Uniformly and nonuniformly elliptic variational equations with gauge invariance

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Abstract

A large class of variational equations for geometric objects is studied. The results imply conformal monotonicity and Liouville theorems for steady, polytropic, ideal flow, and the regularity of weak solutions to generalized Yang-Mills and Born-Infeld systems. 2000 MSC: 58E15.

1 Introduction

The Hodge Theorem asserts the existence of a unique harmonic representative in each cohomology class of $d$-closed forms on a compact Riemannian manifold, where $d$ is the (flat) exterior derivative. (See, e.g., [M, Ch. 7].) This theorem leads to a richly geometric linear model for stationary fields. Nonlinear Hodge theory can be viewed as an attempt to extend the unified geometric interpretation achieved for linear fields to a large class of quasilinear models.

In the nonlinear generalization of Hodge theory, the set of $d$-closed forms in $H^{1,2}$ is replaced by a set of $d$-closed forms having finite energy, the density of which includes a nonlinear function $\rho$ [SS1]. If $\rho$ is identically 1, this energy functional reduces to the Dirichlet energy of the linear theory.

Forms of degree 1 occupy a special place in both the linear and nonlinear Hodge theories, in that $d$-closed 1-forms can be associated to the field of a scalar potential. In Sec. 2 we consider 2-forms which are closed under covariant exterior differentiation $D$. These can in certain circumstances be interpreted as the curvature 2-form derived from a connection 1-form. The nonlinear Hodge equations in this curved-bundle case possess additional nonlinearities and nontrivial gauge invariance which are absent in the conventional flat-bundle case. Sections of the curved bundle which are stationary points of the nonlinear Hodge energy bear the same relation to harmonic curvature in a bundle that stationary sections of the flat bundle bear to harmonic forms on a manifold. This leads to a further generalization of harmonic curvature.

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In applications to particle fields a curvature 2-form represents the field associated to a vector potential. In the special case in which the bundle structure group is the abelian group $U(1)$, $\rho$ can be chosen in such a way that the nonlinear Hodge energy is equivalent to a suitable normalization of the abelian Born-Infeld energy. This functional is of interest in connection with string theory interactions \([G_b]\). It has been studied, partly from a nonlinear Hodge perspective, in \([Y]\). If in addition $\rho \equiv 1$, then we obtain the Maxwell equations for time-independent electromagnetic fields on 4-space. If $G$ is nonabelian and $\rho \equiv 1$, then the variational equations of Sec. 2 reduce to the Yang-Mills equations, the equations for the classical limit of quantum fields. Additional examples of this kind are given in Sec. 1 of \([O_3]\).

The material of Sec. 2 is largely expository and is based on \([O_4]\). The new results are in Sec. 3, in which we consider the general case of section-valued differential forms of arbitrary order, weakly satisfying nonuniformly elliptic equations associated to the nonlinear Hodge energy. The analytic properties derived for such objects are correspondingly weaker than those derived in Sec. 2. The results of Sec. 3 follow from a conformal monotonicity inequality which we prove for the nonlinear Hodge energy. The hypotheses of Sec. 3 are satisfied by certain models of high-speed subsonic flow.

In the estimates that follow we denote by $C$ generic positive constants which generally depend on dimension and which may change in value from line to line. Dependence of the constants $C$ on bounded variables other than dimension, where indicated at all, is noted by subscripts.

## 2 Lie-algebra-valued sections

Denote by $X$ a vector bundle over a manifold $M$. Suppose that $X$ has compact structure group $G \subset SO(m)$, and that $M$ is a smooth, finite, oriented, $n$-dimensional Riemannian manifold. Let $A \in \Gamma(M, adX \otimes T^*M)$ be a connection 1-form on $X$ having curvature 2-form

$$F_A = dA + \frac{1}{2} [A, A] = dA + A \wedge A,$$

where $[ , ]$ is the bracket of the Lie algebra $\mathfrak{g}$, the fiber of the adjoint bundle $adX$. Sections of the automorphism bundle $Aut X$, called gauge transformations, act tensorially on $F_A$ but affinely on $A$; see, e.g., [MM].

Consider energy functionals having the form

$$E(F_A) = \int_M \left( \int_0^Q \rho(s) ds \right) dM,$$

where $Q = |F_A|^2 = \langle F_A, F_A \rangle$ is an inner product on the fibers of the bundle $adX \otimes \Lambda^2(T^*M)$. The inner product on $adX$ is induced by the normalized trace inner product on $SO(m)$ and that on $\Lambda^2(T^*M)$, by the exterior product $*(F_A \wedge *F_A)$, where $*: \Lambda^p \rightarrow \Lambda^{n-p}$ is the Hodge star operator. The function $\rho : \mathbb{R} \cup \{0\} \rightarrow \mathbb{R}^+$ is a bounded $C^{1,\alpha}$ function satisfying
whenever $Q$ is less than a critical value $Q_{\text{crit}}$. As $Q$ tends to $Q_{\text{crit}}$ we retain the right-hand inequality of (2), but allow the middle term to tend to zero.

The functional (1) appears in [SS1] for the special case $Q = |\omega|^2$, where $\omega \in \Gamma (\mathcal{M}, \Lambda^p (T^* \mathcal{M}))$. In that “flat-bundle” case, stationary points with respect to an admissible cohomology class of closed $p$-forms satisfy the \textit{nonlinear Hodge equations}

$$\delta (\rho(Q)\omega) = 0,$$

$$d\omega = 0.$$  \hspace{1cm} (3)

If we choose $X$ to be a bundle having gauge group $U(1)$, making the choices $p = 2$ and

$$\rho(Q) = (1 + Q)^{-1/2},$$

then $\omega$ has an interpretation as the electromagnetic field of a (suitably normalized) Born-Infeld energy. In this case condition (2) fails as $Q$ tends to infinity. Equations of nonlinear Hodge type also figure in elasticity and thermodynamics, including nonrigid-body rotation and capillarity. Applications to magnetic materials and minimal surfaces are given in [O2] and [SS2], respectively.

