THE DWORKE-FROBENIUS OPERATOR ON HYPERGEOMETRIC SERIES

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Abstract. We describe the action of the Dwork-Frobenius operator on certain A-hypergeometric series. As a consequence, we obtain an integrality result for the coefficients of those series. This implies an integrality result for classical hypergeometric series.

1. Introduction

Proving \(p\)-integrality results for hypergeometric series is a well-known problem. Many authors have obtained such results, and we reviewed some of their history and applications in the Introduction to [3]. One of the main applications of \(p\)-integrality is the derivation of \(p\)-adic analytic formulas for zeros of zeta and \(L\)-functions of varieties over finite fields. Such formulas involve \(p\)-adic analytic continuations of ratios of hypergeometric series. Dwork originally obtained such results by studying the action of his Frobenius operator on a solution matrix for the Picard-Fuchs equation of a family of varieties. A necessary condition for the analytic continuation was that the Picard-Fuchs equation have a solution with \(p\)-integral coefficients.

We developed a different method for obtaining such results using A-hypergeometric series [1, 5]. Our method avoids the problem of computing the Picard-Fuchs equation and finding its solution matrix. However, we need to know \(p\)-integrality for not just one hypergeometric series but for an entire class of related hypergeometric series. In [5], the coefficients of the relevant series were multinomial coefficients, hence trivially \(p\)-integral, so this difficulty did not arise. In [1], the relevant series were partial derivatives of a single series, so it sufficed to prove integrality for that one series. This followed from [4, Theorem 6.3].

To apply our idea to more general situations, however, we needed an integrality result for all the series in the class. In the special case where the coefficients of the A-hypergeometric series are ratios of factorials, the desired theorem is [3, Theorem 2.4]. In this article we generalize that result to a wider class of hypergeometric series whose coefficients are ratios of Pochhammer symbols (Theorems 2.21 and 12.15). We expect that this will lead to ratios of hypergeometric series with \(p\)-adic analytic continuation that have arithmetic import.

This paper is organized as follows. In Section 2 we describe the A-hypergeometric series we are considering, establish some notation and basic properties, and state our main result. Section 3 contains a construction of these A-hypergeometric series, as well as some related series that satisfy better \(p\)-adic estimates, by taking products of simpler one-variable series \(\xi_j(t)\) and \(\hat{\xi}_j(t)\). This “separation of variables” allows us to reduce some arguments to the one-variable case. In Section 4, we introduce...
the \( p \)-adic spaces on which the Dwork-Frobenius operator will act, and, in Section 5, we show that the \( \xi_j(t) \) and \( \hat{\xi}_j(t) \) lie in these spaces. Section 6 is used to construct an auxiliary function, an analogue of the \( p \)-adic Gamma function, that will play a role in Section 7. Section 7 applies an idea of Dwork (writing as Boyarsky [3]): we show that the Dwork-Frobenius operator maps each \( \xi_j(t) \) to another \( \xi'_j(t) \) times a simple factor. This result is extended to the \( \hat{\xi}_j(t) \) in Section 8. We extend this result to \( A \)-hypergeometric series in Section 9, showing that \( A \)-hypergeometric series have an “eigenvector-like” property for the Dwork-Frobenius operator (Theorem 9.13). In Section 10 we restrict the Dwork-Frobenius operator to those \( A \)-hypergeometric series that potentially have \( p \)-integral coefficients (Theorem 10.5). This is the key result, and it allows us to give a simple proof of our main result, Theorem 2.21, in Section 11. As an example, we derive in Section 12 an integrality result for classical hypergeometric series (Theorem 12.15).

This paper is completely self-contained except for a reference to Dwork [7] for an estimate for the \( p \)-divisibility of a Pochhammer symbol in Section 5, a reference to an elementary result from [4] in Section 2, and some references to \( p \)-adic estimates from [1] Section 3 in Sections 3 and 6.

We note that the solutions \( F_u(\Lambda) \) of the \( A \)-hypergeometric system with parameter \( u \) described in Corollary 2.16 and studied elsewhere in this paper are formal solutions, they do not necessarily belong to a Nilsson ring or converge anywhere.

**Notation.** It is traditional to express the coefficients of hypergeometric series in terms of Pochhammer symbols. For \( z \in \mathbb{C} \) and \( l \in \mathbb{Z} \), one sets \( (z)_l = \Gamma(z + l)/\Gamma(z) \). This is well-defined except when \( z + l \) is a nonpositive integer and \( z \) is a positive integer, due to the poles of the Gamma function. Otherwise, the functional equation of the Gamma function gives

\[
(z)_l = \begin{cases} 
1 & \text{if } l = 0, \\
\frac{z(z + 1)\cdots(z + l - 1)}{(z - 1)(z - 2)\cdots(z + l)} & \text{if } l > 0, \\
\frac{1}{(z + 1)(z + 2)\cdots(z + l)} & \text{if } l < 0 \text{ and } z \notin \{1, 2, \ldots, -l\}.
\end{cases}
\]

Working with \( A \)-hypergeometric series, we find it more convenient to use the symbol

\[
[z]_l = \begin{cases} 
1 & \text{if } l = 0, \\
\frac{1}{(z + 1)(z + 2)\cdots(z + l)} & \text{if } l > 0 \text{ and } z \notin \{-1, -2, \ldots, -l\}, \\
\frac{z(z - 1)\cdots(z + l + 1)}{(z - 1)\cdots(z + l + 1)} & \text{if } l < 0.
\end{cases}
\]

It has the property that for all \( l \in \mathbb{Z} \)

\[
\frac{d}{dt}[z]_lt^{z+l} = [z]_{l-1}t^{z+l-1},
\]

hence if \( z \) is a \( p \)-integral rational number for some prime number \( p \)

\[
\text{ord}_p [z]_{l_1} \geq \text{ord}_p [z]_{l_2} \text{ if } l_1 \leq l_2.
\]

One checks that \( [z]_l \) is defined if and only if \( (-z)_{-l} \) is defined, in which case

\[
[z]_l = (-1)^l(-z)_{-l}.
\]
For $N$-tuples $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and $l = (l_1, \ldots, l_N) \in \mathbb{Z}^N$ we extend this definition by setting

$$[z]_l = \prod_{j=1}^{N} [z_j]_{l_j},$$

assuming always that $z_j \notin \{-1, -2, \ldots, -l_j\}$ if $l_j > 0$.

2. Statement of results

We develop some theory that will allow us to state the main result. We begin by describing the hypergeometric series we are considering.

Let $A = \{a_j\}_{j=1}^{N}$ with $a_j = (a_{1j}, a_{2j}, \ldots, a_{nj}) \in \mathbb{Z}^n$ for all $j$. We assume that the $a_j$ lie on a hyperplane $w(u) = 1$, where $w(u) = \sum_{i=1}^{n} w_i u_i$ with $w_i \in \mathbb{Q}$ for $i = 1, \ldots, n$. The linear form $w$ defines a function $w : \mathbb{R}^n \to \mathbb{R}$; we refer to $w(u)$ as the weight of $u$. Note that for $u = \sum_{j=1}^{N} c_j a_j$ we have $w(u) = \sum_{j=1}^{N} c_j$.

Let $L$ be the lattice of relations on the set $A$:

$$L = \{ l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \mid \sum_{j=1}^{N} l_j a_j = 0 \}.$$  

For $l \in L$ define the box operator $\Box_l$ by

$$\Box_l = \prod_{l_j > 0} \left( \frac{\partial}{\partial \Lambda_j} \right)^{l_j} - \prod_{l_j < 0} \left( \frac{\partial}{\partial \Lambda_j} \right)^{-l_j}.$$  

(2.1)

The Euler operators for a parameter $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$ are defined by

$$Z_i = \sum_{j=1}^{N} a_{ij} \Lambda_j \frac{\partial}{\partial \Lambda_j} - u_i$$

for $i = 1, \ldots, n$. The $A$-hypergeometric system with parameter $u$ is the system of partial differential equations consisting of the box operators $\Box_l$ for $l \in L$ and the $Z_i$ for $i = 1, \ldots, n$.

We describe the conditions on the parameters that will allow us to prove integrality results. Let $ZA \subseteq \mathbb{Z}^n$ be the abelian group generated by $A$, let $QA \subseteq \mathbb{Q}^n$ be the rational vector space generated by $A$, let $RA \subseteq \mathbb{R}^n$ be the real vector space generated by $A$, and let $CA \subseteq \mathbb{R}^n$ be the real cone generated by $A$, a cone with vertex at the origin. Let $Q \subseteq QA$ and consider the shifted lattice $Q + ZA$.

For a closed face $\sigma$ of the negative cone $-CA$ we denote by $\sigma^\circ$ its relative interior, i. e., $\sigma^\circ$ equals $\sigma$ minus all its proper closed subfaces. Fix a closed face $\sigma$ for which $\mathcal{M} := (Q + ZA) \cap \sigma^\circ$ is nonempty. Then for each $u \in \mathcal{M}$ the closed face $\sigma$ is the smallest closed face of $-CA$ that contains $u$. Our goal is to construct a family of series parameterized by $u \in \mathcal{M}$ which are formal solutions of the $A$-hypergeometric system with parameter $u$ and which have $p$-integral coefficients for all primes $p$ lying in certain residue classes. The first step is to choose a distinguished element of $\mathcal{M}$.

Since $\mathcal{M}$ is discrete and the weight function $w$ is nonpositive on $-CA$ we can choose an element $\beta \in \mathcal{M}$ for which

$$w(\beta) = \max \{ w(u) \mid u \in \mathcal{M} \}.$$  

(2.3)
We examine some consequences of this condition. Since $\beta \in QA \cap (-C(A))$ we can write
\[ \beta = \sum_{j=1}^{N} v_j a_j \]
for some $v_j \in \mathbb{Q}_{<0}$. Put $v = (v_1, \ldots, v_N)$. Note that $-a_j \in \sigma$ if $v_j < 0$. If $v_j = 0$, then $a_j$ may or may not lie on $\sigma$.

**Lemma 2.5.** Suppose that $\beta$ satisfies (2.3). Then $-1 \leq v_j \leq 0$ for $j = 1, \ldots, N$.

**Proof.** Suppose, for example, that $v_1 < -1$. Then
\[ \beta + a_1 = (v_1 + 1)a_1 + \sum_{j=2}^{N} v_j a_j. \]
The coefficients on the right-hand side are nonpositive and are strictly negative whenever $v_j$ is strictly negative. This implies that $\beta + a_1 \in \mathcal{M}$, but $w(\beta + a_1) = w(\beta) + 1 > w(\beta)$. This contradicts the choice of $\beta$ satisfying (2.3). \[ \square \]

**Lemma 2.6.** Suppose that $\beta$ satisfies (2.3). If $J$ is a proper subset of the set \[ \{ j \in \{1, \ldots, N \} \mid v_j = -1 \} \], then
\[ -\sum_{j \in J} a_j - \sum_{\{ j \mid -1 < v_j < 0 \}} a_j \notin \sigma. \]

**Proof.** Suppose \[ -\sum_{j \in J} a_j - \sum_{\{ j \mid -1 < v_j < 0 \}} a_j \in \sigma \] for some proper subset $J \subseteq \{ j \in \{1, \ldots, N \} \mid v_j = -1 \}$. Then
\[ \beta + \sum_{\{ j \mid v_j = -1, j \notin J \}} a_j = -\sum_{j \in J} a_j + \sum_{\{ j \mid -1 < v_j < 0 \}} v_j a_j \in \mathcal{M}. \]
But $w(\beta + \sum_{\{ j \mid v_j = -1, j \notin J \}} a_j) > w(\beta)$, contradicting (2.3). \[ \square \]

Let
\[ E = \{ (l_1, \ldots, l_N) \in \mathbb{Z}^{N} \mid l_j \leq 0 \text{ if } v_j = -1, l_j \geq 0 \text{ if } v_j = 0 \}, \]
\[ E_{+} = \{ (l_1, \ldots, l_N) \in \mathbb{Z}^{N} \mid l_j \geq 0 \text{ if } v_j = 0 \}. \]

Let $u \in \mathcal{M}$ and consider an equation of the form
\[ u = \sum_{j=1}^{N} (v_j + l_j) a_j \text{ with } (l_1, \ldots, l_N) \in E_{+}. \]

**Lemma 2.8.** In Equation (2.7) we have $l_j = 0$ if $-a_j \notin \sigma$.

**Proof.** Let $H \subseteq \mathbb{R}^{n}$ be a hyperplane of support for the face $\sigma$. Then $H$ can be defined by a homogeneous linear equation $h = 0$ with $h(\sigma) = 0$ and $h(x) > 0$ for $x \in (-C(A)) \setminus \sigma$. In particular, $h(\alpha_j) = 0$ if $v_j \neq 0$. If $l_j > 0$ for some $j$ with $v_j = 0$ and $-a_j \notin \sigma$, then (2.7) implies $h(u) > 0$, contradicting the hypothesis that $u \in \mathcal{M}$. \[ \square \]

**Proposition 2.9.** Suppose that $\beta$ satisfies (2.3) and let $u \in \mathcal{M}$ satisfy an equation of the form (2.7). Then $l_j \leq 0$ if $v_j = -1$, i. e., $(l_1, \ldots, l_N) \in E$. 

Proof. Rewrite (2.7) as

\[(2.10) \quad \sum_{\{j, v_j = -1, l_j \leq 0\}} (-1 + l_j) a_j + \sum_{\{j, -1 < v_j < 0, l_j \leq 0\}} (v_j + l_j) a_j = u - \sum_{\{j, -1 < v_j < 0, l_j > 0\}} (v_j + l_j) a_j - \sum_{\{j, v_j = 0\}} l_j a_j.\]

Since \(u \in \mathcal{M}\) and \(l_j = 0\) if \(- a_j \not\in \sigma\) by Lemma 2.8, the right-hand side of (2.10) lies in \(\sigma^\circ\). This implies that

\[\sum_{\{j, v_j = -1, l_j \leq 0\}} (-1 + l_j) a_j + \sum_{\{j, -1 < v_j < 0, l_j \leq 0\}} (v_j + l_j) a_j \in \sigma^\circ.\]

By Lemma 2.6 this would be a contradiction unless \(l_j \leq 0\) for all \(j\) with \(v_j = -1\). □

In the special case \(u = \beta\), Proposition 2.9 reduces to the assertion that \(v\) have minimal negative support, guaranteeing that there is an associated logarithm-free solution of the \(\lambda\)-hypergeometric system with parameter \(\beta\). We show next that in fact we get logarithm-free solutions for all \(u \in \mathcal{M}\) when \(\beta\) satisfies (2.3) (Corollary 2.16 below).

For \(u \in Q + \mathbb{Z} A\) put

\[E(u) = \left\{ l = (l_1, \ldots, l_N) \in E \mid \sum_{j=1}^{N} (v_j + l_j) a_j = u \right\}\]

where the \(v_j\) are as in (2.4) and define

\[(2.11) \quad F_u(\Lambda_1, \ldots, \Lambda_N) = \sum_{l \in E(u)} [v]_l \Lambda^{v+l},\]

where we write \(\Lambda^{v+l}\) for \(\prod_{j=1}^{N} \Lambda_j^{v_j+l_j}\). Note that the restriction to \(l \in E(u)\) in (2.11) guarantees that \([v]_l\) is well-defined.

