A REVERSE ISOPERIMETRIC INEQUALITY AND ITS APPLICATION TO THE GRADIENT FLOW OF THE HELFRICH FUNCTIONAL

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ABSTRACT. We prove a quantitative reverse isoperimetric inequality for embedded surfaces with Willmore energy bounded away from $8\pi$. We use this result to analyze the negative $L^2$ gradient flow of the Willmore energy plus a positive multiple of the enclosed volume. We show that initial surfaces of Willmore energy less than $8\pi$ with positive enclosed volume converge to a round point in finite or infinite time.

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1. INTRODUCTION

The isoperimetric inequality is certainly one of the most famous and oldest inequalities in mathematics with many application in geometry and analysis. Though the origins of the problem go as far back as the founding myth on the ancient town of Carthago it has never stopped to inspire mathematics. An excellent account of the importance and the rich and long history of this inequality can be found in the paper of Osserman [Oss78].

In $\mathbb{R}^3$ the isoperimetric inequality can be put as

$$\text{vol}(\Omega) \leq \frac{1}{6\pi^2} (\mu(\partial \Omega))^2$$

for all nice enough bounded domains $\Omega \subset \mathbb{R}^3$, where vol denotes the volume of $\Omega$ and $\mu(\Omega) = \text{area}(\partial \Omega)$ the surface area of $\partial \Omega$.

Obviously the reverse of the isoperimetric inequality does not hold without extra assumptions. One can just think of a balloon and imagine what happens if one sticks a needle into it. Then of course the volume goes to zero as the air leaves the balloon but the area does not. In this note we will derive a condition that implies a reverse isoperimetric
inequality which we stumbled over while studying the negative gradient flow of a special case of Helfrich’s functional. In its formulation we use the Willmore or elastic energy
\[ W(f) := \int_{\Sigma} \|H_f\|^2 d\mu_f \]
of an immersion \( f : \Sigma \to \mathbb{R}^3 \) with mean curvature \( H_f = \frac{1}{2}(\kappa_1 + \kappa_2) \) and induced surface measure \( \mu_f \). Willmore showed that the absolute minimum of the Willmore energy among closed immersed surfaces is \( 4\pi \) and is attained only by round spheres [Wil67].

For an immersion \( f : \Sigma \to \mathbb{R}^3 \) we denote by \( \text{vol}(f) \) the signed enclosed volume and by \( \mu(f) \) the surface area. We will see that an embedding \( f : \Sigma \to \mathbb{R}^3 \) of Willmore energy less than \( 8\pi \) satisfies the following inequality.

**Theorem 1.1** (Reverse isoperimetric inequality). There is a constant \( C < \infty \) such that
\[ \mu(f)^{\frac{2}{3}} \leq \frac{C}{(8\pi - W(f)^3)|\text{vol}(f)|^{\frac{1}{3}}} \]
for all embeddings \( f : \Sigma \to \mathbb{R}^3 \) with \( W(f) < 8\pi \) of smooth surfaces \( \Sigma \) without boundary.

Note that it was observed by Li and Yau [LY82] that all immersed surfaces without boundary and Willmore energy strictly less than \( 8\pi \) are embedded. So the restriction to embeddings is not essential in the theorem above.

**Remark 1.2.** After we proved Theorem 1.1, we noticed that a non-quantitative version of the isoperimetric inequality above for the special case of spheres can be deduced from the findings of Johannes Schygulla in [Sch12]. Schygulla considered the isoperimetric ratio
\[ I(f) = \left(6 \sqrt{\frac{\pi}{1}}\right)^{\frac{2}{3}} \frac{|\text{vol}(f)|^{\frac{2}{3}}}{\mu(f)^{\frac{2}{3}}} \]
and showed that every isoperimetric ratio in the interval \((0, 1]\) can be realized by an embedded sphere with Willmore energy strictly less than \( 8\pi \). Combining this with an adaptation of the machinery developed by Leon Simon in [Sim93], he proved that the Willmore energy can be minimized among all embedded spheres with fixed isoperimetric ratio \( I(f) \in (0, 1] \) by a surface with Willmore energy strictly less than \( 8\pi \). Furthermore, he shows that these surfaces of minimal energy converge to a doubly covered sphere for \( I(f) \downarrow 0 \) - which of course has Willmore energy \( 8\pi \). But this implies that for \( M < 8\pi \) there is a constant \( C = C(M) < \infty \) such that
\[ \mu(f)^{\frac{2}{3}} \leq C|\text{vol}(f)|^{\frac{1}{3}} \]
for all embeddings \( f : \mathbb{S}^2 \to \mathbb{R}^3 \) with \( W(f) \leq M \). It also shows that no reverse isoperimetric inequality is satisfied by all embeddings \( f : \mathbb{S}^2 \to \mathbb{R}^3 \) satisfying \( W(f) < 8\pi \).

