A parameterized halting problem, $\Delta_0$ truth
and the MRDP theorem*

Yijia Chen
Department of Computer Science
Shanghai Jiao Tong University
yijia.chen@cs.sjtu.edu.cn

Moritz Müller
Faculty of Computer Science and Mathematics
University of Passau
moritz.mueller@uni-passau.de

Keita Yokoyama
Mathematical Institute
Tohoku University
keita.yokoyama.c2@tohoku.ac.jp

Abstract

We study the parameterized complexity of the problem to decide whether a given natural
number $n$ satisfies a given $\Delta_0$-formula $\varphi(x)$; the parameter is the size of $\varphi$. This parameter-
ization focusses attention on instances where $n$ is large compared to the size of $\varphi$. We show
unconditionally that this problem does not belong to the parameterized analogue of $AC^0$.
From this we derive that certain natural upper bounds on the complexity of our parameter-
ized problem imply certain separations of classical complexity classes. This connection is
obtained via an analysis of a parameterized halting problem. Some of these upper bounds
follow assuming that $I\Delta_0$ proves the MRDP theorem in a certain weak sense.

1. Introduction

1.1. The parameterized halting problem. The complexity of the following parameterized
halting problem is still wide open.

$$p\text{-HALT}$$

Instance: $n \in \mathbb{N}$ in unary and a nondeterministic Turing machine $M$.
Parameter: $|M|$, the size of $M$.
Problem: Does $M$ accept the empty input in at most $n$ steps?

The importance of $p\text{-HALT}$ is derived from its close connections to central problems in proof
complexity and descriptive complexity theory \cite{10,28}. Among others, there is a logic for $\text{PTIME}$

*A partial conference version appeared as \cite{12}.
if \( p\text{-H}\text{ALT} \) can be decided by an algorithm in time \( n^f(|M|) \) for some function \( f : \mathbb{N} \to \mathbb{N} \). Sofar, however, such algorithms have been ruled out only under a certain very strong non-standard complexity-theoretic hypothesis and only for computable \( f \) [9, 10].

Thus, lower bounds on \( p\text{-H}\text{ALT} \) are poorly understood and of fundamental interest. A seemingly modest and natural starting point is

**Conjecture 1.1.** \( p\text{-H}\text{ALT} \notin \text{para-AC}^0 \).

Here, para-AC\(^0\) is the analogue of (uniform) AC\(^0\) in the parameterized world. One easily sees that \( p\text{-H}\text{ALT} \) is in a nonuniform version of para-AC\(^0\): for fixed \( k \in \mathbb{N} \), let \( M_{k,0}, \ldots, M_{k,\ell_k-1} \) list all nondeterministic Turing machines of size \( k \) and let \( n_{k,i} \) be the minimal \( n \) such that \( M_{k,i} \) accepts the empty input in \( n \) steps; if there is no such \( n \), let \( n_{k,i} := \infty \). Then, on instances \((1^n, M)\) with parameter \(|M| = k\), \( p\text{-H}\text{ALT} \) is decided by the following family of simple Boolean functions:

\[
F_{n,k}(x_0 \ldots x_{n-1}, y_0 \ldots y_{k-1}) = \bigvee_{i < \ell_k \text{ such that } n_{k,i} \leq n} (x_0 \ldots x_{n-1} = 1^n \land y_0 \ldots y_{k-1} = M_{k,i}).
\]

Observe that \( F_{n,k} \) can be understood as a circuit of depth 2 and size \( O(k \cdot \ell_k \cdot n) \).

Conjecture 1.1 is highly plausible and might appear to be within reach because AC\(^0\) is well-understood and, in particular, [11] establishes (unconditional) para-AC\(^0\) lower bounds for many well-studied parameterized problems. It deserves some genuine interest because its failure implies that AC\(^0\), or equivalently, \((+, \times)\)-invariant FO is captured by some logic. However, we failed to prove the conjecture after years of attempts and only now understand why: it implies that nondeterministic exponential time NE is distinct from the linear time hierarchy LINH. This connection can be further tightened by considering the following variant of \( p\text{-H}\text{ALT} \):

\[
\begin{array}{|l|}
\hline
p\text{-H}\text{ALT}_- \\
\hline
\text{Instance:} & n \in \mathbb{N} \text{ in unary and a nondeterministic Turing machine } M. \\
\text{Parameter:} & |M|. \\
\text{Problem:} & \text{Does } M \text{ accept the empty input in } \text{exactly } n \text{ steps.} \\
\hline
\end{array}
\]

While the classical problems underlying \( p\text{-H}\text{ALT}_- \) and \( p\text{-H}\text{ALT} \) are easily seen to be equivalent, we shall see that their parameterized versions behave quite differently. In fact, \( p\text{-H}\text{ALT}_- \) appears to be harder than \( p\text{-H}\text{ALT} \), e.g., a simple Boolean function family like \( F_{n,k} \) for \( p\text{-H}\text{ALT} \) is not known to exist for \( p\text{-H}\text{ALT}_- \). We refer to Section 7 for some discussion.

We show:

**Theorem 1.2.**

(i) \( p\text{-H}\text{ALT}_- \in \text{para-AC}^0 \) if and only if \( \text{NE} \subseteq \text{LINH} \).

(ii) \( p\text{-H}\text{ALT}_- \in \text{para-AC}^0 \) implies \( p\text{-H}\text{ALT} \in \text{para-AC}^0 \).
1.2. The MRDP theorem. Thus, to settle Conjecture 1.1 one might try to first separate NE from LINH. We tie this question to the provability of the Matiyasevich-Robinson-Davis-Putnam (MRDP) theorem \[13\] in bounded arithmetic. This theorem states that \(\Sigma_1\)-definable sets are Diophantine and it is a long standing open problem whether it is provable in \(I\Delta_0\), i.e., Peano arithmetic with induction restricted to \(\Delta_0\)-formulas.

Wilkie observed \[33\] that a positive answer would imply the collapse of LINH to NLIN (non-deterministic linear time), and therefore \(NP = \text{co-NP}\) and \(NE \not\subseteq LINH\). We derive the latter consequence from an apparently much weaker provability assumption:

**Theorem 1.3.** If \(I\Delta_0\) proves MRDP for small numbers, then \(NE \not\subseteq LINH\).

Roughly, that \(I\Delta_0\) proves MRDP for small numbers means that the equivalence of any \(\Delta_0\)-formula \(\varphi(\bar{x})\) to some Diophantine formula is proved in \(I\Delta_0\) for all \(\bar{x}\) of logarithmic order. Model-theoretically, the equivalence holds in any \(I\Delta_0\)-model for all \(\bar{x}\) from the initial segment of numbers \(x\) such that \(2^x\) exists, while proof-theoretically, we allow an \(I\Delta_0\)-proof to use exponentiation, but only once.

Such limited use of exponentiation has been studied in bounded arithmetic \[26\]. An unlimited use of exponentiation is sufficient: Gaifman and Dimitracopoulos \[20\] showed that \(I\Delta_0 + \forall x \exists y 2^x = y\) does prove MRDP. Kaye \[24\] proved MRDP using only induction for bounded existential formulas plus an axiom stating the totality of a suitable function of exponential growth. It is asked in \[20, p.188\] whether \(I\Delta_0\) plus the totality of \(x^{\log x}\), or of \(x^{\log \log x}\) etc. proves MRDP, and it is easy to see that if the answer to the first (or any) of these questions is positive, then \(I\Delta_0\) proves MRDP for small numbers. Intuitively, this provability is much weaker than \(I\Delta_0\)-provability.

1.3. \(\Delta_0\) truth. Theorem 1.3 follows from our main result concerning the complexity to decide the truth of \(\Delta_0\)-sentences:

| p-\(\Delta_0\)-\textsc{Truth} |
|-------------------------------|
| **Instance:** \(n \in \mathbb{N}\) in unary and a \(\Delta_0\)-formula \(\varphi(x)\). |
| **Parameter:** | \(|\varphi|\), the size of \(\varphi\). |
| **Problem:** \(\mathbb{N} \models \varphi(n)\)? |

Intuitively, taking \(|\varphi|\) as the parameter means shifting attention to inputs where \(n\) is much larger than \(|\varphi|\). This is a natural focus. Classical work of Paris and Dimitracopoulos \[31\] took \(n\) to be nonstandard and related the complexity of truth definitions for \(\Delta_0\)-formulas to the complexity-theoretic hypotheses that LINH or PH does not collapse.

Wilkie proved a weak version of the former hypothesis by showing that \(p-\Delta_0\)-\textsc{Truth} restricted to quantifier-free formula inputs can be decided in space \(f(k) + O(\log n)\) where \(k := |\varphi|\) is the parameter and \(f: \mathbb{N} \to \mathbb{N}\) a computable function \[33, Proof of Lemma 3.1\]. The straightforward algorithm decides \(p-\Delta_0\)-\textsc{Truth} in space \(f(k) \cdot \log n\). Can \(p-\Delta_0\)-\textsc{Truth} be decided in space \(f(k) + O(\log n)\)? Maybe with nondeterminism? Can it be decided in time \(f(k) \cdot n^{O(1)}\)? Maybe with nondeterminism, i.e., is \(p-\Delta_0\)-\textsc{Truth} \(\in\) para-NP?
At present all these questions are wide open. Our main result (Theorem 4.3) shows that such upper bounds on the parameterized complexity of \( p-\Delta_0^\text{-TRUTH} \) imply lower bounds in classical complexity theory. Notably,

**Theorem 1.4.** If \( p-\Delta_0^\text{-TRUTH} \in \text{para-NP} \), then \( \text{NE} \not\subseteq \text{LINH} \).

Theorem 1.4 follows from our analysis of \( p-\text{HALT}_\pm \) and the following unconditional lower bound:

**Theorem 1.5.** \( p-\Delta_0^\text{-TRUTH} \not\in \text{para-AC}^0 \).

The proof is based on diagonalization or, more specifically, the undefinability of truth. Furthermore, it relies on the classical result [6] of descriptive complexity theory that, roughly speaking, equates \( \text{AC}^0 \) and first-order logic with built-in arithmetic.

### 1.4. \( \text{AC}^0 \)-bi-immunity

Could Conjecture 1.1 be false? We give further evidence for its truth by establishing a connection to the existence of \( \text{AC}^0 \)-bi-immune sets in \( \text{NP} \). Recall, a problem \( Q \) is \( \text{AC}^0 \)-bi-immune if neither \( Q \) nor its complement contain an infinite subset in \( \text{AC}^0 \).

**Theorem 1.6.** If \( \text{NP} \) contains an \( \text{AC}^0 \)-bi-immune problem, then \( p-\text{HALT} \not\in \text{para-AC}^0 \).

It is a standard hypothesis that \( \text{NP} \) contains even \( \text{P} \)-bi-immune problems and this follows from the measure hypothesis [27]. Whether \( \text{NP} \) contains at least \( \text{AC}^0 \)-bi-immune problems has been asked once it was realized [21, 1] that deterministic time hierarchy theorems hold with bi-immunity (or, equivalently [5], almost everywhere) while this is open for nondeterministic time [1, 19]. While Zimand [34] obtained some partial positive answers, Allender and Gore [2] showed that this has different answers relative to different oracles.1 This indicates that also refuting Conjecture 1.1 might be non-trivial.

