Biased lattice random walks (BLRW) are used to model random motion with drift in a variety of empirical situations in engineering and natural systems such as phototaxis, chemotaxis or gravitaxis. When motion is also affected by the presence of external borders resulting from natural barriers or experimental apparatuses, modelling biased random movement in confinement becomes necessary. To study these scenarios, confined BLRW models have been employed but so far only through computational techniques due to the lack of an analytic framework. Here, we lay the groundwork for such an analytical approach by deriving the Green’s functions, or propagators, for the confined BLRW in arbitrary dimensions and arbitrary boundary conditions. By using these propagators we construct explicitly the time dependent first-passage probability in one dimension for reflecting and periodic domains, while in higher dimensions we are able to find its generating function. The latter is used to find the mean first-passage passage time for a d-dimensional box, d-dimensional torus or a combination of both. We show the appearance of surprising characteristics such as the presence of saddles in the spatio-temporal dynamics of the propagator with reflecting boundaries, bimodal features in the first-passage probability in periodic domains and the minimisation of the mean first-return time for a bias of intermediate strength in rectangular domains. Furthermore, we quantify how in a multi-target environment with the presence of a bias shorter mean first-passage times can be achieved by placing fewer targets close to boundaries in contrast to many targets away from them.

I. INTRODUCTION

Random walk models have been ubiquitously applied across a variety of disciplines both with continuous space-time variables, i.e. Brownian walks [1], and with discrete variables, i.e. lattice random walks (LRW) [2]. Due to their simplicity, LRW have been used as null models to understand the stochastic dynamics in polymer chains [3], record statistics [4], population genetics [5], foraging behaviour in animals [6], diffusion on the surface of stars [7], energy transfer in molecules [8, 9], and protein transport along DNA [10, 11], to name just a few. LRW have also inspired many theoretical approaches to study coverage times [12, 13], resetting random walks [14] and anomalous dynamics in disordered systems [15].

For many real systems, the use of LRW has provided a convenient way to extract information about the statistics of an important quantity, the so-called first-passage probability, or a related one, the so-called first-return probability. They measure the probability that a random variable has reached or returned to a given value for the first time. These quantities represent a work-horse in random search processes and, more generally, in transport calculations [16–18]. In many empirical scenarios when natural or artificial barriers resulting from experimental apparatuses affect the dynamics, LRW models need to be modified to account for the presence of boundaries. The effects on the first passage statistics become quite significant when the spatial domain is bounded as exemplified by the mean return time (MRT) and mean first-passage time (MFPT) becoming finite as compared to infinite when space is unbounded. Explicit expressions for the MRT have been established long ago [19], those for the MFPT up to 3D for both rectangular and periodic lattices have been known for some time [20, 21], while the analogous ones in higher dimensional cases have been found more recently [22].

Despite the large amount of analytic studies on LRW in confined space and their related first-passage statistics [18, 23, 24], there has been no attempt to generalise the expressions for the MRT or the MFPT when motion is not completely random but possesses a bias in some direction, the so-called biased lattice random walks (BLRW). Similarly there has been no analytic progress for the first-passage and return probability, with studies on confined BLRW having been mainly computational [25–28]. This is somewhat surprising given that there are significant areas of research where BLRW models have been employed. They include biological systems such as cell migration due to concentration gradients (chemotaxis) [29, 30], bacteria drifting towards a light source (phototaxis) [27] or upwards movement of single-celled algae in response to gravity (gravitaxis) [29]. In engineering it is worth mentioning the application of BLRW to study routing protocol for wireless sensor networks [30], to analyse the degradation of pavement [28] and to model field-driven translocation of tracer particles [31].

With the only closed-form results for BLRW in finite domains pertaining to the generating function of the 1D propagator with two absorbing boundaries [32], there is a need to develop a general framework that allows to derive analytically various transport quantities. Here we are able to do so by extending the LRW techniques in reference [22] to construct analytically the confined time-dependent propagator and its generating function for BLRW in arbitrary dimensions and arbitrary boundary conditions. These propagators are then
used to study first-passage and first-return statistics and obtain analytic expressions for the MRT and MFPT.

The remainder of the paper is organised as follows. Section II deals with BLRW in 1D; it develops a symmetrisation procedure that allows to impose different boundary conditions and find the propagator generating functions. The time-dependent propagators are also presented. The derivation of time dependent first-passage probabilities and mean first-passage times using the propagator expressions form Section III. In Section IV we treat the problem in higher dimensions, using a hierarchical procedure to obtain BLRW propagators in arbitrary dimensions and arbitrary boundary conditions. Using these results we derive the MFPT in d-dimension with reflecting boundaries (d-box), periodic boundaries (d-torus) or a mixture of periodic and reflecting boundaries. Lastly, a summary of the findings are presented in Section V.

II. TIME DEPENDENT PROPAGATORS IN ONE DIMENSION

We start by considering the dynamics of a random walker with bias on a 1D infinite lattice. It is conveniently described by utilising two parameters $q$ and $g$. The parameter $q$ controls the ‘diffusivity’, with $q = 0$ representing a walker that never moves, while $q = 1$ a walker that moves at each time step. We take the probability of jumping to the neighbouring site on the left as $\frac{q}{2} (1 + g)$, while the probability of jumping to the right as $\frac{q}{2} (1 - g)$, and $1 - q$ as the probability of not moving. The parameter $g$ controls the strength of the bias. When $g = 0$, the movement is diffusive, whereas the cases $g = 1$ and $g = -1$ are, respectively, the ballistic limit to the left and right. The dynamics are governed by the Master equation

$$P(n, t + 1) = (1 - q)P(n, t) + \frac{q}{2} (1 - g)P(n - 1, t) + \frac{q}{2} (1 + g)P(n + 1, t),$$

with $n$ representing the lattice site and $t$ the discrete time variable. The solution of equation (1) can be obtained by Fourier transforming, $\tilde{P}(\kappa, t) = \sum_{n=-\infty}^{\infty} P(n, t)e^{-\text{i}\kappa n}$, subsequently by finding the generating function and finally by inverse transforming to real space to obtain

$$\tilde{P}_{n_0}(n, z) = \frac{\eta f^{\frac{n-n_0}{2}} \alpha^{-|n-n_0|}}{zq \sinh \left[ \frac{\text{acosh} \left( \frac{\eta}{\beta} \right)}{2} \right]},$$

with $\tilde{P}(n, z) = \sum_{n=0}^{\infty} P(n, t)z^n$ and with $n_0$ indicating the localised initial condition $P(n, 0) = \delta_{n, n_0}$, where $\delta$ is a Kronecker delta. For convenience we have employed the following notation:

$$f = \frac{1 - q}{1 + g}, \quad \eta = \frac{1 + f}{2\sqrt{f}},$$

$$\beta = \frac{zq}{1 - z (1 - q)}, \quad \alpha = \exp \left[ \text{acosh} \left( \frac{\eta}{\beta} \right) \right].$$

and the subscript notation $P_{n_0}$ to denote a Kronecker delta initial condition. The absence of a bias, that is $g \to 0$, implies that $f, \eta \to 1$, and one recovers the expression of the propagator of the so-called lazy lattice walker [22], that is a Polya’s walk where the walker may also stay put at each time step.

A. Symmetrisation Procedure in Presence of Boundaries

When imposing boundary conditions, the method of images is an intuitive and effective technique to solve the Master equation. However, when the dynamics are spatially asymmetric, the method breaks down. If one wishes to employ it, the Master equation needs to be made symmetric first. This can be accomplished using a technique used originally by Montroll [23]. That technique was used to construct the propagator for a biased continuous-time random walk in presence of a single boundary. Here we extend that technique to multiple boundaries. Applying the transformation

$$Q(n, t) = f^{-\frac{z}{2}}P(n, t)\omega - \mu f^{-\frac{z+1}{2}}P(n + 1, t)\omega,$$

to equation (1), or applying its equivalent in z-domain

$$\tilde{Q}(n, z) = f^{-\frac{z}{2}}\tilde{P}(n, z\omega) - \mu f^{-\frac{z+1}{2}}\tilde{P}(n + 1, z\omega),$$

where $\mu \geq 0$ and $\omega^{-1} = 1 - q + \frac{q}{g}$, results in a symmetrised dynamics given by

$$Q(n, t + 1) = \omega (1 - q) Q(n, t) + \frac{q\omega}{2\eta} \left[ Q(n - 1, t) + Q(n + 1, t) \right].$$

To transform back from the symmetric probability $\tilde{Q}(n, z)$ to the original $\tilde{P}(n, z)$, one exploits the recursive nature of transformation [5] to write

$$\tilde{P}(n, z) = f^{\frac{z}{2}} \sum_{m=0}^{\infty} \mu^j \tilde{Q} \left( m + j, \frac{z}{\omega} \right),$$

where $\tilde{Q}(n, z)$ is the general solution to equation (6) in z-domain. The corresponding initial condition of $Q(n, t)$ is related to that of $P(n, t)$ via $Q(n, 0) = f^{-\frac{z}{2}}P(n, 0) - \mu f^{-\frac{z+1}{2}}P(n, 0)$. The general solution of equation (6) is given by

$$\tilde{Q}(n, z) = \sum_{m=-\infty}^{\infty} Q(m, 0)\tilde{H}_m(n, z).$$

where [22]

$$\tilde{H}_m(n, z) = \frac{\eta f^{-|n-n_0|}}{\frac{zq}{\omega q} \sinh \left[ \frac{\text{acosh} \left( \frac{\eta}{\beta} \right)}{2} \right]}. $$

is the propagator of equation [6] and with

$$\zeta = \frac{zq}{\eta [1 - z\omega (1 - q)]}$$

and

$$\varphi = \exp \left[ \text{acosh} \left( \frac{\eta}{\beta} \right) \right].$$
Using the symmetric solution [4], it becomes possible to apply the method of images for various types of boundary conditions. In the following sections, to distinguish the different cases, we use the calligraphic notation, i.e. $P^{(n)}$ and $Q^{(n)}$, for semi-bounded domains, and $P^{(r)}$ and $Q^{(r)}$ for finite domains where $\gamma = a, r, m, p$ represents, respectively, absorbing, reflecting, mixed (one reflecting and one absorbing) and periodic boundary conditions. The unbounded occupation probability is represented by $P$ and $Q$ without any superscript $\gamma$.

B. Semi-bounded Propagators

For semi-infinite domains we consider bias random walks on $\mathbb{Z}^+$. The two straightforward types of boundary conditions that one can impose are a single reflection and a single absorption; they are pictorially represented in figure 1. In both of these cases, the semi-bounded propagator is constructed as a superposition of two unbounded propagators. For a single absorbing boundary at $n = 1$, the requirement $P(1, z) = 0$ corresponds, in the symmetric propagator, to $\tilde{Q}(1, z) = 0$ and with $\mu = 0$. The boundary condition is satisfied using a single mirror image giving the general solution $\tilde{Q}^{(a)}(n, z) = \sum_{m=1}^{\infty} \tilde{Q}^{(a)}(m, 0) \left[ \tilde{H}_m(n, z) - \tilde{H}_{-m}(n, z) \right]$, where the spatial convolution is over the semi-infinite domain and where $\tilde{Q}^{(a)}(m, 0)$ is the initial condition after symmetrisation, that is obtained from equation (4) when $t = 0$. For an initial condition $P^{(a)}(n, 0) = \delta_{n,n_0}$ the propagator with a single absorbing boundary at site $n = 1$ is

$$\tilde{P}^{(a)}_{n_0}(n, z) = \frac{\eta f^{n-n_0}}{zq \sinh \left[ \frac{\beta}{\alpha} \cos h \left( \frac{\beta}{\alpha} \right) \right]} \left( \alpha^{-|n-n_0|} - \alpha^{-|n+n_0+2|} \right).$$

(12)

A reflective boundary condition on the asymmetric propagator requires the flux across the boundary to be zero. With the boundary between site $n = 0$ and $n = 1$, the zero flux condition is given by $f \tilde{P}(0, z) - \tilde{P}(1, z) = 0$. The corresponding conditions on the symmetric propagator are $\mu = f^{-\frac{1}{2}}$ and $\tilde{Q}(0, z) = 0$. With $\mu \neq 0$, under the transformation [5], the space between the lattice in the $P$ domain becomes sites in the $Q$ domain and vice versa. The zero flux boundary condition is transformed into an absorbing one that is satisfied using a single image and a symmetrised initial condition $\tilde{Q}^{(a)}(n, 0)$, i.e. $\tilde{Q}^{(a)}(n, z) = \sum_{m=0}^{\infty} \tilde{Q}^{(a)}(m, 0) \left[ \tilde{H}_m(n, z) - \tilde{H}_{-m}(n, z) \right]$, where once again $\tilde{Q}^{(a)}(m, 0)$ is the initial condition obtained from equation (4) when $t = 0$. Transforming back to the original propagator using equation (7) is quite involved and key steps are given in Appendix A.1. For an initial condition $P^{(r)}(n, 0) = \delta_{n,n_0}$ the propagator with a single reflective boundary between sites $n = 0$ and $n = 1$ is given by

$$\tilde{P}^{(r)}_{n_0}(n, z) = \frac{\eta f^{n-n_0}}{zq \sinh \left[ \frac{\alpha}{\beta} \cos h \left( \frac{\alpha}{\beta} \right) \right]} \left( \alpha^{-|n-n_0|} - \alpha^{-|n+n_0+1|} \right),$$

(13)

where

$$\xi = \frac{j^2 - \alpha}{j^2 - \alpha^{-1}}$$

(14)

In figure 1, we display pictorially the two transformations in the absorbing and reflecting cases, both leading to an absorbing boundary condition in the symmetrised case.

