Abstract

Recent studies of elasto-capillary phenomena have triggered interest in a basic variant of the classical Young-Laplace-Dupré (YLD) problem: The interaction between a liquid drop and a thin solid sheet of low bending stiffness. Here we consider a two-dimensional model where the sheet is subjected to an external tensile load and assume that the solid surface is characterised by a well-defined Young’s contact angle $\theta_Y$. Using a combination of numerical, variational, and asymptotic techniques, we discuss wetting as a function of the applied tension. We find that, for wettable surfaces, with $0 < \theta_Y < \pi/2$, complete wetting is possible below a critical applied tension thanks to the deformation of the sheet. Conversely, for very large applied tensions, the sheet becomes flat and the classical YLD situation of partial wetting is recovered. At intermediate tensions, a vesicle forms in the sheet, which encloses most of the fluid and we provide an accurate asymptotic description of this vesicle. We show that bending stiffness, however small, affects the entire shape of the vesicle. Rich bifurcation diagrams involving partial wetting and “vesicle” solution are found. For moderately small bending stiffnesses, partial wetting can coexist both with the vesicle solution and complete wetting.

Keywords: elasto-capillarity, bendo-capillarity, capillary origami, one-dimensional elasticity, bifurcation, boundary layers

1. Introduction

Elasto-capillary phenomena, namely mechanical deformations of elastic bodies in the presence of capillary forces, are at the focus of a growing attention. Indeed, aside from fundamental interest, this field of research opens new perspectives for fabrication at small scales where surface tension dominates volume weight and can facilitate origami-like folding of sheets into desired three-dimensional objects [1, 2]. Moreover, it appears relevant to the study of budding in biological cells and other biomimetic systems [3, 4]. One of the basic questions in this frame is how the classical Young-Laplace-Dupré (YLD) picture of partial wetting is modified, if the underlying assumptions of a perfectly rigid, semi-infinite solid substrate are relaxed, see Fig. 1.

The YLD theory, which is valid for undeformable solids and fixed volumes of liquid, is condensed in the famous equation

$$\cos \theta_Y = \Delta \gamma / \gamma,$$

where $\theta_Y$ is the contact angle between the solid-liquid and liquid-vapour interfaces while $\gamma_{sv}$, $\gamma_{sl}$, and $\gamma$, are respectively the solid-vapor, solid-liquid, and liquid-vapor surface energies (see for example [5]). Equation (1)
Figure 1: (a) Classical Young-Laplace-Dupré picture of partial wetting on a perfectly rigid substrate, involving liquid-vapor surface tension, $\gamma$, and forces associated with "surface stress", $\gamma_{sv}$, $\gamma_{sl}$, that act parallel to the solid surface. (b) Schematic of our model system. Additional control parameters: applied tension $T$ and bending modulus $B$.

indicates that the angle $\theta_Y$ is an intrinsic material quantity associated to a given solid-liquid-vapour system. For $\Delta \gamma < -\gamma$, surface energy disfavors any liquid-solid contact (non-wetting). Conversely, for $\Delta \gamma > \gamma$, the system is said to be in a complete wetting state, since minimization of surface energy favors maximization of liquid-solid contact. For intermediate values, $-\gamma < \Delta \gamma < \gamma$, that is $0 < \theta_Y < \pi$, the system is in a partial wetting state, whereby the liquid-solid contact has a finite area. In particular, when $0 < \theta_Y < \pi/2$, the substrate is said to be wettable by the liquid.

When the solid substrate is deformable (low elastic Young’s modulus $E$), or bendable (small thickness $t$), one may expect modifications to the ideal YLD picture, due to the nontrivial interplay between the capillary stress and the elastic response of the solid. From an energetic perspective, the partial wetting contact cannot be described any longer through the minimisation of a surface energy $U_{\text{surf}}$ alone; instead, a minimisation of the energy functional $U_{\text{surf}} + U_{\text{elas}}$ is required, where $U_{\text{elas}}$ is the elastic energy associated with deformations of the solid. As a consequence, a couple of questions became the focus of recent studies in this field:

- How do capillary forces modify the flat shape of the solid surface? Specifically, is there a simple, generalized version of the classical YLD law (1), that describes the wetting of deformable or bendable solids?
- Are there other wetting states, in addition to partial wetting, non-wetting, and complete wetting, that may characterize the energetically-favorable state of a liquid volume in contact with solid and vapor phases?

A systematic study of these questions requires one to identify the relevant length scales, and the corresponding dimensionless parameters that govern the physics. When studying the effect of material softness on the partial wetting of thick (unbendable) solid, the simplest model of elastic energy, $U_{\text{elas}}$, is the one of a Hookean, isotropic, thick (3D) solid body which depends on $E$ and Poisson’s ratio $\nu$. In this case, the local deformation of the solid surface is on the order of the elastocapillary length $\ell_{EC} = \gamma/E$, and the relevant dimensionless parameters are, respectively, the ratios, $\ell_{EC}/a$, and $\ell_{EC}/R$, where $a$ is a microscopic (atomic or molecular) length, and $R$ is the characteristic drop size. The rich physics that emerges at various ranges of the parameters $\ell_{EC}/a$ and $\ell_{EC}/R$ has been the subject of theoretical works [6–16] and experiments [17–22] (see also some recent reviews [23, 24]).

For plates of finite thickness, another length scale comes into consideration: the bendo-capillary length $\ell_{BC} = \sqrt{B/\gamma}$, where $B = Et^3/12(1-\nu^2)$ is the bending stiffness [25]. The plate is then easily bent by a liquid drop when $\ell_{BC}$ is small compared to the size of the drop, i.e. when the bendability parameter

\[
\Gamma \equiv R^2 \ell_{BC}^2 = \gamma R^2 B^{-1}
\]

is large. In addition, the local surface deformation is small when $\ell_{EC}/t \ll 1$. This is the situation that we consider in the present study. In terms of $t$, we are thus focusing on the double limit $\ell_{EC} \ll t \ll R^{2/3}\ell_{EC}^{1/3}$. The far edges of the sheet, away from the liquid drop, may be free [2, 5, 26, 29], clamped [30, 32], or subject to
a fixed tensile load by a liquid sub-phase. In this case, the simplest model energy $U_{\text{elas}}$ is the Föpplvon-Kármán energy, which describes the elastic energy of a solid sheet as a sum, $U_{\text{elas}} = U_{\text{strain}} + U_{\text{bend}}$, that penalises in-plane strains and curvatures of the mid-plane respectively.

The parameter regime $\Gamma \gg 1$ and $\ell_{EC}/t \ll 1$ describes sheets that are “highly bendable” yet “nearly inextensible”. In this regime, there is a large contrast between the bending energy, $U_{\text{bend}}$, and the strain energy, $U_{\text{strain}}$, whose high cost must be taken into consideration either explicitly (when studying finite liquid volume $\mathcal{L}_L$), or by imposing an inextensibility constraint, as we will do in this paper for a simplified model system. From a practical perspective, this parameter regime characterizes a large range of solid sheets that are commonly studied in the material science community: from common elastomers ($E \sim$ MPa) with a thickness of few micrometers to stiff polymers ($E \sim$ GPa) with a thickness of few hundreds of nanometers. For a characteristic drop size ranging from few tens to few hundreds of micrometers, $\Gamma$ varies roughly between 1 and $10^6$ whereas $\ell_{EC}/t$ can be as small as $10^{-5}$. Most experiments reported in Refs. $\mathcal{L}_L$ are in this parameter regime. From a pedagogical perspective, this limit provides a simple, yet nontrivial theoretical framework, which highlights essential mechanical-geometrical elements of elasto-capillary phenomena.

In this paper, we consider a two-dimensional model composed of a liquid cylinder of cross-sectional area $A \equiv R^2$ in contact with a rectangular solid sheet of length $L \gg R$ under an applied tension $T$, see Fig. 1(b). We further assume that gravity is negligible. The absence of Gaussian curvature considerably simplifies the analysis compared to more realistic 3D problems and allows us to push the analytical investigation beyond scaling laws. Notice however that such a two-dimensional system can be, to some extend, compared to experiments using a thin elastic filament floating on a fluid surface and wet by a droplet of another immiscible fluid. In contrast to previous works on a related system, we pay close attention to the effect of tensile loads, $T$, exerted on the solid sheet at its far edges, away from the liquid drop and thus, we consider the effect of another dimensionless parameter, $T = T/\gamma$.

Our results are succinctly summarized in the schematic phase diagram, Fig. 2, on which we briefly elaborate below:

1. As $T \to \infty$, the sheet becomes asymptotically flat and is only partially wet, with a contact angle given by the classical value, Eq. $\mathcal{L}_L$. Such a partial wetting state persists for all tensions $T > T^-(\theta_Y, \Gamma)$.
2. In the range $\cos \theta_Y < T < T^+(\theta_Y, \Gamma)$, the sheet can be in self-contact and form a vesicle that wraps most (but not all) of the liquid. Above a certain value of $\Gamma$, the two curves $T^-(\theta_Y, \Gamma)$ merge and in the limit $\Gamma \to \infty$, they asymptote to $\cos^2(\theta_Y/2)$. Analytically, we show that the shape of the vesicle is uniquely determined by the value of $\Gamma(T - \cos \theta_Y)$, a result that is found to hold even for $\Gamma = O(1)$, and we derive the approximate curve $T^+ \approx (\theta_Y, \Gamma)$, in very good agreement with the numerics.
3. Finally, $T < \cos \theta_Y$ is the range of existence of the complete wetting state, whereby the liquid completely wets one side of the sheet. Recalling that in the classical YLD picture complete wetting is obtained only if $\theta_Y = 0$, we see that high bendability enables a complete wetting state even if $\theta_Y > 0$, provided the tensile load is sufficiently small.
4. While the vesicle and complete wetting state are mutually exclusive, the partial wetting state can coexist with the former if $\Gamma < \Gamma^-$ and with both if $\Gamma < \Gamma^+$, paving the way to hysteresic behaviour (a feature that was already reported in the absence of applied tension $\mathcal{L}_L$.)

Thus, for a given value of $0 < \theta_Y < \pi/2$, the partial wetting predicted by the classical YLD law for non-bendable solids separates into three distinct phases – complete wetting, vesicle, and partial wetting – enabled by the floppiness of the solid.

To close this introduction, we must mention the work by Kusumaatmaja and Lipowski, who numerically studied a (3D) axisymmetric bud forming in a membrane under tension and in contact with two
distinct fluids. This budding solution is analogous to the vesicle solution described in the present paper (the authors studied it as a function of the nondimensionalised drop volume, i.e. in the present notation, as a function of $\Gamma^{3/2}$). However, the presence of hoop stress prevented them from obtaining analytical results for $B > 0$ and, in this sense, the present work provides some analytical support to Ref. [41].

The paper is organized as follow. In Sec. 2 we set the stage by discussing the limit of zero bending stiffness ($\Gamma = \infty$). Following Ref. [5], we use the angle $\beta$ of the liquid-vapour interface relative to the horizon as the “order parameter” [see Fig. 1(b)] and monitor it as a function of the non-dimensional applied tension $T$. For $T < \cos \theta_Y$, we find that the energetically-favorable state is complete wetting, where the liquid wets the entire sheet, even if $\theta_Y > 0$, in stark contrast with the classical YLD picture. For $T > \cos^2(\theta_Y/2)$, the system is in a partial wetting state where the liquid has a finite contact length with both the solid and the vapour phase. In that case, the apparent angle of contact differs from $\theta_Y$, as in Ref. [32, 34]. Finally, in the range $\cos \theta_Y < T < \cos^2(\theta_Y/2)$ the system is in an intermediate wetting state where the sheet forms a circular vesicle and wraps the entire liquid area.

The inclusion of bending stiffness starts in Sec. 3, where the governing equations of the system are presented. These are studied numerically in Sec. 4 where we show how the three wetting states identified in Sec. 2 occupy distinct regions in the parameter space spanned by $T$, $\Gamma$, and $\theta_Y$. Furthermore, we show that the shape of the vesicle can be completely altered as soon as $B$ differs from zero. Secs. 5 and 6 are devoted to the asymptotic analysis of the partial wetting and “vesicle” states in the limit $\Gamma \gg 1$. Finally, we conclude in Sec. 7.

2. Inextensible, infinitely bendable sheet

In the 2D model considered here, the energetically costly stretch is eliminated by the inextensibility constraint, whereas the bending cost is expected to be small in comparison to the surface energy for thin
enough sheets. Hence, we start in this section by ignoring the bending stiffness altogether.

We consider an infinitely long rectangular sheet in contact with a cylindrical drop, whose cross-sectional area is $A \equiv R^2$, and we neglect gravity. Upon making contact with the drop, the wet part of the sheet becomes bulged, with a constant radius of curvature, $R_b$, due to the Laplace pressure in the drop, $p = \gamma / R_d$, where $R_d$ is the constant radius of curvature of the liquid-vapour interface, see Fig. 1(b) and Fig. 3.

