LLT POLYNOMIALS, ELEMENTARY SYMMETRIC FUNCTIONS AND MELTING LOLLIPOPS

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Abstract. We conjecture an explicit positive combinatorial formula for the expansion of unicellular LLT polynomials in the elementary symmetric basis. This is an analogue of the Stanley–Stembridge conjecture and previously studied by Panova and the author in 2018. We show that the conjecture for unicellular LLT polynomials implies a similar formula for vertical-strip LLT polynomials.

We prove positivity in the elementary basis in for the class of graphs called “melting lollipops” previously considered by Huh, Nam and Yoo. This is done by proving a curious relationship between a generalization of charge and orientations of unit-interval graphs.

We also provide short bijective proofs of Lee’s three-term recurrences for unicellular LLT polynomials and we show that these recurrences are enough to generate all unicellular LLT polynomials associated with abelian area sequences.

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1. Introduction

1.1. Background on LLT polynomials. LLT polynomials were introduced by Lascoux, Leclerc and Thibon in [LLT97], and are $q$-deformations of products of skew Schur functions. An alternative combinatorial model for the LLT polynomials was later introduced in [HHL+05] while studying Macdonald polynomials. In their paper, LLT polynomials are indexed by a $k$-tuple of skew shapes. In the case each such skew shape is a single box, the LLT polynomial is said to be unicellular LLT polynomial. Such unicellular LLT polynomials are the main topic of this paper.
1.2. **Background on chromatic symmetric functions.** In [CM17] Carlson and Mellit introduced a more convenient combinatorial model for unicellular LLT polynomials, indexed by (area sequences of) Dyck paths. They also highlighted an important relationship using plethysm between unicellular LLT polynomials and the chromatic quasisymmetric functions introduced by Shareshian and Wachs in [SW12].

The chromatic quasisymmetric functions refine the chromatic symmetric functions introduced by Stanley in [Sta95]. The Stanley–Stembridge conjecture [SS93] states that such chromatic symmetric functions associated with unit interval graphs are positive in the elementary symmetric basis, or e-positive for short. Their conjecture was refined and later extended in [SW12, AP18, Ell17], where it is conjectured that the chromatic quasisymmetric functions expanded in the e-basis have coefficients in \( \mathbb{N}[q] \), see Conjecture 13 below. To this date, there is still not even a conjectured combinatorial formula for the e-expansion of the chromatic symmetric functions.

1.3. **Main results.** In [AP18], we stated an analogue of the Stanley–Stembridge conjecture, regarding e-positivity of unicellular LLT polynomials, \( G_a(x; q + 1) \) and proved the conjecture in a few cases. We also provided many similarities between unicellular LLT polynomials and chromatic quasisymmetric functions associated with unit-interval graphs. The problem of e-positivity of unicellular LLT polynomials is the main topic of this article.

The main results are:

- We present a precise conjectured combinatorial formula for the e-expansion of \( G_a(x; q + 1) \). Our conjecture states that the unicellular LLT polynomial \( G_a(x; q) \) is given as
  \[
  G_a(x; q + 1) = \sum_{\theta \in O(a)} q^{asc(\theta)} e_{\pi(\theta)}(x).
  \]

  where \( O(a) \) is the set of orientations of the unit interval graph with area sequence \( a \), and \( \pi(\theta) \) is an explicit partition-valued statistic on such orientation. This formula can be extended to vertical-strip LLT polynomials, and has been verified on the computer for all unit-interval graphs up to 10 vertices. This formula is surprising, as there is still no analogous conjectured formula for chromatic symmetric functions.
  A possible application of (1) is to find a positive combinatorial formula for the Schur-expansion of \( G_a(x; q) \).

- We prove in Corollary 31 that the conjectured formula (1) case implies a generalized formula for so called vertical-strip LLT polynomials. Furthermore, we prove that (1) holds for the family of complete graphs and line graphs.

- Analogous recursions for the unicellular LLT polynomials given by Lee in [Lee18]. We give short bijective proofs of these recurrences and show that all graphs associated with abelian Hessenberg varieties can be computed recursively via Lee’s recurrences, starting from unicellular LLT polynomials associated with the complete graphs.
In Section 5, we prove that the transformed Hall–Littlewood polynomials $H_\lambda(x; q + 1)$ are positive in the complete homogeneous basis. This implies that a corresponding family of vertical-strip LLT polynomials are e-positive.

Note that vertical-strip LLT polynomials appear in diagonal harmonics, see for example [Ber17, Section 4] and [Ber13]. Consequently, [1] provides support for some of the conjectures regarding e-positivity in these references. We remark that e-positivity is very unlikely in reality, see [PvW18] for details.

In Section 6, we prove a curious identity between a generalization of charge, denoted $w_t(a)(T)$, and the set of orientations, $O(a)$, of a unit-interval graph $\Gamma_a$. It states that
\[
\sum_{\lambda \vdash n} \sum_{T \in \text{SYT}(\lambda)} (q + 1)^{w_t(a)(T)} s_\lambda(x) = \sum_{\theta \in O(a)} q^{\text{asc}(\theta)} e_{\sigma(\theta)}(x),
\]
where asc(·) and σ(·) are certain combinatorial statistics on orientations. This version of charge was considered in [HNY18] in order to prove Schur positivity for unicellular LLT polynomials in the melting lollipop graph case.

As a consequence, we get an explicit positive e-expansion the case of melting lollipop graphs which has previously been considered in [HNY18]. The corresponding family of chromatic quasisymmetric functions was considered in [DvW18] where they were proved to be e-positive. Note however that the statistic $\pi(\theta)$ in [1] and $\sigma(\theta)$ in [2] are different.

The paper is organized as follows. We first introduce the family of unicellular–and vertical-strip LLT polynomials and some of their basic properties. In Section 3, we prove several recursive identities for such LLT polynomials. In particular, we show that the recursions by Lee [Lee18] can be used to construct unicellular LLT polynomials indexed by any abelian area sequence.

Some vertical-strip LLT polynomials are closely related to the transformed Hall–Littlewood polynomials. In Section 5, we show that the transformed Hall–Littlewood polynomials $H_\lambda(x; q + 1)$ are h-positive, which gives further support for the main conjecture.

In Section 6, we study the relationship between a type of generalized cocharge introduced in [HNY18] and e-positivity. This provides a proof that unicellular LLT polynomials given by melting lollipop graphs are e-positive.

Finally in Section 7, we describe a possible approach to prove [1] by a comparison in the power-sum symmetric basis.

2. Preliminaries

We use the same notation and terminology as in [AP18]. The reader is assumed to have a basic background on symmetric functions and related combinatorial objects, see [Sta01, Mac95]. All Young diagrams and tableaux are presented in the English convention.

2.1. Dyck paths and unit-interval graphs. An area sequence is an integer vector $a = (a_1, \ldots, a_n)$ which satisfies
• $0 \leq a_i \leq i - 1$ for $1 \leq i \leq n$ and
• $a_{i+1} \leq a_i + 1$ for $1 \leq i < n$.

The number of such area sequences of size $n$ is given by the Catalan numbers. Note that [HNY18] uses a reversed indexing of entries in area sequences.

**Definition 1.** For every area sequence of length $n$, we define a *unit interval graph* $\Gamma_\mathbf{a}$ with vertex set $[n]$ and the directed edges
\[(i - a_i) \to i, \quad (i - a_i + 1) \to i, \quad (i - a_i + 2) \to i, \quad \ldots, \quad (i - 1) \to i\]
for all $i = 1, \ldots, n$. We say that $(u,v)$ with $u < v$ is an *outer corner* of $\Gamma_\mathbf{a}$ if $(u,v)$ is not an edge of $\Gamma_\mathbf{a}$, and either
• $u + 1 = v$
• $(u+1, v)$ and $(u, v-1)$ are edges of $\Gamma_\mathbf{a}$.

