LINEARIZATION AND SOLUTIONS
OF THE DISCRETE PAINLEVÉ-III EQUATION

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Abstract
We present particular solutions of the discrete Painlevé III (d-P_{III}) equation of rational and special function (Bessel) type. These solutions allow us to establish a close parallel between this discrete equation and its continuous counterpart. Moreover, we propose an alternate form for d-P_{III} and confirm its integrability by explicitly deriving its Lax pair.
1. Introduction

The discrete Painlevé III (d-P\textsubscript{III}) equation holds a particular position among all the discrete Painlevé equations. It was the first transcendent that was not discovered “accidentally” in some physical application \cite{1}, but, instead, was derived “on request” \cite{2} using the method of singularity confinement \cite{3}. Given the method of derivation, the Lax pair of d-P\textsubscript{III} was not initially known. It was obtained shortly afterwards \cite{4}, confirming thus the integrability of this equation and consolidating the singularity confinement approach as integrability detector. Another interesting feature of d-P\textsubscript{III} is that its Lax pair is not of the “usual” differential-difference kind but, rather a \textit{q}-difference one of the form: \( \Phi_n(qh) = L_n(h)\Phi_n(h), \Phi_{n+1}(h) = M_n(h)\Phi_n(h) \). This is not a mere novelty but has important implications. Thus when quantizing this discrete Painlevé equations \cite{5} we discovered that the consistent quantization rule was of Weyl, rather than Heisenberg, type i.e. \( xx = qxx \) where \( x \equiv x_n, \bar{x} \equiv x_{n+1} \). Apart from this “natural” appearance of the quantum-line relations, the quantization of d-P\textsubscript{III} revealed interesting factorization properties that will be used in what follows. We are going, in fact, to produce special, Bessel-type, solutions to d-P\textsubscript{III} starting from these factorizable forms. The continuous P\textsubscript{III} equation is also special in the sense that it is traditionally given under two different forms \cite{6}. Although one can go from one to the other through some independent variable transformation both are considered, in a sense, canonical. In the discrete case, only one of these two forms was known up to now. As we will show in this paper, the second canonical form exists as well and moreover its Lax pair can be obtained by generalizing the results of Joshi and collaborators \cite{7} on the discrete d-P\textsubscript{II} equation.

2. The first form of the discrete P\textsubscript{III}

Let us first review briefly what is known about d-P\textsubscript{III}. In \cite{2}, we have obtained its form as:

\[
\bar{xx} = \frac{\nu x^2 - b \mu^n x - d \mu^{2n}}{cx^2 + ax + \nu} \tag{1}
\]

The continuous limit is best obtained if we start with a change of variable \( y_n = \mu^{-n/2}x_n \). Equation (1) is transformed to:

\[
\overline{yy} = \frac{\nu y^2 - b \mu^{n/2}y - d \mu^n}{c\mu^n y^2 + a\mu^{n/2}y + \nu} \tag{2}
\]

and the continuous limit is obtained by taking to \( \nu = -1/\epsilon^2 \) and \( \epsilon \to 0 \). One must take simultaneously \( \mu = 1 + 2\epsilon \) in which case \( \mu^{n/2} \) goes over to \( e^z \) leading to:

\[
x'' = \frac{x'^2}{x} + e^z(ax^2 + b) + e^{2z}(cx^3 + \frac{d}{x}) \tag{3}
\]

The Lax pair of (1) was given in \cite{4}. We implement the compatibility condition as:

\[
M_n(qh)L_n(h) = L_{n+1}(h)M_n(h) \tag{4}
\]
with \( q = \alpha^2 \) and

\[
L = \begin{pmatrix}
\lambda_1 & \lambda_1 + \frac{\kappa}{y} & \frac{\kappa}{y} + \frac{y}{\lambda_3 + y} & 0 \\
0 & \lambda_2 & \lambda_2 + y & 0 \\
h(y + \lambda_2 + \frac{\kappa}{y}) & h\alpha & \lambda_3 + y & 0 \\
\frac{h(\alpha \lambda_1 - \lambda_4)}{\lambda_4 + \alpha \lambda_2} & \lambda_4 + \alpha \lambda_2 & 0 & 0 \\
\end{pmatrix}
\]  