C. Bounded Propagators

Having studied propagators on a semi-infinite domain we now turn to random walks on the finite 1D lattice $1 \leq n \leq N$. We start with the simplest of these cases, a finite domain with two absorbing walls. In addition to the absorbing site $n = 1$ we have an absorbing boundary at $n = N$ giving the further constraint $\tilde{P}(N, z) = 0$, which corresponds to the condition $\tilde{Q}(N, z) = 0$ and $\mu = 0$ in equation [4]. In this case, the bounded solution is constructed with infinite images of the unbounded propagators and the convolution is only over sites within the domain,

$$\tilde{Q}^{(a)}(n, z) = \sum_{m=1}^{N} \sum_{k=-\infty}^{+\infty} \tilde{Q}^{(a)}(m, 0) \left[ \tilde{H}_{m-2k(N-1)}(n, z) - \tilde{H}_{-m-2k(N-1)}(n, z) \right],$$

where $\tilde{Q}^{(a)}(m, 0)$ is the symmetrised initial condition in a finite domain obtained from equation (4). With a localised initial condition, $P^{(a)}(n, 0) = \delta_{n,n_0}$, after computing the double summation the propagator with two absorbing boundaries is

$$\tilde{P}^{(a)}_{n_0}(n, z) = \frac{\eta f^{n-n_0}}{zq \sinh \left[ \frac{\alpha}{\beta} \cos h \left( \frac{\alpha}{\beta} \right) \right]} \left\{ \frac{2 \sinh \left[ (N-n_>) \cos h \left( \frac{\alpha}{\beta} \right) \sinh \left[ (n_>-1) \cos h \left( \frac{\alpha}{\beta} \right) \right] \right]}{\sinh \left[ (N-1) \cos h \left( \frac{\alpha}{\beta} \right) \right]} \right\},$$

(15)

where we use the notation $n_> = \frac{1}{2} (n + n_0 + |n-n_0|)$ and $n_- = \frac{1}{2} (n + n_0 - |n-n_0|)$.

For two reflective boundaries we consider a domain with two impenetrable barriers: the first between the sites $n = 0$ and $n = 1$, the second between the sites $n = N$ and $n = N + 1$, thus imposing the constraints $f \tilde{P}(0, z) - \tilde{P}(1, z) = 0$ and $f \tilde{P}(N, z) - \tilde{P}(N+1, z) = 0$, respectively. With the choice $\mu = f^{-\frac{1}{2}}$ in equation (5), these constraints correspond to the conditions $\tilde{Q}(0, z) = 0$ and $\tilde{Q}(N, z)$ on the symmetric propagator. We follow the same procedure as the absorbing case by constructing the bounded solution with infinite images of the unbounded propagator, $\tilde{Q}^{(a)}(n, z) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{+\infty} \tilde{Q}^{(a)}(m, 0) \left[ \tilde{H}_{m+2kN}(n, z) - \tilde{H}_{-m-2kN}(n, z) \right]$, where once again $\tilde{Q}^{(a)}(m, 0)$ is the initial condition after symmetrisation from equation (4) (for a full derivation see Appendix A.2). With the initial condition $P^{(r)}(n, 0) = \delta_{n,n_0}$, the
barrier between is the first-passage probability of being at site \( n \) a single reflective boundary given in equation (13), and construct the propagator by considering the probability of being at site \( n \) above the boundary \( n \) and having not visited the boundary site \( n \). Under the transformation, both the reflective boundary between sites \( n \) absorbing sites. A reflecting boundary is a constraint imposed on two sites: with the barrier between the sites \( n \) absorbing boundary at \( n \). In so doing the probability of not moving at these sites becomes 1 resulting propagator with two reflective boundaries is

\[
\tilde{P}^{(r)}_{n_0}(n, z) = \frac{f^{n-n_0-1}}{z} \left\{ f^{\frac{n}{2}} \sinh \left[ (N-n) \cosh \left( \frac{z}{\beta} \right) \right] - \sinh \left[ (N+1-n) \cosh \left( \frac{z}{\beta} \right) \right] \right\} \\
\times \left\{ f^{\frac{n}{2}} \sin \left[ \eta \cosh \left( \frac{\eta}{\beta} \right) \right] - \sin \left[ (n-1) \cosh \left( \frac{\eta}{\beta} \right) \right] \right\}. \tag{16}
\]

For the mixed boundary condition (reflecting between \( n = 0 \) and \( n = 1 \) and absorbing at \( n = N \)) we take the propagator with a single reflective boundary given in equation (13), and construct the propagator by considering the probability of being at site \( n \) and having not visited the boundary site \( N \). \( \tilde{P}^{(m)}_{n_0}(n, z) = \tilde{P}^{(r)}_{n_0}(n, z) - \sum_{t=0}^{N} \tilde{F}^{(r)}_{n_0}(N, t') P^{(r)}_{N}(n, t-t') \), where \( \tilde{F}^{(r)}_{n_0}(N, t) \) is the first-passage probability of being at site \( n \) at time \( t \) for a walker that started at site \( n_0 \) in a lattice with an impenetrable barrier between \( n = 0 \) and \( n = 1 \). In \( z \)-domain the relation is simply \( \tilde{F}^{(m)}_{n_0}(n, z) = \tilde{P}^{(r)}_{n_0}(n, z) - \tilde{F}^{(r)}_{n_0}(N, z) \tilde{P}^{(r)}_{N}(n, z) \) where \( \tilde{F}^{(r)}_{n_0}(N, z) \) can be found in equation (17). After some algebra one finds the expression

\[
\tilde{P}^{(m)}_{n_0}(n, z) = \frac{2f^{n-n_0}q \sinh \left[ (N-n) \cosh \left( \frac{\eta}{\beta} \right) \right] \left\{ f^{\frac{n}{2}} \sin \left[ n \cosh \left( \frac{\eta}{\beta} \right) \right] - \sin \left[ (n-1) \cosh \left( \frac{\eta}{\beta} \right) \right] \right\} }{zq \sin \left[ \cosh \left( \frac{\eta}{\beta} \right) \right] \left\{ f^{\frac{n}{2}} \sin \left[ N \cosh \left( \frac{\eta}{\beta} \right) \right] - \sin \left[ (N-1) \cosh \left( \frac{\eta}{\beta} \right) \right] \right\}}. \tag{17}
\]

Lastly, we consider a biased random walk on a 1D periodic domain with \( N \) distinct sites, which implies that \( \tilde{P}(n, z) = \tilde{P}(n+kN, z) \) for any integer \( k \). To satisfy the boundary condition one simply wraps the unbounded propagator \( \tilde{P} \) via the

---

**FIG. 1.** (Colour Online) Schematic diagram showing the effect of the symmetrising transformation 4 or 5 on a single reflecting (leftmost) or a single absorbing (rightmost) boundary with the lattices displayed vertically. The red circles represent sites adjacent to a reflecting boundary shown as a dashed black line and the green circles are absorbing sites. A reflecting boundary is a constraint imposed on two sites: with the barrier between the sites \( n \) absorbing boundary at \( n \). In so doing the probability of not moving at these sites becomes 1 resulting propagator with two reflective boundaries is

\[
\tilde{P}^{(r)}_{n_0}(n, z) = \frac{f^{n-n_0-1}}{z} \left\{ f^{\frac{n}{2}} \sinh \left[ (N-n) \cosh \left( \frac{z}{\beta} \right) \right] - \sinh \left[ (N+1-n) \cosh \left( \frac{z}{\beta} \right) \right] \right\} \\
\times \left\{ f^{\frac{n}{2}} \sin \left[ \eta \cosh \left( \frac{\eta}{\beta} \right) \right] - \sin \left[ (n-1) \cosh \left( \frac{\eta}{\beta} \right) \right] \right\}. \tag{16}
\]

For the mixed boundary condition (reflecting between \( n = 0 \) and \( n = 1 \) and absorbing at \( n = N \)) we take the propagator with a single reflective boundary given in equation (13), and construct the propagator by considering the probability of being at site \( n \) and having not visited the boundary site \( N \). \( \tilde{P}^{(m)}_{n_0}(n, z) = \tilde{P}^{(r)}_{n_0}(n, z) - \sum_{t=0}^{N} \tilde{F}^{(r)}_{n_0}(N, t') P^{(r)}_{N}(n, t-t') \), where \( \tilde{F}^{(r)}_{n_0}(N, t) \) is the first-passage probability of being at site \( n \) at time \( t \) for a walker that started at site \( n_0 \) in a lattice with an impenetrable barrier between \( n = 0 \) and \( n = 1 \). In \( z \)-domain the relation is simply \( \tilde{F}^{(m)}_{n_0}(n, z) = \tilde{P}^{(r)}_{n_0}(n, z) - \tilde{F}^{(r)}_{n_0}(N, z) \tilde{P}^{(r)}_{N}(n, z) \) where \( \tilde{F}^{(r)}_{n_0}(N, z) \) can be found in equation (17). After some algebra one finds the expression

\[
\tilde{P}^{(m)}_{n_0}(n, z) = \frac{2f^{n-n_0}q \sinh \left[ (N-n) \cosh \left( \frac{\eta}{\beta} \right) \right] \left\{ f^{\frac{n}{2}} \sin \left[ n \cosh \left( \frac{\eta}{\beta} \right) \right] - \sin \left[ (n-1) \cosh \left( \frac{\eta}{\beta} \right) \right] \right\} }{zq \sin \left[ \cosh \left( \frac{\eta}{\beta} \right) \right] \left\{ f^{\frac{n}{2}} \sin \left[ N \cosh \left( \frac{\eta}{\beta} \right) \right] - \sin \left[ (N-1) \cosh \left( \frac{\eta}{\beta} \right) \right] \right\}}. \tag{17}
\]

Lastly, we consider a biased random walk on a 1D periodic domain with \( N \) distinct sites, which implies that \( \tilde{P}(n, z) = \tilde{P}(n+kN, z) \) for any integer \( k \). To satisfy the boundary condition one simply wraps the unbounded propagator \( \tilde{P} \) via the
summation \( \tilde{P}_{n_0}^{(p)}(n, z) = \sum_{k=-\infty}^{\infty} \tilde{P}_{n_0}(n + kN, z) \). Evaluating the sum yields

\[
\tilde{P}_{n_0}^{(p)}(n, z) = \frac{\eta f^{\frac{z-n_0}{z}}}{z q \sinh \left( \frac{\pi n_0}{2} \right) \left( \cosh \left( N \cosh \left( \frac{\pi n_0}{2} \right) \right) - \cosh \left( N \cosh (\eta) \right) \right)} \left( \sinh \left( \left( N - |n - n_0| \right) \cosh \left( \frac{\pi n_0}{2} \right) \right) \right) + f^{\frac{z-n_0}{z}} \sinh \left( |n - n_0| \cosh \left( \frac{\pi n_0}{2} \right) \right),
\]

where \( \text{sgn}(n) \) is the signum function, defined as \( \text{sgn}(n) = -1 \) when \( n < 0 \), \( \text{sgn}(n) = 1 \) when \( n > 0 \), and \( \text{sgn}(n) = 0 \) when \( n = 0 \).

Equations [16], [17] and [18] are not known in the literature, even though expressions similar to equations [16] and [17] can be found in reference [35–39], where the continuous time BLRW was derived using an alternative procedure. This procedure was also used in reference [22] to derive equations [12], [13], and [15], for the discrete time BLRW, but only for the case when \( q = 1 \), that is an always moving walker.

