ON LOCAL-IN-TIME STRICHARTZ ESTIMATES FOR THE
SCHRÖDINGER EQUATION WITH POTENTIAL

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Abstract. There have been a lot of works concerning the Strichartz estimates for
the perturbed Schrödinger equation by potential. These can be basically carried
out adopting the well-known procedure for obtaining the Strichartz estimates
from the weighted $L^2$ resolvent estimates for the Laplacian. In this paper we
handle the Strichartz estimates without relying on the resolvent estimates. This
enables us to consider various potential classes such as the Morrey-Campatano
and Kerman-Saywer classes.

1. Introduction

In this paper we are concerned with Strichartz estimates for the Schrödinger equa-
tion with small perturbations by potential,

$$\begin{cases}
  i\partial_t u + \Delta u + V(x)u = 0, \\
  u(x,0) = u_0(x),
\end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. The Strichartz estimates for (1.1) have been extensively studied
over the past several decades.

In the free case $V = 0$, it was first established by Strichartz [20] in connection
with the Tomas-Stein restriction theorem ([21, 19]) in harmonic analysis:

$$\|u\|_{L^2(\mathbb{R}^{n+2})} \lesssim \|u_0\|_{L^2}. \tag{1.2}$$

Since then, there have been developments in extending this estimate to mixed norms
$L^q_t L^r_x$ (see [8, 11, 14]):

$$\|u\|_{L^q_t L^r_x(\mathbb{R}^{n+1})} \lesssim \|u_0\|_{L^2} \tag{1.2}$$

if and only if $(q, r)$ is Schrödinger-admissible; $q \geq 2$, $2/q + n/r = n/2$ and $(q, r, n) \neq (2, \infty, 2)$.

For the perturbed (1.1), the decay $|V| \sim |x|^{-2}$ has been known to be critical for
which the Strichartz estimate (1.2) holds (see [9]), and recent studies have intensively
aimed to get as close as possible to the inverse-square potential. Rodnianski and
Schlag [13] obtained (1.2) for non-endpoint Schrödinger-admissible pairs $(q > 2)$ with
almost critical decay $|V| \lesssim (1 + |x|)^{-2-\varepsilon}$. In [4, 6], this decay assumption is weaken to
the critical $|x|^{-2}$ including the endpoint case $q = 2$. For the non-endpoint case $q > 2$,
these results are recently generalized in [3] to weak $L^{n/2}$ potentials which include $|x|^{-2}$, and generalized earlier in [12] to more larger Morrey-Campanato classes $L^{2-p}(\mathbb{R}^n)$ for $p > (n-1)/2$, $n \geq 3$. In general, the Morrey-Campanato class $L^{\alpha,p}(\mathbb{R}^n)$ is defined for $\alpha > 0$ and $1 \leq p \leq n/\alpha$ by

$$\|V\|_{L^{\alpha,p}} := \sup_{x \in \mathbb{R}^n, r > 0} r^\alpha \left( \frac{1}{r^n} \int_{\mathbb{R}^n} |V(x)|^p dx \right)^{1/p} < \infty.$$ 

Such generalizations with small potentials can be carried out adopting the well-known procedure (cf. [5,3]) for obtaining the Strichartz estimates from the weighted $L^2$ resolvent estimates of the form

$$\|((-\Delta - z)^{-1}f)\|_{L^2(w(x))} \lesssim C(w)\|f\|_{L^2(w(x)^{-1})}$$

(1.3)

where $z \in \mathbb{C} \setminus [0, \infty)$. In particular, we can obtain

$$\|u\|_{L^q_t(L^r_x(\mathbb{R}^n)))} \lesssim \|u_0\|_{L^2}$$

(1.4)

for small potentials $V \in \mathcal{KS}_2(\mathbb{R}^3)$ by making use of (1.3) obtained in [1] with $C(w) \sim \|w\|_{\mathcal{KS}_2(\mathbb{R}^3)}$. In general, the Kerman-Sawyer class $\mathcal{KS}_\alpha(\mathbb{R}^n)$ is defined for $0 < \alpha < n$ by

$$\|V\|_{\mathcal{KS}_\alpha} := \sup_Q \left( \int_Q |V(x)| dx \right)^{-1} \int_Q \int_Q \frac{|V(x)||V(y)|}{|x-y|^{n-\alpha}} dxdy < \infty,$$

where the sup is taken over all dyadic cubes $Q$ in $\mathbb{R}^n$. This class is closely related to the global Kato ($\mathcal{K}$) and Rollnik ($\mathcal{R}$) classes which are fundamental in spectral and scattering theory (cf. [10,18]) and was revealed in [15] for their usefulness for dispersive properties of the Schrödinger equation. They are defined by

$$V \in \mathcal{K} \iff \|V\|_\mathcal{K} := \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{V(y)}{|x-y|^{n-2}} dy < \infty$$

and

$$V \in \mathcal{R} \iff \|V\|_\mathcal{R} = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V(x)V(y)}{|x-y|^2} dxdy \right)^{1/2} < \infty.$$

