Semianalytic Analysis of Primordial Black Hole Formation
During a First-order QCD Phase Transition

Christian Y. Cardall
Department of Physics & Astronomy, State University of New York at Stony Brook, Stony Brook, NY 11794-3800

George M. Fuller
Department of Physics, University of California, San Diego, La Jolla, CA 92039-0319
(January 20, 2022)

Abstract

It has recently been suggested that cosmologically significant numbers of black holes could form during a first-order QCD phase transition. Further, it has been asserted that these black holes would have masses corresponding naturally to the inferred mass (∼1M⊙) of the MACHOs responsible for the observed gravitational microlensing events. In this model, the underlying spectrum of primordial density perturbations provides the fluctuations that give rise to black holes at the epoch of the QCD transition. We employ a simplified model to estimate the reduction in the critical overdensity of a horizon-sized primordial perturbation required for collapse to a black hole. We find that a first-order QCD transition does indeed produce a sharp peak in the black hole mass spectrum, but that this peak corresponds to the horizon mass at an epoch somewhat earlier than the cosmological transition itself. Assuming a COBE normalized primordial density perturbation spectrum with constant spectral index, for the black holes so produced to be cosmologically significant would require an extremely finely tuned “blue” primordial density perturbation spectrum. Specifically, in the context of our simplified model, a spectral index in the range n = 1.37 – 1.42 corresponds to the range Ω ∼ 10^{-5} – 10^3 of the black hole contribution to the present-day density parameter.

97.60.Lf, 12.38.Aw, 95.35.+d, 98.35.Gi

Typeset using REVTeX
I. INTRODUCTION

Recent studies of gravitational microlensing suggest that about half of the Galaxy’s halo mass can be accounted for by massive compact halo objects (MACHOs) with mass $\sim 0.5M_\odot$ [1]. However, a direct search for halo dwarfs in the Hubble Deep Field has limited the halo mass fraction of red dwarfs brighter than magnitude $M_I = 15$ to $f_{RD} < 0.06$, and a halo mass fraction of white dwarfs with $M_I < 15$ to $f_{WD} < 0.33$ [3]. Comparing what a present day white dwarf Galactic halo fraction should have looked like in the past with deep galaxy surveys leads to a limit of $f_{WD} < 0.10$ [4]. A study of the infrared background and Galactic metallicity shows that it is likely that $f_{WD} < 0.25$ [5].

While white dwarf halo mass fractions which exceed these limits might be obtained with specially tailored assumptions [3,4], ultimately it may be necessary to consider more exotic candidates for the MACHOs. In particular, Jedamzik has recently proposed that horizon-size black hole formation from primordial density fluctuations could operate particularly efficiently during a first-order QCD phase transition [8]. A first-order confinement transition involves the coexistence of quark-gluon and hadron phases [9]. On large scales, the pressure response of this “mixed” phase is greatly reduced: when compressed, the energy density does not increase from increases in pressure and temperature, which remain constant. Instead, low density hadron phase is exchanged for high energy density quark-gluon phase. Because of the reduced pressure response, the required overdensity for black hole formation is smaller. For fluctuations entering the horizon during this epoch, statistically more abundant, lower amplitude perturbations could form black holes, leading to a peak in the primordial black hole (PBH) mass spectrum at the QCD epoch horizon mass. Since the horizon mass is

$$M_h(T) \approx 0.87M_\odot \left( \frac{T}{100\text{MeV}} \right)^{-2} \left( \frac{g_*}{51.25} \right)^{-1/2},$$  

(1)

where $g_*$ is the effective number of relativistic degrees of freedom at cosmological temperature $T$, PBHs formed at the QCD transition temperature $T_{QCD} \approx 100$ MeV could be in the right mass range to be MACHOs. This possibility takes on added interest in view of the claim that galactic halos filled with $\sim 0.5M_\odot$ black holes could bring about enough nearby BH-BH mergers to cause a few events/year in the first generation network of gravitational wave interferometers [10].

Other than a general guess that the critical overdensity for collapse would be reduced by a factor of order unity during the QCD transition [8,11], no quantitative estimate of this reduction factor has appeared. A relativistic numerical calculation is in preparation [12]. The purpose of this paper is to use a simplified model to estimate this reduction factor, and to assess the cosmological significance of the resulting peak in the PBH mass spectrum.

