Relationship between the n-tangle and the residual entanglement of even n qubits

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Abstract

We show that $n$-tangle, the generalization of the 3-tangle to even $n$ qubits, is the square of the SLOCC polynomial invariant of degree 2. We find that the $n$-tangle is not the residual entanglement for any even $n \geq 4$ qubits. We give a necessary and sufficient condition for the vanishing of the concurrence $C_{1(2\ldots n)}$. The condition implies that the concurrence $C_{1(2\ldots n)}$ is always positive for any entangled states while the $n$-tangle vanishes for some entangled states. We argue that for even $n$ qubits, the concurrence $C_{1(2\ldots n)}$ is equal to or greater than the $n$-tangle. Further, we reveal that the residual entanglement is a partial measure for product states of any $n$ qubits while the $n$-tangle is multiplicative for some product states.

Keywords: the 3-tangle, the $n$-tangle of even $n$ qubits, the residual entanglement, SLOCC polynomial invariants

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1 Introduction

Quantum entanglement is an important physical resource in quantum information and computation such as quantum teleportation, cloning and encryption. Entanglement phenomenon distinguishes the quantum world from the classical world. Considerable attention has been paid in recent years to the quantification and classification of entanglement. The concurrence was proposed by Wootters in 1998 to quantify entanglement for bipartite systems [1]. For two qubits, the concurrence was defined as \( C_{12} = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \), where \( \lambda_i \) are the eigenvalues, in decreasing order, of \( \rho_{12} \). Here, \( \rho_{12} \) is the density matrix and \( \tilde{\rho}_{12} \) is the “spin-flipped” density matrix of \( \rho_{12} \), i.e., \( \tilde{\rho}_{12} = \sigma_y \otimes \sigma_y \rho_{12}^* \sigma_y \otimes \sigma_y \) [2], where the asterisk denotes complex conjugation in the standard basis. For the state \( |\psi\rangle \) of a bipartite system, the concurrence was also given by [4]

\[
C(\psi) = \sqrt{2(1 - Tr(\rho_{12}^3))}. \tag{1.1}
\]

The definition of the concurrence in Eq. (1.1) was generalized to multipartite systems [5]. Recently, the concurrence was used to study quantum phase transitions [6].

By means of the concurrence, CKW monogamy inequality for three qubits was established. Namely, \( C^2_{12} + C^2_{13} \leq C^2_{123} \leq 2\) [13]. Here \( \rho_{12} \) is obtained from the density matrix \( \rho_{123} \) by tracing out over qubit 3, and \( C^2_{123} = 4\det \rho_1 \), where \( \rho_1 = tr_{23}\rho_{123} \). Note that \( C_{123} \) can be called the concurrence between qubit 1 and the pair of qubits 2 and 3 if qubits 2 and 3 are regarded as a single object. The difference \( (C^2_{123} - (C^2_{12} + C^2_{13})) \) between the two sides of the above CKW monogamy inequality is called “residual entanglement”. The algebraic expression for the residual entanglement is called the 3-tangle (see (20) of [2] for the expression). The expression can also be obtained from Eq. (1.2) by letting \( n = 3 \). The 3-tangle is invariant under permutations of all the qubits [2]. The invariance of entanglement measure under permutations of all the qubits represents a collective property of the qubits. The 3-tangle is also an entanglement monotone [7]. Monotonicity for entanglement measure is a natural requirement.

The 3-tangle was extended to even \( n \) qubits, and the extension was called the \( n \)-tangle [3]. Let the state \( |\psi\rangle = \sum_{i_1i_2...i_n} a_{i_1i_2...i_n} |i_1i_2...i_n\rangle \), where \( i_1, i_2, ..., i_n \in \{0, 1\} \). The \( n \)-tangle was defined as [3]

\[
\tau_{1...n} = 2|S|, \quad S = \sum (a_{i_1i_2...i_n} a_{\beta_1...\beta_n} a_{\gamma_1...\gamma_n} a_{\delta_1...\delta_n} \\
\times \epsilon_{i_1\beta_1} \epsilon_{i_2\beta_2} ... \epsilon_{i_{n-1}\beta_{n-1}} \epsilon_{i_n\beta_n} \\
\times \epsilon_{\gamma_1\delta_1} \epsilon_{\gamma_2\delta_2} ... \epsilon_{\gamma_{n-1}\delta_{n-1}} \epsilon_{\gamma_n\delta_n} \epsilon_{\gamma_n\delta_n}). \tag{1.2}
\]

