UNIQUENESS OF THE MINIMIZER OF THE NORMALIZED VOLUME FUNCTION

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Abstract. We confirm a conjecture of Chi Li which says that the minimizer of the normalized volume function for a klt singularity is unique up to rescaling. This is achieved by defining stability thresholds for valuations, and then showing that a valuation is a minimizer if and only if it is K-semistable, and that K-semistable valuation is unique up to rescaling. As applications, we prove a finite degree formula for volumes of klt singularities and an effective bound of the local fundamental group of a klt singularity.

1. Introduction

Throughout this paper, we work over an algebraically closed field \( k \) of characteristic 0. Given a klt singularity \( x \in (X, \Delta) \), Chi Li introduced in [Li18] the normalized volume function \( \hat{\vol}_{X, \Delta} \) on the space \( \Val_{X,x} \) of real valuations on the function field \( K(X) \) of \( X \) that are centered on \( x \). Motivated by the study of K-stability of Fano varieties, the minimizing valuation of \( \hat{\vol}_{X, \Delta} \) is conjectured to have a number of deep geometric properties, which together comprise the so-called Stable Degeneration Conjecture, see [Li18, LX18].

There has been a lot of progress on the solution of different parts of the Stable Degeneration Conjecture in [Blu18, Li17, LX18, LX16, Xu20]. In particular, it has been known that a minimizing valuation exists (see [Blu18]) and it is always quasi-monomial (see [Xu20]).

1.1. Main Theorems. In this paper, we aim to solve another part of the Stable Degeneration Conjecture, namely, the uniqueness of the minimizing valuation, as conjectured in [Li18, Conjecture 7.1.2].

**Theorem 1.1.** Let \( x \in (X, \Delta) \) be a klt singularity, then up to rescaling, there is a unique minimizer \( v_0 \) of the normalized volume function \( \hat{\vol}_{X, \Delta} \).

We remark that our proof of the theorem does not rely on the fact that the minimizer is quasi-monomial.

An immediate consequence is the following, which is the local version of the K-semistable case of [Zhu20, Theorem 1.1].

**Corollary 1.2.** If a klt singularity \( x \in (X, \Delta) \) admits a group \( G \)-action, then any minimizer \( v_0 \) of \( \hat{\vol}_{X, \Delta} \) is \( G \)-invariant.

Another direct consequence is the finite degree formula for normalized volumes.

**Theorem 1.3** (Finite degree formula). Let \( f: (y \in (Y, \Delta_Y)) \to (x \in (X, \Delta)) \) be a finite Galois morphism between klt singularities such that \( f^*(K_X + \Delta) = K_Y + \Delta_Y \). Then
\[
\hat{\vol}(x, X, \Delta) \cdot \deg(f) = \hat{\vol}(y, Y, \Delta_Y).
\]

CX is partially supported by the NSF (No. 1901849).
Here $\widehat{\text{vol}}(x, X, \Delta)$ denotes the volume of the klt singularity $x \in (X, \Delta)$, see Definition 2.5. We apply this to obtain the following effective bound of the local fundamental group.

**Corollary 1.4.** Let $x \in (X, \Delta)$ be the germ of a klt singularity, then the order of the fundamental group of the smooth locus satisfies

$$\#|\pi_1(x, X^\text{sm})| \leq \frac{n^n}{\text{vol}(x, X, \Delta)},$$

where the equality holds if and only if $\Delta = 0$ and $x \in X$ is étale locally isomorphic to $\mathbb{C}^n/G$ where the action of $G \cong \pi_1(x, X^\text{sm})$ is fixed point free in codimension one.

Combining Corollary 1.4 with the results from [Liu18, BJ20] relating local and global volumes of Fano varieties, we also have the following theorem.

**Theorem 1.5.** Let $(X, \Delta)$ be a log Fano variety. Then for any $x \in (X, \Delta)$, if we denote by $\pi_1^\text{loc}(x, X^\text{sm})$ the local fundamental group of the smooth locus of the germ $x \in (X, \Delta)$, we have the inequality

$$\#|\pi_1^\text{loc}(x, X^\text{sm})| \leq \frac{(n + 1)^n}{\delta(X, \Delta)^n \cdot (- (K_X + \Delta))^{n}}.$$

In particular, the Cartier index of $X$ is bounded from above by the right hand side of the above inequality.

Here $\delta(X, \Delta)$ denotes the stability threshold of the log Fano pair $(X, \Delta)$, see [FO18, Definition 0.2] or [BJ20].

**Remark 1.6.** An interesting application of Theorem 1.5 is that it gives a new proof of the boundedness of K-semistable Fano varieties of a fixed dimension and with volume bounded from below. This was originally proved in [Jia17] as a consequence of the boundedness results proved in [Bir19]. Applying Theorem 1.5, we only need the fact the Fano varieties with fixed Cartier index form a bounded family, which was first proved in [HMXT14, Corollary 1.8].

1.2. **Outline of the proof.** Given a klt singularity $x \in (X = \text{Spec}(R), \Delta)$, the uniqueness of the minimizer $v$ (up to rescaling) of $\widehat{\text{vol}}_{X, \Delta}$ is proved in [LX18] under the assumption that the graded rings associated to the minimizers are finitely generated. The finite generation assumption is used to give a degeneration of the singularity $(X, \Delta)$ to a K-semistable log Fano cone $(X_0, \Delta_0, \xi_v)$, where $X_0 = \text{Spec}(\text{gr}_v(R))$, $\Delta_0$ the degeneration of $\Delta$, and $\xi_v$ is the Reeb vector induced by $v$. This degeneration picture allows one to degenerate any minimizer to $X_0$, and use the strict convexity of the volume function to conclude that $\xi_v$ is the unique $T$-equivariant minimizer on $(X_0, \Delta_0)$ (see [Xu18, Page 823]).

The main aim of this paper is to prove uniqueness of the minimizer without assuming the finite generation property, which still remains a major challenge. For this purpose, a key new input, introduced in Section 3.1, is the K-semistability of a general valuation $v_0 \in \text{Val}_{X,x}$ centered at a klt singularity $x \in (X, \Delta)$. More generally, we will define the stability threshold $\delta(v_0)$ of a valuation $v_0$ with finite log discrepancy. This is done by introducing a local version of basis type divisors. Roughly speaking, a basis type divisor with respect to the chosen valuation $v_0$ is (up to a suitable rescaling factor) a divisor of the form $\{f_1 = 0\} + \cdots + \{f_N = 0\}$ where the images of $f_i$ form a basis of $\mathcal{O}_{X,x}/a_m(v_0)$
(for some integer $m$; here $a_\bullet(v_0)$ denotes the valuation ideals) that is compatible with the filtration induced by $v_0$. Given another valuation $v \in \text{Val}_{X,x}$, we apply the key technical observation from [AZ20] to find basis type divisors that are compatible with both $v_0$ and $v$. This allows us to define the $S$-invariant and $\delta$-invariant of a valuation $v_0$ with respect to another valuation $v$ and to eventually define the local analogue of the stability notions from the global setting. To justify our definition, when $v_0$ is given by a Kollár component $S$, we will show that $\text{ord}_S$ is K-semistable as a valuation if and only if $(S, \Delta_S)$ is K-semistable as a log Fano pair (see Theorem 3.6).

With these new definitions, in the second step we show in Section 3.2 that a K-semistable valuation is always a minimizer, and up to scaling there is a unique K-semistable valuation. The observation here is that the log canonical thresholds (lct) of basis type divisors with respect to a K-semistable valuation $v_0$ is asymptotically computed by $v_0$. On the other hand, the asymptotic expected vanishing order of these basis type divisors along a valuation $v$ is at least $\text{vol}_{X,\Delta}(v)^{-1/n}$, with equality when $v = v_0$. Through the identity

$$\hat{\text{vol}}_{X,\Delta}(v)^{1/n} = \frac{A_{X,\Delta}(v)}{\text{vol}_{X,\Delta}(v)^{-1/n}},$$

minimizing the normalized volume $\hat{\text{vol}}_{X,\Delta}(v)$ can be thought of as finding valuations that compute the lct of basis type divisors. In particular, this implies that K-semistable valuations are minimizers of $\hat{\text{vol}}_{X,\Delta}$ and the uniqueness then follows from an analysis of the equality condition.

