ON THE CAHN-HILLIARD-NAVIER-STOKES EQUATIONS WITH NONHOMOGENEOUS BOUNDARY

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ABSTRACT. The evolution of two isothermal, incompressible, immiscible fluids in a bounded domain is governed by Cahn-Hilliard-Navier-Stokes equations (CHNS System). In this work, we study the well-posedness results for the CHNS system with nonhomogeneous boundary condition for the velocity equation. We obtain the existence of global weak solutions in the two-dimensional bounded domain. We further prove the continuous dependence of the solution on initial conditions and boundary data that will provide the uniqueness of the weak solution. The existence of strong solutions is also established in this work. Furthermore, we show that in the two-dimensional case, each global weak solution converges to a stationary solution.

Key words: Diffuse interface model, Cahn–Hilliard–Navier–Stokes system, long-time behaviour, Lojasiewicz-Simon inequality, Nonhomogeneous boundary conditions.

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1. INTRODUCTION

We consider the motion of an isothermal mixture of two immiscible and incompressible fluids subject to phase separation, which is described by the well-known diffuse interface model. It consists of the Navier-Stokes equations for the average velocity and a convective Cahn-Hilliard equation for the relative concentration, also known as Cahn-Hilliard-Navier-Stokes (CHNS) system or “model H”. A general

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model for such a system is given by:

\[
\begin{align*}
\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi &= \text{div}(m(\varphi)\nabla \mu), \quad \text{in } \Omega \times (0, T), \\
\partial_t \mathbf{u} - \text{div}(\nu(\varphi)D\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi &= \mu \nabla \varphi, \quad \text{in } \Omega \times (0, T), \\
\text{div} \mathbf{u} &= 0, \quad \text{in } \Omega \times (0, T),
\end{align*}
\]

where \(\mathbf{u}(x, t)\) is the average velocity of the fluid and \(\varphi(x, t)\) is the relative concentration of the fluid. Here, \(\Omega\) is a bounded domain in \(\mathbb{R}^2\), with a sufficiently smooth boundary \(\partial \Omega\). The density is taken as matched density, i.e., constant density, which is equal to 1. Moreover, \(m\) is mobility of binary mixture, \(\mu\) is a chemical potential, \(\pi\) is the pressure, \(\nu\) is the viscosity and \(F\) is a double well potential. The symmetric part of the gradient of the flow velocity vector is denoted by \(D\mathbf{u}\), that is, \(D\mathbf{u}\) is the strain tensor \(\frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)\). Furthermore, \(\mu\) is the first variation of the Helmholtz free energy functional

\[
E_1(\varphi) := \frac{1}{2} \int_\Omega |\nabla \varphi|^2 + F(\varphi(x))\, dx,
\]

where \(F\) is a double-well potential of the regular type. A typical example of regular \(F\) is

\[
F(s) = (s^2 - 1)^2, \quad s \in \mathbb{R}.
\]

A physically relevant but singular potential \(F\) is the Flory-Huggins potential given by

\[
F(\varphi) = \frac{\theta}{2}((1 + \varphi) \ln(1 + \varphi) + (1 - \varphi) \ln(1 - \varphi)) - \frac{\theta_c}{2} \varphi^2, \quad \varphi \in (-1, 1),
\]

where \(\theta, \theta_c > 0\).

The model \(\text{H}\) was derived in \([33, 34, 46]\) for matched densities. Whereas, for binary fluids with different densities, more generalized diffuse interface models were proposed in the literature (see, for instance, \([2, 5, 6, 9, 15, 42]\)). There are considerable amount of works devoted to the mathematical analysis of model \((1.1)\) subject to boundary condition

\[
\frac{\partial \varphi}{\partial n} = 0, \quad \frac{\partial \mu}{\partial n} = 0, \quad \mathbf{u} = 0, \quad \text{on } \partial \Omega \times (0, T).
\]

Notably, in \([8]\), the case of \(\Omega \subset \mathbb{R}^d\) being a periodical channel and \(F\) being a suitable smooth double-well potential was investigated, with further insights provided in \([10]\). A more comprehensive mathematical theory concerning the existence, uniqueness, and regularity of solutions for the system \((1.1)\) with \((1.4)\) was developed in \([1]\). Generalizations of the model \((1.1)\) have been discussed in \([2, 3, 4]\). While \([1]\) primarily focused on the case of singular potential, it is worth noting that all these results are applicable for regular potentials as well. For numerical investigations of this model, references such as \([17, 37, 40]\) provide valuable insights. Additionally, a nonlocal version of the model \((1.1)\) was introduced in \([31, 32]\), wherein the chemical potential \(\mu\) is replaced by \(\mu \varphi - J + \varphi + F'(\varphi)\). Mathematical analyses of this nonlocal model have been conducted in \([13, 19, 19, 20]\), among others. In these studies, the boundary conditions for the velocity field \(\mathbf{u}\) have typically been assumed as no-slip or periodic, while the boundary conditions for the phase field variable \(\varphi\) and the chemical potential \(\mu\) are often considered as no-flux. The long time behaviour of the system \((1.1)\) with boundary conditions \((1.4)\) has been investigated in \([1]\), while the existence of global attractors and exponential attractors was explored in \([23]\). Furthermore, the long-term behaviour of the system \((1.1)\) under different boundary conditions, such as, generalized Navier
boundary condition (GNBC) and dynamic boundary conditions have been studied in [50, 51].

The incompressible Navier Stokes equations that describe the motion of a single-phase fluid are well studied in the literature with homogeneous as well as nonhomogeneous boundary data. We refer to [16, 21, 22, 43] for the treatment of nonhomogeneous boundary conditions and references therein. For Cahn Hilliard equations, dynamic boundary conditions which are physically relevant, are analyzed in [14, 29, 38, 41]. For a coupled CHNS model under consideration, the physically relevant boundary conditions derived from the mass and energy conservation equations are generalized Navier Boundary Conditions (GNBC). Such boundary conditions which account for moving contact lines are studied for the existence of weak solutions in [11, 24, 25, 26]. In these papers, the usual boundary conditions are replaced by GNBC which consists of no-flux boundary conditions for \( u \) and \( \mu \), together with a generalized Navier boundary condition for the velocity \( u \) and a dynamic boundary condition with surface convection for \( \varphi \). However if one wants to study a boundary control problem for model H with GNBC the necessary well-posedness results are not available. The boundary optimal control problems are physically realized by suction/blowing which is equivalent to applying a certain force in the part of the boundary. Thus to study boundary control it is necessary to look at a system with general non-homogeneous boundary data.

To the best of our knowledge, the existence of a solution for the model (1.1) with general nonhomogeneous boundary condition for the velocity has not been studied in the literature. Although the numerical results of well-posedness have been established in [28, 35, 36]. In these papers, the authors proved a boundary optimal control problem for a time-discrete CHNS system with nonhomogeneous boundary conditions for the velocity. In this work, we want to extend the work of [11] in the non autonomous case by taking a time dependent boundary condition for velocity. In particular, we consider the system (1.1) with constant mobility; set equal to 1, non constant viscosity and non homogeneous boundary conditions which is given by

\[
\begin{align*}
\partial_t \varphi + u \cdot \nabla \varphi &= \Delta \mu, \quad \text{in } \Omega \times (0, T), \\
\mu &= -\Delta \varphi + F'(\varphi), \\
u_t - \text{div}(\nu(\varphi)D\nu) + (u \cdot \nabla)u + \nabla p &= \mu \nabla \varphi, \quad \text{in } \Omega \times (0, T), \\
\text{div } u &= 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \varphi}{\partial n} &= 0, \quad \frac{\partial \mu}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T), \\
\mu &= \mathbf{h}, \quad \text{on } \partial \Omega \times (0, T), \\
u(0) &= u_0, \quad \varphi(0) = \varphi_0, \quad \text{in } \Omega,
\end{align*}
\]

(1.5)

where \( \mathbf{h} \) is the external force acting on the boundary of the domain. In this work, for the system (1.5) in the two dimensional case, we have established the existence of weak and strong solution and the uniqueness of weak solutions. Furthermore we study the long-time behaviour of solutions by considering constant viscosity. More precisely, we prove that each strong solution of (1.5) converges to a steady state solution when \( \mathbf{h}(t) \) converges to a time independent boundary data \( \mathbf{h}_\infty \).

Before concluding this section, we want to highlight some key aspects of the paper. The system (1.5) is inherently non-autonomous owing to the presence of the time-dependent boundary condition \( \mathbf{h}(t) \). This introduces additional difficulties in subsequent proofs. Firstly, the conventional Galerkin approximation technique for obtaining the existence of weak solutions encounters challenges, as the appropriate
test function space should incorporate discretized boundary data in the trace sense, which is not well-defined. Moreover, the intricate nonlinear coupling prohibits direct application of fixed-point arguments. In the context of single-phase fluid flow, such as Navier-Stokes equations with non-homogeneous boundaries, the use of Dirichlet lifting has been instrumental in proving the existence of solutions, as demonstrated in \[16, 22, 43\]. Thus we introduce a suitable lifting function (cf. \((2.4), (6.1)\) below) for the variable \(u\). Subsequently, using the lifting defined in \((2.4)\) we rewrite the equations for the lifted variable with homogeneous boundary condition and discretize it along with the variable \(\varphi\). Then we study the existence results for the new system for the approximate problem employing Schauder's fixed point theorem. Finally, by deriving the requisite estimates, we proceed to the limit for the approximate problem, thereby establishing the existence of a global weak solution. Another significant challenge is obtaining the energy inequality, which plays a pivotal role in understanding the long-term behaviour of the system. For that, we introduce a lifted energy functional for the lifted system proposed in this paper, which facilitates the derivation of uniform energy estimates for the solution. Further, to study the long time behaviour of the system we wish to use Lojasiewicz-Simon approach (cf. \[45\]), which has been proven to be highly effective in studying the long-time behaviour of nonlinear evolution equations, as demonstrated in prior works such as \[1, 48, 27, 44, 49\]. However, our case differs from these works due to the non-autonomous nature of the system under consideration. To align with the conventional case and apply the available Lojasiewicz-Simon inequality, we convert the non homogeneous system to the homogeneous one by using another type of appropriate lifting. Then, leveraging uniform bounds obtained for solutions and compact embeddings of Sobolev spaces, we establish the convergence of solutions for the lifted system, consequently yielding the convergence of the original system.

The structure of the paper is as follows: In Section 2, we present the requisite functional framework to establish the well-posedness results. Section 3 is dedicated to the establishment of the energy inequality and proof of existence results utilizing a semi-Galerkin approximation. Section 4 focuses on demonstrating the continuous dependence of weak solutions on both initial data and boundary terms, thereby establishing the uniqueness of the weak solution. Moving forward to Section 5, we establish the existence of a strong solution. In Section 6, we analyze the behaviour of the solution in the limit as time \(t \to \infty\) proving that the system converges to a steady state system provided the lifted system converges to its steady state. In the end, we derive some estimates on lifted operators in Section 7.

2. Preliminaries

2.1. Functional Setup. Let \(\Omega\) be a bounded subset of \(\mathbb{R}^2\) with sufficiently smooth boundary \(\partial \Omega\). We introduce the functional spaces that will be useful in the paper.

\[
G_{\text{div}} := \left\{ u \in L^2(\Omega; \mathbb{R}^2) : \text{div } u = 0, \ u \cdot n \big|_{\partial \Omega} = 0 \right\},
\]

\[
\mathbb{V}_{\text{div}} := \left\{ u \in H^1_0(\Omega; \mathbb{R}^2) : \text{div } u = 0 \right\},
\]

\[
\mathbb{H}^s_{\text{div}} := \left\{ u \in H^s(\Omega; \mathbb{R}^2) : \text{div } u = 0 \right\}, \ s \geq 0,
\]

\[
\mathbb{V}^s(\Omega) := \left\{ u \in H^s(\Omega; \mathbb{R}^2) : \text{div } u = 0 \text{ in } \Omega, \ (u \cdot n, 1)_{H^{-\frac{s}{2}}(\partial \Omega), H^{\frac{s}{2}}(\partial \Omega)} = 0 \right\}, \ s \geq 0,
\]

\[
L^2 := L^2(\Omega; \mathbb{R}), \quad H^s := H^s(\Omega; \mathbb{R}), \quad s > 0.
\]
Also, we define boundary spaces \( H^s(\partial \Omega; \mathbb{R}^2) \) in usual trace sense and
\[
\mathbb{V}^s(\partial \Omega) := \left\{ h \in H^s(\partial \Omega; \mathbb{R}^2) : h \cdot n = 0 \right\}, \quad s \geq 0.
\]
With usual convention, the dual space of \( H^s(\Omega) \), \( \mathbb{V}^s(\partial \Omega) \) are denoted by \( H^{-s}(\Omega) \), \( \mathbb{V}^{-s}(\partial \Omega) \) respectively. Let us denote \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) the norm and the scalar product, respectively, on \( \mathbb{V}^0(\Omega) \) and \( \mathbb{G}_{div} \). The duality between any Hilbert space \( \mathcal{X} \) and its dual \( \mathcal{X}' \) will be denoted by \( \langle \cdot, \cdot \rangle \). We know that \( \mathbb{V}_{div} \) is endowed with the scalar product
\[
\langle u, v \rangle_{\mathbb{V}_{div}} = (\nabla u, \nabla v) = 2(Du, Dv), \quad \text{for all } u, v \in \mathbb{V}_{div}.
\]
The norm on \( \mathbb{V}_{div} \) is given by \( \|u\|_{\mathbb{V}_{div}}^2 := \int_{\Omega} |\nabla u(x)|^2 \, dx = |\nabla u|^2 \). Since \( \Omega \) is bounded, the embedding of \( \mathbb{V}_{div} \subset \mathbb{G}_{div} \equiv \mathbb{G}'_{div} \subset \mathbb{V}'_{div} \) is compact.

