Abstract

Relativistic resonances and decaying states are described by representations of Poincaré transformations, similar to Wigner’s definition of stable particles. To associate decaying state vectors to resonance poles of the $S$-matrix, the conventional Hilbert space assumption (or asymptotic completeness) is replaced by a new hypothesis that associates different dense Hardy subspaces to the in- and out-scattering states. Then one can separate the scattering amplitude into a background amplitude and one or several “relativistic Breit-Wigner” amplitudes, which represent the resonances per se. These Breit-Wigner amplitudes have a precisely defined lineshape and are associated to exponentially decaying Gamow vectors which furnish the irreducible representation spaces of causal Poincaré transformations into the forward light cone.
1 Introduction

Stable quantum mechanical states are characterized by one real number – the energy $E_{n,j}$ or, in the relativistic case, the mass $m_{n,j}$ – in addition to definite values of discrete quantum numbers $n$ such as charge, isospin (particle species labels) and by angular momentum or spin (parity) $j$. Quasistable states $D$ are characterized by a pair of real numbers, in addition to the same discrete quantum numbers. For these two numbers one takes either energy and width $(E_R, \Gamma)$ or energy and inverse lifetime $(E_R, \frac{1}{\tau})$ depending upon the way these quantities can be measured. In the relativistic case one takes mass and width $(M_R, \Gamma)$ or mass and (inverse) lifetime $(M_R, \frac{1}{\tau})$.

Lifetime $\tau$ and its inverse $\frac{1}{\tau}$, the initial decay rate of the decay $D \rightarrow \eta$ for any decay channel $\eta$ are measured by fits of the experimental counting rate $\frac{1}{N} \Delta N_\eta(t)$ to the exponential law for the partial decay rate $R_\eta(t)$:

$$\frac{1}{N} \frac{\Delta N_\eta(t)}{\Delta t_i} \approx R_\eta(t) = R_\eta e^{-Rt}. \quad (1.1)$$

(Here $R_\eta(t)$ are the theoretical partial decay rates of the decay $D \rightarrow \eta$ for any decay channel $\eta$ and $\Delta N_\eta(t)$ is the number of decay products $\eta$ registered in the detector during the time interval $\Delta t_i$ around $t_i$).

In contrast, the width $\Gamma$ is measured by fits of the cross section for the resonance scattering process $\eta_0 \rightarrow D \rightarrow \eta$ to the Lorentzian (Breit-Wigner) energy distribution

$$\sigma_j^{BW}(E) \sim |a_j^{BW}(E)|^2 = \left| \frac{r_\eta}{E - (E_R - i\frac{\Gamma}{2})} \right|^2 \sim \frac{1}{(E - E_R)^2 + (\frac{\Gamma}{2})^2}, \quad r_\eta = \sqrt{R_{\eta_0} R_\eta} \quad (1.2)$$

(with $0 \leq E < \infty$). (plus usually some background term $B(E)$). Decay rate $R = \sum_\eta R_\eta$ and resonance width $\Gamma$ are thus different quantities, $R = \frac{1}{\tau}$ is connected with the exponential time evolution (1.1), and $\Gamma$ is connected with the Lorentzian energy distribution. But usually one does not even distinguish between $R$ and the width $\Gamma$; one identifies $\frac{\hbar}{\tau} \equiv R$ with $\Gamma$

$$\Gamma = \frac{\hbar}{\tau} \equiv R \quad (1.3)$$

and uses the words rate and width for either of them. Often one calls a quasistable state a resonance, if the energy $E_R$ (or resonance mass $M_R$) can
be measured from \((1.2)\). This is the case when \(\Gamma / M R \sim 10^{-1} \ldots 10^{-4}\). And one calls the quasistable state a decaying particle when the lifetime \(\tau = 1/R\) can be determined experimentally from \((1.1)\), which is usually for values of \(R/M R \lesssim 10^{-8}\).

In non-relativistic physics the relation \((1.3)\) was justified by the Wigner-Weisskopf approximation \([1]\), which is really not an approximation of standard quantum mechanics (based on the Hilbert space axiom), because in Hilbert space \(\mathcal{H}\) there is no state vector \(\psi^D(t) \in \mathcal{H}\) whose rate for the Born probability \(P_{\eta}(t)\)

\[
R_{\eta}(t) = \frac{d}{dt} P_{\eta}(t) \equiv \frac{d}{dt} \langle \text{Tr}(\Lambda_{\eta} | \psi^D(t) \rangle \langle \psi^D(t) |) \rangle (1.4)
\]

does not obey the exponential law \((1.1)\).

Recently a non-relativistic theory which fulfills

\[
\hbar \frac{\Gamma}{\eta} = \tau.
\]

has been constructed using a slight modification of the standard Hilbert space axiom \([2]\). However in non-relativistic physics there was never much doubt that resonances are not qualitatively different from decaying states but only quantitatively by their value of width \(\Gamma = \text{rate } R\). The equality \((1.3)\) has recently also been confirmed in atomic physics \([3, 4]\).

In relativistic physics, the situation is quite different. Based on the perturbation theoretical definition by the self-energy of the propagators, resonances and decaying states are considered as complicated objects that cannot be described as an exponentially decaying state or as a state characterized by two numbers like \((M, \Gamma)\). This has recently led to some difficulties with the definition of mass and width of the \(Z\)-boson \([5]\).

We shall discuss here the possibility that also relativistic resonances and decaying states are qualitatively the same, described by representations of Poincaré transformations, similarly to Wigner’s theory of relativistic stable particles \([6]\). This will lead to a relativistic Gamow vector which has a “relativistic Breit-Wigner” energy distribution and an exponential decay law fulfilling \((1.3)\). This relativistic Gamow vector and/or the corresponding relativistic Breit-Wigner will represent the resonance per se (without background). It provides the means to precisely define a relativistic resonance and separate the scattering amplitude into a background and the resonance.

3
In an unambiguous way, and therewith define the mass and width of a relativistic resonance.

In Section 2 we review the non-relativistic theory and introduce the Hardy space hypothesis which replaces the Hilbert space axiom of quantum mechanics. Section 3 reviews scattering theory and presents its minor variations based on the new Hardy space hypothesis. The relativistic theory of scattering is presented in Section 4, and Section 5 gives a detailed derivation of the relativistic Gamow vector from the $S$-matrix pole. Some of its properties are deduced here. The transformation of the relativistic Gamow vectors under (causal) Poincaré transformations and its consequences are the subject of the subsequent paper [31].

## 2 Alternative Phenomenological Descriptions and their Mathematical Idealizations

In the non-relativistic phenomenological description of quasistable and stable states, one has two alternative descriptions: the S-matrix description and the Hamiltonian description [7].

In the S-matrix description the partial S-matrix element with angular momentum $j$ (the $j$-th partial S-matrix $S_j(E)$) is an analytic function on a Riemann energy surface cut along the real positive axis from $E_0(=0)$ to $\infty$. The S-matrix is written in terms of the elastic scattering amplitude $a_0^j(E)$ and the reaction amplitude $a_\eta^j(E)$ as

$$S_0^j(E) = 2i a_0^j(E) + 1 \text{ (elastic)}$$

$$S_\eta^j(E) = 2i a_\eta^j(E) \text{ (reaction)}$$

where $\eta_0$ denotes the quantum numbers of the initial state and the elastic channel and $\eta$ denote the reaction channels.

If one has a Hamiltonian given by $H = H_0 + V$, where $H$ is selfadjoint and semi-bounded (stability of matter) by $E_0$, then the S-matrix and Hamiltonian description are related and the scattering amplitude is given by

$$a_j^\eta(E) = -\pi \langle E, j \ldots \eta_0 | V | E, j \ldots \eta^- \rangle$$

where $|E, j \ldots \eta^- \rangle$ are eigenkets of the exact Hamiltonian with real (continuous) eigenvalues $E$, angular momentum $j$, and other quantum numbers indicated by $\ldots$. The superscript ($-$) indicates that they are solutions of the
Lippmann-Schwinger equation fulfilling outgoing boundary conditions \((-i\epsilon\text{ in the denominator})\). The \(|E, j \ldots \eta\rangle\) are eigenkets of the free Hamiltonian \(H_0\).

A stable state is, in the S-matrix description, a pole at \(E = E_n = -|E_n|\) on the first sheet of the Riemann surface. Whereas in the Hamiltonian description, a stable state is an eigenvector with a real discrete eigenvalue corresponding to the equation,

\[
H|E_n, j, \ldots\rangle = E_n|E_n, j, \ldots\rangle
\]  

(2.4)

An unstable particle, in the S-matrix description, is related to a pole at \(E = E_R - i\Gamma/2 \equiv z_R\) on the second sheet of the Riemann Surface. If there is just one resonance with angular momentum \(j\) in the \(\eta\)-channel then

\[
a_j^\eta(E) = \frac{r_\eta}{E - (E_R - i\Gamma/2)} + B(E) \quad r_\eta = \sqrt{R_\eta R_\eta}
\]  

(2.5)

where the “physical” values of \(E\) are \(0 \leq E < +\infty\).

In the Hamiltonian description, since the selfadjoint operator \(H\) of (2.4) cannot have a complex eigenvalue \(z_R\), one devises an “effective”, complex Hamiltonian matrix \(H_{\text{eff}}\) and the decaying state is an eigenvector of \(H_{\text{eff}}\) with a complex eigenvalue

\[
H_{\text{eff}}|f\rangle = (E_R - i\Gamma/2)|f\rangle
\]  

(2.6)

For example, in the neutral Kaon theory, one has two such eigenvectors \(f^{K_S}\) and \(f^{K_L}\) with

\[
H_{\text{eff}}f^{K_{S,L}} = (M_{S,L} - i\Gamma_{S,L}/2)f^{K_{S,L}}
\]  

(2.7)

Since the definition (2.6) is mathematically problematic because it requires the justification of a finite dimensional complex submatrix for a selfadjoint operator \(H\), the definition (2.5) of a resonance as a pole of the S-matrix has become the most universal definition of a resonance state.

A complex eigenvalue of a self-adjoint Hamiltonian, \(H\) is not possible for a vector in the Hilbert Space \(\mathcal{H}\), thus to obtain something like \(|f\rangle\) in (2.6), one must go outside the Hilbert Space. This should not be surprising because Dirac kets (generalized eigenvectors with eigenvalues from the continuous real energy spectrum) are also not elements of \(\mathcal{H}\), and to give them a mathematically rigorous foundation, one has to use a Rigged Hilbert
Space (RHS) (see Appendix A). The ordinary Dirac eigenkets \( |E, j, ..., \eta \rangle \) of \( H_0 \) are usually taken from the RHS, \( \Phi \subset H \subset \Phi^\times \), where the realization of \( \Phi \) is given by the Schwartz Space \( S \), which is the space of smooth functions rapidly decreasing at infinity. This means that every vector \( \psi_{j, \eta} \in \Phi \) (with fixed value of \( j \) and the other quantum numbers \( \eta \)), can be represented according to the Dirac basis vector expansion (nuclear spectral theorem) in terms of the \( |E, j, \eta \rangle \in \Phi^\times \) as
\[
\psi_{j, \eta} = \int_0^\infty dE |E, j, \eta \rangle \langle E, j, \eta | \psi_{j, \eta}
\] (2.8)
where the energy wave functions \( \langle E, j, \eta | \psi_{j, \eta} \rangle \equiv \langle E | \psi \rangle = \psi(E) \) are Schwartz space functions:
\[
\psi \in \Phi \iff \langle E | \psi \rangle \in S_{\mathbb{R}^+}
\] (2.9)

To obtain generalized eigenvectors that fulfill something like (2.6) one follows the same RHS method but one uses in place of the Schwartz Space \( \Phi \), other spaces \( \Phi_+ \) and \( \Phi_- \) which are realized by Hardy functions as explained next.

The eigenkets of the exact Hamiltonian \( H = H_0 + V \) that one uses in scattering theory are not ordinary Dirac kets, i.e. elements of the dual of the Schwartz space \( \Phi^\times \), but they are kets which also have meaning for complex values \( E \pm i\epsilon \) with infinitesimal \( \epsilon > 0 \). In scattering theory, one uses two solutions of the eigenvalue equation
\[
H|E, j, \eta^\mp \rangle = E|E, j, \eta^\mp \rangle, \quad 0 \leq E < \infty
\] (2.10)
where the superscript \( \mp \) refers to the \( \mp i\epsilon \) in the denominator of the Lippmann-Schwinger equation. This indicates that \( |E, j, \eta^\mp \rangle \) must be continued from the real energies into an \( \epsilon \)-strip of the complex energy plane: into the lower half plane for \( (\mp) \) and into the upper half plane for \((+)\). This means the complex conjugate of the wave functions, \( \langle \mp E | \psi^\mp \rangle = \langle \psi^\mp | E^\mp \rangle \), must not only be a smooth function of \( E \) like in (2.9) but they must also be functions that have an analytic continuation into the complex energy plane, in particular \( \langle -\psi | E^- \rangle \) must have an analytic continuation into the lower half plane. Hardy functions, elements of \( \mathcal{H}^2_\mp \cap S_{\mathbb{R}^+} \) have this property. Thus we take for the energy wavefunctions of a scattering system
\[
\langle \mp \psi | E^\mp \rangle \in \left( \mathcal{H}^2_\mp \cap S \right)_{\mathbb{R}^+} \text{ which implies } \langle \mp E | \psi^\mp \rangle \in \left( \mathcal{H}^2_\mp \cap S \right)_{\mathbb{R}^+} \] (2.11±)
This is a new hypothesis which replaces the Hilbert space axiom \( \langle \mp E | \psi \mp \rangle \in L^2(\mathbb{R}_+) \) or the Schwartz space assumption (2.9).

