From Hurwitz numbers to Feynman diagrams: counting rooted trees in log gravity

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Abstract

We show that the partition function of the logarithmic sector of critical topologically massive gravity can be expressed as a sum over rooted trees. Our work appears as a natural extension of recently obtained results on a log partition function that finds itself at the confluence of many theories, relating nonunitary gravity, the plethystic programme, integrable hierarchies of KP soliton equations and \( \tau \)-functions, Hurwitz theory and branched covering of Riemann surfaces to the celebrated Connes-Kreimer Hopf algebra of rooted trees and Feynman diagrams. In particular, the Hurwitz numbers arise as coefficients of isomorphic classes of rooted trees. A parallel is drawn between our findings and established results in the statistical physics literature concerning certain systems with quenched disorder on trees, associated to nonlinear partial differential equations admitting traveling wave solutions. This should be of particular interest in view of a further description of the disorder observed in log gravity.

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1 Introduction

The study of gravity in three dimensions holds the prospect of shedding light on many intricate aspects of classical and quantum gravity. An important result in the study of its asymptotics which anticipated the AdS/CFT correspondence \cite{1}, was the emergence of a Virasoro algebra suggesting a dual conformal
field theory (CFT) in two dimensions at the boundary [2]. Pure Einstein gravity in three dimensions is however locally trivial at the classical level and does not exhibit propagating degrees of freedom, hence the need to modify it. One way of doing so is by introducing a negative cosmological constant. Although the resulting theory still has no propagating degrees of freedom, it has black hole solutions [3]. Another possible modification is to add a gravitational Chern-Simons term. In that case the theory is called topologically massive gravity (TMG), and contains a propagating degree of freedom, the massive graviton [4, 5]. When both cosmological and Chern-Simons terms are included in a theory, it yields cosmological topologically massive gravity (CTMG). Such a theory features both gravitons and black holes.

A proposal to find a CFT dual to Einstein gravity [6] was made in 2007, and was followed by the calculation of the graviton 1-loop partition function [7]. However, discrepancies were found in the results, in particular with the failing in factorization of the left- and right-contributions, therefore clashing with the proposal of [6]. Soon after, a non-trivial slightly modified version of the aforementioned proposal was formulated [8], in which Einstein gravity was replaced by chiral gravity, which can be viewed as a special case of topologically massive gravity at a specific tuning of the couplings, and is asymptotically defined with AdS$_3$ boundary conditions, in the spirit of Fefferman-Graham-Brown-Henneaux [2, 9, 10]. A particular feature of the theory was that one of the two central charges vanishes, whilst the other one can have a non-zero value. This gave an indication that the partition function could factorize. Shortly after the proposal of [8], Grumiller et al. noticed that relaxing the Brown-Henneaux boundary conditions allowed for the presence of a massive mode that forms a Jordan cell with the massless graviton, leading to a degeneracy at the critical point [11]. In addition, it was observed that the presence of the massive mode spoils the chirality of the theory, as well as its unitarity. The non-unitarity arising from the appearance of the Jordan cell is a salient feature in logarithmic conformal field theories (LCFTs), which led to the conjecture that the dual CFT of critical cosmological topologically massive gravity (cTMG) should be an LCFT with central charge $c = 0$ [12, 13, 14]. The massive mode was called the logarithmic partner of the graviton, and the theory gravity with logarithmically relaxed boundary conditions in [11] is also referred to as log gravity.

The proposed non-unitary holography received further support through the calculation of correlation functions in TMG [15, 16], which confirmed the existence of logarithmic correlators of the type that arises in LCFT. Subsequently, the 1-loop graviton partition function of cTMG on the thermal AdS$_3$ background was calculated in [17], resulting in the following expression

$$ Z_{cTMG}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1 - q^n|^2} \prod_{m=2}^{\infty} \prod_{m=0}^{\infty} \frac{1}{1 - q^m \bar{q}^m}, \quad \text{with} \quad q = e^{2i\pi \tau}, \bar{q} = e^{-2i\pi \tau}, \quad (1) $$

where the first product can be identified as the three-dimensional gravity partition function $Z_{0,1}$ in [7]. In search for a better insight into the 1-loop partition function as proposed in [18], it was shown in [19] that the partition function of critical cosmological TMG can be expressed in terms of Bell polynomials. Writing Eq. (1) as

$$ Z_{cTMG}(q, \bar{q}) = Z_{gravity}(q, \bar{q}) \cdot Z_{log}(q, \bar{q}), \quad (2) $$

where

$$ Z_{gravity}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1 - q^n|^2}, \quad \text{and} \quad Z_{log}(q, \bar{q}) = \prod_{m=2}^{\infty} \prod_{m=0}^{\infty} \frac{1}{1 - q^m \bar{q}^m}, \quad (3) $$

it was shown that the log partition function $Z_{log}(q, \bar{q})$ is the generating function of polynomials $Y_n (g_1, \ldots, g_n)$

$$ Z_{log}(q, \bar{q}) = \sum_{n=0}^{\infty} \frac{Y_n}{n!} (q^2)^n, \quad (4) $$

where

$$ g_n = (n - 1)! \sum_{m \geq 0, m \geq 0} q^m q^{m_0} = (n - 1)! \frac{1}{|1 - q^n|^2}. \quad (5) $$
Under the rescaling of variables

\[ g_n(q, \bar{q}) = (n - 1)! \mathcal{G}_n(q, \bar{q}), \]  

(6)

where

\[ \mathcal{G}_n(q, \bar{q}) = \frac{1}{|1 - q^n|^2}, \]  

(7)

the log partition function was also shown to be expressed in the form of the bosonic plethystic exponential \( PEB \) as

\[ Z_{log}(q, \bar{q}) = PEB[\mathcal{G}_1(q, \bar{q})] = \exp \left( \sum_{n=1}^{\infty} \frac{(q^2)^n}{n} \mathcal{G}_n(q, \bar{q}) \right). \]  

(8)

Recently, further results [20] were obtained in terms of properties of the log partition function. On one hand, a structure related to soliton solutions of the KP (Kadomtsev-Petviashvili) hierarchy was found in the log partition function. The KP hierarchy is a completely integrable hierarchy, i.e. a family of partial differential equations which are simultaneously solved, that generalizes the KdV (Korteveg-de-Vries) hierarchy. Using the Sato universal Grassmannian which is the moduli space of the formal power series solutions of the KP hierarchy [21], the log partition function was identified as a \( \tau \)-function of the KP hierarchy. Over the last three decades, a strong interest in the connection between integrable hierarchies and moduli spaces of curves has grown, and \( \tau \)-functions have played an important role in this relationship, starting with Witten’s Conjecture [22] proved by Kontsevich [23], which showed that the \( \tau \)-function of a special solution of the KdV hierarchy generates intersection indices of certain cohomology classes on moduli spaces of curves. On the other hand, the log partition function was found to satisfy the potential form of the Burgers hierarchy [24, 25], which is a family formed by an infinite tower of nonlinear differential equations whose first member is the Burgers equation, originally used to describe the mathematical modelling of turbulence [26].