### 2.2. Linear and Nonlinear Operators.
Let us define the Stokes operator \( A : D(A) \cap \mathbb{G}_{div} \to \mathbb{G}_{div} \) by
\[
A = -P\Delta, \quad D(A) = \mathbb{H}^2(\Omega) \cap \mathbb{V}_{div},
\]
where \( P : \mathbb{L}^2(\Omega) \to \mathbb{G}_{div} \) is the Helmholtz-Hodge orthogonal projection. Note also that, we have
\[
\langle Au, v \rangle_{\mathbb{V}_{div}} = (\nabla u, \nabla v), \quad \text{for all } u \in D(A), v \in \mathbb{V}_{div}.
\]
It should also be noted that \( A^{-1} : \mathbb{G}_{div} \to \mathbb{G}_{div} \) is a self-adjoint compact operator on \( \mathbb{G}_{div} \) and by the classical spectral theorem, there exists a sequence \( \lambda_j \) with \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty \) and a family \( e_j \in D(A) \) of eigenvectors is orthonormal in \( \mathbb{G}_{div} \) and is such that \( A e_j = \lambda_j e_j \). We know that \( u \) can be expressed as \( u = \sum_{j=1}^{\infty} \langle u, e_j \rangle e_j \), so that \( Au = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j \rangle e_j \). Thus, it is immediate that
\[
|\nabla u|^2 = \langle Au, u \rangle = \sum_{j=1}^{\infty} \lambda_j |\langle u, e_j \rangle|^2 \geq \lambda_1 \sum_{j=1}^{\infty} |\langle u, e_j \rangle|^2 = \lambda_1 \|u\|^2,
\]
which is the Poincaré inequality. Now we define some bilinear operators \( B, B_1, B_2 \) and \( B_3 \) as follows:
\( B : \mathbb{V}_{div} \times \mathbb{V}_{div} \to \mathbb{V}'_{div} \) defined by,
\[
\langle B(u, v), w \rangle := b(u, v, w), \quad \text{for all } u, v, w \in \mathbb{V}_{div}.
\]
\( B_1 \) from \( \mathbb{V}_{div} \times \mathbb{H}^1 \) into \( \mathbb{H}^1 \)' defined by,
\[
\langle B_1(u, v), w \rangle := b(u, v, w), \quad \text{for all } u \in \mathbb{V}_{div}, v, w \in \mathbb{H}^1.
\]
Similarly we define \( B_2 : \mathbb{H}^1 \times \mathbb{V}_{div} \to (\mathbb{H}^1)' \) and \( B_3 : \mathbb{H}^1 \times \mathbb{H}^1 \to (\mathbb{H}^1)' \) defined by,
\[
\langle B_2(u, v), w \rangle := b(u, v, w), \quad \text{for all } u, w \in \mathbb{H}^1 \text{ and } v \in \mathbb{V}_{div},
\]
and \( \langle B_3(u, v), w \rangle := b(u, v, w), \quad \text{for all } u, v, w \in \mathbb{H}^1 \) respectively,
where
\[
b(u, v, w) = \int_{\Omega} (u(x) \cdot \nabla)v(x) \cdot w(x) \, dx = \sum_{i,j=1}^{2} \int_{\Omega} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) \, dx.
\]
An integration by parts yields,
\[
\begin{cases}
 b(u, v, v) = 0, & \text{for all } u, v \in \mathbb{V}_{div}, \\
b(u, v, w) = -b(u, v, w), & \text{for all } u, v, w \in \mathbb{V}_{div}.
\end{cases}
\]
For more details about the linear and nonlinear operators, we refer the readers to [27].
We will use the following proposition C.1 from [30] many times later on. So, we state it below,

**Proposition 2.1.** Let Ω be a bounded domain with a smooth boundary in \( \mathbb{R}^2 \). Assume that \( f \in V \) and \( g \in V \). Then, there exists a positive constant \( C \) such that

\[
\|fg\| \leq C\|f\|_V\|g\|_V\|\log(e^{\frac{1}{\|f\|_V}})\|^{\frac{1}{2}}
\]

(2.2)

\[
\|FG\| \leq C\|F\|_V\|G\|_V\|\log(e^{\frac{1}{\|F\|_V}})\|^{\frac{1}{2}}
\]

(2.3)

Let viscosity \( \nu \) and the double well potential \( F \) satisfy following assumptions:

**Assumption 2.2.** (A1) We assume the viscosity coefficient \( \nu \in W^{1,\infty}(\mathbb{R}) \) and for some positive constants \( \nu_1 \) and \( \nu_2 \) satisfies

\[
0 < \nu_1 < \nu(r) < \nu_2,
\]

(A2) \( F \in C^2(\mathbb{R}) \)

(A3) There exist \( C_1 > 0, C_2 \geq 0 \) and \( r \in (1, 2] \) such that \( |F'(s)| \leq C_1|s|^r + C_2, \) for all \( s \in \mathbb{R} \).

(A4) \( F \in C^2(\mathbb{R}) \) and there exists \( C_3 > 0 \) such that \( F''(s) \geq -C_3, \) for all \( s \in \mathbb{R}, \) a.e., \( x \in \Omega \).

(A5) Moreover, there exist \( C_4 > 0, C_4' > 0 \) and \( q > 0 \) such that \( |F'''(s)| \leq C_4|s|^{q-1} + C_4', \) for all \( s \in \mathbb{R}, 1 \leq q < \infty \) and a.e., \( x \in \Omega \).

(A6) \( F \in C^3(\mathbb{R}) \) and there exist \( C_5 > 0, \) \( |F'''(s)| \leq C_5(1 + |s|^{q-2}) \) for all \( s \in \mathbb{R} \) where \( q < +\infty \).

(A7) \( F(\varphi_0) \in L^1(\Omega) \).

**Remark 2.3.** From (A1), we conclude that there exists a positive constant \( \nu \) such that

\[
\nu(\varphi) = \nu_1 \geq \nu.
\]

Now onwards we consider, \( C \) as a non-negative generic constant that may depend on initial data, \( \nu, F, \Omega, T. \) The value of \( C \) may vary even within the proof.

As described in the introduction, to derive the energy estimate and existence of weak solution, we first convert the system (1.5) to the system with homogeneous boundary condition using of the lifting operator.

2.3. **Well-Posedness of Stokes Equations:** Consider the Stokes’ equation (time dependant) with non-homogeneous boundary conditions given by

\[
\begin{cases}
-\nu_1 \Delta u_e + \nabla p = 0, \quad \text{in } \Omega \times (0, T), \\
\text{div } u_e = 0, \quad \text{in } \Omega \times (0, T), \\
u_e = h, \quad \text{on } \partial \Omega \times (0, T).
\end{cases}
\]

(2.4)

Under appropriate regularity of \( h \), we can solve (2.4). We call the operator that maps boundary data \( h \) to the solution of (2.4) as a lifting operator and the solution \( u_e \) as a lifting function.

**Theorem 2.4.** Suppose that \( h \) satisfies the conditions

\[
\begin{align*}
\|h\|_{L^2(0, T; \mathbb{V}^\perp(\partial \Omega))} & \cap L^\infty(0, T; \mathbb{V}^\perp(\partial \Omega)), \\
\|\partial_t h\|_{L^2(0, T; \mathbb{V}^\perp(\partial \Omega))} & \cap L^\infty(0, T; \mathbb{V}^\perp(\partial \Omega)).
\end{align*}
\]

(2.5)

(2.6)

Then equation (2.4) admits a unique weak solution

\[
u_e \in H^{1}(0, T; \mathbb{V}^0(\Omega)) \cap L^\infty(0, T; \mathbb{V}^1(\Omega)) \cap L^2(0, T; \mathbb{V}^2(\Omega)),
\]
such that
\[
\int_0^T \|u_e(t)\|^2_{V^2(\Omega)} \, dt \leq c \int_0^T \|h(t)\|^2_{V^2(\partial\Omega)} \, dt, \tag{2.7}
\]
\[
\int_0^T \|\partial_t u_e(t)\|^2 \, dt \leq c \int_0^T \|\partial_t h(t)\|^2_{V^2(\partial\Omega)} \, dt. \tag{2.8}
\]

**Proof.** The inequality (2.7) can be proved using standard elliptic regularity theory. The time derivative of \(u_e\) in (2.8) is understood in a weak sense. For details, see [43], [47] Proposition 2.2. \(\square\)

By using the lifting function introduced above, system (1.5) can be rewritten as a system with a homogeneous boundary, which will be used to obtain suitable a priori estimates for the solution via the energy method.

Suppose \(u_e\) satisfies the system (2.4). We set,
\[
\overline{u} = u - u_e. \tag{2.9}
\]

Then \((\overline{u}, \varphi)\) solves the system

\[
\begin{aligned}
\partial_t \varphi + \overline{u} \cdot \nabla \varphi + u_e \cdot \nabla \varphi &= \Delta \mu, \quad \text{in } \Omega \times (0, T), \\
\mu &= -\Delta \varphi + F'(\varphi), \quad \text{in } \Omega \times (0, T), \\
\partial_t \overline{u} - \text{div}((\nu_2(\varphi) - \nu_1)D\overline{u}) + ((\overline{u} + u_e) \cdot \nabla)(\overline{u} + u_e) + \nabla \mu &= \mu \Delta \varphi - \partial_t u_e, \quad \text{in } \Omega \times (0, T), \\
\text{div} \overline{u} &= 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \varphi}{\partial n} &= 0, \quad \frac{\partial \mu}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T), \\
\overline{u} &= 0, \quad \text{on } \partial \Omega \times (0, T), \\
\overline{u}(0) = \overline{u}_0 = u_0 - u_e(0), \quad \varphi(0) = \varphi_0 &\quad \text{in } \Omega.
\end{aligned} \tag{2.10}
\]

3. Existence of a weak solution

**Definition 3.1.** Let \(u_0 \in V^0(\Omega), \varphi_0 \in H^1\) and \(0 < T < +\infty\) be given. Let \(F\) satisfies the Assumption 2.2 and \(h\) satisfies (2.6). Let us also assume \(u_e\) solves (2.4). A pair \((u, \varphi)\) is said to be a weak solution of the system (1.5) if \(u = \overline{u} + u_e\) and \((\overline{u}, \varphi)\) obeys

- The following weak formulation:
  \[
  \langle \partial_t \overline{u}(t), v \rangle + \langle (\nu(\varphi) - \nu_1)(\nabla \overline{u}(t), \nabla v) + ((\overline{u}(t) + u_e(t)) \cdot \nabla(\overline{u}(t) + u_e(t)), v) \\
  - (\mu \Delta \varphi, v) - (\partial_t u_e, v), \quad \text{holds for } v \in V_{div}, \text{a.e. in } \Omega \times (0, T), \tag{3.1}
  \]

  \[
  \langle \partial_t \varphi(t), \psi \rangle + \langle (\overline{u}(t) + u_e(t)) \cdot \nabla \varphi(t), \psi \rangle + \langle \nabla \mu(t), \nabla \psi \rangle = 0, \tag{3.2}
  \]

  holds for \(\psi \in H^1, \text{a.e } t \in (0, T)\).

- The initial conditions are satisfied in weak sense:
  \[
  (\overline{u}(0), v) \rightarrow (u_0, v), \quad (\varphi(0), \psi) \rightarrow (\varphi_0, \psi) \text{ as } t \rightarrow 0, \quad \text{for all } v \in V_{div} \text{ and } \psi \in L^2. \tag{3.3}
  \]
\( (\mathbf{u}, \varphi) \) satisfies:

\[
\begin{align*}
\mathbf{u} &\in L^\infty(0, T; \mathbb{G}_{\text{div}}) \cap L^2(0, T; \mathbb{V}_{\text{div}}) \\
\partial_t \mathbf{u} &\in L^2(0, T; \mathbb{V}'_{\text{div}}) \\
\varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \\
\partial_t \varphi &\in L^2(0, T; (H^1)') \\
\mu &\in L^2(0, T; H^1).
\end{align*}
\]  

(3.4)

Remark 3.2. From (3.4) it also follows that \( \mathbf{u} \in C([0, T]; \mathbb{G}_{\text{div}}) \) and \( \varphi \in C([0, T]; L^2) \).

Before proving the existence of weak solution let us derive the energy estimate satisfied by lifted system (2.10).

Energy Estimate: Let us consider the energy functional corresponding to the lifted system (2.10) given by,

\[
\hat{E}(\mathbf{u}(t), \varphi(t)) := \frac{1}{2} \|\mathbf{u}(t)\|^2 + \frac{1}{2} \|\nabla \varphi(t)\| + \int_\Omega F(\varphi(t)).
\]  

(3.5)

Then we can derive the following basic energy inequality for the lifted system (2.10):

Lemma 3.3. Let \( (\mathbf{u}, \varphi) \) be a weak solution equation (2.10) with initial conditions \( (\mathbf{u}_0, \varphi_0) \). Also, let us assume \( \mathbf{h} \) satisfies (2.5), (2.6) and \( F(\varphi_0) \in L^1(\Omega) \). Then, \( (\mathbf{u}, \varphi) \) satisfies the following energy estimate :

\[
\hat{E}(\mathbf{u}(t), \varphi(t)) \leq C\left(\hat{E}(\mathbf{u}_0, \varphi_0) + \int_0^t \|\partial_t \mathbf{h}(\tau)\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} d\tau + \int_0^t \|\mathbf{h}(\tau)\|^2_{1/2(\partial\Omega)} d\tau\right) \exp(\int_0^t \int_\Omega \mathbf{h}(\tau) \mathbf{u}_t \cdot \nabla \mu d\Omega). 
\]  

(3.6)

Proof. Multiplying (2.10) and (2.10) by \( \mathbf{u} \) and \( \mu \) respectively and integrating over \( \Omega \) and adding them together yields

\[
\frac{d}{dt} \left( \frac{1}{2} \|\mathbf{u}\|^2 + \frac{1}{2} \|\nabla \varphi\|^2 + \int_\Omega F(\varphi) \right) + (\nu(\varphi) - \nu_1) \|\nabla \mathbf{u}\|^2 + \|\nabla \mu\|^2
\]

\[
= -b(\mathbf{u}, \mathbf{u}, \mathbf{u}) - b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u}) - \langle \partial_t \mathbf{u}_t, \mathbf{u} \rangle + \int_\Omega (\mathbf{u}_t \cdot \nabla \mu) \varphi. 
\]  

(3.6)

We now estimate each term in the right-hand side of the equations (3.6) using Young’s inequality, Poincaré inequality, Agmon’s inequality, and Sobolev embedding. We have the following estimates.

