LAPLACE EQUATIONS, LEFSCHETZ PROPERTIES AND LINE ARRANGEMENTS

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Abstract. In this note we extend the main result in [6] on artinian ideals failing Lefschetz properties, varieties satisfying Laplace equations and existence of suitable singular hypersurfaces. Moreover we characterize the minimal generation of ideals generated by powers of linear forms by the configuration of their dual points in the projective plane and we use this result to improve some propositions on line arrangements and Strong Lefschetz Property at range 2 in [6]. The starting point was an example in [3]. Finally we show the equivalence among failing SLP, Laplace equations and some unexpected curves introduced in [3].

1. Introduction

Recently there has been an increasing interest both on ideals failing Lefschetz Properties and on arrangements of hyperplanes. In [6] the authors link these two topics. In fact, given a suitable line arrangement in the projective plane, i.e. a finite collection of lines, they relate the unstability of its derivation bundle (see Section 3) to the failure of the Strong Lefschetz Property at range 2 of the ideal generated by suitable powers of the linear forms defining the lines (see Section 2). This linkage is deepened in [3] where the authors add the equivalent condition of existence of a suitable unexpected curve.

Here we recover these arguments in order to improve and generalize [6]. Precisely, in Section 2 we generalize the main theorem in [6] to the case when there exist syzygies of suitable degree (Theorem 13), and in Section 3 we reformulate and improve some results in [6]. More precisely, given an artinian ideal $I$ generated by powers of linear forms, we characterize geometrically the existence of syzygies of degree 0. Let $Z$ be the set of points which are dual to these linear forms, then the existence of 0-syzygies in $I$ is related to the number of aligned points in $Z$ (Theorem 21). In this way we restate some results (Proposition 16 and Conjecture 2) in a more geometric form. The note ends with the equivalence among the existence of unexpected curves, the failure of SLP at range 2 and the existence of Laplace equations in suitable cases (Corollary 25).

2. Lefschetz Properties

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, $R = \mathbb{K}[x_0, \ldots, x_n] = \bigoplus R_t$ be the graded polynomial ring in $n + 1$ variables over $\mathbb{K}$ and $r_t = \dim R_t = \ldots$
\[
\dim H^0(\mathcal{O}_{P^n}(t)) = \binom{t+n}{n}, \text{ when } n \text{ is fixed.}
\]

Let
\[ A = R/I = \bigoplus_{i=0}^{m} A_i \]

be a graded artinian algebra, defined by a homogeneous ideal \( I \). Note that \( A \) is finite dimensional over \( K \). We denote by \( H_A = H_{R/I} \) the Hilbert function of \( R/I \).

First of all, we recall the definition of algebra (or ideal) failing the Lefschetz Property. The artinian algebra \( A \) (or the artinian ideal \( I \)) fails the Weak Lefschetz Property (from now on WLP) if for any linear form \( L \) there exists \( i \) such that the multiplication map by \( L_1 \times L : A_i \to A_{i+k} \), does not have maximal rank (i.e. is neither injective nor surjective). More precisely, \( A \) (or \( I \)) fails WLP if the multiplication map has rank \( \min \{ H_{R/I}(i), H_{R/I}(i+1) \} - \delta \)

Similarly, we define the failure of the Strong Lefschetz Property by \( \delta \).

**Definition 1.** The artinian algebra \( A \) (or the artinian ideal \( I \)) fails the Weak Lefschetz Property (from now on WLP) if for any linear form \( L \) there exists \( i \) such that the multiplication map by \( L_1 \times L : A_i \to A_{i+k} \), has rank \( \min \{ H_{R/I}(i), H_{R/I}(i+1) \} - \delta \).

One of the main examples comes from a classical result of Togliatti: the ideal \( I = (x^3, y^3, z^3, xyz) \) fails the WLP in degree 2 by 1. In [1, Example 3.1] this ideal is studied, but it appeared in [14] in terms of projection center of the Veronese surface (see also [5] for a modern approach and below for some details).

In [6] the authors of this note and J. Vallès characterize artinian ideals failing SLP (see also [5] for a modern approach and below for some details).

**Definition 2.** The artinian algebra \( A \) (or the artinian ideal \( I \)) fails the Strong Lefschetz Property (from now on SLP) at range \( k \) if for any linear form \( L \) the multiplication map \( \times L^k : A_i \to A_{i+k} \), has rank \( \min \{ H_{R/I}(i), H_{R/I}(i+k) \} - \delta \).

First of all, we recall the definition of algebra (or ideal) failing the Lefschetz Property. The artinian algebra (or ideal) failing SLP at range \( k \) by \( \delta \) in terms of suitable projections of Veronese varieties satisfying Laplace equations and of the existence of suitable singular hypersurfaces. In order to state our main result we recall some definitions and results from [5].

