Finite-time Lyapunov fluctuations and the upper bound of classical and quantum out-of-time-ordered expansion rate exponents

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This Letter demonstrates for chaotic maps (logistic, classical and quantum standard maps (SMs)) that the exponential growth rate (Λ) of the out-of-time-ordered four-point correlator (OTOC) is equal to the classical Lyapunov exponent (λ) plus fluctuations (Δ(fluc)) of the one-step finite-time Lyapunov exponents (FTLEs). Jensen’s inequality provides the upper bound Λ ≤ λ for the considered systems. Equality is restored with Λ = λ + Δ(fluc), where Δ(fluc) is quantified by k-higher-order cumulants of the FTLEs. Exact expressions for Λ are derived and numerical results using k = 20 furnish Δ(fluc) ∼ ln (√2) for all maps (large kicking intensities in the SMs).

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Introduction. The interest in the quantum-classical correspondence of classically chaotic systems renewed in the last years due to the conjecture that puts a bound on the exponential growth rate Λ ≤ 2πT (T is the temperature) of an out-of-time-ordered four-point correlator (OTOC) [1]. Introduced in the context of the theory of superconductivity [2], Λ is closely associated with the largest positive asymptotic Lyapunov exponent (LE) of the classical chaotic system for times shorter than the Ehrenfest time (t < tE), for which quantum interference effects did not have time to become relevant. Besides serving as a tool to understand the fundamentals in the quantum-classical relation of classically regular [3–5], quasi-regular [6, 7] and chaotic [6, 8] systems, the behaviour of the OTOC attracted considerable attention in many-body systems [9–17] and in experiments [18–21]. In this context, we refer the readers to the recent interesting review about semiclassical many-body quantum chaos [22].

In general, it is known that even though related, Λ obtained from classical and quantum OTOCs and the LE (λ) are not precisely equal due to distinct order of averaging. While Λ is proportional to ln (E [X]), λ is proportional to E [ln (X)], where X = {x1, x2, ..., xN} is a random variable related to the local finite-time Lyapunov exponent (FTLE) λ(t), calculated at time t (for t → ∞, λ(∞) = λ). Here E[.] is the average over all points in the phase space. In some cases, the relation between both is written as Λ = λ + Δ. For example, in the completely chaotic region of the kicked rotator it was observed numerically that Δ ≈ ln (T) [6] and in the many-body Dick model, Δ ≈ 0.015 [17]. Even though the distinct order of averaging seems to be a mere mathematical property, it has deep interesting physical consequences. The mathematical background of our findings lies in Jensen’s inequality (JI) E[ϕ(X)] ≥ ϕ(E[X]), where ϕ(X) is a convex function. Equality is restored when all variables in X = {x1, x2, ..., xN} are equal or when higher moments of X are taken into account, which is the case considered here.

This Letter demonstrates that fluctuations of the FTLEs lead to the distinction between Λ and λ. Fluctuations of the local one-step FTLEs are well-known properties in dynamical systems (see, for example, [23, 24]). Consequently, our results establish that for times t < tE, the classical and quantum exponential growth rates Λ contain features of classical fluctuations not visible in the asymptotic λ itself. Analytical and numerical results are shown for the chaotic logistic and tend maps and the classical and quantum standard maps. The classical OTOC is defined as

\[ C_\alpha(t) = E \left[ \left\{ x(t), p(0) \right\}^2 \right] = E \left[ \left( \frac{\partial p(t)}{\partial x(0)} \right)^2 \right]. \] (1)

For chaotic systems it is expected that \( C_\alpha(t) \sim e^{2\Lambda t} \) and the exponential growth rate is determined through the numerical computation of \( \Lambda = 1/2 \lim_{t \to \infty} \lim_{\Delta x(0) \to 0} \ln [C_\alpha(t+1)/C_\alpha(t)], \)
where \( \Delta x(0) \) are small initial displacements. For one-dimensional systems \( X = J(t)^2 = e^{2X(t)} \), where \( J(t) \) is the Jacobian at time \( t \). Applying \( J(t) \) to \( -\ln(X) \) (convex) we have \( \ln(\mathbb{E}[X]) \geq \mathbb{E}[\ln(X)] \), which provides the upper bound for the LE, namely that \( \lambda \leq \Lambda \).

