Noncommutative (A)dS and Minkowski spacetimes from quantum Lorentz subgroups

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Abstract

The complete classification of classical \(r\)-matrices generating quantum deformations of the \((3 + 1)\)-dimensional (A)dS and Poincaré groups such that their Lorentz sector is a quantum subgroup is presented. It is found that there exists three classes of such \(r\)-matrices, one of them being a novel two-parametric one. The (A)dS and Minkowskian Poisson homogeneous spaces corresponding to these three deformations are explicitly constructed in both local and ambient coordinates. Their quantization is performed, thus giving rise to the associated noncommutative spacetimes, that in the Minkowski case are naturally expressed in terms of quantum null-plane coordinates, and they are always defined by homogeneous quadratic algebras. Finally, non-relativistic and ultra-relativistic limits giving rise to novel Newtonian and Carrollian noncommutative spacetimes are also presented.

Keywords: quantum groups, cosmological constant, noncommutative spaces, Poisson homogeneous spaces, Galilei and Carroll spacetimes, Minkowski and (A)dS spacetimes, Lie bialgebras

1. Introduction

It is well-known that quantum Poincaré and (anti-)de Sitter (hereafter (A)dS) groups (see [1–11] and references therein) provide Hopf algebra deformations of relativistic symmetries which can be used to construct deformed special relativity (DSR) models [12–17] in which the quantum deformation parameter is assumed to be related to the Planck scale. In this
setting, quantum gravity effects are assumed to be described in a schematic way through the associated noncommutative Minkowski and (A)dS spacetimes. Moreover, in the (A)dS case the interplay between the non-vanishing cosmological constant $\Lambda$ and the Planck scale parameter can be explicitly described.

In this approach, the natural question concerning the complete classification of quantum Poincaré and (A)dS groups in $(3 + 1)$ dimensions arises, since solving this problem will provide the full set of available Hopf algebra deformations of relativistic symmetries. In the Poincaré case, such classification was given in [7] at the quantum group level, while classical $r$-matrices generating all possible quantum deformations were classified in [8]. However, to the best of our knowledge, similar results are not available for the (A)dS case yet. Besides its intrinsic mathematical relevance, the latter result would be indeed relevant for DSR Hopf algebra models dealing with the propagation of particles on quantum spacetimes at cosmological distances.

The aim of this paper is to contribute to fill this gap, at least partially, by providing the complete classification of classical $r$-matrices of the Poincaré and (A)dS groups endowed with an outstanding condition: that their $\mathfrak{so}(3,1)$ Lorentz sector has to be a Hopf subalgebra after the full quantum group is constructed. As we will show, this problem can be fully and explicitly solved by considering that the complete classification of classical $r$-matrices of the $\mathfrak{so}(3,1)$ Lorentz group is known [18], and the answer leads to only three families of $r$-matrices generating quantum Lorentz subgroups. One of them is a novel biparametric $r$-matrix whose associated noncommutative spacetime is constructed and analysed in detail. Remarkably enough, the cosmological constant parameter $\Lambda$ does not appear in any of the three $r$-matrices, which means that these three solutions are invariant under the flat limit $\Lambda \to 0$. Moreover, it is also explicitly shown that there does not exist any non-trivial quantum Poincaré or (A)dS group for which the Lorentz sector is a non-deformed one.

It is worth stressing that the well-known $\kappa$-deformations [1–4, 10, 19, 20] do not belong to the class of quantum groups here studied. Therefore, the new noncommutative Minkowski and (A)dS spacetimes here presented would be worth to be considered in order to construct new DSR models. As we will see in detail, in the Minkowskian case these novel noncommutative spacetimes are defined through a homogeneous quadratic algebra of spacetime operators, and the natural setting for them is to consider noncommutative null-plane coordinates. Consequently, these new noncommutative Minkowskian spacetimes are quite different from the $\kappa$-Minkowski noncommutative spacetime [2] and from its linear-algebraic generalizations [21–23] (see also references therein).

The paper is organized as follows. In the next section the family of $(3 + 1)$-dimensional (A)dS and Poincaré Lie algebras, Lie groups and classical homogeneous spacetimes is presented in a unified setting by making use of the cosmological constant $\Lambda$, where these three Lie groups are collectively denoted by $G_\Lambda$. After recalling in section 3 the basic tools and concepts needed for the paper, we present our main results in section 4 by taking into account that any quantum deformation for $G_\Lambda$ has to be generated by an underlying coboundary Lie bialgebra structure coming from a solution of the modified classical Yang–Baxter equation (mCYBE). In particular, we firstly prove that there does not exist any quantum deformation for the family $G_\Lambda$ with an undeformed Lorentz subgroup. Secondly, we explicitly compute all classical $r$-matrices for the Poincaré and (A)dS groups that present a Lorentz sub-Lie bialgebra structure. It comes out that there only exist three types of such $r$-matrices, which are explicitly obtained: two of them depend on a single deformation parameter, while the other one is a two-parametric deformation. The $(2 + 1)$-dimensional counterpart of this classification is also presented in section 4.1 and the differences that arise between both dimensions are analysed. Furthermore, our results entail a classification of $(3 + 1)$-dimensional (A)dS and Minkowski noncommutative spacetimes associated to deformations with a quantum Lorentz
subgroup, whose semiclassical counterpart corresponds to Poisson homogeneous spaces (PHS) of Poisson Lorentz subgroup type. The latter are deduced in section 5 for the three types of quantum deformations by considering both local and ambient coordinates, and are summarized in table 2. In addition, we perform in section 6 the non-relativistic and ultra-relativistic limits of the above classification, giving rise to Newtonian and Carrollian quantum deformations, for which the relevant role formerly played by the Lorentz subgroup is now replaced by the three-dimensional (3D) Euclidean one generated by rotations and boost transformations. Their corresponding noncommutative spacetimes are also derived and presented in table 3. Finally some remarks and open problems close the paper.

2. (A)dS and Poincaré groups: a joint description

In our framework we will consider the \((3+1)\)D Poincaré and (A)dS Lie algebras expressed in a unified setting as a one-parametric family of Lie algebras denoted by \(\mathfrak{g}_\Lambda\) depending explicitly on the cosmological constant parameter \(\Lambda\). In the usual kinematical basis, spanned by the generators of time translations \(P_0\), spatial translations \(P = (P_1, P_2, P_3)\), boost transformations \(K = (K_1, K_2, K_3)\) and rotations \(J = (J_1, J_2, J_3)\), the commutation relations of \(\mathfrak{g}_\Lambda\) read

\[
\begin{align*}
[L_a, J_b] &= \epsilon_{abc} J_c, \\
[L_a, P_b] &= \epsilon_{abc} P_c, \\
[L_a, K_b] &= \epsilon_{abc} K_c, \\
[J_a, P_0] &= P_a, \\
[K_a, P_0] &= \delta_{ab} P_0, \\
[K_a, K_b] &= -\epsilon_{abc} J_c, \\
[P_0, P_a] &= -\Lambda L_a, \\
[P_0, P_b] &= \Lambda \epsilon_{abc} J_c, \\
[J_a, P_0] &= 0.
\end{align*}
\] (2.1)

From now on sum over repeated indices will be understood unless otherwise stated. Hereafter, Latin indices run as \(a, b, c = 1, 2, 3\), and Greek ones as \(\mu = 0, 1, 2, 3\). The family of Lie algebras \(\mathfrak{g}_\Lambda\) encompasses the dS algebra \(\mathfrak{so}(4,1)\) when \(\Lambda > 0\), the AdS algebra \(\mathfrak{so}(3,2)\) if \(\Lambda < 0\) and the Poincaré one \(\mathfrak{iso}(3,1)\) for \(\Lambda = 0\). As a vector space, \(\mathfrak{g}_\Lambda\) can be decomposed in the form

\[
\mathfrak{g}_\Lambda = \mathfrak{h} \oplus \mathfrak{t}, \quad \mathfrak{h} = \text{span}\{K, J\} = \mathfrak{so}(3,1), \quad \mathfrak{t} = \text{span}\{P_0, P\},
\] (2.2)

where \(\mathfrak{h}\) is the Lorentz subalgebra and \(\mathfrak{t}\) is the translation sector.

A faithful representation of \(\mathfrak{g}_\Lambda\) (2.1), \(\rho: \mathfrak{g}_\Lambda \to \text{End}(\mathbb{R}^5)\), for a generic element \(X \in \mathfrak{g}_\Lambda\) is given by [20]

\[
\rho(X) = x^\mu \rho(P_\mu) + \xi^a \rho(K_a) + \theta^\mu \rho(J_\mu) = \begin{pmatrix}
0 & \Lambda x^0 & -\Lambda x^1 & -\Lambda x^2 & -\Lambda x^3 \\
x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\
x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\
x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\
x^3 & \xi^3 & -\theta^2 & \theta^1 & 0
\end{pmatrix}.
\] (2.3)

The corresponding exponentiation provides a one-parametric family of Lie groups, denoted by \(G_\Lambda\), with Lie algebra \(\mathfrak{g}_\Lambda\). Hence \(G_\Lambda\) contains the dS group \(\text{SO}(4,1)\) for \(\Lambda > 0\), the AdS group \(\text{SO}(3,2)\) for \(\Lambda < 0\), and the Poincaré one \(\text{ISO}(3,1)\) for \(\Lambda = 0\). Note that the exponentiation of (2.3) only recovers the connected component to the identity of these three Lie groups, but the complete description of these groups can be obtained by considering the parity and
time-reversal involutive automorphisms. The family $G_\Lambda$ can be parametrized in terms of local coordinates $\{x^\mu, \xi^\mu, \theta^\mu\}$ in the form
\[
G_\Lambda = \exp \left( x^0 \rho(P_0) \right) \exp \left( x^1 \rho(P_1) \right) \exp \left( x^2 \rho(P_2) \right) \exp \left( x^3 \rho(P_3) \right) \exp \left( \xi^1 \rho(K_1) \right) \\
\times \exp \left( \xi^2 \rho(K_2) \right) \exp \left( \xi^3 \rho(K_3) \right) \exp \left( \theta^1 \rho(J_1) \right) \exp \left( \theta^2 \rho(J_2) \right) \exp \left( \theta^3 \rho(J_3) \right).
\] (2.4)

These coordinates are the so-called exponential coordinates of the second kind on $G_\Lambda$. Therefore $G_\Lambda$ comprises the isometry Lie groups of the $(3+1)$D Minkowski and (A)dS spacetimes, which we denote collectively by $M_\Lambda$ and have a constant sectional curvature given by $-\Lambda$. For each of these three spacetimes, the stabilizer of a point is the Lorentz subgroup $H$ with Lie algebra $\mathfrak{h}$ (2.2). Thus, there exists a global isometry between $M_\Lambda$ and the left coset space $G_\Lambda/H$, so that we write
\[
M_\Lambda = G_\Lambda/H, \quad H = \text{SO}(3,1) = (\mathbf{K}, \mathbf{J}).
\] (2.5)

Hence $M_\Lambda$ is a family of symmetric homogeneous spaces, and we can identify their tangent space at every point $m = gH \in M_\Lambda$, $g \in G_\Lambda$, with the translation sector, i.e.
\[
T_m(M_\Lambda) = T_{\text{eff}}(G_\Lambda/H) \simeq g_\Lambda/\mathfrak{h} \simeq \text{span}\{P_0, P\}.
\] (2.6)
The four Lie group local coordinates $x^\mu$ in (2.4), associated to the translation generators $P_{\mu}$, descend to coordinates on $M_\Lambda$.

