THE EQUILIBRIUM STRUCTURE OF PROLATE MAGNETIZED MOLECULAR CORES

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ABSTRACT

The structure of molecular cloud cores supported by thermal pressure and a poloidal magnetic field is reinvestigated in the magnetohydrostatic and axisymmetric approximation. In addition to oblate configurations found in earlier work, solutions yielding prolate spheroidal shapes have also been obtained for a reference state described by a uniform sphere threaded by a uniform background magnetic field. The solutions for prolate configurations are found to be relevant for lower masses than for their oblate counterparts. Of particular importance is the result that the prolate cloud cores have radii less than a maximum given by 0.25 pc ($a$) / ($c_{s}$) \textsuperscript{2} where $a$ is the sound speed and $P_{\text{ext}}$ is the external pressure of the background medium. The existence of such solutions obviates the presence of toroidal fields in such modeled structures.

Key words: ISM: clouds – magnetic fields – magnetohydrodynamics (MHD) – stars: formation

1. INTRODUCTION

The study of dense molecular cloud cores is of paramount interest because they are the sites where stars form (see Shu et al. 1987; McKee & Ostriker 2007, for reviews). In situations where other physical effects in comparison with gravity and thermal pressure can be neglected, these cores have been modeled as isothermal spheres in hydrostatic equilibrium bounded by an external pressure, known as Bonnor–Ebert spheres (Bonnor 1956; Ebert 1955). For some objects, such a model offers remarkable agreement with observation (Alves et al. 2001; Kirk et al. 2005; Schnee & Goodman 2005; Stutz et al. 2007). However, surveys of large samples of dense cores in dark clouds reveal that spherically symmetric cores are the exceptions rather than the rule. The projected aspect ratios can significantly differ from unity, making the intrinsic geometry even more elongated (Myers et al. 1991; Jijina et al. 1999). If we assume these cores are axisymmetric and are randomly oriented in the sky, statistical analysis suggests many of them are prolate (Myers et al. 1991; Ryden 1996). On the other hand, oblate shapes may fit better with the observed distribution of shapes for intrinsically triaxial cores (Jones et al. 2001; Tassis 2007).

The formation of prolate cores presents certain theoretical challenges. If the formation history of these cores is dominated by quasi-static contraction regulated by ambipolar diffusion, the resulting geometry is a sequence of oblate spheroids (Mouschovias 1976a, 1976b; Nakano 1979; Lizano & Shu 1989). This is a natural outcome since matter can contract freely along the field lines, but must overcome the additional magnetic pressure and tension perpendicular to them. The inclusion of rotation produces even flatter morphologies, but there is little observational evidence supporting the view that rotation is important for cores (Goodman et al. 1993). Tomisaka (1991) and Fiege & Pudritz (2000) invoked a helical magnetic field to explain the prolate cores and were able to produce a large range of aspect ratios consistent with observation. Here, the hoop stress generated by the toroidal field component facilitates the confinement of matter toward the axis. However, the presence of toroidal fields requires a current flowing along the symmetry axis, which is difficult for astrophysical systems to realize. We note that Curry & Stahler (2001) were able to obtain both prolate and oblate cores with only a poloidal field by specifying the shape of the core and solving for the mass-to-flux ratio after an equilibrium had been found. In this case, the mass-to-flux ratios are distinct for the two types of cores.

In this Letter, we reexamine the structure of molecular cloud cores, which are treated as magnetohydrostatic equilibria, with the ultimate goal of providing a grid of models that can be used for interpreting pre-stellar core density structures. We have found that magnetized structures for both prolate and oblate configurations can be produced for a wide range of aspect ratios from a purely poloidal field configuration with a generic functional form of the mass-to-flux ratio. This is in contrast to previous work where prolate configurations were produced only with a toroidal field or a specific mass-to-flux ratio. The findings of this work help to elucidate the physical parameters that dictate the morphologies of dense molecular cores. The theoretical framework including the assumptions and equations of the model are described in Section 2. The method of solution and the numerical results are presented in Section 3. Finally, we summarize and discuss the results in Section 4.

