Stochastic model of hysteresis

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The methods of the probability theory have been used in order to build up a new model of hysteresis which is different from the well-known Preisach model. It is assumed that the system consists of large number of abstract particles in which the variation of an external control parameter (e. g., the magnetic field) may result in transitions between two states \( S^+ \) and \( S^- \). The state of a particle is characterized by the value +1 or −1 of a random variable (e. g., the magnetization direction parallel or antiparallel to the magnetic field). The transitions are governed by two further random variables corresponding to the \( S^- \rightarrow S^+ \) and the \( S^+ \rightarrow S^- \) transitions (e. g., "up switching" and "down switching magnetic field"). The method presented here makes possible to calculate the probability distribution and consequently the expectation value of the number of particles in the \( S^+ \) (or \( S^- \)) state for both increasing and decreasing parameter values, i. e., the hysteresis curves of the transitions can be determined. It turns out that the reversal points of the control parameter are Markov points which determine the stochastic evolution of the process. It has been shown that the branches of the hysteresis loop are converging to fixed limit curves when the number of cyclic back-and-forth variations of the control parameter between two consecutive reversal points is large enough. This convergence to limit curves gives a clear explanation of the accommodation process. The accommodated minor loops show the return-point memory property but this property is obviously absent in the case of non-accommodated minor loops which are not congruent and generally not closed. In contrast to the traditional Preisach model the reversal point susceptibilities are non-zero finite values. The stochastic model can provide a surprisingly good approximation of the Rayleigh quadratic law when the external parameter varies between two sufficiently small values. The practical benefits of the model can be seen in the numerical analysis of the derived equations. On one hand the calculated curves are in good qualitative agreement with the experimental observations and on the other hand, the estimation of the joint distribution function of the up and down switching fields can be performed by using the measured hysteresis curves.

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I. INTRODUCTION

The phenomenon of the hysteresis, understood in a general sense, has been investigated so intensively for many decades that any list of references would be far from complete by any standards. It is very fortunate that in the last few years outstanding monographs have been published in this field, and thus the author does not feel obliged to cite the large amount of old but important references. However, it is considered important to mention two papers of G. Kádár whose work has played a stimulating role in getting to the idea of reconsideration of the hysteresis theory by the present author. There is no doubt that the abstract reformulation of the Preisach model given by Krasnoselskii and A. Pokrovskii and summarized by I. D. Mayergoyz in his book resulted in an improved mathematical clarity in the hysteresis theory, but the stochastic nature of the hysteresis still has not been treated with sufficient mathematical rigor.

The aim of the present paper is to define a stochastic model of hysteresis and to derive exact equations for the probability distribution functions describing the state variations in hysteretic processes as a function of increasing as well as decreasing control parameters. The vocabulary of magnetic hysteresis will be used for convenience from now on, however, the concepts can easily be generalized for any hysteretic phenomenon.

II. DESCRIPTION OF THE MODEL

Let us assume that the unit volume of the system consists of many small abstract regions, called "particles" which are characterized by four random variables: \( \mu, \lambda, \chi_d \) and \( \chi_u \). The absolute value of the particle magnetization is denoted by \( \mu \). If the particle magnetization is parallel (antiparallel) to the external magnetic field \( H \) then the particle
is in the state $S^+(S^-)$ and $\lambda = 1(-1)$. The random value $\chi_d$ corresponds to a local field at which the state $S^+$ jumps to the state $S^-$ and similarly the $S^- \Rightarrow S^+$ transition occurs at the random local field $\chi_u$. For simplicity the $\chi_u$ and $\chi_d$ will be called as U-field and D-field. These two random variables express the obvious fact that each particle "feels" not only the external magnetic field $H$, but also the interaction field due to the adjacent particles and the random field originated from the inhomogeneities of the surrounding environment. These particles characterized by the random variables $\mu, \lambda, \chi_d$ and $\chi_u$ can be regarded as "independent" abstract elements of the system, and they will be called "hysterons".

Figure 1 illustrates a possible realization of transitions $S^+ \iff S^-$ of a hysteron. The transition curves form a random rectangular hysteresis loop which is almost in all cases asymmetrical in the co-ordinate system of magnetization versus external magnetic field since the U- and D-fields are supposed to be random.

Let us denote by $h_1, h_2, \ldots, h_N$ the hysterons in a system, and let $N^+\delta$ be the set of indices and $n^+$ the number of hysterons which are in the state $S^+$ at a given external field $H$. In this case the magnetization of the system is given by the stochastic equation

$$\delta_{n^+} = \sum_{k=1}^{N} \lambda_k \mu_k,$$

where $\mu_k$ is the absolute value of the magnetization of the hysteron $h_k$, while

$$\lambda_k = \begin{cases} 1, & \text{if } k \in N^+, \\ -1, & \text{if } k \not\in N^+. \end{cases}$$

Since the random variables $\mu_1, \mu_2, \ldots, \mu_N$ are mutually independent and have the same probability distribution function

$$P\{\mu_k \leq x\} = L(x), \quad \forall \ k = 1, 2, \ldots, N,$$

it is obvious that the characteristic function of the distribution function

$$P\{\delta_{n^+} \leq x\} = R_{n^+}(x)$$

can be written in the form

$$\Phi_{n^+}(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} dR_{n^+}(x) = \left[\varphi(-\omega)\right]^N \left[\frac{\varphi(\omega)}{\varphi(-\omega)}\right]^{n^+},$$

where

$$\varphi(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} dL(x) = \int_{0}^{+\infty} e^{i\omega x} dL(x).$$

In order to calculate the characteristic function

$$\Phi(H, \omega) = \sum_{n^+ = 0}^{N} \Phi_{n^+}(\omega) \ P_{n^+}(H) = \left[\varphi(-\omega)\right]^N \sum_{n^+ = 0}^{N} \ P_{n^+}(H) \left[\frac{\varphi(\omega)}{\varphi(-\omega)}\right]^{n^+},$$

we need the probability of finding $n^+$ hysterons in the state $S^+$ at the external field $H$ which is the endpoint of a well-defined magnetization prehistory. The determination of this probability and the derivation of the equations for "up" and "down" magnetizations versus magnetic field will be the task of the next section.