### 1.5. Outline

Much of the technical work consists in connecting the dots between results of various subareas of logic and complexity, namely classical, parameterized and descriptive complexity theory and formal arithmetic. Section 2 reviews the results we need and fixes our notation. The technicalities are somewhat subtle, in particular, the move from \( p-\text{HALT} \) to \( p-\text{HALT}_\pm \) is crucial. Section 3 proves Theorem 1.2 and various variants of it via an analysis of \( p-\text{HALT}_\pm \): it is in a strong sense the hardest among what we call almost tally problems in para-NP. Such problems have instances consisting of a natural number in unary notation plus a short binary string where short means having a length effectively bounded in terms of the parameter. Section 4 proves Theorem 1.5. This together with the results in Section 3 implies Theorem 1.4 and various variants. Section 5 derives (a strengthening of) Theorem 1.3 from Theorem 1.4. Section 6 proves Theorem 1.6. The final section discusses the role of uniformity, and exhibits the different behaviours of our parameterized problems \( p-\text{HALT} \), \( p-\text{HALT}_\pm \) and \( p-\Delta_0^\text{-TRUTH} \).

---

1[2] studies \( \text{AC}^0 \)-immunity but their oracle constructions can be adapted to \( \text{AC}^0 \)-bi-immunity.
2. Preliminaries

Standard monographs are [29, 3] for classical complexity theory, [14, 18, 15] for parameterized complexity theory, [22, 25] for formal arithmetic, and [23, 16] for descriptive complexity theory.

2.1. Classical complexity. A (classical) problem is a subset of \( \{0, 1\}^* \), the set of finite binary strings. The length of a binary string \( x \in \{0, 1\}^* \) is denoted \( |x| \). For \( n \in \mathbb{N} \) we let \( 1^n \) denote the binary string consisting of \( n \) many 1’s. We use multitape Turing machines with alphabet \( \{0, 1\} \) as our basic model of computation. When considering \textit{dlogtime} Turing machines, i.e. deterministic machines running in time \( O(\log n) \), it is understood that they access their input via an address tape (see e.g. [6]). As usual, \( P \) and \( NP \) denote deterministic and nondeterministic polynomial time \( n^{O(1)} \), and \( E \) and \( \text{NE} \) denote deterministic and nondeterministic exponential time with linear exponent \( 2^{O(n)} \). The \textit{linear time hierarchy} \( \text{LINH} \) is the set of problems acceptable by alternating Turing machines in linear time \( O(n) \) with \( O(1) \) alternations. \( \text{LINSPE} \) and \( \text{NLINSPE} \) denote deterministic and nondeterministic linear space \( O(n) \). Clearly,

\[
\text{LINH} \subseteq \text{LINSPE} \subseteq \text{NLINSPE} \subseteq E \subseteq \text{NE}.
\]

Following [6] we define (dlogtime uniform) \( \text{AC}^0 \) as the set of problems decided by \( \text{AC}^0 \)-circuit families \( (C_n)_{n \in \mathbb{N}} \):

- \( C_n \) is a circuit (with \( \wedge, \vee, \neg \) gates and unbounded fan-in) with \( n \) variables, size \( \leq n^c \) and depth \( \leq d \), where \( c, d \in \mathbb{N} \) are two constants independent of \( n \);
- there is a dlogtime Turing machine which given \( 1^n, i, b \) where \( n, i \in \mathbb{N} \) and \( b \in \{0, 1\} \) decides whether the \( i \)-th bit of the binary encoding of \( C_n \) is \( b \).

Here, for binary strings \( x = x_0 \cdots x_{|x|-1} \) and \( y = y_0 \cdots y_{|y|-1} \) we use the standard pairing

\[
\langle x, y \rangle := x_0x_0 \cdots x_{|x|-1}x_{|x|-1}01y_0y_0 \cdots y_{|y|-1}y_{|y|-1},
\]

and similarly for more arguments. The above definition is somewhat sensitive to the choice of the binary encoding of a circuit. An appropriate choice would be to encode \( C_n \) by the list of strings in the \textit{direct connection language} corresponding to \( n \); we refer to [6] for details.

For \( n \in \mathbb{N} \) we let \( \text{bin}(n) \in \{0, 1\}^* \) denote the binary expansion of \( n \); it has length \( \lceil \log(n + 1) \rceil \).

For \( x \in \{0, 1\}^* \) let \( \text{num}(x) \) be the natural number with binary expansion \( 1x \), i.e., \( \text{bin}(\text{num}(x)) = 1x \). For a problem \( Q \) let

\[
\text{un}(Q) := \{1^{\text{num}(x)} \mid x \in Q\}.
\]

The last statement of the following is [2, Proposition 5], and the first two are trivial:

\textbf{Proposition 2.1 ([2])}. Let \( Q \) be a problem. Then:

(i) \( Q \in \text{NE} \) if and only if \( \text{un}(Q) \in \text{NP} \).

(ii) \( Q \in \text{E} \) if and only if \( \text{un}(Q) \in \text{P} \).

(iii) \( Q \in \text{LINH} \) if and only if \( \text{un}(Q) \in \text{AC}^0 \).
2.2. Parameterized complexity. A parameterized problem is a pair \((Q, \kappa)\) of an underlying classical problem \(Q \subseteq \{0, 1\}^*\) and a polynomial time computable parameterization \(\kappa : \{0, 1\}^* \to \mathbb{N}\) mapping an instance \(x \in \{0, 1\}^*\) to its parameter \(\kappa(x) \in \mathbb{N}\). E.g., \(p\text{-HALT}\) has underlying classical problem \(\{\langle 1^n, M \rangle \mid \text{the nondeterministic Turing machine } M \text{ accepts the empty input in at most } n \text{ steps}\}\) and a parameterization \(\kappa\) that maps strings of the form \(\langle 1^n, M \rangle\) to \(|M|\) and other strings to, say, 0.

The para-operator \([17]\) turns a classical complexity class into a parameterized one (the most important intractable parameterized classes are not of this form, however). The class \(\text{para-P = FPT}\) contains the parameterized problems \((Q, \kappa)\) that are fixed-parameter tractable, i.e., decidable in deterministic time \(f(\kappa(x)) \cdot |x|^{O(1)}\) for some computable \(f : \mathbb{N} \to \mathbb{N}\). Similarly, para-NP denotes nondeterministic time \(f(\kappa(x)) \cdot |x|^{O(1)}\) (for any computable \(f\)), para-L denotes deterministic space \(f(\kappa(x)) + O(\log |x|)\), and para-NL denotes nondeterministic such space. Clearly,

\[
\text{para-L} \subseteq \text{para-NL} \subseteq \text{FPT} \subseteq \text{para-NP}.
\]

The central parameterized class in this paper is \(\text{para-AC}^0\). It is characterized as follows:

**Proposition 2.2 ([11]).** Let \((Q, \kappa)\) be a parameterized problem such that \(Q\) is decidable and \(\kappa\) is computable by an \(\text{AC}^0\)-circuit family. Then the following are equivalent.

(i) \((Q, \kappa)\) \(\in \text{para-AC}^0\).

(ii) There is a family \((C_{n,k})_{n,k \in \mathbb{N}}\) of circuits such that

- there are a computable \(f : \mathbb{N} \to \mathbb{N}\) and \(c, d \in \mathbb{N}\) such that for all \(n, k \in \mathbb{N}\) the circuit \(C_{n,k}\) has \(n\) inputs, size at most \(f(k) \cdot n^c\), and depth at most \(d\);
- for all \(x \in \{0, 1\}^*\) we have
  \[
  x \in Q \iff C_{|x|,\kappa(x)}(x) = 1;
  \]

- there are a computable \(g : \mathbb{N} \to \mathbb{N}\) and a deterministic Turing machine which given as input \(\langle 1^n, 1^k, i, b \rangle\) where \(n, k, i \in \mathbb{N}\) and \(b \in \{0, 1\}\) decides in time \(g(k) + O(\log n)\) whether the \(i\)-th bit of the binary encoding of \(C_{n,k}\) is \(b\).

(iii) There is a computable \(h : \mathbb{N} \to \mathbb{N}\) and an \(\text{AC}^0\)-circuit family \((C_n)_{n \in \mathbb{N}}\) such that for all \(x \in \{0, 1\}^*\) with \(|x| \geq h(\kappa(x))\) we have

\[
  x \in Q \iff C_{|x|}(x) = 1.
\]

According to the terminology of \([17]\), (iii) states that \((Q, \kappa)\) is eventually in \(\text{AC}^0\).

2.3. Formal arithmetic. We let \(L_{ar} := \{+, \times, 0, 1, <\}\) be the language of arithmetic with binary function symbols \(+, \times\), constants \(0, 1\) and a binary relation symbol \(<\). The standard \(L_{ar}\)-structure, denoted \(\mathbb{N}\), has universe \(\mathbb{N}\) and interprets the symbols in the obvious way. Every \(L_{ar}\)-term \(p\) computes a polynomial with coefficients in \(\mathbb{N}\) and of total degree at most \(|p|\). We do not distinguish terms \(p\) or formulas \(\varphi\) from their binary encodings, so \(|p|\) and \(|\varphi|\) denote the
lengths of these encodings. Writing \( \varphi(\bar{x}) \) for a formula \( \varphi \) means that all free variables of \( \varphi \) are among \( \bar{x} \). A sentence is a formula without free variables.

A \( \Delta_0 \)-formula is an \( L_{\text{ar}} \)-formula obtained from atomic formulas, Boolean connectives, and bounded quantifiers \( \exists x<p, \forall x<p \) where \( p \) is an \( L_{\text{ar}} \)-term not involving \( x \); e.g., \( \exists x<p \varphi \) stands for \( \exists x(x<p \land \varphi) \). \( \Sigma_1 \)- and \( \Pi_1 \)-formulas are obtained from \( \Delta_0 \)-formulas by existential and universal quantification, respectively.

**Theorem 2.3** (MRDP). For every \( \Delta_0 \)-formula \( \varphi(\bar{x}) \) there are \( L_{\text{ar}} \)-terms \( p(\bar{x}, \bar{y}), q(\bar{x}, \bar{y}) \) such that

\[
\mathbb{N} \models \forall \bar{x}(\varphi(\bar{x}) \iff \exists \bar{y} p(\bar{x}, \bar{y})=q(\bar{x}, \bar{y})).
\]

Gödel showed that computable functions are \( \Sigma_1 \)-definable. The MRDP theorem improves this to an existential definition:

**Corollary 2.4.** For every computable \( f : \mathbb{N} \rightarrow \mathbb{N} \) there is a quantifier-free \( L_{\text{ar}} \)-formula \( \varphi_f(x, y, \bar{z}) \) such that for every \( n, m \in \mathbb{N} \)

\[
f(n) = m \iff \mathbb{N} \models \exists \bar{z} \varphi_f(n, m, \bar{z}).
\]

We are mainly concerned with finite arithmetical structures with universe

\[
[n] := \{0, \ldots, n-1\}
\]

for some \( n \in \mathbb{N} \) with \( n \geq 2 \), and therefore consider the relational version

\[
L^r_{\text{ar}}
\]

of \( L_{\text{ar}} \) where \( +, \times \) are ternary relation symbols. The standard \( L^r_{\text{ar}} \)-structure with universe \( \mathbb{N} \), also denoted \( \mathbb{N} \), interprets \( +, \times \) by the graphs of addition and multiplication, respectively. For \( n \in \mathbb{N} \) with \( n \geq 2 \), the standard \( L^r_{\text{ar}} \)-structure with universe \( [n] \), simply denoted \( n \), is the substructure of \( \mathbb{N} \) with universe \( [n] \), i.e., it interprets the symbols in \( L^r_{\text{ar}} \) by \( +^n := \{(k, \ell, m) \in [n]^3 \mid k + \ell = m\}, \times^n := \{(k, \ell, m) \in [n]^3 \mid k \cdot \ell = m\}, 0^n := 0, 1^n := 1 \) and \( <^n := \{(k, \ell) \in [n]^2 \mid k < \ell\} \).

