Odd-mass nuclei in the cluster shell model

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Abstract. In this contribution, we present the cluster shell model which is analogous to the Nilsson model, but for cluster potentials. Special attention is paid to the consequences of the discrete symmetries of three \( \alpha \)-particles in an equilateral triangle configuration. This configuration is characterized by a special structure of the rotational bands which can be used as a fingerprint of the underlying geometric configuration. The cluster shell model is applied to the nucleus \( ^{13}\text{C} \).

1. Introduction
Cluster degrees of freedom are very important for the description of light nuclei, in particular \( k\alpha \) and \( k\alpha + x \) nuclei, due to the large binding energy of the \( ^4\text{He} \) nucleus. Early work on \( \alpha \)-cluster models goes back to the 1930’s with studies by Wheeler [1], and Hafstad and Teller [2], followed by later work by Brink [3, 4] and Robson [5, 6]. Recently, there has been a lot of renewed interest in the structure of \( \alpha \)-cluster nuclei, especially for the nucleus \( ^{12}\text{C} \) [7]. The measurement of new rotational excitations of the ground state [8, 9, 10] and of the Hoyle state [11, 12, 13, 14] has stimulated a large theoretical effort to understand the structure of \( ^{12}\text{C} \) (for a review see e.g. Refs. [7, 15, 16]).

In this contribution, we present a study of cluster states in \( ^{12}\text{C} \) and \( ^{13}\text{C} \) in the framework of the algebraic cluster model and the cluster shell model, respectively.

2. The algebraic cluster model
The algebraic cluster model (ACM) is an interacting boson model to describe the relative motion of cluster systems (see e.g. [17, 18]). The relevant degrees of freedom of a system of \( k \)-body clusters are given by the \( k - 1 \) relative Jacobi coordinates and their conjugate momenta. The building blocks of the ACM consist of a vector boson for each relative coordinate and a scalar boson. Cluster states are then described in terms of a system of \( N \) interacting bosons with angular momentum and parity \( L^P = 1^- \) (vector bosons) and \( L^P = 0^+ \) (scalar bosons). The \( 3(k - 3) \) components of the vector bosons together with the scalar boson span a \( (3k - 2) \)-dimensional space with group structure \( U(3k - 2) \). The many-body states are classified according to the totally symmetric irreducible representation \( [N] \) of \( U(3k - 2) \). The ACM has a very rich symmetry structure. In addition to continuous symmetries like the angular momentum, in case of \( \alpha \)-cluster nuclei the Hamiltonian has to be invariant under the permutation of the \( k \) identical \( \alpha \) particles. Since one does not consider the excitations of the \( \alpha \) particles themselves, the allowed cluster states have to be symmetric under the permutation group \( S_k \).
Table 1. ACM for $k$-body clusters.

| $k$ | $U(3k-2)$ | Symmetry | Geometry       | Nucleus | Ref |
|-----|------------|----------|----------------|---------|-----|
| 2   | $U(4)$    | $Z_2 \sim S_2$ | Dumbbell       | $^8$Be  | [19]|
| 3   | $U(7)$    | $D_{3h} \supset D_3 \sim S_3$ | Equilateral triangle | $^{12}$C | [20, 21]|
| 4   | $U(10)$   | $T_d \sim S_4$ | Regular tetrahedron | $^{16}$O | [22, 23]|

The potential energy surface corresponding to the $S_k$ invariant ACM Hamiltonian gives rise to several possible equilibrium shapes. The ones of special interest for applications to $\alpha$-cluster nuclei, correspond to the geometrical configuration of a dumbbell for $^8$Be ($k = 2$ with $Z_2$ symmetry), an equilateral triangle for $^{12}$C ($k = 3$ with $D_{3h} \supset D_3$ symmetry) and a regular tetrahedron for $^{16}$O ($k = 4$ with $T_d$ symmetry), see Table 1. Even though these cases do not correspond to dynamical symmetries of the ACM Hamiltonian, one can still obtain approximate solutions for the rotation-vibration spectrum and electromagnetic properties [19, 21, 23].

