The $q$-AGT–W Relations Via Shuffle Algebras

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Abstract: We construct the action of the $q$-deformed $W$-algebra on its level $r$ representation geometrically, using the moduli space of $U(r)$ instantons on the plane and the double shuffle algebra. We give an explicit LDU decomposition for the action of $W$-algebra currents in the fixed point basis of the level $r$ representation, and prove a relation between the Carlsson–Okounkov Ext operator and intertwiners for the deformed $W$-algebra. We interpret this result as a $q$-deformed version of the AGT–W relations.

1 Introduction

Fix $r \in \mathbb{N}$. The moduli space $M$ of rank $r$ framed sheaves on $\mathbb{P}^2$ is an algebro-geometric incarnation of the moduli space of $U(r)$ instantons, where the Nekrasov partition function naturally appears. More precisely, it has been known from the work of [10,11,25,34] that the partition function of 5$dU(r)^k$-gauge theory with bi-fundamental hypermultiplets $m_1, \ldots, m_k$, in the presence of full $\Omega$-background, is:

$$Z_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \text{Tr} \left( A_{m_1}(x_1) \circ \cdots \circ A_{m_k}(x_k) \right)_{u^{k+1}=u^i}$$

(1.1)

where the Ext operator (see (4.2) for the precise geometric definition) is:

$$A_{m_i}(x_i) : K_{u^{i+1}} \longrightarrow K_{u^i}$$

(1.2)

and $K_{u^i}$ denotes the equivariant $K$-theory of the moduli space of rank $r$ sheaves, with equivariant weights encoded in the vector of parameters $u^i = (u^i_1, \ldots, u^i_r)$. The partition function of linear quiver gauge theory can be recovered from the operators $A_m(x)$ and their matrix coefficients, as explained in [7]. This partition function has been studied extensively and from many different points of view, see e.g. [26,35].

The main purpose of the present paper is to mathematically state and prove a connection between the rank $r$ Nekrasov partition function and conformal blocks for the
A. Neguț, \( q \)-\( W \)-algebra (more commonly called “deformed \( W \)-algebra”) of type \( \mathfrak{gl}_r \). The proof of our main Theorem 1.1 uses two main mathematical tools: expressing \( q \)-\( W \)-algebras via shuffle algebras, and performing intersection-theoretic computations with the Ext operator (1.2). We interpret our result as a \( q \)-deformed version of the well-known AGT–W relations between gauge theory and conformal field theory (these were introduced by Alday, Gaiotto and Tachikawa and extended by Wyllard in the undeformed case, and formulated in the \( q \)-deformed case by Awata and Yamada, see [4,7,39,40] among other references. The physical literature on the subject is vast, see for example [1,22,36] for other points of view).

The algebra we study is the tensor product of the \( q \)-\( W \)-algebra of type \( \mathfrak{sl}_r \) [3,14] and a \( q \)-Heisenberg algebra. By close analogy with loc. cit., we show in Sect. 5 that the defining currents of our \( q \)-\( W \)-algebra are “elementary symmetric functions”:

\[
W_k(z) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \exp\left[ b^{i_1}(x) \right] \exp\left[ b^{i_2}(x) \frac{x}{q} \right] \cdots \exp\left[ b^{i_k}(x) \frac{x}{q^{k-1}} \right],
\]

in a family of bosonic fields \( b^1(z), \ldots, b^r(z) \) which satisfy the commutation relations (5.1). The nice thing about the \( \mathfrak{gl}_r \) case is that one can send \( r \to \infty \), and the resulting limit can be interpreted as the upper half of the double shuffle algebra (as in Sect. 2). For fixed \( r \), the definition (1.3) implies the following relations:

\[
W_0(x) = 1, \quad W_k(x) = 0 \quad \text{for all } k > r
\]

and:

\[
W_k(x)W_{k'}(y) \cdot f_{kk'}\left( \frac{y}{x} \right) - W_{k'}(y)W_k(x) \cdot f_{kk'}\left( \frac{x}{y} \right) = \sum_{i=\max(0,k'-k)+1}^{k'} \delta \left( \frac{y}{xq^i} \right) \left[ W_{k'-i}(x)W_{k+i}(y) f_{k'-i,k+i}\left( \frac{y}{x} \right) \bigg|_{x=y} \right] - \sum_{i=\max(0,k-k')+1}^{k} \delta \left( \frac{x}{yq^i} \right) \left[ W_{k-i}(x)W_{k'+i}(y) f_{k-i,k'+i}\left( \frac{x}{y} \right) \bigg|_{y=x} \right].
\]

The quantity \( \theta(s) \) is defined for all \( s \in \mathbb{N} \) in (2.63), while the power series \( f_{kk'}(z) \) is defined in (2.64) (note that we always expand it in \( |z| \ll 1 \)). In Proposition 5.7, we will explain how formulas (1.4) differ from those of [3,14].

Our strategy is quite well-known to mathematicians and physicists: to recast the AGT–W relations as a connection between the operator \( A_m(x) \) and intertwiners for the \( q \)-\( W \)-algebra. This starts with Theorem 3.12 below, which states that for arbitrary \( r \in \mathbb{N} \) and generic equivariant parameters \( u = (u_1, \ldots, u_r) \), the \( K \)-theory group \( K_u \) of the moduli space of rank \( r \) sheaves is isomorphic to the Verma module of the \( q \)-\( W \)-algebra, with highest weight prescribed by the equivariant parameters \( u \). Note that our construction and proof are purely geometric, and do not use the isomorphism between the level \( r \) representation and a tensor product of \( r \) Fock spaces (which was used e.g. in [2]). This geometric definition is fruitful because it can be extended to moduli of sheaves on other surfaces, see [32,33]:
Theorem 1.1. For arbitrary \( r \geq 1 \) and collections \( u = (u_1, \ldots, u_r), u' = (u'_1, \ldots, u'_r) \) of equivariant parameters, let \( u = u_1 \ldots u_r \) and \( u' = u'_1 \ldots u'_r \). The Ext operator:

\[ A_m(x) : K_{u'} \longrightarrow K_u \]

is connected by the simple relation (4.27) with the vertex operator:

\[ \Phi_m(x) : K_{u'} \longrightarrow K_u \]

which satisfies the following commutation relations with the \( qW \)-algebra currents:

\[ [\Phi_m(x), W_k(y)]_{m^k} \prod_{i=1}^{k} \left( 1 - \frac{m^r u x}{q^{r-i} u' y} \right) = 0 \] (1.5)

where \([A, B]_s = AB - sBA\) denotes the \( s \)-commutator.

(see Remark 4.8 for a stronger form of relation (1.5), to be proved in [33]). Let us say a few words about our methods. Theorem 3.12 below was proved in the undeformed case in [23,38], by different means. Our proof of the deformed case is new, and uses the embedding of the \( qW \)-algebra into the double shuffle algebra \( \mathcal{A} \) of [9,18,19,37]. More specifically, we recast relations (1.4) in terms of the shuffle algebra, recall the action of the shuffle algebra on \( K_u \) from [29], and show that relations (1.4) hold in \( K_u \) by a shuffle algebra computation. In particular, we prove the following formula for the LDU decomposition of \( qW \)-algebra currents in the basis of fixed points indexed by \( r \)-tuples of partitions \( \lambda = (\lambda^1, \ldots, \lambda^r) \) (note that this basis is quite different, and in a sense “orthogonal”, to the basis arising from the isomorphism \( K_u \cong \text{Fock} \otimes r \) , which we will review in Sect. 5.8):

Theorem 1.2. Let \( D_x \) denote the \( q \)-difference operator \( f(x) \rightsquigarrow f(xq) \). The action of the \( qW \)-algebra on its level \( r \) representation (interpreted as the \( K \)-theory of the moduli space of framed sheaves, as in Sects. 3.1 and 3.2) is given by:

\[ \sum_{k=0}^{\infty} \frac{W_k(x)}{(-y D_x)^k} = T \left( x^{-1}, y D_x \right) \leftarrow E (y D_x) \quad T (xq, y D_x) \rightarrow \] (1.6)

where the three factors are lower triangular, diagonal and upper triangular, respectively (see Remark 2.17 for how to place the non-commuting symbols \( x \) and \( D_x \) in (1.6)). These operators are interpreted geometrically in Sect. 3, when we obtain:

\[ \langle \mu | W_k(x) | \lambda \rangle = \sum_{k} \sum_{\nu} \sum_{\lambda} \langle \mu | T^{-}_{d_{\nu}, k_{\nu}} | \nu \rangle \langle \nu | E_{0,k_{0}} | \nu \rangle \langle \nu | T^{+}_{d_{\nu}, k_{\nu}} | \lambda \rangle q^{(k-1)d_{\nu}} \]

with the three matrix coefficients in the right-hand side given by (3.21), (3.22) and (3.23), respectively. The basis \( | \lambda \rangle \in K \) consists of torus fixed points in \( K \)-theory.

We will prove formula (1.5) by a geometric computation, which takes up most of Sect. 4 and closely follows that in the undeformed case in [27]. Note that in [6,7], the authors computed the commutation relations of the Ext operator with the degree one generators of the double shuffle algebra (some of these relations had appeared in [29], but without any of the very interesting physics studied in [7]). In the present paper, we take the orthogonal viewpoint of computing the commutation relations of the Ext
operator with the $q W$-currents directly. Thus, while the present work uses the double shuffle algebra as a technical step, the final result is presented only in terms of the $q W$-algebra action on $K$-theory. Finally, in Sect. 6, we show how to degenerate the $q W$-algebra to the usual one, in which case the limit of $\Phi_m(x)$ is precisely the vertex operator studied in [12].

Let us note that the three factors that make up the LDU decomposition of the currents (1.6) are purely geometric, and one may ask what the analogous construction means for moduli spaces of sheaves on a more general (projective or toric) surface. Understanding the corresponding $q W$-algebra will be the subject of [32].

2 The Shuffle Algebra and Deformed $W$-Algebras

2.1. Let $q_1, q_2$ be indeterminates, and set $q = q_1 q_2$ and $F = \mathbb{Q}(q_1, q_2)$. One of the basic objects of the present paper is the rational function:

$$\zeta(x) = \frac{(1 - q_1 x)(1 - q_2 x)}{(1 - x)(1 - qx)}$$

(2.1)

which we observe satisfies the relation:

$$\zeta(x) = \zeta\left(\frac{1}{xq}\right)$$

(2.2)

Note that we can write $\zeta$ in the form:

$$\zeta(x) = \exp\left[\sum_{n=1}^{\infty} \frac{(1 - q_1^n)(1 - q_2^n)x^n}{n}\right]$$

(2.3)

Consider an infinite set of variables $z_1, z_2, \ldots$, and take the $F$-vector space:

$$V = \bigoplus_{k \geq 0} F(z_1, \ldots, z_k)^{\text{Sym}}$$

(2.4)

of rational functions which are symmetric in the variables $z_1, \ldots, z_k$, for any $k$. We endow $V$ with an $F$-algebra structure by the shuffle product:

$$R(z_1, \ldots, z_k) \ast R'(z_1, \ldots, z_{k'})$$

$$= \frac{1}{k!k'!} \cdot \text{Sym}\left[R(z_1, \ldots, z_k)R'(z_{k+1}, \ldots, z_{k+k'}) \prod_{i=1}^{k} \prod_{j=k+1}^{k+k'} \zeta\left(\frac{z_i}{z_j}\right)\right]$$

(2.5)

where Sym denotes the symmetrization operator:

$$\text{Sym} (R(z_1, \ldots, z_k)) = \sum_{\sigma \in S(k)} R(z_{\sigma(1)}, \ldots, z_{\sigma(k)})$$

The shuffle algebra $S \subset V$ is defined as the set of rational functions of the form:

$$R(z_1, \ldots, z_k) = \frac{r(z_1, \ldots, z_k)}{\prod_{1 \leq i \neq j \leq k} (z_i - z_j q)}$$

(2.6)
where $r$ is a symmetric Laurent polynomial that satisfies the wheel conditions:

$$r(z_1, \ldots, z_k) \bigg|_{\left\{ \frac{z_i}{z_j}, \frac{z_j}{z_i}, \frac{z_j}{z_l}, \frac{z_l}{z_j} \right\}} = q_1 q_2 \frac{1}{z_i}$$

(2.7)

This condition is vacuous for $k \leq 2$. It is straightforward to show that $S$ is an algebra, and that the shuffle product preserves the grading by the number of variables $k$ in (2.4), and also the grading by the homogeneous degree $d$ of rational functions. We will use the following notation for the graded and bigraded pieces of $S$:

$$S = \bigoplus_{k=0}^{\infty} S_k, \quad S_k = \bigoplus_{d \in \mathbb{Z}} S_{k,d}$$

**Theorem 2.2.** [28] The shuffle algebra $S$ is generated by the degree one elements $R(z) = z^d \in S_1$, as $d$ goes over $\mathbb{Z}$, under the shuffle product (2.5) (when dealing with rational functions in a single variable, we will denote the variable by $z = z_1$).

2.3. The following elements of the shuffle algebra $S$ were constructed in [28] (note that the multiplication in loc. cit. is defined with respect to the function $\xi(x^{-1})^{-1}$ instead of $\xi(x)$, but the fact that the two structures are isomorphic is easily observed). Below, we consider any $k > 0$ and $d \in \mathbb{Z}$, and write $n = \gcd(k, d)$, $a = \frac{k}{n}$:

$$P_{k,d} = \text{Sym} \left[ \frac{n}{k} \sum_{s=0}^{n-1} q^s \frac{z_a(n-s)+1}{z_a(n-1)} \prod_{i < j} (z_i-z_j) \right]$$

(2.8)

$$H_{k,d} = \text{Sym} \left[ \frac{n}{k} \prod_{i < j} (z_i-z_j) \right]$$

(2.9)

$$E_{k,d} = (-q)^{n-1} \cdot \text{Sym} \left[ \frac{n}{k} \prod_{i < j} (z_i-z_j) \right]$$

(2.10)

$$Q_{k,d} = \left( 1 - \frac{1}{q} \right) \cdot \text{Sym} \left[ \frac{n}{k} \sum_{s=0}^{n-1} q^s \frac{z_a(n-s)+1}{z_a(n-1)} \prod_{i < j} (z_i-z_j) \right]$$

(2.11)

It was shown in [28] that $P_{k,d}$ has “minimal degree”, in a sense made explicit in loc. cit., among all symmetric rational functions in $k$ variables which have homogeneous degree $d$ and satisfy the wheel conditions (2.7). As for $H_{k,d}$, $E_{k,d}$ and $Q_{k,d}$, they are in relation to $P_{k,d}$ as complete, elementary and plethystically modified complete symmetric...
functions, respectively, are in relation to power sum functions. Specifically, this means that for all coprime integers \(a\) and \(b\), we have:

\[
\sum_{n=0}^\infty \frac{H_{an, bn}}{x^n} = \exp \left[ \sum_{n=1}^\infty \frac{P_{an, bn}}{nx^n} \right] \tag{2.12}
\]

\[
\sum_{n=0}^\infty \frac{E_{an, bn}(-1)^n}{x^n} = \exp \left[ -\sum_{n=1}^\infty \frac{P_{an, bn}}{nx^n} \right] \tag{2.13}
\]

\[
\sum_{n=0}^\infty \frac{Q_{an, bn}}{x^n} = \exp \left[ \sum_{n=1}^\infty \frac{P_{an, bn}}{nx^n} \cdot (1 - q^{-n}) \right] \tag{2.14}
\]

We henceforth take (2.9), (2.10), (2.11) as the definition of the shuffle elements \(H_{k,d}, E_{k,d}, Q_{k,d}\), and we will prove formulas (2.12), (2.13), (2.14) in the “Appendix”.

2.4. It was shown in [28] that \(S\) has a coproduct and a bialgebra pairing (strictly speaking, this is true only after enlarging the shuffle algebra by adding commuting elements \(a_1, a_2, \ldots\) as below, and we refer the reader to loc. cit. for the details), which allow us to construct its Drinfeld double. The double is defined as the algebra:

\[
\mathcal{A} = \mathcal{A}^\leftarrow \otimes \mathcal{A}^{\text{diag}} \otimes \mathcal{A}^\to \tag{2.15}
\]

where for a central element \(c\) and commuting elements \(\{a_n\}_{n \in \mathbb{Z} \setminus 0}\) we set:

\[
\mathcal{A}^\leftarrow = S \quad \mathcal{A}^{\text{diag}} = \mathbb{F}[, \ldots, a_{-2}, a_{-1}, a_1, a_2, \ldots, c^{\pm 1}] \quad \mathcal{A}^\to = S^{\text{op}}
\]

Therefore, associated to any shuffle element \(R\) as in (2.6), we have two elements \(R^\leftarrow \in \mathcal{A}^\leftarrow\) and \(R^\to \in \mathcal{A}^\to\), which satisfy the following rules for all \(R_1, R_2 \in S\):

\[
(R_1 * R_2)^\leftarrow = R_1^\leftarrow * R_2^\leftarrow \quad (R_1 * R_2)^\to = R_2^\to * R_1^\to
\]

We impose the following relations between the three subalgebras of (2.15):

\[
[R^\leftarrow(z_1, \ldots, z_k), a_n] = R^\leftarrow(z_1, \ldots, z_k)(z_1^n + \cdots + z_k^n) \tag{2.16}
\]

\[
[R^\to(z_1, \ldots, z_k), a_n] = -R^\to(z_1, \ldots, z_k)(z_1^n + \cdots + z_k^n) \tag{2.17}
\]

for all \(R \in S, n \in \mathbb{Z} \setminus 0\), as well as:

\[
\left[(z^d)^\leftarrow, (z^{d'})^\leftarrow\right] = \frac{(1 - q_1)(1 - q_2)}{q^{-1} - 1} \left(A_{d+d'}\delta_{d+d' \geq 0} - \frac{1}{c} \cdot A_{d+d'}\delta_{d+d' \leq 0}\right) \tag{2.18}
\]

where the generators \(A_n\) are defined in terms of the \(a_n\) by:

\[
\sum_{n=0}^\infty \frac{A_{\pm n}}{x^{\pm n}} = \exp \left[ \pm \sum_{n=1}^\infty \frac{a_{\pm n}}{nx^{\pm n}} (1 - q_1^n)(1 - q_2^n)(1 - q^{-n}) \right] \tag{2.19}
\]

According to Theorem 2.2, the \(\mathbb{F}\)-algebras \(\mathcal{A}^\to, \mathcal{A}^\leftarrow\) are generated by the elements \((z^d)^\to, (z^{d'})^\leftarrow\), respectively. Therefore, (2.18) gives an inductive recipe for expressing any product \(R^\to * (R')^\leftarrow\) in terms of products of the form \((\tilde{R})^\leftarrow * a * \tilde{R}^\to\) for various \(\tilde{R}, \tilde{R} \in S\) and \(a \in \mathcal{A}^{\text{ diag}}\). This underlies the decomposition (2.15).
Remark 2.5. In most of the existing literature, the double shuffle algebra is defined as \( \langle A, c^\pm \rangle \), where \( c^\prime \) is a second central element which controls the commutation of the elements \( a_n \) and \( a_{-n} \). Since the central element \( c^\prime \) will act by 1 in all the representations considered in this paper, we will ignore it.

2.6. Since the double shuffle algebra \( A \) contains \( A^- = S \) and \( A^+ = S^{op} \), we have two copies of each of the shuffle elements (2.8)–(2.11) inside it. Specifically, we denote them by:

\[
\begin{align*}
P_{-k,d} &= P_{k,d}, & H_{-k,d} &= H_{k,d}, & E_{-k,d} &= E_{k,d}, & Q_{-k,d} &= Q_{k,d} \in A^- \\
P_{k,d} &= P_{k,d}, & H_{k,d} &= H_{k,d}, & E_{k,d} &= E_{k,d}, & Q_{k,d} &= Q_{k,d} \in A^+
\end{align*}
\]

for all \( k > 0 \) and \( d \in \mathbb{Z} \), and:

\[
P_{0,\pm d} = \pm (1 - q_1^d) (1 - q_2^d) a_{\pm d} \in A^\text{diag}
\]

for all \( d > 0 \). The elements \( P_{k,d} \) for \( (k, d) \in \mathbb{Z}^2 \setminus (0, 0) \) mimic those introduced by Burban and Schiffmann in the elliptic Hall algebra of [9]. More specifically, we claim that the elements \( P_{k,d} \) introduced in the present paper satisfy the relations between the generators studied in loc. cit., namely (2.20) and (2.21) below:

\[
[P_{k,d}, P_{k',d'}] = \delta_{k+k',d+d'}^0 (1 - q_1^n) (1 - q_2^n) \cdot \frac{1 - c^k}{1 - q^{-n}}
\]  

(2.20)

if \( kd' = k'd \) and \( k < 0 \), where \( n = \gcd(k, d) \). The second relation states that whenever \( kd' > k'd \) and the triangle \( T \) with vertices \( (0, 0), (k, d), (k + k', d + d') \) contains no lattice points inside nor on one of the edges, then we have the relation:

\[
[P_{k,d}, P_{k',d'}] = \frac{(1 - q_1^n)(1 - q_2^n)}{q^{-1} - 1} Q_{k+k',d+d'} \cdot \begin{cases} 
 c^k & \text{if } (k, d) \in \mathbb{Z}^-, (k', d') \in \mathbb{Z}^+, (k + k', d + d') \in \mathbb{Z}^+ \\
 c^{-k'} & \text{if } (k, d) \in \mathbb{Z}^-, (k', d') \in \mathbb{Z}^+, (k + k', d + d') \in \mathbb{Z}^- \\
 1 & \text{otherwise}
\end{cases}
\]

(2.21)

where \( n = \gcd(k, d) \gcd(k', d') \) (by the assumption on the triangle \( T \), we note that at most one of the pairs \( (k, d), (k', d'), (k + k', d + d') \) can fail to be coprime), and we divide the lattice plane into its left and right halves:

\[
\begin{align*}
\mathbb{Z}^+ & = \{(k, d) \in \mathbb{Z}^2 \text{ s.t. } k > 0 \text{ or } k = 0, d > 0 \} \\
\mathbb{Z}^- & = \{(k, d) \in \mathbb{Z}^2 \text{ s.t. } k < 0 \text{ or } k = 0, d < 0 \}
\end{align*}
\]

Note that the passage from our generators \( P_{k,d} \) to the generators \( u_{k,d} \) of [9] is:

\[
P_{k,d} = (1 - q_1^n)(1 - q_2^n) c^{-\frac{k}{2}} e^{\frac{k}{2} k_{k,d}} u_{k,d}
\]

for all \( (k, d) \in \mathbb{Z}^2 \setminus (0, 0) \) with \( n = \gcd(k, d) \). Moreover, their \( k_{k,d} \) equals our \( c^k \). The contents of the present subsection may thus be summarized by the following:

Proposition 2.7. (Follows by combining [9, 28, 37]): Relations (2.20) and (2.21) hold between the generators \( \{P_{k,d}\}_{(k,d)\in\mathbb{Z}^2\setminus(0,0)} \) in the algebra \( A \). Moreover, they generate the full ideal of relations, so one may alternatively define \( A \) as the \( \mathbb{F} \)-algebra generated by \( \{P_{k,d}\}_{(k,d)\in\mathbb{Z}^2\setminus(0,0)} \) modulo relations (2.20) and (2.21).
2.8. Let us make several observations about the algebra $\mathcal{A}$. First of all, for each rational slope $\frac{b}{a}$, relation (2.20) implies that there exists a Heisenberg subalgebra:

$$\mathcal{A}_{\frac{b}{a}}^b = \mathbb{F}\left\langle P_{an, bn}, c^{\pm 1}\right\rangle_{n \in \mathbb{Z} \setminus 0} \subset \mathcal{A}$$

with central charge $c^a$. Moreover, the line of slope $\frac{b}{a}$ splits the lattice plane into two open half-planes, and so we will write:

$$\mathcal{A}^{< \frac{b}{a}} = \mathbb{F}\left\langle P_k, d\right\rangle_{kb < da}$$

modulo relations (2.21) between the generators $P_k, d$ for $kb < da$, and:

$$\mathcal{A}^{> \frac{b}{a}} = \mathbb{F}\left\langle P_k, d\right\rangle_{kb > da}$$

modulo relations (2.21) between the generators $P_k, d$ for $kb > da$. It was shown in [9] that the algebra $\mathcal{A}$ exhibits a triangular decomposition:

$$\mathcal{A} = \mathcal{A}^{< \frac{b}{a}} \otimes \mathcal{A}_{\frac{b}{a}}^b \otimes \mathcal{A}^{> \frac{b}{a}}$$

(2.24)

and the decomposition (2.15) is precisely the case when $\frac{b}{a} = \frac{1}{0}$.

**Proposition 2.9.** There exist isomorphisms $\mathcal{A}^{< \frac{b}{a}} \cong S$ and $\mathcal{A}^{> \frac{b}{a}} \cong S^{op}$ for all coprime $a, b$ with $a \geq 0$, and so the decomposition (2.24) reads:

$$\mathcal{A} = S \otimes (q\text{-Heisenberg algebra}) \otimes S^{op}$$

(2.25)

Moreover, if we choose integers $a', b'$ such that $a'b - ab' = 1$, then the relations between the generators $P_{k, d}, P_{k', d'}, P_{k'', d''}$ with $kb < da$, $k'b = d'a$, $k''b > d''a$ are completely determined by the particular cases of relations (2.20) and (2.21) when the indices of the $P$’s that appear in the left-hand side of these relations are:

$$(k, d) \in -(a', b') + \mathbb{Z}(a, b), \quad (k', d') \in \mathbb{Z}(a, b), \quad (k'', d'') \in (a', b') + \mathbb{Z}(a, b)$$

(2.26)

**Proof.** Note that positive/negative multiples of $(a', b')$ plus integer multiples of $(a, b)$ span half of the lattice plane on either side of the line of slope $\frac{b}{a}$. The assignments:

$$P_{-k(a', b') + d(a, b)} \mapsto P_{-k, d}$$

$$P_{k(a', b') + d(a, b)} \mapsto P_{k, d} \cdot c^{(ka' + da)}\delta_{kd' + da < 0}$$

where $k$ ranges over $\mathbb{N}$ and $d$ ranges over $\mathbb{Z}$, induce isomorphisms:

$$\mathcal{A}^{< \frac{b}{a}} \cong \mathcal{A}^< = S$$

and

$$\mathcal{A}^{> \frac{b}{a}} \cong \mathcal{A}^\to = S^{op}$$

because relations (2.21) between the generators $P_{\pm ka' + da, \pm kb' + db}$ are mapped to the exact same relations between the generators $P_{\pm k, d}$ (proving this is a straightforward accounting of powers of $c$, and we leave it to the interested reader; note that the assumption $a \geq 0$ is important). This proves the first statement of the Proposition. Since the shuffle
algebra is generated by its degree 1 part (Theorem 2.2), then for any \( k, k' > 0 \) and \( d, d' \in \mathbb{Z} \), we may write:

\[
P_{-k(a',b')+d(a,b)} = \sum_{e_1, \ldots, e_k \in \mathbb{Z}} \text{constant} \cdot P_{-a', b'}+e_1(a,b) \cdots P_{-a', b'}+e_k(a,b)
\]

\[
P_{k'(a',b')+d'(a,b)} = \sum_{e'_1, \ldots, e''_k \in \mathbb{Z}} \text{constant} \cdot P_{a', b'+e'_1(a,b)} \cdots P_{a', b'+e''_k(a,b)}
\]

Therefore, when expressed as an element of \( A^{<\frac{b}{a}} \cdot A^{\frac{b}{a}} \cdot A^{>\frac{b}{a}} \), the commutator:

\[
[P_{-k(a',b')+d(a,b)}, P_{k'(a',b')+d'(a,b)}]
\]

is an expression in \( P_{-a', b'}+e(a,b), P_{a', b'+e}(a,b) \) and their commutators. This expression must equal the triangular decomposition (2.24) of the right-hand side of relations (2.20)–(2.21), whenever the lattice points \((-k, d)\) and \((k', d')\) satisfy the hypotheses of these relations. Therefore, we conclude that the commutation relations (2.20)–(2.21) in general can be deduced from their particular cases (2.26). \( \square \)

2.10. In this paper, an important role will be played by the case \( \frac{b}{a} = \frac{0}{1} \) of the decomposition (2.24). The corresponding algebras (2.22)–(2.23) will be denoted:

\[
A^\uparrow = A^{<0} = \mathbb{F}\{P_{d,k}\}_{d \in \mathbb{Z}, k > 0} \subset A
\]

\[
A^\downarrow = A^{>0} = \mathbb{F}\{P_{d,k}\}_{d \in \mathbb{Z}, k < 0} \subset A
\]

and \( A^0 = \mathbb{F}\{p_n, c^{\pm1}\}_{n \in \mathbb{Z}\setminus0} \subset A \), where we set:

\[
p_{-n} = P_{-n,0}, \quad p_n = \frac{c^n}{q^n} \cdot P_{n,0}
\]

(2.27)

for all \( n > 0 \). We interpret the symbols \( p_{\pm n} \) as power sum functions, and think of:

\[
h_{-n} = H_{-n,0}, \quad h_n = \frac{c^n}{q^n} \cdot H_{n,0}
\]

(2.28)

as the corresponding complete symmetric functions. The interaction between the positive and negative \( q \)-Heisenberg algebra generators is governed by:

\[
[p_{-n}, p_n] = n(1 - q^n)(1 - q^n) \cdot \frac{1 - c^n}{1 - q^n} \quad \forall n > 0
\]

(2.29)

As an application of Proposition 2.9, we have an isomorphism \( A^\uparrow \cong S \). In other words, to any shuffle element \( R \in S \), there corresponds an element \( R^\uparrow \in A^\uparrow \), determined inductively by Theorem 2.2 and the initial assignment:

\[
(z^d)^\uparrow = P_{d,1} \quad \forall d \in \mathbb{Z}
\]

(2.30)

Finally, we will encounter the subalgebra generated by both \( A^\uparrow \) and \( A^0 \):

\[
A^{\uparrow \text{ext}} = \mathbb{F}\{P_{d,k}, c^{\pm1}\}_{d \in \mathbb{Z}} \subset A
\]

The superscript ext stands for “extended”, and will be used repeatedly throughout the paper, when we will “extend” various algebras by adding the \( q \)-Heisenberg algebra generators (2.27), modulo certain relations defined on a case-by-case basis.
2.11. We call a representation $\mathcal{A} \curvearrowright F$ **good** if the following conditions are met:

