A SCHNEIDER TYPE THEOREM FOR HOPF ALGEBROIDS

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ABSTRACT. Comodule algebras of a Hopf algebroid $H$ with a bijective antipode, i.e. algebra extensions $B \subseteq A$ by $H$, are studied. Assuming that a lifted canonical map is a split epimorphism of modules of the (non-commutative) base algebra of $H$, relative injectivity of the $H$-comodule algebra $A$ is related to the Galois property of the extension $B \subseteq A$ and also to the equivalence of the category of relative Hopf modules to the category of $B$-modules. This extends a classical theorem by H.-J. Schneider on Galois extensions by a Hopf algebra. Our main tool is an observation that relative injectivity of a comodule algebra is equivalent to relative separability of a forgetful functor, a notion introduced and analyzed hereby.

In the first version of this submission, we heavily used the statement that two constituent bialgebroids in a Hopf algebroid possess isomorphic comodule categories. This statement was based on [Brz3, Theorem 2.6], whose proof turned out to contain an unjustified step. In the revised version we return to an earlier definition of a comodule of a Hopf algebroid, that distinguishes between comodules of the two constituent bialgebroids, and modify the statements and proofs in the paper accordingly.

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1. INTRODUCTION

Galois extensions of non-commutative algebras by a Hopf algebra generalize Galois extensions of commutative rings by groups and are known as the algebraic (dual) versions of (non-commutative) principal bundles. By a Hopf Galois extension the following structure is meant. Comodules over a Hopf algebra $H$ form a monoidal category $\mathcal{M}^H$, whose monoids are called comodule algebras. This means an algebra and $H$-comodule $A$, such that the coaction $\rho^A : A \to A \otimes H$ is an algebra map (with respect to the tensor product algebra structure of the codomain). It can be looked at as a notion dual to the action of a group on a manifold. Dualizing the notion of invariant points, coinvariants of $A$ are defined as those elements on which coaction is trivial, i.e. the elements of the subalgebra

$$B := \{ b \in A \mid \rho^A(b) = b \otimes 1_H \}.$$
In this situation the algebra $A$ is called an extension of $B$ by $H$. The algebra extension $B \subseteq A$ is said to be $H$-Galois if in addition the so called canonical map

$$(1.1) \quad \text{can} : A \otimes_k A \to A \otimes H, \quad a \otimes a' \mapsto a \rho^A(a')$$

is bijective (hence an isomorphism of left $A$-modules and right $H$-comodules). This is a dual formulation of the condition that a group action on a manifold is free.

(Right-right) relative Hopf modules are (right) modules for an $H$-comodule algebra $A$ and (right) comodules for the Hopf algebra $H$, satisfying a compatibility condition with the $H$-coaction in $A$. In the case of an $H$-Galois extension $B \subseteq A$, relative Hopf modules are canonically identified with descent data for the extension $B \subseteq A$. Hence if $A$ is faithfully flat as a left $B$-module, it follows by the Faithfully Flat Descent Theorem that the category $\mathcal{M}^H_A$ of right-right relative Hopf modules is equivalent to the category $\mathcal{M}_B$ of right $B$-modules.

In the study of Hopf Galois extensions, important tools are provided by theorems, stating that in appropriate situations surjectivity of the canonical map (1.1) implies its bijectivity. One group of such results (e.g. [K1, Theorem 1.7], [Scha1, Corollary 2.4.8] [SS, Theorem 3.1], [Brz2, Theorem 4.2 and Corollary 4.3]) can be called ‘Kreimer-Takeuchi type’ theorems (as their first representative was proven in [K1, Theorem 1.7]). In this group of theorems projectivity of the regular comodule of the coacting Hopf algebra is assumed. The other group involves ‘Schneider type’ theorems (after [Scha1, Theorem 1], see e.g. [SS, Theorem 4.9], [Brz2, Theorem 4.6], [MM1, Theorem 3.15], [MM2, Theorem 3.9]). Here relative injectivity of the Hopf comodule algebra in question is assumed.

The starting point of our work is an observation that the proofs of all above theorems share a common philosophy. Related to a comodule algebra $A$ of a Hopf algebra $H$ over a commutative ring $k$, there are forgetful functors

$$(1.2) \quad \mathcal{M}^H_A \xrightarrow{R} \mathcal{M}^H \xrightarrow{U} \mathcal{M}_k.$$ 

If $H$ is a projective $k$-module then the codomain of the canonical map (1.1) is a projective $A$-module. Then it follows from the surjectivity of the canonical map that its lifted version

$$(1.3) \quad \text{can} : A \otimes_k A \to A \otimes_k H, \quad a \otimes a' \mapsto a \rho^A(a')$$

has a $k$-linear right inverse, i.e. it is a retraction of $k$-modules. The various Schneider type theorems give sufficient conditions for the forgetful functor $U$ to reflect (certain) retractions. Then bijectivity of the canonical map (1.1) follows by a result of Schauenburg [Scha1, Corollary 2.4.8] stating that – under the additional assumption that all $H$-coinvariants of the obvious right $H$-comodule $A \otimes_k A$ are elements of $A \otimes_k B$ – the canonical map is bijective, provided that its lifted version (1.3) is a retraction of $H$-comodules.

In the present paper we introduce the notion of separability of a functor $U : \mathfrak{A} \to \mathfrak{B}$, relative to a functor $R : \mathfrak{B} \to \mathfrak{A}$ (not to be mixed with separability of the second kind in [CM]). An $R$-relative separable functor $U$ reflects retractions in the sense that, for a morphism $f$ in $\mathfrak{B}$ such that $UR(f)$ is a retraction, $R(f)$ is a retraction. As it turns out, the conditions of all known Schneider type theorems imply the separability of the forgetful functor $U$ in (1.2), relative to $R$.

Our strategy, of tracing back Schneider type theorems to properties of a forgetful functor, can be compared to that of Caenepeel, Ion, Militaru and Zhu, when in [CIMZ] they explained all known Maschke type theorems by the separability of a forgetful functor.

The motivation of our work comes from a wish to prove a Schneider type theorem for more general algebra extensions by a Hopf algebroid, replacing the Hopf algebra $H$ above. A Kreimer-Takeuchi type theorem was proven in [Brz2]. In that paper similar methods have been used as in [SS]: the entwining structure (over a non-commutative base), determined by a comodule algebra of a Hopf algebroid, has been studied. It turns out that this framework is not sufficient to obtain a Schneider type theorem for extensions by Hopf algebroids. Recall that a Hopf algebroid $\mathcal{H}$ consists of two related coring (and bialgebroid) structures, over two different base algebras $L$ and $R$. The proper definition of an $\mathcal{H}$-comodule consists of a compatible pair of comodules, one for each constituent coring. This results in a monoidal category $\mathcal{M}^H$ of $\mathcal{H}$-comodules. By definition,
a right $H$-comodule algebra is an algebra in $\mathcal{M}^H$. As in the Hopf algebra case, right $A$-modules in $\mathcal{M}^H$ are called relative Hopf modules. Their category $\mathcal{M}^H_A$ admits forgetful functors

(1.4) $\mathcal{M}^H_A \xrightarrow{R} \mathcal{M}^H \xrightarrow{U} \mathcal{M}_L$.

The fruitful approach to a Schneider type theorem for Hopf algebroids turns out to be a study of these forgetful functors.

The paper is organized as follows. In Section 2 the notion of a separable functor $U$, relative to functors $\mathbb{L}: \mathcal{L} \to \mathfrak{A}$ and $\mathbb{R}: \mathfrak{R} \to \mathfrak{A}$, is introduced and investigated. Section 3 concerns relative separability of a forgetful functor $\mathcal{M}^D \to \mathcal{M}_L$, associated to an entwining structure $(A, D, \psi)$ over an algebra $L$. If $D$ possesses a grouplike element, relative separability of the forgetful functor is shown to imply relative injectivity of $A$ as a $D$-comodule and, in the case when in addition the entwining map is bijective, also relative injectivity of $A$ as an entwined module (see Theorem 3.3 and Proposition 3.4). In Section 4 separability of the forgetful functor $U: \mathcal{M}^H \to \mathcal{M}_L$, relative to the forgetful functor $R$ from the category of relative Hopf modules to the category of $H$-comodules, is studied, for a Hopf algebroid $H$ and its comodule algebra $A$, cf. (1.4). In the case when the antipode of $H$ is bijective, it is shown to be equivalent to relative injectivity of the $H$-comodule $A$ (see Theorem 4.2). This result enables us to answer a question posed in [Bo2]. That is, in Proposition 4.4 we prove that, in a Galois extension $B \subseteq A$ by a finitely generated and projective Hopf algebroid $H$ with a bijective antipode, $A$ is faithfully flat as a left $B$-module if and only if it is faithfully flat as a right $B$-module. The main result is a Schneider type theorem in Section 5. Recall that Schneider’s classical Theorem I in [Sch] deals with an algebra extension $B \subseteq A$ by a $k$-Hopf algebra $H$ with a bijective antipode. It is assumed that $H$ is a projective $k$-module and the canonical map (1.3) is surjective. Clearly, in this case the lifted canonical map (1.3) is a split epimorphism of $k$-modules. As a proper generalization to an algebra extension $B \subseteq A$ by a Hopf algebroid $H$, in Theorem 5.6 we assume that some lifted canonical map is a split epimorphism of modules for the (non-commutative) base algebra $L$ of $H$. This assumption is related to surjectivity of the canonical map and some projectivity conditions in Remark 5.3. Under the assumption that, for an algebra extension $B \subseteq A$ by a Hopf algebroid $H$ with a bijective antipode, the lifted canonical map is a split epimorphism of $L$-modules, the Galois property of the extension is related to relative injectivity of the $H$-comodule $A$ and to the equivalence of the category $\mathcal{M}^H_A$ of relative Hopf modules to the category of $B$-modules. Section 6 is devoted to a study of (relative) equivariant injectivity and projectivity properties. Preliminary results about entwining structures (over arbitrary non-commutative algebras), coring extensions (in the sense of [Brez]) and Hopf algebroids are collected in Appendix A.

As an experiment, using informality of the arXiv, in this submission corrections (with respect to the first version) are written in blue. We hope it would be helpful to the readers of the original submission.

Throughout this paper the term algebra is used for an associative and unital but not necessarily commutative algebra over a fixed commutative ring $k$. Multiplication is denoted by juxtaposition and the unit element is denoted by $1$. For an algebra $A$, the opposite algebra is denoted by $A^{op}$. The category of right (respectively, left) modules for an algebra $A$ is denoted by $\mathcal{M}_A$ (respectively, $A\mathcal{M}$). The set of morphisms between two $A$-modules $M$ and $M'$ is denoted by $\operatorname{Hom}_A(M, M')$ (respectively, $\mathfrak{A}\operatorname{Hom}(M, M')$). The category of $A$-$A$ bimodules is denoted by $\mathfrak{A}\mathcal{M}_A$ and its Hom sets by $\mathfrak{A}\operatorname{Hom}_A(M, M')$.

For the coproduct in a coring $C$ over an algebra $A$, we use a Sweedler type index notation $c \mapsto c^{(1)} \otimes_A c^{(2)}$, for $c \in C$, where implicit summation is understood. Similarly, for a right $C$-coaction we use an index notation of the form $\varrho^{(1)}(m) = m^{[0]} \otimes_A m^{[1]}$, for $m \in M$. The category of right $C$-comodules is denoted by $\mathcal{M}^C$ and its Hom sets by $\operatorname{Hom}^C(M, M')$. Symmetrical notations are used for left $C$-comodules. The coaction is denoted by $M \varrho(m) = m^{[-1]} \otimes_A m^{[0]}$, for a left $C$-comodule $M$ and $m \in M$. The category of left $C$-comodules is denoted by $\mathcal{M}^C$ and its Hom sets by $\mathcal{M}^C(M, M')$. 


2. Relative separable functors

We start by recalling some material about separable functors. For more information we refer to [15, Chap. IX, page 307-312], [16, Chap. 8, page 279-281] and [CMZ]. Throughout the paper we use the following terminology. A morphism \( f : C_1 \to C_2 \) in a category \( \mathcal{C} \) is said to be a split monomorphism or section if it is cosplit by some morphism \( h : C_2 \to C_1 \) in \( \mathcal{C} \), i.e. \( h \circ f = C_1 \).

Dually, \( f \) is called a split epimorphism or retraction provided that it is split by some morphism \( g : C_2 \to C_1 \) in \( \mathcal{C} \), i.e. \( f \circ g = C_2 \).

**Definition 2.1.** Let \( \mathcal{C} \) be a category and let \( \mathcal{S} \) be a class of morphisms in \( \mathcal{C} \). For a morphism \( f : C_1 \to C_2 \) in \( \mathcal{C} \), an object \( P \in \mathcal{C} \) is called \( f \)-projective if the map \( \text{Hom}_\mathcal{C}(P, f) : \text{Hom}_\mathcal{C}(P, C_1) \to \text{Hom}_\mathcal{C}(P, C_2) \) is surjective. \( P \) is \( \mathcal{S} \)-projective if it is \( f \)-projective for every \( f \in \mathcal{S} \).

Dually, an object \( I \in \mathcal{C} \) is called \( f \)-injective if it is \( f \)-projective in the opposite category \( \mathcal{C}^{\text{op}} \), where \( f : C_2 \to C_1 \) is considered to be a morphism \( \mathcal{C}^{\text{op}} \). \( I \) is called \( \mathcal{S} \)-injective if it is \( f \)-injective for every \( f \in \mathcal{S} \).

All results below about projective objects can be dualized to get their analogues for injective objects.

**Theorem 2.2.** Let \( \mathbb{H} : \mathcal{B} \to \mathfrak{A} \) be a covariant functor and consider a class of morphisms \( \mathcal{E}_\mathbb{H} := \{ g \in \mathcal{B} \mid \mathbb{H}(g) \) is a split epimorphism in \( \mathfrak{A} \} \).

Assume that \( \mathbb{T} : \mathfrak{A} \to \mathcal{B} \) is a left adjoint of \( \mathbb{H} \) and denote by \( \varepsilon : \mathbb{T}\mathbb{H} \to \mathcal{B} \) the counit of the adjunction. Then, for an object \( P \in \mathcal{B} \), the following assertions are equivalent.

1. \( P \) is \( \mathcal{E}_\mathbb{H} \)-projective.
2. \( \varepsilon_P : \mathbb{T}\mathbb{H}(P) \to P \) is a split epimorphism.
3. There is a split epimorphism \( \pi : \mathbb{T}(X) \to P \), for a suitable object \( X \in \mathfrak{A} \).

In particular, all objects of the form \( \mathbb{T}(X) \), for \( X \in \mathfrak{A} \), are \( \mathcal{E}_\mathbb{H} \)-projective.

In [14] also a dual version of Theorem 2.2 can be found. It deals with \( \mathcal{I}_\mathbb{T} \)-injective objects in a category \( \mathfrak{A} \), for a left adjoint functor \( \mathbb{T} : \mathfrak{A} \to \mathcal{B} \), and

\[ \mathcal{I}_\mathbb{T} = \{ f \in \mathfrak{A} \mid \mathbb{T}(f) \) is a split monomorphism in \( \mathcal{B} \} \].

Using the current terminology, relative injective right comodules of an \( A \)-co-ring \( C \), discussed in Section A.2, can be characterized as \( \mathcal{I}_\mathbb{H} \)-injective objects, where \( \mathbb{U} : \mathfrak{M}^C \to \mathfrak{M}_A \) denotes the forgetful functor. As recalled in Section A.2, the forgetful functor \( \mathbb{U} \) possesses a right adjoint, the functor \( \bullet \otimes_A C \). The unit of the adjunction is given by the \( C \)-coaction. Therefore, the dual version of Theorem 2.2 (a)\( \Leftrightarrow \) (b) includes the claim, recalled in Section A.2, that a right \( C \)-comodule \( M \) is relative injective if and only if the coaction \( \varnothing^M \) in it is a split monomorphism in \( \mathfrak{M}^C \).

Since any covariant functor preserves split epimorphisms and split monomorphisms, we immediately have that, for any two functors \( \mathbb{F} : \mathfrak{A} \to \mathcal{B} \) and \( \mathbb{G} : \mathcal{B} \to \mathcal{C} \),

\[ \mathcal{E}_\mathbb{F} \subseteq \mathcal{E}_{\mathbb{F}\mathbb{G}} \quad \text{and} \quad \mathcal{I}_\mathbb{F} \subseteq \mathcal{I}_{\mathbb{F}\mathbb{G}}. \]

As explained in the Introduction, in the area of Schneider type theorems one often faces the following problem. Consider an entwining structure \((A, D, \psi)\) over an algebra \( L \). Assume that some map in \( \mathfrak{M}^D_A(\psi) \) (practically the canonical map) is a retraction in \( \mathfrak{M}_L \). Under what assumptions is it a retraction also in \( \mathfrak{M}^D \)? Putting the question in a more functorial way, we can ask in which cases is \( \mathcal{E}_\mathbb{F} = \mathcal{E}_{\mathbb{F}\mathbb{G}} \), for the forgetful functors \( \mathbb{F} : \mathfrak{M}^D_A(\psi) \to \mathfrak{M}^D \) and \( \mathbb{G} : \mathfrak{M}^D \to \mathfrak{M}_L \). For these particular functors \( \mathbb{F} \) and \( \mathbb{G} \), property 1) in Proposition 2.3 below reduces to a similar (but somewhat weaker) assumption as in a Schneider type theorem [SS, Theorem 5.9] (see also [Brz2, Theorem 4.6]). Properties like in part 2) of Proposition 2.3 are assumed e.g. in [SS, Corollary 4.8].

**Proposition 2.3.** For two functors \( \mathbb{F} : \mathfrak{A} \to \mathcal{B} \) and \( \mathbb{G} : \mathcal{B} \to \mathcal{C} \), \( \mathcal{E}_\mathbb{F} = \mathcal{E}_{\mathbb{F}\mathbb{G}} \) whenever any of the following properties hold.

1. \( \mathbb{F}(A) \) is \( \mathcal{E}_{\mathbb{G}} \)-projective, for every object \( A \in \mathfrak{A} \).
2. \( \mathfrak{A}, \mathfrak{B} \) and \( \mathcal{C} \) are abelian categories, \( \mathbb{G} \) is left exact and reflects epimorphisms, \( \mathbb{F} \) is left exact and \( \mathbb{F}(A) \) is \( \mathcal{I}_{\mathbb{G}} \)-injective, for every object \( A \in \mathfrak{A} \).
Dually, \( \mathcal{I}_G = \mathcal{I}_E \) whenever any of the following properties hold.

1\(^{op} \) \( F(A) \) is \( \mathcal{I}_G \)-injective, for every object \( A \in \mathcal{A} \).
2\(^{op} \) \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) are abelian categories, \( G \) is right exact and reflects monomorphisms, \( F \) is right exact and \( F(A) \) is \( \mathcal{E}_G \)-projective, for every object \( A \in \mathcal{A} \).

**Proof.** 1) Let \( f : A_1 \to A_2 \) be a morphism in \( \mathcal{E}_G \). Then \( F(f) : F(A_1) \to F(A_2) \) belongs to \( \mathcal{E}_G \) and hence, by hypothesis, it is a split epimorphism. Thus \( f \in \mathcal{E}_G \).

2) For \( f \in \mathcal{E}_G \), consider the exact sequence (kernel diagram)

\[
0 \to K \xrightarrow{i} A_1 \xrightarrow{f} A_2
\]

in \( \mathcal{A} \). The left exact functor \( F \) takes it to the exact sequence

\[
0 \to F(K) \xrightarrow{g(i)} F(A_1) \xrightarrow{f} F(A_2)
\]

in \( \mathcal{B} \). Since \( f \) is an element of \( \mathcal{E}_G \), the morphism \( GF(f) \) is a split epimorphism. Since \( G \) is left exact and \( \mathcal{C} \) is an abelian category, the sequence

\[
0 \to GF(K) \xrightarrow{GF(i)} GF(A_1) \xrightarrow{GF(f)} GF(A_2) \xrightarrow{0}
\]

in \( \mathcal{C} \) is split exact. Thus we deduce that \( i \in \mathcal{I}_G \). Moreover, since \( G \) reflects epimorphisms, \( F(f) \) is an epimorphism. So the sequence

\[
0 \to F(K) \xrightarrow{f(i)} F(A_1) \xrightarrow{f} F(A_2) \xrightarrow{0}
\]

in \( \mathcal{B} \) is exact too. Since \( i \) is an element of \( \mathcal{I}_G \), its image \( F(i) \) is in \( \mathcal{I}_G \). By assumption \( F(K) \) is \( \mathcal{I}_G \)-injective hence the monomorphism \( F(i) \) is split. Since \( \mathcal{B} \) is an abelian category, we conclude that \( F(f) \) is a split epimorphism, i.e. that \( f \in \mathcal{E}_G \).

Claims 1\(^{op} \) and 2\(^{op} \) follow by duality. \( \square \)

The most important notions of this section are introduced in the following definition.

**Definition 2.4.** Consider the following diagram of functors.

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{U} & \mathcal{R} \\
\downarrow & & \downarrow \\
\mathcal{L} & \xrightarrow{L} & \mathcal{A}
\end{array}
\]

They give rise to two functors

\[
\hom_{\mathcal{B}}(\mathcal{L}(\bullet), \mathcal{R}(\bullet)) \quad \text{and} \quad \hom_{\mathcal{B}}(\mathcal{U}(\mathcal{L}(\bullet)), \mathcal{U}(\mathcal{R}(\bullet))) : \mathcal{L}^{op} \times \mathcal{R} \to \mathcal{Sets}
\]

and a natural transformation between them

\[
(2.3) \quad \Phi(U, L, R) : \hom_{\mathcal{R}}(\mathcal{L}(\bullet), \mathcal{R}(\bullet)) \to \hom_{\mathcal{B}}(\mathcal{U}(\mathcal{L}(\bullet)), \mathcal{U}(\mathcal{R}(\bullet))), \quad \Phi(U, L, R)_L(f) := \mathcal{U}(f),
\]

for all objects \( L \in \mathcal{L}, R \in \mathcal{R} \) and for every morphism \( f : \mathcal{L}(L) \to \mathcal{R}(R) \). We say that

1) \( U \) is \((\mathcal{L}, \mathcal{R})\)-faithful if \( \Phi(U, L, R)_L \) is injective, for every objects \( L \in \mathcal{L} \) and \( R \in \mathcal{R} \).
2) \( U \) is \((\mathcal{L}, \mathcal{R})\)-full if \( \Phi(U, L, R)_L \) is surjective, for every objects \( L \in \mathcal{L} \) and \( R \in \mathcal{R} \).
3) \( U \) is \((\mathcal{L}, \mathcal{R})\)-separable if \( \Phi(U, L, R) \) is a split natural monomorphism.
4) \( U \) is \((\mathcal{L}, \mathcal{R})\)-coseparable if \( \Phi(U, L, R) \) is a split natural epimorphism.

