UPPER ENDPOINT ESTIMATES AND EXTRAPOLATION FOR COMMUTATORS

KANGWEI LI, SHELDY OMBROSI, AND ISRAEL P. RIVERA-RÍOS

Abstract. In this note we revisit the upper endpoint estimates for commutators following the line by Harboure, Segovia, and Torrea [Illinois J. Math. 41 no. 4 (1997), 676–700]. Relying upon the suitable BMO subspace suited for the commutator that was introduced in Accomazzo’s PhD thesis (2020), we obtain a counterpart for commutators of the upper endpoint extrapolation result by Harboure, Macías and Segovia [Amer. J. Math. 110 no. 3 (1988), 383–397]. Multilinear counterparts are provided as well.

1. INTRODUCTION AND MAIN RESULTS

Extrapolation has been a fruitful area of research since the 80s. The first results in that direction were due to Rubio de Francia [14, 13]. We briefly discuss the general principle behind that kind of results in the following lines.

We say that \( w \) is a weight if it is a non-negative locally integrable function on \( \mathbb{R}^n \). Recall that \( w \in A_p \) for \( 1 < p < \infty \) if

\[
[w]_{A_p} = \sup_Q \left( \int_Q w \left( \frac{1}{w} \right)^{p-1} \right)^{p-1} < \infty
\]

and that \( w \in A_1 \) if

\[
[w]_{A_1} = \left\| \frac{Mw}{w} \right\|_{L^\infty} < \infty,
\]

where \( M \) stands for the Hardy–Littlewood maximal function

\[
Mf(x) = \sup_{Q} \int_{Q} |f(y)| \, dy,
\]

where each \( Q \) is a cube of \( \mathbb{R}^n \) with its sides parallel to the axis.

A fundamental property of the \( A_p \) classes is that they characterize the weighted \( L^p \) boundedness of the Hardy–Littlewood maximal operator and they are good

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weights for a number of operators in the theory such as singular integrals, commu-
tators and some further ones.

The Rubio de Francia extrapolation results say that if $T$ is a sublinear operator
such that for some $1 < p_0 < \infty$
\begin{equation}
\|Tf\|_{L^{p_0}(w)} \leq c_{w,T,p_0} \|f\|_{L^{p_0}(w)}
\end{equation}
for every $w \in A_{p_0}$, then
\begin{equation}
\|Tf\|_{L^p(w)} \leq c_{w,T,p} \|f\|_{L^p(w)}
\end{equation}
for every $w \in A_p$ and every $1 < p < \infty$.

This approach has been extensively studied by a number of authors in a wide
variety of settings. For instance, in the linear setting there are fundamental works
due to Cruz-Uribe, Martell, and Pérez [9, 4, 5, 8, 7, 6], Duoandikoetxea [11, 12],
Dragicevic, Grafakos, Petermichl, Pereyra [10], Harboure, Macías, Segovia [18,
17]. After a number of intermediate results in the multilinear setting (see for
instance [16, 2, 7]) the question was succesfully solved in the last years, in works
such as [26, 25, 28].

A useful development in the area since Rubio de Francia’s pioneering works
consisted in learning that the operator involved in (1.1) and (1.2) actually plays no
role. To be more precise, it can be replaced by a condition on pairs of functions.
Assume that $F$ is a family of pairs of functions such that for some $1 < p_0 < \infty$
\begin{equation}
\|f\|_{L^{p_0}(w)} \leq c_{w,T,p_0} \|g\|_{L^{p_0}(w)}
\end{equation}
for every $(f, g) \in F$ and every $w \in A_{p_0}$, then
\begin{equation}
\|f\|_{L^p(w)} \leq c_{w,T,p} \|g\|_{L^p(w)}
\end{equation}
for every $(f, g) \in F$, for every $w \in A_p$ and every $1 < p < \infty$.

Another line of research would consist in considering the endpoints, namely
$p_0 = \infty$ or $p_0 = 1$ as a “departing” point for extrapolation. For instance, the
following result was obtained in [15, 9]

**Theorem 1.** Let $(f, g)$ be a pair of functions and suppose that
\begin{equation}
\|gw\|_{L^\infty} \leq c_w \|fw\|_{L^\infty}
\end{equation}
holds for all $w$ with $w^{-1} \in A_1$, where $c_w$ depends only on $[w^{-1}]_{A_1}$. Then for all
$1 < p < \infty$ and all $w \in A_p$, we have
\begin{equation}
\|g\|_{L^p(w)} \leq \tilde{c}_w \|f\|_{L^p(w)},
\end{equation}
where $\tilde{c}_w$ depends only on $[w]_{A_p}$.

There are a number of operators that do not map $L^\infty$ into $L^\infty$ such as the
Hilbert transform. However for the Hilbert transform $H$ itself and even for a larger
class of operators, the Calderón–Zygmund operators, it is possible to show that
they map $L^\infty$ into $BMO$. Weighted versions of that result were studied first in [27].
There it was shown that if $w \in A_1$, then
\begin{equation}
\int_Q |Hf - (Hf)_Q| \leq C \left\| \frac{f}{w} \right\|_{L^\infty} \operatorname{ess \ inf} Q \inf_{Q} w(x).
\end{equation}
In view of this estimate, it seems natural to think about extending this result to Calderón–Zygmund operators, and also, within the framework of extrapolation whether it would be possible to extrapolate from that weighted \( L^\infty \to BMO \) bound in order to obtain weighted \( L^p \) estimates. Those questions were answered in the positive in the inspiring paper \([18]\) by Harboure, Macías, and Segovia. In that work the following extrapolation result was settled.

