A DIFFERENTIAL IDEAL OF SYMMETRIC POLYNOMIALS SPANNED BY JACK POLYNOMIALS AT $\beta = -(r - 1)/(k + 1)$

B. FEIGIN, M. JIMBO, T. MIWA AND E. MUKHIN

Abstract. For each pair of positive integers $(k, r)$ such that $k + 1, r - 1$ are coprime, we introduce an ideal $I_n^{(k,r)}$ of the ring of symmetric polynomials $\mathbb{C}[x_1, \cdots, x_n]$. The ideal $I_n^{(k,r)}$ has a basis consisting of Jack polynomials with parameter $\beta = -(r - 1)/(k + 1)$, and admits an action of a family of differential operators of Dunkl type including the positive half of the Virasoro algebra. The space $I_n^{(2,2)}$ coincides with the space of all symmetric polynomials in $n$ variables which vanish when $k + 1$ variables are set equal. The space $I_n^{(2,r)}$ coincides with the space of correlation functions of an abelian current of a vertex operator algebra related to Virasoro minimal series $(3, r + 2)$.

1. Introduction

Let $k$ be a positive integer. Consider the space $F_n^{(k)}$ of all symmetric polynomials $P(x_1, \cdots, x_n)$ with the following property:

$$P(x_1, \cdots, x_n) = 0 \text{ if } x_1 = \cdots = x_{k+1}. \quad (1.1)$$

These polynomials and their analogs originate in the work [FS] where they were used to study integrable modules over affine Lie algebras.

To be specific, let $V$ be the level $k$ vacuum module of $\widehat{sl}_2$, with highest weight vector $|0\rangle$. Let $e_n, f_n, h_n$ be the standard generators of $\widehat{sl}_2$ such that $g_n|0\rangle = 0$ ($n \geq 0$, $g = e, f, h$). Let further $W = U(n)|0\rangle$ be the subspace generated from the highest weight vector by the abelian Lie subalgebra $n = \text{span}_\mathbb{C}\{e_j\}_{j \in \mathbb{Z}}$. Then the dual of the weight $2n$ component $W_n = \{w \in W \mid h_0w = 2nw\}$ is isomorphic to $F_n^{(k)}$. In fact, $F_n^{(k)}$ is nothing but the space of all correlation functions of the current operator $e(x)$

$$\langle \psi | e(x_1) \cdots e(x_n) | 0 \rangle, \quad e(x) = \sum_{n \in \mathbb{Z}} e_n x^{-n-1}, \quad (1.2)$$

where $\langle \psi |$ runs over the dual space $W_n^*$. The condition $(1.1)$ is a consequence of the relation $e(x)^{k+1} = 0$, which holds in any integrable module of level $k$.

This explicit realization of $W_n^*$ was utilized in [FS] to recover a semi-infinite monomial basis of $V$ obtained in [P], and to give a representation theoretical interpretation of fermionic character formulas. We remark that $(1.2)$ as well as the condition $(1.1)$ also appear in the literature on ground state wave functions in the quantum Hall systems [MR, RR].

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In this paper we point out a connection between the space $F_n^{(k)}$ and Jack polynomials which are famous orthogonal polynomials, related to the Calogero-Sutherland $N$-body problem, see [Mac, St].

Let $k, r$ be positive integers such that $r \geq 2$ and $k + 1, r - 1$ are coprime. We consider the Jack polynomials specialized to the value of the coupling constant $\beta = -(r - 1)/(k + 1)$. (For our convention on Jack polynomials, see Section 2.1). In general, Jack polynomials can have poles at a negative rational value of $\beta$. However we observe that if a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ of length at most $n$ satisfies the condition

$$\lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq n - k),$$

then the corresponding Jack polynomial $P_\lambda(x_1, \ldots, x_n)$ does not have a pole at $\beta = -(r - 1)/(k + 1)$. Let $I_n^{(k,r)}$ denote the linear span of $P_\lambda$ satisfying (1.3).

We show that $I_n^{(k,r)}$ is an ideal of the ring of symmetric polynomials $\mathbb{C}[x_1, \cdots, x_n]^{S_n}$, and that it is stable under the action of a family of linear differential operators including the positive half of the Virasoro algebra (Theorem 3.1). Using these properties, we find that $I_n^{(k,2)}$ coincides with $F_n^{(k)}$ discussed above. In other words, the space $F_n^{(k)}$ has a basis consisting of Jack polynomials evaluated at $\beta = -1/(k + 1)$.

We also show that the case $k = 2$ has a similar interpretation via correlation functions. Namely, we verify that $I_n^{(2,r)}$ coincides with the space of correlation functions of an abelian current of a vertex operator algebra studied in [FJM].

The text is organized as follows. In Section 2, after fixing notation, we introduce admissible partitions and examine the regularity of $P_\lambda$. In Section 3 we prove Theorem 3.1 mentioned above. In Section 4 we examine the special cases $r = 2$ and $k = 2$, respectively.

