Level statistics of real eigenvalues in non-Hermitian systems

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Symmetries associated with complex conjugation and Hermitian conjugation, such as time-reversal symmetry and pseudo-Hermiticity, have great impact on eigenvalue spectra of non-Hermitian random matrices. Here, we show that time-reversal symmetry and pseudo-Hermiticity lead to universal level statistics of non-Hermitian random matrices on and around the real axis. From the extensive numerical calculations of large random matrices, we obtain the five universal level-spacing and level-spacing-ratio distributions of real eigenvalues, each of which is unique to the symmetry class. Furthermore, we analyse spacings of real eigenvalues in physical models, such as bosonic many-body systems and free fermionic systems with disorder and dissipation. We clarify that the level spacings in ergodic (metallic) phases are described by the universal distributions of non-Hermitian random matrices in the same symmetry classes, while the level spacings in many-body localized and Anderson localized phases show the Poisson statistics. We also find that the number of real eigenvalues shows distinct scalings in the ergodic and localized phases in these symmetry classes. These results serve as effective tools for detecting quantum chaos, many-body localization, and real-complex transitions in non-Hermitian systems with symmetries.

I. INTRODUCTION

An understanding of spectral correlations under symmetries is useful in classifying phases of matter [1]. In closed quantum systems, the spectral statistics of non-integrable systems typically coincide with those of Hermitian random matrices with symmetries, which serves as an effective tool for detecting quantum chaos [2–5]. The spectral statistics also provide a measure of the Anderson transitions [6–9] and many-body-localization (MBL) transitions [10–13]. When a disordered many-body Hermitian system is in the ergodic phase, statistics of spacing between its eigenenergy levels are described by the Wigner-Dyson distribution of Hermitian random matrices. The Wigner-Dyson distribution is universally classified by time-reversal symmetry (TRS), Hermitian random matrices without TRS and with TRS whose sign is ±1 and −1 respectively belong to the Gaussian unitary, orthogonal, and symplectic ensembles, each of which exhibits the distinct spectral statistics.

While state-of-the-art quantum experiments facilitate probing quantum many-body physics including the MBL [13, 14], energy gain and loss naturally exist in these optical systems, removing the Hermiticity condition from their many-body Hamiltonians. Consequently, open quantum systems described by non-Hermitian operators have attracted growing interest. Researchers have studied the non-Hermitian physics of optical phenomena [15, 24], topological phases [25–28], and Anderson and many-body localization [36, 49]. These works have led to a remarkable advance in spectral properties of non-Hermitian operators [50, 62]. Still, it remains to be fully explored how symmetries influence the universal spectral properties of non-Hermitian operators.

The level statistics analyses of Hermitian systems cannot be directly applied to non-Hermitian systems. Due to the absence of Hermiticity, the 10-fold Hermitian symmetry classification [66] is enriched into 38-fold symmetry classification [67, 68]. Eigenvalues of non-Hermitian systems distribute in the two-dimensional (2D) complex plane. The statistics of complex level spacings sα, defined as the distance with the closest eigenvalues in the complex plane (i.e., sα = minβ |Eα − Eβ| for all complex eigenvalues Eα, Eβ with α ≠ β), were previously studied to capture the spectral correlations of non-Hermitian systems [51, 52, 53, 54, 63]. Non-Hermitian random matrices without any symmetry show a universal distribution of the spacing of complex eigenvalues, known as the Ginibre distribution [50]. An introduction of the transposition version of TRS, $H = U^T \alpha H^\dagger U^\alpha T$, $U^\alpha T U^T = \pm 1$, which is called TRS$^T$, changes the distribution into two distinct distributions, depending on the sign of $U^\alpha T U^T = \pm 1$ [59]. This is similar to the three-fold Wigner-Dyson distribution for Hermitian random matrices. Meanwhile, an introduction of TRS, $H = U^T \alpha H^\dagger U^\alpha T$, $U^\alpha T U^T = \pm 1$, does not alter the spacing distribution away from the real axis [51].

In this paper, we show that TRS leads to universal level statistics on and around the real axis. In addition to TRS, we also identify the relevant symmetries that give rise to universal level statistics of real eigenvalues in non-Hermitian random matrices. The unive-
al level statistics provide an effective tool for detecting quantum chaos in open quantum systems with the symmetries. In the 38-fold symmetry classification of non-Hermitian random matrices, we show that there exist seven symmetry classes in which eigenstates with real eigenvalues preserve all the symmetries of the symmetry class whereas eigenstates away from the real axis break some symmetries. They are a class only with pseudo-Hermiticity (class $A + \eta$; class AIII), classes with TRS whose sign is either $\pm 1$ (classes AI and AII), and classes with both TRS and pseudo-Hermiticity (classes $AI + \eta_\pm$ and AII $+ \eta_\pm$); see Table I. In the last classes, TRS commutes or anti-commutes with pseudo-Hermiticity. The subscript of $\eta_\pm$ denotes the commutation (+) or anti-commutation (−) relation between TRS and pseudo-Hermiticity. Note that random matrices with particle-hole symmetry ($H = -U_0^\dagger H^T U_P$) and/or sublattice symmetry ($H = -U_0^\dagger H U_S$) do not give rise to the universal level statistics of real eigenvalues because only states with zero eigenvalue respect the symmetries.

The density of states (DoS) of non-Hermitian random matrices and physical Hamiltonians is generally defined in the complex plane, $\rho(E \equiv x + iy)$. Based on analytical and numerical analyses, we find that in five symmetry classes out of the seven symmetry classes, the DoS in the complex plane has a delta function peak on the real axis. They are class $A + \eta$, class AI (equivalent to the real Ginibre ensemble [37, 53, 54, 57, 58]), class AI $+ \eta_-$, and class AII $+ \eta_-$. In these symmetry classes, the DoS $\rho(E = x + iy)$ is decomposed into two parts,

$$\rho(E = x + iy) = \rho_c(x, y) + \rho_r(x)\delta(y),$$

(1)

where $\rho_c(x, y)$ is the density of complex eigenvalues away from the real axis and $\rho_r(x)$ is the density of real eigenvalues. Since only the states with real eigenvalues respect the full symmetries in these symmetry classes, $\rho_c(x)$ plays a role similar to the DoS in Hermitian systems. We show that the level statistics of real eigenvalues obtained from non-Hermitian random matrices, such as the level-spacing and level-spacing-ratio distributions, are different from those obtained from Hermitian random matrices and belong to the five distinctive universality classes according to the symmetries. It is also notable that TRS or pseudo-Hermiticity does not necessarily lead to $\rho_r(x) \neq 0$ in the DoS. We find that no real eigenvalues appear generally in class AI, which is consistent with the absence of real eigenvalues in the Ginibre symplectic ensemble [50]. We further generalize the absence of real eigenvalues to class $AI + \eta_-$.

We use random matrix analysis and exact diagonalization to identify universal level statistics of real eigenvalues for the five non-Hermitian symmetry classes. To demonstrate the universality of the level statistics, we apply the analysis to many-body and non-interacting physical Hamiltonians with disorder and non-Hermiticity. In physical systems that belong to the five symmetry classes, a finite density $\rho_r(x)$ of real eigenvalues enables comparison with those of non-Hermitian random matrices. We introduce non-Hermitian terms into interacting spin and hard-core boson models, such that many-body Hamiltonians belong to classes $A + \eta$, AI, and $AI + \eta_\pm$. By the exact diagonalization, we calculate the many-body eigenenergies and their spacing distributions on the real axis. In these four symmetry classes, the level statistics in the ergodic phases follow those of non-Hermitian random matrices in the corresponding symmetry classes. On the other hand, in class AII $+ \eta_+$, the level statistics of a dissipative free fermionic system deviate from those of non-Hermitian random matrices in class AII $+ \eta_+$. We attribute this discrepancy to the unconventional level interaction between real eigenvalues, which is unique to non-Hermitian random matrices in class AII $+ \eta_+$.

The reality of the spectrum in non-Hermitian Hamiltonians was extensively studied [15]. References [54, 50] showed that the number of real eigenvalues is proportional to the square root of the matrix size for non-Hermitian random matrices in class AI, and several previous works [34, 35, 52, 10, 55, 59] found that a non-zero proportion of real eigenvalues can appear in non-Hermitian physical systems with TRS. However, how the number of real eigenvalues scales with the system size in physical systems, and its relationship with random matrix theory are still uncovered. We find that the average number $\bar{N}_{\text{real}}$ of real eigenvalues show distinctive scalings with respect to the dimensions $N$ of Hilbert space in the five symmetry classes. We clarify $\bar{N}_{\text{real}} \propto \sqrt{N}$ in the ergodic (metallic) phase and $\bar{N}_{\text{real}} \propto N$ in the localized phases. Our results show that the level statistics analyses are powerful tools for detecting quantum chaos and MBL in non-Hermitian systems.

This paper is organized as follows. In Sec.II we begin with reviewing the symmetry classification of non-Hermitian matrices and introduce level-spacing and level-spacing-ratio distributions of real eigenvalues for non-Hermitian random matrices. We numerically obtain the real-eigenvalue spacing and spacing-ratio distributions from large non-Hermitian random matrices. Analyzing small random matrices, we clarify the nature of effective interactions between two neighboring eigenvalues on the real axis and use them to explain the behavior of large random matrices. We also show the scaling of the number of real eigenvalues with respect to the dimensions of random matrices. In Sec.III we use a hard-core boson model and interacting spin models to demonstrate the universality of the real-eigenvalue spacing and spacing-ratio distribution functions. We argue that the level statistics of real eigenvalues are useful for detecting different many-body phases in interacting disordered systems. We uncover that in the MBL phase, the number of real eigenvalues shows a non-universal scaling with respect to the dimensions of Hilbert space. We provide an explanation for the non-universal scaling. In Sec.IV we apply the analysis to non-Hermitian non-interacting fermionic models in two and three dimensions. We find that the number of real eigenvalues shows distinctive universal scaling properties
TABLE I. Ten-fold symmetry classification based on time-reversal symmetry (TRS), time-reversal symmetry† (TRS†), and pseudo-Hermiticity (pH). TRS and pH are equivalent to particle-hole symmetry† (PHS†) and chiral symmetry (CS), respectively. For the columns of given symmetry, the blank entries mean the absence of the symmetry. For TRS and TRS†, ±1 stands for the sign of the symmetry. If H belongs to the symmetry class in the first column, iH belongs to the equivalent symmetry class in the second column. The column “soft gap” gives the small y = Im(E) behavior of the density ρc(x, y) of complex eigenvalues if there is a soft gap around the real axis y = 0. The column “δ(y)” indicates whether there is a delta function peak on the real axis y = 0. In the presence of the delta function peak, the column “⟨r⟩” and the column “χ” respectively show the mean spacing ratio and spectral compressibility of real eigenvalues (see Eq. (11) and Eq. (23) for their definitions) obtained from 4000 × 4000 random matrices in the generalized Gaussian ensemble. The standard deviation of ⟨r⟩ is estimated by the bootstrap method [69] and labeled in the parentheses; for example, the standard deviation is 0.0004 for “0.4194(4)”.

| symmetry class | symmetry class (equiv) | TRS (PHS†) | TRS† (CS) | pH | soft gap | δ(y) | ⟨r⟩ | χ |
|----------------|------------------------|------------|-----------|----|----------|------|-----|---|
| A             | A                      |            |           |    |          |      |     |   |
| A + η         | AII                    |            |           |    |          |      |     |   |
| AI            | D†                     | +1         |           |    |          |      |     |   |
| AI†           | AI†                   | -1         |           |    |          |      |     |   |
| AI + η⁺       | BDI†                  | +1         | +1        |    |          |      |     |   |
| AI + η⁻       | DIII†                 | +1         | -1        |    |          |      |     |   |
| AII + η⁺      | CII†                  | -1         | -1        |    |          |      |     |   |
| AII + η⁻      | CII†                  | -1         | +1        |    |          |      |     |   |

with respect to the matrix dimensions in the metal and localized phases. Section V is devoted to the conclusion and discussion.

II. RANDOM MATRICES

A. Non-Hermitian symmetry classes

The 38-fold symmetry class of non-Hermitian Hamiltonians is given by the following anti-unitary symmetries [32],

time-reversal symmetry (TRS) :

\[ U_{\text{TR}} H U_{\text{TR}}^\dagger = H, \quad U_{\text{TR}} U_{\text{TR}}^\dagger = \pm 1, \]

particle-hole symmetry (PHS) :

\[ U_{\text{PH}} H^T U_{\text{PH}}^- = -H, \quad U_{\text{PH}} U_{\text{PH}}^- = \pm 1, \]

time-reversal symmetry† (TRS†) :

\[ U_{\text{TR}} H^T U_{\text{TR}}^\dagger = H, \quad U_{\text{TR}} U_{\text{TR}}^\dagger = \pm 1, \]

pseudo-Hermiticity (pH) :

\[ U_{\eta} H U_{\eta}^\dagger = H, \quad U_{\eta}^2 = 1, \]

chiral symmetry (CS) :

\[ U_{\chi} H U_{\chi}^\dagger = -H, \quad U_{\chi}^2 = 1, \]

sublattice symmetry (SLS) :

\[ U_S H U_S^\dagger = -H, \quad U_S^2 = 1, \]

and unitary symmetries,

\[ U_{\text{TR}}, U_{\text{TR}}^\dagger, U_{\eta}, U_{\chi}, \text{ and } U_S \text{ are unitary matrices. When } H \text{ respects TRS (pH), } iH \text{ respects PHS† (CS), and vice versa. In this sense, TRS and PHS† are unified, so are pH and CS [31]. TRS relates an eigenvalue } z \text{ with its complex conjugate } z^\ast. \text{ If } v \text{ is a right eigenvector of a Hamiltonian } H \text{ with TRS for an eigenvalue } z \text{ (} H v = z v \text{), } U_{\text{TR}} v^\ast \text{ is another right eigenvector of } H \text{ with the eigenvalue } z^\ast \text{ (} H U_{\text{TR}}^\dagger v^\ast = z^\ast U_{\text{TR}}^\dagger v^\ast \text{). Likewise, pseudo-Hermiticity (pH) relates an eigenvalue } z \text{ with its complex conjugate } z^\ast. \text{ PHS† and CS relate an eigenvalue } z \text{ with } -z^\ast, \text{ and PHS and SLS relate an eigenvalue } z \text{ with } -z. \text{ On the other hand, TRS† imposes a constraint on each eigenvector.} \]

When a symmetry relates an eigenvalue } z \text{ with } z' \neq z \text{ and } |z - z'| \text{ is much larger than the mean level-spacing, such symmetry is expected to have no influence on the local eigenvalue correlation around } z. \text{ For example, neither TRS nor PHS changes the nearest-spacing distribution of non-Hermitian random matrices for general complex eigenvalues [59]. This is similar to Hermitian random matrices with pH or CS; for example, the eigenvalue spacing distribution away from zero energy in class D is the same as that in class A [60].} \]

The spectral correlation on or around the real axis depends on TRS, pH, and their combination (TRS†). From TRS, pH, and TRS†, a ten-fold symmetry classification is derived, as shown in Table I. This ten-fold class includes seven symmetry classes that have at least one symmetry associated with complex conjugation (TRS), or Hermitian conjugation (pH): a class with pH (class A + η), classes with TRS whose sign can be ±1 (classes AI and AII), and classes with both pH and TRS, where the sign of TRS is ±1 and TRS commutes with pH (classes AI + η⁺ and AII + η⁻) or TRS anti-commutes with pH (classes AI – η⁻).
\(\eta\) and AII + \(\eta_-\)). According to the 38-fold symmetry classification of non-Hermitian systems \([32]\), these symmetry classes are equivalent to classes AII, D, C, BDI, DIII, CII, and C\(\dagger\). (see Table I). The ten-fold symmetry class in Table I is also equivalent to the Hermitian conjugate of the non-Hermitian Altland-Zirnbauer class (i.e., AZ\(\dagger\) class) in Ref. [32].