In the last step, we show that a minimizing valuation $v_0$ is always K-semistable in Section 3.3. To circumvent the finite generation assumption of $gr_{v_0} R$ in [LX18], we will generalize the derivative argument from [Li17]. Intuitively, given two valuations $v_0, v \in \text{Val}_{X,x}$, we would like to draw a ray between them in the valuation space and use the nonnegativity of the derivative of $\hat{\text{vol}}_{X,\Delta}$ at the minimizer $v_0$ to prove its K-semistability. When $v_0$ and $v$ are quasi-monomial with respect to a common stratum, a natural candidate is given by the line joining them in the corresponding dual complex. However, it is unclear to us how to write down such a ray in general. Our idea is to instead construct a family of graded sequences of ideals that interpolates the valuation ideals of the two given valuations. Combining the derivative formula from [Li17] and an analysis of the log canonical thresholds and multiplicities of these “mixed” ideal sequences, we can then show that if $v_0$ is a minimizer, then $\delta(v_0) \geq 1$, i.e. $v_0$ is K-semistable.

**Acknowledgement:** We want to thank Yuchen Liu for discussions, especially for showing us the preprint [Liu20]. We also would like to thank Harold Blum for helpful comments.

2. Preliminaries

**Notation and Conventions:** We follow the notation as in [KM98, Laz04, Kol13].

We say $x \in (X = \text{Spec}(R), \Delta)$ is a singularity if $R$ is a local ring of essentially finite type over $k$, $\Delta$ is an effective divisor on $X$ and $x \in X$ is the unique closed point.

A filtration $\mathcal{F}^\bullet$ on a finite dimensional vector space $V$ is a decreasing sequence $\mathcal{F}^t V$ ($t \in \mathbb{R}$) of subspaces satisfying $\mathcal{F}^t V \subseteq \mathcal{F}^{t'} V$ whenever $t \geq t'$. It is called an $\mathbb{N}$-filtration if $\mathcal{F}^0 V = V$ and $\mathcal{F}^t V = \mathcal{F}^{[t]} V$ for all $t \in \mathbb{R}$. For any filtration $\mathcal{F}$ on $V$, we define its induced $\mathbb{N}$-filtration $\mathcal{F}_N^\bullet$ by setting $\mathcal{F}_N^t V := \mathcal{F}^{[t]} V$.
A projective klt pair $(X, \Delta)$ is called a log Fano pair if $-K_X - \Delta$ is ample.

2.1. **Graded sequence of ideals.** Let $(R, m)$ be a local ring of essentially finite type over $k \cong R/m$. A graded sequence of ideals (see [JM12]) is a sequence of ideals $a_\bullet = (a_m)_{m \in \mathbb{N}}$ such that $a_m \cdot a_n \subseteq a_{m+n}$. We call it decreasing if $a_{m+1} \subseteq a_m$ for all $m \in \mathbb{N}$. A graded sequence $b_\bullet$ of ideals is said to be linearly bounded by another one $a_\bullet$, if there is a positive integer $C$ such that such that

$$b_{Cm} \subseteq a_m$$

for any $m \in \mathbb{N}$. A finite subset $\{f_1, \ldots, f_N\}$ of $R \setminus \{0\}$ is said to be compatible with a decreasing graded sequence $a_\bullet$ of ideals if for all $m \in \mathbb{N}$, the nonzero images $\bar{f}_i$ of $f_i$ in $R/a_m$ are linearly independent.

The following lemma is a local version of [AZ20] Lemma 3.1.

**Lemma 2.1.** Let $(R, m)$ be a local ring of essentially finite type over $k \cong R/m$, let $a_\bullet$ and $b_\bullet$ be two decreasing graded sequences of $m$-primary ideals and let $m \in \mathbb{N}$. Then there exist some $f_i \in R \setminus \{0\}$ ($1 \leq i \leq N$) whose images in $R/a_m$ form a basis such that $\{f_1, \ldots, f_N\}$ is compatible with both $a_\bullet$ and $b_\bullet$.

**Proof.** Let $V := R/a_m$ which is a finite dimensional linear space. Then $V$ has two filtrations given by

$$F_{a_\bullet}^s V := (a_r + a_m)/a_m \quad \text{and} \quad F_{b_\bullet}^s V := (b_s + a_m)/a_m.$$  

By [AZ20] Lemma 3.1, there exists a basis $\bar{f}_i$ $(1 \leq i \leq N)$ of $V$ that is compatible with both filtrations $F_{a_\bullet}$ and $F_{b_\bullet}$. We can lift each $\bar{f}_i$ to some element $f_i \in R$ such that $\{f_1, \ldots, f_N\}$ is compatible with $b_\bullet$ (it suffices to lift each $\bar{f}_i \in F_{b_\bullet}^s V \setminus F_{b_\bullet}^{s+1} V$ to some $f_i \in b_s$). On the other hand, since $\bar{f}_i$ is compatible with $F_{a_\bullet}$, any such lift is automatically compatible with $a_\bullet$ (i.e. for all $r \leq m$, $f_i \in a_r$ if and only if $f_i \in F_{a_\bullet}^s V$). \qed

2.2. **The space of valuations.**

2.2.1. **Valuations.** Let $X$ be a variety defined over $k$. A real valuation of its function field $K(X)$ is a non-constant map $v : K(X)^* \to \mathbb{R}$, satisfying:

- $v(fg) = v(f) + v(g)$;
- $v(f + g) \geq \min\{v(f), v(g)\}$;
- $v(k^*) = 0$.

We set $v(0) = +\infty$. A valuation $v$ gives rise to a valuation ring

$$\mathcal{O}_v := \{f \in K(X) \mid v(f) \geq 0\}.$$  

We say a valuation $v$ is centered at a scheme-theoretic point $x = c_X(v) \in X$ if we have a local inclusion $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_v$ of local rings. Notice that the center of a valuation, if exists, is unique since $X$ is separated. Denote by $\text{Val}_X$ the set of real valuations of $K(X)$ that admits a center on $X$. For a closed point $x \in X$, we further denote by $\text{Val}_{X,x}$ the set of real valuations of $k(X)$ centered at $x \in X$.

For each valuation $v \in \text{Val}_{X,x}$ and any positive integer $m$, we define the valuation ideal

$$a_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\}.$$  

It is clear that $a_\bullet = \{a_m\}_{m \in \mathbb{N}}$ form a decreasing graded sequence of $m_x$-primary ideals.
Let \((X, \Delta)\) be a pair. We denote by 
\[
A_{X,\Delta} : \text{Val}_X \to \mathbb{R} \cup \{+\infty\}
\]
the log discrepancy function of valuations as in \cite{JM12} and \cite[Theorem 3.1]{BdFFU15} which extends the standard definition of log discrepancies from divisors to all valuations in \text{Val}_X. It is possible that \(A_{X,\Delta}(v) = +\infty\) for some \(v \in \text{Val}_X\), see e.g. \cite[Remark 5.12]{JM12}. We denote by \(\text{Val}_X^*\) the set of valuations \(v \in \text{Val}_X\) with \(A_{X,\Delta}(v) < +\infty\) and set \(\text{Val}_X^* = \text{Val}_X^* \cap \text{Val}_{X,x}\) for a closed point \(x \in X\). Note that \(A_{X,\Delta}\) is strictly positive on \(\text{Val}_X\) if and only if \((X, \Delta)\) is klt.

**Proposition 2.2.** Let \(x \in (X, \Delta)\) be a klt singularity and let \(v_0, v_1 \in \text{Val}_X^*\). Then the graded sequences \(a_\bullet(v_0)\) and \(a_\bullet(v_1)\) of valuation ideals are linearly bounded by each other.

**Proof.** This is a direct consequence of the Izumi type inequalities (see e.g. \cite[Theorem 3.1]{Li18}), which says that \(a_\bullet(v_i)\) and \(\{m^n\}_{m \in \mathbb{N}}\) are linearly bounded by each other. \(\Box\)

**Definition 2.3** (Kollár Components). Let \(x \in (X, \Delta)\) be a klt singularity. A prime divisor \(S\) over \((X, \Delta)\) is a Kollár component if there is a birational morphism \(\pi : Y \to X\) such that \(\pi\) is an isomorphism over \(X \setminus \{x\}\), \(S\) is a prime divisor on \(Y\), \(\pi(S) = \{x\}\), \(-S\) is \(\mathbb{Q}\)-Cartier and \(\pi\)-ample, and \((Y, \pi^{-1}_\Delta + S)\) is plt. The map \(\pi : Y \to X\) is called the plt blowup associated to the Kollár component \(S\). By adjunction (see \cite[Definition 4.2]{Kol13}) we may write
\[
(K_Y + \pi^{-1}_\Delta + S)|_S = K_S + \Delta_S,
\]
where \((S, \Delta_S)\) is a log Fano pair.

### 2.2.2. Local volumes.

**Definition 2.4.** Let \(X\) be an \(n\)-dimensional normal variety and let \(x \in X\) be a closed point. Following \cite{ELS03} we define the volume of a valuation \(v \in \text{Val}_{X,x}\) as
\[
\text{vol}(v) = \text{vol}_{X,x}(v) = \limsup_{m \to \infty} \frac{\ell(\mathcal{O}_{X,x}/a_m(v))}{m^n/n!}.
\]
where \(\ell(\cdot)\) denotes the length of the Artinian module.