2. Regularity of Jack polynomials with negative rational coupling constant

2.1. Jack polynomials. In this section we collect the main facts about the Jack polynomials we use in the paper and fix our notations. For more details on Jack polynomials, see [Mac, St].

Let $S_n$ be the symmetric group on $n$ letters and $\Lambda_n = \mathbb{C}(\beta)[x_1, \cdots, x_n]^{S_n}$ the ring of symmetric polynomials over the field of rational functions $\mathbb{C}(\beta)$. Let $\pi_n$ denote the set of partitions of length at most $n$. To a partition $\lambda = (\lambda_1, \cdots, \lambda_n) \in \pi_n$, where $\lambda_i \geq 0$, we relate the Young diagram which is a subset $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}$ of $\mathbb{Z}^2$. An element $(i, j)$ of the Young diagrams is called a node. We write $|\lambda| = \sum_{i=1}^{n} \lambda_i$.

By $\lambda'$ we denote the partition conjugate to $\lambda$. The dominance ordering $\mu \leq \lambda$ is defined as $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ $(1 \leq i \leq n)$. For $\lambda \in \pi_n$, let $m_\lambda = \sum_{\alpha \in \Lambda_n} x^\alpha$ be the orbit sum, where we use the multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$.

The Dunkl operators are defined by

$$\nabla_i = \partial_i + \beta \sum_{j(\neq i)} \frac{1}{x_i - x_j} (1 - K_{ij}) \quad (1 \leq i \leq n).$$

(2.1)
Here
\[(K_{ij}f)(\cdots, x_i, \cdots, x_j, \cdots) = f(\cdots, x_j, \cdots, x_i, \cdots), \partial_i = \frac{\partial}{\partial x_i}\]
are the operator which exchanges the \(i\)-th and the \(j\)-th variables and the operator of differentiation with respect to the \(i\)-th variable respectively.

We have the commutation relations:
\[
K_{ij} \nabla_m = \nabla_{s_{ij}(m)} K_{ij}, \quad K_{ij} x_m = x_{s_{ij}(m)} K_{ij},
\]
\[
[\nabla_i, \nabla_j] = 0, \quad [x_i, x_j] = 0,
\]
\[
[\nabla_i, x_j] = \delta_{ij} (1 + \beta \sum_{t=1}^{n} K_{it} - \beta K_{ij}) \tag{2.2}
\]
where \(s_{ij}\) signifies the transposition \((i, j) \in S_n\). Let \(D_i = x_i \nabla_i\). The Cherednik operators \(\hat{D}_i\) \((1 \leq i \leq n)\) are defined by
\[
\hat{D}_i = D_i + \beta \sum_{j=i+1}^{n} K_{ij} = x_i \partial_i + \beta \sum_{j(\neq i)} \frac{x_{\max\{i,j\}}}{x_i - x_j} (1 - K_{ij}) + \beta (n - i).
\]
The Cherednik operators also commute with each other
\[
[\hat{D}_i, \hat{D}_j] = 0.
\]
Define the Sekiguchi operator by
\[
S(u, \beta) = \prod_{i=1}^{n} (u + \hat{D}_i). \tag{2.3}
\]
In other words, the Sekiguchi operator is the generating function of the elementary symmetric polynomials in Cherednik operators.

For operators \(A\) and \(B\), we write \(A \sim B\) if \(AP = BP\) for any symmetric function \(P\). For example \(\nabla_i \sim \partial_i\), \(D_i \sim \hat{D}_i \sim x_i \partial_i\).

We have
\[
S(u, \beta) \sim \prod_{i<j} (x_i - x_j)^{-1} \det \left[ x_i^{n-j} \left( x_i \frac{\partial}{\partial x_i} + (n-j)\beta + u \right) \right]_{1 \leq i, j \leq n},
\]
see formula (2.21) in [KN] and VI, § 3, Example 3c in [Ma].

The action of \((2.3)\) is triangular on \(\{m_\lambda\}_{\lambda \in \pi_n}\),
\[
S(u, \beta) m_\lambda = \sum_{\mu \subseteq \lambda} c_{\lambda \mu}(u, \beta) m_\mu, \tag{2.4}
\]
where \(c_{\lambda \mu}(u, \beta)\) is a polynomial in \(u, \beta\). In particular, \(c_{\lambda \lambda}(u, \beta)\) is given by
\[
c_{\lambda \lambda}(u, \beta) = \prod_{i=1}^{n} (u + \lambda_i + (n - i)\beta). \tag{2.5}
\]
The Jack polynomials \( \{ P_\lambda \} \lambda \in \pi_n \) are the unique \( \mathbb{C}(\beta) \)-basis of \( \Lambda_n \) with the following properties:

\[
S(u, \beta) P_\lambda = c_\lambda(u, \beta) P_\lambda, \quad (2.6)
\]

\[
P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda \mu}(\beta) m_\mu \quad (u_{\lambda \mu}(\beta) \in \mathbb{C}(\beta)), \quad (2.7)
\]

see VI, § 4, Example 2 of [Mac]. When necessary we write \( P_\lambda(x; \beta) \) for \( P_\lambda \), exhibiting the \( \beta \)-dependence explicitly.