1. $F = \bigoplus_{n=0}^{\infty} F_n$ is non-negatively graded
2. $P_{k,d} \in \mathcal{A}$ acts on $F$ with degree $-k$ for all $(k, d) \in \mathbb{Z}^2 \setminus (0, 0)$

**Definition 2.12.** The level of an irreducible $\mathcal{A}$-module $F$ is the constant $\log q c$.

All the good representations we will encounter in this paper will have level $\in \mathbb{N}$, and will moreover be quasifinite in the terminology of [17] (we refer the reader to loc. cit. for a thorough treatment of the representation theory of $\mathcal{A}$). In [9,28], it is shown that a linear basis of $\mathcal{A}^\uparrow \cong S$ is given by the products:

$$P_v = P_{d_1,k_1} \cdots P_{d_t,k_t}$$  \hspace{1cm} (2.31)

where:

$$v = \{ (d_1,k_1), \ldots, (d_t,k_t), \ k_i \in \mathbb{N}, d_i \in \mathbb{Z} \}$$ \hspace{1cm} (2.32)

denotes an arbitrary sequence of lattice points, ordered such that:

$$\frac{d_1}{k_1} \leq \cdots \leq \frac{d_t}{k_t}$$ \hspace{1cm} (2.33)

By convention, if we have $d_i/k_i = d_{i+1}/k_{i+1}$ for some $i$, then we place $(d_i,k_i)$ before $(d_{i+1},k_{i+1})$ in the ordering if and only if $k_i < k_{i+1}$ (this convention is not crucial at all, because of (2.20)). Property (2) of a good representation $F$ implies that among the infinitely many compositions (2.31) with $k_1, \ldots, k_t > 0$ and fixed $k_1 + \cdots + k_t$, **only finitely many of them** have $\langle f'|P_v|f \rangle \neq 0$ for any pair of elements $f, f' \in F$. This implies that the action $\mathcal{A} \curvearrowright F$ extends to an action of the completion:

$$\mathcal{\hat{A}}^\uparrow \curvearrowright F$$ \hspace{1cm} (2.34)

where we define:

$$\mathcal{\hat{A}}^\uparrow = \left\{ \text{infinite sums of } P_v \text{'s as in (2.31), for bounded } k_1 + \cdots + k_t \in \mathbb{N} \right\}$$ \hspace{1cm} (2.35)

Analogous remarks apply to the completion $\hat{\mathcal{A}}^{\text{ext}}$ of $\mathcal{A}^{\text{ext}}$, which we define to be spanned by infinite sums of the form:

$$P_{d_1,k_1} \cdots P_{d_t,k_t}$$ \hspace{1cm} (2.36)

going over all $d_i \in \mathbb{Z}, k_i \geq 0$ such that $k_1 + \cdots + k_t$ is bounded above, as well as:

$$-\infty \leq \frac{d_1}{k_1} \leq \cdots \leq \frac{d_t}{k_t} \leq \infty$$

(if $k_i = k_{i+1} = 0$, then we place $P_{d_i,k_i}$ before $P_{d_{i+1},k_{i+1}}$ in (2.36) iff $d_i < d_{i+1}$).
2.13. The basic representation of the algebra $\mathcal{A}$ is the Fock space:

$$F_u = \mathbb{F}[p_{-1}, p_{-2}, \ldots]$$

with action given by $c = q$, as well as:

$$p_{-n}(m) = p_{-n} \cdot m, \quad p_n(m) = -n(1 - q_1^n)(1 - q_2^n) \cdot \partial_{p_{-n}}(m) \quad (2.37)$$

while the actions of $\mathcal{A}^\uparrow$ and $\mathcal{A}^\downarrow$ are completely determined by the formulas:

$$\sum_{d \in \mathbb{Z}} P_{d,1} \cdot x^d = u \cdot \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{nx^n} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{nx^n} \right] \quad (2.38)$$

$$\sum_{d \in \mathbb{Z}} \frac{P_{d,-1}}{q^{db_d}x^d} = \frac{q}{u} \cdot \exp \left[ -\sum_{n=1}^{\infty} \frac{p_{-n}}{nx^n} \right] \exp \left[ -\sum_{n=1}^{\infty} \frac{p_n}{nx^n q^n} \right] \quad (2.39)$$

Indeed, the identifications $\mathcal{A}^\uparrow \cong S$ and $\mathcal{A}^\downarrow \cong S^{\text{op}}$ of Proposition 2.9 when $b/a = 0 \frac{0}{1}$, together with Theorem 2.2, establish the fact that the algebras $\mathcal{A}^\uparrow$ and $\mathcal{A}^\downarrow$ are generated by $P_{d,1}$ and $P_{d,-1}$, respectively. Therefore, formulas (2.37)–(2.39) determine the entire action $\mathcal{A} \curvearrowright F_u$. We must now show this action is well-defined.

**Proposition 2.14.** Formulas (2.37), (2.38), (2.39) define an action $\mathcal{A} \curvearrowright F_u$, in such a way that $F_u$ is a good level 1 representation.

**Proof.** It is clear that the operators (2.37) give rise to an action $\mathcal{A}^0 \curvearrowright F_u$, because relation (2.29) holds for $c = q$. Moreover, we may define an action:

$$\mathcal{A}^\uparrow \cong S \curvearrowright F_u$$

which to any shuffle element $R(z_1, \ldots, z_k) \in S$ associates the operator:

$$R^\uparrow = u^k \oint \ldots \oint \frac{R(z_1, \ldots, z_k)}{\prod_{1 \leq i \neq j \leq k} \xi(z_i, z_j)} \exp \left[ \sum_{i=1}^{k} \sum_{n=1}^{\infty} \frac{p_{-n}}{nz_i^n} \right] \exp \left[ \sum_{i=1}^{k} \sum_{n=1}^{\infty} \frac{p_n}{nz_i^n q^n} \right] \quad (2.40)$$

The contours of integration are defined to be $|z_i| = 1$ for all $1 \leq i \leq k$, and we make the assumption $|q_1|, |q_2| < 1$ in order to avoid poles on the contours of integration from the denominators of $R$ and $\xi$. We leave the following observations as exercises to the interested reader (see [31] for details):

- the composition of the operators (2.40) respects the shuffle product (2.5):

$$R_1^\uparrow \circ R_2^\uparrow = (R_1 \ast R_2)^\uparrow$$

- when $R = z_1^d$, formula (2.40) specializes to (2.38).

Therefore, (2.38) induces an action $\mathcal{A}^\uparrow \curvearrowright F_u$. Analogously, (2.39) induces $\mathcal{A}^\downarrow \curvearrowright F_u$. To prove that these actions give rise to an action of:

$$\mathcal{A} = \mathcal{A}^\uparrow \otimes \mathcal{A}^0 \otimes \mathcal{A}^\downarrow$$

on $F_u$, we must show that the relations between the generators of the three tensor factors are respected. According to the last statement of Proposition 2.9, it is enough to check
relations (2.20) and (2.21) between the generators $P_{d,1}, P_{d,0}, P_{d,-1}$ for all $d \in \mathbb{Z}$. Specifically, these relations state:

$$[P_{d,1}, P_{\pm e,0}] = \pm (1 - q_1^e)(1 - q_2^e)P_{d,\pm e},$$

$$\left[ P_{d,-1,0} \frac{P_{\pm e,0}}{q^{-d\delta_{e<0}}} \right] = \mp (1 - q_1^e)(1 - q_2^e) \frac{P_{d,\pm e,-1}}{q^{(d\pm e)\delta_{d\pm e<0}}}$$

(2.41)

(2.42)

for all $d, d' \in \mathbb{Z}$ and $e \in \mathbb{N}$, as well as:

$$[P_{d,1}, P_{d',-1}] = \frac{(1 - q_1)(1 - q_2)}{1 - q^{-1}} \left( \frac{Q_{d+d',0}}{q^{d+e^d_{d'<0}}} - \delta_{d+d' \leq 0} \frac{Q_{d+d',0}}{q^{d_{d'>0}}} \right)$$

(2.43)

where the operators $Q_{\pm n,0}$ are exponentials of the operators $p_{\pm n}$, given by:

$$\sum_{n=0}^{\infty} \frac{Q_{\pm n,0}}{x^n} = \exp \left[ \sum_{n=1}^{\infty} \frac{p_{\pm n}}{n x^n} (1 - q^{-n}) \right]$$

Apply (2.38) and (2.29) to obtain:

$$\left[ \sum_{d \in \mathbb{Z}} \frac{P_{d,1}}{x^d}, P_{-e,0} \right] = \left[ u \cdot \exp \left( \sum_{n=1}^{\infty} \frac{p_{-n}}{n x^{-n}} \right) \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n x^n} \right), p_{-e} \right]$$

$$\quad = \left[ p_{e, p_{-e}} \right] \cdot u \exp \left( \sum_{n=1}^{\infty} \frac{p_{-n}}{n x^{-n}} \right) \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n x^n} \right)$$

$$\quad = -(1 - q_1^e)(1 - q_2^e) x^{-e} \sum_{d \in \mathbb{Z}} \frac{P_{d,1}}{x^d}.$$

Picking out the coefficient of $x^{-d}$ gives precisely (2.41) when the sign is $\pm = -$. The case when the sign is $\pm = +$, as well as relation (2.42), are analogous and we leave them as an exercise to the interested reader. As for (2.43), we have:

$$\sum_{d \in \mathbb{Z}} \frac{P_{d,1}}{x^d} \cdot \sum_{d' \in \mathbb{Z}} \frac{P_{d',-1}}{q^{d\delta_{d'<0}} y^{d'}}$$

$$= q \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{n x^{-n}} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{n x^n} \right] \exp \left[ -\sum_{n=1}^{\infty} \frac{p_{-n}}{n y^{-n}} \right] \exp \left[ -\sum_{n=1}^{\infty} \frac{p_n}{n (y q)^n} \right]$$

$$= q \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{n (x^n - y^n)} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{n} \left( \frac{1}{x^n} \frac{1}{y^n} \right) \right] \frac{(x - y q_1)(x - y q_2)}{(x - y)(x - y q)}$$

(2.44)

where in the final equality we used the identity:

$$\exp(a) \exp(b) = \exp(b) \exp(a) \exp([a, b]),$$

that holds whenever the commutator $[a, b]$ commutes with both $a$ and $b$. The product of exponentials in the third line of (2.44) is normally-ordered, which means that its matrix coefficients in $F_u$ are Laurent polynomials in $x$ and $y$. Meanwhile, the product of exponentials on the second line only gives a well-defined operator on $F_u$ only if we
expand the power series in the region $|y| \ll |x|$ (see Remark 2.21). We conclude that the equality (2.44) makes sense if we expand the rational function:

$$
\zeta \left( \frac{y}{x} \right) = \frac{(x - yq_1)(x - yq_2)}{(x - y)(x - yq)}
$$

in non-negative powers of $y/x$. Therefore, (2.44) implies:

$$
P_{d,1} P_{d', -1} = q^{1 + d'd' \delta_d \delta_{d'<0}} \int_{|y| \ll |x|} x^d y^{d'} \zeta \left( \frac{y}{x} \right) \exp \left[ \ldots \right] \exp \left[ \ldots \right] \frac{dx}{2\pi i x} \frac{dy}{2\pi i y}
$$

(2.45)

where $\exp \left[ \ldots \right] \exp \left[ \ldots \right]$ denotes the normal-ordered product of exponentials on the third line of (2.44). By definition, the contours over which the variables $x$ and $y$ run in (2.45) are circles centered at the origin, with the former of much larger radius than the latter. Similarly, the product $P_{d', -1} P_{d, 1}$ is given by the same integrand as (2.45), but the contours respect the inequality $|x| \ll |y|$. We conclude that:

$$
[P_{d,1}, P_{d', -1}] = q^{1 + d'd' \delta_d \delta_{d'<0}} \left( \frac{1 - q_1}{1 - q} \right) \left( \frac{1 - q_2}{1 - q} \right)
$$

$$
\times \left( \int y^{d+d'} q^d \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{n y^{n-1}} (q^n - 1) \right] \frac{dy}{2\pi i y} \right)
$$

$$
- \int y^{d+d'} \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{n y^n} (1 - q^{-n}) \right] \frac{dy}{2\pi i y}
$$

(2.46)

By the residue theorem, the value of (2.46) is equal to the sum of the residues of the integrand at poles of the form $x = y\alpha$, for various constants $\alpha \notin \{0, \infty\}$. Since the product of exponentials is normally-ordered, its matrix coefficients in $F_u$ are Laurent polynomials in $x$ and $y$, and therefore do not produce poles at $x = y\alpha$. And in fact, the only poles we encounter in the integral (2.46) come from $\zeta$, and they are $x = y$ and $x = yq$. The corresponding sum of residues is:

$$
P_{d,1} P_{d', -1} = q^{1 + d'd' \delta_d \delta_{d'<0}} \left( \frac{1 - q_1}{1 - q} \right) \left( \frac{1 - q_2}{1 - q} \right)
$$

$$
\times \left( \int \frac{dy}{2\pi i y} \right)
$$

Note that the expression above precisely matches the right-hand side of (2.43). The fact that $F_u$ is good is clear, since the two conditions of Sect. 2.11 are easily seen to be respected. That the level is 1 is precisely restating the fact that $c = q$. □

2.15. Since $z^{d^\uparrow} = P_{d,1}$ are the images of the degree 1 shuffle elements under $S \cong A^\dagger$, the left-hand side of (2.38) plays an important role in the double shuffle algebra $A$:

$$
W_1(x) := \delta \left( \frac{z}{x} \right)^\uparrow = \sum_{d \in \mathbb{Z}} \frac{z^{d^\uparrow}}{x^d} \in A^\dagger[[x^\pm 1]]
$$

(2.47)
will be called the “first \(qW\)-current”. The reason for the terminology in quotes is that we will now define “higher \(qW\)-currents” in terms of the double shuffle algebra \(A\), and in Proposition 2.20 will prove that they satisfy the relations in [3, 14]:

\[
W_k(x) = \eta_k \cdot \text{Sym} \left[ \delta \left( \frac{z_1}{x} \right) \ldots \delta \left( \frac{z_k}{xq^{1-k}} \right) \right]^{\uparrow} \in \hat{A}^{[[x^{\pm 1}]}} \tag{2.48}
\]

where:

\[
\eta_k = \prod_{1 \leq i < j \leq k} \zeta(q^{j-i})
\]

We set \(W_0(x) = 1\) by convention, and often use the following notation:

\[
W_k(x) = \sum_{d \in \mathbb{Z}} \frac{W_{d,k}}{x^d}, \quad k \geq 0 \tag{2.49}
\]

for the coefficients of \(W_k(x)\) as elements of \(\hat{A}^{\uparrow}\). The fact that the \(W_{d,k}\) lie in the completion of \(A^{\uparrow}\) is not immediately obvious, but is implied by the following result (together with (7.20), (7.21), (7.22) and the definition of the completion in (2.35)):

**Proposition 2.16.** In the completion \(\hat{A}^{\uparrow}[[x^{\pm 1}]\), we have the relation:

\[
W_k(x) = \sum_{\substack{d, k \geq 0 \\text{such that } d \rightarrow -k \geq 0 \\text{ and } k \rightarrow -k \geq 0 \\text{ are } \Rightarrow \text{ and } \rangle \geq 0 \\text{ respectively}}} T^{\leftarrow}_{d, k} E_{0, k_0} T^{\rightarrow}_{d, k_0} \cdot q^{(k-1)d} \cdot \left( -1 \right)^{k-1} \zeta_d \cdot \prod_{1 \leq i < j \leq d} \zeta \left( \frac{z_i}{z_j} \right) \in S \tag{2.50}
\]

where we set \(T_{0, k}^{\leftarrow} = T_{0, k} \rightarrow = T_{k, 0}^{\rightarrow} = T_{k, 0} = \delta_{k}^{0}\), while for \(d, k > 0\) we define:

\[
T_{d, k}(z_1, \ldots, z_d) = \text{Sym} \left[ \left( -1 \right)^{k-1} \frac{1}{z_d} \cdot \prod_{1 \leq i < j \leq d} \zeta \left( \frac{z_i}{z_j} \right) \right] \in S \tag{2.51}
\]

Formula (2.50) is more effectively packaged by computing the generating series of all the \(qW\)-currents at the same time. Specifically, define the power series:

\[
W(x, y) = \sum_{k=0}^{\infty} \frac{W_k(x)}{(-y)^k} = 1 + \sum_{d \in \mathbb{Z}} \sum_{k=1}^{\infty} \frac{W_{d,k}}{x^d(-y)^k} \tag{2.52}
\]

If we let \(D_x\) denote the difference operator \(f(x) \rightsquigarrow f(xq)\), then (2.50) implies:

\[
W(x, yD_x) = T \left( x^{-1}, yD_x \right)^{\leftarrow} \cdot E (yD_x) \cdot T (xq, yD_x) \rightarrow \tag{2.53}
\]

---

1 Strictly speaking, our currents differ from those of loc. cit. by certain exponentials in the bosons \(p_{\pm n}\), which we explicitly give in (5.22). The difference, as we will see in Sect. 5, stems from the fact that our \(qW\)-algebra corresponds to \(gl_r\), while that of loc. cit. corresponds to \(sl_r\).
which is precisely formula (1.6), where we set:

\[
T(x, y) = 1 + \sum_{d=1}^{\infty} \text{Sym} \left[ \frac{x^{-d}}{1 - \frac{y}{z_d}} \prod_{i=1}^{d-1} \left(1 - \frac{q z_{i+1}}{z_i} \right) \prod_{1 \leq i < j \leq d} \zeta \left( \frac{z_i}{z_j} \right) \right] \tag{2.54}
\]

\[
E(y) = \sum_{k=0}^{\infty} \frac{E_{0,k}}{(-y)^k} = \exp \left[ -\sum_{n=1}^{\infty} \frac{a_n(1-q^n_1)(1-q^n_2)}{n y^n} \right] \tag{2.55}
\]

**Remark 2.17.** Note that (2.53) is imprecise because \(x\) and \(D_x\) do not commute. In order for this formula to match (2.50), one needs to interpret it as follows: in the expansion of \(W(x, y D_x)\) and \(T(x^{-1}, y D_x)\), place the powers of \(D_x\) after (respectively before) the powers of \(x\). We call (2.53) the LDU decomposition of \(W\)-currents, because the three factors in the right-hand side respectively increase, preserve and decrease degree in any good representation.

In Sect. 3, we will give explicit formulas for the matrix coefficients of the decomposition (2.50) in the level \(r\) representation \(K\) that is the Hilbert space of the gauge theory side of the AGT–W correspondence. The fact that \(W_k(x) = 0\) holds in \(K\) for all \(k > r\) is not apparent from (2.50), and will be proved later.

2.18. Proposition 2.16 will be proved in the “Appendix”, together with the following:

**Proposition 2.19.** Consider the formal series:

\[
W_k(x) \ast W_{k'}(y) = \eta_k \eta_{k'} \prod_{1 \leq i \leq k} \zeta \left( \frac{x q^i}{y q^i} \right) \cdot \text{Sym} \left[ \delta \left( \frac{z_1}{x} \right) \ldots \delta \left( \frac{z_k}{x q^{1-k}} \right) \delta \left( \frac{z_{k+1}}{y} \right) \ldots \delta \left( \frac{z_{k+k'}}{y q^{1-k'}} \right) \right] \tag{2.56}
\]

Then \(W_k(x) \ast W_{k'}(y)\) lies in \(\hat{A}^\uparrow[[x^{\pm 1}, y^{\pm 1}]] \otimes (a\text{ certain explicit rational function})\):

\[
W_k(x) \ast W_{k'}(y) = \frac{\prod_{i=\max(0,k-k')}^{k-1} \zeta \left( \frac{y q^i}{x} \right)^{-1} \sum_v P_v \cdot a_v(x, y)}{\prod_{i=\max(0,k-k')}^{k'} (x - y q^i) \prod_{i=\max(0,k-k')}^{k'} (y - x q^i)} \tag{2.57}
\]

where \(a_v(x, y)\) are certain Laurent polynomials that will not matter to us. In the numerator of (2.57), the sum goes over all ordered sequences \(v\) as in (2.32).

Since all representations considered in this paper are good in the sense of Sect. 2.11, only finitely many of the shuffle elements \(P_v\) will contribute to any given matrix coefficient. Therefore, Proposition 2.19 implies that in a good representation, the operator-valued expression:

\[
W_k(x) W_{k'}(y) = \prod_{i=\max(0,k-k')}^{k-1} \zeta \left( \frac{y q^i}{x} \right)
\]
is a rational function whose only poles are \( x^{\pm 1}, y^{\pm 1}, \) and \( x - y q^*, y - x q^{**} \) for:

\[
* \in \{ \max(0, k - k') + 1, \ldots, k \}
\]
\[
** \in \{ \max(0, k' - k) + 1, \ldots, k' \}
\]

(2.58)  (2.59)

Since the second line of (2.56) is symmetric in \((k, x) \leftrightarrow (k', y)\), we will use the information of the poles (2.58)–(2.59) to prove the commutation relations (2.62).

**Proposition 2.20.** The currents \( \{ W_k(x) \}_{k \geq 1} \in \widehat{A}[[x^{\pm 1}]] \) satisfy the relations:

\[
[W_k(x), p_{-n}] = -\frac{(1 - q_1^n)(1 - q_2^n)(1 - q^{kn})}{1 - q^n} \cdot x^{-n} W_k(x)
\]

(2.60)

\[
[W_k(x), p_n] = \frac{(1 - q_1^n)(1 - q_2^n)(q^{kn} - 1)c^n}{1 - q^n} \cdot x^n W_k(x)
\]

(2.61)

where \( p_n \in A_0 \subset A \) are the bosons of (2.29), and:

\[
W_k(x) W_{k'}(y) \cdot f_{kk'} \left( \frac{y}{x} \right) - W_{k'}(y) W_k(x) \cdot f_{k'k} \left( \frac{x}{y} \right)
\]

\[= \sum_{i=\max(0,k'-k)+1}^{k'} \delta \left( \frac{y}{xq^i} \right) \left[ W_{k' - i}(x) W_{k + i}(y) f_{k' - i, k + i} \left( \frac{y}{x} \right) \bigg|_{x = q^i} \right] - \sum_{i=\max(0,k-k'+1)}^{k} \delta \left( \frac{x}{yq^i} \right) \left[ W_{k - i}(y) W_{k' + i}(x) f_{k - i, k' + i} \left( \frac{x}{y} \right) \bigg|_{y = q^i} \right]
\]

\[\theta(\min(i, k - k' + i)) \quad \theta(\min(i, k' - k + i))
\]

(2.62)

where we set:

\[\theta(s) = \frac{(1 - q_1)(1 - q_2)}{1 - q} \cdot \zeta(q) \cdots \zeta(q^{s-1})
\]

(2.63)

and for all \( k, k' \geq 0 \) define the rational function:

\[f_{kk'}(z) = \exp \left[ \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot \frac{(1 - q_1^n)(1 - q_2^n)(q^{\max(0,k-k')n} - q^{kn})}{1 - q^n} \right]
\]

(2.64)

**Remark 2.21.** Let us give a hands-on explanation of the convention of normal-ordering products of currents. All currents in this paper will be of the form:

\[W(x) = \sum_{d \in \mathbb{Z}} \frac{W_d}{x^d}
\]

where \( W_d \) is an operator that acts in good representations with degree \(-d\). Therefore, the operators \( W_d \) for \( d \gg 0 \) will annihilate any given vector of a good representation, and so the only reasonable way to make sense of compositions such as:

\[
W(x) W'(y) f \left( \frac{y}{x} \right) = \sum_{d, d' \in \mathbb{Z}} \frac{W_d W_{d'}}{x^d y^{d'}} \cdot \sum_{n \geq 0} f_n \frac{y^n}{x^n}
\]
is to expand the analytic function \( f(z) \) in non-negative powers of \( z \). This way, the coefficient of any \( x^a y^b \) in the right-hand side acts by a finite sum locally in all good representations. This is why the first summand of (2.62) must be expanded in the region \( |y| \ll |x| \), while the second summand must be expanded in \( |x| \ll |y| \).

**Remark 2.22.** We may explain relation (2.62) in terms of contour integrals (see [24] for more details). Namely, suppose one wanted to compute the coefficient of \( x^a y^b \) in the two sides of the equation. Naively, the way to do so is consider:

\[
\oint\oint \frac{\text{LHS of (2.62)}}{x^a y^b} \cdot \frac{dxdy}{xy} = \oint\oint \frac{\text{RHS of (2.62)}}{x^a y^b} \cdot \frac{dxdy}{xy}
\]

(2.65)

However, the devil is in the contours: in the LHS, the term \( W_k(x) W_k'(y) f_{kk'}(\frac{x}{y}) \) must be integrated over \( |x| \gg |y| \), while the term \( W_k'(y) W_k(x) f_{kk'}(\frac{y}{x}) \) must be integrated over \( |x| \ll |y| \). Meanwhile, every summand in the right-hand side of (2.62) reduces to a single contour integral, because for any constant \( a \) we have:

\[
\oint\oint \delta\left(\frac{y}{xa}\right) Z(x, y) \frac{dxdy}{xy} = \oint Z(x, xa) \frac{dx}{x}
\]

Therefore, formula (2.65) states that moving the \( y \) contour from \( |x| \gg |y| \) to \( |x| \ll |y| \) picks up precisely the residues that make up the right-hand side of (2.62).

2.23. Note that relation (2.62) matches the \( q \)-\( W \)-algebra relations of [3, 14], modulo the fact that our currents and theirs are not quite equal. Instead, the precise relation between the two conventions is given in (5.22), and this distinction accounts for the fact that our function \( f_{kk'}(z) \) is different from that of [3].

**Definition 2.24.** Define the algebra \( A_r \) as the quotient of \( \hat{A}^\uparrow \) by the relations:

\[
W_k(x) = 0 \quad (2.66)
\]

for all \( k > r \). Define the algebra \( A_r^{\text{ext}} \) as the quotient of \( \hat{A}^{\uparrow \text{ext}} \) by the relations:

\[
W_r(x) = u \left[ \sum_{n=0}^{\infty} \frac{h_{-n}}{x^{-n}} \right] \left[ \sum_{n=0}^{\infty} x^n \right] \quad \text{and} \quad W_k(x) = 0 \quad (2.67)
\]

for all \( k > r \), together with \( c = q^r \). We call \( A_r \) the \( q \)-\( W \)-algebra of type \( \text{gl}_r \).

The parameter \( u \) plays the role of the zero modes of Heisenberg currents from the physics literature (see Sect. 5), and we will abuse notation by ignoring it from our notation. The reader can think of \( u \) as a complex number which will be specialized to the product of equivariant parameters in the representations \( K \) studied in the next section. Note that the inclusion:

\[
\hat{A}^\uparrow \hookrightarrow \hat{A}^{\uparrow \text{ext}}
\]

induces a homomorphism of algebras:

\[
A_r \to A_r^{\text{ext}} \quad (2.68)
\]
Although we will not need this fact, we expect this homomorphism to be injective. Note that it is also “almost” surjective, or it would be if one were able to express the $q$-Heisenberg generators $p_n$ in terms of the coefficients of the current:

$$W_r(x) = u \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{nx^n} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{nx^n} \right] \in A_r^{\text{ext}}$$

This would be possible only if we dropped the phrase “for bounded $k_1 + \cdots + k_t \in \mathbb{N}$” from the definition of the completion in (2.35). Since doing so would be a technical annoyance for us, we accept the fact that the algebras $A_r$ and $A_r^{\text{ext}}$ are a little different, although we note that both act by the same formulas on the representations $K$ defined in the next section. Another feature of this “equivalence”, which we leave as an easy exercise to the interested reader, is that relations (2.60) and (2.61) imply (2.62) for arbitrary $k$ and $k' = r$ (the fact that $W_l(x) = 0$ for $l > r$ in either $A_r$ or $A_r^{\text{ext}}$ implies that the RHS of (2.62) vanishes).

**Proposition 2.25.** The algebra $A_r$ is isomorphic to the $\mathbb{F}$-algebra generated by symbols $\{W_{d,k}\}_{d \in \mathbb{Z}, k \geq 1}$ modulo relations (2.62) and (2.66).

**Remark 2.26.** Similarly, the algebra $A_r^{\text{ext}}$ is isomorphic to the $\mathbb{F}$-algebra generated by symbols $\{W_{d,k}\}_{d \in \mathbb{Z}, k \geq 1}$ and $\{p_n\}_{n \in \mathbb{Z} \setminus 0}$ modulo relations (2.60), (2.61), (2.62), (2.67).

We only sketch the main ideas of the proof, and leave the technical details to [32]. The reader who does not wish to rely on the proof, may replace $A_r$ of Definition 2.24 by the algebra $A'_r$ generated by $W_{d,k}$ modulo relations (2.62) and (2.66). The constructions in the present paper give a map $A'_r \rightarrow A_r$, which the results of loc. cit. imply is an isomorphism. All modules in the present paper are naturally $A_r$-modules, and everything in the present paper continues to apply to them as such.

**Proof.** By definition, $A_r$ is spanned as an $\mathbb{F}$-module by all possible products:

$$\left\{ W_{d_1,k_1} \cdots W_{d_t,k_t}, d_i \in \mathbb{Z}, k_i \in \mathbb{N} \right\}$$

(2.69)

The product (2.69) may be characterized by the following path (broken line):

$$(0, 0), (d_1, k_1), \ldots, (d_1 + \cdots + d_t, k_1 + \cdots + k_t)$$

In [32], we will unpack relation (2.62) using the normal-ordering conventions of [24], and show that it allows us to write the general product (2.69) as a (possibly infinite) linear combination of specific products which satisfy:

$$\frac{d_1}{k_1} \leq \frac{d_2}{k_2} \leq \cdots \leq \frac{d_t}{k_t}$$

In the language of paths, such products correspond to convex lattice paths in the upper half-plane. In other words, repeated applications of (2.62) allows one to “convexify” any path in the upper half-plane. Finally, relation (2.66) shows that we may restrict to the products (2.69) for which $k_1, \ldots, k_t \leq r$. To show that there are no relations among $W_{d,k} \in A_r$ except for (2.62) and (2.66), it is therefore enough to show that the elements (2.69) are linearly independent. To this end, it suffices to show that they are linearly
independent in $\mathring{A}^\uparrow$, in which case the result follows from the fact that the basis (2.69) is upper-triangular in terms of the basis:

$$E_{d_1,k_1} \cdots E_{d_t,k_t}$$  \hspace{1cm} (2.70)

(we show this explicitly in [32], although it is already visible from (7.19)). The fact that the elements (2.70) are linearly independent in $A$ follows from [9], Remark 5.1. □

2.27. Let us now define the main representation of $q$-$W$-algebras that we will study.

**Definition 2.28.** Given parameters $u_1, \ldots, u_r$ with product $u$, the **Verma module** $M_{u_1, \ldots, u_r}$ is the $A_r$-module generated by a single vector $|\emptyset\rangle$ modulo relations:

$$M_{u_1, \ldots, u_r} = A_r |\emptyset\rangle \Big/ W_{d,k} |\emptyset\rangle = 0, \ W_{0,k} |\emptyset\rangle = e_k(u_1, \ldots, u_r) |\emptyset\rangle, \ \forall k \in \{1, \ldots, r\}, \ d > 0$$  \hspace{1cm} (2.71)

where $e_k$ denotes the $k$th elementary symmetric function.