### III. DERIVATION OF THE FUNDAMENTAL EQUATIONS

#### A. Some basic relations

Let us denote by $H(x, y|C)$ the joint distribution function of the random U- and D-fields. From the physical point of view it is quite obvious that the U-field cannot be smaller than the D-field, so the stochastic inequality $\chi_u \geq \chi_d$
must be satisfied. It is easy to show that the joint distribution function of \( \chi_u \) and \( \chi_d \) satisfying the condition \( C = \{ \chi_u \geq \chi_d \} \) can be written in the form

\[
P\{\chi_u \leq x, \chi_d \leq y|\chi_u \geq \chi_d\} = H(x, y|C) = \frac{\int_{-\infty}^{x} dx' \int_{-\infty}^{y} h(x', y') \Delta(x' - y')}{\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} h(x', y') dy'},
\]

where \( \Delta(x) \) is the unit step function. It is clear that the joint density function of the U- and D-fields can be given by

\[
h(x, y|C) = \frac{h(x, y)}{\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} h(x', y') dy'} \Delta(x - y),
\]

provided that the condition \( C \) is valid.

We need in the sequel two conditional probability distribution functions:

\[
P\{\chi_u \leq x|C\} = H(x, \infty|C) = F_u(x|C) = \frac{\int_{-\infty}^{x} dx' \int_{-\infty}^{+\infty} h(x', y') dy'}{\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} h(x', y') dy'}
\]

and

\[
P\{\chi_d \leq y|C\} = H(\infty, y|C) = F_d(y|C) = \frac{\int_{-\infty}^{+\infty} dy' \int_{-\infty}^{y} h(x', y') dx'}{\int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} h(x', y') dx'}.
\]

Evidently \( F_u(x|C) \) is the probability that the U-field of a given hysteron is not larger than \( x \), while \( F_d(y|C) \) is the probability that the D-field is not larger than \( y \) assuming in both cases that the condition \( C \) is fulfilled. By using the Dirichlet’s theorem for changing the sequence of integration it is obvious that

\[
\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{x'} h(x', y) dy' = \int_{-\infty}^{+\infty} dy' \int_{y}^{+\infty} h(x', y') dx'.
\]

For the sake of further considerations it is necessary to introduce two transition probabilities denoted by \( w_u(H_t \uparrow H) \) and \( w_d(H_u \downarrow H) \). Let \( H_t \) be a fixed value of the external magnetic field and let us suppose that the state of a given hysteron is \( S^{(-)} \) at \( H_t \). By using elementary theorems, it can be proved that if the external field increases monotonically from \( H_t \) to \( H \geq H_t \) then

\[
w_u(H_t \uparrow H) = \frac{\int_{H_t}^{H} dx \int_{-\infty}^{x} h(x, y) dy}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} h(x, y) dy}
\]

is the probability of the transition \( S^{(-)} \Rightarrow S^{(+)} \) occurring in the interval \([H_t \uparrow H]\). Similarly, let \( H_u \) be another fixed value and \( S^{(+)} \) the state of a hysteron at \( H_u \). If the external field decreases now monotonically from \( H_u \) to \( H \leq H_u \) then

\[
w_d(H_u \downarrow H) = \frac{\int_{H_u}^{H} dy \int_{-\infty}^{+\infty} h(x, y) dx}{\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} h(x, y) dx}
\]

gives the probability of the transition \( S^{(+)} \Rightarrow S^{(-)} \) occurring in the interval \([H_u \downarrow H]\).

**B. Stochastic magnetizing process**

Let us introduce the "time parameter" \( t \in [0, +\infty] \) and define a real valued, external field function \( H(t) \) which consists of monotone increasing and decreasing sections of different length. Denote by \( H_1, H_2, \ldots, H_j, \ldots \), the extremum values of the function \( H(t) \) belonging to the subsequent time points \( t_1 < t_2 < \cdots < t_j < \cdots \). It is clear that if \( H(t_j) = H_j \) is a local maximum then \( H(t_{j-1}) = H_{j-1} \) and \( H(t_{j+1}) = H_{j+1} \) must be local minimums which are not necessarily equal. In the following the sequence \( \{H_j\} \) will be called magnetizing path and the elements of this sequence are called points of reversal. If the functions \( H^{(1)}(t) \) and \( H^{(2)}(t) \) have the same magnetizing path then
they are said to be equivalent for any magnetizing process irrespective of the form of the time function between the individual extrema. In Fig. 3, two equivalent $H(t)$ functions are seen. The sequence of extrema is the same for both curves, but the time distance and the shape of sections between the consecutive extrema are different.

It is assumed that the magnetizing process which consists of random transitions $S(+) \Leftrightarrow S(-)$ of hysterons does not “feel” the variation speed of $H(t)$ between the consecutive extremum values, i.e., the magnetizing process is static. The evolution of the process in each subinterval $[H_j, H_{j+1}]$, $j = 1, 2, \ldots$, is stochastically determined by the extremum $H_j$ and by the actual values of $H(t)$ following $H_j$, but the process does not depend on the time derivative of $H(t)$. This property is called rate independence in the non-stochastic theory of hysteresis but it will be applied in this stochastic theory too. Denote the maximum reversal fields by odd and the minimum ones by even indices. In

and the other is

$$\mathbb{P}\{\xi_u^{(+)}(H_{2k} \uparrow H) = n_{2k}^{(u)}(H)|\xi_{\text{start}}^{(+)} = 0\} = p_{2k}^{(u)}[H_{2k} \uparrow H, n_{2k}^{(u)}(H)|0],$$