Let \( c_\lambda(\beta) = \prod_{(i,j) \in \lambda} ((\lambda_j' - i + 1)\beta + \lambda_i - j) \), \( (2.8) \)

where the product is over all nodes of partition \( \lambda \). Then the coefficients of the polynomial \( J_\lambda = c_\lambda P_\lambda \) are polynomials in \( \beta \), see Theorem 3.2 in [KN]. In particular, the coefficients \( u_{\lambda \mu}(\beta) \) are free from poles, except possibly at non-positive rational values of \( \beta \). In Section 2.3 we describe the properties of \( \lambda \) which are sufficient for regularity of \( P_\lambda(x; \beta) \) at a negative rational value of \( \beta \).

The Calogero-Sutherland Hamiltonian is given by

\[
H = \sum_{i=1}^n (x_i \partial_i)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_i - x_j \partial_j). \quad (2.9)
\]

We have

\[
H \sim \sum_{i=1}^n D_i^2 \sim \sum_{i=1}^n (\hat{D}_i^2 - (n - 1)\beta \hat{D}_i) + \frac{1}{6} n(n - 1)(n - 2)\beta^2.
\]

Therefore we have

\[
HP_\lambda = \varepsilon_\lambda P_\lambda, \quad \varepsilon_\lambda = \sum_{i=1}^n (\lambda_i + \beta(n + 1 - 2i))\lambda_i. \quad (2.10)
\]

2.2. Admissible partitions. In this section we introduce the main combinatorial object of the paper, the set of admissible partitions.

We call a partition \( \lambda \in \pi_n \) \( (k, r, n) \)-admissible if

\[
\lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq n - k). \quad (2.11)
\]

In particular, (2.11) implies

\[
\lambda_i - \lambda_j \geq \left\lfloor \frac{j - i}{k} \right\rfloor r
\]

for all \( i < j \), where \( \lfloor x \rfloor \) denotes the largest integer not exceeding \( x \).

We will need the following combinatorial lemmas about admissible partitions.

**Lemma 2.1.** Let \( \lambda \) be any admissible partition. Then

\[
(j - i)\beta(k, r) + \lambda_i - \lambda_j \neq 0 \quad (1 \leq i < j \leq n), \quad (2.12)
\]

\[
(\lambda_j' - i + 1)\beta(k, r) + \lambda_i - j \neq 0 \quad (1 \leq i \leq n, \ 1 \leq j \leq \lambda_i). \quad (2.13)
\]
Lemma 2.3. If \( \lambda_i - \lambda_j = (r-1)s \) for some integer \( s \). Then \( j - i = (k+1)s \) and \( \lambda_i - \lambda_j = (r-1)s \) for some integer \( s \). Since \( j > i \), we get \( s > 0 \). We have a contradiction:

\[
(r-1)s = \lambda_i - \lambda_j \geq \left\lceil \frac{j-i}{k} \right\rceil r = \left\lceil \frac{(k+1)s}{k} \right\rceil r \geq sr.
\]

To get (2.13), we suppose \( \lambda'_j - i + 1 = (k+1)s \) and \( \lambda_i - j = (r-1)s \) for some integer \( s \). Since \( \lambda_i \geq j \), we have \( s \geq 0 \). The case \( s = 0 \) means \( j = \lambda_i \), \( \lambda'_j \geq i \) and therefore is impossible. For \( s > 0 \), we have a contradiction:

\[
(r-1)s = \lambda_i - j \geq \lambda_i - \lambda'_j \geq \left\lceil \frac{\lambda'_j - i}{k} \right\rceil r = \left\lceil \frac{(k+1)s - 1}{k} \right\rceil r \geq sr.
\]

\[\square\]

Lemma 2.2. Let \( \lambda \) be \((k,r,n)\)-admissible. If \( \lambda_j < \lambda_{j-1} \) then

\[
(j-i)\beta(k,r) + \lambda_i - \lambda_j \neq 1 \quad (i < j).
\]

Proof. Suppose \((j-i)\beta(k,r) + \lambda_i - \lambda_j = 1\). Then \( j - i = (k+1)s \) and \( \lambda_i - \lambda_j - 1 = (r-1)s \) for some integer \( s \). Since \( j > i \), we get \( s > 0 \). We have a contradiction:

\[
(r-1)s = \lambda_i - \lambda_j - 1 \geq \lambda_i - \lambda_{j-1} \geq \left\lceil \frac{j-i-1}{k} \right\rceil r = \left\lceil \frac{(k+1)s - 1}{k} \right\rceil r \geq sr.
\]

\[\square\]

2.3. Specialization of \( P_\lambda \). In this section we examine the regularity of Jack polynomials when \( \beta \) is negative rational.