B. Level statistics of real eigenvalues for non-Hermitian random matrices

We consider non-Hermitian random matrices \(\mathcal{H}\) in symmetry classes A + \(\eta\), AI, AII, AI + \(\eta_\pm\), and AII + \(\eta_\pm\) in the Gaussian ensemble with the following probability distribution function \(p(\mathcal{H})\):

\[
p(\mathcal{H}) = C_N^{-1} e^{-\beta \text{Tr}(\mathcal{H}^\dagger \mathcal{H})},
\]

where \(\beta\) is a positive constant and \(C_N\) is a normalized constant. Without loss of generality, we choose \(\beta = 1/2\) for the rest of this paper. Non-Hermitian matrices \(\mathcal{H}\) are required to belong to symmetry classes in Table I (see Appendix \(B\) for details). Diagonalizations of large random matrices show that eigenvalues distribute almost uniformly in a circle except around the real axis and its circumference (not shown here). This distribution is consistent with the circular law of the Ginibre ensemble [51].

For non-Hermitian random matrices in classes A + \(\eta\), AI, AI + \(\eta_\pm\), and AII + \(\eta_\pm\), a sub-extensive number of eigenvalues are real, and the DoS \(\rho(x, y)\) has a delta function peak on the real axis:

\[
\rho(E = x + iy) = \rho_+ (x, y) + \rho_+ (x) \delta(y).
\]

In numerical diagonalizations, real eigenvalues and complex eigenvalues are clearly distinguished, although real eigenvalues can artificially have tiny imaginary parts due to machine inaccuracy of a numerical subroutine program. In fact, with proper normalization, the apparent imaginary parts of real eigenvalues of Hermitian matrices are less than a certain error bound [52]. Meanwhile, to avoid regarding real eigenvalues as complex due to the machine inaccuracy, we choose a cut-off \(C\) larger than the error bound. The probability that complex eigenvalues are mistaken as real depends on the dimensions \(N\) of the matrix. With our choice of the cut-off \(C\), this probability is estimated to be negligible for \(N < 10^4\), where \(N\) in this paper is typically less than \(10^4\).

The density \(\rho_+(x, y)\) of complex eigenvalues in all the seven symmetry classes vanishes toward the real axis and hence has a soft gap around the real axis (see Fig. 1). The size of the gap is of the same order as a mean level-spacing of eigenvalues in the complex plane. When \(|y|\) is much smaller than the mean eigenvalue spacing, we have \(\rho_+(x, y) \sim |y|\) in classes AII, A + \(\eta\), AI + \(\eta_-\), AI + \(\eta_+\), and AII + \(\eta_\pm\), while \(\rho_+(x, y) \sim |y|^2\) in class AI and \(\rho_+(x, y) \sim -|y| \log |y|\) in class AII + \(\eta_+\) (Fig. 1). These small \(y\) behaviors are consistent with the small matrix analysis discussed in Sec. [11,13].

The logarithmic correction in class AI + \(\eta_\pm\) seems to be due to TRS\(^\dagger\) with the sign \(+1\) [52, 59].

The number of real eigenvalues of non-Hermitian real random matrices (symmetry class AI) was previously studied \([37, 53, 54, 57, 58]\). However, a systematic study on the other symmetry classes is still lacking. In this paper, we find that the averaged number \(\bar{N}_{\text{real}}\) of real eigenvalues of \(N \times N\) non-Hermitian random matrices is proportional to the square-root of the dimensions of the matrices in all the five symmetry classes (see Fig. 2),

\[
\bar{N}_{\text{real}} \propto \sqrt{N}.
\]

This sub-extensive number of real eigenvalues enables level statistics analyses on the real axis, where the symmetries associated with complex conjugation must have important effects as in the Hermitian case [8].

Furthermore, we obtain the universal distribution functions of spacings of real eigenvalues. Let \(\lambda_1, \lambda_2, \ldots, \lambda_{N_{\text{real}}}\) be all the real eigenvalues of given \(H\) in the descending order. We define a normalized spacing of the real eigenvalues as

\[
s_i = \frac{\lambda_{i+1} - \lambda_i}{\langle \lambda_{i+1} - \lambda_i \rangle},
\]

Here, \(\langle \ldots \rangle\) stands for the average over the ensemble and \(\langle \lambda_{i+1} - \lambda_i \rangle\) is evaluated by the average density of real eigenvalues at \(x = (\lambda_i + \lambda_{i+1})/2\),

\[
\langle \lambda_{i+1} - \lambda_i \rangle = \frac{1}{\mathcal{P}_+(\sqrt{(\lambda_{i+1} + \lambda_i)})},
\]

where \(\mathcal{P}_+(x)\) is the averaged density of real eigenvalues,

\[
\mathcal{P}_+(x) = \sum_{\lambda_i \in \mathcal{R}} \langle \delta(x - \lambda_i) \rangle
\]

with the set \(\mathcal{R}\) of real eigenvalues. Here, \(\mathcal{P}_+(x)\) is estimated by the average over the Gaussian ensemble. To exclude a fluctuation due to finite sampling numbers, we follow Refs. [4, 71] and replace the delta function in Eq. (10) by the Gaussian distribution, \exp[-(x - \lambda_i)^2/(2\sigma^2)]/\(\sqrt{2\pi}\sigma\) with \(\sigma = n\bar{s}\). Here, \(\bar{s}\) is the mean level-spacing on the real axis, and \(n\) is an \(O(1)\) constant.

We verify the validity of this numerical approach by using different \(n\) in the range from 2 to 5 and also replacing the delta function with the uniform distribution

\[
\frac{1}{4\sigma} \times 1_{[\lambda_i - 2\sigma, \lambda_i + 2\sigma]}(x) = \begin{cases} \frac{1}{\pi} & (x \in [\lambda_i - 2\sigma, \lambda_i + 2\sigma]), \\ 0 & (x \notin [\lambda_i - 2\sigma, \lambda_i + 2\sigma]). \end{cases}
\]

We confirm that \(\mathcal{P}_+(x)\) is barely influenced by the approximation scheme, where the maximal difference of \(\mathcal{P}_+(x)\) between the different approximation methods is around or smaller than 1%. Note also that we exclude the real eigenvalues around the edges of the spectrum when...
FIG. 1. Density $\rho_c(x,y)$ of complex eigenvalues of non-Hermitian random matrices in the Gaussian ensemble for classes (a) AI, (b) AII, (c) AI + $\eta_+$, (d) A + $\eta$, (e) AI + $\eta_-$, (f) AII + $\eta_+$, and (g) AII + $\eta_-$. Here, $\rho_c(x,y)$ is shown as a function of $y = \text{Im}(E)$ for fixed $x = \text{Re}(E)$ near the real axis of complex energy $E$ (i.e., $y \approx 0$). For classes AI, AI + $\eta_+$, A + $\eta$, and AII + $\eta_+$, the density of states $\rho(E = x + iy) \equiv \rho_c(x,y) + \delta(y)\rho_s(x)$ is separated into the density $\rho_c(x,y)$ of complex eigenvalues and the density $\rho_s(x)$ of real eigenvalues. For classes AII and AII + $\eta_-$, no real eigenvalues appear, and we have $\rho(x + iy) \equiv \rho_c(x,y)$. The data of $\rho(x + iy)$ are obtained from diagonalizations of 5000 samples of $4000 \times 4000$ random matrices in each symmetry class. Note that $\rho_c(x,y)$ is almost independent of $x = \text{Re}(E)$ when $E$ is away from the boundary of a circle inside which the complex eigenvalues $E$ distribute.

studying the distribution of the spacings of real eigenvalues, because $\overline{\tau}_r(x)$ near the edges changes sharply and the estimated $\overline{\tau}_r(x)$ might have larger error bars.

The spacing ratio of real eigenvalues [10, 11, 72] is also a useful quantity to characterize the level statistics on the real axis. It is defined by

$$r_i \equiv \min\left(\frac{\lambda_{i+1} - \lambda_i}{\lambda_i - \lambda_{i-1}}, \frac{\lambda_{i+1} - \lambda_i}{\lambda_{i-1} - \lambda_i}\right),$$

satisfying $0 \leq r_i \leq 1$. Since $r_i$ is a dimensionless quantity and free from the normalization, it is easier to numerically obtain the distribution of $r_i$ than that of $s_i$.

In each of the five symmetry classes (i.e., classes A + $\eta$, AI, AI + $\eta_+$, AI + $\eta_-$, and AI + $\eta_-$), we numerically calculate the level-spacing distribution $\rho_s(s)$ and the level-spacing-ratio distribution $\rho_r(r)$ of real eigenvalues, both of which converge to the characteristic functions (Fig. 2). Here, $\rho_r(r)$ and $\rho_s(s)$ in class AII + $\eta_+$ converge more slowly than those in the other symmetry classes and do not converge even at the maximal matrix size ($N = 4000$; see Figs. 3(d) and 3(f)).

To improve the convergence, we also introduce a generalized Gaussian ensemble with the following probability distribution function $p'(H)$:

$$p'(H) = C^{-1}_{N,(\beta_1,\beta_2)} e^{-\text{Tr}[\beta_1 (H+H^\dagger)^2 - \beta_2 (H-H^\dagger)^2]},$$

where $\beta_1$ and $\beta_2$ control the fluctuations of Hermitian and anti-Hermitian parts of $H$, respectively. For $\beta_1 = \beta_2$, $p'(H)$ reduces to $p(H)$ in the Gaussian ensemble. For $\beta_1 \neq \beta_2$, the eigenvalues $E = x + iy$ of $H$ distribute almost uniformly in the ellipse in the complex plane,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with $a/b = \beta_2/\beta_1$ [56]. In each symmetry class, random matrices in the generalized Gaussian ensemble show the same universal behaviors, such as soft gaps around the real axis and the square-root scaling of the average
In the Gaussian ensemble with the same matrix size as the standard deviation \( \sigma \) resemble with \( N \) Gaussian ensemble with a larger matrix size. Hermitian random matrices in the Gaussian ensemble as a figure shows the twice of the standard deviation of \( \bar{N} \) plot stand for the twice of the standard deviation of \( \bar{N} \) dimensions of random matrices. The error bars in the sum \( \sum \) for \( \bar{N} \) clearly demonstrates the square-root scaling \( \bar{N} \sim \sqrt{N} \) in all the five symmetry classes.

In each symmetry class, the error bars of \( \bar{N} \) sample \( N \) is at least 5000 for each matrix size. The plot number \( \bar{N} \) real of real eigenvalues. We find that in each symmetry class, for a matrix size \( N \), the average number \( \bar{N} \) real of real eigenvalues in the generalized Gaussian ensemble with \( \beta_1 \) and \( \beta_2 \) is approximately scaled by \( \bar{N} \) real in the Gaussian ensemble with the same matrix size as

\[
\bar{N}'_{\text{real}} \approx \sqrt{\frac{\beta_2}{\beta_1}} \bar{N}_{\text{real}},
\]

for \( \bar{N}, \bar{N}'_{\text{real}} \ll N \). We also find that for \( \bar{N}'_{\text{real}} \approx \bar{N}_{\text{real}} \), \( p(s) \) and \( p_r(r) \) in the two ensembles are close to each other (see Appendix C for details). Thus, \( p(s) \) and \( p_r(r) \) converge much faster in the generalized Gaussian ensemble with \( \beta_2 > \beta_1 \). We choose \( \beta_2/\beta_1 = 16 \) in the following.

In each symmetry class, the error bars of \( p(s) \) and \( p_r(r) \) of non-Hermitian random matrices in the generalized Gaussian ensemble with a larger matrix size \( N > 1000 \) overlap with each other. Both \( p(s) \) and \( p_r(r) \) converge to the characteristic universal functions in the limit of the large matrix size (Figs. 3 and 4). \( p(s) \) for classes \( A + \eta \), \( A \), \( AI + \eta + \), and \( AII + \eta + \) in different ensembles were previously calculated in Ref. 22, although the sizes of the matrices are small and \( p(s) \) in Ref. 23 does not seem to reach the universal function forms shown in Fig. 4. \( p(s) \) and \( p_r(r) \) in the limit of the large matrix size can distinguish the different symmetry classes among the five symmetry classes. We confirm the universality of \( p(s) \) and \( p_r(r) \) in each symmetry class by comparisons with \( p(s) \) and \( p_r(r) \) in the Bernoulli ensemble (see Appendix B for details). This comparison illustrates that both of the level-spacing and level-spacing-ratio distributions of real eigenvalues are universal and unique in each symmetry class. In Secs. III and IV we compare \( p(s) \) and \( p_r(r) \) obtained in the random matrix theory with those obtained from physical models.

Note that the level-spacing distributions \( p(s) \) in the different symmetry classes share the same asymptotic behaviors. For \( s \gg 1 \), we have

\[
\log p(s) \propto -s
\]
in all the five symmetry classes. On the other hand, for \( s \ll 1 \), we have (see the insets of Fig. 4)

\[
p(s) \propto \begin{cases} -s \log s & \text{(class AI + \eta +)}, \\ s & \text{(other four classes)}. \end{cases}
\]

As a comparison, the level-spacing distributions of Hermitian random matrices satisfy \( \log p(s) \propto -s^2 \) for \( s \gg 1 \) and \( p(s) \propto s^\beta \) for \( s \ll 1 \), where the Dyson index \( \beta = 1, 2, 4 \) characterizes the Gaussian orthogonal, unitary, and symplectic ensembles, respectively [1]. The small-\( s \) behavior of \( p(s) \) and its difference with Hermitian random matrices are well understood by analyses of effective small matrices (see Sec. III C for details). In the large \( s \) regime, the tail of \( p(s) \) in the non-Hermitian case is much heavier than that in the Hermitian case. This difference in the large-\( s \) behavior shows that the correlation between two neighboring real eigenvalues decays more quickly in non-Hermitian random matrices than Hermitian ones. When the distance between two neighboring real eigenvalues of a non-Hermitian random matrix is larger, more complex eigenvalues surround them and weaken the correlation between the two real eigenvalues.

For reference, in the Hermitian random matrix theory, the level-spacing-ratio distribution \( p_r(r) \) is well approximated by

\[
p_r(r) \approx \frac{1}{C_\beta} \frac{(r + r^2)\beta}{(1 + r + r^2)^{(1 + \frac{3}{2})\beta}} \theta(1 - r),
\]

where \( \theta(r) \) is the step function, \( C_\beta \) is a normalized constant, and \( \beta = 1, 2, 4 \) is the Dyson index. By contrast, if all real eigenvalues are uncorrelated and follow the Poisson statistics, \( p_r(r) \) is given by

\[
p_r(r) = \frac{2}{(1 + r)^2} \theta(1 - r).
\]

Note that these level-spacing-ratio distributions \( p_r(r) \) of Hermitian matrices can describe any of our universal \( p_r(r) \) of non-Hermitian random matrices in Fig. 5 showing the unique non-Hermitian nature of our universal distribution functions.