### 2.1. Force balance for partial wetting

Denoting the longitudinal tension in the wet part of the sheet by $n_\parallel$, the horizontal and vertical force balances at the contact point are

$$n_\parallel \sin \vartheta = \gamma \sin \beta,$$
$$n_\parallel \cos \vartheta = T - \gamma \cos \beta,$$  \hspace{1cm} (4)

where $\beta$ and $\vartheta$, respectively, denote the angles of the liquid-vapour and liquid-solid interfaces relative to the horizontal $x$-axis. The capillary pressure $p$ inside the drop is balanced by the membrane-type reaction $n_\parallel / R_b$, so that

$$n_\parallel = \gamma R_b / R_d.$$  \hspace{1cm} (5)

Next, the transverse area of the drop is [see Fig. 3(b)]

$$A = A_d + A_b = A(R_d, R_b, \beta, \vartheta) = R_d^2 \left( \beta - \frac{1}{2} \sin 2 \beta \right) + R_b^2 \left( \vartheta - \frac{1}{2} \sin 2 \vartheta \right).$$  \hspace{1cm} (6)

So far, we only have 4 equations for the 5 unknowns $\beta$, $\vartheta$, $n_\parallel$, $R_b$ and $R_d$. The missing piece of information involves $\theta_Y$ and reads (see later why)

$$T = n_\parallel + \gamma \cos \theta_Y.$$  \hspace{1cm} (7)

It is then a simple matter to deduce from Eqs. (4) and (7) that

$$T/\gamma \equiv \mathcal{T} = \frac{\sin^2 \theta_Y}{2(\cos \beta - \cos \theta_Y)},$$
$$\tan \vartheta = \frac{\sin \beta}{T - \cos \beta},$$  \hspace{1cm} (8)

which expresses the rescaled tension $\mathcal{T}$ and the angle $\vartheta$ as functions of $\beta$, see Fig. 4(a). These equations can be inverted to express $\beta$ and $\vartheta$ as a function of $\mathcal{T}$ and $\theta_Y$:

$$\cos \beta = \cos \theta_Y + \frac{\sin^2 \theta_Y}{2 \mathcal{T}},$$
$$\cos \vartheta = 1 - \frac{\sin^2 \theta_Y}{2 \mathcal{T} (T - \cos \theta_Y)}.$$  \hspace{1cm} (9)

In the limit $\mathcal{T} \to \infty$, one tends to the classical partially wet state, with a flat sheet ($\vartheta \to 0$) and Young’s value $\theta_Y$ for the contact angle $\beta + \vartheta$. We therefore call the state given by (8) or (9) the partial wetting state.

As seen from the first of Eqs. (8), the value $\beta = 0$ yields the lower bound of existence of this partial wetting state:

$$T = n_\parallel + \gamma \cos \theta_Y.$$  \hspace{1cm} (10)

This inequality does not have an analog in the classical YLD theory of a drop on a thick (unbendable) solid body; it defines a minimal tensile load that is necessary to maintain a partial wetting contact even if $0 < \theta_Y < \pi / 2$. As $\mathcal{T} \to \mathcal{T}^+$, $\beta \to 0$ and $\vartheta \to \pi$ such that the wet part of the sheet tends to a closed circle and wraps the entirety of the fluid. Such a circular shape satisfies the conditions of static equilibrium at all tensions below that threshold. Henceforth, we call it the vesicle state.

It is immediate from Eq. (8) that $\mathcal{T}$ increases monotonously with $\beta$. Hence, the partial wetting state emerges supercritically from the vesicle state at $\mathcal{T}^+$. A remarkable property of Eq. (9) is that the symmetric case $\beta = \vartheta$ is obtained exactly at $\mathcal{T} = 2 \mathcal{T}^+$. Finally, we note that $\vartheta / (\theta_Y - \beta) \to 1 / 2$ when $\mathcal{T} \to \infty$ as has been discussed elsewhere.

We will see in Sec. 5 that the above relations, notably Eq. (8), provide a faithful picture of the system in the high-bendability limit for $\beta$ positive and not too small. However, the above reasoning does not explain
the origin of the crucial closure equation \( \Theta \). Moreover, it is not known whether the circular vesicle is stable for all \( T < T^* \).

One way to address these issues is to introduce a finite, arbitrarily small amount of bending stiffness as we do from Sec. 3 onwards. This rounds off the corner in the elastic sheet near the triple line [see inset of Fig. 5(a)] and makes Eq. (7) appears as the true force balance at the triple line. In that more realistic picture, the force balance equations \( \Theta \) hold a few elasto-capillary lengths \( \ell_{BC} \) away from the triple line where the “apparent” contact angles \( \vartheta \) and \( \beta \) can be measured. Therefore, the angle \( \vartheta + \beta \) appearing above is only the apparent contact angle, as measured in Ref. [32, 34], but the true contact angle at length scales smaller than \( \ell_{BC} \) remains Young’s angle, \( \theta_Y \), in agreement with recent experiments performed on a related system [44].

However, before entering into these considerations, it is instructive to discuss the problem from an energetic perspective. Indeed, as we show next, minimising the total energy naturally produces Eq. (7) even in the absence of bending stiffness. Moreover, the circular vesicle ceases to be the most energetically favourable state for \( T < T^* = \cos \theta_Y \) and thus looses stability below this threshold.

2.2. Energy approach

In the absence of bending stiffness, the system energy is the sum of the surface energy and the work done by tensile loads at the edges of the sheet.

The surface energy reads as

\[
U_s = \gamma L_d + \gamma_{ad} L_b + (2L - L_b) \gamma_{sv} = \gamma L_d - \Delta \gamma L_b + 2L \gamma_{sv},
\]

where \( L \) is the total length of the sheet, while \( L_b \) and \( L_d \) respectively measure the length of the bulged part of the sheet and of the liquid-vapour interface, see Fig. 3(a). The work performed by \( T \), on the other hand, is

\[
W = 2dT = (L_b - 2x_D)T,
\]

where \( 2x_D \) is the projected length of the liquid-vapour interface along the horizontal \( x \)-axis. Since the liquid-vapour and solid-liquid interfaces are necessarily circular, we have the following geometrical relations

\[
L_d = 2\beta R_d, \quad L_b = 2\vartheta R_b, \quad x_D = R_d \sin \beta = R_b \sin \vartheta,
\]

where the last relation indicates that the two circular segments share the same chord, see Fig. 3. The total energy \( U_{pw} \) of the partial wetting state is then given by

\[
U_{pw} = U_s + W = 2\gamma \beta R_d - \Delta \gamma L_b + 2L \gamma_{sv} + (L_b - 2R_d \sin \beta)T,
\]
where the quantities \( x_D, R_b \) and \( L_d \) have been expressed in terms of \( R_d, \beta \) and \( \theta \). To minimize the total energy under the constraints of fixed area \( A(R_d, \beta, \vartheta) \), given by Eq. (6) with \( R_b = R_d \sin \beta / \sin \vartheta \), and the geometric relation for \( L_b \), we introduce the Lagrangian
\[
\mathcal{L}(\beta, \vartheta, R_d, L_b) = U_{pw} + \mu (A - A) - \eta \left( L_b - \frac{2 \theta R_d \sin \beta}{\sin \vartheta} \right),
\]
where \( \mu \) and \( \eta \) are Lagrange multipliers. The equilibrium equations are found by minimizing \( \mathcal{L} \) with respect to \( \beta \), \( \vartheta \), \( R_d \), and \( L_b \):
\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \beta} &= 2 \gamma R_d - 2 R_d T \cos \beta - \mu \frac{\partial A}{\partial \beta} + 2 \eta \frac{R_d \vartheta \cos \beta}{\sin \vartheta} = 0, \\
\frac{\partial \mathcal{L}}{\partial \vartheta} &= -\mu \frac{\partial A}{\partial \vartheta} + 2 \eta \frac{R_d \sin \beta}{\sin \vartheta} \left( 1 - \frac{\vartheta \cos \vartheta}{\sin \vartheta} \right) = 0, \\
\frac{\partial \mathcal{L}}{\partial R_d} &= 2 \gamma - 2 T \sin \beta - \mu \frac{\partial A}{\partial R_d} + 2 \eta \frac{\vartheta \sin \beta}{\sin \vartheta} = 0, \\
\frac{\partial \mathcal{L}}{\partial L_b} &= -\Delta \gamma + T - \eta = 0.
\end{align*}
\]
\( \Delta \gamma = \gamma \cos \theta \nu \), we recover Eq. (7) which is the missing equation that was necessary to close the system (4)-(6). It is the familiar YLD mechanics law for the stress jump at the contact line.

Combining Eqs. (16a) and (16c)
\[
\frac{\partial A}{\partial \beta} \sin \beta - R_d \frac{\partial A}{\partial R_d} \cos \beta = 2 \gamma R_d (\sin \beta - \cos \beta) - \mu \left[ \frac{\partial A}{\partial \beta} \sin \beta - R_d \frac{\partial A}{\partial R_d} \cos \beta \right] = 0.
\]
and using
\[
\frac{\partial A}{\partial \beta} \sin \beta - R_d \frac{\partial A}{\partial R_d} \cos \beta = 2 R_d^2 (\sin \beta - \cos \beta),
\]
we obtain
\[
\mu \equiv \frac{p}{R_d}, \quad \eta \equiv n_\parallel = \frac{\gamma \sin \beta}{\sin \vartheta} \equiv \gamma \frac{R_b}{R_d},
\]
where we used Eq. (13) in the second equation. Eqs. (21) reveal the Lagrange multiplier \( \eta \) and \( \mu \) as the tension \( n_\parallel \) in the wet part of the sheet and the pressure \( p \) in liquid volume, respectively. We thus recover the first force balance in Eqs. (4) and Eq. (5). Substituting Eqs. (21) into Eqs. (16c) (or Eq. (16a)), we have
\[
\frac{\partial \mathcal{L}}{\partial R_d} = 2 \gamma \beta - 2 T \sin \beta - 2 \gamma \frac{A}{R_d} + 2 \gamma \frac{\vartheta \sin^2 \beta}{\sin^2 \vartheta} = 0,
\]
where we used \( \partial A / \partial R_d = 2 A / R_d \). Using Eqs. (6) and \( R_b = R_d \sin \beta / \sin \vartheta \), we obtain
\[
T = \gamma \cos \beta + \frac{\sin \beta \cos \vartheta}{\sin \vartheta} = \gamma \cos \beta + n_\parallel \cos \vartheta,
\]
which coincide with the second force balance in Eq. (4).
2.3. The vesicle state

When \( T < T^+ = \cos^2(\theta_Y/2) \), the sheet is still only partially wet but it now wraps the entirety of the liquid drop (\( \beta = 0, \vartheta = \pi \)) so that the state is characterized by \( L_d = 0, L_b = 2\pi R_b \), and \( A = \pi R_b^2 \), see Eqs. (6) and (13). The energy (14) becomes

\[
U_{\text{ves}} = 2\gamma_{sv} L + 2\gamma \sqrt{A} \pi (T - \cos \theta_Y)
\]  

(24)

Let us compare this energy with the one of the partial wetting state, in the vicinity of the threshold \( T^+ \). Expanding Eqs. (9) and (14) near \( T = T^+ \), we find

\[
U_{pw} - U_{\text{ves}} = -\frac{32}{3\sin \theta_Y} \sqrt{A} \gamma \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2}),
\]  

(25)

where \( 0 < \epsilon = T - T^+ \ll 1 \). Hence the partial wetting state has lower energy than the vesicle state. The continuity of the contact angles \( \beta \) and \( \vartheta \), in the vicinity of the transition between the two states, reflects the continuity of both the energy, \( U = U_{\text{ves}} \), and its first derivative, \( dU/dT = dU_{\text{ves}}/dT \), at \( T = T^+ \). Hence, in the infinite bendability limit, the transition is a continuous, second order transition. Specifically, we have a pitchfork bifurcation whereby the angles \( \beta \) and \( \vartheta \) vary rapidly with a small increase of the applied force past the bifurcation point. We will see in Sec. 4 how adding bending energy to the model affects the nature of the transition.

2.4. Complete wetting

We now revisit the assumption that the drop shape consists of circular segments. If there is a finite liquid-vapour interface, Laplace’s law implies that it is necessarily a circular arc as well as the rest of the drop’s interface which makes a contact with the sheet. However, for \( T < T^+ \), the liquid in the vesicle state does not have a finite contact length with the vapour. Hence, we must address the possibility that the drop, once fully wrapped by the sheet, is no longer circular.

