The Dynamics of Relativistic Hypersurfaces

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A dynamical theory of hypersurface deformations is presented. It is shown that a \((n+1)\)-dimensional space-time can be always foliated by pure deformations, governed by a non zero Hamiltonian. Quantum deformation states are defined by Schrödinger’s equation constructed with the corresponding deformation Hamiltonian operator, interpreted as the generator of the deformation diffeomorphism group. Applications to quantum gravity and to a modified Kaluza-Klein theory are proposed.

I. INTRODUCTION

Some present problems in theoretical physics, involve local changes of curvature due to density fluctuations in cosmology, changes in the space-time topology due to black and worm holes formation and involve the compactification of subspaces in higher dimensional models. These problems motivate the search for an efficient mechanism capable of describing classical and quantum changes of the geometrical and topological properties of space-time. Considering all possibilities, one such mechanism would be necessarily too complex. However, for some applications it is possible to devise a simpler mechanism. Take for example a system of particles distributed in an neighborhood of a point of an initial hypersurface \(\bar{V}_n\) of some \((n+1)\)-dimensional manifold \(V_{n+1}\) taken as a Cauchy surface. By heating this system, the particles leave \(\bar{V}_n\) but they will not necessarily land all into another hypersurface later on. This will depend on the energy level, interactions and symmetry of the system. A simpler situation occurs when the particles have compatible energies and initial data so that they move from one surface to another. In this case, the evolution of the system, can be described as a dynamical deformation of the original hypersurface.

The purpose of this note is to present a model of dynamical deformations of hypersurfaces and its applications to quantum gravity in four dimensions and to a quantum Kaluza-Klein theory. The concept of classical deformation of a hypersurface as a perturbative process is reviewed in the next section. In section 3 some known geometrical results are applied to show that pure deformations generate a foliation, associated with a dynamical process with a non-zero Hamiltonian. In section 4 the quantum aspect of these deformations is discussed and applied to quantum gravity described by 3-dimensional hypersurfaces in 4-dimensional space-time with some limitations. In section 5 we extend the quantum deformations to higher dimensions obtaining a modified Kaluza-Klein theory.

II. THE GEOMETRY OF DEFORMATIONS

Given a point \(p\) in \(\bar{V}_n\) and an arbitrary vector \(\zeta\) in \(V_{n+1}\) there is a one parameter group of diffeomorphism \(h_s : V_{n+1} \rightarrow V_{n+1}\), whose orbit \(\alpha(s) = h_s(p)\), is an integral curve of \(\zeta\), passing through \(p\). A perturbation of an object \(\bar{\Omega}\) in \(\bar{V}_n\) induced by this diffeomorphism along \(\alpha(s)\) is given by \(\bar{\Omega} = \bar{\Omega} + sL_\zeta \bar{\Omega}\).

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The hypersurface $\bar{V}_n$ containing $p$ is given by the local and isometric embedding $\bar{\mathcal{X}} : \bar{V}_n \to V_{n+1}$ such that

$$
\bar{g}_{ij} = \bar{\mathcal{X}}_i^\mu \bar{\mathcal{X}}_j^\nu \bar{g}_{\mu \nu}, \quad \bar{\mathcal{X}}_i^\mu \delta^\nu_\mu G_{\mu \nu} = 0, \quad \delta^\nu_\mu \bar{g} = -1, \quad \bar{k}_{ij} = -\bar{\mathcal{X}}_i^\mu \bar{\mathcal{X}}_j^\nu \bar{g}_{\mu \nu}
$$

(1)

where $\bar{\eta}$ is the unit vector normal to $\bar{V}_n$ and $\bar{k}_{ij}$ is its extrinsic curvature. A deformation of the hypersurface $\bar{V}_n$ along $\zeta$ is the subset of $V_{n+1}$ described by the coordinate $Z^\mu$ given by the perturbation of the embedding vielbein $\bar{\mathcal{X}}_i^\mu$:

$$
Z^\mu_i(x^i, s) = \bar{\mathcal{X}}_i^\mu + sL\zeta \bar{\mathcal{X}}_i^\mu = \bar{\mathcal{X}}_i^\mu + s[\zeta, \bar{\mathcal{X}}_i^\mu]
$$

(2)

$$
\eta = \bar{\eta} + sL\zeta \bar{\eta} = \bar{\eta} + s[\zeta, \bar{\eta}]
$$

(3)

In order to associate these deformations with a physical process, two conditions are required: In the first place the deformation must be free of coordinate gauges. Secondly, the deformed hypersurface must be again a hypersurface.

