Tensor network approach to 2D Yang-Mills theories

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Abstract

We propose a novel tensor network representation for two-dimensional Yang-Mills theories with arbitrary compact gauge groups. In this method, tensor indices are directly given by group elements with no direct use of the character expansion. We apply the tensor renormalization group method to this tensor network for $SU(2)$ and $SU(3)$, and find that the free energy density and the energy density are accurately evaluated. We also show that the singular value decomposition of a tensor has a group theoretic structure and can be associated with the character expansion.
1. Introduction

The tensor network (TN) method [1–5] is an attractive approach for studying many body systems, because it is free from the sign problem in the first place,\footnote{Recently, significant progress has been made also in the Monte Carlo (MC) approach to the sign problem [26–43], giving rise to a hope that MC simulations can be performed at a reasonable computational cost. Two approaches (TN and MC) may play complementary roles in the future.} and has a potential to precisely investigate critical phenomena in the large volume limit. In field theory, the tensor renormalization group (TRG) method [3] and its variations [6–8] are widely used to study various models such as the Schwinger model [9–12], the Gross-Neveu and NJL models [13,14], scalar field theories [15–18], the Yang-Mills and gauge-Higgs models [19,20], the Wess-Zumino model [21], and other related models [22–24].

For gauge groups $U(1)$ [9–11,22,25] and $SU(2)$ [19,20], the character expansion was employed to represent the partition function with a tensor network. However, since the character expansion becomes a demanding task for higher-rank gauge groups, it remains as a difficult issue to apply the TN method to $SU(N)$ gauge theory for $N \geq 3$ including QCD.

In this paper, we propose a novel method to create a tensor network for two-dimensional
Yang-Mills theory with no direct use of the character expansion. The Haar measure is discretized, and the group integration is replaced by a summation over $K$ randomly generated configurations. Then, the plaquette is regarded as a rank-4 tensor whose index runs from 1 to $K$, and the set of plaquettes constitutes a tensor network. We test our method for $SU(2)$ and $SU(3)$ gauge groups, and find that the free energy density and the energy density agree very well with exact results. We also clarify the mathematical structure behind our method.

This paper is organized as follows. In Sec. 2, we introduce our tensor network representation for two-dimensional Yang-Mills theories with arbitrary compact gauge groups $G$, and discuss its relation with the character expansion. In Sec. 3 we test our method for $G = SU(2)$ and $G = SU(3)$. Section 4 is devoted to summary and discussion. Appendices provide some useful formulas in group theory.

2. Tensor network representations for 2D Yang-Mills theories

In this section, we introduce a new tensor network representation for two-dimensional Yang-Mills theories, and discuss its relation with the character expansion. We exclusively consider pure Yang-Mills theory for simplicity. It is straightforward to extend our method to systems with interacting matter fields.

2.1. Method

We consider the Yang-Mills theory with a compact gauge group $G$ on an infinite lattice $\Gamma \equiv \{ n = (n_1, n_2) \mid n_\mu \in \mathbb{Z} (\mu = 1, 2) \}$. The lattice spacing $a$ is set to $a = 1$ unless otherwise noted, and $\hat{\mu}$ is the unit vector in the $\mu$ direction.

Let $U_\mu(n)$ be the $G$-valued link field on links $(n, n + \hat{\mu})$. The lattice action (the Wilson action) is given by

$$S = \frac{\beta}{N} \sum_{n \in \Gamma} \text{Re} \text{tr} [1 - U_P(n)],$$

(2.1)

where $U_P(n)$ is the plaquette field,

$$U_P(n) = U_1(n) U_2(n + \hat{1}) U_1^\dagger(n + \hat{2}) U_2^\dagger(n).$$

(2.2)

The partition function is defined as $Z = \int DU e^{-S}$, where $DU \equiv \prod_{n \in \Gamma} dU_1(n) dU_2(n)$ with $dU$ the Haar measure of $G$. Note that the partition function $Z$ can be written in the form

\[ Z \approx \exp(i a^2 F_{12}(n)) \]

in the naive continuum limit.