For this purpose, we consider the energy of a vesicle whose shape is not necessarily circular:

\[
2\gamma_{sv} L + \gamma (T - \cos \theta_Y)L_b.
\]  

(26)
For a circular vesicle, $L_b = 2\sqrt{A\pi}$ but otherwise $2\sqrt{A\pi} < L_b \leq L$. Therefore, if
\[ T < T^* = \cos \theta_Y, \]
the energy is minimal for $L_b = L$, i.e. the liquid wets the entire length of the sheet. As a consequence, for $0 < \theta_Y < \pi/2$, a tensile load $T < \gamma \cos \theta_Y$ is not sufficient to stabilize the sheet against a complete wetting by the drop. The energy of the complete wetting state is
\[ U_{cw} = (\gamma_{sl} + \gamma_{sv}) L = 2\gamma_{sl}L - \gamma \cos \theta_Y L. \]

A comparison between Eqs. (28) and (24) shows that, at $T^*$, the system undergoes a discontinuous transition, characterised by a finite energy gap, see Fig. 4(b). This gap can be viewed as a potential barrier which can only be overcome by applying a sufficiently large force ($T \geq \gamma \cos \theta_Y$). Notice that, when $\pi/2 < \theta_Y < \pi$ such a transition requires a compressive force ($T < \gamma \cos \theta_Y < 0$) and complete wetting is therefore unobservable for any tensile load, $T < T^+$.

3. Finite bendability: Model equations

We now consider an elastic sheet with a bending modulus $B > 0$ and set up the mathematical model to describe the partial wetting and vesicle state that are schematically depicted in Fig. 5. By symmetry, we may restrict our attention to $x \geq 0$.

3.1. Partially wet state

Denoting by $\kappa$, $n_\perp$, and $n_\parallel$ the curvature, perpendicular and parallel tractions along the elastic sheet, respectively, the local balance of forces and torques are, in the absence of self contact [25],
\[ B \partial_s \kappa = -n_\perp, \quad \partial_s n_\perp = p - \kappa n_\parallel, \quad \partial_s n_\parallel = \kappa n_\perp, \]
where $s$ is the distance along the sheet and $\partial_s$ denotes derivative with respect to that coordinate. The wet part of the sheet is at $s < D$ and is subjected to the Laplace pressure
\[ p = \gamma/R_d. \]
In the dry part, on the other hand, the pressure is atmospheric: $p = 0$. Given $\kappa$, the local angle with respect to the horizontal direction is found by
\[ \partial_s \theta = \kappa, \]
and the Cartesian coordinates along the sheet solve
\[ \partial_s x = \cos \theta, \quad \partial_s y = \sin \theta. \]
Instead of $n_\parallel$ and $n_\perp$, one may use the Cartesian components $n_x = n_\parallel \cos \theta - n_\perp \sin \theta$ and $n_y = n_\parallel \sin \theta + n_\perp \cos \theta$. This yields
\[ B \partial_s^2 \theta = n_x \sin \theta - n_y \cos \theta, \quad \partial_x n_x = -p \sin \theta, \quad \partial_x n_y = p \cos \theta. \]
Note that the above equations can also be derived through energy minimisation, as detailed in Ref. [5]. In the dry region ($s > D$), we have $(n_x, n_y) = (T, 0)$ everywhere. Hence, multiplying the first of Eqs. (33) by $\partial_s \theta$ and integrating, we obtain, for an infinite domain with $\lim_{s \to \infty} (\theta, \kappa) = (0, 0)$,
\[ \kappa_D = -2(T/B)^{1/2} \sin (\theta_D/2), \]
where $\kappa_D$ and $\theta_D$ respectively denote the curvature and angle at $s = D$. On the wet side of this point, the force balance is
\[ n_x(D) = T - \gamma \cos \beta, \quad n_y(D) = \gamma \sin \beta. \]
Assuming symmetric shapes and imposing YLD law at the contact line, we have,

\[ \theta(0) = 0, \quad \theta_D + \beta = \theta_Y. \]  

(36)

Next, translation invariance allows us to fix the values

\[ x(0) = 0, \quad y(0) = 0. \]  

(37)

Finally, the geometrical constraints (6) and (20) become

\[ A = R_d^2 \left( \frac{\beta - \sin 2\beta}{2} \right) + 2 \int_0^D x(s) \sin \theta(s) \, ds, \quad x_D = R_d \sin \beta. \]  

(38)

In the wet range \( 0 \leq s < D \), we have to solve Eqs. (29)-(32) with the boundary conditions and global constraint in Eqs. (34)-(38). Note that there are 9 conditions because there are 6 differential equations and 3 unknown parameters, \( \beta, \theta_D, \) and \( R_d \). Notice that, integrating the last of Eqs. (33) between 0 and \( D \) and using the first of Eqs. (32) together with the second of Eqs. (35) and Eqs. (38), we obtain

\[ n_y(0) = 0. \]  

(39)

Finally, by virtue of the second geometrical constraint (38) and Eq. (30), the capillary pressure can also be expressed as

\[ p = \gamma x_D^{-1} \sin \beta. \]  

(40)

### 3.2. Vesicle state

In the vesicle state, the sheet is in self-contact, see Fig. 5(b). At the point of self-contact, \( s = \ell \), there is a localised reaction force, \( F_c \), which modifies the second of Eqs. (29) and (33) as

\[ \partial_s n_\perp = p - \kappa n_\parallel + F_c \delta(s - \ell), \quad \partial_s n_x = -p \sin \theta - F_c \delta(s - \ell). \]  

(41)

While the total area \( A \) of the fluid is still given by Eq. (38), it is now split in two parts, \( A_1 \) and \( A_2 \), respectively below and above the contact point. If the contact is such that no fluid is allowed through, both \( A_1 \) and \( A_2 \) are in principle constrained to a fixed value, instead of just \( A \). In response to this new constraint, the pressure \( p \) differs from the capillary pressure inside the vesicle. However, there is no general rule that governs how \( A \) should split between \( A_1 \) and \( A_2 \) and, hence, what the vesicle pressure should be. A complete theory should therefore discuss the solution not only as a function of \( A \), but also of \( A_1 \).
this work, however, we assume that $A_1$ and $A_2$ can freely vary. We hypothesise that the fluid can flow through the contact point, as could happen for instance if the contact is not perfectly realised for all values of the coordinate transverse to the $(x, y)$ plane. If, for some reason, a small gap exists somewhere along the third coordinate, even though the geometry is on average the 2D picture presented so far, the fluid can flow through it and equilibrate the pressure inside the vesicle with the capillary pressure. Hence, we assume that the pressure is everywhere given by the capillary pressure.

Equation (41) brings two new unknown parameters, $F_c$ and $\ell$, into the problem which are fixed by two new boundary conditions:

$$x_\ell = 0, \quad \theta_\ell = \pi/2,$$

where subscript $\ell$ means evaluation at $s = \ell$.

3.3. First integrals and alternative independent variables

In the wet part of the elastic sheet, combining Eqs. (32) with Eqs. (33) and the first and last of Eqs. (29), we obtain

$$\partial_s (x - n_y/p) = 0, \quad \partial_s (y + n_x/p) = 0, \quad \partial_s (B\kappa^2/2 + n_\parallel) = 0.$$  (43)

This allows us, on the one hand, to deduce the shape of the wet part of the sheet once the tractions and $p$ are known and, on the other hand, to write

$$B\kappa^2 + n_\parallel = H,$$  (44)

where $H$ is constant. Evaluating Eq. (44) at $s = D$ and using the boundary conditions (34) and (35) together with the relation between $n_\parallel$ and $n_x$ and $n_y$, one finds

$$H = T - \gamma \cos \theta_Y.$$  (45)

At $s = \ell < D$, where $\theta = \pi/2$ and $x = 0$, we have $n_\parallel = n_y$. Integrating the first of Eqs. (43) between 0 and $\ell$ and using $x(0) = n_y(0) = 0$, we get $n_\parallel(\ell) = 0$. Therefore, evaluating Eq. (44) at $s = \ell$ leads to

$$H = T - \gamma \cos \theta_Y = \frac{B}{2}\kappa^2 \ell,$$  (46)

so that the curvature at the contact point in the vesicle state vanishes when $T = \gamma \cos \theta_Y$, that is $T = T^*$. Since $\kappa^2 \ell \geq 0$, the vesicle state exists only for $T \geq T^*$. Using Eq. (44) to eliminate $n_\parallel$, the second of Eqs. (29) becomes

$$\partial_s n_\perp = p - H \kappa + \frac{B}{2}\kappa^3.$$  (47)

While $s$ appears as the most natural variable to express all the physical quantities along the sheet, one should note that other variables can be more advantageous. In particular, since the differential system is autonomous in $s$, one can reduce its order by using one of the dynamical variables as the independent one and seeking all the others quantities as functions of it. One useful choice is $\theta$. Let us introduce

$$\kappa^2 = 2q(\theta(s)).$$  (48)

The variable $q$ may be regarded as a measure of the density of bending energy. By differentiating each side of Eq. (48) with respect to $s$, one finds that $\partial_s \kappa = \partial_s q$. Hence, the first of Eqs. (29) and Eq. (47) become

$$B \partial_\theta q = -n_\perp, \quad \partial_\theta n_\perp = \frac{p}{\kappa} - H + B q.$$  (49)

Differentiating the first equation above with respect to $\theta$, we thus obtain

$$B(\partial_\theta^2 q + q) + \frac{p}{\kappa} = H.$$  (50)
This last formulation leads to considerable simplification when either \( p/\kappa \) or \( B(\partial^2 q + q) \) dominates the left hand side.

Another useful trick is to treat \( \kappa \) as the independent variable. Indeed, using \( \partial_t n_\perp = (\partial_x n_\perp) \partial_x \kappa \) and the first of Eqs. (29), Eq. (47) becomes

\[
\partial_x n_\perp^2 = 2B \left( -p + H \kappa - \frac{B}{2} \kappa^3 \right),
\]

which is simple to integrate. Together with Eq. (44), this equation yields the tractions, and hence \( x \) and \( y \) directly as functions of \( \kappa \). In what follows, it will be natural to rescale \( H \) with \( \gamma \), giving

\[
H = H/\gamma = T - \cos \theta_Y.
\]

Note that a rather complete treatment of Eqs. (29) in terms of Jacobi functions and elliptic integrals of the first and third kinds was developed in Ref. [45]. While the approach followed here is only asymptotically exact, it has the advantage of involving mostly elementary functions, hence expressions that are easier to interpret (see also the comment at the end of Sec. 6).

3.4. Solution in the dry region

In the dry part of the sheet, where \( n_x = T \) and \( n_y = 0 \), Eq. (33) reduces to

\[
B \partial^2 s \theta - T \sin \theta = 0,
\]

with boundary conditions \( \theta(D) = \theta_D \) and \( \theta(L) = 0 \). Letting \( L \to \infty \), the solution has the exact form

\[
\theta(s) = 4 \arctan \left\{ \tan \left( \frac{\theta_D}{4} \right) \exp \left[ -\left( \frac{T}{B} \right)^{1/2} (s - D) \right] \right\},
\]

whose derivative at \( s = D \) is given by Eq. (34). On the other hand, multiplying Eq. (53) by \( \kappa = \partial_s \theta \) and integrating we have \( \kappa = -2 \left( \frac{T}{B} \right)^{1/2} \sin \left( \frac{\theta}{2} \right) \). Parametrising \( x \) and \( y \) with \( \theta \) and using this expression, Eqs. (32) become

\[
\partial_\theta x = -\left( \frac{B}{T} \right)^{1/2} \frac{\cos \theta}{2 \sin (\theta/2)}, \quad \partial_\theta y = -\left( \frac{B}{T} \right)^{1/2} \frac{\sin \theta}{2 \sin (\theta/2)}.
\]

This yields

\[
x = c_x - \left( \frac{B}{T} \right)^{1/2} \left[ 2 \cos (\theta/2) + \ln (\tan (\theta/4)) \right], \quad y = c_y + 2 \left( \frac{B}{T} \right)^{1/2} \sin (\theta/2),
\]

where \( c_{x,y} \) are constants of integration. This illustrates how all the variables can be expressed in terms of \( \theta \). If we substitute \( \theta \) by the right hand side of Eq. (54), we obtain their explicit dependence on \( s \).

4. Numerical solution

As shown in Sec. 3.4, the shape of the sheet in the dry region, \( D < s \leq L \), is known explicitly in the limit of a long sheet, \( L \gg \sqrt{A} \). To solve the problem in the wet region, \( 0 \leq s \leq D \), for the partial wetting state, we need to integrate numerically Eqs. (33) with the associated boundary conditions (34)-(36) and geometric constraints (38). For the vesicle state, the second of Eqs. (33) is replaced by the second one in Eqs. (41) and we must consider two additional boundary conditions given by Eqs. (42). For this purpose, a shooting method is used where the boundary value problem is transformed into an initial value problem and the unknown initial conditions are varied until the boundary conditions are satisfied. In this way, we may simulate the system for values of \( \Gamma \) ranging from 0 to about 300. Whatever the values of \( \Gamma \) and \( 0 < \theta_Y \leq \pi/2 \), the system is always in a partial wetting state when the applied tension \( T \) is large enough. Indeed, in the limit \( T \to \infty \), the sheet is flat and behaves as an undeformable substrate. In this case, complete wetting is possible only if \( \theta_Y = 0 \).
When the applied tension decreases, transitions towards the vesicle state and complete wetting occur. Figure 6(a)-(c) shows bifurcation diagrams for $\theta_Y = \pi/3$ and three representative values of $\Gamma$ where $\beta$ is used as order parameter and the applied tension as bifurcation parameter. They highlight the existence of three distinctive values of the applied tension.