---

1The absence of short timescale microlensing events restricts the MACHOs to masses $\gtrsim 0.1M_\odot$, unless thick disk/light halo models of the Milky Way are employed. This eliminates brown dwarfs unless, in conjunction with these nonstandard Galactic models, the brown dwarf mass function peaks close to the hydrogen burning limit [1]. It has also been suggested that the gravitational lenses are not in the Galactic halo at all, but rather in “the warped and flaring” Galactic disk [3].
II. CONDITIONS FOR PBH FORMATION

A. Radiation equation of state

A heuristic criterion for gravitational collapse to a black hole of an overdense region of energy density $\rho$ and size $S$ was given by Carr and Hawking [13]. These authors argued that collapse would ensue when the “gravitational energy” $\sim G\rho^2 S^5$ was large enough to overcome the “kinetic energy” of expansion and the pressure forces (here $G$ is the gravitational constant). The overdense region expands more slowly than the average cosmological expansion, and eventually turns around and begins to collapse. Carr and Hawking consider the region at the moment of turnaround (when the kinetic energy of expansion is zero), and account for the pressure forces by comparing the gravitational energy to the “internal energy.” This “internal energy” is approximated as

$$pS^3 \sim w\rho S^3,$$

where $p$ is the pressure and an equation of state of the form $p = w\rho$ is assumed ($w$ is a constant). Then the condition for an overdense region to overcome pressure forces and collapse to a black hole is $G\rho_c S_c^2 \gtrsim w$, where the subscript $c$ indicates that the quantity is to be evaluated at the moment of turnaround when the collapse “begins.” This condition can also be written $S_c \gtrsim (w/G\rho_c)^{1/2} \sim R_J$, where $R_J$ is the Jeans length. However, the overdense region cannot be larger than the horizon size at turnaround, or it will constitute a separate universe, causally disconnected from our own, and would not be relevant to this discussion.

For radiation ($w = 1/3$), the size of the overdense region at turnaround, its Schwarzschild radius, the Jeans length, and the cosmological horizon size are all of the same order of magnitude. Therefore it is expected that an overdense region of radiation satisfying $G\rho_c S_c^2 \gtrsim 1/3$ will form a black hole of order the horizon mass, with little chance for centrifugal or turbulent forces to have a significant effect. For dust ($w = 0$), the Jeans length is much smaller than the horizon size, and centrifugal and turbulent forces could prevent full collapse to a black hole. If a hole does form in this case it could be much smaller than the horizon mass.

The condition for an overdense region to collapse to a black hole is expressed above in terms of the density and size of the region at turnaround. However, cosmological density perturbations are typically classified by specifying their overdensity $\delta = (\rho - \bar{\rho})/\bar{\rho}$ when they enter the horizon, where $\rho$ is the energy density of the overdense region and $\bar{\rho}$ is the average cosmological density. For assessing the cosmological significance of the black hole population, it will therefore be convenient to express the condition for collapse in terms of $\delta$, which in this paper will always refer to the overdensity at horizon crossing.

To make this connection we follow Carr [14]. The line element for the “average” universe can be taken to be the conformally flat Friedmann-Robertson-Walker (FRW) form,

$$ds^2 = -dt^2 + R^2[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)],$$

where the evolution of the time-dependent cosmic scale factor $R$ is given by

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\bar{\rho}R^2.$$
Here the “average” background geometry/evolution is obtained by viewing the universe on a large enough scale that the (nearly critical) mass-energy density can be approximated as homogeneously and isotropically distributed.

We consider “top-hat” fluctuations: spherical, homogeneous overdense regions. Such overdense regions, if taken to have homogeneously distributed internal mass-energy, can be described by the metric of a closed FRW universe,

$$ds^2 = -d\tau^2 + S^2[(1 - \kappa r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)],$$

with the evolution of $S$ given by

$$\left(\frac{dS}{d\tau}\right)^2 = \frac{8\pi G}{3} \rho S^2 - \kappa,$$

where $\rho$ is the energy density in the overdense region. Obviously, at some point in the gravitational collapse of such a fluctuation, pressure gradients will build up and it will not be possible to approximate the internal mass-energy distribution as homogeneous. However, this approximation will not be bad until near turnaround. Here $S$ can be interpreted as a sort of “radius” of the evolving fluctuation. Likewise, $\tau$ is a timelike coordinate characterizing the evolution of the overdense region.

The general relativistic gauge freedom inherent in this model is fixed by setting $\tau_h = t_h$, $S_h = R_h$, and $(dS/d\tau)_h = (dR/dt)_h$, ensuring that the when the overdense region enters the horizon it is only a density perturbation. (The subscript $h$ denotes a quantity evaluated when the overdense region under consideration enters the horizon.) At horizon crossing, the energy density of the fluctuation is $\rho_h = \bar{\rho}_h(1 + \delta)$. Using these conditions, and employing an equation of state $p = w\rho$, the evolution of the overdense region is given by

$$\left(\frac{dS}{d\tau}\right)^2 = \frac{8\pi G}{3} \bar{\rho}_h R_h^{3(1+w)} \left[ \frac{1 + \delta}{S_c^{1+3w}} - \frac{\delta}{R_h^{1+3w}} \right],$$

so at turnaround

$$S_c^{1+3w} = R_h^{1+3w} \left( \frac{1 + \delta}{\delta} \right).$$

The quantity appearing in the collapse condition is

$$G\rho_c S_c^2 = \frac{G\bar{\rho}_h(1 + \delta)R_h^{3(1+w)}}{S_c^{1+3w}} = G\bar{\rho}_h R_h^2 \delta,$$

where we have used Eq. (8). Using Eq. (9) and the fact that the horizon size $\sim t$,

$$G\bar{\rho}_h R_h^2 \sim \left(\frac{dR}{dt}\right)_h \sim \left(\frac{R_h}{t_h}\right)^2 \sim 1,$$

so that the condition for collapse is

$$\delta \gtrsim w,$$

or $\delta \gtrsim 1/3$ for radiation. In other words, in this conventional picture, the minimum fluctuation amplitude $\delta\rho/\rho$ at horizon crossing required for collapse to a black hole is of order unity, i.e. quite large. This large required initial amplitude simply reflects the “hard” equation of state of radiation.