It is important to note that the reversal points (extremum values)

$$H_1, H_2, \ldots, H_{2k-1}, H_{2k}, H_{2k+1}, \ldots,$$

are Markov-points of the stochastic processes $\xi_{u}^{(+)}(H_j \uparrow H)$ and $\xi_{d}^{(+)}(H_j \downarrow H)$, and therefore we can write the following equations:

$$p_{2k+1}^{(d)}[H_{2k+1} \downarrow H, n_{2k+1}^{(d)}(H)|0] = \sum_{n_{2k}^{(u)}(H_{2k+1})=0}^{N} p_{2k}^{(d)}[H_{2k+1} \downarrow H, n_{2k+1}^{(d)}(H)|n_{2k}^{(u)}(H_{2k+1})] \times

\times p_{2k}^{(u)}[H_{2k} \uparrow H_{2k+1}, n_{2k}^{(u)}(H_{2k+1})|0],$$

and

$$p_{2k}^{(u)}[H_{2k} \uparrow H, n_{2k}^{(u)}(H)|0] = \sum_{n_{2k+1}^{(d)}(H_{2k})=0}^{N} p_{2k}^{(u)}[H_{2k} \uparrow H, n_{2k}^{(u)}(H)|n_{2k+1}^{(d)}(H_{2k})] \times

\times p_{2k-1}^{(d)}[H_{2k-1} \downarrow H_{2k}, n_{2k-1}^{(d)}(H_{2k})|0].$$

As it has already been mentioned the hysterons can be regarded as independent of each other particles and, therefore, it is an easy task to determine the probability that the number of $S^{(+)}$-hysteres is exactly equal to a non-negative integer not larger than $N$, at an either decreasing or increasing external field $H$ provided that the number of $S^{(+)}$-hysteres is known at the last reversal point before arriving at $H$.

The probability $p_{2k+1}^{(d)}[H_{2k+1} \downarrow H, n_{2k+1}^{(d)}(H)|n_{2k}^{(u)}(H_{2k+1})]$ can be obtained as a result of the following consideration. If the number of $S^{(+)}$-hysteres at the reversal point $H_{2k+1}$ is equal to $n_{2k}^{(u)}(H_{2k+1})$, then — in order to have $n_{2k+1}^{(d)}(H)$
hysteron in the state \( S^{(+)} \) at the external field \( H \leq H_{2k+1} \) — exactly \( n_{2k}^{(u)}(H_{2k+1}) - n_{2k+1}^{(d)}(H) \) hysteron of state \( S^{(+)} \) have to transform to the state \( S^{(-)} \) in the interval \([H_{2k+1} - H] \). It is obvious that the probability of this event can be given by

\[
p_{2k+1}^{(d)}[H_{2k+1} \downarrow H, n_{2k+1}^{(d)}(H)]n_{2k}^{(u)}(H_{2k+1}) = \left( \frac{n_{2k}^{(u)}(H_{2k+1})}{n_{2k+1}^{(d)}(H)} \right) \times \]

\[
\times \left[ w_d(H_{2k+1} \downarrow H) \right]^{n_{2k}^{(u)}(H_{2k+1}) - n_{2k+1}^{(d)}(H)} [1 - w_d(H_{2k+1} \downarrow H)]^{n_{2k+1}^{(d)}(H)}. \tag{15}
\]

Similarly, to determine the probability \( p_{2k}^{(u)}[H_{2k} \uparrow H, n_{2k}^{(u)}(H)]n_{2k-1}^{(d)}(H_{2k}) \) one has to recognize that if the number of \( S^{(-)} \)-hysteron at the reversal point \( H_{2k} \) is equal to \( N - n_{2k-1}^{(d)}(H_{2k}) \), then — in order to have \( n_{2k}^{(u)}(H) \) hysteron in the state \( S^{(+)} \) at the external field \( H \geq H_{2k} \) — exactly \( n_{2k}^{(u)}(H) - n_{2k-1}^{(d)}(H_{2k}) \) hysteron of state \( S^{(-)} \) have to transform to the state \( S^{(+)} \) in the interval \([H_{2k} \uparrow H] \). The probability of this event is given by

\[
p_{2k}^{(u)}[H_{2k} \uparrow H, n_{2k}^{(u)}(H)]n_{2k-1}^{(d)}(H_{2k}) = \left( \frac{N - n_{2k-1}^{(d)}(H_{2k})}{n_{2k}^{(u)}(H) - n_{2k-1}^{(d)}(H_{2k})} \right) \times \]

\[
\times \left[ w_u(H_{2k} \uparrow H) \right]^{n_{2k}^{(u)}(H) - n_{2k-1}^{(d)}(H_{2k})} [1 - w_u(H_{2k} \uparrow H)]^{N - n_{2k}^{(u)}(H)}. \tag{16}
\]

In order to simplify the further calculations let us introduce the generating functions

\[
\Gamma_{2k+1}^{(d)}(H_{2k+1} \downarrow H, z) = \sum_{n_{2k+1}^{(d)}(H) = 0}^{N} p_{2k+1}^{(d)}[H_{2k+1} \downarrow H, n_{2k+1}^{(d)}(H) \mid 0] z^{n_{2k+1}^{(d)}(H)} \tag{17}
\]

and

\[
\Gamma_{2k}^{(u)}(H_{2k} \uparrow H, z) = \sum_{n_{2k}^{(u)}(H) = 0}^{N} p_{2k}^{(u)}[H_{2k} \uparrow H, n_{2k}^{(u)}(H) \mid 0] z^{n_{2k}^{(u)}(H)}. \tag{18}
\]

By using the Eqs. \([13] \) and \([15] \) we get the first fundamental equation in the form

\[
\Gamma_{2k+1}^{(d)}(H_{2k+1} \downarrow H, z) = \Gamma_{2k}^{(u)}(H_{2k} \uparrow H_{2k+1}, a(H_{2k+1}, H, z)), \tag{19}
\]

where

\[
a(H_{2k+1}, H, z) = w_d(H_{2k+1} \downarrow H) + [1 - w_d(H_{2k+1} \downarrow H)] z. \tag{20}
\]