Similarly to the case of a vanishing bending modulus discussed in Sec. 2, there are $T^* = \cos \theta_Y$ and $T^+(\theta_Y, \Gamma)$ delimiting the domain of existence of the vesicle state. At $T = T^*$, the curvature at the contact point, $s = \ell$, vanishes (see Eq. (40) and Fig. 5(f)). This tension is the smallest for which a vesicle state exists. At $T = T^+(\theta_Y, \Gamma)$, the self-contact occurs with a vanishing contact force, i.e. $F_c = 0$ (see Fig. 5(d)). Beyond this applied tension, there is no longer self-contact.

In addition, for $\Gamma < \Gamma^+(\theta_Y)$, we find that a new special value of the tension, $T^-(\theta_Y, \Gamma)$, shows up for a finite bending modulus. This is the smallest tension for which partial wetting states exist. At the limit point $T = T^-(\theta_Y, \Gamma)$, the curve $\beta(T)$ has a vertical slope [Figs. 5(a,b)] but there is no self-contact in contrast to the case of an infinitely bendable sheet. In addition, the bifurcation diagrams highlight the existence of three remarkable values of the parameter $\Gamma$. These three values are marked by the three vertical dashed lines in Fig. 5(d).

When $\Gamma < \Gamma^0(\theta_Y)$, $T^-(\theta_Y, \Gamma)$ is negative and there is no transition from partial wetting towards other states when a tensile load is applied.

When $\Gamma^0(\theta_Y) < \Gamma < \Gamma^+(\theta_Y)$, as in Fig. 6(a), the system bifurcates at $T = T^-$ from partial wetting to complete wetting as the applied tension decreases. This leads to the surprising situation that the vesicle branch is not reached by decreasing the applied tension from $T = \infty$.

When $\Gamma^+(\theta_Y) < \Gamma < \Gamma^*(\theta_Y)$, as in Fig. 6(b), the transition between vesicle and partial wetting is subcritical and there is a region of applied tension where both states coexist. There are thus discontinuous transitions between both states at $T = T^-(\theta_Y, \Gamma)$ and $T = T^+(\theta_Y, \Gamma)$. In this case, the partial wetting
branch that bifurcates subcritically from the vesicle branch at $T^+$ [blue dashed line in Fig. 6(b)] is unstable; it only becomes stable at the limit point $T^-$. For tensions in the range $T^- < T < T^+$, three values of $\beta$ are possible, each corresponding to a distinct steady state. The middle one, belonging to the blue dashed line in Fig. 6(b) yields a local maximum of the energy and is therefore unstable.

When $\Gamma > \Gamma^+ (\theta Y)$, as in Fig. 6(c), the bifurcation is supercritical with a continuous transition between both states. The transition occurs at $T = T^+ (\theta Y, \Gamma)$, at which value the partial wetting state is stable and there is self-contact with $F_c = 0$. At this point, $\partial_T \beta$ is finite since $\beta \sim \text{const} \times (T - T^+)$ [Fig. 6(c)].

This discussion is summarized in Fig. 6(d), which shows that, as $\Gamma$ increases, the difference between $T^+$ and $T^-$ decreases. When $\Gamma^0 (\theta Y) < \Gamma < \Gamma^+ (\theta Y)$, $T^-$ is smaller than $T^+$ and the bifurcation diagram is similar to the one shown in Fig. 6(a) in this region. When $\Gamma^+ (\theta Y) < \Gamma < \Gamma^+ (\theta Y)$, $T^-$ is larger than $T^*$ while still smaller than $T^+$. The bifurcation diagram in this region is similar to the one shown in Fig. 6(b). When $\Gamma = \Gamma^+ (\theta Y)$, we have the equality $T^- = T^+$. Finally, when $\Gamma > \Gamma^+ (\theta Y)$, partial wetting states, i.e. solutions of Eqs. (33), display self-crossing for $T < T^*$ and must therefore be discarded. Hence, the system is in a vesicle state when $T < T^+$. This corresponds to the bifurcation diagram shown in Fig. 6(c).

The algorithm to compute $\Gamma^+ (\theta Y)$ is described in Appendix A. The result of this computation is gathered in Fig. 6, which shows that when $\theta Y$ is small enough, $\Gamma^+ (\theta Y) \sim \theta Y^{-2}$. Therefore, whatever the value of $\theta Y$ is, there always exist values of $\Gamma$ such that the transition is supercritical. However, this shows that the limit $\Gamma \to \infty$ together with $\theta Y \to 0$ is subtle and will not be considered in the asymptotic theories presented in Secs. 6 and 6. Specifically, we will assume $\Gamma \gg \Gamma^+ (\theta Y)$ with $\theta Y = O(1)$.

It is also possible to compute $\Gamma^0 (\theta Y)$ for which $T^- = 0$ and $\Gamma^* (\theta Y)$ for which $T^- = T^*$. For this purpose, $T^-$ is obtained for given $\theta Y$ and $\Gamma$ and the latter is varied by small increments. For each value of $\Gamma$, $T^-$ is computed until it reaches 0 or $T^* = \cos \theta Y$. The result of this computation is shown in Fig. 7 When $\theta Y$ is small enough, $\Gamma^0 \sim \theta Y$ and $\Gamma^* \sim \theta Y^{-2}$.

Figure 7(a)-(f) shows the evolution of various quantities characterising the system shape as a function of $T$ for two values of $\Gamma$ and $\theta Y = \pi/3$. As the applied tension decreases for a given $\Gamma$, the curvature of the liquid-vapour interface decreases whereas the size of the wet part of the sheet, $D$, increases due to the progressive encapsulation of the drop. When $\Gamma$ increases, the position of the contact point along the $x$-axis, $x_D$, tends to zero at the transition between partial wetting and the vesicle state, i.e. at $T = T^+$, and stays small in the vesicle state. This is consistent with the observation that the length of the sheet forming the vesicle, $\ell$, is close to $D$. Therefore, essentially all the liquid is encapsulated in the vesicle as $\Gamma$ increases. The contact force, $F_c$, vanishes at $T = T^+$ and increases almost linearly when the applied tension decreases and reaches a value $1 - \cos \theta Y$ at $T = T^*$ when $\Gamma \to \infty$, as shown by the asymptotic theory presented in Sec. 6.
Figure 8: (a)-(f) Evolution of various quantities characterising the system shape as a function of $T/\gamma$ for $\theta_Y = \pi/3$ and $\Gamma = 30$ and 200. As in Fig. 6, the blue solid and dashed curves correspond to the stable and unstable branches of the partial wetting state whereas the red curves refer to the vesicle state. Panels (e) and (f) show that $\ell$ and $\kappa_\ell$ are independent on $\Gamma$ when $T = T^*$ and $\beta = 0$. (g)-(i) Influence of $\Gamma$ on the vesicle shapes at three different values of $T$ for $\theta_Y = \pi/3$. The shape of the vesicle is independent on $\Gamma$ when $T \to T^\star$ and $\beta = 0$ whereas it does depend on this parameter at $T = T^+$ where the shape approaches a circular shape of radius $R = (A/\pi)^{1/2}$ as $\Gamma$ increases, see panel (i).

Figure 8(e)-(f) shows that $\ell$ and $\kappa_\ell$ are independent of $\Gamma$ at $T = T^*$. It also shows that the tension at which $\beta = 0$ depends on $\Gamma$ in agreement with the results reported in Fig. 6. However, the values of $\ell$ and $\kappa_\ell$ at $\beta = 0$ are again independent on $\Gamma$. This suggests that the vesicle shape does not depend on $\Gamma$ for some particular values of the tension as confirmed by Fig. 8(g)-(h). However, as shown in Fig. 8(i), the vesicle shape at $T = T^+$ does depend on $\Gamma$ and approaches a circular shape of radius $(A/\pi)^{1/2}$ as $\Gamma \to \infty$ (see also Fig. 12). This striking observation is fully explained by the asymptotic theory, see Sec. 6 which shows that the vesicle has a given shape when $(T - \cos \theta_Y)\Gamma$ is constant. This is obviously the case when $T = T^* \equiv \cos \theta_Y$ and the asymptotic theory shows that this is also the case when $\beta = 0$. However, the product $(T - \cos \theta_Y)\Gamma$ and, hence, the vesicle shape, does change with $\Gamma$ at $T = T^+(\theta_Y, \Gamma)$.

The theory shows that the vesicle state exists only when the tension is larger than $T^*$, see Eq. (46). The numerical results show that this state exists only when the tension is smaller than $T^+(\theta_Y, \Gamma)$, which tends to $(\cos^2 \theta_Y/2)$ as $\Gamma \to \infty$, in agreement with the limit of vanishing bending modulus discussed in Sec. 2.
Eq. (10). For $\Gamma = \infty$, the shape of the vesicle is predicted to be circular with radius $(A/\pi)^{1/2}$ independently of tension. Numerical results, on the other hand, show that the vesicle shape can significantly depart from a circle, as can be seen in Fig. (b) for $\Gamma = 30$. In the vicinity of $T = T^*$, the vesicle has a teardrop shape. The range of tensions for which the vesicle is markedly non-circular shrinks as $\Gamma \to \infty$ but nevertheless remains numerically significant even for $\Gamma = 200$; this is explained by the asymptotic theory of Sec. 6.

A non-vanishing bending modulus has thus a significant impacts on the vesicle shape, and not merely a boundary layer near the self-contact. The vesicle shape is controlled by two length scales. The radius of curvature of the sheet away from the contact point scales like the size of the drop, $\sqrt{A}$. However, the radius of curvature at the contact point does not scale like $\ell_{BC}$ as one could expect. Instead, it scales like $\ell_{TBC} \equiv \ell_{BC}/\sqrt{H}$, as shown by Eq. (46). Since $H = T - T^*$ and $0 \leq \sqrt{H} \leq \sin(\theta_Y/2)$, $\ell_{TBC} \approx \ell_{BC}$ near $T = T^+$. Therefore, $R_\ell \sim \ell_{BC}$, where $R_\ell$ is the radius of curvature at the contact point, and it vanishes as $\Gamma$ diverges, i.e. as $B \to 0$. The shape adopted by the vesicle is thus essentially controlled by the length $\sqrt{A}$ except in the vicinity of the contact point where the curvature is large. The vesicle shape tends thus to a circle as $\Gamma$ diverges near the transition to partial wetting. However, $\ell_{TBC}$ diverges as $T \to T^*$ so that near the transition to complete wetting

$$R_\ell \sim (T - T^*)^{-1/2}. \tag{57}$$

The shape is thus necessarily different from a circle near that transition.

5. Asymptotics of the Partial Wetting Solution

Experiments performed in Ref. [32, 34] show that the bendability parameter $\Gamma$, defined by Eq. (2), can be as large as one million. In this section, we derive an approximate solution in that limit. Even if $1/\Gamma$ appears as the most natural expansion parameter, the ensuing analysis actually rests on the smallness of $\delta$, defined by

$$\delta^2 = B p^2 / H^3 = \Gamma^{-1} \times \frac{A^3}{R^2 H^3} \ll 1. \tag{58}$$

Unlike $1/\Gamma$, $\delta$ is not fixed but varies with the applied tension.

5.1. Low- and large-curvature regions

In the state of partial wetting, we distinguish two regions in the wet part of the sheet (bulge): a region I where $B(\partial_\theta^2 q + q) \ll p/\kappa$ in Eq. (50) and another region III, in the vicinity of the triple line, where $B(\partial_\theta^2 q + q) \gg p/\kappa$, see Fig. (9)(a).