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2We obtain the usual continuum action for $\beta = 2N/(ga)^2$ with $U_P(n) \approx \exp(i a^2 F_{12}(n))$ in the naive continuum limit.
of a tensor network with indices continuously taking values in $G$,
\[
Z = \mathcal{T} \prod_{n \in \Gamma} T_{g(n)h(n)g'(n)h'(n)}, \tag{2.3}
\]
where
\[
T_{g_1g_2g_3g_4} = e^{-(\beta/N) \text{Re tr} (1-g_1g_2g_3g_4^\dagger)} \tag{2.4}
\]
and $\mathcal{T}$ stands for the group integrations for $g(n), h(n) \in G (n \in \Gamma)$ under a proper identification of indices.

We now discretize the Haar measure $dU$ to represent $Z$ as a tensor network with indices in a finite range:
\[
\int dU f(U) \approx \frac{1}{K} \sum_{i=1}^{K} f(U_i), \tag{2.5}
\]
where $\hat{G} = \{U_1, U_2, \ldots, U_K\}$ consists of random points uniformly chosen from the group manifold. Applying Eq. (2.5) to the Haar measures in $DU$ leads to
\[
Z \approx \text{Tr} \prod_{n \in \Gamma} T_{i_nj_ni'_nj'_n}, \tag{2.6}
\]
where
\[
T_{ijkl} = \frac{1}{K^2} e^{-(\beta/N) \text{Re tr} (1-U_iU_jU_kU_l^\dagger)} \tag{2.7}
\]
and Tr stands for the summation over $i_n, j_n = 1, 2, \cdots, K$ for all $n \in \Gamma$ under the same identification of indices as above. As shown in Fig. 1, the tensor is assigned to each plaquette and has four indices corresponding to four links of the plaquette.

Since our method is based on the discrete approximation with finite $K$, we check the convergence of the r.h.s. of Eq. (2.6) for large $K$ in actual numerical computations.

In the tensor network (2.6), a single set $\hat{G}$ is commonly used to discretize all the $U_\mu(n)$-integrations. Actually, we can use a different set for each link. For example, tensors can be decomposed in different ways for even and odd sites \footnote{We make the identifications $g'(n) = g(n + \hat{2})$ and $h'(n) = h(n - \hat{1})$.}, and we can use four different sets $\hat{G}_1, \hat{G}_2, \hat{G}_3, \hat{G}_4$ to discretize the integrations at four links, $U_i, U_j, U_k, U_l$, in Fig. 1. We then have
\[
Z \approx \text{Tr} \prod_{n \in \Gamma_e} T_{i_nj_ni'_nj'_n}^{e} \cdot \prod_{m \in \Gamma_o} T_{i_mj_mi'_mj'_m}^{o}, \tag{2.8}
\]
Figure 1: Two-dimensional square lattice. (a) A plaquette variable consisting of $U_i, U_j, U_k, U_l$. (b) The corresponding tensor (2.7) assigned to the center of the plaquette.

with

\[ T_{ijkl}^e \equiv \frac{1}{K^2} e^{-(\beta/N) \text{Re} \, \text{Tr}(1-U_j^{(1)} U_k^{(3)} U_l^{(4)})}, \]  
\[ T_{ijkl}^o \equiv \frac{1}{K^2} e^{-(\beta/N) \text{Re} \, \text{Tr}(1-U_i^{(3)} U_j^{(4)} U_k^{(1)} U_l^{(2)})}, \]

where $U_i^{(a)} \in \hat{G}_a$ ($a = 1, 2, 3, 4$) and $\Gamma_{e/o}$ are the set of even and odd sites, respectively. The introduction of four different sets significantly improve the precision of the results compared to a single set as presented in section 3.

Once the tensor network is obtained, any TRG method can be applied straightforwardly. In the Levin-Nave TRG, the singular value decomposition (SVD) is employed to decompose the tensors. In general, the SVD of an $n \times n$ matrix $M_{ij}$ is given by

\[ M_{ij} = \sum_{a=1}^{n} \sigma_a U_{ia} V_{ja}^*, \]

where $\sigma_a$ are singular values sorted as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ and $U, V$ are unitary matrices.