**Definition 3.** Let \( I = (F_1, \ldots, F_r) \subset R \) be an artinian ideal generated by forms of degree \( d \). The syzygy bundle \( K \) is defined by the exact sequence

\[
0 \longrightarrow K \longrightarrow \mathcal{O}_{P^n} \longrightarrow \mathcal{O}_{P^n}(d) \longrightarrow 0,
\]

where \( \Phi_I(a_1, \ldots, a_r) = a_1F_1 + \ldots + a_rF_r \).

**Theorem 4.** [6, Theorem 4.1] Let \( I = (F_1, \ldots, F_r) \subset R \) be an artinian ideal generated by forms of degree \( d \) and \( K \) the syzygy bundle. Let \( i \) be a non-negative integer such that \( h^0(K(i)) = 0 \) and \( k \) be an integer such that \( k \geq 1 \). Then \( I \) fails the SLP at the range \( k \) in degree \( d + i - k \) if and only if the induced homomorphism on global sections (denoted by \( H^0(\Phi_{L_k}) \))

\[
H^0(\mathcal{O}_{L_k}(i)) \xrightarrow{\Phi_{L_k}} H^0(\mathcal{O}_{L_k}(i+d))
\]
does not have maximal rank for a general linear form \( L \).

**Definition 5.** Let \( X \subset \mathbb{P}^N \) be a projective \( n \)-dimensional complex variety. For \( m \geq 1 \), the projective \( m \)-th osculating space to \( X \) at a general point \( P \), \( T^m_P(X) \), is the subspace of \( \mathbb{P}^N \) spanned by \( P \) and by all the derivative points of order less than or equal to \( m \) of a local parametrization of \( X \), evaluated at \( P \). Of course, for \( m = 1 \) we get the tangent space \( T_P(X) \).
We remark that the expected dimension of the \( m \)-th osculating space is

\[
\expdim T^m_p(X) = \inf \left( \binom{n+m}{n} - 1, N \right).
\]

**Definition 6.** A \( n \)-dimensional variety \( X \subset \mathbb{P}^N \) satisfies \( \delta \) independent Laplace equations of order \( m \) if the \( m \)-th osculating space at a general point has

\[
\dim T^m_p(X) = \expdim T^m_p(X) - \delta.
\]

If \( N < \binom{n+m}{n} - 1 \), then there are always \( \binom{n+m}{n} - 1 - N \) relations between the partial derivatives. We call these relations “trivial” Laplace equations of order \( m \).

Among the varieties satisfying non trivial Laplace equations, we are interested in rational varieties which are suitable projections of a Veronese variety. For any vector space \( V \), \( V^* = \text{Hom}_K(V, K) \) will be the dual space.

Let \( v_t : [L] \in \mathbb{P}(R^*_t) \hookrightarrow [L'] \in \mathbb{P}(R^*_t) \) be the \( t \)-uple Veronese embedding whose image \( v_t(\mathbb{P}^n) \) is the Veronese \( n \)-fold of order \( t \).

Let \( I = (F_1, \ldots, F_r) \) be an ideal generated by \( r \) forms of degree \( d \) and \( I_h \) be the homogeneous component of degree \( h \) of \( I \) for any \( h \).

**Definition 7.** The apolar space of \( I \) in degree \( d+i \) with \( i \geq 0 \) is

\[
I^\perp_{d+i} = \{ \Delta \in R^*_{d+i} \mid \Delta(F) = 0, \forall F \in I_{d+i} \},
\]

where the canonical basis of \( R^*_{d+i} \) is given by the \( r_{d+i} = \binom{d+i+n}{n} \) derivations

\[
\frac{\partial^{i_0} \cdots \partial^{i_n}}{\partial x_0 \cdots \partial x_n}
\]

with \( i_0 + \cdots + i_n = d + i \).

There is the exact sequence of vector spaces

\[
0 \longrightarrow I^\perp_{d+i} \longrightarrow R^*_{d+i} \longrightarrow I^*_{d+i} \longrightarrow 0
\]

and, by dualizing it, one can identify \( R_{d+i}/I_{d+i} \cong (I^\perp_{d+i})^* \) and write the decomposition \( R_{d+i} = I_{d+i} \oplus (I^\perp_{d+i})^* \).

By denoting the corresponding projection map

\[
\pi_{I_{d+i}} : \mathbb{P}(R^*_{d+i}) \setminus \mathbb{P}(I^*_{d+i}) \rightarrow \mathbb{P}(I^\perp_{d+i}),
\]

we consider the variety \( X := \pi_{I_{d+i}}(u_{d+i}(\mathbb{P}^n)) \).