We do not expect that the dependencies in the above stated reverse isoperimetric inequality above are optimal not to speak of the constant that can in principle be calculated. It should be an interesting and challenging task to craft a more direct proof without relying on the approximate graphic decomposition of Leon Simon in [Sim93]. This might then also point to a sharper estimate.

We will apply Theorem 1.1 to analyze the asymptotic behavior of the negative \( L^2 \)-gradient flow of the Willmore energy plus a positive multiple of the volume. We will see that we can use the scaling of the volume term to show that the flow must shrink to a circle in finite or infinite time. To be more precise, we consider the negative \( L^2 \)-gradient flow of the energy
\[ W_\lambda(f) = W(f) + \lambda \text{vol}(f). \]
This leads to the evolution equation
\begin{equation}
\partial_t f = \nabla_{L^2} W_\lambda(f)
\end{equation}
where
\begin{equation}
\nabla_{L^2} W_\lambda(f) = \Delta f + 2H f (H^2 - K_f) - \lambda v.
\end{equation}
(cf. [Hel73]). Note that the functional $W_\lambda$ is a special case of the Helfrich functional
\begin{equation}
W_{H_0, \lambda_1, \lambda_2}(f) := \int_\Sigma |H - H_0|^2 d\mu(f) + \lambda_1 \mu(f) + \lambda_2 \text{vol}(f)
\end{equation}
with zero spontaneous curvature $H_0$. In [Bla18] under the general assumption $\lambda_1 > 0$ we gave conditions under which the negative gradient flow of $W_{H_0, \lambda_1, \lambda_2}$ converges to a round point in finite time. But we could not deal with the case $\lambda_1 = 0$ which is the content of the present article.

The following theorem in a sense closes the analysis of the negative gradient flow of the Helfrich functional we started in [Bla18].

**Theorem 1.3.** Let $f : S^2 \times [0, T)$ be a maximal solution of the flow (1.3) with $W(f_0) < 8\pi$ and $\text{vol}(f_0) > 0$. If we let
\begin{equation}
c_t := \frac{1}{\mu(f_t)} \int_\Sigma f_t(x) d\mu(f_t(x))
\end{equation}
denote the center of mass of the immersion $f_t$, then the rescaled flow
\begin{equation}
\frac{2\pi}{\mu(f_t)^{\frac{3}{2}}} (f(\cdot, t) - c_t)
\end{equation}
converges to the unit sphere as $t \uparrow T$ as manifolds in the $C^\infty$ topology.

It seems to be an interesting question whether in the situation of Theorem 1.3 the singularities form in finite time or not. Unfortunately, in contrast to the cases discussed in [Bla18], our analysis is not able to answer this question. This is even more surprising keeping in mind the two facts that the rescaled flow converges to the unit sphere and that spheres converge to points in finite time under our evolution equation. Unfortunately, if one scales back the rescaled flow to the original one, one loses control over the gradient of the Willmore energy quite completely.

2. **Proof of the Reverse Isoperimetric Inequality**

2.1. **Preliminaries.** Let us gather some results that essentially go back to Leon Simon’s paper [Sim93]. First of all we will use that there is a universal constant $C < \infty$ such that the density ratio satisfies
\begin{equation}
\frac{\mu(f(\Sigma) \cap B_\rho(x))}{\rho^2} \leq C W(f)
\end{equation}
(cf. [KS04, Equation (A.16)]). The reverse isoperimetric inequality furthermore relies on two well-known facts related to the Willmore energy. On the one hand, we will use the following consequence of the monotonicity formula due to Leon Simon which can be seen as an extension of the inequality of Li and Yau [LY82]. We will use the short hand $B_\rho = B_\rho(0)$ for the ball of radius $\rho > 0$ around the origin.