**Chaotic logistic map (LM).** The map is defined as \( x_{n+1} = 4x_n(1-x_n) \), with discrete times \( n = 1, 2, \ldots \). The LM has an invariant density \( \rho_{LM}(x) = 1/(\pi \sqrt{x(1-x)}) \) and the asymptotic LE is \( \lambda_{LM} = \ln(2) \).

Using the definition of the OTOC for the map, we have

\[
C_{\text{ot}}(n) = \mathbb{E} \left[ \left( \frac{\partial x_n}{\partial x_0} \right)^2 \right] = \mathbb{E} \left[ e^{2 \sum_{n} \lambda_{LM}^{(n)}} \right], \tag{2}
\]

where \( \lambda_{LM}^{(n)} = \ln |J_n| \) are the local one-step FTLEs and \( J_n = \partial x_n/\partial x_{n-1} = 4 - 8x_n(x_n-1) \) is the Jacobian of the map at time \( n \). Note that \( \partial x_n/\partial x_0 = J_n J_{n-1} \ldots J_1 \).

The local one-step FTLEs are fluctuating quantities and are responsible for the emergence of a nontrivial probability density, as observed previously for the LM \([24]\), that asymptotically converges to a delta centered at \( \lambda_{LM} \left( = \lim_{n \to \infty} \lambda_{LM}^{(n)} \right) / N \sum_{n} \lambda_{LM}^{(n)} \). Thus, fluctuations of the FTLEs are expected to be relevant when determining the one-step FTLEs which are relevant to \( \lambda_{LM} \) and \( \lambda_{LM}^{(1)} \) the upper bound for the LE, namely that \( \lambda \leq \Lambda \). To demonstrate the fluctuation properties of the gap, we use a numerical method to determine \( \lambda_{LM}^{(1)} \).

Concerning the term in the middle of Eq. (2), it is easy to show that \( C_{\text{ot}}^{(1)}(1) = \int_0^1 dx_0 \rho_{LM}(x_0) |\partial x_1/\partial x_0|^2 = 8 \), \( C_{\text{ot}}^{(1)}(2) = 64 \), \( C_{\text{ot}}^{(1)}(3) = 512 \), \( C_{\text{ot}}^{(1)}(4) = 4096, \ldots \). From this we obtain the exact OTOC exponent \( \Lambda_{\text{LM}}^{(\text{exact})} = \ln(C_{\text{ot}}^{(1)}(2))/C_{\text{ot}}^{(1)}(1)/2 = \ln(C_{\text{ot}}^{(1)}(3))/C_{\text{ot}}^{(1)}(2)/2 = \ln(C_{\text{ot}}^{(1)}(4))/C_{\text{ot}}^{(1)}(3)/2 = 3 \ln(2)/2 \). These are two iterations processes, namely from \( n = 0 \rightarrow n = 2, n = 1 \rightarrow n = 3 \) and \( n = 2 \rightarrow n = 4 \), respectively. Observe that \( \Lambda_{\text{LM}}^{(\text{exact})} = \lambda_{LM} + \ln(\sqrt{2}) \), so that \( \Delta_{\text{LM}}^{(\text{exact})} = \ln(\sqrt{2}) \) gives the exact gap between both exponents.

In order to connect Eq. (3) with the fluctuations of the one-step FTLEs, we apply the generating function of the cumulants

\[
\ln \left( \mathbb{E} \left[ \exp \left( 2 \lambda_{LM}^{(j)} \right) \right] \right) = \sum_{k=0}^{\infty} \kappa_k^{(j)} \left( \frac{2^k}{k!} \right), \tag{4}
\]

where \( \kappa_k^{(j)} \) are the cumulants related to \( \lambda_{LM}^{(j)} \). Taking into account the first \( k = 20 \) cumulants we determine (see below)

\[
\Lambda_{\text{LM}}^{(1\rightarrow2)} \sim \left[ \lambda_{LM} + 0.347500 \right]. \tag{5}
\]

