ON WEYL MODULES OVER AFFINE LIE ALGEBRAS IN PRIME CHARACTERISTIC

CHUN-JU LAI

Abstract. We construct a family of homomorphisms between Weyl modules for affine Lie algebras in characteristic \( p \), which supports our conjecture on the strong linkage principle in this context. We also exhibit a large class of reducible Weyl modules beyond level one, for \( p \) not necessarily small.

1. Introduction

1.1. The theory of modular representations of reductive algebraic groups is well established [Jan03]. In this context, the Weyl modules play a fundamental role as the Verma modules for the category \( \mathcal{O} \), as Lusztig’s conjecture provides a precise formula expressing the irreducible character in terms of characters of Weyl modules, where the high weights of these Weyl modules are controlled by the so-called strong linkage principle – in other words, if a simple module is a composition factor of a Weyl module then the two high weights are “linked” together.

Explicit homomorphisms between Weyl modules have been constructed by Carter-Lusztig [CL74] and by Carter-Payne [CP80] for type \( A \). For arbitrary types, homomorphisms have been constructed by Franklin [Fra81, Fra88]. Franklin’s work is based on Shapovalov’s construction [Sha72] on homomorphisms between Verma modules in characteristic 0. He first constructed an integral version of Shapovalov elements, which gives rise to homomorphisms between the \( \mathbb{Z} \)-forms of Verma modules. By modifying integral Shapovalov elements, one defines a map between the Weyl modules using reduction modulo \( p \). On one hand, a detailed calculation involving the contravariant form is needed to make sure the map is nonzero. On the other hand, the BGG resolution is needed to ensure that the map is indeed a homomorphism.

We are interested in developing a theory of modular representations of affine Lie algebras. So far there are very few results in literature. For one, Mathieu [Mat96] showed that the Steinberg modules are not irreducible in contrast with the finite types. For another, DeConcini-Kac-Kazhdan [DKK89] showed that the basic representations for untwisted affine ADE types are irreducible if and only if the characteristic \( p \) of the underlying field does not divide the determinant of the Cartan matrix of the underlying finite root system. Chari-Jing [CJ01] further showed that the level one representations are irreducible if \( p \) is coprime to the Coxeter number \( h \). For (possibly twisted) affine ADE types, Brundan-Kleshchev [BK02] computed explicitly the determinant of the Shapovalov form on weight spaces of the basic representations. When combined with [DKK89], it implies that
the basic representation is irreducible if \( p > 3 \) for (possibly twisted) affine \( D \) types and if \( p > h \) for (possibly twisted) affine \( A \) types.

A major difficulty in generalizing modular representation theory of finite types is that in an affine Weyl group there is no longest elements, which play an important role in the finite types. For example, the strong linkage principle for modular finite types can be proved either by the Jantzen sum formula \cite{Jan77} or by Andersen’s approach \cite{And80} on the cohomology on line bundles. In either way, the proof highly relies on the existence of the longest element of a Weyl group. The main observation of this article is that Franklin’s method can be generalized to the affine case as long as one establishes the affine version of the integral Shapovalov elements.

Here we state our main theorem. See \S 2.7 for the definition of \( \gamma \)-mirrored weights, \S 4.1 for “goodness”, and Theorem 5.10 for the restrictions on \( D \).

**Main Theorem.** Assume that \( \lambda, \mu \) are two dominant integral weights such that \( \mu = \lambda - D\gamma \) is \( \gamma \)-mirrored to \( \lambda \) for some positive real root \( \gamma \) and positive integer \( D \). If \( \gamma \) is “good” and \( D \) is “small enough” then there exists a nonzero homomorphism \( V(\mu) \to V(\lambda) \) between Weyl modules.

In this article we also formulate a conjecture (Conjecture 6.1) on the strong linkage principle for Weyl modules over affine Lie algebras. Our main theorem gives rise to a large class of reducible Weyl modules beyond level one for \( p \) not necessarily small, which supports our conjecture.

For type \( \tilde{A}_r \), we also give an upper bond for \( p \) in terms of the level such that the main theorem applies. In particular, for type \( \tilde{A}_1 \) and for the first few primes we give a table showing the lowest level of reducible Weyl modules arise from our main theorem, and the complete list of quasi-simple weights (cf. \S 6.3) of level less than 150.

1.2. The paper is organized as follows. In Section 2 we introduce notations that are needed for the rest of this paper.

In Section 3 we construct the integral Shapovalov elements for affine Lie algebras. These elements, by definition, give rise to nonzero homomorphisms between Verma modules for affine Lie algebras over \( \mathbb{Z} \) if the two highest weights are linked by a reflection in the affine Weyl group.

In Section 4 we show that if the two weights are \( \gamma \)-mirrored for some positive real root \( \gamma \), the integral Shapovalov elements actually give rise to nonzero homomorphisms between Verma modules in the characteristic \( p \), under a condition on the distance between the two highest weights. We then show that condition can be weakened by modifying the integral Shapovalov elements, provided \( \gamma \) is “good”.

In Section 5 we show that if two \( \gamma \)-mirrored weights satisfy an additional condition on their distance, an analogous nonzero homomorphisms between the corresponding Weyl modules can be constructed. The contravariant forms and the BGG resolution for Kac-Moody algebras are used in the proof. Another essential tool here is Lemma 5.2, which is a variant of the Shapovalov determinant formula.
In Section 6 we relate our main theorem to our conjecture on the strong linkage principle. We also describe the candidates for reducible Weyl modules. Using our main theorem, we show that most candidates are reducible except for some rare “quasi-simple weights”. For type $A_1$ we computed for the first few primes a list of reducible Weyl modules that can be detected by our main Theorem; for type $	ilde{A}_r$ we also give an upper bound for $p$ in terms of the level such that our main theorem applies.

**Acknowledgments.** I am grateful to my advisor, Professor Weiqiang Wang, for his guidance and support. I would like to thank Professor Leonard Scott for insightful discussions and bringing Franklin’s paper to our attention. I would like to thank Professor James Franklin for providing his unpublished thesis paper. I would also like to thank Professor Shrawan Kumar for useful comments and conversations.

This research is partially supported by the GRA fellowship of Wang’s NSF grant DMS-1101268. We thank Institute of Mathematics, Academia Sinica, Taipei for providing an excellent working environment and support, where part of this project was carried out.

## 2. Preliminaries

### 2.1. Affine Kac-Moody Algebras.**

Let $g_C = g_C(A)$ be the affine Kac-Moody algebra over $\mathbb{C}$ associated to a symmetrizable generalized Cartan matrix ($GCM$) $A = (a_{ij})_{i,j \in I}$ of affine type and of rank $r$. Following [Kac90] and [Car05], let $\bar{a} = (a_i)_{i \in I}$ and $\bar{c} = (c_i)_{i \in I}$ be the minimal positive integer vectors such that $A\bar{a} = 0 = \bar{c}^t A$. Let $(h_C, (\alpha_i)_{i \in I}, (\lambda_i)_{i \in I})$ be a minimal realization. Here $h_C$ is the $C$-span of $h_0, \ldots, h_r$ and a fixed scaling element $d \in h_C$ such that $\alpha_i(d) = \delta_i 0$. Let $n_C^\pm$ (and $n_C^-$) be the subalgebra of $g_C$ generated by $(e_i)_{i \in I}$ (and $(f_i)_{i \in I}$, respectively). Let $b_C^\pm = h_C \oplus n_C^\pm$ be the corresponding Borel subalgebras. Let $(, ) : g_C \times g_C \to C$ be a non-degenerate invariant symmetric bilinear form and it induces a non-degenerate bilinear form on $b_C^\pm$.

Let $\delta = \sum_{i \in I} a_i \alpha_i$ be the basic imaginary root, let $c = \sum_{i \in I} c_i h_i$ be the central element, and fix a vector $\rho \in b_C^\pm$ satisfying $\rho(h_i) = 1$ for all $i \in I$. Let $h = \sum_{i \in I} a_i$ be the Coxeter number, and let $h^\gamma = \sum_{i \in I} c_i$ be the dual Coxeter number. Let $X = \{ \lambda \in b_C^\pm \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I \}$ be the set of integral weights, and let $X^+ = \{ \lambda \in X \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \}$ be the set of dominant integral weights. Let $\varpi_0, \ldots, \varpi_r \in b_C^*$ be the fundamental weights, that is, each $\varpi_j$ is an element in $b_C^*$ such that $\varpi_j(h_i) = \delta_{i,j}$ and $\varpi_j(d) = 0$. Therefore

$$X^+ = \left\{ \lambda = \sum_{i=0}^r \xi_i \varpi_i + \xi \delta \in b_C^\pm \mid \xi_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I, \xi \in C \right\}.$$  

### 2.2. Affine root systems.

Let $\Phi, \Phi^+, \Phi_{re}, \Phi^+_{re}, \Phi^+_{im}$ and $\Phi^+_{im}$ be the set of roots, positive roots, real roots, positive real roots, imaginary roots and positive imaginary roots of the affine Kac-Moody algebra $g_C$, respectively. Let $\Phi_0$ be the root system generated by $\{\alpha_i\}_{i=1}^r$. Let $Q = \sum_{i \in I} \mathbb{Z} \alpha_i$ (and $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$) be the
set of (non-negative) integer linear combination of roots of $g_C$. For each element $\gamma = \sum_{i \in I} g_i \alpha_i \in Q$ we set

$$h_\gamma = \sum_i g_i^\vee h_i,$$

where $g_i^\vee = (a_i/c_i)g_i$, and let $h \gamma = \sum_{i \in I} g_i$ be the height of $\gamma$. A simple root $\alpha_\eta$ is said to be occurring in $\gamma$ if $g_\eta \neq 0$.

Let $h^*_R$ be the $\mathbb{R}$-span of $\{h_i\}_{i \in I}$ and $d$. To each real root $\gamma \in \Phi_{re}$ we assign a reflection $s_\gamma : h^*_R \to h^*_R$ given by

$$s_\gamma(\lambda) = \lambda - \langle \lambda, \gamma^\vee \rangle \gamma$$

where $\langle \lambda, \gamma^\vee \rangle = \frac{2\langle \lambda, \gamma \rangle}{\langle \gamma, \gamma \rangle}$. In addition, for each $m \in \mathbb{Z}$ we assign an affine reflection $s_{\gamma,m} : h^*_R \to h^*_R$ given by

$$s_{\gamma,m}(\lambda) = s_\gamma(\lambda) + m\gamma.$$ 

The affine reflection $s_{\gamma,m}$ is the reflection with respect to the hyperplane

$$H_{\gamma,m} = \{ \lambda \in h^*_R \mid \langle \lambda + \rho, \gamma^\vee \rangle = m \}.$$ 

2.3. **Affine Weyl groups.** Let $W$ be the Weyl group of $g_C$ generated by $s_\alpha$ with $\alpha \in \Phi_{re}$. It is known that $W$ is also generated by the affine reflections $s_{\alpha,m}$ with $\alpha \in \Phi_0, m \in \mathbb{Z}$. We introduce the dot action of $W$ on $h^*_R$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$ 

In particular, for each real root $\alpha$ we have

$$s_{\alpha,m} \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle - m \alpha.$$ 

Let $\theta$ be the highest root in $\Phi_0$. Then $(W, S)$ is a Coxeter system where the generating set $S = \{s_\theta, s_{\alpha_1}, \ldots, s_{\alpha_r}\}$. A reduced expression for $w \in W$ is a product $w = t_1 \cdots t_N$ of elements $\{t_i\}_{i=1}^N$ in $S$ such that the number $N$ is minimal among all such expressions for $w$. Here $l(w) = N$ is called the length of $w$.

2.4. **Verma modules.** For any Lie algebra $\mathfrak{a}$ let $U(\mathfrak{a})$ be its universal enveloping algebra. Let $U = U(g_C)$ and we denote by $U_Z$ the subring of $U$ generated by $e_i^{(n)}, f_i^{(n)}$ and $(h_i)$ where $i, n$ and $h$ runs over $I, \mathbb{Z}_{\geq 0}$ and the $\mathbb{Z}$-span $h_Z$ of $(h_i)_{i \in I}$ and $d$, respectively. Here $e_i^{(n)} = e_i^n/n!$, $f_i^{(n)} = f_i^n/n!$ and $(h_i) = h(h-1) \cdots (h-n+1)/n!$.