where \( |c| \) is the modulus of the complex number \( c, \alpha_i, \beta_i, \gamma_i \), and \( \epsilon_i \epsilon \{0, 1\} \), and

\[
\epsilon_{00} = \epsilon_{11} = 0 \quad \text{and} \quad \epsilon_{01} = -\epsilon_{10} = 1. \tag{1.3}
\]

The \( n \)-tangle of even \( n \) qubits is invariant under permutations of the qubits, and is an entanglement monotone [3]. In [3], the \( n \)-tangle was proposed as a potential entanglement measure.

The generalized CKW monogamy inequality for \( n \) qubits was given by [18]

\[
C^2_{12} + ... + C^2_{1n} \leq C^2_{12...n}. \tag{1.4}
\]

Here \( \rho_{12} = tr_{3...n} \rho_{12...n} \) i.e., \( \rho_{12} \) is obtained from the density matrix \( \rho_{12...n} \) by tracing out over qubits 3, ..., and \( n \), and \( C^2_{12...n} = 4\det \rho_1 \), where \( \rho_1 = tr_{23...n} \rho_{123...n} \). Note that \( C_{12...n} \) can be called the \( n \)-tangle between qubits 1 and qubits 2, ..., and \( n \) if qubits 2, ..., and \( n \) are regarded as a single object. The difference between the two sides of CKW monogamy inequality in Eq. (1.4) can be considered as a natural generalization of the residual entanglement of three qubits to \( n \) qubits, and was denoted as [18]

\[
\tau_{1(2...n)} = C^2_{12}(2...n) - (C^2_{12} + ... + C^2_{1n}). \tag{1.5}
\]

In this paper, we investigate the relationship between the \( n \)-tangle and the residual entanglement for any even \( n \geq 4 \) qubits. This paper is organized as follows. In Sec. 2, we show that the \( n \)-tangle is the square of the SLOCC polynomial invariant of degree 2. In Sec. 3, we address the relationship between the \( n \)-tangle and the residual entanglement of \( n \) qubits. In Sec. 4, we summarize our results and conclusions.

2 The \( n \)-tangle is the square of the SLOCC polynomial invariant of degree 2

The SLOCC invariants can be used for SLOCC classification and the entanglement measure [8] [9] [10] [11].
For four qubits, four independent SLOCC polynomial invariants: $H$, $L$, $M$, and $D_{xt}$ were given in [9], where $H$ is of degree 2, $L$ and $M$ are of degree 4, and $D_{xt}$ is of degree 6. Very recently, for four and five qubits, SL invariants of degree 2 (for only four qubits), 4, 6, 8, 10, 12 were studied in [14]. The antilinear operators “combs”, which are invariant under $SL(2, C)$, were constructed in [15]. The geometry of four qubit invariants was investigated in [11]. For any even $n$ qubits, the SLOCC polynomial invariant of degree 2 was given in [13]. The SLOCC invariant of degree 4 of odd qubits was discussed in [12, 13]. Note that there are no invariants of degree 2 for odd $n$ qubits [9].

### 2.1 Reduction of the $n$-tangle

The $n$-tangle in Eq. (1.2) is quartic and the computation of the coefficients takes $3 * 2^n$ multiplications. Denote by $\alpha_i^n$ the complement of $\alpha_i$. That is, $\alpha_i^n = 0$ when $\alpha_i = 1$. Otherwise, $\alpha_i^n = 1$. Further, let

$$S_0 = \sum_{\alpha_1 \ldots \alpha_{n-1}} (a_{\alpha_1 \ldots \alpha_{n-1}} \delta_{\alpha_1 \ldots \alpha_{n-1}})$$

$$= \sum_{\alpha_1 \ldots \alpha_{n-1}} \alpha_1 \alpha_2 \ldots \alpha_{n-1} - \sum_{\alpha_1 \ldots \alpha_{n-1}} \alpha_1 \alpha_2 \ldots \alpha_{n-1}$$

