Plane waves in metric-affine gravity

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Abstract

We describe plane-fronted waves in the Yang-Mills type quadratic metric-affine theory of gravity. The torsion and the nonmetricity are both nontrivial, and they do not belong to the triplet ansatz.

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I. INTRODUCTION

Although Einstein’s general relativity theory is satisfactorily supported by experimental tests on a macroscopic level, the gravitational interaction on a microscopic scale is not well understood. The gravitational gauge models provide an alternative description of gravitational physics in the microworld [1]. A variety of models arise within the framework of the gauge approach to gravity (Poincaré, teleparallel, metric-affine, supergravity, to mention but a few), and their corresponding kinematic schemes are well established at present. However, the dynamic aspects of the gauge gravity models have been rather poorly studied up to now. This includes the choice of the basic Lagrangian of the theory, as well as the detailed analysis of possible physical effects. The derivation of a new exact solutions for these models may bring new insight to the understanding of gravitational physics on small scales.

The plane-fronted gravitational waves represent an important class of exact solutions which generalize the basic properties of electromagnetic waves in flat spacetime to the case

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of curved spacetime geometry. The relevant investigation of the gravitational waves in general relativity has a long and rich history, see, e.g., \[2–7\]. The discussion of the possible generalizations of such solutions revealed the exact wave solutions in Poincaré gauge gravity \[8–13\], in teleparallel gravity \[14\], in generalized Einstein theories \[15,16\], in supergravity \[17–21\], as well as, more recently, in superstring theories \[22–28\]. Some attention has also been paid to the higher-dimensional generalizations of the gravitational wave solutions \[29–32\]. It was demonstrated \[33–38\] that gravitational wave solutions are also admitted in the metric-affine theory of gravity (MAG) with the propagating torsion and nonmetricity fields. The latter results are, however, restricted either to the case of torsion waves only, or to the triplet class of solutions with a specific ansatz for torsion and nonmetricity \[39,40\] and for a special form of the Lagrangian.

The aim of this paper is to describe the plane gravitational waves for the general Yang-Mills type quadratic MAG Lagrangian with nontrivial torsion and nonmetricity configurations that do not belong to the triplet ansatz. The motivation is twofold. On the one hand, the systematic study of the space of solutions represents a significant aspect of the development of any field-theoretic model. On the other hand, the wave phenomena as such are of fundamental importance, and the construction and comparison of the wave solutions in different models may clarify the physical contents of and the relations between the microscopic and macroscopic gravitational theories (in particular, general relativity, Poincaré gauge gravity and MAG).

The metric-affine spacetime is described by the metric \(g_{\alpha\beta}\), the coframe 1-forms \(\vartheta^\alpha\), and the linear connection 1-forms \(\Gamma_\beta^\alpha\). These are interpreted as generalized gauge potentials, while the corresponding field strengths are the nonmetricity 1-form \(Q_{\alpha\beta} = -Dg_{\alpha\beta}\) and the 2-forms of torsion \(T^\alpha = D\vartheta^\alpha\) and curvature \(R_\beta^\alpha = d\Gamma_\beta^\alpha + \Gamma_\gamma^\alpha \wedge \Gamma_\beta^\gamma\). The metric-affine geometry reduces to a purely Riemannian one as soon as torsion and nonmetricity both vanish. The teleparallel geometry arises when the curvature is trivial, \(R_\beta^\alpha = 0\), whereas a vanishing nonmetricity \(Q_{\alpha\beta} = 0\) yields the Riemann-Cartan geometry of spacetime. It is well known that for every metric \(g_{\alpha\beta}\) there exists a unique torsion-free and metric-compatible
connection represented by the Christoffel symbols. We will denote this Riemannian connection by \( \tilde{\Gamma}^\alpha_{\beta\gamma} \), and hereafter the tilde will denote purely Riemannian geometrical objects and covariant differentials constructed from them. Our general notations and conventions for the basic geometric objects, the holonomic and anholonomic indices, the choice of the metric signature are that of [1].

The plan of the paper is as follows. In the next Sec. II, we recall the definition of the ordinary electromagnetic wave. This is used then in Sec. III for the description of the corresponding ansatz for a gravitational plane wave in MAG. The properties of the resulting curvature, torsion and nonmetricity are discussed in Sec. IV. Finally, in Sec. V we demonstrate that the proposed ansatz provides the exact solution for the general quadratic MAG model. The conclusions are outlined in Sec VI.