It was also shown that by generating Hurwitz numbers, the log partition function is a generating function of the combinatorics of branched covering maps. Hurwitz theory studies maps of Riemann surfaces by enumerating analytic functions between the Riemann surfaces [27]. The study of analytic functions on Riemann surfaces translates into the study of the geometry of oriented topological surfaces, with the analytic functions being identified as ramified coverings. The number of such functions fixed by the appropriate set of discrete invariants is counted by Hurwitz numbers. The latter thus enumerate ramified coverings of Riemann surfaces.

Ramified coverings naturally induce monodromy representations, i.e. homomorphisms from the fundamental group of the punctured target surface to a symmetric group. The ramifications at the preimages of a point in the base surface is captured by the cycle type of permutations, making it possible to obtain closed formulae for Hurwitz number in terms of characters of the symmetric group. As a consequence, the work in [20] showed an equivalence between the problem of counting the number of ways a genus zero Riemann surface can be covered \( n \)-times with branch points allowed, and computing the partition function of a gauge theory defined on a Riemann surface, with the symmetric group \( S_n \) as the gauge group, where the moduli space \((\mathbb{C}^2)^n / S_n \) is a Hurwitz space of branched coverings. The log partition function can therefore be described as a Hurwitz \( \tau \)-function, on one hand computing Hurwitz numbers and on the other hand being a \( \tau \)-function of the KP hierarchy. Hurwitz \( \tau \)-functions are a relatively new important subject in theoretical physics, having also appeared in the literature as partition functions for HOMFLY polynomials of some knot in the theory of Ooguri-Vafa (OV) [28, 29, 30]. In particular, for any torus knot, the OV partition function is a \( \tau \)-function of the KP hierarchy [31]. This shows a deep connection between Hurwitz \( \tau \)-functions and 3d Chern-Simons theory, which also appears in this Chern-Simons formulation of a non-unitary three-dimensional gravity.

The purpose of this paper is in the first place to show that the log partition function is a sum of contributions indexed by isomorphic classes of rooted trees. In so doing, our work connects the results obtained in [20] to the Connes-Kreimer Hopf algebra of rooted trees and Feynman diagrams, providing further evidence of the relation between the Hopf algebraic structures introduced by Connes and Kreimer and the integrable structure of \( \tau \)-functions [32]. At the same time, it is shown that Hurwitz numbers appear in the partition function as coefficients in the sum over rooted trees. The second purpose of this paper is to
bring a connection between the aforementioned results and statistical physics models with quenched disorder on trees, as a way for further interpretation of the disorder introduced in AdS$_3$ leading to the theory of log gravity. This paper is organized as described below.

We start by giving a presentation of Hopf algebras in section 2. The goal is to give a background of algebraic concepts that are important in the paper. After giving a definition of Hopf algebra, as an example, we mention the Hopf algebra of differential operators with constant coefficients, which shows that the mere fact of calculating the derivative of a product of two functions can intuitively give some insight on Hopf algebras.

In section 3, we give a combinatorial perspective of the Hopf algebraic structure leading to the reformulation of the logarithmic partition function as a sum over rooted trees. Hopf algebras have been used for a long time in an implicit way in statistical physics and quantum field theory. Indeed, in [33] and [34] for instance, the authors made use of a product called the convolution product in the Hopf language. Since the work of Joni and Rota [35], Hopf algebras have also become a sophisticated tool to formalize combinatorics. Indeed, on one hand, the product and the coproduct can capture the actions of composing and decomposing combinatorial objects respectively, and on the other hand, combinatorial objects such as permutations, trees, graphs, posets, or tableaux have natural gradings which allow them to be endowed with a Hopf algebraic structure, such that many interesting invariants can be expressed as Hopf morphisms. In modern physics, phenomena at very small length scales (i.e. at very large energy scales) are described by quantum field theory (QFT). Despite the many successes of QFT, its mathematical construction is plagued by the difficulty of computing quantities in integral form without incurring infinities. A palliative treatment called renormalization has been applied on the perturbative expansions of divergent iterated Feynman integrals to render renormalized values. While looking for the mathematical structure behind the renormalization method of quantum field theory, a remarkable mathematical interpretation of the perturbative renormalization was discovered by Kreimer [36], in arranging Feynman diagrams of the renormalizable QFT into a Hopf algebra. Subsequent works by Kreimer and Connes led to a reformulation of many quantum field constructions such as renormalization in a Hopf algebraic language, placing Hopf algebras at the heart of the noncommutative approach to geometry and physics [37, 38, 39]. In their approach to the renormalization of perturbative QFT, a major role is played by an algebraic structure defined over a set of Feynman diagrams called the Hopf algebra of rooted trees. The latter has a subalgebra whose generators will be used to derive an expression of the log partition function in terms of rooted trees.

Through a morphism between the Hopf algebra of symmetric functions and the Hopf subalgebra of rooted trees, the log partition function previously shown to have a Schur polynomial expansion [40, 20], is also endowed with the Hopf algebraic structure of rooted trees, and can therefore also be expressed as a sum over rooted trees. In section 4, we derive an explicit expression of the log partition function in terms of rooted trees indexed by Hurwitz numbers.

In section 5, we discuss how the derived expression of $Z_{log}$ describes a count of permutations on trees, as well as the relationship between the findings in section 4, and our formerly obtained results [20]. In particular, through Hurwitz theory, we note the one-to-one correspondence between maps of rooted trees and covering maps of Riemann surfaces.

