Semi-analytic numerical method for unsteady flow in leaky confined aquifers

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ABSTRACT

By using advantages of a semi-analytical approach and Laplace transform technique, a Laplace Transform Finite Layer (LTFL) method is presented for the problem of unsteady flow in leaky confined aquifer systems. The finite layer formulation for drawdown response is developed in the Laplace space. Based on the solution in Laplace space, the groundwater drawdown at a given time and an arbitrary point is obtained by using numerical inversion of Durbin algorithm. The validity of the LTFL scheme is verified based on the existing analytical solution and finite difference method, and the applicability for groundwater flow in modeling flow to partially penetrating wells (PPWs) in the anisotropic leaky confined aquifers is demonstrated.

Keywords: leaky confined aquifer, semi-analytical numerical analysis, Laplace transform, Durbin algorithm

1 INTRODUCTION

The semi-analytical numerical method is a mixed numerical method in which some analytical techniques are used to replace the full spatial discretization in some directions. The FLM (finite layer method), which is a typical semi-analytical numerical method for three-dimensional (3D) problems originally applied in static and dynamic analysis of continuous media (Cheung 1976; Cheung and Jiang 2001), was first introduced by Smith et al. (1992) for groundwater flow models in confined aquifers, and the high accuracy and efficiency of the method was demonstrated (Wang et al. 2009). For unsteady flow problems, the discretization procedure of the time span, such as the finite-difference scheme, is commonly employed to approximate the temporal derivative in the groundwater flow equation in the various numerical methods. This always induces high computational cost, especially in long term analysis of the 3D unsteady flow. In order to avoid the computational expense of time discretization, some improved numerical methods for groundwater flow problems are developed by using Laplace transform technique (Liggett and Liu 1983; Tang and Chen 1991; Moridis and Reddell 1991; Alex and Shlomo 2003). However, most traditional numerical approaches (e.g., finite element method and finite difference method) are based on the full discretization of domain, the inverse Laplace transform is available only at the discrete points.

In this paper, a Laplace transform FLM for 3D flow in anisotropic leaky confined aquifers is presented to take the advantages of both approaches, i.e., the partial discretization of FLM in spatial domain and the no discretization of the Laplace transform in time domain. After application of Laplace transform, the resulting system of equation governing flow is firstly solved by using the FLM, and the solution for drawdown is obtained in Laplace space. Then, the numerical inversion of the solution is performed using Durbin algorithm. The validity of the present method is verified through several numerical examples.

2 THE LAPLACE TRANSFORMED EQUATION OF FLOW

Fig. 1 is a schematic diagram of a typical leaky confined aquifer with dimensions 2a, 2b, and c. The aquifer are vertically heterogeneous and anisotropic. The x and y axes are in the horizontal directions and the z axis is in the vertical directions respectively. The origin is at the top of the aquifer. The lateral and bottom boundaries are constant-head and impermeable respectively. We assume that the influence of the storage water from the aquitard on drawdown is negligible, and the initial drawdown is zero. According to Darcy's law and conservation of mass equation, the governing equation and the associated initial and boundary conditions for unsteady flow in leaky confined aquifers are:

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where is drawdown; and are the hydraulic conductivities in the , and directions, respectively; is specific storage; is leakage parameter, and and are the thickness and the hydraulic conductivity of aquitard; the sink/source term give the rate of water withdrawal or injection per unit volume.

After applying the Laplace transform to Eq. (1), the weighted residual equations associated with the unknown coefficients as weight functions, and the weighted residual equations are as follow

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial x} \left(K_\phi \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_\phi \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_\phi \frac{\partial \phi}{\partial z} \right) - \frac{s}{\lambda^2} + \frac{q(x, y, z, t)}{\lambda^2}
\end{array} \right\} = S_\phi \frac{\partial \phi}{\partial t}
\]

where is the drawdown in Laplace space; is the unknown coefficients associated with the nodal plane ; is the standard linear shape function the direction; , , are the hydraulic conductivity matrix, and the tridiagonal entries in are

### 3 LTFL FORMULATION

#### 3.1 Finite-layer model

The layered aquifer can be further discretized horizontally into sublayers shown in Fig. 1. The upper and lower boundary planes of the sublayer with thickness are nodal planes and whose vertical coordinates are and respectively, so that and .

