Abstract. Consider a system \((X, F, \mu, T)\), bounded functions \(f_1, f_2 \in L^\infty(\mu)\) and \(a, b \in \mathbb{Z}\). We show that there exists a set of full-measure \(X_{f_1, f_2}\) in \(X\) such that for all \(x \in X_{f_1, f_2}\) and for every nilsequence \(b_n\), the averages
\[
\frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x)b_n
\]
converge.

1. Introduction

Throughout this paper we denote by a system a probability measure space \((X, F, \mu, T)\) where \(T\) is a measure preserving transformation on it. Without loss of generality we can assume that the system is standard meaning that \(X\) is a compact metric space, \(T\) a homeomorphism and \(F\) the \(\sigma\) field of the Borelian subsets of \(X\). The purpose of this note is to answer a question raised by B. Weiss relative to the extensions of J. Bourgain double recurrence theorem obtained recently in [2], [4] and in [3]. We start by recalling the definition of a nilsequence as given in [12].

Definition 1.1. Let \(a_n\) be a sequence of complex numbers. This sequence is a \(k\) – step basic nilsequence if it can be written as \(F(g^n \Gamma)\), where \(F \in C(X)\), \(X = G/\Gamma\), \(G\) is a \(k\) step nilpotent group, \(\Gamma\) a discrete cocompact subgroup, and \(g \in G\).

Definition 1.2. A sequence \(a_n\) is a \(k\)-step nilsequence if it is a uniform limit of \(k\)-step basic nilsequence.

The interest in nilsequences appears in several papers linked to problems in number theory see for instance the papers by B. Green and T. Tao, [8], [9], [10] and the paper by V. Bergelson, B. Host and B. Kra [6]. Our interest in these sequences comes from the simple observation that sequences of the form \(e^{2\pi in}\) or \(e^{P(n)}\) where \(P\) is a real polynomial with integer coefficients, are nilsequences and that these weights were used to obtain Wiener Wintner extension of J. Bourgain result in [2] and [4]. Our main result is the following

Theorem 1.3. Let \((X, F, \mu, T)\) be a system and \(f_1, f_2\) bounded measurable functions. There exists a set of full measure \(X_{f_1, f_2}\) such that for all \(x \in X_{f_1, f_2}\) and for every nilsequence \(b_n\) the averages
\[
\frac{1}{N} \sum_{n=1}^{N} f_1(T^{an}x) f_2(T^{bn}x)b_n
\]
Thus the answer to B. Weiss question is positive.

2. Proof of the main theorem

The main ingredients in the proof are

(1) Some properties of nilsequences given in [12]

(2) Elements in the proof of the pointwise convergence of the averages along the cubes established in [1]

2.1. Preliminaries. We extract from [12] some properties of nilsequences that we will be using. First we need some notations. Let $a_n$ be a bounded sequence of real numbers. For every $k \geq 1$ points of $\mathbb{Z}^k$ are written $h = (h_1, h_2, ..., h_k)$. For $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_k) \in \{0, 1\}^k$ and $h = (h_1, h_2, ..., h_k) \in \mathbb{Z}^k$, we define $|\epsilon| = \sum_{l=1}^{k} \epsilon_l$ and we denote by $\epsilon . h$ the dot product $\sum_{l=1}^{k} \epsilon_l h_l$. The next lemmas list the properties we seek.

**Lemma 2.1.** Let $a_n$ and $b_n$ two nilsequences of order respectively $k_1$ and $k_2$ then

(1) the sequence $c_n = a_n b_n$ is a nilsequence of order $\max\{k_1, k_2\}$

(2) for each $k$ nilsequence $\alpha_n$ the averages $\frac{1}{N} \sum_{n=0}^{N-1} a_n$ converge.