Thus, the contribution of the fluctuations leads to \( \Delta_{\text{LM}}^{(\text{finite})} \sim 0.347500 \), which is close to the analytical gap \( \Delta_{\text{LM}}^{(\text{analytic})} = \log(\sqrt{2}) \sim 0.346574 \). To determine \( \Delta_{\text{LM}}^{(\text{finite})}(k) \) we integrate numerically the central moments \( \Upsilon_{\text{LM}}^{(\text{finite})}(N) \) (see \([33]\))

\[
\Upsilon_{\text{LM}}^{(k)}(1) = \int_0^1 dx_n \rho_{LM}(x_n) \left( \lambda_{LM}^{(1)} - \lambda_{LM} \right)^k, \quad \Upsilon_{\text{LM}}^{(k)}(2) = \int_0^1 dx_n \rho_{LM}(x_n) \left( \lambda_{LM}^{(2)} + \lambda_{LM}^{(1)} - 2\lambda_{LM} \right)^k.
\]

Results for \( \Delta_{\text{LM}}^{(\text{finite})}(k) \) are presented in Fig. 1 as a function of the cumulants’ order \( k \), and shows that it converges to \( 0.347500 \). Thus, we expect that for \( k \rightarrow \infty \) cumulants, the gap converges to \( \Delta_{\text{LM}}^{(\text{finite})} \rightarrow 0 \sim 0.3465735 \). Worth mentioning is that the cumulant expansion of the individual terms in Eq. (3) increases without bounds, but the difference between them converges following Fig. 1. Furthermore, Eqs. (3) and (5) could be generalized to \( \Lambda_{\text{LM}}^{(n\rightarrow n+1)} \), however, the determination of \( \Upsilon_{\text{LM}}^{(n)}(N) \) becomes harder and harder as \( n \) increases and does not provide new relevant information.
Figure 1: Plotted is $\Delta_{\beta,\Delta}$ as a function of the higher-order cumulants for the two iterations case. Dashed line shows $\ln(\sqrt{2})$ for reference.

Tend map (TM). For the tend map, defined as $x_{n+1} = 2x_n$ for $x_n < 1/2$, and $x_{n+1} = 2(1-x_n)$ for $x_n \geq 1/2$, the asymptotic LE is $\lambda_{\text{TM}} = \ln(2)$. The logistic and tend maps have the same LE [25]. The OTOC growth rate is determined exactly as $\Lambda_{\text{TM}}^{(\text{exact})} = \ln(C_{\text{TM}}(n)/C_{\text{TM}}(1))/2(n-1) = \lambda_{\text{TM}}$. For the TM the invariant density is $f_{\text{TM}}(x) = 1$, the FTLEs are independent of the ICs, and no fluctuations are expected, so that fluctuations of the FTLEs are absent, the OTOC and Lyapunov exponents are identical.

Expressing the OTOC in terms of CLVs. Before discussing results for the classical and quantum SMs, we present an expression for the classical OTOC in terms of CLVs in the two-dimensional continuous case. We write the right-hand side of Eq. (1) as a function of quantities related to the evolution in the tangent space $T_xM \equiv \mathbb{R}^2$, namely in the CLVs basis, $\{v_k\} = \{v_k^{(u)} , v_k^{(s)}\}$, which generate the Oseledec unstable $\{E^{(u)}_k\}$ and stable $\{E^{(s)}_k\}$ subspaces with the properties $D_xf_{\text{TM}}v_k^{(u)} = \lambda_k^{(u)} v_k^{(u)}$, $\lambda_k^{(u)} < 0$, (for $i \neq j$) and $\lim_{t \to -\infty} \frac{1}{t} \log \|D_xf_{\text{TM}}v_k^{(u)}\| = \lambda_k^{(u)}$ being the magnitude of the asymptotic Lyapunov exponent, with $i = u$ or $i = s$. It is possible to show, after straightforward manipulation, that

