BETTI NUMBERS OF CHORDAL GRAPHS AND \( f \)-VECTORS OF SIMPLICIAL COMPLEXES

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Abstract. Let \( G \) be a chordal graph and \( I(G) \) its edge ideal. Let \( \beta(I(G)) = (\beta_0(I), \beta_1(I), \ldots, \beta_p(I)) \) denote the Betti sequence of \( I(G) \), where \( \beta_i \) stands for the \( i \)-th total Betti number of \( I(G) \), and where \( p \) is the projective dimension of \( I(G) \). It will be shown that there exists a simplicial complex \( \Delta \) of dimension \( p \) whose \( f \)-vector \( f(\Delta) = (f_0, f_1, \ldots, f_p) \) coincides with \( \beta(I(G)) \).

Introduction

Let \( S = K[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( K \) with each deg \( x_i = 1 \). The Betti sequence of a homogeneous ideal \( I \subset S \) is the sequence

\[
\beta(I) = (\beta_0(I), \beta_1(I), \ldots, \beta_p(I)),
\]

where each \( \beta_i(I) \) stands for the \( i \)-th total Betti number of \( I \) and where \( p = \text{proj dim } I \) is the projective dimension of \( I \). One has \( \sum_{i=-1}^{p} (-1)^i \beta_i(I) = 0 \) with \( \beta_{-1}(I) = 1 \).

Let \( \Delta \) be a simplicial complex and

\[
f(\Delta) = (f_0, f_1, \ldots, f_{d-1})
\]

its \( f \)-vector, where each \( f_i = f_i(\Delta) \) stands for the number of faces of \( \Delta \) of dimension \( i \) and where \( d-1 \) is the dimension \( \Delta \). Recall that \( \Delta \) is acyclic (over \( K \)) if its reduced homology group \( \tilde{H}_i(\Delta; K) \) with coefficients \( K \) vanishes for all \( i \). Thus in particular if \( \Delta \) is acyclic, then its \( f \)-vector satisfies \( \sum_{i=-1}^{d-1} (-1)^i f_i = 0 \) with \( f_{-1} = 1 \).

Peeva and Velasco \[20\] succeeded in proving that, given an acyclic simplicial complex \( \Delta \), there exists a monomial ideal \( I \) whose Betti sequence \( \beta(I) \) coincides with its \( f \)-vector \( f(\Delta) \). In general, the converse is, however, false. Let \( n = 6 \) and \( I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_1x_6) \). Then \( \dim S/I = 3 \), depth \( S/I = 2 \) and \( p = 4 \). One has \( \beta(I) = (6, 9, 6, 2) \). If a simplicial complex \( \Delta \) possesses 2 faces of dimension 3, then \( \Delta \) possesses at least 7 faces of dimension 2. It then follows that there exists no simplicial complex \( \Delta \) of dimension 3 with \( (6, 9, 6, 2) \) its \( f \)-vector.

On the other hand, in Example \[1.8\] one can find a Cohen–Macaulay monomial ideal \( I \), i.e., \( S/I \) is a Cohen–Macaulay ring, whose Betti sequence is the \( f \)-vector of a simplicial complex, but not the \( f \)-vector of an acyclic simplicial complex.

2000 Mathematics Subject Classification: Primary 13P99, 13F55; Secondary 52B05.

Keywords: monomial ideal, Betti sequence, simplicial complex, \( f \)-vector, chordal graph.

The third author is supported by JSPS Research Fellowships for Young Scientists.
It is natural to ask which monomial ideals $I$ enjoy the property that there exists a simplicial complex (or acyclic simplicial complex) $\Delta$ whose $f$-vector coincides with the Betti sequence of $I$. The purpose of the present paper is to establish the research project on finding a natural class $\mathcal{C}$ of monomial ideals such that, for each ideal $I$ belonging to $\mathcal{C}$, the Betti sequence $\beta(I)$ is the $f$-vector of a simplicial (or an acyclic simplicial) complex.

First, in Section 1, we summarize several answers, which are easily or directly obtained from well-known facts. The topics discussed will include monomial ideals with small projective dimensions, cellular resolutions, componentwise linear ideals and pure resolutions.

Now, Section 2 is the highlight of this paper. Let $G$ be a finite graph on the vertex set $V$ and $E(G)$ the edge set of $G$. We write $S = K[\{x : x \in V\}]$ for the polynomial ring in $|V|$ variables over a field $K$ with each $\deg x = 1$. The edge ideal of $G$ is the ideal $I(G)$ of $S$ generated by those monomials $xy$ with $\{x, y\} \in E(G)$. Recall that a finite graph $G$ is chordal if each cycle of $G$ of length $> 3$ has a chord. Theorem 2.1 guarantees that, for an arbitrary chordal graph $G$, there exists a simplicial complex $\Delta$ whose $f$-vector coincides with $\beta(I(G))$. The recursive-type formula due to Hà and Van Tuyl [12] will be indispensable to achieve the proof of Theorem 2.1.

Finally, in Section 3, we study Gorenstein monomial ideals. It follows that the Betti sequence of a Gorenstein monomial ideal $I$ with $\proj \dim(I) \leq 3$ is the $f$-vector of an acyclic simplicial complex. On the other hand, we can characterize the possible Betti numbers of Gorenstein monomial ideals $I$ with $\proj \dim(I) = 3$. Moreover, it will be proved that, given integers $m \geq 4$ and $p \geq 3$, there exists a Gorenstein monomial ideal $I$ of $K[x_1, \ldots, x_n]$, where $n$ is enough large, with $\beta_0(I) = m$ and $\proj \dim(I) = p$ if and only if $m \geq p + 1$ with $m \neq p + 2$.

1. Betti sequences and acyclic simplicial complexes

The present section is a summary of several answers, which are easily or directly obtained from well-known facts, for the problem of finding a natural class $\mathcal{C}$ of monomial ideals such that, for each ideal $I$ belonging to $\mathcal{C}$, the Betti sequence $\beta(I)$ is the $f$-vector of a simplicial (or an acyclic simplicial) complex.

First, recall a combinatorial characterization of $f$-vectors of acyclic simplicial complexes due to Gil Kalai [17].

