Abstract

The manifestly SU(4)xU(1) super-Poincaré invariant free-field N=2 twistor-string action for the ten-dimensional Green-Schwarz superstring is quantized using standard BRST methods. Unlike the light-cone and semi-light-cone gauge-fixed Green-Schwarz actions, the twistor-string action does not require interaction-point operators at the zeroes of the light-cone momentum, \( \partial z^+ \), which complicated all previous calculations. After defining the vertex operator for the massless physical supermultiplet, as well as two picture-changing operators and an instanton-number-changing operator, scattering amplitudes for an arbitrary number of loops and external massless states are explicitly calculated by evaluating free-field correlation functions of these operators on N=2 super-Riemann surfaces of the appropriate topology, and integrating over the global moduli. Although there is no sum over spin structures, only discrete values of the global U(1) moduli contribute to the amplitudes. Because the spacetime supersymmetry generators do not contain ghost fields, the amplitudes are manifestly spacetime-supersymmetric, there is no multiloop ambiguity, and the non-renormalization theorem is easily proven. By choosing the picture-changing operators to be located at the zeroes of \( \partial z^+ \), these amplitudes are shown to agree with amplitudes obtained using the manifestly unitary light-cone gauge formalism.
I. Introduction

The calculation of scattering amplitudes using the Green-Schwarz formulation of the ten-dimensional superstring is manifestly spacetime supersymmetric, and therefore contains advantages over analogous calculations using the Neveu-Schwarz-Ramond formulation of the superstring. For example, the calculation of scattering amplitudes for external fermions in the Green-Schwarz formulation is no more difficult than the calculation for external bosons, and the divergences that appear before summing over spin structures in the NSR formulation of the superstring are absent in the Green-Schwarz formulation.\(^1\)

Despite this motivation, almost all calculations of superstring scattering amplitudes have been performed using the Neveu-Schwarz-Ramond formulation. The reason is that after gauge-fixing the N=1 worldsheet super-reparameterization invariances of the NSR superstring, the covariant NSR action simplifies to a quadratic free-field action, allowing amplitudes to be calculated by evaluating free-field correlation functions on N=1 super-Riemann surfaces.\(^2,3,4\)

In the Green-Schwarz formulation, however, it has not been possible using the usual superspace variables to gauge-fix the action to a free-field action. In both the light-cone gauge\(^5,6,7\) and semi-light-cone gauge,\(^8,9,10\) the Green-Schwarz action requires non-trivial interaction terms whenever \(\partial_z x^+ = 0\) on the Riemann surface (these zeroes of \(\partial_z x^+ \equiv \partial_z x^0 + \partial_z x^9\) occur at \(2g + N - 2\) points for a \(g\)-loop \(N\)-string scattering amplitude).\(^\dagger\) Because the locations of these interaction points are complicated functions of the momenta of the external strings and of the modular parameters of the Riemann surface, only tree and one-loop scattering amplitudes involving four external massless states have been expressed

\(^\dagger\) In reference 11, it is shown that by adding a counterterm to the free semi-light-cone gauge-fixed Green-Schwarz action, both conformal and Lorentz invariance can be preserved in the effective action. However because their calculations are perturbative around non-zero backgrounds for \(\partial_z x^+\), they can not be used to prove Lorentz invariance near \(\partial_z x^+ = 0\). In fact, since the proposed counterterm vanishes in the light-cone gauge, it is clear that the semi-light-cone gauge-fixed action requires the same non-trivial interaction term as the light-cone gauge-fixed action in order to produce the correct scattering amplitudes.
as Koba-Nielsen-like formulas using these methods.\textsuperscript{1}† An additional problem caused by
the non-trivial interaction terms is that they must be defined in such a way that when two
or more interaction points approach each other, there are no short-distance singularities.
In practice, this requires introducing a contact-term interaction into the light-cone Green-
Schwarz action which further complicates the analysis of scattering amplitudes.\textsuperscript{13,14,7}

The difficulty in gauge-fixing the Green-Schwarz covariant action\textsuperscript{15} to a free-field
action came from the lack of a geometrical interpretation for the fermionic Siegel
symmetries\textsuperscript{16} of the Green-Schwarz superstring when expressed in terms of the usual super-
space variables. Unlike the N=1 superconformal invariance of the NSR superstring, there
was little understanding of the global moduli for these local Green-Schwarz symmetries.\textsuperscript{9}

In a recent paper,\textsuperscript{17} it was shown that by introducing new twistor-like
variables\textsuperscript{18} into the Lorentz-covariant Green-Schwarz heterotic superstring action,
two of the eight fermionic Siegel-transformations can be interpreted as world-
sheet super-reparameterizations,\textsuperscript{19,20} and after gauge-fixing the N=2 worldsheet super-
reparameterization invariance and the remaining six Siegel symmetries, the Green-Schwarz
action simplifies to a free-field action on an N=2 super-Riemann surface. This gauge-fixed
action retains only a manifest SU(4)xU(1) subgroup of the original SO(9,1) target-space
super-Poincaré invariance, however it is manifestly N=2 superconformally invariant on the
worldsheet. ‡

† Restuccia and Taylor were able to analyze properties of the multiloop Green-Schwarz
scattering amplitudes, but only in certain regions of moduli space.\textsuperscript{7} Also, Mandelstam has
proposed a Koba-Nielsen-like formula for the tree-level scattering of N massless states, but
he did not derive it from a functional integral approach.\textsuperscript{12}

‡ The idea of replacing Siegel-transformations with worldsheet super-reparameterizations
originated in the work of Sorokin, Tkach, Volkov, and Zheltukhin on the superparticle,\textsuperscript{21}
although a connection between N=2 worldsheet supersymmetry and spacetime supersym-
metry had already been found by other authors.\textsuperscript{22} In fact it was even conjectured that
the sum over spin structures in the NSR formalism of the superstring might be better
understood as the U(1) moduli of an N=2 surface.\textsuperscript{23}
Although the Lorentz-covariant N=2 twistor-string action is presently only known for the heterotic Green-Schwarz superstring, it is easy to generalize the gauge-fixed free-field action to non-heterotic versions of the Green-Schwarz superstring. In this paper, the free-field action for the Type IIB superstring is quantized using the usual BRST methods, and after bosonizing some of the matter and ghost fields, vertex operators are constructed for the physical massless supermultiplet. Scattering amplitudes with an arbitrary number of loops and external massless states are explicitly calculated by evaluating free-field correlation functions on an N=2 super-Riemann surface of the appropriate topology and integrating over the global super-moduli of the surface.

It is easily shown that these scattering amplitudes satisfy the non-renormalization theorem, that is, all loop amplitudes with less than four external massless states vanish. By choosing light-cone moduli for the super-Riemann surface, it can also be shown that these amplitudes agree with amplitudes obtained using the Green-Schwarz light-cone gauge formalism if one assumes a simple conjecture concerning the contribution of the contact-term interactions to the light-cone gauge amplitudes. A proof of this conjecture would therefore prove the unitarity of the twistor-string scattering amplitudes.

As in the NSR formalism, it is convenient to perform the integration over the anti-commuting moduli by introducing picture-changing operators. Although the matter content of these operators resembles the matter content of the light-cone interaction-point insertions, they differ in the fact that their location on the surface does not affect the scattering amplitudes (the “multiloop ambiguity” will be discussed later in the introduction). For this reason, amplitude calculations using the twistor-string formalism do not require any knowledge about the location of the string interaction points, and are therefore much simpler than calculations using the light-cone gauge or semi-light-cone gauge formalisms.

In addition, it is useful to introduce instanton-number-changing operators in order to evaluate correlation functions on N=2 surfaces of non-zero U(1) instanton number (for external states that transform in a given way under the U(1) subgroup of the target-space SU(4)\times U(1) invariance, only N=2 surfaces of a fixed instanton number contribute to the scattering amplitude). Like the picture-changing operators, the scattering amplitude is independent of the location of the instanton-number-changing operators.
Because the physical states of the Green-Schwarz string are manifestly spacetime supersymmetric, there is no need to sum over spin structures in order to project out unwanted states (in the twistor-string scattering amplitudes, all spin structures contribute equally). However the correlation functions do depend on the global $U(1)$ moduli of the $N=2$ surface,\textsuperscript{24,25} and because of the presence of bosonized matter fields with negative energies (these fields come from bosonizing bosons), these correlation functions contain unwanted poles for all but special discrete values of the $U(1)$ moduli. Fortunately, the twistor-string formalism restricts the region of integration for the global $U(1)$ moduli to coincide with these special values for which no unwanted poles occur.

It is well-known that bosonization of super-reparameterization ghosts also introduces fields with negative energies, and therefore, correlation functions with unwanted poles. Although the residues of these poles are total derivatives in moduli space,\textsuperscript{4} the presence of divergences in the integrands of the scattering amplitudes would force the introduction of cutoffs in the moduli space, possibly creating surface term contributions.\textsuperscript{26} In the NSR formalism, these cutoffs are necessary since before summing over spin structures, the scattering amplitudes are not spacetime supersymmetric and contain divergences.\textsuperscript{27} The fact that the scattering amplitude depends on the choice of the cutoff through the surface term contributions is known as the “multiloop ambiguity”. In the twistor-string formalism, however, there is no multiloop ambiguity since the amplitudes are manifestly spacetime supersymmetric. This fact is easily demonstrated since unlike the spacetime supersymmetry generators in the NSR formalism, the twistor-string spacetime supersymmetry generators are independent of the bosonized ghost fields, and therefore contain no unwanted poles.

Section II of this paper discusses quantization of the gauge-fixed $N=2$ twistor-string action in which the BRST charge is constructed, the $U(1)$-transforming fields are bosonized, and the two picture-changing operators, the instanton-number-changing operator, and the massless physical vertex operators are defined. Section III discusses the calculation of Green-Schwarz superstring scattering amplitudes by describing tree amplitudes, beltrami differentials, and multiloop correlation functions for the various matter and ghost fields. Section IV analyzes the scattering amplitudes, showing that all spin structures contribute
equally, that the non-renormalization theorem is satisfied, and that the scattering am-
plitudes agree with amplitudes obtained using the Green-Schwarz light-cone gauge for-
malism if one assumes a simple conjecture concerning the contribution of the light-cone
gauge contact-term interactions. Section V proposes applications for the results of this pa-
per and discusses possible approaches to Lorentz-covariantizing the scattering amplitudes.
The Appendix reviews the gauge-fixing procedure for the covariant N=(2,0) twistor-string action of the Green-Schwarz heterotic superstring.

II. Quantization of the N=2 Twistor-String

A. The Gauge-Fixed Free-Field Action

It was recently shown that by introducing twistor-like variables,\textsuperscript{16} the Lorentz-
covariant action for the ten-dimensional Green-Schwarz heterotic superstring can be de-

defined on an N=(2,0) super-worldsheet.\textsuperscript{19,20} These new variables allow two of the fermionic
Siegel-symmetries to be replaced with N=(2,0) super-reparameterizations, and after gauge-
fixing the super-reparameterizations and the remaining six Siegel-symmetries, the Lorentz-
covariant twistor-string action reduces to a free-field action with manifest target-space
SU(4)xU(1) super-Poincaré invariance and manifest worldsheet N=(2,0) superconformal
invariance (this gauge-fixing procedure is reviewed in the Appendix).\textsuperscript{17} Unfortunately, at
the present time there are no Lorentz-covariant twistor-string actions for the non-heterotic
Green-Schwarz superstring. Nevertheless, it is straightforward to generalize the gauge-
fixed free-field action of equation (A.10) to the Type IIB Green-Schwarz superstring by
extending the N=(2,0) super-worldsheet to an N=(2,2) super-worldsheet in the following
way:

\begin{equation}
S = \int dz d\bar{z} d\kappa^+ d\bar{\kappa}^- d\bar{\kappa}^- d\kappa^- [X^+ X^- l - \mathcal{W}^+ \Psi^- - \mathcal{W}^- \Psi^+ - \bar{\mathcal{W}}^+ \bar{\Psi}^- - \bar{\mathcal{W}}^- \bar{\Psi}^+] \quad (II.1)
\end{equation}

with the chirality constraints:

\begin{equation}
D_- X^+ l = \bar{D}_- X^+ l = D_+ X^- l = \bar{D}_+ X^- l = 0, \quad (II.2)
\end{equation}
\[ D_+ \Psi^+ = \tilde{D}_- \Psi^+ = D_+ \Psi^- = \tilde{D}_- \Psi^+ = D_- \Psi^+ = \tilde{D}_- \Psi^- = 0, \]
\[ \tilde{D}_- W^+ = \tilde{D}_+ W^- = D_- \tilde{W}^+ = D_+ \tilde{W}^- = 0, \]
the N=(2,2) super-Virasoro constraints:
\[ D_+ W^+ D_- \Psi^- - D_- W^- D_+ \Psi^+ + D_+ X^{+i} D_- X^{-l} = 0, \quad (II.3) \]
\[ \tilde{D}_+ W^+ \tilde{D}_- \Psi^- - \tilde{D}_- W^- \tilde{D}_+ \Psi^+ + \tilde{D}_+ X^{+i} \tilde{D}_- X^{-l} = 0; \]
and the non-local constraint:
\[ \Omega \equiv \int_C dz d\kappa^+ d\kappa^- |\bar{\kappa}^\pm = 0 \Psi^- \Psi^+ + \int_C \bar{dz} d\bar{\kappa}^+ d\bar{\kappa}^- |_{\bar{\kappa}^\pm = 0} \bar{\Psi}^- \bar{\Psi}^+ = 0 \quad (II.4) \]
where \( C \) is any closed curve on the two-dimensional surface (the non-local constraint, \( \Omega \), is the N=(2,2) version of equation (A.14), and imposes restrictions on the U(1) moduli of the surface since it implies that \( \int d\kappa^+ d\kappa^- |_{\kappa^\pm = 0} \Psi^- \Psi^+ \) is a holomorphic one-form with purely imaginary periods when integrated around a non-trivial loop). Note that the N=(2,2) super-worldsheet has been Wick-rotated to Euclidean space with coordinates \([z, \kappa^+, \kappa^-; \bar{z}, \bar{\kappa}^+, \bar{\kappa}^-]\) satisfying \( \bar{z} = z^*, \bar{\kappa}^+ = (\kappa^-)^*, \bar{\kappa}^- = (\kappa^+)^* \) (the action of equation (II.1) is real since after Wick-rotation, the convention \((\Phi_1 \Phi_2)^* = \Phi_1^* \Phi_2^*\) is used for both bosons and fermions); \( D_\pm \equiv \partial_{\kappa^\pm} + \frac{1}{2} \kappa^\pm \partial_z, \tilde{D}_\pm \equiv \partial_{\bar{\kappa}^\pm} + \frac{1}{2} \bar{\kappa}^\pm \partial_{\bar{z}} \) (this definition differs from that of the Appendix); and \( X^{+\bar{l}} = (X^{-l})^*, \bar{W}^\pm = (W^\mp)^*, \bar{\Psi}^\pm = (\Psi^\mp)^*. \) Under the 16 global spacetime-supersymmetry transformations that preserve the gauge-fixing,
\[ \delta X^{+\bar{l}} = \epsilon^{-\bar{l}} \Psi^+ + \epsilon^{-\bar{l}} \Psi^+, \delta X^{-l} = \epsilon^{+l} \Psi^- + \epsilon^{+l} \Psi^-, \quad (II.5) \]
\[ \delta W^+ = -\epsilon^{+l} X^{+\bar{l}}, \delta W^- = -\epsilon^{-\bar{l}} X^{-l}, \delta \bar{W}^+ = -\epsilon^{+l} X^{+\bar{l}}, \delta \bar{W}^- = -\epsilon^{-\bar{l}} X^{-l}, \]
and under the target-space SU(4)xU(1) rotations,
\[ [X^{+\bar{l}}, X^{-l}, \epsilon^{-\bar{l}}, \epsilon^{+l}, \epsilon^{-\bar{l}}, \epsilon^{+l}, W^\pm, \tilde{W}^\pm, \Psi^\pm, \tilde{\Psi}^\pm] \]
transforms like a \([4_+, \tilde{4}_-, 4_-, \tilde{4}_+], 4_+, \tilde{4}_-, 4_-, \tilde{4}_+, 1_{\pm 1}, 1_{\pm 1}, 1_{\pm 1}, 1_{\pm 1}]\) representation (note that the SO(8) anti-chiral spinor is chosen to break into a \([1_{+1}, 6_0, 1_{-1}]\) representation of SU(4)xU(1), rather than the usual choice\(^{5,6,7}\) of the SO(8) vector).
After placing the auxiliary fields on-shell, the free-field action of equation (II.1) takes the following component form:

\[ S = \int dzd\tilde{z}(\partial_{z}x^{l+i}\partial_{\tilde{z}}x^{-l} - \Gamma^{+\tilde{t}}\partial_{z}\Gamma^{-l} - \Gamma^{+\tilde{t}}\partial_{\tilde{z}}\Gamma^{-l}) \]