Details on the construction of the nonabelian variational problem are given in [O4]. The Euler-Lagrange equations for the functional (1) can be written in the form

$$\delta (\rho(Q)F_A) = -\ast [A, *\rho(Q)F_A],$$

where $\delta : \Lambda^p \rightarrow \Lambda^{p-1}$ is the adjoint of the exterior derivative $d$. In addition, we have the Bianchi identity

$$dF_A = -[A, F_A].$$  \hspace{1cm} (6)

If $G$ is abelian, then eqs. (5) reduce to the system

$$\delta \{ \rho (Q(F_A)) F_A \} = \delta \{ \rho (Q(dA)) dA \} = 0.$$  \hspace{1cm} (7)

Equations (6) reduce in the abelian case to the equations

$$d^2 A = 0,$$

which hold automatically on any domain having trivial deRham cohomology.

### 2.1 Uniformly elliptic weak solutions

In this section we derive a Hölder estimate for weak solutions of the variational equations. It is easy to show the existence of weak solutions to (5), (6) by topological arguments, provided that $\rho$ is chosen so that the energy functional is Palais-Smale. An example is given in Corollary 1.2 of [O1].
Theorem 1 Let the pair \((A,F_A)\) weakly satisfy eqs. \((5),(6)\) in a bounded, open, Lipschitz domain \(\Omega \subset \mathbb{R}^n, n > 2\). Suppose that there exist constants \(\kappa_1\) and \(\kappa_2\) such that

\[
0 < \kappa_1 \leq \rho(Q) + 2Q\rho'(Q) \leq \kappa_2 < \infty
\]

and that \(F_A \in L^s(\Omega)\) for some \(s > n/2\). Then \(A\) is equivalent via a continuous gauge transformation to a connection \(\tilde{A}\) such that \(F_{\tilde{A}}\) is Hölder continuous on compact subdomains of \(\Omega\).

Remarks. It is sufficient for \(\Omega\) to be type-A; see, e.g., [Gi]. Theorem 1 was stated and proven in [O4]. Here we expand on the main technical point of the proof, the construction and use of the Hölder continuous comparison 2-form \(d\varphi\). In doing so, we modify somewhat the proof in [O4]. In particular, we show explicitly that a Campanato estimate for \(dA\) in a ball can be derived even if Hölder estimates for \(d\varphi\) do not hold up to the boundary. This is an important point because, while the analogous estimates can be continued up to the boundary in the 1-form case [O5], those arguments fail for higher-degree forms.

Proof. Choose coordinates so that the origin lies in the interior of \(\Omega\). Denote by \(B_r\) a small \(n\)-disc of radius \(r\), lying entirely in the interior of \(\Omega\) and centered at the origin of coordinates. Trivialize \(X\) locally in order to understand the notion of weak solution in the sense of [Si, eq. (1.2b)]. For abelian \(G\), a weak solution of \((5),(6)\) is any curvature 2-form \(F_A\) for which \(\rho(Q)F_A\) is orthogonal in \(L^2\) to the space of \(d\)-closed 2-forms \(d\zeta \in L^2(B_r)\) such that \(\zeta \in \Lambda^1\) has vanishing tangential component on \(\partial B\). For nonabelian \(G\), an obvious extension to inhomogeneous equations allows us to define a weak solution of \((5),(6)\) by the equation

\[
\int_{B_r} \langle d\zeta, \rho(Q)F_A \rangle * 1 = -\int_{B_r} \langle \zeta, * [A, * \rho(Q)F_A] \rangle * 1,
\]

where \(F_A\) is a curvature 2-form.

We can show \(F_A\) to be bounded in a smaller concentric ball of radius \(r/2\). We do this by extending the arguments of [U1, Sec. 1] to equations possessing the nonlinear structure of eqs. \((5),(6)\). Using the \(L^p\) hypothesis for \(F\), we carry out the \(L^\infty\) estimates in a Hodge gauge. Details are given in [O4, proof of Theorem 1 and the Appendix (Lemma 7) to Sec. 4].

As gauge transformations act tensorially on \(F\), the curvature remains bounded under continuous gauge transformations. Choose an exponential gauge in a euclidean \(n\)-disc \(B_R, R < r/2\), centered at the origin of coordinates in \(\mathbb{R}^n\). In such a gauge \(A(0) = 0\) and \(\forall x \in B_R\),

\[
|A(x)| \leq \frac{1}{2} |x| \cdot \sup_{|y| \leq |x|} |F_A(y)|;
\]
see [U2, Sec. 2]. At the origin of coordinates in an exponential gauge, $F_A$ satisfies eqs. (7). Because $X$ has been trivialized in $B$ we can compare $F_A$ to a finite-energy solution $d\varphi$ of the equation

$$\int_{B_R} \left\langle d\zeta, \rho \left( |d\varphi|^2 \right) d\varphi \right\rangle * 1 = 0$$

(9)

for $d\zeta \in L^2(B_R)$, where $\zeta_{\tan} = 0$ on the $(n-1)$-sphere $|x| = R$. Prescribe vanishing tangential data for the 1-form $A - \varphi$. We have

$$\int_{B_R} \int_0^{\|d\varphi\|^2} \rho(s) ds * 1 \geq C \int_{B_R} |d\varphi| * 1$$

