QUANTITATIVE RESULTS ON DIOPHANTINE EQUATIONS
IN MANY VARIABLES

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ABSTRACT. We consider a system of polynomials \( f_1, \ldots, f_R \in \mathbb{Z}[x_1, \ldots, x_n] \) of the same degree with non-singular local zeros and in many variables. Generalising the work of Birch [Bir62] we find quantitative asymptotics (in terms of the maximum of the absolute value of the coefficients of these polynomials) for the number of integer zeros of this system within a growing box. Using a quantitative version of the Nullstellensatz, we obtain a quantitative strong approximation result, i.e. an upper bound on the smallest integer zero provided the system of polynomials is non-singular.

1. INTRODUCTION

Consider a system \( f \) of polynomials \( f_1, \ldots, f_R \in \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}[x] \) of degree \( d \geq 2 \). It was shown by Birch [Bir62], that if these polynomials are homogeneous, they satisfy the smooth Hasse principle providing

\[
n - \dim V^* > R(R + 1)(d - 1)2^{d-1},
\]

where \( V^* \) is the so-called Birch singular locus of the projective variety \( V \) corresponding to \( f \). Let \( V^{sm} \) be the smooth locus of \( V \) (which consists of the points where the Jacobian matrix of \( f \) has rank strictly less than \( R \)). Then, the smooth Hasse principle is that

\[
\prod_{\nu} V^{sm}(Q_{\nu}) \neq \emptyset \quad \text{implies that} \quad V(\mathbb{Q}) \neq \emptyset.
\]

Here, the product is over all places \( \nu \) of \( \mathbb{Q} \) and \( \mathbb{Q}_\infty = \mathbb{R} \).

In this paper we are interested in the distribution and the size of the rational points on \( V \) (or integer points on \( V \) when the system is not assumed to be homogeneous). More specifically, let \( V_\mathbb{Z} \) be an integral model of \( V \). Let \( A_\infty \) be the adele ring of \( \mathbb{Q} \) outside the place \( \infty \) and let \( V_{\mathbb{A}} \) be the base change of \( V_\mathbb{Z} \) to \( \mathbb{A}_\infty \). We say that \( V \) satisfies strong approximation outside \( \infty \) if the image of the diagonal map \( V_\mathbb{Z} \to V_{\mathbb{A}_\infty} \) is dense. Note that the notion of strong approximation outside \( \infty \) implies the smooth Hasse principle. For \( V \) as in Birch’ theorem strong approximation outside \( \infty \) holds. Theorem 3.10 is a quantitative version of this statement, which is a first step in understanding the distribution of the integer zeros of an arbitrary systems of integer polynomials. This result follows directly from our main theorem stated in Section 1.2. In order to obtain this result we generalise the work of Birch [Bir62] to find quantitative asymptotics (in terms of the maximum of the absolute value of the coefficients of these polynomials).
the coefficients of these polynomials) for the number of integer zeros of this system within a growing box. Using a quantitative version of the Nullstellensatz, we obtain an upper bound on the smallest non-trivial common zero of \( f \).

1.1. Related work. There are many improvements on Birch’ result if we restrict to a single form. For example, Heath-Brown showed that a cubic form has a non-trivial integer zero provided only that \( n \geq 14 \) [HB07]. Assuming that the variety \( V \) is non-singular, a form of degree 2, 3, respectively 4 satisfies the smooth Hasse principle provided that \( n \geq 3 \), \( n \geq 9 \), respectively \( n \geq 40 \) [HB96, Hoo88, Han12]. Browning and Prendiville slightly relaxed condition \( (1) \), by showing that for a form of degree \( d \geq 3 \) the smooth Hasse principle holds provided that \( n - \dim V^* \geq \left( d - \frac{1}{2} \sqrt{d} \right) 2^d \) [BP15].

Recent results by Myerson improve on Birch’ result for systems of forms when \( V \) is a complete intersection (which is implied by \( (1) \) in Birch’ theorem). He shows that under this condition one can replace condition \( (1) \) by \( n \geq 9R \) respectively \( n \geq 25R \) for systems of degree 2 respectively 3 [Mye15, Mye17].

Unconditional improvements include the observation that \( \dim V^* \) can replaced by a smaller quantity \( \Delta \), to be defined in \( (2) \), as shown independently by Dietmann and Schindler [Die15, Sch15]. Another improvement is the observation by Schmidt that the assumption that the system of polynomials is homogeneous is not necessary [Sch85]. We will make use of these improvements in this work.

There are known results on the smallest zero of a single form in many variables. Let \( \Lambda(\tilde{f}) \) be the smallest integer zero of a form \( \tilde{f} \in \mathbb{Z}[x_1, \ldots, x_n] \) with coefficients bounded in absolute value by \( C \). If \( d = 2 \), Cassels result

\[
\Lambda(f) \leq c_n C^{\frac{n-1}{2}},
\]

where the constant \( c_n \) is explicit and depends only on \( n \), has the best possible exponent [Cas55, Cas56]. However, for generic quadratic forms one can do much better [BD08]. Recently, Sardari proved an optimal strong approximation theorem for \( \tilde{f} - N \), where \( \tilde{f} \) is a non-degenerate quadratic form and \( N \) a sufficiently large integer [Sar15].

If \( d = 3 \) the best possible exponent is unknown, but asymptotically smaller than the exponent \( (n - 1)/2 \) in Cassels result. Browning, Dietmann and Elliot showed that \( \Lambda(f) \leq cC^{1071} \) for some absolute constant \( c \) provided \( n \geq 17 \), whereas by a result due to Pitman one has for any \( \epsilon > 0 \) and sufficiently large \( n \) that \( \Lambda(f) \leq c_{n,\epsilon} C^{\frac{3d}{2} + \epsilon} \), for some constant \( c_{n,\epsilon} \) [BDE12, Pit68]. In case the hypersurface corresponding to \( f \) has at most isolated ordinary singularities, the former authors provide visibly better bounds, e.g. \( \Lambda(f) \leq cC^{1071} \) for \( n = 17 \). In fact, Browning, Dietmann and Elliot wonder whether their ideas “could be adapted to handle non-singular forms of degree exceeding 3” analogous to “the extension of Birch [Bir62] to higher degree of Davenport’s treatment of cubic forms [Dav63]”. It is the main result of this paper that this is indeed possible, although their method to achieve effective lower bounds for the singular series and integral is completely different than ours.

1.2. Main result. Denote with \( \tilde{f} \) the top degree part of the system \( f \). Denote with \( V \) and \( \tilde{V} \) the affine respectively projective variety corresponding to \( f \) respectively \( \tilde{f} \). Letting \( C \) and \( \tilde{C} \) be the (real) maximum of the absolute value of the coefficients of \( f \) respectively \( \tilde{f} \), we make the work of Birch quantitative in terms of \( C \) and \( \tilde{C} \). We use Birch’ assumption on the number of variables (i.e. equation \( (1) \)) throughout this work, replacing the dimension of the
Birch singular locus by the quantity $\Delta$ of Dietmann and Schindler. For any $b \in \mathbb{Z}^R$, we let $\tilde{f}_b = b_1f_1 + \ldots + b_Rf_R$. For a form $g$ we let $\text{Sing}(g)$ be the singular locus of $g$ in affine space. Define the quantities $\Delta$ and $K$ by

$$\Delta = \max_{\xi \in \mathbb{Z}^R \setminus \{0\}} \left( \dim \text{Sing}(\tilde{f}_b) \right) \quad \text{and} \quad K = \frac{n - \Delta}{2^{d-1}}. $$

Then we assume throughout that

$$K > R(R+1)(d-1).$$

In particular, this implies that $V$ is a complete intersection. Our main theorem is the following, which is proven in Section 3.3.