In region I, Eq. (50) yields

$$p/\kappa \approx H, \quad \theta \approx \theta_1 = \frac{ps}{H}, \quad x \approx x_1 = \frac{H}{p} \sin \theta, \quad y \approx y_1 = \frac{H}{p} (1 - \cos \theta). \tag{59}$$

Conversely, in region III near the triple line, the curvature is large, hence $\theta$ changes rapidly with $s$. In this boundary layer, $p/\kappa$ can be treated as a small perturbation in Eq. (50), leading to the alternative approximation for $q$ and $\kappa$:

$$\frac{B}{H} q_{III} \simeq 1 + c \cos (\theta + \phi) + \delta \int_{\theta_0}^\theta \mathcal{R}(t) \sin (\theta - t) \, dt, \quad \kappa_{III} = -\sqrt{2q_{III}} \simeq -\left( \frac{H}{B} \right)^{1/2} \frac{1}{\mathcal{R}(\theta)} \mathcal{R}(\theta) = \frac{1}{\sqrt{2} \sqrt{1 + c \cos (\theta + \phi)}}. \tag{60a}$$

The function $\mathcal{R}(\theta)$ gives the leading order approximation of the radius of curvature in units of an effective elasto-capillary length $\sqrt{B/H}$, whereas $c$ and $\phi$ are constants of integration which will be determined by
We thus have two solutions according to the sign of $E$:

\[ \frac{n_{||,III}}{H} \simeq -c \cos(\theta + \phi) - \delta \int_{\theta_D}^{\theta} \mathcal{R}(t) \sin(\theta - t) \, dt, \]

\[ \frac{n_{\perp,III}}{H} \simeq c \sin(\theta + \phi) - \delta \int_{\theta_D}^{\theta} \mathcal{R}(t) \cos(\theta - t) \, dt. \]

Hence, the horizontal and vertical components have the local approximations

\[ \frac{n_x,III}{H} \simeq -c \cos \phi + \delta \int_{\theta_D}^{\theta} \mathcal{R}(t) \sin t \, dt, \quad \frac{n_y,III}{H} \simeq c \sin \phi - \delta \int_{\theta_D}^{\theta} \mathcal{R}(t) \cos t \, dt. \]

The region where the above expressions hold is a boundary layer in the $s$ coordinate where $\theta(s)$ varies rapidly over an $O(\sqrt{B/H})$ distance. On the other hand, Eq. (60a) shows no sign of rapid variation of the curvature as a function of $\theta$. Since $|\kappa_{III}|$ is larger than $\kappa_1$ by a $O(\delta^{-1})$ factor, there must be another, internal, boundary layer in the $\theta$ coordinate where the transition between the solutions in regions I and III above takes place. Such a layer is found around the inflexion point, where $\theta$ is maximum. There, the curvature is still small but its variation with $\theta$ is now sufficiently fast that $B\partial_\theta^2 q$ balances $p/\kappa$.

### 5.2. Internal boundary layer

To describe the transition layer, we introduce the new scalings

\[ \kappa = \kappa_{II} = \frac{p}{H} \kappa(\xi), \quad q = q_{II} = \frac{p^2}{H^2} \mathcal{Q}(\xi), \quad \xi = \frac{\theta_* - \theta}{\delta}, \]

where $\theta_*$ is the maximal value of $\theta$ so that $\xi \geq 0$. Using Eqs. (63), Eq. (50) becomes

\[ \partial_\xi^2 \mathcal{Q} + \delta^2 \mathcal{Q} + \frac{1}{\mathcal{K}} - 1 = 0. \]

Recalling (48), Eq. (64) is a second order ODE for the function $\mathcal{Q}(\xi)$. Furthermore, the nonlinearity is in $\mathcal{Q}$ (but not in $\partial_\xi \mathcal{Q}$), and there is no explicit dependence on $\xi$. Consequently, it can be seen as deriving from a Lagrangian $\frac{1}{2}(\partial_\xi \mathcal{Q})^2 - \mathcal{V}(\mathcal{Q})$, such that the corresponding Hamiltonian is conserved [i.e. $\frac{1}{2}(\partial_\xi \mathcal{Q})^2 + \mathcal{V}(\mathcal{Q})$ does not depend on $\xi$]. That is:

\[ \frac{1}{2} (\partial_\xi \mathcal{Q})^2 + \mathcal{V}(\mathcal{Q}) = E \]

\[ \mathcal{V}(\mathcal{Q}) = \delta^2 \mathcal{Q}^2/2 + \text{sgn}(\kappa) \sqrt{2\mathcal{Q} - \mathcal{Q}^2/2} = \delta^2 \mathcal{Q}^2/2 + \kappa - \mathcal{K}^2/2, \]

Henceforth, we neglect the term $\delta^2 \mathcal{Q}^2/2$. The potential $\mathcal{V}(\mathcal{Q})$ with $\delta = 0$ is depicted in Fig. 9(b).

Let us first consider the region of the sheet with positive curvature, $s < s_*$ (blue potential in Fig. 9(b)). The potential is maximum at $\mathcal{Q} = 1/2$, which corresponds to the curvature in region I. Indeed, using the second of Eqs. (63) together with the relation (48) between $\kappa$ and $q$, we obtain the first of Eqs. (59) when $\mathcal{Q} = 1/2$. In order to asymptotically tend to that value in the large-$\xi$ limit, i.e. when $\theta$ differs significantly from its maximal value $\theta_*$ (see Eq. (63)), we must have $\partial_\xi \mathcal{Q} = 0$ at $\mathcal{Q} = 1/2$, that is $E = \mathcal{V}(1/2) = 1/2$. Knowing $E$, Eqs. (65) can be integrated to give

\[ (\mathcal{K} - 1) e^{\mathcal{K} - 1} = -e^{-\mathcal{K} - 1}. \]

We thus have two solutions according to the sign of $\mathcal{K}$:

\[ \mathcal{K}_+(\xi) = 1 + W_0 \left(-e^{-1-\xi}\right) \quad \text{with} \quad \mathcal{K}_+(\xi) \approx 1 - e^{-\mathcal{K} - 1}, \]

\[ \mathcal{K}_-(\xi) = 1 + W_{-1} \left(-e^{-1-\xi}\right) \quad \text{with} \quad \mathcal{K}_-(\xi) \approx -\xi - \frac{2 + \xi}{1 + \xi} \ln(1 + \xi), \]
Figure 9: (a) Schematic of the partial wetting state showing the three regions and areas used in the asymptotic analysis. The dashed curve represents an arc of a circle. $s = s_*$ is the location of the maximum value $\theta_*$ of $\theta$ where the curvature vanishes. (b) Approximate potential in the boundary layer near the inflexion point, see Eq. (65b) with $\delta = 0$. The blue and orange parts, respectively, correspond to $\text{sgn}(K) = 1$ and $\text{sgn}(K) = -1$. (c) Evolution of the rescaled curvature in the transition layer, $K$, given by Eq. (67) as a function of the rescaled angle $\xi$ defined by Eq. (63), together with two asymptotic expressions.

where $W_{0,-1}(z)$ are the Lambert functions defined by [46, p. 111]

$$we^w = z, \quad \Rightarrow \quad w = W_0(z) \quad \text{when} \quad w > -1 \quad \text{or} \quad w = W_{-1}(z) \quad \text{when} \quad w < -1.$$  \hfill (68)

The two solutions $K_{\pm}(\xi)$ smoothly join at $\xi = 0$, see Fig. 9(c). The first one, $K_+(\xi)$, holds on the positive-curvature side of the inflexion point ($s < s_*$) and the comparison between Eqs. (59) and (63) shows that it automatically matches with the solution in region I since $K_+(\xi \to \infty) = 1$. The second one describes the negative curvature on the other side of the inflexion point ($s > s_*$). As $\xi \to \infty$, the solution $K_-(\xi)$ yields

$$\kappa_{Ii} \simeq -\left(\frac{H}{B}\right)^{1/2}(\delta \xi + \delta \ln \xi), \quad \frac{B}{H}q_{II} \simeq \frac{\delta^2 \xi^2}{2} + \delta^2 \xi \ln \xi,$$  \hfill (69)

where we have used Eqs. [58], [63] and [67b] together with the relation [48] between $q$ and $\kappa$, i.e. $Q = \kappa^2/2$. The quadratic behaviour of $q_{II}$ in $\xi$ above can only be matched with Eq. (60a) if

$$c = -1 + \Delta c, \quad \phi = -\theta_* + \Delta \phi$$  \hfill (70)

where $\Delta c, \Delta \phi \ll 1$. Equation (60a) then reads

$$\frac{B}{H}q_{III} \simeq 1 - \cos (\theta_* - \theta) + \Delta c \cos (\theta_* - \theta) - \Delta \phi \sin (\theta_* - \theta) + \delta \int_{\theta_D}^{\theta} \mathcal{R}(t) \sin (\theta - t) \, dt.$$  \hfill (71)

The integral in this expression can be evaluated explicitly at the dominant order, i.e. using $c = -1$ and $\phi = -\theta_*$ in the expression (60a) of $\mathcal{R}(t)$. The result is:

$$\frac{B}{H}q_{III} \simeq 1 - \cos (\delta \xi) + \Delta c \cos (\delta \xi) - \Delta \phi \sin (\delta \xi)$$
$$+ \delta \left[2 \sin \left(\frac{\delta \xi}{2}\right) + 2 \sin \left(\frac{\theta_* - \theta_D}{2} - \delta \xi\right) + \sin \left(\delta \xi\right) \ln \left(\tan \left(\frac{\delta \xi}{4} \cot \frac{\theta_* - \theta_D}{4}\right)\right)\right],$$  \hfill (72)
where $\delta \xi = \theta_* - \theta$. Letting $\delta \to 0$, we obtain

$$B_H q_{\text{III}} \approx \frac{\delta^2 \xi^2}{2} + \delta^2 \xi \log \xi + \Delta c + 2 \delta \sin \left( \frac{\theta_* - \theta_D}{2} \right)$$

$$- \Delta \phi \delta \xi + \Delta^2 \xi \left[ 1 - 2 \cos \left( \frac{\theta_* - \theta_D}{2} \right) - \log \left( 4 \tan \frac{\theta_* - \theta_D}{4} \right) + \log \delta \right].$$

Equation (73) coincides with Eq. (69) if

$$\Delta c = -2 \delta \sin \left( \frac{\theta_* - \theta_D}{2} \right),$$

$$\Delta \phi = \delta \left[ 1 - 2 \cos \left( \frac{\theta_* - \theta_D}{2} \right) - \log \left( 4 \tan \frac{\theta_* - \theta_D}{4} \right) + \log \delta \right].$$

Equations (62) for the horizontal and vertical tractions are now

$$\lim_{s \to \infty} \frac{n_{x, \text{III}}}{H} \approx \cos \theta_* (1 - \Delta c) + \Delta \phi \sin \theta_* + \delta \int_{\beta_D}^{\beta} R(t) \sin t \, dt,$$

$$\lim_{s \to \infty} \frac{n_{y, \text{III}}}{H} \approx \sin \theta_* (1 - \Delta c) - \Delta \phi \cos \theta_* - \delta \int_{\beta_D}^{\beta} R(t) \cos t \, dt.$$}

Imposing the boundary condition (35) at $\theta = \theta_D$, i.e. $s = D$, we obtain

$$\mathcal{T} \approx \sin^2 \theta_Y + 2H^2 \Delta c, \quad \tan \theta_* \approx \frac{\sin \beta}{T - \cos \beta} + \Delta \phi \left( 1 + \tan^2 \theta_* \right).$$

In the limit $\delta \to 0$, these two relations are identical to Eqs. (30) obtained for an infinitely bendable sheet. Finally, the area of the fluid is given by (see Fig. 3a)

$$A = A_d + A_1 + A_{\text{BL}} \approx R_1^2 \left( \frac{\beta - \sin 2\beta}{2} \right) + \kappa_1^{-2} \left( \theta_* - \frac{2\theta_*}{2} \right) + 2 \int_{y_*}^{y_D} x \, dy,$$

$$\approx R_1^2 \left( \frac{\beta - \sin 2\beta}{2} \right) + \frac{H^2}{\beta^2} \left( \theta_* - \frac{2\theta_*}{2} \right),$$

where, in the second line, we neglected the $O(\delta H^2/\beta^2)$ contribution $A_{\text{BL}}$ from the boundary layer. Using Eqs. (30) and (58), Eq. (77) can be written as

$$\delta^2 \approx \frac{1}{2 \Gamma^2} \left[ \beta - \frac{\sin 2\beta}{2} + H^2 \left( \theta_* - \frac{2\theta_*}{2} \right) \right].$$

5.3. First look at the bifurcation diagram

We now have all the information required to plot the branch of partial wetting solutions in the $(\mathcal{T}, \beta)$ plane. Let us first define the values of $\mathcal{T}$ and $\mathcal{H}$ for $\delta = 0$ (i.e. $\Gamma = \infty$) as

$$\mathcal{T}_0(\beta) = \frac{\sin^2 \theta_Y}{2 (\cos \beta - \cos \theta_Y)}, \quad \mathcal{H}_0(\beta) = \mathcal{T}_0(\beta) - \cos \theta_Y.$$

Then Eqs. (74a), and (76) and (78) are asymptotically equivalent, in the small-$\delta$ limit, to

$$\tan \theta_* \approx \frac{\sin \beta}{\mathcal{T}_0(\beta) - \cos \beta},$$

$$\Delta c = -2 \delta \sin \left( \frac{\theta_* - \theta_D}{2} \right), \quad \mathcal{T}(\beta) \approx \mathcal{T}_0(\beta) + \frac{\mathcal{H}_0(\beta)^2}{\cos \beta - \cos \theta_Y},$$

$$\delta^2 \approx \frac{1}{2 \Gamma^2} \left[ \beta - \frac{\sin 2\beta}{2} + \mathcal{H}_0(\beta)^2 \left( \theta_* - \frac{2\theta_*}{2} \right) \right].$$
Evaluating the above expressions in succession for a given $\Gamma$ and of the dimensionless control parameters $T$. The resulting diagram shows that $\theta$ where we recall that $\theta_D = \theta_Y - \beta$. Even though it is natural to seek an expression for the angle $\beta$ in terms of the dimensionless control parameters $T$ and $\Gamma$, it is more convenient to express $T$ in terms of $\Gamma$ and $\beta$. Evaluating the above expressions in succession for a given $\Gamma$ and $\beta$ yields $T(\beta)$ along the branch of partially wet solutions. The resulting diagram shows that $T$ increases monotonously with $\beta$, which excludes the possibility of multiple states for a given value of $T$ in the limit $\delta \to 0$, in agreement with Fig. 7.

