Features of the numerical solution of bi-characteristic equations by methods of computer mathematics

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Abstract. The features of applying the methods of computer mathematics to the numerical solution of the bi-characteristic system of equations describing the propagation of radio waves in the ionospheric plasma are considered. Based on the comparison of the numerical and exact solutions obtained for the parabolic plasma layer, the accuracy of the numerical solution was investigated.

1. Introduction

The bi-characteristic system

To determine the ray trajectories that characterize the magnitude and direction of energy propagation in radio communication and radio sounding problems, it is necessary to construct numerical methods and algorithms for solving the system of bi-characteristic equations

\[
\frac{d\mathbf{r}}{d\tau} = \frac{\partial \Gamma}{\partial \mathbf{k}}, \quad \frac{d\mathbf{k}}{d\tau} = -\frac{\partial \Gamma}{\partial \mathbf{r}}, \quad \frac{dt}{d\tau} = -\frac{\partial \Gamma}{\partial \omega}, \quad \frac{d\omega}{d\tau} = -\frac{\partial \Gamma}{\partial t},
\]

where \( \tau \) – a parameter along the ray path, \( \mathbf{r} \) – ray coordinates, \( t \) – a group time, \( \mathbf{k} \) – a wave vector, \( \omega \) – a circular frequency, \( \Gamma \) – Hamiltonian. This problem has been solved in many investigations in various ways [1-4]. The main feature of this work is the involvement of fourth-level programming languages and the combination of numerical and analytical methods with the use of symbolic computations.

Let us consider the main stages of this approach. Let us dwell on the structure of the bi-characteristic system (1). The system consists of eight ordinary differential equations of the first order. The system is nonlinear. The first three equations determine the derivatives of the ray coordinates \( \mathbf{r} = (x, y, z) \) with respect to the parameter along the trajectory \( \tau \). The second three equations determine the derivatives of the wave vector \( \mathbf{k} = (k_x, k_y, k_z) \) with respect to the parameter along the trajectory, the seventh equation determines the rate of change of the group time \( t \) along the trajectory, and the last eighth equation determines the rate of change of the angular frequency of the signal.

If the propagation of a radio wave on a plane were studied, then the first and second groups of equations would contain two equations each. Such tasks can be set, but they were not considered in this study.

In this paper, we consider a bi-characteristic system with a Hamiltonian of the form:

\[
\Gamma = k_x^2 + k_y^2 + k_z^2 - \varepsilon \frac{\omega^2}{c^2}.
\]
The determining factor for calculations is the form of the effective permittivity $\varepsilon$, which depends on $e$ – the electron charge, $m_e$ – the electron mass and $N$ – electron concentration values. Even in the simplest case:

$$\varepsilon = 1 - \nu, \quad \nu = \left(\frac{\omega_p}{\omega}\right)^2 = \frac{4\pi e^2 N}{m_\nu \omega^2},$$

where $\varepsilon$ depends both on coordinates (as $\varepsilon$ depends on the electron concentration, which in turn depends on the coordinates), and from frequency $\omega$. In the case of a magnetoactive medium, the effective permittivity depends on the wave vector $\mathbf{k}$ too. Such a medium is the ionospheric plasma located in the Earth's magnetic field [5,6]. In addition, the permittivity $\varepsilon$ may depend on the group time $t$, since gravitational waves can propagate in the ionosphere. If the permittivity does not depend on the group time, then, since the frequency $\omega$ remains constant, we can eliminate the parameter $\tau$ and convert the system to a "stationary" form, known as the bi-characteristic Hamilton-Lukin system:

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \Gamma}{\partial \mathbf{k}} / \partial \omega, \quad \frac{d\mathbf{k}}{dt} = \frac{\partial \Gamma}{\partial \mathbf{r}} / \partial \omega.$$ (4)

Then the group time $t$ becomes the parameter of integration.

Before solving system (1) numerically, it is necessary to determine the right-hand sides of the system, that is, knowing the electron concentration and the Earth's magnetic field, differentiate $\varepsilon$ by components of the vector $\mathbf{r}$, by the circular frequency $\omega$, by the components of the wave vector $\mathbf{k}$ and, if necessary, and by the group time $t$. Usually, to solve such problems, either numerical algorithms are used, the accuracy of which is limited, or, in the simplest cases, analytical differentiation, which is unsuitable when the electron density is specified by an interpolation function.