Thanks to the works of \cite{ELS03, LM09, Cut13}, the above limsup is actually a limit.

The following invariant, which was first defined in \cite{Li18}, plays a key role in our study of local stability.

**Definition 2.5** (\cite{Li18}). Let \(x \in (X, \Delta)\) be an \(n\)-dimensional klt singularity. The normalized volume function of valuations \(\text{vol}_{(X,\Delta),x} : \text{Val}_{X,x} \to (0, +\infty)\) is defined as
\[
\text{vol}_{(X,\Delta),x}(v) = \begin{cases} A_{X,\Delta}(v)^n \cdot \text{vol}_{X,x}(v), & \text{if } v \in \text{Val}_{X,x}; \\ +\infty, & \text{if } v \notin \text{Val}_{X,x}. \end{cases}
\]
We often denote it by \(\text{vol}_{X,\Delta}\) or \(\text{vol}\) when \(x \in (X, \Delta)\) is clear from the context. The volume of a klt singularity \((x \in (X, \Delta))\) is defined as
\[
\text{vol}(x, X, \Delta) := \inf_{v \in \text{Val}_{X,x}} \text{vol}_{(X,\Delta),x}(v).
\]
It has been known that the above infimum is indeed a minimum by [Blu18] and that the minimizing valuations are always quasi-monomial by [Xu20]. The study of $\hat{\mathfrak{vol}}_{X,\Delta}$ is closely related to K-stability of log Fano pairs, guided by the so-called Stable Degeneration Conjecture as formulated in [Li18, Conjecture 7.1] and [LX18, Conjecture 1.2]. See [LLX20] for more background. Our Theorem 1.1 settles one part of this conjecture.

The following theorem from [LX18] motivates some of our arguments, although we do not need it in our proof.

**Theorem 2.6.** Let $x \in (X = \text{Spec}(R), \Delta)$ be a klt singularity, and $v^m$ a minimizer of $\hat{\mathfrak{vol}}_{X,\Delta}$. Assume the associated grade ring $\text{gr}_{v^m}(R)$ is finitely generated. Denote by $X_0 = \text{Spec}(\text{gr}_{v^m}(R))$ with the cone vertex $o$, $\Delta_0$ the degeneration of $\Delta$ on $X_0$, $\xi_v$ the Reeb orbit induced by $v$. Then $o \in (X_0, \Delta_0, \xi_v)$ is a K-semistable log Fano cone.

Note that the finite generation assumption always holds when $v$ is a divisorial valuation by [LX16, Blu18].

### 2.3. Log canonical thresholds.

**Definition 2.7.** Given a klt pair $(X, \Delta)$ and a non-zero ideal $a$ on $X$, the log canonical threshold $\text{lct}(X, \Delta; a)$ of $a$ with respect to $(X, \Delta)$ is defined to be

$$\text{lct}(X, \Delta; a) = \max\{t \geq 0 \mid (X, \Delta + at) \text{ is log canonical} \} = \inf_{v \in \text{Val}_X^*} \frac{A_{X,\Delta}(v)}{v(a)}.$$ 

For a graded sequence $a_* = \{a_m\}_{m \in \mathbb{N}}$ of non-zero ideals on a klt pair $(X, \Delta)$, we can also define its log canonical threshold to be

$$\text{lct}(X, \Delta; a_*) := \limsup_m m \cdot \text{lct}(X, \Delta; a_m) \in \mathbb{R}_{>0} \cup \{+\infty\}.$$ 

It is proved in [Liu18, Theorem 27] that

$$\hat{\mathfrak{vol}}(x, X, \Delta) = \inf_{a_*} \text{lct}(X, \Delta; a_*)^n \cdot \text{mult}(a_*),$$ 

where the infimum runs through all graded ideal sequences $a_*$ of $m_x$-primary ideals, and $\text{lct}(X, \Delta; a_*)^n \cdot \text{mult}(a_*)$ is set to be $+\infty$ if $\text{lct}(X, \Delta; a_*) = +\infty$.

### 3. K-semistability of a valuation

This is the main section of this paper. We will first define the notion of K-semistability for valuations. Then we will show that for valuations, being K-semistable is the same as being minimizer of the normalized volume function, and such a K-semistable valuation is unique up to rescaling.

#### 3.1. Definition of K-semistability for a valuation.

In this subsection, we introduce a local version of $S$-invariant on the product of the valuation space $\text{Val}_{X,x}^*$ and use it to define the $\delta$-invariant of valuations, which then naturally give the notion of K-semistability of a valuation.

Let $x \in (X = \text{Spec}(R), \Delta)$ be a klt singularity. Fix a valuation $v_0 \in \text{Val}_{X,x}^*$. By Proposition 2.2 for any valuation $v \in \text{Val}_{X,x}^*$, the graded sequences of ideals $a_* (v)$ and $a_* (v_0)$ are linearly bounded by each other. By Lemma 2.1 for any $m \in \mathbb{N}$ there exist some $f_1, \ldots, f_{N_m} \in R$ (where $N_m = \ell(R/a_m(v_0))$) which are compatible with both $a_* (v_0)$ and
Lemma 3.3. For any measure \( \text{the relative volume} \) by taking a quotient instead of a difference. (logarithmic) relative volume of two norms is given in \([BJ18, \text{Section 3}]\). However, we Remark 3.2. exists. Moreover, we have \( \text{vol}(v_0; v) = 0 \) for all \( t \gg 0 \).

Proof. From the definition we have \( \mathcal{F}^m_v R_m = (a_m(v) + a_m(v_0))/a_m(v_0) \cong a_m(v)/(a_m(v) \cap a_m(v_0)) \)

\[
\ell(\mathcal{F}^m_v R_m) = \ell(R/(a_m(v) \cap a_m(v_0))) - \ell(R/a_m(v)),
\]

thus by \([LM09, \text{Theorem 3.8}]\) we obtain

\[
\lim_{m \to \infty} \frac{1}{m^n/n!} \ell(\mathcal{F}^m_v R_m) = \text{mult}(a_*(v) \cap a_*(v_0)) - \text{mult}(a_*(v)).
\]

Since \( a_*(v_0) \) and \( a_*(v) \) are linearly bounded by each other, we have \( a_{Cm}(v) \subseteq a_m(v_0) \) for some constant \( C > 0 \). Thus \( \mathcal{F}^C_v R_m = 0 \) and

\[
\text{vol}(v_0; v/t) = \lim_{m \to \infty} \frac{(\mathcal{F}^m_v R_m)}{m^n/n!} = 0
\]

for all \( t \geq C \). □

Analogous to the global log Fano case, we set \( \widetilde{S}_m(v_0; v) = \sum_{i=1}^{N_m} [v(f_i)] \), which doesn’t depend on the choice of \( f_i \); indeed it is not hard to check that

\[
\widetilde{S}_m(v_0; v) = \sum_{i=0}^{\infty} i \cdot \ell(\mathcal{F}_v^i R_m/\mathcal{F}_v^{i+1} R_m) = \sum_{i=1}^{\infty} \ell(\mathcal{F}_v^i R_m).
\]

We then define

\[
S_m(v_0; v) := \frac{A_X,\Delta(v_0)}{\widetilde{S}_m(v_0; v_0)} \cdot \widetilde{S}_m(v_0; v),
\]

\[
S(v_0; v) := \frac{n+1}{n} \cdot \frac{A_X,\Delta(v_0)}{\text{vol}(v_0)} \int_0^\infty \text{vol}(v_0; v/t) dt.
\]

Remark 3.2. In the global non-Archimedean setting, a similar construction named the (logarithmic) relative volume of two norms is given in \([BJ18, \text{Section 3}]\). However, we measure “the relative volume” by taking a quotient instead of a difference.

Lemma 3.3. For any \( v_0, v \in \text{Val}_X^\times \), we have \( S(v_0; v) = \lim_{m \to \infty} S_m(v_0; v) \). Moreover, the function \( t \mapsto \text{vol}(v_0; v/t) \) is continuous.

