Initial and Boundary Value Problems for Fractional differential equations involving Atangana-Baleanu Derivative

Fatma Al-Musalhi¹, Nasser Al-Salti¹, and Erkinjon Karimov²

¹Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36 Al-Khoudh, Oman
²Institute of Mathematics named after V.I.Romanovskiy, Academy of Sciences of the Republic of Uzbekistan, Tashkent 125, Uzbekistan.

February 25, 2022

Abstract

Initial value problem involving Atangana-Baleanu derivative is considered. An Explicit solution of the given problem is obtained by reducing the differential equation to Volterra integral equation of second kind and by using Laplace transform. To find the solution of the Volterra equation, the successive approximation method is used and a lemma simplifying the resolvent kernel has been presented. The use of the given initial value problem is illustrated by considering a boundary value problem in which the solution is expressed in the form of series expansion using orthogonal basis obtained by separation of variables.

1 Introduction and Preliminaries

1.1 Introduction and related works

Recently, two newly definitions of fractional derivative without singular kernel were suggested, namely, Caputo-Fabrizio fractional derivative [8] and Atangana-Baleanu fractional derivative [6]. These new derivatives have been applied to real life problems, for example, in the fields of thermal science, material sciences, groundwater modelling and mass-spring system [1, 2, 3].
and have been considered in a number of other recent work, see for example, [4], [7], [9], [10], [11], [14]. The main difference between these two definitions is that Caputo-Fabrizio derivative is based on exponential kernel while Atangana-Baleanu definition used Mittag-leffler function as a non-local kernel. The non-locality of the kernel gives better description of the memory within structure with different scale. These two new derivatives are defined as follows

**Definition 1.1.** Let \( f \in H^1(a, b) \), \( b > a \), \( \alpha \in [0, 1] \). The Caputo-Fabrizio fractional derivative is defined as

\[
\text{CF}_a \ D^\alpha_t f(t) = \frac{B(\alpha)}{1 - \alpha} \int_a^t f'(s) \exp \left[ \frac{-\alpha}{1 - \alpha} (t - s) \right] ds,
\]

and the Atangana-Baleanu fractional derivative is given by

\[
\text{ABC}_a \ D^\alpha_t f(t) = \frac{B(\alpha)}{1 - \alpha} \int_a^t f'(s) E_\alpha \left[ \frac{-\alpha}{1 - \alpha} (t - s)^\alpha \right] ds,
\]

where \( B(\alpha) \) denotes a normalization function such that \( B(0) = B(1) = 1 \) and

\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \text{Re}(\alpha) > 0, \ z \in \mathbb{C},
\]

is the Mittag-leffler function of one parameter [12].

For properties related to these derivatives see [6], [7], [11]. In this paper, we are concerned with solutions to initial and boundary value problems for fractional differential equations involving Atangana-Baleanu derivative. We first recall the Mittag-Leffler of two parameters

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \text{Re}(\alpha) > 0, \ \text{Re}(\beta) > 0, \ z \in \mathbb{C},
\]

and a generalized Mittag-Leffler function

\[
E^\delta_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\alpha n + \beta)} n!,
\]

which is introduced by Prabhakar in [13], where \( \alpha, \beta, \delta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and \( (\delta)_n = \frac{\Gamma(\delta + n)}{\Gamma(\delta)} \) is Pochammer’s symbol. For \( \delta = 1 \), it is reduced to Mittag-Leffler function.

Moreover, Mittag-Leffler function \( E_\alpha(\lambda t^\alpha) \) is bounded (see [13]), i.e,

\[
E_\alpha(\lambda t^\alpha) \leq M,
\]

where \( M \) denotes a positive constant. In [6], Atangana and Baleanu considered the time fractional ordinary differential equation

\[
\text{ABC}_0 \ D^\alpha_t f(t) = u(t),
\]
and on using Laplace transform they found the following solution

\[ f(t) = AB_0 I_t^\alpha u(t) = \frac{1 - \alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t u(s)(t - s)^{\alpha-1} ds, \]

where they have defined

\[ AB_0 I_t^\alpha f(t) = \frac{1 - \alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_a^t f(s)(t - s)^{\alpha-1} ds \]

