MINIMAL EXPONENTIAL GROWTH RATES OF METABELIAN BAUMSLAG-SOLITAR GROUPS AND LAMPLIGHTER GROUPS

MICHELLE BUCHER, ALEXEY TALAMBUTSA

Abstract. We prove that for any prime $p \geq 3$ the minimal exponential growth rate of the Baumslag-Solitar group $BS(1,p)$ and the lamplighter group $L_p = (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ are equal. We also show that for $p = 2$ this claim is not true and the growth rate of $BS(1,2)$ is equal to the positive root of $x^3 - x^2 - 2$, whilst the one of the lamplighter group $L_2$ is equal to the golden ratio $\frac{1 + \sqrt{5}}{2}$. The latter value also serves to show that the lower bound of A. Mann from [8] for the growth rates of non-semidirect HNN extensions is optimal.

1. Introduction

Let $G$ be a finitely generated group. For any finite generating set $S$ of $G$ we can consider the exponential growth rate of $G$ with respect to $S$ which is defined as follows: Any element $g \in G$ can be written as a finite product of elements in $S \cup S^{-1}$ and we define the length $\ell_{G,S}(g)$ of $g$ as the minimum length of such a product. The growth function $F_{G,S}(n)$ counts the number of elements in a ball of radius $n$ centered at the identity, that is the number of elements $g \in G$ for which $\ell_{G,S}(g) \leq n$. Finally the exponential growth rate of $G$ with respect to $S$ is the limit

$$\omega(G,S) = \lim_{n \to \infty} \left( \frac{F_{G,S}(n)}{n} \right)^{\frac{1}{n}} \geq 1.$$  

Note that this limit always exists by submultiplicativity of the growth function (see [6, VI.C.56]).

The exponential growth rate $\omega(G,S)$ clearly depends on the choice of the generating set $S$ and one obtains a group invariant by considering the infimum over all finite generating sets:

$$\Omega(G) = \inf_{|S|<\infty} \{\omega(G,S)\}.$$  

(1.1)

It is now natural to ask if there exists generating sets $S$ for which the equality $\Omega(G) = \omega(G,S)$ is realized. For the free group $F_n$ of rank $n$, Gromov remarked in [4] Example 5.13 that $\Omega(F_n)$ is exactly $2n - 1$ and is realized on any free generating set (with $n$ elements). Except for this example, very few exact values for $\Omega(G)$ have been computed. Known cases include free products $\mathbb{Z}_2 * \mathbb{Z}_p$ [12] (the cases $p^k = 3, 4$ were proven earlier in [8]), the free product $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and the Coxeter group $\text{PGL}(2, \mathbb{Z})$ [2] and a few more examples in the references [2, 12, 8]. But the question of de la Harpe and Grigorchuk whether $\Omega(\pi_1(\Sigma_g))$ is realized on the canonical generators of the fundamental group of a closed surface $\Sigma_g$ with $g \geq 2$ is still open (see [3] p.55). While in many cases, the value $\omega(G,S)$ can be computed for some particular generating set $S$, it is usually much harder to find a generating

Michelle Bucher is supported by Swiss National Science Foundation project PP00P2-128309/1.
Alexey Talambutsa is supported by Russian Science Foundation, project 14-50-00005.
set $S$ such that $\Omega(G) = \omega(G, S)$ and sometimes even impossible due to the existence of groups for which the infimum in (1.1) is not attained (see [10, 14]).

We consider two classes of metabelian groups: Baumslag-Solitar groups $BS(1, n)$ and lamplighter groups $L_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$. The growth functions of the Baumslag-Solitar groups (1.2) $BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle$

with respect to the canonical generating set $S = \{a, t\}$ were computed by Collins, Edjvet and Gill in [3]. The restricted wreath products $L_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$ can be presented as (1.3) $L_n = \langle a, t \mid a^n = 1, [t^k a t^{-k}, a] = 1 (k = 1, 2, \ldots) \rangle$.

To compute the growth function of $L_n$ with respect to the set $\{a, t\}$ one can use formulas given by Parry in [9]. Even though the formulas for the growth functions of $BS(1, n)$ and $L_n$ were obtained by completely different methods and by use of different properties of the groups, we find that remarkably for all odd $n = 2k + 1$ (1.4) $\omega(BS(1, n), \{a, t\}) = \omega(L_n, \{a, t\}) = \omega_k$,

where $\omega_k$ is the unique positive root of $T_k(x) = x^{k+1} - x^k - 2x^{k-1} - \ldots - 2x - 2$,

for $k \geq 1$. This is easily deduced from [9] and [3] in Lemma 11. Interestingly, this equality never holds for even $n$. We will see the case $n = 2$ in more details.

Some inference for the equality (1.4) can be seen in the actions of the groups $BS(1, n)$ and $L_n$ on their corresponding Bass-Serre trees. There is indeed a very strong similarity between these actions, which we exploit to prove the main result of the paper:

**Theorem 1.** Let $p$ be a prime. The minimal growth rate of the Baumslag-Solitar group $BS(1, p)$ and lamplighter groups $L_p$ are realized on the canonical generators $\{a, t\}$:

$\Omega(L_p) = \Omega(BS(1, p)) = \omega_k$, for $p = 2k + 1$,

$\Omega(L_2) = \frac{1 + \sqrt{5}}{2} < \Omega(BS(1, 2)) = \beta$,

where $\beta \sim 1.69572$ is the unique positive root of $z^3 = z^2 - 2$.

The exact computation $\Omega(L_2) = (1 + \sqrt{5})/2$ gives a positive answer to the question of Mann [8] whether the lower bound $\Omega(G) \geq (1 + \sqrt{5})/2$ can be realized on a non-semidirect HNN extension. (The fact that $L_2$ is indeed a non-semidirect HNN extension will be shown in Section 3). Note that it follows from Theorem 1 that this lower bound could never be realized on any of the Baumslag-Solitar groups $\Omega(BS(1, n))$ also for arbitrary integers $n \geq 2$.

The lower bounds for the growth rates in Theorem 1 are obtained by looking at the actions on the corresponding Bass-Serre tree, finding free submonoids using a local variant of the classical ping-pong lemma (Lemma 5 here) and computing their growth with Lemma 10. Interestingly, all the minimal growth rates are in fact realized as the growth rate of some free submonoid. The Bass-Serre trees of $L_p$ and $BS(1, p)$ are both $(p + 1)$-regular trees, but the corresponding actions are of course different. Nevertheless, when $p$ is odd, the same method applies to give the lower bound of Theorem 1 which we abstract in the following theorem:
Theorem 2. Let $G = H * \theta$ be an HNN extension relative to an isomorphism $\theta : A \to B$ with $A = H$ and $B$ a normal subgroup of prime index $p$ in $H$. Then

$$\Omega(G) \geq \frac{1 + \sqrt{5}}{2}, \quad \text{for } p = 2,$$

$$\Omega(G) \geq \omega_k, \quad \text{for } p = 2k + 1.$$ 

Together with the equalities (1.4) proven in Lemma 11 this immediately implies Theorem 1, except in the case of $BS(1, 2)$. For this last group, a finer analysis of its action on its Bass-Serre tree will be needed.