**Lemma 1.1 (Kalai).** A vector $f = (f_0, f_1, \ldots, f_{d-1})$ of positive integers is the $f$-vector of an acyclic simplicial complex of dimension $d - 1$ if and only if there exists a simplicial complex $\Delta'$ of dimension $d - 2$ with $f(\Delta') = (f_0', f_1', \ldots, f_{d-2}')$ such that $f_i = f_i' + f_{i-1}'$ for all $i$, where $f_{d-1}' = 1$ and $f_{d-2}' = 0$.

**(1.1) Monomial ideals with small projective dimensions**

Let $I \subset S$ be a monomial ideal with $\proj \dim(I) \leq 2$ and $\beta(I) = (n, \beta_1, \beta_2)$. One has $1 - n + \beta_1 - \beta_2 = 0$. It follows from the Taylor resolution of $I$ that there
exists an integer $c \geq 0$ such that $\beta_1 = \binom{n}{2} - c$. Thus $\beta(I) = (n, \binom{n}{2} - c, \binom{n-1}{2} - c)$. Since $(n-1, \binom{n-1}{2} - c)$ is the $f$-vector of a simplicial complex, Lemma 1.1 says that $\beta(I) = (n, \binom{n}{2} - c, \binom{n-1}{2} - c)$ is the $f$-vector of an acyclic simplicial complex.

**Theorem 1.2.** Let $I \subset S$ be a monomial ideal with proj dim$(I) \leq 2$. Then $\beta(I)$ is the $f$-vector of an acyclic simplicial complex.

### (1.2) Cellular resolutions

The cellular resolution was introduced by Bayer and Sturmfels [2]. Let $I \subset S$ be a monomial ideal and $F_\bullet$ a $\mathbb{Z}^n$-graded free resolution of $S/I$. The complex $F_\bullet \otimes_S S/(x_1 - 1, \ldots, x_n - 1)$ of $K$-vector spaces is called the frame of $F_\bullet$. We say that $F_\bullet$ is supported by a CW-complex $\Delta$ if its frame is equal to the augmented oriented chain complex of $\Delta$. If a free resolution is supported by a CW-complex $\Delta$, then $\Delta$ must be acyclic ([2, Proposition 1.2]). Thus if a minimal free resolution is supported by a simplicial complex, then its Betti sequence must be the $f$-vector of an acyclic simplicial complex.

A monomial ideal $I \subset S$ is said to be generic if, for all pairs of generators $u = x_1^{a_1} \ldots x_n^{a_n}$ and $v = x_1^{b_1} \ldots x_n^{b_n}$ of $I$, one has $a_k \neq b_k$ or $a_k = b_k = 0$ for all $k$. It was proved by Bayer, Peeva and Sturmfels [3] that a generic monomial ideal has a minimal free resolution which is supported by a simplicial complex.

**Theorem 1.3.** Let $I$ be a generic monomial ideal. Then $\beta(I)$ is the $f$-vector of an acyclic simplicial complex.

We say that a CW-complex $\Delta$ satisfies the intersection property if the intersection of two faces of $\Delta$ is again a face of $\Delta$. For example, all simplicial complexes as well as all polyhedral complexes satisfy the intersection property. Björner and Kalai [5] proved that if $\Delta$ is an acyclic CW-complex satisfying the intersection property, then the $f$-vector of $\Delta$ is the $f$-vector of an acyclic simplicial complex.

**Theorem 1.4.** Suppose that the minimal free resolution of a monomial ideal $I \subset S$ is supported by a CW-complex satisfying the intersection property. Then $\beta(I)$ is the $f$-vector of an acyclic simplicial complex.

Velasco [23] studied minimal free resolutions which are not supported by a CW-complex by means of the nearly scarf ideal introduced in [20]. Let $\Omega$ be a simplicial complex with the vertex set $[n] = \{1, 2, \ldots, n\}$ which is not the boundary of a simplex. The nearly scarf ideal $J_\Omega$ of $\Omega$ is the monomial ideal of the polynomial ring $K[x_\sigma : \sigma \in \Omega \setminus \{\emptyset\}]$ generated by $\{\prod_{\sigma \in \Omega, v \not\in \sigma} x_\sigma : v \in [n]\}$. It is known [20] that the graded Betti numbers of $J_\Omega$ is given by

$$\beta_i(J_\Omega) = f_i(\Omega) + \dim_k \tilde{H}_{i-1}(\Omega; K), \quad i \geq 0.$$  

On the other hand, Björner–Kalai Theorem ([4]), which gives a characterization of the $(f, \beta)$-pairs of simplicial complexes, guarantees that, for an arbitrary simplicial
complex $\Delta$ with $f(\Delta) = (f_0, f_1, \ldots, f_{d-2})$, the vector $(f'_0, \ldots, f'_{d-1})$ defined by setting $f'_i = f_i + \dim_K \tilde{H}_{i-1}(\Delta; K)$ is the $f$-vector of an acyclic simplicial complex.

**Theorem 1.5.** Let $J_\Omega$ be the nearly scarf ideal of $\Omega$. Then $\beta(J_\Omega)$ is the $f$-vector of an acyclic simplicial complex.

(1.3) **Componentwise linear ideals**

One of the most famous classes of monomial ideals for which the formula of graded Betti numbers is known is the class of stable ideals. Recall that a monomial ideal $I \subset S$ is stable if, for all monomials $u \in I$ and for all $1 \leq i < m(u)$, one has $ux_i/x_{m(u)} \in I$, where $m(u)$ is the maximal integer $k$ such that $x_k$ divides $u$. Let $I$ be a stable ideal and $G(I)$ the minimal set of monomial generators of $I$. Write $m_k(I)$ for the number of monomials $u \in G(I)$ with $m(u) = k$. Eliahou and Kervaire [10] proved that

$$\beta_i(I) = \sum_{k=i+1}^{n} m_k(I) \binom{k-1}{i}$$

for all $i \geq 0$.

A homogeneous ideal $I \subset S$ is said to have a $k$-linear resolution if $\beta_{i,j+1}(I) = 0$ whenever $j \neq k$. A homogeneous ideal $I \subset S$ is said to be componentwise linear ([14]) if, for all integers $k \geq 0$, the ideal $I(k)$ which is generated by the homogeneous polynomials of degree $k$ belonging to $I$ has a $k$-linear resolution. A quasi-forest is a simplicial complex $\Delta$ whose Stanley–Reisner ideal $I_\Delta$ has a 2-linear resolution. It is known (Fröberg [11]) that a quasi-forest is the clique complex of a chordal graph.