The first of Eqs. (80a) sheds a new light on the inflexion angle $\theta_*$. Comparison with Eq. (8) shows that $\theta_* \simeq \theta$. It corresponds to the apparent orientation of the sheet at contact and, hence, shows that $\theta_* + \beta$ is the apparent angle of contact of Ref. 32, 34. The difference between the apparent and true angles of contact is thus given, to leading order in $\Gamma^{-1}$, as a function of $\beta$ by

$$\theta_* - \theta_D \simeq \arctan \left( \frac{\sin \beta}{T_0(\beta) - \cos \beta} \right) - \theta_Y + \beta. \quad (81)$$

It is important to emphasize that (i) the deviation of the “apparent” angle from the real angle is non-vanishing in the limit $\Gamma \to \infty$ and (ii) nevertheless, it cannot be computed by a theory that has $\Gamma = \infty$ (i.e. Sec. 2).

5.4. Shape of the sheet: uniform asymptotic expansion and self-crossing

Given $\Gamma$ and $\beta$, the factor

$$\frac{H}{p} = \left( \frac{A}{\delta^3 H_0} \right)^{1/2} \quad (82)$$

appearing in Eq. (59) is known from Eq. (80a). If we write

$$Z = x + iy, \quad (83)$$

we may compactly express the shape in region I as

$$Z_1(\theta) = \frac{-iH}{p} e^{i\theta} \theta \to \theta_* \simeq \frac{-iH}{p} e^{i\theta} \left[ 1 + i(\theta - \theta_*) \right]. \quad (84)$$
In region II, we have
\[ \partial_\xi Z = -\delta \partial_\eta Z = -\frac{\delta}{\kappa} \partial_\eta Z = -\frac{\delta H}{p K(\xi)} e^{i\theta} \sim -\frac{\delta H}{p K(\xi)} e^{i\theta^*}. \]

On the positive-curvature side, we thus have
\[ Z_{\Pi}(\theta) \simeq Z_* - \frac{\delta H}{p} e^{i\theta^*} \int_{\xi_0}^{\xi} \frac{d\xi'}{K_+(\xi')} = Z_* + \frac{\delta H}{p} e^{i\theta^*} \ln |W_0(-e^{-1-\xi})|. \]

As \( \xi \to \infty, \ln |W_0(-e^{-1-\xi})| \to -1 - \xi. \) Matching Eq. (86) with Eq. (84) thus yields
\[ Z_* = \frac{H}{p} e^{i\theta^*} (-i + \delta). \]

By the same token, on the negative-curvature side, we have
\[ Z_{\Pi}(\theta) \simeq Z_* + \frac{\delta H}{p} e^{i\theta^*} \ln |W_{-1}(-e^{-1-\xi})| \simeq \frac{\delta H}{p} e^{i\theta^*} \ln \xi, \]

where the same constant of integration \( Z_* \) is used by continuity of \( Z \) at \( \xi = 0. \) Finally, in the large-curvature region, knowing the tractions, we have
\[ Z_{\Pi}(\theta) \simeq Z_D - \frac{\delta H}{p} \int_0^\theta R(t) e^{it} dt = Z_D - \frac{\delta H}{p} e^{i\theta^*} \left[ 2e^{-it/2} + \ln \left( \tan \left( \frac{t}{4} \right) \right) \right] \theta_* - \theta_D. \]

As \( \theta \to \theta_* \), this tends to
\[ Z_D + \frac{\delta H}{p} e^{i\theta^*} \left[ 2 + \ln (\theta_* - \theta) - \ln \left( 4 \tan \left( \frac{\theta_* - \theta_D}{4} \right) \right) \right] - 2 \frac{\delta H}{p} e^{i(\theta_* + \theta_D)/2}, \]

and matching Eq. (90) with Eq. (88), we find
\[ Z_D = Z_* - \frac{\delta H}{p} e^{i\theta^*} \left[ 2 - \ln \left( 4 \tan \left( \frac{\theta_* - \theta_D}{4} \right) \right) \right] + 2 \frac{\delta H}{p} e^{i(\theta_* + \theta_D)/2}. \]

We may now write a composite approximation that is uniformly valid in the wet region. To this end we sum the approximations inside and outside the boundary layer and retrieve the common part to avoid double counting [77]. On the positive curvature side, with \( 0 < \theta < \theta_* \), the shape is thus given by
\[ Z(\theta) \sim \frac{H}{p} \left[ -ie^{i\theta} + \delta e^{i\theta^*} \left( \ln |W_0(-e^{-1-(\theta_* - \theta)/\delta})| + 1 + \frac{\theta_* - \theta}{\delta} \right) \right]. \]

On the other hand, the negative-curvature side, defined in the range \( \theta_D < \theta < \theta_* \) is described by
\[ Z(\theta) \sim Z_{\Pi}(\theta) + \frac{\delta H}{p} e^{i\theta^*} \left[ \ln |W_{-1}(-e^{-1-(\theta_* - \theta)/\delta})| - \ln \left( \frac{\theta_* - \theta}{\delta} \right) \right]. \]

In Fig. 10 we plot \( \beta(T) \) as obtained from Eq. (80b) for \( \Gamma = 200 \) and \( \theta = \pi/3 \). This plot reveals that \( \beta \) may be considered an “order parameter”, akin to the amplitude of a pattern near a supercritical bifurcation, where the control parameter is the tension \( T \) exerted on the far ends of the sheet. Also plotted is the shape of the sheet. Importantly, self-crossing is found below a certain small value \( \beta = \beta^* > 0 \), which defines a critical tension \( T^+ = T(\beta^*) \) through the formulas [80]. For \( T < T^+ \) a contact force must be included in the model in order to prevent self-crossing. Hence \( T = T^+ \) is a bifurcation point where the partial wetting and the vesicle solution (with vanishing contact force) coincide. The asymptotic estimation of \( T^+ \) agrees well with the numerical simulations. On the other hand, there is a larger relative discrepancy between the
asymptotic and numerical value of $\beta^+$. This comes from the neglect of $2 \int_{y_e}^{y_D} x \, dy$ in Eq. (77), which becomes significant as $\beta \to 0$.

For future reference, let us compute the ratio of the curvatures in the bulge and at the contact point. In the small-$\beta$ limit, $\theta_+ \to \pi$ and $n_{\text{III}}$ vanishes, to leading order, at $\theta = \pi/2$. Hence, using Eq. (44) we find that

$$\kappa_1/|\kappa_{\text{III,contact}}| \simeq \delta/\sqrt{2},$$

so that the parameter $\delta$ may be regarded as a measure of the ratio of the curvatures in the bulge and near the triple line or at self-contact.

6. Asymptotics of the Vesicle Solution

We now analyse the solution depicted in Fig. 5(b). We make the assumption that self-contact at $s = \ell$ takes place in the wet part of the sheet, i.e. $\ell \leq D$. This is numerically verified for $\theta_Y < \pi/2$.

6.1. Beyond the contact point

In the range $\ell < s < D$, one has $\kappa = O(x_D^{-1})$ and numerical solutions indicate that $\beta$ is small, so that $p = \gamma/R_d = \gamma \sin \beta/x_D \approx \gamma \beta/x_D$. In Eq. (50), we have

$$\frac{p/\kappa}{B q} = O \left(\frac{\gamma \beta x_D^2}{\Delta A}\right) \ll 1 \quad \text{if} \quad \beta \ll \frac{A}{\Gamma x_D^2}.$$  \hspace{1cm} (95)

Under this hypothesis, we may neglect $p/\kappa$ in Eq. (50) and obtain

$$B q \approx H [1 + d \cos (\theta + \psi)],$$

where $d$ and $\psi$ are constants of integration. Evaluating this expression at the contact point, where $\theta = \pi/2$, and comparing with Eqs. (46) and (48), we find $\psi = 0$. Next, using the boundary condition (54) and the relation (45) between $H$ and $T$, we get

$$d = \frac{\cos \theta_Y (1 - \cos \theta_D)}{H \cos \theta_D} - 1.$$  \hspace{1cm} (97)

Knowing $q$ and given that $\operatorname{sgn}(\kappa) < 0$, the curvature is readily obtained from Eq. (48):

$$\kappa(\theta) \simeq -\frac{2H}{B} \sqrt{1 + \cos \theta} = -\left(\frac{2H \Gamma}{A}\right)^{1/2} \sqrt{1 + d \cos \theta}.$$  \hspace{1cm} (98)

Hence, with $\partial \theta x = \cos \theta/\kappa$, we obtain

$$x_D \left(\frac{2H \Gamma}{A}\right)^{1/2} = \int_{y_e}^{y_D} \cos \theta \, d\theta \approx \int_{\theta_Y}^{\theta_D} \cos \theta \, d\theta,$$

where we have used the smallness of $\beta$ to approximate the lower bound of the integral with $\theta_Y$. This integral is manifestly an $O(1)$ quantity. Hence the asymptotic limit (95) under which the present derivation holds is simply $\beta \ll 1$. The expression for $x_D$ calls for introducing the rescaled triple line abscissa

$$x_D = \left(\frac{A}{2 \Gamma}\right)^{1/2} \chi_D, \quad \chi_D = \int_{\theta_Y}^{\theta_D} \frac{\cos \theta \, d\theta}{\sqrt{1 + d \cos \theta}}.$$  \hspace{1cm} (100)

On the other hand, the equation $\partial \theta y = \sin \theta/\kappa$ yields $y \simeq y_D + B[\kappa_D - \kappa(\theta)]/(H d)$. Hence, the area $A_2$ is given by

$$A_2 = R_d^2 \left(\beta - \frac{1}{2} \sin 2\beta\right) + 2 \int_{y_e}^{y_D} x \, dy = x_D^2 \left(\frac{\sin 2\beta}{2 \sin^2 \beta}\right) + \frac{2B}{Hd} (\sin \theta_D - 1 - \kappa_D x_D),$$  \hspace{1cm} (101)

where we used integration by parts. It follows that $A_2/A = O(\Gamma^{-1})$.

For future reference, let us finally note that, using Eq. (49), we have

$$\lim_{s \to \ell^+} n_{-\pi/2} = \lim_{\theta \to \pi/2} (-B \partial \theta y) = \gamma \left(\frac{\cos \theta_Y}{\cos \theta_D} - T\right).$$  \hspace{1cm} (102)
6.2. Before the contact point

Inside the vesicle, our numerical solutions indicate that all terms in Eq. (50) are generally of the same order. In this case, the alternative form (51) is more convenient to analyze than Eq. (50). The curvature decreases from a value $\kappa_0 > 0$ at $s = 0$ to $\kappa_\ell < 0$ at $s = \ell$. As pointed out earlier [see Eq. (46)], the introduction of $\kappa_\ell$ allows us to write $H = B\kappa_\ell^2/2$, because $n_\parallel$ vanishes at the contact point. From Eq. (51), we directly get

$$n_\perp^2 = 2Bp(\kappa_0 - \kappa) - BH\left(\kappa_0^2 - \kappa^2\right) + \frac{B^2}{4}\left(\kappa_0^4 - \kappa^4\right),$$

where we used the fact that $n_\perp = \partial_s\kappa = 0$ at $s = 0$. Let us rescale $\kappa, p$, and introduce the parameter $K$ as follow

$$k = \frac{\kappa}{\kappa_0}, \quad p = \left(\frac{2H^3}{B}\right)^{1/2} \mathcal{P}, \quad K = \frac{\kappa_0}{|\kappa_\ell|} = \frac{\kappa_0}{\sqrt{2H/B}}.$$  

Note that the definition of $K$ is similar to the interpretation of $\delta$ as $\beta \to 0$. Importantly, we will allow in what follows $K$ not to be small, even if $\Gamma$ is large. With these new notations, we may rewrite Eq. (103) as

$$n_\perp = B\kappa_0^2\mathcal{N}_\perp (k; \mathcal{P}, K),$$

with $\mathcal{N}_\perp (k; \mathcal{P}, K) = (1 - k)^{1/2}\left[\frac{\mathcal{P}}{K^3} + \frac{1 + k}{4}\left(1 + k^2 - \frac{2}{K^2}\right)\right]^{1/2}$.  

