ALPHONONS AS THE QUANTA OF THE ALFVÉN WAVES

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ABSTRACT
In a magnetized plasma, both the transverse and longitudinal Alfvén waves carry an average energy density that, for each mode of oscillation, resembles the classical energy of a harmonic oscillator with unit mass and angular frequency that depends on the background magnetic field and plasma density. We employ this fact to introduce a pair of canonical variables for the underlying classical system, and then use the canonical quantization procedure to express the Alfvén wave velocity (or the corresponding plasma displacement vector) in terms of creation and annihilation operators. We thus introduce the concept of “Alphonons” as the quanta for the Alfvén wave packets and study their characteristic features. The corresponding quanta (Alphonons) carry spin one for the transverse Alfvén wave and spin zero for the longitudinal Alfvén wave. We obtained behavior of thermodynamic quantities for an Alphonon system (e.g., free energy, entropy, specific heat). We showed that for the Alphonon system the spectral energy density obeys the Planck’s black body radiation law.

1. INTRODUCTION
Alfvén suggested there must be the Alfvén wave extracted from Magnetohydrodynamics (MHD) equations observable in a plasma (Alfvén 1942). Tension within magnetic field lines is the restoring force of the Alfvén wave inside a magnetized plasma. Lundquist detected the Alfvén wave (Lundquist 1949), and Bostick and Levine used a laboratory setup to study the behavior of this wave (Bostick et al. 1952). Jephcott produced the Alfvén wave in a discharged gas (Jephcott 1959). Jephcott (1958), Hollweg (1974), and Roberts et al. (1984) proposed the propagation of the Alfvén wave in space plasma such as intergalactic, interplanetary medium, and solar plasma, respectively. Aschwanden et al. (2011) and Berthold et al. (1960) reported the oscillations with millihertz frequencies of the MHD waves in the solar coronal loops by analyzing the TRACE (Transition Region and Coronal Explorer) extreme-ultraviolet observations. Several attempts have been done in the field of observations and modeling of the Alfvén and MHD waves in the solar atmospheric plasma (see e.g., Safari et al. 2006, 2007; Fathalian et al. 2010; Abedini et al. 2012; Esmaeili et al. 2016; Farahani et al. 2017; Tavabi et al. 2015; Erdélyi et al. 2007; Gruszczki et al. 2007; Kaghashvili et al. 2009; Jess et al. 2009; Srivastava et al. 2017; Pascoe et al. 2017; Andries et al. 2009; Arregui et al. 2007; Moortel et al. 2012).

Most of the physical characteristics of the gaseous plasma can be applied in the solid-state plasma. The primary difference is the number density which is typically large in a solid-state plasma; for example, it is typically $10^{22}$ cm$^{-3}$ in metals, $10^{16}$ cm$^{-3}$ in semiconductors, $10^{12}$ cm$^{-3}$ in a laboratory plasma, and $10^{9}$ cm$^{-3}$ in solar coronal plasma (see e.g., Bowers et al. 1964; Williams et al. 1965). Propagation of the Alfvén wave in the solid-state plasma is detected for example in bismuth (Williams et al. 1965; Kirsch et al. 1962), in graphite (Nakamura et al. 1982), and in metals (Fisher et al. 1995).

Quasiparticles and collective excitations are introduced when the particles in solid or fluid show different treatment with those in free space with weak interactions. Several quasiparticles and collective excitations have been introduced in the physical systems to simplify the complicated interactions between the particles. Chargeon (Mendonça et al. 2001), configuron (Angell et al. 1972), electron quasiparticle (Kaxiras 2003), exciton (Knox 1963; Liang 1970), orbiton (Schlappa et al. 2012), phason (Lubensky et al. 2010), phonon, plasmaron (Bostwick et al. 2010; Tóth et al. 2007), spinon (Jompol et al. 2009), wrinklon (Vandeperre et al. 2011), etc, are a list of the quasiparticles and collective excitations were mostly used in physics.

Song and Lysak defined Alfvénons as the macro (quasi) particle, to give particle like descriptions of both transverse (shear) and longitudinal (compressional) Alfvén waves (Song et al. 1994). They explained the driven reconnections in the solar winds as the interaction of the Alfvénons and embedded magnetic field.
Phonons are the quantum of the vibrational waves (sound) or their equivalent normal modes in a lattice. When a lattice vibrates, atoms or molecules oscillate around their equilibrium locations and these oscillations are well known as the normal modes of the lattice. Phonons are bosonic quasiparticles (Feynman et al. 1972). In three-dimensions, each mode associated with three possible polarizations, two transverse and one longitudinal. We say that longitudinal (acoustic) and transverse (optical) phonons have spin 0 and 1, respectively (Levine 1962).

