that

\[
G_1 = 0 \quad \text{if} \quad \beta > N. \quad (16)
\]

From (15), (16), and (6), we see that \( G_1 > 0 \) if and only if \( q = \beta = N \). \( \square \)

If there does not exist \( \Gamma \) in Proposition 4, then we can expect that the relation between \( q, \beta \) and the positivity of \( G_1 \) depends on the geometry of the boundary \( \partial \Omega \). In order to see it, we consider special cuspidal domains which satisfy the following conditions:

\( (\Omega_1) \): \( \partial \Omega \cap \partial B_R(0) = \{(0, \cdots, 0, -R)\} \).

\( (\Omega_2) \): \( \partial \Omega \) is represented by a graph \( \phi : \mathbb{R}^{N-1} \rightarrow [-R, \infty) \) near the point \( (0, \cdots, 0, -R) \).

Namely, for small \( \delta > 0 \) the following holds true:

\[
Q_\delta := \Omega \cap (\mathbb{R}^{N-1} \times [-R, -R + \delta]) = \{(x', x_N) \in \mathbb{R}^{N-1} \times [-R, -R + \delta] \mid x_N > \phi(x')\}.
\]

\( (\Omega_3) \): there exist \( C_1, C_2 > 0 \) and \( \alpha \in (0, 1) \) such that

\[
C_1 |x'|^\alpha \leq \phi(x') + R \leq C_2 |x'|^\alpha \quad \text{for any} \quad x' \in \mathbb{R}^{N-1}.
\]

\( \alpha \) in \( (\Omega_3) \) expresses the sharpness of the cusp at the point \( (0, \cdots, 0, -R) \). Then we can obtain the following theorem concerned with the positivity and the attainability of \( G_1 \).

**Theorem 2.** Assume that \( \Omega \) satisfies the assumptions \( (\Omega_1) - (\Omega_3) \). Then there exists \( \beta^* = \beta^*(\alpha, q) \in \left[ \frac{N-1}{\alpha} + 1, \frac{N}{\alpha} \right] \) such that \( G_1 = 0 \) for \( \beta > \beta^* \) and \( G_1 > 0 \) for \( \beta < \beta^* \). Furthermore \( G_1 \) is attained for \( \beta \in \left( \frac{N-1}{N} q + 1, \beta^* \right) \).

**Remark 2.** When \( \beta = q = N \) and \( 0 \in \Omega \), \( G_1 \) is not attained for any bounded domain. However, when \( 0 \notin \Omega \), the attainability of \( G_1 \) depends on a geometry of the boundary \( \partial \Omega \). Very recently, Byeon and Takahashi investigate the attainability of \( G_1 \) on cuspidal domains in their article [7] when \( \beta = q = N \).
Proof of Theorem 2. First we shall show that $G_1 = 0$ if $\beta > \frac{N}{\alpha}$. From $(\Omega_3)$, we can observe that $B_{A_1}^\perp(x_{\varepsilon}) \subset \Omega$ for small $\varepsilon > 0$ and small $A > 0$, where $x_{\varepsilon} = (0, \cdots, 0, -R + 2\varepsilon)$. Then we define $w_\varepsilon$ as follows:

$$w_\varepsilon(x) = \begin{cases} v \left( \frac{|x-x_{\varepsilon}|}{A_1} \right) & \text{if } x \in B_{A_1}^\perp(x_{\varepsilon}), \\ 0 & \text{if } x \in \Omega \setminus B_{A_1}^\perp(x_{\varepsilon}), \end{cases}$$

where $v$ is the same function in the proof of Proposition 4. In the same way as the proof of Proposition 4, we have

$$\int_{\Omega} |\nabla_x w_\varepsilon(x)| N \, dx < \infty, \quad \int_{\Omega} |w_\varepsilon(x)|^q |x|^N \left( \log \frac{R}{|x|} \right)^\beta \, dx \geq C \varepsilon^{\frac{N}{\alpha} - \beta} \to \infty$$
as $\varepsilon \to 0$ if $\beta > \frac{N}{\alpha}$. Therefore we have

$$G_1 = 0 \quad \text{at least for } \beta > \frac{N}{\alpha}.$$  \hfill (17)