Proof. We can embed \((X, \Delta)\) into a projective variety \((\bar{X}, \bar{\Delta})\). By \([LM09, \text{Lemma 3.9}]\), we can find a sufficiently ample line bundle \( L \) such that the natural map

\[
H^0(\bar{X}, L^m) \to H^0(\bar{X}, L^m \otimes \mathcal{O}_X/a_{2Cm}(v))
\]
is surjective for all $m \in \mathbb{N}$, where $C$ is a positive integer such that $a_{CM}(v) \subseteq a_m(v)$ and $a_{CM}(v_0) \subseteq a_m(v)$ for all $m \in \mathbb{N}$. Note that this implies that the restriction map
\begin{equation}
  h : H^0(\mathcal{X}, L^m) \to H^0(\mathcal{X}, L^m \otimes \mathcal{O}_\mathcal{X}/a_m(v_0)) \cong R/a_m(v_0)
\end{equation}
is also surjective, where the last isomorphism is given by a trivialization of $L$ near $x$. For such $L$,
\begin{align*}
  W_m &:= H^0(\mathcal{X}, L^m \otimes a_m(v_0)) \quad \text{and} \quad V_m := H^0(\mathcal{X}, L^m)
\end{align*}
defines two graded linear series $W_\bullet, V_\bullet$ that contain ample series. The valuation $v$ induces a filtration $\mathcal{F}_v$ on both $V_\bullet$ and $W_\bullet$ by setting $\mathcal{F}_v^s V_m = \{ s \in H^0(\mathcal{X}, L^m) \mid v(s) \geq \lambda \}$ and $\mathcal{F}_v^s W_m = W_m \cap \mathcal{F}_v^s V_m$.

Through (3.3), the image of $\mathcal{F}_v$ induces a filtration $\mathcal{F}_1$ on $R_m := R/a_m(v_0)$. We claim that it is the same as the filtration $\mathcal{F}_v^0$ on $R_m$. Indeed, given an element $f \in \mathcal{F}_v^s V_m$, it is clear that its image $\tilde{f} \in R_m$ lies in $\mathcal{F}_v^s R_m$. Conversely, if $0 \neq \tilde{f} \in \mathcal{F}_v^s R_m$, then it can be lifted to some $f \in R$ with $v(f) \geq \lambda$. Since
\begin{align*}
  H^0(\mathcal{X}, L^m) \to H^0(\mathcal{X}, L^m \otimes \mathcal{O}_\mathcal{X}/a_{CM}(v))
\end{align*}
is a surjective, there exists some $s \in \mathcal{F}_v^s V_m$ such that $s$ and $f$ has the same image in $R/a_{CM}(v)$. As $a_{CM}(v) \subseteq a_m(v_0)$, we see that the restriction of $s$ in $R_m$ gives $\tilde{f}$. This proves the claim.

Let $W_m^t = \mathcal{F}_v^t W_m$ and $V_m^t = \mathcal{F}_v^t V_m$. Then from the above claim we have $\ell(\mathcal{F}_v^t R_m) = \dim V_m^t - \dim W_m^t$, hence
\begin{align*}
  \text{vol}(v_0; v/t) = \text{vol}(V_m^t) - \text{vol}(W_m^t),
\end{align*}
which, by [3.20, Proposition 2.3], is continuous in $t$ when $0 \leq t \leq C$ since $\text{vol}(V_C^t) \geq \text{vol}(W_C^t) > 0$ by (3.2); on the other hand, $\text{vol}(v_0; v/t) = 0$ when $t \geq C$ as in Lemma 3.1 thus the function $t \mapsto \text{vol}(v_0; v/t)$ is continuous everywhere.

We next prove $S(v_0; v) = \lim_{m \to \infty} S_m(v_0; v)$. We claim that
\begin{equation}
  \lim_{m \to \infty} \sum_{t=1}^\infty \frac{\ell(\mathcal{F}_v^t R_m)}{m^{n+1}/n!} = \int_0^\infty \text{vol}(v_0; v/t) dt.
\end{equation}
By definition, this is equivalent to
\begin{equation}
  \lim_{m \to \infty} \sum_{t=1}^\infty \frac{\ell(\mathcal{F}_v^t R_m)}{m^{n+1}/n!} = \int_0^\infty \text{vol}(v_0; v/t) dt.
\end{equation}
Let $\psi_m(t) = \frac{\ell(\mathcal{F}_v^m R_m)}{m^n/n!}$. Then we may rewrite the expression in the above limit as $\int_0^\infty \psi_m(t) dt$. Notice that $\lim_{m \to \infty} \psi_m(t) = \text{vol}(v_0; v/t)$ and $\psi_m(t) = 0$ for all $t \geq C$ and all $m \in \mathbb{N}$. The equality (3.5) now follows from the dominated convergence theorem.

It is clear that $\text{vol}(v_0; v/t) = \max\{ (1 - t^n) \text{vol}(v_0), 0 \}$ for all $t \geq 0$, thus taking $v = v_0$ in (3.4) we get
\begin{align*}
  \lim_{m \to \infty} \frac{\tilde{S}_m(v_0; v_0)}{m^{n+1}/n!} = \int_0^\infty \text{vol}(v_0; v_0/t) dt = \int_0^1 (1 - t^n) \text{vol}(v_0) dt = \frac{n}{n+1} \text{vol}(v_0),
\end{align*}
hence
\begin{align*}
  \lim_{m \to \infty} \frac{S_m(v_0; v)}{A_X \Delta(v_0)} = \lim_{m \to \infty} \frac{\tilde{S}_m(v_0; v)}{S_m(v_0; v_0)} = \frac{n+1}{n} \cdot \frac{\int_0^\infty \text{vol}(v_0; v/t) dt}{\text{vol}(v_0)}.
\end{align*}
In other words, $S(v_0; v) = \lim_{m \to \infty} S_m(v_0; v)$.  
\hfill \Box
**Definition 3.4.** A valuation $v_0 \in \text{Val}^*_{X,x}$ is said to be $K$-semistable if $A_{X,\Delta}(v) \geq S(v_0; v)$ for all $v \in \text{Val}^*_{X,x}$. We also define the stability threshold $\delta(v_0)$ of a valuation $v_0 \in \text{Val}^*_{X,x}$ as $\delta(v_0) = \inf_v \delta(v_0; v)$ where $\delta(v_0; v) = \frac{A_{X,\Delta}(v)}{S(v_0; v)}$ and the infimum runs over all valuations $v \in \text{Val}^*_{X,x}$.

**Remark 3.5.** The notion of $K$-semistable valuation has been previously defined for valuations which are quasi-monomial, and whose associated graded rings are finitely generated (see [Xu18, Page 819] or [LLX20, Theorem 4.14]). Whereas it is known that minimizers of $\text{vol}_{X,\Delta}$ are quasi-monomial by [Xu20], the finite generation of the associated graded rings remains open. Therefore, while Definition 3.4 is conjecturally equivalent to the previous definition, we circumvent the issue of finite generation.

From the definition it is clear that $\delta(v_0) = \delta(\lambda \cdot v_0)$, thus $v_0$ is $K$-semistable if and only if $\lambda v_0$ is $K$-semistable for some $\lambda > 0$. In the special case of divisorial valuations induced by Kollár components, we have the following equivalent characterization, which serves as the motivation of our definition.

**Theorem 3.6.** Let $S$ be a Kollár component over $x \in (X, \Delta)$ (see Definition 2.3). Then we have $\delta(\text{ord}_S) \geq \min\{1, \delta(S, \Delta_S)\}$ and the valuation $\text{ord}_S$ is $K$-semistable if and only if the log Fano pair $(S, \Delta_S)$ is $K$-semistable.

**Proof.** Let $v_0 = \text{ord}_S$ and let $v \in \text{Val}^*_{X,x}$. We know there is an ample $\mathbb{Q}$-divisor $L \sim_{\mathbb{Q}} -S|_S$ on $S$ such that the short exact sequences

$$0 \to \mathcal{O}_Y(-(m+1)S) \to \mathcal{O}_Y(-mS) \to \mathcal{O}_S(mL) \to 0$$

hold (see e.g. [Kol13, Section 4.1]), where by convention $\mathcal{O}_S(mL) := \mathcal{O}_S([mL])$. Since $R^1\pi_*\mathcal{O}_Y(-mS) = 0$ for all $m \geq 0$ by Kawamata-Viehweg vanishing, we get isomorphisms

$$a_m / a_{m+1} \cong H^0(S, mL)$$

where $a_m := a_m(\text{ord}_S)$. After identifying $\oplus_{m \in \mathbb{N}}a_m / a_{m+1}$ with $R(S, L) := \oplus_{m \in \mathbb{N}}H^0(S, mL)$, the valuation $v$ induces a filtration $\mathcal{F}_v$ on the section ring $R(S, L)$.