When both \( \mathcal{L} \) and \( \mathcal{R} \) are identity functors, we recover the classical definitions of a faithful, full, separable and naturally full (here called coseparable) functor. We are particularly interested in the case when either \( \mathcal{L} \) or \( \mathcal{R} \) is the identity functor. Anyway, some of our results can be stated for the general case.
Theorem 2.4.3) can be reformulated (in the spirit of a characterization of separable functors in \([NV]\)) as follows. A functor \(U : \mathcal{A} \to \mathcal{B}\) is \((L, R)\)-separable, for some functors \(L : \mathcal{C} \to \mathcal{A}\) and \(R : \mathcal{R} \to \mathcal{A}\), if and only if there is a map

\[
\tilde{\Phi}(U, L, R)_{L,R} : \text{Hom}_\mathcal{B}(UL(L), UR(R)) \to \text{Hom}_\mathcal{A}(L(L), R(R)),
\]

for all objects \(L \in \mathcal{C}\) and \(R \in \mathcal{R}\), satisfying the following identities.

1) \(\tilde{\Phi}(U, L, R)_{L,R}(U(f)) = f\), for any \(f \in \text{Hom}_\mathcal{A}(L(L), R(R))\).

2) \(\tilde{\Phi}(U, L, R)_{L,R}(h') : L(L) = R(R) \circ \Phi(U, L, R)_{L,R}(h)\), for every commutative diagram in \(\mathcal{B}\) of the following form.

\[
\begin{array}{ccc}
UL(L) & \xrightarrow{h} & UR(R) \\
\downarrow l & & \downarrow r \\
UL(L') & \xrightarrow{h'} & UR(R')
\end{array}
\]

Remark 2.5. Following \([Raf, page 1446]\), one can prove that Definition 2.4.3) can be reformulated (in the spirit of a characterization of separable functors in \([NV]\)) as follows. A functor \(U : \mathcal{A} \to \mathcal{B}\) is \((L, R)\)-separable, for some functors \(L : \mathcal{C} \to \mathcal{A}\) and \(R : \mathcal{R} \to \mathcal{A}\), if and only if there is a map

\[
\tilde{\Phi}(U, L, R)_{L,R} : \text{Hom}_\mathcal{B}(UL(L), UR(R)) \to \text{Hom}_\mathcal{A}(L(L), R(R)),
\]

for all objects \(L \in \mathcal{C}\) and \(R \in \mathcal{R}\), satisfying the following identities.

1) \(\tilde{\Phi}(U, L, R)_{L,R}(U(f)) = f\), for any \(f \in \text{Hom}_\mathcal{A}(L(L), R(R))\).

2) \(\tilde{\Phi}(U, L, R)_{L,R}(h') : L(L) = R(R) \circ \Phi(U, L, R)_{L,R}(h)\), for every commutative diagram in \(\mathcal{B}\) of the following form.

\[
\begin{array}{ccc}
UL(L) & \xrightarrow{h} & UR(R) \\
\downarrow l & & \downarrow r \\
UL(L') & \xrightarrow{h'} & UR(R')
\end{array}
\]

Remark 2.6. Recall that faithful functors reflect mono, and epimorphisms. Analogously, for an \((L, R)\)-faithful functor \(U\) the following hold true.

1) Assume that \(R\) is surjective on the objects and let \(f : A \to L(L)\) be a morphism in \(\mathcal{A}\).

Then \(f\) is an epimorphism whenever \(U(f)\) is.

2) Assume that \(L\) is surjective on the objects and let \(f : R(R) \to A\) be a morphism in \(\mathcal{A}\).

Then \(f\) is a monomorphism whenever \(U(f)\) is.

In the rest of the section we extend some standard results about separable functors to relative separable functors in Definition 2.4.3). Analogous results can be obtained for coseparable functors by a careful dualization.

Theorem 2.7. Consider the following diagram of functors.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{V} & \mathcal{B} \\
\downarrow U & & \downarrow \Phi \\
\mathcal{C}' & \xrightarrow{L'} & \mathcal{C}
\end{array}
\]

The following assertions hold true.

1) If \(U\) is \((L, R)\)-separable then \(U\) is \((LL', RR')\)-separable.

2) If \(U\) is \((L, R)\)-separable and \(V\) is \((UL, UR)\)-separable then \(VU\) is \((L, R)\)-separable.

3) If \(VU\) is \((L, R)\)-separable then \(U\) is \((L, R)\)-separable.

Proof. The proof is similar to \([CMZ, I.3 Proposition 46 and Corollary 9]\).

1) Since \(U\) is \((L, R)\)-separable, there exists a natural retraction \(\tilde{\Phi}(U, L, R)\) of the natural transformation \((2.3)\). For any objects \(L' \in \mathcal{C}'\) and \(R' \in \mathcal{R}'\), the maps

\[
\tilde{\Phi}(U, L, R)_{L'(L'), R'(R')} : \text{Hom}_\mathcal{B}(UL'(L'), UR'(R')) \to \text{Hom}_\mathcal{A}(LL'(L'), RR'(R'))
\]

define a natural transformation which is a retraction of \(\tilde{\Phi}(U, LL', RR')\), defined analogously to \((2.3)\).

2) The natural transformation

\[
(2.4) \quad \Phi(VU, L, R) : \text{Hom}_\mathcal{B}(LL(\bullet), RR(\bullet)) \to \text{Hom}_\mathcal{C}(VUL(\bullet), VRU(\bullet)), \quad f \mapsto VU(f)
\]

is a composite of the split natural monomorphisms \(\Phi(U, L, R)\), corresponding via \((2.3)\) to the \((L, R)\)-separable functor \(U\), and \(\Phi(V, UL, UR)\), corresponding to the \((UL, UR)\)-separable functor \(V\). Hence \((2.3)\) is a split natural monomorphism too, proving \((L, R)\)-separability of \(VU\).

3) Since the functor \(VU\) is \((L, R)\)-separable, the corresponding natural transformation \((2.3)\) possesses a retraction \(\tilde{\Phi}(VU, L, R)\). The composite \(\Phi(VU, L, R) \circ \Phi(V, UL, UR)\) is a natural retraction of \(\Phi(U, L, R)\) in \((2.3)\). \(\square\)

Theorem 2.8 (Maschke type Theorem). Let \(U : \mathcal{A} \to \mathcal{B}\), \(L : \mathcal{C} \to \mathcal{A}\) and \(R : \mathcal{R} \to \mathcal{A}\) be functors.
1. If \( U \) is \((\mathcal{A}, \mathcal{R})\)-separable then, for any objects \( R \in \mathcal{R} \) and \( A \in \mathcal{A} \), a morphism \( f: R(R) \to A \) is a split monomorphism whenever \( U(f) \) is a split monomorphism. Moreover, in this case \( I_{UR} = I_{UR} \) and \( E_{R} = E_{UR} \).

2. If \( U \) is \((\mathcal{L}, \mathcal{A})\)-separable then, for any objects \( L \in \mathcal{L} \) and \( A \in \mathcal{A} \), a morphism \( f: A \to L(L) \) is a split epimorphism whenever \( U(f) \) is a split epimorphism. Moreover, in this case \( I_{L} = I_{UL} \) and \( E_{L} = E_{UL} \).

Proof. Let \( A, R \) and \( f \) be as in part 1. Let \( \Phi(U, \mathcal{A}, \mathcal{R}) \) be a natural retraction of \( \Phi(U, \mathcal{A}, \mathcal{R}) \) in (2.3). In view of (2.2) in Remark 2.3, any retraction \( \pi \) of \( U(f) \) satisfies

\[
\Phi(U, \mathcal{A}, \mathcal{R})_{A,R}(\pi) \circ f = R(R).
\]

That is, \( f \) is a split monomorphism. In particular, \( f := R(g) \) is a split monomorphism, for any \( g \in I_{UR} \). Together with (2.2) this proves \( I_{UR} = I_{R} \). Next take a morphism \( g: R \to R' \) in \( E_{UR} \), and a section \( \sigma \) of \( U(g) \). Then, by naturality of \( \Phi(U, \mathcal{A}, \mathcal{R}) \),

\[
R(R') = \Phi(U, \mathcal{A}, \mathcal{R})_{R(R'),(UR(R'))} = \Phi(U, \mathcal{A}, \mathcal{R})_{R(R'),(UR(g))}(UR(g)) = \Phi(U, \mathcal{A}, \mathcal{R})_{R(R'),(UR(g))} = \Phi(U, \mathcal{A}, \mathcal{R})_{R(R'),(UR(g))}.
\]

This implies that \( R(g) \) is a split epimorphism, i.e. \( g \in E_{R} \). In view of (2.2), we have \( E_{R} = E_{UR} \) proven.

Part 2) is proven by dual reasoning.

Corollary 2.9. Let \( (T, \mathcal{H}) \) be an adjunction of functors \( T : \mathcal{A} \to \mathcal{B} \) and \( H : \mathcal{B} \to \mathcal{A} \). For any functors \( L : \mathcal{L} \to \mathcal{B} \) and \( R : \mathcal{R} \to \mathcal{A} \), the following hold.

1. If the functor \( H \) is \((\mathcal{L}, \mathcal{B})\)-separable then \( L(L) \) is \( E_{H} \)-projective for every \( L \in \mathcal{L} \).
2. If the functor \( T \) is \((\mathcal{A}, \mathcal{R})\)-separable then \( R(R) \) is \( I_{T} \)-injective for every \( R \in \mathcal{R} \).

Proof. Let \( \eta : \mathcal{A} \to \mathcal{H} \) be the unit and \( \varepsilon : \mathcal{H} \to \mathcal{B} \) be the counit of the adjunction \((T, \mathcal{H})\).

1. For any object \( L \in \mathcal{L} \), the epimorphism \( \mathcal{H}(\varepsilon_{L(L)}) \) is split by \( \eta_{\mathcal{E}_{L}(L)} \). Hence, by Theorem 2.8 2), \( \varepsilon_{L(L)} \) is a split epimorphism in \( \mathcal{B} \). By Theorem 2.2 (b) \( \Rightarrow (a) \), \( L(L) \) is \( E_{H} \)-projective.

2. For any object \( R \in \mathcal{R} \), the monomorphism \( \mathcal{T}(\eta_{R(R)}) \) is split by \( \varepsilon_{\mathcal{I}_{R}(R)} \). Hence the claim follows analogously to part 1), by Theorem 2.8 1) and a dual form of Theorem 2.2.

In the following theorem functors preserving and reflecting relative projective (resp. injective) objects are studied.

Theorem 2.10. Let \( (T, \mathcal{H}) \) and \( (T', \mathcal{H}') \) be adjunctions and consider the following (not necessarily commutative) diagrams of functors.

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Phi} & \mathcal{A}' \\
V & \downarrow & \downarrow V' \\
\mathcal{B} & \xrightarrow{F} & \mathcal{B}'
\end{array}
\quad
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Phi} & \mathcal{A}' \\
\Phi & \downarrow & \downarrow \Phi' \\
\mathcal{B} & \xrightarrow{F} & \mathcal{B}'
\end{array}
\]

If \( T', F' \) and \( F \) are naturally equivalent, then the following hold.

1. If an object \( P \) in \( \mathcal{B} \) is \( E_{H} \)-projective then \( F(P) \) is \( E_{H} \)-projective.
2. Assume that \( F' \) is \((\mathcal{A}, \mathcal{R})\)-projective for some functor \( R : \mathcal{R} \to \mathcal{A} \). If, for an object \( R \in \mathcal{R} \), the object \( F'(R) \) is \( I_{T'} \)-injective, then \( R(R) \) is \( I_{T} \)-injective.

If \( F \) and \( H \) are naturally equivalent, then the following hold.

1. If an object \( I \) in \( \mathcal{A} \) is \( I_{T} \)-injective, then \( F'(I) \) is \( I_{T'} \)-injective.
2. Assume that \( F \) is \((\mathcal{L}, \mathcal{B})\)-separable for some functor \( L : \mathcal{L} \to \mathcal{B} \). If, for an object \( L \in \mathcal{L} \), the object \( F(L) \) is \( E_{H} \)-projective then \( L(L) \) is \( E_{H} \)-projective.

Proof. Denote by \( \eta : \mathcal{A} \to \mathcal{H} \) the unit and by \( \varepsilon : \mathcal{H} \to \mathcal{B} \) the counit of the adjunction \((T, \mathcal{H})\).

1. By Theorem 2.2 (a) \( \Rightarrow (b) \), \( E_{H} \)-projectivity of \( P \) implies that \( \varepsilon_{P} : \mathcal{H}P \to \mathcal{B} \) is a split epimorphism. Hence also \( F(\varepsilon_{P}) : T'F'H(P) \to F\mathcal{H}(P) \) is a split epimorphism. Application of Theorem 2.8 (c) \( \Rightarrow (a) \) to the adjunction \((T', \mathcal{H}') \) completes the proof of \( E_{H} \)-projectivity of \( F(P) \).

2. For any object \( P \) in \( \mathcal{B} \), the monomorphism \( \eta_{R(P)} \) is split by \( \mathcal{H}(\varepsilon_{P}) \). Hence \( F' \mathcal{H}(\varepsilon_{P}) \) is a split epimorphism. By the natural equivalence \( F'\mathcal{H} \sim \mathcal{H}'F \), also \( \mathcal{H}'F(\varepsilon_{P}) \) is a split epimorphism, yielding
that \( \mathbb{F}(\varepsilon_P) : \mathbb{FTH}(P) \rightarrow \mathbb{F}(P) \) belongs to \( \mathcal{E}_{\mathcal{B}} \). In the case when \( \mathbb{F}(P) \) is \( \mathcal{E}_{\mathcal{B}} \)-projective, we conclude that \( \mathbb{F}(\varepsilon_P) \) is a split epimorphism in \( \mathcal{B} \). Now put \( P = L(L) \), such that \( \mathbb{F}L(L) \) is \( \mathcal{E}_{\mathcal{B}} \)-projective as in the claim. Then, by Theorem 2.8, \( \varepsilon_L(L) \) is a split epimorphism in \( \mathcal{B} \) and hence \( L(L) \) is \( \mathcal{E}_{\mathcal{H}} \)-projective by Theorem 2.2 (b) \( \Rightarrow \) (a).

The remaining claims 1(b) and 2(b) follow by dual reasoning. □

Let \((T, H)\) be an adjunction of functors \( T : \mathcal{A} \rightarrow \mathcal{B} \) and \( H : \mathcal{B} \rightarrow \mathcal{A} \). Denote by \( \varepsilon : TH \rightarrow \mathbb{1} \mathcal{A} \) and \( \eta : \mathbb{1} \mathcal{B} \rightarrow HT \) the counit and the unit of the adjunction, respectively. Consider the canonical isomorphism

\[
\phi_{A,B} : \text{Hom}_\mathcal{B}(T(A), B) \rightarrow \text{Hom}_\mathcal{A}(A, H(B)), \quad \phi_{A,B}(f) = H(f) \circ \eta_A
\]

with inverse

\[
\phi_{A,B}^{-1} : \text{Hom}_\mathcal{A}(A, H(B)) \rightarrow \text{Hom}_\mathcal{B}(T(A), B), \quad \phi_{A,B}^{-1}(g) = \varepsilon_B \circ T(g).
\]

In terms of the natural transformations (2.3) and (2.5), for any functors \( L : \mathcal{L} \rightarrow \mathcal{A} \) and \( R : \mathcal{R} \rightarrow \mathcal{A} \), define a natural transformation

\[
\Omega := \phi \circ \Phi(T, L, R) : \text{Hom}_\mathcal{A}(L(\bullet), R(\bullet)) \rightarrow \text{Hom}_\mathcal{A}(L(\bullet), HTL(\bullet), LR(\bullet)).
\]

Then, for every morphism \( f : L(L) \rightarrow R(R) \), one has

\[
\Omega_{L,R}(f) = H(T(f) \circ \eta_{L(L)}) = \eta_{R(R)} \circ f.
\]

Dually, for functors \( L : \mathcal{L} \rightarrow \mathcal{B} \) and \( R : \mathcal{R} \rightarrow \mathcal{B} \), there is a natural transformation

\[
\Omega := \phi^{-1} \circ \Phi(H, L, R) : \text{Hom}_\mathcal{A}(L(\bullet), R(\bullet)) \rightarrow \text{Hom}_\mathcal{A}(THL(\bullet), R(\bullet)),
\]

mapping a morphism \( f : L(L) \rightarrow R(R) \) to

\[
\Omega_{L,R}(f) = \varepsilon_{R(R)} \circ \Theta(f) = f \circ \varepsilon_{L(L)}.
\]

**Lemma 2.11.** On the category of functors and natural transformations consider the following endofunctor \( \alpha \). It maps a functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) to the functor \( \text{Hom}_\mathcal{B}(\bullet, F(\bullet)) : \mathcal{B}^{op} \times \mathcal{A} \rightarrow \mathcal{Sets} \), and it maps a natural transformation \( \sigma \in \text{Nat}(F, G) \) to \( \text{Hom}_\mathcal{B}(\bullet, \sigma(\bullet)) \), i.e.

\[
\alpha(\sigma)_{B,A} : \text{Hom}_\mathcal{B}(B, F(A)) \rightarrow \text{Hom}_\mathcal{B}(B, G(A)), \quad g \mapsto \sigma_B \circ g.
\]

The functor \( \alpha \) is fully faithful.

**Proof.** The bijectivity of the maps

\[
\alpha_{F,G} : \text{Nat}(F, G) \rightarrow \text{Nat}(\text{Hom}_\mathcal{B}(\bullet, F(\bullet)), \text{Hom}_\mathcal{B}(\bullet, G(\bullet))) \quad \sigma \mapsto \alpha(\sigma),
\]

for any functors \( F, G : \mathcal{A} \rightarrow \mathcal{B} \), is proven by constructing the inverse \( (\alpha_{F,G})^{-1}(\sigma) \mid_A := P_{F(A), A}(\sigma) \), for \( P \in \text{Nat}(\text{Hom}_\mathcal{B}(\bullet, F(\bullet)), \text{Hom}_\mathcal{B}(\bullet, G(\bullet))) \) and \( A \in \mathcal{A} \). It is straightforward to check that the naturality of \( P \) (i.e. the identity \( G(a) \circ P_{B,A}(g) \circ b = P_{B',A}(F(a) \circ g \circ b) \), for \( a \in \text{Hom}_\mathcal{B}(A, A') \), \( b \in \text{Hom}_\mathcal{B}(B', B) \) and \( g \in \text{Hom}_\mathcal{B}(B, F(A)) \)) implies the naturality of \( (\alpha_{F,G})^{-1}(P) \). Furthermore, (keeping the notation),

\[
(\alpha_{F,G})^{-1}(\alpha_{F,G}(\sigma)) \mid_A = (\alpha_{F,G}(\sigma))_A = \alpha_{F,G}(\sigma)_{F(A, A}(\mathbb{F}(A)) = \sigma_A,
\]

what completes the proof. □

**Theorem 2.12.** Let \((T, H)\) be an adjunction of functors \( T : \mathcal{A} \rightarrow \mathcal{B} \) and \( H : \mathcal{B} \rightarrow \mathcal{A} \), with unit \( \eta \) and counit \( \varepsilon \). Consider any functor \( R : \mathcal{R} \rightarrow \mathcal{A} \) and a functor \( L : \mathcal{L} \rightarrow \mathcal{A} \) which is surjective on the objects (e.g. the identity functor \( L = \mathbb{1} \)). Then the following assertions hold.

1) \( T \) is \((L, R)\)-faithful if and only if it is \((\mathbb{1}, R)\)-faithful and if and only if \( \eta_{R(R)} \) is a monomorphism, for every object \( R \in \mathcal{R} \).

2) \( T \) is \((L, R)\)-full if and only if it is \((\mathbb{1}, R)\)-full and if and only if \( \eta_{R(R)} \) is a split epimorphism, for every object \( R \in \mathcal{R} \).
3) (Rafael type Theorem) \( T \) is \((\mathfrak{A}, \mathfrak{B})\)-separable if and only if \( \eta_{\mathfrak{B}}(\mathfrak{A}) \) is a split natural monomorphism.

4) (Dual Rafael type Theorem) \( T \) is \((\mathfrak{A}, \mathfrak{B})\)-coseparable if and only if \( \eta_{\mathfrak{B}}(\mathfrak{A}) \) is a split natural epimorphism.

Proof. Recall that the natural transformation \( \phi \) in (2.3) is an isomorphism.

1) \((\mathfrak{L}, \mathfrak{R})\)-faithfulness of \( T \), i.e. injectivity of the natural transformation \( T(L, \mathfrak{L}, \mathfrak{R})_{L,R} \) in (2.3), for every object \( L \in \mathfrak{L} \) and \( R \in \mathfrak{R} \) is equivalent to injectivity of \( \Omega_{L,R} \), for every \( L \in \mathfrak{L} \) and \( R \in \mathfrak{R} \).

Since \( L \) is surjective on the objects, in light of (2.4) this is equivalent to saying that \( \eta_{\mathfrak{R}}(\mathfrak{L}) \) is a monomorphism for every \( R \in \mathfrak{R} \).

2) \((\mathfrak{L}, \mathfrak{R})\)-fullness of \( T \), i.e. surjectivity of the natural transformation \( T(L, \mathfrak{L}, \mathfrak{R})_{L,R} \) in (2.3), for every object \( L \in \mathfrak{L} \) and \( R \in \mathfrak{R} \), is equivalent to surjectivity of \( \Omega_{L,R} \), for every \( L \in \mathfrak{L} \) and \( R \in \mathfrak{R} \). Let us prove that this is equivalent to saying that \( \eta_{\mathfrak{R}}(\mathfrak{L}) \) is a split epimorphism for every \( R \in \mathfrak{R} \).

In fact, since \( L \) is surjective on the objects, for every \( R \in \mathfrak{R} \) there exists an object \( L \in \mathfrak{L} \) such that \( \text{nat}(\mathfrak{R}) = L(L) \). Thus if \( \Omega_{L,R} \) is surjective then, by \( \text{nat}(\mathfrak{R}) \in \text{Hom}_{\mathfrak{L}}(\text{nat}(\mathfrak{R}), \text{nat}(\mathfrak{R})) = \text{Hom}_{\mathfrak{L}}(L(L), \text{nat}(\mathfrak{R})) \) there exists \( \sigma \in \text{Hom}_{\mathfrak{L}}(L(L), \mathfrak{R}(R)) \) such that \( \eta_{\mathfrak{R}}(\mathfrak{L}) \circ \sigma = \text{nat}(\mathfrak{R}) \). Conversely, let \( g \) be any morphism in \( \text{Hom}_{\mathfrak{L}}(L(L), \mathfrak{R}(R)) \), for some \( L \in \mathfrak{L} \) and \( R \in \mathfrak{R} \). Let \( \sigma \) be a section of \( \eta_{\mathfrak{R}}(\mathfrak{L}) \). Define \( f \in \text{Hom}_{\mathfrak{L}}(L(L), \mathfrak{R}(R)) \) by \( f := \sigma \circ g \). Then \( \Omega_{L,R}(f) = \eta_{\mathfrak{R}}(\mathfrak{L}) \circ \sigma = \text{nat}(\mathfrak{R}) \).

3) \((\mathfrak{A}, \mathfrak{R})\)-separability of \( T \), i.e. natural cosplittings of \( T(\mathfrak{T}, \mathfrak{A}, \mathfrak{R}) \), is equivalent to natural cosplittings of \( \Omega \). Note that \( \Omega \) is the image of the natural transformation \( \eta_{\mathfrak{R}}(\mathfrak{L}) \) under the functor \( \alpha \) in Lemma 2.1. Hence the claim follows by Lemma 2.1 as a fully faithful functor preserves and reflects split monomorphisms.