Given two distinct deformations of the same $\bar{V}_n$ corresponding to two slightly distinct directions such that $\zeta' - \zeta = \delta \zeta$, their difference is given by

$$
\delta Z^\mu_i = sL\delta \zeta \bar{\mathcal{X}}_i^\mu = s[\delta \zeta, \bar{\mathcal{X}}_i^\mu].
$$

In the theory of elastic membranes the tangent component of the deformation tension is canceled by the assumption that it is constant, which means invariant under the diffeomorphisms of the membrane. Here we cannot use the same argument. However, if $\bar{V}_n$ is endowed with a general diffeomorphism group then it is always possible to find a coordinate system in which the above Lie bracket vanishes. This means that the deformation can be canceled or transformed away by a choice of coordinate gauge in $\bar{V}_n$ which is of course undesirable for a physically generated deformations. It is possible to filter out these coordinate gauges by imposing condition on the geometry of $\bar{V}_n$. Here we cannot use the ad hoc procedure cannot be applied to generic deformations. Again, comparing with the example of elastic membranes, the fundamental modes of the deformation can be generated by deformations defined along the direction orthogonal to $\bar{V}_n$. These "pure" deformations denoted by $V_n$ are given by

$$
Z^\mu_i(x, s) = \bar{\mathcal{X}}_i^\mu(x) + sL\eta \bar{\mathcal{X}}_i^\mu = \bar{\mathcal{X}}_i^\mu(x) + s\eta^\mu_i(x).
$$

(4)

$$
\eta = \bar{\eta} + sL \bar{\eta} = \bar{\eta}.
$$

(5)

Once obtained a coordinate gauge independent deformation we need to make sure that it represents a hypersurface of $V_{n+1}$. To see that this is true, the embedding equations

$$
g_{ij} = Z^\mu_i Z^\nu_j G_{\mu \nu}, \quad Z^\mu_i \delta^\nu_\mu G_{\mu \nu} = 0, \quad \delta^\nu_\mu \bar{g} = -1, \quad k_{ij} = -Z^\mu_i \delta^\nu_\mu G_{\mu \nu}
$$

(6)

must be satisfied. Indeed, Any pseudo Riemannian manifold $V_{n+1}$ with metric signature $(n, 1)$ is necessarily foliated by pure deformations of a given hypersurface $\bar{V}_n$ with Euclidean signature. This follows from a simple adaptation of some well known embedding theorems based on manifold deformations. From (4), we obtain after applying (3) the metric $g_{ij}$ of $V_n$ in terms of the extrinsic curvature of $\bar{V}_n$:

$$
g_{ij} = Z^\mu_i Z^\nu_j G_{\mu \nu} = \bar{g}_{ij} - 2s\bar{k}_{ij} + s^2 \bar{g}^{mn} \bar{k}_{im} \bar{k}_{jn}.
$$

(7)

On the other hand, from the definition $k_{ij} = -Z^\mu_i \delta^\nu_\mu G_{\mu \nu}$, it follows that

$$
k_{ij} = \bar{k}_{ij} - s \bar{g}^{mn} \bar{k}_{im} \bar{k}_{jn}
$$

(8)

Comparing with (6) we obtain the evolution of the metric as

$$
\frac{dg_{ij}}{ds} = \dot{g}_{ij} = -2k_{ij}.
$$

(9)

1 Small case Latin indices $i, j, k...$ refer to an $n$-dimensional hypersurface $V_n$ and run from 1 to $n$. All Greek indices refer to the $(n + 1)$-dimensional manifold, running from 1 to $n + 1$. The metric of $V_n$ is denoted by $g_{ij}$ and $\nabla_i$ denotes the covariant derivative with respect to this metric. The covariant derivative with respect to the metric of $V_{n+1}$ is denoted by a semicolon and $\eta^\mu_i = \eta^\nu_i \bar{\mathcal{X}}_i^\nu$ denotes the projection of the covariant derivative $\eta^\mu_i$ over the submanifold $V_n$.

2This is known in the modern literature related to quantum gravity as York’s expression where the dot means the derivative with respect to time. To maintain the analogy, we use here the same dot notation: $\dot{\Omega} = d\Omega/ ds$, keeping in mind that $s$ is an independent parameter.
Although \( g = (g_{mn}) \), \( G = (g_{mn}) \) and \( k = (k_{mn}) \), the inverse metric can be given to any order \((k)\) of approximation as

\[
g^{-1} = \left( \sum_{n=0}^{k} (g^{-1})^n \right)^2 g^{-1}, \quad g^{-1} \approx 1 + O(k+1). \tag{10}
\]

From this we obtain \( \dot{g}_{ij} m^j + g_{ij} \dot{m}^j = 0 \). Therefore, to any order of approximation we may define \( k_{ij} = \frac{1}{2} \dot{g}^{ij} \) so that

\[
k_{ij} = g^{mn} g^{jn} k_{mn}
\]

In the limit \( k \to \infty \) we obtain the exact expression

\[
k_{ij} = \frac{2}{2} \dot{g}^{ij}.
\tag{11}
\]