**Example 2.** We can illustrate area sequences and their corresponding unit-interval graphs as *Dyck diagrams*, as is done in [Hag07, AP18]. For example, $(0, 1, 2, 3, 2, 2)$ corresponds to the diagram

![Dyck diagram](image)

where the area sequence specify the number of white squares in each row, bottom to top. The squares on the main diagonal are the vertices of $\Gamma_\mathbf{a}$, and each white square correspond to a directed edge of $\Gamma_\mathbf{a}$. In the second figure we see this correspondence where edge $(i, j)$ is marked as $ij$. The outer corners of $\Gamma_\mathbf{a}$ are $(2, 5)$ and $(3, 6)$.

**Caution:** We do not really distinguish the terms *area sequence*, *Dyck diagram* and *unit interval graph*, as they all relate to the same objects. What term is used depends on context and what features we wish to emphasize.

Let $\Gamma_\mathbf{a}$ be an unit interval graph with $n$ vertices. We let $\mathbf{a}_T$ denote the area sequence of $\Gamma_\mathbf{a}$ where all edges have been reversed, and every vertex $j \in [n]$ has been relabeled with $n + 1 - j$. This operation corresponds to simply transposing the Dyck diagram.

**Lemma 3 (See [AP18]).** The entries in an area sequence $\mathbf{a}$ is a rearrangement of the entries in $\mathbf{a}_T$.

Most results in this paper concerns a few special classes of area sequences.

**Definition 4.** An area sequence of length $n$ is called *rectangular* if either $\mathbf{a} = (0, 1, 2, \ldots, n - 1)$ or there is some $k \in [n]$ such that
\[a_i = i - 1 \text{ for } i = 1, 2, \ldots, k \text{ and } a_j = j - k - 1 \text{ for } j = k + 1, k + 2, \ldots, n.\]
This condition is equivalent with all non-edges forming a $k \times (n - k)$-rectangle in the Dyck diagram. Furthermore, an area sequence $\mathbf{a}'$ is called *abelian* whenever $a_j' \geq a_i$ for some rectangular sequence $\mathbf{a}$. For example, the area sequence in (4) is abelian.
The terminology is motivated by [HP17], where abelian area sequences are associated with abelian Hessenberg varieties.

We will also consider the following families of area sequences:

- The complete graphs, \((0, 1, 2, \ldots, n - 1)\).
- The line graphs \((0, 1, 1, \ldots, 1)\).
- Lollipop graphs, where
  \[
  a_i = \begin{cases} 
  i - 1 & \text{for } i = 1, \ldots, m \\
  1 & \text{for } i = m + 1, \ldots, m + n
  \end{cases}
  \]
  for some \(m, n \geq 1\).
- Melting complete graph,
  \[
  a_i = \begin{cases} 
  i - 1 & \text{for } i = 1, 2, \ldots, n - 1 \\
  n - k - 1 & \text{for } i = n
  \end{cases}
  \]
  where \(0 \leq k \leq n - 1\).
- Melting lollipop graphs, defined as
  \[
  a_i = \begin{cases} 
  i - 1 & \text{for } i = 1, \ldots, m - 1 \\
  m - 1 - k & \text{for } i = m \\
  1 & \text{for } i = m + 1, \ldots, m + n
  \end{cases}
  \]
  for \(m, n \geq 1\) and \(0 \leq k \leq m - 1\).

2.2. Vertical strip diagrams. A vertical strip diagram is a Dyck diagram where some of the outer corners have been marked with \(\rightarrow\). We call such an outer corner a strict edge. These markings correspond to some extra oriented edges of \(\Gamma_n\). We use the notation \(\Gamma_{a,s}\) to denote a directed graph with some additional strict edges \(s\) and refer to the graph \(\Gamma_{a,s}\) as a vertical strip diagram as well.

**Example 5.** Below is an example of a vertical strip diagram.

![Vertical Strip Diagram](image)

The edges (1, 4) and (3, 6) are strict. Note that this is another example of a diagram with an abelian area sequence.

2.3. Vertical strip LLT polynomials. Let \(\Gamma_{a,s}\) be a vertical strip diagram. A valid coloring \(\kappa : V(\Gamma_{a,s}) \to \mathbb{N}\) is a vertex coloring of \(\Gamma_{a,s}\) such that \(\kappa(u) < \kappa(v)\) whenever \((u, v)\) is a strict edge in \(s\). Given a coloring \(\kappa\), an ascent of \(\kappa\) is a (directed) edge \((u, v)\) in \(\Gamma_{a,s}\) such that \(\kappa(u) < \kappa(v)\). Note that strict edges do not count as ascents. Let \(\text{asc}()\) denote the number of ascents of \(\kappa\).
Definition 6. Let $\Gamma_{a,s}$ be a vertical strip diagram. The vertical strip LLT polynomial $G_{a,s}(x; q)$ is defined as

$$G_{a,s}(x; q) := \sum_{\kappa : V(\Gamma_{a,s}) \to \mathbb{N}} x^{\kappa} q^{\text{asc}(\kappa)}$$

where the sum is over valid colorings of $\Gamma_{a,s}$. Whenever $s = \emptyset$, we simply write $G_a(x; q)$ and refer to this as a unicellular LLT polynomial.

As an example, here is $G_{0012}(x; q)$ expanded in the Schur basis:

$$G_{0012}(x; q) = q^3s_{1111} + (q + q^2 + q^3)s_{211} + (q + q^2)s_{22} + (1 + q + q^2)s_{31} + s_4.$$ 

The polynomials $G_{a,s}(x; q)$ are known to be symmetric, and correspond to classical LLT polynomials indexed by $k$-tuples of skew shapes as in [HHL+05]. In fact, the unicellular LLT polynomials correspond to the case when all shapes in the $k$-tuple are skew shapes consisting of a single box, and the vertical strip case correspond to $k$-tuples of skew Young diagrams that are vertical strips. This correspondence is proved in [AP18] and is also done implicitly in [CM17].

Example 7. In the following vertical strip diagram, we illustrate a valid coloring $\kappa$ where we have written $\kappa(i)$ on vertex $i$. That is, $\kappa(1) = 1$, $\kappa(2) = 3$, $\kappa(3) = 2$, etc.

The strict edges and edges contributing to $\text{asc}(\kappa)$ have been marked with $\rightarrow$. Hence, $\kappa$ contributes with $q^5x^3_1x_2x^2_3x_4$ to the sum in (5).

2.4. A conjectured formula.

Definition 8. Let $a$ be an area sequence of length $n$ and $s$ be some strict edges of $\Gamma_a$. Let $O(a, s)$ denote the set of orientations of the graph $\Gamma_a$, together with the extra directed edges in $s$. If $s = \emptyset$, we simply write $O(a)$ for the set of orientations of $\Gamma_a$. Given $\theta \in O(a, s)$, an edge $(u, v)$ is an ascending edge in $\theta$ if it is oriented in the same manner as in $\Gamma_a$. Let $\text{asc}(\theta)$ denote the number of ascending edges in $\theta$. Note that edges in $s$ are not considered to be ascending!