In particular, we note that $\mathcal{K}, \mathcal{R} \subset \mathcal{KS}_2$ and $L^{\alpha,p} \subset \mathcal{KS}_\alpha$ for $p > 1$.

Motivated by these inclusions, we desire to obtain (1.4) for general dimensions. However, the method in [1] is available only in three dimensions and (1.3) is left unsolved when $w \in \mathcal{KS}_3(\mathbb{R}^n)$ for $n \neq 3$. In this regard, we aim to obtain a higher dimensional version of (1.4) without relying on the resolvent estimates. We instead make use of weighted $L^q_tL^r_x$ estimates with $q > 2$ for the Schrödinger flow $e^{it\Delta}$. Our result is the following theorem.

**Theorem 1.1.** Let $n \geq 3$, $n - 1 < \alpha < n$ and $2 < q < a < \infty$. Assume that $u$ is a solution of (1.1) and $\|V\|_{\mathcal{KS}_\alpha}^{\alpha(a-2)/(a-n)} \|V\|_{\mathcal{KS}_\alpha}^{\alpha(a-2)/(a-n)}$ is small enough compared with $T^{-\alpha(a-2)/(a-n)}$. Then we have

$$\|u\|_{L^q_t([0,T];L^r_x(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2}$$

(1.5)

if $n/r = n/2 - 2/q$. 

Remark 1.2. The quantity \( |V^\beta|_{K^S_\alpha}^{1/\beta} \) was already appeared in [16, 17, 13, 17] concerning various problems for the Schrödinger equation as well as the Schrödinger operator. From the inclusion, if \( p \neq 1 \), we also see that
\[
|V|^{\alpha q(a-2)}_{K^{S_\alpha}} \leq |V|^{\alpha q(a-2)}_{L^{\alpha,p}} = |V|^{\alpha q(a-2)}_{L^{\alpha,p}}.
\]
Hence the theorem includes various Morrey-Campanato classes as well.

Throughout this paper, the letter \( C \) stands for a positive constant which may be different at each occurrence. We also denote \( A \lesssim B \) to mean \( A \leq CB \) with unspecified constants \( C > 0 \).

2. Weighted Strichartz estimates

In this section we obtain weighted \( L^q_t L^2_x \) Strichartz estimates with \( q > 2 \) for the free Schrödinger flow, which will be used for the proof of Theorem 1.1 in the next section:

**Proposition 2.1.** Let \( n \geq 3 \) and \( n - 1 < \alpha < n \). If \( 2 < q < a < \infty \) then we have
\[
\|e^{it\Delta}f\|_{L^q_t([0,T];L^2_x(w))} \lesssim T^{\frac{n-2}{\alpha a - 2}} \|w^{\frac{\alpha q(a-2)}{4a(a-2)} - \frac{n}{2a}}\|_{K^{S_\alpha}} \|f\|_{L^2} \tag{2.1}
\]
and
\[
\left\| \int_{-\infty}^t e^{i(t-s)\Delta}F(\cdot,s)ds \right\|_{L^q_t([0,T];L^2_x(w))} \lesssim T^{\frac{2(a-2)}{\alpha a - 2}} \|w^{\frac{\alpha q(a-2)}{4a(a-2)} - \frac{n}{2a}}\|_{K^{S_\alpha}} \|F\|_{L^p_t([0,T];L^2_x(w^{-1})).} \tag{2.2}
\]

**Proof.** To show the first estimate (2.1), we first note that
\[
\|e^{it\Delta}f\|_{L^q_t([0,T];L^2_x)} \leq T^{1/a}\|f\|_{L^2} \tag{2.3}
\]
with \( 2 < a < \infty \) and
\[
\|e^{it\Delta}f\|_{L^q_t([0,T];L^2_x(w))} \lesssim \|w^{\frac{1}{2}}\|_{K^{S_\alpha}}\|f\|_{L^2} \tag{2.4}
\]
for \( n - 1 < \alpha < n \) with \( n \geq 3 \). Estimate (2.3) follows immediately from Hölder’s inequality and the fact that \( e^{it\Delta} \) is isometry on \( L^2 \);
\[
\|e^{it\Delta}f\|_{L^q_t([0,T];L^2_x)} \leq T^{1/a}\|e^{it\Delta}f\|_{L^p_tL^2_x} \leq T^{1/a}\|f\|_{L^2}.
\]
On the other hand, (2.4) was already obtained by the first author in [17]. We now deduce (2.1) from applying the complex interpolation with Lemma 2.2 below, between the two estimates (2.3) and (2.4).