Thus, in spite of the approximation rfgk, the indices of \( k \) are lowered and risen by the metric of \( V_n \), \( g_{ij} \) and \( g^{ij} \) respectively according to \( \left[ \right] \)

\[
g_{im} g_{jn} k_{mn} = g_{im} g_{jn} \frac{1}{2} \dot{g}_{mn} = \frac{1}{2} \frac{d}{ds} \left( g_{im} g_{jn} g^{mn} \right) - \frac{1}{2} \dot{g}_{im} g_{jn} g^{mn} - \frac{1}{2} \dot{g}_{im} g_{jn} g^{mn} = k_{ij}.
\]

To conclude, we make use of the particular reference frame adapted to the foliation, \( \eta^\alpha = \delta^\alpha_{n+1} \) where \( G_{ij} = g_{ij}, G_{n+1} = 0 \) and \( G_{n+1 n+1} = -1 \), and the components of the Riemann tensor are, after making use of \( \left(1\right) \)

\[
R_{\alpha \beta \gamma \delta} Z^\alpha_i Z_j^\beta Z^\gamma_k Z^\delta_l = R_{ijkl} - \frac{1}{4} \left( \dot{g}_{il} \dot{g}_{kj} - \dot{g}_{ik} \dot{g}_{jl} \right) = R_{ijkl} - 2k_{ij} k_{kl} \tag{12}
\]

and

\[
R_{\alpha \beta \gamma \delta} Z^\alpha_i \eta^\beta Z^\gamma_j Z^\delta_k = \frac{1}{2} \left( \frac{\partial \dot{g}_{ik}}{\partial x^j} - \frac{\partial \dot{g}_{ij}}{\partial x^k} \right) - \frac{1}{2} g_{mn} \left( \Gamma_{ikm} \dot{g}_{jn} - \Gamma_{ijm} \dot{g}_{kn} \right) = 2 \nabla_{[i} k_{j]} \tag{13}
\]

Notice that these tensor equations hold in any arbitrary coordinates and they readily seen to be the integrability conditions for a hypersurface of \( V_{n+1} \). Finally, the fundamental theorem of hypersurfaces states that for a given pair of tensors \( g_{ij} \) and \( k_{ij} \) satisfying \( \left(12\right) \) and \( \left(13\right) \), there exists a hypersurface \( V_n \) embedded in \( V_{n+1} \) described by \( Z^\alpha \) and with normal vector \( \eta^\mu \) satisfying \( \left(1\right) \). In other words, the pure deformations given by \( \left(1\right) \) produce the necessary and sufficient conditions to generate a hypersurface through \( \left(1\right) \).

### III. DEFORMATION DYNAMICS

The next question concerns the dynamical aspect of the deformations. To start, we notice that the first equation \( \left(1\right) \) can be solved as a tensor equation to give \( g^{ij} Z_i^\alpha Z_j^\beta = G^{\alpha \beta} + \Psi^{\alpha \beta} \), where \( \Psi^{\alpha \beta} \) is a non null tensor of \( V_{n+1} \) satisfying the condition \( G_{\alpha \beta} \Psi^{\alpha \beta} = -1 \). Applying in the second equation \( \left(1\right) \), it follows that the only solution compatible with \( G_{\alpha \beta} g^{ij} Z_i^\alpha Z_j^\beta = n \) and \( G^{\alpha \beta} G_{\alpha \beta} = n + 1 \) is \( \Psi^{\alpha \beta} = -\eta^\alpha \eta^\beta \), so that

\[
g^{ij} Z_i^\alpha Z_j^\beta = G^{\mu \nu} - \eta^\mu \eta^\nu \tag{14}
\]

Using \( \left(14\right) \), the contractions of \( \left(12\right) \) become

\[
R_{ijk} = g^{il} R_{ijkl} = -k_{ik} k_{lj} - h_{ijk} + R_{\beta \gamma} Z_j^\beta Z_k^\gamma + R_{\alpha \beta \gamma \delta} \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta Z_j^\beta Z_k^\gamma,
\tag{15}
\]

\[
R = -(h^2 - h^2) + R - 2R_{\alpha \beta \eta^\alpha \eta^\beta} \tag{16}
\]

where we have denoted the mean curvature of the deformation \( V_n \) by \( h = g^{ij} k_{ij} \) and \( k^2 = k_{ij} k_{ij} \). On the other hand the contraction of \( \left(13\right) \) gives