**Remark 1.** The toric case is the easiest one: when \( I_d \) is generated by \( r \) monomials of degree \( d \), \( (I_d^*)^* \) is generated by the remaining \( r_d - r \) monomials.

The case of powers of linear forms is very interesting. In [7] it is proved that the apolar of an ideal generated by powers of linear forms is related to the 0-dimensional scheme of the dual points of the linear forms, as in the following Theorem.

**Theorem 8.** [7] Let \( l_1, \ldots, l_r \) be linear forms in \( \mathbb{P}^n \), \( d_1, \ldots, d_r > 0 \) be integers and \( I = (l_1^{d_1}, \ldots, l_r^{d_r}) \). If \( P_i = l_i^* \) denotes the dual point of \( l_i \), then for any \( j \geq \max d_i \),

\[
\dim_K \left( \frac{R}{T} \right)_j \geq \dim_K \bigcap_{i=0, \ldots, r} I_{P_i}^{j-d_i+1}.
\]

**Notation 1.** For any \( i \geq 0, k \geq 1 \), we denote \( N(r, i, k, d) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k}) \), \( N^+ = \sup(0, N(r, i, k, d)) \) and \( N^- = \sup(0, -N(r, i, k, d)) \).
Remark 2. Note that when $h^0(K(i)) = 0$ then $N(r, i, k, d) = H_{R/I}(d + i - k) - H_{R/I}(d + i) = \dim \ker(\times L^k) - \dim \coker(\times L^k)$.

Moreover if $N < \binom{n + d + i - k}{n} - 1$ then the number of trivial equations is exactly $N + \dim I_{d+i-k}$.

The main result in [3] (that generalizes to SLP the result given in [10] on WLP) is the following.

Theorem 9. [3] Theorem 5.1] Let $I = (F_1, \ldots, F_r) \subset R$ be an artinian ideal generated by $r$ homogeneous polynomials of degree $d$. Let $i, k, \delta$ be integers such that $i \geq 0, k \geq 1$. Assume that there is no syzygy of degree $i$ among the $F_j$’s. The following conditions are equivalent:

1. The ideal $I$ fails the SLP at the range $k$ in degree $d + i - k$.
2. For a general linear form $L$ of $\mathbb{P}^n$, there exist $N^+ + \delta$, with $\delta \geq 1$, independent vectors $(G_{1j}, \ldots, G_{rj})_{j=1, \ldots, N^+ + \delta} \in R^{\oplus_{i,j}}$ and $N^+ + \delta$ forms $G_j \in R_{d+i-k}$ such that $G_{1j}F_1 + \ldots + G_{rj}F_r = L^kG_j$.
3. The $n$-dimensional variety $\pi_{L+i}(v_{d+i}(\mathbb{P}^n))$ satisfies $\delta \geq 1$ Laplace equations of order $d + i - k$.
4. For any $L \in R_1$, $\dim_{K}(I_{d+i}^i)^n \cap H^0(\mathbb{T}_{L}\mathcal{I}_{L^i}^{d+i-k+1}(d+i)) \geq N^+ + \delta$, with $\delta \geq 1$.

As noted in [3] the hypothesis on the global syzygy in Theorem 9 is not restrictive in case of syzygies of degree $i \geq 1$.

Lemma 10. [3] Lemma 5.2] Let $I$ be the ideal $(L_1^d, \ldots, L_r^d)$ where the $L_j$’s are general linear forms and $r < r_d$. Let $K$ be its syzygy bundle and $i \geq 1$. Then

\[ h^0(K(i)) = 0 \iff rr_i \leq r_{d+i}. \]

Here we generalize these results in the case in which $H^0(K(i)) \neq 0$ and $H^0(K(i-k)) = 0$. This generalization is interesting as if there are 0-syzygies, i.e. $i = 0$ and $H^0(K) \neq 0$ then the condition $H^0(K(−k)) = 0$ is automatically verified.

Theorem 11. Let $i, k$ be non-negative integers such that $h^0(K(i)) = s$ and $H^0(K(i-k)) = 0$. Denoted by $H^0(\Phi_{L^k})$ the induced homomorphism on global sections

\[ H^0(\mathcal{O}_{L^k}(i)) \xrightarrow{H^0(\Phi_{L^k})} H^0(\mathcal{O}_{L^k}(i + d)) \]

then the homomorphisms $H^0(\Phi_{L^k})$ and $\times L^k$ have the same cokernel, while $\ker(\times L^k) \cong \ker H^0(\Phi_{L^k})$.

