A LINDERSTRAUSS THEOREM FOR SOME CLASSES OF MULTILINEAR MAPPINGS

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Abstract. Under some natural hypotheses, we show that if a multilinear mapping belongs to some Banach multilinear ideal, then it can be approximated by multilinear mappings belonging to the same ideal whose Arens extensions simultaneously attain their norms. We also consider the class of symmetric multilinear mappings.

Introduction

The Bishop-Phelps theorem [6, 7] is an elementary and significant result about continuous linear functionals and convex sets. The most quoted version asserts that the set of linear functionals in $X'$ (the dual space of a Banach space $X$) which attain their supremum on the unit ball of $X$ are norm-dense in $X'$. Lindenstrauss showed that this is not true, in general, for linear bounded operators between two Banach spaces $X$ and $Y$ [22]; while he proved that the set of bounded linear operators whose second adjoints attain their norm, is always dense in the space of all bounded operators. This result was later extended for multilinear operators by Acosta, García and Maestre [4]. These kinds of results are referred to as Lindenstrauss-type theorems.

Regarding multilinear mappings, the Bishop-Phelps theorem fails, in general, even for scalar-valued bilinear forms [2, 17]. In order to handle the study of Lindenstrauss-type multilinear results, the Arens extensions come into scene [5]. The first result in this setting was given by Acosta [1] who proved that the set of bilinear forms on a product on two Banach spaces $X$ and $Y$ such that their Arens extensions are norm attaining is dense in the space of bilinear forms $L(2X \times Y)$. Aron, García and Maestre [3] obtained an improvement by showing that the set of those mappings in $L(2X \times Y)$ such that the two possible Arens extensions attain the norm at the same element of $X \times Y$, is dense in $L(2X \times Y)$.

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last result is stronger than the previous one since there exist bilinear mappings such that only one of their Arens extensions attains the norm \[3\]. In \[4\], the authors prove that the strongest version holds with full generality for multilinear mappings. They also give several positive results of the kind for some multilinear ideals considering the ideal norm instead of the supremum norm.

In this paper, we show that a multilinear Lindenstrauss theorem holds for any ideal of trilinear forms and, as a consequence, we obtain the same result for any ideal of bilinear operators. Then, we prove that an \(N\)-linear Lindenstrauss theorem holds for any \(N \in \mathbb{N}\), for a wide class of multilinear operator ideals which preserve some algebraic structure related to multiplicativity (we say these ideals are stable). Our results include the classes of nuclear, integral, extendible, multiple \(p\)-summing operators \((1 \leq p < \infty)\). Also, if we consider multilinear mappings on Hilbert spaces, the class of Hilbert-Schmidt and, more generally, the multilinear Schatten classes are encompassed. It was observed in \[4\] that any ideal which is dual to an associative tensor norm satisfies the multilinear Lindenstrauss theorem. These ideals are easily seen to be stable, but the converse is not true: the ideal of multiple 2-summing operators is stable, and we show in Proposition \[2.3\] that it cannot be dual to any associative tensor norm. Our list of examples then extends and completes the multilinear ideals treated in \[4\].

In \[15\] we gave an integral representation formula for the duality between tensor products and polynomials on Banach spaces satisfying appropriate hypothesis. Here, we extend \[15\] Theorem 2.2 for the duality between tensor products and symmetric linear operators. As a consequence, we obtain a positive Lindenstrauss-type theorem for symmetric linear operators between Banach spaces satisfying the same appropriate hypotheses.

1. Lindenstrauss theorem in multilinear ideals

Let us fix some notation. Throughout this paper \(X\) and \(Y\) denote Banach spaces, while \(X'\) and \(B_X\) denote respectively the topological dual and the closed unit ball of \(X\). For Banach spaces \(X_1, \ldots, X_N\) we denote the product space by \(X = X_1 \times \cdots \times X_N\) and \(\mathcal{L}(N X; Y)\) stands for the space of continuous \(N\)-linear operators \(\Phi: X \to Y\) endowed with the supremum norm. Recall that the Arens extensions of a multilinear function are obtained by weak-star density. Each extension depends on the order in which the variables are extended. Here we present one of the \(N!\) possible extensions (see \[5\] and \[18\] 1.9]). Given \(\Phi \in \mathcal{L}(N X; Y)\), the
mapping \( \Phi : X_1'' \times \cdots \times X_N'' \rightarrow Y'' \) is defined by

