\( \alpha \)-divergence derived as the generalized rate function in a power-law system

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Abstract—The generalized binomial distribution in Tsallis statistics (power-law system) is explicitly formulated from the precise \( q \)-Stirling’s formula. The \( \alpha \)-divergence (or \( q \)-divergence) is uniquely derived from the generalized binomial distribution in the sense that when \( \alpha \rightarrow -1 \) (i.e., \( q \rightarrow 1 \)) it recovers KL divergence obtained from the standard binomial distribution. Based on these combinatorial considerations, it is shown that \( \alpha \)-divergence (or \( q \)-divergence) is appeared as the generalized rate function in the large deviation estimate in Tsallis statistics.

I. INTRODUCTION

The large deviation principle (LDP for short) has mathematically presented and quantified the asymptotic behavior of the probabilities of rare events in many stochastic phenomena. It has brought about deep significant insights for understanding the exponential decay of rare events in stochastic phenomena with great help of many well-established theoretical results based on “i.i.d. (independent and identically distributed)” for random variables. This assumption leads to the discussion on the exponential decay of rare events in stochastic phenomena with great help of many well-established theoretical results based on “i.i.d.” assumption. This strong “i.i.d.” assumption has been often been to be weakened in many studies. One of the reasons is that actual observations generally do not satisfy i.i.d. assumptions. A typical and well-known example is power-law behavior often observed in strongly correlated systems. In these cases we take Tsallis statistics as one of such power-law systems because its mathematical foundations has been widely explored [8].

Along similar studies on LDP related with Tsallis statistics, there are a few papers such as [9] and [10]. The paper [9] discusses the possibility of LDP for the strongly correlated random variables in Tsallis statistics. They consider the correlated coin tossing model based on the \( q \)-Gaussian distribution and numerically evaluate the possibility of a \( q \)-generalization of LDP for a given \( q \)-divergence. On the other hand, our present paper does not require the \( q \)-Gaussian distribution and the \( q \)-divergence in advance for the large deviation estimate. Our approach is completely analytical starting from the fundamental nonlinear differential equation \( dy/dx = y^q \) only, and the \( q \)-divergence is naturally derived from the formulation of the generalized binomial distribution (\( q \)-binomial distribution), which results in an appearance of the \( q \)-divergence as the rate function in our main result (i.e., Theorem 14). Thus our approach and its results are quite different from [9]. The paper [10] discusses the application of the \( q \)-exponential functions to the standard LDP based on the i.i.d. assumptions, which obviously differs from our present work.

In this paper, we apply our combinatorial formulations in Tsallis statistics to the large deviation estimate in it. This paper consists of the five sections including this introduction. In the section two we briefly review the derivation of Tsallis entropy as the unique entropy corresponding to \( dy/dx = y^q \). In the course of the derivation of Tsallis entropy, the rough \( q \)-Stirling’s formula is derived and applied. In the formulation of the \( q \)-Stirling’s formula, the precise \( q \)-Stirling’s formula has been already given in [9], which is useful for the formulation of the \( q \)-binomial distribution in Tsallis statistics. The derivation of the \( q \)-binomial distribution is given in the section three. The \( q \)-binomial distribution has the nice correspondence with the \( q \)-divergence (or \( \alpha \)-divergence), which is given in the following section four. The one-to-one correspondence between the \( q \)-binomial distribution and the \( q \)-divergence (or \( \alpha \)-divergence) is applied to the large deviation estimate in a combinatorial approach. The final section is devoted to the conclusion.

II. TSALLIS ENTROPY AND \( q \)-STIRLING’S FORMULA UNIQUELY DETERMINED FROM \( dy/dx = y^q \)

Our approach starts from the fundamental nonlinear differential equation \( dy/dx = y^q \) only, instead of Tsallis entropy. This approach provides us with most of the theoretical results in Tsallis statistics shown in [8] [11]. (See [14][15][16] for the review of the studies in Tsallis statistics and the maximum entropy principle approach.)

