A Non-Oracle Quantum Search Algorithm and Its Experimental Implementation

Nanyang Xu, Jin Zhu, Xinhua Peng, Xianyi Zhou, Jiangfeng Du

1Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China

*To whom correspondence should be addressed; E-mail: df@ustc.edu.cn

Grover’s algorithm has achieved great success. But quantum search algorithms still are not complete algorithms because of Grover’s Oracle. We concerned on this problem and present a new quantum search algorithm in adiabatic model without Oracle. We analyze the general difficulties in quantum search algorithms and show how to solve them in the present algorithm. As well this algorithm could deal with both single-solution and multi-solution searches without modification. We also implement this algorithm on NMR quantum computer. It is the first experiment which perform a real quantum database search rather than a marked-state search.

PACS numbers: 03.67.Lx, 89.70.-a, 03.65.-w

Quantum computation is a promising way to solve classical hard problems. Although large-scale quantum hardware has yet been built, quantum computation model in analog to classical circuit is well developed during the last few years. Based on this model, several quantum algorithms have been designed to perform classical algorithms with remarkable speedups. The most splendid one among these is Shor’s Algorithm[1], which can factorize a big number using a running time only polynomial in the size of the number, while all known classical algorithms need a exponential time[2]. Another important algorithm[3], named after its inventor Grover, concerns the problem of searching for a required item in a unsorted database. One common example for this unsorted database search is to find a person’s name in a phone book (the items are sorted by names) with only knowing his phone number. Classically, the only way to achieve this is brute-force search[4] which for $N$ entries in the phone book requires an average of $\frac{N}{2}$ queries. However, if the information is stored in a quantum database, to find the right name with Grover’s algorithm costs only a time of order $\sqrt{N}$, providing a quadratic speedup.

While quantum algorithms are presented in standard circuit model (i.e., using a sequence of discrete quantum gates), a new model of quantum computation show up where the states of quantum computer evolves continuously and adiabatically under a certain time-dependent Hamiltonian. This new idea was firstly brought out by Farhi and co-workers[5]. In this new computation model, a problem Hamiltonian is well designed whose ground state encodes the unknown solution to the problem. Then this adiabatic evolution can be used to switch gradually from an initial Hamiltonian whose ground state is known, to the problem Hamiltonian. If this evolution evolves slowly enough, the system will stay near its instantaneous ground state[6]. So in the end of evolution, the system will on the solution state of the problem. This method has been applied to the database search problem. However, this adiabatic search algorithm results in a complexity of order $N$, which is the same order with classical algorithms. More recently, Roland and Cerf[7] improved the performance of adiabatic search to order $\sqrt{N}$, the same with Grover’s algorithm, by applying adiabatic evolution locally.

Although these quantum search algorithms seems brilliant as they have already done, they are still incomplete algorithms. Grover’s algorithm utilized a Oracle (i.e., a blackbox), which gets an input state $|i\rangle$, checks the quantum database, and changes the state to $-|i\rangle$ if the $i$-th value in the database satisfies the search condition and does nothing otherwise. It is easy to implement such operations in classical database cases, but up to now there’s no efficient universal method to design this Oracle in quantum circuit. And in adiabatic algorithms, the solution of the problem is encoded to the problem Hamiltonian. Since the mechanics of the Oracle remains unknown, the encoding process of the Hamiltonian in the adiabatic algorithm is unclear. Instead, just like what previous experiments[8, 9, 10, 11] of Grover’s algorithm did, the adiabatic search algorithm forms the Hamiltonian directly from the solution state, which means we have to know the state in prior and then perform a algorithm to show it. Obviously this marked-state search algorithm is not a real database search. Thus the main problem with current quantum search algorithms is the existence of Grover’s Oracle.

In this article, we present a new adiabatic algorithm for quantum search. By encoding the database to quantum format and forming the problem Hamiltonian from target value, this adiabatic search algorithm solves Grover’s problem without Oracles. Furthermore, we experimentally implement this non-Oracle quantum search algorithm in NMR quantum computer. Because of the reasons mentioned before, this is the first time implementing a real quantum unsorted database search in experiment. We also analyze the general difficulties in quantum search and show how to solve them in our algorithm.

