We study neutrino oscillations in a rotating spacetime under the weak gravity limit for the neutrino trajectories constrained in the equatorial plane. Using the asymptotic form of the Kerr metric, we show that the gravitational source’s rotation non-trivially modifies the neutrino phase. Further, we find that the flavor transition probability deviates more prominently from the Schwarzschild spacetime results when neutrinos are produced and detected on the same side of the gravitational source, i.e., neutrino propagates only towards or away from the gravitational source. For the neutrino weak gravity lensing in the equatorial plane, we find that the effects of gravitational object spin on the neutrino phase are subdominant compared with the Schwarzschild spacetime neutrino phase. However, the rotational effect can be comparable with the gravitational mass effect in the neutrino lensing if the naked singularity condition that bounds the Kerr rotation is violated. Furthermore, we find that the neutrino phase for the non-radial neutrino emission depends on the relative angular momentum direction of the neutrino and the gravitational source. For an asymptotic observer, we find that the neutrino phase, in the leading orders of gravitational spin corrections, differs from the Schwarzschild spacetime due to the difference in the proper spatial distance of the neutrino trajectory that connects the neutrino source and the detector. In contrast, the higher-order terms, other than those arriving from the proper spatial distance, which lead to the deviation from the Schwarzschild spacetime neutrino phase, are significantly suppressed. Further, we discuss neutrino gravitational decoherence. We find that decoherence length is also sensitive to the relative direction of the neutrino angular momentum and the gravitational source rotation. Finally, we demonstrate results using numerical examples.
I. INTRODUCTION

It is now well established that neutrinos have non-zero mass, and the neutrino flavor eigenstates are not the same as their mass eigenstates \[1,3\]. Due to this, a neutrino can oscillate between their flavors, a phenomenon known as neutrino flavour oscillation. Neutrino oscillations have been studied in the literature in various gravitational physical setups. For example, neutrino flavor oscillations in stochastic gravitational waves have been studied in \[41\], while neutrino decoherence from quantum gravitational stochastic perturbations has been studied in \[5\]. Effects like spin-flip or helicity transitions are also studied in a gravitational setup \[6-9\]. Further, the gravitational effects on neutrino oscillations have been investigated in various backgrounds \[10-17\]. Neutrino oscillations and effects on the beta reactions have been analyzed in the accelerated frames and in the contexts of modified theories of gravity and quintessence \[18-24\]. Gravitational induced neutrino-antineutrino oscillations have also been investigated \[25,26\]. Quantum gravity effects and the possible violation of the equivalence principle are studied in \[27-31\].

In previous work \[32\], neutrino lensing in the Schwarzschild geometry under the weak gravity limit has been studied where it was shown that, unlike in the flat spacetime neutrino, neutrino flavor transition probability in the weak gravity neutrino lensing contains the information about the absolute neutrino mass and their hierarchy. In a curved spacetime, multiple geodesics that connect the neutrino source and the detector can contribute to the massive neutrino wavefunction. Further, these contributions from the multiple paths in the neutrino mass eigenstates wavefunction interfere to produce a neutrino flavor transition pattern that depends on the absolute neutrino mass. In subsequent work \[33\], decoherence effects in neutrino lensing arising due to the introduction of the finite width of the neutrino source and detector were studied, and decoherence length in a gravitational setup was quantified. Further, it was illustrated that even inferring the neutrino decoherence length can reveal vital information about the absolute neutrino mass. This is because, in addition to the neutrino source and detector wavepacket width and peak local momentum, the decoherence length also depends on the absolute neutrino masses.

In this paper, we study neutrino oscillations in the rotating compact gravitational source spacetime under the weak-gravity limit for a possible deviation in the neutrino propagation from the neutrino source to the detector that may significantly change the Schwarzschild spacetime findings of the neutrino mass hierarchy signature in the gravitational lensing and the neutrino flavor transition probability. Although neutrino oscillations in the rotating spacetime have not yet been fully explored, some limited theoretical study of neutrino oscillations has been done in \[34\] for the Kerr-Neumann spacetime. We know that the Kerr spacetime is an exact solution of the Einstein fields equation for a stationary rotating black hole with axial symmetry around the axis of rotation. However, the metric for all rotating compact gravitational sources is not uniquely known and does not uniquely correspond to the Kerr metric. Nevertheless, asymptotically, the metric of all compact rotating gravitational sources approximately reduces to the asymptotic metric form of the Kerr black hole \[35-36\]. Therefore, we study neutrino oscillations in the weak gravity regime using the asymptotic Kerr spacetime. Further, due to the frame-dragging, particles can have non-planar trajectories in the rotating spacetime \[37-39\], unlike in the Schwarzschild spacetime where non-planar trajectories were forbidden due to spherical symmetry. These new non-planar trajectories can now contribute to neutrino’s overall wavefunction through the neutrino phase. Moreover, non-planar trajectories can now interfere with the other trajectories in a non-trivial manner and are likely to modify the neutrino flavor transition probabilities non-trivially.

Further, these deviations are expected to be significant in the spacetime of a highly distorted gravitational object. This paper explores neutrino flavor oscillations in the asymptotic Kerr spacetime in the equatorial plane. We show that for the one-sided propagation, i.e., when neutrinos only propagate either towards or away from the gravitational source, the neutrino phase gets significantly modified by the gravitational source’s rotation. Further, we show that non-radial neutrino emission, unlike in the radial neutrinos emission, is also sensitive to the relative directionality of the neutrino angular momentum and the gravitational source rotation. Furthermore, we study the distinction between the neutrino flavor oscillations in the asymptotics Kerr spacetime and the Schwarzschild spacetime. Further, the choice of \(\theta = \pi/2\) in this study is justifiable in the weak gravity region for neutrinos which only propagate either towards or away from the gravitational source. As the detector and the source located on the one side of the gravitational source in the equatorial plane would only receive neutrinos, which were only produced and propagated in the equatorial plane \[37\]. Furthermore, neutrinos produced in the equatorial plane with initial momentum in the \(\theta = \pi/2\), unlike in the general neutrino propagation, do not change their plane throughout their journey. Though, in neutrino lensing, the choice of the equatorial plane is not justifiable, where non-planar trajectories can reach the detector that lies in the equatorial plane and interfere with other trajectories \[37\]. Nevertheless, for the \(\theta = \pi/2\) neutrino propagation, we have shown that the gravitational rotational parameter does not significantly change the neutrino lensing phase in a weak gravity limit due to the naked singularity constraint on the Kerr rotation. Hence, we argue that the deviation in the neutrino lensing phase due to the non-planar trajectories will not deviate from the Schwarzschild spacetime neutrino lensing results \[32\]. Therefore, it will not disturb the mass hierarchy signature in the neutrino lensing. Further, we discuss neutrino decoherence aspects in the Kerr spacetime. We also estimate decoherence length for the typical numerical examples used in the paper.
where $\Delta \Phi$ produced in weak processes in a flavor eigenstate ($\alpha$ oscillations formalism using the plane wave approximation, which is required for our present study. Neutrinos are applied explicitly in Schwarzschild spacetime for the neutrino lensing. Here, we briefly overview the necessary neutrino and under the weak gravity regime in [32, 40], while the wave packet formalism has been discussed in [33] and is conclude the results. In section V, we discuss weak gravity neutrino lensing the Kerr spacetime. Finally, VI, we neutrino case, which is followed by the discussion on the neutrino flavor decoherence effect due to the finite neutrino

Wherein in the subsection III A, neutrino radial emission by the stationary neutrino source is discussed. In section IV, numerical results are presented for the two flavor ZAMO neutrino source) is discussed, followed by the subsection III B, where non-radial neutrino

