Gravitational wave scattering theory without large-distance asymptotics

Wen-Du Li, a,b,c Shi-Lin Li, b Yu-Jie Chen, b Yu-Zhu Chen, c,b and Wu-Sheng Dai b

a College of Physics and Materials Science, Tianjin Normal University, Tianjin 300387, PR China
b Department of Physics, Tianjin University, Tianjin 300072, P.R. China
c Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin, 300071, P. R. China

ABSTRACT: In conventional gravitational wave scattering theory, a large-distance asymptotic approximation is employed. In this approximation, the gravitational wave is approximated by its large-distance asymptotics. In this paper, we establish a gravitational wave scattering theory without the large-distance asymptotic approximation.

KEYWORDS: Gravitational wave; Scattering; Large-distance asymptotics
1 Introduction

The gravitational wave now has been detected, the first three observations GW150914 [1–3], GW151226 [4], GW170104 [5] by LIGO, the observation by the advanced Virgo detector and the two advanced LIGO detectors [6], and the gravitational wave produced by a binary neutron star merging [7, 8]. There are recent theoretical researches on scattering of gravitational waves, such as long-wavelength gravitational wave scattering [9], rainbow scattering of gravitational plane waves [10], the Regge pole of gravitational wave scattering [11], and scattering of plane-fronted gravitational waves [12]. The gravitational wave and the gravitational wave scattering have been studied for many years, e.g., historically, gravitational radiations at infinity in a asymptotically flat space [13], classical cross sections [14], inelastic cross sections of nonrotating and rotating black holes [15], scattering off a Kerr black hole [16], the weak-field gravitational scattering [17], gravitational wave scattering on a Schwarzschild black hole in the low-frequency limit [18], scattering of small wave amplitudes and weak gravitational fields [19], and differential cross sections of plane gravitational waves scattering from the gravitational field of sources in the weak-field approximation [20].

In observation, there is also an indirect evidence for the existence of gravitational wave [21]. Recently, there are many theoretical studies on the gravitational wave scattering: the scattering and absorption of planar gravitational waves by a Kerr black hole [9], the scattering of a low-frequency gravitational wave by a massive compact body [9], the scattering of weak gravitational waves from a slowly rotating gravitational source [22], the interaction of
a weak gravitational wave with matter [23], the low-energy scattering of gravitons [24], and the scattering of gravitational waves by the weak gravitational fields of lens objects [25].

One important approach to study gravitational waves is to study the weak-field radiative solution of the Einstein equation. Under the weak-field approximation, the metric \( g_{\mu\nu} \) is close to the Minkowski metric \( \eta_{\mu\nu} \):

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]  

(1.1)

with the determinant \( |h_{\mu\nu}| \ll 1 \) [26]. Then the Einstein equation becomes a linear equation \( \Box^2 h_{\mu\nu} = -16\pi G S_{\mu\nu} \) with a source \( S_{\mu\nu} \). When the gravitational radiation comes in from infinity, the gravitational wave approaches a plane wave satisfying

\[
\Box^2 h_{\mu\nu} = 0.
\]  

(1.2)

A plane gravitational wave impinging on a target consists of an incident plane wave and a scattered wave,

\[
h_{\mu\nu} = h_{\mu\nu}^{\text{inc}} + h_{\mu\nu}^{\text{sc}},
\]  

(1.3)

where \( h_{\mu\nu}^{\text{inc}} = e_{\mu\nu} e^{ik\cdot x} e^{-ikt} \) is the incident wave with \( e_{\mu\nu} \) the polarization tensor and \( k^\mu \) the wave vector and \( h_{\mu\nu}^{\text{sc}} \) is the scattered wave.

In conventional gravitational wave scattering theory, a large-distance asymptotic approximation is employed. In this approximation, the incident plane wave \( e^{ik\cdot x} e^{-ikt} \) is replaced by its large-distance asymptotics, \( e^{ikr \cos \theta} \sum_{l=0}^{\infty} (2l + 1) \frac{P_l(\cos \theta)}{2ikr} \left[ e^{ikr} - (-1)^l e^{-ikr} \right] \), and the scattered wave is replaced by a spherical wave, \( h_{\mu\nu}^{\text{sc}} (\hat{x}) \frac{e^{ikr}}{r} e^{-ikt} \), where \( f_{\mu\nu} (\hat{x}) \) is the scattering amplitude [26]. Consequently, in conventional scattering theory, the gravitational wave \( h_{\mu\nu} \) is asymptotically written as

\[
h_{\mu\nu} (x, t) \overset{r \to \infty}{\sim} \left( e_{\mu\nu}^{\text{in}} e^{-ikt} + e_{\mu\nu}^{\text{out}} e^{ikt} \right) e^{-ikt},
\]  