Put

\[E_\sigma = \{ l = (l_1, \ldots, l_N) \in E \mid l_j = 0 \text{ if } -a_j \not\in \sigma \}\]

and

\[E_\sigma(u) = \left\{ l = (l_1, \ldots, l_N) \in E_\sigma \mid \sum_{j=1}^{N} (v_j + l_j) a_j = u \right\}.\]

For \(u \in \mathcal{M}\) we have by Lemma 2.8 the sharper formula

\[(2.12) \quad F_u(\Lambda_1, \ldots, \Lambda_N) = \sum_{l \in E_\sigma(u)} [v]_l \Lambda^{v+l},\]

Lemma 2.13. Suppose that \(\beta\) satisfies (2.3) and let \(u \in \mathcal{M}\). Then

\[\frac{\partial}{\partial \Lambda_k} F_u(\Lambda) = \begin{cases} F_{u-a_k}(\Lambda) & \text{if } a_k \in \sigma, \\ 0 & \text{if } a_k \not\in \sigma. \end{cases}\]

Proof: If \(a_k \not\in \sigma\), then \(\Lambda_k\) does not occur to a nonzero power in (2.12), so \(\partial F_u/\partial \Lambda_k = 0\). A straightforward calculation from (2.12) shows that applying \(\partial/\partial \Lambda_k\) to a term in \(F_u(\Lambda)\) gives either 0 or a term of \(F_{u-a_k}(\Lambda)\). The main point of the proof is to show that every monomial in \(F_{u-a_k}(\Lambda)\) is obtained by applying \(\partial/\partial \Lambda_k\) to some monomial in \(F_u(\Lambda)\).
Suppose that \( a_k \in \sigma \) and consider a monomial \([v]_l \Lambda^{v+l}\) in \( F_{u-a_k}(\Lambda)\). Then

\[ (2.14) \quad \sum_{j=1}^{N} (v_j + l_j) a_j = u - a_k. \]

Suppose first that \( v_k = -1 \). Then (2.14) gives

\[ (2.15) \quad u = (v_k + l_k + 1) a_k + \sum_{j=1, j \neq k}^{N} (v_j + l_j) a_j. \]

By Proposition 2.9 we have \( l_k + 1 \leq 0 \), so if we put

\[ l' = (l_1, \ldots, l_{k-1}, l_k + 1, l_{k+1}, \ldots, l_N) \]

then \( l' \in E_\sigma \) and \( \sum_{j=1}^{N} (v_j + l'_j) a_j = u \) by (2.15), i.e., \( l' \in E_\sigma(u) \). Thus the monomial \([v]_{l'} \Lambda^{v+l'}\) appears in the series (2.12) and applying \( \partial / \partial \Lambda_k \) gives \([v]_{l'} \Lambda^{v+l'}\). Next suppose that \(-1 < v_k \leq 0\). If we again define \( l' = (l_1, \ldots, l_{k-1}, l_k + 1, l_{k+1}, \ldots, l_N) \), then \( l' \in E_\sigma(u) \) and the monomial \([v]_{l'} \Lambda^{v+l'}\) appears in the series (2.12). Applying \( \partial / \partial \Lambda_k \) gives \([v]_{l'} \Lambda^{v+l'}\).

**Corollary 2.16.** Suppose that \( \beta \) satisfies (2.3) and \( u \in \mathcal{M} \). Then \( F_u(\Lambda) \) satisfies the \( A \)-hypergeometric system with parameter \( u \).

**Proof.** It follows from the condition \( \sum_{j=1}^{N} (v_j + l_j) a_j = u \) on the sum in (2.12) that each monomial \( \Lambda^{v+l} \) satisfies the operators \( Z_i \) for the parameter \( u \). Let \( l = (l_1, \ldots, l_N) \in L \). Then

\[ (2.17) \quad \sum_{l_j > 0} l_j a_j = -\sum_{l_j < 0} l_j a_j. \]

This implies that \( l_j > 0 \) for some \( a_j \not\in \sigma \) if and only if \( l_j < 0 \) for some \( a_j \not\in \sigma \). In this case, by Lemma 2.13,

\[ \prod_{l_j > 0} \left( \frac{\partial}{\partial \Lambda_j} \right)^{l_j} F_u(\Lambda) = \prod_{l_j < 0} \left( -\frac{\partial}{\partial \Lambda_j} \right)^{-l_j} F_u(\Lambda) = 0 \]

so \( \Box_l F_u(\Lambda) = 0 \). Otherwise, Lemma 2.13 implies that

\[ \prod_{l_j > 0} \left( \frac{\partial}{\partial \Lambda_j} \right)^{l_j} F_u(\Lambda) = F_{u+\sum_{l_j > 0} l_j a_j}(\Lambda) \]

and

\[ \prod_{l_j < 0} \left( -\frac{\partial}{\partial \Lambda_j} \right)^{-l_j} F_u(\Lambda) = F_{u+\sum_{l_j < 0} l_j a_j}(\Lambda). \]

These two expressions are equal by (2.17), so again \( \Box_l F_u(\Lambda) = 0 \).

Let \( v = (v_1, \ldots, v_N) \) be as in (2.4). Choose a positive integer \( D \) such that \( D v_j \in \mathbb{Z} \) for all \( j \) and choose a positive integer \( h \) such that \( (h, D) = 1 \). For each \( j \) there exists a unique rational number \( v'_j \), \( -1 \leq v'_j \leq 0 \), with \( D v'_j \in \mathbb{Z} \) and \( h v_j - v'_j \in \{0, -1, \ldots, -(h-1)\} \). Furthermore, \( v'_j \) depends only on the residue class of \( h \) (mod \( D \)): if \( \tilde{h} \equiv h \) (mod \( D \)) then both \( h \) and \( \tilde{h} \) define the same map \( v_j \rightarrow v'_j \) (see [4] Section 3). We denote the \( i \)-fold iteration of this map by \( v_j \rightarrow v^{(i)}_j \), i.e.,
\( v_j^{(i)} = (v_j^{(i-1)})' \). Since \((h, D) = 1\), there exists a positive integer \(a\) such that for all \(j\) one has \(v_j^{(a)} = v_j\). We set \(v^{(i)} = (v_1^{(i)}, \ldots, v_N^{(i)})\).

For each \(i\) we define \(\beta^{(i)} = \sum_{j=1}^{N} v_j^{(i)} a_j\). Note that \(\beta^{(i)} \in h^iQ + ZA\). Note also that if \(v_j = -1\) then \(v_j' = -1\) and if \(v_j = 0\) then \(v_j' = 0\). If \(-1 < v_j < 0\), then \(-1 < v_j' < 0\). It follows that \(\sigma\), the smallest closed face of \(-C(A)\) containing \(\beta\), is also the smallest closed face of \(-C(A)\) containing \(\beta^{(i)}\) for all \(i\). Let \(\mathcal{M}^{(i)} = (h^iQ + ZA) \cap \sigma^c\).

For \(u^{(i)} \in h^iQ + ZA\) we define as in (2.11)
\[
F_{u^{(i)}}(\Lambda_1, \ldots, \Lambda_N) = \sum_{l \in E(u^{(i)})} [v^{(i)}]_l A^{v^{(i)}+l}.
\]
And as in (2.12) the sharper formula
\[
F_{u^{(i)}}(\Lambda_1, \ldots, \Lambda_N) = \sum_{l \in E_u^{(i)}} [v^{(i)}]_l A^{v^{(i)}+l}
\]
holds for \(u^{(i)} \in \mathcal{M}^{(i)}\).

Consider the condition analogous to (2.3):
\[
\omega(\beta^{(i)}) = \max\{\omega(v^{(i)}) \mid u^{(i)} \in \mathcal{M}^{(i)}\}.
\]
By Corollary 2.16, this condition implies that for all \(u^{(i)} \in \mathcal{M}^{(i)}\) the series \(F_{u^{(i)}}(\Lambda)\) satisfies the \(A\)-hypergeometric system with parameter \(u^{(i)}\).

The following theorem is the main result of this paper.

**Theorem 2.21.** Suppose that (2.20) holds for \(i = 0, 1, \ldots, a - 1\). Then for \(i = 0, 1, \ldots, a - 1\) and all \(u^{(i)} \in \mathcal{M}^{(i)}\), the hypergeometric series \(F_{u^{(i)}}(\Lambda)\) has \(p\)-integral coefficients for all primes \(p \equiv h \pmod{D}\).

When \(h \equiv 1 \pmod{D}\) we have \(a = 1\), so Theorem 2.21 gives the following corollary.

**Corollary 2.22.** If \(\beta\) satisfies (2.3), then for all \(u \in \mathcal{M}\) the hypergeometric series \(F_u(\Lambda)\) has \(p\)-integral coefficients for all primes \(p \equiv 1 \pmod{D}\).

**Remark.** By Lemma 2.8 one has \(l_j = 0\) in (2.7) for all \(u \in \mathcal{M}\) if \(-a_j \not\in \sigma\) (in which case \(v_j = 0\), thus the corresponding variable \(\Lambda_j\) does not appear in the series \(F_u(\Lambda)\). One can therefore replace the original set \(A\) by the set \(A' := \{a_j \mid a_j \in \sigma\}\), and one then has \(\mathcal{M} = (Q + ZA') \cap (-C(A')^c)\). It thus suffices to prove Theorem 2.21 in the case where \(\sigma = -C(A)\) and \(\mathcal{M} = (Q + ZA) \cap (-C(A)^c)\).

The main application we give, in Section 12, is a condition for the \(p\)-integrality of classical hypergeometric series of the form
\[
\sum_{s_1, \ldots, s_m = 0}^{\infty} \frac{(\theta_1)_{C_1(s)} \cdots (\theta_n)_{C_n(s)} t_1^{s_1} \cdots t_m^{s_m}}{s_1! \cdots s_m!},
\]
where \(C_1, \ldots, C_m\) are homogeneous linear forms with integral coefficients and \(\theta_1, \ldots, \theta_m\) are \(p\)-integral rational numbers in the interval \([0, 1]\).

### 3. Generating series

The proof of Theorem 2.21 will require several steps. We begin with some notation. Fix for the remainder of this article the vector \(Q \in QA\), a vector \(\beta \in (Q + ZA) \cap (-C(A)^c)\), and a rational vector \(v = (v_1, \ldots, v_N) \in Q^N\) whose
coordinates lie in the interval \([-1, 0]\) and such that \(\beta = \sum_{j=1}^{N} v_j a_j\). We thus have \(\mathcal{M} = (Q + Z A) \cap (-C(A)^p)\) and by (2.11)

\[
F_u(A) = \sum_{l \in E(u)} [v] A^{v+l}
\]

for \(u \in Q + Z A\). We do not assume (2.3) or (2.20) until Section 10.

We also fix a prime number \(p\) for which all \(v_j\) are \(p\)-integral. Let \(Z_p\) be the \(p\)-adic integers, and \(Q_p\) the field of \(p\)-adic numbers. We denote by \(C_p\) the completion of an algebraic closure of \(Q_p\). The norm on \(C_p\) is denoted by \(|\cdot|\) and is normalized by the condition \(|p| = 1/p\). The corresponding additive valuation is denoted \(\text{ord}\) and satisfies \(\text{ord} p = 1\). We denote by \(\mathbb{N}\) the nonnegative integers.

It will be useful to have a generating series construction for the \(F_u(A)\). For \(-1 < v_j \leq 0\), define functions of one variable

\[
f_j(t) = \sum_{l=0}^{\infty} [v_j] t^{v_j+l}.
\]

(3.2) Note that in the special case \(v_j = 0\) this simplifies to \(f_j(t) = \exp t\). For \(v_j = -1\), we define

\[
f_j(t) = \sum_{l=0}^{\infty} [v_j] t^{v_j+l} = \sum_{l=0}^{\infty} (-1)^l l! t^{-1-l}.
\]

(3.3) The following proposition is a straightforward calculation from (3.1).

**Proposition 3.4.** We have

\[
\prod_{j=1}^{N} f_j(A_j x^{a_j}) = \sum_{u \in Q + Z A} F_u(A) x^u.
\]

We need to modify this formula to obtain related series with equivalent integrality properties that are easier to study \(p\)-adically. Let \(\pi_0 \in C_p\) be a root of the equation \(\sum_{i=0}^{\infty} t^i/p^i = 0\) satisfying \(\text{ord} \pi_0 = 1/(p - 1)\). We set

\[
\xi_j(t) = f_j(\pi_0 t) = \begin{cases} \sum_{l=-\infty}^{\infty} [v_j]_l (\pi_0 t)^{v_j+l} & \text{if } v_j > -1, \\ \sum_{l=0}^{\infty} (-1)^l l! (\pi_0 t)^{-1-l} & \text{if } v_j = -1, \end{cases}
\]

(3.5) so from Proposition 3.4 we have

\[
\prod_{j=1}^{N} \xi_j(A_j x^{a_j}) = \sum_{u \in Q + Z A} F_u(A) \pi_0^{\mu(u)} x^u.
\]

(3.6) The \(\xi_j(t)\) do not lead to sufficiently strong \(p\)-adic estimates, so we introduce some related series that lead to better estimates.