Notably, the mean value of the level-spacing ratios

\[
\langle r \rangle = \int_0^1 p_r(r) r dr \approx 0.371
\]
The strength of the level repulsion [1] describes the level repulsion \( \eta \) for (a) class A + \( \eta \), (b) class AI, (c) class AI + \( \eta \), (d) class AII + \( \eta \), and (e) class AI + \( \eta \). Level-spacing-ratio distributions \( p(r) \) of real eigenvalues of \( N \times N \) non-Hermitian random matrices in the Gaussian ensemble for (f) class A + \( \eta \), (g) class AI, (h) class AI + \( \eta \), (i) class AII + \( \eta \), and (j) class AI + \( \eta \). For each \( N \) and for each symmetry class, \( p(s) \) and \( p(r) \) are averaged over at least 5000 random matrices in the ensemble. The black points for \( p(s) \) and \( p(r) \) are obtained from \( 4000 \times 4000 \) random matrices, where the standard deviation error bars are evaluated by the bootstrap method [24]. The error bars for the smaller matrices are smaller than the error bars for \( N = 4000 \) and not shown.

In class AII + \( \eta \) is smaller than
\[
\langle r \rangle_{\text{Poisson}} = -1 + \ln 4 \approx 0.386
\]
of uncorrelated levels. By contrast, \( \langle r \rangle \) of non-Hermitian random matrices in the other four symmetry classes are all larger than \( \langle r \rangle_{\text{Poisson}} \) (see Table 1 and Fig. 3). In the Hermitian random matrix theory, the mean values of level-spacing ratios in the Gaussian orthogonal, unitary, and symplectic ensembles are
\[
\langle r \rangle_{\text{GOE}} \approx 0.531, \quad \langle r \rangle_{\text{GUE}} \approx 0.600, \quad \langle r \rangle_{\text{GSE}} \approx 0.674,
\]
all of which are larger than \( \langle r \rangle_{\text{Poisson}} \approx 0.386 \). In addition, \( \langle r \rangle \) increases with the Dyson index \( \beta = 1, 2, 4 \) that describes the strength of the level repulsion [1]. Thus, \( \langle r \rangle < \langle r \rangle_{\text{Poisson}} \) in class AII + \( \eta \) indicates unusual level interactions on the real axis unique to this symmetry class.

To further clarify the nature of the interactions between real eigenvalues in each symmetry class, we also calculate the variance \( \Sigma_2 \) of the number \( N_W \) of real eigenvalues in an interval on the real axis [4],
\[
\Sigma_2 \equiv \langle N_W^2 \rangle - \langle N_W \rangle^2.
\]
We have \( \Sigma_2 \propto \log\langle N_W \rangle \) in Hermitian random matrix theory, while we have \( \Sigma_2 = \langle N_W \rangle \) for uncorrelated real numbers (i.e., Poisson statistics). The spectral compressibility,
\[
\chi \equiv \lim_{N_W \to \infty} \frac{d\Sigma_2}{d\langle N_W \rangle},
\]
quantifies the level interaction between real eigenvalues. For Hermitian random matrices, the level repulsion is stronger, leading to \( \chi = 0 \). On the other hand, we have \( \chi_{\text{Poisson}} = 1 \) for the uncorrelated real spectrum. In the intermediate regime, such as the metal-insulator transition points in Hermitian disordered systems, the level repulsion is weaker than the random matrices but stronger than the Poisson statistics, resulting in \( 0 < \chi < 1 \) [9]. For non-Hermitian random matrices in each of the five symmetry classes, we find
\[
\Sigma_2 \propto \langle N_W \rangle,
\]
meaning that the spectral compressibility \( \chi \) gives a universal constant unique to each symmetry class (see Table 1 and Appendix E for details).

Remarkably, we have \( \chi \approx 1.11 > \chi_{\text{Poisson}} = 1 \) in class AII + \( \eta \), again indicating the unusual level interactions. As shown by \( p(s) \propto s \to 0 \) for \( s \to 0 \), the interaction is repulsive in the small \( s \) regime even for class AII + \( \eta \), although the level repulsion is much smaller than \( p(s) \propto s^4 \) of Hermitian random matrices in class AII (i.e., Gaussian symplectic ensemble). Hence, from our numerical results of \( \langle r \rangle < \langle r \rangle_{\text{Poisson}} \) and \( \chi > \chi_{\text{Poisson}} \), the attractive interaction should appear in the finite \( s \) regime and dominate the repulsive interaction in the small \( s \) regime on average. This is also compatible with the large peak of \( p(s) \) and \( p(r) \) compared with the other symmetry classes (see Figs. 4 and 5). A distinctive feature of class AII + \( \eta \) is the simultaneous presence of TRS and TRS† whose signs are \(-1\). While TRS† with the sign \(-1\) leads to the Kramers degeneracy of generic complex eigenvalues and the consequent strong level repulsion, TRS with the sign \(-1\) leads to the strong level repulsion around the real axis, as shown by the absence of real eigenvalues in
FIG. 4. Level-spacing distributions $p(s)$ of real eigenvalues of $N \times N$ non-Hermitian random matrices in the generalized Gaussian ensemble for (a) all the five symmetry classes, (b) class $\Lambda + \eta$, (c) class AI, (d) class AI + $\eta_+$, (e) class AI + $\eta_-$, and (f) class AI + $\eta_-$. For each $N$ and for each symmetry class, the spacing distribution function is averaged over at least 5000 random matrices in the ensemble. The black points with the error bars are $p(s)$ obtained from 4000 $\times$ 4000 random matrices, where the standard deviation error bars are evaluated by the bootstrap method \[69\]. The error bars for the smaller matrices are smaller than those for $N = 4000$ and not shown. In each symmetry class, $p(s)$ of random matrices with different sizes $N (N > 1000)$ almost overlap with each other. Insets: Asymptotic behaviors of the distribution function for $s \gg 1$ and for $s \ll 1$. For small $s$, the cumulative distribution function $\int_0^s p(s')ds'$ is plotted as a function of either $s^2$ or $-\log(s)s^2$. For $s \gg 1$, $\log(p(s))$ is linear in $s$, indicating the Poisson-like tail. The comparison of $p(s)$ among the five symmetry classes shows that $p(s)$ for classes AI and AI + $\eta_-$ are close to each other, and that $p(s)$ for the other three classes are clearly distinguished from $p(s)$ for classes AI and AI + $\eta_-$. Note also that no real eigenvalues generally appear in classes AII and AII + $\eta_-$. This should be due to TRS with the sign $-1$, which only enforces the Kramers degeneracy on the real axis. If a real eigenvalue is present in these symmetry classes, a Kramers partner with the same real eigenvalue should always appear. While this attractive interaction in class AII + $\eta_+$ are much larger than those of the other symmetry classes.

class AII. The combination of TRS and TRS\$seems to result in the unusual interactions between neighboring levels (Kramers pairs) on the real axis that are repulsive in the small $s$ regime but attractive in the larger $s$ regime. This is a possible reason for $\langle r \rangle < \langle r \rangle_{\text{Poisson}}$ and $\chi > \chi_{\text{Poisson}}$. In Sec. II C, we use effective small matrices to analyze the interactions between neighboring real eigenvalues and find that the degrees of freedom of the
Kramers pair is robust against Hermitian perturbations, it is sensitive to non-Hermitian perturbations and forms a complex-conjugate pair in the complex plane. Consequently, real eigenvalues are unstable in these symmetry classes. On the contrary, in class AII + η+, all eigenvalues including complex ones exhibit the Kramers degeneracy because of the additional presence of TRS$^\dagger$ with the sign −1. Consequently, the Kramers pair on the real axis is robust even against certain degrees of non-Hermitian perturbations, which leads to the sub-extensive number of real eigenvalues. This is different from class AII + η−, where only TRS$^\dagger$ with the sign +1 is present and no such robust Kramers degeneracy is allowed generally. We also discuss the absence of real eigenvalues in classes AII and AII + η− by effective small matrices in Sec. II C.
C. Effective small matrix analysis

Interactions between two neighboring eigenvalues can be qualitatively understood by effective small matrices [59]. When two eigenvalues get close to each other by the change of parameters, the interactions between them can be studied by nearly degenerate perturbation theory [8]. The strength of the interactions is generally determined by symmetry such as TRS and pH. To see the influence of symmetry in each of the seven symmetry classes, we consider the two adjacent eigenvalues that are either both real or complex conjugate to each other. Then, we project the variation of a full Hamiltonian onto a smaller space spanned by eigenvectors that belong to the two adjacent eigenvalues. The small Hamiltonians thus obtained take forms of either 2 × 2 matrix or 4 × 4 matrix, depending on the presence of the Kramers degeneracy. The symmetry classes of the small matrices are the same as the full Hamiltonians.

The small matrices in the seven symmetry classes are of the following forms,

\[ \mathcal{H}_{AI}^{(s)} = \begin{pmatrix} a_0 + a_1 & a_2 + a_3 \\ a_2 - a_3 & a_0 - a_1 \end{pmatrix}, \]

\[ \mathcal{H}_{AI + \eta}^{(s)} = \begin{pmatrix} a_0 + a_1 & a_3 + i a_2 \\ -a_3 + i a_2 & a_0 - a_1 \end{pmatrix}, \]

\[ \mathcal{H}_{AI + \eta^+}^{(s)} = \begin{pmatrix} a_0 + a_1 & a_2 \\ -a_2 & a_0 - a_1 \end{pmatrix}, \]

\[ \mathcal{H}_{AI + \eta^+}^{(s)} = \begin{pmatrix} a_0 + i a_1 & a_3 + i a_2 \\ -a_3 + i a_2 & a_0 - i a_1 \end{pmatrix}, \]

\[ \mathcal{H}_{AI^+}^{(s)} = \begin{pmatrix} a_0 + a_1 & 0 & a_2 + a_5 & a_4 - a_3 \\ 0 & a_0 - a_1 & a_2 - a_5 & -a_2 + a_5 \\ a_2 - a_5 & -a_4 + a_3 & a_0 - a_1 & 0 \\ -a_4 - a_3 & a_2 + a_5 & 0 & a_0 - a_1 \end{pmatrix}, \]

\[ \mathcal{H}_{AI^+ + \eta }^{(s)} = \begin{pmatrix} a_0 + a_1 & a_2 + i a_1 \\ -a_2 + i a_1 & a_0 \end{pmatrix}, \]

\[ \mathcal{H}_{AI^+ + \eta^+}^{(s)} = \begin{pmatrix} a_0 + a_1 & 0 & a_2 + i a_5 & a_4 + i a_3 \\ 0 & a_0 + a_1 & a_4 + i a_3 & a_2 - i a_5 \\ a_2 + i a_5 & a_4 + i a_3 & a_0 - a_1 & 0 \\ -a_4 + i a_3 & a_2 - i a_5 & 0 & a_0 - a_1 \end{pmatrix}, \]

where \( a_0, a_1, \ldots, a_{m+n} \) are real random variables. The eigenvalues of the small matrices are written in a unified form as

\[ \lambda = a_0 \pm \sqrt{X - Y}, \]

\[ X = \begin{cases} \sum_{i=1}^{m} a_i^2 & (m \neq 0), \\ 0 & (m = 0), \end{cases} \]

\[ Y = \sum_{i=m+1}^{m+n} a_i^2, \]

for the seven symmetry classes (see Appendix A for details). The two eigenvalues of each of these matrices are either both real or complex conjugate to each other. For \( m = 0 \) (classes AI and AI + \( \eta^+ \)), they are always complex conjugate to each other, meaning the absence of real eigenvalues. For \( m > 0 \), the probability of two real eigenvalues is finite and equal to the probability for \( X \geq Y \). This explains the presence and absence of the delta function peak on the real axis in the DoS in the seven symmetry classes. In Appendix A, we analytically obtain the level statistics of real eigenvalues and the DoS around the real axis for the above effective small matrices in the seven symmetry classes.

The finite probability of the real eigenvalues of the random matrices leads to the square-root scaling of \( \bar{N}_{\text{real}} \). According to the circular law [50], the uniform distribution of complex eigenvalues within the circle of radius \( R \) suggests that the number of complex eigenvalues near the real axis within an energy window of a mean complex energy spacing \( \bar{s}_c \) is scaled by \( \sqrt{N} \),

\[ N \times \frac{2 R \bar{s}_c}{\pi R^2} = 2 \sqrt{\frac{N}{\pi}} \propto \sqrt{N}, \quad (26) \]

with \( \pi R^2 / \bar{s}_c^2 = N \). The complex eigenvalues near the real axis can be regarded as complex-conjugate pairs of eigenvalues each of which is described by the small random matrices. Due to the finite probability of \( X > Y \), a complex-conjugate pair of the eigenvalues near the real axis is converted into real eigenvalues with a finite probability. This gives the square-root scaling, \( \bar{N}_{\text{real}} \sim \sqrt{N} \).

For \( X < Y \), the two eigenvalues are complex conjugate to each other. Thereby, \( X \) and \( Y \) in Eq. (25) give an attractive and repulsive interaction between the two eigenvalues, respectively. The increase of \( X \) (\( Y \)) will decrease (increase) the distance between the two eigenvalues along the imaginary axis. In symmetry classes A + \( \eta^+ \), AI + \( \eta^+ \), AI + \( \eta^- \), and AI + \( \eta^- \), we have \( m > 0 \), and both attractive and repulsive interactions are present. By contrast, in classes AI and AI + \( \eta^- \), we have \( m = 0 \), and thus no attractive interaction is present (see Table I).

The DoS around the real axis is determined by the
interaction between the two complex-conjugate eigenvalues. When the attractive interaction along the imaginary axis is absent ($m = 0$ in classes $\text{AI}$ and $\text{AI} + \eta$), the two eigenvalues are less likely to appear around the real axis than the other symmetry classes. The repulsion between the two eigenvalues becomes larger for larger $n$. In fact, our analysis of the small matrices in the Gaussian ensemble gives

$$\rho_c(x + iy) \propto \begin{cases} |y|^2 & (\text{class AI}), \\ |y| & (\text{class AI} + \eta). \end{cases}$$

(27)

In the presence of the attractive interaction ($m > 0$ in classes $\text{A} + \eta$, $\text{AI} + \eta$, $\text{AI} + \eta_+$, and $\text{AI} + \eta_-$), the larger attractive interaction converts the complex-conjugate pair of eigenvalues near the real axis onto two real eigenvalues. As a result, a sub-extensive number of eigenvalues of the full matrix appear on the real axis. In fact, our analysis of the small matrices in the Gaussian ensemble gives (see Appendix A for detailed derivations)

$$\rho_c(x, y) \propto \begin{cases} -|y| \log |y| & (\text{class AI} + \eta_+), \\ |y| & (\text{other four classes}), \end{cases}$$

(28)

for small $|y|$. These small-$|y|$ behaviors of $\rho_c(x, y)$ from the small matrix analyses are consistent with the numerical results of large random matrices shown in Fig. 1.