The generalized eigenvectors (2.10) representing out (\(-\)) and in (\(+\)) plane wave solutions of the Lippmann-Schwinger equation are therefore kets in a pair of Rigged Hilbert Spaces:

\[
\Phi_\pm \subset \mathcal{H} \subset \Phi_\pm^X; \quad |E, j, \eta^\mp \rangle \in \Phi_\pm^X
\]  

(2.12)

This means the vectors \( \psi^- \in \Phi_+ \) are given by the Dirac basis vector expansion

\[
\psi^- = \sum_{j, \eta} \int_0^{\infty} dE |E, j, \eta^- \rangle \langle -E, j, \eta| \psi^- \rangle
\]  

(2.13+)

where the energy wave functions \( \langle -E, j, \eta| \psi^- \rangle = \langle -E| \psi^- \rangle \) are Hardy functions analytic in the upper-half plane and the vectors \( \phi^+ \in \Phi_- \) are given by the Dirac basis vector expansion

\[
\phi^+ = \sum_{j, \eta} \int dE |E, j, \eta^+ \rangle \langle +E, j, \eta| \phi^+ \rangle
\]  

(2.13−)

where the \( \langle +E, j, \eta| \phi^+ \rangle = \langle +E| \phi^+ \rangle \) are elements of \( \mathcal{H}^2_\pm \cap \mathcal{S}|_{\mathbb{R}_+} \). In the scattering experiment, the \( \psi^- \in \Phi_+ \) represent the out-states registered by a detector and the \( \phi^+ \in \Phi_- \) represent the in-states prepared by a preparation apparatus (e.g. accelerator), as will be discussed in Section 3.

In the heuristic formulation using the Lippmann-Schwinger equations [18][19], the precise mathematical meaning of the out and in plane wave solutions is usually not stated. It is understood that they are to provide a means to distinguish between in-states \( \phi^+ \) prepared in the past, and out-states \( \psi^- \) registered by the detector in the future after they have passed the interaction region. Such a distinction is meaningless in the Hilbert space where only time symmetric solutions of the Schrodinger or Heisenberg equation- given by the unitary time evolution group- are allowed. Thus there is a contradiction between the Hilbert space axiom of quantum theory and the distinction between in- and out- states in scattering theory. Since the solutions to the Lippmann-Schwinger equation are kets \( |E, j, \eta^\mp \rangle \), they are not elements of the Hilbert space, and one can choose them to be two solutions of the same eigenvalue equation with two different, time asymmetric, boundary conditions. This is precisely what we intend to do [3]. We replace the axiom
of orthodox von Neumann quantum mechanics, which asserts that the set of prepared in-states and the set of detected observables (or out-states) are both equivalent to the Hilbert space

\{\text{set of prepared in-states } \phi\} = \{\text{set of detected out-observables } \psi\} = \mathcal{H} \ (2.14)

by a new axiom.

This axiom states that the prepared states, defined by the preparation apparatus (accelerator), are described by

\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \quad (2.15)

and the registered observables, defined by the registration apparatus (detector) are described by

\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times, \quad (2.16)

where \(\mathcal{H}\) in (2.15) and (2.16) denotes the same Hilbert space but \(\Phi_-\) and \(\Phi_+\) are the two different Hardy spaces which are dense in \(\mathcal{H}\). For the non-relativistic case this axiom is a formulation of time asymmetric boundary conditions for the solutions of the time symmetric Schrodinger and the Heisenberg differential equations, respectively. It is correct, as stated in section 3.2 of [9], that in-states \(\phi^+\) and out-states \(\psi^-\) do not inhabit two different Hilbert spaces. However, in contrast to what is implied in [9], the new hypothesis (2.15), (2.16) postulates that the in- and out-kets (2.10), which are generalized eigenvectors and not in \(\mathcal{H}\), are from two different spaces \(\Phi_-^\times\) because (2.15), (2.16) postulates that the set of in-states \(\{\phi^+\}\), and the set of out-states \(\{\psi^-\}\), are different (dense) subspaces of the same Hilbert space \(\mathcal{H}\). These two dense subspaces are the Hardy spaces \(\Phi_-\) and \(\Phi_+\), whose wave functions have different but complementary analyticity properties.

The possibility of two distinct spaces for the prepared states \(\{\phi^+\}\) and registered observables \(\{\psi^-\}\) is already contained in the historical paper of Feynman [10]. He distinguishes between the state at times \(t' < t_0\) which is defined by the preparation (our prepared states \(\phi^+\)) and what he calls the “state characteristic of the experiment” at time \(t'' > t_0\) (our registered observables \(\psi^-\)). He mentions the possibility that \(\{\psi^-\} \neq \{\phi^+\}\) in Footnote 14, attributing it to H. Snyder, but does not consider it. We implement this possibility by the choice of the two Hardy spaces \(\Phi_-\) and \(\Phi_+\). From the hypothesis (2.15), (2.16) one derives (by Fourier transform, using the Paley-Wiener theorem for Hardy Class functions) the quantum mechanical time asymmetry [21].
The axiom (2.15), (2.16) is the only new hypothesis needed for a time asymmetric quantum theory. The new hypothesis is needed in order to obtain a consistent theory of resonance scattering and decay, that gives the Weisskopf-Wigner approximation if superpositions over the energy continuum are neglected [2]. With this new axiom (2.15) and (2.16), quantum mechanics is no longer a strictly reversible theory, it encapsulates time asymmetry. Except for the new axiom (2.15) and (2.16), all the other basic assumptions (or axioms) of quantum mechanics, including the dynamical differential equations (the Schrödinger equation for the states or the Heisenberg equation for the observables) remain the same, but they will be extended to the new vectors (kets) of $\Phi^\times_+$ and $\Phi^\times_-$. In particular, the Born probability for measuring the observable $\psi^-$ in the state $\phi^+$ is

$$P_{\psi^-}(\phi^+(t)) = |\langle \psi^-, \phi^+(t) \rangle|^2 = |\langle \psi^-(t), \phi^+ \rangle|^2$$

(2.17)

and this axiom will be extended to elements of $\Phi^\times_+$. With the new mathematical concepts of Rigged Hilbert Spaces of Hardy type (2.15) and (2.16), we can also give a precise meaning to such vectors as in (2.6):

We replace the phenomenological Breit-Wigner in (2.5) where $0 \leq E < \infty$ by an “exact” Breit-Wigner for which the energy extends from $-\infty_{II} < E < \infty$. Then one can associate to it an ideal Gamow vector $\psi^G_j$, defined as the continuous superposition of the Lippmann-Schwinger-Dirac kets $|E, j...\rangle$ with an “exact” Breit-Wigner as a wave function.

$$a_{j}^{BW}(E) = \frac{r_n}{E-(E_R-i\Gamma/2)}$$

$$\psi^G_j = \sqrt{2\pi\Gamma}|z_R, j, ...\rangle = \frac{i\sqrt{2\pi\Gamma}}{2\pi} \int_{-\infty_{II}}^{+\infty} dE |E, j, ...\rangle (E-R-z_R)$$

(2.18)

The subscript $II$ in $-\infty_{II}$ indicates that the analytic continuation has been done in the second sheet of the analytic S-matrix where the positions $z_R = E_R - i\Gamma_R/2$ of the resonance poles are located. The Gamow vector $\psi^G_j$ is thus defined as the continuous linear superposition of the $|E, j, ...\rangle$ with Breit-Wigner energy wave functions $\langle -E, j|\psi^G_j \rangle \sim a_{j}^{BW}(E)$. In contrast to the superposition (2.13) for ordinary vectors $\psi^-$, the integration in (2.18) extends over $-\infty_{II} < E < \infty$ and this is only possible if $\psi^G_j$ is a functional

\footnote{The normalization factor $\sqrt{2\pi\Gamma}$ is an inconsequential convention.}
\( \psi^G(\psi^-) = \langle \psi^- | \psi^G \rangle \) over the Hardy space, \( \psi^- \in \Phi_+ \). This definition of \( \psi^G \) by (2.18) has been suggested by the continuous deformation of the integral for the S-matrix [2].

For the vector, \( \psi^G_j \in \Phi^x_+ \) defined in (2.18) (and only if the integral extends to \( -\infty \)) one can derive (using the property of Hardy functions) [2] that

\[
\langle H \psi^-_\eta | \psi^G \rangle \equiv \langle \psi^-_\eta | H^x | \psi^G \rangle = (E_R - i\Gamma/2) \langle \psi^-_\eta | \psi^G \rangle \ \ \forall \ \psi^-_\eta \in \Phi_+ \quad (2.19)
\]

when \( H = H_0 + V \) is self-adjoint (and semibounded). This justifies the notation \( \psi^G_j = \sqrt{2\pi \Gamma} | E_R - i\Gamma/2, j \ldots \rangle \). In Dirac’s notation the arbitrary \( \psi^-_\eta \in \Phi_+ \) is omitted and (2.19) is written as

\[
H^x | E_R - i\Gamma/2, j, \ldots \rangle = (E_R - i\Gamma/2) | E_R - i\Gamma/2, j, \ldots \rangle \quad (2.20)
\]

Dirac also omitted the \( \times \) of \( H^x \) which is uniquely defined as the extension of the operator \( H^\dagger = H \) to \( \Phi^x_+ \) by the first equality in (2.19), c.f. Appendix A. The Gamow ket \( | E_R - i\Gamma/2 \rangle \) is thus a generalized vector like a Dirac ket but with complex eigenvalue \( z_R \), where \( z_R \) is given by the position of the S-matrix pole of the resonance in the lower half plane. However, whereas the usual Dirac ket, is (most of the time) thought of as a functional over the Schwartz space \( \Phi \), the Gamow ket (2.18) is an element of \( \Phi^x_+ \), i.e., a functional over the Hardy space \( \Phi_+ \). The Gamow ket (2.20) and the Lippmann-Schwinger kets (2.10) are vectors which are “more general” than the usual Dirac kets.

The Gamow vector with exact Breit-Wigner energy distribution defined by (2.18) represents the state associated to the Breit-Wigner scattering amplitude (without the background \( B(E) \) of (2.5)) of width \( \Gamma \). For this state vector one derives the exponential time evolution:

\[
\psi^G(t) \equiv e^{-iH^xt} \psi^G = e^{-iE_Rt} e^{-\frac{\Gamma}{2}t} \psi^G; \ \ \text{for} \ t \geq 0 \ \text{only.} \quad (2.21)
\]

Formally (2.21) is just (2.20) applied in the exponent, but for a precise derivation, one needs again the mathematical properties of the Hardy functions [2] and the time asymmetry \( t \geq 0 \) is a consequence of this. Written in the form

\[\text{Physicists usually do not give a mathematical definition of the Dirac ket, but, if they do [11], they define them as functionals over the space } S \text{ or } D. \ \text{The Lippmann-Schwinger kets must be defined over a function space of analytic functions, cf. the remarks leading to (2.11).}\]
\(\text{(A.6)}\), the time evolution (2.21) is also written as
\[
\langle e^{iHt}\psi^-_\eta | E_R - \Gamma/2^- \rangle \equiv \langle \psi^-_\eta | e^{-iH^+t} | E_R - \Gamma/2^- \rangle = e^{-iE_R^t} e^{\frac{-\Gamma}{2}t} \langle \psi^-_\eta | E_R - i\Gamma/2^- \rangle \quad \forall \; \psi^-_\eta \in \Phi_+ \quad \text{and for} \; t \geq 0. \tag{2.22}
\]

Since \(\langle \psi^-_\eta | \psi^G(t) \rangle\) represents—in analogy to (2.17)—the probability amplitude to find the decay product \(\eta\) (described by \(\psi^-_\eta\)) in the state \(\psi^G(t)\) we have derived the exponential law for the probabilities of a transition from the Gamow state \(\psi^G\) into any decay product \(\eta\):
\[
|\langle \psi^-_\eta | \psi^G(t) \rangle|^2 = e^{-\Gamma t} |\langle \psi^-_\eta | \psi^G_j(0) \rangle|^2, \quad \text{for} \; t \geq 0 \tag{2.23}
\]
where \(\Gamma\) is the width of the Breit-Wigner amplitude in (2.18), i.e., \(\Gamma = -2Imz_R\) where \(z_R\) is the resonance pole position. This exponential law shows that the lifetime is given by \(\tau = 1/\Gamma\).