Let \(\text{AH}(t) = \exp(\sum_{i=0}^{\infty} t^i/p^i)\) be the Artin-Hasse series, a power series in \(t\) with \(p\)-integral coefficients. Put \(\theta(t) = \text{AH}(\pi_0 t)\). If we write \(\theta(t) = \sum_{i=0}^{\infty} \theta_i t^i\), then

\[
\text{ord} \theta_i \geq i/(p - 1).
\]

(3.7)
We define \( \hat{\theta}(t) = \prod_{j=0}^{\infty} \theta(t^p) \), which gives \( \theta(t) = \hat{\theta}(t)/\hat{\theta}(t^p) \). We write \( \hat{\theta}(t) = \sum_{i=0}^{\infty} \hat{\theta}_1(\pi_0 t)^i / i! \), and by \([1\text{ Equation (3.8)}\] we have
\[
\text{(3.8)} \quad \text{ord} \hat{\theta}_1 \geq 0.
\]
We also need the series \( \hat{\theta}_1(\pi_0 t) := \hat{\theta}_1(\pi_0 t) / \exp(\pi_0 t) \). We write \( \hat{\theta}_1(\pi_0 t) = \sum_{i=0}^{\infty} \hat{\theta}_1,i(\pi_0 t)^i / i! \) and have by \([1\text{ Equation (3.10)}\]
\[
\text{(3.9)} \quad \text{ord} \hat{\theta}_1,i \geq (p-1)p.
\]
Suppose first that \( v_j > -1 \). Define
\[
\hat{\xi}_j(t) = \xi_j(t) \hat{\theta}_1(t)
\]
and write
\[
\hat{\xi}_j(t) = \sum_{l=-\infty}^{\infty} g(v_j, l)(\pi_0 t)^{v_j+l}.
\]
From (3.5) we have
\[
g(v_j, l) = \sum_{i=0}^{\infty} [v_j]_l+i \cdot \hat{\theta}_1,i / i!,
\]
hence
\[
\text{(3.11)} \quad \frac{g(v_j, l)}{[v_j]_l} = \sum_{i=0}^{\infty} [v_j]_l+i \cdot \hat{\theta}_1,i.
\]
One checks that for all \( l \in \mathbb{Z} \) and all \( i \in \mathbb{Z}_{\geq 0} \) one has
\[
\frac{[v_j]_{l-i}}{[v_j]_l} = \frac{(v_j + l)(v_j + l - 1) \cdots (v_j + l - i + 1)}{i!} = \binom{v_j + l}{i},
\]
which is \( p \)-integral since \( v_j \) is \( p \)-integral. Since the \( i = 0 \) term of the series in (3.11) equals 1, Equation (3.9) implies that \( g(v_j, l)/[v_j]_l \) is \( p \)-integral and
\[
\text{(3.12)} \quad \frac{g(v_j, l)}{[v_j]_l} \equiv 1 \pmod{\pi_0}.
\]
In particular,\[
\text{(3.13)} \quad \text{ord } g(v_j, l) = \text{ord } [v_j]_l.
\]
Let \( \gamma \) be the operator on power series defined by
\[
\gamma \left( \sum_{l=-\infty}^{\infty} c_l t^l \right) = \sum_{l=-\infty}^{-1} c_l t^l.
\]
For \( v_j = -1 \), define
\[
\hat{\xi}_j(t) = \gamma (\xi_j(t) \hat{\theta}_1(t)).
\]
We write
\[
\hat{\xi}_j(t) = \sum_{l=0}^{\infty} g(-1, l)(\pi_0 t)^{-1-l}
\]
where by (3.5)
\[
g(-1, l) = \sum_{i=0}^{\infty} (-1)^{l+i}(l+i)! \hat{\theta}_1,i / i!.
\]
We thus have
\[
g(-1, l) \equiv (-1)^l \pmod{\pi_0}.
\]

(3.15)

In particular,
(3.16) \quad \text{ord } g(-1, l) = \text{ord } l!

For \( u \in Q + ZA \) define
(3.17) \quad G_u(\Lambda_1, \ldots, \Lambda_N) = \sum_{l=(l_1, \ldots, l_N) \in E(u)} \left( \prod_{j=1}^N g(v_j, l_j) \right) \Lambda^{v+l}.

A straightforward calculation gives the analogue of (3.6):
(3.18) \quad \prod_{j=1}^N \xi_j(\Lambda_j x^n) = \sum_{u \in Q + ZA} G_u(\Lambda) \pi_0^{w(u)} x^u.

From (3.13) and (3.16) we have immediately the following result.

**Proposition 3.19.** For \( u \in Q + ZA \) the series \( F_u(\Lambda) \) has \( p \)-integral coefficients if and only if the series \( G_u(\Lambda) \) has \( p \)-integral coefficients.

The point of this proposition will be that it is easier to find good estimates for the coefficients of the \( G_u(\Lambda) \) than for the coefficients of the \( F_u(\Lambda) \). As the next step in this process, we define in the next section spaces that contain the \( \xi_j(t) \) and \( \hat{\xi}_j(t) \). In Sections 7 and 8 we define Dwork-Frobenius operators on these spaces and determine their action on \( \xi_j(t) \) and \( \hat{\xi}_j(t) \).

### 4. Some \( p \)-adic Vector Spaces

We construct some vector spaces over \( \mathbb{C}_p \) on which the Dwork-Frobenius operator will act. Consider a collection of elements \( c_i \in \mathbb{C}_p \) indexed by \( l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \). Let \( |\cdot|_\infty \) denote the usual absolute value on real numbers. We define \( |l|_\infty = \sum_{j=1}^N |l_j|_\infty \). For \( \delta > 0 \), we say that \( |c_l| \delta^{|l|_\infty} \) converges to \( 0 \), \( |c_l| \delta^{|l|_\infty} \to 0 \), if for every \( \epsilon > 0 \) we have \( |c_l| \delta^{|l|_\infty} < \epsilon \) for all but finitely many \( l \). Put
(4.1) \quad C\{t_1, \ldots, t_N\} = \left\{ \xi(t) = \sum_{l=(l_1, \ldots, l_N) \in \mathbb{Z}^N} c_l t_1^{l_1} \cdots t_N^{l_N} \mid |c_l| \delta^{|l|_\infty} \to 0 \text{ for all } \delta < 1 \right\}.

For any subset \( S \subseteq \mathbb{Z}^N \) we set
(4.2) \quad C_S\{t_1, \ldots, t_N\} = \left\{ \xi(t) = \sum_{l=(l_1, \ldots, l_N) \in S} c_l t_1^{l_1} \cdots t_N^{l_N} \mid \xi(t) \in C\{t_1, \ldots, t_N\} \right\}.

The set \( C\{t_1, \ldots, t_N\} \) is not a space of functions. These series need not converge at any point \( t = t_0 \in \mathbb{C}_p \).
Let
\[ H(t) = \sum_{m=(m_1, \ldots, m_N)\in\mathbb{N}^N} d_m t_1^{m_1} \cdots t_N^{m_N} \]
be a power series converging on a polydisk
\[ \{(t_1, \ldots, t_N) \in \mathbb{C}_p^N \mid |t_j| \leq R \text{ for } j = 1, \ldots, N\} \]
for some $R > 1$, i.e.,
\[ |d_m| R^{\sum_{j=1}^N m_j} \to 0. \]

**Proposition 4.4.** Multiplication by $H(t)$ maps $C\{t_1, \ldots, t_N\}$ into itself.

**Proof.** For $\xi(t) = \sum_{l\in\mathbb{Z}^N} c_l t^l \in C\{t_1, \ldots, t_N\}$ we have formally
\[ H(t)\xi(t) = \sum_{k\in\mathbb{Z}^N} e_k t^k, \]
where
\[ e_k = \sum_{m\in\mathbb{N}^N} d_m c_{k-m}. \]

We need to show that the series (4.6) converges and that for each $\delta$, $0 < \delta < 1$, and $\epsilon > 0$ one has
\[ |e_k| \delta^{|k|_\infty} < \epsilon \]
for all but finitely many $k$. By “convergence” of the series (4.6), we mean that the partial sums $\sum_{m\in[0,M]^N} d_m c_{k-m}$ approach a limit as $M \to \infty$. This is equivalent to requiring that for every $\epsilon > 0$, all but finitely many terms of the series (4.6) are less than $\epsilon$.

Choose $r, 1 < r < R$. We have
\[ |d_m c_{k-m}| r^{\sum_{j=1}^N k_j} = (|d_m| r^{\sum_{j=1}^N m_j})(|c_{k-m}| r^{\sum_{j=1}^N (k_j - m_j)}). \]

For all but finitely many terms in the sum (4.6) we have $k_j - m_j \leq 0$ for $j = 1, \ldots, N$. We then have from (4.8)
\[ |d_m c_{k-m}| r^{\sum_{j=1}^N k_j} = (|d_m| r^{\sum_{j=1}^N m_j})(|c_{k-m}| (r^{-1})^{-|k-m|_\infty}) \]
for all but finitely many $m$. The convergence of (4.6) now follows from (4.1) and (4.3).

Choose $\delta, 1/R < \delta < 1$. From (4.6) we have
\[ |e_k| \delta^{|k|_\infty} \leq \sup_{m\in\mathbb{N}^N} (|d_m| \delta^{-\sum_{j=1}^N m_j})(|c_{k-m}| \delta^{|k-m|_\infty} + \sum_{j=1}^N m_j). \]

Since $|k|_\infty + \sum_{j=1}^N m_j \geq |k-m|_\infty$ and $\delta < 1$, we get
\[ |e_k| \delta^{|k|_\infty} \leq \sup_{m\in\mathbb{N}^N} (|d_m| \delta^{-\sum_{j=1}^N m_j})(|c_{k-m}| \delta^{|k-m|_\infty}). \]

Since $\delta^{-1} < R$, condition (4.3) implies that the set $\{|d_m| \delta^{-\sum_{j=1}^N m_j} \mid m\in\mathbb{N}^N\}$ is bounded above, say
\[ |d_m| \delta^{-\sum_{j=1}^N m_j} \leq M_1 \text{ for all } m\in\mathbb{N}^N. \]

Since $\xi(t) \in C\{t_1, \ldots, t_N\}$, the set $\{|c_{k-m}| \delta^{|k-m|_\infty} \mid k\in\mathbb{Z}^N\}$ is bounded above, say
\[ |c_{k-m}| \delta^{|k-m|_\infty} \leq M_2 \text{ for all } k\in\mathbb{Z}^N. \]
Lemma 5.2. is an immediate consequence of (3.5), (3.13), and the following lemma. 

Let $x \in \mathbb{Z}$ be a $p$-integral rational number which is not a negative integer.

\begin{equation}
|d_m|\delta^{-\sum_{j=1}^{m} m_j} < \frac{\epsilon}{M_2} \text{ for } m \notin S_1.
\end{equation}

Since $\xi(t) \in C\{t_1, \ldots, t_N\}$, there exists a finite set $S_2$ such that 

\begin{equation}
|c_k-m|\delta^{k-m}|_\infty < \frac{\epsilon}{M_1} \text{ for } k-m \notin S_2.
\end{equation}

Let $S_1+S_2 = \{s_1+s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}$, a finite set. Suppose that $k \notin S_1+S_2$.  

If $k-m \notin S_2$, then (4.10) and (4.13) imply that 

\begin{equation}
(|d_m|\delta^{-\sum_{j=1}^{m} m_j}(|c_k-m|\delta^{k-m}|_\infty)) < \epsilon.
\end{equation}

If $k-m \in S_2$, i.e., $k \in m+S_2$, then $m \notin S_1$ because $k \notin S_1+S_2$. Equations (4.11) and (4.12) then imply that (4.14) holds.

The proof of (4.17) is then a straightforward calculation.

The products in (4.17) are well-defined by Proposition 4.4 and Lemma 4.15. 

Lemma 4.15. If $\eta_j(t_j) = \sum_{j=1}^{j=\infty} c_{j} t_{j}^{l_j} \in C\{t_j\}$ for $j = 1, \ldots, N$, then 

\begin{equation}
\prod_{j=1}^{N} \eta_j(t_j) = \sum_{l=(l_1, \ldots, l_N) \in \mathbb{Z}^N} \left( \prod_{j=1}^{N} c_{j} \right) \left( \prod_{j=1}^{N} t_{j}^{l_j} \right) \in C\{t_1, \ldots, t_N\}.
\end{equation}

Lemma 4.16. Suppose that for $j = 1, \ldots, N$ the series $H_j(t_j) = \sum_{m_j=0}^{\infty} d_{m_j} t_{j}^{m_j}$ converge for $|t_j| \leq R, R > 1$, and $\eta_j(t_j) = \sum_{l_j=0}^{\infty} c_{j} t_{j}^{l_j} \in C\{t_j\}$. Then 

\begin{equation}
\left( \prod_{j=1}^{N} (H_j(t_j) \eta_j(t_j)) \right) \to \left( \prod_{j=1}^{N} H_j(t_j) \right) \left( \prod_{j=1}^{N} \eta_j(t_j) \right) \in C\{t_1, \ldots, t_N\}.
\end{equation}

Proof. The products in (4.17) are well-defined by Proposition 4.4 and Lemma 4.15. The proof of (4.17) is then a straightforward calculation.

5. $p$-ADIC ESTIMATES FOR $\xi_j(t)$ AND $\hat{\xi}_j(t)$

Proposition 5.1. We have $\xi_j(t), \hat{\xi}_j(t) \in t^v C\{t\}$.

When $v_j = -1$, this result is trivial by (3.5) and (3.16). When $-1 < v_j \leq 0$, it is an immediate consequence of (3.5), (3.13), and the following lemma.

Lemma 5.2. Let $z$ be a $p$-integral rational number which is not a negative integer. Then for all $l \in \mathbb{Z}$

\begin{equation}
\text{ord } \pi_0^l[z] \geq -\log_p(p|l|_\infty).
\end{equation}

The remainder of this section is the proof of Lemma 5.2. For a positive integer $b$ one checks that $\text{ord } (b)_l \geq (l-s_l)/(p-1)$, where $s_l$ denotes the sum of the digits in the $p$-adic expansion of $l$. As a function of $b$, $(b)_l$ is continuous in the $p$-adic topology on $\mathbb{Z}_p$, so this estimate holds on $\mathbb{Z}_p$. If $l$ is a negative integer, it then follows from (1.3) that for $z$ a $p$-integral rational number one has

\begin{equation}
\text{ord } [z]_l \geq \frac{-l-s_{-l}}{p-1}.
\end{equation}
Using the trivial estimate \( s_{-l}/(p - 1) \leq \log_p(-pl) \) we have finally

\[
(5.4) \quad \text{ord} \{ z \} \geq \frac{-l}{p - 1} - \log_p(-pl)
\]

when \( l \) is a negative integer. Estimate (5.4) implies Lemma 5.2 when \( l \) is a negative integer.