For $X > Y$, the two eigenvalues are real. Thereby, $X$ and $Y$ in Eq. (25) respectively give a repulsive and attractive interaction between the pair of two real eigenvalues. The increase of $X$ ($Y$) increases (decreases) the distance between the two eigenvalues along the real axis. We analytically calculate the spacing distribution function of the two real eigenvalues for the small matrices in the five symmetry classes with $m > 0$ (see Appendix A for detailed derivations). For small $s$, the real-eigenvalue spacing distribution is obtained as

$$p(s) \propto \begin{cases} -s \log s & (\text{class AI} + \eta_+), \\ s & (\text{other four classes}). \end{cases}$$

(29)

Note that $p(s)$ from the small matrix analyses and that from large random matrices are not exactly the same, while they share the same asymptotic behavior for small $s$ for each of the five symmetry classes. Note also that in class $\text{AI} + \eta_+$, the degrees of freedom of the attractive interaction $Y$ (i.e., $n = 4$) is much larger than the degrees of freedom $m$ of the repulsive interaction $X$ (i.e., $m = 1$). This is consistent with $\langle r \rangle < \langle r \rangle_{\text{Poisson}}$ and $\chi > 1$ in class $\text{AI} + \eta_+$ for large random matrices (see Sec. II B for details).

III. DISSIPATIVE MANY-BODY SYSTEMS

In the previous section, we study the general behavior of the level statistics of non-Hermitian random matrices in symmetry classes $\text{AI}$, $\text{A} + \eta$, $\text{AI} + \eta_+$, $\text{AI} + \eta_-$, and $\text{AI} + \eta_+$. The DoS in these symmetry classes shows a delta function peak on the real axis. The number of real eigenvalues is scaled by the square-root of the dimensions of the matrices.

In this section, we study many-body disordered Hamiltonians that belong to symmetry classes $\text{AI}$, $\text{A} + \eta$, $\text{AI} + \eta_+$, and $\text{AI} + \eta_-$. We calculate the level-spacing distributions, level-spacing-ratio distributions, and the numbers of real eigenvalues in the weak disorder regime (ergodic phase) and the strong disorder regime (MBL phase). In the weak disorder regime, we show that the level-spacing and level-spacing-ratio distributions of real eigenvalues match well with those of non-Hermitian random matrices in the same symmetry classes. In addition, we find that the number of real eigenvalues in the weak disorder regime is scaled by the square root of the dimensions of the many-body Hamiltonians, which is also consistent with the random matrix theory. In the strong disorder regime, we show that the many-body model in class $\text{AI}$ is characterized by non-universal scalings of the number of real eigenvalues and its standard deviation. We also provide a phenomenological explanation for this non-universal behavior.

A. Hard-core boson system

We consider the following one-dimensional (1D) hard-core boson model with the nonreciprocal hopping [36–41]:

$$\mathcal{H}_{\text{HN}} = \sum_{i=1}^{L} \left\{ t \left( e^{g} c_{i+1}^{\dagger} c_{i} + e^{-g} c_{i}^{\dagger} c_{i+1}^{\dagger} \right) + U n_{i+1} n_{i} + h n_{i} \right\}.$$  

(30)

Here, $c_{i}$ is a boson annihilation operator at site $i$, $n_{i} = c_{i}^{\dagger} c_{i}$ is its number operator, and the periodic boundary conditions are imposed (i.e., $c_{L+1} = c_{1}$). Every site is allowed to be occupied by no more than one boson under the local hard-core boson constraint. $g$ controls the degree of non-reciprocity and non-Hermiticity, and $h$ is the random potential at site $i$ that distributes uniformly in $[-W/2,W/2]$ with $W \geq 0$. On the occupation-number basis, $\mathcal{H}_{\text{HN}}$ satisfies

$$\mathcal{H}_{\text{HN}} = \mathcal{H}_{\text{HN}}^*$$

(31)

and belongs to class $\text{AI}$. This model can be mapped to an interacting spin model with a random magnetic field and realized, for example, in ultra-cold atoms [14, 28–41].

The Hermitian limit ($g = 0$) of the model was previously studied [11–13, 74–77], where the level-spacing distribution obtained by the exact diagonalization is one of the most powerful tools for detecting the MBL. To identify the ergodic and MBL phases in the non-Hermitian case ($g \neq 0$), Ref. [41] used a scaling of the proportion of the number of real eigenvalues, entanglement entropy, and level-spacing distribution in the complex plane. The proportion of the number of real eigenvalues increases as the disorder strength increases. Furthermore, Ref. [41]...
conjectured that complex eigenvalues collapse onto the real axis and that the proportion of real eigenvalues becomes approximately one when the system undergoes a transition from the ergodic to MBL phases. Meanwhile, the scaling relationship between $N_{\text{real}}$ and $N$ in the ergodic phase was not clarified.

We study the weak (strong) disorder regime of this model at the half filling of the boson number with the parameters $t = 1$, $g = 0.1$, $U = 2$, $W = 2$ ($W = 30$). At the half filling, the boson number $M$ is the same as the half of the lattice site number $L_x$, (i.e., $M = L_x/2$). At least 400 different disorder realizations of Eq. (30) are diagonalized for each system size (the maximal system size is $L_x = 16$) and for each disorder strength.

In the weak disorder regime (ergodic phase), we find that $\rho(x+i\gamma)$ has a delta function peak on the real axis, $\rho(x+i\gamma) = \rho_c(x,y) + \delta(y)\rho_c(x)$, and $\rho_c(x,y)$ shows a soft gap $\rho_c(x,y) \propto |y|$ around the real axis $y = 0$ (Figs. 6(a) and 6(b)). We calculate the level-spacing distribution $p(s)$ and level-spacing-ratio distribution $p_c(r)$ of the real eigenvalues from an energy range around the center of the many-body spectrum. We exclude real eigenvalues near the edges of the spectrum from the statistics. For the system sizes $L_x \geq 12$, the error bars of the mean values of the level-spacing ratio with different system size $L_x$ already overlap with each other (see Fig. 6(e)). This indicates that the level statistics in the ergodic phase reach the convergence for $L_x \geq 12$.

We find that the level-spacing distribution and level-spacing-ratio distribution of real eigenvalues are well described by those obtained from non-Hermitian random matrices in class AI (Figs. 6(e) and 6(f)). For reference, we compare the Kolmogorov-Smirnov distances between $p(s)$, $p_c(r)$ of our hard-core boson model in the ergodic phase and those from the non-Hermitian random matrices in the five symmetry classes in Appendix D. The
the square root of the dimensions Hamiltonian, \( \sqrt{N} \) disorder realizations. We find that also calculate the number random matrix theory and the physical Hamiltonian. We tent with the random matrix theory and hence well within the bosonic models in the weak disorder regimes are consistent with the random matrix theory and hence well within the ergodic phase.

The square-root scaling \( \sqrt{N} \) of the many-body bosonic models in the weak disorder regime (\( W = 2 \) or \( W_x = W_y = W_z = W_D = 1 \)). The square-root scaling \( \sqrt{N} \approx \sqrt{N} \) suggests that all the real eigenvalues of the bosonic models in the weak disorder regimes are consistent with the random matrix theory and hence well within the ergodic phase.

comparison further confirms the consistency between the random matrix theory and the physical Hamiltonian. We also calculate the number \( \sqrt{N} \) of all the real many-body eigenenergies and its average \( \sqrt{N} \) over the different disorder realizations. We find that \( \sqrt{N} \) is scaled by the square root of the dimensions \( N \) of the many-body Hamiltonian. \( \sqrt{N} \sim \sqrt{N} \) (Fig. 7), which is also consistent with the random matrix theory.

In the strong disorder regime (MBL phase), by contrast, almost all the eigenvalues are real, and thus we have \( \sqrt{N} \sim N \). While the level-spacing-ratio distribution \( p_o(r) \) of real eigenvalues is the same as \( p_o(r) \) of uncorrelated real eigenvalues (Eq. 13), the level-spacing distribution \( p(s) \) of real eigenvalues is slightly different from the Poisson distribution. This slight difference is due to the finite-size effect and will vanish if the disorder strength or the system size is increased.

### B. Interacting spin system

We consider the following 1D Heisenberg spin models with random magnetic fields, random energy gain (loss), or random imaginary Dzyaloshinskii–Moriya (DM) inter-

action:

\[
H_1 = \sum_{i=1}^{L_x} JS_i \cdot S_{i+1},
\]

\[
H_1 = H_1 + \sum_i \left\{ h_x^{(i)} S_x^{(i)} + i h_y^{(i)} S_y^{(i)} + h_z^{(i)} S_z^{(i)} \right\},
\]

\[
H_2 = H_1 + \sum_i \left\{ i h_x^{(i)} S_x^{(i)} + i h_y^{(i)} S_y^{(i)} + h_z^{(i)} S_z^{(i)} \right\},
\]

\[
H_3 = H_1 + \sum_i \left\{ h_y^{(i)} S_y^{(i)} + h_z^{(i)} S_z^{(i)} \right\},
\]

\[
H_4 = H_1 + \sum_{i=1}^{L_x} \left\{ i D_x^{(i)} \left( S_y^{(i)} S_z^{(i+1)} - S_z^{(i)} S_y^{(i+1)} \right) + i D_z^{(i)} \left( S_y^{(i)} S_z^{(i+1)} - S_z^{(i)} S_y^{(i+1)} \right) \right\}.
\]

Here, \( S_i \equiv (S_x^{(i)}, S_y^{(i)}, S_z^{(i)}) \equiv \frac{1}{2} (\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}) \) is the spin-1/2 operators at site \( i \) with the periodic boundary conditions (i.e., \( S_{L_x+1} = S_1 \)). The lattice site number \( L_x \) for \( H_4 \) is chosen to be an odd integer to respect TRS whose sign is \(-1\) (see also Eq. 40 below). \( h_x^{(i)}, h_y^{(i)}, \) and \( h_z^{(i)} \) are real random numbers that describe random magnetic fields or random energy gain and loss. \( h_{1/2}^{(i)} (\mu = x, y, z) \) distributes independently and uniformly in \([-W_p/2, W_p/2]\) with \( W_p \geq 0 \). \( D_x^{(i)} \) and \( D_z^{(i)} \) are real random numbers (random imaginary DM interactions) that distribute independently and uniformly in \([-W_D/2, W_D/2]\) with \( W_D \geq 0 \). These non-Hermitian terms can be realized, for example, in continuously-measured cold atomic systems [17, 79].

The random spin model \( H_1 \) satisfies

\[
H_1 = H_1^2
\]

and thus belongs to symmetry class AI. \( H_2 \) satisfies

\[
H_2 = \left( \prod_i \sigma_z^{(i)} \right) H_2 \left( \prod_i \sigma_z^{(i)} \right)^* \quad \text{(34)}
\]

and belongs to symmetry class \( \Lambda + \eta_+ \). \( H_3 \) satisfies

\[
H_3 = H_3^3 \quad \text{(35)}
\]

with

\[
\left( \prod_i \sigma_z^{(i)} \right) = \left( \prod_i \sigma_z^{(i)} \right)^* \quad \text{(37)}
\]

and thus belongs to symmetry class \( \Lambda + \eta_- \). \( H_4 \) satisfies

\[
H_4 = H_4^4 \quad \text{(38)}
\]

\[
H_4 = \left( \prod_i \sigma_y^{(i)} \right) H_4 \left( \prod_i \sigma_y^{(i)} \right) \quad \text{(39)}
\]
FIG. 8. (a)-(d) Heat maps of the density $\rho_c(x, y)$ of complex eigenvalues (class AI), (e)-(h) Integrated density of complex eigenvalues, $\bar{\rho}_c(y)$, (i)-(l) Level-spacing distributions $p(s)$ of real eigenvalues in the weak disorder regimes of the non-Hermitian interacting spin models, and their comparison to $p(s)$ from non-Hermitian random matrices in the same symmetry classes. From the heat maps, $-2.5 < E < 2.5$ for $W_x = W_y = W_z = WD = 1$ is well within the ergodic phase for all the spin models. The mean value $\langle r \rangle = \int_0^1 p_r(r)dr$ of each level-spacing-ratio distribution is shown in the figures. The statistics are taken in the weak disorder regime ($W_x = W_y = W_z = WD = 1$ and $-10 < E < 10$ in all the four spin models). The consistency between $p(s)$ ($p_r(r)$) from the spin models and $p(s)$ ($p_r(r)$) from the random matrices justifies that all eigenstates with real energy $-10 < E < 10$ in the weak disorder regime are in the ergodic phases. (i)-(p) The error ranges are evaluated by the bootstrap method. 

\[ \int \rho_c(x, y)dxdy = \int \rho_c(y)dy = \int p(s)ds = \int p_r(r)dr = 1. \]
For different served with the same asymptotic behavior for small $\rho$ shows the square-root scaling with respect to the dimension of the Hilbert space, while single-particle eigenstates in disordered systems. It should be noted that many-body universally holds true in the ergodic phases of interacting spin model $\mathcal{H}_1$ (Figs. 8(i)-8(l) and 8(m)-8(p)). These same distributions as in the random matrices in the same symmetry class (Figs. 8(e)-8(h)). These distributions are consistent with the random matrix theory. We study the weak disorder regimes of the four 1D Heisenberg models with the parameters $J = 1, W_x = W_y = W_z = W_D = 1$ and with the different system sizes (the maximal size is $L_x = 13$). From the DoS $\rho(E = x + iy) = \rho_c(x, y) + \delta(y)\rho_v(x)$, the soft gap of $\rho_c(x, y)$ around the real axis $y = 0$ is universally observed with the same asymptotic behavior for small $y$ (Fig. 8(c)8(h)). For different $L_x$ ($L_x$ of $\mathcal{H}_4$ changes only for odd numbers), the number $\bar{N}_{\text{real}}$ of real eigenvalues shows the square-root scaling with respect to the dimensions $N$ of the many-body Hamiltonians, $\bar{N}_{\text{real}} \propto \sqrt{N}$ (Fig. 7), which is consistent with the random matrix theory.

Both level-spacing statistics and level-spacing-ratio statistics of real eigenvalues for each spin model show the same distributions as in the random matrices in the same symmetry class (Figs. 8(i)8(l) and 8(m)8(p)). These results indicate that the square-root scaling of $\bar{N}_{\text{real}}$ universally holds true in the ergodic phases of interacting disordered systems. It should be noted that many-body eigenstates in the ergodic phases are extended in many-body Hilbert space, while single-particle eigenstates in the metal phase, which are studied in Sec. [V] are extended in the spatial coordinate space.

Moreover, we study the strong disorder regimes of $\mathcal{H}_1$ in class AII with the parameters $J = 1, W_x = W_z = 20, W_y = 2$ or $J = 1, 15 \leq W_x = W_z \leq 40, W_y = 10$. We find that $\mathcal{H}_2$ and $\mathcal{H}_3$ in the strong disorder regimes show the level statistics of real eigenvalues similar to those of $\mathcal{H}_1$ in the strong disorder regimes (not shown). $W_x$ and $W_z$ in $\mathcal{H}_1$ describe Hermitian local disorder while $W_y$ describes anti-Hermitian local disorder. When the Hermitian disorder dominates over the anti-Hermitian disorder ($W_x = W_z = 20, W_y = 2$), almost all the eigenvalues are real, where we have $\bar{N}_{\text{real}} \propto N$. The level-spacings and level-spacing-ratios of real eigenvalues satisfy the Poisson distribution (Fig. 10(a)) and the distribution in Eq. (18) (Fig. 10(c)).