$$C_{\text{cl}}^{(\text{CLV})}(t) = \mathbb{E} \left[ f_{\text{u},x}^{(t)} f_{\text{u},x}^{(t)} e^{2\lambda_k^{(u,t)}} + f_{\text{s},x}^{(t)} f_{\text{s},x}^{(t)} e^{2\lambda_k^{(s,t)}} \right] - 2\mathbb{E} \left[ f_{\text{u},x}^{(t)} f_{\text{s},x}^{(t)} e^{\lambda_k^{(u,t)} + \lambda_k^{(s,t)}} \right], \tag{7}$$

with the amplitudes

$$f_{\text{u},x}^{(t)} = \cos(\phi_x + \theta_x/2) \cos(\phi_{f(x)} - \theta_{f(x)}/2)/\sin(\theta_x),$$

$$f_{\text{s},x}^{(t)} = \cos(\phi_x - \theta_x/2) \cos(\phi_{f(x)} + \theta_{f(x)}/2)/\sin(\theta_x).$$

Equation (7) furnishes explicitly the quantity $C_{\text{cl}}^{(\text{CLV})}(t)$ as a function of: the finite-time CLV $\lambda_k^{(u,t)}$, related to the unstable manifold, the finite-time CLV $\lambda_k^{(s,t)}$, related to the stable manifold, the angle $\theta_x$ between both manifolds, their time derivative $\theta_{f(x)}$, the angle $\phi_x$, which is the angle between $\theta_x/2$ and the horizontal axis, and $\phi_{f(x)}$, its time derivative. We notice that the CLVs $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ are calculated for finite times $t$, and only for $t \to \infty$ they lead to the usual asymptotic LEs $\lambda_k^{(\infty,1)}$ and $\lambda_k^{(\infty,2)}$, respectively. In other words, for short times, the local values of $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$, and their fluctuations, are essential for the behavior of the OTOC. Furthermore, the amplitudes of the exponents provide a clear contribution to the underline dynamics. For example, for $\sin \theta_x \to 0$ an alignment between CLVs occurs and relevant contributions from the amplitudes of Eq. (7) are expected [34]. Recent works in other contexts focus on the role of prefactors to the OTOC [26].

The standard map (SM). The classical dissipative map is defined as $p_{n+1} = \gamma p_n + \frac{\Delta}{2\pi} \sin(2\pi q_n) \mod 1$, and $q_{n+1} = q_n + p_{n+1} \mod 1$, where $(p_n, q_n)$ are conjugate variables, $n = 1, 2, \ldots$ the discrete time, $\gamma$ is the dissipation parameter, and $K$ is the nonlinear parameter. For the analytical analysis of the SM, we use Eq. (1) in the form

$$C_{\text{cl}}^{(n)}(n) = \mathbb{E} \left[ \left( \frac{\partial p_n}{\partial q_0} \right)^2 \right] = \int_0^1 \int_0^1 dq_0 dq_2 \left( \frac{\partial p_n}{\partial q_0} \right)^2, \tag{8}$$

where the integration is over all phase-space initial conditions.

The conservative case ($\gamma = 1$). The analytical LE can be estimated from $\lambda_{\text{SM}}^{(\text{exact})} = \int_0^1 dq \ln |L(q)|$, with $L(q) = 1 + k(q)/2 + \text{sgn}[k(q)] \sqrt{k(q)(1 + k(q)/4)}$ and $k(q) = K \cos(2\pi q)$ [27]. Fluctuations of stability exponents in the SM have been already studied in another context [28]. It is known that for $K > 4$ only one chaotic component lives in the phase space [25]. Therefore, for large values of $K$, a completely chaotic motion is observed, and the asymptotic LE is $\lambda_{\text{SM}} = \ln (K/2)$. Equation (8) furnishes exact expressions, namely $C_{\text{cl}}^{(1)} = K^2/2$ and $C_{\text{cl}}^{(2)} = K^2 + K^4/4$, so that

$$\Lambda_{\text{SM}}^{(1+2)} = \frac{1}{2} \ln \left( \frac{C_{\text{cl}}^{(2)}}{C_{\text{cl}}^{(1)}} \right) = - \ln \sqrt{2} + \ln \left( \frac{4 + K^2}{2} \right). \tag{9}$$