**Theorem 1.1.** Let $f_i \in \mathbb{Z}[x] = \mathbb{Z}[x_1, \ldots , x_n]$ for $i = 1, \ldots , R$ be polynomials of degree $d$ so that $K - R(R+1)(d-1) > 0$, $f$ has zeros over $\mathbb{Z}_p$ for all primes $p$ and $\tilde{f}$ has a real zero. Assume that the affine respectively projective varieties $V$ and $\tilde{V}$ corresponding to $f$ and $\tilde{f}$ are non-singular. Then there exists an $x \in \mathbb{Z}^n$, polynomially bounded by $C$ and $\tilde{C}$, such that $f(x) = 0$, in fact

$$\max_{1 \leq i \leq n} |x_i| \leq c(C^3\tilde{C}^2)^{\frac{4n^3R(Rd)^n}{K-R(R+1)(d-1)}},$$

where the constant $c$ does not depend on $C$ or $\tilde{C}$.

The case that the system $\tilde{f}$ is homogeneous is treated separately in Theorem 3.9.

**Remark.** The bound in above theorem is in no sense believed to be optimal and far worse than known bounds for small degrees which were discussed in Section 1.1. However, we provided an upper bound in a far more general setting; it was not shown that such an upper bound exists in our setting. The main contribution to this bound is due to our lower bound for the singular series and integral, which follows from a quantiative version of the Nullstellensatz by Krick, Pardo and Sombra [KPS01] as discussed on page 10. This theorem, although sharp in general, is not believed to be sharp in the present setting. It would be interesting to explore whether a stronger quantitative version of the Nullstellensatz can be applied in this setting, yielding a significant improvement in the bound (4).

1.3. **Structure of this paper.** In Section 2 we generalise the work of Birch [Bir62] to deduce asymptotics (quantitative in $C$ and $\tilde{C}$) for the number of integer points on $V$ within a box $PB$ for $P \to \infty$. We omit the proofs of lemmas and theorems which are straightforward generalisations of Birch work, but we provide proofs when they are only slightly more involved. We obtain lower bounds for the singular series and integral (introduced in Section 2.4 respectively Section 2.5) in Section 3.1 and Section 3.2. We end with a proof of our main theorems in Section 3.3.

1.4. **Notation.** On the vector space $\mathbb{Q}_p^n$ ($p$ prime or $p = \infty$) we introduce the sup-norm $|a|_p = \max_{1 \leq i \leq n} |a_i|_p$, where $|\cdot|_p$ is the absolute value on $\mathbb{Q}_p$. For $\beta \in \mathbb{R}$, we let $||\beta|| = \min_{\xi \in \mathbb{Z}} |i - \beta|$ and for a point $\alpha \in \mathbb{R}^m$ we write $||\alpha|| = \max_{1 \leq i \leq m} ||\alpha_i||$. If $a \in \mathbb{Z}^m$ and $q \in \mathbb{Z}$, we abbreviate $\gcd(a_1, \ldots , a_m, q)$ by $(a, q)$. For $x \in \mathbb{R}$ we abbreviate $e^{2\pi ix}$ by $e(x)$. For functions $f, g$ defined on a subset of the real numbers we use Vinogradov’s notation $f \ll g$ to mean $f = O(g)$. Without an indication the implied constant may depend on $n, R$ and $d$, but not on $C$ or $\tilde{C}$. 
Denote by \( \mathcal{E} \) the box \([-1, 1]^n\) and let \( \mathcal{B} \) be an \( n \)-dimensional box contained in \( \mathcal{E} \) of side-length at most 1, i.e. there are \( a_j, b_j \in \mathbb{R} \) with \(-1 \leq a_j \leq b_j \leq 1\) and \( 0 < b_j - a_j < 1 \) such that \( \mathcal{B} \) is given by \( \prod_{j=1}^n [a_j, b_j] \).

2. Quantitative asymptotics

2.1. Estimates of exponential sums. Let \( \alpha \in [0, 1)^R \). We obtain estimates for exponential sums

\[
S(\alpha) = \sum_{x \in P^n \cap \mathbb{Z}^n} e(\alpha \cdot f(x)) \quad \text{and} \quad S(\alpha, \nu) = S(\alpha)e(-\alpha \cdot \nu)
\]

depending on \( \alpha_1, \ldots, \alpha_R \) not being too well approximable by rational numbers with small denominators. The following lemma generalises Lemma 2 in [Sch15].

**Lemma 2.1.** Let \( \varepsilon > 0 \) and \( 0 < \theta < 1 \). One of the following holds:

(i) \( |S(\alpha)| \ll P^{n-K\theta+\varepsilon} \); 
(ii) (rational approximation to \( \alpha \) with respect to the parameter \( \theta \)) there are \( \alpha \in (\mathbb{Z}_{\geq 0})^R \) and \( q \in \mathbb{Z}_{> 0} \) such that \( (\alpha, q) = 1 \),

\[
|q \alpha - a| \leq \tilde{C}^{R-1} P^{-d+R(d-1)\theta} \quad \text{and} \quad 1 \leq q \ll \tilde{C}^R P^{R(d-1)\theta}.
\]

**Proof.** Let \( \Gamma_i(x^{(1)}, \ldots, x^{(d)}) \), \( 1 \leq i \leq R \) be the multilinear form associated to \( \tilde{f}_i \), satisfying \( \Gamma_i(a, \ldots, a) = d! \tilde{f}_i(a) \). Let \( N(P^\xi, P^{-\eta}; \alpha) \) be the number of integer vectors \( x^{(i)} \in \mathbb{Z}^n \), \( 2 \leq i \leq d \) with \( |x^{(i)}| \leq P^\xi \) and

\[
\left\| \sum_{i=1}^R \alpha_i \Gamma_i (e_j, x^{(2)}, \ldots, x^{(d)}) \right\| < P^{-\eta}, 1 \leq j \leq n.
\]

Lemma 2.4 in [Bir62] states that if \( |S(\alpha)| > P^{n-k} \), then \( N(P^\theta, P^{-d+(d-1)\theta}; \alpha) \gg P^{(d-1)\theta - 2^{d-1}k - \varepsilon} \), which follows from Weyl’s inequality and Davenport’s application of the the geometry of numbers. This estimate is independent of the coefficients of \( f \), i.e. the implied constant does not depend on \( C \) or \( \tilde{C} \). The lemma now follows with the same proof as Lemma 2 and Theorem 1 in [Sch15] using the estimate \( |\Gamma_i (e_j, x^{(2)}, \ldots, x^{(d)})| \ll \tilde{C} P^{(d-1)}. \)

For \( \alpha \in \mathbb{Z}^R \) and \( q \in \mathbb{Z} \) such that \( (\alpha, q) = 1 \) and \( 1 \leq a_i \leq q \), let

\[
S_{\alpha, q} = \sum_{x \mod q} e(\alpha \cdot f(x)/q) \quad \text{and} \quad S_{\alpha, q}(\nu) = S_{\alpha, q}e(-\alpha \cdot \nu/q).
\]

Here, the summation is over a complete set of residues modulo \( q \) for every vector component of \( \alpha \).