We now define the highest reachable vertex, $\text{hrv}_\theta(u)$ for $u \in [n]$ as the maximal $v$ such that there is a path from $u$ to $v$ in $\theta$ using only strict and ascending edges. Note that $\text{hrv}_\theta(u) \geq u$ for all $u$. The orientation $\theta$ defines a set partition $\pi(\theta)$ of the vertices of $\Gamma_a$, where two vertices are in the same part if and only if they have the same highest reachable vertex. Let $\pi(\theta)$ denote the partition given by the sizes of the sets in $\pi(\theta)$. 
Let \( a \) be an area sequence and \( s \) be some strict edges of \( \Gamma_\alpha \). Define the symmetric function \( \hat{G}_{a,s}(x; q) \) via the relation

\[
\hat{G}_{a,s}(x; q + 1) := \sum_{\theta \in O(a,s)} q^{\text{asc}(\theta)} e_{\pi(\theta)}(x).
\]

**Example 9.** Below, we illustrate an orientation \( \theta \in O(a,s) \), where \( a = (0, 1, 2, 2, 2) \) and \( s = \{(1,4),(2,5)\} \). As before, strict edges and edges contributing to \( \text{asc}(\theta) \) are marked with \( \rightarrow \).

![Diagram](image)

We have that \( hrv_\theta(2) = hrv_\theta(5) = hrv_\theta(6) = 6 \) and \( hrv_\theta(1) = hrv_\theta(3) = hrv_\theta(4) = 4 \). Thus \( \pi(\theta) = \{652,431\} \) and the orientation \( \theta \) contributes with \( q^6 e_{33}(x) \) in \( \hat{G}_{a,s}(x; q + 1) \). The full polynomial \( G_{a,s}(x; q + 1) \) is

\[
(4q^3 + 20q^4 + 41q^5 + 44q^6 + 26q^7 + 8q^8 + q^9)e_6 + (2q^2 + 7q^3 + 9q^4 + 5q^5 + q^6)e_{33} + (2q^2 + 9q^3 + 16q^4 + 14q^5 + 6q^6 + q^7)e_{12} + (4q^2 + 22q^3 + 48q^4 + 53q^5 + 31q^6 + 9q^7 + q^8)e_{51} + (4q + 14q^2 + 18q^3 + 10q^4 + 2q^5)e_{321} + (q + 8q^2 + 20q^3 + 22q^4 + 11q^5 + 2q^6)e_{111} + (1 + 3q + 3q^2 + 3q^3 + q^4)e_{331}.
\]

**Conjecture 10** (Main conjecture). For any vertical-strip LLT polynomial \( G_{a,s}(x; q) \) we have that \( G_{a,s}(x; q) = \hat{G}_{a,s}(x; q) \).

Note that this conjecture implies that \( G_{a,s}(x; q + 1) \) is e-positive, with the expansion given as a sum over all orientations of \( \Gamma_\alpha \). Such a conjecture was first stated in [AP18] but without a precise definition of \( \pi(\theta) \). Conjecture [10] is a natural analogue of the Stanley–Stembridge conjecture, [SS93 Sta95] and the subsequent refinement considered in [SW12 SW16]. There is also a natural generalization of Equation (6) that predicts the e-expansion of the LLT polynomials indexed by circular area sequences considered in [AP18].

### 2.5. Properties of LLT polynomials

We use standard notation and let \( \omega \) be the involution on symmetric functions that sends the complete homogeneous symmetric function \( h_\lambda \) to the elementary symmetric function \( e_\lambda \), or equivalently, sends \( s_\lambda \) to \( s_\lambda' \).

**Proposition 11** (See [AP18]). For any area sequence \( a \) of length \( n \),

\[
\omega G_a(x; q) = q^{a_1 + a_2 + \ldots + a_n} G_{a^T}(x; 1/q)
\]

where \( a^T \) denotes the transpose of the Dyck diagram.

In [AP18], we gave a proof that \( \omega G_{a,s}(x; q + 1) \) is positive in the power-sum basis. It also follows from a much more general result given in [AS18]. Note that if \( f(x) \) is
e-positive, then $\omega f(x)$ is positive in the power-sum basis. Later in Proposition \cite{48} the power-sum expansion of $\omega G_{\mathbf{a}, q}(x; q + 1)$ is stated explicitly.

The following lemma connects the LLT polynomials with the chromatic quasisymmetric functions $X_{\mathbf{a}}(x; q)$ introduced in \cite{SW12}. The function $X_{\mathbf{a}}(x; q)$ is defined exactly as $G_{\mathbf{a}, q}(x; q)$ but the sum in Equation \cite{44} is taken only over proper colorings of $\Gamma_{\mathbf{a}}$, so that no monochromatic edges are allowed.

**Lemma 12.** Let $\mathbf{a}$ be a Dyck diagram. Then

\[(q - 1)^{-n} G_{\mathbf{a}}[x(q - 1); q] = X_{\mathbf{a}}(x; q), \tag{8}\]

where the bracket denotes a substitution using plethysm.

From this formula, together with Conjecture \cite{10} we have a novel conjectured formula for the chromatic quasisymmetric functions:

\[X_{\mathbf{a}}(x; q) = \sum_{\theta \in AO(\mathbf{a})} (q - 1)^{\text{asc}(\theta)} e_{\pi(\theta)}[x(q - 1)] \left(\frac{(q - 1)^n}{(q - 1)^n}\right). \tag{9}\]

Perhaps it is possible to do some sign-reversing involution together with plethysm manipulations to obtain the $e$-expansion of $X_{\mathbf{a}}(x; q)$ and thus find a candidate formula for the Stanley–Stembridge conjecture.

**Conjecture 13** (Stanley–Stembridge). There is some partition-valued statistic $\rho$ on acyclic orientations of $\Gamma_{\mathbf{a}}$, such that

\[X_{\mathbf{a}}(x; q) = \sum_{\theta \in AO(\mathbf{a})} q^{\text{asc}(\theta)} e_{\rho(\theta)}(x). \tag{9}\]

Here $AO(\mathbf{a})$ denotes the set of acyclic orientations of $\Gamma_{\mathbf{a}}$.

Note that the original Stanley–Stembridge conjecture only concerns the $q = 1$ case, and the above refinement was stated in \cite{SW12, SW16}.

**Problem 14.** Prove that the family $\hat{G}_{\mathbf{a}}(x; q)$ defined in \cite{43} fulfills the involution identity \cite{7}.

3. **Recursive properties of LLT polynomials**

We shall now cover several recursive relations for the vertical-strip LLT polynomials. Our proofs are bijective and directly use the combinatorial definition as a weighted sum over vertex colorings. We illustrate these bijections with Dyck diagrams where only the relevant vertices and edges are shown.

The reader thus is encouraged to interpret a diagram as a weighted sum over colorings, where decorations of the diagrams indicate restrictions of the colorings. For example, given an edge $\epsilon$ of $\Gamma_{\mathbf{a}, \mathbf{q}}$, there are two possible cases. Either $\epsilon$ contributes to the number of ascents, or it does not. We can illustrate this simply as

\[
\begin{array}{c}
\begin{array}{c}
\fill \\
\fill \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\fill \\
\fill \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\fill \\
\fill \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\fill \\
\fill \\
\end{array}
\end{array}
\]
\[= \begin{array}{c}
\begin{array}{c}
\fill \\
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\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\fill \\
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\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\fill \\
\fill \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\fill \\
\fill \\
\end{array}
\end{array}
\] + q
\]

where the white box is the edge $\epsilon$ and $\downarrow$ indicates an edge that cannot be an ascent. Note that the vertices shown do not need to have consecutive labels — the
intermediate vertices are simply not shown. Shaded boxes are not edges of $\Gamma_a$ and therefore does not contribute to ascents of the coloring.