**Lemma 2.2** ([2]). Let \( 0 < \theta < 1 \) and \( 1 \leq q_0, q_1 < \infty \). Given two complex Banach spaces \( A_0 \) and \( A_1 \),
\[
(L^{q_0}(A_0), L^{q_1}(A_1))_{[\theta]} = L^q((A_0, A_1)_{[\theta]})
\]
if \( 1/q = (1 - \theta)/q_0 + \theta/q_1 \), and if \( w = w_0^{1-\theta}w_1^{\theta} \)
\[
(L^2(w_0), L^2(w_1))_{[\theta]} = L^2(w).
\]
Here, \((\cdot, \cdot)_{[\theta]}\) denotes the complex interpolation functor.
Indeed, using the complex interpolation between (2.3) and (2.4), we first see
\[ \|e^{it\Delta} f\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))} \lesssim T^{\frac{1-a}{2}} \|w\|_{L_t^{\frac{2}{a}}} \|f\|_{L^2}, \]
and then we make use of Lemma 2.2 to get
\[ \|e^{it\Delta} f\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))} \lesssim T^{\frac{1-a}{2}} \|w\|_{L_t^{\frac{2}{a}}} \|f\|_{L^2} \]
where
\[ \frac{1}{q} = \frac{1-\theta}{a} + \frac{\theta}{2}, \quad 2 \leq q < a < \infty, \quad 0 < \theta < 1, \quad n-1 < \alpha < n, \quad n \geq 3. \]

By substituting \( \theta = \frac{2(a-q)}{a-2} \), we have
\[ \|e^{it\Delta} f\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))} \lesssim T^{\frac{2(a-q)}{a-2}} \|w\|_{L_t^{\frac{2}{a}}} \|f\|_{L^2} \]
under \( 2 \leq q < a < \infty, \quad n-1 < \alpha < n \) and \( n \geq 3 \). Replacing \( w^{\frac{2(a-q)}{a-2}} \) with \( w \), we obtain the desired estimate (2.1).

Next, we show the estimate (2.2). By the standard \( TT^* \) argument, (2.1) implies
\[ \left\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))} \lesssim T^{\frac{2(a-q)}{a-2}} \|w\|_{L_t^{\frac{2}{a}}} \|f\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))}. \]
Since \( q > q' \) from \( q > 2 \), we use the Christ-Kiselev lemma ([6]) to conclude
\[ \left\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))} \lesssim T^{\frac{2(a-q)}{a-2}} \|w\|_{L_t^{\frac{2}{a}}} \|f\|_{L_t^q([0,T];L_x^2(\mathbb{R}^n))}, \]
as desired. \( \Box \)

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1 by making use of the weighted \( L_t^q L_x^2 \) estimates in the previous section.

We first consider the potential term in (1.1) as a source term and then write the solution of (1.1) as the sum of the solution to the free Schrödinger equation plus a Duhamel term, as follows:
\[ u(x,t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} (V(\cdot)u(\cdot,s)) ds. \]

By the classical Strichartz estimates (1.2) for the free Schrödinger flow, we then have
\[ \|u\|_{L_t^q([0,T];L_x^2)} \lesssim \|u_0\|_{L^2} + \left\| \int_0^t e^{i(t-s)\Delta} (V(\cdot)u(\cdot,s)) ds \right\|_{L_t^q([0,T];L_x^2)} \]
for \( q \geq 2 \) and \( 2/q + n/r = n/2 \). Now we are reduced to showing that
\[ \left\| \int_0^t e^{i(t-s)\Delta} V(\cdot)u(\cdot,t) ds \right\|_{L_t^q([0,T];L_x^2)} \lesssim T^{\frac{2(q-a)}{a-2}} \|V\|_{L_t^\infty L_x^\infty} \|u_0\|_{L^2} \]
under the same conditions on \( \alpha, a \) and \((q, r)\) as in Theorem 1.1. By duality, it suffices to prove that

\[
\left\langle \int_0^t e^{i(t-s)\Delta} (Vu) ds, G \right\rangle_{x,t} \lesssim T^{\frac{a(q-2)}{q(q-1)}} \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \| G \|_{L_t^q([0,T];L_x^r)}. \tag{3.2}
\]