When the anti-Hermitian disorder is of the same order as the Hermitian disorders ($W_y = 10, 10 \leq W_x = W_z \leq 40$), the number $\bar{N}_{\text{real}}$ of real eigenvalues fluctuates largely from sample to sample. The standard deviation of $\bar{N}_{\text{real}}, \sigma_{\bar{N}_{\text{real}}} = (\bar{N}_{\text{real}}^{\text{real}} - \bar{N}_{\text{real}})^2$, grows exponentially with the system size $L_x$, and $\sigma_{\bar{N}_{\text{real}}}$ is much larger than $\bar{N}_{\text{real}} = (\bar{N}_{\text{real}})$ (Fig. 9). Notably, we find that the scalings of $\bar{N}_{\text{real}}$ and $\sigma_{\bar{N}_{\text{real}}}$ with respect to the dimensions $N$ of the Hamiltonian are characterized by non-universal powers, such as $\bar{N}_{\text{real}} \sim N^\alpha$ (see Fig. 11). The powers of both $\bar{N}_{\text{real}}$ and $\sigma_{\bar{N}_{\text{real}}}$ increase when the Hermitian disorders become larger.
The non-universal powers $\alpha$ in the scalings of $N_{\text{real}}$ and $\sigma_{N_{\text{real}}}$ can be explained with a hypothesis that the MBL phase in the non-Hermitian case exhibits an emergent integrability as in the Hermitian case \[12, 13\]. Suppose that the many-body non-Hermitian Hamiltonian in the MBL phase can be effectively expanded in terms of $L_x$ mutually-commuting bit operators $\tau_z^{(i)}$ ($i = 1, 2, \cdots, L_x$) as

$$
\mathcal{H}_1^{\text{MBL}} = \sum_{i=1}^{L_x} \tau_z^{(i)} + \sum_{i,j} J_{ij} \tau_z^{(i)} \tau_z^{(j)} + \sum_{ijk} K_{ijk} \tau_z^{(i)\tau_z^{(j)}\tau_z^{(k)}} + \cdots.
$$

Here, we have $[\tau_z^{(i)}, \tau_z^{(j)}] = [\mathcal{H}_1^{\text{MBL}}, \tau_z^{(i)}] = 0$ for all $i$ and $j$. The bit operator $\tau_z^{(i)}$ is a two-by-two non-Hermitian matrix. $\mathcal{H}_1^{\text{MBL}}$ respects TRS and belongs to class AI.

For real numbers $a_\alpha^{(i)}$ ($\alpha = 0, 1, 2, 3$), the real numbers of different $i$ and different components are almost independent of one another in the strong disorder regime. When $\mathcal{H}_1$ is dominated by the on-site random terms ($W_x, W_y, W_z \gg J$), the bit operator is given by the random magnetic fields at each lattice site, $a_0^{(i)} = 0$, $a_1^{(i)} = h_z^{(i)}/2$, $a_2^{(i)} = h_x^{(i)}/2$, and $a_3^{(i)} = h_y^{(i)}/2$. For $(a_1^{(i)})^2 + (a_2^{(i)})^2 > (a_3^{(i)})^2$, the bit operator $\tau_z^{(i)}$ has real eigenvalues,

$$
\lambda_{\pm 1}^{(i)} = a_0 \pm \sqrt{(a_1^{(i)})^2 + (a_2^{(i)})^2 - (a_3^{(i)})^2}.
$$

For $(a_1^{(i)})^2 + (a_2^{(i)})^2 < (a_3^{(i)})^2$, the bit operator has complex eigenvalues,

$$
\lambda_{\pm i}^{(i)} = a_0 \pm i \sqrt{(a_3^{(i)})^2 - (a_1^{(i)})^2 - (a_2^{(i)})^2}.
$$

From them, a many-body eigenvalue of $\mathcal{H}_1^{\text{MBL}}$ is given by

$$
E(\{\beta_j\}) = \sum_{j=1}^{L_x} \lambda_{\beta_j}^{(j)} + \sum_{i,j} J_{ij} \lambda_{\beta_i}^{(i)} \lambda_{\beta_j}^{(j)} + \cdots
$$

with $\beta_j = \pm 1, \pm i$ for $j = 1, 2, \cdots, L_x$.

Let $p$ be a probability of the bit operator of $i$ having real eigenvalues. The probability is independent of $i$ and two bit operators at different $i$ and $j$ are uncorrelated with each other. Thus, a probability of a given many-body eigenvalue being real-valued equals a probability of all $\lambda_{\beta_i}^{(i)}$ being real, which is $p^{L_x}$. The average and standard deviation of the number of real eigenvalues are estimated as

$$
N_{\text{real}} = (2p)^{L_x},
$$

$$
\sigma_{N_{\text{real}}} = \sqrt{\langle N_{\text{real}}^2 \rangle - N_{\text{real}}^2} = 2^{L_x} \sqrt{p^{L_x} - p^{2L_x}}
$$

for $L_x \gg 1$ and $p < 1$. Here, $\langle \cdots \rangle$ means the average over
different disorder realizations. These evaluations lead to the scalings,
\[ N_{\text{real}} \sim N^\alpha, \quad \alpha = 1 + \log_2 p < 1, \]
\[ \sigma_{N_{\text{real}}} \sim N^\beta, \quad \beta = 1 + \frac{1}{2} \log_2 p < 1. \]  

Note that in the MBL phase with \( p < 1 \), \( \sigma_{N_{\text{real}}} \) is much larger than \( N_{\text{real}} \) for large \( N \), being consistent with the numerical observation (Fig. 9). When the on-site disorder in \( H_I \) becomes much more dominant than the Heisenberg interaction \( J \), the probability \( p \) in the scaling forms can be determined only by \( W_x, W_y, \) and \( W_z \) (see \( p_0 \) in the caption of Fig. 11). The numerical data with \( W > \bar{p} < \sigma \) can be determined only by \( W_x \), \( W_y \), and \( W_z \) (see \( p_0 \) in the caption of Fig. 11). The numerical data with \( \bar{p} = N(0), \) which are also consistent with the phenomenological explanation in Eq. 45.

The level-spacing statistics of real eigenvalues show the Poisson distribution in the strong disorder regime. To illustrate this with \( \sigma_{N_{\text{real}}} \gg N_{\text{real}} \), we unfold the level-spacing of many-body eigenvalues by the density \( \rho_{\text{real}}(x, y) \) of complex eigenvalues in each sample of different disorder realizations,
\[ s_i = (\lambda_{i+1} - \lambda_i) \tilde{\rho}_{\text{real}}^{(k)} \left( \frac{\lambda_{i+1} + \lambda_i}{2} \right) \]
with
\[ \tilde{\rho}_{\text{real}}^{(k)}(x) = \frac{N_{\text{real}}^{(k)}}{N_{\text{real}} \sum_{\lambda \in \mathbb{R}} \langle \delta(x - \lambda) \rangle}. \]

Here, \( \lambda_i \) \((i = 1, 2, \ldots, 2L)\) stands for the many-body eigenvalues in the descending order and \( N_{\text{real}}^{(k)} \) is the number of the real eigenvalues in the \( k \)-th sample. The real-eigenvalue spacing thus normalized shows the Poisson distribution in the strong disorder regime with \( \sigma_{N_{\text{real}}} \gg N_{\text{real}} \) (Fig. 10(b)). The spacing-ratio statistics over samples with very different numbers of real eigenvalues inevitably increase statistical errors (Fig. 10(d)).

### IV. DISSIPATIVE FREE FERMIONS

In the previous section, we demonstrate the universal level statistics of real eigenvalues in the ergodic phases of the bosonic many-body Hamiltonians. In this section, we study non-interacting fermionic Hamiltonians with disorder and non-Hermiticity that belong to symmetry classes AI + \( \eta_+ \) and AII + \( \eta_+ \). We calculate the DoS, the level-spacing statistics, and the number of real eigenvalues in the weak and strong disorder regimes. In both regimes, the DoS has a delta function peak on the real axis, \( \rho(E = x + iy) = \rho_0(x, y) + \delta(y) \rho_0(x) \). In the metal phases, we demonstrate that the number of real eigenvalue is scaled by the square root of the dimensions of the Hamiltonians, being consistent with the random matrix theory. We also find that the real-eigenvalue spacings and spacing ratios for class AI + \( \eta_+ \) show the same distributions as those of the random matrices in the same symmetry class while we find discrepancies for class AII + \( \eta_+ \). We discuss possible reasons for these discrepancies. In the localized phases, by contrast, the level-spacings of real eigenvalues show the Poisson distribution, and the number of real eigenvalues is linearly scaled by the dimensions of the Hamiltonians.

#### A. 3D class AI + \( \eta_+ \)

We study a non-Hermitian extension of the Anderson model on the three-dimensional (3D) cubic lattice:
\[ H_{3D} = \sum_i \left( \epsilon_i \sigma_0 + \epsilon'_i \sigma_x \right) c_i + i \omega_i c_i^\dagger \sigma_y c_j + t \sum_{\langle i,j \rangle} \epsilon_i \sigma_0 c_j. \]  

Here, \( \epsilon_i \) and \( \epsilon'_i \) describe the Hermitian disorder potentials that distribute independently and uniformly in \([-W_1/2, W_1/2]\), and \( \omega_i \) describes the anti-Hermitian disordered potential that distributes uniformly in \([-W_2/2, W_2/2]\). The non-Hermitian Hamiltonian \( H_{3D} \) satisfies TRS
\[ H_{3D} = H_{3D}^*, \]
and pH
\[ H_{3D} = \sigma_z H_{3D}^\dagger \sigma_z, \]
where the TRS operator and the pH operator commute with each other. Thus, this model belongs to symmetry class AI + \( \eta_+ \).

We investigate the weak (strong) disorder regime with the parameters \( t = 1, W_1 = 3, W_2 = 1 (W_1 = 60, W_2 = 60) \) and with the periodic boundary conditions. We diagonalize \( H_{3D} \) with 240 different disorder realizations with different system sizes (the maximal system size is 16 x 16 x 16). We find that eigenstates with real energy \( E \) undergo the Anderson transition in the weak disorder regime. An energy region near \( E = 0 (|E| < 4) \) is in the metal and localized phases in the weak and strong disorder regimes, respectively. We calculate the DoS, the level-spacing distribution, and the number of real eigenvalues in the weak and strong disorder regimes. In the weak disorder regime, \( \rho_0(x, y) \) shows a soft gap \( \rho_0(x, y) \propto -|y| \log|y| \) around the real axis \( y = 0 \) (Figs. 13(a) and 13(b)), sharing the same scaling as in the random matrix theory for symmetry class AI + \( \eta_+ \). The mean value of the level-spacing ratios converges for the system size \( L \geq 8 \) (Fig. 13(c)). The level-spacing distribution \( p(s) \) and level-spacing-ratio distribution \( p_c(r) \) of real eigenvalues respectively match well with \( p(s) \) and
The mean value \( \langle r \rangle \) (Figs. 13(c) and 13(f)). The number of real eigenvalues is evaluated as
\[
N_{\text{real}} \sim (L/\xi)^d \sqrt{d^d} \propto N,
\] (50)
which is consistent with the numerical results.

### B. 2D class AII + \( \eta_+ \)

We study a non-Hermitian extension of the disordered SU(2) model on the two-dimensional (2D) square lattice:

\[
\mathcal{H}_{2D} = \sum_i \left( \varepsilon_i c_i^\dagger \sigma_0 c_i + \varepsilon'_i d_i^\dagger \sigma_0 d_i \right) + \sum_{(i,j)} t_1 \left[ \varepsilon_i c_i R(i,j)c_j + d_i^\dagger R'(i,j)d_j \right] + \sum_{(i,j)} t_2 \left[ \varepsilon'_i U(i,j)d_j + d_i^\dagger U'(i,j)c_j \right],
\] (51)

where \( c_i \) and \( d_i \) are annihilation operators for two different orbitals defined on site \( i \). Both operators have
ties, tribute uniformly with respect to the Harr measure of orbital space, satisfying \( \tau \) symmetry class AII +.

The term with \( R \) and \( pH \) is for the metal phase in the weak disorder regime \((|E| \leq 4, W_1 = 3, \text{ and } W_2 = 1) \). The red line is for the localized phase in the strong disorder regime (all the real energy \( E \), \( W_1 = W_2 = 60 \)). For reference, the black lines are \( N_{\text{real}} \propto N \) and \( \bar{N}_{\text{real}} \propto \sqrt{N} \), respectively.

The term with \( t_2 \) is anti-Hermitian and the others are Hermitian. Let \( \tau_z \) be a two-by-two matrix acting on the orbital space, satisfying \( \tau_z(c_i, d_j)^T \equiv (c_i, -d_i)^T \). The Hamiltonian in Eq. (51) satisfies TRS

\[
\mathcal{H}_{2D} = \sigma_y \mathcal{H}_{2D}^{\dagger} \sigma_y
\]

and pH

\[
\mathcal{H}_{2D} = \tau_z \mathcal{H}_{2D}^{\dagger} \tau_z,
\]

where the TRS operator and the pH operator commute with each other. Thus, the Hamiltonian belongs to symmetry class AII + \( \eta_+ \). This Hamiltonian is a minimal model to study the interplay between the spin-orbit coupling [51] and pH.

We investigate the weak (strong) disorder regime of Eq. (51) with the parameters \( t_1 = 1, t_2 = 0.1, W = 0.4 \) \((W = 80)\) and with the periodic boundary conditions.

In the weak disorder regime, we find that eigenstates with real eigenvalues \( E \) undergo the Anderson transition at a certain mobility edge \( E = E_c \). The normalized localization length \( \Lambda(E, L) \equiv \xi(E, L)/L \) shows a scale invariant behavior at \( E = E_c \) (Fig. 15). Here, the localization length \( \xi_x(E, L) \) is calculated along the \( x \) direction by the transfer matrix method in the quasi-1D geometry \( L_x \times \Lambda \) [48, 52, 54]. For the weak disorder regime, \( |E| < E_c \), \( |E| > E_c \) correspond to the metal and localized phases, respectively (Fig. 15). We diagonalize the Hamiltonians in Eq. (51) with 240 different disorder realizations for each system size (the maximal system size is \( 55 \times 55 \) sites), where eigenstates near the mobility edge are excluded from the level statistics of real eigenvalues.