II. ELECTROMAGNETIC PLANE WAVES

An electromagnetic plane wave is described by a 1-form \( u \) which satisfies

\[
\begin{align*}
\ast du &= 0, \\
\ast du &= 0, \\
k \wedge \ast du &= 0, \\
k \wedge du &= 0.
\end{align*}
\]

The propagation 1-form \( k \) is null (i.e., \( k \wedge \ast k = 0 \)) and geodetic, \( k \wedge \ast dk = 0 \), and it corresponds to a congruence with zero shear, expansion and rotation. This is typical for the plane wave, and the equations (2) represent the so-called radiation conditions imposed on the electromagnetic field. With the electromagnetic potential \( A = u \), the electromagnetic field strength \( F = du \) satisfies the vacuum Maxwell equation (1).

We do not specify the spacetime metric (and hence the Hodge operator \( \ast \)) as the flat Minkowski one. It will be convenient not to fix the spacetime geometry at this stage.

Such an electromagnetic wave construction underlies the derivation of the corresponding gravitational wave solutions as described in [41–45], for example. In our study, we will use the similar constructions by extending the Riemannian results to their non-Riemannian counterparts.
Let us denote the local spacetime coordinates as \( x^i = \{\sigma, \rho, z^2, z^3\} \). The upper case Latin indices, \( A, B, \ldots = 0, 1 \), will label the first 2 spacetime dimensions which are relevant to a pp-wave. In particular, \( x^A = \{\sigma, \rho\} \) are the wave coordinates with the wave fronts described by the surfaces of constant \( \sigma \), and \( \rho \) is an affine parameter along the wave vector of the null geodesic. The lower case Latin indices, \( a, b, \ldots = 2, 3 \), refer to the remaining spatial coordinates: \( x^a = \{z^2, z^3\} \). The Greek indices, \( \alpha, \beta, \ldots = 0, \ldots, 3 \), label the local anholonomic (co)frame components. We denote separate frame components by a circumflex over the corresponding index in order to distinguish them from coordinate components.

The can now formulate the wave ansatz for the MAG gravitational potentials \( g_{\alpha\beta}, \psi^\alpha, \) and \( \Gamma^\beta_{\alpha} \) as follows. We choose the half-null metric

\[
g_{\alpha\beta} = \begin{pmatrix} g_{AB} & 0 \\ 0 & g_{ab} \end{pmatrix}, \quad g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_{ab} = \delta_{ab}. \tag{3}\]

The components of the coframe 1-form are given by

\[
\vartheta^0 = -d\sigma, \quad \vartheta^1 = \frac{1}{2} H(\sigma, z^a) d\sigma + d\rho, \quad \vartheta^a = dz^a, \quad a = 2, 3. \tag{4}\]

Finally, the ansatz for the affine connection reads:

\[
\Gamma^{\alpha\beta} = k^{[\alpha} \varphi^{\beta]} k + k^\alpha k^\beta u. \tag{5}\]

The corresponding dual frame basis (such that \( e_{\alpha] \vartheta^\beta = \delta^\beta_\alpha \)) reads:

\[
e^0 = -\partial_\sigma + H \partial_\rho, \quad e^1 = \partial_\rho, \quad e_a = \partial_a. \tag{6}\]

We now make a crucial assumption that the 1-forms \( u \) and \( k \) above fulfill the radiation conditions (1), (2). Moreover, using the local coordinate system adapted for a plane wave and the half-null nature of the coframe (4), we can put \( k = \vartheta^0 \) without loss of generality. As usual, we introduce the components as \( k_\alpha = e_{\alpha] k \). Finally, the components of \( \varphi_\alpha \) are determined by the function \( H \) as follows:
\[ \varphi_0 = 0, \quad \varphi_1 = 0, \quad \varphi_a = \partial_a H. \] (7)

Although this choice looks to be rather ad hoc, it is actually well motivated by the corresponding Riemannian solution, cf. [32]. Indeed, the ansatz (3) and (4) for the metric and the coframe is exactly the same as in the purely Riemannian case, whereas the connection (5) minimally extends the Christoffel connection (given in Appendix of [32]) via the term proportional to the non-Riemannian parameter \( u \). In the next section we demonstrate that both the torsion and the nonmetricity are determined by this 1-form.