In section 6, we draw a parallel between our findings and results well established in the statistical physics literature, concerning disordered systems on trees. In the particular case of the directed polymer on a tree with disorder, it has been shown that the study of such a disordered system reduces to that of nonlinear partial differential equations admitting travelling wave solutions [41]. The analogy is of interest in view of a further elucidation of phenomena associated to the disorder introduced in AdS$_3$ as observed in [11].

In section 7, we give a summary and outlook.
2 Hopf algebras

2.1 Hopf algebra résumé

Following [42], we briefly review the definition of a Hopf algebra. An associative complex algebra \( A \) with unit element \( 1 \) is a structure given for all \( a, b \in A \) and all complex numbers \( z \) by the following maps

\begin{align*}
\text{Multiplication map: } & f(a \otimes b) = ab, & (9a) \\
\text{Unit map: } & g(z) = z1, & (9b) \\
\text{Identity map: } & \text{id}(a) = a. & (9c)
\end{align*}

Using linear extensions, the \( \mathbb{C} \)-linear maps of the algebra \( A \) can be written

\[ f : A \otimes A \rightarrow A, \quad g : \mathbb{C} \rightarrow A, \quad \text{id} : A \rightarrow A, \]

and provided with associativity and unity properties

\begin{align*}
\text{Associativity: } & f(f \otimes \text{id}) = f(\text{id} \otimes f), & (11a) \\
\text{Unity: } & f(g \otimes \text{id}) = f(\text{id} \otimes g) = \text{id}. & (11b)
\end{align*}

The associativity relation \((11a)\) follows from the associative law \( a(bc) = (ab)c \) for all \( a, b, c \in A \), and one can write

\[ f(\text{id} \otimes f)(a \otimes b \otimes c) = f(a \otimes (b \otimes c)) = f(a \otimes bc) = a(bc). \] (12)

Similarly \( f(f \otimes \text{id})(a \otimes b \otimes c) = (ab)c \), which proves the associativity. The unity relation \((11b)\) follows from

\[ f(g \otimes \text{id})(1 \otimes a) = f(g(1) \otimes a) = f(1 \otimes a) = 1a = a. \] (13)

In the same way, \( f(\text{id} \otimes g)(a \otimes 1) = a1 = a \), and the relation \((11b)\) is then proved by identifying \( a \otimes 1 \) and \( 1 \otimes a \) with \( a \), which corresponds to the isomorphisms \( A \otimes \mathbb{C} = A = \mathbb{C} \otimes A \).

The associativity and unity can be endowed with a dual by replacing the symbols \( f \) and \( g \) by \( \Delta \) and \( \varepsilon \) respectively, yielding the dual relations

\begin{align*}
\text{Coassociativity: } & (\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta, & (14a) \\
\text{Counity: } & (\text{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \text{id})\Delta = \text{id}. & (14b)
\end{align*}

The associative unital complex algebra \( A \) becomes a complex bialgebra if there exist two maps

\begin{align*}
\text{Comultiplication: } & \Delta : A \rightarrow A \otimes A, & (15a) \\
\text{Counit: } & \varepsilon : A \rightarrow \mathbb{C}, & (15b)
\end{align*}

satisfying the coassociativity and the counity. The complex bialgebra is called a Hopf algebra if there exists a linear map \( S : A \rightarrow A \) such that

\[ f(S \otimes \text{id})\Delta = f(\text{id} \otimes S)\Delta = g\varepsilon. \] (16)

For all \( a \in A \), the coproduct \( \Delta a \) is contained in the tensor product \( A \otimes A \). Therefore, there are elements \( a_1, \ldots, a_n, b_1, \ldots, b_n \) such that

\[ \Delta a = \sum_{k=1}^{n} a_k \otimes b_k. \] (17)
Using the so-called Sweedler notation, Eq. (17) takes the form
\[
\Delta a = \sum_{a} a_{(1)} \otimes a_{(2)}.
\]
Furthermore, for all \(a, b \in A\), one can write
\[
\Delta a \Delta b = \Delta(ab), \quad \Delta 1 = 1 \otimes 1.
\]

We make the distinction between two coproducts. When \(A\) is a Lie algebra type, the coproduct of the Hopf algebra reads \(\Delta a = a \otimes 1 + 1 \otimes a\), and is usually called a primitive coproduct. When \(A\) is a group algebra, the coproduct of the Hopf algebra reads \(\Delta a = a \otimes a\), and is also called a group-like coproduct.

Let \(\mathbb{C}[t]\) be the ring of formal series on \(\mathbb{C}\), and \(\mathbb{C}[t^{-1}, t]\) the field of Laurent series on \(\mathbb{C}\). A graded Hopf algebra is the direct sum of vector spaces
\[
A = \bigoplus_{n \geq 0} A_n
\]
endowed with a product \(f : A \otimes A \mapsto A\), a coproduct \(\Delta : A \mapsto A \otimes A\), a unit \(\varepsilon : A \mapsto \mathbb{C}\) and an antipode \(S : A \mapsto A\) fulfilling the usual axioms of a Hopf algebra, and such that
\[
A_p A_q \subset A_{p+q}, \quad \Delta(A_n) \subset \bigoplus_{p+q=n} A_p \otimes A_q, \quad S(A_n) \subset A_n
\]

Finally, a graded Hopf algebra \(A\) is connected if \(A_0\) is one-dimensional, i.e. \(A \cong \mathbb{C}\). The Hopf algebras considered in this paper are graded.