2. FORMULATION AND BASIC EQUATIONS

For mathematical tractability, the figures of equilibrium are restricted to be axisymmetric and described in the cylindrical coordinates ($\sigma$, $z$). Consider a cloud core described by an isothermal equation of state $P = a^{2}\rho$, where $P$, $\rho$, and $a$ are the pressure, density, and a constant sound speed, respectively. Let this cloud be embedded in an environment characterized by a large-scale background magnetic field $B_{0} = B_{0}\hat{z}$ and an external pressure $P_{\text{ext}}$. Pressure equilibrium requires the cloud to have density $\rho_{0} = P_{\text{ext}}/a^{2}$ on its surface. From these parameters, along with the gravitational constant, $G$, the units of length, density, pressure, gravitational potential, mass, magnetic field, and magnetic flux are taken as

\[ l_{0} \equiv \frac{a}{\sqrt{4\pi G \rho_{0}}}, \quad \rho_{0}, \quad P_{\text{ext}}, \quad a^{2}, \]
\[ M_{0} \equiv 4\pi \rho_{0} l_{0}^{3}, \quad B_{0}, \quad \Phi_{0} \equiv 2\pi B_{0} l_{0}^{2}, \]

respectively. For typical cores, if we take $a = 0.2$ km s$^{-1}$, $P_{\text{ext}} = 10^{-12}$ dyne cm$^{-2}$, and $B_{0} = 20 \mu G$, then $l_{0} = 0.14$ pc,
The specific enthalpy, we note that the problem. To determine the location of the core boundary and
obtain the force equation can be projected along the magnetic field to obtain $\mathbf{B} \cdot \nabla h = 0$, where
$h \equiv \log \rho + V$ (6)
is the specific enthalpy. This implies that $h(\sigma, z) = h(\Phi)$ is a function of $\Phi$ alone. The component of the force equation perpendicular to the magnetic field, known as the Grad–Shafranov equation, governs the spatial distribution of field lines, and it can now be manipulated to read

$$\nabla \cdot (\sigma^{-2} \nabla \Phi) = -\beta \rho \frac{dh}{d\Phi}. \quad (7)$$

To close the system of equations, we impose the integral constraints that the mass in each flux tube is conserved, and is given by

$$\int_{0}^{Z(\Phi)} \rho \sigma \frac{d\sigma}{d\Phi} \frac{dz}{\sigma} = \frac{dM}{d\Phi}. \quad (8)$$

In the above equation, the $z$ integral is performed over constant $\Phi$, and $dM/d\Phi$ is the known differential mass-to-flux ratio, obtainable from either observation or an evolutionary calculation. The core surface, described by $Z(\Phi)$, is a free internal surface of the problem. To determine the location of the core boundary and the specific enthalpy, we note that $h = V_{z=Z}$ because $\rho = 1$ on the surface. Once $V$ is known, both $h$ and $Z$ can be determined by solving the equation

$$e^{-V} \bigg|_{z=Z} = \frac{dM}{d\Phi} \left[ \int_{0}^{Z(\Phi)} e^{-V} \sigma \frac{d\sigma}{d\Phi} \frac{dz}{\sigma} \right]^{-1}. \quad (9)$$
along each flux tube.

3. NUMERICAL METHODS AND RESULTS

Our numerical scheme for constructing solutions is an iterative procedure similar to that of Mouschovias (1976a) and Tomisaka et al. (1988a). For definiteness, we adopt a differential mass-to-flux ratio corresponding to a reference state consisting of a uniform sphere threaded by the uniform background field. If the total mass of the core is $M_c$, and total trapped flux is $\Phi_c$, then

$$\frac{dm}{d\Phi} = \frac{3M_c}{2\Phi_c} \sqrt{1 - \Phi/\Phi_c}. \quad (10)$$

This configuration was termed “parent cloud” by the authors, and serves as the initial trial solution. Starting with the cloud, $h$ and $Z(\Phi)$ are computed from Equation (9), which allows a calculation of the source functions in Equations (1) and (7). These equations are then solved by successive over-relaxation to obtain intermediate solutions. A corrected solution is constructed by under-relaxation and updated with the next iteration. The procedure continues until the fractional error is reduced to a pre-specified level. Once a converged solution is obtained, higher accuracies are then achieved by a series of mesh refinements.

We were able to reproduce the solutions presented by Mouschovias (1976b) and Tomisaka et al. (1988b) using their input parameters. We compare our result visually to theirs and estimate the fractional error to be within 5%. In particular, for a fixed total trapped flux, there is a maximum mass beyond which no equilibrium solution exists. The numerical value of this maximum mass agrees with earlier work by Tomisaka et al. (1988b) to within 3%. This mass is equivalent to the Bonnor–Ebert mass, the maximum stable mass for a given temperature and external pressure, but modified by the presence of a magnetic field.

