Dispersive qubit readout with machine learning

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Abstract

Open quantum systems can undergo dissipative phase transitions, and their critical behavior can be exploited in sensing applications. For example, it can be used to enhance the fidelity of superconducting qubit readout measurements, a central problem toward the creation of reliable quantum hardware. A recently introduced measurement protocol, named “critical parametric quantum sensing”, uses the parametric (two-photon driven) Kerr resonator’s driven-dissipative phase transition to reach single-qubit detection fidelity of 99.9% [arXiv:2107.04503]. In this work, we improve upon the previous protocol by using machine learning-based classification algorithms to efficiently and rapidly extract information from this critical dynamics, which has so far been neglected to focus only on stationary properties. These classification algorithms are applied to the time series data of weak quantum measurements (homodyne detection) of a circuit-QED implementation of the Kerr resonator coupled to a superconducting qubit. This demonstrates how machine learning methods enable a faster and more reliable measurement protocol in critical open quantum systems.

1 Introduction

A sensing device is a system (weakly) coupled to a second system, the target, characterized by an unknown parameter. By observing the response of the sensor to the coupling, one can estimate the value of the parameter of the target [1]. The larger the response of the sensor, the better the estimation of the unknown parameter. For this reason, quantum systems around criticality have been proposed as sensors, because the diverging susceptibility characterizing second-order phase transition empowers precise parameter estimation. Such a metrological advantage of quantum phase transitions, however, is limited by a practical consideration: a diverging susceptibility is associated...
with a critical slowing down, i.e., the emergence of an infinitely-long timescale. Thus, the price to
pay for increased precision is a diverging measurement time \[2\]. Similarly to thermal and quantum
phase transitions, an open quantum system can develop second-order Dissipative Phase Transitions
(DPTs) \[3, 4\] associated with a diverging susceptibility \[5\]. Very little has been done in investigating
the metrological properties of DPTs \[6, 7\].

Two-photon Kerr resonators are remarkably simple open quantum systems that are at the center
of intense experimental research in quantum optics and information \[8, 9\]. The two-photon Kerr
resonator undergoes a second-order DPT \[10, 11\] where the photon number \(\langle \hat{a}^{\dagger} \hat{a} \rangle\) has a non-analytical
change. In Ref. \[12\], the authors introduced a quantum measurement protocol called “critical
parametric quantum sensing” using the two-photon Kerr resonator. Using this DPT, the state of a
qubit (two-level quantum system) coupled to the sensor can be inferred with very high precision.
Indeed, if the qubit is in the state \(|\uparrow\rangle\), after a quick transient dynamics, the resonator is almost empty,
while if the state is \(|\downarrow\rangle\) the resonator contains many photons. In a first approximation, a homodyne
measurement of the photon field allows assessing the state of a qubit with extremely high fidelity (up
to 99.9%).

This protocol, however, relies on a measurement of the long-time (stationary) properties of the Kerr
resonator, thus ignoring all the information coming from the short-time dynamics of the system. From
a physical point of view, the importance of short-time measurement is clear when considering both
non-ideal dispersive measurement and including qubit dissipation and decoherence (both sources of
noise in current NISQ-era quantum hardware). Thus, in realistic systems, a measurement protocol
must be performed within the qubit coherence time. More generally, growing efforts are dedicated to
the development of sensing protocols based on dynamical properties of critical quantum systems \[13–
18\].

From a theoretical point of view, characterizing the average dynamics of an open quantum system
requires nontrivial computations of the spectral properties of the generator of the dissipative dynamics
\[19\], making it a formidable challenge. This is even more true when discussing individual trajectories,
which describe the outcomes of single experimental realizations. Indeed, the complexity of the
stochastic noise induces nontrivial correlation properties in the dynamics that cannot be captured by
the average dynamics of a Lindblad master equation \[20–22\]. For this reason, an efficient dynamic
critical parametric quantum sensor cannot rely on simple estimation strategies based on average
properties of the system. On the other hand, machine learning algorithms are well suited for such a
task and are known to work well even in the presence of experimental noise \[23\].

In this paper, we present a novel machine learning approach to critical parametric quantum sensing
that improves the protocol metrological power by taking into account the dynamical aspect of the
DPT. The motivation for this improvement is the following. Even though the protocol using the
steady state reaches high fidelity, it is based on the assumption of a perfect qubit, and as such it
requires waiting for a sufficiently long time before the Kerr resonator reaches its steady state. In
actual experimental realizations, however, time is a resource, and the shorter a measurement takes, the
less the qubit is prone to errors. The procedure we propose is a machine learning-based classification
of measurement time series, allowing to extract information on the qubit state from the quantum
fluctuations of the Kerr resonator dynamics, even at short times.