Another result of (2.22), (2.23) is the time asymmetry \(t \geq 0\). It has been called microphysical irreversibility or fundamental quantum mechanical time asymmetry, for further discussions of this subject we refer to the literature [12, 13, 14, 15, 16]. The result (2.22) shows that \(\psi^G\) has only a semigroup time evolution described by the operator \(U^x(t) = e^{-iH^+t}\) in \(\Phi^+_x\) (defined by (A.6)). This is in contrast to the unitary group evolution described by \(U^\dagger(t) = e^{-iH^\dagger t}, -\infty < t < +\infty\) for every vector in the Hilbert space \(\mathcal{H}\). An analogous result is also obtained in the relativistic case from the transformation properties of the relativistic Gamow vectors under Poincaré semigroup transformations into the forward light cone. This is the subject of the subsequent paper [31].

### 3 S-Matrix\(^4\)

In the previous section we used heuristic arguments about the analyticity property of the out- and in- Lippmann-Schwinger energy wave functions \(\langle -E | \psi^- \rangle\) and \(\langle +\phi | E^+ \rangle\) to arrive at the two Hardy spaces \(\Phi_+\) and \(\Phi_-\). Their

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\(^4\)Without replicating the details here, we closely follow chapter 3 of [3] in order to both display the analogy (and comparability) and expose the differences between our development and the standard views in relativistic quantum theory. Our notation transcribes into that of [3] as \(\{\phi^{\text{in/out}}, \psi^{\text{out}}\} \to \Phi_\pm\) and \(\{\phi^+, \psi^\pm\} \to \Psi^\pm_\phi\). In [3], the multi-particle basis vectors are also denoted by \(\Psi^\pm_\alpha\) where \(\alpha = \{p_1\sigma_1 n_1, p_2\sigma_2 n_2, \cdots\}\), \(\sigma\) is the third component of the spin, and \(n\) is the species label.
vectors $\psi^- \in \Phi_+$ and $\phi^+ \in \Phi_-$ were defined as those vectors that could be represented by the Dirac basis vector expansion in terms of the out- and in- Lippmann-Schwinger plane wave kets $|2.13\rangle$ and $|2.13\rangle$ using Hardy functions. Their interpretation was already hinted at by calling $\phi^+$ prepared states and $\psi^-$ registered observables.

Every experiment in quantum physics can be subdivided into a preparation part and a registration part, though this division is not always unique, in particular for complicated experiments [13]. We shall apply this principle to the typical (relativistic) scattering experiment as depicted in Figures 1a–1d. Most approaches in the foundation of quantum mechanics ascribe fundamental importance to the notions of state and of observable and differentiate between them [14]. In the standard formalism of scattering theory however, the basic entities are the interaction-free in-states $\phi^{in}$ which become $\phi^+$ in the interaction region, and the interaction-free out-states $\phi^{out}$ which are the $\phi^-$ in the interaction region. The in-state $\phi^+$ is the state defined by the preparation apparatus (accelerator). The observable $|\psi^-\rangle\langle\psi^-|$ is defined by the registration apparatus (detector); and according to the new hypothesis $|2.13\rangle$ $|2.16\rangle$ states $\{\phi^+\}$ and observables $\{\psi^-\}$ are represented by different dense Hardy subspaces of the same Hilbert space. The entities of quantum theory that are confronted with the experimental data (ratios of large numbers of detector counts) are the Born probabilities $|2.17\rangle$ of an observable $\psi^-$ in a state $\phi^+$. The out-states $\phi^-$ of standard scattering theory do neither represent apparatus controlled states nor observables. Therefore, if one wants to distinguish meticulously between states and observable in scattering experiments, standard scattering theory has to be adapted slightly, in particular the out-state $\phi^-$ as fundamental notion has to be replaced by an out-observable $\psi^-$. In a scattering experiment, the experimentalist prepares the asymptotic state $\phi^{in}$ describing the non-interacting projectile and target beams. This is depicted in Figure 1a for the (fictitious) scattering experiment $\pi^- p \rightarrow \Lambda K^0[15]$. It is assumed that the time translation generator $H$ can be divided into two terms, the “free-particle” Hamiltonian $K (= P_1^0 + P_2^0)$ at rest and an

\[ W^{in} = \sum_{\alpha} w_{\alpha} |\phi^{in}_{\alpha}\rangle\langle\phi^{in}_{\alpha}| \]

and the observables are not given by one-dimensional projection operators $|\psi^{out}_{\beta}\rangle\langle\psi^{out}_{\beta}|$ but by $\Lambda^{out} = \sum_{\beta} \lambda(\beta) |\psi^{out}_{\beta}\rangle\langle\psi^{out}_{\beta}|$. The pion $\pi$ and the proton $p$ are uncorrelated and are thus described by the direct product of density operators $|\pi^{in}\rangle\langle\pi^{in}| \otimes |p^{in}\rangle\langle p^{in}|$ rather than by a vector $\phi^{in} = |\pi p^{in}\rangle$, but the standard discussion in terms of vectors suffices here. 

\[ \text{12} \]
interaction part $V$ (or something similar):

$$H = K + V, \quad (3.1)$$

where the split of $H$ into $K$ and $V$ will be different if different in- and out-particles are involved.

The state vectors $\phi^{\text{in}}(t) = e^{-iKt}\phi^{\text{in}}$ evolve in time (in the Schrödinger picture) according to the free Hamiltonian $K$. When the particles reach their interaction regions, (cf. Figure 1a) the free in-state vector $\phi^{\text{in}}$ changes into an exact state vector $\phi^{\text{+}}$ whose time evolution is governed by the exact Hamiltonian $H = K + V$. This change is usually described by the Moeller wave operator $\Omega^{+}$:

$$\Omega^{+}\phi^{\text{in}}(t) \equiv \phi^{+}(t) = e^{-iHt}\phi^{+} = \Omega^{-}\phi^{\text{out}}(t) \quad (3.2)$$

When the post-interaction particles move apart the exact state vector $\phi^{+}(t)$ changes into the free out state vector $\phi^{\text{out}}$, which is described by the Moeller operator $\Omega^{-}$ in (3.2). Here $t$ is the proper time in the center-of-mass of the projectile and target. The vector $\phi^{\text{out}}$ thus describes a state vector which is determined by the preparation of $\phi^{\text{in}}$ and by the dynamics of the scattering process. This is expressed by:

$$\phi^{\text{out}} = S\phi^{\text{in}}, \quad S = \Omega^{-\dagger}\Omega^{+} \quad (3.3)$$

where $S$ is the operator that describes the dynamical transformation of the asymptotic in state $\phi^{\text{in}}$ into an asymptotic out state $\phi^{\text{out}}$. The vector $\phi^{\text{in}}$ and thus $\phi^{+}$ are determined by the preparation apparatus (accelerator) only and thus $\phi^{+}$ and $\phi^{\text{in}}$ represent apparatus controlled states. The preparation of $\phi^{\text{out}}$ is depicted in Figure 1b. Since it is determined by the preparation apparatus of $\phi^{\text{in}}$ and the interaction (dynamics) described by $V$ or $S$, it does not represent a controlled state [19]. It is not an observable either.

The experimentalist also builds a detector described by the observable $|\psi^{\text{out}}\rangle\langle\psi^{\text{out}}|$ (cf. Figure 1c). For the sake of definiteness we want to consider the reaction

$$\pi^{-}p \to \Lambda K^{0}, \quad K^{0} \to \pi^{-}\pi^{+}. \quad (3.4)$$

The vector $\psi^{\text{out}}$ represents the asymptotically free out-observable (usually also called out-states) which is $|\pi^{+}\pi^{-}\Lambda^{\text{out}}\rangle$ in the case (3.4). The observable vectors also evolve in time (in the Heisenberg picture) according to the free
Hamiltonian, but since they are observables (solutions of the Heisenberg equation) they evolve by the adjoint (or conjugate) operator $e^{iKt}$:

$$\psi_{\text{out}}(t) = e^{iKt}\psi_{\text{out}}$$

A scattering experiment consists of a preparation apparatus and a registration apparatus (detector), as depicted in Figure 1d.

The detector, or generally the registration apparatus, registers an observable $\langle \psi_{\text{out}} | \psi_{\text{out}} \rangle$ outside the interaction region. This observable vector $\psi_{\text{out}}$ comes from a vector $\psi^-$ which, in analogy to (3.2), is given by

$$\psi^- = \Omega^- \psi_{\text{out}}$$

(3.5)

in the interaction region. $\psi_{\text{out}}$ is in the asymptotic region what the observable $\psi^-$ is in the interaction region, and $\phi^\text{in}$ is in the asymptotic region the state which becomes $\phi^+$ in the interaction region. The time evolution of $\psi^-$ and $\phi^+$ is governed by the exact Hamiltonian

$$\psi^-(t) = e^{iHt}\psi^- \equiv U(t)\psi^- \quad \phi^+(t) = e^{-iHt}\phi^+ \equiv U^\dagger(t)\phi^+$$

(3.6)

The observable $\psi_{\text{out}}$ is of course not the same as the state $\phi_{\text{out}}$, since $\phi_{\text{out}}$, like $\phi^+$ and $\phi^\text{in}$, is prepared by the accelerator and $\psi_{\text{out}}$, and $\psi^-$, is defined (or controlled) entirely by the detector. Thus the $\phi^+$ are entirely determined by the accelerator and the $\psi^-$ are determined by the detector only (which is not the case for the $\phi^\text{in} \leftarrow \phi_{\text{out}}$).

Hence the set of vectors $\{\psi^-\}$ could be, and in our case is, distinct from the set of vectors $\{\phi^+\}$. This is the contents of the hypothesis (2.15), (2.16).

The vectors $\phi^+(t)$ fulfill the Schrodinger equation with Hamiltonian $H$ and the vectors $\psi^-(t)$ fulfill the Heisenberg equation of motion. This explains the well known difference in the sign of the exponent in (3.6). This difference is not of great importance in Hilbert space quantum mechanics because with the Hilbert space boundary condition (2.14) the Schrodinger as well as the Heisenberg equation integrates to the unitary group solution with $-\infty < t < \infty$. However, this difference in the sign of the exponent in (3.6) is very important for the Hardy space boundary conditions (2.15) and (2.16), because in the Hardy space the dynamical differential equations integrate to be semigroup solutions of (3.6) with $0 \leq t < \infty$. As a consequence of this

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6 This is a consequence of the specific mathematical properties of the Hilbert space and proven by the Stone-von Neumann theorem [38].
one obtains the time asymmetry $t \geq 0$ for the Born probabilities (2.17) and, in the relativistic case discussed below and in [31], Einstein causality for the quantum mechanical probabilities.

The measured quantities in quantum physics are the Born probabilities, i.e., the probabilities to register an observable $\Lambda^- = |\psi^-\rangle\langle -\psi|$, $\psi^- \in \Phi_-$ in a state $\phi^+ \in \Phi_-$, $W^+ = |\phi^+\rangle\langle ^+\phi|$.

$$P(\phi^+) = Tr(\Lambda^- W^+) = |(\psi^-, \phi^+)|^2$$

(2.17)

The Born probability amplitude $(\psi^-, \phi^+)$ is expressed using the standard notions of scattering theory (3.2), (3.3) and the new (3.5) as the matrix element of the $S$-operator:

$$(\psi^-, \phi^+) = (\Omega^- \psi'^{\text{out}}, \Omega^+ \phi'^{\text{in}}) = (\psi'^{\text{out}}, S \phi'^{\text{in}}) = (\psi'^{\text{out}}, \phi'^{\text{out}})$$

(3.7)

This is essentially the statement of standard scattering theory [19, 9] except that there one speaks of out-states $\phi^-$ instead of out-observables $\psi^-$. But Born probabilities correlate observables and states, not states and other states, and the detector is not built to the specifications of prepared states, but to the specification of observables.

The matrix element (3.7) is the probability amplitude for the observable $\psi^-$ in the state $\phi^+$. It can thus also be given in terms of the asymptotic quantities as the probability amplitude for the observable $\psi'^{\text{out}}$ (e.g., $\psi'^{\text{out}} = |\Lambda \pi^+ \pi^-\rangle$ in the example (3.4)) in the “state” $\phi'^{\text{out}}$.

