Chaos-Assisted Light Squeezing

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We investigate theoretically the dynamics of squeezed state generation in the nonlinear systems possessing the transition from regular to chaotic dynamics in the limit of a large number of photons. As an example, the model of kicked Kerr oscillator is considered. We show the direct correlation of the degree of squeezing and the value of local Lyapunov instability rate in corresponding classical chaotic system.

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In recent years the problem of generation of the squeezed states of electromagnetic field has attracted a great attention of both the theoreticians and experimentalists \cite{1,2}. As a rule, in the experiments on squeezing the quantum fluctuations are small compared to the mean value of the field modes involved into a nonlinear interaction \cite{3,4}. Already for a rather long time it was mentioned that a light squeezing can be increased near the bifurcation points between different dynamical regimes \cite{5,6}. The reason for such enhancing consists in a strong diverging of one of the field quadrature component and corresponding squeezing of another one at the vicinity of instability threshold \cite{6}. Among with well studied parametric media \cite{3}, the increase of squeezing was predicted for the interaction of the field with 2-level atoms inside a high-Q cavity at the dynamical regime corresponding to the separatrix \cite{4,6}. However, all papers devoted to the study of the enhanced squeezing due to the appearance of dynamical instabilities in optical systems dealt with the \textit{integrable or near integrable systems with regular dynamics} \cite{10}.

In this paper, we consider the light squeezing generation by quantum nonintegrable optical system obeying the transition from regular to chaotic dynamics in the classical limit. As for other problems of quantum chaos \cite{12,13}, we consider the semiclassical limit when a great number of quantum levels $N \gg 1$ are involved into the dynamics. Our prime goal is to demonstrate the sufficient increase of the degree of light squeezing at transition to quantum chaos. We also found direct correlation between the degree of local instability in corresponding classical system and the degree of light squeezing.

In spite of our consideration is valid for any nondissipative quantum system with one and half degrees of freedom, we will demonstrate our main results on the enhanced squeezing for some particular model of the nonintegrable optical system – the nonlinear oscillator periodically forced by the classical field. In the interaction picture the Hamiltonian has the form ($\hbar \equiv 1$)

$$H = \Delta b^+ b + \frac{\lambda}{2} b^{+2} b^2 + \varepsilon N^{1/2} (b + b^+) F(t),$$

where Bose operators $b$ and $b^+ \quad (b, b^+ = 1)$ describe a single-mode of the quantum field and $\lambda$ is proportional to third-order nonlinear susceptibility of a nonlinear medium. The last term in \eqref{1} corresponds to a coupling of the oscillator with external classical periodically modulated field containing a large number of photons $N \gg 1$ ($\varepsilon$ is a coupling constant, $F(t)$ is a periodic function of time and $\Delta$ is a detuning of the mode frequency from the carrier frequency of the external field).

The Hamiltonian \eqref{1} describes, for example, a high-Q cavity filled by a medium with Kerr nonlinearity and excited by an external laser field \cite{14}. This effective Hamiltonian may also govern the interaction of a laser field with a high-density exciton system in a semiconductor \cite{13}. The different variants of discussed model are very popular in both quantum optics \cite{10} and quantum chaos \cite{13,17,18} studies.

To investigate the nonlinear dynamics of the model and the dynamics of squeezing, it is natural to use the cumulant expansion technique \cite{21} as a variant of the general $1/N$-expansion method \cite{22}. Recently it was realized that both methods could be applied not only to the integrable \cite{21,22} but also to the nonintegrable quantum systems \cite{18,23,24}. In this paper, we use well-adopted for the problems of quantum optics th $1/N$-expansion method suggested in \cite{21}.

Introduce new normalized operators for annihilation and creation of photons as $a = b/N^{1/2}$, $a^+ = b^+ /N^{1/2}$ with commutation relation $[a, a^+] = 1/N$. In the classical limit ($N \rightarrow \infty$), one has two commuting $c$-numbers. Now the Hamiltonian \eqref{1} can be rewritten as $H = \hat{H} N$, where $\hat{H}$ has the same form as \eqref{1} with an account of replacement $b \rightarrow a$, $b^+ \rightarrow a^+$ and $\lambda \rightarrow g \equiv \lambda N$. It may be shown that the dependence $g \simeq N$ correctly performs the time scale of energy oscillation for Kerr nonlinearity in the classical limit \cite{23}.
Let initially the field is in the coherent state \( |\alpha\rangle = \exp(Na a^+ - N a^+ a) |0\rangle \) corresponding to the mean photon number \( \approx N \). From the Heisenberg equations for \( a, a^\dagger \) and their Hermitian conjugated equations, and adopting the normal operator ordering, we have the following equations of motion for averages over coherent states

