A $\hbar$-deformation of the $W_N$ algebra and its vertex operators

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January 19, 2018

Abstract

In this paper, we derive a $\hbar$-deformation of the $W_N$ algebra and its quantum Miura transformation. The vertex operators for this $\hbar$-deformed $W_N$ algebra and its commutation relations are also obtained.

1 Introduction

Recently, the studies of $q$-deformation of some infinite dimensional algebra—$q$-deformed affine algebra$[4,6,11]$ , $q$-deformed virasoro$[2,21]$ algebra and $W_N$-algebra$[2,3,9,10]$, have attracted much attention of physicist and mathematician. The $q$-deformed affine algebra and its vertex operators provide a powerful method to study the state space and the correlation function of solvable lattice model both in the bulk$[16]$ case and the boundary case$[14]$. However, the symmetry of $q$-deformed affine algebra only correspond to the current algebra (affine Lie algebra) symmetry in the conformal field theory (CFT), not to the Virasoro and $W$ algebra type symmetry. The $q$-deformation of Virasoro and $W$ algebra, which would play the role of symmetry algebra for the solvable lattice model, has been expected for a long time. H.Awata et al also constructed the $q$-deformed $W_N$ algebra (including Virasoro algebra) and associated Miura transformation from studying the Macdonald symmetrical functions$[2]$. On the other hand, E.Frenkel and N.Reshetikin succeeded in constructing the $q$-deformed classical $W_N$ algebra and corresponding Miura transformation in analysis of the $U_q(\hat{sl}_N)$ algebra at the critical level$[10]$. Then, B.Feigin and E.Frenkel obtained the quantum version of this $q$-deformed classical $W_N$ algebra, i.e the $q$-deformed $W_N$ algebra$[9]$. The $q$-deformed Virasoro algebra was also given by S.Lukyanov et al in studying the Bosonization for ABF model$[21]$. The bosonization for vertex operators of $q$-deformed Virasoro$[17,21]$ and $W_N$ algebra$[2,1]$ have been constructed.

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However, there exists another important deformation of infinite dimensional algebra, which plays an important role in the completely integrable field theories (in order to comparison with q-deformation, we call it as $\hbar$-deformation). This deformation for affine algebra was originated by Drinfeld in studies of Yangian [7]. It has been shown that the Yangian ($DY(\hat{sl}_2)$) is the dynamical non-abelian symmetry algebra for SU(2)-invariant Thirring model [13,18,19,23]. Naively, the $\hbar$-deformed affine algebra (or Yangian) would play the same role in the integrable field theories as the q-deformed affine algebra in solvable lattice model. Naturally, the $\hbar$-deformed Virasoro and $W$ algebra, which would play the role of symmetry algebra of some integrable field model, are expected. We have succeeded in constructed $\hbar$-deformed Virasoro algebra in Ref.[12] and shown that this $\hbar$-deformed Virasoro algebra is the dynamical symmetry algebra of the Restricted sine-Gordon model. In this paper, we construct the $\hbar$-deformed $W_N$ algebra (including the $\hbar$-deformed Virasoro algebra as a special case of N=2), corresponding quantum Miura transformation and its vertex operators. The $\hbar$-deformed $W_N$ algebra will become the usual no-deformed $W_N$ algebra [8] with some center charge which is related to parameter $\xi$, when $\hbar \rightarrow 0$ and $\xi$ and $\beta$ are fixed.

This paper is arranged as follows: In section 2, we define the $\hbar$-deformed $W_N$ algebra and its Miura transformation. The screening currents and vertex operators are derived in section 3 and 4.

2 $\hbar$-deformation of $W_N$ algebra

In this section, we start with defining a $\hbar$-deformed $W_N$ algebra by quantum Miura transformation.

2.1 $A_{N-1}^{(1)}$ type weight

In this subsection, we shall give some notation about $A_{N-1}^{(1)}$ type weight which will be used in the following parts of this paper. Let $\epsilon_\mu(1 \leq \mu \leq N)$ be the orthonormal basis in $\mathbb{R}^N$, which is supplied with the inner product $\langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu\nu}$. Set

$$\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon, \quad \epsilon = \frac{1}{N} \sum_{\mu=1}^{N} \epsilon_\mu$$

(1)

The $A_{N-1}^{(1)}$ type weight lattice is the linear space of the

$$P = \sum_{\mu=1}^{N} Z \bar{\epsilon}_\mu$$

Note that $\sum_{\mu=1}^{N} \bar{\epsilon}_\mu = 0$. Let $\omega_\mu(1 \leq \mu \leq N - 1)$ be the fundamental weights:

$$\omega_\mu = \sum_{\nu=1}^{\mu} \bar{\epsilon}_\nu$$
and the simple roots $\alpha_\mu (1 \leq \mu \leq N - 1)$:

$$\alpha_\mu = \tau_\mu - \tau_{\mu + 1} = \epsilon_\mu - \epsilon_{\mu + 1}$$  \hspace{1cm} (2)

An ordered pair $(b, a) \in \mathbb{P}^2$ is called admissible if only if there exists $\mu \in (1 \leq \mu \leq N - 1)$ such that

$$b - a = \tau_\mu$$

An ordered set of four weight $\begin{pmatrix} c & d \\ b & a \end{pmatrix} \in \mathbb{P}^4$ is called an admissible configuration around a face if and only if the pairs $(b,a),(c,b),(d,a)$ and $(c,d)$ are all admissible pairs. To each admissible configuration around a face we shall associate a Boltzmann weight in section 4.

2.2 Quantum Miura transformation

Let us consider free bosons $\lambda_i(t)$ ($i = 1, \ldots, N$) with continuous parameter $t \in \{\mathbb{R} - 0\}$ which satisfy

$$[\lambda_i(t), \lambda_i(t')]=\frac{4 \sinh \left(\frac{(N-1)\hbar t}{2}\right) \sinh \frac{\hbar t}{2} \sinh \frac{\hbar (\xi + 1) t}{2}}{t \sinh \frac{N\hbar t}{2}} \delta(t + t')$$  \hspace{1cm} (3)

$$[\lambda_i(t), \lambda_j(t')]=\frac{4 \sinh \left(\frac{\hbar t}{2}\right) \sinh \frac{\hbar t}{2} \sinh \frac{\hbar (\xi + 1) t}{2} \epsilon \text{sign}(j-i) \frac{N\hbar}{2}}{t \sinh \frac{N\hbar t}{2}} \delta(t + t') \quad i \neq j$$  \hspace{1cm} (4)

with the deformed parameter $\hbar$ and a generic parameter $\xi$ and $\lambda_i(t)$ should be subject to the following condition

$$\sum_{i=1}^{N} \lambda_i(t)e^{i\hbar t} = 0$$  \hspace{1cm} (5)

One can check that the restricted condition would be compatible with Eq.(3) and Eq.(4).