The second fundamental equation follows from the relations \([14] \) and \([16] \). We have

\[
\Gamma_{2k}^{(u)}(H_{2k} \uparrow H, z) = [c(H_{2k}, H, z)]^{N} \Gamma_{2k-1}^{(d)}[H_{2k-1} \downarrow H_{2k}, b(H_{2k}, H, z)], \tag{21}
\]

where

\[
c(H_{2k}, H, z) = 1 - (1 - z) w_u(H_{2k} \uparrow H), \tag{22}
\]

and

\[
b(H_{2k}, H, z) = \frac{z}{c(H_{2k}, H, z)}. \tag{23}
\]

Now we will derive the characteristic function of the probability that the system magnetization is not larger than \( x \) at a decreasing external field \( H \) which follows the last reversal point \( H_{2k+1} \). Introducing the notation

\[
\psi(\omega) = \frac{\varphi(\omega)}{\varphi(-\omega)}, \tag{24}
\]
and by using the relation (17) we obtain from Eq. (4)

$$\Phi_d(H_{2k+1} \downarrow H, \omega) = [\varphi(-\omega)]^N \Gamma^{(d)}_{2k+1}[H_{2k+1} \downarrow H, \psi(\omega)].$$

(25)

Similarly, if we use Eq. (18) then the characteristic function of the probability that the magnetization of the system is not larger than \( x \) at an increasing external field \( H \) after the last reversal point \( H_{2k} \) can be obtained from Eq. (4) in the form

$$\Phi_u(H_{2k} \uparrow H, \omega) = [\varphi(-\omega)]^N \Gamma^{(u)}_{2k}[H_{2k} \uparrow H, \psi(\omega)].$$

(26)

These two characteristic functions describe completely the stochastic behaviour of the magnetizing process in both increasing and decreasing external magnetic fields. It is apparent from the above considerations that the stochastic model developed by us has been built up without any reference to a particular nature of hysteresis and therefore, its generality is at least as high as that of the Krasnoselskii and Pokrovskii\(^8\) model.

C. Calculation of the hysteresis curves

The expectation value of the magnetic moment \( \mu_k \) due to the \( k \)-th hysteron can be given by

$$E\{\mu_k\} = i^{-1} \left[ \frac{d\varphi(\omega)}{d\omega} \right]_{\omega=0} = M_s.$$ 

By using this expression we can write the expectation value of the magnetization of the system at the decreasing external field \( H \) following the reversal point \( H_{2k+1} \) in the form

$$i^{-1} \left[ \frac{d\Phi_d(H_{2k+1} \downarrow H, \omega)}{d\omega} \right]_{\omega=0} = M^{(d)}_{2k+1}(H_{2k+1} \downarrow H|0) = 2M_s N^{(d)}_{2k+1}(H_{2k+1} \downarrow H|0) - N M_s,$$

(27)

where

$$N^{(d)}_{2k+1}(H_{2k+1} \downarrow H|0) = \left[ \frac{d\Gamma_{2k+1}^{(d)}(H_{2k+1} \downarrow H, z)}{dz} \right]_{z=1}.$$ 

(28)

From the fundamental Eq. (19) we obtain

$$N^{(d)}_{2k+1}(H_{2k+1} \downarrow H|0) = N^{(u)}_{2k}(H_{2k} \uparrow H_{2k+1}|0) \left[ 1 - w_d(H_{2k+1} \downarrow H) \right].$$

(29)

The expectation value of the magnetization of the system at the external magnetic field \( H \) increasing after the reversal field \( H_{2k} \) can be obtained from the equation

$$i^{-1} \left[ \frac{d\Phi_u(H_{2k} \uparrow H, \omega)}{d\omega} \right]_{\omega=0} = M^{(u)}_{2k}(H_{2k} \uparrow H|0) = 2M_s N^{(u)}_{2k}(H_{2k} \uparrow H|0) - N M_s,$$

(30)

where

$$N^{(u)}_{2k}(H_{2k} \uparrow H|0) = \left[ \frac{d\Gamma_{2k}^{(u)}(H_{2k} \uparrow H, z)}{dz} \right]_{z=1}.$$ 

(31)

The recursive relation

$$N^{(u)}_{2k}(H_{2k} \uparrow H|0) = N w_u(H_{2k} \uparrow H) + N^{(d)}_{2k-1}(H_{2k-1} \downarrow H_{2k}|0) \left[ 1 - w_u(H_{2k} \uparrow H) \right]$$

(32)

follows from the other fundamental Eq. (21).
By introducing the relative magnetizations

\[ m_{2k+1}^{(d)}(H_{2k+1} \downarrow H|0) = \frac{1}{N} M_{2k+1}^{(d)}(H_{2k+1} \downarrow H|0), \]  

(33)

and

\[ m_{2k}^{(u)}(H_{2k} \uparrow H|0) = \frac{1}{N} M_{2k}^{(u)}(H_{2k} \uparrow H|0), \]  

(34)

from Eqs. (27), (28) and (30), (32) after elementary calculations the following recursive relations are obtained:

\[ m_{2k+1}^{(d)}(H_{2k+1} \downarrow H|0) = \]

\[ = \left[ 1 + m_{2k}^{(u)}(H_{2k} \uparrow H_{2k+1}|0) \right] \left[ 1 - w_d(H_{2k+1} \downarrow H) \right] - 1, \]  

(35)

and

\[ m_{2k}^{(u)}(H_{2k} \uparrow H|0) = \]

\[ = 2 w_u(H_{2k} \uparrow H) + \left[ 1 + m_{2k-1}^{(d)}(H_{2k-1} \downarrow H_{2k}|0) \right] \left[ 1 - w_u(H_{2k} \uparrow H) \right] - 1. \]  

(36)