(c.f. [U1, (1.3)']), so $d\varphi$ lies in the space $L^2(B_R)$ by ellipticity and finite energy. In an exponential gauge $F_A = dA$, so the boundedness of $F_A$ implies that $dA$ is bounded, and thus is certainly in $L^2(B_R)$. Because $d(A - \varphi)$ is in $L^2$, we can choose $\zeta = A - \varphi$ in (9). The resulting weak Dirichlet problem is solvable by Proposition 4.3 of [Si]; see also [ISS]. The 2-form $d\varphi$ is Hölder continuous in the interior of $B_R$ by Proposition 4.4 of [Si], which is derived from [U1]. Thus the Campanato Theorem (Theorem III.1.2 of [Gi]) implies that

$$\int_{B_{R/2}} |d\varphi - (d\varphi)(R/2,0)|^2 * 1 \leq CR^{n+\alpha}$$

for some $\alpha \in (0, 2]$, where $(f)_{\tau, \sigma}$ denotes the mean value of $f$ in an $n$-disc of radius $\tau$ centered at the point $\sigma \in \mathbb{R}^n$. Combining (8) and (9), we have

$$\int_{B_R} \left\langle d(A - \varphi), \rho \left( |F_A|^2 \right) F_A - \rho \left( |d\varphi|^2 \right) d\varphi \right\rangle * 1 =$$

$$- \int_{B_R} \left\langle A - \varphi, * \left[ A, * \rho \left( |F_A|^2 \right) F_A \right] \right\rangle * 1.$$

The application of a generalized mean-value formula to this equation, as in Lemma 1.1 of [Si], leads to the inequality

$$\int_{B_R} |d(A - \varphi)|^2 * 1 \leq C_{\rho} \left( \int_{B_R} (|F_A| + |d\varphi|)|x| * 1 + \int_{B_R} |A - \varphi||A||F_A| * 1 + \int_{B_R} |d(A - \varphi)||A|^2 * 1 \right)$$

$$\equiv C(i_1 + i_2 + i_3).$$

(10)

Denote by $\varepsilon$ a small, positive number.

$$i_1 = \int_{B_R} (|F_A| + |d\varphi|)|x| * 1 \leq \int_{B_R} (|F_A| + |d(d\varphi - A)| + |dA|)|x| * 1$$

$$\leq \int_{B_R} (|F_A| + |dA|)|x| * 1 + \int_{B_R} |d(d\varphi - A)||x| * 1 \leq$$
\[ C_{\|F_A\|_{\infty}} \int_0^R |x|^n d|x| + \varepsilon \int_{B_R} |d(\varphi - A)|^2 \ast 1 + C_{\|F_A\|_{\infty}} \int_0^R |x|^{n+1} d|x|. \]

Denote by \( \|\cdot\|_p \) the \( L^p \)-norm over \( B_R \). Using the properties of an exponential gauge, we have by the Sobolev Theorem

\[ i_2 = \int_{B_R} |A - \varphi| |A| |F_A| \ast 1 \leq \varepsilon \int_{B_R} |A - \varphi|^2 |F_A| \ast 1 + \varepsilon^{-1} C_{\|F_A\|_{\infty}} \int_{B_R} |A|^2 \ast 1 \]

\[ \leq \varepsilon \|F_A\|_{n/2} \left\| A - \varphi \right\|_{2n/(n-2)}^2 + C_{\|F_A\|_{\infty}} \int_0^R |x|^{n+1} d|x| \]

\[ \varepsilon C_{\text{Sobolev},\|F_A\|_{n/2}} \int_{B_R} |d(\varphi - A)|^2 \ast 1 + C_{\|F_A\|_{\infty}} R^{n+2}, \]

and

\[ i_3 = \int_{B_R} |d(A - \varphi)| |A|^2 \ast 1 \leq \varepsilon \int_{B_R} |d(A - \varphi)|^2 \ast 1 + \varepsilon^{-1} \int_{B_R} |A|^4 \ast 1 \]

\[ \leq \varepsilon \int_{B_R} |d(A - \varphi)|^2 \ast 1 + C R^{n+4}. \]

Substituting the estimates for \( i_1, i_2, \) and \( i_3 \) into (10) and collecting small terms on the left, we obtain

\[ \int_{B_R} |d(A - \varphi)|^2 \ast 1 \leq C R^{n+1}. \]

Then of course

\[ \int_{B_{R/2}} |d(A - \varphi)|^2 \ast 1 \leq C R^{n+1}. \]

The minimizing property of the mean value with respect to location parameters implies that

\[ \int_{B_{R/2}} |dA - (dA)_{R,x_0}|^2 \ast 1 \leq \int_{B_{R/2}} |dA - (d\varphi)_{R,x_0}|^2 \ast 1 \]

\[ \leq \int_{B_{R/2}} |dA - d\varphi|^2 \ast 1 + \int_{B_{R/2}} |d\varphi - (d\varphi)_{R,x_0}|^2 \ast 1 \]

\[ \leq C R^{n+\ell} \] (11)

for some \( \ell > 0 \). Again using the properties of the exponential gauge and the boundedness of \( F_A \), we find that we can replace \( dA \) by \( F_A \) in estimate (11). Because the arguments leading to (11) hold for any radius \( \tau \in (0, R/2] \), we
apply Campanato’s Theorem in the form [Gi, Theorem III.1.3] to conclude that $F_A$ is Hölder continuous in $B_{R/2}$.