Next we shall show that $G_1 > 0$ if $\beta < \frac{N-1}{\alpha} + 1$. For $u \in W_0^{1,N}(\Omega)$, we divide the domain $\Omega$ into three parts as follows:

$$\int_{\Omega} |u(x)|^q \, dx = \int_{\Omega \cap B_{\frac{R}{2}}(0)} + \int_{\Omega \setminus (B_{\frac{R}{2}}(0) \cup Q_0)} + \int_{Q_0} =: I_1 + I_2 + I_3. \hfill (18)$$

From Theorem A, we obtain

$$I_1 \leq C \left( \int_{\Omega} |\nabla u|^N \, dx \right)^{\frac{q}{N}}. \hfill (19)$$

Since the potential function $|x|^{-N}(\log \frac{R}{|x|})^{-\beta}$ does not have any singularity in $\Omega \setminus \left( B_{\frac{R}{2}} \cup Q_0 \right)$, the Sobolev inequality yields that

$$I_2 \leq C \int_{\Omega} |u|^q \, dx \leq C \left( \int_{\Omega} |\nabla u|^N \, dx \right)^{\frac{q}{N}}. \hfill (20)$$

Finally, we shall derive a estimate of $I_3$ from above. Since $\log t \geq \frac{1}{t}(t-1) (1 \leq t \leq 2)$, we obtain

$$I_3 \leq C \int_{Q_0} \frac{|u(x)|^q}{(R - |x|)^\beta} \, dx \leq C \int_{z_N=0}^{z_N=\delta} \int_{z_N \geq C_1|z'|^\mu} \frac{|\tilde{u}(z', z_N)|^q}{|z|^\beta} \, dz,$$  \hfill (21)
where $u(x) = \tilde{u}(z) (z = x + (0, \cdots, 0, R))$. If $\beta < \frac{N-1}{\alpha} + 1$, then there exists $\varepsilon > 0$ and $p > \frac{N}{N-\varepsilon}$ such that $(\beta - \varepsilon)p < \frac{N-1}{\alpha} + 1$. By using the Hölder inequality and the Sobolev inequality, we have

$$
\int_{z_N=0}^{\infty} \int_{z_N \geq C_1 |z|^p} |\tilde{u}(z', z_N)|^q \frac{dz}{|z|^p} = \int \int \frac{|\tilde{u}|^q}{|z|^p} |\tilde{u}|^{q-\varepsilon} |z|^{\beta-\varepsilon} dz
$$

$$
\leq \left( \int \int \frac{|\tilde{u}|^q}{|z|^N} dz \right)^{\frac{1}{q}} \left( \int \int |\nabla \tilde{u}|^N dz \right)^{1-q} \left( \int \int z_N^{-(\beta-\varepsilon)p} dz \right)^{\frac{1}{p}}
$$

$$
\leq C \left( \int \int \frac{|\tilde{u}|^q}{|z|^N} dz \right)^{\frac{1}{q}} \left( \int \int |\nabla \tilde{u}|^N dz \right)^{1-q} \left( \int_{z_N=0}^{\infty} z_N^{-(\beta-\varepsilon)p} dz \right)^{\frac{1}{p}}.
$$