In order to solve this system of recursive equations we need the formula for the starting branch of the relative magnetization. Since the negative saturation has been chosen as the initial state of the system it follows from Eq. (1) that if \( H_f \Rightarrow H_0 = -\infty \), then

\[ w_u(H_f \uparrow H) \Rightarrow F_u(H|C), \]

and so the equation for the starting branch will be

\[ m_0^{(u)}(-\infty \uparrow H|0) = 2 F_u(H|C) - 1. \]  

(37)

This branch can be also called the limiting ascending branch because there is no branch below it. If the positive saturation would be the initial state then it is easy to show that the limiting descending branch can be written in the form

\[ m_0^{(d)}(\infty \downarrow H|0) = 2 F_d(H|C) - 1, \]  

(38)

where \( F_d(H|C) \) is defined by the expression (5), and at the same time it is obvious that the limiting descending branch has the property that there is no other branch above it. The two limiting curves form the major hysteresis loop which defines an area where all other loops should be located.

By using the expression (37) for the starting branch and Eq. (33) we can obtain the first descending branch

\[ m_1^{(d)}(H_1 \uparrow H|0) = 2 F_u(H_1|C) \left[ 1 - w_d(H_1 \uparrow H) \right] - 1, \]  

(39)

which is attached to the limiting ascending branch at the point \( H_1 \). This descending branch is called by Mayergoyz\[ the first-order transition curve. The field \( H_1 \) where the first-order transition curve starts from, will be called start field. Denote by \( H_2 \) the next reversal point where the magnetizing field begins again to increase. The corresponding ascending branch, i. e., the second-order transition curve is given by the formula:

\[ m_2^{(u)}(H_2 \uparrow H|0) = 2 w_u(H_2 \uparrow H) + \left[ 1 + m_1^{(d)}(H_1 \downarrow H_2) \right] \left[ 1 - w_u(H_2 \uparrow H) \right] - 1. \]  

(40)

This procedure can be continued and it is seen that there is no need to take into account any special requirement in order to describe the field dependence of the average magnetization since the Markov points of the magnetizing field determine exactly the stochastic behaviour of the process.
D. Stationarity of hysteresis loops

Let us investigate now the variation of the magnetization for a special sequences of reversal fields. Let as suppose that $H_{2k+1} = H_u$, $\forall \ k = 0, 1, \ldots$, while $H_{2k} = H_d$, $\forall \ k = 1, 2, \ldots$, and $H_u \geq H_d$, i.e. the magnetizing field is varying between two extreme values $H_u$ and $H_d$. The field variation which starts with a decrease of the the external magnetic field $H$ from the reversal point $H_u$ until it reaches the next reversal point $H_d$ and then turns to increase to the nearest $H_u$ value, is called magnetizing cycle. The magnetizing cycle results in a hysteresis loop called minor hysteresis loop. The first cycle corresponds to the variation of the external field between the reversal points $H_1 \Rightarrow H_2 \Rightarrow H_3$, where $H_1 = H_3 = H_u$ and $H_2 = H_d$, while the $k$-th cycle is done by the variation of the magnetizing field between the reversal points $H_{2k-1} \Rightarrow H_{2k} \Rightarrow H_{2k+1}$, where $H_{2k-1} = H_{2k+1} = H_u$ and $H_{2k} = H_d$ for $k = 1, 2, \ldots$. For the descending branch of the $k$-th minor loop one can obtain from Eq. (35) the following expression:

$$m_{2k-1}^{(d)}(H_u \downarrow H|0) = [1 + m_{2k-2}^{(u)}(H_d \uparrow H_u|0)] [1 - w_d(H_u \downarrow H)] - 1. \quad (41)$$

while for the ascending branch of the $k$-th minor loop the relation

$$m_{2k}^{(u)}(H_d \uparrow H|0) = 2 w_u(H_d \uparrow H) + [1 + m_{2k-1}^{(d)}(H_u \downarrow H_d|0)] [1 - w_u(H_d \uparrow H)] - 1 \quad (42)$$
can be derived from Eq. (30). By using Eq. (37) we have

$$m_1^{(d)}(H_u \uparrow H|0) = 2 F_u(H_u|C) [1 - w_d(H_u \downarrow H)] - 1, \quad (43)$$
for the descending branch of the first minor loop and

$$m_2^{(u)}(H_d \uparrow H|0) = 2 F_u(H_u|C) [1 - w_d(H_u \downarrow H_d)] [1 - w_u(H_d \uparrow H)] + 2 w_u(H_d \uparrow H) - 1 \quad (44)$$
for the ascending branch of the same loop. It is to note that according to the Mayergoyz’s terminology the first minor loop consists of a first-order descending and a second-order ascending transition curves. Following the reversal points of the magnetizing field this procedure can be continued and we can obtain both the descending and ascending branches of relative magnetization for any minor loop.

One can prove a very important limit theorem, namely, there exist two limit curves

$$\lim_{k \to \infty} m_{2k-1}^{(d)}(H_u \downarrow H|0) = m_d(H_u \downarrow H), \quad (45)$$
and

$$\lim_{k \to \infty} m_{2k}^{(u)}(H_d \uparrow H|0) = m_u(H_u \downarrow H), \quad (46)$$
which are determining a closed minor loop. In other words, the magnetizing process becomes stationary with increasing number of cycles. It means the system "forgets" gradually its initial state by repeating the magnetizing cycle. This forgetting process can be related to the well-known accommodation process. The original Preisach model results in an immediate formation of the minor hysteresis loop after only one cycle of back-and-forth variation of the input between any two consecutive extremum values. However, this consequence of the Preisach model contradicts to many experimental facts indicating the accommodation plays an important role in magnetizing processes. In order to describe the accommodation process the traditional Preisach model was modified rather artificially in the "moving" and the "product" models. In contrast to these the stochastic model contains the phenomenon of accommodation inherently and so there is no need for any additional improving the model.