**Theorem 2.** Let \( T \) be a sublinear operator defined on \( \mathcal{C}_0^\infty (\mathbb{R}^n) \) satisfying that for any cube \( Q \subset \mathbb{R}^n \) and any \( w \in A_1 \)

\[
\int_Q |Tf - (Tf)_Q| \leq c_{w,T} \left\| \frac{f}{w} \right\|_{L^\infty} \essinf_Q w.
\]

Then for every \( 1 < p < \infty \) and every \( w \in A_p \) we have that

\[
\|Tf\|_{L^p(w)} \leq c_w \|f\|_{L^p(w)}.
\]

Quite recently in \([3]\), a quantitative version of this result was obtained. In that paper it was shown that if \( \delta \in (0, 1) \) and

\[
\inf_{c \in \mathbb{R}} \left( \int_Q |Tf - c|^\delta \right)^{\frac{1}{\delta}} \leq c_{n,\delta,T} \varphi([w]_{A_1}) \left\| \frac{f}{w} \right\|_{L^\infty} \essinf_Q w,
\]

then

\[
\|Tf\|_{L^p(w)} \leq c_n \varphi(\|M\|_{L^p(w)}) \|M\|_{L^{p'}(\sigma)} \|f\|_{L^p(w)},
\]

where \( \sigma = w^{-\frac{1}{p-1}} \). Note that since \( \|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p'}} \) such an estimate yields

\[
\|Tf\|_{L^p(w)} \leq c_n \varphi([w]_{A_p}^{\frac{1}{p'}}) [w]_{A_p} \|f\|_{L^p(w)}.
\]

In the same paper it is shown that for Calderón–Zygmund operators

\[
\inf_{c \in \mathbb{R}} \left( \int_Q |Tf - c|^\delta \right)^{\frac{1}{\delta}} \leq c_{n,\delta,T} [w]_{A_1} \left\| \frac{f}{w} \right\|_{L^\infty} \essinf_Q w \tag{1.4}
\]

namely \( \varphi(t) = t \) and hence the sharp exponent for the \( A_p \) constant max \( \{1, \frac{1}{p-1}\} \) for that class is not recovered. Such a fact is not surprising since the current best known extrapolation argument from the lower endpoint neither recovers the sharp estimate. At this point we would like to note that a way more general version of the aforementioned extrapolation result, replacing \( L^p(w) \) spaces by function Banach spaces and the \( A_p \) constant by suitable boundedness constants of the maximal function over those spaces, was obtained very recently in \([29]\). Also a quantitative multilinear result in that direction was provided in \([28, Corollary 4.14]\).

Now we turn our attention to our contribution in this work. We recall that given \( b \in \text{BMO} \), the Coifman–Rochberg–Weiss commutator is defined as

\[
[b, T]f(x) = b(x)Tf(x) - T(bf)(x).
\]
It is well known that \([b, T]\) is bounded on \(L^p(w)\) and that, as Pérez showed in [30], \([b, T]\) is not of weak type \((1, 1)\) but it satisfies the following estimate instead:

\[
w(|\{ |[b, T]f(x)| > t\}|) \lesssim [w]^2_{A_1} \log(e + [w]_{A_1}) \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{t}\right) w,\]

where \(\Phi(t) = t \log(e + t)\). The quantitative dependence was obtained in [23].

In view of (1.3) and (1.4) one may wonder what can be said about commutators. In [19, Theorem A], Harboure, Segovia, and Torrea provided the following result.

**Theorem 3.** Let \(T\) be a Calderón–Zygmund operator and let \(b \in BMO\). Then the following statements are equivalent:

1. For every ball \(B\) and every \(f \in L^\infty_c(\mathbb{R}^n)\),
\[
\int_B |[b, T]f(x) - ([b, T]f)_B| dx \lesssim \|f\|_{L^\infty} . \tag{1.5}
\]
2. The function \(b\) satisfies the following condition. For any cube \(Q\) and \(u \in Q\)
\[
\left(\int_Q |b - b_Q| T(f\chi_{(2Q)^c})(u) \right) \leq C\|f\|_{L^\infty(\mathbb{R}^n)}
\]
for every \(f \in L^\infty_c(\mathbb{R}^n)\).

Also in [19] the authors point out that if \(T\) is the Hilbert transform and any of the conditions in the preceding Theorem are satisfied, then necessarily \(b\) is constant and hence \([b, H] = 0\). This fact leads us to think about the possibility of considering a “smaller” oscillation in the left hand side of (1.5).