In our case, regarding $T_{ijkl}^e$ (resp. $T_{ijkl}^o$) as a matrix with the column $ij$ (resp. $jk$) and the row $kl$ (resp. $li$), we have

\[ T_{ijkl}^e = \sum_{A=1}^{K^2} \sigma_A^e U_{ij,A}^e V_{kl,A}^{e*}, \]
\[ T_{ijkl}^o = \sum_{A=1}^{K^2} \sigma_A^o U_{jk,A}^o V_{li,A}^{o*}. \]

Figure 2 shows these decompositions. We again arrive at the tensor network of two-dimensional square lattice by defining the renormalized tensor $T^{(1)}$ with bond dimension $D$ as

\[ T^{(1)}_{A_1A_2A_3A_4} = \sqrt{\sigma_{A_1}^e \sigma_{A_2}^e \sigma_{A_3}^o \sigma_{A_4}^o} \sum_{i,j,k,l=1}^{D} U_{ij,A_1}^e V_{jk,A_2}^o V_{kl,A_3}^e U_{li,A_4}^o, \]

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The tensor network is repeatedly renormalized in this way. Since the bond dimension of the initial tensors [Eqs. (2.9) and (2.10)] is \( K \), the cost of the first SVD scales with \( O(K^6) \). Once the tensors are renormalized, the bond dimension changes to \( D \). The cost of the subsequent iterations then scales with \( O(D^6) \).

### 2.2. Relation with the character expansion

To understand the group theoretic structure of the SVD in the previous subsection, we consider the limit \( K \to \infty \), i.e., the case where the tensor indices continuously take all the values in \( G \). See appendix A for a mathematical material necessary for the argument below.

Let \( R \) be an irreducible unitary representation of \( G \) with dimension \( d_R \), and \( D_R(U) = (D^R_{rs}(U)) \ (r, s = 1, 2, \ldots, d_R) \) the representation matrix of \( U \). Denoting the character of \( R \) by \( \chi_R(U) \), the function \( e^{-\beta/N \text{Re} \text{tr}(1 - U)} \) can be expanded as

\[
e^{-\beta/N \text{Re} \text{tr}(1 - U)} = \sum_R d_R \lambda_R(\beta) \chi_R(U).
\]  

(2.15)

Here and hereafter, \( \sum_R \) stands for the summation over the irreducible representations \( R \). The coefficients \( \lambda_R(\beta) \) are given by

\[
\lambda_R(\beta) = \frac{1}{d_R} \int dU e^{-\beta/N \text{Re} \text{tr}(1 - U)} \chi_R(U^{-1}),
\]  

(2.16)

as can be shown by using Eq. (A.8).

We again consider the infinite dimensional rank-4 tensor \( \mathcal{T}_{g_1 g_2 g_3 g_4} \) [see Eq. (2.1)]. By using Eq. (2.15), this can be written as \( \sum_R d_R \lambda_R \chi_R(g_1 g_2 g_3^{-1} g_4^{-1}) \) and decomposed in two ways:

\[
\mathcal{T}_{g_1 g_2 g_3 g_4} = \sum_{A=(R,r,s)} \mathcal{U}^e_{(g_1, g_2), A} \lambda_R \mathcal{Y}^{e^*}_{(g_3,g_4), A} = \sum_{A=(R,r,s)} \mathcal{U}^o_{(g_2, g_3), A} \lambda_R \mathcal{Y}^{o^*}_{(g_4,g_1), A}
\]  

(2.17)

with

\[
\mathcal{U}^e_{(g_1, g_2), A} = \mathcal{Y}^{e}_{(g_2, g_1), A} = \mathcal{U}^o_{(g_1, g_2^{-1}), A} = \mathcal{Y}^{o}_{(g_2, g_1^{-1}), A} = \sqrt{d_R D^R_{rs}(g_1 g_2)}.
\]  

(2.18)
The Peter-Weyl theorem (see appendix A) states that the matrix \( W_{g,A} \equiv \sqrt{d_R D_R^{rs}}(g) \) is unitary. Thus, together with the inequality \( \lambda_R \geq 0 \) we find that the decompositions (2.17) are actually SVDs. Then, the new tensor \( \mathfrak{T}^{(1)}_{A_1A_2A_3A_4} \) [Eq. (2.14) with \( D = \infty \)] is calculated by following Eqs. (2.12)–(2.14), and is found to be

\[
\mathfrak{T}^{(1)}_{A_1A_2A_3A_4} = \frac{\lambda^2_{R_1}}{d_{R_1}} \delta_{R_1R_2R_3R_4} \delta_{s_1s_2} \delta_{r_2r_3} \delta_{r_3r_4} \delta_{s_4s_1}. \tag{2.19}
\]

