Lagrangian perturbation theory of nonrelativistic rotating superfluid stars

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ABSTRACT

We develop a Lagrangian perturbation framework for rotating non-relativistic superfluid neutron stars. This leads to the first generalization of classic work on the stability properties of rotating stars to models which account for the presence of potentially weakly coupled superfluid components. Our analysis is based on the standard two-fluid model expected to be relevant for the conditions that prevail in the outer core of mature neutron stars. We discuss the implications of our results for dynamical and secular instabilities of a simple neutron star model in which the two fluids are allowed to assume different (uniform) rotation rates.

Key words: stars: neutron – hydrodynamics – instability – gravitational waves

1 INTRODUCTION

Investigations into the stability properties of rotating self-gravitating bodies are of obvious relevance to astrophysics. By improving our understanding of the relevant issues we can hope to shed light on the nature of the various dynamical and secular instabilities that may govern the spin-evolution of rotating stars. The relevance of such knowledge for neutron star astrophysics may be highly significant. It is for example not clear (i) whether instabilities may affect nascent neutron stars leading to slow birth spin rates inferred for radio pulsars, (ii) to what extent gravitational-wave driven instabilities may provide the explanation for the observed clustering of spin periods in Low-Mass X-ray Binaries, and (iii) if compact stars are likely to evolve through phases where unstable oscillations lead to the emission of detectable gravitational-wave signal (see the review article by Andersson (2003) for a discussion of these issues and further references).

The aim of this (and a subsequent) paper is to develop a Lagrangian perturbation framework for rotating non-relativistic superfluid neutron stars. This leads to the first generalization of Friedman and Schutz’s classic work (Friedman & Schutz 1978a,b) on the stability properties of rotating stars to models which account for the presence of potentially weakly coupled superfluid components. We believe that our extension of the, by now 25 years old, single fluid model for neutron stars to models which account for the presence of potentially weakly coupled superfluid components is also an exciting addition to the standard model, because the additional degree of freedom may lead to new phenomena. An interesting example of this is the recently discovered superfluid two-stream instability (Andersson et al. 2003; Preis et al. 2004; Andersson et al. 2002).

The layout of this paper is as follows: We begin by introducing the two-fluid model for superfluid stars and the Lagrangian perturbation theory that we will employ (Section 2). We then revisit (in Section 3) what should be well-known results for a single (barotropic) fluid model. We also revise those results in such a way that the main concepts will remain relevant for the two-fluid problem. Yet they are easier to introduce in the single fluid case. Having developed the necessary tools we then move on to the two-fluid case (Section 4). As we will see, the extension of the single fluid formulae is relatively straightforward provided that we only allow for the direct chemical coupling between the two fluids [for the moment neglecting the entrainment effect (Andreev & Bashkin 1975; Borumand et al. 1996; Comer & Joynt 2003; Comer 2004), the role of which will be the main focus of a subsequent paper]. Section 5 then provides a discussion of some implications of our results, which is followed in Section 6 by specific implications for the r-mode instability and gravitational wave emission. Finally, in Section 7 we briefly look ahead to the work that will be presented in a subsequent paper.

Our analysis is based on the standard two-fluid model (Cartel 1985; Comer & Langelo 1994; Carter & Langelo 1998; Langelo et al. 1998; Carter & Langelo 1998; Sedrakian & Wasserman 2003; Andersson & Comer 2001; Comer 2002; Lee & Yoshida 2003; Yoshida & Lee 2003a,b; Preis 2004; Carter & Chamel 2003b).
2 TWO KEY INGREDIENTS

We begin our discussion by introducing the two main ingredients of our analysis: The two-fluid model for neutron stars, and the Lagrangian perturbation theory developed by Friedman & Schutz (1978a,b).

2.1 The two-fluid model for superfluid stars

A mature neutron star is likely to contain several superfluid components. One would certainly expect superfluid neutrons to coexist with the crystal lattice of nuclei that makes up the outer kilometer or so of the star. In the outer parts of the fluid core superfluid neutrons are expected to coexist with superconducting protons, while the deep core may contain exotic states of matter like superfluid hyperons and perhaps even colour-superconducting deconfined quarks. The modelling of a star comprising all these components is far beyond our current means. In fact, the parameters governing the possible states are uncertain to say the least. Given this, most investigations into the dynamics of superfluid neutron stars have considered a much simplified scenario that accounts for the presence of two distinct fluid components.

The two-fluid picture of superfluid neutron stars is based on the notion that the outer core is dominated by superfluid neutrons, superconducting protons and electrons. Since the charged components couple electromagnetically on a short timescale they are assumed to move together. This assumption should be valid as long as we consider dynamics that takes place on a timescale long compared to that associated with the electromagnetic interaction. The outcome is a model describing the motion of the neutrons and the “protons” (a conglomerate of the charged components).

The equations that describe the two-fluid model are derived from an energy functional $E(n_n, n_p, w^2)$ where $n_n$ are the two number densities. Given the two fluid velocities $v_x^n$ the quantity $w_x^n = v_x^n - v_x^p$ represents the relative velocity between the two fluids, for ease of notation we define $w_x = w_x^n w_x^p$. Throughout this paper we will use the constituent indices $x = n, \ p$ and $y \neq x$. This allows us to condense the various equations significantly, but in order to avoid confusion we should stress that repeated constituent indices never imply summation, while repeated vector component indices always do. It is also worth pointing out that all calculations will be carried out in a coordinate basis. Hence, we will distinguish between co- and contravariant vectors etcetera.

Once we are supplied with the energy functional (the equation of state) a straightforward variation yields

$$dE = \sum_{x=n, p} \mu_x dn_x + \alpha dw^2.$$ (1)

This leads to the definition of the two chemical potentials

$$\mu_x = \left( \frac{\partial E}{\partial n_x} \right)_{n_y, w^2}. (2)$$

For later convenience, we introduce the notation $\mu_x = \mu_x/m_B$, where we take the two nucleon masses to be equal such that $m_B = m_n = m_p$. We also see that the entrainment coefficient $\alpha$ follows from

$$\alpha = \left( \frac{\partial E}{\partial w^2} \right)_{n_{x}, n_{y}}. (3)$$

The dynamical equations that govern the two coupled fluids can be derived either from a Newtonian variational principle (Prix 2004) or from the Newtonian limit (Andersson & Comer 2001) of the fully relativistic equations derived by Carter and collaborators (Carter 1989, Comer & Langlois 1994, Carter & Langlois 1998, Langlois et al 1998). The end result is two continuity equations

$$\partial_t n_x + \nabla_i (n_x v_x^i) = 0$$ (4)

and two coupled Euler equations

$$\begin{align*}
(\partial_t + v_x^i \nabla_j) (v_x^j + \varepsilon_x w_x^{jy} ) + \nabla_i (\Phi + \tilde{\mu}_x) + \varepsilon_x w_x^{jy} \nabla_i v_x^j = 0 .
\end{align*} (5)$$

Here we have defined $\varepsilon_x = 2\alpha/\mu_x$. We also have the standard Poisson equation for the gravitational potential $\Phi$;

$$\nabla^2 \Phi = 4\pi m_B G (n_n + n_p) . (6)$$

We assume the background to be stationary and axisymmetric, with the two fluids rotating around the $z$-axis with rates $\Omega_0$ and $\Omega_p$ respectively. Hence we have

$$v_x^z = \Omega_0 \varphi^1 ,$$

and

$$w_x^{zy} = (\Omega_0 - \Omega_p) \varphi^1$$

with $\varphi^1$ given by

$$\varphi^1 \partial_i = \partial \varphi^1 . (8)$$

In spherical coordinates, i.e. $x^i \in \{ r, \theta, \varphi \}$, this vector has the components $\varphi^i = (0, 0, 1)$, and its norm is $\varphi^i \varphi_i = r^2 \sin^2 \theta$.