2.2 An example: the Hopf algebra of differential operators

Fix the dimension \(N = 1, 2, \ldots\), then denote the points of the space \(\mathbb{R}^N\) by \(x = (x_1, \ldots, x_N)\) the space of smooth complex-valued functions \(f : \mathbb{R}^n \mapsto \mathbb{C}\). The algebra \(A\) of linear differential operators with respect to \(N\) arguments and constant coefficients is generated by the partial derivatives \(\partial_i = \partial/\partial x_i\) [43]. The product emerges naturally when \(A\) acts on functions. For smooth functions \(f\), the product is the composition of derivatives \(\partial_i \partial_j f = \partial_j \partial_i f\), for \(i, j = 1, \ldots, N\). Furthermore, \(A\) can be endowed with the unit element \(1\) such that, for any \(D \in A\), \((1D)f = (D1)f = Df\). Thus \(A\) is an associative, commutative, and unital complex algebra. In order to make it a Hopf algebra, three objects are required: the coproduct, the counit and the antipode. The coproduct appears when the action of \(D\) on a product of smooth functions \(fg\) is considered. For instance
\[
1(fg) = fg = (1f)(1g),
\]
\[
\partial_i(fg) = (\partial_i f)g + f(\partial_i g) = (\partial_i f)(1g) + (1f)(\partial_i g).
\]
By convention, writing the action of an element \(D\) of \(A\) on \(fg\) as \(D(fg) = \sum (D_{(1)}f) (D_{(2)}g)\) where \(D_{(1)}, D_{(2)} \in A\), the coproduct arises by omitting reference to the functions \(f\) and \(g\), and defining the linear map \(\Delta A \mapsto A \otimes A\) thus obtaining the so-called Sweedler notation \(\Delta D = D_{(1)} \otimes D_{(2)}\). In this case, Eqs. (22) give
\[
\Delta 1 = 1 \otimes 1,
\]
\[
\Delta \partial_i = \partial_i \otimes 1 + 1 \otimes \partial_i.
\]
The mathematical correspondence of the processes of fusion and splitting of physical states observed in nature is the product and coproduct of Hopf algebras, respectively. Here, the coproduct can then be seen as a procedure to split a differential operator \(D\) into two operators \(D_{(1)}\) and \(D_{(2)}\) such that \(D_{(1)}D_{(2)} = D\). A pictorial representation of the process is given below
An iterated application yields of the product rule in Eq. \((22b)\) yields

\[
\partial_i \partial_j (fg) = (\partial_i \partial_j f)g + \partial_i f \partial_j g + \partial_i f \partial_j g + f \partial_i \partial_j g,
\]

from which we get

\[
\Delta(\partial_i \partial_j) = (\partial_i \partial_j \otimes 1)(fg) + (\partial_j \otimes \partial_i)(fg) + (\partial_i \otimes \partial_j)(fg) + (1 \otimes \partial_i \partial_j)(fg).
\]

Hence, the coproduct

\[
\Delta(\partial_i \partial_j) = \partial_i \partial_j \otimes 1 + \partial_j \otimes \partial_i + \partial_i \otimes \partial_j + 1 \otimes \partial_i \partial_j
\]

(26)
corresponds to the four possible ways of splitting of the product \(\Delta(\partial_i \partial_j)\). Using the more compact Sweedler notation, if \(\Delta D = D_{(1)} \otimes D_{(2)}\) and \(\Delta D' = D'_{(1)} \otimes D'_{(2)}\), then the relation between the product and the coproduct in \(\mathcal{A}\) is \(\Delta(DD') = (\Delta D) (\Delta D') = \sum \sum (D_{(1)} D'_{(1)}) \otimes (D_{(2)} D'_{(2)})\).

The coassociativity of the coproduct \(\Delta\) comes from the associative law \(f(gh) = (fg)h\) for functions \(f, g, h\), from which follows \(D(f(gh)) = D((fg)h)\). In turn, from \(D(fgh) = \sum (D_{(1)} f) (D_{(2)} g) (D_{(3)} h)\), one can write

\[
\sum D_{(1)} \otimes D_{(2)} \otimes D_{(3)} = (\Delta \otimes \text{id}) \Delta D = (\text{id} \otimes \Delta) \Delta D.
\]

(27)
The remaining ingredients for \(\mathcal{A}\) to be defined as a Hopf algebra are the counit \(\varepsilon\) and the antipode \(S\). In the case of the algebra of differential operators the counit can be defined as the linear map \(\varepsilon : \mathcal{A} \rightarrow \mathbb{C}\) such that \(D1 = \varepsilon(D)1\), where 1 is the constant function 1. From this definition we deduce that \(\varepsilon(1) = 1\) (where 1 here is the real number 1), and \(\varepsilon(\partial_{i_1} \cdots \partial_{i_n}) = 0\) for \(n > 0\). It can be verified that \(\varepsilon\) indeed satisfies the defining property of a counit as \(D = \sum \varepsilon(D_{(1)}) D_{(2)} = D_{(1)} \varepsilon(D_{(2)})\). Finally, the antipode is a linear map \(S : \mathcal{A} \rightarrow \mathcal{A}\) such that \(\sum S(D_{(1)}) D_{(2)} = D_{(1)} S(D_{(2)}) = \varepsilon(D)1\). It can be verified that \(S(1) = 1\) and \(S(\partial_{i_1} \cdots \partial_{i_n}) = (-1)^n (\partial_{i_1} \cdots \partial_{i_n})\) for \(n > 0\).

The Hopf algebra of differential operators can be quite effective in practice. It has for instance been used to describe the hierarchy of Green functions in quantum field theory [44].

3 Feynman graphs and Hopf (sub)algebra of rooted trees

Graphical techniques using trees have been used in many algebraic constructions leading to important developments in various fields. In high energy physics particularly, a groundbreaking achievement in connection with the process of renormalization in perturbative quantum field theory was reached when Kreimer realized that the underlying structure of BPHZ renormalization is captured by a Hopf algebra of combinatorial nature [36]. This accomplishment opened the way to a fertile interaction between mathematics and physics, marked by the seminal work of Connes and Kreimer, who introduced the Connes-Kreimer Hopf algebra of rooted trees thus giving a sturdy algebraic framework for the BPHZ renormalization [37].

The intricacy of renormalization can be considered in the following way [45, 43]: an integral is attached to certain graphs called Feynman graphs, according to certain rules called the Feynman rules (see Fig. 1).
These integrals turn out to be divergent, because of the presence of loops in the Feynman graphs; each loop creates a subdivergence in the associated integral (see Fig. 2).

The renormalization procedure [46] is used to make sense of these integrals. In order to renormalize the Feynman graphs associated to integrals, one must remove the subdivergences in a complicated way known as Zimmermann's forest formula [12]. In 1998, Kreimer discovered [36] that this formula could be understood as a Hopf algebra over rooted trees. Then, in the Connes-Kreimer algebraic setting, the renormalization consists in associating to each Feynman graph a rooted tree that describes the structure of the subdivergences of the graph (see Fig. 3).

After a regularization step, the Feynman rules induce an algebra morphism from the algebra of rooted trees to the algebra of functions. Through algebra morphism, it is eventually possible to express the log partition function as a sum over rooted trees.