An important application of phonons in the artificial black holes (e.g. acoustic black hole) is given by Unruh Unruh (1981). Since this pioneering work, several attempts have been made to detect the Hawking radiation emitted at the horizon of a sonic black hole. In a sonic black hole the phonons (related to the acoustic wave in super-sonic flow) cannot escape the flow at the horizon of the black hole and Hawking radiation is emitted from the horizon which is made by phonons (Garay et al. 2000; Barceló et al. 2001; Recati et al. 2009; Zapata et al. 2011; Unruh et al. 2003; Rousseaux et al. 2008; Weinfurtner et al. 2011; Steinhauer et al. 2014).

In a similar analysis, Gheibi, Safari, and Innes introduced the Alfvénic black hole based on the magnetohydrodynamics approach (Gheibi et al. 2018). They showed that similar to sonic black holes, which trap phonons and emit Hawking radiation at the sonic horizon where the flow speed changes from super- to sub-sonic, in the horizon of Alfvénic black holes, the Alfvén waves will be trapped and emit Hawking radiation made of quantized vibrations similar to phonons, for which they coined the name “Alphonons”. They also defined the magnephonons as a new quasi-particle for Hawking radiation emitted from the horizon of the magnetooacoustic black hole.

Here, we investigate the physical properties of the Alphonons as the quanta of the Alfvén waves. To do this end, we first observe that each mode of oscillation of the Alfvén waves may be identified with a harmonic oscillator with unit mass and an angular frequency that depends on background magnetic field and plasma density. This allows us to use the machinery of canonical quantization to express the corresponding velocity field (or equivalently the plasma displacement) in term of creation and annihilation operators that define the quanta related to the Alfvén wave packet.

The rest of the paper is organized as follows. We briefly review the basic equations of Magnetohydrodynamics. We first linearize the basic ideal MHD equations to write down a wave equation for velocity perturbation. Then, we recall some basic properties of its wave solutions, i.e., MHD waves that includes sound and Alfvén waves. We employ the canonical quantization scheme to introduce the concept of Alphonon as the quantum of the transverse and longitudinal Alfvén wave, we study the thermodynamics of an Alphonon system. Finally, we present our conclusion.

THE BASIC EQUATIONS OF MHD

Magnetohydrodynamics is a fluid theory that provides a theoretical framework to study the macroscopic behavior of plasmas. In this framework, the behavior of a continuous plasma with mass density ($\rho$), pressure ($p$), velocity ($v$), and current density ($J$) is governed by the set of coupled equations: “a simplified form of Maxwell’s equations, Ohm’s Law, a gas law and equations of mass continuity, motion and energy” (Priest 2014). If we neglect the gravitational and all other dissipative forces from our analysis, the ideal MHD equations in the non-relativistic limit are given by (Priest 2014; Benz 2002)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{1}
\]

\[
\rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla)v + \nabla p = \frac{1}{\mu} (\nabla \times B) \times B, \tag{2}
\]

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B), \tag{3}
\]

\[
\nabla \cdot B = 0, \tag{4}
\]

\[
\rho \rho^\gamma = \text{const}, \tag{5}
\]

where $B$, $\mu$, and $\gamma$ are the magnetic induction, magnetic permeability, and the adiabatic exponent, respectively. Throughout the article, all vectors are denoted as boldface. These coupled equations serve to determine $\rho$, $p$, $v$ and $B$ that can be used to find other variables in plasma. For example, for the current density ($J$) and the electric field ($E$) we have, respectively, $J = \nabla \times B/\mu$ and $E = -v \times B$.

MHD WAVES IN A UNIFORM UNBOUNDED MAGNETIZED PLASMA

We recall that for a uniformly magnetized medium the equilibrium quantities such as mass density ($\rho_0$), pressure ($p_0$), temperature ($T_0$), and magnetic field ($B_0$) are constants. The ideal MHD equations (1)–(5), can be linearized
about the equilibrium state. Indeed, the equilibrium quantities can be perturbed as

\[ B = B_0 + B_1(x,t), \]
\[ p = p_0 + p_1(x,t), \]
\[ \rho = \rho_0 + \rho_1(x,t), \]
\[ v = v_1(x,t), \]

where the perturbed mass density \( \rho_1 \), pressure \( p_1 \), magnetic field \( B_1 \), and velocity \( v_1 \) are assumed to be small and the plasma initial flow is considered in the static condition \( v_0 = 0 \). Also, in our analysis below, the magnetic field is assumed to be in the \( z \)-direction, i.e., \( B_0 = B_0 z \) (see Figure 1).