4
B. First-order QCD transition

It is not at all clear at this point whether or not there is a first order cosmic vacuum phase transition associated with the QCD epoch. The chiral symmetry breaking event remains the prime candidate for such a transition, though the lattice gauge calculation results on this issue remain unsettled. For illustrative purposes, we will assume that there is a first order phase transition associated with this epoch in the early universe. The QCD era is promising for primordial black hole production, since phase separation at this time implies that the duration of a first order phase transition will be (essentially) a gravitational timescale.

A bag-like picture employing a QCD vacuum energy provides a simple description of a first order QCD phase transition with critical transition temperature \( T_{QCD} \). The pressure and energy density of the hadron phase \((T \leq T_{QCD})\) are approximately

\[
p_{\text{had}}(T) = \frac{1}{3} \rho_{\text{had}}, \quad \rho_{\text{had}}(T) = \frac{\pi^2}{30} g_{\text{had}} T^4,
\]

with, for example, \( g_{\text{had}} \approx 17.25 \). The quark-gluon phase \((T \geq T_{QCD})\) is described by

\[
p_{\text{qg}}(T) = \frac{1}{3} \frac{\pi^2}{30} g_{\text{qg}} T^4 - B, \quad \rho_{\text{qg}}(T) = \frac{\pi^2}{30} g_{\text{qg}} T^4 + B,
\]

where \( g_{\text{qg}} \approx 51.25 \) and \( B \) is the QCD vacuum energy density. During the transition \((T \approx T_{QCD})\), the requirement \( p_{\text{had}}(T_{QCD}) = p_{\text{qg}}(T_{QCD}) \) yields

\[
B = \frac{1}{3} \frac{\pi^2}{30} (g_{\text{qg}} - g_{\text{had}}) T_{QCD}^4.
\]

Averaged over the coexisting phases, the energy density during the transition is

\[
\langle \rho \rangle = f_{\text{qg}} \rho_{\text{qg}}(T_{QCD}) + (1 - f_{\text{qg}}) \rho_{\text{had}}(T_{QCD}),
\]

where \( f_{\text{qg}} \) is the volume fraction in quark-gluon phase. The evolution of the average energy density during the transition as a function of cosmic scale factor \( R \) can be described as “dust-like” \((\rho \sim R^{-3})\)

\[
\langle \rho \rangle(R) = \left( \frac{R_1}{R} \right)^3 \left( \rho_{\text{qg}} + \frac{1}{3} \rho_{\text{had}} \right) - \frac{1}{3} \rho_{\text{had}},
\]

where \( R_1 \) is the scale factor when the cosmic energy density reaches \( \rho_1 \equiv \rho_{\text{qg}}(T_{QCD}) \). This is the start of the phase transition.

For the purpose of assessing the probability of black hole formation during the QCD epoch, we will consider the following further caricature of the behavior of the plasma through the transition. For energy densities greater than \( \rho_1 \), take the plasma to be radiation with \( g_* = g_{\text{qg}} \). This ignores the vacuum energy density \( B \), which at \( T = T_{QCD} \) contributes about 18% of the total. For energy densities in the range \( \rho_2 \equiv \rho_{\text{had}}(T_{QCD}) < \rho < \rho_1 \), we can treat the plasma as dust. Here \( \rho_1 \) and \( \rho_2 \) correspond to the energy densities in the plasma at the start and conclusion of the phase transition, respectively. Even though the pressure itself is not zero, the almost negligible pressure response of the plasma during the transition leads
to the dust-like evolution noted above. For densities less than \( \rho_2 \), we treat the plasma as radiation with \( g_* = g_{\text{had}} \).

The duration of the transition can be obtained by comparing the cosmic scale factor at the beginning \( (R_1) \) and end \( (R_2) \) of the transition. From the conservation of co-moving entropy density we obtain \( (R_2/R_1) = (g_{\text{eq}}/g_{\text{had}})^{1/3} \approx 1.44 \), so the transition is fairly short. An overdense region that goes through the transition during its collapse does not correspond to the cases discussed in the previous subsection, i.e. either radiation or dust throughout the collapse. How should the condition for collapse to a black hole be modified in this case? Based on the above caricature of the behavior of the plasma around the transition, we suggest modifying the “internal energy” of Eq. (2) to \((1/3)\rho_c S_c^3(1-f)\), where \( f \) is the fraction of the overdense region’s “evolution volume” spent in the dust-like transition epoch.