It should be emphasized that, although in the case when the electron concentration $N$ is given analytically, the calculation of the derivatives in the traditional way is possible, for really complex dependencies such a procedure becomes not only very time consuming (expressions for the derivatives can take dozens of pages), but inevitably leads to errors, the probability of which is increases with increasing computational complexity. Moreover, even if the problem is solved correctly, the author cannot be sure of his results.

In this work, to find the derivatives, symbolic calculations were used, provided in the fourth generation programming languages, which is equally suitable both in the case when the electron concentration and the magnetic field are given in the form of formulas (practically of any level of complexity), and the case when the electronic concentration and magnetic field of the Earth are given in the form of interpolation functions.

The last step, which finally forms the Cauchy problem for the bi-characteristic system of equations, is the setting of the initial conditions. In this work, it is assumed that the radiation source is point and stationary. At least, the speed of its movement is much less than the speed of signal propagation, and within the framework of this problem, its motion can be neglected. In this way,

$$\mathbf{r}|_{t=0} = (x_0, y_0, z_0)$$

– initial conditions for the first group of equations.

For the second group of equations, it is necessary to set the initial values of the wave vector:

$$k_x(0) = \frac{\alpha}{c} \sqrt{\varepsilon_0} \cos \zeta \cos \eta, \quad k_y(0) = \frac{\alpha}{c} \sqrt{\varepsilon_0} \sin \zeta \cos \eta, \quad k_z(0) = \frac{\alpha}{c} \sqrt{\varepsilon_0} \sin \eta.$$ (6)

In fact, the initial value of the effective permittivity $\varepsilon_0$ itself depends on the initial value of the wave vector, but since the effective permittivity depends on the wave vector through $\cos^2 \alpha$, and
\[\cos^2 \alpha = \frac{(H_0, k_x + H_0, k_y + H_0, k_z)^2}{H_0^2 |\mathbf{k}|^2}, \quad (7)\]

the absolute value of the wave vector at the initial moment in the expression for \( \epsilon_0 \) can be chosen arbitrarily. The last remark applies to the case when the source is in a magnetoactive plasma. Angles \( \zeta \) and \( \eta \) define the components of the wave vector in a spherical coordinate system.

If the bi-characteristic system is stationary (see (4)), then six initial conditions (5) and (6) are sufficient. They are determined at \( t=0 \). For non-stationary tasks, two more conditions must be specified:

\[\omega(0) = \omega_0 \quad (8)\]

– the initial value of the circular frequency, and

\[t(0) = t_0. \quad (9)\]

The latter is especially important for frequency-modulated signals (pulses), when, in accordance with the concepts of space-time geometric optics [7-12], each frequency is emitted at its own moment in time.

After completing the formation of the system, you can proceed to the numerical integration of the system of ordinary differential equations, which in the fourth-level programming languages is performed by a standard well-debugged procedure with a given accuracy. The solution is obtained in the form of an interpolation function from the integration parameter, which can be used for plotting graphs, generating tables, or in further calculations.

The advantage of this approach is simplicity and guaranteed accuracy of the problem solution. The disadvantage is also obvious: it is impossible to interfere with the process of constructing a solution and obtain additional information directly during the integration of the system, which is actively used in traditional approaches to solving shooting tasks, calculating divergence, detecting caustics, etc. Therefore, to solve all these problems, it is necessary to develop special algorithms.

Figure 1 shows a typical (basic) algorithm for constructing a solution of a bi-characteristic system.

2. Parabolic model of electron concentration. Rays

Consider testing the solution of a bi-characteristic system of equations obtained using the numerical methods, algorithms and programs described above. Unfortunately, at present, relatively few models of the plasma layer are known for which an analytical solution of the bi-characteristic system can be obtained, and they are quite simple. There are even fewer such solutions for ionospheric plasma layers. In this work, we have chosen a parabolic model for testing:

\[N = \begin{cases} \omega_0^2(1-\theta^2), & z \in [z_0, 2z_0 + z_h] \\ 0, & z \notin [z_h, 2z_0 + z_h] \end{cases}, \quad \theta = \frac{z - z_0 - z_h}{z_0}. \quad (10)\]

In formula (1) \( \omega_0 \) is the circular plasma frequency at the maximum of the layer, \( z_0 \) – a half-thickness of the layer, and \( z_h \) – a height from the surface of the earth to the lower boundary of the layer (see figure 2). Here and below, it was assumed that \( z_h =100 \) km and \( z_0 =200 \) km.