Remark: The free bosons with continuous parameter in the case of $N=2$, was first introduced by M. Jimbo et al in studying Massless XXZ mode[15]. This kind bosons could be used to construct the bosonization of Yangian double with center $DY(\hat{sl}_N)$.

Let us define the fundamental operators $\Lambda_i(\beta)$ and the $\hbar$-deformed $W_N$ algebra generators $T_i(\beta)$ for $i=1,\ldots,N$ as follows

$$\Lambda_i(\beta) = :e^{i \int_{-\infty}^{\infty} \lambda_i(t)e^{i\hbar t} dt}:$$  \hspace{1cm} (6)

$$T_i(\beta) = \sum_{1 \leq j_1 < j_2 \ldots < j_k \leq N} \Lambda_{j_1}(\beta + i \frac{l - 1}{2}\hbar) \Lambda_{j_2}(\beta + i \frac{l - 3}{2}\hbar) \ldots \Lambda_{j_k}(\beta - i \frac{l - 1}{2}\hbar)$$  \hspace{1cm} (7)

and $T_0(\beta) = 1$. Here $O$ stands for the usual bosonic normal ordering of some operator $O$ such that the bosons $\lambda_i(t)$ with non-negative mode $t > 0$ are in the right. The restricted condition for bosons $\lambda_i(t)$ in
Eq.(5) results in $T_N(\beta) = 1$. Actually, the generators $T_i(\beta)$ are obtained by the following quantum Miura transformation

$$:(e^{i\hbar \partial_\beta} - \Lambda_1(\beta))(e^{i\hbar \partial_\beta} - \Lambda_2(\beta - i\hbar)) \cdots (e^{i\hbar \partial_\beta} - \Lambda_N(\beta - i(N-1)\hbar)) :$$

$$= \sum_{l=0}^{N} (-1)^l T_l(\beta - i\frac{l-1}{2}\hbar) e^{i(N-l)\hbar \partial_\beta}$$

(8)

**Remark:** $e^{i\hbar \partial_\beta}$ is the $\hbar$-shift operator such that

$$e^{i\hbar \partial_\beta} f(\beta) = f(\beta + i\hbar)$$

If we take the limit of $\xi \rightarrow -1$, the above generators $T_i(\beta)$ reduce to the classical version of $\hbar$-deformed $W_N$ algebra, which can be obtained by studying the Yangian Double with center $DY(\hat{sl}_N)$ at the critical level (i.e., $l = -2$). For the case of $N=2$, the corresponding classical $\hbar$-deformed $W_2$ (Virasoro) algebra has been given by Hou et al [5]. Thus, we call the limit ($\xi \rightarrow -1$ with $\hbar, \beta$ fixed) as the classical limit.

Let us consider another limit: $\hbar \rightarrow 0$ with fixed $\xi$. Then we have $\Lambda_i(\beta) = 1 + i\hbar \chi_i(\beta) + o(\hbar)$ and $e^{i\hbar \partial_\beta} = 1 + i\hbar \partial_\beta + o(\hbar)$. Hence the right hand side of (8) in this limit becomes

$$:(i\hbar)^N (\partial_\beta - \chi_1(\beta))(\partial_\beta - \chi_2(\beta)) \cdots (\partial_\beta - \chi_N(\beta)) : + o(\hbar^N)$$

(9)

and we obtain the normally ordered Miura transformation corresponding to the non-deformed $W_N$ algebra introduced by V. Fateev and S. Lukyanov [8]. Therefore, the non-deformed $W_N$ algebra (the ordinary one) with the center charge $(N - 1) - \frac{N(N+1)}{2(\xi+1)}$ can be obtained by taking this kind limit. In this sense, we call this limit ($\hbar \rightarrow 0$ with fixed $\xi$ and $\beta$) as the conformal limit.

### 2.3 Relations of $\hbar$-deformed $W_N$ algebra

In order to get the commuting relations for bosonic operators, we should give a comment about regularization. When one compute the exchange relation of bosonic operators, one often encounter an integral as follows

$$\int_0^{\infty} F(t)dt$$

which is divergent at $t = 0$. Hence we adopt the regularization given by Jimbo et al [15]. Namely, the above integral should be understood as the contour integral

$$\int_C F(t) \frac{log(-t)}{2i\pi} dt$$

(10)
where the contour \( C \) is chosen as the same as in the Ref.[15]. From the definition of fundamental operators \( \Lambda_i(\beta) \) and the the commutation relations of bosons \( \lambda_i(t) \), we can derive the following OPEs

\[
\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i=i}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_i(\beta_2) : \\
\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i<j}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_j(\beta_2) : \quad \text{if } i < j \\
\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i>j}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_j(\beta_2) : \quad \text{if } i > j
\]

\[
\phi_{i=i}(\beta) = \begin{cases} 
\frac{\Gamma(i\beta \pm \xi N^2 \beta + 1 - \beta N^2 \beta)}{\Gamma(i\beta \pm \xi N^2 \beta + 1)\Gamma(i\beta \pm \xi N^2 \beta + 1 + \beta N^2 \beta)} & \text{if } i = j \\
\frac{\Gamma(i\beta \pm \xi N^2 \beta + 1 - \beta N^2 \beta)}{\Gamma(i\beta \pm \xi N^2 \beta + 1 + \beta N^2 \beta)} & \text{if } i < j \\
\frac{\Gamma(i\beta \pm \xi N^2 \beta + 1 - \beta N^2 \beta)}{\Gamma(i\beta \pm \xi N^2 \beta + 1 + \beta N^2 \beta)} & \text{if } i > j
\end{cases}
\]

To calculate the general OPEs, the integral representation for \( \Gamma \)-function is very useful

\[
\Gamma(z) = \exp\left\{ \int_0^\infty \left( e^{-zt} - e^{-t} \right) \frac{dt}{t} \right\}, \quad \text{Re}(z) > 0
\]

**Remark:** The above OPEs can be considered as the operator scaling limit of \( q \)-deformed \( \Lambda_i(z) \) given by B. Feigin and E. Frenkel in studying the \( q \)-deformed \( W_N \) algebra [9](the very similar \( \Lambda_i(z) \) was also constructed by H. Awata et al [2]). The scaling limit is taken as following way

\[
z = p^{\frac{-i\beta}{\hbar}}, \quad q = p^{-\xi}, \quad \Lambda_i(\beta) = \lim_{p \to 1} \Lambda_i(z) \equiv \lim_{p \to 1} \Lambda_i(p^{-\frac{i\beta}{\hbar}})
\]

