How to compute one-loop Feynman diagrams in lattice QCD with Wilson fermions

Giuseppe Burgio\textsuperscript{a}, Sergio Caracciolo\textsuperscript{b,∗} and Andrea Pelissetto\textsuperscript{c}

\textsuperscript{a}Dipartimento di Fisica and INFN, Università degli Studi di Parma, Parma 43100, ITALIA
\textsuperscript{b}Dipartimento di Fisica and INFN, Università degli Studi di Lecce, Lecce 73100, ITALIA
\textsuperscript{c}Dipartimento di Fisica and INFN, Università degli Studi di Pisa, Pisa 56100, ITALIA

We describe an algebraic algorithm which allows to express every one-loop lattice integral with gluon or Wilson-fermion propagators in terms of a small number of basic constants which can be computed with arbitrary high precision. Although the presentation is restricted to four dimensions the technique can be generalized to every space dimension. We also give a method to express the lattice-free propagator for Wilson fermions in coordinate space as a linear function of its values in eight points near the origin. This is an essential step in order to apply the recent methods of Lüscher and Weisz to higher-loop integrals with fermions.

In \cite{1} we presented a general technique which allows to express every one-loop bosonic integral at zero external momentum in terms of two unknown basic quantities which could be computed numerically with high precision. This method allows the complete evaluation of every diagram with gluon propagators.

We have recently generalized \cite{2} this technique to deal with integrals with both gluonic and fermionic propagators. For the fermions we use the Wilson action \cite{3}. Notice that our method depends only on the structure of the propagator and thus it can be applied in calculations with bosonic and fermionic propagators. The advantage of this procedure is twofold: first of all every Feynman diagram can be computed in a completely symbolic way making it easier to perform checks and verify cancellations; moreover the basic constants can be easily computed with high precision and thus the numerical error on the final result can be reduced at will. Although the presentation is restricted to four dimensions the technique can be generalized to every space dimension.

Define
\begin{align}
F_{\delta}(p, q, n_x, n_y, n_z, n_t) &= \int_{-\pi}^{\pi} d^4k \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{(2\pi)^4 \hat{D}_F(k, m_f) \hat{D}_B(k, m_b)} \tag{1}
\end{align}

where \(n_i\) are positive integers, \(p\) and \(q\) are arbitrary integers (not necessarily positive), \(k_\mu = 2 \sin(k_\mu/2)\)

\begin{align}
\hat{D}_B(k, m_b) &= \hat{k}^2 + m_b^2 = \sum_i \hat{k}_i^2 + m_b^2 \tag{2}
\end{align}

\begin{align}
\hat{D}_F(k, m_f) &= \sum_i \sin^2 k_i + \frac{r_W^2}{4} (\hat{k}^2)^2 + m_f^2 \tag{3}
\end{align}

is the denominator appearing in the propagator for Wilson fermions\textsuperscript{2}. In the following when one

\textsuperscript{2}To be precise, \(D_F(k, m_f)\) is the denominator in the propagator for Wilson fermions only for \(m_f = 0\). For \(m_f \neq 0\) the correct denominator would be

\begin{align}
\hat{D}_F(k, m_f) &= \sum_i \sin^2 k_i + \left(\frac{r_W^2}{2} (\hat{k}^2)^2 + m_f^2\right) \tag{4}
\end{align}

However in our discussion \(m_f\) will only play the role of an infrared regulator and thus it does not need to be the true fermion mass. The definition \(\hat{D}_F\) is easier to handle than

\textsuperscript{∗}Speaker at the conference.
of the arguments \( n_i \) is zero it will be omitted as an argument of \( \mathcal{F} \). The parameter \( \delta \) is used in the intermediate steps of the calculation and will be set to zero at the end.

To simplify the discussion we have only considered the case \( r_W = 1 \) but the technique can be applied to every value of \( r_W \). Moreover we have restricted our attention to the massless case, i.e. we have considered the integrals \( \mathcal{F}_\delta \) in the limit \( m_b = m_f \equiv m \to 0 \).