Since $\frac{N-1}{\alpha} - (\beta - \varepsilon)p > -1$, $\int_0^\delta z_N^{-(\beta-\varepsilon)p} dz_N < \infty$. Furthermore, applying the Hardy inequality on the half space $\mathbb{R}^N_+ := \{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} | x_N > 0 \}$:

$$
\left(\frac{r-1}{r}\right)^r \int_{\mathbb{R}^N_+} \frac{|u|^r}{|x_N|^r} dx \leq \int_{\mathbb{R}^N_+} |\nabla u|^r dx \quad (1 \leq r < \infty)
$$

yields that

$$
\int_{z_N=0}^{\infty} \int_{z_N \geq C_1 |z|^\beta} |\tilde{u}(z', z_N)|^q \frac{dz}{|z|^\beta} \leq C \left( \int_{\Omega + (0, \cdots, 0, R)} |\nabla \tilde{u}|^N dz \right)^{\frac{q}{p}}.
$$

(22)

By (21) and (22), we have

$$
I_3 \leq C \left( \int_{\Omega} |\nabla u|^N dx \right)^{\frac{q}{p}}.
$$

(23)

Therefore, from (18), (19), (20), and (23), for all $u \in W_{0}^{1,N}(\Omega)$,

$$
C \left( \int_{\Omega} \frac{|u(x)|^q}{|x|^N (\log \frac{R}{|x|})^\beta} dx \right)^{\frac{N}{q}} \leq \int_{\Omega} |\nabla u|^N dx.
$$

15
Hence
\[ G_1 > 0 \quad \text{at least for} \quad \beta < \frac{N-1}{\alpha} + 1. \tag{24} \]

From (17) and (24), there exists \( \beta^* \in \left[ \frac{N-1}{N}q+1, \frac{N}{\alpha} \right] \) such that \( G_1 > 0 \) for \( \beta < \beta^* \) and \( G_1 = 0 \) for \( \beta > \beta^* \).

Lastly we shall show that \( G_1 \) is attained for \( \beta \in \left( \frac{N-1}{N}q+1, \beta^* \right) \). In order to show it, we show that the continuous embedding \( W_{0}^{1,N}(\Omega) \hookrightarrow L^{q}(\Omega; f_i(x)dx) \) is compact if \( \frac{N-1}{N}q+1 < \beta < \beta^* \). Let \( (u_m)_{m=1}^{\infty} \subset W_{0}^{1,N}(\Omega) \) be a bounded sequence. Then there exists a subsequence \( (u_{m_k})_{k=1}^{\infty} \) such that
\[ u_{m_k} \rightharpoonup u \quad \text{in} \quad W_{0}^{1,N}(\Omega), \quad u_{m_k} \rightarrow u \quad \text{in} \quad L^{r}(\Omega) \quad \text{for all} \quad 1 \leq r < \infty. \tag{25} \]

We divide the domain into two parts as follows:
\[ \int_{\Omega} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{R}{|x|})^\beta} dx = \int_{\Omega \setminus Q_\delta} + \int_{Q_\delta} =: J_1(u_{m_k} - u) + J_2(u_{m_k} - u). \tag{26} \]

Since \( \log \frac{R}{|x|} \geq C \log \frac{aR}{|x|} \) for any \( x \in \Omega \setminus Q_\delta \) for some \( a > 1 \) and \( C > 0 \), it holds that
\[ J_1(u_{m_k} - u) \leq C \int_{\Omega \setminus Q_\delta} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \leq C \int_{\Omega \setminus Q_\delta} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx. \]

Note that the continuous embedding \( W_{0}^{1,N}(\Omega) \hookrightarrow L^{q}(\Omega; f_i(x)dx) \) is compact for \( \beta > \frac{N-1}{N}q+1 \) from Lemma 1, we obtain
\[ J_1(u_{m_k} - u) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \tag{27} \]

On the other hand, for any \( \epsilon > 0 \), we take \( \gamma > 0 \) which satisfies \( \beta < \gamma < \beta^* \) and \( (\log \frac{R}{|x|})^{\gamma-\beta} < \epsilon \) for \( x \in Q_\delta \) (If necessary, we take small \( \delta > 0 \) again.). Then we have
\[ J_2(u_{m_k} - u) \leq \epsilon \int_{Q_\delta} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{R}{|x|})^\gamma} dx \leq C \epsilon \left( \int_{\Omega} |\nabla(u_{m_k} - u)|^N dx \right)^{\frac{q}{N}} \leq C \epsilon. \tag{28} \]