The following recursive relationship allows us to express vertical-strip LLT polynomials as linear combinations of unicellular LLT polynomials. Later in Proposition 30, we prove that the polynomials in Equation (6) satisfy the same recursion. We use the notation $a \cup \{\epsilon\}$ to describe the area sequence of the unit interval graph where the edge $\epsilon$ has been added to the edges of $\Gamma_a$. The notation $s \cup \{\epsilon\}$ for strict edges is interpreted in a similar manner.

**Proposition 15.** If $\Gamma_{a,s}$ is a vertical strip diagram, and $\epsilon$ is a non-strict outer corner of $\Gamma_{a,s}$, then

$$G_{a \cup \{\epsilon\},s}(x; q + 1) = G_{a,s}(x; q + 1) + qG_{a,s \cup \{\epsilon\}}(x; q + 1).$$  \hspace{1cm} (10)

**Proof.** By shifting the variable $q$, the identity can be restated as

$$G_{a \cup \{\epsilon\},s}(x; q) + G_{a,s \cup \{\epsilon\}}(x; q) = qG_{a,s \cup \{\epsilon\}}(x; q) + G_{a,s}(x; q),$$  \hspace{1cm} (11)

which in (as sum over colorings) diagram form can be expressed as follows. The two vertices shown are the vertices of $\epsilon$.

The first and last diagram can be expanded into subcases,

$$\begin{align*}
\begin{array}{c}
\square + \blacktriangle = q \\
\blacktriangle + \square = q \\
\end{array}
\end{align*}
$$

and here it is evident that both sides agree. \hfill \square

The above recursion seem to relate to certain recursions on Catalan symmetric functions, see [BMPS18, Prop. 5.6]. Catalan symmetric functions are very similar in nature to LLT polynomials.

### 3.1. Lee’s recursion

In Proposition 18 below, we prove a recursion on certain LLT polynomials. We then show that this relation is equivalent to Lee’s recursion, given in [Lee18, Thm 3.4].

**Definition 16.** Let $a$ be an area sequence of length $n \geq 3$. An edge $(i,j) \in E(\Gamma_a)$, $3 \leq j \leq n$, is said to be admissible if the following four conditions hold:

- $i = j - a_j$
- $j = n$ or $a_j \geq a_{j+1} + 1$
- $a_j \geq 2$
- $a_i + 1 = a_{i+1}$

The last condition is automatically satisfied if the first three are true and $a$ is abelian. Note that if $(i,j)$ is admissible, then for all $k < i$ or $k > i + 1$ we have

$$(k,i) \in E(\Gamma_a) \iff (k,i+1) \in E(\Gamma_a) \text{ and } (i,k) \in E(\Gamma_a) \iff (i+1,k) \in E(\Gamma_a).$$  \hspace{1cm} (12)

These properties are crucial in later proofs.
Example 17. For the following diagram $a$, the edge $(2, 5)$ is admissible.

Let $e_j$ denote the $j$th unit vector.

Proposition 18. Suppose $(i, j)$ is an admissible edge of the area sequence $a$, set $a^1 := a - e_j$ and $a^2 := a - 2e_j$ and $s^1 := \{(i, j)\}$, $s^2 := \{(i + 1, j)\}$. Then

$$G_{a^1, s^1}(x; q) = qG_{a^2, s^2}(x; q).$$

Proof. We use the diagram-type proof as before, now only showing the vertices $i$, $i + 1$ and $j$. The identity we wish to show is then presented as

Both sides are considered as a weighted sum over colorings with restrictions indicated by $\rightarrow$. Subdividing these sums into subcases by forcing additional inequalities gives

$$q \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} = q \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} \begin{array}{c}
\end{array}.$$

Two terms cancel and additional inequalities follows by transitivity. It therefore suffices to prove the following.

$$q \begin{array}{c}
\end{array} = q \begin{array}{c}
\end{array}.$$

Note that the additional $q$ in the left hand side appears due to the ascent $(i, i + 1)$.

There is now a simple $q$-weight-preserving bijection between colorings on the diagram on the left hand side, and colorings of the diagram on the right hand side. For a coloring $\kappa$ in the left hand side, we have $\kappa(i) < \kappa(j) \leq \kappa(i + 1)$, while on the right hand side, we have $\kappa(i + 1) < \kappa(j) \leq \kappa(i)$. Hence, vertex $i$ and vertex $i + 1$ are never assigned the same color.

The bijection is to simply swap the colors of the adjacent vertices $i$ and $i + 1$. The property in Equation (12) ensures that the number of ascending edges are preserved under this swap.

Corollary 19 (Local linear relation [Lee18, Thm 3.4]). Let $a$ be an area sequence for which $(i, j)$ is admissible, and let $a^0 := a$, $a^1 := a - e_j$ and $a^2 := a - 2e_j$. Then

$$G_{a^0}(x; q) - G_{a^1}(x; q) = q(G_{a^1}(x; q) - G_{a^2}(x; q)).$$

Proof. We see that the left hand side of (14) can be rewritten in diagram form using Equation (10):

$$\text{LHS} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = (q - 1)\begin{array}{c}
\end{array}.$$
The right hand side is treated in a similar manner:

\[
\text{RHS} = q \left( \binom{-1}{q} - \binom{-2}{q} \right) = q(q - 1)
\]

The identity in (13) now implies that LHS = RHS. \(\square\)

**Example 20.** As an illustration of Corollary [19] we have \((i, j) = (2, 5)\) and the following three Dyck diagrams:

\[a^0 = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} \quad a^1 = \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} \quad a^2 = \begin{array}{c}
\text{Diagram 7} \\
\text{Diagram 8} \\
\text{Diagram 9}
\end{array}\]

### 3.2. The dual Lee recursion.

There is a “dual” version of Corollary [19] obtained by applying \(\omega\) to both sides of (14), and then apply the relation in (7). We shall now state this in more detail.

**Definition 21.** Let \(a\) be an area sequence of length \(n \geq 3\). An edge \((i, j)\) is said to be **dual-admissible** if the edge \((n + 1 - j, n + 1 - i)\) is admissible for \(a^T\).

We can then formulate the dual versions of Proposition [18] and Corollary [19]

**Proposition 22 (The dual Lee recursion).** Let \(a\) be an area sequence for which \((i, j)\) is dual-admissible and let \(a^0 := a\), \(a^1 := a - e_j\) and \(a^2 := a - e_j - e_{j-1}\). Then

\[
G_{a^1, s^1}(x; q) = q G_{a^2, s^2}(x; q)
\]

and

\[
G_{a^0}(x; q) - G_{a^1}(x; q) = q(G_{a^1}(x; q) - G_{a^2}(x; q))
\]

where \(s^1 := \{(i, j)\}\) and \(s^2 := \{(i, j - 1)\}\).

**Proof sketch.** We can either prove these identities by applying \(\omega\) as outlined above, or bijectively using diagrams. We leave out the details. \(\square\)

**Example 23.** Proposition [18] applies in the following generic situation. Here, the edge \((x, z)\) is an admissible edge. The crucial condition in (12) states that the area of the rows with vertices \(x\) and \(y\) in the diagram differ by exactly one.

\[\begin{array}{c}
(a^1, s^1) = \begin{array}{c}
\text{Diagram 10} \\
\text{Diagram 11} \\
\text{Diagram 12}
\end{array} \\
(a^2, s^2) = \begin{array}{c}
\text{Diagram 13} \\
\text{Diagram 14} \\
\text{Diagram 15}
\end{array}
\end{array}\]
Similarly, the dual recursion in Equation (15) applies in the following situation, where \((x, z)\) is a dual-admissible edge:

\[
(a_1^s, s^1) = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1}
\end{array}
\quad \text{and} \quad
(a_2^s, s^2) = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array}
\tag{18}
\]

### 3.3. Recursion in the complete graph case.

We end this section by recalling a recursion for LLT polynomials in the complete graph case.

