The electrodynamics of inhomogeneous rotating media and the Abraham and Minkowski tensors. I. General theory

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This is paper I of a series of two papers, offering a self-contained analysis of the role of electromagnetic stress–energy–momentum tensors in the classical description of continuous polarizable perfectly insulating media. While acknowledging the primary role played by the total stress–energy–momentum tensor on spacetime we argue that it is meaningful and useful in the context of covariant constitutive theory to assign preferred status to particular parts of this total tensor, when defined with respect to a particular splitting. The relevance of tensors, associated with the electromagnetic fields that appear in Maxwell’s equations for polarizable media, to the forces and torques that they induce has been a matter of some debate since Minkowski, Einstein and Laub, and Abraham considered these issues over a century ago. The notion of a force density that arises from the divergence of these tensors is strictly defined relative to some inertial property of the medium. Consistency with the laws of Newtonian continuum mechanics demands that the total force density on any element of a medium be proportional to the local linear acceleration field of that element in an inertial frame and must also arise as part of the divergence of the total stress–energy–momentum tensor. The fact that, unlike the tensor proposed by Minkowski, the divergence of the Abraham tensor depends explicitly on the local acceleration field of the medium as well as the electromagnetic field sets it apart from many other terms in the total stress–energy–momentum tensor for a medium. In this paper, we explore how electromagnetic forces or torques on moving media can be defined covariantly in terms of a particular 3-form on those spacetimes that exhibit particular Killing symmetries. It is shown how the drive-forms associated with translational Killing vector fields lead to explicit expressions for the electromagnetic force densities in stationary media subject to the Minkowski constitutive relations and these are compared with other models involving polarizable media in electromagnetic fields that have been considered in the recent literature.

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1. Introduction

The interaction of matter with the electromagnetic field has played a dominant role in the development of our understanding of Nature. In classical Newtonian continuum mechanics one is concerned with dynamic processes involving the interaction of (bounded) classical continua with external forces and torques in space (e.g. Kovetz 2000). In relativistic continuum mechanics the theory is generalized to incorporate the concept of the spacetime manifold and formulated in terms of tensors on this manifold (e.g. Maugin 1978). In both formulations the histories of configuration variables must be compatible with balance laws associated with possible translational and rotational symmetries of a metric structure and an associated balance law of ‘energy’. If electromagnetic fields are involved these laws are supplemented with the macroscopic Maxwell equations in media. As a result of cohesive forces originating at the molecular level the interaction of a material with these (and gravitational influences) results in locally induced strains. These strains determine material stresses that are encoded into various ‘stress tensors’. These in turn describe the local distribution of force and torque densities in the medium. The mathematical structure of such tensors characterizes the response of different materials to external influences. Constitutive relations are relations that are used together with field equations, equations of motion and boundary conditions to fix the dynamic evolution of the independent dynamical variables. If the configuration of the system (open or closed) involves thermodynamic variables the classical laws of thermodynamics can be used to constrain these relations for real media. In the Newtonian formulation one uses the Maxwell and Cauchy stress tensors (Landau et al. 1984), together with certain equations of state. In the relativistic formulation a primary role is played by the total stress–energy–momentum tensor for the system. The determination of these tensors, together with appropriate constitutive relations for different types of polarizable uncharged media, often requires input from experiment. The electromagnetic constitutive properties and the appropriate electromagnetic stress–energy–momentum for light in moving media has been a subject of debate (and possible confusion) for over a century. Difficulties arise in properly accounting for the local nature of the wave–matter interaction in terms of experimentally measurable effects. Even for static electromagnetic fields there are (often non-linear) subtle interactions that induce changes in shape or volume in deformable media dependent on their thermodynamic state.

There is a considerable body of opinion suggesting that in some sense the choice between different stress–energy–momentum tensors describing interactions of a material medium with an electromagnetic field is a matter of convenience and that different choices simply provide alternative descriptions of the same overall interaction1 (Ginzburg 1973; Israel 1977; Maugin 1980; Pfeifer et al. 2007). Since there is no preferred tensor partition of the total stress–energy–momentum tensor into subtensors describing the behaviour of interacting subsystems there can be no unique definition of a stress–energy–momentum tensor describing an interacting subsystem. Thus for electromagnetic fields interacting with an electrically neutral bounded continuum it is always possible to redefine integrated electromagnetic forces and torques on the medium associated with

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1The genesis of this idea of classical duality may have its origins in the analogy with wave–particle duality in quantum mechanics.
subtensors according to taste, particularly if the medium is composed of piecewise inhomogeneous material subsystems or the fields are time dependent. However, in order to model the total interacting system the choice of a total stress–energy–momentum tensor is necessary. If one has decided how to model a neutral but polarizable medium in the absence of externally applied electromagnetic fields then different choices of a stress–energy–momentum tensor for electromagnetic fields in the medium to describe the additional interactions with such fields will inevitably lead in general to different predictions for the interacting system.

Although the laws of classical electromagnetism in the vacuum are firmly established in a relativistic context there is no general agreement on how best to accommodate the dynamics of deformable media as a self-consistent theory on spacetime (e.g. Frauendiener & Kabobel 2007). This makes any rigorous formulation of relativistic continuum mechanics of inhomogeneous dispersive polarizable media interacting with electromagnetic and gravitational fields difficult even if one contemplates using it for systems in non-relativistic motion in some frame of reference.

Despite these shortcomings there has recently been a resurgence of interest in the so-called Abraham–Minkowski controversy and its relevance to the use of either the Abraham or the Minkowski form of stress–energy–momentum tensor in interpreting experiments involving electromagnetic fields in media (Obukhov & Hehl 2003, 2008; Brevik & Ellingsen 2009; Hinds & Barnett 2009). Part of this difficulty is no doubt the result of the complex nature of material responses to forces in general. From our perspective such experiments are seeking constitutive relations involving particular material systems and the macroscopic Maxwell fields in media. As such it should come as no surprise that different systems might yield different responses, particularly if the competing effects of electro- or magneto-striction mentioned above are contributing differently in different experiments.

A number of historic experiments have sought to detect the discriminating Abraham force (Walker et al. 1975; Walker & Walker 1977). This is strictly discriminatory only for homogeneous non-dispersive stationary media, so calls into question those experiments that involve media in motion. Indeed the fundamental difference between the forces or torques induced by the divergence of the Abraham and Minkowski electromagnetic stress–energy–momentum tensors is that the former, unlike the latter, can depend explicitly on the acceleration of the medium. From this observation it is our contention that, for any specified electromagnetic constitutive relation, there remain experimental avenues offering new means to discriminate between alternative proposals for the form of the electromagnetic stress–energy–momentum tensor in media, despite the attendant inherent material constitutive complexities. In particular, we argue that, in principle, the observed dependence of a time-averaged electromagnetic wave-induced torque on the angular speed of a materially isotropic but inhomogeneous uniformly rotating electrically neutral medium could be used to discriminate between the electromagnetic wave interactions described by the Abraham tensor from those described by the symmetrized version of the tensor introduced by Minkowski, and possibly those proposed by others.

It is therefore of interest to calculate how such a torque depends on geometric properties of a cylindrical insulator, its speed of rotation and its electromagnetic constitutive properties. Since the medium will be assumed electrically polarizable
and magnetizable (but non-conducting) one is immediately confronted with a number of subtleties associated with the form of this tensor and the material constitutive properties of the medium. These issues have a bearing on how one formulates the classical electromagnetic force on a macroscopic body particularly one that is accelerating. Since this has led to a number of related questions in the recent literature (Brevik 1970; Barnett & Loudon 2006; Mansuripur 2008), this article attempts to make explicit our perspective. After reviewing approaches in the Newtonian framework the use of a fully covariant special relativistic framework is advocated. The essentials of this formulation, which are described below, should be read in conjunction with the expository material on electromagnetic theory in the language of differential forms contained in the electronic supplementary material. In paper II (Goto et al. in press), applications to rotating media are discussed in this context.