**Proof.** The first part follows immediately from the nilpotent structure of the product of two homogeneous spaces generating the sequences $a_n$ and $b_n$. The second part is a consequence of the unique ergodicity of the system associated with the $k$ step nilsequence $\alpha_n$. \hfill \Box

**Lemma 2.2.** Let us fix $k \in \mathbb{N}$. Assume that the real bounded sequence $a_n$ is such that

$$c_h = \lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{\epsilon \in \{0,1\}^k} a_{n+h.\epsilon}$$

exists. Then

(1) $\lim_{H} \frac{1}{H} \sum_{h_1, ..., h_k=0}^{H-1} c_h$ exists and is nonnegative. Therefore $\|a\|_k = \left( \lim_{H} \frac{1}{H} \sum_{h_1, ..., h_k=0}^{H-1} c_h \right)^{1/2}$ is well defined.

(2) if $\|a\|_k = 0$ then for any $k - 1$ step nilsequence $b_n$ we have $\lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} a_n b_n = 0$.

**Proof.** It is a consequence of the Proposition 2.2 and Corollary 2.14 in [12]. In that paper the authors defined what they call local seminorms with respect to a sequence of intervals $I_j$ in $\mathbb{Z}$ with length tending to infinity. In this paper we only focus on the sequence of intervals in $\mathbb{N}$ of the form $[0, N - 1]$. So the seminorm $\|a\|_k$ in our lemma corresponds to the local semi norm
\[ \|a\|_{I,k} \] with \( I = (I_j) \) where \( I_j = [0, j - 1] \). Proposition 2.2 in [12] says that part (1) is true, while Corollary 2.14 from the same reference tells us that \( \limsup_N \left| \frac{1}{N} \sum_{n=0}^{N-1} a_n b_n \right| \leq c.0 + \delta \|a\|_\infty = \delta \|a\|_\infty \), for any \( \delta > 0 \). From this, part 2 of the lemma follows.

**Remarks** In order to eliminate possible confusion between the local seminorms \( \|a\|_k \) and the similar notation for the Gowers Host Kra seminorms we will denote by \( \|a\|_k \) the local seminorm and by \( ||f||_k \) the GHK seminorms of a function \( f \).

### 2.2. Joinings

First we need a lemma allowing to define the limit in Bourgain double recurrence theorem as the integral of the functions with respect to a joining.

**Lemma 2.3.** Given a standard dynamical system \((X, F, \mu, T)\) and \( a, b \in \mathbb{Z}, a \neq b \), for \( \mu \) a.e. \( x \) there exists a joining \( \mu_x \) defined on \(((X \times X), F^2)\) which is \( T^a \times T^b \) invariant such that for any continuous function \( f \otimes g \) we have

1. \( \lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^a n x) g(T^b n x) = \int f \otimes g d\mu_x \)
2. There exists a joining \( \omega \) on \(((X \times X), F^2)\) such that

\[
\int \lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^a n x) g(T^b n x) d\mu = \int f \otimes g d\mu_x (x) = \int f \otimes g d\omega.
\]

3. Furthermore if \( \mathcal{I} \) denotes the \( \sigma \) field of the \( T^b-a \) invariant subsets of \( X \) there exists a kernel \( K(x, y) \) such that we have

\[
\int f \otimes g d\omega = \int \mathbb{E}[f|\mathcal{I}] \mathbb{E}[g|\mathcal{I}] d\mu = \int \int K(x, y) f(x) f(y) d\mu \otimes \mu
\]

**Proof.** By using a countably dense set of continuous functions \( F_j \) on \( X^2 \) we can find a set \( \tilde{X} \) of full measure in \( X \) on which the averages \( \frac{1}{N} \sum_{n=0}^{N-1} F_j(T^a n x, T^b n x) \) converge for each \( j \). By approximation we can conclude that on the same set \( \tilde{X} \) we have the convergence of these averages for each continuous function \( F \) on \( X^2 \). By Riesz representation theorem we can find a measure \( \mu_x \) on \((X \times X, F^2)\) such that

\[
\lim_N \frac{1}{N} \sum_{n=0}^{N-1} F(T^a n x, T^b n x) = \int F d\mu_x.
\]