\[
\Phi(x_1'', \ldots, x_N'') = w^* - \lim_{\alpha_1} \ldots \lim_{\alpha_N} \Phi(x_{1,\alpha_1}, \ldots, x_{N,\alpha_N})
\]

where \((x_{j,\alpha_j})_{\alpha_j} \subseteq X\) is a net \(w^*\)-convergent to \(x_j'' \in X_j''\), \(j = 1, \ldots, N\). For \(N = 1\) this recovers the definition of the bitranspose of a continuous operator. Now, we recall the definition of a multilinear operator ideal.

**Definition 1.1.** A normed ideal of \(N\)-linear operators is a pair \((U, \| \cdot \|_U)\) such that for any \(N\)-tuple of Banach spaces \(X = X_1 \times \cdots \times X_N\) satisfies

(i) \(U(X; Y) = U \cap L^N(X; Y)\) is a linear subspace of \(L(X; Y)\) for any Banach space \(Y\) and \(\| \cdot \|_U\) is a norm on it.

(ii) For any \(N\)-tuple of Banach spaces \(Z = Z_1 \times \cdots \times Z_N\), any Banach space \(W\) and operators \(T_i \in L(Z_i; X_i), 1 \leq i \leq N\), \(S \in L(Y; W)\) and \(\Phi \in U(X; Y)\) the \(N\)-linear mapping \(S \circ \Phi \circ (T_1, \ldots, T_n) : Z \rightarrow W\) given by

\[
S \circ \Phi \circ (T_1, \ldots, T_N)(z_1, \ldots, z_N) = S(\Phi(T_1(z_1), \ldots, T_n(z_N)))
\]

belongs to \(U(Z; W)\) with \(\|S \circ \Phi \circ (T_1, \ldots, T_N)\|_U \leq \|S\| \|\Phi\|_U \|T_1\| \cdots \|T_N\|\).

(iii) \((z_1, \ldots, z_N) \mapsto z_1 \cdots z_N\) belongs to \(U(C^N; C)\) and has norm one.

If the pair \((U, \| \cdot \|_U)\) is closed, we say that \((U, \| \cdot \|_U)\) is a Banach ideal of \(N\)-linear mappings.

For \(N \in \mathbb{N}\) and \(j = \{j_1, \ldots, j_p\}\) a subset of the initial set \(\{1, \ldots, N\}\), \(j_1 < j_2 < \cdots < j_p\), we define \(P_j : X \rightarrow X\) the projection given by

\[
P_j(x_1, \ldots, x_N) := (y_1, \ldots, y_N), \quad y_k = \begin{cases} x_k & \text{if } k \in j \\ 0 & \text{if } k \notin j \end{cases}
\]

If \(j^c\) denotes the complement of \(j\) in \(\{1, \ldots, N\}\), then \(P_j + P_{j^c} = Id\) the identity map.

Let us define a rather natural property for multilinear ideals which will ensure the validity of a Lindenstrauss-type theorem. Let us say that the ideal of \(N\)-linear forms \(U\) is stable at \(X\) if there exists \(K > 0\) such that for all \(a = (a_1, \ldots, a_N) \in X\) and all \(j \subset \{1, \ldots, N\}\), the operator \(V_{j,a} : L^N(X) \rightarrow L^N(X)\) defined by

\[
V_{j,a}(\Phi)(x) = \Phi(P_j(x) + P_{j^c}(a))\Phi(P_j(a) + P_{j^c}(x))
\]

satisfies

\[
V_{j,a}(\Phi) \in U(X) \quad \text{for all } R \in U(X) \quad \text{and} \quad \|V_{j,a}(\Phi)\|_U \leq K\|\Phi\|^2_U\|a_1\| \cdots \|a_N\|.
\]
In order to see that being stable is a natural property, take $N = 4$ and $j = \{1, 2\}$. In this case, what we are imposing to a 4-linear form $\Phi \in U(X)$ is that the mapping 

$$(x_1, x_2, x_3, x_4) \mapsto \Phi(x_1, x_2, a_3, a_4) \Phi(a_1, a_2, x_3, x_4)$$

also belongs to $U(X)$, for any $(a_1, \cdots, a_4)$ with some control on the norm. The next result extends [4, Theorem 2.1] and [4, Corollary 2.5].