\[ -w^{-}\partial_{z}\lambda^{+} - \varepsilon^{-}\partial_{z}\psi^{+} - w^{+}\partial_{\tilde{z}}\lambda^{-} - \varepsilon^{+}\partial_{\tilde{z}}\bar{\psi}^{-} - \bar{w}^{-}\partial_{\tilde{z}}\bar{\lambda}^{-} - \varepsilon^{-}\partial_{\tilde{z}}\bar{\psi}^{-} \]

with the N=(2,2) super-Virasoro constraints:\n
\[ \partial_{z}x^{l+i}\partial_{\tilde{z}}x^{-l} - \frac{1}{2}(\Gamma^{+\tilde{t}}\partial_{z}\Gamma^{-l} + \Gamma^{+\tilde{t}}\partial_{\tilde{z}}\Gamma^{-l}) \]

\[ -\frac{1}{2}(w^{-}\partial_{z}\lambda^{+} - \lambda^{+}\partial_{z}w^{-}) - \varepsilon^{-}\partial_{z}\psi^{+} - \frac{1}{2}(w^{+}\partial_{\tilde{z}}\lambda^{-} - \lambda^{-}\partial_{\tilde{z}}w^{+}) - \varepsilon^{+}\partial_{\tilde{z}}\bar{\psi}^{-} = \]

\[ \partial_{z}x^{l+i}\Gamma^{-l} + \varepsilon^{+}\lambda^{-} - w^{-}\partial_{z}\psi^{+} = \partial_{\tilde{z}}x^{-l}\Gamma^{+\tilde{t}} + \varepsilon^{-}\lambda^{+} - w^{+}\partial_{\tilde{z}}\bar{\psi}^{-} = \Gamma^{+\tilde{t}}\Gamma^{-l} + w^{+}\lambda^{-} - w^{-}\lambda^{+} = 0, \]

\[ \partial_{z}x^{l+i}\Gamma^{-l} + \varepsilon^{+}\lambda^{-} - w^{-}\partial_{z}\psi^{+} = \partial_{\tilde{z}}x^{-l}\Gamma^{+\tilde{t}} + \varepsilon^{-}\lambda^{+} - w^{+}\partial_{\tilde{z}}\bar{\psi}^{-} = \Gamma^{+\tilde{t}}\Gamma^{-l} + w^{+}\lambda^{-} - w^{-}\lambda^{+} = 0, \]

and the non-local Ω constraint:

\[ \int_{C} dz(\lambda^{-}\lambda^{+} - \frac{1}{2}\psi^{-}\partial_{z}\psi^{+} - \frac{1}{2}\bar{\psi}^{+}\partial_{z}\bar{\psi}^{-}) + \int_{C} d\tilde{z}(\bar{\lambda}^{-}\bar{\lambda}^{+} - \frac{1}{2}\bar{\psi}^{-}\partial_{\tilde{z}}\bar{\psi}^{+} - \frac{1}{2}\bar{\psi}^{+}\partial_{\tilde{z}}\bar{\psi}^{-}) = 0, \]

where at \( \kappa^{\pm} = \bar{\kappa}^{\pm} = 0 \),

\[ X^{l+i} \equiv x^{l+i} + ix^{l+4}, \ D_{\mp}X^{+\tilde{t}} = \Gamma^{+\tilde{t}}, \ D_{\mp}X^{-\tilde{t}} = \Gamma^{-\tilde{t}}, \]

\[ X^{-l} \equiv x^{-l} - ix^{-l+4}, \ D_{\mp}X^{-\tilde{t}} = \Gamma^{-\tilde{t}}, \ D_{\mp}X^{+\tilde{t}} = \Gamma^{+\tilde{t}}, \]

\[ D_{\pm}w^{\pm} = w^{\pm}, \quad D_{\mp}D_{\pm}W^{\pm} = \varepsilon^{\pm}, \quad \Psi^{\pm} = \psi^{\pm}, \quad D_{\pm}\Psi^{\pm} = \lambda^{\pm}, \]

\[ D_{\pm}\bar{W}^{\pm} = \bar{w}^{\pm}, \quad D_{\mp}D_{\pm}\bar{W}^{\pm} = \bar{\varepsilon}^{\pm}, \quad \bar{\Psi}^{\pm} = \bar{\psi}^{\pm}, \quad D_{\mp}\bar{\Psi}^{\pm} = \bar{\lambda}^{\pm}. \]

For the rest of this paper, only the closed oriented chiral Green-Schwarz superstring (type IIB) will be discussed, although it should be straightforward to generalize the discussion to the open and heterotic types (the Type IIA closed string may present special problems since the two spacetime supersymmetries transform differently under the SU(4)\times U(1)
subgroup). To conserve space, most equations will be written only for the right-handed sector of the Type IIB superstring, and the corresponding equations for the left-handed sector can be obtained by complex conjugation.

B. Construction of the BRST Charge

The action for the N=(2,2) super-Virasoro ghosts, \([B, C]\) and \([\bar{B}, \bar{C}]\), is:

\[
S_{\text{ghost}} = \int dzd\bar{z}d\kappa^+ d\kappa^- d\bar{\kappa}^+ d\bar{\kappa}^- [BC + \bar{B}\bar{C}] \tag{II.10}
\]

with the chirality constraints, \(\bar{D}_- B = D_+ C = 0\), and the super-Virasoro constraints,

\[
D_+ BD_+ C + D_- BD_- C + \partial_z (BC) = 0.
\]

Placing the auxiliary fields on-shell, this action in component form is:

\[
S_{\text{ghost}} = -\int dzd\bar{z}(b\partial_z c + \beta^+ \partial_z \gamma^- + \beta^- \partial_z \gamma^+ + v\partial_z u + b\partial_z c + \bar{b}\partial_z \bar{c} + \bar{\beta}^+ \partial_z \bar{\gamma}^- + \bar{\beta}^- \partial_z \bar{\gamma}^+ + \bar{v}\partial_z \bar{u})
\]

with the super-Virasoro constraints:

\[
b\partial_z c + \beta^+ \partial_z \gamma^- + \beta^- \partial_z \gamma^+ + v\partial_z u + \partial_z (bc + \frac{1}{2}(\beta^+ \gamma^- + \beta^- \gamma^+)) = 0
\]

\[
(b + \frac{1}{2}\partial_z v)\gamma^+ - \beta^+ (u + \frac{1}{2}\partial_z c) - \partial_z (v\gamma^+ + \beta^+ c) = (b - \frac{1}{2}\partial_z v)\gamma^- + \beta^- (u - \frac{1}{2}\partial_z c) + \partial_z (v\gamma^- - \beta^- c) = \beta^+ \gamma^- - \beta^- \gamma^+ + \partial_z (cv) = 0,
\]

where at \(\kappa^\pm = 0\),

\[
B = v + \kappa^+ \beta^- - \kappa^- \beta^+ + \kappa^+ \kappa^- b, \quad C = c + \kappa^+ \gamma^- + \kappa^- \gamma^+ + \kappa^+ \kappa^- u. \tag{II.12}
\]

Using the N=(2,2) super-Virasoro constraints of equation (II.11), a BRST charge can be constructed in the following way:\[29\]

\[
Q = \int dzd\kappa^+ d\kappa^- |_{\kappa^\pm = 0} [C(D_+ X^+ D_- X^- l + D_+ W^+ D_- \Psi^- - D_- W^- D_+ \Psi^+) + C\partial_z CB - D_4 CD_- CB] \tag{II.13}
\]
\[ + \int d\bar{z} \bar{\kappa}^+ d\kappa^- |_{\kappa^\pm = 0} [\bar{C}(\bar{D}_+ X^{-l} \bar{D}_- X^l + \bar{D}_+ \bar{W}^+ \bar{D}_- \bar{\Psi} - \bar{D}_- \bar{W}^- \bar{D}_+ \bar{\Psi}^+) \]

\[ + C \partial_z CB - D_+ CD_+ CB]. \]

It is easy to check that \([Q, B]\) at \(\bar{\kappa}^\pm = 0\) is the sum of the matter and ghost super-stress-energy tensors of equations (II.7) and (II.11), that \([Q, \Omega]\) = 0 where \(\Omega\) is defined in equation (II.8), and that \(Q\) is nilpotent including normal-ordering effects, since the central charge contribution of the matter fields is \((4 \times 2) + (4 \times 1) - (2 \times 2) - (2 \times 1) = +6\), while the contribution of the N=2 ghost fields is \(-26 + (2 \times 11) - 2 = -6\).

Physical vertex operators, \(V\), can now be defined by the conditions \([Q, V] = [\Omega, V] = 0\) and \(V \neq [Q, B]\) for any \(B\).

C. Bosonization of the U(1) Current

In order to construct the physical vertex operator, it is useful to first bosonize matter and ghost fields that appear in the U(1) current,

\[ J_{U(1)} = \Gamma^{+\bar{I}} \Gamma^{-l} + w^+ \lambda^- - w^- \lambda^+ + \beta^+ \gamma^- - \beta^- \gamma^+ + \partial_z (cv). \quad (II.14) \]

As in the NSR formalism of the superstring, an unfortunate consequence of bosonization is that the worldsheet superfields must be broken into their individual components. The super-reparameterization ghosts, \(\beta^\pm\) and \(\gamma^\pm\), are bosonized in the following standard way:

\[ \beta^+ = e^{-\phi^-} \partial_z \xi^+, \beta^- = e^{-\phi^+} \partial_z \xi^-, \quad (II.15) \]

\[ \gamma^+ = e^{\phi^+} \eta^+, \gamma^- = e^{\phi^-} \eta^- \]

where all expressions are normal-ordered, and as \(y \to z\), \(\partial_y \phi^-(y) \partial_z \phi^-(z)\) and \(\partial_y \phi^+(y) \partial_z \phi^+(z)\) \(\to -(y - z)^{-2}\) and therefore have negative energies, \(\eta^-(y) \partial_z \xi^+(z) \to (y - z)^{-2}\), \(\eta^+(y) \partial_z \xi^-(z) \to (y - z)^{-2}\), and all other operator products are non-singular.

The \(\Gamma^{+\bar{I}}\) and \(\Gamma^{-l}\) matter fields are also bosonized in the standard way\(^3\) as:

\[ \Gamma^{+\bar{I}} = e^{\sigma_{\bar{l}}}, \Gamma^{-l} = e^{-\sigma_l} \quad (II.16) \]

where as \(y \to z\), \(\partial_y \sigma_l(y) \partial_z \sigma_m(z) \to \delta_{l,m}(y - z)^{-2}\).
Finally, the bosonization of the $\lambda^\pm$ and $w^\pm$ matter fields is less straightforward, but the following formulas can be shown to have the correct operator-product expansions:

$$
\lambda^+ = (\partial_z x^+ + \frac{1}{2}\psi^+ \partial_z \psi^- + \frac{1}{2}\psi^- \partial_z \psi^+)e^{h^+} + e^{-h^-}, \quad \lambda^- = e^{-h^+}
$$

$$
w^+ = e^{h^+}(\partial_z h^+ + \partial_z h^- + x^-(\partial_z x^+ + \frac{1}{2}\psi^+ \partial_z \psi^- + \frac{1}{2}\psi^- \partial_z \psi^+)) + x^- e^{-h^-}, \quad w^- = x^- e^{-h^+},
$$

where as $y \to z$, $\partial_y x^+(y) \partial_z x^-(z) \to (y-z)^{-2}$, $\partial_y h^+(y) \partial_z h^-(z) \to (y-z)^{-2}$, and the $+$ and $-$ indices of $x^+$ and $x^-$ refer to the target-space light-cone indices $x^9 \pm 0$ (the SO(9,1) metric is $[-+++-++-++]$). Since $\partial_y h^1(y) \partial_z h^1(z) \to (y-z)^{-2}$ and $\partial_y h^2(y) \partial_z h^2(z) \to -(y-z)^{-2}$ where

$$
h^+ \equiv \frac{1}{\sqrt{2}}(h^1 + h^2) \quad \text{and} \quad h^- \equiv \frac{1}{\sqrt{2}}(h^1 - h^2),
$$

$h^1$ and $h^2$ describe two chiral bosons that take values on a circle of radius $\sqrt{2}$, one with positive energy and the other with negative energy. Note that by shifting the scalar fields $h^+$ and $h^-$ by a constant, the relative coefficients of the two terms in $\lambda^+$ and $w^+$ can be changed without affecting the operator-product expansions.

In order to give the correct conformal weights for the un bosonized fields, the bosonized scalar fields $[\phi^\pm, \sigma_l, h^1, h^2, x^\pm]$ must have screening charges $q = [+2, 0, +\sqrt{2}, 0, 0]$ and the $[\xi^\pm, \eta^\pm]$ fields must have conformal weight $[0,1]$. It is easy to check using the formula $c = 1 \mp 3q^2$ that the total contribution to the central charge of the un bosonized fields is equal to the total contribution of the bosonized fields.

By defining $x^+$ and $x^-$ to be real quantities (i.e., the same $x^+$ and $x^-$ appear in the bosonizations of $[\lambda^\pm, w^\pm]$ and $[\vec{\lambda}^\pm, \vec{w}^\pm]$), this bosonization would seem to guarantee that the $\Omega$ constraint of equation (II.8) is satisfied (note that $e^{-h^-}(y)e^{-h^+}(z) \to 0$ as $y \to z$). However, it was shown in reference 17 that the $\Omega$ constraint restricts the global U(1) moduli of the N=2 surface since only for certain special values of the U(1) moduli is it possible to find holomorphic fields, $\lambda^+$ and $\lambda^-$, such that the real part of $\int_C dz(\lambda^- \lambda^+ - \frac{1}{2}\psi^- \partial_z \psi^+ - \frac{1}{2}\psi^+ \partial_z \psi^-)$ vanishes around all non-contractible loops. As will be shown in Section III.C., the bosonization prescription of equation (II.17) also imposes a restriction on the U(1)
moduli since correlation functions of the $e^{\pm h^+}$ and $e^{\pm h^-}$ fields are only well-defined (i.e., do not have unwanted poles) for special values of the U(1) moduli.

Because $\varepsilon^\pm$ and $\bar{\varepsilon}^\pm$ must commute with $\lambda^\pm$ and $\bar{\lambda}^\pm$, they should commute with $\partial_z x^+ + \frac{1}{2} \psi^+ \partial_z \psi^- + \frac{1}{2} \psi^- \partial_z \psi^+$ and $\partial_z x^+ + \frac{1}{2} \bar{\psi}^+ \partial_z \bar{\psi}^- + \frac{1}{2} \bar{\psi}^- \partial_z \bar{\psi}^+$, but not with $x^+$. It is therefore convenient to define new fields,

$$\hat{\varepsilon}^\pm \equiv \varepsilon^\pm - \frac{1}{2} \partial_z x^- \psi^+ - x^- \partial_z \psi^+, \quad \hat{\bar{\varepsilon}}^\pm \equiv \bar{\varepsilon}^\pm - \frac{1}{2} \partial_z x^- \bar{\psi}^+ - x^- \partial_z \bar{\psi}^+,$$

(II.19)

which no longer commute with $\lambda^\pm$ and $\bar{\lambda}^\pm$, but which do commute with $x^+$. When expressed in terms of the bosonized fields and $\hat{\varepsilon}^\pm$, it is easy to check that the BRST charge is invariant under constant shifts of $x^+$ and $x^-$.  

D. Picture-Changing Operators

As in the NSR formalism for the superstring, it is useful to define operators that change the ghost number (or picture) of a physical vertex operator (right and left-handed ghost number is defined as $\int dz (cb + uv + \partial_z \phi^+ - \partial_z \phi^-)$ and $\int d\bar{z} (\bar{c}b + \bar{u}v + \partial_z \bar{\phi}^+ - \partial_z \bar{\phi}^-)).$ These operators are constructed in the usual way by commuting the BRST charge, Q, with the $\xi^\pm$ fields that appear in the bosonized $[\beta^\pm, \gamma^\pm]$ system of equation (II.15):

$$Z^+ \equiv [Q, \xi^+] = e^{\phi^-} [\partial_z x^- i \Gamma^+ i + \varepsilon^- \lambda^+ - w^+ \partial_z \psi^- + (b - \frac{1}{2} \partial_z v) \gamma^+ - v \partial_z \gamma^+ + c \xi^+],$$

(II.20)

$$Z^- \equiv [Q, \xi^-] = e^{\phi^+} [\partial_z x^+ i \Gamma^- i + \varepsilon^+ \lambda^- - w^- \partial_z \psi^+ + (b + \frac{1}{2} \partial_z v) \gamma^- + v \partial_z \gamma^- + c \xi^-].$$

Like the N=1 case, $\partial_z Z^\pm$ is BRST-trivial so changing the location of the picture-changing operators changes the integrand of the scattering amplitude by a total derivative in the moduli space. As was already mentioned in the introduction, these total derivatives are harmless for the twistor-string because the integrands are manifestly spacetime supersymmetric and therefore contain no divergences. Note that unlike the N=1 case, there are no inverse picture-changing operators in the N=2 cohomology since there are no terms in $Z^\pm$ that are proportional to $e^{2\phi^\mp}$.  