For the particular case where \( F = f \otimes g \) we derive the equality

\[
\int \lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^a n x) g(T^b n x) d\mu = \int f \otimes g d\mu_x d\mu (x)
\]
by integration with respect to the measure $\mu$. It remains to identify the measure $\omega$. Simple computations, using the measure preserving property of the map $T'$, show that

$$\int \lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^{an} x) g(T^{bn} x) d\mu = \int \mathbb{E}[f|\mathcal{I}] \mathbb{E}[g|\mathcal{I}] d\mu$$

where $\mathcal{I}$ is the $\sigma-$algebra of invariant subsets for the map $T^b-a$. In other words $\omega$ is the relatively independent joining over the $\sigma$ algebra $\mathcal{I}$. The last part of the theorem follows from Lemma 5.2 in [2].

2.3. Proving that $\| (g_1(T^{an} x) g_2(T^{bn} x))_n \|_k = 0$. Starting with two functions in $L^\infty(\mu)$ that we can assume to be bounded by one we can decompose them into the sum of their projections onto the Host-Kra-Ziegler factor [11], [14] $\mathcal{Z}_{k-1}$ and onto $\mathcal{Z}_{k-1}^\perp$. We denote by $g_1$ and $g_2$ the projections of these functions onto $\mathcal{Z}_{k-1}^\perp$. Our goal in this section is to prove the following lemma.

**Lemma 2.4.** With the notations of the previous lemma we have for $\mu$ a.e. $x \in X$,

$$\| (g_1(T^{an} x) g_2(T^{bn} x))_n \|_k = 0.$$

**Proof.** First we need to check that $c_b(x)$ exists for $a_n(x) = g_1(T^{an} x), g_2(T^{bn} x)$. This follows from J. Bourgain a.e. double recurrence result [7]. Indeed the quantity $\prod_{\epsilon \in \{0,1\}} a_{n+\epsilon}(x)$ can be written as the product of two functions $G^1_{h_1, h_2, \ldots, h_k}(T^{an} x) G^2_{h_1, h_2, \ldots, h_k}(T^{bn} x)$. Therefore for $\mu$ a.e. $x$ the limit of the averages

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} G^1_{h_1, h_2, \ldots, h_k}(T^{an} x) G^2_{h_1, h_2, \ldots, h_k}(T^{bn} x)$$

exists. To be more explicit and for the simplicity of the notation we can look at the case $k = 3$. The same ideas will give the proof for the case $k > 3$. The product $\prod_{\epsilon \in \{0,1\}} a_{n+\epsilon}(x)$ is equal to the product of

$$G^1_{h_1, h_2, h_3}(T^{an} x)$$

$$= g_1(T^{an} x) g_1(T^{a(n+h_1) x}) g_1(T^{a(n+h_2) x}) g_1(T^{a(n+h_3) x}) g_1(T^{a(n+(h_1+h_2)) x})$$

and

$$G^2_{h_1, h_2, h_3}(T^{bn} x)$$

$$= g_2(T^{bn} x) g_2(T^{b(n+h_1) x}) g_2(T^{b(n+h_2) x}) g_2(T^{b(n+h_3) x}) g_2(T^{b(n+(h_1+h_2)) x})$$

$$g_2(T^{b(n+(h_1+h_2)) x}) g_2(T^{b(n+(h_1+h_2+h_3)) x}) g_2(T^{b(n+(h_1+h_2+h_3)) x}).$$
By Lemma 2.2 we have

1. \( \lim_{H} \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} c_h(x) \geq 0 \)

2. \( \|g_1(T^{a_n}x)g_2(T^{b_n}x)\|_3 = \left( \lim_{H} \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} c_h(x) \right)^{1/3} \)

Our goal is to show that

\[ \limsup_{H} \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} c_h(x) \right) = 0. \]