The solution to \( dy/dx = y^q \) yields the \( q \)-exponential \( \exp_q (x) \) as the inverse function of the \( q \)-logarithm \( \ln_q x \), which are respectively defined as follows:

**Definition 1:** (\( q \)-logarithm, \( q \)-exponential) The \( q \)-logarithm \( \ln_q x : \mathbb{R}^+ \rightarrow \mathbb{R} \) and the \( q \)-exponential \( \exp_q (x) : \mathbb{R} \rightarrow \mathbb{R} \) for \( x \in \mathbb{R} \) satisfying \( 1 + (1 - q) x > 0 \) are respectively defined by

\[
\ln_q x := \frac{x^{1-q} - 1}{1-q}, \quad (1)
\]
\[
\exp_q (x) := [1 + (1 - q) x]^\frac{1}{1-q}. \quad (2)
\]
Then a new product $\otimes_q$ to satisfy the following identities as the $q$-exponential law is introduced.

$$
\ln_q (x \otimes_q y) = \ln_q x + \ln_q y, \quad \exp_q (x \otimes_q y) = \exp_q (x + y).
$$

For this purpose, the new multiplication operation $\otimes_q$ is introduced in [17] [18] (See also [19]). The concrete forms of the $q$-logarithm and $q$-exponential are given in [1] and [3], so that the above requirement (3) or (4) as the $q$-exponential law leads to the definition of $\otimes_q$, between two positive numbers.

**Definition 2:** ($q$-product) For $x, y \in \mathbb{R}^+$ satisfying $x^{1-q} + y^{1-q} - 1 > 0$, the $q$-product $x \otimes_q y$ is defined by

$$
x \otimes_q y := \left[ x^{1-q} + y^{1-q} - 1 \right]^{1-q}. \tag{5}
$$

The $q$-product recovers the usual product such that

$$
\lim_{q \to 1} (x \otimes_q y) = xy.
$$

By means of the $q$-product $\otimes_q$, the $q$-factorial is naturally defined in the following form [8].

**Definition 3:** ($q$-factorial) For a natural number $n \in \mathbb{N}$ and $q \in \mathbb{R}^+$, the $q$-factorial $n!_q$ is defined by

$$
n!_q := 1 \otimes_q \cdots \otimes_q n. \tag{6}
$$

Thus, we concretely compute $q$-Stirling’s formula.

**Theorem 4:** (rough $q$-Stirling’s formula) Let $n!_q$ be the $q$-factorial defined by (6). The rough $q$-Stirling’s formula $\ln_q (n!_q)$ is computed as follows:

$$
\ln_q (n!_q) = \begin{cases} 
n \ln_q n - n - \frac{2-q}{2-q} & \text{if } q \neq 2, \\
n - n \ln n + O(1) & \text{if } q = 2. \end{cases} \tag{7}
$$

See [3] for the proof. Similarly to the $q$-product, $q$-ratio is introduced from the requirements:

$$
\ln_q (x \otimes_q y) = \ln_q x - \ln_q y, \quad \exp_q (x \otimes_q y) = \exp_q (x - y). \tag{9}
$$

Then we define the $q$-ratio as follows.

**Definition 5:** ($q$-ratio) For $x, y \in \mathbb{R}^+$ satisfying $x^{1-q} - y^{1-q} + 1 > 0$, the inverse operation to the $q$-product is defined by

$$
x \otimes_q y := \left[ x^{1-q} - y^{1-q} + 1 \right]^\frac{1}{1-q}. \tag{10}
$$

which is called $q$-ratio in [18].

The $q$-product, $q$-factorial and $q$-ratio are applied to the definition of the $q$-multinomial coefficient [8].

**Definition 6:** ($q$-multinomial coefficient) For $n = \sum_{i=1}^{k} n_i$ and $n_i \in \mathbb{N} (i = 1, \cdots, k)$, the $q$-multinomial coefficient is defined by

$$
\begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q := (n!_q) \otimes_q \left[ (n_1!_q) \otimes_q \cdots \otimes_q (n_k!_q) \right]. \tag{11}
$$

From the definition (11), it is clear that

$$
\lim_{q \to 1} \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q = \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix} = \frac{n!}{n_1! \cdots n_k!}. \tag{12}
$$

Throughout this paper, we consider the $q$-logarithm of the $q$-multinomial coefficient to be given by

$$
\ln_q \left[ \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q \right] = \ln_q (n!_q) - \ln_q (n_1!_q) - \cdots - \ln_q (n_k!_q). \tag{13}
$$

Based on these fundamental formulas, we obtain the one-to-one correspondence [14] between the $q$-multinomial coefficient and Tsallis entropy as follows [8].