![Figure 1](image)

**Figure 1.** The sketch of the magnetic field lines, and the direction of the wave vector (\( k \)), and the plasma eigendisplacement vector for (a) transverse Alfvén wave, and (b) longitudinal Alfvén wave.

Inserting equations (6)–(9) into equations (1)–(5), we get (Priest 2014; Benz 2002)

\[ \frac{\partial p_1}{\partial t} + \rho_0 (\nabla \cdot v_1) = 0, \]  
\[ \rho_0 \frac{\partial v_1}{\partial t} + \nabla p_1 = \frac{1}{\mu} (\nabla \times B_1) \times B_0, \]  
\[ \frac{\partial B_1}{\partial t} = \nabla \times (v_1 \times B_0), \]  
\[ \nabla \cdot B_1 = 0, \]  
\[ p_1 = c_s^2 \rho_1, \]

where \( c_s = \left( \frac{2 p_0}{\rho_0} \right)^{1/2} \) is the acoustic speed or sound speed. These equations can be combined to obtain the following wave equation for the disturbed velocity \( v_1 \)

\[ \frac{\partial^2 v_1}{\partial t^2} = c_s^2 \nabla (\nabla \cdot v_1) + \{ \nabla \times [\nabla \times (v_1 \times B_0)] \} \times \frac{B_0}{\mu \rho_0}, \]

which supports plane-wave solutions of the form

\[ v_1(x,t) = \hat{v}_{1k} e^{i(k \cdot x - \omega t)}, \]

where \( \hat{v}_{1k} \), \( k \), and \( \omega \) represent the Fourier amplitude of \( v_1 \), wave vector, and angular frequency, respectively. Indeed, equation (15) together with equations (10)–(14) provide a ground to study the basic characteristics of the sound and MHD waves (Alfvén, fast, and slow magnetoacoustic waves) in a homogenous plasma that can be found in the literature (see, for example, (Priest 2014; Benz 2002)). Here, to fix the notation, we briefly recall some basic properties of these main types of wave solution.

**SOUND WAVE**

For a compressible plasma with no background magnetic field \( (B_0 = 0) \), the only restoring force is the pressure gradient and the wave equation (15) is reduced to the well-known sound wave equation

\[ \frac{\partial^2 v_1}{\partial t^2} = c_s^2 \nabla (\nabla \cdot v_1). \]

Substituting the plane-wave solution equation (16) in equation (17) we find

\[ \omega^2 \hat{v}_{1k} = c_s^2 k (k \cdot \hat{v}_{1k}), \]
which, for \( \mathbf{k} \cdot \mathbf{v}_{1k} \neq 0 \), yields the dispersion relation for acoustic waves or sound waves, i.e., \( \omega^2 = c_s^2 k^2 = \frac{k^2}{\mu} \left( \frac{\rho_i}{\rho_0} \right) \). Therefore, both the phase speed (\( v_{ph} = \frac{\omega}{k} \)) and group velocity (\( v_g = \frac{\partial \omega}{\partial k} \)) are equal to \( \pm c_s \). Expectedly, as can be seen from equation (18), sound waves are longitudinal (i.e., \( \mathbf{v}_{1k} \parallel \mathbf{k} \)).

The energy density (per unit volume) of a sound wave with frequency \( \omega \) is given by

\[
\mathcal{U} = \frac{1}{2} \rho_0 v_1^2 = \frac{1}{2} \rho_0 \omega^2 \xi^2,
\]

where \( \xi \) is the eigendisplacement of the wave equation \( \mathbf{v}_1 = \frac{\partial \xi}{\partial t} \).

**TRANSVERSE ALFVÉN WAVE**

In an incompressible magnetized plasma (\( \nabla \cdot \mathbf{v}_1 = 0 \)), if we make the assumption that the pressure remains uniform (\( \nabla p = 0 \)), we can restrict ourselves to the waves propagating along the background magnetic field (\( \mathbf{k} \parallel \mathbf{B}_0 \)). If we take \( \mathbf{B}_1 = \mathbf{B}_{1k} e^{i(k \cdot \mathbf{x} - \omega t)} \), equations (11) and (12) imply that the vectors \( \mathbf{v}_1 \) and \( \mathbf{B}_1 \) are perpendicular to \( \mathbf{B}_0 \) (and also to \( \mathbf{k} \)). Therefore, equation (15) can be reduced to

\[
\left( \frac{k^2 B_0^2}{\mu} - \rho_0 \omega^2 \right) \mathbf{v}_{1k} = 0,
\]

which, for non-trivial solutions (\( \mathbf{v}_{1k} \neq 0 \)), gives

\[
v_{ph}^2 = \frac{v_A^2}{k^2} = \frac{B_0^2}{\mu \rho_0} = v_A^2,
\]

where \( v_{ph} \) and \( v_A \) are the phase speed and Alfvén speed, respectively.