4) \((\mathfrak{A}, \mathfrak{R})\)-coseparability of \( T \), i.e. natural splitting of \( T(\mathfrak{T}, \mathfrak{A}, \mathfrak{R}) \), is equivalent to natural splitting of \( \Omega \). Hence this claim follows by the same argument as 3) does, as a fully faithful functor preserves and reflects split epimorphisms as well.

Dually, one proves the following result.

Theorem 2.13. Let \( (\mathfrak{T}, \mathfrak{H}) \) be an adjunction of functors \( \mathfrak{H} : \mathfrak{A} \to \mathfrak{B} \) and \( \mathfrak{T} : \mathfrak{B} \to \mathfrak{A} \), with unit \( \eta \) and counit \( \varepsilon \). Consider any functor \( \mathfrak{L} : \mathfrak{L} \to \mathfrak{A} \) and a functor \( \mathfrak{R} : \mathfrak{R} \to \mathfrak{A} \) which is surjective on the objects (e.g. the identity functor \( \mathfrak{R} = \mathfrak{A} \)). Then the following assertions hold.

1) \( \mathfrak{H} \) is \((\mathfrak{L}, \mathfrak{R})\)-faithful if and only if it is \((\mathfrak{L}, \mathfrak{A})\)-faithful and if and only if \( \varepsilon_{\mathfrak{L}(L)} \) is an epimorphism for every object \( L \in \mathfrak{L} \).

2) \( \mathfrak{H} \) is \((\mathfrak{L}, \mathfrak{R})\)-full if and only if it is \((\mathfrak{L}, \mathfrak{A})\)-full and if and only if \( \varepsilon_{\mathfrak{L}(L)} \) is a split monomorphism for every object \( L \in \mathfrak{L} \).

3) (Rafael type Theorem) \( \mathfrak{H} \) is \((\mathfrak{L}, \mathfrak{A})\)-separable if and only if \( \varepsilon_{\mathfrak{L}(\mathfrak{A})} \) is a split natural epimorphism.

4) (Dual Rafael type Theorem) \( \mathfrak{H} \) is \((\mathfrak{L}, \mathfrak{A})\)-coseparable if and only if \( \varepsilon_{\mathfrak{L}(\mathfrak{A})} \) is a split natural monomorphism.

A notion somewhat reminiscent to our relative separability of a functor was introduced in [CM] under the name of separability of the second kind. Our next task is to find a relation between the two notions.

Definition 2.14. Let \( \mathfrak{R} : \mathfrak{A} \to \mathfrak{A} \) and \( \mathfrak{T} : \mathfrak{A} \to \mathfrak{B} \) be covariant functors. Following [CM] Definition 2.1 and using the notation introduced in (2.3), \( \mathfrak{T} \) is called \( \mathfrak{R} \)-separable of the second kind if the natural transformation \( \Phi(\mathfrak{T}, \mathfrak{A}, \mathfrak{A}) \) factors through \( \Phi(\mathfrak{T}, \mathfrak{A}, \mathfrak{A}) \).

Proposition 2.15. Let \( (\mathfrak{T}, \mathfrak{H}) \) and \( (\mathfrak{T}', \mathfrak{H}') \) be adjunctions with respective units \( \eta \) and \( \eta' \). Consider the following diagrams of functors.

\[
\begin{array}{c}
\mathfrak{A} & \xrightarrow{\mathfrak{R}} & \mathfrak{A}' \\
\mathfrak{B} & \xrightarrow{\mathfrak{T}} & \mathfrak{B}' \\
\mathfrak{A} & \xrightarrow{\mathfrak{R}} & \mathfrak{A}' \\
\mathfrak{B} & \xrightarrow{\mathfrak{H}} & \mathfrak{B}' \\
\end{array}
\]

The following assertions are equivalent.

(a) \( \mathfrak{T} \) is \( \mathfrak{R} \)-separable of the second kind.
(b) There exists a natural transformation $\nu : \mathbb{R}HT \to \mathbb{R}$, satisfying $\nu_A \circ \mathbb{R}(\eta_A) = \mathbb{R}(A)$, for any $A \in \mathfrak{A}$.

Assume that there exists a natural equivalence $\xi : \mathbb{R}T' \mathbb{R} \to \mathbb{R}HT$ such that

$$\xi \circ \eta_{\mathbb{R}H} = \mathbb{R}(\eta).$$

Then the following assertion is also equivalent to the foregoing ones.

(c) $T'$ is $(\mathfrak{A}', \mathbb{R})$-separable.

Proof. (a) $\Leftrightarrow$ (b) This equivalence was proven in [CM, Theorem 2.7].

(b) $\Leftrightarrow$ (c) This equivalence follows by Theorem 2.12, in view of (2.7). \hfill $\square$

3. Application to entwining structures

As it is recalled in Section A.3, a coring $D$ over an algebra $L$ is said to be a right extension of a coring $C$ over an algebra $A$ provided that $C$ is a $C-D$ bicomodule, via the left regular coaction. Under the additional assumption that the coring extension is pure (cf. Section A.3), there exists a $k$-linear functor $R : M^C \to M^D$, making the following diagram, involving four forgetful functors, commutative.

![Diagram](image)

The functor $R$ was explicitly constructed in [Brz2], cf. Section A.3. In this section we study pure coring extensions, especially those ones which arise from entwining structures, cf. Section A.3. We focus on the problem of $(M^D, \mathbb{R})$-separability of the functor $U^D$ in Figure (3.1).

The following first result is an easy generalization of [Brz1, Corollary 3.6] to pure coring extensions.

**Proposition 3.1.** Consider an $L$-coring $D$ which is a pure right extension of an $A$-coring $C$, and the corresponding functors in Figure (3.1). The forgetful functor $U^D$ is $(M^D, \mathbb{R})$-separable if and only if the right $D$-coaction $\tau_C$ in $C$ is a split monomorphism of left $C$-comodules and right $D$-comodules.

Proof. The functor $U^D$ possesses a right adjoint, the functor $\bullet \otimes_L D : M_L \to M^D$ (cf. [BW, 18.13]). The unit of the adjunction is given by the $D$-coaction $\tau$. Hence, by Theorem 2.12, $U^D$ is $(M^D, \mathbb{R})$-separable if and only if there exists a natural retraction $\nu$ of $\tau_{\mathbb{R}H}$. Therefore if $U^D$ is $(M^D, \mathbb{R})$-separable then in particular $\tau_C$ possesses a right $D$-linear retraction $\nu_C$. We claim that $\nu_C$ is also left $C$-linear. Indeed, for any right $A$-module $N$ and $n \in N$, the map $C \to N \otimes_A C$, $c \mapsto n \otimes A c$ is right $C$-linear. Hence by the naturality of $\nu$,

$$\nu_{N \otimes A} c (n \otimes c \otimes d) = n \otimes \nu_C (c \otimes d),$$

for $n \in N$, $c \in C$ and $d \in D$. In particular, taking $N = A$, we conclude on the left $A$-linearity of $\nu_C$. Furthermore, a right $C$-coaction $\delta^M : m \mapsto A m^{[0]} \otimes_A m^{[1]}$ (being coassociative) is $C$-linear, hence the naturality of $\nu$ implies

$$\rho^M (\nu_M(m \otimes d)) = \nu_M \otimes_A (\rho^M(m) \otimes d),$$

for any $C$-comodule $M$, $m \in M$ and $d \in D$. Therefore

$$\nu_M (m \otimes d)^{[0]} \otimes_A \nu_M (m \otimes d)^{[1]} = m^{[0]} \otimes_A \nu_C (m^{[1]} \otimes d).$$

Taking $M = C$ we have the left $C$-linearity of $\nu_C$ proven.
Conversely, let \( \tilde{\nu} \) be a left \( C \)-colinear right \( D \)-colinear retraction of \( \tau_C \). The natural transformation \( \nu \) is constructed as follows. For any right \( C \)-comodule \( M \), put
\[
(3.2) \quad \nu_M : M \otimes L D \to M, \quad m \otimes d \mapsto m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d).
\]
Its naturality is obvious. It follows by the \( D \)-colinearity of a \( C \)-coaction \( \rho^M \) that \( \nu_M \) is a retraction of \( \tau_M : m \mapsto m[l_0] \otimes_L m[l_1] \). Indeed,
\[
\nu_M \circ \tau_M(m) = m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes m[l_1]) = m[l_0] \epsilon_C \circ \tilde{\nu} \circ \tau_C(m[l_1]) = m.
\]
It remains to check the \( D \)-colinearity of \( \nu_M \). For \( m \otimes L d \in M \otimes_L D \),
\[
\tau_M \circ \nu_M(m \otimes d) = \left( m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d) \right) [0] \otimes (m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d))[1]
\]
\[
= \left( m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d) \right)[0] \epsilon_C \left( \left( m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d) \right)[1] \right) \otimes \left( m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d) \right)[1]
\]
\[
= m[l_0] \epsilon_C \left( \tilde{\nu}(m[l_1] \otimes d)[0] \right) \otimes \tilde{\nu}(m[l_1] \otimes d)[1]
\]
\[
= m[l_0] \epsilon_C \circ \tilde{\nu}(m[l_1] \otimes d)[1] \otimes d[2] = (\nu_M \otimes D)( \circ (M \otimes \Delta_D) \otimes d),
\]
where the second equality follows by the explicit form of the functor \( R \), relating \( \tau_M \) to \( g^M \), cf. (A.6), the third one follows by the right \( A \)-linearity of a \( C \)-coaction, and the fourth and fifth equalities follow by the left \( C \)-colinearity and the right \( D \)-colinearity of \( \tilde{\nu} \), respectively.

If the two coreings \( C \) and \( D \) are equal and \( R \) is the identity functor, then Proposition 3.3 reduces to [Brz1, Corollary 3.6]. More generally, if \( C \) and \( D \) are coreings over the same base algebra \( A \) and the right \( A \)-actions of the \( A \)-coring \( C \) and the right \( D \)-comodule \( D \) coincide, then \( D \) is a right extension of \( C \) if and only if there exists a homomorphism \( \kappa : C \to D \) (in terms of which the \( D \)-coaction on \( C \) is given by \( \tau_C := (C \otimes_A \kappa) \circ \Delta_C \), cf. [BB3, Corrigendum]. In this case, using the same methods in [Brz1, Corollary 3.6], the map \( \tau_C \) is checked to be a split monomorphism of left \( C \)-comodules and right \( D \)-comodules (i.e. the functor \( U^D \) in Figure (3.1) is checked to be \( (2R^D, \mathcal{R}) \)-separable) if and only if there exists an \( A \)-\( A \) bimodule map \( \tilde{\nu} : C \otimes_A D \to A \), such that
\[
\tilde{\nu} \circ C \otimes_A \kappa \circ \Delta_C = \epsilon_C \quad \text{and} \quad \kappa \circ C \otimes_A \tilde{\nu} \circ (\Delta_C \otimes_A D) = (\tilde{\nu} \otimes_A D) \circ (C \otimes_A \Delta_D).
\]
This extends [Brz1, Theorem 3.5]. On the other hand, for an arbitrary pure corening \( D \) of \( C \), [Brz1, Corollary 3.6] together with Theorem 8.1(1) implies that if \( D \) is a coseparable coring then the functor \( U^D \) in Figure (3.1) is \( (2R^D, \mathcal{R}) \)-separable. This fact follows alternatively by Proposition 3.1: if \( \zeta \) is a \( D \)-\( D \) bicolinear retraction of \( \Delta_D \), then \( (C \otimes_L \epsilon_D \otimes \zeta) \circ (\tau_C \otimes_L D) \) is a \( C \)-\( D \) bicolinear retraction of \( \tau_C \).

Note that, by Corollary 2.9(2), for any pure corening \( D \) of \( C \), \( (2R^D, \mathcal{R}) \)-separability of \( U^D \) implies in particular that every right \( C \)-comodule is relative injective as a right \( D \)-comodule. In what follows we turn to analyzing more consequences of \( (2R^D, \mathcal{R}) \)-separability of \( U^D \), for corening extensions arising from entwining structures \( (A, D, \psi) \) over an algebra \( L \). As the main results of the section, Theorem 3.3 and Proposition 3.4 show that if \( \psi \) is bijective and there exists a grouplike element in \( D \), then \( (2R^D, \mathcal{R}) \)-separability of \( U^D \) implies that \( A \) is relative injective also as an entwined module. A key notion of our study is the following generalization of Doi’s total integral in [Doi].

**Definition 3.2.** Let \( (A, D, \psi) \) be an entwining structure over an algebra \( L \). Assume that \( D \) possesses a grouplike element \( e \) so that \( A \) is a right \( D \)-comodule with coaction \( a \mapsto \psi(e \otimes_L a) \), cf. (A.7). A right \( D \)-comodule map \( j : D \to A \), satisfying the normalization condition \( j(e) = 1_A \), is called a right total integral.

In a bijective entwining structure \( (A, D, \psi) \) over an algebra \( L \), such that \( D \) possesses a grouplike element \( e \), a left total integral is defined as a right total integral in the \( L^e \)-entwining structure \( (A^{op}, D_{cap}, \psi^{-1}) \). This is the same as a left \( D \)-comodule map \( j : D \to A \), with respect to the coaction \( a \mapsto \psi^{-1}(a \otimes_L e) \), cf. (A.8), satisfying the normalization condition \( j(e) = 1_A \).
Consider an entwining structure \((A, D, \psi)\) over an algebra \(L\), and denote by \(C\) the associated \(A\)-coring \(A \otimes_L D\) (cf. Section 3.3). Consider the following diagrams of functors

\[
\begin{array}{ccc}
\mathcal{M}^C & \cong & \mathcal{M}^D_A(\psi) \\
\downarrow \mathcal{T} = \mathcal{U} & & \downarrow \mathcal{T}' = \mathcal{U}^D \\
\mathcal{M}_A & & \mathcal{M}_A
\end{array}
\]

where \(\mathcal{T} = \mathcal{U}^C, \mathcal{T}' = \mathcal{U}^D\) and \(\mathcal{R}\) are forgetful functors (cf. Figure (3.1)). Note that \((\mathcal{T}, \mathcal{H})\) and \((\mathcal{T}' , \mathcal{H}')\) are adjunctions and the respective units \(\eta\) and \(\eta'\) are given by the right \(D\)-coaction, in both cases (cf. Section A.2). Hence they satisfy \(\mathcal{R}(\eta) = \eta'_R\).

**Theorem 3.3.** Let \((A, D, \psi)\) be an entwining structure over an algebra \(L\). Consider the functors in Figure (3.3). The following assertions are equivalent.

(a) \(\mathcal{T} = \mathcal{U}^C\) is \(\mathcal{R}\)-separable of the second kind.

(b) There exists a natural transformation \(\nu : \mathcal{RHT} \rightarrow \mathcal{R},\) satisfying \(\nu_M \circ \mathcal{R}(\eta_M) = \mathcal{R}(M)\), for any \(M \in \mathcal{M}^A_\mathcal{H}(\psi)\).

(c) \(\mathcal{T}' = \mathcal{U}^D\) is \((\mathcal{M}^D, \mathcal{R})\)-separable.

(d) There exists a morphism \(\theta : \mathcal{L}^\mathcal{H}(\mathcal{D} \otimes_L \mathcal{D}, A)\) satisfying, for all \(d, d' \in \mathcal{D}\),

\[
\theta(d \otimes d'(1)) \otimes d'(2) = \psi(d(1) \otimes \theta(d(2) \otimes d')) \quad \text{and} \quad \theta(d(1) \otimes d(2)) = \eta_A \circ \epsilon_D(d).
\]

If these equivalent conditions hold, and in addition there exists a grouplike element in \(D\), then there exists a right total integral in the \(L\)-entwining structure \((A, D, \psi)\).

**Proof.** The equivalence of assertions (a), (b) and (c) is a consequence of Proposition 2.15.

The equivalence of assertions (a) and (d) is proven by an extension to non-commutative base of arguments in [CM, Proposition 4.12], about entwining structures over commutative rings.

Assume that there exists a grouplike element \(e \in D\), hence \(A\) is a right \(D\)-comodule with coaction \((A, L, \Delta_D)\). In this situation the map

\[
j : D \rightarrow A \quad d \mapsto \theta(e \otimes d)
\]

is right \(D\)-colinear and satisfies the normalization condition \(j(e) = 1_A\). That is, \(j\) is a right total integral in the sense of Definition 3.2. \(\square\)

Note that, following the proof of [CM, Proposition 4.12], a bijective correspondence can be obtained between maps \(\theta\) as in (3.4) and left \(C = (A \otimes_L D)\)-colinear right \(D\)-colinear retraction of the \(D\)-coaction \(A \otimes_L \Delta_D\). The explicit relation is given by the same formulae as in [CM, in the paragraph preceding Proposition 4.12]. Since in view of Proposition 3.3 the existence of a left \(C = (A \otimes_L D)\)-colinear right \(D\)-colinear retraction of the \(D\)-coaction \(A \otimes_L \Delta_D\) is equivalent to assertion (c) in Theorem 3.3, in [CM, Proposition 4.12] implicitly also the equivalence of assertions (a) and (c) in Theorem 3.3 is proven.

In contrast to [CM, in the current paper the term total integral is used only in the more restricted sense of Definition 3.2.

The following proposition extends [BB1, Proposition 4.2]. It clarifies the role of total integrals in bijective entwining structures with a grouplike element. For the notion of coinvariants, with respect to a grouplike element in a coring, consult Section A.2.

**Proposition 3.4.** Consider a bijective entwining structure \((A, D, \psi)\) over an algebra \(L\), such that there exists a grouplike element \(e \in D\). Let \(C := A \otimes_L D\) be the associated \(A\)-coring. The following assertions are equivalent.

(a) \(A\) is a relative injective right (resp. left) \(C\)-comodule.

(b) \(A\) is a relative injective right (resp. left) \(D\)-comodule.

(c) There exists a right (resp. left) total integral in the entwining structure \((A, D, \psi)\).

If these equivalent conditions hold then \(B := A^{coC} = A^{coD}\) is a direct summand of \(A\) as a right (resp. left) \(B\)-module.
Proof. \((a) \Rightarrow (b)\) For a relative injective right \(C\)-comodule \(M\), the right \(D\)-coaction has a right \(A\)-linear right \(D\)-colinear retraction. Hence it is a relative injective right \(D\)-comodule.

\((b) \Rightarrow (a)\) Assume that \(A\) is a relative injective right \(D\)-comodule. Similarly to the proof of [SS, Lemma 4.1], in terms of a right \(D\)-colinear retraction \(\nu_A\) of the \(D\)-coaction \((A, \rho_A)\) in \(A\), a right \(C\)-colinear retraction is given by

\[\mu_A \circ [\nu_A(1_A \otimes \bullet) \otimes A] \circ \psi^{-1} : A \otimes D \rightarrow A.\]

The equivalence \((b) \Leftrightarrow (c)\) was proven in [BB1, Proposition 4.2], as follows. To a right \(D\)-colinear retraction \(\nu_A\) of the \(D\)-coaction \((A, \rho_A)\) in \(A\), one associates a right total integral \(j : d \mapsto \nu_A(1_A \otimes L d)\). Conversely, in terms of a right total integral \(j\), a right \(D\)-colinear retraction of the \(D\)-coaction \((A, \rho_A)\) in \(A\) is constructed as \(\nu_A := \mu_A \circ (j \otimes L A) \circ \psi^{-1}\).

It remains to prove the last statement. By property \((a)\), the right \(C\)-coaction in \(A\) is a split monomorphism in \(\mathfrak{M}^C\). Taking the \(C\)-coinvariants part (with respect to the grouplike element \(1_A \otimes L e\)) of its retraction, we obtain right \(B\)-linear retraction of the inclusion \(B \rightarrow A\).

In order to prove the claim about the left comodule structures, the same arguments can be applied to the entwining structure \((A^{op}, C_{cap}, \psi^{-1})\) over the algebra \(L^{op}\).

The following Lemma is a simple generalization of [SS, Lemma 4.1].

**Lemma 3.5.** Let \(C\) be an \(A\)-coRING possessing a grouplike element \(g\). Assume that \(A\) is a relative injective left \(C\)-comodule via the coaction \(a \mapsto a g\), determined by \(g\). Denote by \(B := A^{coC}\) the coinvariants of \(A\) with respect to \(g\). Then the unit of the adjunction \((\bullet \otimes_B A, (\bullet)^{coC})\), i.e. the natural transformation \((A, \rho_A)\), is an isomorphism.

**Proof.** Let \(M\) be a relative injective left \(C\)-comodule and \(\nu_M\) be a left \(C\)-colinear retraction of the coaction \(M \rho\). Introduce a further map \(\xi^M : M \rightarrow C \otimes_A M, m \mapsto g \otimes_A m\). We claim that

\[\xymatrix{M \ar[r]^{M \rho} & C \otimes_A M \ar[r]^{\nu_M} & C \otimes_M \xi^M}

is a contractible pair in \(B \mathfrak{M}\). Clearly, all morphisms \(M \rho, \xi^M\) and \(\nu_M\) are left \(B\)-linear. By definition \(\nu_M \circ M \rho = M\). Hence we conclude after observing that, for \(m \in M\),

\[M \rho \circ \nu_M \circ \xi^M(m) = \nu_M(g \otimes m^{[-1]} \otimes \nu_M(g \otimes m)^{[0]} = g \otimes \nu_M(g \otimes m) = \xi^M \circ M \rho \circ \xi^M(m),\]

where in the second equality the left \(C\)-linearity of \(\nu_M\) has been used. In particular we deduce that the equalizer of \(A^C : A \rightarrow C, a \mapsto a g\) and \(\xi^A : A \rightarrow C, a \mapsto a g\) cosplits in \(B \mathfrak{M}\). Hence it is preserved by the functor \(N \otimes_B \bullet : B \mathfrak{M} \rightarrow B \mathfrak{M}\), for any right \(B\)-module \(N\). Recall that \(A\) is a right \(C\)-comodule with coaction \(\xi^A\) and \(N \otimes_B A\) is a right \(C\)-comodule with coaction \(N \otimes_B \xi^A\). Therefore

\[(N \otimes_B A)^{coC} = \text{Ker}(N \otimes_B A^C - N \otimes_B \xi^A) = N \otimes \text{Ker}(A^C - \xi^A) = N \otimes B \cong N.\]

This proves that \((A, \rho_A)\) is a natural isomorphism, as stated.

The following proposition formulates a functorial criterion for a coring with a grouplike element to be a Galois coring.

**Proposition 3.6.** Let \(C\) be an \(A\)-coRING possessing a grouplike element \(g\). Denote by \(B := A^{coC}\) the coinvariants of \(A\) with respect to \(g\). Consider the adjunction \((\bullet \otimes_B A, (\bullet)^{coC})\) in Section \(A.2\) and the canonical map \(\text{can} : A \otimes_B A \rightarrow C\) in \((A, \rho_A)\). The following statements hold.

1) can is an epimorphism if and only if the functor \((\bullet)^{coC}\) is \((\bullet \otimes_A C, \mathfrak{M}^C\)-faithful.

2) can is a split monomorphism if and only if the functor \((\bullet)^{coC}\) is \((\bullet \otimes_A C, \mathfrak{M}^C\)-full.

In particular, \(C\) is a Galois coring if and only if the functor \((\bullet)^{coC}\) is \((\bullet \otimes_A C, \mathfrak{M}^C\)-fully faithful.