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E. Instanton-Number-Changing Operators

After bosonizing the fields that transform under the worldsheet U(1)-transformations, the U(1) current of equation (II.14) can be written as:

$$J_{U(1)} = \partial_z \left( \sum_{l=1}^{4} \sigma_l - h^+ + h^- - \phi^+ + \phi^- + cv \right). \quad (II.21)$$

Although \( \int dz J_{U(1)} \) is not a well-defined operator since it is only defined up to \( 2\pi \), the operator

$$I^n \equiv \exp \left[ n \left( \sum_{l=1}^{4} \sigma_l - h^+ + h^- - \phi^+ + \phi^- + cv \right) \right] \quad (II.22)$$

is a well-defined operator when \( n \) is an integer, and \( I \) will be called the right-handed instanton-number-changing operator. Since \( J_{U(1)} = [Q,v], \partial_z I^n = n J_{U(1)} I^n = [Q,nvI^n] \), and therefore, \( \partial_z I^n \) is BRST-trivial. So \( I^n \) shares the property of the picture-changing operators that it is in the BRST cohomology, but amplitudes do not depend on its location.

It is easy to check that evaluating a correlation function on an N=2 surface with instanton number \([n,\bar{n}]\) (i.e., \( n = \int dz d\bar{z} \partial_z A_z \) and \( \bar{n} = \int dz d\bar{z} \partial_{\bar{z}} A_{\bar{z}} \), where \( A_z \) and \( A_{\bar{z}} \) are the two components of the U(1) gauge field) is equivalent to evaluating the correlation function on an N=2 surface with vanishing instanton number, but with an insertion of the operator \( I^n \bar{I}^\bar{n} \).

By adding \( I^{n(g-1)} \bar{I}^{\bar{n}(g-1)} \) to the background charge, it is easily seen that “twisting” the conformal weights of the fields by redefining the Virasoro generators \( L \rightarrow L + \frac{m}{2} \partial_z J_{U(1)} \) and \( \bar{L} \rightarrow \bar{L} + \frac{n}{2} \partial_{\bar{z}} J_{U(1)} \), is equivalent to integrating over a genus g surface with its instanton number shifted by \([m(g-1),n(g-1)]\).

F. Massless Physical Vertex Operators

The massless supermultiplet for the closed oriented chiral superstring consists of 256 physical states, half fermionic and half bosonic. Using SU(4)xU(1) super-Poincaré invariant notation, this supermultiplet,

$$G(x^\mu, \theta^{-I}, \bar{\theta}^{-\bar{I}}) = g^{-I} (x^\mu) + \theta^{-I} g^{-I,-}(x^\mu) + \bar{\theta}^{-\bar{I}} g^{-\bar{I},-}(x^\mu) + ..., \quad (II.23)$$
can be expressed as a function of the ten spacetime bosonic coordinates and 8 of the 32 spacetime fermionic coordinates, where \( \partial_{x^\mu} \partial_{x^\nu} G = 0 \). Half of the 32 spacetime supersymmetry transformations are realized linearly on \( G \) by

\[
S^{+l} G = \partial_{\theta^{-l}} G, \quad S^{-\bar{l}} G = -i\theta^{-\bar{l}} \partial_{x^+} G, \quad S^{+\bar{l}} G = \partial_{\bar{\theta}^{-\bar{l}}} G, \quad S^{-l} G = -i\bar{\theta}^{-l} \partial_{x^+} G. \tag{II.24}
\]

It is convenient to choose to break the target-space SO(9,1) Lorentz invariance down to SU(4)xU(1) in such a way that the SO(8) vector breaks into \( 4_+ \) and \( 4_- \) representations of SU(4)xU(1), the chiral SO(8) spinor breaks into \( 4_+ \) and \( 4_- \) representations of SU(4)xU(1), and the anti-chiral SO(8) spinor breaks into \( 1_+ \), \( 6_0 \), and \( 1_- \) representations of SU(4)xU(1). Although this choice of breaking SO(9,1) down to SU(4)xU(1) is not the usual one\(^5,6,7\) in which the SO(8) vector breaks into \( 1_+ \), \( 6_0 \), and \( 1_- \) representations of SU(4)xU(1), it is related to the usual choice by SO(8) triality.

With this unconventional choice, the \( \theta^{-\bar{l}} = \bar{\theta}^{-l} = 0 \) component of \( G \), \( g^{-\bar{l}}(x^\mu) \), is one component of a direct product of two anti-chiral SO(8) spinors, rather than one component of a direct product of two SO(8) vectors. The vertex operator for this state with momentum \( p^\mu \) and ghost number \((-2,-2)\) is:

\[
V_{-, -} = \left| (p^+)^{-2} c \psi^+ \psi^- \exp(-h^+ - \phi^+ - 2\phi^-) \right|^2 \exp(ip^{+\bar{l}} x^{-\bar{l}} + ip^{-l} x^{+l} + ip^- x^+ + ip^+ x^-) \tag{II.25}
\]

where \( p^\mu p_\mu = 0 \) and \( p^+ \) is assumed to be non-zero.

It is straightforward to check that this state is in the BRST cohomology and that if any of the \( \psi \)'s are removed, the state becomes BRST-trivial since

\[
c \psi^+ \exp(-h^+ - \phi^+ - 2\phi^- + ip^+ x^-) = [Q, \partial_\xi^- \psi^- c \psi^+ \exp(-2\phi^+ - 2\phi^- + ip^+ x^-)],
\]

\[
c \psi^- \exp(-h^+ - \phi^+ - 2\phi^- + ip^+ x^-) = [Q, (p^+)^{-1} \partial_\xi^+ \psi^+ c \psi^- \exp(-2h^+ - \phi^+ - 3\phi^- + ip^+ x^-)].
\]

By attaching an arbitrary number of picture-changing operators to \( V_{-, -} \), other “pictures” for the vertex operator can be constructed. For example, one picture for the vertex operator of \( g^{-\bar{l}}(x^\mu) \), with ghost number \((-1,-1)\) is:

\[
Z^+ \bar{Z}^+ V_{-, -} = \left| (p^+)^{-1} c \psi^- \exp(-\phi^+ - \phi^-) \right|^2 \exp(ip^{+\bar{l}} x^{-\bar{l}} + ip^{-l} x^{+l} + ip^- x^+ + ip^+ x^-). \tag{II.26}
\]

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Pictures for the vertex operator of $g^{-\bar{-}}$ with lower ghost number than $(-2,-2)$ can be obtained by starting from

$$W_{-,-} = \left(p^+\right)^{-M-1}c \prod_{m=0}^{M-1} \partial_x^m \psi^- \partial_{\bar{x}}^m \psi^+ \exp(-h^+ - M\phi^+ - (M + 1)\phi^-) \right)^2 \exp(ip^\mu x_\mu),$$

(II.27)

which has ghost number $(-2M, -2M)$ and satisfies $V_{-,-} = |Z^+Z^-|^{2(M-1)}W_{-,-}$.

The easiest way to obtain the vertex operators for the other states in the massless supermultiplet, $G$, is to first construct the 16 spacetime supersymmetry generators of equation (II.24) that act linearly on G. These generators are constructed out of the twistor-string matter fields as follows:

$$S^{+l} = \int dz (\Gamma^{-l} \lambda^+ - \partial_x^{-l} \psi^+)$$

$$= \int dz (\partial_x^+ \psi^- + \frac{1}{2} \psi^- \partial_x^+ \psi^- + \frac{1}{2} \bar{\psi}^- \partial_{\bar{x}}^+ \psi^-) \exp(-\sigma_l + h^+) + \exp(-\sigma_l - h^-) - \partial_x^{-l} \psi^+,$n

$$S^{-\bar{l}} = \int d\bar{z} (\Gamma^{+\bar{l}} \bar{\lambda}^- - \partial_{\bar{x}}^{+\bar{l}} \bar{\psi}^-) = \int d\bar{z} (\exp(\sigma_l - h^+) - \partial_{\bar{x}}^{+\bar{l}} \bar{\psi}^-),$$

$$\bar{S}^{+l} = \int d\bar{z} (\bar{\Gamma}^{-l} \bar{\lambda}^+ - \partial_{\bar{x}}^{-l} \bar{\psi}^+)$$

$$= \int d\bar{z} (\partial_{\bar{x}}^+ \bar{\psi}^- + \frac{1}{2} \bar{\psi}^- \partial_{\bar{x}}^+ \bar{\psi}^- + \frac{1}{2} \bar{\psi}^- \partial_{\bar{x}}^+ \bar{\psi}^-) \exp(-\bar{\sigma}_l + \bar{h}^+) + \exp(-\bar{\sigma}_l - \bar{h}^-) - \partial_{\bar{x}}^{-l} \bar{\psi}^+,$n

$$\bar{S}^{-\bar{l}} = \int d\bar{z} (\bar{\Gamma}^{+\bar{l}} \bar{\lambda}^- - \partial_{\bar{x}}^{+\bar{l}} \bar{\psi}^-) = \int d\bar{z} (\exp(\bar{\sigma}_l - \bar{h}^+) - \partial_{\bar{x}}^{+\bar{l}} \bar{\psi}^-).$$

Note that these generators commute with the BRST charge and have the anti-commutation relations, $\{S^{-\bar{l}}, S^{+m}\} = 2\delta_{l,m} \int dz \partial_x^+ x^+, \{S^{-\bar{l}}, S^{+m}\} = 2\delta_{l,m} \int d\bar{z} \partial_{\bar{x}}^+ x^+.$

Unlike the NSR case, these spacetime supersymmetry generators contain no ghost fields and therefore do not have unwanted poles coming from correlation functions of the $\phi^\pm$ fields. This lack of unwanted poles for $S$ and $\bar{S}$ means that spacetime supersymmetry is manifest and there is no multiloop ambiguity in the scattering amplitudes.

The vertex operator for the other states in $G$ can now be constructed by commuting various combinations of $S^{+l}$ and $S^{+l}$ with $V_{-,-}$ (note that $g^{-\bar{-}}(x^\mu)$ is the lowest component of the superfield $G$, so $[S^{-\bar{l}}, V_{-,-}] = [\bar{S}^{-\bar{l}}, V_{-,-}] = 0$). Since $S^{+l}$ and $\bar{S}^{+l}$ commute with $Z^\pm$ and $\bar{Z}^\pm$, this construction can be carried out in any picture of $V_{-,-}$.
The vertex operator for the supermultiplet $G(x^\mu, \theta^-\bar{I}, \bar{\theta}^-\bar{I})$ with momentum $p^\mu$ ($p^\mu p_\mu = 0$ and $p^+ \neq 0$) is therefore:

$$V_{G(x^\mu, \theta^-\bar{I}, \bar{\theta}^-\bar{I})} = \exp(\theta^-\bar{I}S^+\bar{I} + \bar{\theta}^-\bar{I}\bar{S}^+\bar{I})V_{-, -} \quad (\text{II.29})$$

where the contours of $S^+\bar{I}$ and $\bar{S}^+\bar{I}$ surround the vertex operator $V_{-, -}$. The overall factor of $(p^+)^{-4}$ in $V_{-, -}$ can easily be checked by calculating the vertex operator for the $(l, -m)$ component of the graviton state in the ghost-number $(1, 1)$ picture and setting all fermion fields to zero:

$$V_{(l, -m)} = S^+\bar{S}^+m \left| (Z^+)\bar{Z} \right|^2 \left| V_{-, -} \right| = \quad (\text{II.30})$$

$$c\bar{c}(\partial_z x^- l - \frac{p_-^l}{p^+} \partial_z x^+) (\partial_z x^- m - \frac{p_-^m}{p^+} \partial_z x^+) \exp(ip^\mu x_\mu).$$

The corresponding vertex operators for the other components of the graviton when all fermion fields are set to zero can be obtained by using the appropriate combinations of picture-changing and instanton-number-changing operators, e.g.,

$$V_{(\bar{l}, -m)} = (p^+)^{-1} \epsilon_{ijkl} S^+ i S^+ j S^+ k \bar{S}^+ m I(\bar{Z}^-)^2 Z^+ I(\bar{Z}^+)^2 V_{-, -} = \quad (\text{II.31})$$

$$c\bar{c}(\partial_z x^+ \bar{l} - \frac{p^+ \bar{l}}{p^+} \partial_z x^+) (\partial_z x^- m - \frac{p_-^m}{p^+} \partial_z x^+) \exp(ip^\mu x_\mu).$$

### III. Calculation of Green-Schwarz Scattering Amplitudes

#### A. Tree Amplitudes

Tree-level scattering amplitudes for N massless states of the Green-Schwarz superstring are calculated by evaluating free-field correlation functions on the sphere of the operators $V_G, Z^\pm, \bar{Z}^\pm, I,$ and $\bar{I}$. The locations of $N - 3$ of the vertex operators are integrated over the sphere, whereas the locations of the other operators are arbitrary.

The number of $Z^\pm$ and $I$ operators that need to be inserted on the sphere can be determined from the rule that

$$< \left| \exp(-2\phi^+ - 2\phi^- - \sqrt{2}h^1)\psi^- \psi^+ c\partial_z c \partial_z c \right|^2 >_{\text{sphere}} = 1. \quad (\text{III.1})$$
Note that the background charges for the fields $\xi^\pm$ and $u$ do not appear in the normalization rule. This does not violate BRST invariance since after bosonization, the zero modes of these fields do not appear in the BRST charge. Since $V_G \sim |\exp(-\phi^+ - 2\phi^-)|^2$, $Z^\pm \sim \exp(\phi^\mp)$, and $I \sim \exp(\phi^- - \phi^+)$,

$$n_{Z^+} = 2N - 2 - n_I, \quad n_{Z^-} = N - 2 + n_I. \quad (III.2)$$

A final relation for $n_I$ can be derived from the fact that the charge

$$K \equiv \int dz( : \bar{\epsilon}^+ \psi^- - \bar{\epsilon}^- \psi^+ + w^+ \chi^- - w^- \chi^+ : ) = \int dz( : \bar{\epsilon}^+ \psi^- - \bar{\epsilon}^- \psi^+ : + \partial_z h^- - \partial_z h^+) \quad (III.3)$$

commutes with the BRST charge and has the following commutation relations with the other operators:

$$[K, V_G] = (\theta^{-\bar{I}} \partial_{\theta^{-\bar{I}}} - 1) V_G, \quad [K, Z^\pm] = 0, \quad [K, I] = -2I. \quad (III.4)$$

The fact that $K$ commutes with the background charge implies that for the component of the scattering amplitude with $Y \ \theta^{-\bar{I}}$'s ($0 \leq Y \leq 4N$),

$$n_I = \frac{1}{2}(Y - N), \quad (III.5)$$

where the different $\theta^{-\bar{I}}$ components of the amplitude correspond to the scattering of different components of $G$. Since $Y - N$ is even by fermion number conservation, this equation always has a solution with integer-valued $n_I$.

Using equations (III.2) and (III.5), one finds

$$n_{Z^+} = \frac{1}{2}(5N - Y - 4), \quad n_{Z^-} = \frac{1}{2}(N + Y - 2), \quad n_I = \frac{1}{2}(Y - N). \quad (III.6)$$

Since $Z^\pm$ are the only operators with $\bar{\epsilon}^\pm$, there must be at least $N - 1$ of each of these picture-changing operators in order to cancel all but one of the $\psi^+ \psi^-$ terms that come from the $V_G$'s (one of the terms must remain to provide the background charge). From equation (III.2) and the complex conjugate equation, this implies that $1 \leq n_I \leq N - 1$ and $1 \leq \bar{n}_I \leq N - 1$, and therefore, the only non-zero components in the tree-level scattering
amplitude have between \((2 + N)\) and \((3N - 2)\) \(\theta^{-\tilde{l}}\)'s and between \((2 + N)\) and \((3N - 2)\) \(\bar{\theta}^{-\tilde{l}}\)'s.

So the tree-level scattering amplitude for \(N\) massless states is:

\[
A_{\text{sphere}} = \left(\text{III.7}\right)
\]

\[
\prod_{r=4}^{N} \int dz_r d\bar{z}_r < \hat{V}_{G,r}(z_r, \bar{z}_r) \prod_{s=1}^{3} V_{G,s}(z_s, \bar{z}_s) \left| \sum_{n=1}^{N-1} I^n (Z^+)^{2N-2-n} (Z^-)^{N-2+n} \right|^2 >_{\text{sphere}}
\]

where \(V_{G,r} \equiv c\bar{c} \hat{V}_{G,r}\), and \(A_{\text{sphere}}\) is a function of \(p^\mu_r, \theta^{-\tilde{l}}_r, \) and \(\bar{\theta}^{-\tilde{l}}_r\) for \(\mu = 0\) to 9, \(l = 1\) to 4, and \(r = 1\) to \(N\). Note that since terms with a different number of instanton-number-changing operators only contribute to components of \(A_{\text{sphere}}\) with a different number of \(\theta^{-\tilde{l}}\)'s, the instanton contribution to the action of \(e^{i\vartheta n_l}\) (\(\vartheta\) is the instanton “theta-parameter”) can be cancelled by shifting \(\theta^{-\tilde{l}} \rightarrow e^{-\frac{1}{2}i\vartheta} \theta^{-\tilde{l}}\), or equivalently, by rotating by \(e^{i\vartheta}\) the U(1) subgroup of the manifest SU(4)xU(1) target-space invariance.\(^{30}\)

After expressing the operators in equation (III.7) in terms of the bosonized fields, the free-field correlation functions are easy to evaluate using the following formula\(^3\) for chiral bosons, \(\phi\), of screening charge \(q\):

\[
< \prod_{i=1}^{n} \exp(c_i \phi(z_i)) >_{\text{sphere}} = \delta_{-q, \Sigma c_i} \prod_{i<j}(z_i - z_j)^{\pm c_i c_j} \quad \text{(III.8)}
\]

where the + sign is taking for positive-energy chiral bosons and the - sign is taken for negative-energy chiral bosons. The only exception to this formula is for the screening charge of the \((\eta^\pm, \xi^\pm)\) and \((u, v)\) fields, which is taken to be zero.