**Lemma 2.1** (A Li-Yau type inequality [Sim93, Lemma 1.4]). Suppose $M \subset \mathbb{R}^3$ is a compact surface without boundary, $\partial B_\rho$ intersects $M$ transversely, and $M \cap B_\rho$ contains disjoint subsets $M_1, M_2$ with $\partial M_j \subset \partial B_\rho$, $M_1 \cap B_\rho \neq \emptyset$, and $|\partial M_j| \leq \beta \rho$, $j = 1, 2$, where $\theta \in (0, \frac{1}{4})$ and $\beta > 0$. Then
\begin{equation}
W(M) > 8\pi - C\beta \theta
\end{equation}
where $C$ does not depend on $M, \beta, \theta.$
The second main ingredient to the proof of the reverse isoperimetric inequality is Leon Simon’s approximate graphical decomposition lemma for surfaces with small Willmore energy. For the convenience of the reader we restate this result in the version we are going to use later on. Roughly speaking this lemma tells us that if the Willmore energy of an embedded surface in a ball is small it consists mainly of graphs of Lipschitz functions with small Lipschitz constant. More precisely one has:

**Lemma 2.2 ([Sim93, Lemma 2.1])**. For any $\beta > 0$ there is an $\varepsilon_0 = \varepsilon_0(n, \beta)$ such that if $M \subset \mathbb{R}^n$ is a compact surface such that for an $\varepsilon \in (0, \varepsilon_0]$ and a ball $B_\rho = B_\rho(0)$ we have $\partial M \cap \overline{B}_\rho = \emptyset$, $0 \in M$, $|M \cap \overline{B}_\rho| \leq \beta \rho^2$, and $\int_{M \cap \partial B_\rho} |A|^2 \leq \varepsilon \rho$, then the following holds:

There are pairwise disjoint closed sets $P_1, \ldots, P_N \subset M$ with

$$\sum_{j=1}^N \text{diam} P_j \leq C \varepsilon^\frac{1}{2} \rho$$

and

$$M \cap B_{\rho/2} \setminus \left( \bigcup_{j=1}^N P_j \right) = \left( \bigcup_{i=0}^M \text{graph } u_i \right) \cap B_{\rho/2}$$

where each $u_i \in C^\infty(\Omega, \mathbb{R}^n)$ where $\Omega_i$ is a plane in $\mathbb{R}^n$, $\Omega_i$ a smooth bounded connected domain in $\Omega_i$ of the form $\Omega_i = \Omega_i^0 \setminus \bigcup \{ d_{i,k} \}$, where $\Omega_i^0$ is simply connected and $d_{i,k}$ are pairwise disjoint closed discs in $\Omega_i$ which do not intersect $\partial \Omega_i^0$ with graph $u_i$ connected, and with

$$N \leq C\beta \quad \text{and} \quad \rho^{-1} \sup_{\Omega} |u_i| + \sup_{\Omega} |Du_i| \leq C \varepsilon^{-\frac{1}{2}} \rho.$$

Furthermore, for any $\sigma \in [\frac{2}{3}, \frac{7}{8}]$ with $\partial B_{\rho} \cap \left( \bigcup_j P_j \right) = \emptyset$, we have

$$M \cap \overline{B}_\sigma \subseteq \bigcup D_{\sigma,i}$$

where each $D_{\sigma,i}$ is a topological disc with graph($u_i$) \ cap \ overline{B}_\sigma \ subset D_{\sigma,i}$, and $D_{\sigma,i} \ setminus \text{graph } u_i$ is the union of a subcollection of the $P_j$ and each $P_j$ is topologically a disc.

2.2. **The Proof of Theorem 1.1.** The idea behind the proof of Theorem 1.1 is first to pick a ball in which in conclusion of the approximate graphical decomposition lemma of Leon Simon, Lemma 2.2 holds. A simple covering argument will help us to pick such a ball centered on the manifold whose radius is comparable to the size of the manifold. The approximate graphical decomposition lemma then tells us that inside this ball the manifold consists mainly of Lipschitz graphs and at least one of these graphs is getting close to the center of the manifold. If a second of these graphs would get close to the center we will get a contradiction using Lemma 2.1 as the Willmore energy is known to be strictly smaller than $8\pi$. So all other parts of manifold have a certain distance to the center – which shows that there must also be a certain amount of enclosed volume in our ball.