Suppose now that \( l > 0 \) and \( z \) is a \( p \)-integral rational number. Since \( \text{ord} \{ z \} = 1/(z+1)_l \), finding a lower bound for \( \text{ord} \{ z \} \) is equivalent to finding an upper bound for \( \text{ord} \{ z + 1 \} \). For this we use Dwork \([7, \text{Equation (1.3)}]\). Write

\[
(5.5) \quad z + 1 = -\sum_{i=0}^{\infty} \sigma_i p^i \quad \text{with} \quad 0 \leq \sigma_i \leq p - 1 \quad \text{for all} \quad i
\]

and set

\[
(5.6) \quad \phi(z + 1) = -\sum_{i=0}^{\infty} \sigma_{i+1} p^i.
\]

Then

\[
p\phi(z + 1) - (z + 1) = \sigma_0.
\]

Write \( l = \sum_{i=0}^{k-1} l_ip^i \) and put \( l^{(m)} = l_{m+1} + \sum_{j=0}^{m} l_{j}p^{j-k-1} \). For a real number \( x \) define \( \rho(x) = 0 \) if \( x \leq 0 \) and define \( \rho(x) = 1 \) if \( x > 0 \). From \([7, \text{Equation (1.3)}]\) we have

\[
(5.7) \quad \text{ord} \{ \phi^m(z + 1) \} - (1 + \text{ord} \{ \phi^{m+1}(z + 1) \}) = l^{(m+1)} + (1 + \text{ord} \{ \phi^{m+1}(z + 1) \}) \rho(l_m - \sigma_m).
\]

Summing (5.5) over \( m = 0, 1, \ldots, k - 1 \) gives

\[
(5.8) \quad \text{ord} \{ z + 1 \} = \frac{l - s_l}{p - 1} + \sum_{m=0}^{k-1} (1 + \text{ord} \{ l^{(m+1)} + \phi^{m+1}(z + 1) \}) \rho(l_m - \sigma_m).
\]

For a given \( m \in \{0, \ldots, k - 1\} \) the contribution to the sum on the right-hand side is zero unless \( l_m > \sigma_m \). When \( l_m > \sigma_m \) the contribution is

\[
1 + \text{ord} \{ l^{(m+1)} + \phi^{m+1}(z + 1) \}.
\]

We have

\[
l^{(m+1)} = l_{m+1} + l_{m+2}p + \cdots + l_{k-1}p^{k-1-m},
\]

\[
\phi^{m+1}(z + 1) = -\sigma_{m+1} - \sigma_{m+2}p - \cdots.
\]

It follows that \( \text{ord} \{ l^{(m+1)} + \phi^{m+1}(z + 1) \} \geq s \) if and only if

\[
l_r = \sigma_r \quad \text{for} \quad r = m + 1, m + 2, \ldots, m + s.
\]

Call a collection of consecutive terms \( (l_m, l_{m+1}, \ldots, l_{m+s}) \) of \( (l_0, \ldots, l_{k-1}) \) good if \( l_m > \sigma_m \), \( l_r = \sigma_r \) for \( r = m + 1, \ldots, m + s \), and \( l_{m+s+1} \neq \sigma_{m+s+1} \). Clearly any two good collections are disjoint. If we let \( \ell \) be the total number of terms of \( (l_0, \ldots, l_{k-1}) \) that appear in some good collection of consecutive terms, then

\[
(5.7) \quad \text{ord} \{ z + 1 \} = \frac{l - s_l}{p - 1} + \ell,
\]

and hence

\[
(5.8) \quad \text{ord} \{ z \} = -\frac{l - s_l}{p - 1} - \ell.
\]
when \( l \) is a positive integer. In particular, the disjointness of good subsequences implies the estimate

\[
(5.9) \quad \text{ord } [z]_l \geq -\frac{l - s_l}{p - 1} - k.
\]

Since \( k \leq \log_p (pl) \) we get finally

\[
(5.10) \quad \text{ord } [z]_l \geq -\frac{l}{p - 1} - \log_p (pl)
\]

when \( l \) is a positive integer. Estimate (5.10) implies Lemma 5.2 when \( l \) is a positive integer.

6. An auxiliary function

The purpose of this section is to introduce a function that will appear in formulas for the Dwork-Frobenius action. We use the following variation of the Dwork exponential function:

\[
\sigma(t) := \exp(\pi_0 t - \pi_0 t^p) = \sum_{i=0}^{\infty} \sigma_it^i.
\]

From the definition of \( \hat{\theta}_1(t) \) we have \( \exp(\pi_0 t) = \hat{\theta}(t)/\hat{\theta}_1(t) \), so

\[
\exp(\pi_0 t - \pi_0 t^p) = \frac{\hat{\theta}(t)\hat{\theta}_1(t^p)}{\hat{\theta}(t^p)\hat{\theta}_1(t)} = \theta(t)\hat{\theta}_1(t^p)\hat{\theta}_1(t).
\]

By (3.7) and (3.9) the \( \sigma_i \) in the series expansion satisfy

\[
(6.1) \quad \text{ord } \sigma_i \geq \frac{i(p - 1)}{p^2}.
\]

Consider the series

\[
H(z) = \sum_{l=0}^{\infty} [-z]_l \sigma_pl\pi_0^{-l} = \sum_{l=0}^{\infty} (-1)^l(z)_l\sigma_pl\pi_0^{-l}.
\]

The estimate (6.1) implies that this series is an analytic function of \( z \) for \( \text{ord } z > -(p - 1)/p + 1/(p - 1) \). The set of analyticity includes \( \mathbb{Z}_p \) for \( p \geq 3 \) but not for \( p = 2 \). However, since \( (z)_l \) is a continuous function on \( \mathbb{Z}_p \), the estimates of (6.1) and Lemma 5.2 show that the series \( H(z) \) converges on \( \mathbb{Z}_p \) to a continuous function for \( p \geq 2 \).

**Lemma 6.2.** The function \( H(z) \) assumes unit values for \( z \in \mathbb{Z}_p \).

**Proof.** It follows from the definition that

\[
\sigma_pl = (-\pi_0)^l\sum_{i=0}^{l} (-1)^i\pi_0^{(p-1)i}/(pi)!/(l-i)!.
\]

This gives

\[
H(z) = \sum_{l=0}^{\infty} (z)_l \sum_{i=0}^{l} (-1)^i\pi_0^{(p-1)i}/(pi)!/(l-i)!.
\]
We also have \((z)_l = (-1)^l l! \left(\frac{-z}{l}\right)\), so

\[
H(z) = \sum_{l=0}^{\infty} (-1)^l l! \left(\frac{-z}{l}\right) \sum_{i=0}^{l} \frac{(-1)^i \pi_0^{(p-1)i}}{(pi)!((l-i)!)}. \tag*{(6.1)}
\]

Let \(r\) be a positive integer. Then

\[
H(-r) = \sum_{l=0}^{r} (-1)^l \binom{r}{l} \sum_{i=0}^{l} \frac{(-1)^i \pi_0^{(p-1)i}}{(pi)!}. \tag*{(6.2)}
\]

Using the relation \((r)_l \binom{l}{i} = \binom{r}{i} \binom{r-i}{l-i}\) this becomes

\[
H(-r) = \sum_{l=0}^{r} \sum_{i=0}^{l} (-1)^{i+l} \binom{r}{i} \frac{\pi_0^{(p-1)i}}{(pi)!}. \tag*{(6.3)}
\]

Interchange the order of summation:

\[
H(-r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\pi_0^{(p-1)i}}{(pi)!} \sum_{l=i}^{r} (-1)^l \binom{l}{i}. \tag*{(6.4)}
\]

The inner sum in (6.3) collapses:

\[
\sum_{l=i}^{r} (-1)^l \frac{r-i}{(l-i)!} = \begin{cases} (-1)^r & \text{if } i = r, \\ 0 & \text{if } i < r, \end{cases}
\]

so (6.3) reduces to

\[
(6.4) H(-r) = r! \pi_0^{(p-1)r} / (pr)!. \tag*{(6.4)}
\]

From [11] unnumbered equation between Equations (3.23) and (3.24) we have \(\pi_0^{p-1} = -p + up^p\), where \(\text{ord } u \geq 0\), so

\[
H(-r) = \frac{r!(-p + up^p)^r}{(pr)!} = \frac{r! \sum_{i=0}^{r} \binom{r}{i} (-p)^r (up^p)^i}{(pr)!}. \tag*{(6.5)}
\]

For \(i = 0\), the contribution to the summation on the right-hand side of (6.5) is \(r!(-p)^r/(pr)!\), which is easily checked to be a \(p\)-adic unit. For \(1 \leq i \leq r\), the contribution to the summation on the right-hand side of (6.5) is

\[
\frac{r! \binom{r}{i} (-p)^r (up^p)^i}{(pr)!}. \tag*{(6.6)}
\]

The \(p\)-ordinal of this numerator is greater than or equal to

\[
\text{ord } r! \frac{p^{r-i}}{p-1} = \frac{r - s_r}{p-1} + r + (p-1)i, \tag*{(6.7)}
\]

while the \(p\)-ordinal of this denominator equals

\[
\text{ord } (pr)! = r + \frac{r - s_r}{p-1}. \tag*{(6.8)}
\]

Thus for \(1 \leq i \leq r\) expression (6.6) has \(p\)-ordinal greater than 0, hence (6.5) implies that \(H(-r)\) is a \(p\)-adic unit for every positive integer \(r\). Lemma 6.2 follows because the negative integers are dense in \(\mathbb{Z}_p\) and \(H\) is continuous on \(\mathbb{Z}_p\).

\[\square\]
Remark. The Dwork exponential function is defined by replacing $\pi_0$ by $\pi$, where $\pi^{p-1} = -p$, in the definition of $\sigma(t)$. If we replaced $\pi_0$ by $\pi$ in the definition of $H(z)$, Equation (6.5) would hold with $u = 0$. But from the functional equation for the $p$-adic gamma function $\Gamma_p$ we have $\Gamma_p(-pr) = r!(-p)^r/(pr)!$, so we would get $H(z) = \Gamma_p(pz)$. This would lead to simpler formulas below, but the $p$-adic estimates arising from the use of $\pi$ in place of $\pi_0$ are not strong enough to prove Theorem 2.21.

7. The Dwork-Frobenius operator, part 1

Since each $v_j$ is $p$-integral, for each $v_j$ there exists a unique rational number $v_j'$ in $[-1, 0]$ such that $pv_j - v_j' \in \{0, -1, \ldots, -(p-1)\}$. Recall that if $v_j = -1$ then $v_j' = -1$ and if $v_j = 0$ then $v_j' = 0$. Set $v' = (v_1', \ldots, v_N')$.

We first construct for $-1 < v_j \leq 0$ a Dwork-Frobenius map $\alpha : t^{\nu_j} C\{t\} \to t^{\nu_j} C\{t\}$. Let $\Phi : t^{\nu_j} C\{t\} \to t^{\nu_j} C\{t\}$ be the map that replaces $t$ by $t^p$. We denote by $\sigma(t) : t^{\nu_j} C\{t\} \to t^{\nu_j} C\{t\}$ the map “multiplication by $\sigma(t)$.” Now define

$$\alpha = \sigma(t) \circ \Phi : t^{\nu_j} C\{t\} \to t^{\nu_j} C\{t\}.$$ 

Set

$$C^-\{t\} = \left\{ \xi(t) = \sum_{i=-\infty}^{0} c_i t^i \mid \xi(t) \in C\{t\} \right\}.$$ 

For $v_j = -1$ we construct a Dwork-Frobenius map $\alpha : t^{-1} C^-\{t\} \to t^{-1} C^-\{t\}$ by defining

$$\alpha = \gamma \circ \sigma(t) \circ \Phi : t^{-1} C^-\{t\} \to t^{-1} C^-\{t\}.$$ 

Let $\xi_j(t)$ be the series obtained by replacing $v_j$ by $v_j'$ in the definition of $\xi_j(t)$ (Equation (3.5)). The following result is a variation of Boyarsky [6].

Theorem 7.1. For $j = 1, \ldots, N$ we have

$$\alpha(\xi_j(t)) = \pi_0^{-\ell(p-1)v_j} \frac{H(-v_j)}{[v_j']^{pv_j-v_j'}} \xi_j'(t).$$

For $v_j = 0$ we have $v_j' = 0$ and the assertion reduces to $\alpha(\xi_j(t)) = \xi_j'(t)$ which is trivial from the definitions. For $-1 \leq v_j < 0$ we need a lemma. Let $D$ be the differential operator

$$D = \frac{d}{dt} - \pi_0 t,$$

which maps $t^{\nu_j} C\{t\}$ (resp. $t^{\nu_j} C\{t\}$) to itself.

Lemma 7.2. (a). For $-1 < v_j \leq 0$, we have $D \circ \alpha = p\alpha \circ D$ as maps from $t^{\nu_j} C\{t\}$ to $t^{\nu_j} C\{t\}$.

(b). For $v_j = -1$ we have $(\gamma \circ D) \circ \alpha = p\alpha \circ (\gamma \circ D)$ as maps from $t^{-1} C^-\{t\}$ to $t^{-1} C^-\{t\}$.

Proof. The operators $D$ and $\alpha$ are additive, i. e., if $\xi(t) = \sum_{l \in \mathbb{Z}} c_l t^{\nu_j+l} \in t^{\nu_j} C\{t\}$, then

$$D(\xi(t)) = \sum_{l \in \mathbb{Z}} c_l D(t^{\nu_j+l})$$

and

$$\alpha(\xi(t)) = \sum_{l \in \mathbb{Z}} c_l \alpha(t^{\nu_j+l}).$$
It thus suffices to verify the relation on monomials \( t^{v_j+l} \).

Suppose \(-1 < v_j \leq 0\). On monomials, \( \alpha \) factors formally as

\[
\alpha = \exp(\pi_0 t) \circ \Phi \circ \exp(-\pi_0 t)
\]

and \( D \) factors formally as

\[
D = \exp(\pi_0 t) \circ t \frac{d}{dt} \circ \exp(-\pi_0 t).
\]

This reduces the assertion of part (a) to the obvious equality

\[
(7.3)
\]

\[
t \frac{d}{dt} \circ \Phi = \rho \Phi \circ t \frac{d}{dt}.
\]

If \( I \) denotes the identity operator on \( t^{-1}C\{t\} \), then it is straightforward to check that \( \gamma \circ D \circ (I - \gamma) = 0 \) and \( \gamma \circ (\sigma(t) \circ \Phi) \circ (I - \gamma) = 0 \) as operators on \( t^{-1}C\{t\} \). These equalities imply that \( \gamma \circ D \circ \gamma = \gamma \circ D \) and

\[
\gamma \circ (\sigma(t) \circ \Phi) \circ \gamma = \gamma \circ (\sigma(t) \circ \Phi).
\]

Using these relations, one reduces the assertion of part (b) to (7.3). \( \square \)

We determine the kernel of \( D \) on \( t^{v_j}C\{t\} \) for \(-1 < v_j \leq 0\). Let

\[
\xi(t) = \sum_{l=-\infty}^{\infty} c_l(\pi_0 t)^{v_j+l} \in t^{v_j}C\{t\}.
\]

Then

\[
D(\xi) = \sum_{l=-\infty}^{\infty} ((v_j + l)c_l - c_{l-1})(\pi_0 t)^{v_j+l}.
\]

It follows that \( D \) has a one-dimensional kernel generated by \( \xi_j(t) \).

For \( v_j = -1 \) we determine the kernel of \( \gamma \circ D \) on \( t^{-1}C^{-}\{t\} \). Let

\[
\xi(t) = \sum_{l=-\infty}^{0} c_l(\pi_0 t)^{-1+l} \in t^{-1}C^{-}\{t\}.
\]

Then

\[
\gamma \circ D(\xi) = \sum_{l=-\infty}^{0} ((-1+l)c_l - c_{l-1})(\pi_0 t)^{-1+l}.
\]

It follows that \( \gamma \circ D \) has a one-dimensional kernel again generated by \( \xi_j(t) \).

It follows from Lemma 7.2 that \( \alpha \) maps the kernel of \( D \) (resp. \( \gamma \circ D \)) in \( t^{v_j}C\{t\} \) (resp. \( t^{-1}C^{-}\{t\} \)) to the kernel of \( D \) (resp. \( \gamma \circ D \)) in \( t^{v_j'}C\{t\} \) (resp. \( t^{-1}C^{-}\{t\} \)). Since these kernels are all one-dimensional we have for all \( j \)

\[
(7.4)
\]

\[
\alpha(\xi_j(t)) = \mu_j \xi_j'(t)
\]

for some \( \mu_j \in \mathbb{C}_p \). To finish the proof of Theorem 7.1 we need to compute \( \mu_j \).