It is worthwhile to note that the wave 1-form is closed, \( dk = 0 \), whereas the wave covector is covariantly constant,

\[ Dk_\alpha = -\Gamma_\alpha^{\beta\gamma} k_\beta = 0, \quad Dk^\alpha = \Gamma_{\beta}^{\alpha} k^\beta = 0. \] (8)

Here we used the fact that this covector is null, \( k^\alpha k_\alpha = 0 \) and orthogonal to \( \varphi_\alpha \), i.e. \( k^\alpha \varphi_\alpha = 0 \).

IV. NONMETRICITY, TORSION AND CURVATURE

Given the above ansatz for the MAG potentials – metric (3), coframe (4) and linear connection (5) – it is straightforward to find the corresponding gauge field strengths. The nonmetricity, torsion and curvature read, explicitly:

\[ Q_{\alpha\beta} = 2k_\alpha k_\beta u, \] (9)

\[ T^\alpha = k^\alpha u \wedge k, \] (10)

\[ R^{\alpha\beta} = 2\gamma^{[\alpha} k^{\beta]} \wedge k + k^\alpha k^\beta du. \] (11)

Here the covector-valued 1-form is defined by \( \gamma_\alpha = -\frac{1}{2} d\varphi_\alpha \), where the differential \( d \) is taken with respect to the \( z^a \) coordinates only. This 1-form has the obvious properties: \( k^\alpha \gamma_\alpha = 0 \), \( \varphi^\alpha \wedge \gamma_\alpha = 0 \) and \( e_A [\gamma_\alpha = 0, \quad A = 0, 1. \)

When \( u = 0 \), we find zero torsion and nonmetricity. In this sense, we may consider the nontrivial \( u \) to represent the true post-Riemannian geometric structures which we are primarily interested in.
Let us compute the irreducible parts of the curvature 2-form. It is well known [1] that the curvature for the general linear connection can be decomposed into 11 irreducible parts. Following [1], we first decompose the curvature 2-form into the skew-symmetric and symmetric forms,

\[ W^{\alpha\beta} = R^{[\alpha\beta]} = 2\gamma^{[\alpha} k^{\beta]} \wedge k, \quad Z^{\alpha\beta} = R^{(\alpha\beta)} = k^{\alpha} k^{\beta} \, du. \]  

Then we have to calculate the interior products with the frame and exterior products with the coframe. In tensor language, this corresponds to computing the various contractions of the curvature tensor.

All the contractions of the symmetric curvature are trivial. Namely, \( Z = Z_{\alpha} = 0 \) in view of the nullity of the wave vector \( (k_{\alpha} k^{\alpha} = 0) \), whereas

\[ e_\alpha \lrcorner Z^{\alpha\beta} = -k^\beta \ast (k \wedge \ast du) = 0, \quad \vartheta_\alpha \wedge Z^{\alpha\beta} = k^\beta k \wedge du = 0, \]  

due to the fact that \( k_\alpha \vartheta^\alpha = k \) and using the radiation conditions (2). As a result, all the irreducible parts of the symmetric curvature form are zero except for the first piece:

\[ Z_{\alpha\beta} = (1) Z_{\alpha\beta}. \]  

In tensor language this means that all the contractions of the symmetric part of the curvature tensor are trivial, i.e., this tensor is totally trace-free and dual trace-free. The symmetric part of the curvature has no Riemannian counterpart, this is a totally post-Riemannian object.

For the skew-symmetric curvature we find straightforwardly:

\[ e_\alpha \lrcorner W^{\alpha\beta} = (e_\alpha \lrcorner \gamma^\alpha) k^\beta k, \quad \vartheta_\alpha \wedge W^{\alpha\beta} = 0. \]  

Accordingly, if we demand that the zero-form \( e_\alpha \lrcorner \gamma^\alpha \) vanishes, then all the contractions of the skew-symmetric curvature are also trivial. This condition imposes the partial differential equation on the unknown function \( H \):

\[ e_\alpha \lrcorner \gamma^\alpha = -\frac{1}{2} \partial_\alpha \partial^\alpha H = 0. \]
Provided that $H(\sigma, z^a)$ is a solution of the Laplace equation (16), we ultimately find that all the irreducible parts of the skew-symmetric curvature form are zero except for the first piece

$$W_{\alpha\beta} = (1)W_{\alpha\beta}.$$  \hfill (17)

This is again the pure tensor part which is totally trace-free and dual trace-free, in complete analogy with the symmetric curvature. The 2-form $W_{\alpha\beta}$ is a direct non-Riemannian generalization of the Weyl tensor.