**Theorem 1** The generators \( T_1(\beta) \) and \( T_m(\beta) \) of \( \hbar \)-deformed \( W_N \) algebra satisfy the following relations

\[
f_{1m}^{-1}(\beta_2 - \beta_1)T_1(\beta_1)T_m(\beta_2) - f_{1m}^{-1}(\beta_1 - \beta_2)T_m(\beta_2)T_1(\beta_1) = -2i\pi\{ih\xi(\xi + 1) \times (T_{m+1}(\beta_2 + \frac{i\hbar}{2}\delta(\beta_1 - \beta_2 - i\frac{m+1}{2}\hbar)) - T_{m+1}(\beta_2 - \frac{i\hbar}{2}\delta(\beta_1 - \beta_2 + i\frac{m+1}{2}\hbar)))
\]

where

\[
f_{1m}(\beta) = \frac{\Gamma(i\beta \pm \xi N^2 \beta + 1 - \frac{m+1}{2}\hbar)\Gamma(i\beta \pm \xi N^2 \beta + 1 + \frac{m+1}{2}\hbar)}{\Gamma(i\beta \pm \xi N^2 \beta + 1 + \frac{m+1}{2}\hbar)\Gamma(i\beta \pm \xi N^2 \beta + 1 - \frac{m+1}{2}\hbar)}
\]

**Proof** Using the OPEs Eq.(11)-Eq.(13), we obtain that when \( \text{Im}\beta_2 < \text{Im}\beta_1 \)

\[
\Lambda_i(\beta_1) : \Lambda_i(\beta_2 + i\frac{m-1}{2}\hbar) : \Lambda_{j_1}(\beta_2 + i\frac{m-1}{2}\hbar) : \Lambda_{j_2}(\beta_2 + i\frac{m-1}{2}\hbar) : \Lambda_{j_m}(\beta_2 + i\frac{m-1}{2}\hbar) :
\]

is equal to

\[
f_{1m}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_{j_1}(\beta_2 + i\frac{m-1}{2}\hbar) : \Lambda_{j_2}(\beta_2 + i\frac{m-1}{2}\hbar) : \Lambda_{j_m}(\beta_2 + i\frac{m-1}{2}\hbar) :
\]
if \( l = j_k \) for some \( k \in \{1, \ldots, m\} \); and
\[
f_{1m}(\beta_2 - \beta_1) \frac{(i \bar{\beta}_2 - \bar{\beta}_1 - \frac{\xi}{N} - \frac{1}{2N} + 2k-m)(i \bar{\beta}_2 - \bar{\beta}_1 + \frac{\xi}{N} + \frac{1}{2N} + 2k-m)}{(i \bar{\beta}_2 - \bar{\beta}_1 - \frac{\xi}{N} - \frac{1}{2N} + 2k-m)(i \bar{\beta}_2 - \bar{\beta}_1 + \frac{\xi}{N} + \frac{1}{2N} + 2k-m)} \cdot \Lambda_l(\beta_1)
\]
\[
\times \Lambda_{j_1}(\beta_2 + i \frac{m-1}{2} \hbar) \ldots \Lambda_{j_m}(\beta_2 - i \frac{m-1}{2} \hbar) : \Lambda_l(\beta_1)
\]
if \( j_k < l < j_{k+1} \). Here and below the case \( l < j_1 \) corresponds to \( k = 0 \) and the case \( l > j_m \) corresponds to \( k = m \). On the other hand, when \( \text{Im}\beta_2 >> \text{Im}\beta_1 \),
\[
\Lambda_{j_1}(\beta_2 + i \frac{m-1}{2} \hbar) \ldots \Lambda_{j_m}(\beta_2 - i \frac{m-1}{2} \hbar) : \Lambda_l(\beta_1)
\]
is equal to
\[
f_{1m}(\beta_1 - \beta_2) : \Lambda_{j_1}(\beta_2 + i \frac{m-1}{2} \hbar) \ldots \Lambda_{j_m}(\beta_2 - i \frac{m-1}{2} \hbar) \Lambda_l(\beta_1) :
\]
if \( l = j_k \) for some \( k \in \{1, \ldots, m\} \); and
\[
f_{1m}(\beta_1 - \beta_2) \frac{(i \bar{\beta}_1 - \bar{\beta}_2 - \frac{\xi}{N} - \frac{1}{2N} - 2k+m)(i \bar{\beta}_1 - \bar{\beta}_2 + \frac{\xi}{N} + \frac{1}{2N} - 2k+m)}{(i \bar{\beta}_1 - \bar{\beta}_2 - \frac{\xi}{N} - \frac{1}{2N} - 2k+m)(i \bar{\beta}_1 - \bar{\beta}_2 + \frac{\xi}{N} + \frac{1}{2N} - 2k+m)} : \Lambda_{j_1}(\beta_2 + i \frac{m-1}{2} \hbar)
\]
\[
\times \ldots \Lambda_{j_m}(\beta_2 - i \frac{m-1}{2} \hbar) \Lambda_l(\beta_1) :
\]
if \( j_k < l < j_{k+1} \). Noting that
\[
\lim_{\epsilon \to 0^+} \left( \frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = -2i\pi \delta(x)
\]
we can obtain the commutation relations Eq.(17) for \( T_1(\beta_1) \) and \( T_m(\beta_2) \) after some straightforward calculation. Therefore, we complete the proof of Theorem 1.

In fact, the commutation relations for the generators of \( \hbar \)-deformed \( W_N \) algebra have already been defined from the commutation relations of fundamental operators \( \Lambda_i(\beta) \) in Eq.(6) and the corresponding quantum Miura transformation in Eq.(7). So, one can also derive some similar commutation relations between \( T_i(\beta) \) and \( T_j(\beta) \) with \( i, j > 1 \) using the same method as that in proof of the Theorem 1. These commutation relations are quadratic, and involve products of \( T_{i-r}(\beta) \) and \( T_{j+r}(\beta) \) with \( r = 1, \ldots, \min(i, j) - 1 \).

In the case of \( N = 2 \), this \( \hbar \)-deformed \( W_2 \) algebra becomes \( \hbar \)-deformed Virasoro algebra, which has been studied by us in the Ref.[12]. Here, we give an example for the case \( N = 3 \). The generators of this case are
\[
T_1(\beta) = \Lambda_1(\beta) + \Lambda_2(\beta) + \Lambda_3(\beta)
\]
\[
T_2(\beta) = : \Lambda_1(\beta + \frac{i\hbar}{2}) \Lambda_2(\beta - \frac{i\hbar}{2}) : + : \Lambda_1(\beta + \frac{i\hbar}{2}) \Lambda_3(\beta - \frac{i\hbar}{2}) : + : \Lambda_2(\beta + \frac{i\hbar}{2}) \Lambda_3(\beta - \frac{i\hbar}{2}) :.
\]
The commutation relations for these two generators are