It should be noted that this model could be difficult for numerical implementation in direct application, since the electron concentration is represented by a piecewise continuous function, to which the methods of symbolic computations (in particular, differentiation) are further applied. However, as can be seen from the test results, this does not happen.
Figure 1. A basic algorithm for constructing a bi-characteristic system solution
It can be shown [13] that the ray equation as a function of \( z(x) \) can be represented in the form:

\[
z = \begin{cases} 
  z_0 + z_h + \frac{z_0}{a} \left( C \text{sh} W - a \text{ch} W \right), & 0 \leq x < x_i \\
  2z_h - \frac{z_h}{a} \ln \left| \frac{a - e}{a + e} \right| - \frac{x}{s} c, & x_i \leq x \leq x_p \\
  x, & x_p < x \leq x_p + x_i 
\end{cases}
\]  

(11)

where

\[
W = \frac{a}{z_0} \left( \frac{x - z_h}{s} - c \right), \quad c = \sin(\eta), \quad s = \cos(\eta), \quad a = \omega_0 / \omega, \quad \omega_0 = \sqrt{\frac{4\pi e^2 N_0}{m}}.
\]  

(12)

Figure 2. Dependence of electron concentration on height

In the formulas (12) \( \omega = 2\pi f \) is a circular operating frequency, \( f \) – an operating frequency, \( N_0 \) – the value of the electron concentration at the maximum of the layer, \( \eta \) – ray exit angle with horizontal \( x \)-axis, that is, the angle between the vector \( \vec{k} \) in the radiation source and the positive direction of the \( x \)-axis. In the calculations, it was assumed that \( N_0 = 2 \times 10^6 \text{ cm}^{-3}, f_0 = \omega_0 / (2\pi) = 12.6979 \text{ MHz}, f = 14 \text{ MHz}, \ a = 0.906993. \)

In figure in red the ray structure in the parabolic layer in the plane \( (x, z) \) is shown. In the region that can be conditionally limited along the \( x \)-axis by the interval \( (300 \text{ km}, 500 \text{ km}) \), and along the \( z \)-axis by the interval \( (80 \text{ km}, 180 \text{ km}) \), concentration of rays is observed [14], which corresponds to caustic focusing of the \( A_5 \)-type («butterfly») [15,16] in accordance with the wave theory of catastrophes [7,17,18].
3. Simulation and testing

Four rays were selected for testing: a gentle ray (exit angle 30°), a standard ray reflected from the ionosphere (exit angle 60°), a standard ray passing through the ionosphere (exit angle 70°), and a ray reflected from the ionosphere, but close to the rays passing through the ionospheric layer (exit angle 64°) (see figure 4).

![Figure 3. Ray structure in a parabolic layer](image)

Calculations show that, with graphic accuracy, distinguish between the exact solution and the solution obtained by integrating the bi-characteristic system, within the framework of figure 4 is not possible.

![Figure 4. Ray trajectories with exit angles: 30° – green, 60° – blue, 64° – purple, 70° – dark yellow.](image)
Therefore, the deviation ($\Delta z$ or $\Delta x$) of the solution obtained by the bi-characteristic method from the exact solution was calculated in this work. In this case, the following algorithm was used. The interval to which the parameter belongs in the bi-characteristic system along the ray $t$ or $\tau$ was divided into a finite number of points ($t_j$ or $\tau_j$), and at these points the values of the solution of the bi-characteristic system $x_j$ and $z_j$ were calculated. Further, two variants are possible: either knowing $z_j$, the value of the coordinate $x_j$ ($\hat{x}_j$) is determined for the exact solution and then the horizontal deviation $\Delta x_j = x_j - \hat{x}_j$ is found, or knowing $x_j$, the value of the coordinate $z_j$ ($\hat{z}_j$) is determined for the exact solution and then the vertical deviation $\Delta z_j = z_j - \hat{z}_j$ is found.

Figure 5 shows the deviation $\Delta z$ in meters for a ray with an exit angle of $30^\circ$.