This paper is organized as follows. In section II, the formalism for neutrino oscillations in the curved spacetime is briefly reviewed. In section III we discuss the asymptotic form of the Kerr metric in the Boyer-Lindquist coordinates. Wherein in the subsection III A, neutrino radial emission by the stationary neutrino source ( local zero angular momentum observer (ZAMO) neutrino source) is discussed, followed by the subsection III B, where non-radial neutrino emission by the stationary neutrino source is discussed. In section IV numerical results are presented for the two flavor neutrino flavor mixing unitary matrix, $U$ is neutrino flavor mixing unitary matrix, $\alpha$ and $\tau$ and $i = 1, 2, 3$. As, in a general curved spacetime, a neutrino wavefunction can have contribution from multiple geodesics which connect the neutrino source and the detector; therefore, a neutrino flavor state propagation from source $S$ to detector $D$, located at $x_S$ and $x_D$ respectively, considering all the paths, is given as

$$|\nu_\alpha(t_D, x_D)\rangle = \sum_i U_{\alpha i}^* \sum_m N \psi_i^m(x_D, x_S)|\nu_i(t_S, x_S)\rangle,$$

where $m$ is the index for neutrino trajectory, which connects neutrino source and detector, N is the normalisation which, in general, depends on the path $m$ and $\psi_i^m(x_D, x_S)$ is the wavefunction for the $i$th massive neutrino which is evolved between the time of production ($t_S$) and detection ($t_D$). Under the plane wave approximation, i.e.,

$$\psi_i^m(x_D, x_S) = e^{-i\Phi_i^m(x_D, x_S)},$$

the neutrino flavor transition probability from initial produced $\alpha$ flavor to $\beta$ flavor at the detection point is given by

$$P_{\alpha \beta} = |\langle \nu_\beta | \nu_\alpha(t_D, x_D)\rangle|^2 = |N|^2 \sum_{i,j} U_{\beta j} U_{\alpha j}^* U_{\alpha i}^* \sum_{m,n} \exp(-i\Delta \Phi_{ij}^{mn}),$$

where,

$$|N|^2 = \left(\sum_i |U_{\alpha i}|^2 \sum_{m,n} \exp(-i\Delta \Phi_{ii}^{mn})\right)^{-1},$$

$\Delta \Phi_{ij}^{mn} = \Phi_i^m - \Phi_j^n$, and $\Phi_i = \int_S^D p_{\mu}^{(i)} dx^\mu$ is covariant phase and $p_{\mu}^{(i)}$ is the canonical conjugate momentum to the coordinate $x^\mu$ for the $i$th neutrino mass. For a particular case, i.e., in a scenario when there is only one classical available path $m = 1$ for the neutrino, the flavor transition probability from $\nu_\alpha \rightarrow \nu_\beta$ at the detection point becomes

$$P_{\alpha \beta} \equiv |\langle \nu_\beta | \nu_\alpha(t_D, x_D)\rangle|^2 = \sum_{i,j} U_{\beta j} U_{\alpha j}^* U_{\alpha i}^* \exp(-i(\Phi_i - \Phi_j)),$$

where the normalisation is $N = 1$. Further, for the two flavor case, the neutrino flavor transition probability from the initially produced flavor $e$ and to the flavor $\mu$ is given as

$$P_{e \mu} = \sin^2 2\alpha \sin^2 \left(\frac{\Delta \Phi_{12}}{2}\right),$$

where $\Delta \Phi_{12} = \Phi_1 - \Phi_2$ and the lepton mixing matrix is parametrised by the mixing angle $\alpha$ as

$$U = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$
III. NEUTRINO OSCILLATIONS IN ROTATING SPACETIME

We would like to obtain the neutrino phase for rotating compact star’s background. However, the metric for rotating compact stars is not uniquely known as there is no unique metric that corresponds for all rotating spacetime. Nevertheless, asymptotically, the metric of all compact rotating gravitational sources approximately reduces to the asymptotic metric form of the Kerr black hole \[[53, 56]\]. Therefore, one can study neutrino oscillations in the weak gravity regime using the asymptotic Kerr spacetime. Hence, we will use the asymptotic form of the Kerr metric to study neutrino oscillations in the weak gravity region. The metric for the Kerr geometry in the Boyer-Lindquist coordinates is written as

\[
ds^2 = \left(1 - \frac{2GMa}{\rho^2}\right) dt^2 + \frac{4GMar^2}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} d\rho^2 - \rho^2 d\theta^2 - \frac{\Lambda \sin^2 \theta}{\rho^2} d\phi^2,
\]

where \(\Delta(r) = r^2 - 2GMr + a^2, \rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta, \Lambda(r, \theta) = (r^2 + a^2)^2 - a^2 \sin^2 \theta \) and \(a = J/M\). \(G\) is the Newtonian constant, \(M\) is the mass and \(J\) is the angular momentum of the black hole, respectively. Further, the naked singularity constrains an upper bound on the \(a\)’s values, i.e., \(a < GM\). Furthermore, considering neutrino propagation only in the \(\theta = \pi/2\) plane (\(p^\theta = 0\)), the metric, in the limit \(a/r << 1\), ignoring \(O(a/r)^2\), has the following form

\[
ds^2 \simeq B(r) dt^2 + 2h(r) dt d\phi - \frac{1}{B(r)} dr^2 - r^2 d\phi^2,
\]

where \(B(r) = 1 - 2GM/r\) and \(h(r) = 2GJ/r\). Now, we shall discuss relativistic neutrino propagation, which is constrained in the \(\theta = \pi/2\) plane. Keeping with the standard null trajectory approximation \[[40]\] in literature, the integral equation for neutrino phase change from the source location \(r_\text{S}\) to the detector location \(r_\text{D}\) for the \(i\)th massive neutrino, which is produced with conserved energy \((E_0)\) and angular momentum \((L)\), is given by, see Appendix C

\[
\Phi_i = \frac{m_i^2}{2} \int_{r_\text{S}}^{r_\text{D}} \frac{h(r)E_0 + LB(r)}{(h(r)E_0 + LB(r)) \sqrt{(B(r)\frac{E_i(r)^2 - h(r)L}{h(r)E_0 + LB(r)})^2 + 2h(r)B(r)\frac{E_i(r)^2 - h(r)L}{h(r)E_0 + LB(r)} - r^2B(r)}} dr,
\]

where \(m_i\) is mass of \(i\)th neutrino. Although, in general, taking the null trajectory approximation in the neutrino phase, in the leading order of neutrino mass \(m_i\), gives only a half of the neutrino phase contribution as compared to the massive neutrino trajectory, i.e., \(2\Phi_i^{null} = \Phi_i^{massive}\), see Appendix A, this study will not change qualitatively if the neutrino trajectories are taken as massive trajectories. Further, in the curved spacetime, the energy and momentum of a particle are frame-dependent, and different observers perceive different four-momenta of the particle. In a locally specified observer’s frame, the local four momenta \(p^\mu\) of the particle is related to the coordinate four momenta \(p^\mu\) through local frame fields \(e^\mu_\nu\), known as tetrad fields, by the following relation \(p^\mu = e^\mu_\nu p^\nu\). Now, we shall first discuss a case when neutrinos are emitted radially by the stationary neutrino source. Then we will discuss when neutrinos are emitted non-radially.

