Elliptic solutions of the semi-discrete BKP equation

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Abstract

We consider elliptic solutions of the semi-discrete BKP equation and derive equations of motion for their poles. The basic tool is the auxiliary linear problem for the wave function.

1 Introduction

The dynamics of poles of singular solutions to nonlinear integrable equations is a well known subject in mathematical physics [1] [2] [3] [4]. In particular, it was shown that the poles of singular solutions to the Kadomtsev-Petviashvili (KP) equation move as particles of the integrable Calogero-Moser many-body system [5] [6] [7] [8]. Rational, trigonometric and elliptic (doubly periodic in the complex plane) solutions correspond respectively to rational, trigonometric or elliptic Calogero-Moser systems. In the most general elliptic case, the equations of motion are

$$\ddot{x}_i = 4 \sum_{k \neq i} \wp'(x_i - x_k)$$

(1)

(\wp is the Weierstrass \wp-function).

The method suggested by Krichever consists in substituting the pole ansatz not in the KP equation but in the auxiliary linear problem for it, which has the form of the non-stationary Schrödinger equation for the wave function \(\psi\):

$$\partial_t \psi = \partial_x^2 \psi + 2U \psi, \quad U = - \sum_i \wp(x - x_i) + \text{const.}$$

(2)

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Elliptic solutions to the semi-discrete KP equation (and, more generally, to the matrix 2D Toda equation) were investigated in \[9\]. The corresponding linear problem has the form of the differential-difference equation
\[
\partial_t \psi(x) = \psi(x + \eta) + u(x) \psi(x).
\] (3)

In this case the dynamics of poles is given by equations of motion of the integrable elliptic Ruijsenaars-Schneider system \([10]\) (a relativistic version of the Calogero-Moser system):
\[
\ddot{x}_i = \sum_{j \neq i} \frac{\varphi_j'(x_i - x_j)}{\varphi(\eta) - \varphi(x_i - x_j)}. \tag{4}
\]

Elliptic solutions to the B-version of the KP equation (BKP) \([11, 12, 13, 14, 15]\) were recently studied in the paper \([16]\), see also \([17]\). It was shown that their poles move as particles of a previously unknown many-body system with equations of motion
\[
\ddot{x}_i + 6 \sum_{j \neq i} (\dot{x}_i + \dot{x}_j) \varphi_j'(x_i - x_j) - 72 \sum_{j \neq k \neq i} \varphi(x_i - x_j) \varphi_j'(x_i - x_k) = 0. \tag{5}
\]

The Hamiltonian structure and integrability of this system are to be further investigated.

In this paper we study elliptic solutions of the semi-discrete BKP equation (with a discrete space variable and continuous time variable). We derive the equations of motion for the dynamics of the poles (zeros of the tau-function). Just as equations (5) are in some sense \(B\)-version of the Calogero-Moser system, so the equations of motion in the semi-discrete case are a \(B\)-version of the Ruijsenaars-Schneider system. We also derive the commutation representation for the equations of motion which is a sort of the Manakov’s triple representation \([18]\). Like in the KP case, the main tool is the auxiliary linear problem for the wave function.

\section{The discrete BKP equation}

We begin with the continuous BKP hierarchy. Let \(t = \{t_1, t_3, t_5, \ldots\}\) be an infinite set of continuous “times”, the independent variables of the hierarchy. The dependent variable is the tau-function \(\tau(t)\). The infinite BKP hierarchy is encoded in the basic bilinear relation for the tau-function \([12]\)
\[
\oint_{C_\infty} \frac{dz}{2\pi i z} e^{\xi(t,z) - \xi(t',z)} \tau(t - 2[z^{-1}]) \tau(t' + 2[z^{-1}]) = \tau(t)\tau(t') \tag{6}
\]
valid for any \(t, t'\). Here we use the notation
\[
\xi(t, z) = \sum_{k \geq 1, k \text{ odd}} t_k z^k, \quad t \pm 2[z^{-1}] = \left\{t_1 \pm \frac{2}{z}, t_3 \pm \frac{2}{z^3}, t_5 \pm \frac{2}{z^5}, \ldots\right\}.
\]
The contour \(C_\infty\) is a big circle around infinity.