### 3.1 Hopf algebra of rooted trees

We introduce fundamental definitions concerning rooted trees and their (Connes-Kreimer) Hopf algebra. A (non planar) rooted tree is either the empty set, or a finite connected oriented graph without loops in
which every vertex has exactly one incoming edge, except for a distinguished vertex (the root) which has no incoming edge, but only outgoing edges. The set of edges and vertices of a rooted tree $T$ is denoted $E(T)$ and $V(T)$, respectively. Let $T_R$ be the set of (isomorphism classes of) rooted trees. We list below all elements of this set up to degree 4

\[
\ldots, \quad 1, \quad 1, V, \quad \{1, V, V\}, \quad \{1, V, V, V\}\]

The commutative, unital, associative $\mathbb{C}$-algebra of rooted trees $A_R$ is the polynomial algebra generated by the symbols $T$, one for each isomorphism class of rooted trees. The unit denoted by 1 is the empty tree, and the product of rooted trees is written as the concatenation of their symbols. The grading of $A_R$ is defined in terms of the number of vertices of $\#(T) = |V(T)|$, which is extended to monomials (i.e. products of rooted trees) also called rooted forests, by $\#(T_1 T_2 \cdots T_n) = \sum_{i=1}^n \#(T_i)$, turning $A_R = \bigoplus_{n \geq 0} A_R^{(n)}$ into a graded, connected, unital, commutative, associative $\mathbb{C}$-algebra.

The Connes-Kreimer Hopf algebra $H_R = (A_R, \Delta, \epsilon)$ is the algebra $A_R$ endowed with the counit $\epsilon : H_R \rightarrow \mathbb{C}$ defined by $\epsilon(1) := 1$ and $\epsilon(T_1 T_2 \cdots T_n) = 0$ if $T_1 T_2 \cdots T_n$ are trees, as well as the coproduct $H_R \rightarrow H_R \otimes H_R$ defined in terms of admissible cuts on the rooted trees as follows \[\text{(28)}\]. First consider a rooted forest $F$, and impose a path from a root to $y$ passing through $x$. The set $V(F)$ of vertices of the forest $F$ is then endowed with a partial order defined by $x \leq y$. Any subset $B$ of $V(F)$ defines a subforest $F|_B$ of $F$ by keeping the edges of $F$ which link two elements of $B$. The coproduct is then defined by

$$\Delta(F) = \sum_{A:B=V(F)} F|_A \otimes F|_B.$$

A couple $(A, B)$ is also called an admissible cut, with crown (or pruning) $F|_A$, and trunk $F|_B$. In \[\text{(29)}\], the coproduct of $H_R$ was given as

$$\Delta 1 := 1 \otimes 1, \quad \Delta T = T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T),$$

where $C(T)$ is the list of admissible cuts, $P_c(T)$ is the pruning, i.e. the subforest formed by the edges above the cut $c$, and $R_c(T)$ is the subforest formed by the edges under $c$. For instance

$$\Delta (\{\}) = 1 \otimes 1 + 1 \otimes 1 + \ldots.$$ \[\text{(30)}\]

$$\Delta (\{V\}) = V \otimes 1 + 1 \otimes V + 2 \cdot 1 \otimes \ldots.$$ \[\text{(31)}\]

The linear operator $B_+ : H_R \mapsto H_R$ also known as the grafting operator is a map that takes any forest to a tree, by connecting the roots in the monomials of rooted trees making the forest to a new adjoined root. For example

$$B_+ (\{\}) = V$$ \[\text{(32)}\]

3.2 The ladder tree Hopf algebra

The Connes-Kreimer Hopf algebra of rooted trees contains a subalgebra called the Hopf algebra of rooted ladder trees denoted $H_L$ \[\text{[48, 49, 50]}\], which is generated by rooted ladder trees

\[
\ldots, \quad 1, \quad \{1, V, V\}, \quad \{1, V, V, V\}, \quad \ldots
\]
Denoting the ladder trees with \( n \) vertices in the above type of linearly ordered set by \( l_n \in H_L \subset H_R \), the coproduct of \( H_L \) then becomes

\[
\Delta(l_n) = \sum_{i=0}^{n} l_i \otimes l_{n-i}. \tag{33}
\]

This is a commutative, cocommutative Hopf algebra, isomorphic to the Hopf algebra of symmetric functions.

Having established the necessary background, we show in the next section how the log partition function generates rooted trees indexed by Hurwitz numbers.

4 Hurwitz numbers and maps of rooted trees

A Hurwitz number counts the number of non-equivalent branched coverings of a surface with a prescribed set of branch points and branched profile. Although branched coverings first appeared in [51], their enumeration was studied in a systematic way by Hurwitz who observed that the counting of branched coverings could be interpreted in terms of permutation factorizations [52, 53]. Ever since, Hurwitz numbers have been an important subject in mathematics and physics, with an enormous amount of literature dedicated to them [54, 55, 56, 57, 58, 59, 60, 61]. They have been found notably in the context of string theory after a crucial observation made in [62, 63] from which many works followed, in integrable systems with early works [64, 65], or in matrix models [66, 67, 68, 69, 70].

Recalling the definition of Hurwitz numbers [27], let \( Y \) be a connected Riemann surface of genus \( g \). Define the set \( B = \{y_1, \ldots, y_d\} \in Y \), and let \( \lambda_1, \ldots, \lambda_d \) be partitions of the positive integer \( n \). Then the Hurwitz number can be defined as the sum

\[
\mathcal{H}_{X \to Y}(\lambda_1, \ldots, \lambda_d) = \sum_{|f| |\text{Aut}(f)|} \frac{1}{|\text{Aut}(f)|} \tag{34}
\]

that runs over each isomorphism class of \( f : X \mapsto Y \) where

1. \( f \) is a holomorphic map of Riemann surfaces;
2. \( X \) is connected and has genus \( h \);
3. the branch locus of \( f \) is \( B = \{y_1, \ldots, y_d\} \);
4. the ramification profile of \( f \) at \( y_i \) is \( \lambda_i \).

Hurwitz numbers arise in two different flavors, depending on whether the covering space \( X \) of \( Y \) is connected or not. Although we will start with the above definition of the connected Hurwitz number, our focus is on the disconnected theory, so we also give the general formula for disconnected Hurwitz numbers, mentioning beforehand that we will restrict our attention to the target space with genus \( g = 0 \). The problem at hand is then attacked by using the representation theory of the symmetric group.