(5)

\[
M = \begin{pmatrix}
\frac{(\alpha \lambda_1 - \lambda_4)y}{\lambda_4 y + \alpha \kappa} & \frac{\lambda_1 y + \kappa}{\lambda_2 y + \kappa} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \frac{\lambda_4 - q \lambda_2}{y + q \lambda_2} & \frac{y + \lambda_3}{y + \lambda_4} \\
h & 0 & 0 & 0 \\
\end{pmatrix}
\]

(6)

where \( \lambda_1 = \text{cnst.}, \lambda_3 = \text{cnst.}, \lambda_2 = \lambda \alpha^{n-1}, \lambda_4 = \lambda \alpha^n, \kappa = C \alpha^n \).

We obtain d-P_{III} in the form [4,5]:

\[
\bar{y}y = \frac{\alpha \kappa (y + \lambda_3)(\kappa + \lambda_2 y)}{(\kappa + \lambda_1 y)(y + \lambda_4)}
\]  

(6)

Equation (6) is just (2) with \( \mu = 1/\alpha \) and the remaining parameters easily identifiable. The important point is that (6) is written in a factorized form. This suggests that the right-hand-side of (6) can be considered as a product of two homographic mappings, and this is precisely the key for the obtention of special solutions. It is based on the fact that the homographic mapping is the discrete form of the Riccati equation [8] and that the special solutions of the continuous Painlevé equations are obtained through linearizations via Riccati equations [9].

The simplest way to find these solutions is to assume a form for the homographic mapping:

\[
\xi = -\frac{\alpha x + \beta}{\gamma x + \delta}
\]  

(7a)

and thus

\[
\xi = -\frac{\delta x + \beta}{\gamma x + \alpha}
\]  

(7b)

and substitute in (1) (rather than (6)). Moreover we can freely choose the normalization for the mapping (7) by taking \( \gamma = 1 \). Expressions become much simpler if instead of working with (1) we take \( c = 1 \) (by simple division):

\[
\bar{x}x = \frac{\nu x^2 - b \mu^n x - d \mu^{2n}}{x^2 + ax + \nu}
\]  

(8)

It is well known that one can freely scale-out two of the parameters of P_{III} and the same is true for d-P_{III}. Equating terms between (8) and the product of (7a)(7b) we find:

\[
\delta + \alpha = a
\]

\[
\delta \alpha = \alpha \delta = \nu
\]

\[
\beta \delta + \beta \alpha = -b \mu^n
\]  

(9)
\[ \beta \beta = -d \mu^{2n} \]

Hence, \( \alpha \) and \( \delta \) are constant with sum equal to \( a \) and product \( \nu \) and \( \beta = \beta_0 \mu^n \), with \( \beta_0 = \sqrt{-d \mu} \) (which can be scaled to 1). Moreover, one constraint must be satisfied:

\[ \delta + \frac{\alpha}{\mu} = b \]  \hspace{1cm} (10)

Substituting back into (7a) we find:

\[ \Pi = -\frac{\alpha x + \mu^n}{x + \delta} \]  \hspace{1cm} (11)

that can be linearized by the substitution \( x = P/Q \). We find thus:

\[ \overline{P} = -\alpha P - \mu^n Q \]
\[ \overline{Q} = P + \delta Q \]  \hspace{1cm} (12)

and finally:

\[ \overline{Q} + (\alpha - \delta)\overline{Q} + (\mu^n - \alpha \delta)Q = 0 \]  \hspace{1cm} (13)

This is just a discrete form of Bessel’s equation. In fact, it is essentially the same as the \( q \)-Bessel function that was derived in [10] in relation to the \( q \)-discrete Toda model and which can be also found in the classical monograph of Exton [11]. The continuous limit of (13) is just Bessel’s equation. First, we introduce \( Q = \epsilon^{-n}R \) and rewrite (13) as:

\[ \overline{R} + \epsilon(\alpha - \delta)\overline{R} + \epsilon^2(\mu^n - \alpha \delta)R = 0 \]  \hspace{1cm} (14)