**Theorem 1.2.** If the ideal of $N$-linear forms $U$ is stable at $X = X_1 \times \cdots \times X_N$, then the set of $N$-linear forms in $U(X)$ whose Arens extensions attain the supremum-norm at the same $N$-tuple is $\| \cdot \|_{U}$-dense in $U(X)$.

**Proof.** Fix $\Phi \in L(NX)$. By the proof of [4, Theorem 2.1], there exist a sequence of multilinear mappings $(\Phi_n)_n$ given recursively by

$$\Phi_1 = \Phi, \quad \Phi_{n+1} = \Phi_n + \sum_j C_n V_j a^n(\Phi_n),$$

where $(C_n)_n$ is a sequence of positive numbers, $(a^n)_n \subset X$ and $V_j a^n$ is defined as in (2) for all $j \subset \{1, \ldots, N\}$. In [4, Theorem 2.1], it is shown that given $\varepsilon > 0$, $(C_n)_n$ and $a^n$ can be chosen so that $(\Phi_n)_n$ converges to an element $\Psi \in L(NX)$ whose Arens extensions attain their norm at the same $N$-tuple and $\| \Phi - \Psi \| < \varepsilon$.

The stability of $U$ implies that every $\Phi_n$ belongs to $U(X)$ whenever $\Phi \in U(X)$. Now, the control of the norms given in (3) together with a careful reading of the proof of [4, Corollary 2.5] ensure that $(C_n)_n$ and $a^n$ can be chosen so that $(\Phi_n)_n$ converges to an element $\Psi = \| \cdot \|_U - \lim \Phi_n$ and $\| \Phi - \Psi \|_U < \varepsilon$. \qed

Given an $(N + 1)$-linear form $\Phi: X_1 \times \cdots \times X_N \times X_{N+1} \to \mathbb{K}$, we define the associate $N$-linear operator $\tilde{\Phi}: X_1 \times \cdots \times X_N \to X'_{N+1}$ as usual:

$$\tilde{\Phi}(x_1, \ldots, x_N)(x_{N+1}) = \Phi(x_1, \ldots, x_N, x_{N+1}).$$

Now, given an ideal of $N$-linear operators $U$ we define the ideal of $N + 1$-linear forms $\tilde{U}$ by

$$\Phi \in \tilde{U} \quad \text{if and only if} \quad \tilde{\Phi} \in U$$

and

$$\| \Phi \|_{\tilde{U}} = \| \tilde{\Phi} \|_U.$$

It should be noted that if $\Phi$ in the proof of the previous theorem is $w^*$-continuous in the last variable, then so is $\Psi$. As a consequence, we can proceed as in [4, Theorem 2.3] to obtain the following.
Corollary 1.3. With the notation above, if $\tilde{U}$ is stable at $X_1 \times \cdots \times X_N \times Y'$, then the set of $N$-linear operators in $\mathcal{U}(X; Y)$ such that their Arens extensions attain the supremum-norm at the same $N$-tuple is $\| \cdot \|_{\tilde{U}}$-dense in $\mathcal{U}(X; Y)$.

We will see that most of the known examples of multilinear ideals are stable and, then, satisfy a Lindenstrauss-type theorem. First, let us see that this property is fulfilled for every ideal of trilinear forms. Hence, we have a Lindenstrauss theorem for ideals of trilinear forms and of bilinear operators.

Corollary 1.4. Let $(\mathcal{U}, \| \cdot \|_U)$ be a Banach ideal of 3-linear forms. Then, for every $X = X_1 \times X_2 \times X_3$, the set of trilinear forms in $\mathcal{U}(X)$ whose Arens extensions attain the supremum-norm at the same 3-tuple is $\| \cdot \|_{\tilde{U}}$-dense in $\mathcal{U}(X)$.