The question of Mann mentioned above was prompted by his proof of the lower bound $\Omega(G) \geq (1 + \sqrt{5})/2$ for any non-semidirect HNN extension $G$ (see [8], using the cute algebraic observation that a hyperbolic element and a nontrivial conjugate of it generate a free monoid with growth rate equal to the golden ratio. Our proof for the case $p = 2$ of Theorem 2 also holds for any non-semidirect HNN extension and gives an alternative geometric proof to Mann’s inequality.

Finally, as an application of Theorem 1, we can compute the minimal growth rate of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Indeed, as was already noted by Shukhov in [11], one can deduce from (1.5) that

$$\lim_{n \to \infty} \omega(BS(1, n), \{a, t\}) = 1 + \sqrt{2}.$$ 

Since the wreath product $\mathbb{Z} \wr \mathbb{Z}$ can be viewed as an extension of the groups $\mathcal{L}_p$, combining Theorem 2 and Parry’s computations for $\mathbb{Z} \wr \mathbb{Z}$, we obtain

Corollary 3. The minimal growth rate of the restricted wreath product $\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid [t^k at^{-k}, a] = 1 \ (k = 1, 2, \ldots) \rangle$ is realized on the set $\{a, t\}$ and

$$\Omega(\mathbb{Z} \wr \mathbb{Z}) = \omega(\mathbb{Z} \wr \mathbb{Z}, \{a, t\}) = 1 + \sqrt{2}.$$ 

Acknowledgements. We thank Murray Elder for helpful discussions in the preparation of this work and Tatiana Smirnova-Nagnibeda for pointing out some useful references.

2. Bass-Serre tree for an HNN extension

Let $G = H * \theta$ be the HNN extension of $H$ relative to the isomorphism $\theta : A \to B$ between the two subgroups $A, B$ of $H$. Following [5] we call $H * \theta$ a non-semidirect HNN-extension if at least one of the subgroups $A$ or $B$ is a proper subgroup in $H$.

If $H = \langle S_H \mid R_H \rangle$ is a presentation of $H$, then $G$ admits the presentation

$$G = \langle S_H, t \mid R_H, tat^{-1} = \theta(a) \forall a \in A \rangle.$$ 

There is a natural surjection $\varphi : G \to \mathbb{Z}$ defined by sending the generators $S_H$ to 0 and $t$ to 1.

The vertices of the associated Bass-Serre tree $T$ of $G$ are the right cosets of $G$ by $H$ and the edges are the right cosets of $G$ by $B$,

$$T^0 = G/H, \quad T^1 = G/B.$$ 

The edge $gB \in T^1$ has vertices $gH$ and $gtH$. This is a tree of valency $[H : A] + [H : B]$. The group $G$ acts on $T$ by left multiplication.

Since the natural surjection $\varphi : G \to \mathbb{Z}$ is trivial on $H$, it induces a map $\overline{\varphi} : T^0 \to \mathbb{Z}$ which sends vertices $v, w$ of an edge of $T^1$ to images satisfying $|\overline{\varphi}(v) - \overline{\varphi}(w)| = 1$. 

This allows us to define an orientation on the edges by giving an edge from \( v \) to \( w \) with \( \varphi(w) - \varphi(v) = 1 \) the positive orientation. This allows us to distinguish between two types of neighbors to any vertex \( v \): the \([H : A]\) vertices \( w \) such that \( \varphi(w) = \varphi(v) - 1 \) which we call the direct ascendants of \( v \), and the \([H : B]\) vertices \( w \) such that \( \varphi(w) = \varphi(v) + 1 \), which we call the direct descendants of \( v \). We further call a vertex \( z \) a ascendant, respectively an descendant, of \( v \) if there is a sequence \( v = w_0, w_1, \ldots, w_\ell = z \) such that \( w_i \) is a direct ascendant, resp. direct descendant, of \( w_{i-1} \) for \( 1 \leq i \leq \ell \). In our examples, \([H : A]\) = 1, which means that there is only one direct ascendant to any vertex. We will also use the terminology that a vertex \( v \) is above, respectively below, a vertex \( w \) if \( v \) is an ascendant, resp. descendant, of \( w \).

Since the action of \( G \) on \( T \) preserves the orientation on the edges defined above, it is immediate that \( G \) acts on \( T \) without inversions. Thus there are two types of elements: elliptic elements \( g \in G \) have a fixed point on \( T \) and are thus conjugated to \( H \), and hyperbolic elements \( g \in G \) have no fixed point and possess a unique invariant geodesic \( L_g \), called the axis of \( g \), on which \( g \) acts by translation. Note that any element \( g \in G \) which is not in the kernel of \( \varphi : G \to \mathbb{Z} \) necessarily is hyperbolic, so in particular, any generating set of \( G \) contains a hyperbolic element. Such hyperbolic elements will be called positive, respectively negative according to their image in \( \mathbb{Z} \) being positive or negative.

The orientation on \( T \) induced by the surjection \( \varphi : G \to \mathbb{Z} \) allows us to distinguish two types of neighbors to any vertex \( v \).

Let us look at the first of our two main examples: the Baumslag-Solitar group \( BS(1, n) \). (The example of the lamplighter groups is postponed to the next section where we will also first prove that it can be seen as an HNN extension of type \((n, 1)\)). The Baumslag-Solitar group \( BS(1, n) \) is an HNN extension for \( H = A = \mathbb{Z} \), \( B = n\mathbb{Z} \) and \( \varphi : \mathbb{Z} \to n\mathbb{Z} \) given by multiplication by \( n \),

\[
BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle.
\]

Its Bass-Serre tree is depicted in Figure 2.1.

**Lemma 4.** Let \( G \) be an HNN extension such that \( A = H \) and \( B \) is a normal subgroup of \( H \) of odd prime index \( p = 2k + 1 \). Let \( g \in G \) be an elliptic element. For any vertex \( v \) of the Bass-Serre tree \( T \) either \( g(v) = v \) or the \( p = 2k + 1 \) vertices

\[
g^{-k}(v), \ldots, g^{-1}(v), v, g(v), \ldots, g^k(v)
\]

are distinct.
Proof. Let \( a \in A = H \) be any element not in the kernel of the natural surjection \( A \to A/B \cong \mathbb{Z}_p \). Then \( A = \bigcup_{k=0}^{p-1} a^kB \). In the Bass-Serre tree of \( G \), the \( p \) direct descendants of the vertex \( A \) are the vertices \( a^{-k}tA, \ldots, tA, \ldots, a^k tA \) and are joined to \( A \) through the edges \( a^{-k}B, \ldots, B, \ldots, a^kB \) respectively. Observe that since \( B \) is normal in \( A \), any element \( b \in B \) acts trivially on the direct descendants of the vertex \( A \). Furthermore, \( a \) and any of its powers \( a^j \) where \( p \) does not divide \( j \) obviously acts cyclically on the first descendants of \( A \).