Aiming for a dual of the Hardy spaces for commutators studied by Pérez [30] and Ky [21], Accomazzo introduced in [1] the spaces \(BMO_q^b\) which are defined as follows. Given a function \(b\), and \(q \in [1, \infty)\) we have that \(f \in BMO_q^b\) if

\[
\|f\|_{BMO_q^b} := \sup_B \left( \inf_{c_0, c_1 \in \mathbb{R}} \int_B |f(x) - c_0 + c_1 b(x)|^q \right)^{\frac{1}{q}}.
\]

Note that \(\|f\|_{BMO_q^b} = 0\) if and only if \(f = \alpha + \beta b\) and hence in order to consider \(\|f\|_{BMO_q^b}\) as a norm one needs to take quotient by the subspace \(\langle 1, b \rangle\) (the space of linear combinations of 1 and \(b\)). It readily follows from the definition that \(BMO \subset BMO_q^b\) for every \(q\). It is also easy to show that if \(b \in BMO\) then \(b^2 \in BMO_q^b\). And for instance choosing \(b(x) = \log(x)\), we have that \(\log(x)^2 \in BMO_q^b \setminus BMO\).

Inspired by the definition of \(BMO_q^b\) we provide the following result for commutators.

**Theorem 4.** Let \(b \in BMO\) and \(T\) a Calderón–Zygmund operator satisfying a log-Dini regularity condition. Then for every ball \(B\), if \(\delta \in (0, 1)\) and \(r > 1\) we have that

\[
\inf_{c_1 \in \mathbb{R}} \left( \int_B |[b, T]f(x) - c_1 - T(f\chi_{(2B)^c})(c_B)b(x)|^\delta dx \right)^{\frac{1}{\delta}} \lesssim r' \|f\|_{L^\infty} \|b\|_{BMO} \inf_{z \in B} M_r w(z).
\]

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Consequently,

\[
\inf_{c_1, c_2 \in \mathbb{R}} \left( \int_B \left| [b, T]f(x) - c_1 - c_2 b(x) \right|^\delta dx \right)^{\frac{1}{\delta}} \lesssim r' \left\| f \right\|_{L^\infty} \left\| b \right\|_{\text{BMO}} \inf_{z \in B} M_v w(z).
\]

The next natural question would be whether it is possible to extrapolate from the condition above. We show that that is the case under some additional conditions.

**Theorem 5.** Let \( T \) be a linear operator such that for every \( b \in \text{BMO} \) and every \( w \in A_1 \)

\[
\inf_{c_1 \in \mathbb{R}} \left( \int_B \left| [b, T]f(x) - c_1 - T (f \chi_{(2B)^c})(x) \right|^\delta dx \right)^{\frac{1}{\delta}} \leq c_{w, T} \left\| f \right\|_{L^\infty} \left\| b \right\|_{\text{BMO}} \inf_{z \in B} w(z)
\]

and such that Lerner’s grand maximal operator

\[
\mathcal{M}_T f(x) = \sup_{z \in B} \text{ess sup}_{z \in B} T (f \chi_{(2B)^c})(z)
\]

is bounded on \( L^p(v) \) for some \( p \in (1, \infty) \) and some \( v \in A_p \). Then

\[
\left\| [b, T]f \right\|_{L^p(v)} \leq c_{v, T} \left\| b \right\|_{\text{BMO}} \left\| f \right\|_{L^p(v)}.
\]

Observe that the operator \( \mathcal{M}_T \) was introduced in [22] in order to study sparse domination. There it was shown that in the case of \( T \) being a Calderón–Zygmund operator

\[
\mathcal{M}_T f(x) \lesssim Mf(x) + T^* f(x),
\]

where \( T^* \) stands for the maximal Calderón–Zygmund operator. Since both \( M \) and \( T^* \) are bounded on \( L^p(w) \) for \( w \in A_p \), the result above combined with the estimate in Theorem [1.4] allows to provide an alternative proof of the weighted \( L^p \) boundedness of the commutator \([b, T]\).

Here we just presented the results in the linear setting. However results in the multilinear setting are feasible as well and will be obtained in Section 4.

The remainder of the paper is organized as follows. In Section 2 we gather some preliminaries.

In Section 3 we settle Theorems 4 and 5. Finally in Section 4 we present and settle the multilinear counterparts of the main results.

## 2. Preliminaries

We recall that \( T \) is a Calderón–Zygmund operator if \( T \) is a linear operator that is bounded on \( L^2 \) and it admits a representation in terms of a kernel \( K \)

\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \quad x \notin \text{supp} f,
\]

where \( K \) satisfies the following properties:
• Size condition: $|K(x, y)| \leq C_K |x - y|^{-n}$.

• Smoothness condition: Provided that $|x - y| \geq 2|x - z|$, 
  
  $|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq \omega \left( \frac{|x - z|}{|x - y|} \right) \frac{1}{|x - y|^n}$, 
  
  where $\omega$ is a continuous subadditive function such that 
  
  $\int_0^1 \omega(t) \log \left( \frac{1}{t} \right) \frac{dt}{t} < \infty$.