We have used the exponential gauge at the origin of coordinates. In order to remove this limitation, suppose that a map $\gamma \in AutX$ is continuous at each point $x \in B_r(\sigma)$, an $n$-disc of sufficiently small radius $r$ centered at a point $\sigma$. Using the continuity of $\gamma$ and the fact that $\gamma$ is unitary, we have for small $r$,

$$
\left\| \gamma^{-1}(x)F_A(x)\gamma(x) - \left[ \gamma^{-1}(x)F_A(x)\gamma(x) \right]_{r,\sigma} \right\|_2 \approx
\left\| \gamma^{-1}(x)F_A(x)\gamma(x) - \left[ \gamma^{-1}(\sigma)F_A(x)\gamma(\sigma) \right]_{r,\sigma} \right\|_2 \leq
\left\| F_A(x)\left( (\gamma(x)\gamma^{-1}(\sigma) - I) + (I - \gamma(x)\gamma^{-1}(\sigma)) \right) [F_A(x)]_{r,\sigma} \right\|_2 + \left\| F_A(x) - [F_A(x)]_{r,\sigma} \right\|_2,
$$

where $I$ is the identity transformation and the $L^2$-norms are taken over $B_r(\sigma)$. This implies the inequality

$$
\left\| \gamma^{-1}(x)F_A(x)\gamma(x) - \left[ \gamma^{-1}(x)F_A(x)\gamma(x) \right]_{r,\sigma} \right\|_2 \leq \left\| (\gamma(x)\gamma^{-1}(\sigma) - I) \left( F_A(x) - [F_A(x)]_{r,\sigma} \right) \right\|_2 + Cr^{(n+1)/2} \leq C' r^{(n+1)/2},
$$

which allows us to complete the proof of Theorem 1 by a covering argument.

### 3 A conformal monotonicity formula

In this section we extend the results of [O2] concerning conformal monotonicity properties of 2-forms satisfying an ellipticity condition to forms of arbitrary degree. In place of the ellipticity condition on the variational equations, we impose pointwise monotonicity and noncavitation hypotheses on the energy density. A standard construction [P] allows the result to be applied to solutions of the gauge-invariant equations studied in the preceding section.

Recall that a functional is said to be $r$-stationary [A] if it is stationary with respect to compactly supported $C^1$ reparametrizations of its domain.

**Theorem 2** Let $\omega$ be a form of order $q \geq 1$ and let the associated nonlinear Hodge energy $E(\omega)$ be $r$-stationary on a domain $\Omega$ of $\mathbb{R}^n$ containing the unit $n$-disc, $n > 2q$. Suppose that the (conformally weightless) energy density $\rho$ satisfies $\rho'(s) \leq 0 \forall s \in [0,s_{\text{crit}}]$. Then for almost every point of $\Omega$ and for $0 < r_1 < r_2$, we have

$$
r_1^{2q-n}E_{|B_{r_1}} \leq r_2^{2q-n}E_{|B_{r_2}},
$$

where $B_r$ is an $n$-disc, of radius $r$, completely contained in the interior of $\Omega$.
Remarks. If \(q = 1\), then Theorem 2 has an interpretation as a monotonicity formula for steady, polytropic, ideal flow. The theorem is true for the case in which \(\omega\) is the differential of a map into a Riemannian manifold \(N\); see, e.g., [T] for a proof in the constant-density case. If \(\omega\) is the curvature of a Lie-algebra-valued connection 1-form, the result holds for suitably lifted r-variations of \(\omega\); this is illustrated by [P] in the constant-density case and [O2] in the case of nonlinear mass density. The flatness of \(\Omega\) is of somewhat more than notational significance: this hypothesis simplifies the proof by permitting the use of the expansion (12); but it is otherwise of little importance (c.f. [P]). Certain other hypotheses of Theorem 2 can be weakened. For example, we will show the theorem to be true if \(E(\omega)\) is r-stationary only on \(\Omega/\Sigma\), where \(\Sigma\) is a compact subset of sufficiently small Hausdorff dimension (Theorem 6). The theorem also remains true if \(E(\omega)\) is not quite r-stationary but satisfies a certain integral inequality on \(\Omega/\Sigma\); see Sec. 2 of [O2] for the special case of 2-forms. Because [O2] places no assumption on the sign of \(\rho'\), that result depends on the ellipticity coefficients, which our result does not.

Proof of Theorem 2. Denote by \(\psi^t\) a 1-parameter family of compactly supported diffeomorphisms of \(\Omega\) such that

\[
\psi^s \circ \psi^t = \psi^{s+t},
\]

and \(\psi^0 = \text{identity}\). The r-variations of \(E(\omega)\) are given by

\[
\delta_r E(\omega) = \frac{d}{dt} \big|_{t=0} E\left(\psi^{t*}\omega\right).
\]

The first step of the proof is to obtain an explicit expression for this quantity. Write

\[
f \equiv \psi^t(x) = x + t\xi(x) + O(t^2),
\]

(12)

where

\[
\xi(x) = \frac{d}{dt} \big|_{t=0} \psi^t(x)
\]

is the variation vector field, which will be chosen later in the proof.