In this contribution, we present the results for three-body clusters with a triangular configuration. The rotation-vibration spectrum is given by

$$E = \omega_1 \left( v_1 + \frac{1}{2} \right) + \omega_2 (v_2 + 1) + \kappa L(L+1) .$$

(1)

The rotational structure of the ground-state band, $(v_1, v_2) = (0, 0)$, and the fundamental vibrations, $(1, 0)$ and $(0, 1)$, depends on the $D_{3h}$ point group symmetry of the equilateral triangle configuration and is summarized in Fig. 1. The ground-state band is characterized by a rotational sequence involving both positive and negative parity states, $L^P = 0^+, 2^+, 3^-, 4^\pm, 5^-, \ldots$, all of which have been observed in $^{12}$C. The so-called Hoyle band has the same structure, but so far only the positive parity states have been observed. The rotational bands in $^{12}$C are shown in Fig. 2.

![Figure 1](image-url)  

**Figure 1.** Schematic spectrum of a triangular configuration. The rotational bands are labeled by $(v_1, v_2)$ and $t$ (bottom). All states are symmetric under $S_3 \sim D_3$. 
The matter and charge distributions are usually taken as a Gaussian distribution

\[ \rho(\vec{r}) = \left( \frac{\alpha}{\pi} \right)^{3/2} \sum_{i=1}^{3} \exp \left[ -\alpha (\vec{r} - \vec{r}_i)^2 \right], \tag{2} \]

where the \( \alpha \)-particles are located at a distance \( \beta \) from the center of mass with \( \vec{r}_i = (\beta, \theta_i, \phi_i) \). The charge density is obtained by multiplying Eq. (2) by \( Ze/3 \) and the matter density by \( Am/3 \).

Electromagnetic transition probabilities can be obtained from the transition form factors, which are the matrix elements of the Fourier transform of the charge distribution. For transitions along the ground-state band the transition form factors can be obtained from the transition form factors, which are the matrix elements of the Fourier transform of the charge distribution. For transitions along the ground-state band the transition form factors are given in terms of a product of a spherical Bessel function and an exponential factor arising from a Gaussian distribution of the electric charges [21]

\[ F(0^+ \rightarrow L^P; q) = c_L J_L(q\beta) e^{-q^2/4\alpha}. \tag{3} \]

The transition form factors depend on the parameters \( \alpha \) and \( \beta \). The value of \( \alpha \) is determined from the radius of the \( \alpha \)-particle to be \( \alpha = 0.56 \text{ fm}^{-2} \) [24], and the value of \( \beta \) from the first minimum of the elastic form factor of \( ^{12}\text{C} \) to be \( \beta = 1.74 \text{ fm} \) [21].

The transition probabilities \( B(EL) \) along the ground-state band can be extracted from the form factors in the long wavelength limit

\[ B(EL; 0^+ \rightarrow L^P) = \frac{(Ze\beta^L c_L)^2}{4\pi}, \tag{4} \]

with

\[ c_L^2 = \frac{2L + 1}{3} \left[ 1 + 2P_L(-\frac{1}{2}) \right] \tag{5} \]

and the charge radius from the slope of the elastic form factor in the origin

\[ \langle r^2 \rangle^{1/2} = \sqrt{\beta^2 + 3/2\alpha}. \tag{6} \]
Eq. (4) shows that all electric transitions are related to one another, and only depend on the value of $\beta$. For example, quadrupole, octupole and hexadecupole transitions are related by

$$ \frac{B(E3; 3^-_1 \rightarrow 0^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} = \frac{5}{2} \beta^2, $$

$$ \frac{B(E4; 4^+_1 \rightarrow 0^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} = \frac{9}{16} \beta^4. $$

(7)

The quadrupole moment can be calculated as [25]

$$ Q_{2^+} = \frac{2}{7} Z e \beta^2, $$

(8)

a positive value, as is to be expected for a planar configuration of three $\alpha$-particles.