In the definition of commutators we used $BMO$ functions. We recall that $b \in BMO$ if 

$$\|b\|_{BMO} = \sup_B \int_B |b - b_B| < \infty.$$ 

A fundamental property of this space of functions is the well-known John–Nirenberg that says that the integrability of the oscillations self-improves to exponential integrability, namely, there exist constants $\lambda, c > 0$ such that for every ball $B$ and every $BMO$ function 

$$|\{x \in B : |b - b_B| > \lambda\}| \lesssim e^{-c\lambda/\|b\|_{BMO}} |B|.$$ 

Note that this in turn implies that 

$$\int_B |b - b_B|^\alpha \lesssim \max\{\alpha, 1\} \|b\|_{BMO}$$ (2.1) 

for every $\alpha > 0$. Another fact that we will use in what follows is that if $B$ is a ball then 

$$|b_{2jB} - b_B| \lesssim j \|b\|_{BMO}.$$ (2.2) 

We remit the interested reader to [20] for more details on $BMO$.

Quite related to the definition of $BMO$ is that of the sharp maximal function. Given $\delta > 0$, we define 

$$M_{\delta}(f)(x) = \sup_{x \in B} \inf_{c \in \mathbb{R}} \left( \int_B |f - c|^\delta \right)^{\frac{1}{\delta}}.$$ 

We would like to end this preliminaries section by gathering some basic facts about multilinear theory. We recall that a linear operator $T$ is an $m$-linear Calderón–Zygmund operator if $T : L^{p_1} \times \cdots \times L^{p_m} \to L^p$ for some $1 < p_1, \ldots, p_m < \infty$ with $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$, and it admits the following representation: 

$$T(f)(x) = \int_{\mathbb{R}^m} K(x, y_1, \ldots, y_m) f(y_1) \cdots f(y_m) dy_1 \cdots dy_m,$$ 

where $x \not\in \text{supp}(f_i)$ for any $i \in \{1, \ldots, m\}$, in terms of a kernel $K$ that satisfies the following properties:
• Size condition: \(|K(x, \vec{y})| \leq C_K(\sum_{i=1}^{m} |x - y_i|)^{-m}n\).

• Smoothness condition: Given \(\omega\) a continuous subadditive function such that \(\int_0^1 \omega(t) \log\left(\frac{1}{t}\right) \frac{dt}{t} < \infty\), the following conditions hold

\[
|K(x, \vec{y}) - K(z, \vec{y})| \leq \omega\left(\frac{|x - z|}{\max_{i \in \{1, \ldots, m\}} |x - y_i|}\right) \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^{mn}}
\]

provided that \(\max_{i \in \{1, \ldots, m\}} |x - y_i| \geq 2|x - z|\), and also, for any \(j \in \{1, \ldots, m\}\)

\[
|K(x, y_1, \ldots, y_j, \ldots, y_m) - K(x, y_1, \ldots, y_j', \ldots, y_m)| \\
\leq \omega\left(\frac{|y_j - y_j'|}{\max_{i \in \{1, \ldots, m\}} |x - y_i|}\right) \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^{mn}},
\]

where \(\max_{i \in \{1, \ldots, m\}} |x - y_i| \geq 2|y_j - y_j'|\).

Note that in this context the commutator \([b, T]_f\) is defined as

\([b, T]_f(x) = b(x)T(f)(x) - T(f_1, \ldots, f_jb, \ldots, f_m)\).

Note that the definition is essentially equivalent whichever index we commute in. Hence throughout the remainder of this work we will consider just the case \([b, T]_1\).

Let us also recall that we say \(\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{p}}\), if

\[
\sup_Q \left(\int_Q w^p\right)^{\frac{1}{p}} \prod_{i=1}^{m} \left(\int_Q w_i^{-p_i'}\right)^{\frac{1}{p_i'}} < \infty, \quad w := \prod_{i=1}^{m} w_i,
\]

where \(\vec{p} = (p_1, \ldots, p_m)\) with \(1 \leq p_i \leq \infty\) and \(1/p = 1/p_1 + \cdots + 1/p_m\). A consequence of the multilinear extrapolation result that appeared first in \([28]\) Theorem 4.12 (see also \([25]\)) states the following.

**Theorem 6.** Let \((f, f_1, \ldots, f_m)\) be an \((m + 1)\)-tuple of functions. Suppose that

\[
||f w||_{L^\infty} \leq c_{\vec{w}} \prod_{i=1}^{m} ||f_i w_i||_{L^\infty}
\]

holds for all \(\vec{w} \in A_{(\infty, \ldots, \infty)}\), where \(c_{\vec{w}}\) depends only on \([\vec{w}]_{A_{(\infty, \ldots, \infty)}}\). Then for all \(\vec{p}\) with \(p_i > 1, i = 1, \ldots, m\), and all \(\vec{w} \in A_{\vec{p}}\), we have

\[
||f w||_{L^{p_i}} \leq c_{\vec{w}} \prod_{i=1}^{m} ||f_i w_i||_{L^{p_i}},
\]

where \(c_{\vec{w}}\) depends only on \([\vec{w}]_{A_{\vec{p}}\).
\[
\left( \int_B \left[ [b, T] f(x) - c_1 - c_2 b(x) \right] \delta dx \right)^{\frac{1}{\beta}} \\
= \left( \int_B \left[ [b - \lambda, T] f(x) - c_1 - c_2 b(x) \right] \delta dx \right)^{\frac{1}{\beta}} \\
\lesssim \left( \int_B \left| (b(x) - \lambda)T f(x) + \lambda T f_2(c_B) - c_2 b(x) \right| \delta dx \right)^{\frac{1}{\beta}} \\
+ \left( \int_B \left| T((b - \lambda)f)(x) - T((b - \lambda)f_2)(c_B) \right| \delta dx \right)^{\frac{1}{\beta}} \\
=: L_1 + L_2.
\]