Once this expression is obtained, one can perform the TRG iteration (see Appendix C) to obtain

\[
Z = \sum_R \lambda_R(\beta)^V. \tag{2.20}
\]

Recall that the TN representation [Eqs. (2.8)–(2.10)] is a discretization of Eqs. (2.3) and (2.4). Note that the singular values of the tensor \( \mathfrak{T}_{g_1g_2g_3g_4} \) have a degeneracy of \( d^2_R \) for each \( R \) because both \( r \) and \( s \) in \( W_{g,A} = \sqrt{d_R D_R^{rs}}(g) \) take \( d_R \) values. This means that the singular values \( \sigma_A \) of our tensor \( T_{ijkl} \) [Eqs. (2.12) and (2.13)] must have this degeneracy approximately. We actually find this approximate degeneracy in numerical calculations presented in the next section.

3. Numerical results

In this section, we apply our method to the Yang-Mills theory with gauge group \( G = SU(N) \) \((N = 2, 3)\) on a periodic square lattice. We construct the tensor network with four different sets \( \tilde{G}_a (a = 1, 2, 3, 4) \) of \( K \) random link variables [see the discussion after Eqs. (2.8)–(2.10)]. We evaluate the free energy density \( f(\beta) \equiv (1/V) \ln Z(\beta) \) with the Levin-Nave TRG, and the energy density \( e(\beta) \equiv -(\partial/\partial \beta)f(\beta) \) by taking numerical derivatives. Note that estimates have statistical errors in addition to the systematic errors coming from the finiteness of \( K \) and bond dimension \( D \). The statistical errors to be given below are obtained from five independent trials.

\[\text{This can be proved by rewriting Eq. (2.16) to the form}
\[
e^\beta d_R \lambda_R = \int dU e^{(\beta/(2N)) [\tau U + \tau U^{-1}]} \chi_R(U^{-1}) = \sum_{m,n=0}^{\infty} \beta^{m+n} \frac{C_R^{(m,n)}}{m! n!}.
\]

In fact, \( C_R^{(m,n)} \equiv \int dU [\tau U]^m [\tau U^{-1}]^n \chi_R(U^{-1}) \) is the multiplicity of \( R \) in the product representation \( N^\otimes m \otimes \bar{N}^\otimes n \), and thus is a nonnegative integer. \((N \text{ and } \bar{N} \text{ are the fundamental and anti-fundamental representations, respectively.)}\)

\[\text{We use the symbol } \delta_{R_1R_2...R_k} \equiv \delta_{R_1R_2} \delta_{R_2R_3} \cdots \delta_{R_{k-1}R_k}.
\]
3.1. $SU(2)$

We first make a detailed analysis for $SU(2)$.

Figure 3 shows $f(\beta)$ for various volumes $V = L^2$ ($L = 4, 8, 16, 32, 64$) with $\beta/V$ fixed to 0.01. The exact values are indicated by the gray dashed line. Figure 4 shows the relative errors to the exact values for the same calculation. We see that the numerical results agree well with the exact values. We also see that as $V$ (and thus $\beta$) is increased, larger $K$ and $D$ are required to decrease the systematic errors. Figures 5 and 6 show the $K$, $D$ dependences of the free energy density at $V = 64^2$ ($\beta = 40.96$). We confirm that the numerical estimates approach the exact value in the limit $K \to \infty$ and $D \to \infty$.

Having obtained the estimates for several values of $K$, we can make use of extrapolation to obtain a better estimate. Figure 7 shows the $\chi^2$ fit to the obtained data for $D = 60$ with the scaling ansatz $g(K) \equiv \mu + \alpha K^{-p}$. Here, the fitting parameters $\alpha$, $\mu$ and $p$ are determined.
by minimizing the cost function
\[ \chi^2(\mu, \alpha, p) \equiv \sum_{K=20,30,\ldots,90} \frac{[f(\beta; K) - g(K)]^2}{[\delta f(\beta; K)]^2}, \]  
where \( f(\beta; K) \) is the obtained value for each \( K \), and \( \delta f(\beta; K) \) the statistical error. The value of \( \mu \) is then used as the final estimate of \( f(\beta) \).