This last equation would certainly suffice to prove Lemma 2.4. To establish (3) we will show that

\[ \int \limsup_{H} \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} c_h(x) \right) d\mu = 0. \]

To this end we use Lemma 2.3

\[ \int \limsup_{H} \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} c_h(x) \right) d\mu = \int \lim \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} c_h(x) \right) d\mu \]

\[ = \int \lim \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} \int G_{h_1,h_2,h_3}^1 \otimes G_{h_1,h_2,h_3}^2 d\mu_0 \right) d\mu(x) \]

\[ = \lim \int \int \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} G_{h_1,h_2,h_3}^1(x)G_{h_1,h_2,h_3}^2(x) d\omega \text{ by Lemma 2.3} \]

\[ = \lim \int \int K(x,y) \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} G_{h_1,h_2,h_3}^1(x)G_{h_1,h_2,h_3}^2(y) d\mu \otimes d\mu(x,y) \]

The quantities \( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} G_{h_1,h_2,h_3}^1(x)G_{h_1,h_2,h_3}^2(y) \) represent averages along cubes of order 3. The pointwise estimates obtained for these averages in Lemma 6 in [1] gives us the inequality

\[ \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} G_{h_1,h_2,h_3}^1(x)G_{h_1,h_2,h_3}^2(y) \right)^2 \]

\[ \leq C \frac{1}{H} \sum_{h_2=0}^{H-1} \sup_{t} \left| \frac{1}{H} \sum_{h_1=0}^{2(H-1)} g_1(T^{a_1}x)g_2(T^{b_1}x)g_1(T^{a(h_1+h_2)}x)g_2(T^{b(h_1+h_2)}x)e^{2\pi ih_1t} \right|^2 \]

\[ \leq 2C \frac{1}{H} \sum_{h_2=0}^{H-1} \sup_{t} \left| \frac{1}{2(H-1)} \sum_{h_1=0}^{2(H-1)} g_1(T^{a_1}x)g_2(T^{b_1}x)g_1(T^{a(h_1+h_2)}x)g_2(T^{b(h_1+h_2)}x)e^{2\pi ih_1t} \right|^2 \]

where \( C \) is an absolute constant. We would like to prove that under the assumption made on the functions \( g_1 \) and \( g_2 \), the last term converge to zero. To this end we use some of the estimates made in [1]. As \( x \) and \( y \) will be fixed throughout these estimates, to simplify the notations we
simply write \( \alpha_{h_1} = g_1(T^{ah_1}x)g_2(T^{bh_1}y) \) and so \( \alpha_{h_1+h_2} = g_1(T^{a(h_1+h_2)}x)g_2(T^{b(h_1+h_2)}y) \). We use the van der Corput lemma (see [13]) to derive that for \( (K+1)^2 < H \) we have

\[
\sup_t \left| \frac{1}{H} \sum_{h_1}^{H-1} \alpha_{h_1}e^{2\pi i h_1 t} \right|^2 \leq \frac{C}{K} + \frac{C}{K} \sum_{k=1}^{K} \left| \frac{1}{H} \sum_{h_2=0}^{H-1} \alpha_{h_1+k}e^{2\pi i h_2 t} \right| \leq \frac{C}{K} \sum_{k=1}^{K} \left( \frac{1}{H} \sum_{h_2=0}^{H-1} \frac{1}{H} \sum_{h_1=0}^{H-1} \alpha_{h_1+k}e^{2\pi i h_2 t} \right)^{1/2}.
\]

(by Cauchy Schwarz inequality)

Now we can apply part 2 of the remarks 3 in [11]. It gives us the following estimate

\[
\left( \frac{1}{H} \sum_{h_2=0}^{H-1} \left| \frac{1}{H} \sum_{h_1=0}^{H-1} \alpha_{h_1+k}e^{2\pi i h_2 t} \right|^2 \right)^{1/2} \leq C \sup_t \left| \frac{1}{H} \sum_{h_1=0}^{2(H-1)} \alpha_{h_1+k}e^{2\pi i h_1 t} \right|
\]