In order to simplify the analysis, we restrict our attention to models with uniform rotation, i.e. take $\Omega_0$ to be constants. Furthermore, in order to elucidate the details we will only consider the case of vanishing entrainment. That is, we let $\alpha = 0$. The motivation for this is simple: If we could not make progress even in this restricted case, it would be pointless to consider the much more complicated problem which includes the entrainment. However, as we will see, the case of vanishing entrainment works out very neatly. Hence there is every reason for optimism, and we will return to the case $\alpha \neq 0$ in a subsequent paper.

2.2 Lagrangian perturbation theory

Our aim is to derive conserved quantities for the perturbations of the system of equations described above. By doing this we hope to be able to provide criteria that can be used to decide when the oscillations of a rotating superfluid neutron star are unstable. The need for such criteria is clear given that i) the astrophysical relevance of the gravitational-wave driven $r$-mode instability may to a large extent depend on whether the star contains superfluid components (Andersson 2003), and ii) the dynamical two-stream instability may set in above a critical relative rotation rate (Andersson et al 2003, Prix et al 2004, Andersson et al 2005). A detailed analysis of both these problems clearly requires an improved understanding of the stability properties of superfluid stars.

Following Friedman & Schutz (1978a,b), we analyze the problem within the Lagrangian perturbation formalism. The Lagrangian variation $\delta Q$ of a quantity $Q$ is related to the Eulerian variation $\delta Q$ by

$$\Delta Q = \delta Q + \mathcal{L}_Q \delta Q$$ (9)

where the Lie derivative $\mathcal{L}_Q$ has the meaning

$$\mathcal{L}_Q f = \xi^i \nabla_i \xi^i$$ (10)

for scalars,

$$\mathcal{L}_Q v^i = \xi^i \nabla_j v^i - v^j \nabla_j \xi^i$$ (11)

for contravariant vectors, and
\[ L_\xi v_i = \xi^j \nabla_j v_i + v_j \nabla_i \xi^j \] (12)
for covariant vectors.

The Lagrangian change in the fluid velocity follows from
\[ \Delta v^i = \partial_t \xi^i \] (13)
where \( \xi^i \) is the Lagrangian displacement. Given this, and
\[ \Delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i \] (14)
we have
\[ \Delta v_i = \partial_t \xi_i + v^j \nabla_j \xi_i - \nabla_i v^j \] (15)
It is also useful to note that the Eulerian variations are given by
\[ \delta v^i = \partial_t \xi^i + v^j \nabla_j \xi^i - \xi^i \nabla_j v^j \] (16)
and
\[ \delta v_i = \partial_t \xi_i + v^j \nabla_j \xi_i - \xi^i \nabla_j v_i \] (17)
(quoted obviously, since \( v_i = g_{ij} v^j \) and \( \nabla_k g_{ij} = 0 \)).

### 3 REVISITING AND REVISING THE SINGLE FLUID PROBLEM

In order to lay the foundation for our analysis of the superfluid problem, it is useful to revisit the analysis of an ordinary perfect fluid star. In doing this we want to stay as close as possible to the equations used to describe the two-fluid problem. We know from previous work that the relevant model to compare to is a barotropic perfect fluid, e.g., since the g-modes are absent from the pulsation spectrum of a non-rotating superfluid model (Lee 1995; Andersson & Comer 2001; Comer 2002). Furthermore, we prefer to work with the number density \( n \), the fluid velocity \( v_i \) and the chemical potential \( \mu \) rather than the pressure \( P \).

#### 3.1 The perturbation equations

From thermodynamic principles we know that, for a barotropic ordinary fluid we have \( E = E(n) \), and
\[ d\mu = -\frac{1}{n} dP \] (18)
This allows us to write the standard Euler equation as
\[ \left( \partial_t + v^j \nabla_j \right) v_i + \nabla_i (\tilde{\mu} + \Phi) = 0 \] (19)
where \( \tilde{\mu} = \mu/m_B \) as before. In addition we have the continuity equation
\[ \partial_t n + \nabla_i (nv^i) = 0 \] (20)
and the Poisson equation for the gravitational potential
\[ \nabla^2 \Phi = 4\pi m_B G n \] (21)

We now want to perturb these equations. First of all, conservation of mass for the perturbations is readily expressed as
\[ \Delta n = -n \nabla_i \xi^i \rightarrow \delta n = -\nabla_i (n \xi^i) . \] (22)
Consequently, the perturbed gravitational potential follows from
\[ \nabla^2 \delta \Phi = 4\pi G m_B \delta n = -4\pi G m_B \nabla_i (n \xi^i) . \] (23)

In order to perturb the Euler equations we first rewrite Eq. (19) as
\[ (\partial_t + L_v) v_i + \nabla_i \left( \tilde{\mu} + \Phi - \frac{1}{2} v^2 \right) = 0 . \] (24)
This form is particularly useful since the Lagrangian variation commutes with the operator \( \partial_t + L_v \), i.e.
\[ \Delta (\partial_t + L_v) v_i = (\partial_t + L_v) \Delta v_i . \] (25)
Perturbing (24) we thus have
\[ (\partial_t + L_v) \Delta v_i + \nabla_i \left( \Delta \tilde{\mu} + \Delta \Phi - \frac{1}{2} \Delta (v^2) \right) = 0 . \] (26)

We want to rewrite this equation in terms of the displacement vector \( \xi \). After some algebra one finds that
\[ \partial_t^2 \xi_i + 2v^j \nabla_j \partial_t \xi_i + (v^j \nabla_j)^2 \xi_i + \nabla_i \delta \Phi + \xi^i \nabla_j \Phi - (\nabla_i \xi^i) \nabla_j \tilde{\mu} + \nabla_i \Delta \tilde{\mu} = 0 . \] (27)
Finally, we need
\[ \Delta \tilde{\mu} = \delta \tilde{\mu} + \xi^i \nabla_i \tilde{\mu} = \left( \frac{\partial \tilde{\mu}}{\partial n} \right) \delta n + \xi^i \nabla_i \tilde{\mu} \]
\[ = - \left( \frac{\partial \tilde{\mu}}{\partial n} \right) \nabla_i (n \xi^i) + \xi^i \nabla_i \tilde{\mu} . \] (28)
Given this, we have arrived at the following form for the perturbed Euler equation
\[ \partial_t^2 \xi_i + 2v^j \nabla_j \partial_t \xi_i + (v^j \nabla_j)^2 \xi_i + \nabla_i \delta \Phi + \xi^i \nabla_j \Phi - (\nabla_i \xi^i) \nabla_j \tilde{\mu} - \nabla_i \left[ \left( \frac{\partial \tilde{\mu}}{\partial n} \right) \nabla_j (n \xi^i) \right] = 0 . \] (29)
This equation should be compared to Eq. (15) of Friedman & Schutz (1978a).

#### 3.2 Conserved quantities: The canonical energy/angular momentum

Having derived the perturbed Euler equations, we want to construct conserved quantities that can be used to assess the stability of the system. To do this, we first multiply Eq. (19) by the number density \( n \), and then write the result (schematically) as
\[ A \partial_t^2 \xi + B \partial_t \xi + C \xi = 0 . \] (30)
We are omitting the indices, since there should be little risk of confusion.