The representation of $G_\Lambda$ (2.4), coming from (2.3), allows us to consider $G_\Lambda$ as the isometry group of the 5D linear space $(\mathbb{R}^5, I_\Lambda)$, with canonical linear ambient coordinates $(s^4, s^\mu)$, such that $I_\Lambda$ is the bilinear form given by
\[
I_\Lambda = \text{diag}(+1, -\Lambda, \Lambda, \Lambda, \Lambda),
\] (2.7)
fulfilling that $G^2_\Lambda I_\Lambda G_\Lambda = I_\Lambda$. The point with ambient coordinates $O = (1, 0, 0, 0, 0)$ is invariant under the action of the Lorentz subgroup $H$, and will be taken as the origin of $M_\Lambda$. The orbit passing through $O$ corresponds to the $(3+1)$D spacetime $M_\Lambda$ (2.5) defined by the pseudosphere
\[
\Sigma_\Lambda \equiv (s^4)^2 - \Lambda (s^0)^2 + \Lambda (s^1)^2 + (s^2)^2 + (s^3)^2 = 1,
\] (2.8)
determined by $I_\Lambda$ (2.7). In the flat limit $\Lambda \to 0$, the Minkowski spacetime will be identified with the hyperplane $s^4 = +1$ containing $O$. We remark that the coordinates
\[
q^\mu = \frac{s^\mu}{s^4}
\] (2.9)
are just the Beltrami projective coordinates in $M_\Lambda$ which can be obtained through the projection with pole $(0, 0, 0, 0) \in \mathbb{R}^3$ of a point with ambient coordinates $(s^4, s^\mu)$ onto the projective hyperplane with $s^4 = +1$ (see [24] for details).

Now the set of spacetime local coordinates $x^\mu$ can be introduced through the following action onto the origin $O$ of the one-parameter subgroups of $G_\Lambda$ (2.4) [20]
\[
(s^4, s^\mu)^T = \exp \left( x^0 \rho(P_0) \right) \exp \left( x^1 \rho(P_1) \right) \exp \left( x^2 \rho(P_2) \right) \exp \left( x^3 \rho(P_3) \right) O^T,
\] (2.10)
thus yielding
\[ s^4 = \cos(\eta x^0) \cosh(\eta x^1) \cosh(\eta x^2) \cosh(\eta x^3), \]
\[ s^0 = \frac{\sin(\eta x^0)}{\eta} \cosh(\eta x^1) \cosh(\eta x^2) \cosh(\eta x^3), \]
\[ s^1 = \frac{\sinh(\eta x^1)}{\eta} \cosh(\eta x^2) \cosh(\eta x^3), \]
\[ s^2 = \frac{\sinh(\eta x^2)}{\eta} \cosh(\eta x^3), \]
\[ s^3 = \frac{\sinh(\eta x^3)}{\eta}, \]

where the parameter \( \eta \) is defined by
\[ \eta^2 := -\Lambda. \] (2.12)

Thus \( \eta \) is real for the AdS space and a pure imaginary number for the dS one. The four space-time coordinates \( x^\mu \) are called geodesic parallel coordinates [25], and can be regarded as a generalization of the flat Cartesian coordinates to non-zero curvature. In fact, under the vanishing cosmological constant limit, \( \eta \to 0 \), the ambient (2.11) and Beltrami (2.9) coordinates reduce to the usual Cartesian ones in the Minkowski spacetime:
\[ (s^4, s^\mu) \equiv (1, q^\mu) \equiv (1, x^\mu). \] (2.13)

In ambient coordinates, the time-like metric on the \((3 + 1)D \) \( M_\Lambda \) spacetime comes from the pseudosphere (2.8) and turns out to be [24]
\[ d\sigma^2_\Lambda = \frac{1}{-\Lambda} \left( (ds^4)^2 - \Lambda (ds^0)^2 - (ds^1)^2 - (ds^2)^2 - (ds^3)^2 \right) \bigg|_{|s_\Lambda} \]
\[ = -\Lambda \left( s^0 ds^0 - s^1 ds^1 - s^2 ds^2 - s^3 ds^3 \right)^2 \]
\[ + \frac{1}{1 + \Lambda \left( (s^0)^2 - (s^1)^2 - (s^2)^2 - (s^3)^2 \right)} \left( ds^0)^2 - (ds^1)^2 - (ds^2)^2 - (ds^3)^2 \right). \] (2.14)

From this expression the metric in terms of Beltrami coordinates (2.9) can be obtained [24], and in geodesic parallel coordinates (2.11) the metric reads [20]
\[ d\sigma^2_\Lambda = \cosh^2(\eta x^1) \cosh^2(\eta x^2) \cosh^2(\eta x^3) (dx^0)^2 \]
\[ - \cosh^2(\eta x^2) \cosh^2(\eta x^3) (dx^1)^2 - \cosh^2(\eta x^3) (dx^2)^2 - (dx^3)^2. \] (2.15)

Indeed, when \( \eta \to 0 \) expressions (2.14) and (2.15) reduce to the usual metric on the Minkowski spacetime:
\[ d\sigma^2_0 = (ds^0)^2 - (ds^1)^2 - (ds^2)^2 - (ds^3)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \] (2.16)

3. Lie bialgebras and Poisson homogeneous spaces

Let us briefly recall the basic notions needed for the paper, and set up the notation; for more details we refer to [25]. A Lie bialgebra is a pair \((\mathfrak{g}, \delta)\) where \( \mathfrak{g} \) is a Lie algebra and
\[ \delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} \] (3.1)
is a linear map called cocommutator and satisfying the following conditions

$$\sum_{\text{cycl}} (\delta \otimes \text{id}) \circ \delta(X) = 0, \quad \forall \ X \in \mathfrak{g}. \quad (3.2)$$

$$\delta([X, Y]) = \text{ad}_X \delta(Y) - \text{ad}_Y \delta(X), \quad \forall \ X, Y \in \mathfrak{g}. \quad (3.2)$$

The first expression is called co-Jacobi condition since this is equivalent to requiring that the transpose map $\delta^T : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies the Jacobi identity. The second relation states that the map $\delta$ is a one-cocycle in the Chevalley–Eilenberg cohomology with values in $\mathfrak{g} \wedge \mathfrak{g}$. Given a Lie bialgebra $(\mathfrak{g}, \delta)$ and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the pair $(\mathfrak{h}, \delta|_{\mathfrak{h}})$ is said to be a sub-Lie bialgebra of $(\mathfrak{g}, \delta)$ in the case that $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$.

For some Lie bialgebra structures the cocommutator map can be completely defined in terms of an element $r \in \mathfrak{g} \wedge \mathfrak{g}$. In these cases, called one-coboundaries, the map

$$\delta_r(X) = \text{ad}_X(r) \quad (3.3)$$

defines a Lie bialgebra structure if and only if $r$ fulfills the mCYBE

$$\text{ad}_X[[r, r]] = 0, \quad \forall \ X \in \mathfrak{g}, \quad (3.4)$$

where the algebraic Schouten bracket is defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \quad (3.5)$$

Therefore an element $r \in \mathfrak{g} \wedge \mathfrak{g}$ satisfies the mCYBE (3.4) if and only if

$$[[r, r]] \in \bigwedge^3_{\mathfrak{g}} \mathfrak{g}, \quad (3.6)$$

i.e. if its algebraic Schouten bracket is a $\mathfrak{g}$-invariant element of $\bigwedge^3 \mathfrak{g}$. Particular solutions of the mCYBE are those which fulfill the classical Yang–Baxter equation (CYBE)

$$[[r, r]] = 0, \quad (3.7)$$

that is, the algebraic Schouten bracket vanishes identically. Solutions of the CYBE are called triangular (or nonstandard) classical $r$-matrices, while solutions of the mCYBE that are not solutions of the CYBE are called quasitriangular (or standard) classical $r$-matrices. A Lie bialgebra $(\mathfrak{g}, \delta)$ is then called a coboundary one if its cocommutator $\delta$ is of the form (3.3) for some solution of the mCYBE (3.4).

A Poisson–Lie (PL) group is the global object integrating a Lie bialgebra structure. More explicitly, a PL group is a pair $(G, \Pi)$ where $G$ is a Lie group and $\Pi$ is a Poisson structure such that the Lie group multiplication $\mu : G \times G \to G$ is a Poisson map with respect to $\Pi$ on $G$ and the product Poisson structure $\Pi_{G \times G} = \Pi \oplus \Pi$ on $G \times G$. The relation between the Poisson bivector field and the Poisson bracket is given by

$$(df_1 \otimes df_2) \Pi = \{f_1, f_2\}_\Pi. \quad (3.8)$$

A Lie subgroup $H$ of $G$ is said to be a PL subgroup of $(G, \Pi)$ if $(H, \Pi|_H)$ is a Poisson submanifold of $(G, \Pi)$. A PL group is called coboundary if its tangent Lie bialgebra is a coboundary
Let \((G, \Pi)\) be a coboundary PL group with tangent Lie bialgebra \((g, \delta)\) \((3.3)\), then the Poisson bivector on \(G\) is given by the Sklyanin bivector

\[\Pi = r^{ij} (X^L_i \otimes X^L_j - X^R_i \otimes X^R_j),\]

where \(X^L_i\) and \(X^R_i\) denote the left- and right-invariant vector fields, respectively.

A Poisson manifold \((M, \pi)\) is a manifold \(M\) endowed with a Poisson structure \(\pi\) on \(M\). A PHS for a PL group \((G, \Pi)\) is a Poisson manifold \((M, \pi)\) which is endowed with a transitive group action \(\alpha : (G \times M, \Pi \oplus \pi) \to (M, \pi)\) which is a Poisson map. When the manifold is a homogeneous space, \(M = G/H\), there does exist a distinguished class of PHS for which the Poisson structure \(\pi\) on \(M\) can be obtained by canonical projection of the PL structure \(\Pi\) on \(G\). In terms of the underlying Lie bialgebra \((g, \delta)\) of \((G, \Pi)\), this requirement corresponds to imposing the so-called coisotropy condition for the cocommutator \(\delta\) with respect to the isotropy subalgebra \(h\) of \(H\) that is given by \([27–29]\)

\[\delta(h) \subset h \wedge g.\]

A particular case fulfilling the above condition is obtained when the Lie subalgebra \(h\) is also a sub-Lie bialgebra:

\[\delta(h) \subset h \wedge h,\]

which implies that the PHS is constructed through an isotropy subgroup \(H\) such that \((H, \Pi|_H)\) is a PL subgroup of \((G, \Pi)\) and this is called a ‘PHS of Poisson subgroup type’.

We recall that since a quantum group is the quantization of a PL group \((G, \Pi)\), the quantization of a coisotropic PHS \((M = G/H, \pi)\) fulfilling \((3.10)\) provides a quantum homogeneous space (or noncommutative space) on which the quantum group co-acts covariantly \([30]\). The coisotropy condition \((3.10)\) ensures that the commutation relations that define the noncommutative space at the first-order in all the quantum coordinates close on a Lie subalgebra which is just the annihilator \(h^\perp\) of \(h\) on the dual Lie algebra \(g^\ast\) \([28, 31, 32]\).

By taking into account the above concepts, we would like to remark that in the next section we shall deduce all the possible Lie bialgebra structures for the family \(g^\Lambda (2.2)\) such that the Lorentz subalgebra \(h\) is a sub-Lie bialgebra of \((g^\Lambda, \delta)\). We will find three types of such bialgebras being all of them coboundaries \((3.3)\), so coming from solutions \(r \in g^\Lambda \wedge g^\Lambda\) of the mCYBE \((3.4)\). Consequently, each of them provides a PL group \((G^\Lambda, \Pi)\) for the family \(G^\Lambda (2.4)\) through the Sklyanin bivector \((3.9)\) and, by construction, the Lorentz subgroup \(H\) becomes a PL subgroup \((H, \Pi|_H)\) of \((G^\Lambda, \Pi)\). Furthermore, each of these PL groups leads to a PHS \((M^\Lambda = G^\Lambda/H, \pi)\) where the Poisson structure \(\pi\) on \(M^\Lambda\) is obtained by canonical projection of the PL structure \(\Pi\) on \(G^\Lambda\). In section 5 we will first compute explicitly all such PHS and, secondly, we will provide their complete quantum versions, thus obtaining all the \((3 + 1)D\) noncommutative (A)dS and Minkowski spacetimes which are covariant under the only quantum (A)dS and Poincaré groups for which the Lorentz subalgebra has a quantum subgroup structure.