**Proof.** Denote the counit of the coring \(C\) by \(\epsilon\). For any right \(A\)-module \(M\), the counit \(\epsilon\) of the adjunction \((\bullet \otimes_B A, (\bullet)^{coC})\) (cf. \((A, \rho_A)\)) is subject to the equality of maps \((M \otimes_A C)^{coC} \otimes_B A \rightarrow M \otimes_A C,\)

\[(M \otimes \text{can}) \circ (M \otimes_A \epsilon \otimes_B A) = n_{M \otimes_A C}.\]
Since the restriction of $M \otimes_A \epsilon$ is an isomorphism $(M \otimes_A C)^{coC} \rightarrow M \otimes A A$, the claims follow by Theorem 2.13 1) and 2), respectively.

4. COMODULE ALGEBRAS OF HOPF ALGEBROIDS

Consider a Hopf algebroid $H$, with constituent left bialgebroid $H_L$ over the base algebra $L$ and right bialgebroid $H_R$ over $R$ (cf. Section A.10), and a right $H$-comodule algebra $A$ (cf. Appendix A.14). Recall (from Section A.10) that latter means a right $H_R$-comodule algebra and right $H_L$-comodule algebra $A$, with coactions $a \mapsto a[0] \otimes_R a[1]$ and $a \mapsto a[0] \otimes_L a[1]$, respectively, related as in (A.16). Recall (from Appendix A.18) that if the antipode of $H$ is bijective, then the right $H$-comodule algebra structure of $A$ is equivalent to a left $H$-comodule algebra structure of $A^{op}$.

Consider the forgetful functors

$$(4.1) \quad M_A^H \xrightarrow{\mathbb{R}} M^H \xrightarrow{\mathbb{V}} M^L \xrightarrow{\mathbb{U}} M_L.$$ 

In this section we study relative separability of $U$ with respect to $\mathbb{V}\mathbb{R}$ and relative separability of $\mathbb{U}$ with respect to $\mathbb{R}$.

Theorem 4.1. Consider a Hopf algebroid $H$, with constituent left bialgebroid $H_L$, right bialgebroid $H_R$ and antipode $S$. For a right $H$-comodule algebra $A$, the following assertions are equivalent.

(a) There exists a right total integral in the (bijective) $L$-entwining structure (A.19) with grouplike element $1_H$, i.e. a morphism $j \in \text{Hom}^{H_L}(H, A)$, normalized as $j(1_H) = 1_A$.

(b) $A \in M^H_L$ is $\mathbb{R}$-injective (i.e. $A$ is a relative injective right $H_L$-comodule).

(c) Any object in the image of $\mathbb{V}\mathbb{R}$ is $\mathbb{R}$-injective (i.e. injective with respect to $U$).

(d) The functor $\mathbb{U}$ is $(\mathcal{M}^H_L, \mathcal{V}\mathbb{R})$-separable.

If the antipode of $H$ is bijective then the following statements are also equivalent to the foregoing ones.

(e) There exists a left total integral in the (bijective) $R$-entwining structure (A.12) with grouplike element $1_H$, i.e. a left $H_R$-colinear map $j^{op} : H \rightarrow A$, normalized as $j^{op}(1_H) = 1_A$.

(f) $A$ is a relative injective left $H_R$-comodule.

(g) Any object of $\mathcal{M}^H_A$ is a relative injective right $H_R$-comodule.

(h) The forgetful functor $\mathcal{M}^H_R \rightarrow \mathcal{M}^H$ is $(\mathcal{M}^H_R, (\mathcal{V}\mathbb{R})^{op})$-relative separable, where $(\mathcal{V}\mathbb{R})^{op}$ denotes the forgetful functor $\mathcal{M}^H_A \rightarrow \mathcal{M}^H_R$.

If the antipode of $H$ is bijective then the following statements are also equivalent to each other (but not necessarily to the foregoing ones).

(i) There exists a left total integral in the $R$-entwining structure (A.12) with grouplike element $1_H$, i.e. a morphism $j^{op} \in \text{Hom}^{H_R}(H, A)$, normalized as $j^{op}(1_H) = 1_A$.

(j) $A$ is a relative injective right $H_R$-comodule.

(k) Any object of $\mathcal{M}^H_R$ is a relative injective right $H_R$-comodule.

(l) The forgetful functor $\mathcal{M}^{H_R} \rightarrow \mathcal{M}_R$ is $(\mathcal{M}^{H_R}, (\mathcal{V}\mathbb{R})^{op})$-relative separable, where $(\mathcal{V}\mathbb{R})^{op}$ denotes the forgetful functor $\mathcal{M}^H_A \rightarrow \mathcal{M}^{H_R}$.

(m) There exists a left total integral in the (bijective) $L$-entwining structure (A.19) with grouplike element $1_H$, i.e. a left $H_L$-colinear map $j_{cop} : H \rightarrow A$, normalized as $j_{cop}(1_H) = 1_A$.

(n) $A$ is a relative injective left $H_L$-comodule.

(o) Any object of $\mathcal{M}^H_R$ is a relative injective left $H_L$-comodule.

(p) The forgetful functor $\mathcal{M}^H_L \rightarrow \mathcal{M}_R$ is $(\mathcal{M}^H_L, (\mathcal{V}\mathbb{R})_{cop})$-relative separable, where $(\mathcal{V}\mathbb{R})_{cop}$ denotes the forgetful functor $\mathcal{M}^H_A \rightarrow \mathcal{M}^H_L$.

Proof. (a) $\Leftrightarrow$ (b) This equivalence follows by Proposition 3.4. (b) $\Leftrightarrow$ (c), since the entwining map (A.19) is bijective.

(a) $\Rightarrow$ (d) In light of Theorem 2.12 3), we need to construct a right $H_L$-colinear natural retraction $\nu_M$ of the $H_L$-coaction, for any $M \in M^H_A$. In terms of the map $j$ in part (a), it is given by the well defined maps

$$(4.2) \quad \nu_M : M \otimes_L H \rightarrow M, \quad m \otimes_L h \mapsto m[0]j(S(m[1])h),$$
Hence also the right $\mathcal{H}_R$-coaction on $M$.

(d) $\Rightarrow$ (c) The forgetful functor $\mathfrak{M}^H_L \to \mathfrak{M}_L$ has a right adjoint, the functor $\bullet \otimes_L H$, cf. Section A.2. Hence the claim follows by Corollary 2.9 2).

(c) $\Rightarrow$ (b) This implication is trivial as $A$ itself is an object in $\mathfrak{M}_L^H$.

If the antipode is bijective then implications (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) follow by applying (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) to the opposite-coopposite Hopf algebroid $\mathcal{H}_{\text{op}}$ and its right comodule algebra $A$ (with $(\mathcal{H}_R)_{\text{op}}^\text{cop}$-coaction $a \mapsto a [0] \otimes_{\text{op}} S(a [1])$ and $(\mathcal{H}_L)_{\text{cop}}^\text{cop}$-coaction $a \mapsto a [0] \otimes_{\text{op}} S(a [1])$).

(a) $\Leftrightarrow$ (e) In terms of $j$ in part (a), a normalized left $\mathcal{H}_R$-comodule map is given by $j \circ S^{-1}$. Clearly, it yields a bijective correspondence.

The other sequence of equivalences (i)-(p) follows by applying the proven result to the opposite (or coopposite) Hopf algebroid and its right comodule algebra $A^{\text{op}}$.

\[\Box\]

**Theorem 4.2.** Let $\mathcal{H}$ be a Hopf algebroid with constituent left bialgebroid $(H, L, s_L, t_L, \gamma_L, \pi_L)$, right bialgebroid $(H, R, s_R, t_R, \gamma_R, \pi_R)$, and antipode $S$. For a right $\mathcal{H}$-comodule algebra $A$, the following assertions are equivalent.

(a) There exists a normalized right $\mathcal{H}$-comodule map $j : H \to A$.

(b) $A \in \mathfrak{M}^H$ is $I_{UV}$-injective.

(c) Any object in the image of $R$ is $I_{UV}$-injective.

(d) The functor $UV$ is $(\mathfrak{M}^H, \mathfrak{R})$-separable.

If the antipode of $\mathcal{H}$ is bijective, then these equivalent statements are equivalent also to the existence of a normalized left $\mathcal{H}$-comodule map $H \to A$, hence the symmetrical counterparts of (b)-(d).

\[\Box\]

**Proof.** (d) $\Rightarrow$ (c) follows by statement 1) in Appendix A.12 and Corollary 2.9 2).

(c) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a) Denote by $\eta : R \to A$ the unit of the $R$-ring $A$. Since $\eta \circ \pi_R \circ t_L : L \to A$ and $t_L : L \to H$ are $\mathcal{H}$-comodule maps and $t_L$ is a split monomorphism of right $L$-modules, using $I_{UV}$-injectivity of $A$, $j$ is constructed as the unique $\mathcal{H}$-comodule map for which $j \circ t_L = \eta \circ \pi_R \circ t_L$.

(a) $\Rightarrow$ (d) We need to construct a natural retraction $\nu_M$ of the $\mathcal{H}_L$-coaction, for any object $M$ in $\mathfrak{M}_L^H$. In terms of the map $j$ in part (a), it is given by the same formula (4.2). Since $j$ is an $\mathcal{H}$-comodule map, so in $\nu_M$.

If the antipode is bijective then any (normalized) right $\mathcal{H}$-comodule map $j : H \to A$ determines a (normalized) left $\mathcal{H}$-comodule map $j \circ S^{-1} : H \to A$. This correspondence is clearly bijective. \[\Box\]

Obviously, if the equivalent statements in Theorem 4.2 hold then also the equivalent statements in Theorem 4.1 hold.

**Lemma 4.3.** Let $\mathcal{H}$ be a Hopf algebroid with constituent left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$, right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$, and a bijective antipode $S$. Assume that $H$ is a projective right $\mathcal{H}_R$-coaction for the $R$-coring $(H, \gamma_R, \pi_R)$ via $\gamma_R$. Then $H$ is a projective left $L$-module via left multiplication by $s_L$.

\[\Box\]

**Proof.** By [BW] 18.20(1)], projectivity of $H$ as a right $\mathcal{H}_R$-comodule implies that $H$ is a projective right $R$-module via the action

\[H \otimes R \to H, \quad h \otimes r \mapsto h s_R(r).\]

By bijectivity of the antipode, the right $R$-module (4.3) is isomorphic to the right $R$-module $H$, with action

\[H \otimes R \to H, \quad h \otimes r \mapsto t_R(r) h.\]

Hence also the right $R$-module (4.3) is projective. Furthermore, the algebra isomorphism $\pi_R \circ s_L : L^{\text{op}} \to R$ induces a category isomorphism $\mathfrak{M}_R \cong L \mathfrak{M}$. This isomorphism takes the projective right $R$-module (4.3) to the projective left $L$-module $H$, with action

\[L \otimes H \to H, \quad l \otimes h \mapsto t_R \circ \pi_R \circ s_L(l) h = s_L(l) h.\]

\[\Box\]
Theorem 4.2 makes us able to answer a question which was left open in \[\text{[Bö2]}\]. Consider a Hopf algebroid \(H\) with a bijective antipode and a right \(H\)-comodule algebra \(A\). Denote \(B := A^{\text{co} H} = A^{\text{co} H_L}\), cf. \([A,12]\). By Appendix \([A,19]\) one can associate to \(A\) four (anti-) isomorphic corings. Clearly, if any of them is a Galois coring (with respect to the grouplike element determined by the unit elements in \(A\) and \(H\)), then all of them are Galois corings. In other words, the four properties that \(B \subseteq A\) is a left or right Galois extension by \(H_R\) or \(H_L\) are all equivalent to each other. In Proposition 4.4 below, \(H\)-comodule algebras \(A\) are studied, such that these equivalent Galois conditions hold.

**Proposition 4.4.** Let \(H\) be a Hopf algebroid with constituent left bialgebroid \(H_L = (H, L, s_L, t_L, \gamma_L, \pi_L)\), right bialgebroid \(H_R = (H, R, s_R, t_R, \gamma_R, \pi_R)\), and a bijective antipode \(S\). Assume that \(H\) is a projective left \(R\)-module via \(t_R\) and a projective right \(R\)-comodule for the \(R\)-coaction \((H, \gamma_R, \pi_R)\) via \(\gamma_R\). (These assumptions hold e.g. if \(H\) is a finitely generated and projective both as a right and left \(L\)-module and also as a right and left \(R\)-module, cf. \([Bö2, \text{Section } 4]\).) Then \(\mathcal{M}^{H_L} \cong \mathcal{M}^H \cong \mathcal{M}^{H_R}\) and \(\mathcal{H}^L \mathcal{M} \cong \mathcal{H} \mathcal{M} \cong \mathcal{H}^R \mathcal{M}\) as monoidal categories. Moreover, for a right \(H\)-comodule algebra \(A\), such that \(B := A^{\text{co} H} \subseteq A\) is a right \(H_R\)-Galois extension, the following assertions are equivalent:

1. \(A\) is a faithfully flat right \(B\)-module.
2. \(B\) is a direct summand of the right \(B\)-module \(A\).
3. The functors \(A \otimes_B \bullet : B \mathcal{M} \rightarrow A \mathcal{M}\) and \(c^H(\bullet) : A \mathcal{M} \rightarrow B \mathcal{M}\) are inverse equivalences and \(H \otimes_R A\) is a flat right \(A\)-module.
4. \(A\) is a projective generator in \(A \mathcal{M}\) and \(H \otimes_R A\) is a flat right \(A\)-module.
5. \(A\) is a generator of right \(B\)-modules.
6. \(B\) is a direct summand of the left \(B\)-module \(A\).
7. The functors \(\bullet \otimes_B A : B \mathcal{M} \rightarrow A \mathcal{M}\) and \(c^H(\bullet) : A \mathcal{M} \rightarrow B \mathcal{M}\) are inverse equivalences.
8. \(A\) is a projective generator in \(A \mathcal{M}\).
9. \(A\) is a generator of left \(B\)-modules.
10. The equivalent conditions in Theorem 4.2 hold.

**Proof.** Since \(H\) is a projective left \(R\)-module by assumption, it is in particular flat. As a left \(L\)-module, \(H\) is projective hence flat by Lemma 4.3. Thus the monoidal isomorphisms \(\mathcal{M}^H \cong \mathcal{M}^L \cong \mathcal{M}^{H_R}\) follow by \([A,13]\) and \([A,14]\). By Appendix \([A,18]\) bijectivity of the antipode implies strict anti-monoidal isomorphisms \(\mathcal{H}^R \mathcal{M} \cong \mathcal{M}^{H_L}\), \(\mathcal{H} \mathcal{M} \cong \mathcal{M}^{H_R}\) and \(\mathcal{H} \mathcal{M} \cong \mathcal{M}^H\). Hence also \(\mathcal{H} \mathcal{M} \cong \mathcal{H}^L \mathcal{M} \cong \mathcal{H}^R \mathcal{M}\). Note that this implies in particular \(\mathcal{H} \mathcal{M} \cong A \mathcal{M}\) and \(\mathcal{H}^R \mathcal{M} \cong A \mathcal{M}\).

(a) \(\iff\) (b) and (f) \(\iff\) (g) These equivalences follow by \([\text{Row}, \text{2.11.29}]\), as \(A\) is a projective left and right \(B\)-module by \([\text{Bö2}, \text{Proposition } 4.2]\).

(b) \(\Rightarrow\) (k) and (k) \(\Rightarrow\) (b) Note that in the current case assertion (k) is equivalent to relative injectivity of \(A\) as a right \(H_R\)-comodule, or as a right \(H_L\)-comodule, or as a left \(H_R\)-comodule. Since a comodule algebra for a Hopf algebroid with a bijective antipode determines bijective entwining structures \([A,12]\) and \([A,19]\), these implications follow by \([\text{BB1}, \text{Proposition } 4.1]\) (which is a simple generalization of \([\text{SS}, \text{Remark } 4.2]\) to the case of non-commutative base algebras).

(k) \(\Rightarrow\) (b) and (k) \(\Rightarrow\) (g) These assertions follow by Proposition 4.4.

(a) \(\iff\) (d) \(\iff\) (c) Since \(\mathcal{H} \mathcal{M} \cong \mathcal{H}^R \mathcal{M}\) is isomorphic to the category of left comodules for the \(A\)-coaction \(H \otimes_R A\) (cf. Appendix \([A,8]\), these equivalences follow by the Galois Coring Structure Theorem \([\text{BW}, \text{28.19 (2)}]\)).

(f) \(\iff\) (i) \(\iff\) (h) Since \(H\) is a projective left \(R\)-module by assumption, \(A \otimes_R H\) is a projective (hence flat) left \(A\)-module. Therefore also these equivalences follow by the Galois Coring Structure Theorem \([\text{BW}, \text{28.19 (2)}]\), as \(\mathcal{M}^H_A \cong \mathcal{M}^{H_R} \cong \mathcal{M}^{A \otimes_R H}\).

(b) \(\Rightarrow\) (e) and (g) \(\Rightarrow\) (j) These implications are trivial.

(e) \(\Rightarrow\) (b) and (j) \(\Rightarrow\) (g) A is a generator of right (respectively, left) \(B\)-modules if and only if there exist finite sets \(\{a_i\}\) in \(A\) and \(\{a_i\}\) in \(\text{Hom}_B(A, B)\) (respectively, in \(B\text{Hom}(A, B)\)), satisfying \(\sum_i a_i(a_i) = 1_B\). In terms of these elements, a right \(B\)-linear retraction of the inclusion \(B \rightarrow A\) is...
given by the map $a \mapsto \sum_i \alpha_i(a_i a)$ (respectively, a left $B$-linear retraction of the inclusion $B \to A$ is given by the map $a \mapsto \sum_i \alpha_i(a a_i)$).

Applying Proposition 4.4 to the co-opposite Hopf algebroid $H_{\mathsf{cop}}$, we see that the claims in Proposition 4.4 — with the only modification that claims (h) and (i) need to be supplemented by the assertion that $A \otimes_R H$ is a flat left $A$-module — can be proven alternatively by replacing the assumptions about the projectivity of $H$ a left $R$-module (via $t_H$) and a right $H_{\mathsf{op}}$-comodule (via the coproduct of $H_{\mathsf{R}}$) with the assumptions that it is a projective right $R$-module (via $s_R$) and a projective left $H_{\mathsf{R}}$-comodule (via the coproduct of $H_{\mathsf{R}}$).

5. A Schneider type theorem

This section contains the main result of the paper, Theorem 5.6. The starting point of our study is the following result [BTW, Theorem 2.1]. Recall that a right comodule $P$ for an $A$-coring $C$, which is a finitely generated and projective right $A$-module, is a Galois comodule if the canonical map

$$\text{(5.1)} \quad \text{can} : \text{Hom}_A(P,A) \otimes_S P \to C, \quad \phi \otimes p \mapsto \phi(p[0])p[1]$$

is bijective, where $S := \text{End}^C(P)$. Assume that $S$ is a $T$-ring (e.g. $T$ is a subalgebra of $S$). Denote $P^* = \text{Hom}_A(P,A)$. A symmetrical (and slightly extended) version of [BTW, Theorem 2.1], formulated for right comodules, is the following.

**Theorem 5.1.** The canonical map (5.1) is bijective and $P^*$ is a $T$-relative projective right $S$-module provided that the following conditions hold true.

(i) The map $P^* \otimes_T S \to \text{Hom}^C(P,P^* \otimes_T P)$, $\xi \otimes_T s \mapsto \left( p \mapsto \xi \otimes_T s(p) \right)$ is an isomorphism (of right $S$-modules);

(ii) The lifted canonical map,

$$\text{(5.2)} \quad \widetilde{\text{can}}^T : P^* \otimes_T P \to C, \quad \phi \otimes p \mapsto \phi(p[0])p[1]$$

is a split epimorphism of right $C$-comodules.

Motivated by this result, in the present section we investigate how one can use $(\mathfrak{AH}, \mathbb{R})$-separability of the functor UV in (3.1) to derive properties i) and ii) in Theorem 5.1 for a coring $A \otimes_R H$ associated to a right comodule algebra $A$ of a Hopf algebroid $H$.

**Remark 5.2.** In the particular case when the right $C$-comodule $P$ in Theorem 5.1 is equal to the base algebra $A$, property i) reduces to $(A \otimes_T A)^{\mathsf{coC}} = A \otimes_T A^{\mathsf{coC}}$. Let us investigate this condition. Note that, for an $A$-coring $C$ possessing a grouplike element $g$, and any right $T$-module $V$, $V \otimes_T A$ is a right $C$-comodule via the comodule structure of $A$. There is an obvious map $V \otimes_T A^{\mathsf{coC}} \to (V \otimes_T A)^{\mathsf{coC}}$, which is an isomorphism in appropriate situations: e.g. if $V$ is a flat $T$-module, or in the situation described in Lemma 3.3. Indeed, in the last case, by applying Lemma 3.3 to a right $B := A^{\mathsf{coC}}$-module $V \otimes_T B$, for a right $T$-module $V$, we conclude that $(V \otimes_T A)^{\mathsf{coC}} = V \otimes_T B$ whenever $A$ is a relative injective left $C$-comodule.

As it is explained in Appendix A.19, to a right comodule algebra $A$ of a Hopf algebroid $H$ (with constituent left and right bialgebroids $(H, L, s_L, t_L, \gamma_L, \pi_L)$ and $(H, R, s_R, t_R, \gamma_R, \pi_R)$, and a bijective antipode $S$) one associates two isomorphic $A$-corings, on the $k$-modules $A \otimes_R H$ and $H \otimes_R A$, and two isomorphic $A^{\mathsf{op}}$-corings, on the $k$-modules $A \otimes_R H$ and $H \otimes_R A$. These $A$- and $A^{\mathsf{op}}$-corings are anti-isomorphic, cf. (A.23). The grouplike element $1_H$, in the $L$- and $R$-corings underlying $H$, determines grouplike elements in all associated $A$- and $A^{\mathsf{op}}$-corings (preserved by the coring (anti-) isomorphisms (A.12), (A.19) and (A.23) between them). That is, $A$ (or $A^{\mathsf{op}}$) is a right comodule in each case. Corresponding to the four corings, there are four canonical maps of the type (5.1), which differ by the respective coring (anti-) isomorphisms in Section A.19.

Following Theorem 5.6 is formulated in terms of the $A$-coring $A \otimes_R H$ and the corresponding canonical map (A.13). Certainly, all claims can be reformulated in terms of any of the other three (anti-) isomorphic corings.
Let \( \mathcal{H} \) be a Hopf algebroid over base algebras \( L \) and \( R \). Recall that a right \( \mathcal{H} \)-comodule algebra \( A \) is an \( R \)-ring. Assume that the coinvariant subalgebra \( B := A^{\mathcal{H}\text{co}R} \) is a \( T \)-ring (e.g. \( T \) is a \( k \)-subalgebra of \( B \)). Consider the lifted version of the canonical map (A.13)

\[
(A.13) \quad \text{can}^T : A \otimes_T A \to A \otimes_R H, \quad a \otimes a' \mapsto aa' \otimes a'^{[0]} \otimes a'^{[1]}.
\]

Note that it is right \( L \)-linear with respect to the module structures

\[
(a \otimes a')l = a \otimes \pi_R \circ t_L(l)a' \quad \text{and} \quad (a \otimes h)l = a \otimes t_L(l)h,
\]

for \( a \otimes a' \in A \otimes_T A, a \otimes h \in A \otimes_R H \) and \( l \in L \). Moreover, the lifted canonical map (5.3) is also left \( L \)-linear with respect to the module structures

\[
l(a \otimes a') = a \pi_R \circ s_L(l) \otimes a' \quad \text{and} \quad l(a \otimes h) = a \otimes s_L(l)h,
\]

for \( a \otimes a' \in A \otimes_T A, a \otimes h \in A \otimes_R H \) and \( l \in L \).