**Theorem 1.6.** Let $\beta = (\beta_0, \beta_1, \ldots, \beta_p)$ with $p \leq n - 1$ be a sequence of integers. The following conditions are equivalent:

(i) There exists a componentwise linear ideal $I \subset S$ with $\proj \dim(I) = p$ such that $\beta(I) = \beta$;

(ii) There exists a stable ideal $I \subset S$ with $\proj \dim(I) = p$ such that $\beta(I) = \beta$;

(iii) There exists a sequence $c_1, \ldots, c_p+1$ of positive integers with $c_1 = 1$ such that $\beta_i = \sum_{k=1}^{p+1} c_k \binom{k-1}{i}$ for all $i \geq 0$;

(iv) There exists an acyclic quasi-forest $\Delta$ of dimension $p$ such that $\beta = f(\Delta)$.

**Proof.** First, (i) \(\Leftrightarrow\) (ii) is known ([13, Lemma 1.4]). Second, (ii) \(\Rightarrow\) (iii) follows from Eliahou–Kervaire formula and the fact that if $I$ is a stable ideal and $m_k(I) \neq 0$ for some $k > 0$, then $m_k(I) \neq 0$ for all $1 \leq \ell < k$ ([15, Lemma 1.3]). Third, to prove (iii) \(\Rightarrow\) (ii), we introduce the monomial ideal $I$ generated by

$$\bigcup_{i=1}^{p+1} \{ (x_1^{c_1} \cdots x_{i-2}^{c_{i-2}}) x_{i-1}^{c_{i-1}} x_i^{c_i+1-k} : k = 1, \ldots, c_i \},$$

where $c_{p+2} = 0$. It follows that $I$ is stable and $(m_1(I), \ldots, m_{p+1}(I)) = (c_1, \ldots, c_{p+1})$. 


Finally, (iii) \( \Leftrightarrow \) (iv) will be shown. It is known \([16]\) that \( f = (f_0, f_1, \ldots, f_p) \) is the \( f \)-vector of a quasi-forest of dimension \( p - 1 \) if and only if there exists a sequence of positive integers \( b_1, \ldots, b_p \) such that \( f_{i-1} = \sum_{k=1}^{p} b_k \binom{k-1}{i-1} \) for all \( i \geq 1 \). If \( \Delta \) is a quasi-forest, then it follows from \([13]\) Theorem 7.1] that its algebraic shifted complex \( \Sigma \) is again a quasi-forest. If \( \Delta \) is acyclic then \( \Sigma \) must be a cone \([17]\). However, if a quasi-forest \( \Sigma \) is a cone, then it must be a cone of a quasi-forest. These facts guarantee that \( f = (f_0, f_1, \ldots, f_p) \) is the \( f \)-vector of an acyclic quasi-forest of dimension \( p - 1 \). The latter condition is equivalent to saying that there exists a sequence of positive integers \( b_1, \ldots, b_p \) such that \( f_{i-1} = \sum_{k=1}^{p} b_k \binom{k-1}{i-1} \) for all \( i \geq 2 \) and \( f_0 = 1 + \sum_{k=1}^{p} b_k \). Set \( c_1 = 1 \) and \( c_k = b_{k-1} \) for \( k = 2, 3, \ldots, p + 1 \). Then the sequence \( c_1, \ldots, c_{p+1} \) satisfies the conditions of (iii), as desired. \( \square \)

(1.4) Pure resolutions

We discuss the question whether Betti sequences of monomial ideals with pure resolutions are \( f \)-vectors of simplicial complexes. We say that a homogeneous ideal \( I \subset S \) has a pure resolution if its minimal free resolution is of the form

\[
0 \rightarrow S(-c_p)^{b_p} \rightarrow S(-c_{p-1})^{b_{p-1}} \rightarrow \cdots \rightarrow S(-c_0)^{b_0} \rightarrow I \rightarrow 0.
\]

Let \( v > d \geq 1 \) and \( C(v, d) \) the cyclic polytope \([21]\) p. 59] of dimension \( d \) with \( v \) vertices. Since \( C(v, d) \) a simplicial polytope, its boundary \( \partial C(v, d) \) defines a simplicial complex \( \Delta(C(v, d)) \), called the boundary complex of \( C(v, d) \). It is known \([22]\) Proposition 3.1] that, when \( d \) is even, the Stanley–Reisner ideal \( I_{\Delta(C(v, d))} \) \([21]\) p. 53]) of \( \Delta(C(v, d)) \) has a pure resolution.

**Example 1.7.** Let \( v = 7 \) and \( d = 2 \). Then the Betti sequence of \( I_{\Delta(C(7,2))} \) is \((14, 35, 35, 14, 1)\). In particular \((14, 35, 35, 14, 1)\) is the Betti sequence arising from a pure resolution. However, it turns out that \((14, 35, 35, 14, 1)\) cannot be the Betti sequence arising from a linear resolution.

**Example 1.8.** In \([7]\) it is shown that there exists a simplicial complex \( \Delta \) such that (i) \( I_{\Delta} \) has a pure, but not a linear resolution; (ii) the Betti sequence of \( I_{\Delta} \) is \( \beta(I_{\Delta}) = (14, 21, 14, 6) \); (iii) the Stanley–Reisner ring \( K[\Delta] = S/I_{\Delta} \) \([21]\) p. 53]) is Cohen–Macaulay. Now, Kruskal–Katona theorem \([21]\) p. 55] says that \((14, 21, 14, 6)\) is the \( f \)-vector of a simplicial complex. However, by using Lemma \([11]\) it turns out that \((14, 21, 14, 6)\) cannot be the \( f \)-vector of an acyclic simplicial complex.

**Theorem 1.9.** If \( d \) is even, then the Betti sequence of \( I_{\Delta(C(v, d))} \) is the \( f \)-vector of a simplicial complex.