Table 2 shows a comparison between the experimental values of electromagnetic properties of $^{12}$C and the ACM. According to Eqs. (4)-(8), the ACM results only depend on the value of $\alpha$ and $\beta$ which have been determined from other observables, *i.e.* the radius of the $\alpha$-particle and the first minimum of the elastic form factor. Therefore, the good agreement with the experimental values as shown in Table 2 provides evidence that the $2^+_1$ and $3^-_1$ states indeed belong to the same ground-state rotational band of an equilateral triangle [21]. In particular, the large $B(E3)$ value cannot be obtained in shell-model calculations without the introduction of large effective charges.

|          | ACM       | Exp          |
|----------|-----------|--------------|
| $B(E2; 2^+_1 \rightarrow 0^+_1)$ | 8.4       | 7.61 ± 0.42  |
| $B(E3; 3^-_1 \rightarrow 0^+_1)$ | 73        | 104 ± 14     |
| $B(E4; 4^+_1 \rightarrow 0^+_1)$ | 44        | $e^2fm^8$    |
| $Q_{2^+}$ | 5.2       | 5.3 ± 4.4    |
| $\langle r^2 \rangle_{ch}^{1/2}$ | 2.389     | 2.468 ± 0.012 fm |

3. The cluster shell model

For the description of odd-cluster nuclei, the cluster shell model (CSM) was developed recently [18, 19, 29]. The CSM combines cluster and single-particle degrees of freedom, and is very similar in spirit as the Nilsson model [30], but in the CSM the odd nucleon moves in the deformed field generated by the (collective) cluster degrees of freedom. The Hamiltonian is written as

$$ H = T + V(\vec{r}) + V_{so}(\vec{r}) + V_C(\vec{r}), $$

(9)

*i.e.* the sum of the kinetic energy, a central potential obtained by convoluting the density $\rho(\vec{r})$ of Eq. (2) with the interaction between the $\alpha$-particle and the nucleon, a spin-orbit interaction and, for an odd proton, a Coulomb potential [29]. Fig. 3 shows the splitting of single-particles levels of a neutron moving in the deformed potential generated by a triangular configuration of three $\alpha$-particles as a function of $\beta$. The solutions of the CSM Hamiltonian, $\chi_\Omega$ with energy $\epsilon_\Omega$, are
Figure 3. Single-particle energies in a cluster potential with \( D'_{3h} \) triangular symmetry. The single-particle levels are labeled by \( E^{(+)}_{1/2} \) (black), \( E^{(-)}_{1/2} \) (red) and \( E_{3/2} \) (blue). For \( \beta = 0 \) the ordering of the single-particle orbits is \( 1s_{1/2}, 1p_{3/2}, 1p_{1/2} \) and (almost degenerate) \( 1d_{5/2}, 2s_{1/2} \). labeled by the irreducible representations of the double group \( D'_{3h} \): \( \Omega = E^{(+)}_{1/2}, E^{(+)}_{1/2} \) and \( E_{3/2} \), each of which is doubly degenerate. The resolution of single-particle levels into representations of \( D'_{3h} \) is shown in Table 3.

In a study of \( ^{12}\text{C} \) the value of \( \beta \) was determined from the first minimum of the elastic form factor to be \( \beta = 1.74 \text{ fm} \) [21]. Inspection of Fig. 3 shows that for this value of \( \beta \), the first 6 neutrons occupy the intrinsic states with \( \Omega = E^{(+)}_{1/2} \) (black), \( E_{3/2} \) (blue) and \( E^{(-)}_{1/2} \) (red), so that the last neutron in \( ^{13}\text{C} \) occupies the intrinsic state with \( E^{(-)}_{1/2} \) (red), followed by \( E^{(+)}_{1/2} \) (black).

The representations \( E^{(+)}_{1/2}, E^{(-)}_{1/2} \) and \( E_{3/2} \) can be further decomposed into values of \( K \) [31].

\[
\begin{align*}
\Omega &= E^{(+)}_{1/2} : \quad K^P = \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}, \ldots \\
\Omega &= E^{(-)}_{1/2} : \quad K^P = \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}, \ldots \\
\Omega &= E_{3/2} : \quad K^P = \frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \ldots (10)
\end{align*}
\]

with \( n = 1, 2, 3, \ldots \), and \( K > 0 \). The angular momenta are given by \( J = K, K + 1, K + 2, \ldots \).
Table 3. Resolution of single-particle levels into irreducible representations of $D'_{3h}$. Each $E$ level is doubly degenerate.