(2.1)

Note that $S_0$ is of degree 2. Then, $S$ in Eq. 1.2 can be reduced to $S = 2S_0^2$ (see (A) of Appendix A for the proof). This leads to

$$\tau_{1 \ldots n} = |2S_0|^2$$

(2.2)

### 2.2 The $n$-tangle is the square of the SLOCC polynomial invariant of degree 2

Let $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$ and $|\psi'\rangle = \sum_{i=0}^{2^n-1} b_i |i\rangle$ be any states of $n$ qubits. Two states $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC equivalent if and only if there exist invertible local operators $A_1, A_2, \ldots, A_n$, such that

$$|\psi'\rangle = A_1 \otimes A_2 \otimes \ldots \otimes A_n |\psi\rangle$$

(2.3)

The entanglement measure of the state $|\psi\rangle$ of even $n$ qubits was proposed as [13] [16]

$$\tau'(\psi) = 2 |I'(a, n)|$$

(2.4)

where

$$I'(a, n) = \sum_{l=0}^{2^n-2} (-1)^{N(l)}$$

$$\times (a_{2l}a_{(2^n-1)-2l} - a_{2l+1}a_{(2^n-2)-2l})$$

(2.5)

Here we take $N(l)$ to be the number of the occurrences of “1” in $l_{n-1} \ldots l_0$, which is a $n$-bit binary representation of $l$, i.e., $l = l_{n-1}2^{n-1} + \ldots + l_02^0$. In [13], it was proven that if $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC equivalent then

$$I'(b, n) = \frac{I'(a, n) \det(A_1) \ldots \det(A_n)}{\det(A_1) \ldots \det(A_n)}$$

(2.6)

where $I'(b, n)$ is obtained from $I'(a, n)$ by replacing $a$ in $I'(a, n)$ with $b$, and $I'(a, n)$ was called the SLOCC polynomial invariant of degree 2 of even $n$ qubits.

Note that $S_0$ is just $I'(a, n)$ (see (B) in Appendix A for the proof). By virtue of Eqs. (2.2) and (2.4), we have $\tau_{1 \ldots n} = (\tau'(|\psi\rangle))^2$. It then follows from Eqs. (2.4) and (A) that

$$\tau_{1 \ldots n} = 4 \left| \sum_{l=0}^{2^n-1} (-1)^{N(l)}a_{2l}a_{(2^n-1)-2l} \right|^2$$

(2.7)

In Eq. (2.7), computing the coefficients requires $(2^{n-1} + 2)$ multiplications. The $n$-tangle $\tau_{1 \ldots n}$ is not considered as the SLOCC polynomial invariant of degree 4 though $\tau_{1 \ldots n}$ is quartic and satisfies the equation $\tau_{1 \ldots n}(|\psi'\rangle) = \tau_{1 \ldots n}(|\psi\rangle) \det(A_1) \ldots \det(A_n)$. However, the square root of the $n$-tangle is the SLOCC polynomial invariant of degree 2. The square root of the $n$-tangle turns out to be $\tau'(|\psi\rangle)$. Using the properties of $\tau'(|\psi\rangle)$ [13] [16], the square root is also an entanglement monotone, and invariant under permutations of all the qubits.

### 3 Relationship between the n-tangle and the residual entanglement

#### 3.1 The $n$-tangle is not the residual entanglement for any even $n \geq 4$ qubits.

To illustrate the relationship between $n$-tangle and residual entanglement, we consider the following examples. For the $n$-qubit state $a_1|0\ldots 0\rangle + a_2|0\ldots 010\rangle + \ldots + a_n|10\ldots 0\rangle$, equality in Eq. (1.3) holds [21] [20], i.e. the residual entanglement $\tau_{1\ldots n} = 0$. According to Eq. (2.7), it is easy to see that the $n$-tangle $\tau_{1\ldots n} = 0$. It follows that $\tau_{1\ldots n} = \tau_{1\ldots n}$. This is particularly true for the $n$-qubit state $|W\rangle$ [3]. For the state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^n + |1\rangle^n)$, the residual entanglement $\tau_{1\ldots n} = 1$ [20], and the $n$-tangle $\tau_{1\ldots n} = 1$ [3]. Thus, $\tau_{1\ldots n} = \tau_{1\ldots n}$ for the state $|GHZ\rangle$. Here is another example which gives $\tau_{1\ldots n} = \tau_{1\ldots n}$ =
$4|\alpha\gamma|^2$ for the state of four qubits: $\alpha|0011\rangle + \beta|0110\rangle + \gamma|1100\rangle$ by utilizing Eq. (2.4).