A more rigorous way to define $M_{u_1, \ldots, u_r}$ is to say that it is the quotient of $A_r$ by the right ideal $(W_{d,k}, W_{0,k} - e_k(u_1, \ldots, u_r))_{d > 0, k > 0}$, endowed with the induced left action of $A_r$. Therefore, we could also describe the Verma module as:

$$M_{u_1, \ldots, u_r} \cong \frac{\mathring{A}^\uparrow}{\text{two-sided ideal of relations (2.67), right ideal (} p_k, W_{d,k}, W_{0,k} - e_k(u_1, \ldots, u_r))_{d > 0}}$$

as linear maps of $\mathbb{F}$-vector spaces.

**Proposition 2.29.** When $r = 1$, we have $M_u \cong F_u$ of Sect. 2.13.

**Proof.** When $k' = 1$, relation (2.62) states that:

$$W_k(x)W_1(y)\xi\left(\frac{x}{yq^k}\right) - W_1(y)W_k(x)\xi\left(\frac{y}{xq}\right) = \frac{(1 - q)(1 - q^2)}{1 - q} \left[ \delta\left(\frac{y}{xq}\right)W_{k+1}(y) - \delta\left(\frac{x}{yq\xi}\right)W_{k+1}(x) \right]$$  \hspace{1cm} (2.72)

Taking the series coefficient of $x^{-a} y^{-b}$ allows one to express $W_{k+1,a+b}$ as a $\mathbb{F}$-linear combination of $W_k,d$’s and $W_1,e$’s. In order to show that the action $A \curvearrowright F_u$ of Proposition 2.14 factors through its quotient $A_1$, we need to show that $W_k(x)$ acts by 0 for all $k > 1$. By iterating (2.72), it is enough to prove that $W_2(x)$ acts by 0. Again in virtue of (2.72), this boils down to showing that:

$$W_1(x)W_1(y)\xi\left(\frac{x}{yq}\right) = W_1(y)W_1(x)\xi\left(\frac{y}{xq}\right)$$  \hspace{1cm} (2.73)
holds in the module $F_u$. By (2.38), this identity reduces to the expression:

$$u^2 \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{nx^{-n}} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{nx^n} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{ny^{-n}} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{ny^n} \right] \zeta \left( \frac{x}{yq} \right)$$

being symmetric in $x$ and $y$, as an endomorphism of the module $F_u$. This is a straightforward computation, that one can derive from $[p_{-n}, p_n] = n(1 - q_1^n)(1 - q_2^n)$, in the same way that the second line of (2.44) implies the third line of (2.44).

Having shown that $A_1$ acts on $F_u$, let us show that $M_u \cong F_u$ as $A_1$-modules. The equation right before the statement of the Proposition gives a $\mathbb{F}$-linear map:

$$M_u \phi \rightarrow \hat{A}_1^{\text{ext}} \big/ (I, J)$$

(2.74)

where $I$ is the two-sided ideal generated by $c - q$ and the coefficients of relations:

$$W_k(x) = \begin{cases} u \exp \left[ \sum_{n=1}^{\infty} \frac{p_{-n}}{nx^{-n}} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{p_n}{nx^n} \right] & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

(2.75)

and $J$ is the right ideal generated by $p_k$ for all $k > 0$. Note that the right-hand side of (2.74) is the Fock space $F_u$, because the ideal of relations allows one to reduce any element to a product of $p_{-1}, p_{-2}, \ldots$. The fact that the action matches that of (2.37) and (2.38) is forced upon us by relations (2.29), $c = q$ and (2.75).

Thus, we obtain a map of $\mathbb{F}$-vector spaces $M_u \rightarrow F_u$. However, both of these vector spaces are non-negatively graded with the vacuum vector in degree zero. The dimension of $F_u$ in degree $k$ is equal to the number of partitions of size $k$. Meanwhile, the dimension of $M_u$ in degree $k$ is less than or equal to the number of partitions of size $k$, because the collection $\{W_{-d_1,1} \ldots W_{-d_s,1}|\emptyset\} \{d_1 \geq \ldots \geq d_s \geq 0\}$ is a spanning set of $M_u$ over $\mathbb{F}$ (this follows from the proof of Proposition 2.25, or by (2.73)). Thus, to prove that the map $\phi$ is an isomorphism, it is enough to show that it is surjective.

Recall that $h_{\pm n}$ are in relation to $p_{\pm n}$ as complete symmetric functions are in relation to power sum functions. As $d_1 \geq \ldots \geq d_s$ range over the natural numbers, the monomials $p_{-d_1} \ldots p_{-d_s}|\emptyset|$ form a $\mathbb{F}$-basis of $F_u$, which is upper triangular in the basis consisting of monomials $h_{-d_1} \ldots h_{-d_s}|\emptyset|$. Meanwhile, (2.75) implies that:

$$W_{-d,1} = uh_{-d} + u \sum_{i=1}^{\infty} h_{-d-i}h_i$$

If $d_1 \geq \ldots \geq d_s > 0$, we may iterate the equality above to obtain:

$$W_{-d_1,1} \ldots W_{-d_s,1} = u^s h_{-d_1} \ldots h_{-d_s} + u^s \sum_{i_1 \ldots i_s \in \mathbb{N} \cup \{0\}} h_{-d_1-i_1}h_{i_1} \ldots h_{-d_s-i_s}h_{i_s}$$

(2.76)

The commutation relation (2.29) implies that for all $a, b > 0$, the product $h_a h_{-b}$ is a linear combination of $h_{a+j}h_{a-j}$ as $j \geq 0$. Therefore, Eq. (2.76) implies:

$$W_{-d_1,1} \ldots W_{-d_s,1} = u^s h_{-d_1} \ldots h_{-d_s} + \sum_{f_1, \ldots, f_s \geq 0} \text{constant} \cdot h_{-e_1} \ldots h_{-e_s}h_{f_1} \ldots h_{f_s}$$
where $e_1 \geq d_1$, if $e_1 = d_1$ then $e_2 \geq d_2$, if $e_1 = d_1$ and $e_2 = d_2$ then $e_3 \geq d_3$ etc. We conclude that if $d_1 \geq d_2 \geq \cdots \geq d_r > 0$, we have the following equality in $M_u$:

$$W_{-d_1,1} \cdots W_{-d_r,1}|[\emptyset] = h_{-d_1} \cdots h_{-d_r}|[\emptyset] + \text{higher order terms}$$

where we say that the monomial $h_{-e_1} \cdots h_{-e_t}$ (with $e_1 \geq \cdots \geq e_t > 0$) has higher order than $h_{-d_1} \cdots h_{-d_r}$ if $e_1 > d_1$, or if $e_1 = d_1$ and $e_2 > d_2$, or if $e_1 = d_1$ and $e_2 = d_2$ and $e_3 > d_3$ etc. Therefore, the basis $W_{-d_1,1} \cdots W_{-d_r,1}|[\emptyset]$ is upper triangular in the basis $h_{-d_1} \cdots h_{-d_r}|[\emptyset]$, and this concludes the surjectivity of the map $M_u \rightarrow F_u$. □

3 The Moduli Space of Sheaves on $\mathbb{P}^2$

3.1. In this section, we construct level $r$ good representations of $A$, in the guise of the $K$-theory of the moduli space of rank $r$ sheaves. To introduce this algebraic variety, consider the projective plane and fix a line $\infty \subset \mathbb{P}^2$. We let $\mathcal{M}$ denote the moduli space of rank $r$ torsion free sheaves $\mathcal{F}$ on $\mathbb{P}^2$, together with an isomorphism:

$$\mathcal{F}|_{\infty} \cong \mathcal{O}(\infty)^{\oplus r} \tag{3.1}$$

This latter condition forces $c_1(\mathcal{F}) = 0$, but $c_2(\mathcal{F})$ is still free to range over the non-negative integers. For $n \geq 0$, we denote by $\mathcal{M}_n \subset \mathcal{M}$ the connected component of rank $r$ sheaves of second Chern class $n \cdot [pt]$. Its tangent spaces are given by:

$$\Tan_\mathcal{M}_n = \text{Ext}^1(\mathcal{F}, \mathcal{F}(-\infty)) \tag{3.2}$$

by the Kodaira–Spencer isomorphism. Using the Riemann–Roch theorem, one can easily prove that $\mathcal{M}_n$ is smooth of dimension $2rn$. We have a universal sheaf $\mathcal{U}_n$ on $\mathcal{M}_n \times \mathbb{P}^2$, and its first derived direct image under the standard projection $pr_1 : \mathcal{M}_n \times \mathbb{P}^2 \rightarrow \mathcal{M}_n$ is called the **tautological vector bundle**:

$$\mathcal{V}_n = R^1pr_1(\mathcal{U}_n(-\infty)) \tag{3.3}$$

on $\mathcal{M}_n$. The twist in formula (3.3) is by the pull-back of the divisor $\infty \subset \mathbb{P}^2$ to $\mathcal{M}_n \times \mathbb{P}^2$, and it forces $R^0$ and $R^2$ to vanish. Therefore, $\mathcal{V}_n$ is a vector bundle, and a standard application of the Riemann–Roch Theorem shows that it has rank $n$.

3.2. Consider the torus $T = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r$, which acts on the moduli space $\mathcal{M}$ in the following way: the first two factors rescale the coordinate directions of $\mathbb{P}^2$ and keep the line $\infty$ invariant, and the torus $(\mathbb{C}^*)^r$ acts by left multiplication on the framing isomorphism (3.1). We will consider the equivariant $K$-theory groups $K_T(\mathcal{M}_n)$, which are all modules over:

$$K_T(pt) = \mathbb{Z}[q_1^\pm 1, q_2^\pm 1, u_1^\pm 1, \ldots, u_r^\pm 1]$$

Here, $q_1$ and $q_2$ are equivariant parameters in the factors of $\mathbb{C}^* \times \mathbb{C}^*$ (which will be identified with the homonymous parameters of Sect. 2.1), and $u_i$ are equivariant parameters of the maximal torus of $GL_r$ (which will be identified with the parameters of Sect. 2.27). It will be convenient to localize these groups:

$$K_n = K_T(\mathcal{M}_n) \otimes_{K_T(pt)} \text{Frac}(K_T(pt))$$
and work with them all together:

\[ K = \bigoplus_{n \geq 0} K_n \]

In the remainder of this section, we will recall the action \( \mathcal{A} \curvearrowright K \) which makes \( K \) into a good level \( r \) representation. Before doing so, let us describe the spaces \( K_n \) using tautological classes. Consider the assignment:

\[
\gamma_n \mapsto \tilde{f}_n := f \left( e_\text{Chern roots of } \mathcal{V}_n \right) \in K_n
\]  

(3.4)

When \( f = e_k(x_1, x_2, \ldots) \) is an elementary symmetric function, we have \( \tilde{f}_n = \left[ \wedge^k \mathcal{V}_n \right] \).

Since the assignment \( \gamma_n \) is additive and multiplicative, any class in its image is a linear combination of products of exterior powers of \( \mathcal{V}_n \). As \( \mathcal{V}_n \) is the tautological bundle, elements in the image of \( \gamma_n \) will be called tautological classes.

**Proposition 3.3** (See, for example, [29]). For all \( n \geq 0 \), the map \( \gamma_n \) is surjective, i.e. the \( \mathbb{Q}(q_1, q_2, u_1, \ldots, u_r) \)-algebra \( K_n \) is generated by tautological classes.

### 3.4

In this subsection, we will use the language of partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \). To any such partition, we can associate its Young diagram, which is a collection of lattice squares in the first quadrant. For example, the following is the Young diagram of the partition \( \lambda = (4, 3, 1) \):

![Young Diagram](image)

**Fig. 3.4**

Given two partitions, we will write \( \lambda \geq \mu \) if the Young diagram of \( \lambda \) completely contains that of \( \mu \). In this case, the collection of boxes \( \lambda \setminus \mu \) will be called a skew partition. If such a skew partition \( \lambda \setminus \mu \) has \( n \) boxes, then a labeling of these boxes with the numbers 1, \ldots, \( n \) is called a standard Young tableau if the numbers decrease as we go up and to the right in the partition.

For a fixed natural number \( r \), we will use the term \( r \)-partition:

\[ \lambda = (\lambda_1^1, \ldots, \lambda_r^r) \]

for an ordered collection of ordinary partitions. The size of \( \lambda \) is defined to be \( |\lambda| = |\lambda_1^1| + \cdots + |\lambda_r^r| \), and we mimic the notation of usual partitions by writing \( \lambda \vdash n \) for “\( \lambda \) is an \( r \)-partition of size \( n \)”.

A box in an \( r \)-partition is determined not only by its position
in the plane, but also by which of $\lambda^1, \ldots, \lambda^r$ it lies in. We collect this information in the weight of the box, namely the monomial:

$$\chi_{\Box} = u_k q^i_1 q^j_2$$

(3.5)

for the box $\Box \in \lambda^k$ whose bottom left corner lies at coordinates $(i, j)$. We will say that $\lambda \geq \mu$ if we have $\lambda^i \geq \mu^i$ for all $i \in \{1, \ldots, r\}$. If this is the case, the $r$-tuple $\lambda \setminus \mu = \{\lambda^1 \setminus \mu^1, \ldots, \lambda^r \setminus \mu^r\}$ will be called a skew $r$-partition. If such a skew $r$-partition consists of $n$ boxes, then a labeling of these boxes with the numbers $1, \ldots, n$ is called a standard Young tableau if the numbers decrease as we go up and to the right in each constituent partition $\lambda^i \setminus \mu^i, i \in \{1, \ldots, r\}$. Note that there is no restriction between the numbers placed in $\Box \in \lambda^i \setminus \mu^i$ and $\Box' \in \lambda^j \setminus \mu^j$ if $i \neq j$.

3.5. The reason why we introduced the language of the previous subsection is that the torus fixed points of the action $T \curvearrowright \mathcal{M}$ are naturally indexed by $r$-partitions $\lambda = (\lambda^1, \ldots, \lambda^r)$. Specifically, fixed points are of the form:

$$\mathcal{I}_\lambda := \mathcal{I}_{\lambda^1} \oplus \cdots \oplus \mathcal{I}_{\lambda^r}$$

where for any usual partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$, we write $\mathcal{I}_\mu \subset \mathcal{O}_{\mathbb{P}^2}$ for the sheaf corresponding to the ideal $I_\mu = (x^{\mu_1}, x^{\mu_2} y, x^{\mu_3} y^2, \ldots) \subset \mathbb{C}[x, y]$. The $K$-theory class of the skyscraper sheaf at $\mathcal{I}_\lambda$ will be denoted by $[\lambda] \in K$, and for convenience we will renormalize it to:

$$|\lambda\rangle := \frac{[\lambda]}{[\wedge^\bullet \text{Tan}_\lambda \mathcal{M}_n]} \in K_n$$

(3.6)

if $n = |\lambda| \Leftrightarrow \lambda \vdash n$. The symbol $\wedge^\bullet$ is defined in (3.34) below. The reason for the normalization (3.6) is a consequence of the Thomason localization theorem, also known as the $K$-theoretic version of Atiyah–Bott localization, which reads:

$$c = \sum_{\lambda \vdash n} c|\lambda\rangle \cdot |\lambda\rangle \quad \forall c \in K_n$$

(3.7)

The situation we will mostly be concerned with is when $c = \bar{f}_n$ is a tautological class, for some symmetric polynomial $f$. In this case, we have:

$$\bar{f}_n|\lambda = f(\lambda) := f(\ldots, \chi_{\Box}, \ldots)_{\Box \in \lambda}$$

The notation above means that we plug the weights of the boxes of the $r$-partition $\lambda$, which are defined in (3.5), into the arguments of the symmetric polynomial $f$. Combining this with (3.7), we conclude that:

$$\bar{f}_n = \sum_{\lambda \vdash n} f(\lambda) \cdot |\lambda\rangle$$
3.6. We are now ready to define the action $\mathcal{A} \curvearrowright K$, following [29] (note that this action also appears in [19,37], in either case presented in a different language). Let us write $Z = z_1 + \cdots + z_k$ for the finite alphabet of variables that appears in the definition of shuffle elements, and $X = x_1 + x_2 + \cdots$ for the infinite alphabet of variables that appears in the definition of tautological classes. Also write:

$$\tau(z) = \prod_{i=1}^{r} \left( 1 - \frac{z}{u_i} \right)$$

(3.8)

We will often use the language of plethysms when it comes to symmetric polynomials. Specifically, given a symmetric polynomial in infinitely many variables $f(x_1, x_2, \ldots)$ and a finite collection of variables $Z = z_1 + \cdots + z_k$, we will write:

$$f(X + Z) = f(x_1, x_2, \ldots, z_1, \ldots, z_k)$$

and think of it as a symmetric polynomial in the $x_i$’s with coefficients depending on the $z_j$’s. The notation $f(X - Z)$ does not have as straightforward a presentation as above, but we may rigorously define the assignments $f(X) \sim f(X \pm Z)$ as the homomorphisms defined on the ring of symmetric polynomials in $X$ by setting:

$$\sum_{i=1}^{\infty} x_i^n \sim \sum_{i=1}^{\infty} x_i^n \pm (z_1^n + \cdots + z_k^n)$$

for all $n \in \mathbb{N}$. Since power sum functions generate the ring of symmetric polynomials, this completely defines the notation $f(X \pm Z)$ for any symmetric $f$.

**Theorem 3.7** [29]. For any $k, d > 0, n \geq 0$, any symmetric polynomial $f(X) = f(x_1, x_2, \ldots)$ and any shuffle element $R(Z) = R(z_1, \ldots, z_k)$, the assignments $c = q^r$,

$$P_{0, \pm d} \cdot \tilde{f}_n = \pm f \cdot \left[ \sum_{i=1}^{r} u_i^{\pm d} q^{\delta_i - d} - (1 - q_1^d)(1 - q_2^d) \sum_{j=1}^{\infty} x_j^{\pm d} \right]$$

(3.9)

$$R^{-} \cdot \tilde{f}_n = \int_{+} f(X - Z) \zeta \left( \frac{Z}{X} \right) \frac{R(Z) \tau(qZ)}{\zeta \left( \frac{Z}{Z} \right)}$$

(3.10)

$$R^{-} \cdot \tilde{f}_n = q^{k(1-r)} \int_{-} f(X + Z) \zeta \left( \frac{X}{Z} \right)^{-1} \frac{R(Z) \tau(Z)^{-1}}{\zeta \left( \frac{Z}{Z} \right)}$$

(3.11)

give rise to an action $\mathcal{A} \curvearrowleft K$. The integrals $\int_{+}, \int_{-}$ will be defined in Remark 3.8 below. The right-hand sides of (3.10) and (3.11) use the multiplicative notation:

$$\zeta \left( \frac{X}{Z} \right) := \prod_{i=1}^{k} \prod_{j=1}^{k} \zeta \left( \frac{x_i}{z_j} \right), \quad \zeta \left( \frac{Z}{Z} \right) := \prod_{1 \leq i \neq j \leq k} \zeta \left( \frac{z_i}{z_j} \right)$$

(3.12)

If $n < k$, then the right-hand side of (3.11) is defined to be 0.
Remark 3.8. The normal ordered integrals \( \int_{\pm} \) are defined in [29], but we will only need to invoke them when \( R \) is a shuffle element of the form:

\[
R(z_1, \ldots, z_k) = \text{Sym} \left[ \frac{\rho(z_1, \ldots, z_k)}{\prod_{i=1}^{k-1} \left( 1 - \frac{qz_{i+1}}{z_i} \right)} \prod_{1 \leq i < j \leq k} \zeta \left( \frac{z_i}{z_j} \right) \right]
\]

(3.13)

for a Laurent polynomial \( \rho \). For this choice of \( R \), the right-hand sides of (3.10) and (3.11) are defined as the following contour integrals (write \( Dz = \frac{dz}{z\pi i} \)):

\[
\int_+ = \int_{z_k > \cdots > z_1 > X} f(X - \sum_{i=1}^k z_i) \prod_{i=1}^k \left[ \zeta \left( \frac{z_i}{X} \right) \tau(qz_i) \right] \frac{\rho(z_1, \ldots, z_k) Dz_1 \cdots Dz_k}{\prod_{i=1}^{k-1} \left( 1 - \frac{qz_{i+1}}{z_i} \right) \prod_{i < j} \zeta \left( \frac{z_i}{z_j} \right)}
\]

(3.14)

\[
\int_- = \int_{z_1 > \cdots > z_k > X} f(X + \sum_{i=1}^k z_i) \prod_{i=1}^k \left[ \zeta \left( \frac{X}{z_i} \right) \tau(z_i) \right] \frac{\rho(z_1, \ldots, z_k) Dz_1 \cdots Dz_k}{\prod_{i=1}^{k-1} \left( 1 - \frac{qz_{i+1}}{z_i} \right) \prod_{i < j} \zeta \left( \frac{z_i}{z_j} \right)}
\]

In the right-hand side, the notation \( z > X \) means that the variable \( z \) runs over a contour that surrounds all of the \( X \) variables. More generally, in the integral \( \int_- \), the contour of \( z_1 \) surrounds the contour of \( z_2 \), which surrounds the contour of \( z_3, \ldots \), which surrounds the contour of \( z_k \), which in turn surrounds all the \( X \) variables. 0 and \( \infty \) must not be contained in any of the contours.

Remark 3.9. Formulas (3.10) and (3.11) continue to hold if we multiply \( f(X) \) by a product of the form \( \prod_j \prod_{i=1}^\infty (x_i - t_j)^{\pm1} \) for some formal symbols \( t_j \). Then the notation \( f(X \pm Z) \) may pick up poles from various factors \( \prod_j \prod_{i=1}^k (x_i - t_j)^{\pm1} \), and in this case, the normal-ordered integrals \( \int_{\pm} \) must be defined such that the complex numbers \( \{t_j\} \), together with 0 and \( \infty \), lie outside the \( z_1, \ldots, z_k \) contours.

3.10. In [29], we proved that formulas (3.9)–(3.11) imply the following relations in the basis of fixed points \( |\lambda\rangle \), and are in fact equivalent to them upon computing the integrals (3.14) by a residue computation:

\[
\langle \mu | P_{0,\pm d} | \lambda \rangle = \pm \delta^{\mu}_{\lambda} \sum_{i=1}^r u_i^{\pm d} q^{\delta_{d}}(1 - q_1^d)(1 - q_2^d) \prod_{\square \in \lambda} \chi^\pm_{\square}
\]

(3.15)

\[
\langle \lambda | R^- | \mu \rangle = R(\lambda \setminus \mu) \prod_{\square \in \lambda \setminus \mu} \left[ \frac{(1 - q_1)(1 - q_2)}{1 - q} \zeta \left( \frac{\chi_{\square \setminus \mu}}{\chi_{\mu}} \right) \tau(q \chi_{\square})^{-1} \right]
\]

(3.16)

\[
\langle \mu | R^- | \lambda \rangle = R(\lambda \setminus \mu) \prod_{\square \in \lambda \setminus \mu} \left[ \frac{(1 - q_1)(1 - q_2)}{q \zeta^{-1}(1 - q)} \zeta \left( \frac{\chi_{\lambda \setminus \mu}^{-1}}{\chi_{\mu}} \right) \tau(\chi_{\square})^{-1} \right]
\]

(3.17)

In formulas (3.15)–(3.17), \( \zeta \left( \frac{\chi_{\square}}{z} \right) \) is multiplicative notation for \( \prod_{\square \in \lambda} \zeta \left( \frac{\chi_{\square}}{z} \right) \). If \( \lambda \nless \mu \), then the right-hand sides of (3.16) and (3.17) are defined to be zero. In particular, if \( R \)
is a shuffle element of the form (3.13), then it is easy to see that:

\[ \langle \lambda | R \leftarrow | \mu \rangle = \sum_{\text{tableau of shape } \lambda \setminus \mu} \frac{\rho(\chi_1, \ldots, \chi_k)}{\prod_{i=1}^{k-1} 1 - q^{Z_i+1}/\chi_i} \prod_{1 \leq i < j \leq k} \zeta \left( \frac{\chi_i}{\chi_j} \right) \prod_{i=1}^{k} \left[ (1 - q) \left( 1 - q^{2} \right) \zeta \left( \frac{\chi_i}{\chi_\mu} \right) \tau \left( q \chi_i \right) \right] \]  

(3.18)

\[ \langle \mu | R \rightarrow | \lambda \rangle = \sum_{\text{tableau of shape } \lambda \setminus \mu} \frac{\rho(\chi_1, \ldots, \chi_k)}{\prod_{i=1}^{k-1} 1 - q^{Z_i+1}/\chi_i} \prod_{1 \leq i < j \leq k} \zeta \left( \frac{\chi_i}{\chi_j} \right) \prod_{i=1}^{k} \left[ (1 - q) \left( 1 - q^{2} \right) \zeta \left( \frac{\chi_\lambda}{\chi_i} \right)^{-1} \tau \left( \chi_i \right)^{-1} \right] \]  

(3.19)

where \( \chi_1, \ldots, \chi_k \) denote the weights of the boxes labeled 1, \ldots, k in the standard Young tableau \( T \). The argument for why formulas (3.16)–(3.17) imply (3.18)–(3.19) is explained in detail in loc. cit.: the main idea is that, up to a product of linear factors, (3.16) and (3.17) are given by evaluating a shuffle element \( R \) at the weights of a skew \( r \)-partition \( \lambda \setminus \mu \). When \( R \) is presented as a symmetrization \( \text{Sym} \ldots \) as in (3.13), this evaluation may be computed by adding together the specializations of \( \ldots \) at all labelings of the boxes of \( \lambda \setminus \mu \) with the numbers 1, \ldots, k. However, because \( \zeta(q^{-1}) = \zeta(q_2^{-1}) = 0 \), such a labeling produces a non-zero contribution if and only if the labels decrease as we go up and to the right in the \( r \)-partition. This precisely says that the labeling must be a standard Young tableau.

3.11. Formulas (3.18) and (3.19) allow us to write down the operators \( W_{d,k} \in \hat{A}^1 \) in the basis \( \{ |\lambda\rangle, \lambda \text{ } r\text{-partition} \} \) of the level \( r \) representation \( K \). We will do so by invoking formula (2.50):

\[ W_{d,k} = \sum_{k-\leftrightarrow, k-\rightarrow} T_{d-\leftrightarrow, k-\rightarrow} E_{0,k_0} T_{d-\rightarrow, k-\rightarrow} \cdot q^{(k^1)d-} \]  

(3.20)

where the sum goes over \( d-\leftrightarrow, d-\rightarrow \in \mathbb{N} \) and \( k-\leftrightarrow, k_0, k-\rightarrow \in \mathbb{N} \cup \{0\} \). Since the shuffle elements \( T_{d,k} \) are given by (2.51), then (3.18)–(3.19) imply:

\[ \langle \mu | T_{d-\leftrightarrow, k-\rightarrow} \rangle = \sum_{\text{tableau of shape } \mu \setminus \nu} (-1)^{k_{\mu} - 1} \chi_{d-\leftrightarrow} \prod_{i=1}^{d_{\leftrightarrow}-1} 1 - q^{Z_i+1}/\chi_i \prod_{1 \leq i < j \leq d_{\leftrightarrow}} \zeta \left( \frac{\chi_i}{\chi_j} \right) \prod_{i=1}^{d_{\rightarrow}} \left[ (1 - q) \left( 1 - q^{2} \right) \zeta \left( \frac{\chi_i}{\chi_\mu} \right) \tau \left( q \chi_i \right) \right] \]  

(3.21)
\[ \langle \nu | T_{d_{\to}, k_{\to}} | \lambda \rangle = \sum_{\text{tableau of shape } \lambda \setminus \nu} (-1)^{k_{\to} - 1} \chi_{d_{\to}}^{-1} \prod_{i=1}^{d_{\to} - 1} \left( 1 - \frac{q \chi_{i+1}}{\chi_i} \right) \prod_{1 \leq i < j \leq d_{\to}} \zeta_{( \chi_i \chi_j )} \left( \frac{\chi_{j}}{\chi_{i}} \right) \prod_{i=1}^{d_{\to}} \left[ (1 - q_1)(1 - q_2) \zeta_{\tau \chi_i} - 1 \right] \]

(3.22)

for any skew partitions \( \mu \setminus \nu \) and \( \lambda \setminus \nu \) of size \( d_{\to} \) and \( d'_{\to} \), respectively. Finally, because of (3.15), we have:

\[ \langle \nu' | E_{0, k_0} | \nu \rangle = \delta_{\nu \nu'} \text{Res}_{y = \infty} \left[ (-1)^{k_0} \prod_{i=1}^{r} \left( 1 - \frac{u_i}{y} \right) \prod_{\Box \in \nu} \zeta_{\Box} \left( \frac{\chi_{\Box}}{y} \right) \frac{dy}{y} \right] \]

(3.23)

Therefore, we may restate (3.20) by saying that the matrix coefficient \( \langle \mu | W_{d, k} | \lambda \rangle \) is a sum over all \( r \)-partitions \( \nu \subset \lambda \cap \mu \) of products of the matrix coefficients as in (3.21), (3.22), (3.23), thus proving the second formula in Theorem 1.2. This formula will allow us to show that the action \( A \backsim K \) factors through an action of the \( q \)-W-algebra \( A_r \), as introduced in Definition 2.24.

**Theorem 3.12.** There is an action \( A_r \backsim K \), with respect to which the latter is isomorphic to the Verma module \( M_{u_1, \ldots, u_r} \) of (2.71). The parameters \( u_1, \ldots, u_r \) are defined as the equivariant parameters of \( K \), hence there is no abuse of notation.