For example, letting \( S_1 \) and \( S_2 \) respectively denote the size of the initially expanding overdense region at the beginning and end of the phase transition, \( f = (S_2^3 - S_1^3)/S_c^3 \) for an overdense region that has completed the transition before turnaround. Put another way, for an overdense region that obeys \( p = w\rho \) for much of its evolution and a dust-like phase for the rest, we define an effective Jeans length \( R_{\text{eff}} = (w/G\rho_c)^{1/2}(1-f)^{1/2} \) at turnaround. In terms of the overdensity of the fluctuation at horizon crossing, the condition for collapse can be expressed

\[
\delta \gtrsim w(1-f).
\]

Therefore, the larger the fraction of a fluctuation’s evolution time which is spent in the mixed phase regime, the smaller the required initial horizon crossing perturbation amplitude for collapse to a black hole. The evaluation of this collapse criterion will be the subject of Sec. IV. Note that in our analysis we have assumed that at no time is the universe ever dominated by the QCD vacuum energy \( B \). Were such a vacuum-dominated condition to arise, then a kind of “mini-inflation” occurs, where the scale factor can increase exponentially with time. However, in this case the nucleation rate for bubbles of low temperature phase will increase very rapidly with time and decreasing temperature, with the result that the fraction \( f \) will likely be smaller than in the non-vacuum-dominated case.

### III. CLASSIFYING PERTURBATIONS IN THE QCD TRANSITION EPOCH

“Top-hat” perturbations are completely specified by \( \delta \)—their overdensity when they enter the horizon—and by the cosmological time at which horizon crossing occurs. Here we will find it convenient to use the average cosmological energy density at horizon crossing \( \bar{\rho}_h \) rather than the time of horizon crossing. In particular, we use the variable \( x \) to identify the epoch of horizon crossing, where \( x \equiv \bar{\rho}_h/\rho_1 \). Again, \( \rho_1 \) is the energy density above which the plasma is pure quark-gluon phase, and \( \rho_2 \) is the density below which the plasma is pure hadron phase. We define the constant \( \beta \equiv \rho_1/\rho_2 \). Then perturbations with \( x > 1 \), \( \beta^{-1} < x < 1 \), and \( x < \beta^{-1} \) enter the horizon when the average cosmological background—not the overdense region itself—is in quark-gluon, mixed quark-gluon/hadron, and hadron phase, respectively.

In addition, our description of the collapse of an overdense region involves two important events: (1) the region’s horizon crossing, and (2) its turnaround, when it stops expanding.
and begins to recollapse. The state of the matter at these two events constitute various classes of perturbations; these are listed in Table I.

Given the characteristics of an overdense region, δ and x, we can specify the class A-F to which it belongs. The classes of perturbations that exist for various epochs of horizon crossing and values of δ are shown in Table I. From the information in this table two curves can be drawn in the (x, δ) plane (Fig. 1). Classes A, B, and C are separated from classes D and E by the curve δ = x^{-1} - 1, and classes D and E are separated from class F by the curve δ = β^{-1}x^{-1} - 1. These divisions are based on the state of matter in the overdense region when it enters the horizon.

In contrast, the curves in the (x, δ) plane separating classes A and B, B and C, and D and E are based on the state of matter in the overdense region at turnaround. For example, the boundary between classes A and B is determined by ρ_c = ρ_1, that is, by the condition that an overdense region entering the horizon in the quark-gluon phase just reaches the density for the phase transition to begin when it begins to recollapse. The condition

\[ \rho_c = \frac{\bar{\rho}_h(1 + \delta)R_h^4}{S_c^4} = \rho_1, \]

using Eq. (8) with \( w = \frac{1}{3} \) yields the curve defined by

\[ \frac{\delta^2}{(1 + \delta)} = x^{-1}. \]

The other curves can be obtained in a similar manner, with the complication that changes in the density evolution between radiation (\( \rho \propto S^{-4} \)) and dust (\( \rho \propto S^{-3} \)), and/or vice-versa, must be followed when the density reaches \( \rho_1 \) and/or \( \rho_2 \) respectively. The analysis leading to Eq. (8) can then be generalized in a straightforward manner. The curve defining the boundary between classes B and C is

\[ \frac{\delta^3}{(1 + \delta)^{3/2}} = \beta^{-1}x^{-3/2}, \]

and the boundary between classes D and E is defined by

\[ \frac{\delta^3}{(1 + \delta)^2} = \beta^{-1}x^{-1}. \]

These last two curves are valid only in the regions above and below the curve separating classes A, B, and C from classes D and E, respectively.

Fig. 1 shows the (x, δ) plane divided into regions that constitute classes A-F. Class A has no dust-like phase, and δ > 1/3 for collapse. Class D does not exist for δ < \( \sim 1.8 \), so perturbations of this class would be expected to collapse even without the dust-like phase. Therefore, interesting reductions in the critical value of δ for collapse to a black hole are possible for classes B, C, E, and F.