**Proof.** Let \( d = 2d' \) and \( \beta(I_{\Delta(C(v, d))}) = (\beta_0, \ldots, \beta_{v-2d'-1}) \). It follows from \([22]\) that

\[
\beta_i = \left( \frac{v - d' - 1}{d' + i + 1} \right) \left( \frac{d' + i}{d'} \right) + \left( \frac{v - d' - 1}{i} \right) \left( \frac{v - d' - i - 2}{d'} \right)
\]
By using the formula (1) it follows easily that
\[ d \]
\[ f \]
\[ ♭ \]
and suppose that there exists a simplicial complex \( Γ \) such that \( f(Γ(v - 1)) = β(Γ(Δ(C(v, 2)))) \).

Let \( x_0 \) be a new vertex and write \( \{x_0\} * Γ(v - 1) \) for the cone of \( Γ(v - 1) \) over \( x_0 \).

In other words,
\[ \{x_0\} * Γ(v - 1) = \{\{x_0\} \cup F : F ∈ Γ(v - 1)\} \cup Γ(v - 1). \]

By using the formula (1) it follows easily that
\[
β_i(Γ(Δ(C(v, 2)))) = \begin{cases} 
  f_0(\{x_0\} * Γ(v - 1)) + v - 3, & i = 0, \\
  f_i(\{x_0\} * Γ(v - 1)) + \binom{v - 2}{i + 1}, & 1 ≤ i ≤ v - 5, \\
  f_{v-4}(\{x_0\} * Γ(v - 1)) + v - 3, & i = v - 4, \\
  1, & i = v - 3.
\end{cases}
\]

Let \( x_1, \ldots, x_{v-3} \) be new vertices and \( Γ' \) the simplicial complex consisting of all subsets of \( \{x_0, x_1, \ldots, x_{v-3}\} \). We then introduce the simplicial complex \( Γ(ν) \) by setting
\[
Γ(ν) = (\{x_0\} * Γ(ν - 1)) \cup (Γ' \setminus \{\{x_0, x_1, \ldots, x_{v-3}\}, \{x_1, \ldots, x_{v-3}\}\}).
\]

Since \( f_{v-3}(\{x_0\} * Γ(ν - 1)) = f_{v-4}(Γ(ν - 1)) = 1 \), one has \( β_i(I_Δ(C(v, 2))) = f_i(Γ(ν)) \) for all \( i \), as desired.

Next, let \( d' > 1 \). Again, we show that, by using induction on \( v \), the Betti sequence \( β(Γ(Δ(C(v,d)))) \) is the \( f \)-vector of a simplicial complex. When \( v = d + 2 \), the Betti sequence of \( I_Δ(C(v,d)) \) is \( (2, 1) \), which is the \( f \)-vector of a 1-simplex.

Let \( v > d + 2 \) and suppose that there exists a simplicial complex \( Γ^2 = Γ(ν - 1, d) \) such that \( f(Γ^2) = β(Γ(Δ(C(v-1,d)))) \). On the other hand, since we are working on induction on \( d' \), it follows that there exists a simplicial complex \( Γ^3 = Γ(v-2, d-2) \) such that \( f(Γ^3) = β(Γ(Δ(C(v-2,d-2)))) \). We will assume that the vertex set of \( Γ^2 \) and that of \( Γ^3 \) are disjoint.

Let \( x_0 \) be a new vertex. Again, by using the formula (1) it follows easily that
\[
β_i(Γ(Δ(C(v,d)))) = \begin{cases} 
  f_0(\{x_0\} * Γ^2) + f_0(Γ^3) - 1, & i = 0, \\
  f_i(\{x_0\} * Γ^2) + f_i(Γ^3), & 1 ≤ i ≤ v - d - 3, \\
  f_{v-d-2}(\{x_0\} * Γ^2) + f_{v-d-2}(Γ^3) - 1, & i = v - d - 2, \\
  1, & i = v - d - 1.
\end{cases}
\]
In other words,
\[
\beta_i(I_{\Delta(C(v,d))}) = f_i(\{x_0\} \ast \Gamma^d) + f_i(\Gamma^v) - 1, \quad i = 0, v - d - 2, v - d - 1.
\]

Let \( y_0 \) be a vertex of \( \Gamma^v \). Let \( F \in \Gamma^v \) be the unique face of dimension \( v - d - 1 \) and \( G \) a maximal proper subset of \( F \). Then the simplicial complex
\[
\Gamma(v,d) = (\{y_0\} \ast \Gamma^d) \cup (\Gamma^v \setminus \{F,G\})
\]
satisfies \( \beta_i(I_{\Delta(C(v,d))}) = f_i(\Gamma(v,d)) \) for all \( i \), as desired. \( \square \)

**Conjecture 1.10.** The Betti sequence arising from a pure resolution of a monomial ideal is the \( f \)-vector of a simplicial complex.

### 2. Edge ideals of chordal graphs

Let \( V \) be the vertex set and \( G \) a finite graph on \( V \) having no loop and no multiple edge. Let \( E(G) \) denote the edge set of \( G \). We write \( S = K[\{x : x \in V\}] \) for the polynomial ring in \( |V| \) variables over a field \( K \) with each \( \deg x = 1 \). The *edge ideal* of \( G \) is the ideal \( I(G) \) of \( S \) generated by those monomials \( xy \) with \( \{x, y\} \in E(G) \).

We cannot escape from the temptation to ask if the Betti sequence of the edge ideal of a finite graph can be the \( f \)-vector of a simplicial complex. Unfortunately, as was stated explicitly in Introduction, the Betti sequence of the edge ideal of the cycle of length 6 cannot be the \( f \)-vector of a simplicial complex. However, it turns out to be true that the Betti sequence of the edge ideal of a finite chordal graph can be the \( f \)-vector of a simplicial complex (Theorem 2.1). Recall that a finite graph \( G \) is *chordal* if each cycle of \( G \) of length \( > 3 \) has a chord.

**Theorem 2.1.** Given an arbitrary chordal graph \( G \), there exists a simplicial complex \( \Delta \) whose \( f \)-vector \( f(\Delta) \) coincides with the Betti sequence \( \beta(I(G)) \) of the edge ideal \( I(G) \).

The recursive-type formula ([12, Theorem 5.8]) due to Hà and Van Tuyl will be indispensable to achieve the proof of Theorem 2.1.