The dispersion relation (equation (21)) and the propagation of the wave along the magnetic field are two important characteristics of the transverse or shear Alfvén wave. Using equations (10), (12), and (14), we obtain

\[
\rho_1 = 0, \quad p_1 = 0, \quad \mathbf{B}_1 = -\frac{1}{\omega} \mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0) = -\frac{\text{sign}(k) k B_0}{\omega} \mathbf{v}_1,
\]

where “sign” is the sign function. Equation (21) shows that the shear Alfvén wave is a purely magnetic wave and, in view of equation (11), its restoring force is magnetic tension that depends on the perturbed magnetic field (Priest 2014; Benz 2002).

The Alfvén wave carries energy at the Alfvén speed along the magnetic field. The average (over a period of oscillation) kinetic energy density carried by the Alfvén wave is

\[
\mathcal{U}_{\text{kin.}} = \frac{1}{2} \rho_0 \langle \mathbf{v}_1^2 \rangle = \frac{1}{4} \rho_0 \omega^2 \xi^2.
\]

The time-averaged magnetic energy density related to the Alfvén wave is given by

\[
\mathcal{U}_{\text{mag.}} = \frac{1}{2\mu} (\mathbf{B}_1^2) = \frac{B_0^2}{2\mu} \frac{\langle \mathbf{v}_1^2 \rangle}{v_A^2} = \frac{B_0^2}{4\mu} \frac{\omega^2}{v_A^2} \xi^2.
\]

Using the definition of the Alfvén velocity (\( v_A \)), we observe that \( \mathcal{U}_{\text{kin.}} = \mathcal{U}_{\text{mag.}} \). Therefore, each mode of the transverse Alfvén waves, with a given wave vector \( \mathbf{k} \) and frequency \( \omega_k \), may be treated as an independent harmonic oscillator whose total average energy density can be expressed by

\[
\mathcal{U}_k = \mathcal{U}_{\text{kin.}} + \mathcal{U}_{\text{mag.}} = \frac{1}{2} \rho_0 \omega_k^2 \xi^2.
\]

**LONGITUDINAL ALFVÉN WAVE**

In a compressible magnetized plasma (\( \nabla \cdot \mathbf{v}_1 \neq 0 \)), both the magnetic pressure and plasma pressure gradient forces exist. These two restoring forces can act together to drive longitudinal magnetoacoustic waves (\( \mathbf{k} \parallel \mathbf{v}_1 \)) that propagate perpendicular to the background magnetic field \( \mathbf{B}_0 \) (i.e., \( \mathbf{k} \perp \mathbf{B}_0 \)) (Priest 2014; Benz 2002). For longitudinal plane wave solutions (16), equation (15) can be reduced to

\[
\left( -\rho_0 \omega^2 + \frac{\gamma p_0 k^2}{\mu} + \frac{k^2 B_0^2}{\mu} \right) \mathbf{v}_{1k} = 0,
\]
which implies
\[ v_{ph}^2 = \frac{\omega^2}{k^2} = c_s^2 + v_A^2. \] (28)

This dispersion relation represents a fast longitudinal MHD wave which propagates perpendicular to the background magnetic field.

For small sound speed \((c_s \to 0)\) the phase speed in equation (28) is equal to the Alfvén velocity, and then the corresponding wave is called the longitudinal or compressional Alfvén wave (Benz 2002). In a similar manner, the same relation as given by equation (26) can be obtained for the average energy density carried by each mode of the longitudinal Alfvén waves.

**ALPHONONS AS THE QUANTA OF THE ALFVÉN WAVES**

**TRANSVERSE ALPHONONS**

A classical dynamical theory of field with a set of well-defined canonically conjugate variables can be quantized by promoting the phase space variables to the Hermitian quantum operators. Then, the field and the Hamiltonian of the system can be expressed in terms of the creation and annihilation operators that defines the excitations (quanta) of the filed. The tensor product of the one-particle states make a basis for the Hilbert space (Fock space) of the quantum theory.

