IDEAL GAMES AND RAMSEY SETS

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Abstract. It is shown that Matet’s characterization of the Ramsey property relative to a selective co-ideal \( H \), in terms of games of Kastanas, still holds if we consider semiselectivity instead of selectivity. Moreover, we prove that a co-ideal \( H \) is semiselective if and only if Matet’s game-theoretic characterization of the \( H \)-Ramsey property holds. This lifts Kastanas’s characterization of the classical Ramsey property to its optimal setting, from the point of view of the local Ramsey theory, and gives a game-theoretic counterpart to a theorem of Farah, asserting that a co-ideal \( H \) is semiselective if and only if the family of \( H \)-Ramsey subsets of \( \mathbb{N}^{[\infty]} \) coincides with the family of those sets having the abstract \( H \)-Baire property. Finally, we show that under suitable assumptions, for every semiselective co-ideal \( H \) all sets of real numbers are \( H \)-Ramsey.

1. Introduction

Let \( \mathbb{N} \) be the set of nonnegative integers. Given an infinite set \( A \subseteq \mathbb{N} \), the symbol \( A^{[\infty]} \) (resp. \( A^{[<\infty]} \)) represents the collection of the infinite (resp. finite) subsets of \( A \). Let \( A^{[n]} \) denote the set of all the subsets of \( A \) with \( n \) elements. If \( a \in \mathbb{N}^{[<\infty]} \) is an initial segment of \( A \in \mathbb{N}^{[\infty]} \), then we write \( a \sqsubseteq A \). Also, let \( A/a := \{ n \in A : \max(a) < n \} \) and write \( A/n \) to mean \( A/\{ n \} \).

Let \((P,\leq)\) be a poset. A subset \( D \subseteq P \) is dense in \( P \) if for every \( p \in P \), there is \( q \in D \) with \( q \leq p \). The subset \( D \subseteq P \) is open if \( p \in D \) and \( q \leq p \) imply \( q \in D \). We say \( P \) is \( \sigma \)-distributive if the intersection of countably many dense open subsets of \( P \) is dense. \( P \) is \( \sigma \)-closed if every decreasing sequence of elements of \( P \) has a lower bound.

**Definition 1.1.** A family \( \mathcal{H} \subseteq \wp(\mathbb{N}) \) is a co-ideal if it satisfies the following:

(i) \( A \subseteq B \) and \( A \in \mathcal{H} \) implies \( B \in \mathcal{H} \) and 
(ii) \( A \cup B \in \mathcal{H} \) implies \( A \in \mathcal{H} \) or \( B \in \mathcal{H} \).

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The complement \( I = \varphi(N) \setminus H \) is the **dual ideal** of \( H \). We will suppose that co-ideals differ from \( \varphi(N) \). Also, we say that a nonempty family \( F \subset H \) is **\( H \)-disjoint** if for every \( A, B \in F \), \( A \cap B \notin H \). We say that \( F \) is a **maximal \( H \)-disjoint family** if it is \( H \)-disjoint and it is not properly contained in any other \( H \)-disjoint family.

A subset \( X \) of \( \mathbb{N}^{[\omega]} \) is **Ramsey** if for every \( [a, A] \neq \emptyset \) with \( A \in \mathbb{N}^{[\omega]} \) there exists \( B \in [a, A] \) such that \( [a, B] \subseteq X \) or \( [a, B] \cap X = \emptyset \). Some authors have used the term “completely Ramsey” to express this property, reserving the term “Ramsey” for a weaker property. Galvin and Prikry [4] showed that all Borel subsets of \( \mathbb{N}^{[\omega]} \) are Ramsey, and Silver [12] extended this to all analytic sets. Mathias in [10] showed that if the existence of an inaccessible cardinal is consistent with \( \text{ZF}^+ \), then it is consistent with \( \text{ZF} + \text{DC} \), that every subset of \( \mathbb{N}^{[\omega]} \) is Ramsey. Mathias introduced the concept of a selective co-ideal (or a happy family), which has turned out to be of wide interest. Ellentuck [2] characterized the Ramsey sets as those having the Baire property with respect to the exponential topology of \( \mathbb{N}^{[\omega]} \).

A game-theoretic characterization of the Ramsey property was given by Kastanas in [6], using games in the style of Banach-Mazur with respect to Ellentuck’s topology.

In this work we will deal with a game-theoretic characterization of the following property:

**Definition 1.2.** Let \( H \subset \mathbb{N}^{[\omega]} \) be a co-ideal. \( X \subset \mathbb{N}^{[\omega]} \) is **\( H \)-Ramsey** if for every \( [a, A] \neq \emptyset \) with \( A \in H \) there exists \( B \in [a, A] \cap H \) such that \( [a, B] \subseteq X \) or \( [a, B] \cap X = \emptyset \). We say \( X \) is **\( H \)-Ramsey null** if for every \( [a, A] \neq \emptyset \) with \( A \in H \) there exists \( B \in [a, A] \cap H \) such that \( [a, B] \cap X = \emptyset \).