Let, as before, \( G \) be a finite graph on \( V \) and \( E(G) \) its edge set. Given a subset \( W \subset V \), the *restriction* \( G \) to \( W \) is the finite graph \( G_W \) on \( W \) whose edges are those edges \( e \in E(G) \) with \( e \subset W \). The *neighborhood* of a vertex \( v \) of \( G \) is the subset \( N(v) \subset V \) consisting of those vertices \( u \) of \( G \) with \( \{u, v\} \in E(G) \). We write \( G \setminus e \), where \( e \in E(G) \), for the subgraph of \( G \) which is obtained by removing \( e \) from \( G \).

The *distance* \( \text{dist}_G(e, e') \) of two edges \( e, e' \in E(G) \) is the smallest integer \( \ell \geq 0 \) for which there is a sequence \( e = e_0, e_1, \ldots, e_\ell = e' \), where each \( e_i \in E(G) \), with \( e_i \cap e_{i+1} \neq \emptyset \) for all \( i \).

A *complete graph* on \( V \) is the finite graph on \( V \) such that \( \{x, y\} \) is its edge for all \( x, y \in V \) with \( x \neq y \).
Lemma 2.2 (Hà and Van Tuyl). Let $G$ be a chordal graph and $E(G)$ its edge set. Suppose that $e = \{u,v\}$ is an edge of $G$ such that $G_{N(v)}$ is a complete graph. Let $t = |N(u) \setminus \{v\}|$ and $G'$ the subgraph of $G$ with
\[
E(G') = \{e' \in E(G) : \text{dist}_G(e,e') \geq 3\}.
\]
Then each of $G \setminus e$ and $G'$ is chordal and
\[
(2) \quad \beta_i(I(G)) = \beta_i(I(G \setminus e)) + \sum_{\ell=0}^{i} \binom{t}{\ell} \beta_{i-\ell-1}(I(G'))
\]
for all $i \geq 0$, where $\beta_{-1}(I(G')) = 1$.

Remark 2.3. (a) In Dirac [9] it is proved that a finite graph $G$ is chordal if and only if $G$ possesses a “perfect elimination ordering.” This fact guarantees the existence of a vertex $v$ of a chordal graph $G$ such that $N(v)$ is a complete graph.

(b) Let $N(u) = \{v, x_1, \ldots, x_t\}$. Since $N(v)$ is complete, if $\{v,z\} \in E(G)$, then $\{u,z\} \in E(G)$. In particular, if $z \notin \{u,v, x_1, \ldots, x_t\}$, then $\{v,z\} \notin E(G)$. Thus an edge $e'$ of $G$ satisfies $\text{dist}_G(e,e') \leq 2$ if and only if $e' \cap \{u,v, x_1, \ldots, x_t\} \neq \emptyset$. Let $W$ denote the subset of $V$ consisting of those vertices $z$ such that there is $e' \in E(G')$ with $z \in e'$. In particular $W \subset V \setminus \{u,v, x_1, \ldots, x_t\}$. Obviously $G' \subset G_W$. Since none of the vertices $u,v,x_1,x_2,\ldots,x_t$ belongs to $W$, one has $\text{dist}_G(e,e') \geq 3$ for $e' \in E(G_W)$. Hence $G_W \subset G'$. Thus $G' = G_W$.

Example 2.4. Let $G$ be the chordal graph on $\{y_1, \ldots, y_6\}$ drawn below. Let $v = y_1$, $u = y_2$, and $e = \{u,v\}$. Then $G_{N(v)}$ is a complete graph, $N(u) \setminus \{v\} = \{y_3, y_4, y_5\}$, $t = 3$ and $G' = G_{\{y_6, y_7, y_8\}}$.

![Diagram](attachment:image.png)

The Betti sequences of $I(G)$, $I(G \setminus e)$ and $I(G')$ are
\[
\beta(I(G)) = (13, 36, 47, 34, 13, 2),
\beta(I(G \setminus e)) = (12, 30, 33, 18, 4), \quad \beta(I(G')) = (3, 2).
\]

We can easily check that these Betti sequences satisfy the formula (2) due to Hà and Van Tuyt. For example, since $47 = 33 + 2 \cdot \binom{3}{0} + 3 \cdot \binom{3}{1} + 1 \cdot \binom{3}{2}$, one has
\[
\beta_2(I(G)) = \beta_2(I(G \setminus e)) + \binom{3}{0} \beta_1(I(G')) + \binom{3}{1} \beta_0(I(G')) + \binom{3}{2} \beta_{-1}(I(G')).
\]
Lemma 2.5. Let $G$ be an arbitrary graph on $V = V(G)$ and $W$ a subset of $V$. Then one has
\[ \beta_i(I(G)) \geq \beta_i(I(G_W)) \]
for all $i$.

Proof. Since $I(G)$ and $I(G_W)$ are squarefree monomial ideals, there exist simplicial complexes $\Delta$ on $V$ and $\Delta'$ on $W$ such that $I_\Delta = I(G)$ and $I_{\Delta'} = I(G_W)$. Hochster’s formula \cite[Corollary 4.9, p. 64]{21} says that
\[
\beta_i(I(G)) = \beta_i(I_\Delta) = \sum_{U \subseteq V} \dim_K \tilde{H}_{|U|-i-2}(\Delta_U; K),
\]
\[
\beta_i(I(G_W)) = \beta_i(I_{\Delta'}) = \sum_{U \subseteq W} \dim_K \tilde{H}_{|U|-i-2}(\Delta'_U; K).
\]
What we must prove is that $\Delta_U = \Delta'_U$ whenever $U \subseteq W$.

Let $F \in \Delta_U$. Then, for all $\{x, y\} \subseteq F$, one has $\{x, y\} \not\in E(G)$. In particular $\{x, y\} \not\in E(G_W)$. Thus $F \in \Delta'$ and $F \in \Delta'$. Conversely, let $F \in \Delta'_U$. Then, for all $\{x, y\} \subseteq F$, one has $\{x, y\} \not\in E(G)$. Hence $F \in \Delta$ and $F \in \Delta_U$, as desired. $\Box$

Lemma 2.6. Let $S$ be a polynomial ring over a field $K$.