(1.4)

with

\[
e_{\mu\nu}^{\text{in}} = -\frac{1}{2ikr} \sum_{l=0}^{\infty} e_{\mu\nu} (2l + 1) (-1)^l P_l(\cos \theta),
\]  

(1.5)

\[
e_{\mu\nu}^{\text{out}} = \frac{1}{2ikr} \sum_{l=0}^{\infty} e_{\mu\nu} (2l + 1) P_l(\cos \theta) + f_{\mu\nu} (\hat{x}) \frac{1}{r},
\]  

(1.6)

where the superscript \( \infty \) denotes the large-distance asymptotics. All of the observable quantities are described by the scattering amplitude \( f_{\mu\nu} (\hat{x}) \) which is independent of the distance \( r \).

The large-distance asymptotic approximation, however, losses the information of the distance. In this paper, we establish a rigorous gravitational wave scattering theory without the large-distance asymptotic approximation.

In section 2, we establish a gravitational wave scattering theory without large-distance asymptotics. In section 3, we show how this scattering theory recovers conventional scattering theory when taking the large-distance asymptotic approximation. In section 4, we consider the gravitational wave with a source. The conclusion is given in section 5.
2 Scattering theory without large-distance asymptotics

In this section, we provide an expression of $h_{\mu\nu}$ without the large-distance asymptotic approximation.

2.1 Gravitational wave

Incident plane wave. The incident plane wave $e^{ikx}$ can be exactly expanded as $e^{ikx} = e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l + 1) i^l j_l (kr) P_l (\cos \theta)$. In the large-distance asymptotic approximation, the Bessel function $j_l (kr)$ is approximately replaced by its large-distance asymptotics $j_l (z) \sim \frac{e^{iz} - (-1)^l e^{-iz}}{2iz}$. In the following, we show that, by use of the relations $j_l (z) = \frac{1}{2} \left( h_l^{(1)} (z) + h_l^{(2)} (z) \right)$ and $h_l^{(1,2)} (z) = i^{l+1} e^{\mp iz} y_l \left( \mp \frac{1}{2} \right)$, where $h_l^{(1)} (z)$ and $h_l^{(2)} (z)$ are the spherical Hankel functions of first and second kinds and $y_l (z) = \sum_{k=0}^{l} \frac{(i+k)!}{k!(l-k)!} \left( \frac{1}{z} \right)^k$ is the Bessel polynomial [27–29], the expansion of the plane wave can be exactly rewritten as

$$e^{ikx} = \frac{e^{-ikr}}{2ikr} \sum_{l=0}^{\infty} (2l + 1) (-1)^l y_l \left( \frac{1}{ikr} \right) P_l (\cos \theta) + e^{ikr} \frac{e^{-ikr}}{2ikr} \sum_{l=0}^{\infty} (2l + 1) y_l \left( -\frac{1}{ikr} \right) P_l (\cos \theta).$$

The first term here describes the ingoing wave propagating inward along the radial direction and the second term describes the outgoing wave propagating outward along the radial direction.

The information of the distance is preserved since there is no large-distance asymptotic approximation.

Scattered wave. In the large-distance asymptotic approximation, the scattered wave is approximated as a spherical wave. Nevertheless, the exact scattered wave at a finite distance is not spherical. The scattered wave $h_{sc \mu\nu}$ without large-distance asymptotics can be written as [28]

$$h_{sc \mu\nu} = \sum_{l=0}^{\infty} a_{l \mu\nu} (\theta) h_l^{(1)} (kr) e^{-ikt},$$

where $a_{l \mu\nu} (\theta)$ is the partial wave scattering amplitude.

Gravitational wave. The incident plane wave and the scattered wave are now expressed without the large-distance asymptotic approximation. Then the gravitational wave (1.3) can be expressed as

$$h_{\mu\nu} = \left( e_{\mu\nu}^{in} e^{-ikt} + e_{\mu\nu}^{out} e^{ikt} \right) e^{-ikt},$$

where

$$e_{\mu\nu}^{in} = -\frac{1}{2ikr} \sum_{l=0}^{\infty} e_{\mu\nu} (2l + 1) (-1)^l y_l \left( \frac{1}{ikr} \right) P_l (\cos \theta),$$

$$e_{\mu\nu}^{out} = \frac{1}{2ikr} \sum_{l=0}^{\infty} \left[ e_{\mu\nu} (2l + 1) P_l (\cos \theta) + 2 (-i)^l a_{l \mu\nu} (\theta) \right] y_l \left( -\frac{1}{ikr} \right).$$

Comparing Eqs. (1.4) and (2.3), we can see that the influence of the distance $r$ is embodied in the Bessel polynomial in Eq. (2.1). The gravitational wave $h_{\mu\nu}$ here depends
on the distance \( r \) and when \( r \to \infty \) it recovers the result given by conventional scattering theory.