By conjugation, we can suppose that our elliptic element is in fact \( h = a^j b \in H = A \), with \( b \in B \) and \(-k \leq j \leq k \). If \( j = 0 \) then \( h \) acts trivially on the direct descendants of \( A \), while if \( j \neq 0 \) then \( h \) acts as a cyclic permutation of order \( p \).

This implies the lemma.\( \square \)

The following Lemma is an immediate application of the classical ping-pong lemma for semigroups [3] Proposition VII.2] taking as ping-pong sets, the descendants of \( x_i v \), for every \( i \):

**Lemma 5** (Ping-Pong Lemma). Let \( x_1, x_2, \ldots, x_r \in BS(1, p) \) act as positive hyperbolic automorphisms on the corresponding Bass-Serre tree \( T \). Suppose that there exists a vertex \( v \in T^0 \) such that \( \{ x_1 v, x_2 v, \ldots, x_r v \} \) are leaves of a tree rooted at \( v \). Then the set \( \{ x_1, \ldots, x_r \} \) freely generates a free monoid.

### 3. Lamplighter groups viewed as non-semidirect HNN extensions

The standard presentation for a restricted wreath product \( G \wr \mathbb{Z} \) is also an HNN-extension, but the subgroups \( A, B \) are both equal to \( G \), so the corresponding Bass-Serre tree is a line, and the corresponding action of \( G \) on a line is not useful for our goals. It has already been pointed out in [13] that there exists another HNN-extension presentation of any wreath product \( G \wr \mathbb{Z} \) with indices \( |G| \) and 1 so that the corresponding Bass-Serre tree is a regular tree of valency \( |G| + 1 \). For completeness, we include a proof of this fact for \( \mathcal{L}_p = (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z} \):

**Lemma 6.** The lamplighter group \( \mathcal{L}_p = (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z} \) can be decomposed as an HNN-extension \( H \rtimes \theta \) with indices of the subgroups \( |H : A| = 1, |H : B| = p \).

**Proof.** First, we find some useful presentation of the lamplighter group \( \mathcal{L}_p \). Start with the standard presentation

\[
\mathcal{L}_p = \langle a, t \mid a^p = 1, [t^a t^{-m}, t^a t^{-n}] = 1, m, n \in \mathbb{Z} \rangle,
\]

which can economically be rewritten as

\[
\mathcal{L}_p = \langle a, t \mid a^p = 1, [a, t^k t^{-k} a] = 1, k \in \mathbb{Z} \rangle
\]

since the element \([t^a t^{-m}, t^a t^{-n}]\) can be obtained via conjugation of \([a, t^k t^{-k}]\) by a suitable power of \( t \). Now we reduce the set of relations in (3.1) further to get

\[
\mathcal{L}_p = \langle a, t \mid a^p = 1, [a, t^k t^{-k}] = 1, k \in \mathbb{N} \rangle,
\]

which is possible because \([a, t^k t^{-k}]\) is a consequence of a conjugate of \([a, t^k t^{-k}]\).

Now we show how to decompose the group \( \mathcal{L}_p \) as a non-semidirect HNN-extension. Consider the infinite direct sum \( D = \oplus_{\mathbb{N}_0} (\mathbb{Z}/p\mathbb{Z}) \) canonically generated by the set of elements \( \{ a_0, a_1, a_2, \ldots \} \). The presentation of this group is

\[
D = \langle a_0, a_1, a_2, \ldots \mid a_0^p = 1, [a_m, a_n] = 1, m, n \in \mathbb{N}_0 \rangle.
\]
Consider the HNN extension of $D$ given by the subgroups $H = D$ and $K = \langle a_1, a_2, \ldots \rangle$ and the isomorphism $f(a_i) = a_{i+1}$. Note that $[D : K] = p$. The HNN extension $D * f$ then has the presentation

$$D * f = \langle t, a_0, a_1, a_2 \ldots | a_m^p = 1, [a_m, a_n] = 1, ta_m t^{-1} = a_{m+1}, m, n \in \mathbb{N}_0 \rangle.$$ 

The relations $a_m^p = 1$ with $m \geq 1$ can be excluded from this presentation because they follow from the relation $a_0^p = 1$ and the series of relations $ta_m t^{-1} = a_{m+1}$. Then, repeatedly using the series $ta_m t^{-1} = a_{m+1}$ we substitute the letters $a_i$ in the commutators so that we get

$$D * f = \langle t, a_0, a_1, a_2 \ldots | a_0^p = 1, [t^m a_0 t^{-m}, t^n a_0 t^{-n}] = 1, ta_m t^{-1} = a_{m+1}, m, n \in \mathbb{N}_0 \rangle.$$ 

Now we repeatedly remove the generators $a_m$ for all $m \geq 1$ and get the presentation

$$D * f = \langle t, a_0 | a_0^p = 1, [t^m a_0 t^{-m}, t^n a_0 t^{-n}] = 1, m, n \in \mathbb{N}_0 \rangle.$$ 

Again, the relation $[t^m a_0 t^{-m}, t^n a_0 t^{-n}] = 1$ with $n \geq m$ follows from the relation $[a_0, t^m a_0 t^{-m}] = 1$, so we obtain that

$$D * f = \langle t, a_0 | a_0^p = 1, [a_0, t^k a_0 t^{-k}] = 1, k \in \mathbb{N} \rangle,$$

which is equivalent to the presentation (3.2) of the lamplighter group $L_p$. \hfill \Box

It is quite obvious that the groups $L_p$ tend to $\mathbb{Z} \wr \mathbb{Z}$ when $p$ tends to $\infty$. Actually, the following nice fact is also true:

**Proposition 7.** The groups $BS(1, n)$ are factor groups of the wreath product $\mathbb{Z} \wr \mathbb{Z}$.

**Proof.** As seen above, the group $L_2 = \mathbb{Z} \wr \mathbb{Z}$ can be presented as

$$\mathbb{Z} \wr \mathbb{Z} = \langle a, t | [a, t^k a t^{-k}] = 1, k \in \mathbb{N} \rangle. \tag{3.4}$$

The presentations (1.3) and (3.4) prove the Proposition, since according to (1.3), for every positive $k$ the element $t^k a t^{-k}$ is a power of $a$, hence it commutes with $a$ so that the corresponding relation in (3.2) holds true. \hfill \Box

We will see later that $\lim_{p \to +\infty} \omega(BS(1, p)), \{a, t\}) = 1 + \sqrt{2} = \omega(\mathbb{Z} \wr \mathbb{Z}, \{a, t\})$, which is some further evidence for the fact that $\mathbb{Z} \wr \mathbb{Z}$ is a limit of the groups $BS(1, n)$.