Lemma 3

\[
\frac{d}{dt} \bigg|_{t=0} \omega_{i_1 \cdots i_q}(f) df^{i_1} \cdots df^{i_q} =
\]

\[
\frac{d}{dt} \bigg|_{t=0} \omega_{i_1 \cdots i_q}(f) A_{\ell_1 \cdots \ell_q}^{i_1 \cdots i_q}(q,\xi,x,t) dx^{\ell_1} \cdots dx^{\ell_q},
\]

where

\[
A_{\ell_1 \cdots \ell_q}^{i_1 \cdots i_q}(q,\xi,x,t) = \delta_{\ell_1}^{i_1} \cdots \delta_{\ell_q}^{i_q} + \sum_{j=1}^q \left(\delta_{\ell_1}^{i_1} \cdots \hat{\delta}_{\ell_j}^{i_j} \cdots \delta_{\ell_q}^{i_q}\right) t \frac{\partial \xi_j}{\partial x_{\ell_j}} + O(t^2).
\]
We prove Lemma 3 by induction on the case $q = 2$, the simplest nontrivial case. We have

$$\frac{d}{dt}_{t=0} \omega_{ij}(f)df^i df^j = \frac{d}{dt}_{t=0} \omega_{ij}(f) \frac{\partial f^i}{\partial x^k} \frac{\partial f^j}{\partial x^m} dx^m =$$

$$\frac{d}{dt}_{t=0} \omega_{ij}(f) \left( \delta^i_k + t \frac{\partial \xi^i}{\partial x^k} \right) dx^k \left( \delta^j_m + t \frac{\partial \xi^j}{\partial x^m} \right) dx^m =$$

$$\frac{d}{dt}_{t=0} \omega_{ij}(f) \left( \delta^i_k \delta^j_m + \delta^i_k t \frac{\partial \xi^j}{\partial x^m} + \delta^j_m t \frac{\partial \xi^i}{\partial x^k} + O(t^2) \right) dx^k dx^m. \quad (13)$$

Assume that eq. (13) holds for $q = m$. We show that the formula must also hold for $q = m + 1$. We have

$$\frac{d}{dt}_{t=0} \omega_{i_1 \cdots i_{m+1}}(f) df^{i_1} \cdots df^{i_{m+1}} =$$

$$\frac{d}{dt}_{t=0} \omega_{i_1 \cdots i_{m+1}}(f) \left( \delta^{i_1}_{\ell_1} + \frac{\partial \xi^{i_1}}{\partial x^{\ell_1}} \right) dx^{\ell_1}, \cdots \left( \delta^{i_{m+1}}_{\ell_{m+1}} + \frac{\partial \xi^{i_{m+1}}}{\partial x^{\ell_{m+1}}} \right) dx^{\ell_{m+1}} =$$

$$\frac{d}{dt}_{t=0} \omega_{i_1 \cdots i_{m+1}}(f) A_{\ell_1 \cdots \ell_{m+1}}^{i_1 \cdots i_{m+1}}(m, \xi, x, t) dx^{\ell_1}, \cdots dx^{\ell_{m+1}}.$$

By the induction hypothesis

$$A_{\ell_1 \cdots \ell_m}^{i_1 \cdots i_m}(m, \xi, x, t) =$$

$$[\delta^{i_1}_{\ell_1} \cdots \delta^{i_m}_{\ell_m} + t \frac{\partial \xi^{i_1}}{\partial x^{\ell_1}} \delta^{i_2}_{\ell_2} \cdots \delta^{i_m}_{\ell_m} + \sum_{j=2}^{m-1} \left( \delta^{i_1}_{\ell_1} \cdots \delta^{i_j}_{\ell_j} \cdots \delta^{i_m}_{\ell_m} \right) t \frac{\partial \xi^{i_j}}{\partial x^{\ell_j}}]$$

$$+ \delta^{i_1}_{\ell_1} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} t \frac{\partial \xi^{i_{m+1}}}{\partial x^{\ell_{m+1}}} + O(t^2) =$$

$$\delta^{i_1}_{\ell_1} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} + \left( \delta^{i_1}_{\ell_1} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} \right) t \frac{\partial \xi^{i_{m+1}}}{\partial x^{\ell_{m+1}}} + O(t^2) =$$

$$\delta^{i_1}_{\ell_1} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} + t \frac{\partial \xi^{i_1}}{\partial x^{\ell_1}} \delta^{i_2}_{\ell_2} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} + \sum_{j=2}^{m-1} \left( \delta^{i_1}_{\ell_1} \cdots \delta^{i_j}_{\ell_j} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} \right) t \frac{\partial \xi^{i_j}}{\partial x^{\ell_j}}$$

$$+ \delta^{i_1}_{\ell_1} \cdots \delta^{i_{m+1}}_{\ell_{m+1}} t \frac{\partial \xi^{i_{m+1}}}{\partial x^{\ell_{m+1}}} + O(t^2).$$

This proves the lemma.

Make the coordinate transformation $x \to y$, where

$$y = (\psi^t)^{-1}(x).$$

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Lemma 4 In terms of \( y \),

\[
\frac{d}{dt}_{|t=0} \omega_{i_1 \ldots i_q} (f) df^{i_1} \ldots df^{i_q} = \\
\omega_{i_1 \ldots i_q} (x) \sum_{j=1}^{q} \frac{\partial \xi^{i_j}}{\partial x^{i_j}} dx^{i_j} \ldots dx^{i_q} = \\
q \omega_{i_1 i_2 \ldots i_q} (x) \frac{\partial \xi^{i_1}}{\partial x^{i_1}} dx^{i_1} dx^{i_2} \ldots dx^{i_q}. \tag{14}
\]

Proof. Notice that if \( q = 2 \),

\[
\frac{d}{dt}_{|t=0} \omega_{ij} (f) df^i df^j = \omega_{ij} (x) \left( \frac{\partial \xi^j}{\partial x^m} dx^m + \frac{\partial \xi^i}{\partial x^k} dx^k dx^j \right) = \\
2 \omega_{ij} (x) \frac{\partial \xi^i}{\partial x^k} dx^k dx^j, \tag{15}
\]

where in the last identity we used the fact that both \( \omega_{ij} \) and \( dx^i dx^j \) are antisymmetric in \( i \) and \( j \). We show that this reasoning extends, with mainly notational complications, to forms of arbitrary order.

