Finite Quantum Gravity in (A)dS

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We hereby study the properties of a large class of weakly non-local gravitational theories around the (anti-) de Sitter spacetime background. In particular we explicitly prove that the kinetic operator for the graviton field has the same structure as the one in Einstein-Hilbert theory around any maximally symmetric spacetime. Therefore, the perturbative spectrum is the same of standard general relativity, while the propagator on any maximally symmetric spacetime is a mere generalization of the one from Einstein’s gravity derived and extensively studied in several previous papers. At quantum level the range of theories here presented is super-renormalizable or finite when proper (non affecting the propagator) terms cubic or higher in curvatures are added. Finally, it is proven that for a large class of non-local theories, which in their actions do involve neither the Weyl nor the Riemann tensor, the theory is classically equivalent to the Einstein-Hilbert one with cosmological constant by means of a metric field redefinition at any perturbative order.

I. INTRODUCTION

In previous studies it has been extensively shown that a class of weakly non-local theories of gravity is unitary (ghost-free) and perturbatively super-renormalizable or finite in the framework of quantum field theory \[1–11\]. These works mostly concentrated on the perturbative theory around the flat Minkowski spacetime. The very foundations of the theory are the following: (i) general covariance; (ii) weak non-locality (or quasi-polynomiality) \[12\]; (iii) unitarity (ghost freedom); (iv) super-renormalizability or finiteness at quantum level. The new class of generally covariant theories differs from Einstein’s gravity because of the weak non-locality, which makes possible to achieve unitarity and super-renormalizability at the same time, and at any order in the perturbative loop expansion. Nevertheless, the theory is not unique and all the freedom is mainly encoded in one, two, or three form-factors (entire functions) with very specific asymptotic limits in the ultraviolet (UV) and infrared (IR) regimes in order to have a well-defined quantum field theory.

We here study the same range of weakly non-local theories around maximally symmetric spacetimes (MSS) applying exactly the same logic so successfully implemented for theories around the Minkowski vacuum. In particular, we show that the kinetic operator for the gravitational fluctuation $h$ resumes exactly the Einstein-Hilbert one up to some multiplicative factors. For this achievement, we explicitly show the results for the expansion of the action at the second order in $h$ around a MSS. Therefore, all the results concerning the propagator on (A)dS spaces for the Einstein-Hilbert action can be exported and applied to the quasi-polynomial theories too. In particular, for one out of the two classes of theories, which we extensively study in this paper, we prove by the meaning of a field redefinition that at perturbative level, but to all perturbative orders in the field redefinition, the non-local action is classically equivalent to the Einstein-Hilbert one in the presence of a cosmological constant. The proof is based on a field redefinition theorem that was already applied in \[13\] to the theory around the Minkowski vacuum.

There are several good theoretical as well as observational reasons to study the class of gravitational theories around
MSS and not only around flat spacetime vacuum. Primarily, the true gravitational vacuum in quantum field theory is not precisely located as suggested by the cosmological constant problem. This has to do in other disguise with the gravitational effect of the zero modes of the simple quantized harmonic oscillator. (The last one works as a toy model for any perturbative QFT, when it is treated as a theory of free propagating excitations). Therefore, the flat Minkowski spacetime may not be the correct gravitational vacuum and in some theories this state may even decay (via the spontaneous production of ghosts like in higher derivative models of gravity). It may happen that the perturbative calculus around such false vacua is very fast divergent and not reliable due to the presence of different type of instabilities like ghost (negative norm states) or tachyons (negative mass states). A rescue could be to look for another vacuum state and to study quantum perturbations around the new vacuum. The MSS is the only other type of instabilities like ghost (negative norm states) or tachyons (negative mass states). A rescue could be to look perturbative calculus around such false vacua may be perturbatively unreachable from the original one, therefore, by studying quantum theories around (A)dS spacetimes we actually do non-perturbative physics from the flat spacetime perspective. Additionally, different backgrounds can be viewed as a resummation of collective gravitational fluctuations around an initial background.

On the other hand, the inclusion of background spacetimes of constant curvature is a very mild modification that can be treated exactly without tremendous efforts in computations. Therefore, it is an interesting laboratory to study perturbative implications of the same theory, but on different maximally symmetric backgrounds. For example, we can play with the value of the curvature radius of the background and easily we can check the claims about background-independence of non-local theories. Since AdS spacetimes gained in the last two decades a lot of attention, mainly due to the AdS/CFT conjecture, it is also highly desirable to have a gravitational version of non-local theories formulated on general AdS backgrounds. This could be viewed as a first step towards the investigation of the gauge-gravity duality in a class of weakly non-local gravitational theories consistent at quantum level.

The cosmological constant $\Lambda_A$ appearing in the tree-level action should be understood as a new coupling constant of gravitational character, and not like a special matter source. Moreover, the cosmological constant term is an IR completion of the theory when we introduce all possible operators with a fixed number of derivatives. In the case of the cosmological constant we actually add a generally covariant term with no derivatives. And then the following issue arises that for consistency we should quantize physical theory around on-shell backgrounds, i.e. such that solves exactly classical equations of motion. If we have the cosmological constant in the action, then the flat spacetime is not a solution anymore and we have to study the theory around Einstein spaces. De Sitter and anti-de Sitter serve as examples of such background spacetimes.

However, here we want to remark that it is also possible to pursue a different idea that the cosmological constant $\Lambda_{cc}$ may not be present in the action. The background does not have to be on-shell with respect to the equations for the perturbations, and only at the end the physical theory for the on-shell Minkowski background without $\Lambda_{cc}$ should be considered. If $\Lambda_{cc}$ is in the action then the fluctuations must be analyzed around on-shell (A)dS spacetimes. On the other hand from the mathematical point of view it is consistent to have off-shell backgrounds on which one can have whatever theory describing the propagation and interactions of fluctuating modes. We only want these fluctuations to be small (kind of a probe theory) to do not influence the background too much (back-reaction is neglected). For example, we can study the quantum fluctuations around the flat background in a theory that incorporates the cosmological constant term too. Indeed, we can always consider contributions to the propagator and vertices coming from the cosmological constant term even on a flat spacetime background.

Last but not least, we must look for a theory consistent on a MSS spacetime background because the cosmological observations suggest that we are living in an exponentially expanding de-Sitter-like universe. Despite that for all Earth- and solar system-based gravitational experiments we can safely neglect the effect of being in the dS phase, it is crucial that the theory, which have very good quantum properties around the Minkowski background, can be also formulated around any other MSS without major obstructions. Hereby, we show that such theory exists and is well-defined and it has the same analogous good quantum and UV properties as the theory previously studied on the flat spacetime background. The theory around any MSS possesses the following virtues: is generally-covariant, background-independent, perturbatively unitary, and in the quantum domain can be easily selected to be super-renormalizable or UV-finite. It is easily seen that the presence of one constant parameter in this fundamental theory does not destroy, but rather only generalizes, the amazing structure already known around the flat background. Indeed, on a MSS the theory is only slightly modified with respect to the theory on the flat background. The MSS backgrounds are very well-behaved: they are for example constant with respect to covariant derivatives and the commutators of derivatives can be traced back to some correction proportional to $\Lambda_{cc}$. Moreover, the background curvature tensors can be completely written using only the metric tensor and the parameter $\Lambda_{cc}$. Therefore, we are going to include the cosmological constant in all the operators present in the action. In particular, the form-factors could be selected to be functions of $\Lambda_{cc}$ or the Ricci scalar. In the former case, as an additional advantage we can
easily recover the flat spacetime results, by taking the limit $\Lambda_c \to 0$.

At the level of classical solutions the gravitational potential in the class of non-local theories is singularity-free and approaches a constant at $r = 0$, regardless of the particular form-factor appearing in the action [14–25]. This was found in the context of approximate solutions. On the other hand regular bouncing solutions and the Starobinsky’s cosmological solution have been shown to solve exactly the equations of motion of the non-local theory [26]. However, Ricci-flat spacetimes and the FRW spacetime in the presence of radiation are still exact solutions of the weakly non-local theory [21]. This issue has also to do with the question of localization of non-local theories as addressed in [28]. Therefore, any form of non-locality is not enough to smear out the singularities. However, at the present stage we cannot exclude that a special non-local theory could have only non-singular solutions. Moreover we have evidences that in this class of theories with infinitely many derivatives the black hole entanglement entropy is completely regularized and takes only finite values [29, 30].

In section two we review the perturbative weakly non-local gravitational theory around the Minkowski space: the propagator, power-counting super-renormalizability, and finiteness at quantum level. In section three we propose two classes of weakly non-local theories on (A)dS and we explicitly prove that the action at the second order in the graviton fluctuation has the same structure of the Einstein-Hilbert one. In section four we prove that the theory is finite at quantum level, while in section five we prove that for one out of the two classes of theories a perturbative field redefinition allows to map the non-local theory in the Einstein-Hilbert theory plus cosmological constant. In the last section we propose and study the most general weakly non-local field theory.

Most of the results obtained in this paper can be easily exported to Lee-Wick gravitational theories [31–36] just replacing the non-local form-factors with appropriate polynomials.

II. NON-LOCAL GRAVITATIONAL THEORIES ON MINKOWSKI VACUUM

The most general $D$-dimensional theory weakly non-local (or quasi-local) and quadratic in curvature reads [11],

\[
\mathcal{L}_\kappa = -2\kappa_D^{-2} \sqrt{|g|} \left[ R + R g_\gamma(\square) R + \text{Ric} g_\gamma(\square) \text{Ric} + \text{Riem} g_\gamma(\square) \text{Riem} + \mathcal{V} \right].
\]

The above Lagrangian density of the theory consists of a kinetic weakly non-local operator quadratic in curvature, three entire functions $g_\gamma(\square)$, $g_2(\square)$, $g_4(\square)$, and a set of local terms $\mathcal{V}$ cubic or higher in curvature. The latter consists of operators with a properly chosen number of derivatives to not spoil the good quantum properties of the theory. Moreover, $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d’Alembertian (or box) operator, while the entire functions $g_\gamma(\square)$ are defined in terms of exponentials of entire functions $H_\ell(z)$ ($\ell = 0, 2$), namely

\[
\begin{align*}
\gamma_0(\square) &= -\frac{(D-2)(e^{H_0(\square)} - 1) + D(e^{H_2(\square)} - 1)}{4(D-1)\square} + \gamma_4(\square), \\
\gamma_2(\square) &= \frac{e^{H_2(\square)} - 1}{\square} - 4\gamma_4(\square),
\end{align*}
\]

while $g_4(\square)$ stays arbitrary. It is only constrained by renormalizability to have the same asymptotic UV behaviour as the other two form-factors $g_\ell(\square)$ ($\ell = 0, 2$). The minimal choice compatible with unitarity and super-renormalizability corresponds to retaining only two out of three form-factors, i.e. we can choose $g_4(\square) = 0$.