A. Radial neutrino emission by the stationary neutrino source

Unlike in the Schwarzschild spacetime, in general, a particle can not propagate only along the radial direction in Kerr spacetime, even if produced with zero angular momentum \(L = 0\), with an exception for particles traveling along \(\theta = 0\) line, where the metric effectively reduces to the Schwarzschild metric in the weak gravity limit. For instance, in the asymptotic Kerr spacetime, null trajectory in \(\theta = \pi/2\) plane gives

\[
\frac{d\phi}{dr} = \frac{2GJ}{\sqrt{\left(1 - \frac{2GM}{r}\right)(r^6 - 2GMr^5 + 4G^2r^2J^2)}},
\]

which is non-zero even for the \(L = 0\) case. The trajectory gets dragged along the \(\phi\) coordinate due to the rotation of the gravitational source. Note the particle gets dragged in the same direction of gravitational source rotational. Further, neutrinos created with zero angular momentum as seen by the asymptotic observer will have, in general, non-zero angular momentum as seen by the local observer. This can be seen from the relation between \(p^\phi\) and conserved quantity \(E_0\) and \(L\), which is given as

\[
p^\phi = \left. \frac{E_0h(r) + B(r)L}{h(r)^2 + r^2B(r)} \right|_{L=0} = \frac{E_0h(r)}{h(r)^2 + r^2B(r)}.
\]
As can be seen for the $L = 0$, $p^\phi$ is not zero and depends on neutrino source location, except for the asymptotic observer. However, we can make the connection between the local energy and momenta ($p^a$) of the produced neutrino with the conserved energy and angular momentum of the neutrino using local tetrad fields $e^a_\mu$. Further, if neutrinos are emitted radially in the neutrino source’s rest frame, then only $p^a (a = 1, 2, 3)$ which is along the radial direction will be non-zero. For example, in the $L = 0$ case, the local frame source which satisfies such conditions, in the Kerr spacetime, is known as zero angular momentum observer (ZAMO) or stationary neutrino source. These are the special class of observers whose angular momentum ($L_O$) vanishes. Explicitly, the ZAMO tetrads [42, 43], in the weak gravity limit can be found as

$$\left( \begin{array}{c} e^0_\mu \\ e^1_\mu \\ e^2_\mu \\ e^3_\mu \end{array} \right) = \left( \begin{array}{cccc} \rho \sqrt{\Delta} & 0 & 0 & 0 \\ 0 & \sqrt{\Delta} & 0 & 0 \\ 0 & 0 & \rho & 0 \\ \frac{2Jr \sin \theta}{\rho \sqrt{\Delta}} & 0 & 0 & \sin \theta \sqrt{\Delta} \end{array} \right) \approx \left( \begin{array}{cccc} \sqrt{B(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{B(r)}} & 0 & 0 \\ 0 & 0 & r & 0 \\ \frac{2JG \sin \theta}{r^2} & 0 & 0 & r \sin \theta \end{array} \right),$$

(14)

where, $\mu$ can take values from $t, r, \theta, \phi$. These tetrad fields satisfies the following condition

$$g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu,$$

(15)

where $\eta_{ab}$ is dig($1, -1, -1, -1$) 4 × 4 matrix and $g_{\mu \nu}$ is the metric tensor. We note that the asymptotic observer also falls under the class of ZAMO observers. Further, using $p^i$ and $p^\phi$ in terms of conserved quantities $E_0$ and $L$, i.e.,

$$p^i = \frac{E_0 r^2 - h(r)L}{h(r)^2 + r^2 B(r)},$$

$$p^\phi = \frac{E_0 h(r) + B(r)L}{h(r)^2 + r^2 B(r)}.$$

(16)

We get local 4-momentum for the neutrino which is constrained in $\theta = \pi/2$ plane and created with $L = 0$ as

$$E_{loc}(r_S) \equiv p^0 = \sqrt{B(r_S)} p^i(r_S),$$

$$p^i = \frac{p^i(r_S)}{\sqrt{B(r_S)}}, \quad p^2 = 0, \quad p^3_{loc} \equiv p^3 = 0,$$

(17)

where $E_{loc}, p^1, p^2$ and $p^3_{loc}$ are the local energy, local momentum along the radial direction, local momentum perpendicular to $\theta = \pi/2$ plane and local tangential momentum of the neutrino respectively. To move further, taking $L = 0$ in Eq. [11] we get the phase change for the $i^{th}$ neutrino, considering $r_D > r_S$, see Appendix [C]

$$\Phi_i = \frac{m_i^2}{2E_0} \int_{r_S}^{r_D} \frac{(2J^2 - \rho^2 (1 - \frac{2GM}{r}))}{\sqrt{(\frac{2GM}{r^3} + 2GM^2r^3 + 4G^2J^2)}} dr,$$

(18)

where, using Eq. [17], $E_0$ is related to $E_{loc}$ as

$$E_0 = \frac{E_{loc} h(r_S)^2 + r_S^2 B(r_S)}{r_S^2 \sqrt{B(r_S)}}.$$

(19)

Further, the neutrino phase change, in the leading order of $J$ and $M$ after ignoring higher terms such as $O \left( G^4 M^2 J^2 / r^5 \right)$, is given by

$$\Phi_i \approx \frac{m_i^2}{2E_0} \left[ (r_D - r_S) - \frac{2GMJ^2}{3} \left( \frac{1}{r_D^3} - \frac{1}{r_S^3} \right) - G^3 MJ^2 \left( \frac{1}{r_D^4} - \frac{1}{r_S^4} \right) + O \left( \frac{G^4MJ^2}{r_S^5} \right) \right].$$

(20)

Here, we note that the neutrino phase depends explicitly on the gravitational source’s mass if $J \neq 0$. This is in contrast to the Schwarzschild case, where the neutrino phase, for the zero angular momentum case, depends only on the radial coordinate difference, and the gravitational dependence appeared only due to proper spatial distance of the neutrino trajectory [33] [11]. Now, the phase written in term of proper spatial distance ($L_p$) of the neutrino trajectory, under null approximation, can be written as

$$\Phi_i \approx \frac{m_i^2}{2E_0} \left[ L_p - \frac{R_s}{2} \ln \frac{r_D}{r_S} - \frac{G^3MJ^2}{4} \left( \frac{1}{r_D^4} - \frac{1}{r_S^4} \right) + O \left( \frac{G^4MJ^2}{r_S^5} \right) \right],$$

(21)
where $L_p$ is given

$$L_p = \int_{r_S}^{r_D} \sqrt{\frac{1}{B(r)} + \left(\frac{d\phi}{dr}\right)^2} \, dr \simeq r_D - r_S + \frac{r_D}{2} \ln \frac{r_D}{r_S} - \frac{2G^2J^2}{3} \left(\frac{1}{r_D^3} - \frac{1}{r_S^3}\right) - \frac{3G^3MJ^2}{4} \left(\frac{1}{r_D^4} - \frac{1}{r_S^4}\right) + \mathcal{O}\left(\frac{G^4M^2J^2}{r_{S,D}^6}\right),$$

(22)

We notice that the neutrino phase differs from the neutrino phase in Schwarzschild geometry in two aspects (1) The spatial distance ($L_p$) between the neutrino source and the detector, which is connected by the neutrino trajectory, is more as compared to the Schwarzschild spacetime due to the frame dragging along the $\phi$ coordinate, (2) The phase, Eq. (21), itself gets modified by the Kerr rotational terms $MJ^2/r^4$, whereas the logarithmic term in Eq. (21) remains the same as before for the Schwarzschild case. As both contributions in the phase are positive, for the $P_{loc} = 0$ case, both aspects compel the neutrino phase change in the Kerr geometry to be more than in the Schwarzschild geometry. Though the neutrino phase gets non-trivially modified due to gravitational source rotation, this effect, however, is of order $\mathcal{O}((GJ)^2/r^4)$, and hence, phenomenologically expected to be a minor correction in weak gravity.

B. Non-radial neutrino emission by the stationary neutrino source

![Diagram](image)

**FIG. 1.** Diagrammatic representation of the neutrino source and the detector setup, when neutrinos are produced with a non-zero angular momentum $L \neq 0$ and propagated away from the gravitational source. Here, $J$ and $L$ have different direction of angular momentum.