Now we can show that the classic lamplighter $L_2$ gives the answer to Mann’s question about growth of non-semidirect HNN-extensions (see [8, Problem 1]), proving a part of the Theorem 1, which we state as

**Proposition 8.** The minimal growth rate $\Omega(L_2)$ of the lamplighter group $L_2$ is realized on the generating set $\{a, t\}$ and it is equal to the golden ratio $\varphi = (1 + \sqrt{5})/2$.

**Proof.** For the group $G \wr \mathbb{Z}$ one can compute the exact growth series using the following formula of W. Parry from [9, Corollary 3.3]. If $f_G(x)$ is the growth series of a finitely generated group $G$ then the growth series of $G \wr \mathbb{Z}$ can be obtained as

$$f_{G \wr \mathbb{Z}}(x) = \frac{f_G(x)(1 - x^2)^2(1 + xf_G(x))}{(1 - x^2 f_G(x))^2(1 - xf_G(x))}. \tag{3.5}$$

We use this formula to compute the growth series for $L_2$.

$$f_{L_2}(x) = \frac{(1 + x)(1 - x^2)^2(1 + x(1 + x))}{(1 - x^2(1 + x))(1 - x(1 + x))} = \frac{(1 + x)(1 - x^2)^2(1 + x + x^2)}{(1 - x^2 + x^3)(1 - x - x^2)}.$$
The factors in nominator have roots on the unit circle, whilst the factors of the denominators give two roots inside the unit circle, whose reciprocals are the golden ratio $\varphi = (1 + \sqrt{5})/2$ and the so-called “plastic number”$^1$. Since $\varphi > \psi$, we get $\omega(\mathcal{L}_2, \{a, t\}) = \varphi$. As $\mathcal{L}_2$ is a non-semidirect HNN extension due to Lemma$^6$ we may apply the Theorem 1 from$^5$ to get the lower bound $\omega(\mathcal{L}_2) \geq \varphi$ and finally conclude that $\omega(\mathcal{L}_2) = \varphi$. \hfill \Box

The equality $\omega(\mathcal{L}_2, \{a, t\}) = \varphi$ was also mentioned in$^7$, p.1997 by Lyons-Pemantle-Peres, and follows from the observation that there is a subtree in the Cayley graph of $\mathcal{L}_2$ which is a Fibonacci tree.

Remark 9. It would be interesting to find a natural (maybe geometric) reason for the group $\mathcal{L}_2$ to have the “second biggest growth rate” equal to the plastic number $\psi$.

4. Growth rates computations and estimates

We collect in this section some explicit computations and estimates on growth rates. Lemma$^{10}$ which is proved in$^2$, Lemma 6, will be used extensively in the proofs of Theorems$^1, \, 2$ and will combine with our Ping-Pong Lemma$^5$. The exact growth rates of some Baumslag-Solitar groups and lamplighters groups are computed in Lemma$^{11}$ and the last Lemma$^{12}$ allows us to compare some particular roots.

Lemma 10. Let $G$ be a group generated by a finite set $S$. Suppose that there exists a set $\{x_1, \ldots, x_k\} \subset G$ generating a free monoid inside $G$. Set $\ell_i = \ell_{G,S}(x_i)$, for $i = 1, \ldots, k$, and $m = \max\{\ell_1, \ldots, \ell_k\}$. Then $\omega(G, S)$ is greater or equal to the unique positive root of the polynomial

$$Q(z) = z^m - \sum_{i=1}^{k} z^{m - \ell_i}. \quad (4.1)$$

As mentioned in the introduction, we can easily compute the growth rate of the lamplighters and Baumslag-Solitar group with respect to the canonical generators from the growth functions found by Parry$^9$ and Collins, Edjvet and Gill$^3$ respectively. Recall that for any integer $k \geq 1$ we consider the polynomial

$$T_k(x) = x^{k+1} - x^k - 2x^{k-1} - \ldots - 2x - 2.$$ 

Due to Descartes rule of signs, $T_k$ has single positive root, which we denote by $\omega_k$.

Lemma 11. (a) The growth rate $\omega(\mathcal{L}_2, \{a, t\})$ is equal to $\frac{1 + \sqrt{5}}{2}$.

(b) For any $k \geq 1$ we have that $\omega(BS(1, 2k + 1), \{a, t\}) = \omega(\mathcal{L}_{2k+1}, \{a, t\}) = \omega_k$, where $\kappa$ is the smallest positive zero of the function $1 - x f_{G,S}(x)$. Taking $f_{G,S}(x) = 1 + 2x + 2x^2 + \ldots + 2x^{k-1}$ we get that $\omega(\mathcal{L}_{2k+1}, \{a, t\}) = 1/\kappa_k$, where $\kappa_k$ is the smallest positive zero of the

$^1$Notably $\psi = \Omega(GL(2, \mathbb{Z})) = \Omega(PGL(2, \mathbb{Z}))$, see$^2$ for more information about this number.
polynomial $R_k(x) = 1 - x - 2x^2 - \ldots - 2x^{k+1}$. The polynomials $R_k$ and $T_k$ are reciprocal, so we indeed get that $\omega(C_{2k+1}, \{a, t\}) = 1/\omega_k$.

To prove that $\omega(BS(1, 2k + 1), \{a, t\}) = \omega_k$ we use the following explicit formula from [3], which gives a power series $\Sigma_k(x) = \sum_{m=0}^{\infty} f(m)x^m$ for the growth function $f(m) = f_{BS(1, n), \{a, t\}}(m)$. For the case $n = 2k + 1$ they obtain

\begin{equation}
\Sigma_n(x) = \frac{(1 + x^2 - 2x^{k+2})(1 + x - 2x^{k+2})(1 + x)^2(1 - x)^3}{(1 - x - x^2 - x^3 + 2x^{k+3})^2(1 - 2x - x^2 + 2x^{k+2})}.
\end{equation}

Then the growth rate $\omega(BS(1, 2k + 1), \{a, t\})$ is equal to $1/\alpha$, where $\alpha$ is the smallest positive pole of the function $\Sigma_n(x)$. Since $1 < \omega(BS(1, 2k + 1), \{a, t\}) < 3$, we have bounds $1/3 < \alpha < 1$. We will first prove that $\alpha = \gamma_2$, where $\gamma_2$ is the smallest positive root of the second factor $Q_2(x) = 1 - 2x - x^2 + 2x^{k+2}$ of the denominator of (4.2). Let $\gamma_1$ be the smallest positive root of the first factor $Q_1(x) = 1 - x - x^2 - x^3 + 2x^{k+3}$. Note that $Q_1(0) = Q_2(0) = 1$ and $Q_1(1) = Q_2(1) = 0$, so the numbers $\gamma_1, \gamma_2$ are well defined and $0 < \gamma_1, \gamma_2 \leq 1$.