We denote by $U^-_Z$ and $U^+_Z$ the subrings of $U$ generated by $f_i^{(n)}$ and by $e_i^{(n)}$, respectively. Let $g_Z = g_C \cap U_Z, n_Z^+ = n_C^+ \cap U_Z^+$ and $b_Z^+ = n_C^+ \oplus h_Z$.

Let $\mathbb{K}$ be an algebraically closed field, let $g_\mathbb{K}, n_\mathbb{K}^+, h_\mathbb{K}, U_\mathbb{K}$ and $U_\mathbb{K}^+$ be the tensor product of $\mathbb{K}$ with $g_Z, n_Z^+, h_Z, U_Z$ and $U_Z^+$, respectively. For each $\lambda \in h_\mathbb{K}^*$ we denote by $M(\lambda)_\mathbb{K} = U_\mathbb{K}/I(\lambda)_\mathbb{K}$ the Verma module as a $U_\mathbb{K}$-module, where $I(\lambda)_\mathbb{K} = I(\lambda)_Z \otimes \mathbb{K}$ and

$$I(\lambda)_Z = \sum_{j \in I, m \geq 1} U_Z e_j^{(m)} + \sum_{j \in I, m \geq 1} U_Z(b_j^-)^{\frac{h_j - \lambda(h_j)}{m}}.$$
Let $v_{\lambda, k}^+ = 1 + I(\lambda)_k$ be a highest weight vector of $M(\lambda)_k$. The Verma module $M(\lambda)_C$ contains a unique maximal submodule $N(\lambda)_C$ and a unique simple quotient $L(\lambda)_C = M(\lambda)_C / N(\lambda)_C$.

In [KK79] Theorem 2] Kac and Kazhdan proved the linkage principle for symmetrisable Kac-Moody algebras.

**Proposition 2.1.** $L(\mu)_C$ is a composition factor of $M(\lambda)_C$ if and only if $\mu = \lambda$ or $\mu = \mu_0 \uparrow \cdots \uparrow \mu_N = \lambda$ where
\[
x \uparrow y \iff \text{there exists } \begin{cases} n \in \mathbb{Z}_{>0}, \\ \beta \in \Phi^+ \end{cases} \text{ such that } \begin{cases} y - x = n\beta, \\ n(\beta, \beta) = 2(y + \rho, \beta). \end{cases} \tag{2.1}
\]

2.5. **Integral bases for** $U_z$. Based on Garland’s work [Gar78] on the loop algebras, Mitzman [Mit85] constructed a PBW-type basis for all (possibly twisted) affine Lie algebras.

Fix a total ordering $\leq$ on the multiset $\Phi^+_M$ of positive roots taken with their multiplicities, i.e., each positive root $\gamma$ occurred $\dim g_{\mathbb{C}, \gamma}$ times in $\Phi^+_M$. Let $\mathcal{P}$ be the set of functions mapping from $\Phi^+_M$ to $\mathbb{Z}_{>0}$ with finite support. For each $\pi \in \mathcal{P}$ with finite support $\{\gamma_1 < \cdots < \gamma_m\}$ we denote by $\deg \pi = \sum_{i=1}^m \pi(\gamma_i) \in \mathbb{Z}_{>0}$ the degree of $\pi$ and denote by $\omega \pi = \sum_{i=1}^N \pi(\gamma_i) \gamma_i \in \mathbb{Q}^+$ the weight of $\pi$.

For now we fix root vectors $f_\gamma \in g_{\mathbb{C}, \gamma}$ and $e_\gamma \in g_{\mathbb{C}, \gamma}$ for each $\gamma \in \Phi^+$. In particular, let $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$ for each $i \in I$. For each $\pi \in \mathcal{P}$, let $F_\pi = \pi(\gamma_1) \cdots \pi(\gamma_m) \in U(n^-_C)$, $E_\pi = e^{\pi(\gamma_1)} \cdots e^{\pi(\gamma_m)} \in U(n^+_C)$, $f_\pi = f^{\pi(\gamma_1)} \cdots f^{\pi(\gamma_m)} \in U_Z^-$, and $e_\pi = e^{\pi(\gamma_1)} \cdots e^{\pi(\gamma_m)} \in U_Z^+$.

Similarly, for each tuple $\varphi = (\varphi_i)_{i \in I} \in \mathbb{Z}_{>0}^{I+1}$ we assign $H_\varphi = h^{\varphi_0}_0 \cdots h^{\varphi_i}_{\mathbb{Z}_+} \in U(h_C)$, and $h_\varphi = (h^{\varphi_0}_0 \cdots h^{\varphi_i}_{\mathbb{Z}_+}) \in U(h_{\mathbb{Z}})$.

Mitzman proved [Mit85, Theorem 4.2.6] that $U_{\mathbb{Z}}$ has a Garland-Kostant-type $\mathbb{Z}$-basis $\{f_\gamma h_\varphi e^{\pi'} \mid \pi, \pi' \in \mathcal{P}, \varphi \in \mathbb{Z}_{>0}^{I+1}\}$, which also implies that $U$ has a PBW-type $\mathbb{C}$-basis $\{F_\pi H_\varphi E^{\pi'} \mid \pi, \pi' \in \mathcal{P}, \varphi \in \mathbb{Z}_{>0}^{I+1}\}$.

For a fixed simple root $\alpha$, $m \in \mathbb{Z}_{>0}$, and a monomial $F_\omega \in U(n^-_C)$, we define $C_{\omega, \pi, i, m} \in \mathbb{C}$ be the coefficients such that
\[
f_\alpha F_\omega = \sum_{\pi(\alpha) = 0} \sum_{i \in \mathbb{Z}} F_\pi f^{-i}_\alpha C_{\omega, \pi, i, m}.
\]

**Proposition 2.2.** We can rescale the root vectors $e_\gamma$ and $f_\gamma$ so that the $C_{\omega, \pi, i, m}$ defined above satisfy $C_{\omega, \pi, i, m} = C_{\omega, \pi, i}(m)$ for some $C_{\omega, \pi, i} \in \mathbb{C}[x]$. Also, $\deg C_{\omega, \pi, i} = j$ where $j \in \mathbb{Z}$ be such that $\pi = \omega + j\alpha$.

**Proof.** The proof is exactly the same as in the finite case (see [Fra88, Lemma 4.1]). \hfill \Box

From now on, by a little abuse of notation, let $\{e_\gamma, f_\gamma\}_{\gamma \in \Phi^+_M}$ be a fixed set of root vectors as in Proposition 2.2. For each $u \in U(b^-_C)$, $u = \sum_\pi F_\pi Q_\pi$ where $Q_\pi \in U(h_C) \simeq \mathbb{C}[h_0, \ldots, h_{r+1}, d]$. In this case, let
\[
u(\lambda) = \sum F_\pi Q_\pi(\lambda)
\]
denote the evaluation of \( u \) at \( \lambda \in \mathfrak{b}_C^\ast \).

2.6. **Weyl modules.** From now on let \( \mathbb{K} \) be an algebraically closed field of characteristic \( p > 0 \). For each \( \lambda \in X^+ \), recall that \( M(\lambda)_C = U \sigma^+_\lambda \mathbb{K} \), so that we have

\[
L(\lambda)_C = U \sigma^+_\lambda \mathbb{K} = \sigma^+_\lambda \mathbb{K} + N(\lambda)_C.
\]

Then we define its \( \mathbb{Z} \)-form \( L(\lambda)_\mathbb{Z} = U \pi^+_\lambda \mathbb{Z} \) and define the **Weyl module**

\[
V(\lambda) = L(\lambda)_\mathbb{Z} \otimes \mathbb{K}.
\]

2.7. **Nearest lower reflections.** Fix a positive real root \( \gamma \in \Phi^+_\mathfrak{b}_{te} \) and a positive integer \( e \), for each \( \lambda \in \mathfrak{b}_\mathbb{K}^+ \) we set \( D' = \langle \lambda + \rho, \gamma^\vee \rangle \). There are uniquely determined integers \( M, D \in \mathbb{Z} \) such that \( D' = Mp^e + D \) such that \( 0 \leq D < p^e \). Set \( R_{\gamma,e} = s_{\gamma,Mp^e} \), hence

\[
R_{\gamma,e} \cdot \lambda = s_{\gamma,Mp^e} \cdot \lambda = \lambda - D\gamma.
\]

In words, \( R_{\gamma,e} \) is the **nearest lower \( p^e \)-reflection**. That is, among those \( p^e \)-hyperplanes of the form \( H_{\gamma,Mp^e} \) with \( M \in \mathbb{Z} \), \( R_{\gamma,e} \) is the reflection with respect to the hyperplane that is nearest to \( \lambda \).

Let \( \lambda, \mu \in X^+ \) be two dominant integral weights. \( \mu \) is said to be \((\gamma,e)\)-**mirrored** to \( \lambda \) if \( \mu = R_{\gamma,e} \cdot \lambda \) and we write \( \mu \uparrow_{\gamma}^e \lambda \). Finally, \( \mu \) is said to be \( \gamma \)-**mirrored** to \( \lambda \) if \( \mu \) is \((\gamma,e)\)-**mirrored** to \( \lambda \) for some positive integer \( e \) and we write \( \mu \uparrow_{\gamma} \lambda \).

**Remark 2.3.** In modular finite case, the blocks were classified by Donkin \[Don80\] using the theory of \( G_nT \)-modules. Here \( G_n \) is the \( n \)th Frobenius kernel in the algebraic group \( G \) and \( T \) is a maximal torus in \( G \). In the representation theory for \( G_nT \), the building blocks of strong linkage are given by the \( \gamma \)-mirrored condition. The strong linkage principle in this context is proved by Doty \[Dot89\] and he showed that it implies the strong linkage principle for the modular finite case. This is why our seemingly strange \( \gamma \)-mirrored condition is the reasonable format.

### 3. Constructing of Integral Shapovalov Elements

In this section we construct an integral version of Shapovalov elements for the affine case. The argument is adapted from Franklin’s construction \[Fra88\] for the finite case. The idea is to study closely the coefficients in every Shapovalov equation in terms of the PBW-type basis for affine Lie algebras.

3.1. **Generic Shapovalov Elements.** As in \[KK79\], it is easy to verify that if \( \chi \in H_{\gamma,D} \) then any nonzero homomorphism \( M(\chi-D\gamma)_C \rightarrow M(\chi)_C \) is an embedding and is of the form \( v_\chi^{+D\gamma} \mapsto uv_\chi^{+D\gamma} \) for some \( u \in \mathfrak{u}(\mathfrak{b}_C^\ast)_{-D\gamma} \).

**Definition 3.1.** An element \( S \in \mathfrak{u}(\mathfrak{b}_C^\ast)_{-D\gamma} \) is called a \((\text{generic})\) **Shapovalov element** if there is a fixed positive real root \( \gamma = \sum_{i \in I} g_i \alpha_i \in \Phi^+_\mathfrak{b}_{te} \) and a positive integer \( D \in \mathbb{Z}_{>0} \) such that the conditions (S1)-(S3) below hold:

- (S1) \( S \in \mathfrak{u}(\mathfrak{b}_C^\ast)_{-D\gamma} \);
- (S2) \( e_i S \cdot v_\chi = 0 \) for all \( i \in I \) and \( \chi \in H_{\gamma,D} \);
- (S3) The highest degree PBW monomial of \( S \) is a product of \( f_i^{Dg_i} \) for all \( i \in I \).
Lemma 3.2. Let $D$ be a positive integer, and let $\gamma$ be a positive real root. There is a Zariski-dense subset $\Theta$ of the hyperplane $H_{\gamma,D}$ such that for each $\chi \in \Theta$ there exists an element $S^\chi = S^\chi(\gamma, D)$ in $U(n^-_C)$ satisfying (S1')-(S3') below:

- $(S1')$ $S^\chi = \sum F_\pi Q_\pi^\chi$ where $\deg \pi = D\gamma$ and $Q_\pi^\chi \in \mathbb{C}$;
- $(S2')$ $S^\chi \cdot v^\chi_\chi$ is a singular vector in $M(\chi)_C$;
- $(S3')$ The highest degree PBW monomial of $S^\chi$ is a product of $f_i^{D\rho_i}$ for all $i \in I$;

Moreover, $\Theta$ satisfies a polynomial condition below:

- $(S4')$ There are polynomials $Q_\pi \in \mathbb{C}[h_0, \ldots, h_r]$ such that $Q_\pi(\chi) = Q_\pi^\chi$ for all $\chi \in \Theta$.