**Remark 5.3.** Consider a Hopf algebroid \( \mathcal{H} \) and a right \( \mathcal{H} \)-comodule algebra \( A \). The lifted canonical map (5.3) is a split epimorphism of right \( L \)-modules i.e., using the notations in (1.4), it belongs to \( \mathcal{E}_{\mathcal{UVR}} \) in various situations.

1) If the (right \( L \)-linear) canonical map (A.13) is surjective and \( A \otimes_R H \) is a projective right \( L \)-module. The latter condition holds provided that \( H \) is a projective right \( L \)-module (via \( t_L \)) and \( A \) is a projective right \( R \)-module.

2) If the (right \( A \)-linear) canonical map (A.13) is surjective, \( A \otimes_R H \) is a projective right \( A \)-module (e.g. \( H \) is a projective right \( R \)-module via \( s_R \)) and \( \mathcal{E}_{\mathcal{UC}} \subseteq \mathcal{E}_{\mathcal{UVR}} \), where \( \mathcal{UC} \) denotes the forgetful functor \( \mathcal{M}_A^H \to \mathcal{M}_A \).

The condition \( \mathcal{E}_{\mathcal{UC}} \subseteq \mathcal{E}_{\mathcal{UVR}} \) holds whenever dealing with a comodule algebra \( A \) of a Hopf algebroid \( \mathcal{H} \) over a commutative ring \( k \). Indeed, in this case \( \mathcal{V} \) is the identity functor \( \mathcal{M}^H \), and the functors \( \mathcal{UC}, \mathcal{R} \) and \( \mathcal{U} \) are forgetful functors. A fourth forgetful functor \( \mathcal{M}_A \to \mathcal{M}_k \) makes the following diagram commutative.

\[
\begin{array}{ccc}
\mathcal{M}_A^H & \xrightarrow{\mathcal{R}} & \mathcal{M}_A^H \\
\mathcal{UC} & \searrow & \mathcal{U} \\
\mathcal{M}_A & \xrightarrow{=} & \mathcal{M}_k
\end{array}
\]

This proves that in this case \( \mathcal{E}_{\mathcal{UC}} \subseteq \mathcal{E}_{\mathcal{UVR}} \), thus assumptions 2) hold e.g. in Schneider’s theorem [Schn, Theorem I].

**Lemma 5.4.** Consider a right comodule algebra \( A \) of a Hopf algebroid \( \mathcal{H} \) as a right comodule algebra of the constituent right bialgebroid in \( \mathcal{H} \). Then the following maps are morphisms in \( \mathcal{M}_A^H \).

1) the entwining map in (A.12);
2) the lifted canonical map (5.3).

**Proof.** \( H \otimes_R A \) is an object in \( \mathcal{M}_A^H \) via the \( A \)-action induced by the multiplication in \( A \) and the diagonal \( \mathcal{H}_R^\ast \) and \( \mathcal{H}_L^\ast \)-coactions. \( A \otimes_R H \) is an object in \( \mathcal{M}_A^H \) via the \( A \)-action \((a \otimes_R h)aa' = aa' \otimes_R h a'^{[0]} \otimes a'^{[1]} \), (where \( a \to a^{[0]} \otimes_R a^{[1]} \) denotes the \( \mathcal{H}_R \)-coaction) and \( \mathcal{H}_R^\ast \) and \( \mathcal{H}_L^\ast \)-coactions induced by the respective coproducts in \( \mathcal{H} \). \( A \otimes_T A \) is an object in \( \mathcal{M}_A^H \) via the relative Hopf module structure of the second factor. It is left to the reader to check that both maps in the lemma are compatible with these structures. \(\square\)

**Lemma 5.5.** Let \( \mathcal{H} \) be a Hopf algebroid with constituent left bialgebroid \((H, L, s_L, t_L, \gamma_L, \pi_L)\), right bialgebroid \((H, R, s_R, t_R, \gamma_R, \pi_R)\), and a bijective antipode \( S \). Let \( A \) be a right \( \mathcal{H} \)-comodule algebra such that there exists a normalized right \( \mathcal{H} \)-comodule map \( j : H \to A \). Assume that \( A^{\mathcal{H}\text{co}R} \) is a \( T \)-ring (e.g. \( T \) is a subalgebra of \( A^{\mathcal{H}\text{co}R} \)). Assume furthermore that the lifted canonical map (5.3)
possesses a right $L$-module section $\zeta^T_0$ (with respect to the module structures (5.4)). Then (5.3) possesses a section in $\mathcal{M}_A^H$, given as
\[
\zeta^T : A \otimes_R H \to A \otimes_T A,
\]
where $\alpha \mapsto a_0 \otimes_L A[a_1]$ and $\alpha \mapsto a_0 \otimes_R A[a_1]$ are the $\mathcal{H}_L$- and $\mathcal{H}_R$-coactions in $A$, respectively, related via (A.11), and $\gamma_L(h) = h(1) \otimes L h(2)$, for $h \in H$.

Proof. By Theorem 4.2, the forgetful functor $\mathcal{M}_H \to \mathcal{M}_L$ is $(\mathcal{M}_H, \mathcal{R})$-separable, where $\mathcal{R}$ is the forgetful functor $\mathcal{M}_A^H \to \mathcal{M}_A^H$. By Theorem 2.3 this implies that (5.3) is a split epimorphism in $\mathcal{M}_A^H$. Furthermore, by Lemma 5.1, the object $A \otimes_R H \in \mathcal{M}_A^H$ is isomorphic to $H \otimes_R A$. Since by definition $\mathcal{M}_A^H$ is the category of modules for the monad $- \otimes_R A : \mathcal{M}_H \to \mathcal{M}_H$, the forgetful functor $\mathcal{M}_A^H \to \mathcal{M}_H$ possesses a left adjoint $- \otimes_R A : \mathcal{M}_H \to \mathcal{M}_A^H$ (where for any right $\mathcal{H}$-comodule $M$, $M \otimes_R A$ is a relative Hopf module via the $A$-action on the second factor and the diagonal coactions). Hence $H \otimes_R A$ (thus $A \otimes_R H$) is $\mathcal{E}_R$-projective by Theorem 2.3. This proves that the lifted canonical map (5.3) is a split epimorphism in $\mathcal{M}_A^H$.

A section can be explicitly constructed as follows. A right $\mathcal{H}$-comodule section $\zeta^T$ can be constructed using arguments in the proof of the Rafael type theorem 4.2. Indeed, in terms of the unit of the adjunction of the forgetful functor $\mathcal{M}_A^H \to \mathcal{M}_H$,
\[
(\zeta^T_0)(1_A \otimes h(1))S^{-1}(a_1(1)_1)(1)_1j \left( S(\zeta^T_0)(1_A \otimes h(1))S^{-1}(a_1(1)_1)(1)_1h(2)S^{-1}(a_1(1)_1) \right) a_0,
\]
where $a \mapsto a_0 \otimes_L A[a_1]$ and $a \mapsto a_0 \otimes_R A[a_1]$ are the $\mathcal{H}_L$- and $\mathcal{H}_R$-coactions in $A$, respectively, related via (A.11), and $\gamma_L(h) = h(1) \otimes L h(2)$, for $h \in H$.

The natural retraction $\nu$ was constructed in (1.2). Thus a right $A$-module right $\mathcal{H}$-comodule section of the lifted canonical map (5.3) is given by
\[
\zeta^T = (A \otimes \mu) \circ (\zeta^T_0 \otimes A) \circ (\psi_R \otimes A) \circ (H \otimes_R \eta \otimes R) \circ \psi_R^{-1},
\]
where $\eta : R \to A$ and $\mu : A \otimes_R A \to A$ are unit and multiplication maps in the $R$-ring $A$.

The map $\zeta^T$ comes out explicitly as in the claim. □

**Theorem 5.6.** Let $\mathcal{H}$ be a Hopf algebroid with constituent left bialgebroid $H(L, s_L, t_L, \gamma_L, \tau_L)$, bialgebroid $H(R, s_R, t_R, \gamma_R, \tau_R)$, and a bijective antipode $S$. Let $A$ be a right $\mathcal{H}$-comodule algebra and put $B := A^{\text{co}\mathcal{H}_R}$. Assume that $B$ is a $T$-ring (e.g. $T$ is a $k$-subalgebra of $B$).

1) If the lifted canonical map (5.3) is a split epimorphism of right $L$-modules (with respect to the module structures (5.4)) then the following assertions are equivalent.
   (a) The canonical map $A \otimes_B A \to A \otimes_R H$ in (A.13) is bijective and $B$ is a direct summand of $A$ as a right $B$-module (hence $A$ is a generator of right $B$-modules).
   (b) The equivalent conditions in Theorem 4.3 hold.
   (c) The functor $\text{co}\mathcal{H}_R(\bullet) : \mathcal{M}_A^H \to \mathcal{M}_B^H$ is an equivalence, with inverse $A \otimes_B \bullet$, and $B$ is a direct summand of $A$ as a right $B$-module.

Furthermore, if these equivalent statements hold then $A$ is a $T$-relative projective right $B$-module.

2) If the lifted canonical map (5.3) is a split epimorphism of left $L$-modules (with respect to the module structures (5.5)) then the following assertions are equivalent.
   (a) The canonical map (A.13) is bijective and $B$ is a direct summand of $A$ as a left $B$-module.
   (b) The equivalent conditions in Theorem 4.3 hold.
   (c) The functor $(\bullet)^{\text{co}\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B^H$ is an equivalence, with inverse $(\bullet) \otimes_B A$, and $B$ is a direct summand of $A$ as a left $B$-module.

Furthermore, if these equivalent statements hold then $A$ is a $T$-relative projective left $B$-module.
Proof. Recall from Section A.10 that bijectivity of the antipode \( S \) in the Hopf algebroid \( \mathcal{H} \) implies that also the entwining map \( (A,13) \) is bijective.

\((a) \Rightarrow (b)\) In terms of the canonical map \( (A,13) \), introduce the index notation \( h^{(1)}_R \odot B h^{(2)}_R := \text{can}^{-1}(1_A \otimes_R h) \) for \( h \in \mathcal{H} \) (implicit summation is understood). Using a right \( B \)-module retraction \( p \) of the inclusion \( B \to A \), a normalized right \( \mathcal{H} \)-comodule map is given by \( j : H \to A, h \mapsto p(h^{(1)}_R)h^{(2)}_R \).

\((b) \Rightarrow (a)\) The lifted canonical map \( (5.3) \) is a split epimorphism in \( \mathfrak{M}^H_A \) by Lemma 3.3. Hence it is in particular a split epimorphism of right \( \mathcal{H} \)-comodules for the \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \). By considerations in Section A.3, coinvariants of the right \( \mathcal{H} \)-comodules \( A \) and \( A \otimes_T \mathcal{A} \) (latter one defined via the second tensorand) for the \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \) coincide with the \( \mathcal{H} \)-coinvariants in \( A \) and \( A \otimes_T \mathcal{A} \), respectively. By Theorem 4.1 A (with coaction \( a \mapsto S^{-1}(a_{(1)}) \otimes_R a_{(0)} \)) is a relative injective left \( \mathcal{H} \)-comodule for \( \mathcal{H} \). So that, by Proposition 3.4 \((b) \Rightarrow (a)\), \( A \) is a relative injective left \( \mathcal{H} \)-comodule for the \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \). Taking Remark 5.2 into account, it follows that \( (A \otimes_T \mathcal{A})^\mathcal{H}_R = A \otimes_T B \), hence all assumptions in Theorem 5.1 hold. Therefore the canonical map \( (A,13) \) is bijective and \( A \) is a \( T \)-relative projective right \( B \)-module by Theorem 5.1. It follows by Proposition 3.4 that the right regular \( B \)-module is a direct summand in \( A \).

\((b) \Rightarrow (c)\) By Theorem 4.1 and Proposition 3.4 \((b) \Rightarrow (a)\), assertion \( (b) \) in the claim implies that \( A \) is a relative injective right \( \mathcal{H} \)-comodule for the \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \). The \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \) possesses a grouplike element \( 1_A \otimes \mathcal{H}_R \), hence the unit of the antipode \( \mathcal{H}_R \), that is \( (A,22) \), one checks that \( (A,22) \) is a relative injective left \( \mathcal{H} \)-comodule for \( \mathcal{H} \). So that, by Proposition 3.4 \((b) \Rightarrow (a)\), \( A \) is a relative injective left \( \mathcal{H} \)-comodule for the \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \). Taking Remark 5.2 into account, it follows that \( (A \otimes_T \mathcal{A})^\mathcal{H}_R = A \otimes_T B \), hence all assumptions in Theorem 5.1 hold. Therefore the canonical map \( (A,13) \) is bijective and \( A \) is a \( T \)-relative projective right \( B \)-module by Theorem 5.1. It follows by Proposition 3.4 that the right regular \( B \)-module is a direct summand in \( A \).

Let us construct the inverse of the counit,
\[
n_M : A \otimes_R \mathcal{H}_R M \to M, \quad a \otimes m \mapsto am,
\]
for \( M \in \mathfrak{M}^H_A \). Denote the left \( \mathcal{H}_R \)-, and \( \mathcal{H}_L \)-coactions on \( M \) by \( m \mapsto m_{[-1]} \otimes_R m_0 \) and \( m \mapsto m_{[-1]} \otimes_L m_0 \), respectively. The canonical map \( (L,13) \) is bijective by part \( (a) \). Consider the map
\[
(5.6) \quad M \to A \otimes_R M, \quad m \mapsto \text{can}^{-1}(1_A \otimes m_{[-1]} m_0).
\]
By Lemma 3.3, the lifted canonical map \( (5.3) \) has a section \( \zeta_T \) in \( \mathfrak{M}^H_A \). The map \( (L,13) \) is equal to the composite of
\[
(5.7) \quad M \to A \otimes_R M, \quad m \mapsto \zeta_T(1_A \otimes_R m_{[-1]} m_0)
\]
and the canonical epimorphism \( A \otimes_T M \to A \otimes_B M \). We claim that the range of \( (5.7) \) is in \( A \otimes_T \mathcal{H} \)-comodules. By Theorem 4.1, the left \( \mathcal{H}_L \)-coaction in \( M \) has a retraction in \( \mathcal{H} \)-comodules. The \( \mathcal{H}_R \)-coinvariants part' of this retraction yields a \( k \)-linear retraction of the inclusion \( \mathcal{H}_R \)-comodules. Explicitly, in terms of a normalized right \( \mathcal{H} \)-comodule map \( j : H \to A \) (cf. Theorem 4.1 \((a)\)), we obtain an idempotent map
\[
E_M : M \to \mathcal{H}_R M, \quad m \mapsto j(m_{[-1]} m_0).
\]
Consider the right \( L \)-module \( A \) with action \( al := \pi_R \circ \tau_L(l)a \), for \( l \in L \) and \( a \in A \). Take a normalized right \( \mathcal{H} \)-comodule map \( j : H \to A \) as in part \( (a) \) of Theorem 4.1 and introduce a left \( B \)-module map
\[
P_M : A \otimes_R M \to M, \quad a \otimes m \mapsto a_0 j(S(a_1 m_{[-1]} m_0)).
\]
It is well defined by the right \( L \)-linearity of the right \( \mathcal{H}_R \)-coaction in \( M \) and the left \( L \)-linearity of the left \( \mathcal{H}_L \)-coaction in \( M \), and module map properties of \( S \) and \( j \). Making use of the relative Hopf module structure of \( M \), that is \( (A,22) \), one checks that \( E_M \circ P_M = P_M \). This means that the range of \( P_M \) is within \( \mathcal{H}_R \)-comodules. Since the section \( \zeta_T \) in Lemma 3.3 of the lifted canonical map \( (L,13) \) satisfies, for \( m \in M \),
\[
\zeta_T(1_A \otimes_R m_{[-1]} m_0) = \left( (A \otimes_T P_M) \circ (\zeta_T \otimes M) \right)(1_A \otimes_R m_{[-1]} m_0),
\]
the range of \( (5.7) \) is in \( A \otimes_T \mathcal{H}_R M \). This implies that the range of \( (5.6) \) is in \( A \otimes_B \mathcal{H}_R M \). The proof is completed by showing that the corestriction of \( (5.6) \) to a map \( n_M : M \to A \otimes_B \mathcal{H}_R M \)
yields the inverse of \(n_M\). Indeed, since \((A \otimes_R \pi_R) \circ \text{can}(a \otimes_B a') = aa'\), for \(a, a' \in A\), and can is bijective, \(n_M \circ n_M (m) = \pi_B (m^{-1}[1]) m[0] = m\) for any \(m \in M\). On the other hand, since \(M\) is an object in \(\mathcal{H}/M\), it follows by (A.22) that \(n_M \circ n_M (a \otimes_B m) = (\text{can}^{-1}(1_A \otimes_R S^{-1}(a[1])) a[0]) m\), for \(a \otimes_B m \in A \otimes_B \mathcal{H}/M\). The right \(A\)-linearity of \(a\) can implies that \(\text{can}^{-1}(1_A \otimes_R S^{-1}(a[1])) a[0] = a \otimes_B 1_A\), for \(a \in A\), which proves \(n_M \circ n_M (a \otimes_B m) = a \otimes_B m\).

\((c) \Rightarrow (a)\) Observe that \(H \otimes_R A\) is an object in \(\mathcal{H}/M\), with \(A\)-action
\[
a' \circ (h \otimes_R a) = h S^{-1}(a[1]) \otimes_R a[0]a',
\]
where \(a \mapsto a[0] \otimes_L a[1]\) denotes the \(\mathcal{H}\)-coaction on \(A\), and \(\mathcal{H}_L\) and \(\mathcal{H}_R\)-coactions induced by the respective coproducts. The counit of the adjunction \((A \otimes_B \bullet : B \mathcal{H} \to \mathcal{H}, \mathcal{H}_{\text{cop}}(\bullet) : \mathcal{H} \mathcal{M} \to \mathcal{H} \mathcal{M})\), evaluated at the object \(H \otimes_R A\), is the isomorphism
\[
n_H \otimes_R A : A \otimes_R A \to H \otimes_R A, \quad a \otimes_R a' \mapsto S^{-1}(a[1]) \otimes_R a[0]a'.
\]
The canonical map (A.13) is a composite of isomorphisms, can = \(\psi_R \circ n_H \otimes_R A\), where \(\psi_R\) is the bijective entwining map (A.12). This proves bijectivity of the canonical map (A.13).

In view of Theorem 4.2 (part 2) follows by applying part 1) to the \(\mathcal{H}\)-coaction on \(A\), and \(\mathcal{H}_L\) and \(\mathcal{H}_R\)-coactions induced by the respective coproducts. The counit of the adjunction \((A \otimes_B \bullet : B \mathcal{H} \to \mathcal{H}, \mathcal{H}_{\text{cop}}(\bullet) : \mathcal{H} \mathcal{M} \to \mathcal{H} \mathcal{M})\), evaluated at the object \(H \otimes_R A\), is the isomorphism
\[
n_H \otimes_R A : A \otimes_R A \to H \otimes_R A, \quad a \otimes_R a' \mapsto S^{-1}(a[1]) \otimes_R a[0]a'.
\]
The canonical map (A.13) is a composite of isomorphisms, can = \(\psi_R \circ n_H \otimes_R A\), where \(\psi_R\) is the bijective entwining map (A.12). This proves bijectivity of the canonical map (A.13).
6. Equivariant injectivity and projectivity

The notion of equivariant projectivity of a Hopf Galois extension was introduced in the papers [DGH] and [HM]. Equivariant projectivity of a Hopf Galois extension is a crucial property from the non-commutative geometric point of view, as it turns out to be equivalent to the existence of a strong connection – a non-commutative formulation of local triviality of a principal bundle (see [H]). In the context of Galois extensions $B \subseteq A$ by corings (or bialgebroids or Hopf algebroids), equivariant projectivity relative to some subalgebra of $B$ was shown to be equivalent to the existence of more general strong connections in the paper [BB1].

In this section we look for conditions on a Galois extension by a Hopf algebroid, under which it obeys (relative) equivariant injectivity and projectivity properties. Recall that having a Hopf algebra $H$ over a commutative ring $k$ and a right $H$-comodule algebra $A$, which is a relative injective right $H$-comodule, $A$ was shown to be a $B(=A^{\text{co}H})$-equivariantly injective $H$-comodule in [SS, Theorem 5.6]. (This result is extended to algebra extensions by Hopf algebroids in Theorem 6.3 below.) What is more, using the proven $B$-equivariant injectivity of a relative injective $H$-comodule algebra $A$, it was also shown in [SS, Theorem 5.6] that the $B$-module $A$ is $H$-equivariantly projective if and only if it is $k$-relative projective. If $A$ is a relative injective right comodule algebra of a Hopf algebra $H$ with a bijective antipode, with coinvariants $B$, and the lifted canonical map (1.3) is a split epimorphism of $k$-modules, then $A$ is an $H$-Galois extension of $B$ and the $B$-module $A$ is relative projective (cf. Theorem 5.6). Hence the $B$-module $A$ is also $H$-equivariantly projective by the quoted result in [SS, Theorem 5.6]. A most naive generalization of this result to Hopf algebroid Galois extensions seems not to hold. The reason is that – if working with a Hopf algebroid $H$ over different non-commutative base algebras $L$ and $R$ – relative projectivity of the $B$-module $A$ is not enough to prove its (relative) $H$-equivariant projectivity. One needs more: (relative) $L$-equivariant projectivity (see Theorem 6.4 below). As a matter of fact, for a relative injective right comodule algebra $A$ of a Hopf algebroid with a bijective antipode, with coinvariants $B$, we were not able to deduce relative $L$-equivariant projectivity of the $B$-module $A$ form the splitting of the lifted canonical map (5.3) as a left, or right $L$-module map, as assumed in Theorem 5.6. We needed a stronger assumption: splitting of the lifted canonical map (5.3) as an $L$-$L$ bimodule map (see Proposition 5.5 below).

**Definition 6.1.** Let $D$ be an $L$-coring and $B$ a $T$-ring. A left $B$-module and right $D$-comodule $V$, with left $B$-linear right $D$-coaction, is called a $T$-relative $D$-equivariantly projective left $B$-module if the left action $B \otimes_T V \rightarrow V$ is an epimorphism split by a left $B$-module right $D$-comodule map. We call $V$ a $B$-equivariantly injective right $D$-comodule if the right coaction $V \rightarrow V \otimes_L D$ is a monomorphism split by a left $B$-module right $D$-comodule map.

Analogous notions for right $B$-modules and left $D$-comodules, with a right $B$-linear left $D$-coaction, are defined symmetrically.

Considering an algebra $L$ as a trivial $L$-coring $L$, a $T$-relative $L$-equivariantly projective left $B$-module $V$ is called simply $T$-relative $L$-equivariantly projective. Clearly, for an $L$-coring $D$ and a $T$-ring $B$, a $T$-relative $D$-equivariantly projective left $B$-module $V$ is necessarily $T$-relative $L$-equivariantly projective.