to be the fractional integral associated with the fractional derivative \[2\]. In this paper, we consider the following initial value problem (IVP)

\[ ABC_0 D_t^\alpha u(t) - \lambda u(t) = f(t), \quad t \geq 0, \]
\[ u(0) = u_0, \]

where \( \lambda, u_0 \in \mathbb{R} \). The solution of this IVP is obtained by two different methods, namely, by reducing it to Volterra integral equation of the second kind and by using Laplace transform. The use of such IVP is illustrated by considering a boundary value problem in which the solution is expressed in the form of series expansion using orthogonal basis obtained by separation of variables. The rest of the paper is organized as follows: at the end of this section, we present a lemma which is important for simplifying the resolvent kernel of the Volterra equation. Then, section 2 is devoted for our main result which is the explicit solution of the IVP \[4\]. We conclude this paper by considering a boundary value problem where we have utilized the solution of the IVP \[4\].

### 1.2 Preliminaries

As mentioned earlier, one way to solve the IVP \[4\] is to reduce it to a Volterra integral equation and in order to simplify our calculations, namely the resolvent kernel of the Volterra equation, we have established the following Lemma:

**Lemma 1.2.** Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha < 1 \) with \( \lambda \neq \frac{B(\alpha)}{1 - \alpha} \), then

\[ \sum_{i=1}^{\infty} \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^i \left( \frac{\alpha}{1 - \alpha} \right)^i (t - \xi)^{\alpha-1} E_{\alpha,\alpha} \left( \frac{-\alpha}{1 - \alpha}(t - \xi)^\alpha \right) \]

\[ = \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \frac{\alpha}{1 - \alpha} (t - \xi)^{\alpha-1} E_{\alpha,\alpha} \left[ \frac{\alpha \lambda}{B(\alpha) - \lambda(1 - \alpha)} (t - \xi)^\alpha \right]. \]
Proof. We begin by expanding the following series:

$$\sum_{i=1}^{\infty} \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^i \left( \frac{\alpha}{1 - \alpha} \right)^i (t - \xi)^{i-1} F_{\alpha,\alpha} (\frac{-\alpha}{1 - \alpha} (t - \xi)^{\alpha})$$

$$= \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^1 \left( \frac{\alpha}{1 - \alpha} \right) (t - \xi)_{\alpha-1} E_{\alpha,\alpha} (\frac{-\alpha}{1 - \alpha} (t - \xi)^{\alpha}) + \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^2 \left( \frac{\alpha}{1 - \alpha} \right)^2 (t - \xi)^{2_{\alpha-1}} E_{\alpha,2\alpha} (\frac{-\alpha}{1 - \alpha} (t - \xi)^{2\alpha}) + \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 (t - \xi)^{3_{\alpha-1}} E_{\alpha,3\alpha} (\frac{-\alpha}{1 - \alpha} (t - \xi)^{3\alpha}) + \cdots.$$  

Using the definition of Mittag-Leffler functions, it can be written as follows:

$$\left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^1 \left( \frac{\alpha}{1 - \alpha} \right) (t - \xi)^{\alpha-1} \sum_{n=0}^{\infty} \left( \frac{1}{\Gamma(\alpha + \alpha n)} \right)^n + \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^2 \left( \frac{\alpha}{1 - \alpha} \right)^2 (t - \xi)^{2_{\alpha-1}} \sum_{n=0}^{\infty} \left( \frac{2}{\Gamma(\alpha + 2\alpha n)} \right)^n + \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 (t - \xi)^{3_{\alpha-1}} \sum_{n=0}^{\infty} \left( \frac{3}{\Gamma(\alpha + 3\alpha n)} \right)^n + \cdots,$$

and expanding the series representations of Mittag-Leffler functions, gives

$$\left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^1 \left( \frac{\alpha}{1 - \alpha} \right) (t - \xi)^{\alpha-1} \left[ \frac{1}{\Gamma(\alpha)} - \left( \frac{\alpha}{1 - \alpha} \right) \frac{(t - \xi)^{\alpha}}{\Gamma(2\alpha)} + \left( \frac{\alpha}{1 - \alpha} \right)^2 \frac{(t - \xi)^{2\alpha}}{\Gamma(3\alpha)} + \cdots \right]$$