For example, the tree-level scattering amplitude for three massless states is calculated as follows:

\[
A_{\text{sphere}} =< \prod_{s=1}^{3} V_{G,s}(z_s, \bar{z}_s) \left| I(Z^+)\right|^3 (Z^-)^2 + I^2(Z^+)\right|^3 \geq_{\text{sphere}} \quad \text{(III.9)}
\]

\[
= \prod_{\mu=0}^{9} \left( \sum_{s=1}^{3} p^\mu_s \right) \prod_{l=1}^{4} \left( \sum_{s=1}^{3} p^\mu_s \theta^{-\tilde{l}}_s \right) \left| (p_1^+ p_2^+ p_3^+)^{-1} P^{-a} \Theta^{-\bar{a}} + \epsilon^{abcd} P^\mu{\bar{a}} \Theta^{-\bar{b}} \Theta^{-\bar{c}} \Theta^{-\bar{d}} \right|^2
\]

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where \( P^\mu \equiv p_1^+ p_2^- - p_2^+ p_1^- \) and \( \Theta^{-\bar{a}} \equiv (p_3^+)^{-1}(\theta_1^{-\bar{a}} - \theta_2^{-\bar{a}}) \). Note that \( P^\mu \) and \( \Theta^{-\bar{a}} \), when multiplied by the delta-functions, are invariant up to a sign when the labels of the strings are interchanged.\(^{12}\)

**B.N=2 Super-Beltrami Differentials**

Because N=2 super-Riemann surfaces of non-zero genus are not all conformally equivalent, beltrami differentials need to be introduced to distinguish the different surfaces.\(^{24}\) For a surface of genus \( g \) with \( N \) punctures and instanton number \([n, \bar{n}]\), the complex beltrami differentials, \( M_i^j \), describe shifts in the bosonic Teichmuller parameters, \( m_i^T \) for \( i = 1 \) to \( 3g - 3 + N \), the complex differentials, \( M_j^{U(1)} \), describe shifts in the bosonic U(1) moduli, \( m_j^{U(1)} \) for \( j = 1 \) to \( g \), the complex differentials, \( M_k^+ \), describe shifts in the fermionic supermoduli, \( m_k^+ \) for \( k = 1 \) to \( 2g - 2 + N - n \), and the complex differentials, \( M_l^- \), describe shifts in the fermionic supermoduli, \( m_l^- \) for \( l = 1 \) to \( 2g - 2 + N + n \). The contribution of the instanton number, \( n \), to the relative numbers of fermionic moduli can be understood from the fact that shifting the conformal weights of the fields by \( L \rightarrow L + \frac{n}{2(g-1)} \theta_2 J_{U(1)} \) is equivalent to shifting the instanton number of the surface by \( n \).

Using the notation of reference 4, the g-loop scattering amplitude for \( N \) massless states is:

\[
A_g = \sum_{n=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} \prod_{i=1}^{3g-3+N} dm_i^T \prod_{j=1}^{g} dm_j^{U(1)} \prod_{k=1}^{2g-2+N-n} dm_k^+ \prod_{l=1}^{2g-2+N+n} dm_l^- \left| \int d\rho_{a_j} d\rho_{b_j} \int_{R} DX^+ DX^- D\Psi^+ D\Psi^- DBDC \right|^2
\]

\[
\left| \delta(<M_i^j|b>)\delta(<M_j^{U(1)}|v>)\delta(<M_k^+|\beta^+>)\delta(<M_l^-|\beta^->) \right|^2
\]

\[
\exp(\rho_{a_j} \Omega_{a_j} + \rho_{b_j} \Omega_{b_j}) \exp(S_{\text{matter}} + S_{\text{ghost}}) \prod_{r=1}^{N} V_{G,r}(z_r, \bar{z}_r) |Z^+(z_r)|^2,
\]

where \( R \) is an N=2 super-Riemann surface of genus \( g \) and instanton number \([n, \bar{n}]\), \( V_{G,r}|Z^+|^2 \) is in the picture with ghost number \((-1, -1)\), \( \rho_{a_j} \) and \( \rho_{b_j} \) are Lagrange multipliers for the \( \Omega \) constraint of equation (II.8) (for \( \Omega_{a_j} \), the loop \( C \) goes around the \( a_j \)-cycle,
while for $\Omega_{b_j}$, the loop goes around the $b_j$-cycle, $S_{\text{matter}}$ and $S_{\text{ghost}}$ are defined in equations (II.1) and (II.10), and the $\delta(<M|B>)$ terms come from the $N=2$ superconformal gauge-fixing.

As in the NSR formalism, the integration over the anti-commuting moduli, $m^+_k$ and $m^-_l$, can be easily performed if one chooses the fermionic beltrami differentials, $M^+_k$ and $M^-_l$, to have the form:

$$M^+_k(z) = \partial \overline{z} \left( \frac{1}{z - w^+_k} \right) = \delta(z - w^+_k), \quad M^-_l(z) = \partial \overline{z} \left( \frac{1}{z - w^-_l} \right) = \delta(z - w^-_l) \quad (III.11)$$

where $w^+_k$ and $w^-_l$ are independent of the $N=2$ super-moduli. Since the only dependence on $m^+_k$ and $m^-_l$ comes from the action,

$$S = S|_{m^\pm = \bar{m}^\pm = 0} + m^+_k < M^+|[Q, \beta^+] > + \bar{m}^+_k < M^+|[Q, \bar{\beta}^+] > \quad (III.12)$$

$$+ m^-_l < M^-|[Q, \beta^-] > + \bar{m}^-_l < M^-|[Q, \bar{\beta}^-] >,$$

integration over $m^\pm$ and $\bar{m}^\pm$, when combined with the $|\delta(<M|\beta^\pm >)|^2$ factors, introduces the picture-changing operator insertions,

$$\left| \prod_{k=1}^{2g-2+N-n} \delta(\beta^+(w^+_k)) [Q, \beta^+(w^+_k)] \prod_{l=1}^{2g-2+N+n} \delta(\beta^-(w^-_l)) [Q, \beta^-(w^-_l)] \right|^2$$

$$= \left| \prod_{k=1}^{2g-2+N-n} Z^+(w^+_k) \prod_{l=1}^{2g-2+N+n} Z^-(w^-_l) \right|^2. \quad (III.13)$$

It is convenient to choose $N$ of the Teichmuller parameters to be the locations, $z_r$, of the vertex operators. This implies that the beltrami differentials, $M^i_T$ for $i = 3g - 2$ to $3g - 3 + N$, are $\partial z_H(\epsilon - |z - z_r|)$ where $H$ is the Heavyside step-function and $\epsilon$ is small.

With this choice, the effect of the $\left| \prod_{i=3g-2}^{3g-N} dm^-_i \delta(<M^-_i|b>) \right|^2$ term is to replace the $N$ vertex operators, $V_{G,r}(z_r, \bar{z}_r)$, with $\int dz_r d\bar{z}_r \tilde{V}_{G,r}(z_r, \bar{z}_r)$ where $V_{G,r} \equiv c\bar{c} \tilde{V}_{G,r}$.

For the $g$ U(1) moduli, it is convenient to choose

$$m^U(1)_j = \frac{1}{2\pi} \int_{b_j} dz A_z, \quad \bar{m}^U(1)_j = \frac{1}{2\pi} \int_{b_j} d\bar{z} A_{\bar{z}}, \quad (III.14)$$
where $\int_{a_j} dz A_z$ and $\int_{a_j} d\bar{z} A_{\bar{z}}$ have been gauge-fixed to zero using the N=2 superconformal transformations ($A_z$ and $A_{\bar{z}}$ are the two components of the U(1) gauge field and $a_j, b_j$ are the 2g non-trivial loops of the surface with intersection properties $a_i \cap b_j = \delta_{i,j}$). The corresponding beltrami differentials are

$$M^U_j(z) = \partial \bar{z} \int_{a_j} dy_j \left( \frac{1}{z - y_j} \right), \quad (III.15)$$

so $<M^U_j|v> = \int_{a_j} dy_j v(y_j)$.

As in the tree-level amplitudes, conservation of the charge $K \equiv \int dz(\hat{e}^+ \psi^- - \hat{e}^- \psi^+ : +\partial_z h^- - \partial_z h^+) \text{ of equation (III.3) implies that the instanton number } n \text{ must equal } \frac{1}{2}(Y - N) \text{ for the component of the scattering amplitude with } Y \theta^{-i_l}'s. \text{ Furthermore, only the picture-changing operators } Z^\pm \text{ contain } \hat{e}^\pm, \text{ and since } \hat{e}^\pm \text{ contains } g - 1 \text{ more zero modes than } \psi^\pm, \text{ there must be at least } (N + g - 1) Z^+ \text{'s and } (N + g - 1) Z^- \text{'s (each } V_{G,r} \text{ contains one } \psi^- \text{ and one } \psi^+). \text{ Since there are } (2N + 2g - 2 - n) Z^+ \text{'s and } (N + 2g - 2 + n) Z^- \text{'s, this implies that } 1 - g \leq n \leq N - 1 + g, \text{ and therefore, only components of the amplitude with between } (2 - 2g + N) \text{ and } (2g - 2 + 3N) \theta^{-i_l}'s \text{ are non-zero.}

Using this information, the g-loop amplitude, $A_g$, can be written as:

$$A_g = \left| \prod_{i=1}^{3g-3} d\rho_i \prod_{j=1}^g dm^T_i \prod_{j=1}^g dm^U_j(z) \right|^2 \int_R DX^+DX^-D\Psi^-D\Psi^+DBDC \left| \int_{a_j} dy_j (y_j) \frac{1}{z - y_j} \delta(<M^U_j|b>) \right|^2$$

$$\prod_{j=1}^g d\rho_{a_j} d\rho_{b_j} \left| \frac{1}{z - y_j} \right|^{N-1+g} \sum_{n=1-g}^{N} \prod_{j=1}^{3g-3} d\rho_i \prod_{j=1}^g dm^T_i \prod_{j=1}^g dm^U_j(z) \left( Z^+ \right)^{2g-2+2N-n} \left( Z^- \right)^{2g-2+2N+n} \exp(\rho_{a_j} \Omega_{a_j} + \rho_{b_j} \Omega_{b_j}) \exp(S_{\text{matter}} + S_{\text{ghost}}) \prod_{r=1}^{N} \int d\bar{z}_r d\bar{z}_r \hat{V}_{G,r}(z_r, \bar{z}_r), \quad (III.16)$$

where the locations of the $I$'s and $Z^\pm$'s are arbitrary (changing their locations changes the integrand by a total derivative in the moduli).

The above functional integral can be explicitly evaluated once the free-field correlation functions on the surface, $R$, for the matter and ghost fields have been determined.
C. Correlation Functions for the $W^\pm$ and $\Psi^\pm$ Superfields

After bosonizing the $[\lambda^\pm, w^\pm]$ fields as in equation (II.17), the $\Omega$ constraint,
\[
\int_C dz (\lambda^\pm - \frac{1}{2} \partial_z \psi^* - \frac{1}{2} \partial_z \bar{\psi}^*) + \int_{\bar{C}} d\bar{z} (\bar{\lambda}^\pm + \frac{1}{2} \partial_{\bar{z}} \bar{\psi}^* + \frac{1}{2} \partial_{\bar{z}} \psi^*) = 0,
\]
is trivially solved. However, the restriction on the U(1) moduli that is imposed by this constraint does not disappear. Since the bosonized fields include fields with negative energy, demanding analyticity of these fields (i.e., that their correlation functions do not have unwanted poles) will impose a similar restriction on the U(1) moduli.

By expressing the unbosonized fields, $[\lambda^\pm, w^\pm, \psi^\pm, \varepsilon^\pm]$, in terms of the bosonized fields, $[x^\pm, h^1, h^2, \psi^\pm, \hat{\varepsilon}^\pm]$, all correlation functions can be performed using the results of references 31 and 32 (because the zero-mode of $x^+$ is a well-defined field on the surface, there is no analog for the special treatment needed to handle the $\xi^\pm$ zero-mode in the bosonization of the $[\beta^\pm, \gamma^\pm]$ system). The anomolous contributions to these correlation functions from the conformal factor can be safely ignored since vanishing of the total central charge implies that these anomolous contributions will cancel out in the final scattering amplitude.

The $[\psi^+, \hat{\varepsilon}^-]$ fields can be represented by chiral scalar bosons with screening charge $q = 1$ which take values on a circle of radius 1, so their correlation functions are:

\[
< \prod_{i=1}^{m} \hat{\varepsilon}^-(y_i) \prod_{j=1}^{n} \psi^+(z_j) >_{\tau} = Z([\sum_{i=1}^{m} y_i - \sum_{j=1}^{n} z_j - \Delta], \tau) \tag{III.17}
\]

where $\Delta$ is the Riemann class, $\tau$ is the period matrix of the surface and is a complex symmetric $g \times g$ matrix with positive-definite imaginary part,

\[
Z([\sum_{i=1}^{n} c_i z_i - q\Delta], \tau) = \delta_{q(g-1), \Sigma c_i} \prod_{i<j} E(z_i, z_j)^{c_i c_j} \prod_{i=1}^{n} \sigma(z_i)^{q c_i} (Z_1(\tau))^{-\frac{1}{2}} \Theta([\sum_{i=1}^{n} c_i z_i - q\Delta], \tau), \tag{III.18}
\]

$Z_1(\tau)$ is a normalization for $Z$ such that $Z([\sum_{i=1}^{q} z_i - y - \Delta], \tau) = Z_1(\tau) \det_{i,j} \omega_i(z_j)$, $\omega_i$ are the $g$ canonical holomorphic one-forms satisfying $\int_{a_j} \omega_i = \delta_{i,j}$ and $\int_{b_j} \omega_i = \tau_{ij}$, $E(y, z)$ is the prime form,

\[
\frac{\sigma(y)}{\sigma(z)} = \frac{\Theta([y - \sum_{i=1}^{q} p_i + \Delta], \tau)}{\Theta([z - \sum_{i=1}^{q} p_i + \Delta], \tau)} \prod_{j=1}^{q} \frac{E(y, p_j)}{E(z, p_j)} \tag{III.19}
\]
for arbitrary $p_i$ (the final amplitudes will contain equal powers of $\sigma$ in the numerator and denominator because of the vanishing conformal anomaly), \[
\sum_{i=1}^{n} (y_i - z_i) \equiv \prod_{i=1}^{n} \int z_i^y \omega_j
\] is an element in the Jacobian variety $C^g/(Z^g + \tau Z^g)$, and $\Theta(z, \tau) \equiv \sum_{n \in Z^g} \exp(i\pi n_j \tau_j n_k + 2\pi i n_j z_j)$ which satisfies $\Theta(z + \tau n + m, \tau) = \exp(-i\pi n_j \tau_j n_k - 2\pi i n_j z_j)\Theta(z, \tau)$ for $z \in C^g$ and $n, m \in Z^g$. For a brief but sufficient review of these objects, see Chapter 3 of reference 31.

The $x^+$ and $x^-$ fields are non-chiral scalar bosons which take values on the real line, so their correlation functions are:

\[
< \prod_{j=1}^{n} \exp(ip_j^+ x^-(z_j) + ip_j^- x^+(z_j)) >_{\tau} \tag{III.20}
\]

\[
= \delta(\sum_{j=1}^{n} p_j^+) \delta(\sum_{j=1}^{n} p_j^-) (\det \text{Im} \tau)^{-1} |Z_1(\tau)|^{-2} \prod_{j \neq k} F(z_j, z_k) p_j^+ p_k^-, \]

where $F(y, z) = \exp(-2\pi \text{Im}[y - z](\text{Im} \tau)^{-1} \text{Im}[y - z])|E(y, z)|^2$. Note that this non-holomorphic correlation function can be expressed as:

\[
\delta(\sum_{j=1}^{n} p_j^+) \delta(\sum_{j=1}^{n} p_j^-) \prod_{j=1}^{g} \int_{-\infty}^{\infty} dk_j^+ dk_j^- \tag{III.21}
\]

\[
< \exp[\int_{b_j} dy_j (ik_j^+ \partial_{y_j} x^- (y_j) + ik_j^- \partial_{y_j} x^+ (y_j))] \prod_{j=1}^{n} \exp(ip_j^+ x^- (z_j) + ip_j^- x^+ (z_j)) >_{\tau} \left| \right|^2
\]

where \[
< \prod_{j=1}^{n} \exp(ip_j^+ x^- (y_j) + ip_j^- x^+ (y_j)) >_{\tau} \equiv (Z_1(\tau))^{-1} \prod_{j \neq k} E(y_j, y_k) p_j^+ p_k^- ,
\]

and $(k_j^+)^* \equiv -k_j^-$, $(p_j^+)^* \equiv -p_j^-$ in the above formula. The integrations over $k_j^+$ and $k_j^-$ in the above formula bear a close resemblance to the Lagrange multipliers, $\rho_{a,j}$ and $\rho_{b,j}$, of equation (III.10).