Mathias considered sets that are \( H \)-Ramsey with respect to a selective co-ideal \( H \) and generalized Silver’s result to this context. Matet [9] used games to characterize sets which are Ramsey with respect to a selective co-ideal \( H \). These games coincide with the games of Kastanas if \( H \) is \( \mathbb{N}^{[\omega]} \) and with the games of Louveau [5] if \( H \) is a Ramsey ultrafilter.

Given a co-ideal \( H \subset \mathbb{N}^{[\omega]} \), the collection \( \{ [a, A] : (a, A) \in \mathbb{N}^{<\omega} \times H \} \) is not, in general, a basis for a topology on \( \mathbb{N}^{[\omega]} \), but the following abstract version of the Baire property and related concepts will be useful.

**Definition 1.3.** Let \( H \subset \mathbb{N}^{[\omega]} \) be a co-ideal. \( X \subset \mathbb{N}^{[\omega]} \) has the abstract **\( H \)-Baire property** if for every \( [a, A] \neq \emptyset \) with \( A \in H \) there exists \( [b, B] \subseteq [a, A] \) with \( B \in H \) such that \( [b, B] \subseteq X \) or \( [b, B] \cap X = \emptyset \). We say \( X \) is **\( H \)-nowhere dense** if for every \( [a, A] \neq \emptyset \) with \( A \in H \) there exists \( [b, B] \subseteq [a, A] \) with \( B \in H \) such that \( [b, B] \cap X = \emptyset \). Also, \( X \) is **\( H \)-meager** if it is the union of countably many \( H \)-nowhere dense sets.

Given a decreasing sequence \( A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \) of infinite subsets of \( \mathbb{N} \), a set \( B \) is a **diagonalization** of the sequence (or \( B \) **diagonalizes** the sequence) if and only if \( B/n \subseteq A_n \) for each \( n \in B \). A co-ideal \( H \) is **selective** if and only if every decreasing sequence in \( H \) has a diagonalization in \( H \).

A co-ideal \( H \) has the **\( Q^+ \)-property** if for every \( A \in H \) and every partition \( (F_n)_n \) of \( A \) into finite sets, there is \( S \in H \) such that \( S \subseteq A \) and \( |S \cap F_n| \leq 1 \) for every \( n \in \mathbb{N} \).

**Proposition 1.4** ([10]). A co-ideal \( H \) is selective if and only if the poset \( (H, \subseteq^\ast) \) is \( \sigma \)-closed and \( H \) has the **\( Q^+ \)-property**.
Given a co-ideal $\mathcal{H}$, recall that a set $D \subseteq \mathcal{H}$ is dense open in the ordering $(\mathcal{H}, \subseteq)$ if (a) for every $A \in \mathcal{H}$ there exists $B \in D$ such that $B \subseteq A$ and (b) for every $A, B \in \mathcal{H}$, if $B \subseteq A$ and $A \in D$, then $B \in D$. Please notice that we will also consider the ordering $(\mathcal{H}, \subseteq^*)$, where $A \subseteq^* B$ if and only if $A \setminus B$ is a finite set, but any reference to “dense open” in this paper will be only with respect to the ordering $(\mathcal{H}, \subseteq)$.

Given a sequence $\{D_n\}_{n \in \mathbb{N}}$ of dense open sets in $(\mathcal{H}, \subseteq)$, a set $B$ is a diagonalization of $\{D_n\}_{n \in \mathbb{N}}$ if and only if $B/n \in D_n$ for every $n \in B$. A co-ideal $\mathcal{H}$ is semiselective if for every sequence $\{D_n\}_{n \in \mathbb{N}}$ of dense open subsets of $\mathcal{H}$, the family of its diagonalizations is dense in $(\mathcal{H}, \subseteq)$.

**Proposition 1.5 (3).** A co-ideal $\mathcal{H}$ is semiselective if and only if the poset $(\mathcal{H}, \subseteq^*)$ is $\sigma$-distributive and $\mathcal{H}$ has the $Q^+$-property.

Since $\sigma$-closedness implies $\sigma$-distributivity, then semiselectivity follows from selectivity, but the converse does not hold (see [3] for an example).

In Section 2 we list results of Ellentuck, Mathias and Farah that characterize topologically the Ramsey property and the local Ramsey property. In Section 3 we define a family of games and present the main result, which states that a co-ideal $\mathcal{H}$ is semiselective if and only if the $\mathcal{H}$-Ramsey sets are exactly those for which the associated games are determined. This generalizes results of Kastanas [6] and Matet [9]. The proof is given in Section 4. In Section 5 we show that in Solovay’s model, for every semiselective co-ideal $\mathcal{H}$ all sets of real numbers from $L(\mathbb{R})$ are $\mathcal{H}$-Ramsey.

### 2. Topological characterization of the Ramsey property

The following are the main results concerning the characterization of the Ramsey property and the local Ramsey property in topological terms.

**Theorem 2.1** (Ellentuck). Let $\mathcal{X} \subseteq \mathbb{N}[\omega]$ be given.

(i) $\mathcal{X}$ is Ramsey if and only if $\mathcal{X}$ has the Baire property with respect to Ellentuck’s topology.

(ii) $\mathcal{X}$ is Ramsey null if and only if $\mathcal{X}$ is meager with respect to Ellentuck’s topology.

**Theorem 2.2** (Mathias). Let $\mathcal{X} \subseteq \mathbb{N}[\omega]$ and let a selective co-ideal $\mathcal{H}$ be given.