$$+ \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^2 \left( \frac{\alpha}{1 - \alpha} \right)^2 (t - \xi)^{2_{\alpha-1}} \left[ \frac{1}{\Gamma(2\alpha)} - 2 \left( \frac{\alpha}{1 - \alpha} \right) \frac{(t - \xi)^{\alpha}}{\Gamma(3\alpha)} + \cdots \right]$$

$$+ \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 (t - \xi)^{3_{\alpha-1}} \left[ \frac{1}{\Gamma(3\alpha)} + \cdots \right].$$

Now, combining like terms, we have

$$\left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^1 \left( \frac{\alpha}{1 - \alpha} \right) (t - \xi)^{\alpha-1} \frac{1}{\Gamma(\alpha)} + \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^2 \left( \frac{\alpha}{1 - \alpha} \right)^2 (t - \xi)^{\alpha-1} \frac{1}{\Gamma(2\alpha)}$$

$$+ \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 (t - \xi)^{\alpha-1} \frac{1}{\Gamma(3\alpha)} + \cdots,$$

and simplifying further gives

$$\left( \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} \right)^1 \left( \frac{\alpha}{1 - \alpha} \right) (t - \xi)^{\alpha-1} \frac{1}{\Gamma(\alpha)} + \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1-\alpha)} \right) (t - \xi)^{\alpha} \frac{1}{\Gamma(2\alpha)}$$

$$+ \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1-\alpha)} \right)^2 (t - \xi)^{2\alpha} \frac{1}{\Gamma(3\alpha)} + \cdots.$$
2 Main Result

2.1 Initial value problem

Here, we consider the following problem:

Find a solution \( u(t) \in H^1(0, T) \) that satisfies the following equation

\[
ABC_0D^\alpha_t u(t) - \lambda u(t) = f(t), \quad 0 \leq t \leq T,
\]

and the initial condition

\[
u(0) = u_0,
\]

where \( \lambda, u_0 \in \mathbb{R} \). The solution of this initial value problem is formulated in the following theorem:

**Theorem 2.1.** If \( \lambda \neq \frac{B(\alpha)}{1-\alpha} \), \( f(t) \in C(0, T) \) and \( f(0) = -\lambda u_0 \), then the solution of the initial value problem (6) – (7) is given by

\[
u(t) = \frac{B(\alpha)u_0}{B(\alpha) - \lambda(1-\alpha)E_\alpha}\left[\frac{-\alpha}{1-\alpha}(t-s)^\alpha\right] ds - \lambda u(t) = f(t),
\]

which on integrating by parts leads to

\[
\left(\frac{B(\alpha)}{1-\alpha} - \lambda\right) u(t) - \frac{B(\alpha)}{1-\alpha} \int_0^t \frac{d}{ds}\left(E_\alpha\left[\frac{-\alpha}{1-\alpha}(t-s)^\alpha\right]\right) u(s)ds = f(t) + \frac{B(\alpha)u_0}{1-\alpha} E_\alpha\left[\frac{-\alpha}{1-\alpha}t^\alpha\right].
\]

For \( \lambda \neq \frac{B(\alpha)}{1-\alpha} \), one can write the above equation as a Volterra integral equation of the second kind

\[
u(t) - \int_0^t u(s)K(t, s)ds = \hat{f}(t),
\]

where

\[
K(t, s) = \frac{B(\alpha)}{B(\alpha) - \lambda(1-\alpha)} dE_\alpha\left[\frac{-\alpha}{1-\alpha}(t-s)^\alpha\right],
\]

and

\[
\hat{f}(t) = \frac{B(\alpha)u_0}{B(\alpha) - \lambda(1-\alpha)} E_\alpha\left[\frac{-\alpha}{1-\alpha}t^\alpha\right] + \frac{1-\alpha}{B(\alpha) - \lambda(1-\alpha)} f(t).
\]
To solve equation (9), we use successive approximation method starting with \( u_0(t) = \hat{f}(t) \). Then,