4. (A)dS and Poincaré bialgebras with a Lorentz sub-bialgebra

We proceed to obtain all the PL groups \((G^\Lambda, \Pi)\) associated with Lie bialgebras \((g^\Lambda, \delta)\) for which the Lorentz Lie algebra has a sub-Lie bialgebra structure. With this aim, let us recall that

**Remark 1.** All possible Lie bialgebra structures for the family \(g^\Lambda (2.1)\) are coboundary ones.
This fact is straightforward for \( \mathfrak{so}(3, 2) \) and \( \mathfrak{so}(4, 1) \) since it is well-known that all Lie bialgebra structures for semisimple Lie algebras are coboundaries. For the Poincaré algebra \( \mathfrak{iso}(3, 1) \) this property was proven in [5, 7, 8], which is a particular case of the more general result stating that all Lie bialgebras for inhomogeneous pseudo-orthogonal Lie algebras \( \mathfrak{iso}(p, q) \) with \( p + q \geq 3 \) are coboundaries [33].

Since PL groups \( (G_\Lambda, \Pi) \) are in one-to-one correspondence to Lie bialgebras and for \( g_\Lambda \) these are always coboundaries \( (g_\Lambda, \delta_r) \) (3.3), we thus face the problem consisting in obtaining all the possible solutions \( r \) of the mCYBE (3.4) such that the Lorentz sub-bialgebra condition (3.11) is fulfilled, namely that \( \delta_r(\mathfrak{h}) \subset \mathfrak{h} \cap \mathfrak{h} \).

We start by noting that the 45D vector space \( g_\Lambda \wedge g_\Lambda \) admits the following \( \text{ad}_h \)-invariant decomposition coming from (2.2):

\[
g_\Lambda \wedge g_\Lambda = (\mathfrak{h} \wedge \mathfrak{h}) \oplus (\mathfrak{h} \wedge \mathfrak{t}) \oplus (\mathfrak{t} \wedge \mathfrak{t}),
\]

(4.1)

Hence any element \( r \in g_\Lambda \wedge g_\Lambda \) can be expressed in the form

\[
r = r^{hh} + r^{ht} + r^{tt},
\]

(4.2)

\[
such that\]

\begin{align*}
r^{hh} & \in \mathfrak{h} \wedge \mathfrak{h}, & r^{ht} & \in \mathfrak{h} \wedge \mathfrak{t}, & r^{tt} & \in \mathfrak{t} \wedge \mathfrak{t},
\end{align*}

so that the general element \( r \in g_\Lambda \wedge g_\Lambda \) can be written in the kinematical basis (2.1) as

\[
r = \sum_{a < b} r^{K_a K_b} K_a \wedge K_b + \sum_{a < b} r^{J_a J_b} J_a \wedge J_b + \sum_{a, b} r^{K_a J_b} K_a \wedge J_b
\]

\[
+ \sum_{\mu \neq \nu} r^{L_{\mu \nu} P_\mu} K_a \wedge P_\mu + \sum_{\mu \neq \nu} r^{L_{\mu \nu} P_\mu} J_a \wedge P_\mu + \sum_{\mu < \nu} r^{P_\mu P_\nu} P_\mu \wedge P_\nu.
\]

(4.3)

Therefore, \( r \) initially depends on 45 parameters \( r^{XY} \). From this expression, we can directly compute the cocommutator map \( \delta_r : g_\Lambda \to g_\Lambda \wedge g_\Lambda \) using (3.3), which defines a Lie bialgebra if and only if \( r \) is a solution of the mCYBE.

Obviously, the simplest case to be studied is whether there exists a non-trivial Lie bialgebra structure for the family \( g_\Lambda \) such that the Lorentz subalgebra has a trivial sub-Lie bialgebra structure \( \delta_r(\mathfrak{h}) = 0 \). By direct computation we find the following negative result:

**Proposition 1.** The only PL group \( (G_\Lambda, \Pi) \) such that \( \Pi|_H = 0 \) is the trivial one.

**Proof.** At the Lie bialgebra level, the condition \( \Pi|_H = 0 \) reads \( \delta_r(\mathfrak{h}) = 0 \). Hence the six cocommutators \( \delta_r(K_a) \) and \( \delta_r(J_a)(a = 1, 2, 3) \) obtained from the generic \( r \) (4.3) by means of (3.3) must vanish. This amounts to solve a system of \( 6 \times 45 = 270 \) linear equations with 45 unknowns \( r^{XY} \). With the aid of the Wolfram Mathematica software system, it is found that their unique solution is \( r = 0 \).

As a straightforward consequence we find that

**Corollary 1.** The only PHS \( (M_\Lambda = G_\Lambda / H, \pi) \) of Poisson Lorentz subgroup type such that \( \Pi|_H = 0 \) is the trivial one.

Therefore the relevant question about the existence of (non-trivial) quantum symmetries that maintain the Lorentz sector undeformed is solved. Since these symmetries are necessarily quantizations (formal deformations) of the PL ones, we have proven that they cannot exist.

Next we investigate the existence of PL structures for the Poincaré and (A)dS groups with a non-trivial Poisson Lorentz subgroup. This requirement implies that \( \mathfrak{h} \) has to be endowed with a sub-Lie bialgebra structure (3.11) with \( \delta_r(\mathfrak{h}) \neq 0 \). Under this condition the cocommutators \( \delta_r(K_a) \) and \( \delta_r(J_a)(a = 1, 2, 3) \) provided by \( r \) (4.3) lead to a system of 180 linear equations whose
solution is \( r^{\mu \nu \rho} = r^{\nu \rho \mu} = r^{\rho \mu \nu} = 0 \) for all \( a \in \{1, 2, 3\} \) and \( \mu, \nu \in \{0, 1, 2, 3\} \). In this way, \( r \) reduces to \( r = r^{bb} \in \mathfrak{h} \wedge \mathfrak{h} \), which means that in the kinematical basis we are using, only the terms \( r^{\mu \nu} \), \( r^{\nu \mu} \), and \( r^{\mu \nu} \) from (4.3) do not vanish. Moreover, these terms have to be further constrained by the mCYBE. This result can be summarized as follows:

**Lemma 1.** Lie bialgebra structures for \( \mathfrak{g}_\Lambda \) such that \( \delta_\Lambda (\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h} \) are in one-to-one correspondence with elements \( r = r^{bb} \in \mathfrak{h} \wedge \mathfrak{h} \) satisfying the mCYBE (3.4), i.e.

\[
[r, r] = [[r^{bb}, r^{bb}]] \in \left( \begin{array}{c} \mathfrak{h} \\ \mathfrak{g}_\Lambda \end{array} \right), \tag{4.4}
\]

We stress that the classification of the solutions of the mCYBE for the Lorentz algebra \( so(3, 1) = \text{span}(\mathbf{K}, \mathbf{J}) \) in the kinematical basis (2.1).

**Table 1.** [18] Classification of solutions of the mCYBE for the Lorentz algebra \( so(3, 1) = \text{span}(\mathbf{K}, \mathbf{J}) \) in the kinematical basis (2.1).

| Type | \( r_\Lambda \) | \( r_B \) | \( r_C \) | \( r_D \) |
|------|----------------|--------------|--------------|--------------|
| A    | \( \alpha(-K_2 \wedge K_1 + J_2 \wedge J_3) + \beta(-K_2 \wedge J_3 + J_3 \wedge J_2) + \frac{\gamma}{2}K_1 \wedge J_1 \) | \( 2(K_1 \wedge K_2 + K_1 \wedge J_3 - K_2 \wedge J_1 - J_1 \wedge J_2) - \frac{\gamma}{4}(K_2 \wedge K_3 - K_2 \wedge J_2 - K_3 \wedge J_1 + J_1 \wedge J_3) \) | \( -(\gamma + \frac{\mu}{2})K_2 \wedge K_1 + (\gamma - \frac{\mu}{2})J_2 \wedge J_3 - \gamma K_1 \wedge J_1 + \frac{\mu}{2}(K_2 \wedge J_2 + K_3 \wedge J_3) \) | \( 3(\chi(K_1 \wedge K_2 + K_1 \wedge J_3 - J_1 \wedge J_2) - 2(\chi(K_1 \wedge J_3 - J_1 \wedge J_3) + J_1 \wedge J_3) \) |

The previous lemma together with the classification of solutions of the mCYBE for the Lorentz algebra [18] allow us to state the following main result.

**Proposition 2.** There exist three classes of PHSs \( (M_\Lambda = G_\Lambda / H, \pi) \) for each of the maximally symmetric spacetimes of constant curvature (Minkowski and (A)dS) such that the isotropy Lorentz subgroup \( H \) is a PL subgroup of \( (G_\Lambda, \Pi) \). All of them are obtained from coboundary PL structures on their respective isometry group \( G_\Lambda \) which are determined, up to \( \mathfrak{g}_\Lambda \)-isomorphisms, by the classical \( r \)-matrices

\[
\begin{align*}
  r_1 &= z(K_1 \wedge K_2 + K_1 \wedge J_3 - K_3 \wedge J_1 - J_1 \wedge J_2), \\
  r_\Pi &= zK_1 \wedge J_1, \\
  r_\Pi &= z(K_1 \wedge K_2 + K_1 \wedge J_3),
\end{align*}
\]

where \( z \) and \( z' \) are free parameters. These three \( r \)-matrices are solutions of the CYBE.

**Proof.** A PHS of Poisson subgroup type \( (G_\Lambda / H, \pi) \) can be obtained by canonical projection from a PL group \( (G_\Lambda, \Pi) \). We already know that any PL structure II on \( G_\Lambda \) is a coboundary one, so that it is completely determined by a solution of the mCYBE whose algebraic Schouten bracket is \( \mathfrak{g}_\Lambda \)-invariant (3.4). Therefore, using lemma 1 and the standard inclusion \( \mathfrak{h} \hookrightarrow \mathfrak{g}_\Lambda \) it follows that there exists a one-to-one correspondence between PHS of Poisson Lorentz subgroup type and the elements \( r \in \mathfrak{h} \wedge \mathfrak{h} \) in table 1 fulfilling (3.4) and (4.4). The triangular classical \( r \)-matrices \( r_B \) and \( r_D \) already satisfy this condition, since their Schouten bracket
vanishes, and lead to the solutions $r_1$ and $r_{II}$, respectively. On the other hand, the quasitriangular classical $r$-matrix $r_C$ verifies the relations (3.4) and (4.4) if and only if $\gamma = 0$ reducing to the particular case of $r_1$ with $z = 0$. Finally, $r_\lambda$ fulfills (3.4) and (4.4) whenever $\alpha = \beta = 0$, thus yielding $r_{II}$ which is also a solution of the CYBE (3.7).

Consequently, we have proven that there only exist three types of $(3 + 1)$D Poincaré and (A)dS quantum deformations endowed with a non-trivial quantum Lorentz subgroup, whose underlying Lie bialgebra are determined by $r_1$, $r_0$ and $r_{II}$. The three classes correspond to triangular or nonstandard deformations. Types II and III are one-parametric deformations, while type I is a two-parametric one with arbitrary deformation parameters $z$ and $z'$.