One might wonder if the two generalizations, which are the $n$-tangle $\tau_{1...n}$ and the residual entanglement $\tau_{1(2...n)}$, are equal. However, this is not always the case as the following example will show. Consider, for example, the $n$-qubit symmetric Dicke states with $l$ excitations ($1 \leq l \leq (n-1)$) [21]

$$|l, n\rangle = \sum_{i} P_i |1_i1_2...1_l0_{l+1...n}\rangle,$$  
(3.1)

where $\{P_i\}$ is the set of all the distinct permutations of the qubits. For the Dicke state $|(n/2), n\rangle$ with $(n/2)$ excitations of any even $n \geq 4$ qubits, Eq. (2.7) yields the $n$-tangle $\tau_{1...n} = 1$. In this case, $\rho_{12}\rho_{12}$ has only three nonzero eigenvalues ($\frac{n}{2(n-1)}$), ($\frac{n}{2(n-1)}$), ($\frac{n(n-1)}{2}$) (double). We then get the concurrence

$$C_{12}^2 = \frac{1}{(n-1)^2}.$$  

The symmetry of the Dicke state leads to $C_{1i}^2 = C_{2i}^2$, $i = 3, \ldots, n$. Calculating $C_{1(2...n)}$ further gives $C_{1(2...n)} = 1$. In light of Eq. (3.3), the residual entanglement $\tau_{1(2...n)} = \frac{1}{n-1}$. It says that for the Dicke state $|(n/2), n\rangle$, the $n$-tangle $\tau_{1...n}$ is greater than the residual entanglement $\tau_{1(2...n)}$ and the difference is given by $\frac{1}{n-1}$.

### 3.2 A necessary and sufficient condition for the vanishing of the concurrence $C_{1(2...n)}$

For the state $|\psi\rangle = \sum_{i=0}^{2^n} a_i|i\rangle$ of $n$ qubits, the concurrence $C_{1(2...n)}$ can be written as

$$C_{1(2...n)}^2 = \frac{4}{n-1} \sum_{0 \leq i < j \leq 2^{n-1} - 1} |a_i a_{j+2^{n-1}} - a_{i+2^{n-1}} a_j|^2.$$  

(3.2)

The right hand side of Eq. (3.2) turns out to be the sum of squared moduli (see Appendix B for the proof).

In view of Eq. (3.2), any $n$-qubit concurrence $C_{1(2...n)}$ vanishes if and only if the state is a product of a state of one qubit and a state of $(n-1)$ qubits, i.e., the state is of the form $|\phi\rangle \otimes |\varphi\rangle_{2...n}$ (see Appendix B for the proof). This allows one to understand how the concurrence $C_{1(2...n)}$ measures the entanglement of a state. In other words, the concurrence $C_{1(2...n)}$ is always positive unless the state is a product of a state of one qubit and a state of $(n-1)$ qubits. In particular, this is true for any entangled state of any $n$ qubits. That is, there exist $i$ and $j$ with $0 \leq i < j \leq 2^{n-1} - 1$, such that $a_i a_{j+2^{n-1}} \neq a_{i+2^{n-1}} a_j$. It is worthwhile pointing out that the $n$-tangle vanishes for some entangled states [16].

### 3.3 The concurrence $C_{1(2...n)} \geq \tau_{1...n}$

A closer examination of Eqs. (3.2) and (2.5) reveals that for even $n$ qubits, the concurrence $C_{1(2...n)}$ is equal to or greater than the $n$-tangle $\tau_{1...n}$ (see Appendix B for the proof). We immediately have the following corollaries:

1. For any state $|\psi\rangle$ of even $n$ qubits, if the concurrence $C$ vanishes then, clearly, so does the $n$-tangle.
2. If the $n$-tangle $\tau_{1...n}$ of even $n$ qubits is positive, then the concurrence $C_{1(2...n)}$ is also positive.