\[
i \frac{d}{dt} \langle \alpha \rangle = \langle V \rangle, \quad i \frac{d}{dt} \langle (\delta \alpha)^2 \rangle = 2 \langle V \delta \alpha \rangle + \langle W \rangle, \quad i \frac{d}{dt} \langle \delta \alpha^* \delta \alpha \rangle = -\langle V^* \delta \alpha \rangle + \langle \delta \alpha^* V \rangle, \tag{2}\]

where \( V = \frac{dW}{d\alpha} = W = \frac{1}{2} \frac{\partial W}{\partial \alpha^*} \), \( (\langle \delta \alpha \rangle^2) \equiv \langle a^2 \rangle - \langle a \rangle^2 \), \( \langle \delta \alpha^* \delta \alpha \rangle \equiv \langle a^+ a \rangle - \langle a^+ \rangle \langle a \rangle \). However, the set of equations (2) is not closed and actually is equivalent to the infinite dynamical hierarchy system for moments and cumulants [12]. To truncate it up to the cumulants of the second order, we made the substitution \( a \rightarrow \langle a \rangle + \delta \alpha \), where at least initially the mean \( \langle \alpha \rangle \approx 1 \) and the quantum correction \( |\delta \alpha(t = 0)| \approx N^{-1/2} \ll 1 \). Using the Taylor expansion of the functions \( V \) and \( W \), we have, in the first order of \( 1/N \) and after some algebra, the following self-consistent system of equations for the mean value and the second order cumulants [2]

\[
i \dot{z} = \langle V \rangle z + q, \tag{3a}\]

\[
i \dot{C} = 2 \left( \frac{\partial V}{\partial \alpha} \right)_z C + 2 \left( \frac{\partial V}{\partial \alpha^*} \right)_z B, \tag{3b}\]

\[
i \dot{B} = - \left( \frac{\partial V^*}{\partial \alpha} \right)_z C + \left( \frac{\partial V}{\partial \alpha^*} \right)_z C^*, \tag{3c}\]

where \( B \equiv \langle \delta \alpha^* \delta \alpha \rangle + \frac{1}{2} C \), \( C \equiv \langle (\delta \alpha)^2 \rangle \), \( z \equiv \langle \alpha \rangle \) and subscript \( z \) means that the values of \( V \) and its derivatives are calculated at mean value \( z \). Involved to the equation (3b) the small quantum correction \( q \approx 1/N \) has the form of the second differential of \( V \) as follows

\[
q = \frac{1}{2} \frac{d^2 V}{dz^2} |z| \left( C + \frac{1}{2} \left( \frac{\partial^2 V}{\partial \alpha^2} \right)_z C^* + \left( \frac{\partial^2 V}{\partial \alpha \partial \alpha^*} \right)_z \left( B - \frac{1}{2N} \right) \right).
\]

The initial conditions for the system (3) are \( B(0) = \frac{1}{2N}, C(0) = 0 \) and some arbitrary \( z(0) \) which is of the order of unity.

It is easy to see that the equations of motion (3b) and (3c) for the cumulants can be obtained from the classical equations by linearization near \( z \) (substitution \( z \rightarrow z + \Delta z, |\delta \alpha| \ll |z| \)), if one writes the dynamical equations for the variables \( (\Delta \alpha)^2 \) and \( |\Delta \alpha|^2 \). But there still exists the principal difference between the linearization of the classical motion equations and the equations for quantum cumulants (3b): it is impossible to obtain the initial conditions for \( C \) and \( B \) from only initial conditions for linearized classical equations.

Define the general field quadrature as \( X_\theta = a \exp(-i\theta) + a^\dagger \exp(i\theta) \), where \( \theta \) is a local oscillator phase. A state is said to be squeezed if there is some angle \( \theta \) for which the variance of \( X_\theta \) is smaller then the variance for a coherent state or the vacuum [2]. Minimizing the variance of \( X_\theta \) over \( \theta \), one can determine the minimum half-axis of the quantum noise ellipse. Then, the condition of the principal squeezing is \( S \equiv 1 + 2N(\langle |\Delta \alpha|^2 \rangle - \langle (\delta \alpha)^2 \rangle) = 2N(B - |C|) < 1 \) [2]. The determination of the principal squeezing \( S \) is very useful because it gives the maximal squeezing measurable by the homodyne detection [2].