|          | $E_{1/2}^{(+)}$ | $E_{1/2}^{(-)}$ | $E_{3/2}$ |
|----------|-----------------|-----------------|-----------|
| $s_{1/2}$ | 1               | 0               | 0         |
| $p_{1/2}$ | 0               | 1               | 0         |
| $p_{3/2}$ | 0               | 1               | 1         |
| $d_{3/2}$ | 1               | 0               | 1         |
| $d_{5/2}$ | 1               | 1               | 1         |
| $f_{5/2}$ | 1               | 1               | 1         |
| $f_{7/2}$ | 2               | 1               | 1         |

As a result, the rotational sequences for each one of the irreducible representations of $D'_{3h}$ are given by (see also Table 3)

\[
\Omega = E_{1/2}^{(+)} : \quad J^P = \frac{1}{2}^+; \frac{3}{2}^+; \frac{5}{2}^\pm; \left(\frac{7}{2}\right)^2; \frac{9}{2}^+; \left(\frac{9}{2}\right)^2 \ldots
\]

\[
\Omega = E_{1/2}^{(-)} : \quad J^P = \frac{1}{2}^-; \frac{3}{2}^\pm; \left(\frac{7}{2}\right)^2; \frac{7}{2}^-; \left(\frac{9}{2}\right)^2; \frac{9}{2}^- \ldots
\]

\[
\Omega = E_{3/2} : \quad J^P = \frac{3}{2}^\pm; \frac{5}{2}^\pm; \left(\frac{9}{2}\right)^2 \ldots
\]

(11)

The angular momentum structure of each one of the representations of $D'_{3h}$ is shown in Fig. 4.

The rotational energy spectra can be analyzed with

\[
E_{\text{rot}}(\Omega, K, J) = \varepsilon_{\Omega} + A_{\Omega} \left[J(J+1) + b_{\Omega} K^2 + a_{\Omega} (-1)^{J+1/2} (J + 1/2) \delta_{K,3/2}\right],
\]

(12)

where $\varepsilon_{\Omega}$ is the intrinsic energy, $A_{\Omega} = \hbar^2 / 2I$ the inertial parameter, $b_{\Omega}$ a Coriolis term, and $a_{\Omega}$ the decoupling parameter. The latter term applies only to representations $\Omega = E_{1/2}^{(\pm)}$ and $K^P = 1/2^\pm$. Eq. (12) is the same energy formula as used by Nuhn in Ref. [32] in a description of the bandheads of $^{13}\text{C}$ in the context of a two-center shell model description of the system $^{13}\text{C} + ^{16}\text{O} \rightarrow ^{29}\text{Si}$. Nuhn used an axially symmetric potential, in contrast to a cluster potential with triangular $D_{3h}$ symmetry in the CSM. As a consequence, in the CSM the $K^P = 1/2^-$ and $5/2^+$ bands belong to the same configuration $\Omega = E_{1/2}^{(-)}$ (see Eq. (10) and Fig. 4), whereas in the axially symmetric case they represent separate rotational bands.

Fig. 5 shows the rotational bands of $^{13}\text{C}$. The ground-state band has $K^P = 1/2^-$ and is assigned to the representation $\Omega = E_{1/2}^{(-)}$ of $D'_{3h}$ (blue lines and filled circles) arising from the coupling of the ground-state band in $^{12}\text{C}$ to the intrinsic state with $E_{1/2}^{(-)}$. According to Eq. (10), this representation contains also $K^P = 5/2^+$ and $7/2^+$ bands, both of which appear to have been observed. In the shell model, positive parity states are expected to occur at much higher energies since they come from the $s$-$d$ shell. The first excited rotational band has $K^P = 1/2^+$.
| $D_{3h} : A'_{1}$ | $D'_{3h} : E_{1/2}^{(-)}$ | $D'_{3h} : E_{1/2}^{(+)}$ | $D'_{3h} : E_{3/2}$ |
|-----------------|-----------------|-----------------|-----------------|
| $- 4^+ - 4^-$   | $- 9^- - 9^+ - 9^+$ | $- 9^+ - 9^- - 9^-$ | $- 9^± - 9^±$ |
| $- 3^-$         | $- 7^- - 7^+ - 7^+$ | $- 7^+ - 7^- - 7^-$ | $- 7^±$ |
| $- 2^+$         | $- 5^- - 5^+$     | $- 5^+ - 5^-$     | $- 5^±$ |
| $- 0^+$         | $- 3^- - 1^- $    | $- 3^+ - 1^+$     | $- 3^±$ |
| $0^+ 3^-$       | $1^- 5^+ 7^+$     | $1^+ 5^- 7^-$     | $3^± 9^±$ |