V. FIELD EQUATIONS: QUADRATIC MAG MODEL

Let us consider the general Yang-Mills type (curvature quadratic) Lagrangian for the MAG model which was studied recently in the literature (see, for example, [33–38]):

$$V_{\text{MAG}} = -\frac{1}{2} R^{\alpha\beta} \wedge \left( \sum_{I=1}^{6} w_I (W_{\alpha\beta} + w_7 \theta_\alpha \wedge (\epsilon_\gamma)^{(5)} W^{\gamma\beta}) + \sum_{I=1}^{5} z_I (Z_{\alpha\beta} + z_6 \theta_\gamma \wedge (\epsilon_\alpha)^{(2)} Z^{\gamma\beta}) + \sum_{I=7}^{9} z_I \theta_\alpha \wedge (\epsilon_\gamma)^{(I-4)} Z^{\gamma\beta} \right).$$  \hfill (18)

The 16 dimensionless coupling constants $w_1, \ldots, w_7, z_1, \ldots, z_9$ describe the contributions of all possible quadratic invariants which can be constructed from the components of the curvature in a general MAG theory [46].

The vacuum gravitational field equations of the MAG theory read [1]:

$$DH_\alpha - E_\alpha = 0,$$  \hfill (19)

$$DH^{\alpha}_\beta - E^{\alpha}_\beta = 0.$$  \hfill (20)

The gravitational gauge field momenta are introduced by partial differentiation,

$$H_\alpha = -\frac{\partial V}{\partial T^\alpha}, \quad H^{\alpha}_\beta = -\frac{\partial V}{\partial R^{\alpha}_\beta}, \quad M^{\alpha\beta} = -2 \frac{\partial V}{\partial Q^{\alpha\beta}},$$  \hfill (21)

whereas the canonical gauge field currents of the gravitational energy–momentum and of the hypermomentum, respectively, are defined as the following expressions, linear in the Lagrangian and in the gauge field momenta:
\[ E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha [V + (e_\alpha J^\beta) \wedge H_\beta + (e_\alpha J R^\gamma) \wedge H_\gamma + \frac{1}{2}(e_\alpha J Q_{\beta\gamma}) M^{\beta\gamma}, \quad (22) \]
\[ E_{\alpha \beta} := \frac{\partial V}{\partial \Gamma_{\alpha \beta}} = -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma} M^{\alpha\gamma}. \quad (23) \]

For the purely curvature quadratic Lagrangian (18), we obviously have \( H_\alpha = 0 \) and \( M^{\alpha\beta} = 0 \). Hence \( E_{\alpha \beta} = 0 \), and as a result the field equations (19) and (20) reduce to

\[ E_\alpha = e_\alpha [V + (e_\alpha J R^\gamma) \wedge H_\gamma] = 0, \quad (24) \]
\[ D H_{\alpha \beta} = 0. \quad (25) \]

For the above gravitational wave ansatz, we have verified that all the irreducible parts of the curvature are trivial except for the pure tensor pieces (14) and (17). Accordingly, the direct computation of the gravitational hypermomentum 3-form then yields

\[ H_{\alpha \beta} = * \left( w_1^{(1)} W^{\alpha \beta} + z_1^{(1)} Z^{\alpha \beta} \right). \quad (26) \]

Let us now demonstrate that the gravitational wave ansatz above provides an exact solution for the the gravitational field equations (24) and (25). The following two facts are crucial for this. The first one is the property of the curvature 2-form (11)

\[ k_\alpha R_{\alpha \beta} = 0, \quad k_\beta R_{\alpha \beta} = 0, \quad (27) \]

which is obviously satisfied due to the null nature of the wave vector \( k \) and its orthogonality to the 1-form \( \gamma_\alpha \). The second fact concerns the well-known double duality property for the first irreducible part of the curvature:

\[ *^{(1)} W_{\alpha \beta} = \frac{1}{2} \eta_{\alpha \beta \mu \nu}^{(1)} W^{\mu \nu}, \quad (28) \]

We begin with the first equation (24). The property (27) obviously yields for the gravitational wave configuration \( V_{\text{MAG}} = -\frac{1}{2} R^{\alpha\beta} \wedge H_{\alpha\beta} = 0 \) as well as the contraction \((e_\alpha J R^\gamma) \wedge H_\gamma = 0 \). Hence, our ansatz solves the first equation.