(i) $\mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if $\mathcal{X}$ has the abstract $\mathcal{H}$-Baire property.

(ii) $\mathcal{X}$ is $\mathcal{H}$-Ramsey null if and only if $\mathcal{X}$ is $\mathcal{H}$-meager.

**Theorem 2.3** (Farah, Todorcevic). Let $\mathcal{H}$ be a co-ideal. The following are equivalent:

(i) $\mathcal{H}$ is semiselective.

(ii) The $\mathcal{H}$-Ramsey subsets of $\mathbb{N}[\omega]$ are exactly those sets having the abstract $\mathcal{H}$-Baire property, and the following three families of subsets of $\mathbb{N}[\omega]$ coincide and are $\sigma$-ideals:

(a) $\mathcal{H}$-Ramsey null sets,

(b) $\mathcal{H}$-nowhere dense sets, and

(c) $\mathcal{H}$-meager sets.
In the next section we state results by Kastanas [6] and Matet [9] (Theorems 3.1 and 3.2 below) which are the game-theoretic counterparts of Theorems 2.1 and 2.2, respectively, and we also present our main result (Theorem 3.3 below), which is the game-theoretic counterpart of Theorem 2.3.

3. Characterizing the Ramsey Property with Games

The following is a relativized version of a game due to Kastanas [6], employed to obtain a characterization of the family of completely Ramsey sets (i.e. $\mathcal{H}$-Ramsey for $\mathcal{H} = \mathbb{N}[\leq \omega]$). The same game was used by Matet in [9] to obtain the analog result when $\mathcal{H}$ is selective.

Let $\mathcal{H} \subseteq \mathbb{N}[\leq \omega]$ be a fixed co-ideal. For each $\mathcal{X} \subseteq \mathbb{N}[\leq \omega], A \in \mathcal{H}$ and $a \in \mathbb{N}[\leq \omega]$ we define a two-player game $G_\mathcal{H}(a, A, \mathcal{X})$ as follows: player I chooses an element $A_0 \in \mathcal{H} \upharpoonright A$: II answers by playing $n_0 \in A_0$ such that $a \subseteq n_0$, and $B_0 \in \mathcal{H} \cap (A_0/n_0)[\leq \omega]$; then I chooses $A_1 \in \mathcal{H} \cap B_0[\leq \omega]$; II answers by playing $n_1 \in A_1$ and $B_1 \in \mathcal{H} \cap (A_1/n_1)[\leq \omega]$; and so on. Player I wins if and only if for every $a \cup \{n_j : j \in \mathbb{N}\} \in \mathcal{X}$:

\[
\begin{array}{cccc}
I & A_0 & A_1 & \cdots & A_k & \cdots \\
II & n_0, B_0 & n_1, B_1 & \cdots & n_k, B_k & \cdots \\
\end{array}
\]

A strategy for a player is a rule that tells him (or her) what to play based on the previous moves. A strategy is a winning strategy for player I if player I wins the game whenever he (or she) follows the strategy, no matter what player II plays. Analogously, it can be defined what a winning strategy for player II is. The precise definitions of strategy for two-player games can be found in [7,11].

Let $s = \{s_0, \ldots, s_k\}$ be a nonempty finite subset of $\mathbb{N}$, written in its increasing order, and let $\overrightarrow{B} = \{B_0, \ldots, B_k\}$ be a sequence of elements of $\mathcal{H}$. We say that the pair $(s, \overrightarrow{B})$ is a legal position for player II if $(s_0, B_0), \ldots, (s_k, B_k)$ is a sequence of possible consecutive moves of II in the game $G_\mathcal{H}(a, A, \mathcal{X})$, respecting the rules. In this case, if $\sigma$ is a winning strategy for player I in the game, we say that $\sigma(s, \overrightarrow{B})$ is a realizable move of player I according to $\sigma$. Notice that if $r \in B_k/s_k$ and $C \in \mathcal{H} \upharpoonright B_k/s_k$, then $(s_0, B_0), \ldots, (s_k, B_k), (r, C)$ is also a sequence of possible consecutive moves of II in the game. We will sometimes use the notation $(s, \overrightarrow{B}, r, C)$ and say that $(s, \overrightarrow{B}, r, C)$ is a legal position for player II and $\sigma(s, \overrightarrow{B}, r, C)$ is a realizable move of player I according to $\sigma$.

We say that the game $G_\mathcal{H}(a, A, \mathcal{X})$ is determined if one of the players has a winning strategy.

**Theorem 3.1** (Kastanas). $\mathcal{X}$ is Ramsey if and only if for every $A \in \mathbb{N}[\leq \omega]$ and $a \in \mathbb{N}[\leq \omega]$ the game $G_{\mathbb{N}[\leq \omega]}(a, A, \mathcal{X})$ is determined.

**Theorem 3.2** (Matet). Let $\mathcal{H}$ be a selective co-ideal. $\mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if for every $A \in \mathcal{H}$ and $a \in \mathbb{N}[\leq \omega]$ the game $G_\mathcal{H}(a, A, \mathcal{X})$ is determined.