2. Points of departure

In order to motivate our method of analysis leading to a computation of the electromagnetic torque on an inhomogeneous rotating uncharged insulator, it is useful to place our methodology in the context of the recent literature in this subject. Some of this literature is devoted to the derivation of expressions for the classical force (and torque) induced by electromagnetic fields on electrically neutral polarizable continua based on an underlying discrete model of electromagnetic sources. A traditional non-relativistic approach is to take as a point of departure the classical vacuum Maxwell equations for the fields $e^U(r, t)$ and $b^U(r, t)$ in some inertial frame (here labelled $U$) and moving point sources$^2$ together with the Newtonian equations of motion in $\mathbb{R}^3$ for the sources. For a collection of $N$ point charges where the $\alpha$-th point has mass $m_\alpha$ and Newtonian velocity $v_\alpha(t)$ at time $t$, the motion of each particle is given by a solution to the $N$ coupled ordinary differential equations

$$\frac{d}{dt}(m_\alpha v_\alpha(t)) = \hat{F}_\alpha(t) + \sum_{\beta \neq \alpha} F_{\alpha \beta}(t), \quad \alpha, \beta = 1, \ldots, N,$$

where $F_{\alpha \beta}(t)$ is the interparticle force and $\hat{F}_\alpha(t)$ the resultant force on the $\alpha$-th particle owing to all other influences. The non-relativistic interaction of the ambient Maxwell fields $e^U$ and $b^U$ with any point particle with charge $q_\alpha$ located at $r = r_\alpha(t)$ with Newtonian velocity $v_\alpha(t) = \dot{r}_\alpha(t)$ is derived from the electromagnetic force $F^\text{EM}_\alpha(t)$ on that particle with

$$F^\text{EM}_\alpha(t) = q_\alpha(e^U(r_\alpha(t), t) + v_\alpha(t) \times b^U(r_\alpha(t), t))$$

in the Gibbs vector notation. The fields themselves are in turn derived from the vacuum microscopic Maxwell equation with singular sources. In the Gibbs vector field notation these are the equations

$^2$We assume throughout that free magnetic charge is absent in Nature.
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\[ \text{curl } \mathbf{e}^U = -\frac{\partial \mathbf{b}^U}{\partial t}, \quad (2.3) \]
\[ \text{div } \mathbf{b}^U = 0, \quad (2.4) \]
\[ \mu_0^{-1} \text{curl } \mathbf{b}^U = \mathcal{J}^U + \epsilon_0 \frac{\partial \mathbf{e}^U}{\partial t}, \quad (2.5) \]
and
\[ \epsilon_0 \text{ div } \mathbf{e}^U = \hat{\rho}^U, \quad (2.6) \]
respectively, where \( \mu_0^{-1} = \frac{c_0^2}{\epsilon_0}, \) \( c_0 \) denotes the speed of light in vacuo, \( \hat{\rho}^U = \sum_a q_a \delta^D(\mathbf{r} - \mathbf{r}_a(t)) \) and \( \mathcal{J}^U = \sum_a q_a v_a(t) \delta^D(\mathbf{r} - \mathbf{r}_a(t)) \) in terms of the singular Dirac distribution \( \delta^D. \) In these equations only the fundamental electromagnetic fields \( \mathbf{e}^U \) and \( \mathbf{b}^U \) play a role. Multi-pole magnetization and polarization sources are defined in terms of the attributes of the charged particles and their motion as limits in a multi-pole expansion. In non-relativistic models retardation effects due to the Maxwell displacement current are often ignored. It is straightforward to verify that the total linear particle momentum \( \sum_a m_a v_a(t) \) is a constant of the motion provided \( \sum_a \hat{\mathcal{F}}_a(t) = 0 \) and \( \mathcal{F}_{ab}(t) = -\mathcal{F}_{ba}(t). \) From these fundamental assumptions one may approach a continuum description from a number of distinct directions including a multi-pole expansion of the electromagnetic fields about some arbitrary point in space followed by a spatial smoothing procedure for the singular sources, in order to generate a balance law for smoothed out total non-relativistic linear momentum for the total interacting system. This yields an Euler continuum description from the fundamental particle-field description. While the resulting overall continuum balance equation may not be sensitive to the precise nature of the smoothing functions the interpretation of individual forces in the Eulerian balance relation may be (de Groot & Sutterp 1972; Murdoch & Bedeaux 1994).

An alternative approach to a continuum description has been to assume that the point particles are electrically neutral (atoms or molecules) but are endowed with elementary electric dipole moments and/or magnetic dipoles or elementary current loops. This requires that equation (2.2) be changed to reflect this modification. Subsequent smoothing would lead to a balance law involving force terms different from the point charge model even in the static limit. This should come as no surprise since the underlying microscopic models are different.

In comparing the predictions of different continuum models one must be aware that they may differ only in their effects at the boundaries of spatially compact media. Terms that may be discarded during an integration by parts in the development of the modelling process may well contribute to boundary interactions depending on the nature of the boundary conditions.

To illustrate some of these points consider the typical modelling of a homogeneous dispersion-free electrically neutral stationary medium (Schwinger et al. 1998) based on a collection of point charges yielding, in some continuum limit, a time-dependent volume force density on \( \mathbb{R}^3 \) given by
\[ \mathbf{F}_t = \frac{1}{2} \nabla (\mathbf{e}^U \cdot \mathbf{p}^U + \mathbf{b}^U \cdot \mathbf{m}^U) + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{p}^U \times \mathbf{b}^U) = F_{1t} \mathbf{i} + F_{2t} \mathbf{j} + F_{3t} \mathbf{k}, \quad (2.7) \]
expressed in terms of time-dependent electromagnetic vector fields on \( \mathbb{R}^3 \) in a global orthonormal Cartesian frame with basis \( \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \} \) and Cartesian coordinates \( \{ x^1, x^2, x^3 \}. \) The force vector on a finite volume \( \mathcal{V}_t \) of the medium is then
\[
\tilde{F}_t = \int_{V_t} F_t \, dx^1 \, dx^2 \, dx^3.
\]

In equation (2.7), the \( p^U \) and \( m^U \) are spatially smoothed time-dependent electric and magnetic dipole fields on \( \mathbb{R}^3 \) obtained by truncating a particle force multipole expansion. Prior to smoothing they can be expressed in terms of the charge, position and velocity of the electrically charged constituents of the medium relative to some arbitrary point and the bulk velocity in the medium. They are superscripted to indicate that these fields are referred to an inertial (laboratory) reference frame \( U \). In principle, \( p^U \) and \( m^U \) are determined from the continuum limit of the particle equations of motion. In practice, this is difficult so one resorts to constitutive relations and assumes a form for the bulk motion of the continuum. With these closure relations the vacuum Maxwell system absorbs the smoothed polarization and magnetization sources into the phenomenological fields \( p^U \), \( m^U \), \( d^U \) and \( h^U \).