\[
\nabla_i k^k_{kj} - h_{ij} = R_{\alpha \beta} Z_i^\alpha \eta^\beta
\]
Therefore, the Einstein-Hilbert Lagrangian as derived directly from (14) in arbitrary coordinates is
\[ \mathcal{L} = \mathcal{R} \sqrt{-G} = \left[ R + \kappa^2 - h^2 \right] + 2R_{\alpha \beta} \eta^\alpha \eta^\beta \sqrt{-G} \quad (17) \]
Before discussing the Euler-Lagrange equations, let us proceed with the definition of the canonical momenta conjugated to the metric \( G_{\alpha \beta} \) with respect to the deformation parameter \( s \):
\[ \pi^{\alpha \beta} = \frac{\partial \mathcal{L}}{\partial (\pi^{\alpha \beta})}. \]
The corresponding Hamiltonian is given by the Legendre transformation
\[ \mathcal{H} = \mathcal{L} - \pi^{\alpha \beta} \dot{G}_{\alpha \beta} \quad (18) \]
where we notice that no specific metric decomposition (as in the ADM formulation) was used. In generic coordinates the metric \( G_{\alpha \beta} \) of \( V_{n+1} \) can be split as
\[ G_{\alpha \beta} = \left( \begin{array}{cc} g_{ij} & G_{i n+1} \\ G_{n+1 i} & G_{n+1 n+1} \end{array} \right), \]
Here \( G_{ij} \) corresponds to the metric of \( V_n \) and the index \( n+1 \) does not refer to the deformation parameter \( s \) but rather to just one of the coordinates \( x^{n+1} \) of \( V_{n+1} \). Assuming that the metric \( g_{ij} \) of \( V_n \) is written in arbitrary coordinates, we may identify without loss of generality \( G_{ij} = g_{ij} \), while keeping the remaining components \( G_{i n+1} \) and \( G_{n+1 n+1} \) completely arbitrary and unspecified. Using this notation, the momentum components corresponding to \( g_{ij} \) can be written as
\[ \pi^{ij} = -(k^{ij} - h g^{ij}) \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial \dot{g}_{ij}} \sqrt{-G} \quad (19) \]
Again, in the particular coordinates adapted to the deformation where \( \eta^\alpha = \delta^\alpha_{n+1} \), the scalar \( R_{\alpha \beta} \eta^\alpha \eta^\beta \) is given by is
\[ R_{\alpha \beta} \eta^\alpha \eta^\beta = \Gamma^\alpha_{n+1 \alpha, n+1} - \Gamma^\alpha_{n+1 n+1, n} + \Gamma^\alpha_{n+1, n+1, \alpha} \Gamma^\beta_{n+1 n+1, \beta} - \Gamma^\alpha_{n+1 n+1, \beta} - \Gamma^\alpha_{n+1, n+1, \beta} = \kappa^2 - h \quad (20) \]
Using (19), it follows that
\[ \theta^{ij} \overset{\text{def}}{=} \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial k_{ij}} = 2k^{ij} - \dot{g}^{ij} = 0, \]
Now \( g_{ij} \) and \( k_{ij} \) are independent quantities and \( \theta^{ij} \) is a tensor so that in any other coordinate system we also have
\[ \theta^{ij} = \frac{\partial x^i}{\partial x^n} \frac{\partial x^j}{\partial x^m} \theta^{mn} = k^{ij} - \dot{g}^{ij} = 0 \]
Therefore, (19) assumes a more familiar form in generic coordinates
\[ \pi^{ij} = -\frac{1}{2} \frac{\partial \mathcal{L}}{\partial k_{ij}} = -(k^{ij} - h g^{ij}) \sqrt{-G}, \quad (21) \]
Contrarily to York's relation (14), the pure deformation (4) does not prescribe the evolution of \( G_{i n+1} \) and \( G_{n+1 n+1} \). This means that the values of the corresponding momenta \( \pi^{i n+1} \) and \( \pi^{n+1 n+1} \) remain arbitrary and therefore their values should be given as constraints to the deformations. Since these expressions are all derived from the same basic equations (12), (13) and (1), they hold along the entire foliation and as suggested by (20) we set
\[ \pi^{i n+1} = -2 \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial G_{i n+1}} \sqrt{-G} = 0, \quad (22) \]
\[ \pi^{n+1 n+1} = -2 \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial G_{n+1 n+1}} \sqrt{-G} = 0 \quad (23) \]
Equation (22) ensures that the deformation does not have tangent components, a condition already imposed to guarantee a coordinate gauge independent deformation. Equation (23) tells that evolution of the system is given by the parameter \( s \) with fixed lapse.
As a consequence of the above constraints, the indices of $\pi^{ij}$ may be risen and lowered with the metric of $V_n$ alone, without the need to use a supermetric as required in other formulations \[2\]

$$\pi_{ij} = \mathcal{G}_{\alpha\beta} \pi^{\alpha\beta} = g_{im}g_{jn}\pi^{mn} = (hg_{ij} - k_{ij})\sqrt{-g}.$$ 

Using the definition $\pi = \mathcal{G}_{\alpha\beta} \pi^{\alpha\beta}$ and the above constraints we obtain the familiar expressions

$$k_{ij} = \frac{1}{\sqrt{-g}}\left(\pi \frac{1}{2} g_{ij} - \pi_{ij}\right), \quad \hbar = \frac{\pi}{2\sqrt{-g}} \quad \text{and} \quad k^2 - h^2 = \frac{\pi^2}{g} - \pi^{ij}\pi_{ij}.$$ 

and the Hamiltonian \[18\] can be written as

$$\mathcal{H} = R\sqrt{-g} - \frac{1}{\sqrt{-g}} \left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right) - 2\mathcal{R}_{\alpha\beta} \eta^\alpha \eta^\beta \sqrt{-g}.$$ \[24\]