\[
\begin{align*}
&f_{11}^{-1}(\beta_2 - \beta_1)T_1(\beta_1)T_1(\beta_2) - f_{11}^{-1}(\beta_1 - \beta_2)T_1(\beta_2)T_1(\beta_1) = -2i\pi \{i\hbar \xi (\xi + 1) \\
&\times (T_2(\beta_2 + \frac{i\hbar}{2})\delta(\beta_1 - \beta_2 - i\hbar) - T_2(\beta_2 - \frac{i\hbar}{2})\delta(\beta_1 - \beta_2 + i\hbar))\} \\
&f_{12}^{-1}(\beta_2 - \beta_1)T_1(\beta_1)T_2(\beta_2) - f_{12}^{-1}(\beta_1 - \beta_2)T_2(\beta_2)T_1(\beta_1) = -2i\pi \{i\hbar \xi (\xi + 1) \\
&\times (\delta(\beta_1 - \beta_2 - i\hbar) - \delta(\beta_1 - \beta_2 + i\hbar))\} \\
&f_{22}^{-1}(\beta_2 - \beta_1)T_2(\beta_1)T_2(\beta_2) - f_{22}^{-1}(\beta_1 - \beta_2)T_2(\beta_2)T_2(\beta_1) = -2i\pi \{i\hbar \xi (\xi + 1) \\
&\times (T_1(\beta_2 + \frac{i\hbar}{2})\delta(\beta_1 - \beta_2 - i\hbar) - T_1(\beta_2 - \frac{i\hbar}{2})\delta(\beta_1 - \beta_2 + i\hbar))\}
\end{align*}
\]

(22)

where the coefficient function \(f_{ij}(\beta)\) are

\[
\begin{align*}
f_{11}(\beta) &= \frac{\Gamma\left(\frac{i\beta}{\sqrt{N}} + 1 - \frac{1}{N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{1}{N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{2}{N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{3}{N}\right)}{\Gamma\left(\frac{i\beta}{\sqrt{N}} + 1 + \frac{1}{N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{2}{N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{3}{N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{4}{N}\right)} \\
f_{12}(\beta) &= \frac{\Gamma\left(\frac{i\beta}{\sqrt{N}} + 1 - \frac{3}{2N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + 1 - \frac{1}{2N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{3}{2N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{1}{2N}\right)}{\Gamma\left(\frac{i\beta}{\sqrt{N}} + 1 + \frac{1}{2N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{3}{2N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{1}{2N}\right)\Gamma\left(\frac{i\beta}{\sqrt{N}} + \frac{3}{N}\right)} \\
f_{11}(\beta) &= f_{22}(\beta)
\end{align*}
\]

3 Screening currents

In this section, we will consider the screening currents for the \(h\)-deformed \(W_N\) algebra. First, we introduce some zero mode operators. To each vector \(\alpha \in \mathbf{P}\) (the \(A^{(1)}_{N-1}\) type weight lattice defined in section 2.1), we associate operators \(P_\alpha, Q_\alpha\) which satisfy

\[
[iP_\alpha, Q_\beta] = \langle \alpha, \beta \rangle , \quad (\alpha, \beta \in \mathbf{P})
\]

(25)

We shall deal with the bosonic Fock spaces \(F_{l,k}\) \((l, k \in \mathbf{P})\) generated by \(\lambda_i(-t)\) \((t > 0)\) over the vacuum states \(|l, k>\). The vacuum states \(|l, k>\) are defined by

\[
\begin{align*}
\lambda_i(t)|l, k> &= 0 \quad \text{if} \quad t > 0 \\
P_\beta|l, k> &= \langle \beta, \alpha_+ l + \alpha_- k | l, k> \\
|l, k> &= e^{i\alpha_+ Q_1 + i\alpha_- Q_-} |0, 0>
\end{align*}
\]

where \(\alpha_\pm\) are some parameters related to \(\xi\)

\[
\alpha_+ = -\sqrt{\frac{1 + \xi}{\xi}} , \quad \alpha_- = \sqrt{\frac{\xi}{1 + \xi}}
\]

(26)

and we also introduce \(\alpha_0\)

\[
\alpha_0 = \sqrt{\xi(1 + \xi)}
\]

(27)
To each simple root $\alpha_j \ (j = 1, \ldots, N - 1)$, let us introduce two series bosons $s^+_j(t)$ which are defined by

$$s^+_j(t) = \frac{e^{i\beta t}}{2s_h \frac{(1 + \xi) \hbar t}{2}} (\lambda_j(t) - \lambda_{j+1}(t))$$

(28)

$$s^-_j(t) = \frac{e^{i\beta t}}{2s_h \frac{(1 + \xi) \hbar t}{2}} (\lambda_j(t) - \lambda_{j+1}(t))$$

(29)

By these simple root bosons, we can define the screening currents as follows

$$S^+_j(\beta) = \exp\{-\int_{-\infty}^{\infty} s^+_j(t) e^{i\beta t} dt\} : e^{-i\alpha_j Q_{\alpha_j}}$$

(30)

$$S^-_j(\beta) = \exp\{-\int_{-\infty}^{\infty} s^-_j(t) e^{i\beta t} dt\} : e^{-i\alpha_j Q_{\alpha_j}}$$

(31)

Then we have

**Theorem 2** The screening currents $S^+_j(\beta)$ satisfy

$$[ (e^{i\hbar \partial_{\beta_j}} - \Lambda_1(\beta))(e^{i\hbar \partial_{\beta_j}} - \Lambda_2(\beta - i\hbar)) \ldots (e^{i\hbar \partial_{\beta_j}} - \Lambda_N(\beta - i(N - 1)\hbar)) : , S^+_j(\sigma) ]$$

$$= i\hbar(1 + \xi) \{ (e^{i\hbar \partial_{\beta_j}} - \Lambda_1(\beta)) \ldots (e^{i\hbar \partial_{\beta_j}} - \Lambda_{j-1}(\beta - i(j - 2)\hbar))$$

$$\times D_{\sigma, i\hbar \xi}(2\pi i \delta(\sigma - \beta - i\frac{j + \xi}{2} \hbar) A^+_j(\sigma))$$

$$\times e^{i\hbar \partial_{\beta_j}} (e^{i\hbar \partial_{\beta_j}} - \Lambda_{j+2}(\beta - i(j + 1)\hbar)) \ldots (e^{i\hbar \partial_{\beta_j}} - \Lambda_N(\beta - i(N - 1)\hbar)) :$$

(32)

with

$$A^+_j(\sigma) = : \Lambda_j(\sigma - i\frac{j + \xi}{2} \hbar) S^+_j(\sigma) :$$

and the operator $D_{\sigma, i\hbar \xi}$ is a difference operator with variable $\sigma$

$$D_{\sigma, i\hbar \xi} f(\sigma) \equiv f(\sigma) - f(\sigma + \eta)$$

**Proof** From the formulas Eq.(28), we obtain the following commutation relations

$$[\lambda_j(t), s^+_j(t')] = -\frac{2e^{\frac{1(1 - 2)}{t} \hbar \frac{(1 + \xi) \hbar t}{2}}}{t^2} \delta(t + t')$$

$$[\lambda_{j+1}(t), s^+_j(t')] = \frac{2e^{-\frac{1(1 - 2)}{t} \hbar \frac{(1 + \xi) \hbar t}{2}}}{t^2} \delta(t + t')$$

$$[\lambda_j(t), s^-_j(t')] = 0 \quad \text{if} \quad |j - l| > 1$$

From these commutation relations and the formula Eq.(15), we can derive the following OPEs

$$\Lambda_j(\beta_1) S^+_j(\beta_2) = f^+_j(\beta_2 - \beta_1) : \Lambda_j(\beta_1) S^+_j(\beta_2) :$$