In this section, we consider neutrinos produced at $\vec{r}_S$ as a plane wave with conserved energy $E_0$ and angular momentum $L$ and detected by detector at $\vec{r}_D$, which lies on the same side as the neutrino source, see Fig. **FIG. 1**. As previously noted, angular momentum of a particle is observer dependent and related to the local tetrad fields. In the case when neutrinos are produced with non-zero conserved angular momentum ($L \neq 0$), a source frame tetrads, in which the neutrino is produced only along the radial direction, can be found using tetrad fields constraints imposed by the following relations

\[ p^\phi = e^\phi_\mu p^\mu = 0, \]

\[ p^3 = e^3_\mu p^\mu = 0, \]

(23)

whereas $e^1_\mu$ is such that, if $p^r = 0$, then this should imply $p^1 = 0$. This ensures the $e^1_\mu$ is locally directed along the radial direction only. For instance, if a local neutrino source has the following tetrad fields $\vec{e}^0_\mu = (e^0_t, 0, 0, e^0_\phi)$, $\vec{e}^1_\mu = (0, e^1_t, 0, 0)$, $\vec{e}^2_\mu = (0, 0, e^2_\phi, 0)$, and $\vec{e}^3_\mu = (e^3_t, 0, 0, e^3_\phi)$. Then for the neutrino trajectories constrained in $\theta = \pi/2$ plane ($p^\theta = 0$), we have the following constraint

\[ p^3 = 0 = e^3_\phi p^\phi + e^3_t p^t \Rightarrow \frac{e^3_\phi}{e^3_t} = \frac{p^t}{p^\phi} = -\frac{E_0 r^2 - h(r)L}{E_0 h(r) + B(r)L} \bigg|_{r=r_D}, \]

(24)
which can be written as

\[
\frac{L}{E_0} = \frac{e_3^2 r^2 + e_3^2 h(r)}{h(r)e_3^2 - B(r)e_3^2}.
\]  
(25)

The above equation relates the source’s frame with the conserved neutrino energy and angular momentum. Further, these new tetrads are related to ZAMO tetrads by a local lorentz boost along \(x^3\) direction \(e_3^a = \Lambda_{ij}^a e_i^a\). Lorentz matrix elements \((\Lambda_{ij}^a)\) can be found out from Eq. (25) in terms of \(L\) and \(E_0\). Moreover, all observers’ satisfying Eq. (25) at their location will perceive neutrinos as if they are emitted only along the radial direction. We, however, take the ZAMO observer’s frame (stationary neutrino source) as the local neutrino source frame. Now, the phase change for the massive neutrino \(i\) during the trajectory from \(\vec{r}_S\) to \(\vec{r}_D\) (\(\vec{r}_D > \vec{r}_S\)), and up to the first leading terms in mass \(M\) and the angular momentum \(J\) is given by, see Appendix C,

\[
\Phi_i \approx \frac{m_i^2}{2E_0} \left[ \sqrt{r_D^2 - \left( \frac{L}{E_0} \right)^2} - \sqrt{r_S^2 - \left( \frac{L}{E_0} \right)^2} + \left( GM - \frac{2GJE_0}{L} \right) \left( \frac{r_D}{\sqrt{r_D^2 - \left( \frac{L}{E_0} \right)^2}} - \frac{r_S}{\sqrt{r_S^2 - \left( \frac{L}{E_0} \right)^2}} \right) \right].
\]  
(26)

Now \(E_0\) and \(L/E_0\) are given in terms of neutrino local energy \(E_{\text{loc}}\) and local tangential momentum \(p^3 \equiv p^T_{\text{loc}}\) as

\[
E_0 = E_{\text{loc}} \left( h(r) r^2 + r^2 B(r) \right) \sqrt{B(r)} \left( r^2 - h(r) \frac{L^2}{E_0} \right),
\]

\[
\frac{L}{E_0} = \frac{r^3 p^T_{\text{loc}} \sqrt{B(r)} \left( r^2 h(r) E_{\text{loc}} + r \sqrt{B(r)} p^T_{\text{loc}} \right)}{B(r) E_{\text{loc}} r^2 + h(r) \left( h(r) E_{\text{loc}} + r \sqrt{B(r)} p^T_{\text{loc}} \right)},
\]  
(27)

whereas the relation \(p^i = p^T / \sqrt{B(r)}\) remains same. We notice that phase Eq. (26), unlike in the Schwarzschild geometry, is not invariant under \(p^T_{\text{loc}} \rightarrow -p^T_{\text{loc}}\). This statement is equivalently true in terms of \(L\). Consequently, the neutrino flavor transition probability will depend on whether neutrinos are produced with angular momentum \(L\) aligned or anti-aligned with the rotation of Kerr black hole. Further, writing the phase in term of the proper spatial distance of the neutrino trajectory, under the null approximation, we get

\[
\Phi_i = \frac{m_i^2}{2E_0} \left[ L_p - GM \left( \tanh^{-1} \sqrt{1 - \left( \frac{L}{E_0 r_D} \right)^2} - \tanh^{-1} \sqrt{1 - \left( \frac{L}{E_0 r_S} \right)^2} \right) \right],
\]  
(28)

where

\[
L_p = \int_{r_S}^{r_D} \frac{1}{\left( B(r) + \left( \frac{r \, d\phi}{dr} \right)^2 \right)} \, dr \simeq \sqrt{r_D^2 - \left( \frac{L}{E_0} \right)^2} - \sqrt{r_S^2 - \left( \frac{L}{E_0} \right)^2} + GM \left( \tanh^{-1} \sqrt{1 - \left( \frac{L}{E_0 r_D} \right)^2} - \tanh^{-1} \sqrt{1 - \left( \frac{L}{E_0 r_S} \right)^2} \right) - \frac{r_D}{\sqrt{r_D^2 - \left( \frac{L}{E_0} \right)^2}} \frac{r_S}{\sqrt{r_S^2 - \left( \frac{L}{E_0} \right)^2}} - \frac{2GJE_0}{L} \left( \frac{r_D}{\sqrt{r_D^2 - \left( \frac{L}{E_0} \right)^2}} - \frac{r_S}{\sqrt{r_S^2 - \left( \frac{L}{E_0} \right)^2}} \right),
\]  
(29)

we used the following relation to calculate \(L_p\)

\[
\frac{d r}{d \phi} = \sqrt{\left( B(r) \frac{p^T}{p^\phi} \right)^2 + 4B(r) \frac{GJ}{r} \frac{p^T}{p^\phi} - B(r) r^2}.
\]  
(30)

Further, we note that \(L_p\) is not invariant under \(p^T_{\text{loc}} \rightarrow -p^T_{\text{loc}}\) (using \(L/E_0\) relation from Eq. (27), i.e., now, the spatial distance travelled by the particle depends on the the particle tangential momentum direction. Furthermore, we note that neutrino phase’s first explicit dependence on \(J\), other than from \(L_p\), appears at \(O \left( G^2 J M E_0^2 / \left( L^2 \right) \right)\), and given as,

\[
\Phi_i = \frac{m_i^2}{2E_0} \left[ L_p - GM \left( \tanh^{-1} \sqrt{1 - \left( \frac{L}{E_0 r_D} \right)^2} - \tanh^{-1} \sqrt{1 - \left( \frac{L}{E_0 r_S} \right)^2} \right) - \frac{2G^2 J M E_0^2}{L^2} \left( \sin^{-1} \left( \frac{r_D}{E_0} \right) \left( \frac{r_D}{E_0} \right)^2 - \frac{r_D}{E_0} \left( \frac{r_D}{E_0} \right) \right) \right],
\]  
(31)
FIG. 2. Probability $\nu_e \rightarrow \nu_\mu$ conversion in the two flavor neutrino case as a function of $c p^T_{loc}/E_{loc}$ for different values of the rotational parameter $a \equiv J/cM$, taking $R_s = 30$ km. Other parameters are taken as $r_S = 10^5$ km, $\alpha = \pi/4$, $r_D = 10^{11}$ km, $E_{loc} = 10$ MeV, $m_1 = 0$ eV/c$^2$ and $\Delta m^2 = 10^{-3}$ eV$^2/c^4$.