The trial function of drawdown in Laplace space can be represented in series form as by

\[
\Phi(x, y, z, p) = \sum_{j=1}^{L+1} \sum_{i=1}^{M} \sum_{n=1}^{N} \phi_{mnj}(p)A_{mn}(x, y)N_j(z)
\]

where is the drawdown in Laplace space; is the unknown coefficients associated with the nodal plane ; is the standard linear shape function the direction; , , are the hydraulic conductivity matrix, and the tridiagonal entries in are

\[
[ \tilde{G} ]_{mn}^f \{ \tilde{\phi} \}^f_{mn} + \{ \tilde{G} \}^f_{mn} = 0
\]

where is the unknown coefficient vector; is the flow vector corresponding to in Eq. (4); is the conductivity matrix, and the tridiagonal entries in are
For fully penetrating wells (FPWs) at \((x_0, y_0)\), the entries \(\overline{q}\) of \(\{\overline{G}\}\) can be obtained from the following equation

\[
\overline{q} = \frac{c_i}{2p} q(x_0, y_0) \sin[k_x(a + x_0)] \sin[k_y(b + y_0)]
\]

where \(q(x_0, y_0) = Q(x_0, y_0)/c\), \(Q(x_0, y_0)\) is the pumping rate of the well.

### 3.3 The global finite layer equations

On the basis of all conductivity matrices \([\overline{G}]_{mn}\) and flow vectors \([\overline{Q}]_{mn}\) obtained above, the global conductivity matrix \([\bar{G}]_{mn}\) and the global flow vector \([\bar{Q}]_{mn}\) can be obtained (Zhu et al. 2008), and hence the global finite layer equation with each \((m, n)\) can be obtained

\[
[\bar{G}]_{mn} [\bar{\phi}]_{mn} + [\bar{Q}]_{mn} = 0
\]  

(12)

After solving the tridiagonal systems of linear equations (12), the drawdown \(\bar{\pi}(x, y, z, t)\) in Laplace space can be consequently obtained.

### 4 The Numerical Inversion of the Laplace Transform Solution

In order to obtain the drawdown solution in the original time domain, the inverse Laplace transform should be conducted. The results in original time domain are obtained numerically by various existing inverse methods such as Stehfest algorithm (1970), de Hoog algorithm (1982) and Durbin algorithm (1973). In this work, the inverse transforms is conducted by Durbin algorithm. Based on the works of Dubner and Abate, and in order to decrease computation time, Drubin developed the following inversion formula via the fast Fourier transform (13)

\[
f(t_o) = \frac{2e^{\text{max}}}{T} \left(-\frac{1}{2} \text{Re}[F(a)] + \text{Re}\left(\sum_{k=0}^{N-1} (A_k + iB_k)\left(\cos\frac{2\pi}{T} + i\sin\frac{2\pi}{T}\right)^k\right)\right)
\]

(13)

in which

\[
A_k = \sum_{l=0}^{N} \text{Re}[F(a + ik(l + N))\frac{2\pi}{T}]
\]

(14)

\[
B_k = \sum_{l=0}^{N} \text{Im}[F(a + ik(l + N))\frac{2\pi}{T}]
\]

(15)

where \(T = 2t_{\text{max}}\), \(t_o = \eta\Delta t = \eta T/N\), \(\eta = 0, 1, L, l - 1\); \(\text{Re}\) and \(\text{Im}\) denote the real and imaginary, respectively.

The drawdown in original time domain can be represented by trial function in Eq. (5)

\[
s(x, y, z, t) = L^{-1}\left(\bar{\pi}(x, y, z, p)\right)
\]

\[
= \sum_{l=1}^{L} \sum_{m=1}^{N} L^{-1}\left(\bar{\phi}_{mn}(a)\right)A_{mn}N_j
\]

\[
= \sum_{l=1}^{L} \sum_{m=1}^{N} \phi_{mn}(t)A_{mn}N_j
\]

(16)

where \(L^{-1}\) denote Laplace inverse transformation; \(\phi_{mn}(t)\) is the inverse transform coefficients of \(\bar{\phi}_{mn}(p)\).

According to Eq. (16), the inverse transform of \(\bar{\pi}\) is, in essence, the inverse transform of \(\phi_{mn}(p)\). Using Durbin algorithm, the detailed method to obtain the original drawdown is as follows:

1. Assign \(a\) an value from the interval \([5/T, 10/T]\) and \(L \times N\) a value from 50 to 5000;
2. After substituting \(a\), \(\bar{\phi}_{mn}(a)\) can be obtained by solving Eq. (12);
3. For a specific \(k\), the parameter \(\beta_{il}\) can be determined from the following equation

\[
\beta_{il} = a + ik(l + N)\frac{2\pi}{T} \quad (l = 0, 1, L, L)
\]

(17)

4. By substituting \(\beta_{il}\) into Eq. (12), \(\bar{\phi}_{mn}(\beta_{il})\) can be obtained in the same way as step (2), and \(A_{k}, B_{k}\) can be further obtained.
5. The numerical inversion of \(\phi_{mn}(t)\) is given by (Durbin 1973)

\[
\phi_{mn}(t) = \frac{2e^{\text{max}}}{T} \left(-\frac{1}{2} \text{Re}[F(a)] + \text{Re}\left(\sum_{k=0}^{N-1} (A_k + iB_k)\left(\cos\frac{2\pi}{T} + i\sin\frac{2\pi}{T}\right)^k\right)\right)
\]

(18)

where \(\eta\) can be selected according to \(t\), if \(t = \text{t}_{\text{max}},\) then \(\eta = T/(2\Delta t)\).
6. The solution \(s(x, y, z, t)\) to Eq. (1) in time domain can be obtained by substituting \(\phi_{mn}(t)\) into Eq. (16).