Suppose first that \(-1 < v_j < 0\). To find \( \mu_j \), we compute the coefficient of \( t^{p v_j} \) on each side of (7.4). From the definition of \( \xi_j'(t) \), the coefficient of \( t^{p v_j} \) on the right-hand side of (7.4) is

\[
(7.5)
\]

\[
\mu_j \pi^{p v_j'}_0 [v_j]_{p v_j - v_j'}.
\]
It follows that we take $l$.

This proves Theorem 7.1 for $-1 < v_j < 0$.

Now suppose that $v_j = -1$, in which case $v_j' = -1$ also. To compute $\mu_j$ we compute the coefficient of $t^{-p}$ on each side of (7.4). For the right-hand side of (7.4) we take $l = (p - 1)$ in the series expansion of $\xi_j(t)$ (see (3.5)) to get

$$
\mu_j (-1)^{p-1} (p-1)! \pi_0^{-p}.
$$

For the left-hand side of (7.4), a calculation from the definitions gives

$$
\pi_0^{-1} \sum_{k=0}^{\infty} (-1)^{-k} \sigma_{p_0} \pi_0^{-k} = \pi_0^{-1} H(1).
$$

It follows that $\mu_j = (-\pi_0)^{p-1} H(1)/(p-1)!$, so

$$
\alpha_j(\xi_j(t)) = (-\pi_0)^{p-1} H(1) \xi_j(t).
$$

Furthermore,

$$
(-\pi_0)^{p-1} \frac{H(1)}{(p-1)!} = \pi_0^{- (p-1)(-1)} \frac{H(1)}{[-1]_{-p+1}},
$$

proving Theorem 7.1 when $v_j = -1$.

8. The Dwork-Frobenius operator, part 2

The purpose of this section is to prove a result analogous to Theorem 7.1 for the $\xi_j(t)$. For $-1 < v_j \leq 0$, we define $\hat{\alpha} : t^{v_j} C[t] \to t^{v_j} C[t]$ by $\hat{\alpha} = \theta(t) \circ \Phi$.

For $v_j = -1$ we define $\hat{\alpha} : t^{-1} C^{-1} \to t^{-1} C^{-1}$ by $\hat{\alpha} = \gamma \circ \theta(t) \circ \Phi$. Let $\hat{\theta}_1(t) : t^{v_j} C[t] \to t^{v_j} C[t]$ and $\hat{\theta}_1(t) : t^{v_j} C[t] \to t^{v_j} C[t]$ be the maps “multiplication by $\hat{\theta}_1(t)$.”

**Proposition 8.1.** (a): We have $\hat{\alpha} \circ \hat{\theta}_1(t) = \hat{\theta}_1(t) \circ \alpha$ as maps from $t^{v_j} C[t]$ to $t^{v_j} C[t]$.

(b): We have $\hat{\alpha} \circ (\gamma \circ \hat{\theta}_1(t)) = (\gamma \circ \hat{\theta}_1(t)) \circ \alpha$ as maps from $t^{-1} C^{-1}$ to $t^{-1} C^{-1}$.

**Proof.** All these operators are additive so it suffices to verify these relations on monomials. On monomials, we can apply formal factorizations of these operators.
From the definitions
\[ \hat{\alpha} \circ \hat{\theta}_1(t) = \theta(t) \circ \Phi \circ \hat{\theta}_1(t) \]
\[ = \hat{\theta}(t) \circ \Phi \circ \frac{1}{\theta(t)} \circ \hat{\theta}_1(t) \]
\[ = \hat{\theta}_1(t) \exp(\pi_0 t) \circ \Phi \circ \frac{1}{\exp(\pi_0 t)} \]
\[ = \hat{\theta}_1(t) \sigma(t) \circ \Phi \]
\[ = \hat{\theta}_1(t) \circ \alpha, \]
where the second equality follows from \( \theta(t) = \hat{\theta}(t)/\hat{\theta}(t^p) \) and the third equality follows from \( \theta(t) = \hat{\theta}_1(t) \exp(\pi_0 t) \). This proves part (a).

From the definitions
\[ \hat{\alpha} \circ \gamma \circ \hat{\theta}_1(t) = \gamma \circ \theta(t) \circ \Phi \circ \gamma \circ \hat{\theta}_1(t). \]
We have \( \gamma \circ \theta(t) \circ \Phi \circ (J - \gamma) = 0 \) as operators on \( t^{-1}C^- \{ t \} \), so
\[ \hat{\alpha} \circ \gamma \circ \hat{\theta}_1(t) = \gamma \circ \theta(t) \circ \Phi \circ \hat{\theta}_1(t). \]

Similarly,
\[ \gamma \circ \hat{\theta}_1(t) \circ \gamma \circ \sigma(t) \circ \Phi = \gamma \circ \hat{\theta}_1(t) \sigma(t) \circ \Phi. \]
The assertion of part (b) now follows from part (a). \( \Box \)

Let \( \hat{\xi}_j(t) \) be defined by replacing \( v_j \) by \( v'_j \) in the definition of \( \hat{\xi}_j(t) \). Proposition 8.1 implies the analogue of Theorem 7.1.

**Theorem 8.2.** For \( j = 1, \ldots, N \) we have
\[ \hat{\alpha}(\hat{\xi}_j(t)) = \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v'_j]_{pv_j-v_j'}} \hat{\xi}'_j(t). \]

**Proof.** Suppose first that \(-1 < v_j \leq 0\). We have
\[ \hat{\alpha}(\hat{\xi}_j(t)) = \hat{\alpha}(\hat{\theta}_1(t) \xi_j(t)) \]
\[ = \hat{\theta}_1(t) \alpha(\xi_j(t)) \]
\[ = \hat{\theta}_1(t) \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v'_j]_{pv_j-v_j'}} \xi'_j(t) \]
\[ = \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v'_j]_{pv_j-v_j'}} \hat{\xi}'_j(t), \]
where the first and fourth equalities follow from (3.10), the second equality follows from Proposition 8.1(a), and the third equality follows from Theorem 7.1.

Now suppose that \( v_j = -1 \). We have
\[ \hat{\alpha}(\hat{\xi}_j(t)) = \hat{\alpha}(\gamma(\hat{\theta}_1(t) \xi_j(t))) \]
\[ = \gamma(\hat{\theta}_1(t) \alpha(\xi_j(t))) \]
\[ = \gamma(\hat{\theta}_1(t) \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v'_j]_{pv_j-v_j'}} \xi'_j(t)) \]
\[ = \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v'_j]_{pv_j-v_j'}} \hat{\xi}'_j(t), \]
By (4.15), (8.3), and a calculation we have

\[
\gamma_j \left( \sum_{l \in \mathbb{Z}} c_l t^l_j \right) = \sum_{l = -\infty}^{-1} c_l t^l_j.
\]

Replace \( \hat{a} \) by \( \hat{\alpha}_j \), where \( \hat{\alpha}_j = \theta(t_j) \circ \Phi_j \) for \(-1 < v_j \leq 0\) and \( \hat{\alpha}_j = \gamma_j \circ \theta(t_j) \circ \Phi_j \) for \( v_j = -1 \). Then Theorem 8.2 becomes

\[
(8.3) \quad \hat{\alpha}_j(\hat{\xi}_j(t_j)) = \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v_j']^{p
u_j-v_j'}} \hat{\xi}_j(t_j).
\]

9. DWORK-FROBENIUS ON HYPERGEOMETRIC SERIES

Put \( \hat{\xi}_v(t_1, \ldots, t_N) = \prod_{j=1}^{N} \hat{\xi}_j(t_j) \). Then \( \hat{\xi}_v(t_1, \ldots, t_N) \in t^v C_E \{ t_1, \ldots, t_N \} \) by Lemma 5.2 (\( C_E \{ t_1, \ldots, t_N \} \) is defined in (4.2)). We also define

\[
\hat{\xi}_v(t_1, \ldots, t_N) = \prod_{j=1}^{N} \hat{\xi}_j(t_j) \in t^v C_E \{ t_1, \ldots, t_N \}.
\]

Let \( \Phi_N : t^v C \{ t_1, \ldots, t_N \} \to t^{v'} C \{ t_1, \ldots, t_N \} \), be the operator that replaces each \( t_j \) by \( t^v_j \). Put \( \theta(t_1, \ldots, t_N) = \prod_{j=1}^{N} \theta(t_j) \). Multiplication by \( \theta(t_1, \ldots, t_N) \) defines a map

\[
\theta(t_1, \ldots, t_N) : t^v C_E \{ t_1, \ldots, t_N \} \to t^{v'} C_{E_+} \{ t_1, \ldots, t_N \}.
\]

Let \( \gamma_E : t^{v'} C \{ t_1, \ldots, t_N \} \to t^{v'} C_E \{ t_1, \ldots, t_N \} \) be defined by

\[
\gamma_E \left( t^{v'} \sum_{l \in \mathbb{Z}^N} c_l t^l \right) = t^{v'} \sum_{l \in E} c_l t^l.
\]

Finally we define \( \hat{\alpha}_N : t^v C \{ t_1, \ldots, t_N \} \to t^{v'} C_E \{ t_1, \ldots, t_N \} \) by

\[
\hat{\alpha}_N = \gamma_E \circ \theta(t_1, \ldots, t_N) \circ \Phi_N.
\]

By (4.15), (8.3), and a calculation we have

\[
(9.1) \quad \hat{\alpha}_N(\hat{\xi}_v(t_1, \ldots, t_N)) = \prod_{j=1}^{N} \hat{\alpha}_j(\hat{\xi}_j(t_j))
\]

\[
= \left( \prod_{j=1}^{N} \pi_0^{-(p-1)v_j} \frac{H(-v_j)}{[v_j']^{p
u_j-v_j'}} \right) \hat{\xi}_v(t_1, \ldots, t_N).
\]

Since \( \sum_{j=1}^{N} v_j a_j = \beta \) and \( w(a_j) = 1 \) for all \( j \) we have \( \sum_{j=1}^{N} v_j = w(\beta) \), so this equation simplifies to

\[
(9.2) \quad \hat{\alpha}_N(\hat{\xi}_v(t_1, \ldots, t_N)) = \pi_0^{-(p-1)w(\beta)} \left( \prod_{j=1}^{N} \frac{H(-v_j)}{[v_j']^{p
u_j-v_j'}} \right) \hat{\xi}_v(t_1, \ldots, t_N).
\]

Note that the factor \( \prod_{j=1}^{N} \frac{H(-v_j)}{[v_j']^{p
u_j-v_j'}} \) is a p-adic unit. The factors \( H(-v_j) \) are p-adic units by Lemma 6.2 and the \( [v_j']^{p
u_j-v_j'} \) are p-adic units by a direct calculation.
From (3.18) we have

\begin{equation}
\hat{\xi}_\nu(\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}) = \sum_{u \in \beta + \mathbb{Z} A} G_u(\Lambda) \pi_0^u(x^u).
\end{equation}

We set

\begin{equation}
G_u(\Lambda, x) = \sum_{u \in \beta + \mathbb{Z} A} G_u(\Lambda) \pi_0^u(x^u).
\end{equation}

Put \( \beta' = \sum_{j=1}^N \nu_j \alpha_j \). As in (3.18) we have

\begin{equation}
\prod_{j=1}^N \hat{\xi}_j(\Lambda x^{a_j}) = \sum_{u' \in \beta' + \mathbb{Z} A} G_{u'}(\Lambda) \pi_0^{u'}(x^{u'}),
\end{equation}

where

\begin{equation}
G_{u'}(\Lambda_1, \ldots, \Lambda_N) = \sum_{l \in E(u')} [\nu']_l \Lambda^{\nu'} + l.
\end{equation}

As in (9.3) we have

\begin{equation}
\hat{\xi}_{\nu'}(\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}) = \sum_{u' \in \beta' + \mathbb{Z} A} G_{u'}(\Lambda) \pi_0^{u'}(x^{u'}),
\end{equation}

and as in (9.4) we set

\begin{equation}
G_{\nu'}(\Lambda, x) = \sum_{u' \in \beta' + \mathbb{Z} A} G_{u'}(\Lambda) \pi_0^{u'}(x^{u'}).
\end{equation}

For \( S \subseteq \mathbb{Z}^N \) we set

\begin{equation}
\text{CS}\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\} = \left\{ \xi = \sum_{l=(l_1, \ldots, l_N) \in S} c_l (\Lambda_1 x^{a_1})^{l_1} \cdots (\Lambda_N x^{a_N})^{l_N} \mid c_l |\delta|_\infty \to 0 \text{ for all } \delta < 1 \right\}.
\end{equation}

The change of variable \( t_j \to \Lambda_j x^{a_j} \) induces a \( \mathbb{C}_p \)-isomorphism

\begin{equation}
t^\nu C_E\{t_1, \ldots, t_N\} \to \Lambda^\nu x^\beta C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\}.
\end{equation}

We let \( \alpha^* : \Lambda^\nu x^\beta C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\} \to \Lambda^{\nu'} x^{\beta'} C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\} \) be the map corresponding to \( \hat{\alpha}_N \) under this isomorphism. Like \( \hat{\alpha}_N \), the map \( \alpha^* \) can be written as a composition. Let

\begin{equation}
\Phi^* : \Lambda^\nu x^\beta C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\} \to \Lambda^{\nu'} x^{\beta'} C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\}
\end{equation}

be the map which replaces each \( \Lambda_j \) by \( \Lambda_j^p \) and each \( x_i \) by \( x_i^p \). Put

\begin{equation}
\theta(\Lambda, x) = \prod_{j=1}^N \theta(\Lambda_j x_j^a).
\end{equation}

Multiplication by \( \theta(\Lambda, x) \) defines a map

\begin{equation}
\theta(\Lambda, x) : \Lambda^{\nu'} x^{\beta'} C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\} \to \Lambda^{\nu'} x^{\beta'} C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\}.
\end{equation}

Then

\begin{equation}
\alpha^* = \gamma_E \circ \theta(\Lambda, x) \circ \Phi^*,
\end{equation}

where
Lemma 10.2. Suppose that $l \not\in v\Lambda$.

Consider a monomial $C$ be the

Proposition 2.9 implies that $\beta$ (10.1)

$w$ $l\not\in E$ then $\gamma_E(\xi) = 0$ so $I - \gamma_E(\xi) = 0$. If $l \not\in E$ then $\gamma_{E'}(\xi) = 0$ so $I - \gamma_{E'}(\xi) = 0$. But since $\beta'$ satisfies (10.1) and $l \not\in E$, Proposition 2.9 implies that $\beta' + \sum_{j=1}^{N} l_j a_j \not\in \mathcal{M}'$, hence $\gamma_{E'}(\xi) = 0$. $\square$

10. Restriction to $\mathcal{M}$

In general, not all the hypergeometric series that appear in the generating series $G_v(\Lambda, x)$ will have $p$-integral coefficients. We need to truncate $G_v(\Lambda, x)$ to include only those series that are potentially $p$-integral and we need a version of Theorem 9.13 for these truncations.