For every \( L^r_{\text{ar}} \)-formula \( \varphi(\bar{x}) \) and \( \bar{n}, n \in \mathbb{N} \) with \( 1 \leq \bar{n} < n \) we have

\[
\mathbb{N} \models \varphi^{<n}(\bar{n}) \iff n \models \varphi(\bar{n}),
\]

where \( \varphi^{<n} \) is obtained from \( \varphi \) by replacing all quantifiers \( \exists z, \forall z \) by \( \exists z < y, \forall z < y \).

**Remark 2.5.** Corollary 2.4 holds for a quantifier-free \( L^r_{\text{ar}} \)-formula \( \varphi_f(\bar{x}, y, \bar{z}) \). Indeed, it is straightforward to express an \( L_{\text{ar}} \)-term equality by an existential \( L^r_{\text{ar}} \)-formula.

### 2.4. Descriptive complexity

A binary string \( x = x_0 \cdots x_n-1 \in \{0, 1\}^* \) of length \( n \geq 2 \) is often identified with the **string structure** \( S(x) \) defined as the \( L^r_{\text{ar}} \cup \{\text{ONE}\} \)-expansion of the standard \( L^r_{\text{ar}} \)-structure \( n \) that interprets the unary relation symbol \( \text{ONE} \) by

\[
\text{ONE}^x := \{i \in [n] \mid x_i = 1\},
\]

i.e., \( S(x) = ([n], +^n, \times^n, 0^n, 1^n, <^n, \text{ONE}^x) \). We shall work with the following descriptive characterization of (dlogtime uniform) \( \text{AC}^0 \).
Theorem 2.6 ([6]). A problem \( Q \) is in \( \text{AC}^0 \) if and only if there is an \( L_{\text{ar}}^r \cup \{ \text{ONE} \} \)-sentence \( \varphi \) such that for every \( x \in \{0,1\}^* \) with \( |x| \geq 2 \):

\[
x \in Q \iff S(x) \models \varphi.
\]

This result and Proposition 2.2 (iii) imply what is to be our working definition of para-\( \text{AC}^0 \): the parameterized problems that are eventually definable.

Corollary 2.7. Let \( (Q, \kappa) \) be a parameterized problem such that \( Q \) is decidable and \( \kappa \) is computable by an \( \text{AC}^0 \)-circuit family. Then \((Q, \kappa)\) is in para-\( \text{AC}^0 \) if and only if \((Q, \kappa)\) is eventually definable: there are a computable \( h : \mathbb{N} \to \mathbb{N} \) and an \( L_{\text{ar}}^r \cup \{ \text{ONE} \} \)-sentence \( \varphi \) such that for all \( x \in \{0,1\}^* \) with \( |x| \geq h(\kappa(x)) \):

\[
x \in Q \iff S(x) \models \varphi.
\]

In descriptive complexity the role of reductions is played by interpretations. Let \( L, L' \) be languages consisting of relation symbols and constants. Let \( w \in \mathbb{N} \) with \( w \geq 1 \). An interpretation \( I \) of \( L' \) in \( L \) (of width \( w \)) is given by an \( L \)-formula \( \varphi_{\text{uni}}(\overline{x}) \), an \( L \)-formula \( \varphi_R(\overline{x}_0, \ldots, \overline{x}_{r-1}) \) for each \( r \)-ary relation symbol \( R \in L' \), and an \( L \)-formula \( \varphi_c(\overline{x}) \) for every constant \( c \in L' \); here, \( \overline{x}, \overline{x}_i \) are \( w \)-tuples of variables. Given an \( L \)-structure \( A \) define the \( L' \)-structure \( A^I \) as follows. It has universe \( A^I := \{ \overline{a} \in A^w \mid A \models \varphi_{\text{uni}}(\overline{a}) \} \), interprets an \( r \)-ary \( R \in L' \) by \( \{ (\overline{a}_0, \ldots, \overline{a}_{r-1}) \in (A^I)^r \mid A \models \varphi_R(\overline{a}_0, \ldots, \overline{a}_{r-1}) \} \), and a constant \( c \in L' \) by the unique \( \overline{a} \in A^I \) satisfying \( \varphi_c(\overline{x}) \) in \( A \). If this uniqueness is violated or if the universe \( A^I \) is empty, then \( A^I \) is not defined. If \( B \cong A^I \) for some \( I \), we say \( B \) is interpretable in \( A \). The following is standard.

Lemma 2.8. Let \( I \) an interpretation of \( L' \) in \( L \) of width \( w \) and \( I' \) an interpretation of \( L'' \) in \( L' \) of width \( w' \). Further let \( A \) be an \( L \)-structure such that \( A^I \) is defined.

(i) For every \( L' \)-formula \( \varphi(x, y, \ldots) \) there is an \( L \)-formula \( \varphi^I(\overline{x}, \overline{y}, \ldots) \) where \( \overline{x}, \overline{y}, \ldots \) are \( w \)-tuples of variables such that for all \( \overline{a}, \overline{b}, \ldots \in A^I \):

\[
A^I \models \varphi(\overline{a}, \overline{b}, \ldots) \iff A \models \varphi^I(\overline{a}, \overline{b}, \ldots).
\]

(ii) There is an interpretation \( I' \circ I \) of \( L'' \) in \( L \) of width \( w \cdot w' \) such that if \( (A^I)^{I'} \) is defined, then so is \( A^{I' \circ I} \) and

\[
A^{I' \circ I} \cong (A^I)^{I'}.
\]

The following is folklore, and a proof can be found in [32, Appendix].

Lemma 2.9. Let \( d \in \mathbb{N} \).

(i) For every \( n \geq 2 \) the standard \( L_{\text{ar}}^d \)-structure \( n^d \) is interpretable in the standard \( L_{\text{ar}}^1 \)-structure \( n \). In fact, there is an interpretation \( I_d \) of width \( d \) such that \( n^d \cong n^I \) for every \( n \geq 2 \), and the isomorphism maps each \( a < n^d \) to the length \( d \) representation of \( a \) in base \( n \).

(ii) There is an \( L_{\text{ar}}^1 \)-formula \( \text{BIT}(x, y) \) such that for every \( n \geq 2 \) and all \( i, j \in [n] \):

\[
n \models \text{BIT}(i, j) \iff \text{the } j \text{-th bit of } \text{bin}(i) \text{ is } 1.
\]
3. p-HALT and NE versus LINH

In this section we first introduce a workable notion of reduction that preserves para-AC⁰, prove Theorem 1.2, and then consider some generalizations and variants that will be instrumental later in Section 4 for the proof of Theorem 1.4 and its variants.

3.1. Eventually definable reductions. A parameterized reduction from a parameterized problem \((Q, \kappa)\) to another \((Q', \kappa')\) is a reduction \(r : \{0,1\}^* \to \{0,1\}^*\) from \(Q\) to \(Q'\) such that \(\kappa' \circ r \leq f \circ \kappa\) for some computable function \(f : \mathbb{N} \to \mathbb{N}\).

Definition 3.1. Let \(\kappa\) be a parameterization. A function \(r : \{0,1\}^* \to \{0,1\}^*\) is \(\kappa\)-eventually definable if there are a computable \(h : \mathbb{N} \to \mathbb{N}\) and an interpretation \(I\) such that

\[ S(x)^I \cong S(r(x)) \]

for all \(x \in \{0,1\}^*\) with \(|x| \geq h(\kappa(x))\).

Example 3.2. The function

\[ (1^n, x) \mapsto 1^{num}(\langle bin(n), x \rangle) \]

where \(n \in \mathbb{N}, x \in \{0,1\}^*\) is \(\kappa\)-eventually definable where \(\kappa\) maps \((1^n, x)\) to \(|x|\) (both functions map arguments that are not of the required form to, say, 0).

Proof: Note \(num(\langle bin(n), x \rangle) < 2^{\langle bin(n), x \rangle} + 1 \leq 2^{O(n + |x|)}\). Choose a constant \(d \in \mathbb{N}\) and a computable \(h : \mathbb{N} \to \mathbb{N}\) such that \(num(\langle bin(n), x \rangle) < n^d\) and \(num(x) < n\) whenever \(n \geq h(|x|)\). We describe an interpretation of \(S(1^{num(\langle bin(n), x \rangle)})\) in \(S((1^n, x))\) whenever \(n \geq h(|x|)\). It will be clear that the interpretation does not depend on \(n, x\).

Let \((n, num(x))\) be the expansion of the standard \(L^r_{ar}\)-structure \(n\) that interprets a new constant by \(num(x) \notin [n]\). This is interpretable in \(S((1^n, x))\) using \(BIT\). By Lemma 2.9, also \((n^d, num(x))\) is interpretable in \(S((1^n, x))\). But this structure defines \((n)\) and \(num(\langle bin(n), x \rangle) \in [n^d]\) using \(BIT\). Thus, \(S(1^{num(\langle bin(n), x \rangle)})\) is interpretable in \(S((1^n, x))\) as claimed.

Recall, a function \(r : \{0,1\}^* \to \{0,1\}^*\) is honest if \(|r(x)| \geq |x|^\Omega(1)|.

Lemma 3.3. Assume that \(r, r' : \{0,1\}^* \to \{0,1\}^*\) are \(\kappa\)- and \(\kappa'\)-eventually definable, respectively, that \(\kappa' \circ r \leq f \circ \kappa\) for some computable \(f : \mathbb{N} \to \mathbb{N}\), and that \(r\) is honest. Then \(r' \circ r\) is \(\kappa\)-eventually definable.

Proof: Choose \(I, h\) for \(r\) and \(I', h'\) for \(r'\) according to Definition 3.1. We can assume that \(h'\) is nondecreasing. Choose \(n_0, c \in \mathbb{N}\) such that \(|r(x)| \geq |x|^{1/c}\) for all \(x \in \{0,1\}^*\) with \(|x| \geq n_0\).

Define \(g : \mathbb{N} \to \mathbb{N}\) by

\[ g(k) := \max\{h'(f(k))^c, h(k), n_0\}. \]

We claim that \(I' \circ I\) and \(g\) witness that \(r' \circ r\) is \(\kappa\)-eventually definable. To verify this let \(x \in \{0,1\}^*\) satisfy \(|x| \geq g(k)\) where \(k := \kappa(x)\). Then \(|r(x)| \geq |x|^{1/c} \geq h'(f(k)) \geq h'(\kappa'(r(x)))\) using that \(h'\) is nondecreasing. Hence \(S(r(x))^{I'} \cong S(r'(r(x)))\). Then \(S(x)^{I' \circ I} \cong S(r'(r(x)))\) because \(S(x)^I \cong S(r(x))\), because \(|x| \geq h(k)\). □
Definition 3.4. Let \((Q, \kappa)\) and \((Q', \kappa')\) be parameterized problems. An eventually definable reduction from \((Q, \kappa)\) to \((Q', \kappa')\) is a parameterized reduction from \((Q, \kappa)\) to \((Q', \kappa')\) that is honest and \(\kappa\)-eventually definable.

Remark 3.5. A parameterized problem is in \(\text{para-AC}^0\) if and only if there is an eventually definable reduction from \((Q, \kappa)\) to a trivial problem, say, \((Q_0, \kappa_0)\) for \(Q_0\) the set of strings starting with 0 and \(\kappa_0\) is constantly 0.

It is straightforward to check that this reducibility is transitive and preserves membership in \(\text{para-AC}^0\):

Lemma 3.6. Assume there is an eventually definable reduction from \((Q, \kappa)\) to \((Q', \kappa')\).