FIG. 3. Probability $\nu_e \rightarrow \nu_\mu$ conversion in the two flavor neutrino case as a function of $c p^T_{loc}/E_{loc}$. (1) $a \neq 0$, and (2) $a = 0$, i.e., Schwarzschild case. Here $R_s = 40$ km, while other parameters are kept same as in the Fig. 2.

where the corresponding proper spatial distance $\tilde{L}_p$ is given by

$$\tilde{L}_p = L_p - \frac{2G^2 J M E_0^2}{L^2} \left[ \frac{L \left( 4r^2 - 5 \frac{L^2}{E_0} \right)^{3/2}}{E_0 \left( r^2 - \frac{L^2}{E_0} \right)^{3/2}} \right] \left[ 4 \sin^{-1} \left( \frac{L}{E_0 r^2} \right) \right]^{1/2}.$$

(32)

Note, $O \left( G^2 J M E_0^2/(L^2) \right)$ is subdominant and suppressed term in the weak gravity. Hence, effectively in the weak gravity limit, for an asymptotic observer, the distinction between the neutrino phase in the Schwarzschild and the asymptotic Kerr spacetime emerges only from the difference in the proper spatial length along the neutrino trajectory which connects the neutrino source and detector.

IV. NUMERICAL RESULTS

In this section, we demonstrate previously discussed neutrino phase modification in the Kerr background using numerical examples and compare it with the Schwarzschild background. For this purpose, we consider two neutrino flavor case and compare neutrino flavor transition probability in the aforementioned spacetimes. The probability expression is given as

$$P_{e\mu} = \sin^2 2\alpha \sin^2 \left( \frac{\Delta \Phi_{12}}{2} \right).$$

(33)
In the expected appearance of the mu neutrino flavor transition probability at the detector. Further, we also see that neutrinos produced with different neutrino source locations \(r_S\) give different appearance probability of neutrino flavor detected at the detector location for, (1) \(\alpha = 0\) and (2) the Schwarzschild case \(\alpha = 0\) are clearly distinguishable. Further, in Fig. 4 we have plotted

\[
\Delta P_{e\mu} \equiv P_{e\mu}(J \neq 0) - P_{e\mu}(J = 0) = \sin^2 2\alpha \left[ \sin^2 \left( \frac{\Delta \Phi_{12}}{2} \right) - \sin^2 \left( \frac{\Delta \Phi_{12}}{2} \right) \right]_{J=0},
\]

as a function of rotational parameter \(a\) for the different neutrino source location \(r_S\). Here, \(\Delta P_{e\mu}\) is the difference in the expected appearance of the \(\mu\) flavor neutrino in the asymptotic Kerr \((J \neq 0)\) and Schwarzschild \((J = 0)\)
background by the detector. The left panel shows the plot for $R_s = 10$ km, whereas the right panel is plotted for the $R_s = 20$ km. Further, $\Delta P_\mu > 0$ means more chance for the $\mu$ flavor neutrino to be detected by the detector in the asymptotic Kerr spacetime in comparison to the Schwarzschild spacetime while for the $\Delta P_\mu < 0$ opposite statement will be true. In the $\Delta P_\mu = 0$ case, we cannot distinguish rotating spacetime from the Schwarzschild using neutrino flavor flux received at the detector. The plots show that the neutrino $\mu$ flavor detected by the detector can be either more or less than the Schwarzschild spacetime and depends on the neutrino source location and the rotational parameter. For example, in the left panel, we see that for the neutrino source located at $r_s = 10^6$ km in the Kerr geometry, the detector will detect more $\mu$ flavor neutrino as compared with the case when neutrinos are produced in the Schwarzschild background. In contrast, for the neutrino source located at $r_s = 5 \times 10^5$ opposite is true. Similarly, we see the same conclusion for $R_s = 20$ km in the right panel. However, now neutrino transition probability $\nu_e \rightarrow \nu_\mu$ has been drastically changed compared to the left panel.

Up to now, we have considered neutrino oscillations in the asymptotic Kerr spacetime assuming neutrinos as plane waves. However, neutrinos are produced and detected in a real-world scenario as wave packets. Introducing wave packet formalism introduces a new length scale $L_D$, after which neutrino oscillations cease to exist. This new length scale is known as decoherence length. For the Gaussian profile, the decoherence length is given by the following relation [33].

$$D_{21} = \frac{\sigma^2}{2} \left( |\vec{X}_2|^2 - |\vec{X}_1|^2 \right) \geq 1,$$

(37)

where $\vec{X}_1 = \partial_\vec{p}' \Phi_i (\vec{p} = \vec{p}_i^0)$, $\sigma^2 = \sigma_1^2 \sigma_2^2 / (\sigma_0^2 + \sigma_0^2)$ and $D_{21}$ is the decoherence factor. Here, $\vec{p}_i^0$ is the peak momentum of the neutrino in the local stationary frame, whereas $\sigma_0$ and $\sigma_D$ are the standard deviations in the momentum space for the neutrino and the detector wave profiles. Further, the peak momentum of the detector profile $\vec{p}_D$ and the source $\vec{p}_i^0$ are assumed to be equal in the local frame, i.e., $\vec{p}_i^0 = \vec{p}_D = \vec{p}_S^0$. Now, to estimate decoherence length, the neutrino phase (Eq. 28) for a large proper spatial distance between the neutrino source and detector can be approximated as

$$\Phi_i = \frac{m_i^2}{2E_0} L_p \left( 1 - \frac{GM}{L_p} \left( \tanh^{-1} \left( 1 - \left( \frac{L}{E_0 r_D} \right)^2 \right) - \tanh^{-1} \left( 1 - \left( \frac{L}{E_0 r_s} \right)^2 \right) \right) \right) \approx \frac{m_i^2}{2E_0} L_p.$$  

(38)

Then, using Eq. [27, 37], the proper spatial distance ($L_D$), in the leading correction to the Schwarzschild spacetime, which neutrinos will travel before they decohere, is given by

$$L_D \gtrsim 2\sqrt{2} \left( \sqrt{B(r_s)} + \frac{4GJp_{loc}^T}{r_s^2 E_{loc}} \right) \frac{E_{loc}^2}{\sigma \sqrt{m_1^2 - m_1^2}}. $$

(39)

We see coherence distance $L_D$ is also sensitive to the neutrino tangential momentum $p_{loc}^T$ (equivalently $L$) and the gravitational source rotation. Now, for the typical parameters used in the paper $\Delta m^2 = 10^{-3}$ eV$^2$, $m_i = 0$, $p_{loc}^T/E_{loc} = 0.1$, $E_{loc} = 10$ MeV, $R_s = 30$ km, $a = 10$ km and $r_s = 10^5$ km, the decoherence length $L_D$ for $\sigma_D/E_{loc} = \sigma_S/E_{loc} \approx 10^{-12}$ is of order $L_D \approx 10^{13}$ km. Hence, after this characteristic length $L_D$ between the source and the detector, all the neutrino oscillations effects will be washed away. Therefore, one requires precise information of the neutrino and the detector profile to estimate the observability of these effects in the neutrino oscillation experiments.