We have developed a procedure that allows to compute iteratively a generic \( \mathcal{F} \), which is \( \mathcal{F}_\delta \) in the limit \( \delta \to 0 \), in terms of a finite number of them: precisely every \( \mathcal{F}(p, q; n_x, n_y, n_z, n_t) \) with \( p > 0 \) and \( q \leq 0 \) (purely fermionic integrals) can be expressed in terms of \( \mathcal{F}(1, 0), \mathcal{F}(1, -1), \mathcal{F}(1, -2), \mathcal{F}(2, 0), \mathcal{F}(2, -1), \mathcal{F}(2, -2), \mathcal{F}(3, -2), \mathcal{F}(3, -3) \) and \( \mathcal{F}(3, -4) \): the integral \( \mathcal{F}(2, 0) \) appears only in infrared-divergent integrals and we write it as

\[
\mathcal{F}(2, 0) = -\frac{1}{16\pi^2} \left( \log m^2 + \gamma_E - F_0 \right) + Y_0 \tag{5}
\]

where \( Y_0 \) is a numerical constant. If \( q > 0 \) the result contains three additional constants which we have chosen to be

\[
\begin{align*}
Y_1 &= \frac{1}{8} \mathcal{F}(1, 1; 1, 1, 1) \\
Y_2 &= \frac{1}{16} \mathcal{F}(1, 1; 1, 1, 1, 1) \\
Y_3 &= \frac{1}{16} \mathcal{F}(1, 2; 1, 1, 1) \tag{6}
\end{align*}
\]

and together with the bosonic quantities \( Z_0, Z_1 \) and \( F_0 - \gamma_E \).

As a first step in our procedure we express each integral \( \mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) \) in terms of integrals of the form \( \mathcal{F}_\delta(r, s) \) only. This is achieved by the use of the identity

\[
\sum_{i=1}^{4} \hat{k}_i^2 = D_B(k, m) - m^2 \tag{7}
\]

in the case one of the \( n_i \)’s is 1. In the case one of the \( n_i \)’s is 2 we use instead

\[
\sum_{i=1}^{4} \hat{k}_i^4 = 4 \left[ D_B(k, m) - D_F(k, m) \right] + \left[ D_B(k, m) - m^2 \right]^2 \tag{8}
\]

In all other cases we use an integration by parts. It is here that we found convenient to have a non-vanishing \( \delta \), otherwise integration by parts in the case in which there is only one fermionic propagator would produce a \( \log D_F \).

As a result we have that

\[
\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \sum_{r=p-k+1}^{p} \sum_{s=q-k}^{q+k} a_{rs}(m, \delta) \mathcal{F}_\delta(r, s) \tag{9}
\]

where \( k = (n_x + n_y + n_z + n_t) \), and \( a_{rs}(m, \delta) \) is a polynomial in \( m^2 \).

At the second step we express every \( \mathcal{F}_\delta(p, q) \) in terms of a finite number of them. We start by inserting the trivial identity

\[
0 = 1 - \sum \hat{k}_i^2 + m^2 \tag{10}
\]

into the integral \( \mathcal{F}_\delta(p, q; 1, 1, 1) \) to get

\[
0 = I_1(p, q) \equiv \mathcal{F}_\delta(p, q; 1, 1, 1, 1) - 4 \mathcal{F}_\delta(p, q + 1; 2, 1, 1, 1)
- m^2 \mathcal{F}_\delta(p, q + 1; 1, 1, 1, 1) \tag{11}
\]

Using the previous recursions we obtain from (11) a non trivial relation among the \( \mathcal{F}_\delta(r, s) \). Starting from the analogous relation obtained from the fermionic propagator we get a new set of identities \( I_2(p, q) \) which can be used to provide additional relations between the remaining integrals. We end up with the result we have quoted above.