Since the chiral scalar bosons, $h^1$ and $h^2$, take values on a circle of radius $\sqrt{2}$, their correlation functions are not as straightforward to calculate. Nevertheless, these functions can be determined using the results of reference 32. The $h^1$ fields have screening charge $q = \sqrt{2}$, and so their correlation functions are:
\[
\langle \prod_{i=1}^{n} \exp(c_i^1 h^1(z_i)) \rangle_{\tau} = f_1(\tau) \delta_{\sqrt{2}(g-1),\Sigma c_i^1} \prod_{i<j} E(z_i,z_j)^{c_i^1 c_j^1} \prod_{i=1}^{n} \sigma(z_i) \sqrt{2} c_i^1 \quad (III.22)
\]

\[
\Theta([\sum_{i=1}^{n} \sqrt{2} c_i^1 z_i - 2\Delta], 2\tau),
\]

where \( f_1(\tau) \) is an unknown overall normalization factor. These correlation functions for chiral bosons that take values on a circle of radius \( \sqrt{2} \) have the strange property that they are periodic when any of the operators is taken once around an \( a_j \)-cycle, or \textit{twice} around a \( b_j \)-cycle.

The \( h^2 \) fields also take values on a circle of radius \( \sqrt{2} \), but differ from the \( h^1 \) fields in three ways. Firstly, the \( h^2 \) field undergoes a shift of \( 2\sqrt{2} \pi i m U_j(1) \) when it goes around a \( b_j \)-cycle, and therefore a shift of \( 4\sqrt{2} \pi i m U_j(1) \) when it goes twice around. Secondly, the energy of \( h^2 \) is negative, so the correlation functions are inverted. And thirdly, \( h^2 \) has no screening charge. The correlation functions for the \( h^2 \) fields are therefore:

\[
\langle \prod_{i=1}^{n} \exp(c_i^2 h^2(z_i)) \rangle_{\tau} = f_2(\tau) \delta_{0,\Sigma c_i^2} \left[ \prod_{i<j} E(z_i,z_j)^{c_i^2 c_j^2} \Theta([\sum_{i=1}^{n} \sqrt{2} c_i^2 z_i] - 2mU^{(1)}, 2\tau) \right]^{-1}. \quad (III.23)
\]

Putting together the correlation functions of equations (III.22) and (III.23), one finds:

\[
\langle \prod_{i=1}^{n} \exp(c_i^- h^+ (z_i) + c_i^+ h^- (z_i)) \rangle_{\tau} = f_1(\tau) f_2(\tau) \delta_{g-1,\Sigma c_i^1} \delta_{g-1,\Sigma c_i^2} \quad (III.24)
\]

\[
\prod_{i \neq j} E(z_i,z_j)^{c_i^- c_j^+} \prod_{i=1}^{n} \sigma(z_i)^{c_i^- + c_i^+} \frac{\Theta([\sum_{i=1}^{n} \sqrt{2} c_i^1 z_i - 2\Delta], 2\tau)}{\Theta([\sum_{i=1}^{n} \sqrt{2} c_i^2 z_i] - 2mU^{(1)}, 2\tau)},
\]

where \( h^\pm = \frac{1}{\sqrt{2}} (h^1 \pm h^2) \) and where \( c_i^\pm = \frac{1}{\sqrt{2}} (c_i^1 \pm c_i^2) \).

Since \( \Theta([\sum_{i=1}^{n} \sqrt{2} c_i^1 z_i] - 2mU^{(1)}, 2\tau) \) has zeroes as a function of \( z_i \), the correlation functions of the \( h^+ \) and \( h^- \) fields will have unwanted poles unless the zeroes of
\( \Theta(\sum_{i=1}^{n} \sqrt{2}c_i^1 z_i - 2\Delta, 2\tau) \) occur at the same locations as the zeroes of \( \Theta(\sum_{i=1}^{n} \sqrt{2}c_i^2 z_i - 2m^{U(1)}_j, 2\tau) \). This is possible only if

\[
m^{U(1)}_j = \left[ \frac{1}{\sqrt{2}} \sum_{i=1}^{n} (c_i^2 \pm c_i^1) \mp \Delta \right]_j + \alpha_j, \tag{III.25}
\]

where \( \alpha_j \) is 0 or \( \frac{1}{2} \), and \( m^{U(1)}_j \) is defined such that \( 0 \leq \text{Re}(m^{U(1)}_j) < 1 \) and \( 0 \leq \text{Im}(m^{U(1)}_j)/(\text{Im} \tau)^{-1} < 1 \) (this definition chooses one point in the Jacobian variety, \( C^g/(Z^g + \tau Z^g) \), however as will be shown in Section IV.A., the total scattering amplitude is independent of this choice).

The ambiguity in \( m^{U(1)}_j \) comes from the fact that \( \Theta(z, 2\tau) = \Theta(-z, 2\tau) \) and \( \Theta(z + 1, 2\tau) = \Theta(z, 2\tau) \), and can be fixed by analyzing the following correlation function:

\[
F(y^+, y^-) \equiv \langle \prod_{i=1}^{n} \exp(c_i^- h^+(z_i) + c_i^+ h^-(z_i))\lambda^+(y^+)\lambda^-(y^-) \rangle_\tau \tag{III.26}
\]

where \( \sum_{i=1}^{n} c_i^- = \sum_{i=1}^{n} c_i^+ = g - 1 \). From equations (II.17) and (III.24),

\[
F(y^+, y^-) = \langle \prod_{i=1}^{n} \exp(c_i^- h^+(z_i) + c_i^+ h^-(z_i))\rangle \\
= \langle (\partial x^+ + \frac{1}{2} \psi^+ \partial z \psi^- + \frac{1}{2} \psi^- \partial z \psi^+) (y^+) \exp(h^+(y^+) - h^+(y^-)) \rangle_\tau \\
= C(\partial x^+ + \frac{1}{2} \psi^+ \partial z \psi^- + \frac{1}{2} \psi^- \partial z \psi^+) (y^+) \prod_{i=1}^{n} \frac{E(y^+, z_i) c_i^+ \sigma(y^+)}{E(y^-, z_i) c_i^+ \sigma(y^-)} \tag{III.27}
\]

where \( C \) is independent of \( y^\pm \), and therefore, \( F(y^+, y^-) \to \exp(2\pi i [\sum_{i=1}^{n} c_i^+ - \Delta]_j) F(y^+, y^-) \) when \( y^+ \) goes around the \( b_j \)-cycle. Since \( \lambda^+(y^+) \to \exp(2\pi i m^{U(1)}_j) \lambda^+(y^+) \) when \( y^+ \) goes around the \( b_j \)-cycle, the correct choice for the \( U(1) \) moduli is:

\[
m^{U(1)}_j = [\sum_{i=1}^{n} c_i^+ - \Delta]_j. \tag{III.28}
\]

So finally, the correlation function for the \( h^\pm \) fields is given by:

\[
\langle \prod_{i=1}^{n} \exp(c_i^- h^+(z_i) + c_i^+ h^-(z_i)) \rangle_\tau = \tag{III.29}
\]

25
\[ N(\tau) \delta_{g-1,\Sigma_i e_i} \delta_{g-1,\Sigma_i e_i^+} \prod_{i \neq k} E(z_i, z_k) c_i^- c_k^+ \prod_{i=1}^n \sigma(z_i) c_i^- + c_i^+ \prod_{j=1}^g \delta(m_j^{U(1)} - \sum_{i=1}^n c_i^+ - \Delta_j) \]

where \( N(\tau) \) is an overall measure factor that is independent of the locations of the fields.

D. Correlation Functions for \( X \) Superfields

The correlation functions for each pair of \( x^{+l} \) and \( x^{-l} \) fields are the same as for the \( x^\pm \) fields of equation (III.20), that is:

\[
< \prod_{j=1}^n \exp[ip_j^{+l} x^{-l}(z^j) + ip_j^{-l} x^{+l}(z_j)] >_{\tau} = \] (III.30)

\[
\delta^2(\sum_{j=1}^n p_j^{+l}) \delta^2(\sum_{j=1}^n p_j^{-l}) (\det \text{Im} \tau)^{-1} |Z_1(\tau)|^{-2} \prod_{j \neq k} F(z_j, z_k) p_j^{+l} p_k^{-l},
\]

where \( F(y, z) = \exp(-2\pi \text{Im}[y - z] \text{Im} \tau)^{-1} \text{Im} \tau |E(y, z)|^2. \)

The correlation functions for the \( \Gamma^{-l} \) and \( \Gamma^{+l} \) fields are also straightforward, with the only subtlety coming from the U(1) shift of \( \exp(2\pi i m_j^{U(1)}) \) when \( \Gamma^{+l} \) goes around the \( b_j \)-cycle. Since \( \Gamma^{-l} \) and \( \Gamma^{+l} \) can be represented by chiral bosons, \( \sigma_l \), with no screening charge that take values on a circle of radius 1, their correlation functions are:

\[
< \prod_{i=1}^n \exp(c_i \sigma_l(z_i)) >_{\tau} = Z([\sum_{i=1}^n c_i z_i] - m_j^{U(1)}, \tau), \] (III.31)

where \( Z \) is defined in equation (III.18) and \( m_j^{U(1)} = [\sum_{i=1}^n c_i^l - \Delta_j]. \) There is no need to include other spin structures for the theta-function since, as will be shown in Section IV.A., all spin structures contribute equally to the total scattering amplitude.

E. Correlation Functions for Ghosts

The correlation functions for the \( b \) and \( c \) Virasoro ghosts with screening charge \( q = 3 \) is:

\[
< \prod_{i=1}^m b(y_i) \prod_{j=1}^n c(z_j) >_{\tau} = Z([\sum_{i=1}^m y_i - \sum_{j=1}^n z_j - 3\Delta], \tau). \] (III.32)
For the $\beta^\pm$ and $\gamma^\pm$ fields, the only difference with the NSR treatment of the bosonized super-reparameterization ghosts comes from the contribution of the U(1) moduli (note that in equation 36 of reference 4, a factor of $(Z_1)^{1/2}$ was mistakenly omitted). Because the zero mode of $\xi^+$ does not appear in any of the operators, an extra field, $\xi^+(x_0)$, needs to be introduced into correlation functions of the $\beta^+$ and $\gamma^-$ fields. Since the screening charge of the bosonized field, $\phi^-$, is +2, these correlation functions are:\(^4\)

\[
< \prod_{i=0}^p \xi^+(x_i) \prod_{j=1}^q \eta^-(y_j) \prod_{k=1}^r \exp(c_k \phi^-(z_k)) >_\tau = (III.33)
\]

Note that the correlation functions are independent of $x_0$ since only the zero mode of $\xi^+(x_0)$ contributes.

Because the $u$ ghost does not appear in either the vertex operators, the picture-changing operators, or the instanton-number-changing operator, the correlation functions for the $u$ and $v$ ghosts must introduce an extra $u(x_0)$ field, just as the $\beta^+$ and $\gamma^-$ correlation functions required an extra $\xi^+(x_0)$ field. Also, since the U(1) beltrami differentials of equation (III.15) already introduce $g$ $v$ fields, there can be no further contributions of $v$ fields from the other operators (there must be $g - 1$ more $v$ fields than $u$ fields to get a non-zero amplitude, since the screening charge is +1).

The relevant correlation function for the $u$ and $v$ ghosts, using equation (III.17), is therefore:

\[
< u(x_0) \prod_{i=1}^g \int_{a_i} dy_i v(y_i) >_\tau = \prod_{i=1}^g \int_{a_i} dy_i Z(\sum_{j=1}^g y_j - x_0 - \Delta], \tau) (III.34)
\]

\[
= Z_1(\tau) \prod_{i=1}^g \int_{a_i} dy_i \text{det} \omega_j(y_k) = Z_1(\tau) \text{ since } \int_{a_i} dy_i \omega_j(y_i) = \delta_{ij}.
\]

The overall measure factor, $N(\tau)$, from the $h^\pm$ correlation function of equation (III.29) can now be fixed by requiring that correlation functions without any $h^-$ fields, when
integrated over the U(1) moduli, are normalized to one. This normalization prescription will be shown in Section IV.C. to give amplitudes which agree with amplitudes obtained using the light-cone gauge formalism, and therefore is the correct unitary prescription.

Since the background charge normally requires \((g-1)\) \(h^-\) fields, these correlation functions should be evaluated on surfaces with their instanton number shifted by \((g-1)\) (recall that this shifts the conformal weight of \(\lambda^-\) from \(\frac{1}{2}\) to 0, and therefore shifts the screening charge of \(h^2\) from 0 to \(\sqrt{2}\)). On such surfaces, the correlation function of equation (III.29) in the absence of \(h^-\) fields is simply \(N(\tau) \prod_{i=1}^{g} \delta(m_j^{U(1)})\), where \(\sum c_i h_i^+\) is assumed to be equal to \(2(g-1)\). So integration over the global U(1) moduli of this correlation function, when combined with the gauge-fixing contribution coming from the U(1) ghosts, \(u\) and \(v\), gives \(Z_1(\tau) N(\tau)\). Therefore, normalization to one prescribes that

\[
N(\tau) = [Z_1(\tau)]^{-1}. \tag{III.35}
\]

This normalization prescription is consistent with the vanishing of the conformal anomaly since after shifting the screening charge of \(h^2\), the contribution of the \(h^1\) and \(h^2\) fields to the central charge is +2 (the partition function of a \(c = 1\) chiral boson is \((Z_1)^{-\frac{3}{2}}\)).

**IV. Analysis of the Scattering Amplitudes**

**A. Equivalence of Different Spin Structures**

By expressing the vertex operators and picture-changing operators in terms of the bosonized free-fields and using the results of Sections III.C., D., and E. for evaluating their correlation functions, the scattering amplitude of equation (III.16) can be calculated as follows:

\[
A_g = \left| \prod_{i=1}^{3g-3} \int dm_i^{T} \prod_{j=1}^{g} \int dm_j^{U(1)} \delta(<M_i^T|b>) \int_{a_j} dy_j v(y_j) \right|^2 \tag{IV.1}
\]

\[
\xi^+ \xi^- \sum_{n=1-g}^{N-1+g} \frac{n}{2} (Z^+)^{2g-2+2N-n}(Z^-)^{2g-2+N+n} \int_{r=1}^{N} dz_r dz_r^{\ast} \hat{V}_{G,r}(z_r, z_r^{\ast}) > \tau,
\]
where the locations of the $I$'s, $Z^\pm$'s, $\xi^\pm$, and $u$ are arbitrary. Note that for each combination of $h^-$ fields occurring in the scattering amplitude, only one value of $m_j^{U(1)}$ contributes.

Because all operators in the correlation function are $U(1)$ singlets,

$$
\sum_{l=1}^{4} c_i^{\sigma^l} + c_i^{h^+} - c_i^{h^-} + c_i^{\phi^+} - c_i^{\phi^-} = 0 \quad (IV.2)
$$

for each $z_i$ that appears in the scattering amplitude. This property implies the cancellation of all terms involving $E(y, z_i)$ and $\sigma(y)$ where $y$ is the location of an instanton-number-changing operator, $I(y)$. Furthermore, all theta-functions in the amplitude are independent of $y$ since for $U(1)$-transforming fields, the contribution to the argument of the theta function from the $U(1)$ moduli is $\mp y$ (see equation (III.28)) while the contribution to the argument from the fields is $\pm y$. Therefore, the integrand of $A_g$ is completely independent of the locations of the instanton-number-changing operators (this differs from the picture-changing operators, since only $A_g$, and not the integrand of $A_g$, is independent of the locations of the $Z^\pm$'s).