**Lemma 2.2** (Generalisation of [Bir62], Lemma 5.4). For every \( \varepsilon > 0 \) we have

\[
|S_{\alpha, q}| \ll \tilde{C}^{K/(d-1)} q^{n-K/R(d-1)+\varepsilon}.
\]
2.2. Minor arcs. Given \(q \in \mathbb{Z}^R, q \in \mathbb{Z}_{>0}\) and \(0 < \theta \leq 1\), define a major arc as

\[
M_{a,q}(\theta) = \prod_{i=1}^{R} \left[ \frac{a_i}{q} - \tilde{C} R - 1 \frac{P - d + R (d-1) \theta}{2q}, \frac{a_i}{q} + \tilde{C} R - 1 \frac{P - d + R (d-1) \theta}{2q} \right].
\]

Then, define the major arcs to be

\[
M(\theta) = \bigcup_{1 \leq q \leq \tilde{C} R P} \bigcup_{1 \leq a_i \leq q} M_{a,q}(\theta).
\]

Modulo 1 we have that \(M(\theta)\) consists of all \(\alpha\) satisfying (ii) in Lemma 2.1. Define the minor arcs by \(m = [0, 1] \setminus M\) modulo 1.

**Lemma 2.3** (Generalisation of [Bir62, Lemma 4.2]). There exists an \(\varepsilon > 0\) such that \(M(\theta)\) has volume at most

\[
\tilde{C} R^2 P^{-Rd + R (R+1) (d-1) \theta - \varepsilon}.
\]

For small enough \(\theta\) the minor arcs are disjoint.

**Lemma 2.4** (Generalisation of [Bir62, Lemma 4.1]). If \(d > 2R (d-1) \theta + (2R - 1) \log_P(\tilde{C})\), then \(M(\theta)\) is given as a disjoint union of \(M_{a,q}(\theta)\) by (6).

Now, take major arcs \(M(\theta_0)\), where \(\eta, \theta_0, \delta\) are such that

\[
\eta = R (d-1) \theta_0, \\
1 > \eta + R \log_P(\tilde{C}), \\
\frac{K}{R (d-1)} - (R + 1) > \delta \eta^{-1}.
\]

Observe that assumption (9) is a quantitative version of our main assumption (3). Note that (8) implies that the major arcs \(M_{a,q}(\theta_0)\) are disjoint. Later, we will choose \(\eta\) and \(\delta\) satisfying (8) and (9).

We now use Birch’ idea of a sliding scale to bound \(S(\alpha, \nu)\) on the minor arcs.

**Lemma 2.5.**

\[
\int_{m} |S(\alpha, \nu)| \, d\alpha = O(\tilde{C} R^2 P^{n-Rd-\delta}),
\]

where \(O\) does not depend on \(C\) or \(\tilde{C}\).

**Proof.** First, observe that \(|S(\alpha, \nu)| = |S(\alpha)|\). Let \(\varepsilon > 0\) be small. Now, define a sequence

\[
\theta_T > \theta_{T-1} > \ldots > \theta_1 > \theta = \theta_0 > 0
\]

such that

\[
2d = (R + 1)(d-1) \theta_T, \\
\varepsilon \delta > R (R + 1) (d-1) (\theta_{t+1} - \theta_t) \quad \text{for } 0 \leq t \leq T - 1.
\]

Then \(T \ll P^{\varepsilon \delta}\) for \(P\) large enough (independent of \(\tilde{C}\)).

By Lemma 2.1 and as \(-K \theta_T + \varepsilon < -2Rd\) by (3), we find

\[
\int_{\alpha \notin M(\theta_T)} |S(\alpha, \nu)| \, d\alpha \ll P^{n-2Rd}.
\]
By Lemma 2.3 and Lemma 2.1 we have
\[
\int_{\mathfrak{M}(\theta_{t+1})}^{} |S(\underline{\alpha}, \underline{\nu})| \ d\underline{\alpha} \ll \tilde{C} R^2 P^{-Rd+R(R+1)(d-1)\theta_{t+1}} P^{n-K\theta_t-2\delta\varepsilon}
\ll \tilde{C} R^2 P^{n-Rd-(K-R(R+1)(d-1)\theta_t-\delta\varepsilon} \quad \text{(by (11))}
\ll \tilde{C} R^2 P^{n-Rd-\delta-\delta\varepsilon}. \quad \text{(by (9), (7) and (11))}
\]
Therefore,
\[
\int_{\alpha \not\in \mathfrak{M}(\theta_0)} |S(\underline{\alpha}, \underline{\nu})| \ d\underline{\alpha} = \int_{\alpha \not\in \mathfrak{M}(\theta_0)} |S(\underline{\alpha}, \underline{\nu})| \ d\underline{\alpha} + \sum_{t=0}^{T-1} \int_{\mathfrak{M}(\theta_{t+1})}^{} |S(\underline{\alpha}, \underline{\nu})| \ d\underline{\alpha}
\ll P^{n-2Rd} + P^{\delta\varepsilon} \tilde{C} R^2 P^{n-Rd-\delta-\delta\varepsilon}
\ll \tilde{C} R^2 P^{n-Rd-\delta}.
\]
\[\square\]
Denote the number of integer points in a box \(PB\) satisfying \(f(x) = \underline{\nu}\) by
\[
M(P, \underline{\nu}) = \int_{\underline{\nu} \in [0,1]^R} S(\underline{\alpha}, \underline{\nu}) \ d\underline{\alpha}.
\]

Corollary 2.6 (Generalisation of Lemma 4.5 in [Bir62]).
\[
M(P, \underline{\nu}) = \sum_{1 \leq q \leq \tilde{C} R P} \sum_{1 \leq a_i \leq q} \int_{\mathfrak{M}_{a_i q}(\theta_0)} S(\underline{\alpha}, \underline{\nu}) \ d\underline{\alpha} + O(\tilde{C} R^2 P^{n-Rd-\delta}),
\]
where \(O\) does not depend on \(\tilde{C}\).

2.3. Approximating exponential sums by integrals. Write \(\mathfrak{M}_{\underline{a}, q}\) for \(\mathfrak{M}_{\underline{a}, q}(\theta_0)\). Let \(\underline{\alpha} \in \mathfrak{M}_{\underline{a}, q}\) and define \(\underline{\beta} = \underline{\alpha} - \underline{a} / q\).

Letting \(\underline{x} = \underline{z} + q\underline{y}\) we find that
\[
S(\underline{\alpha}, \underline{\nu}) = \sum_{\underline{z} \mod q} \sum_{\underline{z} + q\underline{y} \in PB \cap \mathbb{Z}^R} e(\underline{a} \cdot (f(\underline{z} + q\underline{y}) - \underline{\nu}))
= \sum_{\underline{z} \mod q} e(\underline{a} \cdot (f(\underline{z} + q\underline{y}) - \underline{\nu}) / q) \sum_{\underline{z} + q\underline{y} \in PB \cap \mathbb{Z}^R} e(\underline{\beta} \cdot (f(\underline{z} + q\underline{y}) - \underline{\nu})).
\]

We wish to replace the sum \(\sum_{\underline{z} + q\underline{y} \in PB \cap \mathbb{Z}^R} e(\underline{\beta} \cdot f(\underline{z} + q\underline{y}))\) by the integral \(\int_{\underline{z} + q\underline{y} \in PB} e(\underline{\beta} \cdot f(\underline{z} + q\underline{y})) \ d\underline{\omega}\). For a measurable subset \(\mathcal{C}\) of \(\mathcal{E}\) and \(\underline{\gamma} \in \mathbb{R}^R\), we write
\[
(12) \quad I(\mathcal{C}, \underline{\gamma}) = \int_{\underline{\zeta} \in \mathcal{C}} e(\underline{\gamma} \cdot \tilde{f}(\underline{\zeta})) \ d\underline{\zeta}.
\]