The vectors $\phi^+_\alpha \in \Phi_+$ are the prepared in-states and the detected out-observable vectors $\psi^-_\beta \in \Phi_+$ are often also called the out-states [9] and the array of complex amplitudes $(\psi^-_\alpha, \phi^+_\beta)$ is called the $S$-matrix. The labels $\alpha$ and $\beta$ stand for a whole collection of discrete quantum numbers. The $S$-matrix is also defined when $\alpha$ and/or $\beta$ are continuous labels, only then the $S$-matrix does not represent probability but a probability density. These continuous labels appear because not all quantum numbers are discrete, in particular the scattering energy $E$ is continuous. We obtain these continuous labels when we insert the Dirac basis vector expansions (2.13$^+_\pm$) for $\psi^-$ and (2.13$^-\pm$) for $\phi^+$ into (3.7).

$$(\psi^-, \phi^+) = \sum_\alpha \sum_\beta \int dE_\alpha dE_\beta \langle \psi^- | E_\alpha \alpha^- \rangle \langle -E_\alpha \alpha| E_\beta \beta^+ \rangle \langle ^+E_\beta \beta| \phi^+ \rangle$$

(3.8)

$$= (\psi'^{\text{out}}, S \phi'^{\text{in}}) = \sum_\alpha \sum_\beta \int dE_\alpha dE_\beta \langle \psi'^{\text{out}} | E_\alpha \alpha \rangle \langle E_\alpha \alpha | S | E_\beta \beta \rangle \langle E_\beta \beta | \phi'^{\text{in}} \rangle$$

(3.9)
In the second expansion we have used the Dirac basis expansion with respect to eigenkets of the free Hamiltonian $K$ of (2.8) and the same for the observables. For instance if the quantum numbers are

$$E_\alpha, \alpha = E_\alpha, j, j_3, n$$

(3.10)

then the two complete systems of commuting observables are

$$K, J^2, J_3, N \quad H, J^2, J_3, N$$

(by $N$ we denote the operator of the particle label quantum number $n$ e.g., charge operators)

In the mathematically heuristic formulation [19] (i.e., where the kets $|E_\alpha\alpha\pm\rangle$ and $|E_\alpha\alpha\rangle$ are not mathematically defined as functionals) one obtains the identities

$$\langle E_\alpha\alpha|S|E_\beta\beta\rangle = \langle -E_\alpha\alpha|E_\beta\beta^+\rangle$$

(3.12)

and

$$\langle E|\psi^{out}\rangle = \langle -E|\psi^-\rangle; \quad \langle E|\phi^{in}\rangle = \langle +E|\phi^+\rangle$$

(3.13)

Since we are here not interested in the problem of expressing the exact $|E_\alpha\alpha^-\rangle$ in terms of the interaction free $|E_\alpha\rangle$, e.g., by a perturbation series for the Lippmann-Schwinger equation (3.13a), we do not want to work here any further with the asymptotically free eigenstates $|E_\alpha\rangle$. We shall only work with the interaction incorporating states $\phi^+$ and $\psi^-$ and the basis vectors $|E_\alpha\alpha^-\rangle$ which are basis vectors of the representation spaces for the exact Poincaré transformations [1]. In this spirit, we shall use the equality (3.12) only as a definition of the $S$-matrix on the left hand side in terms of the exact eigenvectors $|E_\alpha, \alpha^-\rangle$ and $|E, \beta^+\rangle$ on the right hand side. And we shall use the equalities (3.13) only to state that the energy wave functions $\langle -E|\psi^-\rangle$, 

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7From the Lippmann-Schwinger equation

$$|E_\alpha^\pm\rangle = \left(1 + \frac{1}{E_\alpha - H \pm i0}\right)|E_\alpha\rangle \equiv \Omega^\mp|E_\alpha\rangle$$

(3.13a)

and from $\Omega^-\Omega^- = 1$ one obtains $\langle \psi^-|E^-\rangle = \langle \Omega^-\psi^{out}|\Omega^-|E\rangle = \langle \psi^{out}|\Omega^-\Omega^-|E\rangle = \langle \psi^{out}|E\rangle$ and similar for $\Omega^+\Omega^+ = 1$. But $\Omega^-\Omega^-$ and $\Omega^-\Omega^+$ must somehow be defined as conjugate operators $(\Omega^-)^\times$ and $(\Omega^-\Omega^-)^\times$ in some spaces; see also [24].
\[ \langle +E|\phi^+ \rangle, \text{ (which by hypothesis (2.15), (2.16) are mathematically described by Hardy functions analytic in the upper and lower energy plane, respectively)} \]

are physically interpreted as the energy distribution of the observable \[ \psi \] and of the state \[ \phi \] in the asymptotic regions.

Thus in terms of energy wave functions, (2.15) and (2.16) say:

\[ \langle -E|\psi^- \rangle \equiv \langle E|\psi^{\text{out}} \rangle \in S \cap \mathcal{H}^2_+|_{R+} \quad (3.14) \]

\[ \langle +E|\phi^+ \rangle \equiv \langle E|\phi^{\text{in}} \rangle \in S \cap \mathcal{H}^2_-|_{R-} \quad (3.15) \]

where

\[ |\langle E|\psi^{\text{out}} \rangle|^2 = \text{energy resolution of the detector in the asymptotic region} \quad (3.14a) \]

and

\[ |\langle E|\phi^{\text{in}} \rangle|^2 = \text{energy distribution of the prepared beam in the asymptotic region} \quad (3.15a) \]

The \[ \langle -E|\psi^- \rangle, \langle +E|\phi^+ \rangle \] describe thus the particular experimental setup and (2.15), (2.16) is the axiom that states that the choice of allowed energy wave function is (much) more limited than under the standard axiom (2.14).

In our interpretation based on hypothesis (2.15), (2.16) the prepared in-state \( \phi^+ \) is controlled by the preparation apparatus for \( \langle E|\phi^{\text{in}} \rangle \) at the asymptotic in-region and the registered out-observable \( \psi^- \) is controlled by the registration apparatus (detector) for \( \langle E|\psi^{\text{out}} \rangle \) at the asymptotic out-region.

In addition to the \( \phi^{\text{in}} \) and \( \psi^{\text{out}} \) we mentioned above also the out-state \( \phi^{\text{out}} = S\phi^{\text{in}} \). This out-state is uncontrolled. Therefore the out-going wave \( \langle E|\phi^{\text{out}} \rangle \) would be uncontrolled. It would be a state in which an observable \( |\psi^{\text{in}}\rangle \langle \psi^{\text{in}}| \) could have been measured in the distant past (asymptotic in-region). Such entities have no physical meaning because a state must be prepared first before an observable can be measured in it (causality). Thus, (3.14), (3.15) is all that is needed for (causal) physics. It is roughly half of what one conventionally uses.

\[ ^8 \text{It excludes the version of [19] (page 188) which is called not appealing to our physical intuition with regard to the notion of causality, and which we here exclude by axiom (2.15), (2.16).} \]
The statement \((3.14), (3.15)\) for the wave functions does not tell us anything about the spaces of the \(\psi^{\text{out}}, \phi^{\text{in}}\) and \(|E\rangle\), and we do not need them. It is sufficient to think of the in-states \(\phi^{+} \in \Phi_{-}\) and the out-observables \(\psi^{-} \in \Phi_{+}\), and the elements \(|E, \alpha^{\pm}\rangle\) of the conjugate spaces \(\Phi_{\pm}^{\ast}\). \((3.14), (3.15)\) tell us that their energy wave function can be prepared or determined in the asymptotic region, and \((3.12)\) tells us that the dynamics (particle interaction), encapsulated in the \(S\)-matrix, is given by the matrix elements of the two eigenkets \(|\alpha, E^{-}\rangle = \Phi_{+}^{\ast}\) and \(|\beta, E^{+}\rangle \in \Phi_{-}^{\ast}\) of the exact Hamiltonian \(H\) with the same eigenvalue, \((2.8)\), and different boundary conditions.

The objects of our theory are thus the interaction incorporating states \(\phi^{+}\), observables \(\psi^{-}\) and the corresponding kets \(|E^{\pm}\rangle\) which are eigenvectors of the full Hamiltonian \(H\). The interaction is encapsulated in the \(S\) operator or the \(S\)-matrix \((3.12)\), of which we do not have any specific knowledge unless we know (in the non-relativistic case) \(V\) and can calculate it from \((2.3)\).

With the use of symmetry conditions we can reduce the \(S\)-matrix in \((3.8) (3.9)\) to a (much) smaller number of reduced matrix elements and then simplify the integrals on the r.h.s of \((3.9)\). In particular, from energy conservation and angular momentum conservation it follows
\[
\langle E', j'_{3}, n'|S|E, j_{3}, n\rangle = \delta(E' - E)\delta_{j'_{3} j_{3}}\delta_{j'_{3} j_{3}}\langle n'|S|E\rangle|n\rangle
\] (3.16)
so that \((3.9)\) becomes
\[
\langle \psi^{-}, \phi^{+}\rangle = \sum_{j} \int dE \sum_{j_{3} n} \sum_{n' n''} \langle \psi^{-}|E, j_{3} n\langle n' |S_{j}(E)|n\rangle
\] (3.17)
and our non-knowledge is now encapsulated in the much smaller number of the \(j\)-th partial matrix elements \(S_{j}^{n''}(E)\) (some of the dependence on the \(n'\) and \(n\) can in a similar way be reduced using intrinsic (e.g., isospin) symmetries).

One defines a resonance with spin \(j\) by the pole of the \(j\)-th partial \(S\)-matrix element \(S_{j}(E)\) on the lower complex semiplane of the second sheet. From this definition one obtains – starting from the exact basis vectors with real energy – the state vector of the resonance, which we called Gamow vectors (or Gamow kets) \([2]\). In the relativistic case symmetry properties (with respect to interaction-incorporating Poincaré transformations) are even more important than in the non-relativistic case \([4]\). We shall now turn to the derivation of the relativistic Gamow vector from the first order pole of the relativistic \(S\)-matrix.

\[9\] In non relativistic quantum mechanics one has some model Hamiltonian for which
4 Relativistic Scattering and the Definition of the Relativistic In- and Out-Plane Waves

The relativistic theory of resonances does not require the non-relativistic theory as a backdrop but can be developed from the relativistic S-matrix theory in an analogous way. All that has to be done is to use in place of the non-relativistic angular momentum vectors of \((2.8)\) and \((2.13)\), relativistic basis vectors which span the representation spaces of Poincaré transformations

\[
\left| E_{j3}n \right\rangle \rightarrow \left| \hat{p}_{j3}[s]n \right\rangle \quad \text{where } \sqrt{s} = E^{cm} \text{ is the center of mass energy. (4.1)}
\]

\[
\left| E_{j3}n^{\pm} \right\rangle \rightarrow \left| \hat{p}_{j3}[s]n^{\pm} \right\rangle \quad (4.2)
\]

(Here we use not the momentum \(p\) but the space component of the 4-velocity \(\hat{p} = p/m\) as additional label of the relativistic basis kets, cf. below after \((4.17)\) and \([26]\).

The two sets of bases are again thought of as being related by the Moeller wave operators

\[
\left| \hat{p}_{j3}[w = \sqrt{s}, j]^{\pm} \right\rangle = \Omega^{\pm} \left| \hat{p}_{j3}[w = \sqrt{s}, j] \right\rangle. \quad (4.3)
\]

In the non-relativistic case one usually separates off the center of mass motion and ignores it; this has been done in the preceding sections. Including the center of mass motion in the non-relativistic case, one has \(|pE_{j3}n\rangle = |p\rangle \otimes |E_{j3}n\rangle\) in place of \(|E_{j3}n\rangle\), where \(p\) is the center of mass momentum. In the relativistic case this cannot be done, the center of mass energy squared (or invariant mass squared) \(s\) is related to energy and momentum by

\[
s^2 = E^2 - p^2 = (p_1 + p_2 + \cdots + p_N)_{\mu}(p_1 + p_2 + \cdots + p_N)^{\mu} \quad (4.4)
\]

where \(p_1, p_2, \cdots\) are the 4-momenta of the \(N\) particles involved in the scattering process.

The vectors on the r.h.s. of \((4.1)\) are the basis vectors of a representation of Poincaré transformations which are irreducible for fixed values of \([sj]\). For one can solve equation \((2.9)\) with a \(V\) that includes the interaction causing decay (e.g., square well \([25]\)). In the relativistic case one always starts with the free \(V \rightarrow 0\) asymptotic solution \(|Ejn\rangle\) and constructs “exact” solutions \(|Ejn^{-}\rangle\) in terms of the asymptotic \(|Ejn\rangle\) using perturbation theory, which does not seem to work for resonances, cf. remark in \([20]\).
the sake of definiteness, we restrict ourselves to two particle scattering (with possible formations of a resonance $R$)

$$a + b \rightarrow R \rightarrow c + d$$ (4.5)

The non-interacting in-states $\phi^{in}$ are characterized by their (two-) particle contents which is described by the two particle space given by the direct product of one-particle spaces $\mathcal{H}^a, \mathcal{H}^b$. The latter are irreducible representation spaces of the Poincaré group $\mathcal{H}^a = \mathcal{H}(m_a, j_a, n_a)$, where $m_a, j_a$ are mass and spin and $n_a$ is the particle species label of particle $a$.

$$\mathcal{H}^{in} = \mathcal{H}^a \otimes \mathcal{H}^b = \mathcal{H}(m_a j_a n_a) \otimes \mathcal{H}(m_b j_b n_b)$$ (4.6)

The basis vectors are product basis vectors

$$|a\rangle \otimes |b\rangle = |p^a_{a j^a_a}(m_a j_a n_a)\rangle \otimes |p^b_{b j^b_b}(m_b j_b n_b)\rangle$$ (4.7)

The same holds for the detected out states $\psi^{out}$

$$\mathcal{H}^{out} = \mathcal{H}^c \otimes \mathcal{H}^d = \mathcal{H}(m_c j_c n_c) \otimes \mathcal{H}(m_d j_d n_d)$$ (4.8)

and their basis vectors

$$|c\rangle \otimes |d\rangle = |p^c_{c j^c_c}(m_c j_c n_c)\rangle \otimes |p^d_{d j^d_d}(m_d j_d n_d)\rangle$$ (4.9)

In place of labeling the basis of $\mathcal{H}^{in}$ by the momenta $p_a, p_b$ and spins $j_a, j_b$, etc., of the individual particles, one can combine the two vectors $|a\rangle$ and $|b\rangle$ into an eigenvector of the total 4-momentum operator

$$P^{\mu\text{free}} = P^a_\mu + P^b_\mu$$ (4.10)

and the total angular momentum $j$.