Proof. By Theorem 1.2, it suffices to prove that $\mathcal{U}(X)$ is stable. Take $\Phi \in \mathcal{U}(X)$, $a = (a_1, a_2, a_3) \in X$ with $\|a_k\| = 1$ for $1 \leq k \leq 3$ and take $j \subset \{1, 2, 3\}$. We proceed to show that (3) is satisfied for $j = \{1, 2\}$, the other cases are analogous.

Consider the linear operator $T_{\Phi}: X_3 \to X_3$ defined by $T_{\Phi}(x_3) = \Phi(a_1, a_2, x_3) a_3$. Then $\|T_{\Phi}\| \leq \|\Phi\| \leq \|\Phi\|_{\mathcal{U}}$ and

\[
V_{j,a}(\Phi)(x) = \Phi(x_1, x_2, a_3) \Phi(a_1, a_2, x_3) = \Phi(x_1, x_2, T_{\Phi}(x_3)) = \Phi \circ (I, I, T_{\Phi})(x_1, x_2, x_3).
\]

Since $(\mathcal{U}, \| \cdot \|_U)$ is a Banach ideal, $V_{j,a}(\Phi) \in \mathcal{U}(X)$ and $\|V_{j,a}(\Phi)\|_{\tilde{U}} \leq \|\Phi\|_{\mathcal{U}}^2$. □

Now the following corollary is immediate.

Corollary 1.5. Let $(\mathcal{U}, \| \cdot \|_U)$ be a Banach ideal of bilinear operators and let $X_1, X_2, Y$ be Banach spaces. The set of bilinear operators of $\mathcal{U}(X_1 \times X_2; Y)$ attaining their norm at the same pair is $\| \cdot \|_{\tilde{U}}$-dense in $\mathcal{U}(X_1 \times X_2; Y)$.

2. Examples of multilinear ideals satisfying a Lindenstrauss theorem

Let us start with the simplest examples. Any finite type multilinear form on $X_1 \times \cdots \times X_N$ (i.e., any linear combinations of products of linear forms), has a unique Arens extension to $X'_1 \times \cdots \times X'_N$ which is weak-star continuous on each coordinate. By the Banach-Alaogu’s theorem, this extension must attain its supremum norm. As a consequence, every Banach ideal in which finite type multilinear operators are dense will satisfy the multilinear Lindenstrauss theorem. This is the case, for instance, of the ideal of nuclear multilinear operators.
More generally, if $U$ is a minimal ideal of multilinear forms, the finite type multilinear operators are dense in $U$ [20, 21] and Lindenstrauss theorem trivially holds.

As observed in [4], if $U$ is an ideal of multilinear forms which is dual to an associative tensor norm (such as the injective or projective tensor norms $\varepsilon$ and $\pi$), then $U$ satisfies the multilinear Lindenstrauss theorem. We can rephrase their remark in our terminology: ideals which are dual to associative tensor norms are always stable. In [4], it is mentioned that the ideal of multiple summing multilinear operators satisfies the Lindenstrauss theorem. We will see that this is truly the case, although this ideal is not dual to any associative tensor norm as we show in Example 2.3. Actually, it is not very usual for ideals of multilinear forms to be dual to associative tensor norms. Fortunately, in order to satisfy a multilinear Lindenstrauss theorem (in fact, in order to be stable), a much weaker property is sufficient. Coherent and multiplicative ideals of polynomials have been studied (also with different terminologies) in [13, 9]. Here, we present a multilinear version of these properties (see [10], where similar properties for multilinear mappings are considered). To simplify the definitions, we restrict ourselves to symmetric ideals, although this is clearly not necessary.

Fix $N \in \mathbb{N}$ and $X = X_1 \times \cdots \times X_N$. If $\theta$ is a permutation of $\{1, \ldots, N\}$, we write $X_\theta = X_{\theta(1)} \times \cdots \times X_{\theta(N)}$ and $x_\theta = (x_{\theta(1)}, \ldots, x_{\theta(N)})$. We say that the ideal of multilinear operators $U_N$ is symmetric if for any $\Phi \in U_N(X; Y)$ and every permutation $\theta$ of $\{1, \ldots, N\}$, the $N$-linear mapping $\theta\Phi: X_\theta \to Y$, $\theta\Phi(x_\theta) = \Phi(x)$

belongs to $U_N(X_\theta; Y)$ with $\|\theta\Phi\|_{U_N} = \|\Phi\|_{U_N}$.