Let \( \lambda_1, \ldots, \lambda_d \) be partitions of the positive integer \( n \). Recall from the representation theory of the symmetric group that \( \mathfrak{S}(\mathbb{C}[S_n]) \) is a vector space with dimension equal to the number of partitions of \( n \) and basis indexed by conjugacy classes of permutations. Denoting the basis element associated to the corresponding conjugacy class by \( C_{\lambda_i} \) for every \( i \in [1; d] \), the genus zero disconnected Hurwitz number takes the form

\[
\mathcal{H}^d_{X \to \emptyset}(\lambda_1, \ldots, \lambda_d) = \frac{1}{n!} [C_e] C_{\lambda_d} \cdots C_{\lambda_2} C_{\lambda_1}, \tag{35}
\]

where \([C_e] C_{\lambda_d} \cdots C_{\lambda_2} C_{\lambda_1}\) is the coefficient of \( C_e = \{e\} \) after writing the product as a linear combination of the basis element \( C_{\lambda_i} \).

After restricting the genus of the base and target Riemann surfaces to be zero, we further impose \( d = 2 \) and \( \lambda_1 = \lambda_2 = (n) \). The expression of connected Hurwitz numbers becomes
As a subalgebra of the Hopf algebra of rooted trees $H_R$, the ladder tree Hopf algebra $H_L$ is also known to be isomorphic to the Hopf algebra of polynomials [48]. Let us consider the $\mathbb{C}$-algebra of polynomials in $n$ variables $\mathbb{C}[G_1, \ldots, G_n]$. According to the plethystic program [71], the Hilbert series

$$G(q, \bar{q}) = \frac{1}{(1 - q)(1 - \bar{q})}$$

is the generating function of basic single-trace invariants of the theory. To find the multi-trace invariants, i.e. the unordered products of the single-trace invariant objects, we take the plethystic exponential

$$Z(q, \bar{q}) = \text{PE} \{G(q, \bar{q})\} = \exp \left\{ \sum_{n=1}^{\infty} \frac{G_n(q, \bar{q})}{n} \right\} = \prod_{k,l} \frac{1}{1 - q^k \bar{q}^l}.$$  

Due to the fact that $H_L$ is isomorphic to $\mathbb{C}[G_1, \ldots, G_n]$, there is a one-to-one correspondence between the coordinates $G_1, \ldots, G_n$ and the generators of $H_L$ which we will denote $l_1, \ldots, l_n$, obtained by making use of iterated action of the operator $B_+$ on the empty tree $1$, the first three of which read

$$l_1 = B_+ (1) = 1, \quad l_2 = B_+ (B_+ (1)) = 1, \quad l_3 = B_+ (B_+ (B_+ (1))) = 1.$$ 

This implies that the sum $F(G_1, \ldots, G_n)$ that is subjected to exponentiation in Eq. (38) can be expressed as a sum of ladder rooted trees

$$F(l_1, \ldots, l_n) = \sum_{n=1}^{\infty} \frac{1}{n} l_n = 1 + \frac{1}{2} 1 + \frac{1}{3} 1 + \frac{1}{4} 1 + \frac{1}{5} 1 + \cdots.$$  

From Eq. (36), the first part of our derivation is

$$F(l_1, \ldots, l_n) = \sum_{n=1}^{\infty} \frac{1}{n} l_n = \sum_{n=1}^{\infty} \left[ H_{0 \to 0} ((n), (n)) \right] l_n,$$

where we see that the connected Hurwitz numbers $H_{0 \to 0} ((n), (n))$ are the coefficients of the generators of the Hopf algebra of ladder rooted trees $H_L$.

Because connected and disconnected Hurwitz generating functions are related by exponentiation, disconnected Hurwitz numbers appear in the $(q^2)$ parameter-inserted version of the exponentiation of the function $F(l_1, \ldots, l_n)$ that yields $Z_{log}$ as a generating function of rooted trees. The derivation is as follows.

Let $n = 1, 2, \ldots$ and $k = 1, \ldots, n$. Moreover, let $p(n, k)$ denote the tuple of nonnegative integer solutions $j := j_1, \ldots, j_n$ of the system

$$\left\{ \begin{array}{l} j_1 + j_2 + \cdots + j_n = k, \\ j_1 + 2j_2 + \cdots + nj_n = n. \end{array} \right.$$  

The generalized binomial coefficient defined as

$$\binom{n}{j_1, \ldots, j_k} = \frac{n!}{j_1! j_2! \cdots j_n! (1!)^{j_1} (2!)^{j_2} \cdots (n!)^{j_n}}.$$
can be interpreted in terms of partitions by considering partitions of the set \( \{1, 2, \ldots, n\} \) into \( k \) blocks of \( j_i \) elements subsets such that the system (41) holds. Then the number of all partitions of this type is equal to the binomial coefficient. Then the partition function

\[
Z_{\text{log}} (l_1, \ldots, l_n) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{k=1}^{n} \left( \frac{1}{n!} \int_{j_1, \ldots, j_k} \right) B_+ \left( \frac{t_{i_1}^1 t_{i_2}^2 \cdots t_{i_n}^n}{\mathcal{L}^n} \right) \right) (q^n)^n
\]

(43a)

with its final expression as

\[
Z_{\text{log}} (l_1, \ldots, l_n) = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \left( \frac{1}{n!} \int_{j_1, j_2, \ldots, j_n} \right) \frac{1}{(1)^{j_1} (2)^{j_2} \cdots (n)^{j_n}} B_+ \left( \frac{t_{i_1}^1 t_{i_2}^2 \cdots t_{i_n}^n}{\mathcal{L}^n} \right) \right) (q^n)^n,
\]

(43b)

and the disconnected Hurwitz numbers expressed as

\[
H^*_{0, a_0} \left( (1)^{j_1} (2)^{j_2} \cdots (n)^{j_n}, (1)^{j_1}, (2)^{j_2}, \ldots, (n)^{j_n} \right) = \frac{1}{j_1! j_2! \cdots j_n! (1)^{j_1} (2)^{j_2} \cdots (n)^{j_n}},
\]

(45)

where the \((1)^{j_1}, (2)^{j_2}, \ldots\) associated to the trees \( B_+ \left( \frac{t_{i_1}^1 t_{i_2}^2 \cdots t_{i_n}^n}{\mathcal{L}^n} \right) \) are such that \([i]^{j_i} = 1, \ldots, i\).