In the Poincaré case, the correspondence between the results given by proposition 2 with the 21 multiparametric PL structures presented in [8] can be easily established. The two-parametric deformation of type I for the Poincaré PL group $(G_{\Lambda=0}, \Pi)$ is just case 5 of table 1 in [8]. The classical $r$-matrix $r_{II}$ is provided by a Reshetikhin twist, since both generators in $r_{II}$ (4.5) do commute, which can be identified with case 1 with parameters $\alpha = \hat{\alpha} = 0$ in the same table. Finally, the deformation of type III comes from the lower dimensional Lorentz subalgebra $\mathfrak{so}(2, 1)$ spanned by $\{J_3, K_1, K_2\}$ and $r_{II}$ corresponds to case 6 with parameters $\beta_1 = \beta_2 = 0$ in [8].

4.1. The $(2 + 1)$-dimensional case

For the sake of completeness, it is worth comparing the $(2 + 1)$D case with the $(3 + 1)$D one characterized by propositions 1 and 2. As it will be shown in the following, the classification in $(2 + 1)$D is significantly different.

Let us consider the $(2 + 1)$D counterpart $\mathfrak{g}_2^{2+1}$ of the family of Lie algebras $\mathfrak{g}_\Lambda$ (2.1). Hence $\mathfrak{g}_\Lambda^{2+1}$ is spanned by the six generators $\{P_0, P_1, P_2, K_1, K_2, J_3\}$ in such a manner that the commutation rules are given by (2.1) setting the indices $a, b, c = 1, 2$ and fixing $c = 3$. Thus $\mathfrak{g}_\Lambda$ comprises the $(2 + 1)$D dS algebra $\mathfrak{so}(3, 1)$ for $\Lambda > 0$, the AdS algebra $\mathfrak{so}(2, 2)$ when $\Lambda < 0$ and the Poincaré one $\mathfrak{iso}(2, 1)$ for $\Lambda = 0$. The corresponding Lorentz subalgebra is given by

$$\mathfrak{h}^{2+1} = \text{span}\{K_1, K_2, J_3\} = \mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R}).$$  (4.6)

We now follow a procedure similar to the above one, so that we consider the general element $r^{2+1} \in \mathfrak{h}_\Lambda^{2+1} \wedge \mathfrak{g}_\Lambda^{2+1}$ written as

$$r^{2+1} = a_1 J_3 \wedge P_1 + a_2 J_3 \wedge K_1 + a_3 P_0 \wedge P_1 + a_4 P_0 \wedge K_1 + a_5 P_1 \wedge K_1
\begin{eqnarray*}
+ a_6 P_1 \wedge K_2 + b_1 J_3 \wedge P_2 + b_2 J_3 \wedge K_2 + b_3 P_0 \wedge P_2 + b_4 P_0 \wedge K_2
\end{eqnarray*}
\begin{eqnarray*}
+ b_5 P_2 \wedge K_2 + b_6 P_2 \wedge K_1 + c_1 J_3 \wedge P_0 + c_2 K_1 \wedge K_2 + c_3 K_1 \wedge K_2 + c_4 P_1 \wedge P_2.  \ (4.7)
\end{eqnarray*}

Therefore, $r^{2+1}$ now depends on 15 parameters: $a$’s, $b$’s and $c$’s. We recall that the explicit expressions for the general cocommutator $\delta_{2+1}$ (3.3) along with the equations coming from the mCYBE (3.4) were presented in [37] in equation (4.1), such that the notation corresponds here to set $J \equiv J_3$ and $\omega = -\Lambda$. In what follows we assume such results.

Firstly, if we require that the Lorentz subalgebra $\mathfrak{h}^{2+1}$ (4.6) remains as a trivial sub-Lie bialgebra structure, that is, $\delta_{2+1}(\mathfrak{h}^{2+1}) = 0$, we obtain that the only non-vanishing parameters in $r^{2+1}$ (4.7) are: $a_0 = -c_1$, $b_0 = c_1$, and $c_1$. Then, $r^{2+1}$ reduces to the one-parameter classical $r$-matrix given by

$$r^{2+1} = c_1 (J_3 \wedge P_0 - P_1 \wedge K_2 + P_2 \wedge K_1). \ (4.8)$$
And the mCYBE provides a single equation:

\[ \Lambda c_1^2 = 0. \]  

(4.9)

Consequently, when \( \Lambda \neq 0 \) the solution is the trivial one \( r = 0 \) as in proposition 1. Nevertheless, in the Poincaré case \( \mathfrak{so}(2, 1) \) with \( \Lambda = 0 \), there arises \( r^{2+1} \) (4.8) as the single non-trivial solution that keeps the Lorentz subalgebra underformed. Remarkably enough, this solution is just class (IV) in the classification of equivalence classes (under automorphisms) of \( r \)-matrices for the \( (2 + 1)D \) Poincaré algebra presented in [38]. In fact, it is mentioned in remark 2 in [38] that such a solution has no counterpart in higher dimensions. Furthermore \( r^{2+1} \) (4.8) also appears in the classification of Drinfel’d double structures of the \( (2 + 1)D \) Poincaré group performed in [11] as the case 0 (the ‘trivial’ Drinfel’d double structure). And \( r^{2+1} \) (4.8) has also been obtained in [36] (see equations (4.50) and (4.52)) through a contraction approach from \( \mathfrak{so}(3, 1) \) and \( \mathfrak{so}(2, 2) \).

Secondly, if we impose that \( \delta_{2+1}(h^{2+1}) \subset h^{2+1} \land h^{2+1} \neq 0 \) we find that the non-vanishing parameters in \( r^{2+1} \) (4.7) are: \( a_2, b_2, c_2, a_6 = -c_1, b_6 = c_1, \) and \( c_1 \). Hence, \( r^{2+1} \) reads

\[ r^{2+1} = a_2 J_3 \land K_1 + b_2 J_3 \land K_2 + c_2 K_1 \land K_2 + c_1 (J_3 \land P_0 - P_1 \land K_2 + P_2 \land K_1). \]  

(4.10)

And the mCYBE [37] leads to the constraint

\[ c_1^2 - a_2^2 - b_2^2 - 4\Lambda c_1^2 = 0. \]  

(4.11)

In the Poincaré case with \( \Lambda = 0 \), the solution can be reduced, via automorphisms, to \( b_2 = 0 \) and \( a_2 = -c_2 \):

\[ r^{2+1} = c_2 K_1 \land (K_2 + J_3) + c_1 (J_3 \land P_0 - P_1 \land K_2 + P_2 \land K_1). \]  

(4.12)

This two-parametric solution is just class (I) in the classification [38] and corresponds to the Drinfel’d double structure of case 1 in [11]. For \( c_1 = 0 \) one recovers the solution \( r_{III} \) in proposition 2, which has also been obtained via contraction in [36] (see equations (4.27) and (4.41)).

Finally, from the classification of classical \( r \)-matrices for \( \mathfrak{so}(3, 1) \) obtained in [18] and displayed in table 1, it follows for \( \Lambda = +1 \) that, under automorphisms, \( r^{2+1} \) (4.12), subjected to the constraint (4.11), gives rise to two solutions:

- If \( b_2 = 0, a_2 = -c_2, \) then \( \Lambda c_1^2 = c_2^2 = 0. \) Hence

\[ r^{2+1} = c_2 K_1 \land (K_2 + J_3), \]  

(4.13)

which is type D in table 1 and \( r_{III} \) in proposition 2.

- If \( a_2 = b_2 = 0, \) then \( c_2^2 - 4c_1^2 \) and \( c_2 = 2c_1. \) This leads to

\[ r^{2+1} = c_1 (2K_1 \land K_2 + J_3 \land P_0 - P_1 \land K_2 + P_2 \land K_1), \]  

(4.14)

which is the particular case of type C in table 1 for \( \gamma = \chi'/2 \equiv c_1 \) and has no \((3 + 1)D\) counterpart as shown in proposition 2.
Table 2. The three types of (A)dS and Minkowski PHS with Poisson Lorentz subgroups expressed in geodesic parallel $x^\mu$ and ambient $s^\mu$ spacetime coordinates (2.11) such that the cosmological constant $\Lambda = -\eta^2$. The ambient coordinate $s^4$ always Poisson-commutes with $s^\mu$.

| Type | $f_1 = z(K_1 \land K_2 + K_1 \land J_1 - K_1 \land J_1 - J_1 \land J_2 - J_1 \land J_2)$ |
|------|----------------------------------------------------------------------------------------------------------------------------------|
| Type I | $\{x^0, x^1\} = zA \cos(\eta x^0) \tanh(\eta x^1)$ |
| Type II | $\{x^0, x^1\} = -z A \cos(\eta x^0) \left( \frac{\tanh(\eta x^0)}{\eta} \right) - z A \cos(\eta x^0) \cosh(\eta x^1) \tanh(\eta x^1)$ |
| Type III | $\{x^0, x^1\} = -z A \cos(\eta x^0) \left( \frac{\tanh(\eta x^0)}{\eta} \right) + z A \cos(\eta x^0) \sinh(\eta x^1) \tanh(\eta x^1)$ |

5. Noncommutative (A)dS and Minkowski spacetimes

In this section we first present explicitly the three non-isomorphic PHS defined by proposition 2 and afterwards we perform their quantization.

The three PL structures ($G_{III}$, II) provided by (4.5) can be explicitly obtained by computing the left- and right-invariant vector fields on $G_{\Lambda}$ (2.4) and then constructing the Sklyanin bivector $\Pi$ (3.9). The pushforward of II on $G_{\Lambda}$ by the canonical projection $p: G_{\Lambda} \rightarrow G_{\Lambda} / H$ gives rise to the fundamental Poisson brackets for the PHS in terms of the geodesic parallel coordinates $x^\mu$. From them the corresponding expressions in terms of ambient coordinates $(s^4, s^\mu)$ (2.11) can be deduced. The resulting fundamental Poisson brackets for the three classes of PHS are displayed in table 2 both in terms of local (geodesic parallel) $x^\mu$ and ambient coordinates $(s^4, s^\mu)$, where the latter are subjected to the pseudosphere constraint (2.8).

Nevertheless, it turns out that the ambient coordinate $s^4$ is always a central element for all three Poisson structures

$$\{s^4, s^\mu\} = 0.$$ (5.1)
This fact means that all the PHS can be defined in terms of the \((3 + 1)\) spacetime coordinates \(s^\mu\) and that the pseudosphere condition \((2.8)\) can be rewritten as

\[
(s^0)^2 - \left( (s^1)^2 + (s^2)^2 + (s^3)^2 \right) = \frac{(s^4)^2 - 1}{\Lambda}.
\] (5.2)

Hence this relation, automatically, yields a common quadratic Casimir for the three types of PHS in table 2, which is given by

\[
\mathcal{C} = (s^0)^2 - (s^1)^2 - (s^2)^2 - (s^3)^2.
\] (5.3)

Although the (A)dS PHS in the local coordinates \(x^\mu\) present involved expressions, which do depend explicitly on the curvature/cosmological constant parameter \(\eta\) \((2.12)\), we stress that these are homogeneously quadratic and \(\eta\)-independent for the three types of PHS in the ambient coordinates \(s^\mu\). Consequently, the expressions for the PHS in terms of \(s^\mu\) are formally the same in the \((3 + 1)\) Beltrami projective variables \(q^\mu\) \((2.9)\) since the latter can simply be obtained from the former dividing by \((s^4)^2\) (see \((5.1)\)).