Proof. The proof is exactly the same as in the finite case (see [Hum08, Sect. 4.13]).

□

Corollary 3.3. For fixed $\gamma \in \Phi^+_{re}$ and $D \in \mathbb{Z}_{>0}$, there exists a (generic) Shapovalov element $S = S(\gamma, D)$. Moreover, $S$ is unique up to modulo the left $U(b^-_C)$-ideal $J(\gamma, D) = U(b^-_C)(h_\gamma + \rho(h_\gamma) - D)$.

Proof. Let $S = \sum F_\pi Q_\pi$ where the polynomials $Q_\pi$ are defined in Lemma 3.2. For each $\chi \in \Theta$ the evaluation $S(\chi)$ equals to the element $S^\chi$ constructed in Lemma 3.2. By the density of $\Theta$, the evaluation $S(\chi)$ also satisfies Conditions $(S1') - (S3')$ for each $\chi \in H_{\gamma,D}$ and hence $S$ satisfies Conditions $(S1) - (S3)$. □

3.2. Integral Shapovalov Elements. The above-mentioned generic Shapovalov elements give rise to nonzero homomorphisms between Verma modules in characteristic 0. Franklin ([Fra81]) introduced the integral Shapovalov elements for finite types and showed that such elements give rise to nonzero homomorphisms between the $\mathbb{Z}$-forms of Verma modules for finite types.

We generalize his idea to the affine types and showed that the integral Shapovalov elements give rise to nonzero homomorphisms between the $\mathbb{Z}$-forms of Verma modules for affine types. In this section we mainly follow the outline given in [Fra88], while some adaptations are needed for the affine type.

Definition 3.4. An element $Z \in U(b^-_C)$ is called an integral Shapovalov element if there is a fixed positive real root $\gamma \in \Phi^+_{re}$ and a positive integer $D \in \mathbb{Z}_{>0}$ such that the conditions (Z1)-(Z3) below hold:

- $(Z1)$ $Z = \sum F_\pi Q_\pi$ where $\deg \pi = D\gamma$ and $Q_\pi \in \mathbb{Z}[h_0, \ldots, h_r]$.
- $(Z2)$ $e_i^{(n)} Z \in I(\gamma, D)_Z = \sum_{j \in I, m \geq 1} U_{\mathbb{Z}} e_i^{(m)} + \sum_{m \geq 1} U(b^-_C)(h_\gamma + \rho(h_\gamma) - D)$ for $i \in I, n > 0$.
- $(Z3)$ The highest degree PBW monomial in $Z$ is a product of $f_i^{D\rho_i}$ for all $i \in I$.

Theorem 3.5. Let $D$ be a positive integer, and let $\gamma$ be a positive real root. There is a dense subset $\Theta \subset H_{\gamma,D}$ such that for each $\chi \in \Theta$ there exists $Z^\chi = Z^\chi(\gamma, D) \in U(n^-_C)$ satisfying $(Z1')-(Z3')$ below:

- $(Z1')$ $Z^\chi = \sum F_\pi Q_\pi^\chi$ where $\deg \pi = D\gamma$ and $Q_\pi^\chi \in \mathbb{Z}$.
- $(Z2')$ $e_i^{(n)} Z^\chi \in I(\chi)_Z$ for all $i \in I, n \in \mathbb{Z}_{>0}$. (cf. [Z2])
(Z3') The highest degree PBW monomial in $Z^\chi$ is a product of $f_i^{D\gamma}$ for all $i \in I$.
Moreover, $\Theta$ satisfies a polynomial condition below:

(Z4') There are polynomials $Q_\pi \in \mathbb{Z}[h_0, \ldots, h_r]$ such that $Q_\pi(\chi) = Q_\pi^\chi$ for $\chi \in \Theta$.
Moreover, $Z^\chi$ can be described recursively by the Shapovalov equation

\[
\begin{align*}
Z^\chi(\gamma, D)f^q_\alpha &= f^{q + D\delta}_\alpha Z^{s\alpha}(s\alpha, D) & \text{if } \text{ht} \gamma > 1, \\
Z^\chi(\gamma, D) &= f^D_\eta & \text{if } \text{ht} \gamma = 1.
\end{align*}
\]

where $q = -\langle \chi + \rho, \alpha^\vee \rangle > 0$ and $b = \langle \gamma, \alpha^\vee \rangle > 0$.

Before proceeding to the proof of Theorem 3.5 we need some lemmas.

**Lemma 3.6.** If $\gamma \in \Phi_\text{re}^+$ then there is a simple root $\alpha_\eta$ occurring (cf. §2.2) in $\gamma$, a sequence of simple roots $\epsilon_1, \ldots, \epsilon_{n-1}$ and distinct positive roots $\gamma = \gamma_0 > \gamma_1 > \cdots > \gamma_{n-1} = \alpha_\eta$

where $\gamma_i = s_{\epsilon_i} \cdots s_{\epsilon_1}(\gamma) = s_{\epsilon_i}(\gamma_{i-1})$.

**Proof.** This is done by induction on the height of $\gamma$. \qed

**Remark 3.7.** For any $\lambda \in H_{\gamma,M^\rho + D}$, we have a commutative diagram of weights via dot-actions:

\[
\begin{array}{ccccccc}
\lambda = \lambda_0 & \epsilon_1 & \lambda_1 & \epsilon_2 & \cdots & \epsilon_{n-1} & \lambda_{n-1} \\
\gamma = \gamma_0 & & \gamma_1 & & \cdots & & \gamma_{n-1} \\
\mu = \mu_0 & \epsilon_1 & \mu_1 & \epsilon_2 & \cdots & \epsilon_{n-1} & \mu_{n-1}
\end{array}
\]

where $\lambda_i = s_{\epsilon_i} \lambda_{i-1}, \mu_i = s_{\epsilon_i} \mu_{i-1} = \lambda_i - D\gamma, \mu = \lambda - D\gamma$.

Note that we reserve the freedom to choose a suitable dense subset of $H_{\gamma,M^\rho + D}$
to determine the order of the $\lambda_i$'s and $\mu_i$'s, and hence the direction of the corre-
sponding Verma module inclusions.

**Lemma 3.8.** Let $\chi \in H_{\gamma,D}$, $\gamma \in \Phi_\text{re}^+$, and let $\alpha$ be a simple root such that $\beta = s\alpha(\gamma) < \gamma$. If $Z(\beta, D)$ exists, then there is a dense subset $\Theta$ of the hyperplane $H_{\gamma,D}$ satisfying that each $\chi \in \Theta$ has a unique $Z^\chi \in U(n^-_{\gamma,D})$ making the following diagram of Verma module homomorphisms commutes.

\[
\begin{array}{ccc}
M(\chi)_C & \xrightarrow{f^q_\alpha} & M(\nu)_C \\
Z^\chi & & Z(\beta, D)(\nu) \\
M(\chi - D\gamma)_C & \xrightarrow{f^q + D\delta}_\alpha & M(\nu - D\beta)_C
\end{array}
\]
Here \( b = \langle \gamma, \alpha^\vee \rangle \), \( q = -\langle \chi + \rho, \alpha^\vee \rangle \) and \( Z(\beta, D)(\nu) \) is the evaluation of \( Z(\beta, D) \) at \( \nu \) (cf. [2,5]).

In particular, the element \( Z^\chi \) is determined by the Shapovalov equation

\[
Z^\chi f^q = f^{q+Db} Z(\beta, D)(\nu).
\]

**Proof.** Let \( \Theta = H_{\gamma, D} \cap X \cap H_{\alpha_0, 0} \) where \( H_{\alpha, m} = \{ \chi \in \mathfrak{h}_R^* | \langle \chi + \rho, \alpha^\vee \rangle < m \} \). A direct calculation shows that \( \nu - D\beta = s_\beta \cdot \nu = s_\alpha \cdot (\chi - D\gamma) \), and hence by the strong linkage principle (Proposition 2.1) we have a commuting diagram of nonzero homomorphisms below.

\[
\begin{array}{ccc}
M(\chi)_C & \xleftarrow{i_3} & M(\nu)_C \\
\downarrow i_1 & & \downarrow i_4 \\
M(\chi - D\gamma)_C & \xrightarrow{i_2} & M(\nu - D\beta)_C
\end{array}
\]

Here \( i_k \) is the left multiplication of a nonzero element \( u_k \in U(n^-_\gamma) \) for \( k = 1, \ldots, 4 \). Since \( f^q_\alpha \in C u_3 \), \( f^{q+Db}_\alpha \in C u_2 \) and by assumption that \( Z(\beta, D)(\nu) \in C u_4 \), so there is a unique element \( Z^\chi \in C u_1 \) such that the Shapovalov equation holds. \( \square \)

**Lemma 3.9.** Under the same assumption as in Lemma 3.8, there is a dense subset \( \Theta \subset H_{\gamma, D} \) such that each \( \chi \in \Theta \) has a unique \( Z^\chi \in U(n^-_\gamma - D\gamma) \) such that the following diagram of Verma module homomorphisms commutes.

\[
\begin{array}{ccc}
M(\chi)_C & \xleftarrow{f^{-q}_\alpha} & M(\nu)_C \\
\downarrow Z^\chi & & \downarrow Z(\beta, D)(\nu) \\
M(\chi - D\gamma)_C & \xrightarrow{f^{q+Db}_\alpha} & M(\nu - D\beta)_C
\end{array}
\]

In particular, the element \( Z^\chi \) is determined by the Shapovalov equation

\[
Z^\chi = f^{q+Db}_\alpha Z(\beta, D)(\nu) f^{-q}_\alpha.
\]

**Proof.** Let \( \Theta = H_{\gamma, D} \cap X \cap H_{\alpha_0, 0} \cap H_{\alpha, -D\beta} \) where \( H_{\alpha, m} = \{ \chi \in \mathfrak{h}_R^* | \langle \chi + \rho, \alpha^\vee \rangle > m \} \) for all \( m \in \mathbb{Z} \). The rest is similar to Lemma 3.8 \( \square \)

Now we are in a position to prove Theorem 3.5.

**Proof of Theorem 3.5.** The proof is done by induction on the height of \( \gamma \). For the initial case \( \gamma = \alpha_\eta \), we set \( \Theta = H_{\gamma, D} \cap X \) and \( Z^\chi = f^D_\eta \) so that the conditions (Z1'), (Z3') and (Z4') follow immediately from definition. Finally (Z2') follows from the useful formula of Kostant below.

\[
e^{(n)}_i f^{(D)}_\eta = \delta_{i, \eta} \sum_{k=0}^{\min(n, D)} f^{(D-k)}_i \binom{n-D+2k}{k} e^{(n-k)}_i.
\]
For the inductive case \( \text{ht} \gamma > 1 \), we apply Lemma 3.8 with \( \alpha = \epsilon_1 \) and obtain a Shapovalov equation \( Z^\gamma f_{\alpha}^q = f_{\alpha}^{q+Db}Z(\beta, D)(\nu) \). Let \( Z(\beta, D)(\nu) = \sum_\pi F_\pi Q'_\pi \) where \( Q'_\pi \in \mathbb{Z} \). Solving this leads to an explicit expression:

\[
Z^\gamma = \sum_\pi \sum_{\pi|\pi(\alpha)=0} F_\pi f^\alpha_{\pi} \left( \sum_\omega C_{\omega, \pi, i}(q + Db)Q'_\omega \right) \\
= \sum_\pi F_\pi Q^\gamma_\pi \text{ where } Q^\gamma_\pi = \sum_\omega C_{\omega, \pi - \pi(\alpha), Db - \pi(\alpha)}(q + Db)Q'_\omega.
\]

It is routine to check that the element \( Z^\gamma \) obtained this way satisfies (Z1')-(Z3'). For (Z4'), by the inductive hypothesis for each \( \pi \) there is a polynomial \( P_\pi \in \mathbb{Z}[h_0, \ldots, h_r] \) such that \( Q'_\pi = P_\pi(\nu) \). Let \( \alpha = \alpha_k \). Note that for each \( i \in I \) we have

\[
h_i(\nu) = \langle \nu, \alpha_i^\gamma \rangle = \langle \chi, \alpha_i^\gamma \rangle - \langle \chi + \rho, \alpha_i^\gamma \rangle = (h_i - a_{ik}(h_k + 1))(\chi).
\]