Since the difference function

$$Q_1(x) - Q_2(x) = x - x^3 + 2x^{k+2} - 2x^{k+3} = x(1 - x^2) + 2x^{k+1}(1 - x)$$

is non-negative on $[0, 1]$, we obtain that $\gamma_1 \geq \gamma_2$.

To show that $\alpha = \gamma_2$ we are left to prove that $\gamma_2$ is not a root of the nominator. The factors $(1 + x)^2$ and $(1 - x)^3$ do not have roots on the interval $I = (1/3, 1)$, and we will check that $P_1(x) = 1 + x^2 - 2x^{k+2}$ and $P_2(x) = 1 + x - 2x^{k+2}$ have no common roots with $Q_2(x)$ on $I$. This is true, since otherwise either $Q_2(x) + P_1(x) = 2 - 2x$ or $Q_2(x) + P_2(x) = (2 + x)(1 - x)$ would have a root on $(1/3, 1)$, which is false.

We can factorize $Q_2(x)$ as $(1 - x)Z(x)$ with $Z(x) = 1 - x - 2x^2 - \ldots - 2x^{k+1}$. Since the polynomial $Z(x)$ is reciprocal to the polynomial $T(x)$ from the statement, the part (b) of Lemma is proved.

(c) Here we use another formula from [3] that is

$$\Sigma_2(x) = \frac{(1 - x)^2(1 + x)^2H(x)}{(1 - x - x^3)(1 - x^2 - 2x^5)^2},$$

where $H(x) = 1 + 3x + 8x^2 + 12x^3 + 16x^4 + 20x^5 + 22x^6 + 16x^7 + 14x^8 + 12x^9 + 4x^{10}$.

We follow the same strategy as in the part (b), and first make sure that the positive root of the polynomial $Q_1(x) = 1 - x - 2x^3$ is smaller than the one of $Q_2(x) = 1 - x^2 - 2x^5$, because $Q_2(x) - Q_1(x) = x(1 - x) + 2x^3(1 - x^2) > 0$ on $(0, 1)$. Then, making tedious computations or using a computer, one gets that $\text{GCD}(H(x), Q_1(x)) = 1$, so the smallest pole of $\Sigma_2(x)$ indeed comes from $Q_1(x)$. Again, $Q_1(x)$ is reciprocal to $x^3 - x^2 - 2$, and the part (c) is also proved.

The next lemma will allow us to compare $\omega_k$ with the growth rate of some free monoid in the proof of Theorem [2]

**Lemma 12.** Let $k \geq 1$ be an integer and $\delta_k$ be the unique positive root of the polynomial $D_k(x) = x^{2k+1} - 2x^{2k} - 2x^{2k-2} - \ldots - 2x^2 - 2$. Then

$$\frac{1 + \sqrt{5}}{2} \leq \omega_k \leq \delta_k < 1 + \sqrt{2}.$$
Proof. The inequality \((1 + \sqrt{3})/2 \leq \omega_k\) may be proven directly, but actually we already know that \(\omega(BS(1, 2k + 1), \{a, t\}) = \omega_k\) and \(\Omega(BS(1, 2k + 1)) \geq (1 + \sqrt{3})/2\) as proved by Mann.

Since \(T_k(1), P_k(1) < 0\) and \(T_k(+\infty) = P_k(+\infty) = +\infty\) we get \(\delta_k, \omega_k > 1\). Consider the polynomials \(D(x) = (x^2 - 1)D_k\) and \(T(x) = (x^2 - 1)T_k = (x + 1)(x - 1)T(x)\). After a simple calculation we get
\[
D(x) = x^{2k+3} - 2x^{2k+2} - x^{2k+1} + 2, \\
T(x) = x^{k+3} - x^{k+2} - 3x^{k+1} - x^k + 2x + 2.
\]

As \((x^2 - 1) > 0\) on \((1, +\infty)\) and \(D(1 + \sqrt{2}) = 2 > 0\), we get that \(\delta_k \in (1, 1 + \sqrt{2})\).

Since \(T(1) = D(1) = 0\) and \(T(1 + \varepsilon), D(1 + \varepsilon) > 0\) for small \(\varepsilon\), in order to show the inequality \(\omega_k \leq \delta_k\) it suffices to show that \(T(x) \geq D(x)\) on the interval \((1, 1 + \sqrt{2})\).

Consider the difference function
\[
D(x) - T(x) = x^{2k+3} - 2x^{2k+2} - x^{2k+1} - x^{k+3} + 3x^{k+1} + x^k - 2x \\
= (x^k - 1)(x^{k+1} - 1)(x^2 - 2x - 1) - (x^2 - 1).
\]

Since the polynomials \(x^k - 1\) and \(x^{k+1} - 1\) are positive on \((1, +\infty)\) and \(x^2 - 2x - 1\) is negative on \((1, 1 + \sqrt{2})\), we indeed have that \(D(x) - T(x) < 0\) on \((1, 1 + \sqrt{2})\), which proves the lemma.

5. Proofs of Theorems 1 and 2

Proof of theorem 2. Let \(G = H \ast_\theta\) be an HNN extension relative to an isomorphism \(\theta : A \to B\) with \(A = H\) and \(B\) a normal subgroup of prime index \(p\) in \(H\). Let \(S\) be any generating set for \(G\). We need to show that \(\omega(G, S) \geq (1 + \sqrt{3})/2\) for \(p = 2\) and \(\omega(G, S) \geq \omega_k\) for \(p = 2k + 1\).

As explained above (see Section 2), the natural surjection \(\varphi : G \to \mathbb{Z}\) ensures the existence of a hyperbolic element in \(S\). Furthermore, upon replacing \(x\) by \(x^{-1}\) we can suppose that \(x\) is a positive element. Since the action of \(G\) is transitive on its \((p + 1)\)-regular Bass-Serre tree, there exists an element in \(S\) not preserving the axis \(L_x\) of \(x\). We distinguish two cases according to this element being elliptic or hyperbolic.

Case 1 (elliptic). There exists an elliptic element \(z \in S\) such that \(z(L_x) \neq L_x\).

For \(p = 2\), we consider the set
\[
M = \{x, zx\},
\]
while for odd primes \(p = 2k + 1\),
\[
M = \{x, zx, z^2x, \ldots, z^kx, z^{-1}x, z^{-2}x, \ldots, z^{-k}x\}.
\]

In either cases, we will show that \(M\) freely generates a free monoid.