Fix a negative rational noninteger number and write it in the form

\[
-\frac{r-1}{k+1} =: \beta(k,r),
\]

where \( k, r \) are positive integers such that \( k+1 \) and \( r-1 \) are coprime and \( r \geq 2 \).

Lemma 2.3. If \( \lambda \) is \((k,r,n)\)-admissible then \( P_\lambda \) has no pole at \( \beta = \beta(k,r) \).

Proof. By Lemma 2.1, we have \( c_\lambda(\beta(k,r)) \neq 0 \), where \( c_\lambda \) is given by (2.8). The lemma follows. \[\square\]

The condition \( c_\lambda(\beta(k,r)) \neq 0 \) is sufficient for \( P_\lambda \) being regular at \( \beta = \beta(k,r) \), but not necessary. The point is the following. Presumably, if the number of variables in \( P_\lambda \) is sufficiently large, then the order of the pole of \( P_\lambda \) in \( \beta \) is given exactly by the order of zero of \( c_\lambda \). However, we have

\[P_\lambda(x_1, \ldots, x_n, 0) = P_\lambda(x_1, \ldots, x_n),\]

and in some cases the order of the pole of \( P_\lambda(x_1, \ldots, x_n) \) is smaller than that of \( P_\lambda(x_1, \ldots, x_n, x_{n+1}) \) and therefore smaller than the order of zero of \( c_\lambda \). This is the case we deal with in Proposition 2.4 below. To prove the regularity of \( P_\lambda \) in such a situation we use a different method.

Before proving the main result of this section, Proposition 2.4, we establish a couple of technical lemmas.
Lemma 2.4. Suppose $P_\lambda$ has a pole at $\beta = \beta_0$. Then there exists a partition $\nu < \lambda$ such that
\begin{equation}
    c_{\lambda\lambda}(u, \beta_0) = c_{\nu\nu}(u, \beta_0).
\end{equation}
Proof. Substituting (2.24), (2.27) into (2.26) and equating coefficients of $m_{\nu}$, we obtain
\begin{equation}
    (c_{\nu\nu}(u, \beta) - c_{\lambda\lambda}(u, \beta))u_{\lambda\nu}(\beta) + \sum_{\nu < \mu < \lambda} u_{\lambda\mu}(\beta)c_{\nu\mu}(u, \beta) + c_{\lambda\nu}(u, \beta) = 0
\end{equation}
for all $\nu < \lambda$. From the assumption, the set of $\mu$ for which $u_{\lambda\mu}(\beta)$ has a pole is non-empty. A maximal element $\nu$ in this set has the required property (2.13). \qed

Lemma 2.5. If $P_\lambda$ has a pole at $\beta = \beta(k, r)$, then there exists a partition $\nu < \lambda$ and a permutation $w \in S_n$, $w \not\equiv 1$, with the properties
\begin{align}
    
    \nu_i &= \lambda_{w(i)} + (w(i) - i)\frac{r - 1}{k + 1}, \quad (1 \leq i \leq n), \tag{2.16} \\
    w(i) &\equiv i \mod k + 1 \quad (1 \leq i \leq n). \tag{2.17}
\end{align}
Proof. In view of the formula (2.23) for $c_{\lambda\lambda}(u, \beta)$, the first assertion is an immediate consequence of the previous lemma. Since $\nu_i - \lambda_{w(i)} \in \mathbb{Z}$, the second assertion follows. \qed

Proposition 2.6. Let partition $\lambda$ be obtained from a $(k, r, n)$-admissible partition $\mu$ either by adding or by removing one node. Then $P_\lambda$ has no pole at $\beta = \beta(k, r)$.
Proof. We have
\begin{equation}
    \lambda_j - \lambda_{j'} \geq \mu_j - \mu_{j'} - 1 \tag{2.18}
\end{equation}
for all $j < j'$. Suppose $P_\lambda$ has a pole, and take $\nu$ and $w \not\equiv 1$ as in Lemma 2.5. We claim that if $i$ satisfies $w(i) > w(i + 1)$, then
\begin{equation}
    w(i) = w(i + 1) + k, \quad \lambda_{w(i+1)} - \lambda_{w(i)} = \mu_{w(i+1)} - \mu_{w(i)} = r - 1. \tag{2.19}
\end{equation}
Indeed, setting $m = w(i) - w(i + 1) > 0$ we have $m \equiv k \mod k + 1$. From (2.16) we obtain
\begin{equation}
    \frac{m + 1}{k + 1}(r - 1) \geq \lambda_{w(i+1)} - \lambda_{w(i)}. \tag{2.20}
\end{equation}
Since $\mu$ is admissible, we also have
\begin{equation}
    \mu_{w(i+1)} - \mu_{w(i)} \geq \left\lceil \frac{m}{k} \right\rceil r. \tag{2.21}
\end{equation}
It follows from (2.18), (2.20) and (2.21) that
\begin{equation}
    \frac{m + 1}{k + 1}(r - 1) \geq \left\lceil \frac{m}{k} \right\rceil r - 1,
\end{equation}
which is possible only when $m = k$ and (2.13) holds.