As a new quantum computation model, adiabatic algo-
Three adiabatic algorithms have been designed for solving computational hard problems[12, 13]. And a simple proof has been given to show that adiabatic model is equivalent to circuit mode in quantum computation[14]. Moreover, since adiabatic computation only involves the ground state, it keeps the system at a low temperature. Thus the system appears lower sensitive to some perturbations and have a improved robustness against dephasing, environmental noise and some unitary control errors[15, 16].

As mentioned before, the key part of an adiabatic algorithm is how to describe the solution to a specific problem in the problem Hamiltonian. Here let’s focus on the database search problem. To be simplified, we now consider a phone book which contains $N$ (assume $N = 2^n$) entries with each entry a pair of telephone number and person’s name. Usually, the entries are sorted by name. The database search problem here is to find a specific name in the book whose telephone number is given. To solve this problem in our model, the names are encoded to $n$-qubit states and the phone numbers represented as integers (in fact, any real numbers are permit). An example for $N = 4$ is shown in Table I. We encode the names and the phone numbers as in Table II. Thus the database could be stored by the state-integer pairs like $\{(|0\rangle, 4), (|1\rangle, 3), (|2\rangle, 1), (|3\rangle, 2)\}$. If we want to find the name which connect to the number 3601003 which is encoded as 3, state $|1\rangle$ should be returned from our quantum search machine.

| Name | Number |
|------|--------|
| Alex | 3601004 |
| Bob  | 3601003 |
| Cherry | 3601001 |
| David | 3601002 |

After encoding classical database to quantum database, next step is to design the problem Hamiltonian $H_p$. For this problem, $H_p$ is:

$$H_p \equiv \left( \sum_{i=0}^{N-1} number_i |name_i\rangle \langle name_i| - target \cdot I^\otimes n \right)^2$$

where $target$ is the code of the phone number which we want to search for. Obviously, the ground state of $H_p$ is the state which connects to the target number. In general, for an encoded database where entries are pairs as $(i, value_i)$ and sorted by $i$, we write $H_p$ as

$$H_p = \left( \sum_{i=0}^{N-1} value_i |i\rangle \langle i| - target \cdot I^\otimes n \right)^2$$

where $D = \sum_{i=0}^{N-1} value_i |i\rangle \langle i|$ is the database operator and could be formulated separately.

Next, we will choose an initial Hamiltonian $H_i$. Conventionally[12], $H_i$ is chosen to be noncommutative with $H_p$ to avoid crossing of energy levels. Thus we write $H_i$ as:

$$H_i = g(\sigma_x^0 + \sigma_x^1 + \cdots + \sigma_x^{n-1})$$

which means the qubits coupling with a magnetic field at the $x$-direction and the coupling strength is $g$. The ground state of this Hamiltonian is simple and they are,

$$\begin{align*}
|\psi_0\rangle &= \frac{|0^{(n-1)}\rangle - |1^{(n-1)}\rangle}{\sqrt{2}} \otimes \frac{|0^{(n-2)}\rangle - |1^{(n-2)}\rangle}{\sqrt{2}} \otimes \cdots \\
&\otimes \frac{|0^0\rangle - |1^0\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{N}} \sum_{j=0}^{N} (-1)^{b(j)} |j\rangle,
\end{align*}$$

where $b(j)$ is the binary representation of $j$. The eigenvalue of $H_i$ are $0$ and $2g$, and the eigenvalue of $H_p$ are $\sum_{i=0}^{N-1} value_i |i\rangle \langle i| - target$ and the rest of the Hamiltonian's spectrum. The adiabatic theorem[16] states that if the adiabatic condition is satisfied:

$$\frac{d}{dt} \frac{\langle \psi(t)|H(t)|\psi(t)\rangle}{\langle \psi(t)|\psi(t)\rangle} = 0$$

then the instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian’s spectrum.

The idea of adiabatic quantum computation is straightforward. First find a complex Hamiltonian whose ground state describes the solution to our problem, and the monotonic function $s(t)$ fulfills $s(0) = 0$ and $s(T) = 1$.