- \( |b(\mathbf{u}, \mathbf{u}_t, \mathbf{u})| \leq \|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{u}_t\| \|\mathbf{u}\|_{L^2} \leq \frac{1}{2} \|\nabla \mathbf{u}_t\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}\|^2, \)
- \( |b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u})| \leq \|\mathbf{u}_t\|_{L^\infty} \|\nabla \mathbf{u}_t\| \|\mathbf{u}_t\| \leq \frac{1}{2} \|\nabla \mathbf{u}_t\|^2 + \frac{\nu}{2} \|\mathbf{u}_t\|^2_{1/2(\Omega)}, \)
- \( \langle \partial_t \mathbf{u}_t, \mathbf{u} \rangle \leq \frac{1}{2} \|\partial_t \mathbf{u}_t\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}\|^2, \)
- \( \int_\Omega (\mathbf{u}_t \cdot \nabla \mu) \varphi \leq \|\mathbf{u}_t\|_{L^2} \|\nabla \mu\| \|\varphi\|_{L^\infty} \leq \frac{1}{2} \|\nabla \mu\|^2 + \frac{\nu}{2} \|\mathbf{u}_t\|^2_{1/2(\Omega)} \|\varphi\|_{L^2(\Omega)}^2. \)

Considering the above estimates and using Remark 2.3 we get the following from (3.6):

\[
\frac{d}{dt} \left( \|\mathbf{u}\|^2 + \|\nabla \varphi\|^2 + 2 \int_\Omega F(\varphi) \right) + \nu \|\nabla \mathbf{u}\|^2 + \|\nabla \mu\|^2
\]

\[
\leq C(\|\partial_t \mathbf{u}_t\|^2 + \|\mathbf{u}_t\|^2_{1/2(\Omega)} + C \|\mathbf{u}_t\|^2_{1/2(\Omega)} \left( \|\mathbf{u}\|^2 + \|\nabla \varphi\|^2 \right) + 2 \int_\Omega F(\varphi)). 
\]  

(3.7)
By integrating (3.7) from 0 to t and using (2.7) and (2.8) we get
\[
\dot{E}(\mathbf{u}(t), \varphi(t)) \leq \dot{E}(\mathbf{u}(0), \varphi(0)) + C \left( \int_0^t \|\partial_t \mathbf{h}(\tau)\|_{\mathcal{V}^1(\partial \Omega)}^2 \, d\tau + \int_0^t \|\mathbf{h}(\tau)\|_{\mathcal{V}^0(\Omega)}^2 \, d\tau + \int_t^0 \int_0^1 C \|u_n(\tau)\|_{\mathcal{V}^1(\partial \Omega)} \dot{E}(\mathbf{u}(\tau), \varphi(\tau)) \, d\tau \right) + \int_0^t \int_0^1 C \|u_n(\tau)\|_{\mathcal{V}^1(\partial \Omega)} \dot{E}(\mathbf{u}(\tau), \varphi(\tau)) \, d\tau.
\]

Then applying Gronwall’s inequality in (3.8) we obtain
\[
\dot{E}(t) := \dot{E}(\mathbf{u}(t), \varphi(t)) \leq C \left( \dot{E}(\mathbf{u}(0), \varphi(0)) + \int_0^t \|\partial_t \mathbf{h}(\tau)\|_{\mathcal{V}^1(\partial \Omega)}^2 \, d\tau + \int_0^t \|\mathbf{h}(\tau)\|_{\mathcal{V}^0(\Omega)}^2 \, d\tau + \exp \left( \int_0^t \|\mathbf{h}(\tau)\|_{\mathcal{V}^0(\Omega)}^2 \, d\tau \right) \right).
\]

**Theorem 3.4.** Let \( \mathbf{u}_0 \in G_{\text{div}} \) and \( \varphi_0 \in H^1 \). Let \( F \) satisfy the assumption (2.2) and \( h \) satisfies (2.5)–(2.6). Moreover, let us assume the compatibility condition
\[
u_0|_{\partial \Omega} = h|_{t=0}. \tag{3.9}
\]
Then, for a given \( 0 < T < +\infty \), there exists a weak solution \((\mathbf{u}, \varphi)\) to the system (2.10) in the sense of Definition 3.1.

The existence of a global weak solution for the homogeneous system obtained in Theorem 3.4 guarantees a global weak solution \((\mathbf{u}, \varphi)\) for the non-homogeneous system (1.5). We now state the existence of a weak solution result for the non-homogeneous system.

**Theorem 3.5.** Let \( \mathbf{u}_0 \in \mathcal{V}^0(\Omega) \), \( \varphi_0 \in H^1 \) and \( 0 < T < +\infty \) be given. Let \( F \) satisfies the Assumption (2.2) and \( h \) satisfies (2.5)–(2.6). Then there exists a global weak solution \((\mathbf{u}, \varphi)\) to the system (1.5) such that
\[
\begin{align*}
\mathbf{u} &\in L^\infty(0, T; \mathcal{V}^0(\Omega)) \cap L^2(0, T; \mathcal{V}^1(\Omega)), \\
\varphi &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \\
\partial_t \mathbf{u} &\in L^2(0, T; (\mathcal{V}^1(\Omega))'), \\
\partial_t \varphi &\in L^2(0, T; (H^1)'), \\
\mu &\in L^2(0, T; H^1).
\end{align*}
\]

**Proof.** From the Theorem 3.4 we have a global weak solution \((\mathbf{u}, \varphi)\) for the homogeneous system (2.10). The theorem 2.4 proves the existence of \( \mathbf{u}_n \). Hence in the sense of Definition 3.1 we can prove the existence of a weak solution \( \mathbf{u}, \varphi \) for the system (1.5).

**Proof.** [Proof of Theorem 3.4] We will prove the existence by a Semi-Galerkin approximation. We use the Faedo-Galerkin approximation only for the concentration equation. Hence we consider \( \{\psi_k\}_{k \geq 1} \) which are eigenfunctions of Neumann operator \( B_N = -\Delta + I \) in \( H^1 \). Let \( \Psi_n = \langle \psi_1, \ldots, \psi_n \rangle \) be \( n \)-dimensional subspace, \( \overline{\Psi}_n = P_{\Psi_n} \) be orthogonal projector of \( \Psi_n \) in \( L^2 \). We complete the proof in several steps. The overall idea of the proof is explained below and in the following steps 1-4, we show the calculations for relevant estimates to conclude the theorem.

First we will find an approximate solution \( \mathbf{u}^n \in L^\infty(0, T; G_{\text{div}}) \cap L^2(0, T; \mathcal{V}_{\text{div}}) \) and \( \varphi^n \in H^1(0, T; \Psi_n) \) that satisfies the following weak formulation:
\[
\langle \partial_t \mathbf{u}^n(t), \nu \rangle + (\nu(\varphi) - \nu_1)(\nabla \mathbf{u}^n(t), \nabla \nu) + ((\nabla \mathbf{u}^n(t) + \nabla \mathbf{u}(t)) \cdot \nabla)(\mathbf{u}^n(t) + \mathbf{u}_n(t), \nu).
\]
\[
(\mu^n(t)\nabla \varphi^n(t), \nu) - \langle \partial_t \mathbf{u}_n(t), \nu \rangle, \quad \text{holds } \forall \nu \in \mathbb{V}_{\text{div}}, \ \text{a.e. } t \in (0, T),
\]
(3.11)

and
\[
\langle \partial_t \varphi^n(t), \psi \rangle + ((\mathbf{u}^n(t) + \mathbf{u}_e(t)) \cdot \nabla \varphi^n(t), \psi) + (\nabla \mu^n(t), \nabla \psi) = 0,
\]
holds \forall \psi \in \Psi_n, \ \text{a.e. } t \in (0, T), \ \text{where}
\[
\mu^n = \mathcal{T}_n(-\Delta \varphi_n + F' \varphi_n)
\]

(3.12)

which satisfies (3.11). Then for this \(\mathbf{u}^n\) we find \(\varphi^n \in H^1(0, T; \Psi_n)\) which satisfy (5.12). This gives us approximate solution \((\mathbf{u}^n, \varphi^n)\). Finally, showing appropriate convergences, we can find the solution for (2.10).

More precisely, for given
\[
\mathbf{u}^n(x, t) = \sum_{i=1}^{n} c_i(t)\psi_i \in C([0, T]; \Psi_n), \quad \text{and} \quad \varphi^n(x, t) = \sum_{i=1}^{n} d_i(t)\psi_i \in C([0, T]; \Psi_n),
\]
we find
\[
\mathbf{u}^n \in L^\infty(0, T; \mathcal{G}_{\text{div}}) \cap L^2(0, T; \mathcal{V}_{\text{div}}),
\]
which satisfies (3.11). Then for this \(\mathbf{u}^n\) we find \(\varphi^n \in H^1(0, T; \Psi_n)\) which satisfy (5.12). This gives us approximate solution \((\mathbf{u}^n, \varphi^n)\). Finally, showing appropriate convergences, we can find the solution for (2.10).

Further for solution \(\mathbf{u}^n \in L^\infty(0, T; \mathcal{G}_{\text{div}}) \cap L^2(0, T; \mathcal{V}_{\text{div}})\) of (3.14), we will find \(\varphi^n \in H^1(0, T; \Psi_n)\) that satisfies the following equations
\[
\begin{cases}
\langle \partial_t \varphi^n(t), \psi \rangle + (\mathbf{u}^n(t) + \mathbf{u}_e(t) \cdot \nabla \varphi^n(t), \psi) + (\nabla \mu^n(t), \nabla \psi) = 0, \quad \forall \psi \in \Psi_n, \\
\mu^n = \mathcal{T}_n(-\Delta \varphi_n + F' \varphi_n) \\
\varphi^n(x, 0) = \varphi^n_0 \in \Omega.
\end{cases}
\]
(3.16)

Here, \(c_i, d_i \in C([0, T])\) which implies,
\[
\sup_{t \in [0, T]} \sum_{i=1}^{n} |c_i(t)|^2 \leq M \quad \text{and} \quad \sup_{t \in [0, T]} \sum_{i=1}^{n} |d_i(t)|^2 \leq M
\]
for a positive number \(M\), which depends on \(\varphi_0\) and \(|\Omega|\). Therefore we have
\[
\sup_{t \in [0, T]} ||\mathbf{u}^n(x, t)||^2 \leq M.
\]

**Step1:** For a given \(\mathbf{u}^n\) and \(\varphi^n\) and for given \(0 < T < \infty\) we find the solution of (3.14), \(\mathbf{u}^n \in L^\infty(0, T; \mathcal{G}_{\text{div}}) \cap L^2(0, T; \mathcal{V}_{\text{div}})\) by applying Theorem 1.6 of [7]. Now
we derive an estimate of $\mathbf{u}^n$ such that it continuously depends on given $\mathbf{v}^n$, $\mathbf{v}^n$ and initial data $\mathbf{v}_0$.

Therefore substituting $\mathbf{v}$ as $\mathbf{v}^n$ in (3.14) we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}^n\|^2 + (\nu(\varphi) - \nu_1) \|\nabla \mathbf{u}^n\|^2 = -b(\mathbf{u}^n, \mathbf{u}_c, \mathbf{v}^n) - b(\mathbf{u}_c, \mathbf{u}_c, \mathbf{v}^n) - \int_\Omega \mathbf{u}^n \cdot (\nabla \mathbf{v}^n \cdot \mathbf{v}^n) - \langle \partial_t \mathbf{u}_c, \mathbf{v}^n \rangle.$$  

(3.17)

We estimate the R.H.S. of (3.17). By using integration by parts and Hölder inequality we obtain

$$\int_\Omega \mathbf{u}^n \cdot (\nabla \mathbf{v}^n \cdot \mathbf{v}^n) \leq \|\nabla \mathbf{u}^n\|_{L^4} \|\mathbf{v}^n\|_{L^4} \leq 2M^2|\Omega|^2 + \frac{1}{2} \|\nabla \mathbf{v}^n\|^2.$$  

(3.18)

In the above, we have used the expression of $\mathbf{v}^n$, $\mathbf{v}^n$ and the fact that $\psi_i$ are eigenfunctions of $B_N$. Furthermore, we have some straightforward estimates on the trilinear operator

- $|b(\mathbf{u}^n, \mathbf{u}_c, \mathbf{v}^n)| \leq \|\mathbf{u}^n\|_{L^4} \|\nabla \mathbf{u}_c\| \|\mathbf{v}^n\|_{L^4} \leq \frac{1}{2} \|\nabla \mathbf{u}_c\|^2 \|\mathbf{v}^n\|^2 + \frac{1}{2} \|\nabla \mathbf{v}^n\|^2$,

- $|b(\mathbf{u}_c, \mathbf{u}_c, \mathbf{v}^n)| \leq \|\mathbf{u}_c\|_{L^4} \|\nabla \mathbf{u}_c\| \|\mathbf{v}^n\| \leq \frac{1}{2} \|\nabla \mathbf{u}_c\|^2 \|\mathbf{v}^n\|^2 + \frac{1}{2} \|\mathbf{u}_c\|_{L^{2}\Omega}^2.$

Using Theorem 2.4, we also have

$$\langle \partial_t \mathbf{u}_c, \mathbf{v}^n \rangle \leq \|\partial_t \mathbf{u}_c\| \|\mathbf{v}^n\| \leq \frac{1}{2} \|\partial_t \mathbf{u}_c\|^2 + \frac{1}{2} \|\mathbf{v}^n\|^2$$

Now from (3.17) and using Remark 2.3 we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}^n\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}^n\|^2 \leq 2M^2|\Omega|^2 + \frac{1}{2} \|\partial_t \mathbf{u}_c\|^2 + C \|\mathbf{u}_c\|_{L^{2\Omega}}^2 C(1 + \|\nabla \mathbf{u}_c\|^2) \|\mathbf{v}^n\|^2.$$  

(3.19)

Integrating (3.19) from 0 to $t$ and using Gronwall’s inequality finally we deduce

$$\|\mathbf{u}^n\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}^n\|^2 \leq C \left( \|\mathbf{u}_0^n\|^2 + \int_0^t M^2|\Omega|^2 + \int_0^t \|\partial_t \mathbf{u}_c\|^2 + C \|\mathbf{u}_c\|_{L^{2\Omega}}^2 \right) e^{\int_0^t C \|\mathbf{h}\|_{L^{2\Omega}}^2}, \forall t \in [0, T),$$  

(3.20)

which implies that

$$\|\mathbf{u}^n\|_{L^\infty(0,T;\mathbb{G}_{div}) \cap L^2(0,T;\mathbb{V}_{div})} \leq L(T, M)$$  

(3.21)

Let us define the operator $I_n : C(0,T;\Psi_n) \rightarrow L^\infty(0,T;\mathbb{G}_{div}) \cap L^2(0,T;\mathbb{V}_{div})$ given by $I_n(\mathbf{u}^n) = \mathbf{u}^n$. Observe that $I_n$ is continuous since from (3.20) it can be easily seen that $\mathbf{u}^n$ continuously depends on initial data and the fixed vector $\mathbf{v}^n$ and $\mathbf{v}^n$.