From (26), (27), and (28), we have
\[ \int_{\Omega} \frac{|u_{m_k} - u|^q}{|x|^N (\log \frac{R}{|x|})^\beta} dx \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]
Therefore the continuous embedding $W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega; f\alpha(x)dx)$ is compact if $\frac{N-1}{N} q + 1 < \beta < \beta^*$. In conclusion, we have showed that $G_1$ is attained if $\frac{N-1}{N} q + 1 < \beta < \beta^*$. □

Next we extend Theorem 1 to general bounded domains.

**Theorem 3.** Let $a > 1$. Then the followings hold.
(i) If $\beta > \frac{N-1}{N} q + 1$, then $G_a$ is attained for any bounded domains $\Omega$.
(ii) If $\beta = \frac{N-1}{N} q + 1$, $q > N$, and $a \geq e^\frac{q}{N}$, then $G_a = G_{\text{rad}}$ and $G_a$ is not attained for any bounded domain $\Omega$.
(iii) If $\beta > \frac{N-1}{N} q + 1$, $q > N$, and $\Omega$ satisfies either $(\Omega_4)$ or $(\Omega_5)$, where

$$(\Omega_4) : \partial \Omega \text{satisfies the Lipschitz condition at some point } x_0 \in \Omega \cap B_R(0),$$

$$(\Omega_5) : \Omega \text{satisfies } (\Omega_1) - (\Omega_3) \text{ and } \alpha \in (\Omega_3) \text{ is greater than } \frac{N}{\beta},$$

then there exists $a_* \in (1, e^\frac{q}{N}]$ such that $G_a$ is attained for $a \in (1, a_*)$ and $G_a$ is not attained for $a > a_*$.

In order to show Theorem 3(iii), we need the continuity of $G_a$ with respect to $a$ at $a = 1$. Under the assumptions $(\Omega_4), (\Omega_5)$, we can show the continuity of $G_a$ at $a = 1$ as follows.

**Lemma 4.** Let $\beta > N$. If $\Omega$ satisfies either $(\Omega_4)$ or $(\Omega_5)$, then $G_1 = \lim_{a \searrow 1} G_a = 0$.

Lemma 4 follows from the following proposition.

**Proposition 5.** Let $a > 1$. If $\Omega$ satisfies either $(\Omega_4)$ or $(\Omega_5)$, then there exists $C > 0$ such that for a close to 1, $G_a \leq C \alpha N - \frac{\alpha}{2} (a - 1) - \frac{N}{\beta}$, where $\alpha$ is regarded as 1 if $\Omega$ satisfies $(\Omega_4)$.

Proof of Proposition 5. Let $x_a = R(2-a)^\frac{a}{[\alpha]}$ and $\phi \in C_c^\infty(B_1(0))$. Here $x_0$ is regarded as $(0, \cdots, 0, -R)$ if $\Omega$ satisfies $(\Omega_5)$. Then $B_{c(a-1)}(x_a) \subset \Omega$ for $a$ close to 1 and for sufficiently small $c > 0$. Set $\phi_a(x) = \phi \left( \frac{x-x_a}{c(a-1)^\frac{a}{2}} \right)$. Since $\log \frac{1}{t} \leq \frac{1-t}{2} \log t$ for $t$ close to 1, we have the followings for $a$ close to 1.

$$\int_{\Omega} |\nabla \phi_a|^N dx = \int_{\Omega} |\nabla \phi|^N d\nu < \infty,$$

$$\int_{\Omega} \frac{|\phi_a|^q}{|x|^N} d\nu \geq c^N (a-1)^\frac{N}{2} ||\phi||_{L^q}^q \left( \log \frac{aR}{R(2-a)-c(a-1)^\frac{a}{2}} \right)^{-\beta} \geq C(a-1)^\frac{N}{\beta} - \frac{\alpha}{2} (a-1) - \frac{N}{\beta}. □$$
Proof of Theorem 3. (i) We can check that Lemma 1 holds true for any bounded domains $\Omega$. Therefore (i) follows from the compactness of the embedding $W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$. We omit the proof.