### 3.4 The residual entanglement is a partial measure for product states

In this section, we show that for product state $|\psi\rangle_{1...l} \otimes |\phi\rangle_{(l+1)...n}$ of any $n$ qubits, where $|\psi\rangle$ is the state of the first $l$ qubits, the residual entanglement $\tau_{1(2...n)}$ for the product state is reduced to the residual entanglement $\tau_{1(l...n)}$ for the state $|\psi\rangle$. First we observe that $\rho_l (|\psi\rangle \otimes |\phi\rangle) = \rho_l (|\psi\rangle |\psi\rangle) = \rho_l (|\psi\rangle |\psi\rangle)$. By the definition of the concurrence,

$$C_{1(l...n)} (|\psi\rangle \otimes |\phi\rangle) = C_{1(2...l)} (|\psi\rangle).$$  

(3.3)

That is, the concurrence $C_{1(2...n)}$ for the product state $|\psi\rangle \otimes |\phi\rangle$ is just the concurrence $C_{1(l...n)}$ for the state $|\psi\rangle$. It tells us that the concurrence $C_{1(2...n)}$ only measures the entanglement of the state $|\psi\rangle$.

Likewise, the concurrence $C_{1k}$ for the state $|\psi\rangle_{1...l} \otimes |\phi\rangle_{(l+1)...n}$ is just the concurrence $C_{1k}$ for the state $|\psi\rangle_{1...l}$, $k = 2, \ldots, l$. Since qubits 1 and $k$ are not entangled, the concurrence $C_{1k}$ for the state $|\psi\rangle_{1...l} \otimes |\phi\rangle_{(l+1)...n}$ vanishes for $k > l$. This can be seen as follows. After some algebra, we find $\rho_{1(l+1)} \rho_{l(l+1)} = cI$, where $c$ is a constant. It implies that the concurrence $C_{1(l+1)} = 0$. In a similar manner we can show that the concurrence $C_{1k} = 0$ for $k \geq (l+2)$. This leads to

$$C_{1k} (|\psi\rangle_{1...l} \otimes |\phi\rangle_{(l+1)...n}) = \begin{cases} C_{1k} (|\psi\rangle_{1...l}), & 2 \leq k \leq l \\ 0, & l < k \leq n. \end{cases}$$  

(3.4)

Eqs. (3.3) and (3.4) together with the definition of the residual entanglement give

$$\tau_{1(l...n)} (|\psi\rangle \otimes |\phi\rangle) = \tau_{1(2...l)} (|\psi\rangle).$$  

(3.5)

This shows that the residual entanglement $\tau_{1(2...n)}$ for the product state $|\psi\rangle \otimes |\phi\rangle$ is reduced to the residual entanglement $\tau_{1(2...l)}$ for the state $|\psi\rangle$. It tells us that
are different for any even $n$ of even $n$ qubits, when $|\psi\rangle$ is a state of even $n$ qubits, the $n$-tangle is multiplicative. That is, $\tau_{12...n}(|\psi\rangle \otimes |\phi\rangle) = \tau_{12..i}(|\psi\rangle) \times \tau_{12...(n-i)}(|\phi\rangle)$ \[ \text{(A3)}. \]

The following example shows that the residual entanglement $\tau_{1(2...n)}$ is not the $n$-way entanglement measure. For the product state $(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle))^\otimes 2k$, by Eq. \[ \text{(3.5)}, \] the residual entanglement $\tau_{1(2...6k)} = 1$. It is worth noting that the $n$-tangle is not the $n$-way entanglement measure either \[ \text{[3]}. \]

4 Conclusion

In summary, we have shown that the $n$-tangle is the square of the SLOCC polynomial invariant of degree 2. We have found that the two generalizations, namely the $n$-tangle and the residual entanglement of $n$-qubits, are different for any even $n \geq 4$ qubits. We have also proven that the concurrence $C_{1(2...n)}$ vanishes if and only if the state is a product of a state of one qubit and a state of $(n-1)$ qubits. In other words, the concurrence $C_{1(2...n)}$ is always positive unless the state is a product of a state of one qubit and a state of $(n-1)$ qubits. Furthermore, we have argued that the concurrence $C_{1(2...n)}$ is equal to or greater than the $n$-tangle, and that the residual entanglement is a partial measure for product states of any $n$ qubits.