Now we state our main result:

**Theorem 3.3.** Let $\mathcal{H}$ be a co-ideal. The following are equivalent:

1. $\mathcal{H}$ is semiselective.
2. $\forall \mathcal{X} \subseteq \mathbb{N}[\leq \omega], \mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if for every $A \in \mathcal{H}$ and $a \in \mathbb{N}[\leq \omega]$ the game $G_\mathcal{H}(a, A, \mathcal{X})$ is determined.
So Theorem 3.3 is a game-theoretic counterpart to Theorem 2.3 in the previous section, in the sense that it gives us a game-theoretic characterization of semiselectivity. Obviously, it also gives us a characterization of the $\mathcal{H}$-Ramsey property, for semiselective $\mathcal{H}$, which generalizes the main results of Kastanas in [6] and Matet in [9] (Theorems 3.1 and 3.2 above).

It is known that every analytic set is $\mathcal{H}$-Ramsey for $\mathcal{H}$ semiselective (see Theorem 2.2 in [3] or Lemma 7.18 in [11]). Assuming $AD_\mathbb{R}$, i.e., the axiom of determinacy for games over the reals (see [7] or [11]), we obtain the following from Theorem 3.3:

**Corollary 3.4.** Assume $AD_\mathbb{R}$. If $\mathcal{H}$ is a semiselective co-ideal, then every subset of $\mathbb{N}^{[\infty]}$ is $\mathcal{H}$-Ramsey.

## 4. Proof of the Main Result

Throughout the rest of this section, fix a semiselective co-ideal $\mathcal{H}$. Before proving Theorem 3.3 in Propositions 4.1 and 4.2 below we will deal with winning strategies of players in a game $G_\mathcal{H}(a, A, X)$.

**Proposition 4.1.** For every $X \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$, I has a winning strategy in $G_\mathcal{H}(a, A, X)$ if and only if there exists $E \in \mathcal{H} \upharpoonright A$ such that $[a, E] \subseteq X$.

**Proof.** Suppose $\sigma$ is a winning strategy for I. We will suppose that $a = \emptyset$ and $A = \mathbb{N}$ without loss of generality.

Let $A_0 = \sigma(\emptyset)$ be the first move of I using $\sigma$. We will define a tree $T$ of finite subsets of $A_0$, and for each $s \in T$ we will also define a family $M_s \subseteq A_0^{[\infty]}$ and a family $N_s \subseteq (A_0^{[\infty]})^{[s]}$, where $|s|$ is the length of $s$. Put $\{p\} \in T$ for each $p \in A_0$ and let

$$M_{\{p\}} \subseteq \{\sigma(p, B) : B \in \mathcal{H} \upharpoonright A_0\}$$

be a maximal $\mathcal{H}$-disjoint family (see paragraph after Definition 1.1), and set

$$N_{\{p\}} = \{\{B\} : \sigma(p, B) \in M_{\{p\}}\}.$$

Suppose we have defined $T \cap A_0^{[n]}$ and we have chosen a maximal $\mathcal{H}$-disjoint family $M_s$ of realizable moves of player I of the form $\sigma(s, \vec{B})$ for every $s \in T \cap A_0^{[n]}$. Let

$$N_s = \{\vec{B} : \sigma(s, \vec{B}) \in M_s\}.$$

Given $s \in T \cap A_0^{[n]}$, $\vec{B} \in N_s$ and $r \in \sigma(s, \vec{B})/s$, we put $s \cup \{r\} \in T$. Then choose a maximal $\mathcal{H}$-disjoint family

$$M_{s \cup \{r\}} \subseteq \{\sigma(s, \vec{B}, r, C) : \vec{B} \in N_s, C \in \mathcal{H} \upharpoonright \sigma(s, \vec{B})/r\}.$$

Put

$$N_{s \cup \{r\}} = \{(\vec{B}, C) : \sigma(s, \vec{B}, r, C) \in M_{s \cup \{r\}}\}.$$

Now, for every $s \in T$, let

$$U_s = \{E \in \mathcal{H} : (\exists F \in M_s) E \subseteq F\} \quad \text{and} \quad V_s = \{E \in \mathcal{H} : (\forall F \in M_s \setminus \{\max(s)\}) \max(s) \in F \rightarrow F \cap E \notin \mathcal{H}\}.$$

**Claim 4.2.** For every $s \in T$, $U_s \cup V_s$ is dense open in $(\mathcal{H} \upharpoonright A_0, \subseteq)$.
Proof. Fix $s \in T$ and $A \in \mathcal{H} \upharpoonright A_0$. If $(\forall F \in M_s \setminus \{\max(s)\}) \max(s) \in F \rightarrow F \cap A \notin \mathcal{H}$ holds, then $A \in \mathcal{V}_s$. Otherwise, fix $F \in M_s \setminus \{\max(s)\}$ such that $\max(s) \in F$ and $F \cap A \in \mathcal{H}$. Let $\overrightarrow{B} \in N_s \setminus \{\max(s)\}$ be such that $\sigma(s \setminus \{\max(s)\}, \overrightarrow{B}) = F$. Notice that since $\max(s) \in F$, then

$$(s \setminus \{\max(s)\}, \overrightarrow{B}, \max(s), F \cap A/\max(s))$$

is a legal position for player II. Then, using the maximality of $M_s$, choose $\hat{F} \in M_s$ such that

$$E := \sigma(s \setminus \{\max(s)\}, \overrightarrow{B}, \max(s), F \cap A/\max(s)) \cap \hat{F}$$

is in $\mathcal{H}$. So $E \in \mathcal{U}_s$ and $E \subseteq A$. This completes the proof of Claim 4.2. \qed

Claim 4.3. There exists $E \in \mathcal{H} \upharpoonright A_0$ such that for every $s \in T$ with $s \subseteq E$, $E/s \in \mathcal{U}_s$.