Several adiabatic algorithms have been designed for solving computational hard problems[12, 13]. And a simple proof has been given to show that adiabatic model is equivalent to circuit mode in quantum computation[14]. Moreover, since adiabatic computation only involves the ground state, it keeps the system at a low temperature. Thus the system appears lower sensitive to some perturbations and have a improved robustness against dephasing, environmental noise and some unitary control errors[15, 16].

As mentioned before, the key part of an adiabatic algorithm is how to describe the solution to a specific problem in the problem Hamiltonian. Here let’s focus on the database search problem. To be simplified, we now consider a phone book which contains $N$ (assume $N = 2^n$) entries with each entry a pair of telephone number and person’s name. Usually, the entries are sorted by name. The database search problem here is to find a specific name in the book whose telephone number is given. To solve this problem in our model, the names are encoded to $n$-qubit states and the phone numbers represented as integers (in fact, any real numbers are permit). An example for $N = 4$ is shown in Table I. We encode the names and the phone numbers as in Table II. Thus the database could be stored by the state-integer pairs like $\{(|0\rangle, 4), (|1\rangle, 3), (|2\rangle, 1), (|3\rangle, 2)\}$. If we want to find the name which connect to the number 3601003 which is encoded as 3, state $|1\rangle$ should be returned from our quantum search machine.

### Table I: phone book example

| Name  | Number   |
|-------|----------|
| Alex  | 3601004  |
| Bob   | 3601003  |
| Cherry| 3601001  |
| David | 3601002  |

### Table II: encoder table

| Name | State | Number | Integer |
|------|-------|--------|---------|
| Alex | 0     | 3601001 | 1       |
| Bob  | 1     | 3601002 | 2       |
| Cherry | 2   | 3601003 | 3       |
| David | 3     | 3601004 | 4       |
where \( b(j) \) is the Hamming distance between \( j \) and 0.

In the adiabatic evolution, the system Hamiltonian interpolates from \( H_i \) to \( H_p \) (i.e., see Eq. 1) and the state of the system evolves according to the Schrödinger equation:

\[
\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle
\]

(6)

\[
|\psi(0)\rangle = |\psi_0\rangle.
\]

(7)

If this evolution acts slow enough (i.e., the total evolution time \( T \) is large enough), the Adiabatic Theorem ensures the system will always stay on the ground state of \( H(t) \) and in the end the solution of our problem will show up.

Again, we take the phone book in Table I as example. If we want to find number 3601002 in the database and using the encoder in Table II, we will get the Hamiltonians as follows,

\[
H_i = g(\sigma_x^0 + \sigma_x^1)
\]

\[
H_p = (4|0\rangle\langle 0| + 3|1\rangle\langle 1| + 1|2\rangle\langle 2| + 2|3\rangle\langle 3| - 2I)^2
\]

\[
= \frac{3}{2}I + \sigma_z^{(0)} + \sigma_z^{(1)} + \frac{\sigma_z^{(1)} \otimes \sigma_z^{(0)}}{2},
\]

(8)

And the eigenvalues of time dependent Hamiltonian \( H(t) \) (see Eq. (1)) are plotted in FIG. 1. By the adiabatic theorem, the state will stay on the lowest energy level during the adiabatic evolution. And finally we will get the state on the basis \(|3\rangle\). After measurement and decoding we will get the name connecting to the number 3601002 which is David.

To demonstrate this Non-Oracle search algorithm, we selected \(^{13}\text{C}\)-labeled CHCl\(_3\) as a physical system for our experiments. The two qubits are represented by \(^{13}\text{C}\) and \(^{1}\text{H}\). Its natural Hamiltonian in the multiply rotating frame is

\[
H_{\text{sys}} = \omega_1 I_x^1 + \omega_2 I_x^2 + 2\pi J I_x^1 I_x^2,
\]

(9)

where \( \omega_1 \) and \( \omega_2 \) are Larmour frequencies, \( J \) is the spin-spin coupling constant \( J = 214.5 \text{Hz} \). Experiments were performed at room temperature using a standard 400MHz NMR spectrometer (AV-400 Bruker instrument).