Furthermore, all Minkowski PHS can be derived straightforwardly via the limit \(\eta \to 0\). Obviously, these are also homogeneous quadratic in terms of Cartesian coordinates and, therefore, quite different from the well-studied \(\kappa\)-Minkowski spacetime\[ 2\], \[\hat{x}_0, \hat{x}_a\] = \(-1/\kappa\), \([\hat{x}_a, \hat{x}_b]\) = 0, (5.4)

and from their Lie-algebraic generalizations obtained in \([21–23]\) (see also references therein). In this respect, it should be noted that, to the best of our knowledge, only two noncommutative quadratic Minkowski spacetimes have been considered so far:

- The \((3 + 1)\)D quantum Minkowski space constructed in \([39]\) from a twisted Poincaré group that corresponds to type II.
- The \((2 + 1)\)D Poisson Minkowski spacetime of case 1 in \([11]\), which was obtained from a Drinfel’d double structure of the \((2 + 1)\)D Poincaré group, which can be identified with type III.

We also recall that Lie-algebraic deformations for the \((3 + 1)\)D and \((2 + 1)\)D Minkowski spaces can also be found in \([11, 39]\), respectively.

In the following, we carry out the quantization for each class of PHS, thus giving rise to the corresponding noncommutative spacetimes. We construct the noncommutative Minkowski spaces in quantum Cartesian coordinates \(\hat{x}^\mu\) in an explicit manner. From them, noncommutative (A)dS spacetimes can be straightforwardly obtained in terms of the quantum spacetime coordinates \(\hat{s}^\mu\), since they are defined formally as the same Poisson quadratic algebra as the corresponding Minkowski spacetimes.

5.1. Type I spacetimes

This class corresponds to a two-parametric family of PHS, with arbitrary deformation parameters \(z\) and \(z'\), and exhibit complicated expressions. As a shorthand notation, we have introduced the functions \(A(x^0, x^1)\) and \(B(x^0, x^1)\) given in table 2.

The vanishing cosmological constant limit \(\eta \to 0\) of the (A)dS brackets in geodesic parallel coordinates \(x^\mu\) gives the Minkowski PHS. Notice that the limit of the functions \(A(x^0, x^1)\) and \(B(x^0, x^1)\) reads

\[
\lim_{\eta \to 0} A = x^0 + x^1, \quad \lim_{\eta \to 0} B = 1.
\] (5.5)
Hence the explicit Minkowski PHS is defined by the following quadratic Poisson algebra:

\[
\begin{align*}
\{x^0, x^1\} &= z(x^0 + x^1)x^2, \\
\{x^0, x^2\} &= -z[(x^0 + x^1)x^1 + (x^3)^2] - \zeta(x^0 + x^1)x^3, \\
\{x^0, x^3\} &= z^2 x^3 + \zeta(x^0 + x^1)x^2, \\
\{x^1, x^2\} &= -z[(x^0 + x^1)x^0 - (x^3)^2] + \zeta(x^0 + x^1)x^3, \\
\{x^1, x^3\} &= -z^2 x^3 - \zeta(x^0 + x^1)x^2, \\
\{x^2, x^3\} &= z(x^0 + x^1)x^3 + \zeta(x^0 + x^1)x^2,
\end{align*}
\]  

(5.6)

which is formally identical with the (AdS expressions given in table 2 in ambient variables \(s^\underline{\mu}\) (2.13) (recall that \(s^4\) does not appear in the Poisson brackets). The quadratic Casimir \(C\) (5.3) for this Poisson algebra yields

\[
C = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2,
\]

(5.7)

for any \(z\) and \(\zeta\).

Although this two-parametric family is quite involved, it can be regarded as the superposition of two particular ‘subfamilies’ that we proceed to quantize separately.

5.1.1. Subfamily with \(z = 0\). The Minkowski PHS (5.6) with \(z = 0\) can be described in a natural way by introducing null-plane coordinates [40]. We consider the null-plane \(N^0_0\) orthogonal to the light-like vector \(n = (1, 1, 0, 0)\) in the classical Minkowski spacetime with Cartesian coordinates \(x = (x^0, x^a)\) and we define

\[
x^+ = x^0 + x^1, \quad x^- = n \cdot x = x^0 - x^1 = \tau.
\]

(5.8)

A point \(x \in N^0_0\) is labelled by the coordinates \((x^+, x^2, x^3)\), while \(\tau = \tau\) will play the role of a ‘time’ or evolution parameter, and the chosen null-plane is associated with the boost generator \(K_1\). It can be checked that under the action of the boost transformation generated by \(K_1\) (2.4), the initial null-plane \(N^0_0\) (\(\tau = 0\)) is invariant, the transverse coordinates \((x^2, x^3)\) remain unchanged, and \(\exp(\xi\rho(K_1))\) maps \(x^+\) into \(e^\xi x^+\). For our purposes, let us also recall that the generators of the Poincaré algebra \(g_0\) (2.1) can be casted into three different classes according to their commutator with \(K_1\),

\[
[K_1, X] = \gamma X, \quad X \in g_0,
\]

(5.9)

where the parameter \(\gamma\) is called the ‘goodness’ of the generator \(X\) [40]; these are

\[
\begin{align*}
\gamma = +1 : & \quad P_+ = P_0 + P_1, \quad K_2 - J_3, \quad K_3 + J_2, \\
\gamma = 0 : & \quad K_1, \quad J_1, \quad P_2, \quad P_3, \\
\gamma = -1 : & \quad P_- = P_0 - P_1, \quad K_2 + J_3, \quad K_3 - J_2.
\end{align*}
\]

(5.10)

From the representation (2.3) it can be checked that the seven generators with \(\gamma = +1\) and \(\gamma = 0\) span the stability group of the initial null-plane \(N^0_0\), while the three remaining ones with \(\gamma = -1\) move \(N^0_0\). In particular, the transformations generated by \((K_2 + J_1)\) and \((K_3 - J_2)\) rotate \(N^0_0\), and \(\exp(\xi\rho(P_2))\) translates \(N^0_0\) into \(N^2_+\). Hence the latter generators, which span an abelian subgroup, determine the dynamical evolution of \(N^0_0\) with time \(\tau\).

The quantization of the Minkowski PHS (5.6) with \(z = 0\) can be immediately realised in terms of the noncommutative coordinates \((\hat{x}^\pm = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2, \hat{x}^3)\) since the quantum coordinate
\( \dot{x}^+ \) becomes a central element; namely in such null-plane coordinates this noncommutative spacetime reads
\[
[\dot{x}^-, \dot{x}^3] = -2z' \dot{x}^+ \dot{x}^3, \quad [\dot{x}^-, \dot{x}^3] = 2z' \dot{x}^+ \dot{x}^3, \quad [\dot{x}^2, \dot{x}^3] = z'(\dot{x}^+)^2, \quad [\dot{x}^+, \cdot] = 0. \tag{5.11}
\]
And the quantum counterpart of the Casimir (5.7) is
\[
\hat{C} = \dot{x}^- \dot{x}^+ - (\dot{x}^2)^2 - (\dot{x}^3)^2. \tag{5.12}
\]

At this point, it is worth calculating the cocommutator \( \delta_c \) coming from \( r_1 \) (4.5) with \( z = 0 \) through the relation (3.3) and analyse its structure in relation with the above null-plane framework. Explicitly, \( \delta_c \) is given by:
\[
\delta_c(P_0) = z'P_2 \wedge (K_3 - J_2) - z'P_3 \wedge (K_2 + J_3),
\delta_c(P_1) = z'P_2 \wedge (K_3 - J_2) - z'P_3 \wedge (K_2 + J_3),
\delta_c(P_2) = z'(P_0 - P_1) \wedge (K_3 - J_2),
\delta_c(P_3) = -z'(P_0 - P_1) \wedge (K_2 + J_3),
\delta_c(K_1) = 2z'(K_2 + J_3) \wedge (K_3 - J_2),
\delta_c(K_2) = -z'K_1 \wedge (K_3 - J_2) - z'J_1 \wedge (K_2 + J_3),
\delta_c(K_3) = z'K_1 \wedge (K_2 + J_3) - z'J_1 \wedge (K_3 - J_2),
\delta_c(J_1) = 0,
\delta_c(J_2) = z'K_1 \wedge (K_2 + J_3) - z'J_1 \wedge (K_3 - J_2),
\delta_c(J_3) = z'K_1 \wedge (K_3 - J_2) + z'J_1 \wedge (K_2 + J_3). \tag{5.13}
\]

These expressions clearly exhibit the underlying null-plane symmetry determined by the boost generator \( K_1 \) according to the goodness \( \gamma \) (5.10). It can be checked that, besides \( J_1 \), the three generators with \( \gamma = -1 \), have a vanishing cocommutator:
\[
\delta_c(P_-) = \delta_c(K_2 + J_3) = \delta_c(K_3 - J_2) = 0. \tag{5.14}
\]
Moreover, in this null-plane basis \( r_1 \) (4.5) with \( z = 0 \) adopts the simple form
\[
r_1 = z'(K_3 - J_2) \wedge (K_2 + J_3). \tag{5.15}
\]
Consequently, \( r_1 \) corresponds to a Reshetikhin twist generated by two commuting operators, which means that the quantum Poincaré algebra \( U_q(\mathfrak{g}_0) \) can be easily constructed. It can also be checked from \( \delta_c \) (5.13) that
\[
\delta_c(t) \subset t \wedge h, \quad \delta_c(h) \subset h \wedge h \neq 0, \tag{5.16}
\]
as it should be. In addition, the relations (5.16) mean that \( \delta_c \) does not contain any term in \( t \wedge t \), i.e. \( P_\alpha \wedge P_\beta \), and by quantum duality this fact implies that all the commutators \([\dot{x}^\mu, \dot{x}^\nu]\) vanish at the first-order in the quantum coordinates \( \dot{x}^\mu \), although higher-order terms could exist. The latter is exactly the case here with the noncommutative Minkowski space (5.11) which is defined by a homogeneous quadratic algebra. We stress that the relations (5.16) and so the first-order brackets \([\dot{x}^\mu, \dot{x}^\nu]\) = 0 are satisfied by the three types I, II and III of deformations and hold for any value of \( \Lambda \).
5.1.2. Subfamily with $z' = 0$. We now consider the Minkowski PHS (5.6) with $z' = 0$ together with the same null-plane coordinates (5.8) associated with $K_1$. By taking the ordered monomials $(\hat{x}^-)^I(\hat{x}^+)^J(\hat{x}^3)^K$ the corresponding noncommutative Minkowski space is found to be

\[
[\hat{x}^-, \hat{x}^+] = 2z\hat{x}^+\hat{x}^2, \quad [\hat{x}^-, \hat{x}^3] = z\hat{x}^+ - 2z(\hat{x}^3)^2, \quad [\hat{x}^-, \hat{x}^3] = 2z\hat{x}^3\hat{x}^2, \quad [\hat{x}^+, \hat{x}^3] = 0, \quad [\hat{x}^+, \hat{x}^3] = 0, \quad [\hat{x}^2, \hat{x}^3] = z\hat{x}^+ \hat{x}^3,
\]

where associativity is ensured by the Jacobi identity. Although the resulting expressions are more complicated than in the previous case, the quantum null-plane coordinates $(\hat{x}^+, \hat{x}^2, \hat{x}^3)$ again close a subalgebra and the quantum Casimir (5.12) is the same.

The cocommutator $\delta$ can be obtained from $r_1$ (4.5) with $z' = 0$ applying (3.3) and its structure turns out to be naturally adapted to the null-plane Poincaré generators (5.10). However, the generators $J_1$ and $P_\pm$ are no longer primitive and only $(K_2 + J_1)$ and $(K_3 - J_2)$ have a vanishing cocommutator. Again $r_1$ (4.5) with $z' = 0$ has a simple expression in this null-plane basis:

\[
r_1 = zK_1 \wedge (K_2 + J_1) + zJ_1 \wedge (K_3 - J_2).
\]

Observe that this $r$-matrix is not defined through Reshetikhin twists and therefore is a more complicated solution of the CYBE than the previous case.