Going back to the functions \( g_1 \) and \( g_2 \) we have obtained the estimate

\[
\left( \frac{1}{H^2} \sum_{h_1,h_2,h_3=0}^{H-1} G^1_{h_1,h_2,h_3}(x) G^2_{h_1,h_2,h_3}(y) \right)^2 \leq 2C \sup_t \left| \frac{1}{2(H-1)} \sum_{h_1=0}^{2(H-1)} g_1(T^{ah_1}x)g_2(T^{bh_1}y)g_1(T^{a(h_1+h_2)}x)g_2(T^{b(h_1+h_2)}y)e^{2\pi i h_1 t} \right|^2 \leq \frac{C}{K} \sum_{k=1}^{K} \sup_t \left| \frac{1}{2(H-1)} \sum_{h_1=0}^{2(H-1)} g_1(T^{ah_1}x)g_1(T^{a(h_1+k)}x)g_2(T^{b(h_1)}y)g_2(T^{b(h_1+k)}y)e^{2\pi i h_1 t} \right| \]

for each \( K \) such that \( (K+1)^2 < 2(H-1) \).

Now we can conclude with the path leading to Uniform Wiener Wintner theorem obtained in [5] (see Lemma 6 and Lemma 7 in this paper). We can use the same method to show that for
each function $V_1, V_2$ bounded by one
\[
\int \int \limsup_h \sup_t \left| \frac{1}{H} \sum_{h_1=0}^{H-1} V_1(T^{an}x) V_2(T^{bn}y) e^{2\pi i n t} \right|^2 d\mu \otimes \mu \leq a, b \min_{i=1,2} ||V_i||_3^2.
\]
Therefore we have
\begin{equation}
\int \int \limsup_h \sup_t \left| \frac{1}{H} \sum_{h_1=0}^{H-1} V_1(T^{an}xV_2(T^{bn}y) e^{2\pi i n t} \right|^2 d\mu \otimes \mu \leq a, b \min_{i=1,2} ||V_i||_3^2.
\end{equation}

As a consequence of (5) we derive the following estimate.
\[
\int \int \limsup_h \left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} G^1_{h_1,h_2,h_3}(x) G^2_{h_1,h_2,h_3}(y) \right)^2 d\mu \otimes \mu
\leq \frac{C}{K} + \frac{C}{K} \int \int \sum_{k=1}^{K} \limsup_h \sup_t \left| \frac{1}{(2(H-1))^{2(2(H-1)-1)}} \sum_{h_1=0}^{2(2(H-1)-1)} g_1(T^{a(h_1+k)x}) g_2(T^{b(h_1+k)x}) e^{2\pi i h_1 t} \right| d\mu \otimes \mu
\leq \frac{a, b}{K} \sum_{k=1}^{K} \min \left( \left\| g_1 \cdot g_1(T^{a(k)}) \right\|_3, \left\| g_2 \cdot g_2(T^{b(k)}) \right\|_3 \right)
\leq \frac{a, b}{K} \left( \sum_{k=1}^{K} \min \left( \left\| g_1 \cdot g_1(T^{a(k)}) \right\|_3^8, \left\| g_2 \cdot g_2(T^{b(k)}) \right\|_3^8 \right) \right)^{1/8}
\]
By taking the limit with respect to $K$ we obtain the upper bound $\min(\|g_1\|_4, \|g_2\|_4)$. Thus if $g_1$ or $g_2$ belongs to $\mathbb{Z}_3^+$ we have shown that the sequence
\[
\left( \frac{1}{H^3} \sum_{h_1,h_2,h_3=0}^{H-1} G^1_{h_1,h_2,h_3}(x) G^2_{h_1,h_2,h_3}(y) \right)^2
\]
converge a.e. to zero. The dominated convergence theorem allows us to end the proof of this lemma for $k = 3$. The general case $k \geq 4$ follows similar steps.