Defining the standard inner product
\[ \langle \eta^i, \xi_i \rangle = \int \eta^i \xi_i dV \] (31)
where the asterisk denotes complex conjugation, one can readily show that
\[ \langle \eta, A \xi \rangle = \langle \xi, A^* \eta \rangle \quad \text{and} \quad \langle \eta, B \xi \rangle = -\langle \xi, B^* \eta \rangle . \] (32)
The latter requires the background relation \( \nabla_i (n v^i) = 0 \), and holds provided that \( n \rightarrow 0 \) at the surface of the star. A slightly more involved calculation leads to
\[ \langle \eta, C \xi \rangle = \langle \xi, C^* \eta \rangle . \] (33)
In particular, we use
\[ \int \eta n^i \nabla_i \left[ \left( \frac{\partial \tilde{\mu}}{\partial n} \right) \nabla_j (n \xi^j) \right] dV = \int \eta n^i \nabla_i \left[ \left( \frac{\partial \tilde{\mu}}{\partial n} \right) \nabla_j (n \xi^j) \right] dV \] (34)
which (again) holds as long as \( n \rightarrow 0 \) at the surface, and
\[ \int \eta n^i \nabla_i \delta \Phi dV = \frac{1}{4\pi m_B} \int g^{ij} \delta \Phi \nabla_i \nabla_j \Phi^* dV \]
This leads us to define the
\[ W(\eta, \xi) = \left\langle \eta, A\phi \xi + \frac{1}{2}B\xi \right\rangle - \left\langle A\phi \eta + \frac{1}{2}B\eta, \xi \right\rangle \] (36)
where \( \eta \) and \( \xi \) both solve the perturbed Euler equation. Given this, it is straightforward to show that \( W(\eta, \xi) \) is conserved, i.e. \( \partial_t W = 0 \). This leads us to define the canonical energy of the system as
\[ E_c = \frac{m_\beta}{2} W(\partial_x \xi, \xi) = \frac{m_\beta}{2} \left\{ (\partial_x \xi, A\phi \xi) + (\xi, C\xi) \right\} . \] (37)
After some manipulations, we arrive at the following explicit expression
\[ E_c = \frac{1}{2} \int \left\{ \rho |(\partial_t \xi)|^2 - \rho |v|^2 \nabla_j \xi_j|^2 + \rho \xi^i \xi^j \nabla_i \nabla_j (\hat{\mu} + \Phi) \right\} + \left( \frac{\eta_0}{\beta n} \right) |\delta n|^2 \frac{1}{2} |\nabla \delta \Phi|^2 \right\} dV \] (38)
which can be compared to Eq. (45) of Friedman & Schutz (1978a).

In the case of an axisymmetric system, e.g. a rotating star, we can also define a canonical angular momentum as
\[ J_c = -\frac{m_\beta}{2} \int (\partial_t \xi, \xi) = -\text{Re} \left\{ (\partial_t \xi, A\phi \xi) + \frac{1}{2}B\xi \right\} . \] (39)
The proof that this quantity is conserved relies on the fact that (i) \( W(\eta, \xi) \) is conserved for any two solutions to the perturbed Euler equations, and (ii) \( \partial_t \) commutes with \( \rho v^j \nabla_j \) in axisymmetry, which means that if \( \xi \) solves the Euler equations then so does \( \partial_t \xi \).

As elucidated by Friedman & Schutz (1978b), the stability analysis is complicated by the presence of so-called “trivial” displacements. These trivialities can be thought of as “integration constants” representing a relabeling of the physical fluid elements. A trivial displacement \( \xi^i \) leaves the physical quantities unchanged, i.e. such that \( \delta n = \delta \xi^i = 0 \). This means that we must have
\[ \nabla_i (\rho \xi^i) = 0, \] (40)
\[ (\partial_t + L_v) \xi^i = 0 . \] (41)
The solution to the first of these equations can be written
\[ \rho \xi^i = e^{ijk} \nabla_k \chi^j \] (42)
where, in order to satisfy the second equations, the vector \( \chi_j \) must have time-dependence such that
\[ (\partial_t + L_v) \chi^j = 0 . \] (43)
This means that the trivial displacement will remain constant along the background fluid trajectories. Or, as Friedman & Schutz (1978a) put it, the “initial relabeling is carried along with the unperturbed motion.”

The trivialities may cause trouble because they affect the canonical energy. Before one can use the canonical energy to assess the stability of a rotating configuration one must deal with this “gauge problem.” The way to do this is to ensure that the displacement vector \( \xi \) is orthogonal to all trivialities. A prescription for doing this is provided by Friedman & Schutz (1978a). In particular, they show that the required canonical perturbations preserve the vorticity of the individual fluid elements. Most importantly, one can also prove that a normal mode solution is orthogonal to the trivialities. Thus, normal mode solutions can serve as canonical initial data, and be used to test stability.

### 3.3 Example: Instabilities of rotating perfect fluid stars

The importance of the canonical energy stems from the fact that it can be used to test the stability of the system. In particular, we note that:

- Dynamical instabilities are only possible for motions such that \( E_c = 0 \). This makes intuitive sense since, the amplitude of a mode which \( E_c \) vanishes can grow without bounds and still obey the conservation laws.
- If the system is coupled to radiation (e.g. gravitational waves) which carries positive energy away from the system (which should be taken to mean that \( \partial_t E_c < 0 \)) then any initial data for which \( E_c < 0 \) will lead to an unstable evolution.

Consider a real frequency normal-mode solution to the perturbation equations, a solution of form \( \xi = \xi e^{i(\omega t + m\Omega)} \). One can readily show that the associated canonical energy becomes
\[ E_c = \omega \left[ \langle \xi, A\xi \rangle - \frac{i}{2} \langle \xi, B\xi \rangle \right] . \] (44)
where the expression in the bracket is real valued. For the canonical angular momentum we get
\[ J_c = -m \left[ \omega \langle \xi, A\xi \rangle - \frac{i}{2} \langle \xi, B\xi \rangle \right] . \] (45)
Combining Eq. (44) and Eq. (45) we see that, for real frequency modes we will have
\[ E_c = -m \omega J_c = \sigma_p J_c \] (46)
where \( \sigma_p \) is the pattern speed of the mode.

Now notice that Eq. (45) can be rewritten as
\[ \left\langle \xi, \rho \xi^i \right\rangle = -m \omega + m \left\langle \xi, \rho \xi^i \nabla \xi \right\rangle \] (47)
Using cylindrical coordinates, and \( v^i = \Omega \rho \xi^i \), one can show that
\[ -\iota \xi^i \nabla_j \xi^j = \rho \Omega m |\xi|^2 + i(\xi^* \times \xi) \iota . \] (48)
But
\[ |\xi^* \times \xi| \leq |\xi|^2 \] (49)
and we must have (for uniform rotation)
\[ \sigma_p = \Omega \left( 1 + \frac{1}{m} \right) \leq J_c / m \leq \sigma_p - \Omega \left( 1 - \frac{1}{m} \right) . \] (50)

Eq. (50) forms an integral part of the Friedman & Schutz (1978b) proof that rotating perfect fluid stars are generically unstable in the presence of radiation. The argument is as follows: Consider modes with finite frequency in the \( \Omega \rightarrow 0 \) limit. Then Eq. (50) implies that co-rotating modes (with \( \sigma_p > 0 \)) must have \( J_c > 0 \), while counter-rotating modes (for which \( \sigma_p < 0 \)) will have \( J_c < 0 \). In both cases \( E_c > 0 \), which means that both classes of modes are stable. Now consider a small region near a point where \( \sigma_p = 0 \) (at a finite rotation rate). Typically, this corresponds to a point where the initially counter-rotating mode becomes co-rotating. In this region \( J_c < 0 \). However, \( E_c \) will change sign at the point where \( \sigma_p \) (or, equivalently, the frequency \( \omega \)) vanishes. Since the mode was stable in the non-rotating limit this change of sign indicates the onset of instability at a critical rate of rotation.