The results of the fitting are summarized in Table 1. We obtain \( \mu = -5.8045^{+0.0040}_{-0.0029} \) (exact), which agrees well with the exact value \( f_{\text{exact}}(\beta) = -5.8040 \). Since the estimate without extrapolation is given by \( f(\beta; K = 90) \approx -5.81365 \pm 0.00032 \), we see that the extrapolation significantly improves the accuracy.

We now show the results for the energy density \( e(\beta) \). In Fig. 8, we plot the estimates of \( e(\beta) \) for various \( V \) with \( \beta/V = 0.01 \) fixed, and in Fig. 9 the relative errors to the exact values.
Figure 7: $\chi^2$ fit of the free energy densities $f(\beta)$ for various $K$ with $\beta/V = 0.01$, $V = 64^2$ and $D = 60$ for $SU(2)$.

We again see good agreements, suggesting the effectiveness of our method. In Figs. 11 and

Figure 8: Volume dependence of $e(\beta)$ with $\beta/V = 0.01$ for $SU(2)$.

the $K$ and $D$ dependences are shown for $V = 64^2$ ($\beta = 40.96$), from which we again
confirm that the numerical estimates approach the exact value in the limit $K \to \infty$ and
$D \to \infty$. We can make use of extrapolation to improve the accuracy. Figure 12 shows the
$\chi^2$ fit to the obtained data with the cost function (3.1) with $f(\beta)$ replaced by $e(\beta)$. The
results of the fitting are also given in Table 1.

Figure 13 shows the singular values $\sigma_A$ of the initial tensor for $\beta = 2$ with $K = 90$. For
$SU(2)$, $\lambda_R$ in Eq. (2.15) takes the following form (see appendix B):

$$\lambda_{R-n}(\beta) = \frac{2}{\beta} e^{-\beta} I_{n}(\beta).$$

(3.2)

Here, $n$ is the $n$-dimensional irreducible representation of $SU(2)$, and $I_n(z)$ is the modified
Bessel function of the first kind. According to the discussion in Sec. 2.2, there will be
$d_R^2 = n^2$ degenerate singular values in the limit $K \to \infty$ for each representation $R = n$. In
the figure, we clearly observe this degeneracy even for finite $K$ ($K = 90$ here).
3.2. $SU(3)$

We make a similar analysis for $SU(3)$ with $\beta/V = 0.005$, $V = 64^2$ and $D = 90$. The irreducible representations $R$ of $SU(3)$ are labeled by two nonnegative integers, $R = [q_1, q_2]$ (see appendix [B]), for which the dimension is given by $d_R = (q_1 + 1)(q_2 + 1)(q_1 + q_2 + 2)/2$. The coefficients $\lambda_R(\beta)$ are given by the formula (B.3). One can show that they are ordered
Figure 11: $D$ dependence of $e(\beta)$ with $\beta/V = 0.01$ for $SU(2)$.

Figure 12: Fitting of the energy densities $e(\beta)$ for various $K$ with $\beta/V = 0.01$, $V = 64^2$ and $D = 60$ for $SU(2)$.

\[ \lambda_1 > \lambda_3 = \lambda_3 > \lambda_8 > \lambda_6 = \lambda_6 > \lambda_{15} = \lambda_{15} > \lambda_{10} = \lambda_{10} > \cdots. \]  

(3.3)

In Fig. 15, we plot the free energy densities $f(\beta)$ and the energy densities $e(\beta)$ against various values of $K$. We make the $\chi^2$ fit to the obtained data at $K = 70, 80, \ldots, 120$ again with the scaling ansatz $g(K) \equiv \mu + \alpha K^{-p}$. A similar analysis is performed for $e(\beta)$. The obtained results of the fitting are summarized in Table 2.

As for the free energy density $f(\beta)$, we obtain the estimate $\mu = -9.4400^{+0.0019}_{-0.0043}$, which agrees well with the exact value $f_{\text{exact}}(\beta) = -9.4323$. As for the energy density $e(\beta)$, we obtain the estimate $\mu = 0.1941^{+0.0017}_{-0.0008}$, which also agrees well with the exact value $e_{\text{exact}}(\beta) = 0.1923$. These good agreements show that our method also works for $SU(3)$.