**Definition 2.1.** Let $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ be a sequence where $U_n$ is a symmetric Banach ideal of $n$-linear forms for each $n$. We say that $\mathcal{U}$ is multiplicative if there exist positive constants $C$ and $D$ such that, for any $N \in \mathbb{N}$ and any $X = X_1 \times \cdots \times X_N$:

(i) If $\Phi \in U_N(X)$ and $a_N \in X_N$, the $(N-1)$-linear form $\Phi_{a_N}$ given by $\Phi_{a_N}(x_1, \ldots, x_{N-1}) = \Phi(x_1, \ldots, x_{N-1}, a_N)$,

belongs to $U_{N-1}(X_1 \times \cdots \times X_{N-1}; Y)$ and $\|\Phi_{a_N}\|_{U_{N-1}} \leq C \|\Phi\|_{U_N} \|a_N\|$.

(ii) If $\Phi \in U_k(X_1 \times \cdots \times X_k)$, and $\Psi \in U_{N-k}(X_{k+1} \times \cdots \times X_N)$ the $N$-linear form $\Phi \cdot \Psi$ given by $(\Phi \cdot \Psi)(x_1, \ldots, x_N) = \Phi(x_1, \ldots, x_k)\Psi(x_{k+1}, \ldots, x_N)$
belongs to $\mathcal{U}_N(X)$ and $\|\Phi \cdot \Psi\|_{\mathcal{U}_N} \leq D^N \|\Phi\|_{\mathcal{U}_k} \|\Psi\|_{\mathcal{U}_{N-k}}$.

It is rather easy to see that if $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ is a multiplicative sequence of multilinear ideals, then $\mathcal{U}_n$ is stable at any Banach space, for any $n \in \mathbb{N}$. What makes this concept interesting in our framework is that most of the usual ideals of multilinear forms have been proven to be multiplicative. For example, the ideals of nuclear, integral, extendible, multiple $p$-summing multilinear forms ($1 \leq p < \infty$) are multiplicative (see [12, 14, 24] for the proof in the polynomial case, the multilinear one being analogous). Also, if we consider multilinear forms on Hilbert spaces, the class of Hilbert-Schmidt and, more generally, the multilinear Schatten classes are multiplicative. As a consequence, since Hilbert spaces are reflexive, all these ideals satisfy a multilinear Bishop-Phelps theorem.

We end this section by showing that the ideal of multiple summing operators is not dual to any associative tensor norm.

**Definition 2.2.** Let $1 \leq p < \infty$. A multilinear form $\Phi : X_1 \times \cdots \times X_N \to \mathbb{K}$ is multiple $p$-summing if there exists $K > 0$ such that for any sequences $(x_j^i)_{i_j=1}^{m_j} \subseteq X_j$, $j = 1, \ldots, N$, we have

\[
\left( \sum_{i_1, \ldots, i_N=1}^{m_1, \ldots, m_N} |T(x_{i_1}^1, \ldots, x_{i_N}^N)|^p \right)^{1/p} \leq K \prod_{j=1}^{N} \| (x_j^i)_{i_j=1}^{m_j} \|_p.
\]

The least constant $K$ satisfying the inequality is the $p$-summing norm of $T$ and is denoted by $\pi_p(T)$. We write $\Pi_p^N(X_1 \times \cdots \times X_N)$ for the space of multiple $p$-summing forms.

Recall that $\Pi_p^N(X_1 \times \cdots \times X_N)$ is the dual of the tensor product $X_1 \otimes \cdots \otimes X_N$ endowed with the tensor norm $\tilde{\alpha}_p(u)$ [23, Proposition 3.1] (see also [23]) where

\[
\tilde{\alpha}_p(u) = \inf \left\{ \sum_{m=1}^{M} \| (\lambda_{m,i_1^m, \ldots, i_N^m})_{i_1^m, \ldots, i_N^m=1}^{I_1^m, \ldots, I_N^m} \|_{p'} \cdot \|(x_{m,i_1^m}^1)_{i_1^m=1}^{I_1^m} \|_p \cdots \|(x_{m,i_N^m}^N)_{i_N^m=1}^{I_N^m} \|_p \right\}
\]

whith $\frac{1}{p'} + \frac{1}{p} = 1$ and the infimum is taken over all the representations of the form

\[
u = \sum_{m=1}^{M} \sum_{i_1^m, \ldots, i_N^m=1}^{I_1^m, \ldots, I_N^m} \lambda_{m,i_1^m, \ldots, i_N^m} x_{m,i_1^m}^1 \otimes \cdots \otimes x_{m,i_N^m}^N.
\]

Although we know that the $(\Pi_2^n)_n$ is a multiplicative sequence, we have the following.