2 Two-photon Kerr resonator

The Hamiltonian of a two-photon driven Kerr-resonator coupled to a qubit is

\[
\hat{H}_{Kerr} = \omega \hat{a}^{\dagger} \hat{a} + \omega_q \hat{\sigma}^{\dagger} \hat{\sigma} - \frac{\epsilon}{2} (\hat{a}^{\dagger 2} + \hat{a}^2) + \chi \hat{a}^{\dagger 2} \hat{a}^2 + g (\hat{a}^{\dagger} \hat{\sigma} - \hat{\sigma}^{\dagger} + \hat{a} \hat{\sigma}^{\dagger}),
\]

where \(\hat{a}(\hat{a}^{\dagger})\) is the bosonic annihilation(creation) operator and \(\hat{\sigma}^{\dagger}(\hat{\sigma})\) is the lowering(raising)
spin operator. This model can be realized in various photonic platforms, such as circuit-QED
implementation, where a driven resonator is coupled with a superconducting quantum interference
device (SQUID) element \[24, 25\]. We define the pump-resonator detuning \(\omega\), the effective pump-
power \(\epsilon\), and the SQUID-induced non-linearity \(\chi\).

Photons will continuously escape the Kerr resonator, and the number of photons emitted is directly
proportional to the number of photons in the Kerr resonator. This is described by the Lindblad master
equation for the system density matrix [1]

\[
\frac{\partial}{\partial t} \hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] + \Gamma[2\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \hat{\rho}(t)\}]
\]

(2)

where \( \Gamma \geq 0 \) is the dissipation rate, and it is used as a characteristic timescale.

Dispersive-readout protocols assume the qubit-resonator coupling to be in the linear dispersive regime, where the qubit-resonator detuning \( \Delta = |\omega_q - \omega_r| \gg |\omega_q + \omega_r|/2 \). In such a regime, the passage of excitation between the qubit and the resonator is suppressed, and the overall effect of the qubit on the resonator is to introduce an effective shift on the frequency of the resonator by \( \delta \omega = g^2/\Delta \).

Indeed, if the qubit is \(|\uparrow\rangle\) (|\downarrow\rangle), a photon inside the resonator has energy \( \omega + \delta \omega/2 \) (\( \omega - \delta \omega/2 \)). In this dispersive regime, a transition occurs at a critical frequency, such that \( \omega = \sqrt{\epsilon^2 - \Gamma^2} \).

As such, the small change \( \delta \omega \) can determine if the system is in the phase with a few photons or in that with many photons, as shown in Fig. 1, where on the left we show the photon number inside the cavity and on the right we plot the qubit state (see also the discussion in [12]). Consequently, by collecting the number of photon emitted, we can determine the state of the qubit.

Figure 1: The number of photons and the qubit state averaged over 1000 quantum trajectories for the two different initial qubit states at \( \epsilon = 1.67 \) and \( \delta \omega = 2.3 \). The steady-state solution is reached around \( t\Gamma \approx 6 \).

There is, however, another unwanted effect stemming from the coupling to the Kerr resonator. Indeed, the Kerr resonator induces an effective dissipation on the qubit, introducing additional unwanted errors in the measurement. Since the probability that a quantum jump induced by the qubit-resonator coupling increases with time, the shorter the required measurement time, the better the outcome. For example, in Fig. 2 we show that the qubit flips due to the coupling to the cavity. Therefore, if we are to consider only the long-time dynamics, in both cases we would wrongly guess the initial state of the qubit. This is the first reason to consider the Kerr resonator dynamics at short time.

Furthermore, from a metrological point of view, the main theoretical reason for this work is the fact that, contrary to quantum phase transitions, the critical slowing down of DPTs only affects certain observables. For instance, at the critical point of the two-photon Kerr resonator, the critical slowing down only affects \( \langle \hat{a} \rangle \), but leaves unaffected \( \langle \hat{a}^\dagger \hat{a} \rangle \). This is a common feature of DPTs, and it was demonstrated also for the XYZ model [21]. Since the protocol proposed in Ref. [12] uses photon number to estimate, e.g., the state of a qubit, it is possible to obtain metrological advantages from the diverging susceptibility of a DPT avoiding the critical slowing down. Consequently, the system should display critical fluctuations even at short times, allowing to extract information on the qubit state from the emitted field of the resonator.

3 Methods

These critical fluctuations can be clearly seen in actual experimental realizations (or their numerical simulations) as shown in Fig. 2 at short time. By comparing this single-shot dynamics in the left panel, we see important fluctuations of the Kerr-resonator state, far larger than those in the averaged dynamics (for instance, a small peak around \( t\Gamma = 1 \) when the qubit is initialized in \(|\downarrow\rangle = |0\rangle\)). This

\[1\] To be precise, the DPT occurs in the “thermodynamic” limit \( \chi \to 0 \) [11]. Nevertheless, for small \( \chi \), a sharp crossover occurs.
information, however, can be quite challenging to extract from an analytical point of view, because it would require to obtain the Liouvillian superoperator eigenvalues and eigenmatrices (i.e., the spectral decomposition of the generator of the dynamics) \[11\]. For this reason, we will employ a machine learning algorithm to guess the qubit state exploiting these dynamical critical fluctuations.