\[
    u_1(t) = \hat{f}(t) + \int_0^t u_0(s) K(t, s) ds
    = \hat{f}(t) + \int_0^t \hat{f}(s) K(t, s) ds,
\]

and similarly we obtain \( u_2(t) \)

\[
    u_2(t) = \hat{f}(t) + \int_0^t u_1(s) K(t, s) ds
    = \hat{f}(t) + \int_0^t \left( \hat{f}(s) + \int_0^s \hat{f}(\xi) K(s, \xi) d\xi \right) K(t, s) ds
    = \hat{f}(t) + \int_0^t \hat{f}(s) K(t, s) ds + \int_0^t \hat{f}(\xi) d\xi \int_\xi^t K(s, \xi) K(t, s) ds
\]

Set \( K_2(t, \xi) = \int_\xi^t K(t, s) K(s, \xi) ds \), so that

\[
    u_2(t) = \hat{f}(t) + \int_0^t \hat{f}(s) K(t, s) ds + \int_0^t \hat{f}(\xi) K_2(t, \xi) d\xi
    = \hat{f}(t) + \int_0^t \hat{f}(\xi) \left[ K_1(t, \xi) + K_2(t, \xi) \right] d\xi.
\]

Continuing the same process, the \( n^{\text{th}} \) term will have the following form

\[
    u_n(t) = \hat{f}(t) + \int_0^t \hat{f}(\xi) \sum_{i=1}^n K_i(t, \xi) d\xi,
\]

where

\[
    K_1(t, \xi) = K(t, \xi), \quad K_i(t, \xi) = \int_\xi^t K(t, s) K_{i-1}(s, \xi) ds, \quad i = 2, 3, \ldots,
\]

which can be derived using mathematical induction.

To obtain the general expression for the kernel \( K_i(t, \xi) \), we substitute for \( K(t, s) \) and start with

\[
    K_2(t, \xi) = \int_\xi^t K(t, s) K_1(s, \xi) ds
    = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^2 \int_\xi^t \frac{d}{ds} \left( E_{\alpha} \left[ \frac{-\alpha}{1 - \alpha} (t - s)^\alpha \right] \right) \frac{d}{d\xi} \left( E_{\alpha} \left[ \frac{-\alpha}{1 - \alpha} (s - \xi)^\alpha \right] \right) ds
    = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^2 \left( \frac{-\alpha}{1 - \alpha} \right)^2 \int_\xi^t (t - s)^{\alpha-1} E_{\alpha,\alpha} \left[ \frac{-\alpha}{1 - \alpha} (t - s)^\alpha \right] (s - \xi)^{\alpha-1} E_{\alpha,\alpha} \left[ \frac{-\alpha}{1 - \alpha} (s - \xi)^\alpha \right] ds,
\]

whereupon using Theorem 5. in [13], \( K_2(t, \xi) \) reduces to

\[
    K_2(t, \xi) = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^2 \left( \frac{\alpha}{1 - \alpha} \right)^2 (t - \xi)^{2\alpha-1} E_{\alpha,2\alpha}^2 \left[ \frac{-\alpha}{1 - \alpha} (t - \xi)^\alpha \right].
\]
According to Lemma 1.2, we get the following

\[ K_3(t, \xi) = \int_\xi^t K(t, s)K_2(s, \xi)ds \]

\[ = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 \int_\xi^t \frac{d}{ds} \left( \frac{\alpha}{1 - \alpha} (t - s)^3 \right) ds \]

\[ = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 \int_\xi^t (t - s)^{3 - \alpha} E_{\alpha, 3\alpha} \left( \frac{-\alpha}{1 - \alpha} (t - s)^\lambda \right) ds, \]

and end with

\[ K_3(t, \xi) = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^3 \left( \frac{\alpha}{1 - \alpha} \right)^3 (t - \xi)^{3 - \alpha} E_{\alpha, 3\alpha} \left( \frac{-\alpha}{1 - \alpha} (t - \xi)^\lambda \right). \]

Consequently, the general expression for the kernel is given by

\[ K_i(t, \xi) = \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right)^i \left( \frac{\alpha}{1 - \alpha} \right)^i (t - \xi)^{i - 1} E_{\alpha, i\alpha} \left( \frac{-\alpha}{1 - \alpha} (t - \xi)^\lambda \right), i = 1, 2, 3, \ldots. \]