For reference, the expressions for the turnaround radius \( S_c \) for the various classes are given in Table III. We also give expressions for \( S_1 \) and \( S_2 \), the values of S for which the density in the overdense region reaches \( \rho_1 \) and \( \rho_2 \) respectively. These are
\[ S_1 = R_h x^{1/4} (1 + \delta)^{1/4}, \]  
(22)  
which is valid (and useful) for classes B and C,

\[ S_2 = R_h (\beta x)^{1/3} (1 + \delta)^{1/3}, \]  
(23)  
valid (and useful) for class E, and

\[ S_2 = R_h (\beta x)^{1/4} (1 + \delta)^{1/4}, \]  
(24)  
for class F.

IV. COMPUTING CRITICAL OVERDENSITIES

In the framework of our model, overdense regions with

\[ \delta \gtrsim \frac{1}{3} (1 - f) \]  
(25)  
will collapse to a black hole, where \( f \) is the fraction of “evolution volume” that the overdense region spends in the dust-like phase. For class B, turnaround occurs while the overdense region is still in the mixed phase:

\[ f = \frac{S_3^c - S_1^3}{S_3^c} = 1 - \left( \frac{S_1}{S_c} \right)^3 \]  
(Class B),  
(26)  
and \( S_1 \) and \( S_c \) are given in Eq. (22) and Table III respectively, yielding

\[ \left( \frac{S_1}{S_c} \right)^3 = \frac{x^{3/2} \delta^3}{(1 + \delta)^{3/2}}. \]  
(27)  
For classes C, E, and F, the transition to hadron phase is completed by turnaround, so

\[ f = \frac{S_3^c - S_1^3}{S_3^c} \]  
(Classes C, E, F).  
(28)  
For class C, it is convenient to write this in the form

\[ f = \left( \frac{S_1}{S_c} \right)^3 \left[ \left( \frac{S_2}{S_1} \right)^3 - 1 \right] \]  
(Class C),  
(29)  
where \( (S_2/S_1)^3 = \rho_1/\rho_2 = \beta \), and

\[ \left( \frac{S_1}{S_c} \right)^3 = \frac{x^{3/4} \delta^{3/2}}{\beta^{1/2} (1 + \delta)^{3/4}} \]  
(Class C).  
(30)  
For classes E and F it is convenient to write Eq. (28) in the form

\[ f = \left( \frac{S_2}{S_c} \right)^3 \left[ 1 - \left( \frac{S_1}{S_2} \right)^3 \right] \]  
(Classes E, F),  
(31)
where \((S_1/S_2)^3 = \beta^{-1}\) and
\[
\frac{(S_2)}{S_c}^3 = \frac{(\beta x)^{1/2} \delta^{3/2}}{(1 + \delta)} \quad \text{(Class E)}
\]
\[
\frac{(S_2)}{S_c}^3 = \frac{(\beta x)^{3/4} \delta^{3/2}}{(1 + \delta)^{3/4}} \quad \text{(Class F)},
\]
using Eqs. (23,24) and Table III.

For a given \(x\)—that is, for overdense regions entering the horizon at a given time—the values of \(\delta\) for which collapse to a black hole will occur can be found by plotting both \((1 - f)/3\) and \(\delta\) itself as functions of \(\delta\), taking care to use the expression for \(f\) appropriate to the class specified by \((x, \delta)\) (see Fig. 1). Figure 2 shows such a plot for \(x = 2\), from which it is clear that collapse will occur for all values of \(\delta\) above a certain critical value. However, as seen in Fig. 3, for larger values of \(x\) \((x = 15\) in Fig. 3) an interesting thing occurs: the range of \(\delta\) for which \(\delta > (1 - f)/3\) has an upper as well as lower bound. Figure 4 indicates the region in the \((x, \delta)\) plane for which collapse to a black hole occurs. The range of \(x\) for which there is a reduction in the critical overdensity for collapse has an upper bound; and while it has no lower bound, the reduction becomes slight at small values of \(x\).

V. COSMOLOGICAL SIGNIFICANCE

The potential cosmological significance of primordial black holes lies in the fact that the energy density of a black hole population redshifts as \(R^{-3}\), in contrast with radiation, whose energy density redshifts as \(R^{-4}\). Thus the relative contribution of black holes to the cosmological density parameter \(\Omega\) increases as the universe expands. See, for example, the work by Crawford and Schramm [16].

For black holes formed at a particular epoch, the contribution to the present-day density parameter is
\[
\Omega_{\text{BH}} \approx 2 \times 10^7 \epsilon(T) \left(\frac{T_0}{2.75\text{K}}\right)^3 \left(\frac{T}{100\text{MeV}}\right)^2 h^{-2},
\]
where \(\epsilon(T)\) is the fraction of radiation energy density converted into black holes at cosmological temperature \(T\), \(T_0\) is the present-day microwave background temperature, and \(h\) is the present epoch Hubble parameter in units of 100 km s\(^{-1}\) Mpc\(^{-1}\). Assuming that the mass of the black holes is the horizon mass at \(T\), we can write
\[
\epsilon(T) = \int_{\delta_a}^{\delta_b} F(\delta, T) \, d\delta,
\]
where \(F(\delta, T)\) is the probability of a horizon volume at \(T\) to have overdensity \(\delta\), and \(\delta_a\) and \(\delta_b\) delimit the range of overdensities that contribute to the black hole population.