From the assumption, there is one and only one $i$ which violates the condition $w(i) < w(i+1)$. We have then $w(j) \geq j$ for $1 \leq j \leq i$. Moreover, since $w(i) \equiv i \mod k + 1$ and $w(i + 1) = w(i) - k$ we have $w(i) \geq i + k + 1$, $w(i + 1) \geq i + 1$. Since $w(j) < w(j + 1)$ holds for $j \geq i + 1$ we have also $w(j) \geq j$ for $j \geq i + 1$. Therefore $w(j) \geq j$ for all $j$. This is a contradiction. \qed
Note that if $\lambda$ is as in Proposition 2.6 and if $\lambda$ is not $(k, r, n)$-admissible then $c_\lambda$ has a zero of order one at $\beta = \beta(k, r)$.

Also note that the proof of Proposition 2.6 with $\mu = \lambda$ gives an alternative proof of Lemma 2.3.

3. Ideal $I^{(k, r)}_n$ and its properties

3.1. The ideal $I^{(k, r)}_n$. In this section we introduce our main object of the study, the space $I^{(k, r)}_n$ and describe its properties.

Recall Lemma 2.3 which states that, when $\lambda$ is a $(k, r, n)$-admissible partition, the specialization $P_\lambda$ to $\beta = \beta(k, r)$ given in (2.14) is well-defined as an element of $\mathbb{C}[x_1, \ldots, x_n]S_n$. Clearly these polynomials are linearly independent. Let $I^{(k, r)}_n$ be their $\mathbb{C}$-linear span,

$$I^{(k, r)}_n = \text{span}_\mathbb{C}\{P_\lambda(x_1, \ldots, x_n; \beta(k, r)) \mid \lambda \text{ is } (k, r, n)\text{-admissible}\}. \quad (3.1)$$

Set

$$p_m = \sum_{j=1}^{n} x_j^m \quad (m \geq 1),$$

$$l_m = \sum_{j=1}^{n} x_j^{m+1} \partial_j \quad (m \geq -1),$$

$$w^{(t)}_m = \sum_{j=1}^{n} x_j^{m+t-1} \nabla_j^{t-1} \quad (t \geq 2, \ m \geq -t + 1),$$

where $\nabla_j$ are the Dunkl operators (2.1).

The operators $\{l_m\}_{m \geq -1}$ constitute the positive half of the Virasoro algebra

$$[l_m, l_n] = (n - m)l_{m+n}, \quad (3.2)$$

and we have

$$l_m \sim w^{(2)}_m.$$

The operator $w^{(3)}_0$ is related to the Calogero-Sutherland Hamiltonian (2.3) as

$$w^{(3)}_0 \sim H + (\beta - 1)l_0 - \beta p_1l_{-1}. \quad (3.3)$$

**Theorem 3.1.**

(i) $I^{(k, r)}_n$ is an ideal of $\mathbb{C}[x_1, \ldots, x_n]S_n$, $p_ml^{(k, r)}_n \subset I^{(k, r)}_n \ (m \geq 1)$.

(ii) $w^{(t)}_ml^{(k, r)}_n \subset I^{(k, r)}_n \ (t \geq 2, \ m \geq -t + 1)$.

We defer the proof of Theorem 3.1 to Section 3.2 and mention here an immediate consequence.

**Proposition 3.2.** Let $P \in I^{(k, r)}_n$. Then $(\partial_j^j P)(x_1, \ldots, x_{n-1}, 0) \in I^{(k, r)}_{n-1}$ for all $j \geq 0$. 

Proof. Set \((\rho P)(x_1, \cdots, x_{n-1}) = P(x_1, \cdots, x_{n-1}, 0)\). Since \(\rho(P_\lambda)\) is a Jack polynomial for the same partition \(\lambda\) in \((n - 1)\) variables, \(\rho\) gives rise to a map \(I_n^{(k,r)} \to I_{n-1}^{(k,r)}\). The assertion follows from Theorem 3.1 and the relation

\[
\rho \circ \partial_j n = \rho \circ \partial_j - 1 \circ l_{n-1} - l_{n-1} \circ \rho \circ \partial_j - 1,
\]

where we set \(l_{n-1} = \sum_{j=1}^n \partial_j\).