Next, we expand \( R, \overline{R} \) up to second order in \( \epsilon \) and taking into account that \( \alpha \delta \approx -\frac{1}{\epsilon^2} \), we obtain in the limit \( \epsilon \to 0 \):

\[ R'' + (\epsilon^{2z} - \frac{a^2}{4})R = 0 \]  \hspace{1cm} (15)

i.e. precisely Bessel’s equation for \( J_{\frac{a}{2}}(\epsilon^z) \) [12]. From this basic solution one can construct higher ones, just as in the case of d-P \( \Pi \) [13] in close parallel to the continuous \( P_{\Pi} \). Bessel function solutions are not the only elementary solutions for d-P \( \Pi \). Rational solutions exist as well. For example, when \( d = 0 \) we find readily \( x = 0 \). Other rational solutions can be found also: when \( d = 0, b = 0 \) we obtain \( x = -a/c \). As in the continuous case, a duality under \( x \to 1/x \) exists, resulting in an exchange of \( (b, d) \) and \( (a, c) \). Thus, when \( a = c = 0 \) we find \( x = -\frac{d}{\lambda} \mu^n \) as a solution. This a very particular case of the general solution that can be obtained in closed form for \( a = c = 0 \). Putting \( x = X\mu^n \) we can reduce (1) to the autonomous form: \( \overline{XX} = X^2 - bX - d \) (where we have put \( \nu = 1 \) by simple division). The solution is found by a straightforward calculation:

\[ X = A\lambda^n + B\lambda^{-n} + C \]

where \( C(\sqrt{\lambda} - 1/\sqrt{\lambda})^2 + b = 0 \) and \( AB(\lambda - 1/\lambda)^2 + bC + d = 0 \). Thus \( \lambda \) and one of the \( A, B \) are free in general. In the particular case \( b = 0 \) we have \( C = 0 \) but the solution involves still two free constants.
Thus d-P$_{III}$ exhibits all the richness, as far as its particular solutions are concerned, as its continuous counterpart.

3. An alternate form of the discrete P$_{III}$

As we explained in the introduction, the continuous P$_{III}$ is given usually under two different canonical forms and even, sometimes, a third one. So, besides P$_{III}$ written in the form (3) we have also:

\[
x'' = \frac{x'^2}{x} - \frac{x'}{z} + \frac{1}{z} (ax^2 + b) + cx^3 + \frac{d}{x}
\]

(16)

and

\[
x'' = \frac{x'^2}{x} - \frac{x'}{z} + \frac{1}{4z^2} (ax^2 + cx^3) + \frac{b}{4z} + \frac{d}{4x}
\]

(17)

All these forms (3,16,17) are, of course, equivalent within a simple change of variables. For example, (17) and (16) are related through the transformation \(z \rightarrow z^2\) and \(x \rightarrow zx\).

As we have pointed out in section 2, the discrete form of P$_{III}$ corresponds naturally to the form (3). This does not mean that (3) is the only continuous limit of (2). Indeed one can recover any form of P$_{III}$ by taking the continuous limit in the appropriate way.

Usually, one relates the continuous variable \(z\) to the discrete one \(n\) through some constant-step discretization \(z = nc\) and thus \(\bar{z} - z = z - z = \epsilon\). Changes in the discrete independent variable can be represented by a nonconstant step in \(z\). We start from:

\[
\bar{z} - z = \epsilon f_+
\]

\[
z - \bar{z} = \epsilon f_-
\]

(18)

where \(f_+\) and \(f_-\) are slowly varying functions (in particular, \(f_+ - f_- = \mathcal{O}(\epsilon)\), and thus \(f_+^2 + f_-^2 = 2f_+f_- + \mathcal{O}(\epsilon^2)\)). We have:

\[
\bar{x} = x + \epsilon f_+ x' + \epsilon^2 f_+^2 x''/2 + \ldots
\]

\[
\bar{x} = x - \epsilon f_- x' + \epsilon^2 f_-^2 x''/2 + \ldots
\]

(19)