As an example, the set \( \{1, 2, 3\} \) has five partitions. Three of these have two blocks, namely \( \{1, 2\} \{3\}, \{1, 3\} \{2\} \) and \( \{2, 3\} \{1\} \). With data \( k = 2, j_1 = j_2 = 1 \), we associate the rooted tree

\[
\sqrt{\hfill}
\]

to each of them. Then the Hurwitz number can be computed as

\[
H^*_{0, a_0} \left( (1, 2) \right) = \frac{1}{3} \cdot \frac{1}{111(1)^2} = \frac{1}{2}.
\]

(46)

This result can be obtained using Eq. (35) known in the literature, by considering the basis element \( C_{(1, 2)} = (12) + (13) + (23) \) of the class algebra \( \mathcal{H} (\mathbb{C} [S_3]) \). Then

\[
[(12) + (13) + (23)][(12) + (13) + (23)] = 3e + 3(123) + 3(132) = 3C_e + 3C_{(3)},
\]

(47)

and

\[
H^*_{0, a_0} \left( (1, 2) \right) = \frac{1}{3} \cdot 3 = \frac{1}{2}.
\]

(48)

Similarly, one of the five partitions of the set \( \{1, 2, 3\} \) has three blocks, namely \( \{1\} \{2\} \{3\} \). With data \( k = 3, j_1 = 3, j_2 = j_3 = 0 \), we associate to the three-block partition the rooted tree

\[
\sqrt{\hfill} \cdot \sqrt{\hfill}
\]

and the corresponding Hurwitz number is computed as

\[
H^*_{0, a_0} \left( (1, 1, 1) \right) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{6}.
\]

(49)

Finally, one of the five partitions of the set \( \{1, 2, 3\} \) has one block, namely \( \{1, 2, 3\} \) itself. With data \( k = 1, j_1 = j_2 = 0, j_3 = 1 \), we associate to the one-block partition the rooted tree

\[
\sqrt{\hfill}
\]
and the corresponding Hurwitz number is computed as

$$H_{0^\infty0}^\bullet (3, (3)) = \frac{1}{3}. \quad (50)$$

The last case illustrates the general fact that when $k = 1$, i.e for one block partitions,

$$H_{0^\infty0}^\bullet ((n), (n)) = H_{0^\infty0}^\bullet ((n), (n)). \quad (51)$$

5 From covering maps to maps of rooted trees, counting permutations

In [72], a theory of universal covers for posets was developed. In particular, considering a partially ordered set $P$ to be the poset of rooted trees, the map $\pi : \tilde{P} \to P$ from the universal cover of $P$ denoted $\tilde{P}$ to $P$ was developed, such that the rank-1 elements of $P$ are permutations $\rho = s_1 s_2 \cdots s_n$ of $\{1, 2, \ldots, n\}$ associated with labelled rooted trees, and the map $\pi(\rho)$ is just the rooted tree obtained by forgetting the labels. In the theoretical physics literature, on one hand, holomorphic covering maps have been associated to Feynman diagrams in proposals of worldsheet duals for AdS spaces, where string worldsheets corresponding to covering maps are related to gauge theory Feynman diagrams through the Strebel parametrization of the moduli space of Riemann surfaces, which allows an interpretation of Feynman diagrams in terms of moduli spaces of Riemann surfaces (see [73] for early work on the subject). On the other hand, simple Hurwitz spaces have also been defined as the space of holomorphic maps from worldsheet to target space [67], and it has been shown that the Riemann surfaces appearing as covering spaces, and equivalently the Feynman diagrams corresponding to Hurwitz classes consist of string worldsheets. This shows that Hurwitz numbers have a very natural interpretation in terms of a string worldsheet. The derivation of $Z_{\log}$ as a sum of rooted trees presented in the previous section is inspired by these works to show that the Hurwitz numbers generated by $Z_{\log}$ not only count Riemann surfaces but also enumerate maps of rooted trees. In fact, regardless of these different objects, what are really being counted are permutations. Below we take a look at permutations on trees, as they apply to the counting generated by $Z_{\log}$.

From the tree-level derivation of $Z_{\log}$, we see that the trees of the log partition function are composed of ladder trees whose roots are connected to an added common root. This turns the ladder trees into subtrees of the trees with an added root. The next step is to introduce symmetry groups on the trees. We then introduce the group of permutations of $\{1, 2, \ldots, n\}$ in the vertices of the ladder trees, by assigning a label between 1 and $n$ to each vertex, and apply permutations on the labels, such that each ladder tree carries a cycle decomposition of the permutations. We only consider isomorphic classes of rooted trees, and choose a representation in each isomorphism class. As an example, the trees

\[ \begin{array}{ccc}
1 & 2 & 3 \\
\text{,} & \text{,} & \text{,} \\
4 & 3 & 2 \\
\text{,} & \text{,} & \text{,} \\
1 & 2 & 3
\end{array} \]

represent the same rooted tree. Then, forgetting the labels, the series

$$Z_n (l_1, \ldots, l_n) = \sum_{k=1}^n \left[ H_{0^\infty0}^\bullet (([1]^{j_1}, [2]^{j_2}, \ldots), ([1]^{j_1}, [2]^{j_2}, \ldots)) \right] B_+ \left( \prod_{i=1}^n l_i^{j_i} \right) \quad (52)$$

is a generating function that enumerates the $S_n$ permutations associated to the isomorphic classes of labelled rooted trees. To illustrate this, we consider below, the cases for $n \in [2, 4]$.

For $n = 2$, we consider the permutations $\{(1)(2), (12)\}$ of labelled vertices in the subtrees to give
and we write $Z_2$ as

$$Z_2(\star, \star) = \frac{1}{2!} \left( \mathcal{V} + \mathcal{I} \right).$$  \hspace{1cm} (53)$$

For $n = 3$, we consider the permutations $\{(1)(2)(3), (12)(3), (13)(2), (23)(1), (123), (132)\}$ of labelled vertices in the subtrees to give

and we write $Z_3$ as

$$Z_3(\star, \star, \star) = \frac{1}{3!} \left( \mathcal{V} + 3 \mathcal{V} + 2 \mathcal{I} \right).$$ \hspace{1cm} (54)$$

For $n = 4$, the subtrees' labelled vertices associated to the permutation elements of $S_4$ are

and we write $Z_4$ as

$$Z_4(\star, \star, \star, \star) = \frac{1}{4!} \left( \mathcal{V} + 6 \mathcal{V} + 4 \mathcal{V} + 4 \mathcal{I} \right).$$
and we write $Z_4$ as

$$Z_4 \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right) = \frac{1}{4!} \left( \begin{array}{c} \downarrow \\ 6 \\ \downarrow \\ 3 \\ \downarrow \\ 8 \\ \downarrow \\ 6 \end{array} \right).$$

(55)

This can easily be verified to be consistent at higher order in $n$.