3.2. Proof of Theorem 3.1. We prove Theorem 3.1 in several steps. First we use the following special case of the Pieri formula ([Mac], VI,(6.7')):

\[
p_1 P_\mu = \sum_{\lambda} \psi'_{\lambda/\mu} P_\lambda.
\]

The sum ranges over partitions \(\lambda\) obtained by adding one node to \(\mu\). If \(j\) is such that \(\lambda_j = \mu_j + 1\) holds, then \(\psi'_{\lambda/\mu}\) is given by

\[
\psi'_{\lambda/\mu} = \prod_{i=1}^{j-1} \frac{(j - i - 1)\beta + \mu_i - \mu_j}{(j - i)\beta + \mu_i - \mu_j - 1},
\]

Equation (3.4) is an identity in \(\mathbb{C}(\beta)[x_1, \cdots, x_n]^{S_n}\).

Proposition 3.3. Let \(\mu\) be \((k,r,n)\)-admissible. Then the formula (3.4) remains true at \(\beta = \beta(k,r)\), where we retain only \((k,r,n)\)-admissible \(\lambda\) in the sum. In particular

\[
p_1 l_n^{(k,r)} \subset I_n^{(k,r)}.
\]

Proof. Let \(\lambda\) be a partition appearing in the sum (3.4), and let \(j\) be such that \(\lambda_j = \mu_j + 1\).

Note that \(P_\lambda\) does not have a pole at \(\beta = \beta(k,r)\) by Proposition 2.6.

Clearly \(\mu_j - 1 > \mu_j\). Then the denominators in (3.5) do not vanish at \(\beta = \beta(k,r)\) by Lemmas 2.1 and 2.2.

Suppose in addition that \(\lambda\) is not \((k,r,n)\)-admissible. Then \(\mu_j - k = \mu_j + r\) and the numerator of the second factor in (3.5) with \(i = j - k\) has a zero at \(\beta = \beta(k,r)\). The proof is over.

Proposition 3.4. \(l_{\pm 1} l_n^{(k,r)} \subset I_n^{(k,r)}\).

Proof. We use the following identities in \(\mathbb{C}(\beta)[x_1, \cdots, x_n]^{S_n}\) due to [Las]:

\[
l_1 P_\mu = \sum_{\lambda} \psi''_{\lambda/\mu} P_\lambda,
\]

\[
l_{-1} P_\mu = \sum_{\lambda} \psi'_{\lambda/\mu} P_\lambda.
\]

In (3.6) (resp. (3.7)), the sum is taken over \(\lambda\) obtained from \(\mu\) by adding (resp. removing) one node. Note that all \(P_\lambda\) appearing in (3.6), (3.7) have no pole at \(\beta = \beta(k,l)\) by Proposition 2.6.

In the case (3.6),

\[
\psi''_{\lambda/\mu} = \psi'_{\lambda/\mu} \times (- (j - 1)\beta + \mu_j)
\]
where \( \lambda_j = \mu_j + 1 \) and \( \psi'_{\lambda/\mu} \) is given by (3.3). Hence the assertion follows from the proof of Proposition 3.3.

In the case (3.7), the formula for \( \psi'_{\lambda/\mu} \) reads

\[
\psi'_{\lambda/\mu} = \frac{1}{\beta} \left((n-i)\beta + \mu_i\right)\left((n-i+1)\beta + \mu_i - 1\right) \times \\
\prod_{j=i+1}^{n} \frac{(j-i-1)\beta + \mu_j - \mu_j - \mu_j}{(j-i)\beta + \mu_i - \mu_j} \prod_{j=1}^{\mu_i-1} \frac{(\mu_j - i)\beta + \mu_i - j}{(\mu_j - i+1)\beta + \mu_i - j},
\]

(3.8)

where \( i \) is such that \( \lambda_i = \mu_i - 1 \).

The denominators in (3.8) do not vanish at \( \beta = \beta(k, r) \) by Lemma 2.1. If in addition \( \lambda \) is not \( (k, r, n) \)-admissible, then \( \mu_{i+k} = \mu_i - r \). Since the node \((i, \mu_i)\) is removable, we have \( \mu_i > \mu_{i+1} \). Now the admissibility of \( \mu \) forces \( \mu_{i+k+1} < \mu_{i+k} \). In particular, \( \mu'_{\mu_{i+k}} = i + k \) and the numerator of the factor with \( j = \mu_{i+k} \) in the second product of (3.8) vanishes.

\[\square\]

**Proposition 3.5.** \( l_m I_n^{(k,r)} \subset I_n^{(k,r)} \) \( (m \geq -1) \).

**Proof.** We have \( 2l_0 = [l_{-1}, l_{1}] \). Therefore \( l_0 I_n^{(k,r)} \subset I_n^{(k,r)} \) by Proposition 3.4.

From (3.3) and (2.10) we get \( w_0^{(3)} I_n^{(k,r)} \subset I_n^{(k,r)} \).

Using (2.2) we find

\[
[w_0^{(3)}], p_2 \sim 4l_2 + 2(n-1)\beta + 1)p_2.
\]

Therefore \( l_0 I_n^{(k,r)} \subset I_n^{(k,r)} \).

The proposition then follows from the commutation relations (3.2).

\[\square\]

**Proof of Theorem 3.4.** Statement (i) follows from Propositions 3.3, 3.4, and the relation \( [l_1, p_m] = mp_{m+1} \) \( (m \geq 1) \).