(a) Let $I \subseteq S$ be a squarefree monomial ideal and $x$ a variable of $S$. Then
\[
\beta_i(I) \geq \beta_i(I : x)
\]
for all $i$.

(b) Let $I$ and $J$ be monomial ideals of $S$ and $G(I)$ (resp. $G(J)$) the minimal system of monomial generators of $I$ (resp. $J$). Suppose that $\text{supp}(u) \cap \text{supp}(v) = \emptyset$ for all $u \in G(I)$ and for all $v \in G(J)$, where $\text{supp}(u)$ is the set of variables $x \in V$ which divides $u$. Then, for all $i$, one has
\[
\beta_i(S/(I + J)) = \sum_{m=0}^{i} \beta_{i-m}(S/I) \beta_m(S/J).
\]

Proof. (a) Let $S = K[x_1, x_2, \ldots, x_n]$, $x = x_1$, and $R = K[x_2, \ldots, x_n]$. Since the variable $x$ does not appear in the minimal system of monomial generators of $I : x$, it follows that $J = (I : x) \cap R$ has the same minimal set of monomial generators as that of $I : x$. Hence $\beta_i^R(J) = \beta_i^S(I : x)$, where $\beta_i^R(J)$ is the $i$th total Betti number of $J \subseteq R$ and $\beta_i^S(I : x)$ is that of $I : x \subseteq S$. We claim $\beta_i^S(I) \geq \beta_i^R(J)$.

Let $F_\bullet$ be a minimal graded free resolution of $S/I$ on $S$. Since $x - 1$ is a non-zero divisor of $S/I$, by using \cite[Proposition 1.1.5]{6}, it follows that $F_\bullet \otimes S/(x - 1)$ is a free resolution of $S/I \otimes S/(x - 1)$ on $S \otimes S/(x - 1)$. Since
\[
R/J \cong S/I \otimes_S S/(x - 1), \quad R \cong S/(x - 1) \cong S \otimes S/(x - 1),
\]
$F_\bullet \otimes S/(x - 1)$ is a free resolution of $R/J$ on $R$.\[9\]
(b) Let $F_\bullet$ (resp. $G_\bullet$) be a minimal graded free resolution of $S/I$ (resp. $S/J$) and $T_\bullet^{(I)}$ (resp. $T_\bullet^{(J)}$) the Taylor resolution of $S/I$ (resp. $S/J$). Then $T_\bullet^{(I)} \otimes T_\bullet^{(J)}$ is isomorphic to the Taylor resolution of $S/(I + J)$. Thus

$$H_i(F_\bullet \otimes G_\bullet) \cong \text{Tor}_i^S(S/I, S/J) \cong H_i(T_\bullet^{(I)} \otimes T_\bullet^{(J)}) = 0.$$ 

Hence $F_\bullet \otimes G_\bullet$ is a graded free resolution of $S/I \otimes S/J \cong S/(I + J)$. In particular $F_\bullet \otimes G_\bullet$ is minimal. □

**Lemma 2.7.** Let $G$ be an arbitrary graph on $V$ and let $W$ be a subset of $V$. Suppose that $G \setminus W$ contains edges

$$\{u, x_1\}, \{u, x_2\}, \ldots, \{u, x_t\},$$

where $t \geq 1$ is an integer and where $u, x_1, x_2, \ldots, x_t$ are distinct vertices of $G$. If $\{u, z\} \notin E(G)$ for all $z \in W$, then

$$\beta_i(I(G)) \geq \sum_{m=0}^{i+1} \binom{t}{m} \beta_{i-m}(I(G_W)),$$

for all $i \geq 0$, where $\beta_{-1}(I(G_W)) = -1$.

**Proof.** Set $V' = \{u, x_1, \ldots, x_t\}$. Lemma 2.5 together with Lemma 2.6 (a) says that

$$\beta_i(I(G)) \geq \beta_i(I(G_{V' \cup W})) \geq \beta_i(I(G_{V' \cup W}) : u).$$

Since $\{u, z\} \notin E(G)$ for all $z \in W$, it follows that

$$I(G_{V' \cup W}) : u = I(G_W) + (x_1, x_2, \ldots, x_t).$$

Then, since $V' \cap W = \emptyset$, by using Lemma 2.6 (b), one has

$$\beta_i(I(G_W) + (x_1, x_2, \ldots, x_t)) = \beta_{i+1}(S/(I(G_W) + (x_1, x_2, \ldots, x_t)))$$

$$= \sum_{m=0}^{i+1} \beta_{i+1-m}(S/I(G_W)) \beta_m(S/(x_1, x_2, \ldots, x_t))$$

$$= \sum_{m=0}^{i+1} \binom{t}{m} \beta_{i-m}(I(G_W)),$$

as required. □

Let $\Delta$ be a simplicial complex on the vertex set $V$ and let $x$ be a new vertex. The cone of $\Delta$ over $x$ is the simplicial complex

$$\text{cone}(\Delta) = \{\{x\} \cup F : F \in \Delta\} \cup \Delta$$

on $V \cup \{x\}$. Moreover, by setting $\text{cone}^0(\Delta) = \Delta$, the $t$th cone of $\Delta$ is defined recursively by

$$\text{cone}^t(\Delta) = \text{cone}(\text{cone}^{t-1}(\Delta)).$$
It follows that
\[ f_i(\text{cone'}(\Delta)) = \sum_{\ell=0}^{i+1} \binom{t}{\ell} f_{i-\ell}(\Delta) \]
for all \(i\).

We are now in the position to give a proof of Theorem 2.1. Recall that the Stanley–Reisner ideal \(I_\Delta \subset S\) is squarefree lexsegment ([1]) if, for all monomials \(u\) and \(v\) of \(S\) with \(\deg u = \deg v\) and with \(v <_{\text{lex}} u\) such that \(v \in I_\Delta\), one has \(u \in I_\Delta\), where \(<_{\text{lex}}\) is the lexicographic order induced by a (fixed) ordering of the variables of \(S\). Given a simplicial complex \(\Delta\), there is a unique simplicial complex \(\Delta_{\text{lex}}\) such that \(I_{\Delta_{\text{lex}}}\) is squarefree lexsegment with \(f(\Delta) = f(\Delta_{\text{lex}})\).

