Some open 3-manifolds and 3-orbifolds without locally finite canonical decompositions

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Abstract
We give examples of open 3-manifolds and 3-orbifolds that exhibit pathological behavior with respect to splitting along surfaces (2-suborbifolds) with nonnegative Euler characteristic.

Introduction

Much of the theory of compact 3-manifolds relies on decompositions into canonical pieces, in particular the Kneser-Milnor prime decomposition [12, 16], and the Jaco-Shalen-Johannson characteristic splitting [10, 11]. These have led to important developments in group theory [22, 7, 9, 24], and form the background of W. Thurston’s geometrization conjecture, which has recently been proved by G. Perelman [19, 20, 21].

For open 3-manifolds, by contrast, there is not even a conjectural description of a general 3-manifold in terms of geometric ones. Such a description would be all the more useful that noncompact hyperbolic 3-manifolds are now increasingly well-understood, thanks in particular to the recent proofs of the ending lamination conjecture [17, 4] and the tameness conjecture [5, 11].

The goal of this paper is to present a series of examples which show that naive generalizations to open 3-manifolds of the canonical decomposition theorems of compact 3-manifold theory are false. This contrasts with the positive results of [15, 13] which give decompositions under various hypotheses.

We now describe our examples and their properties in more detail. All manifolds and orbifolds in the following discussion are connected, orientable, and without boundary.

An embedded 2-sphere $S$ in a 3-manifold $M$ is called compressible if $S$ bounds a 3-ball in $M$. If all 2-spheres in $M$ are compressible, we say that
$M$ is irreducible. A *spherical decomposition* $\mathcal{S}$ of a 3-manifold $M$ is a locally finite collection of (possibly nonseparating) pairwise disjoint embedded 2-spheres in $M$ such that the operation of cutting $M$ along $\mathcal{S}$ and gluing a ball to each boundary component of the resulting manifold yields a collection of irreducible manifolds. Note that if $\mathcal{S}$ is a spherical decomposition, then the collection of spheres obtained by removing compressible spheres of $\mathcal{S}$ is still a spherical decomposition.

Kneser’s theorem is equivalent to the statement that every compact 3-manifold has a spherical decomposition. The first two examples in this paper show that this result does not generalize to open manifolds. The first relevant example was given by P. Scott [23]. Our example $M_1$ is simpler, and has additional properties: for instance, it is a graph manifold, and has only one end. Our second example $M_2$ is closer in spirit to Scott’s; its main purpose is to lead to example $M_3$ below.

The remaining examples are concerned with generalizing the toric splitting of Jaco-Shalen and Johannson. The correct definition of a JSJ-splitting for open 3-manifolds is still open to debate; however we make a few observations. Call an embedded torus $T$ in a 3-manifold $M$ *incompressible* if it is $\pi_1$-injective, and *canonical* if it is incompressible, and any incompressible torus in $M$ is homotopically disjoint from $T$. The version of the toric splitting theorem proved in [18] asserts that if one takes a collection $\mathcal{T}$ of pairwise disjoint representatives of all homotopy classes of canonical tori in $M$ (which is always possible, for instance by taking least area surfaces in some generic Riemannian metric), then $\mathcal{T}$ splits $M$ into submanifolds that are either Seifert-fibered or atoroidal. This approach can be generalized to 3-orbifolds [2, Chapter 3]. Here tori are replaced by *toric* 2-orbifolds, i.e. finite quotients of tori. Those include *pillows*, i.e. spheres with four cone points of order 2.

Let $N_1$ be an orientable 3-manifold whose boundary is an annulus, which does not contain any essential tori and open annuli, and is not homeomorphic to $S^1 \times \mathbb{R} \times [0, +\infty)$. Let $N_2$ be the product of $S^1$ with an orientable surface of infinite genus whose boundary is a line. Since $N_2$ is a Seifert fiber space of infinite topological complexity, it contains many incompressible tori, neither of which is canonical. By gluing $N_1$ and $N_2$ along their boundaries, one obtains an open 3-manifold $M$ containing again many incompressible tori, neither of which are canonical. It is easy to construct infinite families of disjoint incompressible tori which cannot be made locally finite by any isotopy. However, there is in our view nothing pathological about this example: the “JSJ-splitting” in this case should consist of the single open annulus $A$, which splits $M$ into a Seifert part and an atoroidal part.

This discussion makes plausible the idea that every irreducible open 3-ma-
nifold (or 3-orbifold) $M$ could have a JSJ-splitting consisting of a representative of each class of canonical tori (toric suborbifolds), plus some properly embedded, incompressible open annuli (annular suborbifolds), splitting $M$ into pieces which are either atoroidal, or maximal Seifert submanifolds (orbifolds). Furthermore, each incompressible torus (toric suborbifold) in $M$ should be homotopic into some Seifert piece.