Concerning the 'superposition' of these two subfamilies, it is clear that the two-parametric noncommutative Minkowski space just comes out by considering altogether (5.11) and (5.17) since $z$ and $z'$ are arbitrary and we have considered the same quantum coordinates for both spaces. Notice also that the order in (5.11) is trivially compatible with that of (5.17). The complete $r_1$ is just the addition of (5.15) and (5.18), which would further lead to a new two-parametric null-plane quantum Poincaré algebra $U_{z',z}(\mathfrak{g}_0)$ determined by the two commuting generators $(K_2 + J_1)$ and $(K_3 - J_2)$.

In this respect, it is worth stressing that $U_{z',z}(\mathfrak{g}_0)$ would be quite different from the known null-plane quantum Poincaré algebra formerly obtained in [6], which is determined by the generator $P_\pm$. In order to compare both quantum deformations, we recall that the noncommutative Minkowski spacetime for the latter was obtained in [41] from its universal quantum $R$-matrix in the above null-plane basis (ruled now by the generator $K_1$ instead of $K_3$ [41]). This quantum spacetime has non-vanishing commutators given by

\[
[\hat{x}^+, \hat{x}^-] = -2z\hat{x}^-\hat{x}^2, \quad [\hat{x}^+, \hat{x}^3] = -2z\hat{x}^2\hat{x}^3, \quad [\hat{x}^+, \hat{x}^3] = -2z\hat{x}^3.
\]

Similarly to the $\kappa$-Minkowski space (5.4), this null-plane noncommutative Minkowski space is also a linear-algebraic deformation and for which a quadratic quantum Casimir (5.12) does not exist. For linear-algebraic generalizations of (5.19) we refer to [23].

Concerning the corresponding noncommutative (A)dS spacetimes, they can be straightforwardly obtained in terms of the ambient quantum variables $(\hat{x}^i = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2, \hat{x}^3)$, whose commutators are given by expressions of the same form as (5.11) and (5.17). We also point out that the cocommutator $\delta_{z,z'}$ obtained from $r_1$ is $\Lambda$-independent, and thus holds for the whole family $\mathfrak{g}_\Lambda$, so that the quantum algebra $U_{z',z}(\mathfrak{g}_\Lambda)$ would be endowed with a common coproduct for the (A)dS and Poincaré algebras with primitive generators $(K_2 + J_1)$ and $(K_3 - J_2)$.

Obviously, the commutation rules for the quantum (A)dS algebras will be a $\Lambda$-deformation of the corresponding Poincaré ones.
5.2. Type II spacetimes

The limit $\eta \to 0$ of the (A)dS PHS of type II in coordinates $x^\mu$ given in table 2 leads to the following Minkowski PHS:

$$\begin{align*}
\{x^0, x^1\} &= 0, \\
\{x^0, x^2\} &= z x^1 x^3, \\
\{x^0, x^3\} &= -z x^1 x^2, \\
\{x^1, x^2\} &= 0, \\
\{x^1, x^3\} &= z x^0 x^3, \\
\{x^2, x^3\} &= -z x^0 x^2.
\end{align*}$$

(5.20)

The noncommutative Minkowski spacetime is directly deduced by considering the ordered monomials $(\hat{x}^0)^k (\hat{x}^1)^l (\hat{x}^2)^m (\hat{x}^3)^n$ and reads

$$\begin{align*}
[\hat{x}^0, \hat{x}^1] &= 0, \\
[\hat{x}^0, \hat{x}^2] &= z \hat{x}^1 \hat{x}^3, \\
[\hat{x}^0, \hat{x}^3] &= -z \hat{x}^1 \hat{x}^2, \\
[\hat{x}^1, \hat{x}^2] &= 0, \\
[\hat{x}^1, \hat{x}^3] &= z \hat{x}^0 \hat{x}^3, \\
[\hat{x}^2, \hat{x}^3] &= -z \hat{x}^0 \hat{x}^2.
\end{align*}$$

(5.21)

This structure can be regarded as a nonlinear generalization of two coupled 2D quadratic Euclidean algebras, for which $\hat{x}^0$ and $\hat{x}^1$ play the role of rotation operators on the two-vector $(\hat{x}^2, \hat{x}^3)$. The behaviour of the quantum coordinate pairs $(\hat{x}^0, \hat{x}^1)$ and $(\hat{x}^2, \hat{x}^3)$ is somehow inherited from the Reshetikhin twist defined by $r_{II}$ (4.5) which is given by the commuting generators $K_1$ and $J_1$. It can be checked that these are the only primitive generators within the Lie bialgebra $(\mathfrak{g}_\Lambda, \delta_{II})$.

We remark that (5.21) corresponds to the quadratic noncommutative Minkowski space constructed in [39] by following a different approach which consists of starting with a representation of the universal quantum $R$-matrix for the twisted quantum Poincaré group and then applying the FRT procedure.

Note also that the obtention of the corresponding quantum (A)dS algebra $U_z(\mathfrak{g}_\Lambda)$ is straightforward via the Reshetikhin twist associated to $r_{II}$.

5.3. Type III spacetimes

The third type of PHS are written in table 2 in local coordinates $x^\mu$ by using the function $A(x^0, x^1)$ as a shorthand notation. Before performing its quantization, let us relate these (A)dS PHS with $\eta \not= 0$ to some results already obtained in the literature.

Since $x^3$ Poisson-commutes with all the remaining coordinates, we are dealing in fact with a $(2 + 1)$D (A)dS PHS which can be expressed in a more symmetric manner as

$$\begin{align*}
\{x^0, x^1\} &= z \frac{\tanh(\eta x^0)}{\eta} \Theta, \\
\{x^0, x^2\} &= -z \frac{\tanh(\eta x^1)}{\eta} \Theta, \\
\{x^1, x^2\} &= -z \frac{\tanh(\eta x^0)}{\eta} \Theta,
\end{align*}$$

(5.22)

where we have introduced the function $\Theta = \cos(\eta x^0) A(x^0, x^1)$:

$$\Theta(x^0, x^1) = \cos(\eta x^0) \left( \frac{\sin(\eta x^0)}{\eta} \cosh(\eta x^1) + \frac{\sinh(\eta x^1)}{\eta} \right).$$

(5.23)

In this form, the PHS (5.22) deeply resembles the $(2 + 1)$D AdS PHS constructed in [42] by considering the AdS Lie algebra $\mathfrak{so}(2, 2)$ as a Drinfel’d double arising from the
Drinfel’d–Jimbo deformation of \( \mathfrak{sl}(2, \mathbb{R}) \) and taking into account the results formerly obtained in [43]. In the notation of [42] the AdS PHS reads

\[
\{x^0, x^1\} = -\frac{\xi \tanh(\eta x^2)}{\eta} \Upsilon, \quad \{x^0, x^2\} = \frac{\xi \tanh(\eta x^1)}{\eta} \Upsilon, \quad \{x^1, x^2\} = \frac{\xi \tanh(\eta x^0)}{\eta} \Upsilon.
\]

(5.24)

such that \( \xi \) is the deformation parameter, \( \eta^2 = -\Lambda > 0 \), and

\[
\Upsilon(x^0, x^1) = \cos(\eta x^0) \left( \cos(\eta x^0) \cosh(\eta x^1) + \sinh(\eta x^1) \right).
\]

(5.25)

The apparent similarity between (5.22) and (5.24) disappears when both PHS are expressed in ambient coordinates \( s^\mu \), and this fact becomes more evident when the Minkowski limit \( \eta \to 0 \) is calculated: in that case \( \Theta \to (x^0 + x^1) \) (see (5.5)), while \( \Upsilon \to 1 \). In particular, the Minkowski PHS obtained from (5.24) is given by [42]

\[
\{\hat{x}^0, \hat{x}^1\} = -\xi \hat{x}^2, \quad \{\hat{x}^0, \hat{x}^2\} = \xi \hat{x}^1, \quad \{\hat{x}^1, \hat{x}^2\} = \xi \hat{x}^0.
\]

(5.26)

Hence this structure provides a Lie-algebraic deformation of the \((2 + 1)\)D Minkowski space which corresponds to the Lie algebra \( \mathfrak{so}(2, 1) \). We remark that such a noncommutative Minkowski space was formerly considered in [44] in a \((2 + 1)\) quantum gravity framework. This also appears as the case 0 in the classification of Drinfel’d double structures for the \((2 + 1)\)D Poincaré group performed in [11]. The noncommutative Minkowski space \( \mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R}) \) (5.26), linked to the \((2 + 1)\)D Poincaré group, together with its Euclidean counterpart corresponding to the Lie algebra \( \mathfrak{so}(3) \simeq \mathfrak{su}(2) \), associated with the 3D Euclidean group, have been extensively studied in \((2 + 1)\)-gravity [44–49] (see also references therein).

In contrast to (5.26), the limit \( \eta \to 0 \) of the \((A)dS\) PHS (5.22) gives the following quadratic \((2 + 1)\)D Minkowski PHS

\[
\{\hat{x}^0, \hat{x}^1\} = z(\hat{x}^0 + \hat{x}^1)^2, \quad \{\hat{x}^0, \hat{x}^2\} = -z(\hat{x}^0 + \hat{x}^1) \hat{x}^1, \quad \{\hat{x}^1, \hat{x}^2\} = -z(\hat{x}^0 + \hat{x}^1) \hat{x}^0,
\]

(5.27)

which is naturally adapted to the null-plane description associated with \( K_1 \) that was presented in section 5.1. This Poisson structure has been obtained previously within the case 1 of the classification of \((2 + 1)\)D Poincaré Drinfel’d double structures in [11].

If we now consider the noncommutative coordinates \( (\hat{x}^\mu = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2, \hat{x}^3) \) and take the ordered monomials \( (\hat{x}^\mu)^n(\hat{x}^+)^p(\hat{x}^3)^q \) we get the noncommutative Minkowski spacetime given by

\[
[\hat{x}^2, \hat{x}^+] = z(\hat{x}^+)^2, \quad [\hat{x}^2, \hat{x}^-] = -z\hat{x}^- \hat{x}^+, \quad [\hat{x}^-, \hat{x}^+] = 2z\hat{x}^+ \hat{x}^2, \quad [\hat{x}^3, \hat{x}^\mu] = 0.
\]

(5.28)

The quantization of the Casimir (5.7) reads

\[
\hat{C} = \hat{x}^- \hat{x}^+ - (\hat{x}^3)^2,
\]

(5.29)

which is the \((2 + 1)\)D version of (5.12), where obviously one could trivially add the central term \((\hat{x}^3)^2\). Hence \( \hat{C} \) is related to a \((1 + 1)\)D D3 space associated with the Lorentz algebra \( \mathfrak{so}(2, 1) \) in agreement with the \((2 + 1)\)D character of this deformation. The primitive generators with
vanishing cocommutator of the Lie bialgebra \((g_\Lambda, \delta_{\text{III}})\), determined by \(r_{\text{III}} (4.5)\), are \((K_2 + J_3)\) and, as expected, \(P_3\).

Finally, we stress that the quantum deformation determined by \(r_{\text{III}} (4.5)\), coming from the lower dimensional Lorentz subalgebra \(\mathfrak{so}(2, 1)\) spanned by \(\{J_3, K_1, K_2\}\) (see section 4.1), can be related to the well-known nonstandard (or Jordanian) quantum deformation of \(\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 1)\) [50–54]. Explicitly, if we define new generators in such subalgebra of \(g_\Lambda\) as

\[
A = -2K_1, \quad A_\pm = K_2 \pm J_3,
\]

we find that they close the commutation relations of \(\mathfrak{sl}(2, \mathbb{R})\) while \(r_{\text{III}} (4.5)\) turns out to be the so-called Jordanian twist for \(\mathfrak{sl}(2, \mathbb{R})\) [51], namely

\[
[A, A_\pm] = \pm 2A_\pm, \quad [A_+, A_-] = A, \quad r_{\text{III}} = -\frac{z}{2} A \wedge A_+.
\]

In this respect, it is worth noting that the nonstandard quantum algebra \(U_z(\mathfrak{sl}(2, \mathbb{R}))\) has already been considered in [55, 56] as the cornerstone to obtain higher-dimensional quantum (A)dS algebras by imposing to keep \(U_z(\mathfrak{sl}(2, \mathbb{R}))\) as a Hopf subalgebra, which is exactly the situation here. Such results were deduced by working in a conformal basis, and they would provide the complete quantum algebra \(U_z(g_\Lambda)\) through an appropriate change of basis from the conformal to the kinematical one.