**Proof.** We will actually show that \( A_r^{\text{ext}} \) acts on \( K \), hence the desired action will be a consequence of the homomorphism (2.68). In fact, we have actions of:

\[ \tilde{A}^\uparrow \subset \tilde{A}^{\text{ext}} \subset \text{a completion of } A \]

on \( K \), all induced by the fact that Theorem 3.7 provides an action \( A \backsim K \), for which the latter is a good representation. To show that the action factors through the subquotient \( A_r^{\text{ext}} \), we need to show that this action respects the relations:

\[ W_r(x) = u \left[ \sum_{n=0}^{\infty} \frac{h_n}{x^n} \right] \left[ \sum_{n=0}^{\infty} \frac{h_R}{x^n} \right] \]

(3.24)

\[ W_k(x) = 0 \quad \forall \ k > r \]

(3.25)

where \( u = u_1 \ldots u_r \). First of all, let us prove that \( K \) is generated by the coefficients of the \( q \)-\( W \)-currents acting on the vacuum vector \( |\emptyset\rangle \) (here, \( \emptyset \) denotes the empty \( r \)-partition). The reason for this is that the operators \( P_{n,1} \in A \) lie in the \( q \)-\( W \)-algebra, according to the case \( k = 1 \) of (2.48). From formula (2.21), we see that the operators \( P_{n,1} \) generate the entire upper shuffle algebra \( A^\uparrow \) (alternatively, this follows from Theorem 2.2 and Proposition 2.9), and the upper shuffle algebra acting on \( |\emptyset\rangle \) generates \( K \) (a quick argument for this is to observe that \( P_{0,d} \in A^\uparrow \), and \( \{P_{0,d}\}_{d>0} \) form a commuting family of operators which are diagonal in the basis \( |\lambda\rangle \) with distinct spectra). Thus, proving (3.24) and (3.25)
reduces to proving that:

\[
W_r(x) \prod_{(d,k')} \text{various } W_{d,k'} |\emptyset\rangle = u \left[ \sum_{n=0}^{\infty} \frac{h_{-n}}{x^{-n}} \right] \left[ \sum_{n=0}^{\infty} \frac{h_n}{x^n} \right] \prod_{(d,k')} \text{various } W_{d,k'} |\emptyset\rangle \quad \text{(3.26)}
\]

\[
W_k(x) \prod_{(d,k')} \text{various } W_{d,k'} |\emptyset\rangle = 0 \quad \forall k > r \quad \text{(3.27)}
\]

**Claim 3.13.** Formulas (3.26) and (3.27) reduce to the fact that:

\[
W_r(x) |\emptyset\rangle = u \left[ \sum_{n=0}^{\infty} h_{-n} \right] x^{-n} |\emptyset\rangle \quad \text{and} \quad W_k(x) |\emptyset\rangle = 0 \quad \forall k > r \quad \text{(3.28)}
\]

**Proof of Claim 3.13.** Suppose that (3.28) holds. Then one may prove that the left-hand side of (3.27) vanishes by commuting the current \( W_k(x) \) past the various \( W_{d,k'} \)'s. This uses the commutation relations (2.62), which express the commutator of \( W_k(x) \) and \( W_{k'}(y) \) in terms of currents \( W_{k''}(x) W_{k'''}(y) \) with \( k''' > \max(k, k') \). The same argument shows that (3.28) \( \Rightarrow \) (3.26), once one shows that \( W_r(x) \) and:

\[
u \left[ \sum_{n=0}^{\infty} h_{-n} \right] x^{-n} |\emptyset\rangle
\]

have the same commutation relations with \( W_1(y), \ldots, W_r(y) \). This is an immediate consequence, which we leave as an exercise to the interested reader, of (2.60)–(2.62) and the already proved fact that \( W_k(x) = 0 \) for \( k > r \). \( \square \)

It therefore remains to prove (3.28), so let us recall the LDU decomposition (3.20). Since \( T_{d,k} \) annihilates the vacuum vector unless \( d = 0 \), we have:

\[
W_{d,k} |\emptyset\rangle = \sum_{k_k, k_0 \geq 0} T_{d,k} E_{0,k_0} |\emptyset\rangle
\]

We may sum the above expression over all \( d \in \mathbb{Z} \), and obtain the formula:

\[
W_k(x) = \text{Res}_{y=\infty} \left[ T(x, y)^{-\leftarrow} E(y) |\emptyset\rangle \cdot (-y)^k \frac{dy}{y} \right] \quad \text{(3.29)}
\]

where \( E(y) = \sum_{k=0}^{\infty} (-1)^k \frac{E_{0,k}}{y^k} \) and:

\[
T(x, y) = \sum_{d=0}^{\infty} \text{Sym} \left[ \frac{x^d}{(1-y/z_d)^{d-1}} \prod_{1 \leq i < j \leq d} \left( 1 - \frac{q_{i,j}}{z_{i,j}} \right) \right] \quad \text{(3.30)}
\]

Recall that (3.23) implies that \( E(y) |\emptyset\rangle = \prod_{i=1}^{r} \left( 1 - \frac{u_i}{y} \right) |\emptyset\rangle \). Thus (3.29) becomes:

\[
W_k(x) |\emptyset\rangle = \text{Res}_{y=\infty} \left[ T(x, y)^{-\leftarrow} |\emptyset\rangle \cdot (-y)^k \prod_{i=1}^{r} \left( 1 - \frac{u_i}{y} \right) \frac{dy}{y} \right]
\]
Now we will use (3.18), or equivalently (3.21), to compute the matrix coefficients of the above vector in terms of the basis $|\lambda\rangle$. Letting $|\lambda\rangle = d$, we have:

$$\langle \lambda | W_k(x) | \emptyset \rangle = \text{Res}_{y=\infty} (-y)^k \prod_{i=1}^r \left( 1 - \frac{u_i}{y} \right) T \text{ a standard Young tableau of shape } \lambda \sum \left( -y \right)^k \prod_{i=1}^r \left( 1 - \frac{u_i}{y} \right) \sum \zeta \left( \frac{\chi_i}{\chi_j} \right) \prod_{i=1}^d \left( 1 - \frac{q\chi_i+1}{\chi_i} \right) \prod_{1 \leq i < j \leq d} \zeta \left( \frac{\chi_i}{\chi_j} \right) \prod_{i=1}^d \left( 1 - \frac{q\chi_i+1}{\chi_i} \right) \prod_{i=1}^d \left( 1 - \frac{q\chi_i+1}{\chi_i} \right) \prod \frac{dy}{y}$$

The integral of (3.31) is equal to the sum of its residues at the poles $y = 0$ and $y = \chi_d$. However, note that the integrand does not actually have a pole at $y = \chi_d$, because in any standard Young tableau of shape $\lambda$, we have $\chi_d \in \{u_1, \ldots, u_r\}$, and thus the pole at $y = \chi_d$ is canceled by one of the factors on the first line of (3.31). Moreover, when $k > r$ there is also no pole at $y = 0$, and hence the integral is 0. This implies that $W_k(x) | \emptyset \rangle = 0$ for all $k > r$, as in (3.28). By the same reason, when $k = r$ the integral is equal to the residue at the simple pole at $y = 0$:

$$\langle \lambda | W_r(x) | \emptyset \rangle = \text{Res}_{y=\infty} \left[ \frac{(-y)^k E(y)}{y^{N-1}} \right] = e_k(u_1, \ldots, u_r)|\emptyset\rangle$$

and this implies (3.28). Now that we have shown that $A_r \rightarrow A_r^{\text{ext}} \lhd K$, let us show that the representation $K$ is isomorphic to the Verma module $M_{u_1, \ldots, u_r}$. The map:

$$M_{u_1, \ldots, u_r} \rightarrow K, \quad |\emptyset\rangle \mapsto |\emptyset\rangle$$

is well-defined because the currents $W_{d,k}$ for $d > 0$ annihilate the vacuum vector for degree reasons. Meanwhile, the LDU decomposition (3.29) implies:

$$W_{0,k}|\emptyset\rangle = \text{Res}_{y=\infty} \left[ (-y)^k E(y)|\emptyset\rangle \frac{dy}{y} \right] = e_k(u_1, \ldots, u_r)|\emptyset\rangle$$

Moreover, the map (3.32) is surjective, because as we have shown in the beginning of the proof, the vector space $K$ is generated by the $q$-$W$-algebra acting on the vacuum. Therefore, it is enough to show that the graded dimension of $M_{u_1, \ldots, u_r}$ is less than or equal to that of $K$. As a consequence of the fact that the monomials (2.69) linearly span the $q$-$W$-algebra, the monomials:

$$W_{d_1,k_1} \cdots W_{d_t,k_t}|\emptyset\rangle$$

(3.33)
with \( \frac{d_1}{k_1} \leq \cdots \leq \frac{d_r}{k_r} \) and any \( d_i < 0, k_i \in \{1, \ldots, r\} \), linearly generate \( M_{u_1, \ldots, u_r} \). Monomials (3.33) are therefore in 1-to-1 correspondence with unordered collections:

\[
\left( \frac{d_a_1, 1}{a_1 \geq \cdots \geq a_e > 0}, \ldots, \left( \frac{d_c_1, r}{c_1 \geq \cdots \geq c_g > 0} \right) \right)
\]

The number of such collections of any total degree \( n = \sum d_i \) is equal to the number of \( r \)-partitions of size \( r \), which is also equal to the dimension of the degree \( n \) piece of \( K \). This proves that the surjective map (3.32) is bijective, hence an isomorphism. \( \Box \)

3.14. Given an equivariant vector bundle \( \mathcal{V} \) on a space \( X \) with an action of a torus \( T \), we will define two operations on it. The first is the total exterior power:

\[
\wedge^\bullet \mathcal{V} = \sum_{i=0}^{\text{rank } \mathcal{V}} (-1)^i \left[ \wedge^i \mathcal{V}^\vee \right] \in K_T(X) \quad (3.34)
\]

Note the dual sign in the right-hand side, which is placed there to ensure that \( \wedge^\bullet \) naturally deforms the Euler class of the vector bundle \( \mathcal{V} \), in the same way \( 1 - e^{-x} \) deforms \( x \). The total exterior power is multiplicative in \( \mathcal{V} \). The second operation on vector bundles is \( T \)-equivariant Euler characteristic, which is additive in \( \mathcal{V} \):

\[
\chi_T(X, \mathcal{V}) = \sum_{i=0}^{\infty} (-1)^i \left[ T\text{-character of } H^i(X, \mathcal{V}) \right] \in \text{Rep}(T)
\]

We will now show that for any shuffle element \( R \in S \), the operators \( R^\leftarrow, R^\rightarrow \) of (3.10), (3.11) are (up to a constant) adjoint with respect to the inner product:

\[
(\alpha, \beta) = \chi_T \left( \mathcal{M}_n, \frac{\alpha \otimes \beta}{(\det \mathcal{V}_n)^{\otimes r}} \right) \prod_{i=1}^{r} (-u_i)^n \quad (3.35)
\]

The basis \( |\lambda\rangle \) is orthogonal with respect to this pairing:

\[
(|\lambda\rangle, |\mu\rangle) = \delta_{\mu}^{\lambda} [\wedge^\bullet \text{tan}_\lambda \mathcal{M}_n] \quad (3.36)
\]

where \( n = |\lambda| \) and:

\[
\wedge^\bullet \text{tan}_\lambda \mathcal{M}_n = \wedge^\bullet \text{tan}_\lambda \mathcal{M}_n \otimes \frac{(\det \mathcal{V}_{|\lambda|})^{\otimes r}}{\prod_{i=1}^{r} (-u_i)^{|\lambda|}}
\]

We will use the following expression (see, for example, [29]) for the \( K \)-theory class of the tangent space to \( \mathcal{M}_n \):

\[
[Tan \mathcal{M}_n] = \sum_{i=1}^{r} \left( \frac{\mathcal{V}_n}{u_i} + \frac{u_i}{q \mathcal{V}_n} \right) - \left( 1 - \frac{1}{q_1} \right) \left( 1 - \frac{1}{q_2} \right) \frac{\mathcal{V}_n}{\mathcal{V}_n} \quad (3.37)
\]
In the right-hand side of (3.37) we employ the shorthand notation:

$$\frac{\mathcal{V}}{\mathcal{W}} := [\mathcal{V}] \otimes [\mathcal{W}^\vee]$$

for any vector bundles (or $K$-theory classes) $\mathcal{V}$ and $\mathcal{W}$. If we recall that the restriction of $\mathcal{V}_n$ to the fixed point $\lambda$ is given by $\mathcal{V}_n|_\lambda = \sum_{\square} x_\square$, we obtain:

$$[\wedge^i \cdot \text{Tan}_\lambda \mathcal{M}_n] = \prod_{\square \in \lambda} \left(1 - \frac{u_i}{x_\square}\right) \prod_{\square, \square' \in \lambda} \zeta \left(\frac{x_\square}{x_\square'}\right)$$

$$\Rightarrow [\wedge^i \cdot \text{Tan}_\lambda \mathcal{M}_n] = \prod_{\square \in \lambda} \left(1 - \frac{x_\square}{u_i}\right) \prod_{\square, \square' \in \lambda} \zeta \left(\frac{x_\square}{x_\square'}\right)$$

We conclude that (3.36) reads:

$$(|\lambda\rangle, |\mu\rangle) = \frac{\delta_\mu^\lambda}{\prod_{\square \in \lambda} \tau(x_\square) \tau(q x_\square) \prod_{\square, \square' \in \lambda} \zeta \left(\frac{x_\square}{x_\square'}\right)}$$

Together with (3.16) and (3.17), this implies that:

$$(R^\rightarrow)^\dagger = q^{(1-r) \deg R} \cdot R^\leftarrow \quad \forall R \in S$$

where adjoint is defined with respect to the inner product (3.35).

3.15. Although not apparently clear from formulas (3.10) and (3.11), the operators $R^\leftarrow$ and $R^\rightarrow$ are given by geometric correspondences. To be precise, this is only true for shuffle elements $R$ of the form (3.13), which we will assume for the remainder of this section. Consider the so-called fine correspondence defined in [29]:

$$\mathcal{Z}_{n+k,n} \subset \mathcal{M}_{n+k} \subset \mathcal{M}_n$$

for any $k > 0$ and any natural number $n$, which is defined as the locus:

$$\mathcal{Z}_{n+k,n} = \left\{ \text{flags of sheaves } \mathcal{F}_{n+k} \subset \cdots \subset \mathcal{F}_n \text{ s.t. } \mathcal{F}_{n+i-1}/\mathcal{F}_{n+i} \text{ for } 1 \leq i \leq k \text{ are all length 1 skyscraper sheaves supported at the same point} \right\}$$

We will write $\mathcal{Z}_k = \sqcup_{n \in \mathbb{N}} \mathcal{Z}_{n+k,n}$. Note that $\mathcal{Z}_k$ is endowed with line bundles:

$$\mathcal{L}_1, \ldots, \mathcal{L}_k \in \text{Pic}_T(\mathcal{Z}_k)$$

with fibers given by:

$$\mathcal{L}_i|_{\mathcal{F}_{n+k} \subset \cdots \subset \mathcal{F}_n} = \Gamma \left(\mathbb{P}^2, \mathcal{F}_{n+k-i}/\mathcal{F}_{n+k-i+1}\right)$$
In (3.40), the maps $\pi_+$ and $\pi_-$ send a flag $F_{n+k} \subset \cdots \subset F_n$ to the sheaves $F_{n+k}$ and $F_n$, respectively. In loc. cit., we defined a virtual class on $Z_k$ which allows us to make sense of push-forward maps $\pi_{++}$ and $\pi_{--}$ on $K$-theory. For convenience, let us renormalize these maps as:

$$\tilde{\pi}_{++}(c) = \pi_{++}(c), \quad \tilde{\pi}_{--}(c) = \pi_{--}(c \cdot L_1^{-r} \cdots L_k^{-r}) \prod_{i=1}^{r} (-u_i)^{k}$$

This normalization was done precisely to ensure that the pairs $(\tilde{\pi}_{\pm \pm}, \pi_{\pm +})$ are adjoint with respect to the inner product (3.35) on $K_T(M_n)$ and the inner product:

$$(\alpha, \beta) = \chi_T \left( Z_{n+k,n}, \frac{\alpha \otimes \beta}{(\det V_{n+k})^{\otimes r}} \right) \prod_{i=1}^{r} (-u_i)^{n+k} \quad (3.43)$$

on the $K$-theory of $Z_k$. We have the following:

**Theorem 3.16** [29]. For a shuffle element $R$ as in (3.13), we have:

$$R^{\leftarrow} = \tilde{\pi}_{++} \left( \rho(L_1, \ldots, L_k) \cdot \pi_{++} \right)$$

$$R^{\rightarrow} = \tilde{\pi}_{--} \left( \rho(L_1, \ldots, L_k) \right) \frac{q^{k(r-1)}}{\pi_{++}}$$

This theorem establishes two things: first of all, the operators $R^{\leftarrow}$ and $R^{\rightarrow}$ of Theorem 3.7 are geometric in nature. Secondly, relation (3.39) on the adjointness of these operators follows from the adjointness of the correspondences $\tilde{\pi}_{++} \pi_{++}$ and $\tilde{\pi}_{--} \pi_{++}$. In the next section, we will make use of the following Proposition, whose proof follows that of Proposition 3.52 of [27] almost word-by-word, so we omit it.

**Proposition 3.17.** Assume $\rho(z_1, \ldots, z_k)$ is a Laurent polynomial, divided by linear factors of the form $z_i - y$, for constants $y$ in a certain finite set $Y$. Then:

$$\tilde{\pi}_{++} (\rho(L_1, \ldots, L_k)) = \int_{X' > z_1 > \ldots > z_k > Y \cup [0, \infty)} \frac{\rho(z_1, \ldots, z_k) \prod_{i=1}^{k} \left[ \xi \left( \frac{z_i}{\xi} \right) \tau(q z_i) Dz_i \right]}{\prod_{i=1}^{k-1} \left( 1 - qz_i \frac{1}{z_i} \right) \prod_{i < j} \xi \left( \frac{z_j}{z_i} \right)}$$

$$\tilde{\pi}_{--} (\rho(L_1, \ldots, L_k)) = \int_{Y \cup [0, \infty)] > z_1 > \ldots > z_k > X^\leftarrow} \frac{\rho(z_1, \ldots, z_k) \prod_{i=1}^{k} \left[ \xi \left( \frac{z_i}{\xi} \right) \right]}{\prod_{i=1}^{k-1} \left( 1 - qz_i \frac{1}{z_i} \right) \prod_{i < j} \xi \left( \frac{z_j}{z_i} \right)}$$

where the notation $z > z'$ is defined in Remark 3.8.

Because of the normal weights of $\pi_-$, the analogous formula for $\pi_{--}$ instead of $\tilde{\pi}_{--}$ involves replacing the factors:

$$\prod_{i=1}^{k} \frac{1}{\tau(z_i)} = \prod_{i=1}^{k} \prod_{j=1}^{r} \left( 1 - \frac{z_i}{u_j} \right)^{-1}$$

by

$$\prod_{i=1}^{k} \prod_{j=1}^{r} \left( 1 - \frac{u_j}{z_i} \right)^{-1}$$

The discrepancy between the two expressions is precisely the factor $\prod_{i=1}^{k} \prod_{j=1}^{r} \frac{-u_j}{z_i}$, which accounts for the discrepancy between the operators $\tilde{\pi}_{--}$ and $\pi_{--}$.
Remark 3.18. Let us explain the notation in Proposition 3.17, since it will feature in the following section. Let $X^+$ and $X^-$ denote tautological classes on the spaces $M_{n+k}$ and $M_n$ in (3.40), respectively, as well as their pull-backs to $Z_{n+k,n}$. For example, the notation $e_d(X^+)e_p(X^-)$ will refer to the class:

$$\wedge^a (\pi_n^* V_{n+k}) \cdot \wedge^b (\pi_n^* V_n) \in K_T (Z_{n+k,n})$$

There will be a slight ambiguity as to whether the notation $e_d(X^+)$ refers to a $K$-theory class on $M_{n+k}$ or its pull-back to $Z_{n+k,n}$, but the situation will always be clear from context. For example, because the right-hand sides of the formulas in Proposition 3.17 are $K$-theory classes in the targets of $\tilde{\Pi}_{++}$ and $\tilde{\Pi}_{--}$, respectively, they have no choice but to live on the spaces $M_{n+k}$ and $M_n$, respectively.

Remark 3.19. The contour that gives the integral for $\tilde{\Pi}_{--}$ in Proposition 3.17 precisely matches that of $\int_0$ in Remark 3.8. However, the contour in the integral for $\tilde{\Pi}_{++}$ is ordered as $X^+ > z_1 > \ldots > z_k > Y \cup [0, \infty]$, while the corresponding contour that defines $\int_0$ in Remark 3.8 would be $Y \cup [0, \infty] > z_k > \ldots > z_1 > X_+$. The discrepancy between the two contours is merely cosmetic, since it just involves looking at the Riemann sphere from “behind”, i.e. changing the variable $z \leftrightarrow 1/z$.

4 The Ext Operator

4.1. Keep the natural number $r$ fixed. We will now consider two different moduli spaces of rank $r$ sheaves, of degrees $n$ and $n'$, respectively:

$$M_{n,u} \quad \text{and} \quad M_{n',u'}$$

We assume that these moduli spaces are acted on by two different rank $r$ tori, whose equivariant parameters will be denoted by $u = (u_1, \ldots, u_r)$ and $u' = (u'_1, \ldots, u'_r)$, respectively. Therefore, we will write $K_{n,u} = K_{\mathbb{C}^r \times \mathbb{C}^r} (M_{n,u})$, with the understanding that the rank $r$ torus $(\mathbb{C}^*)^r$ has equivariant parameters $u$. With this in mind, consider the vector bundle $E_{n,n'}$ on $M_{n,u} \times M_{n',u'}$ with fibers given by:

$$E_{n,n'}|_{F,F'} = \text{Ext}^1 (F', F(-\infty))$$

The fact that $E_{n,n'}$ is a vector bundle follows from the vanishing of the corresponding Hom and Ext groups. In fact, an easy application of the Riemann–Roch theorem allows one to see that the rank of $E_{n,n'}$ is $r(n + n')$. Its $K$-theory class is given by:

$$[E_{n,n'}] = \sum_{i=1}^r \left( \frac{V_n}{u_i} + \frac{u_i}{qV_{n'}} \right) - \left( 1 - \frac{1}{q_1} \right) \left( 1 - \frac{1}{q_2} \right) \frac{V_n}{V_{n'}} \quad (4.1)$$

where the vector bundles $V_n$ and $V_{n'}$ are the pull-backs of the tautological bundles from $M_{n,u}$ and $M_{n',u'}$, respectively, to the product $M_{n,u} \times M_{n',u'}$ (see formula 6.16 of [29]). We use the $K$-theory class of $E_{n,n'}$ as a correspondence:
Definition 4.2. For a formal parameter \( m \), consider the operators:

\[
A_m|_n'^{\lambda'} K_{n',u'} \longrightarrow K_n,u
\]

\[
A_m|_n = \tilde{p}_{1*} \left( \tilde{\wedge}^* (\mathcal{E}_{n,n'} \otimes m) \cdot p_2^* \right)
\]  

(4.2)

where we define the “tilde” quantities as small modifications of the usual ones:

\[
\tilde{p}_{1*}(c) = p_{1*} \left( c \otimes \frac{\prod_{i=1}^r (-u_i')^{n'}}{(\det V_n')^{\otimes r}} \right)
\]

\[
\tilde{\wedge}^* (\mathcal{E}_{n,n'} \otimes m) = \wedge^* (\mathcal{E}_{n,n'} \otimes m) \otimes (\det V_n)^{\otimes r} \prod_{i=1}^r \left( -\frac{m}{u_i} \right)^n
\]

Note that the latter formula may be recast using (4.1) as:

\[
\left[ \tilde{\wedge}^* (\mathcal{E}_{n,n'} \otimes m) \right] = \tau' (mX) \tau \left( \frac{qX'}{m} \right) \zeta \left( \frac{X'}{mX} \right)
\]  

(4.3)

where:

\[
\tau(z) = \prod_{i=1}^r \left( 1 - \frac{z}{u_i} \right) \quad \text{and} \quad \tau'(z) = \prod_{i=1}^r \left( 1 - \frac{z}{u_i'} \right)
\]

In formula (4.3) and throughout the remainder of this paper, \( X \) and \( X' \) denote tautological classes on the two factors of \( \mathcal{M}_{n,u} \times \mathcal{M}_{n',u'} \), on which \( \mathcal{E}_{n,n'} \) is defined.

4.3. Formula (4.1) implies the following expression in the basis of fixed points:

\[
\mathcal{E}|_{\lambda,\lambda'} := \mathcal{E}|_{\lambda,\lambda'}|_{\mathcal{I}_\lambda,\mathcal{I}_{\lambda'}} = \sum_{1 \leq i \leq r} \frac{X_i^\lambda}{u_i^\lambda} + \sum_{1 \leq i \leq r} \frac{u_i'}{qX_i'^{\lambda'}} - \sum_{\square \in \lambda} \left( 1 - \frac{1}{q_1} \right) \left( 1 - \frac{1}{q_2} \right) \frac{X_i^\lambda}{X_i'^{\lambda'}}
\]

One understands the above formula by interpreting the right-hand side as the character of \( \mathbb{C}^r \times \mathbb{C}^r \times (\mathbb{C}^*)^r \times (\mathbb{C}^*)^r \) in the fiber of \( \mathcal{E} \) above the fixed point indexed by \( r \)-partitions \( \lambda, \lambda' \). The two factors \( (\mathbb{C}^*)^r \) act with equivariant parameters \( u \) and \( u' \), respectively. If we let \( n = |\lambda| \) and \( n' = |\lambda'| \), then the definition of \( A_m|_n'^{\lambda'} \) as a correspondence in (4.2) allows us to compute its matrix coefficients in the basis of renormalized fixed points:

\[
\langle \lambda | A_m|_n'^{\lambda'} | \lambda' \rangle = \frac{\tilde{\wedge}^* \left( \mathcal{E}|_{\lambda,\lambda'} \otimes m \right)}{\tilde{\wedge}^* (\text{Tan} \mathcal{M}_{n';u'})} \prod_{1 \leq i \leq r} \left( 1 - \frac{mX_i^\lambda}{u_i^\lambda} \right) \prod_{1 \leq i \leq r} \left( 1 - \frac{qX_i'^{\lambda'}}{mu_i'} \right) \prod_{\square \in \lambda} \frac{X_i^\lambda}{X_i'^{\lambda'}} \zeta \left( \frac{X_i^\lambda}{mX_i'^{\lambda'}} \right)
\]

(4.4)

To group all of the operators \( A_m|_n'^{\lambda'} \) together, we introduce the generating current:

\[
A_m(x) = \sum_{n,n' \geq 0} A_m|_n'^{\lambda'} \cdot x^{n-n'}
\]  

(4.5)
and note that this was the main actor in the Introduction. Then (4.4) implies the following formula for the Nekrasov partition function (1.1):

\[
Z_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = \text{Tr} \left( A_{m_1}(x_1) \ldots A_{m_k}(x_k) \right) = \sum_{\lambda_1, \ldots, \lambda_k} \prod_{a=1}^{r} \chi_{\lambda_a}^{r-|\lambda_{a+1}|} |\lambda_{a+1}| \prod_{1 \leq i \leq r} \left( 1 - \frac{\mu^{|\lambda|}}{u_i^{r+1}} \right) \prod_{1 \leq i \leq r} (1 - q X_i^{r+1}) \prod_{1 \leq i \leq r} \zeta \left( \frac{x_i^r}{m X_i} \right) \prod_{1 \leq i \leq r} \zeta \left( \frac{x_i^{r+1}}{m X_i} \right)
\] (4.6)

where \( u^a = (u_1^a, \ldots, u_r^a) \) is the collection of equivariant parameters corresponding to the moduli space of sheaves whose fixed points are indexed by \( r \)-partitions \( \lambda_a \), as in the Introduction, and we identify \( u^k = u^1, \lambda_{k+1} = \lambda_1 \). The right-hand side of (4.6) is the partition function of gauge theory with matter on a length \( k \) cyclic quiver, and justifies physical interest in the operators \( A_m(x) \). Note that in our definition, there is a single rank \( r \) involved in the definition of the operators \( A_m(x) \), which amounts to the fact that all the vertices in the cyclic quiver are given gauge group \( U(r) \). If one wanted to study quiver gauge theory for various groups \( U(r_1), \ldots, U(r_k) \), one would have to consider the operators \( A_m(x) \) for some integer \( r \geq r_1, \ldots, r_k \) and then send some of the equivariant parameters in (4.6) to \( \infty \).

4.4. For brevity, when we will work with a single operator \( A_m(x) \), we will set \( x = 1 \) without losing any information (the variable \( x \) comes in handy when computing compositions of vertex operators, as we will see in Sect. 4.6). We will now turn to the main purpose of this section, which is to compute the commutation relations of \( A_m := A_m(1) \) with the operators \( P_{k,0} \) and \( P_{k,1} \) (let us remark that in principle, our method allows us to compute the commutation relations with any \( P_{k,d} \), but the answer becomes more complicated as \( d \) grows larger).

**Theorem 4.5.** For any rank \( r \), we have the following relations between the Ext operator \( A_m \) and the generators of the algebra \( \mathcal{A} \):

\[
[A_m, p_{-k}] = A_m \left[ \left( \frac{q^r u^r}{m^r u^r} \right)^k - 1 \right], \quad [A_m, p_k] = A_m \left[ 1 - \left( \frac{m^r u^r}{u^r} \right)^k \right]
\] (4.7)

for all \( k > 0 \), and:

\[
A_m P_{k,1} - \frac{m^r u^r}{q^r u^r} A_m P_{k-1,1} = \frac{m}{q} P_{k,1} A_m - \frac{m^{r+1} u}{q^r u^r} P_{k-1,1} A_m
\] (4.8)

for all \( k \in \mathbb{Z} \). Here we use the notation \( u = u_1 \ldots u_r \) and \( u' = u_1' \ldots u_r' \).