The discrete BKP hierarchy is a subhierarchy of the continuous one. The simplest discrete BKP equation is obtained as follows. Put
\[
\tau(l, m, n) = \tau(t - 2l[a^{-1}] - 2m[b^{-1}] - 2n[c^{-1}]), \tag{7}
\]
then setting $t'_k = t_k - 2a^{-k}/k - 2b^{-k}/k - 2c^{-k}/k$ in the bilinear relation (6) one can calculate the integral using the residue calculus (it should be taken into account that the poles at the points $a, b, c$ are outside the contour and the contour should be shrunk to infinity). The result is that $\tau(l, m, n)$ satisfies the discrete BKP equation \[19\]

\[
(a + b)(a + c)(b - c)\tau(l + 1)\tau(m + 1, n + 1) \\
- (a + b)(b + c)(a - c)\tau(m + 1)\tau(l + 1, n + 1) \\
+ (a + c)(b + c)(a - b)\tau(n + 1)\tau(l + 1, m + 1) \\
= (a - b)(a - c)(b - c)\tau\tau(l + 1, m + 1, n + 1).
\]

Here we explicitly write only those arguments that undergo shifts. Taking the limit $c \to \infty$, we get the semi-discrete BKP equation

\[
\tau\tau(l + 1, m + 1) \left(1 + \frac{1}{a + b} \partial_{t_1} \log \frac{\tau(l + 1, m + 1)}{\tau}\right) \\
= \tau(l + 1)\tau(m + 1) \left(1 + \frac{1}{a - b} \partial_{t_1} \log \frac{\tau(l + 1)}{\tau(m + 1)}\right).
\]

The wave function $\psi(l, m; z)$ is introduced by the formula

\[
\psi(l, m; z) = e^{\xi(t, z)} \left(1 + \frac{1}{a - b} \partial_{t_1} \log \frac{\tau(l + 1)}{\tau^t(m - 1)}\right).
\]

It follows from (8) that the wave function $\psi = \psi(l, m; z)$ satisfies the following linear equation:

\[
\psi(m + 1) - \psi(l + 1) = \frac{a - b}{a + b} u(\psi(l + 1, m + 1) - \psi),
\]

where

\[
u = \frac{\tau\tau(l + 1, m + 1)}{\tau(l + 1)\tau(m + 1)}.
\]

Tending $b \to \infty$, one obtains from (11) the linear problem

\[
\partial_{t_1} \left(\psi(l) + \psi(l + 1)\right) = (v(l) + a)\left(\psi(l) - \psi(l + 1)\right),
\]

where

\[
v(l) = \partial_{t_1} \log \frac{\tau(l + 1)}{\tau(l)}.
\]

3 Elliptic solutions

Our aim is to study double-periodic (elliptic) in the variable $x = l\eta$ solutions of the semi-discrete BKP equation. Here $\eta$ is a parameter (a “lattice spacing”). For such solutions the tau-function is an “elliptic polynomial” in the variable $x$:

\[
\tau = Ce^{Cx^2 + Bx t_1} \prod_{i=1}^{N} \sigma(x - x_i)
\]
with some constants $C, B$, where
\[ \sigma(x) = \sigma(x|\omega, \omega') = x \prod_{s \neq 0} \left(1 - \frac{x}{s} \right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2\omega_1 m_1 + 2\omega_2 m_2 \quad \text{with integer } m_1, m_2, \]
is the Weierstrass $\sigma$-function with quasi-periods $2\omega_1, 2\omega_2$ such that $\text{Im}(\omega_2/\omega_1) > 0$. It is connected with the Weierstrass $\zeta$- and $\wp$-functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. The roots $x_i = x_i(t_1)$ are assumed to be all distinct.

We put $t_1 = t$ and write the linear problem (13) in the form
\[ \partial_t \left( \psi(x) + \psi(x + \eta) \right) = (v(x) + \mu) \left( \psi(x) - \psi(x + \eta) \right), \quad v(x) = \partial_t \log \frac{\tau(x + \eta)}{\tau(x)}, \quad (16) \]
where $\mu$ is a parameter (the former $a$ in (13)). For elliptic solutions
\[ v(x) = B\eta + \sum_i \left( \dot{x}_i \zeta(x-x_i) - \dot{x}_i \zeta(x+\eta-x_i) \right), \quad (17) \]
where dot means the $t$-derivative. It is an elliptic function of $x$. Since the coefficient function $v$ is double-periodic, one can find double-Bloch solutions $\psi(x)$, i.e., solutions such that $\psi(x + 2\omega_\alpha) = B_\alpha \psi(x)$ ($\alpha = 1, 2$) with some Bloch multipliers $B_\alpha$.