### D. Time Dependent Propagators with Finite Domains

In order to find the time dependence of the propagators one must evaluate the integral (inverse \( z \) transform) \( P_{n_0}^{(\gamma)}(n, t) = (2\pi i)^{-1} \oint \tilde{P}_{n_0}^{(\gamma)}(n, z) z^{-t-1} dz \), with \( |z| < 1 \) and where the integration contour is counterclockwise. Equivalently, one can find time dependent solution more directly by solving the matricial Master equation

\[
\hat{P}(t + 1) = \mathbf{A} \cdot \hat{P}(t),
\]

where

\[
\mathbf{A} = \begin{bmatrix}
1 - q + \varepsilon & \frac{1}{2}(1 + g) & \cdots & \sigma \\
\frac{1}{2}(1 - g) & 1 - q & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{1}{2}(1 - g) & 1 - q + \delta
\end{bmatrix}.
\]

The different types of boundary conditions are accounted for by a relevant size of \( \mathbf{A} \) and appropriately chosen parameters \( \varepsilon, \delta, \sigma \) and \( \nu \); reflective boundaries with \( \varepsilon = \frac{1}{2}(1 + g) \), \( \delta = \frac{1}{2}(1 - g) \), \( \nu = \sigma = 0 \) and \( \mathbf{A}_{N \times N} \); absorbing boundaries with \( \varepsilon = \delta = \nu = \sigma = 0 \) and \( \mathbf{A}_{(N-2) \times (N-2)} \); mixed boundaries with \( \varepsilon = \frac{1}{2}(1 + g) \), \( \delta = \nu = \sigma = 0 \) and \( \mathbf{A}_{(N-1) \times (N-1)} \); and periodic boundaries with \( \nu = \frac{1}{2}(1 + g) \), \( \sigma = \frac{1}{2}(1 - g) \), \( \varepsilon = \delta = 0 \) and \( \mathbf{A}_{N \times N} \). By diagonalising the matrix \( \mathbf{A} \), the solution can be written as \( \hat{P}(t) = \mathbf{LE}'\mathbf{R}\hat{\mathbf{P}}(0) \) where respectively, \( \mathbf{L} \) and \( \mathbf{R} \) are matrices containing the left and right normalised eigenvectors, while \( \mathbf{E} \) is the diagonal matrix of eigenvalues. The spatial dependence is determined by the eigenvectors while the eigenvalues give the time dependence. These eigenvalues and eigenvectors are known explicitly for the absorbing, reflecting and periodic cases [27–39], while for the mixed boundary condition we exploit the properties of Chebyshev polynomials to write a propagator with time dependent coefficients known numerically (see Appendix B for details). To represent the time dependent propagator a convenient notation is

\[
P_{n_0}^{(\gamma)}(n, t) = \sum_{k \in \mathbb{Z}} h_k^{(\gamma)}(n, n_0) \left[ 1 + s_k^{(\gamma)} \right]^t,
\]

where \( w^{(p)} = 0 \) and \( W^{(p)} = N - 1 \) for the periodic case; \( w^{(r)} = 0 \) and \( W^{(r)} = N - 1 \) for the reflecting case; \( w^{(a)} = 1 \) and \( W^{(a)} = N - 2 \) for the absorbing case; and \( w^{(m)} = 1 \) and \( W^{(m)} = N - 1 \) for the mixed case. The time dependence is defined by \( \left[ 1 + s_k^{(\gamma)} \right]^t \) with

\[
s_k^{(\gamma)} = \begin{dcases}
q \cos \left( \frac{k\pi}{N} \right) + i qg \sin \left( \frac{k\pi}{N} \right) - q, & \gamma = p, \\
\frac{2}{\eta} \cos \left( \frac{k\pi}{N} - \frac{\pi}{2} \right) - q, & \gamma = a, \\
\frac{2}{\eta} \cos \left( \theta_k - \frac{k\pi}{N} \right) - q, & \gamma = m, \\
\frac{2}{\eta} \cos \left( \frac{k\pi}{N} - \frac{\pi}{2} \right) - q, & k \neq 0, \\
0, & k = 0,
\end{dcases}
\]

where \( \cos(\theta_k) \) is the \( k \)th root of the orthogonal polynomial \( \frac{1}{2} \sum_{n=0}^{N-1} \cos(n\theta) \) and where \( U_n \) is an \( n \)th order Chebyshev polynomial of the second kind. The spatial dependence in equation [21] is

\[
\text{FIG. 2. (Colour Online) One dimensional propagator with absorbing boundaries at sites 1 and 101. The localised initial condition is at } n_0 = 2 \text{ and the bias and diffusive parameters are, respectively, } g = -0.3 \text{ and } q = 0.8. \text{ Each of the curves represent the probability at different times, with the left most curve being at } t = 40, \text{ the right most being at } t = 280 \text{ and } \Delta t = 40 \text{ between each of them. The dots are from equation [21] with } \gamma = a, \text{ whereas the solid lines are obtained by solving iteratively equation [19].}
\]
probability can be written as equation (D1)) by considering the case when \( n > n \). Using the propagator (16), the generating function of the first-passage probability is written in a compact manner (see to reach a target site \( \gamma \) absorbing boundaries (\( \gamma \)). While the first-passage dynamics of a diffusive walker in

\[
\gamma = p,
\]

\[
\gamma = a,
\]

\[
\gamma = m,
\]

\[
\gamma = r.
\]

In figure 2, we plot the propagator in equation (21) with two absorbing boundaries (\( \gamma = a \)) at \( n = 1 \) and at \( n = N \) and with a negative bias, \( g = -0.3 \). The drift to the right is evident from the movement of the peak of the probability, while the broadening of the overall shape is due to diffusion. Using the 1D propagators in equation (21) it is possible to recover known solutions to the bounded drift-diffusion equation. In Appendix C we outline the limiting procedure to obtain the space-time continuous propagators for the four boundary conditions studied.

III. FIRST-PASSAGE PROCESSES IN ONE DIMENSION

An important quantity in transport calculations, already introduced in Section II C, is the first-passage probability, \( F_{n_0}(n, t) \), to reach a target site \( n \) from site \( n_0 \) at time \( t \). It is directly related to the propagator through the renewal relation in z-domain \( F_{n_0}(n, z) = \tilde{F}_{n_0}(n, z)/\tilde{P}_n(n, z) \). We consider first the reflective domains and subsequently the periodic domain. Using the propagator \( \tilde{F}(n, z) \), the generating function of the first-passage probability is written in a compact manner (see equation (11)) by considering the case when \( n > n_0 \) and vice versa. Through a z-inversion the time dependent first-passage probability can be written as

\[
F^{(r)}_{n_0}(n, t) = \frac{qf^{n-n_0}}{\eta} \sum_{k=1}^{N-n} \sin(\theta_k) \left\{ \frac{f \sin([n-n_0]\theta_k) - \sin([n_0-1]\theta_k)}{N(1-p+\frac{\eta}{2}\cos(\theta_k))} \right\}^{t-1} \quad n > n_0
\]

\[
F^{(r)}_{n_0}(n, t) = \frac{qf^{n-n_0}}{\eta} \sum_{k=1}^{N-n} \sin(\psi_k) \left\{ \frac{f \sin([n-n_0]\psi_k) - \sin([n_1-n_0]\psi_k)}{N(1-p+\frac{\eta}{2}\cos(\psi_k))} \right\}^{t-1} \quad n < n_0,
\]

with \( F^{(r)}_{n_0}(n, 0) = 0 \), and where \( \cos(\theta_k) \) and \( \cos(\psi_k) \) are, respectively, the \( k \)th roots of the orthogonal polynomial \( f + U_{n-1} \cos(\theta_k) - U_{n-2} \cos(\theta_k) \) and \( f + U_{n-1-n} \cos(\psi_k) - U_{n-2-n} \cos(\psi_k) \) with \( U_{-1} = 0 \) for the periodic case a similar procedure gives a compact expression in equation (12) by treating \( n > n_0 \) and \( n < n_0 \) separately. Using the signum function the time dependence can be written conveniently as the following single expression:

\[
F^{(p)}_{n_0}(n, t) = \frac{qf^{n-n_0}}{\eta N} \sum_{k=1}^{N-1} (-1)^{k+1} \sin \left( \frac{k\pi}{N} \right) \left\{ \sin \left[ (N-|n-n_0|) \frac{k\pi}{N} \right] \right\}^{t-1} + \sin \left[ |n-n_0| \frac{k\pi}{N} \right] f^{n-n_0} \left\{ 1 - q + \frac{\eta}{2} \cos \left( \frac{k\pi}{N} \right) \right\}^{t-1},
\]

with \( F^{(p)}_{n_0}(n, 0) = 0 \).

In the case of periodic domains, an interesting feature is the appearance of two peaks in the first-passage probability. While the first-passage dynamics of a diffusive walker in a periodic domain is monomodal, in the presence of a bias one can find bimodal features. To display these features we plot \( F^{(p)}_{n_0}(n, t) \) in figure 3. The panels (a)-(d) depict the first-
passage probability to the same target at \( n = 4 \) starting from \( n_0 = 2 \) but each panel represents a stronger bias from left to right. The panels (e)-(h) have the same bias, \( g = 0.35 \), but the target locations are displaced away from the starting site \( n_0 = 2 \). In the absence of a bias, i.e. figure 3, one finds a monomodal probability function characteristic of diffusive processes. As the bias is increased to positive values (first row), the walker is more likely to travel leftwards taking the longer route to reach the target (via the site \( Nf \)) and opposed to the direction of the bias, the first peak is gradually lost while the second becomes more prominent. With the target site close to \( n_0 \), the distance to travel against the bias to reach \( n \) is small enough such that there is still a high probability of reaching \( n \) from \( n_0 \) without visiting \( N \). As one increases the distance between \( n_0 \) and \( n \), with \( n > n_0 \), the likelihood of a walker travelling this distance against the bias decreases resulting in the progressive loss of the first peak.

\[ T_{n_0 \to n}^{(p)} = \frac{1}{q} \frac{(f+1)}{(f-1)(f^N-1)} \left\{ (n-n_0)(f^N-1) + Nf_{\frac{f}{2}}[1-\text{sgn}(n-n_0)] \right\} \]  

(27)

In the left (right) ballistic limit, that is \( q \to 1 \) and \( f \to 0 \) (\( f \to \infty \)) of equation (26), we find the MFPT to be \(|n-n_0|\) if the target is in the direction of the bias or infinite if the target is against the bias. On the other hand, the MFPT with periodic boundaries in equation (27) will always be finite: with \( n > n_0 \) in the left ballistic limit \( T_{n_0 \to n}^{(p)} = N - |n-n_0| \), while in the right ballistic limit \( T_{n_0 \to n}^{(p)} = |n-n_0| \) and vice versa. In the diffusive limit, i.e. \( f \to 1 \), equations (26) and (27) reduce to [22],

\[ T_{n_0 \to n}^{(r)} = \frac{1}{q} \left[ N |n-n_0| + (n-n_0)(n+n_0-1-N) \right] \]

and

\[ T_{n_0 \to n}^{(p)} = \frac{1}{q} \left( N - |n-n_0| \right) |n-n_0|, \]

respectively.

IV. DYNAMICS IN HIGHER DIMENSIONS

To find propagators in higher dimensions we need both the series solution and the compact solution from the method of images. The procedure for finding propagators in higher dimension is a slight variation of the eight step method introduced by one of the present authors [22]. To illustrate this new procedure we first present the case of a walker in a 2D domain with reflective boundary conditions.
A. Two dimensional propagator with reflective boundaries

We start by considering the dynamics of a walker on a 2D lattice that is bounded along the first dimension whilst unbounded in the second. The probability of stepping left or right along the first dimension are, respectively, \( \frac{q_1}{4} \) and \( \frac{q_2}{4} \). Similarly stepping left or right along the second dimension are, respectively, \( \frac{q_1}{4} \) and \( \frac{q_2}{4} \). In the bulk of the domain, the probability of remaining at a site is \( 1 - \frac{q_1}{2} - \frac{q_2}{2} \), while along the left (or right) boundary at \( n_1 = 1 \) (or at \( n_1 = N_1 \)), is \( 1 - \frac{q_1}{4} (1 - g_1) - \frac{q_2}{4} (1 + g_2) \) (or \( 1 - \frac{q_1}{4} (1 + g_1) - \frac{q_2}{4} \)). The dynamics in the bulk of the domain are governed by the Master equation

\[
P(n_1, n_2, t+1) = \left[ 1 - \frac{q_1}{2} - \frac{q_2}{2} \right] P(n_1, n_2, t)
+ \frac{q_1}{4} \left( (1-g_1) P(n_1-1, n_2, t) + (1+g_1) P(n_1+1, n_2, t) \right)
+ \frac{q_2}{4} \left( (1-g_2) P(n_1, n_2-1, t) + (1+g_2) P(n_1, n_2+1, t) \right),
\]

along the left boundary by

\[
P(1, n_2, t+1) = \left[ 1 - \frac{q_1}{4} (1-g_1) - \frac{q_2}{2} \right] P(1, n_2, t)
+ \frac{q_1}{4} \left( (1-g_2) P(1, n_2-1, t) + (1+g_2) P(1, n_2+1, t) \right)
+ \frac{q_1}{4} (1+g_1) P(2, n_2, t),
\]

and along the right boundary by

\[
P(N_1, n_2, t+1) = \left[ 1 - \frac{q_1}{4} (1+g_1) - \frac{q_2}{2} \right] P(N_1, n_2, t)
+ \frac{q_1}{4} \left( (1-g_2) P(N_1-1, n_2, t) + (1+g_2) P(N_1, n_2+1, t) \right)
+ \frac{q_1}{4} (1-g_1) P(N_1-1, n_2, t),
\]

Note that \( \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} P(n_1, n_2, t) = 1 \), which indicates that equations (28), (29) and (30) represent a probability preserving Master equation.