**Proposition 24** ([AP18, Prop.5.8]). Let \(G_{K_n}(x; q)\) denote the LLT polynomial for the complete graph, where the area sequence is \((0, 1, 2, \ldots, n-1)\). Then

\[
G_{K_n}(x; q) = \sum_{i=0}^{n-1} G_{K_i}(x; q)e_{n-i}(x) \prod_{k=i+1}^{n-1} [q^k - 1], \quad G_{K_0}(x; q) = 1. \tag{19}
\]

**Lemma 25.** If \(a\) is rectangular and the non-edges form a \(k \times (n-k)\)-rectangle in the Dyck diagram, then \(G_a(x; q) = G_{K_k}(x; q) G_{K_{n-k}}(x; q)\).

**Proof.** The unit-interval graph \(\Gamma_a\) is a disjoint union of two smaller complete graphs, so this now follows immediately from the definition of unicellular LLT polynomials. \(\square\)

For the remaining of this section, it will be more convenient to use the notation in [Lee18], and index unicellular LLT polynomials of degree \(n\) with partitions \(\lambda\) that fit inside the staircase \((n-1, n-2, \ldots, 2, 1, 0)\). We fix \(n\) and let the area sequence \(a\) correspond to the partition \(\lambda\) where \(\lambda_i = n - i - a_{n+1-i}\). Hence, \(\lambda\) is exactly the shape of the (shaded) non-edges in the Dyck diagram. By definition, \(\lambda\) is abelian if it fits inside some \(k \times (n-k)\)-rectangle.

**Lemma 26** (Follows from [HNY18, Thm. 3.4]). Let \(\lambda\) be abelian with \(\ell \geq 2\) parts such that \(\lambda_\ell < \lambda_{\ell-1}\). Let

\[
\mu = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell-1}) \quad \text{and} \quad \nu = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell-1}, \lambda_\ell + 1).
\]

Then there are rational functions \(c(q)\) and \(d(q)\) such that

\[
G_\lambda(x; q) = c(q)G_\mu(x; q) + d(q)G_\nu(x; q).
\]

**Proof.** Use Corollary [19] repeatedly on row \(\ell\) of \(\mu\). \(\square\)

**Example 27.** To illustrate Lemma 26, we have the following three partitions:

\[
\lambda = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{example1}
\end{array}, \quad \mu = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{example2}
\end{array}, \quad \nu = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{example3}
\end{array}
\]

**Proposition 28.** Every \(G_\lambda(x; q)\) where \(\lambda\) is abelian, can be expressed recursively via Lee’s recurrences, as a linear combination of some \(G_{\mu^i}(x; q)\) where the \(\mu^i\) are rectangular.
Proof. Let $\lambda$ be abelian partition with exactly $\ell$ parts, so that it fits in a $\ell \times (n - \ell)$-rectangle. We shall do a proof by induction over $\lambda$, and in particular its number of parts.

1. **Case $\lambda = \emptyset$.** This is rectangular by definition.
2. **Case $\lambda = (n - 1)$.** This is rectangular.
3. **Case $\ell = 1$.** Use Lemma 26 to reduce to Case (1) and Case (2).
4. **Case $\ell > 1$ and $\lambda_i \leq \ell - i$ for some $i \in [\ell]$.** The conditions imply that it is possible to remove a $2 \times 1$ or a $1 \times 2$-domino from $\lambda$ and obtain a new partition. Hence we can use Lee’s recursion to express $G_\lambda(x; q)$ using polynomials indexed by two smaller partitions. For example, this case applies in the following situation:

$$
\lambda = \begin{array}{ccc}
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
\quad \text{and}
\begin{array}{ccc}
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
$$

(20)

5. **Case $\ell > 1$ and $\lambda_i > \ell - i$ for all $i \in [\ell]$.** Three things can happen here, and it is easy to see that this list is exhaustive.
   - $\lambda$ is rectangular and we are done.
   - We can add a $2 \times 1$ or $1 \times 2$-domino to $\lambda$ without increasing $\ell$ and still obtain a partition. Similar to Case (4), we can therefore reduce to cases where $|\lambda|$ has increased by 1 and 2.
   - Lemma 26 can be applied, thus reducing $\lambda$ to a case where $\ell$ has strictly been decreased, and a case where $\lambda$ has increased by one box.

Notice that Case (4) reduces only back to Case (4), or a case where $\ell$ is decreased, and the same goes for Case (5). There are therefore no circular dependencies amongst these cases and the induction is valid. □

4. **Recurcions for the conjectured formula**

In this section, we prove that $\hat{G}(x; q)$ also fulfills the recursion in Proposition 15. We use similar bijective technique as in Section 3, but *diagrams now represent weighted sums over orientations* as in Equation (6). Note that we now also consider the shifted polynomial $\hat{G}_{a,s}(x; q + 1)$.

**Example 29.** Suppose the following diagram illustrates the entire graph. The diagram represents the weighted sum over all orientations of the non-specified edges $(x, y)$ and $(y, z)$. The edge $(x, z)$ is strict, and $(z, w)$ is forced to be ascending. Remember that each ascending edge contributes with a $q$-factor.

$$
\begin{array}{c}
x \\
y \\
z \\
w
\end{array}
\quad \rightarrow \\
\begin{array}{c}
\rightarrow \\
y \\
z \\
w
\end{array}
$$
There are four orientations in total,

\[
\begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
\]

which according to (6) give the sum \(q_{e_1} + q^2_{e_2} + q^2_{e_3} + q^3_{e_4}\).

In the diagrams below, only relevant vertices of the graphs are included.

**Proposition 30.** If \(\Gamma_{n,s}\) is a vertical-strip graph, with \(\epsilon\) being a non-strict outer corner, then

\[
\hat{G}_{n,\cup{\epsilon}}(x; q+1) = q\hat{G}_{n,\cup{\epsilon}}(x; q + 1) + \hat{G}_{n,\epsilon}(x; q + 1).
\]  

(21)

**Proof.** In diagram form, this amounts to showing that orientations of the diagram in the left hand side can be put in \(q\)-weight-preserving bijection with the disjoint sets of orientations in the right hand side, while also preserving the \(\pi(\cdot)\)-statistic. Thus we wish to prove that

\[
\begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
= q
\begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
+ \begin{array}{c|c|c}
\hline
\downarrow & \downarrow & \downarrow \\
\hline
x & y & x \\
\hline
\end{array}
\]

Consider an orientation in the left hand side. There are two cases to consider:

- The edge \((x, y)\) is ascending. We map the orientation to an orientation of the first diagram in the right hand side, by preserving the orientation of all other edges.
- The edge \((x, y)\) is non-ascending. We map this to the second diagram, by preserving the orientation of all other edges.

In both cases above, note that both the \(q\)-weight and \(\pi(\cdot)\) is preserved under this map. \(\square\)

**Corollary 31.** If Conjecture 10 holds in the unit-interval case, it holds in the vertical-strip case.

**Proof.** We can use Proposition 30 and Proposition 15 to recursively remove all strict edges. Since both families satisfy the same recursion, we have that the unicellular case of Conjecture 10 implies the vertical-strip case. \(\square\)

### 4.1. The complete graph recursion and line graphs.

Analogous to Proposition 24, we have a recursion for the corresponding \(\hat{G}_{K_n}(x; q)\), where we again consider the complete graph case. Here, \(\binom{[n]}{k}\) denotes the set of \(k\)-subsets of \(\{1, \ldots, n\}\).