**Step 2:** Now we want to find $\varphi^n(x,t) = \sum_{i=1}^n \mathbf{u}_i(t)\psi_i$ and $\mu^n(x,t) = \sum_{i=1}^n \mathbf{v}_i(t)\psi_i$ which satisfies (3.16) for the $\mathbf{u}^n$ that we determined in Step 1. By taking $\psi = \psi_i$ in (3.19), we get a system of non-linear ODEs in $\mathbf{u}_i(t)$ given by

$$\mathbf{u}_i(t) + \sum_{j=1}^n \mathbf{u}_j(t) \int_\Omega (\mathbf{u}^n \cdot \nabla) \mathbf{u}_j dx = \mathbf{u}_i^2 + \int_\Omega \psi_i F' \left( \sum_{i=1}^n \mathbf{u}_i(t)\psi_i \right) - \sum_{j=1}^n \mathbf{u}_j(t) \int_\Omega (\mathbf{u}_c \cdot \nabla) \mathbf{u}_j \psi_i$$  

(3.22)

for all $i = 1, \ldots, n$. By Cauchy - Peano existence theorem the system of ODEs (3.22) admits a unique local solution in $[0,T^n]$ for some $T_n \in (0,T]$ such that $\mathbf{u}_i \in H^1(0,T_n)$ ( $\mathbf{u}_i$ can be found by solving (3.16) ). Thus we have $\varphi^n(t,x) \in H^1(0,T_n;\Psi_n)$. 
Now we will show that $\varphi^n$ is bounded in $L^\infty(0,T_n; H^1) \cap L^2(0,T_n; H^2)$. By replacing $\psi$ by $\mu^n + \varphi^n$ in (3.16), we derive
\[
\frac{d}{dt} \left( \frac{1}{2} \| \varphi^n \|^2 + \frac{1}{2} \| \nabla \varphi^n \|^2 + \int_\Omega F(\varphi^n) \right) + \| \nabla \mu^n \|^2 = -((\varphi^n + u_0) \cdot \nabla \varphi^n, \mu^n) - ((\varphi^n + u_0) \cdot \nabla \varphi^n, \varphi^n) - (\nabla \mu^n, \nabla \varphi^n).
\] (3.23)

Estimating the R.H.S. of (3.23) we have
\[
\frac{d}{dt} \left( \frac{1}{2} \| \varphi^n \|^2 + \frac{1}{2} \| \nabla \varphi^n \|^2 + \int_\Omega F(\varphi^n) \right) + \| \nabla \mu^n \|^2 \leq \| \nabla \varphi^n \| \| \nabla \varphi^n \|_L^4 + \| u_0 \|_{L^2(\Omega)} \| \nabla \mu^n \| \| \varphi^n \| + \frac{1}{4} \| \nabla \mu^n \|^2 + \| \nabla \varphi^n \|^2.
\]
After some straightforward calculations, we deduce,
\[
\frac{d}{dt} \left( \frac{1}{2} \| \varphi^n \|^2 \right) + \frac{1}{4} \| \nabla \mu^n \|^2 \leq C(\| \nabla \varphi^n \|^2 + \| u_0 \|^2_{L^2(\Omega)} + 1) \| \varphi^n \|^2_{H^1}.
\] (3.24)

Integrating (3.24) from 0 to $T_n$, we obtain
\[
\| \varphi^n \|^2_{H^1} + 2 \int_0^{T_n} F(\varphi^n) + \frac{1}{2} \int_0^{T_n} \| \nabla \mu^n \|^2 \leq \| \varphi^n \|^2_{H^1} + 2 \int_0^{T_n} F(\varphi^n) + C \int_0^{T_n} (\| \nabla \varphi^n \|^2 + \| u_0 \|^2_{L^2(\Omega)} + 1) \| \varphi^n \|^2_{H^1}.
\]

Now we will show $\varphi^n \in L^2(0,T_n; H^2)$. We have
\[
\frac{1}{4} \| \nabla \mu^n \|^2 + \| \nabla \varphi^n \|^2 \geq (\mu^n, -\Delta \varphi^n) \geq \| \Delta \varphi^n \|^2 - C_3 \| \nabla \varphi^n \|^2,
\]
which implies
\[
\frac{1}{2} \| \nabla \mu^n \|^2 \geq \| \Delta \varphi^n \|^2 - (C_3 + 1) \| \nabla \varphi^n \|^2.
\] (3.25)

Combining (3.24) and (3.25) we obtain
\[
\frac{d}{dt} \left( \| \varphi^n \|^2_{H^1} + 2 \int_0^{T_n} F(\varphi^n) + \| \Delta \varphi^n \|^2 + \frac{1}{4} \| \nabla \mu^n \|^2 \leq C(\| \nabla \varphi^n \|^2 + \| u_0 \|^2_{L^2(\Omega)} + C_3 + 1) \| \varphi^n \|^2_{H^1}.
\] (3.26)

By integrating (3.26) from 0 to $t$, $0 \leq t \leq T_n$, and then using Gronwall’s inequality and the estimate (3.20) yields
\[
\| \varphi^n(t) \|^2_{H^1} + 2 \int_0^t F(\varphi^n) + \int_0^t \| \Delta \varphi^n \|^2 + \frac{1}{4} \int_0^t \| \nabla \mu^n \|^2 \leq C \left( \| \varphi^n(t) \|^2_{H^1} + 2 \int_0^t F(\varphi^n) \right) \exp(T_n + L + \| h \|^2_{L^2(0,T_n; \nabla g(\partial \Omega))}).
\] (3.27)

Let $M > 0$ be such that $\| \varphi^n(t) \|^2_{H^1} + 2 \int_0^t F(\varphi^n) < \frac{M}{R}$, where $R > 0$ is a constant. Then from (3.20) and (3.27), if we consider $\| \varphi^n(t) \|^2_{H^1} < M$ for all $t \in [0,T_n]$, we obtain $\| \varphi^n(t) \|^2_{H^1} < M$ for all $t \in [0,T_n]$ where $0 < T_n < T$ is sufficiently small.

**Step 3:** From (3.27), it is easy to see that $\varphi^n$ depends continuously on initial data $\varphi_0$ and fixed $\nabla \varphi$. Therefore by solving (3.16) we get another continuous operator $J_n : L^\infty(0,T_n; G_{div}) \cap L^2(0,T_n; \nabla_{div}) \rightarrow H^1(0,T_n; \Psi_n)$ such that $J_n(\nabla \varphi) = \varphi^n$.

Therefore, the composition map
\[
J_n \circ I_n : C([0,T_n]; \Psi_n) \rightarrow H^1(0,T_n; \Psi_n), \quad J_n \circ I_n(\varphi) = \varphi^n,
\]
is continuous. Since $\Psi_n$ is a finite dimensional space, compactness of $H^1(0,T_n; \Psi_n)$ into $C([0,T_n]; \Psi_n)$ gives that $J_n \circ I_n$ is a compact operator from $C([0,T_n]; \Psi_n)$ to
itself. We can choose \( T_n' \in (0, T_n) \) such that \( \sup_{t \in [0, T'_n]} \| \varphi^n(t) \|_{H^1} < M \). Then by Schauder’s fixed point theorem there exists a fixed point \( \varphi^n \) in the set

\[
\left\{ \varphi^n \in C([0, T'_n]; \Psi_n) \mid \sup_{t \in [0, T'_n]} \| \varphi^n(t) \|_{H^1} < M \text{ with } \varphi^n(0) = \mathbb{T}_n\varphi_0 \right\}
\]

such that \( \varphi^n \in H^1(0, T'_n; \Psi_n) \) and \( \mathbb{T}_n \in L^\infty(0, T'_n; G_{div}) \cap L^2(0, T'_n; \mathcal{V}_{div}) \) and satisfies the variational formulation

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t \mathbb{T}_n(t), \nu) + \left( \nu (\mathbb{T}_n) - \nu_1 \right) (\nabla \mathbb{T}_n(t), \nabla \nu) + \left( (\mathbb{T}_n(t) + u_e(t)) \cdot \nabla \right) (\mathbb{T}_n(t) + u_e(t), \nu) \\
= (\mathbb{T}_n(t), \nabla \varphi^n(t), \nu) - \langle \partial_t u_e(t), \nu \rangle, \quad \forall \nu \in \mathcal{V}_{div}, \text{ a.e. } t \in (0, T_n'), \end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t \varphi^n(t), \psi) + \left( (\mathbb{T}_n + u_e)(t) \cdot \nabla \varphi^n(t), \psi \right) + (\nabla \mu^n(t), \nabla \psi) = 0, \quad \text{for all } \psi \in \Psi_n, \text{ a.e. } t \in (0, T_n').
\end{array} \right.
\]

**Step 4:** Now we want to prove that \( T_n' = T \). We know that \( (\mathbb{T}_n, \varphi^n) \) satisfies (3.11)-(3.13). From (3.11) and (3.12), by performing some straightforward estimates we deduce

\[
\begin{aligned}
\| \varphi^n \|^2 + \| \mathbb{T}_n \|^2 + \frac{\nu}{2} \int_0^{T_n} \| \nabla \mathbb{T}_n \|^2 + \int_0^{T_n} \| \Delta \varphi^n \|^2 + \frac{1}{2} \int_0^{T_n} \| \nabla \mu^n \|^2 \\
\leq C(\| h \|_{L^2(0, T; V^2(\Omega))}, \| \partial_t h \|_{L^2(0, T; V^2(\Omega))}, \varphi_0, \mathbb{T}_0).
\end{aligned}
\]

Thus, we get the uniform bound for the approximate solution \( (\mathbb{T}_n, \varphi^n) \), which is independent of \( n \). Thus, we can extend the solution to \( [0, T] \), and \( \mathbb{T}_n \) and \( \varphi^n \) are

\[
\begin{aligned}
&\mathbb{T}_n \text{ uniformly bounded in } L^\infty(0, T; G_{div}) \cap L^2(0, T; \mathcal{V}_{div}), \\
&\text{and } \varphi^n \text{ uniformly bounded in } L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \\
&\text{and } \mu^n \text{ uniformly bounded in } L^2(0, T; H^1).
\end{aligned}
\]

Now we will estimate \( \partial_t \mathbb{T}_n \in L^2(0, T; \mathcal{V}_{div}) \) and \( \partial_t \varphi^n \in L^2(0, T; (H^1)' \cap (L^2)' \cap (H^2)' \cap (L^\infty)' \) for that let us first recall that \( \{ e_i \} \) are eigenfunctions of \( A \) and define it’s \( n \)-dimensional subspace \( \mathbb{E}_n = \{ e_1, e_2, \ldots, e_n \} \) and consider the orthogonal projection on the subspace \( G_{div}^n := P_n \mathbb{E}_n \). Then we can write (3.14) as

\[
\partial_t \mathbb{T}_n + P_n(div(\nu(\varphi) - \nu_1)D\mathbb{T}_n) + B(\mathbb{T}_n, \mathbb{T}_n) + B(u_e, u_e) + B(u_e, u_e) - \mu^n \nabla \varphi^n + \partial_t u_e = 0.
\]

We multiply the equation (3.32) by \( u = P_n u^n + (I - P_n) u^n \) and integrate by parts and use the fact that \( P_n u^n \) is orthogonal to \((I - P_n) u^n\) and estimate every term in (3.32) as follows:

- \( \| P_n(div(\nu(\varphi) - \nu_1)D\mathbb{T}_n) \|_{\mathcal{V}_{div}} \leq C(\nu_2 - \nu_1) \| \mathbb{T}_n \|_{\mathcal{V}_{div}} \),
- \( \| P_n B(\mathbb{T}_n, u_e) \|_{\mathcal{V}_{div}} \leq \| P_n B(\mathbb{T}_n, u_e) \|_{(V^1(\Omega))'} \leq C \| \mathbb{T}_n \|_{\mathcal{V}_{div}} \| u_e \|_{(V^1(\Omega))'} \),
- \( \| P_n B(u_e, u_e) \|_{\mathcal{V}_{div}} \leq \| P_n B(u_e, u_e) \|_{(V^1(\Omega))'} \leq C \| u_e \|_{(V^1(\Omega))'}^2 \),
- \( \| P_n(\mu^n \nabla \varphi^n) \|_{\mathcal{V}_{div}} \leq \| \varphi^n \|_{H^1} \| \nabla \mu^n \| \),
- \( \| P_n(\partial_t u_e) \|_{\mathcal{V}_{div}} \leq C \| \partial_t u_e \| \).

Taking these estimates in (3.32) we get the bound on time derivative of \( \mathbb{T}_n \) as

\[
\begin{aligned}
&\int_0^T \| \partial_t \mathbb{T}_n(t) \|_{\mathcal{V}_{div}}^2 \leq C \left( \| \nu_2 - \nu_1 \| \int_0^T \| \mathbb{T}_n(t) \|_{\mathcal{V}_{div}}^2 dt + \int_0^T \| h(t) \|_{V^2(\Omega)}^2 dt \right) + \sup_{t \in [0, T]} \| \varphi^n(t) \|_{H^1} \int_0^T \| \nabla \mu^n(t) \|_{\mathcal{V}_{div}}^2 dt + \sup_{t \in [0, T]} \| u_e(t) \|_{(V^1(\Omega))'} \left( \int_0^T \| \mathbb{T}_n(t) \|_{\mathcal{V}_{div}}^2 dt + \int_0^T \| u_e(t) \|_{(V^1(\Omega))'} dt \right) \).\]

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Similarly, we can write (3.16) as

\[ \partial_t \varphi^n + \mathcal{T}_n \cdot (u^n_e \cdot \nabla \varphi^n - \Delta \mu^n) = 0, \]  

(3.34)

Let us consider \( \psi = \psi_1 + \psi_2 \) where \( \psi_1 = \mathcal{P}_n \psi = \sum_{i=1}^{n} (\psi, \psi_i) \psi_i \in \Psi_n \) and \( \psi_2 = (I - \mathcal{P}_n)\psi = \sum_{i=n+1}^{\infty} (\psi, \psi_i) \psi_i \in \Psi_n^\perp \) as a test function. Now taking duality pairing of (3.34) with \( \psi \) and using the fact that \( \psi_1 \) and \( \psi_2 \) are orthogonal in the space \( V \), we obtain

\[ \begin{align*}
&\int_{\Omega} \mathcal{T}_n (u^n_e \cdot \nabla \varphi^n) \psi_1 \leq C \|u^n_e\|_{V_{div}} \|\nabla \varphi^n\| \|\psi\|_{H^1}, \\
&\int_{\Omega} \mathcal{T}_n (u^n_e \cdot \nabla \varphi^n) \psi_1 \leq C \|\varphi^n\|_{V^1(\Omega)} \|\nabla \varphi^n\| \|\psi\|_{H^1}, \\
&\int_{\Omega} \mathcal{T}_n (\Delta \mu^n) \psi_1 \leq C \|\Delta \mu^n\| \|\psi\|_{H^1}.
\end{align*} \]

Therefore, from (3.34) we have

\[ \int_0^T \|\partial_t \varphi^n(t)\|_{(H^1)^*}^2 \leq C \sup_{t \in [0,T]} \|\varphi^n(t)\|^2_{H^1} \int_0^T \|u^n_e(t)\|^2_{V_{div}} + \|\varphi^n(t)\|^2_{V^1(\Omega)} dt + \int_0^T \|\nabla \mu^n(t)\| dt. \]  

(3.35)

From (3.29), (3.31), (3.33), (3.35) and by using Banach-Alaoglu theorem we can extract subsequences \( \bar{\varphi}^n \), \( \varphi^n \) and \( \mu^n \) that satisfy

\[ \begin{align*}
\bar{\varphi}^n &\to \bar{\varphi} \text{ in } L^\infty(0,T;G_{div}) \text{ weak-*}, \\
\bar{u}^n &\to \bar{u} \text{ in } L^2(0,T;V_{div}) \text{ weak}, \\
\bar{\varphi}^n &\to \bar{\varphi} \text{ in } L^2(0,T;G_{div}) \text{ strong}, \\
\partial_t \bar{\varphi}^n &\to \partial_t \bar{\varphi} \text{ in } L^2(0,T;V'_{div}) \text{ weak}, \\
\varphi^n &\to \varphi \text{ in } L^\infty(0,T;H^1) \text{ weak-*}, \\
\varphi^n &\to \varphi \text{ in } L^2(0,T;H^2) \text{ weak}, \\
\varphi^n &\to \varphi \text{ in } L^2(0,T;H^1) \text{ strong}, \\
\partial_t \varphi^n &\to \partial_t \varphi \text{ in } L^2(0,T;(H^1)') \text{ weak}, \\
\mu^n &\to \mu \text{ in } L^2(0,T;H^1) \text{ weak}.
\end{align*} \]

(3.36)

With this convergence in hand our ultimate objective is to proceed to the limit in the equations (3.28). The process of passing to limit in the linear terms of (3.28) is straightforward. Additionally, leveraging the strong convergence outlined earlier, the process of passing to limit in nonlinear terms follows standard procedures as elaborated comprehensively in [13]. Since \( \bar{\varphi} \in L^2(0,T;V_{div}) \) and \( \bar{\varphi} \in L^2(0,T;V'_{div}) \), it implies \( \bar{\varphi} \in C(0,T;G_{div}) \) using Aubin-Lions’ compactness lemma. Similarly, \( \varphi \in L^\infty(0,T;H^1) \) and \( \partial_t \varphi \in L^2(0,T;(H^1)') \), which gives \( \varphi \in C(0,T;H) \).