(ii) Note that $W^{1,N}_{0,rad}(B_\varepsilon(0)) \subset W^{1,N}_0(\Omega) \subset W^{1,N}_{0,rad}(B_R(0))$ for small $\varepsilon > 0$ by zero extension. Then we have

$$
\inf_{u \in W^{1,N}_{0,rad}(B_\varepsilon(0))} \frac{\int_\Omega |\nabla u|^N \, dx}{\left( \int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} \, dx \right)^{\frac{N}{q}}} \geq G_a \geq \inf_{u \in W^{1,N}_0(B_R(0))} \frac{\int_\Omega |\nabla u|^N \, dx}{\left( \int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} \, dx \right)^{\frac{N}{q}}}
$$

where the last equality comes from $a \geq e^{\frac{\beta}{N}}$. From the proof of Proposition 5 (ii), we can observe that $G_{rad}$ does not vary even if we replace $W^{1,N}_{0,rad}(B_R(0))$ to $W^{1,N}_{0,rad}(B_\varepsilon(0))$ for any small $\varepsilon > 0$. Thus the right hand side and the left hand side of (29) take same value, that is $G_{rad}$. Therefore we have $G_a = G_{rad}$. Furthermore if we assume that $G_a$ is attained by $u \in W^{1,N}_0(\Omega)$, then $u \in W^{1,N}_0(B_R(0))$ is also a minimizer on a ball. This contradicts Theorem 1 (ii) in §2. Hence $G_a$ is not attained for any bounded domains $\Omega$.

(iii) Note that $G_a$ is continuous with respect to $a \in (1, \infty)$, and is monotone increasing with respect to $a \in [1, \infty)$ for any bounded domains. From Lemma 4 and Theorem 3 (ii), we can show that there exists $a_* \in (1, e^{\frac{\beta}{N}})$ such that $G_a < G_{rad}$ for $a \in (1, a_*)$ and $G_a = G_{rad}$ for $a > a_*$ in the same way as the proof of Theorem 1 (ii). The remaining parts of the proof are similar to the proof of Theorem 1 (ii).

\hfill \Box

5. Symmetry breaking

In this section, we consider radially symmetry of the minimizers of $G_a$ when $\Omega = B_R(0)$. We can show that any minimizer of $G_a$ has axial symmetry by using spherical symmetric rearrangement, see [23]. Namely, for any minimizer $u_\beta$ of $G_a$ there exists some $\xi \in \mathbb{S}^{N-1}$ such that the restriction of $u_\beta$ to any sphere $\partial B_r(0)$ is symmetric decreasing with respect to the distance to $r\xi$. See also [32]. The last result is as follows.

Theorem 4. Let $\beta > \frac{N-1}{N}q + 1$, $a > 1$, and $u_\beta$ be a minimizer of $G_a$ in Theorem 7 (i). Then the followings hold true.
(i) For fixed \( q > N \), there exists \( \beta_* \) such that \( u_\beta \) is non-radial for \( \beta > \beta_* \).
(ii) \( u_\beta \) is radial for any \( \beta \) and \( q \leq N \).

In order to show Theorem 4 (i), we need two lemmas concerning growth orders of \( G_a \) and \( G_{a,\text{rad}} \) with respect to \( \beta \).