The formulas for the limit curves defined by Eqs. (45) and (46) can be obtained by some elementary calculations in following forms:

$$m_d(H_u \downarrow H) = 2 Q(H_d, H_u) w_u(H_d \uparrow H_u) [1 - w_d(H_u \downarrow H)] - 1, \quad (47)$$
and

$$m_u(H_d \uparrow H) = 1 - 2 Q(H_d, H_u) w_d(H_u \downarrow H_d) [1 - w_u(H_d \uparrow H)], \quad (48)$$

where

$$Q(H_d, H_u) = [w_d(H_u \downarrow H_d) + w_u(H_d \uparrow H_u) - w_d(H_u \downarrow H_d) w_u(H_d \uparrow H_u)]^{-1}.$$
From these Eqs. two important relations can be derived, namely
\[ m_d(H_u \downarrow H_u) = m_u(H_d \uparrow H_u), \quad \text{and} \quad m_d(H_u \downarrow H_d) = m_u(H_d \uparrow H_d), \]
which show that the return-point memory property is fulfilled for the accommodated minor hysteresis loops. It is also obvious, that the accommodated minor loops due to the same pair of reversal fields \( H_d \) and \( H_u \geq H_d \) are not only congruent but identical since the field values \( H_d \) and \( H_u \) unambiguously determine the branches of stationary loops. One has to mention that the accommodated branches \( m_d(\infty \downarrow H) \) and \( m_u(-\infty \uparrow H) \) are exactly identical with the limiting descending and ascending branches which indicates the consistency of the theory.

The explicit form of the expressions \( m_d(H_u \downarrow H) \) and \( m_u(H_d \uparrow H) \) which describe the descending and the ascending branches of the stationary minor loop between two reversal fields \( H_d \) and \( H_u \geq H_d \) has a great advantage in numerical calculations in comparison with the well-known Everett integral. It is to be noted that the expressions (47) and (48) are suitable to describe not only symmetrical but \textit{asymmetrical hysteresis loops} too and it is easy to show that symmetrical hysteresis loops can be obtained only if the function \( h(x, y) \) has a \textit{mirror symmetry} expressed by Eq.
\[ h(x, y) = h(-y, -x). \]  
(49)

In the following the mirror symmetry of \( h(x, y) \) will be assumed.

It is worthwhile to derive the formula for the \textit{virgin curve of the magnetization} depending on the parameters of the density function \( h(x, y|C) \). After some simple manipulations we obtain
\[ m_0(H) = 2 \frac{s_1(H)}{s_1(H) + s_2(H) - s_1(H) s_2(H)} - 1, \]  
(50)
where
\[ s_1(H) = \frac{\int_{-H}^{+H} dx \int_{-\infty}^{\infty} h(x, y) dy}{\int_{-H}^{+H} dx \int_{-\infty}^{\infty} h(x, y) dy}, \]  
(51)
and
\[ s_2(H) = \frac{\int_{-H}^{+H} dy \int_{-\infty}^{\infty} h(x, y) dx}{\int_{-H}^{+H} dy \int_{-\infty}^{\infty} h(x, y) dx}. \]  
(52)

If the function \( h(x, y) \) satisfies the symmetry relation \( (13) \) then it is easy to prove that
\[ \lim_{H \to 0} m_0(H) = 0 \]
and the \textit{initial susceptibility} defined by Eq.
\[ \chi_a = \lim_{H \to 0} \frac{d m_0(H)}{dH} = \frac{\int_{-\infty}^{+\infty} h(x, 0) dx}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{\infty} h(x, y) dy} \]  
(53)
is different from zero in contrary to the classical Preisach model which gives a nonrealistic zero slope of the virgin curve at \( H = 0 \).

\section*{IV. NUMERICAL CALCULATIONS AND DISCUSSION}

In order to compute the magnetization vs. field curves we have to know the joint density function \( h(x, y|C) \) of the \textit{U} and \textit{D}-fields. Since these fields are the sum of many small random components it is reasonable to assume that the central limit theorem is approximately valid and so the function \( h(x, y) \) in \( h(x, y|C) \) can be chosen in the form
\[ h(x, y) = \frac{1}{2\pi \sigma^2 \sqrt{1 - C_r^2}} \]
\[ \times \exp \left\{ -\frac{1}{2\sigma^2 (1 - C_r^2)} \left[ (x - H_c)^2 - 2 C_r (x - H_c) (y + H_c) + (y + H_c)^2 \right] \right\}, \]  
(54)
where the meaning of the constants $H_c$, $\sigma$ and $C_r$ is clear from the elements of the probability theory. Figure 3 shows the contour plot of $h(x,y|C)$ defined by (1) for the parameters $H_c = 0.2$, $\sigma = 0.6$ and $C_r = 0.5$. The contours correspond to the following values of $h(x,y|C)$: 0.1, 0.2, 0.3, 0.5, 0.65 and 0.682. The last one is slightly smaller than $\max_{x,y} h(x,y|C) = 0.682923$. The discontinuity along the line $y = x = 0$ can be clearly seen in the figure. In the sequel to this formula will be used in all of our numerical calculations provided that the correlation coefficient $C_r$ is equal to zero, i.e. $h(x,y) = f(x)f(-y)$ where $f$ is the density function of the normal distribution. This case corresponds to the product model introduced by G. Biorci and D. Pescetti and used consequently by G. Kádár.

The relative magnetization vs. field curves are shown in Fig. 4. The parameter values used for the calculation are $H_c = 0.4$, $\sigma = 0.6$ and $C_r = 0$. The curves $\text{LA}_1$ and $\text{LD}_1$ correspond to the limiting ascending and descending branches, while the curves indexed in the figure by 1, 2, 3, 4 are the first-, second-, third- and fourth-order transition curves defined by the reversal points $H_1 = 1.2$, $H_2 = -0.8$, $H_3 = 0.6$, $H_4 = -0.6$. It is worth noting that the all information about the past history of the magnetizing process is transferred by the state of the system in the last reversal point. For example, the fourth-order transition curve 4 which is plotted in the field interval $[-0.6,1.5]$, is determined by the state in the reversal point $H_4 = -0.6$.