### 2.2 Power of gravitational wave

The power of the gravitational wave emitting out of a sphere of radius \( r \) is \( P_{\text{out}} = \int \langle \epsilon_{\mu \nu}^{\text{out}} \rangle r^2 \hat{t}_i d\Omega \), where \( \langle t_{\mu \nu} \rangle = \frac{k_{h_{\mu \nu}}}{16\pi G} \left( e^{\lambda \mu \nu} e_{\lambda \rho} - \frac{1}{2} |e_\lambda|^2 \right) \) is the average energy-momentum [26]. Then the power

\[
P_{\text{out}} = \frac{k^2 r^2}{16\pi G} \int d\Omega \left[ \left( \epsilon_{\mu \nu}^{\text{out}} \right)^* \epsilon_{\mu \nu}^{\text{out}} - \frac{1}{2} \left| \epsilon_{\mu \nu}^{\text{out}} \right|^2 \right]. \tag{2.6}
\]

By Eq. (2.5) we can calculate the terms in Eq. (2.6), respectively:

\[
\begin{align*}
\left( \epsilon_{\mu \nu}^{\text{out}} \right)^* \epsilon_{\mu \nu}^{\text{out}} &= e_{\mu \nu}^{\text{out}} e^{(1)} h_i (kr) ^2 \\
&+ \frac{1}{k^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l + 1) P_l (\cos \theta) \text{Im} \left[ (-i)^{l'-1} e^{\mu \nu} a_{l, l'}^{\mu \nu} (\theta) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right) \right] \\
&+ \frac{1}{k^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} s_{l, l'} (\theta) a^{\mu \nu} (\theta) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right).
\end{align*}
\]

\[
\left| \epsilon_{\mu \nu}^{\text{out}} \right|^2 = \left| e_{\mu \nu}^{(1)} h_i (kr) \right|^2 + \frac{1}{k^2 r^2} \sum_{l=0}^{\infty} \left( -i \right)^l a_{l, l'}^{\mu \nu} (\theta) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right)
\]

\[
\begin{align*}
&+ \frac{1}{k^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l + 1) P_l (\cos \theta) \text{Im} \left[ (-i)^{l'-1} e^{\mu \nu} a_{l, l'}^{\mu \nu} (\theta) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right) \right].
\end{align*}
\]

The power \( P_{\text{out}} \) includes three parts:

\[
P_{\text{out}} = P_{\text{sc}} + P_{\text{int}} + P_{\text{inc}}, \tag{2.9}
\]

where

\[
P_{\text{sc}} = \frac{1}{16\pi G} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} s_{l, l'} (\theta) a_{l, l'}^{\mu \nu} (\theta) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right)
\]

\[
- \frac{1}{32\pi G} \int d\Omega \left[ \sum_{l=0}^{\infty} \left( -i \right)^l a_{l, l'}^{\mu \nu} (\theta) y_l \left( \frac{1}{ikr} \right) \right]^2 \tag{2.10}
\]

describes the contribution of the scattering wave,

\[
P_{\text{inc}} = \frac{k^2 r^2}{16\pi G} \int d\Omega \left( e^{\mu \nu} e_{\mu \nu} \left| h_i^{(1)} (kr) \right|^2 + \left| e_{\mu \nu}^{(1)} h_i (kr) \right|^2 \right) \tag{2.11}
\]

describes the contribution of the incident plane wave, and

\[
P_{\text{int}} = \frac{1}{16\pi G} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l + 1)
\]

\[
\times \int d\Omega P_l (\cos \theta) \text{Im} \left[ (-i)^{l'-1} \left( e^{\mu \nu} a_{l, l'}^{\mu \nu} (\theta) - \frac{1}{2} e^{\mu \nu} a_{l, l'}^{\mu \nu} (\theta) \right) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right) \right] \tag{2.12}
\]
describes the contribution of the interference between the incident plane wave and the scattered wave.

Noting that a quantity containing \( e_{\mu\nu} \) is the contribution from the incident wave, a quantity containing \( a_{l\mu\nu}(\theta) \) is the contribution from the scattered wave, and a quantity containing the cross term of \( e_{\mu\nu} \) and \( a_{l\mu\nu}(\theta) \) is the contribution from the interference.