In the magnetically regulated star formation paradigm, a key parameter is the total mass-to-flux ratio, $\lambda = \beta^{1/2} M_c/\Phi_c$ (or in conventional units $\lambda = 2\pi G^{1/2} M_c/\Phi_c$). The supercritical clouds (with $\lambda > 1$) are capable of continued contraction leading to star formation, while the subcritical clouds (with $\lambda < 1$) are not. Through a process of natural selection, modern-day molecular clouds are most likely to be in a marginally critical state (Shu et al. 2004). Clouds with $\lambda \gg 1$ would have collapsed to form stars, while those with $\lambda \ll 1$ would have evolved to the diffuse interstellar medium. These theoretical arguments are consistent with the findings of Troland & Crutcher (2008) who estimated $\lambda \sim 2$ in the cores. Thus, we argue that instead of fixing the total flux, as in Tomisaka et al. (1986b) and Tomisaka et al. (1988b), a more convenient parameterization would be to vary the total mass while maintaining an order unity value of $\lambda$.

The solution space is three dimensional, parameterized by $\lambda$, $\beta$, and $M_c$. However, for values of $\lambda \in [0.1, 7]$ and $M_c < 20$, the solution is approximately degenerate. With $\alpha \equiv \lambda^{1/2}/M_c/\Phi_c$ held fixed, varying $\beta$ by 2 orders of magnitude only introduces less than a 1% change to the equilibrium solutions, indicating the relative insensitivity of the magnetic field strength in this parameterization. We shall thus use $\alpha$ and $M_c$ as our basic parameters when discussing the numerical results.

Of primary interest in this study is the core shape, and the aspect ratios of the cores are illustrated in Figure 1 as functions of $M_c$ for several values of $\alpha$. Here, the aspect ratio is defined as the ratio of the radial extent to the vertical extent of the cloud core surface. An important feature to note is that for fixed $\alpha$, there exists a critical mass $M_{\text{crit}}$ below which the core takes a prolate shape. It can be seen that the prolate configurations are restricted to low core masses. Specifically, for $\alpha = 1$ and
Figure 1. Core aspect ratios as functions of cloud mass for $\lambda = 1$, and $\alpha = 0.5, 0.75, 1.0, 2.0$, and $3.0$. The dashed line indicates the critical mass at which the aspect ratio is unity.

Figure 2. Typical prolate core solution is illustrated in the cylindrical coordinates ($\varpi, z$). The solid curves are density contours, the dashed curves represent magnetic field lines, and the dash-dotted curves are contours of the gravitational potential. For this particular solution, $\lambda = 1$, $\alpha = 2$, and $M_c = 1$. It has an aspect ratio of 0.582, a central density of $\rho_c = 2.5$, and a radius of $R = 1$.

Typical core parameters (see above), the critical mass is 0.96 $M_\odot$. Furthermore, as $\lambda$ increases for fixed $\beta$ or as $\beta$ decreases with fixed $\lambda$, the critical core mass increases. The density and magnetic field structure for a typical prolate solution is displayed in Figure 2.

The solutions for prolate cores are also found to exhibit a maximum radius, beyond which such solutions do not exist. In particular, the parent cloud radius $R_c$ and the mass $M_c$ that leads to a final core of unit aspect ratio are shown as functions of $\alpha$ in Figure 3. Here, the final core radius is virtually indistinguishable from that of the parent cloud. It is evident that the radius is a non-monotonic function of $\alpha$, in contrast to the core mass, exhibiting a maximum at a radius of 1.75 (or about 0.25 pc for typical core parameters) for $\alpha \sim 3$.

4. SUMMARY AND DISCUSSION

We have reexamined the structure of dense molecular cloud cores supported, in part, by magnetic forces and thermal pressure. In most of the parameter space that we have surveyed, our results agree with previous studies that exist in the literature. Of particular interest is the discovery of solutions for which the cores assume prolate shapes. As we have shown (see Figure 1), for a fixed mass-to-flux ratio, prolate configurations are more relevant for smaller masses, while the oblate ones are for larger masses.