Therefore Hamilton’s equations describing the foliation of $V_{n+1}$ with respect to $s$ are

\begin{align*}
\frac{dg^{ij}}{ds} &= \frac{2}{\sqrt{-g}} \left(\frac{\pi}{2} g^{ij} - \pi^{ij}\right) \sqrt{-g}, \\
\frac{d\pi^{ij}}{ds} &= -\left(R^{ij} - \frac{1}{2} R g^{ij}\right) \sqrt{g} + \frac{1}{\sqrt{-g}} \left[\pi^{ij} - 2\pi^{im}\pi_m^j + \frac{1}{2} \left(\frac{\pi^2}{2} - \pi^{mn}\pi_{mn}\right)g^{ij}\right].
\end{align*} \[25\], \[26\]

where the total derivative terms in the right hand side were dropped. Notice that the first equation coincide with York’s relation \[1\], meaning that the perturbative process which define the foliation is consistent with a canonical formulation of the theory.

The Euler-Lagrange equations derived from \[15\] with respect to $\mathcal{G}_{\alpha\beta}$ and to the deformation parameter $s$ gives the (vacuum) Einstein’s equations $\mathcal{R}_{\alpha\beta} = 0$ for $V_{n+1}$. If we wish, general relativity may be implemented by identifying $s$ as a time coordinate and by imposing the remaining postulates of that theory. The identification of $s$ with a coordinate can be made by the choosing a foliation based coordinate system where $\eta^\alpha = \delta^\alpha_{s+1}$ so that we have a $(n+1)$-dimensional general relativity written in a special frame. Of course, the imposition of general covariance eventually will mix $s$ with the remaining coordinates of $V_{n+1}$ and the undesirable coordinate gauges will appear in the deformations. On the other hand, we may decide to work with the special frame where $s$ is identified with a coordinate time. In this case the above constraints simply tell that Einstein’s equations in this frame assume a simpler form $\partial \mathcal{L}/\partial g_{s+1} = 0$ and $\partial \mathcal{L}/\partial g_{s+1} = 0$, which corresponds to Dirac’s canonical formulation in a fixed frame.

**IV. QUANTUM DEFORMATIONS**

To the deformation Hamiltonian $\mathcal{H}$ given by \[22\] we may now associate an Hermitian operator $\hat{\mathcal{H}}$ acting on a Hilbert space of the wave functions solutions of Schrödinger’s equation with respect to the deformation parameter $s$:

$$i\hbar \frac{d\Psi}{ds} = \hat{\mathcal{H}}\Psi.$$ \[27\]

The solution of this equation describes what can be called a quantum deformation state when the usual interpretations of quantum mechanics are given and when its semi classical limit corresponds to a small deformation of the hypersurface $V_n$. The commutators involving $\hat{\mathcal{H}}$, $\hat{\pi}^{ij}$ and $\hat{g}_{ij}$ correspond to the relevant Poisson brackets of the quantum geometry.

It should be remembered that not all postulates of general relativity are being imposed here. However, it is useful to compare the above formulation with the ADM metric decomposition procedure \[2\], trivially extended to $n+1$ dimensions. Using the same notation as before the metric of $V_{n+1}$ decomposes as a ADM as

$$G^{\alpha\beta} = \left(\begin{array}{cc}
g_{ij} & \frac{N_i N^j}{N^2} \\
\frac{N^i}{N^2} & -\frac{1}{N^2}
\end{array}\right)$$ \[28\]

where $N$ is the lapse function and $N_i$ the shift vector and where $s$ assumes the role of a coordinate time. The Einstein-Hilbert Lagrangian obtained directly from this metric is

$$\mathcal{L} = R\sqrt{-g} = -(kg^{ij} - h^2)\sqrt{g} \frac{dg^{ij}}{ds} - (N\mathcal{H}_o + N^i\mathcal{H}_i) - 2\frac{d}{ds}(h\sqrt{g}) - \nabla_i \phi^i.$$ \[29\]
Here $\mathcal{H}_0 = [R - (k^2 - h^2)]\sqrt{g}$ is the super Hamiltonian and $\mathcal{H}_i = 2[\nabla^i k_{ij} - h_{ij}]\sqrt{g}$ is the super momentum. Excluding the divergence and total derivatives in (29), the effective Lagrangian can be written as

$$\mathcal{L} = \pi^{ij} \frac{dg_{ij}}{ds} - \mathcal{H}_{ADM}$$

where $\pi^{ij} = -(k^{ij} - h^{ij})\sqrt{g}$ is identified with the momentum conjugate to $g_{ij}$ and $\mathcal{H}_{ADM} = N\mathcal{H}_0 + N^i\mathcal{H}_i$ is identified with the Hamiltonian. Taking the variation of the action with respect to $N$ and $N^i$ respectively we obtain