The initial condition also satisfies in the weak sense since \( \bar{\varphi} \) and \( \varphi \) are right continuous at 0.

\[ \square \]

4. CONTINUOUS DEPENDENCE

Let \( (u_1, \varphi_1) \) and \( (u_2, \varphi_2) \) be two weak solutions of the system (1.5) with non-homogeneous boundaries \( h_1 \) and \( h_2 \) and initial conditions \( u_{i0}, \phi_{i0} \), for \( i = 1, 2 \) respectively. Let \( u_{e1} \) and \( u_{e2} \) be the corresponding lifting functions respectively. Then \( \bar{u}_1 = u_1 - u_{e1} \) and \( \bar{u}_2 = u_2 - u_{e2} \) is the solution of the system with homogeneous boundary data. Let us denote the differences \( u = \bar{u}_1 - \bar{u}_2 \), \( \varphi = \varphi_1 - \varphi_2 \).
Also, note that proceeding similarly as in (3.25) and using (A2) we obtain
\[ \|\Delta \varphi\|^2 = 0 \]
which satisfies the equation (2.4) with the boundary conditions.

\[ \tilde{\mu} \]
\[ \|\nabla \varphi\|^2 + \|\nabla \tilde{\mu}\|^2 = - \int_\Omega [(u \cdot \nabla \varphi_1)\tilde{\mu} + (u_2 \cdot \nabla \varphi)\tilde{\mu} + (u_1 \cdot \nabla \varphi_2)\tilde{\mu} + (u_{e2} \cdot \nabla \varphi)\tilde{\mu}], \]

and,
\[ \frac{1}{2} \frac{d}{dt}\|u\|^2 + (\nu(\varphi_1) - \nu(\varphi_2))(\nabla u_1, \nabla u) + \nu(\varphi_2)\|\nabla u\|^2 = -b(u, u_1, u) - b(u_2, u_e, u) - b(u_3, u_3, u) + \langle \partial_t u, u \rangle + \int_\Omega (u \cdot \nabla \varphi_1)\tilde{\mu} + \int_\Omega (u \cdot \nabla \varphi)\mu_2. \]

Also, note that proceeding similarly as in (3.25) and using (A2) we obtain
\[ \|\Delta \varphi\|^2 \leq \left( \frac{1}{2} + C_3 \right)\|\nabla \varphi\|^2 + \frac{1}{8}\|\nabla \tilde{\mu}\|^2 \]
Through a systematic calculation employing well-established inequalities such as Hölder, Ladyzhenskaya, Young’s, Poincaré, Gagliardo-Nirenberg, Agmon’s inequalities, and Sobolev embedding, we have calculated the right-hand side of (4.3) and (4.4). The resulting estimates are as follows:

- \( (\nu(\varphi_1) - \nu(\varphi_2))(\nabla u_1, \nabla u) \leq (\nu_2 - \nu_1)(\nabla u_1, \nabla u) \leq \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla u_1\|^2 + \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla \varphi\|^2, \)
- \( \int_\Omega (u \cdot \nabla \varphi_1)\tilde{\mu} \leq \|\nabla \tilde{\mu}\|\|u\|\frac{\nu}{2}\|\nabla u\|^2 + \nu^2\|\nabla \varphi_1\|_{H^1} \leq \frac{1}{2}\|\nabla \tilde{\mu}\|^2 + C\|\varphi_1\|_{H^1}^2\|u\|^2 + \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla u\|^2, \)
- \( \int_\Omega (u_2 \cdot \nabla \varphi)\tilde{\mu} \leq \frac{1}{2}\|\nabla \tilde{\mu}\|^2 + C\|u_2\|^2\|\nabla \varphi\|^2, \)
- \( \int_\Omega (u_1 \cdot \nabla \varphi_2)\tilde{\mu} \leq \frac{1}{2}\|\nabla \tilde{\mu}\|^2 + C\|u_1\|^2\|\nabla \varphi_2\|^2, \)
- \( \int_\Omega (u_3 \cdot \nabla \varphi)\tilde{\mu} \leq \frac{1}{2}\|\nabla \tilde{\mu}\|^2 + \frac{1}{2}\|u_3\|^2\|\varphi_1\|_{H^1}^2, \)
- \( \langle \partial_t u, u \rangle \leq \frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}\|u\|^2, \)
- \( \int_\Omega (u \cdot \nabla \varphi_2)\mu_2 \leq \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla u\|^2 + \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla \varphi\|^2, \)
- \( |b(u, u_1, u)| \leq \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla u\|^2 + \frac{\nu_2 - \nu_1}{2\nu_1}\|\nabla u_1\|^2\|u\|^2, \)
Similarly, we can estimate \( b(u_2, u, u) \) and \( b(u_2, u_2, u) \) by adding (4.5), (4.4) and (4.3) and using above estimates we obtain
\[
d\|u\|^2 + \frac{1}{2}\|\nabla \varphi\|^2 + C_3\|\varphi\|^2 + \frac{\mu_2}{2}\|u\|_{H_1}^2 + \|\Delta \varphi\|^2
\leq C(\|\nabla u_1\|^2 + \|u_e\|_{H_2}^2 + \|\partial_t u_e\|^2 + \|\nabla u_e\|^2 + \|u_e\|_{H_2}^2 + \|\partial_t u_e\|^2 + \|\nabla u_e\|_{H_2}^2 + \|\partial_t u_e\|_{H_2}^2 + \|\varphi_1\|_{H_1}^2 + \|\varphi_1\|_{H_1}^2)
\]

Let us choose \( k = \min\{C_3, \frac{1}{2}\} > 0 \). Now integrating (4.6) from 0 to \( t \) we get
\[
k\left(\|u\|^2 + \|\varphi\|_{H_1}^2\right) + \frac{\mu_2}{2}\int_0^t \|u\|_{H_2}^2 + \int_0^t \|\Delta \varphi\|^2 \leq \left(\|u_0\|^2 + \|\varphi_0\|_{H_1}\right)
\]

Applying Gronwall's inequality yields
\[
\|u\|_{L^\infty(0,T;H^1)} + \|\varphi\|_{L^\infty(0,T;H^1)} \leq C(\|u_0\|_{L^2(0,T;V^2)} + \|\varphi_0\|_{L^2(0,T;V^1)}),
\]

for all \( t \in [0,T] \).

We summarize the above discussion in the following lemma:

**Lemma 4.1.** Let \((u_i, \varphi_i)\) be two weak solutions of the system (1.5) with boundary data \( h_i \) and initial data \((u_{0i}, \varphi_{0i})\) for \( i = 1, 2 \). Let us denote the difference of solution as \( u = u_1 - u_2 \) and \( \varphi = \varphi_1 - \varphi_2 \), then \((u, \varphi)\) satisfy the estimate (4.8). As a consequence, if \( h_1 = h_2 \) and \((u_{01}, \varphi_{01}) = (u_{20}, \varphi_{20})\) then \((u_1, \varphi_1) = (u_2, \varphi_2)\) and weak solution of the system (1.5) is unique.

### 5. Strong Solution

**Definition 5.1.** Let \( 0 < T < +\infty \). We say a pair \((u, \varphi)\) is a strong solution of the system (1.5) if it is a weak solution and moreover
\[
\begin{align*}
\text{a.e. } x \in \Omega, T > 0,
\end{align*}
\]



\[
\begin{align*}
\varphi & \in L^\infty(0, T; H^2) \\
\varphi & \in L^\infty(0, T; H^2),
\end{align*}
\]



\[
\begin{align*}
\begin{cases}
h \in L^2(0, T; \mathbb{V}^2(\partial \Omega)) & \land & L^\infty(0, T; \mathbb{V}^2(\partial \Omega)), \\
\partial_t h \in L^2(0, T; \mathbb{V}^2(\partial \Omega)).
\end{cases}
\end{align*}
\]

and \( F \) satisfies assumption (2.2). Then there exists a unique global strong solution of the system (1.5) in the sense of Definition 5.1.
Proof. Let $h$ satisfies (5.1). Then the Stokes equations (2.4) admits a strong solution
\[ u_c \in H^1(0, T; \Psi^1(\Omega)) \cap L^2(0, T; \Psi^2(\Omega)), \] (5.2)
such that
\[ \int_0^T \|u_c(t)\|_{\Psi_1(\Omega)}^2 dt \leq c \int_0^T \|h(t)\|_{H_0^1(\Omega)}^2 dt, \quad \text{and} \quad \int_0^T \|\partial_t u_c(t)\|_{\Psi_2(\Omega)}^2 dt \leq c \int_0^T \|\partial_t h(t)\|_{H_0^1(\Omega)}^2 dt. \]
Let us take $\Phi = u - u_c$, where $(u, \varphi)$ is a weak solution of the system (1.5). Then $(\Phi, \varphi)$ be a weak solution of the system (2.10). Taking test function $v$ as $\Lambda \Phi$ and $\psi$ as $\Delta^2 \varphi$ in (5.1) and (5.2) respectively, and using Remark 2.3 we get
\[ \frac{1}{2} \frac{d}{dt} \|\nabla \Phi\|^2 + \nu \|\Delta \Phi\|^2 \leq \frac{1}{2} \nu \|\nabla \varphi\|^2 + \nu \|\Delta \varphi\|^2 \leq C \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2, \] (5.3)
and
\[ \frac{1}{2} \frac{d}{dt} \|\Delta \varphi\|^2 \leq \frac{1}{2} (\nu \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 \leq C \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2, \] (5.4)
By applying the Hölder, Ladyzhenskaya, Young’s, Poincaré, Agmon’s inequalities, and Sobolev embedding, we conduct estimations for all the terms in the R.H.S. of equation (5.3). We have the following list of estimates:
- $|I_1| \leq \|\nabla \Phi\|_{L^2(\Omega)} \|\Delta \Phi\| \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2 \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2$,
- $|I_2| \leq \|\nabla \Phi\|_{L^2(\Omega)} \|\Delta \Phi\| \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2$,
- $|I_3| \leq \|\nabla \Phi\|_{L^2(\Omega)} \|\Delta \Phi\| \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2$,
- $|I_4| \leq \|\nabla \Phi\|_{L^2(\Omega)} \|\Delta \Phi\| \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2$,
- $|I_5| \leq C \|\mu \nabla \Phi\| \|\Delta \varphi\| \leq C \|\mu \nabla \varphi\| \|\Delta \varphi\| \leq C \|\mu \nabla \varphi\| \|\Delta \varphi\|^2$,

Now substituting all the above estimates of $I_1$ to $I_6$ into the equation (5.3) we get the following expression
\[ \frac{d}{dt} \|\nabla \Phi\|^2 + \nu \|\Delta \Phi\|^2 \leq C (\|\nabla \varphi\|^2 \|\Delta \varphi\|^2 + \|\nabla \varphi\|^2 \|\Delta \varphi\|^2 + \|\partial_t \varphi\|^2) \] (5.5)
Utilizing a combination of Hölder, Ladyzhenskaya, Young’s, Poincaré, Gagliardo-Nirenberg, Agmon’s inequalities, and Sobolev embeddings, we proceed to estimate all terms appearing on the R.H.S. of equation (5.4).

\[ |J_1| \leq \|\nabla \Phi\|_{L^2(\Omega)} \|\Delta \varphi\| \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2 \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2 \] (5.6)
\[ |J_2| \leq \|\nabla \Phi\|_{L^2(\Omega)} \|\Delta \varphi\| \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2 \leq C \|\nabla \varphi\|^2 \|\Delta \varphi\|^2 \] (5.7)
\[ J_3 = (\Delta \mu, \Delta \varphi) = (\Delta (\varphi + F'(\varphi)), \Delta \varphi) = -\|\Delta \varphi\|^2 + (\Delta F'(\varphi), \Delta \varphi) \]
Combining (5.6)-(5.8) and subsequently substituting them into (5.4) we arrive at following inequality:

\[
\frac{1}{2} \frac{d}{dt} \| \Delta \varphi \|^2 + \frac{1}{2} \| \Delta^2 \varphi \|^2 \leq C \| \nabla \mathbf{u} \|^2 \| \varphi \|_{L^2} + C \| \mathbf{u}_e \|_{L^2(\Omega)}^2 \| \nabla \varphi \|^2 + (F'''(\varphi) \nabla \varphi, \Delta^2 \varphi) + (F''(\varphi) \Delta \varphi, \Delta^2 \varphi).
\]  

(5.9)

Now, we estimate the last two terms on the R.H.S. of (5.9) using (A5), (A6) and Gagliardo-Nirenberg inequality:

\[
(F'''(\varphi) \nabla \varphi, \Delta^2 \varphi) \leq \| F'''(\varphi) \|_{L^\infty} \| (\nabla \varphi)^2 \| \| \Delta^2 \varphi \|
\leq C_5 (1 + \| \varphi \|_{L^2}^2) \| \nabla \varphi \|_{L^2} \| \Delta \varphi \| + C \| \varphi \|_{H^2}^2 \| \Delta \varphi \|_{H^2}^2
\]

\[
(5.10)
\]

Furthermore,

\[
(F''(\varphi) \Delta \varphi, \Delta^2 \varphi) \leq \| F''(\varphi) \|_{L^\infty} \| \Delta \varphi \| \| \Delta^2 \varphi \|
\leq C (1 + \| \varphi \|_{L^2}^{-1}) \| \Delta \varphi \| \| \Delta^2 \varphi \|
\]

\[
(5.11)
\]

Now substituting (5.10)–(5.11) in (5.9) we finally get,

\[
\frac{1}{2} \frac{d}{dt} \| \Delta \varphi \|^2 + \frac{1}{2} \| \Delta^2 \varphi \|^2 \leq C \| \nabla \mathbf{u} \|^2 \| \varphi \|_{L^2}^2 + C \| \mathbf{u}_e \|_{L^2(\Omega)}^2 \| \nabla \varphi \|^2 + C \| \nabla \varphi \|^{10} + C \| \varphi \|_{H^1}^{4+2} \| \varphi \|_{H^2}^{2q-4} + C \| \varphi \|_{H^1}^{q+1} \| \varphi \|_{H^2}^{q+1} + C (1 + \| \varphi \|_{H^1}^{2q+2}) \| \Delta \varphi \|^2.
\]  

(5.12)