In the weak disorder regime, the density \( \rho_c(x, y) \) of complex eigenvalues shows the soft gap \( \rho_c(x, y) \propto |y| \) in the metal phase \( \langle |x| = |\text{Re}(E)| \rangle < E_c \) for \( y = \text{Im}(E) \) much smaller than the mean level-spacing (Figs. 17(a) and 17(b)). The soft-gap behavior is consistent with the DoS of non-Hermitian random matrices in class AII + \( \eta_+ \). The number of real eigenvalues within the metal phase, \( N_{\text{real}} \equiv \int_{-E_c}^{E_c} dx \rho_c(x) \), shows the square-root scaling with respect to the dimensions \( N \) of the Hamiltonian, \( N_{\text{real}} \propto \sqrt{N} \) (Fig. 10), which is also consistent with the random matrix theory. On the other hand, the number of real eigenvalues in the localized phase,
A possible reason for the discrepancies is unusual level interactions in non-Hermitian random matrices in class AII + \( \eta_+ \). As discussed in Sec. \( \text{III} \), the small level-spacing ratio \( \langle r \rangle < \langle r \rangle_{\text{Poisson}} \) and the large spectral compressibility \( \chi > \chi_{\text{Poisson}} \), which are also supported by the small random matrix analyses, suggest that the attractive interaction between real eigenvalues in the finite \( s \) regime dominates the repulsive interaction in the small \( s \) regime on average. We speculate that this unconventional level attraction makes the level statistics of the random matrices in the finite \( s \) regime unstable against details of physical models. Only for level spacing \( s \ll 1 \), the behaviors between the physical model and random matrices are consistent. It is unclear whether \( p(s) \) and \( p_r(r) \) of physical models in class AII + \( \eta_+ \) are universal or not, and we leave this issue for future study. It is also interesting to investigate whether the non-Hermitian generalization of other random matrix ensembles, such as the power-law random banded matrix ensemble \( \text{Ref. [86]} \) and the Rosenzweig-Porter random-matrix ensemble \( \text{Ref. [87, 88]} \), can describe the physical models in class AII + \( \eta_+ \).

In the strong disorder regime (\( W = 80 \)), the soft gap in the DoS around the real axis \( y = 0 \) disappears (not shown here). The level-spacings of real eigenvalues show the Poisson distribution (Fig. 17(f)), and the level-spacing ratios of real eigenvalues show the same distribution as uncorrelated real numbers in Eq. \( \text{(18)} \) (Fig. 17(h)). The average number of all the real eigenvalues becomes proportional to \( N \) (Fig. 17(g)). In this phase, the Hamiltonian is dominated by the Hermitian part and almost all the eigenvalues are real, leading to the linear scaling \( \tilde{N}_\text{real} \propto N \). To confirm that this linear scaling is a general property of the Anderson localized phase, we generalize the model in Eq. \( \text{(51)} \) and study the following model \( \mathcal{H}^\prime_{2D} \):

\[
\mathcal{H}^\prime_{2D} = \mathcal{H}_{2D} + \Delta \mathcal{H},
\]

\[
\Delta \mathcal{H} = \sum_i \omega_i c_i^\dagger \sigma_0 d_i - \omega_i d_i^\dagger \sigma_0 c_i. \tag{57}
\]

Adding \( \Delta \mathcal{H} \) does not change the symmetry class of \( \mathcal{H}_{2D} \) (class AII + \( \eta_+ \)). Here, \( \omega_i \) distributes uniformly in \([-W'/2, W'/2]\). We calculate the DoS, the level-spacing distribution \( p(s) \), and the number of real eigenvalues with the parameter \( W' = 40 \). Thereby, \( \rho_c(x, y) \) has no soft gap around \( y = 0 \) (Fig. 17(c)), \( p(s) \) is consistent with the Poisson distribution, and \( \tilde{N}_\text{real} = \int_{-\infty}^{+\infty} dx \rho_r(x) \) remains linear in \( N \) (Fig. 17(h)).

V. CONCLUSIONS

Non-Hermitian Hamiltonians in the seven symmetry classes (classes \( A + \eta, \text{AI}, \text{AI} + \eta_\pm, \text{AII}, \text{and AII} + \eta_\pm \)) have TRS or pH. Real eigenvalues respect these symmetries with complex or Hermitian conjugation. We find that a sub-extensive number of eigenenergies of non-Hermitian random matrices in classes \( A + \eta, \text{AI}, \text{AI} +\)
$\eta$, and AI + $\eta_+$ are real, where the DoS on the complex plane has a delta function peak on the real axis. We clarify the universal level-spacing distributions on the real axis in these five symmetry classes for non-Hermitian random matrices, as well as bosonic many-body Hamiltonians in the ergodic phases and fermionic non-interacting Hamiltonians in the metal phases. In classes A + $\eta$, AI, and AI + $\eta$, the level statistics of real eigenvalues show good agreement between the ergodic physical models and random matrices, while we find discrepancies in class AI + $\eta_+$. The average number of real eigenvalues in the ergodic phases universally shows the square-root scaling with respect to the dimensions of the Hamiltonians. We explain the universal asymptotic behaviors of the DoS around the real axis and the level-spacing distributions on the real axis by small random matrix analyses. We also clarify the level statistics of the physical models in the Anderson localized and MBL phases by extensive numerical calculations together with the phenomenological interpretations.

The universal level-spacing and level-spacing-ratio distributions of real eigenvalues and the scaling relationship of the number of real eigenvalues obtained in this paper provide powerful methods of studying level statistics of non-Hermitian systems with TRS and/or pH. They are also useful for detecting quantum chaos, many-body localization, and real-complex transitions in non-Hermitian systems with the symmetries.

For example, the level statistics of real eigenvalues help us answer fundamental questions on non-Hermitian disordered systems. Analyzing level spacings or level-spacing ratios of real eigenvalues in the different energy windows, we can determine the presence of mobility edges in non-Hermitian many-body systems. By the finite-size scaling analysis of spacing ratios, we can also evaluate the critical exponents of the Anderson transitions in non-Hermitian systems with TRS and/or pH. Comparisons of the critical exponents with systems without
TR or pH tell us whether TRS and/or pH change the universality classes of MBL transitions in non-Hermitian systems. We believe that with the help of the results in this paper, a number of spectral analysis method and quantum chaotic study used in Hermitian systems are ready to be applied to non-Hermitian systems with TRS and/or pH.

The real spectrum is related to the stability of non-Hermitian systems in their long-time dynamics. The square-root scaling of the number of real eigenvalues hints at possible dynamical instability in ergodic non-Hermitian systems. In contrast, the linear scaling of the Anderson localized phase may imply that disorder can stabilize the dynamics of non-Hermitian systems. The non-universal scaling of the number of real eigenvalues, together with our explanation, also serves as evidence for the emergent integrability in the MBL phase of non-Hermitian systems.

The eigenstates on the real (imaginary) axis can only respect TRS (PHS), TRS, and pH (CS) among all the symmetries, as long as they are away from zero in the complex plane. Thus, for all the 38 symmetry classes, we believe that the level-spacing distributions of real (purely imaginary) eigenvalues away from zero satisfy one of the five universal level spacing distributions found in this paper. For example, for non-Hermitian random matrices in symmetry class BDI (i.e., with TRS whose sign is +1 and PHS whose sign is +1), PHS affects the spectral statistics only around zero, and hence the real-eigenvalue spacing distribution away from zero should be the same as that in class AI. We leave testing this for all the 38 symmetry classes for future work.

While we have focused on non-Hermitian Hamiltonians in this paper, our results should also be relevant to Lindbladians, which govern the open quantum dynamics of the Markovian master equation. Similarly to closed quantum systems, it was conjectured that the level-spacing statistics of Lindbladians obey those of non-Hermitian random matrices in the non-integrable phases and the Poisson statistics in the integrable phases. This conjecture was previously verified for generic complex eigenvalues of Lindbladians away from the real axis. Notably, Lindbladians always respect TRS, where time reversal is effectively defined by the combination of complex conjugation and the swap operation between the bra and ket spaces. Thus, we expect that our results of the level statistics of real eigenvalues for non-Hermitian random matrices should coincide with those of non-integrable Lindbladians in the same symmetry classes. It is also notable that the quadratic Lindbladians can be classified by the $A_2^I$ symmetry class, which is the same as the ten-fold symmetry class studied in this paper.

Before concluding this paper, we note in passing that a non-Hermitian extension of the disordered Su-Schrieffer-Heeger model was shown to exhibit the level-spacing statistics of the Gaussian orthogonal ensemble. This is due to an additional symmetry that enables a similarity transformation between this non-Hermitian model and the Hermitian Su-Schrieffer-Heeger model. Thus, in this non-Hermitian model, only the Hermitian degrees of freedom are present, and all eigenvalues are real, which are different from generic non-Hermitian random matrices studied in this paper.

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Appendix A: Analyses of small random matrices

Small non-Hermitian random matrices in the seven symmetry classes are written in the following unified form,

$$\mathcal{H}^{(s)} = a_0 I + \sum_{i=1}^{m} a_i L_i + \sum_{i=m+1}^{m+n} i a_i L_i. \quad (A1)$$

Here, $I$ is the identity matrix, $L_1, L_2, \ldots, L_{m+n}$ are anti-commuting Hermitian traceless matrices, $\{L_i, L_j\} = 2\delta_{i,j} I$, and $a_0, a_1, \ldots, a_{m+n}$ are real numbers. Note that

$$\text{Tr}(\mathcal{H}^{(s)} \mathcal{H}^{(s)\dagger}) = \left( \sum_{i=0}^{m+n} a_i^2 \right) \text{Tr}(I). \quad (A2)$$

The real numbers $a_0, a_1, \ldots, a_{m+n}$ are independent of each other and obey the identical standard Gaussian distribution. Note that $\sum_{i=1}^{k} a_i^2$ obeys the $\chi^2$ distribution with the degree $k$. The probability of $\sum_{i=1}^{k} a_i^2 = X$ is given by

$$p(X; k) = \frac{\chi^{\frac{2k}{2}}(-\frac{X}{2})^{-\frac{k}{2}}}{2^{k} \Gamma^2(\frac{k}{2})} \text{ for } X \geq 0,$$

$$\frac{1}{2^{k} \Gamma^2(\frac{k}{2})} \text{ for } X < 0. \quad (A3)$$

The square of the traceless part of $\mathcal{H}^{(s)}$ is proportional to $I$,

$$\left( \sum_{i=1}^{m} a_i L_i + \sum_{i=m+1}^{m+n} i a_i L_i \right)^2 = \left( \sum_{i=1}^{m} a_i^2 - \sum_{i=m+1}^{m+n} a_i^2 \right) I. \quad (A4)$$
Thus, eigenvalues of $\mathcal{H}^{(s)}$ are given by

$$\lambda = a_0 \pm \sqrt{X - Y}, \quad X \equiv \sum_{i=1}^{m} a_i^2, \quad Y \equiv \sum_{i=m+1}^{m+n} a_i^2. \quad (A5)$$

The probability that the eigenvalues are real is given by the probability of $X \geq Y$:

$$p_{\lambda = \lambda^*} = \int_{-\infty}^{\infty} dX \int_{-\infty}^{+\infty} dY \theta(X - Y) p(X; m) p(Y; n), \quad (A6)$$

where $\theta(u)$ is the step function satisfying $\theta(u) \equiv 1$ for $u \geq 0$ and $\theta(u) \equiv 0$ for $u < 0$.

In terms of Pauli matrices $\sigma_\mu$ and $\tau_\mu$ ($\mu = 0, x, y, z$) and their Kronecker products $\sigma_\mu \sigma_\nu$, the traceless parts of the small random matrices in the seven symmetry classes are given as

$$\begin{align*}
\mathcal{H}^{(s)}_{A_1} &= a_1 \sigma_x + a_2 \sigma_y + ia_3 \sigma_y, \\
\mathcal{H}^{(s)}_{A_1+\eta} &= a_1 \sigma_x + ia_2 \sigma_x + ia_3 \sigma_y, \\
\mathcal{H}^{(s)}_{A_1+\eta^*} &= a_1 \sigma_x + a_2 \sigma_y, \\
\mathcal{H}^{(s)}_{A_1+\eta} &= (\tilde{a}_1 \tau_z + a_2 \tau_x + a_3 \tau_y \sigma_y + ia_4 \tau_y \sigma_x + ia_5 \tau_y \sigma_z, \\
\mathcal{H}^{(s)}_{A_1+\eta} &= a_1 \tau_z + a_2 \tau_x + a_3 \tau_y \sigma_y + ia_4 \tau_y \sigma_x + ia_5 \tau_y \sigma_z. \\
\end{align*} \quad (A7)$$

In fact, the small matrix in each class respects the following symmetries,

$$\begin{align*}
\mathcal{H}^{(s)}_{A_1} &= \mathcal{H}^{(s)*}_{A_1}, \\
\mathcal{H}^{(s)}_{A_1+\eta} &= \mathcal{H}^{(s)*}_{A_1+\eta}, \\
\mathcal{H}^{(s)}_{A_1+\eta^*} &= \mathcal{H}^{(s)*}_{A_1+\eta^*}, \\
\mathcal{H}^{(s)}_{A_1+\eta} &= \mathcal{H}^{(s)*}_{A_1+\eta}, \\
\mathcal{H}^{(s)}_{A_1+\eta^*} &= \mathcal{H}^{(s)*}_{A_1+\eta}. \\
\end{align*} \quad (A8)$$

The number $m$ of the real degrees of freedom and the number $n$ of the imaginary degrees of freedom for the small random matrices in each symmetry class are summarized in Table II. The probability of real eigenvalues is finite for $\mathcal{H}^{(s)}$ in classes $A_1$, $A + \eta$, $A_1 + \eta_+$, $A_1 + \eta_-$, and $A_1 + \eta_+$ because of $m \neq 0$; on the other hand, the probability of real eigenvalues is zero for $\mathcal{H}^{(s)}$ in classes $A_1$ and $A_1 + \eta_-$ because of $m = 0$.