Symmetrising the dynamics and Fourier transforming along the second dimension results in an effective 1D problem, analogous to equation [19],

\[
\hat{Q}(n_1, \kappa_2, t+1) = \sum_{\ell=1}^{N_1} B_{n_1, \ell} \hat{Q}(\ell, \kappa_2, t)
\]

where \( B \) is a tridiagonal matrix with elements on the upper and lower diagonal being, respectively, \( \frac{q_1}{2} \) and \( \frac{q_2}{2} \) (1 + \( g_1 \)) \( \omega^\ell = 1 - \frac{q_1}{2} + \frac{q_2}{2} \). The elements along the diagonal are \( B_{\ell, \ell} = \omega \left[ 1 - \frac{q_1}{2} (1 + g_1) - \frac{q_2}{2} \frac{\cos(\kappa_2)}{2} \right] \), when \( \ell \neq 1, N_1 \), \( B_{1,1} = \omega \left[ 1 - \frac{q_1}{4} (1 - g_1) - \frac{q_2}{2} + \frac{q_2}{2} \frac{\cos(\kappa_2)}{2} \right] \), \( B_{N_1, N_1} = \omega \left[ 1 - \frac{q_1}{4} (1 + g_1) - \frac{q_2}{2} + \frac{q_2}{2} \frac{\cos(\kappa_2)}{2} \right] \). After supplementing the initial conditions \( \hat{Q}(n_1, \kappa_2, 0) = \delta_{n_1,n_0} e^{-ikn_0} (1 - f_2^{-1}) \), equation (31), due to the comparable structure with equation [19], can be solved explicitly in \( z \) and Fourier domains. Subsequently, inverse Fourier transforming the second dimension, applying the method of images, reversing to the asymmetric propagator and finally, after inverse \( z \) transforming, one obtains the exact spatio-temporal dependence (the calculation is outlined in Appendix E). Knowledge of the identity equation (G1), allows us to write the time dependent solution to the 2D random walks with independent bias in each dimension as

\[
P_{\tilde{n}_0, \tilde{n}_2}(n_1,n_2,t) = \lambda_1 \lambda_2^{N_1-1} \sum_{k_1=1}^{N_2-1} \sum_{k_2=1}^{N_2} h^{(r_1)}_{k_1}(n_1,n_0) h^{(r_2)}_{k_2}(n_2,n_0) \left[ 1 + \frac{s_{k_1}^{(r_1)}}{2} + \frac{s_{k_2}^{(r_2)}}{2} \right]^t
+ \lambda_1 \sum_{k_2=1}^{N_2-1} \sum_{k_1=1}^{N_1} k^{(r_2)}_{k_2}(n_2,n_0) \left[ 1 + \frac{s_{k_2}^{(r_2)}}{2} \right]^t
+ \lambda_2 \sum_{k_1=1}^{N_1-1} \sum_{k_2=1}^{N_2} k^{(r_1)}_{k_1}(n_1,n_0) \left[ 1 + \frac{s_{k_1}^{(r_1)}}{2} \right]^t,
\]

where

\[
\lambda_i = \frac{f_i^{n_1-1} (1 - f_i)}{1 - f_i^{N_1}}.
\]

A similar procedure can be applied for the case of the absorbing, periodic and mixed boundary conditions, although in the latter case, as mentioned earlier, the analytic solution is not fully explicit, but based on the numerical roots of the orthogonal polynomials of the form \( f^z U_{s-1}(x) - U_{s-2}(x) \).

In figure 4, we plot \( P_{\tilde{n}_0, \tilde{n}_2}(n_1,n_2,t) \) for a specific time value with the left-downward bias \( \tilde{\eta} = (0.1, 0.1) \). A feature worth pointing out is the appearance of two saddle points that emerge at intermediate times. They appear due to the
steady-state probability at the boundary being higher than the transient peak. In 1D this results in the appearance of a local minimum.

B. Propagator in Arbitrary dimensions and Arbitrary Boundary Conditions

We use a hierarchical procedure to construct bias lattice walk propagators of any dimensions by generalising the procedure used to derive the 2D random walk propagator equation (32) (the summary of the procedure can be found in Appendix F). The resulting analytic propagators are

\[
P_{\vec{g}_{\vec{n}}}^{(\gamma)}(\vec{n}, t) = \sum_{k_1=W^{(\gamma)}} W^{(\gamma_2)} \cdots \prod_{j=1}^{d} h^{(\gamma_j)}(n_j, n_{0_j}) \times \left[ 1 + \frac{s_{k_1}}{d} + \cdots + \frac{s_{k_d}}{d} \right]^t, \tag{33}
\]

with \(s^{(\gamma)}_k\) and \(h^{(\gamma)}_k(n, n_{0})\) defined, respectively, in equations (22) and (23), and with \(W^{(\gamma)}\) and \(W^{(\gamma_2)}\) defined after equation (21). Using equation (33), one can derive first-passage (or first-return) probability and mean-first passage times in higher dimensions with an arbitrary combination of reflecting and periodic boundaries which were previously unknown. Such expressions enable one to study transport processes that were, until now, only possible through numerical means. In the following subsections we employ equation (33) to reveal an intricate bias dependence on the time dependent first-return probability, and we study the effect of bias on the mean first-passage times in a multi-target environment.

1. First-Return Processes in Higher Dimensions

A useful quantity in studying search processes is the probability of the first recurrence of an event, that is the probability of a lattice walker returning to the starting location for the first time. The first-return probability, and henceforth, the return probability, is derived via the renewal equation and in z-domain it is given by \(\tilde{R}^{(\gamma)}(\vec{n}, z) = 1 - \left[ \tilde{P}^{(\gamma)}(\vec{n}, z) \right]^{-1}\). The study of the return probability on lattice random walks has a long history [11, 42]. Used originally for unbounded \(d\)-dimensional lattices where it is found that a walker returns with certainty to the starting location in 1D and 2D while for higher dimensions there is a finite probability that the walker does not return. Although the walker is bound to return to its initial position in unbounded 1D and 2D domains, the mean return time (MRT) is always infinite. In bounded domains, on the other hand, the MRT is finite and is equal to the reciprocal of the steady-state probability at the site [19]. For a LRW (without bias) the steady-state probability is uniform and the MRT reduces to the domain size. In the presence of a non uniform steady-state, as is the case with BLRW with reflecting boundaries, the MRT, \(R_{\vec{n}}^{(\gamma)}\), becomes site-dependent and the return dynamics may be rather complex. Namely, given an off-centre lattice site, one finds the MRT to be minimised for a bias with a specific direction (see Appendix H).

However, the MRT may hide the nuances of the temporal dynamics. In order to examine the dynamics of the return probability, we use the starting location \(\vec{n} = (5, 18)\), and track the return probability, \(R^{(\gamma)}(\vec{n}, t)\), at different times. We do this in figure 5 by plotting \(R^{(\gamma)}(\vec{n}, t)\) as a function \(\vec{g}\). We use known numerical methods [43] to invert the generating function and plot in each panel the return probability for progressively longer times from (a) to (e).

At short times (figure 5a) one finds the return probability to be independent of the bias direction as any bias pushes the walker away from the starting location lowering the likelihood of return. With \(t = 10^2\) in panel (b) we observe greater return probabilities for certain values of \(g_1 > 0\) and \(g_2 < 0\). Since the time \(t\) is comparable to the shortest MRT (see Appendix H), one expects the likelihood of returning at \(t = 10^2\) to be greater for the bias that yields the shortest MRT. A further increase in time (\(t = 10^3\)) results in the appearance of a void. The void represents an area around a local minimum of \(R^{(\gamma)}(\vec{n}, t)\). Its appearance indicates that a large number of the trajectories for which the bias has values inside, have already returned when compared to those with bias outside of the void. Moreover, as the time scale is considerably larger than the one corresponding to the minimum of \(R^{(\gamma)}_{\vec{n}}\), there is no optimal bias to return. We thus observe an arched area of high return probability in panel (c) compared to the area around a maximum in panel (b). Increasing time further in panels (d) and (e) results in the expansion of the void as stronger biases are necessary to increase the probability of returning at longer times. One also observes the radial stretching of the area of high return probability as the dependence on the bias direction is progressively lost. In particular, the dependence on the bias direction is progressively lost. In particular, the dependence on the bias direction is progressively lost. In particular, the dependence on the bias direction is progressively lost. In particular, the dependence on the bias direction is progressively lost. In particular, the dependence on the bias direction is progressively lost.
\( g = p \); where \( s_k^{(\gamma)} \) and \( h_k^{(\gamma)}(n,n_0) \) are defined, respectively, in equation \((22)\) and \((23)\); and with \( W^{(\gamma)} \) defined after equation \((21)\).

The first-passage dynamics becomes very rich in the presence of multiple targets as the bias towards a specific target influences dramatically the time it takes to reach either of the targets. We show this dependence by plotting in figure 6 the MFPT to either of three targets as a function of the position of the first target in a 2D box with reflective boundaries. We use equation \((34)\) and the MFPT expression to either of three targets from reference \((22)\). Figure 5(a) depicts the schematic diagram of the lattice, the biases and the position of the targets. The position of the targets \( \vec{n}_2 \) and \( \vec{n}_3 \) are fixed while the position of the first target \( \vec{n}_1 = (m,m) \) is slid along the diagonal.

The bias \( \vec{g}_1 \) shows the least dependence on the position of the first target. With the bias \( \vec{g}_1 \), the walker always has a high probability of reaching the second or third target regardless of the position of the first. The shorter MFPT in this case occurs when the line connecting \( \vec{n}_0 \) to \( \vec{n}_1 \) is parallel to \( \vec{g}_1 \), i.e. when \( m = 11 \). The diffuse case, \( \vec{g}_2 \), shows a slightly stronger dependence on \( m \) than \( \vec{g}_1 \), with its shortest MFPT value being attributed to when the first target is closer to \( \vec{n}_0 \), that is when \( m = 14 \). As the direction of the bias \( \vec{g}_3 \) is away from the targets, the MFPT decreases as the first target is moved. It gives the minimum MFPT in correspondence to the shortest distance that the walker travels against the bias to reach \( \vec{n}_1 \).

With \( \vec{g}_2 \) and \( \vec{g}_4 \) being opposite to each other and parallel to any of the positions of the first target, the corresponding MFPT displays similar characteristics. When \( m = 1 \) (\( m = 41 \)), corresponding with the first target located in the bottom-left (top-right) corner, the bias \( \vec{g}_4 \) (\( \vec{g}_2 \)) exhibits the shortest MFPT due to the bias pushing trajectories towards the corner. As \( m \) is increased from \( m = 1 \), the MFPT of \( \vec{g}_4 \) increases due to \( \vec{n}_1 \) moving out of the bottom-left corner. Analogously, as \( m \) is decreased from \( m = 41 \), \( \vec{n}_1 \) moves out of the top-right corner causing the MFPT of \( \vec{g}_2 \) to increase. The difference in the high values of the MFPT of \( \vec{g}_4 \) when \( m > 30 \), and the MFPT of \( \vec{g}_2 \) when \( m < 10 \), is due to the position of the second and third targets. With \( \vec{n}_2 \) and \( \vec{n}_3 \) being closer to the bottom-left than the top-right corner, one expects the largest MFPT of \( \vec{g}_4 \) to be smaller than the largest MFPT of \( \vec{g}_2 \), and vice versa for the shortest MFPT.