Lemma 3 implies that

\[
\frac{d}{dt}_{|t=0} \omega_{i_1 \ldots i_q} (f) df^{i_1} \ldots df^{i_q} = \\
\omega_{i_1 \ldots i_q} \frac{\partial \xi^{i_1}}{\partial x^m} dx^m dx^{i_2} \ldots dx^{i_q} + \sum_{j=2}^{q-1} dx^{i_1} \ldots dx^{i_j} \frac{\partial \xi^{i_j}}{\partial x^m} dx^m \ldots dx^{i_q} \\
+ dx^{i_1} \ldots dx^{i_q-1} \frac{\partial \xi^{i_q}}{\partial x^m} dx^m \equiv i_1 + i_2 + i_3. \tag{16}
\]

Here

\[
i_1 + i_3 = \omega_{i_1 \ldots i_q} \frac{\partial \xi^{i_1}}{\partial x^m} dx^m \wedge (dx^{i_2} \ldots dx^{i_q}) + \\
\omega_{i_1 \ldots i_q} (dx^{i_1} \ldots dx^{i_{q-1}}) \wedge \frac{\partial \xi^{i_q}}{\partial x^m} dx^m = i_1 + \\
\omega_{i_1 \ldots i_q} \frac{\partial \xi^{i_q}}{\partial x^m} dx^m \wedge (dx^{i_1} \ldots dx^{i_{q-1}}) = 2i_1 \tag{17}
\]

by a relabelling of indices in the second term of the sum.

\[
i_2 = \omega_{i_1 \ldots i_q} \sum_{j=2}^{q-1} (dx^{i_1} \ldots dx^{i_{j-1}}) \wedge \frac{\partial \xi^{i_j}}{\partial x^m} dx^m \wedge dx^{i_{j+1}} \ldots dx^{i_q} = \\
- \omega_{i_1 \ldots i_{j-1} i_j i_{j+1} \ldots i_q} \sum_{j=2}^{q-1} \frac{\partial \xi^{i_j}}{\partial x^m} dx^m \wedge (dx^{i_1} \ldots dx^{i_{j-1}}) \wedge (dx^{i_{j+1}} \ldots dx^{i_q}) = 
\]
\[
\omega_{ij_1 \cdots j_q} \sum_{j=2}^{q-1} \frac{\partial \xi_i^{j_j}}{\partial x^m} dx^m \wedge (dx^{i_1} \cdots dx^{i_{j-1}}) \wedge (dx^{i_{j+1}} \cdots dx^{i_q})
\]
\[
= (q-2) \omega_{i_1 \cdots i_q} \frac{\partial \xi_i}{\partial x^m} dx^m dx^{i_2} \cdots dx^{i_q} = (q-2) i_1.
\] (18)

The last identity results from a relabelling of indices. Substituting identities (17) and (18) into the right-hand side of (16) extends (15) to the case of arbitrary \( q \). This proves Lemma 4.

If \( J \) is the Jacobian of the transformation \( x \to y \), then from (12) we obtain

\[
\frac{d}{dt} \bigg|_{t=0} J \left[ (\psi^t)^{-1} \right] = \frac{d}{dt} \bigg|_{t=0} \left| \frac{dx}{df} \right| = -\text{div} \xi.
\] (19)

Define

\[
e(Q) = \int_0^Q \rho(s) \, ds.
\]

By hypothesis,

\[
0 = \delta_r E(\omega) = \frac{d}{dt} \bigg|_{t=0} \int_\Omega e \left( (\psi^t \omega, \psi^t \omega) \right) J \left[ (\psi^t)^{-1} \right] \ast 1 = \\
\int_\Omega e(Q) \frac{d}{dt} \bigg|_{t=0} J \left[ (\psi^t)^{-1} \right] \ast 1 + \\
\int_\Omega e'(Q) 2 \left( \frac{d}{dt} \bigg|_{t=0} \omega_{i_1 \cdots i_q}(f) df^{i_1} \cdots df^{i_q}, \omega_{\ell_1 \cdots \ell_q}(f) df^{\ell_1} \cdots df^{\ell_q} \right) \ast 1.
\] (20)

(The weightlessness of \( \rho \) is used on the right-hand side of this expression.) Substituting (14) and (19) into (20) yields

\[
\int_\Omega e(Q) \text{div} \xi \ast 1 = \\
2q \int_\Omega e'(Q) \omega_{i_1 \cdots i_q}(x) \frac{\partial \xi_1^{i_1}}{\partial x^{i_1}} dx^{i_1} dx^{i_2} \cdots dx^{i_q}, \omega_{\ell_1 \cdots \ell_q}(f) df^{\ell_1} \cdots df^{\ell_q} \ast 1.
\] (21)

Choose an orthonormal basis

\[
\{ u_i \}_{i=1}^n = \left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_2}, \ldots, \frac{\partial}{\partial \theta_n} \right\}
\]

and let \([P]\)

\[
\xi = \eta(r) r \cdot \frac{\partial}{\partial r},
\]

where \( r \) is the radial coordinate in a curvilinear system; \( \eta(r) \in C^\infty_0 [0, 1] ; \eta'(r) \leq 0; \eta(r) = v(r/\tau) = 1 \) for \( r \leq \tau \), where \( \tau \) is a number in the interval \( (0, 1) \); there
is a positive number \( \delta \) for which \( \eta(r) = 0 \) whenever \( r \) exceeds \( \tau + \delta \). For this choice of \( \xi \),

\[
div \xi = \nabla \frac{\partial}{\partial r} \cdot \xi + \nabla \frac{\partial}{\partial \theta_m} \cdot \xi, \quad m = 2, \ldots, n.
\]