Let

$$\gamma_{\mathcal{M}} : \Lambda^\alpha x^\beta C_{\mathcal{E}_v} \{\Lambda_1 x^{\alpha_1}, \ldots, \Lambda_N x^{\alpha_N}\} \to \Lambda^{\alpha'} x^{\beta'} C_{E_v} \{\Lambda_1 x^{\alpha_1}, \ldots, \Lambda_N x^{\alpha_N}\}$$

be the $\mathbb{C}_p$-linear map defined by

$$\gamma_{\mathcal{M}}(\Lambda^{\alpha+l} x^{\sum_{j=1}^{N} (\nu_j + l_j) a_j}) = \begin{cases} \Lambda^{\alpha+l} x^{\sum_{j=1}^{N} (\nu_j + l_j) a_j} & \text{if } \sum_{j=1}^{N} (\nu_j + l_j) a_j \in \mathcal{M}, \\ 0 & \text{if } \sum_{j=1}^{N} (\nu_j + l_j) a_j \not\in \mathcal{M}. \end{cases}$$

Note that the image of $\gamma_{\mathcal{M}}$ lies in $\Lambda^\alpha x^\beta C_{\mathcal{E}_v} \{\Lambda_1 x^{\alpha_1}, \ldots, \Lambda_N x^{\alpha_N}\}$ by Proposition 2.9.

Similarly we define

$$\gamma_{\mathcal{M}'} : \Lambda^{\alpha'} x^{\beta'} C_{\mathcal{E}_v} \{\Lambda_1 x^{\alpha_1}, \ldots, \Lambda_N x^{\alpha_N}\} \to \Lambda^{\alpha'} x^{\beta'} C_{E_v} \{\Lambda_1 x^{\alpha_1}, \ldots, \Lambda_N x^{\alpha_N}\}.$$

Put $G_v^M(\Lambda, x) = \gamma_{\mathcal{M}}(G_v(\Lambda, x))$ and $G_{v'}^{M'}(\Lambda, x) = \gamma_{\mathcal{M}'}(G_{v'}(\Lambda, x))$. Thus (see (9.4) and (9.8))

$$G_v^M(\Lambda, x) = \sum_{u \in \mathcal{M}} G_u(\Lambda) \pi_0^{w(u)} x^u$$

and

$$G_{v'}^{M'}(\Lambda, x) = \sum_{u' \in \mathcal{M}'} G_{u'}(\Lambda) \pi_0^{w(u')} x^{u'}.$$
Define
\[ \alpha^*_{\mathcal{M}'} := \gamma_{\mathcal{M}'} \circ \Theta(\Lambda, x) \circ \Phi^* : \Lambda^v x^\beta C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\} \]
\[ \rightarrow \Lambda' x^{\beta'} C_E\{\Lambda_1 x^{a_1}, \ldots, \Lambda_N x^{a_N}\}. \]

**Corollary 10.3.** Suppose that \( \beta' \) satisfies (10.1). Then
\[ \alpha^*_{\mathcal{M}'}(G_v(\Lambda, x)) = \pi_0^{-(p-1)w(\beta)}\left( \prod_{j=1}^N \frac{H(-v_j)}{[v_j]_p v'_j} \right) G_{v'}(\Lambda, x). \]

**Proof.** It follows from Lemma 10.2 that \( \gamma_{\mathcal{M}'} = \gamma_{\mathcal{M}} \circ \gamma_E \). The assertion of the corollary then follows from Equation (9.14) by applying \( \gamma_{\mathcal{M}'} \) to both sides. \( \square \)

We can now prove the main result of this section.

**Theorem 10.5.** Suppose that \( \beta' \) satisfies (10.1). Then
\[ \alpha^*_{\mathcal{M}'}(G_v^M(\Lambda, x)) = \pi_0^{-(p-1)w(\beta)}\left( \prod_{j=1}^N \frac{H(-v_j)}{[v_j]_p v'_j} \right) G_{v'}^M(\Lambda, x). \]

**Proof.** From (9.11) we have
\[ \theta(\Lambda, x) = \sum_{\nu \in \mathbb{N}A} \theta_{\nu}(\Lambda) x^\nu \]
with
\[ \theta_{\nu}(\Lambda) = \sum_{\sum_{m=1}^N m \Lambda^m = \nu} \theta_m \Lambda^m. \]

The right-hand sides of (10.4) and (10.6) are identical, so to prove (10.6) we need to show the left-hand side of (10.4) equals the left-hand side of (10.6). Since \( G_v^M(\Lambda, x) = \gamma_{\mathcal{M}}(G_v(\Lambda, x)) \) it suffices to show that \( \alpha^*_{\mathcal{M}'} \) annihilates all monomials of the form \( \Lambda^{v+l} x^u \) with \( \sum_{j=1}^N (v_j + l_j) a_j = u \notin \mathcal{M} \).

Let \( \Lambda^{v+l} x^u \) be such a monomial. Then \( \Phi^*(\Lambda^{v+l} x^u) = \Lambda^{p(v+l)} x^{pu} \), and since \( u \notin \mathcal{M} \) we have \( pu \notin \mathcal{M}' \). It follows from (10.7) and (10.8) that the monomials in \( \Theta(\Lambda, x) \Phi^*(\Lambda^{v+l} x^u) \) are of the form \( \Lambda^{p(v+l)+m} x^{pu+\nu} \) where \( m \in \mathbb{N}^N \) and \( \sum_{j=1}^N m_j a_j = \nu \). We thus have
\[ \sum_{j=1}^N (p(v_j + l_j) + m_j) a_j = pu + \nu. \]

We need to show that \( pu + \nu \notin \mathcal{M}' \).

Note that \( pl + m \in E_+ \). If \( pl_j + m_j \neq 0 \) for some \( j \) with \( -a_j \notin \sigma \), then Lemma 2.8 tells us that \( p(v+l) + m \notin \mathcal{M}' \) and we are done. So suppose that \( pl_j + m_j = 0 \) for all \( j \) for which \( -a_j \notin \sigma \). We noted earlier that \( -a_j \notin \sigma \) implies that \( v_j = 0 \), hence \( l_j \geq 0 \) because \( l \in E \). But \( m_j \geq 0 \) for all \( j \), so if \( pl_j + m_j = 0 \) then \( l_j = m_j = 0 \).

It follows that if some \( m_j > 0 \) in (10.9), then \( -a_j \in \sigma \), so
\[ -\nu = - \sum_{j=1}^N m_j a_j \in \sigma. \]
If $pu + \nu$ were in $\mathcal{M}'$, then adding this expression to (10.9) would give

$$\sum_{j=1}^{N} p(v_j + l_j)a_j = pu \in \mathcal{M}'.$$

But this contradicts the assumption that $u \not\in \mathcal{M}$.

\[\square\]

11. PROOF OF THEOREM 2.21

We suppose that our prime $p$ satisfies $p \equiv h \pmod{D}$ and that (2.20) holds for $i = 0, \ldots, a - 1$. By Proposition 3.19 it suffices to prove that all the corresponding series $G_{u^{(i)}}(\Lambda)$ have $p$-integral coefficients. By our assumption (2.20), Theorem 10.5 holds for all $\beta^{(i)}$ so we have

\[(11.1) \quad \alpha_{\mathcal{M}^{(i+1)}}(G_{u^{(i)}}^{M^{(i)}}(\Lambda, x)) \]

\[= \pi_0^{-(p-1)w(\beta^{(i)})} \left( \prod_{j=1}^{N} \frac{H(-v^{(i)}_j)}{[\nu_j^{(i+1)}]^{[\nu_j^{(i)}]} - \nu_j^{(i+1)}} \right) G_{u^{(i+1)}}^{M^{(i+1)}}(\Lambda, x).\]

We begin by computing the left-hand side of (11.1) directly from the definition. We have

$$\Phi^*(G_{u^{(i)}}^{M^{(i)}}(\Lambda, x)) = \sum_{u^{(i)} \in \mathcal{M}^{(i)}} G_{u^{(i)}}(\Lambda^p)\pi_0^{w(u^{(i)})}x^{pu^{(i)}}.$$

Thus from (10.7)

$$\theta(\Lambda, x)\Phi^*(G_{u^{(i)}}^{M^{(i)}}(\Lambda, x)) = \sum_{\rho \in E_+} \left( \sum_{\nu \in \mathcal{N}, u^{(i)} \in \mathcal{M}^{(i)}} \theta_\nu(\Lambda)G_{u^{(i)}}(\Lambda^p)\pi_0^{w(u^{(i)})} \right) x^\rho.$$

Finally, applying $\gamma_{\mathcal{M}^{(i+1)}}$, we get

\[(11.2) \quad \alpha_{\mathcal{M}^{(i+1)}}(G_{u^{(i+1)}}^{M^{(i+1)}}(\Lambda, x)) \]

\[= \sum_{u^{(i+1)} \in \mathcal{M}^{(i+1)}} \left( \sum_{\nu \in \mathcal{N}, u^{(i)} \in \mathcal{M}^{(i)}} \theta_\nu(\Lambda)G_{u^{(i)}}(\Lambda^p)\pi_0^{w(u^{(i)})} \right) \pi_0^{-(u^{(i)})}x^{u^{(i+1)}}.\]

Using the definition of $G_{u^{(i+1)}}^{M^{(i+1)}}(\Lambda, x)$ the right-hand side of (11.1) equals

\[(11.3) \quad \pi_0^{-(p-1)w(\beta^{(i)})} \left( \prod_{j=1}^{N} \frac{H(-v^{(i)}_j)}{[\nu_j^{(i+1)}]^{[\nu_j^{(i)}]} - \nu_j^{(i+1)}} \right) G_{u^{(i+1)}}^{M^{(i+1)}}(\Lambda) \pi_0^{-(u^{(i)})}x^{u^{(i+1)}}.\]

By (11.1), the coefficients of $\pi_0^{w(u^{(i+1)})}x^{u^{(i+1)}}$ on the right-hand sides of (11.2) and in (11.3) must be equal for all $u^{(i+1)} \in \mathcal{M}^{(i+1)}$.

\[(11.4) \quad \pi_0^{-(p-1)w(\beta^{(i)})} \left( \prod_{j=1}^{N} \frac{H(-v^{(i)}_j)}{[\nu_j^{(i+1)}]^{[\nu_j^{(i)}]} - \nu_j^{(i+1)}} \right) G_{u^{(i+1)}}(\Lambda) \]

\[= \sum_{\nu \in \mathcal{N}, u^{(i)} \in \mathcal{M}^{(i)}} \theta_\nu(\Lambda)G_{u^{(i)}}(\Lambda^p)\pi_0^{w(u^{(i)})}x^{u^{(i+1)}}.\]
Solve this equation for \( G_{u(i+1)}(\Lambda) \):

\[
(11.5) \quad G_{u(i+1)}(\Lambda) = \pi_0^{(p-1)w(\beta^{(i)})} \left( \prod_{j=1}^{N} \frac{H(-v_j^{(i)})}{[v_j^{(i+1)}]_{p^{\nu_j^{(i)}} - v_j^{(i+1)}}} \right)^{-1} \sum_{\nu \in \mathbb{N}\Lambda, u(i) \in \mathcal{M}^{(i)}} \theta_{\nu}(\Lambda) G_{u(i)}(\Lambda^p) \pi_0^{w(\nu^{(i)}) - u^{(i+1)}}.
\]

In Equation (10.8) we have for \( m \in \mathbb{N}^N \) with \( \sum_{j=1}^{N} m_j a_j = \nu \) that \( \theta_m = \prod_{j=1}^{N} \theta_{m_j} \). Hence by (3.7)

\[
\text{ord} \theta_m = \sum_{j=1}^{N} \text{ord} \theta_{m_j} \geq \frac{\sum_{j=1}^{N} m_j}{p-1} = \frac{w(\nu)}{p-1}.
\]

We can thus write \( \theta_{\nu}(\Lambda) = \pi_0^{w(\nu)} \tilde{\theta}_{\nu}(\Lambda) \) where \( \tilde{\theta}_{\nu}(\Lambda) \) has \( p \)-integral coefficients and (11.5) can be rewritten as

\[
(11.6) \quad G_{u(i+1)}(\Lambda) = \pi_0^{(p-1)w(\beta^{(i)})} \left( \prod_{j=1}^{N} \frac{H(-v_j^{(i)})}{[v_j^{(i+1)}]_{p^{\nu_j^{(i)}} - v_j^{(i+1)}}} \right)^{-1} \sum_{\nu \in \mathbb{N}\Lambda, u(i) \in \mathcal{M}^{(i)}} \tilde{\theta}_{\nu}(\Lambda) G_{u(i)}(\Lambda^p) \pi_0^{w(\nu^{(i)}) - u^{(i+1)}}.
\]

Since \( \nu + pu^{(i)} = u^{(i+1)} \) we have \( \nu + u^{(i)} - u^{(i+1)} = -(p-1)u^{(i)} \) so (11.6) becomes

\[
(11.7) \quad G_{u(i+1)}(\Lambda) = \left( \prod_{j=1}^{N} \frac{H(-v_j^{(i)})}{[v_j^{(i+1)}]_{p^{\nu_j^{(i)}} - v_j^{(i+1)}}} \right)^{-1} \sum_{\nu \in \mathbb{N}\Lambda, u(i) \in \mathcal{M}^{(i)}} \pi_0^{(p-1)w(\beta^{(i)} - u^{(i)})} \tilde{\theta}_{\nu}(\Lambda) G_{u(i)}(\Lambda^p).
\]

In this equation, \( \left( \prod_{j=1}^{N} \frac{H(-v_j^{(i)})}{[v_j^{(i+1)}]_{p^{\nu_j^{(i)}} - v_j^{(i+1)}}} \right)^{-1} \) is a \( p \)-adic unit and all coefficients of the polynomial \( \pi_0^{(p-1)w(\beta^{(i)} - u^{(i)})} \tilde{\theta}_{\nu}(\Lambda) \) are \( p \)-integral by (2.20).

We now apply Equation (11.7) to deduce \( p \)-integrality of the series coefficients of \( G_{u(i)}(\Lambda) \). Recall the series expansion (see Equation (3.17))

\[
(11.8) \quad G_{u(i)}(\Lambda) = \sum_{I = (I_1, \ldots, I_N) \in E(u(i))} \left( \prod_{j=1}^{N} g(v_j^{(i)}, I_j) \right) \Lambda^{v_j^{(i)} + I},
\]

where the \( g(v_j^{(i)}, I_j) \) satisfy

\[
(11.9) \quad \text{ord} g(v_j^{(i)}, I_j) = \text{ord} [v_j^{(i)}]_{I_j}
\]

by (3.13) and (3.16).

Since \( I \in E(u(i)) \), if \( v_j^{(i)} = -1 \) then \( I_j \in \mathbb{Z}_{\leq 0} \) so \( \prod_{j=1}^{N} [v_j^{(i)}]_{I_j} \) is \( p \)-integral.