Finally, we turn to the second equation (25). Substituting (26), we notice that the second term vanishes,
due to the property (8) and the radiation conditions (1). As a result, the second equation (25) reduces to \( w_1 D^* \left( (1) W^\alpha_\beta \right) = 0 \). Since \((1) W^\alpha_\beta = g^{\alpha\gamma} (1) W_{\gamma\beta} \), we have \( D^* \left( (1) W^\alpha_\beta \right) = g^{\alpha\gamma} D^* \left( (1) W_{\gamma\beta} \right) + Q^{\alpha\gamma} \wedge ^* \left( (1) W_{\gamma\beta} \right) \). Using the explicit form the nonmetricity (9), we prove that the last term vanishes. Thus we find, in the end,

\[ w_1 D^* \left( (1) W^\alpha_\beta \right) = 0. \tag{30} \]

Now we can use the double duality identity (28) and obtain

\[ \frac{w_1}{2} \eta_{\alpha\beta\mu\nu} D (1) W^{\mu\nu} = 0, \tag{31} \]

where we used the fact that the covariant derivative of the Levi-Civita tensor \( D\eta_{\alpha\beta\mu\nu} = -2Q \eta_{\alpha\beta\mu\nu} = 0 \) vanishes due to the absence of the Weyl covector, \( Q = \frac{1}{4} Q^{\alpha} = 0 \).

In order to demonstrate that our ansatz solves (31), we can use the MAG Bianchi identity which reads \( DR^\alpha_\beta = 0 \). We find \( D \left( g_{\alpha\gamma} R^\gamma_\beta \right) = g_{\alpha\gamma} DR^\gamma_\beta - Q_{\alpha\gamma} \wedge R^\gamma_\beta \). Because of (9) and (11), the last term vanishes for the gravitational wave configuration. Thus, in view of the Bianchi identity, the gravitational wave curvature satisfies \( DR^\alpha_\beta = 0 \). It now remains to use the explicit formulas (11), (12), (14) and (17) to verify that

\[ D(1) W^\alpha_\beta = 0, \tag{32} \]

since the covariant derivative of the symmetric curvature vanishes identically \( D(1) Z^\alpha_\beta = k^\alpha k^\beta ddu \equiv 0 \).

Consequently, (31) is satisfied by the gravitational wave ansatz, and this completes the proof that such a configuration is, indeed, an exact solution of the MAG field equations.

**VI. DISCUSSION AND CONCLUSION**

The extension of the Riemannian geometry of Einstein’s general relativity to the post-Riemannian structures of the metric-affine gravity can be motivated by a number of reasons.
Among them, we mention the problem of quantization (see the discussion of the renormalizable MAG models in [47,48]), the theory of defects in the continuous media with microstructure (for a overview, see [49] and [1]), the physics of hadrons in terms of extended structures (see [50–52] and more details and references in [1]), the study of the early universe (in particular, relating the post-Riemannian structures to the dark matter problem, see [53–55]). Finally, one can show that the MAG models may arise as the effective theories in the context of the dilaton-axion-metric low-energy limit of the string theory (see, e.g., [56–59]). The study of the exact solutions of the MAG field equations is important for understanding and development of the physical aspects mentioned above.