**Proposition 2.3.** The ideal $\Pi_2^N$ ($N \in \mathbb{N}$) of multiple 2-summing forms is not dual to any associative tensor norm.
Proof. Take $N = 4$ and $X_1 = \cdots = X_4 = c_0$. Suppose that $\beta$ is an associative tensor norm of order 2 such that

$$\Pi^4_2(c_0 \times \cdots \times c_0) \simeq \left( (c_0 \hat{\otimes}_\beta c_0) \hat{\otimes}_\beta (c_0 \hat{\otimes}_\beta c_0) \right)'.$$

By [8, Theorem 3.1], every multilinear form on $c_0$ is multiple 2-summing. As a consequence, $\pi$ and the tensor norm predual to the multiple 2-summing forms should be equivalent on $c_0 \otimes \cdots \otimes c_0$. Using this fact first for the 4-fold and then for the 2-fold tensor products, we have

$$(c_0 \hat{\otimes}_\pi c_0) \hat{\otimes}_\pi (c_0 \hat{\otimes}_\pi c_0) \simeq (c_0 \hat{\otimes}_\beta c_0) \hat{\otimes}_\beta (c_0 \hat{\otimes}_\beta c_0) \simeq (c_0 \hat{\otimes}_\pi c_0) \hat{\otimes}_\pi (c_0 \hat{\otimes}_\pi c_0).$$

In [11] it is shown that $c_0 \hat{\otimes}_\pi c_0$ has uniformly complemented copies of $\ell^n_2$. Then, the isomorphisms given above imply that

$$\ell^n_2 \otimes_\pi \ell^n_2 \simeq \ell^n_2 \otimes_\beta \ell^n_2,$$

uniformly in $n \in \mathbb{N}$. The Density Lemma [13, 13.4] then gives

$$\ell_2 \otimes_\pi \ell_2 \simeq \ell_2 \otimes_\beta \ell_2,$$

which means that every bilinear form on $\ell_2 \times \ell_2$ is multiple 2-summing. But in Hilbert spaces, multiple 2-summing and Hilbert-Schmidt multilinear forms coincide, and clearly there are bilinear forms which are not Hilbert-Schmidt. This contradiction completes the proof. \qed

3. INTEGRAL REPRESENTATION AND LINDENSTRAUSS THEOREM FOR MULTILINEAR SYMMETRIC MAPPINGS

In [15] we showed a Lindenstrauss theorem for homogeneous polynomials between Banach spaces satisfying some appropriate hypotheses. In the same lines, we briefly sketch the proof of the corresponding result for symmetric multilinear mappings.

Let us recall that a Banach space $X$ whose dual $X'$ is separable and enjoys the approximation property admits a bounded sequence of finite rank operators $(T_n)_n$ on $X$ such that both $T_n \rightarrow Id_X$ and $T_n' \rightarrow Id_{X'}$ in the strong operator topology [16, p.288-289]. This is a consequence of the principle of local reflexivity. The sequence $(T_n)_n$ clearly satisfies

$$(6) \quad T_n''(x'') \subseteq J_X(X) \quad \text{and} \quad T_n''(x'') \xrightarrow{w^*} x'' \text{ for all } x'' \in X'',$$

where $J_X : X \rightarrow X''$ is the canonical inclusion.
In the sequel, $X_1, \ldots, X_N, Y$ will be Banach spaces whose duals are separable spaces with the approximation property. We will denote by $(T^j_n)_n$ a sequence of finite rank operators satisfying (6) for $X_j$, $j = 1, \ldots, N$.