Next, using the first of Eqs. (29), we have $\kappa = \partial_s\kappa = \partial_s\kappa_\ell\partial_s\theta = -(B\kappa_0)^{-1}n_\perp \partial_s\theta$. Hence, using this last relation and Eq. (105a), we obtain

$$\theta(k; K) = \int_k^1 \frac{k'}{\mathcal{N}_\perp (k'; \mathcal{P}, K)} \, dk', \quad \left(106\right)$$

where we used the fact that $\theta$ vanishes when $k = 1$, i.e. when $s = 0$. Evaluating this expression at the contact point, where $k = -1/K$, yields a condition on $\mathcal{P}$:

$$\int_{-1/K}^1 \frac{k}{\mathcal{N}_\perp (k; \mathcal{P}, K)} \, dk = \frac{\pi}{2}.$$  

(107)

The solution of this equation is universal and denoted by $\mathcal{P} = \mathcal{P}(K)$, see Fig. 11(a). Note in particular that it doesn’t depend on $\Gamma$. The formula above implies that the curvature varies monotonously from $\kappa_0$ to $\kappa_\ell$. This is only true up to $K \approx 3.9207$. For larger values, $\mathcal{N}_\perp$ necessarily passes by zero and the integral in the left hand side of Eq. (107) must be split into two parts. We ignore this difficulty in this section and give details in Appendix B. A good numerical fit of $\mathcal{P}(K)$, valid for all $K$, is given by

$$\mathcal{P}(K) \approx (K_0 - K) \left(\frac{K}{K_0} + \frac{3}{14} \frac{K^3}{K + 0.846}\right), \quad K_0 = 1.77842.$$  

(108)

The value $K_0$ is such that $\mathcal{P}$, and hence $\beta$, vanishes. Once the function $\theta(k; K)$ is known, the equation for $x$ can be rewritten as

$$\cos \theta = \partial_x x = \partial_s\kappa\partial_s x = -(\kappa_0 B)^{-1}n_\perp \partial_s x$$

and similarly for $y$. Knowing that $x = y = 0$ at $k = 1$, i.e. at $s = 0$, and using Eqs. (109) and (105a), the shape of the vesicle is thus given by the double quadrature

$$\kappa_0 x(k; K) = \int_k^{1} \frac{\cos \theta(k'; K)}{\mathcal{N}_\perp (k'; \mathcal{P}, K)} \, dk', \quad \kappa_0 y(k; K) = \int_k^{1} \frac{\sin \theta(k'; K)}{\mathcal{N}_\perp (k'; \mathcal{P}, K)} \, dk'.$$  

(110)
It turns out that, once Eq. (107) is satisfied, \( x \) automatically vanishes at the contact point, so that no new constraint results from that condition. Finally, the area of the vesicle is computed as

\[
A_1 = 2 \int_{0}^{y_c} x \, dy = -2 \int_{-1/K}^{1} x \, (\partial_k y) \, dk = \frac{2}{k_0^2} \int_{-1/K}^{1} \frac{k_0 x(k; K) \sin \theta(k; K)}{N_{\perp}(k; P, K)} \, dk. \tag{111}
\]

Having determined previously that \( A_2/A = O(\Gamma^{-1}) \), we have \( A_1 \approx A \) and we obtain

\[
\kappa_0^2 A \approx \phi(K), \quad \phi(K) = 2 \int_{-1/K}^{1} \frac{k_0 x(k; K) \sin \theta(k; K)}{N_{\perp}(k; P, K)} \, dk, \quad K \lesssim 3.9207. \tag{112}
\]

Like \( P(K) \), the function \( \phi(K) \) is universal and independent of \( \Gamma \), see Fig. [11]a. Using the result of Appendix B, it is well fitted over all \( K \) by

\[
\phi(K) \approx \pi \left[ \frac{1 + 3.373 K^2 + 0.606 K^4}{1 + 1.819 K^2 + 0.276 K^4} \right]. \tag{113}
\]

We are now able to simultaneously parametrise \( \beta \) and \( \mathcal{T} \) with \( K \). Indeed, using the last of Eqs. (104) and the first of Eqs. (112), we obtain \( \mathcal{H} \) as a function of \( K \) and Eq. (45) gives then \( \mathcal{T} \) as a function of \( K \):

\[
\mathcal{T}(K) = \cos \theta_Y + \mathcal{H}(K), \quad \mathcal{H}(K) = \frac{\phi(K)}{2 \Gamma K^2}. \tag{114}
\]

The expression of \( \mathcal{X}_D(K) \) is obtained by using Eq. (97) with \( \theta_D \approx \theta_Y \) and the first of Eqs. (114) in Eq. (100). Finally, the expression of \( \beta \) as a function of \( K \) is obtained by using Eq. (100) and the second of Eqs. (104) together with \( \beta \approx \sin \beta \) in Eq. (40)

\[
\beta(K) = \mathcal{X}_D(K) \mathcal{H}(K)^{3/2} \mathcal{P}(K), \quad \mathcal{X}_D(K) = \int_{\theta_Y}^{\pi/2} \frac{\cos \theta \, d\theta}{\sqrt{\mathcal{H}(K) + [1 - \mathcal{T}(K)] \cos \theta}}. \tag{115}
\]

Equations (114) and (115) imply that \( \beta = O(\Gamma^{-3/2}) \) if \( K = O(1) \) and that \( \beta = O(\Gamma^{-1/2}) \) if \( \mathcal{H} = O(1) \), that is if \( K = O(\Gamma^{-1/2}) \). Hence, the assumption \( \beta \ll 1 \) made in deriving the solution above the contact point.
Finally, the shape of the vesicle is obtained by combining Eqs. (110) with the first of Eqs. (112) provided that this family of curves does not depend explicitly on $\Gamma$ and is therefore valid for arbitrary bending stiffness, where $\gamma$ decreases and $K$ increases. Additionally, the second of Eqs. (114) indicates that a given vesicle shape, identified by the single number $\lambda_D$, is obtained by solving $F_\ell = 0$ using Eq. (118a) together with the expressions (115) and (114) of $\beta$ and $\lambda_D$ with $\Gamma = 200$ and $\theta_Y = \pi/6$.

is verified in the large-$\Gamma$ limit. On the other hand, the radius of the liquid-vapour interface is obtained by using the second of Eqs. (104) and (114) in Eq. (30)

$$\frac{R_\ell}{A} = \frac{2\Gamma K^3}{\phi(K)^{3/2} \mathcal{P}(K)}, \quad \text{(116)}$$

Finally, the shape of the vesicle is obtained by combining Eqs. (110) with the first of Eqs. (112)

$$Z(k; K) = \sqrt{\frac{A}{\phi(K)}} \int_k^1 e^{i\beta(k'; K)} \mathcal{N}_\perp(k'; P, K) \, dk', \quad -1/K < k < 1, \quad \text{(117)}$$

where $Z = x + iy$. For a given value of $K$, the above expression yields the shape of the vesicle. Strikingly, this family of curves does not depend explicitly on $\Gamma$ and is therefore valid for arbitrary bending stiffness, provided that $\beta \ll 1$. Nor does it depend on $\theta_Y$, which is understandable if there is no triple line within the vesicle. Additionally, the second of Eqs. (114) indicates that a given vesicle shape, identified by the single number $K$, is achieved over the locus of a constant product $H\Gamma$ where the product in question is $\phi(K)/2K^2$, see Fig. 12(b).

Two particular vesicle shapes stand out in Fig. 12(b). One is at $K = K_0$, where the capillary pressure vanishes. This implies that the liquid-vapour interface is flat, i.e. that $\beta = 0$. This corresponds to the unique curve $Z(k; K_0)$, with $-1/K_0 < k < 1$, in agreement with the numerical curves of Fig. 8(h). The second corresponds to the limit $K \to \infty$, where $H \to 0$ so that $T \to \cos \theta_Y$ and $\kappa_\ell \to 0$, see Eqs. (46) and (114). For lower tensions, i.e. $T < T^+ \equiv \cos \theta_Y$, the vesicle does not exist, in agreement with the no-bending limit. The fact that $\kappa_\ell$ tends to zero suggest that the disappearance of the vesicle state occurs through a lengthening and thinning of region of contact.

6.3. Contact force

In the absence of adhesion, the existence of the vesicle state requires that $F_\ell > 0$. The tension $T^+$ for which $F_\ell$ vanishes is also the bifurcation point with the partial wetting solution. From Eqs. (41), we have
\[ F_c(K) = \lim_{s \to 0} [n_{\parallel}(\ell + \epsilon) - n_{\parallel}(\ell - \epsilon)]. \] The expression of \( n_{\parallel} \) at \( s = \ell^+ \) is obtained from Eq. (102) with \( \theta_D = \theta_Y - \beta \). The expression of \( n_{\parallel} \) at \( s = \ell^- \) is obtained by using Eqs. (105) with \( k = -1/\ell \) and the first of Eqs. (112) and (114).

The expression of the contact force as a function of \( K \) reads then

\[
\frac{F_c(K)}{\gamma} = \frac{\cos \theta_Y}{\cos (\theta_Y - \beta)} - \cos \theta_Y - \frac{\mathcal{F}(K)}{\Gamma},
\]

where \( \mathcal{F}(K) = \frac{\phi(K)}{2K^2} \left[ 1 + \sqrt{4(1 + K)^2 \mathcal{P} + (K^2 - 1)^2} \right] \). \hfill (118a)

Numerically solving \( F_c(K) = 0 \) yields the root \( K^+ \) and therefore, from Eqs. (114), \( T^+ \) as a function of \( \theta_Y \) and \( \Gamma \), in very good agreement with the numerical simulations (see the curve \( T^*_\alpha \) in Fig. 2). In the large-\( \Gamma \) limit, we have \( \beta \ll 1 \) and \( K^+ = O(\Gamma^{-1/2}) \). It is easy to find, with the aid of Eqs. (108) and (113), that

\[ K^+ \simeq \left( \frac{\pi/\Gamma}{1 - \cos \theta_Y} \right)^{1/2}, \quad \text{corresponding to} \quad T^+ \simeq \cos^2(\theta_Y/2). \] \hfill (119)

A more detailed calculation yields

\[ T^+ \simeq \cos^2(\theta_Y/2) - C^+(\theta_Y) \Gamma^{-1/2}, \] \hfill (120)

where \( C^+(\theta_Y) \) is a complicated function that is approximated in the range \( 0 < \theta_Y < \pi/2 \) by

\[ C^+ \approx \frac{\sqrt{\pi/8 \theta_Y}}{1 - 0.296 \theta_Y + 0.235 \theta_Y^2}. \] \hfill (121)

Note that \( K^+ \ll 1 \) when \( \Gamma \gg 1 \). This is consistent with our earlier remark that \( K \) is equivalent to \( \delta \), the latter being small in the partial wetting state. The function \( T^+(\theta_Y, \Gamma) \) is depicted in Fig. 2 (see curve \( T^*_\alpha \)).

6.4. Limiting shape at \( T = T^* \)

We close this section by noting that, as \( T \to T^* \), the boundary conditions of Eqs. (29) in the range \( |s| < \ell \) are \( \kappa = n_{\parallel} = 0 \) at \( s = \ell \), so that the problem is mathematically equivalent to the one studied by Mora et al. in Ref. [48] using the method of Ref. [15] (with corrected boundary conditions). The study in Ref. [48] concerned the shape of a fishing line deformed by the surface tension of a soap film. At the particular point \( T = T^* \), corresponding to \( K \to \infty \) in our theory, the vesicle shape is thus given, up to a scale factor, by the solution explicitly given in Ref. [48]. A similar shape is also found in portions of solutions reported in Refs. [2] [55]. In Appendix B.1 we provide an alternative formulation of the solution reported in Ref. [48] based on the present theory.

7. Conclusions and perspectives

In this paper, we have presented a comprehensive picture of the bending of a thin elastic sheet under the opposite actions of capillary forces and an external tension \( T \). In order to elucidate the essential mechanisms, we have focused on a sheet that is much larger than the drop size, so that it can be modelled as being infinitely long. When \( T = 0 \), the system has been used as a 2D model for “capillary origamis” and configurations of complete wetting had already be reported in this context [2]. However, applying an external tension significantly modifies the folding dynamics. Now, the conformation of the system depends on \( T \) and upon varying this parameter, we found the possibility of wrapping most of the liquid inside a vesicle, which corresponds to the “budding transition” described by Kusumaatmaja and Lipowsky [41]. In this regard, one of the most dramatic results of our study is that, at the vicinity of the transition from the wrapped vesicle state to the complete wetting, the vesicle shape is universal — being independent on the explicit value of the bending rigidity \( B \), but nevertheless distinct from a the circular shape that is obtained for \( B = 0 \). This is surprising for one may expect a small \( B \) to manifest itself only in boundary layers whose
size is comparable to the bendocapillary length $\ell_{BC} = \sqrt{B/\gamma}$. One intuitive explanation for this, which is motivated by the asymptotic analysis of Secs. 5 and 6, is that the effective bendocapillary length is not $\ell_{BC}$ but rather the tension-dependent length scale $\ell_{TBC} = \sqrt{B/(T - \gamma \cos \theta_Y)}$. As the denominator of this expression tends to zero, the balance of bending and capillary forces is pronounced in the whole vesicle.