It is well-known that in the traditional Preisach model the minor loops which describe the cyclic change of the magnetization with back-and-forth variation of the magnetizing field between the same two limiting values are congruent and the formation of a closed minor loop is realized in one cycle, i.e., the accommodation process is absent. In contrast to this the stochastic model contains inherently the accommodation process which is clearly demonstrated in Fig. 3. For the sake of orientation the limiting ascending branch $\text{LA}_1$ is also plotted in Fig. 3 where it is seen that the descending branch of the first minor loop starts from the point $A$ due to the first reversal field $H_1 = 0.8$ and after reaching the reversal point $H_2 = H_4 = -0.2$ it turns to increase to the point $B$ which corresponds to the next reversal field $H_3 = H_4 = 0.8$. One can observe that the first minor loop is not closed, the point $B$ where the decreasing branch of the second minor loop starts from, occupies a higher position than the point $A$ and the end point $C$ of the increasing branch of the second minor loop is found above the point $A$ but the distance between the points $C$ and $B$ is smaller than that between the points $B$ and $A$. By repeating the magnetizing cycle between the reversal fields $H_2 = -0.2$ and $H_4 = 0.8$ the difference between the branches of the same type becomes gradually negligible, i.e., the branches converge to limit curves which form finally a closed stationary hysteresis loop denoted by $\text{LC}_1$. The magnetizing curves measured by Carter and Richards on silicon steel (4.3%/Si) are surprisingly similar to that plotted in Fig. 3.

In order to demonstrate the speed of the convergence, the non-accommodated relative magnetizations have been calculated in the reversal point $H_5 = -0.8$ for the subsequent cycles. Figure 3 shows that the stationary (i.e., the limit) value of the magnetization can be very well approached by repeating the cycle 8-9 times in the case of parameter values $H_c = 0.4$, $\sigma = 0.6$ and $H_u = 0.3$.

The non-accommodated minor loops due to the same pair of reversal fields are evidently not congruent and generally are not closed. However, this non-congruency has nothing in common with that introduced and discussed in details by Kádár. The non-congruency of the non-accommodated minor loops bounded by the same field limits has a quite different origin in the stochastic model, namely, the non-equilibrium response of the system for the cyclic back-and-forth variation of the external magnetic field between two consecutive reversal points. It is obvious consequence of the non-stationarity of minor loops that the return-point memory property is absent in these loops.

In order to study the properties of non-congruency of this type the first minor loops belonging to different start fields are calculated. Denote by $\Delta H = H_u - H_d$ the difference between the consecutive reversal fields. For the characterization of the non-accommodated first minor loops due to different start fields $H_1$ let us introduce two parameters defined by

$$W = W(H_1, \Delta H) = \max_{H_1 - \Delta H \leq H \leq H_1} [m_1(H_1 \downarrow H) - m_2(H_1 - \Delta H \uparrow H)],$$

and

$$O = O(H_1, \Delta H) = m_2(H_1 - \Delta H \uparrow H_1) - m_1(H_1 \downarrow H_1).$$

The dependence of these parameters on the start field $H_1$ is shown in Fig. 3 for the parameter values: $H_d = -0.2$, $\Delta H = 1$. The author of the present paper is far not convinced whether the experimental data contradict or support the non-congruency of this type because of the lack of careful measurements.

It seems to be useful to investigate the remanence properties of systems described by the stochastic model. In Fig. 3 the first-order descending curves which start from different points of the ascending limiting branch $\text{LA}_1$ can be seen. The curves starting from the points due to the field values $H_1 = 1.6$, $H_2 = 1.4$, $H_3 = 1.2$, $H_4 = 1$ are plotted to the points of remanences $\text{R}_1$, $\text{R}_2$, $\text{R}_3$, $\text{R}_4$ which are obviously different from the stationary (i.e., the accommodated) values. The non-accommodated NR and stationary remanences SR versus start field are shown in Fig. 3. As it
is seen the non-accommodated remanences can be negative below a critical start field $CR$ since the initial negative saturation has a significant effect on the first-order transition curves. The stationary remanence curve $SR$ calculated from the equations (47) and (48) is non-negative in all points of the start field interval.

The influence of the parameter $\sigma$ on the shape of the major hysteresis loop can be seen in Fig. 10. As it is expected the larger is the parameter $\sigma$ the wider is the hysteresis loop, i.e. the larger non-homogeneity in a system (e.g. in a magnetic sample) results in a higher "coercive force".

It seems to be useful to calculate the accommodated (stationary) hysteresis loops for different pairs of reversal points $H_d$ and $H_u \geq H_d$. The hysteresis loops plotted in Fig. 13 correspond to the reversal points: $H_d = -1.5$, $H_u = 1.5$ (loop ML1), $H_d = -1$, $H_u = 1$ (loop ML2), $H_d = -0.5$, $H_u = 0.5$ (loop ML3). For the calculation we used the parameter values: $H_c = 0.4$ and $\sigma = 0.6$. For the sake of completeness the virgin curve $VC$ calculated by (50) and the major loop $LL$ bounded by the limiting ascending and descending curves are also shown in the figure. The stochastic model clearly shows that all accommodated minor loops corresponding to cyclic inputs between the same two consecutive extremum values are not only congruent but simply identical.

In Fig. 12 three accommodated first-order minor loops denoted by 1, 2, 3 can be seen. The descending branches of the loops are started from the field values $H = 0.5, 0.3, 0$, and each of the ascending branches returns exactly to the same point that the corresponding descending branch left. The returning curves have an apparent slope discontinuity with regard to the major loop $ALA$.