We claim that

$$S(v_0; v) = A_{X,\Delta}(v_0) \cdot S(L; \mathcal{F}_v),$$

where $S(L; \mathcal{F}_v)$ denotes the $S$-invariant of a filtration as in [BJ20, Section 2.5-2.6] (we will also use its approximated versions $S_m(L; \mathcal{F}_v)$ from loc. cit.). To see this, we note that

$$\tilde{S}_m(v_0; v) = \sum_{i=1}^{\infty} \sum_{j=1}^{m} l(\mathcal{F}_v^i(a_{j-1}/a_j)) = \sum_{j=1}^{m} j \cdot h^0(S, jL) \cdot S_j(L; (\mathcal{F}_v)_|\mathbb{N}).$$

By [BJ20, Corollary 2.12], we have $S_j(L; (\mathcal{F}_v)_|\mathbb{N}) \to S(L; (\mathcal{F}_v)_|\mathbb{N}) = S(L; \mathcal{F}_v)$ ($j \to \infty$), thus as $h^0(S, jL) = (L^{n-1})(n-1)! + O(j^{n-2})$, we obtain

$$\lim_{m \to \infty} \frac{\tilde{S}_m(v_0; v)}{m^{n+1/(n+1)!}} = n(L^{n-1}) \cdot S(L; \mathcal{F}_v).$$

Thus

$$\frac{S(v_0; v)}{A_{X,\Delta}(v_0)} = \lim_{m \to \infty} \frac{\tilde{S}_m(v_0; v)}{S_m(v_0; v)} = \frac{S(L; \mathcal{F}_v)}{S(L; \mathcal{F}_v)}.$$
On the other hand, it is clear from the definition that $S(L; \mathcal{F}_{v_0}) = 1$ (the filtration $\mathcal{F}_{v_0}$ satisfies $\mathcal{F}_{v_0}^j H^0(S, mL) = H^0(S, mL)$ if $j \leq m$ and $\mathcal{F}_{v_0}^j H^0(S, mL) = 0$ when $j \geq m + 1$), which proves (3.6).

Since $-(K_S + \Delta_S) \sim (K_Y + \pi_*^{-1} \Delta + S)|_S \sim A_{X, \Delta}(v_0) \cdot L$, we may rewrite (3.6) as

$$(3.7) \quad S(v_0; v) = S(-(K_S + \Delta_S); \mathcal{F}_v).$$

Let $m \in \mathbb{N}$ be a sufficiently divisible integer and let $f_1, \cdots, f_N \in \mathcal{A}_{Am}$ (where $A := A_{X, \Delta}(v_0)$) be the lift of a basis $\{f_i\}$ of $H^0(S, -(m(K_S + \Delta_S))) = H^0(S, mL)$. Let

$$N := \dim H^0(S, mL) \quad \text{and} \quad D = \frac{1}{mN} \sum_{i=1}^N \{f_i = 0\}.$$ 

Then we have $\pi^* D = A \cdot S + \widetilde{D}$ where $\widetilde{D}|_S$ is an $m$-basis type $\mathbb{Q}$-divisor of the log Fano pair $(S, \Delta_S)$ (see [FO18, BJ20]). Let $\delta_m := \min \{1, \delta_m(S, \Delta_S)\}$. From the definition of stability thresholds, we know that the pair $(S, \Delta_S + \delta_m \widetilde{D}|_S)$ is lc, thus $(Y, S + \pi_*^{-1} \Delta + \delta_m \widetilde{D})$ is also lc by inversion of adjunction. We have

$$K_Y + S + \pi_*^{-1} \Delta + \delta_m D \geq \pi^*(K_X + \Delta + \delta_m D),$$

hence $(X, \Delta + \delta_m D)$ is lc, which implies that $A_{X, \Delta}(v) \geq \delta_m \cdot v(D)$ for any $v \in \Val_{X,x}$ and any $D$ as above.

If we choose $f_i$ to be compatible with the filtration $\mathcal{F}_v$, then $v(D) = S_m(-(K_S + \Delta_S); \mathcal{F}_v)$ and we obtain

$$A_{X, \Delta}(v) \geq \delta_m \cdot S_m(-(K_S + \Delta_S); \mathcal{F}_v).$$

Letting $m \to \infty$, we deduce $\delta(v_0) \geq \min \{1, \delta(S, \Delta_S)\}$ using (3.7). In particular, if $(S, \Delta_S)$ is K-semistable, then $v_0 = \ord_S$ is K-semistable.

Conversely, if $v_0$ is K-semistable, then we have

$$A_{X, \Delta}(v) \geq S(-(K_S + \Delta_S); \mathcal{F}_v)$$

for any $v \in \Val_{X,x}$. Let $c := v(\mathcal{A}_v(v_0))$. We may shift the filtration $\mathcal{F}_v$ by $c$ to get a new filtration $\mathcal{F}$ on $R(S, L)$, i.e., $\mathcal{F}^l H^0(S, mL) := \mathcal{F}_v^{l+c} H^0(S, mL)$. It satisfies $\mathcal{F}^l H^0(S, mL) = H^0(S, mL)$ as $v(\mathcal{A}_v) \geq cm$ for all $m \in \mathbb{N}$. By [BJ20, Corollary 2.10], there exists some $\epsilon_m$ with $\lim_{m \to \infty} \epsilon_m = 1$ such that for all $m \in \mathbb{N}$ and any $v \in \Val_{X,x}$,

$$\epsilon_m \cdot S_m(-(K_S + \Delta_S); \mathcal{F}) \leq S(-(K_S + \Delta_S); \mathcal{F}) - A_{X, \Delta}(v_0) \cdot v(\mathcal{A}_v(v_0)) \leq A_{X, \Delta}(v) - A_{X, \Delta}(v_0) \cdot v(\mathcal{A}_v(v_0)).$$

For sufficiently divisible integer $m$ and with $\{f_i\}$, $D$ and $\widetilde{D}$ as before, this means that $(Y, S + \pi_*^{-1} \Delta + \epsilon_m \widetilde{D})$ is lc. By adjunction we see that $(S, \Delta_S + \epsilon_m \widetilde{D}|_S)$ is lc. Since $\widetilde{D}|_S$ can be any $m$-basis type $\mathbb{Q}$-divisor of $(S, \Delta_S)$, we conclude that $\delta_m(S, \Delta_S) \geq \epsilon_m$. Letting $m \to \infty$ we obtain $\delta(S, \Delta_S) \geq 1$, i.e. $(S, \Delta_S)$ is K-semistable. \(\Box\)

In general, if $(S, \Delta_S)$ is not K-semistable, then the inequality in Theorem 3.6 could be strict.
3.2. K-semistable valuation is the unique minimizer. In this subsection, we show that if $\text{Val}_{X,x}^*$ contains a K-semistable valuation, then it is the unique minimizer of $\text{vol}_{X,\Delta}$ up to rescaling.

**Theorem 3.7.** Let $x \in (X = \text{Spec}(R), \Delta)$ be a klt singularity and let $v_0 \in \text{Val}_{X,x}^*$. Assume that $v_0$ is K-semistable. Then

1. $v_0$ is a minimizer of $\text{vol}_{X,\Delta}$, i.e., $\text{vol}(x, X, \Delta) = \text{vol}(v_0)$;
2. if $v_1 \in \text{Val}_{X,x}^*$ is another minimizer of $\text{vol}_{X,\Delta}$, then $v_1 = \lambda v_0$ for some $\lambda > 0$.

For the proof we need some auxiliary calculation. For each valuation $v \in \text{Val}_{X,x}^*$ and every integer $m > 0$, we set

$$w_m(v) := \min_{1} \sum_{i=1}^{m} |v(f_i)|$$

where the minimum runs over all $f_1, \ldots, f_m \in R \setminus \{0\}$ that are compatible with $a_*(v)$. Clearly the minimum is achieved by some $f_1, \ldots, f_m$ that are compatible with $a_*(v)$, if and only if for the unique integer $r$ satisfying $\ell(R/a_{r+1}(v)) > m \geq \ell(R/a_r(v))$, $f_1, \ldots, f_m$ span $R/a_r(v)$ and form a linearly independent set in $R/a_{r+1}(v)$.