The pole ansatz for the wave function is
\[ \psi = e^{tz} \left( \frac{\mu - z}{\mu + z} \right)^{x/\eta} \sum_{i=1}^N c_i \Phi(x - x_i, \lambda), \quad (18) \]
where the coefficients $c_i$ do not depend on $x$ (but do depend on $z$ and $t$). Here the function $\Phi$ is defined as
\[ \Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda) \sigma(x)} e^{-\zeta(\lambda)x} \]
($\zeta$ is the Weierstrass $\zeta$-function). It has a simple pole at $x = 0$ with residue 1:
\[ \Phi(x, \lambda) = \frac{1}{x} - \frac{1}{2} \varphi(\lambda) x + \ldots, \quad x \to 0. \]
The parameters $z$ and $\lambda$ are spectral parameters. Using the quasiperiodicity properties of the function $\Phi$,
\[ \Phi(x + 2\omega_\alpha, \lambda) = e^{2(\zeta(\omega_\alpha) x - \zeta(\lambda) \omega_\alpha)} \Phi(x, \lambda), \]
one can see that the wave function given by (18) is indeed a double-Bloch function with Bloch multipliers $B_\alpha = e^{2(\zeta(\omega_\alpha) x - \zeta(\lambda) \omega_\alpha)}$. We will often suppress the second argument of $\Phi$ writing simply $\Phi(x) = \Phi(x, \lambda)$. We will also need the $x$-derivative $\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)$.

4  Equations of motion for poles of elliptic solutions

Let us substitute (17) and (18) into the linear problem (16). The expression has obvious poles at $x = x_i$ and $x = x_i - \eta$. One should impose conditions on the coefficients $c_i$
which ensure cancellation of the poles. The second order poles cancel identically. From cancellation of the first order poles at \( x = x_i - \eta \) and \( x = x_i \) we obtain the conditions

\[
\begin{cases}
(z + \mu + B\eta)c_i + \dot{c}_i &= \frac{z + \mu}{z - \mu} \sum_k c_k \Phi(x_i - x_k - \eta) + \dot{x}_i \sum_{k \neq i} c_k \Phi(x_i - x_k) \\
&- c_i \sum_k \dot{x}_k \zeta(x_i - x_k - \eta) + c_i \sum_{k \neq i} \dot{x}_k \zeta(x_i - x_k) \\
(z - \mu - B\eta)c_i + \dot{c}_i &= \frac{z - \mu}{z + \mu} \sum_k c_k \Phi(x_i - x_k + \eta) + \dot{x}_i \sum_{k \neq i} c_k \Phi(x_i - x_k) \\
&- c_i \sum_k \dot{x}_k \zeta(x_i - x_k + \eta) + c_i \sum_{k \neq i} \dot{x}_k \zeta(x_i - x_k)
\end{cases}
\]  
(19)

which should be valid for all \( i = 1, \ldots, N \).

Let us introduce the \( N \times N \) matrices \( A^\pm, A^0 \) with matrix elements

\[ A_{ik}^\pm = \Phi(x_i - x_k \pm \eta), \quad A_{ik}^0 = (1 - \delta_{ik}) \Phi(x_i - x_k) \]

and diagonal matrices \( X, D^\pm, D^0 \) with matrix elements \( X_{ik} = \delta_{ik} x_i, \)

\[ D_{ik}^\pm = \delta_{ik} \sum_j x_j \zeta(x_i - x_j \pm \eta), \quad D_{ik}^0 = \delta_{ik} \sum_{j \neq i} x_j \zeta(x_i - x_j), \]

then the conditions (19) can be written in the matrix form as a system of linear equations for the column vector \( c = (c_1, \ldots, c_N)^T \):

\[
\begin{cases}
Lc = 2\tilde{\mu}(z^2 - \mu^2)c \\
\dot{c} = M c,
\end{cases}
\]  
(20)

where \( \tilde{\mu} = \mu + B\eta \) and the matrices \( L, M \) have the form

\[ L = (z + \mu)^2 \dot{X} A^{-} - (z - \mu)^2 \dot{X} A^{+} + (z^2 - \mu^2)(D^{+} - D^{-}), \]  
(21)

\[ M = -(z - \tilde{\mu})I + \frac{z - \mu}{z + \mu} \dot{X} A^{+} + \dot{X} A^{0} - D^{+} + D^{0} \]  
(22)

(here \( I \) is the unity matrix).

The system (20) is overdetermined. As a simple calculation shows, the compatibility condition is

\[ (\tilde{L} + [L, M])c = 0. \]  
(23)