Since any vertex has only one direct ascendant, if a vertex is in the fixed point set of \(z\), then all its ascendants are. For the same reason, any two ascending rays meet, so there exists a vertex of the axis of \(x\) which is fixed by \(z\). Let \(v\) be the lowest vertex on \(L_x \cap \text{Fix}(z)\). Then \(x(v)\) is a descendant of \(v\), which is not in the set \(\text{Fix}(z)\), hence the vertices
\[
x(v), zx(v), \text{ for } p = 2,
\]
and by Lemma [4], the vertices
\[ x(v), \, xx(v), \ldots, z^k x(v), \, z^{-1} x(v), \ldots, z^{-k} x(v), \] for odd \( p = 2k + 1 \),
are all distinct leaves of a tree rooted at \( v \), so \( M \) freely generates a free monoid
due to the Ping-Pong Lemma 5. Lemma 10 now implies that \( \omega(G, S) \) is greater or equal to
the unique positive root of
\[ z^2 - 2z - 1, \] for \( p = 2 \),
which is precisely the golden ratio \((1 + \sqrt{5})/2\), while for \( p = 2k + 1 \), it is greater or
equal to the unique positive root of
\[ T_k(z) = z^{k+1} - z^k - 2z^{k-1} - \ldots - 2z - 2, \]
which is \( \omega_k \) by definition.

**Case 2 (Hyperbolic).** There exists a hyperbolic element \( y \in S \) such that \( y(L_x) \neq L_x \).
Upon replacing \( y \) by its inverse, we can suppose that \( y \) is a positive hyperbolic.
Since \( y \) preserves its axis \( L_y \), this implies that the axes \( L_x \) and \( L_y \) are different.
This already implies that \( \omega(BS(1, p), S) \geq 2 \) (see [1, Lemma] or Lemma 10 with \( \ell_1 = \ell_2 = 1 \)).
Since for \( p = 2, 3 \) we have
\[ \omega(BS(1, 2), \{a, t\}) < \omega(BS(1, 3), \{a, t\}) = 2, \]
we can suppose that \( p \geq 5 \), and again \( p = 2k + 1 \).

We consider four subcases, according to the situations when
A. \( \ell(x) = \ell(y) \), B. \( 2\ell(y) < \ell(x) \), C. \( \ell(x) = 2\ell(y) \) and D. \( \ell(y) < \ell(x) < 2\ell(y) \).

**Case 2A.** \( \ell(x) = \ell(y) \). Note that the element \( xy^{-1} \) is elliptic and \( xy^{-1}(L_x) \neq L_x \).
We can apply the claim of **Case 1** to \( x \) and \( z = xy^{-1} \) to conclude that the set
\[ \{x, y, xy^{-1} y, \ldots, (xy^{-1})^{k-1} y, xy^{-1} x, \ldots, (xy^{-1})^k x\} \]
freely generates a free monoid. Then Lemma 10 shows that \( \omega(BS(1, 2k+1), S) \geq \delta_k \),
where \( \delta_k \) is the single positive root of the polynomial \( D_k(x) = x^{2k+1} - 2 \sum_{m=0}^{k} x^{2m} \).
Finally, Lemma 12 gives the desired inequality \( \omega(BS(1, 2k + 1)) \geq \delta_k \geq \omega_k \).

We can now suppose that \( \ell(y) < \ell(x) \) and distinguish three further subcases:

**Case 2B.** \( 2\ell(y) < \ell(x) \)
We will show that the infinite family
\[ \{y^{-2} x, y^{-2} x, x, xy, y^2 x, \ldots, y^s x, \ldots, y x^{-1} y, y^2 x^{-1} y x, \ldots, y^s x^{-1} y x, \ldots\} \]
which is maybe better described as
\[ \{y^s x \mid s \geq -2\} \cup \{y^s x^{-1} y x \mid s \geq 1\} \]
freely generates a free monoid. Then, taking as free generators only the \( 2k + 1 \) elements
\[ x, xy, y^2 x, \ldots, y^k x, y^{-1} x, y^{-2} x, y x^{-1} y, y^2 x^{-1} y x, \ldots, y^{k-2} x^{-1} y x \]
we get that \( \omega(G, S) \) is by Lemma 10 greater or equal to the unique positive root of
\[ T_k(z) = z^{k+1} - z^k - 2z^{k-1} - \ldots - 2z - 2, \]
which is \( \omega_k \) by definition.

To prove that the above infinite family freely generates a monoid, let \( v_0 \) be the
lowest vertex on \( L_x \cap L_y \) and let \( v_x \in L_x \) and \( v_y \in L_y \) be the corresponding direct
descendants of \( v_0 \). We aim at applying the Ping-Pong Lemma 5 to the vertex
\( w = x^{-1}(v_x) \), see Figure 5.1.
First notice that since \( v_x \notin L_y \), the translates \( y^s x(w) = y^s(v_x) \) are all distinct, branching from \( L_y \) at \( y^s(v_0) \). Furthermore, for \(-2 \leq s \leq 0\), the highest such translate is \( y^{-2}x(w) = y^{-2}(v_x) \) which is strictly below \( y^{-2}(v_0) \) by construction. Now \( w = x^{-1}(v_x) \) is equal or above \( y^{-2}(v_0) \) since \( 2\ell(y) < \ell(x) \). This already implies that the infinite subfamily \( \{ y^s x \mid -2 \leq s \} \) freely generates a free monoid.

Second consider the vertex \( y(v_x) \). It is branching from \( L_x \) at \( v \) and the first vertex from \( L_x \cap L_y \) to \( y(v_x) \) is \( v_y \). It follows that \( x^{-1}y(v_x) \) does not belong to \( L_x \) either and is branching at \( x^{-1}(v) \) from \( L_x \) and hence also from \( L_y \). It follows that all the translates \( y^s x^{-1} yx(w) = y^s x^{-1} y(v_x) \) belong to different branches of \( L_y \), branching at \( y^s x^{-1}(v_0) \). Since \( \ell(y) \geq 1 \), for \( 1 \leq s \) the branch points are below or equal to \( w = x^{-1}(v_x) \).

If \( \ell(x) \) is not a multiple of \( \ell(y) \) the two families of branching points are different and we are done. If \( \ell(x) = m\ell(y) \) for some \( m > 2 \) we need to check that \( y^{n+m}x^{-1}(v_y) \neq y^n v_x \) and it is enough to check it for \( n = 0 \). Consider the elliptic element \( y^m x^{-1} \). It fixes \( v_0 \), sends \( v_x \) to \( v_y \) and \( v_y \) to \( y^m x^{-1}(v_x) \) which cannot be equal to \( v_x \) otherwise the action on the direct descendants of \( v_0 \) of the elliptic element \( y^m x^{-1} \) would not be transitive, contradicting Lemma 4.

**Case 2c.** \( \ell(x) = 2\ell(y) \).

It is enough to show that the set
\[
\{ x, y, xy^{-1}x, xy^{-2}x, xy^{-1}y^{-1}x, y^2 x^{-1} y, y x y^{-1} y \}
\]
freely generates a free monoid. Then, using Lemma 10 we get that \( \omega(BS(1, k)) \) is at least \( \gamma \), where \( \gamma \) is the root of the polynomial \( F(x) = x^5 - 2x^4 - x^2 - 3x - 1 \). Since \( F(x) = (x - 2)(x + 1)(x^3 + x + 1) \), we get that \( \gamma = 1 + \sqrt{7} \), and again Lemma 12 gives the desired inequality \( \omega(G, S) \geq \omega_k \).