### 3.4 Example: The r-mode instability

In order to further demonstrate the usefulness of the canonical energy, let us prove the instability of the single-fluid r-modes.
For a general inertial mode we have (cf. Lockitch & Friedman [1999] who provide a discussion of the single fluid problem using notation which closely resembles the one we adopt here)

$$\vec{v} \sim \delta \vec{v} \sim \vec{\xi} \sim \Omega \quad \text{and} \quad \delta \Phi \sim \delta n \sim \Omega^2.$$ (51)

If we also assume axial-led modes, like the r-modes, then we have $\delta v_r \sim \Omega^2$ and the continuity equation leads to

$$\nabla \cdot \delta \vec{v} \sim \Omega^2 \to \nabla \cdot \vec{\xi} \sim \Omega^2.$$ (52)

Under these assumptions we find that $E_c$ becomes (to order $\Omega^2$)

$$E_c \approx \frac{1}{2} \int \rho \left[ \left| \partial_t \vec{\xi} \right|^2 - |v \cdot \nabla |\vec{\xi}|^2 + \xi^\theta \xi^\theta + \nabla j \right] dV.$$ (53)

We can rewrite the last term using the equation governing the axisymmetric equilibrium. Keeping only terms of order $\Omega^2$ we have

$$\xi^\theta \xi^\theta \nabla j \sim \frac{1}{2} \Omega^2 \xi^\theta \xi^\theta \nabla j(r^2 \sin^2 \theta).$$ (54)

A bit more work then leads to

$$\frac{1}{2} \Omega^2 \xi^\theta \xi^\theta \nabla j(r^2 \sin^2 \theta) = \Omega^2 r^2 \left[ \cos^2 \theta |\xi^\theta|^2 + \sin^2 \theta |\xi^\theta|^2 \right]$$ (55)

and

$$|v \cdot \nabla j|^2 \sim \Omega^2 \left\{ m^2 |\xi|^2 - 2 m^2 \sin \theta \cos \theta \left[ \xi^\theta \xi^\theta - \xi^\varphi \xi^\varphi \right] + m^2 \cos^2 \theta |\xi|^2 + \sin^2 \theta |\xi|^2 \right\}$$ (56)

which means that the canonical energy can be written in the form

$$E_c \approx \frac{1}{2} \int \rho \left\{ (m \Omega - \omega)(m \Omega + \omega)|\xi|^2 - 2 m \sin^2 \theta \cos \theta \left[ \xi^\theta \xi^\theta - \xi^\varphi \xi^\varphi \right] \right\} dV$$ (57)

for an axial-led mode.

Introducing the axial stream function,

$$\xi^\theta = - \frac{i U}{r \sin \theta} \partial_r Y^m_l e^{i \omega t},$$ (58)

$$\xi^\varphi = \frac{2 i U}{r^2 \cos \theta} Y^m_l e^{i \omega t},$$ (59)

where $Y^m_l = Y^m_l (\theta, \varphi)$ are the standard spherical harmonics, we have

$$|\xi|^2 = \frac{|U|^2}{r^2} \left[ \frac{1}{\sin \theta} \partial_r Y^m_l |^2 + \partial \partial Y^m_l |^2 \right]$$ (60)

and

$$2 r^2 \sin \theta \cos \theta \left[ \xi^\theta \xi^\varphi - \xi^\varphi \xi^\varphi \right] = \frac{1}{r^2 \sin \theta} m |U|^2 \left[ Y^m_l \partial_r Y^m_l + Y^m_l \partial \partial Y^m_l \right].$$ (61)

After performing the angular integrals, we find that

$$E_c = -\frac{l(l+1)}{2} \left\{ (m \Omega - \omega)(m \Omega + \omega) - \frac{2 m^2 \Omega^2}{l(l+1)} \right\} \int \rho |U|^2 dV.$$ (62)

Combining this with the $r$-mode frequency Lockitch & Friedman [1999]

$$\omega = m \Omega \left[ 1 - \frac{2}{l(l+1)} \right]$$ (63)

we see that $E_c < 0$ for all $l > 1$ r-modes, i.e. they are all unstable. The $l = m = 1$ r-mode is a special case, leading to $E_c = 0$.

### 4 THE SUPERFLUID PROBLEM

In this Section we generalise the Lagrangian perturbation analysis to the two-fluid model for superfluid neutron stars. In order to simplify matters we only consider the case of vanishing entrainment. The, significantly more complicated, general case will be discussed in a subsequent paper.

#### 4.1 The perturbation equations

As in the single-fluid problem, we begin by deriving the equations governing Lagrangian perturbations of the system. Assuming vanishing entrainment, i.e. letting $\alpha = 0$, we have the Euler equations

$$\partial_t v_i + L_{\text{rot}} v_i + \nabla \left( \frac{\Phi + \mu}{c^2} - \frac{1}{2} \nabla \cdot \vec{v} \right) = 0$$ (64)

where we recall that $x = n$ or $p$. Clearly, we must introduce two distinct Lagrangian displacement vectors $\vec{\xi}_n$. To distinguish between the two possibilities we use variations $\Delta \xi$ such that

$$\Delta \xi = \delta \xi_n,$$ (65)

which leads to the perturbed continuity equations taking the form

$$\Delta \xi_n = -n \nabla \xi_n \rightarrow \delta \xi_n = -\nabla \left( n \xi_n \right).$$ (66)

With these definitions, it is very easy to derive the perturbed Euler equations. Simply comparing Eq. (24) to Eq. (64) we see that we must have

$$\partial_t^2 \xi_n + 2 v^r \nabla_j \partial_n \xi_n + (v^r \nabla_j)^2 \xi_n + \nabla_i \Phi + \xi_n \nabla_i \nabla_j \Phi \nabla \xi_n \nabla_j \nabla \xi_n + \nabla_i \partial_n \xi_n = 0.$$ (67)

To express this in terms of the displacement vectors we need

$$\Delta \xi = \delta \xi_n + \xi_n \nabla \delta v_n,$$ (68)

and we arrive at the following form for the perturbed Euler equations

$$\partial_t^2 \xi_n + 2 (v^r \nabla_j ^2 \xi_n) + \nabla_i \Phi + \xi_n \nabla_i \nabla_j \Phi \xi_n \nabla_j \nabla_j \xi_n + \nabla_i \partial_n \xi_n - \nabla_i \nabla_j \xi_n \nabla_j \nabla_j \xi_n = 0.$$ (69)

From this equation it is clear that the two fluids are coupled. In order to proceed we need to understand the nature of this coupling better. In particular, we note that the perturbed gravitational potential depends on both displacement vectors. We have

$$\nabla^2 \Phi = 4 \pi m_B G (\delta n_n + \delta n_p) = -4 \pi m_B G \nabla_i \left( n_n \xi_n + n_p \xi_p \right).$$ (70)

Since this is a linear equation we can write the solution as

$$\delta \Phi = \delta \Phi_n + \delta \Phi_p = \sum_{x=n,p} \delta \Phi_x$$ (71)

where we define
\[\nabla^2 \Phi_k = 4\pi m_B G \delta n_k = -4\pi m_B G \nabla \cdot \langle n_k \xi_k \rangle \]  
(73)

In analogy with the single fluid case, we can write the perturbed Euler equations in the schematic form (after multiplying Eq. \(60\) by \(n_k\))

\[A_k \partial_t \xi_k + B_k \partial_x \xi_k + C_k \xi_k + D_k \xi = 0 \]  
(74)

It should be noted that the first three terms are obvious generalisations of the single fluid case. Now they pertain to each of the two fluids. The last term is new, and describes the coupling between the fluids. Explicitly, it takes the form

\[D_k \xi = -n_k \nabla \left( \frac{\partial \mu_k}{\partial n_k} \right) \nabla \left( n_k \xi_k \right) + n_k \nabla \xi \cdot \delta \Phi_k \]  
(75)

From this we see that the fluids are coupled (i) “chemically” through the equation of state, and (ii) “gravitationally” because of the fact that variations in one of the number densities affects the gravitational potential, which then influences the other fluid.