The power spectrum (spatial Fourier transform) of density perturbations is typically assumed to have a power law form, \(|\delta_k|^2 \propto k^n\), where \(k\) is the wavenumber and \(n\) is the spectral index. Density perturbations generated during inflation typically have \(n \approx 1\), with a normal distribution of overdensities.
\[
F = \frac{1}{\sqrt{2\pi}\sigma(M)} \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right),
\]
where \(\sigma(M)\) is the variance at horizon mass \(M\) and can be taken to be \(17\)
\[
\sigma(M) = 9.5 \times 10^{-5} \left(\frac{M}{10^{22} M_\odot}\right)^{(1-n)/4}
\]
when normalized by COBE measurements of the microwave background anisotropy.

The critical value of \(\delta\) above which collapse to a black hole can occur \([\delta_a\text{ in Eq. (35)}]\) is given by the curve in Fig. 4, taking the lowest branch when it becomes multivalued at the higher values of \(x\). Noting that perturbations with \(\delta \gtrsim 1/3\) make negligible contributions to \(\Omega\), we take \(\delta_b = 1/3\), except for the higher values of \(x\) for which the curve in Fig. (4) is multivalued. For this region, \(\delta_b\) is given by the middle branch of the curve in Fig. (4).

It is convenient to change from our variable \(x = \bar{\rho}_h/\rho_1\), in terms of which the curve in Fig. (4) was calculated, to the horizon mass \(M\). This connection is made as follows. We account for the fact that we have ignored the vacuum energy density \(B\) in the quark-gluon phase by defining an effective transition temperature \(T_{\text{QCD,eff}}\) by
\[
\frac{\pi^2}{30} g_* T_{\text{QCD}}^4 + B = \frac{\pi^2}{30} g_{\text{rg}} T_{\text{QCD,eff}}^4.
\]
Using this equation and Eqs. (1) and (14), we then have the following relation between \(x\) and \(M\):
\[
x = \frac{g_* T_{\text{QCD,eff}}^4}{g_{\text{rg}} T_{\text{QCD,eff}}^4} = \left(\frac{4}{3} g_{\text{rg}} - \frac{1}{3} g_{\text{had}}\right)^{-1} \left(\frac{M}{0.87 M_\odot}\right)^2 \left(\frac{T_{\text{QCD}}}{100\text{MeV}}\right)^{-4}.
\]
For a given value of the spectral index \(n\), \(\epsilon(M)\) can be computed, where the horizon mass \(M\) has replaced \(T\) in Eq. (35) as the variable indicating the cosmological epoch.

For the values of \(n\) we will have to consider \((n \approx 1.4)\), \(\epsilon(M)\) is a very strongly peaked function, justifying the assumption in Eq. (34) of black holes being formed at a single epoch. This sharp peak comes about as follows. For \(n > 1\), there is increasing power on small scales [Eq. (37)]. Also, \(\delta_a\) decreases with decreasing \(M\) (increasing \(x\), see Fig. 4). Both of these tend to make \(\epsilon(M)\) increase rapidly with decreasing \(M\). However, when \(M\) gets too small, the range of overdensities contributing to black holes is cut off completely [the cusp at \(x \approx 50\) in Fig. 4], and \(\epsilon(M)\) drops abruptly.

To obtain \(\epsilon\) for \(\epsilon\) use in Eq. (34), we choose a value of \(n\) and find the peak value of \(\epsilon(M)\). The mass of the black holes is the value of \(M\) for which \(\epsilon\) has its peak value. Figures (5) and (6) show \(\Omega_{\text{BH}}\) as a function of \(n\) for two choices of transition temperature \(T_{\text{QCD}}\). The fine tuning in \(n\) required for \(\Omega_{\text{BH}} \sim 1\) is immediately apparent [11]. For a given \(T_{\text{QCD}}\) the masses of the black holes do not vary significantly over the ranges of \(n\) displayed: \(m_{\text{BH}} \simeq 0.11 M_\odot\) for \(T_{\text{QCD}} = 100\text{MeV}\), and \(m_{\text{BH}} \simeq 0.03 M_\odot\) for \(T_{\text{QCD}} = 200\text{MeV}\).

**VI. CONCLUSIONS**

Given the potential difficulties in accounting for the observed MACHO population with red or white dwarfs, it is interesting to follow up on the suggestion [8] that the MACHOs
could be black holes formed during a first-order QCD confinement transition. We have performed a semianalytic analysis using a simplified model. This model involves consideration of spherical “top-hat” overdense regions and the use of the Friedmann equation to calculate their evolution. A caricature of the behavior of the equation of state near the QCD transition is employed, along with an extension of the condition given by Carr and Hawking [13] for primordial overdense regions to collapse to black holes.