1. Real-eigenvalue spacing

With a finite probability, the small random matrices in classes $A_1$, $A + \eta$, $A_1 + \eta_+$, and $A_1 + \eta_-$ have a pair of real eigenvalues. The distance $s$ between the two real eigenvalues is given by

$$s = 2 \sqrt{\sum_{i=1}^{m} a_i^2 - \sum_{i=m+1}^{m+n} a_i^2}. \quad (A9)$$

The probability of $(s/2)^2 = X$ under the condition that the eigenvalues are real is calculated as follows,

$$P(X) = \frac{\int_{-\infty}^{\infty} dX' \int_{-\infty}^{\infty} dY' \delta(X' - Y') p(X'; m) p(Y'; n)}{\int_{-\infty}^{\infty} dX' \int_{-\infty}^{\infty} dY' \theta(X' - Y') p(X'; m) p(Y'; n)}$$

$$= \frac{\int_{0}^{\infty} dx (x + X) \frac{1}{2} e^{-\frac{x}{2}} X^{-1} e^{-\frac{X}{2}}}{\int_{0}^{\infty} dx (x + X) \frac{1}{2} e^{-\frac{x}{2}} X^{-1} e^{-\frac{X}{2}}} = \begin{cases} 
\frac{1}{2} e^{-\frac{x}{2}} & \text{(m, n) = (2, 1)}, \\
\frac{1}{2(e^{\sqrt{2}-1})} e^{-\frac{X}{2}} \text{erfc}(\sqrt{X}) & \text{(m, n) = (1, 2)}, \\
\frac{1}{2(2^{\sqrt{2}-1})\sqrt{\pi}} e^{-\frac{X}{2}} (\text{erf}(\sqrt{X}) + 2\text{erf}(\sqrt{X})) & \text{(m, n) = (1, 1)}, \\
\frac{1}{2\sqrt{\pi}(4\sqrt{2}-5)} e^{-\frac{X}{2}} (\text{erf}(\sqrt{X}) + 2\text{erf}(\sqrt{X})) & \text{(m, n) = (3, 2)}, \\
\frac{1}{2\sqrt{\pi}(4\sqrt{2}-5)} e^{-\frac{X}{2}} (\text{erf}(\sqrt{X}) + 2\text{erf}(\sqrt{X})) & \text{(m, n) = (4, 1)}, \\
\end{cases}$$

with $\int_{0}^{\infty} dX P(X) = 1$. Here, $\text{erfc}(u) \equiv \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-t^2} dt$ is the complementary error function and $K_0(u)$ is the modified Bessel function of the second kind. Then, the probability of the distance being $s$ under the condition that the eigenvalues are real is given by

$$P_s(s) = \frac{s}{2} P \left( \left( \frac{s}{2} \right)^2 \right), \quad (A10)$$

with $\int_{0}^{\infty} P_s(s) ds = 1$. Note that the real-eigenvalue spacing distribution function $p(s)$ in the main text is defined for the distance normalized by the mean value of the distance: $\int_{0}^{\infty} p(s) s ds = 1$. Thus, we have

$$p(s) \equiv \tilde{s} P_s(\tilde{s}s), \quad (A11)$$

with the mean value

$$\tilde{s} \equiv \int_{0}^{\infty} s P_s(s) ds. \quad (A12)$$

From Eqs. (A10) and (A11), we obtain the probability distribution functions of the normalized distances between the two real eigenvalues as follows,
with
\[
\bar{s}_2 = \frac{2(1 + \sqrt{2})(2 - \sqrt{2}\sinh^{-1}(1))}{\sqrt{\pi}} \approx 2.053, \quad (A14)
\]
\[
\bar{s}_3 = \frac{6 - \sqrt{2}\sinh^{-1}(1)}{(2\sqrt{2} - 1)\sqrt{\pi}} \approx 1.467, \quad (A15)
\]
\[
\bar{s}_4 = \frac{20 - 14\sqrt{2}\sinh^{-1}(1)}{(4\sqrt{2} - 5)\sqrt{\pi}} \approx 2.190. \quad (A16)
\]
Note that \(p(s)\) takes the following asymptotic forms for \(s \ll 1\),
\[
p(s) \sim \begin{cases} 
  \bar{s}, & (m, n) = (2, 1), (1, 2), (3, 2), (1, 4), \\
  -s \log(s), & (m, n) = (1, 1). 
\end{cases} \quad (A17)
\]

2. Density of states (DoS)

We calculate the DoS of the small random matrices. For \(m \neq 0\), the DoS \(\rho(E = x + iy)\) is decomposed into the density \(\rho_c(x, y)\) of complex eigenvalues and the density \(\rho_r(x)\) of real eigenvalues, \(\rho(E) = \rho_c(x, y) + \delta(y)\rho_r(x)\), while for \(m = 0\), the density of real eigenvalues always vanishes, \(\rho(E) = \rho_c(x, y)\). The density \(\rho_c(x, y)\) of complex eigenvalues is the probability of \(E\) being \(x + iy\). For \(m \neq 0\), it is given by
\[
\rho_c(x, y) = \frac{2|y|e^{-\frac{x^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY \delta(Y - X - y^2) \\
  \times p(Y; n) p(X; m)
\]
\[
= \begin{cases} 
  \sqrt{2}|y|e^{-\frac{x^2}{4}} \text{erfc}(|y|), & (m, n) = (2, 1), \\
  \sqrt{2}|y|e^{-\frac{x^2}{4}} + \frac{1}{2}, & (m, n) = (1, 2), \\
  \sqrt{2}|y|K_0(x^2) e^{-x^2}, & (m, n) = (1, 1), \\
  \sqrt{2}|y|e^{-\frac{x^2}{4}} - \frac{1}{2}, & (m, n) = (3, 2), \\
  \frac{1}{2}(2y^2 + 1)|y|e^{-\frac{x^2}{4}} - \frac{1}{2}, & (m, n) = (1, 4), 
\end{cases} \quad (A18)
\]
with the normalization \(\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho_c(x, y) = 2\).
Here, the constant 2 is the number of different eigenvalues of small matrices. Note also that we have
\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho_c(x, y) \neq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x, y) = 2 \text{ under this normalization condition.}
\]
For \(m = 0\), the DoS in the complex plane \(\rho(x, y) = \rho_c(x, y)\) is given by
\[
\rho_c(x, y) = \frac{2|y|e^{-\frac{x^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dY \delta(Y - y^2) p(Y; n)
\]
\[
= \frac{1}{2\sqrt{2\pi}} \sqrt{\Gamma(\frac{n}{2})} |y|^{n-1} e^{-\frac{x^2 - y^4}{2}} \quad (m, n) = (0, 2),
\]
\[
\frac{1}{2}|y|^{2} e^{-\frac{x^2 - y^4}{2}} \quad (m, n) = (0, 3),
\]
with the same normalization \(\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho_c(x, y) = 2\).
For \(|y| \ll 1\), the density \(\rho_c(x, y)\) of complex eigenvalues takes the following asymptotic forms for each symmetry class,
\[
\rho_c(x, y) \sim \begin{cases} 
  |y|, & (m, n) = (2, 1), (1, 2), (3, 2), (1, 4), \\
  |y|^2, & (m, n) = (0, 3), \\
  -|y| \log(|y|), & (m, n) = (1, 1). 
\end{cases} \quad (A20)
\]

Appendix B: Random matrix ensembles

1. Class AI

For a random matrix in class AI, let \(U_T\) be the identity matrix \(I\). In this choice, \(H\) is a real matrix. The probability distribution function in the Gaussian ensemble is given by
\[
p(H) dH = C_N \exp \left( -\beta \sum_{ij} H_{ij}^2 \right) \prod_{ij} dH_{ij}, \quad (B1)
\]
where \(\beta\) is a constant and \(C_N\) is a normalization constant. With \(U_T = I\), the probability distribution function in the Bernoulli ensemble is given by
\[
H_{ij} = \begin{cases} 
  1 & \text{with the probability } 1/2, \\
  -1 & \text{with the probability } 1/2.
\end{cases} \quad (B2)
\]
Notably, the probability distribution in the Bernoulli ensemble is not invariant under unitary transformations.
2. Class $\text{AI} + \eta^+$

For a random matrix in class $\text{AI} + \eta^+$, let us choose $\mathcal{U}_n = I$ and $\mathcal{U}_y = \sigma_z \otimes I_{\frac{N}{2} \times \frac{N}{2}}$ with the identity matrix $I_{\frac{N}{2} \times \frac{N}{2}}$. Then, $\mathcal{H}_{\text{AI} + \eta^+}$ generally takes

$$\mathcal{H}_{\text{AI} + \eta^+} = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix},$$

where $A, B, C$ are $\frac{N}{2} \times \frac{N}{2}$ real matrices with

$$A_{ij} = A_{ji}, C_{ij} = C_{ji}. \quad \text{(B4)}$$

The probability distribution function in the Gaussian ensemble is

$$p(\mathcal{H}) d\mathcal{H} = C_N \exp \left\{ -\beta \left[ 2 \sum_{i,j} B_{ij}^2 + 2 \sum_{i > j} (A_{ij}^2 + C_{ij}^2) \right] + \sum_i \left( A_{ii}^2 + C_{ii}^2 \right) \right\} \prod_{i,j} dA_{ij} dC_{ij}. \quad \text{(B5)}$$

The probability distribution function in the Bernoulli ensemble is

$$B_{ij}, A_{ij}(i \geq j), C_{ij}(i \geq j)$$

$$= \begin{cases} 1 & \text{with the probability } 1/2, \\ -1 & \text{with the probability } 1/2, \end{cases} \quad \text{(B6)}$$

with Eq. (B3).

3. Class $\text{AI} + \eta^-$

For a random matrix in class $\text{AI}^+ \eta^-$, let us choose $\mathcal{U}_n = I$ and $\mathcal{U}_y = \sigma_y \otimes I_{\frac{N}{2} \times \frac{N}{2}}$. Then, $\mathcal{H}_{\text{AI} + \eta^-}$ takes a form of

$$\mathcal{H}_{\text{AI} + \eta^-} = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix},$$

where $A, B, C$ are $\frac{N}{2} \times \frac{N}{2}$ real matrices satisfying

$$B_{ij} = -B_{ji}, C_{ij} = -C_{ji}. \quad \text{(B8)}$$

The probability distribution function in the Gaussian ensemble is

$$p(\mathcal{H}) d\mathcal{H} = C_N \exp \left\{ -2\beta \left[ \sum_{i,j} A_{ij}^2 + \sum_{i > j} (B_{ij}^2 + C_{ij}^2) \right] \right\} \prod_{i,j} dA_{ij} dB_{ij} dC_{ij}. \quad \text{(B9)}$$

The probability distribution function in the Bernoulli ensemble is

$$A_{ij}, B_{ij}(i > j), C_{ij}(i > j)$$

$$= \begin{cases} 1 & \text{with the probability } 1/2, \\ -1 & \text{with the probability } 1/2, \end{cases} \quad \text{(B10)}$$

with Eq. (B8).

4. Class $\text{A} + \eta^+$

For a random matrix in class $\text{A} + \eta$ with $\mathcal{U}_n$ be $\sigma_z \otimes I_{\frac{N}{2} \times \frac{N}{2}}$. Then, $\mathcal{H}_{\text{A} + \eta}$ is given by

$$\mathcal{H}_{\text{A} + \eta} = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}, \quad \text{(B11)}$$

where $A, B, C$ are $\frac{N}{2} \times \frac{N}{2}$ matrices satisfying

$$A_{ij} = A_{ji}^*, C_{ij} = C_{ji}^*. \quad \text{(B12)}$$

The probability distribution function in the Gaussian ensemble is given by

$$p(\mathcal{H}) d\mathcal{H} = C_N \exp \left\{ -\beta \left[ \sum_i (A_{ii}^2 + C_{ii}^2) \right] + 2 \sum_{i > j} (|A_{ij}|^2 + |C_{ij}|^2) + \sum_{i,j} |B_{ij}|^2 \right\} \prod_{i,j} dA_{ij} dA_{ij}^* dC_{ij} dC_{ij}^* \prod_i dA_{ii} dC_{i} \prod_{i,j} dB_{ij} dB_{ij}^*. \quad \text{(B13)}$$

The probability distribution function in the Bernoulli ensemble is given by

$$A_{ii}, C_{ii} = \begin{cases} 1 & \text{with the probability } 1/2, \\ -1 & \text{with the probability } 1/2, \end{cases} \quad \text{(B14)}$$

and

$$A_{ij}(i > j), C_{ij}(i > j), B_{ij}$$

$$= \begin{cases} 1 + i & \text{with the probability } 1/4, \\ -1 + i & \text{with the probability } 1/4, \\ 1 - i & \text{with the probability } 1/4, \\ -1 - i & \text{with the probability } 1/4, \end{cases} \quad \text{(B15)}$$

and Eq. (B8).

5. Class $\text{AII}$

For a random matrix in class $\text{AII}$, let $\mathcal{U}_n$ be $\sigma_y \otimes I_{\frac{N}{2} \times \frac{N}{2}}$. Then, $\mathcal{H}_{\text{AII}}$ is given by

$$\mathcal{H}_{\text{AII}} = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \quad \text{(B16)}$$

with $\frac{N}{2} \times \frac{N}{2}$ matrices $A$ and $B$. The probability distribution function in the Gaussian ensemble is

$$p(\mathcal{H}) d\mathcal{H} = C_N \exp \left\{ -2\beta \sum_{ij} (|A_{ij}|^2 + |B_{ij}|^2) \right\} \prod_{ij} dA_{ij} dA_{ij}^* dB_{ij} dB_{ij}^*. \quad \text{(B17)}$$
The probability distribution function in the Bernoulli ensemble is
\[
A_{ij}, B_{ij} = \begin{cases} 
1 + i & \text{with the probability } 1/4, \\
-1 + i & \text{with the probability } 1/4, \\
1 - i & \text{with the probability } 1/4, \\
-1 - i & \text{with the probability } 1/4.
\end{cases} 
\] (B18)

6. Class AII + $\eta_+$

For a random matrix in class AII + $\eta_+$, let us choose $\mathcal{U}_\tau = \tau_0 \sigma_0 \otimes \mathbb{1}_{N \times \bar{N}}$ and $\mathcal{U}_\eta = \tau_\eta \sigma_0 \otimes \mathbb{1}_{\bar{N} \times \bar{N}}$ with the identity matrix $\mathbb{1}_{\bar{N} \times \bar{N}}$. Then, $\mathcal{H}_{\text{AII} + \eta_+}$ generally takes
\[
\mathcal{H}_{\text{AII} + \eta_+} = \begin{pmatrix} 
A^1 & A^2 & B^1 & B^2 \\
-A^2 & A^1 & -B^2 & B^1 \\
-B^1 & B^2 & C^1 & C^2 \\
B^2 & -B^1 & C^2 & C^1 
\end{pmatrix}, 
\] (B19)

where $A^\mu, B^\mu, C^\mu$ ($\mu = 1, 2$) are $\bar{N} \times \bar{N}$ matrices satisfying
\[
A^1_{ij} = A^{1*}_{ji}, C^1_{ij} = C^{1*}_{ji}, A^2_{ij} = -A^{2*}_{ji}, C^2_{ij} = -C^{2*}_{ji}. 
\] (B20)

The probability distribution function in the Gaussian ensemble is
\[
p(\mathcal{H})d\mathcal{H} = C_N \exp \left\{ -\beta \left( 2 \sum_i (A^1_{ii})^2 + (C^1_{ii})^2 \right) + 4 \sum_{\mu=1,2} \sum_{i>j} (|A^\mu_{ij}|^2 + |C^\mu_{ij}|^2) \right\} \prod_{i>j} dA^1_{i<j} dA^2_{i<j} dA^1_{i>j} dA^2_{i>j} dC^1_{i<j} dC^2_{i<j} dC^1_{i>j} dC^2_{i>j} 
\] (B21)

The probability distribution function in the Bernoulli ensemble is
\[
A^1_{ii}, C^1_{ii} = \begin{cases} 
1 & \text{with the probability } 1/2, \\
-1 & \text{with the probability } 1/2.
\end{cases} 
\] (B22)

FIG. 18. Density $\rho_c(x, y)$ of complex eigenvalues for non-Hermitian random matrices in the Bernoulli ensemble for classes (a) AI, (b) AII, (c) AI + $\eta_+$, (d) A + $\eta$, (e) AI + $\eta_-$, (f) AII + $\eta_+$, and (g) AII + $\eta_-$. Here, $\rho_c(x, y)$ is shown as a function of $y = \text{Im}(E)$ for fixed $x = \text{Re}(E)$ near the real axis $y = 0$ of complex energy $E$. The data are obtained from diagonalizations of 5000 samples of $4000 \times 4000$ random matrices in each symmetry class. Note that $\rho_c(x, y)$ is almost independent of $x$ as long as $E$ is away from the boundary of a circle inside which the complex eigenvalues $E$ distribute.
The probability distribution function in the Gaussian ensemble is given by

$$p(H)dH = C_N \exp \left\{ -\beta \left( \sum_i 2(A_{ii}^2 + |B_{ii}|^2) 
+ 4 \sum_{i>j} (|A_{ij}|^2 + |B_{ij}|^2) \right) \right\} \prod_{i>j} dA_{ij}d\beta_{ij}dA_{ij}^*d\beta_{ij}^*.$$  

The probability distribution function in the Bernoulli ensemble is given by

$$A_{ii} = \begin{cases} 
1 & \text{with the probability } 1/2, \\
-1 & \text{with the probability } 1/2, 
\end{cases} \quad (B27)$$

$$A_{ij}(i > j), B_{ij}(i \geq j) = \begin{cases} 
1+i & \text{with the probability } 1/4, \\
-1+i & \text{with the probability } 1/4, \\
1-i & \text{with the probability } 1/4, \\
-1-i & \text{with the probability } 1/4, 
\end{cases} \quad (B28)$$

with Eq. $(B25)$.