Therefore, the gap is $\Delta_{\text{SM}}^{(\text{exact})} = \Lambda_{\text{SM}}^{(1+2)} - \lambda_{\text{SM}} = \ln \sqrt{2} + \ln \left( \frac{4 + K^2}{2} \right)$, for which $K > 4$ reduces to $\Delta_{\text{SM}}^{(\text{exact})} \approx 2\ln \sqrt{2}$. Amazingly, this is the same gap $\Delta_{\text{LM}}^{(\text{exact})}$ obtained for the LM, which is a dissipative system. We present results for the SM with $K \geq 4$, since for smaller values of $K$ the dynamic is mixed (regular and chaotic), and the classical and quantum averages lead to additional difficulties which, besides being of general interest, are not essential for the goal of the present work.

For the numerical results, we initially compared the time evolution of $C_{\text{cl}}^{(\text{CLV})}(n)$ from Eq. (7) with $C_{\text{cl}}(n)$ from Eq. (1), obtained using some small initial displacements $\Delta(x(0))$. Both results are in full agreement. However, the $C_{\text{cl}}^{(\text{CLV})}(n)$ from Eq. (7) is much superior in terms of the stability for longer iterations, since it does not depend on $\Delta(x(0))$. Thus, to reckon the exponent $\Lambda_{\text{SM}}^{(\text{CLV})}(n)$ we use
Eq. (7). To obtain the fluctuations of the CLVs, we determine numerically the distributions of $\lambda_{SM}^{(i)}$ at the Ehrenfest time [35], and obtain the $k = 20$ first central moments directly from these distributions. Such higher moments lead again to $\Delta_{SM}^{(fluc)}$, whose convergences are similar to those obtained for the LM in Fig. 1(a). Figure 2(a) shows results for $\Delta_{SM}^{(fluc)}$ (blue squares), $\Delta_{SM}^{(CLV)} - \Delta_{SM}^{(exact)}$ (red crosses) for distinct values of $K$, together with the exact results $\Delta_{SM}^{(exact)}$ (dark-pink squares). For $K \geq 8$, $\Delta_{SM}^{(CLV)}$ and $\Delta_{SM}^{(exact)}$ are indistinguishable, and approach the value 0.346 for $K = 1000$. The values of $\Delta_{SM}^{(fluc)}$ are very close, even though a bit smaller. The discrepancy between distinct curves for smaller values of $K$ is surely a consequence of the larger amount of dynamical fluctuations due to sticky motion [25], which strongly depend on appropriate averages.

Before we proceed to the quantum analysis, some information must be given. The quantum OTOC is obtained numerically from $C_2(n) = \mathbb{E}\left\{ |\hat{q}(n), \hat{p}(0)|^2 \right\}$, where $(\hat{q}, \hat{p})$ are the corresponding position and momentum operators, and $[,]$ denoting the commutator. The quantum SM problem is described using the kicked Hamiltonian operator (dimensionless units) $\hat{H} = \hat{p}^2/2 + K/(4\pi^2) \cos(2\pi \hat{q}) \sum_{n=0}^{\infty} \delta(t - n\tau)$, and $C_{SM}^{(Q)}(n)$ is obtained from the numerical integration of the corresponding Schrödinger equation. The associated OTOC exponent is named $\Lambda_{SM}^{(Q)}$. We use individual angular-momentum eigenstates $|\Psi(0)\rangle = \sum_{n=-\infty}^{\infty} \delta_{n0} |n\rangle$ and Gaussian wave packets $\phi_{n0}^{(0)} = \exp(-K_{\sigma}^2 (n-n_0)^2)/(2\pi\sigma)$, where $n_0 = \frac{\text{num}}{\text{eff}}$, for the numerical integration we use $p_0 = 0$, $\sigma = 4$ and $|\Psi\rangle$, represented in a finite basis of eigenstates $|n\rangle$, $n \in [-N,N-1]$. Functions of $\hat{p}$ are applied on this basis, and functions of $\hat{q}$ are applied in the Fourier-transformed representation. We use an adaptive grid with $2\text{num} \times 2\text{eff} N \in [2^{12}, 2^{16}]$ [6]. The quantum and classical OTOC exponents for the conservative SM were already considered recently [6], and our exponents are in good agreement with those presented in Fig. 2 from [6], for the considered $K$ values. We choose not to repeat such a figure but, instead, display results for $\Delta_{SM}$, our main interest. Numerical results for $\Delta_{SM} = \Lambda_{SM}^{(Q)} - \Lambda_{SM}^{(exact)}$ as a function of $K$ are shown as black stars in Fig. 2(a).