**Proof of Theorem 2.1.** Our proof will proceed by using induction on the number of edges of \(G\). If \(G\) possesses only one edge \(\{x, y\}\), then \(I(G) = (xy)\) and
\[
\beta_i(I(G)) = \begin{cases} 
1, & i = 0, \\
0, & i \neq 0.
\end{cases}
\]
Thus its Betti sequence is equal to the \(f\)-vector of a 0-simplex.

Now, suppose that \(G\) possesses at least two edges and that, for an arbitrary chordal graph \(\Gamma\) with \(|E(\Gamma)| < |E(G)|\), the Betti sequence \(\beta(I(\Gamma))\) is the \(f\)-vector \(f(\Delta_{\Gamma})\) of a simplicial complex \(\Delta_{\Gamma}\).

Let \(e = \{u, v\}\) be an edge of \(G\) such that \(G_{N(v)}\) is complete. Work with the same notation as in Lemma 2.2 and in Remark 2.3 (b). One has
\[
\beta_i(I(G)) = \beta_i(I(G \setminus e)) + \sum_{\ell=0}^{i} \binom{t}{\ell} \beta_{i-\ell-1}(I(W_G)),
\]
Since each of \(G \setminus e\) and \(W_G\) is a subgraph of \(G\) with \(e \notin E(G \setminus e)\) and \(e \notin E(W_G)\), the hypothesis of induction guarantees the existence of simplicial complexes \(\Delta_{G \setminus e}\) and \(\Delta_W\) such that
\[
f_i(\Delta_{G \setminus e}) = \beta_i(I(G \setminus e)), \quad f_i(\Delta_W) = \beta_i(I(W_G)).
\]
Thus what we must prove is the existence of a simplicial complex \(\Delta\) with
\[
f_i(\Delta) = f_i(\Delta_{G \setminus e}) + \sum_{\ell=0}^{i} \binom{t}{\ell} f_{i-\ell-1}(\Delta_W).
\]
(3)

It follows from Lemma 2.7 that
\[
\beta_i(I(G \setminus e)) \geq \sum_{m=0}^{i+1} \binom{t}{m} \beta_{i-m}(I(W_G)).
\]

In other words,
\[
f_i(\Delta_{G \setminus e}) \geq \sum_{m=0}^{i+1} \binom{t}{m} f_{i-m}(\Delta_W) = f_i(\text{cone'}(\Delta_W)).
\]
Thus, by choosing $\Delta_{G\setminus e}$ for which $I_{\Delta_{G\setminus e}}$ is squarefree lexsegment, we assume that $\Delta_{G\setminus e}$ contains a subcomplex $\Delta'$ whose $f$-vector coincides with that of cone($\Delta_{G_W}$).

We introduce the simplicial complex $\Delta$ by setting

$$\Delta = \Delta_{G\setminus E} \cup \text{cone}(\Delta'),$$

where the new vertex of cone($\Delta'$) cannot be a vertex of $\Delta_{G\setminus E}$. Then

$$f_i(\Delta) - f_i(\Delta_{G\setminus E}) = f_i(\text{cone}(\Delta')) - f_i(\Delta') = f_{i-1}(\Delta') = f_{i-1}(\text{cone}(\Delta_{G_W})) = \sum_{\ell=0}^i \binom{i}{\ell} f_{i-\ell-1}(\Delta_{G_W}).$$

Thus the simplicial complex satisfies the equality (3), as desired. 

\[ \square \]

3. Gorenstein Monomial Ideals

We now turn to the discussion on Betti sequences of Gorenstein monomial ideals. Let, as before, $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over a field $K$ with each deg $x_i = 1$. Recall that a homogeneous ideal $I \subset S$ is Gorenstein if $S/I$ is a Gorenstein ring. If $I \subset S$ is Gorenstein, then its Betti sequence $\beta(I) = (\beta_0(I), \beta_1(I), \ldots, \beta_p(I))$ is symmetric, that is, $\beta_i(I) = \beta_{p-i}(I)$ for all $i$, where $p = \text{proj dim}(I)$ and where $\beta_{-1}(I) = 1$.

Let $I \subset S$ be a Gorenstein monomial ideal with proj dim$(I) = p$. If $p = 1$, then $\beta(I) = (2, 1)$ by the Hilbert–Burch theorem [9, Theorem 1.4.17]. If $p = 2$, then there exists an odd integer $m \geq 3$ such that $\beta(I) = (m, m, 1)$ by the structure theorem due to Buchsbaum and Eisenbud ([6, Theorem 3.4.1]). In fact, these facts characterize the Betti numbers of Gorenstein (monomial) ideals with proj dim$(I) \leq 2$. For example, to prove the sufficiency, let $I$ be the Stanley–Reisner ideal of the boundary complex of the cyclic $2m$-polytope with $2m + 3$ vertices. Then $I$ is a Gorenstein ideal with $\beta(I) = (2m + 3, 2m + 3, 1)$ for all $m \geq 1$ by the formula (4).

Let $p = 3$. Let $I \subset S$ be a Gorenstein monomial ideal with proj dim$(I) = 3$. Since $(\beta_{-1}(I), \beta_0(I), \beta_1(I), \beta_2(I), \beta_3(I))$, where $\beta_{-1}(I) = 1$, is symmetric and since $\sum_{i=1}^{3}(-1)^i\beta_i(I) = 0$, it follows that there exists an integer $m$ such that $\beta(I) = (m + 1, 2m, m + 1, 1)$. Since $I$ is a monomial ideal, the Taylor resolution of $I$ says that $m = \beta_0(I) - 1 \geq \text{proj dim}(I) = 3$. Since $(m, m, 1)$ is the $f$-vector of a simplicial complex for $m \geq 3$, it follows from Lemma 1.1 that $\beta(I)$ is the $f$-vector of an acyclic simplicial complex.

Example 3.1. Let $I = (x_1x_4, x_1x_5, x_2x_6, x_3x_7, x_4x_6, x_4x_7, x_2x_3x_5)$. Then $I$ is Gorenstein and $\beta(I) = (7, 12, 7, 1) = (6 + 1, 2 \times 6, 6 + 1, 1)$. 