Lemma 2.7. Given \(\underline{z}, \underline{\beta} \in \mathbb{Z}^R\) and \(q \in \mathbb{Z}_{>0}\), we have
\[
(13) \quad \sum_{\underline{z} + q\underline{y} \in PB \cap \mathbb{Z}^R} e(\underline{\beta} \cdot f(\underline{z} + q\underline{y})) = q^{-n} P^n I(B, P^d \underline{\beta}) + O((C |P^d \underline{\beta}| + 1) q^{1-n} P^{n-1}).
\]

Proof. For the system of polynomials \(\underline{r} = f - \tilde{f}\) of degree at most \(d - 1\) we have
\[
|e(\underline{\beta} \cdot \underline{r}(\underline{z} + q\underline{y})) - 1| \ll |\underline{\beta}||\underline{r}(\underline{z} + q\underline{y})| \ll |\underline{\beta}| \cdot CP^{d-1},
\]
and
\[
\int_{\underline{\zeta} \in \mathcal{C}} e(\underline{\gamma} \cdot \tilde{f}(\underline{\zeta})) \ d\underline{\zeta}.
\]
where we assumed that $z + qy \in PB$. There are $O((P/q)^n)$ values of $y$ in the sum, hence
\[
\sum_{z + qy \in PB \cap \mathbb{Z}^R} e(\beta \cdot f(z + qy)) = \sum_{z + qy \in PB \cap \mathbb{Z}^R} e(\beta \cdot \tilde{f}(z + qy)) + O(|\beta| \cdot Cq^{-n} P^{n+d-1}).
\]

Next, we replace the sum in the right-hand side by the integral
\[
(14) \quad \int_{z + q\omega \in PB} e(\beta \cdot \tilde{f}(z + q\omega)) \, d\omega.
\]
The edges of the cube of summation and integration have length $P/q$. In the replacement of the sum by the integral, we have an error of at most $\ll (P/q)^{n-1}$ coming from the boundaries. The variation in $e(\beta \cdot f(z + qy))$ results in an error of at most $O(|\beta|qC^{-1} P^{n+d-1}(P/q)^n)$. Hence, the total error in (13) is
\[
\ll |\beta|Cq^{-n} P^{n+d-1} + |\beta|\tilde{C}q^{-1} P^{n+d-1} + q^{1-n} P^{n-1} \ll (C |P^d \beta| + 1) q^{1-n} P^{n-1}.
\]

Applying the substitution $z + q\omega = P\zeta$ to (14) gives the desired result.

**Corollary 2.8.** Given $z \in \mathbb{Z}^R$ and $\alpha \in \mathcal{M}_{\zeta q}$ so that $\beta = \alpha - a/q$, we have
\[
\sum_{z + qy \in PB \cap \mathbb{Z}^R} e(\beta \cdot f(z + qy)) = q^{-n} P^n I(B, P^d \beta) + O(C\tilde{C}^{-1} P^{-n+\eta-1}).
\]

**Proof.** Estimate the error term in Lemma 2.7 by observing that for $\alpha \in \mathcal{M}_{\zeta q}$, it holds that $q \leq \tilde{C} R P^{\eta}$ and $|\beta| \leq \tilde{C} R^{-1} q^{-1} P^{-d+\eta}$. \hfill \square

**Corollary 2.9 (Generalisation of [Bir62, Lemma 5.1]).** Let $\alpha = a/q + \beta \in \mathcal{M}_{\zeta q}$. Then,
\[
S(\alpha, \nu) = P^n q^{-n} S_{\zeta q}(\nu) \cdot I(B, P^d \beta) \cdot e(-\beta \cdot \nu) + O(C\tilde{C}^{-1} P^{n+\eta-1}).
\]

### 2.4. Singular series

Define the **singular series** as
\[
\mathcal{G}(\nu) = \sum_{q=1}^{\infty} q^{-n} \sum_{\alpha \mod q \atop (\alpha, q) = 1} S_{\zeta q}(\nu),
\]
where $S_{\zeta q}(\nu)$ is defined by (5). The singular series converges absolutely under assumption (9) on $K$. This is made quantitative in the following lemma:

**Lemma 2.10 (Generalisation of [Bir62, p.256]).** For all $\tau \geq 0$ we have
\[
\sum_{P^{\tau q} \lessdot q < \infty} \sum_{\alpha \mod q \atop (\alpha, q) = 1} q^{-n} |S_{\zeta q}(\nu)| \ll \tilde{C}^{K/(d-1)} P^{-\tau \delta}.
\]

**Proof.** Observe $|S_{\zeta q}(\nu)| = |S_{\zeta q}|$. We have that
\[
\sum_{P^{\tau q} \lessdot q < \infty} \sum_{\alpha \mod q \atop (\alpha, q) = 1} q^{-n} |S_{\zeta q}(\nu)| \leq \sum_{P^{\tau q} \lessdot q < \infty} \sum_{\alpha \mod q \atop (\alpha, q) = 1} q^{-n} \tilde{C}^{K/(d-1)} q^{n-K/R(d-1)+\epsilon} \quad (\text{by Lemma 2.2})
\]
\[
\ll \tilde{C}^{K/(d-1)} \sum_{P^{\tau q} \lessdot q < \infty} q^{-1-\delta n^{-1}} \quad (\text{by } (9))
\]
\[
\ll \tilde{C}^{K/(d-1)} P^{-\tau \delta}. \quad \square
\]
As usual, for each prime \( p \) define the local density at \( p \) to be

\[
\mathcal{G}_p(\nu) = \sum_{r=0}^{\infty} \sum_{a \text{mod } p^r \atop \left(\frac{a}{p}\right) = 1} p^{-rn} S_{a,p^r}(\nu).
\]

Then, by multiplicativity of \( S_{a,q} \) we can factorize the singular series as a product over the local densities, i.e., \( \mathcal{G}(\nu) = \prod_{p \text{ prime}} \mathcal{G}_p(\nu) \).

2.5. Singular integral.

**Lemma 2.11** (Generalisation of [Bir62 Lemma 5.2]). One has

\[
|I(B; \gamma)| \ll \min(1, (\tilde{C}^{1-R}|\gamma|)^{-R-1-\delta \eta^{-1}} (\tilde{C}|\gamma|)^{\varepsilon}),
\]

where \( I(B; \gamma) \) is defined by (12).

**Proof.** \( |I(B; \gamma)| \ll 1 \) follows directly. Therefore, in proving the second part of the inequality we may assume that

(15) \( \tilde{C}^{1-R}|\gamma| > 1 \).

Take \( \tilde{P} = \tilde{C}|\gamma|((\tilde{C}^{1-R}|\gamma|)^{K/R(d-1)} \cdot ) \). By (15) and \( d \geq 2 \) we find that \( \tilde{P} > (\tilde{C}|\gamma|^{2})^{1/d} \). Hence, for \( \alpha = P^{-d}|\gamma| \) we have that \( |\alpha| < (\tilde{C}P^{d})^{-1/2} \). This implies, as can be found by generalising [Bir62 p. 252], we obtain

(16) \( |S(\alpha)| \ll P^{n+\varepsilon} (\tilde{C}^{1-R}P^{d}|\alpha|)^{-K/R(d-1)} \).

On the other hand, by Lemma 2.7 with \( \tilde{z} = 0, q = 1, a = 0 \), we obtain

(17) \( S(\alpha) = \sum_{\nu \in PB^{c} \cap R} e(\alpha \cdot f(\nu)) = P^{n} I(B, P^{d} \alpha) + O \left( (\tilde{C}|P^{d} \alpha| + 1) P^{n-1} \right) \).