We notice that, except for specific values of $K$, $\Delta_{SM}^{(Q)}$ and $\Delta_{SM}^{(CLV)}$ are in relatively good agreement. Differences between both gaps are related to quantum averages and the number of eigenstates for each value of $K$. The determination of $\Lambda_{SM}^{(Q)}$ revealed to be a difficult numerical issue. Summarizing, Fig. 2(a) demonstrates that the gaps between the distinct OTOCs and the classical LE are close to each other, and all quantities approach $\sim \ln(\sqrt{2})$ for large $K$ values, as accurately explained by the fluctuations of the finite-time CLVs.

The dissipative case ($\gamma = 3/5$). We could not obtain an analytical expression for the OTOC exponent using arbitrary values of $\gamma$. However, for $\gamma = 3/5$, we attain $\Lambda_{SM}^{(1\rightarrow2)} = \frac{1}{2} \ln \left\{ \frac{34}{25} + \frac{K^2}{2} - \sqrt{5-\sqrt{3}} \frac{(J_0(K)-J_1(K))}{K\pi \sqrt{2}} \right\}$,

with $J_i(K) (i = 0,1)$ being the Bessel function of first type. Other specific values of $\gamma$ could be used. Furthermore, Eq. (7) is used to reckon $\Lambda_{SM}^{(i)}$, and the FTLE $\lambda_{SM}^{(num)}$ is determined numerically, as usual. Dissipation in the quantum model is introduced between the kicks by coupling the main system to a zero-temperature environment. The density operator is determined as an ensemble mean over pure states obtained from the quantum state diffusion [29].

Ito-stochastic Schrödinger equation $|d\psi\rangle = -\hat{H}|\psi\rangle dt + \sum_k (L_k - \langle L_k \rangle)|\psi\rangle d\xi_k - 1/2 \sum_k (L_k^\dagger L_k^\dagger - 2(L_k^\dagger L_k^\dagger) L_k^\dagger + |\langle \psi |^2 |\psi\rangle dt$. $L_k$ are the Lindblad operators with $k = 1, 2$ and $\langle \rangle$ stands for the expectation value. The Lindblad operators induce a damping $-\nu^\dagger \hat{p}$, and the dissipation parameter becomes $\gamma = e^{-\nu^\dagger \hat{p}}$, where the kicking time $\tau = \hbar\alpha$ is the effective Planck’s constant [29]. For details, we refer to [30]. Figure 2(b) summarizes our results for the dissipative case with $\gamma = 3/5$. Plotted is $\Delta_{SM} = \Lambda_{SM}^{(i)} - \lambda_{SM}^{(num)}$ for the distinct calculated $\Lambda_{SM}^{(i)}$. Figure 2: Plotted is $\Delta_{SM} = \Lambda_{SM}^{(i)} - \lambda_{SM}^{(num)}$ for the distinct calculated $\Lambda_{SM}^{(i)}$, namely the analytical result $\Lambda_{SM}^{(Q)} = \Lambda_{SM}^{(1\rightarrow2)}$ (black line), $\Lambda_{SM}^{(Q)} = \lambda_{SM}^{(CLV)}$ (red crosses), $\Lambda_{SM}^{(SM)} = \lambda_{SM}^{(fluc)}$ (blue square), and $\Lambda_{SM}^{(i)} = \lambda_{SM}^{(Q)}$ (black stars). (a) For the conservative case and (b) for the dissipative case. Upper borders of the light green rectangles show $\ln(\sqrt{2})$ for reference to be compared when $K$ is large.
namely the analytical result $\Lambda^{(i)}_{LM} = \Lambda^{(exact)}_{LM}$ (dark-pink squares), $\Lambda^{(i)}_{SM} = \Lambda^{(CLV)}_{SM}$ (blue crosses), $\Lambda^{(i)}_{LM} = \Lambda^{(Q)}_{SM}$ (blue square), and $\Lambda^{(i)}_{SM} = \Lambda^{(SM)}_{SM}$ (black stars). As for the conservative case, all quantities lead to a gap $\Delta_{LM} \sim \ln(\sqrt{2})$ for larger $K$ values, nicely explained by the fluctuations of the one-step finite-time CLVs.