**Lemma 32.** The polynomial \(\hat{G}_{K_n}(x; q)\) satisfy \(\hat{G}_{K_0}(x; q) := 1\) and \(\hat{G}_{K_n}(x; q + 1)\) is equal to

\[
\sum_{i=0}^{n-1} \hat{G}_{K_i}(x; q + 1)e_{n-i}(x) \left( \sum_{S \subseteq \binom{[n-1]}{n-1-i}} \prod_{j=1}^{n-1-i} (q + 1)^{s_j-j}[(q + 1)^j - 1] \right). \tag{22}
\]
Proof. We first give an argument for the recursion in (22). Given an orientation \( \theta \) of \( K_i \), we can construct a new orientation \( \theta' \) of \( K_n \) by inserting a new part of size \( n - i \) in the vertex partition where vertex \( n \) is a member. Choose an \( i \)-subset of \([n - 1]\) and let \( \theta \) define the orientation of the edges in \( \theta' \) on these vertices. The remaining \( n - i - 1 \) vertices will be in the new part — let us call this set of vertices \( S \). Each element \( s_j \in S \) must have at least one ascending edge to either vertex \( n \), or to another member in \( S \) larger than \( s_j \), but all other choices of ascending edges are allowed. It then follows that that for such a subset \( S = \{s_1, \ldots, s_{n-i}\} \), there are

\[
\prod_{j=1}^{n-1-i} (q + 1)^{s_j - j} [(q + 1)^j - 1]
\]

asc(\( \cdot \))-weighted ways of choosing subsets of ascending edges in \( \theta' \) so that all vertices in \( S \) has \( n \) as highest reachable vertex. Hence,

\[
\sum_{S \subset \binom{[n-1]}{i}} \prod_{j=1}^{n-1-i} (q + 1)^{s_j - j} [(q + 1)^j - 1]
\]

is the asc(\( \cdot \))-weighted count of the number orientation of \( K_n \), where the part of the vertex-partition containing \( n \) has exactly \( n - i \) members. \( \Box \)

We shall now prove that \( \mathcal{G}_{K_n}(x; q) = G_{K_n}(x; q) \). By using Lemma 32 and Proposition 24, this follows from the following lemma.

**Lemma 33.** For all \( n \) and \( 1 \leq i \leq n - 1 \), we have that

\[
\prod_{k=i+1}^{n-1} [q^k - 1] = \sum_{S \subset \binom{[n]}{i-1}} \prod_{j=1}^{n-1-i} (q + 1)^{s_j - j} [(q + 1)^j - 1].
\]

Proof. By changing the indices, this is equivalent to

\[
\prod_{k=1}^{i} [q^{n-k+1} - 1] = \sum_{S \subset \binom{n}{i}} \prod_{j=1}^{i} q^{s_j} [1 - q^{-j}].
\]

This can be restated as

\[
\prod_{k=1}^{i} \frac{q^{n+1} - q^k}{q^k - 1} = \sum_{S \subset \binom{n}{i}} q^{|S|}
\]

where \(|S|\) denotes the sum of the entries in \( S \). We can subdivide the right hand sum depending on if \( n \in S \) or not,

\[
\sum_{S \subset \binom{n}{i}} q^{|S|} = \sum_{S \subset \binom{n}{i-1}} q^{|S|} + q^n \sum_{S \subset \binom{n-1}{i-1}} q^{|S|}.
\]

By induction over \( n \) and \( i \) it suffices to show that

\[
\prod_{k=1}^{i} \frac{q^{n+1} - q^k}{q^k - 1} = \prod_{k=1}^{i} \frac{q^n - q^k}{q^k - 1} + q^n \prod_{k=1}^{i-1} \frac{q^n - q^k}{q^k - 1}
\]

and this is easy to verify. \( \Box \)
The case of line graphs follows immediately from [AP18, Prop. 5.18].

**Proposition 34.** Let \( a = (0, 1, 1, \ldots, 1) \) be a line graph. Then \( \hat{G}_a(x; q) = G_a(x; q) \).

4.2. **On Lee’s recursion for orientations.** We would also like to prove that the \( \hat{G}(x; q) \) fulfill Lee’s recursions. However, this is a surprisingly challenging and we are unable to show this at the present time. A proof that Lee’s recursions hold \( \hat{G}(x; q) \) would imply that \( G_a(x; q) = \hat{G}_a(x; q) \) at least for all abelian area sequences \( a \). Computer experiment with \( n \leq 7 \) confirms that the polynomials \( \hat{G}_a(x; q) \) indeed satisfy these recurrences.

The class of melting lollipop graphs can be constructed recursively from the complete graphs and the line graphs by just using the recursion in Corollary 19. This is in fact done in [HNTY18], so we simply sketch a proof of this statement. Recall that a melting lollipop graph \( a \) is given by

\[
\begin{align*}
a_i &= \begin{cases} 
  i - 1 & \text{for } i = 1, \ldots, m - 1 \\
  m - 1 - k & \text{for } i = m \\
  1 & \text{for } i = m + 1, \ldots, m + n
\end{cases}
\end{align*}
\]

for some \( m, n \geq 1 \) and \( 0 \leq k \leq m - 1 \). A melting lollipop graph with \( m = 7, k = 3 \) and \( n = 3 \) is shown here.

We can use the recursion in Corollary 19 to express the corresponding LLT polynomial as a linear combination of LLT polynomials where \( k = 0 \) and \( k = m - 1 \). When \( k = m - 1 \), the graph \( \Gamma_a \) is a disjoint union of a complete graph and a line graph, which are base cases. Furthermore, when \( k = 0 \), we obtain a melting lollipop graph with the new parameters \( m' = m + 1, k' = 1 \) and \( n' = n - 1 \), which are dealt with by induction.

5. **The Hall–Littlewood case**

In [HHL+05], the modified Macdonald polynomials \( \tilde{H}_\lambda(x; q, t) \) are expressed as a positive sum of certain LLT polynomials. The modified Macdonald polynomials specialize to modified Hall–Littlewood polynomials at \( q = 0 \), which in turn are closely related to the transformed Hall–Littlewood polynomials.

**Definition 35** (See [DLT94] [TZ03] for a background). Let \( \lambda \) be a partition. The transformed Hall–Littlewood polynomials are defined as

\[
H_\lambda(x; q) = \sum_{\mu} K_{\lambda\mu}(q) s_\lambda(x)
\]

where \( K_{\lambda\mu}(q) \) are the Kostka–Foulkes polynomials.

The \( H_\lambda \) are sometimes denoted \( Q'_\lambda \) and is the adjoint basis to the Hall–Littlewood \( P \) polynomials for the standard Hall scalar product, see [DLT94]. A more convenient
definition of the transformed Hall–Littlewood polynomials is the following. For \( \lambda \vdash n \) we have
\[
H_\lambda(x; q) = \prod_{1 \leq i < j \leq n} \frac{1 - R_{ij}}{1 - qR_{ij}} h_\lambda(x)
\] (23)
where \( R_{ij} \) are raising operators acting on the partitions (or compositions) indexing the complete homogeneous symmetric functions as
\[
R_{ij} h_{(\lambda_1, \ldots, \lambda_n)}(x) = h_{(\lambda_1, \ldots, \lambda_i+1, \ldots, \lambda_j, \ldots, \lambda_n)}(x).
\]
Note that \( s_\lambda(x) = H_\lambda(x; 0) \), and (23) gives
\[
s_\lambda(x) = \prod_{i<j} (1 - R_{ij}) h_\lambda(x)
\]
which is just the Jacobi–Trudi identity for Schur functions in disguise. Furthermore, note that (23) immediately implies that
\[
H_\lambda(x; q) = h_\lambda(x) + \sum_{\mu \triangleright \lambda} c_\mu(q) h_\lambda(x), \quad c_\mu(q) \in \mathbb{Z}[q]
\] (24)
where \( \triangleright \) denotes the dominance order, since the raising operators \( R_{ij} \) can only make partitions larger in dominance order.