(i) If there is an eventually definable reduction from \((Q', \kappa')\) to \((Q'', \kappa'')\), then there is one from \((Q, \kappa)\) to \((Q'', \kappa'')\).

(ii) If \((Q', \kappa')\) \(\in\) \(\text{para-AC}^0\), then \((Q, \kappa)\) \(\in\) \(\text{para-AC}^0\).

Proof: (i) follows from Lemma 3.3. (ii) follows from (i) and Remark 3.5. \(\square\)

3.2. The complexity of \(p\)-HALT\(_=\). It is known that the question whether \(p\)-HALT\(_=\) is fixed-parameter tractable is closely related to the relationship of \(E\) and \(\text{NE}\):

Theorem 3.7 ([4, 7]). \(p\)-HALT\(_=\) \(\in\) \(\text{FPT}\) if and only if \(\text{NE} \subseteq E\).

Theorem 1.2 (i) is a \(\text{para-AC}^0\)-analogue of this result.

Proof of Theorem 1.2: (ii) follows easily from the equivalence that a nondeterministic Turing machine \(M\) accepts the empty input in at most \(n\) steps if and only if \(M\) accepts the empty input in exactly \(n'\) steps for some \(n' \leq n\).

To prove (i), first assume \(\text{NE} \subseteq \text{LINH}\) and let \(Q\) be the classical problem underlying \(p\)-HALT\(_=\) but with input \(n\) encoded in binary:

| \(Q\) | Instance: \(n \in \mathbb{N}\) in binary and a nondeterministic Turing machine \(M\). |
| Problem: Does \(M\) accept the empty input in exactly \(n\) steps? |

Clearly, \(Q \in \text{NE}\), so by assumption and Proposition 2.1 (iii) we have \(un(Q) \in \text{AC}^0\). Recall

\[
\text{un}(Q) = \left\{ 1^{\text{num}(\text{bin}(n), M)} \mid \text{the nondeterministic Turing machine } M \text{ accepts the empty input in exactly } n \text{ steps} \right\}.
\]

By Example 3.2 the map \(\langle n^* \rangle, M \mapsto 1^{\text{num}(\text{bin}(n), M)}\) is eventually definable with respect to the parameterization of \(p\)-HALT\(_=\). It is a honest parameterized reduction to \((\text{un}(Q), \kappa)\) where \(\kappa\) maps \(1^{\text{num}(\text{bin}(n), M)}\) to \(|M|\). Since \((\text{un}(Q), \kappa) \in \text{para-AC}^0\), Lemma 3.6 implies \(p\)-HALT\(_=\) \(\in\) \(\text{para-AC}^0\).

Conversely, assume \(p\)-HALT\(_=\) \(\in\) \(\text{para-AC}^0\). Let \(Q \subseteq \{0, 1\}^*\) be a problem in \(\text{NE}\). To show that \(Q \in \text{LINH}\), it suffices to prove \(\text{un}(Q) \in \text{AC}^0\) again by Proposition 2.1 (iii).

As \(Q \in \text{NE}\) there is a nondeterministic Turing machine \(M\) and a constant \(c \in \mathbb{N}\) such that \(M\) accepts \(Q\) in time at most \(\text{num}(x)^c - 2|x| - 2\). Consider the nondeterministic Turing machine \(M^*\) that started with the empty input runs as follows:
1. guess $y \in \{0, 1\}^*$
2. simulate $M$ on $y$
3. if $M$ rejects, then reject
4. make dummy steps such that so far the total running time is $num(y)^c$
5. accept.

Line 1 takes exactly $2|y| + 2$ many steps by moving the head forth and back on some tape, so the dummy steps in line 4 are possible. Since $num$ is injective, we have

$$x \in Q \iff M^* \text{ accepts the empty input tape in exactly } num(x)^c + 1 \text{ many steps.} \quad (2)$$

Since $M^*$ is a fixed machine, $p$-$\text{HALT}_=$ $\in$ para-$\text{AC}^0$ implies that the classical problem

$$Q' := \left\{ 1^n \mid M^* \text{ accepts the empty input tape in exactly } n + 1 \text{ many steps} \right\}$$

is in $\text{AC}^0$. Choose a first-order sentence $\phi$ for $Q'$ according to Theorem 2.6. Lemma 2.9 gives an interpretation $I$ such that $S(1^n)^I \cong S(1^n^c)$ for all $n \geq 2$. Then $1^{n^c} \in Q'$ is equivalent to $S(1^n) \models \phi^I$. Thus the r.h.s. in (2) is equivalent to $S(1^{num(x)}) \models \phi^I$ provided $num(x) \geq 2$, i.e., $x$ is non-empty. The l.h.s. in (2) is equivalent to $1^{num(x)} \in un(Q)$. Thus $\phi^I$ witnesses that $un(Q) \in \text{AC}^0$ according to Theorem 2.6.

**Remark 3.8.** The direction from left to right only required a $\text{AC}^0$-circuit family for instances of $p$-$\text{HALT}_=$ with the fixed machine $M^*$. This implies that the assertions in Theorem 1.2 (i) are equivalent to $p$-$\text{HALT}_= \in X\text{AC}^0$ (see Definition 7.1).

### 3.3. Almost tally problems.

Recall that a classical problem $Q \subseteq \{0, 1\}^*$ is tally if $Q \subseteq \{1\}^*$. All parameterized problems mentioned in the introduction are almost tally in the following sense:

**Definition 3.9.** A parameterized problem $(Q, \kappa)$ is almost tally if

$$Q \subseteq \{ 1^n, x \mid n \in \mathbb{N}, x \in \{0, 1\}^* \}$$

and there is a computable $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}, x \in \{0, 1\}^*$

$$|x| \leq f(\kappa(1^n, x)).$$

Theorem 1.2 (ii) holds not only for $p$-$\text{HALT}$ but for every almost tally problem in para-$\text{NP}$. In fact, $p$-$\text{HALT}_=$ is the hardest almost tally problem in para-$\text{NP}:

**Lemma 3.10.** For every almost tally problem in para-$\text{NP}$ there is an eventually definable reduction to $p$-$\text{HALT}_-$.
Proof: Let \((Q, \kappa) \in \text{para-NP}\) be almost tally. The identity is a parameterized reduction from \((Q, \kappa)\) to its re-parameterization \((Q, \kappa')\) where \(\kappa'(\langle 1^n, x \rangle) := |x|\) for all \(n \in \mathbb{N}, x \in \{0, 1\}^*\). We can therefore assume that \(\kappa = \kappa'\).

Let \(M\) be a nondeterministic Turing machine that accepts \(Q\) and on input \(\langle 1^n, x \rangle\) runs in time at most \(f(k) \cdot n^c\) where \(c \in \mathbb{N}\), \(f : \mathbb{N} \to \mathbb{N}\) is a computable function, and \(k := |x|\).

Define \(g : \mathbb{N}^2 \to \mathbb{N}\) by
\[
g(m, k) := m^{c+1} + 2m + 2k + 2.
\]

For \(x \in \{0, 1\}^*\) with \(k := |x|\), consider the nondeterministic Turing machine \(M_x\) that on the empty input runs as follows:

1. nondeterministically write \(\langle 1^m, x \rangle\) for some \(m \in \mathbb{N}\)
2. simulate \(M\) on \(\langle 1^m, x \rangle\)
3. if \(M\) does not halt or rejects, then reject
4. make dummy steps such that so far the total running time is \(g(m, k)\)
5. accept.

Step 1 can be implemented to take exactly \(2 + 2m + 2 + 2k\) many steps (recall (1)), so the dummy steps in line 4 are possible if \(m > f(k)\). Note that for each \(k\), the function \(m \mapsto g(m, k)\) is injective. Thus, if \(n > f(k)\), we have
\[
\langle 1^n, x \rangle \in Q \iff \langle 1^{g(n,k)+1}, M_x \rangle \in \text{p-HALT}_x.
\]

Choose a parameterized honest reduction from \((Q, \kappa)\) to \(\text{p-HALT}_x\) that maps \(\langle 1^n, x \rangle\) with \(n > f(k)\) to \(\langle 1^{g(n,k)+1}, M_x \rangle\). We verify that it is eventually definable.

Choose a computable \(h : \mathbb{N} \to \mathbb{N}\) and \(d \in \mathbb{N}\) such that \(n \geq h(k)\) implies \(n > f(k)\) and \(n^d > g(n, k) + 1, \text{num}(x), \text{num}(M_x)\) for all \(x \in \{0, 1\}^*\) of length \(|x| = k\).

Let \(\langle 1^n, x \rangle\) satisfy \(n > h(k)\) where \(k := |x|\). By Lemma 2.9, \(S(\langle 1^n, x \rangle)\) interprets the expansion \((n^d, k, \text{num}(x))\) of the standard \(L^{\text{sat}}\)-structure \(n^d\) by two constants \(c_0\) and \(c_1\) denoting \(k\) and \(\text{num}(x)\). We are left to show that \((n^d, k, \text{num}(x))\) interprets \(S(\langle 1^{g(n,k)+1}, M_x \rangle)\). Using \(\text{BIT}\), it suffices to show that both \(g(n, k) + 1\) and \(\text{num}(M_x)\) are definable in \((n^d, k, \text{num}(x))\).

The former being clear, we show the latter. Consider a computable function \(M : \mathbb{N} \to \mathbb{N}\) that maps \(\text{num}(x)\) to \(\text{num}(M_x)\). Choose a quantifier-free \(L_{\text{sat}}\)-formula \(\varphi_M(x, y, \bar{z})\) according to Remark 2.5. We can assume that \(h\) grows fast enough so that for all \(\ell \in \mathbb{N}\)
\[
\mathbb{N} \models \exists \bar{z}<h(\ell) \varphi_M(\ell, M(\ell), \bar{z}).
\]

Then \(\exists \bar{z} \varphi_M(c_1, y, \bar{z})\) defines \(\text{num}(M_x)\) in the structure \((n^d, k, \text{num}(x))\).

It is straightforward to infer from Proposition 2.1 that \(\text{NE} \subseteq \text{LINH}\) if and only if every tally problem in \(\text{NP}\) is in \(\text{AC}^0\). We don’t know of a similarly easy proof of the following parameterized variant of this observation. Instead, our proof relies on our analysis of \(\text{p-HALT}_x\):
Corollary 3.11. NE \subseteq LINH if and only if every almost tally problem in para-NP is in para-AC^0.

Proof: The l.h.s. is equivalent to \( p\text{-HALT}_\leq \subseteq \text{para-AC}^0 \) by Theorem 1.2 (i). And by Lemmas 3.10 and 3.6, \( p\text{-HALT}_\leq \subseteq \text{para-AC}^0 \) is equivalent to the r.h.s.. \( \square \)

3.4. Variants. For the optimistic reader, Corollary 3.11 is an approach to separate NE from LINH. From this perspective, it is of interest to ask whether finding an almost tally problem outside para-AC^0 but in a natural subclass of para-NP implies stronger separations of natural complexity classes. We verify the following variants of Corollary 3.11:

Lemma 3.12.

(i) \( E \subseteq LINH \) if and only if every almost tally problem in FPT is in para-AC^0.

(ii) \( \text{NLINSPACE} \subseteq LINH \) if and only if every almost tally problem in para-NL is in para-AC^0.

(iii) \( \text{LINSPACE} \subseteq LINH \) if and only if every almost tally problem in para-L is in para-AC^0.