An interesting observation is that when the first target is positioned in the top-right corner with \( m > 35 \), the MFPT of \( \vec{g}_2 \) and \( \vec{g}_1 \) are comparable. It highlights the strong dependence of the MFPT on the positioning of the targets relative to the boundary corner. In the presence of a bias, one can achieve shorter or similar MFPTs by positioning fewer targets close to the corner and in the direction of the bias (\( \vec{g}_2 \) case with high \( m \)) as opposed to many targets away from it (\( \vec{g}_1 \) case).

Our observations are particularly relevant in the domain of field-driven translocation in channels with periodic corrugation. Here, one is interested in the first-passage times of tracer particle moving under an external bias. Recent numerical analysis [21], reveal that particles travel close to the boundaries as they pass through a funnel. For further work, it would be interesting to reaffirm such results, by studying the MFPT using a similar setup to figure 5, but with targets concentrated in the corner and by changing the initial position instead of a target position. With the two orthogonal boundaries acting like a funnel one expects similar results to those observed numerically.

V. CONCLUSIONS

We conclude, by reminding that while the continuous-time BLRW in confined domains has been studied extensively in 1D [35, 36], a thorough treatment of the analogous discrete time case was missing from the literature and only the propagator for the case \( q = 1 \) and with absorbing boundaries was known [32]. Compared to reference [32], here we have derived the generating function of the 1D propagator in finite domains by employing an alternative procedure yielding both finite series and compact expressions, with the latter used to create the generating function for the first-passage probability in reflecting and periodic domains. In order to find the time-dependent propagators, we have used known results for tridiagonal matrices with perturbed corners [39, 44] instead of inverting the generating function of the propagators via a contour integral. From the finite series time-dependent propagators we have recovered the known solutions to the
FIG. 6. (Colour Online) MFPT in a 2D domain of size $\vec{N} = (41, 41)$ with reflective boundary conditions to either of three targets. The walker is initially at $\vec{n}_0 = (21, 1)$ with diffusion parameter $\vec{q} = (0.8, 0.8)$. The coordinates of the first target are $\vec{n}_1 = (m, m)$ with $1 \leq m \leq 41$, while that of the second and third targets are static with respective positions $\vec{n}_2 = (1, 31)$ and $\vec{n}_3 = (6, 26)$. The different biases considered are $\vec{g}_1 = (0.1, -0.1)$, $\vec{g}_2 = (-0.1, -0.1)$, $\vec{g}_3 = (-0.1, 0.1)$, $\vec{g}_4 = (0.1, 0.1)$ and the fully diffusive case $\vec{g}_5 = (0, 0)$. The panel (a) shows a schematic diagram of the setup with the positions of the targets, the initial condition and the directions of the different biases. In panel (b) for each of the bias we plot the MFPT as a function of the position of the third target.

drift-diffusion equation, linking the movement parameters of a BLRW with the drift velocity and the diffusion coefficient for a Brownian walker.

By exploiting the properties of Chebyshev polynomials, the generating function of the first-passage probability with periodic and reflective boundaries was inverted explicitly to yield the exact time dependence. Surprisingly, the periodic case was shown to display bimodal features when its analog without bias is known to be monomodal. Lastly, by employing a hierarchical dimensional reduction we have derived time-dependent propagators for the confined BLRW in any number of dimensions and with arbitrary boundary conditions. The propagators were then used to find explicit expressions for the mean first-passage time in a $d$-dimensional box, torus or an arbitrary combination of both. The generating function of the propagators have also highlighted the influence a bias may have on the time dependence of the return probability.

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Chapter 5: Probability Theory

Appendix A: Derivation of Propagators in 1D with Reflective Boundaries

1. Single Reflective Boundary

The Kronecker delta initial condition for the propagator \( \mathcal{P}^{(r)}_{n,0}(n,0) = \delta_{n,0} \) gives an initial condition for the symmetric propagator with \( \mu = f^{2} \) in equation [4] equal to \( \mathcal{Q}^{(r)}(n,0) = f^{2} \delta_{n,0} - f^{-2} \delta_{n+1,0} \). Convoluting this initial condition with the symmetric propagator \([9]\) and accounting for the contribution of the image of the initial condition via \( \mathcal{Q}^{(r)}(n,z) = \)?

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∑_{m=0}^{∞} Q^{(a)}(m, 0) \left[ \tilde{H}_m(n, z) - \tilde{H}_{-m}(n, z) \right] \text{ gives }
\bar{Q}^{(a)}(n, z) = \frac{\varphi^{-|n-n_0|} - \varphi^{-|n+n_0|}}{[1 - z \omega (1 - q)] \sqrt{1 - \zeta^2}} - f^{-\frac{1}{2}} \frac{\varphi^{-|n-n_0+1|} - \varphi^{-|n+n_0-1|}}{[1 - z \omega (1 - q)] \sqrt{1 - \zeta^2}},
(A1)
where \( \zeta \) and \( \varphi \) are defined in equation \((10)\) and equation \((11)\) respectively. Using equation \((A1)\) in equation \((7)\) yields
\bar{P}^{(r)}_{n_0}(n, z) = \frac{1}{[1 - z (1 - q)] \sqrt{1 - \zeta^2}} \sum_{j=0}^{∞} f^{-\frac{j}{2}} \left( \alpha^{-|n-n_0+j|} - \alpha^{-|n+n_0+j|} \right) - f^{-\frac{j+1}{2}} \left( \alpha^{-|n-n_0+j+1|} - \alpha^{-|n+n_0+j-1|} \right).
(A2)
To assist in evaluating equation \((A2)\), it is useful to consider the full summation as differences of two series. The difference involving \(\alpha^{-|n+n_0|}\) terms produces
\[ \sum_{j=0}^{∞} f^{-\frac{j}{2}} \alpha^{-|n-n_0+j|} - \sum_{j=0}^{∞} f^{-\frac{j+1}{2}} \alpha^{-|n-n_0+j+1|} = \alpha^{-|n-n_0|}, \]
(A3)
as the only surviving term is the \(j = 0\) term, whereas evaluating the sum with the terms \(\alpha^{-|n+n_0|}\) gives
\[ \sum_{j=0}^{∞} f^{-\frac{j+1}{2}} \alpha^{-|n+n_0+j|} - \sum_{j=0}^{∞} f^{-\frac{j}{2}} \alpha^{-|n+n_0+j+1|} = \frac{\alpha^{-|n+n_0|} (\alpha - f^{\frac{1}{2}})}{f^{\frac{1}{2}} - \alpha^{-1}}. \]
(A4)
Hence the propagator with a single reflective boundary between the sites \(n = 0\) and \(n = 1\) in \(z\)-domain is
\[ \bar{P}^{(r)}_{n_0}(n, z) = \frac{1}{[1 - z (1 - q)] \sqrt{1 - \zeta^2}} \left( \alpha^{-|n-n_0|} + \frac{\alpha^{-|n+n_0|} (\alpha - f^{\frac{1}{2}})}{f^{\frac{1}{2}} - \alpha^{-1}} \right). \]
(A5)
With some simple algebra one then obtains equation \((13)\) in the main text.

2. Two Reflective Boundaries
With the domain being finite, one must construct the bounded propagator with an infinite number of images of the propagator \((9)\) with shifted initial conditions. Similar to the case with a single reflecting boundary, the initial condition \(P^{(r)}_{n_0}(n, 0) = \delta_{n,n_0}\) translates into \(\mu = f^{-\frac{1}{2}}\) and \(Q^{(a)}(n, 0) = f^{-\frac{1}{2}} \delta_{n,n_0} - f^{-\frac{j+1}{2}} \delta_{n+j+1,n_0}\). Convolution of this initial condition with the general solution constructed with infinite images of the unbounded propagator \((9)\) via \(\bar{Q}^{(a)}(n, z) = \sum_{m=0}^{∞} \sum_{k=-∞}^{∞} Q^{(a)}(m, 0) \left[ \tilde{H}_{m+2kN}(n, z) - \tilde{H}_{-m+2kN}(n, z) \right] \) gives
\[ \bar{Q}^{(a)}(n, z) = \sum_{k=-∞}^{∞} \frac{\varphi^{-|n-n_0+2kN|} - \varphi^{-|n+n_0+2kN|}}{[1 - z \omega (1 - q)] \sqrt{1 - \zeta^2}} - f^{-\frac{1}{2}} \sum_{k=-∞}^{∞} \frac{\varphi^{-|n-n_0+1+2kN|} - \varphi^{-|n+n_0-1+2kN|}}{[1 - z \omega (1 - q)] \sqrt{1 - \zeta^2}}. \]
(A6)
Summing the images yields the solution to the symmetric propagator with absorbing boundaries at \(n = 0\) and \(n = N\)
\[ \bar{Q}^{(a)}(n, z) = \frac{\varphi^{-|n-n_0|} + \varphi^{-N-|n-n_0|} - \varphi^{-N+|n+n_0|} - \varphi^{-N-|n+n_0|}}{(\varphi^{N} - \varphi^{-N}) [1 - z \omega (1 - q)] \sqrt{1 - \zeta^2}}. \]
(A7)
To transform back to the symmetric propagator we apply the transformation \((7)\) to equation \((A7)\) and we obtain
\[ \bar{P}^{(r)}_{n_0}(n, z) = \sum_{j=0}^{∞} f^{-\frac{j}{2}} \left\{ \frac{\alpha^{N-|n-n_0|} + \alpha^{-N+|n-n_0|} - \alpha^{N-|n+n_0|} - \alpha^{-N+|n+n_0|}}{(\alpha^{N} - \alpha^{-N}) [1 - z (1 - q)] \sqrt{1 - \frac{n^2}{3}}} \right\} - \sum_{j=0}^{∞} f^{-\frac{j+1}{2}} \left\{ \frac{\alpha^{N-|n-n_0+1|} + \alpha^{-N+|n-n_0+1|} - \alpha^{N-|n+n_0-1|} - \alpha^{-N+|n+n_0-1|}}{(\alpha^{N} - \alpha^{-N}) [1 - z (1 - q)] \sqrt{1 - \frac{n^2}{3}}} \right\}. \]
(A8)
We proceed in a similar fashion as before by considering pairwise differences of the series. The $\alpha^{\pm N \mp |n-n_0|}$ terms result in the difference of two geometric series, namely,

$$\alpha^{\pm N \mp |n-n_0|} = \sum_{j=0}^{\infty} f^{-j/2} \alpha^{\pm N \mp |n-n_0+j|} - \sum_{j=0}^{\infty} f^{-j+1/2} \alpha^{\pm N \mp |n-n_0+j+1|},$$  \hspace{2cm} (A9)

while the $\alpha^{\pm N \mp |n+n_0|}$ terms produce a difference of the following geometric series

$$\alpha^{\pm N \mp |n+n_0|} \left[ \frac{\alpha - f^{1/2}}{f^{1/2} - \alpha^{-1}} \right]^{\pm 1} = \sum_{j=0}^{\infty} f^{-j+1} \alpha^{\pm N \mp |n+n_0+j-1|} - \sum_{j=0}^{\infty} f^{-j} \alpha^{\pm N \mp |n+n_0+j|},$$ \hspace{2cm} (A10)

Putting everything together we find

$$\tilde{P}^{(m)}_{n_0}(n, z) = \frac{\eta f^{n-n_0}/2}{\sinh \left[ \text{acosh} \left( \frac{\eta}{n} \right) \right]} \left\{ \alpha^{N-|n-n_0|} + \alpha^{N+|n-n_0|} - \alpha^{N-|n+n_0|} \xi - \alpha^{N+|n+n_0|} \xi^{-1} \right\},$$ \hspace{2cm} (A11)

and with some further algebra we obtain equation (16) in the main text.