### 4.2 Conserved quantities in the superfluid case

We want to derive conserved quantities similar to those in the single-fluid case. To do this we again use the inner product. Given the results from Sec. \(4\) it is easy to show that we have the following symmetries

\[\langle \eta, A_x \xi \rangle = \langle \xi, A_x \eta \rangle^* \]  
(76)

\[\langle \eta, B_x \xi \rangle = -\langle \xi, B_x \eta \rangle^* \]  
(77)

\[\langle \eta, C_x \xi \rangle = \langle \xi, C_x \eta \rangle^* \]  
(78)

where \(\eta_x\) can (at this point) be any vector field.

Next we want to introduce symplectic structures that would be natural generalisations of the one we used to construct the canonical energy and angular momentum in the single fluid problem. To do this we consider two sets of solutions \([^\xi_u, \xi_v]\) and \([\eta_u, \eta_v]\) to our perturbation equations. Then we define

\[W_u(\eta_u, \xi_u) = \langle \eta_u, A_x \partial_x \xi_u + \frac{1}{2} B_x \xi_u \rangle - \langle A_u \partial_x \eta_u + \frac{1}{2} B_u \eta_u, \xi_u \rangle \]  
(79)

Given this definition and the above symmetry relations, it is straightforward to show that

\[\partial_t W_u = -\langle \eta_u, D_x \xi_u \rangle + \langle D_u \eta_u, \xi_u \rangle \neq 0 \]  
(80)

Analogously we introduce

\[W_p(\eta_p, \xi_p) = \langle \eta_p, A_p \partial_p \xi_p + \frac{1}{2} B_p \xi_p \rangle - \langle A_p \partial_p \eta_p + \frac{1}{2} B_p \eta_p, \xi_p \rangle \]  
(81)

which leads to

\[\partial_t W_p = -\langle \eta_p, D_p \xi_p \rangle + \langle D_p \eta_p, \xi_p \rangle \neq 0 \]  
(82)

Intuitively one would expect the sum \(W_u + W_p\) to be conserved. That is, the coupling terms in Eq. \(76\) should facilitate non-dissipative energy transfer between the two fluids. We will now prove that this is, indeed, the case.

Explicitly we have

\[\partial_t W_u = \int n_u \eta_u^* \nabla \left( \frac{\partial \mu_u}{\partial \n_u} \right) \nabla \left( n_u p_u \xi_u \right) dV \]

\[\partial_t W_p = \int n_p \eta_p^* \nabla \left( \frac{\partial \mu_p}{\partial \n_p} \right) \nabla \left( n_p p_u \xi_p \right) dV \]

and assuming that \(n_u\) and \(n_p\) both vanish at the surface\(^1\)

\[\int n_p \eta_p^* \nabla \left( \frac{\partial \mu_p}{\partial \n_p} \right) \nabla \left( n_u p_u \xi_u \right) dV = \int n_u \eta_u^* \nabla \left( \frac{\partial \mu_u}{\partial \n_u} \right) \nabla \left( n_p p_p \xi_p \right) dV \]

\[\int n_p \eta_p^* \nabla \left( \frac{\partial \mu_p}{\partial \n_p} \right) \nabla \left( n_u p_u \xi_u \right) dV = \int n_u \eta_u^* \nabla \left( \frac{\partial \mu_u}{\partial \n_u} \right) \nabla \left( n_p p_p \xi_p \right) dV \]

This means that, when the two expressions Eq. \(84\) and Eq. \(85\) are added, the first two terms of each expression will cancel each other.

To rewrite the terms involving the gravitational potentials in Eq. \(84\) and Eq. \(85\) we need to use

\[\int n_u \eta_u^* \nabla \left( \delta \eta_u \Phi_u \right) dV = \int n_p \eta_p^* \nabla \left( \delta \eta_p \Phi_p \right) dV \]

\[\partial_t W = W_u(\eta_u, \xi_u) + W_p(\eta_p, \xi_p) \]

(83)

is a conserved quantity.

By analogy with the single fluid case, it now makes sense to define the canonical energy of the system as

\[E_c = \frac{m_B}{2} [W_u(\partial_t \xi_u, \xi_u) + W_p(\partial_t \xi_p, \xi_p)] \]

(91)

\(^1\) Somewhat artificially, we assume that the rotating background is such that the two fluids have a common surface. In reality, the outer layers of a neutron star will not be superfluid and one would have to add a single fluid envelope to our model. The analysis of such composite models is beyond the scope of the present analysis.
We can derive the superfluid equations of motion (74) from a variational principle whose action is (for clarity assuming that the displacement vectors are real, the generalisation to the complex case is straightforward)

\[
E_c = \frac{1}{2} \int \left( \rho_n |\partial_t \xi_n|^2 + \rho_p |\partial_t \xi_p|^2 - \rho_n |\nabla_j \xi_n|^2 \right) \, dV
\]

\[
= -\rho_p \left[ \sum_{\ell=1}^{2} \Phi_{\ell} \nabla_j \xi^*_{\ell} \nabla_j \Phi_{\ell} + n_n \xi_n \xi_n \nabla_j \mu_n \right. \\
\left. + n_p \xi_p \xi_p \nabla_j \mu_p + \left( \frac{\partial \mu_n}{\partial n_p} \right)_{n_p} |\delta n_p|^2 + \left( \frac{\partial \mu_p}{\partial n_p} \right)_{n_p} |\delta n_p|^2 \\
- \frac{1}{4\pi G} \left| \nabla_j \Phi \right|^2 + \left( \frac{\partial \mu_n}{\partial n_p} \right)_{n_p} |\delta n_n \delta n_p^* + \delta n_n \delta n_p| \right) 
\]

and some manipulations, we arrive at the following final explicit form

\[
E_c = \frac{1}{2} \int \left\{ \rho_n |\partial_t \xi_n|^2 + \rho_p |\partial_t \xi_p|^2 - \rho_n |\nabla_j \xi_n|^2 \right\} \, dV
\]

\[
= -\rho_p \left[ \sum_{\ell=1}^{2} \Phi_{\ell} \nabla_j \xi^*_{\ell} \nabla_j \Phi_{\ell} + n_n \xi_n \xi_n \nabla_j \mu_n \right. \\
\left. + n_p \xi_p \xi_p \nabla_j \mu_p + \left( \frac{\partial \mu_n}{\partial n_p} \right)_{n_p} |\delta n_p|^2 + \left( \frac{\partial \mu_p}{\partial n_p} \right)_{n_p} |\delta n_p|^2 \\
- \frac{1}{4\pi G} \left| \nabla_j \Phi \right|^2 + \left( \frac{\partial \mu_n}{\partial n_p} \right)_{n_p} |\delta n_n \delta n_p^* + \delta n_n \delta n_p| \right) 
\]