Because the particles in transverse Alfvén waves oscillate in a plane perpendicular to the background magnetic field (and also to the direction of propagation), each mode of oscillation with a given wave vector \(\mathbf{k}\) may be described by two distinct polarization vectors \(\xi^1_k\) with \(\lambda = 1, 2\). This is identical to the physical polarizations of electromagnetic fields (Schwartz 2014; Ryder 1985). If we take \(z\)-direction as the direction of propagation, i.e., \(\mathbf{k} = k\mathbf{z}\), then two transverse linear polarization vectors can be read as
\[ \xi^1_k = (1, 0, 0), \quad \xi^2_k = (0, 1, 0). \] (29)

Consequently, for each Alfvén mode with a given \(\mathbf{k}\), the plasma displacement vector \(\xi\) and hence the disturbed velocity \(\mathbf{v}_1\) may be expressed, respectively, by \(\xi = \xi^1_k \xi^1_k + \xi^2_k \xi^2_k\) and \(\mathbf{v}_1 = v_{k,1} \xi^1_k + v_{k,2} \xi^2_k\).

Next, we observe that equation (26) can be rewritten as
\[ \mathcal{U}_k = \frac{1}{2} (P^2_{k,\lambda} + \omega^2_k Q^2_{k,\lambda}), \] (30)
provided that we introduce a pair of canonical variables, \(Q_{k,\lambda}\) and its conjugate momentum \(P_{k,\lambda}\), such that
\[ Q_{k,\lambda} = \sqrt{\rho_0} (\xi_{k,\lambda} + \xi^*_{k,\lambda}), \] (31)
\[ P_{k,\lambda} = \frac{dQ_{k,\lambda}}{dt} = -i\omega_k \sqrt{\rho_0} (\xi_{k,\lambda} - \xi^*_{k,\lambda}), \] (32)
where \("*" stands for the complex conjugation. Equation (30) represents the energy of a harmonic oscillator with unit mass, position variable \(Q_{k,\lambda}\) and momentum \(P_{k,\lambda}\).

To quantize the velocity field (or equivalently the displacement vector), we promote the canonical variables \(Q_{k,\lambda}\) and \(P_{k,\lambda}\) to Hermitian operators and introduce the annihilation operator
\[ a_{k,\lambda} = \frac{1}{\sqrt{2\hbar \omega_k}} (\omega_k Q_{k,\lambda} + iP_{k,\lambda}), \] (33)
for the plane wave modes with the wave vector \(\mathbf{k}\) and frequency \(\omega_k\). The adjoint of equation (33) defines the creation operator \(a_{k,\lambda}^\dagger\). The creation and annihilation operators \(a_{k,\lambda}\) and \(a_{k,\lambda}^\dagger\) satisfy the (bosonic fields) communication relations
\[ [a_{k,\lambda}, a_{k',\lambda'}^\dagger] = \delta_{\lambda,\lambda'} \delta(|\mathbf{k} - \mathbf{k}'|), \] (34)
\[ [a_{k,\lambda}, a_{k',\lambda'}] = [a_{k,\lambda}^\dagger, a_{k',\lambda'}^\dagger] = 0, \]
where \(\delta(|\mathbf{k} - \mathbf{k}'|)\) denotes the Dirac delta function. In terms of these operators, the energy density (30) can be expressed by the Hamiltonian
\[ \mathcal{H}_k = \sum_{\lambda=1}^2 \hbar \omega_k \left( a_{k,\lambda}^\dagger a_{k,\lambda} + \frac{1}{2} \right). \] (35)
This suggests that the state with \( n_{k,\lambda} \) quanta of Alfvén wave in \((k, \lambda)\)-mode can be read as

\[
|k, \lambda, n_{k,\lambda}\rangle = \frac{1}{\sqrt{n_{k,\lambda}}} (a_{k,\lambda}^\dagger)^{n_{k,\lambda}} |k, 0\rangle,
\]

where \(|k, 0\rangle\) is the vacuum state for a given mode \(k\). These are also the eigenstates of the Alphonon occupation number density operator defined by \( \hat{N}_{k,\lambda} = a_{k,\lambda}^\dagger a_{k,\lambda} \). We then note that the expectation value of \( \hat{N}_{k,\lambda} \) given by

\[
\langle k \rangle = \sum_{\lambda=1}^{2} \langle a_{k,\lambda}^\dagger a_{k,\lambda} \rangle = \frac{\rho_0}{2\hbar} |\xi_k\rangle^2 = \frac{\rho_0}{2\hbar} |\hat{v}_{1k}\rangle^2,
\]

that gives Alphonon occupation number, can be used to characterize the Alphonons concentration density. Equation (37) shows that a single Alphonon corresponds to a plasma fluctuation \( \xi_k \) (or equivalently, velocity fluctuation \( v_k \)), with an amplitude given by

\[
|\xi_k|_{\text{min}} = \left( \frac{2\hbar}{\rho_0 \omega_k} \right)^{1/2} \Longleftrightarrow |\hat{v}_{1k}|_{\text{min}} = \left( \frac{2\hbar \omega_k}{\rho_0} \right)^{1/2},
\]

where \( \omega_k = k v_A = k B_0 / \sqrt{\mu_0} \) is the Alfvén wave frequency. This establishes the characteristic scale of the displacement (or velocity) quantization in a magnetized plasma.