Combining (16) and (17) we obtain

\[
|I(B, \gamma)| \ll (\tilde{C}^{1-R}|\gamma|)^{-K/R(d-1)} (\tilde{C}|\gamma|)^{\varepsilon}.
\]

Estimating \( K/R(d-1) \) by \( R + 1 + \delta \eta^{-1} \) using (9) completes the proof. \( \square \)

For \( \underline{\nu} \in \mathbb{Z}^{R} \) and \( \Phi \in \mathbb{R}_{\geq 0} \), write

\[
J(\underline{\nu}, \Phi) = \int_{|\gamma| \leq \Phi} I(B, \gamma) e(-\gamma \cdot \underline{\nu}) \, d\gamma,
\]

where \( I(B; \gamma) \) is defined by (12). Define the singular integral to be \( J(\underline{\nu}) = \lim_{\Phi \to \infty} J(\underline{\nu}, \Phi) \). This limit exists and \( J(\underline{\nu}) \) is continuous. The following lemma estimates the speed of convergence.

**Lemma 2.12** (Generalisation of [Bir62 Lemma 5.3]). For all \( \Phi > 0 \) we have

\[
|J(\underline{\nu}) - J(\underline{\nu}, \Phi)| \ll \tilde{C}^{R^{2} - 1 + (R - 1)\delta \eta^{-1}} \Phi^{1 - \delta \eta^{-1}}.
\]

**Lemma 2.13.** For all \( \underline{\nu} \in \mathbb{Z}^{R} \) it holds that

\[
|J(\underline{\nu})| \ll \tilde{C}^{R(R - 1)}.
\]

**Proof.** By the trivial bound in Lemma 2.11 we have \( |J(\underline{\nu}, \tilde{C}^{R - 1})| \ll \tilde{C}^{R(R - 1)} \). The previous lemma implies that \( |J(\underline{\nu}) - J(\underline{\nu}, \tilde{C}^{R - 1})| \ll \tilde{C}^{R^{2} - 1 + (R - 1)\delta \eta^{-1}} \tilde{C}^{R(R - 1)} \), which implies the lemma. \( \square \)
2.6. **Major arcs.** We are now ready to give an asymptotic for the number of integer points in a box \( PB \):

**Lemma 2.14** (Generalisation of [Bir62] Lemma 5.5).

\[
\frac{M(P; \nu)}{P^{n-Rd}} = \mathcal{S}(\nu) J(P^{-d} \nu) + O\left( \widetilde{C}^{R^2-R} P^R \nu^R \left( C \widetilde{C}^{R^2+2R-1} P^{-1+2(R+1)} \nu + \widetilde{C} K/(d-1) P^{-\delta} \right) \right).
\]

**Proof.** By Corollary 2.6 we have that

\[
M(P; \nu) = \sum_{1 \leq q \leq C^R P^n} \sum_{1 \leq a_i \leq q \atop (a_i, q) = 1} \int \sum_{\nu} S(\alpha, \nu) \, d\alpha + O(\widetilde{C}^{R^2} P^{n-Rd-\delta})
\]

\[
= \sum_{1 \leq q \leq C^R P^n} \sum_{1 \leq a_i \leq q \atop (a_i, q) = 1} \int \sum_{\nu} S(\alpha, \nu) \, d\beta + O(\widetilde{C}^{R^2} P^{n-Rd-\delta}),
\]

where in the second integral it is understood that \( \alpha = a/q + \beta \). As \( S_{a_1, q}(\nu) \leq q^n \) and there are at most \( (\widetilde{C}^R P^n)^{R+1} \) choices for \( a \) and \( q \), we find using Corollary 2.9 that

\[
M(P; \nu) = P^{n-Rd} \sum_{1 \leq q \leq C^R P^n} q^{-n} \sum_{1 \leq a_i \leq q \atop (a_i, q) = 1} S_{a_1, q}(\nu) \int |\gamma| \leq \widetilde{C} R^{-1} P^n \, I(\mathcal{B}; \gamma) \cdot e(-\gamma \cdot P^{-d} \nu) \, d\gamma + \mathcal{O},
\]

where

\[
\mathcal{O} = O(C \widetilde{C}^{2R^2+R-1} P^{n-Rd-1+2(R+1)} \nu) + O(\widetilde{C} K/(d-1) + R^2 R P^{n-Rd-\delta}).
\]

Using Lemma 2.12 and Lemma 2.10 for \( \tau = 0 \) and afterwards Lemma 2.10 for \( \tau = 1 \) and Lemma 2.13 we find

\[
\frac{M(P; \nu)}{P^{n-Rd}} = \sum_{1 \leq q \leq C^R P^n} q^{-n} \sum_{1 \leq a_i \leq q \atop (a_i, q) = 1} S_{a_1, q}(\nu) J(P^{-d} \nu) + O \left( P^{n-Rd} \widetilde{C} K/(d-1) \widetilde{C} R^2 R P^{-\eta-\delta} \right) + \mathcal{O}
\]

\[
= \left( \mathcal{S}(\nu) + O(\widetilde{C}^{K/(d-1)} P^{-\delta}) \right) J(P^{-d} \nu) + \mathcal{O}
\]

\[
= \mathcal{S}(\nu) J(P^{-d} \nu) + \mathcal{O}.
\]

**Theorem 2.15.**

\[
M(P; \nu) = P^{n-Rd} \mathcal{S}(\nu) J(P^{-d} \nu) + O(C \widetilde{C}^{K/(d-1)+R^2-1} P^{n-Rd-\delta}),
\]

where

\[
\delta < \frac{K - R(R + 1)(d - 1)}{K + R(R + 1)(d - 1)}.
\]

**Proof.** Let \( \varepsilon > 0 \) be given and take

\[
\delta < \left( \frac{K}{R(d - 1)} - (R + 1) \right) \eta, \quad \eta = \frac{1}{K/R(d - 1) + R + 1}.
\]

It follows directly that \( \mathcal{S} \) and \( \mathcal{O} \) are satisfied. Moreover, \( P^{-1+2(R+1)} \nu < P^{-\delta} \). The statement now follows directly from Lemma 2.14. \( \square \)
3. Quantitative strong approximation

3.1. Lower bound for the singular series.

Lemma 3.1. If there exists a non-singular solution \( x_0 \in \mathbb{Z}_p^n \) to \( f(x) = \nu \), then
\[
\mathcal{S}_p(\nu) \geq (p^{-1} \max_I |\Delta_I(x_0)|_p^2)^{n-R}.
\]

Proof. We have that
\[
p^{N(n-R)} \sum_{r=0}^{N} \sum_{\alpha \mod p^r \mod \mathbb{Z}_p} p^{-rn} S_{\alpha, p^r}(\nu)
\]
is the number of points satisfying \( f(x) = \nu \mod p^N \). So,
\[
\mathcal{S}_p(\nu) = \lim_{N \to \infty} \sum_{r=0}^{N} \sum_{\alpha \mod p^r \mod \mathbb{Z}_p} p^{-rn} S_{\alpha, p^r}(\nu) = \lim_{N \to \infty} p^{N(R-n)} \# \{ x \mod p^N | f(x) \equiv \nu \mod p^N \}.
\]

Now, take \( e \in \mathbb{Z} \) such that \( p^{-e} = \max_I |\Delta_I(x_0)|_p \) and assume that \( N > 2e + 1 \). The non-singular solution \( x_0 \in \mathbb{Z}_p^n \) gives a non-singular solution modulo \( p^{2e+1} \). Using Hensel’s lemma, we can lift this solution to at least \( p^{(n-R)(N-2e-1)} \) non-singular solutions of \( f(x) \equiv \nu \mod p^N \). Hence, \( \mathcal{S}_p(\nu) \geq (p^{-1} \max_I |\Delta_I(x_0)|_p^2)^{n-R} \) as desired. \( \square \)

Let \( I \) be a subset of \( \{1, \ldots, n\} \) of size \( R \) and let \( \Delta_I(x) \) be the \( R \times R \)-minor of the Jacobian matrix of \( f \) (of dimensions \( R \times n \)) with columns given by the elements of \( I \). Similarly, let \( \tilde{\Delta}_I(x) \) be the \( R \times R \)-minor of the Jacobian matrix of \( \tilde{f} \) with columns given by the elements of \( I \).