**Lemma 3.8.** We have

$$\lim_{m \to \infty} \frac{w_m(v)}{m^{\frac{1}{n}}} = \frac{n}{n+1} \cdot \left(\frac{n!}{\text{vol}(v)}\right)^{1/n}.$$

**Proof.** Let $a_* = a_*(v)$. From the above description we have

$$0 \leq w_m(v) - \sum_{i=0}^{r-1} i \cdot \ell(a_i/a_{i+1}) \leq r \cdot \ell(a_r/a_{r+1})$$

for all integers $r, m > 0$ with $\ell(R/a_r) \leq m < \ell(R/a_{r+1})$. Note that this implies

$$\lim_{r \to \infty} \frac{m}{r^{n/n!}} = \text{vol}(v).$$

We also have

$$\lim_{r \to \infty} \frac{r \cdot \ell(R/a_{r+1}) - \sum_{i=0}^{r-1} \ell(R/a_i)}{r^{n+1/n!}} = \left(1 - \frac{1}{n+1}\right) \text{vol}(v).$$

Thus

$$\lim_{r \to \infty} \frac{w_m(v)}{m^{\frac{1}{n}}} = \frac{n}{n+1} \text{vol}(v)$$

and

$$\lim_{m \to \infty} \frac{w_m(v)}{m^{\frac{1}{n}}} = \lim_{r \to \infty} \left(\frac{w_m(v)}{r^{n/n!}} \cdot \frac{r^{n/n!}}{m^{1/n!}} \cdot \frac{r}{m^{1/n}}\right) = \frac{n}{n+1} \cdot \left(\frac{n!}{\text{vol}(v)}\right)^{1/n}.$$

Proof of Theorem 3.7. We first prove that $v_0$ is a minimizer of $\text{vol}_{X,\Delta}$, i.e. $\text{vol}(v) \geq \text{vol}(v_0)$ for every valuation $v \in \text{Val}_{X,x}^*$. Without loss of generality we may assume that $A_{X,\Delta}(v_0)$
$A_{X, \Delta}(v) = 1$. Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_{N_m}$ be an $(m, v)$-basis with respect to $v_0$ (where $N_m = \ell (R/\mathfrak{a}_m(v_0))$). Since $v_0$ is K-semistable, we have

$$1 = A_{X, \Delta}(v) \geq S(v_0; v). \quad (3.8)$$

From the definition it is clear that $\tilde{S}_m(v_0; v_0) = w_{N_m}(v_0)$ and $\tilde{S}_m(v_0; v) \geq w_{N_m}(v)$, hence by Lemma 3.8 we get

$$S(v_0; v) \geq \lim_{m \to \infty} \frac{w_{N_m}(v)}{w_{N_m}(v_0)} = \left( \frac{\text{vol}(v_0)}{\text{vol}(v)} \right)^{\frac{1}{n}}.$$

Combined with (3.8) we immediately have

$$\hat{\text{vol}}(v) = \text{vol}(v) \geq \text{vol}(v_0) = \hat{\text{vol}}(v_0),$$

i.e. $v_0$ minimizes the normalized volume function $\hat{\text{vol}}_{X, \Delta}$.

Now assume $\text{vol}(v_0) = \text{vol}(v)$. We claim that

$$\text{vol}(v_0) = \text{vol}(v) = \text{mult}(\mathfrak{a}_\bullet(v_0) \cap \mathfrak{a}_\bullet(v)). \quad (3.9)$$

Suppose this is not the case, then $\text{vol}(v_0; v) > 0$ by (3.1). Thus by the continuity part of Lemma 3.8, there exists some $\epsilon > 0$ such that

$$\gamma := \text{vol} \left( v_0; \frac{v}{1 + 2\epsilon} \right) = \lim_{m \to \infty} \frac{\ell (F_{\nu}^{(1+2\epsilon)^m}(R/\mathfrak{a}_m(v_0)))}{m^n/n!} > 0.$$

For each $m \in \mathbb{N}$, let $k_m$ be the unique integer $k$ determined by

$$\ell (R/\mathfrak{a}_{k-1}(v)) \leq N_m < \ell (R/\mathfrak{a}_k(v)).$$

Since $\text{vol}(v_0) = \text{vol}(v)$, we have $\lim_{m \to \infty} \frac{k_m}{m} = 1$ and thus $k_m < (1 + \epsilon)m$ for sufficiently large $m$. Let $g_1, \ldots, g_{N_m} \in R \setminus \{0\}$ be a sequence that’s compatible with $\mathfrak{a}_\bullet(v)$ such that

$$w_{N_m}(v) = \sum_{i=1}^{N_m} \lfloor v(g_i) \rfloor.$$

Then by construction we have $v(g_i) \leq k_m$ for all $1 \leq i \leq N_m$ and the inequality

$$\tilde{S}_m(v_0; v) = \sum_{i=1}^{N_m} \lfloor v(f_i) \rfloor \geq w_{N_m}(v)$$

can be upgraded as

$$\sum_{i=1}^{N_m} \min \{ \lfloor v(f_i) \rfloor, k_m \} \geq \sum_{i=1}^{N_m} \lfloor v(g_i) \rfloor = w_{N_m}(v).$$
In particular, for sufficiently large \( m \) we get
\[
\sum_{i=1}^{N_m} (v(f_i)) = \sum_{j=0}^{\infty} j \cdot \ell(F_v^j R_m / F_v^{j+1} R_m)
\]
\[
\geq \sum_{j=0}^{\infty} \min\{j, k_m\} \cdot \ell(F_v^j R_m / F_v^{j+1} R_m) + ((1 + 2\varepsilon)m - k_m) \cdot \ell(F_v^{(1+2\varepsilon)m} R_m)
\]
\[
\geq \sum_{j=0}^{\infty} \min\{j, k_m\} \cdot \ell(F_v^j R_m / F_v^{j+1} R_m) + \epsilon m \cdot \gamma m^{n+1}/n!
\]
\[
= \sum_{i=1}^{N_m} \min\{[v(f_i)], k_m\} + \epsilon \gamma m^{n+1}/n!
\]
\[
\geq w_{N_m}(v) + \epsilon \gamma m^{n+1}/n!,
\]
where \( R_m = R/a_m(v_0) \). Dividing by \( \sum [v_0(f_i)] = \tilde{S}_m(v_0; v_0) = w_{N_m}(v_0) = O(m^{n+1}) \) and letting \( m \to \infty \), we obtain
\[
1 \geq S(v_0; v) = \lim_{m \to \infty} \frac{\tilde{S}_m(v_0; v_0)}{w_{N_m}(v_0)} > \lim_{m \to \infty} \frac{w_{N_m}(v)}{w_{N_m}(v_0)} = \left( \frac{\text{vol}(v_0)}{\text{vol}(v)} \right)^{1/n}
\]
where the last equality follows from Lemma 3.8 hence \( \text{vol}(v) > \text{vol}(v_0) \), a contradiction. This proves the claim (3.9). By the following Lemma 3.9 it implies \( v = v_0 \) and we are done.

The following result, which is an improvement of [LX16, Proposition 2.7], is used in the above proof.

**Lemma 3.9.** Let \( x \in X = \text{Spec}(R) \) be a singularity and let \( v_0, v_1 \in \text{Val}_{X,x}^* \). Assume that
\[
\text{vol}(v_0) = \text{vol}(v_1) = \text{mult}(a_*(v_0) \cap a_*(v_1)) > 0.
\]

Then \( v_0 = v_1 \).

**Proof.** We prove by contradiction. Assume that \( v_0(f) \neq v_1(f) \) for some \( f \in R \). Without loss of generality we may assume that \( v_0(f) = \ell_0 > \ell_1 = v_1(f) \). Replacing \( f \) by \( f^k \) for some \( k \in \mathbb{N} \) we may further assume that \( \ell_0 \geq \ell_1 + 1 \). For \( v \in \text{Val}_{X,x}^* \) and \( r \geq 0 \), let \( a_r(v) = \{ f \in R \mid v(f) \geq r \} \). Let \( b_r = a_r(v_0) \cap a_r(v_1) \) and \( c_r = a_r(v_0) \cap a_{2r}(v_1) \) where \( r \geq 0 \).

For every \( m \in \mathbb{N} \) and every \( s \in b_m \), we have \( v_0(f^m s) = m \cdot v_0(f) + v_0(s) \geq m(\ell_0 + 1) \), thus multiplication by \( f^m \) induces a map
\[
b_m \xrightarrow{f^m} a_{m(\ell_0+1)}(v_0) \to a_{m(\ell_0+1)}(v_0)/b_{m(\ell_0+1)}
\]
whose kernel is contained in \( c_m \) (since \( v_1(f^m s) \geq m(\ell_0 + 1) \) implies \( v_1(s) \geq m(\ell_0 + 1) - m\ell_1 \geq 2m \)). It follows that
\[
\ell(a_{m(\ell_0+1)}(v_0)/b_{m(\ell_0+1)}) \geq \ell(b_m/c_m)
\]
for all \( m \in \mathbb{N} \). By [LX16, Proposition 2.7], there exists some \( 0 \neq g \in m_x \) such that \( \ell_2 = v_1(g) > v_0(g) > 0 \). For every \( m \in \mathbb{N} \) and every \( s \in c_m \), we then have
\[
v_1(g^m s) = m \cdot v_1(g) + v_1(s) \geq m(\ell_2 + 2),
\]
Thus multiplication by \( g^m \) induces a map
\[
c_m \xrightarrow{g^m} a_{m(\ell_2+2)}(v_1) \to a_{m(\ell_2+2)}(v_1)/b_{m(\ell_2+2)}
\]
whose kernel is contained in \( b_{2m} \) (if \( v_0(g^m s) \geq m(\ell_2 + 2) \) then as \( v_0(g) \leq \ell_2 \) we get \( v_0(s) \geq 2m \)). It follows that
\[
\ell(a_{m(\ell_2+2)}(v_1)/b_{m(\ell_2+2)}) \geq \ell(c_m/b_{2m}) \tag{3.11}
\]
for all \( m \in \mathbb{N} \). Combining (3.10) and (3.11) we see that
\[
(\ell_0 + 1)^n(m(b_*) - \text{vol}(v_0)) + (\ell_2 + 2)^n(m(b_*) - \text{vol}(v_1))
\]
\[
= \lim_{m \to \infty} \frac{\ell(a_{m(\ell_0+1)}(v_0)/b_{m(\ell_0+1)})}{m^n/n!} + \lim_{m \to \infty} \frac{\ell(a_{m(\ell_2+2)}(v_1)/b_{m(\ell_2+2)})}{m^n/n!}
\]
\[
\geq \lim_{m \to \infty} \frac{\ell(b_m/c_m) + \ell(c_m/b_{2m})}{m^n/n!} = \lim_{m \to \infty} \frac{\ell(b_m/b_{2m})}{m^n/n!}
\]
\[
= (2^n - 1)m(b_*) > 0,
\]
which contradicts our assumption. Thus \( v_0(f) = v_1(f) \) for all \( f \in R \) as desired. \( \square \)

3.3. Every minimizer is K-semistable. In this subsection, we show that every valuation that minimizes the normalized volume function is K-semistable. Combined with Theorem 3.10, this proves the uniqueness of the minimizer.