Proof. For each $n \in \mathbb{N}$, let

$$\mathcal{D}_n = \bigcap_{\max(s) = n} \mathcal{U}_s \cup \mathcal{V}_s,$$

$$\mathcal{U}_n = \bigcap_{\max(s) = n} \mathcal{U}_s$$

(if there is no $s \in T$ with $\max(s) = n$, then we put $\mathcal{D}_n = \mathcal{U}_n = \mathcal{H} \upharpoonright A_0$). By Claim 4.2, every $\mathcal{D}_n$ is dense open in $(\mathcal{H} \upharpoonright A_0, \subseteq)$. Using semiselectivity, choose a diagonalization $\hat{E} \in \mathcal{H} \upharpoonright A_0$ of the sequence $(\mathcal{D}_n)_n$. Let

$$E_0 := \{n \in \hat{E} : \hat{E}/n \notin \mathcal{U}_n\} \text{ and } E_1 := \hat{E} \setminus E_0.$$ 

Let us prove that $E_1 \notin \mathcal{H}$. Suppose $E_1 \in \mathcal{H}$. By the definitions, $(\forall n \in E_1) \hat{E}/n \notin \mathcal{U}_n$. Let $n_0 = \min(E_1)$ and fix $s_0 \in \hat{E}$ such that $\max(n_0) = n_0$, satisfying, in particular, the following:

$$(\forall F \in M_{s_0 \setminus \{n_0\}}) n_0 \in F \rightarrow F \cap E_1/n_0 \notin \mathcal{H}.$$ 

Notice that $|s_0| > 1$, by the construction of the $M_s$'s. 

Now, let $m = \max(s_0 \setminus \{n_0\})$. Then $m \in E_0$ and therefore $\hat{E}/m \in \mathcal{U}_m \subseteq \mathcal{U}_{s_0 \setminus \{n_0\}}$. So there exists $F \in M_{s_0 \setminus \{n_0\}}$ such that $\hat{E}/m \subseteq F$. Since $m < n_0$, $n_0 \in F$. But $F \cap E_1/n_0 = E_1/n_0 \in \mathcal{H}$, a contradiction. Hence, $E_1 \notin \mathcal{H}$ and therefore $E_0 \in \mathcal{H}$. Then $E := E_0$ is as required. \qed

Claim 4.4. Let $E$ be as in Claim 4.3 and let $s \cup \{r\} \in T$ with $s \subseteq E$ and $r \in E/s$. If $E/s \subseteq \sigma(s, \overrightarrow{B})$ for some $\overrightarrow{B} \in N_s$, then there exists $C \in \mathcal{H} \upharpoonright \sigma(s, \overrightarrow{B})/r$ such that $E/r \subseteq \sigma(s, \overrightarrow{B}, r, C)$ and $(\overrightarrow{B}, C) \in N_{s \cup \{r\}}$.

Proof. Fix $s$ and $r$ as in the hypothesis. Suppose $E/s \subseteq \sigma(s, \overrightarrow{B})$ for some $\overrightarrow{B} \in N_s$. Since $E/r \in \mathcal{U}_{s \cup \{r\}}$, there exists $(\overrightarrow{D}, C) \in N_{s \cup \{r\}}$ such that $E/r \subseteq \sigma(s, \overrightarrow{D}, r, C)$. Notice that $E/r \subseteq \sigma(s, \overrightarrow{B}) \cap \sigma(s, \overrightarrow{D})$. Since $M_s$ is $\mathcal{H}$-disjoint, then $\sigma(s, \overrightarrow{D})$ is necessarily equal to $\sigma(s, \overrightarrow{B})$ and therefore $\sigma(s, \overrightarrow{B}, r, C) = \sigma(s, \overrightarrow{D}, r, C)$. Hence $(\overrightarrow{B}, C) \in N_{s \cup \{r\}}$ and $E/r \subseteq \sigma(s, \overrightarrow{B}, r, C)$. \qed

Claim 4.5. Let $E$ be as in Claim 4.3. Then $[0, E] \subseteq \mathcal{X}$. 

Proof. Let \( \{k_i\}_{i \geq 0} \subseteq E \) be given. Since \( E/k_0 \in U(k_0) \), there exists \( B_0 \in N(k_0) \) such that \( E/k_0 \subseteq \sigma(k_0, B_0) \). Thus, by the choice of \( E \) and applying Claim \( 4.4 \) iteratively, we prove that \( \{k_i\}_{i \geq 0} \) is generated in a run of the game in which player I has used his winning strategy \( \sigma \). Therefore \( \{k_i\}_{i \geq 0} \in \mathcal{X} \). \( \square \)

The converse is trivial. This completes the proof of Proposition \( 4.7 \). \( \square \)

Now we turn to the case when player II has a winning strategy. The proof of the following is similar to the proof of Proposition 4.3 in [9]. First we show a result we will need in the sequel. It should be compared with Lemma 4.2 in [9].