Finally, we note that a general transverse Alfvén wave packet can be obtained by superimposing many plane-wave solutions. In view of equations (31) and (32), we observe that

\[
\xi_{k,\lambda} = \left( \frac{\hbar}{2\rho_0 \omega_k} \right)^{1/2} a_{k,\lambda}, \quad \xi_{k,\lambda}^* = \left( \frac{\hbar}{2\rho_0 \omega_k} \right)^{1/2} a_{k,\lambda}^\dagger.
\]

Thus, we can express a general displacement vector \( \xi(x, t) \), describing a general Alfvén wave packet, according to (Note that for transverse Alfvén mode we take \( k = k_2 \) in the direction of background magnetic field \( B_0 \).)

\[
\xi(x, t) = \int \frac{dk}{\sqrt{2\pi}} \left( \frac{\hbar}{2\rho_0 \omega_k} \right)^{1/2} \sum_{\lambda=1}^{2} \epsilon_{k,\lambda} a_{k,\lambda} e^{i(k \cdot x - \omega_k t)},
\]

which yields the velocity fluctuation

\[
v_{1}(x, t) = -i \int \frac{dk}{\sqrt{2\pi}} \left( \frac{\hbar \omega_k}{2\rho_0} \right)^{1/2} \sum_{\lambda=1}^{2} \epsilon_{k,\lambda} a_{k,\lambda} e^{i(k \cdot x - \omega_k t)}.
\]

For this wave packet we may introduce a quantum field \( Q(x, t) \) and its conjugate momentum \( P(x, t) \) according to

\[
Q(x, t) = \left( \frac{\hbar \rho_0}{2} \right)^{1/2} \int \frac{dk}{\sqrt{2\pi}} \left( \frac{1}{\omega_k} \right)^{1/2} \sum_{\lambda=1}^{2} \left( \epsilon_{k,\lambda} a_{k,\lambda} e^{i k \cdot x} + \epsilon_{k,\lambda}^* a_{k,\lambda}^\dagger e^{-i k \cdot x} \right),
\]

\[
P(x, t) = -i \left( \frac{\hbar \rho_0}{2} \right)^{1/2} \int \frac{dk}{\sqrt{2\pi}} \left( \frac{1}{\omega_k} \right)^{1/2} \sum_{\lambda=1}^{2} \left( \epsilon_{k,\lambda} a_{k,\lambda} e^{i k \cdot x} - \epsilon_{k,\lambda}^* a_{k,\lambda}^\dagger e^{-i k \cdot x} \right),
\]

where, \( k \cdot x = k \cdot x - \omega_k t \). Making use of equation (34), we can show that these fields satisfy canonical commutation relation, i.e.,

\[
[Q_i(x, t), P_j(x', t)] = i\hbar \delta_{ij} \delta(x - x').
\]

**LONGITUDINAL ALPHONONS**

As discussed in Subsec. 1, for the longitudinal Alfvén wave, the direction of plasma oscillations is parallel to the direction of wave propagation (the wave vector \( k \)). Therefore, a single polarization vector \( \epsilon_k \) along the wave vector \( k \) is required to express the oscillation amplitude. More explicitly, we take \( \epsilon_k = k/k \) and write \( \xi = \xi_k \epsilon_k \), and \( v_1 = v_{1k} \epsilon_k \).

To quantize the longitudinal Alfvén wave or its corresponding velocity field \( v_1 \), we define a pair of canonical variables, \( Q_k \) and \( P_k \), by

\[
Q_k = \sqrt{\rho_0} (\xi_k + \xi_k^*),
\]

\[
P_k = \frac{dQ_k}{dt} = -i \omega_k \sqrt{\rho_0} (\xi_k - \xi_k^*).
\]
Making use of these variables into equation (26), we find $\mathcal{U}_k = \frac{1}{2} (p_k^2 + \omega_k^2 q_k^2)$ that describes a harmonic oscillator with unit mass and frequency $\omega_k$. Now, it is convenient to introduce the annihilation operator

$$a_k = \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k q_k + i p_k),$$

(47)

that together with its adjoint satisfy the commutation relations

$$[a_k, a_k^\dagger] = \delta(k - k'), \quad [a_k, a_k'] = [a_k^\dagger, a_k^\dagger] = 0.$$  