The present work nicely fits with recently obtained results [20] where, in order to specify the genus zero Hurwitz numbers that feature in the log partition function, we considered the map $f : (C^2)^n \mapsto (C^2)^n / S_n$, and related these Hurwitz numbers to combinatorial properties of the symmetric group $S_n$ by describing an appropriate Hurwitz cover with monodromies specified by $S_n$-permutations of the $n$-sheets of Riemann surfaces. The Hurwitz numbers were therefore equivalent to the monodromies weighted by the inverse of the number of automorphisms of the Hurwitz cover, i.e $n!$. We can see from the present work that if we now consider isomorphic classes of rooted trees, according to a specific choice of permutations in the subtrees forming the rooted trees, the same Hurwitz numbers appear now as the sum of these isomorphic classes of rooted trees weighted by the inverse of the number of automorphisms of the Hurwitz cover. This highlights the fact that via Hurwitz theory, ramified coverings naturally provide an interesting correspondence between trees and (genus zero) Riemann surfaces. At level $n = 2$ for instance, this correspondence is pictured in the figures below.

Figure 4: Riemann surfaces and rooted tree associated to permutation (1)(2) $\in S_2$

In Fig. (4), the representations of covering maps are associated to the identity permutation expressed in cycle notation as (1)(2), and in Fig. (5), the branched covering representations are associated with the nontrivial (12) permutation.
6 Trees, traveling waves and disorder

Beyond the combinatorial aspect of the work exposed above, an important objective is also to bring a
connection between the introduction of disorder in $\text{AdS}_3$ as observed in [11] and the theory of disorder on
trees.

From the previous sections, it appears that the partition function $Z_{\text{log}}$ describes a Fock space geometry
living on rooted trees. This result, coupled to the fact that $Z_{\text{log}}$ is also a $\tau$-function of the KP integrable
hierarchy of nonlinear partial differential equations is of interest in analogy with works that have appeared
in the statistical physics literature concerning disordered models defined on trees, and their relations to
traveling waves. A specific case is given by the directed polymer in a random medium, whose discrete version
is formulated with the lattice taken to be the Cayley tree $[74, 75]$. Such a system, away (yet not far) from thermal equilibrium exhibit a time-scale hierarchical structure with quenched randomness
at the microscopic level, and its study can be reduced to the classical statistical mechanics problem of a
one-dimensional string-like object, the directed polymer on the Cayley tree (DPCT).

In this particular example, it was discovered that traveling waves appear in disordered models on trees
[41], where the Cayley tree is closely connected to traveling wave solutions of a certain nonlinear partial
differential equation called the Kolmogorov-Petrovsky-Piscounov (KPP) equation (also called the Fisher
equation) $[81]$.

Our work on log gravity has been driven by the desire to have a better understanding of the combinatorics
of the multi-particle excitations of the logarithmic partner, and by the resolution to unveil hidden phenomena
in the theory, encoded by the partition function. With respect to that, we also note that representation
of trees in the Fock space of multiparticle states has also appeared in the literature $[82]$. The analogies
mentioned in this section highlight nontrivial correspondences between log gravity and aspects of statistical
physics related to disorder systems, which deserve further investigation.

7 Summary and outlook

In this work, we gave a tree-level description of the logarithmic contribution of the partition function of
topologically massive gravity at the critical point. Through the isomorphism between the symmetric algebra
and the Hopf algebra of rooted ladder trees, we showed that $Z_{\text{log}}$ can be expressed as a sum over rooted
trees indexed by Hurwitz numbers. It was also shown how the count of factorizations of permutations by
the Hurwitz numbers appears on the rooted trees.

Our work illustrates the relation between the algebraic structures introduced by Connes and Kreimer,
and integrability. The thread running through this relationship is that the global form of the KP hierarchy
appearing in the log partition function is given by the Hirota equations $[83]$, whose bilinearity is related
to the properties of comultiplication and to the basic relation $\Delta(g) = g \otimes g$ for group elements $[84]$. $Z_{\text{log}}$
therefore provides additional evidence of the nontrivial integrable structure behind the standard formalism
of perturbative quantum field theory.

The relationship between rooted trees and Hurwitz numbers has been discussed in various places in the
mathematical literature $[85, 86, 87, 88, 89, 90]$. A mention of the relationship has recently appeared in
theoretical physics, in the study of a non-compact topological symmetric orbifold conformal field theory
$[91]$. From a holographic perspective, it would be interesting to further probe what appears as a topological
subsector of non-unitary holography.

The partition function of the log sector of TMG at the critical point is at the confluence of many theories,
among which nonunitary gravity, the plethystic programme, integrable hierarchies of soliton equations and
$\tau$-functions, Hurwitz theory and branched covering of Riemann surfaces and the Connes-Kreimer Hopf
algebra of rooted trees and Feynman diagrams. The links between on one hand integrable hierarchies, Schur polynomials, Hurwitz numbers and matrix models [66, 67, 68, 69, 70, 92, 93, 94], and on the other hand the work of Connes and Kreimer within the formalism of [32] well suited for applications to matrix models [95, 96], naturally brings us to question whether the fact that all the aforementioned objects that appear in the log partition function can lead to a matrix model interpretation of the counting problem in the log sector. We hope to shed some light on this matter in the future.

A more physical motivation for this work is given by the analogy between our results and the problem of the directed polymer on a tree with disorder, a notable type of disordered system which can be reduced to the study nonlinear partial differential equations that admit travelling wave solutions. The analogy comes from the fact that a hierarchical tree-like structure captured by $Z_{\text{log}}$ encodes the geometry of the Fock space of multiparticle states, bringing a relationship between trees in a disordered landscape and traveling wave solutions (of KP solitonic type in our case). This calls our attention to the fact that the sector of the theory counted by $Z_{\text{log}}$ appears as a random medium with partial equilibrium, and suggests that there might be a need to incorporate a discussion about nonequilibrium phenomena in log gravity. We would like to report on this in the near future.

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