Note that for \( L_1 \), choosing \( \lambda = b_{2B} \) we have that for \( \delta < \varepsilon < 1 \), calling \( f_1 = f\chi_{2B} \),

\[
L_1 = \left( \int_B \left| (b(x) - b_{2B})T f_1(x) + b_{2B}T f_2(c_B) - c_2 b(x) \right| \delta dx \right)^{\frac{1}{\beta}} \\
\lesssim \left( \int_B \left| (b(x) - b_{2B})T f_1(x) \right| \delta dx \right)^{\frac{1}{\beta}} \\
+ \left( \int_B \left| (b(x) - b_{2B})T f_2(x) + b_{2B}T f_2(c_B) - c_2 b(x) \right| \delta dx \right)^{\frac{1}{\beta}} \\
=: L_{11} + L_{12}.
\]

First we focus on \( L_{11} \). We argue as follows:

\[
L_{11} = \left( \int_B \left| (b(x) - b_{2B})T f_1(x) \right| \delta dx \right)^{\frac{1}{\beta}} \\
\leq \left( \int_B \left| b(x) - b_{2B} \right|^{\delta(\frac{1}{\beta})} dx \right)^{\frac{1}{\delta}} \left( \int_B \left| T f_1 \right| \delta dx \right)^{\frac{1}{\beta}} \\
\lesssim \|b\|_{BMO} \int_{2B} |f| = \|b\|_{BMO} \int_{2B} \frac{|f|}{w} \\
\leq \|b\|_{BMO} \left\| \frac{f}{w} \right\|_{L^\infty \inf_{z \in B} M w}.
\]

Now we turn to \( L_{12} \). Choosing \( c_2 = T f_2(c_B) \) we have that

\[
L_{12} = \left( \int_B \left| (b(x) - b_{2B})T f_2(x) - T f_2(c_B)b(x) + b_{2B}T f_2(c_B) \right| \delta dx \right)^{\frac{1}{\beta}} \\
= \left( \int_B \left| (b(x) - b_{2B})T f_2(x) - (b(x) - b_{2B})T f_2(c_B) \right| \delta dx \right)^{\frac{1}{\beta}} \\
\leq \left( \int_B \left| b(x) - b_{2B} \right| \left| T f_2(x) - T f_2(c_B) \right| \delta dx \right)^{\frac{1}{\beta}}.
\]
From this point taking into account the smoothness condition of the kernel we may argue as follows:

\[
\left( \int_B |b(x) - b_{2B}||Tf_2(x) - Tf_2(c_B)|dx \right) \\
\leq \int_B |b(x) - b_{2B}| \int_{\mathbb{R}^n \setminus 2B} |K(x, y) - K(c_B, y)||f(y)|dydx \\
\leq \int_B |b(x) - b_{2B}| \int_{\mathbb{R}^n \setminus 2B} \frac{1}{|x - y|^n} \omega \left( \frac{|x - c_B|}{|x - y|} \right) |f(y)|dydx \\
\leq \int_B |b(x) - b_{2B}| \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{1}{|x - y|^n} \omega \left( \frac{|x - c_B|}{|x - y|} \right) |f(y)|dydx \\
\leq \int_B |b(x) - b_{2B}| \ |dy| \sum_{j=1}^{\infty} \frac{1}{2^n l_B^n} \omega \left( \frac{l_B}{2^n l_B} \right) \int_{2^{j+1}B \setminus 2^j B} |f(y)|dy \\
\lesssim \|b\|_{BMO} \sum_{j=1}^{\infty} \omega(2^{-j}) \int_{2^{j+1}B} |f(y)|dy \\
\lesssim \|b\|_{BMO} \left\| \frac{f}{w} \right\|_{L^\infty} \inf_{z \in B} Mw.
\]

We continue bounding \( L_2 \). Note that

\[
L_2 \lesssim \left( \int_B |T((b - b_{2B})f_1(x))|^\delta dx \right)^{\frac{1}{\delta}} \\
+ \left( \int_B |T((b - b_{2B})f_2(x) - T((b - b_{2B})f_2(x))|^\delta dx \right)^{\frac{1}{\delta}} \\
= L_{21} + L_{22}.
\]