The experiments was divided into three parts, shown as Fig.2: the first part consists of preparation of the state of the initial Hamiltonian. The second part is the adiabatic evolution, and the third part is the tomography of the resultant state. To prepare the initial ground state [Eq. 5], we first created a pseudopure state (PPS) \([00] + \epsilon|01\rangle \langle 01|00\rangle\), where \( \epsilon \approx 10^{-9} \) describes the thermal polarization of the system and I is a unit matrix, using the method of spatial averaging. Then \( \pi/2 \) pulses along the \(-y\) axis was applied to prepare the ground state.

Discretizing a continuous Hamiltonian is a straightforward process and changes the run time \( T \) of the adiabatic algorithm only polynomially. Simply, let the discrete time Hamiltonian \( H(s) \) be a linear interpolation from some beginning Hamiltonian \( H(0) = H_0 \) to some final problem Hamiltonian \( H(S) = H_1 \) such that

\[
H(s) = (s/S)H_1 + (1 - s/S)H_0.
\]

The unitary evolution of the discrete algorithm can be written as

\[
U = \prod_s U_s = \prod_s e^{-iH(s)\tau},
\]

(10)

where \( \tau = T/(S + 1) \), \( T \) is the total duration of the adiabatic passage and \( S + 1 \) is the total number of discretization steps. When both \( T, S \to \infty \) and \( \tau \to 0 \), the adiabatic limit is achieved.

For our example shown in Eq. 8, an optimized set of parameters was set as \( g = 1, T = 10.45 \) and \( S = 10 \), so \( \tau = T/(S + 1) = 0.95 \). This set of parameters yields an adiabatic evolution that finds the solution in a relatively efficient way. Using the Trotter formula, we can
approximate \( U_s \) to second order
\[
U_s \approx U'_s = e^{-i(1-\frac{1}{2})H_0 \tau} e^{-i\theta H_p \tau} e^{-i(1-\frac{1}{2})H_0 \tau} + O(\tau^2),
\]
the fidelity of \( U_s \rightarrow U'_s \) is all above 0.996 and overall fidelity is 0.991. For the implementation of \( U'_s \), \( e^{-i(1-\frac{1}{2})H_0 \tau} \) can be simply realized using a \( \theta_s \) pulses around \( \pi \) axis for both H and C nuclei, \( \theta_s = \tau (1-s/S) \), shown in Fig. 2. And the evolution under \( H_1 \) can be simulated by a free evolution \( \tau_s \) under the Hamiltonian \( H_{sys} \), the identity term of \( H_1 \) does not cause any evolution of the state and so it can be omitted, \( \tau_s = \tau \times (s/S)/\pi J \).

![Diagram](Image)

**FIG. 3:** The tomography of theoretically expected and experimentally obtained density matrices for the search states in adiabatic search algorithm. The density matrices consist of just a real term on the diagonal corresponding to the population of the state that has been searched.

The third stage of the experiment is the tomography of the final density matrix after the adiabatic evolution. The result was shown as Fig. 3. Theoretically, the four state \( |00\rangle, |01\rangle, |10\rangle \) and \( |11\rangle \) should be find at the probability of 0, 0.014, 0.014 and 0.972. Our experiments show that the probability is 0.037, 0.032, 0.006 and 0.925. The fidelity of the experiment is 0.985.

The errors in adiabatic algorithms may be caused by three parts. Firstly, the total time of evolution in adiabatic algorithms should be infinite. Actually the evolution is terminated when the state is supposed to reach our expected high probability. Secondly, the error is due to neglect of \( O(\tau^2) \) terms in the Trotter Formula (Eq. 11). The third part of error is due to decoherence effects of the NMR system and imperfect pulses.

Unlike previous standard quantum algorithms only using qubits as registers to store information, our algorithm represents the value field by the strength of interactions in the operator and the index field by qubits. This is because that if both the fields are represented in qubits, \( 2n \) qubits are needed for a database with \( N \) items, which result in the failure that the optimal running time would be scaled from order \( \sqrt{N} \) to \( N \), the same performance as classical algorithms. Since the construction function is simply quadratic, the interaction strength in problem Hamiltonian grows with the database’s size. For further consideration, the algorithm may be improve to suppress the interaction strength in the problem Hamiltonian by choosing a better construction function in Eq. 3.