However, if one wishes to stick to locally finite splittings, this simple idea does not work, as examples 3–5 show. Our example $M_3$ is an irreducible open 3-manifold which contains an infinite collection $\{T'(v)\}$ of pairwise non-isotopic canonical tori and a compact set $X'$ that meets every torus isotopic to some $T'(v)$. In particular, it is impossible to select a representative in each isotopy class of canonical tori to form a locally finite collection.

The manifold $M_3$ is constructed as a finite cover of a 3-orbifold $O_3$ which contains infinitely many isotopy classes of canonical pillows, but such that no infinite collection of pairwise nonisotopic canonical pillows is locally finite. Moreover, all incompressible toric 2-suborbifolds in $O_3$ are pillows, and they are all canonical.

The next example $O_4$ is another open, irreducible 3-orbifold with the property that there are infinitely many isotopy classes of canonical pillows, but no infinite, locally finite collection of representatives. Again, all of its incompressible toric 2-suborbifolds are pillows. However, unlike $O_3$, it also contains infinitely many classes of non-canonical incompressible pillows. Such pillows go in pairs and can be used to produce Seifert suborbifolds bounded by canonical pillows, which also accumulate in an essential way, and hence do not give a maximal Seifert suborbifold which would contain all incompressible toric 2-suborbifolds up to isotopy. Since the underlying space of $O_4$ is $\mathbb{R}^3$, it is somewhat easier to visualize than $M_3$ and $O_3$.

Lastly, the example $O_5$ is an open, irreducible 3-orbifold which contains infinitely many isotopy classes of non-canonical incompressible pillows, but not a single canonical toric 2-suborbifold. Its pathological character comes from the fact that it is impossible to find a Seifert suborbifold that contains all incompressible pillows up to isotopy. Instead, one finds an infinite collection of incompatible Seifert suborbifolds which can be made pairwise disjoint, but all intersect essentially some fixed compact subset of $O_5$. This shows that strange things can occur even without canonical toric suborbifolds.

For terminology and background on 3-orbifolds and their geometric decompositions, we refer to [2].

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1 Example 1

Our first example is a one-ended open 3-manifold $M_1$ that does not have any spherical decomposition.

Let $F$ be the orientable surface with one end, infinite genus and one boundary component homeomorphic to the circle. Set $N := S^1 \times F$. Let $M_1$ be the 3-manifold obtained by gluing a solid torus $V \cong D^2 \times S^1$ to $N$ so that the boundary of the $D^2$ factor of $V$ is glued to the $S^1$ factor of $N$, and the $S^1$ factor of $V$ is glued to $\partial F$.

We observe that the universal cover of $\text{Int} \ N$ is $\mathbb{R}^3$. We know from Alexander’s theorem that $\mathbb{R}^3$ is irreducible, so by an elementary argument, $N$ is also irreducible.

Let us prove by contradiction that $M_1$ cannot have a spherical decomposition $\mathcal{S}$. We may assume that there are no compressible spheres in $\mathcal{S}$. Since $N$ is irreducible, all spheres in $\mathcal{S}$ must intersect $V$. Since $V$ is compact and $\mathcal{S}$ is locally finite, we deduce that $\mathcal{S}$ must be finite.

Hence there is a compact subsurface $X \subset F$ such that every sphere in $\mathcal{S}$ lies in $V \cup S^1 \times X$. Since $F$ has infinite genus, we can find a properly embedded arc $\alpha \subset F$ and a simple closed curve $\beta \subset F - X$ which intersect transversally in a single point. We then obtain an embedded 2-sphere $S \subset M$ by taking the annulus $S^1 \times \alpha$ and gluing a meridian disk to each boundary component.

We may assume that $S$ is in general position with respect to $\mathcal{S}$. After finitely many isotopies and surgeries along disks in $\bigcup \mathcal{S}$, we get a finite collection $S_1, \ldots, S_n$ of embedded 2-spheres in $M_1$ such that $[S] = \sum_i [S_i] \in H_2(M_1)$ and each $S_i$ is disjoint from $\bigcup \mathcal{S}$. Since $\mathcal{S}$ is a spherical decomposition, each $S_i$ either bounds a ball, or cobounds a punctured 3-sphere with some members of $\mathcal{S}$.

As a result, the homology class $[S] \in H_2(M_1)$ can be written as a finite sum of classes $[S_j]$ with $S_j \in \mathcal{S}$. Now the intersection number of each $S_j$ with $\beta$ is zero, while that of $S$ with $\beta$ is one. This is a contradiction.

Remarks.

- The first example of an open 3-manifold without a spherical decomposition was given by P. Scott. His construction is quite intricate and his example is simply-connected. Our example is simpler; it is far from simply-connected however, in fact its fundamental group is an infinitely generated free group. Our example has one end; it is easy to modify the construction to give any number of ends.