(33)

$$S^+_j(\beta_1) \Lambda_j(\beta_2) = f^+_j(\beta_2 - \beta_1) : S^+_j(\beta_1) \Lambda_j(\beta_2) :$$

(34)

$$\Lambda_{j+1}(\beta_1) S^+_j(\beta_2) = f^+_{j+1}(\beta_2 - \beta_1) : \Lambda_{j+1}(\beta_1) S^+_j(\beta_2) :$$

(35)

$$S^+_j(\beta_1) \Lambda_{j+1}(\beta_2) = f^+_{j+1}(\beta_2 - \beta_1) : S^+_j(\beta_1) \Lambda_{j+1}(\beta_2) :$$

(36)

$$S^+_j(\beta_1) \Lambda_l(\beta_2) = S^+_j(\beta_1) \Lambda_l(\beta_2) := \Lambda_l(\beta_2) S^+_j(\beta_1) : \quad \text{if} \quad |j - l| > 1$$

(37)
and

\[ f^+_j(\beta) = \frac{i\beta}{N\hbar} - \frac{i\xi}{N\hbar} - \frac{1}{2N} + \frac{j}{2N} \]

\[ f^+_{j+1}(\beta) = \frac{i\beta}{N\hbar} - \frac{1+i\xi}{N\hbar} + \frac{j}{2N} \]

The formula Eq.(37) means that

\[
\text{LHS of Eq.}(32) = \left([e^{i\hbar\partial_\beta} - \Lambda_1(\beta)]...[e^{i\hbar\partial_\beta} - \Lambda_{j-1}(\beta - i(j-2)\hbar)]
\times[(e^{i\hbar\partial_\beta} - \Lambda_j(\beta - i(j-1)\hbar))(e^{i\hbar\partial_\beta} - \Lambda_{j+1}(\beta - i(j)\hbar)), S^+_j(\sigma)]
\times(e^{i\hbar\partial_\beta} - \Lambda_{j+2}(\beta - i(j+1)\hbar))...[e^{i\hbar\partial_\beta} - \Lambda_N(\beta - i(N-1)\hbar)) : (38)\]

Therefore, it is sufficient to consider the commutation relation

\[
[\Lambda_j(\beta - i(j-1)\hbar), S^+_j(\sigma)] = 0
\]

According to the OPEs Eq.(33)—Eq.(36), we can derive that

\[
[\Lambda_j(\beta - i(j-1)\hbar), S^+_j(\sigma)] = 0
\]

Now we only need to consider the commutation relations between the term \(\Lambda_j(\beta - i(j-1)\hbar) + \Lambda_{j+1}(\beta - i(j-1)\hbar)\) and the screening current \(S^+_j(\sigma)\). From the OPEs Eq.(33)—Eq.(36), the formula Eq.(19), noting that

\[
[\Lambda_j(\beta - i(j-1)\hbar) + \Lambda_{j+1}(\beta - i(j-1)\hbar), S^+_j(\sigma)] = \hbar(1+\xi)D_{\sigma,\hbar\xi}(2\pi i\delta(\sigma - \beta - i\frac{j + \xi}{2}\hbar) : A^+_j(\sigma)) : (39)
\]

Therefore, the Eq.(32) has been obtained. □

Using the same method, we can have

**Theorem 3** The second series screening currents \(S^-_j(\sigma)\) satisfy

\[
[\Lambda_j(\beta - i(j-1)\hbar) + \Lambda_{j+1}(\beta - i(j-1)\hbar), S^-_j(\sigma)] = \hbar(1+\xi)D_{\sigma,\hbar\xi}(2\pi i\delta(\sigma - \beta - i\frac{j + \xi}{2}\hbar) : A^-_j(\sigma)) : (39)
\]
with
\[ A_j^{-} (\sigma) =: \Lambda_j (\sigma + j + \xi + \frac{1}{2} \hbar) S_j^{-} (\sigma) : \]

Therefore, the screening currents \( S_j^\pm (\beta) \) commute with any \( \hbar \)-deformed \( W_N \) algebra generators up to total difference.

**Remark:** Let us take the conformal limit (\( \hbar \to 0 \) and with \( \xi \) and \( \beta \) fixed), the screening currents \( S_j^\pm (\beta) \) will become the ordinary screening current [8].

Theorem 2 and Theorem 3 imply that one can construct the intertwining operators (namely, vertex operators) for the \( \hbar \)-deformed \( W_N \) algebra using the screening currents \( S_j^\pm (\beta) \). In the next section we shall construct the vertex operators for the \( \hbar \)-deformed \( W_N \) algebra.

### 4 The vertex operators and its exchange relations

In this section, we construct the type I and type II vertex operators for this \( \hbar \)-deformed \( W_N \) algebra through the two series screening currents \( S_j^\pm (\beta) \). Firstly, we set

\[ \hat{\pi}_\mu = \alpha_0 P_\mu , \quad \hat{\pi}_{\mu \nu} = \hat{\pi}_\mu - \hat{\pi}_\nu \] (40)

and

\[ \hat{\pi}_{\mu \nu} F_{l, k} = \langle \epsilon_\mu - \epsilon_\nu, -(1 + \xi) l + \xi k \rangle F_{l, k} \] (41)

Note that

\[ e^{-i\alpha_+ Q_\gamma} \hat{\pi}_\sigma e^{i\alpha_+ Q_\gamma} = \hat{\pi}_\sigma + \alpha_0 \alpha_+ < \sigma, \gamma > \] (42)

and this formula is very useful for calculating commutation relations of vertex operators. To each fundamental weight of \( \omega_j (j = 1, ..., N - 1) \), let us introduce two series bosons \( a_j (t) \) and \( a'_j (t) \) which are defined by

\[ a_j = \sum_{k=1}^{j} e^{-\frac{\hbar (j - 2k + 1) t}{2}} \lambda_k (t) , \quad a'_j = \sum_{k=1}^{j} e^{-\frac{\hbar (j + 2k - 1) t}{2}} \lambda_k (t) \] (43)

and also define

\[ U_{\omega_j} (\beta) =: \exp \left\{ \int_{-\infty}^{\infty} a_j (t) e^{i\beta t} \, dt \right\} : e^{i\alpha_+ Q_{-j}} : , \quad U'_{\omega_j} (\beta) =: \exp \left\{ \int_{-\infty}^{\infty} a'_j (t) e^{i\beta t} \, dt \right\} : e^{i\alpha_- Q_{-j}} \] (44)