The same reasoning applies to the role of the bendability parameter $\Gamma = \gamma A/B$. Values that may appear large, such as $\Gamma = 30$ or even $\Gamma = 200$ may not be sufficient that the true asymptotic parameter of the partial wetting state, namely $\delta$ [Eq. (58)], be small everywhere in the bifurcation diagram. Indeed, while $\delta$ scales as $\Gamma^{-1/2}$, it also depends on $T$, just as the effective bendo-capillary length $\ell_{TBC}$ does, and in some cases can become not so small even for a value $\Gamma = 200$. Experimentally, $\Gamma$ can reach one million. Values in that range are challenging to simulate numerically but make $\delta$ truly small for all tensions and render the asymptotic analysis all the more reliable and useful.

One of the questions that motivated the introduction of a finite $B$ in the model was whether this would induce a snapping transition between the partial wetting and vesicle states, i.e. would the former emerge asymptotic analysis all the more reliable and useful.

Numerical simulations with lower values of $\Gamma$ unexpectedly revealed that the partial wetting state can exist at applied tensions $T$ significantly smaller than the threshold for complete wetting $T^* = \gamma \cos \theta_Y$, see Fig. 6(a). In that scenario, the partial wetting state disappears upon decreasing $T$ without exhibiting the vesicle state. It gives way, at a limit point $T^-$ close to $T = 0$, to the complete wetting state. If, subsequently, $T$ is increased, one expects the vesicle state to emerge from the complete wetting state at $T^*$. Further, at $T^+$, the vesicle state opens and the system jumps discontinuously to the partial wetting state.

For an infinitely long sheet, the complete wetting state is not a steady state: if $T < T^*$, the vesicle state disappears and the tension is not sufficient to counteract capillary forces. Hence, an infinitely long stretch of sheet is entrained by capillarity, in a never-ending process. Consistently with this, $T^*$ is precisely the value at which the curvature at self-contact vanishes in the vesicle state. Geometrically, this allows the self-contact point to become a segment of line of arbitrary length. Interestingly, the corresponding limiting shape coincides with that of a fishing line or hair that collapses onto itself when dipped in a soapy solution \([L]([8]). This is a particular case in our theory and we thus provide an alternative formula for what these authors call a “tennis racket” loop, see Appendix B.1.

We have not studied Young’s angles in the range $\pi/2 < \theta_Y < \pi$. The threshold for complete wetting, $T^* = \gamma \cos \theta_Y$ suggests in that case that a complete wetting state may be realized only if the sheet is under compression. This amounts to completely modify the mechanics of the problem: for one thing, an infinitely long sheet would buckle at arbitrarily small compressive stresses. Thus, considering this range of Young’s angle would require us to abandon our simplifying hypothesis and include the sheet length, $L$, as a key parameter. These two aspects, $\pi/2 < \theta_Y < \pi$ and finite $L$, open interesting research perspectives on this basic physical setting.

Finally, in our model, we simplified the discussion by assuming that the pressure inside the vesicle is governed by surface tension outside it. We proposed some circumstances under which this could be true in practice. Should the self-contact be impermeable, the enclosed area in the vesicle would be fixed and arbitrary and the pressure would differ from the capillary pressure. The techniques used in this paper would remain applicable but the analysis of such a scenario introduces another level of complexity that we hope will be addressed by future works.

Acknowledgement

The research leading to these results has received funding from NSF-CAREER Grant No. DMR 11-51780 (BD), visitor grant from the Fonds de la Recherche Scientifique - FNRS (BD) and W. M. Keck Foundation (BD, FB). GK is a Research Associate of the Fonds de la Recherche Scientifique - FNRS.
We are grateful to R. Govindarajan, J. Hanna, N. Menon, S. Neukirch, S. Walker and D. Vella for useful discussions. BD benefited from stimulating discussions with participants of the program “Geometry, elasticity, fluctuations, and order in 2D soft matter”, held in winter 2016 at the Kavli Institute for Theoretical Physics, UCSB.

Appendix A. Algorithm to find $\Gamma_c^+$

By definition, $\Gamma^+(\theta_Y)$ is the value of $\Gamma$ at which the self-contact state with a vanishing contact force ($F_c = 0$) switches from the unstable to the stable branch of the partial wetting state at a given $\theta_Y$. Self-contact solutions of Eqs. (33) can be computed by adding two additional shooting parameters, $T$ and $\ell$, which are fixed thanks to two additional boundary conditions, $\theta(\ell) = 0$ and $x(\ell) = 0$. The particular value of the tension at which such a state is found is, by definition, $T^+$. This procedure allows the self-contact state with a vanishing contact force to be computed for given values of $\Gamma$ and $\theta_Y$.

To determine $\Gamma^+$ numerically, $\Gamma$ is increased by small steps for a given value of $\theta_Y$. When $\Gamma < \Gamma^+$, the self-contact state belongs to the unstable branch of the partial wetting state and the transition is subcritical and, when $\Gamma > \Gamma^+$, it belongs to the stable branch and the transition is supercritical.

To determine at which branch the self-contact state belongs to, the self-contact solution is perturbed at each value of $\Gamma$ by slightly increasing $T$, i.e. $T = T^+ + \Delta T$ with $0 < \Delta T / T^+ \ll 1$. If the self-contact solution belongs to the unstable branch, then the perturbed solution will feature some self-crossing, i.e. $\min_s x(s)$ of the perturbed solution near $s = \ell$ is negative. If $\min_s x(s) > 0$, then the self-contact solution belongs to the stable branch. At a given $\theta_Y$, $\Gamma^+$ corresponds thus to the value of $\Gamma$ at which $\min_s x(s)$ changes its sign.

Appendix B. Vesicle with $K \gtrsim 3.9207$

In Sec. 6.2 we assumed that $\kappa$ decreases monotonously from $\kappa_0 > 0$ to $\kappa_\ell < 0$. This assumption ceases to hold for the range $3.9207 \lesssim K < \infty$. In that range of values, $n_\perp$ necessarily vanishes somewhere along the curve and the formulas of Sec. 6.2 must be revised. The change of sign of $\partial_\kappa \kappa$ happens when $\mathcal{N}_\perp$ vanishes, that is at $k = k_{\min}$, solution of

$$\frac{\mathcal{P}}{K^3} + \frac{1 + k_{\min}}{4} \left( 1 + \frac{k_{\min}^2}{\kappa^2} - \frac{2}{\kappa^2} \right) = 0. \quad (B.1)$$

Since the above equation is of third order, a closed form expression can be written for $k_{\min}$ in terms of $\mathcal{P}$ and $K$:

$$k_{\min} = -\frac{1}{3} + \frac{W^{1/3}}{3} - \frac{2}{W^{1/3}} \frac{1}{3 \kappa^2}, \quad (B.2)$$

where

$$\frac{WK^3}{2} = 9K - 5K^3 - 27\mathcal{P} + \sqrt{27} \sqrt{-2 + 5K^2 - 4K^4 + K^6} - 18K\mathcal{P} + 10K^3\mathcal{P} + 27\mathcal{P}^2. \quad (B.3)$$

Starting from the lowermost point of the vesicle, $k$ first decreases from 1 to $k_{\min} < 0$, before increasing again from $k_{\min}$ to $-1/K$. Starting from $k = 1$ and as long as $n_\perp > 0$, the function $\theta(k; K)$ has the expression

$$\theta_1(k) = \int_1^k \frac{k'}{\mathcal{N}_\perp(k'; \mathcal{P}, K)} \, dk', \quad k_{\min} < k < 1 \quad (B.4)$$

Once $k_{\min}$ is reached, $n_\perp$ changes sign and, subsequently,

$$\theta(k; K) = \theta_2(k) = \theta_1(k_{\min}) + \int_{k_{\min}}^k \frac{k'}{\mathcal{N}_\perp(k'; \mathcal{P}, K)} \, dk', \quad k_{\min} < k < -1/K, \quad (B.5)$$
where $\mathcal{N}_\perp$ is still the positive function defined in (105b). Note that $\theta_2(k) = 2\theta_1(k_{\text{min}}; K) - \theta_1(k)$, so that only $\theta_1(k)$ needs to be evaluated in practice. The equation that yields $P(K)$ is now

$$\int_{k_{\text{min}}}^{1} \frac{k'}{\mathcal{N}_\perp (k'; P, K)} \text{d}k' + \int_{k_{\text{min}}}^{-1/K} \frac{k'}{\mathcal{N}_\perp (k'; P, K)} \text{d}k' = \frac{\pi}{2}.$$  

(B.6)

Once $P(K)$ is determined, we may compute the complex coordinates

$$\eta_1(k) = \int_{k_{\text{min}}}^{1} \frac{e^{i\theta_1(k)}}{\mathcal{N}_\perp (k'; P, K)} \text{d}k', \quad k_{\text{min}} < k < 1,$$  

(B.7)

of which $\kappa_0 x_1(k)$ and $\kappa_0 y_1(k)$ are the real and imaginary parts, respectively. Similarly,

$$\eta_2(k) = \eta_1(k_{\text{min}}) + \int_{k_{\text{min}}}^{k} \frac{e^{i\theta_2(k)}}{\mathcal{N}_\perp (k'; P, K)} \text{d}k', \quad k_{\text{min}} < k < -1/K.$$  

(B.8)

Only the function $\eta_1(k)$ needs to be evaluated, for we have

$$\eta_2(k) = \eta_1(k_{\text{min}}) + e^{2i\theta_1(k_{\text{min}})} [\eta_1^*(k_{\text{min}}) - \eta_1^*(k)].$$  

(B.9)

Combining the expressions just obtained, one may derive

$$\phi(K) = A\kappa_0^2 = -2\int_{k_{\text{min}}}^{1} \Re[\eta_1(k)]\Im[\eta_1^*(k)] \text{d}k + 2\int_{k_{\text{min}}}^{-1/K} \Re[\eta_2(k)]\Im[e^{2i\theta_1(k_{\text{min}})}\eta_1^*(k)] \text{d}k,$$

(B.10)

where $\Re[\cdot]$ and $\Im[\cdot]$ denote real part and imaginary part, respectively. To close this section, let us compute the length of the curve that makes up the vesicle. One has $\partial_k k = -n_\perp/k_0 B$. Hence $\partial_k s = -k_0 B/n_\perp$. From this, and bearing in mind the change of sign of $n_\perp$, one obtains

$$l(K) = \sqrt{A/\phi(K)} \left( \int_{k_{\text{min}}}^{1} \frac{1}{\mathcal{N}_\perp (k; P, K)} \text{d}k + \int_{k_{\text{min}}}^{-1/K} \frac{1}{\mathcal{N}_\perp (k; P, K)} \text{d}k \right).$$  

(B.11)

**Appendix B.1. “Tennis racket” shape**

We conclude by giving the solution as $K \to \infty$, which is an alternative formulation of the solution of Ref. [48]. In that limit, $P \sim -3K^3/14$ and

$$\mathcal{N}_\perp \rightarrow \sqrt{1 - k^2} \sqrt{1 + 7k + 7k^2 + 7k^3/\sqrt{28}}, \quad W \rightarrow \left(11 + 3\sqrt{57}\right)/7, \quad k_{\text{min}} \rightarrow -0.165785\ldots$$  

(B.12)
The profile is obtained by evaluating Eq. (B.4) together with the real and imaginary parts of the functions (B.7) and (B.9) and using \( \phi_m = \phi(K \to \infty) \approx 6.89495 \). Note that, contrary to Ref. [48], no root-finding is necessary to obtain the solution.

Here is a Mathematica code to draw the curve, with output displayed in Fig. [B.13]:

```
Set[n, Sqrt[1-k] Sqrt[1+7k+7k^3]/Sqrt[28];
Set[theta1, Interpolation[Table[{kt, NIntegrate[Exp[I \theta1[kt]]/n, {kt, kmin, 1}]}, {kt, 0, 1, .025}]], InterpolationOrder \rightarrow 1];
Set[z1, Interpolation[Table[{kt, NIntegrate[Exp[I \theta1[kt]]/n, {kt, k, kt, 1}]}/Sqrt[phim], {kt, 0, 1, .025}], InterpolationOrder \rightarrow 1];
Set[z2[k], z1[kmin] + Exp[2 I \theta1[kmin]] Conjugate[z1[kmin] - z1[k]]

Show[ParametricPlot[{Re[z1[k]], Im[z1[k]]}, {k, kmin, 1}],
ParametricPlot[{Re[z2[k]], Im[z2[k]]}, {k, kmin, 1}],
ParametricPlot[{Re[z1[k]], Im[z1[k]]}, {k, kmin, 1}],
ParametricPlot[{Re[z2[k]], Im[z2[k]]}, {k, kmin, 1}]]
```

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