Lemma 4.6. Let \( B \in \mathcal{H} \), \( f : \mathcal{H} \mid B \rightarrow \mathbb{N} \), and \( g : \mathcal{H} \mid B \rightarrow \mathcal{H} \mid B \) be given such that \( f(A) \in A \) and \( g(A) \subseteq A \setminus f(A) \). Then there is \( E_{f,g} \in \mathcal{H} \mid B \) with the property that for each \( p \in E_{f,g} \) there exists \( A \in \mathcal{H} \mid B \) such that \( f(A) = p \) and \( E_{f,g} \setminus p \subseteq g(A) \).

Proof. For each \( n \in \{f(A) : A \in \mathcal{H} \mid B\} \), let

\[
U_n = \{ E \in \mathcal{H} \mid B : (\exists A \in \mathcal{H} \mid B) (f(A) = n \land E \subseteq g(A)) \}
\]

and

\[
V_n = \{ E \in \mathcal{H} \mid B : (\forall A \in \mathcal{H} \mid B) (f(A) = n \rightarrow (g(A) \setminus E = \emptyset)) \}.
\]

The set \( D_n = U_n \cup V_n \) is dense open in \( \mathcal{H} \mid B \). Choose \( E \in \mathcal{H} \mid B \) such that for each \( n \in E \), \( E/n \in D_n \). Let

\[
E_0 = \{ n \in E : E/n \in U_n \} \text{ and } E_1 = \{ n \in E : E/n \in V_n \}.
\]

Now, suppose \( E_1 \in \mathcal{H} \). Then, for each \( n \in E_1 \), \( E_1/n \in V_n \). Let \( n_1 = f(E_1) \). So \( n_1 \in E_1 \) by the definition of \( f \). But, by the definition of \( g \), \( g(E_1) \subseteq n_1 \) and so \( E_1/n_1 \not\in V_{n_1} \), a contradiction. Therefore, \( E_1 \not\in \mathcal{H} \). Hence \( E_0 \in \mathcal{H} \), since \( \mathcal{H} \) is a co-ideal. The set \( E_{f,g} := E_0 \) is as required. \( \square \)

Proposition 4.7. For every \( \mathcal{X} \subseteq \mathbb{N}^{<\infty} \), \( A \in \mathcal{H} \) and \( a \in \mathbb{N}^{<\infty} \), II has a winning strategy in \( G_{\mathcal{H}}(a, A, \mathcal{X}) \) if and only if \( \forall A' \in \mathcal{H} \mid A \) there exists \( E \in \mathcal{H} \mid A' \) such that \( [a, E] \cap \mathcal{X} = \emptyset \).

Proof. Let \( \tau \) be a winning strategy for II in \( G_{\mathcal{H}}(a, A, \mathcal{X}) \) and let \( A' \in \mathcal{H} \mid A \) be given. We are going to define a winning strategy \( \sigma \) for I, in \( G_{\mathcal{H}}(a, A', \mathbb{N}^{<\infty} \setminus \mathcal{X}) \), in such a way that we will get the required result by means of Proposition 4.1. So, in a play of the game \( G_{\mathcal{H}}(a, A', \mathbb{N}^{<\infty} \setminus \mathcal{X}) \), with II's successive moves being \( (n_j, B_j) \), \( j \in \mathbb{N} \), define \( A_j \in \mathcal{H} \) and \( E_{f_j,g_j} \) as in Lemma 4.6, for \( f_j \) and \( g_j \) such that

1. for all \( \hat{A} \in \mathcal{H} \mid A' \),

\[
(f_0(\hat{A}), g_0(\hat{A})) = \tau(\hat{A});
\]

2. for all \( \hat{A} \in \mathcal{H} \mid B_j \cap g_j(A_j) \),

\[
(f_{j+1}(\hat{A}), g_{j+1}(\hat{A})) = \tau(A_0, \cdots, A_j, \hat{A});
\]

3. \( A_0 \subseteq A' \) and \( A_{j+1} \subseteq B_j \cap g_j(A_j) \);

4. \( n_j = f_j(A_j) \) and \( E_{f_j,g_j}/n_j \subseteq g_j(A_j) \).
Now, let \( \sigma(\emptyset) = E_{f_{g_{0}}} \) and \( \sigma((n_{0}, B_{0}), \ldots, (n_{j}, B_{j})) = E_{f_{g_{j+1}}} \).

Conversely, let \( A_{0} \) be the first move of I in the game. Then there exists \( E \in \mathcal{H} \upharpoonright A_{0} \) such that \([a, E] \cap X = \emptyset\). We define a winning strategy for player II by letting him (or her) play \((\min E, E \setminus \{\min E\})\) at the first turn and arbitrarily from there on. \( \square \)

We are ready now for the following:

Proof of Theorem 3.3. If \( \mathcal{H} \) is semiselective, then part (2) of Theorem 3.3 follows from Propositions 4.1 and 4.7.