Appendix A The $n$-tangle is the square of the SLOCC polynomial invariant.

(A). Proof of $S = 2S_0^2$

In view of Eq. \[ \text{(1.3)}, \] we only need to consider $\beta_i = \frac{\alpha_i}{\gamma_i}, \delta_i = \frac{\gamma_i}{\alpha_i}, i = 1, ..., (n-1), \gamma_n = \overline{\alpha_n}$, and $\delta_n = \overline{\beta_n}$. Thus, Eq. \[ \text{(1.2)} \] becomes

$$S = \sum (a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n}), \text{ (A1)}$$

We distinguish two cases.

Case 1. $\beta_n = \alpha_n$.

In this case, $\epsilon_{\alpha_n\overline{\alpha_n}}\epsilon_{\beta_n\overline{\beta_n}} = 1$. Let

$$S' = \sum (a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n} - \delta_n). \text{ (A2)}$$

Then, Eq. \[ \text{(A1)} \] becomes

$$S = \sum (a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n} - \delta_n) \times S'). \text{ (A3)}$$

To compute $S'$, we assume that $\overline{\alpha_n}$ is fixed in $S'$. For each term

$$t = a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n} \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n} - \delta_n),$$

we have $S' = 1 - \delta_n \tau_{1\gamma_1}...\tau_{1\gamma_n}$. Thus, $t = t'$ and so $S' = 0$. Hence, $S = 0$.

Case 2. $\beta_n = \overline{\alpha_n}$.

In this case, $\epsilon_{\alpha_n\overline{\alpha_n}}\epsilon_{\beta_n\overline{\beta_n}} = -1$. Eq. \[ \text{(A1)} \] becomes

$$S = - \sum (a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n} - 1). \text{ (A4)}$$

Let

$$S_i = \sum (a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n} - 1), \text{ (A5)}$$

where $i = 0, 1$. Thus,

$$S = -2S_0S_1. \text{ (A6)}$$

Next we verify that $S_1 = -S_0$. By the condition in Eq. \[ \text{(1.3)}, \] $\epsilon_{\alpha_i\overline{\alpha_i}} = -\epsilon_{\alpha_i\overline{\alpha_i}}, i = 1, ..., n$. Then,

$$S_1 = \sum (a_{\alpha_1...\alpha_{n-1}}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}a_{\gamma_1...\gamma_n}) \times \epsilon_{\gamma_1...\gamma_n}(\tau_{1\gamma_1}...\tau_{1\gamma_n} - 1),$$

where $i = 0, 1$. Thus,

$$S = -2S_0S_1. \text{ (A6)}$$

Together the latter two equations yield the desired result.
Let \( C \mid \) becomes and

\[ \sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} a_2(a_{(2n-1)-2i}) \]  

(A8)

Let \( l_{n-1...l_1} \) be the \((n-1)\)-bit binary number of \( l \). Then, it follows from Eq. (A8) that

\[ T^*(a, n) = \sum_{l_{n-1...l_1}} (-1)^{N(i)} a_{l_{n-1...l_1}} \]

\[ = \sum_{l_{n-1...l_1}} (a_{l_{n-1...l_1}}) \times \epsilon_{l_{n-1...l_1}}(\epsilon_{l_{n-1...l_1}}) \]

\[ = S_0. \]  

(A9)

The second equality follows by noting that

\[ (-1)^{N(i)} = \epsilon_{l_{n-1...l_1}} \epsilon_{l_{n-1...l_1}}. \]

**Appendix B. Concurrency \( C_{1(2...n)} \)**

**Result 1.** Let the state \( |\psi\rangle = \sum_{i=0}^{2^{n-1}} a_i|i\rangle \) be any state of any \( n \) qubits. Then

\[ C_{1(2...n)}^2 = 4 \sum_{0 \leq i < j \leq 2^{n-1}-1} |a_i a_j + a_j a_i - a_i a_j| \]

(B1)