Proof. The proof exploits the same ideas as in [3] Theorem 4.1]. Let us consider the canonical exact sequence

\[ 0 \longrightarrow K(i-k) \xrightarrow{\times L^k} K(i) \longrightarrow K \otimes \mathcal{O}_{L^k}(i) \longrightarrow 0. \]

As $H^0(K(i-k)) = 0$ and $A_{d+i} = H^0(K(i))$ for any $i \in \mathbb{Z}$ ([11] Proposition 2.1]), we obtain the long exact sequence of cohomology

\[ 0 \rightarrow H^0(K(i)) \rightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \rightarrow \]

\[ \rightarrow H^1(K \otimes \mathcal{O}_{L^k})(i) \rightarrow H^2(K(i-k)) \rightarrow 0. \]

Moreover, by tensoring the exact sequence defining the bundle $K$ by $\mathcal{O}_{L^k}(i)$

\[ 0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^r \xrightarrow{\Phi_i} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0, \]
we get in cohomology
\[ 0 \rightarrow \mathbb{H}^0(K \otimes \mathcal{O}_L(k)) \rightarrow \mathbb{H}^0(\mathcal{O}_L(k))^{r_{t}^{(\Phi_{L,k})}} \rightarrow \mathbb{H}^0(\mathcal{O}_L(k)) \rightarrow \mathbb{H}^1(\mathcal{O}_L(k))^{r_{t}^{(\Phi_{L,k})}} \rightarrow \mathbb{H}^1(\mathcal{O}_L(k)) \rightarrow 0. \]

Moreover \( \mathbb{H}^2(K(i-k)) = 0 = \mathbb{H}^1(\mathcal{O}_L(k)) = 0 \) when \( n > 2 \); while when \( n = 2 \) we have \( \mathbb{H}^2(K(i-k)) = K' \) with \( t = r_{k-i-2} - r_{k-i-3} \) and \( h^1(\mathcal{O}_L(k)) = h^2(\mathcal{O}_L^2(i-k)) = r_{k-i-3} \). Hence, the kernel of both maps \( \mathbb{H}^0(\Phi_{L,k}) \) is the same and between the kernels there is the sought isomorphism.

Following Remark 2 it is natural to generalize the integers \( N(r, i, k, d), N^+, N^- \) in case of syzygies.

**Definition 12.** Let \( h^0(K(i)) = s \), we define:

- \( N_s = N(r, i, k, d, s) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k}) - s; \)
- \( N^+_s := \max(0, N_s); \)
- \( N^-_s := \max(0, -N_s). \)

**Remark 3.** Note that when \( h^0(K(i-k)) = 0 \) we have \( N_s = H_{R/I}(d + i - k) - H_{R/I}(d + i). \)

Now we give the main theorem of this note, that generalizes [6] Theorem 5.1.

**Theorem 13.** Let \( I = (F_1, \ldots, F_r) \subset R \) be an artinian ideal generated by \( r \) homogeneous polynomials of degree \( d \). Let \( i, k, \delta \) be non negative integers such that there is no syzygy of degree \( i-k \) among the \( F_j \)'s. The following conditions are equivalent:

1. The ideal \( I \) fails the SLP at the range \( k \) in degree \( d + i - k \) by \( \delta; \)
2. \( \dim \ker(\times L^k) = N^+_s + \delta > N^-_s; \)
3. \( \dim \coker(\times L^k) = N^+_s - \delta > N^-_s; \)
4. The \( n \)-dimensional variety \( \pi_{L+1}(v_{d+i}(\mathbb{P}^n)) \) satisfies \( \delta \geq 1 \) non trivial Laplace equations of order \( d + i - k \) and no Laplace equation of smaller order;
5. For any \( L \in R_1, \dim_{\mathbb{K}}((I_{d+i}^k)^* \otimes \mathbb{H}^0(T_{L'}^n+1)(d+i)) = N^+_s + \delta, \) with \( \delta \geq 1. \)

**Proof.** By Theorem 1 the equivalence between (2) and (3) is an easy calculation. Moreover, it is obvious that (2) and (3) imply (1).

Let us assume that \( I \) fails the SLP at the range \( k \) in degree \( d + i - k \). Let \( D = \dim \ker(\times L^k) \) and suppose that \( 0 < D \leq N^+_s \). Denoting \( h^0 = \dim \ker(\Phi_{L,k}), \) by Theorem 3 \( D = h^0 - s \) and \( \dim \coker(\times L^k) = D - N_s. \) Now, if \( N_s \geq 0, \) then \( N^+_s = N_s = N - s \) and \( 0 < D = h^0 \leq N - s \) so \( h^0 \leq N. \) On the other hand, \( 0 < \dim \ker(\times L^k) = D - N_s = h^0 - s - N + s = h^0 - N \) so \( h^0 > N \) that is a contradiction. If \( N_s < 0, \) then \( N^+_s = 0 \) and \( 0 < D \leq 0 \) is a contradiction. So (1) \( \iff \) (2) \( \iff \) (3).