**Lemma 5.** For fixed \( q > N \), there exists \( C > 0 \) such that for sufficiently large \( \beta \) the following estimate holds true.

\[
G_a \leq C \beta^{\frac{n^2}{q}} (\log a)^{n/2}.
\]

**Proof of Lemma 5.** Let \( u \in C^\infty_c(B_R(0)) \). Following [33], we consider \( u_\beta(x) := u(\beta(x-x_\beta)) \) for \( x \in B_{\beta^{-1}}(x_\beta) \), where \( x_\beta := (R - \beta^{-1}, 0, \ldots, 0) \in B_R(0) \). Then for sufficiently large \( \beta \) we obtain

\[
\int_{B_{\beta^{-1}}(x_\beta)} |\nabla u_\beta(x)|^N dx = \int_{B_R(0)} |\nabla u(y)|^N dy,
\]

\[
\int_{B_{\beta^{-1}}(x_\beta)} \frac{|u_\beta(x)|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \geq (R - 2\beta^{-1})^{-N} \left( \log \frac{aR}{R - 2\beta^{-1}} \right)^{-\beta} \int_{B_R(0)} |u(y)|^q dy.
\]

We set \( f(\beta) := (R - 2\beta^{-1})^{-N} (\log \frac{aR}{R - 2\beta^{-1}})^{-\beta} \). Since \( \log \frac{1}{1-x} \leq 2x \) for all \( x \in [0, \frac{1}{2}] \), for large \( \beta \) we have

\[
f(\beta) \geq \frac{1}{2} \left( \log a + \log \frac{1}{1 - 2\beta^{-1}} \right)^{-\beta} \geq \frac{1}{2} (\log a + 4\beta^{-1} R^{-1})^{-\beta} = \frac{1}{2} (\log a)^{-\beta} \left( 1 + \frac{4}{\beta R \log a} \right)^{-\beta}
\]

which yields that

\[
f(\beta) \geq C (\log a)^{-\beta} \quad \text{for large } \beta.
\]

From (30), (31), and (32), we obtain

\[
G_a \leq \frac{\int_{B_R(0)} |\nabla u_\beta(x)|^N dx}{\left( \int_{B_R(0)} \frac{|u_\beta(x)|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{q}{N}}} \leq C \beta^{n^2} (\log a)^{n/2}.
\]

\[\Box\]
Lemma 6. For fixed $q > N$, there exists $C > 0$ such that for sufficiently large $\beta$ the following estimate holds true.

$$G_{a, rad} \geq C \beta^{N-1+N \over q} (\log a)^{Nq \over q} (N-1)^{N+1 \over q}.$$

Proof of Lemma 6. For $u \in W^{1,N}_{0,rad}(B_R(0))$ we define $v \in W^{1,N}_{0,rad}(B_R(0))$ as follows:

$$v(s) = u(r),$$

where $$(\log a)^{A^{-1}} \log a \leq \frac{aR}{s} = \left(\log \frac{aR}{r}\right)^A$$ and $A = \frac{N(\beta - 1)}{(N - 1)q}$.

Direct calculation shows that

$$\int_{B_R(0)} \frac{|u|^q}{|x|^N (\log aR)^{1 \over q}} dX = \omega_{N-1} \int_0^R |u(r)|^q (\log aR)^{-\beta} r \frac{dr}{r}$$

$$= \omega_{N-1} A^{-1} (\log a)^{A^{-1}(1-\beta)} \int_0^R \frac{|v(s)|^q}{s \log aR} \left(\frac{aR}{s}\right)^{A^{-1}(1-\beta)} ds$$

$$= A^{-1} (\log a)^{A^{-1}(1-\beta)} \int_{B_R(0)} \frac{|v|^q}{|y|^N (\log aR)^{Nq \over q} + 1} dy.$$