2.3 Scattering amplitude

In the above, the gravitational wave (1.3) is represented by the partial wave scattering amplitude \( a_{l\mu\nu}(\theta) \) in Eq. (2.3). In order to compare the exact result, Eq. (2.3), with the asymptotic approximation given by conventional scattering theory, Eq. (1.4), we rewrite the exact result, Eq. (1.3), as

\[
h_{\mu\nu} = e_{\mu\nu}e^{ikr}e^{-ikt} + f_{\mu\nu}(r, \theta) \frac{e^{ikr}}{r}e^{-ikt},
\]

where \( f_{\mu\nu}(r, \theta) \) plays the role of the scattering amplitude in conventional scattering theory. By Eq. (2.2) and the relation \( h_{l}^{(1,2)}(kr) = i^{\mp l} e^{\pm ikr} y_l (\mp \frac{1}{ikr}) \), we obtain

\[
f_{\mu\nu}(r, \theta) = \frac{1}{ik} \sum_{l=0}^{\infty} (-i)^l a_{l\mu\nu}(\theta) y_l \left( -\frac{1}{ikr} \right).
\]

Strictly speaking, \( f_{\mu\nu}(r, \theta) \) here is not the scattering amplitude like its analogue \( f_{\mu\nu}(\hat{x}) \) in conventional scattering theory. In the rigorous scattering theory, there is only the partial wave scattering amplitude \( a_{l\mu\nu}(\theta) \) [28].

3 Large-distance asymptotics

The rigorous result recovers the result given by conventional gravitational wave scattering theory when \( r \to \infty \). In the rigorous gravitational wave scattering theory, the scattering is described by a series of partial wave scattering amplitudes given by Eq. (2.2). We now show that when taking large-distance asymptotics, the scattered wave reduces to a spherical wave and \( f_{\mu\nu}(r, \theta) \) given by Eq. (2.14) reduces to the conventional scattering amplitude \( f_{\mu\nu}(\hat{x}) \).

3.1 Scattering amplitude

For \( r \to \infty \), the ingoing wave (2.4) asymptotically reduces to

\[
c_{\mu\nu}^{in} \sim \frac{e_{\mu\nu}}{2ikr} \sum_{l=0}^{\infty} (2l + 1) (-1)^l P_l (\cos \theta)
= -\frac{e_{\mu\nu}}{ikr} \delta (1 + \cos \theta)
\]

and the outgoing wave (2.5) asymptotically reduces to

\[
c_{\mu\nu}^{out} \sim \frac{1}{2ikr} \sum_{l=0}^{\infty} \left[ (2l + 1) P_l (\cos \theta) e_{\mu\nu} + 2(-i)^l a_{l\mu\nu}(\theta) \right]
= \frac{e_{\mu\nu}}{ikr} \delta (1 − \cos \theta) + \frac{1}{ikr} \sum_{l=0}^{\infty} (-i)^l a_{l\mu\nu}(\theta),
\]
where the relation $\sum_{l=0}^{\infty} (2l + 1) P_l (x) = 2 \delta (1 - x) \ [30]$ is used.

Consequently, the gravitational wave (2.3) when $r \to \infty$ reduces to

$$h_{\mu\nu} = \left\{ -\frac{e_{\mu\nu}}{ikr} \delta (1 + \cos \theta) e^{-i kr} + \left[ \frac{e_{\mu\nu}}{ikr} \delta (1 - \cos \theta) + \frac{1}{ikr} \sum_{l=0}^{\infty} (-i)^l a_{l\mu\nu} (\theta) \right] e^{ikr} \right\} e^{-ikr}. \tag{3.3}$$

Comparing with the asymptotic gravitational wave in conventional scattering theory, Eq. (1.4), we obtain the scattering amplitude in conventional scattering theory:

$$f_{\mu\nu} (\hat{x}) = \frac{1}{ik} \sum_{l=0}^{\infty} (-i)^l a_{l\mu\nu} (\theta). \tag{3.4}$$

It can be seen directly from Eq. (2.14) that $f_{\mu\nu} (\hat{x})$ is just the large-distance asymptotics of $f_{\mu\nu} (r, \theta)$, i.e., $f_{\mu\nu} (r, \theta) \sim r^{-\infty} f_{\mu\nu} (\hat{x})$.

### 3.2 Power of gravitational wave

Similarly, the large-distance asymptotics of the powers $P_{\text{sc}}$ and $P_{\text{int}}$ can be obtained directly.