- Our example is a graph-manifold: this is of some interest since those
manifolds arise in the theory of collapsing sequences of manifolds in Riemannian geometry.

• There is an alternative description of $M_1$ as the double of the manifold $H = I \times F$. Any properly embedded arc in $F$ gives a properly embedded 2-disk in $H$, which gives a sphere in the double. One readily sees that all those spheres have to intersect the annulus $I \times \partial F$. I owe this remark to Saul Schleimer.

2 Example 2

Here we give another example of a manifold without any spherical decomposition. The main interest of this construction is that a simple modification of it will give a manifold which behaves pathologically with respect to essential tori.

First we define an open 3-orbifold $O_2$. In the following construction, all local groups will be cyclic of order 2. We let $X$ be a 3-ball with singular locus a trivial 2-tangle, whose components are denoted by $\sigma_l$ and $\sigma_r$. Let $Y$ be a thrice punctured 3-sphere with singular locus consisting of six unknotted arcs, as in Figure 1. The boundary components of $Y$ are denoted by $\partial_u Y$, $\partial_l Y$, and $\partial_r Y$ (where the letters $u$, $l$, $r$ stand for “up”, “left” and “right” respectively). There are two arcs $\sigma^1_l, \sigma^2_l$ connecting $\partial_u Y$ to $\partial_l Y$, two arcs $\sigma^1_r, \sigma^2_r$ connecting $\partial_u Y$ to $\partial_r Y$, and two arcs $\sigma^1_m, \sigma^2_m$ connecting $\partial_l Y$ to $\partial_r Y$.

Then we take a countable collection of copies of $Y$ indexed by the vertices of the regular rooted binary tree $T$. We glue them together according to the following rule: each copy $Y_u$ of $Y$ has two sons, a left son $Y_l$ and a right son $Y_r$. Then we glue $\partial Y_u$ to $\partial Y_l$ so that $\sigma^i_l(Y_u)$ is glued to $\sigma^i_l(Y_l)$ and $\sigma^i_m(Y_u)$ is glued to $\sigma^i_r(Y_l)$ for $i = 1, 2$. Likewise we glue $\partial_r Y_u$ to $\partial_u Y_r$ so that $\sigma^i_r(Y_u)$ is glued to $\sigma^i_r(Y_r)$ and $\sigma^i_m(Y_u)$ is glued to $\sigma^i_l(Y_r)$ for $i = 1, 2$.

In this way we get a noncompact 3-orbifold $N$ with a single boundary component which is the upper boundary of the ancestor $Y_0$. We glue in a copy of $X$ so that $\sigma_l(X)$ is glued to $\sigma^i_l(Y_0)$ and $\sigma_r(X)$ is glued to $\sigma^i_r(Y_0)$. We call $O_2$ the resulting open 3-orbifold.

For future reference, we note:

Lemma 2.1. The orbifold $N$ is irreducible.

Proof. Since the gluing of the various copies of $Y$ occurs along incompressible suborbifolds, we only need to show that $Y$ is irreducible. Arguing by contradiction, let $S$ be an incompressible spherical 2-suborbifold of $Y$. Since any incompressible sphere in $|Y|$ meets the singular locus $\Sigma Y$ in at least four
Figure 1: $Y$

points, $|S|$ is compressible in $|Y|$, i.e. bounds a 3-ball $B$. Since $Y \setminus \Sigma_Y$ is irreducible, the only possibility is that $B$ intersects $\Sigma_Y$ in a knotted arc $\alpha$. One could then extend $\alpha$ to produce a knotted component of $\Sigma_Y$, contradicting the definition of $Y$.

We claim that $\mathcal{O}_2$ is homeomorphic to a connected sum of two copies of itself. Indeed, let $D_1 \subset X$ be a properly embedded nonsingular 2-disk separating $\sigma_l$ from $\sigma_r$. Then $\partial D_1$ bounds a 2-disk $D_2 \subset Y_0$ intersecting the singular locus transversally in two points, one on $\sigma_m^1$ and one on $\sigma_m^2$. The union of $D_1$ and $D_2$ is a football $S$. Splitting $\mathcal{O}_2$ along $S$ and capping off the corresponding discal 3-orbifolds, we get two copies of $\mathcal{O}_2$.

We now define our manifold $M_2$: by repeatedly applying the Seifert-
van Kampen Theorem [2, Corollary 2.3], we see that the fundamental group of \(O_2\) has an infinite presentation with generators \(m_1, m_2, \ldots\) and relations \(r_1, r_2, \ldots\) as follows: each generator \(m_i\) corresponds to a meridian; for each \(i\) one has a relation \(m_i^2 = 1\); all other relations are of the form \(m_im_jm_km_l = 1\) corresponding to some boundary component of some copy of \(Y\). Hence there is a well-defined group epimorphism \(\phi : \pi_1O_2 \to \mathbb{Z}/2\mathbb{Z}\) sending each \(m_i\) to the generator of \(\mathbb{Z}/2\mathbb{Z}\). The kernel of \(\phi\) is a torsion free index two subgroup of \(\pi_1O_2\). Let \(M_2\) be the corresponding covering space of \(O_2\) and \(p\) be the covering map.