All practical quantum search algorithms must face the problem that the database is unsorted, thus quantum operators would traverse all the items in the database to learn the complete information and after measurement all information in the states are destroyed. It is a hard problem to efficiently implement the quantum operators concerning the database. The first effort was reported by Ju and coworkers\(^4\) when they tried to implement Grover’s Oracle in quantum circuit. They have to spend \( N \) steps to build the relation in the database to the states for each query, such that the circuit design is not efficient. In this algorithm, we describe the database information in a single operator (\( D \) in Eq. 3), thus this operator may be analyzed and formulate separately for each database. On the other side, if it could not be formulated efficiently in some cases, approximate implementation is another possible solution according to recent works on geometric quantum computation\(^{19}\), of which detailed consideration is beyond the scope of this article.

As an end of this section, we will give a simple analysis of the multi-solution search case in our algorithm. If there’re \( m > 1 \) entries in the database satisfying the search condition, the problem Hamiltonian will have ground states with \( m \) times degenerated. And because of symmetry, the state would finally evolve to an average superposition of all the ground states. Thus without any modification, our algorithm could also deal with multi-solution search.

To be concluded, we present a new kind of adiabatic search algorithm to solve Grover’s problem without Oracles and give a demonstrative experiment on NMR quantum computer. The result of experiment agrees well with theoretical expectation. This is a new style of quantum search algorithm which utilize both quantum registers and interaction strength to store information. This algorithm aims at general difficulties of quantum search algorithms and give a promising way to solve them ultimately.

We thank Zeyang Liao for initial discussion and help. This work was supported by National Nature Science Foundation of China, the CAS, Ministry of Education of PRC, the National Fundamental Research Program, and the DFG through Su 192/19-1. For this article, any comment is welcome.

[1] P. Shor, p. 124 (1994).
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge Univ. Press, Cambridge, 2000).
[3] L. K. Grover, Phys. Rev. Lett. 79, 325 (1997).
[4] Y. L. Ju, I. M. Tsai, and S. Y. Kuo, IEEE Transactions on Circuits and Systems I-Regular Papers 54, 2552 (2007).
[5] E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, and D. Preda, Science 292, 472 (2001).

[6] A. Messiah, Quantum Mechanics (Wiley, New York, 1976).

[7] J. Roland and N. J. Cerf, Physical Review A 65, 6 (2002).

[8] I. L. Chuang, N. Gershenfeld, and M. Kubinec, Physical Review Letters 80, 3408 (1998).

[9] J. L. Dodd, T. C. Ralph, and G. J. Milburn, Physical Review A 68, 42328 (2003).

[10] L. M. K. Vandersypen, M. Steffen, M. H. Sherwood, C. S. Yannoni, G. Breyta, and I. L. Chuang, Applied Physics Letters 76, 646 (2000).

[11] K. A. Brickman, P. C. Haljan, P. J. Lee, M. Acton, L. Deslauriers, and C. Monroe, Physical Review A 72, 50306 (2005).

[12] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, quant-ph 0001106 (2000).

[13] X. Peng, Z. Liao, N. Xu, G. Qin, X. Zhou, D. Suter, and J. Du, quant-ph arXiv:0808.1935v1 (2008).

[14] A. Mizel, D. A. Lidar, and M. Mitchell, Phys Rev Lett 99, 070502 (2007).

[15] A. M. Childs, E. Farhi, and J. Preskill, Physical Review A 65, 10 (2002).

[16] J. Roland and N. J. Cerf, Physical Review A 71, 032330 (2005).

[17] I. L. Chuang, N. Gershenfeld, M. G. Kubinec, and D. W. Leung, Proc. R. Soc. Lond. A 454, 447 (1998).

[18] D. G. Cory, A. F. Fahmy, and T. F. Havel, 94, 1634 (1997).

[19] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, Science 311, 1133 (2006).