But

\[
\nabla \frac{\partial}{\partial \theta_m} \cdot \xi = [r \eta(r)] \nabla \frac{\partial}{\partial r} \frac{\partial}{\partial \theta_m} = \eta(r) \frac{\partial}{\partial r},
\]

so

\[
div \xi = (r \eta)' + (n - 1) \eta.
\]

Initially, let \( q = 2 \). We compute, for our choice of \( \xi \), and \( i, j = 1, \ldots, n \),

\[
\langle \omega(\nabla_i \xi, u_j), \omega(u_i, u_j) \rangle = \frac{d}{dr} (r \eta) \left\langle \omega \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_m} \right), \omega \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_m} \right) \right\rangle \\
+ \eta(r) \left\langle \omega \left( \frac{\partial}{\partial \theta_\ell}, \frac{\partial}{\partial \theta_m} \right), \omega \left( \frac{\partial}{\partial \theta_\ell}, \frac{\partial}{\partial \theta_m} \right) \right\rangle |_{\ell \neq m},
\]

where \( \ell, m = 2, \ldots, n \); we have used in computing eq. (22) the facts that \( dr \wedge dr = 0 \) and the basis vectors are orthonormal. Evaluating \((r \eta)'\) by the product rule, we can write eq. (21) in the form

\[
\int_\Omega e(Q) [n \eta + r \eta' + (n - 1) \eta] * 1 =
\]

\[
4 \int_\Omega \rho(Q) \left\{ (\eta + r \eta') \sum_{m=2}^n \left| \omega \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_m} \right) \right|^2 + \eta \sum_{\ell,m=2,\ell \neq m}^n \left| \omega \left( \frac{\partial}{\partial \theta_\ell}, \frac{\partial}{\partial \theta_m} \right) \right|^2 \right\} * 1.
\]

The extension to arbitrary \( q \) follows by obvious notational alterations for \( \omega = \omega(u_1, \ldots, u_q) \).

We thus obtain

\[
\int_\Omega e(Q) (n \eta + r \eta') * 1 =
\]

\[
2q \int_\Omega Q \rho(Q) \eta * 1 + 2q \int_\Omega \rho(Q) r \eta' \left| \frac{\partial}{\partial r} \right| * 1, \quad (23)
\]

Our hypothesis on the sign of \( \rho' \) implies that

\[
Q \rho(Q) = \int_0^Q \frac{d}{ds} (s \rho(s)) \, ds
\]

\[
= \int_0^Q [s \rho(s) \rho'(s) + \rho(s)] \, ds \leq \int_0^Q \rho(s) \, ds = e(Q).
\]
Substituting this into (23) yields
\[
\int_{\Omega} e(Q) \left( n \eta - 2q \eta + r \eta' \right) * 1 \leq 
\]
\[
2q \int_{\Omega} \rho(Q) r \eta' \left| \frac{\partial}{\partial r} \omega \right|^2 * 1. 
\]
By construction
\[
r \eta'(r) = -\tau \frac{\partial}{\partial \tau} v \left( \frac{r}{\tau} \right) \leq 0.
\]
This yields
\[
0 \leq 2q \int_{\Omega} \rho(Q) \tau \frac{\partial}{\partial \tau} v \left( \frac{r}{\tau} \right) \left| \frac{\partial}{\partial r} \omega \right|^2 * 1 \leq 
\]
\[
\int_{\Omega} e(Q) \left[ (2q - n) \eta + \tau \frac{\partial}{\partial \tau} v \left( \frac{r}{\tau} \right) \right] * 1.
\]
As \( \delta \) tends to zero we obtain
\[
0 \leq \left( 2q - n + \tau \frac{\partial}{\partial \tau} \right) \int_{B_r} e(Q) * 1.
\]
Multiplying this last inequality by the integrating factor \( \tau^{2q-(n+1)} \), the proof of Theorem 2 is completed by integration over \( \tau \) between \( r_1 \) and \( r_2 \).

3.1 An application and an extension

**Corollary 5** Assume the hypotheses of Theorem 2 for \( \Omega = \mathbb{R}^n \) and suppose that
\[
E_{|B_r} \leq C r^k
\]
as \( r \) tends to infinity for sufficiently small \( k \). Suppose that \( \rho(Q) \) is bounded below away from zero. Then \( Q(x) \) is zero for almost every \( x \in \Omega \).

This result was proven for the case \( \rho \equiv 1 \) in [P].