From now on assume that \( V \) and \( \tilde{V} \) are non-singular over \( \overline{\mathbb{Q}} \) as affine respectively projective varieties. Consider the polynomials \( f \) and all \( R \times R \)-minors \( \Delta_I \). As \( V \) is non-singular, these polynomials have no common zero over \( \overline{\mathbb{Q}} \). Hence, by the Nullstellensatz, the ideal generated by these polynomials equals \( \overline{\mathbb{Q}}[x] \). This is made quantitative in Theorem 1 of [KPS01]: there exists an \( N \in \mathbb{Z}_{>0} \) and polynomials \( g_1, \ldots, g_R \) and \( g_I \) in \( \mathbb{Z}[x] \) for all \( I \subset \{n\} \) with \( |I| = R \) such that
\[
\sum_{i=1}^{R} f_i(x)g_i(x) + \sum_I \Delta_I(x)g_I(x) = N,
\]
satisfying the estimate
\[
\log(N) \ll 4n(n+1)D^n \log(C^R),
\]
where \( D \) is such that \( \deg f_i \leq D \) and \( \deg \Delta_I \leq D \). Taking \( D = \max(R(d-1), d) \), we find
\[
N \ll C^{4n(n+1)R} \max(R(d-1), d)^n = \mathfrak{C},
\]
where the above equation defines \( \mathfrak{C} \).

Lemma 3.2. For all primes \( p \) for which there exists a solution \( x_0 \in \mathbb{Z}_p^n \) of \( f(x) = 0 \) we have
\[
\max_I |\Delta_I(x_0)|_p \geq |N|_p.
\]
Proof. Let \( p \) be a prime such that there exists an \( \underline{x}_0 \in \mathbb{Z}_p^n \) with \( f(\underline{x}_0) = 0 \), so that the first set of terms on the left-hand side of (18) vanish for \( \underline{x} = \underline{x}_0 \). Then taking \( p \)-adic absolute values in (18) shows that
\[
\max_i |\Delta_I(\underline{x}_0)|_p \max_j |g_I(\underline{x}_0)|_p \geq |N|_p.
\]
As \( g_I \in \mathbb{Z}[\underline{x}] \) we obtain \( \max_I |\Delta_I(\underline{x}_0)|_p \geq |N|_p. \)

\[\square\]

Lemma 3.3. If \( p \) is prime such that \( p \nmid d \) and \( p \nmid N \), then
\[
(20) \quad \mathcal{G}_p(0) - 1 \ll p^{-n/2+R+\varepsilon}.
\]

Proof. Suppose \( V \) is singular over \( \mathbb{F}_p \). Then, there exists an \( \underline{x} \in \mathbb{F}_p^n \) such that \( f(\underline{x}) = 0 \) and \( \Delta_I(\underline{x}) = 0 \) over \( \mathbb{F}_p \) for all \( I \subset [n] \) with \( |I| = R \). Considering (18) over \( \mathbb{F}_p \), it follows that \( N \equiv 0 \mod p \). This contradicts our assumption, so \( V \) is non-singular over \( \mathbb{F}_p \).

As pointed out by Schmidt [Sch84], a result of Deligne, worked out in the appendix of [Ser77], then shows that
\[
\#V_{\mathbb{F}_p}(0) = p^{n-R} + O(p^{n/2+\varepsilon})
\]
provided \( p \nmid d \), where the implied constant depends at most on \( n \) and \( d \). Observe that if \( \underline{x} \in \mathbb{Z}^n \) is a solution of \( f(\underline{x}) = 0 \) mod \( p^e \) for some \( e \in \mathbb{Z}_{>0} \), then \( \underline{x} \) reduces to a non-singular point on \( V_{\mathbb{F}_p} \). Hence, \( \underline{x} \mod p^e \) can be obtained by lifting a point of \( V_{\mathbb{F}_p} \). We conclude that
\[
\#\{x \mod p^N | f_i(x) \equiv v_i \mod p^N \text{ for all } i = 1, \ldots, R\} = p^{N(n-R)} + O(p^{(n-R)(N-1)+n/2+\varepsilon}).
\]

Similar to the proof of Lemma 3.1 we obtain (20). \[\square\]

Proposition 3.4. Suppose that for all primes \( p \) there exists a non-singular solution \( \underline{x}_0 \in \mathbb{Z}_p^n \) to \( f(\underline{x}_0) = 0 \). Then
\[
\mathcal{G}(0) \gg N^{-3(n-R)}.
\]

Proof. Let \( S \) be the finite set of primes for which \( p \mid dN \). Applying Lemma 3.1 and Lemma 3.2 and using the product formula for \( |\cdot|_p \) we obtain
\[
\prod_{p \in S} \mathcal{G}_p(0) \geq \prod_{p \in S} (p^{-1}|N|_p^2)^{-R} \gg (N^{-1}N^{-2})^{n-R} = N^{3(R-n)}.
\]

It follows from Lemma 3.3 that
\[
\prod_{p \notin S} \mathcal{G}_p(0) = \prod_{p \notin S} 1 + O(p^{-n/2+R+\varepsilon}) \gg 1,
\]
where the implied constant does not depend on \( C \). Therefore, we conclude that
\[
\mathcal{G}(0) = \prod_{p \in S} \mathcal{G}_p(0) \prod_{p \notin S} \mathcal{G}_p(0) \gg N^{-3(n-R)}.
\]
\[\square\]

3.2. Lower bound for the singular integral. The following lemma, which is a quantitative version of the inverse function theorem, is the real analogue of Lemma 3.1. Recall that \( \tilde{\Delta}_I(\underline{x}) \) is the \( R \times R \)-minor of the Jacobian of \( \tilde{f} \) with columns determined by \( I \). Abbreviate \( \tilde{\Delta}_{\{1,2,\ldots,R\}}(\underline{x}) \) by \( \tilde{\Delta}(\underline{x}) \).
Lemma 3.5. Given $x_0 \in \mathbb{R}^n$ with $|x_0| \leq \Lambda$ with $\Lambda \geq 1$, assume that $M := \max_{I \subseteq [n], |I| = R} |\Delta_I(x_0)| = |\Delta(x_0)| > 0$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$g : x \mapsto (\tilde{f}_1(x), \ldots, \tilde{f}_R(x), x_{R+1}, \ldots, x_n).$$

Then there are open subsets $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ with $x_0 \in U$ and $g(x_0) \in W$ such that $g$ is a bijection from $U$ to $W$ and has differentiable inverse $g^{-1}$ on $W$ with $\det((g^{-1})') \geq M^{-1}$ on $W$. Furthermore, one may choose

$$W = \left\{ y \in \mathbb{R}^n : |g(x_0) - y| \leq \frac{M^2}{C2^R \Lambda R(d-1)-1} \right\}.$$