Then \(M_2\) is a manifold which is a connected sum of two copies of itself. Now \(N\) is irreducible (Lemma 2.1) hence by [14, Theorem 10.1], \(p^{-1}(N)\) is irreducible. Since \(M_2 \setminus p^{-1}(N)\) has compact closure, we conclude that any spherical decomposition of \(M_2\) with only essential spheres would have to be finite.

Next we prove that there is no such spherical decomposition: let \(Z\) be a compact submanifold of \(M_2\) containing all spheres in a putative spherical decomposition. Let \(v\) be a vertex of \(T\) which is lower than any vertex \(v'\) such that \(p(Z) \cap Y_{v'} \neq \emptyset\). There is a football \(F \subset O_2\) intersecting \(Y_v\) in a disk with two cone points, one on \(\sigma_l\) and the other on \(\sigma_r\), each \(Y_{v'}\) in a nonsingular annulus (for \(v'\) above \(v\)) and \(X\) in a nonsingular disk. Then \(S := p^{-1}(F)\) is a 2-sphere embedded in \(M_2\). One can find a properly embedded line \(L \subset M_2\) missing \(Z\) and hitting \(S\) transversally in a single point. As before we get a contradiction.

### 3 Example 3

Here is an example of an irreducible open 3-manifold which contains infinitely many isotopy classes of incompressible tori, all of which are canonical, but such that there is no infinite, locally finite collection of canonical tori.

Let \(O_3\) be the open 3-orbifold obtained by the following modification of the previous construction, where every singular arc is “doubled”.

Let \(X\) be a 3-ball with singular locus a trivial 4-tangle, whose components are denoted by \(\sigma_l, \sigma_l', \sigma_r, \sigma_r'\). Let \(Y\) be a thrice punctured 3-sphere with singular locus consisting of twelve unknotted arcs: four arcs \(\sigma_1^1, \sigma_1^2, \sigma_1^{1'}, \sigma_1^{2'}\) connecting \(\partial_lY\) to \(\partial_rY\), four arcs \(\sigma_r^1, \sigma_r^2, \sigma_r^{1'}, \sigma_r^{2'}\) connecting \(\partial_uY\) to \(\partial_rY\), and four arcs \(\sigma_m^1, \sigma_m^2, \sigma_m^{1'}, \sigma_m^{2'}\) connecting \(\partial_lY\) to \(\partial_uY\).

We remark (cf. [6]) that \(X\) and \(Y\) are irreducible and atoroidal, \(Y\) has incompressible boundary, and the only essential annular 2-suborbifolds in \(Y\) are nonsingular annuli connecting two distinct boundary components.

As before, let \(T\) be the regular rooted binary tree. To each vertex \(v\) of
we assign a copy $Y(v)$ of $Y$. With the same notation as in the previous section, we glue $\partial_n Y_u$ to $\partial_n Y_l$ and $\partial_n Y_u$ $\partial_n Y_r$ so that $\sigma'_1(Y_u)$ (resp. $\sigma'_2(Y_u)$) is glued to $\sigma'_1(Y_l)$ (resp. $\sigma'_1(Y_l)$), etc. We again call $N$ the resulting 3-orbifold with boundary. We glue in a copy of $X$ so that $\sigma_l(X)$ (resp. $\sigma'_l(X)$) is glued to $\sigma_l(Y_0)$ (resp. $\sigma'_l(Y_0)$), and $\sigma_r(X)$ (resp. $\sigma'_r(X)$) is glued to $\sigma_r(Y_0)$.

We call $O_3$ the resulting open 3-orbifold, and $P$ the union of all boundaries of all $Y(v)$’s.

**Proposition 3.1.**

i. $O_3$ is irreducible.

ii. There are infinitely many incompressible pillows up to isotopy.

iii. All incompressible toric 2-suborbifolds in $O_3$ are pillows. Furthermore, they are all canonical, and they all meet the compact 3-suborbifold $X$.

**Proof.** (i) Suppose there is an essential 2-suborbifold $S \subset O_3$ with positive Euler characteristic. Take among all of them one that intersects $P$ minimally. Since $X$ and $Y$ are irreducible, $S \cap P$ cannot be empty. Let $v$ be a vertex of $T$ such that $S$ meets $Y(v)$, but avoids all $Y(v’)$’s with $v’$ below $v$. Let $F$ be a component of $S \cap Y(v)$. By minimality of $\#S \cap P$, $F$ must be essential. However, $Y$ does not contain any essential 2-suborbifold whose boundary is nonempty and contained in $\partial_n Y(v)$ (this can be seen by embedding $Y$ into a product orbifold and applying [6, Prop. 5].) This is a contradiction.