$$\mathcal{H}_0 = (R - (k^2 - h^2))\sqrt{g} = 0, \quad \mathcal{H}_i = 2(\nabla^i k_{ij} - h_{ij})\sqrt{g} = 0$$

so that $\mathcal{H}_{ADM} = 0$, up to surface terms. It is a simple matter to see that $\mathcal{H}_0$ corresponds to the double trace of $\mathcal{H}_i$ and that $\mathcal{H}_i$ corresponds to the trace of $\mathcal{H}_i$. Therefore $\mathcal{H}$ is constrained to zero over all hypersurfaces $V_n$. This is solved as Dirac constrained system over hypersurfaces, where $N$ and $N^i$ play the role of the Lagrange multipliers. However, when we attempt to canonically quantize the theory, the constraint algebra do not propagate as expected. In fact, the Poisson bracket algebra,

$$[\mathcal{H}_i(x), \mathcal{H}_j(x')] = -\mathcal{H}_j(x)\frac{\partial \delta(x,x')}{\partial x^i} + \mathcal{H}_i(x)\frac{\partial \delta(x,x')}{\partial x^j}$$

$$[\mathcal{H}_i(x), \mathcal{H}_0(x')] = \mathcal{H}_0(x)\frac{\partial \delta(x,x')}{\partial x^i}$$

$$[\mathcal{H}_0(x), \mathcal{H}_0(x')] = g^{ij}(x)\mathcal{H}_i(x)\frac{\partial \delta(x,x')}{\partial x^j} - g^{ij}(x')\mathcal{H}_i(x')\frac{\partial \delta(x,x')}{\partial x^j}$$

does not remain closed as a Lie algebra from one hypersurface to another.

The ADM formalism can also be described in terms of a parametric foliation of $V_{n+1}$ generated by deformations of hypersurfaces along the direction $s$ constructed with the components $N$ and $N^i$. However, due to the general covariance of $V_{n+1}$, the tangent component of this deformation characterized by the shift vector $N_i$ cannot be dispensed with. As we have already seen, this implies in the emergence of coordinate gauges and that the Hamiltonian becomes constrained to zero. We notice again that a basic difference from the present hypersurface dynamics and the ADM formulation is that the deformation parameter $s$ is not a coordinate and therefore it is not subjected to general coordinate transformations. As it was shown by Dirac, if we chose specific coordinates in the ADM formulation then the constraint on the Hamiltonian is also removed. However, admittedly it appears to be very difficult to conciliate general covariance with a non constrained canonical formalism of general relativity. A possible solution for this conflict may be obtained using another theory in which the postulates concerning symmetries are different from those required in general relativity. In the next section we examine one of these possibilities.

V. QUANTUM KALUZA THEORY

Kaluza-Klein theory represents a very intuitive scheme for the unification of fundamental interactions based on the Einstein-Hilbert principle. Unfortunately, it has become somewhat stagnant in face of two major difficulties. One of these problems is the inability of the theory to generate the light chiral fermions that are expected to be present at the electroweak level. The second and perhaps more serious problem is that it inherits the non-renormalizability of Einstein’s general relativity. In that theory, dimensional reduction uses the spontaneous compactification of the extra dimensions so as to render them unobservable at the lower energy levels. As a result, the physical space would have a topology like $V_4 \times B_N$ where $B_N$ is some compact space with a maximal number of Killing vector fields. Unlike general relativity, the ground state of the theory $M_4 \times B_N$ is not flat. A theorem due to Lichnerowicz implies that in this case the mas of fermions would be proportional to the inverse of the curvature radii of the compactified dimension. Additionally, we would end up with an infinite tower of massive states in four dimensional space-time which could still be present today. Many different proposals have been presented to modify the non-Abelian

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3 $M_4$ denotes Minkowski’s space-time. $V_D$ is a higher dimensional taken to be the physical space solution of the vacuum Einstein’s equations, with $D \geq 4$. Capital Latin indices run from 4 to $D_4$.

4 The old 5-dimensional theory does not present the same problem because its ground state $M_4 \times S^1$ happens to be flat. However as we know today, that theory does not have sufficient degrees of freedom to promote the intended unification of gravitation with electromagnetism.
Kaluza-Klein theory, but without a satisfactory solution of the above problems [12–17]. In this section we apply the deformation program to a non-compact version of Kaluza-Klein theory where the compactification hypothesis is replaced by the more general concept of breaking the translational symmetry along the extra dimensions.