So there is a polynomial \( P'_\pi \in \mathbb{Z}[h_0, \ldots, h_r] \), obtained from \( P_\pi \) by replacing \( h_i \) by \( h_i - a_{ik}(h_k + 1) \) for each \( i \), such that \( P'_\pi(\chi) = P_\pi(\nu) = Q'_\pi \) and hence by defining \( Q_\pi = \sum_\omega C_{\omega, \pi - \pi(\alpha), Db - \pi(\alpha)}(q + Db)P'_\pi(\chi) \) one has

\[
Q_\pi(\chi) = \sum_\omega C_{\omega, \pi - \pi(\alpha), Db - \pi(\alpha)}(q + Db)P'_\pi(\chi) \\
= \sum_\omega C_{\omega, \pi - \pi(\alpha), Db - \pi(\alpha)}(q + Db)Q'_\pi = Q^\gamma_\pi.
\]

\( \square \)

**Corollary 3.10.** For each \( \gamma \in \Phi_\mathbb{R}^+ \) and \( D \in \mathbb{Z}_{>0} \) there exists an integral Shapovalov element \( Z = Z(\gamma, D) \in U(b^-_Z) \). Moreover, \( Z \) is unique up to modulo the left \( U(b^-_Z) \)-ideal

\[
J(\gamma, D)_Z = \sum_{m \geq 1} U(b^-_Z) \left( h_\gamma + \rho(h_\gamma) - D \right) / m.
\]

**Proof.** The existence part is similar to Corollary 3.9. The uniqueness part follows from that \( e_\gamma^{(n)} J(\gamma, D)_Z \subset U(b^-_Z)e_\gamma^{(n)} + \sum_{m \geq 1} U(b^-_Z)(h_\gamma + \rho(h_\gamma) - D) / m \subset I(\gamma, D)_Z \). \( \square \)

**Corollary 3.11.** Assume that \( \lambda \in H_{\gamma, D} \) for some \( \gamma \in \Phi_\mathbb{R}^+ \) and \( D \in \mathbb{Z}_{>0} \). Let \( \mu = \lambda - D\gamma \). There is a nonzero homomorphism \( \tilde{M}(\mu)_Z \to \tilde{M}(\chi)_Z \) given by

\[
v_\mu^+ \mapsto Zv_\chi^+,
\]

where \( Z = Z(\gamma, D) \) is the integral Shapovalov element.

**Example 3.12.** Consider the affine Kac-Moody algebra of type \( \tilde{A}_1 \) with GCM

\[
A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.
\]

Fix a positive real root \( \gamma = \alpha_0 + \delta \) and a positive integer \( D = 1 \). Then the sequence of positive roots described in Lemma 3.6 is given by

\[
\alpha_0 + \delta = \gamma_0 > \gamma_1 = s_{\alpha_0}(\gamma_0) = \alpha_1.
\]
Then the dense subset \( \Theta \subset H_{\gamma,D} \) as in Theorem 3.5 is given by
\[
\Theta = \{ \chi = \xi_0 \omega_0 - 2(\xi_0 + 1) \omega_1 + \xi_\delta \mid \xi_0 \in \mathbb{Z}_{<0} \} \\
= \{-(q + 1) \omega_0 + 2q \omega_1 + \xi_\delta \mid q \in \mathbb{Z}_{\geq 0} \} \text{ where } q = -\langle \chi + \rho, \alpha_0^\vee \rangle.
\]
For each weight \( \chi \in \Theta \), we have a Shapovalov equation
\[
Z^\chi f_0^q = f_0^{q+2} f_1.
\]
Solving this, we get
\[
Z^\chi = f_1 f_0^2 f_{\delta,1} - f_1 f_0 (h_0 - 1) - f_\gamma \left( \frac{h_0}{2} \right).
\]

4. Homomorphisms between Verma modules in characteristic \( p \)

For finite types, Franklin ([Fra88]) showed that if the corresponding positive root for an integral Shapovalov element is not “exceptional”, there is a “modified” Shapovalov element that gives rise to a nonzero homomorphism between Verma modules in characteristic \( p \), without any restriction. As for the five “exceptional” roots, additional restrictions on the distance between the two highest weights are required.

In this section we generalize Franklin’s idea to affine types. In our viewpoint, the restriction for “exceptional” roots is in fact a generic phenomenon. We first show that under a restriction on the distance, integral Shapovalov elements give rise to nonzero homomorphisms between Verma modules in the characteristic \( p \).

We then show that the restriction is the coarsest if the corresponding positive real root is “good”.

4.1. \( \eta \)-goodness.

**Definition 4.1.** For a fixed prime number \( p \) and \( \eta \in \mathbb{I} \), a positive real root \( \gamma \in \Phi_{re}^+ \) is called \( \eta \)-good if the multiplicity of \( h_\eta \) occurring in \( h_\gamma \) is coprime to \( p \), and the simple root \( \alpha_\eta \) has the same length as \( \gamma \).

In other words, assume that \( h_\gamma = \sum g^\gamma_i h_i \) for some \( g^\gamma_i \in \mathbb{Z} \). \( \gamma \) is \( \eta \)-good if \( \gcd(g^\gamma_i, p) = 1 \) and \( (\alpha_\eta, \alpha_\eta) = (\gamma, \gamma) \).

**Lemma 4.2.** If \( \gamma \) is \( \eta \)-good then
\[
Z[h_0, \ldots, h_r] = Z[h_0, \ldots, \hat{h}_\eta, \ldots, h_r, h_\gamma + \rho(h_\gamma) - D] \otimes \mathbb{Z}(p).
\]
Here \( \hat{h}_\eta \) represents the omission of variable and \( \mathbb{Z}(p) = \{ r/q \in \mathbb{Q} \mid \gcd(p, q) = 1 \} \) is the localization of \( \mathbb{Z} \) at \( p\mathbb{Z} \).

**Proof.** It follows from that \( h_\eta = \frac{1}{g^\gamma_\eta}(h_\gamma - \sum_{i \neq \eta} g^\gamma_i h_i) \) and \( p \nmid g^\gamma_\eta \). \( \square \)

**Lemma 4.3.** Let \( D \) be a positive integer, and let \( \gamma \) be a positive real root such that \( \alpha_\eta \) is occurring in \( \gamma \). There is an element \( Z_\eta = Z_\eta(\gamma, D) \in U(\mathfrak{b}_\mathbb{Z}) \) satisfying the following conditions (G1)-(G4):

1. \( Z_\eta = \sum_\pi F_\pi Q_{\eta,\pi} \) where \( \deg \pi = D \gamma \) and \( Q_{\eta,\pi} \in \mathbb{Z}[h_0, \ldots, h_r] \).
(G2) \( e_i^{(n)} Z_\eta \in I(\gamma, D)_Z \) for all \( i \in I \) and \( n > 0 \).

(G3) The highest degree PBW monomial in \( Z_\eta \) is a product of \( f_i^{D_\eta} \) for all \( i \in I \) times the scalar multiple \( g_\eta^\vee \).

(G4) \( \text{deg}_{h_\eta} Z_\eta = 0 \).

Here \( \text{deg}_{h_\eta} Z_\eta \) is the degree of \( Z_\eta \in Z[h_0, \ldots, \hat{h}_\eta, \ldots, h_r] [h_\eta] \) in the variable \( h_\eta \).

**Proof.** Let \( N = \text{deg}_{h_\eta} Z_\eta \) and for each \( m \) let \( u_{\eta, m} \in U(b^-_Z) \) be the coefficient of \( (h_\eta)_m \) in \( (g_\eta^\vee)^N Z - \sum_N \eta_{\eta, j} (h_\gamma + \rho(h_\gamma) - D) \). Finally we define

\[
Z_\eta = (g_\eta^\vee)^N Z - \sum_{m=0}^N u_{\eta, m} (h_\gamma + \rho(h_\gamma) - D).
\]

By construction (G1), (G2) and (G4) are satisfied. (G3) follows from the fact that \( e_i^{(n)} (h_\gamma + \rho(h_\gamma)) \in U \subset I(\gamma, D)_Z \) and hence

\[
e_i^{(n)} Z_\eta = (g_\eta^\vee)^N e_i^{(n)} Z - \sum_{m=0}^N u_{\eta, m} e_i^{(n)} (h_\gamma + \rho(h_\gamma) - D) \in I(\gamma, D)_Z.
\]

**Example 4.4.** Recall that in Example 3.12 we have \( \gamma = 2\alpha_0 + \alpha_1 \) and hence \( h_\gamma = 2h_0 + h_1 \). Therefore \( \gamma \) is 0-good for any odd prime and is 1-good for any prime. Since \( g_1 = 1 \), we have

\[
Z_1 = Z = f_1 f_0^2 - f_{\delta, 1} f_0 (h_0 - 1) - f_\gamma \left( \frac{h_0}{2} \right).
\]

As for \( \eta = 0 \), we have \( g_0 = 2 \) and so

\[
Z_0 = 4 f_1 f_0^2 + 2 f_{\delta, 1} f_0 (h_1 + 4) - f_\gamma \left( \frac{h_1 + 3}{2} \right).
\]

4.2. **Projection onto** \( U(b^-_Z) \). Let \( P_{b^-} : U \to U(b^-_Z) \) be the projection corresponding to the direct sum decomposition \( U = \sum_{\alpha \in \Phi^+} U e_\alpha \oplus U(b^-_Z) \). It is easy to see that \( uv_\lambda^+ = P_{b^-}(u) v_\lambda^+ \) for each \( \lambda \in \mathfrak{h}_C, u \in U \).

**Lemma 4.5.**

1. Fix a positive integer \( n \in \mathbb{Z}_{>0} \) and \( i \in I \), and let \( \Psi_{\pi, \omega} = \Psi_{\pi, \omega}(i, n) \in U(h_C) \) be such that \( P_{b^-} \left( e_i^{(n)} F_\pi \right) = \sum_{\omega \in \mathcal{P}} F_\omega \Psi_{\pi, \omega} \). Then \( \Psi_{\pi, \omega} \in \mathbb{Z}[h_i] \) and \( \text{deg}_{h_i} \Psi_{\pi, \omega} \leq \pi(\alpha_i) \).

2. Let \( D \) be a positive integer, and let \( \gamma = \sum g_i \alpha_i \) be a positive real root such that \( \alpha_\gamma \) is occurring in \( \gamma \). Let \( Z_\eta = \sum F_\omega Q_{\eta, \pi} \) where \( Q_{\eta, \pi} \in \mathbb{Z}[h_0, \ldots, h_r] \) as in Lemma 4.3. Let \( \Psi_{\omega} = \sum_{\pi} \Psi_{\pi, \omega} Q_{\eta, \pi} \) that is,

\[
P_{b^-} \left( e_i^{(n)} Z_\eta \right) = \sum_{\omega \in \mathcal{P}} F_\omega \Psi_{\omega}.
\]

Then \( \Psi_{\omega} \in \mathbb{Z}[h_0, \ldots, h_r] \) and \( \text{deg}_{h_\eta} \Psi_{\omega} \leq D g_\eta \).
Proof. The first part is exactly the same as in the finite case [Fra88, Lemma 4.3]. Also,
\[
\deg_{h_\eta} \Psi'_\omega = \max \left\{ \deg_{h_\gamma} \Psi_{\pi,\omega} + \deg_{h_\eta} Q_{\eta,\pi} \mid \text{wt} \pi = D_\gamma \right\} \\
\leq \delta_{i,\eta} \max \left\{ \pi(\alpha_\eta) \mid \text{wt} \pi = D_\gamma \right\} + \deg_{h_\eta} Z_\eta \\
\leq Dg_\eta.
\]
This proves (2). \qed

Proposition 4.6. Let $Z_\eta = Z_\eta(\gamma, D)$. If $\gamma = \sum g_i \alpha_i \in \Phi^+_\text{re}$ is $\eta$-good then
\[
P_{b^-} \left( e_i^{(n)} Z_\eta \right) = \sum_{m=1}^{Dg_\eta} u_m \left( h_\gamma + \rho(h_\gamma) - D \right)
\]
for some $u_m \in U(n^-) \otimes \mathbb{Z}[h_0, \ldots, \hat{h}_\eta, \ldots, h_r] \otimes \mathbb{Z}(p)$.