As \( n \to \infty \), the approximations of \( u_n(t) \) converges to the solution \( u(t) \)

\[ u(t) = \hat{f}(t) + \int_0^t \hat{f}(\xi) \sum_{i=1}^{\infty} K_i(t, \xi)d\xi. \]

According to Lemma 1.2 we get the following

\[ u(t) = \hat{f}(t) + \left( \frac{B(\alpha)}{B(\alpha) - \lambda(1 - \alpha)} \right) \left( \frac{\alpha}{1 - \alpha} \right) \int_0^t \hat{f}(\xi)(t - \xi)^{\alpha - 1} E_{\alpha, \alpha} \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1 - \alpha)} (t - \xi)^\lambda \right) d\xi, \]

simplifying the above integral using formula (1.107) in [12] and properties of Mittag-Leffler function, we obtain the desired solution given by [8].

An alternative way of finding the solution of the initial value problem (3) - (7) is using Laplace transform method. So, by applying Laplace transform to both sides of equation (6), we have

\[ \frac{B(\alpha)}{1 - \alpha} s^\alpha U(s) - \frac{B(\alpha)}{1 - \alpha} s^{\alpha - 1} u(0) - \lambda U(s) = F(s), \]

where \( U(s) = \mathcal{L}\{u(t)\}(s) and

\[ \mathcal{L}\{_0^\alpha D t^\alpha u(t)\}(s) = \frac{B(\alpha)}{1 - \alpha} s^\alpha U(s) - \frac{B(\alpha)}{1 - \alpha} s^{\alpha - 1} u(0). \]

Simplifying and solving for \( U(s) \), we get

\[ U(s) = \frac{B(\alpha)s^{\alpha - 1}u_0}{s^\alpha(B(\alpha) - \lambda(1 - \alpha)) - \lambda \alpha} + \frac{(1 - \alpha)s^\alpha + \alpha}{s^\alpha(B(\alpha) - \lambda(1 - \alpha)) - \lambda \alpha} F(s), \]
which can be rewritten as

\[
U(s) = \frac{B(\alpha)s^{\alpha-1}u_0}{(B(\alpha) - \lambda(1-\alpha)) \left[ s^\alpha - \frac{\lambda \alpha}{B(\alpha) - \lambda(1-\alpha)} \right] + \frac{\lambda \alpha}{(1-\alpha)s^\alpha + \alpha}} F(s).
\]

Since the Laplace transform of Mittag-Leffler function is given by

\[
\mathcal{L}\{\beta E_{\alpha,\beta}(\lambda t^\alpha)\}(s) = \frac{s^\alpha - \beta}{s^\alpha - \lambda},
\]

then, applying Laplace inverse gives

\[
u(t) = \frac{B(\alpha)u_0}{B(\alpha) - \lambda(1-\alpha)} E_{\alpha,1} \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1-\alpha)} t^\alpha \right) + \frac{1-\alpha}{B(\alpha) - \lambda(1-\alpha)} f(t) + \frac{\alpha}{B(\alpha) - \lambda(1-\alpha)} \left( t^{\alpha-1} E_{\alpha,\alpha} \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1-\alpha)} t^\alpha \right) * f(t) \right).
\]

Consequently,

\[
u(t) = \frac{B(\alpha)u_0}{B(\alpha) - \lambda(1-\alpha)} E_{\alpha,1} \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1-\alpha)} t^\alpha \right) + \frac{(1-\alpha)}{\alpha B(\alpha)} f(t) + \frac{\alpha}{B(\alpha) - \lambda(1-\alpha)} \left( t^{\alpha-1} E_{\alpha,\alpha} \left( \frac{\alpha \lambda}{B(\alpha) - \lambda(1-\alpha)} t^\alpha \right) * f(t) \right),
\]

which is the same as the solution obtained by successive iterations.

**Remark 2.2.** For the case \( \lambda = 0 \) and \( u(0) = 0 \), we get

\[
u(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t f(\xi)(t-\xi)^{\alpha-1} d\xi,
\]

which coincides with the result obtained in [6].