**Figure 4.** Structure of rotational bands for a triangular configuration of $\alpha$ particles in even-cluster nuclei (first panel) and odd-cluster nuclei with $E_{1/2}^{(-)}$, $E_{1/2}^{(+)}$ and $E_{3/2}$ symmetry (second, third and fourth panel). Each rotational band is labeled by the quantum numbers $K^P$.

![Graph](image)

**Figure 5.** Rotational bands in $^{13}\text{C}$ [31].

which can be assigned to $\Omega = E_{1/2}^{(+)}$ (black line and filled squares) arising from the coupling of the ground-state band in $^{12}\text{C}$ to the excited intrinsic state with $E_{1/2}^{(+)}$. In contrast to the ground-state band, this excited band has a large decoupling parameter. In addition, Fig. 5 shows evidence for the occurrence of a rotational band at an energy slightly higher than that of the Hoyle state in $^{12}\text{C}$ which is interpreted as the coupling of the Hoyle band in $^{12}\text{C}$ to the ground-state intrinsic
Further evidence for the occurrence of $D'_{3h}$ symmetry in $^{13}$C is provided by electromagnetic transition rates. The $B(EL)$ values in $^{13}$C are related to those in $^{12}$C [31] (see also Fig. 6),

$$
B(E2; 3/2^- \rightarrow 1/2^-) = 0.99 \times B(E2; 2^- \rightarrow 0^+),
B(E2; 5/2^- \rightarrow 1/2^-) = 0.66 \times B(E2; 2^- \rightarrow 0^+),
B(E3; 5/2^- \rightarrow 1/2^-) = 0.57 \times B(E3; 3^- \rightarrow 0^+). \quad (13)
$$

The numerical factors arise from combinations of Clebsch-Gordan coefficients. The results are shown in Table 4. The good agreement with the experimental $B(EL)$ values in $^{13}$C shows that the states $3/2^-, 5/2^-$ and $5/2^+$ belong to the same rotational band with $\Omega = E_(1/2)^{(-)}$. Just as in the case of $^{12}$C there is a large octupole transition which is obtained in the CSM without the need to introduce effective charges.

**Table 4.** $B(EL)$ values in $^{13}$C. Experimental data taken from [33], and the CSM values from [18, 31].

|               | CSM     | Exp        |
|---------------|---------|------------|
| $B(E2; 3/2^- \rightarrow 1/2^-)$ | 8.3     | 6.4 ± 1.5  |
| $B(E2; 5/2^- \rightarrow 1/2^-)$ | 5.5     | 5.6 ± 0.4  |
| $B(E3; 5/2^+ \rightarrow 1/2^-)$ | 42      | 100 ± 40   |

4. **Summary and conclusions**

In this contribution, we have presented a combined analysis of the rotation-vibration spectra and electromagnetic transition rates in $^{12}$C and $^{13}$C. A comparison with the theoretical predictions of the algebraic cluster model ($^{12}$C) and the cluster shell model ($^{13}$C) provides strong evidence for the occurrence of triangular symmetry in these nuclei. A characteristic feature of the triangular
symmetry is the appearance of rotational bands consisting of both positive and negative parity states. The quadrupole and octupole transitions in $^{12}$C and $^{13}$C are strongly correlated and only depend on a single coefficient $\beta$ whose value was determined independently from the first minimum in the elastic form factor of $^{12}$C. The good agreement between theory and experiment supports the interpretation of the nucleus $^{13}$C as a system of three $\alpha$-particles in a triangular configuration plus an additional neutron moving in the deformed field generated by the cluster.

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