This means one takes the direct product of the irreducible single particle representations $[m_a j_a]$ and $[m_b j_b]$ of the Poincaré group and reduces it in terms of irreducible representations of the Poincaré transformations labeled by $[s j]^{10}$.

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\(^{10}\)This is done in analogy to the irreducible representations of the rotation group when one couples two angular momenta $j_a$ and $j_b$ to the total angular momentum $j$ with $|j_a - j_b| \leq j \leq |j_a + j_b|$ which is accomplished by the $SO(3)$ Clebsch-Gordon coefficients. For the Poincaré transformations one needs the Clebsch-Gordon coefficients of the Poincaré group which are more complicated but also known \(^{23}\).
In this way one obtains for \( H_{\text{in}} \) (and similarly for \( H_{\text{out}} \) with \( a, b \to c, d \)) the basis vectors

\[
|p_j[s]lsn_an_b\rangle \quad (4.11)
\]

where \( p_{\mu} = (p_a + p_b)_{\mu} \) is the eigenvalue of \( P_{\mu}^{\text{free}} \) in the space \( (4.10) \) and \( s = p_{\mu}p^{\mu} \) is the center of mass energy squared \( (4.4) \). \( j \) is the total angular momentum and \( j_3 \) is its third component. For a given \( [m_aj_a] \) and \( [m bj_b] \), the \( [s_j] \) of the vectors \( (4.11) \) that span \( H_{\text{in}} \) take the following values:

\[
(m_a + m_b)^2 \leq s < \infty ; \quad (4.12a)
\]

and

\[
\begin{align*}
&j = 0, 1, 2, \ldots, |j_a + j_b| \text{ if } j_a + j_b = \text{integer} \\
&j = 1/2, 3/2, \ldots, |j_a + j_b| \text{ if } j_a + j_b = \text{half-odd-integer} \quad (4.12b)
\end{align*}
\]

The same value \([sj]\) can occur in the combination of particle \( a = [m_aj_a] \) and particle \( b = [m_bj_b] \) (for fixed values of \( n_a \) and \( n_b \)) more than once. This multiplicity is labeled by the degeneracy quantum numbers \( l \) and \( s \) which take the values

\[
s = j_a + j_b, j_a + j_b - 1, \ldots, |j_a - j_b| \\
(4.13)
\]

\[
l = j + s, j + s - 1, \ldots, |j - s| \\
(4.14)
\]

These degeneracy quantum numbers \( l, s \), which distinguish the identical irreducible Poincaré group representations \([sj]\), have the interpretation of total orbital angular momentum \( l \) and total spin \( s \), respectively.

We shall take these two-particle vectors \((4.11)\) with a specific Poincaré transformation property \([sj]\), as the asymptotically free in-state basis vectors \((4.1)\), with the two particle label \( n = n_a, n_b \). The general asymptotic in-state is the continuous superposition of \((4.11)\)

\[
\phi_{\text{in}} = \sum_j \int_{(m_a + m_b)^2}^{\infty} ds \sum_{l,s} \int \frac{d^3p}{2p^0} |p_j[s]lsn_an_b\rangle \langle p_j[s]lsn_an_b|\phi_{\text{in}} \rangle \quad (4.15)
\]

This reduction of the direct product representations of the Poincaré group into its irreducible components \([sj]\) is discussed in details in [23]. A similar
result holds for the asymptotic out-state $\psi^{\text{out}}$. If the incoming (outgoing) particles have $j_a = j_b = 0$ ($j_c = j_d = 0$), then $s = 0$ and $l = j$, and there is no degeneracy. We shall therefore in the following suppress the degeneracy labels $l, s$ (or consider the case $j_a = j_b = 0$) and use the notation of \([11.3]\).

Instead of using as basis vectors of the irreducible representation spaces of \([s_j]\) the momentum eigenvectors $|p_{j_3}[s_j]\rangle$ and $|p_{a,j_3}[m_{a,j_a}]\rangle$ same for $b, c, d$ \(4.16\)

one could as well use eigenvectors of the 4-velocity

$$\hat{p}_\mu = \frac{p_\mu}{\sqrt{s}}, \quad s = p^\mu p_\mu; \quad \hat{p}_{a\mu} = \frac{p_{a\mu}}{m_a}, \text{ etc.}, \quad 4.17$$

as basis vectors of an irreducible representation \([s_j]\). These 4-velocity basis vectors $|\hat{p}_{j_3}[s_j]\rangle$ have the property

$$\hat{P}_\mu |\hat{p}_{j_3}[s_j]\rangle = \hat{p}_\mu |\hat{p}_{j_3}[s_j]\rangle \quad 4.18$$

where

$$\hat{P}_\mu = P_\mu M^{-1}, \quad M = (P_\mu P^\mu)^{1/2} \quad 4.19$$

is the 4-velocity operator and $M$ is the invariant square mass operator whose eigenvalue is the invariant mass $\sqrt{s}$. The normalization of the velocity basis kets is chosen to be

$$\langle \hat{p}_{j'_3}'[s'_{j'}'] | \hat{p}_{j_3}[s_j]\rangle = 2\hat{p}^0 \delta(s' - s) \delta^3(\hat{p}' - \hat{p})\delta_{j_3,j'_3}\delta_{j,j'} \quad 4.20$$

and the integration measure for this normalization is $d^4\hat{p}/2\hat{p}^0$, in place of the $d^4p/2p^0$ of \([1.13]\).

The reduction of the direct product of the two irreducible representations $[m_{a,j_a}]$ and $[m_{b,j_b}]$ into a direct sum of irreducible representations $[s_j]$:

$$\mathcal{H}([m_{a,j_a}] \otimes \mathcal{H}([m_{b,j_b}]) = \sum_j \int_{(m_a + m_b)^2}^\infty ds \mathcal{H}([s_j]) \quad 4.21$$

and the calculation of the Clebsch-Gordan coefficients can be done in terms of the velocity basis vectors \([1.18]\) in the same way as has been done in \([23]\) for the momentum basis vectors, this has been shown in \([26]\).
The velocity kets $|\hat{p}j_3[sj]|$ provide as valid a basis for the representations of the Poincaré group as Wigner’s momentum kets. Moreover, the 4-velocity kets are often more useful for physics reasons because the four-velocity operators (4.19) may commute with intrinsic symmetries when the four-momentum does not. Further, the 4-velocities seem to fulfill to a rather good degree “velocity super selection rules” which the momenta do not [27]. Our use of velocity kets for relativistic resonances was motivated by a remark of Zwanziger [28]. The use of velocity kets will become important when we analytically extend the basis vectors from the values (4.12a) (the “physical” values of the scattering energies) into the complex plane, as explained below.

Although the interaction free theory can be some guidance for how the exact theory is to be constructed, the transition from the interaction-free theory into the exact theory incorporating interactions is not entirely clear. But we shall use the correspondence between the free and exact theories that is usually assumed to be provided by a pair of Moeller operators $\Omega^\pm$:

| interaction free theory | exact theory |
|-------------------------|--------------|
| in-states               | $\phi^{\text{in}}$ | $\phi^+ = \Omega^+\phi^\text{in}$ |
| out-observables         | $\psi^\text{out}$ | $\psi^- = \Omega^-\psi^\text{out}$ |
| basis vectors           | $|\hat{p}j_3[sj]|$ | $|\hat{p}j_3[sj]|^\pm = \Omega^\pm|\hat{p}j_3[sj]|$ |
| generators              | $P^\text{free}_\mu, J_{\mu\nu}^\text{free}$ | $P_\mu, J_{\mu\nu}$ |

(4.22)

From now on, based on the arguments in Section 3, we will only be concerned with the exact quantities $P_\mu, J_{\mu\nu}$, etc. The exact basis vectors are chosen to be eigenvectors of the complete system of commuting observables (csco):

$$\hat{P}_\mu = P_\mu M^{-1}, \quad M = (P_\mu P^\mu)^{1/2}, \quad -\hat{w}^\mu \hat{w}_\mu, \quad U(L(\hat{p}))(\hat{w}_3 U^{-1}(L(\hat{p}))) ,$$

(4.23)

where $\hat{w}^\mu = \epsilon^{\mu\nu\rho\sigma} \hat{P}_\nu J_{\rho\sigma}$. The eigenvalues of the csco (4.23) are

$$\hat{p}_\mu, \ s, j(j + 1), j_3$$

(4.24)

The Dirac basis vector expansion for the free states $\phi^{\text{in}}$ and $\psi^{\text{out}}$ is as in (4.15). In the theory that incorporates interactions, the Dirac basis vector

11This $P^{\text{free}}_\mu$ is the generator in (4.10) when there is no interaction between particles $a$ and $b$. 

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expansion for the in-state $\phi^+ \in \Phi_-$ is

$$
\phi^+ = \sum_{j,j_3} \int_{s_0}^{\infty} ds \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{p} j_3[s_j]^+\rangle \langle \hat{p} j_3[s_j]|\phi^+ \rangle
$$

and for an out-observable $\psi^- \in \Phi_+$ it is:

$$
\psi^- = \sum_{j,j_3} \int_{s_0}^{\infty} ds \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{p} j_3[s_j]|^-\rangle \langle \hat{p} j_3[s_j]|\psi^- \rangle,
$$

where $\hat{p}^0 = \hat{E}(\hat{p}) = \sqrt{1 + \hat{p}^2}$.

In (4.25) and (4.26), the wave functions of $\phi^+$ and $\psi^-$, i.e., their components $\langle \hat{p} j_3[s_j]|\phi^+ \rangle$ and $\langle \hat{p} j_3[s_j]|\psi^- \rangle$ along the basis vectors, are functions of the continuous variables $s$ and $\hat{p}$.

To simplify the notation we often omit the labels $\hat{p}, j_3, \eta = \hat{p}, j_3, l, s, n$ (where $n$ are degeneracy and particle species labels) and write the multiple dimensional projection operator on the irreducible representation space of $[s_j]$ as

$$
\sum_{j,j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{p} j_3[s_j]^+\rangle \langle \hat{p} j_3[s_j]| = |[s_j]^+\rangle \langle [s_j]| = |s^+\rangle \langle s^+|
\tag{4.27}
$$

Then the basis vector expansion (4.25a) can be written as

$$
\phi^+ = \sum_j \int_{s_0}^{\infty} ds |s_j|^+\rangle \langle s_j|\phi^+ \rangle = \int_{s_0}^{\infty} ds |s^+\rangle \langle s^+|\phi^+ \rangle
\tag{4.25a}
$$

and (4.26) for the out-observable $\psi^-$ is written as

$$
\psi^- = \sum_j \int_{s_0}^{\infty} ds |s_j|^-\rangle \langle s_j|\psi^- \rangle = \int_{s_0}^{\infty} ds |s^-\rangle \langle s^-|\psi^- \rangle
\tag{4.26a}
$$

For the wave function, now considered just as a function of the center of mass energy squared $s$, we thus use the notation

$$
\langle \hat{p} j_3[s_j]|\psi^- \rangle \to \langle s|\psi^- \rangle
\tag{4.25b}
$$
\[ \langle \hat{p} j_3 | s_j \rangle | \phi^+ \rangle \rightarrow \langle s | \phi^+ \rangle . \]  
\textit{(4.25c)}

We also write for the basis vectors

\[ | \hat{p} j_3 | s_j \rangle \rightarrow | s^- \rangle \in \Phi^+ \times \Phi^- \]  
\textit{(4.25d)}

We shall make the hypothesis that these wave functions have the same analyticity properties in the invariant mass squared \( s = (E_{\text{cm}})^2 \) as the energy wave functions in \((3.14) (3.15)\). However, in the relativistic case, due to the mathematical requirement of the invariance of the subspaces \( \Phi^\pm \) under the action of the generators of the Poincaré group, we cannot use exactly the Schwartz space \( \mathcal{S} \) of \((3.14) (3.15)\) but have to consider a closed subspace \( \tilde{\mathcal{S}} \) of \( \mathcal{S} \). This subspace \( \tilde{\mathcal{S}} \), constructed in \[29\], is the space of Schwartz functions which vanish at zero faster than any polynomial. This requirement also assures that the zero mass states do not contribute to the Gamow vector. This avoids the difficulty that the four velocity operators, which is central for our construction of Gamow vectors, cannot be meaningfully defined in the zero-mass case in any obvious way. The features of the space \( \tilde{\mathcal{S}} \) which are needed for the construction of the relativistic Gamow vectors are as follows:

**Proposition 4.1.** [29] The triplets

\[ \tilde{\mathcal{S}} \cap \mathcal{H}_2^\pm |_{\mathbb{R}_{s_0}} \subset L^2(\mathbb{R}_{s_0}) \subset \left( \tilde{\mathcal{S}} \cap \mathcal{H}_2^\pm |_{\mathbb{R}_{s_0}} \right)^\times \]  
\textit{(4.28)}

form a pair of Rigged Hilbert Spaces. This means the validity of the Dirac basis vector expansion \((4.25)\) and \(\textit{(1.24)}\) is assured.