**Proof.** We will actually prove slightly different, but equivalent, formulas than (4.7) and (4.8). For example, since the \( p_{\pm k} = P_{\pm k,0} \) generate a copy \( \Lambda_\pm \) of the ring of symmetric polynomials, then we may recast formula (4.7) in terms of the following commutation relations involving the corresponding complete symmetric functions:

\[
A_m h_{-k} - h_{-k} A_m = \frac{q^r u'}{m^r u} A_m h_{-k+1} - h_{-k+1} A_m
\] (4.9)

\[
A_m h_k - h_k A_m = A_m h_{k-1} - \frac{m^r u}{u'} h_{k-1} A_m
\] (4.10)
The way we prove that (4.7) is equivalent to (4.9)–(4.10) is to observe that both are particular cases of the following formulas:

\[ A_m x^{(1)}_\pm \cdot \varphi_{\frac{q}{m} u'} \left( x^{(2)}_\pm \right) = x^{(1)}_\pm A_m \cdot \varphi_1 \left( x^{(2)}_\pm \right) \]  \hspace{1cm} (4.11)

\[ A_m x^{(1)}_+ \cdot \varphi_1 \left( x^{(2)}_+ \right) = x^{(1)}_+ A_m \cdot \varphi_{\frac{mr}{u'} u} \left( x^{(2)}_+ \right) \]  \hspace{1cm} (4.12)

for all \( x_\pm \in \Lambda_\pm \), where \( \Delta(x_\pm) = x^{(1)}_\pm \otimes x^{(2)}_\pm \) denotes the usual coproduct on \( \Lambda_\pm \) in Sweedler notation (recall that this coproduct has \( p_{\pm k} \) as primitive elements and \( \sum_{k=0}^{\infty} h_{\pm k} z^k \) as a group-like element). The functionals \( \varphi_s : \Lambda_\pm \rightarrow \mathbb{Q}(q, m, u, u') \) are ring homomorphisms defined by either of the equivalent sets of assignments:

\[
 \varphi_s(h_{\pm k}) = \begin{cases} 
 1 & \text{if } k = 0 \\
 -s & \text{if } k = 1 \\
 0 & \text{otherwise}
\end{cases} \Rightarrow \varphi_s(p_{\pm k}) = -s^k
\]

Therefore, formulas (4.11)–(4.12) are multiplicative in \( x_\pm \): if they hold for \( x_\pm \) and \( y_\pm \), then they also hold for \( (xy)_\pm \). Therefore, proving these formulas for \( x_\pm = p_{\pm k} \) is equivalent to proving them for \( x_\pm = h_{\pm k} \), hence (4.7) is equivalent to (4.9)–(4.10).

Meanwhile, by changing the index \( k \leftrightarrow -k + 1 \), one can easily see that formula (4.8) is equivalent to the two following relations for all \( k > 0 \):

\[ A_m P_{-k,1} - m P_{-k,1} A_m = \frac{q' u'}{m' u} \left( A_m P_{-k+1,1} - \frac{m}{q} P_{-k+1,1} A_m \right) \]  \hspace{1cm} (4.13)

\[ A_m P_{k,1} - \frac{m}{q} P_{k,1} A_m = \frac{m' u}{q' u'} \left( A_m P_{k-1,1} - m P_{k-1,1} A_m \right) \]  \hspace{1cm} (4.14)

To prove (4.9) and (4.13), consider the following diagrams of spaces and arrows:

\[
\begin{align*}
\mathcal{M}_{n,u} \times \mathcal{M}_{n',u'} & \xrightarrow{\text{Id} \times \pi_-} \mathcal{M}_{n,u} \times M_{n'+k,u'} \\
\mathcal{M}_{n,u} \times \mathcal{M}_{n'+k,u'} & \xrightarrow{\pi_+ \times \text{Id}} \mathcal{M}_{n,u} \\
\mathcal{M}_{n,u} \times \mathcal{M}_{n',u'} & \xrightarrow{\pi_- \times \text{Id}} \mathcal{M}_{n,u} \times M_{n'+k,u'} \\
\mathcal{M}_{n,u} \times \mathcal{M}_{n'+k,u'} & \xrightarrow{\text{Id} \times \pi_+} \mathcal{M}_{n,u} \times M_{n'+k,u'}
\end{align*}
\]  \hspace{1cm} (4.15)
As a consequence of (2.9) and Theorem 3.16, we have:

\[ H_{-k,0} = \tilde{\pi}_+^* \left( \pi^- \right) = \tilde{\pi}'_+^* \left( \pi'^- \right) \]

Then the usual composition of operators gives us the following formulas:

\[ A_m H_{-k,0} = \tilde{\rho}_1^* (\Gamma_k \cdot p_2^*) \quad \text{and} \quad H_{-k,0} A_m = \tilde{\rho}'_1^* (\Gamma'_k \cdot p'^_2^*) \quad (4.17) \]

where \( \tilde{\rho}_1^*, \tilde{\rho}'_1^* \) are the renormalizations of \( p_1^*, p'_1^* \) defined in subsection 4.1, and:

\[ \Gamma_k = (\text{Id} \times \tilde{\pi}_-) \left( \prod_{i=1}^k \tau' \left( m U \right) \tau \left( \frac{q U}{m} \right) \zeta \left( \frac{X}{m X} \right) \right) \]

\[ \Gamma'_k = (\tilde{\pi}'_+ \times \text{Id}) \left( \prod_{i=1}^k \tau' \left( m U' \right) \tau \left( \frac{q X'}{m} \right) \zeta \left( \frac{X'}{m U'} \right) \right) \]

Using (4.3), we may rewrite \( \Gamma_k \) and \( \Gamma'_k \) in terms of tautological classes:

\[ \Gamma_k = (\text{Id} \times \tilde{\pi}_-) \left( \prod_{i=1}^k \tau' \left( m X \right) \tau \left( \frac{q U}{m} \right) \zeta \left( \frac{U}{m X} \right) \right) \]

\[ \Gamma'_k = (\tilde{\pi}'_+ \times \text{Id}) \left( \prod_{i=1}^k \tau' \left( m U' \right) \tau \left( \frac{q X'}{m} \right) \zeta \left( \frac{X'}{m U'} \right) \right) \]

where the alphabets of variables \( X, X' \) denote tautological classes on the spaces \( \mathcal{M}_{n,u}, \mathcal{M}_{n',u'} \), respectively, and \( U, U' \) denote tautological classes on the middle spaces of the bottom row of (4.15), (4.16), respectively. We may identify:

\[ U = X' + L_1 + \ldots + L_k \quad \text{and} \quad U' = X - L_1 - \ldots - L_k \]

because the classes \( U \) and \( U' \) do not just live on the product \( \mathcal{M}_{n+k} \times \mathcal{M}_n \), but on the subvariety \( \mathfrak{Z}_k \), which comes endowed with the line bundles \( L_1, \ldots, L_k \) of (3.42). Thus:

\[ \Gamma_k = (\text{Id} \times \tilde{\pi}_-) \left( \prod_{i=1}^k \tau \left( q U_i \right) \zeta \left( \frac{U_i}{m X} \right) \right) \]

and:

\[ \Gamma'_k = (\tilde{\pi}'_+ \times \text{Id}) \left( \prod_{i=1}^k \tau' \left( m U_i \right) \zeta \left( \frac{X'}{m U'_i} \right)^{-1} \right) \]
where:
\[
\Upsilon = \tau' \left( \frac{qX'}{m} \right) \tau' \left( \frac{X'}{mX} \right)
\]
(4.18)

Note that in each of the formulas above, \( \Upsilon \) is pulled back from \( \mathcal{M}_{n,u} \times \mathcal{M}_{n',w} \), so we can slide it in front of the direct image. Therefore, Proposition 3.17 implies:
\[
\Gamma_k = \Upsilon \cdot \int_{X \cup \{0,\infty\} > z_1 > \ldots > z_k > X'} \frac{\prod_{i=1}^{k} \xi \left( \frac{z_i}{mX} \right) \xi' \left( \frac{X'}{m} \right)}{\prod_{i=1}^{k-1} \left( 1 - \frac{q z_i + 1}{z_i} \right)} \prod_{i<j} \xi \left( \frac{z_i}{z_j} \right)
\]
(4.19)

and:
\[
\Gamma_k' = \Upsilon \cdot \int_{X > z_1 > \ldots > z_k > X' \cup \{0,\infty\}} \frac{\prod_{i=1}^{k} \xi \left( \frac{z_i}{m z_i} \right) \xi' \left( \frac{X'}{m} \right)}{\prod_{i=1}^{k-1} \left( 1 - \frac{q z_i + 1}{z_i} \right)} \prod_{i<j} \xi \left( \frac{z_i}{z_j} \right)
\]
(4.20)

Let us write:
\[
I_k(z_1, \ldots, z_k) = \frac{\prod_{i=1}^{k} \xi \left( \frac{z_i}{mX} \right) \xi' \left( \frac{X'}{m} \right)}{\prod_{i=1}^{k-1} \left( 1 - \frac{q z_i + 1}{z_i} \right)} \prod_{i<j} \xi \left( \frac{z_i}{z_j} \right)
\]

The change of variables \( z_i \mapsto z_i m \) implies that \( \Gamma_k - \Gamma_k' = \)
\[
= \Upsilon \left[ \int_{X \cup \{0,\infty\} > z_1 > \ldots > z_k > X'} I_k \prod_{i=1}^{k} Dz_i - \int_{X > z_1 > \ldots > z_k > X' \cup \{0,\infty\}} I_k \prod_{i=1}^{k} Dz_i \right]
\]
(4.21)

The two integrals have the same integrand, so their difference consists of the residues when one of the variables \( z_1, \ldots, z_k \) passes over either 0 or \( \infty \). However, because \( \lim_{w \to 0} \xi(w) = \lim_{w \to \infty} \xi(w) = 1 \) we have the limits:
\[
\begin{align*}
\lim_{z_1 \to 0} I_k(z_1, \ldots, z_k) &= 0, \\
\lim_{z_1 \to \infty} I_k(z_1, \ldots, z_k) &= \frac{q^r u'}{m^r u} I_{k-1}(z_2, \ldots, z_k) \\
\lim_{z_k \to 0} I_k(z_1, \ldots, z_k) &= \lim_{z_k \to \infty} I_k(z_1, \ldots, z_k) = 0 \quad \forall \ s \in \{2, \ldots, k-1\} \\
\lim_{z_1 \to 0} I_k(z_1, \ldots, z_k) &= I_{k-1}(z_1, \ldots, z_{k-1}), \\
\lim_{z_k \to \infty} I_k(z_1, \ldots, z_k) &= 0
\end{align*}
\]
(4.22) - (4.24)

where \( u = u_1 \ldots u_s \) and \( u' = u_1' \ldots u_s' \). Therefore, the only two residues which appear in the difference (4.21) are when \( z_1 \) passes over \( \infty \) and when \( z_k \) passes over 0. We conclude that the difference equals:
\[
\Gamma_k - \Gamma_k' = \Upsilon \left[ \int_{X \cup \{0,\infty\} > z_2 > \ldots > z_k > X'} \frac{q^r u'}{m^r u} I_{k-1}(z_2, \ldots, z_k) \prod_{i=2}^{k} Dz_i - \int_{X > z_1 > \ldots > z_{k-1} > X' \cup \{0,\infty\}} I_{k-1}(z_1, \ldots, z_{k-1}) \prod_{i=1}^{k-1} Dz_i \right] = \frac{q^r u'}{m^r u} \cdot \Gamma_{k-1} - \Gamma_{k-1}'
\]
which is precisely (4.9). Similarly, we have:

$$A_m P_{-k, 1} = \tilde{p}_1 (\tilde{r}_k \cdot p_2^*) \quad \text{and} \quad P_{-k, 1} A_m = \tilde{p}_1^* (\tilde{r}_k^* \cdot p_2^*)$$

where:

$$\tilde{r}_k = (\text{Id} \times \pi_-)_* \left[ \mathcal{L}_k \otimes \tilde{\pi}^\bullet ((\text{Id} \times \pi_+)^* \mathcal{E}_{n, n' + k} \otimes m) \right]$$

$$\tilde{r}_k^* = (\tilde{\pi}_+^* \times \text{Id})_* \left[ \mathcal{L}_k \otimes \tilde{\pi}^\bullet ((\pi'_- \times \text{Id})^* \mathcal{E}_{n - k, n'} \otimes m) \right]$$

By analogy with (4.19) and (4.20), we have:

$$\tilde{r}_k = \gamma \cdot \int_{X \cup [0, \infty) > z_1 > \ldots > z_k > X'} z_k \cdot I_k(z_1, \ldots, z_k) \prod_{i=1}^{k} Dz_i$$

and:

$$\tilde{r}_k^* = \gamma \int_{X > z_1 > \ldots > z_k > X' \cup [0, \infty]} z_k \cdot I_k(z_1 m, \ldots, z_k m) \prod_{i=1}^{k} Dz_i$$

where in the last equality we changed the variable \(z_i \mapsto z_i m\). Computing the difference between the expressions above, we see that \(\tilde{r}_k - m\tilde{r}_k^* = \)

$$= \gamma \left[ \int_{X \cup [0, \infty) > z_1 > \ldots > z_k > X'} z_k I_k \prod_{i=1}^{k} Dz_i - \int_{X > z_1 > \ldots > z_k > X' \cup [0, \infty]} z_k I_k \prod_{i=1}^{k} Dz_i \right]$$

(4.25)

To compute the difference, we must once again sum the residues of the integrand as one of the variables passes over 0 or \(\infty\). Formulas (4.22) and (4.23) continue to hold for \(I_k\) replaced by \(z_k I_k\), but formula (4.24) must be replaced by:

$$\lim_{z_k \to 0} z_k I_k = 0, \quad \lim_{z_k \to \infty} z_k I_k(z_1, \ldots, z_k) = -\frac{q^{r-1} u'}{m' u} z_{k-1} I_{k-1}(z_1, \ldots, z_{k-1})$$

(4.26)

With this in mind, (4.25) yields:

$$\tilde{r}_k - m\tilde{r}_k^* = \gamma \left[ \int_{X \cup [0, \infty) > z_2 > \ldots > z_k > X'} \frac{q^r u'}{m' u} z_k I_{k-1}(z_2, \ldots, z_k) \prod_{i=2}^{k} Dz_i ight.$$

$$- \left. \int_{X > z_1 > \ldots > z_{k-1} > X' \cup [0, \infty]} \frac{q^{r-1} u'}{m' u} z_{k-1} I_{k-1}(z_1, \ldots, z_{k-1}) \prod_{i=1}^{k-1} Dz_i \right]$$

$$= \frac{q^r u'}{m' u} \left( \tilde{r}_{k-1} - m\tilde{r}_k^* \right)$$

Note that this relation is precisely (4.13), written in terms of correspondences. The proofs of (4.10) and (4.14) are completely analogous, so we leave them as exercises to the interested reader. More formally, they follow from (4.9) and (4.13) by transposition,
since $h_k, P_{k,1}$ are the adjoints of $h_{-k}, P_{-k,1}$ under the inner product (3.35) and Serre duality implies that $A_{\frac{m}{\delta}}$ is the adjoint of $A_m$. To see the latter claim explicitly, one can also combine formulas (3.38) and (4.4) to obtain:

$$\langle \lambda' | A_{\frac{m}{\delta}} | \lambda \rangle = \langle \lambda | A_m | \lambda' \rangle \cdot \frac{((|\lambda\rangle, |\lambda\rangle))}{(|\lambda'\rangle, |\lambda'\rangle)}$$

\[\square\]

4.6. Let us now use formulas (4.7)–(4.8) to compute the commutation relations of $A_m$ with the $q W$-algebra currents. We revert to the notation $A_m(x)$ for the generating series (4.5). To assure uniformity in our formulas, it will be convenient to study instead of $A_m(x)$ the operator:

$$\Phi_m(x) : K_u \to K_u$$

given by:

$$\Phi_m(x) = A_m(x) \exp \left[ -\sum_{n=1}^{\infty} \frac{p_n}{nx^n} \cdot \left( \frac{u'}{mr u} \right)^n \cdot \frac{1 - q^n}{(1 - q_1^n)(1 - q_2^n)} \right] \quad (4.27)$$

**Proposition 4.7.** The operator $\Phi_m(x)$ has the following commutation relations with the first $q W$-algebra generating current:

$$\Phi_m(x) W_1(y) \cdot \left( 1 - \frac{m' u x}{q^{r-1} u' y} \right) = W_1(y) \Phi_m(x) \cdot m \left( 1 - \frac{m' u x}{q^{r-1} u' y} \right) \quad (4.28)$$

Moreover, it makes sense to ask for the commutation relations between $\Phi_m$ and the $q$-Heisenberg generators $p_n \in A_{\mathbb{F}}^\text{ext}$, which take the form:

$$[\Phi_m(x), p_{\pm k}] = \pm \Phi_m(x)x^{\pm k} \left[ 1 - \left( \frac{m' u}{u'} \right)^{\pm k} \right] \quad (4.29)$$

**Proof.** Formula (4.29) is equivalent to (4.7) when the sign is +. When the sign is −, we must supplement (4.7) with the computation of the commutator of:

$$Z_m(x) := \exp \left[ -\sum_{n=1}^{\infty} \frac{p_n}{nx^n} \left( \frac{u'}{mr u} \right)^n \cdot \frac{1 - q^n}{(1 - q_1^n)(1 - q_2^n)} \right] = A_m(x)^{-1} \Phi_m(x) \quad (4.30)$$

with $p_{-n}$. To do so, we directly apply (2.29):

$$[Z_m(x), p_{-k}] = Z_m(x)x^{-k} \left[ \left( \frac{u'}{mr u} \right)^k - \left( \frac{q^{r-1} u'}{mr u} \right)^k \right]$$

and then use (4.7) to obtain (4.29). As for (4.28), let us rewrite relation (4.8) as:

$$A_m(x) \left( \sum_{k \in \mathbb{Z}} \frac{P_{k,1}}{y^k} \right) \left( 1 - \frac{m' u x}{q^{r-1} u' y} \right) = \left( \sum_{k \in \mathbb{Z}} \frac{P_{k,1}}{y^k} \right) A_m(x) \left( \frac{m}{q} - \frac{m^{r+1} u x}{q^{r} u' y} \right)$$
Using (4.27) and (2.48), we obtain:

\[ \Phi_m(x) Z_m(x)^{-1} W_1(y) \left( 1 - \frac{m'u x}{q^r u'y} \right) \]

\[ = W_1(y) \Phi_m(x) Z_m(x)^{-1} \left( \frac{m}{q} - \frac{m^{r+1} u x}{q^r u'y} \right) \quad (4.31) \]

Since \( Z_m(x) \) of (4.30) is an exponential in the annihilation bosons \( p_n \), in any normal ordered expression it should be placed at the very right. This means that in the left-hand side of (4.31), we must move this exponential past \( W_1(y) \). To this end, recall that formula (2.21) and the fact that \( p_n = q^{n(r-1)} P_n,0 \) imply:

\[ \left[ \sum_{k \in \mathbb{Z}} \frac{P_{k,1}}{y^k}, p_n \right] = (1 - q_1^n)(1 - q_2^n)q^{n(r-1)} y^n \sum_{k \in \mathbb{Z}} \frac{P_{k,1}}{y^k} \]

Exponentiating this relation, we conclude that \( Z_m(x)^{-1} W_1(y) \) equals:

\[ W_1(y) Z_m(x)^{-1} \exp \left[ \sum_{n=1}^{\infty} \frac{y^n}{n x^n} \left( q^r u'y \right)^n \right] = W_1(y) Z_m(x)^{-1} \frac{1 - q_1^{r-1} u'y}{1 - q^r u'y} \]

Plugging this formula into (4.31) gives us precisely (4.28). \( \square \)

**Proof of Theorem 1.1.** We will prove (1.5) by induction on \( k \), whose base case \( k = 1 \) is precisely (4.28). For the induction step, assume that (1.5) is proved for some \( k \) and let us prove it for \( k + 1 \). Applying relation (1.4) for \( k' = 1 \) gives us:

\[ W_k(y') W_1(y) \zeta \left( \frac{y'}{y q^k} \right) - W_1(y) W_k(y') \zeta \left( \frac{y'}{y q} \right) \]

\[ = \frac{(1 - q_1)(1 - q_2)}{1 - q} \left[ \delta \left( \frac{y'}{y q} \right) W_{k+1}(y) - \delta \left( \frac{y'}{y q^k} \right) W_{k+1}(y') \right] \quad (4.32) \]

Since \( \delta(z)(1 - z) = 0 \), we may isolate \( W_{k+1}(y) \) in the right-hand side by multiplying both sides of the expression above with \( 1 - \frac{y'}{y q^k} \):

\[ W_k(y') W_1(y) \zeta \left( \frac{y'}{y q^k} \right) \left( 1 - \frac{y'}{y q^k} \right) - W_1(y) W_k(y') \zeta \left( \frac{y'}{y q^k} \right) \left( 1 - \frac{y'}{y q^k} \right) \]

\[ = \frac{(1 - q_1)(1 - q_2)}{1 - q} \delta \left( \frac{y}{y q^k} \right) W_{k+1}(y) \left( 1 - \frac{1}{q^{k+1}} \right) \quad (4.33) \]

Therefore, one can obtain the series \( W_{k+1}(y) \) by taking the constant term in \( y' \) of the series in the left-hand side of formula (4.33). Take the identity of commutators:

\[ [\Phi_m(x), W_k(y') W_1(y)]_{m^{k+1}} = [\Phi_m(x), W_k(y')]_{m^k} W_1(y) + m^k W_k(y') [\Phi_m(x), W_1(y)]_m \]

and multiply it by:

\[ \left( 1 - \frac{m'u x}{q^{r-1} u'y} \right) \prod_{i=1}^{k} \left( 1 - \frac{m'u x}{q^{r-i} u'y} \right) \quad (4.34) \]
Then the induction hypothesis implies that \([\Phi_m(x), W_k(y') W_1(y)]_{m+1}\) multiplied by (4.34) vanishes. The same reasoning implies that \([\Phi_m(x), W_1(y) W_k(y')]_{m+1}\) multiplied by (4.34) vanishes, so we conclude that the same must be true for the right-hand side of (4.33):

\[
0 = \frac{(1 - q_1)(1 - q_2)(1 - q^{-k-1})}{1 - q} \delta \left( \frac{y}{y' q} \right) [\Phi_m(x), W_{k+1}(y)]_{m_k+1} \quad \text{(expression (4.34))}
\]

Because of the \(\delta\) function, we may replace \(y'\) by \(\frac{y}{q}\) in (4.34) and obtain:

\[
0 = \frac{(1 - q_1)(1 - q_2)(1 - q^{-k-1})}{1 - q} \delta \left( \frac{y}{y' q} \right) [\Phi_m(x), W_{k+1}(y)]_{m_k+1} \prod_{i=1}^{k+1} \left( 1 - \frac{m' u x}{q^{r-i} u' y} \right)
\]

Taking the constant term in \(y'\) of this expression establishes (1.5) for \(k + 1\).

**Remark 4.8.** In [33], we will improve formula (1.5) by showing that:

\[
[\Phi_m(x), W_k(y)]_{m_k} \cdot \left( 1 - \frac{m' u x}{q^{r-k} u' y} \right) = 0 \quad \text{(4.35)}
\]

for all \(k \geq 1\), and that relations (4.35) uniquely determine the operator \(\Phi_m(x)\) up to constant multiple. In fact, we leave it as an exercise to the interested reader to prove (4.35) when \(k = r\) using the tools we already have on hand, specifically (2.67) and (4.29), and observe that in this case (4.35) is already stronger than (1.5).

## 5 The Quantum Miura Transformation

5.1. The original definition of the \(q\)-\(W\)-algebra is through the **quantum Miura transformation**. In this section, we will define the \(q\)-\(W\)-algebra of type \(gl_r\) by analogy with [3,14], and show that it matches \(\mathcal{A}_r\) of Definition 2.24. Consider the following deformed Heisenberg algebra of type \(gl_r\):

\[
\mathcal{H}_r = \mathbb{F} \left\{ b^i_n \right\}_{1 \leq i \leq r, n \in \mathbb{Z} \setminus 0}
\]

modulo the commutation relations:

\[
[b^i_{-n}, b^j_n] = n(1 - q_1^n)(1 - q_2^n) \cdot \begin{cases} 1 - q^{-n} & \text{if } i < j \\ 1 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases} \quad \text{(5.1)}
\]

for all \(n > 0\). All other commutators are defined to be zero. Consider the elements:

\[
p_n = \sum_{i=1}^{r} b^i_n q^{n(i-1)} \in \mathcal{H}_r \quad \text{(5.2)}
\]

for all \(n \in \mathbb{Z} \setminus 0\). Let us form currents out of these generators:

\[
b^i(x) = \sum_{n \in \mathbb{Z} \setminus 0} b^i_n \frac{x^n}{|n| x^n}, \quad p(x) = \sum_{n \in \mathbb{Z} \setminus 0} \frac{p_n}{|n| x^n} = \sum_{i=1}^{r} b \left( \frac{x}{q^{i-1}} \right)
\]
and consider the normal ordered exponentials:

\[ \Lambda^i(x) = u_i : \exp \left[ b^i(x) \right] : = u_i \exp \left[ \sum_{n=1}^{\infty} \frac{b_n}{nx^n} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{b_n}{nx^n} \right] \]

Then by analogy with [3,14], we define the \( q \)-algebra currents as the expressions in \( \Lambda^1(x), \ldots, \Lambda^r(x) \) given by the following equality of difference operators:

\[
\sum_{k=0}^{\infty} (-1)^k W_k(x) D_x^{r-k} = \left( D_x - \Lambda^1(x) \right) \left( D_x - \Lambda^2 \left( \frac{x}{q} \right) \right) \ldots \left( D_x - \Lambda^r \left( \frac{x}{q^{r-1}} \right) \right) ;
\] (5.3)

Here, \( D_x \) denotes the difference operator \( f(x) \mapsto f(xq) \), and it commutes past the currents \( \Lambda^i(x) \) according to the rule:

\[ D_x \Lambda^i(x) = \Lambda^i(xq) D_x \]

Then one makes sense of (5.3) by foiling out the right-hand side and moving all of the difference operators to the right of each summand. After doing so, we are left with the following formulas, which are equivalent to (5.3):

\[
W_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} : \Lambda^{i_1}(x) \Lambda^{i_2} \left( \frac{x}{q} \right) \ldots \Lambda^{i_k} \left( \frac{x}{q^{k-1}} \right) :
\] (5.4)

and the normal ordered product simply means that in all expressions, the creation operators \( \{ b_n \}_{n<0} \) must be placed to the left of the annihilation operators \( \{ b_n \}_{n>0} \). Very roughly, one may interpret the \( q \)-currents of (5.4) as normal ordered “elementary symmetric functions” in the bosonic fields \( \Lambda^i(x) \).

Note that, as a \( F \)-vector space, we have:

\[
\mathcal{H}_r = \bigoplus_{s_1 \geq \cdots \geq s_k > 0} \bigoplus_{t_1 \geq \cdots \geq t_l > 0} \bigoplus_{s'_1 \geq \cdots \geq s'_{k'} > 0} \bigoplus_{t'_1 \geq \cdots \geq t'_{l'} > 0} F \cdot b_{-s_1} \cdots b_{-s_k} \cdots b'_{-s'_{1}} \cdots b'_{-s'_{k'}} \cdots b'_{-t_1} \cdots b'_{-t_l} \cdots b'_{-t'_{1}} \cdots b'_{-t'_{l'}}
\] (5.5)

Let us define:

\[
\tilde{\mathcal{H}}_r \supset \mathcal{H}_r
\]

as the completion consisting of infinite sums of the basis vectors (5.5), for finite \( k + \cdots + l + k' + \cdots + l' \). Note that the coefficients of \( W_k[[x^{\pm 1}]] \) lie in this completion.

5.2. One observes several things from (5.4). First of all, we have \( W_0(x) = 1 \), while:

\[
W_r(x) = \Lambda^1(x) \Lambda^2 \left( \frac{x}{q} \right) \ldots \Lambda^r \left( \frac{x}{q^{r-1}} \right) := u_1 \cdots u_r : \exp [p(x)] : \] (5.6)

and \( W_k(x) = 0 \) for \( k > r \). We give the following definition, by analogy with [3,14]:
Definition 5.3. Define the deformed $W$-algebra $\mathcal{B}_r \subset \mathcal{H}_r$ to be generated by:

$$\left\{ W_{d,k} \right\}_{d \in \mathbb{Z}}^{1 \leq k \leq r} \quad \text{where} \quad W_k(x) = \sum_{d \in \mathbb{Z}} \frac{W_{d,k}}{x^d}$$

Proposition 5.4. The elements $W_{d,k} \in \mathcal{H}_r$ and $p_n \in \mathcal{H}_r$ satisfy relations (2.60), (2.61) and (2.62) with $c = q^r$.