Let as above \( v \) be the lowest vertex on \( L_x \cap L_y \). We aim at applying the Ping-Pong Lemma 5 to the vertex \( v \). Let \( v_x \in L_x \) and \( v_y \in L_y \) be the corresponding direct descendants of \( v_0 \).

The elliptic transformation \( b = y^2 x^{-1} \) fixes \( v \) and takes \( v_x \) to \( v_y \). Thus its action on the direct descendants of \( v \) is nontrivial and hence transitive. Since we assume \( p \geq 4 \), it follows by Lemma 4 that the image \( v_{+} = y^2 x^{-1}(v_y) \) of \( v_y \) and the preimage \( v_{-} := y^{-2}(v_x) \) of \( v_x \) give four distinct direct descendants of \( v_0 \) as depicted in Figure 5.2.

Observe that \( y^2 x^{-1} y(v) \) is on the branch through \( v \) and \( v_{+} \), while \( xy^{-2} x(v) \) is on the branch through \( v_0 \) and \( v_{-} \). Thus the four elements \( x v, y v, xy^{-2} x(v) \) and \( y^2 x^{-1} y(v) \) have distinct geodesics to \( v \).
We now forget about $xy^{-2}x(v)$ and look at the image of the tree rooted at $v$ of the three remaining elements through the hyperbolic transformation $xy^{-1}$. The root $v$ is mapped on the segment from $v$ to $x(v)$. The vertex $y(v)$ is mapped to $x(v)$, and the two remaining leaves are sent to vertices branching from $L_x$ at $xy^{-1}(v)$.

Iterating this procedure but only on $xy^{-1}(v), x(v)$ and $xy^{-1}x(v)$ shows that $xy^{-1}$ is branching from the segment between $xy^{-1}(v)$ and $xy^{-1}x(v)$. We have thus proven that the seven vertices are leaves of a tree rooted at $v$, as illustrated in Figure 5.3, which finishes the proof of this case.

**Case 2d.** $ℓ(y) < ℓ(x) < 2ℓ(y)$.

We will show that the set

$$\{x, y, xy^{-1}x, xy^{-2}x, yx^{-1}y\}$$

freely generates a free monoid. Since the corresponding polynomial $x^4 - 2x^3 - 2x - 1 = x(x^2 + 1)(x^2 - 2x - 1)$ has only one positive root $1 + \sqrt{2}$, this will prove this case.

Set $a = ℓ(x)$ and $b = ℓ(y)$. The proof decomposes in the two cases $b < a \leq (3/2)b$ and $(3/2)b \leq a < 2b$ with an additional small argument needed in the equality case.

In case $b < a \leq (3/2)b$ we aim at applying the Ping-Pong Lemma 5 to the vertex $w = xy^{-2}(v)$. (See Figure 5.4.) This vertex is on the intersection of the axes $L_x \cap L_y$ at distance $2b - a$ above $v$. Of the five images of $w$, only $x(w)$ is on the axis $L_x$, at distance $a$ below $w$ and hence $2(a - b)$ below $v$. The four other images are not in $L_x$ and we will determine their projection on $L_x$.

The image $y(w)$ is on the axis $L_y$ at distance $b$ below $w$ and hence at distance $a - b$ from its projection $v \in L_x$. Since the axis of the hyperbolic transformation $xy^{-2}$ contains $L_x \cap L_y$ and at least the vertex $v_y \in L_y$, the segment $[v, x(w)]$, which
intersects $L_{xy^{-2}}$ at least at the vertex $v$ and hence at distance $a - b$ from both $v$ and $x(v)$. Finally, the axis of $yx^{-1}$ contains $L_x \cap L_y$ and at least the vertex $v_y \in L_y$, so that the hyperbolic transformation $yx^{-1}$ takes the segment $[v, x(v)]$ to the segment $[xy^{-1}(v), xy^{-1}x(v)]$ which intersects $L_{xy^{-1}}$ and hence $L_x$ precisely in $xy^{-1}(v)$ which is at distance $a - b$ from $v$ and $x(v)$. The ordered pair $(v, x(v)) = (a, b)$ sends the ordered pair $(b, a)$ to $(3b - 2a, 0)$ below $w$. If the inequality is strict, the claim immediately follows from the Ping-Pong Lemma 5. If $3b - 2a = 0$, we will see below how to show that the segments $[yx^{-1}(v), yx^{-1}y(w)]$ and $[w, xy^{-2}x(w)]$ only intersect at $w = yx^{-1}(v)$.

If $(3/2)b \leq a < 2b$ the argument is completely analogous, except that the vertex $yx^{-1}(v)$ is above or equal to $w = xy^{-2}(v)$. Thus we want to replace $w$ by $w' := yx^{-1}(v)$ and apply the Ping-Pong Lemma 5 to this vertex $w'$. (See Figure 5.5. This vertex is on the intersection of the axes $L_x \cap L_y$ at distance $a - b$ above $v$. Of the five images of $w'$, only $x(w')$ is on the axis $L_x$, at distance $a$ below $w$ and hence below $v$. The four other images are not in $L_x$ and we will determine their projection on $L_x$.

The image $y(w')$ is on the axis $L_y$ at distance $b$ below $w$ and hence at distance $2b - a$ from its projection $v \in L_x$. For the three other image points, the proof is identical to the above case, replacing $w$ by $w'$.

In the equality case the two vertices $w = w'$ agree. Let $v_1$, respectively $v_2$ be the first vertex after $w$ on the geodesic to $xy^{-2}(w)$, respectively $yx^{-1}y(w)$. We need to show that $v_1 \neq v_2$. Let $v_a$ be the direct descendant of $w$ on the geodesic to $v$. The ordered pair $(v_1, v_a)$ is mapped to $(v_a, v_y)$ by $y^2 x^{-1}$, which are further mapped to $(v_a, v_2)$ by $yx^{-1}$. Thus the elliptic element $yx^{-1}y^2 x^{-1}$ sends the ordered pair $(v_1, v_a)$ to $(v_a, v_2)$ and since $p \geq 3$ and elliptic elements act either trivially or
transitively on direct descendants of a fixed point by Lemma 4, it follows that $v_1 \neq v_2$, which finishes the proof of this case and of the theorem.

**Proof of theorem** In view of Lemma 11, Theorem 1 follows immediately from Theorem 2 except in the case of $BS(1, 2)$ where we need a better understanding of its action on the Bass-Serre to obtain the accurate lower bound of $\omega(BS(1, 2), \{a, t\}) = \beta$, where $\beta$ is the unique real root of $x^3 - x^2 - 2$.

Let $S$ be a generating set for $BS(1, 2)$. As in the proof of Theorem 2, the case where $S$ contains two hyperbolic elements with different axes immediately gives the lower bound of $\omega(BS(1, 2), S) \geq 2 > \beta$. We thus only have to treat the corresponding elliptic case, that is, there exists a positive hyperbolic element $x \in S$ with axis $L_x$ and an elliptic element $z \in S$ such that $z(L_x) \neq L_x$.