In the same way as above, we have

$$\int_{B_R(0)} |\nabla u|^N dX = A^{N-1} (\log a)^{-A^{-1} \over q} \int_{B_R(0)} |\nabla v|^N \left(\log \frac{aR}{|y|}\right)^{A^{-1}} dy$$

$$\geq A^{N-1} \int_{B_R(0)} |\nabla v|^N dy.$$

Therefore we have

$$\frac{\int_{B_R(0)} |\nabla u|^N dX}{\left(\int_{B_R(0)} \frac{|u|^q}{|x|^N (\log aR)^{1 \over q}} dX\right)^{\frac{N}{q}}} \geq A^{N-1} (\log a)^{Nq \over q} (N-1)^{N+1 \over q} \inf_v \frac{\int_{B_R(0)} |\nabla v|^N dy}{\left(\int_{B_R(0)} \frac{|v|^q}{|y|^N (\log aR)^{Nq \over q} + 1} dy\right)^{\frac{N}{q}}}$$

which yields that

$$G_{a, rad} \geq \left(\frac{N(\beta - 1)}{(N - 1)q}\right)^{N-1+N \over q} (\log a)^{Nq \over q} (N-1)^{N+1 \over q} \inf_v \frac{\int_{B_R(0)} |\nabla v|^N dy}{\left(\int_{B_R(0)} \frac{|v|^q}{|y|^N (\log aR)^{Nq \over q} + 1} dy\right)^{\frac{N}{q}}}.$$
Therefore, for sufficiently large $\beta$ we have
\[
G_{a, \text{rad}} \geq C \beta^{N-1+\frac{N}{\tau}} (\log a)^{\frac{N\tau}{\tau}-(N-1+\frac{N}{\tau})}.
\]

Finally we shall show Theorem 4.

**Proof of Theorem 4.** (i) It is enough to show that $G_a < G_{a, \text{rad}}$. By Lemma 5 and Lemma 6, for fixed $q > N$ there exists $\beta_*$ such that for $\beta > \beta_*$
\[
G_a \leq C \beta^{N^2} (\log a)^{\frac{N\tau}{\tau}} < C \beta^{N-1+\frac{N}{\tau}} (\log a)^{\frac{N\tau}{\tau}-(N-1+\frac{N}{\tau})} \leq G_{a, \text{rad}},
\]
since $\frac{N^2}{q} < N - 1 + \frac{N}{q}$. Therefore we see that $G_a < G_{a, \text{rad}}$.

(ii) Let $x = r\omega (r = |x|, \omega \in S^{N-1})$ for $x \in B_{R}(0)$. For $u \in W^{1,N}_0(B_{R}(0))$ we consider the following radial function $U$:
\[
U(r) = \left(\omega^{-1}_{N-1} \int_{S^{N-1}} |u(r\omega)|^N dS_{\omega} \right)^{\frac{1}{N}}.
\]

Then we have
\[
U'(r) \leq \left(\omega^{-1}_{N-1} \int_{S^{N-1}} \left| \frac{\partial}{\partial r} u(r\omega) \right|^N dS_{\omega} \right)^{\frac{1}{N}}
\]
which yields that
\[
\int_{B_{R}(0)} |\nabla U|^N dx \leq \int_{B_{R}(0)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \leq \int_{B_{R}(0)} |\nabla u|^N dx.
\]

(33)

On the other hand, we have
\[
\int_{B_{R}(0)} \frac{|U|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx = \omega^{-1}_{N-1} \int_0^R \left(\omega^{-1}_{N-1} \int_{S^{N-1}} |u(r\omega)|^N dS_{\omega} \right)^{\frac{q}{N}} \frac{dr}{r (\log \frac{aR}{|x|})^\beta} \geq \int_0^R \int_{S^{N-1}} |u(r\omega)|^q dS_{\omega} \frac{dr}{r (\log \frac{aR}{|x|})^\beta} = \int_{B_{R}(0)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx
\]

(34)

where the inequality follows from Jensen’s inequality and $q \leq N$. From (33) and (34), we obtain $G_{a, \text{rad}} \leq G_a$. Therefore $G_{a, \text{rad}} = G_a$ for any $q \leq N$ and $\beta$. Moreover we observe that any minimizers of $G_a$ must be radial from the equality condition of (33).
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