For \( L_{21} \) by Kolmogorov’s inequality,

\[
L_{21} = \left( \int_B |T((b - b_{2B})f_1(x))|^\delta dx \right)^{\frac{1}{\delta}} \lesssim \int_{2B} |b - b_{2B}||f|dx \\
\leq \left\| \frac{f}{w} \right\|_{L^\infty} \int_{2B} |b - b_{2B}|wdx \lesssim r' \left\| \frac{f}{w} \right\|_{L^\infty} \|b\|_{BMO} \inf_{z \in B} M_r w(z),
\]

where in the last step we have used Hölder’s inequality and (2.1). For \( L_{22} \), we have that, using the smoothness condition of the kernel,

\[
L_{22} \leq \int_B \int_{\mathbb{R}^n \setminus 2B} |K(x, y) - K(c_B, y)||b(y) - b_{2B}||f(y)|dydx \\
\leq \int_B \int_{\mathbb{R}^n \setminus 2B} \omega \left( \frac{|x - c_B|}{|x - y|} \right) \frac{1}{|x - y|^n} |b(y) - b_{2B}||f(y)|dydx
\]
Observe that if we call
by hypothesis we have that for all 1
then by Theorem 1 we have that
and hence by Theorem 1 we have that
where in the last step we have used Hölder’s inequality, (2.1) and (2.2). This ends the proof.

3.2. Proof of Theorem 5. Let us fix a ball \( B \) and \( x \in B \). Following the same notation as that in the proof of Theorem 4 if we choose \( c = c_1 + Tf_2(c_B)b_{2B} \),

\[
\left( \int_B \| [b, T] f(y) - c \|^{\delta} dy \right)^{\frac{1}{\delta}}
\]

\[
= \left( \int_B \| [b, T] f(y) - c_1 - Tf_2(c_B)b(y) - Tf_2(c_B)b_{2B} + Tf_2(c_B)b(y) \|^{\delta} dy \right)^{\frac{1}{\delta}}
\]

\[
\lesssim \left( \int_B \| [b, T] f(y) - c_1 - Tf_2(c_B)b(y) \|^{\delta} dy \right)^{\frac{1}{\delta}} + |Tf_2(c_B)| \left( \int_B |b(y) - b_{2B}|^{\delta} dy \right)^{\frac{1}{\delta}}
\]

\[
\lesssim \left( \int_B \| [b, T] f(y) - c_1 - Tf_2(c_B)b(y) \|^{\delta} dy \right)^{\frac{1}{\delta}} + \mathcal{M}_T f(x) \| b \|_{\text{BMO}}.
\]

Note that this yields

\[
M_{x,\delta}([b, T] f)(x) \lesssim \sup_{x \in B, c_1 \in \mathbb{R}} \left( \int_B \| [b, T] f(y) - c_1 - Tf_2(c_B)b(y) \|^{\delta} dy \right)^{\frac{1}{\delta}} + \mathcal{M}_T f(x) \| b \|_{\text{BMO}}.
\]

Observe that if we call

\[
g(x) := \sup_{x \in B, c_1 \in \mathbb{R}} \left( \int_B \| [b, T] f(y) - c_1 - Tf_2(c_B)b(y) \|^{\delta} dy \right)^{\frac{1}{\delta}},
\]

by hypothesis we have that

\[
\| gw \|_{L^\infty} \leq c_w \| b \|_{\text{BMO}} \| f \|_{L^\infty}
\]

and hence by Theorem 1 we have that for all \( 1 < q < \infty \) and every \( w \in A_q \)

\[
\| g \|_{L^q(w)} \leq c_w \| b \|_{\text{BMO}} \| f \|_{L^q(w)}.
\]

Since by hypothesis as well we know that

\[
\| \mathcal{M}_T f \|_{L^p(v)} \leq c_v \| f \|_{L^p(v)}.
\]

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We have that combining the estimates above,
\[ \|M_{2,\delta}([b,T]f)\|_{L^p(v)} \leq \tilde{c}_v\|b\|_{BMO}\|f\|_{L^p(v)}. \]
Then the desired estimate
\[ \|[b,T]f\|_{L^p(v)} \leq \tilde{c}_v\|b\|_{BMO}\|f\|_{L^p(v)} \]
follows from the Fefferman–Stein’s inequality.

4. Multilinear counterparts

In this section we present multilinear versions of the results presented above. We begin providing a counterpart of Theorem 4.