Because the vertex operators associated with each fundamental weight \( \omega_j (= 2, ..., N - 1) \) can be constructed from the skew-symmetric fusion of the basic ones \( U_{\omega_1} (\beta) \) and \( U'_{\omega_1} (\beta) \) [1], it is sufficient to only
deal with the vertex operators corresponding to the fundamental weight \( \omega_1 \). In order to calculate the exchange relations of the vertex operators, we first derive the following commutation relations

\[
[a_j(t), s_j^+(t')] = -\frac{sh \frac{\hbar(1+\xi) \omega}{2}}{tsh \frac{\hbar \xi}{2}} \delta_j, i \delta(t + t'), \quad [a_j(t), s_j^-(t')] = -\frac{1}{t} \delta_j, i \delta(t + t')
\]

\[
[a'_j(t), s_j^-(t')] = -\frac{sh \frac{\hbar(1+\xi) \omega}{2}}{tsh \frac{\hbar \xi}{2}} \delta_j, i \delta(t + t'), \quad [a'_j(t), s_j^+(t')] = -\frac{1}{t} \delta_j, i \delta(t + t')
\]

\[
[a_1(t), a_1(t')] = -\frac{sh \frac{(N-1) \hbar t}{2} sh \frac{(1+\xi) \hbar t}{2}}{tsh \frac{N \hbar t}{2} sh \frac{\hbar \xi t}{2}} \delta(t + t')
\]

\[
[a_1(t), a'_1(t')] = -\frac{sh \frac{(N-1) \hbar t}{2}}{tsh \frac{N \hbar t}{2} sh \frac{\hbar \xi t}{2}} \delta(t + t')
\]

\[
[a'_1(t), a'_1(t')] = -\frac{sh \frac{(N-1) \hbar t}{2} sh \frac{\hbar \xi t}{2}}{tsh \frac{N \hbar t}{2} sh \frac{\hbar \xi t}{2}} \delta(t + t')
\]

From the above relations, and taking the regularization in section 2.3, we can derive the following exchange relation

\[
U_{\omega_1}(\beta_1)U_{\omega_1}(\beta_2) = r_1(\beta_1 - \beta_2)U_{\omega_1}(\beta_2)U_{\omega_1}(\beta_1)
\]

\[
U'_{\omega_1}(\beta_1)U'_{\omega_1}(\beta_2) = r'_1(\beta_1 - \beta_2)U'_{\omega_1}(\beta_2)U'_{\omega_1}(\beta_1)
\]

\[
S^+_j(\beta_1)S^+_{j+1}(\beta_2) = -f(\beta_1 - \beta_2, 0)S^+_j(\beta_2)S^+_1(\beta_1)
\]

\[
S^-_j(\beta_1)S^-_{j+1}(\beta_2) = -f'(\beta_1 - \beta_2, 0)S^-_j(\beta_2)S^-_1(\beta_1)
\]

\[
U_{\omega_1}(\beta_1)S^+_1(\beta_2) = -f(\beta_1 - \beta_2, 0)S^+_1(\beta_2)U_{\omega_1}(\beta_1)
\]

\[
U'_{\omega_1}(\beta_1)S^-_1(\beta_2) = -f'(\beta_1 - \beta_2, 0)S^-_1(\beta_2)U'_{\omega_1}(\beta_1)
\]

\[
U_{\omega_1}(\beta_1)S^-_1(\beta_2) = -U_{\omega_1}(\beta_1)S^+_1(\beta_2), \quad U'_{\omega_1}(\beta_1)S^+_1(\beta_2) = -U'_{\omega_1}(\beta_1)S^-_1(\beta_2)
\]

(45)

where the fundamental function \( f(\beta, w) \) and \( f'(\beta, w) \) (which play a very important role in constructing the vertex operators) are defined by

\[
f(\beta, w) = \frac{\sin\pi(\frac{i\beta}{\hbar} - \frac{\beta}{N} - \frac{w}{\hbar})}{\sin\pi(\frac{i\beta}{\hbar} + \frac{\beta}{N})}, \quad f'(\beta, w) = \frac{\sin\pi(\frac{i\beta}{\hbar} + \frac{1}{2(1+\xi)}) + \frac{w}{\hbar}}{\sin\pi(\frac{i\beta}{\hbar} + \frac{1}{2(1+\xi)})}
\]

(46)

and

\[
r(\beta) = \exp\{-\int_0^\infty \frac{2sh \frac{(N-1) \hbar t}{2} sh \frac{(1+\xi) \hbar t}{2} sh \beta t}{tsh \frac{N \hbar t}{2} sh \frac{\hbar \xi t}{2}} dt\}
\]

(47)

\[
r'(\beta) = \exp\{-\int_0^\infty \frac{2sh \frac{(N-1) \hbar t}{2} sh \frac{\hbar \xi t}{2} sh \beta t}{tsh \frac{N \hbar t}{2} sh \frac{(1+\xi) \hbar t}{2}} dt\}
\]

(48)

\[
\tau(\beta) = \frac{\sin\pi(\frac{1}{2N} - \frac{i\beta}{\hbar})}{\sin\pi(\frac{1}{2N} + \frac{i\beta}{\hbar})}
\]

(49)
Now let us define the type I vertex operators $Z^\mu_\mu(\beta)$ and the type II vertex operators $Z^\mu_\mu(\beta)$ ($\mu = 1, \ldots, N$)

$$Z^\mu_\mu(\beta) = \int C_1 \prod_{j=1}^{\mu-1} \eta_j U_{\omega^j}(\beta) S^{\mu}_{\mu}(\eta_1)S^{-\mu}_{\mu}(\eta_2) \cdots S^{\mu}_{\mu}(\eta_{\mu-1}) \prod_{j=1}^{\mu-1} f(\eta_j - \eta_{j-1}, \hat{\pi}_{\mu j})$$ (50)

$$Z_\mu(\beta) = \int C_2 \prod_{j=1}^{\mu-1} \eta_j U_{\omega^j}(\beta) S^{\mu}_{\mu}(\eta_1)S^{\mu}_{\mu}(\eta_2) \cdots S^{\mu}_{\mu}(\eta_{\mu-1}) \prod_{j=1}^{\mu-1} f(\eta_j - \eta_{j-1}, \hat{\pi}_{j\mu})$$ (51)

It is easy to see that the vertex operator $Z^\mu_\mu(\beta)$ and $Z^\mu_\mu(\beta)$ are some bosonic operator intertwining the Fock spaces $F_{l,k}$

$$Z^\mu_\mu(\beta) : F_{l,k} \rightarrow F_{l,k+\tau_\mu}, \ Z^\mu_\mu(\beta) : F_{l,k} \rightarrow F_{l+\tau_\mu,k}$$ (52)

Here we set $\eta_0 = \beta$, the integration contour $C_1$ is chosen as : the contour corresponding to the integration variable $\eta_j$ enclose the poles $\eta_{j-1} + i\frac{\pi}{2} - i\hbar \xi n (0 \leq n)$, and the other integration contour $C_2$ is chosen as : the contour corresponding to the integration variable $\eta_j$ enclose the poles $\eta_{j-1} - i\frac{\pi}{2} - i(1 + \xi)\hbar n (0 \leq n)$.