Appendix B: Time Dependent Solution with Mixed Boundary Condition

We rewrite the mixed propagator (17) in terms of Chebyshev polynomials of the second kind,

$$\tilde{P}^{(m)}_{n_0}(n, z) = \frac{2\eta f^{n-n_0}/2}{\sinh \left[ \text{acosh} \left( \frac{\eta}{n} \right) \right]} \left\{ \{f^2 U_{N-1} - 1\} \right\} \left\{ \alpha^{N-|n-n_0|} + \alpha^{N+|n-n_0|} - \alpha^{N-|n+n_0|} \xi - \alpha^{N+|n+n_0|} \xi^{-1} \right\},$$ \hspace{2cm} (B1)

To find the inverse z transform of equation (17), we first find the roots of the orthogonal polynomial $f^2 U_{N-1}(\sigma) - U_{N-2}(\sigma)$. Defining $\cos(\theta_k)$ as the roots, the time dependent solution is then written as

$$2f^{n-n_0} \sum_{k=1}^{N-1} \lim_{\sigma \to \cos(\theta_k)} \frac{1}{\sin(\theta_k)} \left\{ \frac{[1 - p + \frac{\eta}{n} \cos(\theta_k)]}{\{f^2 U_{N-1} - 1\}} \left[ \alpha^{N-|n-n_0|} + \alpha^{N+|n-n_0|} - \alpha^{N-|n+n_0|} \xi - \alpha^{N+|n+n_0|} \xi^{-1} \right] \right\},$$ \hspace{2cm} (B2)

To evaluate the limit we apply L’Hôpital’s rule to obtain

$$2f^{n-n_0} \sum_{k=1}^{N-1} \lim_{\sigma \to \cos(\theta_k)} \frac{1}{\sin(\theta_k)} \left\{ N \left\{ f^2 T_{N-1}(\sigma) - T_{N-2}(\sigma) \right\} \right\},$$ \hspace{2cm} (B3)

and with some further algebraic manipulation we obtain equation (21) with $\gamma = m$.

Appendix C: Continuous Time and Spatial Limits of the One Dimensional Propagators

The continuous space and time propagators of the biased lattice walk in finite domains, that is the drift-diffusion bounded propagator can be recovered by appropriate limiting procedures. We consider first the continuous-time discrete-space analog of the 1D propagators (i.e. equation (21) in the main text) given by $C^{(m)}_{n_0}(n, \tau) = \sum_{s=0}^{\infty} W(n, \tau) P^{(m)}_{n_0}(n, s)$, where $W(n, \tau)$ is the probability of $s$ jumps to occur in (continuous) time $\tau$. With $\psi(\tau)$, the probability of a jump event to occur at time $\tau$, one can construct $W(n, \epsilon) = \frac{1 - W(\epsilon)}{\epsilon}$, where $\mathcal{F}(\epsilon) = \int_0^\infty e^{-\epsilon t} f(t)dt$ is the Laplace transform of $f(t)$. Laplace transforming and evaluating the geometric sum yields

$$C^{(m)}_{n_0}(n, \epsilon) = \sum_{k=0}^{\infty} \frac{h_k^{(1)}(n, n_0)}{\epsilon \left[ 1 - \psi(\epsilon) \left[ 1 + s_k^{(m)} \right] \right]}.$$ \hspace{2cm} (C1)

Defining $\psi(\tau) = 2Re^{-2R\tau}$ where $R$ is a rate, and its Laplace transform $\mathcal{F}(\epsilon) = 2R/(\epsilon + 2R)$, one can inverse Laplace transform equation (C1) obtaining the continuous-time discrete-space biased random walk in finite domains

$$C^{(m)}_{n_0}(n, \tau) = \sum_{k=0}^{\infty} \frac{h_k^{(1)}(n, n_0)}{\epsilon \left[ 1 - \psi(\epsilon) \left[ 1 + s_k^{(m)} \right] \right]} e^{2R\tau s_k^{(m)}}$$ \hspace{2cm} (C2)
To take the continuous spatial limit of equation (C2), we consider a lattice spacing \( b \) with \( b, g \rightarrow 0 \) and \( R, N, n, n_0 \rightarrow +\infty \), such that \( x = bn, x_0 = bn_0, L = Nb, Rq b^2 \rightarrow D \), and \( s / b \rightarrow \gamma / 2D \), where \( L \) is the domain size \( (0 \leq x, x_0 \leq L) \), \( D \) the diffusion constant and \( v \) the drift velocity. Evaluation of these limits requires different steps for each of the boundary conditions which are outlined in the following sections.

1. Absorbing Boundaries

From equation (C2), the continuous-time discrete-space propagator with absorbing boundaries is given by

\[
C^{(a)}_{n_0}(n, \tau) = \sum_{k=1}^{N-1} 2 f^{n-n_0} \sin \left( \frac{n-1}{N-1} k\pi \right) \sin \left( \frac{n_0-1}{N-1} k\pi \right) \times \exp \left\{ -2Rq \tau \left[ 1 - \frac{1}{\eta} \cos \left( \frac{k\pi}{N-1} \right) \right] \right\}. \tag{C3}
\]

The term \( f^{n-n_0} \) in equation (C3) needs to be rewritten as \( \exp \left[ \frac{1}{2} (n - n_0) \ln (f) \right] \) before Taylor expansion of \( \ln (f) \) with \( f = \frac{1 - g}{1 - g} \) to obtain

\[
\exp \left[ \frac{1}{2} (n - n_0) \ln (f) \right] \rightarrow \exp \left[ \frac{v (x_0 - x)}{2D} \right], \tag{C4}
\]

where \( bn \rightarrow x, bn_0 \rightarrow x_0 \) and \( g/b \rightarrow v/(2D) \). The time dependent term in equation (C2) requires a Taylor expansion of the \( \cos(\theta) \) term around \( \theta = 0 \) and \( \eta = \left( \sqrt{1 - g^2} \right)^{-1} \) around \( g = 0 \). With \( Nb \rightarrow L \), one then has

\[
2Rq \left[ 1 - \frac{1}{\eta} \cos \left( \frac{k\pi}{N-1} \right) \right] \rightarrow \frac{v^2}{4D} + \frac{D^2 \pi^2 k^2}{L^2}. \tag{C5}
\]

Combining these results we recover the continuous space-time solution to the drift-diffusion equation with absorbing boundaries (see e.g. equation (1.1.4-7) in reference [18]),

\[
C^{(a)}_{x_0}(x, \tau) = \frac{2}{L} \sum_{k=1}^{\infty} \sin \left( \frac{k\pi x}{L} \right) \sin \left( \frac{k\pi x_0}{L} \right) \times \exp \left[ -\frac{D^2 \pi^2 \tau}{L^2} + \frac{2v (x_0 - x) - \tau v^2}{4D} \right]. \tag{C6}
\]

2. Reflecting Boundaries

Using equation (C2), the continuous-time discrete-space propagator with two reflective boundaries is

\[
C^{(r)}_{n_0}(n, \tau) = \frac{f^{n-1} (1 - f)}{1 - f^N} + \frac{2f^{n-n_0}}{N} \sum_{k=1}^{N-1} \left( f^{\frac{k}{2}} \sin \left[ \frac{nk\pi}{N} \right] - \sin \left[ (n - 1) \frac{k\pi}{N} \right] \right) \left( f^{\frac{k}{2}} \sin \left[ \frac{n_0k\pi}{N} \right] - \sin \left[ (n_0 - 1) \frac{k\pi}{N} \right] \right) \times \exp \left\{ -2Rpr \left[ 1 - \frac{1}{\eta} \cos \left( \frac{k\pi}{N} \right) \right] \right\}. \tag{C7}
\]

For the continuous spatial limit the procedure is analogous to the case with two absorbing boundaries. The important differences with the absorbing case are the steady-state term,

\[
\lim_{\tau \rightarrow \infty} C^{(r)}_{n_0}(n, \tau) = \frac{f^{n-1} (1 - f)}{1 - f^N}, \tag{C8}
\]

and the \( \frac{1}{2} (1 + f) - f^{\frac{k}{2}} \cos \left( \frac{k\pi}{N} \right) \) term in equation (C7). Starting with the steady state term and rewriting

\[
\frac{f^{n-1} (1 - f)}{1 - f^N} = \frac{(1 - f) \exp [(n - 1) \ln (f)]}{1 - \exp [N \ln (f)]}, \tag{C9}
\]

one has to expand the \( \ln (f) \) term, as done for the absorbing case, before taking the limits. The steady-state probability density in the continuous case becomes

\[
\lim_{t \rightarrow \infty} C_{x_0}(x, \tau) = \frac{v \exp \left( \frac{vx}{D} \right)}{D \left[ 1 - \exp \left( \frac{vx}{D} \right) \right]}. \tag{C10}
\]
For the terms inside the summation, it is convenient to expand the \( n \) and \( n_0 \) dependence first and rewrite

\[
\left( f^\frac{1}{2} \sin \left[ \frac{n\pi}{N} \right] - \sin \left[ \frac{(n-1)\pi}{N} \right] \right) \left( f^\frac{1}{2} \sin \left[ \frac{n_0\pi}{N} \right] - \sin \left[ \frac{(n_0-1)\pi}{N} \right] \right)
\]

\[
= \frac{1 + f - 2f^\frac{1}{2} \cos \left( \frac{k\pi}{N} \right)}{1 + f - 2f^\frac{1}{2} \cos \left( \frac{k\pi}{N} \right)} \times \left\{ \sin \left[ \frac{n_0k\pi}{N} \right] \left( f^\frac{1}{2} \cos \left( \frac{k\pi}{N} \right) - \cos \left( \frac{k\pi}{N} \right) \right) + \cos \left[ \frac{n_0k\pi}{N} \right] \left( f^\frac{1}{2} \cos \left( \frac{k\pi}{N} \right) - \cos \left( \frac{k\pi}{N} \right) \right) \right\}. \tag{C11}
\]

With a Taylor expansion of \( f^\frac{1}{2} \cos [\theta] - \cos [\theta] \), around \( \theta = 0 \), where \( \theta = \frac{k\pi}{N} \) is the expansion variable, we find the limits

\[
\lim_{N \to \infty} \left\{ f^\frac{1}{2} \cos \left[ \frac{k\pi}{N} \right] - \cos \left[ \frac{k\pi}{N} \right] \right\} = -\frac{vL}{2Dk\pi}, \tag{C12}
\]
similarly,

\[
\lim_{N \to \infty} \left\{ (1 + f) \cos^2 \left[ \frac{k\pi}{N} \right] + 2f^\frac{1}{2} \cos \left[ \frac{k\pi}{N} \right] \cos \left[ \frac{k\pi}{N} \right] \right\} = \frac{v^2L^2}{4D^2(2\pi)^2}. \tag{C13}
\]

We finally recover the continuous space-time propagator with two reflective boundaries

\[
C^{(c)}(x, \tau) = \frac{v \exp \left( \frac{vx}{N} \right)}{D \left[ 1 - \exp \left( \frac{vx}{N} \right) \right]} + \frac{2}{L} \exp \left[ \frac{2v(x_0 - x) - \tau v^2}{4D} \right] \times \sum_{k=1}^{\infty} \left\{ \cos \left[ \frac{k\pi x}{L} \right] - \mu_k \sin \left[ \frac{k\pi x}{L} \right] \right\} \left\{ \cos \left[ \frac{k\pi x_0}{L} \right] - \mu_k \sin \left[ \frac{k\pi x_0}{L} \right] \right\} e^{-\frac{\rho^2}{2} \frac{x^2}{L^2}}, \tag{C14}
\]

where \( \mu_k = vL/(2Dk\pi) \), (e.g. see (1.1.4-8) in reference [48]).