**Proof.** Without loss of generality, take \( k \) to be nonnegative. We can write the growth condition in the form
\[
r^{2q-n} E_{|B_r} \leq C r^{2q+k-n},
\]
where \( 2q + k - n < 0 \) for sufficiently small \( k \). The right-hand side of (26) tends to zero as \( r \) tends to infinity. The left-hand side is nonnegative by construction. Thus the conformal energy \( r^{2q-n} E_{|B_r} \) tends to zero on \( \mathbb{R}^n \). Because by Theorem 2 the conformal energy is nondecreasing for increasing \( r \), we conclude that \( E \) is identically zero on \( \mathbb{R}^n \). The vanishing of the energy on a ball of infinite radius implies the pointwise vanishing of \( Q \) almost everywhere by the inequality
\[
\frac{1}{2} \int_{\Omega} \int_0^Q \rho(s) ds * 1 \geq \frac{1}{2} \min_{s \in [0,Q]} \rho(s) \int_{\Omega} \int_0^Q ds * 1 \geq C \int_{\Omega} Q * 1,
\]
13
which follows from our assumption that $\rho(Q)$ is bounded below away from 0. For example, if $\omega$ is the 1-form canonically associated by the inner product to the velocity of a steady, polytropic, ideal flow we have, by noncavitation,

$$
\int_0^Q \rho(s) \, ds = \frac{2}{\gamma} \left[ 1 - \left( 1 - \frac{\gamma - 1}{2} Q \right)^{\gamma/(\gamma - 1)} \right] \\
\geq \frac{2}{\gamma} \left[ 1 - \left( 1 - \frac{\gamma - 1}{2} Q \right) \right] = \frac{\gamma - 1}{\gamma} Q,
$$

where $\gamma > 1$ is the adiabatic constant of the medium. This inequality holds even for transonic flow, provided $Q$ is exceeded by the number $2/(\gamma - 1)$.

**Theorem 6** Let $\Sigma$ be a compact singular set, of codimension $m \in (2, n]$, completely contained in a sufficiently small ball which is itself completely contained in the interior of $\Omega$, such that $\partial \Sigma$ is Lipschitz. If the domain $\Omega$ is replaced by the domain $\Omega/\Sigma$ in Theorem 2 and $\mathbb{R}^n$ is replaced by the domain $\mathbb{R}^n/\Sigma$ in Corollary 5, then the assertions of these propositions continue to hold provided we add the hypothesis that $Q(\omega) \in L^{m/(m-1)}(\Omega)$. If $\Sigma$ is a point, then the $L^p$ condition on $Q$ can be replaced by a hypothesis of finite energy.

**Remark.** For certain choices of $\rho$ the $L^p$ condition on $Q$ can of course be obtained from finite energy. If $\partial \Sigma$ is not Lipschitz, then the result is true under a slightly stronger $L^p$ hypothesis (c.f. [O2]).

**Proof.** Replace the variation vector field in the proof of Theorem 1 by the quantity [O2]

$$
\xi = \left( 1 - \chi^{(\nu)} \right) \eta(r) r \cdot (\partial/\partial r),
$$

where $\chi^{(\nu)}$ denotes a sequence of functions of $r$ such that $\chi^{(\nu)} \in [0, 1]$ and $\chi^{(\nu)}$ is equal to 1 in a neighborhood of the singular set $\Sigma$. If $\Sigma$ has zero $s$-capacity with respect to $\Omega$ for $1 \leq s \leq n$, then $\chi^{(\nu)}$ can be chosen so that as $\nu$ tends to infinity, $\chi^{(\nu)} \to 0$ a.e. and $\nabla \chi^{(\nu)} \to 0$ in $L^s$ [Se, Lemma 2 and p. 73]. Because $\rho$ is noncavitating, we have by (24),

$$
\int_{\Omega} \rho(Q) \chi^{(\nu)'}(r) \eta \ast 1 \geq -C \left\| \nabla \chi^{(\nu)} \right\|_{L^m} \| Q \|_{L^{m/(m-1)}}. \tag{28}
$$

By a result of Carlson [W], a Lipschitz set of codimension $m$ has vanishing $m$-capacity, so the right-hand side of this inequality tends to zero as $\nu$ tends to infinity. Similarly,

$$
\int_{\Omega} e(Q) \chi^{(\nu)'}(r) \eta \ast 1 \geq -C \left\| \nabla \chi^{(\nu)} \right\|_{L^m} \| Q \|_{L^{m/(m-1)}}. \tag{29}
$$

The analogue of the left-hand and right-hand sides of inequality (25) that arises from taking the variation vector field given to equal (27) will contain terms that can be estimated by inequalities (28) and (29), respectively. This completes the proof of Theorem 6 in the case $\dim(\Sigma) > 0$. 

14
If $\Sigma$ is a point, then we can choose coordinates in which the singularity lies at the origin of $\mathbb{R}^n$ and define [L]

$$\xi = \zeta \eta (r) \cdot (\partial / \partial r),$$

(30)

where $\zeta = 0$ in a ball $B_\sigma$ of radius $\sigma$ about the singularity, $\zeta (|x|) = 1$ for $|x|$ exceeding $2\sigma$, and

$$\zeta'(|x|) \leq C \sigma^{-1}$$

(31)

for $x \in B_{2\sigma}/B_\sigma$. We obtain

$$\int_{\Omega} e(Q) \zeta'(r) r \eta * 1 \geq -C \int_{B_{2\sigma}/B_\sigma} e(Q) \eta * 1$$

(32)

and

$$\int_{\Omega} \rho(Q) \zeta'(r) r \eta \left| \frac{\partial}{\partial r} \omega \right|^2 * 1 \leq C \int_{B_{2\sigma}/B_\sigma} \rho(Q) \eta \left| \frac{\partial}{\partial r} \omega \right|^2$$

$$\leq C \int_{B_{2\sigma}/B_\sigma} \rho(Q) \eta Q * 1 \leq C \int_{B_{2\sigma}/B_\sigma} e(Q) \eta * 1.$$

(33)

Locally finite energy implies that the right-hand sides of both (32) and (33) tend to zero as $\sigma$ tends to zero. Considering these inequalities in evaluating the extra terms introduced into (25) by choosing $\xi$ as in (30) and (31) completes the proof for the case $\dim(\Sigma) = 0$.

This completes the proof of Theorem 6.

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