Proof. Since for each $\chi \in H_\gamma, D$ we have $0 = e_i^{(n)} Z_\eta v^\chi_\alpha$ and hence $P_{b^-} \left( e_i^{(n)} Z_\eta \right) v^\chi_\alpha = 0$ for all $\chi \in H_\gamma, D$. Thus $h_\gamma + \rho(\gamma) - D$ divides $P_{b^-} \left( e_i^{(n)} Z_\eta \right) = \sum_\omega F_\omega \Psi'_\omega$, in the sense that $\mathbb{Z}[h_0, \ldots, h_r] \cong \mathbb{Z}[h_0, \ldots, \hat{h}_\eta, \ldots, h_r, h_\gamma + \rho(h_\gamma) - D] \otimes \mathbb{Z}(p)$ by Lemma 4.2. Therefore $h_\gamma + \rho(\gamma) - D$ must divide each $\Psi'_\omega$ and so $m \geq 1$.

On the other hand, by Lemma 4.3 we know that $m \leq Dg_\eta$. \qed

4.3. Homomorphisms between Verma modules. Now we are in a position to construct the homomorphisms in detail. Note that the naive map $M(\mu)_K \to M(\lambda)_K$ sending $v^\mu_\mu \otimes 1_K$ to $Z_\eta v^\chi_\omega \otimes 1_K$ is not necessarily nonzero since $p$ may divide $Z_\eta(\lambda)$ in $U_Z$. In our construction we send $v^\mu_\mu \otimes 1_K$ to $(Z_\eta v^\chi_\omega)/p^f \otimes 1_K$ instead, where $p^f$ is the highest $p$-power dividing $Z_\eta(\lambda)$ in $U_Z$.

Proposition 4.7. If $N \in \mathbb{Z}_{>0}$ divides $Z_\eta(\lambda)$ in $U_Z$ then $N$ also divides each $u_m$ defined in Proposition 4.6.

Proof. It is exactly the same as in the finite case [Fra88, Proposition 5.1]. \qed

Corollary 4.8. Assume that $\mu \uparrow, \lambda$ (cf. §2.2.1) and $\gamma = \sum g_i \alpha_i \in \Phi^+_\text{re}$ is $\eta$-good. If $Dg_\eta < p^e$ then there is a nonzero homomorphism $M(\mu)_K \to M(\lambda)_K$.

Proof. Let $p^f$ be the highest $p$-power dividing $Z_\eta(\lambda)$ in $U_Z$. Applying Lemma 4.6 and Proposition 4.7 with $N = p^f$, we have
\[
e_i^{(n)} \frac{Z_\eta(\lambda)}{p^f} v^\lambda_\lambda = \sum_{m=1}^{Dg_\eta} \frac{u_m}{p^f} \left( \frac{\lambda + \rho, \gamma^\vee - D}{m} \right) v^\chi_\omega.
\]
The assumption then implies $m < p^e$ and hence each $\frac{\lambda + \rho, \gamma^\vee - D}{m} = \frac{M_{p^e}}{m}$ is a multiple of $p$. Therefore $e_i^{(n)} \frac{Z_\eta(\lambda)}{p^f} v^\chi_\omega \otimes 1_K = 0$ and the map $M(\mu)_K \to M(\lambda)_K$ sending $v^\mu_\mu \otimes 1_K$ to $\frac{Z_\eta(\lambda)}{p^f} v^\chi_\omega \otimes 1_K$ is a homomorphism. \qed
5. Homomorphisms between Weyl modules

Now we start to deal with the Weyl modules using reduction modulo $p$. To construct an analogous homomorphism as in Section 4, we have to make sure that $Z_{\eta \bar{\lambda}}$ is nonzero in $L(\lambda)_{\mathbb{Z}}$, which is not obvious. This is done by using the contravariant form and the Shapovalov factor formula (= Lemma 5.2). Finally we prove the existence of nonzero homomorphisms between Weyl modules whose highest weights are $\gamma$-linked if the two highest weights are close enough. In this section we mainly follow the outline given in [Fra81], while some adaptations are needed for the affine type.

5.1. Contravariant Forms. In this section we review some basic properties of the contravariant form mentioned in [Hum08, Sect. 3.14, 3.15] and [Fra81, §8]. A symmetric bilinear form $C : U \times U \rightarrow U(h_C)$ is called contravariant if

$$C(u \cdot v, v') = C(v, \tau(u) \cdot v')$$

for all $u \in U, v, v' \in U$.

**Proposition 5.1.**

(a) Let $P_h : U \rightarrow U(h_C)$ be the projection. Then $C(u, v) = P_h(\tau(u)v)$ is a nonzero contravariant form.

(b) The weight spaces are orthogonal to each other with respect to $C$, i.e., $C(U_{\lambda}, U_{\mu}) = 0$ if $\lambda \neq \mu$.

(c) Let $C^\lambda : U \times U \rightarrow \mathbb{C}$ be the evaluation of $C$ at $\lambda$. Then

$$\text{Rad} C^\lambda = \{u \in U, u \bar{v}_\lambda = 0\} = \text{Ann}_U \bar{v}_\lambda.$$

(d) $C$ is non-degenerate on $U(b_{\gamma})$.

(e) $\deg C(F_\pi, F_\omega) \leq \min(\deg F_\pi, \deg F_\omega)$.

5.2. Shapovalov Factor Formula. Recall that for arbitrary Kac-Moody algebras, the Shapovalov determinant formula [KK79, Theorem 1] computes the determinant of the contravariant form on $U_{D\gamma}$ as follows.

$$\det C \bigg|_{U_{D\gamma}} = \prod_{\alpha \in \Phi^+_{re}} \prod_{n=1}^{\infty} (h_\alpha + \rho(h_\alpha) - n)^{P(D\gamma-na)} \prod_{\alpha \in \Phi^+_{im}} (h_\alpha + \rho(h_\alpha))^{\sum_{j=1}^{\infty} P(D\gamma-na)}.$$

Here $P$ is the Kostant partition function. In this section we prove a variant that describes the common factor for the column corresponding to an integral Shapovalov element $Z(\gamma, D)$.

**Lemma 5.2.** Fix a positive real root $\gamma$ and a positive integer $D$, and let $Z = Z(\gamma, D)$. Define $\epsilon_i, \gamma_i$ as in Lemma 3.6. Let $b_i \in \mathbb{Z}_{\geq 0}$ be such that $\gamma = \sum b_i \epsilon_i$, and let $\beta_i = s_{\epsilon_1} \cdots s_{\epsilon_{i-1}}(\epsilon_i) \in \Phi^+_{re}$. Then for any $u \in U$ we have

$$C(u, Z) \in \mathbb{C} \prod_{i=1}^{n} \prod_{j=1}^{D_{b_i}} (h_{\beta_i} + \rho(h_{\beta_i}) - j).$$
Proof. We want to show that for fixed \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, D_{bi}\} \) that \( h_{\beta_i} + \rho(h_{\beta_i}) - j \) divides \( C(u, Z) \). By Proposition 5.1 one needs to check that \( C^{\lambda}(u, Z) = 0 \) for all \( \lambda \in H_{\beta_i,j} \). So it remains to check that there is a dense subset \( \Theta \) of \( H_{\beta_i,j} \) such that

\[
Zv^{+}_\lambda \in N(\lambda)_C = \sum_{\alpha \in \Phi^+} M(s_\alpha \cdot \lambda)_C \text{ for all } \lambda \in \Theta.
\]

Now \( q_i = -\langle \lambda + \rho, \beta_i^\vee \rangle = -j < 0 \). By assumption \( q_i + D_{bi} = D_{bi} - j \geq 0 \), and hence by Lemma 3.9 there is a commutative diagram of Verma module inclusions

\[
M(\lambda_i-1)_C \xrightarrow{f_{\epsilon_i}^{-q_i}} M(\lambda_i)_C
\]

\[
Z(\gamma_{i-1}, D)(\lambda_i-1) \xrightarrow{f_{e_i}^{q_i+D_{bi}}} Z(\gamma_i, D)(\lambda_i)
\]

\[
M(\mu_i-1)_C \xleftarrow{f_{\epsilon_i}^{q_i}} M(\mu_i)_C
\]

Recall that as in Remark 3.7 we can choose a dense subset of \( H_{\gamma, MP^e}^+D \) such that for any \( \lambda \) in the subset we have the following commutative diagram of weights.

\[
\lambda = \lambda_0 \xleftarrow{e_1} \lambda_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{i-1}} \lambda_{i-1}
\]

\[
\gamma = \gamma_0 \xrightarrow{e_1} \gamma_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{i-1}} \gamma_{i-1}
\]

\[
\mu = \mu_0 \xleftarrow{e_1} \mu_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{i-1}} \mu_{i-1}
\]

By Lemma 3.8 the above diagram leads to a commutative diagram of Verma module inclusions.

\[
M(\lambda_0)_C \xrightarrow{f_{e_1}^{q_1}} M(\lambda_1)_C \xrightarrow{f_{e_2}^{q_2}} \cdots \xrightarrow{f_{e_{i-1}}^{q_{i-1}}} M(\lambda_{i-1})_C
\]

\[
Z(\gamma, D) \xrightarrow{f_{e_1}^{q_1-D_{bi}}} Z(\gamma_1, D)(\lambda_1) \xleftarrow{f_{e_2}^{q_2-D_{bi}}} Z(\gamma_{i-1}, D)(\lambda_{i-1})
\]

\[
M(\mu_0)_C \xleftarrow{f_{e_1}^{q_1}} M(\mu_1)_C \xleftarrow{f_{e_2}^{q_2}} \cdots \xleftarrow{f_{e_{i-1}}^{q_{i-1}-D_{bi}}} M(\mu_{i-1})_C
\]

Finally, calculations show that if \( i \neq n \) then for a suitable choice of dense subset of \( H_{\gamma, MP^e}^+D \) there is a commutative diagram of weights as the following with \( \lambda'_{i-1} = \lambda_i, \lambda'_k = s_{\epsilon_{k+1}} \cdot \lambda'_{k+1}, \mu'_i = \mu_i, \mu'_k = s_{\epsilon_{k+1}} \cdot \mu'_{k+1}, \beta'_k = s_{\epsilon_{k+1}} \cdots s_{\epsilon_{i-1}}(\epsilon_i), \)
\(\gamma'_{i-1} = \gamma_i\) and \(\gamma'_k = s_{\epsilon_{k+1}}(\gamma_{k+1})\).

Now let \(Q_k = \langle \gamma'_k, \epsilon'_k \rangle\). Again by Lemma 3.8 the above diagram leads to a commu-
tative diagram of Verma module inclusions.

\[
\begin{align*}
M(\lambda_0) & \xleftarrow{f^q_{\epsilon_1}} M(\lambda_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\lambda_{i-1}) \\
S(\beta_i, j) & \\
M(\lambda'_0) & \xleftarrow{f^q_{\epsilon_1}} M(\lambda'_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\lambda'_{i-1}) \\
Z(\gamma_i, D)(\lambda'_0) & \\
M(\mu_0) & \xleftarrow{f^q_{\epsilon_1}} M(\mu_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\mu_{i-1}) \\
S(\beta_i, D\mu_i - j)(\mu'_0) & \\
M(\mu'_0) & \xleftarrow{f^q_{\epsilon_1}} M(\mu'_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\mu'_{i-1}) \\
Z(\gamma'_{i-1}, D)(\lambda'_{i-1}) & \\
M(\mu'_0) & \xleftarrow{f^q_{\epsilon_1}} M(\mu'_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\mu'_{i-1}) \\
S(\beta_i, D\mu_i - j)(\mu'_0) & \\
M(\mu'_0) & \xleftarrow{f^q_{\epsilon_1}} M(\mu'_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\mu'_{i-1}) \\
Z(\gamma'_{i-1}, D)(\lambda'_{i-1}) & \\
M(\mu_0) & \xleftarrow{f^q_{\epsilon_1}} M(\mu_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\mu_{i-1}) \\
S(\beta_i, D\mu_i - j)(\mu'_0) & \\
M(\mu'_0) & \xleftarrow{f^q_{\epsilon_1}} M(\mu'_1) \xrightarrow{f^q_{\epsilon_2}} \cdots \xrightarrow{f^q_{\epsilon_{i-1}}} M(\mu'_{i-1}) \\
Z(\gamma'_{i-1}, D)(\lambda'_{i-1}) &
\end{align*}
\]
Combining Diagrams (5.1), (5.2) and (5.3) and use the fact that \( U \) has no zero divisors, we obtain another commutative diagram below:

\[
\begin{array}{ccc}
M(\lambda) & \xrightarrow{S(\beta, j)} & M(\lambda') \\
Z(\gamma, D) & \xrightarrow{S(\beta, D\gamma - j)(\mu')} & Z(\gamma, D)(\lambda') \\
M(\mu) & \xleftarrow{S(\beta, D\gamma - j)(\mu')} & M(\mu') \\
\end{array}
\]

This shows that \( Zv^+ = S(\beta, D\gamma - j)(\mu')(\gamma, D)(\lambda')S(\beta, j)v^+ \in M(s_{\beta} \cdot \lambda) \).