We can thus focus attention on the \([v_j^{(i)}]_{I_j} \) for \(-1 < v_j^{(i)} \leq 0\). We prove that all monomials \( \Lambda^{v_j^{(i)} + I} \) in all \( G_{u(i)}(\Lambda) \), \( u^{(i)} \in \mathcal{M}^{(i)} \), and \( i = 0, 1, \ldots, a-1 \), have \( p \)-integral coefficients by induction on \( d(v^{(i)} + l) := \sum_{-1 < v_j^{(i)} \leq 0} \max \{0, v_j^{(i)} + I_j\} \). If
Under these assumptions, the series \( (12.2) \) can be written as
\[
\sum_{v} \prod_{d} \theta_{v_{d}} \prod_{s_{j}} \frac{t_{s_{j}}^{1}}{s_{j}!} = \sum_{v} \prod_{d} \theta_{v_{d}} \prod_{s_{j}} \frac{t_{s_{j}}^{1}}{s_{j}!}.
\]

In this section we give a formula for the coefficient of \( \Lambda_{v} \) in the right-hand side of \((11.7)\) one gets a formula for the coefficient of \( \Lambda_{v} \) in \( G_{u(i)} \).

Let \( c_{jk} \in \mathbb{Z} \) for \( 1 \leq j \leq n \) and \( 1 \leq k \leq m \) and let
\[
C_{j}(s) := C_{j}(s_{1}, \ldots, s_{m}) = \sum_{k=1}^{m} c_{jk} s_{k}.
\]
To avoid trivial cases, we assume that no \( C_{j} \) is identically zero. We also assume that for each \( k \) there are at least two values of \( j \) such that \( c_{jk} \neq 0 \), i.e., each \( s_{k} \) appears in at least two \( C_{j} \) with nonzero coefficient. We also assume that
\[
(12.1) \sum_{j=1}^{n} c_{jk} = 1 \quad \text{for } k = 1, \ldots, m.
\]
This condition will guarantee that the associated \( A \)-hypergeometric system is regular holonomic.

Many classical hypergeometric equations have a solution of the form (Dwork-Loeser Appendix)

\[
(12.2) \sum_{s_{1}, \ldots, s_{m}=0}^{\infty} (\theta_{1})_{C_{1}(s)} \cdots (\theta_{n})_{C_{n}(s)} \frac{t_{s_{1}}^{1} \cdots t_{s_{m}}^{m}}{s_{1}! \cdots s_{m}!}.
\]
In this section we give a \( p \)-integrality criterion for such series when all \( \theta_{i} \) are \( p \)-integral rational numbers in the interval \([0, 1]\).

We impose two additional conditions. If \( \theta_{i} = 1 \), then \( (\theta_{i})_{C_{i}(s)} \) is undefined for \( C_{i}(s) < 0 \), so we assume that all \( c_{ik} \) are nonnegative when \( \theta_{i} = 1 \). If \( \theta_{i} = 0 \), then \( (\theta_{i})_{C_{i}(s)} = 0 \) for \( C_{i}(s) > 0 \), so we assume that all \( c_{ik} \) are nonpositive when \( \theta_{i} = 0 \). Under these assumptions, the series \((12.2)\) can be written as

\[
(12.3) \sum_{s_{1}, \ldots, s_{m}=0}^{\infty} \frac{\prod_{\theta_{i}=1} C_{i}(s)!}{\prod_{\theta_{i}=0} (-1)^{C_{i}(s)}(-C_{i}(s))!} \prod_{0<\theta_{i}<1} (\theta_{i})_{C_{i}(s)} \frac{t_{s_{1}}^{1} \cdots t_{s_{m}}^{m}}{s_{1}! \cdots s_{m}!}.
\]

For the study of such series, the assumption above that each \( s_{k} \) appears in at least two \( C_{i}(s) \) is not restrictive. If, for example, \( s_{1} \) appears in only one \( C_{i}(s) \) we can multiply the series \((12.3)\) by \( \frac{s_{1}!}{s_{1}!} \) and the assumption will be satisfied. For
instance, this would replace the series \( F_0(\theta_1; t_1) \) by the series \( F_1(\theta_1, 1; 1; t_1) \), which has identical coefficients.

We describe the \( A \)-hypergeometric system for which a solution \( F_\beta(A) \) has coefficients identical (up to sign) to those of (12.2). Let \( a_1, \ldots, a_n \) be the standard unit basis vectors in \( \mathbb{R}^n \) and for \( k = 1, \ldots, m \) let

\[
a_{n+k} = (c_{1k}, \ldots, c_{nk}).
\]

Our hypothesis that for a fixed \( k \) at least two \( c_{jk} \) are nonzero implies that \( a_1, \ldots, a_{n+m} \) are all distinct. Put \( N = n + m \) and let \( A = \{ a_i \}_{i=1}^N \subseteq \mathbb{Z}^n \). Condition (12.1) implies that the elements of the set \( A \) all lie on the hyperplane \( \sum_{i=1}^n u_i = 1 \) in \( \mathbb{R}^n \).

Let \( \Theta = (\theta_1, \ldots, \theta_n) \) be a sequence of \( p \)-integral rational numbers in the interval \([0, 1]\). Take

\[
v = (-\theta_1, \ldots, -\theta_n, 0, \ldots, 0)
\]

where 0 is repeated \( m \) times. This is a sequence of \( p \)-integral rational numbers in the interval \([-1, 0]\), and we take

\[
\beta = \sum_{i=1}^N v_i a_i = (-\theta_1, \ldots, -\theta_n).
\]

We show that \( \beta \) lies in \( -C(A)^p \).

Put \( c_i = \sum_{k=1}^m c_{ik} \). We begin with the relation

\[
(12.4) \quad \sum_{i=1}^n c_i a_i = \sum_{k=1}^m a_{n+k}.
\]

We rewrite this as

\[
(12.5) \quad \sum_{i: c_i > 0} c_i a_i = \sum_{i: c_i < 0} (-c_i) a_i + \sum_{k=1}^m a_{n+k}.
\]

Let \( \sigma \) be the smallest closed face of \( C(A) \) containing \( A_1 := \{ a_i \mid c_i > 0 \} \). All coefficients on both sides of (12.5) are positive, so the face \( \sigma \) also contains the set \( A_2 := \{ a_i \mid c_i < 0 \} \) and the set \( A_3 := \{ a_{n+k} \mid k = 1, \ldots, m \} \). Our hypothesis implies that if \( \theta_i = 0 \) then \( c_i < 0 \), so all \( a_i \) with \( \theta_i = 0 \) lie in \( A_2 \) and hence lie on \( \sigma \). The only elements of \( A \) that may not lie on \( \sigma \) are those in the set \( A_4 := \{ a_i \mid \theta_i > 0 \) and \( c_i = 0 \} \), so \( \sigma \cup A_4 \) contains the set \( A \). This implies that

\[
\sum_{a_i \in A_1} a_i + \sum_{a_i \in A_4} a_i \in C(A)^p.
\]

Since \( A_1 \) and \( A_4 \) are subsets of \( \{ a_i \mid \theta_i > 0 \} \), it follows that \( \sum_{\theta_i > 0} \theta_i a_i \in C(A)^p \).

Equivalently, \( \beta \) is an interior point of \( -C(A) \).

From the definition of the set \( E \) we have

\[
(12.6) \quad E = \{ l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \mid l_j \leq 0 \text{ if } \theta_j = 1, \quad \quad l_j \geq 0 \text{ if } \theta_j = 0 \text{ or if } j = n + 1, \ldots, n + m \}.
\]

For \( u \in \beta + \mathbb{Z}A \) we have

\[
(12.7) \quad E(u) = \left\{ l = (l_1, \ldots, l_N) \in E \mid \sum_{j=1}^N (v_j + l_j) a_j = u \right\}
\]
and
\[ F_u(\Lambda_1, \ldots, \Lambda_N) = \sum_{l \in E(u)} [v] l^{\nu+l}. \]

We take \(u = \beta\) in these formulas to verify that the series \(F_\beta(\Lambda)\) has the same coefficients (up to sign) as the series (12.2). From (12.7) we have
\[ E(\beta) = \left\{ l = (l_1, \ldots, l_N) \in E \mid \sum_{j=1}^N l_j a_j = 0 \right\}. \]

We need to solve
\[ 0 = \sum_{j=1}^N l_j a_j = (l_1 + C_1(l_{n+1}, \ldots, l_{n+m}), \ldots, l_n + C_n(l_{n+1}, \ldots, l_{n+m})) \]
with \(l \in E\). From (12.9) we must have for \(i = 1, \ldots, n\)
\[ l_i = -C_i(l_{n+1}, \ldots, l_{n+m}). \]

From (12.6) we must have \(l_{n+k} \geq 0\) for \(k = 1, \ldots, m\). If \(\theta_i = 1\), then \(C_i\) has nonnegative coefficients, so the \(l_i\) given by (12.10) is nonpositive. If \(\theta_i = 0\), then \(C_i\) has nonpositive coefficients, so the \(l_i\) given by (12.10) is nonnegative. It follows that
\[ E(\beta) = \left\{ (-C_1(s_1, \ldots, s_m), \ldots, -C_n(s_1, \ldots, s_m), s_1, \ldots, s_m) \mid s_1, \ldots, s_m \in \mathbb{Z}_{\geq 0} \right\}. \]

Equation (12.8) with \(u = \beta\) gives
\[ F_\beta(\Lambda) = \sum_{s_1, \ldots, s_m = 0}^{\infty} [v] s \Lambda^{\beta+s} \]
\[ = \sum_{s_1, \ldots, s_m = 0}^{\infty} \prod_{i=1}^n [-\theta_i]^{s_i} C_i(s_1, \ldots, s_m) \prod_{k=1}^m [0]_{s_k} \Lambda^{\beta+s} \]
\[ = \Lambda^\beta \sum_{s_1, \ldots, s_m = 0}^{\infty} \prod_{i=1}^n (-1)^{s_i} C_i(s_1, \ldots, s_m) \prod_{i=1}^n \Lambda_{i}^{-s_i} \prod_{k=1}^m \Lambda_{n+k}^{s_k} s_1! \ldots s_m!, \]
where the last equality follows from (1.3). Thus the coefficients of \(F_\beta(\Lambda)\) equal (up to sign) the corresponding coefficients of (12.2).

We can thus apply Theorem 2.21 to this \(A\)-hypergeometric system to get an integrality condition for the classical hypergeometric series (12.2) and all the related series \(F_u(\Lambda)\) with \(u \in \mathcal{M}\). Fix a positive integer \(D\) such that \(D\theta_j \in \mathbb{Z}\) for all \(j\) and fix a positive integer \(h\) with \((h, D) = 1\). In Section 2, we used \(h\) to define a map \(v_j \rightarrow v_j'\) for \(-1 \leq v_j \leq 0\). We denoted the \(i\)-fold iteration of this map by \(v_j \rightarrow v_j^{(i)}\), and, if \(h^a \equiv 1 \pmod{D}\), then \(v_j^{(a)} = v_j\). We define \(\theta_j^{(i)}\) by the formula
\[ \theta_j^{(i)} = -(-\theta_j)^{(i)}. \]
The \(\theta_j^{(i)}\) lie in the interval \([0, 1]\). We set \(\Theta^{(i)} = (\theta_1^{(i)}, \ldots, \theta_n^{(i)})\) and
\[ \nu^{(i)} = (-\theta_1^{(i)}, \ldots, -\theta_n^{(i)}, 0, \ldots, 0), \]
where 0 is repeated \( m \) times.

We have \( \beta^{(i)} = \sum_{j=1}^{N} v_j^{(i)} a_j \) and \( M^{(i)} = (\beta^{(i)} + ZA) \cap (-C(A)^p) \). For \( u^{(i)} \in M^{(i)} \) we have

\[
F_{u^{(i)}}(A_1, \ldots, A_N) = \sum_{l \in E(u^{(i)})} [v^{(i)}]_l A^{u^{(i)}+1}.
\]

Note that

\[
w(\beta^{(i)}) = -\sum_{j=1}^{n} \theta_j^{(i)}.
\]

Define a step function on \( \mathbb{R}^m \)

\[
\rho(\Theta; x_1, \ldots, x_m) = \sum_{j=1}^{n} [1 - \theta_j + C_j(x_1, \ldots, x_m)].
\]

**Theorem 12.15.** Suppose that for each \( i = 0, 1, \ldots, a - 1 \) we have

\[
\rho(\Theta^{(i)}; x_1, \ldots, x_m) \geq 0
\]

for all \( x_1, \ldots, x_m \in [0, 1) \). Then for \( i = 0, 1, \ldots, a - 1 \) and all \( u^{(i)} \in M^{(i)} \) the series

\[
F_{u^{(i)}}(\Lambda) \text{ has } p \text{-integral coefficients for all primes } p \equiv h \pmod{D}.
\]

**Remark.** The special case of Theorem 12.15 where each \( C_i \) has either all coefficients nonnegative or all coefficients nonpositive and where \( u^{(i)} = \beta^{(i)} \) is [4, Theorem 5.6]. Theorem 12.15 extends that result to all \( u^{(i)} \) in all \( M^{(i)} \) and allows each \( C_i \), when \( \theta_i \neq 0, 1 \), to have both positive and negative coefficients. The special case of Theorem 12.15 where each \( \theta_i \) equals either 0 or 1 and where each \( C_i \) has either all coefficients nonnegative or all coefficients nonpositive is [3, Theorem 2.6].

Theorem 12.15 is an immediate consequence of Theorem 2.21 and the following result.

**Lemma 12.16.** Let \( i \in \{0, 1, \ldots, a - 1\} \). Equation (2.20) holds if and only if

\[
\rho(\Theta^{(i)}; x_1, \ldots, x_m) \geq 0
\]

for all \( x_1, \ldots, x_m \in [0, 1) \).

**Proof.** It suffices to prove this in the case \( i = 0 \) as the other cases are similar. Since \( w(\beta) = -\sum_{j=1}^{n} \theta_j \), We need to show that for all \( u \in M \) we have

\[
w(u) \leq -\sum_{i=1}^{n} \theta_i
\]

if and only if

\[
\sum_{i=1}^{n} [1 - \theta_i + C_i(x_1, \ldots, x_m)] \geq 0
\]

for all \( x_1, \ldots, x_m \in [0, 1) \).

Fix \( \hat{u} \in M \) such that \( w(\hat{u}) = \max\{w(u) \mid u \in M\} \) and choose \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_n) \in \mathbb{Z}^n \) such that \( \hat{u} = \beta + \hat{u} \). Since \( w(\hat{u}) = w(\beta) + w(\hat{u}) = -\sum_{i=1}^{n} \theta_i + \sum_{i=1}^{n} \hat{u}_i \), we need to show that

\[
\sum_{i=1}^{n} \hat{u}_i \leq 0
\]
if and only if (12.18) holds for all \(x_1, \ldots, x_m \in [0, 1)\).