6. Noncommutative Newtonian and Carrollian spacetimes

So far we have obtained all the PHS \((G_\Lambda/H, \pi)\) such that the Lorentz subgroup \(H\) is a PL subgroup of \((G_\Lambda, \Pi)\) and then we have constructed the corresponding noncommutative (A)dS and Minkowski spacetimes. In this context it is rather natural to analyse which are their non-relativistic and ultra-relativistic limits leading to noncommutative Newtonian and Carrollian spacetimes, respectively. Recall that under such limits the Lorentz subgroup \(H\) becomes isomorphic to the 3D Euclidean group ISO(3). In what follows we study these two limits separately.

6.1. Non-relativistic limit

We consider the commutation relations of the family \(g_\Lambda (2.1)\), we apply the map defined by

\[
P \rightarrow c^{-1} P, \quad K \rightarrow c^{-1} K,
\]

where \(c\) is the speed of light, and then take the limit \(c \rightarrow \infty\) obtaining the Lie brackets given by

\[
\begin{align*}
[J_a, J_b] &= \epsilon_{abc} J_c, & [J_a, P_b] &= \epsilon_{abc} P_c, & [J_a, K_b] &= \epsilon_{abc} K_c, \\
[K_a, P_0] &= P_a, & [K_a, P_b] &= 0, & [K_a, K_b] &= 0, \\
[P_0, P_a] &= -\Lambda K_a, & [P_0, P_b] &= 0, & [P_0, J_a] &= 0.
\end{align*}
\]

The three contracted Lie algebras constitute the family of non-relativistic or Newtonian Lie algebras \(n_\Lambda\): expanding Newton–Hooke (NH) \(n_+\) for \(\Lambda > 0\), oscillating NH \(n_-\) for \(\Lambda < 0\) and Galilei \(n_0\) for \(\Lambda = 0\) [24, 57–64].
The three associated \((3 + 1)D\) Newtonian spacetimes are constructed as the coset spaces
\[ N_\Lambda / H, \quad H = \text{ISO}(3) = \{ K, J \}, \] (6.3)
where \(N_\Lambda\) is the Lie group with Lie algebra \(n_\Lambda\) and \(H\) is the isotropy subgroup of rotations and (commuting) Newtonian boosts, which is isomorphic to the 3D Euclidean group ISO(3) (instead of the Lorentz one SO(3, 1)). These Newtonian spacetimes are also of constant sectional curvature equal to \(\eta^2 = -\Lambda\) but the time-like metric becomes degenerate providing ‘absolute-time’. Hence there arises an invariant foliation under the Newtonian group action whose leaves are defined by a constant time which is determined by a ‘subsidiary’ 3D non-degenerate Euclidean spatial metric \(g'\) restricted to each leaf of the foliation (see, e.g. [24] for explicit metric models).

By using Lie group duality, it is found that the contraction map (6.1) leads to the following contraction map for geodesic parallel and ambient coordinates
\[ \begin{align*}
x_0 & \rightarrow x_0^0, \quad x_a' \rightarrow c x_a', \\
s_0' & \rightarrow s_0^0, \quad s_a' \rightarrow c s_a', \quad s_4' \rightarrow s_4^4.
\end{align*} \] (6.4)

By introducing these maps into (2.11) and next applying the limit \(c \rightarrow \infty\) we obtain the corresponding relations between Newtonian geodesic parallel and ambient coordinates:
\[ s_4^4 = \cos(\eta x_0^0), \quad s_0^0 = \frac{\sin(\eta x_0^0)}{\eta}, \quad s_4' = x_4', \] (6.5)
such that
\[ (s_4^4)^2 + \eta^2 (s_0^0)^2 = 1, \] (6.6)
to be compared to (2.8). In the flat Galilei space both sets of coordinates coincide: \(s_4' \equiv x_4'\).

The main time-like metric and the subsidiary 3D non-degenerate Euclidean spatial metric \(g'\) restricted to each leaf of the foliation turn out to be [24]
\[ \begin{align*}
d\sigma^2_A &= \frac{(dx_0^0)^2}{1 + \Lambda (s_0^0)^2} = (dx_0^0)^2, \\
d\sigma^2 &= (dx_4^4)^2 + (dx_0^2)^2 + (dx_3^3)^2 = (dx_4^4)^2 + (dx_0^2)^2 + (dx_3^3)^2 \quad \text{with } s_0^0, x_0^0 \text{ constants.}
\end{align*} \] (6.7)

With all of these ingredients we analyse the contractions of the three types of Lie bialgebras determined by proposition 2. We apply the Lie bialgebra contraction (LBC) approach introduced in [65]: starting from a given Lie algebra contraction, here (6.1), the transformation of the quantum deformation parameter \((z, z')\) that ensures a well-defined and non-trivial Lie bialgebra structure after contraction has to be found. Such a transformation should be studied for \(r\) and \(\delta\) separately, since they could behave differently. In our case, the transformation of the deformation parameter is exactly the same for the three types of \(r\) (4.5) and their cocommutators \(\delta\) (3.3). This means that we obtain fundamental and coboundary LBC in such a manner that the contracted \(\delta\) coincides with the one provided by the contracted \(r\) through the relation (3.3). The resulting LBC are given by

Type I: \(z \rightarrow c^2 z, \quad z' \rightarrow c^2 z', \quad n_I = z K_1 \wedge K_2 - z' K_2 \wedge K_1.\)
Type II: \(z \rightarrow c z, \quad r_{II} = z K_1 \wedge J_1.\)
Type III: \(z \rightarrow c^2 z, \quad r_{III} = z K_1 \wedge K_2.\)
consistent with the contraction (6.4) of the Casimir (5.3) which reduces to reminiscences of the null-planenoncommutative Minkowskispaces (5.11) and (5.17). Hence the Newtonian deformation of type III is the particular case of type I with $\zeta = 0$. Therefore, all of them are just twisted Reshetikhin deformations (the boosts now commute) and the type II is the only one that remains invariant under contraction with respect to proposition 2.

The Newtonian PHS in local and ambient coordinates (6.5) can be deduced either by contracting the (A)dS and Minkowski PHS given in table 2 by applying the maps (6.4) and (6.8), or by computing the left- and right-invariant vector fields and then using the Sklyavin bivector (3.9) with $r_1$ and $r_2$. The resulting PHS can be directly quantized since there are no ordering ambiguities. Obviously, the contractions can also be applied to the noncommutative spaces obtained in sections 5.1 and 5.2, but observe that for the type I deformation this should be done by working in the basis $\hat{x}^\mu$ instead of the null-plane basis, since the latter is not applicable in the non-relativistic spaces. Such quantum Newtonian spaces are presented in table 3.

It is worth stressing that the quantum time coordinate $\hat{t}^0$ becomes a central element (and so $\hat{x}^0$ as well) reminding the ‘time-absolute’ character of the Newtonian spaces. This fact is consistent with the contraction (6.4) of the Casimir (5.3) which reduces to $\hat{C} = (\hat{x}^0)^2$. However the noncommutative spaces do not split into $\hat{r}^0$ plus a quantum three-space subalgebra $\hat{s}^\mu$, since the latter involves $\hat{x}^0$.

Let us briefly comment on the results presented in table 3 by writing the explicit noncommutative spaces for the Galilei case corresponding to the limit $\eta \to 0$, such that $\hat{s}^\mu \equiv \hat{\xi}^\mu$. The non-vanishing commutators of the type I space read

\begin{equation}
[\hat{x}^1, \hat{x}^2] = -z(\hat{x}^0)^2, \quad [\hat{x}^2, \hat{x}^3] = \zeta'(\hat{x}^0)^2, \quad (6.9)
\end{equation}

which, surprisingly enough, can be regarded as two coupled Heisenberg–Weyl algebras sharing the quantum space coordinate $\hat{x}^2$, and whose central element is determined by the square of the quantum time coordinate $\hat{x}^0$. For each subfamily with either $z$ or $\zeta'$ equal to zero the coupling disappears leaving a single Heisenberg–Weyl algebra. Thus (6.9) provides the non-relativistic reminiscences of the null-plane noncommutative Minkowski spaces (5.11) and (5.17).
The noncommutative Galilei space of type II is given by
\[
[\hat{x}^1, \hat{x}^2] = z \hat{x}^0 \hat{x}^3, \quad [\hat{x}^1, \hat{x}^3] = -z \hat{x}^0 \hat{x}^2, \quad [\hat{x}^2, \hat{x}^3] = 0, \quad (6.10)
\]
which for a given eigenvalue of $\hat{x}^0$ can be interpreted as a quadratic 2D Euclidean algebra with $\hat{x}^1$ playing the role of a rotation on the quantum two-space $(\hat{x}^2, \hat{x}^3)$. Hence the quadratic Casimir for this space is given by
\[
\hat{C} = (\hat{x}^2)^2 + (\hat{x}^3)^2. \quad (6.11)
\]
Thus, under the non-relativistic limit, the noncommutative Minkowski spacetime (5.21) is decoupled at the central quantum time coordinate $\hat{x}^0$ and the noncommutative space (6.10).

Finally, regarding the Newtonian Lie bialgebras determined by (6.8) note that, as a result of our approach, both types I and II have a non-trivial Euclidean sub-Lie bialgebra $h = \text{span}\{K, J\} = \mathfrak{iso}(3)$, i.e. $\delta(h) = h \wedge h \neq 0$. For the two-parametric Lie bialgebra of type I $(n_{\Lambda}, \delta_I)$ the primitive generators are $P$ and $K$, while for the type II $(n_{\Lambda}, \delta_{II})$ these are $\{K_1, J_1, P_1\}$. Their quantum algebras can be straightforwardly constructed via the corresponding Reshetikhin twists.

### 6.2. Ultra-relativistic limit

We now introduce the speed of light $c$ in the family $g_{\Lambda}$ (2.1) through the map \[57, 66\]
\[
P_0 \rightarrow cP_0, \quad K \rightarrow cK. \quad (6.12)
\]
The ultra-relativistic limit $c \rightarrow 0$ gives rise to the contracted commutation relations
\[
[J_{\alpha}, J_{\beta}] = \epsilon_{\alpha\beta\gamma} J_{\gamma}, \quad [J_{\alpha}, P_{\beta}] = \epsilon_{\alpha\beta\gamma} P_{\gamma}, \quad [J_{\alpha}, K_{\beta}] = \epsilon_{\alpha\beta\gamma} K_{\gamma},
\]
\[
[K_{\alpha}, P_{\beta}] = 0, \quad [K_{\alpha}, K_{\beta}] = 0, \quad (6.13)
\]
corresponding to the family of three Carrollian algebras $c_{\Lambda}$. This comprises the para-Euclidean $c_{\Lambda} \simeq \mathfrak{iso}(4)$ for $\Lambda > 0$, the para-Poincaré $c_{\Lambda} \simeq \mathfrak{iso}(3,1)$ for $\Lambda < 0$, and the (proper) Carroll algebra $c_0$ for $\Lambda = 0$ \[24, 57, 61–64, 66–74\].