In view of the covariant language used in the rest of this paper, it is useful to write equation (2.7) in tensor form using the Killing symmetry of the Euclidean structure defined by the metric of three-dimensional space. The reader should consult the electronic supplementary material for a self-contained formulation of the Maxwell system in terms of the frame-dependent differential 1-forms \( e^U, b^U, d^U, h^U, p^U \) and \( m^U \) on space and their definition in terms of a unit time-like vector field \( U \) and the Maxwell and excitation 2-forms \( F \) and \( G \), respectively, on four-dimensional spacetime endowed with the Lorentzian metric tensor field\(^3 \) \( g \). In a Cartesian coordinate system on \( \mathbb{R}^3 \), the covariant and contra-variant metric tensor fields are

\[
g = \sum_{i=1}^{3} dx^i \otimes dx^i \quad \text{with} \quad g^{-1} = \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i},
\]

and in these coordinates the associated translational (Killing) vector fields are \( \{\partial/\partial x^i\} \) (for all time) satisfying \( \mathcal{L}_{\partial/\partial x^i} g = 0 \). If \( K \) is any of these Killing fields, then\(^4 \) \( \tilde{K} = dx^i \), for some Cartesian coordinate \( x^i \), \( i = 1, 2, 3 \). In terms of \( \kappa = \#\tilde{K} \), with \( \#1 = dx^1 \wedge dx^2 \wedge dx^3 \), a Euclidean covariant representation of (2.7) that meshes with the conventions to be established below is given in terms of the equivalent force 3-form for direction \( K \),

\[
\mathcal{F}_K = \frac{1}{2} \left[ d(e^U \cdot p^U + b^U \cdot m^U) + \#\mathcal{L}_{\partial_i}(p^U \wedge b^U) \right] \wedge \kappa,
\]

where we define \( \alpha \cdot \beta = g^{-1}(\alpha, \beta) = g^{-1}(\alpha, \beta) \) for any spatial 1-forms \( \alpha, \beta \) and \( d \) denotes the spatial exterior derivative\(^5 \) on \( \mathbb{R}^3 \). Then the Cartesian component of the vector force in the Cartesian direction \( i \) on a volume \( V_i \) owing to its surroundings and ambient fields in the medium is given by

\(^3\)All electromagnetic tensors in this article have dimensions constructed from the Système International (SI) dimensions \([M], [L], [T] \text{ and } [Q]\) where \([Q]\) has the unit of the Coulomb in this system. We adopt \([g] = [L]^2\), \([G] = [L] = [Q]\), \([F] = [Q]/[e_0]\) where the permittivity of free space \( e_0 \) has the dimensions \([Q^2T^2M^{-1}L^{-3}]\). Note that the operators \( d \) and \( \nabla \) defined in the electronic supplementary material preserve the physical dimensions of tensor fields but with \([g] = [L]^2\), for \( p \)-forms \( \alpha \) in four dimensions, one has \([\star \alpha] = [\alpha][L^{4-2p}]\).

\(^4\)For any metric tensor field \( G \), we define the \( G \)-dual of any 1-form \( \alpha \) to be the vector field \( \tilde{\alpha} = G^{-1}(\alpha, -) \). Similarly, for any vector \( V \), we define the \( G \)-dual 1-form \( \tilde{V} = G(V, -) \).

\(^5\)See the electronic supplementary material for the definition of the spatial exterior derivative \( d \) in terms of the exterior derivative \( \partial \) on spacetime, and for definitions of other spatial operators.
\[ \mathcal{F}_i[V_i] = \int_{V_i} \mathcal{F}_{\partial_i}. \]

Note that in the static field situation the force 3-form (2.8) is expressed in terms of the gradient of the interaction scalar \( \frac{1}{2}(e^U \cdot p^U + b^U \cdot m^U) \). Using the macroscopic Maxwell equations

\[ \underline{de}^U = -B^U, \]
\[ \underline{dB}^U = 0, \]
\[ \underline{dh}^U = J^U + \dot{D}^U \]
\[ \underline{dD}^U = \rho^U, \]

and

where \( D^U = \# d^U \), \( B^U = \# b^U \), \( \dot{\alpha} = c_0 \mathcal{L}_{\partial_i} \alpha \) for any \( p \)-form \( \alpha \) and, with the definitions

\[ \mathcal{J}^U_p = -i_{U^p} \rho^U = \frac{1}{c_0} (\dot{P}^U - \underline{d}m^U) \]

and

\[ \rho^U_p = -(i_{U^p} \star j^p) \star \tilde{U} = -\underline{d}P^U, \]

one may write,\(^6\) with \( P^U = \# p^U \) and \( E^U = \# e^U \),

\[ \mathcal{F}_K = \frac{1}{2}[\rho^U_p e^U(K) + i_b \mathcal{J}^U_p \wedge \kappa + \mathcal{L}_b \mathcal{L}^U p^U \wedge \kappa + \mathcal{L}_K p^U \wedge E^U + \underline{d}(e^U(K)P^U)]. \]

Although the overall charge of the medium is taken to be zero the first two terms in the equation for \( \mathcal{F}_K \) above constitute a local Lorentz force density\(^7\) generated by the *induced polarization* charge \( \rho^U_p \) and current \( \mathcal{J}^U_p \). The exact 3-form in the last term will contribute to forces on the spatial boundary of the medium unless they happen to be zero as a result of boundary conditions satisfied by the fields there. The remaining terms describe local forces depending on inhomogeneities arising from the spatial rate of change of polarization and magnetization in the system.

The above derivation of the structure of a local Newtonian force density in a neutral macroscopic continuum in an electromagnetic field starts with a particular non-relativistic model in an inertial frame. We stress that in the absence of dynamical information about the polarization and magnetization the cogency of equation (2.14) demands a knowledge of supplementary constitutive relations to accommodate the response of the medium (via \( p^U \) and \( m^U \)) to the fundamental fields \( e^U \) and \( b^U \). One could proceed to express this force density as the sum of the spatial divergence of a time-dependent Maxwell second rank stress tensor on \( \mathbb{R}^3 \) and a time derivative modulo boundary term.\(^8\) Adding the Cauchy tensor describing the medium in the absence of electromagnetic forces to such a Maxwell stress tensor would give the total Cauchy stress for the medium (in the presence of fields) that enters into the local balance law

\(^6\)See the electronic supplementary material for the definition of the Hodge maps \( \# \) and \( \star \), and further details of this calculation.

\(^7\)The factor \( \frac{1}{2} \) arises since the polarization is smoothly distributed in the medium.

\(^8\)There is no unique way to perform such a split.
for non-relativistic linear momentum for the complete interacting system. The classical local dynamics of this system follows as a solution to such a balance law and requires implementation of interface conditions at media boundaries or interfaces where material properties of the continuum are discontinuous. The latter constraints benefit from a distributional reformulation that has been discussed more fully elsewhere (Tucker 2009, in press; Tucker & Walton 2009).

An alternative point of departure for the modelling process is to start with a fully covariant total symmetric stress–energy–momentum tensor $T_{\text{tot}}$ for a medium interacting with classical fields on a general spacetime. The fully covariant local classical equations of motion of the medium are then postulated to be given by

$$\nabla \cdot T_{\text{tot}} = 0$$

(2.15)

together with equations for the fields. Here the divergence operator $\nabla \cdot$ is defined with respect to the Levi-Civita connection $\nabla$ on spacetime. The fundamental property of $T_{\text{tot}}$ is that it acts as a source of Einstein gravitation. In most classical considerations one demands $T_{\text{tot}}(U, U) > 0$ for all future pointing time-like unit vector fields $U$, reflecting the attractive nature of gravitation, and $T_{\text{tot}}(U, U)$ is then identified with local mass-energy density in the frame $U$. In a source-free region of spacetime containing only electromagnetic fields this condition is maintained and identifies field energy.

In a spacetime with local isometries generated by a set of Killing vector fields associated with the spacetime metric $g$, $T_{\text{tot}}$ can be used to generate a set of closed 3-forms on spacetime. The vanishing of the exterior derivative of each form in this set can in turn give rise to a conservation law when integrated over a regular four-dimensional domain of spacetime provided the forms are free of singularities there. Then a class of Killing vectors that generate spatial translations can be used to construct conservation laws that reduce to the balance laws for the components of Newtonian linear momentum in some non-relativistic Newtonian limit.

Thus if $T$ is any symmetric rank 2 symmetric tensor on spacetime so $T_{ab} = T_{ba}$ where

$$T = T_{ab} e^a \otimes e^b$$

in any cobasis of 1-forms $\{e^a\}$ with a dual basis of vector fields $\{X_b\}$ one defines the drive 3-form associated with $T$ and $K$

$$\tau_K = -\zeta_K T(K, X_a) \star e^a$$

(2.16)

for any vector field $K$ on spacetime where $\zeta_K = \pm 1$ will be defined below to conform with the physical interpretation of the different components of the drive-form. Given a frame defined by the unit time-like (future pointing) observer field$^{10}$ $U$, one may decompose $\tau_K$ into a spatial 2-form $J^U_K$ and spatial 3-form $\rho^U_K$ relative to $U$ on spacetime:

$$\tau_K = J^U_K \wedge \tilde{U} + \rho^U_K$$

(2.17)

$^9$In the SI system the tensor $T_{\text{tot}}$ has the physical dimension of a force, i.e. ML/T$^2$.