For a (left or right) bialgebroid $S$ over an algebra $L$, and a $T$-ring $B$, a $T$-relative $S$-equivariantly projective $B$-module means a $B$-module which is $T$-relative equivariantly projective for the $L$-coring underlying $S$.

In the same spirit, relative $H$-equivariant projectivity for a Hopf algebroid $H$ can be introduced:

**Definition 6.2.** Consider a Hopf algebroid $H$ and a $T$-ring $B$. A left $B$-module and right $H$-comodule $V$, such that the left $B$-action on $V$ is a right $H$-comodule map, is said to be $T$-relative $H$-equivariantly projective if the action $B \otimes_T V \rightarrow V$ is an epimorphism split by a left $B$-linear and right $H$-colinear map.
The following two theorems extend [SS, Theorem 5.6] to non-commutative base algebras.

**Theorem 6.3.** Let \( \mathcal{H} \) be a Hopf algebroid with constituent left bialgebroid \( \mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L) \), right bialgebroid \( \mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \), and antipode \( S \). Let \( A \) be a right \( \mathcal{H} \)-comodule algebra and denote \( B := A^{co\mathcal{H}_R} \). If the equivalent conditions in Theorem 4.2 hold then the \( \mathcal{H}_L \)-coaction on \( A \) possesses a left \( B \)-linear and right \( \mathcal{H} \)-colinear retraction.

**Proof.** Using a method in [SS, Lemma 4.1], one constructs a left \( B \)-linear right \( \mathcal{H} \)-colinear retraction \( \phi \) of the \( \mathcal{H}_L \)-coaction \( a \mapsto a^{[0]} \otimes_L a^{[1]} \) on \( A \), in terms of an \( \mathcal{H} \)-colinear retraction \( \nu \). Explicitly,

\[
\phi : A \otimes H \to A, \quad a \otimes h \mapsto a^{[0]} \nu \left( 1_A \otimes S(a^{[1]}h) \right),
\]

where \( a \mapsto a^{[0]} \otimes_R a^{[1]} \) denotes the right \( \mathcal{H}_R \)-coaction on \( A \). Note that since \( \nu \) is an \( \mathcal{H} \)-module map it is left \( R \)-linear and hence the map \( \phi \) is well defined. \( \square \)

**Theorem 6.4.** Let \( \mathcal{H} \) be a Hopf algebroid with constituent left bialgebroid \( \mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L) \), right bialgebroid \( \mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \), and antipode \( S \). Let \( A \) be a right \( \mathcal{H} \)-comodule algebra and \( B := A^{co\mathcal{H}_R} \). Let \( T \) be an algebra such that \( B \) is a \( T \)-ring (e.g. \( T \) is some \( k \)-subalgebra of \( B \)). If the right \( \mathcal{H}_L \)-coaction on \( A \) possesses a left \( B \)-linear and right \( \mathcal{H} \)-colinear retraction then the following assertions are equivalent.

1. \( A \) is a \( T \)-relative \( \mathcal{H} \)-equivariantly projective left \( B \)-module.
2. \( A \) is a \( T \)-relative \( \mathcal{L} \)-equivariantly projective left \( B \)-module.

**Proof.** If \( A \) is a \( T \)-relative \( \mathcal{H} \)-equivariantly projective left \( B \)-module then it is obviously \( T \)-relative \( \mathcal{H} \)-equivariantly projective. In order to see the converse implication, take a \( B \)-module bimodule section \( \chi^T \) of the multiplication map \( B \otimes_T A \to A \) and a left \( B \)-linear and right \( \mathcal{H} \)-colinear retraction \( \phi \) of the right \( \mathcal{H}_L \)-coaction \( \tau_A : A \to A \otimes_L H \) in \( A \). It determines a left \( B \)-linear and \( \mathcal{H} \)-colinear map

\[
\chi^T := (B \otimes \phi) \circ (\chi^T_0 \otimes H) \circ \tau_A : A \to B \otimes_T A.
\]

It follows by the left \( B \)-linearity of \( \phi \) that \( \chi^T \) is a section of the multiplication map \( B \otimes_T A \to A \). \( \square \)

The message of Theorem 6.3 and Theorem 6.4 is to look for situations, in which the \( T \)-relative \( \mathcal{L} \)-equivariant projectivity condition in Theorem 5.4 (b) holds, for a right comodule algebra of a Hopf algebroid, obeying the conditions in Theorem 6.2.

It is discussed in Appendix A.18 that if the antipode of a Hopf algebroid \( \mathcal{H} \) (over base algebras \( L \) and \( R \)) is bijective then a right comodule algebra \( A \) has a canonical left \( \mathcal{H}^{coR} \)-comodule algebra structure, with coactions \( a \mapsto S^{-1}(a^{[1]}) \otimes_R a^{[0]} \) \( a \mapsto S^{-1}(a^{[1]}) \otimes_L a^{[0]} \). Recall that this left \( \mathcal{H}_L \)-coaction corresponds to the left \( L \)-module structure of \( A \), which is related to its right \( R \)-module structure via

\[
la = a \pi_R \circ s_L, \quad \text{for } l \in L, \ a \in A.
\]

Since the right actions in \( A \) by \( R \) and \( B := A^{co\mathcal{H}_R} \) commute (cf. Section [A.7]), \( A \) is an \( L \)-\( B \) bimodule via the left \( L \)-action \((6.1)\) and the obvious right \( B \)-action. The following proposition concerns equivariant projectivity of this \( L \)-\( B \) bimodule \( A \).

**Proposition 6.5.** Let \( \mathcal{H} \) be a Hopf algebroid with constituent left bialgebroid \( \mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L) \), right bialgebroid \( \mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \), and a bijective antipode \( S \). Let \( A \) be a right \( \mathcal{H} \)-comodule algebra. Set \( B := A^{co\mathcal{H}_R} \), and assume that \( B \) is a \( T \)-ring (e.g. \( T \) is a \( k \)-subalgebra of \( B \)). Assume that the lifted canonical map \((5.3)\) is a split epimorphism of \( L \)-\( B \) bimodules (with respect to the module structures \((6.2)\) and \((5.5)\)). If the equivalent conditions in Theorem 6.2 hold then \( A \) is a \( T \)-relative \( \mathcal{H} \)-equivariantly projective right \( B \)-module.

**Proof.** Let \( \zeta^T_0 \) be an \( L \)-\( B \) bimodule section of the lifted canonical map \((5.3)\). By Lemma 5.3 the map \((5.3)\) is split by the right \( A \)-module and right \( \mathcal{H} \)-comodule map \( \zeta^T \), explicitly given in Lemma 5.3. From the proof of implication \((b) \Rightarrow (a)\) in Theorem 5.6 we have that \( (A \otimes T) \zeta^{co\mathcal{H}_R} = \)
$A \otimes_T B$. Hence taking the ‘$\mathcal{H}_R$-coinvariants part’ of $\zeta^T$, we obtain a right $B$-module section of the multiplication map $A \otimes_T B \rightarrow A$,

$$\chi_0 : A \rightarrow A \otimes_T B, \quad a \mapsto \zeta^T \left(1_A \otimes R S^{-1}(a_{(1)})(1)\right) \left(S(\zeta_0^T(1_A \otimes R S^{-1}(a_{(1)})(1)) S^{-1}(a_{(1)})(2)) a_{(0)}\right).$$

Consider the left $L$-module structure (6.1) of $A$. The right $\mathcal{H}_L$-coaction in $A$ is left $L$-linear in the sense that $(a \pi_R \circ s_L(l))_{(0)} \otimes L (a \pi_R \circ s_L(l))_{(1)} = a_{(0)} \otimes L a_{(1)} \pi_R \circ s_L(l)$, for $l \in L$ and $a \in A$. The antipode satisfies $S^{-1}(h_{s_R} \circ \pi_R \circ s_L(l)) = 1_R \circ \pi_R \circ s_L(l) S^{-1}(h) = s_L(l) S^{-1}(h)$, for $l \in L$ and $h \in H$. The coproduct $\gamma_L$ is left $L$-linear, i.e. $(s_L(l) h_{(1)} \otimes L (s_L(l) h_{(2)})) = S^{-1}(s_L(l) S^{-1}(h_{(1)})) \otimes L h_{(2)}$, for $l \in L$ and $h \in H$. The map $\zeta_0^T$ is left $L$-linear by assumption, with respect to the left $L$-module structures in (6.5). All these considerations together verify the left $L$-linearity of $\chi_0^T$ with respect to the left $L$-module structure (6.3) in $A$. Hence $\chi_0^T$ is an $L$-$B$-bilinear section of the multiplication map $A \otimes_T B \rightarrow A$, which proves $T$-relative $L$-equivariant projectivity of the right $B$-module $A$.

Let $A$ be a right comodule algebra of a Hopf algebroid $\mathcal{H}$ with a bijective antipode $S$. Recall that in this case the conditions (a)-(c) in Theorem 6.3 are equivalent also the the analogous conditions for the right $\mathcal{H}_{\text{cop}}$-comodule algebra $A^{op}$, with $(\mathcal{H}_{\text{cop}})$-coaction $a \mapsto a_{(0)} \otimes L a_{(1)}^{op} S^{-1}(a_{(1)}^{(1)})$ and $(\mathcal{H}_{\text{cop}})$-coaction $a \mapsto a_{(0)}^{op} \otimes L a_{(1)} S^{-1}(a_{(1)}^{(1)})$. Hence Proposition 6.3 can be applied to the right $\mathcal{H}_{\text{cop}}$-comodule algebra $A^{op}$. It yields a result about the equivariant projectivity of $A$ as a $B$-$L$ bimodule, with obvious left $B$-action, and right $L$-action related to the left $R$-action via

$$al = \pi_R \circ t_L(l) a, \quad \text{for } l \in L, \ a \in A.$$

**Corollary 6.6.** In the setting of Proposition 6.3, $A$ is a $T$-relative $L$-equivariantly projective left $B$-module.

The following corollary is the main result of this section. It formulates sufficient conditions on a Galois extension $B \subseteq A$ by a Hopf algebroid $\mathcal{H}$ with a bijective antipode, under which $A$ is a $T$-relative $L$-equivariantly projective left and right $B$-module, for an algebra $T$ such that $B$ is a $T$-ring.

**Corollary 6.7.** Let $\mathcal{H}$ be a Hopf algebroid with constituent left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$, right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$, and a bijective antipode $S$. Let $A$ be a right $\mathcal{H}$-comodule algebra. Denote $B := A^{ov} \mathcal{H}_R$ and assume that $B$ is a $T$-ring (e.g. $T$ is a subalgebra of $B$). Assume that the lifted canonical map (6.3) is a split epimorphism of $L$-$L$-bimodules (with respect to the module structures (6.4) and (6.5)). If the equivalent conditions in Theorem 6.3 hold then $A$ is a $T$-relative $L$-equivariantly projective left and right $B$-module. Moreover, in this case $B \subseteq A$ is a $T$-relative $\mathcal{H}_R$-Galois extension.

**Proof.** The Galois property, i.e. bijectivity of the canonical map (6.13), follows by virtue of Theorem 7.6 (b) $\Rightarrow$ (a).

The right $\mathcal{H}_L$-coaction on $A$ possesses a left $B$-module right $\mathcal{H}$-comodule retraction and the left $\mathcal{H}_L$-coaction on $A$ possesses a right $B$-module left $\mathcal{H}$-comodule retraction, by Theorem 7.6, and its application to the co-opposite Hopf algebroid $\mathcal{H}_{\text{cop}}$ and the right $\mathcal{H}_{\text{cop}}$-comodule algebra $A^{op}$, respectively. By Proposition 6.3, $A$ is a $T$-relative $L$-equivariantly projective right $B$-module. By Corollary 6.4, $A$ is a $T$-relative $L$-equivariantly projective left $B$-module. Hence $A$ is a $T$-relative $L$-equivariantly projective left and right $B$-module by Theorem 6.4, and its application to the co-opposite Hopf algebroid $\mathcal{H}_{\text{cop}}$ and the right $\mathcal{H}_{\text{cop}}$-comodule algebra $A^{op}$, respectively. □

By [BB1, Theorem 3.7] we conclude that there exists a strong $T$-connection for an extension $B \subseteq A$ as in Corollary 6.7 whenever $T$ is a $k$-subalgebra of $B$. In [BB1, Theorem 5.14] conditions are formulated for the independence of the corresponding relative Chern-Galois character of the choice of a strong $T$-connection. Note that in the case when in Corollary 6.7 the $k$-algebra $T$ is equal to $k$, these conditions reduce to the assumption that $A$ is a locally projective $k$-module.
Example 6.8. Cleft extensions by Hopf algebroids were introduced in [BB2, Definition 3.5], as follows. Let \( \mathcal{H} \) be a Hopf algebroid with constituent left bialgebroid \( \mathcal{H}_L = (H, L, s, t, c, \psi_L, \pi_L) \), right bialgebroid \( \mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \), and antipode \( S \). Let \( A \) be a right \( \mathcal{H} \)-comodule algebra with coinvariants \( B := A^{co\mathcal{H}_R} \). Denote the unit map of the corresponding \( R \)-ring \( A \) by \( \eta_R : R \rightarrow A \). The algebra extension \( B \subseteq A \) is called \( \mathcal{H} \)-cleft provided that the following conditions hold:

a) \( A \) is an \( L \)-ring (with some unit map \( \eta_L : L \rightarrow A \)) and \( B \) is an \( L \)-subring of \( A \).

b) There exist morphisms \( j \in L \text{Hom}^H(H, A) \) and \( \tilde{j} \in R \text{Hom}_L(H, A) \), satisfying

\[
\mu \circ (\tilde{j} \otimes \tilde{j}) \circ \gamma_L = \eta_R \circ \pi_R \quad \text{and} \quad \mu \circ (j \otimes j) \circ \gamma_R = \eta_L \circ \pi_L,
\]

where \( \mu \) denotes the multiplication in \( A \), both as an \( L \)-ring and as an \( R \)-ring. The bimodule structures in \( H \) are given by

\[
lhr := s_L(l)h s_R(r) \quad \text{and} \quad rhl = t_L(l)h t_R(r), \quad \text{for} \quad l \in L, \ r \in R, \ h \in H.
\]

The bimodule structures in \( A \) are given by

\[
lar := \eta_L(l)a \eta_R(r) \quad \text{and} \quad ral = \eta_R(r)a \eta_L(l), \quad \text{for} \quad l \in L, \ r \in R, \ a \in A.
\]

In an \( \mathcal{H} \)-cleft extension \( B \subseteq A \) the map \( \tilde{j}(1_H) \) satisfies \( \tilde{j} : H \rightarrow A \) is right \( \mathcal{H} \)-colinear and normalized. Hence the equivalent conditions in Theorem 4.2 hold. By definition \( B \) is an \( L \)-ring. The lifted canonical map

\[
\tilde{c}an^L : A \otimes A \rightarrow A \otimes H, \quad a \otimes a' \mapsto \tilde{a}^{(0)} \otimes \tilde{a}^{(1)}
\]

possesses an \( L \)-\( L \) bilinear section (with respect to the module structures (5.3) and (5.5)):

\[
(6.2) \quad \tilde{c}an^L : A \otimes H \rightarrow A, \quad a \otimes h \mapsto \tilde{a}^{(1)} j(h_{(1)}) \otimes j(h_{(2)}).
\]

The map (6.2) is well defined by the module map properties of \( j \) and \( \tilde{j} \). It is left \( L \)-linear by the identity \( \tilde{j}(t_R h) = \tilde{j}(h) \eta_R(r) \), for \( r \in R \) and \( h \in H \), see [BB2, Lemma 3.7]. Right \( L \)-linearity of (6.2) follows by the left \( R \)-linearity of (the right \( \mathcal{H} \)-comodule map) \( j \), i.e., \( j(s_R r h) = \eta_R(r) j(h) \), for \( r \in R \) and \( h \in H \). In view of Corollary 6.7 all these considerations together imply that a cleft extension \( B \subseteq A \) by a Hopf algebroid \( \mathcal{H} \) with a bijective antipode is an \( \mathcal{H}_R \)-Galois extension which is an \( L \)-relative \( \mathcal{H} \)-equivariantly projective left and right \( B \)-module, cf. [BB2, Lemma 5.1].

Appendix A. Coring extensions, entwining structures and Hopf algebroids

A.1. For a \( k \)-algebra \( A \), an \( A \text{-ring} \) \( T \) means an algebra (or monoid) in the monoidal category of \( A \text{-}\)modules. More explicitly, it consists of an \( A \text{-}\)bimodule \( T \), equipped with a bilinear associative product \( \mu : T \otimes A \rightarrow T \) and a bilinear unit map \( \eta : A \rightarrow T \). An \( A \text{-ring} \) \( T \) is equivalent to a \( k \)-algebra \( T \) and a \( k \)-algebra map \( \eta : A \rightarrow T \).

For an \( A \text{-ring} \) \( (T, \mu, \eta) \), the opposite means the \( A^{op} \text{-ring} \) \( T^{op} \), with \( A^{op} \text{-}\)\( A^{op} \) bimodule structure

\[
A^{op} \otimes T \otimes A^{op} \rightarrow T, \quad a \otimes t \otimes a' \mapsto a'ta,
\]

product \( t \otimes A^{op} t' \mapsto t't \) and unit \( \eta : A^{op} \rightarrow T^{op} \).

An \( A \text{-ring} \) \( T \) determines a monad \( \bullet \otimes A \) \( T \) on \( \mathcal{M} A \). By right \( T \)-modules we mean algebras for this monad. This notion coincides with that of right modules for the \( k \)-algebra \( T \). Left modules are defined symmetrically.

A.2. A coring over an algebra \( A \) means a coalgebra (or comonoid) in the monoidal category of \( A \text{-}\)modules. More explicitly, it consists of an \( A \text{-}\)bimodule \( C \), equipped with a bilinear coassociative coproduct \( \Delta : C \rightarrow C \otimes A \) and a bilinear counit map \( \epsilon : C \rightarrow A \). This extends the notion of a coalgebra.

For an \( A \text{-coring} \) \( (C, \Delta, \epsilon) \), the co-opposite means the \( A^{op} \text{-coring} \) \( C^{op} \), with \( A^{op} \text{-}\)\( A^{op} \) bimodule \( C^{op} \),

\[
A^{op} \otimes C \otimes A^{op} \rightarrow C, \quad a \otimes c \otimes a' \mapsto a'ca,
\]

coproduct \( \Delta^{op} : c \mapsto c^{(2)} \otimes A^{op} c^{(1)} \) and counit \( \epsilon \).

An \( A \text{-coring} \) \( C \) determines a comonad \( \bullet \otimes A \) \( C \) on \( \mathcal{M} A \). By right \( C \)-comodules we mean coalgebras for this comonad. That is, a right \( C \)-comodule is a right \( A \)-module \( M \), equipped with a right
A-linear coassociative and counital coaction \( \varrho^M \). Right \( C \)-comodule maps are right \( A \)-linear maps which are compatible with the coactions. Left \( C \)-comodules are defined symmetrically.

The forgetful functor \( \mathfrak{M}^C \rightarrow \mathfrak{M}_A \) possesses a right adjoint, the functor \( \bullet \otimes_A \mathcal{C} : \mathfrak{M}_A \rightarrow \mathfrak{M}^C \). The unit of the adjunction is given by the right coaction \( \varrho^M : M \rightarrow M \otimes_A \mathcal{C} \), for \( M \in \mathfrak{M}^C \), and the counit of the adjunction is given in terms of the counit \( \varepsilon \) of \( \mathcal{C} \) as \( N \otimes_A \varepsilon : N \otimes_A \mathcal{C} \rightarrow N \), for \( N \in \mathfrak{M}_A \). cf. [BW], 18.13 (2).

A right comodule \( M \) for an \( A \)-coring \( \mathcal{C} \) is called relative injective if any \( C \)-comodule map of domain \( M \), which is a split monomorphism of \( A \)-modules, is a split monomorphism of \( C \)-comodules, too. By [BW], 18.18, \( M \) is a relative injective \( C \)-comodule if and only if the coaction \( \varrho^M \) is a section of \( C \)-comodules.

By [Brz], Lemma 5.1, the right regular \( A \)-module extends to a comodule for an \( A \)-coring \( \mathcal{C} \) if and only if there exists a grouplike element \( g \) in \( C \) (meaning that \( \Delta(g) = g \otimes_A g \) and \( \varepsilon(g) = 1_A \)). A bijective correspondence between grouplike elements \( g \) in \( C \) and right \( C \)-coactions \( \varrho^A \) in \( A \) is given by \( g \mapsto (\varrho^A : a \mapsto ga) \). A similar equivalence holds between grouplike elements and left \( C \)-comodule structures in \( A \).

For an \( A \)-coring \( \mathcal{C} \) with a grouplike element \( g \), the coinvariants with respect to \( g \) of a right \( C \)-comodule \( M \) are defined as the elements of the set
\[
M^{\text{co}} = \{ m \in M \mid \varrho^M(m) = m \otimes_A g \}.
\]

Coinvariants of left \( C \)-comodules are defined symmetrically. In particular, the coinvariants of \( A \), both as a right \( C \)-comodule and a left \( C \)-comodule, are the elements of the subalgebra
\[
B = \{ b \in A \mid gb = bg \}.
\]

A grouplike element \( g \) in \( C \) determines an adjunction \( (\bullet \otimes_A (\bullet)^{\text{co}}) \), between the categories \( \mathfrak{M}_B \) and \( \mathfrak{M}^C \). The unit and counit are given by the maps
\[
\begin{align*}
(A.1) & \quad u_N : N \rightarrow (N \otimes_B (\bullet)^{\text{co}}), \quad x \mapsto x \otimes_B 1_A, \quad \text{and} \\
(A.2) & \quad n_M : M^{\text{co}} \otimes_B (\bullet)^{\text{co}} \rightarrow M, \quad y \otimes_B a \mapsto ya,
\end{align*}
\]
respectively, for any \( N \in \mathfrak{M}_B \) and \( M \in \mathfrak{M}^C \), cf. [BW], 28.8. There is a symmetrical adjunction between the categories \( \mathfrak{M}_C \) and \( \mathfrak{M}^B \).

An \( A \)-coring \( \mathcal{C} \) with a grouplike element \( g \) is called a Galois coring if the canonical map
\[
\begin{align*}
(A.3) & \quad \text{can} : A \otimes_A (\bullet)^{\text{co}} \rightarrow \mathcal{C}, \quad \text{a} \otimes \text{a}' \mapsto \text{aga}'
\end{align*}
\]
is bijective. For more information about corings we refer to the monograph [BW].