Another consequence of equation (IV.2) is that changing the spin structures of the $U(1)$-transforming fields does not affect the integrand of $A_g$. This fact is not surprising since changing the spin structures from $[0]$ to $[\alpha]$ is equivalent to changing the gauge-fixing condition on the $U(1)$ gauge field, $A_z$, to $\int_{a_j} dz A_z = 2\pi i \alpha_{a_j}$ and $\int_{b_j} dz A_z = 2\pi i (m_j^{U(1)} + \alpha_{b_j})$, where $\alpha \in (Z_2)^{2g}$. This affects equation (III.28) for the $U(1)$ moduli since the theta-function for $h^2$ in equation (III.23) now carries spin structure $[\alpha]$. It is straightforward to check that the zeroes of this theta-function coincide with the zeroes of the theta-function for the $h^1$ field if

$$
m_j^{U(1)} = \sum_{i=1}^{n} c_i^{h^-} - \Delta + \tau_{jk}\alpha_{a_k} + \alpha_{b_j}. \quad (IV.3)
$$

Using the relation

$$
\Theta([\alpha], z - \tau\alpha_a - \alpha_b, \tau) = \exp(-\pi i \alpha_{a_j}\tau_{jk}\alpha_{a_k} + 2\pi i \alpha_{a_j} z_j)\Theta(z, \tau) \quad (IV.4)
$$

and equation (IV.2), one can show that all phase factors in the total scattering amplitude cancel out (the $\tau$ dependent factor cancels since the $\sigma_l$ correlation functions contribute
exp(\(-4\pi i \alpha_{a_j} \tau_{jk} \alpha_{a_k}\)), the \(\phi^+\) and \(\phi^-\) correlation functions contribute \(\exp(2\pi i \alpha_{a_j} \tau_{jk} \alpha_{a_k})\), and the \(h^2\) correlation function contributes \(\exp(2\pi i \alpha_{a_j} \tau_{jk} \alpha_{a_k})\).

Similarly, one can show using equation (IV.2) and the periodicity properties of \(\Theta(z, \tau)\) that shifting the U(1) moduli, \(m^{U(1)}_j \to m^{U(1)}_j + \tau_{jk} p_k + q_j\) for \(p_j\) and \(q_j \in \mathbb{Z}\), does not affect the integrand of \(A_g\), and it is therefore unnecessary to choose a region in the Jacobian variety, \(C_g/(Z^g + \tau Z^g)\), when defining \(m^{U(1)}_j\).

B. Proof of the Non-Renormalization Theorem

The non-renormalization theorem for the superstring states that all loop amplitudes with three or less massless particles vanish. In the Green-Schwarz light-cone gauge\(^7\) and semi-light-cone gauge\(^9\) formalisms, this theorem can only be proven by explicitly assuming Lorentz covariance for the scattering amplitudes. In the Neveu-Schwarz-Ramond formalism for the superstring, proof of the non-renormalization theorem is complicated by the possible contribution of surface-terms from cutoffs in the moduli space (these cutoffs are necessary since the NSR amplitudes diverge before summing over spin structures, but were ignored in the proof of reference 35).

Using the expression for \(A_g\) in equation (IV.1), it will now be shown that \(A_g\) vanishes for \(g \geq 1\) when there are three external massless states. Since the spacetime supersymmetry generators of equation (II.28), \(S^{+l}\), are analytic everywhere on the surface except at the locations of the vertex operators, \(z_i\) for \(i = 1\) to \(3\), the \(S^{+l}\)'s that encircle \(z_3\) in \(V_{G,3}(z_3, \bar{z}_3)\) can be pulled off until they encircle either \(z_1\) or \(z_2\). This implies that \(A_g\) can only contain terms proportional to \((\theta_{1}^{\bar{l}} - \theta_{3}^{\bar{l}})\) and \((\theta_{2}^{\bar{l}} - \theta_{3}^{\bar{l}})\), and by fermion number conservation it must contain a total odd amount of these factors. Supposing that the components of \(A_g\) that we are examining have an even number of \((\theta_{1}^{\bar{l}} - \theta_{3}^{\bar{l}})\) factors and an odd number of \((\theta_{2}^{\bar{l}} - \theta_{3}^{\bar{l}})\) factors, choose all the picture-changing operators, \(Z^\pm\), and instanton-number-changing operators, \(I^n\), to be located at \(z_1\).

For these components of \(A_g\), the contributing components of \(\hat{V}_{G,2}(\hat{z}_2, \hat{z}_2)\) are

\[\hat{V}_{-l,-} = S^{+l} \hat{V}_{-, -} = p_2^+ \exp(-\sigma_l + h^+) \hat{V}_{-, -}\]  \hspace{1cm} (IV.5)
\[ \hat{V}_{+l,-} = \epsilon_{lmnq} S^{+m} S^{+n} S^{+q} V_{-, -} = \epsilon_{lmnq} (p_2^+)^3 \exp(-\sigma_m - \sigma_n - \sigma_q + 3h^+) \hat{V}_{-, -} \]

\[ + \epsilon_{lmnq} (p_2^+)^2 : \exp(-\sigma_m - \sigma_n - \sigma_q + 2h^+ - h^-) \hat{V}_{-, -} : , \quad (IV.6) \]

where the \( \psi^+ \) term in \( S_{+l} \) does not contribute since \( \hat{V}_{-, -} \) of equation (II.25) is proportional to \( \psi^+ \). Note that the last term of equation (IV.6) requires normal-ordering because of singularities between \( h^- \) and \( h^+ \).

For the component \( \hat{V}_{-l,-} \), the correlation function of equation (III.31) for the \( \Gamma^{+m} \) and \( \Gamma^{-m} \) fields where \( m \neq l \) is proportional to \( \Theta((g-1)z_1 - \Delta], \tau) \) since all of the \( \sigma_m \) and \( h^- \) fields are located at \( z_1 \). But this is zero by Riemann’s vanishing theorem when \( g \geq 1 \), since \( \Theta(\sum_{i=1}^{g-1} y_i - \Delta], \tau) = 0 \) for arbitrary \( y_i \). Similarly for the first term in the component, \( \hat{V}_{+l,-} \), the correlation function for the \( \Gamma^{+l} \) and \( \Gamma^{-l} \) fields is proportional to \( \Theta((g-1)z_1 - \Delta], \tau) \), and is therefore zero. For the second term in the component, \( \hat{V}_{+l,-} \), the correlation function for the \( \Gamma^{+m} \) and \( \Gamma^{-m} \) fields where \( m \neq l \) is proportional to \( \Theta((g-1)z_1 - \Delta], \tau) = 0 \), since the argument of the theta-function receives \(-z_2\) from the \( \sigma_m \) field and \(+z_2\) from the \( h^- \) field.

To prove the vanishing of \( A_{g} \) for two massless states whose vertex operators are located at \( z_1 \) and \( z_2 \), pull all of the \( S^{+l} \)'s off of \( z_2 \) and encircle them around \( z_1 \), and then place all of the picture-changing and instanton-number-changing operators at \( z_1 \). Then the correlation function for the \( \Gamma^{+l} \) and \( \Gamma^{-l} \) fields is proportional to \( \Theta((g-1)z_1 - \Delta], \tau) = 0 \) for all \( l \). Finally, for one massless state or no states, choose the locations of all of the picture-changing-operators and instanton-number-changing operators to coincide with the vertex operator (or at any point, \( z_1 \), if there are no states). Then once again, the correlation function for the \( \Gamma^{+l} \) and \( \Gamma^{-l} \) fields is proportional to \( \Theta((g-1)z_1 - \Delta], \tau) = 0 \) for all \( l \).

So the superstring amplitudes, \( A_{g} \), that were calculated using the twistor-string formalism of the Green-Schwarz superstring, have been proven to satisfy the non-renormalization theorem.

C. Agreement of \( A_{g} \) with Light-Cone Gauge Amplitudes

The light-cone gauge formalism for calculating Green-Schwarz superstring scattering amplitudes was first developed by Green, Schwarz,\(^5\) and Mandelstam,\(^6\) and more recently,
by Restuccia and Taylor.\(^7\) As discussed in reference 17, the Type IIB light-cone gauge action on a Wick-rotated two-dimensional worldsheet parameterized by \(\rho\) and \(\bar{\rho} \equiv (\rho)^*\) is:

\[
S_{LC} = \int d\rho d\bar{\rho} (\partial_x x^i \partial_y x^i - s^\alpha \partial_x s^\alpha - \bar{s}^\alpha \partial_y \bar{s}^\alpha).
\] (IV.7)

After breaking SO(8) down to SU(4)xU(1) in such a way that the SO(8) vector, \(x^i\), splits into \(\bar{4}_{\pm\frac{1}{2}}\) and \(4_{\pm\frac{1}{2}}\) representations of SU(4)xU(1) while the SO(8) chiral spinor, \(s^\alpha\), splits into \(4_{\pm\frac{1}{2}}\) and \(\bar{4}_{-\frac{3}{2}}\) representations of SU(4)xU(1), the light-cone gauge interaction term (ignoring contact terms) is \(|H^-(\bar{\rho}_a) + H^+(\bar{\rho}_a)|^2\), where

\[
\begin{align*}
H^{-}(\bar{\rho}_a) & \equiv \lim_{\rho \to \bar{\rho}_a} (\rho - \bar{\rho}_a)(\partial_x x^{-l} s^{-l}) = (\frac{\partial^2 \rho}{\partial z^2})^{-1}(\partial_x x^{-l} s^{-l})(\bar{z}_a), \quad (IV.8) \\
H^{+}(\bar{\rho}_a) & \equiv \lim_{\rho \to \bar{\rho}_a} (\rho - \bar{\rho}_a)^2 \epsilon^{klmn} (\partial_x x^k \bar{s}^{-l} \bar{s}^{-m} \bar{s}^{-n}) = (\frac{\partial^2 \rho}{\partial z^2})^{-2} \epsilon^{klmn} (\partial_x x^k \bar{s}^{-l} \bar{s}^{-m} \bar{s}^{-n})(\bar{z}_a), \quad (IV.9)
\end{align*}
\]

are conformal weight one and zero fields\(^7\) as functions of \(z\) (this implies that \(s^{-l}(\rho(z))\) has zeroes at the punctures, \(z_r\) for \(r = 1\) to \(N\), and poles at the interaction points, \(\bar{z}_a\) for \(a = 1\) to \(2g - 2 + N\), whereas \(s^{+l}(\rho(z))\) is regular at these points), \(\bar{\rho}_a \equiv \rho(\bar{z}_a)\) are the \((2g - 2 + N)\) interaction points where \(\partial_x \rho(\bar{z}_a) = 0\), and \(\rho(z)\) is the unique meromorphic function that maps the \(g\)-loop string diagram onto a genus \(g\) surface with \(N\) punctures such that \(Re(\rho)\) is single-valued and \(\partial_x \rho\) has poles with residue \(p_i^+\) at the points \(z_r\) for \(r = 1\) to \(N\).

If the contributions from the contact term interactions are ignored, the \(g\)-loop light-cone gauge scattering amplitude for \(N\) massless states is:

\[
A_g^{LC} = \int_0^{2g - 2} d\rho d\bar{\rho} \prod_{a=1}^{g} d\alpha_id\phi_I \int Dx^{+l} D^{+l} \prod_{r=1}^{N} V_{G,r}^{LC}(z_r) \exp\left[ \int dz d\bar{z} (\partial_x x^{+l} \partial_x x^{-l} - \bar{s}^{+l} \partial_x \bar{s}^{-l} - \bar{s}^{+l} \partial_x \bar{s}^{-l}) \right] \prod_{a=1}^{2g - 2 + N - n} H^+(\bar{\rho}_{a+}) \prod_{a=-1}^{2g - 2 - n} H^-(\bar{\rho}_{a-})^2,
\] (IV.10)

\[
\sum_{n=0}^{2g - 2 + N} a_+, a_- \in a a^+ = 1 H^+(\bar{\rho}_{a+}) \prod_{a=-1}^{2g - 2 - n} H^-(\bar{\rho}_{a-})^2,
\]
where $\tilde{\rho}_a, \tilde{\bar{\rho}}_a, \alpha_I$, and $\phi_I$ are the $(6g - 6 + 2N)$ real light-cone moduli of interaction-point locations, internal $p^+$'s, and twists, $\sum_{a+, a- \in a}$ means that the $2g - 2 + N$ interaction points, $\tilde{\rho}_a$, should be split up into two subsets, $\tilde{\rho}_a^+$ and $\tilde{\rho}_a^-$, in all possible ways, and

$$V_{G,r}^{LC}(z_r) = \left| (p_r^+)\left( \exp\left( p_r^+ \theta_r^{-T} S^+ (z_r) \right) \right) \right|^2 \exp\left( i p_r^- x^+ (z_r) + i p_r^+ x^- (z_r) + p_r^- \text{Re}[\rho(z_r)] \right)$$

(IV.11)

(the light-cone vertex operator, $V_{G}^{LC}$, has been normalized such that it agrees with the matter part of the vertex operator, $V_G$, in the ghost-number zero picture when the non-light-cone fields, $\psi^\pm$ and $h^\pm$ have been set to zero, and $\Gamma^{-l}$ has been identified with $\hat{s}^{+l}$).

Note that the locations of the $\tilde{\rho}_a$'s are determined by the $p^+_r$'s and $z_r$'s, but in a very complicated way. It is this complicated dependence that makes Green-Schwarz light-cone gauge amplitudes difficult to explicitly evaluate.

The first step in comparing $A_g$ of equation (IV.1) with $A_{g}^{LC}$ of equation (IV.10) is to choose light-cone moduli for $A_g$ (these light-cone moduli depend not only on the surface, but also on the $p^+_r$ momenta of the N vertex operators), and to insert $(2g - 2 + N) Z^+$'s and $(2g - 2 + N) Z^-$'s at the interaction points of the string diagram where $\partial_z \rho = 0$.

In order to ensure that there are enough picture-changing operators available to do this, the picture for the vertex operators should be chosen to have ghost number $(-4, -4)$, so the vertex operators are:

$$W_{G,r}(z_r, \bar{z}_r) = (IV.12)$$

$$| (p_r^+)^{-3} \psi^- \partial_z \psi^+ \partial_z \psi^+ \exp(\text{h} - 2\phi^+ - 3\phi^-) \exp(\theta_r^{-T} S^+ l) |^2 \exp(i p^\mu x_\mu)(z_r, \bar{z}_r).$$

It is easy to check that $V_{G,r} = |Z^+ Z^-|^2 W_{G,r}$, so the number of available $Z^+$ picture-changing operators is $2g - 2 + 3N - n_I$ and the number of available $Z^-$ picture-changing operators is $2g - 2 + 2N + n_I$. Since the instanton number, $n_I$, satisfies $n_I = \frac{1}{2}(Y - N)$ where $0 \leq Y \leq 4N$, there are enough available picture-changing operators to insert one of each type at all the interaction points. The extra $(2N - n_I) Z^+$’s and $(N + n_I) Z^-$’s can be inserted anywhere on the string diagram.

Since $A_{g}^{LC}$ ignores the contribution of the light-cone contact-term interactions (these contact-terms are necessary in the light-cone formalism in order to cancel the non-Lorentz-invariant divergences that occur when two interaction-points approach each other), $^{13,14,7}$
one should not expect that $A^L_{g}$ is precisely equal to $A_{g}$. However, a simple conjecture is that the contribution of the light-cone contact-term interactions is equivalent to the contribution of the fields

$$e^{\phi}[-w^{\pm}\partial_{z}\psi^{\mp} + (b \mp \frac{1}{2}\partial_{z}v)\gamma^{\pm} \mp v\partial_{z}\gamma^{\pm} + c\xi^{\pm}] \quad (IV.13)$$

in the picture-changing operators $Z^{\pm}$ of equation (II.20), plus the contribution from the moduli dependence of the interaction-point locations (this moduli dependence of the locations of the picture-changing operators implies that the beltrami differentials for the teichmuller parameters, $M_{T}^{i}$, depend on the fermionic moduli, $m^{\pm}_{k}$)\(^{4}\). Therefore ignoring the contact-term interactions in $A^L_{g}$ is conjectured to be equivalent to using the following truncated form of the picture-changing operators of equation (II.20):

$$\hat{Z}^{+} \equiv e^{\phi}[\partial_{z}x^{-I}\Gamma^{+I} + \hat{\epsilon}^{+}\lambda^{+}] \quad \text{and} \quad \hat{Z}^{-} \equiv e^{\phi}[\partial_{z}x^{+I}\Gamma^{-I} + \hat{\epsilon}^{-}\lambda^{-}], \quad (IV.14)$$

and ignoring the dependence of $M_{T}^{i}$ on the fermionic moduli.

It is easily checked that an analogous conjecture for the Neveu-Schwarz-Ramond string is correct. This conjecture states that ignoring the contribution of the NSR light-cone contact-term interactions is equivalent to ignoring the dependence of $M_{T}^{i}$ on the NSR fermionic moduli and using the truncated NSR picture-changing operator, $\hat{Z} \equiv e^{\phi}[\partial_{z}x^{i}\Gamma^{i} + \partial_{z}x^{+}\Gamma^{-}]$ at the interaction points ($i = 1$ to $8$), rather than the full BRST-invariant operator, $Z = e^{\phi}\partial_{z}x^{\mu}\Gamma_{\mu} + e^{2\phi}\partial_{z}\eta^{b} + \partial_{z}(e^{2\phi}\eta^{b}) + c\partial_{z}\xi$.