The dimension of the kernel of the map \( \times L^k \) i.e. the dimension of \( \mathbb{H}^0(K \otimes \mathcal{O}_L(k)) - s, \) written in a geometric way, is

\[ N^+_s + \delta = \dim([P(I_{d+i}) \cap T_{L_{d+i}} d+i} v_{d+i}^n] + 1, \delta \geq 0 \]

where the projective dimension is \(-1\) if the intersection is empty. The number \( \delta \) is the number of (non trivial) Laplace equations. Indeed, the dimension of the \((d + i - k)\)-th osculating space to \( \pi_{L+1}(v_{d+i}(\mathbb{P}^n)) \) is \( r_{d+i-k} - N^+_s - \delta, \) since the \((d + i - k)\)-th osculating space to \( v_{d+i}^n(\mathbb{P}^n) \) meets the center of projection along a \( \mathbb{P}^{N^+_s + \delta - 1} \). In other words, the \( n \)-dimensional variety \( \pi_{L+1}(v_{d+i}(\mathbb{P}^n)) \) satisfies \( \delta \)
Laplace equations.

The image by \( \pi_{d+1} \) of the \((d+i-k)\)-th osculating space to the Veronese \( v_{d+1}(\mathbb{P}^n) \) at a general point has codimension \( h^0(K \otimes \mathcal{O}_{L^k}(i)) - N_s^+ \) in \( \mathbb{P}(I_{d+1}^{d+i}) \).

The codimension corresponds to the number of hyperplanes in \( \mathbb{P}(I_{d+1}^{d+i}) \) containing the osculating space to \( \pi_{d+1}(v_{d+1}(\mathbb{P}^n)) \). These hyperplanes are images by \( \pi_{d+1} \) of hyperplanes in \( \mathbb{P}(R_{d+1}) \) containing \( \mathbb{P}(I_{d+1}^i) \) and the \((d+i-k)\)-th osculating space to \( v_{d+1}(\mathbb{P}^n) \) at the point \([L^{d+i}]\). In the dual setting it means that these hyperplanes define forms of degree \( d+i \) in \((I_{d+1}^d)^*\) with multiplicity \((d+i-k+1)\) at \([L^d]\).

To summarize, the number of non-trivial Laplace equations is \( h^0(K \otimes \mathcal{O}_{L^k}) - s - N_s^+ \) and \( \text{coker}(H^0(\Phi_{I,L})) \simeq (I_{d+1}^d)^* \cap H^0(I_{L^d}^{d+i-k+1})(d+i) \).

\[ \square \]

3. Remarks on failing SLP at the range 2 and line arrangements on \( \mathbb{P}^2 \)

In this section we explain the connection with certain line arrangements in the plane and we add some remarks on [6, Section 7].

A line arrangement is a collection of distinct lines in the projective plane. Arrangements of lines, and more generally of hyperplanes, have long been an important topic of study (see [2] or [11] for a good introduction).

**Definition 14.** Given a line arrangement, let us denote by \( f = 0 \) the equation of the union of lines of this arrangement. The vector bundle \( D_0 \) defined as the kernel of the jacobian map:

\[
0 \rightarrow D_0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow (\mathcal{O}_{\mathbb{P}^2}(d-1)
\]

is called derivation bundle (or logarithmic bundle) of the line arrangement (see [12] and [13] for an introduction to derivation bundles). One can consider the lines of the arrangement in \( \mathbb{P}^2 \) as a set of distinct points \( Z \) in \( \mathbb{P}^{2s} \). Then we will denote by \( D_0(Z) \) the associated derivation bundle.

The arrangement of lines is said free with exponents \((a,b)\) if its derivation bundle splits on \( \mathbb{P}^2 \) as a sum of two line bundles, more precisely if

\[
D_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b),
\]

while \((a,b)\) is called the general splitting type if it corresponds to the splitting of \( D_0(Z) \) over a general line \( l \subset \mathbb{P}^2 \).

The general splitting type \((a,b)\) is related to the existence of curves of degree \( a+1 \) passing through \( Z \), having multiplicity \( a \) at \( l^{\vee} \in \mathbb{P}^{2s} \). More precisely,

**Lemma 15.** ([15, [9] Proposition 2.1]) Let \( Z \subset \mathbb{P}^{2s} \) be a set of \( a + b + 1 \) distinct points with \( 1 \leq a \leq b \) and \( l \) be a general line in \( \mathbb{P}^2 \). Then the following conditions are equivalent:

(1) \( D_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b) \).