A.3. Let \( \mathcal{D} \) be a coring over a base \( k \)-algebra \( L \) and \( \mathcal{C} \) a coring over a \( k \)-algebra \( A \). Assume that \( \mathcal{C} \) is a \( \mathcal{C} \)-\( \mathcal{D} \) bicomodule with the left regular \( \mathcal{C} \)-coaction \( \Delta_C \) and some right \( \mathcal{D} \)-coaction \( \tau_C \). By definition [BW, 22.1], this means that \( \tau_C \) is left \( A \)-linear (hence \( \mathcal{C} \otimes_A \mathcal{C} \) is also a right \( \mathcal{D} \)-comodule with coaction \( \mathcal{C} \otimes_A \tau_C \)) and the coproduct \( \Delta_C \) is right \( \mathcal{D} \)-colinear. Equivalently, the coproduct \( \Delta_C \) is right \( L \)-linear (hence \( \mathcal{C} \otimes_L \mathcal{D} \) is a left \( \mathcal{C} \)-comodule with coaction \( \Delta_C \otimes_L \mathcal{D} \)) and the \( \mathcal{D} \)-coaction \( \tau_C \) is left \( \mathcal{C} \)-colinear. In this case, following [Brz], Definition 2.1, we say that \( \mathcal{D} \) is a right extension of \( \mathcal{C} \). For a right extension \( \mathcal{D} \) of \( \mathcal{C} \), assume that the equalizer
\[
\begin{align*}
(A.4) & \quad M \xrightarrow{\varrho^M} M \otimes_A \mathcal{C} \xrightarrow{\rho^M \otimes_A \rho_C} M \otimes_A \mathcal{C} \otimes_A \mathcal{C}
\end{align*}
\]
in \( \mathfrak{M}_L \) is \( \mathcal{D} \otimes_L \mathcal{D} \)-pure, i.e. it is preserved by the functor \( \varrho^M \otimes_L \mathcal{D} \otimes_L \mathcal{D} : \mathfrak{M}_L \rightarrow \mathfrak{M}_L \), for any right \( \mathcal{C} \)-comodule \( M \). If this condition holds, we say that \( \mathcal{D} \) is a pure right coring extension of \( \mathcal{C} \). By [BW, 22.3] and its Erratum, for pure coring extension \( \mathcal{D} \) of \( \mathcal{C} \), there is a functor \( \mathcal{R} := \otimes_C \mathcal{C} : \mathfrak{M}^C \rightarrow \mathfrak{M}^D \), given by a cotensor product, that renders diagram (3.1) commutative (up to the natural isomorphism \( M \cong M \otimes_C \mathcal{C} \), for \( M \in \mathfrak{M}^C \)). The explicit form of the functor \( \mathcal{R} \) is computed in [Brz], Theorem 2.6 (mind the missing purity condition in the journal version). Using the right \( \mathcal{D} \)-coaction \( \tau_C : c \mapsto c_{[0]} \otimes_L c_{[1]} \), for \( c \in \mathcal{C} \) (note our convention to use character \( \varphi \) for \( \mathcal{D} \)-coactions.
and lower indices of the Sweedler type to denote components of the coproduct and coactions of \( \mathcal{D} \), any right \( \mathcal{C} \)-comodule \( M \) is equipped with a right \( \mathcal{D} \)-comodule structure with right \( L \)-action

\[
\tau_M : M \to M \otimes \mathcal{D}, \quad m \mapsto m[0] \otimes m[1] := m[0] \epsilon_C(m[1][0]) \otimes m[1][1],
\]

for \( m \in M \). and \( \mathcal{D} \)-coaction

\[
(A.6) \quad \tau_M : M \to M \otimes \mathcal{D}, \quad m \mapsto m[0] \otimes m[1] := m[0] \epsilon_C(m[1][0]) \otimes m[1][1], \quad \text{for } m \in M,
\]

where \( \phi^M : m \mapsto m[0] \otimes_A m[1] \) denotes the \( \mathcal{C} \)-coaction on \( M \) (note our convention to use character \( \phi \) for \( \mathcal{C} \)-coactions and upper indices of the Sweedler type to denote components of the coproduct and coactions of \( \mathcal{C} \)). With this definition any right \( \mathcal{C} \)-comodule map is \( \mathcal{D} \)-colinear. In particular, a right \( \mathcal{C} \)-coaction, being \( \mathcal{C} \)-colinear by coassociativity, is \( \mathcal{D} \)-colinear.

\begin{enumerate}
\item \textbf{A.4. Any right coring extension of a coseparable \( A \)-coring \( \mathcal{C} \) is pure.} Indeed, \( [A,A] \) is a split equalizer in \( \mathcal{M}_A \) (split by the right \( A \)-module map \( M \otimes_A C \otimes_A \epsilon_C \)). By separability of the functor \( \mathcal{M}^C \to \mathcal{M}_A \), it is a split equalizer also in \( \mathcal{M}^C \). If \( \mathcal{D} \) is an \( L \)-coring that is a right extension of \( \mathcal{C} \), then taking cotensor products with the \( \mathcal{C} \)-\( \mathcal{D} \) bicomodule \( \mathcal{C} \) defines a functor \( - \otimes_C : \mathcal{M}^C \to \mathcal{M}_L \), equipping any right \( \mathcal{C} \)-comodule \( M \cong M \otimes_C \mathcal{C} \) with a right \( L \)-action. By right \( L \)-linearity of any \( \mathcal{C} \)-comodule map, splitting of the equalizer \( [A,A] \) in \( \mathcal{M}^C \) implies that it splits in also in \( \mathcal{M}_L \). Hence the purity condition holds.

\item \textbf{A.5. An entwining structure} over a (not necessarily commutative) algebra \( L \), with multiplication \( \mu \) and unit \( \eta \), an \( L \)-coring \( \mathcal{D} \), with comultiplication \( \Delta \) and counit \( \epsilon \), and an \( L \)-\( L \)-bilinear map \( \psi : \mathcal{D} \otimes \mathcal{L} \mathcal{A} \to \mathcal{A} \otimes \mathcal{D} \), satisfying the following compatibility conditions.

\[
\psi \circ (\mathcal{D} \otimes \mu) = (\mu \otimes \mathcal{D}) \circ (\psi \otimes \mathcal{A}) \quad \psi \circ (\mathcal{D} \otimes \eta) = \eta \otimes \mathcal{D}
\]

\[
(A \otimes \Delta) \circ \psi = (\psi \otimes \mathcal{D}) \circ (\Delta \otimes \mathcal{A}) \quad (A \otimes \epsilon) \circ \psi = \epsilon \otimes A.
\]

In complete analogy with [Bez1], Proposition 2.2], \( A \otimes \mathcal{D} \mathcal{A} \) is an \( A \)-coring. Its bimodule structure is given by

\[
a_1(\mathcal{d} \otimes \mathcal{a})a_2 = a_1a \psi(a_2 \otimes d), \quad \text{for } a_1, a_2 \in \mathcal{A}, \ a \otimes d \in \mathcal{A} \otimes \mathcal{D}.
\]

The coproduct is equal to \( \mathcal{A} \otimes \mathcal{L} \Delta : \mathcal{A} \otimes \mathcal{D} \to \mathcal{A} \otimes \mathcal{D} \otimes \mathcal{D} \cong (\mathcal{A} \otimes \mathcal{D}) \otimes \mathcal{A} (\mathcal{A} \otimes \mathcal{D}) \) and the counit is \( \mathcal{A} \otimes \mathcal{L} \epsilon : \mathcal{A} \otimes \mathcal{D} \to \mathcal{A} \). Via the canonical isomorphism \( \mathcal{M} \otimes \mathcal{A} \mathcal{A} \otimes \mathcal{D} \mathcal{D} \cong \mathcal{M} \otimes \mathcal{D} \mathcal{D} \), for any right \( A \)-module \( M \), right comodules for the \( A \)-coring \( \mathcal{A} \otimes \mathcal{D} \) are identified with entwined modules. A right-right entwined module means a right \( A \)-module and right \( \mathcal{D} \)-comodule \( M \), with coaction \( \tau_M : M \to m[0] \otimes \mathcal{L} m[1] \), such that

\[
m \eta(l) = ml \quad \text{and} \quad \tau_M(ma) = m[0] \psi(m[1] \otimes a), \quad \text{for } m \in M, \ l \in L, \ a \in A.
\]

Morphisms of entwined modules are \( A \)-linear and \( \mathcal{D} \)-colinear maps. The category of right-right entwined modules is denoted by \( \mathcal{M}_A^D(\psi) \).

Entwining structures \( (A, \mathcal{D}, \psi) \) over an algebra \( L \) provide examples of coring extensions. Namely, the associated \( A \)-coring \( \mathcal{C} := (\mathcal{A} \otimes \mathcal{D}, \mathcal{A} \otimes \Delta, \mathcal{A} \otimes \epsilon) \) is a right \( \mathcal{D} \)-comodule with coaction \( \tau_C := \mathcal{A} \otimes \mathcal{D} : \mathcal{A} \otimes \mathcal{D} \to \mathcal{A} \otimes \mathcal{D} \otimes \mathcal{D} \). By the coassociativity of the coproduct \( \Delta \) in \( \mathcal{D} \), \( \tau_C \) is left \( \mathcal{C} \)-colinear. This means that the \( \mathcal{L} \)-coaction \( \mathcal{D} \) is right extension of the \( A \)-coring \( \mathcal{C} \).

Note that any coring extension arising from an \( A \)-coring \( \mathcal{D} \) is pure. Use again that \( [A,A] \) is a split equalizer in \( \mathcal{M}_A \). Thus the existence of a forgetful functor \( \mathcal{M}_A \to \mathcal{M}_L \) implies that \( [A,A] \) is a split equalizer in \( \mathcal{M}_L \), hence it is preserved by any functor of domain \( \mathcal{M}_L \). In this situation the functor \( \mathcal{R} \) in Figure [8.4] can be identified with the forgetful functor \( \mathcal{M}^P \cong \mathcal{M}_A^D(\psi) \to \mathcal{M}^P \).

Let \( (A, \mathcal{D}, \psi) \) be an entwining structure over an algebra \( L \) and \( \mathcal{C} := A \otimes \mathcal{D} \) the associated \( A \)-coring. If \( e \) is a grouplike element in \( \mathcal{D} \) then \( 1_A \otimes_L e \) is a grouplike element in \( \mathcal{C} \). In this case \( A \) is a right \( \mathcal{C} \)-comodule hence a right-right entwined module. The \( \mathcal{D} \)-coaction in \( A \) comes out as

\[
(A.7) \quad A \to A \otimes \mathcal{D}, \quad a \mapsto \psi(e \otimes a).
\]
The coinvariants of a right \( C \)-comodule (i.e. entwined module) \( M \) with respect to \( 1_A \otimes_L e \) can be identified with \( \text{Hom}^D(A, M) \), and the coinvariants of \( M \) as a right \( D \)-comodule with respect to \( e \) can be identified with \( \text{Hom}^D(L, \mathbb{R}(M)) \) (cf. [BW, 28.4]). Since in this case the forgetful functor \( \mathbb{R} : \mathcal{M} \rightarrow \mathbb{R}(M) \) possesses a left adjoint, \( \bullet \otimes_L A \), it follows that \( \text{Hom}^D(L, \mathbb{R}(M)) \cong \text{Hom}^D(A, M) \). That is to say, the coinvariants of a right \( C \)-comodule (i.e. entwined module) with respect to \( 1_A \otimes_L e \) are the same as its coinvariants as a right \( D \)-comodule with respect to \( e \).

If the entwining map \( \psi \) is bijective then it induces an \( A \)-coring structure in \( D \otimes_L A \). Its left comodules are identified with left \( A \)-modules and left \( D \)-comodules, satisfying a compatibility condition with \( \psi \). If there exists a grouplike element \( e \) in \( D \) then the corresponding left \( D \)-coaction in \( A \) is given by

\[
A \rightarrow D \otimes L, \quad a \mapsto \psi^{-1}(a \otimes e).
\]

(A.8)

A.6. The notion of a **bialgebroid** over an algebra \( L \) was introduced by Takeuchi in [T] under the original name \( \times_L \)-**bialgebra**. Takeuchi’s definition was shown by Brzeziński and Militaru in [BM] to be equivalent to the structure introduced in [Lu]. As a \( k \)-bialgebra consists of compatible algebra and coalgebra structures on the same \( k \)-module, an \( L \)-bialgebroid comprises compatible \( L \otimes_k \ell \)-**ring** and \( L \)-**coring** structures. More explicitly, a **left bialgebroid** is given by the data \( (H, L, s_L, t_L, \gamma_L, \pi_L) \). Here \( H \) and \( L \) are \( k \)-algebras and \( s_L : L \rightarrow H \) and \( t_L : L^{\text{op}} \rightarrow H \) are algebra maps, called the source and target maps, respectively. The map

\[
L \otimes_k L^{\text{op}} \rightarrow H, \quad l \otimes l' \mapsto s_L(l)t_L(l')
\]

is required to be an algebra map, equipping \( H \) with the structure of an \( L \otimes_k L^{\text{op}} \)-**ring**. The \( L \)-\( L \) bimodule \( H \), with actions

\[
lhl' = s_L(l)t_L(l')h, \quad \text{for } l, l' \in L, \ h \in H,
\]

(A.9) is required to be an \( L \)-coring with coproduct \( \gamma_L \) and counit \( \pi_L \). For the coproduct we use a Sweedler type index notation with **lower** indexes, \( \gamma_L(h) = h_1 \otimes_k h_2 \), for \( h \in H \), where implicit summation is understood. The compatibility axioms between the \( L \otimes_k \ell \)-**ring** and \( L \)-**coring** structures are the following. Consider the subset of the \( L \)-module tensor square of the bimodule \( (A.4) \), the so-called **Takeuchi product**

\[
H \times_L H = \left\{ \sum_i h_i \otimes l_i \in H \otimes L \mid \forall l \in L \sum_i h_i t_L(l) \otimes h_i' = \sum_i h_i \otimes h_i' s_L(l) \right\}.
\]

Note that \( H \times_L H \) is an \( L \otimes_k L^{\text{op}} \)-**ring**, with factorwise multiplication and unit map

\[
L \otimes_k L^{\text{op}} \rightarrow H \times_L H, \quad l \otimes l' \mapsto s_L(l) \otimes t_L(l').
\]

The first bialgebroid axiom asserts that the coproduct corestricts to a map of \( L \otimes_k L^{\text{op}} \)-**rings** \( H \rightarrow H \times_L H \). The requirement, that the range of the coproduct lies within \( H \times_L H \), is referred to as the **Takeuchi axiom**. Further axioms require the counit to preserve the unit and satisfy

\[
\pi_L(hs_L \circ \pi_L(h')) = \pi_L(hh') = \pi_L(ht_L \circ \pi_L(h')),
\]

for all \( h, h' \in H \).

The \( L \)-\( L \) bimodule \( (A.4) \) is defined in terms of left multiplication by the source and target maps. Symmetrically, one defines **right bialgebroids** by interchanging the roles of left and right multiplications. Explicitly, a right bialgebroid is given by the data \( (H, R, s_R, t_R, \gamma_R, \pi_R) \), where \( H \) and \( R \) are \( k \)-algebras and \( s_R : R \rightarrow H \) and \( t_R : R^{\text{op}} \rightarrow H \) are algebra maps, called the source and target maps, respectively. \( H \) is required to be an \( R \otimes_k R^{\text{op}} \)-**ring** with unit

\[
R \otimes_k R^{\text{op}} \rightarrow H, \quad r \otimes r' \mapsto s_R(r)t_R(r'),
\]

and an \( R \)-**coring**, with bimodule structure

\[
rhr' = hs_R(r')t_R(r), \quad \text{for } r, r' \in R, \ h \in H,
\]

(A.10)
coproduct $\gamma_R$ and counit $\pi_R$. For the coproduct we use a Sweedler type index notation with upper indices, $\gamma_R(h) = h^{(1)} \otimes_R h^{(2)}$, for $h \in H$, where implicit summation is understood. The coproduct is required to be a map of $R \otimes_k R^{op}$-rings from $H$ to the Takeuchi product

$$H \times_R H = \{ \sum_i h_i \otimes h'_i \in H \otimes H \mid \forall r \in R \sum \frac{s_R(r)}{r} h_i \otimes h'_i = \sum_i h_i \otimes t_R(r) h'_i \},$$

where the $R$-module tensor product is taken with respect to the bimodule structure \cite{A.10}. The counit is defined to preserve the unit and satisfy

$$\pi_R(s_R \circ \pi_R(h)h') = \pi_R(hh') = \pi_R(t_R \circ \pi_R(h)h'),$$

for all $h, h' \in H$. For more details we refer to \cite{KSz}.

The co-opposite $(H, L^{op}, t_L, s_L, \gamma^{op}_L, \pi_L)$ of a left bialgebroid $(H, L, s_L, t_L, \gamma_L, \pi_L)$ is a left bialgebroid too. The opposite $(H^{op}, L, t_L, s_L, \gamma_L, \pi_L)$ is a right bialgebroid.

A.7. A right comodule of a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$ means a right comodule of the $R$-coring $(H, \gamma_R, \pi_R)$ (with bimodule structure \cite{A.10}). The category of right $\mathcal{H}_R$-comodules is denoted by $\mathcal{M}^{H,R}_R$. In Section A.2, a right $\mathcal{H}_R$-comodule was defined to be in particular a right $R$-module. Using the bialgebroid structure of $\mathcal{H}_R$ (not its coring structure alone), one can introduce also a left $R$-module structure in a right $\mathcal{H}_R$-comodule $M$,

\begin{equation}
rm := m^{[0]} \pi_R(s_R(r)m^{[1]}), \quad \text{for } m \in M, \ r \in R.
\end{equation}

This makes $M$ an $R$-$R$ bimodule such that the (so called Takeuchi) identity

$$rm^{[0]} \otimes m^{[1]} = m^{[0]} \otimes t_R(r)m^{[1]}$$

holds, for all $m \in M$ and $r \in R$. Any $\mathcal{H}_R$-comodule map is $R$-$R$ bilinear. This amounts to saying that there is a forgetful functor $\mathcal{M}^{H,R}_R \to R\mathcal{M}_R$. It was observed in \cite[Proposition 5.6]{Scha2} that the forgetful functor $\mathcal{M}^{H,R}_R \to R\mathcal{M}_R$ is strict monoidal. That is, $\mathcal{M}^{H,R}_R$ is a monoidal category via the $R$-module tensor product. The coaction in the product $M \otimes_R N$ of two right $\mathcal{H}_R$-comodules $M$ and $N$ is

$$(m \otimes n)^{[0]} \otimes (m \otimes n)^{[1]} = (m^{[0]} \otimes n^{[0]}) \otimes m^{[1]} n^{[1]}, \quad \text{for } m \otimes n \in M \otimes N.$$

The monoidal unit is $R$ with coaction given by the source map $s_R$. A right $\mathcal{H}_R$-comodule algebra is an algebra in the monoidal category $\mathcal{M}^{H,R}_R$ (hence it is in particular an $R$-ring). Explicitly, it means an $R$-ring and right $\mathcal{H}_R$-comodule $A$ whose coaction $\theta^A$ satisfies

$$\theta^A(1_A) = 1_A \otimes 1_H, \quad \theta^A(aa') = a^{[0]} \otimes a'^{[0]} \otimes a^{[1]} a'^{[1]}, \quad \text{for } a, a' \in A.$$

The $R$-coring $(H, \gamma_R, \pi_R)$ underlying $\mathcal{H}_R$ possesses a grouplike element $1_H$. Coinvariants of a right $\mathcal{H}_R$-comodule are meant always with respect to the distinguished grouplike element $1_H$. By the $R$-$R$ bilinearity of the coaction $\theta^A$ in a right $\mathcal{H}_R$-comodule algebra $A$, for any element $r$ in $R$ and any coinvariant $b$ in $A$, the unit map $\eta : R \to A$ satisfies

$$\theta^A(b \eta(r)) = b \otimes s_R(r) = \theta^A(\eta(r)b).$$

Hence the elements $b \in A^{co\mathcal{H}_R}$ and $\eta(r)$, for $r \in R$, commute in $A$.

Left modules of a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$ (i.e. of the $R$-coring $(H, \gamma_R, \pi_R)$) are treated symmetrically. Their category is denoted by $\mathcal{H}_R \mathcal{M}$. A left $\mathcal{H}_R$-comodule $M$ (which is a priori a left $R$-module) can be equipped with an $R$-$R$ bimodule structure with right $R$-action

$$mr = \pi_R(s_R(r)m)[-1]m^{[0]} \quad \text{for } m \in M, \ r \in R.$$
It is in particular an \( R^\text{op} \)-ring. Explicitly, a left \( \mathcal{H}_R \)-comodule algebra is an \( R^\text{op} \)-ring and left \( \mathcal{H}_R \)-comodule \( A \), whose coaction \( A \mapsto \) satisfies
\[
A \mapsto (1_A) = 1_H \otimes 1_A, \quad A \mapsto (\alpha a') = a'[-1]_R \otimes a'[0]_R, \quad \text{for } a, a' \in A.
\]
Coinvariants of left \( \mathcal{H}_R \)-comodules are meant always with respect to the distinguished grouplike element \( 1_H \).

Comodules of left bialgebroids can be described symmetrically. For a right bialgebroid \( \mathcal{H}_R \), the categories \((\mathcal{H}_R)^{op} \mathfrak{M} \) and \( \mathfrak{M}^{\mathcal{H}_R} \) are monoidally isomorphic. The categories \( \mathfrak{M}^{\mathcal{H}_R}{^R} \) and \( \mathfrak{M}^{\mathcal{H}_R} \) are anti-monoidally isomorphic.

A.8. Let \( \mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \) be a right bialgebroid and \( A \) a right \( \mathcal{H}_R \)-comodule algebra with coaction \( a \mapsto a[0]_R \otimes_R a[1]_R \). Recall from Section A.7 that \( A \) possesses an \( R \)-ring structure. The \( R \)-ring \( A \), the \( R \)-coacting \((H, \gamma_R, \pi_R)\) and the \( R \)-\( R \) bimodule map
\[
(A.12) \quad \psi_R : H \otimes_R A \to A \otimes_R H, \quad h \otimes a \mapsto a[0]_R \otimes_R ha[1]_R
\]
form an entwining structure over \( R \). This implies that \( A \otimes_R H \) is an \( A \)-coacting, with bimodule structure
\[
a_1(a \otimes h) a_2 = a_1 a a_2[0]_R \otimes_R h a_2[1]_R, \quad \text{for } a_1, a_2 \in A \text{ and } a \otimes h \in A \otimes_R H,
\]
coproduct \( A \otimes_R \gamma_R : A \otimes_R H \to A \otimes_R H \otimes_R H \cong (A \otimes_R H) \otimes_A (A \otimes_R H) \) and counit \( A \otimes_R \pi_R : A \otimes_R H \to A \). Via the canonical isomorphism \( M \otimes_A (A \otimes_R H) \cong M \otimes_R H \), for \( M \in \mathfrak{M}_A \), right comodules for the \( A \)-coacting \((A \otimes_R H, A \otimes_R \gamma_R, A \otimes_R \pi_R)\) can be identified with right-right entwined modules for the entwining structure \((A, H, \psi_R)\). Such entwined modules are also called right-right \((A, \mathcal{H}_R)\)-relative Hopf modules. They can be described equivalently as right modules for the algebra \( A \) in the category of right \( \mathcal{H}_R \)-comodules. That is, right \( A \)-modules and right \( \mathcal{H}_R \)-comodules \( M \), such that the \( A \)-action is \( \mathcal{H}_R \)-colinear, in the sense that the compatibility condition
\[
(m a)[0]_R \otimes_R (m a)[1]_R = m[0]_R a[0]_R \otimes_R m[1]_R a[1]_R
\]
holds, for \( m \in M, a \in A \). The category of right-right \((A, \mathcal{H}_R)\)-relative Hopf modules will be denoted by \( \mathfrak{M}^A_{\mathcal{H}_R} \). As it is explained in Section A.8, in the \( R \)-entwining structure \((A, H, \psi_R)\) the \( R \)-coacting \((H, \gamma_R, \pi_R)\) is a right extension of the \( A \)-coacting \((A \otimes_R H, A \otimes_R \gamma_R, A \otimes_R \pi_R)\). For this coacting extension, the functor \( \mathcal{R} \) on Figure 3.1 can be identified with the forgetful functor \( \mathfrak{M}^A_{\mathcal{H}_R} \to \mathfrak{M}^{\mathcal{H}_R} \).