**Proof of Theorem 1.1.** Let $\varepsilon > 0$ be so small that Lemma 2.2 applies. For $r > 0$, Bessonovich’s covering lemma yields that there is a collection of pairwise disjoint balls $B_r(x_i)$, $i = 1, \ldots, M$ centered on $f(B_r^2)$ such that

$$\sum_{i=1}^M \mu(B_r(x_i) \cap f(B_r^2)) \geq c \mu(f)$$

where $c$ is a universal constant. As (2.1) together with $W(f) < 8\pi$ implies

$$\mu(B_r(x_i) \cap f(B_r^2)) \leq C r^2$$

for a constant $C < \infty$ we deduce that

$$M \geq cr^{-2} \mu(f)$$

(2.2)
for a suitably changed $c > 0$.

Using that the balls $B_r(x_i)$ are pairwise disjoint and (2.2), we deduce that there is a ball centered at an $x \in f(\Sigma)$ with radius $r$ such that

$$\text{W}(f^{-1}(B_r(x))) \leq \frac{\text{W}(f)}{c \mu(f)} r^2$$

as otherwise

$$\text{W}(f) \geq \sum_{i=1}^{M} \text{W}(f^{-1}B_r(x_i)) > M \frac{\text{W}(f)}{c \mu(f)} r^2 \geq \text{W}(f).$$

We pick $r = \varepsilon \sqrt{\frac{\mu(f)}{\text{W}(f)}}$ to get

\begin{equation}
(2.3) \quad \text{W}(f^{-1}(B_r(x))) \leq \varepsilon^2.
\end{equation}

Let us for the following assume that $x = 0$ and $\varepsilon < \varepsilon_0$ where $\varepsilon_0$ is the constant from the approximate graphical decomposition lemma, Lemma 2.2. Then there is a radius $\rho \in [\frac{\varepsilon}{c}, r]$ such that the assumptions of the approximate graphical decomposition lemma of Leon Simon are fulfilled and

$$|\Sigma \cap B_\rho(x)| \leq C \rho.$$

Hence, Lemma 2.1 tells us that for $\varepsilon = \frac{8\pi - \text{W}(f)}{c}$ only one of the connected components $D_j$ of $\Sigma \cap B_\rho(0)$ can intersect $B_\rho(x)$ as otherwise

$$\text{W}(f) \geq 8\pi - C \theta.$$

Let us assume that $D_1$ is this unique component intersecting $B_\rho(x)$. After some rotation of the surrounding space we can assume that $L_1$ is parallel to the $(x_1, x_2)$-plane. As by assumption $0 \in \Sigma$, the component $D_1$ must contain 0 and consists of parts of the graph of a Lipschitz function $u_1$ with Lipschitz constant less that $C \varepsilon^\frac{1}{2}$ up to some pimples whose sum of diameters can be estimated by $C \varepsilon^\frac{1}{2} \rho$. Hence, we have for all $x = (x_1, x_2, x_3) \in D_1 \cap B_\rho(x)$

$$|x_3| \leq C(\varepsilon^\frac{1}{2} \theta + \varepsilon^\frac{1}{2}) \rho.$$

In other words

$$D_1 \subset \{|x_2| \leq C(\varepsilon^\frac{1}{2} \theta + \varepsilon^\frac{1}{2}) \rho|\}.$$

Let us now chose $\varepsilon = \min[\varepsilon_0, (4C)^{-6}, \theta^2(4C)^{-2}]$ so that $\varepsilon \geq c \theta^2$ for a suitable constant $c > 0$. Then the inclusion above implies that either $B_\rho \cap \{x > \frac{\theta}{2}\}$ or $B_\rho \cap \{x < -\frac{\theta}{2}\}$ is contained in the domain bounded by $f(\Sigma)$ and hence using $\rho \in [\frac{\varepsilon}{c}, r]$, $r = \varepsilon \sqrt{\frac{\mu(f)}{\text{W}(f)}}$, $\varepsilon \geq c \theta^2$, and that $\theta = \frac{8\pi - \text{W}(f)}{c \text{W}(f)}$ we get

$$|\text{vol}(f)| \geq |B_\rho \cap \{x > \frac{\theta}{2}\}| \geq c(\theta \rho)^3 \geq c \theta^3 \rho^3 \geq c(8\pi - \text{W}(f))^9 \mu(f) \frac{1}{2}.$$  

This proves the reverse isoperimetric inequality (1.2), \, \Box

3. Application to the Constrained Willmore Flow

3.1. Vanishing of the Flow for $T = \infty$. We will use the reverse isoperimetric inequality to prove that solutions to (1.3) with positive enclosed volume must vanish if they exist for all times in the sense that its enclosed volume, surface, and diameter goes to zero. In the proof we will use that Peter Topping [Top98, Lemma 1] showed that

\begin{equation}
(3.1) \quad \text{diam}(f) \leq \frac{2}{\pi} \sqrt{\frac{\mu(f)}{\text{W}(f)}}.
\end{equation}

This estimate is a sharpened version of [Sim93, Lemma 1.1].