Statement (ii) for \( t = 2 \) follows from Proposition 3.3. It remains to show

\[
w_m^{(t)} I_n^{(k,r)} \subset I_n^{(k,r)} \quad (t > 2, m \geq -t + 1).
\]

(3.9)

Using (2.2) one verifies the following relations.

\[
[l_{-1}, w_0^{(3)}] = 2w_0^{(3)},
\]

\[
[w_{-p+1}^{(t)}, w_{-1}^{(t+1)}] = (t-1)w_{-t}^{(t+1)} \quad (t \geq 2),
\]

\[
[w_m^{(t+1)}, p_2] \sim 2tw_m^{(t+1)} + t(t-1)(1-\beta)w_{m+2}^{(t-1)} + 2\beta \sum_{i=0}^{t-2} (t-1-i)w_{m+t-i}^{(t-1-i)}w_{t+2+i}^{(t-1-i)}.
\]

In the last formula \( t \geq 2 \) and \( m \geq -t \) and we set \( w_m^{(1)} = p_m \). Since \( w_{-1}^{(2)} \sim l_{-1} \), the first two formulas imply (3.9) for \( m + t = 1 \). The general case follows from the third formula by induction on \( m + t \).

\[\square\]
4. Special cases

In this section we examine the two special cases $r = 2$ and $k = 2$, and identify $I^{(k,r)}_n$ with some spaces of correlation functions. For a graded subspace $U = \bigoplus_{d \geq 0} U_d \subset \mathbb{C}[x_1, \ldots, x_n]^{S_n}$, the formal character is defined to be $\text{ch} U = \sum_{d \geq 0} (\dim U_d) q^d$. Thus the character of (3.1) is

$$\text{ch} I^{(k,r)}_n = \sum_{\lambda} q^{[\lambda]},$$

the sum being taken over $(k, r, n)$-admissible partitions $\lambda$.

4.1. The case $r = 2$. Consider the subspace of symmetric polynomials

$$F^{(k)}_n = \{ P \in \mathbb{C}[x_1, \ldots, x_n]^{S_n} \mid P = 0 \text{ if } x_1 = \cdots = x_{k+1} \}. \quad (4.1)$$

**Proposition 4.1.** $I^{(k,r)}_n \subset F^{(k)}_n \quad (n \geq 0)$.

**Proof.** The case $n \leq k$ being obvious, we assume that $n \geq k + 1$.

To see the assertion in the case $n = k + 1$, we use the following specialization formula (see [Mac], VI,(6.11')) for Jack polynomials:

$$P_\lambda(1, \cdots, 1; \beta) = \prod_{(i,j) \in \lambda} \frac{(n-i+1)\beta+j-1}{(\lambda'_j-i+1)\beta+\lambda_j-j}.$$  

Here the product is taken over all nodes of $\lambda$.

Let $\lambda$ be $(k, r, n)$-admissible and $n = k + 1$. Since $\lambda_1 - \lambda_{k+1} \geq r$, the node $(1, r)$ is contained in $\lambda$. This means that the numerator has a zero at $\beta = \beta(k, r)$. The denominator does not vanish by Lemma 2.1. Therefore $P_\lambda = 0$ holds for $\beta = \beta(k, r)$ and $x_1 = \cdots = x_{k+1}$.

By induction, suppose we have proved the proposition for $n - 1$, with $n \geq k + 2$. Clearly $P \in F^{(k)}_n$ if and only if $(\partial_1 P)(x_1, \cdots, x_{n-1}, 0) \in F^{(k)}_{n-1}$ for any $j \geq 0$. The assertion then follows from Proposition 3.2. $\square$

**Theorem 4.2.** $F^{(k)}_n = I^{(k,2)}_n$.

**Proof.** It is known (see [FS], Proposition 2.6.1’ and Theorem 2.7.1) that the dimension of $(F^{(k)}_n)_d$ is given by the number of $(k, 2, n)$-admissible partitions such that $|\lambda| = d$, and hence $\text{ch} F^{(k)}_n = \text{ch} I^{(k,2)}_n$. Now Theorem 4.2 follows from Proposition 4.1. $\square$

4.2. The case $k = 2$. Let us consider another special case $k = 2$. In [FJM], a vertex operator algebra related to the Virasoro minimal series $(3, r + 2)$ was studied. The main objects in $\mathfrak{FJM}$ are an abelian current $a(x)$ which plays a role analogous to that of $e(x)$ in $\mathfrak{s}l_2$, and the ‘principal subspace’ $W$ created from the highest weight vector $|0\rangle$ by $a(x)$. Denote the space of correlation functions by

$$C^{(r)}_n = \text{span}_C \{ \langle \psi | a(x_1) \cdots a(x_n) | 0 \rangle \mid \langle \psi \rangle \in W^* \}.$$  

The following properties are known:

(i) $\text{ch} C^{(r)}_n = \text{ch} I^{(2,r)}_n$, 

(ii) $\text{ch} C^{(r)}_n |_{\lambda} = \sum_{\beta} q^{[\lambda]}$, where $\sum_{\beta}$ is taken over all $\lambda$.