Proof. Formula (5.1) and the definition of $p_n$ in (5.2) implies that:

$$[b^i_m, p_{-n}] = -\delta^0_{m-n} \cdot n(1 - q^r_1)(1 - q^r_2)$$  (5.7)

$$[b^i_m, p_n] = \delta^0_{m+n}(1 - q^r_1)(1 - q^r_2)q^n(r-1)$$  (5.8)

for all $n \geq 0$. An easy consequence of these relations is the fact that:

$$\left[ \Lambda^i(x), p_{-n} \right] = -(1 - q^r_1)(1 - q^r_2) \cdot x^{-n} \Lambda^i(x)$$  (5.9)

$$\left[ \Lambda^i(x), p_n \right] = (1 - q^r_1)(1 - q^r_2)q^{n(r-1)}x^n \Lambda^i(x)$$  (5.10)

for all $i$. Applying formula (5.4), formulas (5.9)–(5.10) precisely imply (2.60)–(2.61) with $c = q^r$, respectively. Meanwhile, note that (5.1) implies the following relations:

$$\Lambda^i(x) \Lambda^j(y) = u_i^2 \exp \left[ \sum_{n>0} \frac{b^i_{-n}}{nx^{-n}} \right] \exp \left[ \sum_{n>0} \frac{b^j_n}{ny^{n}} \right] \exp \left[ \sum_{n>0} \frac{b^j_n}{nx^{n}} \right] \exp \left[ \sum_{n>0} \frac{b^i_{-n}}{ny^{n}} \right]$$

$$= u_i^2 \exp \left[ \sum_{n>0} \frac{b^i_{-n}}{n(x+y)^n} \right] \left[ \sum_{n>0} \frac{b^j_n}{n(x+y)^n} \left( \frac{1}{x^n} + \frac{1}{y^n} \right) \right] \frac{1}{\zeta \left( \frac{x}{\lambda} \right)}$$

while:

$$\Lambda^i(x) \Lambda^j(y) = u_i u_j \exp \left[ \sum_{n>0} \frac{b^i_{-n}}{nx^{-n}} + \frac{b^j_{-n}}{ny^{-n}} \right] \left[ \sum_{n>0} \frac{b^i_n}{nx^{n}} + \frac{b^j_n}{ny^{n}} \right] = \Lambda^i(x) \Lambda^j(y) : \frac{1}{\zeta \left( \frac{x}{\lambda} \right)}$$

for $i < j$ and:

$$\Lambda^i(x) \Lambda^j(y) = u_i u_j \exp \left[ \sum_{n>0} \frac{b^i_{-n}}{nx^{-n}} + \frac{b^j_{-n}}{ny^{-n}} \right] \left[ \sum_{n>0} \frac{b^i_n}{nx^{n}} + \frac{b^j_n}{ny^{n}} \right] \exp \left[ \sum_{n>0} \frac{b^j_{-n}}{ny^{n}} \right]$$

$$= u_i u_j \exp \left[ \sum_{n>0} \frac{b^i_{-n}}{nx^{-n}} + \frac{b^j_{-n}}{ny^{-n}} \right] \left[ \sum_{n>0} \frac{b^i_n}{nx^{n}} + \frac{b^j_n}{ny^{n}} \right] \frac{\zeta \left( \frac{y}{xq} \right)}{\zeta \left( \frac{x}{\lambda} \right)}$$

$$= \Lambda^i(x) \Lambda^j(y) : \frac{\zeta \left( \frac{x}{\lambda} \right)}{\zeta \left( \frac{y}{\lambda} \right)}$$

for $i > j$. Therefore, we conclude that:

$$\Lambda^i(x) \Lambda^j(y) \zeta \left( \frac{y}{x} \right) - \Lambda^j(y) \Lambda^i(x) \zeta \left( \frac{x}{y} \right) = 0$$  (5.11)
because the normal-ordered product is symmetric under \( x \leftrightarrow y \), while:

\[
\Lambda^i(x) \Lambda^j(y) \zeta \left( \frac{y}{x} \right) - \Lambda^j(y) \Lambda^i(x) \zeta \left( \frac{x}{y} \right)
\]

\[
= \left( : \Lambda^i(x) \Lambda^j(y) : \zeta \left( \frac{y}{x} \right) \right) \text{ for } |y| \ll |x|
\]

\[
- \left( : \Lambda^j(y) \Lambda^i(x) : \zeta \left( \frac{x}{y} \right) \right) \text{ for } |x| \ll |y|
\]

\[
= \frac{(1 - q_1)(1 - q_2)}{1 - q} \delta \left( \frac{x}{y} \right) \Lambda^i(x) \Lambda^j(x) : 
\]

\[
- \frac{(1 - q_1)(1 - q_2)}{1 - q} \delta \left( \frac{y}{x} \right) : \Lambda^i(x) \Lambda^j \left( \frac{x}{q} \right) : 
\]

(5.12)

for \( i < j \) and:

\[
\Lambda^i(x) \Lambda^j(y) \zeta \left( \frac{y}{x} \right) - \Lambda^j(y) \Lambda^i(x) \zeta \left( \frac{x}{y} \right)
\]

\[
= \left( : \Lambda^i(x) \Lambda^j(y) : \zeta \left( \frac{y}{x} \right) \right) \text{ for } |y| \ll |x|
\]

\[
- \left( : \Lambda^j(y) \Lambda^i(x) : \zeta \left( \frac{x}{y} \right) \right) \text{ for } |x| \ll |y|
\]

\[
= - \frac{(1 - q_1)(1 - q_2)}{1 - q} \delta \left( \frac{x}{y} \right) : \Lambda^i(y) \Lambda^j(y) : 
\]

\[
+ \frac{(1 - q_1)(1 - q_2)}{1 - q} \delta \left( \frac{y}{x} \right) \Lambda^i \left( \frac{y}{q} \right) \Lambda^j(y) : 
\]

(5.13)

for \( i > j \). In either formula (5.12) or (5.13), the equality between the second and third lines follows from a general fact about rational functions, which we now explain. Since \( : \Lambda^i(x) \Lambda^j(y) : \) is a Laurent polynomial with coefficients \( \in B_r \), then:

\[
R(x, y) = \Lambda^i(x) \Lambda^j(y) : \zeta \left( \frac{x}{y} \right) \in B_r[[x^{\pm 1}, y^{\pm 1}]] \cdot \frac{(x - y/q_1)(x - y/q_2)}{(x - y)(x - y/q)}
\]

Therefore, the second line of (5.13) is the difference between the expansions of \( R(x, y) \) at \( |y| \ll |x| \) and at \( |x| \ll |y| \). The third line of (5.13) is equal to:

\[
\sum_{x \neq 0, \infty} \delta \left( \frac{y}{x} \right) \text{Res} R(x, y)
\]

and so the left and right-hand sides of (5.13) are equal. By (5.4), we have \( W_1(x) = \Lambda^1(x) + \cdots + \Lambda^r(x) \), and so relations (5.11), (5.12), (5.13) imply the following relation:

\[
W_1(x) W_1(y) \zeta \left( \frac{y}{x} \right) - W_1(y) W_1(x) \zeta \left( \frac{x}{y} \right) = \frac{(1 - q_1)(1 - q_2)}{1 - q} .
\]

\[
\sum_{1 \leq i < j \leq r} \left[ \delta \left( \frac{y}{x} \right) : \Lambda^j \left( \frac{y}{q} \right) \Lambda^i(y) : - \delta \left( \frac{x}{y} \right) : \Lambda^i(x) \Lambda^j \left( \frac{x}{q} \right) : \right] = \frac{(1 - q_1)(1 - q_2)}{1 - q} \left[ \delta \left( \frac{y}{x} \right) W_2(y) - \delta \left( \frac{x}{y} \right) W_2(x) \right]
\]
This is precisely (2.62) for \( k = k' = 1 \). As for higher values of \( k \) and \( k' \), the computation follows the same lines as in the proof above, and is presented in detail in section 2.2 of [24]. We refer the reader to loc. cit. for the remainder of the computation, as it is simply an exercise in commuting bosonic currents. \( \square \)

**Proposition 5.5.** Relations (2.62) generate the ideal of relations between the elements \( \{ W_{d,k} \}_{d \in \mathbb{Z}}^{k \in \{1,\ldots,r\}} \in B_r \). Combining this with Proposition 2.25, we infer that:

\[
B_r \cong A_r
\]

hence we refer to either of these algebras as “the deformed \( W \)-algebra of type \( \mathfrak{gl}_r \)”.

**Proof.** Propositions 2.25 and 5.4 yield an epimorphism \( A_r \to B_r \). As shown in Proposition 2.25, a basis of \( A_r \) as a \( \mathbb{F} \)-module is given by the elements (2.69). Thus it is enough to show that these elements are independent in \( B_r \subset \widehat{\mathcal{H}}_r \). Since \( \widehat{\mathcal{H}}_r \) is the completion of a free \( \mathbb{F} \)-module with basis given by the normal-ordered products of \( \{ b_i^{1 \leq i \leq r} \}_{i \in \mathbb{N}} \), it suffices to show that the elements (2.69) are independent in \( \widehat{\mathcal{H}}_r \). To this end, we claim that it suffices to show independence when we specialize \( q_1 = 1 \) and leave \( q = q_2 \) generic (the reader may object to the latter claim, given that \( \mathcal{H}_r \) is an infinite-dimensional \( \mathbb{F} \)-module, but one can repeat the contents of this paragraph with the field \( \mathbb{F} = \mathbb{Q}(q_1, q_2) \) replaced with the ring \( \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \otimes \mathbb{Z}[q] \mathbb{Q}(q) \), over which all of our algebras are well-defined; all we are saying in this sentence is that \( 1 - q_1 \) never appears in the denominator of any commutation relation).

Therefore, setting \( q_1 = 1 \), the algebras \( \mathcal{H}_r \subset \widehat{\mathcal{H}}_r \) become commutative, and:

\[
W_{d,k} = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \lambda_{e_1}^{i_1} \cdots \lambda_{e_k}^{i_k} \cdot q^\sum_{i=1}^k (i-1)e_i
\]

(5.14)

where:

\[
\lambda_{e_i}^i = \text{coefficient of } x^{-e} \text{ in } u_i \exp \left[ \sum_{n=1}^{\infty} \frac{b_{-n}}{nx_n} \right] \exp \left[ \sum_{n=1}^{\infty} \frac{b_n}{nx_n} \right]
\]

Since unordered monomials in the \( b_i^{1 \leq i \leq r} \) form a linear basis of the commutative algebra \( \mathcal{H}_r|_{q_1=1} \), unordered monomials in the \( \lambda_{e_i}^i \) are linearly independent in \( \widehat{\mathcal{H}}_r|_{q_1=1} \). We will consider unordered products in the \( W_{d,k} \) (since the algebra is commutative, the order is immaterial) and define the following total ordering:

\[
\prod_{i=1}^{r} W_{d_{1,i},i} \cdots W_{d_{n_i,i},i} \quad \text{is greater than} \quad \prod_{i=1}^{r} W_{e_{1,i},i} \cdots W_{e_{n_i,i}}
\]

(5.15)

(we always write such products assuming that \( d_{1,i}^1 \leq \cdots \leq d_{n_i,i}^{n_i} \) and \( e_{1,i}^1 \leq \cdots \leq e_{n_i,i}^{n_i} \) if the transpose of the partition \( \mu = 1^{m_1} \cdots r^{m_r} \) is greater than the transpose of the partition \( \nu = 1^{n_1} \cdots r^{n_r} \) in lexicographic ordering. If the two partitions \( \mu \) and \( \nu \) are equal, then we impose the order (5.15) if \( (d_{m_i}^{m_r}, \ldots, d_{1,i}^1, \ldots, d_{1,i}^{m_1}, \ldots, d_{1,i}^{m_r}) \) is greater than \( (e_{n_i}^{n_r}, \ldots, e_{1,i}^1, \ldots, e_{n_i}^{n_1}, \ldots, e_{n_i}^{n_r}) \) lexicographically. If there exists a linear relation between the unordered products of \( W_{d,k} \) in the algebra \( \widehat{\mathcal{H}}_r|_{q_1=1} \), we may write it as:

\[
\prod_{i=1}^{r} W_{d_{1,i},i} \cdots W_{d_{n_i,i},i} \in \sum_{\text{products}} \mathbb{Q}(q) \prod_{i=1}^{r} W_{e_{1,i},i} \cdots W_{e_{n_i,i}}
\]

(5.16)
The contradiction to the existence of such a relation is that the term:

\[
\lambda_1 d_1 \lambda^{1-M} \lambda_2 d_2^{+M} \lambda^{2-M} \ldots \lambda_m d_m^{+M(r-1)} \lambda^{M(r-1)}
\]

(for some \(-M \leq d_i \forall i, j\)) appears in the left-hand side of (5.16) upon applying (5.14), but does not appear in the right-hand side in virtue of our choice of ordering. \(\square\)

5.6. Let us recall from [3,14] the construction of the \(q\)-\(W\)-algebra of type \(\mathfrak{sl}_r\). Consider the deformed Heisenberg algebra of type \(\mathfrak{sl}_r\):

\[
\tilde{H}_r = \mathbb{F} \left\{ h_{n}^{i} \right\}_{n \in \mathbb{Z}, i \leq r}
\]

subject to the commutation relations:

\[
[h_{-n}^{i}, h_{n}^{j}] = n(1 - q_{1}^{n})(1 - q_{2}^{n}) \cdot \frac{1 - q^{(r\delta_{ji} - 1)n}}{1 - q^{rn}} q^{rn h_{i,j}} \quad (5.17)
\]

for all \(n > 0\), the fact that all other commutators vanish, and the linear relation:

\[
\sum_{i=1}^{r} h_{n}^{i} q^{n(i-1)} = 0
\]

for all \(n \in \mathbb{Z} \setminus \{0\}\). Then the \(q\)-\(W\)-algebra of type \(\mathfrak{sl}_r\), denoted by \(\tilde{B}_r \subset \tilde{H}_r\), is defined to be generated by the coefficients of currents \(\tilde{W}_{k}(x)\) from the following formula:

\[
: (D_{x} - \tilde{\Lambda}^{1}(x)) \ldots (D_{x} - \tilde{\Lambda}^{r}(\frac{x}{q^{r-1}})) : = \sum_{k=0}^{\infty} (-1)^{k} \tilde{W}_{k}(x) D_{x}^{r-k} \quad (5.18)
\]

where:

\[
\tilde{\Lambda}^{i}(x) = u_{i} : \exp \left[ h_{i}(x) \right] := u_{i} \exp \left[ \sum_{n>0} \frac{h_{i,n}^{i}}{n x^{n}} \right] \exp \left[ \sum_{n>0} \frac{h_{i,n}^{i}}{n x^{n}} \right]
\]

By analogy with (5.4), we may rewrite formula (5.18) as:

\[
\tilde{W}_{k}(x) = \sum_{1 \leq i_{1} < \ldots < i_{k} \leq r} : \tilde{\Lambda}^{i_{1}}(x) \tilde{\Lambda}^{i_{2}}(\frac{x}{q}) \ldots \tilde{\Lambda}^{i_{k}}(\frac{x}{q^{k-1}}) : \quad (5.19)
\]

for any \(k \in \{1, \ldots, r\}\).

**Proposition 5.7.** The map \(h_{n}^{i} \mapsto b_{n}^{i} - p_{n} \cdot \frac{1 - q_{1}^{n}}{1 - q^{rn}}\) gives rise to a homomorphism:

\[
\tilde{B}_r \rightarrow B_r \quad (5.20)
\]

Moreover, the \(q\)-Heisenberg generators \(p_{m}\) commute with the image of (5.20), and therefore we obtain a homomorphism:

\[
\tilde{B}_r \otimes_{q} \text{Heisenberg} \rightarrow B_r
\]
In terms of currents, the map (5.20) takes the form:

$$\tilde{W}_k(x) = \exp \left[ -\sum_{n=1}^{\infty} \frac{p_{-n}}{nx^n} \frac{1 - q^{kn}}{1 - q^{-rn}} \right] W_k(x) \exp \left[ -\sum_{n=1}^{\infty} \frac{p_n}{nx^n} \frac{1 - q^{kn}}{1 - q^{rn}} \right]$$ \hspace{1cm} (5.22)

for all $k \in \{1, \ldots, r\}$.

We note a slight imprecision in the definition of the homomorphisms (5.20) and (5.21), and we will leave it as is, because we will not revisit the issue in the present paper. These homomorphisms are well-defined only if the $q$-Heisenberg generators $\{p_n\}_{n \in \mathbb{Z} \setminus 0}$, as well as arbitrary normal-ordered exponentials of these generators, lie in $B_r$. This is not automatically obvious, because although the series coefficients $\{W_{r,d}\}_{d \in \mathbb{Z}}$ of the normal-ordered exponential (5.6) lie in $B_r$, the individual $p_n$ can be obtained from the $W_{r,d}$ only if we suitably complete $B_r$.

**Proof.** It is easy to see that if the $b^i_j$ satisfy the commutation relations (5.1) and the $p_m$ are defined as the linear combinations (5.2), then:

$$h^i_n := b^i_n - p_n \cdot \frac{1 - q^n}{1 - q^{rn}}$$ \hspace{1cm} (5.23)

satisfy the commutation relations (5.17). This induces an embedding $\tilde{H}_r \subset \mathcal{H}_r$, and it is clear that $p_m$ commute with the image of this map, because:

$$[p_m, h^i_n] = \left[ \sum_{j=1}^{r} b^i_n q^{m(j-1)}, b^i_n - \sum_{j=1}^{r} b^i_n q^{n(j-1)} \frac{1 - q^n}{1 - q^{rn}} \right] = \delta^0_{m+n}(1 - q^n)(1 - q^2)$$

\[ \times \left( q^{(1-i)} + \sum_{j=1}^{i-1} (1 - q^{-n}) q^{(1-j)} - \sum_{j=1}^{r} \frac{1 - q^n}{1 - q^{rn}} \right) \]

\[ \sum_{j < j'} (1 - q^{-n}) q^{(j'-j)} \frac{1 - q^n}{1 - q^{rn}} = 0 \]

Therefore, (5.20) follows from the fact that formulas (5.4) and (5.19) are connected by (5.22). In more detail, we note that as a consequence of (5.19), we have:

$$\tilde{W}_k(x) = \sum_{1 \leq i_1 < \ldots < i_k \leq r} : \tilde{\Lambda}^{i_1}(x) \tilde{\Lambda}^{i_2} \left( \frac{x}{q} \right) \ldots \tilde{\Lambda}^{i_k} \left( \frac{x}{q^{k-1}} \right)$$

\[ := \sum_{1 \leq i_1 < \ldots < i_k \leq r} \prod_{j=1}^{k} \exp \left[ -\sum_{n>0} \frac{p_{-n}}{nx^n} q^{-n(j-1)} \frac{1 - q^n}{1 - q^{-rn}} \right] \Lambda^{i_j} \left( \frac{x}{q^{j-1}} \right) \exp \]

\[ \times \left[ -\sum_{n>0} \frac{p_n}{nx^n} q^{n(j-1)} \frac{1 - q^n}{1 - q^{rn}} \right] : \]

\[ = \exp \left[ -\sum_{n>0} \frac{p_{-n}}{nx^n} \frac{1 - q^{kn}}{1 - q^{-rn}} \right] \sum_{1 \leq i_1 < \ldots < i_k \leq r} \prod_{j=1}^{k} \Lambda^{i_j} \left( \frac{x}{q^{j-1}} \right) \]

\[ \times \exp \left[ -\sum_{n>0} \frac{p_n}{nx^n} \frac{1 - q^{kn}}{1 - q^{rn}} \right] \] \hspace{1cm} (5.24)
5.8. Let us now recall the construction of the stable basis isomorphism introduced by Maulik–Okounkov [23] in the context of symplectic resolutions. In the case of the moduli space of rank $r$ sheaves on the plane, this construction takes the form:

$$\text{Stab} : F_{u_1} \otimes \cdots \otimes F_{u_r} \xrightarrow{\cong} K$$

(5.25)

where the RHS is the $K$-theory group over equivariant parameters $u = (u_1, \ldots, u_r)$. The algebra $\mathcal{H}_r$ acts on the left-hand side via:

$$b_{-n}^i = p_{-n}^{(i)}, \quad b_n^i = p_n^{(i)} + \sum_{j=1}^{i-1} (1 - q^{-n}) p_n^{(j)}$$

(5.26)

where $p_{\pm n}^{(i)}$ denotes the operators (2.37) acting in the $i$th factor of the tensor product in (5.25). Since the deformed $W$-algebra maps into the completion of $\mathcal{H}_r$ according to Definition 5.3 and Proposition 5.5, we obtain actions of the deformed $W$-algebra on both sides of relation (5.25). The hallmark of (5.25) is that it is an isomorphism of deformed $W$-algebra modules. Let us describe this isomorphism.

Let $\{s_\lambda \in F_u\}_{\lambda}$ partition denote the basis of plethystically modified Schur functions. We will shortly recall the definition of stable basis $\{s_\lambda \in K\}_{\lambda}$ a $r$-tuple of partitions from loc. cit., but let us first state the main idea of the present subsection:

**Claim 5.9.** The assignment of (5.25), which is explicitly defined as:

$$\text{Stab} (s_{\lambda^{1}} \otimes \cdots \otimes s_{\lambda^{r}}) = s_{(\lambda^{1},\ldots,\lambda^{r})}$$

(5.27)

intertwines the deformed $W$-algebra action on the LHS defined by (5.4) and (5.26), with the action of the same algebra on the RHS that we defined in Sect. 3.

The proof of Claim 5.9 involves comparing the Hopf algebra that Maulik–Okounkov associate to the moduli space of rank $r$ sheaves, with the algebra $A_r$ that we construct as a subquotient of the completed double shuffle algebra. Their construction is detailed in the cohomological case in [23], but the $K$-theoretic version has not yet been published. Therefore, we do not attempt a proof of Claim 5.9, and include its statement only for the sake of our exposition. We believe showing Claim 5.9 would be a good challenge for anyone interested in studying [23].

Let us recall the definition of stable bases in the greater generality in which they were defined by [23]. Consider any symplectic resolution $\mathcal{M}$ acted on by a generic rank 1 torus $\mathbb{C}^*$, and suppose for simplicity that the fixed point set $\mathcal{M}^{\mathbb{C}^*}$ is isolated and indexed by symbols $\lambda$. The fixed point set is partially ordered by the transitive closure of the relation which has $\mu \preceq \lambda$ if the fixed point $\mu$ is in the closure of the attracting locus of $\lambda$ under the one-parameter subgroup generated by the action $\mathbb{C}^* \curvearrowright \mathcal{M}$. Then loc. cit. define the **stable basis** corresponding to this data as:

$$\{s_\lambda\}_{\lambda \text{ fixed points}} \in K_{\mathbb{C}^*}(\mathcal{M})$$

which is upper triangular in the basis of fixed points:

$$s_\lambda = \sum_{\mu \prec \lambda} c^{\mu}_{\lambda}(t) |\mu\rangle$$
where $t$ denotes the equivariant parameter of $\mathbb{C}^* \curvearrowright \mathcal{M}$. The coefficients $c^\mu_\lambda(t) \in \mathbb{Z}[t^{\pm 1}]$ are uniquely determined by the following conditions:

$$c^\lambda_\lambda(t) = [\wedge \cdot \text{Tan}_{\lambda}^{\text{attr}} \mathcal{M}] \in K_{\mathbb{C}^*}(\text{pt}) = \mathbb{Z}[t^{\pm 1}]$$

for all $\lambda$, while for all $\mu \prec \lambda$ we require that:

$$c^\mu_\lambda(t) = t^\#_{\text{min}} \text{coefficient} + \cdots + t^\#_{\text{max}} - 1 \text{coefficient}$$

for certain coefficients that do not depend on $t$, where $\#_{\text{min}}$ and $\#_{\text{max}}$ denote the smallest and largest power of $t$ that appears in the Laurent polynomial $[\wedge \cdot \text{Tan}_{\mu}^{\text{attr}} \mathcal{M}]$. When $\mathcal{M}$ is the space of framed rank 1 sheaves (i.e. the Hilbert scheme of points on $\mathbb{C}^2$), loc. cit. observe that the $K$-theory classes $s_\lambda$ thus defined coincide with plethystically modified Schur functions under the isomorphism $\oplus_{n=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^*}(\mathbb{C}^2)^n \cong F_u$ given by the Bridgeland–King–Reid–Haiman equivalence. In the higher rank case, the stable basis is described by Claim 5.9.

Remark 5.10. The definition of the $K$-theoretic stable basis of Maulik and Okounkov also depends on an extra “slope” parameter, which in this paper is equal to 0. In the setup of the moduli spaces of rank $r$ sheaves on the plane, the slope is a real number with which we shift the numbers $\#_{\text{min}}$ and $\#_{\text{max}}$ in relation (5.29). This results in an infinite family of stable bases, as were studied in [20,30]. Finally, let us reiterate the fact that the $K$-theoretic stable basis would make (5.25) into an isomorphism of deformed $W$-algebra representations (see [23] for the cohomological version). The corresponding coproduct on the algebra $A$, as well as the action of the $W$-currents on the left-hand side of (5.25), have been studied in [16]. In the present paper, we take the opposite point of view from loc. cit. and define the level $r$ representation as the right-hand side of (5.25).

6 The Classical Limit

6.1. The usual $W$-algebra is the “classical limit” of the $q$ $W$-algebra. Specifically, this limit means that we specialize the parameters studied in this paper to:

$$q_1 = e^{\varepsilon h_1}, \quad q_2 = e^{\varepsilon h_2}, \quad q = e^{\varepsilon h} \quad \text{where} \quad h = h_1 + h_2$$
$$u = e^{\varepsilon \bar{u}}, \quad u_i = e^{\varepsilon \bar{u}_i}, \quad m = e^{\varepsilon \bar{m}}, \quad y = e^{\varepsilon \bar{y}}$$

and we take the leading term of all our formulas in the limit $\varepsilon \to 0$. This is quite natural, since many of our formulas are sums of products of expressions:

$$1 - q_1^{-k_1} q_2^{-k_2} \cdots = 1 - e^{-\varepsilon (k_1 h_1 + k_2 h_2 + \cdots)} = \varepsilon (k_1 h_1 + k_2 h_2 + \cdots) + O(\varepsilon^2)$$

for various integers $k_1, k_2, \ldots$. As a matter of notation, a bar over a symbol means the classical limit of that symbol from the $q$-deformed theory. Let us recall the additive shuffle algebra studied in [27]:

$$\mathcal{S} = \bigoplus_{k=0}^{\infty} \left\{ \text{symmetric rational functions} \frac{r(\bar{z}_1, \ldots, \bar{z}_k)}{\prod_{1 \leq i < j \leq k} (\bar{z}_i - \bar{z}_j - \bar{h})} \right\}$$
that satisfy the wheel conditions \( r | \bar{z}_1 - \bar{z}_2, \bar{z}_2 - \bar{z}_3, \bar{z}_3 - \bar{z}_1 | = (h_1, h_2, -h) = 0 \). The shuffle product is defined as in (2.5), with the rational function \( \zeta \) replaced by its additive analogue:

\[
\bar{\zeta}(\bar{z}) = \lim_{\varepsilon \to 0} \zeta(e^{\varepsilon \bar{z}}) = \frac{\bar{z} + h_1)(\bar{z} + h_2)}{\bar{z} + h} \quad (6.1)
\]

The analogue of our level \( r \) modules in the case of the additive shuffle algebra is the equivariant cohomology of the same moduli spaces \( M_n \) of rank \( r \) sheaves on \( \mathbb{P}^2 \):

\[
H = \bigoplus_{n=0}^{\infty} H^*_T(M_n) \otimes \text{Frac}(H^*_T(pt))
\]

This vector space has a basis of fixed points \(|\bar{\lambda}\rangle\), also indexed by \( r \)-tuples of partitions. By analogy with Theorem 3.7, we have two opposite \( \bar{S} \) actions on \( H \):

\[
\langle \bar{\lambda} | \bar{R}^{-} | \bar{\mu} \rangle = \bar{R}(\bar{\lambda} \setminus \bar{\mu}) \prod_{\square \in \bar{\lambda} \setminus \bar{\mu}} \left[ \frac{h_1 h_2}{-h} \bar{\zeta}(\bar{\chi}_\square - \bar{\chi}_\mu) \bar{\tau}(\bar{\chi}_\square + h) \right] \quad (6.2)
\]

\[
\langle \bar{\mu} | \bar{R}^{+} | \bar{\lambda} \rangle = \bar{R}(\bar{\lambda} \setminus \bar{\mu}) \prod_{\square \in \bar{\lambda} \setminus \bar{\mu}} \left[ \frac{h_1 h_2}{-h} \bar{\zeta}(\bar{\chi}_\lambda - \bar{\chi}_\square)^{-1} \bar{\tau}(\bar{\chi}_\square)^{-1} \right] \quad (6.3)
\]

where \( \bar{R}(\bar{\lambda} \setminus \bar{\mu}) = \bar{R}(\ldots, \bar{\chi}_\square, \ldots) \square \in \bar{\lambda} \setminus \bar{\mu} \) and:

\[
\bar{\chi}_\square = \bar{u}_k + i h_1 + j h_2
\]

for a square \( \square \) lying at coordinates \((i, j)\) in the \( k \)th constituent partition of the \( r \)-tuple \( \lambda \). Moreover, we write \( \bar{\tau} \) for the additive analogue of the function \( \tau \) of (3.8):

\[
\bar{\tau}(\bar{z}) = \prod_{i=1}^{r} (\bar{u}_i - \bar{z}) \quad (6.4)
\]

Finally, we have a series of diagonal operators on \( H \), with matrix coefficients:

\[
\langle \bar{\mu} | \bar{E}(\bar{y}) | \bar{\nu} \rangle = \delta^\mu_\nu \prod_{i=1}^{r} (\bar{y}_i - \bar{u}_i) \prod_{\square \in \bar{\nu}} \bar{\zeta}(\bar{\chi}_\square - \bar{y}) \quad (6.5)
\]

6.2. To take the classical limit \( S \rightarrow \bar{S} \) at the shuffle algebra level, simply set the variables to \( z_i = e^{\varepsilon \bar{z}_i} \) and take the leading order term as \( \varepsilon \to 0 \). Explicitly, for a shuffle element \( R \) of the form (3.13), its classical limit is:

\[
R(z_1, \ldots, z_k) \rightarrow \bar{R}(\bar{z}_1, \ldots, \bar{z}_k) = \text{Sym} \left[ \frac{l.o.}{\prod_{i=1}^{k} (\bar{z}_i - \bar{z}_{i+1} - h)} \prod_{i < j} \bar{\zeta}(\bar{z}_i - \bar{z}_j) \right] \quad (6.6)
\]

where “l.o.” stands for the leading order term in \( \varepsilon \). Note that we do not claim that the assignment (6.6) has any property whatsoever (it is not a homomorphism). In particular, the classical limit of the shuffle elements (3.30) is:
$T(x, y) \rightsquigarrow \tilde{T}(x, \tilde{y}) = \sum_{d=0}^{\infty} \text{Sym} \left[ \frac{x^d}{(\tilde{z}_d - \tilde{y}) \prod_{i=1}^{d-1} (\tilde{z}_i - \tilde{z}_{i+1} - \hbar)} \prod_{i<j} \tilde{z}_i \tilde{z}_j \right]$ \hfill (6.7)

To give meaning to the procedure $\rightsquigarrow$ outlined above, we will connect the action of $T(x, y)$ on $K$ with the action of $\tilde{T}(x, \tilde{y})$ on $H$. Define the classical limit $K \rightsquigarrow H$ by:

$$|\lambda\rangle \rightsquigarrow \lim_{\varepsilon \to 0} e^{\varepsilon^\lambda}|\lambda\rangle =: |\bar{\lambda}\rangle$$ \hfill (6.8)

Because of this renormalization of the basis vectors, the classical limit of Theorem 1.2 is given by the following limit of the operators (1.6):

**Proposition 6.6.** Let:

$$\tilde{W}(x, \tilde{y}) = \tilde{T}(x, y^{-1})^{\sim} \cdot \tilde{E}(\tilde{y}) \cdot \tilde{T}(x, y)^{\sim}$$ \hfill (6.9)

Then we have $W(x, yD_x) \rightsquigarrow \tilde{W}(x, \tilde{y})$, by which we mean the fact that:

$$\langle \mu | W(x, yD_x) | \lambda \rangle = \varepsilon^r \langle \bar{\mu} | \tilde{W}(x, \tilde{y}) | \bar{\lambda} \rangle + O(\varepsilon^{r+1})$$ \hfill (6.10)