Proof: The proof of (i) is analogous to the proof of Corollary 3.11 using the subproblem of \( p\text{-HALT}_\leq \) where the input machine \( M \) is deterministic. Similarly the proof of (iii) is analogous to the proof of (ii). We show how (ii) is proved by modifying the proof of Corollary 3.11.

Consider the following variant of \( p\text{-HALT}_\leq \):

\[
p\text{-HALT}_*\leq
\]

**Instance:** \( n, m \in \mathbb{N} \) in unary with \( n \leq m \) and a nondeterministic Turing machine \( M \).

**Parameter:** \( |M| \).

**Problem:** Does \( M \) accept the empty input in exactly \( n \) steps and space at most \( \lceil \log m \rceil \)?

It is clear that this problem is in para-NL.

Claim 1. \( p\text{-HALT}_*\leq \subseteq \text{para-AC}^0 \) if and only if \( \text{NLINSPACE} \subseteq LINH \).

Proof of the Claim 1. Assume \( \text{NLINSPACE} \subseteq LINH \) and let \( Q \) be the classical problem underlying \( p\text{-HALT}_*\leq \) but with the inputs \( n, m \) encoded in binary. Clearly, \( Q \in \text{NLINSPACE} \subseteq LINH \), so \( \text{un} (Q) \in \text{AC}^0 \) by Proposition 2.1 (iii). Similarly as Example 3.2 one sees that \( \langle 1^n, 1^m, M \rangle \mapsto 1\text{num}(\langle n, m, M \rangle) \) is eventually definable. Then \( p\text{-HALT}_*\leq \subseteq \text{para-AC}^0 \) follows as before.

Conversely, assume \( p\text{-HALT}_*\leq \subseteq \text{para-AC}^0 \) and let \( Q \in \text{NLINSPACE} \). Choose a nondeterministic Turing machine \( M \) accepting \( Q \) that on input \( x \in \{0, 1\}^* \) runs in time at most

\[
\frac{\text{num}(x)c}{10c(|x| + 2)} - 10c(|x| + 2) - |x|
\]

and uses space at most \( c \cdot |x| \); here \( c \in \mathbb{N} \) is a suitable constant. Define \( M^* \) as in page 10 but with the following implementation details. For the simulation in line 2, first initialize a length \( c(|y| + 2) \) binary counter using exactly \( 10c(|y| + 2) \) steps, and increase it using exactly \( 10c(|y| + 2) \).
many steps for each simulated step of $\mathbb{M}$. In line 4 continue increasing the counter in this way until it reaches $num(y)^c/(10c(|y| + 2))$. For long enough $y$, the binary representation of this number can be computed in time at most $num(y)$ and space $O(|y|)$ (where the constant in the O-notation depends on $c$). This computation can be done in parallel to the simulation in lines 2 and 4. Hence, line 5 completes exactly $num(y)^c + 1$ steps, and uses space at most $d \cdot |y|$ for a suitable $d \geq c$.

Thus, we arrive at the following variant of (2). For long enough $x \in \{0, 1\}^*$:

\[
x \in Q \iff \mathbb{M}^* \text{ accepts the empty input in exactly } num(x)^c + 1 \text{ many steps and space at most } \lceil \log(num(y)^d) \rceil.
\]

Our assumption $p$-HALT$_\ast \subseteq \text{para-AC}^0$ implies that the classical problem

\[
Q' := \{ \langle 1^n, 1^m \rangle \mid n \leq m \text{ and } \mathbb{M}^* \text{ accepts the empty input in exactly } n + 1 \text{ many steps and space at most } \lceil \log m \rceil \}
\]

is in $\text{AC}^0$. Now $un(Q) \in \text{AC}^0$ (and hence $Q \in \text{LINH}$) follows as before using an interpretation $I$ such that $S(\langle 1^n, 1^m \rangle)^I \cong S(\langle 1^n, 1^m \rangle)$. ⊥

Claim 2. For every almost tally problem in $\text{para-NL}$ there is an eventually definable reduction to $p$-HALT$_\ast$.

Proof of the Claim 2. Let $(Q, \kappa) \in \text{para-NL}$ be almost tally and $\mathbb{M}$ be a nondeterministic Turing machine that accepts $Q$ and that on input $\langle 1^n, x \rangle$ runs in time at most $f(k) \cdot n^c$ and space at most $f(k) + c \cdot \log n$ where $c \in \mathbb{N}$, $f : \mathbb{N} \to \mathbb{N}$ is a computable function, and $k := \kappa(\langle 1^n, x \rangle) = |x|$.

For $x \in \{0, 1\}^*$ with $k := |x|$, define the nondeterministic Turing machine $\mathbb{M}_x$ as in page 12 but with a different $g$ (chosen below) and line 1 changed to nondeterministically write some $m \in \mathbb{N}$ in binary in exactly $2 \lceil \log (m+1) \rceil + 2$ steps. The simulation in line 2 is done as in the previous claim maintaining a length $(c + 1) \lceil \log (m+1) \rceil$ binary counter. It further maintains the position of $\mathbb{M}$’s head on the input tape (which we can assume to be at most $|\langle 1^m, x \rangle| + 1$) and uses it to compute the currently scanned bit. If $k < m$, both the maintenance of the counter and position can be done in exactly $10c \lceil \log (m+1) \rceil$ steps. In line 4 the binary counter is updated until it reaches $m^{c+1}$. Hence line 4 is completed after exactly $g(m, k) := m^{c+1} \cdot 10c \lceil \log (m+1) \rceil + 2 \lceil \log (m+1) \rceil + 2$ steps. The dummy steps in line 4 are possible if $m > f(k)$. In this case the computation takes space at most $d \log m$ for suitable $d \in \mathbb{N}$. Thus, if $n > f(k)$, we have

\[
\langle 1^n, x \rangle \in Q \iff \langle 1^{g(n,k)+1}, 1^{nd}, \mathbb{M}_x \rangle \in p$-HALT$_\ast$.
\]

Similarly as seen in the proof of Lemma 3.10, this implies the claim. ⊥

It now suffices to show that $p$-HALT$_\ast \subseteq \text{para-AC}^0$ if and only if every almost tally problem in $\text{para-NL}$ is in $\text{para-AC}^0$. The forward direction follows from Claim 2 and Lemma 3.6. And if $p$-HALT$_\ast \not\subseteq \text{para-AC}^0$, then we get an almost tally problem in $\text{para-NL} \setminus \text{para-AC}^0$ by rewriting inputs $\langle 1^n, 1^m, \mathbb{M} \rangle$ of $p$-HALT$_\ast$ to $\langle 1^{m,n}, \mathbb{M} \rangle$ where $\langle n, m \rangle$ is a pairing function on $\mathbb{N}$. □

We find it worthwhile to explicitly point out the following direct corollary concerning the parameterized halting problem for deterministic Turing machines:
Corollary 3.13. If $p$-DHALT $\not\in$ para-AC$^0$, then $E \not\subseteq$ LINH.

| $p$-DHALT |
|------------|
| **Instance:** $n \in \mathbb{N}$ in unary and a deterministic Turing machine $\mathbb{M}$. |
| **Parameter:** $|\mathbb{M}|$. |
| **Problem:** Does $\mathbb{M}$ accept the empty input in at most $n$ steps? |

4. On the parameterized complexity of $p$-$\Delta_0$-TRUTH

This section first observes that $p$-$\Delta_0$-TRUTH is “the same” as a basic parameterized model-checking problem, uses this to prove the lower bound $p$-$\Delta_0$-TRUTH $\not\in$ para-AC$^0$ (Theorem 1.5), and finally, based on the previous section, infers consequences from upper bounds on the parameterized complexity of $p$-$\Delta_0$-TRUTH, including Theorem 1.4.

4.1. Model-checking arithmetic. We observe that $p$-$\Delta_0$-TRUTH is “the same” as the parameterized model-checking problem for first-order logic over finite standard $L_{ar}$-structures:

| $p$-MC($L_{ar}^r$) |
|---------------------|
| **Instance:** $n \geq 2$ in unary and an $L_{ar}^r$-sentence $\varphi$. |
| **Parameter:** $|\varphi|$. |
| **Problem:** $n \models \varphi$? |

Lemma 4.1. There is a computable function that maps every $\Delta_0$-formula $\varphi(x)$ to an $L_{ar}^r$-sentence $\psi$ such that for all $n \in \mathbb{N}$ with $n \geq 2$:

$$\mathbb{N} \models \varphi(n) \iff n \models \psi.$$ (3)

Further, there is a computable function that maps every $L_{ar}^r$-sentence $\psi$ to a $\Delta_0$-formula $\varphi(x)$ such that (3) holds all $n \in \mathbb{N}$ with $n \geq 2$.

Proof: For the second assertion define $\varphi(x)$ as $\psi^{<x}$ with atoms rewritten in the functional language $L_{ar}$. The first assertion is folklore, see [20, Proposition 2.2]. We give a brief sketch for completeness. It is routine to compute, given a $\Delta_0$-formula $\varphi(\bar{x})$, a constant $c_\varphi > 1$ and an $L_{ar}^r$-formula $\psi_0(\bar{x})$ such that

$$\mathbb{N} \models \varphi(\bar{n}) \iff \mathbb{N} \models \psi_0^{<m}(\bar{n})$$

for all $\bar{n}, m \in \mathbb{N}$ with $m \geq \max\{\bar{n}, 2\}^{c_\varphi}$. Hence, for $n \geq 2$, the truth of $\varphi(n)$ is equivalent to $n^{c_\varphi} \models \psi_0(n)$. Since the number $n$ is definable in the standard $L_{ar}^r$-structure $n^{c_\varphi}$ (as the minimal element whose $c_\varphi$-th power does not exist), we can replace $\psi_0(n)$ by some sentence $\psi_1$. Then set $\psi := \psi_1^{<n}$ for the interpretation $I_{c_\varphi}$ from Lemma 2.9. \hfill \Box

4.2. A lower bound. In this subsection we prove Theorem 1.5. We fix a proper elementary extension $M$ of the standard $L_{ar}^r$-model $\mathbb{N}$, and a nonstandard element $a \in M \setminus \mathbb{N}$. We let $<^M$ denote the interpretation of $<$ in $M$. We need a simple lemma:
Lemma 4.2. Let \( f : \mathbb{N} \to \mathbb{N} \) be a computable function. Then there is an \( L^c_{ar} \)-formula \( \chi_f(x, y) \) such that for every \( k \in \mathbb{N} \) and every \( b <^M a \):

\[
f(k) = b \iff M \models \chi_f^a(k, b).
\]

Proof: Let \( \exists \varphi(x, y, z) \) define \( f \in \mathbb{N} \) where \( \varphi \) is \( \Delta_0 \). As in the proof of Lemma 4.1 let \( \psi(x, y, z) \) be an \( L^c_{ar} \)-formula such that for all \( k, \ell, m, n \in \mathbb{N} \) with \( k, \ell, m < n \):

\[
\mathbb{N} \models \varphi(k, \ell, m) \iff \mathbb{N} \models \psi^n(k, \ell, m)
\]  

(4)

Here, \( \mathbb{N} \) ambiguously denotes the standard model in the respective languages. Set

\[
\chi_f(x, y) := \exists z \psi(x, y, z).
\]

If \( f(k) = b \), then \( b \in \mathbb{N} \) and there is \( m \in \mathbb{N} \) such that \( \mathbb{N} \models \psi^n(k, b, m) \) for all \( n > k, b, m \), so \( M \models \chi_f^a(k, b) \) as \( k, b, m <^M a \), and \( M \models \chi_f^a(k, b) \). Conversely, assume \( M \models \chi_f^a(k, b) \) for \( b <^M a \) and \( f(k) \neq b \); then \( \exists u, y, z(k, y, z < u \land \psi^n(k, y, z) \land y \neq f(k)) \) holds in \( M \) and hence in \( \mathbb{N} \); by (4), \( \mathbb{N} \models \varphi(k, \ell, m) \) for some \( \ell, m \in \mathbb{N} \) with \( \ell \neq f(k) \); this contradicts the fact that \( \exists z \varphi(x, y, z) \) defines \( f \) in \( \mathbb{N} \).