For the special case \( i = n \), Diagram (5.3) is replaced by

\[
\begin{array}{ccc}
M(\lambda_n) & \xrightarrow{f_{\epsilon_1}^{q_1}} & M(\lambda_1) & \xrightarrow{f_{\epsilon_2}^{q_2}} & \cdots & \xrightarrow{f_{\epsilon_{i-1}}^{q_{i-1}}(\mu_{i-1})} M(\lambda_{i-1}) \\
S(\gamma, j) & & & & & f_{\epsilon_i}^{q_i} \\
M(\lambda'_n) & \xrightarrow{f_{\epsilon_1}^{q_1-jb_1}} & M(\lambda'_1) & \xrightarrow{f_{\epsilon_2}^{q_2}} & \cdots & \xrightarrow{f_{\epsilon_{i-1}}^{q_{i-1}-j}} M(\lambda'_{i-1}) \\
M(\mu_0) & \xleftarrow{f_{\epsilon_1}^{q_1-jb_1}} & M(\mu_1) & \xleftarrow{f_{\epsilon_2}^{q_2}} & \cdots & \xleftarrow{f_{\epsilon_{i-1}}^{q_{i-1}-j}} M(\mu_{i-1}) \\
\end{array}
\]

Similarly we have \( Zv^+_n \in M(s_{\beta_n} \cdot \lambda) = M(s_{\gamma} \cdot \lambda) \).

Note that the \( \beta_i \)'s are distinct and none of it is a multiple of another, so the factors \( h_{\beta_i} + \rho(h_{\beta_i}) - j \) are relatively prime. Using Proposition 5.1(e), one can show that those are all the factors, and hence each \( C(U, Z) \) is a scalar multiple of the product of these factors.

\[ \square \]

**Corollary 5.3.** If \( \gamma \) is \( \eta \)-good then \( Z_{\eta}v^+_\lambda \neq 0 \) in \( L(\lambda)_{\mathbb{Z}} \).

**Proof.** Suppose that \( Z_{\eta}v^+_\lambda = \overline{0} \) in \( L(\lambda)_{\mathbb{Z}} \), then \( Zv^+_\lambda = \overline{0} \) in \( L(\lambda)_{\mathbb{Z}} \) as well. Therefore \( C^\lambda(u, Z(\lambda)) = 0 \) for all \( u \in U \) and hence \( C^\lambda(u, Z) = 0 \) for all \( u \in U \). By Proposition 5.1 we know that \( C \) is non-degenerate on \( U(b_\overline{\mathbb{C}}) \) and hence the fact \( Z \neq 0 \) implies \( C(u, Z) \neq 0 \) for some \( u \in U \).

Applying Lemma 5.2 we have \( \prod_{i=1}^n \prod_{j=1}^{ Db_i}(h_{\beta_i} + \rho(h_{\beta_i}) - j)(\lambda) = 0 \), and hence \( \langle \lambda + \rho, \beta_i' \rangle = j \) for some \( 1 \leq i \leq n \) and \( 1 \leq j \leq Db_i \). Therefore both \( j \) and \( Db_i - j \) are non-negative so that \( \lambda_i \) and \( \mu_i \) are on the opposite sides of the hyperplane \( H_{\epsilon_i0} \). This shows that they are in different chambers, which is a contradiction. \[ \square \]

### 5.3. BGG resolution.

Kumar has generalized the strong BGG resolution over any Kac-Moody algebras.
**Proposition 5.4.** [Kum90, Theorem 3.20] For each $\lambda \in X^+$ there is an exact sequence of $\mathfrak{g}_C$-modules:

$$\cdots \to \mathcal{C}_i \to \cdots \to \mathcal{C}_1 \to M(\lambda)_C \to L(\lambda)_C \to 0$$

where $\mathcal{C}_i = \bigoplus_{w \in W, l(w) = i} M(w \cdot \lambda)_C$.

The BGG resolution together with the following lemma describe bases of weight spaces of $L(\lambda)_C$.

**Lemma 5.5.** Assume that $\lambda \in X^+, w \in W$ has a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_N}$ in which $s_{\alpha_\eta}$ occurred. For any $\beta = \sum m_i \alpha_i \in Q^+$, if $\langle \lambda + \rho, \alpha_\eta^\vee \rangle > m_\eta$ then $\lambda - \beta$ is not a weight of $M(w \cdot \lambda)_C$.

**Proof.** Suppose that $\lambda - \beta$ is a weight of $M(w \cdot \lambda)_C$, then

$$\lambda - \beta < w \cdot \lambda = \lambda - \sum_{j=1}^{N} \langle \lambda + \rho, \alpha_j^\vee \rangle \beta_j$$

where $\beta_j = s_{\alpha_{i_j}} \cdots s_{\alpha_{i_{j+1}}}(\alpha_{i_j})$ are distinct positive roots. Since $s_{\alpha_\eta}$ occurs in $w$, there is an integer $L$ which is the largest one satisfying $\alpha_\eta = \alpha_{i_L}$, that is,

$$\beta_L = s_{\alpha_{i_N}} \cdots s_{\alpha_{i_{L+1}}}(\alpha_{i_N}) = \alpha_{i_N} + \sum_{k \neq \eta} n_k \alpha_k$$

for some $n_k \geq 0$.

Hence,

$$\lambda - \beta < \lambda - \langle \lambda + \rho, \alpha_L^\vee \rangle \beta_L < \lambda - \langle \lambda + \rho, \alpha_\eta^\vee \rangle \alpha_\eta,$$

and therefore $\langle \lambda + \rho, \alpha_\eta^\vee \rangle < m_\eta$, which is a contradiction. \square

**Lemma 5.6.** Let $\lambda \in X^+, \beta = \sum m_i \alpha_i \in Q^+$. Assume that $\{B_{i, \pi^+_\lambda}\}_{i \in \Lambda}$ is a basis of $L(\lambda)_{C, \lambda - \beta}$. If $\langle \lambda + \rho, \alpha_\eta^\vee \rangle > m_\eta$ then $\{B_{i, \pi^+_\lambda + k\tau_\eta}\}_{i \in \Lambda}$ is a basis of $L(\lambda + k\tau_\eta)_{C, \lambda + k\tau_\eta - \beta}$ for $k \in \mathbb{Z}_{\geq 0}$.

**Proof.** By Lemma 5.5 the condition $\langle \lambda + \rho, \alpha_\eta^\vee \rangle > m_\eta$ shows that each $\mathcal{C}_L$ (cf. Proposition 5.4) has weight space

$$(\mathcal{C}_L)_{\lambda - \beta} = \bigoplus_{l(w) = L, w \in W_\eta} M(w \cdot \lambda)_{C, \lambda - \beta}.$$

Here $W_\eta$ is the subgroup of $W$ consisting of those elements with no $s_{\alpha_\eta}$ in any reduced expression. For such $w \in W_\eta$ we have $w(\tau_\eta) = \tau_\eta$ and hence

$$w \cdot (\lambda + k\tau_\eta) = w \cdot \lambda + w(k\tau_\eta) = w \cdot \lambda + k\tau_\eta$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Let $w = s_1 \cdots s_N$ be a reduced expression, and let $\beta_j = s_{\alpha_{i_{j+1}}} \cdots s_{\alpha_{i_N}}(\alpha_{i_j})$ for each $j = 1, \ldots, N$. The inclusions $M(w \cdot \lambda)_C \subset M(s_i \cdots s_N \cdot \lambda)_C \subset \cdots \subset M(\lambda)_C$ suggest that $M(w \cdot \lambda)_{C, \lambda - \beta}$ has a basis $\{F_{\pi} u_w v_\pi^+ \mid wt(\pi) = w \cdot \lambda - (\lambda - \beta)\}$, where

$$u_w = f^{(\lambda + \rho, \beta_1^\vee)}_{\alpha_{i_1}} \cdots f^{(\lambda + \rho, \beta_N^\vee)}_{\alpha_{i_N}} \in U(\mathfrak{n}_C)_{-w \cdot \lambda}.$$
Similarly, \( \{ F_{\pi u_w} v_{\lambda + k\omega}^+ \mid \text{wt}(\pi) = w \cdot \lambda - (\beta) \} \) is a basis of \( M(w \cdot \lambda + k\omega) \subset \lambda + k\omega - \beta = M(w \cdot (\lambda + k\omega)) \). Combined with the BGG resolution, this gives us another basis \( \{ B_j^+ \} \) of \( L(\lambda) \) with the property
\[
y_j^+ = \sum C_j^i B_j^i v_\lambda^+ = \sum C_j^i B_j^i v_{\lambda + k\omega}^+ \quad \text{for all } k \in \mathbb{Z}_{\geq 0},
\]
(5.5)
since the \( F_{\pi u_w} \) is independent of \( k \).

Plugging in \( y = \sum_{i \in A} C_i B_i \) for arbitrary \( C_i \in \mathbb{C} \) to (5.5), we obtain that
\[
\sum_{i \in A} C_i B_i v_\lambda^+ = \sum_{j \in A'} C_j^i B_j^i v_\lambda^+
\]
for some \( C_j^i \in \mathbb{C} \), and hence
\[
\sum_{i \in A} C_i B_i v_{\lambda + k\omega}^+ = \sum_{j \in A'} C_j^i B_j^i v_{\lambda + k\omega}^+ = 0
\]
if and only if all the \( C_j^i \)'s are zero, which is equivalent to that all the \( C_i \)'s are zero. Therefore \( \{ B_j^+ v_{\lambda + k\omega}^+ \}_{j \in A} \) is a basis of \( L(\lambda + k\omega) \) for all \( k \in \mathbb{Z}_{\geq 0} \).

**Lemma 5.7.** Assume \( \lambda \in X^+, \beta = \sum m_i \alpha_i \in Q^+ \), and \( \{ B_i v_\lambda^+ \}_{i \in A} \) is a basis of \( L(\lambda) \). If \( \langle \lambda + \rho, \alpha_i^\vee \rangle > m_i \) and if \( y v_\lambda^+ = \sum C_i B_i v_\lambda^+ \) with \( C_i \in \mathbb{C} \) then
\[
y v_{\lambda + k\omega}^+ = \sum C_i B_i v_{\lambda + k\omega}^+ \quad \text{for } k \in \mathbb{Z}_{\geq 0}.
\]
Moreover, if \( y v_\lambda^+ \in L(\lambda) \) then
\[
P_{\hat{h}}(y) = \sum f_\omega y_\omega, \quad \text{where } y_\omega \in \mathbb{C}[h_0, \ldots, \hat{h}_\eta, \ldots, h_r].
\]

**Proof.** It is easy to show that the basis \( \{ B_i v_\lambda \} \) also has the property (5.5) in Lemma 5.6 using elementary linear algebra, which concludes the first part. Let \( P_{\hat{h}}(y) = \sum f_\omega y_\omega \) where \( y_\omega \in \mathbb{C}[h_0, \ldots, h_r] \). By the first part we have
\[
y v_{\lambda + k\omega}^+ = \sum f_\omega y_\omega (\lambda + k\omega) v_{\lambda + k\omega}^+, \quad \text{for } k \in \mathbb{Z}_{\geq 0}.
\]
Hence \( y_\omega \) are the same for all \( k \geq 0 \) and so \( y_\omega \in \mathbb{C}[h_0, \ldots, \hat{h}_\eta, \ldots, h_r] \).

**5.4. Homomorphisms between Weyl Modules.** In this section let \( \lambda, \mu \in X^+ \) such that \( \mu \uparrow^e \gamma \) for some \( \eta \)-good positive real root \( \gamma = \sum g_i \alpha_i \in \Phi_{te}^+ \). Let \( \{ B_a v_\lambda \}_{a \in B_1} \) and \( \{ B_b v_\lambda \}_{b \in B_2} \) be fixed bases of \( L(\lambda) \) and \( L(\mu) \) respectively. We start with a corollary of Lemma 5.6.