V. NEUTRINO WEAK GRAVITY LENSING IN THE ASYMPTOTIC KERR SPACETIME

In the weak gravitational neutrino lensing in the asymptotic Kerr spacetime, we find that lensing in the $\theta = \pi/2$ plane does not significantly change the Schwarzschild spacetime neutrino lensing results. In our earlier work, we had shown that neutrino lensing contains the signature of the neutrino mass hierarchy [32]. Further, the lensed neutrino phase in Kerr spacetime, under weak gravity limit, is found to be, see Appendix C3

$$\Phi_i^m = \frac{m_i^2}{2E_0} (r_S + r_D) \left( 1 - \frac{b_m^2}{2r_S r_D} + \frac{2GM}{r_S + r_D} - \frac{4GJ}{b_m(r_S + r_D)} \right).$$

(40)

where $b_m$ is the impact parameter, which is now constrained by the geometrical parameters $(r_S, r_D, R_s, a)$ and on the deflection angle $\delta$ of the neutrino in the weak gravity. Although the neutrino phase gets modified by the gravitation source’s rotational term $4GJ/b_m(r_S + r_D)$, this new term does not significantly affect the Schwarzschild lensing results. This is due to the following reasons:
- The rotational term $4GJ/b_m(r_s + r_D)$ in the neutrino phase is subdominant term. This can be seen from the ratio of $|4GJ/b_m(r_s + r_D)|$ and $2GM/(r_s + r_D)$, which is equals to $2a/b_m$. Further, $a$ is bounded by the naked singularity condition, i.e., $a < GM$. Therefore, in the weak gravity limit, we conclude following

$$\left|\frac{4GJ}{b_m(r_s + r_D)}\right| \ll \left|\frac{2GM}{r_s + r_D}\right| \implies \left|\frac{2a}{b_m}\right| \ll 1. \quad (41)$$

- Further, the impact parameter $b_m$ can be shown to be equal to the deflection angle $\delta$ in the weak gravity limit as $\delta \simeq -4GM/b + O(GJ/b^2) \ [38,39]$. Therefore, we see $\delta$ will be modified in the order $GJ/b^2$, which is again a very suppressed term as $|a/b| \ll 1$. Hence, effectively $\delta$ remains the same as in the weak gravity limit of Schwarzschild spacetime. Consequently, the impact parameter would not significantly modify the neutrino phase due to the gravitational source’s rotation in the weak gravity limit.

Now, for the numerical comparison, we take typical parameters used in the Schwarzschild neutrino lensing paper for the Sun and the Earth system $[32]$, i.e., $r_s + r_D \approx 10^{13}$ km, while $|b_m|$ was typically of order $10^3$ km. Further, we take extremal rotational parameter for the Sun $a = R_s/2$, where $R_s = 3$ km. Then, we have $|2a/b_m| \approx 10^{-3} \ll 1$. Hence, the Phenomenological results remain insignificant even if we consider the Sun’s rotation. Further, $L = 0$ trajectories do not contribute to weak gravity lensing even if they have non-zero $dr/d\phi$. This is because these trajectories, even if started in the weak gravity region, will not remain in the weak gravity region, see Appendix C.3.a. Furthermore, the rotational effect can be comparable with the gravitational mass effect in the weak gravity limit neutrino lensing if the naked singularity condition that bounds the Kerr rotation is violated arbitrarily, i.e., if $a$ is chosen such that $|2a/b_m| \sim R_s$. Here, we stress that thought neutrino lensing results remained unchanged for the neutrino trajectory in the $\theta = \pi/2$ plane. However, the neutrino lensing is fundamentally different in the Kerr spacetime if we consider all the neutrino trajectory that connects the neutrino source and the detector. As particles can have non-planar trajectories due to frame dragging (or broken spherical symmetry) in the Kerr spacetime, i.e., particles can now change planes during their journey. These non-planar trajectories of particles can now reach the detector, which was impossible in the Schwarzschild spacetime, where particles can take only planar trajectories. However, as for the $\theta = \pi/2$ plane, the deviation from the Schwarzschild spacetime was not phenomenologically significant. Hence, non-planar trajectories in Kerr spacetime should not significantly deviate the results in the weak gravity neutrino lensing. Nevertheless, one must account for all neutrino trajectories in the neutrino lensing in the broken spherical symmetry spacetimes, such as highly asymmetric mass distribution spaces, where non-planar trajectories are expected to play a significant role.

### VI. CONCLUSION

Neutrino oscillation has been studied in the literature under various gravitational setups $[32,33]$. This study has examined neutrino flavor oscillations in the equatorial plane for the following cases in the rotating spacetime (1) when neutrinos are produced and detected on the same side of the gravitational source, i.e., neutrino propagates only towards or away from the gravitational source, and (2) when the neutrinos are lensed by the gravitational source in the weak gravity region. Using the asymptotic form of the Kerr metric, we showed that the neutrino phase gets non-trivially modified due to the gravitational source rotation. Further, due to the frame-dragging effect in the Kerr spacetime, neutrinos emitted radially by the stationary neutrino source get an explicit modification in the neutrino phase from the gravitational source angular momentum ($J$) and mass ($M$) at the level of radial coordinate. This contrasts with Schwarzschild’s spacetime, where, in the radial neutrino emission case, the neutrino phase did not depend explicitly on the gravitational mass at the level of the radial coordinate $[32,33,40,41]$. Furthermore, we find that the neutrino phase in the non-radial emission of neutrino depends on the relative direction of the neutrino angular momentum and the gravitational source’s rotation, as the neutrino phase is not invariant under $L \rightarrow -L$, or equivalently $p_{loc}^T \rightarrow -p_{loc}^T$. For an asymptotic observer, we find that the neutrino phase, in the leading orders of gravitational spin corrections, differs from the Schwarzschild spacetime due to the difference in the proper spatial distance of the neutrino trajectory that connects the neutrino source and the detector, as the higher-order terms are significantly suppressed. Further, we explicitly demonstrated the effects of the gravitational source rotation on the neutrino flavor transition probabilities using numerical examples for the two flavor neutrino case. It is seen that neutrino flavor transition probability is also sensitive to the relative gravitational source rotation and neutrino angular momentum direction, as was anticipated from the neutrino phase. Furthermore, we showed that neutrino flavor appearance probability could be either equal, more, or less than as compared with the Schwarzschild spacetime, depending on the neutrino source and detector location. Further, we discussed decoherence effects due to the introduction of the finite width wave packet. We found that decoherence length in the rotating spacetime is sensitive to the relative direction of the neutrino angular momentum and the gravitational source rotation. Furthermore, for the lensed neutrino, we find that the gravitational source rotation does not substantially modify neutrino lensing results of the Schwarzschild spacetime $[32]$. We argued...
that the gravitational source’s rotational effect in the neutrino phase is subdominant due to the naked singularity constraint on the rotational parameter \[36\]. However, the rotational effect can be comparable with the gravitational mass effect in the weak gravity neutrino lensing if the naked singularity condition that bounds the Kerr rotation is arbitrarily violated, i.e., if \(a\) is chosen such that \(|2a/b_m| \sim R_s\). Finally, we discuss the distinction in the neutrino lensing for a broken spherical symmetric spacetime and the Schwarzschild spacetime. In spacetimes where the spherical symmetric is broken, particles generally have non-planar geodesics, which was not possible in the Schwarzschild spacetime due to the spherical symmetry of the spacetime. Therefore, in the Kerr spacetime, the non-planar paths connecting the neutrino source and detector will also contribute to the neutrino wavefunction via the neutrino phase. However, these non-planar trajectories in asymptotic Kerr spacetime should not significantly change the Schwarzschild results in the weak gravity neutrino lensing like in the \(\theta = \pi/2\) plane, where the deviation in the neutrino phase due to the gravitational object rotation is not significant enough to change the Schwarzschild spacetime results. However, these non-planar trajectories are generally expected to play a considerable role in a spacetime with highly distorted mass distribution. Therefore, in neutrino lensing, one has to take account of all the trajectories to find the neutrino flavor transition probability, and the effects of non-planar trajectories should be assessed before discarding them.

VII. ACKNOWLEDGEMENT

HS thanks Kinjalk Lochan and Ketan M. Patel for the invaluable comments and suggestions during the manuscript’s preparation and careful readings of the manuscript. HS would also like to thank Council of Scientific & Industrial Research (CSIR), India, for the financial support through research fellowship award no. 514856. This research is also partially supported by the Department of Science and Technology, India, through a project grant under DST/INSPIRE/04/2016/000571.