$^{10}$The frame is inertial if $\nabla U = 0$. 

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with \( i_U J^U_K = i_U \rho^U_K = 0 \). When \( K \) is a Killing vector field \((\mathcal{L}_K g = 0)\) it is a mathematical identity (Benn 1982; Benn & Tucker 1988) that\(^{11}\)

\[
d\tau_K = -\zeta_K (\nabla \cdot T)(K) * 1. \tag{2.18}
\]

Hence, if \( \nabla \cdot T = 0 \) then

\[
d\tau_K = 0. \tag{2.19}
\]

In terms of \( J^U_K \) the conservation equation (2.19) becomes

\[
dJ^U_K + \mathcal{L}_U \tau_K = 0.
\]

If \( K \) is a space-like translational Killing vector field with open integral curves then

\[
J^U_K \equiv -i_U \tau_K
\]

is a linear momentum current (stress) 2-form in the frame \( U \) and

\[
\rho^U_K \equiv -(i_U * \tau_K) * \tilde{U}
\]

is the associated linear momentum density 3-form in the frame \( U \). If \( K \) is a space-like rotational Killing vector field generating \( SO(3) \) group isometries with closed integral curves, then \( J^U_K \) is an angular-momentum current (torque stress) 2-form and \( \rho^U_K \) is the associated angular-momentum density 3-form in the frame \( U \). If \( K \) is a time-like translational Killing vector field, then \( J^U_K \) is an energy current (power) 2-form and \( \rho^U_K \) is the associated energy density 3-form in the inertial frame field \( U \). In the following, attention will be directed mainly to particular translational and rotational space-like Killing vectors \( K \) of flat spacetime and the computation of integrals of \( J^U_K \) and \( \rho^U_K \) for a particular contribution to \( \tau_K \) associated with electromagnetic fields in media undergoing various states of motion observed in an inertial frame defined by \( U \).

As noted above when \( T^{\text{tot}} \) describes a domain of vacuum spacetime, free of matter but containing electromagnetic fields, one requires the electromagnetic field energy in any local frame \( U \) to be positive, i.e. \( \# \rho^U_K > 0 \) in terms of \( \rho^U_K \). Furthermore we shall require that, for any space-like Killing vector field \( K \), \( \# \rho^U_K > 0 \). This ensures that when \( K \) generates spatial translations in Minkowski spacetime and \( U \) defines an inertial frame then the time-averaged integral of \( \rho^U_K \) over a finite region of space can be identified with the time-averaged component of physical linear momentum associated with a harmonic plane wave in the direction of its propagation. These conditions are ensured if

\[
\zeta_K = \frac{g(K, K)}{|g(K, K)|}, \tag{2.20}
\]

i.e. \( \zeta_K = 1 \) if \( K \) is space-like and \( \zeta_K = -1 \) if \( K \) is time-like.

If the spacetime admits a foliation by hypersurfaces \( t = \text{constant} \) then equation (2.19) takes the form adapted to the frame \( U \),

\[
dJ^U_K + \dot{\rho}^U_K = 0.
\]

\(^{11}\)More generally if \( T \) is an arbitrary \((2, 0)\) tensor, \( T = T_{ab} e^a \otimes e^b \), of no particular symmetry, one has for any vector field \( W \) the identity \( \frac{1}{2}(\mathcal{L}_W g)(X_a, X_b) T^{[ab]} \star 1 = -\zeta_W d\tau_W - (\nabla \cdot (\text{Sym } T)) \tilde{W} \star 1 \) where \( -\zeta_W \tau_W = (\text{Sym } T)(W, X_a) \star e^a \) and \( \text{Sym } T \) is the symmetric part of \( T \) with components \( T_{[ab]} = \frac{1}{2}(T_{ab} + T_{ba}) \).
In this spacetime framework, a basic postulate is that the history of a material continuum interacting with gravity and electromagnetic fields can be determined from equation (2.15) and the Maxwell system (Tucker & Walton 2009),

$$dF = 0 \quad \text{and} \quad d \star G = j, \quad (2.21)$$

where the excitation 2-form $G$ depends on the interaction with the medium and the 3-form electric 4-current $j$ encodes the electric charge and current source. Such an electric 4-current describes both (mobile) electric charge and effective (ohmic) currents in a conducting medium. To close this system in a background gravitational field, electromagnetic constitutive relations relating $G$ and $j$ to $F$ are necessary. These relations may also depend upon properties of the medium, including its state of motion.

If one makes the arbitrary split $T^{\text{tot}} = T^{\text{matter, EM}} + T^{\text{EM}}$ then in general $\nabla \cdot T^{\text{matter, EM}} = -\nabla \cdot T^{\text{EM}} \neq 0$ so, as in the non-relativistic modelling situation, any interpretation of various terms in the decomposition of $\nabla \cdot T^{\text{tot}}$ must be understood to be with respect to a particular splitting. This is of particular relevance in situations where $\nabla \cdot T^{\text{EM}}$ contains a coupling of electromagnetic fields and mass-energy to the bulk local acceleration field of a medium.

In a fully coupled system that consistently incorporates the gravitational interactions using Einstein’s gravitational field equations with a symmetric Einstein tensor, equation (2.15) becomes an identity. In descriptions with non-dynamic gravitation equation (2.15) is part of the coupled system for the remaining dynamic fields. Since the effects of gravity will be neglected in the following we assume henceforth a background Minkowski spacetime and all observers will be inertial with $\nabla U = 0$.

Consider a material continuum macroscopic model in which $T^{\text{tot}}$ contains the symmetric electromagnetic stress–energy–momentum tensor

$$T^{\text{SM}} = \frac{1}{2} (i_a F \otimes i^a G + i_a G \otimes i^a F + \star (F \wedge \star G) g). \quad (2.22)$$

The excitation 2-form $G$ in this expression must be specified in terms of $F$ and other properties of the medium. For a simple non-dispersive isotropic medium, one has the constitutive relation

$$G = \varepsilon_0 \varepsilon_r i_V F \wedge \widetilde{V} - \frac{\varepsilon_0}{\mu_r} \star (i_V \star F \wedge \widetilde{V}) = \varepsilon_0 \left( \frac{\varepsilon_r - 1}{\mu_r} \right) i_V F \wedge \widetilde{V} + \frac{\varepsilon_0}{\mu_r} F, \quad (2.23)$$

where $V$ is a unit, time-like 4-velocity field describing the bulk motion of the medium and $\varepsilon_r$ and $\mu_r$ are relative permittivity and permeability scalars on spacetime. For a spatially inhomogeneous medium these will not be constants: $d \varepsilon_r \neq 0$, $d \mu_r \neq 0$. Furthermore, if the medium is accelerating then $\nabla_V V \neq 0$. For any Killing vector field $K$, the tensor $T^{\text{SM}}$ gives rise to the Killing drive 3-form

$$\tau^{\text{SM}}_K = -\frac{\zeta_K}{2} (i_K G \wedge \star F - F \wedge i_K \star G), \quad (2.24)$$

where the forms $F$ and $G$ are required to satisfy the Maxwell equations (2.21). For an uncharged non-conducting medium $j = 0$. The stress–energy–momentum tensor (2.22) is that obtained by symmetrizing the one proposed by Minkowski to describe electromagnetic stresses and energy balance in a medium.
3. Covariant forces and torques in media

In general, the total stress–energy–momentum for a medium may contain singularities and discontinuities in its material properties. To facilitate the following it proves convenient to restrict to a bounded domain of spacetime containing (the smooth history of) the medium immersed in the vacuum (see figure 1). Thus we explicitly leave out of the discussion sources of stress that may arise from singularities or discontinuities in the medium history. One may regard the immersion as the material body and it is assumed here to have a topologically trivial structure with a total interior Killing drive form \( \tau_K^{\text{in}} \) with interior support including the smooth boundary of the body. The exterior of the body in spacetime is assigned a total Killing\textsuperscript{12} drive \( \tau_K^{\text{out}} \) with exterior support excluding the boundary of the body.