Some notation: for \( n \in \mathbb{N} \) define the \( L^c_{ar} \)-formula “\( x = n \)” by “\( x = 0 \)” := \( x = 0 \) and “\( x = (n + 1) \)” := \( \exists y(“y = n” \land +)(y, 1, x) \). For an \( L^c_{ar} \)-formula \( \varphi(y, \bar{x}) \) set \( \varphi(n, \bar{x}) := \exists y(“y = n” \land \varphi(y, \bar{x})); \) we understand \( \varphi^{<n}(n, \bar{x}) \) as \( \varphi(n, \bar{x})^{<n} \). If \( n < m \), then both \( (“x = n”^{<m} \land “x = n” \land \varphi(n, \bar{x})^{<m} \) and \( “x = n” \) define \( n \) in \( \mathbb{N} \), so \( \varphi^{<m}(n, \bar{x}) \) and \( \varphi^{<m}(n, \bar{x}) \) are equivalent in \( \mathbb{N} \). In particular, for every \( n \in \mathbb{N} \):

\[
M \models \varphi^{<a}(n, \bar{x}) \iff \varphi^{<a}(n, \bar{x})
\]

(5)

Proof of Theorem 1.5: For contradiction, assume otherwise, so \( p \)-MC(\( L^c_{ar} \)) \( \in \) para-AC\( ^0 \) by Lemma 4.1. By Corollary 2.7, there is an increasing computable function \( h : \mathbb{N} \to \mathbb{N} \) and a sentence \( sat \) such that for every \( n \in \mathbb{N} \) and every \( L^c_{ar} \)-sentence \( \varphi \) with \( n > h(num(\varphi)) \) we have

\[
n \models \varphi \iff S(\langle 1^n, \varphi \rangle) \models sat.
\]

(6)

For \( k < n \), let \( (n, k) \) denote the expansion of the standard \( L^c_{ar} \)-structure \( n \) that interprets a constant \( c \) by \( k \). It is clear that there is an interpretation \( I \) (independent of \( n, \varphi \)) such that \( (n, num(\varphi))I \cong S(\langle 1^n, \varphi \rangle) \) for all \( \varphi \) with \( num(\varphi) < n \). Replacing in \( satI \) the constant \( c \) by a new variable \( x \) gives an \( L^c_{ar} \)-formula \( true(x) \) such that for \( n > h(num(\varphi)) \)

\[
S(\langle 1^n, \varphi \rangle) \models sat \iff n \models true(num(\varphi)) \iff \mathbb{N} \models true^{<n}(num(\varphi)),
\]

(7)

where \( \mathbb{N} \) is the standard \( L^c_{ar} \)-model. Since \( h : \mathbb{N} \to \mathbb{N} \) is computable, there is an \( L^c_{ar} \)-formula “\( h(x) < y \)” with the obvious meaning. Note the l.h.s. of (6) is equivalent to \( \mathbb{N} \models \varphi^{<n} \). Combining (6) and (7) we get

\[
\mathbb{N} \models “h(num(\varphi)) < y” \to (\varphi^{<y} \iff true^{<y}(num(\varphi)))
\]

16
for every $L_{ar}$-sentence $\varphi$. But $M \models \text{“} h(\text{num}(\varphi)) < a \text{”}$, hence
\[
M \models \varphi^{<a} \leftrightarrow \text{true}^{<a}(\text{num}(\varphi)) \tag{8}
\]
for every $L_{ar}$-sentence $\varphi$. As stated in [31, proof of Proposition 3] this contradicts Tarski’s undefinability of truth. We include the details as they are omitted in [31].

The function which for every $L_{ar}$-formula $\varphi(x)$ maps $\text{num}(\varphi)$ to $\text{num}(\varphi(\text{num}(\varphi)))$ is computable. So by Lemma 4.2, there is a formula $\text{sub}(x,y)$ such that for every formula $\varphi(x)$ and every $b \in M$ with $b <^M a$:
\[
b = \text{num}(\varphi(\text{num}(\varphi))) \iff M \models \text{sub}^{<a}(\text{num}(\varphi), b) \tag{9}
\]
Define $\chi(x) := \forall y (\text{sub}(x, y) \rightarrow \neg \text{true}(y))$ and $\theta := \chi(\text{num}(\chi))$, and note
\[
\text{num}(\theta) = \text{num}(\chi(\text{num}(\chi))). \tag{10}
\]
We arrive at the desired contradiction:
\[
M \models \theta^{<a} \iff M \models \forall y < a (\text{sub}^{<a}(\text{num}(\chi), y) \rightarrow \neg \text{true}^{<a}(y)) \text{ by (5)}
\]
\[
\iff M \models \text{sub}^{<a}(\text{num}(\chi), b) \rightarrow \neg \text{true}^{<a}(b) \text{ for all } b <^M a \text{ by (9) and (10)}
\]
\[
\iff M \models \neg \text{true}^{<a}(\text{num}(\theta)) \text{ by (8)}
\]
\[
\iff M \not\models \theta^{<a} \text{ by (8)}. \quad \square
\]

4.3. Upper bounds. Based on our analysis of halting problems in Section 3, we now see that various upper bounds on the complexity of $p$-$\Delta_0$-TRUE imply separations of classical complexity classes from LINH. This is our main result. The first assertion is Theorem 1.4:

Theorem 4.3.

(i) If $p$-$\Delta_0$-TRUE $\in \text{para-NP}$, then $\text{NE} \not\subseteq \text{LINH}$.

(ii) If $p$-$\Delta_0$-TRUE $\in \text{FPT}$, then $\text{E} \not\subseteq \text{LINH}$.

(iii) If $p$-$\Delta_0$-TRUE $\in \text{para-NL}$, then $\text{NLINSPACE} \not\subseteq \text{LINH}$.

(iv) If $p$-$\Delta_0$-TRUE $\in \text{para-L}$, then $\text{LINSPACE} \not\subseteq \text{LINH}$.

Proof: Since $p$-$\Delta_0$-TRUE is an almost tally problem, (i) follows from Theorem 1.5 and Corollary 3.11. The other assertions follow using Lemma 3.12. \quad \square

5. Provability of the MRDP theorem

In this section we prove Theorem 1.3 as a corollary to Theorem 1.4 via Parikh’s theorem [30]:

Theorem 5.1. Let $T$ be a $\Pi_1$-theory and $\varphi(\bar{x}, \bar{y})$ a $\Delta_0$-formula. If $T$ proves $\exists \bar{y} \varphi(\bar{x}, \bar{y})$, then $T$ proves $\exists \bar{y} < p(\bar{x}) \varphi(\bar{x}, \bar{y})$ for some term $p(\bar{x})$. 

17
Here, a \textit{theory} is a set of sentences, and a \textit{\Pi_1-theory} is a set of \Pi_1-sentences. For example, \(I\Delta_0\) is (equivalent to) a \Pi_1-theory.

\textbf{Definition 5.2.} A theory \(T\) \textit{proves MRDP} if for every \(\Delta_0\)-formula \(\varphi(\bar{x})\) there are \(L_{ar}\)-terms \(p(\bar{x}, \bar{y})\) and \(q(\bar{x}, \bar{y})\) such that \(T\) proves
\[
\varphi(\bar{x}) \iff \exists \bar{y} \ p(\bar{x}, \bar{y})=q(\bar{x}, \bar{y}).
\]

As mentioned in the introduction it is a long standing open problem whether \(I\Delta_0\) proves MRDP but it is known that adding exponentiation suffices. Intuitively, the following concept asks whether MRDP can be proved using exponentiation only once.

\textbf{Definition 5.3.} A theory \(T\) \textit{proves MRDP for small numbers} if for every \(k \in \mathbb{N}\) and every \(\Delta_0\)-formula \(\varphi(\bar{x}) = \varphi(x_0, \ldots, x_{k-1})\) there are \(L_{ar}\)-terms \(p(\bar{x}, \bar{y})\) and \(q(\bar{x}, \bar{y})\) such that \(T\) proves
\[
\bigwedge_{i < k} 2^{x_i} \leq z \rightarrow \left( \varphi(\bar{x}) \iff \exists \bar{y} \ p(\bar{x}, \bar{y})=q(\bar{x}, \bar{y}) \right).
\]

Here, \(2^{x_i} \leq y\) stands for a well-known \(\Delta_0\)-formula \([22, \text{Section V.3.(c)}]\). The following strengthens Theorem 1.3:

\textbf{Theorem 5.4.} Let \(T\) be a true \(\Pi_1\)-theory. Moreover, assume that \(T\) is computably enumerable. If \(T\) proves MRDP for small numbers, then \(p-\Delta_0\text{-}TRUTH \in \text{para-NP}\) and thus \(\text{NE} \not\subseteq \text{LINH}\).

\textit{Proof}: Assume \(T\) proves (11) for \(\varphi(x)\), and hence
\[
2^{x_i} \leq z \land \varphi(x) \rightarrow \exists \bar{y} \ p(x, \bar{y})=q(x, \bar{y}).
\]

By Theorem 5.1 \(\exists \bar{y}\) can be replaced by \(\exists \bar{y} < r(x, z)\) for some term \(r(x, z)\). But since \(T\) proves (11) for \(\varphi(x)\), \(T\) proves
\[
2^{x_i} \leq z \rightarrow \left( \varphi(x) \iff \exists \bar{y} < r(x, z) \ p(x, \bar{y})=q(x, \bar{y}) \right).
\]

Since \(T\) is computably enumerable, such terms \(p, q, r\) can be computed from \(\varphi\). Given an instance \(\langle 1^n, \varphi \rangle\) of \(p-\Delta_0\text{-}TRUTH\), compute \(p, q, r\) as above, guess \(\bar{m} < r(n, 2^n)\) and check \(p(n, \bar{m})=q(n, \bar{m})\). Note the length of the guess \(\bar{m}\) is \(O(|r| \cdot |\bar{y}| \cdot n)\). The check can be done in time \((|p| \cdot |q| \cdot |r| \cdot n)^{O(1)}\). Thus, \(p-\Delta_0\text{-}TRUTH \in \text{para-NP}\). Now apply Theorem 1.4. \(\square\)

It would be interesting to find variants of the this result that infer \(p-\Delta_0\text{-}TRUTH \in \text{FPT}\) or \(p-\Delta_0\text{-}TRUTH \in \text{para-NL}\) from certain provabilities of MRDP or other arithmetical statements. Note this implies stronger separations of complexity classes by Theorem 4.3.

\section{6. \(p\text{-HALT} \text{ and a universal } AC^0\text{-easy set in NP}\)}

In this section we prove Theorem 1.6. We use the following technical lemma stating, roughly, that every computable function is dominated by a computable injection which is \(AC^0\)-invertible.
Lemma 6.1. Let \( f : \mathbb{N} \to \mathbb{N} \) be computable. Then there is an increasing \( h : \mathbb{N} \to \mathbb{N} \) with the following properties.

(i) \( h(n) \geq f(n^2) \) for every \( n \in \mathbb{N} \).

(ii) \( 1^n \mapsto 1^h(n) \) is computable in time \( h(n)^{O(1)} \).