Appendix A: Neutrino massive trajectory vs null trajectory approximation

Ignoring spin structure of neutrino, the relevant phase for the \(i^{th}\) massive neutrino is given by

\[
\Phi_i = \int p^i_\mu dx^\mu. \tag{A1}
\]

However, using standard neutrino phase under the null trajectory approximation for a relativistic neutrino is shown to be \[40\]

\[
\Phi^{null}_i = \int p^i_\mu dx^\mu|_{null} = \int p^i_\mu p^\mu_{null} d\lambda = \frac{m_i^2}{2} \int d\lambda = \frac{m_i^2}{2} \lambda, \tag{A2}
\]

where \(\lambda\) is affine parameter along the null geodesic. Whereas if we take a massive trajectory for the neutrino, we get from Eq. \[A1\]

\[
\Phi^{massive}_i = \int p^i_\mu dx^\mu|_{massive} = \int p^i_\mu \frac{dx^\mu}{d\tau_i} d\tau_i, \tag{A3}
\]

where \(\tau_i\) is the proper time along the neutrino massive geodesic. Now, using \(p_\mu = m dx^\mu/d\tau\) and on mass shell condition \(p_\mu p^\mu = m^2\), we get

\[
\Phi^{massive}_i = m_i \int d\tau_i. \tag{A4}
\]

Now, we can parameterize the geodesic with any one of the coordinates \(t(\tau), r(\tau), \theta(\tau)\) or \(\phi(\tau)\) provided it has a non-zero first derivative w.r.t. \(\tau\) in the region. Let us parameterize geodesics, for the sake of the calculation, with radial coordinate \(r\). Then, we can rewrite Eq. \[A2\] \[A3\] as

\[
\Phi^{null}_i = \frac{m_i^2}{2} \int \frac{dr}{(p^i_1)^{null}}, \tag{A2}
\]

\[
\Phi^{massive}_i = m_i^2 \int \frac{dr}{(p^i_1)^{massive}}. \tag{A5}
\]
Maintaining neutrino phase only in the leading power of \( m \), we can expand \( (p^r)^{\text{massive}} \) as
\[
(p^r_{\text{massive}}) = (p^r)_{\text{null}} + \text{corrections in } m_i,
\]
and putting it in \( \Phi_i^{\text{massive}} \), we get
\[
\Phi_i^{\text{massive}} = m_i^2 \int \frac{dr}{(p^r)^{null}_{\text{null}}} + \text{corrections of higher order in } m_i.
\]
which is twice of the phase we get from null trajectory approximation, i.e., \( 2 \Phi_i^{\text{null}} = \Phi_i^{\text{massive}} = m_i \tau_i \), in the leading order of neutrino mass \( m_i \).

Appendix B: Critical geodesic solution for \( \theta(\tau) \) in \( \theta = \pi/2 \) plane

Geodesic equation for \( p_\theta \) is given as
\[
m \frac{dp_\theta}{d\tau} = \frac{1}{2} \partial_\theta g_{\mu\nu} p^\mu p^\nu.
\]
For the following metric,
\[
ds^2 = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2,
\]
where \( g_{tt} = 1 - R_s/r \), \( g_{rr} = -1/g_{tt} \), \( g_{\theta\theta} = -r^2 \), \( g_{\phi\phi} = -r^2 \sin^2 \theta \), and \( g_{t\phi} = 2GJ \sin^2 \theta/r \), the Eq. \( B1 \) after some manipulation, can also be written as
\[
m \frac{dp_\theta}{d\tau} = p^\phi (g_{tt} p^\phi + 2 g_{t\phi} p^t) \cot \theta.
\]
Now, using conserved quantities \( p_t \) and \( p_\phi \), i.e., \( p_\phi = g_{\phi\phi} p^\phi + g_{t\phi} p^t \), \( p_t = g_{tt} p^\phi + g_{t\phi} p^t \), and \( p_\theta = m g_{\theta\theta} d\theta/d\tau \), we can rewrite Eq. \( B3 \) as
\[
m^2 g_{\theta\theta} \frac{d^2 \theta}{d\tau^2} + m^2 g_{\theta\theta} \frac{d\theta}{d\tau} - \left( p_\phi + g_{\phi\phi} (g_{\phi\phi} p_t - g_{t\phi} p_\phi) \right) \left( \frac{g_{tt} p_\phi - g_{t\phi} p_t}{g_{\phi\phi} g_{tt} - g_{t\phi}^2} \right) \cot \theta = 0.
\]
As all coefficients in above D.E. are well defined at \( \theta = \pi/2 \), we see that the above D.E. for the initial values \( \theta = \pi/2 \) and \( d\theta/d\tau = 0 \) has a critical solution, i.e., for these initial values we get \( d^2 \theta/d\tau^2 = 0 \). So, the D.E. has a solution \( \theta(\tau) = \pi/2 = \text{constant} \) along the trajectory. Hence, all the particles which are produced with initial values \( \theta = \pi/2 \) and \( d\theta/d\tau = 0 \) remain in the \( \theta = \pi/2 \) plane throughout its journey.

Appendix C: Neutrino phase change in asymptotic Kerr geometry under null trajectory approximation

Let us parametrize geodesic with radial coordinate; then we can rewrite neutrino phase Eq. \( A2 \) as
\[
\Phi_i = \frac{m_i^2}{2} \int \frac{d\phi}{p^\phi} dr.
\]
Using null condition \( (p_\mu p^\mu)_{null} = 0 \), we get
\[
\frac{dr}{d\phi} = \pm \sqrt{B(r) \left( \frac{p_t}{p^\phi} - \frac{B(r) GJ p_t}{r \sin^2 \theta} \right)}.
\]
Now, two conserved quantities \( p_t = E_0 \) and \( p_\phi \equiv -L \) along the geodesic are related to \( p^t \) and \( p^\phi \) by the following relation
\[
p^t = \frac{E_0 r^2 - L h(r) L}{h(r)^2 + r^2 B(r)},
\]
and
\[
p^\phi = \frac{E_0 h(r) + B(r) L}{h(r)^2 + r^2 B(r)},
\]
where \( h(r) = 2GJ/r \), \( B(r) = 1 - 2GM/r \).
1. Neutrino produced with non-zero angular momentum ($L \neq 0$) (The source and detector lie on the same side of the gravitational mass)

For $L \neq 0$, using Eq. (C2, C4, C6) in Eq. (C1) we get for $r_D > r_S$

$$
\Phi_i = \frac{m_i^2}{2} \int_{r_D}^{r_S} \frac{h(r)^2 + r^2 B(r)}{(h(r)E_0 + LB(r)) \sqrt{\left( B(r) \frac{E_0 r^2 - h(r)L}{h(r)E_0 + LB(r)} \right)^2 + 2h(r)B(r) \frac{E_0 r^2 - h(r)L}{h(r)E_0 + LB(r)} - r^2 B(r)} } \, dr. \tag{C5}
$$

Now, expanding $\Phi_i$ keeping the leading order of $GM/r$, $GJ/r^2$ and the cross term $G^2JM/r^3$ under weak field limit, we find

$$
\Phi_i \simeq \frac{m_i^2}{2E_0} \int_{r_D}^{r_S} \left( \frac{r}{\sqrt{r^2 - \frac{L^2}{E_0^2}}} + \frac{2GJL}{E_0} - \frac{GML^2}{E_0^2} \right) \left( \frac{1}{(r^2 - \frac{L^2}{E_0^2})^{3/2}} - \frac{6G^2JML^3}{E_0^3} \right) \left( \frac{1}{r} \left( \frac{r}{\sqrt{r^2 - \frac{L^2}{E_0^2}}} \right)^{5/2} \right) \, dr. \tag{C6}
$$

Integrating for condition $r > L/E_0$ we get

$$
\Phi_i \simeq \frac{m_i^2}{2E_0} \left( \frac{3r_D^2 - 4L^2}{E_0^2} \right)^{3/2} - \frac{3r_S^2 - 4L^2}{E_0^2} \right)^{3/2} - \frac{6G^2JML^3}{E_0^3} \left( \sin^{-1} \left( \frac{L}{r_D E_0} \right) - \sin^{-1} \left( \frac{L}{r_S E_0} \right) \right). \tag{C7}
$$