### 2.2 Boundary value problem

Now, we consider a direct problem of determining \( u(x, t) \) in a rectangular domain \( \Omega = \{(x, t) : 0 < x < 1, 0 < t < T\} \), such that \( u \in C^2(0, 1) \times H^1(0, T) \) and satisfies the following initial-boundary value problem:

\[
\begin{align*}
&ABC \quad \frac{D^\alpha_0}{D^\alpha} u(x, t) - u_{xx}(x, t) = f(x, t), \quad (x, t) \in \Omega \quad \text{(10)} \\
&u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T, \quad \text{(11)} \\
&u(x, 0) = 0, \quad 0 \leq x \leq 1, \quad \text{(12)}
\end{align*}
\]
where \( f(x, t) \) is a given function. We begin by using separation of variables method to solve the homogeneous equation corresponding to equation (10) along with the boundary conditions (11). Thus, we obtain the following spectral problem:

\[
\begin{aligned}
X'' + \lambda X &= 0, \\
X(0) &= 0, \\
X(1) &= 0.
\end{aligned}
\]  

(13)

which is self adjoint and has the following eigenvalues

\[
\lambda_k = (k\pi)^2, \quad k = 1, 2, 3, \ldots.
\]

The corresponding eigenfunctions are

\[
X_k = \sin(k\pi x) \quad k = 1, 2, 3, \ldots.
\]  

(14)

Since the system of eigenfunctions (14) forms an orthogonal basis in \( L^2(0, 1) \), we can then write the solution \( u(x, t) \) and the given function \( f(x, t) \) in the form of series expansions as follows:

\[
u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x),
\]

(15)

\[
f(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x),
\]

(16)

where \( u_k(t) \) is the unknown to be found and \( f_k(t) \) is given by \( f_k(t) = 2 \int_0^1 f(x, t) \sin(k\pi x) dx \).

Substituting (13) and (16) into (10) and (12), we obtain the following fractional differential equation

\[
ABCD_0^\alpha u_k(t) + k^2\pi^2 u_k(t) = f_k(t),
\]

(17)

along with the following condition

\[
u_k(0) = 0.
\]

(18)

Whereupon using Theorem 2.1, the solution is given by

\[
u_k(t) = \frac{1 - \alpha}{B(\alpha) + k^2\pi^2(1 - \alpha)} f_k(t) + \frac{\alpha B(\alpha)}{(B(\alpha) + k^2\pi^2(1 - \alpha))^2} \int_0^t f_k(\xi)(t - \xi)^{\alpha-1} E_{\alpha,\alpha} \left[ \frac{-\alpha k^2\pi^2}{B(\alpha) + k^2\pi^2(1 - \alpha)} (t - \xi)^{\alpha} \right] d\xi,
\]

with \( f_k(0) = 0 \), which is achieved by assuming \( f(x, 0) = 0 \). Thus, the solution \( u(x, t) \) can now be written as

\[
u(x, t) = \sum_{k=1}^{\infty} \left( \frac{1 - \alpha}{B(\alpha) + k^2\pi^2(1 - \alpha)} f_k(t) + \frac{\alpha B(\alpha)}{(B(\alpha) + k^2\pi^2(1 - \alpha))^2} \int_0^t f_k(\xi)(t - \xi)^{\alpha-1} E_{\alpha,\alpha} \left[ \frac{-\alpha k^2\pi^2}{B(\alpha) + k^2\pi^2(1 - \alpha)} (t - \xi)^{\alpha} \right] d\xi \right) \sin(k\pi x).
\]
In order to complete the proof of existence, we need to show the uniform convergence of the series representations of

\[ u(x, t), \, u_x(x, t), \, u_{xx}(x, t), \, ^{ABC}_0 D_t^\alpha u(x, t). \]