In the context of a COBE normalized primordial density perturbation spectrum characterized by a power law with constant spectral index $n$, we find that for the black holes so produced to be cosmologically significant would require $n \sim 1.4$, with extreme fine tuning (see Figs. 5-6) [8,11]. While such a spectral index is not in conflict with observations of microwave background anisotropies [18,19], it conflicts with limits on $n$ based on, for example, evaporation of lower mass black holes formed at earlier epochs (e.g., Refs. [17,20] and references therein). In addition, we find that the holes most efficiently formed by this mechanism entered the horizon at an epoch somewhat earlier than the time the average universe goes through the QCD transition. Accordingly, we find black holes masses in the range $0.11 - 0.03M_\odot$ for QCD transition temperatures of $100 - 200$ MeV. This mass range is somewhat low for the estimated $0.5M_\odot$ MACHO mass.

Our model involves some gross simplifications. The condition for collapse to a black hole is heuristic. The use of spherical “top-hat” perturbations is not realistic to begin with, and using the Friedmann equation for their evolution after they enter the horizon is even less realistic. The bag-like model is not an overly detailed description of the behavior of matter near the confinement transition, and we have even employed a caricature of the bag model. We have assumed, in keeping with the standard lore, that the mass of black holes formed during a radiation dominated era will be roughly the horizon mass at the epoch the overdense region enters the horizon.

The results from our simplified model are not particularly encouraging for this mechanism of black hole formation, and refinements in the treatment would probably make matters worse. Early numerical treatments of black hole collapse had different results, with one set of calculations [21] finding black holes to have about the horizon mass, while other calculations [22] found that the holes formed with masses somewhat smaller than the horizon mass. More recent calculations [23] suggest that, contrary to the usual assumption, the masses of primordial black holes formed in a radiation dominated era may be significantly smaller than the horizon mass. These same calculations also found that the overdensities necessary for collapse in a radiation era are 2-3 times higher than the $\delta \approx 1/3$ from the standard lore. (Larger values of $\delta$ are also obtained if one keeps various factors of order unity in the heuristic analysis leading to the condition $\delta > 1/3$ for collapse.) Finally, in assessing the cosmological significance of the black holes we have assumed a Gaussian distribution of overdensities. Non-gaussian effects on the large overdensity tail are likely to be skew-negative [11], although they do not weaken the effect on the needed value of the spectral index too much [17].

The shortcomings of our model may make it unsuitable for precise quantitative predictions, but we believe two conclusions can be drawn from it that would be confirmed with refined treatment: 1) The perturbations gaining the most “benefit” from a first-order QCD transition enter the horizon somewhat earlier than the time the averaged universe goes through the transition. (This effect is also being seen in numerical calculations [24].) This
means that the mass of the black holes formed by this mechanism will be somewhat smaller than the horizon mass when the averaged universe goes through the transition. 2) If density perturbations can be characterized as a COBE-normalized power law with constant spectral index, a very finely tuned, blue spectrum would be required for the black holes produced by this mechanism to be cosmologically significant [8][9].

ACKNOWLEDGMENTS

We thank Karsten Jedamzik for communications regarding ongoing calculations. We also thank Mitesh Patel and Jim Wilson for useful conversations. This work was supported by grant DOE-FG02-87ER40317 at SUNY at Stony Brook, and by NSF PHY95-03384 and NASA NAG5-3062 at UCSD.
REFERENCES