Since we are interested in simulating actual experimental measurement outcomes, we resort to homodyne quantum trajectories. Instead of the average dynamics of the system in Eq. (2), we simulate a set of stochastic Schrödinger equations where noise is sampled at each time step \[1\]. For a fixed set of parameters in \( H_{\text{Kerr}} \) we simulate 1000 quantum trajectories of the system starting from the two different states of the qubit \(|0\rangle = |↓\rangle\) and \(|1\rangle = |↑\rangle\) using QuTiP \[26, 27\]. We save the homodyne quantum evolution for several observables related to the resonator (sensor), such as the number of photons \( \langle \hat{a}^\dagger \hat{a} \rangle \), and to the qubit (target), such as \( \langle \hat{\sigma}_z \rangle \).

\[\text{Figure 2: The number of photons and the qubit state for a single quantum trajectory where one of the qubit states flips (fixed } \epsilon = 1.67 \text{ and } \delta \omega = 2.3). \text{ A measurement protocol based on the long-time (stationary) properties would fail to recognize that one qubit was in the } |↑\rangle \text{ state. Machine learning classifiers can be accurate even with short-time data, when qubits have not had enough time to be altered by the environment.}\]

In circuit-QED experiments, the parameter \( \Gamma \) is fixed by the manufacturing process, the non-linearity \( \chi \) is set by the SQUID characteristics, while the detuning \( \Delta \) and the drive intensity \( \epsilon \) can be easily tuned in real time. We choose to save data with a maximal time resolution of \( \Delta t = 10^{-3}/\Gamma \) and up to \( t\Gamma = 15 \), corresponding to state-of-the-art homodyne measurement frequencies. The other relevant time scales, in units of \( \Gamma \), are the total time of measurement \( t_f \) and the instrument measurement smoothing scale \( \tau \), corresponding to a time interval over which individual values can not be obtained and are averaged over by the measurement instrument. These two time parameters can be optimized by the protocol to achieve the highest fidelity.

We choose to analyze the quantum trajectories of the observable \( \hat{x} \) using two simple machine learning classifiers for time series. Both classifiers employ a Support Vector Machine algorithm (SVC) with radial basis function kernels (rbf) to discover non-linearity in the feature space. The first classifier (named TAB+SVC(rbf)) is applied to a feature space identified with all the time points of the trajectory, therefore neglecting the time ordering and time correlations. On the other hand, the second classifier (named RIFE+SVC(rbf)) acts on a meta-feature space which is constructed from random interval features (RIFE) \[28\], such as the average value or the slope of a random time interval of the trajectory. To estimate the error on the classification accuracy we utilize repeated cross-validation folds, with 100 repetitions and 5 folds.

Our choice of classifiers is limited and dictated by simplicity. Similar results should hold irrespective of the classifier algorithm as long as the time series dynamics can be captured. Ideally, classifiers requiring less data to achieve high accuracy would be preferred because that would improve the experimental setup. Higher accuracy can be achieved by including careful feature engineering and feature selection steps, which we have left for future investigations.

4 Results

We want to tune the Kerr resonator to be the best possible measurement instrument. In Candia et al. \[12\], such an optimal point was inferred from analytical considerations on the steady state. Here
instead we consider the dynamical properties. We scanned the parameter set \{\epsilon, \omega\} (which can be easily controlled in standard experimental implementations) and verified the accuracy provided by our choice of classifiers for the state of a qubit initialized either in \(|\uparrow\rangle\) or \(|\downarrow\rangle\), where the accuracy is the ratio between the number of correct results over the total number of simulations (the variability of the accuracy in our cross-validation procedure is always of the order of 0.002 or less). This procedure is repeated for different total measurement time \(t_f\Gamma\) and for different measurement smoothing scales \(\tau\Gamma\): note that this procedure amounts to changing the quality and quantity of samples used in the supervised learning classification task.

In Fig. 3, we plot the results obtained at various \(t_f\Gamma\). In the left panel, where \(\tau = 10^{-3}/\Gamma\), we see that for very short times the algorithm is capable of determining the qubit state in a very short time \(t_f\Gamma = 2\) and very high accuracy (errors \(\simeq 1 - 9 \times 10^{-4}\)). If we increase \(\tau = 10^{-1}/\Gamma\), we can obtain similar accuracy \((\simeq 1 \times 10^{-3})\), but we need to wait for longer times \((t_f\Gamma = 9)\).

These results demonstrate that, exploiting critical fluctuations, it is possible to obtain a performing measurement instrument, capable of determining the state of a qubit with extreme high precision and in very short time. In the future, we plan to further investigate the properties of critical open quantum system on a more mathematical ground, and to test this protocol on actual experimental data, including measurement noise and qubit dissipation effects.

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