Theorem 3.10. Let \( x \in (X = \text{Spec}(R), \Delta) \) be a klt singularity and let \( v_0 \in \text{Val}^*_X \) be a minimizer of the normalized volume function \( \text{vol}^*_X \). Then \( v_0 \) is K-semistable.

In other words, we will show that \( A_X(\Delta) \geq S(v_0; v) \) for every valuation \( v \in \text{Val}^*_X \). Inspired by the argument of [Li17], we consider a family \( b_{*, t} \) \((t \in \mathbb{R}_{\geq 0})\) of graded sequences of ideals that interpolate the valuation ideal sequences of \( v_0 \) and \( v \), defined as follows: we set \( b_{*, 0} = a_*(v_0) \); when \( t > 0 \), we set
\[
b_{m, t} = \sum_{i=0}^{m} a_{m-i}(v_0) \cap a_i(t v).
\]
Roughly speaking, the ideal \( b_{m, t} \) is generated by elements \( f \in R \) with \( v_0(f) + t \cdot v(f) \geq m \). By (2.1), we have
\[
lct(b_{*, t})^n \cdot \mult(b_{*, t}) \geq \text{vol}(v_0) = lct(b_{*, 0})^n \cdot \mult(b_{*, 0}).
\]
To relate this to the K-semistability of \( v_0 \), the idea is to take the derivative of the above normalized multiplicities at \( t = 0 \), which was a technique introduced in [Li17]. To do so we analyze the log canonical thresholds and multiplicities of \( b_{*, t} \).

3.3.1. Log canonical threshold of summation. We first establish an inequality for the log canonical thresholds of graded sequences of ideals. Given two graded sequences of ideals \( a_* \) and \( b_* \), we define \( c_* := a_* \boxplus b_* \) by setting
\[
c_m = (a \boxplus b)_m = \sum_{i=0}^{m} a_i \cap b_{m-i}.
\]
It is easy to verify that $c_*$ is also a graded sequence of ideals. Note that our definition differs from the usual sum of ideal sequences (see e.g. [Mus02]) since we use intersections of ideals rather than taking product.

**Theorem 3.11.** Under the above notation, assume $a_*$ and $b_*$ are graded sequences of $m_x$-primary ideals. Then we have

$$\text{lct}(c_*) \leq \text{lct}(a_*) + \text{lct}(b_*)$$

We denote by $J(a^t)$ the multiplier ideal of a fractional ideal and similarly by $J(a_*^t)$ the asymptotic multiplier ideal of a graded sequence of ideals $a_*$ with exponent $t$ (see [Laz04] for details). The above inequality will follow from a summation formula of multiplier ideals.

**Lemma 3.12.** For any two graded sequences of ideals $a_*$, $b_*$ and any $t > 0$, we have

$$J(c_*^t) \subseteq \sum_{\lambda + \mu = t} J(a_*^\lambda) \cap J(b_*^\mu)$$

where $c_* = a_* \boxplus b_*$.  

**Proof.** We follow the proof of [Tak06, Proposition 4.10]. Let $m$ be a sufficiently large and divisible integer such that $J(c_*^t) = J(c_*^{t/m})$. By the summation formula of multiplier ideals (see [Tak06, Theorem 0.1(2)]), which says that for any two ideals $a$ and $b$, 

$$J((a + b)^t) = \sum_{t_1 + t_2 = t} J(a^{t_1} \cdot b^{t_2}),$$

we have

$$J(c_*^{t/m}) = J\left(\sum_{i=0}^{m} a_i \cap b_{m-i}^{t/m}\right) = \sum_{t_0 + \cdots + t_m = t/m} J\left(\prod_{i=0}^{m} (a_i \cap b_{m-i})^{t_i/m}\right).$$

(The right hand side is a finite sum.) Since $a_i^{mt_i/m} \subseteq a_{mt_i}$, each individual term in the above right hand side is contained in

$$J\left(\prod_{i=0}^{m} a_i^{t_i}\right) \subseteq J\left(\prod_{i=0}^{m} a_{mt_i}^{t_i/m}\right) = J\left(a_{m^t}^{\lambda/m^t}\right) \subseteq J(a_*^\lambda)$$

where $\lambda = \sum_{i=0}^{m} it_i$. By symmetry, it is also contained in $J(b_*^\mu)$ where $\mu = \sum_{i=0}^{m} (m-i)t_i$. Note that $\lambda + \mu = \sum_{i=0}^{m} mt_i = m \cdot \frac{t}{m} = t$, thus every

$$J\left(\prod_{i=0}^{m} (a_i \cap b_{m-i})^{t_i}\right) \subseteq J(a_*^\lambda) \cap J(b_*^\mu)$$

is contained in the right hand side of (3.13). This completes the proof.

**Proof of Theorem 3.11** Let $\alpha = \text{lct}(a_*)$, $\beta = \text{lct}(b_*)$ and let $t = \alpha + \beta$. For any $\lambda, \mu \geq 0$ with $\lambda + \mu = t$ we have either $\lambda \geq \alpha$ or $\mu \geq \beta$, therefore $J(a_*^\lambda) \cap J(b_*^\mu) \subseteq m_x$. By Lemma 3.12 we see that $J(c_*^t) \subseteq m_x$ and hence $\text{lct}(c_*) \leq t = \text{lct}(a_*) + \text{lct}(b_*)$. \qed
3.3.2. Multiplicities of a family of graded sequences of ideals. We next derive a formula for the multiplicities of $b_{*,t}$.

**Lemma 3.13.** $\text{mult}(b_{*,t}) = \text{vol}(v_0) - (n+1) \int_0^\infty \text{vol}(v_0; v/u) \frac{t du}{(1+tu)^{n+2}}$.

**Proof.** By definition, we have

$$\text{mult}(b_{*,t}) = \lim_{m \to \infty} \frac{\ell(R/b_{m,t})}{m^n/n!}.$$ 

However, to derive the statement of the lemma, it is better to use a different formula, which follows from the above equality:

$$(3.14) \quad \text{mult}(b_{*,t}) = \lim_{m \to \infty} \frac{\sum_{j=1}^m \ell(R/b_{j,t})}{m^{n+1}/(n+1)!}.$$ 

For ease of notation, let $a_* = a_*(v_0)$. We have

$$(a_{j-\ell-1} \cap a_{\ell+1}(tv)) \cap \sum_{i=0}^{\ell} (a_{j-i} \cap a_i(tv)) = a_{j-\ell} \cap a_{\ell+1}(tv)$$

for all $0 \leq \ell < j$ and we get short exact sequences

$$0 \to \frac{a_{j-\ell-1} \cap a_{\ell+1}(tv)}{a_{j-\ell} \cap a_{\ell+1}(tv)} \to \frac{R}{\sum_{i=0}^{\ell} a_{j-i} \cap a_i(tv)} \to \frac{R}{\sum_{i=0}^{\ell+1} a_{j-i} \cap a_i(tv)} \to 0.$$ 

Thus from the definition of $b_{*,t}$, we get

$$\ell(R/b_{j,t}) = \ell(R/a_j) - \sum_{i=1}^j \ell(F^{i/t}_v (a_{j-i}/a_{j-i+1})).$$ 