It is worth noting that the symmetries of our system of equations imply the existence of a quadratic Lagrangian for the perturbations, which in turn implies the conservation of $W = W_n + W_p$. We can derive the superfluid equations of motion (74) from a variational principle whose action is (for clarity assuming that the displacement vectors are real, the generalisation to the complex case is straightforward)

\[
I = \int L \, dV = \frac{MB}{2} \left[ \left\{ \xi_n, A_n \xi_n \right\} + \left\{ \xi_p, A_p \xi_p \right\} + \left\{ \xi_n, B_n \xi_n \right\} \\
+ \left\{ \xi_p, B_p \xi_p \right\} - \left\{ \xi_n, C_n \xi_n \right\} - \left\{ \xi_p, C_p \xi_p \right\} - \left\{ \xi_n, D_n \xi_n \right\} - \left\{ \xi_p, D_p \xi_p \right\} \right] . 
\]

Then the momentum conjugate to $\dot{\xi}_n$ follows from $\partial L / \partial \dot{\xi}_n$, while the equations of motion (74) can be derived from the standard Euler-Lagrange equations

\[
\partial_t \left( \frac{\partial L}{\partial \dot{\xi}_n} \right) - \frac{\partial L}{\partial \xi_n} = 0 . 
\]

Finally, the canonical energy for the system is

\[
E_c = \int \left[ \dot{\xi}_n \frac{\partial L}{\partial \dot{\xi}_n} + \dot{\xi}_p \frac{\partial L}{\partial \dot{\xi}_p} - L \right] \, dV . 
\]

One can readily verify that these formulas lead to the given results.

In an axisymmetric system we can also define a conserved angular momentum;

\[
J_c = -\frac{MB}{2} W_n(\partial_\phi \xi_n, \xi_n) - \frac{MB}{2} W_p(\partial_\phi \xi_p, \xi_p) \\
- m_B \Re \left\{ \left( \partial_\phi \xi_n, A_n \partial_\phi \xi_n + \frac{1}{2} B_n \xi_n \right) \\
+ \left( \partial_\phi \xi_p, A_p \partial_\phi \xi_p + \frac{1}{2} B_p \xi_p \right) \right\} . 
\]

Now one can readily use Eq. (92) and Eq. (97) to prove that, for a normal mode solution to the problem, $\dot{\xi}_c = \xi_c e^{i(m_\phi \omega_c + \omega t)}$, the canonical energy and angular momentum will still be related by Eq. (45). Furthermore, it follows that we must have $E_c = J_c = 0$ for dynamically unstable (complex frequency) modes, just like in the single fluid case. This is hardly surprising, but it could turn out to be a very useful result. One could, for example, hope to be able to use our expressions for the canonical energy and angular momentum to derive necessary criteria for the superfluid two-stream instability [Andersson et al 2003; Prix et al 2004; Andersson et al 2002].

### 4.3 Trivial displacements

Before we close this section, let us address the issue of trivial displacements in the two-fluid problem. We clearly need to deal with two sets of trivial displacements, one for each fluid degree of freedom. Fortunately, the analysis of these trivials is essentially identical to that of the single barotropic fluid case discussed in Sec. 8 see [Friedman & Schutz 1978a] for further details.

In the superfluid problem, the trivial displacements are such that

\[
\delta n = \delta p = 0 . 
\]

It is easy to see from the superfluid perturbation equations that the two sets of equations that determine the functional form of the trivials are identical to the corresponding single fluid equations. This means that the single fluid result can be adapted to the superfluid problem: we simply have one trivial displacement ($\xi_\infty$) per fluid.

In our case, we can take the requirement that the canonical displacements $\xi_\infty$ must be “orthogonal” to the trivials to mean that we should have

\[
W_n(\xi_\infty, \xi_n) = W_p(\xi_\infty, \xi_p) = 0 . 
\]

This condition leads to the trivial displacements having identical form to those of the single-fluid problem. However, it is worth noticing that we could in principle permit the somewhat less restrictive condition

\[
W_n(\xi_\infty, \xi_n) + W_p(\xi_\infty, \xi_p) = 0 
\]

and still ensure that the trivial displacements do not affect our conserved quantities. We have not yet investigated the implications of this possibility.

Finally, and most importantly, one can readily extend the calculation of [Friedman & Schutz 1978a] to prove that normal modes are (usually) orthogonal to the trivials also in the present case. In other words, normal modes may serve as canonical initial data. This is extremely useful as we hope to use the canonical energy and angular momentum to assess the stability of various superfluid normal mode solutions.

### 5 INSTABILITIES OF ROTATING SUPERFLUID STARS

The main motivation for the present investigation was the lack of proper instability criteria for rotating superfluid stars. The need for such criteria is clear given that it has long been acknowledged that the astrophysical relevance of the gravitational-wave driven CFS instability (of both f- and r-modes) may depend on the extent to which superfluid dissipation [like mutual friction (Lindblom & Mendell 1995, 2000)] counteracts the growth of the unstable mode. It would seem obvious that, before worrying about such issues, one ought to establish that the instability is actually present once the star becomes superfluid. We are not aware of any such proof, despite the number of investigations of unstable oscillations of superfluid stars that exist in the literature. It should be clear that this is a non-trivial issue given the simple fact that the two fluids are only weakly coupled, and may in fact rotate at different rates.

In this section we provide the first applications of our conserved canonical energy and angular momentum for the two fluid problem. We first present an argument in favour of the simple single fluid criterion for CFS instability—that the instability sets in when the pattern speed of an originally backwards moving mode passes...
through zero in the inertial frame—holding also in the superfluid problem. Having done this, we discuss the superfluid r-mode instability.

5.1 The superfluid CFS instability

The main question here is: Does the simple criterion that a counter-rotating mode becomes unstable when the pattern speed changes sign remain valid also in the two-fluid problem? Intuitively, one might expect this to be the case, but it nevertheless warrants a proof.

We approach the problem as in the single fluid case. Assuming a real frequency mode solution to the perturbation equations, Eq. (27) leads to

\[ J_c = -m \left\{ \omega \left[ (\xi_n, \rho_\alpha \xi_n) + (\xi_p, \rho_\beta \xi_p) \right] - \left[ (\xi_n, i \rho_\alpha \vec{v}_n \cdot \nabla \xi_n) + (\xi_p, i \rho_\beta \vec{v}_p \cdot \nabla \xi_p) \right] \right\}. \tag{101} \]

Divide through to get

\[ \langle \xi_n, \rho_\alpha \xi_n \rangle + (\xi_p, \rho_\beta \xi_p) \]

\[ = -m \omega + m \frac{\langle (\xi_n, i \rho_\alpha \vec{v}_n \cdot \nabla \xi_n) + (\xi_p, i \rho_\beta \vec{v}_p \cdot \nabla \xi_p) \rangle}{\langle \xi_n, \rho_\alpha \xi_n \rangle + (\xi_p, \rho_\beta \xi_p)}. \tag{102} \]

Using cylindrical coordinates, we see from the results in Sec. 43 that

\[ -i \rho_\alpha \xi_n^* \nabla_j \xi_n^* = \rho_\alpha \Omega_n [m |\xi_n|^2 + i (\xi_n \times \xi_n)_x] . \tag{103} \]