The constructive form of type I (type II) vertex operators of ours seems to be similar as that of vertex operators for $A^1_{N-1}$ face model given by Y. Asai et al, but with different bosonic operators and “coefficient parts” function $f(\beta, w)$ ($f'(\beta, w)$). Thus the same trick [1] can be used to calculate the commutation relations for our vertex operators. Using the method which was presented by Y. Asai et al in the appendix B of the Ref.[1], we can derive the commutation relations for vertex operators $Z^\mu_\mu(\beta)$ and $Z_\mu(\beta)$

$$Z^\mu_\mu(\beta_1) Z^\mu_\mu(\beta_2) = \sum_{\mu' \nu'} Z^\mu_\mu(\beta_2) Z^\mu_\mu(\beta_1) \hat{W}' \left( \hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu, \hat{\pi} + \hat{\tau}_{\mu'} + \hat{\tau}_{\nu'} \mid \beta_1 - \beta_2 \right)$$ (53)

$$Z^\mu_\mu(\beta_1) Z^\mu_\mu(\beta_2) = \sum_{\mu' \nu'} Z^\mu_\mu(\beta_2) Z^\mu_\mu(\beta_1) \hat{W}' \left( \hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu, \hat{\pi} + \hat{\tau}_{\mu'} + \hat{\tau}_{\nu'} \mid \beta_1 - \beta_2 \right)$$ (54)

$$Z^\mu_\mu(\beta_1) Z^\mu_\mu(\beta_2) = Z^\mu_\mu(\beta_2) Z^\mu_\mu(\beta_1) \hat{W}_r (\beta_1 - \beta_2)$$ (55)

and the braid matrices $\hat{W}_r \left( \hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu, \hat{\pi} + \hat{\tau}_{\mu'} + \hat{\tau}_{\nu'} \mid \beta \right)$ and $\hat{W}_l \left( \hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu, \hat{\pi} + \hat{\tau}_{\mu'} + \hat{\tau}_{\nu'} \mid \beta \right)$ are some functions taking the value on operators $\hat{\pi}_{\mu\nu}$ like

$$\hat{W}_r \left( \hat{\pi} + 2\hat{\tau}_\mu, \hat{\pi} + \hat{\tau}_\mu \mid \beta \right) = r(\beta)$$ (56)

$$\hat{W}_r \left( \hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu, \hat{\pi} + \hat{\tau}_{\mu'} + \hat{\tau}_{\nu'} \mid \beta \right) = -r(\beta) - \frac{\sin \pi \pi \sin \pi (\frac{\beta \mu}{\xi} - \frac{\beta \nu}{\xi})}{\sin \pi (\frac{\pi \mu}{\xi}) \sin \pi (\frac{\pi \nu}{\xi} + \frac{1}{\xi})}$$ (57)

$$\hat{W}_l \left( \hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu, \hat{\pi} + \hat{\tau}_{\mu'} + \hat{\tau}_{\nu'} \mid \beta \right) = r(\beta) - \frac{\sin \pi (\frac{\beta \mu}{\xi}) \sin \pi (\frac{\beta \nu}{\xi} + \frac{1}{\xi})}{\sin \pi (\frac{\pi \mu}{\xi}) \sin \pi (\frac{\pi \nu}{\xi} + \frac{1}{\xi})}$$ (58)

$$\hat{W}\left( \hat{\pi} + 2\hat{\tau}_\mu, \hat{\pi} + \hat{\tau}_\mu \mid \beta \right) = r'(\beta)$$ (59)
\[
\hat{W}\left(\frac{\hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu}{\hat{\pi} + \hat{\tau}_\nu} \mid \beta\right) = -r'(\beta) \frac{\sin\frac{\pi}{1 + \xi} \sin\pi(\frac{i\beta}{1 + \xi} + \frac{\hat{\pi} \mu}{1 + \xi})}{\sin\pi(\frac{l + \tau_\mu}{1 + \xi}) \sin\pi(\frac{\tau_\mu}{1 + \xi} - \frac{l}{1 + \xi})}
\]

(60)

\[
\hat{W}\left(\frac{\hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu}{\hat{\pi} + \hat{\tau}_\nu} \mid \beta\right) = r'(\beta) \frac{\sin\frac{\pi}{1 + \xi} \sin\pi(\frac{i\beta}{1 + \xi} + \frac{\hat{\pi} \mu}{1 + \xi})}{\sin\pi(\frac{l + \tau_\mu}{1 + \xi}) \sin\pi(\frac{\tau_\mu}{1 + \xi} - \frac{l}{1 + \xi})}
\]

(61)

Therefore, these braid matrices are not commutate with the vertex operators and the exchange relations should be written in that order as Eq.(53) and Eq.(54). Noting that

\[
r(-\beta)\xi \rightarrow 1 + \xi = r'(\beta) \Delta_N(\beta)
\]

\[
\Delta_N(\beta) = \frac{\sin\pi(\frac{i\beta}{N} + \frac{1}{N})}{\sin\pi(\frac{1}{N} - \frac{1}{N})}
\]

we find that the braid matrices \(\hat{W}\left(\frac{\hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu}{\hat{\pi} + \hat{\tau}_\nu} \mid \beta\right)\) and \(\hat{W}'\left(\frac{\hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu}{\hat{\pi} + \hat{\tau}_\nu} \mid \beta\right)\) are related to each other as follows

\[
\hat{W}'\left(\frac{\hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu}{\hat{\pi} + \hat{\tau}_\nu} \mid -\beta\right)\mid_{\xi \rightarrow 1 + \xi} = \Delta_N(\beta) \hat{W}\left(\frac{\hat{\pi} + \hat{\tau}_\mu + \hat{\tau}_\nu}{\hat{\pi} + \hat{\tau}_\nu} \mid \beta\right)
\]

When \(N = 2\), the factor \(\Delta_N(\beta) = -1\), it occurred in studying the \(\hbar\)-deformed Virasoro algebra [12]. If both sides of Eq.(53) and Eq.(54) are acted on the special Fock space \(F_{l,k}\), noting that \(l_{\mu\nu}\) and \(k_{\mu\nu}\)

\[
l_{\mu\nu} = <\tau_\mu - \tau_\nu, l >, \quad k_{\mu\nu} = <\tau_\mu - \tau_\nu, k >
\]

are all some integer, we have

\[
Z'_\mu(\beta_1)Z'_\nu(\beta_2)|_{F_{l,k}} = \sum_{\mu'\nu'} Z'_\mu(\beta_2)Z'_\nu(\beta_1)|_{F_{l,k}} \hat{W}'\left(\frac{l + \tau_\mu + \tau_\nu}{l + \tau_\nu} \mid \beta_1 - \beta_2\right)
\]