Since Mittag-Leffler functions is bounded, then the uniform convergence of the series representation of \( u(x, t) \) is ensured by assuming \( f(x, \cdot) \in C(0, T) \). Now, the series representation of \( u_{xx}(x, t) \) is given by

\[
\begin{align*}
    u_{xx}(x, t) &= -\sum_{k=1}^{\infty} \left( \frac{k^2 \pi^2 (1 - \alpha)}{B(\alpha) + k^2 \pi^2 (1 - \alpha)} f_k(t) + \frac{k^2 \pi^2 \alpha B(\alpha)}{(B(\alpha) + k^2 \pi^2 (1 - \alpha))^2} \right) \\
    &\quad \cdot \left( \int_0^t f_k(\xi)(t - \xi)^{a-1} E_{\alpha, \alpha} \left[ \frac{-\alpha k^2 \pi^2}{B(\alpha) + k^2 \pi^2 (1 - \alpha)} (t - \xi)^{\alpha} \right] d\xi \right) \sin(k\pi x) \\
    &= \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x) + \sum_{k=1}^{\infty} \frac{B(\alpha)}{B(\alpha) + k^2 \pi^2 (1 - \alpha)} \sin(k\pi x) \\
    &\quad \cdot \int_0^t f'_k(\xi) E_{\alpha, 1} \left[ \frac{-\alpha k^2 \pi^2}{B(\alpha) + k^2 \pi^2 (1 - \alpha)} (t - \xi)^{\alpha} \right] d\xi.
\end{align*}
\]

Assuming \( f_t(x, t) \) is integrable, it is clear that the second term of the above series converges uniformly. For convergence of the first term, we assume \( f(0, t) = f(1, t) = 0 \) and use integration by parts to get

\[
\left| \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x) \right| = \left| \sum_{k=1}^{\infty} \frac{1}{k\pi} f_{1k}(t) \sin(k\pi x) \right| \leq \sum_{k=1}^{\infty} \frac{1}{k\pi} |f_{1k}(t)|,
\]

where

\[
f_{1k}(t) = 2 \int_0^1 f_x(x, t) \cos(k\pi x) \, dx.
\]

Using the inequality \( ab \leq \frac{1}{2}(a^2 + b^2) \) and the Bessel’s inequality, we then have the following estimate

\[
\left| \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x) \right| \leq \sum_{k=1}^{\infty} \frac{1}{2k^2 \pi^2} \left( \frac{1}{k^2 \pi^2} + |f_{1k}|^2 \right) \leq \sum_{k=1}^{\infty} \frac{1}{2k^2 \pi^2} + \frac{1}{2} \|f_x(x, t)\|_{L^2(0,1)}^2.
\]

Therefore, the expression of \( u_{xx}(x, t) \) is uniformly convergent. Finally, the uniform convergence of \( ^{ABC}_0 D_t^\alpha u(x, t) \), follows from equation (10).

The uniqueness of solution can be shown using the completeness properties of the system \( \{\sin(k\pi x)\} \).

The main result for the direct problem can be summarized in the following theorem:

**Theorem 2.3.** Assume \( f(x, t) \in C[0, 1] \times C[0, T] \) such that \( f(x, 0) = 0, \, f(0, t) = f(1, t) = 0, \, f_t(x, t) \in L[0, T] \) and \( f_x(x, t) \in L^2[0, 1] \), then the problem (11) - (12) has a unique solution.
\[ u(x,t) \text{ given by} \]
\[ u(x,t) = \sum_{k=1}^{\infty} \left( \frac{1 - \alpha}{B(\alpha) + k^2\pi^2(1 - \alpha)} f_k(t) + \frac{\alpha B(\alpha)}{(B(\alpha) + k^2\pi^2(1 - \alpha))^2} \right) \]
\[ \int_{0}^{t} f_k(\xi)(t - \xi)^{\alpha-1}E_{\alpha,\alpha} \left[ \frac{-\alpha k^2\pi^2}{B(\alpha) + k^2\pi^2(1 - \alpha)}(t - \xi)^\alpha \right] d\xi \] \[ \sin(k\pi x). \]

where,
\[ f_k(t) = 2 \int_{0}^{1} f(x,t) \sin(k\pi x) \, dx. \]

Acknowledgements. The first two authors acknowledge financial support from The Research Council (TRC), Oman. This work is funded by TRC under the research agreement no. ORG/SQU/CBS/13/030.

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