Again we assume that physical space $\mathcal{V}_D$, $D \geq 4$ is a D-dimensional pseudo-Riemannian manifold, with metric signature $(3, D - 3)$, solution of the $D$-dimensional vacuum Einstein’s equations $R_{\mu\nu} = 0$. Dimensional reduction corresponds to a loss of energy of the system and consequently a loss of some of these degrees of freedom. It is reasonable to suppose that this happened in sequence, one after the other. That is, $V_D$ reduces to $V_{D-1}$, then to $V_{D-2}$... and so on, till reaching the present day $V_4$ which remains as a submanifold of $V_D$. In each of these steps, the particles of the reduced system, stay at a certain hypersurface $V_n$ of $V_{n+1}$, for $n = 4, 5, \cdots D - 1$ so that we may apply the deformation dynamics of last sections. To complete, we follow the same principle of general relativity to stabilize the ground state to be the D-dimensional Minkowski space $M_D$, instead of the traditional $M_4 \times B_N$. With this choice the consequences of the compactification are removed and in principle large mass fermions will not appear.

In the original Kaluza-Klein theory the total amount of energy required to compactify the extra degrees of freedom down to Planck’s wavelength of $10^{-33}$ cm is Planck’s energy $\approx 10^{19}$ Gev proportional to the ratio of the coupling constants in the unified Lagrangian. Therefore, it is reasonable to suppose here that this corresponds to the energy lost in the breaking of the translational symmetry. Reciprocally, the same of energy is required to restore that symmetry. Therefore the range of definition of the theory is that of Planck’s energy, which definitively denounces a quantum affair. In accordance with this, we use quantum deformations to access the extra dimension $\eta^A$ by the translation generated by the deformation Hamiltonian, according to Schrödinger’s equation corresponding to $s^A$:

$$i\hbar \frac{d\Psi_A}{ds^A} = \hat{H}_A \Psi_A, \quad A = n + 1...D$$

where the operator $\hat{H}_A$ corresponds to the deformation Hamiltonian along the extra dimension $\eta_A$. The corresponding dimensional reduction occurs when $\hat{H}_A$ becomes constrained to zero. In the application to general relativity this results from the imposition of general covariance to the extra dimensions $\eta_A$ but in a Kaluza-Klein scheme this condition is not required. This represents a contrast with Klein’s compactification hypothesis which enabled the harmonic expansion of the fields in terms of the internal variables, producing a hybrid theory where a quantum sized compact geometry was described by a classical metric.

The choice of ground state $M_D$ and its signature $(3, D - 3)$ implies that the complementary space orthogonal to the space-time is Euclidean, with a compact internal group $SO(D - 4)$. From the embedding point of view this seems to be too restrictive. In fact, it is well known that any four dimensional space-time can be analytically embedded in $M_{10}$. However, the theorem of last section dispenses with the analyticity of the embedding (which in terms of high energy physics sounds like a luxury anyway) and we have in fact differentiable embeddings. This increases the upper limit of D to 14 dimensions [13], making the model compatible with an $SO(10)$ gauge group. The use of Euclidean signature in the orthogonal space may also be a cause for concern because of the dependence of the fermion chirality on the signature of the space [13]. Yet, we must keep in mind the possibility of that the signature of that space may change as a result of the quantum dynamics.

To end, we will write the equivalent to the classical Kaluza-Klein metric. Consider the ground state $M_D$ containing the space-time $V_4$ as a subspace and that we have sufficient energy so that all dimensions are freely accessible. Since we have no memory on which dimension was reduced first and which one should be restored first, we define a single direction $\eta$ orthogonal to $V_4$ as a linear combination of all $(D - 4)$ extra dimensions:

$$s\eta = \sum_{5}^{D-4} s^A \eta_A, \quad s \neq 0$$

The pure deformation along this direction with parameter $s$ is given by

$$2^\mu = \tilde{X}_i^\mu + s\eta_i^\mu = \tilde{X}_i^\mu + \sum_A x^A \eta_A,i^\mu.$$ 

This corresponds to the superposition of $D - 4$ pure deformations, one for each normal $\eta_A$ given by (31). In the case of the chosen signature, the isometric embedding equations for each of these deformations are now given by

$$g_{ij} = 2_{\mu,i}^\mu G_{\mu
u}, \quad 2^\mu \eta_A^\nu G_{\mu
u} = s^M A_{i,M,A}, \quad \eta_A^\mu \eta_B^\nu G_{\mu
u} = g_{AB} = -\delta_{AB}$$

where

$$k_{ij,A} = -2_{\mu,i}^\mu \eta_A^\nu G_{\mu
u}, \quad A_{i,AB} = \eta_A,i^\mu \eta_B^\nu G_{\mu
u},$$

$$\quad (32)$$

$$\quad (33)$$

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As compared with \( \Box \), we notice the emergence of a new geometrical object, the twisting vector \( A_{iAB} \), which appears in the derivative of \( \eta_A \)

\[
\eta^\mu_A = -g^{mn}k_{im}Z^\mu_{jn} + g^{MN}A_{iMA}
\]