The formation of prolate cores can be understood in an evolutionary sense not too different from that of oblate ones. For our choice of the reference state, and a fixed $\alpha$, the core mass is related to the density, $\rho_c$, by

$$M_c = \left(\frac{\alpha}{2}\right)^3 \left(\frac{3}{\rho_c}\right)^2.$$  \hspace{1cm} (11)

Cores with sufficiently small mass represent overdense regions, and they must expand to achieve equilibrium. Because of the additional external pressure exerted by the magnetic field, the gas experiences less resistance in the direction parallel to the field lines than perpendicular to them. The subsequent evolution naturally leads to a prolate figure. Instead of an hourglass-shaped magnetic field commonly seen accompanying oblate cores in either theoretical calculations (Mouschovias 1976b; Tomisaka et al. 1988b) or observations (Girart et al. 2006, 2009), the magnetic field lines bow outward in the midplane. However, for our particular choice of the parent cloud, because the overall magnetic energy is much larger than thermal energy for small masses, the outward bowing of the field lines is not pronounced in general. Of all the prolate-shaped models we have constructed, the fractional distortion of the magnetic field lines only occurs at a $10^{-4}$ level. Other forms of $dm/d\Phi$ may lead to more drastic outward bowing of the field lines for prolate cores (see, e.g., Curry & Stahler 2001).

It is instructive to inquire why previous studies, such as Mouschovias (1976b) and Tomisaka et al. (1988b), did not find such kind of prolate cores with just poloidal fields, since they have surveyed a large portion of the parameter space spanning many orders of magnitude in both $\beta$ and central density $\rho_c$. The answer lies in the total flux trapped by the core. Tomisaka et al. (1988b) used a uniform sphere with various mass-to-flux distributions as a parent cloud to initiate their iterative calculation. Our particular choice of parent cloud is threaded by a uniform field, which corresponds to $N = 1$ model.
Its radius is related to the trapped flux by \( R_{c} = (2\Phi_{c})^{1/2} \) in dimensionless variables. As we have shown in Section 3, all prolate cores must evolve from parent clouds of radius \( R \approx 1.75 \), but other authors chose radii larger than 2.4. Because mass and flux scale differently with radius, a sufficiently large parent cloud with fixed \( \alpha \) will always be dominated by gravity, and hence evolve to an oblate configuration. Whether adopting a different mass-to-flux distribution within the cloud will change this conclusion remains to be examined.

The fact that there exists a maximum radius, \( R_{\text{max}} \), for prolate cores has interesting observational consequences. In conventional units, this critical radius is given by

\[
R_{\text{max}} = 0.25 \text{ pc} \left( \frac{a}{0.2 \text{ km} \text{ s}^{-1}} \right)^{2} \left( \frac{P_{\text{ext}}}{10^{-12} \text{ dyne cm}^{-2}} \right)^{-1/2}.
\]

(12)

For typical molecular cloud cores, this value is consistent with those studied by Myers et al. (1991) and Ryden (1996). We may infer more detailed parameters of the cores if the properties of the environment can be constrained by observations. In particular, if we are able to obtain both the values of \( \lambda \) and

\[
\beta = 0.13 \frac{P_{\text{ext}}}{10^{-12} \text{ dyne cm}^{-2}} \left( \frac{B_{0}}{10 \mu \text{G}} \right)^{-2},
\]

then we may compute the value of \( \alpha \). Inspection of Figure 3 provides an upper limit on the mass of a prolate core. As remarked earlier, the masses for prolate configurations are predicted to be lower than for oblate configurations and, hence, such cloud cores are expected to be characterized by a lower visual extinction.

Our analysis is restricted to axisymmetric models with only poloidal fields. These assumptions are justified if the cores were formed via gradual contraction regulated by ambipolar diffusion (however, see Basu & Ciolek 2004; Ciolek & Basu 2006, for discussions on the growth of non-axisymmetric modes in sheet-like structures). Relaxation of these restrictions yields additional possibilities, which may shed light on the initial conditions and the formation process of the molecular cores. In order for magnetic confinement to yield prolate cores, the parent cloud must be over dense and try to expand initially. How to achieve such an initial state is an intriguing question. For instance, numerical simulations suggest that collisions between turbulent clouds would leave behind triaxial dense cores that tend to be prolate (Gammie et al. 2003; Li et al. 2004; Offner & Krumholz 2009). One could consider the possibility that the colliding clouds created the high-density region in the first place.

After the turbulence had decayed, the subsequent expansion against an anisotropic external magnetic pressure may facilitate the formation of the low-mass prolate cores. We reserve such a consideration for a future study.

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