**Corollary 5.8.** Fix \( i \in I \) and \( n \in \mathbb{Z}_{\geq 0} \). If \( \langle \lambda + \rho, \alpha_i^\vee \rangle > D g_\eta \) then for each \( a \in B_1 \) we have
\[
e_i^{(n)} B_a v_\lambda^+ = \sum_{m \geq 0} \sum C_{a,b,m}(\lambda) \left( \langle \lambda + \rho, \alpha_i^\vee \rangle - D \right)^m B_b v_\lambda^+
\]
for some \( C_{a,b,m} \in \mathbb{C}[h_0, \ldots, \hat{h}_\eta, \ldots, h_r] \).

**Proof.** It follows from Lemma 5.6 with \( \beta = D \gamma \) and \( y = e_i^{(n)} B_a \).
Note that \( e^{(n)}_i Z^n_1 \xi^+_\lambda \in L(\lambda)_{C,\mu+n\alpha} \) can be written as a combination of \( \{B'_b\xi^+_\lambda\} \).

Applying Proposition 4.6 for fixed \( i \in I \) and \( n > 0 \) we have

\[
e^{(n)}_i Z^n_1 \xi^+_\lambda = \sum_{m=1}^{Dg_\eta} u_m(\lambda) \left( \langle \lambda + \rho, \gamma^\vee \rangle - D \right) \xi^+_m \mathcal{V}_\lambda^+ \]

for some polynomials \( u_{b,m} \in \mathbb{C}[h_0, \ldots, \hat{h}_n, \ldots, h_r] \). Now let \( Z^n_1 \xi^+_\lambda = \sum_a C_a B_a \xi^+_\lambda \) with \( C_a \in \mathbb{C} \). Hence, each number \( N \) dividing \( Z^n_1 \xi^+_\lambda \) in \( L(\lambda)_Z \) also divides each \( C_a \) in \( \mathbb{Z} \).

**Lemma 5.9.** Fix \( i \in I \) and \( n \in \mathbb{Z}_{\geq 0} \). If \( \langle \lambda + \rho, \alpha^\vee \rangle > Dg_\eta \) then

\[
e^{(n)}_i Z^n_1 \xi^+_\lambda = \sum_{m=1}^{Dg_\eta} C_a C_{a,b,m}(\lambda) \left( \langle \lambda + \rho, \gamma^\vee \rangle - D \right) B'_b \xi^+_\lambda.
\]

**Proof.** It remains to show that, for each \( u_{b,m} \) defined in Corollary 5.8 that \( u_{b,m}(\lambda) = \sum_a C_a C_{a,b,m}(\lambda) \) for all \( \lambda \). By Corollary 5.8 we have

\[
e^{(n)}_i Z^n_1 \xi^+_\lambda = \sum_a C_a B_a \xi^+_\lambda = \sum_{m=0}^{Dg_\eta} \sum_{a,b} C_a C_{a,b,m}(\lambda) \left( \langle \lambda + \rho, \gamma^\vee \rangle - D \right) B'_b \xi^+_\lambda.
\]

On the other hand, by (5.6) we have

\[
e^{(n)}_i Z^n_1 \xi^+_\lambda = \sum_{m=1}^{Dg_\eta} \sum_{b,m} u_{b,m}(\lambda) \left( \langle \lambda + \rho, \gamma^\vee \rangle - D \right) B'_b \xi^+_\lambda.
\]

By Lemma 5.6 for each \( k \in \mathbb{Z}_{\geq 0} \) we have

\[
e^{(n)}_i Z^n_{1+k\eta} \xi^+_\lambda = \sum_{m=0}^{Dg_\eta} \sum_{a,b} C_a C_{a,b,m}(\lambda + k\omega_\eta) \left( \langle \lambda + k\omega_\eta + \rho, \gamma^\vee \rangle - D \right) B'_b \xi^+_\lambda
\]

Equating the coefficient of \( B'_b \xi^+_\lambda \), one gets

\[
\sum_{m=0}^{Dg_\eta} \sum_{a} C_a C_{a,b,m}(\lambda + k\omega_\eta) \left( \langle \lambda + k\omega_\eta + \rho, \gamma^\vee \rangle - D \right)
\]

\[
= \sum_{m=1}^{Dg_\eta} u_{b,m}(\lambda + k\omega_\eta) \left( \langle \lambda + k\omega_\eta + \rho, \gamma^\vee \rangle - D \right) \text{ for all } k \in \mathbb{Z}_{\geq 0}.
\]

Since the right hand side has finite degree in \( k \), this is an equation of polynomials in \( k \) with infinitely many solutions. We may equate the coefficients of powers of \( k \),
and hence
\[ \sum_a C_a C_{a,b,m} = u_{b,m}. \]

Now we can prove our main theorem.

**Theorem 5.10.** Assume that \( \mu \uparrow^e \lambda \) where \( \gamma = \sum g_i \alpha_i \in \Phi_+^e \) is \( \eta \)-good. If \( D_{g_\eta} < \langle \lambda + \rho, \alpha_\eta^\vee \rangle \) and \( D_{g_\eta} < p^e \) then there exists a nonzero homomorphism \( V(\mu) \rightarrow V(\lambda) \) between Weyl modules.

**Proof.** Let \( p^g \) be the highest \( p \)-power dividing \( Z_{\eta} \overline{\tau}_\lambda^+ \) in \( L(\lambda)_{\mathbb{Z}} \). We shall show that the map sending \( \overline{\tau}_\mu^+ \otimes 1_{\mathbb{K}} \) to \( \frac{Z_{\eta} \overline{\tau}_\lambda^+}{p^g} \otimes 1_{\mathbb{K}} \) is a homomorphism. Since each number \( N \) dividing \( Z_{\eta} \overline{\tau}_\lambda^+ \) in \( L(\lambda)_{\mathbb{Z}} \) also divides each \( C_a \) in \( \mathbb{Z} \), in particular this is true for \( N = p^g \). Applying Lemma 5.9, we have
\[ e_i^{(n)} \frac{Z_{\eta} \overline{\tau}_\lambda^+}{p^g} = \sum_{m=1}^{D_{g_\eta}} \sum_{a,b} \frac{C_a}{p^g} C_{a,b,m}(\lambda) \left( \frac{\langle \lambda + \rho, \gamma^\vee \rangle - D}{m} \right) B_b \overline{\tau}_\lambda^+. \]

The assumption that \( D_{g_\eta} < p^e \) then implies \( m < p^e \) and hence \( p \) divides each \( \frac{\langle \lambda + \rho, \gamma^\vee \rangle - D}{m} \). Therefore \( e_i^{(n)} \frac{Z_{\eta} \overline{\tau}_\lambda^+}{p^g} \otimes 1_{\mathbb{K}} = 0 \) in \( V(\lambda) \). \qed

**Example 5.11.** In Example 3.12 we have \( \gamma = 2\alpha_0 + \alpha_1 \) is \( \eta \)-good for any prime if \( \eta = 1 \), \( g_1 = 1 \) and \( \lambda = 2\omega_0 + \omega_1 \), \( Z_1 = f_1 f_0^2 - f_1 f_0 h_0 - 3 \). Hence \( p = 7 \) is the only possible prime. Also, the assumptions are satisfied since \( \langle \lambda + \rho, \alpha_\eta^\vee \rangle = 2 > D_{g_1} = 1 \) and \( D_{g_1} = 1 < 7^e \). Since \( Z_1 \overline{\tau}_\lambda^+ = 3 \left( 2f_1 f_0^2 - f_1 f_0 \right) \overline{\tau}_\lambda^+ \), so \( g = 0 \) and there is a nonzero homomorphism \( V(3\omega_1 - 2\delta) \rightarrow V(2\omega_0 + \omega_1) \) given by \( \overline{\tau}_\mu^+ \otimes 1_{\mathbb{K}} \mapsto Z_1 \overline{\tau}_\lambda^+ \otimes 1_{\mathbb{K}} \).

6. A CONJECTURAL STRONG LINKAGE PRINCIPLE

6.1. **Formulation of a conjecture.** Having in mind the strong linkage principles for classical cases, we formulate a conjecture (joint with W. Wang) on the strong linkage principle for the modular representation of affine Lie algebras.

**Conjecture 6.1** (Strong linkage principle). Let \( \mu, \lambda \in X^+ \), and let \( L(\mu) \) be the unique irreducible highest weight \( U_{\mathbb{K}} \)-module with highest weight \( \mu \). If \( L(\mu) \) is a composition factor of the Weyl module \( V(\lambda) \), then \( \mu = \lambda \) or \( \mu = \mu_0 \uparrow \cdots \uparrow \mu_N = \lambda \) for some \( N \) where
\[
\begin{cases}
    n \in \mathbb{Z}_{>0}, \\
    m \in \mathbb{Z}, \\
    \beta \in \Phi^+ \\
\end{cases}
\begin{align*}
    x \uparrow y &\iff \\
    m &\text{ such that } \\
    \beta &\text{ such that } \\
    y - x = (n - mp)\beta, \\
    n(\beta, \beta) &= 2(y + \rho, \beta).
\end{align*}
\]

**Remark 6.2.** If we interpret \( p = 0 \), condition (6.1) coincides with Condition (2.1). For finite types, Condition (6.1) describes exactly the strong linkage for modular
finite case below:

\[ x \uparrow y \Leftrightarrow \text{there exists } \begin{cases} m \in \mathbb{Z}, \\ \beta \in \Phi^+ \end{cases} \text{ such that } x = s_{\beta,mp} \cdot y < y. \quad (6.2) \]

**Remark 6.3.** The formulation of Conjecture [6.1] also makes sense for all symmetric Kac-Moody algebras.

### 6.2. Candidates for reducible Weyl modules

Now we focus on the reducibility problem for Weyl modules. Note that we do not know the irreducibility for any Weyl module, but our main theorem is able to detect if a Weyl module is reducible.

The candidates for the high weights of reducible Weyl modules are those dominant integral weights which are \( \gamma \)-mirrored by another dominant integral weight. Note that if \( \lambda, \mu \in X^+ \) such that \( \mu \uparrow_\gamma \lambda \) for some positive real root \( \gamma \), there is a unique weight \( \lambda + t\delta \in X^+ \) such that \( \mu + t\delta \uparrow_\gamma \lambda + t\delta \) and \( \xi = 0 \) if we express \( \lambda + t\delta = \sum_i \xi_i \omega_i + \xi \delta \). Without loss of generality, we only need to consider the set defined below:

\[ Y^+ = \left\{ \lambda \in \sum_i \mathbb{Z}_{\geq 0} \omega_i \mid \exists \mu \in X^+ \text{ s.t. } \mu \uparrow_\gamma \lambda \text{ for some } \gamma \in \Phi^+_\text{re} \right\}. \]

**Lemma 6.4.** For an arbitrary affine type, the set of positive real roots can be described as the following.

\[ \Phi^+_\text{re} = \bigcup_{i=1}^3 \{ \gamma_0 + t\delta \mid \gamma_0 \in \Phi^+_i, t \in i\mathbb{Z}_{\geq 0} \}, \]

where \( \Phi^+_i \) are subsets of \( \Phi^+ \) for \( i = 1, 2, 3 \).

In particular, \( \Phi^+_3 = \emptyset \) unless for type \( E_4^{(3)} \). \( \Phi^+_2 = \emptyset \) unless for type \( A_r^{(2)}, D_r^{(2)} \) and \( \hat{E}_6^{(2)} \). For untwisted types we have \( \Phi^+_1 = \{ \alpha, \delta - \alpha \mid \alpha \in \Phi_0^+ \} \).

**Proof.** This is done by a case-by-case analysis on the data of root systems. \( \square \)

**Proposition 6.5.** Let \( \lambda \in X^+ \) be a dominant integral weight of fixed level \( \ell \), and let \( \gamma_0 \in \Phi^+_i \) for some \( i \) as defined in Lemma [6.4]. Let \( C_i = \frac{2i}{(\gamma_0, \gamma_0)}(\ell + h^\vee) \). If \( \lambda - \gcd(C_i, p)\gamma_0 \in X^+ \) then \( \lambda \in Y^+ \).