**Theorem 7.** Let \( b \in BMO \) and \( T \) be an \( m \)-linear CZO. Then for every ball \( B \), if \( \delta \in (0,1/m) \) then
\[
\inf_{c_1 \in \mathbb{R}} \left( \int_B \|[b,T]_1(\tilde{f})(y) - c_1 - \left( T(\tilde{f}) - T(\tilde{f}\chi_{2B}) \right) b(y) \right) dy \right]^\frac{1}{2} \leq c_w\|b\|_{BMO} \prod_{i=1}^\infty \|f_iw_i\|_{L^\infty} \inf_{x \in B} \frac{1}{w(x)},
\]
where \( (w_1, \ldots, w_m) \in A_{\infty, \ldots, \infty} \) and \( w = \prod_{i=1}^m w_i \). Consequently, the following inequality holds as well
\[
\inf_{c_1, c_2 \in \mathbb{R}} \left( \int_B \|[b,T]_1(\tilde{f})(y) - c_1 - c_2 b(y) \right) dy \right]^\frac{1}{2} \leq c_w\|b\|_{BMO} \prod_{i=1}^\infty \|f_iw_i\|_{L^\infty} \inf_{x \in B} \frac{1}{w(x)}.
\]
**Proof of Theorem 7.** For notational convenience we may denote
\[ T_B(\tilde{f}) = T(\tilde{f}) - T(\tilde{f}\chi_{2B}). \]
Let
\[ c_1(B) = -b_{2B}T_B(\tilde{f})(c_B) - T_B((b - b_{2B})f_1, f_2, \ldots, f_m)(c_B) \]
and
\[ c_2(B) = T_B(\tilde{f})(c_B). \]
Having that notation in mind we have that
\[
\left( \int_B \|[b,T]_1(\tilde{f})(y) - c_1(B) - c_2(B)b(y) \right) dy \right)^\frac{1}{2} = \left( \int_B \|[b - b_{2B}, T]_1(\tilde{f})(y) - c_1(B) - c_2(B)b(y) \right) dy \right)^\frac{1}{2} \leq \left( \int_B |(b - b_{2B})T(\tilde{f})(y) - (b - b_{2B})T_B(\tilde{f})(c_B) | \right) dy \right)^\frac{1}{2} + \left( \int_B |T((b - b_{2B})f_1, \ldots, f_m)(y) - T_B((b - b_{2B})f_1, \ldots, f_m)(c_B) | \right) dy \right)^\frac{1}{2} =: I_1 + I_2.
\]
For $I_1$, we have
\[
I_1 \lesssim \left( \int_B |b - b_{2B}|^\delta \cdot |T_B(\tilde{f})(y) - T_B(\tilde{f})(c_B)|^\delta \, dy \right)^{\frac{1}{\delta}}
\]
\[+ \left( \int_B |b - b_{2B}|^\delta \cdot |T(\tilde{f} \chi_{2B})(y)|^\delta \, dy \right)^{\frac{1}{\delta}}
\]
\[\lesssim \|b\|_{BMO}(\sup_{y \in B} |T_B(\tilde{f})(y) - T_B(\tilde{f})(c_B)| + \|T(\tilde{f} \chi_{2B})\|_{L^\infty(B, \frac{w}{|B|})})
\]
\[\lesssim \|b\|_{BMO}\left( \sum_{k=1}^\infty \omega(2^{-k}) \prod_{i=1}^m \int_{2^k B} |f_i| + \prod_{i=1}^m \int_{2B} |f_i| \right)
\]
\[\leq \|b\|_{BMO}[\tilde{w}]_{A(\infty, \ldots, \infty)} \left( \prod_{i=1}^m \|f_i w_i\|_{L^\infty} \right) \text{ess inf}_{x \in B} \frac{1}{w(x)},
\]
where we have used the weak endpoint estimate of $T$. Now we turn to estimate $I_2$.

We argue as follows:
\[
I_2 \lesssim \left( \int_B |T_B((b - b_{2B})f_1, \ldots, f_m)(y) - T_B((b - b_{2B})f_1, \ldots, f_m)(c_B)|^\delta \, dy \right)^{\frac{1}{\delta}}
\]
\[+ \left( \int_B |T((b - b_{2B})f_1 \chi_{2B}, \ldots, f_m \chi_{2B})(y)|^\delta \, dy \right)^{\frac{1}{\delta}}
\]
\[=: I_{21} + I_{22}.
\]

The estimate of $I_{22}$ can be handled similarly as before, that is, we use Kolmogorov’s inequality and then the weak type endpoint estimate
\[
I_{22} \lesssim \left( \int_{2B} |b - b_{2B}| |f_1| \right) \prod_{i=2}^m \int_{2B} |f_i|
\]
\[\leq [\tilde{w}]_{A(\infty, \ldots, \infty)} \left( \prod_{i=1}^m \|f_i w_i\|_{L^\infty} \right) \text{ess inf}_{x \in B} \frac{1}{w(x)} \cdot \frac{1}{w_1^{-1}(2B)} \int_{2B} |b - b_{2B}| w_1^{-1}
\]
\[\lesssim \|b\|_{BMO}[\tilde{w}]_{A(\infty, \ldots, \infty)} \left( \prod_{i=1}^m \|f_i w_i\|_{L^\infty} \right) \text{ess inf}_{x \in B} \frac{1}{w(x)},
\]
where the last inequality holds since $w_1^{-1} \in A_\infty$.

It remains to consider $I_{21}$. Similarly as before, we have
\[
I_{21} \lesssim \sum_{k=1}^\infty \omega(2^{-k}) \int_{2^k B} |b - b_{2B}| |f_1| \prod_{i=2}^m \int_{2^k B} |f_i|
\]
\[\leq \sum_{k=1}^\infty \omega(2^{-k}) \int_{2^k B} |b - b_{2B}| |f_1| \prod_{i=2}^m \int_{2^k B} |f_i|
\]
\[+ \sum_{k=1}^\infty \omega(2^{-k}) |b_{2B} - b_{2kB}| \prod_{i=1}^m \int_{2^k B} |f_i|.
\]
\[ \lesssim \|b\|_{BMO} \mathcal{A}(\infty, \ldots, \infty) \left( \prod_{i=1}^{m} \|f_{i}w_{i}\|_{L^{\infty}} \right) \text{ess inf}_{x \in B} \frac{1}{w(x)}, \]

where we have used that
\[ |b_{2B} - b_{2kB}| \lesssim k\|b\|_{BMO}. \]

This requires that the kernel satisfies the log-Dini condition. This completes the proof. \(\square\)

Having the Theorem above at our disposal we can obtain the following result.