(2) \( h^0((J_Z \otimes J_{l^{\vee}})(a+1)) \neq 0 \) and \( h^0((J_Z \otimes J_{l^{\vee}})(a)) = 0 \).

So the general splitting type is related to the degree of suitable singular curves through \( Z \) and, thanks to the result of Emsalem-Iarrobino ([7]) here stated in Theorem 3 in [6, Proposition 7.2] an equivalence between unstability of the derivation bundle and the failing of SLP at range 2 is given. Actually, in the statement of Proposition 7.2 and its corollaries in [6] the hypothesis on the non-existence of 0-syzygies, that is implicitly used in the proofs, is missing. Here we give a precise
statement, in a more general form with respect to the number of lines and the splitting type of the derivation bundle. In this way we determine an interval of possible degrees of generators of ideals failing SLP.

**Proposition 16.** Let I \( \subset R = \mathbb{K}[x, y, z] \) be an artinian ideal generated by \( 2d+1+n \) polynomials \( \ell_1^d, \ldots, \ell_{2d+1+n}^d \) where \( n \geq 0 \) and \( \ell_i \) are distinct linear forms in \( \mathbb{P}^2 \). Let \( Z = \{ \ell_1^y, \ldots, \ell_{2d+1+n}^y \} \) be the corresponding set of points in \( \mathbb{P}^{2V} \). If the ideal is minimally generated in degree \( d \), then the following conditions are equivalent:

1. The ideal \( I \) fails the SLP at the range 2 in degree \( d-2 \).
2. The derivation bundle \( D_0(Z) \) is non-balanced with splitting type \( (d-s, \ldots, d+s+n) \), with \( s \geq 1 \).

Moreover, if \( n \) is even the following third condition is equivalent to the previous two:
3. The derivation bundle \( D_0(Z) \) is unstable with splitting type \( (d-s, d+s+n) \), with \( s \geq 1 \).

**Proof.** It is enough to argue as in [3 Proposition 7.2]. The ideal \( I \) fails the SLP at the range 2 in degree \( d-2 \) if and only if there exists a curve of degree \( d \) through \( Z \) and with multiplicity \( d-1 \) at a general point \( P \). By Lemma 15, this condition is equivalent to asking that \( D_0(Z) \) has splitting type \( (a, b) \) with \( a \leq d-1 \), so we can write \( a = d-s \) with \( s \geq 1 \). As the number of points in a line arrangement is always \( a + b + 1 \) we get \( b = 2d+n-a = d+s+n \) and the difference \( |b-a| = 2s \geq 2 \), so \( D_0(Z) \) has to be non-balanced. The last part comes from the equivalence between unstability and the non-balanced condition \( |b-a| \geq 2 \) that holds when the first Chern class \( a+b \) is even. \( \square \)

Now, the corollary in [3] becomes the following.

**Proposition 17.** Let \( A = \{ l_1, \ldots, l_{a+b+1} \} \) be a free line arrangement with exponents \( (a, b) \) such that \( a \leq b, b-a \geq 2 \), let \( Z = \{ l_1^y, \ldots, l_{a+b+1}^y \} \) be the corresponding set of points in \( \mathbb{P}^{2V} \). For every integer \( d \) such that \( a+1 \leq d \leq \lceil \frac{a+b}{2} \rceil \) if the ideals defined below are minimally generated in degree \( d \), then

1. If \( a+b \) is even, \( I = (l_1^d, \ldots, l_{a+b+1}^d) \) fails the SLP at the range 2 and degree \( d-2 \).
2. If \( a+b \) is odd, let \( P \) be a point in general position with respect to \( Z \), then \( I = (l_1^d, \ldots, l_{a+b+1}^d, P^y) \) fails the SLP at the range 2 and degree \( d-2 \).

Now, we have to explain how we recognized that in [6] the hypothesis on the minimality of generators has been forgotten. We thank the authors of [3] for the following example.

**Example 18.** Let \( A = x y z (x+z)(x+2z) \prod_{j=1}^{12} (y+jz) \). It is a free arrangement of splitting type \( (3, 13) \). The derivation bundle is unstable because it is non-balanced and the first Chern class is even, but the ideal \( I = (x^8, y^8, z^8, (x+z)^8, (x+2z)^8, (y+jz)^8 | 1 \leq j \leq 12 ) \) has the SLP at range 2.

Actually, by investigating on the geometric meaning of 0-syzygies in artinian ideals generated by powers of linear forms, we recognized that they are equivalent to the existence of a suitable number of aligned points in \( Z \). So examples analogous to Example [18] are the only possibility in order to get 0-syzygies. In order to state

\[ 1 \text{By } [x] \text{ we denote the minimum integer greater or equal than } x \]
precisely and prove this result, we recall the following one by Ellia and Peskine [9] Proposition pag. 112 related to the numerical character of $Z$.