We write the irreducible representations $R = [q_1, q_2]$ (see appendix B) as

\[
\begin{align*}
[0, 0] &= 1, & [1, 0] &= 3, & [0, 1] &= 3, & [1, 1] &= 8, & [2, 0] &= 6, & [0, 2] &= \overline{6}, \\
[2, 1] &= 15, & [1, 2] &= \overline{15}, & [3, 0] &= 10, & [0, 3] &= \overline{10}, & \cdots.
\end{align*}
\]
Figure 13: Singular values $\sigma_A$ of the initial tensors $T^{(e)}$ [Eq. (2.9)] with $\beta = 2$ and $K = 90$ for $SU(2)$. Horizontal lines indicate the exact values of $\lambda_R(\beta = 2)$, and vertical lines the points at which the exact values change discontinuously.

Figure 14: Dependence of the estimate of $f(\beta)$ on the number of $\hat{G}_a$'s ($\beta/V = 0.04$, $V = 16^2$, $K = 20$, $D = 20$) for $SU(2)$. The statistical errors decrease as the number increases.

The singular values of the initial tensor agree with the character expansion coefficients $\lambda_R(\beta)$ also for $SU(3)$. Figure 16 shows the singular values $\sigma_A$ for $\beta = 2$ with $K = 120$. We see that the coefficients are well reproduced with the correct degeneracies, reconfirming the group theoretical structure discussed in Sec. 2.2.

4. Summary and discussion

We have proposed a novel tensor network representation for two-dimensional Yang-Mills theories with arbitrary compact gauge groups, which makes no direct use of the character expansion. The numerical results for $SU(2)$ and $SU(3)$ gauge groups show that our method properly works. Although this paper focuses on pure Yang-Mills theories, it is straight-
forward to include the dynamical degrees of freedom of fermions and scalar fields into the tensor.

As a future project, it should be important to investigate whether the precision is improved by applying other renormalization algorithms to our tensor network, such as the higher-order tensor renormalization group (HOTRG). It should be also interesting to develop a method to optimally choose group elements from the group manifold, as the Gauss-Hermite quadrature for a field space with flat geometry. The extension of the framework to higher-dimensional Yang-Mills theories should also be one of the next steps to be considered. A study in this direction is now in progress and will be reported elsewhere.

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**Table 2**: Results of the $\chi^2$ fit for SU(3).

|        | (exact) | $\mu$       | $\alpha$ | $p$       | $\chi^2$/DOF |
|--------|---------|-------------|----------|-----------|--------------|
| $f(\beta)$ | -9.4323 | $-9.4400^{+0.0019}_{-0.0043}$ | $-0.3^{+0.2}_{-1.7} \times 10^{10}$ | $5.31^{+0.44}_{-0.01}$ | 0.21         |
| $e(\beta)$ | 0.1923  | $0.1941^{+0.0017}_{-0.0008}$ | $2.2^{+5.6}_{-1.6} \times 10^{10}$ | $5.88^{+0.29}_{-0.01}$ | 1.18         |

Figure 15: $K$ dependences of $f(\beta)$ (left panel) and $e(\beta)$ (right panel) with $\beta/V = 0.005$, $V = 64^2$ and $D = 90$ for SU(3).
A. Mathematical formulas

In this appendix, we summarize useful formulas for the integration over a compact group $G$.

For a unitary representation $R$ (not necessarily irreducible) with dimension $d_R$, we denote the representation matrix of $U \in G$ by $D_R(U) = (D^R_{rs}(U))$ ($r, s = 1, \ldots, d_R$) and the character by $\chi_R(U) = \text{tr} D_R(U)$. Note that $\chi_R(1) = d_R$. Hereafter we use the term “representation” as meaning “representation class”, and fix a representative $R$ for each representation class. Note that for a unitary representation, we have $D^R_{rs}(U^{-1}) = [D^R_{sr}(U)]^*$ and $\chi_R(U^{-1}) = [\chi_R(U)]^*$.