The large-distance asymptotics of the scattering part of the power given by Eq. (2.10) reads

$$P_{\text{sc}} \sim \frac{1}{16\pi G} \sum_{l=0}^{\infty} \sum_{l' = 0}^{\infty} (i)^l l' \int d\Omega \left( a_{l l'}^{\mu\nu} (\theta) a_{l'\mu\nu} (\theta) - \frac{1}{2} a_{l \mu}^{\mu\nu} (\theta) a_{l'\mu}^{\mu\nu} (\theta) \right) \tag{3.5}$$

and the large-distance asymptotics of the interference part of the power given by Eq. (2.12) reads

$$P_{\text{int}} \sim \frac{1}{16\pi G} \sum_{l=0}^{\infty} \sum_{l' = 0}^{\infty} (2l + 1) \int d\Omega P_l (\cos \theta) \text{Im} \left[ (-i)^l (e^{\mu\nu} a_{l'\mu\nu} (\theta) - \frac{1}{2} e_{l \mu}^{\mu\nu} a_{l'\mu}^{\mu\nu} (\theta)) \right]. \tag{3.6}$$

Using Eq. (3.4), we can rewrite Eqs. (3.5) and (3.6) as

$$P_{\text{sc}} \sim \frac{k^2}{16\pi G} \int d\Omega \left( \frac{1}{ik} \sum_{l=0}^{\infty} l a_{l l'}^{\mu\nu} (\theta) \frac{1}{ik} \sum_{l' = 0}^{\infty} (-i)^l a_{l'\mu\nu} (\theta) - \frac{1}{2} \frac{1}{ik} \sum_{l=0}^{\infty} l a_{l \mu}^{\mu\nu} (\theta) \frac{1}{ik} \sum_{l' = 0}^{\infty} (-i)^l a_{l'\mu}^{\mu\nu} (\theta) \right) = \frac{k^2}{16\pi G} \int d\Omega \left( f_{\lambda\rho\lambda} (\hat{x}) f_{\lambda\rho\lambda} (\hat{x}) - \frac{1}{2} \left| f_{\mu}^{\lambda\mu} (\hat{x}) \right|^2 \right), \tag{3.7}$$

$$P_{\text{int}} \sim \frac{k^2}{16\pi G} \int d\Omega \text{Im} \left[ -\frac{e^{\mu\nu}}{k} \sum_{l=0}^{\infty} (2l + 1) P_l (\cos \theta) \frac{1}{ik} \sum_{l' = 0}^{\infty} (-i)^l a_{l'\mu\nu} (\theta) \right. \left. + \frac{1}{2} \frac{e_{\mu}^{\mu}}{k} \sum_{l=0}^{\infty} (2l + 1) P_l (\cos \theta) \frac{1}{ik} \sum_{l' = 0}^{\infty} (-i)^l a_{l'\mu}^{\mu\nu} (\theta) \right] = \frac{k^2}{16\pi G} \int d\Omega \left( 1 - \hat{k} \cdot \hat{x} \right) \text{Im} \left\{ -\frac{1}{k} \left[ e^{\mu\nu} f_{\mu\nu} (\hat{x}) - \frac{1}{2} e_{\mu}^{\mu} f_{\mu}^{\mu} (\hat{x}) \right] \right\}, \tag{3.8}$$

where $\sum_{l=0}^{\infty} (2l + 1) P_l (x) = 2 \delta (1 - x)$ with $\cos \theta = \hat{k} \cdot \hat{x}$ is used.
These are just the powers given by conventional gravitational wave scattering theory [26]. The leading modification of the powers to conventional scattering theory is

\[
\Delta P_{\text{sc}} = \frac{1}{64\pi Gk^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} l (l+1) l' (l'+1) \int d\Omega \left( a^\mu_{l,\mu} (\theta) a_{l',\mu} (\theta) - \frac{1}{2} a^\mu_{l,\mu} (\theta) a_{l',\mu} (\theta) \right),
\]

\[
\Delta P_{\text{int}} = \frac{1}{64\pi Gk^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l+1) l (l+1) l' (l'+1) \times \int d\Omega P_l (\cos \theta) \text{Im} \left[ (-i)^{l'-1} \left( e^{\mu\nu} a_{l',\mu} (\theta) - \frac{1}{2} e_{\mu}^{\nu} a_{l',\mu} (\theta) \right) \right].
\]

4 Gravitational wave with source

4.1 Gravitational wave

In the above, we consider the gravitational wave without sources, which is determined by the homogeneous equation (1.2). This result can be directly applied to the gravitational wave with sources.

A gravitational wave with a source is determined by the inhomogeneous equation

\[
\square^2 h_{\mu\nu}^{\text{source}} = -16\pi G S_{\mu\nu},
\]

where the source

\[
S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^\lambda_\lambda
\]

with \(T_{\mu\nu}\) the energy-momentum tensor.

The inhomogeneous equation (4.1) has a retarded potential solution:

\[
h_{\mu\nu}^{\text{source}} (x, t) = 4G \int \frac{d^3 x'}{|x - x'|} S_{\mu\nu} (x', t - |x - x'|).
\]

Furthermore, a solution of the inhomogeneous equation (4.1) plus a solution of the homogeneous equation (1.2) (under the harmonic coordinate condition \(\partial_{\nu} h^\mu_{\nu} = \frac{\partial}{\partial x^\mu} h^\mu_{\nu}\)), which describes the gravitational wave at infinity, is still a solution of the inhomogeneous equation (4.1).