8. Level statistics in the Bernoulli ensemble

We diagonalize non-Hermitian random matrices in the Bernoulli ensemble for the seven symmetry classes and obtain the universal properties of the DoS as well as the level statistics of real eigenvalues in the Bernoulli ensemble. The DoS in each symmetry class shows the same property as the DoS in the Gaussian ensemble. The soft-gap behaviors of the density $p_c(x, y)$ of complex eigenvalues around the real axis ($y = 0$) are consistent with the DoS in the Gaussian ensemble (Fig. 18). For non-Hermitian random matrices in classes $\text{AI} + \eta, \text{AII} + \eta_\pm, \text{AII} + \eta_\mp$, no real eigenvalues appear. By contrast, the average numbers of real eigenvalues in classes $\text{AI} + \eta, \text{AII} + \eta_\pm, \text{AII} + \eta_\mp$ are proportional to $\sqrt{N}$ in the Bernoulli ensemble (Fig. 19). In these five symmetry classes, both of the level-spacing distribution $p(s)$ and level-spacing-ratio distribution $p_r(r)$ of real eigenvalues in the Bernoulli ensemble are consistent with the distribution functions in the Gaussian ensemble (Figs. 20 and 21). These results demonstrate the universality of the soft gap of the DoS around the real axis, the level-spacing and level-spacing-ratio distributions of real eigenvalues, and the scaling relation of the average number of real eigenvalues.
Thus, we can generalize the Gaussian ensemble in each symmetry class as
\[ p(s) = \frac{1}{2\pi} \exp \left( -\frac{s^2}{4} \right) \]

The inverse transform of Eq. (C1) is given by
\[ \mathcal{H} = \frac{1}{2x} (\mathcal{H}' + \mathcal{H}'^\dagger) + \frac{1}{2(1-x)} (\mathcal{H}' - \mathcal{H}'^\dagger). \] 

As Eq. (C1) is a linear transform, the Jacobian matrix of the transform depends only on \( x \),
\[ d\mathcal{H}' = C_x d\mathcal{H}, \]

where \( C_x \) is an \( x \)-dependent constant. The probability distribution in the generalized Gaussian ensemble is given by
\[ p'(\mathcal{H}')d\mathcal{H}' = p(\mathcal{H})d\mathcal{H} \]
\[ = C_N^{-1} e^{-\beta Tr(\mathcal{H}'\mathcal{H}')/2} \mathcal{H}' d\mathcal{H}' \]
\[ = C_N^{-1} C_x^{-1} e^{-\beta Tr\left[ \frac{1}{2} (\mathcal{H}' + \mathcal{H}')^2 - \frac{\beta}{4} (\mathcal{H}' - \mathcal{H}')^2 \right]} \mathcal{H}' d\mathcal{H}' \]
\[ = C_N^{-1} C_x^{-1} e^{-\beta Tr\left[ \frac{1}{2} (\mathcal{H}' + \mathcal{H}')^2 - \frac{\beta}{4(1-x)} (\mathcal{H}' - \mathcal{H}')^2 \right]} \mathcal{H}' d\mathcal{H}', \] 

which reduces to Eq. (12) with \( C_{N,(\beta_1,\beta_2)}^{-1} = C_N^{-1} C_x^{-1} \), \( \beta_1 \equiv \frac{\beta}{4x^2} \), and \( \beta_2 \equiv \frac{\beta}{4(1-x)^2} \).

2. Level statistics

We show that level-spacing and level-spacing ratio distributions, \( p(s) \) and \( p_r(r) \), in the generalized Gaussian ensemble (GGE) with \( \beta_2 > \beta_1 \) converge faster than those in the Gaussian ensemble (GE). Figures 3(d) and 3(i) show that level-spacing and level-spacing ratio distributions in class AI + \( \eta_+ \) converge more slowly than those in the other symmetry classes. Thus, we focus on \( p(s) \) and \( p_r(r) \) in class AII + \( \eta_+ \) and compare their convergence in the GGE and GE. For the other four symmetry classes, random matrices in the GGE have the same properties.
with respect to the dimensions $N$ as shown in Fig. 23(e). The square-root scaling of the soft gap of density of complex eigenvalues around the real and nonreal axis are also universally observed in random matrices of the GGE (not shown).

**Appendix D: Kolmogorov-Smirnov distance**

It is more feasible to use cumulative distribution functions than probability distribution functions, when comparing ergodic phases of physical systems with random matrices. Here, we calculate the Kolmogorov-Smirnov (KS) distance from the cumulative spacing distribution function $\int_0^r \rho(s') ds'$ and cumulative spacing-ratio distribution function $\int_0^r p_r(s') ds'$ among physical systems and random matrices in different symmetry classes (Tables III and IV). In the calculations, we first obtain the empirical cumulative distribution function $F_n(x)$ from a set of real random variables $\{x_1, x_2, \ldots, x_n\}$ for the spacing and spacing ratio,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \theta(x_i - x),$$

with the step function $\theta$. This function $F_n(x)$ corresponds to the cumulative level-spacing and level-spacing-ratio distribution function of real eigenvalues. The KS distance between the two empirical cumulative distribution functions $F_{c1}(x)$ and $F_{c2}(x)$ is defined by the maximum...
The plot clearly demonstrates $N'_{\text{real}}$ as a function of $\beta_2/\beta_1$. For random matrices with different sizes and in the other four symmetry classes, the same scaling relation also holds true.

value of the difference between the two functions over all $x$,

$$D_{c_1,c_2} \equiv \sup_x |F_{c_1}(x) - F_{c_2}(x)|.$$  \hfill (D2)

Tables III (Table IV) summarizes the KS distance among $p(s)$ ($p_r(r)$) from 4000 $\times$ 4000 random matrices in the generalization Gaussian ensemble with $\beta_2/\beta_1 = 16$ for the five symmetry classes. Table III (Table IV) also give the KS distance between $p(s)$ ($p_r(r)$) from the physical systems in the ergodic phases with the maximal system size and $p(s)$ ($p_r(r)$) from the random matrices in the five symmetry classes. Tables III and IV show that in classes A, AI, $\text{AI} + \eta_\pm$, and $\text{A} + \eta$, the probability distribution functions $p(s)$ and $p_r(r)$ from the physical systems in the ergodic phases have the shortest distance to $p(s)$ and $p_r(r)$ from the random matrices in the same symmetry class, and the shortest distances are less than 0.01.

Tables III and IV also show that $p(s)$ and $p_r(r)$ of the physical systems in class AI $+ \eta_-$ have the shortest KS distance with $p(s)$ and $p_r(r)$ of the random matrices in class A $+ \eta$ but have the larger distance ($> 0.05$) with $p(s)$ and $p_r(r)$ of the random matrices in class AI $+ \eta_+$.  

### Appendix E: Number variance and spectral compressibility

For an ensemble of non-Hermitian random matrices, we count the number $N_W(E)$ of real eigenvalues in an energy window $[-E, E]$ with $E \geq 0$ in each sample. We evaluate the mean value $\langle N_W(E) \rangle$ and the variance $\Sigma_2(E) = \langle N_W(E) \rangle^2 - \langle N_W(E) \rangle^2$ of the number in the energy window for different $E$. In our evaluation, only less than 50% of all the real eigenvalues are included in the energy window. Note also that in classes AI $+ \eta_-$ and AI $+ \eta_+$, we regard each Kramers pair as one real eigenvalue and characterize the level interaction between neighboring Kramers pairs by $\Sigma_2(E)$ and $\langle N_W(E) \rangle$.

In all random matrix ensembles studied in this paper (i.e., generalized Gaussian ensemble with $\beta_2/\beta_1 = 16$, Gaussian ensemble, and Bernoulli ensemble), we have the scaling relation

$$\Sigma_2(E) \simeq \chi N_W(E)$$  \hfill (E1)

for all the five symmetry classes (Fig. 23). The spectral compressibility $\chi$ in each symmetry class takes the same value for the different ensembles, suggesting the universality of $\chi$. While $\chi$ is less than $\chi_{\text{Poisson}} = 1$ in classes AI, AI $+ \eta_\pm$, and A $+ \eta_+$, $\chi$ is larger than $\chi_{\text{Poisson}} = 1$ in class AI $+ \eta_+$. This unusual relation $\chi > \chi_{\text{Poisson}} = 1$ in class AI $+ \eta_+$ indicates that attractive interactions are more dominant than repulsive interactions in this symmetry class, which has no analogs in Hermitian random matrices and also non-Hermitian random matrices in the other four symmetry classes.
FIG. 23. (a)-(d) Comparison between level-spacing-ratio distributions $p_r(r)$ and level-spacing distributions $p(s)$ of real eigenvalues obtained from $N' \times N'$ non-Hermitian random matrices in the generalized Gaussian ensemble (GGE) and $N \times N$ non-Hermitian random matrices in the Gaussian ensemble (GE) for class AII + $\eta_+$. (e) Mean value of the level-spacing ratio $\langle r \rangle = \int_0^1 p_r(r) dr$ as a function of the average number $\bar{N}_{\text{real}}$ of real eigenvalues in the GE and GGE for class AII + $\eta_+$. For $\bar{N}_{\text{real}} \gtrsim 70$ ($N' > 1000$), the error bars of $\langle r \rangle$ for different sizes overlap with one another. The error ranges are evaluated by the bootstrap method [69]. The error ranges of the distributions of the GGE random matrices are much smaller than those of the GE random matrices and not shown here (see also Figs. 4 and 5).

FIG. 24. Variance $\Sigma_2$ and mean value $\langle N_{W} \rangle$ of the number of real eigenvalues of non-Hermitian random matrices in the five symmetry classes. $\Sigma_2$ and $\langle N_{W} \rangle$ within different energy windows are obtained from $4000 \times 4000$ non-Hermitian random matrices in the (a) generalized Gaussian ensemble with $\beta_2/\beta_1 = 16$, (b) Gaussian ensemble, and (c) Bernoulli ensemble. The scaling relation $\Sigma_2 \simeq \chi \langle N_{W} \rangle$ holds for all the five symmetry classes. The spectral compressibility $\chi$ is obtained by the linear fitting of the data points for each symmetry class.
TABLE IV. Kolmogorov-Smirnov (KS) distance among level-spacing-ratio distribution functions $p_r(r)$ of real eigenvalues obtained from $4000 \times 4000$ non-Hermitian random matrices in the five symmetry classes (classes A+$\eta$, AI, AI + $\eta_\pm$, and AII + $\eta_\pm$), and KS distance between $p_r(r)$ obtained from the physical systems in the ergodic phases and $p_r(r)$ obtained from the random matrices in the five classes. The notation is the same as Table III.

| system class | AI | AI + $\eta_+$ | AI + $\eta_-$ | AI + $\eta_+ \eta_-$ | AI + $\eta_+$ | AI + $\eta_-$ | AI + $\eta_+ \eta_-$ |
|--------------|----|---------------|---------------|----------------------|---------------|---------------|----------------------|
| RM AI        | 0  | 0.070         | 0.114         | 0.016                | 0.194         |                |                      |
| RM AI + $\eta_+$ | 0.070 | 0             | 0.044         | 0.085                | 0.125         |                |                      |
| RM A + $\eta_+$ | 0.114 | 0.044         | 0             | 0.129                | 0.083         |                |                      |
| RM AI + $\eta_-$ | 0.016 | 0.085         | 0.129         | 0                    | 0.209         |                |                      |
| RM AII + $\eta_+$ | 0.194 | 0.125         | 0.083         | 0.209                | 0             |                |                      |
| $\mathcal{H}_{1D}$ AI | 0.007 | 0.067         | 0.110         | 0.021                | 0.191         |                |                      |
| $\mathcal{H}_{2D}$ AI + $\eta_+$ | 0.003 | 0.071         | 0.114         | 0.017                | 0.195         |                |                      |
| $\mathcal{H}_{3D}$ AI + $\eta_-$ | 0.069 | 0.006         | 0.045         | 0.084                | 0.126         |                |                      |
| $\mathcal{H}_{4D}$ AI + $\eta_+ \eta_-$ | 0.067 | 0.004         | 0.047         | 0.083                | 0.128         |                |                      |
| $\mathcal{H}_{4D}$ A + $\eta_+$ | 0.119 | 0.049         | 0.006         | 0.134                | 0.081         |                |                      |
| $\mathcal{H}_{4D}$ AI + $\eta_-$ | 0.015 | 0.085         | 0.128         | 0.004                | 0.208         |                |                      |
| $\mathcal{H}_{4D}$ AII + $\eta_+$ | 0.138 | 0.069         | 0.025         | 0.154                | 0.062         |                |                      |
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[70] Tiny apparent imaginary parts of real eigenvalues stem from the machine inaccuracy of a numerical subroutine program. We can estimate their upper bound (error bound) by numerical diagonalization of Hermitian random matrices based on the same subroutine program. By the diagonalization, we observe that when eigenvalues of an N-by-N Hermitian random matrix in the Gaussian ensemble are in a range of [−Λ, Λ] with Λ = O(1), the upper bound of the imaginary parts of the eigenvalues is of the order of 10^{−15} − 10^{−14} for 100 ≤ N ≤ 4000. Based on this observation, we set the cut-off C of the imaginary part to C = 10^{−13}. To be specific, we first normalize an N-by-N non-Hermitian random matrix in the Gaussian ensemble such that its complex eigenvalues E are distributed within a circle of the order of 1 (i.e., |E| ≤ O(1)). Then, we regard any complex eigenvalue of the random matrix whose imaginary parts are less than C = 10^{−13} as real. The cut-off C is much larger than the error bound, so that it shall not miss any real eigenvalues of the matrix. The probability P of complex eigenvalues being mistaken as real depends on N and is negligible for N < 10^3. To evaluate this probability P, let us suppose that complex eigenvalues E of the non-Hermitian random matrix are distributed uniformly within the circle (i.e., |E| ≤ O(1)). Then, the probability p that a given complex eigenvalue of the random matrix is mistaken as real is of the order of C = 10^{−13}, and the probability P that complex eigenvalues are mistaken as real is estimated as P = 1 − (1 − p)^N ≈ Np. This probability is negligible for N < 10^6 owing to p ∼ C = 10^{−13}. This justifies our setting of the cut-off C in this paper, where N is typically less than 10^4.
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