(62)

\[
Z_\mu(\beta_1)Z_\nu(\beta_2)|_{F_{l,k}} = \sum_{\mu'\nu'} Z_\mu(\beta_2)Z_\nu(\beta_1)|_{F_{l,k}} \hat{W}\left(\frac{k + \tau_\mu + \tau_\nu}{k + \tau_\nu} \mid \beta_1 - \beta_2\right)
\]

(63)

and

\[
\hat{W}'\left(\frac{l + 2\tau_\mu}{l + \tau_\mu} \mid \beta\right) = r(\beta)
\]

\[
\hat{W}'\left(\frac{l + \tau_\mu + \tau_\nu}{l + \tau_\nu} \mid \beta\right) = r(\beta) \frac{\sin\frac{\pi}{2} \sin\pi(\frac{i\beta}{2} + \frac{l}{\frac{l}{2}} \frac{k}{\xi})}{\sin\pi(\frac{l}{2} \frac{k}{\xi}) \sin\pi(\frac{\tau_\mu}{2} \frac{k}{\xi})}
\]

\[
\hat{W}'\left(\frac{l + \tau_\mu + \tau_\nu}{l + \tau_\nu} \mid \beta\right) = r(\beta) \frac{\sin\pi(\frac{i\beta}{2} \frac{k}{\xi}) \sin\pi(\frac{\tau_\mu}{2} \frac{k}{\xi} - \frac{l}{\frac{l}{2}} \frac{k}{\xi})}{\sin\pi(\frac{l}{2} \frac{k}{\xi}) \sin\pi(\frac{\tau_\mu}{2} \frac{k}{\xi} + \frac{l}{\frac{l}{2}} \frac{k}{\xi})}
\]

\[
\hat{W}\left(\frac{k + 2\tau_\mu}{k + \tau_\mu} \mid \beta\right) = r'(\beta)
\]

\[
\hat{W}\left(\frac{k + \tau_\mu + \tau_\nu}{k + \tau_\nu} \mid \beta\right) = r'(\beta) \frac{\sin\frac{\pi}{2} \sin\pi(\frac{i\beta}{2} - \frac{k}{\frac{k}{2}} \frac{l}{\xi})}{\sin\pi(\frac{l}{2} \frac{k}{\xi}) \sin\pi(\frac{i\beta}{2} - \frac{k}{\frac{k}{2}} \frac{l}{\xi})}
\]
\[ W \left( k + \tau_\mu, k + \tau_\nu \right) = r'(\beta) \frac{\sin\pi \left( \frac{i\beta}{\mu + \xi} \right) \sin\pi \left( -\frac{1}{1 + \xi} + \frac{k + \mu}{\mu + \xi} \right)}{\sin\pi \left( \frac{k + \mu}{\mu + \xi} \right) \sin\pi \left( \frac{i\beta}{\mu + \xi} - \frac{1}{1 + \xi} \right)} \]

It can be check that the Boltzmann weights \( W \left( c, d | \beta \right) \) and \( W \left( c, d | \beta \right) \) satisfy the star-triangle equations (or Yang-Baxter equation)

\[
\sum_g W \left( d, e | \beta_1 \right) W \left( c, g | \beta_2 \right) W \left( e, f | \beta \right) = \sum_g W \left( g, f | \beta_1 \right) W \left( d, e | \beta_2 \right) W \left( d, g | \beta_1 - \beta_2 \right)
\]

and unitary relation

\[
\sum_g W \left( c, g | -\beta \right) W \left( c, d | \beta \right) = \delta_{bd}
\]

Actually, the Yang-Baxter equation Eq.(64) and the unitary relation Eq.(65) are direct results of the exchange relation of vertex operators in Eq.(62) and Eq.(63) (the associativity of algebra \( Z_\mu(\beta) \) and \( Z'_\mu(\beta) \))

**Remark:** In fact, we have construct the bosonization of the vertex operators for the trigonometric SOS model of \( A^{(1)}_{N-1} \) type.

**Discussions**

We have constructed a \( \hbar \)-deformed \( W_N \) algebra and its quantum Miura transformation. The \( \hbar \)-deformation of \( W_N \) algebra can be obtained by two way: one can first derive the classical version of \( \hbar \)-deformed \( W_N \) algebra from studying the Yangian double with center \( DY(\hat{sl}_N) \) at the critical level as E.Frenkel and N.Reshetikin in studying the \( U_q(\hat{sl}_N) \) at critical level [10], then one construct the (quantum) \( \hbar \)-deformed \( W_N \) algebra by quantizing the classical one; Another way to construct the \( \hbar \)-deformed \( W_N \) algebra is by taking some scaling limit of \( q \)-deformed \( W_N \) algebra like Eq.(16). Actually, the same phenomena also occur in studying Yangian double with center \( DY(\hat{sl}_N) \): the \( DY(\hat{sl}_N) \) can be considered as some scaling limit of \( U_q(\hat{sl}_N) \) algebra.

We only consider the \( \hbar \)-deformed \( W_N \) algebra for generic \( \xi \). When \( \xi \) is some rational number (\( \xi = \frac{p}{q} \), \( p \) and \( q \) are two coprime integers), the realization of \( \hbar \)-deformed \( W_N \) algebra in the Fock space \( F_{l,k} \) would be highly reducible and we have to throw out some states from the Fock space \( F_{l,k} \) to obtain the irreducible component \( H_{l,k} \). (Here we choose the same symbol as that in [1]). For \( N=2 \), the irreducible space \( H_{l,k} \) (i.e \( L_{l,k} \) in [12]) can be obtained by some BRST cohomology [12]. Unfortunately, the construct of the BRST complex and the calculation of cohomology for \( 3 \leq N \) is still an open problem.
We also construct the vertex operators of type I and type II. These vertex operators satisfy some Fadeev-Zamolodchikov algebra with the face type Boltzmann weight as its constructure constant. In order to obtain the correlation functions as the traces of products of these vertex operators, we should introduce a boost operators $H_1$

$$ H_1 = \int_0^{\infty} - \frac{t \sinh \frac{N \hbar t}{2} \sinh \frac{\xi \hbar t}{2}}{\sinh \frac{(N-1) \hbar t}{2} \sinh \frac{(1+\xi) \hbar t}{2}} a_1(-t) a_1(t) dt \quad (66) $$

which enjoys the property

$$ e^{x H_1} a_1(t) e^{-x H_1} = e^{-xt} a_1(t) $$

Moreover, using the skew-symmetric fusion of $N$'s vertex operators, we can obtain some invertibility for our vertex operators of the form like (3.19) and (c.20) in Ref.[1]. Then the correlation function can be described by the following trace function

$$ G(\beta_1, \ldots, \beta_{Nn})_{\mu_1, \ldots, \mu_{Nn}} = \frac{\text{tr}(e^{-xH_1} Z'_{\mu_1}(\beta_1) \cdots Z'_{\mu_{Nn}}(\beta_{Nn}))}{\text{tr}(e^{-xH_1})} \quad (67) $$

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