Let us now compare the dynamics of the principal squeezing for regular and chaotic motion. Initially Gaussian wave packet spreads when it propagates through nonlinear medium. But it still exists the time interval of the well-defined quantum-classical correspondence during which the wave packet center follows path in the phase space governed by semiclassical equations of motion (3b). Moreover, because our equations of motion for cumulants (3b) and (3c), in fact, coincide with the equations in the definition of the maximal Lyapunov exponent [12,3], we can apply simple physical arguments on strong deformation of the classical phase volume at chaos for prediction of the strong squeezing of the noise ellipse at quantum chaos in the semiclassical limit. For chaotic dynamics, the distance in the phase space between two initially very closed trajectories \( D \) grows exponentially with time \( D(t) \approx \exp(\lambda t) \), where \( \lambda \) is the maximal Lyapunov exponent. Due to the presence of local strong (exponential) local instability inherent for underlying classical chaotic dynamics, a quantum noise ellipse may be strongly stretched in one direction and squeezed in another direction. As a result, the value of the principal squeezing in average exponentially decreases in time at chaotic dynamics. The stretching and squeezing of noise ellipse at quantum chaos is much stronger than for the case of regular and stable dynamics, when the distance between two initially closed trajectories in phase space increases in time power-wise resulting in only power-wise decreasing of the principle squeezing in time.
In order to illustrate this general picture, we return to Hamiltonian (1) and choose the form of $F(t)$ as a periodic sequence of kicks: $F(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$, where $\delta(t)$ is the Dirac $\delta$-function. In an experiment, a sequence of short light pulses can be generated by a mode locked laser. Now, by means of standard technique [3], we obtain from the differential equations (3) the coupled maps for the mean value and for the second order cumulants. In this paper, we present only the form of the resulting map for the mean value 

$$z_{n+1} = e^{-iT(\Delta+g|z_n-i\epsilon|^2)}(z_n - i\epsilon) + q_n(C_n, B_n, z_n),$$

where the subscript $n$ indicates the value of a function just before the action of the $n$-th kick. The use of the maps instead of the differential equations sufficiently simplifies the computation and reduces numerical errors.

In the classical limit $N \to \infty$ and $q \to 0$, the map (4) possesses the transition from regular to chaotic dynamics. Fig. 1 illustrates the behavior of the classical map: the phase portrait and the time-dependence of intensity $|z_n|^2$ for regular (Fig. 1a,b) and chaotic (Fig. 1c,d) dynamics.

Define the distance in the phase space between two initially very close trajectories as $D(t) = [(\delta x)^2 + (\delta y)^2]^{1/2}$, $\delta x = Re z$ and $\delta y = Im z$. The Fig. 2 shows the time-dependence of the logarithms of the principal squeezing $S$ (Fig. 2a), the normalized distance between two initially very closed trajectories $d = ND$ (Fig. 2b), and the normalized value of quantum correction $Q = Nq$ (Fig. 2c) for large but finite number of quanta $N = 10^9$. In this figure, curve 1 corresponds to the regular motion, and curves 2 and 3 - to chaotic motion with slightly different values of the Lyapunov exponent. As it is evident from this figure, the most strong local instability determines the highest degree of squeezing. It should be noticed that the difference in the magnitude of principal squeezing for chaotic and regular motion may achieve of several orders during only several kicks.

As follows from (3), for chaos, both cumulants $B$ and $C$ increase also exponentially in average, resulting in an exponential growth of the quantum correction $q$. All our analysis is valid, if and only if the values of the quantum correction and of the second order cumulants are much less than the mean values: $q, |C|, B \ll |z| \geq 1$. Under the conditions of chaos, this means that the equations (3) are correct only during the time interval $t \ll t^* \simeq \lambda^{-1} \ln N$. The time interval $t^*$ coincides with time scale determining the well-defined quantum-classical correspondence in chaotic systems [17]. Nowadays the time scale $t^*$, during which classical chaos is revealed as a quantum transient, is established for the different models of quantum chaos [13,18,24]. Moreover, only during this time interval it is possible to define the Lyapunov exponent for a quantum system [29]. Our numerical experiments on the model of kicked Kerr oscillator demonstrate, that for the chaotic motion at $gT = 7$, $\Delta/g = 1$ and $\epsilon = 0.1$, the value of the principal squeezing $S$ for $N = 10^7$ is practically indistinguishable from its value for $N = \infty$ up to the six kicks. The increase of the number of photons $N$ results in the corresponding increase of the time interval for applicability of our description of squeezed dynamics. On other hand, the number of photons $N \geq 10^7$ initially pumped to the system is quite realistic for contemporary experiments on light squeezing [24].

In summary, we have showed that among with very narrow class of the integrable systems near threshold of an instability, there is another wide class of potentially effective systems for the enhanced light squeezing - the systems with quantum chaos operating during the time scale of the well-defined quantum-classical correspondence.