**Definition 19.** Let $S = \mathbb{K}[x_0, x_1]$ and $Z$ be a 0–dimensional scheme in the projective plane $\mathbb{P}^2$. The numerical character of $Z$ is the sequence $(n_0, \ldots, n_{s-1})$ with $n_0 \geq \cdots \geq n_{s-1}$ such that

$$0 \to \bigoplus_{i=0}^{s-1} S(-n_i) \to \bigoplus_{i=0}^{s-1} S(-i) \to \frac{\mathbb{K}[x_0, x_1, x_2]}{I_Z} \to 0$$

is a minimal resolution and

1. $s$ is the minimal degree of a curve containing $Z$;
2. $n_i \geq s$, for each $i = 0, \ldots, s-1$;
3. $\text{Def}(a)_+ = \max\{a, 0\}$, $h^1(\mathcal{I}_Z(n)) = \deg(Z) - H_Z(n) = \sum_{i=0}^{s-1} (n_i - n - 1)_+ - \sum_{i=0}^{s-1} (i - n - 1)_+$, in particular $\deg(Z) = \sum_{i=0}^{s-1} (n_i - i)$.

**Lemma 20.** Let $Z \subset \mathbb{P}^2$ a set of points with numerical character $(n_i)_{i=0,\ldots,s-1}$. If $t$ is an integer such that $n_{t-1} > n_t + 1$, then there exists a curve $C$ of degree $t$ such that the points of $Z$ contained in $C$ form a set with numerical character $(n_0, \ldots, n_{t-1})$.

**Theorem 21.** Let $l_1, \ldots, l_{2d+1}$ be linear forms in $\mathbb{K}[x_0, x_1, x_2]$. Then

$$\dim_{\mathbb{K}}(l_1^d, \ldots, l_{2d+1}^d) < 2d + 1$$

if and only if in the set $Z = \{l_1^\vee, \ldots, l_{2d+1}^\vee\} \subset \mathbb{P}^{2*}$ there are at least $d + 2$ aligned points.

**Proof.** In the Veronese map $v_d$, aligned points $l_1^\vee, \ldots, l_{2d+1}^\vee \in L$ go to points $l_1^d, \ldots, l_{2d+1}^d \in v_d(L) = C_d$, where $C_d \subset \mathbb{P}^d$ is the rational normal curve of degree $d$; so $d + 1$ of them are linearly independent, while $d + 2$ are dependent. So if there are $d + 2$ aligned points we get $\dim_{\mathbb{K}}(l_1^d, \ldots, l_{2d+1}^d) < 2d + 1$.

Conversely, let $\dim_{\mathbb{K}}(l_1^d, \ldots, l_{2d+1}^d) < 2d + 1$. By Emsalem-Iarrobino result ([7], here Theorem 5) we get

$$\dim_{\mathbb{K}} I_{Z,d} = \dim_{\mathbb{K}} \left( \frac{R}{l_1^d, \ldots, l_{2d+1}^d} \right).$$

So the Hilbert function of $Z$ in degree $d$ is $H_Z(d) = \dim_{\mathbb{K}}(l_1^d, \ldots, l_{2d+1}^d) < 2d + 1$ and, by [3] in Definition 19

$$h^1(\mathcal{I}_Z(d)) = \deg(Z) - H_Z(d) > 2d + 1 - (2d + 1) = 0.$$
2d + 2 > 2d + 1 = \deg(Z) \geq n_0 + n_1 - 1 \geq d + 2 + d + 1 - 1 = 2d + 2 and this is a
contradiction. So \( n_1 < n_0 - 1 \) and we can apply Lemma \[20\] with \( t = 1 \) in order to
get a line containing a subset \( Z' \subset Z \) consisting of \( \deg Z' = n_0 \geq d + 2 \) points. \( \Box \)

So the hypothesis of non-existence of 0-syzygies is equivalent to the condition
that there are at most \( d + 1 \) aligned points in \( Z \). In Example \[13\] there are too many
aligned points, actually 14 that are exactly \( 14 = d + 1 + s \). Precisely, for \( d = 8 \)
in order to avoid linear dependence among \( l_1^b, \ldots, l_r^b \), we can have at most \( r = 9 \)
aligned points in the dual set \( l_1^\vee, \ldots, l_r^\vee \). But there are 14 aligned points so there
are 5 more, corresponding to 14-9=5 independent 0-syzygies.

With this interpretation of 0-syzygies we can restate \[6\] Proposition 7.2 and its
corollaries in a nicer geometric way.