Conversely, suppose part (2) holds and \( (\mathcal{D}_{n})_{n} \) is a sequence of dense open sets in \((\mathcal{H}, \subseteq)\). For every \( a \in \mathbb{N}^{<\infty} \), let \( X_{a} = \{ B \in [a, \mathbb{N}]: B/a \text{ diagonalizes some decreasing } (A_{n})_{n} \text{ such that } (\forall n) \ A_{n} \in \mathcal{D}_{n} \} \) and define \( \mathcal{X} = \bigcup_{a \in \mathbb{N}^{<\infty}} X_{a} \).

Fix \( A \in \mathcal{H} \) and \( a \in \mathbb{N}^{<\infty} \) with \([a, A] \neq \emptyset\), and define a winning strategy \( \sigma \) for player I in \( G_{\mathcal{H}}(a, A, \mathcal{X}) \), as follows: let \( \sigma(\emptyset) \) be any element of \( \mathcal{D}_{0} \) such that \( \sigma(\emptyset) \subseteq A \). At stage \( k \), if II’s successive moves in the game are \((n_{j}, B_{j})\), \( j \leq k \), let \( \sigma((n_{0}, B_{0}), \ldots, (n_{k}, B_{k})) \) be any element of \( \mathcal{D}_{k+1} \) such that \( \sigma((n_{0}, B_{0}), \ldots, (n_{k}, B_{k})) \subseteq B_{k} \). Notice that \( a \cup \{n_{0}, n_{1}, n_{2}, \ldots\} \in X_{a} \).

So the game \( G_{\mathcal{H}}(a, A, \mathcal{X}) \) is determined for every \( A \in \mathcal{H} \) and \( a \in \mathbb{N}^{<\infty} \) with \([a, A] \neq \emptyset\). Then, by our assumptions, \( \mathcal{X} \) is \( \mathcal{H} \)-Ramsey. So given \( A \in \mathcal{H} \), there exists \( B \in \mathcal{H} \upharpoonright A \) such that \( B^{<\infty} \subseteq \mathcal{X} \) or \( B^{<\infty} \cap \mathcal{X} = \emptyset \). The second alternative does not hold, so \( \mathcal{X} \cap \mathcal{H} \) is dense in \((\mathcal{H}, \subseteq)\). Hence, \( \mathcal{H} \) is semiselective. \( \square \)

5. The Ramsey property in Solovay models

Recall that the Mathias forcing notion \( \mathbb{M} \) is the collection of all the sets of the form

\( [a, A] := \{ B \in \mathbb{N}^{<\infty} : a \sqsubseteq B \subseteq A \} \),

ordered by \([a, A] \leq [b, B]\) if and only if \([a, A] \subseteq [b, B] \).

If \( \mathcal{H} \) is a co-ideal, then \( \mathbb{M}_{\mathcal{H}} \), the Mathias partial order with respect to \( \mathcal{H} \), is the collection of all the \([a, A] \) as above but with \( A \in \mathcal{H} \), ordered in the same way.

A co-ideal \( \mathcal{H} \) has the Mathias property if it satisfies the following: if \( x \) is \( \mathbb{M}_{\mathcal{H}} \)-generic over a model \( M \), then every \( y \in x^{<\infty} \) is \( \mathbb{M}_{\mathcal{H}} \)-generic over \( M \). Also, \( \mathcal{H} \) has the Prikry property if for every \([a, A] \in \mathbb{M}_{\mathcal{H}} \) and every formula \( \varphi \) of the forcing language of \( \mathbb{M}_{\mathcal{H}} \), there is \( B \in \mathcal{H} \upharpoonright A \) such that \([a, B]\) decides \( \varphi \).

Theorem 5.1 ([3], Theorem 4.1]). For a co-ideal \( \mathcal{H} \) the following are equivalent:

1. \( \mathcal{H} \) is semiselective.
2. \( \mathbb{M}_{\mathcal{H}} \) has the Prikry property.
3. \( \mathbb{M}_{\mathcal{H}} \) has the Mathias property.

Suppose \( M \) is a model of ZFC and there is an inaccessible cardinal \( \lambda \) in \( M \). The Levy partial order \( \text{Col}(\omega, < \lambda) \) produces a generic extension \( M[G] \) of \( M \) where \( \lambda \) becomes \( \aleph_{1} \). Solovay’s model (see [13]) is obtained by taking the submodel of \( M[G] \) formed by all the sets hereditarily definable in \( M[G] \) from a sequence of ordinals (see [10] or [3]).
In [10], Mathias shows that if $V = L$, $\lambda$ is a Mahlo cardinal and $V[G]$ is a generic extension obtained by forcing with $Col(\omega, < \lambda)$, then every set of real numbers defined in the generic extension from a sequence of ordinals is $H$-Ramsey for $H$ a selective co-ideal. This result can be extended to semiselective co-ideals under a suitable large cardinal hypothesis.

**Theorem 5.2.** Suppose $\lambda$ is a weakly compact cardinal. Let $V[G]$ be a generic extension by $Col(\omega, < \lambda)$. Then, if $H$ is a semiselective co-ideal in $V[G]$, every set of real numbers in $L(\mathbb{R})$ of $V[G]$ is $H$-Ramsey.