**Figure 5 a** Deflection $\Delta z$, ray exit angle $30^\circ$, stationary system

**Figure 5 b** Deflection $\Delta z$, ray exit angle $30^\circ$, complete system

Here and below, two versions of the bi-characteristic system are considered: a stationary system of six equations (a) and a complete system of eight equations (b). In this case, the deviation is positive. In figure 5a, three sections can be distinguished: the initial section (before the ray enters the layer), in which the deviation is practically zero, the middle section (the ray in the layer), in which the deviation sharply increases (the error accumulates), and the final section (the ray leaves the layer) where the deviation decreases. In the case of a complete bi-characteristic system (figure 5 b), the dependence of the error is different: in the first section, the error is almost constant and close to zero, in the second section it oscillates, and in the last section it increases linearly. Moreover, the maximum error value is an order of magnitude less. Thus, the full bi-characteristic system is preferable for shallow rays.

Figure 6 shows the relative deviation $\Delta z / z$, showing what real accuracy (selected "automatically") instead of 16 bits, the built-in function that integrates the bi-characteristic system provides.

In figure 7 the deviation $\Delta z$ in meters for a ray with an exit angle of $60^\circ$ is shown. As in the previous case, in figure 7a, three sections can be distinguished: the initial section (before the ray enters the layer), in which the deviation is practically zero, the middle section (the ray in the layer), in which the deviation sharply increases (the error accumulates), and the final section (the ray leaves the layer), at which the deviation increases much more slowly.

For the complete system (cf. figure 7a and figure 7b) the dependence in the first and second sections is similar with the only difference that now the deviation is negative and the word “increases” should be added “modulo”. In the last section, the modulus deviation decreases. The absolute values themselves differ by approximately an order of magnitude, but already in favor of the stationary bi-characteristic system.

In figure 8 the deviation $\Delta z$ in meters for a ray with an exit angle of $64^\circ$ is shown. This angle is close to the critical angle at which the ray does not leave the layer, but asymptotically approaches the line $z=z_0$. At angles larger than the critical one, the rays pass through the ionospheric layer. In this case,
for both bi-characteristic systems, the deviations are less than zero, and the absolute values are somewhat larger and reach 21 m and 43 m, respectively.

Figure 6 a. Relative deviation $\Delta z / z$, ray exit angle 30°, stationary system

Figure 6 b. Relative deviation $\Delta z / z$, ray exit angle 30°, complete system

Figure 7 a. Deflection $\Delta z$, ray exit angle 60°, stationary system

Figure 7 b. Deflection $\Delta z$, ray exit angle 60°, complete system

Finally, figure 9 shows the deflections for a ray with an exit angle of 70°, passing through the ionospheric layer, for a stationary bi-characteristic system.

Figure 8 a. Deflection $\Delta z$, ray exit angle 64°, stationary system

Figure 8 b. Deflection $\Delta z$, ray exit angle 64°, complete system
The maximum deviation has become smaller, and the deviation is again positive. In figure 9b the deviation along the x-axis for a stationary bi-characteristic system is shown too.

\[ \Delta x, m \]

\[ x, \text{km} \]

**Figure 9 a.** Ray exit angle 70°, stationary system, deviation Δz

**Figure 9 b.** Ray exit angle 70°, stationary system, deviation Δx

4. Conclusion
The question arises: is the accuracy of the calculations sufficient for the problems devoted to modeling the propagation of radio waves in the Earth's ionosphere [19]. This question should be answered in the affirmative, since the model of the ionospheric plasma is typically known to an accuracy of several kilometers above [5, 20], and the maximum deviation of the order of 40 m, that is, two wavelengths in vacuum (\( \lambda = 21.4138 \) m). However, if there is a need to increase the accuracy of calculations, then first of all, one should pay attention to the procedure that ensures the solution of the bi-characteristic system, since the loss of the declared accuracy arises precisely during the numerical solution of the system of ordinary differential equations. In figure 10 the values of deviations at an angle of exit of the ray of 64° with increased accuracy is shown. The parameter "working precision" has been increased to 32. We see that the accuracy of calculations has increased dramatically and is now about 1 mm.

\[ \Delta z, m \]

\[ x, \text{km} \]

**Figure 10 a.** Deflection Δz, ray exit angle 64°, stationary system, (working) precision 32

**Figure 10 b.** Deflection Δz, ray exit angle 64°, complete system, (working) precision 32

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