Equations (32) can also be seen as an expression of the metric components of \( M_D \) written in terms of the coordinate basis defined by the multiparameter deformation. In order to make a distinction from \( \Box \), we denote it by \( \gamma_{\alpha\beta} \) with separate components:

\[
\begin{align*}
\gamma_{ij} &= Z^\mu_i Z^\mu_j G_{\mu\nu} = \tilde{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A s^B (\tilde{g}^{mn}\bar{k}_{imA}\bar{k}_{jnB} + g^{MN}A_{iMA}A_{jNB}) \\
\gamma_{iA} &= Z^\mu_i \eta^\mu_{A\nu} G_{\mu\nu} = s^M A_{iMA} \\
\gamma_{AB} &= \eta^\mu_{A\nu} \eta^\mu_{B\nu} G_{AB} = g_{AB}
\end{align*}
\]

or, after denoting

\[
\begin{align*}
g_{ij} &= \tilde{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A s^B g^{MN}\bar{k}_{imA}\bar{k}_{jnB} \\
A_{iA} &= s^M A_{iMA}
\end{align*}
\]

this metric can be written as

\[
\gamma_{\alpha\beta} = \begin{pmatrix} g_{ij} + g^{MN}A_{iMA}A_{jNB} & A_{iA} \\ A_{jB} & g_{AB} \end{pmatrix}
\]  

(34)

which has the same appearance as the Kaluza-Klein metric ansatz, but in fact it contains some relevant differences:

Firstly, notice that the metric \( g_{ij} \) is not the metric of the initial hypersurface but rather the metric of a deformation. The other relevant difference is that although there are no compact spaces, the twisting vector \( A_{iAB} \) transforms like Yang-Mills potential over the space-time, relative to the group of isometries of the orthogonal space \([19]\). The metric \( g_{AB} = -\delta_{AB} \) is the metric of the extra dimensional space which is naturally written in its Killing basis of the group \( SO(D - 4) \). Finally, the expansion of the Lagrangian calculated from this metric in terms of \( s^A \) produces

\[
\mathcal{L} = \mathcal{R}\sqrt{-\gamma} = R\sqrt{-g} + \frac{1}{4} \text{tr} F^2 \sqrt{-g} + \text{extra terms}
\]

where \( F^2 = F_{\mu\nu} F^{\mu\nu} \) and \( F_{\mu\nu} \) is constructed with the potentials \( A_{iA} = s^A A_{iMA} \) of \( V_n \) \([20]\).

Comparing (34) with the ADM metric we also notice an analogy between the Yang-Mills potential \( A_{iAB} \) and the shift vector \( N^i \). However, this is only apparent because contrarily to (23), the deformation given by (34) is pure so that it is coordinate gauge free and the dynamical system can be constructed in a non constrained way. On the other hand the deformation implicit in (28) contains a genuine transverse component associated with the shift vector \( N_i \).

In resume, we have started with perturbative deformations of a hypersurface which is consistently associated with a classical dynamical process. Since the corresponding canonical formulation is not constrained, the system can be quantized in a straightforward manner, where the fundamental modes are solutions of Schrödinger’s equation relative to the deformation parameter and to the deformation Hamiltonian.

As a possible application, we considered general relativity as described by 3-dimensional hypersurface deformations with respect to an independent parameter \( s \) regarded as a time parameter in the sense of Liebnitz, characterized within each dynamical process. In this case, the quantum deformations could be applied to general relativity by using a coordinate transformation to the system adapted to the foliation, construct \( \mathcal{H} \) and then transform back to general coordinates \([21]\).

However, in the four dimensional formulation of general relativity it appears to be no room for an extra time parameter and at the end, for practical purposes we need to identify \( s \) with a coordinate time. The result is similar to the well known canonical formulation of the gravitational field in a fixed frame. Thus, in the context of general relativity this approach has some objectionable limitations and it cannot be generally acknowledged as a valid solution to the quantum gravity problem \([11]\).

Perhaps a more realistic alternative is to consider the deformation parameter as a independent variable in a higher dimensional theory such as a modified version of Kaluza-Klein theory. This has the advantage over general relativity of not imposing general covariance on the extra dimensions while keeping all properties of the four dimensional space-time intact.
In this quantum Kaluza-Klein model based on deformation dynamics the extra dimensions are physically accessible by translational motions at high energies, generated by the deformation Hamiltonian operator in Schrödinger’s equation

\[ i\hbar \frac{\partial \Psi}{\partial s} = \hat{\mathcal{H}} \Psi \]

where the resulting quantum state of the deformation is a superposition of \( D - 4 \) independent states \( \Psi_A \). The result is quantization of the four dimensional geometry as opposed to the usual three dimensional approach \[22,23\], with the advantage that no restrictions are imposed on general relativity. Moreover, the resulting multiparameter quantum deformation can produce signature and topological changes, which are relevant for the expected results of a quantum theory of gravity \[24,25\].

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