### 3. Mixed Boundaries

The discrete-space continuous-time propagator with mixed boundary conditions is

\[
C^{(m)}(n, \tau) = \frac{2f^{N-n_0}}{N-1} \sum_{k=1}^{N-1} \frac{2f^{n-n_0}}{N-1} \sin \left[ (N - n_0) \theta_k \right] \left\{ f^\frac{1}{2} \sin \left[ n_0 \theta_k \right] - \sin \left[ (n_0 - 1) \theta_k \right] \right\} \frac{1 - \cos \left[ N(\theta_k - 1) \right]}{N(\theta_k - 1) \cos \left[ N(\theta_k - 1) \right]} \exp \left\{ -2Rq\tau \right\}. \tag{C15}
\]

Before taking the limits on the spatial dependence, it is necessary to study first the effect of the limits on the relationship

\[
f^\frac{1}{2} U_{N-1}(\sigma) - U_{N-2}(\sigma) = 0. \tag{C16}
\]

We rewrite the Chebyshev polynomials in equation (C16) using their trigonometric definition to yield

\[
f^\frac{1}{2} \sin \left[ N\cos(\sigma) \right] - \sin \left[ (N - 1)\cos(\sigma) \right] = 0. \tag{C17}
\]

Expanding the \( \sin \left[ (N - 1)\cos(\sigma) \right] \) results in the relationship

\[
\tan \left[ N\cos(\sigma) \right] = \frac{\sin \left[ \cos(\sigma) \right]}{\cos \left[ \cos(\sigma) \right] - f^\frac{1}{2}}. \tag{C18}
\]

Defining \( \theta_k = b\rho_k = \cos(\sigma) \) and substituting we find

\[
\frac{\tan \left( L\rho_k \right)}{\rho_k \cos \left( b\rho_k \right)} = \frac{\sin \left( b\rho_k \right)}{\rho_k \left[ \cos^2 \left( b\rho_k \right) - f^\frac{1}{2} \cos \left( b\rho_k \right) \right]}, \tag{C19}
\]

which, in the limit \( b \to 0 \) and \( g \to 0 \), results in the transcendental equation

\[
\frac{\tan \left( L\rho_k \right)}{\rho_k} = \frac{2D}{v}. \tag{C20}
\]
Expanding the numerator and denominator in the spatial dependence \( h_k^{(m)}(n,n_0) \), and with the help of equation (C18) we find

\[
h_k^{(m)}(n,n_0) = \frac{2f^{\frac{n-n_0}{2}}}{B_k} \left( \tan(N\theta_k) \cos(n\theta_k) - \sin(n\theta_k) \right) \left( \tan(N\theta_k) \cos(n\theta_k) - \sin(n\theta_k) \right). \tag{C21}
\]

With the \( n_\theta \) dependence being equivalent to the \( n_\varphi \) dependence in equation (C21), we rewrite

\[
h_k^{(m)}(n,n_0) = \frac{f^{\frac{n-n_0}{2}}}{B_k} \left[ \tan(N\theta_k) \cos(n\theta_k) - \sin(n\theta_k) \right] \left[ \tan(N\theta_k) \cos(n_0\theta_k) - \sin(n_0\theta_k) \right]. \tag{C22}
\]

where

\[
B_k = \frac{(N-1) \cos(\theta_k) + \tan(N\theta_k) \sin(\theta_k) - Nf^{\frac{1}{2}}}{2 \left[ \cos(\theta_k) - f^{\frac{1}{2}} \right]}.
\]

After substituting \( bp_k = \theta_k \), in the continuous limit we find

\[
h_k^{(m)}(x,x_0) = \frac{1}{A_k} \exp \left[ \frac{v(x_0 - x)}{2D} \right] \left[ \tan(L\rho_k) \cos(x\rho_k) - \sin(x\rho_k) \right] \left[ \tan(L\rho_k) \cos(x_0\rho_k) - \sin(x_0\rho_k) \right]. \tag{C23}
\]

where

\[
A_k = \lim_{b \to 0} b B_k = \lim_{b \to 0} \left\{ \frac{(L-b) \cos(b\rho_k) + \tan(L\rho_k) \sin(b\rho_k) - Lf^{\frac{1}{2}}}{2 \left[ \cos(b\rho_k) - f^{\frac{1}{2}} \right]} \right\} = -\frac{2D}{v} + \frac{L}{\cos^2(L\rho_k)}. \tag{C24}
\]

Putting everything together we recover the continuous-time continuous-space solution (see Equation (1.1.4-9) in [48]) with mixed boundary conditions

\[
C^{(m)}(x,\tau) = \exp \left[ \frac{2v(x_0 - x) - \tau v^2}{4D} \right] \sum_{k=1}^{\infty} \frac{1}{A_k} \left[ \tan(L\rho_k) \cos(x\rho_k) - \sin(x\rho_k) \right] \times \left[ \tan(L\rho_k) \cos(x_0\rho_k) - \sin(x_0\rho_k) \right], \tag{C25}
\]

where \( \rho_k \) are the roots of equation (C20).

4. Periodic Boundaries

Starting with the discrete-space continuous-time propagator

\[
C^{(p)}(n,\tau) = \frac{1}{N} \sum_{k=0}^{N-1} \exp \left\{ \frac{2k\pi i}{N} (n-n_0) - 2R\rho \tau \left[ 1 - \cos \left( \frac{2k\pi}{N} \right) - ig \sin \left( \frac{2k\pi}{N} \right) \right] \right\}, \tag{C26}
\]

one needs to expand the \( \cos \) and \( \sin \) terms before taking the limits resulting in the following continuous space-time propagator

\[
C^{(p)}(x,\tau) = \frac{1}{L} \sum_{k=0}^{\infty} \exp \left\{ \frac{2k\pi i}{L} (x - x_0 + \tau v) - \frac{4Dk^2\pi^2\tau^2}{L^2} \right\}, \tag{C27}
\]

which clearly satisfies the periodic boundary condition. For higher dimensions, the limiting procedure can be carried through to give the continuous space-time analog which is the product of the one dimensional propagators (equations (C6), (C14), (C25) and (C27) along each direction.

Appendix D: First-Passage Probability and Related Quantities with Reflecting and Periodic One Dimensional Domain

The first-passage probability, being the ratio of propagators in the \( z \) domain, can be constructed, respectively, for reflecting and periodic domains from equation (16) and equation (18) yielding

\[
\bar{F}(n,z) = \frac{\bar{F}^{(r)}(n,z)}{\bar{F}^{(v)}(n,z)} = \begin{cases} \frac{f^{\frac{n-n_0}{2}} \left( f^{\frac{1}{2}} \sinh \left[ n_0 \cosh \left( \frac{1}{2} \right) \right] - \sin \left( \left[ n_0 - 1 \right] \cosh \left( \frac{1}{2} \right) \right) \right)}{f^{\frac{1}{2}} \sinh \left[ n \cosh \left( \frac{1}{2} \right) \right] - \sin \left( \left[ n-1 \right] \cosh \left( \frac{1}{2} \right) \right)}, & n > n_0 \\ \frac{f^{\frac{n-n_0}{2}} \left( f^{\frac{1}{2}} \sinh \left[ (N-n_0) \cosh \left( \frac{1}{2} \right) \right] - \sin \left( \left[ (N-n_0) - 1 \right] \cosh \left( \frac{1}{2} \right) \right) \right)}{f^{\frac{1}{2}} \sinh \left[ (N-n) \cosh \left( \frac{1}{2} \right) \right] - \sin \left( \left[ (N-n) - 1 \right] \cosh \left( \frac{1}{2} \right) \right)}, & n < n_0. \end{cases} \tag{D1}
\]
and

\[
\bar{F}^{(p)}_{n_0}(n, z) = \frac{\bar{P}^{(p)}_{n_0}(n, z)}{\bar{F}^{(p)}_n(n, z)} = \begin{cases} 
\frac{f^{n-n_0}}{2} \left( f^{\frac{1}{2} \sinh \left[ N-n+n_0 \right] \cosh \left( \frac{\pi}{2} \right) \right) + f^{\frac{1}{2} \sinh \left[ (n-n_0) \cosh \left( \frac{\pi}{2} \right) \right]} \sinh \left[ N \cosh \left( \frac{\pi}{2} \right) \right], & n < n_0 \\
\frac{f^{n-n_0}}{2} \left( f^{\frac{1}{2} \sinh \left[ N-n-n_0 \right] \cosh \left( \frac{\pi}{2} \right) \right) + f^{\frac{1}{2} \sinh \left[ (n-n_0) \cosh \left( \frac{\pi}{2} \right) \right]} \sinh \left[ N \cosh \left( \frac{\pi}{2} \right) \right], & n > n_0.
\end{cases}
\]

(D2)

Using equations (D1) and (D2) yields the MFPT expressions in (26) and (27) in the main text.

From the generating function of the return probability, \( \bar{R}^{(r)}(n, z) = 1 - \left( \bar{P}_n(n, z) \right)^{-1} \), one can compute the MRT

\[
\mathcal{R}_n^{(r)} = \frac{1 - f_N}{f^{n-1}_n(1 - f)} \quad \text{and} \quad \mathcal{R}_n^{(p)} = N.
\]

(D3)

which is, as expected from Kac’s theorem, the reciprocal of the steady-state probability at the site.

1. **First-Passage Probability when \( g \pm 1 \) and \( q \neq 1 \) with Periodic Boundary**

Restricting the walker to move only forward or to remain at a site, that is no back-tracking, involves taking the limits \( g \rightarrow \pm 1 \), alternatively, \( f \rightarrow 0 \) or \( f \rightarrow \infty \) in the first-passage probability expressions for the periodic domain. Although it is trivial to find the periodic propagator in equation (21) for such limits, the same cannot be said of the first-passage probability in equation (23) due to the term \( f^{n-n_0} \). In the latter case it is much easier to derive the first-passage probability by separating the cases \( n < n_0 \) and \( n > n_0 \), and constructing the expression combinatorially. One can show that when \( g = 1 \) and \( n < n_0 \), or when \( g = -1 \) and \( n > n_0 \), the first-passage probability becomes

\[
F^{(p)}_{n_0}(n, t) = \left( \frac{t - 1}{t - |n - n_0|} \right) q^{n-n_0} (1 - q)^{t - |n - n_0|} \Theta \left[ t - |n - n_0| \right],
\]

(D4)

while when \( g = 1 \) and \( n > n_0 \), or when \( g = -1 \) and \( n < n_0 \), it is

\[
F^{(p)}_{n_0}(n, t) = \left( \frac{t - 1}{t - (N - |n - n_0|)} \right) q^{N-n-n_0} (1 - q)^{t - (N-|n-n_0|)} \Theta \left[ t - (N - |n - n_0|) \right].
\]

(D5)

When equation (D4) applies, the probability of reaching the site is zero if the number of time steps is smaller than the displacement between the target and initial site, i.e. when \( t < |n-n_0| \). At \( t = |n-n_0| \), the relation \( F^{(p)}_{n_0}(n, t) = q^{n-n_0} \) indicates that the walker may reach the target by always moving with each step giving a contribution equal to \( q \) to the probability. When \( t > |n-n_0| \), the walker has a choice of remaining at any of the sites between \( n \) and \( n_0 \) (excluding \( n \)). In that case the coefficient \( (t - |n-n_0|) \) represents all the possible combinations with which the walker can reach \( n \) from \( n_0 \) by making \( |n-n_0| \) steps and \( t - |n-n_0| \) pauses along the way. On the other hand when equation (D5) applies, the walker travels around the domain with the length of the shortest possible path from \( n \) to \( n_0 \) being \( N - |n-n_0| \), and the meaning of the terms are analogous to the case in (D4). The case \( q = 1 \) corresponds to a Kronecker delta in time: \( F^{(p)}_{n_0}(n, t) = \delta_{t,|n-n_0|} \) or \( F^{(p)}_{n_0}(n, t) = \delta_{t,N-|n-n_0|} \).

**Appendix E: Derivation of the Two Dimensional Propagator**

Solving the effective 1D Master equation equation (31), and taking the \( z \) transform gives

\[
\bar{Q}(n_1, n_2, z) = \frac{\lambda_1 \left[ e^{i \pi z n_0} f_2^{-n_0} (1 - f_2^{-1}) \right]}{1 - 2 \omega \left( 1 - \frac{q_1}{2} + \frac{q_2}{2 \eta_1} \cos \left[ \kappa_2 \right] \right)} + \frac{f_1^{-n-n_0}}{2} \left( f_2^{\frac{1}{2} \sinh \left[ n_0 \cosh \left( \frac{\pi}{2} \right) \right]} \right) \sinh \left[ N \cosh \left( \frac{\pi}{2} \right) \right] \sum_{k=1}^{N-1} \left\{ f_1^{\frac{1}{2} \sinh \left[ n_0 k_1 \pi \right]} \sin \left[ (n_1 - 1) k_1 \pi \right] \right\}
\]

\[
\times \left\{ f_1^{\frac{1}{2} \sinh \left[ (n_0 k_1 \pi) \right]} - \sin \left[ (n_1 - 1) k_1 \pi \right] \right\} \left\{ q_1 - \cos \left[ k_1 \pi \right] \right\}^{-1}
\]

\[
x \left( 1 - z \left( 1 - \frac{q_1}{2} - \frac{q_2}{2} + \frac{q_1}{2 \eta_1} \cos \left[ k_1 \pi \right] \right) \right) \left\{ q_2 - \frac{q_1}{2 \eta_2} \cos \left[ k_2 \right] \right\}^{-1},
\]

(E1)
where

\[
f_2 = \frac{1 - g_2}{1 + g_2}, \quad \eta_2 = \frac{1 + f_2}{2f_2^2} \quad \text{and} \quad \lambda_i = \frac{f_i^{n_i-1}(1 - f_i)}{1 - f_i^{N_i}}.
\]