**Theorem 3.1.** If the flow $f_t$ exists for all $t \in [0, \infty)$ and satisfies $\text{vol}(f_t) > 0$ for all $t \in [T, \infty)$, then we have $\text{vol}(f_t) \to 0$, $\mu(f_t) \to 0$, and $\text{diam}(f_t) \to 0$ as $t \to \infty$. 

Proof. Let us start with an observation for a general immersion \( f : \Sigma \to \mathbb{R}^3 \). We consider the dilations
\[
f_a = a(f - p)
\]
around a fixed point \( p \) in \( f(\Sigma) \). Note that the Willmore energy stays constant while the volume behaves like \( a^3 \text{vol}(f) \). By the definition of the \( L^2 \) gradient we hence find
\[
\int_{\Sigma} \nabla_{L^2} W_d(f)(f - p) d\mu = \frac{d}{da} \|W_d(f_a)\|_{L^2} = 3\lambda \text{vol}(f).
\]
Using the Cauchy-Schwartz inequality together with (3.1), we get
\[
\int_{\Sigma} \nabla_{L^2} W_d(f)(f - p) d\mu \leq \|\nabla_{L^2} W_d(f)\|_{L^2(\mu)} \cdot \|f - p\|_{L^2(\mu)}
\]
\[
\leq \|\nabla_{L^2} W_d(f)\|_{L^2(\mu)} \text{diam}(f) \sqrt{\mu(f)}
\]
\[
\leq \frac{2}{\pi} \sqrt{\text{vol}(f)} \|\nabla_{L^2} W_d(f)\|_{L^2(\mu)} \mu(f).
\]
Combining equation (3.2) with the estimate (3.3) we get from the reverse isoperimetric inequality (1.2)
\[
\|\nabla_{L^2} W_d(f)\|_{L^2(\mu)}^2 \geq \frac{9\pi^2 d^2 \text{vol}(f)^2}{4W(f) \mu(f)^2} \geq C \text{vol}(f)^\#.
\]
Let us now assume that the maximal time of existence satisfies \( T = \infty \). Then the inequality above shows that \( \liminf_{t \to \infty} \text{vol}(f_j) = 0 \) as otherwise the energy would become negative eventually. Applying the reverse isoperimetric inequality again, using (3.1) and the fact that the Willmore energy is uniformly bounded along the flow, we get
\( \mu(f_j) \) and \( \text{diam}(f_j) \to 0 \)
as \( t \to \infty \). \( \square \)

3.2. Construction of a Blowup Limit. For the convenience of the reader let us briefly repeat the essence of the blowup construction at a singularity in finite or infinite time [MW16] at the beginning of the proof of Theorem 1.4 on page 25. This result extends the construction of a blowup by Kuwert and Schätzle for the Willmore flow [KS01, KS04]. Exchanging the a-priori estimates for the Willmore flow by the a-priori estimates for the constrained Willmore flow of McCoy and Wheeler one finds that the following is true.

**Theorem 3.2.** There are constants \( \varepsilon_0 > 0 \) and \( c_0 > 0 \) such that the following holds: Let \( f : \Sigma \times [0, T) \to \mathbb{R}^3 \) be the maximal solution to (1.3). Then there is a sequence of times \( t_j \uparrow T \), of radii \( r_j > 0 \) and points \( x_j \in \mathbb{R}^n \) such that the rescaled flows
\[
f_j : \Sigma \times [0, c_0] \to \mathbb{R}^3, f_j(p, t) := \frac{1}{r_j} (f(p, t_j + r_j^4 t) - x_j)
\]
satisfy
\[
\int_{f_j^{-1}(B_j(0))} \|A_{f_j}\|^2 d\mu_{f_j} \geq \varepsilon_0
\]
and converge smoothly locally to a smooth family of proper immersions
\( \tilde{f} : \Sigma \times [0, c_0] \to \mathbb{R}^3 \)
in the following sense: We can represent
\[
f_j(f_j, t) = \tilde{f} + u_j(\cdot, t)
\]
where
- \( \phi_j : f_j^{-1}(B_j(0)) \to U_j \) is a diffeomorphism,
\[ f_j^{-1}(B_\epsilon) \subset U_j \text{ for } j \geq j(R), \]
\[ u_j \in C^\infty(\Sigma \times [0, c_0], \mathbb{R}^n) \text{ is normal along } \bar{f}, \]
\[ \|\nabla^2 u_j\|_{L^2(f^{-1}(B_\epsilon), 0)} \to 0 \text{ as } j \to 0. \]