**Proof.** Let $H$ be a semiselective co-ideal in $V[G]$. Let $A$ be a set of reals in $L(\mathbb{R})^V[G]$; in particular, $A$ is defined in $V[G]$ by a formula $\varphi$ from a sequence of ordinals. Let $[a, A]$ be a condition of the Mathias forcing $M_H$ with respect to the semiselective co-ideal $H$. Finally, let $H$ be a name for $H$. Notice that $H \subseteq V_\lambda$.

Since $V[G]$ satisfies that $H$ is semiselective, the following statement holds in $V[G]$:

For every sequence $D = (D_n : n \in \omega)$ of open dense subsets of $H$ and for every $x \in H$ there is $y \in H$, $y \subseteq x$, such that $y$ diagonalizes the sequence $D$.

Therefore, there is $p \in G$ such that, in $V$, the following statement holds:

$$\forall D \forall \tau (p \Vdash_{Col(\omega, < \lambda)} (\dot{D} \text{ is a name for a sequence of dense open subsets of } \dot{H} \text{ and } \tau \in \dot{H}) \rightarrow (\exists x (x \in H, x \subseteq \tau, x \text{ diagonalizes } \dot{D}))).$$

Notice that every real in $V[G]$ has a name in $V_\lambda$, and names for subsets of $H$ or countable sequences of subsets of $H$ are contained in $V_\lambda$. Also, the forcing $Col(\omega, < \lambda)$ is a subset of $V_\lambda$. Therefore the same statement is valid in the structure $(V_\lambda, \in, H, Col(\omega, < \lambda))$. This statement is $\Pi^1_1$ over this structure, and since $\lambda$ is $\Pi^1_1$-indescribable, there is $\kappa < \lambda$ such that in $(V_\gamma, \in, H \cap V_\gamma, Col(\omega, < \lambda) \cap V_\gamma)$

$$\forall D \forall \tau (p \Vdash_{Col(\omega, < \kappa)} (\dot{D} \text{ is a name for a sequence of dense open subsets of } \dot{H} \cap V_\kappa \text{ and } \tau \in \dot{H} \cap V_\kappa) \rightarrow (\exists x (x \in H \cap V_\kappa, x \subseteq \tau, x \text{ diagonalizes } \dot{D}))).$$

We can get $\kappa$ inaccessible, since there is a $\Pi^1_1$ formula expressing that $\lambda$ is inaccessible. Also, $\kappa$ is such that $p$ and the names for the real parameters in the definition of $A$ and for $A$ belong to $V_\kappa$.

If we let $G_\kappa = G \cap Col(\omega, < \kappa)$, then $G_\kappa \subseteq Col(\omega, < \kappa)$ and it is generic over $V$. Also, $p \in G_\kappa$. Here $H \cap V_\kappa$ is a $Col(\omega, < \kappa)$-name in $V$ which is interpreted by $G_\kappa$ as $H \cap V[G_\kappa]$; thus $H \cap V[G_\kappa] \in V[G_\kappa]$. Moreover, since every subset (or sequence of subsets) of $H \cap V[G_\kappa]$ which belongs to $V[G_\kappa]$ has a name contained in $V_\kappa$, we have that, in $V[G_\kappa]$, $H \cap V_\kappa$ is semiselective, and in consequence it has both the Prikry and the Mathias properties.

Now the proof can be finished as in [10]. Let $\dot{r}$ be the canonical name of an $M_{H \cap V[G_\kappa]}$-generic real and consider the formula $\varphi(\dot{r})$ in the forcing language of $V[G_\kappa]$. By the Prikry property of $H \cap V[G_\kappa]$, there is $A' \subseteq A$, $A' \in H \cap V[G_\kappa]$, such that $[a, A']$ decides $\varphi(\dot{r})$. Since $2^{2\omega}$ computed in $V[G_\kappa]$ is countable in $V[G]$, there is (in $V[G]$) an $M_{H \cap V[G_\kappa]}$-generic real $x$ over $V[G_\kappa]$ such that $x \in [a, A']$. 

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To see that there is such a generic real in $\mathcal{H}$ we argue as in 5.5 of [10] using the semiselectivity of $\mathcal{H}$ and the fact that $\mathcal{H} \cap V[G_\kappa]$ is countable in $V[G_\kappa]$, every $y \in [a, x \setminus a]$ is also $M_{\mathcal{H} \cap V[G_\kappa]}$-generic over $V[G_\kappa]$, and also $y \in [a, A']$. Thus $\varphi(x)$ if and only if $[a, A'] \models \varphi(\check{r})$, if and only if $\varphi(y)$. Therefore, $[a, x \setminus a]$ is contained in $A$ or is disjoint from $A$. □

As in [10], we obtain the following.

**Corollary 5.3.** If ZFC is consistent with the existence of a weakly compact cardinal, then so is the statement that for every semiselective co-ideal $\mathcal{H}$ all sets of real numbers from $L(\mathbb{R})$ are $\mathcal{H}$-Ramsey.

Eisworth ([1]) showed that the hypothesis of the existence of a Mahlo cardinal in Mathias’s result cannot be weakened.

**Question.** Can the weakly compact cardinal hypothesis in the statement of Theorem 5.2 be weakened? Would a Mahlo cardinal suffice?

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