We introduce the Haar measure $dU$, which is two-side invariant and normalized:

\[
\int dU f(g_1 U g_2) = \int dU f(U) \quad (\forall g_1, g_2 \in G),
\]

\[
\int dU f(U^{-1}) = \int dU f(U),
\]

\[
\int dU 1 = 1.
\]

We also introduce the invariant delta function $\delta(U, V)$ associated with the Haar measure:

\[
\int dU \delta(U, V) f(U) = f(V),
\]

\[
\delta(g_1 U g_2, g_1 V g_2) = \delta(U, V) \quad (\forall g_1, g_2 \in G),
\]

\[
\delta(U^{-1}, V^{-1}) = \delta(U, V).
\]

We write the set of irreducible unitary representations by $\text{Irrep} = \{R : \text{irreducible}\}$.
Then, we have the following formula for $R_1, R_2 \in \text{Irrep}$,

$$
\int dU \, D_{r_1s_1}^{R_1}(U)D_{r_2s_2}^{R_2}(U^{-1}) = \frac{\delta_{R_1R_2}}{d_{R_1}} \delta_{r_1s_2} \delta_{s_1r_2},
$$
(A.7)

from which we readily obtain the formulas for the integration of characters,

$$
\int dU \, \chi_{R_1}(g_1U) \chi_{R_2}(U^{-1}g_2) = \frac{\delta_{R_1R_2}}{d_{R_1}} \chi_{R_1}(g_1g_2),
$$
(A.8)

$$
\int dU \, \chi_{R}(g_1Ug_2^{-1}U^{-1}) = \frac{1}{d_{R}} \chi_{R}(g_1) \chi_{R}(g_2).
$$
(A.9)

The characters of irreducible representations $\{\chi_{R}(U)\} (R \in \text{Irrep})$ form a linear basis of the set of class functions $\{f(U)\}$ that satisfy $f(gUg^{-1}) = f(U) \, (\forall g \in G)$. In particular, as can be easily proved, $\delta(U,1)$ is expanded as $\sum_{R \in \text{Irrep}} d_R \chi_{R}(U)$, and thus we have

$$
\delta(U,V) = \sum_{R \in \text{Irrep}} d_R \chi_{R}(UV^{-1}) = \sum_{R \in \text{Irrep}} d_R \chi_{R}(VU^{-1}).
$$
(A.10)

From this equation readily follows the Peter-Weyl theorem, which states that the infinite dimensional matrix

$$
W_{U,A} \equiv \sqrt{d_R} \, D_{r,s}^{R}(U) \quad [A = (R, r, s)]
$$
(A.11)

is unitary:

$$
\int dU \, W_{U,A}^* W_{U',A'} = \delta_{AA'}, \quad \sum_A W_{U,A} W_{U',A}^* = \delta(U,U')
$$
(A.12)

with $\delta_{AA'} \equiv \delta_{RR'} \delta_{rr'} \delta_{ss'}$ and $\sum_A \equiv \sum_{R \in \text{Irrep}} \sum_{r=1}^{d_R} \sum_{s=1}^{d_R}$.

**B. $\lambda_R(\beta)$ for $G = SU(N)$**

For $G = SU(N)$, the irreducible representation $R = [q_1, \ldots, q_{N-1}]$ ($q_i \in \mathbb{Z}_{\geq 0}$: Dynkin labels) can be labeled by a Young diagram $Y = (f_1, f_2, \ldots, f_{N-1}) \,(f_1 \geq f_2 \geq f_{N-1} \geq 0)$ with the relations $f_i \equiv \sum_{j=1}^{N-1} q_j$ (see Fig. [17]). The dimension $d_R$ is given by

$$
d_R = \Delta(\ell_1, \ell_2, \ldots, \ell_{N-1}, \ell_N)/\Delta(N-1, N-2, \ldots, 1, 0),
$$
(B.1)

From this equation, one can show the formula

$$
\int dU \, D_{r_1s_1}^{R_1}(g_1U)D_{r_2s_2}^{R_2}(U^{-1}g_2) = \int dU \, D_{r_1s_1}^{R_1}(g_1U^{-1})D_{r_2s_2}^{R_2}(Ug_2) = \frac{\delta_{R_1R_2}}{d_{R_1}} \delta_{s_1s_2} D_{r_1s_2}^{R_1}(g_1g_2).
$$
where $\ell_i \equiv f_i + N - i$ with $f_N \equiv 0$ and $\Delta(x_1, \ldots, x_N) \equiv \prod_{i<j}(x_i - x_j)$. One can show that the coefficients $\lambda_R(\beta)$ can be expressed as (see, e.g., [44])

$$\lambda_R(\beta) = \frac{e^{-\beta}}{d_R} \sum_{Q \in \mathbb{Z}} \det[I_{f_j+i-j+Q}(\beta/N)] \ [G = SU(N)],$$  \hspace{1cm} (B.2)

where $I_n(z)$ are the modified Bessel functions of the first kind.