**Theorem 7:** When $n \in \mathbb{N}$ is sufficiently large, the $q$-logarithm of the $q$-multinomial coefficient coincides with Tsallis entropy in the following correspondence:

$$
\ln_q \left[ \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q \right] \approx \begin{cases} 
n_2^{1-q} \cdot S_{2-q} \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right) & \text{if } q > 0, q \neq 2 \\
-S_1 (n) + \sum_{i=1}^{k} S_1 (n_i) & \text{if } q = 2 \end{cases} \tag{14}
$$

where $S_q$ is Tsallis entropy defined by

$$
S_q (p_1, \ldots, p_k) := \frac{1 - \sum_{i=1}^{k} p_i^q}{q - 1}. \tag{15}
$$

and $S_1 (n) := \ln n$.

See [3] for the proof. In this way, Tsallis entropy is determined as the unique entropy corresponding to the fundamental nonlinear differential equation $dy/dx = y^q$. Clearly, the additive duality $“q \leftrightarrow 2 – q”$ appears in the above one-to-one correspondence [14]. Other dualities such as the multiplicative duality $“q \leftrightarrow 1/q”$, $q$-tripllet and multifractal triplet appears as special cases of the more generalized correspondence [11]. Apart from these derivations, a typical and fundamental application of the $q$-product is the derivation of the $q$-Gaussian distribution through the maximum likelihood principle [12].

### III. The Generalized Binomial Distribution Derived from the Precise $q$-Stirling’s Formula

In the previous section, the rough $q$-Stirling’s formula (7) is applied to the derivation of Tsallis entropy as the unique entropy corresponding to $dy/dx = y^q$. In order to define the generalized binomial distribution, the precise $q$-Stirling’s formula is required.

**Theorem 8:** (precise $q$-Stirling’s formula) Let $n!_q$ be the $q$-factorial defined by (6). The precise $q$-Stirling’s formula $\ln_q n!_q$ is computed by:

$$
\ln_q n!_q \simeq \begin{cases} 
n - \frac{1}{2} \ln n - \frac{1}{2} - \delta_2 & \text{if } q = 2 \\
n \frac{1}{q - 2} + \frac{1}{2} \ln n - \frac{n}{2-q} + c_q \quad (q \neq 2) \end{cases} \tag{16}
$$

where $c_q := \frac{1}{2-q} - \delta_q$, $\delta_q$ is a function of $q$ only, and $\delta_1 = 1 - \ln \sqrt{2\pi}$. The proof is given in [3]. Note that the terms $c_q$ and $\frac{1}{2}$ do not depend on $n$, so that the precise $q$-Stirling’s formula (16) recovers the rough $q$-Stirling’s formula (7) if $c_q$ and $\frac{1}{2}$ are ignored.
Proposition 9:
\[
\ln_q \left[ \binom{n}{k} \right]_q \simeq -c_q + \frac{1}{2} \left( \ln_q n - \ln_q k - \ln_q (n-k) \right) + \frac{n^{2-q}}{2-q} S_{2-q} \left( \frac{k}{n}, 1 - \frac{k}{n} \right)
\]
(17)

This is easily proved by the straightforward computation using the precise \( q \)-Stirling’s formula \([16]\). For easy understanding of the derivation of the generalized binomial distribution, consider the case \( q = 1 \) in (17).

\[
\ln \left[ \binom{n}{k} \right] \simeq -n R \sqrt{2\pi} + \frac{1}{2} \ln \left( \frac{n}{k(n-k)} \right) + n S_1 \left( \frac{k}{n}, 1 - \frac{k}{n} \right)
\]
(18)

\[
= \ln \frac{1}{\sqrt{2\pi R}} \sqrt{\frac{n}{k(n-k)}} + \left( -k \ln \frac{k}{n} - (n-k) \ln \left( 1 - \frac{k}{n} \right) \right)
\]
(19)

That is, we have
\[
\binom{n}{k} \left( \frac{k}{n} \right)^k \left( 1 - \frac{k}{n} \right)^{n-k} \simeq \frac{1}{\sqrt{2\pi R}} \sqrt{\frac{n}{k(n-k)}} \]
(20)

The left side of (20) is the special form of the standard binomial distribution \( \binom{n}{k} \) in the sense of \( R = \frac{k}{n} \). For generalization of the standard binomial distribution, the most important observation in this computation is that the term:
\[
n S_1 \left( \frac{k}{n}, 1 - \frac{k}{n} \right) = -k \ln \frac{k}{n} - (n-k) \ln \left( 1 - \frac{k}{n} \right)
\]
(21)

in (18) and (19) corresponds to \( R^k (1-R)^{n-k} \) in the standard binomial distribution \( \binom{n}{k} \) by replacement \( R = \frac{k}{n} \). More precisely, (21) coincides with
\[
-k \ln (1-R) = -k \ln r - (n-k) \ln \left( 1 - \frac{k}{n} \right)
\]
(22)
by the replacement. (Compare the right sides in (21) and (22).)