We now connect the transformed Hall–Littlewood polynomials with certain vertical strip LLT polynomials.

**Definition 36.** Given a partition \( \mu \vdash n \), let \( s_i \) be defined as
\[
s_i := \mu_1 + \cdots + \mu_i,
\]
with \( s_0 := 1 \). From \( \mu \), we construct a vertical strip diagram \( \Gamma_\mu \) on \( n \) vertices with the following edges:

(a) for each \( j = 1, \ldots, \ell(\mu) \), let the vertices \( \{s_{j-1}, \ldots, s_j\} \) constitute a complete subgraph of \( \Gamma_\mu \),
(b) for each \( j = 2, \ldots, \ell(\mu) \), we also have the edges
\[
s_j - k + 1 \to s_j - i \text{ for any } \{(k, j) : 1 \leq k \leq \mu_j \text{ and } 1 \leq i \leq \mu_j - k\}.
\]
Furthermore, all outer corners are taken as strict edges, see Example 38 below. As before, let \( O(\Gamma_\mu) \) denote the set of orientations of the edges of \( \Gamma_\mu \).

**Proposition 37.** Let \( \mu \) be a partition and let \( \Gamma_\mu \) be the vertical strip diagram constructed from \( \mu \) and let \( G_\mu(x; q) \) be the corresponding LLT polynomial. Then
\[
\omega G_\mu(x; q) = q \sum_{\ell \geq 2} \binom{n}{\ell} H_{\mu'}(x; q).
\] (25)

**Brief proof sketch.** We use [Hag07, A.59] which states that for any partition \( \lambda \), the coefficient of \( t^\mu(\lambda) \) in the modified Macdonald polynomial \( \tilde{H}_\lambda(x; t, q) \) is almost a transformed Hall–Littlewood polynomial:
\[
\omega \tilde{H}_\lambda(x; t, q) \big|_{t^\mu(\lambda)} = H_{\lambda'}(x; q).
\]
The \( \tilde{H}_\lambda(x; t, q) \) is a sum over certain LLT polynomials and in particular, the coefficient of \( t^\mu(\lambda) \) is a single vertical-strip LLT polynomial, multiplied with \( q^{-A} \), where \( A \) is the sum of arm lengths in the diagram \( \lambda \). Unraveling the definitions in [Hag07, A.14], we arrive at the identity in (25).
Example 38. The Hall–Littlewood polynomial $H_{3321}(x; q)$ is related to the vertical strip diagram $\Gamma_{432}$ in (25).

\[ \Gamma_{432} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 0 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array} \]  \hspace{1cm} (26)

The edges marked with a dot are the edges in item (b). There are $\sum_{i \geq 2} \binom{\mu_i}{2}$ such edges. Notice that the vertex partition of this orientation is \{974, 863, 52, 1\}. Furthermore, it is fairly straightforward to see that for any orientation $\theta$ of $\Gamma_{\mu}$, we must have that the partition $\pi(\theta)$ dominates $\mu'$.

We can now easily give some strong support for Conjecture 10.

Corollary 39. For any partition $\mu$, the vertical-strip LLT polynomial $G_{\mu}(x; q + 1)$ is $e$-positive.

Proof. Using (25), it suffices to prove that $H_{\lambda'}(x; q + 1)$ is $h$-positive. From (23), we have that

\begin{align*}
H_{\mu'}(x; q + 1) &= \prod_{i < j} \frac{1 - R_{ij}}{1 - (q + 1)R_{ij}} h_{\mu'}(x) \\
&= \prod_{i < j} (1 - R_{ij})(1 + (q + 1)R_{ij} + (q + 1)^2 R_{ij}^2 + \cdots) h_{\mu'}(x) \\
&= \prod_{i < j} (1 + qR_{ij} + q(q + 1)R_{ij}^2 + q(q^2 + 1)R_{ij}^3 + \cdots) h_{\mu'}(x) \\
&= \prod_{i < j} \left(1 + \sum_{t \geq 1} q(1 + q)^t R_{ij}^t\right) h_{\mu'}(x).  \hspace{1cm} (30)
\end{align*}

This proves positivity. \hfill \Box

Problem 40. Find a bijective proof that $\hat{G}_{\mu}(x; q + 1)$ is equal to $G_{\mu}(x; q + 1)$, by interpreting each term in Equation (30), and combine with (25).

It is tempting to believe that summing over the orientations of $\Gamma_{\mu}$ in Definition 36 where all edges in condition (b) are oriented in a non-descending manner would give exactly $\omega H_{\mu'}(x; q + 1)$. However, this fails for $\mu = 222$.

6. Generalized cocharge and e-positivity

In [HNY18], the authors consider a certain classes of unicellular LLT polynomials that can be expressed in a particularly nice way. These are polynomials indexed by complete graphs, line graphs and a few other families. In this section, we prove that the corresponding LLT polynomials are positive in the elementary basis. In fact, we
do this by giving a rather surprising relationship between a type of cocharge and orientations.

For a semi-standard Young tableau $T$, the reading word is formed by reading the boxes of $\lambda$ row by row from bottom to top, and from left to right within each row. The descent set of a standard Young tableau $T$ is defined as

$$\text{Des}(T) := \{i \in [n-1] : i+1 \text{ appear before } i \text{ in the reading word}\}.$$ 

Given a Dyck diagram $a$, we define the weight as

$$\text{wt}_a(T) = \sum_{i \in \text{Des}(T)} a_{n+1-i}. \quad (31)$$

The weight here is also known as cocharge whenever $a$ is the complete graph $(0,1,2,\ldots,n-1)$, see for example [Hag07]. If we let $T'$ denote the transposed tableau, then for any $T$ and $a$, we have

$$\text{Des}(T') = [n-1] \setminus \text{Des}(T) \quad \text{and} \quad \text{wt}_a(T') = (a_1 + \cdots + a_n) - \text{wt}_a(T).$$

It will be convenient to define

$$\widetilde{\text{wt}}_a(T) := \text{wt}_a(T') = \sum_{i \in \text{Des}(T)} a_{n+1-i}. \quad (32)$$

**Example 41.** Let $a = (0,1,2,3,2,3,3)$ and $T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 \2 & 6 \3 \end{array}$. The reading word of $T$ is $75268134$, $\text{Des}(T) = \{1,4,6\}$ so $\text{wt}_a(T) = a_8 + a_5 + a_3 = 7$ and $\widetilde{\text{wt}}_a(T) = 9$.

**Definition 42.** Given an area sequence $a$ of length $n$, we define the polynomial

$$\tilde{G}_a(x; q) := \sum_{\lambda \vdash n} \sum_{T \in \text{SYT}^{\lambda}(\gamma)} q^{\text{wt}_a(T)} s_{\lambda}(x). \quad (33)$$

From this definition, it follows that

$$\omega \tilde{G}_a(x; q) = \sum_{\lambda \vdash n} \sum_{T \in \text{SYT}^{\lambda}(\gamma)} q^{\widetilde{\text{wt}}_a(T)} s_{\lambda}(x). \quad (34)$$

The following proposition is a collection of results in [HNY18].

**Proposition 43.** We have that $\tilde{G}_a(x; q) = G_a(x; q)$ for the families of graphs listed in Section 2.1: the complete graphs, line graphs, lollipop graphs, melting complete graphs and melting lollipop graphs.