**Proof.** By applying (1.6), we see that:

$$\langle \mu | W(x, yD_x) | \lambda \rangle = \langle \mu | W(x, e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) | \lambda \rangle$$

$$= \sum_{\nu \in \lambda \cap \mu} \langle \mu | T(x^{-1}, e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) \nu | \nu \rangle \langle \nu | E(e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) \nu | \nu \rangle \langle \nu | \tilde{T}(x e^{\varepsilon \hbar}, e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) \sim | \lambda \rangle$$

$$= \varepsilon^r \sum_{\nu \in \lambda \cap \mu} \langle \bar{\mu} | \tilde{T}(x^{-1}, \bar{y})^{\sim} \nu | \nu \rangle \langle \nu | \bar{E}(\bar{y}) \nu | \nu \rangle \langle \nu | \tilde{T}(x, \bar{y})^{\sim} | \bar{\lambda} \rangle + O(\varepsilon^{r+1}) = \text{RHS of (6.10)}$$

The next-to-last equality holds because the leading powers of $\langle \mu | T(x^{-1}, e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) \nu | \nu \rangle$, $\langle \nu | E(e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) \nu | \nu \rangle$, $\langle \nu | \tilde{T}(x e^{\varepsilon \hbar}, e^{\varepsilon \bar{y} + \varepsilon \hbar x} \partial_x) \sim | \lambda \rangle$ are $\varepsilon^0$, $\varepsilon^r$, $\varepsilon^0$, respectively, due to the normalization (6.8). The leading order terms are given by the corresponding coefficients of $\tilde{T}$, $\tilde{E}$, $\tilde{T}$, respectively, as in the equation above. \hfill $\Box$

6.4. Formulas (6.2)–(6.3) may be interpreted as the classical limit of the algebra $\mathcal{A}_r$ acting on $H$, but this action had already been presented in a different language in [23,38]. In [27], we asked about a geometric interpretation of the action of the usual $W$-currents on the cohomology group $H$. Since the operators $\tilde{T}(x, \bar{y})^{\sim}$ and $\tilde{T}(x, \bar{y})^{\sim}$ are given by the correspondences $3_3$ as $d$ ranges over $\mathbb{N}$ (this is the content of the additive version of Theorem 3.16, see [27]), formulas (6.9) and (6.10) would give a manifestly geometric answer, if we could relate the usual $W$-currents with $\tilde{W}(x, \bar{y})$. We will prove this in Proposition 6.6 and Theorem 6.7, but let us start by recalling the classical limit of the quantum Miura transformation (5.3). Specifically, set the bosons of Sect. 5.1 equal to:

$$b^i(x) = \varepsilon \partial_x (\mathcal{B}^i(x)) + O(\varepsilon^2)$$

in such a way that relation (5.1) yields in the limit $\varepsilon \to 0$:

$$[\mathcal{B}^i_m, \mathcal{B}^j_n] = \delta^i_j \delta^{0}_{m+n} m \hbar_1 \hbar_2$$ \hfill (6.11)
Note that:

\[ \Lambda^i(x) = 1 + \varepsilon \tilde{\Lambda}^i(x) + O(\varepsilon^2) \quad \text{where} \quad \tilde{\Lambda}^i(x) = \tilde{u}_i - \sum_{n \in \mathbb{N}} \frac{\tilde{b}_n^i}{x^n} + \sum_{n \in \mathbb{N}} \frac{\tilde{b}_n^i}{x^{-n}} \]

Then the quantum Miura transformation (5.3) reads:

\[
\sum_{k=0}^{r} (-1)^k W_k(x)e^{\varepsilon \hbar (r-k)x \partial_x} := \prod_{i=1}^{r} \left( e^{\varepsilon \hbar x \partial_x} - 1 - \varepsilon \tilde{\Lambda}^i(x) - O(\varepsilon^2) \right)
\]

\[
= \varepsilon^r : \prod_{i=1}^{r} (\hbar x \partial_x - \tilde{\Lambda}^i(x)) : + O(\varepsilon^{r+1}) = \varepsilon^r \sum_{k=0}^{r} (-1)^k \tilde{W}_k(x)(\hbar x \partial_x)^{r-k} + O(\varepsilon^{r+1})
\]

(6.12)

The right-hand side of (6.12) defines the currents of the $W$-algebra of type $\mathfrak{gl}_r$ (our convention differs from the usual one by a sign). We expect this algebra to be the tensor product of a Heisenberg algebra and the usual $W$-algebra of type $\mathfrak{sl}_n$ (defined in [13], see also [8] for reference). Specifically, the Heisenberg subalgebra is generated by:

\[
\tilde{W}_1(x) = \sum_{i=1}^{r} \tilde{\Lambda}^i(x) = \tilde{u}_1 + \cdots + \tilde{u}_r - \sum_{n \in \mathbb{N}} \frac{B_n}{x^n} + \sum_{n \in \mathbb{N}} \frac{B_n}{x^{-n}}
\]

Explicitly, we have $B_n = \tilde{b}_n^1 + \cdots + \tilde{b}_n^r$ and so $[B_m, B_n] = \delta_{m+n}^0 rm \hbar_1 \hbar_2$.

6.5. Formula (6.12) has an interesting consequence: the left-hand side vanishes to order $\varepsilon^r$. Identifying the coefficients of $(\hbar x \partial_x)^{r-i} \varepsilon^{r-i+j}$ for all $0 \leq j \leq i \leq r$ in the two sides of Eq. (6.12) gives us:

\[
\text{coefficient of } \varepsilon^j \text{ in } \sum_{k=0}^{r} (-1)^k W_k(x) \frac{(r-k)^{r-i}}{(r-i)!} = \delta_j^i (-1)^i \tilde{W}_i(x)
\]

\[
\Rightarrow \sum_{k=0}^{r} (-1)^k W_k(x) \frac{(r-k)^{r-i}}{(r-i)!} = \varepsilon^i (-1)^i \tilde{W}_i(x) + O(\varepsilon^{i+1})
\]

Since this relation holds for all $0 \leq i \leq r$, we infer that:

\[
\sum_{k=0}^{r} (-1)^k W_k(x) \pi (r-k) = \varepsilon^i (-1)^i \tilde{W}_i(x) + O(\varepsilon^{i+1})
\]

for any polynomial $\pi$ of degree exactly $r-i$, with leading term $1/(r-i)!$. We will take this polynomial to be $\pi(n) = \binom{n}{r-i}$, so we obtain the relation:

\[
\tilde{W}_i(x) = \lim_{\varepsilon \to 0} \varepsilon^{-i} \sum_{k=0}^{r-i} (-1)^k W_k(x)(r-k) \binom{r-k}{r-i}
\]

(6.13)

Recall the notation $W(x, y) = \sum_{k=0}^{r} W_k(x)(-y)^{-k}$ from Sect. 2.15. With it, we conclude the following formula for the classical limit of $q$-$W$-currents:
Proposition 6.6. The currents of the usual $W$-algebra are given by:

$$\bar{W}_i(x) = \lim_{\epsilon \to 0} \epsilon^{-i} \cdot \frac{\partial^{r-i}}{(r-i)!} \left[ y' W(x, y) \right] \mid_{y=1}$$  \hspace{1cm} (6.14)

Set $y = e^{\epsilon \bar{y}}$ and recall from (6.10) that $W(x, y)$ is of order $\epsilon^r$ when degenerating the representation on $K$-theory into the representation on cohomology $H$ (the operator $D_x$ is of order 0 in $\epsilon$, so it does not contribute anything to the leading order term). The $r-i$ derivatives in (6.14) can bring down no more than $r-i$ powers of $\epsilon^{-1}$, so we conclude that only the leading term of (6.10) contributes to the limit (6.14). Explicitly, we obtain:

Theorem 6.7. The action of the $W$-algebra on the level $r$ module $H$ is given by:

$$\bar{W}_i(x) = \frac{\partial^{r-i}}{(r-i)!} \left[ \bar{W}(x, \bar{y}) \right] \mid_{\bar{y}=0}$$  \hspace{1cm} (6.15)

Combining this result with (6.9), we obtain the following factorization for the currents of the classical $W$-algebra in the level $r$ representation $H$:

$$\bar{W}_i(x) = \frac{\partial^{r-i}}{(r-i)!} \left[ T(x^{-1}, \bar{y})^{\leftarrow} \cdot \bar{E}(\bar{y}) \cdot T(x, \bar{y}) \rightarrow \right] \mid_{\bar{y}=0}$$  \hspace{1cm} (6.16)

The expression $\bar{W}(x, \bar{y})$ is interpreted as a rational function in $\bar{y}$, via (6.9).

6.8. The classical analogue of $A_m(x)$ is given by the Chern polynomial of the Ext bundle, used as a correspondence on cohomology [27]. We will denote it by:

$$\tilde{A}_m(x) : H_{\tilde{u}} \rightarrow H_{\tilde{u}}$$

where $H_{\tilde{u}}$ and $H_{\tilde{u}'}$ are the cohomology groups of two different moduli spaces of sheaves, acted on by two different rank $r$ tori, with equivariant parameters given by $\tilde{u}$ and $\tilde{u}'$, respectively. All we need to use is the classical limit of formula (4.4):

$$\langle \lambda | A_m(x) | \lambda' \rangle = \langle \tilde{\lambda} | \tilde{A}_m(x) | \tilde{\lambda}' \rangle + O(\epsilon)$$  \hspace{1cm} (6.17)

since the operator $\tilde{A}_m(x)$ can be easily seen to have the following matrix coefficients (recall the notation of (6.1), (4.4) and (6.8)):

$$\langle \lambda | \tilde{A}_m(x) | \lambda' \rangle = \prod_{\square \in \lambda} \tilde{\varphi}'(\tilde{m} + \tilde{\square}) \prod_{\square' \in \lambda'} \tilde{\varphi}'(\tilde{\chi}' + \tilde{m}) \prod_{\square \in \lambda} \tilde{\varphi}'(\tilde{\chi} - \tilde{\square})$$

As in relation (4.27), we will replace the study of $\tilde{A}_m(x)$ with the closely related operator $\Phi_{\tilde{m}}(x) : H_{\tilde{u}} \rightarrow H_{\tilde{u}}$, defined by the formula:

$$\Phi_{\tilde{m}}(x) = \Lambda_{\tilde{m}}(x) \exp \left[ \sum_{n=1}^{\infty} \frac{\tilde{p}_n}{x^n} \cdot \frac{h}{h_1 h_2} \right]$$

where $\tilde{p}_n = -\frac{B_n}{n}$ is the leading order term of the classical limit of the bosons $p_n$. The operator $\Phi_{\tilde{m}}(x)$ was connected with vertex operators for the Toda conformal field theory.
in [12]. In [27], we used geometric techniques similar to those of Sect. 4 to show that when \( r = 2 \), the operator \( \Phi_m(x) \) is (up to conjugation by certain explicit exponentials in the Heisenberg subalgebra) equal to the Liouville vertex operator. In arbitrary rank \( r \), we may obtain a similar result by taking the classical limit of formula (1.5). Specifically, this formula implies that:

\[
\Phi_m(x) \left[ \sum_{k=0}^{i} (-1)^k W_k(y) \left( \frac{r - k}{r - j} \right) \right] \prod_{j=1}^{i} \left( 1 - \frac{m' u x}{q^{-j} u' y} \right) = \frac{m' u x}{q^{-j} u' y} \Phi_m(x) \prod_{j=1}^{i} \left( 1 - \frac{m' u x}{q^{-j} u' y} \right)
\]

By (6.13) and (6.17), the leading order term (namely \( \tilde{v}^j \)) of the above relation is:

\[
\tilde{\Phi}_m(x) \tilde{W}_i(y)(x - y)^j = \tilde{W}_i(y) \tilde{\Phi}_m(x)(x - y)^j \tag{6.18}
\]

which yields the locality of the fields \( \tilde{\Phi}_m(x) \) and \( \tilde{W}_i(y) \) (note that \( m^k = 1 + O(\varepsilon) \) in the right-hand side does not contribute anything to the leading order term). Up to the fact that our conventions are somewhat non-standard, formula (6.18) is of the same nature as the realization of \( \tilde{\Phi}_m(x) \) as a vertex operator in [12].

### 7 Appendix

**Proof of formulas** (2.12), (2.13), (2.14). Fix any pair of coprime integers \( a, b \), and recall that the subalgebra \( \Lambda_{b/a} = \mathbb{F}[P_{a,b}, P_{2a,2b}, \ldots] \) is commutative. Moreover, as shown in [28], \( \Lambda_{b/a} \) is actually a bialgebra with respect to the coproduct:

\[
\Delta_{b/a}(R) = \sum_{i=0}^{na} \lim_{\xi \to \infty} \frac{R(z_1, \ldots, z_i \otimes \xi z_{i+1}, \ldots, \xi z_{na})}{\xi^{b(na-i)/a}}
\]

for any \( R \in \mathcal{S}_{na,nb} \cap \Lambda_{b/a} \). The fact that \( P_{na,nb} \) is primitive with respect to \( \Delta_{b/a} \):

\[
\Delta_{b/a}(P_{na,nb}) = P_{na,nb} \otimes 1 + 1 \otimes P_{na,nb}
\]

was established in *loc. cit.* Moreover, it was shown therein that any other primitive element of \( \Lambda_{b/a} \) is a constant multiple of \( P_{na,nb} \). Because of this, the elements \( H_{na,nb}, E_{na,nb}, Q_{na,nb} \) are exponentials in the shuffle elements \( P_{na,nb} \) (times constant multiples) if and only if they are group-like with respect to \( \Delta_{b/a} \):

\[
\Delta_{b/a}(H_{na,nb}) = \sum_{m=0}^{n} H_{na,nb} \otimes H_{(n-m)a,(n-m)b} \tag{7.1}
\]

and the analogous formulas for \( E_{na,nb} \) and \( Q_{na,nb} \). These formulas follow from relation (6.10) of *loc. cit.*, in the notation of which we have:

\[
H_{na,nb} = X_{na,nb}^{(0, \ldots, 0)} \tag{7.2}
\]

\[
E_{na,nb} = (-q)^{n-1} X_{na,nb}^{(1, \ldots, 1)} \tag{7.3}
\]

\[
Q_{na,nb} = \left( 1 - \frac{1}{q} \right) \left( X_{na,nb}^{(0, \ldots, 0)} + X_{na,nb}^{(0, \ldots, 0, 1)} + \ldots + X_{na,nb}^{(0,1, \ldots, 1)} + X_{na,nb}^{(1, \ldots, 1)} \right) \tag{7.4}
\]
Indeed, the fact that (7.2) implies (7.1) is an immediate application of (6.10) of loc. cit., as is the analogous statement for $E_{na,nb}$. The fact that the same holds for (7.4) is proved analogously with Proposition 6.5 of loc. cit., and we leave the details to the interested reader.\[2\] It is a well-known fact concerning the bialgebra structure on the ring of symmetric functions that relation (7.1) implies that:

$$
\sum_{n=0}^{\infty} \frac{H_{an,bn}}{x^n} = \exp \left( \sum_{n=1}^{\infty} \frac{P_{an,bn}}{nx^n} \cdot \alpha_n \right)
$$

(7.5)

for some constants $\alpha_1, \alpha_2, \ldots$. Similarly, we have formulas analogous to (7.5) for $E_{na,nb}$ and $Q_{na,nb}$, with respect to other sets of constants $\{\alpha_n\}_{n \in \mathbb{N}}$. Therefore, to prove (2.12), (2.13), (2.14), it remains to compute these constants. As in loc. cit., we will do so by considering the linear functional:

$$
\phi : \Lambda_{b/a} \to \mathbb{F}, \quad \phi(R(z_1, \ldots, z_{na})) = \frac{R(1, q_1^{-1}, \ldots, q_1^{-na+1})}{\prod_{1 \leq i < j \leq na} \zeta(q_1^{j-i})} \cdot q_1^{\frac{n^2ab-nb+na-n}{2}} (1 - q_2)^{na}
$$

It is shown in (6.12) of loc. cit. that the functional $\phi$ is multiplicative. Moreover, from (2.8)–(2.11), we see that:

$$
\phi(P_{na,nb}) = 1 - q_2^n \quad \quad \phi(H_{na,nb}) = 1 - q_2
$$

$$
\phi(E_{na,nb}) = (1 - q_2)(-q_2)^{n-1} \quad \quad \phi(Q_{na,nb}) = \frac{(1 - q^{-1})(1 - q_2)(1 - q_1^{-n})}{1 - q_1^{-1}}
$$

The reason for these very simple formulas is that $\zeta(q_1^{-1}) = 0$, and so only the identity permutation survives evaluation at $z_i = q_1^{-i+1}$ in formulas (2.8)–(2.11). For brevity, we leave out the elementary manipulations involving integer parts that have produced the formulas above. Then applying the multiplicative function $\phi$ to (7.5), we conclude that:

$$
1 + \sum_{n=1}^{\infty} \frac{1 - q_2}{x^n} = \exp \left( \sum_{n=1}^{\infty} \frac{\alpha_n (1 - q_2^n)}{nx^n} \right) \quad \Rightarrow \quad \alpha_n = 1
$$

for all $n$. Then (7.5) leads to (2.12). Formulas (2.13)–(2.14) are proved similarly. \[ \square \]

**Proof of Proposition 2.16.** Let us recall the following pairing on $S$ from [28]:

$$
\langle \cdot, \cdot \rangle : S \otimes S \to \mathbb{F}
$$

$$
\langle R, R' \rangle = \int : R(z_1, \ldots, z_k) R' \left( \frac{1}{z_1}, \ldots, \frac{1}{z_k} \right) \prod_{1 \leq i \neq j \leq n} \zeta \left( \frac{z_i}{z_j} \right) Dz_1 \ldots Dz_k
$$

(7.6)

The normal-ordered integral $: \int :$ is defined for all Laurent polynomials $\rho$ and:

$$
R' = \text{Sym} \left[ \rho(z_1, \ldots, z_k) \prod_{1 \leq i < j \leq k} \zeta \left( \frac{z_i}{z_j} \right) \right]
$$

\[2\] The proof is actually in Subsection 5.7 of the first version of the paper loc. cit., which is unpublished, but can be found at arXiv:1209.3349v1.
by the formula:

$$\langle R, R' \rangle := \int_{|z_1| \gg \cdots \gg |z_k|} R(z_1, \ldots, z_k) \rho \left( \frac{1}{z_1}, \ldots, \frac{1}{z_k} \right) \frac{Dz_1 \ldots Dz_k}{\prod_{1 \leq i < j \leq n} \xi \left( \frac{z_i}{z_j} \right)}$$

(7.7)

for all $R \in \mathcal{S}$. Let us now consider the isomorphism $\mathcal{S} \cong \mathcal{A}^\uparrow$, under which the shuffle element $z^d$ in a single variable corresponds to the generator $P_{d,1}$. The situation we will mostly be concerned with is when:

$$\rho(z_1, \ldots, z_k) = \delta \left( \frac{z_1}{x} \right) \ldots \delta \left( \frac{z_k}{x q^{1-k}} \right)$$

in which case (7.7) implies:

$$\langle R(z_1, \ldots, z_k), W_k(x) \rangle = \frac{R \left( \frac{1}{x}, \frac{q}{x}, \ldots, \frac{q^{k-1}}{x} \right)}{\prod_{1 \leq i < j \leq k} \xi \left( \frac{z_i}{z_j} \right)}$$

(7.8)

because integrating against a $\delta$ function is equivalent to evaluation. A priori, the right-hand side of (7.8) is an undefined expression of the form $\frac{\infty}{\infty}$, but we make it precise as the linear functional $\varphi^k_x : \mathcal{S}_k \rightarrow \mathbb{F}[x^{\pm 1}]$ defined by:

$$\varphi^k_x(R) = \lim_{z_k \mapsto q^{k-1}/x} \cdots \lim_{z_1 \mapsto 1/x} \left( \frac{R(z_1, \ldots, z_k)}{\prod_{1 \leq i < j \leq k} \xi \left( \frac{z_i}{z_j} \right)} \right)$$

(7.9)

Indeed, each successive limit is well-defined for all rational functions $R$ of the form (2.6), because at the $i$th step, the pole $z_i - q^{i-1}/x$ of $R$ is precisely canceled by a zero of $\xi^{-1}$. Therefore, (7.8) and (7.9) imply that:

$$\langle R(z_1, \ldots, z_k), W_k(x) \rangle = \varphi^k_x(R)$$

(7.10)

**Claim 7.1.** The functional $\varphi_x$ satisfies the following multiplicativity property:

$$\varphi^k_{x_1} \ast k_2 (R_1 \ast R_2) = \varphi^k_{x_1}(R_1) \varphi^k_{x_2/q} (R_2)$$

(7.11)

for any shuffle elements $R_1, R_2$ of degrees $k_1, k_2$.

**Claim 7.2.** For any $k > 0$ and $d \in \mathbb{Z}$ with greatest common divisor $n$, we have:

$$\varphi^k_x(P_{d,k}) = \frac{q^{\alpha(k,d)}}{x^d} \cdot \frac{(1 - q^n)(1 - q^2)(1 - q^{-1})^k}{(1 - q_1)^k(1 - q_2)^k(1 - q^{-n})}$$

(12.12)

where $\alpha(k, d) = \frac{kd + k - d - n}{2}$. We think of $P_{d,k}$ as lying in $\mathcal{A}^\uparrow \cong \mathcal{S}$. 


We will prove these two claims after we show how they allow us to complete the proof of Proposition 2.16. For any ordered collection $v$ as in (2.32) and $P_v$ defined as in (2.31), relations (7.11) and (7.12) imply that:

$$\varphi_S(P_v) = \frac{q^{\alpha(v)}}{x^d} \prod_{i=1}^t \frac{(1 - q_1^{n_i})(1 - q_2^{n_i})(1 - q^{-1})^{k_i}}{(1 - q_1)^{k_i}(1 - q_2)^{k_i}(1 - q^{-n_i})}$$ (7.13)

where we write $n_i = \gcd(k_i, d_i), k = k_1 + \cdots + k_t, d = d_1 + \cdots + d_t$, and set:

$$\alpha(v) = \sum_{1 \leq i < j \leq t} k_id_j + \sum_{i=1}^t \frac{k_id_i + k_i - d_i - n_i}{2}$$ (7.14)

The products $P_v$ for ordered sequences $v$ are known [9,28] to form an orthogonal basis of the algebra $S$ with respect to the inner product (7.6):

$$\langle P_v, P_{v'} \rangle = \delta_{v,v'} \cdot z_v \prod_{i=1}^t \frac{(1 - q_1^{n_i})(1 - q_2^{n_i})(q^{-1} - 1)^{k_i}}{(1 - q_1)^{k_i}(1 - q_2)^{k_i}(q^{-n_i} - 1)}$$ (7.15)

where:

$$z_v = \prod_{(k,d)} (# \text{ of } (k,d) \text{ in } v)! \prod_{i=1}^t n_i$$ (7.16)

Therefore, (7.10), (7.13) and (7.15) imply the following formula:

$$W_k(x) = \sum_{t \geq 1} (-1)^{k-t} \sum_{v = \left\{ d_1 < \cdots < d_t \right\}} P_v \cdot \frac{q^{\alpha(v)}}{x^d} \frac{z_v}{z_{\lambda}}$$ (7.17)

Using relation (2.20), we conclude that the generators $P_{na, nb}$ for fixed coprime $(a, b)$ and all $n > 0$ commute. Therefore, the multiplicative map that assigns to $P_{na, nb}$ the $n$th power sum function $P_n$ also assigns to $E_{na, nb}$ the $n$th elementary symmetric function $e_n$. Therefore, the well-known identity of symmetric functions:

$$e_n = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)} (-1)^{n_t} \cdot \frac{P_{\lambda}}{z_{\lambda}}$$

where $P_{\lambda} = P_{n_1} \cdots P_{n_t}$

implies the following identity of shuffle elements for any slope $\frac{b}{d}$ with $\gcd(a, b) = 1$:

$$E_{na, nb} = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)} (-1)^{n_t} \cdot \frac{P_{\lambda(a,b)}}{z_{\lambda}}, \quad \text{where } P_{\lambda(a,b)} = P_{n_1 a, n_1 b} \cdots P_{n_t a, n_t b}$$ (7.18)

where we use the well-known constant $z_{\lambda} = \prod_{i}(\# \text{ of } i \text{ in } \lambda)! \prod_{i=1}^t n_i$ defined for any partition $\lambda$ by analogy with (7.16). Formula (7.18) allows us to collect all terms $P_{d,k}$ of the same slope together in formula (7.17), and rewrite it as:

$$W_k(x) = \sum_{v = \left\{ d_1 < \cdots < d_t \right\}} \frac{E_v}{x^d} \cdot \frac{q^{\alpha(v)}}{z_{\lambda}} (-1)^{\sum_{i=1}^t k_i - \gcd(k_i, d_i)}$$ (7.19)
where we still denote $E_v = E_{d_1,k_1} \cdots E_{d_r,k_r}$. Since $E_{d,k}$ acts with degree $-d$ in any good representation, we may consider the LDU decomposition of (7.19):

$$W_k(x) = \sum_{k_{-d}+k_0+k_{-d} = k}^{d_{-d}+d_{-d} = 0} \frac{L_{d_{-d}+k_{-d}} \cdot E_{0,k_{0}} \cdot U_{d_{-d}+k_{-d}}}{x^{d_{-d}+d_{-d}}} \cdot q^{(k_{-d}+k_0)d_{-d}}$$

where we set $L_{0,k} = U_{0,k} = \delta_k^0$ and:

$$L_{d,k} = \sum_{t,k_i,d_i \in \mathbb{N}}^{t,k_i+\cdots+k_i = k} E_v \cdot q^{\alpha(v)} (-1)^{i} \sum_{i=1}^{t} k_i - \gcd(k_i,d_i) \quad \text{(7.20)}$$

$$U_{d,k} = \sum_{t,k_i,d_i \in \mathbb{N}}^{t,k_i+\cdots+k_i = k} E_v \cdot q^{\alpha(v)} (-1)^{i} \sum_{i=1}^{t} k_i - \gcd(k_i,d_i) \quad \text{(7.21)}$$

for $d > 0$. The crucial fact is that $L_{d,k} \in A^\uparrow \cap A^{\leftarrow}$ and $U_{d,k} \in A^\uparrow \cap A^{\rightarrow}$. Up until now, we have thought about these operators as lying in the upper shuffle algebra, but from now on we wish to think about them in the left/right shuffle algebras:

**Claim 7.3.** For any $k, d > 0$, we have the following equalities in the left and right shuffle algebras $A^{\leftarrow} \cong S$ and $A^{\rightarrow} \cong S^{op}$, respectively:

$$L_{d,k} = T_{d,k}^{\leftarrow}, \quad U_{d,k} = q^{d(k-1)} T_{d,k}^{\rightarrow} \quad \text{(7.22)}$$

This completes the proof of Proposition 2.16, modulo Claims 7.1, 7.2 and 7.3, to which we now turn. To establish the first of these claims, note that the shuffle product (2.5) implies that:

$$\varphi_{x}^{k_1+k_2} (R_1 \ast R_2) = \sum_{A_1 \cup A_2 = [1, \ldots, k_1+k_2]}^{k_1 = k_2} R_1 \left( \left\{ \frac{q^{i-1}}{x} \right\}_{i \in A_1} \right) R_2 \left( \left\{ \frac{q^{j-1}}{x} \right\}_{j \in A_2} \right) \prod_{i \in A_1} \xi(q^{i-j}) \prod_{1 \leq i < j \leq k_1+k_2} \xi(q^{i-j}) \quad \text{(7.23)}$$

Since $R_i$ is of the form (2.6) for each of $i \in \{1, 2\}$, the number of poles produced by the evaluation of $R_i$ as in (7.23) equals the number of pairs of consecutive indices in $A_i$. Therefore, the total number of poles in each summand of (7.23) equals:

$$\# \text{(pairs of consecutive numbers in } A_1 \text{)} + \# \text{(pairs of consecutive numbers in } A_2 \text{)}$$

$$- \# \text{( } i \in \{1, \ldots, k_1+k_2-1\} \text{ and } i \notin A_1 \text{ or } i+1 \notin A_2 \text{)} \quad \text{(7.24)}$$

It is elementary to see that the number in (7.24) is always strictly negative, unless $A_1 = \{1, \ldots, k_1\}$ and $A_2 = \{k_1+1, \ldots, k_1+k_2\}$, in which case it is equal to zero. In the former case, the corresponding term in (7.23) vanishes. In the latter case, it produces precisely the right-hand side of (7.11), completing the proof of Claim 7.1.