In \((4.28)\), \(\mathbb{R}_{s_0}\) is the set of physical values of the scattering energy \( s \) for the process \((4.3)\) \( \mathbb{R}_{s_0} = [(m_1 + m_2)^2, \infty) \).

**Proposition 4.2.** [29] The space \( \tilde{\mathcal{S}} \) is endowed with a nuclear Fréchet topology such that multiplication by \( s^n \),

\[ s^n : \tilde{\mathcal{S}} \cap \mathcal{H}_2^\pm \rightarrow \tilde{\mathcal{S}} \cap \mathcal{H}_2^\pm , \quad n = 1, 2, 3, \ldots \]  

is a continuous linear operator in the topology of \( \tilde{\mathcal{S}} \).

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Thus the relativistic characterization of $\Phi_{\pm}$ analogous to (3.14) and (3.15) is:

\begin{align*}
\psi^- &\in \Phi_+ \quad \text{if and only if} \quad \langle -\hat{p}_j|s_j|\psi^- \rangle \in \tilde{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_0} \times \mathcal{S}(\mathbb{R}^3) \quad (4.29a) \\
\phi^+ &\in \Phi_- \quad \text{if and only if} \quad \langle +\hat{p}_j|s_j|\phi^+ \rangle \in \tilde{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_0} \times \mathcal{S}(\mathbb{R}^3) ; \quad (4.29b)
\end{align*}

where $\mathbb{R}^3$ is the space of components of the 4-velocity and the Hilbert space $\mathcal{H}$ of (2.15) (2.16) is realized by the function space $L^2(\mathbb{R}_0, ds) \times L^2(\mathbb{R}^3, \frac{d^3\hat{p}}{2\hat{p}^0})$. (4.30)

In the truncated notation of (4.25a) and (4.26), the definition (4.29a) and (4.29b) of the space $\Phi_-$ of prepared in-states $\phi^+$ and of the space $\Phi_+$ of the detected out-observables $\psi^-$ is

\begin{align*}
\psi^- &\in \Phi_+ \quad \text{in and only if} \quad \langle -s|\psi^- \rangle \in \tilde{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_0} \quad (4.29c) \\
\phi^+ &\in \Phi_- \quad \text{in and only if} \quad \langle +s|\phi^+ \rangle \in \tilde{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_0} \quad (4.29d)
\end{align*}

In mathematics one calls the space of Lebesgue square integrable functions $L^2(\mathbb{R}_0, ds)$ a realization of the abstract Hilbert space. In the same way we will call the triplet of function spaces

\begin{align*}
\tilde{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_0} \otimes \mathcal{S}(\mathbb{R}^3) \subset L^2(\mathbb{R}_0, ds) \otimes L^2(\mathbb{R}^3, \frac{d^3\hat{p}}{2\hat{p}^0}) \subset \left( \tilde{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_0} \otimes \mathcal{S}(\mathbb{R}^3) \right) \times
\end{align*}

realizations of the abstract rigged Hilbert spaces

\begin{align*}
\Phi_{\pm} \subset \mathcal{H} \subset \Phi_{\pm}^\times \quad (4.30a)
\end{align*}

if the abstract spaces are algebraically and topologically (i.e., their meaning of convergence) equivalent to the corresponding function spaces \(\Phi_{\pm}\).

The energy wave function $\langle -s|\psi^- \rangle \in \tilde{S} \cap \mathcal{H}^2_+$ is a mathematical realization of the vector $\psi^- \in \Phi_+$ and the vector $\psi^-$ is a mathematical representation of
the physical observable \( \psi^- \) which is physically defined by the apparatus (detector) that registers the observable, and similarly for \( \phi^+ \in \Phi_- \). Therefore, we have the following correspondences:

registration apparatus \( \Leftrightarrow \) observable \( |\psi^-\rangle\langle \psi^-| \Leftrightarrow \) wave function \( \psi^-(s) = \langle -s|\psi^- \rangle \)

preparation apparatus \( \Leftrightarrow \) state vector \( \phi^+ \Leftrightarrow \) wave function \( \phi^+(s) = \langle +s|\phi^+ \rangle \)

The modulus of the energy wave function \( |\langle -s|\phi^+ \rangle|^2 \) describes the energy distribution in the beam and the energy wave function \( |\langle -s|\psi^- \rangle|^2 \) describes the energy resolution of the detector.

We shall call all the function spaces that are intersections with the spaces of Hardy class \( \mathcal{H}^2_\pm|_{\mathbb{R}_0} \) Hardy (function) spaces and we call all (abstract) spaces \( \Phi_\pm \) that have realization by Hardy function spaces also (abstract) Hardy spaces.

In (4.30) as in (3.14) (3.15), \( \mathcal{H}^2_+ \) means the functions of Hardy class analytic in the upper half of the second sheet of the \( s \)-plane and \( \mathcal{H}^2_- \) means the functions of Hardy class analytic in its lower half. Specifically, the physical values of \( \langle +s - i0|\phi^+ \rangle \) and of \( \langle \psi^-|s - i0^- \rangle = \langle -s + i0|\psi^- \rangle \) are the boundary values of functions analytic in the lower half of the second sheet \( \mathcal{H}^2_- \). These analyticity properties on the second sheet of the complex \( s \)-Riemann surface will turn out to be important because the resonance poles of the \( S \)-matrix are located on the second Riemann sheet.

By virtue of Proposition 11.2, the operator of total momentum \( P_\mu = P_{\alpha\mu} + P_{b\mu} \) and the invariant mass square operator \( M^2 = P_\mu P^\mu \) are \( \tau_{\Phi_\pm} \)-continuous; hence their conjugates \( P^\mu_\pm \) and \( M^{2\times}_\pm \), are well defined on \( \Phi_\pm^{\times} \). This can be seen by considering the realization, for instance, of the vectors \( P_\mu^+\psi^- \) and

\[^{12}\text{if } f \in \mathcal{H}^2_+, \text{ then its complex conjugate function } \overline{f} \in \mathcal{H}^2_- \text{ and vice versa. Therefore for the matrix elements } \langle \psi^-|\phi^+ \rangle \text{ we only require the lower half of the complex energy plane (2nd sheet).}\]

\[^{13}\text{defined by (A.3) in Appendix A}\]

\[^{14}\text{To be precise, we should label } P_\mu \text{ etc by the space that it acts on. } P_{\mu+} \text{ is the restriction to the space } \Phi_+ \text{ of the selfadjoint operator } \tilde{P}_\mu \text{ in } \mathcal{H} \text{ and } P_{\mu-} \text{ is the restriction to the space } \Phi_- \text{ of the selfadjoint operator } \tilde{P}_\mu \text{ in } \mathcal{H}. \text{ } P^{\times}_\mu_+ \text{ is the conjugate operator of } P_{\mu+} \text{ in } \Phi_+^{\times} \text{ which is a unique extension of } P_{\mu+} = \tilde{P}_\mu \text{ in } \mathcal{H} \text{ to } \Phi_+^{\times}. \text{ } P^{\times}_\mu_- \text{ is the conjugate operator of } P_{\mu-} \text{ in } \Phi_-^{\times} \text{ which is a unique extension of } P_{\mu-} = \tilde{P}_\mu \text{ in } \mathcal{H} \text{ to } \Phi_-^{\times}.\]
\[ M^2 \psi^- : \]
\[
\langle P \mu \psi^- | \hat{\mathbf{p}} j_3 | s j \rangle = \langle \psi^- | P^x \hat{\mathbf{p}} j_3 | s j \rangle = \sqrt{s} \hat{\mathbf{p}} \mu \langle \psi^- | \hat{\mathbf{p}} j_3 | s j \rangle, \tag{4.31a}
\]
\[
\langle M^2 \psi^- | \hat{\mathbf{p}} j_3 | s j \rangle = \langle \psi^- | M^2 \times \hat{\mathbf{p}} j_3 | s j \rangle = s \langle \psi^- | \hat{\mathbf{p}} j_3 | s j \rangle. \tag{4.31b}
\]

According to Proposition 4.2 and the definition of the wave functions \( \langle \psi^- | \hat{\mathbf{p}} j_3 | s j \rangle \) given in (4.29a), the multiplication operators by \( \sqrt{s} \hat{\mathbf{p}} \mu \) and by \( s \) which appear in the right hand side of (4.31a) and (4.31b) are \( \tau_{\Phi^+} \)-continuous. Consequently, \( P \mu \) and \( M^2 \) are \( \tau_{\Phi^+} \)-continuous operators, and the conjugate operators \( M^2 \times \) and \( P^x \times \mu \) that appear in (4.31) are everywhere defined, continuous operators on \( \Phi^+ \). Hence, (4.31a) and (4.31b) define, according to (A.6) (A.7), the functionals \( \langle \hat{\mathbf{p}} j_3 | s j \rangle \) as generalized eigenvectors of \( P \mu \) and \( M^2 \). The same discussion applies for the space \( \Phi^- \).

We can re-express the generalized eigenvalues of the momentum operator in terms of the three velocity \( v \) by noting that \( \hat{\mathbf{p}} = \gamma \mathbf{v} = \frac{\mathbf{v}}{\sqrt{1 - v^2}} \), and \( 1 + \hat{\mathbf{p}}^2 = \frac{1}{1 - v^2} = \gamma^2 \). Hence, the eigenvalues in (4.33) can be rewritten as
\[
H^\times \hat{\mathbf{p}} j_3 | s j \rangle = \gamma \sqrt{s} \hat{\mathbf{p}} j_3 | s j \rangle, \tag{4.33a}
\]
\[
P^\times \hat{\mathbf{p}} j_3 | s j \rangle = \gamma \sqrt{s} v \hat{\mathbf{p}} j_3 | s j \rangle. \tag{4.33b}
\]

Note that here \( \sqrt{s} \) is not only restricted to the “physical” scattering energies (4.12a) but can be a complex value; in particular for the kets \( | \hat{\mathbf{p}} j_3 | s j \rangle \), \( s \) can be any value in the lower half complex plane (second sheet).

For the branch of \( \sqrt{s} \) in (4.31), (4.33) and (4.34), we choose
\[
-\pi \leq \text{Arg} s < \pi. \tag{4.35}
\]

This choice of branch, even though irrelevant for the physical values of \( s \), will be needed since we will analytically continue the kets \( | \hat{\mathbf{p}} j_3 | s j \rangle \) to the second Riemann sheet as described in Section 5.
We have now a well defined mathematical theory in which the momentum and energy operators $P_\mu$ (and the other generators $J_{\mu\nu}$) of the Poincaré group have a well defined mathematical meaning. We denote the triplet whose wave functions are given by the Hardy function spaces \( \Phi_- \) again by

$$\Phi_- \subset \mathcal{H} \subset \Phi_-$$

for the prepared in-states $\phi^+$

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+$$

for the detected out-observables $\psi^-$

(4.36)

In addition to the apparatus prepared states $\phi^+ \in \Phi_-$ and the apparatus detected observables $\psi^- \in \Phi_+$ one has the generalized state vectors $F^\pm \in \Phi_{\pm\mp}$. An example of these are the $F^\pm = |\hat{p}_j|_3^{\pm} \rangle$. The standard interpretation of these kets is as out-going and in-coming plane waves (in analogy to the non-relativistic case of Section 2). The bra-kets $|\langle s|\phi^+\rangle|^2$ represent the energy distribution in the prepared in-state, i.e., the probability density for the CM-energy $\sqrt{s}$ and for the momentum $p = \sqrt{s}p$ in the state $\phi^+$, as prepared by the preparation apparatus (accelerator). The bra-ket $|\langle s|\psi^-\rangle|^2$ represents the energy resolution of the detector $\psi^-$. The interpretation of the Born probability $|\langle \psi^-, \phi^+\rangle|^2$ is thus extended to the $|\langle s|\phi^+\rangle|^2$ (probability density for the beam $\phi^+$, i.e., the probability of the particles $a+b$ to have the energy $\sqrt{s}$) and to the $|\langle \psi^-|s\rangle|^2$ (probability for the detector to register the particles $c+d$ with energy $\sqrt{s}$).

If one uses the standard relationship (4.22) between interaction-free and exact quantities, one can – heuristically – justify (as in Section 3 for the non-relativistic case) that:

$$|\langle \phi^+|s^+\rangle| = |\langle \phi^{in}|s\rangle|$$

(4.37a)

i.e., the energy distribution in the beam is measured in the asymptotic interaction-free region. Similarly

$$|\langle \psi^-|s^-\rangle| = |\langle \psi^{out}|s\rangle|$$

(4.37b)

i.e., the detector’s energy resolution is the resolution of the asymptotic region. Since we want to work in our mathematical theory only with the interaction incorporating exact quantities, we only conclude from (4.37) that $|\langle \phi^+|s^+\rangle|^2$ and $|\langle \psi^-|s^-\rangle|^2$ are as measured in the asymptotic in and out region.