Here the sum is over all $\lambda$ with some spaces of correlation functions. For a graded subspace $U = \bigoplus_{d \geq 0} U_d \subset$ 

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n}$, the formal character is defined to be $\text{ch} U = \sum_{d \geq 0} (\dim U_d) q^d$. Thus the character of (3.1) is

$$\text{ch} I^{(k,r)}_n = \sum_{\lambda} q^{[\lambda]},$$

the sum being taken over $(k, r, n)$-admissible partitions $\lambda$.
(ii) $G_n^{(r)}$ is generated from the non-zero homogeneous component of lowest degree by the action of $p_m$ ($m \geq 1$) and $l_m$ ($m \geq -1$),
(iii) $P \in G_n^{(r)}$ if and only if $(\partial^\alpha P)(x_1, x_2, 0, \cdots, 0) \in G_3^{(r)}$ holds for all $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n-3}$ where $\partial^\alpha = \partial_4^{\alpha_4} \cdots \partial_n^{\alpha_n}$, see Theorem 2.8, Proposition 3.1 and Proposition 4.4 in [FJM] respectively.

**Theorem 4.3.** $G_n^{(r)} = I_n^{(2,r)}$.

**Proof.** Since both spaces have the same characters, it suffices to show the inclusion relation.

First, consider the case $n = 3$. Let $\varphi_3 \in G_3^{(r)}$ be a non-trivial element of the smallest degree $d = r$. The corresponding element of $I_3^{(2,r)}$ is $\tilde{\varphi}_3 = P_\lambda$, with $\lambda = (r, 0, 0)$. For this polynomial we have $l_0 \tilde{\varphi}_3 = r \tilde{\varphi}_3$ and $H \tilde{\varphi}_3 = \varepsilon_\lambda \tilde{\varphi}_3$. Since removing a node from $\lambda$ leads to non-admissible partitions, we have also $l_{-1} \tilde{\varphi}_3 = 0$. These equations are the same as those known for $\varphi_3$ (see proof of Proposition 3.2 in [FJM]). Since their polynomial solution is unique up to a constant multiple, we see that $\varphi_3 \in I_3^{(2,r)}$. Property (ii) together with Theorem 3.1 then imply $G_3^{(r)} \subset I_3^{(2,r)}$, and hence $G_3^{(r)} = I_3^{(2,r)}$.

Now, in the general case $n \geq 4$ the inclusion $I_n^{(2,r)} \subset G_n^{(r)}$ follows from Proposition 3.2 and Property (iii). This completes the proof.

4.3. **Generalizations.** We conclude with two remarks about generalizations of the results of this paper.

The first remark is about an extension to the case of Macdonald polynomials.

For an indeterminate $s$ we set

$q = s^{k+1}, \quad t = s^{-(r-1)}$.

Let

$\tilde{\mathcal{F}}_n^{(k)} = \{ P \in \mathbb{C}(s)[x_1, \cdots, x_n]^{S_n} \mid P = 0 \text{ if } x_j = t^{j-1} \text{ for } j = 1, \cdots, k + 1 \}$,

$\tilde{I}_n^{(k,r)} = \text{span}_{\mathbb{C}(s)} \{ P_\lambda(x; q, t) \mid \lambda \text{ is } (k, r, n)-\text{admissible} \}$.

Here $P_\lambda(x; q, t)$ denotes the ‘monic’ Macdonald polynomial, see [Mac], VI,(4.7).

All the working in Subsection 4.1 can be extended straightforwardly to get the following theorem.

**Theorem 4.4.** $\tilde{\mathcal{F}}_n^{(k)} = \tilde{I}_n^{(k,2)}$.

The second remark is about the case of general $k, r$.

It seems natural to anticipate a relation similar to Theorem 4.3. Namely we expect that $I_n^{(k,r)}$ coincides with the space of correlation functions of an abelian current in a vertex operator algebra, associated with the minimal series $(k + 1, k + r)$ of the $W_k$ algebra. We hope to address this subject in the future.

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BF: LANDAU INSTITUTE FOR THEORETICAL PHYSICS, CHERNOGOLOVKA, 142432, RUSSIA
E-mail address: feigin@feigin.mccme.ru

MJ: GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, TOKYO 153-8914, JAPAN
E-mail address: jimbomic@ms.u-tokyo.ac.jp

TM: DIVISION OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN
E-mail address: tetsuji@kusm.kyoto-u.ac.jp

EM: DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY-PURDUE UNIVERSITY-INDIANAPOLIS, 402 N. BLACKFORD ST., LD 270, INDIANAPOLIS, IN 46202
E-mail address: mukhin@math.iupui.edu