For $G = SU(2)$, the irreducible representation $R = [q]$ corresponds to the spin $j = q/2$ representation with $d_R = q + 1 = 2j + 1$, for which the infinite series (B.2) can be summed up to a simple form,

$$\lambda_R(\beta) = (2/\beta) e^{-\beta} I_{2j+1}(\beta) \ [G = SU(2)].$$  \hspace{1cm} (B.3)

Thus, the free energy density and the energy density can be expressed as

$$f(\beta) = \frac{1}{V} \log \left[ \sum_{n=1}^{\infty} \left( \frac{2}{\beta} e^{-\beta} I_n(\beta) \right)^V \right],$$  \hspace{1cm} (B.4)

$$e(\beta) = - \frac{\sum_{n=1}^{\infty} I_n^{V-1}(\beta) \left[ (I_{n+1}(\beta) + I_{n-1}(\beta))/2 - I_n(\beta)/\beta \right]}{\sum_{n=1}^{\infty} I_n^V(\beta)} + 1.$$  \hspace{1cm} (B.5)

### C. TN derivation of the exact partition function

The well-known formula (2.20) can be easily derived from the TN representation of the partition function with the infinite dimensional tensor, Eq. (2.19):

$$\Xi^{(1)}_{A_1 A_2 A_3 A_4} = \alpha_1 \delta_{R_1 R_2 R_3 R_4} \delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{r_3 r_4} \delta_{s_4 r_1}. \hspace{1cm} (C.1)$$

Here, $A_i = (R_i, r_i, s_i)$, $\delta_{R_1 R_2 R_3 R_4}$, $\delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{r_3 r_4} \delta_{s_4 r_1}$ are factors located at vertices. Figure 18 shows a graphical representation of $\Xi^{(1)}$. 

It is straightforward to evaluate the value of $Z$ as shown in Fig. 19. Figure 19 (b) is obtained from Fig. 19 (a) where $\alpha_1$ is replaced by $\alpha_1 d_R = \lambda^2_R$ because $d_R$ is provided from the inner loop. The final expression is immediately obtained because the remaining tensors in Fig. 19 (b) are diagonal with respect to the $R$ indices 20.
Instead, we can use the TRG iterations to evaluate $Z$. Omitting the tensor indices, we write

$$ T^{(1)} = 0. $$

Then, we decompose $T^{(1)}$ in two ways as

$$ T^{(1)} = \sqrt{\alpha_1} = \frac{\lambda_R^{1}}{\sqrt{\alpha_1}} = \frac{\lambda_R^{2}}{\sqrt{\alpha_1}} = \frac{\lambda_R^{3}}{\sqrt{\alpha_1}}. $$

where rank-3 tensors are defined in a manner similar to Eq. (C.1). These decompositions correspond to the SVDs given in Fig. 2. With these rank-3 tensors, we construct the second
tensor as

\[ \mathbf{T}^{(2)} = \begin{array}{c}
\sqrt{\alpha_1} \\
\sqrt{\alpha_1} \\
\sqrt{\alpha_1}
\end{array} \begin{array}{c}
\alpha_2 \\
\alpha_2 \\
\alpha_2
\end{array} \]  \quad (C.4)

Note that we have not made any truncation. Repeating this procedure, we have the \( n \)-th tensor

\[ \mathbf{T}^{(n)} = \begin{array}{c}
\alpha_n \\
\alpha_n \\
\alpha_n
\end{array} \]  \quad (C.5)

from which the partition function \( Z \) with volume \( V = 2^n \) is calculated as

\[ Z = \begin{array}{c}
\alpha_n \\
\alpha_n \\
\alpha_n
\end{array} = \sum_R \lambda_R(\beta)^V. \quad (C.6)\]

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