[1] C. Alcock et al. (MACHO Collaboration), Astrophys. J. 486, 697 (1997).
[2] N. W. Evans, G. Gyuk, M. S. Turner, and J. J. Binney, Report No. astro-ph/9711224, 1997 (unpublished).
[3] C. Flynn, A. Gould, and J. N. Bahcall, Astrophys. J. 466, L55 (1996).
[4] S. Charlot and J. Silk, Astrophys. J. 445, 124 (1995).
[5] F. C. Adams and G. Laughlin, Astrophys. J. 468, 586 (1996).
[6] G. Chabrier, L. Segretain, and D. Mera, Astrophys. J. 468, L21 (1996).
[7] B. D. Fields, G. J. Mathews, and D. N. Schramm, Astrophys. J. 483, 625 (1997).
[8] K. Jedamzik, Phys. Rev. D 55, R5871 (1997).
[9] E. Witten, Phys. Rev. D 30, 272 (1984); J. H. Applegate and C. Hogan, Phys. Rev. D 31, 3037 (1985); G. M. Fuller, G. J. Mathews, and C. R. Alcock, Phys. Rev. D 37, 1380 (1988).
[10] T. Nakamura, M. Sasaki, T. Tanaka, and K. S. Thorne, Astrophys. J. 487, L139 (1997).
[11] J. S. Bullock and J. R. Primack, Phys. Rev. D 55, 7423 (1997).
[12] J. R. Wilson, G. M. Fuller, and C. Y. Cardall, in preparation (1997).
[13] B. J. Carr and S. W. Hawking, Mon. Not. R. Ast. Soc. 168, 399 (1974).
[14] B. J. Carr, Astrophys. J. 201, 1 (1975).
[15] C. R. Alcock, G. M. Fuller, and G. J. Mathews, Astrophys. J. 320, 439 (1987).
[16] M. Crawford and D. N. Schramm, Nature 298, 538 (1982).
[17] A. M. Green and A. R. Liddle, Report No. SUSSEX-AST 97/4-2, astro-ph/9704251, 1997 (unpublished).
[18] C. L. Bennett et al., Astrophys. J. 464, L1 (1996).
[19] W. Hu, D. Scott, and J. Silk, Astrophys. J. 430, L5 (1994).
[20] B. J. Carr, J. H. Gilbert, and J. E. Lidsey, Phys. Rev. D 50, 4853 (1994).
[21] G. V. Bicknell and R. N. Henriksen, Astrophys. J. 232, 670 (1979).
[22] D. K. Nadezhin, I. D. Novikov, and A. G. Polnarev, Sov. Astron. 22, 129 (1978); I. D. Novikov and A. G. Polnarev, in Sources of Gravitational Radiation, Proceedings of the Battelle Seattle Workshop, ed. L. L. Smarr (Cambridge University Press, Cambridge, 1979), 173.
[23] J. C. Niemeyer and K. Jedamzik, Report No. astro-ph/9709072, 1997 (unpublished).
[24] K. Jedamzik, private communication.
FIG. 1. Regions in the \((x, \delta)\) plane corresponding to the classes of perturbations listed in Table I. The variable \(x\) identifies the epoch a perturbation enters the horizon: \(x > 1\) \((x < 1)\) corresponds to overdense regions that enter the horizon before \(\text{after}\) the average cosmological density begins the transition from quark-gluon to hadron phase. The quantity \(\delta\) is the overdensity \(\delta \rho / \bar{\rho}\) of the perturbation at horizon crossing.

FIG. 2. Solid curve: \((1 - f) / 3\), for \(x = 2\). Dashed curve: \(\delta\). Collapse to a black hole occurs for values of \(\delta\) for which the dashed line is above the solid curve. The intersection corresponds to the value of \(\delta\) at \(x = 2\) in the curve of Fig. [4].
FIG. 3. Same as Fig. 2, but for $x = 15$. The three intersection points correspond to the values of $\delta$ at $x = 15$ in the curve of Fig. 2.

FIG. 4. The curve in the $(x, \delta)$ plane indicating which parameter values lead to collapse to a black hole. Without the phase transition, this would be a straight line at $\delta = 1/3$. 
FIG. 5. The contribution to the present day density parameter by black holes formed during the QCD transition, as function of the spectral index of the primordial density fluctuations. The transition temperature is taken to be $T_{\text{QCD}} = 100 \text{ MeV}$, and the resulting black holes have mass $\approx 0.11 M_\odot$.

FIG. 6. Same as Fig. 5, but for transition temperature $T_{\text{QCD}} = 200 \text{ MeV}$. The black holes in this case have mass $\approx 0.03 M_\odot$. 
TABLES

TABLE I. Classification of overdense regions according to the state of matter at horizon crossing and turnaround.

| Class | Horizon crossing$^a$ | Turnaround$^a$ |
|-------|----------------------|----------------|
| A     | $gq$                 | $gq$           |
| B     | $gq$                 | $m$            |
| C     | $gq$                 | $h$            |
| D     | $m$                  | $m$            |
| E     | $m$                  | $h$            |
| F     | $h$                  | $h$            |

$^a$ $gq$: quark-gluon phase. $m$: mixed phase, i.e. coexistent quark-gluon and hadron phases. $h$: hadron phase.

TABLE II. Classes of perturbations that exist for various epochs of horizon crossing and ranges of $\delta$.

| Epoch of horizon crossing | Overdensity | Class |
|--------------------------|-------------|-------|
| $x > 1$                  | $\delta > 0$ | A, B, C |
| $\beta^{-1} < x < 1$     | $\delta > x^{-1} - 1$ | A, B, C |
|                           | $0 < \delta < x^{-1} - 1$ | D, E |
| $x < \beta^{-1}$         | $\delta > x^{-1} - 1$ | A, B |
|                           | $\beta^{-1}x^{-1} < \delta < x^{-1} - 1$ | D, E |
|                           | $0 < \delta < \beta^{-1}x^{-1}$ | F |

TABLE III. Radius of the overdense region at turnaround.

| Class | Radius at turnaround |
|-------|----------------------|
| A     | $S_c = R_h \left( \frac{1+\delta}{\delta} \right)^{1/2}$ |
| B     | $S_c = R_h x^{-1/4} (1+\delta)^{3/4} / \delta$ |
| C     | $S_c = R_h \beta^{1/6} \left( \frac{1+\delta}{\delta} \right)^{1/2}$ |
| D     | $S_c = R_h (1+\delta)^{-1/2}$ |
| E     | $S_c = R_h (\beta x)^{1/6} (1+\delta)^{2/3}$ |
| F     | $S_c = R_h \left( \frac{1+\delta}{\delta} \right)^{1/2} \beta^{2/3}$ |