**Lemma 44.** Let $\lambda \vdash n$ and let $\gamma$ be a composition of $n$ with $\ell$ parts. Then the standardization map

$$\text{std} : \{S \in \text{SSYT}(\lambda, \gamma)\} \to \{T \in \text{SYT}(\lambda) : \text{Des}(T) \subseteq D(\gamma)\}$$

is a bijection.
Proof. This is straightforward from the definition of standardization and descents, see for example [Hag07, p. 5].

We shall now introduce a different statistic on orientations. Given \( \theta \in O(\Gamma_a) \), we say that a vertex \( v \) is a bottom of \( \theta \) if there is no \( u < v \) such that \((u,v)\) is ascending in \( \theta \). Let \( s_1, \ldots, s_k \) be the bottoms ordered decreasingly and let \( s_0 := n + 1 \). By definition, vertex 1 is always a bottom. Let \( \sigma(\theta) \) be defined as the composition of \( n \) with the parts given by \( \{ s_{i-1} - s_i : i = 1, \ldots, k \} \) and note that \( D(\sigma(\theta)) = \{ n + 1 - s_i : i = 1, 2, \ldots, k - 1 \} \).

**Example 45.** The orientation \( \theta \) in (35) has vertices 1, 3 and 6 as bottoms. Furthermore, \( \sigma(\theta) = (1, 3, 2) \) and \( D(\sigma(\theta)) = \{ 1, 4 \} \).

\[35\]

Note that \( \pi(\theta) = (5, 1) \) so \( \sigma \) and \( \pi \) are indeed very different.

The following theorem was proved for the complete graph and the line graph in [AP18]. We can now generalize it to all unit interval graphs.

**Theorem 46.** Let \( a \) be an area sequence of length \( n \). Then
\[
\tilde{G}_a(x; q + 1) = \sum_{\theta \in O(\Gamma_a)} q^{\text{asc}(\theta)} e_{\sigma(\theta)}(x).
\] (36)

Proof. We apply \( \omega \) on both sides of Equation (36), so it suffices to prove that
\[
\omega \tilde{G}_a(x; q + 1) = \sum_{\theta \in O(\Gamma_a)} q^{\text{asc}(\theta)} h_{\sigma(\theta)}(x).
\] (37)

Recall, in e.g. [Mac95], the standard expansion
\[
h_\nu(x) = \sum_\lambda K_{\lambda, \nu} s_\lambda(x),
\] (38)

where \( K_{\lambda, \nu} = |\text{SSYT}(\lambda, \nu)| \) are the Kostka coefficients. Thus, comparing both sides of (37) in the Schur basis, it suffices to show that for every partition \( \lambda \),
\[
\sum_{T \in \text{SYT}(\lambda)} (1 + q)^{\text{wt}_a(T)} = \sum_{\theta \in O(\Gamma_a)} q^{\text{asc}(\theta)} K_{\lambda, \sigma(\theta)}.
\]

Using Lemma 44 in the right hand side and unraveling the definition in the left hand side, it is enough to prove that
\[
\sum_{T \in \text{SYT}(\lambda)} \prod_{i \notin \text{Des}(T)} (1 + q)^{a_{n+1-i}} = \sum_{T \in \text{SYT}(\lambda)} \sum_{\theta \in O(\Gamma_a) \atop \text{Des}(T) \subseteq D(\sigma(\theta))} q^{\text{asc}(\theta)}.
\]

It then suffices to prove that for a fixed \( T \in \text{SYT}(\lambda) \) we have
\[
\prod_{i \notin \text{Des}(T)} (1 + q)^{a_{n+1-i}} = \sum_{\theta \in O(\Gamma_a) \atop \text{Des}(T) \subseteq D(\sigma(\theta))} q^{\text{asc}(\theta)}. \] (39)
Both sides may now be interpreted as a weighted sum over all orientations of $\Gamma_a$ where no ascending edges end in $\{i : n + 1 - i \in \text{Des}(T)\}$.

\begin{corollary}
All families of unicellular LLT polynomials $G_a(x; q + 1)$ indexed by complete graphs, line graphs, lollipop graphs and melting lollipop graphs are $e$-positive.
\end{corollary}

Notice that the formula in (36) is different from the conjectured formula in Conjecture 10, since $\pi(\theta)$ and $\sigma(\theta)$ are different. This is not surprising as $G_a(x; q)$ and $\tilde{G}_a(x; q)$ are not equal for general $a$. However, it is rather remarkable that Conjecture 10 implies that (36) and Equation (6) agree whenever $G_a(x; q) = \tilde{G}_a(x; q)$.

7. A possible approach to settle the main conjecture

In [AS18], we gave a formula for the power-sum expansion of all vertical-strip LLT polynomials. It is straightforward to expand (6) in the power-sum basis, so to settle Conjecture 10 it suffices to show that $\omega G_a(x; q + 1) = \omega \tilde{G}_a(x; q + 1)$ for all $a$. By comparing coefficients of $p_\lambda / z_\lambda$. We shall now introduce the necessary terminology from [AS18] to state Conjecture 10 in this form.

For any subset $S \subseteq E(\Gamma_a)$, let $P(S)$ denote the poset given by the transitive closure of the edges in $S$. Given a poset $P$ on $n$ elements, let $O(P)$ be the set of order-preserving surjections $f : P \to [k]$ for some $k$. The type of a surjection $f$ is defined as

$$\alpha(f) := (|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(k)|),$$

and this is a composition of $n$ with $k$ parts. Let $O_a(P) \subseteq O(P)$ be the set of surjections of type $\alpha$. Finally, let $O^*_a(P) \subseteq O_a(P)$ be the set of surjections $f \in O_a(P)$ such that for each $j \in [k]$, $f^{-1}(j)$ is a subposet of $P$ with a unique minimal element.

\begin{proposition}[See [AS18] Thm. 5.6, Thm. 7.10]
The power-sum expansion of $\omega G_a(x; q + 1)$ is given as

$$\omega G_a(x; q + 1) = \sum_{\theta \in O(a)} q^{\text{asc}(\theta)} \sum_{\lambda \vdash n} \frac{p_\lambda(x)}{z_\lambda} |O^*_\lambda(P(\theta))|$$

(40)

where $P(\theta)$ is the poset on $[n]$ and edges given by the transitive closure of the ascending edges in $\theta$.

The family of functions $\tilde{G}_a(x; q + 1)$ has a similar expansion in terms of the power-sum symmetric functions.

\begin{lemma}
The power-sum expansion of $\omega \tilde{G}_a(x; q + 1)$ is given as

$$\omega \tilde{G}_a(x; q + 1) = \sum_{\theta \in O(a)} q^{\text{asc}(\theta)} \sum_{\lambda \vdash n} \frac{p_\lambda(x)}{z_\lambda} |O^*_\lambda(B(\theta))|$$

(41)

where $B(\theta)$ is the poset consisting of a disjoint union of chains with lengths given by $\pi(\theta)$.
Proof. This follows easily from the definition of \( \hat{G}_a(x; q + 1) \) and the expansion of the elementary symmetric functions into power-sum symmetric functions, see \[ER91\] and \[AS18\] Section 7. \qed

**Conjecture 50** (Equivalent with Conjecture 10). For any area sequence \( a \) of length \( n \) and partition \( \lambda \vdash n \),
\[
\sum_{\theta \in O(a)} q^{asc(\theta)}|O_\lambda^a(P(\theta))| = \sum_{\theta \in O(a)} q^{asc(\theta)}|O_\lambda^a(B(\theta))|.
\]

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