(ii) Pick any vertex $v$ of $T$. Let $v = v_0, v_1, v_2, \ldots, v_n$ be a path in $T$ connecting $v$ to the ancestor. We define a pillow $T(v) \subset O_3$ as follows: Let $F \subset Y(v)$ be a disk with four cone points, with boundary in $\partial_n Y(v)$, and intersecting each of $\sigma^1_m, \sigma^2_m, \sigma^1'_m, \sigma^2'_m$ exactly once. Define inductively a family $\{A_i\}_{1 \leq i \leq n}$ such that:

i. $A_i$ is an essential nonsingular annulus in $Y(v_i)$;

ii. One boundary component of $A_1$ is $\partial F$;

iii. For every $i$, one boundary component of $A_i$ is equal to some boundary component of $A_{i+1}$;

iv. One boundary component of $A_n$ lies in $\partial_n Y(v_n)$.

Finally let $D$ be a disk in $X$ with $\partial D$ equal to the other boundary component of $A_n$. Then $F \cup A_1 \cup \cdots \cup A_n \cup D$ is an embedded pillow in $O_3$, which we denote by $T(v)$.

The four singular points of $T(v)$ belong to four distinct components $L_1, L_2, L_3, L_4$ of $\Sigma_{O_3}$, which are properly embedded lines. There exists a
properly embedded nonsingular annulus \( A \) which separates \( \mathcal{O}_3 \) in two components \( Z_1, Z_2 \) such that \( Z_1 \) contain the \( L_i \)'s and is homeomorphic to the product of a disk with four cone points of order two with the real line. By the Seifert-van Kampen theorem, \( \pi_1 \mathcal{O}_3 \) is an amalgamated product of \( \pi_1 Z_1 \) with \( \pi_1 Z_2 \) over \( \pi_1 A \). The fundamental group of \( Z_1 \) can be expressed as the free product of four order two cyclic subgroups generated by meridians \( m_1, m_2, m_3, m_4 \) of \( L_1, L_2, L_3, L_4 \) respectively. Now \( \partial Y(v) \) intersects \( A \) in a circle, which is essential on \( A \), but bounds a nonsingular disk in \( \bar{Z}_2 \). Hence \( \pi_1 A \), has trivial image in \( \pi_1 \mathcal{O}_3 \). As a result, the image of \( \pi_1 Z_1 \) in \( \pi_1 \mathcal{O}_3 \) is the quotient of \( \pi_1 Z_1 \) by the single relation \( m_1 m_2 m_3 m_4 = 1 \). This implies that \( T(v) \) is \( \pi_1 \)-injective, hence incompressible.

If \( v \neq v' \), then up to exchanging \( v \) and \( v' \), we can find a line in \( \Sigma_{\mathcal{O}_3} \) which intersects \( T(v) \) transversally in a single point, and does not meet \( T(v') \). As a consequence, \( T(v) \) and \( T(v') \) are not isotopic.

(iii) Let \( T \) be an incompressible toric 2-suborbifold of \( \mathcal{O}_3 \). Then \( T \) could be a nonsingular torus, a pillow, or a Euclidean turnover. Any sphere in \( |\mathcal{O}_3| \) which is transverse to \( \Sigma_{\mathcal{O}_3} \) meets it in an even number of points, so \( \mathcal{O}_3 \) does not contain any turnover. Let \( T \) be a nonsingular torus in \( \mathcal{O}_3 \). Since \( T \) is compact, there is a compact suborbifold \( Z \subset \mathcal{O}_3 \) consisting of \( X \) and finitely many \( Y(v) \)'s such that \( T \subset Z \). The orbifold \( Z \) is homeomorphic to \( S^3 \) minus a finite union of disjoint balls, with planar singular locus, so \( \pi_1(|Z| \setminus \Sigma_Z) \) is a free group. Thus \( T \) is compressible in \( |Z| \setminus \Sigma_Z \), hence in \( \mathcal{O}_3 \). As a consequence, all incompressible toric 2-suborbifolds of \( \mathcal{O}_3 \) are pillows.

Our next goal is to prove that the collection \( \{T(v)\}_{v \in T} \) actually contains all incompressible pillows up to isotopy. Indeed, let \( T \) be an incompressible pillow. Assume after isotopy that \( T \) intersects \( P \) minimally. Since \( X \) is atoroidal, \( T \) meets some \( Y(v) \). Choose \( v_0 \) so that \( T \) meets \( Y(v_0) \) and does not meet any \( Y(v) \) with \( v \) below \( v_0 \). Then the argument used to prove assertion (i) shows that \( T \cap Y(v_0) \) must consist of a disk \( D \) intersecting each of \( \sigma_1, \sigma_2, \sigma_1', \sigma_2' \) in exactly one point. Let \( v_1 \) be the father of \( v_0 \). The intersection of \( T \) with \( Y(v_1) \) must be an essential nonsingular annulus \( A \), one of whose boundary components is \( \partial D \).