We then know that

\[ \rho_\alpha \Omega_n (m - 1) |\xi_n|^2 \leq -i \rho_\alpha \xi_n^* v^j \nabla_j \xi_n^* \leq \rho_\alpha \Omega_n (m + 1) |\xi_n|^2 \tag{104} \]

and it is easy to see that we will have

\[ \langle \xi_n, i \rho_\alpha \vec{v}_n \cdot \nabla \xi_n \rangle \geq -(m + 1) \Omega_n \alpha_x \tag{105} \]

where we have defined \( \alpha_x = \langle \xi_n, \rho_\alpha \xi_n \rangle > 0 \). We also get

\[ \langle \xi_n, i \rho_\alpha \vec{v}_n \cdot \nabla \xi_n \rangle \leq -(m - 1) \Omega_n \alpha_x . \tag{106} \]

These results can be summarised as

\[ \sigma_p(a_n + a_p) - \left(1 + \frac{1}{m}\right) [\Omega_n a_n + \Omega_p a_p] \leq J_c/m^2 \]

\[ \leq \sigma_p(a_n + a_p) - \frac{1}{2} \left(1 - \frac{1}{m}\right) [\Omega_n a_n + \Omega_p a_p] . \tag{107} \]

This is the relation we need. Provided that \( \Omega_n a_n + \Omega_p a_p > 0 \) (which is obviously true if the fluids both rotate in the positive direction) we easily show that

- if we let \( \Omega_n, \Omega_p \to 0 \) while \( \omega \) is finite, then for co-rotating modes we have \( \sigma_p > 0 \), which means that \( J_c > 0 \). In contrast, for counter-rotating modes \( \sigma_p < 0 \) and \( J_c < 0 \). In both cases, we will get \( E_c > 0 \) which indicates that all (finite frequency) modes are stable.
- if we consider a region near \( \omega = 0 \) for finite rotation rates, then \( \sigma_p = 0 \) implies that \( J_c < 0 \). This means that the mode is stable as long as \( \sigma_p < 0 \), but when \( \sigma_p \) changes sign (and the mode becomes co-rotating) we will have \( E_c < 0 \) and an instability.

This concludes the proof that the criterion for the onset of radiation driven instabilities of modes that have a finite frequency limit as \( \omega \to 0 \) remains as in the single fluid case. Modes become unstable when the (inertial frame) pattern speed changes sign.

6 THE SUPERFLUID R-MODE INSTABILITY

To conclude this paper we will discuss some aspects of the r-mode instability for rotating superfluid stars. Although we are not yet at a point where we can discuss the general problem (since we did not include entrainment in our derivation), we can still learn quite a lot about the issues that arise when we consider two coupled fluids. Furthermore, we are not aware of any previous proof of the presence of an instability in the case when the two fluids rotate at different rates. The nature of the various inertial modes of oscillation (of which the r-modes form a sub-class) of a superfluid star has, however, been discussed by several authors. We will draw on these investigations for information concerning the nature of the r-modes in both (i) the general case of a background star such that the two fluids rotate at different rates (although with respect to the same axis), and (ii) the special case of co-rotating fluids.

6.1 The case with relative rotation

The general case, in which \( \Omega_n \neq \Omega_p \), was recently discussed by [Prix et al. 2004]. From that study we learn that, in absence of entrainment, the r-mode fluid motion must be such that only one of the fluids oscillates. This means that we will have two classes of modes, corresponding to

\[ \delta v_\xi \neq 0 \Rightarrow \delta \vec{v}_r = 0 \Rightarrow \omega = m \Omega_n \left[1 - \frac{2}{l(l+1)}\right] . \tag{108} \]

Since this implies that only one of the two displacements vectors is non-vanishing, it is straightforward to show that the canonical energy Eq. (23) reduces to

\[ E_c = \frac{1}{2} \int \left\{ \rho_\alpha |\partial_\xi \xi_n|^2 - \rho_\alpha |v^i \nabla_j \xi_n^*|^2 + \rho_\alpha \xi_n^* \nabla_j \nabla_j (\Phi + \tilde{\mu}_n) \right\} dV \tag{109} \]

where we have assumed that \( \delta v_\xi \) and \( \delta \Phi \) are of higher order in the slow-rotation scheme (as in the single fluid case), and also used \( v^i_n = v^i \). Noticing the close resemblance of this result to the expression for the case of a barotropic single fluid, Eq. (23), we readily infer that both these classes of modes will be unstable due to the emission of gravitational radiation.

However, this result is likely of very limited relevance. Even though it is expected that the two rotation rates will be slightly different in astrophysical neutron stars, the entrainment coupling will affect the r-modes significantly [as elucidated by [Prix et al. 2004]], naturally leading to both displacements being non-zero. This means that, despite being of conceptual interest, the case we have discussed here is pathological.

6.2 The co-rotating case

In the special case of \( \Omega_n = \Omega_p = \Omega \) we know (Andersson & Comer 2001; Comer 2002; Lee & Yoshida 2003; Yoshida & Lee 2003b; Prix et al. 2004) that there will exist two classes of r-modes. One is such that the two fluids move in phase, while the other has the two fluids counter-moving. Furthermore, for
non-stratiﬁed stars one can show that these two degrees of freedom decouple. Given the relative simplicity of this case, we will focus our attention on it. The general case essentially follows as a linear combination of the two results we present.

We begin by introducing the two classes of displacements

\[
\xi^+ = \frac{n_n \xi_n^+ + n_p \xi_p^+}{n_n + n_p} \quad (110)
\]

and

\[
\xi^- = \xi_n^+ - \xi_p^+. \quad (111)
\]

These are clearly such that, when \(\xi_n^+ = \xi_p^+\) and the two ﬂuids move together only \(\xi^+\) is present, while the total momentum ﬂux vanishes when \(\xi^+ = 0\).

Now consider purely axial r-mode solutions such that \(\xi^- = 0\). In this situation the canonical energy Eq. (93) can be written as

\[
E_c \approx \frac{1}{2} \left\{ \rho |\partial_t \xi^+|^2 - \rho |v^i \nabla_j \xi^+|^2 + \rho \xi^+ \xi^* \nabla_j (\Phi + \tilde{\mu}) \right\} dV \quad (112)
\]

where we (again) neglect the higher order contributions from \(\delta n_x\) and \(\delta \Phi\).

In this expression, the last term can be rewritten using the fact that we must have

\[
\nabla_j \mu_n = \nabla_j \mu_p = \nabla_j \mu^\alpha \quad (113)
\]

if the two ﬂuids rotate at the same rate, cf Eq. (64). This immediately leads to

\[
n_n \nabla_i \nabla_j \mu_n + n_p \nabla_i \nabla_j \mu_p = n \nabla_i \nabla_j \mu \quad (114)
\]

(where \(n = n_n + n_p\) and

\[
E_c \approx \frac{1}{2} \rho \left\{ (\partial_t \xi^+)^2 - |v^i \nabla_j \xi^+|^2 + \xi^+ \xi^* \nabla_j (\Phi + \tilde{\mu}) \right\} dV \quad (115)
\]

Clearly, this result is (provided that we identify \(\tilde{\xi}^+\) with \(\tilde{\xi}\)) identical to that of the single ﬂuid problem, Eq. (53). Hence, the instability of pure \(\tilde{\xi}^+\) modes follows from the calculation in Section 3.4. This is not very surprising given that the degree of freedom we are considering is such that the two ﬂuids move together.