By adding (5.5) and (5.12) we further have,

\[
\frac{d}{dt} \left( \| \nabla \mathbf{u} \|^2 + \| \Delta \varphi(t) \|^2 + \frac{\nu}{2} \int_0^t \| \Delta \varphi \|^2 \right) + \nu \| \Delta \varphi \|^2 + \| \Delta^2 \varphi \|^2 \leq C \left( \| \nabla \mathbf{u} \|^2 + \| \mathbf{u}_e \|^2_{L^2(\Omega)} + \| \mathbf{u}_e \|^2_{L^2(\Omega)} + \| \varphi \|^2_{H^1} \right) \| \nabla \mathbf{u} \|^2
\]

\[
+ C (1 + \| \varphi \|_{H^1}^{2q+2}) \| \Delta \varphi \|^2 + C \int_0^t \left( \| \mathbf{u}_e \|^2_{L^2(\Omega)} + \| \mu \|^2_{H^1} \| \nabla \varphi \|^2 + \| \partial_t \mathbf{h} \|^2_{L^2(\Omega)} \right)
\]

\[
+ C \int_0^t \| \mathbf{u}_e \|^2_{L^2(\Omega)} \| \nabla \varphi \|^2 + C \| \nabla \varphi \|^{10} + C \| \varphi \|_{H^1}^{q+2} \| \varphi \|_{H^2}^{2q-4} + C \| \varphi \|_{H^1}^{q-1} \| \varphi \|_{H^2}^{q+1}.
\]  

(5.13)

By integrating (5.13) from 0 to t we get,

\[
\| \nabla \mathbf{u} \|^2 + \| \Delta \varphi(t) \|^2 + \frac{\nu}{2} \int_0^t \| \Delta \varphi \|^2 + \frac{1}{2} \int_0^t \| \Delta^2 \varphi \|^2 \leq \left( \| \nabla \mathbf{u}_0 \|^2 + \| \Delta \varphi_0 \|^2 \right)
\]

\[
+ C \int_0^t \left( \| \mathbf{u}_e \|^2_{L^2(\Omega)} + \| \mu \|^2_{H^1} \| \nabla \varphi \|^2 + \| \partial_t \mathbf{h} \|^2_{L^2(\Omega)} \right)
\]

\[
+ C \int_0^t \| \mathbf{u}_e \|^2_{L^2(\Omega)} \| \nabla \varphi \|^2 + C \| \nabla \varphi \|^{10} + C \| \varphi \|_{H^1}^{q+2} \| \varphi \|_{H^2}^{2q-4} + C \| \varphi \|_{H^1}^{q-1} \| \varphi \|_{H^2}^{q+1}.
\]  

(5.14)

Now, by using Gronwall inequality, we obtain,

\[
\| \nabla \mathbf{u} \|^2 + \| \Delta \varphi(t) \|^2 + \nu \int_0^t \| \Delta \varphi \|^2 + \frac{1}{2} \int_0^t \| \Delta^2 \varphi \|^2 \leq \left( \| \nabla \mathbf{u}_0 \|^2 + \| \Delta \varphi_0 \|^2 \right)
\]

\[
+ C \int_0^t \| \mathbf{u}_e \|^2_{L^2(\Omega)} + \| \mu \|^2_{H^1} \| \nabla \varphi \|^2 + \| \partial_t \mathbf{h} \|^2_{L^2(\Omega)}
\]
\begin{align}
&+ C \int_0^t \|u_c\|_{L^2(\Omega)}^2 \|\nabla \varphi\|^2 + C\|\nabla \varphi\|^{10} + C \|\varphi\|_{H_1^1}^{4q+2} \|\varphi\|_{H_1^1}^{2q-4} + C\|\varphi\|_{H_1^1}^{-1} \|\varphi\|_{H_1^2}^{2q+1})
&\quad \exp \int_0^t C \left(\|\tau\|^2 + \|\nabla u\|^2 + \|u_c\|_{L^2(\Omega)}^2 + \|u_c\|^2_{L^2(\Omega)} + \|\varphi\|_{H_1^2}^2 + (1 + \|\varphi\|_{H_1^2}^{2q+1})\right),
\end{align}

for all \( t \in (0, T) \). The R.H.S. of (5.15) is finite since \((\tau, \varphi)\) satisfies (3.4) and \(u_c\) satisfies (5.2). Thus we have,

\[
\|\nabla \tau(t)\|^2 + \|\Delta \varphi(t)\|^2 + \nu \int_0^t \|\Delta \tau\|^2 + \int_0^t \|\Delta^2 \varphi\|^2 \leq C, \quad \forall t \in [0, T],
\]

which gives \( \tau \in L^\infty(0, T; \mathbb{V}_{diss}) \cap L^2(0, T; \|q\|_2(\Omega)) \). Thus we have \( u \in L^\infty(0, T; \mathbb{V}_1(\Omega)) \cap L^2(0, T; \mathbb{V}_2(\Omega)) \). Furthermore, together with the no-flux boundary condition on \( \varphi \) we get \( \varphi \in L^\infty(0, T; H^4) \). Next we want to show that \( \varphi \in L^2(0, T; H^4) \). Thus it is enough to prove that \( \nabla(\Delta \varphi) \in L^2(0, T; L^4) \).

By taking gradient on the second equation of (1.5), we get,

\[
\nabla \mu = -\nabla \Delta \varphi + \nabla F'(\varphi).
\]

Taking the \( L^2 \)-norm on both sides of the above equation, we further get,

\[
\|\nabla(\Delta \varphi)\|^2 \leq \|\nabla \mu\|^2 + \|\nabla(F'(\varphi))\|^2.
\]

Now, from (A5) and Young’s inequality we have,

\[
\|\nabla F'(\varphi)\|^2 = \|F''(\varphi) \nabla \varphi\|^2 = \int_\Omega (C_4 |\varphi|^{2q-2} + C_4') |\nabla \varphi|^2
\]

\[
\leq C_4 \int_\Omega |\varphi|^{2q-2} |\nabla \varphi|^2 + C_4' \|\nabla \varphi\|^2
\]

\[
\leq C_4 \|\varphi\|_{L^{2q}}^{2q} + C_4' \left(\frac{q-1}{q}\right)^{(q-1)} \|\nabla \varphi\|_{L^{2q}}^{2q} + C_4' \|\nabla \varphi\|^2
\]

\[
\leq C_4 \|\varphi\|_{H_1^1}^{2q} + C_4' \|\nabla \varphi\|^2 + C_4 \|\varphi\|_{H_1^2}^{2q}
\]

\[
\leq C(\|\varphi\|_{H_1^1}^{2q} + \|\nabla \varphi\|^2),
\]

for all \( 1 \leq q < \infty \). Now, by substituting (5.19) in (5.18) we get

\[
\|\nabla(\Delta \varphi)\|^2 \leq \|\nabla \mu\|^2 + C(\|\varphi\|_{H_1^2}^2 + \|\nabla \varphi\|^2).
\]

Integrating both sides from 0 to \( T \) we obtain,

\[
\int_0^T \|\nabla(\Delta \varphi)\|^2 \leq \int_0^T \|\nabla \mu\|^2 + C \int_0^T \|\varphi\|_{H_1^2}^2 \leq C.
\]

Together with (5.16), (5.21) and the no-flux boundary condition on \( \mu \), we obtain \( \varphi \in L^2(0, T; H^4) \). Thus we finally have \( u \in L^\infty(0, T; \mathbb{V}_1(\Omega)) \cap L^2(0, T; \mathbb{V}_2(\Omega)) \) and \( \varphi \in L^\infty(0, T; H^2) \cap L^2(0, T; H^4) \). This completes the proof.

6. Convergence to Equilibrium

In this section we establish that any global strong solution of (1.5) converge to a stationary solutions as \( t \to +\infty \) by using Lojasiewicz–Simon inequality. For that
first let us introduce the lifting operator of the parabolic type

\[
\begin{align*}
\begin{cases}
\partial_t u_p - \nu_1 \Delta u_p + \nabla p &= 0, &\text{in } \Omega \times \mathbb{R}^+, \\
\text{div } u_p &= 0, &\text{in } \Omega \times \mathbb{R}^+, \\
u_1 u_p &= h, &\text{on } \partial \Omega \times \mathbb{R}^+, \\
u_1 u_p(0) &= u_{e0}, &\text{in } \Omega,
\end{cases}
\end{align*}
\]

(6.1)

where \( u_{e0} \) can be viewed as an elliptic lifting function of initial data \( u_0 \):

\[
\begin{align*}
\begin{cases}
-\nu_1 \Delta u_{e0} + \nabla p &= 0, &\text{in } \Omega, \\
\text{div } u_{e0} &= 0, &\text{in } \Omega, \\
u_1 u_{e0} &= u_0, &\text{on } \partial \Omega.
\end{cases}
\end{align*}
\]

(6.2)

Remark 6.1. Since we have assumed \( u_0|_{\partial \Omega} = h|_{t=0} \) [cf. (3.9)], the elliptic lifting (2.4) and (6.2) are compatible at \( t = 0 \) i.e. \( u_{e0} = u_0 \).

Now we state the existence of a solution of (6.1) (cf. [43, Theorem 3.1]). We refer [43] for more details about the system (6.1).

Lemma 6.2. Let \( T > 0 \) be given and assume that the boundary data \( h \) satisfies

\[
\begin{align*}
\begin{cases}
h \in L^2(0, +\infty; \mathbb{V}^2(\partial \Omega)) \cap L^\infty(0, +\infty; \mathbb{V}^1(\partial \Omega)), \\
\partial_t h \in L^2(0, +\infty; \mathbb{V}^2(\partial \Omega)) \cap L^1(0, +\infty; \mathbb{V}^1(\partial \Omega)).
\end{cases}
\end{align*}
\]

(6.3)

Then the equation (6.1) has a unique solution \( u_p \in H^1(0, T; \mathbb{V}^0(\Omega)) \cap L^\infty(0, T; \mathbb{V}^1(\Omega)) \cap L^2(0, T; \mathbb{V}^2(\Omega)).

Now, set \( \overline{u}_p = u - u_p \), then \((\overline{u}_p, \varphi)\) satisfy

\[
\begin{align*}
\begin{cases}
\partial_t \varphi + \overline{u}_p \cdot \nabla \varphi + u_p \cdot \nabla \varphi &= \Delta \mu, &\text{in } \Omega \times (0, T), \\
\mu &= \Delta \varphi + \mathcal{F}(\varphi), &\text{in } \Omega \times (0, T), \\
\partial_t \overline{u}_p - \nu \Delta \overline{u}_p + ((u_p + u_p) \cdot \nabla)(\overline{u}_p + u_p) + \nabla \pi &= \mu \nabla \varphi, &\text{in } \Omega \times (0, T), \\
\text{div } \overline{u}_p &= 0, &\text{in } \Omega \times (0, T), \\
\partial \mu \overline{u}_p &= 0, &\text{on } \partial \Omega \times (0, T),
\end{cases}
\end{align*}
\]

(6.4)

Proposition 6.3. Let \( 0 < T < +\infty \) and \((u_0, \varphi_0) \in \mathbb{V}^1(\Omega) \times H^2 \). Furthermore, \( F \) satisfies the Assumptions [2.2] If \((\overline{u}_p, \varphi)\) is a smooth solution of (2.10), then it satisfies the following higher-order energy estimate:

\[
\frac{d}{dt} \mathcal{A}(t) + B(t) \leq C_1 \mathcal{A}(t)^2 + C_2 \mathcal{G}(t) + C_3,
\]

(6.5)

where

\[
\begin{align*}
\mathcal{A}(t) &= \|\nabla u_p(t)\|^2 + \|\Delta \varphi(t)\|^2 + \|\mu(t)\|^2, \\
B(t) &= \nu \|\nabla \pi_p(t)\|^2 + \|\Delta^2 \varphi(t)\|^2 + \|\Delta \mu(t)\|^2, \\
\mathcal{G}(t) &= \|u_p(t)\|_{H^1(\Omega)}^4 + \|u_p(t)\|_{H^2(\Omega)}^2 \|u_p(t)\|_{H^1(\Omega)}^2 + \|\nabla \mu(t)\|^2 \|\Delta \varphi(t)\|^2 + \|\nabla \pi_p(t)\|_{H^1(\Omega)}^2 \|\varphi(t)\|^2 \\
&+ \|u_p(t)\|^2 \|\Delta \varphi(t)\|^2 + \|\varphi(t)\|^2 + \|\varphi(t)\|_{H^{1/2}}^4 \|\varphi(t)\|_{H^{1/4}}^4 + \|\varphi(t)\|_{H^{1/2}}^4 \|\varphi(t)\|^2_{H^{2}} + \|\varphi(t)\|^2_{H^{1/2}} \|\varphi(t)\|^2_{H^{1/4}} + \|\varphi(t)\|_{H^{1/2}}^2 \|\varphi(t)\|_{H^{1/4}}^2 + \|\varphi(t)\|_{H^{1/2}} \|\varphi(t)\|_{H^{1/4}}.
\end{align*}
\]
Proof. Taking the time derivative of \(A(t)\), we obtain from a direct calculation that

\[
\frac{d}{dt}A(t) + B(t) = -b(\mathbf{u}_p, \mathbf{u}_p, \mathbf{A}_p) - b(\mathbf{u}_p, \mathbf{u}_p, \mathbf{A}_p) - b(\mathbf{u}_p, \mathbf{u}_p, \mathbf{A}_p) \\
+ \langle \mu \nabla \varphi, \mathbf{A}_p \rangle - \langle \mathbf{u}_p \cdot \nabla \varphi, \Delta^2 \varphi \rangle - \langle \mathbf{u}_p \cdot \nabla \varphi, \Delta^2 \varphi \rangle + (F''(\varphi)|\nabla \varphi|^2 + \Delta^2 \varphi) \\
+ (F''(\varphi)|\Delta \varphi|^2) + \langle \mathbf{u}_p \cdot \nabla \varphi, \Delta \mu \rangle + (\mathbf{u}_p \cdot \nabla \varphi, \Delta \mu) + (F''(\varphi)|\mathbf{A}_p \cdot \nabla \varphi, \mu) \\
+ (F''(\varphi) u_p \cdot \nabla \varphi, \mu) = \sum_{i=1}^{13} L_i. 
\]

(6.6)