Finally, the entire functions $V_\ell^{-1}(z) \equiv \exp(H_\ell(z))$ ($z \equiv -\nabla^2 \equiv -\square/\Lambda^2$) (for $\ell = 0, 2$) introduced in [3] and [2] satisfy the following general conditions [3, 57]:

(i). $V_\ell^{-1}(z)$ is real and positive on the real axis and it has no zeros on the whole complex plane $|z| < +\infty$. This requirement implies that there are no gauge-invariant poles other than the transverse massless physical graviton pole;

(ii). $|V_\ell^{-1}(z)|$ has the same asymptotic behaviour along the real axis at $\pm\infty$;

(iii). There exist $\Theta > 0$, $\Theta < \pi/2$ and positive integer $\gamma$, such that asymptotically

\[
|V_\ell^{-1}(z)| \to |z|^\gamma + N + 1, \text{ when } |z| \to +\infty \text{ with } \gamma > \frac{D_{\text{even}}}{2} \text{ or } \gamma > \frac{D_{\text{odd}} - 1}{2},
\]

---

1 Definitions — The metric tensor $g_{\mu\nu}$ has signature $(- + \cdots +)$ and the curvature tensors are defined as follows: $R^\mu_{\nu\rho\sigma} = -\partial_\nu B^\mu_{\rho\sigma} + \ldots$, $R^\mu_{\nu} = R^\rho_{\mu\rho\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$. With symbol $R$ we generally denote one of the above curvature tensors.
for the complex values of $z$ in the conical regions $C$ defined by:

$$C = \{ z \mid -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta \}.$$  

The last condition is necessary to achieve the maximum convergence of the theory in the UV regime. The necessary asymptotic behaviour is imposed not only on the real axis, but also on the conical regions, that surround it. In an Euclidean spacetime, the condition (ii) is not strictly necessary if (iii) applies. In (iii) the capital N is defined to be the following function of the spacetime dimension $D$: $2N + 4 = D_{odd} + 1$ in odd dimensions and $2N + 4 = D_{even}$ in even dimensions. Moreover, by $\Lambda$ we denote the scale of non-locality of the theory (not to be confused with the cosmological constant: $\Lambda_{cc}$).

One example of such entire function due to Tomboulis [3] is:

$$V^{-1}(z) = e^{\frac{1}{2}[\Gamma(0,p(z)^2)+\gamma_E+\log(p(z)^2)]}, \quad (5)$$

where $\Gamma(0, x)$ is the incomplete Gamma function with its first argument put to zero, $p(z)$ is a polynomial of degree $\gamma + N + 1$ and $\gamma_E$ is the Euler-Mascheroni mathematical constant. To achieve (super-)renormalizability the degrees of the polynomials appearing in the definitions of $V_0^{-1}(z)$ and $V_2^{-1}(z)$ must be equal. In the rest of the paper we will denote the common degree by $\gamma + N + 1$ ($N = 0$ in $D = 4$).

Few comments are in order here.

- First, it is obvious that the Minkowski spacetime is indeed a solution of the background EOM corresponding to the above action [1]. Other terms than the Einstein-Hilbert in the original Lagrangian are at least quadratic in curvature and as such vanish when evaluated for the Minkowski metric. Even though the EOM are not used in our present analysis the reader can find them in [27, 38, 39]. A cosmological constant term cannot be introduced here as it would lead to a constant non-trivial curvature at least.

- Second, the action [4] is written exactly as it is above because we want to highlight below the structure of the gravity propagator. Since the propagator can be read from a quadratic variation of the background action we worry about terms at most quadratic in curvatures. Higher curvature corrections vanish as upon the second variation as long as the background curvature itself is zero (as it is the case in Minkowski flat background).

- The requirement that the form-factors $\gamma_\ell$ are entire functions deserves a little bit more explanation. The object of our consideration are weakly non-local theories. This means that we have analytic function in the whole complex plane with in particular a smooth limit when momenta tend to zero. The way to think about this is to introduce a scale of gravity modification $\Lambda$ with the dimension of mass and carefully write everywhere $\Box/\Lambda^2$. As such the low energy limit is when the non-locality scale goes to infinity. From here we find out that the form-factors must be at least analytic in the origin. One may wonder why we need them to be entire functions i.e. analytic everywhere. It can be shown that the propagators of canonical variables in ADM formalism (which are observable quantities during inflation, for instance) feature a propagator with the form-factor $\gamma_0$ in the denominator. As such, if the function $\gamma_0$ has some pole, it will become a new pole for the canonical variable and the quantum properties of the theory would be spoiled w.r.t. our expectations for the measurements. The mathematical details of this arguments can be found in a parallel study [40]. However, we make a statement that indeed the functions $\gamma_\ell$ must be entire.

- The advertised above form of the form-factors $\gamma_\ell$ and the comment that only two out of three of these functions are essential is a consequence of the structure of the propagator and this is the matter of the succeeding analysis. It is however worth mentioning that the formulae [2], [3] do not guarantee that $\gamma_\ell$ are entire functions even though the functions $H_{0,2}$ are. This should be checked independently.

We additionally remark here the reason to call the term $\mathcal{V}$ appearing in the action [11] “curvature potential”. First of all, we argue that for any gauge theory as well as for gravity the strict distinction between the kinetic term and the potential of interaction does not exist. This is due to gauge invariance that connects interactions also with standard terms responsible for the propagator. The nomenclature we have adopted here is that by kinetic terms we mean terms that do contribute to the propagator around flat spacetime. Around the flat spacetime typically the kinetic terms are operators up to quadratic in curvature, while in the “curvature potential” we put all the terms cubic and higher in the curvature. The counting above is insensitive to the number of covariant derivatives appearing in the term under consideration. This is the only meaningful difference between the two parts of the action. On MSS operators cubic and higher in the curvature can contribute to the propagator. However, we can suitably modify the potential to make it compatible with the above definition around the Minkowski flat spacetime. Moreover, the locality of $\mathcal{V}$ is not a must, while the weak non-locality is not required by the unitarity.
Finally, since in the gravitational case the notion of local energy density of the gravitational field is not well-defined (strictly this is not a gauge-invariant observable with respect to the diffeomorphism group) we can not sensibly speak about the potential energy for the gravitational Lagrangian case. We want to emphasize that even in the case of finite QED (studied in [41]) the role of the potential \( V \) is different from the standard role ascribed to it in classical mechanics or in other field theory models, so it is a little inappropriate to call it like that.

### A. Propagator and unitarity around the Minkowski spacetime

Splitting the spacetime metric into the flat Minkowski background and the dimensionful fluctuation \( h_{\mu\nu} \) defined by \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa_D h_{\mu\nu} \) (here and above \( \kappa_D \) is proportional to the square root of the gravitational Newton constant), we can expand the action \([11]\) to the second order in \( h_{\mu\nu} \). The result of this expansion together with the usual harmonic gauge-fixing term reads \([42]\)

\[
\mathcal{L}_{\text{quad}} + \mathcal{L}_{\text{GF}} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\rho\sigma} h_{\rho\sigma},
\]

where the operator \( \mathcal{O} \) is made out of two terms, one coming from the quadratization of \([11]\) and the other from the following gauge-fixing term, \( \mathcal{L}_{\text{GF}} = \xi^{-1} \partial^\mu h_{\mu\nu} \omega(-\Box_L) \partial_\nu h_{\rho\sigma} \), where \( \omega(-\Box_L) \) is a weight functional \([43, 44]\). The d’Alembertian operator in \( \mathcal{L}_{\text{quad}} \) and the gauge fixing term must be conceived on the flat spacetime. Inverting the operator \( \mathcal{O} \) \([42]\) and making use of the form-factors \([2]\) and \([3]\), we find the two-point function in the harmonic gauge \( \partial^\mu h_{\mu\nu} = 0 \),

\[
\mathcal{O}^{-1} = \frac{\xi (2 P^{(1)} + \bar{P}^{(0)})}{2k^2 \omega(k^2/\Lambda^2)} + \frac{P^{(2)}}{k^2 e H_2(k^2/\Lambda^2)} - \frac{P^{(0)}}{(D-2)k^2 e H_0(k^2/\Lambda^2)}.
\]

We omitted the tensorial indices for the propagator \( \mathcal{O}^{-1} \) and the projectors \( \{ P^{(0)}, P^{(2)}, P^{(1)}, \bar{P}^{(0)} \} \) defined in \([42, 43]\).

The propagator \([7]\) is the most general one compatible with unitarity. It propagates no other degree of freedom besides the standard massless transverse spin-2 graviton. This follows from the fact that exponents of entire functions are special entire functions with no zeros. So we technically avoid new poles which means we avoid new physical degrees of freedom (d.o.f.). Returning to the comment in the previous subsection we see that the structure of the propagator advocates the form of form-factors \( \gamma_\ell \) as the absence of new d.o.f. was exactly the requirement behind formulae \([2]\) and \([3]\). We also note that in order to have a well-behaved propagator we need to get a correct form of only two factors corresponding to spin-0 and spin-2 parts. This explains why one function out of three \( \gamma_\ell \) can be put zero from the point of view of unitarity. Further, the unitarity is manifest, because the optical theorem at tree-level is trivially satisfied, namely

\[
2 \text{Im} \{ T(k)^{\mu\nu} \mathcal{O}^{-1}_{\mu\nu,\rho\sigma} T(k)^{\rho\sigma} \} = 2\pi \text{Res} \{ T(k)^{\mu\nu} \mathcal{O}^{-1}_{\mu\nu,\rho\sigma} T(k)^{\rho\sigma} \} \big|_{k^2=0} > 0,
\]

where \( T^{\mu\nu}(k) \) is the Fourier transform of the conserved energy tensor of a matter source.

### B. Power-counting in a nutshell

We now review \([11, 45, 53, 46, 47]\) power-counting analysis of the quantum divergences. We remark that the divergences do not depend on the choice of the background spacetime metric, therefore, the results in this subsection apply equally well to the case of theories studied around general MSS backgrounds. In the high energy regime, the above propagator \([7]\) in momentum space schematically scales as

\[
\mathcal{O}^{-1}(k) \sim \frac{1}{k^{2\gamma+D}} \text{ in the UV}.
\]

We have also replaced \(-\Box\) by \( k^2 \) in the quadratized action.

\[\text{2}\] The standard projectors are defined by:

\[
P_{\mu\nu,\rho,\sigma}^{(2)}(k) = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho} - \frac{1}{D-1} \theta_{\mu\nu} \theta_{\rho\sigma}), \quad P_{\mu\nu,\rho,\sigma}^{(1)}(k) = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}),
\]

\[
P_{\mu\nu,\rho,\sigma}^{(0)}(k) = \frac{1}{D-1} \theta_{\mu\rho} \theta_{\nu\sigma}, \quad P_{\mu\nu,\rho,\sigma}^{(0)}(k) = \omega_{\mu\rho} \omega_{\nu\sigma}, \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k\nu k\rho}{k^2}, \quad \omega_{\mu\nu} = \frac{k\nu k\rho}{k^2}.
\]

We also note that in order to have a well-behaved propagator we need to get a correct form of only two factors corresponding to spin-0 and spin-2 parts. This explains why one function out of three \( \gamma_\ell \) can be put zero from the point of view of unitarity. Further, the unitarity is manifest, because the optical theorem at tree-level is trivially satisfied, namely

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\]
The vertices can be collected in different sets, that may or may not involve the entire functions exp $H_i(z)$. However, to find a bound on the quantum divergences it is sufficient to concentrate on the leading operators in the UV regime. These operators scale as the propagator giving the following upper bounds on the superficial degree of divergence of any graph $G_1$.

$$\omega(G) = DL + (V - I)(2\gamma + D),$$  \hspace{1cm} (11)$$

in a spacetime of even or odd dimension respectively. We simplify the above relation further to

$$\omega(G) = D - 2\gamma(L - 1).$$  \hspace{1cm} (12)$$

In (12), we used the topological relation between the numbers of vertices $V$, internal lines $I$ and the number of loops $L$: $I = V + L - 1$. Thus, if $\gamma > D/2$, in the theory only 1-loop divergences survive. Therefore, the theory is super-renormalizable $[\text{1, 3, 5, 6}]$ and only a finite number of operators of mass dimension up to $M^D$ has to be included in the action in even dimension for the purpose of renormalization.