The other component of \( \partial A \) cannot be on \( \partial Y(v_1) \) or \( \partial Y(v_1) \), for otherwise by carrying on the same argument, we would get a string of nonsingular annuli going down the tree, and we would never be able to close up. Hence the other component of \( \partial A \) lies on \( \partial Y(v_1) \). We can repeat the argument until we arrive at \( Y_0 \). The upshot is that \( T \) is isotopic to \( T(v_0) \). Finally, since the \( T(v)'s \) can be realized so as to be pairwise disjoint, it follows that they are all canonical.

As before, we consider the homomorphism \( \phi : \pi_1 \mathcal{O}_3 \to \mathbb{Z}/2\mathbb{Z} \) which
sends meridians to the generator. The corresponding regular cover $M_3$ is a good orbifold with torsion-free fundamental group, hence a manifold. We let $p : M_3 \to O_3$ denote the covering map, and set $X' := p^{-1}(X)$, $P' := p^{-1}(P)$, and $Y'(v) := p^{-1}(Y(v))$ for all $v \in T$.

**Proposition 3.2.**

i. $M_3$ is irreducible;

ii. There is an infinite collection $\{T'(v)\}_{v \in T}$ of pairwise nonisotopic canonical tori, all of which essentially intersect $X'$.

**Proof.** (i) follows from irreducibility of $O_3$ and [14, Theorem 10.1].

To establish (ii), we define $T'(v) = p^{-1}(T(v))$. This gives us an infinite collection of tori in $M_3$. If two of them were isotopic, then they would be homologous, and so would be the underlying spaces of their projections to $O_3$, which is not the case. By the equivariant Dehn Lemma, each $T'_v$ is incompressible.

Fix $v \in T$. To show that $T'(v)$ is canonical, it suffices to prove that every incompressible annulus properly embedded in $M_3$ cut along $T'(v)$ is parallel to an annulus in $T'(v)$. We hence need to deduce this from the corresponding property for $T(v) \subset O_3$. This can probably done by elementary topology, but we prefer to use a direct minimal surface argument suggested by Peter Scott. Notice first that for all $v, v'$, the pairs $(M_3, T'(v))$ and $(M_3, T'(v'))$ are homeomorphic. Hence we only give the proof when $v$ is the ancestor $v_0$.

Let $Z$ be a component of $M_3$ cut along $T'(v_0)$, and let $A$ be an incompressible annulus properly embedded in $Z$. Assume that $A$ has been isotoped so as to intersect $P'$ minimally.

If $A$ intersects $P'$ at all, then for some $v$ one component of $A \cap Y'(v)$ is an essential annulus whose boundary is contained in a single component of $\partial Y'(v)$. However, we have the following lemma:

**Lemma 3.3.** Every essential annulus in $Y'(v)$ meets two different components of $\partial Y'(v)$.

**Proof.** To shorten notation set $Y := Y(v)$ and $Y' := Y'(v)$. Put a sufficiently convex Riemannian metric on $Y$ and give $Y'$ the lifted metric. Assume the lemma is false, so that there is a component $U$ of $\partial Y'$ and an essential, properly embedded annulus $A \subset Y'$ with both boundary components in $U$. By [8], one finds an annulus $A_0$ of least area with this property. Let $A'_0$ be the translate of $A_0$ by the deck transformation group of the double cover $p : Y' \to Y$. If $A_0 = A'_0$ or $A_0 \cap A'_0 = \emptyset$, then $p(A_0)$ is a properly embedded, essential, annular 2-suborbifold of $Y$ with both boundary components in $p(U)$. This is impossible.
So generically \( A_0 \) and \( A'_0 \) intersect in a finite family of curves and arcs. By standard exchange/roundoff arguments (cf. \([8, 14]\)) one obtains a contradiction. This proves Lemma 3.3.

We return to the proof of Proposition 3.2. By Lemma 3.3 our annulus \( A \), does not intersect \( P' \). Hence it is contained in \( X' \cap Z \). But this manifold does not contain any essential annulus with both boundary components in \( X' \cap T'_v \), by an argument entirely similar to that used in the proof of Lemma 3.3. This contradiction completes the proof of Proposition 3.2.

4 Just another brick in the wall

Let \( O_4 \) be the orbifold whose underlying space is \( \mathbb{R}^3 \) and whose singular locus is the trivalent graph shown in Figure 2, where all edges should be labeled with the number 2.

\[ \begin{array}{c|c|c|c} \hline & & & \\
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Figure 2: The singular locus of \( O_4 \)

A key property of the graph \( \Sigma_{O_4} \) is that it is planar. In particular, every properly embedded line in \( \Sigma_{O_4} \) is unknotted as a subset of \( \mathbb{R}^3 \). We choose an unknotted arc \( \alpha \) connecting the two components of \( \Sigma_{O_4} \) as in Figure 3.