For any observer field \( U \), spacetime domain \( \mathcal{M} \) and total \( K \)-drive \( \tau_K^{\mathcal{M}} \) on \( \mathcal{M} \), one has from equation (2.17)

\[
J_{K}^{U,\mathcal{M}} = -i_U \tau_K^{\mathcal{M}} \quad \text{and} \quad \rho_{K}^{U,\mathcal{M}} = -(i_U \star \tau_K^{\mathcal{M}}) \star \tilde{U}.
\]

If one writes

\[
\tau_K^{\mathcal{M}} = -\zeta_K c_0^2 \hat{\rho}_{\text{me}} g(V, K) \star \tilde{V} + \hat{\tau}_K^{\mathcal{M}}
\]

to describe a medium with bulk 4-velocity \( V \) on \( \mathcal{M} \) and proper inertial mass-energy density scalar\textsuperscript{13} \( \hat{\rho}_{\text{me}} \), given the conservation law \( d(\hat{\rho}_{\text{me}} \star \tilde{V}) = 0 \), the equation of motion \( d\tau_K^{\mathcal{M}} = 0 \) becomes

\[
\zeta_K \int_{\Sigma_t} \mu_U \tilde{A}(K) = f_K^{\mathcal{M}},
\]

where \( f_K^{\mathcal{M}} \equiv -\star \hat{\tau}_K^{\mathcal{M}} \) and the 4-acceleration vector field of \( \mathcal{M} \) is \( \hat{A} \equiv \nabla_V V \). Contracting the \( \star \) Hodge dual of this equation of motion with \( U \) and integrating the resulting 3-form over any \( U \)-orthogonal space-like hypersurface \( \Sigma_t \) that intersects the domain \( \mathcal{M} \) yields

\[
\zeta_K \int_{\Sigma_t} \mu_U \tilde{A}(K) = f_K^{U,\mathcal{M}}[\Sigma_t],
\]

where the mass-energy 3-form \( \mu_U \equiv -c_0^2 \hat{\rho}_{\text{me}} \star \tilde{U} \) and the total instantaneous integrated \( K \)-drive on \( \Sigma_t \) at time \( t \) in the \( U \) frame is

\[
f_K^{U,\mathcal{M}}[\Sigma_t] \equiv \int_{\Sigma_t} i_U \dd \hat{\tau}_K^{\mathcal{M}}.
\]

Thus, for any part \( \tau_K^{\mathcal{M},j} \) of the total \( K \)-drive \( \tau_K^{\mathcal{M}} = \sum_j \tau_K^{\mathcal{M},j} \) on \( \mathcal{M} \) we call the 3-form \( -i_U \dd \tau_K^{\mathcal{M},j} \) the \( j \)-th part of the instantaneous \( K \)-drive density 3-form.

Using this notation consider the two regions \( \mathcal{M}_1 = \text{in} \) and \( \mathcal{M}_2 = \text{out} \) with \( \tau_K^{\text{in}} \) on \( \mathcal{M}_1 \) describing the interaction of a medium with electromagnetic fields according to \( d\tau_K^{\text{in}} = 0 \) and \( \tau_K^{\text{out}} \) describing electromagnetic fields in the vacuum according to \( d\tau_K^{\text{out}} = 0 \) on \( \mathcal{M}_2 \). The interface between these regions will be denoted \( \Sigma_{12} \) regarded

\textsuperscript{12}When the effects of gravity are neglected the interior and exterior spacetimes admit the same set of Killing vector fields.

\textsuperscript{13}The first term on the right-hand side in the above equation arises from a contribution \( c_0^2 \hat{\rho}_{\text{me}} \tilde{V} \otimes \tilde{V} \) to the total stress–energy–momentum tensor.
as an immersion in spacetime. In the following the difference of $J^K_{U,\text{in}}$ and $J^K_{U,\text{out}}$ across $\Sigma_{12}$ will be encountered. This discontinuity jump is defined in terms of the pull-back map $\Sigma_{12}^*$ on these forms to $\Sigma_{12}$,

$$\Delta_{12}(J^K_U) = \Sigma_{12}^*(J^K_{U,\text{in}} - J^K_{U,\text{out}}).$$

With a split given by $j = \{\text{matter, EM}\}$ write

$$\tau^K_{\text{in}} = \tau^K_{\text{in,matter}} + \tau^K_{\text{in,EM}},$$

where $\tau^K_{\text{in,matter}}$ describes matter without permanent polarization or magnetization in the absence of external electromagnetic fields, the equation $d\tau^K_{\text{in}} = 0$ yields

$$f^K_{U,\text{in,matter}}[\Sigma_t] + f^K_{U,\text{in,EM}}[\Sigma_t] = 0. \quad (3.1)$$

As noted in the introduction the electromagnetic constitutive relation may imply a coupling of the electromagnetic fields in the medium to its deformation tensor and hence may contribute to its strains. For an uncharged medium, if there are no electromagnetic fields to polarize the medium, the second term on the left-hand side in equation (3.1) will be zero. In this case one may identify it with the $K$-drive on the medium due to an applied electromagnetic field. Such a $K$-drive will in general give rise to a non-zero bulk acceleration field $A = \nabla V V$ on $\mathcal{M}_1$. In principle\(^{14}\) one could maintain any prescribed state of acceleration $A_0$ with $V = W_0$ by the addition of a further drive $f^K_{in,ext}$ of some nature, provided

$$\zeta_K \int_{\Sigma_t} \mu_U A_0(\vec{K}) = f^K_{in,\text{elast}}[\Sigma_t] + f^K_{in,\text{EM}}[\Sigma_t] + f^K_{in,ext}[\Sigma_t],$$

where $f^K_{in,\text{elast}}$ isolates possible drives in the medium owing to internal non-electromagnetically induced stresses and friction with the environment. In particular, if the medium is maintained in a state of uniform rotation in the absence of interaction with electromagnetic fields by mechanical torques

\(^{14}\)This would be difficult in practice if electromagnetic stresses in a deformable medium led to significant strains producing a dynamic change in its shape or volume.
then the presence of any electromagnetic induced torques can (in principle) be compensated by additional mechanical torques in order to maintain a state of uniform rotation. Using the orthogonal decompositions with respect to $U$ above and the relation $i_U \tau_{in,EM}^K = dJ_{in,EM}^K + \mathcal{L}_{U} \rho_{K}^{in,EM}$, such an integrated instantaneous external $K$-drive may be written

$$f_{K,\text{in,ext}}^{U} \{ \Sigma_t \} = - \int_{\Sigma_t} i_U d\tau_{K}^{in,EM} = - \int_{\Sigma_t} \mathcal{L}_{U} \rho_{K}^{U,in,EM} - \int_{\partial \Sigma_t} J_{K}^{U,in,EM},$$

(3.2)

where the Stokes theorem has been used in the last term. The precise nature of this compensating drive will depend of course not only on the form of $\tau_{in,EM}^K$ but also on the electromagnetic constitutive relation for the medium.