We have:

\[
\tilde{L} + [L, M] = (z + \mu)^2 \left( \dot{X} A^{-} + \dot{X} A^{+} + [\dot{X} A^{-}, \dot{X} A^{0} - D^{+} + D^{0}] \right) \\
- (z - \mu)^2 \left( \dot{X} A^{+} + \dot{X} A^{-} + [\dot{X} A^{+}, \dot{X} A^{0} - D^{-} + D^{0}] \right) \\
+ (z^2 - \mu^2) \left( \dot{D}^{+} - \dot{D}^{-} - [\dot{X} A^{+}, \dot{X} A^{-}] + [D^{+} - D^{-}, \dot{X} A^{0}] \right).
\]
With the help of the identities
\[ \Phi(x, \lambda)\Phi(y, \lambda) = \Phi(x + y, \lambda)(\zeta(x) + \zeta(y) - \zeta(x + y + \lambda) + \zeta(\lambda)), \] (24)
\[ \Phi(x, \lambda)\Phi(-x, \lambda) = \varphi(\lambda) - \varphi(x) \] (25)
one can see, by a straightforward calculation, that
\[ \dot{X} \dot{A}^\pm + [\dot{X} A^\pm, \dot{X} A^0 - D^\mp + D^0] = (D^+ + D^- - 2D^0)\dot{X} A^\pm, \]
\[ \dot{D}^+ - \dot{D}^- - [\dot{X} A^+, \dot{X} A^-] + [D^+ - D^-, \dot{X} A^0] = W^+ - W^-, \]
where \( W^\pm \) are diagonal matrices with matrix elements
\[ W^\pm_{ii} = \sum_j \ddot{x}_j \zeta(x_i - x_j \pm \eta) + \sum_j \dddot{x}_j \varphi(x_i - x_j \pm \eta). \]
Therefore, we have the matrix identity
\[ \dot{L} + [L, M] = R\left(L - 2\tilde{\mu}(z^2 - \mu^2)I\right) + (z^2 - \mu^2)P, \] (26)
where \( R, P \) are the diagonal matrices
\[ R = \dot{X} \dot{X}^{-1} + D^+ + D^- - 2D^0, \] (27)
\[ P = W^+ - W^- - R(D^+ - D^- - 2\tilde{\mu}I). \] (28)
We see that the compatibility condition \( [23] \) means that all elements of the diagonal matrix \( P \) should be equal to zero \( (P_{ii} = 0 \) for all \( i \)). This yields the equations of motion for the \( x_i \)'s.

Using the standard identities for the Weierstrass functions, one can bring the equations of motion to the form
\[ \sum_{j \neq i}(\dddot{x}_i \dddot{x}_j - \dddot{x}_i \dddot{x}_j) \left( \frac{\varphi'(\eta)}{\varphi(x_{ij}) - \varphi(\eta)} - 2\zeta(\eta) \right) \]
\[ + \sum_k \sum_{j \neq i,k} \dddot{x}_i \dddot{x}_j \dddot{x}_k \frac{\varphi'(x_{ij})}{\varphi(x_{ij}) - \varphi(\eta)} \left( \frac{\varphi'(\eta)}{\varphi(x_{ik}) - \varphi(\eta)} - 2\zeta(\eta) \right) \]
\[ + 2\tilde{\mu} \dddot{x}_i + 2\tilde{\mu} \sum_{j \neq i} \dddot{x}_i \dddot{x}_j \frac{\varphi'(x_{ij})}{\varphi(x_{ij}) - \varphi(\eta)} = 0, \] (29)
where \( x_{ij} = x_i - x_j \). In contrast to the equations of motion \( [14], [15] \), this system of linear equations is not resolved with respect to the \( \dddot{x}_i \)'s. In the rational limit \((\varphi(x) \to 1/x^2)\) the equations of motion for poles of rational solutions are
\[ \sum_{j \neq i} \frac{\dddot{x}_i \dddot{x}_j - \dddot{x}_i \dddot{x}_j}{x_{ij}^2 - \eta^2} + \sum_k \sum_{j \neq i,k} \frac{2\eta^2 \dddot{x}_i \dddot{x}_j \dddot{x}_k}{x_{ij}(x_{ij}^2 - \eta^2)(x_{ik}^2 - \eta^2)} + \frac{\tilde{\mu}}{\eta} \dddot{x}_i + \tilde{\mu} \sum_{j \neq i} \frac{2\eta^2 \dddot{x}_i \dddot{x}_j}{x_{ij}(x_{ij}^2 - \eta^2)} = 0. \] (30)

The commutation representation of the equations of motion follows from \( [26] \). It is
\[ \dot{L} + [L, M] = R\left(L - 2\tilde{\mu}(z^2 - \mu^2)I\right) \] (31)
which is a sort of the Manakov’s triple representation \([18]\).
5 Concluding remarks

The equations of motion (29) for poles of elliptic solutions to the semi-discrete BKP equation together with their Manakov’s triple representation (31) are the main results of the paper. The resulting system of equations of motion is not resolved with respect to accelerations $\ddot{x}_i$. Such a non-resolved form of equations of motion was previously known for elliptic solutions to the Novikov-Veselov equation [17]. However, in this case the system admits an explicit solution in the rational limit. We do not know whether the system (30) admits an explicit solution for $\ddot{x}_i$. It is not also clear how to construct integrals of motion. These are problems for future investigation.

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