2. Neutrino produced with zero angular momentum ($L = 0$)

Now, similarly for $L = 0$, we get for $r_D > r_S$

$$
\frac{d\phi}{dr} = \frac{2GJ}{\sqrt{(1 - \frac{2GM}{r})(r^6 - 2GMr^5 + 4G^2r^2J^2)}}. \tag{C8}
$$

Using Eq. (C5) and Eq. (C4) in Eq. (C1) and integrating from $r_S$ to $r_D$, we get phase in the leading order after ignoring order $O(G^2M^2J^2/r^4)$

$$
\Phi_i \simeq \frac{m_i^2}{2E_0} \left[ (r_D - r_S) - \frac{2G^2J^2}{3} \left( \frac{r_D}{r_D} - \frac{1}{r_S} \right) \right. - \frac{G^3MJ^2}{r_D^2} \left( \frac{1}{r_D} - \frac{1}{r_S} \right) \left. + O \left( \frac{G^4M^2J^2}{r_S^3} \right) \right]. \tag{C9}
$$

3. Neutrino phase change in the weak gravitational lensing in the $\theta = \pi/2$ plane

In the weak gravitational lensing, one usually defines impact parameter $b$ in the asymptotic observer frame as $b \equiv L/E_0$, where $L$ and $E_0$ is the angular momentum and energy of the particle as measured by the asymptotic observer. Now, replacing $L$ and $E_0$ in term of $b$ in the Eq. (C5) we get

$$
\Phi_i = \frac{m_i^2}{2E_0} \int_{r_D}^{r_S} \frac{h(r)^2 + r^2 B(r)}{(h(r) + bB(r)) \sqrt{\left( B(r) \frac{r^2 - h(r)b}{h(r) + bB(r)} \right)^2 + 2h(r)B(r) \frac{r^2 - h(r)b}{h(r) + bB(r)} - r^2 B(r)} } \, dr. \tag{C10}
$$

Now, limit of integral is broken into two regions of neutrino propagation (1) when neutrino is propogating towards gravitational source, and (2) when neutrino is propogating away from the gravitational source, i.e., $(r_S \to r_0 \to r_D)$,

$$
\Phi_i^{\text{lens}} = \frac{m_i^2}{2E_0} \left( \int_{r_D}^{r_0} - \int_{r_S}^{r_0} \right) \left[ \frac{h(r)^2 + r^2 B(r)}{(h(r) + bB(r)) \sqrt{\left( B(r) \frac{r^2 - h(r)b}{h(r) + bB(r)} \right)^2 + 2h(r)B(r) \frac{r^2 - h(r)b}{h(r) + bB(r)} - r^2 B(r)} } \, dr. \tag{C11}
$$
The $-ve$ sign accounts for the neutrino traveling towards the gravitational source ($dr$ decreasing). Where $r_0$ is the radius of closest approach and can be found in terms of $b$ using following equation

$$\frac{dr}{d\phi} = 0 = \sqrt{\left(B(r) \frac{p^l}{p^0}\right)^2 + 4B(r) \frac{GJ p^l}{r p^0} - B(r)r^2}$$  \hspace{1cm} (C12)$$

Solving Eq. \textbf{C12} in the leading order of $M$ and $J$, we find

$$b \simeq r_0 + GM - \frac{2GJ}{r_0},$$

$$r_0 \simeq b - GM + \frac{2GJ}{b}. \hspace{1cm} (C13)$$

Now plugging $b$ in terms of $r_0$ in Eq. \textbf{C10} and expanding Eq. \textbf{C10} up to the leading orders of $GM/r$ and $GJ/r^2$. We get after integration

$$\Phi^\text{lens}_i \simeq \frac{m^2_i}{2E_0} \left[ \sqrt{r^2_D - r^2_0} + \sqrt{r^2_S - r^2_0} + \left( GM - \frac{2GJ}{r_0} \right) \left( \sqrt{\frac{r_D - r_0}{r_D + r_0}} + \sqrt{\frac{r_S - r_0}{r_S + r_0}} \right) \right]. \hspace{1cm} (C14)$$

Now replacing $r_0$ in terms of $b$ from Eq. \textbf{C13}, we get in the weak field limit $b/r_{S,D} << 1$ (ignoring higher order in $b^2/r^2_{S,D}$, $GM/r$ and $GJ/r^2$)

$$\Phi^\text{lens}_i \simeq \frac{m^2_i}{2E_0} (r_S + r_D) \left[ 1 - \frac{b^2}{2r_S r_D} + \frac{2GM}{r_S + r_D} - \frac{4GJ}{b(r_S + r_D)} \right] \hspace{1cm} (C15)$$

\textbf{a. $L = 0$ trajectories do not contribute in the weak gravity lensing}

To see that neutrinos produced with $L = 0$ do not contribute to the weak gravity lensing, we find radial coordinate ($r_t$) at which these trajectories turn around, i.e., when $dr/d\phi = 0$, then using Eq. \textbf{C8} we get

$$\left(1 - \frac{2GM}{r_t}\right) (r_t^6 - 2GMr_t^5 + 4G^2r_t^2J^2) = 0. \hspace{1cm} (C16)$$

This gives following solutions

$$r_t = 0,$$

$$r_t = R_s,$$

$$r_t^4 \left(1 - \frac{2GM}{r_t}\right) + 4G^2J^2 = 0 \hspace{1cm} (C17)$$

The first two solutions violate the weak gravity limit, while the third equation has no real solution. Hence, the $L = 0$ case is irrelevant for the weak gravity lensing.

\begin{thebibliography}{9}
\bibitem{Capozzi2014} F Capozzi, GL Fogli, E Lisi, A Marrone, D Montanino, and A Palazzo. Status of three-neutrino oscillation parameters, circa 2013. \textit{Physical Review D}, 89(9):093018, 2014.
\bibitem{Fernandez2018} Pablo Fernández de Salas, DV Forero, Christoph Andreas Ternes, Mariam Tórtola, and José WF Valle. Status of neutrino oscillations 2018: 3$\sigma$ hint for normal mass ordering and improved cp sensitivity. \textit{Physics Letters B}, 782:633–640, 2018.
\bibitem{Esteban2019} Ivan Esteban, Maria Concepción González-García, Alvaro Hernandez-Cabezudo, Michele Maltoni, and Thomas Schwetz. Global analysis of three-flavour neutrino oscillations: synergies and tensions in the determination of $\theta_{23}$, $\delta$ cp, and the mass ordering. \textit{Journal of High Energy Physics}, 2019(1):1–35, 2019.
\bibitem{Dvornikov2019} Maxim Dvornikov. Neutrino flavor oscillations in stochastic gravitational waves. \textit{Physical Review D}, 100(9):096014, 2019.
\bibitem{Stuttard2020} Thomas Stuttard and Mikkel Jensen. Neutrino decoherence from quantum gravitational stochastic perturbations. \textit{Physical Review D}, 102(11):115003, 2020.
\bibitem{Sorge2007} Francesco Sorge and Silvio Zilio. Neutrino spin flip around a schwarzschild black hole. \textit{Classical and Quantum Gravity}, 24(10):2653, 2007.
\end{thebibliography}
Savitri V Iyer and Edward C Hansen. Light’s bending angle in the equatorial plane of a kerr black hole. *Physical Review D*, 80(12):124023, 2009.

Christian Y Cardall and George M Fuller. Neutrino oscillations in curved spacetime: A heuristic treatment. *Physical Review D*, 55(12):7960, 1997.

N Fornengo, C Giunti, CW Kim, and J Song. Gravitational effects on the neutrino oscillation. *Physical Review D*, 56(4):1895, 1997.

Daniela Kunst, Tomáš Ledvinka, Georgios Lukes-Gerakopoulos, and Jonathan Seyrich. Comparing hamiltonians of a spinning test particle for different tetrad fields. *Physical Review D*, 93(4):044004, 2016.

Enrico Barausse, Etienne Racine, and Alessandra Buonanno. Hamiltonian of a spinning test particle in curved spacetime. *Physical Review D*, 80(10):104025, 2009.