Notice that the power counting analysis can be done in Minkowski spacetime because any smooth spacetime is locally flat and the divergences are related to the UV coincidence limit in the correlation functions. Therefore, we can expand around whatever background and we will always end up with the same divergent contributions to the quantum effective action, namely we will always get the same beta functions.

C. The theory in Weyl basis

We can equally consider a different action, which will be written by re-shuffling quadratic in curvature terms in (1). The following action is equivalent to (11) for everything about unitarity (the propagator is given again by (7)) and super-renormalizability or UV-finiteness and its Lagrangian density reads

$$\mathcal{L}_C = -2\kappa D^{-2} \sqrt{|g|} \left[ R + C \gamma C(\square)C + R \gamma S(\square)R + \text{Riem} \gamma R(\square)\text{Riem} + \mathcal{V} \right],$$  \hspace{1cm} (13)$$

where $C$ is the Weyl tensor and all the form-factors $\gamma_\ell$ are defined in (2) and (3).

To start with, we recall that only two form-factors are needed to have appropriate propagator. We thus put $\gamma_R = 0$ and the theory (14) reduces to:

$$\mathcal{L}_C = -2\kappa D^{-2} \sqrt{|g|} \left[ R + C \gamma C(\square)C + R \gamma S(\square)R + \mathcal{V}(C) \right],$$  \hspace{1cm} (15)$$

where

$$\gamma_C = \frac{D - 2}{4} e^{H_2} - 1 - \frac{2(D - 1)}{4}\gamma_2, \hspace{1cm} \gamma_2 = \frac{D - 2}{4} e^{H_0} - 1, \hspace{1cm} \gamma = \frac{D - 2}{4} e^{H_0} - 1 \hspace{1cm} (16)$$

In $D = 4$ it is enough to include $\mathcal{V}$ made out of two Weyl killers to end up with a completely finite quantum gravitational theory at any perturbative order in the loop expansion. For example we can choose the following two operators,

$$\mathcal{V}(C) = s_w^{(1)} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \square^{\gamma} - 2 C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + s_w^{(2)} C_{\mu\nu\rho\sigma} C^{\alpha\beta\gamma\delta} \square^{\gamma} - 2 C_{\alpha\beta\gamma\delta} C^{\mu\nu\rho\sigma},$$  \hspace{1cm} (17)$$

The Gauss-Bonnet (GB) operator does not contribute to the divergent part of the quantum effective action in $D = 4$ when the manifold has a trivial topology, namely the spacetime is topologically equivalent to the Minkowski one or the Euclidean space (see for example [1]). However, in the rest of the paper we will deal with the (A)dS space and we will have to take care of the divergence proportional to the GB too.

The beta functions for the two couplings in front of terms quadratic in curvature can be only linear in the front coefficients $s_w^{(1)}$ and $s_w^{(2)}$, then we can always find a solution to the equations $\beta R_2 = 0$ and $\beta R_{\text{ricc}} = 0$ regardless of the energy scale and the loop order. The integral of the Gauss-Bonnet operator is in this section identically zero because we assume the space to be topologically equivalent to the Minkowski spacetime. Later we will be forced to give up this hypothesis in (A)dS.

As pointed out in the introduction the weak non-locality is not sufficient to solve the singularity issue that plagues the Einstein-Hilbert gravitational theory. In particular, for the theory in the Weyl basis presented in this section the FRW metrics for conformal matter ($T_{\text{matter}} = 0$) solve exactly the non-local EOM [27]. This means that the Big-Bang singularity shows up in an exact solution of our non-local quantum gravity. However, if the gravitational sector also
enjoys conformal invariance, then any FRW singular spacetime is conformally equivalent to the flat spacetime by a conformal rescaling and the singularity turns out to be unphysical [48]. Notice, that the presence of singularities in particular non-local theories does not rule out that it may exist non-local theories singularity-free. However, the naive non-locality by itsel itself is not enough [27, 48].

The crucial ingredient here is the conformal symmetry, which allows for rescalings like described above. Moreover, scale invariance helps with singularity of geodesics [48, 49] and also with non-locality by itself is not enough [27, 48].

III. NON-LOCAL GRAVITY IN (A)DS VACUUM

A generalization to a constant curvature background is rather straightforward. We here provide the expansion of the action to the second order in the gravitational fluctuations around an (A)dS spacetime and we will infer about the stability properties of the theory around any maximally symmetric vacuum. We retrace the path followed for the case of the Minkowski vacuum to mimic as much as possible the Einstein-Hilbert theory. Therefore, we will end up with a quadratic operator that reproduces the one from Einstein’s gravity on the same background up to, at most, two multiplicative form-factors that do not change the structure of the classical two-point function [52].

Technically, with a quadratic operator that reproduces the one from Einstein’s gravity on the same background up to, at most, two multiplicative form-factors that do not change the structure of the classical two-point function [52]. Technically, we will use the previous computations published in [26, 53, 54] (see also the appendix). For definiteness, we will first focus on the non-local theories that we identify as “theories in the Weyl basis” and theories in the “Ricci basis”.

A. A class of theories in the Weyl basis

To see how things work we stick to \( D = 4 \), make use of the Weyl basis, and consider the case \( \gamma_R = 0 \),

\[
\mathcal{L}_{CR} = -2k^{-2} \sqrt{|g|} \left[ R - 2\lambda_{cc} + C \gamma_{C(\Box)}C + R\gamma_{S(\Box)}R + \mathcal{V}(C) \right] \tag{18}
\]

where the translations of the covariant box operators (by the amount proportional to \( R \)) in comparison to the form-factors given in (10) will be shortly clear. Notice that the form-factors (14) turn in (19) when the formal limit \( R \to 0 \) in (14) is taken. Moreover, the ordering of the operators could be relevant in (19) for some choices of the asymptotic polynomials. Indeed, the arguments of the entire functions \( H_0 \) and/or \( H_2 \) can in general differ from the denominators in (19) so that they do not commute due to the Ricci scalar curvatures appearing with different numerical coefficients.

In taking the quadratic part of the action (13) in the graviton fluctuation \( h_{\mu\nu} \) we use the following decomposition of the graviton field,

\[
h_{\mu\nu} = h^\perp_{\mu\nu} + \nabla_{(\mu}A^\perp_{\nu)} + \left( \nabla_\mu \nabla_\nu - \frac{1}{4}g_{\mu\nu} \Box \right) B + \frac{1}{4}g_{\mu\nu} h,
\]

where the spin-two fluctuation \( h^\perp_{\mu\nu} \) contains 5 degrees of freedom because it satisfies \( \nabla^\nu h^\perp_{\mu\nu} = g^{\mu\nu} h^\perp_{\mu\nu} = 0 \). The transverse vector \( A^\perp_{\mu} \), satisfying \( \nabla_\mu A^\perp_\mu = 0 \), is accounting for three degrees of freedom. Finally, \( B \) and \( h \) are two real

\[\]
scalars. However, $A^2_{\mu\nu}$ automatically drops out of the second variation of the action and out of the two scalars only the following combination $\phi = \Box B - h$ appears there.

We end up with the following second order variation of the action [24, 53, 54],

$$S^{(2)\text{CR}}_{(\lambda)\text{dS}} = \frac{1}{2} \int d^4 x \sqrt{|g|} \left\{ \tilde{h}^{\perp}_{\mu\nu} \left( \Box - \frac{\tilde{R}}{6} \right) \left[ 1 + 2\gamma_S(0)\tilde{R} + 2 \left( \Box - \frac{\tilde{R}}{3} \right) \gamma_C \left( \Box + \frac{\tilde{R}}{3} \right) \right] \tilde{h}^{\perp}_{\mu\nu} - \tilde{\phi} \left( \Box + \frac{\tilde{R}}{3} \right) \left[ 1 + 2\gamma_S(0)\tilde{R} - 6 \left( \Box - \frac{\tilde{R}}{3} \right) \gamma_S(\Box) \right] \tilde{\phi} \right\}, \tag{22}$$

where we introduced the canonically normalized fields $\tilde{h}^{\perp}_{\mu\nu} = M_P h^{\perp}_{\mu\nu}/2$, $\tilde{\phi} = \sqrt{3/32} M_P \phi$, and $M_P^2 = 4\kappa_4^{-2}$. In this part of the section the bar operators $\bar{O}$ denote any background quantity. Moreover, $\gamma_S(0)$ can be read out of the following general expansion,

$$\gamma_S(\Box) = \sum_{i=0}^{\infty} c_{S,i} \left( (\Box + X)^n \right)^i, \quad n_1, n_2 \in \mathbb{N}, \tag{23}$$

where $X$ is an operator proportional to the background Ricci scalar $R$. The chosen order in [23] is consistent with the polynomial given below in [24]. The detailed expressions for the most general second order variations of various actions on MSS are collected in the appendix. We can assume $\gamma_S(0) = 0$ (i.e. $c_{S,0} = 0$ in [19], [23]) because this is consistent with the requirements for the special entire function $H(z)$ [2]. Therefore, replacing the form-factors [19] in the variation [22], we end up with the following result,

$$S^{(2)\text{CR}}_{(\lambda)\text{dS}} = \frac{1}{2} \int d^4 x \sqrt{|g|} \left\{ \tilde{h}^{\perp}_{\mu\nu} \left( \Box - \frac{\tilde{R}}{6} \right) e^{H_2(\Box - \frac{3}{2} \tilde{R})} \tilde{h}^{\perp}_{\mu\nu} - \tilde{\phi} \left( \Box + \frac{\tilde{R}}{3} \right) e^{H_0(\Box, \tilde{R})} \tilde{\phi} \right\}. \tag{24}$$

The condition $\gamma_S(0) = 0$ and the locality of counterterms force us to select the following entire function (we here consider the $\gamma = 3$ case),

$$H_0(\Box, R) = \frac{1}{2} \left\{ \gamma_E + \Gamma \left( 0, \left[ p_S(\Box, R) \right]^2 \right) + \log \left[ p_S(\Box, R) \right]^2 \right\}, \tag{25}$$

$$p_S(\Box, R) = \frac{1}{\Lambda^8} \left( \Box + \frac{R}{3} \right)^2 \Box^2. \tag{26}$$

For the form-factor $\gamma_C$ we can take the following entire function $H_2$,

$$H_2 \left( \Box - \frac{2}{3} \tilde{R} \right) = \frac{1}{2} \left\{ \gamma_E + \Gamma \left( 0, \left[ p_C \left( \Box - \frac{2}{3} \tilde{R} \right) \right]^2 \right) + \log \left[ p_C \left( \Box - \frac{2}{3} \tilde{R} \right) \right]^2 \right\}, \tag{27}$$

$$p_C \left( \Box - \frac{2}{3} \tilde{R} \right) = \frac{1}{\Lambda^8} \left( \Box - \frac{2}{3} \tilde{R} \right)^4. \tag{28}$$

We notice here that the choice of polynomials is fixed only by the UV behaviour of the propagator and as such we have a lot of freedom in choosing them as long as basic principles are obeyed.