To apply the result of gravitational waves without sources to gravitational waves with sources, we Fourier expand the source \(T_{\mu\nu} (x, t)\) as

\[
T_{\mu\nu} (x, t) = \sum_k T_{\mu\nu} (x, k) e^{-ikt}.
\]

First consider a source with only one Fourier component, i.e.,

\[
T_{\mu\nu} = T_{\mu\nu} (x, k) e^{-ikt}.
\]

Then the gravitational wave with the source (4.5), by Eq. (4.3), reads

\[
h_{\mu\nu}^{\text{source}} (x, t) = 4G \int \frac{d^3 x'}{|x - x'|} S_{\mu\nu} (x', k) e^{-ikt + ik|x - x'|}.
\]
For \( r = |x| \to \infty \), we have \( |x-x'| \sim r - x' \cdot \frac{x}{r} \). We then arrive at

\[
h_{\mu\nu}^{\text{source}}(x, t) \sim e^{-ikt+ikr} \left[ 4G \int d^3 x' \frac{S_{\mu\nu}(x', k)}{r} e^{-i k x' \cdot \frac{x}{r}} \right].
\] (4.7)

For large \( kr \), this gravitational wave with the source (4.5) can be regarded as a plane wave

\[
h_{\mu\nu}^{\text{source}}(x, t) = e_{\mu\nu}(x, t) e^{-ikt+ikr}
\] (4.8)

with the polarization tensor

\[
e_{\mu\nu}(x, t) = 4G \int d^3 x' \frac{S_{\mu\nu}(x', k)}{r} e^{-ik x' \cdot \frac{x}{r}}.
\] (4.9)

Now we consider an arbitrary source.

Fourier expanding the source \( T_{\mu\nu}(x, t) \) as Eq. (4.4), by Eq. (4.2), we then have the Fourier expansion of \( S_{\mu\nu}(x, t) \):

\[
S_{\mu\nu}(x, t) = \sum_k S_{\mu\nu}(x, k) e^{-ikt}.
\] (4.10)

Then the gravitational wave with the source \( T_{\mu\nu}(x, t) \), by Eq. (4.3), reads

\[
h_{\mu\nu}^{\text{source}}(x, t) = 4G \sum_k \int d^3 x' \frac{S_{\mu\nu}(x', k)}{|x-x'|} e^{-ikt+ik|x-x'|}.
\] (4.11)

Substituting Eq. (4.8) into Eq. (4.11) gives

\[
h_{\mu\nu}^{\text{source}}(x, t) = \sum_k \left[ 4G \int d^3 x' \frac{S_{\mu\nu}(x', k)}{r} e^{-ik x' \cdot \frac{x}{r}} \right] e^{-ikt+ikr}
= \sum_k e_{\mu\nu}(x, t) e^{-ikt+ikr}.
\] (4.12)

By the exact result (2.1), the gravitational wave with the source \( T_{\mu\nu}(x, t) \) can be expressed as

\[
h_{\mu\nu}^{\text{source}}(x, t) = \sum_k e_{\mu\nu}(x, t) e^{-ikt} \frac{1}{2ikr} \sum_{l=0}^{\infty} \frac{1}{2l+1} \\left( -1 \right)^l \frac{1}{ikr} P_l(\cos \theta)
+ \sum_k e_{\mu\nu}(x, t) e^{-ikt} \frac{1}{2ikr} \sum_{l=0}^{\infty} \frac{1}{2l+1} \\left( -1 \right)^l \frac{1}{ikr} P_l(\cos \theta).
\] (4.13)

This result recovers the large-distance asymptotically approximate result in the conventional scattering theory when \( y_l \left( \frac{1}{ikr} \right) \to 1 \).

### 4.2 Power of gravitational wave

In this section, we consider the power of the gravitational wave with a source.
For a source $T_{\mu\nu}(x,t)$, by the incident wave (4.12), the scattering power of the gravitational wave can be obtained by summing over the powers of the single frequency plane gravitational waves, Eqs. (2.10) and (2.12):

$$P_{\text{source}} = \sum_k e_{\mu\nu}(x,t) \left\{ \frac{1}{16\pi G} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \int d\Omega a_{l}^{\mu\nu s}(\theta) a_{l'}^{\mu\nu}(\theta) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right) \right\}, \quad (4.14)$$

$$P_{\text{int}} = \sum_k e_{\mu\nu}(x,t) \left\{ \frac{1}{32\pi G} \sum_{l=0}^{\infty} (2l + 1) \int d\Omega P_l(\cos \theta) \text{Im} \left[ (-i)^{l'-1} \left( e^{\mu\nu s} a_{l'}^{\mu\nu}(\theta) - \frac{1}{2} e^{\mu s} \epsilon_{l,l'}^{\mu}(\theta) \right) y_l \left( \frac{1}{ikr} \right) y_{l'} \left( -\frac{1}{ikr} \right) \right] \right\}. \quad (4.15)$$