In addition to the eigenkets $|\hat{p}_j|_3^{\pm} \rangle \in \Phi_+^\times$ with real positive energy $\sqrt{s}$, the spaces $\Phi_+^\times$ contain many other generalized vectors $F^\pm$. In particular,

$$\Phi_+^\times$$

contains eigenkets $|s^+\rangle$ with complex eigenvalue $s$ of $\text{Im}s \geq 0$ (4.38a)
Φ⁺ contains eigenkets \(|s^-\rangle\) with complex eigenvalue \(s\) of \(\text{Im} s \leq 0\) (4.38b)

This is the important advantage that the functionals \(|s^\pm\rangle \in \Phi_\pm\) of the Hardy spaces \(\Phi_\pm\) have over the ordinary Dirac kets which (if defined at all) are defined as functionals over the Schwartz space and have therefore only real generalized eigenvalues.

As a consequence of (4.38), the quantities \(\langle \psi^-|s^-\rangle\) and \(\langle s^+|\phi^+\rangle\) can be analytically continued from the real positive energy axis into the lower half complex energy plane \([15]\). The kets \(|s^\pm\rangle\) are thus more generalized vectors than the ordinary Dirac kets because the \(\langle \psi^-|s^-\rangle\) and \(\langle s^+|\phi^+\rangle\) are not only smooth rapidly decreasing functions of \(s\) but can also be analytically extended to complex \(s\). This analytic property of the \(|s^\pm\rangle\) is already contained in its infinitesimal form in the plane-wave solutions of the Lippmann-Schwinger equations, where one describes the incoming and outgoing boundary conditions by an infinitesimal complex energy \(s \pm i\epsilon\) \([16]\). To go from infinitesimal analyticity to the analyticity requirement for the whole semi-plane contained in the Hardy space hypothesis (4.38) may appear a big jump. But since one cannot experimentally distinguish between an energy resolution of a detector described by smooth functions \(|\langle s|\psi\rangle|^2\) of energy squared \(s\) and a smooth \(|\langle -s|\psi^-\rangle|^2\) that can be analytically continued into the complex \(s\)-plane, the hypothesis is not in conflict with observations. And, as we shall see below, the consequences of the hypothesis have many physical aspects which the Hilbert space axiom is not capable of describing.

After we have established (4.37) in analogy to the usual heuristic arguments for the non-relativistic case invoking the Moeller wave operators \(\Omega^\pm\), we know that the energy distribution of the prepared (beam) state \(|\langle +s|\phi^+\rangle|^2\) and the energy resolution of the detector \(|\langle -s|\psi^-\rangle|^2\) is observed in the asymptotic in- and out-regions, respectively. If we use this as postulate and define the relativistic \(S\)-matrix (with 4-velocity basis) in analogy to (3.11) by

\[
\langle \hat{p}j_3|s|j_3'\rangle \equiv \langle -\hat{p}j_3|s|s'j_3'\rangle^+ \tag{4.39}
\]

\(15\)From \(|\langle -s|\psi^-\rangle\rangle \in \mathcal{H}^2_-\) follows that \(\langle \psi^-|s^-\rangle \equiv \langle -s|\psi^-\rangle \equiv \langle -\bar{s}|\psi^-\rangle \in \mathcal{H}^2_.\)

\(16\)The conventional Lippmann-Schwinger integral equation uses actually an infinitesimal imaginary part of the non-Lorentz invariant energy \(p^0\), \(s = (p^0 \pm i\epsilon)^2 - \hat{p}^2 = s \pm i2p^0\epsilon = s \pm i\epsilon'\), which for infinitesimal \(\epsilon\) is equivalent to using an infinitesimal Lorentz invariant \(i\epsilon\). We prefer to make the analytic extension in the invariant energy squared \(s\), since it is the preferred variable used in the relativistic \(S\)-matrix.

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we have no further need for the interaction free in-states $\phi^{in}$, out-observables $\psi^{out}$ and basis vectors $|\hat{p}^j[sj]\rangle$ nor for the interaction free Poincaré generators $P^\mu_{\text{free}} = \Omega^{-1}_\mp P^\mu\Omega_\mp$, $J^{\mu\nu}_{\text{free}} = \Omega^{-1}_\mp J^{\mu\nu}\Omega_\mp$. We can work entirely with the states $\phi^+ \in \Phi_-$, observables $\psi^- \in \Phi_+$, the basis vectors $|\hat{p}^j[sj]^{\pm}\rangle \in \Phi^\times_\mp$ and the exact generators $P^\mu$ and $J^{\mu\nu}$ of Poincaré transformations which incorporate the interactions [9]. The Poincaré transformations $U_+^{\Lambda}(a)$ generated by $P^\mu$, $J^{\mu\nu}$, in the space of observables $\Phi_+$ and the Poincaré transformations $U_-^{\Lambda}(a)$ in the space of in-states $\Phi_-$ do not describe kinematic translations and rotations like the interaction free transformations but dynamical evolutions, e.g., $U_+^{\Lambda}(a) = e^{iP^0t}$ describes time evolution of the observable $\psi^-$ (Heisenberg picture) and $U_-^{\Lambda}(a) = e^{-iP^0t}$ describes the time evolution of the state $\phi^+$ (Schrodinger picture) by the amount $t$ (in the rest frame of the state). The property of the interaction is encapsulated in the (reduced) $S$-matrix element. For instance, the property that a resonance is formed in the reaction (4.5) is described by a pole in the second sheet of the $S$-matrix at the complex energy $s = s_R = (M - i\Gamma/2)^2$.

In the following section we shall obtain the decaying state vectors, the Gamow vectors, from the $S$-matrix pole. Our theoretical frame based on the new hypothesis (4.36) allows us to describe relativistic resonances as decaying states by vectors associated to irreducible representations of Poincaré transformations, this is similar to Wigner description of stable relativistic particles. In a subsequent paper [31], we shall derive their transformation properties under Poincaré transformations. The result will look similar to but will turn out to be very different from Wigner’s Poincaré transformations, since it will distinguish the forward light cone and a direction of time. This transformation property will lead to an unambiguous definition of resonance mass and width, which has been an open problem [5].

5 Derivation of the Relativistic Gamow Vectors from an $S$-matrix Pole

To obtain the Gamow vector from the first order $S$-matrix pole [7], we start from the $S$-matrix element $(\psi^- , \phi^+)$, use in it the expansions (4.23), (4.26)\footnote{The same procedure can be used for higher order $S$-matrix poles which leads to Jordan blocks of Gamow vectors, cf. [32], for the non-relativistic case}.
and the relativistic analogy of the definition (3.12):
\[
\langle \hat{p}_{j3}[s_j][\eta]|S|\hat{p}'_{j3}'[s'_j'][\eta']\rangle \equiv \langle -\hat{p}_{j3}[s_j][\eta]|\hat{p}'_{j3}'[s'_j'][\eta']\rangle^+.
\]
Then we obtain
\[
(\psi^-,\phi^+) = \sum_{jj_3} \int \frac{d^3\hat{p}}{2\hat{E}} \sum_{j'j'_3} \int \frac{d^3\hat{p}'}{2\hat{E}'} \psi^-| -\hat{p}_{j3}[s_j]\rangle \langle \hat{p}'_{j3}'[s'_j'][\phi^+]
\]
where \(\hat{E} = \hat{E}(\hat{p}) \equiv \sqrt{1 + \hat{p}^2} = \hat{p}^0\).

Using the invariance of the \(S\) operator with respect to space time translations one can show that the \(S\)-matrix can be written in the following way
\[
\langle \hat{p}_{j3}[s_j][\eta]|S|\hat{p}'_{j3}'[s'_j'][\eta']\rangle = \delta(p - p')\delta(p_0 - p'_0) \langle \hat{p}_{j3}[s_j][\eta]|\tilde{S}|\hat{p}'_{j3}'[s'_j'][\eta']\rangle
\]
where \(\langle \tilde{S} \rangle\) is a reduced \(S\)-matrix element defined by (5.2). For later considerations it is important to note that the invariance does not have to be considered for the whole group of transformations; it is sufficient to consider a subsemigroup to obtain (5.2). The equation (5.2) can also be written
\[
\langle \hat{p}_{j3}[s_j][\eta]|S|\hat{p}'_{j3}'[s'_j'][\eta']\rangle = 2\hat{E}(\hat{p})\delta(\hat{p} - \hat{p}')\delta(s - s') \langle \hat{p}_{j3}[s_j][\eta]|\tilde{S}|\hat{p}'_{j3}'[s'_j'][\eta']\rangle
\]
where \(\langle \tilde{S} \rangle\) is another reduced matrix element defined by (5.3). In (5.2) and (5.3) we include explicitly the degeneracy quantum numbers and the species labels which we denote collectively by \(\eta\) for purposes of clarity and completion, but we will omit it below for the sake of notational convenience. The form (5.3) follows from (5.2) by the defining identities \(\hat{p} = \frac{p}{\sqrt{s}}, \hat{p}^0 = \frac{p^0}{\sqrt{s}}\).

We now use the invariance of the \(S\)-matrix with respect to Lorentz transformations either in the form
\[
(U(\Lambda)\psi^-)(U(\Lambda)\phi^+) = (\psi^-,\phi^+),
\]
\(^{18}\)This is just another more familiar way of writing the \(S\)-matrix elements on the right hand side.
or in the form

\[ U_{\text{free}}^\dagger(\Lambda)SU_{\text{free}}(\Lambda) = S \]

(5.4b)

With this we further simplify (5.3). We first choose \( \Lambda = L^{-1}(\hat{p}) \) where \( L(\hat{p}) \) is the boost (rotation free Lorentz transformation)

\[
L^\kappa_\nu = \begin{pmatrix}
\frac{p^\mu}{m} & -\frac{p_\mu}{m} \\
\frac{p_\mu}{m} & \frac{\delta_\kappa_\nu - \frac{p_\mu p_\nu}{1 + \frac{E^2}{m^2}}}{m}
\end{pmatrix}
\]  

(5.5)

\( L(\hat{p}) \) depends upon the parameter \( \hat{p} = \frac{\hat{p}}{m} \in \mathbb{R} \) and has the property

\[
L^{-1}(\hat{p})^\mu_\nu p_\nu = \begin{pmatrix}
m \\
0 \\
0 \\
0
\end{pmatrix}
\]  

(5.6)

From (5.4b) follows

\[
\langle \langle \hat{p}j_3[sj]|U^\dagger(L^{-1}(\hat{p})))SU(L^{-1}(\hat{p})))|\hat{p}j_3'[sj'] \rangle \rangle = \langle \langle 0j_3[sj]|S|0j_3'[sj'] \rangle \rangle = \langle \langle j_3[sj]|S|j_3'[sj] \rangle \rangle
\]

(5.7)

for all \( \hat{p} \in \mathbb{R}^3 \). This means the reduced \( S \)-matrix element is the same for all \( \hat{p} \) as in the center of mass frame, i.e., for \( \hat{p} = 0 \).

Invariance with respect to rotations, \( \Lambda = \mathcal{R} \) in the center of mass frame shows then by analogous arguments for the discrete quantum numbers \( j_3 \) and \( j \) that the reduced matrix element is proportional to \( \delta_{j_3j_3'}\delta_{jj'} \) and independent of \( j_3 \). Since Poincaré transformations do not change the Poincaré invariants \( s \) and \( j \), the reduced matrix element can still depend upon \( s \) and \( j \). Thus we have

\[
\langle \langle \hat{p}j_3[sj]|S|\hat{p}'j_3'[sj']\eta' \rangle \rangle = 2\hat{E}(\hat{p})\delta(\hat{p} - \hat{p}')\delta(s - s')\delta_{j_3j_3'}\delta_{jj'}
\]

\[
\langle \eta \parallel S_j(s) \parallel \eta' \rangle
\]

(5.8)

If there are no degeneracy quantum numbers, or if we suppress the particle species label and channel numbers and restrict ourselves to the case without spins (like for the \( \pi^+\pi^- \) system), then the reduced matrix element can be written as

\[
\langle \eta \parallel S_j(s) \parallel \eta' \rangle = S_j(s) = \begin{cases} 
2ia_j(s) + 1 \text{ for elastic scattering } \eta = \eta' \\
2ia_j^\dagger(\eta) (s) \text{ for a reaction from state } \eta' \text{ to } \eta
\end{cases}
\]

(5.9)
where \( j \) is the total angular momentum in the center of mass, and \( a_j(s) \) is the \( j \)-th partial wave amplitude for elastic scattering and \( a_j^0(s) \) is the amplitude for inelastic scattering (from \( \eta' \) into channel \( \eta \)). We insert (5.8) (or (5.9)) into (5.1) and integrate over \( \hat{\mathbf{p}} \) and \( s \) to obtain for the \( S \)-matrix element

\[
\langle \psi^-, \phi^+ \rangle = \sum_j \int_{s_0}^\infty ds \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} (\psi^- | \hat{p} j_3[s_j]^- \rangle S_j(s) \langle + \hat{p} j_3[s_j] | \phi^+ \rangle \ (5.10)
\]

Resonances are usually associated with a fixed value of \( j \) or \( (j, \text{parity}) \). This means, in the partial wave analysis of the experimental (differential cross section) data, a resonance is identified by one (or a superposition of several) Breit-Wigner amplitudes (2.5) in a partial wave amplitude \( a_j(s) \). The value \( j \) for this partial wave amplitude is then reported as the spin (or \( j^P \)) of the resonance [22]. In the \( S \)-matrix theory, resonances are defined as poles of the \( j \)-th partial \( S \)-matrix, \( S_j(s) \), at a complex value \( s_R \). Thus, by today's theoretical definition and experimental analysis, a resonance is assigned to one partial wave with definite value of angular momentum \( j \). Therefore, we consider of (5.10) only the term with the resonating partial wave \( j = j_R \) (e.g., \( j_R = 1 \) for \( \pi^+ \pi^- \rightarrow \rho^0 \rightarrow \pi^+\pi^- \)), i.e., we restrict ourselves to the subspace with \( j = j_R (s = 0, l = j, n = n_{\rho}, n_{\pi\pi}) \). This means that we consider only the term with \( j = j_R \) in the sum on the right hand side of (5.10) and call \( S_{j_R}(s) = S_j(s) \).