We will call such an immersion \( \bar{f} \) a blowup limit. Note that due to Theorem 3.1 we know even in the case that \( T = \infty \) the radii \( r_j \) must converge to to zero. This will be crucial in our further analysis of the blowup limits.

### 3.3 Convergence to a Round Point

The next theorem shows that possible blowup limits are stationary and parametrize Willmore surfaces. It is an extension of Theorem 4.4 in [MW16]. McCoy and Wheeler have shown the result only under the assumption that the energy of the initial surface is close to the Willmore energy of a sphere \( 4\pi \).

**Theorem 3.3.** Let \( f : \Sigma \times [0, T) \to \mathbb{R}^3 \) be the maximal solution to (1.3) with \( \text{vol}(f(t)) > 0 \) for all \( t \in [0, T) \). Then the blowup limit \( \bar{f} : \Sigma_\infty \to \mathbb{R}^3 \) constructed above does not depend on time and parametrizes a Willmore surface.

**Proof.** Using that \( f \) satisfies equation (1.3) together with
\[ \Delta_j f_r + = r_j^3 \Delta f_r, \]
\[ \nu_j = \nu_j, \]
as
\[ \partial_t f_r = r_j^3 \partial_t f \]
we get from (1.3) that the \( f_j \) satisfy
\[ \partial_t f_r = \nabla_{L}^2 W(f_r) + r_j^3 \nu_j. \]

Since \( f_j \) converges to \( \bar{f} \) locally smoothly and \( r_j \to 0 \), this implies
\[ \bar{f} \text{ is proper. We consider the images } \Sigma \subset \mathbb{R}^3 \text{ of the } \Sigma_j \text{ with energy below } 8\pi \text{ and thus is parametrizing a round sphere by the classification result of Bryant [Bry84].} \]

We now lead the case that \( \Sigma_\infty \) is not compact to a contradiction as in [KS04]. We can assume without loss of generality that \( 0 \notin f(\Sigma_\infty) \) since \( \bar{f} \) is proper. We consider the images
of the $f_j$ under the inversion on the standard sphere $I : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$, $x \mapsto \frac{x}{|x|^2}$, which is well-defined for large enough $j \in \mathbb{N}$. The embeddings $\tilde{f}_j = I \circ f_j$ converge locally smoothly to the embedding $I \circ f$ in $\mathbb{R}^3 \setminus \{0\}$ and due to the Möbius invariance of the Willmore energy

$$W(I \circ \tilde{f}) \leq \liminf_{j \to \infty} W(\tilde{f}_j) = \liminf_{j \to \infty} W(f_j) < 8\pi.$$ 

The Möbius invariance of the Willmore energy also implies that $I \circ \tilde{f}$ is a Willmore surface away from 0. Due to the point removability result of Kuwert and Schätzle, $\tilde{f}_j$ can be extended to a Willmore sphere. Hence, due to a result of Bryant, it must parametrize a round sphere. But this would imply that $\tilde{f}$ was a plane - which would contradict

$$\int_{\tilde{\Sigma}} \|A_{\tilde{f}}\|^2 d\mu_{\tilde{f}} > 0.$$ 

Hence, $\tilde{\Sigma}$ must be compact which concludes the proof.

3.4. Proof of Theorem 1.3. To complete the proof of Theorem 1.3 i.e. to get from the subconvergence proven in Corollary 3.4 to convergence, we will use the following a-priori estimates proven by Kuwert and Schätzle for the Willmore flow and by McCoy and Wheeler for the flow of the Helfrich functional.