**Proof.** For any positive real root of the form \( \gamma = \gamma_0 + it\delta \) we have \( \langle \gamma, \gamma \rangle = \langle \gamma_0, \gamma_0 \rangle \), and hence for any \( \lambda \in X^+ \) we have \( \langle \lambda + \rho, \gamma \rangle = \langle \lambda + \rho, \gamma_0 \rangle + C_i t \). Therefore for \( m, e \in \mathbb{Z}_{\geq 0} \) we have

\[ s_{\gamma, Mp^e} \cdot \lambda = s_{\gamma_0, C_it + Mp^e} \cdot \lambda - (\langle \lambda + \rho, \gamma \rangle - Mp^e)t\delta. \]

Therefore, we can always choose \( t, M \in \mathbb{Z}_{\geq 0} \) large enough such that

\[ s_{\gamma_0 + it\delta, Mp^e} \cdot \lambda = \lambda - \gcd(C_i, p^e)\gamma_0 - (\langle \lambda + \rho, \gamma \rangle - Mp^e)t\delta. \]

By assumption \( s_{\gamma_0 + t\delta, Mp^e} \cdot \lambda \in X^+ \) and hence \( \lambda \in Y^+ \). \( \square \)
Using Proposition 6.5, one can describe for each type and for each prime $p$ the lowest level weights in $Y^+$. For most types, every weight in $Y^+$ has level $\geq 2$. $Y^+$ contains a level one weight if and only if the following conditions hold:

(1) $p$ is an odd prime that does not divide $h^\vee + 1$.
(2) $g_C$ is of type $\tilde{B}_r$, $\tilde{C}_r$, $\tilde{F}_4$ or $\tilde{G}_2$.

In this case, the level one weights are given by Table 1 below.

**Table 1.** The complete list of weights of level one in $Y^+$.

| Type       | $\tilde{B}_r; r \geq 3$ | $\tilde{C}_r; r \geq 2$ | $\tilde{F}_4$ | $\tilde{G}_2$ |
|------------|-------------------------|--------------------------|---------------|--------------|
| weight     | $\varpi_0, \varpi_1$    | $\varpi_0, \varpi_r$    | $\varpi_0$    | $\varpi_0, \varpi_2$ |

This shows that for (possibly twisted) affine ADE types, there is no dominant integral weights that are $\gamma$-mirrored to $\varpi_0$ for any $\gamma \in \Phi^+_r e$. One might expect that the lack of $\gamma$-mirrored weights would imply that the basic representation is irreducible. However, the table in [BK02, §1] shows that for each (possibly twisted) affine ADE type, there exists a reducible basic representation for some prime $p \leq h$.

6.3. **Quasi-simple weights.** Our main theorem shows that the corresponding Weyl module of a weight in $Y^+$ is reducible if the restrictions in Theorem 5.10 are satisfied. Here we shall demonstrate that these restrictions are actually quite mild.

**Definition 6.6.** A weight $\lambda \in Y^+$ is called quasi-simple if Theorem 5.10 does not apply for any $\mu \in X^+$ that is $\gamma$-mirrored to $\lambda$.

Note that every weight of level one described in Remark 6.5 is quasi-simple. Unfortunately, it seems that there is not an obvious pattern for the quasi-simple weights in general.

6.4. **Type $\tilde{A}_1$.** Assume that $\gamma = \alpha_0 + t\delta$ with $t \geq 0$. Let $(\xi_0, \xi_1)$ be the shorthand notation of the weight $\xi_0 \varpi_0 + \xi_1 \varpi_1$. Our main theorem has the following corollary.

**Corollary 6.7.** For type $\tilde{A}_1$ and a fixed prime $p$, a weight $\lambda = (\xi_0, \ell - \xi_0)$ of level $\ell$ lies in $Y^+$ if and only if there are integers $M, D, e \in \mathbb{Z}_{>0}$ and $t \in \mathbb{Z}_{\geq 0}$ satisfying the following conditions.

1. $(\ell + 2)t + \xi_0 + 1 = Mp^e + D$.
2. $2D \leq \xi_0 \leq \ell$.

In particular, if $\lambda \in Y^+$, the corresponding Weyl module $V(\lambda)$ is reducible if either Condition (i) or (ii) below holds:

(i) $\gcd(t + 1, p) = 1$ and $D(t + 1) < \min(\xi_0 + 1, p^e)$.
(ii) $\gcd(t, p) = 1$ and $Dt < \min(\ell - \xi_0 + 1, p^e)$.
Proof. This is a reformulation of Theorem 5.10 in type $\tilde{A}_1$. □

For a prime $p$, we denote by $\ell\ell(p)$ the lowest level of weight $\lambda$ such that $V(\lambda)$ is a reducible Weyl module arise from our main theorem. Note that there could be reducible Weyl modules that cannot be detected by our main Theorem with level lower than $\ell\ell(p)$. Table 2 below describes the first few $\ell\ell(p)$ and the possible high weights $\lambda$ of reducible Weyl modules detected by Corollary 6.7. One can observe that $\ell\ell(p)$ grows much slower than $p$.

| $p$ | $\ell\ell(p)$ | possible $\xi_0$ |
|-----|----------------|------------------|
| 2   | 2              | $\{0,2\}$       |
| 3   | 2              | $\{0,2\}$       |
| 5   | 4              | $\{0,1,3,4\}$   |
| 7   | 7              | $\{1,2\}$       |
| 11  | 3              | $\{0,4\}$       |
| 13  | 4              | $\{0,3\}$       |
| 17  | 3              | $\{2,3\}$       |
| 19  | 5              | $\{0,5\}$       |

| $p$ | $\ell\ell(p)$ | possible $\xi_0$ |
|-----|----------------|------------------|
| 23  | 5              | $\{2,3\}$       |
| 29  | 6              | $\{1,5\}$       |
| 31  | 7              | $\{3,4\}$       |
| 37  | 6              | $\{1,5\}$       |
| 41  | 7              | $\{2,5\}$       |
| 43  | 7              | $\{0,7\}$       |
| 47  | 8              | $\{2,5\}$       |
| 53  | 8              | $\{3,5\}$       |

The quasi-simple weights for $p = 2, 3, \text{and} 5$ up to level 150 are given in Table 3 below. In other words, any weight in $Y^+$ that is not in the table corresponds to a reducible Weyl module. This supports that almost every Weyl module is reducible.

| $p = 2$ | $p = 3$ | $p = 5$ |
|---------|---------|---------|
| $\ell$ | $(\xi_0, \ell - \xi_0)$ | $\ell$ | $(\xi_0, \ell - \xi_0)$ | $\ell$ | $(\xi_0, \ell - \xi_0)$ |
| 3      | $(0,3)$, $(3,0)$ | 3      | $(1,2)$, $(2,1)$ | 2      | $(0,2)$, $(2,0)$ |
| 8      | $(1,7)$, $(7,1)$ | 6      | $(1,5)$, $(5,1)$ | 4      | $(2,2)$ |
| 18     | $(3,15)$, $(15,3)$ | 13     | $(5,8)$, $(8,5)$ | 6      | $(3,3)$ |
| 38     | $(7,31)$, $(31,7)$ | 22     | $(5,17)$, $(17,5)$ | 18     | $(4,14)$, $(14,4)$ |
| 78     | $(15,63)$, $(63,15)$ | 43     | $(17,26)$, $(26,17)$ | 28     | $(14,14)$ |
|        |          | 70     | $(17,53)$, $(53,17)$ | 38     | $(19,19)$ |
|        |          |        |                        | 98     | $(24,74)$, $(74,24)$ |
|        |          |        |                        |        | $(74,74)$ |

6.5. Type $\tilde{A}_r : r \geq 2$. Assume that $\gamma = \alpha_0 + t\delta$ with $t \geq 0$. Let $(\xi_0, \xi_1, \ldots, \xi_r)$ be the shorthand notation of the weight $\sum_{i=0}^{r} \xi_i \omega_i$. Our main theorem leads to the following corollary.

**Corollary 6.8.** For type $\tilde{A}_r, r \geq 2$ and a fixed prime $p$, a weight $\lambda = (\xi_0, \xi_1, \ldots, \xi_{r-1}, \ell - (\xi_0 + \ldots + \xi_{r-1}))$ of level $\ell$ lies in $Y^+$ if and only if there are integers $M, D, e \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}_{\geq 0}$ satisfying the following conditions.
\[(\ell + h^\vee)t + \xi_0 + 1 = Mp^e + D.\]
\[2D \leq \xi_0 \leq \ell - (\xi_1 + \ldots + \xi_{r-1}).\]

In particular, if \(\lambda \in Y^+\), the corresponding Weyl module \(V(\lambda)\) is reducible if either Condition (i),(ii) or (iii) below holds:

(i) \(\gcd(t + 1, p) = 1\) and \(D(t + 1) < \min(\xi_0 + 1, p^e)\).
(ii) \(\gcd(t, p) = 1\) and \(Dt < \min(\xi_i + 1, p^e)\) for some \(i = 1, \ldots, r - 1\).
(iii) \(\gcd(t, p) = 1\) and \(Dt < \min(\ell - (\xi_0 + \ldots + \xi_{r-1}) + 1, p^e)\).

Remark 6.9. We observe that, for a fixed type, every weight of level \(\ell\) seems to be quasi-simple if \(p \gg \ell\). In particular, for type \(\tilde{A}_r\), every weight in \(Y^+\) of level \(\ell\) is quasi-simple if \(p > \ell(\ell + h^\vee) - h^\vee\).

References

[And80] H. H. Andersen, The strong linkage principle, J. Reine Angew. Math. 315 (1980), 53–59.
[BK02] J. Brundan and A. Kleshchev, Cartan determinants and Shapovalov forms, Math. Ann. 324 (2002), 431–449.
[Car05] R. W. Carter, Lie algebras of finite and affine type, Cambridge Studies in Advanced Mathematics 96, Cambridge University Press, Cambridge, 2005.
[CJ01] V. Chari and N. Jing, Realization of level one representations of \(U_q(\hat{g})\) at a root of unity, Duke Math. J. 108 (2001), 183–197.
[CL74] R. W. Carter and G. Lusztig, On the modular representations of the general linear and symmetric groups, Math. Z. 136 (1974), 193–242.
[CP80] R. W. Carter and M. T. J. Payne, On homomorphisms between Weyl modules and Specht modules, Math. Proc. Cambridge Philos. Soc. 87 (1980), 419–425.
[DKK89] C. DeConcini, V. G. Kac, and D. A. Kazhdan, Boson-Fermion correspondence over \(Z\), Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), Adv. Ser. Math. Phys. 7, World Sci. Publ., Teaneck, NJ, 1989, pp. 124–137.
[Don80] S. Donkin, The blocks of a semisimple algebraic group, J. Algebra 67 (1980), 36–53.
[Dot89] S. Doty, The strong linkage principle, Amer. J. Math. 111 (1989), 135–141.
[Fra81] J. Franklin, Homomorphisms between Verma modules and Weyl modules in characteristic \(p\), Ph.D. thesis, University of Warwick, 1981.
[Fra88] _____. Homomorphisms between Verma modules in characteristic \(p\), J. Algebra 112 (1988), 58–85.
[Gar78] H. Garland, The arithmetic theory of loop algebras, J. Algebra 53 (1978), 480–551.
[Hum08] J. E. Humphreys, Representations of semisimple Lie algebras in the BGG category \(\mathcal{O}\), Graduate Studies in Mathematics 94, American Mathematical Society, Providence, RI, 2008.
[Jan77] J. C. Jantzen, Darstellungen halbeinfacher Gruppen und kontravariante Formen, J. Reine Angew. Math. 290 (1977), 117–141.
[Jan03] _____. Representations of algebraic groups, second ed., Mathematical Surveys and Monographs 107, American Mathematical Society, Providence, RI, 2003.
[Kac] V. G. Kac, Infinite-dimensional Lie algebras, third ed., Cambridge University Press, Cambridge, 1990.
[KK79] V. G. Kac and D. A. Kazhdan, Structure of representations with highest weight of infinite-dimensional Lie algebras, Adv. in Math. 34 (1979), 97–108.
[Kum90] S. Kumar, Bernstein-Gelfand-Gelfand resolution for arbitrary Kac-Moody algebras, Math. Ann. 286 (1990), 709–729.
[Mat96] O. Mathieu, On some modular representations of affine Kac-Moody algebras at the critical level, Compositio Math. 102 (1996), 305–312.
[Mit85] D. Mitzman, *Integral bases for affine Lie algebras and their universal enveloping algebras*, Contemporary Mathematics 40, American Mathematical Society, Providence, RI, 1985.

[Sha72] N. N. Shapovalov, *On a bilinear form on the universal enveloping algebra of a complex semisimple lie algebra*, Funct. Anal. Appl. 6 (1972), 307–312.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904

E-mail address: cl8ah@virginia.edu