A right \( \mathcal{H}_R \)-comodule algebra \( A \) is called an \( \mathcal{H}_R \)-Galois extension of its coinvariant subalgebra \( B \) if the associated \( A \)-coacting \((A \otimes_R H, A \otimes_R \gamma_R, A \otimes_R \pi_R)\), with grouplike element \( 1_A \otimes_R H \), is a Galois coacting, i.e. the canonical map
\[
(A.13) \quad \text{can} : A \otimes_R A \to A \otimes_R H, \quad a \otimes a' \mapsto aa'[0]_R \otimes_R a'[1]_R
\]
is bijective.

For a left comodule algebra \( A' \) for a left bialgebroid \( \mathcal{H}_L \) one defines left-left \((A', \mathcal{H}_L)\)-relative Hopf modules in a symmetrical way.

A.9. A Hopf algebra \([\text{Sz}], [\text{Bo1}]\) is a triple \( \mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S) \). It consists of a left bialgebroid \( \mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L) \) and a right bialgebroid \( \mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \) on the same total algebra \( H \). They are subject to the following compatibility axioms
\[
(A.14) \quad s_R \circ \pi_R \circ t_L = t_L \quad s_L \circ \pi_L \circ t_R = t_R \\
\quad t_R \circ \pi_R \circ s_L = s_L \quad t_L \circ \pi_L \circ s_R = s_R
\]
and
\[
(A.15) \quad (\gamma_R \otimes_R H) \circ \gamma_L = (H \otimes_R \gamma_L) \circ \gamma_R \quad (\gamma_L \otimes_R H) \circ \gamma_R = (H \otimes_R \gamma_R) \circ \gamma_L.
\]
The \( k \)-linear map \( S : H \to H \) is called the antipode. It is required to be \( R-L \) bilinear in the sense that
\[
S(t_L(l)ht_R(r)) = s_R(r)S(h)s_L(l), \quad \text{for } l \in L, r \in R, h \in H.
\]
The antipode axioms read as
\[ \mu \circ (H \otimes S) \circ \gamma_R = s_L \circ \pi_L, \quad \mu \circ (S \otimes H) \circ \gamma_L = s_R \circ \pi_R, \]
where \( \mu \) denotes the multiplication both in the \( L \)-ring \( s_L : L \to H \) and the \( R \)-ring \( s_R : R \to H \).

In a Hopf algebroid there are two bialgebroid (hence two coring) structures present. Throughout this paper we insist on using upper indices of the Sweedler type to denote components of the coproduct and coactions of the right bialgebroid \( H_R \), and lower indices in the case of the left bialgebroid \( H_L \).

Similarly to the case of Hopf algebras, the antipode of a Hopf algebroid \( H = (H_L, H_R, S) \) is an anti-algebra map on the total algebra \( H \). That is,
\[ S(1_H) = 1_H \quad \text{and} \quad S(hh') = S(h')S(h), \quad \text{for } h, h' \in H. \]

It is also an anti-coring map \( H_L \to H_R \) and \( H_R \to H_L \). That is,
\[ \pi_R \circ S = \pi_R \circ s_L \circ \pi_L \quad \text{and} \quad S(h)^{(1)} \otimes S(h)^{(2)} = S(h(2)) \otimes S(h(1)), \]
\[ \pi_L \circ S = \pi_L \circ s_R \circ \pi_R \quad \text{and} \quad S(h(1)) \otimes S(h)^{(2)} = S(h^{(2)}) \otimes S(h^{(1)}), \quad \text{for } h \in H. \]

For a Hopf algebroid \( H = (H_L, H_R, S) \), also the opposite-co-opposite \( H_{\text{cop}} = ((H_R)^{\text{op}}, (H_L)^{\text{op}}, S) \) is a Hopf algebroid. If the antipode \( S \) is bijective then so are the opposite \( H^\text{op} = ((H_R)^{\text{op}}, (H_L)^{\text{op}}, S^{-1}) \) and the co-opposite \( H_{\text{cop}} = ((H_L)^{\text{cop}}, (H_R)^{\text{cop}}, S^{-1}) \), too.

A.10. The following definition was proposed in [BB2, Definition 3.2] and [BaSz, Section 2.2].

A right comodule of a Hopf algebroid \( H \) is a right \( L \)-module as well as a right \( R \)-module \( M \), together with a right coaction \( \varrho_R : M \to M \otimes_R H \) of the constituent right bialgebroid \( H_R \) and a right coaction \( \varrho_L : M \to M \otimes_L H \) of the constituent left bialgebroid \( H_L \), such that \( \varrho_R \) is an \( H_L \)-comodule map and \( \varrho_L \) is an \( H_R \)-comodule map. Explicitly,
\[ (\varrho_R \otimes L) \circ \varrho_R = (\varrho_R \otimes L H) \circ \varrho_R \quad \text{and} \quad (M \otimes_L \gamma_R) \circ \varrho_L = (\varrho_L \otimes_R H) \circ \varrho_R. \]

Morphisms of \( H \)-comodules are \( H_R \)-comodule maps as well as \( H_L \)-comodule maps. The category of right \( H \)-comodules is denoted by \( \mathfrak{M}^H \).

Note that any right \( H \)-comodule is a right \( R \otimes L \)-module.

The category \( \mathfrak{M}^H \) of left \( H \)-comodules is defined symmetrically.

A.11. Since a comodule \( M \) of a Hopf algebroid \( H \) is a comodule of both constituent bialgebroids \( H_L \) and \( H_R \), we can consider the coinvariants \( M^{\text{co}H_R} \) and \( M^{\text{co}H_L} \) in the sense of Appendix A.6. By [BB2, Corrigendum], for any \( H \)-comodule \( M \), \( M^{\text{co}H_R} \subseteq M^{\text{co}H_L} \). If the antipode of \( H \) is bijective then an equality holds.

A.12. For any Hopf algebroid \( H \) the following hold.

1) The forgetful functor \( \mathfrak{M}^H \to \mathfrak{M}_L \) possesses a right adjoint \( - \otimes L H \).

2) The forgetful functor \( \mathfrak{M}^H \to \mathfrak{M}_R \) possesses a right adjoint \( - \otimes R H \).

Proof. 1) The unit of the adjunction is given by the \( H_L \)-coaction \( M \to M \otimes L H \), for any right \( H \)-comodule \( M \). It is an \( H \)-comodule map by definition. Counit is given by \( N \otimes L \pi_L : N \otimes L H \to N \), for any right \( L \)-module \( N \). Part 2) is proven symmetrically.

A.13. [BB2, Corrigendum] Consider a Hopf algebroid \( H \). Denote by \( F_R \) and \( F_L \) the forgetful functors \( \mathfrak{M}^H \to \mathfrak{M}_L \) and \( \mathfrak{M}^H \to \mathfrak{M}_R \), respectively.

1) If the equalizer
\[ M \xrightarrow{\varrho_R} M \otimes_R H \xrightarrow{\varrho_R \otimes R H} M \otimes_R H \otimes_R H \]
in \( \mathfrak{M}_L \) is \( H \otimes L H \)-pure, i.e. it is preserved by the functor \( - \otimes L H \otimes L H : \mathfrak{M}_L \to \mathfrak{M}_L \), for any right \( H_R \)-comodule \( (M, \varrho_R) \), then there exists a functor \( U : \mathfrak{M}^H \to \mathfrak{M}^{H_L} \), such that \( F_L \circ U = F_R \). Moreover, in this case the forgetful functor \( G_R : \mathfrak{M}^H \to \mathfrak{M}^{H_R} \) is fully faithful.
Moreover, the following diagram is commutative and all occurring forgetful functors are monoidal. Moreover, in this case the forgetful functor $H$ is a right $H$-bialgebroid $M$-algebra. This means in particular that $H$ is a right $H$-bialgebroid $M$-algebra. Moreover, in this case the forgetful functor $G_L : M^H \to M^{H_L}$ is fully faithful.

2) If the equalizer

$$\theta_\ell : N \to N \otimes_L H \xrightarrow{\theta_L \otimes_L \varepsilon_L} N \otimes_L H \otimes_L H$$

in $M_R$ is $H \otimes_R H$-pure, i.e. it is preserved by the functor $- \otimes_R H \otimes_R H : M_R \to M_R$, for any right $H_L$-comodule $(N, \theta_L)$, then there exists a functor $V : M^{H_L} \to M^{H_R}$, such that $F_R \circ V = F_L$. Moreover, in this case the forgetful functor $G_L : M^H \to M^{H_L}$ is fully faithful.

3) If both purity assumptions in parts 1) and 2) hold, then the forgetful functors $G_R : M^H \to M^{H_R}$ and $G_L : M^H \to M^{H_L}$ are isomorphisms. Moreover, $G_L \circ G_R^{-1} = U$ and $G_R \circ G_L^{-1} = V$, hence $U$ and $V$ are inverse isomorphisms.

A.14. [BB3, Corrigendum] For any Hopf algebroid $H$, the category $M^H$ of right $H$-comodules is monoidal. Moreover, the following diagram is commutative and all occurring forgetful functors are strict monoidal.

$$\begin{array}{ccc}
M^H & \longrightarrow & M^{H_R} \\
\downarrow & & \downarrow \\
M^{H_L} & \longrightarrow & R M_R.
\end{array}$$

In light of this observation, the following definition can be made.

A right comodule algebra of a Hopf algebroid $H$ is an algebra in the monoidal category $M^H$. Right/left modules of a right $H$-comodule algebra $A$ in $M^H$ are termed (right-right/left-right) relative Hopf modules. Their categories are denoted by $M^H_A$ and $A M^H$, respectively.

A.15. For a right comodule algebra $A$ of a Hopf algebroid $H$, denote $B := A^{coH_R}$. Then there is an adjunction

$$- \otimes_B A : M_B \to M^H_A \quad (-)^{coH_R} : M^H_A \to M_B.$$

Proof. For any right $B$-module $N$, the unit of the adjunction is given by

$$N \mapsto (N \otimes_B A)^{coH_R}, \quad n \mapsto n \otimes_B 1_A.$$

For any relative Hopf module $M \in M^H_A$, counit is given by

$$M^{coH_R} \otimes_B A \to M, \quad m \otimes_B a \mapsto ma.$$

Obviously, it is a right $A$-module map. In light of [A.11], it is also a morphism of $H$-comodules. Verification of the adjunction relations is a routine computation.

A.16. For a Hopf algebroid $H$ and a right $H$-comodule algebra $A$, denote $B := A^{coH_R}$.

1) The functor $- \otimes_B A : M_B \to M^H_A$ is fully faithful if and only if the functor $- \otimes_B A : M_B \to M^H_A$ is fully faithful.

2) If the functor $- \otimes_B A : M_B \to M^H_A$ is an equivalence then also the functor $- \otimes_B A : M_B \to M^H_A$ is an equivalence.

Proof. Consider the adjunction in Appendix A.15 and the adjunction

$$(A.18) \quad - \otimes_B A : M_B \to M^H_A \quad (-)^{coH_R} : M^H_A \to M_B,$$

cf. ([1],[2]). Both statements follow by noticing that the units of the two adjunctions coincide and counit of the adjunction in Appendix A.15 is equal to the restriction of the counit of the adjunction (A.18) to the objects of $M^H_A$.

A.17. Let $H$ be a Hopf algebroid with constituent left bialgebroid $H_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$, right bialgebroid $H_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$, and antipode $S$, and let $A$ be a right $H$-comodule algebra. This means in particular that $A$ is a right comodule algebra for the right $R$-bialgebroid $H_R$, with coaction $a \mapsto a^{[0]} \otimes_R a^{[1]}$. What is more, since $A$ is a right comodule algebra for the left bialgebroid $H_L$ as well, with coaction $a \mapsto a^{[0]} \otimes_L a^{[1]}$, related to the $H_R$-coaction as in (A.16), the opposite algebra $A^{op}$ is a right comodule algebra for the right $L$-bialgebroid $(H_L)^{op}$. Hence in addition to the $R$-entwining structure (A.12), $A$ determines also an $L$-entwining structure. It
consists of the $L$-ring $A^{op}$ (with unit, expressed in terms of the unit $\eta$ of the $R$-ring $A$ as $\eta \circ \pi_R \circ t_L$), the $L$-coring $(H, \gamma_L, \pi_L)$, and the entwining map

$$\psi_L : H \otimes L \to A \otimes L, \quad h \otimes a \mapsto a_{[0]} \otimes a_{[1]} h.$$  

Therefore there is an associated $A^{op}$-coring structure on $A \otimes L H$.

Note that the entwining map (A.19) is bijective with inverse $\psi_L^{-1}(a \otimes_L h) = S(a_{[1]})h \otimes a_{[0]}$. Hence $H \otimes_L A$ has a unique $A^{op}$-coring structure such that (A.19) is an isomorphism of corings. Clearly, by the existence of grouplike elements, $A^{op}$ is a left comodule for the $A^{op}$-corings $A \otimes_L H \cong H \otimes_L A$.

A.18. The antipode $S$ of a Hopf algebroid induces strict anti-monoidal functors $H R \mathfrak{M} \rightarrow \mathfrak{M} H_L$, $H L \mathfrak{M} \rightarrow \mathfrak{M} H_R$ and $H \mathfrak{M} \rightarrow \mathfrak{M} H$: Let $M$ be a left $H_R$-comodule with coaction $m \mapsto m_{[-1]} \otimes_R m_{[0]}$. Then $M$ has a right $H_L$-comodule structure with right $L$-action $ml := \pi_R \circ t_L(l)m$, for $l \in L$ and $m \in M$, and coaction

$$m \mapsto m_{[0]} \otimes S(m_{[-1]}).$$

If $M$ is a left $H_L$-comodule with coaction $m \mapsto m_{[-1]} \otimes_L m_{[0]}$, then $M$ has a right $H_R$-comodule structure, with right $R$-action $mr := \pi_L \circ t_R(r)m$, for $r \in R$ and $m \in M$, and coaction

$$m \mapsto m_{[0]} \otimes S(m_{[-1]}).$$

If $M$ is a left $H$-comodule then the $H_R$-coaction (A.20) and the $H_L$-coaction (A.21) are checked to constitute a right $H$-comodule structure on $M$.

Clearly, if $S$ is bijective, then all these functors are isomorphisms. Therefore, $A$ is a right $H$-comodule algebra if and only if the opposite algebra $A^{op}$ possesses a left $H$-comodule algebra structure. For a right comodule algebra $A$ of a Hopf algebroid $H$ with a bijective antipode, left/right $A^{op}$-modules in $H \mathfrak{M}$ are called (right-left/left-left) relative Hopf modules. Their categories are denoted by $H \mathfrak{M}_A$ and $H_A \mathfrak{M}$, respectively.

In particular, left-left relative Hopf modules are left $A$-modules and left $H$-comodules, subject to the compatibility conditions

$$(am)_{[-1]}^{\otimes_R} (am)_{[0]}^{\otimes_R} = m_{[-1]}^{\otimes_R} S^{-1}(a_{[1]})_{\otimes_R} a_{[0]} m_{[0]}^{\otimes_R}$$

and

$$(am)_{[-1]}^{\otimes_L} (am)_{[0]}^{\otimes_L} = m_{[-1]}^{\otimes_L} S^{-1}(a_{[1]})_{\otimes_L} a_{[0]} m_{[0]}^{\otimes_L}$$

for $a \in A, m \in M$.

A.19. Let $H$ be a Hopf algebroid with constituent left bialgebroid $H_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$, right bialgebroid $H_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$, and a bijective antipode $S$. Let $A$ be a right $H$-comodule algebra with $H_R$-coaction $a \mapsto a_{[0]}^{\otimes_R} a_{[1]}^{\otimes_R}$ and $H_L$-coaction $a \mapsto a_{[0]}^{\otimes_L} a_{[1]}^{\otimes_L}$, related via (A.10). The two isomorphic $A^{op}$-corings $A \otimes_L H \cong H \otimes_L A$ in Section A.17 are anti-isomorphic to the $A$-corings $A \otimes_R H \cong H \otimes_R A$. An anti-isomorphism is given by a bijection in [Boe, Lemma 3.3],

$$A \otimes_R H \rightarrow A \otimes_L H, \quad a \otimes_R h \mapsto a_{[0]}^{\otimes_L} a_{[1]}^{\otimes_L} S(h),$$

and an isomorphism $H \otimes_R A \rightarrow A \otimes_R H$ is given by the entwining map (A.12), with inverse

$$a \otimes_R h \mapsto h S^{-1}(a_{[1]})_{\otimes_R} a_{[0]}^{\otimes_R},$$

cf. [Boe, Lemma 4.1]. By the existence of grouplike elements (given by the units in $A$ and $H$), $A$ is a left comodule for all these corings.

Acknowledgements

This paper was written while A. Ardizzoni and C. Menini were members of G.N.S.A.G.A. with partial financial support from M.I.U.R.. Work of G. Böhm is supported by the Hungarian Scientific Research Fund OTKA T043159 and the Bolyai János Fellowship. Her stay, as a visiting professor at University of Ferrara, was supported by I.N.D.A.M.. She would like to express her gratitude to the members of the Department of Mathematics at University of Ferrara for a very warm hospitality.
References

[Ar] A. Ardizzoni, Separable Functors and formal smoothness, Journal of K-Theory 1 (2008), 535–582.
[BaSz] I. Bălint and K. Szlachányi, Finitary Galois extensions over noncommutative bases, J. Algebra 296 (2006), 529–560.
[Bö1] G. Böhm, Integral theory for Hopf algebroids, Alg. Rep. Theory 8 (2005), 563–599. Corrigendum, to be published. See also http://arxiv.org/abs/math/0403120, to be replaced by version 4.
[Bö2] G. Böhm, Galois theory for Hopf algebroids, Ann. Univ. Ferrara - Ser. VII - Sc. Mat., Vol LI (2005), 233–262. A corrected version is available at http://arxiv.org/abs/math/0409513v2.
[BB1] G. Böhm and T. Brzeziński, Strong connections and the relative Chern-Galois character for corings, Int. Math. Res. Not. 2005, 42 (2005), 2579–2625.
[BB2] G. Böhm and T. Brzeziński, Cleft extensions of Hopf algebroids, Appl. Categorical Structures 14 (2006), 431–469. Corrigendum, to be published. See also http://arxiv.org/abs/math/0510253v2.
[BB3] G. Böhm and T. Brzeziński, Pre-torsors and equivalences, J. Algebra 317 (2007), 544–580. Corrigendum, J. Algebra 319 (2008), 1339–1340.
[BSz] G. Böhm and K. Szlachányi, Hopf algebroids with bijective antipodes: axioms, integrals and duals, J. Algebra 274 (2004), 708–750.
[Bro] K. S. Brown, Cohomology of groups. Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982.
[Brz1] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorems, and Frobenius and Galois-type properties, Alg. Rep. Theory 5 (2002), 389–410.
[Brz2] T. Brzeziński, Galois comodules, J. Algebra 290 (2005), 503–537.
[Brz3] T. Brzeziński, A note on coring extensions, Ann. Univ. Ferrara - Ser. VII - Sc. Mat., Vol LI (2005), 15–27. A corrected version is available at http://arxiv.org/abs/math/0410020v2.
[BM] T. Brzeziński and G. Militaru, Bialgebroids, ×₂-bialgebras and duality, J. Algebra 251 (2002), 279–294.
[BTW] T. Brzeziński, R.B Turner and A.P. Wrightson, The structure of weak coalgebra-Galois extensions, Comm. Algebra 34 (2006), 1489–1519.
[BW] T. Brzeziński and R. Wisbauer, Corings and Comodules, Cambridge University Press, Cambridge, 2003. Erratum: http://www.maths.swan.ac.uk/staff/tb/corimerr.pdf.
[CM] S. Caenepeel and G. Militaru, Maschke functors, semisimple functors and separable functors of the second kind. Applications, J. Pure Appl. Algebra 178 (2003), 131–157.
[CIMZ] S. Caenepeel, B. Ion, G. Militaru and S. Zhu, Separable functors applied to Doi Hopf modules. Applications, Adv. Math. 145 (1999), 239–290.
[CMZ] S. Caenepeel, G. Militaru and Shenglin Zhu, Frobenius Separable Functors for Generalized Module Categories and Nonlinear Equations. LNM 1787 (2002), Springer-Verlag, Berlin - New York.
[DGH] L. Dabrowski, H. Grosse and P.M. Hajac, Strong connections and Chern-Connes pairing in the Hopf-Galois theory, Comm. Math. Phys. 220 (2001), 301–331.
[Doi] Y. Doi, Algebras with total integrals, Comm. Algebra 13 (1985), 2137–2159.
[H] P.M. Hajac, Strong connections on quantum principal bundles, Comm. Math. Phys. 182 (1996), 579–617.
[HM] P.M. Hajac and S. Majid, Projective module description of the q-monopole, Comm. Math. Phys. 206 (1999), 247–264.
[HS] P.J. Hilton and U. Stambach, A course in Homological algebra. Graduate Text in Mathematics 4, Springer, New York, 1971.
[KSz] L. Kadison and K. Szlachányi, Bialgebroid actions on depth two extensions and duality, Adv. Math. 179 (2003), 75–121.
[KT] H. F. Kreimer and M. Takeuchi, Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J. 30 (1981), 675–692.
[Lu] J.-H. Lu, Hopf algebroids and quantum groupoids, Int. J. Math. 7 (1996), 47–70.
[MM1] C. Menini and G. Militaru, Integrials, quantum Galois extensions and the affineness criterion for quantum Yetter-Drinfel’d modules, J. Algebra 247 (2002), 467–508.
[MM2] C. Menini and G. Militaru, The affineness criterion for Doi-Koppinen modules, in: ”Hopf algebras in non-commutative geometry and physics”, S. Caenepeel and F. Van Oystaeyen (eds.), Lecture Notes in Pure and Applied Mathematics 239 pp 215–228, Marcel Dekker, New York, 2004.
[NVV] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, Separable functors applied to graded rings, J. Algebra 123 (1989), 397–413.
[Raf] M. D. Rafael, Separable Functors Revisited, Comm. Algebra 18 (1990), 1445–1459.
[Row] L.H. Rowen, Ring theory. Vol. I, Academic Press, Boston (1988).
[Sch1] P. Schauenburg, Hopf-Galois and bi-Galois extensions, in: Galois theory, Hopf algebras, and semiabelian categories, Fields Inst. Commun. 43, AMS 2004, pp. 469–515.
[Sch2] P. Schauenburg, Bialgebroids over noncommutative rings and a structure theorem for Hopf bimodules, Appl. Categorical Structures 6 (1998), 193–22.
[SS] P. Schauenburg and H.-J. Schneider, On generalized Hopf Galois extensions, Journal of Pure and Applied Algebra 202 (2005), 168–194.
[Schn] H.-J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras*, Israel J. Math. 72 (1990), 167–195.

[Ta] M. Takeuchi, *Groups of algebras over $A \otimes \mathcal{A}$*, J. Math. Soc. Japan 29 (1977), 459–492.

[We] C. Weibel, *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.