Given $\Phi \in \mathcal{L}(\bigotimes X; Y')$, for each $(n_1, \ldots, n_N) \in \mathbb{N}^N$ we define the finite type multilinear mapping

$$\Phi_{n_1, \ldots, n_N} = \Phi \circ (T^1_{n_1}, \ldots, T^n_{n_N}).$$

Proceeding as in [15, Lemma 2.1], we can see that the Arens extension of $\Phi$ is given by

$$\Phi_{n_1, \ldots, n_N}(x''_1, \ldots, x''_N) = \Phi_{n_1, \ldots, n_N}(x''_1, \ldots, x''_N) = \lim_{n_1 \to \infty} \cdots \lim_{n_N \to \infty} \Phi_{n_1, \ldots, n_N}(x''_1, \ldots, x''_N)(y'').$$

We claim that for each $u \in (\bigotimes X_n)_{\pi,s} Y$, there exists a regular Borel measure $\mu_u$ on $(B_{X''_1} \times \cdots \times B_{X''_N} \times B_{Y''}, w^*)$ such that $\|u\| \leq \|u\|_{\pi}$ and

$$\langle u, \Phi \rangle = \int_{B_{X''_1} \times \cdots \times B_{X''_N} \times B_{Y''}} \Phi^n(x''_1, \ldots, x''_N)(y'') d\mu_u(x''_1, \ldots, x''_N, y''),$$

for all $\Phi \in \mathcal{L}(\bigotimes X; Y')$.

Indeed, using the Riesz representation theorem for $C((B_{X''_1}, w^*) \times \cdots \times (B_{X''_N}, w^*) \times (B_{Y''}, w^*))$, we can find a measure $\mu$ satisfying (8) for finite type multilinear maps. Then, we can use (7), the Dominated convergence theorem and the density of linear combination of elementary tensors to show that (8) holds for every continuous multilinear map.

Our aim now is to show that the Lindenstrauss theorem holds for symmetric multilinear mappings into a dual space. Since symmetric $N$-linear operators are defined on $X$ for $X_1 = \cdots = X_N = X$, we simply write $\mathcal{L}_{s}(\bigotimes X; Y')$ to denote the space of those mappings (with values on a dual space $Y'$). Also we denote by $\bigotimes X_n^{N,s}$ the $N$-fold symmetric tensor product of $X$ endowed with the (full, not symmetric) projective tensor norm $\pi$. We refer to [19] for general theory on symmetric tensor products. Take $\Psi \in \mathcal{L}_{s}(\bigotimes X; Y')$ and $\varepsilon > 0$. We consider the associated linear function

$$L_{\Psi} \in \left(\bigotimes X_n^{N,s} Y\right)'.$$ 

By the Bishop-Phelps theorem, there exists a norm attaining functional $L = L_{\Phi}$ such that $\|\Psi - \Phi\| = \|L_{\Psi} - L_{\Phi}\| < \varepsilon$, for some $\Phi \in \mathcal{L}_{s}(\bigotimes X; Y')$. 

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Take \( u \in (\tilde{\otimes}_{\pi}^{N,s} X) \tilde{\otimes}_{\pi} Y \) with \( \|u\|_{\pi} = 1 \) such that \( |L_\Phi(u)| = \|L_\Phi\| = \|\Phi\| \) and take a regular Borel measure \( \mu_u \) satisfying (8). Then,
\[
\|\Phi\| = |L_\Phi(u)| \leq \int_{B_{X''} \times \cdots \times B_{X''} \times B_{Y''}} |\tilde{\Phi}(x''_1, \ldots, x''_N)(y'')| d|\mu_u|(x''_1, \ldots, x''_N, y'') \leq \|\tilde{\Phi}\| \|\mu_u\| \leq \|\Phi\|.
\]
As a consequence, \( |\tilde{\Phi}(x''_1, \ldots, x''_N)(y'')| = \|\Phi\| \) almost everywhere (for \( \mu_u \)) and \( \tilde{\Phi} \) attains its norm. We summarize the previous discussion in the following statement.

**Theorem 3.1.** Let \( X \) and \( Y \) be Banach spaces such that \( X' \) is separable and has the approximation property. Then, every symmetric multilinear mapping in \( \mathcal{L}_s(NX;Y') \) can be approximated by symmetric multilinear mappings whose Arens extensions attain the supremum-norm at the same \( N \)-tuple.

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