**Theorem 8.** Let \(T\) be an \(m\)-linear operator such that for every \(b \in BMO\) and every \(\vec{w} \in \mathcal{A}(\infty, \ldots, \infty)\),
\[
\inf_{c_{1} \in \mathbb{R}} \left( \int_{B} |[b, T]_{1}(\vec{f})(y) - c_{1} - \left( T(\vec{f}) - T(\vec{f} \chi_{2B}) \right)(y)b(y)|^{\delta} dy \right)^{\frac{1}{\delta}} \lesssim c_{w}\|b\|_{BMO} \prod_{i=1}^{\infty} \|f_{i}w_{i}\|_{L^{\infty}} \text{ess inf}_{x \in B} \frac{1}{w(x)},
\]

and such that Lerner’s grand maximal operator
\[ \mathcal{M}_{T}(\vec{f})(x) = \sup_{x \in B} \text{ess sup}_{z \in B} T(\vec{f} \chi_{(2B)^{v}})(z) \]
satisfies
\[ \|\mathcal{M}_{T}(\vec{f})v\|_{L^{p}} \leq c_{\vec{p}, T}\|b\|_{BMO} \prod_{i=1}^{m} \|f_{i}v_{i}\|_{L^{p_{i}}}. \]

for some \(\vec{p}\) with \(p_{i} > 1, i = 1, \ldots, m\), and some \(\vec{v} \in \mathcal{A}_{\vec{p}}\). Then
\[ \|([b, T]_{1}(\vec{f})v\|_{L^{p}} \leq c_{\vec{v}, T}\|b\|_{BMO} \prod_{i=1}^{m} \|f_{i}v_{i}\|_{L^{p_{i}}}. \]

**Proof.** The argument is analogous to the one given for the linear case. Let us define
\[ g(x) = \sup_{B \ni x} \inf_{c_{1} \in \mathbb{R}} \left( \int_{B} |[b, T]_{1}(\vec{f})(y) - c_{1} - \left( T(\vec{f}) - T(\vec{f} \chi_{2B}) \right)(y)b(y)|^{\delta} dy \right)^{\frac{1}{\delta}}; \]
then, arguing as in the linear case, we have that
\[ M_{2, \delta}([b, T]_{1}(\vec{f}))(x) \leq \sup_{B \ni x} \left( \int_{B} |[b, T]_{1}(\vec{f})(y) - c(B)|^{\delta} dy \right)^{\frac{1}{\delta}} \lesssim g(x) + \sup_{B \ni x} \left( \int_{B} |b(y) - b_{2B}|^{\delta} dy \right)^{\frac{1}{\delta}} |T_{B}(\vec{f})(c_{B})| \lesssim g(x) + \|b\|_{BMO} \mathcal{M}_{T}(\vec{f})(x). \]

To deal with \(g\) we argue as in the linear setting. \(\mathcal{M}_{T}\) is bounded by hypothesis. Hence we are done. \(\square\)
Note that in the case of $T$ being a Calderón–Zygmund operator, in the bilinear case, a careful calculus to bound $\mathcal{M}_T$ was presented in [24]. Such an estimate, that we recall in the following line, can be extended to the multilinear case directly. Therefore, we have

$$\mathcal{M}_T(\vec{f})(x) \lesssim M(\vec{f})(x) + M_s(T(\vec{f}))(x)$$

for every $0 < s < \frac{1}{m}$. Of course, since $M_s$ is increasing with $s$, the inequality holds for all $s > 0$. In particular, we can choose $s = \frac{1}{m}$, which in turn allows us to show that

$$\|M_{1/m}(T(\vec{f}))w\|_{L^p} = \|M(T(\vec{f})^{1/m})\|_{L^{mp}(w^p)} \lesssim \|T(\vec{f})w\|_{L^p} \lesssim \prod_{i=1}^{m} \|f_iw_i\|_{L^p},$$

where we have used the fact that if $(w_1, \ldots, w_m) \in A_P$ then $w^p \in A_{mp}$ (when $p < \infty$). This ends the argument and completes an alternative proof of the boundedness of $[b,T]_i$ in the multilinear setting.

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Kangwei Li  
Center for Applied Mathematics, Tianjin University, Tianjin 300072, The People’s Republic of China  
kli@tju.edu.cn

Sheldy Ombrosi  
Departamento de Análisis Matemático y Matemática Aplicada. Facultad de Ciencias Matemáticas. Universidad Complutense (Madrid, Spain). Departamento de Matemática e Instituto de Matemática de Bahía Blanca. Universidad Nacional del Sur - CONICET (Bahía Blanca, Argentina).  
sombrosi@ucm.es

Israel P. Rivera Ríos  
Departamento de Análisis Matemático, Estadística e Investigación Operativa y Matemática Aplicada. Facultad de Ciencias. Universidad de Málaga (Málaga, Spain). Departamento de Matemática. Universidad Nacional del Sur (Bahía Blanca, Argentina).  
israelpriverarios@uma.es

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