For instance we can have “commutative” form-factors [20] such that

$$p_C(\Box, \Lambda_{cc}) = \frac{1}{\Lambda^8} \left( \Box - \frac{8}{3} \Lambda_{cc} \right)^2, \tag{28}$$

$$p_S(\Box, \Lambda_{cc}) = \frac{1}{\Lambda^8} \left( \Box + \frac{4}{3} \Lambda_{cc} \right)^2. \tag{29}$$

Notice that the above polynomials are zero for $\Box = 0$, which is crucial to secure $\gamma_S(0) = 0$. Indeed, [29] is a polynomial in $\Box$ and if we do not multiply by $\Box^2$ we get a constant dimensionless contribution proportional to $(\Lambda_{cc}/\Lambda^2)^2 \propto c_{S,0} \neq 0$.

The following special choice of the polynomial $p_{\gamma+1}$ ($\gamma + 1 = 8$) makes also consistent the identification of $H_2$ with $H_0$,

$$p_S(\Box, R) = \frac{1}{\Lambda^{16}} \left( \Box + \frac{R}{3} \right)^2 \Box^2 \left( \Box - \frac{2}{3} \tilde{R} \right)^4. \tag{30}$$
It is now clear why the fraction $1/(\Box - \frac{2}{3} R)$ is located on the right in the definition of $\gamma_C(\Box)$ in (19). This in turn results in that the second variation of the action simplifies to,

$$S_{(2)\text{AdS}}^{(2)\text{CR}} = \frac{1}{2} \int d^4x \sqrt{|g|} \left\{ h_{\parallel \mu \nu} \left( \Box - \frac{R}{6} \right) e^{H_S(\Box + \frac{4}{3})} \gamma_C(h_{\parallel \mu \nu} - \phi) \left( \Box + \frac{R}{3} \right) e^{H_S(\Box + \frac{4}{3})} \phi \right\},$$

(31)

where $H_S$ is in the class of entire functions \[3\] with the polynomial $p(z)$ substituted by the above definition of the polynomial $p_S(\Box + \frac{R}{3})$ evaluated on the (A)dS background of curvature $R$.

With a globally well-defined field redefinition we can now completely remove the form-factor and the kinetic operator turns into the one of Einstein-Hilbert theory with cosmological constant. The interactions will get modified by the field redefinition as well, but the Feynman diagrams will stay the same. However, in doing so we do not really need to equate the form-factors in front of different spin modes. Therefore, for the moment the choice (30) is just to make the second order variation of the non-local theory as much similar as we can to the Einstein-Hilbert one. Details about such a field redefinition are explained in Section V.

**Analysis of the “non-commutative” form-factors**

In this quite technical subsection we study some properties of the form-factors that will turn out to be crucial in Section IV about quantum finiteness. Let us remind that the exponentials of one, two, or multiple matrices are defined by means of power series, namely

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k,$$

$$e^{X+Y} = \sum_{k=0}^{\infty} \frac{1}{k!} (X+Y)^k,$$

$$e^{X_1+X_2+X_3+\cdots+X_N} = \sum_{k=0}^{\infty} \frac{1}{k!} (X_1 + X_2 + X_3 + \cdots + X_N)^k.$$  

(32)

For the form-factors defined in (19) with polynomials (26) and (27) we can make explicit the above formula (32) as follows,

$$e^{H_{0,2}(\Box, R)} = \sum_{n=0}^{\infty} \frac{1}{n!} H(\Box, R)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \frac{1}{2} \left[ \Gamma(0, p_S, C(\Box, R)^2) + \gamma_E + \log(p_S, C(\Box, R)^2) \right] \right\}^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m!} \frac{p_S, C(\Box, R)^{2m}}{2m} \right\}^n = \sum_{s=0}^{\infty} c_s p_S, C(\Box, R)^{2s},$$

(33)

(34)

where the coefficients $c_s$ are obtained by comparing the last two sums above. Notice that $p(\Box, R)$ certainly commutes with itself, but its arguments, namely $\Box$ and $R$, do not commute. Therefore, it is still easily possible to apply the structure of the vertex functions found in [3] to the case of a binomial like (27) because the form-factor $\gamma_C$ is a function of only one polynomial, namely

$$\gamma_C = \sum_{s=0}^{\infty} \tilde{c}_s \left( \Box - \frac{2}{3} R \right)^{8s-1} = \tilde{c}_1 \left( \Box - \frac{2}{3} R \right)^7 + \tilde{c}_3 \left( \Box - \frac{2}{3} R \right)^{23} + \cdots$$  

or  

$$\gamma_C = \sum_{r=0}^{\infty} a_r \left( \Box - \frac{2}{3} R \right)^r,$$

(35)

for a proper and fixed choice of the coefficients $a_r$ given the coefficients $\tilde{c}_s$. Now we can apply the formula presented in [1] and rigorously proved in [3, 46] to the operator $\Box = \Box - 2R/3$. In particular we can introduce the following notation,

$$\gamma_C = \sum_{r} a_r (\Box_M + I)^r, \quad I = \Box - \Box_M, \quad \Box_M = \eta^\mu^\nu \nabla_\mu \nabla_\nu$$

(36)
We also remind that around fixed Minkowski background the form-factors in momentum space are the Fourier transform of

$$\gamma_C = \sum_{r=0}^{\infty} a_r (\Box_M)^r = \frac{1}{2} \frac{1}{\Box_M} \left( e^{H_0(\Box_M)} - 1 \right),$$  \hspace{1cm} (37)

while the gravitons vertices come only from the perturbative expansion of $I$.

Less trivial is to apply the formula in [46] to $\gamma_S$ that we can express as follows,

$$\gamma_S = \frac{1}{6} \frac{1}{\Box + \frac{R^2}{3}} \sum_{r=0}^{\infty} c_r p_{2s}^2 (\Box, R) = -\frac{1}{6} \frac{1}{\Box + \frac{R^2}{3}} \sum_{s=1}^{\infty} c_s \left( \frac{1}{\Lambda^8} \left( \Box + \frac{R^2}{3} \right)^{2s} \right)^{2s-1},$$

$$= \frac{1}{6} \left( \Box^3 + \frac{R^2}{3} \right) \sum_{s=1}^{\infty} c_s \frac{1}{\Lambda^{16s}} \left( \left( \Box + \frac{R^2}{3} \right)^{2s} \right)^{2s-1},$$

$$= \frac{1}{6} \left( \Box^3 + \frac{R^2}{3} \right) \sum_{s=0}^{\infty} a_s \left( \left( \Box + \frac{R^2}{3} \right)^{2s} \right)^r. \hspace{1cm} (38)$$

The coefficients $c_s$ are fixed using the definition [34], while the coefficients $a_r$ can be derived comparing the last two expressions in [38]. Now we can apply the derivation in [46] to

$$\sum_{r=0}^{\infty} a_r (\Box^4_M + I)^r, \hspace{1cm} I = \Box^4 - \Box^4_M + O(R). \hspace{1cm} (39)$$

When we expand in the graviton field we get interaction vertices coming from the variation of the binomial on the left of the sum in (38) and other vertices come from the variation of the sum. However, the full non-local contribution resulting from the variation of (38) will reconstruct the same incremental ratios as defined in [46], but for the form-factor in Minkowski space rescaled by 1/\Box^4, namely

$$\frac{\gamma_S(\Box_M)}{\Box_M} = -\frac{1}{6} \frac{e^{H_0(\Box_M)} - 1}{\Box_M^4} \Lambda^8. \hspace{1cm} (40)$$

We can forget the non-locality to evaluate the divergent contributions to the quantum effective action.

### B. A class of theories in the Ricci basis

As a second example we consider the following action involving the Ricci tensor, but not the Weyl tensor in the quadratic in curvature part of the action, namely

$$\mathcal{L}_{SR} = -2\kappa^{-2}_4 \sqrt{|g|} \left[ R - 2\Lambda_{cc} + S_{\gamma S_2(\Box)} S + R_{\gamma S(\Box)} R + \mathcal{V}(\mathcal{C}) \right] , \hspace{1cm} (41)$$

where the rank-two tensor $S$ is defined by

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}. \hspace{1cm} (42)$$

In $D = 4$ it is identically zero when evaluated on an (A)dS background and, moreover, it is completely trace-free. The form-factors in the action (41) are defined by:

$$\gamma_{S_2(\Box)} = \frac{1}{\Box - \frac{4}{3}} \left( e^{H_{S_2}(\Box)} - 1 \right), \hspace{1cm} (43)$$

$$\gamma_S(\Box) = -\frac{1}{6} \frac{1}{\Box + \frac{4}{3}} \left( e^{H_0(\Box, R)} - 1 \right) - \frac{1}{12} \left( \Box + \frac{4}{3} \right) \left( e^{H_{S_2}(\Box + \frac{4}{3}) - 1} \right) \frac{1}{(\Box + \frac{4}{3})}. \hspace{1cm} (44)$$
We should here clarify how the entire functions defined above depend on their arguments. Let us start with $H_{S2}$ in $\gamma_{S2}(\square)$, that is defined to be the following entire function of the polynomial $p_{S2}$ of a fourth degree in $\square$,

$$H_{S2}\left(\left(\square - \frac{R}{6}\right)\left(\square - \frac{R}{3}\right)\right) = \frac{1}{2} \left(\gamma_E + \Gamma \left(0, p_{S2}(\square, R)\right) + \log \left(p_{S2}(\square, R)\right)\right),$$

$$p_{S2}(\square, R) = \left(\square - \frac{R}{6}\right)^2 \left(\square - \frac{R}{3}\right)^2. \tag{45}$$

Therefore, the exponentiated entire function $H_{S2}\left(\left(\square + \frac{R}{2}\right)\left(\square + \frac{R}{2}\right)\right)$ in the second analytic operator in (44) is obtained translating the operator $\square$ of the amount $\frac{2}{3}R$, namely

$$H_{S2}\left(\left(\square + \frac{R}{2}\right)\left(\square + \frac{R}{2}\right)\right) := H_{S2}\left(\left(\square - \frac{R}{6}\right)\left(\square - \frac{R}{3}\right)\right) \bigg|_{\square \rightarrow \square + \frac{2}{3}R}. \tag{46}$$

Notice how the translated $\square$ operators at the denominator in (44) have been placed in order to avoid ordering issues. Moreover, taking the “formal” limit $R \to 0$ in the form-factors (44) and assuming $H_{S2} = H_0$ the above Lagrangian (41) turns into

$$\mathcal{L}_E = -2\kappa_D^{-2} \sqrt{|g|} \left[ R + G_{\mu\nu} \gamma_G(\square) R^{\mu\nu} + \mathcal{V} \right],$$

$$\gamma_G = e^{H_2 - 1}. \tag{49}$$

Finally, the second order variation of the action for the Lagrangian (41) reads

$$S^{(2)CSR}_{(A)dS} = \frac{1}{2} \int d^4x \sqrt{|g|} \left\{ \mathring{h}^{\perp\mu\nu} \left(\square - \frac{R}{6}\right) \left[ 1 + 2\gamma_S(0) \tilde{R} + \left(\square - \frac{R}{6}\right) \gamma_{S2}(\square) \right] \mathring{h}^{\perp}_{\mu\nu} - \tilde{\phi} \left(\square + \frac{R}{3}\right) \left[ 1 + 2\gamma_S(0) \tilde{R} - 6 \left(\square + \frac{R}{3}\right) \gamma_S(\square) - \frac{1}{2} \gamma_{S2}(\square) \left(\square + \frac{2}{3}R\right) \tilde{\phi} \right] \right\}. \tag{50}$$