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[1] D. F. Walls and G. J. Milburn, Quantum Optics (Springer-Verlag, Berlin, 1994).
[2] J. Peřina, Quantum Statistics of Linear and Nonlinear Optical Phenomena ( Kluwer Academic Publishers, Dordrecht, 1991).
[3] S. Reynaud, A. Heidmann, E. Giacobino and C. Fabre, in Progress in Optics XXX, edited by E. Wolf (Elsevier, Amsterdam, 1992), p. 1.
[4] C. Fabre, Phys. Rep. 219 (1992) 215.
[5] L. A. Lugiato, P. Galatola, and L. M. Narducci, Opt. Commun. 76 (1990) 276.
[6] A. Heidmann, J. M. Raimond, and S. Reynaud, Phys. Rev. Lett. 54 (1985) 326.
[7] A. Heidmann, J. M. Raimond, S. Reynaud, and N. Zagury, Opt. Commun. 54 (1985) 189.
[8] K. N. Alekseev, Opt. Commun. 116 (1995) 468.
[9] Very nice discussion of large squeezing near bifurcation points is presented in ref. [4], sec. 3.3.1., see also references cited therein.
[10] The only exceptions are the papers [8,11]. In [8] the attempt was made to study the light squeezing in some particular system – generalized Tavis-Cummings model, which is chaotic in the classical limit. In contrast, the results of the present paper are valid for an arbitrary single-mode quantum system with the time-dependent Hamiltonian. Strong squeezing of a wave-packet during time scale of well-defined quantum-classical correspondence also has been briefly discussed in [11].
[11] B. V. Chirikov, The uncertainty principle and quantum chaos, in Proc. of 2nd Intern. Workshop on Squeezed States and Uncertainty Relations, Moscow 25-29 May, 1992, NASA Conference Publication 3219 (1993) 317.
[12] Chaos and Quantum Physics, edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin, Les Houches Session LIL 1989 (Elsevier, Amsterdam, 1991).
[13] G. M. Zaslavsky, Phys. Rep. 80 (1981) 175.
[14] J. R. Kuklinski, Phys. Rev. Lett. 64 (1990) 2507.
[15] G. S. Agarwal, Phys. Rev. A 51 (1995) R2711, and references therein.
[16] For extensive references see sec.5.4 of ref. [4] and R. Tanaś, A. Miranowicz, and S. Kielich, Phys. Rev. A 43 (1991) 4014.
[17] G. P. Berman and G. M. Zaslavsky, Physica A 91 (1978) 450; M. Berry, N. Balazs, M. Tabor, and A. Voros, Ann Phys. 122 (1979) 26.
[18] G. P. Berman, E. N. Bulgakov and D. D. Holm, Crossover-Time in Quantum Boson and Spin Systems (Springer-Verlag, Berlin, 1994).
[19] P. Szlachetka, K. Grygiel, and J. Bajer, Phys. Rev. E 48 (1993) 101.
[20] K. N. Alekseev and G. P. Berman, Sov. Phys.-JETP 61 (1985) 569.
[21] R. Schack and A. Schentzle, Phys Rev. A 41 (1990) 3847.
[22] L. G. Yaffe, Rev. Mod. Phys. 54 (1982) 407.
[23] R. F. Fox and J. C. Eidson, Phys. Rev. A 36 (1987) 4321.
[24] B. Sundaram and P. W. Milonni, Phys. Rev. E 51 (1995) 1971.
[25] K. N. Alekseev et al., J. Mod. Opt. 37 (1990) 41.
[26] The same self-consistent system can be obtained obeying another truncation schemes. For example, if one makes the decoupling corresponding to the hypothesis that during evolution a quantum state appears which is the superposition of the coherent signal and the quantum noise, then the resulting system of equations for the forced nonlinear oscillator coincides with our system of equations up to the terms of O(1/N^2).
[27] The formulas describing coupled maps for the cumulants B and C and the quantum correction q are too long, therefore we will present them together with the detailed description of computation in our future full-size paper.
[28] The map may be also derived from time-dependent integrals of motion, which have been obtained for Hamiltonian \( \hat{H} \) in V. I. Man'ko and F. Haake, Ann. der Physik 1 (1992) 302.
[29] F. Haake, H. Wiedemann, and K. Życzkowski, Ann. der Physik 1 (1992) 531; B. L. Lan, Phys. Rev. E 50 (1994) 764.
FIG. 1. The nonlinear dynamics of kicked classical nonlinear oscillator: phase portrait and time-dependence of the intensity $|z_n|^2$ for the regular behavior at $gT = 3$ (a, b) and for the chaotic motion at $gT = 10$ (c, d). Initial condition is $z_0 = 1$ and $\varepsilon = 0.1, \Delta/g = 1$.

FIG. 2. The time-dependence of the principal squeezing (a), of the distance between two initially closed trajectories (b), and of the quantum correction (c). Curve 1 corresponds to the regular dynamics at $gT = 3$, curve 2 – to the mild chaos ($gT = 7$) and curve 3 – to the hard chaos ($gT = 10$). The average photon number is $N = 10^9$, initial condition and other parameters are the same as in Fig. 1.