The modifications of the power to the conventional scattering theory are

$$\Delta P_{\text{sc}} = \sum_k e_{\mu\nu}(x,t) \left\{ \frac{1}{64\pi G k^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} l(l+1) l'(l'+1) i^{l-l'} \right\}, \quad (4.16)$$

$$\Delta P_{\text{int}} = \sum_k e_{\mu\nu}(x,t) \left\{ \frac{1}{64\pi G k^2 r^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l + 1) l(l+1) l'(l'+1) \right\}. \quad (4.17)$$

The leading modification is the $p$-wave modification:

$$\Delta P_{\text{sc}}^{p\text{-wave}} = \sum_k \frac{1}{k^2 r^2} e_{\mu\nu}(x,t) \frac{1}{16\pi G} \int d\Omega \left( a_{l}^{\mu\nu s}(\theta) a_{l'}^{\mu\nu}(\theta) - \frac{1}{2} a_{l}^{\mu s}(\theta) a_{l'}^{\mu}(\theta) \right), \quad (4.19)$$

$$\Delta P_{\text{int}}^{p\text{-wave}} = \sum_k \frac{1}{k^2 r^2} e_{\mu\nu}(x,t) \frac{3}{16\pi G} \int d\Omega \cos \theta \text{Im} \left[ \left( e^{\mu\nu s} a_{l}^{\mu\nu}(\theta) - \frac{1}{2} e^{\mu s} \epsilon_{l,l'}^{\mu}(\theta) \right) \right]. \quad (4.20)$$

As a comparison, the $p$-wave contributions in the conventional scattering theory, by Eqs. (3.5) and (3.6), are

$$P_{\text{sc}}^{\text{conventional} \ p\text{-wave}} = \sum_k e_{\mu\nu}(x,t) \frac{1}{16\pi G} \int d\Omega \left( a_{l}^{\mu\nu s}(\theta) a_{l'}^{\mu\nu}(\theta) - \frac{1}{2} a_{l}^{\mu s}(\theta) a_{l'}^{\mu}(\theta) \right) \quad (4.21)$$

$$P_{\text{int}}^{\text{conventional} \ p\text{-wave}} = \sum_k e_{\mu\nu}(x,t) \frac{3}{16\pi G} \int d\Omega \cos \theta \text{Im} \left[ \left( e^{\mu\nu s} a_{l}^{\mu\nu}(\theta) - \frac{1}{2} e^{\mu s} \epsilon_{l,l'}^{\mu}(\theta) \right) \right]. \quad (4.22)$$
5 Conclusion

In summary, the conventional weak-field gravitational wave scattering theory is a large-distance asymptotically approximate theory. In this paper, we establish a rigorous gravitational wave scattering theory without the large-distance asymptotic approximation. In the rigorous scattering theory, the information of the distance is preserved.

There is also an important issue in gravitational wave scattering: the calculation of the scattering amplitude and the partial wave scattering amplitude. In quantum mechanical scattering theory, the scattering amplitude is described by the scattering phase shift. We have developed a method for the calculation of the scattering phase shift based on the heat kernel theory [31, 32] and the integral equation method of scalar scattering in curved spacetime [33]. These methods can also be used to calculate the scattering amplitude in gravitational wave scattering.

Acknowledgments

We are very indebted to Dr G. Zeitrauman for his encouragement. This work is supported in part by Special Funds for theoretical physics Research Program of the NSFC under Grant No. 11947124, Nankai Zhide foundation, and NSFC under Grant Nos. 11575125 and 11675119.

References

[1] B. P. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, et al., Observation of gravitational waves from a binary black hole merger, Physical review letters 116 (2016), no. 6 061102.

[2] B. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, et al., Astrophysical implications of the binary black hole merger gw150914, The Astrophysical Journal Letters 818 (2016), no. 2 L22.

[3] B. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, et al., Binary black hole mergers in the first advanced ligo observing run, Physical Review X 6 (2016), no. 4 041015.

[4] B. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, et al., Gw151226: Observation of gravitational waves from a 22-solar-mass binary black hole coalescence, Physical Review Letters 116 (2016), no. 24 241103.

[5] L. Scientific, B. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, et al., Gw170104: Observation of a 50-solar-mass binary black hole coalescence at redshift 0.2, Physical Review Letters 118 (2017), no. 22 221101.