We also selected out a form-factor such that $\gamma_S(0) = 0$. For this purpose, after looking at formula (44), it is sufficient to take the following asymptotic polynomial $p_{\gamma+1}$ (for $\gamma + 1 = 3 + 1$) as an argument of $H_0$,

$$p_{S}(\square, R) = \left(\square + \frac{R}{3}\right)^2. \tag{51}$$

Therefore, after plugging the form-factors (44) in the second order variation (50) we end up again with (44), but with $H_2$ replaced by $H_{S2}$, namely

$$S^{(2)CR}_{(A)dS} = \frac{1}{2} \int d^4x \sqrt{|g|} \left\{ \mathring{h}^{\perp\mu\nu} \left(\square - \frac{R}{6}\right) e^{H_{S2}\left(\left(\square - \frac{R}{6}\right)\left(\square - \frac{R}{6}\right)\right)} \mathring{h}^{\perp}_{\mu\nu} - \mathring{\phi} \left(\square + \frac{R}{3}\right) e^{H_0(\square, R)} \mathring{\phi} \right\}. \tag{52}$$

In order to end up with the same form-factor in the spin-two as long as in the spin-zero graviton sectors we slightly modify the polynomial in (45) and we replace the curvature $R$ with the cosmological constant $\Lambda_{cc}$, namely

$$\tilde{p}(\square; \Lambda_{cc}) = \left(\square - \frac{R}{6}\right)^2 \left(\square + \frac{R}{3}\right)^2 \left(\square - \frac{R}{3}\right)^2 \bigg|_{R \rightarrow 4\Lambda_{cc}} \times \square^2 = \left(\square - \frac{2}{3}\Lambda_{cc}\right)^2 \left(\square + \frac{4}{3}\Lambda_{cc}\right)^2 \left(\square - \frac{4}{3}\Lambda_{cc}\right)^2 \square^2. \tag{53}$$

---

4 We can use here different definitions of the form-factors to avoid the ordering problems of the denominators versus the exponential form-factors, namely

$$\gamma_{S2}(\square) = \frac{e^{H_{S2}\left(\left(\square - \frac{2}{3}\Lambda_{cc}\right)\left(\square - \frac{4}{3}\Lambda_{cc}\right)\right)} - 1}{\square - \frac{2}{3}\Lambda_{cc}}, \tag{47}$$

$$\gamma_S(\square) = -\frac{1}{6} \frac{e^{H_0(\square, \frac{2}{3}\Lambda_{cc})} - 1}{\square + \frac{2}{3}\Lambda_{cc}} - \frac{1}{12} \frac{e^{H_{S2}\left(\left(\square + \frac{2}{3}\Lambda_{cc}\right)\left(\square + \frac{4}{3}\Lambda_{cc}\right)\right)} - 1}{\square + \frac{4}{3}\Lambda_{cc}}. \tag{48}$$

The ordering is now irrelevant, because $\Lambda_{cc}$ is a numerical constant.
where we technically replaced the Ricci scalar $R$ with $4\Lambda_{cc}$ to end up with a form-factor without ordering issues. This replacement does not mean that we evaluate the form-factor on the background, but just that the form-factor has a particular (a posteriori) dependence on the constant $\Lambda_{cc}$. In the untranslated $\Box$ on the right secures that $\gamma_S(0) = 0$. The form-factors now read:

$$
\gamma_{S2}(\Box) = \frac{e^{H_{S2}\left(\Box - \frac{4}{3}\Lambda_{cc}\right)} - 1}{\Box - \frac{4}{3}\Lambda_{cc}},
$$

(54)

$$
\gamma_S(\Box) = -\frac{1}{6} e^{H_{S2}\left(\Box - \frac{4}{3}\Lambda_{cc}\right)} \left(\Box - \frac{4}{3}\Lambda_{cc}\right) - \frac{1}{12} \left(\Box - \frac{4}{3}\Lambda_{cc}\right) \left(\Box - \frac{4}{3}\Lambda_{cc}\right) - 1.
$$

(55)

Therefore, we end up with the form-factors already introduced in the footnote above, but with a new polynomial as an argument of the entire function $H_{S2}$. Moreover, now $H_0 = H_{S2}$ and the second variation of the action [52] turns into

$$
S^{(2)}_{CR}(\Lambda) \delta S = \frac{1}{2} \int d^4x \sqrt{|g|} \left\{ \tilde{h}^{\mu\nu} \left(\Box - \frac{\tilde{R}}{3}\right) e^{H_{S2}(\tilde{p})} \left(\Box - \frac{\tilde{R}}{3}\right) e^{H_{S2}(\tilde{p})} \tilde{\phi} \right\}.
$$

(56)

The second order variation (56) has the same form-factor to multiply the tensorial as long as the scalar perturbations. Once more we point out that the replacement of $R$ with $\Lambda_{cc}$ is an off-shell operation just as in Einstein-Hilbert theory in the presence of a cosmological constant.

IV. QUANTUM FINITENESS

In this section we study two classes of theories involving respectively the Ricci scalar and the off-shell cosmological constant in the form-factors. In the first subsection we study the theory [18] with form-factors [19], while in the second subsection the theory [18] with form-factors [20].

A. Analysis of the theory [18] with form-factors [19]

In agreement with the analysis in the previous section the polynomial appearing in the ultraviolet limit of the form-factor can also contain powers of the Ricci scalar, while the non-local structure only gives contributions to the finite part of the quantum effective action. Therefore, a quite general polynomial giving contribution to the beta functions in $D = 4$ is:

$$
p(z, \mathcal{R}) = a_{\gamma+1}^{(0)} z^{\gamma+1} + a_{\gamma+1}^{(1)} z^{\gamma+2} \mathcal{R} + a_{\gamma+1}^{(2)} z^{\gamma+3} \mathcal{R}^2 + \ldots + a_{\gamma}^{(0)} z^{\gamma} + a_{\gamma}^{(1)} z^{\gamma+1} \mathcal{R} + \ldots + a_{\gamma}^{(0)} z^{\gamma+1} \mathcal{R} + \ldots .
$$

(57)

For the theories presented in this paper $\mathcal{R}$ can only be the Ricci scalar. The ellipsis ($\ldots$) also include terms arising from commutators of the $\Box$ operator with covariant derivatives and curvatures.

To have a finite theory at quantum level (or better conformally invariant) we have to make vanish the beta functions for the following six operators,

$$
\sqrt{|g|}, \sqrt{|g|} \mathcal{R}, \sqrt{|g|} \mathcal{R}^2, \sqrt{|g|} \text{Ric}^2, \sqrt{|g|} \text{GB}, \sqrt{|g|} \Box \mathcal{R},
$$

(58)

where $\text{GB}$ is the Gauss-Bonnet operator. The beta functions $\beta_{\mathcal{R}^2}, \beta_{\mathcal{R}^2}, \beta_{\text{GB}}$ (in the basis [18]) get contributions also from the following killers, if they are added to the action,

$$
\mathcal{V}(C) = s^{(1)}_w C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \Box^{-2} C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + s^{(2)}_w C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \Box^{-2} C_{\alpha\beta\gamma\delta} C^{\mu\nu\rho\sigma}.
$$

(59)

These killers do not spoil the structure of the kinetic operator nor the propagator because the Weyl tensor evaluated on any homogeneous and isotropic spacetime is identically zero and the second order variation of the action (59) is at least quadratic in the Weyl tensor. Moreover, they are enough to make zero the two beta functions $\beta_{\mathcal{R}^2}$ and $\beta_{\mathcal{R}^2}$. Indeed, the contribution of (59) can only be linear in the front coefficients $s^{(1)}_w$ and $s^{(2)}_w$ as has been shown in [6] by a direct implementation of the background field method.
If we want to use killers that do not change the structure of the kinetic operator around (A)dS one option is to build them using only hatted quantities like in the footnote no. five (so with the background value of the tensor subtracted). Other viable killers, which possess the same property, are:

\[ s_s^{(1)} S_{\mu \nu} S^\rho \square \gamma^{-2} S_{\rho \sigma} S_{\mu \sigma} , \quad s_s^{(2)} S_{\mu \nu} S^\rho \square \gamma^{-2} S_{\rho \sigma} S_{\mu \sigma} \], \quad \text{where} \quad S_{\mu \nu} \quad \text{was defined in (12).} \quad (60)

On any MSS background the GB operator is non-vanishing, while \( \square R \) always vanishes. Regarding the contributions to the divergent part of the quantum effective action (under an integral) GB and \( \square R \) can be neglected as total derivatives on MSS. The reason to kill these two more divergences has eventually to do with the conformal invariance of the theory, but not with finiteness.

The beta function for the Newton constant \( \beta_R \) can be made zero using following example of the killer operator \(^5\),

\[ S_{\mu \nu} s_s^{\mu \rho} \square \gamma^{-2} S^\nu \rho . \quad (62) \]

Finally, to have a finite theory we need to make vanishing the beta function for the cosmological constant. For this achievement we need to explicitly evaluate the divergent contributions to the one-loop effective action that do not contain any curvature tensor. This result was derived for the first time in \([33]\) and also successfully attained by our group \([55]\). Given the polynomial \([57]\), only the monomials independent on the curvatures can contribute to the \( R^0 \) divergence. Therefore, the beta function can only depend on the coefficients \( a_{\gamma +1}^{(0)}, a_{\gamma}^{(0)}, a_{\gamma -1}^{(0)} \) in \([57]\). For the sake of simplicity we here only consider the theory in Weyl basis \([18]\) with form-factors \([19]\). Moreover, we take \( H_2 = H_0 \), but we replace the polynomial \([60]\) with

\[ p_{12} = \left( \square + \frac{R}{3} \right)^2 (c_1 \square^3 + c_2 \square^2 + c_3 \square)^2 \left( \square - \frac{2}{3} R \right)^4 = a_{\gamma +1}^{(0)} \square^{12} + a_{\gamma}^{(0)} \square^{11} + a_{\gamma -1}^{(0)} \square^{10} + O(R) , \quad (63) \]

and comparing with \([57]\): \( a_{\gamma +1}^{(0)} = c_1, \quad a_{\gamma}^{(0)} = 2c_1c_2, \quad a_{\gamma -1}^{(0)} = 2c_1c_3 \). Note that with the polynomial \([63]\) we surely avoid the issue of non-local counterterms because it is definite positive on the real axis (namely \( \sqrt{p_{12}} = p_{12} \)). Therefore, \( c_1, c_2 \) and \( c_3 \) can be selected to be positive, negative or zero (at least one of the \( c_i \) must be non-zero).