Equation (1) becomes:

\[
\bar{x}x - x^2 = \epsilon^2 [(xx'' - x'^2)f_+f_- + xx'f_+ - f_-

\]

\[
\epsilon \frac{1}{\nu} (cx^4 + ax^3 + b\mu^n x + d\mu^{2n}) + \mathcal{O}(\frac{1}{\nu^2})
\]

(20)

and with \(\nu = -\frac{1}{\epsilon^2}\) we can obtain various forms of P$_{III}$ in the continuous limit. In order to find precisely P$_{III}'$ in the form (17) we take \(z = (1 + \epsilon)^n\), \(f_+ = z\), \(f_- = \frac{z}{1+\epsilon}\), leading to \(\frac{f_+ - f_-}{\epsilon} = \frac{z}{1+\epsilon}\) and we choose \(\epsilon\) so as to have \(\mu^n = z\). Had we wished to find P$_{III}$, eq.(16), we should have started from eq.(2) for \(y\) and then take \(\epsilon\) so that \(\mu^{n/2} = z\).

However this limiting procedure, although interesting, may appear somewhat artificial since it incorporates the change of variables that we need in order to find the
desired form of P_{III}. Thus one may wonder whether there exists a “natural” discrete analog to P_{III} (16) or P_{III}' (17). Interestingly, the answer is: yes. The hint lies in the results we obtained in [14] where the mapping:

\[
\frac{z + \frac{x}{x}}{x + \frac{x}{z}} + \frac{z + \frac{x}{x}}{x + \frac{x}{z}} = k(1 + \frac{1}{x^2}) + \frac{2z}{x}
\]  

was identified as a discrete form of P_{III} (although with restricted coefficients). Its continuous limit can be obtained through \( k = \epsilon^2 a \), and at the limit \( \epsilon \to 0 \) we find:

\[
x'' = \frac{x'^2}{x} - \frac{x'}{z} - \frac{2a}{z}(x^2 + 1)
\]  

Equation (21) can also be rewritten as a system, whereupon one realizes that the missing coefficients, that would lead to a full P_{III} can be easily introduced. The final result is:

\[
x + \overline{x} = \frac{\zeta y + \theta}{y^2 - 1} \tag{23a}
\]

\[
y + \overline{y} = \frac{\eta x + \kappa}{x^2 - 1} \tag{23b}
\]

with constant \( \theta \) and \( \kappa \), and \( \zeta, \eta \) related through \( 2\zeta = \eta + \overline{\eta}, 2\eta = \zeta + \overline{\zeta} \). Thus \( \zeta \) and \( \eta \) are linear in \( n \) and staggered i.e. \( \eta(n) = \zeta(n - 1/2) \).

This is a remarkable result, since each of the equations of the system has the form of d-P_{II}. In fact, this form has been known from the outset [2] since the general form of d-P_{II} has been obtained as:

\[
x + \overline{x} = \frac{(\alpha n + \beta)x + \gamma + \delta(-1)^n}{x^2 - 1} \tag{24}
\]

where the \( \delta(-1)^n \) term indicates an even-odd dependence that can be used in order to write system (23a,b). However in these first treatments of discrete Painlevé equations the even-odd dependence was discarded since it was considered that it should disappear in the continuous limit. While this is true when one considers a single equation, this is not so for the full system. Indeed we will show that (23a,b) leads to P_{III} in the form (17). First we introduce a more convenient scaling:

\[
x + \overline{x} = \frac{zy + \epsilon^2 a}{y^2 - \epsilon^2 c} \tag{25a}
\]

\[
y + \overline{y} = \frac{(z - \epsilon/2)x + \epsilon^2 b}{x^2 - \epsilon^2 d} \tag{25b}
\]

whereupon the continuous limit becomes straightforward. We take:

\[
y = \frac{z}{2x} - \frac{zx'}{dx^2} + \left( \frac{x'}{8x^2} + \frac{b}{2x^2} + \frac{dz}{2x^3} \right) \epsilon^2
\]

and find for \( x \) (at the limit \( \epsilon \to 0 \))