For unbounded media the notion of a total integrated drive may not be meaningful. In these circumstances one may be able to deduce contributions to $K$-drive pressures from the last term in equation (3.2) or by integrating discontinuities of $J_{K}^{U}$ over surfaces instead of integrating contributions to $i_U d\tau_{K}$ over volumes. This follows simply from the bounded medium case above by writing

$$\int_{\partial \Sigma_t} J_{K}^{U,in,EM} = \int_{\partial \Sigma_t} J_{K}^{U,out,EM} + \int_{\partial \Sigma_t} \Delta_{12}(J_{K}^{U}),$$

where the discontinuity $\Delta_{12}$ is across the boundary of the history in spacetime. Both integrals now require a knowledge of the pull-back to the interface of $J_{K}^{U,out}$ in the vacuum region. However, one may relax the condition that the medium be bounded and assert that the instantaneous $K$-drive exerted on any finite spatial volume $V_t$ in an infinite history having an interface $S_t = \partial V_t$ with the rest of the medium and the vacuum, by drive 3-forms that contain discontinuities over part of $S_t$, is given by

$$f_{K,\text{in,ext}}^{U} \{ V_t \} = - \int_{V_t} \mathcal{L}_{U} \rho_{K}^{U,in,EM} - \int_{S_t} J_{K}^{U,out,EM} - \int_{S_t} \Delta_{12}(J_{K}^{U}).$$

(3.3)

In static situations the first integral on the right-hand side is zero and in some cases the second integral can be evaluated in terms of fields or their sources in the vacuum region.

4. The Abraham and symmetrized Minkowski Killing drives in non-accelerating media

As noted in the introduction there have been a number of different proposals for the electromagnetic stress–energy–momentum tensor. In tables 1 and 2 in appendix A, the properties of a number of these (Minkowski 1908; Abraham 1909) are displayed using the notation introduced above. For a recent derivation of these tensors from a variational approach see Dereli et al. (2006, 2007). They permit a ready evaluation of the $K$-drives on media with bulk 4-velocity field $V$ and specified constitutive properties. For convenience, separate entries for the Abraham tensor have been provided for media at rest relative to a frame $U$ (i.e. $V = U$). To calculate forces and torques it is only necessary to recognize that they are defined by particular space-like Killing vectors $K$. So in an inertial frame $\tilde{U}$ with $\tilde{U}(K) = 0$ many entries will simplify. Separate entries are also given for situations when $K$ is time-like and equal to $U$ since this facilitates computation
of energies and powers. In the calculation of the divergence of the Abraham stress–energy–momentum we have made explicit those terms that depend on the bulk acceleration field \( A \equiv \nabla_V V \) of the medium. Such terms are of course absent for media at rest or moving with arbitrary linear velocity in an inertial frame \((\nabla U = 0)\). It is only in special circumstances that there are simple relations between the Minkowski and Abraham tensors. These relations arise mainly for simple media at rest in an inertial frame. Thus if

\[
d^U = \epsilon_0 e^e U \quad \text{and} \quad b^U = \mu_0 \mu_r h^U
\]  

in a stationary homogeneous medium (with constant scalars \( \epsilon_r \) and \( \mu_r \)), then, for all \( K \), the \( J^K_U \) are identical for \( T^{SM} \), \( T^M \) and \( T^{AB} \) (see tables 1 and 2 in appendix A). However, the corresponding \( p^K_U \) are all different. In problems with time-harmonic electromagnetic fields the \( L_U^r p^K_U \) have zero averages over time, implying that the corresponding time-averaged \( K \)-drives \( \langle f^K_U \rangle \) are the same.

In a simple non-accelerating \(( A = 0)\) medium that may be inhomogeneous there is a simple relation between \( \tau^K_A \) and \( \tau^K_{SM} \) (see tables 1 and 2 in appendix A) for arbitrary time-dependent fields. Given a space-like translational Killing vector \( K \) one sees from tables 1 and 2 that (since \( \tilde{\Theta}^U(K) = 0 \))

\[
\tau^K_A = \tau^K_{SM} - \frac{1}{2} s^U(K) \star \tilde{U},
\]

where the 1-form

\[
s^U = \star \left( \frac{1}{c_0} e^e U \wedge h^U \wedge \tilde{U} - c_0 d^U \wedge b^U \wedge \tilde{U} \right).
\]

Using the simple constitutive relations (4.1) this 1-form can be expressed in terms of the spatial fields \( e^e U \), \( h^U \) and the form \( \tilde{U} \)

\[
s^U(K) \star \tilde{U} = -\frac{1}{c_0} (1 - N^2) i_U (e^e U \wedge h^U \wedge \tilde{U} \wedge \tilde{K}) = \frac{1}{c_0} (1 - N^2) (e^e U \wedge h^U \wedge \tilde{K})
\]

and so

\[
\tau^K_A = \tau^K_{SM} + \frac{1}{2 c_0} (N^2 - 1) (e^e U \wedge h^U \wedge \tilde{K}).
\]

By contracting the exterior derivative of this expression with \( U = (1/c_0) \partial_t \) (defining a non-accelerating reference frame) one deduces a relation between the 3-form force density \( F^{U,A}_K = i_U d\tau^K_A \) associated with the Abraham drive 3-form, and \( F^{U,SM}_K = i_U d\tau^K_{SM} \) associated with the symmetrized Minkowski drive 3-form,

\[
F^{U,A}_K = F^{U,SM}_K + \frac{1}{2 c_0^2} L_{\partial_t} ((N^2 - 1) e^e U \wedge h^U) \wedge \tilde{K},
\]

using

\[
d\phi = d\phi - \tilde{U} \wedge L_U \phi = d\phi + dt \wedge L_{\partial_t} \phi,
\]
and noting that \((N^2 - 1)(e^U \wedge h^U \wedge \tilde{K})\) is a spatial 3-form. The second term on the right-hand side in equation (4.4) has been termed the Abraham force. In static field situations it is zero and the two force densities are equal. This is also the situation in the presence of time-periodic fields if one takes the time-average of this equation over a time period. For smooth pulsed fields the second term will in general yield a finite contribution to a total force when equation (4.4) is integrated over a finite-time interval.

Returning to the constitutive case for an arbitrarily moving medium (2.23) one can use the results in the previous section to facilitate a comparison between the Newtonian force density (2.14) and that based on the choice with \(t^M_{EM}K = t^M_{SM}K\). Thus one must calculate \(iUd\tau^SM_K\) for a Killing field \(K\) that generates spatial translations and express the result in terms of the polarization and magnetization 1-forms \(p^U\) and \(m^U\), respectively. A local constitutive relation between \(G\) and \(F\) will induce a local constitutive relation between \(p^U\) and \(m^U\) and the electromagnetic fields \(e^U\) and \(b^U\) and the permittivity and permeability of the medium. Since \(dF = 0\),

\[
2d\tau^SM_K = -(di_KG \wedge \star F - i_K G \wedge d \star F - F \wedge d i_K \star G).
\]

Writing \(G = \epsilon_0F + \Pi\), one has

\[
d i_K G \wedge \star F = \mathcal{L}_K G \wedge \star F - i_K dG \wedge \star F = F \wedge \star \mathcal{L}_K G - i_K d(\epsilon_0 F + \Pi) \wedge \star F
\]

\[
= F \wedge \star \mathcal{L}_K G - i_K d\Pi \wedge \star F = F \wedge i_K d \star G + F \wedge d i_K \star G - F \wedge \star i_K d\Pi
\]

\[
= F \wedge i_K \mathcal{J} + F \wedge d i_K \star G - F \wedge \star i_K d\Pi.
\]

Hence

\[
2d\tau^SM_K = -F \wedge i_K \mathcal{J} + F \wedge \star i_K d\Pi + i_K G \wedge d \star F.
\]

Since

\[
d \star G = \epsilon_0d \star F + d \star \Pi = j \quad \text{or} \quad d \star F = \frac{1}{\epsilon_0}(j - d \star \Pi),
\]

and

\[
i_K G \wedge d \star F = -G \wedge i_K d \star F = -\frac{1}{\epsilon_0}G \wedge i_K (j - d \star \Pi)
\]

\[
= -\frac{1}{\epsilon_0}G \wedge i_K j + \frac{1}{\epsilon_0}G \wedge i_K d \star \Pi
\]

\[
= -F \wedge i_K j - \frac{\Pi}{\epsilon_0} \wedge i_K j + G \wedge i_K d \star \Pi \frac{\epsilon_0}{\epsilon_0},
\]

one has

\[
2d\tau^SM_K = -(2F + \frac{\Pi}{\epsilon_0}) \wedge i_K \mathcal{J} + F \wedge \star i_K d\Pi + G \wedge i_K d \star \Pi \frac{\epsilon_0}{\epsilon_0}.
\]