**Theorem 3.5 ([MW16, Theorem 3.1, Theorem 3.11]).** Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion. There are absolute constants $\epsilon_0 > 0$ and $c_0 < \infty$ such that if $\rho > 0$ is chosen with

$$\int_{f^{-1}(B_\rho(x))} |A_f|^2 d\mu_f \leq \epsilon < \epsilon_0$$

for any $x \in \mathbb{R}^3$, then the maximal time $T$ of smooth existence of the flow (1.3) satisfies

$$T \geq \frac{1}{c_0} \rho^4 := \delta$$

and for $t \in (0, \delta)$ we have

$$\|\nabla^k A\|_{L^\infty} \leq C_k \sqrt{\rho} \frac{1}{\rho^{k-1}}$$

where $C_k$ is a universal constant only depending on $k$.

Now we can finally prove Theorem 1.3

**Proof of Theorem 1.3.** Let us first note, that $\text{vol}(f_0) > 0$ together with the fact that $W_{\lambda}(f_0) < 8\pi$ implies that the enclosed volume stays positive along the flow. Otherwise there would be a first time $t_0$ for which we have $\text{vol}(f_{t_0}) = 0$. But then the immersion $f_{t_0}$ cannot be an embedding and hence due to the inequality of Li and Yau we must have $W_{\lambda}(f_{t_0}) = W(f_{t_0}) > 8\pi$.

As the blowup limit must be a round sphere, we know that there is a sequence of times $t_j \uparrow T$, radii $0 < r_j \downarrow 0$ and points $x_k$ such that

$$f_j := \frac{1}{r_j}(f_{t_j} - x_j)$$

converges to a round sphere smoothly as manifolds. Hence,

$$W_{\lambda}(f_j) = W(f_j) + \frac{1}{r_j} \text{vol}(f_j) \to 4\pi$$

as $j \to 0$. As $\text{vol}(f_j) \geq 0$, $W(f_j) \geq 4\pi$ and $r_j \downarrow 0$ this implies

$$\lim_{j \to \infty} W_{\lambda}(f_j) = \lim_{j \to \infty} W_{\lambda}(f_t) = 4\pi$$

Using again that $\text{vol}(f_j) \geq 0$ and $W(f_j) \geq 4\pi$ we get

$$W(f_j) \to 4\pi$$
and \( \text{vol}(f_t) \to 0 \)

for \( t \to \infty \).

We hence get using the scaling invariance of the Willmore energy that

\[
\int_{S^2} \|A_{f_t}^0\|^2 d\mu_f \to 0
\]

as \( t \to T \). As \( \mu(f_t) = 4\pi \) we can apply the rigidity result of De Lellis and Müller [DLM05] to obtain

\[
\|A - Id\|_{L^2(S^2)} \leq C\|A_{f_t}^0\|_{L^2(S^2)}.
\]

But together with estimate of the density ratio (??) this implies that there is a radius \( r > 0 \) independent of \( t \) such that

\[
\sup_{x \in B_r(x)} \int_{B_r(x)} \|A_{f_t}^0\|^2 d\mu_{f_t} < \varepsilon
\]

for all \( t \in [0, T) \). Hence we can apply the a-priori estimate as stated in Theorem 3.5 to deduce that

\[
(3.7) \quad \|\nabla^2 A\|_{L^\infty} \leq C_k
\]

for a constant \( C_k < \infty \) not depending on time.

We will show that \( f_t \) converges smoothly in the sense of manifolds to the unit sphere \( S^2 \) by showing that every sequence of times \( t_j \to T \) has a subsequence \( t_{j_k} \) such that \( f_{t_{j_k}} \) after converges smoothly to a unit sphere in the sense of manifolds.

But this can be seen as follows. One first applies the uniform estimates (3.7) to deduce that a subsequence converges to a compact smooth embedded manifold as in the proof of Theorem 5.2 in [KS04]. Using the different scalings of the terms building the gradient as in the proof of Theorem 3.3 one then gets that \( f : \Sigma \to \mathbb{R}^3 \) is a Willmore sphere with energy less than \( 8\pi \) and hence a round sphere by the classification result of Bryant [Bry84].

\[\square\]

**Remark 3.6.** Note that although we know that the rescaled flow converges to a round sphere, we cannot deduce from this fact that the volume is non-increasing close to a singularity simply from reversing the scaling. So a different technique is needed to show the vanishing of the flow in the case that the flow exists till \( T = \infty \).

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