To show this, we first write

\[
\left\langle \int_0^t e^{i(t-s)\Delta} Vu ds, G \right\rangle_{x,t} = \int \int_0^t \left\langle Vu, e^{-i(t-s)\Delta} G \right\rangle_x ds dt
\]

and then use Hölder’s inequality to bound (3.3) by

\[
\|u\|_{L_t^q([0,T];L_x^r(|V|))} \int_0^\infty e^{-i(t-s)\Delta} G dt \left\| L_x^r([0,T];L_x^r(|V|)) \right\|
\]

We will then show that

\[
\|u\|_{L_t^q([0,T];L_x^r(|V|))} \lesssim T^{\frac{(a-1)(q-2)}{q(q-1)}} \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \tag{3.4}
\]

and

\[
\left\| \int_0^\infty e^{i(t-s)\Delta} G ds \right\|_{L_t^q([0,T];L_x^r(|V|))} \lesssim T^{\frac{a-2}{q(q-1)}} \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \| G \|_{L_t^q([0,T];L_x^r')} \tag{3.5}
\]

to yield the desired estimate (3.2).

For the first estimate (3.4), we use Hölder’s inequality in time and then apply Proposition 2.1 to (3.1) with \( w = |V| \) to get

\[
\| u \|_{L_t^{\alpha'}([0,T];L_x^r(|V|))} \lesssim T^{1-2/q} \| u \|_{L_t^q([0,T];L_x^r(|V|))} \lesssim T^{1-2/q} \left( T^{\frac{q-2}{q(q-1)}} \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \tag{3.6}
\]

for \( 2 < q < a < \infty, n - 1 < a < n \) and \( n \geq 3 \). Since we are assuming that \( \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \) is small enough compared with \( T^{-\frac{a(q-2)}{q(q-1)}} \), the last term on the right-hand side of (3.6) can be absorbed into the left-hand side. Hence, we obtain the first estimate (3.4). To show the second estimate (3.5), we use the weighted estimate (2.1) and its dual estimate to see that

\[
\left\| \int_{-\infty}^\infty e^{i(t-s)\Delta} G ds \right\|_{L_t^q([0,T];L_x^r(|V|))} = \left\| e^{i\Delta} \int_{-\infty}^\infty e^{-i\Delta} G ds \right\|_{L_t^q([0,T];L_x^r(|V|))} \lesssim T^{\frac{q-2}{q(q-1)}} \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \| G \|_{L_t^q([0,T];L_x^r')}
\]

for \( 2 < q < a < \infty, n - 1 < a < n \) and \( n \geq 3 \). Since we are assuming that \( \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \) is small enough compared with \( T^{-\frac{a(q-2)}{q(q-1)}} \), the last term on the right-hand side of (3.6) can be absorbed into the left-hand side. Hence, we obtain the first estimate (3.4). To show the second estimate (3.5), we use the weighted estimate (2.1) and its dual estimate to see that

\[
\left\| \int_{-\infty}^\infty e^{i(t-s)\Delta} G ds \right\|_{L_t^q([0,T];L_x^r(|V|))} = \left\| e^{i\Delta} \int_{-\infty}^\infty e^{-i\Delta} G ds \right\|_{L_t^q([0,T];L_x^r(|V|))} \lesssim T^{\frac{q-2}{q(q-1)}} \| V \|_{\frac{\alpha(q-2)}{q(q-1)}} \| u_0 \|_{L^2} \| G \|_{L_t^q([0,T];L_x^r')}
for $2 < q < a < \infty$, $n - 1 < \alpha < n$ and $n \geq 3$. Since $q > q'$ from $q > 2$, we then use the Christ-Kiselev lemma ([6]) to conclude

$$
\left\| \int_\infty^t e^{i(t-s)\Delta} G ds \right\|_{L^q_t([0,T];L^2_\mathbb{R})(|V|)} \lesssim T^{\frac{q-2}{\alpha(q-2)}} \left\| V^{\frac{1}{q} \left( \frac{q}{a} - 1 \right)} \right\|_{KS_\alpha} \left\| G \right\|_{L^q_t([0,T];L^q_\mathbb{R})},
$$

which implies (3.5) by changing some variables. This completes the proof.

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