The form-factor \( \gamma_C \) and \( \gamma_S \) in the UV are respectively:

\[ \begin{align*}
\gamma_C &= \frac{e^{\gamma_C/2}}{2} \left( \square + \frac{R}{3} \right)^2 (c_1 \square^3 + c_2 \square^2 + c_3 \square)^2 \left( \square - \frac{2}{3} R \right)^4 , \\
\gamma_S &= -\frac{e^{\gamma_S/2}}{6} \left( \square + \frac{R}{3} \right) (c_1 \square^3 + c_2 \square^2 + c_3 \square)^2 \left( \square - \frac{2}{3} R \right)^4 .
\end{align*} \quad (64, 65) \]

Moreover, the operators \( O(R) \) do not give contribution to the beta function for the cosmological constant. Finally, we need to explicitly evaluate the beta function for the cosmological constant \( \langle \beta_{\Lambda_{cc}} \rangle \) and select the parameters \( a_{\gamma +1}^{(0)}, a_{\gamma}^{(0)}, a_{\gamma -1}^{(0)} \) to make zero \( \beta_{\Lambda_{cc}} \). Once more, the parameters \( a_{\gamma +1}^{(0)}, a_{\gamma}^{(0)}, a_{\gamma -1}^{(0)} \) do not run because all of them appear in front of higher dimensional operators of dimension higher than four.

For the theory \([18]\) with form-factors \([19]\) we can explicitly show the finiteness of the theory because the beta function \( \beta_{\Lambda_{cc}} \) has been computed in \([33]\) and \([55]\) for the following prototype theory

\[ S_N = \int d^4 x \sqrt{|g|} (\omega_{N,R} R \square N R + \omega_{N,C} C \square N C) . \quad (66) \]

From the divergent part of the quantum effective action we can read the beta function. The outcome of the computation is \([33]\):

\[ \Gamma_{cc, \text{div}}^{(1)} = -\frac{1}{2(4\pi)^2} \frac{1}{\epsilon} \int d^4 x \sqrt{|g|} \left( \frac{\omega_{N-2,C}}{\omega_{N,C}} + \frac{\omega_{N-2,R}}{\omega_{N,R}} - \frac{5\omega_{N-2,R}^2}{2\omega_{N,R}^2} - \frac{\omega_{N-1,R}^2}{2\omega_{N,R}} \right) \equiv -\frac{1}{\epsilon} \int d^4 x \sqrt{|g|} \beta_{\Lambda_{cc}} . \quad (67) \]

\(^5\) Additionally, we can make to vanish the beta functions \( \beta_{R^2}, \beta_{R_{\mu \nu}^2} \), and \( \beta_R \) also introducing the following terms in \( V \),

\[ \begin{align*}
& s_s^{(1)} \hat{R}_{\mu \nu} \hat{R}^{\mu \rho} \square \gamma^{-2} \hat{R} \hat{R}^{\rho \sigma} s_s^{(2)} \hat{R}^2 \square \gamma^{-2} \hat{R}^2 , \quad \text{where} \quad \hat{R}_{\mu \nu} = R_{\mu \nu} - \Lambda_{cc} g_{\mu \nu} \quad \text{and} \quad \hat{R} = R - 4\Lambda_{cc} , \\
& \hat{R}_{\mu \nu} \hat{R}^{\mu \rho} \square \gamma^{-2} \hat{R}^2 .
\end{align*} \quad (61) \]

However, the operators \([61]\) must be used more carefully because the cosmological constant can be present in the beta functions.
Finally, we have to compare the non-running coefficients $\omega_{i,C}$ and $\omega_{i,R}$ ($i = N + 1, N, N - 1$), which appear in front of the operators quadratic in the Weyl tensor and in the Ricci scalar in (66), with the parameters in front of the same operators resulting in the action (15) with asymptotic form-factors (64) and (65).

Since the issue with UV-divergences is probing the UV limit of the theory this can also be thought in the following way. The divergences arise because of the coincidence limit of points used as arguments of Green’s functions. When points do come closer the spacetime is effectively flat and they do not see such effect like the (A)dS curvature radius. That is why all divergences on MSS are the same as on the flat spacetime. Finally, the UV-divergences in QFT do not depend on the background and, therefore, we have backgroup independence of super-renormalizability or finiteness. In other words, if the theory is UV-finite around the flat spacetime, then it is also finite around any other background, in particular this applies to MSS backgrounds.

### B. Analysis of the theory (15) with form-factors (20)

We hereby consider the theory (15) with form-factors (20). These form-factors (and also (17), (18)) depend explicitly on the cosmological constant $\Lambda_{cc}$ that in general could appear in the beta functions making the search for a finite quantum gravity much more involved. However, it is sufficient to select out polynomials that in the UV regime do not involve the cosmological constant at least in the coefficients $\omega_{i,C}$ and $\omega_{i,R}$ for $i = N + 1, N, N - 1$. Given the theory (15) with form-factors (20) we can select the following asymptotic polynomials,

$$p_C(\Box; \Lambda_{cc}) = \frac{1}{\Lambda^8} \Box^2 \left( \Box - \frac{8}{3} \Lambda_{cc} \right)^2 \left( \Box^2 + \frac{8}{3} \Lambda_{cc} \Box + \left( \frac{8}{3} \Lambda_{cc} \right)^2 \right), \quad (68)$$

$$p_S(\Box; \Lambda_{cc}) = \frac{1}{\Lambda^8} \Box^2 \left( \Box + \frac{4}{3} \Lambda_{cc} \right)^2 \left( \Box^2 - \frac{4}{3} \Lambda_{cc} \Box + \left( \frac{4}{3} \Lambda_{cc} \right)^2 \right). \quad (69)$$

Notice that the two parabolic trinomials on the right sides in (68) and (69) are positive for any value of $\Box$ and $\Lambda_{cc} > 0$. For the above selected polynomials (65) and (66), $\omega_{N-1,C(R)} = 0$ and $\omega_{N-2,C(R)} = 0$. Indeed,

$$\gamma_C(\Box) = \frac{1}{2} \left( e^{H_2(\Box - \frac{4}{3} \Lambda_{cc}) - 1} \right) \rightarrow \frac{1}{2} \frac{1}{\Lambda^8} \Box^2 \left( \Box - \frac{8}{3} \Lambda_{cc} \right) \left( \Box^2 + \frac{8}{3} \Lambda_{cc} \Box + \left( \frac{8}{3} \Lambda_{cc} \right)^2 \right) = \frac{1}{2 \Lambda^8} \left( \Box^5 - \frac{512 \Lambda_{cc}^2 \Box^2}{27} \right),$$

$$\gamma_S(\Box) = -\frac{1}{6} \left( e^{H_0(\Box, \Lambda_{cc}) - 1} \right) \rightarrow -\frac{1}{6} \frac{1}{\Lambda^8} \Box^2 \left( \Box + \frac{4}{3} \Lambda_{cc} \right) \left( \Box^2 - \frac{4}{3} \Lambda_{cc} \Box + \left( \frac{4}{3} \Lambda_{cc} \right)^2 \right) = -\frac{1}{6 \Lambda^8} \left( \Box^5 + \frac{64 \Lambda_{cc}^2 \Box^2}{27} \right).$$

Therefore, there is no contribution to the beta functions $\beta_{\Lambda_{cc}}$ and $\beta_\kappa$. More importantly, the cosmological constant does not appear in any beta function. In general, we only need the beta function for the cosmological constant to be independent on $\Lambda_{cc}$ itself to achieve one-loop exact super-renormalizability or finiteness because $\kappa_4$ does not appear in the form-factors and, therefore, in the beta functions. Let us expand a little on this point. If the beta functions do not depend on any of the running couplings then we can make them zero at any energy scale and at any loop order by adding suitably selected killer operators because the super-renormalizability implies that the beta functions are one-loop exact.

### V. FIELD REDEFINITION & TREE-LEVEL PERTURBATIVE TRIVIALITY

In this section we explicitly show that for a large class of theories involving neither the Riemann nor the Weyl tensor, a field redefinition theorem provides an explanation for the stability of MSS in weakly non-local theories. Namely all these theories are tree-level equivalent to the Einstein-Hilbert theory in the presence of cosmological constant. Let us consider the theory (11) with $\gamma_C = 0$, namely

$$S_{\text{NLA}} = -2\kappa_4^2 \int d^4x \sqrt{|g|} \left[ R - 2\Lambda_{cc} + S_{\gamma_2}(\Box) S + R_{\gamma_2}(\Box) R + V(\Box, \text{Ric}, R) \right]. \quad (70)$$

We can now recast the above action in a way that explicitly shows the Einstein’s gravitational EOM in the presence of a cosmological constant, i.e.

$$E_{\mu\nu} = G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu}, \quad R_{\mu\nu} = E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E^\alpha_\alpha + \Lambda_{cc} g_{\mu\nu}, \quad S_{\mu\nu} = E_{\mu\nu} - \frac{1}{4} g_{\mu\nu} E^\alpha_\alpha, \quad R = - E^\alpha_\alpha + 4\Lambda_{cc}. \quad (71)$$
Making use of the EOM (71), the action now equivalently turns into
\[
S_{\text{NLA}} = -2\kappa_4^{-2} \int d^4x \sqrt{|g|} \left[ R - 2\Lambda_{cc} + \left( E_{\mu\nu} - \frac{1}{4} E g_{\mu\nu} \right) \gamma_S(\Box, E, \Lambda_{cc}) \left( E^{\mu\nu} - \frac{1}{4} E g^{\mu\nu} \right) \right. \\
+ \left. \left( E - 4\Lambda_{cc} \right) \gamma_S(\Box, E, \Lambda_{cc}) (E - 4\Lambda_{cc}) + \mathcal{V}(\Box, E, E, \Lambda_{cc}) \right],
\] (72)
where \( E \) stays for \( E_{\mu\nu} \) and \( E = E_{\mu\nu}^\mu \). The non-local form-factor \( \gamma_S \) satisfies the property \( \gamma_S(0) = 0 \) (see for example (55) with the polynomial (53)). Therefore, we can rewrite the action in the following simplified form,
\[
S_{\text{NLA}} = -2\kappa_4^{-2} \int d^4x \sqrt{|g|} \left[ R - 2\Lambda_{cc} + E_{\mu\nu} F^{\mu\nu,\rho\sigma} E_{\rho\sigma} \right],
\] (73)
\[
F^{\mu\nu,\rho\sigma} \equiv \gamma_S(\Box, E, \Lambda_{cc}) \left( g^{\mu\rho} g^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} \right) + \gamma_S(\Box, E, \Lambda_{cc}) g^{\mu\nu} g^{\rho\sigma} + \mathcal{V}^{\mu\nu,\rho\sigma}(\Box, E, E, \Lambda_{cc}),
\] (74)
where the potential \( \mathcal{V}(\Box, E, E, \Lambda_{cc}) \) must be at least quadratic in the EOM \( E_{\mu\nu} \), namely
\[
\mathcal{V}(\Box, E, E, \Lambda_{cc}) = E_{\mu\nu} \tilde{\mathcal{V}}^{\mu\nu,\rho\sigma}(\Box, E, E, \Lambda_{cc}) E_{\rho\sigma}.
\] (75)
In the view of the restructured action (73), we are now ready to implement the following general theorem in the presence of a cosmological constant. An analogous theorem was previously proved and applied to the case without cosmological constant (13).

**Theorem.** By making use of a proper analytic field redefinition the action (73) can be recast in the Einstein-Hilbert form with the presence of a cosmological constant term, i.e.
\[
\mathcal{L}_{\text{EHA}} = -2\kappa_4^{-2} \sqrt{|g|} (R - 2\Lambda_{cc}),
\] (76)
provided that \( \mathcal{V} \) has the structure given in (15), namely it is at least quadratic in \( E_{\mu\nu} \) and/or \( E_{\alpha}^\alpha \) and does not contain any Riemann or Weyl tensor explicitly. Therefore, the theorem does not apply to the theory with \( \gamma_C \neq 0 \).