Now we estimate each term on the right-hand side of (6.6) by using Hölder inequality, Ladyzhenskaya inequality, Young’s inequality, Poincaré inequality, Gagliardo-Nirenberg inequality, and assumptions on \(F\) given in Assumption 2.2 and obtain

\[
|L_1| \leq \|\mathbf{A}_p\|^2 + C||\nabla \mathbf{u}_p||^2, \\
|L_2| \leq ||\mathbf{u}_p||^2 + ||\nabla \mathbf{u}_p||^2 + \|\mathbf{A}_p\|, \\
|L_3| \leq ||\mathbf{u}_p||^2 + C\|\mathbf{u}_p\|^4 + \|\nabla \mathbf{u}_p\|^4 + \|\mathbf{A}_p\|^4, \\
|L_4| \leq ||\mathbf{u}_p||^2 + ||\nabla \mathbf{u}_p||^2 + \|\mathbf{A}_p\|^2, \\
|L_5| \leq ||\mathbf{u}_p||^2 + ||\mathbf{u}_p\|^4 + \|\Delta \varphi\|^4 + \|\mathbf{u}_p\|^2 ||\varphi||^2, \\
|L_6| \leq ||\mathbf{u}_p||^2 + ||\mathbf{u}_p\|^4 + ||\mathbf{A}_p||^2, \\
|L_7| \leq \|\mathbf{u}_p\|^2 ||\nabla \mathbf{u}_p\|^2 ||\varphi\|^2 ||\Delta \varphi\|^2, \\
|L_8| \leq \|\mathbf{u}_p\|^2 ||\nabla \mathbf{u}_p\|^2 ||\varphi\|^2 ||\Delta \varphi\|^2, \\
|L_9| \leq ||\mathbf{u}_p||^2 ||\nabla \mathbf{u}_p\|^2 ||\varphi\|^2 ||\Delta \varphi\|^2, \\
|L_{10}| \leq \|\mathbf{u}_p\|^2 ||\nabla \mathbf{u}_p\|^2 ||\varphi\|^2 ||\Delta \varphi\|^2 ||\Delta \mu\|, \\
|L_{11}| \leq \|\mathbf{u}_p\|^2 ||\nabla \mathbf{u}_p\|^2 ||\varphi\|^2 ||\Delta \varphi\|^2 ||\Delta \mu\|, \\
|L_{12}| \leq C(||\mathbf{u}_p||^4 + ||\mathbf{A}_p||^4 + ||\nabla \mathbf{u}_p||^4 + ||\mathbf{A}_p||^4 + ||\nabla \mathbf{u}_p||^4 + ||\mathbf{A}_p||^4 + ||\nabla \mathbf{u}_p||^4 + ||\mathbf{A}_p||^4), \\
|L_{13}| \leq C(||\mathbf{u}_p||^4 + ||\mathbf{A}_p||^4 + ||\nabla \mathbf{u}_p||^4 + ||\mathbf{A}_p||^4 + ||\nabla \mathbf{u}_p||^4 + ||\mathbf{A}_p||^4 + ||\nabla \mathbf{u}_p||^4 + ||\mathbf{A}_p||^4). 
\]
Combining all the estimates of $L_1$ to $L_{13}$ in (6.6) and taking $\epsilon, \delta,$ and $\gamma$ sufficiently small, we have the differential inequality (6.5). \hfill \square

Next we recall the following lemma (see [44]).

**Lemma 6.4.** For a given $T$ with $0 < T \leq +\infty$ suppose that $y(t)$ and $g(t)$ be two prescribed continuous function on $[0, T]$ and satisfying\[ \frac{dy}{dt} \leq c_1 y^2 + c_2 g + c_3, \]
with $\int_0^T y(t)dt \leq c_4$, $\int_0^T g(t)dt \leq c_5$, where $c_i (i = 1, \cdots, 5)$ are some non-negative constants. Then for any $r \in (0, T)$ we have\[ y(t + r) \leq (c_4 r^{-1} + c_2 r + c_3)e^{c_1 r}, \quad \forall t \in [0, T - r]. \]
Moreover, if $T = +\infty$, then $\lim_{t \to +\infty} y(t) = 0$.

If $(\mathbf{u}_p, \varphi)$ is the strong solution of (2.10), then we have\[ \int_0^T G(t)dt < +\infty. \]

**Remark 6.5.** Note that we had introduced elliptic lifting (2.4) to prove the existence of weak and strong solutions [see (2.4)]. The purpose of introducing parabolic lifting (6.1) in this section is to study the long-term behaviour of the solution. We can prove the existence results using parabolic lifting for the system (1.5) as in Theorem 3.4 and Theorem 5.2 as well.

In the next theorem we state a result on the regularity of the weak solution of (1.5) which can be proved by using Proposition 6.3 and Lemma 6.4.

**Theorem 6.6.** Let $T > 0$ and $(\mathbf{u}_0, \varphi_0) \in \mathbb{V}^1(\Omega) \times \mathbb{H}^2$. Furthermore, $F$ satisfies the Assumptions 2.2. Then the problem (1.5) admits a unique global strong solution satisfying\[ \|\mathbf{u}(t)\|_{\mathbb{V}^1(\Omega)} \leq C, \quad \|\varphi(t)\|_{\mathbb{H}^2} \leq C, \quad \|\mu(t)\| \leq C, \quad \forall t \geq 0, \quad (6.7) \]
\[ \int_0^t \left( \|\mathbf{u}(\tau)\|_{\mathbb{V}^2(\Omega)}^2 + \|\varphi(\tau)\|_{\mathbb{H}^4}^2 + \|\mu(\tau)\|_{\mathbb{H}^2}^2 \right) d\tau \leq CT, \quad \forall t \in [0, T]. \]

**Assumption 6.7.** In order to study the long-time behaviour of the global solution, we need some decay condition on time-dependent boundary data $h$. We assume that for any $t \geq 0$\[ \int_t^{t+\infty} \|\partial_1 h(\tau)\|_{\mathbf{V}^{-2}(\partial \Omega)}^2 d\tau \leq C(1 + t)^{-1-\gamma}, \]
\[ \int_t^{t+\infty} \|\partial_1 h(\tau)\|_{\mathbf{V}^2(\partial \Omega)}^2 d\tau \leq C(1 + t)^{-1-\gamma}, \]
\[ \int_t^{t+\infty} \|h(\tau)\|_{\mathbf{V}^2(\partial \Omega)}^2 d\tau \leq C(1 + t)^{-1-\gamma}, \quad (6.8) \]
where $C$ and $\gamma$ are given positive constants.

We will show the long time behaviour results using Lojasiewicz-Simon inequality. Since the energy inequality holds for lifted energy functional $\hat{E}$ (cf. Lemma 3.3), we will write the Lojasiewicz-Simon inequality for the lifted energy functional. We write the energy functional $\mathbf{mathcal{E}}$ given in (5.5) as\[ \hat{E}(t) = E_1(t) + E_2(t), \]
where \( E_1(t) = \frac{1}{2} \int_\Omega |\nabla u(t)|^2 \, dx \) and \( E_2(t) = \int_\Omega \left( \frac{1}{2} |\nabla \varphi(t)|^2 + F(\varphi(t)) \right) \, dx \).

The following lemma follows directly.

**Lemma 6.8.** If \( \bar{w} \in G_{\text{div}} \) is a critical point of the functional \( E_1 \), then \( \bar{w} = 0 \).

**Lemma 6.9.** If \( \psi \in H^1 \) is a weak solution of the elliptic problem

\[
\begin{cases}
-\Delta \psi + F'(\psi) = 0, & \text{in } \Omega, \\
\frac{\partial \psi}{\partial n} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

(6.9)

then \( \psi \) is a critical point of the functional \( E_2 \). Conversely, if \( \psi \) is a critical point of \( E_2 \), then \( \psi \) is a weak solution of (6.9).

Note that, if \( u_\infty, \varphi_\infty \) is critical point of \( E_1 \) and \( E_2 \) respectively, then \( (u_\infty, \varphi_\infty) \) is a critical point of \( \hat{E} \). Thus we have the following Lojasiewicz-Simon inequality for the energy functional \( \hat{E} \) (see [1, Proposition 3]).

**Proposition 6.10.** Suppose that \( (u_\infty, \varphi_\infty) \) is a critical point of \( \hat{E} \). Then there exist constants \( \theta \in (0, \frac{1}{2}], C, \rho \in (0, 1) \) such that

\[
|\hat{E}(\bar{\psi}, \varphi) - \hat{E}(u_\infty, \varphi_\infty)|^{1-\theta} \leq C(\|\bar{\psi}\| + \|F'(\varphi) - \Delta \varphi\|_{(H^1, \gamma)}).
\]

for all \( (\bar{\psi}, \varphi) \) satisfying \( \bar{\psi} \in G_{\text{div}}, \|\varphi - \varphi_\infty\|_{H^1} \leq \rho \).

We confine ourselves to considering strong solutions. Notice that, from Theorem 6.6 we have uniform estimates for any given global strong solution \( (u, \varphi) \). Thus, the \( \omega \)-limit set corresponding to the initial data \( (u_0, \varphi_0) \) is nonempty. Next, we characterize the \( \omega \)-limit set in the following definition.

**Definition 6.11.** For any initial data \( (u_0, \varphi_0) \in V^1(\Omega) \times H^2 \), we define the \( \omega \)-limit set

\[
\omega(u_0, \varphi_0) = \{ (u_\infty, \varphi_\infty) \in (H^2 \cap V^1(\Omega)) \times H^2 : \exists \{t_n\} \nearrow +\infty \text{ such that up to a subsequence,} \}
\]

\[
\{t_j \subset \{t_n\} \text{ we have } \lim_{j \to +\infty} (u(t_j), \varphi(t_j)) = (u_\infty, \varphi_\infty) \text{ in } L^2 \times H^1 \}.
\]

Let us define the set

\[
N = \{ u \in H^2 \cap V^1(\Omega) : -\nu_1 \Delta u + \nabla p = 0 \text{ in } \Omega, \ u|_{\partial \Omega} = h_\infty \}
\]

where \( h_\infty \in V^+(\partial \Omega) \) is given.

**Proposition 6.12.** Let \( (u_0, \varphi_0) \in V^1(\Omega) \times H^2 \) and \( F \) satisfies the Assumptions 2.2. Then the \( \omega \)-limit set \( \omega(u_0, \varphi_0) \) is a subset of

\[
S = \{ (u_\infty, \varphi_\infty) \in N \times H^2 : -\Delta \varphi_\infty + F'(\varphi_\infty) = 0 \text{ in } \Omega, \ \frac{\partial \varphi_\infty}{\partial n} = 0 \text{ on } \partial \Omega \}.
\]

**Proof.** We note that, as \( \partial_t h \in L^1(0, +\infty; \mathbb{V}^+(\partial \Omega)) \), \( h(t) \) converges to a function \( h_\infty \) as \( t \to \infty \), namely, \( \|h(t) - h_\infty\|_{\mathbb{V}^+(\partial \Omega)} \leq \int_t^{+\infty} \|\partial_t h(\tau)\|_{\mathbb{V}^+(\partial \Omega)} \, d\tau \to 0 \) as \( t \to +\infty \) using (6.8). From (6.5), we obtain that

\[
\lim_{t \to +\infty} A(t) = 0, \quad (6.10)
\]

Which implies

\[
\lim_{t \to +\infty} \|\nabla \bar{u}_{\infty}(t)\|^2 = 0.
\]

Thus we have \( u(t) \to u_\infty(t) \) in \( V^1(\Omega) \) as \( t \to +\infty \). Now, we will show that \( u_\infty(t) \to u_\infty \) in \( V^1(\Omega) \) as \( t \to +\infty \), where \( u_\infty \) satisfies

\[
\{ -\nu_1 \Delta u_\infty + \nabla p_\infty = 0 \text{ in } \Omega, \ div u_\infty = 0 \text{ in } \Omega, \ u_\infty|_{\partial \Omega} = h_\infty \}. \quad (6.11)
\]
Let us take \( v(t) = u_p(t) - u_\infty \). Thus \( v \) solves
\[
\{ -\nu_1 \Delta v + \nabla \pi_1 = -\partial_t u_p, \quad \text{in } \Omega, \quad \text{div } v = 0, \quad \text{in } \Omega, \quad v|_{\partial \Omega} = h - h_\infty. \quad (6.12)
\]
From the regularity of Stokes theorem (see [47]), we obtain
\[
\|v(t)\|_{V^1(\Omega)} + \|\pi_1(t)\| \leq \|\partial_t u_p(t)\| + \|h(t) - h_\infty\|_{V^1(\Omega)}.
\]
Therefore by using Lemma 7.1 we have \( \lim_{t \to \infty} \|v(t)\|_{V^1(\Omega)} = 0 \). In summary, we proved that for any \((u_\infty, \varphi_\infty) \in \omega(u_0, \varphi_0)\), we have
\[
\|u(t) - u_\infty\|_{V^1(\Omega)} \leq \|u(t) - u_p(t)\|_{V^1(\Omega)} + \|u_p(t) - u_\infty\|_{V^1(\Omega)} \to 0 \text{ as } t \to +\infty.
\]
(6.13)

Now we will derive the boundary condition satisfied by \( u_\infty \). Since \( V^1(\partial \Omega) \subset L^2(\partial \Omega) \), from asymptotic behaviour of boundary data \( h \) we have
\[
\|u_\infty|_{\partial \Omega} - h_\infty\|_{L^2(\partial \Omega)} \leq \|u_\infty|_{\partial \Omega} - h(t_j)|_{L^2(\partial \Omega)} + \|h(t_j) - h_\infty\|_{L^2(\partial \Omega)}
\leq \|u_\infty|_{\partial \Omega} - u_p|_{\partial \Omega}\|_{V^1(\Omega)} + \|u_p - u(t_j)\|_{V^1(\Omega)} + \|h(t_j) - h_\infty\|_{V^1(\Omega)} \to 0, \quad j \to +\infty.
\]

It is known that for \( h_\infty \in V^1(\partial \Omega) \), the solution of the system (6.11) satisfies \((u_\infty, p_\infty) \in V^1(\Omega) \times L^2(\Omega) \) (we can use better regularity of \( h_\infty \) to get better solution \( u_\infty \)). As a consequence \( u_\infty \in N \). On the other hand from (6.10), it also follows that
\[
\lim_{t \to +\infty} \|\varphi(t) - F'(\varphi(t))\| = 0.
\]
(6.14)

Considering the uniform estimate (6.7), we have limit function \( \varphi_\infty \in H^2 \) and by using compactness
\[
\lim_{j \to +\infty} \|\varphi(t_j) - \varphi_\infty\|_{H^1} = 0.
\]
(6.15)

Finally, by using (6.14), (6.15) and using (A4), yields that for any \( \psi \in H^1 \)
\[
\left| \int_{\Omega} (-\Delta \varphi_\infty + F'(\varphi_\infty)) \cdot \psi \, dx \right|
\leq \left| \int_{\Omega} (-\Delta \varphi_\infty + \Delta \varphi(t_j)) \cdot \psi \, dx \right| + \left| \int_{\Omega} (F'(\varphi_\infty) - F'(\varphi(t_j))) \cdot \psi \, ds \right|
+ \left| \int_{\Omega} (\varphi(t_j) - F'(\varphi(t_j))) \cdot \psi \, dx \right| \to 0 \text{ as } j \to +\infty.
\]

Thus, we get \((u_\infty, \varphi_\infty) \in S \). This completes the proof. \( \Box \)

Now we will prove the convergence of lifted energy function \( \hat{E} \).

**Proposition 6.13.** Let \((u_0, \varphi_0) \in V^1(\Omega) \times H^2 \) and \( F \) satisfies the Assumptions 2.2. Moreover, the boundary data \( h \) satisfy the decay conditions (6.8). Then the lifted energy functional \( \hat{\mathcal{E}} \) define on (3.5) is constant on the \( \omega \)-limit set \( \omega(u_0, \varphi_0) \).