This NSR conjecture can be proven by not integrating out the anti-commuting moduli in the BRST formalism, and comparing the resulting amplitudes with the supersheet formalism of the light-cone gauge amplitudes,\(^{36}\) in which the contact-term interactions are automatically included. Since the BRST amplitudes coincide (without using the conjecture)\(^{37}\) with amplitudes obtained from the light-cone supersheet formalism, and also coincide (using the conjecture) with amplitudes obtained from the ordinary component form of the light-cone formalism,\(^{38}\) the NSR conjecture must be valid. If a similar proof could be found for the Green-Schwarz conjecture, it would prove that the twistor-string amplitudes, $A_{g}$, agree with amplitudes obtained using the Green-Schwarz light-cone formalism, and that they are therefore unitary.
With this conjecture, the truncated twistor-string amplitude, \( \hat{A}_g \), is:

\[
\hat{A}_g = \left\langle \int \prod_{a=1}^{2g-2} d\tilde{\rho}_a d\tilde{\rho} \prod_{l=1}^{g} d\alpha_l d\phi_l \prod_{j=1}^{g} dm_j^{(1)} \prod_{i=1}^{3g-3+N} \delta(<M_T^I|b>) \int d y_j v(y_j) \right\rangle^2
\]

\[
(IV.15)
\]

where \( \hat{Z}^\pm \) is defined in equation (IV.14), and \( W_{G,r} \) is defined in equation (IV.12).

Since the vertex operators \( W_{G,r} \) contribute 2\( N \) \( \psi^+ \)'s and 2\( N \) \( \psi^- \)'s, there must be at least \( (2N + g - 1) \) \( \hat{\epsilon}^- \)'s and \( (2N + g - 1) \) \( \hat{\epsilon}^+ \)'s coming from the truncated picture-changing operators, \( \hat{Z}^+ \) and \( \hat{Z}^- \). This means that the \( \hat{Z}^+ \hat{Z}^- (\tilde{z}_a) \) factors at the interaction points must contribute at least \( (g - 1 + n_I) \) \( \hat{\epsilon}^- \)'s and \( (g - 1 + N - n_I) \) \( \hat{\epsilon}^+ \)'s. But since \( x^- \) appears only at the vertex operators, \( \partial_z x^+ \) has the same poles and residues as \( \partial_z \rho \), implying that \( \partial_z x^+(\tilde{z}_a) = \partial_z \rho(\tilde{z}_a) = 0 \) in correlation functions of the \( x^- \) and \( x^+ \) fields. Therefore, no factor of \( \hat{Z}^+ \hat{Z}^- (\tilde{z}_a) \) at the interaction points can contribute \( \hat{\epsilon}^- \hat{\epsilon}^+ \), since any such term would be proportional to \( \lambda^- \lambda^+ \) (recall that \( \lambda^- \lambda^+ = \partial_z x^+ + \frac{1}{2} \psi^+ \partial_z \psi^- + \frac{1}{2} \psi^- \partial_z \psi^+ \), and if more \( \psi^+ \psi^- \) terms are introduced, one needs even more \( \hat{\epsilon}^- \hat{\epsilon}^+ \)'s).

So the only way to have enough \( \hat{\epsilon}^\pm \)'s is if \( (g - 1 + n_I) \) interaction-point factors at \( z = \tilde{z}_a^+ \) contribute \( \hat{\epsilon}^- \lambda^+ \exp(\phi^+ + \phi^-) \partial_z x^+ \Gamma^{-I} \), the other \( (g - 1 + N - n_I) \) interaction-point factors at \( z = \tilde{z}_a^- \) contribute \( \hat{\epsilon}^+ \lambda^- \exp(\phi^+ + \phi^-) \partial_z x^- \Gamma^{+I} \), and the remaining \( (2N - n_I) \) \( \hat{Z}^+ \)'s and \( (N + n_I) \) \( \hat{Z}^- \)'s contribute \( \hat{\epsilon}^- \lambda^+ \exp(\phi^-) \) and \( \hat{\epsilon}^+ \lambda^- \exp(\phi^+) \).

In order to compare with \( A_g^{LC} \), it is convenient to insert one instanton-number-changing operator, \( I(z) \), at each of the \( (g - 1 + n_I) \) interaction-points, \( z = \tilde{z}_a^+ \) (recall that none of the correlation functions depend on the locations of the instanton-number-changing operators, so there is no problem with making the locations of the \( I(z) \)'s change with different choices for the \( \tilde{z}_a^+ \)'s). Since this inserts \( (g - 1) \) more instanton-number-changing operators than is necessary, it shifts the conformal weights of the U(1)-transforming fields by \( \pm \frac{1}{2} \) \( \left[ \Gamma^{-I}, \Gamma^{+I}, \lambda^-, \lambda^+, w^-, w^+, \gamma^-, \gamma^+, \beta^-, \beta^+ \right] \) now have conformal weights \( [0, 1, 0, 1, 0, 1, -1, 0, 1, 2] \).
After making these modifications and using equations (II.22) and (IV.14), $\hat{A}_g$ takes the form:

$$\hat{A}_g = -\prod_{a=1}^{2g-2} d\bar{\rho}_a d\bar{\rho}_a \prod_{I=1}^g d\alpha_I d\phi_I \prod_{j=1}^g \int dm^{U(1)}_{j} \prod_{i=1}^{3g-3+N} \delta(<M_T^i|b>) \int_{a_j} dy_j v(y_j)^2 \quad \text{(IV.16)}$$

$$\sum_{n_I=0}^{2g-2+N} \sum_{a^+,a^- \in a} n_I \prod_{a^+} \left[ \exp(-h^+ + 2\phi^-) \tilde{\varepsilon}^- e^{klmn} (\partial z^+ k \Gamma e^{-l} \Gamma + m \Gamma + n) \right] (\tilde{z}_{a^+})$$

$$\sum_{n_I=0}^{2g-2+N-n_I} \prod_{a^-=1} \left[ \exp(-h^+ + \phi^+ + \phi^-) \tilde{\varepsilon}^+ \partial z^{-l} \Gamma e^{-l} \Gamma \right] (\tilde{z}_{a^-})$$

$$\xi^+ \xi^- u(e^{\phi^-} \varepsilon^- \lambda^+) 2^{N+g-1-n_I} (e^{\phi^+} \varepsilon^+ \lambda^-)^{-N+g+n_I} (w) \prod_{r=1}^N W_{G,T}(z_r, \bar{z}_r) > \tau,$$

where the location of $w$ is arbitrary.

Since there are no $e^{+h^-}$ terms and since the screening charge of $h^-$ is now zero, none of the $e^{-h^-}$ terms can contribute. Also because there are no extra $\tilde{\varepsilon}^\pm$ fields, no unnecessary $\psi^\pm$ fields can contribute to the amplitude. The $U(1)$ moduli, $m^{U(1)}_j$, is therefore fixed to zero, and after performing the correlation functions for the $u, v, \xi^\pm, \eta^\pm, \phi^\pm, \varepsilon^\pm$, and $\psi^\pm$ fields, one obtains

$$\hat{A}_g = -\int \prod_{a=1}^{2g-2} d\bar{\rho}_a d\bar{\rho}_a \prod_{I=1}^g d\alpha_I d\phi_I \prod_{i=1}^{3g-3+N} \delta(<M_T^i|b>) \quad \text{(IV.17)}$$

$$\sum_{n_I=0}^{2g-2+N} \sum_{a^+,a^- \in a} n_I \prod_{a^+} \left[ \exp(-h^+ + 2\phi^-) \tilde{\varepsilon}^- e^{klmn} (\partial z^+ k \Gamma e^{-l} \Gamma + m \Gamma + n) \right] (\tilde{z}_{a^+})$$

$$\sum_{n_I=0}^{2g-2+N-n_I} \prod_{a^-=1} \left[ \exp(-h^+ + \phi^+ + \phi^-) \tilde{\varepsilon}^+ \partial z^{-l} \Gamma e^{-l} \Gamma \right] (\tilde{z}_{a^-})$$

$$d^{2g-2} \prod_{r=1}^N [p_r^+]^{\frac{5}{2}} \prod_{a^+} \left[ \frac{\partial^2 \rho}{\partial z^{2+}} (z_{a^+}) \right]^{-\frac{1}{2}}$$

$$\prod_{r=1}^N \int dz_r d\bar{z}_r |(p_r^+)^{-3} \exp(p_r^+ \theta_r^{-l} \Gamma e^{-l} \Gamma) |^2 \exp(i p_r^+ p_r^\dagger (z_r, \bar{z}_r)) > \tau,$$

where the equations $\partial^2 \rho (z_{a^+}) = \lim_{z \to z_{a^+}} (\partial z \rho) [E(z, z_{a^+})]^{-1}$ and $p_r^+ = \lim_{z \to z_r} (\partial z \rho) E(z, z_r)$ have been used, and

$$d = \frac{\partial z \rho(z) \prod_{r=1}^N E(z, z_r)}{(\sigma(z))^2 \prod_{a=1}^{2g-2+N} E(z, z_{a})} \quad \text{(IV.18)}$$

36
is independent of $z$ because it is a single-valued function with no zeroes or poles on the surface ($E(y, z)$ and $\sigma(z)$ are defined in equation (III.19)).

The correlation function for the $b$ and $c$ fields is the same calculation as for the bosonic string, and therefore can be obtained by comparing the known covariant and light-cone bosonic string amplitudes.\textsuperscript{39,40} Equivalence of these bosonic amplitudes implies that

$$
< | \prod_{i=1}^{3g-3+N} \delta(< M_i^i | b >) \prod_{r=1}^{N} c(z_r)|^2 > = \det(Im \tau) |Z_1(\tau)|^{2g-2} \prod_{r=1}^{n} (p_r^+)^{-\frac{1}{2}} \prod_{a=1}^{2g-2+N} \left( \frac{\partial^2 p}{\partial z_a} \right)^{-\frac{1}{2}}.
$$

(IV.19)

Note that under rescalings $p_r^+ \rightarrow Cp_r^+$, this correlation function scales like $C^{3-3g-N}$ which cancels the rescaling of the light-cone moduli.

The correlation function for the $x^+$ and $x^-$ in $\hat{A}_g$ simply substitutes $Re(\rho)$ everywhere for $x^+$ and introduces an extra factor of

$$
\det(Im \tau)^{-1} |Z_1|^{-2}
$$

from the partition function.

After multiplying together equations (IV.17), (IV.19), and (IV.20), $\hat{A}_g$ takes exactly the same form as $A_g^{LC}$ of equation (IV.10), but with $s^+l$ and $s^-\bar{l}$ replaced by $\Gamma^{-l}$ and $\Gamma^{+\bar{l}}$. Because these two sets of fields have identical conformal weights, their correlation functions are equivalent, implying that the truncated twistor-string scattering amplitude, $\hat{A}_g$, agrees with the light-cone gauge scattering amplitude, $A_g^{LC}$. Note that the conformal anomaly contribution to the light-cone gauge amplitude is zero since the contribution from the $x^{-l}$ and $\partial_z x^{+\bar{l}}$ fields cancels the contribution from the $s^{+l}$ and $s^{-\bar{l}}$ fields.

**VI. Concluding Remarks**

In this paper, the gauge-fixed N=(2,0) twistor-string action was used to calculate Type IIB Green-Schwarz superstring amplitudes with an arbitrary number of loops and external massless states. The manifest spacetime supersymmetry of these amplitudes gives them advantages over superstring amplitudes calculated using the Neveu-Schwarz-Ramond formalism. As was mentioned in the introduction, NSR amplitudes contain divergences before
summing over spin structures, giving rise to “multiloop ambiguities”\textsuperscript{4,26} and complicating the analysis of finiteness.\textsuperscript{27} Furthermore, the ghost contributions to the fermionic vertex operator\textsuperscript{3} and the necessity of performing a GSO projection makes scattering in fermionic backgrounds difficult to describe in the NSR formulation of the superstring.

Since the Green-Schwarz superstring scattering amplitudes derived in this paper do not suffer from these problems, they may be useful in providing a better understanding of the finiteness properties of superstrings, and in allowing superstring scattering amplitudes to be calculated in fermionic backgrounds. Another possible application of the results in this paper is to find new relations between theta-functions on higher-genus Riemann surfaces. By comparing Green-Schwarz multiloop scattering amplitudes with their NSR counterparts, one might discover generalizations of the well-known Jacobi identity that relates theta-functions of different spin-structures on the torus.\textsuperscript{1}

An obvious disadvantage of the scattering amplitude calculations in this paper is that they are manifestly invariant under only an SU(4)xU(1) subgroup of the full SO(9,1) super-Poincaré transformations. One possibility for covariantizing the amplitude calculations is to quantize a recently proposed version of the twistor-string with N=8 worldsheet supersymmetry.\textsuperscript{41,42} Since this twistor-string version of the Green-Schwarz superstring replaces all eight of the Siegel-symmetries with worldsheet super-reparameterizations, it might be possible to gauge-fix the N=8 twistor-string action without breaking the SO(9,1) super-Poincaré invariance. However, at this point, it is not known how to gauge-fix the N=8 action to a free-field action, and even though the N=8 twistor-string has been shown to be classically equivalent to the ten-dimensional Green-Schwarz superstring, it is unclear if the equivalence remains after quantization (similar statements can be made about the N=1 and N=4 twistor-string versions of the ten-dimensional Green-Schwarz superstring\textsuperscript{43,44}).

A second possibility for covariantizing the amplitude calculations is to quantize the Lorentz-covariant version of the N=2 twistor-string action without explicitly breaking the SO(9,1) super-Poincaré invariance (at the present time, this possibility exists only for the N=(2,0) twistor-string version of the heterotic Green-Schwarz superstring, since Lorentz-covariant twistor-string actions are not yet known for the non-heterotic superstrings). It is likely that pure spinors (the definition of a pure spinor in ten dimensions is a non-zero
sixteen-component complex Weyl SO(9,1) spinor, $\lambda^\alpha$, satisfying $\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta$ for $\mu=0$ to 9, where $\lambda^\alpha$ is defined up to a complex projective transformation) would play a fundamental role in any covariant quantization of the N=2 twistor-string since breaking SO(9,1) down to SU(4)xU(1) is equivalent to selecting out a pure spinor (the only non-zero component of this pure spinor is the $1_{+1}$ component of the anti-chiral SO(8) spinor).

It is interesting that pure spinors arise naturally in the covariant N=(2,0) twistor-string since $D_+ \Theta^\alpha \gamma^\mu_{\alpha\beta} D_- \Theta^\beta = 0$ is implied by $D_{\pm} X^\mu = D_{\pm} \Theta^\alpha \gamma^\mu_{\alpha\beta} \Theta^\beta$, which is an on-shell equation of motion for the superfields. Using Howe’s observation that the on-shell supergravity and super-Yang-Mills classical equations of motion are related to the existence of pure spinors in loop superspace, Tonin was able to show that the classical N=(2,0) twistor-string action can be consistently coupled (i.e., coupled in a manner that preserves the local gauge symmetries of the two-dimensional action) to a supergravity and super-Yang-Mills background if the background fields satisfy their classical equations of motion.\(^{19}\) If covariant quantization of the N=(2,0) twistor-string were possible, one could see how the classical supergravity and super-Yang-Mills equations of motion are modified by requiring the full quantum action to be consistently coupled to the background fields.

**Appendix: Gauge-Fixing of the N=(2,0) Twistor-String**

This appendix will review Section III.C of reference 17, in which the Lorentz-covariant twistor-string action for the Green-Schwarz heterotic superstring is gauge-fixed to a free-field action.