Summing over $j = 0, 1, \ldots, m$ we obtain

$$\sum_{j=0}^m \ell(R/b_{j,t}) = \sum_{j=1}^m \ell(R/a_j) - \sum_{1 \leq i \leq j \leq m} \ell(F^{i/t}_v (a_{j-i}/a_{j-i+1}))$$

$$= \sum_{j=1}^m \ell(R/a_j) - \sum_{i=1}^m \ell(F^{i/t}_v (R/a_{m-i+1})).$$

Combining with (3.14), we deduce that

$$(3.15) \quad \text{mult}(b_{*,t}) = \text{vol}(v_0) - (n+1) \cdot \lim_{m \to \infty} \frac{W_m}{m^{n+1}/n!}$$

where $W_m := \sum_{i=1}^m \ell(F^{i/t}_v (R/a_{m-i+1}))$. To analyze the limit in the above expression, we set (c.f. the proof of Lemma 3.3 or the argument in [Li17, Section 4.1.1])

$$\phi_m(y) = \frac{\ell(F^{[my]/t}_v (R/a_{m-[my]+1}))}{m^n/n!}$$

$$= \frac{\ell(F^{[my]/t}_v (R/a_{m-[my]+1}))}{(m-[my]+1)^n/n!} \cdot \frac{(m-[my]+1)^n}{m^n}.$$
where $0 < y < 1$. It is not hard to check that
\[
\lim_{m \to \infty} \phi_m(y) = g \left( \frac{y}{t(1-y)} \right) (1-y)^n
\]
where $g(u) = \text{vol}(v_0; v/u)$, hence by the dominated convergence theorem we have
\[
\lim_{m \to \infty} \frac{W_m}{m^{n+1}/n!} = \lim_{m \to \infty} \int_0^1 \phi_m(y) dy
\]
\[
= \int_0^1 g \left( \frac{y}{t(1-y)} \right) (1-y)^n dy = \int_0^\infty g(u) \frac{tdu}{(1+tu)^{n+2}}.
\]
Together with (3.15) this implies the statement of the lemma. \hfill \Box

We are now ready to give the proof of Theorem 3.10.

**Proof of Theorem 3.10.** Let $v \in \text{Val}_{X,x}^*$. Up to rescaling, we may assume that $A_{X,\Delta}(v_0) = A_{X,\Delta}(v) = 1$. Define $b_{*,t} (t \geq 0)$ as in (3.12), and let
\[
f(t) := (1+t)^n \cdot \text{mult}(b_{*,t}).
\]
Clearly $f(0) = \widehat{\text{vol}}(v_0)$. By Theorem 3.11 we have
\[
\text{lct}(b_{*,t}) \leq \text{lct}(a_*(v_0)) + \text{lct}(a_*(tv)) \leq \frac{A_{X,\Delta}(v_0)}{v_0(a_*(v_0))} + \frac{A_{X,\Delta}(v)}{v(a_*(tv))} \leq 1 + t.
\]
Hence for all $t \geq 0$,
\[
f(t) \geq \text{lct}(b_{*,t})^n \cdot \text{mult}(b_{*,t}) \geq \widehat{\text{vol}}(v_0) = f(0),
\]
where the second inequality follows from (2.1) and the assumption that $v_0$ is a minimizer of $\widehat{\text{vol}}_{X,\Delta}$. Thus $f'(0) \geq 0$. Using Lemma 3.13 we find
\[
f'(0) = n \cdot \text{vol}(v_0) - (n+1) \int_0^\infty \text{vol}(v_0; v/u) du,
\]
thus
\[
A_{X,\Delta}(v) = 1 \geq \frac{n+1}{n} \cdot \frac{n}{n} \int_0^\infty \frac{\text{vol}(v_0; v/u) du}{\text{vol}(v_0)} = S(v_0; v).
\]
Since $v \in \text{Val}_{X,x}^*$ is arbitrary, it follows that $v_0$ is K-semistable. \hfill \Box

**Remark 3.14.** If we combine together Theorems 3.9, 3.7 and 3.10, we get a proof of the fact that a Kollár component is a minimizer if and only if it is K-semistable, which was first established in [Li17, LX16]. While in the proof of Theorem 3.10 we still use a version of the derivative formula introduced in [Li17], we do not need it for the converse.

4. Applications

In this section, we prove the results mentioned in the introduction.

**Proof of Theorem 1.1.** By [Blu18] (see also [Xu20, Remark 3.8]), there exists $v_0 \in \text{Val}_{X,x}^*$ such that $\widehat{\text{vol}}(v_0) = \widehat{\text{vol}}(x, X, \Delta)$. By Theorem 3.10, $v_0$ is K-semistable, thus by Theorem 3.7 it is the unique minimizer of the normalized volume function up to scaling. \hfill \Box
Proof of Corollary 1.2. If \( v_0 \) is a minimizer of \( \hat{\text{vol}}_{X,\Delta} \), then for any \( g \in G \), the valuation \( g \cdot v_0 \) defined by \( (g \cdot v_0)(s) = v_0(g^{-1} \cdot s) \) is also a minimizer of \( \hat{\text{vol}}_{X,\Delta} \). By Theorem 1.1 we have \( g \cdot v_0 = \lambda v_0 \) for some \( \lambda > 0 \); but since \( A_{X,\Delta}(v_0) = A_{X,\Delta}(g \cdot v_0) \), we must have \( \lambda = 1 \), hence \( v_0 = g \cdot v_0 \) is \( G \)-invariant. □

Theorem 1.3 is then an easy consequence of Corollary 1.2.

Proof of Theorem 1.3. Let \( G = \text{Aut}(Y/X) \) be the Galois group. By Corollary 1.2, the minimizer \( v_0 \) of \( \hat{\text{vol}}_{Y,\Delta_Y} \) is \( G \)-invariant, hence \( \hat{\text{vol}}(y,Y,\Delta_Y) = \hat{\text{vol}}^G (y,Y,\Delta_Y) \), where

\[
\hat{\text{vol}}^G (x,X,\Delta) := \inf_{v \in \text{Val}^G_{X,x}} \hat{\text{vol}}_{(X,\Delta),x}(v)
\]

as the infimum runs over all valuations \( v \in \text{Val}_{X,x} \) that are invariant under the \( G \)-action.

By [LX19, Theorem 2.7(1)], we get \( \hat{\text{vol}}^G (y,Y,\Delta_Y) = |G| \cdot \hat{\text{vol}}(x,X,\Delta) \) (in loc. cit. it is assumed that \( \Delta_Y = 0 \) and \( f \) is étale in codimension one, but the proof applies in general since these assumptions are only used to guarantee that \( \Delta = 0 \)). Thus

\[
\hat{\text{vol}}(y,Y,\Delta_Y) = |G| \cdot \hat{\text{vol}}(x,X,\Delta) = \deg(f) \cdot \hat{\text{vol}}(x,X,\Delta).
\]

□

In fact, the above argument implies the finite degree formula for any quasi-étale (i.e. étale in codimension one) finite morphism \( Y \to X \), as we can pass to the Galois closure of \( Y/X \), which is also quasi-étale. However, if there is a branched divisor, then the pull back of \( K_X + \Delta \) to the Galois closure of \( Y/X \) might have negative coefficients.

Proof of Corollary 1.4. For the germ of a klt singularity \((X,\Delta)\), by [Xu14, Bra20] (see also [TX17]), the fundamental group \( \pi_1(x,X^{\text{sm}}) \) of the smooth locus \( X^{\text{sm}} \) of is finite.

Let \( f: (Y,y) \to (X,x) \) be the universal cover of \( X^{\text{sm}} \) and let \( \Delta_Y = f^* \Delta \). Then we have \( K_Y + \Delta_Y = f^*(K_X + \Delta) \), hence by Theorem 1.3 we get \( \hat{\text{vol}}(y,Y,\Delta_Y) = \deg(f) \cdot \hat{\text{vol}}(x,X,\Delta) \). By [LX19, Theorem A.4], we also have \( \hat{\text{vol}}(y,Y,\Delta_Y) \leq n^n \) with equality if and only if \( y \in X \) is smooth and \( \Delta_Y = 0 \). It follows that

\[
\deg(f) = \#|\pi_1(x,X^{\text{sm}})| \leq \frac{n^n}{\hat{\text{vol}}(x,X,\Delta)}
\]

and the equality holds if and only if \((y \in (Y,\Delta_Y)) \cong (0 \in \mathbb{C}^n) \) (étale locally), i.e., \( \Delta = 0 \) and \((x \in X) \) is étale locally isomorphic to \( \mathbb{C}^n/G \) where \( G \cong \pi_1(x,X^{\text{sm}}) \) and the action of \( G \) is fixed point free in codimension one. □

Proof of Theorem 1.5. By [Bri20, Theorem D], we have

\[
\hat{\text{vol}}(x,X,\Delta) \geq \left( \frac{n}{n+1} \right)^n \cdot \delta(X,\Delta)^n \cdot (-(K_X + \Delta))^n.
\]

Thus the result follows immediately from Corollary 1.4. □
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