15If one replaces \(U\) by \(V\) with \(\nabla_V V \neq 0\) in equation (4.2) one is dealing with accelerating media and the Abraham drive \(\mathcal{F}^{U,A}_K\) then acquires additional terms in equation (4.4). Such additional terms are often overlooked when considering the effects of the so-called Abraham force (or torque). This may lead to inconsistencies in processes involving accelerating media.
This can be decomposed in terms of $e^U, b^U, p^U, m^U$, and real and induced polarization and magnetization sources, yielding

$$-2d\tau^\text{SM}_K = \left( \rho^U_p e^U(K) + \frac{1}{\epsilon_0} \left( \frac{1}{\epsilon_0} \rho^U_p + \rho^U_p \right) p^U(K) \right) \wedge \tilde{U}$$

$$+ \left( \mu_0 i_{m}^U (J^U + J_p^U) \wedge \kappa + 2 \rho^U e^U(K) + 2i_{b}^U J^U \wedge \kappa \right)$$

$$+ \frac{1}{c_0} \left( i_{m}^U J^U_m \wedge \kappa - c_0 \rho^U_m b^U(K) \right) \wedge \tilde{U}$$

$$- 2c_0 B^U \wedge i_K \rho^U - \frac{1}{\epsilon_0 c_0} M^U \wedge i_K \rho^U$$

$$- i_K \rho^U_m \wedge E^U + c_0 i_K B^U \wedge \rho^U_p + \frac{1}{\epsilon_0 c_0} i_K M^U \wedge \rho^U_p,$$

where $M^U = \# m^U$. Contracting with $U$ yields

$$\mathcal{F}^U_{K, \text{SM}} = -i_U d\tau^\text{SM}_K = \rho^U e^U(K) + \frac{1}{\epsilon_0} \left( \frac{1}{\epsilon_0} \rho^U_p + \rho^U_p \right) e^U(K)$$

$$+ \frac{1}{2} i_{b}^U J^U_p \wedge \kappa + \frac{1}{2c_0} i_{m}^U J^U_m \wedge \kappa - \frac{1}{2} c_0 \rho^U_m b^U(K)$$

$$+ \frac{1}{2\epsilon_0} \left( \rho^U + \rho^U_p \right) p^U(K) + \frac{1}{2} \mu_0 i_{m}^U (J^U + J_p^U) \wedge \kappa. \quad (4.6)$$

The first two terms on the right-hand side constitute the Lorentz force density owing to any real charge $\rho^U$ and electric current $J^U$ in the medium. The third and fourth terms constitute the Lorentz force density owing to induced polarization charge $\rho^U_p$ and electric current $J_p^U$. These are precisely the force densities that arise in the non-relativistic electrically neutral smoothed model (2.14) above based on the motion of point charges. The fifth and sixth terms constitute the Lorentz force density owing to magnetization charge $\rho^U_m$ and magnetization current $J_m^U$. These do not arise in the Newtonian model (2.14) above since the point charges in that model were not endowed with intrinsic magnetic moments.

5. Conclusions

The general theory of drive-forms has been developed starting from the vanishing divergence of a total stress–energy–momentum in an arbitrary spacetime using the language of exterior systems. It has been shown that a decomposition of a drive 3-form, relative to a unit time-like vector field, on a spacetime with sufficient Killing vectors yields forms that can be associated with force and torque densities in continuous media. For material subject to the electromagnetic constitutive relations proposed by Minkowski it has also been shown how the computation of the electromagnetic force density on a non-accelerating polarizable perfectly insulating medium in an electromagnetic field can be effectively carried out in terms of a particular split of the total stress–energy–momentum into parts describing its inertial and electromagnetic properties. In paper II this theory is applied to homogeneous and inhomogeneous dielectric media where it is argued

$^{16}$See the electronic supplementary material for further details of this calculation.
that a means of discriminating between a split into the electromagnetic stress–
ergy–momentum tensors proposed by Abraham and Minkowski (and possibly
others) can be explored experimentally using rotating media.

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Appendix A. Summary of electromagnetic stress–energy–momentum tensors

In tables 1 and 2 we have introduced the 1-form
\[ s^V = \star (i_V F \wedge i_V G \wedge \tilde{V} - i_V G \wedge i_V F \wedge \tilde{V}) \]
\[ = \star \left( \frac{1}{c_0} e^V \wedge h^V \wedge \tilde{V} - c_0 d^V \wedge b^V \wedge \tilde{V} \right), \]
where the spatial fields \( e^V, b^V, d^V, h^V \) are defined by the orthogonal splits of \( F \) and \( G \) relative to the 4-velocity field \( \tilde{V} \) of the medium. Furthermore, we have
defined \( H^U = \# h^U \). The vector fields \( U \) and \( V \) on spacetime are time-like, unit,
normalized with a metric \( g \) of signature \((-1,+1,+1,+1)\) so that \( U(U) = V(V) = -1 \),
and describe the state of motion of the observer and medium, respectively. It
is also convenient to introduce, in terms of the vector field \( U \) and any vector field
\( Y \) on spacetime, the projection \( Y^\perp = Y + \tilde{U}(Y) U \), which induces the projection
operator on \( p \)-forms \( \Pi^p = \mathbb{I} + \tilde{U} \wedge i_U \). The spatial Hodge map \( \# \) with respect to
\( U \) is defined by \( \star 1 = \tilde{U} \wedge \# 1 \), and maps spatial \( p \)-forms to spatial \((3-p)\)-forms.
For any Killing vector field \( K \) on spacetime (i.e. \( \mathcal{L}_K g = 0 \)) we denote the 2-form
\( \# \tilde{K} \) by \( \kappa \). If \( K \) is a Killing vector then so is \( \xi_0 K \), where \( \xi_0 \) is any constant. Hence
the physical dimensions of quantities that depend on the Killing vector field \( \xi_0 K \)
will depend on the physical dimensions of \( \xi_0 \).

In tables 1 and 2 we have isolated those terms in the divergence that depend
explicitly on the bulk medium acceleration \( A \equiv \nabla_Y V \). Among the remaining terms
there is a set that can be expressed in terms of a \((2,0)\)-tensor field \( \Psi \) on spacetime,
defined, for any vector fields \( X, Y \), by \( \Psi(X, Y) = (\Psi(X))(Y) \), where
\[ \Psi(X) \equiv (\nabla_X F)(i_X \tilde{G}) + F(i_X \nabla_X G) - (\nabla_X G)(i_X \tilde{F}) - G(i_X \nabla_X F) - \frac{1}{2} (\mathcal{L}_X F)(i_X \tilde{G}) \]
\[ - \frac{1}{2} F((\mathcal{L}_X g^{-1})(i_X G)) - \frac{1}{2} F(i_X \mathcal{L}_X G) + \frac{1}{2} (\mathcal{L}_X G)(i_X \tilde{F}) + \frac{1}{2} G((\mathcal{L}_X g^{-1})(i_X F)) \]
\[ + \frac{1}{2} G(i_X \mathcal{L}_X F) + \frac{1}{2} (\mathcal{L}_X g)(F(i_X G)) - \frac{1}{2} (\mathcal{L}_X g)(G(i_X F)) \]
with \( F(Y) \equiv i_Y F \) and \( G(Y) \equiv i_Y G \). If \( \{ X_a \} \) denotes an orthonormal frame with
dual cobasis \( \{ e^b \} \) one has \( g^{-1} = \eta^{ab} X_a \otimes X_b \), \( g = \eta_{ab} e^a \otimes e^b \) where \( \eta^{ab} \) and \( \eta_{ab} \) are
both diagonal matrices with \( \eta^{00} = \eta_{00} = -1 \) and \( \eta_{ij} = \eta^{ij} = \delta_{ij} \) for \( i, j = 1, 2, 3 \). The
contraction operator \( i_X \) abbreviated \( i_a \) and \( i^a \equiv \eta^{ab} i_b \). The spacetime Hodge
map is denoted \( \star \) and the canonical 4-form on spacetime is \( \star 1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \)
in an orthonormal basis and is \( \sqrt{|\det g|} d^4x \) in any coordinate system \( \{ x^a \} \).