**Proof.** We know from Definition 6.11 that, for any \((u_1^i, \varphi_1^i) \), \((u_\infty, \varphi_\infty) \in \omega(u_0, \varphi_0) \), there exists unbounded sequences \((t_j^i)_{j \in \mathbb{N}} \), \((t_j^2)_{j \in \mathbb{N}} \) such that
\[
\lim_{j \to +\infty} (u(t_j^i), \varphi(t_j^i)) = (u_\infty, \varphi_\infty) \in L^2 \times H^1,
\]
for \( i = 1, 2 \). Thus, we get
\[
\lim_{j \to +\infty} \hat{\mathcal{E}}(t_j^i) = E_1(u_\infty) + E_2(\varphi_\infty) := \hat{\mathcal{E}}_i.
\]

Then for any \( t_1 > t_2 > 0 \), we obtain from the energy estimate (3.7) that
\[
|\hat{\mathcal{E}}(t_1) - \hat{\mathcal{E}}(t_2)| \leq \int_{t_2}^{t_1} \frac{d}{dt} \hat{\mathcal{E}}(t) \, dt \leq \int_{t_2}^{t_1} r(t) \, dt + C_1 \int_{t_2}^{t_1} \|h(t)\|_{V^1(\partial \Omega)}^2 \, dt,
\]
and this completes the proof.
we can find an integer \( j \)

We only need to show the convergence of \( \phi \)

converges to an equilibrium.

Moreover, we define a functional on \( h \).

This yields,

\[ \lim_{t \to +\infty} \phi(t) = \phi_{\infty}. \]

This yields, \( \lim_{t \to +\infty} \hat{\phi}(t) = \hat{\phi}_{\infty}. \)

In the next theorem we will prove the convergence of solutions to the equilibrium. We will prove the theorem along the lines of [12, 27].

**Theorem 6.14.** Let \((u_0, \varphi_0) \in \mathbb{V}^1(\Omega) \times H^2 and F satisfies the Assumptions 2.2**. In addition, \( h \) satisfies the decay condition \((6.8) \). Then any strong solution \((u(t), \varphi(t))\) converges to an equilibrium \((u_{\infty}, \varphi_{\infty})\) strongly in \( H^1 \times L^2 \) as \( t \to +\infty \).

**Proof.** Firstly, from \((6.13)\) we have

\[ \|u(t) - u_{\infty}\|_{\mathbb{V}^1(\Omega)} \to 0, \text{ as } t \to +\infty. \]

We only need to show the convergence of \( \varphi(t) \to \varphi_{\infty} \) as \( t \to +\infty \). Notice that, we can find an integer \( j_0 \) such that for all \( j \geq j_0, \|\varphi(t) - \varphi_{\infty}\|_{H^1} < \frac{\eta}{2}, \) where \( \eta \in (0, 1) \) is a constant given in Proposition \((6.10)\).

Consequently, we define

\[ s(t_j) = \sup\{t \geq t_j : \|\varphi(t) - \varphi_{\infty}\|_{L^2} < \eta\}. \]

Since \( \varphi \in C([0, +\infty), H^1), \) we see that \( s(t_j) > t_j \) for any \( j \geq j_0 \). Also from the energy estimate \((5.7)\) and Proposition \((6.13)\) we have

\[ \hat{\phi}(t) - \hat{\phi}_{\infty} \geq \min\{\nu, 1\} \int_t^{+\infty} C(t) \, dt - \int_t^{+\infty} r_1(t) \, dt, \]

where \( C(t) = \|\nabla \varphi(t)\| + \|\nabla \mu\| \) and \( r_1(t) = C_1(\|\partial_t u_e\|^2 + \|u_e\|^2_{H^2(\Omega)}) + \|u_e\|^2_{H^1(\Omega)} \)

with constant \( C_1 \) depending on \( \hat{\phi}(0), \|\partial_t h\|_{L^2(0, +\infty, \mathbb{V}_{\mathbb{V}^1(\Omega)})^*}, \|h\|_{L^2(0, +\infty, \mathbb{V}^1(\Omega))}, \)

\[ \|h\|_{L^2(0, +\infty, \mathbb{V}_{\mathbb{V}^1(\Omega)})^*}. \]

Note that since \( h \) satisfy decay conditions \((6.8), r_1(t)\) also has some decay, namely,

\[ \int_t^{+\infty} r_1(t) \, dt \leq C_1(1 + t)^{-1 - \gamma}, \quad \forall t \geq 0. \]

Let \( \theta \) be defined as in Proposition \((6.10)\) we choose \( \theta' \) such that \( 0 < \theta' \leq \theta \) satisfying \( 0 < \theta' < \frac{1}{1 + \gamma} \). For any fix \( t_j, j \geq j_0 \) we set

\[ K_j = [t_j, s(t_j)], \quad K_j^{(1)} = \{t \in K_j : C(t) \geq (1 + t)^{-((1-\theta')(1+\gamma))}\}, \quad K_j^{(2)} = K_j \setminus K_j^{(1)}. \]

Moreover, we define a functional on \( K_j \) as following

\[ \Phi(t) = \hat{\phi}(t) - \hat{\phi}_{\infty} + \int_t^{s(t_j)} r_1(\tau) \, d\tau, \quad \forall t \in K_j. \]

Then it follows from the convergence of energy and from the decay condition on \( r_1(t) \) that

\[ \lim_{j \to +\infty} \Phi(t_j) = 0. \]
Moreover, we have from (3.7)
\[ \frac{d}{dt} \| \Phi(t) \|^\theta \text{sgn} \Phi(t) \leq -\theta' \min \{ \nu, 1 \} \| \Phi(t) \|^{\theta' - 1} C(t) \leq 0. \]  
(6.16)

This implies \( |\Phi(t)|^{\theta'} \text{sgn} \Phi(t) \) is decreasing on \( K_j \). Now, by using Proposition 6.10 we obtain that
\[
|\Phi(t)|^{1-\theta'} \leq |\hat{\mathcal{E}}(t) - \hat{\mathcal{E}}_\infty|^{1-\theta'} + C \left( \int_t^{+\infty} r_1(\tau) d\tau \right)^{1-\theta'}
\]
\[
\leq C(\| \mathbf{u}(t) \|^{1-\theta'} + \| -\Delta \varphi + F'(\varphi) \|_{W_2^1}) + C \left( \int_t^{+\infty} r_1(\tau) d\tau \right)^{1-\theta'}
\]
\[
\leq C(\| \nabla \mathbf{u}(t) \| + \| \nabla \mu \|) + C(1 + t)^{-(1-\theta')(1+\gamma)}. \]  
(6.17)

Thus, on \( K_j^{(1)} \) we have
\[
|\Phi(t)|^{1-\theta'} \leq C \langle t \rangle,
\]
which together with (6.16) gives
\[
-\frac{d}{dt}(\| \Phi(t) \|^{\theta'} \text{sgn} \Phi(t)) \geq \theta' \min \{ \gamma, 1 \} \langle t \rangle = C \langle t \rangle.
\]

Therefore, on \( K_j^{(1)} \) we obtain
\[
\int_{K_j^{(1)}} C(t) \, dt \leq -C \int_{K_j^{(1)}} \frac{d}{dt}(\| \Phi(t) \|^{\theta'} \text{sgn} \Phi(t)) \, dt
\]
\[
\leq C(\| \Phi(t_j) \|^{\theta'} + \| \Phi(s(t_j)) \|^{\theta'}) < +\infty, \]  
(6.18)

since \( \Phi(s(t_j)) = 0 \), when \( s(t_j) = +\infty \). On \( K_j^{(2)} \), we have
\[
\int_{K_j^{(2)}} C(t) \, dt \leq C \int_{t_j}^{+\infty} (1 + t)^{-(1-\theta')(1+\gamma)} \leq \frac{C}{-\gamma \theta' - \theta' + \gamma}(1 + t_j)^{\gamma \theta' + \theta' - \gamma},
\]
(6.19)
due to the choice of \( \theta' \). Therefore the inequality (6.18) and (6.19) yields
\[
\int_{K_j^{(1)}} C(t) \, dt = \int_{K_j^{(1)}} C(t) \, dt + \int_{K_j^{(2)}} C(t) \, dt < +\infty \quad \text{for any } j. \]  
(6.20)

From (5.35), we get
\[
\int_{K_j} \| \partial_t \varphi(t) \|_{W_2^{1,\gamma}}^2 dt \leq C \int_{K_j} C(t) \, dt + \int_{K_j} \| \mathbf{h}(t) \|_{W_2^{1,\gamma}}^2 \, dt
\]
\[
\leq C(\| \Phi(t_j) \|^{\theta'} + \| \Phi(s(t_j)) \|^{\theta'}) + C(1 + t_j)^{\gamma \theta' + \theta' - \gamma}. \]  
(6.21)

Now, we claim that there exists an integer \( j_1 \geq j_0 \) such that \( s(t_{j_1}) = +\infty \). As a consequence
\[
\| \varphi(t) - \varphi_\infty \|_{L^2} < \rho, \quad \forall t \geq t_{j_1}.
\]

Indeed, if this is not true, then suppose that for any \( j \geq j_0 \), we have \( s(t_j) < +\infty \). Then by definition, we have
\[
\| \varphi(s(t_j)) - \varphi_\infty \|_{L^2} = \rho > 0.
\]

Then, by using (6.21) and the fact that \( (u_\infty, \varphi_\infty) \in \omega(u_0, \varphi_0) \), it follows that
\[
\| \varphi(s(t_j)) - \varphi_\infty \|_{W_2^{1,\gamma}} \leq \| \varphi(s(t_j)) - \| \varphi(t_j) \|_{W_2^{1,\gamma}} + \| \varphi(t_j) - \varphi_\infty \|_{W_2^{1,\gamma}} \]
\[
\leq \int_{t_j}^{s(t_j)} \| \partial_t \varphi(t) \|_{W_2^{1,\gamma}}^2 \, dt + \| \varphi(t_j) - \varphi_\infty \|_{W_2^{1,\gamma}} \to 0 \text{ as } j \to +\infty. \]  
(6.22)
Then, by using interpolation inequality, we obtain
\[
\|\varphi(s(t_j)) - \varphi\|_{L^2} \leq \|\varphi(s(t_j)) - \varphi\|_{H^1}^{\frac{1}{2}} \|\varphi(s(t_j)) - \varphi\|_{H^2}^{\frac{1}{2}} \to 0 \text{ as } j \to +\infty,
\]
which leads to a contradiction. This completes the proof. \(\square\)

**Remark 6.15.** One can show from (6.22) that \(\|\varphi(t) - \varphi\|_{H^1} \to 0 \text{ as } t \to +\infty.\)

Then using uniform bound on \(\varphi\) in \(H^2\), we can easily get that \(\varphi(t) \to \varphi\) weakly in \(H^2\).

### 7. Appendix

In this section, we establish some estimates on lifting functions \(u_t\) and \(u_p\) that we have used in previous sections.

**Lemma 7.1.** Let \(u_0 \in \mathbb{V}^2(\Omega).\) Also, suppose that \(h\) satisfy (5.9) and \(\partial_t h \in L^2([0, +\infty); \mathbb{V}^2(\partial\Omega)).\)

Then, for any \(t > 0\) we have the following estimates
\[
\|u_p(t) - u_c(t)\|_{V^1(\Omega)}^2 + \int_0^t \|\Delta(u_p(s) - u_c(s))\|^2 ds \leq C \int_0^t \|\partial_t h(s)\|^2 \|\nabla \cdot (\partial_t h)\|_{V^2(\partial\Omega)} ds.
\]

(7.1)

In addition, if \(\partial_t h \in L^2([0, +\infty); \mathbb{V}^2(\partial\Omega)),\) then
\[
\lim_{t \to +\infty} \|\partial_t u_p\| = 0.
\]

(7.2)

**Proof.** It follows from (2.4) and (6.1) that
\[
\begin{aligned}
\nu_1 \Delta(u_p - u_c) - \partial_t u_p, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
\text{div} \ (u_p - u_c) = 0, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
u_1 \Delta(u_p - u_c) = 0, \quad &\text{in } \partial\Omega \times \mathbb{R}^+, \\
u_1 \Delta(u_p - u_c) = 0, \quad &\text{in } \Omega.
\end{aligned}
\]

(7.3)

and
\[
\begin{aligned}
\partial_t(u_p - u_c) - \nu_1 \Delta(u_p - u_c) = 0, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
\text{div} \ (u_p - u_c) = 0, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
\partial_t(u_p - u_c) = 0, \quad &\text{in } \partial\Omega \times \mathbb{R}^+, \\
\partial_t(u_p - u_c) = 0, \quad &\text{in } \Omega.
\end{aligned}
\]

(7.4)

Multiplying the above equation with \(-\Delta(u_p - u_c)\) and integrating by parts and using Poincare inequality we have
\[
\frac{d}{dt} \|\nabla(u_p - u_c)\|^2 + \nu_1 \|\Delta(u_p - u_c)\|^2 \leq \|\partial_t u_c\|^2.
\]

Integrating from 0 to \(t\) and using the estimate (2.3) and Poincare inequality we have (7.1).

Applying now the Laplacian to the equation (7.4) we get,
\[
\begin{aligned}
\partial_t \Delta(u_p - u_c) - \nu_1 \Delta^2(u_p - u_c) = 0, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
\Delta \text{div} \ (u_p - u_c) = 0, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
\Delta(u_p - u_c) = 0, \quad &\text{in } \partial\Omega \times \mathbb{R}^+, \\
\Delta(u_p - u_c)|_{t=0} = 0, \quad &\text{in } \Omega.
\end{aligned}
\]

(7.5)

Multiplying the first equation of (7.5) by \(\Delta(u_p - u_c)\) and integrating by parts we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\Delta(u_p - u_c)\|^2 + \nu_1 \|\nabla \Delta(u_p - u_c)\|^2 = (\partial_t (\nabla p), \Delta(u_p - u_c)) + \int_{\partial\Omega} \partial_n \Delta(u_p - u_c) \partial_t h
\]
\[
\leq \|\partial_t \Delta (u_p - u_e)\|_{V^{-\frac{1}{2}}(\partial \Omega)} \|\partial_t h\|_{V^{\frac{1}{2}}(\partial \Omega)}
\leq \frac{1}{2} \|\Delta (u_p - u_e)\|^2 + \frac{1}{2} \|\partial_t h\|_{V^{\frac{1}{2}}(\partial \Omega)}^2.
\]

Now, by using the estimate (7.1) and the Lemma 6.4 it follows that
\[
\lim_{t \to +\infty} \|\Delta (u_p(t) - u_e(t))\| = 0.
\]

Now, from (7.3) and using the fact \(\|\partial_t u_p(t)\| = \|\Delta (u_p(t) - u_e(t))\|\), we can get
\(7.2\). This completes the proof. \(\Box\)

8. DECLARATIONS

Ethical Approval: This declaration is not applicable for the current manuscript.

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