(48)

These simplify the Hamiltonian of the $k$-mode longitudinal Alfvén wave to the form

$$\mathcal{H}_k = \hbar \omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right),$$

(49)

which has the general eigenvector $|k, n_k\rangle$ represented by

$$|k, n_k\rangle = \frac{1}{\sqrt{n_k!}} \left( a_k^\dagger \right)^{n_k} |k, 0\rangle.$$  

(50)

These states occupy $n_k$ quanta (Alphonons) of the longitudinal Alfvén wave with frequency $\omega_k$ and longitudinal polarization ($\varepsilon_k||\mathbf{k}$). Accordingly, the longitudinal Alphonon occupation number and the corresponding characteristic velocity fluctuation can be read, respectively, by the same expressions as given in equations (37) and (38).

THERMODYNAMICS OF AN ALPHONON SYSTEM

We assume that the system of Alphonons is in thermal contact with the environment at temperature $T$. Also, we suppose both the volume of the system and the number of particles (number density) are constant. A collection of such systems with several copies is called the canonical ensemble. The canonical partition function depends on the degree of freedom of the system in both classical and quantum mechanics (Landau et al. 1996; Pathria et al. 1996; Huang et al. 1963). For the canonical ensemble, the partition function is given by

$$Z = \sum_k \exp \left( -\frac{E_k}{k_B T} \right)$$  

(51)

$$Z = \sum_{n_1=0}^\infty \cdots \sum_{n_{\text{max}}=0}^\infty \exp \left( -\frac{1}{k_B T} \sum_{k=1}^{\text{max}} \hbar \omega_k \left( n_k + \frac{1}{2} \right) \right)$$  

$$Z = \prod_{k=1}^{\text{max}} \frac{\exp \left( -\frac{\hbar \omega_k}{k_B T} \right)}{1 - \exp \left( -\frac{\hbar \omega_k}{k_B T} \right)}.$$  

where $\hbar$, $k_B$, $T$, and $k_{\text{max}}$ are, respectively, the Planck’s constant, Boltzmann constant, temperature, and maximum wave number of the Alphonon in mode $k$.

Using equation (51), the Helmholtz free energy $F$ is defined by

$$F = -k_B T \ln Z$$

(52)

$$F = \sum_k \frac{\hbar \omega_k}{2} + k_B T \sum_k \ln \left( 1 - \exp \left( -\frac{\hbar \omega_k}{k_B T} \right) \right).$$  

The first term in the right hand side is the zero-temperature energy of the system. For more simplicity we remove the zero-temperature from our analysis. Moreover, the specific energy per volume is introduced by

$$f = \frac{1}{L^3} F = \frac{k_B T}{L^3} \sum_k \ln \left( 1 - \exp \left( -\frac{\hbar \omega_k}{k_B T} \right) \right)$$  

(53)

$$f \approx k_B T \int_0^\infty D(\omega) \ln \left( 1 - \exp \left( -\frac{\hbar \omega}{k_B T} \right) \right) d\omega,$$

where the density of states in frequency $D(\omega)$ is related to the mode occupies the volume $\frac{2\pi}{L^2}$ in the wave vector space by

$$D(\omega) = D(k) \frac{dk}{d\omega}, \quad D(k)dk = \frac{k^2}{2\pi^2} dk.$$  

(54)
Using the dispersion relation $\frac{d\omega}{dk}$, the density of state can be read as
\[ D(\omega) = \frac{\omega^2}{2\pi^2} \left( \frac{\rho_0 \mu}{B_0^2} \right)^\frac{3}{2}. \] (55)

For the entropy per volume $s$ of the Alphonon system we have
\[ s = -\frac{\partial f}{\partial T} = -k_B \int_0^\infty \frac{\omega^2}{2\pi^2} \left( \frac{\rho_0 \mu}{B_0^2} \right)^\frac{3}{2} \ln \left( 1 - \exp \left( \frac{-\hbar \omega k_B}{k_B T} \right) \right) + \frac{\hbar \omega \exp \left( \frac{-\hbar \omega}{k_B T} \right)}{k_B T (\exp \left( \frac{-\hbar \omega}{k_B T} \right) - 1)} d\omega. \] (56)

Also, the internal energy $u$ is defined by
\[ u = f + T.s = \int_0^\infty \frac{\omega^2}{2\pi^2} \left( \frac{\rho_0 \mu}{B_0^2} \right)^\frac{3}{2} \frac{\hbar \omega}{\exp \left( \frac{\hbar \omega}{k_B T} \right) - 1} d\omega. \] (57)

According to the expression $u = \int_0^\infty u(\omega) d\omega$, equation (57) gives the spectral energy density as
\[ u(\omega) = \frac{\omega^2}{2\pi^2} \left( \frac{\rho_0 \mu}{B_0^2} \right)^\frac{3}{2} \frac{\hbar \omega}{\exp \left( \frac{\hbar \omega}{k_B T} \right) - 1}. \] (58)

Equation (58) is equivalent to the Planck’s black body radiation law for an Alfvén wave system.