**Proof.** The proof is based on a perturbative field redefinition to all orders in the Taylor expansion with respect to the redefinition of the metric. First we assume that we have given two general weakly non-local action functionals \( S'(g) \) and \( S(g') \), respectively defined in terms of the metric fields \( g \) and \( g' \), such that
\[
S'(g) = S(g) + E_i(g) F_{ij}(g) E_j(g),
\] (77)
where \( F \) can contain derivative operators and \( E_i = \delta S/\delta g_i \) are the EOM of the theory with the action \( S(g) \) \( ^6 \). The statement of the theorem is that there exists a field redefinition
\[
g'_i = g_i + \Delta g_i E_{ij} \quad \Delta_{ij} = \Delta_{ji},
\] (78)
such that, perturbatively in \( F \), but to all orders in powers of \( E \), we have the equivalence
\[
S'(g) = S(g').
\] (79)
Above \( \Delta_{ij} \) is a possibly non-local operator acting linearly on the EOM \( E_{ij} \), with indices \( i \) and \( j \) in the field space, and it is defined perturbatively in powers of the operator \( F_{ij}(g) \), namely \( \Delta_{ij} = F_{ij}(g) + \ldots \). Let us consider the first order in the Taylor expansion for the functional \( S(g') \), which reads
\[
S(g') = S(g + \delta g) \approx S(g) + \frac{\delta S}{\delta g_i} \delta g_i = S(g) + E_i \delta g_i.
\] (80)
If we can find a weakly non-local expression for \( \delta g_i \) such that \( S'(g) = S(g) + E_i \delta g_i \) (note that the argument of the functionals \( S' \) and \( S \) is now the same), then there exists a field redefinition \( g \to g' \) satisfying (79). Hence the two actions \( S'(g) \) and \( S(g') \) are tree-level equivalent. □

\( ^6 \) Here we use a compact deWitt notation and with the indices \( i, j \) on fields we encode all Lorentz, group indices, and the spacetime dependence of the fields. Additionally, we assume that the field space is flat and we do not need to raise indices in sums there.
As it is obvious from above, in the proof of our theorem it was crucial to use the classical EOM $E_i$. In the theory \(73\) this implies $E = 0$ (here no matter source is present).

Now we can explicitly apply the above field redefinition theorem to our class of theories \(73\), where we do not include terms with the Riemann tensor $R_{\mu\nu\rho\sigma}$ nor the Weyl tensor $C_{\mu\nu\rho\sigma}$ in the action. Since we are interested in $S(g') \equiv S_{\text{EHA}}(g')$ and $S'(g) \equiv S_{\text{NL}}(g)$, the relation \(77\) reads

$$S(g') = S_{\text{EHA}}(g) - 2\kappa_4^2 \int d^4 x \sqrt{|g|} E_{\mu\nu}(g) F^{\mu\nu,\rho\sigma}(g) E_{\rho\sigma}(g) = S'(g),$$

where $E_{\mu\nu}$ is given in \(71\), $F^{\mu\nu,\rho\sigma}(g)$ is defined in \(74\), and $\mathcal{V}$ compatible with the field redefinition has been introduced in \(75\).

As a particular implication of the theorem we can always make a field redefinition to turn the kinetic operator and the propagator for the gravitational fluctuations of the non-local theory into the one of Einstein’s gravity plus the cosmological constant. Moreover, when we can properly define asymptotic graviton’s states in a MSS, all the tree-level on-shell $n$-point functions for the weakly non-local theory \(73\) are exactly the same as the ones for the Einstein-Hilbert-cc gravity \(76\).

Finally, in view of the theorem proved here, it is clear why at tree-level a class of weakly non-local theories and the local Einstein-Hilbert theory with the cosmological constant have the same spectrum and the same $n$-point functions, ergo this range of weakly non-local theories is actually local at classical perturbative level. However, we can not push further the outcome of the theorem because in a theory with an infinite number of derivatives at the moment we do not know the number of non-perturbative degrees of freedom, in contrast to the Einstein-Hilbert theory where the ADM formulation ensures that there are only two degrees of freedom at perturbative and non-perturbative level and around any background.

**VI. MORE ON PROPAGATORS IN WEAKLY NON-LOCAL THEORIES**

In this section we are going to extend our own construction of propagators in weakly non-local theories. Let us consider a simple example of a weakly non-local scalar field theory and its propagator, namely

$$S = \int d^D x \varphi f(\Box)(\Box + m^2)\varphi \quad \implies \quad \Pi = \frac{1}{f(\Box)(\Box + m^2)}. \quad (82)$$

The above theory in most cases comes as a generalization of a local second order theory whose action and propagator respectively read:

$$S = \int d^D x \varphi(\Box + m^2)\varphi \quad \implies \quad \Pi = \frac{1}{\Box + m^2}. \quad (83)$$

One of the most often situations is the wish to constraint $f(\Box)$ such that both theories have the same physical excitations. In this example this means that each theory describes only a single scalar and the mathematical requirement for this is that $f(\Box)$ has no zeros and hence the propagator in \(82\) has no extra new poles on the whole complex plane besides the one at $-m^2$. That is $f(z) \neq 0$ for all $|z| < \infty$, $z \in \mathbb{C}$, where $z = \Box$.

A usual way used to achieve no extra poles in the propagator in \(82\) is to require:

$$f(z) = e^{\alpha(z)} \quad \text{where } \alpha(z) \text{ is an entire function.} \quad (84)$$

This indeed works and propagates from \(12\) through all known to us papers on the subject of weakly non-local theories. By definition an entire function is a function analytic on the whole complex plane. As such it has no poles in any finite region of the complex plane. The exponent of any finite (and zero) argument is always a non-zero complex number. Actually, the exponent of an entire function is a special entire function with no zeros on the whole complex plane. As a result $f(z)$ in \(84\) is always non-zero. Moreover, a particular setup may be required to preserve the normalization of the propagator in the low-energy limit. This implies $f(0) = 1$ or equivalently $\alpha(0) = 0$. Physically this can be understood as follows: given there is a scale $\Lambda$ which defines the characteristic scale of $f(\Box)$ such that this function is truly $f(\Box/\Lambda^2)$, one may want to see the modified model \(82\) returning to its local counterpart \(83\) when $\Lambda \to \infty$. The latter is the local theory limit.

At this point we put the following claim.

**Claim:** The form of $f(\Box)$ given by \(84\) is overrestricted and is not necessary as long as the number of degrees of freedom is concerned.

Instead we can prove the following:
Proposition: The less restrictive form that is still compatible with the requirements (i) to avoid a generation of new degrees of freedom, (ii) to keep the original normalization in the local limit, and (iii) to preserve and/or improve the UV behaviour of the propagator is:

\[ f(z) = \frac{e^{\alpha(z)}}{\beta(z)} \quad \text{where } \alpha(z), \beta(z) \text{ are entire functions.} \quad (85) \]

Indeed, substituting this in the propagator in (82) one gets

\[ \Pi = \frac{\beta(\Box)}{e^{\alpha(\Box)}(\Box + m^2)}. \quad (86) \]

The exponent in the denominator works exactly as it worked before when the form (84) was used. The crucial thing to understand is that the new function \( \beta(\Box) \) does not change neither of required properties (i)-(iii) as long as it is an entire function. This is a trivial consequence of the definition of an entire function that says that it has no poles on the whole complex plane and as such, our propagator has no new poles as well. The normalization in the local limit can always be preserved by the demand \( \beta(0) \exp(-\alpha(0)) = 1 \). The UV behaviour is subject to a particular choice of the functions \( \alpha(\Box) \) and \( \beta(\Box) \), which are almost unrestricted so far in any way.

A point of worry is instead the EOM, which we are going to consider in more detail. The EOM can be written as

\[ \frac{e^{\alpha(\Box)}(\Box + m^2)}{\beta(\Box)} \varphi = 0. \quad (87) \]

We start with reminding that thanks to the Weierstrass factorization theorem [56] any entire function \( \beta(z) \) can be represented as

\[ \beta(z) = e^{\tilde{\beta}(z)} \prod_I (z - z_I)^{m_I}, \quad (88) \]

where \( \tilde{\beta}(z) \) is again an entire function, \( z_I \) are roots of \( \beta(z) \) and \( m_I \) are their multiplicities. First of all we stress that \( \beta(z) \) in the condition of the theorem is an entire function and as such in general \( 1/\beta(z) \) factor in EOM (87) cannot be presented like this. Consequently, and not surprisingly, we do not gain new factors in the numerator of EOM. Having \( \alpha(z) \) and \( \beta(z) \) both entire functions we can join them into a redefined function \( \tilde{\alpha}(z) = \alpha(z) - \beta(z) \). So, without any assumptions we can write the EOM (87) as

\[ \frac{e^{\tilde{\alpha}(\Box)}(\Box + m^2)}{\prod_I (\Box - z_I)^{m_I}} \varphi = 0. \quad (89) \]

We assume that by construction neither of \( z_I \) coincides with \( -m^2 \). Otherwise we would immediately write another EOM and propagator. Then a canonical solution originates from the mode

\[ (\Box + m^2) \varphi = 0. \quad (90) \]

Further, it was shown in [57] that the exponent operator does not generate new solutions and we can drop it from the consideration of solutions of the EOM. A simple way to see that the denominator does not provide new solutions, which could be associated with new degrees of freedom, is to notice that we can use the Schwinger and Feynman parametrization to achieve

\[ \frac{1}{\prod_I (\Box - z_I)^{m_I}} \varphi = \frac{\Gamma(\sum_I m_I)}{\prod_I \Gamma(m_I)} \int_0^1 \left( \prod_I du_I \right) \frac{\delta(1 - \sum_I u_I)}{\prod_I u_I^{m_I - 1}} \varphi = \]

\[ = \frac{1}{\prod_I \Gamma(m_I)} \int_0^1 \left( \prod_I du_I u_I^{m_I - 1} \right) \delta(1 - \sum_I u_I) \int_0^\infty ds s^{\sum_I m_I - 1} e^{s \sum_I u_I z_I} e^{-s \sum_I u_I \Box} \varphi. \quad (91) \]

This is again an exponential of the d’Alembertian operator acting on the scalar field \( \varphi \). Therefore, we can say that no new solutions are generated as long as we can change the order of differentiation and integration. The latter is true as long as a Laplace transform of the scalar field function can be defined. The classic field in turn has to have a well-defined Laplace transform in order to be properly quantized.
The things become trickier when we have to define and solve for Green function that is defined as a solution to the fundamental equation

\[ e^{\tilde{\alpha}(\Box)}(\Box + m^2) \prod_I (\Box - z_I)^{m_I} G(x, x') = \delta(x - x'), \tag{92} \]

with appropriate boundary conditions (retarded, advanced, causal, etc.). Here we can act in analogy with the treatment of \(1/\Box\) operator in gravity theories like in \[58\]. However, a consistent treatment exists for the single inverse d'Alembertian only. We do not need Green functions of this (or any other) kind to proceed, but we very much hope to see this question solved in future works.

Two more comments are in order here. First, the new form (85) is definitely a significant extension of the class of possible form-factors which can enter in weakly non-local theories. Second, it will be shown below that such an extension is crucial to guarantee the no-ghost conditions in both regimes: quantum gravity and inflation.