So we have shown that we can find a set of full measure $X_1$ such that if one of the functions $f$ or $g$ in the statement of our main theorem belongs to $\mathbb{Z}_k^+$ then the averages $\frac{1}{N} \sum_{n=0}^{N-1} f(T^{an}x) g(T^{bn}x) c_n$ converge to zero for each bounded sequence $c_n$ which is a $k$ step nilsequence. It remains the case where both functions belong to $\mathbb{Z}_k^-$.

2.4. The functions $f$ and $g$ belong to $\mathbb{Z}_k^-$. We can assume that each function $f$ and $g$ is continuous. It turns out that for each $x$ the sequences $f(T^{an}x)$ and $g(T^{bn}x)$ are themselves $k$ step nilsequences. As the product of two $k - 1$ step nilsequences is also a $k - 1$ step nilsequence we can conclude that the product $f(T^{an}x) g(T^{bn}x) c_n$ is also a $k$-step nilsequence and therefore the
convergence of the averages $\frac{1}{N} \sum_{n=0}^{N-1} f(T^{an}x)g(T^{bn}x)c_n$ is immediate in this case. In the general case the functions are simply in $L^\infty$. A simple approximation argument allows to derive the same conclusion from the case where the functions are continuous.

Combining the results in each subsection we have obtained a proof of our main theorem.

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Remarks A similar result was announced very recently by P. Zorin-Kranich by using a different method [15].

References

[1] I. Assani. Pointwise convergence of ergodic averages along cubes. J. Analyse Math., 110:241–269, 2010.
[2] I. Assani, D. Duncan, and R. Moore. Pointwise characteristic factors for Wiener-Wintner double recurrence theorem. Ergod. Th. and Dynam. Sys., 2015. Available on CJO 2015 doi:10.1017/etds.2014.99.
[3] I. Assani and R. Moore. A good universal weight for nonconventional ergodic averages in norm. Preprint, available on arXiv:1503.08863, submitted, March 2015.
[4] I. Assani and R. Moore. Extension of Wiener-Wintner double recurrence theorem to polynomials. Available on http://www.unc.edu/math/Faculty/assani/WWDR_poly_final_abSept19.pdf submitted, September 2014.
[5] I. Assani and K. Presser. Pointwise characteristic factors for the multiterm return times theorem. Ergod. Th. and Dynam Sys., 32:341–360, 2012.
[6] V. Bergelson, B. Host, and B. Kra with appendix by I Ruzsa. Multiple recurrence and nilsequences. Invent. Math., 160:261–303, 2005.
[7] J. Bourgain. Double recurrence and almost sure convergence. J. reine angew. Math., 404:140–161, 1990.
[8] B. Green and T. Tao. The primes contain arbitrarily long arithmetic progressions. Annals of Math, 167:481–547, 2008.
[9] B. Green and T. Tao. Quadratic uniformity of the mobius function. Annales de l’Institut Fourier (Grenoble), 58(6):1863–1935, 2008.
[10] B. Green and T. Tao. Linear equations in the primes. Annals of Math, 171(3):1753–1850, 2010.
[11] B. Host and B. Kra. Nonconventional ergodic averages and nilmanifolds. Ann. of Math., 161:387–488, 2005.
[12] B. Host and B. Kra. Uniformity seminorms on $\ell^\infty$ and applications. J. Anal. Math, 108:219–276, 2009.
[13] L. Kuipers and H. Niederreiter. Uniform Distribution of Sequences. John Wiley and Sons, 1974.
[14] T. Ziegler. Universal characteristic factors and Furstenberg averages. J. Amer. Math. Soc., 20(1):53–97, 2006.
[15] P. Zorin-Kranich. A nilsequence wiener wintner theorem for bilinear ergodic averages. Preprint available on arXiv:1504.04647, 2015.