**Proof.** By direct calculation we find

\[ \det \rho_1 = \sum_{i,j=0}^{2^{n-1}-1} a_i a_j + a_j a_i - a_i a_j \]

where \( a_i^* \) is the complex conjugate of \( a_i \). By switching \( i \) and \( j \), the term \( a_i a_j + a_j a_i - a_i a_j \)

becomes \( a_j a_{i+2^{n-1}} - a_i a_{j+2^{n-1}} \). Then

\[ a_i a_j + a_j a_i - a_i a_j = \sum_{j=0} a_j |j\rangle \]

Conversely, if \( |\psi\rangle = |\phi\rangle \otimes |\varphi\rangle \), then it is readily verified that \( C_{1(2...n)} = 0. \)

**Result 3.** For even \( n \) qubits, the concurrence \( C_{1(2...n)} \) is equal to or greater than the \( n \)-tangle \( \tau_{1...n} \).

**Proof.** We rewrite Eq. (2.3) as

\[ \tau^*(a, n) = \sum_{k=0}^{2^{n-2}-1} \left| (a_k a_{2n-1-k} - a_{2n-1-k} a_{2n-1+k}) \right| \]

(B7)

To prove this, we note that \( \tau^*(a, n) \) can be written as (see 10)

\[ \tau^*(a, n) = \sum_{k=0}^{2^{n-1}-1} (-1)^{N(k)} a_k a_{2n-1-k} \]

(B8)
From Eq. (B8),
\[ T(a,n) = \sum_{k=0}^{2^{n-2}-1} (-1)^{N(k)} a_k a_{2^{n-1}-k} \]
\[ + \sum_{k=2^{n-1}-1}^{2^n-2} (-1)^{N(k)} a_k a_{2^{n-1}-k}. \]  
\( \text{(B9)} \)

Let \( k = 2^{n-1}-1-i \), in which case \( N(k)+N(i) = n-1 \). Then, the second sum of the above equation becomes \(-\sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_{2^{n-1}-1-i} a_{2^{n-1}+i} \). Thus, Eq. (B17) holds.

For any \( n \) qubits, we may write Eq. (B1) as
\[ C_{1(2...n)}^2 = 4 \left\{ \sum_{0 \leq i \leq 2^{n-2}-1} |a_{i+2^{n-1}} - a_{i+2^{n-1}+1}|^2 \right. 
\[ + \sum_{2^{n-2} \leq i < j \leq 2^{n-1}-1} |a_{i} a_{j+2^{n-1}} - a_{i+2^{n-1}} a_{j}|^2 \]
\[ + \sum_{i=0}^{2^{n-2}-1} |a_{i} a_{2^{n-1}+i} - a_{2^{n-1}-1-i} a_{2^{n-1}+i}|^2 \}. \]
\( \text{(B10)} \)

For even \( n \) qubits, from Eq. (B7) it holds that
\[ \tau_{1...n} \leq 4 \left\{ \sum_{k=0}^{2^{n-2}-1} |a_k a_{2^{n-1}-k} - a_{2^{n-1}-1-k} a_{2^{n-1}+k}|^2 \right. 
\[ \left. + \sum_{2^{n-2} \leq i < j \leq 2^{n-1}-1} |P(i,j)|^2 + \sum_{i=0}^{2^{n-2}-1} |P(i,j)|^2 \}. \]
\( \text{(B11)} \)

Let, for brevity,
\[ Z_k = |a_k a_{2^{n-1}-k} - a_{2^{n-1}-1-k} a_{2^{n-1}+k}| \quad \text{and} \quad P(i,j) = a_i a_{j+2^{n-1}} - a_{i+2^{n-1}} a_j. \]  
To show \( C_{1(2...n)}^2 \geq \tau_{1...n} \), from Eqs. (B10) and (B11), it is enough to prove
\[ \sum_{0 \leq i \leq 2^{n-2}-1} |P(i,j)|^2 + \sum_{2^{n-2} \leq i < j \leq 2^{n-1}-1} |P(i,j)|^2 \]
\[ \geq 2 \sum_{0 \leq k \leq 2^{n-2}-1} Z_k Z_m. \]  
\( \text{(B12)} \)

Observe that in Eq. (B12), the first, second, and third sums contain \( 3 \times 2^{n-3}(2^{n-2}-1) \) different terms \( |P(i,j)|^2 \), \( 2^{n-3}(2^{n-2}-1) \) different terms \( |P(i,j)|^2 \), and \( 2^{n-3}(2^{n-2}-1) \) different terms \( Z_k Z_m \), respectively. Next we show that for each term \( Z_k Z_m \) on the right side of Eq. (B12), there exist four different corresponding terms \( |P(i,j)|^2 \) on the left side of Eq. (B12) such that their sum is equal to or greater than \( 2Z_k Z_m \).