For the case of gravitational theories the propagator (82), especially the new higher derivative factors must obey several conditions which were first formulated in \[1\], \[3\] and are given above in Section \[1\]. These conditions are aimed at achieving the maximum convergence of loop integrals still preserving the power law fall-off of the integrands at infinity. The latter is important to preserve the locality of counterterms and as such to maintain the renormalizability of the theory \[5\], \[6\], \[26\]. Prior the current analysis the conditions in question were considered for the function \(f(z)\) as it is given in (84). However, it is easy to see that no extra complications arise when we have to satisfy the above conditions using the function \(f(z)\) in (85) in kinetic operators for theory not involving gravity or any other non-abelian gauge theory.

Going further one can easily understand that the form (85) is again not an ultimate non-local factor. That factor was constructed under the assumption that we do not alter the already existing pole at \(\Box = -m^2\) in the propagator in (82). Under the assumption the non-local factor is still maximally general. However, this requirement can in principle be relaxed unless we have some external reasons to maintain this property. Having said this, we understand that we can suggest a function

\[ f(z) = \frac{e^{\alpha(z)} z + \mu^2}{\beta(z) z + m^2} \text{ where } \alpha(z), \beta(z) \text{ are entire functions,} \tag{93} \]

which being substituted in the propagator (82) results in

\[ \Pi = \frac{\beta(\Box)}{e^{\alpha(\Box)}(\Box + \mu^2)}. \tag{94} \]

This clearly propagates only a scalar with a new mass square \(\mu^2\) while all other properties remain the same. From here actually no further generalization is seen as long as we preserve the number of poles. The latter property is indeed very much important because new poles will be ghosts due to the Ostrogradski instability \[59\].

In an extreme case we can have a very special function

\[ f(z) = \frac{\alpha(z)}{\beta(z) z + m^2} \text{ where } \alpha(z) \text{ and } \beta(z) \text{ are entire functions,} \tag{95} \]

which being substituted in the propagator (82) results in

\[ \Pi = \frac{\beta(\Box)}{e^{\alpha(\Box)}}. \tag{96} \]

This is a clear analog of a Lagrangian of a \(p\)-adic theory which has no poles and as such no propagating degrees of freedom in the perturbative vacuum at all.

The following comment follows. Our generalized construction is transparent and obviously valid as long as everything but gravity is concerned. As an immediate example, what we have just discussed, helps in understanding the behaviour of quantum perturbations during inflation as will be explained in details in \[40\]. This is because the propagator for perturbations explicitly and generically has a numerator factor which is \(\beta(\Box)\) above. Concerning gravity, the propagator (96) comes together with strongly non-local vertices. The role of these new vertices in the perturbative unitarity is at the moment not under control and deserves much more investigation.

**VII. CONCLUSIONS**

In this paper we explicitly proved that all the weakly non-local gravitational theories consistent at quantum level have exactly the same classical properties as Einstein's gravity at linear level when studied perturbatively around any
maximally symmetric spacetime background. These theories differ only for the presence or not of the Weyl tensor in the non-local operators quadratic in curvatures, but the outcome is always the same. Namely, the quadratic action at the second order in the graviton perturbation around any MSS can be recast in the form of the Einstein-Hilbert quadratized action up to exponential form-factors in front of the corresponding projectors for the spin-two and spin-zero components. For one out of the two ranges of theories, namely the one without Weyl or Riemann tensor in the action, we proved, making use of a field redefinition theorem, that the theory is perturbatively (in the entire function defining the field redefinition) equivalent to the Einstein-Hilbert action in the presence of a cosmological constant. This statement holds to all orders in the Taylor expansion in the field redefinition of the metric tensor. Moreover, defining the field redefinition theorem, when the graviton’s asymptotic states on a MSS are properly defined, endorses that all tree-level $n$-point scattering amplitudes in the weakly non-local theory coincide with the ones of Einstein-Hilbert gravity with cosmological constant on the same MSS background.

At quantum level, for one out of the two classes of theories (namely the one in Weyl’s basis) we explicitly proved that all the beta functions can be made to vanish. Therefore, the quantum theory is finite (in DIMREG scheme) on any MSS. Certainly, also the theory in the Ricci basis enjoys the same convergence properties.

We can finally claim that the weakly non-local theories are perturbatively well-defined, unitary (as long as the Einstein-Hilbert is), and finite at quantum level on any maximally symmetric space. Having an ultraviolet complete theory for gravity in the quantum field theory framework, we can now study the implications and/or applications in the AdS/CFT domain.

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Appendix: Variations

Here we collect the results about variations on a maximally symmetric background of operators quadratic in curvature. We write them in a manifestly self-adjoint form. First the variation of action written with Weyl tensors reads,

$$\frac{1}{2}\delta^2 \left( \int d^D x \sqrt{|g|} C_{\mu\nu\rho\sigma} \mathcal{F} (\Box) O^{\mu\nu\rho\sigma} \right)_{\text{MSS}} = \int d^D x \sqrt{|g|} \left[ h_{\mu\nu} \left( \frac{2(D-3)}{(D-2)(D-1)} \Lambda_{cc}^2 - \frac{(D-3)(D+2)}{(D-2)(D-1)} \Lambda_{cc}\Box + \frac{D-3}{D-2} \Box^2 \right) + h_{\mu\nu} \nabla^\mu \nabla^\nu \left( \frac{2(D-3)}{(D-2)(D-1)} \Lambda_{cc} + \frac{D-3}{D-2} \Box \right) \right] \mathcal{F} (\Box + 4\Lambda_{cc}) h^{\mu\nu}$$

Next we introduce the definition $\tilde{R} = R - D\Lambda_{cc} = R - \mathcal{R}$ and we evaluate the following variation,

$$\frac{1}{2}\delta^2 \left( \int d^D x \sqrt{|g|} \tilde{R} \mathcal{F} (\Box) \tilde{R} \right)_{\text{MSS}} = \int d^D x \sqrt{|g|} \left[ h (\Lambda_{cc} + \Box)^2 \mathcal{F} (\Box) h - h_{\mu\nu} \nabla^\mu \nabla^\nu (\Lambda_{cc} + \Box) \mathcal{F} (\Box) h - h_{\mu\nu} \nabla^\mu \nabla^\nu \mathcal{F} (\Box) \nabla^\rho \nabla^\sigma h^{\rho\sigma} \right].$$
On a MSS we have the following values of the background curvatures,

$$R_{\mu\nu\rho\sigma} = \frac{2\Lambda_{cc}}{D-1}g_{\mu\nu}g_{\rho\sigma}, \quad \hat{R}_{\mu\nu} = \Lambda_{cc}g_{\mu\nu}, \quad \hat{R} = D\Lambda_{cc}. \quad (99)$$

A useful formula is:

$$\hat{R}_\Box = \hat{R} = R\Box + 2DF_0\Lambda_{cc}R + D^2F_0\Lambda_{cc}^2, \quad \text{where} \quad F_0 = F(0). \quad (100)$$

The variations of the cosmological constant $a_\lambda$ and the Einstein-Hilbert actions read as follows [60].

$$\frac{1}{2} \delta^2 \left( \int d^Dx \sqrt{|g|} a_\lambda \right) = \int d^Dx \sqrt{|g|} \left[ \frac{1}{8} h^2 - \frac{1}{4} h_{\mu\nu}h^{\mu\nu} \right] a_\lambda, \quad (101)$$

$$\frac{1}{2} \delta^2 \left( \int d^Dx \sqrt{|g|} R \right) = \int d^Dx \sqrt{|g|} \left[ \frac{1}{4} h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{4} h \Box h + \frac{1}{4} h \nabla_\mu \nabla_\nu h^{\mu\nu} + \frac{1}{4} h_{\mu\nu} \nabla^\rho h^{\mu\nu} \nabla_\rho h \right. \left. - \frac{1}{2} h_{\mu\nu} \nabla^\rho \nabla_\rho h^{\mu\nu} + \Lambda_{cc} \left( - \frac{D^2 - 3D + 4}{4(D-1)} h_{\mu\nu}h^{\mu\nu} + \frac{D^2 - 5D + 8}{8(D-1)} h^2 \right) \right]. \quad (102)$$

For completeness we give also the expression for the variation (in normal form) in $D = 4$, where we introduced a short notation for the form-factor with a translated argument,

$$F_2 \equiv F \left( \Box + \frac{4}{3} \Lambda_{cc} \right). \quad (103)$$

We observe an interesting fact, that all dependence on form-factor in [103] is only via shifted one $F_2$. The variation of the non-local Weyl square operator on a MSS background can be written also in a manifestly self-adjoint form in $D = 4$, namely

$$\frac{1}{2} \delta^2 \left( \int d^4x \sqrt{|g|} C_{\mu\nu\rho\sigma} F(\Box) C^{\mu\nu\rho\sigma} \right)_{\text{MSS}} = \int d^4x \sqrt{|g|} \left[ h_{\mu\nu} \left( \frac{4}{9} \Lambda_{cc}^2 - \Lambda_{cc}\Box - \frac{1}{2}\Box^2 \right) F \left( \Box + \frac{4}{3} \Lambda_{cc} \right) h_{\mu\nu} + \right.$$

$$\left. + h \left[ \left( - \frac{1}{9} \Lambda_{cc}^2 + \frac{1}{3} \Lambda_{cc}\Box - \frac{1}{8}\Box^2 \right) F \left( \Box + \frac{4}{3} \Lambda_{cc} \right) - \left( \frac{1}{12} \Lambda_{cc}\Box + \frac{1}{24}\Box^2 \right) F (\Box + 4\Lambda_{cc}) \right] h + \right.$$

$$\left. + h_{\mu\nu} \nabla_\rho h_{\nu\sigma} \left( \frac{1}{3} \Lambda_{cc} + \frac{1}{6} \right) F (\Box + 4\Lambda_{cc}) h + h \left( \frac{1}{3} \Lambda_{cc} + \frac{1}{6} \right) F (\Box + 4\Lambda_{cc}) \nabla_\rho h_{\nu\sigma} - \right.$$

$$\left. - h_{\mu\nu} \nabla_\rho \left( \frac{1}{3} \Lambda_{cc} + \Box \right) F (\Box + 3\Lambda_{cc}) \nabla_\rho h_{\nu\sigma} + \frac{1}{2} h_{\mu\nu} \nabla_\rho h_{\nu\sigma} F (\Box + 4\Lambda_{cc}) \nabla_\rho \nabla_\sigma h_{\rho\sigma} \right]. \quad (104)$$

It can be easily seen that in this self-adjoint form we encounter 3 different shifts of the argument of the form-factor by $4/3\Lambda_{cc}$, $3\Lambda_{cc}$ and $4\Lambda_{cc}$ respectively.

The above results for the second variations were checked using various methods. First, all expressions can be put in the self-adjoint form of the operator of the second order variational derivative. Second, all the variations, except for the term with cosmological constant only, are invariant under the substitution $h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\rho (\xi_\nu) h^{\mu\nu}$ for all left or right instances of the fluctuations of metric, where $\xi^\alpha$ is an arbitrary vector field and when on-shell background is used. This is the statement of gauge-invariance of the action with respect to general coordinate transformations. Last but not less important, the second variations were checked against conformal invariance of the action with two
Weyl tensors in $D = 4$. More precisely, it had been checked that for metric fluctuations of the form $h_{\mu\nu} = \omega^2(x)g_{\mu\nu}$ the second variation of such action on any MSS background vanishes.
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