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Table 1. Minkowski and symmetrized Minkowski drive forms and their orthogonal decompositions.

|                  | Minkowski                                                                 | symmetrized Minkowski                                     |
|------------------|---------------------------------------------------------------------------|----------------------------------------------------------|
| $T_{EM}$         | $i_α F \otimes i^α G + \frac{1}{2} *(F \wedge \ast G)g$                  | $T_{SM} = \frac{1}{2} (i_α F \otimes i^α G + i_α G \otimes i^α F + *(F \wedge \ast G)g)$ |
| $r_{EM}^K$      | $-\frac{ζ_K}{2} (i_K F \wedge \ast G - F \wedge i_K \ast G)$             | $-\frac{ζ_K}{2} (i_K G \wedge \ast F - F \wedge i_K \ast G)$ |
| $\nabla \cdot T_{EM}$ | $\frac{1}{2} d \ast (F \wedge \ast G) - (\nabla X G)(i^α F) - G(\nabla \wedge F)$ | $\frac{1}{2} (d \ast (F \wedge \ast G) - (\nabla X G)(i^α F) - (\nabla X F)(i^α G)$ |
| $J_K^U$         | $-ζ_K \left[ e^U(K)D^U + h^U(K)B^U - \frac{1}{c_0} \tilde{U}(K)e^U \wedge h^U \right.$ | $-ζ_K \left[ e^U(K)D^U + d^U(K)E^U + h^U(K)B^U + b^U(K)H^U \right.$ |
|                 | $\left. - \frac{1}{2} (e^U \cdot d^U + b^U \cdot h^U)K \right]$         | $\left. -(e^U \cdot d^U + b^U \cdot h^U)K - \tilde{U}(K) \left( \frac{1}{c_0} e^U \wedge h^U + c_0 d^U \wedge b^U \right) \right] \right]$ |
| $ρ_K^U$        | $ζ_K \left[ c_0 d^U \wedge b^U \wedge \tilde{K} \right.$               | $ζ_K \left[ \left( \frac{1}{c_0} e^U \wedge h^U + c_0 d^U \wedge b^U \right) \right.$ |
|                 | $\left. + \frac{1}{2} \tilde{U}(K)(e^U \cdot d^U + b^U \cdot h^U) \right\} \#1 \right]$ | $\left. + \tilde{U}(K)(e^U \cdot d^U + b^U \cdot h^U) \right\} \#1 \right]$ |
| $J_U^U$        | $\frac{1}{c_0} e^U \wedge h^U$                                          | $\frac{1}{2} \left( \frac{1}{c_0} e^U \wedge h^U + c_0 d^U \wedge b^U \right)$ |
| $ρ_U^U$        | $\frac{1}{2} (e^U \cdot d^U + b^U \cdot h^U) \#1$                       | $\frac{1}{2} (e^U \cdot d^U + b^U \cdot h^U) \#1$       |
Table 2. Abraham drive forms and their orthogonal decompositions.

|                  | Abraham ($U \neq V$)                                                                 | Abraham ($U = V$)                                                                 |
|------------------|------------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| $T_{EM}^U$       | $\frac{1}{2} (i_a F \otimes i^a G + i_a G \otimes i^a F + \star (F \wedge G) g - s V \otimes \widetilde{V} - \widetilde{V} \otimes s V)$ | $\frac{1}{2} (i_a F \otimes i^a G + i_a G \otimes i^a F + \star (F \wedge G) g - s U \otimes \widetilde{U} - \widetilde{U} \otimes s U)$ |
| $T_{K}^U$       | $\frac{-\zeta_K}{2} (i_K G \wedge \star F - F \wedge i_K \star G - s V (K) \star \widetilde{V} - \widetilde{V} (K) \star s V)$ | $\frac{-\zeta_K}{2} (i_K G \wedge \star F - F \wedge i_K \star G - s U (K) \star \widetilde{U} - \widetilde{U} (K) \star s U)$ |
| $\nabla \cdot T_{EM}$ | $-\frac{1}{2} ((\nabla \cdot s V) \widetilde{V} + (\nabla \cdot \widetilde{V}) s V)$ | $\nabla \cdot T_{SM} - \frac{1}{2} ((\nabla \cdot s U) \widetilde{U} + (\nabla \cdot \widetilde{U}) s U) - \Psi (U)$ |
| $J_{K}^U$       | $\frac{-\zeta_K}{2} \left[ e^U (K) D^U + d^U (K) E^U + h^U (K) B^U \right.$ | $-\frac{1}{2} (e^U (K) D^U + d^U (K) E^U + h^U (K) B^U + b^U (K) H^U)$ |
|                 | $+ b^U (K) H^U - (e^U \cdot d^U + b^U \cdot h^U) \kappa$ | $- \frac{1}{2} (e^U \cdot d^U + b^U \cdot h^U) \kappa - \frac{1}{\alpha_0} \widetilde{U} (K) e^U \wedge h^U \right]$ |
|                 | $- \widetilde{U} (K) \left( \frac{1}{\alpha_0} [ e^U \wedge h^U + c_0 d^U \wedge b^U \right)$ | $- \frac{1}{2} \left( e^U \cdot d^U + b^U \cdot h^U \right) \kappa - \frac{1}{\alpha_0} \widetilde{U} (K) e^U \wedge h^U \right]$ |
|                 | $\left. - i_U (i_K \wedge i_V \wedge s V) + 2 \widetilde{V} (K) i_U \wedge s V \right]$ | $\left. - \frac{1}{\alpha_0} \widetilde{U} (K) e^U \wedge h^U \right]$ |
| $\rho_{K}^U$    | $\frac{\zeta_K}{2} \left[ \left( \frac{1}{\alpha_0} [ e^U \wedge h^U + c_0 d^U \wedge b^U \right) \wedge \widetilde{K}^\perp \right.$ | $\zeta_K \left[ \frac{1}{\alpha_0} e^U \wedge h^U \wedge \widetilde{K}^\perp + \frac{1}{2} \widetilde{U} (K) (e^U \cdot d^U + b^U \cdot h^U) \# 1 \right]$ |
|                 | $\left. - \widetilde{U} (K) \wedge s V + \widetilde{U} (K) (e^U \cdot d^U + b^U \cdot h^U) \# 1 \right]$ | $\left. - \frac{1}{\alpha_0} \widetilde{U} (K) e^U \wedge h^U \right]$ |
|                 | $\left. + \widetilde{U} (K) i_U \wedge s V \wedge \widetilde{K} + \widetilde{V} (K) i_U \wedge s V \right]$ | $\left. + \frac{1}{\alpha_0} \widetilde{U} (K) (e^U \cdot d^U + b^U \cdot h^U) \# 1 \right]$ |
| $J_{U}^U$       | $\frac{1}{2} \left( \frac{1}{\alpha_0} e^U \wedge h^U + c_0 d^U \wedge b^U \right)$ | $\frac{1}{2} e^U \wedge h^U$ |
|                 | $\left. + \frac{1}{2} \Pi U i_U \wedge s V + \widetilde{U} (V) i_U \wedge s V \right]$ | $\left. + \frac{1}{2} \Pi U i_U \wedge s V + \widetilde{U} (V) i_U \wedge s V \right]$ |
| $\rho_{U}^U$    | $\frac{1}{2} (e^U \cdot d^U + b^U \cdot h^U) \# 1 - \widetilde{U} (V) \Pi U \wedge s V$ | $\frac{1}{2} (e^U \cdot d^U + b^U \cdot h^U) \# 1$ |
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