This correspondence is also applied to (17). The last term on the right hand of (17) is computed as
\[
\frac{n^{2-q}}{2-q} S_{2-q} \left( \frac{k}{n}, 1 - \frac{k}{n} \right) = \frac{1}{2-q} \left( -k^{2-q} \ln_{2-q} k - (n-k)^2 \ln_{2-q} \left( 1 - \frac{k}{n} \right) \right).
\]
(23)

Applying the replacement \( R = \frac{k}{n} \) in this formula to (17) as similarly as the replacement \( R = \frac{k}{n} \) in the standard case (19), the generalized binomial distribution \( b_q (k; n, r) \) is defined.

Definition 10: For given \( n, k \leq n \in \mathbb{N} \) and \( r \in (0, 1) \), if the \( q \)-logarithm of the probability mass function \( b_q (k; n, r) \) is given by
\[
\ln_q b_q (k; n, r) = \ln_q \left[ \binom{n}{k} \right]_q + \frac{1}{2-q} \left( k^{2-q} \ln_{2-q} r + (n-k)^2 \ln_{2-q} \left( 1 - r \right) \right) + C_q
\]
(24)
with
\[
\sum_{k=0}^{n} b_q (k; n, r) = 1, \quad 1 + (1-q) C_q > 0 \quad \text{and} \quad C_1 = 0,
\]
(25)
\( b_q (k; n, r) \) is the \( q \)-binomial distribution.

The \( q \)-multinomial distribution \( m_q (n_1, \ldots, n_k; n_1, r_1, \ldots, r_k) \) is easily defined in a similar way.
\[
\ln_q m_q (n_1, \ldots, n_k; n_1, r_1, \ldots, r_k) = \ln_q \left[ \begin{array}{c} n_1 \\ \vdots \\ n_k \end{array} \right]_q + \frac{1}{2-q} \sum_{i=1}^{k} n_i^{2-q} \ln_{2-q} r_i + C_q
\]
(26)
where \( n = \sum_{i=1}^{k} n_i, \quad n_i \in \mathbb{N} \) \( (i = 1, \ldots, k) \), and \( 1 = \sum_{i=1}^{k} r_i \).

The explicit form of the \( q \)-binomial distribution \( b_q (k; n, r) \) can be written, but the form of \( \ln_q b_q (k; n, r) \) in (24) is much simpler and more useful for applications. Note that the \( q \)-binomial distribution \( b_q (k; n, r) \) includes the scaling effect \( \exp_q (C_q) \left( \frac{1-q}{1+q} \right) = 1 + (1-q) C_q (\geq 0) \) in itself. Of course, such a scaling effect disappears when \( q \to 1 \).

IV. \( \alpha \)-DIVERGENCE AND LARGE DEVIATION ESTIMATE DERIVED FROM THE GENERALIZED BINOMIAL DISTRIBUTION

The \( q \)-divergence is explicitly derived from the definition (24) of the \( q \)-binomial distribution \( b_q (k; n, r) \).

Theorem 11: For the \( q \)-binomial distribution \( b_q (k; n, r) \) defined by (24), we have
\[
\ln_q b_q (k; n, r) \simeq -\frac{n^{2-q}}{2-q} D_{2-q} (p || r) + C_q
\]
(27)
for large \( n \in \mathbb{N} \) where \( D_q (p || r) \) is the \( q \)-divergence defined by
\[
D_q (p || r) := \sum_{i=0}^{1} p_i \ln_{q} \left( p_i r_i \right) = \frac{1 - \sum_{i=0}^{1} p_i^q r_i^{1-q}}{1-q}
\]
(28)
and
\[
p := (p_0, p_1) = \left( \frac{k}{n}, 1 - \frac{k}{n} \right),
\]
(29)
\[r := (r_0, r_1) = (r, 1-r).
\]
(30)

Proof: A straightforward computation using (14) yields the right side of (27). 