**Proposition 4.1.**

i. \( O_4 \) is irreducible.

ii. All incompressible toric 2-suborbifolds are pillows, and intersect \( \alpha \).

iii. There are infinitely many isotopy classes of canonical pillows.

**Proof.** (i) Let \( S \) be a general position 2-sphere in \( \mathbb{R}^3 \). Let \( B \subset \mathbb{R}^3 \) be the 3-ball bounded by \( S \). If \( S \) avoids \( \Sigma_{O_4} \), then so does \( B \). Otherwise \( S \) intersects \( \Sigma_{O_4} \) in at least two points. If \( \#S \cap \Sigma_{O_4} = 2 \), then the two intersection points lie on the same edge of \( \Sigma_{O_4} \) and \( B \) intersects \( \Sigma_{O_4} \) in an arc. This arc cannot be knotted inside \( B \) since it extends to a properly embedded singular line in \( \Sigma_{O_4} \), which cannot be knotted in \( \mathbb{R}^3 \) as remarked above. If \( \#S \cap \Sigma_{O_4} = 3 \),
then $B$ intersects $\Sigma_{O_4}$ in a Y-shaped graph, which must be unknotted for a similar reason. In each case, the 2-suborbifold whose underlying sphere is $S$ is compressible.

(ii) Recall (cf. [2, Chapter 2]) that all Euclidean turnovers have at least a cone point of order different from 2. Hence $O_4$ contains no Euclidean turnovers. Any nonsingular torus $T \subset O_4$ lies in a 3-ball intersecting $\Sigma_{O_4}$ in a planar graph. Hence one can find a handlebody in $|O_4 \setminus \Sigma_{O_4}|$ containing $T$. This shows that $T$ is compressible in $|O_4 \setminus \Sigma_{O_4}|$, hence a fortiori in $O_4$.

Let $P$ be a pillow in $O_4$. If $P$ misses $\alpha$, then one can find two edges $e_1, e_2$ of $\Sigma_{O_4}$ such that $P \cap \Sigma_{O_4}$ consists of two points of $e_1$ and two points of $e_2$. Thus the compact 3-suborbifold whose boundary is a 3-ball $P$ is a solid pillow, which implies that $P$ is compressible.

![Figure 3: Two canonical pillows in $O_4$](image)

(iii) Let $P_1$ be the pillow depicted on Figure 3. Let $X_1, X_2$ be the 3-suborbifolds bounded by $P_1$. Then any 2-disk properly embedded in $|X_1|$ or $|X_2|$ intersects $\Sigma_{O_4}$ in at least two points unless it is parallel rel $\Sigma_{O_4}$ to some disk in $P_1$. Hence $P_1$ is incompressible. If there were an incompressible pillow $P_2$ meeting $P_1$ essentially, then after isotopy $X_1 \cap P_2$ and $X_2 \cap P_2$ would consist of essential annular 2-orbifolds with underlying space a 2-disk and two singular points. Now the only such annular suborbifolds are shown on Figure 3; their boundaries are not isotopic, hence they cannot be glued together to give a pillow that would intersect $P_1$ essentially. This shows that $P_1$ is canonical.

The same argument works for the pillow $P_2$ on the same figure. It is easy to see that one can find in this way an infinite family of canonical pillows $P_1, P_2, P_3, \ldots$, where $P_{n+1}$ is separated from $P_n$ by three vertical bars.

Remark. This orbifold $O_4$ also contains noncanonical incompressible pillow
For instance, let $X$ be the 3-suborbifold bounded by $P_1 \cup P_2$ and observe that $P_2$ can be obtained from $P_1$ by moving to the right and `crossing' three vertical arcs in $\Sigma_{O_4}$. If one `crosses' only one of these arcs, one obtains another incompressible pillow, consisting of an annular suborbifold of $P_1$ together with one of the dotted annular suborbifolds in $X$. There are two such pillows. With a little more work, one can show that $X$ has a Seifert fibration where those two pillows are vertical, and their projections to the base orbifold intersect essentially. Hence by $\ref{3}$ those two pillows intersect essentially. In particular, they are noncanonical.

5 Jacob’s nightmare

In order to motivate the example $O_5$ constructed later in this section, we first consider an example of a 3-orbifold which we do not view as pathological. Its underlying space is $\mathbb{R}^3$ and its singular locus is as in Figure 4, all meridians having order 2. It consists of two connected trivalent graphs looking like bi-infinite ladders (which we shall call `Jacob ladders'.)