The three $(3 + 1)$D Carrollian spacetimes are identified with the coset spaces
\[
C_{\Lambda}/H, \quad H = \mathfrak{iso}(3) = (K, J), \quad (6.14)
\]
where $C_{\Lambda}$ is the Lie group with Lie algebra $c_{\Lambda}$ and $H$ is the isotropy subgroup isomorphic to $\mathfrak{iso}(3)$ spanned by rotations and (commuting) Carrollian boosts. We stress that the main metric for the Carrollian spacetimes is again degenerate and has constant sectional curvature equal to $-\eta^2 = +\Lambda$ \[24\], instead of $\eta^2 = -\Lambda$ (as in the Lorentzian and Newtonian spacetimes), providing a 3D ‘absolute-space’; this is therefore spherical in the para-Euclidean space with $\Lambda > 0$ and hyperbolic in the para-Poincaré one with $\Lambda < 0$. There does also exist an invariant foliation under the Carrollian group action characterized by a ‘subsidiary’ 1D time metric $g'$ restricted to each leaf of the foliation \[24\].

In this case the contraction map (6.12) yields the following transformations for geodesic parallel and ambient coordinates
\[
\begin{align*}
x^0 & \rightarrow c^{-1}x^0, \quad x^a \rightarrow x^a, \\
x^0 & \rightarrow c^{-1}x^0, \quad x^a \rightarrow x^a, \quad s^4 \rightarrow s^4. \quad (6.15)
\end{align*}
\]
By introducing them into (2.11) and applying the limit \( c \to 0 \) we find the relations between Carrollian geodesic parallel and ambient coordinates, namely

\[
\begin{align*}
    s^4 &= \cosh(\eta x^1) \cosh(\eta x^2) \cosh(\eta x^3), \\
    s^0 &= x^0 \cosh(\eta x^1) \cosh(\eta x^2) \cosh(\eta x^3), \\
    s^1 &= \frac{\sinh(\eta x^1)}{\eta} \cosh(\eta x^2) \cosh(\eta x^3), \\
    s^2 &= \frac{\sinh(\eta x^2)}{\eta} \cosh(\eta x^3), \\
    s^3 &= \frac{\sinh(\eta x^3)}{\eta},
\end{align*}
\]

(6.16)

fulfilling

\[
(s^4)^2 - \eta^2 ((s^1)^2 + (s^2)^2 + (s^3)^2) = 1, \quad \eta^2 = -\Lambda, \tag{6.17}
\]

(see (2.8)), showing the spherical (\( \eta \) imaginary) or hyperbolic (\( \eta \) real) nature of the underlying three-space. Observe that for the flat Carroll space both sets of coordinates coincide \( s^\mu \equiv x^\mu \).

The 3D spatial main metric in ambient coordinates \( s^\mu \) was introduced in [24] and by means of the parametrization (6.16) this can also be written in terms of geodesic parallel coordinates \( x^\mu \) as

\[
\begin{align*}
    d\sigma^2 &= \frac{\Lambda}{1 - \Lambda} \left( (s^1)^2 + (s^2)^2 + (s^3)^2 \right) + (ds^1)^2 + (ds^2)^2 + (ds^3)^2 \\
    &= \cosh^2(\eta x^1) \cosh^2(\eta x^2) (dx^1)^2 + \cosh^2(\eta x^3) (dx^2)^2 + (dx^3)^2. \tag{6.18}
\end{align*}
\]

The metric structure on the Carrollian spacetime is completed with the ‘subsidiary’ 1D time metric \( g' \) restricted to each leaf of the foliation (the ‘absolute-space’) and reads

\[
    d\sigma'^2 = (ds^0)^2 = \cosh^2(\eta x^1) \cosh^2(\eta x^2) \cosh^2(\eta x^3) (dx^0)^2 \quad \text{on} \quad s^\mu, x^\mu = \text{constant}. \tag{6.19}
\]

Now we proceed to study the ultra-relativistic limit of the solutions of the mCYBE from proposition 2 by analysing their LBC. We again obtain simultaneous fundamental and coboundary LBC [65] which are defined by

\[
\begin{align*}
    \text{Type I:} \quad &z \to z/c^2, \quad z' \to z'/c^2, \quad r_1 = z K_1 \wedge K_2 - z' K_2 \wedge K_3, \\
    \text{Type II:} \quad &z \to z/c, \quad r_{II} = z K_1 \wedge J_1, \tag{6.20} \\
    \text{Type III:} \quad &z \to z/c^2, \quad r_{III} = z K_1 \wedge K_2.
\end{align*}
\]

Consequently, the result is formally the same as in the Newtonian cases (6.8), so that, again, the type III deformation is included in the type I for \( z' = 0 \). The corresponding Carrollian PHS in local and ambient coordinates (6.16) can be derived through contraction from the Lorentzian PHS in table 2 by taking into account the transformations (6.15) and (6.20). They can also be constructed by means of the left- and right-invariant vector fields and the Sklyanin bivector (3.9). The resulting PHS have no ordering problems so that their quantization is immediate. The same result follows by applying the contractions (6.15) and (6.20) to the noncommutative spaces given in sections 5.1 and 5.2. The noncommutative Carrollian spaces are presented in table 3.
The noncommutative Carrollian spaces of type I are trivial ones, since all their Poisson brackets vanish. Hence there are no Heisenberg–Weyl algebras with central quantum time coordinate $\hat{s}^0$, a fact that could be expected since, in the classical picture, time is no longer absolute and its role is replaced by space.

In the noncommutative spaces of type II, all the spatial coordinates commute, thus reminding the 'absolute-space' feature of the Carrollian spaces. Moreover, the quantum spatial coordinate $\hat{s}^1$ becomes a central generator (such a role was played by the quantum time coordinate $\hat{s}^0$ in the noncommutative Newtonian spaces). In the proper Carroll case with $\eta \to 0$ we find that

$$[\hat{x}^0, \hat{x}^2] = z\hat{x}^1 \hat{x}^3, \quad [\hat{x}^0, \hat{x}^3] = -z\hat{x}^1 \hat{x}^2, \quad [\hat{x}^0, \hat{x}^1] = 0.$$  \hspace{1cm} (6.21)

Hence for a fixed eigenvalue of $\hat{s}^1$ this noncommutative space can be seen as a quadratic 2D Euclidean algebra with the quantum time coordinate $\hat{x}^0$ behaving as a rotation on the quantum two-space $(\hat{x}^2, \hat{x}^3)$. Thus (6.21) has a quadratic Casimir given by

$$\hat{C} = (\hat{x}^2)^2 + (\hat{x}^3)^2.$$  \hspace{1cm} (6.22)

Therefore, under the ultra-relativistic limit, the noncommutative Minkowski space (5.21) is now decoupled as a central quantum spatial coordinate $\hat{s}^1$ plus the noncommutative Carroll space (6.21). Notice that the contraction (6.15) of the Casimir (5.3) gives $C = (s^1)^2 + (s^2)^2 + (s^3)^2$ but here $s^1$ is a central element.

As far as the Carrollian Lie bialgebras provided by (6.20) are concerned, we remark that they are again endowed with a non-trivial Euclidean sub-Lie bialgebra $\delta(h) = h \wedge h \neq 0$. For the two-parametric Lie bialgebra of type I ($\epsilon_A, \delta_h$) the primitive generators are $P_0$ and $K$, while for the type II ($\epsilon_A, \delta_{\eta}$) these are $\{K_1, J_1, P_0\}$. Again, the corresponding quantum algebras can be easily deduced through twist operators.

7. Concluding remarks

The fate of Lorentz invariance in the context of deformed symmetries is a long-standing question. In this paper, we have studied this problem from a quantum group approach. Namely, we have proven that there are no possible quantum group deformations of the classical spacetime symmetries for the three maximally symmetric spacetimes of constant curvature such that the Lorentz subgroup is kept undeformed. Furthermore, we have proved that there are only three families of non-isomorphic quantum group deformations that keep the Lorentz sector as a quantum subgroup. This directly implies that there is no (non-trivial) covariant noncommutative spacetime with an undeformed Lorentz isotropy subgroup, and that there are three families of non-isomorphic noncommutative spacetimes endowed with a quantum Lorentz subgroup. These results differ from the $(2 + 1)$D case for which we have shown that only for the Poincaré group there exists a deformation keeping the Lorentz subgroup undeformed.

Moreover, we have explicitly constructed the Poisson version of these three noncommutative spacetimes by using both ambient and local coordinates. In terms of ambient coordinates the three Poisson bivectors give rise to PHS which are formally identical for any value of the cosmological constant, something that in our kinematical approach is indeed related to the fact that the cosmological parameter $\Lambda$ does not appear explicitly on the Lorentz subgroup. However, by using local coordinates the spacetime curvature explicitly arises in the noncommutative structures thus distinguishing the (A)dS and Minkowski cases. We stress that in the flat Minkowski space the three noncommutative spacetimes that we obtain are quadratic homogeneous, in striking difference with respect to the well-known $\kappa$-Minkowski spacetime. In
addition, we have studied the non-relativistic and ultra-relativistic limits, obtaining the Newtonian and Carrollian noncommutative spacetimes in which, respectively, their absolute time and absolute space structure is preserved.

An interesting open problem is indeed the construction of the full Hopf algebra structures associated with the three families of quantum groups here presented, together with their corresponding quantum $R$-matrices. In fact, family II can be straightforwardly quantized through the corresponding Reshetikhin twist operator. On the other hand, a Hopf algebra structure of the type III was constructed in [53, 54] for $\mathfrak{so}(2, 1)$ and for higher dimensional (A)dS algebras in [55, 56]. The quantum algebra corresponding to the biparametric $r$-matrix of type I will be indeed more involved from a technical viewpoint, but this problem is worth to be faced since its associated DSR Hopf algebra model would be a novel one that would be relevant in situations when null-plane symmetry is physically relevant.

Furthermore, in order to make use of the results here presented in a general relativistic framework, it would be essential to face the issue of general covariance for a noncommutative field theory in which any of the three novel noncommutative spacetimes here introduced is assumed as the basic local structure of the noncommutative spacetime, and whose ten-dimensional local covariance would be provided by the corresponding quantum Hopf algebras here characterized (see [75] and references therein). In this sense, the fact that the three spacetimes here introduced come from classical $r$-matrices which are skewsymmetric solutions to the CYBE makes it possible to construct their associated Hopf algebras from Drinfel’d twist operators $F$ [76] and, from the latter, to obtain the corresponding $\ast$-products for both the noncommutative spacetimes and the full quantum groups of local covariance [77]. This would open the path to the quantization-deformation of the complete group of diffeomorphisms for a general relativistic noncommutative theory by following the approach presented in [75], where the hypersurface deformation algebroid is constructed, and where the $\ast$-product on the background noncommutative spacetime plays an essential role in order to define Drinfel’d twists of spacetime diffeomorphisms and their deformed actions. In particular, the building block for this approach in the deformation of type III would be given by the so-called Jordanian twist operator, whose $\ast$-product was explicitly given in [50]. Moreover, the $\ast$-products for the noncommutative Minkowski spacetimes of both the first subfamily of type I and the type II deformation should be relatively simple to construct, since both deformations are generated by a Reshetikhin twist. Also, an alternative route to an emergent Gravity theory with general covariance in a noncommutative setting would be provided by the construction of matrix models of the type [78] (see also [79] and references therein) where the covariant Euclidean quantum spaces provided by 4D fuzzy spheres would be substituted by the Lorentzian noncommutative spacetimes with quantum group symmetry here presented. Indeed, within this approach the complete study of the representation theory of the chosen spacetime algebra and of the associated covariance quantum group would be necessary as the first step in order to analyse the feasibility of the full construction of noncommutative gauge and matter fields on this background.

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Data availability statement

No new data were created or analysed in this study.

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