Given \( Z_k Z_m \) with \( 0 \leq k < m \leq 2^{n-2}-1 \). We first choose two different terms \( |P(k,2^{n-1}-1-m)|^2 \) and \( |P(m,2^{n-1}-1-k)|^2 \) from the first sum in Eq. (B12). It is trivial that
\[ |P(k,2^{n-1}-1-m)|^2 + |P(m,2^{n-1}-1-k)|^2 \]
\[ \geq 2|P(k,2^{n-1}-1-m)||P(m,2^{n-1}-1-k)|. \]  
\( \text{(B13)} \)

We then choose the term \( |P(k,m)|^2 \) from the first sum in Eq. (B12) and the term \( |P(2^{n-1}-1-m,2^{n-1}-1-k)|^2 \) from the second sum in Eq. (B12). It is trivial that
\[ |P(k,m)|^2 + |P(2^{n-1}-1-m,2^{n-1}-1-k)|^2 \]
\[ \geq 2|P(k,m)||P(2^{n-1}-1-m,2^{n-1}-1-k)|. \]  
\( \text{(B14)} \)

Now, using the fact that \( |x| + |y| \geq |x-y| \), from Eqs. (B13) and (B14), we establish the inequality
\[ |P(k,2^{n-1}-1-m)||P(m,2^{n-1}-1-k)| \]
\[ + |P(k,m)||P(2^{n-1}-1-m,2^{n-1}-1-k)| \]
\[ \geq Z_k Z_m. \]  
\( \text{(B15)} \)

and this implies the desired result Eq. (B12). This completes the proof.

References

[1] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[2] V. Coffman et al., Phys. Rev. A 61, 052306 (2000).
[3] A. Wong and N. Christensen, Phys. Rev. A 63, 044301 (2001).
[4] P. Rungta et al., Phys. Rev. A 64, 042315 (2001).
[5] L. Aolita et al., Phys. Rev. Lett. 97, 050501 (2006).
[6] A. Osterloh et al., Nature (London) 416, 608 (2002).
[7] W. Dür, G. Vidal and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[8] A. Sudbery, J. Phys. A: Math. Gen 34, 643 (2001) M. Grass et al., Phys. Rev. A 58, 1853 (1998); N. Linden et al., Phys. Rev. Lett. 83, 243 (1999).
[9] Jean-Gabriel Luque et al., Phys. Rev. A 67, 042303 (2003); Jean-Gabriel Luque et al., Math. Struct. in Comp Science (2007) vol. 17, pp1133-1157, 2007.

[10] M. S. Leifer et al., Phys. Rev. A 69, 052304 (2004).

[11] P. Lévy, J. Phys. A: Math. Gen. 39, 9533-9545 (2006).

[12] Dafa Li et al., Phys. Lett. A 359 428-437 (2006).

[13] Dafa Li et al., Phys. Rev. A 76, 032304 (2007); Dafa Li et al., e-print quant-ph/0704.2087.

[14] D. Z. Dokovic and A. Osterloh, J. Math. Phys. 50, 033509 (2009).

[15] A. Osterloh and J. Siewert, Phys. Rev. A 72, 012337 (2005); A. Osterloh and J. Siewert, Int. J. Quant. Inf. 4, 531 (2006).

[16] Dafa Li et al., J. Math. Phys. 50, 012104 (2009).

[17] F. Verstraete et al., Phys. Rev. A 68, 012103 (2003).

[18] Chang-shui Yu and He-shan Song, Phys. Rev. A 71, 042331 (2005).

[19] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).

[20] Youg-Cheng Ou and Heng Fan, Phys. Rev. A 75, 062308 (2007).

[21] J.K. Stockton et al., Phys. Rev. A 67, 022112 (2003).