Let us now prove Claim 7.2 by induction on $k$. The task reduces to proving the claims contained in the two bullets below:
the induction hypothesis implies the formula:
\[
\varphi_x^k(Q_{d,k}) = \frac{q^{a(k,d)}}{x^d} \cdot \frac{(1 - q^{-1})^k}{(1 - q_1)(1 - q_2)^k} \cdot \frac{(1 - q_1)(1 - q_2)(1 - q^n)}{1 - q} \tag{7.25}
\]

• formula (7.25) and the induction hypothesis imply (7.12)

Let us prove the first bullet. Choose numbers \(k_1 + k_2 = k\) and \(d_1 + d_2 = d\) such that \(k_1d_2 - k_2d_1 = n\), and \(\gcd(k_1, d_1) = \gcd(k_2, d_2) = 1\) (the existence of such numbers comes down to the existence of a lattice triangle of minimal area with \((0, 0), (k, d)\) as an edge, which is always the case). Then (2.21) and the linearity of \(\varphi_x^k\) imply:
\[
\varphi_x^k(Q_{d,k}) = \frac{1 - q^{-1}}{(1 - q_1)(1 - q_2)} \left[ \varphi_x^k(P_{d_1,k_1} * P_{d_2,k_2}) - \varphi_x^k(P_{d_2,k_2} * P_{d_1,k_1}) \right]
\]

Then formula (7.11) and the induction hypothesis of (7.12) imply that \(\varphi_x^k(Q_{d,k}) = \frac{1 - q^{-1}}{(1 - q_1)(1 - q_2)} \cdot \frac{q^{a(k_1,d_1)+a(k_2,d_2)}}{x^d} \cdot \frac{(1 - q_1)^2(1 - q_2)^2(1 - q^{-1})^k}{(1 - q_1)^k(1 - q_2)^k(1 - q^{-1})^2} \cdot (q^{k_1d_2} - q^{d_1k_2})\)

which proves (7.25) (note that one needs the elementary formulas \(k_1d_2 = k_2d_1 + n\) and \(\alpha(k, d) = \alpha(k_1, d_1) + \alpha(k_2, d_2) + k_2d_1 + 1\). Let us now prove the second bullet, to which end we write \(d = na, k = nb\) for \(a\) and \(b\) coprime. Formula (2.14) implies:
\[
Q_{na,nb} = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)}^{\lambda \vdash n} \frac{P_{\lambda(a,b)}}{z_{\lambda}} \prod_{i=1}^{t} (1 - q^{-n_i})
\]
where we use the notation in (7.18). Applying the linear map \(\varphi_x^{nb}\) gives us:
\[
\varphi_x^{nb}(Q_{na,nb}) = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)}^{\lambda \vdash n} \frac{q^{\sum_{i \leq j \leq t} n_in_jab}}{z_{\lambda}} \prod_{i=1}^{t} \varphi_x^{n_i b}(P_{n_ia,n_ib})(1 - q^{-n_i})
\]
where we used \(\varphi_x^{nb}(P_{n_1a,n_1b} \cdots P_{n_ta,n_tb}) = \varphi_x^{n_1 b}(P_{n_1a,n_1b}) \cdots \varphi_x^{n_t b}(P_{n_ta,n_tb})\)
\(q^{\sum_{i < j} n_in_jab}\), which is a consequence of (7.11). Since the second bullet tells us that we may assume formula (7.25) for the left-hand side and formula (7.12) for all summands in the right-hand side except for \(\lambda = (n)\), proving formula (7.12) for \(P_{na,nb}\) boils down to establishing the identity:
\[
\frac{q^{a(nb,na)}}{x^{na}} \cdot \frac{(1 - q^{-1})^{nb}}{(1 - q_1)^{nb}(1 - q_2)^{nb}} \cdot \frac{(1 - q_1)(1 - q_2)(1 - q^n)}{1 - q} = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)}^{\lambda \vdash n} \frac{q^{\sum_{i < j \leq t} n_in_jab}}{z_{\lambda}} \prod_{i=1}^{t} \left[ \frac{q^{a(n_i b, n_i a)}}{x^{n_i a}} \cdot \frac{(1 - q_1^{n_i})(1 - q_2^{n_i})(1 - q^{-1})^{n_i b}}{(1 - q_1)^{n_i b}(1 - q_2)^{n_i b}(1 - q^{-n_i})(1 - q^{-n_i})} \right]
\]

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Note that $\alpha(nb, na) = \sum_{i=1}^t \alpha(n_i b, n_i a) + \sum_{1 \leq i < j \leq t} n_i n_j ab$, as follows from (7.14).

After canceling common factors, the required equality states:

$$\frac{(1 - q_1)(1 - q_2)(1 - q^n)}{1 - q} = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)}^{\lambda \vdash n} \prod_{i=1}^t (1 - q_{1_i}^n)(1 - q_{2_i}^n)$$

for all $n > 0$. We will prove this by establishing the identity of generating series:

$$1 + \sum_{n=1}^{\infty} \frac{(1 - q_1)(1 - q_2)(1 - q^n) y^n}{1 - q} = \sum_{\lambda = (n_1 \geq \cdots \geq n_t)}^{\lambda \vdash n} \prod_{i=1}^t (1 - q_{1_i}^n)(1 - q_{2_i}^n) y^{n_i}$$

Summing the two sides of the equation above gives us:

$$\frac{(1 - q_1 y)(1 - q_2 y)}{(1 - y)(1 - q y)} = \exp \left[ \sum_{n=1}^{\infty} \frac{(1 - q_1^n)(1 - q_2^n) y^{n}}{n} \right]$$

which is a straightforward identity.

Finally, let us prove Claim 7.3. We will only prove the statement for $L_{d,k}$, as the statement for $U_{d,k}$ follows by applying the anti-automorphism $P_{-d,k} \mapsto P_{d,k}$ (the power of $q$ in the formula for $U_{d,k}$ stems from the difference between $\alpha(v)$ and $\alpha(v^\dagger)$, where for a collection of lattice points $v$, its reflection across the vertical axis is denoted by $v^\dagger$)

From formula (2.10), we see that:

\[
E_{-d_1,k_1} \cdots E_{-d_t,k_t} (-1)^{\sum_{i=1}^t k_i - n_i} = q^{\sum_{i=1}^t n_i - d_i} (-1)^{\sum_{i=1}^t k_i - d_i} \cdot \left[ \prod_{s=1}^{d_t} z_{i_1 + d_1 + \cdots + d_s - 1} \right] \prod_{i=1}^{d_t - 1} \left( 1 - \frac{z_i}{q z_{i+1}} \right) \prod_{1 \leq i < j \leq d} \left( \frac{z_i}{z_j} \right) \]

where $d = d_1 + \cdots + d_t$ and $n_s = \gcd(k_s, d_s)$. Meanwhile, it is elementary to prove that $\alpha((-d_1, k_1), \ldots, (-d_t, k_t))$ from formula (7.14) equals:

\[
= -k (d - 1) + \sum_{s=1}^t d_s - n_s + \sum_{i=1}^{k_s} (i + d_1 + \cdots + d_{s-1} - 1) \left( \left\lfloor \frac{ik_s}{d_s} \right\rfloor - \left\lfloor \frac{(i-1)k_s}{d_s} \right\rfloor \right)
\]

If we let $w_i = z_i q^{i-1}$, then the two formulas above together imply that:

\[
L_{d,k} = q^{-k (d-1)} (-1)^k d \sum_{1 \leq j \leq q} \frac{\left( \frac{w_j q^j}{w_i q^i} \right)}{1 - \frac{w_j}{w_i + 1}} \prod_{s=1}^{d_t} \left( \prod_{i=1}^{d_1} \left\lfloor \frac{ik_s}{d_s} \right\rfloor - \left\lfloor \frac{(i-1)k_s}{d_s} \right\rfloor \right) \prod_{i=1}^{d_t - 1} \left( 1 - \frac{w_i d_1 + \cdots + d_{s-1}}{w_i d_1 + \cdots + d_s + 1} \right)
\]
Therefore, formula (7.22) follows from the identity of Laurent polynomials:

\[
\sum_{k_1+\cdots+k_t=k} \prod_{1\leq i\leq d_i} w_{\frac{ik_i}{d_i}} \left[ \frac{(i-1)k_i}{d_i} \right] \prod_{s=1}^{t-1} \left( 1 - \frac{w_{d_1+\cdots+d_s}}{w_{d_1+\cdots+d_s+1}} \right) = w_1 w_d^{k-1} \tag{7.26}
\]

Note that the summands in the left-hand side are indexed by convex piecewise-linear paths \((0,0), (d_1,k_1), (d_1+d_2,k_1+k_2), \ldots, (d,k)\) between the origin and the point \((d,k)\), which lie in the first quadrant minus the coordinate axes. Since we will encounter this terminology often, we will refer to such paths as broken paths. We will prove (7.26) by counting how many times a monomial \(w_1^{a_1} \cdots w_d^{a_d}\) with \(a_1 + \cdots + a_d = k\) appears in the left-hand side (the point is to show that the answer should be 0, unless the monomial is \(w_1 w_d^{k-1}\), when the answer should be 1). Such monomials are in one-to-one correspondences with non-decreasing collections:

\[
0 \leq s_1 \leq \cdots \leq s_{d-1} < k, \quad \text{where} \quad s_i + 1 = a_1 + \cdots + a_i
\]

(we also make the convention that \(s_0 = -1\) and \(s_d = k - 1\)). Note that we can assume that \(a_1 \geq 1\), because all monomials in (7.26) have \(w_1\) raised to positive powers. We identify \((s_1, \ldots, s_{d-1})\) with the collection of lattice points \((1,s_1), \ldots, (d-1,s_{d-1})\), and refer to this set of points as a collection of bullets.

Claim 7.4. The coefficient of \(\prod_i w_i^{s_i-s_i-1}\) in the left-hand side of (7.26) counts the number of broken paths \(P\) between the lattice points \((0,0)\) and \((d,k)\), which:

1. Intersect each line \(x = i\) between heights \(y = s_i\) and \(y = s_i + 1\), \(\forall 1 \leq i < d\)
2. Can only pass through the point \((i,s_i)\) if the broken path bends at this point; in this case, we will call such an \((i,s_i)\) a hinge, and note that this terminology depends on both the broken path and the collection of bullets.
3. Are counted with sign \((-1)^\text{# hinges}\) in the left-hand side of (7.26)

For a given collection of bullets \(C\), we denote by \(\mathcal{P}(C)\) the collection of broken paths with the above properties. Then identity (7.26) reduces to the fact that:

\[
\sum_{P \in \mathcal{P}(C)} (-1)^\text{# hinges of } P = 0 \tag{7.27}
\]

Remark 7.5. Based on (7.27), we need to explain why the right-hand side of (7.26) is \(1 \cdot w_1 w_d^{k-1}\) instead of \(0 \cdot w_1 w_d^{k-1}\). There are only two broken paths corresponding to the collection of bullets \(\{(1,0), \ldots, (d-1,0)\}\): one is \((0,0), (d-1,1), (d,k)\) and the other is \((0,0), (1,0), (d-1,1), (d,k)\). The first contributes sign + to (7.27) and the second contributes sign − to (7.27). However, only the first path contributes to the left-hand side of (7.26), because of the fact that \(k_1, \ldots, k_t\) therein must be > 0.

Claim 7.4 is an elementary bijection between the sums in (7.26) and (7.27). Indeed, as we have mentioned immediately after (7.26), there is a bijection between summands in the left-hand side of (7.26) and broken paths. Moreover, to every subset \(S = \{u_1,u_2, \ldots\} \subset \{1, \ldots, t-1\}\), we may associate the monomial:

\[
\prod_{1 \leq i \leq d_i} w_i^{\frac{ik_i}{d_i}} \left[ \frac{(i-1)k_i}{d_i} \right] \prod_{u \in S} \left( -\frac{w_{d_1+\cdots+d_u}}{w_{d_1+\cdots+d_u+1}} \right)
\]
to the collection of bullets \((1, s_1), \ldots, (d - 1, s_{d-1})\) with:

\[
s_{d_1 + \cdots + d_u - 1 + i} = k_1 + \cdots + k_u - 1 + \left\lfloor \frac{ik_u}{d_u} \right\rfloor - 1 + \delta_u \in S^j_{d_u} \quad \forall i \in \{1, \ldots, d_u\}, \forall u \in \{1, \ldots, t\}
\]

To any monomial in (7.26), this procedure associates a term \(\pm 1\) in (7.27), and we leave it to the interested reader to show that this assignment is a bijection. The challenging part is the proof of statement (7.27), to which we now turn. For any collection of bullets \(C\), consider the assignment:

\[
P(C) \xrightarrow{\Xi_C} \{ \text{subsets of } \{1, \ldots, d - 1\} \}
\]

which sends a path to the set of \(i\)'s such that \((i, s_i)\) is a hinge.

**Claim 7.6.** The assignment \(\Xi_C\) is injective. Its image is the collection of subsets \(H = \{1, \ldots, d - 1\}\) with the property that for all \(0 \leq a < b < c \leq d\) we have:

\[
\frac{s_c - s_b + 1}{c - b} > \frac{s_b - s_a - 1}{b - a} \quad \text{(7.28)}
\]

\[
\frac{s_c - s_b + 1}{c - b} > \frac{s_b - s_a}{b - a} \quad \text{if } a \in H \quad \text{(7.29)}
\]

\[
\frac{s_c - s_b}{c - b} > \frac{s_b - s_a - 1}{b - a} \quad \text{if } c \in H \quad \text{(7.30)}
\]

\[
\frac{s_c - s_b}{c - b} > \frac{s_b - s_a}{b - a} \quad \text{if } a, c \in H \quad \text{(7.31)}
\]

By convention, we write \(s_0 = -1\) and \(s_d = k - 1\). The four conditions above can be represented pictorially by requiring that the following 4 situations do not occur:

- (7.28)

- (7.29)

- (7.30)

- (7.31)

where the full circles depict the lattice points \((i, s_i)\) and the hollow circles depict the lattice points \((i, s_i + 1)\). We conclude that \(\text{Im} \, \Xi_C\) is an abstract simplicial complex, namely a collection of sets with the property that if some \(H\) belongs to the collection, so do all of the subsets of \(H\). Formula (7.27) then becomes a statement about the reduced Euler characteristic of this abstract simplicial complex:

\[
\sum_{H \in \text{Im} \, \Xi_C} (-1)^{|H|} = 0 \quad \text{(7.32)}
\]
Proof. It is clear that properties (7.28)–(7.31) are necessary in order for a broken path passing through the set of hinges $H$ to be convex. Conversely, let us consider a set of hinges $H$ with properties (7.28)–(7.31), and let us show that they correspond to a broken path $P$ with the correct properties. First of all, the hinges $\{(i, s_i), i \in H\}$ must themselves form a convex path, otherwise we would violate condition (7.28). In between two consecutive hinges $(i, s_i)$ and $(j, s_j)$, property (1) of Claim 7.4 forces the piecewise-linear path $P$ to be the convex hull of the points:

\[(i + 1, s_{i+1} + 1), \ldots, (j - 1, s_{j-1} + 1)\] (7.33)

In particular, the path $P$ is uniquely determined in between any two consecutive hinges, which implies the injectivity of $\Xi_C$. The path $P$ thus constructed is convex between any two consecutive hinges, so we must also show that it is convex at each hinge $(i, s_i)$. In other words, we must show that the part of $P$ to the right of the hinge has slope greater than the part of $P$ that is to the left. This is guaranteed by formulas (7.28)–(7.31).

Finally, we must show that the convex piecewise-linear path $P$ thus constructed satisfies property (1) of Claim 7.4 (since property (2) holds automatically). Assume that the path $P$ intersects the vertical line $x = b$ at height $y$. If $y > s_b + 1$, then we contradict the fact that $P$ is the convex hull of the points (7.33) between any two consecutive hinges. If $y \leq s_b$, then we contradict one of (7.28)–(7.31). \(\square\)

Therefore, we need to prove formula (7.32) for the abstract simplicial complex consisting of subsets $H \subset \{1, \ldots, d - 1\}$ satisfying properties (7.28)–(7.31). We may assume that (7.28) is never violated, otherwise the abstract simplicial complex would be empty and (7.32) would hold trivially. Therefore, let us consider the set:

\[H_0 := \{1, \ldots, d - 1\} \setminus \{\{a\text{’s as in (7.29)}\} \cup \{c\text{’s as in (7.30)}\}\} \] (7.34)

We claim that for all $a < b < c$ with $a, c \in H_0$, we have:

\[
\frac{s_c - s_b}{c - b} \geq \frac{s_b - s_a}{b - a}
\] (7.35)

Indeed, assume for the purpose of contradiction that the opposite inequality holds. Then because $a, c \in H_0$, the points $(a, s_a)$ and $(c, s_c)$ cannot be in the situations of (7.29) and (7.30). Therefore, the lattice point $(b, s_b)$ must be strictly inside the bottom triangle in the picture below. We assume $b$ is chosen such that the distance from the point $(b, s_b)$ to the line $l = \{(a, s_a), (c, s_c)\}$ is maximal.

![Diagram](image-url)
Assume without loss of generality that $b$ is closer to $a$ than to $c$, and consider the lattice point $(e, y)$ with $e = 2b - a$ and $y = 2s_b - s_a$. We have three options:

- If $s_e < y$ then we contradict the fact that $a \in H_0$ because the triple $(a, b, e)$ violates condition (7.29).
- If $s_e = y$ and $(e, y)$ lies in the bottom triangle, then we violate the choice of $(b, s_b)$ as having maximal possible distance from the line $l$.
- If $s_e = y$ and $(e, y)$ lies in the right triangle, or if $s_e > y$, then we contradict the fact that $c \in H_0$ because the triple $(a, e, c)$ violates condition (7.30).

Inequality (7.35) proves that the lattice points $\{(i, s_i), i \in H_0\}$ form a convex path $R$ (we will call these lattice points marks) and moreover, that there are no other lattice points $(j, s_j)$ strictly above $R$. Recall from Claim 7.6 that the abstract simplicial complex $\Xi_C$ consists of all subsets $H \subset H_0$ such that situation (7.31) does not occur. Because there are no points $(j, s_j)$ above the convex path $R$, we conclude that the abstract simplicial complex consists of those subsets $H$ of the set of marks $H_0$, such that $H$ does not contain non-adjacent marks on a side of the convex path $R$. Then (7.32) follows from the more general statement below:

**Claim 7.7.** For any convex path $R$ with marked points $p_1, \ldots, p_n$, the abstract simplicial complex consisting of $H \subset \{1, \ldots, n\}$ such that $H$ does not contain non-adjacent marked points on a side of $R$, has reduced Euler characteristic $\sum_H (−1)^{|H|} = 0$.

The claim is proved by induction on the number $n$. Assume that the rightmost marked point $p_n$ is on the last edge $E \subset R$. Then the set of subsets in question can be partitioned into groups:

- those $H$’s which do not contain $p_n$
- those $H$'s which contain $p_n$ and no other points on the edge $E$
- those $H$’s which contain $p_n$ and $p_{n−1}$ and no other points on the edge $E$

By the induction hypothesis, each of the three collections of $H$’s above has Euler characteristic zero, unless the convex path $R$ only consists of the edge $E$. If it happens that $R = E$, then the second and third bullets contribute 1 and $−1$ to the reduced Euler characteristic, and hence their contributions cancel each other out.

**Proof of Proposition 2.19.** By the property of $\delta$ functions, observe that:

$$W_k(x) * W_k'(y) = \text{Sym} \left[ \delta \left( \frac{z_1}{x} \right) \ldots \delta \left( \frac{z_k}{xq^{1−k}} \right) \delta \left( \frac{z_{k+1}}{y} \right) \ldots \delta \left( \frac{z_{k+k'}}{yq^{1−k'}} \right) \prod_{i<j} \xi \left( \frac{z_i}{z_j} \right) \right]$$

and therefore, the analogue of formula (7.10) states that:

$$(R(z_1, \ldots, z_{k+k'}), W_k(x) * W_k'(y)) = \varphi_{x,y}^{k,k'} (R) \quad (7.36)$$

where the functional $\varphi_{x,y}^{k,k'} : S_{k+k'} \rightarrow \mathbb{F}(x, y)$ is defined by:

$$\varphi_{x,y}^{k,k'} (R) = \frac{R \left( \frac{1}{x}, \ldots, \frac{q^{k−1}}{x}, \frac{1}{y}, \ldots, \frac{q^{k−1}}{y} \right) \prod_{1 \leq i < j \leq k} \xi(q^{i−j}) \prod_{1 \leq j < k'} \xi(q^{j−j'}) \prod_{1 \leq i < j \leq k'} \xi(q^{j−j}) \prod_{1 \leq i < j \leq k'} \xi \left( \frac{yq^j}{x^q} \right)}{\lim_{z_k \to \frac{q^{k−1}}{y}} \lim_{z_{k+1} \to \frac{1}{y}} \left( \lim_{z_k \to q^{k−1}} \left( \ldots \lim_{z_1 \to \frac{1}{x}} \left( \frac{R(z_1, \ldots, z_{k+k'})}{\prod_{1 \leq i < j \leq k'} \xi \left( \frac{z_i}{z_j} \right)} \right) \ldots \right) \right)} \quad (7.37)$$
By analogy with the proof of Proposition 2.16, we claim that it suffices to prove the following statement. For any \( k, k' \geq 0 \) and any shuffle element \( R \) of degree \( k + k' \), we have:

\[
\varphi_{x,y}^{k,k'}(R) = \prod_{i = \max(0,k'-k)+1}^{k'} \frac{1}{y - xq^i} \prod_{i = -k+1}^{\min(0,k'-k)} \frac{y - xq^i}{(y - xq_1q^{i-1})(y - xq_2q^{i-1})} \ldots
\]

(7.38)

where \( \ldots \) stands for a Laurent polynomial in \( x, y \). Indeed, combined with Claim 7.2 and (7.15), formula (7.38) applied to \( R = P_v \) implies that the basis element \( P_v \) of the shuffle algebra appears in the decomposition of \( W_k(x) \ast W_{k'}(y) \) with a coefficient which is a rational function of \( x, y \) as in the right-hand side of (7.38).

Therefore, it remains to prove (7.38). By the argument in Proposition 2.9 of [28], which closely follows a key Lemma of [15], the wheel conditions (2.7) imply that the evaluation:

\[
R \left( \frac{1}{x}, \ldots, \frac{q^{k-1}}{x}, \frac{1}{y}, \ldots, \frac{q^{k'-1}}{y} \right)
\]

(7.39)

is divisible by the linear factors:

\[
\left\{ \begin{array}{ll}
\frac{q^{a-1}}{x} = \frac{q^{a-1}}{y} & \text{for } 1 \leq a \leq k, \ 1 \leq b < k' \ \text{if } k \leq k' \\
\frac{q^{b-1}}{y} = \frac{q^{b-1}}{x} & \text{for } 1 \leq a < k, \ 1 \leq b \leq k' \ \text{if } k \geq k'
\end{array} \right.
\]

Dividing (7.39) by \( \prod_{1 \leq i \leq k} \prod_{1 \leq j \leq k'} \xi \left( \frac{yq^i}{xq^j} \right) \) as in (7.37) yields:

\[
\varphi_{x,y}^{k,k'}(R) = \frac{1}{\prod_{i = -k+1}^{\min(0,k'-k)} (y - xq_1q^{i-1})(y - xq_2q^{i-1})}
\]

where \( \ldots \) refers to a Laurent polynomial times linear factors other than those of the form \( y - xq_1q^* \) and \( y - xq_2q^* \). Meanwhile, as far as the factors \( y - xq^* \) are concerned, let us write \( R \) as a shuffle element (2.6):

\[
\varphi_{x,y}^{k,k'}(R) = \frac{R \left( \frac{1}{x}, \ldots, \frac{q^{k-1}}{x}, \frac{1}{y}, \ldots, \frac{q^{k'-1}}{y} \right)}{\prod_{1 \leq i \leq k} (y - xq_1q^{i-1})(y - xq_2q^{i-1}) \prod_{1 \leq j \leq k'} (y - xq_2q^{j-1})(y - xq_1q^{j-1})}
\]

\[
\ldots = \ldots \left( \left\{ \frac{y - xq_1^{k-1}}{y - xq_1^{j-1}} \frac{y - xq_2^{k'}q_1^{j-1}}{y - xq_2^{k'}q_1^{j-1}} \frac{y - xq_2^{k'}q_1^{j-1}}{y - xq_2^{k'}q_1^{j-1}} \frac{y - xq_1^{k-1}}{y - xq_1^{j-1}} \right\} \right)
\]

(7.40)

where \( \ldots \) denote Laurent polynomials and linear factors other than those of the form \( y - xq^* \). This completes the proof of (7.38), and with it, Proposition 2.19.

Proof of Proposition 2.20. As a consequence of (2.21), we have:

\[
[P_{d,1}, P_{\pm n,0}] = \pm (1 - q^n_1)(1 - q^n_2) P_{d \pm n,1}
\]

\[
\Rightarrow \left[ \frac{z}{x}, P_{\pm n,0} \right] = \pm (1 - q^n_1)(1 - q^n_2) \cdot x^{\pm n} \delta \left( \frac{z}{x} \right)
\]

(7.40)
which proves (2.60)–(2.61). Let \( \beta_n = \frac{(1-q^n)(1-q^{2n})}{n} \) and consider the functions:

\[
f_{kk'}(z) = \prod_{i=\max(0,k-k')}^{k-1} \zeta(qz^i) = \exp \left[ \sum_{n=1}^{\infty} z^n \beta_n \frac{q^{\max(0,k-k')n} - q^{kn}}{1 - q^n} \right]
\]

\[
g_{kk'}(z) = \prod_{1 \leq i \leq k} \zeta(qz^{i-j-1}) = \exp \left[ \sum_{n=1}^{\infty} z^n \beta_n \frac{(1-q^{kn})(1-q^{k'n})}{q^n(1-q^n)(1-q^{-n})} \right]
\]

\[
h_{kk'}(z) = f_{kk'}(z) g_{kk'}(z)
\]

\[
= \exp \left[ \sum_{n=1}^{\infty} z^n \beta_n \frac{q^{\max(0,k-k')n} + q^{\min(0,k-k')n-n} - q^{kn} - q^{-(k+1)n}}{(1-q^n)(1-q^{-n})} \right]
\]

Recall formula (2.56):

\[
W_k(x) W_{k'}(y) = S_{k,k'}(x, y) g_{kk'} \left( \frac{y}{x} \right)
\]

(7.41)

where:

\[
S_{k,k'}(x, y) = \eta_k \eta_{k'} \text{Sym} \left[ \delta \left( \frac{z_1}{x} \right) \ldots \delta \left( \frac{z_k}{x q^{1-k}} \right) \delta \left( \frac{z_{k+1}}{y} \right) \ldots \delta \left( \frac{z_{k+k'}}{y q^{1-k'}} \right) \right]
\]

(7.42)

Moreover, formula (2.57) claims that the expression:

\[
W_k(x) W_{k'}(y) f_{kk'} \left( \frac{y}{x} \right) = S_{k,k'}(x, y) h_{kk'} \left( \frac{y}{x} \right)
\]

(7.43)

is a rational function with poles given by (2.58)–(2.59). Note that we have the equality of formal sums \( S_{k,k'}(x, y) = S_{k',k}(y, x) \) by symmetry, while:

\[
h_{kk'} \left( \frac{y}{x} \right) = h_{k'k} \left( \frac{x}{y} \right)
\]

by using formulas (2.2) and (2.3). Therefore, we can switch \((k, x) \leftrightarrow (k', y)\) in (7.43) at the cost of picking up the residues at the aforementioned poles:

\[
S_{k,k'}(x, y) h_{kk'} \left( \frac{y}{x} \right) - S_{k',k}(y, x) h_{k'k} \left( \frac{x}{y} \right)
\]

\[
= \sum_{i=\max(0,k'-k)+1}^{k'} \delta \left( \frac{y}{x q^i} \right) \text{Res}_{x=\frac{y}{q^i}} \left[ S_{k',k}(x, y) h_{kk'} \left( \frac{y}{x} \right) \frac{dx}{x} \right]
\]

\[
- \sum_{i=\max(0,k-k')+1}^{k} \delta \left( \frac{x}{y q^i} \right) \text{Res}_{y=\frac{x}{q^i}} \left[ S_{k',k}(y, x) h_{k'k} \left( \frac{x}{y} \right) \frac{dy}{y} \right]
\]

(7.44)

From the definition of \( S_{k,k'}(x, y) \) in (7.42), we see that:

\[
S_{k,k'}(x, y) \bigg|_{x=\frac{y}{q^i}} = \frac{\eta_k \eta_{k'}}{\eta_{k+i} \eta_{k'-i}} \cdot S_{k'-i,k+i}(x, y) \bigg|_{x=\frac{y}{q^i}}
\]
and so we may rewrite (7.44) as:

\[
S_{k,k'}(x, y)h_{kk'}\left(\frac{y}{x}\right) - S_{k',k}(y, x)h_{k'k}\left(\frac{x}{y}\right) = \sum_{i=\max(0,k-k')+1}^{k'} \delta\left(\frac{y}{xq^i}\right) \text{Res}_{x=\frac{y}{xq^i}} \left[ S_{k-i, k+i}(x, y)h_{kk'}\left(\frac{y}{x}\right) \frac{dx}{x} \right] \frac{\eta_k \eta_{k'}}{\eta_{k+i} \eta_{k'-i}} \\
- \sum_{i=\max(0,k-k')+1}^{k} \delta\left(\frac{x}{yq^i}\right) \text{Res}_{y=\frac{x}{yq^i}} \left[ S_{k-i, k'+i}(y, x)h_{k'k}\left(\frac{x}{y}\right) \frac{dy}{y} \right] \frac{\eta_k \eta_{k'}}{\eta_{k-i} \eta_{k'+i}}
\]

(7.45)

Let us rewrite this equality as:

\[
S_{k,k'}(x, y)h_{kk'}\left(\frac{y}{x}\right) - S_{k',k}(y, x)h_{k'k}\left(\frac{x}{y}\right) = \sum_{i=\max(0,k-k')+1}^{k'} \delta\left(\frac{y}{xq^i}\right) \text{Res}_{x=\frac{y}{xq^i}} \left[ S_{k-i, k+i}(x, y)h_{k'k}\left(\frac{y}{x}\right) \frac{\eta_k \eta_{k'} \cdot dx}{x} \right] \frac{h_{kk'}\left(\frac{y}{x}\right)}{h_{k'k}\left(\frac{x}{y}\right)} \\
- \sum_{i=\max(0,k-k')+1}^{k} \delta\left(\frac{x}{yq^i}\right) \text{Res}_{y=\frac{x}{yq^i}} \left[ S_{k-i, k'+i}(y, x)h_{kk'}\left(\frac{x}{y}\right) \frac{\eta_k \eta_{k'} \cdot dy}{y} \right] \frac{h_{kk'}\left(\frac{x}{y}\right)}{h_{k'k}\left(\frac{x}{y}\right)}
\]

(7.46)

Using the definition of \( h_{kk'}(z) \) and \( \eta_k \), we conclude that \( \frac{h_{kk'}(q^i) \eta_k \eta_{k'}}{h_{k'k}(q^i) \eta_{k+i} \eta_{k'-i}} \) equals:

\[
\exp \left[ \sum_{n=1}^{\infty} \beta_n \left( \frac{q^{[\text{max}(0,k-k')+i]n} + q^{[\text{min}(0,k-k')+i-1]n} - q^{(k+i)n} - q^{(i-k'-1)n}}{(1-q^n)(1-q^{-n})} \right) \right] = \exp \left( \sum_{n=1}^{\infty} \beta_n \cdot \frac{1 - q^{\text{min}(i,k-k'+i)n}}{1 - q^n} \right) = \prod_{s=0}^{\text{min}(i,k-k'+i)-1} \zeta(q^s)
\]

(7.47)

Note that in the first equality in (7.47), we have made generous use of the identity (2.2), which in exponential notation takes the form:

\[
\exp \left( \sum_{n=1}^{\infty} \beta_n \cdot q^{na} \right) = \exp \left( \sum_{n=1}^{\infty} \beta_n \cdot q^{(a-1)n} \right)
\]

for any integer \( a \). Also note that \( \zeta(1) \) has a factor of 1–1 in the denominator, and this factor is precisely the reason why the summands in the right-hand side of (7.46) have a simple pole at \( y = xq^i \) and \( x = yq^i \), respectively. Without this factor, expression (7.47) equals \( \theta(\text{min}(i,k-k'+i)) \) of (2.63). Therefore, changing all \( S \)'s to \( W \)'s via relation (7.43) converts formula (7.46) into (2.62).
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