Since \(\hat{u}\) is an interior point of \(-C(A)\), by [2, Lemma 1] we may write

\[
(12.20) \quad \hat{u} = \sum_{i=1}^{N} z_i a_i
\]

with \(z_i \leq 0\) for all \(i\) and \(z_i < 0\) for \(i = 1, \ldots, n\). Note that since the coordinates of each \(a_i\) sum to 1, we have

\[
(12.21) \quad w(\hat{u}) = \sum_{i=1}^{N} z_i.
\]

We must have \(z_i \geq -1\) for all \(i\). For if some \(z_{i_0} < -1\), then

\[
(12.22) \quad \hat{u} + a_{i_0} = (z_{i_0} + 1)a_{i_0} + \sum_{i=1 \atop i \neq i_0}^{N} z_i a_i
\]

is an element of \(M\) since all coefficients on the right-hand side are less than or equal to 0 and every \(a_i\) that occurs with a negative coefficient in (12.20) also occurs with a negative coefficient in (12.22). But \(w(\hat{u} + a_{i_0}) > w(\hat{u})\), contradicting the choice of \(\hat{u}\).

We claim that \(z_i > -1\) for \(i = n + 1, \ldots, N\). If \(z_{i_0} = -1\) for some \(i_0 \in \{n + 1, \ldots, N\}\), then (12.22) becomes

\[
\hat{u} + a_{i_0} = \sum_{i=1 \atop i \neq i_0}^{N} z_i a_i
\]

But since \(z_i < 0\) for \(i = 1, \ldots, n\), the point \(\hat{u} + a_{i_0}\) is an element of \(M\), and again \(w(\hat{u} + a_{i_0}) > w(\hat{u})\), contradicting the choice of \(\hat{u}\).

In summary, we have proved that in the representation (12.20) one has

\[
(12.23) \quad z_i \in [-1, 0) \quad \text{for} \quad i = 1, \ldots, n,
\]

\[
(12.24) \quad z_i \in (-1, 0] \quad \text{for} \quad i = n + 1, \ldots, N.
\]

We now examine (12.20) coordinatewise. For \(i = 1, \ldots, n\) we have

\[
(12.25) \quad \hat{u}_i = -\theta_i + \tilde{u}_i = z_i + C_i(z_{n+1}, \ldots, z_{n+m}).
\]

This equation shows that

\[
 z_i \equiv -\theta_i - C_i(z_{n+1}, \ldots, z_{n+m}) \pmod{Z},
\]

and since \(-1 \leq z_i < 0\) this implies that

\[
(12.26) \quad z_i = -1 - \theta_i - C_i(z_{n+1}, \ldots, z_{n+m}) - [\theta_i - C_i(z_{n+1}, \ldots, z_{n+m})].
\]

This implies by (12.25) that

\[
(12.27) \quad \tilde{u}_i = -1 - [\theta_i - C_i(z_{n+1}, \ldots, z_{n+m})].
\]

It follows that

\[
(12.28) \quad \sum_{i=1}^{n} \tilde{u}_i = -\sum_{i=1}^{n} [1 - \theta_i - C_i(z_{n+1}, \ldots, z_{n+m})] = -\sum_{i=1}^{n} [1 - \theta_i + C_i(-z_{n+1}, \ldots, -z_{n+m})].
\]
It is clear from this equation that if (12.18) holds for all \(x_1, \ldots, x_m \in [0, 1]\) then (12.19) holds.

Conversely, suppose there exist \(x_1, \ldots, x_m \in [0, 1]\) such that

\[
(12.29) \quad \sum_{i=1}^{n} [1 - \theta_i + C_i(x_1, \ldots, x_m)] < 0.
\]

For \(i = 1, \ldots, m\), define \(z_{n+i} = -x_i\), so \(-1 < z_{n+i} \leq 0\). Then for \(i = 1, \ldots, n\) define \(z_i\) by (12.26) and define \(\tilde{u}_i\) by (12.27). Then \(-1 \leq z_i < 0\) for \(i = 1, \ldots, n\), so (12.20) shows that \(\tilde{u}\) is an interior point of \(-C(A)\). Equation (12.28) holds by the definition of the \(\tilde{u}_i\). Equations (12.28) and (12.29) then imply that \(\sum_{i=1}^{n} \tilde{u}_i > 0\). This shows that the failure of (12.18) implies the failure of (12.19), which in turn implies that \(w(\tilde{u}) > w(\beta)\), i.e., Equation (2.3) fails. □

As an application of Theorem 12.15, consider the two-variable Horn series

\[
(12.30) \quad G_1(\theta_1, \theta_2, \theta_3; t_1, t_2) = \sum_{s_1, s_2=0}^{\infty} (\theta_1)_{s_1} (\theta_2)_{s_2} (\theta_3)_{s_1-s_2} \frac{t_1^{s_1} t_2^{s_2}}{s_1! s_2!}
\]

with \(0 < \theta_1, \theta_2, \theta_3 < 1\). As above, we let \(a_i\) for \(i = 1, 2, 3\) be the standard unit basis vectors for \(\mathbb{R}^3\) and we take \(a_1 = (1, -1, 1)\) (the coefficients of \(s_1\) in the subscripts for the \(\theta_i\)) and \(a_3 = (1, 1, -1)\) (the coefficients of \(s_2\) in the subscripts for the \(\theta_i\)). We have \(C_1(x_1, x_2) = x_1 + x_2\), \(C_2(x_1, x_2) = x_2 - x_1\), and \(C_3(x_1, x_2) = x_1 - x_2\). To apply Theorem 12.15, we need to consider expressions of the form

\[
[1 - \theta_1 + x_1 + x_2] + [1 - \theta_2 + x_2 - x_1] + [1 - \theta_3 + x_1 - x_2]
\]

and determine when they are nonnegative for all \(x_1, x_2 \in [0, 1]\).

**Lemma 12.31.** We have

\[
(12.32) \quad [1 - \theta_1 + x_1 + x_2] + [1 - \theta_2 + x_2 - x_1] + [1 - \theta_3 + x_1 - x_2] \geq 0
\]

for all \(x_1, x_2 \in [0, 1]\) if and only if either \(\theta_2 + \theta_3 \leq 1\) or both \(\theta_1 + \theta_2 \leq 1\) and \(\theta_1 + \theta_3 \leq 1\).

**Proof.** On the left-hand side of (12.32), the first term is always nonnegative and the last two terms are greater than or equal to \(-1\). Since

\[
(1 - \theta_2 + x_2 - x_1) + (1 - \theta_3 + x_1 - x_2) = 2 - \theta_2 - \theta_3 \geq 0,
\]

at most one of the last two terms on the left-hand side of (12.32) can be negative. Thus the only way the left-hand side of (12.32) can be negative is if either the first two terms equal 0 and the third term equals \(-1\) or the first and third terms equal 0 and the second term equal \(-1\).

If \(\theta_2 + \theta_3 \leq 1\), then

\[
(1 - \theta_2 + x_2 - x_1) + (1 - \theta_3 + x_1 - x_2) \geq 1,
\]

which implies that the sum of the last two terms on the left-hand side of (12.32) is nonnegative, hence (12.32) holds for all \(x_1, x_2 \in [0, 1]\).

Suppose that \([1 - \theta_2 + x_2 - x_1] = -1\) and \([1 - \theta_3 + x_1 - x_2] \geq 0\). If \(\theta_1 + \theta_2 \leq 1\) then

\[
(1 - \theta_1 + x_1 + x_2) + (1 - \theta_2 + x_2 - x_1) = 2 - \theta_1 - \theta_2 + 2x_2 \geq 1,
\]

which implies that the sum of the first two terms on the left-hand side of (12.32) is nonnegative. A similar argument applies to the case \([1 - \theta_3 + x_1 - x_2] = -1\) and \([1 - \theta_2 + x_2 - x_1] = 0\) if \(\theta_1 + \theta_3 \leq 1\).
To prove the converse, suppose that \( \theta_2 + \theta_3 > 1 \) and \( \theta_1 + \theta_2 > 1 \). Then
\[
(12.33) \quad \theta_1 > 1 - \theta_2 \text{ and } \theta_3 > 1 - \theta_2.
\]
Take \( x_2 = 0 \) and choose \( x_1 \) so that \( \theta_1, \theta_3 > x_1 > 1 - \theta_2 \). Then the first and third terms on the left-hand side of (12.32) equal 0 while the second term on the left-hand side of (12.32) equals -1. A similar argument applies when \( \theta_2 + \theta_3 > 1 \) and \( \theta_1 + \theta_2 > 1 \).

From Lemma 12.31 and Theorem 12.15 we get the following.

**Corollary 12.34.** Suppose for each \( i = 0, 1, \ldots, a - 1 \) we have either \( \theta_1^{(i)} + \theta_3^{(i)} \leq 1 \) or both \( \theta_1^{(i)} + \theta_2^{(i)} \leq 1 \) and \( \theta_1^{(i)} + \theta_3^{(i)} \leq 1 \). Then the series \( G_1(\theta_1^{(i)}, \theta_2^{(i)}, \theta_3^{(i)}; t_1, t_2) \) have \( p \)-integral coefficients for all primes \( p \equiv h \pmod{D} \).

By Equation (2.12) the series \( F_{\beta^{(i)}}(\Lambda) \), with \( \beta = (-\theta_1, -\theta_2, -\theta_3) \) and \( \{a_i\}_{i=1}^5 \) as in this example, have the same coefficients (up to sign) as the \( G_1(\theta_1^{(i)}, \theta_2^{(i)}, \theta_3^{(i)}; t_1, t_2) \).

By Theorem 12.15, the hypothesis of Corollary 12.34 implies \( p \)-integrality for not just one series but for all the series \( F_{u^{(i)}}(\Lambda) \) with \( u^{(i)} \in M^{(i)} \). We describe these series. To compute the series \( F_{u^{(i)}}(\Lambda) \) given by (12.8) we need to find the set \( E(u^{(i)}) \) of (12.9). For \( u^{(i)} \in \beta^{(i)} + Z\Lambda \) write
\[
u^{(i)} = \beta^{(i)} + \tilde{u}^{(i)}
\]
with \( \tilde{u}^{(i)} = (\tilde{u}_1^{(i)}, \tilde{u}_2^{(i)}, \tilde{u}_3^{(i)}) \in Z^3 \). To find \( E(u^{(i)}) \) we need to solve the equation
\[
(12.35) \quad \sum_{k=1}^{3} (-\theta_k^{(i)} + l_k) a_i + l_4 a_4 + l_5 a_5 = (-\theta_1^{(i)} + \tilde{u}_1^{(i)}, -\theta_2^{(i)} + \tilde{u}_2^{(i)}, -\theta_3^{(i)} + \tilde{u}_3^{(i)}),
\]
subject to the condition that \( (l_1, \ldots, l_5) \in E \), i.e., \( l_k \in Z \) for \( k = 1, 2, 3 \) and \( l_4, l_5 \in Z_{\geq 0} \) for \( k = 4, 5 \).

Equation (12.35) simplifies to the system
\[
l_1 + l_4 + l_5 = \tilde{u}_1^{(i)}, \quad l_2 - l_4 + l_5 = \tilde{u}_2^{(i)}, \quad l_3 + l_4 - l_5 = \tilde{u}_3^{(i)}.
\]
The solution in \( E \) is to take \( l_4, l_5 \in Z_{\geq 0} \) arbitrary, then \( l_1 = -l_4 - l_5 + \tilde{u}_1^{(i)} \), \( l_2 = l_4 - l_5 + \tilde{u}_2^{(i)} \), and \( l_3 = -l_4 + l_5 + \tilde{u}_3^{(i)} \). Substituting these values into (12.8) and using the relation (1.3) shows that the series
\[
(12.36) \quad \sum_{s_1, s_2 \geq 0} \left( \theta_1^{(i)} \right)^{s_1} \left( \theta_2^{(i)} \right)^{s_2} \left( \theta_3^{(i)} \right)^{s_1 - s_2 - \tilde{u}_2^{(i)}} \left( \theta_2^{(i)} \right)^{s_2 - s_1 - \tilde{u}_2^{(i)}} \left( \theta_3^{(i)} \right)^{s_1 - s_2 - \tilde{u}_2^{(i)}} \frac{t_1^{s_1+t_2^{s_2}}}{s_1!s_2!}
\]
has \( p \)-integral coefficients for all primes \( p \equiv h \pmod{D} \) when \( u^{(i)} \in M^{(i)} \) and the hypothesis of Corollary 12.34 is satisfied.

Choosing \( u = \beta \) gives \( \tilde{u} = 0 \) and (12.36) becomes the series \( G_1(\theta_1^{(i)}, \theta_2^{(i)}, \theta_3^{(i)}; x_1, x_2) \). We give another example of \( u \in M \). The faces of the cone \( C(A) \) lie in the planes \( x_1 = 0, x_2 + x_3 = 0, x_1 + x_2 = 0, \) and \( x_1 + x_3 = 0 \). The interior points of \( -C(A) \) are the points where these four linear forms take on negative values. Suppose that \( \theta_2^{(i)} + \theta_3^{(i)} \leq 1 \) for \( i = 0, 1, \ldots, a - 1 \), so the hypothesis of Corollary 12.34 is satisfied, but that \( \theta_1 + \theta_2 > 1 \). Then \( u = (-\theta_1, -\theta_2+1, -\theta_3-1) \) is an interior point of \( -C(A) \),
so the series corresponding to $u$ also has $p$-integral coefficients. Thus $\tilde{u} = (0, 1, -1)$ and substitution into (12.36) gives
\[ \sum_{s_1, s_2 = 0}^{\infty} (\theta_1)_{s_1 + s_2} (\theta_2)_{s_2 - s_1 - 1} (\theta_3)_{s_1 - s_2 + 1} \frac{t_1^{s_1} t_2^{s_2}}{s_1! s_2!}, \]
a series with $p$-integral coefficients for all primes $p \equiv h \pmod{D}$.

References

[1] A. Adolphson and S. Sperber. Distinguished-root formulas for generalized Calabi-Yau hypersurfaces. Algebra Number Theory 11 (2017), no. 6, 1317–1356.
[2] A. Adolphson and S. Sperber. On the integrality of factorial ratios and mirror maps. Integers 20 (2020), art. A10, 15pp.
[3] A. Adolphson and S. Sperber. On the integrality of hypergeometric series whose coefficients are factorial ratios. Acta Arith. 200 (2021), no. 1, 39–59.
[4] A. Adolphson and S. Sperber. On integrality properties of hypergeometric series. Funct. Approx. Comment. Math. 65 (2021), no. 1, 7–31.
[5] A. Adolphson and S. Sperber. $A$-hypergeometric series and a $p$-adic refinement of the Hasse-Witt matrix. Abh. Math. Semin. Univ. Hambg. 91 (2021), no. 2, 225–256.
[6] M. Boyarsky. $p$-adic gamma functions and Dwork cohomology. Trans. Amer. Math. Soc. 257 (1980), no. 2, 359–369.
[7] B. Dwork. $p$-adic cycles. Inst. Hautes Études Sci. Publ. Math. No. 37, (1969), 27–115.
[8] B. Dwork and F. Loeser. Hypergeometric series. Japan. J. Math. (N. S.) 19 (1993), no. 1, 81–129.