In this paper, we have derived a new plane wave solution of the general Yang-Mills type (curvature quadratic) metric-affine theory of gravity. This extends the previous study of the waves in the Yang-Mills type models of the Poincaré gauge gravity [8–13]. As compared to the other exact wave solutions available in the literature [33–38], the new configuration has the following characteristic properties: (i) the spacetime metric is not a flat Minkowski one but the metric of the Riemannian gravitational plane wave determined by the single harmonic function $H(\sigma, z^a)$, (ii) there are not only torsion waves present but the nonmetricity has a nontrivial wave behavior as well, (iii) the post-Riemannian sector of the torsion and nonmetricity does not belong to the triplet ansatz. It is worthwhile to note that the triplet ansatz might be considered as a useful tool which helps to avoid a possible problem of the well-posedeness of the field equations by reducing them to the effective Einstein-Maxwell system of equations. However, this ansatz is applicable only to a quite narrow class of the MAG models, namely to those Lagrangians (18) where the only nontrivial coupling constant $z_4 \neq 0$ is allowed. Our results apply to the general case with all the 16 nontrivial coupling constants $w_1, \ldots, w_7, z_1, \ldots, z_9$. The well-posedness of the general MAG model was never studied in the literature, and this question clearly represents an open potentially interesting and important problem within the metric-affine approach to gravity.

The results obtained have a number of interesting mathematical and physical applications. To begin with, the curvature quadratic Lagrangian is potentially important for a
quantized theory of gravity. Furthermore, the long-distance character of the wave solutions makes them a convenient tool for the tests of the additional properties of matter besides the mass (energy-momentum), namely, the hypermomentum which includes the spin and the dilaton/shear charges. This is of particular interest for the study of the elastic media with defects, and for the physics of hadrons (see the references quoted above).

It is worthwhile to stress that the new solutions are obtained as the direct generalization of the general-relativistic wave solutions. When the 1-form $u$ vanishes, the post-Riemannian geometric quantities disappear. On the other hand, the Riemannian (metric) sector of the solution has the same form as in general relativity, and thus all the earlier mathematical and physical analyses [2–7] are directly applicable to our case. In physical terms this means that the usual general relativistic detectors (with mass as the only gravitational charge) will not distinguish between the gravitational waves of the Einstein theory and the new MAG waves. This is in complete agreement with the correspondence principle which underlies the dynamical structure of the metric-affine gravity: MAG is not supposed to replace the general relativity theory in the well established macroscopic domain, but rather to extend the latter in the microscopic domain by taking into account the additional physical properties of matter (such as the spin, dilation and hypermomentum currents). The above conclusion is based on the fact that the equations of motion in MAG for the test particles without the spin, dilaton and proper hypermomentum exactly coincide with the equations of motion of the test massive matter in general relativity [60]. Furthermore, it is worthwhile to recall that within the Poincaré gauge gravity the equations of motion of matter are also known to coincide with the general-relativistic equations of motion for the bodies with the trivial average spin value [61]. The corresponding generalization is expected to be valid in MAG for the macroscopic bodies with vanishing average spin, dilaton and proper hypermomentum, although the detailed relevant analysis is still missing in the literature.

The new solution gives a natural generalization of the definitions of a gravitational plane-fronted wave. In accordance with [42], a gravitational wave is defined by the existence of a null, geodetic, shear-, twist, and expansion-free vector field $k$ and the Weyl 2-form subject
to the algebraic conditions

\[(1) W^\alpha_{\beta k} \kappa_{\beta} = 0, \quad (1) W[\alpha^\beta k^\gamma] = 0. \tag{33}\]

According to [10], a gravitational wave is defined by a covariantly constant null field and a quadratic algebraic condition on the components of the curvature tensor

\[R_{\mu\rho\alpha}^\beta R_{\rho\sigma\beta}^\alpha = 0. \tag{34}\]

As we can immediately verify, the non-Riemannian curvature (as well as its Riemannian constituent) satisfies both (33) and (34).

In [35–38] the notion of the pseudo-instanton solutions of the MAG field equations was introduced. The latter are described by a metric-compatible linear connection for which only one of the eleven irreducible parts of the curvature is nontrivial. Our gravitational wave solution provides a minimal generalization of the pseudo-instanton, in the sense that the nonmetricity does not vanish and that the curvature has two purely tensor irreducible parts (14) and (17).

The applications mentioned above refer to the fundamental MAG theory. However, we recall that MAG also arises as an effective theory within the framework of the dilaton-axion-metric low energy limit of the string models. Accordingly, one can use our solution as a technical tool to construct the exact wave configurations in the string motivated models where the plane waves play essential role [22–28]. This construction will be described in detail elsewhere.

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