Of course, for the \( q \)-multinomial distribution \( m_q (n_1, \ldots, n_k; n_1, r_1, \ldots, r_k) \) defined by (26), we easily obtain the similar result,
\[
\ln_q m_q (n_1, \ldots, n_k; n_1, r_1, \ldots, r_k) \simeq -\frac{n^{2-q}}{2-q} D_{2-q} (p || r) + C_q
\]
(31)
for large \( n \in \mathbb{N} \) where \( D_q (p || r) \) is the \( q \)-divergence defined by
\[
D_q (p || r) := \sum_{i=1}^{k} p_i \ln_{q} \left( p_i r_i \right) = \frac{1 - \sum_{i=1}^{k} p_i^q r_i^{1-q}}{1-q}
\]
(32)
and \( p := (p_1, \cdots , p_k) = (\frac{n_1}{n}, \cdots , \frac{n_k}{n}) \) and \( r := (r_1, \cdots , r_k) \). \((31)\) recovers the standard case in the limit \( q \to 1 \):

\[
\ln m_1 (n_1, \cdots , n_k; n, r_1, \cdots , r_k) \simeq -nD_1(p \| r) \tag{33}
\]

where \( D_1 (p \| r) \) is Kullback-Leibler (KL) divergence defined by

\[
D_1 (p \| r) := \sum_{i=1}^{k} p_i \ln \frac{p_i}{r_i} \tag{34}
\]

The \( q \)-divergence is known to have the simple relation with the \( \alpha \)-divergence \((22)\).

**Proposition 12:** The \( \alpha \)-divergence \( D^{(\alpha)} (p \| r) \) defined by

\[
D^{(\alpha)} (p \| r) := \begin{cases} \frac{4}{1-\alpha^2} \left( 1 - \sum_i p_i^{\frac{\alpha}{1-\alpha}} r_i^{\frac{1}{1-\alpha}} \right) & (\alpha \neq \pm 1) \\ \sum_i r_i \ln \frac{p_i}{r_i} & (\alpha = 1) \\ \sum_i p_i \ln \frac{p_i}{r_i} & (\alpha = -1) \end{cases}
\]

has the following simple relation with the \( q \)-divergence.

\[
D^{(\alpha)} (p \| r) = \frac{1}{q} D_{q} (p \| r) \quad (q \neq 0, 1) \tag{36}
\]

where

\[
q = \frac{1-\alpha}{2} \quad (\alpha \neq \pm 1). \tag{37}
\]

The simple transformation \((37)\) of the parameters \( q \) and \( \alpha \) reveals that the \( q \)-divergence is mathematically equivalent to the \( \alpha \)-divergence. This means that the \( \alpha \)-divergence is derived from the \( q \)-binomial distribution.

**Corollary 13:** For the \( q \)-binomial distribution \( b_q(k; n, r) \) defined by \((24)\), we have

\[
\ln_q b_q(k; n, r) \simeq -\frac{3}{2} D^{(-2-q)} (p \| r) + C'_{\alpha} \tag{38}
\]

where \( D^{(\alpha)} (p \| r) \) is the \( \alpha \)-divergence defined by \((35)\) and \( \lim_{n \to 1} C'_{\alpha} = 0 \).

The \( q \)-divergence was introduced as the relative entropy in Tsallis statistics \([20][21]\), independently from the \( \alpha \)-divergence and the family of the \( j \)-divergence. In fact, no references and comments on these divergences were given in \([20][21]\). On the other hand, the \( \alpha \)-divergence has much longer history than the \( q \)-divergence and was originally introduced in the evaluation of the classification errors \([23]\). Later, the \( \alpha \)-divergence has been studied in the information geometry for providing the geometrical structures of the manifold of probability measures which is well consistent with the fundamentals in statistics \([24]\).

Finally, using \((27)\), the large deviation estimate will be presented by means of the \( q \)-divergence (or the \( \alpha \)-divergence) in a combinatorial way.

**Theorem 14:** Let \( X_i \ (i = 1, \cdots , n) \) be a random variable taking values in \( \{0, 1\} \) with probability

\[
P(X_i = 0) = r, \quad P(X_i = 1) = 1 - r. \tag{39}
\]

If a sum of the random variables \( \sum_{i=1}^{n} X_i \) follows the \( q \)-binomial distribution \( b_q(k; n, r) \), for \( 0 < x < r \) and \( 0 < q < 2 \) we have

\[
\frac{1}{n^{2-q}} \ln_q P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \simeq -\frac{1}{2-q} D_{2-q} (x \| r). \tag{40}
\]

**Proof:**

\[
P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) = P \left( \sum_{i=1}^{n} X_i < nx \right) = \sum_{k=0}^{\lfloor nx \rfloor} b_q(k; n, r) \tag{41}
\]

where \( [a] := \max \{ m \in \mathbb{Z} \mid m \leq a \} \).