As observed in the elliptic case of the proof of Theorem 2, the intersection of $L_x$ with the fixed point set of $z$ is nonempty. Upon conjugating the generating set $S$, we can suppose that the lowest vertex on $L_x$ fixed by $z$ is $A$, which implies that $z$ belongs to $A$. Since $z$ does not fix the direct descendants $tA$ and $atA$ it must be an odd power of $A$.

Consider the action of $a$ on the second generation of descendants of $A$, that is $t^2A, tatA, at^2A$ and $atatA$. The action has order four, mapping $t^2A \mapsto at^2A \mapsto a^2t^2A = tatA \mapsto atatA \mapsto a^2tatA = t^2A$. The action of $z$, as an odd power of $A$ is thus necessarily equal to the action of $a$ or $a^{-1}$ on these second generation descendants. It follows that $xA, zxA$ and $z^{-1}xA$ are leaves of a tree rooted at $A$, and hence $x, zx^2, z^{-1}x^2$ generate a free monoid by the Ping-Pong Lemma 5. Since these elements have lengths 1, 3, and 3 respectively, we can invoke 10 to conclude that the grow rate of $BS(1, 2)$ with respect to $S$ is greater or equal to the greatest and unique real root of $x^3 - x^2 - 2$. Finally, Lemma 11 gives

$$\omega(BS(1, 2), S) \geq \omega(BS(1, 2), \{a, t\}),$$

which finishes the proof of the theorem.

The next lemma will be needed to prove Corollary 3.
Lemma 13. The limit \( \lim_{k \to \infty} \omega_k = 1 + \sqrt{2} \) exists.

Proof. From Lemma \[12\] and the definition of \( \omega_k \) we know that \( \omega_k \) is a single positive root of the polynomial \( Z_k(x) \), and \((1 + \sqrt{2})/2 < \omega_k < 1 + \sqrt{2} \) for every \( k \geq 1 \). Then the reciprocal polynomial \( R_k(x) = 1 - x - 2x^2 - \ldots - 2x^k - 2x^{k+1} \) has a single positive root \( 1/\omega_k \) which belongs to the interval \( I = (1/3, 2/3) \). Consequently the polynomial

\[
R_k'(x) = (1 - x)R_k = (1 - x)^2 - 2x^2(1 - x^k) = 1 - 2x - x^2 + 2x^{k+2}
\]

also has two positive roots: 1 and \( 1/\omega_k \). Obviously, for \( k \to \infty \) the polynomials \( 2x^{k+2} \) uniformly converge to the zero function on the enlarged interval \( I' = (1/4, 3/4) \). For this reason the roots \( 1/\omega_k \) of \( R_k'(x) \) on \( I' \) converge to the root of the polynomial \( 1 - 2x - x^2 \) on \( I \), and the latter root is equal to \( \sqrt{2} - 1 = 1/(1 + \sqrt{2}) \), which proves the lemma.

Proof of Corollary \[3\]. We use Parry’s formula \[32\] to compute the series \( \Sigma(x) \) for the growth function \( \mathbb{Z} \wr \mathbb{Z} \) with respect to the generating set \( \{a, t\} \):

\[
\Sigma(x) = \frac{(1 - x^2)^2(1 + x^2)}{(1 - x^2 - x^3)(1 - 2x - x^2)} = \frac{(1 + x)^2(1 - x^2)^2(1 + x^2)}{(1 - x^2)^2(1 - 2x - x^2)}.
\]

All the roots of the nominator and the denominator lie on the unit circle except for the roots of \( 1 - 2x - x^2 \). The reciprocal of the smallest root is equal to \( \sqrt{2} + 1 \), hence this is the value for \( \omega(\mathbb{Z} \wr \mathbb{Z}, \{a, t\}) \).

Now we will show that \( \Omega(\mathbb{Z} \wr \mathbb{Z}) = 1 + \sqrt{2} \). We already know that \( \Omega(\mathbb{Z} \wr \mathbb{Z}) \leq 1 + \sqrt{2} \). Suppose that \( \omega(\mathbb{Z} \wr \mathbb{Z}) = 1 + \sqrt{2} - \varepsilon \), where \( \varepsilon > 0 \). As any group \( L_p \) is a factor group of the group \( \mathbb{Z} \wr \mathbb{Z} \), then for any prime \( p \) we have \( \omega(L_p) \leq 1 + \sqrt{2} - \varepsilon \) which contradicts the limit equality \( \lim_{p \to \infty} \omega(L_p) = \lim_{k \to \infty} \omega_k = 1 + \sqrt{2} \) proven in Lemma \[13\].

References

[1] M. Bucher, P. de la Harpe, Free products with amalgamation and HNN extensions of uniformly exponential growth, Math. Notes 67 (2000) 686–689, translated from Mat. Zametki 67 (2000) 811–815.
[2] M. Bucher, A. Talambutsa Exponential growth rates of free and amalgamated products, to be published in Israel Journal of Mathematics, 2015.
[3] D.J. Collins, M. Edjvet, C.P. Gill Growth series for the group \( \langle x, y \mid x^{-1}yx = y' \rangle \), Arch. Math., 1994, Vol.62, pp.1–11.
[4] M. Gromov Structures metriques pour les varietes rimanniennes, 1981, CEDIC,Paris.
[5] R. Grigorchuk, P. de la Harpe - On problems related to growth, entropy and spectrum in group theory, Journal of dynamical and control systems, 1997, Vol. 3, No. 1, pp.51–89.
[6] P. de la Harpe, Topics in Geometric Group Theory, The University of Chicago Press, Chicago, 2000.
[7] R. Lyons, R. Pemantle and Y. Peres Random Walks on the Lamplighter Group The Annals of Probability, Vol. 24, No. 4 (Oct., 1996), pp. 1993-2006.
[8] A. Mann, The growth of free products, Journal of Algebra 326, no. 1 (2011) 208–217.
[9] W. Parry Growth series of some wreath products, Transactions of the AMS, 1992, Vol. 331, N.2, pp.751–759.
[10] A. Sambusetti Growth tightness of free and amalgamated products, Ann. Scient. Éc. Norm. Sup.(4), Vol. 235, pp. 477–488, 2002.
[11] A.G. Shukhov On the dependence of the growth rate on the length of the defining relator, Mathematical Notes, 1999, Vol.65(4), pp.510–555.
[12] A. Talambutsa, Attainability of the minimal exponential growth rate for free products of finite cyclic groups, Proc. of the Steklov Mathematical Institute Vol. 274, No. 1 (2011), pp. 289–302.
[13] Y. Stalder and A. Valette, Wreath products with the integers, proper actions and Hilbert space compression, Geom. Dedicata, 124 (2007), pp. 199–211.

[14] J.S. Wilson, On exponential growth and uniformly exponential growth for groups, Invent. Math. 155 (2004) 287–303.