To find the analytic expression for the 2D random walker with bias and reflective boundaries in the first dimension, while diffusive and unbounded in the second dimension, one needs to inverse Fourier transform the second dimension to obtain

\[
Q(n_1, n_2 z) = \lambda_i \left\{ \frac{2\eta_2 f_2^2}{2\omega q_2 \sinh \left( \frac{\eta_2}{2} \right)} \left[ \frac{z\omega q_2}{z\omega q_2 (1 - z\omega (1 - \frac{q_2}{2}))} \right] \left[ \alpha^{N_2-n_2-n_0} + \alpha^{-N_2+n_2-n_0} - \alpha^{N_2-n_2+n_0} - \alpha^{-N_2+n_2+n_0} \right] \right\},
\]

\[
= \frac{1}{N_1} \sum_{k_1=1}^{N_1-1} \left( \frac{f_2^2}{2} \sin \left( \frac{n_1 k_1 \pi}{N_1} \right) - \sin \left( (n_1 - 1) \frac{k_1 \pi}{N_1} \right) \right) \left( \frac{f_2^2}{2} \sin \left( \frac{n_1 k_1 \pi}{N_1} \right) - \sin \left( (n_1 - 1) \frac{k_1 \pi}{N_1} \right) \right) \left[ \alpha^{N_2-n_2-n_0} + \alpha^{-N_2+n_2-n_0} - \alpha^{N_2-n_2+n_0} - \alpha^{-N_2+n_2+n_0} \right] \left[ \sinh \left( \frac{\eta_2}{2} \right) \right] \right\},
\]

\[
\times \left( \frac{z\omega q_2}{z\omega q_2 \sinh \left( \frac{\eta_2}{2} \right)} \left( \frac{z\omega q_2}{z\omega q_2 (1 - z\omega (1 - \frac{q_2}{2}))} \right) \right) \left[ \alpha^{N_2-n_2-n_0} + \alpha^{-N_2+n_2-n_0} - \alpha^{N_2-n_2+n_0} - \alpha^{-N_2+n_2+n_0} \right] \left[ \sinh \left( \frac{\eta_2}{2} \right) \right] \right\},
\]

(E2)

where (redefining)

\[
\zeta = \frac{z\omega q_2}{2\eta_2 \left( 1 - z\omega (1 - \frac{q_2}{2}) \right)}, \quad \zeta = \frac{z\omega q_2}{2\eta_2 \left( 1 - z\omega \left[ 1 - \frac{q_2}{2} - \frac{q_2}{2} + \frac{q_2}{2m} \cos \left( \frac{k_1 \pi}{N_1} \right) \right] \right)}, \quad \varphi = \exp \left[ \sinh \left( \frac{\eta_2}{2} \right) \right]
\]

and

\[
\tilde{\varphi} = \exp \left[ \sinh \left( \frac{\eta_2}{2} \right) \right].
\]

On equation (E2), we use the method of images to impose the boundary condition following the procedure outlined in Appendix [A2] and asymmetrise the second dimension to yield

\[
\tilde{P}_{n_1, n_2 z} = \frac{2\lambda_1 \eta_2 f_2^2}{z\omega q_2 \sinh \left( \frac{\eta_2}{2} \right)} \left\{ \alpha^{N_2-n_2-n_0} + \alpha^{-N_2+n_2-n_0} - \alpha^{N_2-n_2+n_0} - \alpha^{-N_2+n_2+n_0} \right\},
\]

\[
= \frac{1}{N_1} \sum_{k_1=1}^{N_1-1} \left( \frac{f_2^2}{2} \sin \left( \frac{n_1 k_1 \pi}{N_1} \right) - \sin \left( (n_1 - 1) \frac{k_1 \pi}{N_1} \right) \right) \left( \frac{f_2^2}{2} \sin \left( \frac{n_1 k_1 \pi}{N_1} \right) - \sin \left( (n_1 - 1) \frac{k_1 \pi}{N_1} \right) \right) \left[ \alpha^{N_2-n_2-n_0} + \alpha^{-N_2+n_2-n_0} - \alpha^{N_2-n_2+n_0} - \alpha^{-N_2+n_2+n_0} \right] \left[ \sinh \left( \frac{\eta_2}{2} \right) \right] \right\},
\]

(E3)

where (redefining)

\[
\beta = \frac{z\omega q_2}{2 \left[ 1 - z \left( 1 - \frac{q_2}{2} \right) \right]}, \quad \tilde{\beta} = \frac{2}{2 \left( 1 - z \left[ 1 - \frac{q_2}{2} - \frac{q_2}{2} + \frac{q_2}{2m} \cos \left( \frac{k_1 \pi}{N_1} \right) \right] \right)}, \quad \alpha = \exp \left[ \sinh \left( \frac{\eta_2}{2} \right) \right],
\]

\[
\tilde{\alpha} = \exp \left[ \sinh \left( \frac{\eta_2}{2} \right) \right], \quad \xi = \frac{f_2^2 - \alpha}{f_2^2 - 1/\alpha}, \quad \text{and} \quad \tilde{\xi} = \frac{f_2^2 - \tilde{\alpha}}{f_2^2 - 1/\tilde{\alpha}}.
\]

Employing the general identity equation (G1) (below) before inverse $z$ transforming results in the time-dependent 2D propagator equation (E2) found in the main text.

**Appendix F: Constructing Propagators of Higher Dimensions**

To build a $d$-dimensional confined lattice random walk with bias, one first considers a semi-confined LRW where the first $d-1$ dimensions are bounded while the final $d^{th}$ dimension is unbounded. Symmetrising the dynamics in the $d^{th}$ dimension,
yields a biased confined LRW in the $d - 1$ dimension while being diffusive and unconfined in the $d^{th}$ dimension. By Fourier transforming along the $d^{th}$ dimension reduces the problem to an effective $d - 1$ dimensional biased LRW whose solution is known. Solving for the dynamics in the (diffusive) $d^{th}$ dimension in the Fourier-$z$-domain and imposing boundary condition via the method of images, gives the solution to a confined random walk with bias in $d - 1$ dimensions and no bias in the $d^{th}$ dimension. Inverting the symmetrisation procedure along the $d^{th}$ dimension yields the confined BLRW in $d$-dimensions in $z$-domain. Finally, with the use of the identities (G1), (G2), equation (G4) or (G3) one inverts the propagator from the $z$-domain to the time domain. With such a procedure one can build propagators with arbitrary dimensions and arbitrary boundary conditions.

**Appendix G: Identities of Finite Trigonometric Series**

For the derivation of the higher dimensional propagators analytic identities can be obtained by equating the $z$-transform of equation (21) for each of the different boundary cases with the corresponding equations (15), (16) and (18). For the reflecting condition we find

$$
\left\{ f^{±}U_{M-1-m>}(\frac{y}{\gamma}) - U_{M-m>}(\frac{y}{\gamma}) \right\} \left\{ f^{±}U_{m<1}(\frac{y}{\gamma}) - U_{m<2}(\frac{y}{\gamma}) \right\} = \frac{\mu U_{M-1}(\frac{y}{\gamma})}{\mu (1 - f^M)} + \frac{1}{M} \sum_{k=1}^{M-1} \left\{ \left( f^{±} \sin \left( \frac{m_k \pi}{N} \right) - \sin \left( (m_1 - 1) \frac{\pi}{M} \right) \right) \left( f^{±} \sin \left( \frac{m_k \pi}{N} \right) - \sin \left( (m_2 - 1) \frac{\pi}{M} \right) \right) \right\},
$$



for the absorbing case we generate

$$
\frac{U_{M-1-m>}(\frac{y}{\gamma}) - U_{m<2}(\frac{y}{\gamma})}{U_{M-2}(\frac{y}{\gamma})} = \frac{1}{M - 1} \sum_{k=1}^{M-1} \left\{ \frac{\sin \left( \frac{(m-1) \pi}{M} \right)}{\frac{y}{\gamma} - \cos \left( \frac{\pi k}{M} \right)} \right\},
$$

and for the periodic domain we obtain

$$
\frac{f^{±} \left[ U_{M-1-m}(\frac{y}{\gamma}) + U_{m-1}(\frac{y}{\gamma}) \right] - f^{-M \text{sign}(m)}}{T_M(\frac{y}{\gamma}) - T_M(\eta)} = \frac{1}{M} \sum_{k=0}^{M-1} \frac{\exp \left( \frac{2k \pi \text{im}}{M} \right)}{\frac{y}{\gamma} - \cosh \left( \frac{2k \pi \text{im}}{M} \right) - \frac{2}{M} \Gamma \left( f^M \right)}.
$$

In equations (G1), (G2) and (G3) the meaning of the symbols are as follows: $\gamma$ and $\mu$ are complex constants; $M, m$ and $n$ are integers with $1 \leq m_1, m_2 \leq N$; $f > 0$; $m_\perp = \frac{1}{2} (m_1 + m_2 + |m_1 - m_2|)$ and $m_\parallel = \frac{1}{2} (m_1 + m_2 - |m_1 - m_2|)$; and $\eta = \frac{1}{2} (1 + f) f^{-\frac{1}{2}}$. The validity of equations (G1) and (G2) is based on the known general identity (E1) in reference [22], while equation (G3) is a new identity that reduces to (E3) in reference [22] when $f, \eta = 1$. There is also a relation (numerical identity) that can be obtained from the mixed scenario using the procedure in Appendix E given by

$$
\frac{U_{M-1-m>}(\frac{y}{\gamma}) - U_{m<2}(\frac{y}{\gamma})}{f^{±}U_{M-1}(\frac{y}{\gamma}) - U_{M-2}(\frac{y}{\gamma})} = \sum_{k=1}^{M-1} \frac{\sin \left( (M - m_\parallel) \theta_k \right)}{\left( M - 1 \right) \cos \left( (M - 1) \theta_k \right) - M f^{±} \cos \left( M \theta_k \right)} \left\{ \frac{\sin \left( m_\parallel \theta_k \right)}{\frac{y}{\gamma} - \cos \left( \theta_k \right)} \right\},
$$

where $\cos(\theta_k)$ is the $k^{th}$ (numerical) root of the orthogonal polynomial $f^{±}U_{M-1}[\cos(\theta)] - U_{M-2}[\cos(\theta)]$.

**Appendix H: Mean First-Return Times in Higher Dimensions**

A hint of the non-trivial dependence of the return dynamics can be evinced by studying how different initial positions affect the MRT. We display for this purpose in figure [H.1] the reciprocal of the MRT in a 2D domain with reflecting boundaries with different starting locations, $\vec{n}$, as a function of the bias $\vec{g}$. Each panel from (a)-(d) represents a different starting location which is progressively closer to the top-left corner. In the presence of a bias, the walker is pushed away from the starting site. For an initial location at the centre of the domain it results in a weak dependence on the bias direction as shown in panel (a). A strong dependence when the starting location is off-centre is instead shown in panels (b)-(d). With the shift in the starting sites from panel (b) to (c) to (d), there is a shorter MRT the stronger the bias is directed towards that corner ($g_1 < 0, g_2 > 0$) with instead long MRT for all other bias directions. In panel (b), one may also notice an asymmetry with respect to the diagonal which is not present in panels (c) and (d). It is due to the starting site being closer to the top boundary at $n_2 = N_2$ than the left boundary at $n_1 = 1$.

All panels in figure [H.1] display dependence on the bias strength. When the starting location is near the centre, the bias towards a corner yields long MRTs when compared with a diffusive walker ($g_1 = g_2 = 0$) which has a natural tendency to stay
FIG. H.1. (Colour Online) The reciprocal of the MRT, \( (\mathcal{R}^{(\mathbf{r})}_{\mathbf{n}})^{-1} \), of a 2D BLRW with domain size \( \mathbf{N} = (20, 20) \) and diffusion parameter \( \mathbf{q} = (0.8, 0.8) \) as a function of the bias \( \mathbf{g} \). Each panel from (a) to (d) represents different starting sites, respectively, \( \mathbf{n} = (10, 10), (5, 18), (2, 19) \) and \( (1, 20) \). A positive (negative) \( g_1 \) indicates a drift to the left (right), while a positive (negative) \( g_2 \) indicates a drift downwards (upwards).

near the starting location. Conversely, with the starting location at a corner, panel (d), one finds the shortest MRT when the walker is kept at the starting location with the bias \( \mathbf{g} = (1, -1) \). Interestingly, when the starting location is off-centre and not at the boundary corner (panels (b) and (d)) the MRT is minimised for an intermediate bias strength. The latter is strong enough to reduce the number of trajectories travelling right or downwards from the starting site whilst weak enough to allow the walker to travel against the bias when near the top-left corner. The precise location of the minimum can be computed numerically for arbitrary dimensions from the explicit definition

\[
\mathcal{R}^{(\mathbf{r})}_{\mathbf{n}} = \left[ \prod_{j=1}^{d} h^{(\mathbf{r})}_{\mathbf{n}j}(n_j, n_j) \right]^{-1}.
\]