(iii) There is an \( L_{ar}^r \)-sentence \( \varphi_h \) such that for every \( x \in \{0, 1\}^* \) with \( |x| \geq 2 \):

\[
S(x) \models \varphi_h \iff x = 1^h(n) \text{ for some } n \in \mathbb{N}.
\]

(iv) There is an \( L_{ar}^r \)-formula \( \varphi(x) \) that defines \( n \in S(1^h(n)) \) for every \( n \geq 2 \).

Proof: Given a deterministic Turing machine \( M \) and \( x \in \{0, 1\}^* \) we let \( y_{M,x} \in \{0, 1\}^* \) encode the computation of \( M \) on \( x \). This encoding can be chosen so that:

(a) \( x \mapsto y_{M,x} \) is computable in time \( |y_{M,x}|^{O(1)} \).

(b) \( \{ \langle x, y_{M,x} \rangle \mid x \in \{0, 1\}^* \} \in AC^0 \).

Now let \( M_f \) be a Turing machine that computes \( 1^n \mapsto 1^{f(n)} \). Let \( M \) be the machine that on input \( 1^n \) runs \( M_f \) on \( 1^d \) for every \( d \leq n \). Define the increasing function \( h : \mathbb{N} \to \mathbb{N} \) by

\[
h(n) = \text{num}(\langle 1^n, y_{M,1^n} \rangle)
\]

Clearly, the string \( y_{M_f,1^n} \) encoding the computation of \( M_f \) on input \( 1^n \) has length at least \( f(n^2) \). Similarly, \( |y_{M,1^n}| \geq f(n^2) \). Thus \( h(n) \geq f(n^2) \) for every \( n \in \mathbb{N} \), i.e., (i) holds.

(ii) holds by (a). To show (iii), Theorem 2.6 and (b) imply that there is an \( L_{ar}^r \)-sentence \( \varphi \) that holds precisely in the string structures of the form \( S(\text{bin}(h(n))) \) for \( n \in \mathbb{N} \). Using \( \text{BIT} \), there is an interpretation \( I \) such that \( S(1^n)^I = S(\text{bin}(m)) \) for every \( m > 2 \), so \( \varphi_{h} := \varphi^I \) holds precisely in the string structures of the form \( S(1^h(n)) \) for \( n \in \mathbb{N} \) (we have \( h(n) > 2 \) for all \( n \in \mathbb{N} \)).

Trivially, \( n \) is definable in \( S(\text{bin}(h(n))) \), so (iv) follows using the interpretation \( I \) above. \( \square \)

Theorem 1.6 is an easy consequence of the following stronger result, and we view it as good evidence for the truth of Conjecture 1.1.

Theorem 6.2. Assume \( \text{p-HALT} \in \text{para-AC}^0 \). Then there is an infinite tally problem \( X \) such that for every \( Q \in \text{NP} \) we have \( Q \cap X \in \text{AC}^0 \).

Proof of Theorem 1.6 from Theorem 6.2: Assume \( \text{p-HALT} \in \text{para-AC}^0 \) and let \( Q \in \text{NP} \). Let \( X \) be as stated in Theorem 6.2. Then either \( Q \cap X \) or \( (\{0, 1\}^* \setminus Q) \cap X \) is infinite. By Theorem 6.2 they are both in \( \text{AC}^0 \). Hence \( Q \) is not \( \text{AC}^0 \)-bi-immune. \( \square \)

Proof of Theorem 6.2: By Corollary 2.7 there is a computable increasing function \( f : \mathbb{N} \to \mathbb{N} \) and an \( L_{ar}^r \)-sentence \( \varphi \) such that for every \( \langle 1^n, M \rangle \) with \( n \geq f(|M|) \):

\[
S(\langle 1^n, M \rangle) \models \varphi \iff M \text{ accepts the empty input tape in at most } n \text{ steps.} \tag{13}
\]
Now let $h : \mathbb{N} \to \mathbb{N}$ be the increasing function as stated in Lemma 6.1. In particular, there is a deterministic Turing machine $M_h$ and a constant $c \geq 1$ such that on input $1^m$ the machine $M_h$ outputs the string $1^{h(m)}$ in time $h(m)^c$. The desired set $I$ is defined by

$$X := \{1^{h(m)} \mid m \geq 2\}.$$ 

By Lemma 6.1 (iii) the sentence $\varphi_h$ witnesses $X \in \text{AC}^0$ according to Theorem 2.6. 

Now let $Q \subseteq \{0, 1\}^*$ be a problem in NP. In particular, there is a nondeterministic Turing machine $M_Q$ and a constant $d \geq 1$ such that $M_Q$ accepts $x$ in time $|x|^d$.

Define the nondeterministic Turing machine $M_{Q,h,m}$ to run $M_h$ on $1^m$ to produce output $1^{h(m)}$ and then run $M_Q$ on $1^{h(m)}$. This machine runs in time

$$n(m) := h(m)^c + h(m)^d.$$ 

Choose a constant $e \in \mathbb{N}$ such that $m \geq |M_h| + |M_Q| + e$ implies $m^2 \geq |M_{Q,h,m}|$. Then

$$n(m) \geq h(m) \geq f(m^2) \geq f(|M_{Q,h,m}|).$$ 

Hence, by (13), for $m \geq |M_h| + |M_Q| + e$:

$$1^{h(m)} \in Q \iff M_{Q,h,m} \text{ accepts the empty input in at most } n(m) \text{ steps} \iff S(\langle 1^{n(m)}, M_{Q,h,m} \rangle) \models \varphi. \quad (14)$$

Lemma 6.1 (iv) implies that there is an interpretation $I$ such that for every $m \in \mathbb{N}$

$$S(1^{h(m)})^I = S(\langle 1^{n(m)}, M_{Q,h,m} \rangle).$$

By Theorem 2.6 it suffices to show that for every $x \in \{0, 1\}^*$ with $|x| \geq h(|M_h| + |M_Q| + e)$:

$$x \in Q \cap X \iff S(x) \models \varphi_h \land \varphi_f.$$ 

Assume $x \in Q \cap X$. Then $x = 1^{h(m)}$ for some $m \geq 2$ and $S(x) \models \varphi_h$. Since $|x| = h(m) \geq h(|M_h| + |M_Q| + e)$ and $h$ is increasing, we have $m \geq |M_h| + |M_Q| + e$. Thus $x = 1^{h(m)} \in Q$ implies $S(\langle 1^{n(m)}, M_{Q,h,m} \rangle) \models \varphi$ by (14), and $S(1^{h(m)}) \models \varphi^I$ follows.

Conversely, assume $S(x) \models \varphi_h \land \varphi_f$. By $S(x) \models \varphi_h$, we have $x \in X$, so $x = 1^{h(m)}$ for some $m \geq 2$. By $S(1^{h(m)}) \models \varphi_f$ we have $S(\langle 1^{n(m)}, M_{Q,h,m} \rangle) \models \varphi$. This implies $x = 1^{h(m)} \in Q$ by (14) because, as above, $m \geq |M_h| + |M_Q| + e$. \hfill $\square$

7. Problem comparison

7.1. The role of uniformity. Our proof of the lower bound $p$-$\Delta_0$-$\text{TRUTH} \notin \text{para-AC}^0$ (Theorem 1.5) makes crucial use of the uniformity condition in the definition of para-AC$^0$. To shed some light on this dependence, we relax the uniformity condition as follows.
Definition 7.1. Let \((Q, \kappa)\) be a parameterized problem and \(d, k \in \mathbb{N}\). The \(k\)-th slice of \((Q, \kappa)\) is the classical problem \(\{ x \in Q \mid \kappa(x) = k \}\). The class \(\text{XAC}^0\) contains \((Q, \kappa)\) if and only if \(\text{AC}^0\) contains every slice of \((Q, \kappa)\). The class \(\text{XAC}^0_d\) contains \((Q, \kappa)\) if and only if \(\text{AC}^0_d\) contains every slice of \((Q, \kappa)\); here, \(\text{AC}^0_d\) denotes the class of problems decided by dlogtime uniform circuit families of polynomial size and depth \(d\).

Clearly,
\[
\text{para-AC}^0 \subseteq \bigcup_{d \in \mathbb{N}} \text{XAC}^0_d \subseteq \text{XAC}^0
\]
and \(\text{XAC}^0_1 \not\subseteq \text{para-AC}^0\) as witnessed, e.g., by an undecidable problem with parameterization \(\text{num}\). The class \(\text{XAC}^0\) is important in our context because it is a natural upper bound on \(p\)-\(\Delta_0\)-\(\text{TRUTH}\):

**Proposition 7.2.** \(p\)-\(\Delta_0\)-\(\text{TRUTH} \in \text{XAC}^0\).

**Proof:** It suffices to show that for every \(\Delta_0\)-formula \(\varphi(x)\) the problem \(\{1^n \mid \mathbb{N} \models \varphi(n)\}\) belongs to \(\text{AC}^0\). But this problem is \(\text{un}(Q)\) for \(Q := \{ x \in \{0,1\}^* \mid \mathbb{N} \models \varphi(\text{num}(x))\}\). Clearly \(Q \in \text{LINH}\), so \(\text{un}(Q) \in \text{AC}^0\) follows from Proposition 2.1. \(\square\)

We show that it is likely difficult to improve Theorem 1.5 to \(p\)-\(\Delta_0\)-\(\text{TRUTH} \not\subseteq \bigcup_{d \in \mathbb{N}} \text{XAC}^0_d\). This somewhat artificial class also exhibits the different behaviors of the parameterized problems \(p\)-\(\text{HALT}\), \(p\)-\(\text{HALT}^=\), and \(p\)-\(\Delta_0\)-\(\text{TRUTH}\).

**Theorem 7.3.**

(i) \(p\)-\(\text{HALT} \in \text{XAC}^0_2\).

(ii) \(p\)-\(\text{HALT}^= \in \text{XAC}^0_d\) for some \(d \in \mathbb{N}\) if and only if \(\text{NE} \subseteq \text{LINH}\).

(iii) \(p\)-\(\Delta_0\)-\(\text{TRUTH} \in \text{XAC}^0_d\) for some \(d \in \mathbb{N}\) if and only if \(\text{LINH}\) collapses.

**Proof:** (i) has been observed in the introduction and (ii) follows from Remark 3.8 and (15). To see (iii), assume \(\text{LINH}\) collapses. Paris and Dimitracopolous [31, Proof of Proposition 4] showed that this implies the following. There is an \(L^\varphi\)-formula \(\lambda(x, y)\) such that for every \(\Delta_0\)-formula \(\varphi(x)\) there are \(c_\varphi, d_\varphi, e_\varphi \in \mathbb{N}\) such that for all \(n \geq c_\varphi\)
\[
\mathbb{N} \models \varphi(n) \iff n^{d_\varphi} \models \lambda(n, e_\varphi)
\]
For each fixed \(\varphi\) there is an \(\text{AC}^0\)-family that given \(1^n\) decides whether \(n\) satisfies the the r.h.s.. The size of this family is bounded \(n^{f_\varphi}\) for some \(f_\varphi \in \mathbb{N}\) depending on \(\varphi\), but the depth of this family is determined by the quantifier alternation rank of \(\lambda\) and, in particular, does not depend on \(\varphi\). This implies \(p\)-\(\Delta_0\)-\(\text{TRUTH} \in \text{XAC}^0_d\) for some \(d \in \mathbb{N}\).