Next we consider the canonical energy for counter-rotating modes, which are such that

\[
n_n \xi_n^+ + n_p \xi_p^+ = 0 \quad (116)
\]

In this situation only \(\xi^-\) is present, and we have

\[
\xi^+ = \frac{n_n}{n} \xi_n^- \quad (117)
\]

\[
\xi^- = - \frac{n_n}{n} \xi_n^+ - \frac{n_p}{n} \xi_p^- . \quad (118)
\]

Incidentally, the latter of these relations emphasizes the fact that pure \(\tilde{\xi}^-\) modes can only exist for non-stratiﬁed stars. We know from the results of Prix et al. (2003) that \(\xi_n^+\) and \(\xi_n^-\) will have the same functional dependence on the radial coordinate \(r\). This means that, in order for both displacements to be non-zero, they must be proportional. This is clearly only possible if \(n_n/n_p\) is constant.

Using the above relations for the expression for the canonical energy Eq. (93), one can show that

\[
E_c = \frac{1}{2} \rho_n x_p \left\{ (\partial_t \xi^-)^2 + |v^i \nabla_j \xi^-|^2 + \xi^+ \xi^* \nabla_j (\Phi + \tilde{\mu}) \right\} dV \quad (119)
\]

where \(x_p = n_p/n\) is the proton fraction. Since the expression in the bracket has the same form as in the single ﬂuid case, and the prefactor \(\rho_n x_p\) is positive deﬁnite, it is easy to prove that \(E_c < 0\) also for these counter-rotating modes.

6.3 Gravitational-wave emission

At this point it is appropriate to discuss how efﬁcient the oscillations of a superﬂuid star are as a source of gravitational waves. After all, we have shown that the superﬂuid r-modes generally lead to \(E_c < 0\), and should therefore be driven unstable by radiation. The relevance of the instability is then largely dependent on the rate at which the motion generates gravitational radiation.

To address this problem, we consider a source with weak internal gravity, and focus our attention on a single pulsation mode with time-dependence \(\exp(i \omega t)\). We also assume that the background is such that the two ﬂuids rotate at different rates: \(\Omega_n\) and \(\Omega_p\), respectively.

The gravitational-wave luminosity follows from e.g. Thorne (1980).

\[
d\mathcal{E} = \sum_{l=2}^{\infty} N_l \omega^{2l+2} \left( |\delta D_{lm}|^2 + |\delta J_{lm}|^2 \right) , \quad (120)
\]

where

\[
N_l = \frac{4 \pi k^2}{c^2 (l+1) (l+2) (l+3)} \left[ (2l+1)!! \right] . \quad (121)
\]

The ﬁrst term in the bracket of Eq. (120) represents radiation due to the mass multipoles. These are, quite generally, determined from

\[
\delta D_{lm} = \int T_{00} Y_{lm}^* r^4 dV \quad (122)
\]

where \(T_{00}\) is the contribution to stress-energy tensor associated with non-axisymmetric motion in the source. The second term in the bracket of Eq. (120) corresponds to the current multipoles, which follow from

\[
\delta J_{lm} = \int (-T_{ij}) Y_{lm}^* r^4 dV . \quad (123)
\]

where \(\tilde{Y}_{lm}^* \propto \hat{r} \times \nabla Y_{lm}^*\) are the magnetic multipoles (Thorne 1980).

By taking the Newtonian limit of the relativistic stress-energy tensor (see, for example, Comet (2002))

\[
T_{\mu\nu}^N = \Psi \delta_{\mu\nu} + p^\nu \chi_\mu + n^\nu \mu_\nu \quad (124)
\]

[which is easily done using formulas given in Appendix A of Andersson & Comet (2001)]], one can show that

\[
T_{00} \approx \rho_n + \rho_p \quad (125)
\]

and

\[
T_{ij} \approx \rho_n \delta v_i + \rho_p \delta v_p \quad (126)
\]

for a Newtonian source. Perturbing these expressions we ﬁnd that

\[
\delta D_{lm} = \int (\delta \rho_n + \delta \rho_p) r^4 Y_{lm}^* dV \quad (127)
\]

and

\[
\delta J_{lm} = - \frac{2}{c} \sqrt{\frac{t}{t+1}} \int r^4 (\rho_n \delta v_i + \rho_p \delta v_p) \cdot \tilde{Y}_{lm}^* dV \quad . \quad (128)
\]
As discussed in the previous section, it is sometimes instructive to express the superfluid formulas in terms of the variables

\[
\delta \vec{v}^+ = \frac{\rho \delta \vec{v}_n + \rho_p \delta \vec{v}_p}{\rho}
\] (129)

and

\[
\delta \vec{v}^- = \delta \vec{v}_p - \delta \vec{v}_n .
\] (130)

Using these, together with

\[
\delta \rho = \delta \rho_n + \delta \rho_p
\] (131)

we get

\[
\delta D_{lm} = \int \delta \rho^l Y_{lm}^* dV .
\] (132)

and

\[
\delta J_{lm} = \frac{2}{c} \sqrt{\frac{l}{l+1}} \int r^l [\rho \delta \vec{v}^+ + \delta \rho \delta \vec{v}_p]
\]

\[
+ \delta \rho_n (\vec{\Omega}_n - \vec{\Omega}_p) \cdot \vec{Y}_{lm}^* dV .
\] (133)

When written in this form, the formulas closely resemble the standard single-fluid results. The only new feature is the last term in the bracket of Eq. 133.

An interesting question concerns whether it is possible to have oscillations in a superfluid star that (at least at this post-Newtonian level) do not radiate gravitationally. From the above equations we immediately deduce that in the case of a non-rotating star (with \( \Omega_n = \Omega_p = 0 \)) or a co-rotating star (when \( \Omega_n = \Omega_p \)) we must have \( \delta \rho = \delta \vec{v} = 0 \) in order not to have any gravitational radiation emission. This result is quite intuitive since, as discussed in the previous section, the co-moving degree of freedom represents the total momentum flux. It is not surprising to find that motion which corresponds to zero momentum flux does not radiate gravitationally.

However, when combined with the results obtained from our superfluid canonical energy, these results illustrate some of the complexities associated with a discussion of gravitational-wave driven instabilities in a superfluid star. In particular, we conclude that even though they are formally unstable (\( \delta E < 0 \)) any purely counter-moving modes (for which \( \vec{\xi}^+ = 0 \)) will not grow (\( dE/dt = 0 \)).

7 CONCLUDING REMARKS

With this paper we have taken the first steps towards a Lagrangian perturbation framework for rotating non-relativistic superfluids. The primary motivation for this work, and the key application concerns the stability properties of rotating superfluid neutron stars. Our analysis generalises the classic work of Friedman & Schutz (1978) to the case of stars which require a multi-fluid description. We have applied our framework to the problem of dynamical and secular instabilities of a simplified superfluid neutron star model in which the two fluids are allowed to have different uniform rotation rates around the same axis, and where the entrainment effect is neglected. We have demonstrated that the criterion for the onset of radiation driven instabilities for modes that have a finite frequency as the background rotation vanishes remains unchanged from the ordinary fluid case. We have also considered the superfluid analogue of the r-mode instability, and found that both co- and counter-moving superfluid r-modes have negative canonical energy and will therefore be driven unstable by gravitational-wave emission.

Of course, our neglect of the entrainment is a serious limitation of the formalism. However, as we stated earlier, if we could not make progress in the case of vanishing entrainment it would be pointless to proceed to the general case. Naturally, we are encouraged by the progress we have made, which provides crucial benchmarks for the onset of instability in multi-fluid systems. In the near future, we will aim to generalize our results to include entrainment. This is absolutely essential for the study of gravitational-wave driven instabilities, since the primary superfluid damping mechanism is the mutual friction which depends crucially on the entrainment. The inclusion of entrainment will also be important for future studies of the superfluid two-stream instability, since entrainment may be the dominant coupling to drive the instability in the interiors of neutron stars.

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