Because the drawdown is represent by continuous functions in the horizontal planes, the drawdown at any position and time can be obtained after the
numerical inverse of \( \phi_{mon}(t) \) is achieved. The limitations of traditional full-discretization method that the inverse Laplace transform can only be performed at discrete points can be overcome.

5 NUMERICAL EXAMPLES

In order to verify the validity of the present LTFL scheme for unsteady flow in leaky confined aquifers, several specific numerical tests have been conducted. The LTFL numerical results are compared with the published analytical solutions and the numerical solutions obtained using the finite difference method (FDM). The hydraulic and geometric parameters defining the numerical examples are shown in Table 1. It should be noted that the analytical solutions were obtained under the assumption of infinite lateral extent. In order to approximate this infinite boundary, large horizontal dimensions \((a\equiv b\equiv 2000\text{m})\) are used in the following numerical tests. For the effect of finite horizontal size on drawdown, see Zhu et al. (2008).

Table 1. The hydraulic and geometric parameters

| \( c/m \) | \( S_a/(m^3) \) | \( K/(m/d) \) | \( Q/(m^3/d) \) | \( \lambda/(m^2) \) | \( d/(m^2) \) |
|---|---|---|---|---|---|
| FPWs | 60 | 3.3\times10^{-3} | 5,5,5 | 2000 | 100-1000 |
| | | | 5,2,5 | | 500 |
| | | | 5,5,1 | | |
| PPWs | 50 | 2.0\times10^{-3} | 5,5,5 | 1000 | 500 |
| | | | 5,5,1 | | |

5.1 The drawdown in the isotropic leaky aquifers

The radial flow to a fully penetrating well in the isotropic leaky aquifers is considered using the present method, and the obtained results are shown in Fig. 3. Good overall agreement among FLM, FDM and analytical results (Jacob 1946) can be observed, and the validity of the present method is verified. Because the LTFL method involves no discretization in time, it not only improves computation efficient but also avoids error propagation in the time-stepping methods. The numerical results for different values of the leakage parameter \( \lambda \) are presented and compared with the analytical and FDM solutions in Fig. 4. As \( \lambda \) increasing from 100 to 1000, the effect of leakage on the drawdown response in the aquifers becomes negligible and the drawdown curve for leaky confined aquifers tend to the one for confined aquifers without leakage (Wang et al. 2008).

5.2 The drawdown in the anisotropic leaky aquifers

The hydraulic anisotropy of the aquifers can significantly affect the flow behaviour. The LTFL solutions which accounts for the effect of anisotropy \((K_x/K_y=2.5)\) is presented, and the new numerical solutions agree well with published analytical and FDM solutions in all cases. As shown in Fig. 5, the drawdown in the \( x \) direction with larger permeability is significantly smaller than the one in the \( y \) directions. In order to further investigate the influence of anisotropy on 3D flow pattern, the case of PPWs with well screen length 20m has been considered. The numerical results at different elevations are presented in Fig. 6 in which the comparison with the FDM and previous analytical solution (Hantush 1964) is also carried out. It can be shown that the drawdown at both well screen top and aquifer bottom is sensitive to the hydraulic anisotropy of aquifers, especially near the partially penetrating well. Because the analytical approach can only be applied to the simple idealized cases, the present method can provide a useful way to analyze the 3D flow problems in layered anisotropic aquifers.
Figure 5. The distance-drawdown curves for FPWs in the anisotropic leaky aquifers

Figure 6. The distance-drawdown curves for PPWs in the anisotropic leaky aquifers

6 CONCLUSIONS

A semi-analytical numerical approach, i.e. Laplace transform FLM, has been developed for groundwater flow in anisotropic leaky confined aquifers by the Laplace transform and the numerical Laplace inversion algorithm of Durbin. Through comparison with analytical solutions of previous studies and FDM solutions, the validity of the present method is shown. The applicability of the present method for the 3D flow in anisotropic leaky aquifers is further demonstrated. The present LTFL method can overcome the limitations of traditional numerical method in the pointwise inverse Laplace transform, and the unsteady drawdown at any time can be obtained without discretization and iteration procedure in the time domain.

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