Proof. We explicitly find a small open neighbourhood of $x_0$ in which the implicit function theorem is applicable, following the proof of [Spi63 Theorem 2.11] or [PSW16 Lemma 9.3]. Note that $M \ll \tilde{C}R \Lambda R(d-1)$. Let $U$ be the closed rectangle given by

$$U = \left\{ x \in \mathbb{R}^n : |x - x_0| \leq a \frac{M}{\tilde{C}R \Lambda R(d-1)-1} \right\},$$

for a sufficiently small constant $a \in \mathbb{R}$ depending only on $d, n$ and $R$. Then for $x \in U$ we have that $|x| \leq |x - x_0| + |x_0| \ll \Lambda$. Observe that $\frac{\partial g_k}{\partial x_i x_k}(x)$ for all $1 \leq i, j, k \leq n$ is a polynomial with maximal coefficient $\ll \tilde{C}$ of degree at most $d - 2$. Hence, $\frac{\partial g_k}{\partial x_i x_k}(x) \ll \tilde{C} \Lambda^{d-2}$. It follows that

$$\left| \frac{\partial g_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(x_0) \right| \ll \tilde{C} \Lambda^{d-2} |x - x_0| \ll a \tilde{C}^{-R+1} M.$$

As

$$\frac{\partial (g(x) - Dg(x_0) \cdot x_0)}{\partial x_j} = \frac{\partial g_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(x_0),$$

for $x, x_0 \in U$ we have

$$|g(x) - Dg(x_0) \cdot x_0 - g(x_0) + Dg(x_0) \cdot x_0| \ll a \tilde{C}^{-R+1} M |x_0 - x_0|.$$

Let $A$ be an invertible $n \times n$-matrix, denote with $|A| = \max_{i,j} |A_{i,j}|$ the max norm and assume that $|A| \ll 1$. For all $h \in \mathbb{R}^n$ one has $|h| \ll \frac{1}{\det A} |Ah|$. Let $A = \tilde{C}^{-1} Dg(x_0)$ (for which indeed $|A| \ll 1$ with an implied constant depending on $\Lambda, n$ and $d$, but not on $C$ or $\tilde{C}$). Since $M = |\Delta(x_0)| = |Dg(x_0)|$, we find that for $x_1, x_2 \in U$ we have

$$|\tilde{C}^{-1} Dg(x_0)(x_1 - x_2)| \gg \det(\tilde{C}^{-1} Dg(x_0)) |x_1 - x_2| = \tilde{C}^{-R} M |x_1 - x_2|.$$

Hence,

$$|g(x_1) - Dg(x_0) \cdot x_1 + g(x_0) - g(x_2) + Dg(x_0) \cdot x_2| \geq |Dg(x_0)(x_1 - x_2)| \gg \tilde{C}^{-R+1} M |x_1 - x_2|.$$

Therefore, using (21) for $a$ small enough, we find for all $x_1, x_2 \in U$ that

$$|g(x_1) - g(x_2)| \gg \tilde{C}^{-R+1} M |x_1 - x_2|.$$

This implies that if $x$ is on the boundary of $U$ we have

$$|g(x) - g(x_0)| \gg \tilde{C}^{-R+1} M |x - x_0| = a \frac{M^2}{\tilde{C}2^R \Lambda R(d-1)-1}.$$
Set \( b \gg a \frac{M^2}{C_2R^{-1}\Lambda R(d-1)} \) so that for \( x \) on the boundary of \( U \) it holds that \( |g(x) - g(x_0)| \gg b \) and define
\[
W = \{ y \in \mathbb{R}^n : |y - g(x_0)| < \frac{1}{2}b \}.
\]
The proof of [PSW16, Lemma 9.3] ensures that \( W \) has the required properties (after shrinking \( U \)).

**Theorem 3.6.** Suppose that \( x_0 \in \mathbb{R}^n \) with \( |x_0| \leq \Lambda \) satisfies \( \tilde{f}(x_0) = 0 \) and \( \Lambda \geq 1 \) such that
\[
M = \max_{I \subset [n], |I| = R} |\Delta_I(x_0)| > 0.
\]
Then, we have
\[
J(\emptyset) \gg M^{-1} \left( \frac{M^2}{C_22R^{-1}\Lambda R(d-1)} \right)^{n-R}.
\]

**Proof.** In Paragraph 11 of [Sch82] it is shown that for \( \mu \in \mathbb{R}^R \) we have
\[
J(\mu) = \lim_{t \to \infty} t^R \int_{|f(x) - \mu| \leq t^{-1}} \prod_{i=1}^{R} (1 - t|f_i(x) - \mu_i|) \, dx.
\]
Let \( \mathbb{1}_{1/2t} : \mathbb{R} \to \{0, 1\} \) be the characteristic function of the interval \([-\frac{1}{2t}, \frac{1}{2t}]\). Let \( U, W, g \) as in Lemma 3.3. Then,
\[
J(\emptyset) \geq \lim_{t \to \infty} \left( \frac{t}{2} \right)^R \int_U \prod_{i=1}^{R} \mathbb{1}_{1/2t} \circ \tilde{f}_i(x) \, dx.
\]
Applying the change of variables as in Lemma 3.3 we obtain
\[
\int_U \prod_{i=1}^{R} \mathbb{1}_{1/2t} \circ \tilde{f}_i(x) \, dx = \int_W |\det((g^{-1})')| \prod_{i=1}^{R} \mathbb{1}_{1/2t}(y_i) \, dy \geq \int_W M^{-1} \prod_{i=1}^{R} \mathbb{1}_{1/2t}(y_i) \, dy.
\]
For \( t \) sufficiently large, so that \( \mathbb{1}_{1/2t} \equiv 0 \) outside \( W \), the theorem follows.

Recall that we assumed that \( \widetilde{V} \) is non-singular. For the projective variety \( \widetilde{V} \) we have a similar reasoning as on page 10 for every affine patch of \( \widetilde{V} \) obtained by setting one of the coordinates \( x_j \) equal to 1. Let \( 1 \leq j \leq n \) be given. Because \( \widetilde{V} \) is non-singular over \( \overline{\mathbb{Q}} \), we find \( \widetilde{N}_j \in \mathbb{Z}_{>0} \) and polynomials \( \widetilde{g}_{1,j}, \ldots, \widetilde{g}_{R,j} \) and \( \widetilde{g}_{I,j} \in \mathbb{Z}[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \) for all \( I \subset [n] \) with \( |I| = R \) such that
\[
\sum_{i=1}^{R} \tilde{f}_i(x) \widetilde{g}_{i,j}(x) + \sum_I \Delta_I(x) \widetilde{g}_{I,j}(x) = \widetilde{N}_j
\]
for all \( x \) with \( x_j = 1 \). Denote with \( \|g\|_\infty \) the height of a polynomial \( g \), that is \( \|g\|_\infty \) is the maximum of the absolute values of the coefficients of \( g \). Then, by Theorem 1 in [KPS01], equation (22) satisfies the following estimate:
\[
\log \|\widetilde{g}_{I,j}\|_\infty \ll 4n(n-1)D^{n-1} \log(\widetilde{C}^R),
\]
for all \( I \subset [n] \) with \( |I| = R \). Here, we again take \( D = \max(R(d-1), d) \) so that
\[
\|\widetilde{g}_{I,j}\|_\infty \ll \widetilde{C}^{4n(n-1)R \max(R(d-1), d)^{n-1}} = \widetilde{C},
\]
where the above equation defines \( \widetilde{C} \). Also, let \( \widetilde{N} = \min_j \widetilde{N}_j \).
Lemma 3.7. Let $x_0 \in \mathbb{R}^n$ be such that $|x_0| = 1$ and $\tilde{f}(x_0) = 0$. Then for some $1 \leq j \leq n$ one has
\[
\max_i |\tilde{\Delta}_i(x_0)| \gg \tilde{c}^{-1} \tilde{N}_j.
\]