![Figure 4: An orbifold with two Jacob ladders as singular locus](image)

The same arguments as for $O_4$ show that this orbifold is irreducible, and that its only incompressible toric 2-suborbifolds are pillows. Furthermore, around each ladder one can find a properly embedded bi-infinite annulus that
bounds a Seifert 3-suborbifold $U$: one chooses a vertical band $Z$ containing
the ladder and foliates $|Z|$ by intervals such that the rungs are leaves. Then
those intervals, viewed as 1-suborbifolds, are mirrored intervals, which are
fibers of the Seifert fibration on $U$, the other fibers being circles wrapped
around $Z$ (cf. [2, p. 33].) Thus the orbifold under consideration has a natural
‘JSJ splitting’ consisting of two annuli.

Of course there could be more ladders, or even infinitely many of them.
This we still do not consider pathological, since the corresponding infinite
collection of annuli would be locally finite.

However the 3-orbifold $O_5$ in Figure 4 fails to have such a decomposition.
Its underlying space is again $\mathbb{R}^3$, and its singular locus consists of an infinite
sequence of Jacob ladders plus another component $J$. The precise shape of $J$
is unimportant; the only relevant feature is that it should not create unwanted
incompressible 2-suborbifolds. For example, we can take it to be a planar
‘brickwall’ graph similar to each component of $\Sigma O_4$, also with all labels equal
to 2, but with five horizontal half-lines instead of four, so that there is an
unknotted, properly embedded, nonsingular plane in $O_5$ separating $J$ from
the rest of the singular locus, and having the property that the component
containing $J$ is irreducible and does not contain any toric 2-suborbifold.

Most important is the position of $J$ with respect to the ladders: any finite
set of ladders can be separated from $J$ by a properly embedded nonsingular
plane, but there is an arc $\alpha$ (see Figure 4) connecting $J$ to $L_0$, and which
can be extended to a properly embedded line $\Lambda \subset \mathbb{R}^3$ by adding a half-line
in $J$ and a half-line in $L_0$, such that for all negative $n$, one can find a circle
$Q_n \subset L_n$ (consisting of four arcs: a subarc of each upright plus two ru
connections them) such that the linking number of $Q_n$ and $\Lambda$ is 1.

Also, any two ladders are separated by a properly embedded nonsingular
plane. For each ladder we assign to each rung an integer so that going up
the ladder corresponds to increasing the numbers.

By arguments similar to those of the previous section, one shows that
$O_5$ is irreducible and that its only incompressible toric 2-suborbifolds are
pillows, and meet a single component of the singular locus. By choice of $J$,
this component must be a ladder. More precisely, each incompressible pillow
is obtained in the following way: fix a ladder $L_n$, an integer $p \geq 2$ and a
finite sequence of consecutive rungs $r_1 < \cdots < r_p$ of $L_n$. Then take a sphere
$S$ intersecting $\Sigma O_5$ only on $L_n$, and such that $S \cap L_n$ consists of four points
on the uprights of $L_n$, two immediately below $r_1$ and two immediately above
$r_p$. Observe that this pillow is not canonical, since for instance, the pillow
associated to the sequence $(r_1 - 1), r_1$ will intersect it essentially. Thus $O_5$
has no canonical toric 2-suborbifolds at all.

Further observe that for all $n < 0$, there is a pillow $T_n$ associated to

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$L_n$ such that the intersection number of $T_n$ with $\alpha$ is 1. (In fact, there are infinitely many.) As a consequence, $T_n$ intersects $\alpha$ essentially, i.e. any pillow isotopic to $T_n$ still meets $\alpha$.

**Proposition 5.1.** There exists no Seifert 3-suborbifold $U \subset O_5$ such that all incompressible pillows can be isotoped into $U$.

This is a consequence from the following Claim:

**Claim** Let $T, T'$ be two incompressible pillows associated to distinct ladders, and $U$ be a Seifert 3-suborbifold containing $T \cup T'$. Then $T, T'$ belong to distinct components of $U$.

Indeed, applying the Claim to the infinite family of pillows $\{T_n\}_{n<0}$ described above, we see that the compact set $\alpha$ would have to meet infinitely many distinct connected components of $U$, which is impossible.

Lastly, we prove the claim: if $T$ and $T'$ are contained in a connected Seifert suborbifold $V$, then they are vertical by [3, Theorem 4]. Therefore, there exist closed curves $c \subset T$ and $c' \subset T'$ such that $c$ and $c'$ are freely isotopic in $V$, hence freely isotopic in $O_5$. Now since the ladders are unlinked, there exists a properly embedded nonsingular plane $P \subset O_5$ separating $T$.
from $T'$. Hence by the Seifert-van Kampen theorem, $\pi_1 O_5$ can be expressed as a free product, with $\pi_1 T$ and $\pi_1 T'$ belonging to distinct factors. Hence the elements of $\pi_1 O_5$ represented by $c$ and $c'$ cannot be conjugate. This is a contradiction.

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