Conclusions. Time fluctuations of the one-step FTLEs in the LM and the one-step finite-time CLVs in the SM are demonstrated to be the origin of the distinction between the classical and quantum OTOC exponential growth rate ($\Lambda$) and the classical LE ($\lambda$). The fluctuations are quantified by higher-order cumulant expansions corrections $\Delta^{(n)}_{\text{fluc}}$, so that the upper bound $\Lambda = \lambda + \Delta^{(n)}_{\text{fluc}}$ is reached. Comparing the LM, and the SM for $K \geq 4$, the correction is $\Delta^{(4)}_{\text{LM}} \sim \Delta^{(4)}_{\text{SM}} \sim \ln(\sqrt{2})$. Such approximated equality is intriguing: the statistical properties of the one-step FTLEs from the dissipative chaotic attractor of the LM are equal to those (CLVs) of the chaotic component of the conservative and dissipative SMs. For the tend map, no fluctuations of the FTLEs are observed, leading to $\Delta^{(i)}_{\text{fluc}} = 0$. Thus, the quantum-classical correspondence regarding the exponential growth of instabilities in the SMs, becomes clear and uniquely described for $t < t_E$ when taking into account the dynamical fluctuations of the one-step finite-time CLVs. On account for the fact that the tend, logistic and standard maps are paradigmatic models describing a huge number of dynamical systems in distinct physical contexts, we are confident that the finite-time Lyapunov fluctuations producing the gap $\Delta^{(n)}_{\text{fluc}}$ should be a generic property. Finally, it would be interesting to investigate the gap in many-body systems.

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[31] For the present analysis it is not adequate to use $\mathbb{E}\left[\epsilon^2 \Sigma_n (\lambda_{LM}^{(n)})^2\right] = \mathbb{E}\left[\epsilon^{2N\lambda_{LM}}\right]$ due to the relevance of each one-step FTLE.
[32] Here we use the joint cumulant of the variables $\lambda_{LM}^{(1)}$ and $\lambda_{SM}^{(1)}$, namely $\kappa_{(2)} = \mathbb{E}\left[\lambda_{LM}^{(1)}\lambda_{SM}^{(1)}\lambda_{LM}^{(1)}\right] - \mathbb{E}\left[\lambda_{LM}^{(1)}\right] \mathbb{E}\left[\lambda_{SM}^{(1)}\right] = \mathbb{E}\left[\lambda_{LM}^{(1)}\lambda_{SM}^{(1)} - \lambda_{LM}^{(2)}\right]$, and so on for higher moments.
[33] The cumulants in terms of the central moments can be obtained from expansions of the incomplete Bell polynomials. As an example, the first terms are $\kappa_{2n} = (\sum^{2n})_2 = (\sum^{2n-1})_3 = (\sum^{2n-2})_4 = \cdots = (\sum^1)^{2n} = \cdots = (\sum^1^{2n-1})_2$. For larger $K$ values, nicely explained by the fluctuations of the one-step finite-time CLVs.

For the present analysis it is not adequate to use $\mathbb{E}\left[\epsilon^2 \Sigma_n (\lambda_{LM}^{(n)})^2\right] = \mathbb{E}\left[\epsilon^{2N\lambda_{LM}}\right]$ due to the relevance of each one-step FTLE.
$Y^{(5)} - 10Y^{(3)}Y^{(2)}, \ldots$. To not confuse the readers we did not use $n = 1$ here.

[34] More details of these contributions will be considered elsewhere.

[35] The Ehrenfest time depends on $K$. For details and specific values of the Ehrenfest times, we refer the reader to Ref. [6].