First, consider the upper bound. Each term in the sum is bounded by \( b_q(k; n, r) \mid_{k=[nx]} = b_q([nx]; n, r) \) and so

\[
P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \leq (\lfloor nx \rfloor + 1) b_q ([nx]; n, r) \tag{42}
\]

Let \( A_n \) be defined by the right side of this inequality

\[
A_n := (\lfloor nx \rfloor + 1) b_q ([nx]; n, r). \tag{43}
\]

(i) If \( q = 1 \), that is, the random variables \( X_i \) are i.i.d., \( \frac{1}{n} \ln A_n \) can be computed as

\[
\frac{1}{n} \ln A_n = \frac{1}{n} \ln (\lfloor nx \rfloor + 1) + \frac{1}{n} \ln b_q ([nx]; n, r) \tag{44}
\]

\[
\simeq -D_1 (x \| r) \quad (n \gg 0). \tag{45}
\]

Thus, we obtain the standard upper bound:

\[
\frac{1}{n} \ln P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \leq -D_1 (x \| r). \tag{46}
\]

(ii) If \( 0 < q < 1 \), we have

\[
\frac{1}{n^{2-q}} \ln_q A_n \tag{47}
\]

\[
= \ln_q b_q ([nx]; n, r) + b_q ([nx]; n, r)^{1-q} \ln_q ([nx] + 1) \tag{48}
\]

\[
\simeq -\frac{1}{2-q} D_{2-q} (x \| r) \quad (n \gg 0) \tag{49}
\]

\[
\left( \cdot : \frac{1}{n^{2-q}} \ln_q ([nx] + 1) \simeq 0 \right). \tag{50}
\]

Thus, we obtain the upper bound:

\[
\frac{1}{n^{2-q}} \ln_q P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \leq -\frac{1}{2-q} D_{2-q} (x \| r). \tag{51}
\]

(iii) If \( 1 < q < 2 \), using \( \ln_q a < \ln a \), we have

\[
\frac{1}{n^{2-q}} \ln_q A_n < \frac{1}{n^{2-q}} \ln A_n \tag{52}
\]

\[
= \frac{1}{n^{2-q}} (\ln ([nx] + 1) + \ln b_q ([nx]; n, r)) \tag{53}
\]

\[
\simeq \frac{1}{n^{2-q}} \ln \exp_q \left( -\frac{n^{2-q}}{2-q} D_{2-q} (x \| r) + C_q \right) \quad (n \gg 0) \tag{54}
\]

\[
= \frac{1}{n^{2-q}} \ln \left[ 1 + (q-1) \left( \frac{n^{2-q}}{2-q} D_{2-q} (x \| r) + C_q \right) \right] \tag{55}
\]

\[
< -\frac{1}{2-q} D_{2-q} (x \| r) \quad (\cdot : \ln a < a - 1) \tag{56}
\]

Thus, we obtain the upper bound:

\[
\frac{1}{n^{2-q}} \ln_q P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \leq -\frac{1}{2-q} D_{2-q} (x \| r). \tag{57}
\]
Then, immediately we find main result (40).

Then, immediately we find

\[ P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \geq b_q \left( \lfloor nx \rfloor ; n, r \right). \]  

(58)

Then, immediately we find

\[ \frac{1}{n^{2-q}} \ln_q P \left( \frac{1}{n} \sum_{i=1}^{n} X_i < x \right) \geq \frac{1}{n^{2-q}} \ln_q b_q \left( \lfloor nx \rfloor ; n, r \right) \]

\[ \simeq -\frac{1}{2-q} D_{2-q} (x \| r). \]  

(59)

The upper bound (57) and lower bound (59) lead to the main result (40).

V. CONCLUSION

Starting from the fundamental nonlinear differential equation \( dy/dx = y^q \) only, we obtain the large deviation estimate in Tsallis statistics by means of the \( q \)-divergence (or the \( \alpha \)-divergence) as the generalized rate function. The present approach provides us with important formulas for a power-law system such as \( q \)-logarithm, \( q \)-exponential, \( q \)-product, \( q \)-Gaussian distribution, \( q \)-Stirling’s formula, \( q \)-multinomial coefficient, Tsallis entropy, and \( q \)-binomial distribution. We use analytical derivations only, which recover the standard case when \( q \to 1 \). Therefore, our present results reveal that there exists a fundamental and novel mathematical structure for power-law system recovering the standard case (i.e., i.i.d. case in probability theory, Shannon information theory, Boltzmann-Gibbs statistics in statistical physics) as a special case. Of course, some important theorems in a power-law system are still missing. However, the present strategy will definitely provide us with fruitful applications in any related areas. Such some applications will be presented in the near future.

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