\[
x'' = \frac{x'^2}{x} - \frac{x'}{z} + \frac{1}{2^2}(16cx^3 + 8ax^2) - \frac{4b}{z} - \frac{4d}{x}
\]
precisely the equation $P_{III}'$, eq.(17) up to an unimportant rescaling of $a$, $b$, $c$ and $d$. While this is an interesting result in itself, i.e. that $d$-$P_{II}$ can be extended so as to give $d$-$P_{III}$, we shall go one step further and show that the Lax pair for (25) can also be obtained in the same spirit. In fact the Lax pair for $d$-$P_{II}$ has been obtained in [4] and [15] but what will interest us particularly is the one obtained by Joshi and collaborators [7] in the framework of the discrete AKNS theory [16]. In [7] it was shown that the discrete problem:

$$\frac{\partial \Phi_n}{\partial \zeta} = M_n(\zeta) \Phi_n, \quad \Phi_{n+1} = L_n(\zeta) \Phi_n$$

where

$$L = \begin{pmatrix} \zeta & x \\ x & \frac{1}{\zeta} \end{pmatrix}$$

$$M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

starting from simple ansätze for $A$, $B$ and $C$, has a solution of the form

$$A = \frac{\kappa}{\zeta^3} + \frac{-2\kappa x + n + \nu}{\zeta} + \kappa \zeta$$

$$B = -\frac{2\kappa x}{\zeta^2} + 2\kappa x, \quad C = -\frac{2\kappa x}{\zeta^2} + 2\kappa x$$

with $\kappa$, $\nu$ constants, leading to $d$-$P_{II}$

$$x + \overline{x} = \frac{(\alpha n + \beta)x}{x^2 - 1}$$

with $\alpha = \frac{1}{\kappa}$, $\beta = \frac{(\nu + 1/2)}{\kappa}$.

This is not, however, the most general solution of this AKNS problem that can be reached with the same ansatz for $A$, namely $A = \frac{\kappa}{\zeta^3} + \frac{2}{\zeta} + r \zeta$, (without fixing the form of $B$ and $C$). Eliminating $B$ and $C$ in terms of $A$ we obtain an equation for the latter that reads:

$$\zeta^2 \overline{x}(A - \overline{A}) + \zeta \overline{x} x + x \overline{x}(A(x^2 + 1) + A(x^2 - 1)) - x \overline{x}(A(\overline{x}^2 + 1) + \overline{A}(\overline{x}^2 - 1))$$

$$-x(x + \overline{x}) \zeta^{-1} + \overline{x} \overline{x}(A - \overline{A}) \zeta^{-2} + \overline{x} \overline{x} \zeta^{-3} = 0$$

The solution for $A$ is exactly the same as in Eq. (30) (but $B$ and $C$ are more complicated). The equation for $x$ can be written as

$$\overline{W} - W = 0 \quad \text{where} \quad W = (\overline{x} + x)(x^2 - 1) - x(\alpha n + \beta)$$

Integrating (33) we find $W = \gamma + \delta(-1)^n$ leading to exactly equation (24) above which, as we saw, is equivalent to $d$-$P_{III}$, Eq. (23). The result of reference [7] corresponds to taking $W = 0$ as a solution of (33), which is the only choice leading to simple expressions for $B$ and $C$. 
Thus the alternate discrete $d$-$P_{III}$ equation has a Lax pair, a fact that establishes its integrability. Contrary to (5), this eigenvalue problem is a standard one and not of $q$-difference type.

4. Conclusion

In the previous sections we have shown that the discrete $P_{III}$ equation is the perfect analog to its continuous counterpart. Not only does it possess special solutions (in terms of rational or discrete Bessel functions) but, also, it exists under two different canonical forms that correspond to the two canonical forms of the continuous $P_{III}$. Furthermore, from the case of $q$-deformed $P_{III}$ (4-6), we conjecture the existence of an interesting class of new special functions which are the $q$-deformed difference analogs of the $P_{III}$ transcendent.

The fact that most features of integrable continuous equations can be extended to the discrete case offers us a handle on integrable discrete systems. Following the discovery of the singularity confinement property (that allows one to detect integrable discrete systems) the fact that there exists a continuous/discrete parallel opens a new road in the study of discrete integrability.

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