**A. The Lorentz-Covariant N=(2,0) Twistor-String Action**

The Lorentz-covariant action for the N=(2,0) twistor-string defined on an N=(2,0) super-worldsheet with Minkowski metric is:\(^{10,20,43}\)

$$\int dz d\bar{z} d\kappa^+ d\kappa^- \left\{-i (P^+_{\mu} \tilde{\Pi}^\mu_{\kappa^+} - P^-_{\mu} \tilde{\Pi}^\mu_{\kappa^-}) + \frac{1}{2} \tilde{\Phi}^+ q \tilde{\Phi}^q \right\}$$

$$- \frac{1}{2} \kappa^+ \left[ \partial_{\bar{z}} X^\mu (\hat{D}_+ \Theta^\alpha \gamma^\mu_{\alpha\beta} \Theta^\beta) - \hat{D}_+ X^\mu (\partial_{\bar{z}} \Theta^\alpha \gamma^\mu_{\alpha\beta} \Theta^\beta) \right]$$

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$$+\frac{1}{2}\kappa^-[\partial_z X_\mu (\hat{D}_- \Theta^\alpha \gamma^\mu_{\alpha\beta} \Theta^\beta) - \hat{D}_- X_\mu (\partial_z \Theta^\alpha \gamma^\mu_{\alpha\beta} \Theta^\beta)]$$

with the chirality constraints, \( \hat{D}_- \hat{\Phi}^{+q} = \hat{D}_+ \hat{\Phi}^{-q} = 0 \),

and the covariant derivatives, \( \hat{D}_\pm \equiv \partial_{\kappa^\pm} + i\kappa^\mp [\partial_z + e(z, \bar{z})\partial_z + (\partial_{\bar{z}} e(z, \bar{z}))M] \),

where \([z, \kappa^+, \kappa^-, \bar{z}]\) are the coordinates for the Minkowski-space worldsheet (note that \(\kappa^- = (\kappa^+)^*\), but \(\bar{z} \neq z^*\)), \(e(z, \bar{z})\) is a real component field independent of \(\kappa^\pm\) and is the only remnant of the two-dimensional super-vielbein \(\hat{D}_z \equiv -i\frac{1}{2} \{\hat{D}_+, \hat{D}_-\} = \partial_z + e\partial_z + (\partial_{\bar{z}} e)M\), \(M\) is the generator of two-dimensional Lorentz rotations that measures the conformal weight with respect to \(\partial_{\bar{z}}\) (i.e., \(M\) commutes with everything except for \([M, \partial_z] = \partial_z\) and \([M, \hat{\Phi}] = \frac{i}{2} \hat{\Phi}\)), \(\hat{\Pi}_\kappa^\pm \equiv \hat{D}_\pm X^\mu - i(\hat{D}_\pm \Theta^\alpha \gamma^\mu_{\alpha\beta} \Theta^\beta)\), \(P_\mu^\pm\) are Lagrange multipliers for \(\hat{\Pi}_\kappa^\pm\), and \(X^\mu, \Theta^\alpha, \hat{\Phi}^{+q}, \hat{\Phi}^{-q}\) are N=(2,0) superfields whose \(\kappa^\pm = 0\) components are the usual superspace variables of the Green-Schwarz heterotic superstring, \(x^\mu, \theta^\alpha, \phi^{+q}, \phi^{-q}\) for \(q=1\) to 16. Note that because the Wess-Zumino term multiplying \(\kappa^+\) (or \(\kappa^-\)) is chiral (or anti-chiral) when \(\hat{\Pi}_\kappa^\pm = 0\), the action will be super-reparameterization invariant after shifting \(P_\mu^+\) and \(P_\mu^-\) appropriately. \(^\dagger\)

The equations of motion one gets from varying the unconstrained superfields are:

\[
\hat{D}_+ \hat{\Phi}^{+q} = \hat{D}_- \hat{\Phi}^{-q} = \hat{\Pi}_\kappa^\pm = \kappa^+\kappa^- (\hat{D}_+(P_\mu^+ \Pi_\kappa^\mu) + \hat{D}_-(P_\mu^- \Pi_\kappa^\mu) - \frac{i}{2}(\hat{\Phi}^{+q} \partial_z \hat{\Phi}^{-q} + \hat{\Phi}^{-q} \partial_z \hat{\Phi}^{+q}))
\]

\[
= \hat{D}_+ P_\mu^+ + i\kappa^+ (\hat{D}_+ \Theta^\alpha \gamma^\mu_{\alpha\beta} \partial_z \Theta^\beta) - \hat{D}_- P_\mu^- - i\kappa^- (\hat{D}_- \Theta^\alpha \gamma^\mu_{\alpha\beta} \partial_z \Theta^\beta)
\]

\[
= (P_\mu^+ + \frac{1}{2}\kappa^+ \Pi_\kappa^\mu) (\gamma^\mu_{\alpha\beta} \hat{D}_+ \Theta^\beta) - (P_\mu^- + \frac{1}{2}\kappa^- \Pi_\kappa^\mu) (\gamma^\mu_{\alpha\beta} \hat{D}_- \Theta^\beta) = 0.
\]

These equations imply that \(\hat{D}_\pm \Theta^\alpha \gamma^\mu_{\alpha\beta} \hat{D}_\pm \Theta^\beta = 0\) (i.e., \(\hat{D}_\pm \Theta^\alpha\) are “pure spinors”), and that

\[
\hat{D}_+ P_\mu^+ + \frac{1}{2}\Pi_\kappa^\mu + i\kappa^+ (\hat{D}_+ \Theta^\alpha \gamma^\mu_{\alpha\beta} \partial_z \Theta^\beta) = \hat{D}_- P_\mu^- + \frac{1}{2}\Pi_\kappa^\mu + i\kappa^- (\hat{D}_- \Theta^\alpha \gamma^\mu_{\alpha\beta} \partial_z \Theta^\beta)
\]

\(^\dagger\) It has recently been shown\(^{41}\) that the second and third lines of equation (A.1) can be written in a manifestly super-reparameterization invariant form by introducing a superfield \(E^\pm\) in the place of the coordinate \(\kappa^\pm\). Since this superfield can be gauge-fixed on-shell to be proportional to \(\kappa^\pm\), these two actions are classically equivalent, where the proportionality constant is interpreted as the string tension.
\[ A(\hat{D}_+ \Theta^\alpha \gamma_\alpha^\mu \hat{D}_- \Theta^\beta) \]  

(A.3)

for some real N=2 superfield A. Therefore,

\[ \hat{D}_\pm (\Pi_\pm^\mu - A \hat{\Pi}_\pm^\mu) = (\hat{\Pi}_\pm^\mu)^2 = \kappa^+ \kappa^- [(\Pi_\pm^\mu - A \hat{\Pi}_\pm^\mu)^2 + \frac{i}{2} (\Phi^+ \partial_z \Phi^{-q} + \Phi^{-q} \partial_z \Phi^+ \partial_\bar{q} \Phi^+ \partial_\bar{q} \Phi^+ \partial_\bar{q})] = 0, \]  

(A.4)

where \( \hat{\Pi}_\pm^\mu \equiv \partial_\pm X^\mu - i \partial_\pm \Theta^\alpha \gamma_\alpha^\mu \Theta^\beta \hat{D}_+ \Theta^\alpha \gamma_\alpha^\mu \hat{D}_- \Theta^\beta \). In addition to implying the usual superstring equations of motion for the component fields, \( X^\mu, \Theta^\alpha, \) and \( \phi^p \),

\[ \partial_\pm \phi^p = \partial_\pm \pi_\pm^\mu = \pi_\pm \mu (\gamma_\alpha^\mu \partial_\pm \Theta^\beta) = (\pi_\pm^\mu)^2 + i \phi^p \partial_\pm \phi^p = 0, \]  

(A.5)

where \( \pi_\pm^\mu \equiv \partial_\pm X^\mu - i (\partial_\pm \Theta^\alpha \gamma_\alpha^\mu \Theta^\beta), \partial_\pm \equiv \partial_z + e \partial_\bar{z} \) and \( \partial_\pm \equiv (1 - ae) \partial_z - a \partial_\bar{z} \), these superfield equations fix the values of the auxiliary fields in \( X^\mu, \Theta^\alpha, P^\pm_\mu, \Phi^+_q, \) and \( \Phi^{-q} \).

B. The Gauge-Fixing Procedure

With the appropriate transformations of \( P^\pm_\mu, \Phi^+_q, \) and \( \Phi^{-q} \), the action of equation (A.1) is invariant under the N=2 super-reparameterizations,

\[ \delta z = 2 R - \kappa^+ \hat{D}_+ R - \kappa^- \hat{D}_- R, \quad \delta \kappa^\pm = - i \hat{D}_\mp R, \quad \delta \bar{z} = r + e \delta z \]  

(A.6)

where \( R(z, \kappa^+, \kappa^-, \bar{z}) \) is a real N=2 superfield and \( r(z, \bar{z}) \) is a real component field independent of \( \kappa^\pm \) (from this super-reparameterization, \( \delta \hat{D}_+ = -i(\hat{D}_+ \hat{D}_- R) \hat{D}_+ \) and \( \delta \hat{D}_- = -i(\hat{D}_- \hat{D}_+ R) \hat{D}_- \) where \( \delta e = - \partial_z r - e \partial_\bar{z} r + r \partial_\bar{z} e \)), under the six independent \( K^\beta \)-transformations,

\[ [\delta \Theta^\alpha = (\hat{D}_+ \Theta^\gamma \gamma_\gamma^\mu \hat{D}_- \Theta^\delta) (\gamma_\gamma^\alpha \kappa^\beta), 2 \hat{D}_+ \Theta^\alpha (\hat{D}_- \Theta^\beta K^\beta) - 2 \hat{D}_- \Theta^\alpha (\hat{D}_+ \Theta^\beta K^\beta), \]

\[ \delta X^\mu = i(\delta \Theta^\alpha \gamma_\alpha^\mu \Theta^\beta) \]  

(A.7)

(only six are independent since \( \delta \Theta^\alpha \gamma_\alpha^\mu \hat{D}_\pm \Theta^\beta = 0 \) on-shell), and under the five independent complex \( C^\alpha \)-transformations,

\[ [\delta \Theta^\alpha = \delta X^\mu = 0, \delta P^+\mu = \hat{D}_+ C^\alpha \gamma_\alpha^\mu \hat{D}_+ \Theta^\beta, \delta P^-\mu = \hat{D}_- C^\alpha \gamma_\alpha^\mu \hat{D}_- \Theta^\beta] \]  

(A.8)

(only five are independent since \( \delta P^\pm \gamma_\mu^\alpha \hat{D}_\pm \Theta^\beta = \hat{D}_\pm \delta P^\pm_\mu = 0 \) on-shell).
In order to write the action in terms of free fields, it is necessary to use the six $K_{\beta}$-transformations to gauge-fix to zero $\gamma^{+}_{\dot{a}\beta}\Theta^\beta$ for $\dot{a}=1$ to 6, and to use the five $C^\alpha$-transformations to gauge-fix the non-auxiliary components of $P^{\pm}_{\mu}$. Since none of these gauge transformations involve derivatives on $K_{\beta}$ or $C^\alpha$, there are no propagating ghosts coming from this gauge fixing. Furthermore, the N=(2,0) super-reparameterizations, $r(z,\bar{z})$ and $R(z,\kappa^\pm,\bar{z})$, should be used to locally gauge-fix $e(z,\bar{z})$ and $A(z,\kappa^\pm,\bar{z})$ to zero, giving rise to the usual right and left-moving fermionic reparameterization ghosts of conformal weight +2, two right-moving bosonic ghosts of conformal weight $+\frac{3}{2}$, and one right-moving fermionic ghost of conformal weight +1.

Because six components of $\Theta^\alpha$ have been gauge-fixed to zero, only an SU(4)xU(1) subgroup of the SO(9,1) Lorentz invariance remains manifest in this N=(2,0) superconformal gauge. Under this SU(4)xU(1) subgroup, the SO(8) anti-chiral spinor, $(\gamma^+\Theta)^{\dot{a}}$, can be chosen to break up into a $(1_{+1},6_0,1_{-1})$ representation, in which case the SO(8) chiral spinor, $(\gamma^-\Theta)^{a}$, breaks up into a $(4_{\pm \frac{1}{2}},\bar{4}_{-\frac{1}{2}})$ representation, and the SO(9,1) vector, $X^\mu$, breaks up into a $(1_0,1_0,4_{-\frac{1}{2}},\bar{4}_{\frac{1}{2}})$ representation.

Since the constraint $D_+\Theta^\alpha\gamma^\mu_{\alpha\beta}D_+\Theta^\beta=0$ implies that $(\gamma^+D_+\Theta)^{\dot{a}}(\gamma^+D_+\Theta)^{\dot{a}}=0$, it can be assumed that $D_+[\gamma^+(\gamma^+\Theta)^7-i(\gamma^+\Theta)^8]=0$ without loss of generality (if $D_+[\gamma^+(\gamma^+\Theta)^7+i(\gamma^+\Theta)^8]=0$, simply exchange $\kappa^+$ with $\kappa^-$ everywhere). After making this choice, the constraints $\Pi^\mu_{\kappa^\pm}=0$ can be used to combine the $X^\mu$ and $\Theta^\alpha$ real superfields into the following chiral and anti-chiral complex superfields:

$$
\Psi^\pm \equiv (\gamma^+\Theta)^7 \pm i(\gamma^+\Theta)^8, \quad S^{+l} \equiv (\gamma^-\Theta)^l + i(\gamma^-\Theta)^{l+4}, \quad S^{-\bar{l}} \equiv (\gamma^-\Theta)^l - i(\gamma^-\Theta)^{l+4}, \\
X^{+\bar{l}} \equiv X^l + iX^{l+4} + i\Psi^+S^{-\bar{l}}, \quad X^{-l} \equiv X^l - iX^{l+4} + i\Psi^-S^{+l}, \quad X^{\pm} \equiv X^0 \pm X^9,
$$

where $D_-\Psi^+ = D_-S^{+l} = D_-X^{+\bar{l}} = D_+\Psi^- = D_+S^{-\bar{l}} = D_+X^{-l} = 0$, \hspace{1cm} (A.9)

$$(\Psi^+)^* = \Psi^-, \quad (S^{+l})^* = S^{-\bar{l}}, \quad (X^{+\bar{l}})^* = X^{-l}, \quad (X^{+})^* = X^+, \quad (X^-)^* = X^-,$$

and $(\Psi^+, \Psi^-, S^{+l}, S^{-\bar{l}}, X^{+\bar{l}}, X^{-l}, X^+, X^-)$ transforms like a

$(1_{+1},1_{-1},4_{+\frac{1}{2}},\bar{4}_{-\frac{1}{2}},\bar{4}_{+\frac{1}{2}},4_{-\frac{1}{2}},1_0,1_0)$ representation of SU(4)xU(1) for $l=1$ to 4.
C. The Gauge-Fixed Free-Field N=(2,0) Twistor-String Action

In terms of these complex superfields, the action of equation (A.1) in N=(2,0) superconformal gauge takes the following simple form:

\[ S = \int dzd\bar{z}d\kappa^+d\kappa^- \left[ \frac{i}{4}(X^{+\bar{i}}\partial_{\bar{z}}X^{-\bar{l}} - X^{-\bar{l}}\partial_{\bar{z}}X^{+\bar{i}}) + W^-\partial_{\bar{z}}\Psi^+ - W^+\partial_{\bar{z}}\Psi^- + \frac{1}{2}\Phi^{+\bar{q}}\Phi^{-q} \right] \]

(A.10)

with the constraints:

\[ D_-W^- - D_-\Psi^-(X^- + iS^{+l}S^{-\bar{l}}) = D_+W^+ - D_+\Psi^+(X^- - iS^{+l}S^{-\bar{l}}) = \]

(A.11)

\[ \kappa^+\kappa^-[\partial_{\bar{z}}X^{+\bar{i}}\partial_{\bar{z}}X^{-\bar{l}} - \partial_{\bar{z}}X^{-\bar{l}}\partial_{\bar{z}}X^{+\bar{i}} + \frac{i}{2}(\Phi^{+\bar{q}}\partial_{\bar{z}}\Phi^{-q} + \Phi^{-q}\partial_{\bar{z}}\Phi^{+\bar{q}})] = \]

\[ D_+X^+ - i\Psi^-D_+\Psi^+ = D_-X^- - i\Psi^+D_-\Psi^- = \]

\[ D_+X^- - iS^{-\bar{l}}D_+S^{+l} = D_-X^+ - iS^{+l}D_-S^{-\bar{l}} = \]

\[ D_+X^{+\bar{i}} - 2iS^{-\bar{l}}D_+\Psi^+ = D_-X^{+\bar{i}} = D_+X^{-\bar{l}} = D_-X^{-\bar{l}} = 2iS^{+l}D_-\Psi^- = \]

\[ D_-\Psi^+ = D_-S^{+l} = D_-\Phi^{+\bar{q}} = D_+\Psi^- = D_+S^{-\bar{l}} = D_+\Phi^{-q} = 0. \]

Note that the action is invariant under \( W^\pm \rightarrow W^\pm + D_\pm \Lambda^\pm \) and that the constraints on \( W^\pm \) are solved by \( W^\pm = \Psi^\pm(X^- \mp iS^{+l}S^{-\bar{l}}) \).

It is easy to check that the only effect of the right-moving constraints on the superfields in the action is to fix their chiralities through

\[ D_-X^{+\bar{i}} = D_+X^{-\bar{l}} = D_-\Psi^+ = D_+\Psi^- = D_-\Phi^{+\bar{q}} = D_+\Phi^{-q}; \]

(A.12)

to relate \( W^+ \) and \( W^- \) through the condition

\[ D_+W^+D_-\Psi^- - D_-W^-D_+\Psi^+ + \frac{i}{2}D_+X^{+\bar{i}}D_-X^{-\bar{l}} = 0; \]

(A.13)

and to require that

\[ D_+\Psi^+D_-\Psi^- - i\Psi^+\partial_{\bar{z}}\Psi^- - i\Psi^-\partial_{\bar{z}}\Psi^+ = \partial_{\bar{z}}X^+ \text{ for some real superfield } X^+. \]

(A.14)
Furthermore, the $\kappa^+ = \kappa^- = 0$ component of $\partial_\bar{z} X^+$ should equal the $\partial_\bar{z} x^+$ component field that appears in the left-moving Virasoro constraint (this implies that $\int_C dz d\kappa^+ d\kappa^- (\Psi^+ \Psi^-) = \int_C d\bar{z} (\partial_\bar{z} x^+)$ for any closed curve C).

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