In order to numerically evaluate the eight basic purely-fermionic integrals we considered the integrals

\[
J_q = \mathcal{F}(1, -q) \quad 6 \leq q \leq 13 \tag{12}
\]

which can be analytically expressed in terms of the elements in the fermionic basis. We computed

\[
J_{q, L} = \frac{1}{L^4} \sum_k \frac{D_B(k, 0)^q}{D_F(k, 0)} \tag{13}
\]

where \( k \) runs over the points \( k = (2\pi/L) (n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}, n_4 + \frac{1}{2}) \), \( 0 \leq n_i < L \) for various values of \( L \) between 50 and 100 and extrapolated the results to infinite volume by using

\[
J_{q, L} \approx J_q \left( 1 + \frac{A}{L^{2q+2}} \right) \tag{14}
\]
The inversion of this set of equations provides a very accurate estimate of the elements in the basis (see Table 1). Analogously to compute $Y_0$, $Y_1$, $Y_2$ and $Y_3$, we computed numerically $\mathcal{F}(1, 1, 8)$, $\mathcal{F}(2, 1, 9)$, $\mathcal{F}(3, 1, 10)$ and $\mathcal{F}(5, 2, 11)$ and then solved the corresponding equations.

A second important application of our method is connected with the use of coordinate-space methods for the evaluation of higher-loop Feynman diagrams. This technique, introduced by Lüscher and Weisz \cite{5}, is extremely powerful and allows a very precise determination of two- and higher-loop integrals. One of the basic ingredients of this method is the computation of the free propagator in coordinate space. But this can be achieved by the use of our relations. Consider

\[
\begin{align*}
G(p, q; x) &= \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{D_F(p, k)D_B(q, m)} \quad (15) \\
&= \int \frac{dk}{(2\pi)^4} \frac{\prod \cos k_\mu x_\mu}{D_F(p, k)D_B(q, m)} \quad (16)
\end{align*}
\]

Then express $\cos(k_\mu x_\mu)$ as a polynomial in $k_\mu^2$. It follows that $G$ is a linear combination of the $\mathcal{F}$ and therefore they can be expressed on our basis.

Here, like in the bosonic case, the expression of $G$ in terms of the basis become numerically unstable for $|x| \to \infty$: numerical errors in the basis become amplified! This problem has a standard way out: if the expressions are unstable going outward from the origin, they will be stable in the opposite direction: thus, if we want to compute the propagator for $|x| < d$, for some fixed $d$, we choose eight points with $|x| \approx d$ (say $y_1, \ldots, y_8$) and then we express the propagator for $|x| < d$ in terms of $G(p, q; y_i), i = 1, \ldots, 8$. The new expressions are numerically stable: the numerical error on $G(p, q; y_i)$ gets reduced when we compute the propagator for $|x| \to 0$. As noticed in \cite{6} the instability of the recursion can also be used to provide precise estimates for the basic constants.

We have thus used this method to obtain an independent numerical estimate of the eight purely fermionic and infrared finite constants in our basis, considering the set of eight points $X^{(n)} = \{ (n, [0-3], 0, 0), (n + 1, [0-3], 0, 0) \}$.

Table 1

| $\mathcal{F}$ | $F(1, 0)$ | $0.08539036359532067913516702879$
| | $0.08539036359532067913516702676$ |
| $\mathcal{F}(1, -1)$ | $0.46936331002696944753475397003$ |
| | $0.4693633100269694475347539758$ |
| $\mathcal{F}(1, -2)$ | $3.395690736771300586086896873$ |
| | $3.39569073677130058608689071$ |
| $\mathcal{F}(2, -1)$ | $0.05188019503901136636490228763$ |
| | $0.05188019503901136636490228706$ |
| $\mathcal{F}(2, -2)$ | $0.238773756341478520233613767$ |
| | $0.238773756341478520233613767$ |
| $\mathcal{F}(3, -2)$ | $0.0344764443803223145396188143$ |
| | $0.0344764443803223145396188128$ |
| $\mathcal{F}(3, -3)$ | $0.132027212278129308531473096$ |
| | $0.132027212278129308531473035$ |
| $\mathcal{F}(3, -4)$ | $0.75167199030295682253543148585$ |
| | $0.75167199030295682253543148379$ |

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