Finally, for the total internal energy $u$ we obtain
\[ u = \left( \frac{\rho_0 \mu}{B_0^2} \right)^\frac{3}{2} \frac{\pi^2 k_B^3 T^4}{30 \hbar^2}. \] (59)

Then, using equation (59), the specific heat per volume is given by
\[ c_V = \frac{\partial u}{\partial T} = \left( \frac{\rho_0 \mu}{B_0^2} \right)^\frac{3}{2} \frac{2\pi^2 k_B^4}{15 \hbar^2} T^3. \] (60)

CONCLUSION

In a gaseous plasma or a solid-state plasma, the propagation of the Alfvén waves is a significant characterization of the disturbed magnetic field. The Alfvén waves have been detected in astrophysical, laboratory, and solid-state plasmas. The magnetic pressure and tension are the restoring forces of the longitudinal and transverse Alfvén waves, respectively. Both the transverse and longitudinal Alfvén waves carry an average energy density that is, for each mode of oscillation, proportional to the square of the velocity perturbation (and hence to the square of the plasma displacement vector) and is reminiscent of the classical energy of a harmonic oscillator with unit mass. This observation suggests that each mode of the Alfvén wave may be treated as an independent harmonic oscillator that can be canonically quantized. In particular, we introduce a pair of canonical variables that specify the phase space of the underlying classical system and use the Dirac’s canonical quantization prescription to write down the Alfvén wave velocity field (or the corresponding plasma displacement vector) in terms of the creation and annihilation operators. This defines the concept of “Alphonons”, the elementary excitations of the Alfvén wave velocity in a magnetized plasma that was first introduced in Gheibi et al. (2018).

We studied the characteristic features of the Alfvén wave quanta, the Alphonons. Specifically, we showed that two polarization vectors (perpendicular to the wave vector) are required for the quantization of the transverse Alfvén wave. These two physical polarization vectors and the corresponding dispersion relation (see equation (21)) closely resemble those of the photon. This fact allows us to assign spin one to the Alphonon as a quantum of the transverse Alfvén wave. We also used a single polarization vector (parallel to the wave vector) to quantize the longitudinal Alfvén wave. This means that the longitudinal Alfvén wave quanta can be specified with a quantized scalar filed (the amplitude of the velocity field or the displacement vector along the wave vector). Therefore, these quanta have zero spin. This shows that the Alphonons obey the bosons behavior. Indeed, any number of identical excitations (normal modes of the Alfvén waves related to the perturbations of the magnetic fields and the reason for the oscillatory motions for particles in plasma) can be created by the repeated application of the creation operator $a^\dagger$. For a given state of the system the creation operator increases the number of particles by one in that state. This is equivalent to the excitation of new Alfvén mode in the system (new Alphonon).

Finally, we also studied the thermodynamic properties of a system of Alphonons. Assuming that the system with a constant volume and fixed number of quasi-particles is in a thermal contact with the environment at temperature
$T$, we can consider the canonical ensemble and write down the expressions for the Helmholtz free energy. From the dispersion relation we determined the density of states, the entropy, the total internal energy. Then the spectral energy density of the interstellar medium, etc) are examples that the concept of Alphonons may be useful to study the collective behavior of such systems.

An immediate application of the concept of Alphonon was made by Gheibi et al. (2018), who showed that in the horizon of an Alfvénic black hole, Alfvén waves will be trapped and emit Hawking radiation that comprises the Alphonons. Detecting the Hawking radiation from the gravitational black holes is an important task for scientists. Because the temperatures predicted by Hawking effect is far less than the cosmic microwave background radiation temperature, it makes difficult to verify the black hole evaporation. Nevertheless, the concept of the artificial black holes (e.g., sonic, Alfvénics, magnetooacoustic) is recently investigated to study the properties of Hawking radiations. On the other hand, the solid-state plasma, solar and stellar atmospheres, the space plasma (e.g., intergalactic, interplanetary, interstellar medium, etc) are examples that the concept of Alphonons may be useful to study the collective behavior of such systems.

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