Conversely, assume \(p\)-\(\Delta_0\)-\(\text{TRUTH} \in \text{XAC}^0_d\) and let \(Q \in \text{LINH}\). It is well known (see e.g. [22, Ch.V, Lemma 2.13]) that there is a \(\Delta_0\)-formula that is satisfied by \(\text{num}(x)\) if and only if \(x \in Q\). Fixing this formula in the input to \(p\)-\(\Delta_0\)-\(\text{TRUTH}\), the assumption implies that there is a dlogtime uniform circuit family \((C_n)_n\) of polynomial size and depth \(d\) such that for all \(x \in \{0,1\}^*\):
\[
x \in Q \iff C_{\text{num}(x)}(1^{\text{num}(x)}) = 1.
\]
It suffices to show that, given \( x \), the r.h.s. can be checked by an alternating machine in linear time with \( d \) alternations. This is straightforward by guessing a path through \( C_{\text{num}}(x) \). E.g., if the output gate is a \( \lor \)-gate, the machine existentially guesses an input gate \( g_1 \) to it, and if it is a \( \land \)-gate it universally guesses \( g_1 \). Depending on the type of \( g_1 \) it either existentially or universally guesses an input gate \( g_2 \) to \( g_1 \), and so on. When reaching (with \( g_{d-1} \) or earlier) an input gate or a negation thereof, the machine checks it is satisfied by the corresponding bit of \( x \). Each guess requires \( O(|x|) \) bits. Checking that e.g. \( g_2 \) is an input to \( g_1 \) can be done in time logarithmic in the size of \( C_{\text{num}}(x) \), that is, in time \( O(|x|) \). We omit further details. \( \square \)

7.2. Reducibilities. In this section we draw some corollaries concerning how our problems \( p\text{-HALT} \), \( p\text{-HALT}_= \) and \( p\Delta_0\text{-TRUTH} \) compare with respect to our notion of reducibility. Saying that a (parameterized) problem is reducible to another means that there is an eventually definable reduction. Two problems are equivalent if they are reducible to one another.

The picture is as follows: an arrow indicates reducibility, \( \equiv \) means equivalence.

\[
p\text{-SPEC} \rightarrow p\text{-HALT}_= \quad \not\equiv \quad p\Delta_0\text{-TRUTH} \\
p\text{-HALT} \uparrow
\]

In particular, we show unconditionally that \( p\text{-HALT}_= \) and \( p\Delta_0\text{-TRUTH} \) are not equivalent and both are reducible to yet another almost tally problem of central importance to mathematical logic, namely the following parameterized version of the spectrum problem:

| \( p\text{-SPEC} \) |
| --- |
| **Instance:** \( n \in \mathbb{N} \) in unary and a first-order sentence \( \varphi \). |
| **Parameter:** \( |\varphi| \). |
| **Problem:** Does \( \varphi \) have a model of size \( n \)? |

Recall that having a model of size \( n \) means that \( n \) belongs to the spectrum of \( \varphi \).

We start comparing \( p\text{-HALT} \) and \( p\text{-HALT}_= \). Clearly, \( p\text{-HALT} \) is reducible to \( p\text{-HALT}_= \). Concerning the converse we note that Theorem 7.3 (i), (ii) implies:

**Corollary 7.4.** If \( p\text{-HALT}_= \) is reducible to \( p\text{-HALT} \), then \( \text{NE} \subseteq \text{LINH} \).

Adapting a mode of speech from [8], call an almost tally problem \((Q, \kappa)\) slicewise monotone if \( (1^n, x) \in Q \) implies \( (1^m, x) \in Q \) for all \( x \in \{0, 1\}^* \) and all \( n, m \in \mathbb{N} \) with \( n < m \). One can show that \( p\text{-HALT} \) is the hardest such problem in \( \text{para-NP} \). This is an easy modification of the proof Lemma 3.10 and strengthens [8, Proposition 11]:

**Corollary 7.5.** Every almost tally problem in \( \text{para-NP} \) that is slicewise monotone is reducible to \( p\text{-HALT} \).

We turn to \( p\text{-HALT}_= \) and \( p\Delta_0\text{-TRUTH} \).
Corollary 7.6.

(i) If $p\Delta_0$-Truth is reducible to $p$-HALT$\infty$, then NE $\nsubseteq$ LINH.

(ii) If $p$-HALT$\infty$ is reducible to $p\Delta_0$-Truth, then NE $\subseteq$ LINH.

(iii) $p\Delta_0$-Truth and $p$-HALT$\infty$ are not equivalent.

Proof: (iii) follows from (i) and (ii). For (i), assume $p\Delta_0$-Truth is reducible to $p$-HALT$\infty$. Then $p\Delta_0$-Truth $\in$ para-NP and NE $\nsubseteq$ LINH follows by Theorem 1.4.

For (ii), assume $p$-HALT$\infty$ is reducible to $p\Delta_0$-Truth. Then $p$-HALT$\infty$ $\in$ XAC$^0$ by Proposition 7.2 and hence NE $\subseteq$ LINH by Remark 3.8. \hfill $\square$

Finally, we turn to $p$-SPEC:

Proposition 7.7. Both $p$-HALT and $p\Delta_0$-Truth are reducible to $p$-SPEC.

Proof: It is straightforward to compute from a nondeterministic Turing machine $M$ a first-order sentence $\varphi_M$ that has a model of size $n$ if and only if $M$ accepts the empty input in exactly $n$ steps.

Concerning $p\Delta_0$-Truth, by Lemma 4.1, it suffices to show that $p$-MC($L^r_{ar}$) is reducible to $p$-SPEC: map an instance $(1^n, \varphi)$ of $p$-MC($L^r_{ar}$) to $(1^n, \varphi \wedge \psi)$ where $\psi$ is an $L^r_{ar}$-sentence whose finite models are exactly those isomorphic to some standard finite $L^r_{ar}$-structure. \hfill $\square$

Observe $p$-SPEC can be solved in nondeterministic time $n^{f(k)}$ for some computable $f : \mathbb{N} \to \mathbb{N}$ where $k := |\varphi|$ is the parameter. Can the parameter be moved out of the exponent? We find in worthwhile to explicitly point out the following direct corollary of the previous proposition and Theorem 1.4:

Corollary 7.8. If $p$-SPEC $\in$ para-NP, then NE $\nsubseteq$ LINH.

References

[1] E. Allender, R. Beigel, U. Hertrampf, and S. Homer. Almost-everywhere complexity hierarchies for nondeterministic time. *Theoretical Computer Science*, 115(2):225–241, 1993.

[2] E. Allender and V. Gore. On strong separations from AC$^0$. In Jin-Yi Cai, editor, *Advances in Computational Complexity Theory*, volume 13 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 21–38, 1990.

[3] Sanjeev Arora and Boaz Barak. *Computational Complexity - A Modern Approach*. Cambridge University Press, 2009.

[4] Y. Aumann and Y. Dombb. Fixed structure complexity. In *3rd International Workshop on Parameterized and Exact Computation (IWPEC’08)*, pages 30–42, 2008.

[5] J. L. Balcázar and U. Schöning. Bi-immune sets for complexity classes. *Mathematical Systems Theory*, 18(1):1–10, 1985.
[6] D. A. M. Barrington, N. Immerman, and H. Straubing. On uniformity within NC$^1$. *Journal of Computer and System Sciences*, 41(3):274–306, 1990.

[7] Y. Chen and J. Flum. A logic for PTIME and a parameterized halting problem. In *Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science*, *(LICS’09)*, pages 397–406, 2009.

[8] Y. Chen and J. Flum. On slicewise monotone parameterized problems and optimal proof systems for TAUT. In *Proceedings of the 24th International Workshop Computer Science Logic *(CSL 2010)**, volume 6247 of *Lecture Notes in Computer Science*, pages 200–214. Springer, 2010.

[9] Y. Chen and J. Flum. On the complexity of Gödel’s proof predicate. *The Journal of Symbolic Logic*, 75(1):239–254, 2010.

[10] Y. Chen and J. Flum. From almost optimal algorithms to logics for complexity classes via listings and a halting problem. *Journal of the ACM*, 59(4):17:1–17:34, 2012.

[11] Y. Chen and J. Flum. Some lower bounds in parameterized AC$^0$. *Information and Computation*, 267:116–134, 2019.

[12] Y. Chen, M. Müller, and K. Yokoyama. A parameterized halting problem, the linear time hierarchy, and the MRDP theorem. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, *(LICS’18)*, pages 235–244, 2018.

[13] M. Davis. Hilbert’s tenth problem is unsolvable. *The American Mathematical Monthly*, 80(3):233–269, 1973.

[14] Rodney G. Downey and Michael R. Fellows. *Parameterized Complexity*. Monographs in Computer Science. Springer, 1999.

[15] Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[16] Heinz-Dieter Ebbinghaus and Jörg Flum. *Finite model theory*. Perspectives in Mathematical Logic. Springer, 1995.

[17] J. Flum and M. Grohe. Describing parameterized complexity classes. *Information and Computation*, 187(2):291–319, 2003.

[18] Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006.

[19] Lance Fortnow and Rahul Santhanam. New mon-uniform lower bounds for uniform classes. In *Proceedings of the 31st Conference on Computational Complexity (CCC 2016)*, volume 50 of *LIPIcs*, pages 19:1–19:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.
[20] H. Gaifman and C. Dimitracopoulos. Fragments of arithmetic and the MRDP theorem. In *Logic and algorithmic*, volume 30 of *Monographie de L’Enseignement Mathematique*, pages 187–206, 1982.

[21] J. G. Geske, D. T. Huynh, and J. I. Seiferas. A note on almost-everywhere-complex sets and separating deterministic-time-complexity classes. *Information and Computation*, 92(1):97–104, 1991.

[22] Petr Hájek and Pavel Pudlák. *Metamathematics of first-order arithmetic*. Perspectives in Mathematical Logic. Oxford University Press, 1998. Second printing.

[23] Neil Immerman. *Descriptive complexity*. Graduate texts in computer science. Springer, 1999.

[24] R. Kaye. Diophantine induction. *Annals of Pure and Applied Logic*, 46:1–40, 1990.

[25] Richard Kaye. *Models of Peano Arithmetic*. OxfordLogic Guides. Springer-Verlag, Berlin, 1991.

[26] J. Krajíček. Exponentiation and second order bounded arithmetic. *Annals of Pure and Applied Logic*, 48(3):261–276, 1990.

[27] Elvira Mayordomo. Almost every set in exponential time is p-bi-immune. *Theor. Comput. Sci.*, 136(2):487–506, 1994.

[28] A. Nash, J. B. Remmel, and V. Vianu. PTIME queries revisited. In *Proceedings of the 10th International Conference of Database Theory (ICDT’05)*, pages 274–288, 2005.

[29] Christos H. Papadimitriou. *Computational complexity*. Addison-Wesley, 1994.

[30] R. Parikh. Existence and feasibility in arithmetic. *The Journal of Symbolic Logic*, 36:494–508, 1971.

[31] J. B. Paris and C. Dimitracopoulos. Truth definitions for $\Delta_0$ formulae. In *Logic and algorithmic*, volume 30 of *Monographie de L’Enseignement Mathematique*, pages 317–329, 1982.

[32] N. Schweikardt. Arithmetic, first-order logic, and counting quantifiers. *ACM Transactions on Computational Logic*, 6(3):634–671, 2005.

[33] A. J. Wilkie. Applications of complexity theory to $\Sigma_0$-definability problems in arithmetic. In *Model Theory of Algebra and Arithmetic*, volume 834 of *Lecture Notes in Mathematics*, pages 363–369, 1980.

[34] M. Zimand. Large sets in $\text{AC}^0$ have many strings with low Kolmogorov complexity. *Information Processing Letters*, 62(3):165–170, 1997.