Proof. This is essentially the same proof as the proof of Lemma 3.2. Substitute $x = x_0$ in (22) for a choice of $j$ such that $(x_0)_j = |x_0| = 1$. Then the first sum vanishes and we find that
\[
\max_i |\tilde{\Delta}_i(x_0)| \gg \tilde{N}_j.
\]
Note that $\tilde{g}_{I,j}(x_0) \ll \|\tilde{g}_{I,j}\|_\infty \ll \tilde{c}$. This implies that
\[
\max_i |\tilde{\Delta}_i(x_0)| \gg \tilde{c}^{-1} \tilde{N}_j.
\]

Corollary 3.8. Suppose $\tilde{V}$ is non-singular and $\tilde{f}$ has a non-singular real zero. Then
\[
J(0) \gg \tilde{c}^{-(2(n-R)-1)} \tilde{c}^{(-2R+1)(n-R)} \tilde{N}^{2(n-R)-1}.
\]

Proof. Observe that by homogeneity of $\tilde{f}$ we can assume that the non-singular real zero $x_0$ satisfies $|x_0| = 1$. The corollary then follows directly from Theorem 3.6 and Lemma 3.7. □

3.3. Main theorems.

Proof of Theorem 1.7. From Theorem 2.15 it follows that for $P$ satisfying
\[
P \gg \left(\frac{C \tilde{c}^{K/(d-1)+R^2-1}}{\mathcal{S}(0)J(0)}\right)^{1/\delta}
\]
we have that $M(P,0) > 0$ (if the implied constant is large enough). Hence, there exists an integer zero $x$ of $f$ with $|x| \leq P$.

By Proposition 3.4, Corollary 3.8, (19) and $\tilde{N} \geq 1$ it follows that
\[
\mathcal{S}(0)J(0) \gg \tilde{c}^{-(2(n-R)-1)} \tilde{c}^{(-2R+1)(n-R)} \left(\frac{\tilde{N}^2}{\tilde{N}^3}\right)^{n-R} \tilde{N}^{-1}
\]
\[
\gg \tilde{c}^{-(2(n-R)-1)} \tilde{c}^{(-2R+1)(n-R)}.
\]
Using that $(n+1)(n-R) < n^2$, one finds that one can take
\[
P = c(C^3 \tilde{c}^2)^{4n^3 R(Rd)^n} \frac{K^{R(R+1)(d-1)}}{K^{-R(R+1)(d-1)}}.
\]
where $c$ is a constant not depending on $C$ and $\tilde{c}$. □

We can do slightly better in case we add the assumption that the polynomials $f_i$ are homogeneous:

Theorem 3.9. Suppose $f_i \in \mathbb{Z}[x]$ for $i = 1, \ldots, R$ are homogeneous polynomials of degree $d$ so that $K - R(R+1)(d-1) > 0$, $f$ has zeros over $\mathbb{Z}_p$ for all primes $p$ and a non-singular real zero. Assume that the corresponding projective variety $\tilde{V}$ is non-singular. Then there exists an $x \in \mathbb{Z}^n \setminus \{0\}$, polynomially bounded by $C$ and $\tilde{c}$, such that $\tilde{f}(x) = 0$, namely
\[
|x| \ll C^{12n^3 R(Rd)^n} \frac{K^{R(R+1)(d-1)}}{K^{-R(R+1)(d-1)}}.
\]
Proof. As in the proof of Theorem 1.1 (with $C = \tilde{C}$) we use that for $P$ satisfying

$$P \gg \left( \frac{\tilde{C}^{K/(d-1)+R^2}}{\mathcal{S}(0),J(0)} \right)^{1/\delta}$$

we have that $M(P,\mathcal{O}) > 0$ (if the implied constant is large enough). The quantitative version of the Nullstellensatz for $\tilde{f}$ given in (23) does still hold. Hence, mutatis mutandis, the proof of Proposition 3.4 applies and we find that $\mathcal{S}(\mathcal{O}) \geq \tilde{N}^{-3(n-R)}$. Together with Corollary 3.8 and (19) it follows that

$$\mathcal{S}(0)J(0) \gg \tilde{c}^{-(2(n-R)-1)\tilde{C}(-2R+1)(n-R)\tilde{N}^{-n}+R-1} \gg \tilde{c}^{-(3(n-R)-2)\tilde{C}(-2R+1)(n-R)}.$$ 

One finds that one can take

$$P = c\tilde{C}^{12n^3(Rd)^n}.K^{K+R(R+1)(d-1)}$$

where $c$ is a constant not depending on $C$ and $\tilde{C}$. \hfill \square

Remark. In case of one homogeneous form of degree $d = 3$ with $\Delta = 0$, so that $K = n/4$, one obtains the upper bound

$$|x| \ll \tilde{C}^{12n^3.\frac{n+16}{n-16}}.$$ 

This is visibly worse than the bound found in Theorem 1 of [BDE12]. However, (24) also applies when $d > 3$.

As already indicated in the introduction, we provide a quantitative strong approximation theorem for systems $\tilde{f}$ satisfying the same conditions as in Theorem 1.1

**Theorem 3.10.** Let $m, M \in \mathbb{Z}^n$. Suppose $f_i \in \mathbb{Z}[x]$ for $i = 1, \ldots, R$ are polynomials of degree $d$ so that $K - R(R+1)(d-1) > 0$ and the corresponding varieties $V$ and $\tilde{V}$ are non-singular affine respectively projective varieties. Suppose that a zero $y \in \mathbb{Z}_p$ of $f$ satisfying $y_i \equiv m_i \mod M_i$ exists for every prime $p$ and suppose $\tilde{f}$ has a real zero. Then, there exists an $x \in \mathbb{Z}^n$, polynomially bounded by $C$ and $\tilde{C}$, such that

$$f(x) = 0 \quad \text{and} \quad x_i \equiv m_i \mod M_i$$

and

$$|x| \ll \left( |M|^{5dC^3\tilde{C}^2} 2^{4n^3R(Rd)^n}.K^{K+R(R+1)(d-1)} \right)^{n/(Rn+1)(d-1)};$$

where the implied constant does not depend on $C, \tilde{C}, m$ or $M$.

Proof. Let

$$g(y) = f(My + m) \quad \text{and} \quad \tilde{g}(y) = f(My + m) = \tilde{f}(My),$$

where $(My)_i = M_i y_i$. Observe that over $\mathbb{Q}$ we have that $f$ or $\tilde{f}$ is non-singular if and only if $g$ respectively $\tilde{g}$ is non-singular. Moreover, the condition on the existence of zeros of $\tilde{f}$ ensures that $g$ has zeros over $\mathbb{Z}_p$ for all primes $p$ and that $\tilde{g}$ has a zero over $\mathbb{R}$. Hence, we can apply Theorem 1.1 to $g$. The theorem follows by noting that the maximal coefficient of $g$ and $\tilde{g}$ is $\ll |M|^dC$, respectively $|M|^d\tilde{C}$ as we can assume without loss of generality that $|m_i| \leq |M_i|$ for all $i = 1, \ldots, n$. \hfill \square
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