**Proposition 22.** Let \( I \subset R = \mathbb{K}[x, y, z] \) be an artinian ideal generated by \( 2d + 1 \)
polynomials \( l_1^b, \ldots, l_{2d+1}^b \) where \( l_i \) are distinct linear forms in \( \mathbb{P}^2 \). Let \( Z = \{l_1^\vee, \ldots, l_r^\vee\} \)
be the corresponding set of points in \( \mathbb{P}^{2r} \). If there are no more than \( d+1 \) aligned
points in \( Z \), then the following conditions are equivalent:

1. The ideal \( I \) fails the SLP at the range 2 in degree \( d - 2 \).
2. The derivation bundle \( D_0(Z) \) is unstable with splitting type \( (d - s, d + s) \),
   with \( s \geq 1 \).

We recall also Terao’s conjecture.

**Definition 23.** Let \( \mathcal{A} \) be a line arrangement, then the combinatorics of \( \mathcal{A} \) is the
intersection lattice with reverse order.

**Conjecture 1.** (Terao) Let \( \mathcal{A} \) be a free arrangement and \( \mathcal{A}' \) an arrangement with
the same combinatorics as \( \mathcal{A} \). Then \( \mathcal{A}' \) is free, too (i.e. the freeness is a combinatorial
property).

This conjecture for line arrangements has been proved up to 12 lines \([9]\). In \[6\],
where freeness and unstability are related with failing SLP, an equivalent conjecture
was given in terms of SLP, that here we complete with the missed hypothesis on
syzygies.

**Conjecture 2.** \([6]\) Let \( \ell_1 \cdots \ell_{2b+1} \) and \( h_1 \cdots h_{2b+1} \) be two arrangements with
the same combinatorics and such that the dual sets of points have at most \( b + 1 \) aligned
points. If \( I = (\ell_1^b, \ldots, \ell_{2b+1}^b) \) has SLP at range 2 in degree \( b - 2 \) then also \( J =
(h_1^b, \ldots, h_{2b+1}^b) \) has SLP at range 2 in degree \( b - 2 \).

Now, we end this note with a remark that links Example \[18\] with Theorem \[13\]
If there is no 0-syzygy, \( N^- = 0 \) and the existence of a suitable singular curve is
equal to the failure of SLP. Moreover, this hypothesis on 0-syzygies is necessary,
otherwise the unstability is not enough to get an ideal that fails SLP: in Example
\[18\] \( H_{R/J}(8) = 33 = 45 - (17 - 5) \) that means that there are \( s = 5 \) independent
0-syzygies and the ideal has SLP and, moreover, the difference between Hilbert
functions is \( H_{R/J}(8) - H_{R/I}(6) = -5 \). From this we got the idea to generalize
Theorem \[9\] when there are \( s \) syzygies in degree \( i \) and no syzygy in degree \( i - k \) just
by replacing the integer \( N(r, i, k, d) \) defined in \[9\] with \( N(r, i, k, d, s) \) introduced in
Definition \[12\]. In Example \[18\] we get \( N^- = 5 \) and the existence of singular curves is
expected (in the sense of \[3\]) and is not enough to the failure of SLP, by applying
Theorem \[13\]

We recall here the definition of unexpected curve.
Definition 24. ([3, Definition 2.1]) We say that a reduced finite set of points $Z \subset \mathbb{P}^2$ admits an unexpected curve if there is an integer $j > 0$ such that, for a general point $P$, $jP$ fails to impose the expected number of conditions on the linear system of curves of degree $j + 1$ containing $Z$. That is, $Z$ admits an unexpected curve of degree $j + 1$ if

$$h^0((I_Z \otimes I_P^j)(j + 1)) > \max \left\{ h^0(I_Z(j + 1)) - \binom{j + 1}{2} ; 0 \right\}.$$ 

A very simple calculation shows that for $k = 2, i = 0, d = j + 1$ we have

$$N_s^- = \max \left\{ h^0(I_Z(j + 1)) - \binom{j + 1}{2} ; 0 \right\},$$

so that we have the following special case of [3, Theorem 6.5].

Corollary 25. Let $I = (l_{d1}^1, \ldots, l_{d2d+1}^j) \subset R = \mathbb{K}[x, y, z]$ be an artinian ideal generated by $2d + 1$ distinct powers of linear forms in $\mathbb{P}^2$. Let $Z = \{l_{d1}^1, \ldots, l_{d2d+1}^j\}$ be the corresponding set of points in $\mathbb{P}^{2v}$. If there are no more than $d + 1$ aligned points in $Z$, then the following are equivalent:
1. $Z$ has an unexpected curve of degree $d$;
2. $I$ fails the SLP in range 2 and degree $d - 2$;
3. the variety $\pi_{I_d}(v_d(\mathbb{P}^2))$ satisfies at least one non trivial Laplace equation of order $d - 2$.

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