At this point it is worth to make a remark of somewhat historical nature. As it is well-known Preisach’s idea for his model was originated from the quadratic Rayleigh relation which can be easily obtained assuming a uniform distribution of the $U$- and $D$-fields over the “Preisach triangle”. It is interesting to note that in the stochastic model the calculated hysteresis loops almost perfectly coincide with that calculated by the Rayleigh formula when the reversal fields $H_d$ and $H_u \geq H_d$ and so the magnetizing field $H \in [H_d, H_u]$ are sufficiently small. The hysteresis loop $R$ defined by reversal points $H_u = 0.5$ and $H_d = -0.5$ in Fig. 13 can be very well approximated by the equations

$$m^d_0(0.5 \downarrow H) = C_0^{(d)} + C_1^{(d)} H + C_2^{(d)} H^2,$$
$$m^u_0(0.5 \uparrow H) = C_0^{(u)} + C_1^{(u)} H + C_2^{(u)} H^2,$$

where

$$C_0^{(d)} = -C_0^{(u)} = 0.13035...,$$
$$C_1^{(d)} = C_1^{(u)} = 0.79301...,$$
$$C_2^{(d)} = -C_2^{(u)} = -0.54251...$$

in the case of parameter values $H_c = 0.2$ and $\sigma = 0.6$. In Fig. 13 the squares correspond to the values calculated by the quadratic equations. The excellent agreement with the curves of the stochastic model indicates that the Rayleigh law can be reproduced in a straightforward way in the stochastic model.

This model differs from the original Preisach model in a very essential point in relation to the reversal point susceptibility. Namely, the non-zero initial susceptibility at the turning points is an inherent property of the stochastic model, while the traditional Preisach model can produce positive initial slope only if the Preisach function is supposed to have a Dirac-delta like singularity along the boundary of the Preisach triangle, and that is a rather artificial requirement introduced by Mayergoyz. The susceptibility vs. magnetizing field is seen in Fig. 14 for the parameter values: $H_c = 0.2$ and $\sigma = 0.6$. The shape of the calculated curve can be expected on the basis of physical considerations and corresponds to those found experimentally.

The estimation of the joint density function $h(x, y|C)$ from measured hysteresis curves was beyond the scope of our present theoretical consideration. Of course, one may attempt in simple cases to estimate the parameters of a plausible density function (e. g., (54)) by an appropriate data evaluation procedure.

**V. CONCLUSIONS**

It has been shown that the Preisach model of hysteresis can be replaced by a new model based on exact concepts of the probability theory. In this model the phenomenon of hysteresis has been described as a stochastic process defined on a set of all possible values of the control parameter the reversal (turning) points of which are Markov points of the process. The one dimensional distribution function of the stochastic process has been exactly determined and the magnetizations versus up and down magnetic fields have been calculated as expectation values of the stochastic process. It has been proven that the magnetizing process becomes stationary with increasing number of magnetizing cycles. It means that for the description of the accommodation process there is no need of any artificial auxiliary
assumption since the stochastic model contains the phenomenon of accommodation inherently. In general case the model is able to describe the symmetric as well as the asymmetric hysteresis. In relatively small magnetizing fields the quadratic Rayleigh law can be easily obtained from the equations of the stochastic model. It is important to note that the turning point susceptibilities have non-zero finite values in contrary to the traditional Preisach model which does not take consequently into account the random nature of the elementary switching process. Finally, the stochastic model shows that all stationary loops corresponding to the same two limiting values of the magnetizing field are equivalent but the non-stationary loops are non-congruent and in general not closed.

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FIG. 1. A possible realization of the transition $S^{(+)} \leftrightarrow S^{(-)}$.

FIG. 2. Two equivalent $H(t)$ curves.

FIG. 3. Contour plot of $h(x, y; C)$ defined by Eq. (6) where $h(x, y)$ is given by Eq. (57) with parameter values: $H_c = 0.2, \sigma = 0.6$ and $C_r = 0.5$.

FIG. 4. Limiting branches LA, LD and magnetization versus field curves starting from the reversal points $H_1 = 1.2, H_2 = -0.8, H_3 = 0.7$ and $H_4 = -0.6$. The magnetization curves are indicated by 1, 2, 3 and 4.

FIG. 5. Accommodation process of the minor loop in consecutive magnetizing cycles between the reversal points $H_d = -0.2$ and $H_u = 0.8$. The loop LC is the accommodated minor loop.

FIG. 6. Convergence of the relative magnetization in the reversal point $H_u = 0.8$ with increasing number of cycles to the limit (i. e., the stationary) value.
FIG. 7. Width $W$ of the first-order minor loops and the difference $O$ between the values of the descending and ascending branches in the reversal point $H_u = 0.8$ versus start field.

FIG. 8. The limiting ascending branch $L_A$ and four first-order descending curves ending in non-accommodated remanences denoted by $R_1, R_2, R_3$ and $R_4$.

FIG. 9. The non-accommodated $N_R$ and the stationary $S_R$ relative remanences versus start field due to different points of the ascending limiting branch.

FIG. 10. Influence of the parameter $\sigma$ on the shape of the major hysteresis loop in the case of $H_c = 0.4$.

FIG. 11. The virgin curve $V_C$, the major loop $L_L$ and three accommodated hysteresis loops $M_{L1}, M_{L2}, M_{L3}$ calculated for different pairs of reversal points in the case of parameter values $H_c = 0.4$ and $\sigma = 0.6$.

FIG. 12. Three accommodated first-order minor loops denoted by $1, 2, 3$ and the shifted ascending branch $A_L A$.

FIG. 13. The hysteresis loop between "small" reversal points and the quadratic Rayleigh curves denoted by $\Box$.

FIG. 14. The irreversible susceptibility versus magnetizing field $H$. 
LIMIT VALUE: 0.7635...

$H_c = 0.4 \quad H_d = -0.2$

$\sigma = 0.6 \quad H_u = 0.8$
\( \sigma = 0.6 \)
$\sigma = 0.6$

$H_c = 0.2$

**Magnetizing Field vs. Susceptibility**

- X-axis: Magnetizing Field
- Y-axis: Susceptibility