[6] B. P. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, V. Adya, et al., Gw170814: A three-detector observation of gravitational waves from a binary black hole coalescence, Physical Review Letters 119 (2017), no. 14 141101.
[7] B. P. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, V. Adya, et al., *Gw170817: observation of gravitational waves from a binary neutron star inspiral*, Physical Review Letters **119** (2017), no. 16 161101.

[8] B. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, V. Adya, et al., *Gravitational waves and gamma-rays from a binary neutron star merger: Gw170817 and grb 170817a*, The Astrophysical Journal Letters **848** (2017), no. 2 L13.

[9] S. R. Dolan, *Scattering and absorption of gravitational plane waves by rotating black holes*, Classical and Quantum Gravity **25** (2008), no. 23 235002.

[10] T. Stratton and S. R. Dolan, *Rainbow scattering of gravitational plane waves by a compact body*, Physical Review D **100** (2019), no. 2 024007.

[11] A. Folacci and M. O. El Hadj, *Regge pole description of scattering of gravitational waves by a schwarzschild black hole*, Physical Review D **100** (2019), no. 6 064009.

[12] F. Pretorius and W. E. East, *Black hole formation from the collision of plane-fronted gravitational waves*, Physical Review D **98** (2018), no. 8 084053.

[13] R. Torrence and A. Janis, *Approach to gravitational radiation scattering*, Journal of Mathematical Physics **8** (1967), no. 7 1355–1366.

[14] P. J. Westervelt, *Scattering of electromagnetic and gravitational waves by a static gravitational field: comparison between the classical (general-relativistic) and quantum field-theoretic results*, Physical Review D **3** (1971), no. 10 2319.

[15] R. Matzner and M. Ryan Jr, *Scattering of gravitational radiation from vacuum black holes*, The Astrophysical Journal Supplement Series **36** (1978) 451–481.

[16] F. Handler and R. A. Matzner, *Gravitational wave scattering*, Physical Review D **22** (1980), no. 10 2331.

[17] W. K. De Logi and S. J. Kovács Jr, *Gravitational scattering of zero-rest-mass plane waves*, Physical Review D **16** (1977), no. 2 237.

[18] R. A. Matzner and M. P. Ryan Jr, *Low-frequency limit of gravitational scattering*, Physical Review D **16** (1977), no. 6 1636.

[19] P. C. Peters, *Index of refraction for scalar, electromagnetic, and gravitational waves in weak gravitational fields*, Physical Review D **9** (1974), no. 8 2207.

[20] P. Peters, *Differential cross sections for weak-field gravitational scattering*, Physical Review D **13** (1976), no. 4 775.

[21] R. A. Hulse and J. H. Taylor, *Discovery of a pulsar in a binary system*, The Astrophysical Journal **195** (1975) L51–L53.

[22] F. Sorge, *On the gravitational scattering of gravitational waves*, Classical and Quantum Gravity **32** (2015), no. 3 035007.

[23] A. Cetoli and C. Pethick, *Interaction of gravitational waves with matter*, Physical Review D **85** (2012), no. 6 064036.

[24] E. Guadagnini, *Graviton scattering from classical matter*, Classical and Quantum Gravity **25** (2008), no. 9 095012.

[25] R. Takahashi, T. Suyama, and S. Michikoshi, *Scattering of gravitational waves by the weak gravitational fields of lens objects*, Astronomy & Astrophysics **438** (2005), no. 1 L5–L8.
[26] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*. John Wiley & Sons, 2004.

[27] T. Liu, W.-D. Li, and W.-S. Dai, *Scattering theory without large-distance asymptotics*, *Journal of High Energy Physics* **2014** (2014), no. 6 1–12.

[28] W.-D. Li and W.-S. Dai, *Scattering theory without large-distance asymptotics in arbitrary dimensions*, *Journal of Physics A: Mathematical and Theoretical* **49** (2016), no. 46 465202.

[29] C.-C. Zhou, W.-D. Li, and W.-S. Dai, *Acoustic scattering theory without large-distance asymptotics*, *Journal of Physics Communications* **2** (2018), no. 4 041002.

[30] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST handbook of mathematical functions*. Cambridge University Press, 2010.

[31] H. Pang, W.-S. Dai, and M. Xie, *Relation between heat kernel method and scattering spectral method*, *The European Physical Journal C* **72** (2012), no. 5 1–13.

[32] W.-D. Li and W.-S. Dai, *Heat-kernel approach for scattering*, *The European Physical Journal C* **75** (2015), no. 6.

[33] W.-D. Li, Y.-Z. Chen, and W.-S. Dai, *Scalar scattering in schwarzschild spacetime: Integral equation method*, *Physics Letters B* (2018).