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More precisely, we can characterize the Betti numbers of Gorenstein monomial ideals $I$ with $\text{proj dim}(I) = 3$. Recall that a monomial ideal $I \subset S$ is strongly stable if, for all monomials $u \in I$ and for all $j < i$ such that $x_i$ divides $u$, one has $ux_j/x_i \in I$.

**Theorem 3.2.** Let $\beta = (m + 1, 2m, m + 1, 1)$, where $m$ is an integer with $m \geq 3$. Then there exists a Gorenstein monomial ideal $I$ of a polynomial ring with $\beta(I) = \beta$ if and only if $m = 3$ or $m \geq 5$.

**Proof.** ("If") Let $m \geq 3$ be odd. Then there exists a Gorenstein monomial ideal $J \subset S$ with $\beta(J) = (m, m, 1)$. Let $y$ be a new variable and $S' = S[y]$. Then the ideal $I = J + (y)$ is a Gorenstein monomial ideal with $\text{proj dim}(I) = 3$ and $\beta_0(I) = m + 1$.

Let $m \geq 6$ be even. Example 3.1 yields an example of $m = 6$. Now, let $m = 2k + 6 \geq 8$ be even. Given a strongly stable ideal $J \subset R = K[x_1, \ldots, x_p]$ such that $R/J$ is of finite length, it follows from [18, Theorem 9.6] and [19, Theorem 5.3] that there exists a Gorenstein squarefree monomial ideal $I_{(J)}$ for which $\beta_i(S/I_{(J)}) = \beta_i(R/J) + \beta_{p+1-i}(R/J)$ for all $i$. Let $J$ be the strongly stable ideal

$$J = (x_1^2, x_1x_2, x_1x_3, x_2^{k+1}, x_2x_3, \ldots, x_3^{k+1}) \subset K[x_1, x_2, x_3].$$

Eliahou–Kervaire formula says that $\beta_0(I_{(J)}) = \beta_0(J) + \beta_2(J) = 2k + 7$, as required.

("Only If") We show, in general, that if $I \subset S$ is a Gorenstein monomial ideal with $\text{proj dim}(I) = p - 1 \geq 3$, then $\beta_0(I) \neq p + 1$. Suppose, on the contrary, that there exists a Gorenstein monomial ideal with $\text{proj dim}(I) = p - 1 \geq 3$ and $\beta_0(I) = p + 1$. By taking the polarization ([16, Lemma 4.2.16]) of $I$, we assume that $I$ is squarefree. Without loss of generality, we assume that no variable $x_i$ is contained in $I$. Let $\Delta$ (resp. $\Delta'$) denote the simplicial complex whose Stanley–Reisner ideal is $I$ (resp. the $I : x_i$). Then $\Delta'$ is the star ([16, Definition 5.3.4]) of $\Delta$ of the face $\{i\}$. Hence $I : x_i$ is a Gorenstein ideal with $\text{dim}(S/I) = \text{dim}(S/(I : x_i))$. In particular $\text{proj dim}(I) = \text{proj dim}(I : x_i)$. Thus, in case of $\beta_0(I) = \beta_0(I : x_i)$, we replace $I$ with $I : x_i$. Hence, for each variable $x_k$ which appears in the minimal system of monomial generators of $I$, we assume that $\beta_0(I : x_k) < p + 1$. On the other hand, since $\text{proj dim}(I) = \text{proj dim}(I : x_k) \leq \beta_0(I : x_k) - 1$, it follows that $\beta_0(I : x_k) = p$ and $I : x_k$ is a complete intersection.

Let $G(I) = \{u_1, \ldots, u_{p+1}\}$ be the minimal system of monomial generators of $I$. Say, $x_1$ divides $u_1$ and, since $\beta_0(I : x_1) = p$, $u_1/x_1$ divides $u_{p+1}$. Let $u_1 = x_1x_F$ and $u_{p+1} = x_Fx_G$, where $x_F = \prod_{i \in F} x_i$ with $F \subset [n]$. Then

$$I : x_1 = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_p),$$

where $\tilde{u}_k = u_k/x_1$ (resp. $\tilde{u}_k = u_k$) if $x_1$ divides (resp. does not divide) $u_k$. In particular $\tilde{u}_1 = x_F$. Since $I : x_1$ is a complete intersection, it follows that

$$\text{supp}(\tilde{u}_s) \cap \text{supp}(\tilde{u}_t) = \emptyset.$$
if \( s \neq t \), where \( \text{supp}(\tilde{u}_s) \) stands for the set of variables \( x_k \) which divides \( \tilde{u}_s \). If there is \( 2 \leq k \leq p \) with \( u_k = \tilde{u}_k \), then, since \( \beta_0(I : x_j) = p \) for all \( x_j \in \text{supp}(u_k) \), it follows from (4) that \( u_k \) must divide \( u_{k+1} \), a contradiction. Thus \( \tilde{u}_k = u_k / x_1 \) for each \( 1 \leq k \leq p \). Let \( j \in F \). Then, by (4), \( x_j \not\in \text{supp}(u_k) \) for \( k = 2, \ldots, p \). Since \( \beta_0(I : x_j) = p \), there is \( k \) with \( 2 \leq k \leq p \) such that either \( u_1 / x_j \) or \( u_{p+1} / x_j \) must divide \( u_k \). If \( u_1 / x_j \) divides \( u_k \), then \( u_1 = x_1 x_j \) by (4). Thus \( \beta_0(I : x_j) = p = 2 \), a contradiction. If \( u_{p+1} / x_j \) divides \( u_k \), then, again by (4), one has \( u_1 = x_1 x_F = x_1 x_j \) and \( p = 2 \), a contradiction. \[ \square \]

The technique appearing in the “If” part of the proof of Theorem 3.2 together with the result shown in the “Only If” part of Theorem 3.2 yields the following

**Corollary 3.3.** Fix integers \( m \geq 4 \) and \( p \geq 3 \). Then there exists a Gorenstein monomial ideal \( I \) of \( K[x_1, \ldots, x_n] \), where \( n \) is enough large, with \( \beta_0(I) = m \) and \( \text{proj dim}(I) = p \) if and only if \( m \geq p + 1 \) with \( m \neq p + 2 \).

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