We can further simplify our notation. After we employed the 4-velocity basis vectors \(|\hat{p} j_3[s_j] \eta^-\rangle\) to make use of Poincaré invariance, we can ignore the quantum numbers \( j, \eta \) because they are fixed, and we can also suppress the quantum numbers \( \hat{p}, j_3 \) because they are summed (integrated) over in (5.10). We therefore use again the notation (4.25)

\[
\langle -\hat{p} j_3[s_j] | \psi^- \rangle \rightarrow \langle -s | \psi^- \rangle \in \mathcal{S} \cap \mathcal{H}_-^2 |_{R_0} \ (5.11)
\]
\[
\langle +\hat{p} j_3[s_j] | \phi^+ \rangle \rightarrow \langle +s | \phi^+ \rangle \in \mathcal{S} \cap \mathcal{H}_-^2 |_{R_0} \ (5.12)
\]

and also write for the basis vectors the short form (4.25d)

\[
|\hat{p}[s_j]^{-\top} \rangle \rightarrow |s^- \rangle \in \Phi_+ \ (5.13)
\]

We assume now that our following analytic extension in the variable \( s \) does not effect the values of the quantum numbers \( j_3 \) and \( \hat{p} = \frac{p}{\sqrt{s}} \) which will be justified below. Then we can write the \( j \)-th partial \( S \)-matrix element of
as

\[
(\psi^-, \phi^+) = \int_{s_0}^\infty ds \sum_j \int \frac{d^3\hat{p}}{2p_0} \langle \psi^- | \hat{p} j_3 [s j]^- \rangle S_j(s) \langle \hat{p} j_3 [s j] | \phi^+ \rangle
\]

\[
\equiv \int_{s_0}^\infty ds \langle \psi^- | s^- \rangle S_j(s) \langle s | \phi^+ \rangle
\]  

(5.14)

According to the standard analyticity assumptions \[34\] of the \( j \)-th partial \( S \)-matrix, \( S_j(s) \) is an analytic function on the first ("physical") sheet. The boundary values of this analytic function on the real axis \( s + i\epsilon \), \( \epsilon \to 0 \) are the "physical" values that appear in the integral (5.14). There may be poles on the real axis for values \( s < s_0 \), i.e., below the elastic scattering threshold \( s_0 = (m_a + m_b)^2 \). We ignore here such stable, bound state poles, if they exist. The two sheeted Riemann surface (in the simplest case that we consider here) has a cut that starts at \( s = s_0 \). To reach the second sheet one burrows through the cut, Figure 2. The integration contour of the integral (5.14) extends along the lower edge of the first sheet, right above the cut. If there is no further cut, which is the case we want to consider for the time being, then \( S(s + i\epsilon) = S(s - i\epsilon_{II}) \) along the cut \( s, s_0 \leq s < \infty \) where \( s - i\epsilon_{II} \) is on the second sheet. Thus we can as well extend the integration along the upper edge of the second sheet just below the cut. The second sheet of \( S(s) \) can contain singularities. We want to consider the case that there are only pole-singularities of the \( S \)-matrix.

In particular, we shall consider only first order poles and for the sake of definiteness we shall assume the case that there are 2 (or \( N \)) first order poles. Poles of higher order can be treated in a similar way and lead to Gamow states described by non diagonalizable density operators (and Jordan blocks) \[32\]. Cut-singularities in the lower half-plane can also be accommodated by additional background integrals, which we also do not want to consider here.

The \( S \)-matrix definition of a resonance is a first order pole on the second Riemann sheet at \( s_R = (M_R - i\Gamma_R/2)^2 \). This definition is of practical importance only for values of \( \frac{\Gamma_R}{M_R} \leq 10^{-1} \). Unstable states with \( \frac{\Gamma_R}{M_R} \approx 10^{-3} - 10^{-1} \) are usually called relativistic resonances, while those with \( \frac{\Gamma_R}{M_R} \approx 10^{-8} - 10^{-16} \) are called decaying relativistic particles (cf. Section 4).

The particular parameterization of the complex pole position \( s_R \) in terms of \( M_R \) and \( \Gamma_R \) is still arbitrary and will be given a physical meaning by our subsequent considerations.
Hermitian analyticity (symmetry relation of the $S$-matrix $S(s - i\epsilon) = S^*(s + i\epsilon)$) implies that when there is a pole $P$ at the complex position $s_R$ then there must also be a pole $P'$ at the complex conjugate position $s'_R = (M_R + \frac{i}{2}\Gamma_R)^2$ on the second sheet reached by burrowing through the cut from the lower half plane of the physical sheet. Thus a scattering resonance is defined by a pair of poles on the second sheet of the analytically continued $S$-matrix located at positions that are complex conjugates of each other. The pole $P'$ corresponds to the time-reversed situation which we do not want to discuss here. There may exist other resonance poles located at the same or other physical sheets, but we will restrict ourselves here to $N = 2$ poles in the lower half plane second sheet at $s_{R_1}$ and $s_{R_2'}$ as shown in Figure 2.

The contour of integration parallel to the real axis in the second sheet over $s_0 < s < \infty$ in (5.14) can now be deformed into the contour shown in Figure 2: an integral from $s_0$ to $-\infty$, then along the infinite semicircle $C_\infty$, around the two poles $C_1, C_2$ and again along the infinite semicircle $C_\infty$. After this contour deformation, the integral in (5.14) becomes (dropping the $j$ notation):

$$\langle \psi^-, \phi^+ \rangle = \int_{s_0}^{-\infty} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle s^+ | \phi^+ \rangle + \int_{C_1} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle s^+ | \phi^+ \rangle + \int_{C_2} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle s^+ | \phi^+ \rangle + \int_{C_\infty} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle s^+ | \phi^+ \rangle$$

(5.15)

where $C_i$ is the circle (in the negative direction) around the pole at $s_{R_i}$. The first integral in (5.13) extends along the negative real axis in the second sheet (indicated by $-\infty_{II}$). The fourth integral is along the infinite semicircle $C_\infty$. In order for this path deformation to be possible, the integrand in (5.14) and (5.15) must be well defined in the area into which the path will be deformed. This will be in our case the whole real line at the upper edge of the second sheet, i.e., for

$$\langle \psi^- | s_{II} - i\epsilon^- \rangle S_{II}(s_{II} - i\epsilon) \langle s_{II} - i\epsilon^+ | \phi^+ \rangle, \quad -\infty < s_{II} < \infty$$

and the entire lower plane of the second sheet. That $S(s)$ is well defined on the whole Riemann surface except for the singularities discussed above, and
that it is bounded by a polynomial, i.e., that there is a polynomial $P(s)$ such that

$$|S_{II}(s)| \leq |P(s)| \quad \text{for large } |s|,$$

are the standard assumptions of the $S$-matrix theory \[14\].

The functions $\langle \psi^-|s^- \rangle$ and $\langle ^{+}s| \phi^+ \rangle$ are known for the physical values $s = s_I + i\epsilon = s_{II} - i\epsilon$, $\mathbb{R}_{s_0} = \{ s : s_0 \leq s_{II} < \infty \}$. Our new hypothesis (4.29), (4.36) tells us that they are Hardy functions on $\mathbb{R}_{s_0}$, and since we are concerned with the second sheet for resonances, (4.29) or (5.11) (5.12) must precisely mean Hardy with respect to the second sheet. From the van-Winter theorem ((B.2), Appendix B) it follows that every Hardy function is completely determined from its values on a half axis of the real line. In other words, there exists a bijective mapping

$$\theta : \tilde{S} \cap \mathcal{H}_1^2 \rightarrow \left( \tilde{S} \cap \mathcal{H}_1^2 \right)|_{\mathbb{R}_{s_0}}$$

(5.17)

This means that the Hardy functions on the negative real axis $-\infty < s_{II} < s_0$ are already completely determined from their values for $s \in \mathbb{R}_{s_0}$ (scattering energies). Thus the $\langle -\psi|s^- \rangle$ and $\langle ^{+}s| \phi^+ \rangle$ are known for the entire real axis, second sheet. From this they can be determined on the entire semiplane using Titchmarsh theorem ((B.1), Appendix B). As a consequence of these remarkable properties of the Hardy functions, the integrand of (5.14) is uniquely defined on the whole lower semiplane second sheet and we can do the contour deformation throughout the lower semiplane, second sheet, of the complex energy Riemann surface.

Further, the integral along $C_\infty$ in (5.15) vanishes. To see this, notice that from (5.16)

$$\int_{C_\infty} |ds\langle \psi^-|s^- \rangle S(s)\langle ^{+}s| \phi^+ \rangle| \leq \int_{C_\infty} |ds\langle \psi^-|s^- \rangle P(s)\langle ^{+}s| \phi^+ \rangle|$$

(5.18)

From Proposition 4.2, it follows that $P(s)\langle ^{+}s| \phi^+ \rangle \in \tilde{S} \cap \mathcal{H}_1^2$. Hence, a straightforward application of Hölder’s inequality shows that

$$\langle \psi^-|s^- \rangle P(s)\langle ^{+}s| \phi^+ \rangle \in \mathcal{H}_1^-$$

(5.19)

With (5.18) and (5.19), the vanishing of the integral on $C_\infty$ follows then from Corollary B.1, Appendix B.
We shall now consider each of the remaining integrals in (5.15). The first integral has nothing to do with any of the resonances; it is the non-resonant background term

\[ \int_{s_0}^{\infty} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle +s | \phi^+ \rangle = \langle \psi^- | \phi^{bg} \rangle \]  

(5.20)

which we express as the matrix element of \( \psi^- \) with a generalized vector \( \phi^{bg} \) that is defined by it. It will not be further discussed in the present section.

In the integrals along the circles \( C_i \) around the poles \( s_{R_i} \) we use the expansion

\[ S(s) = \frac{R^{(i)}}{s - s_{R_i}} + R_0 + R_1(s - s_{R_i}) + \cdots \]  

(5.21)

for each of the two (or \( N \)) integrals separately. The integrals around the poles, the pole terms, are calculated in the following way

\[ \langle \psi^- , \phi^+ \rangle_{\text{pole term}} = \int_{C_i} ds \langle \psi^- | s^- \rangle S(s) \langle +s | \phi^+ \rangle \]  

(5.22)

\[ = \int_{C_i} ds \langle \psi^- | s^- \rangle \frac{R^{(i)}}{s - s_{R_i}} \langle +s | \phi^+ \rangle \]  

(5.23)

\[ = -2\pi i R^{(i)} \langle \psi^- | s_{R_i}^- \rangle \langle +s_{R_i} | \phi^+ \rangle \]  

(5.24)

\[ = \int_{-\infty}^{\infty} ds \langle \psi^- | s^- \rangle \frac{R^{(i)}}{s - s_{R_i}} \langle +s | \phi^+ \rangle \]  

(5.25)

To get from (5.23) to (5.24), the Cauchy theorem has been applied. To get from (5.23) to (5.25), the contour \( C_i \) of each integral has been separately deformed into the integral along the real axis from \(-\infty_{II} < s < \infty \) (and an integral along the infinite semicircle, which vanishes because of the Hardy class property). The equality (5.24) and (5.25) is the Titchmarsh theorem for Hardy class functions (B.1, Appendix B).

The integral (5.23) extends from \( s = -\infty \) in the second sheet along the real axis to \( s = s_0 \) and then from \( s = s_0 \) to \( s = \infty \) in either sheet. (It does not matter whether we take the second part of the integral over the physical values \( s, s_0 \leq s < \infty \) immediately below the real axis in the second sheet or in the first sheet immediately above the real axis). The major contribution to the integral comes from the physical values \( s_0 \leq s < \infty \), if the pole position
s_{R_i} is not too far from the real axis. The integral in (5.25) contains the Breit-Wigner amplitude

$$a_{j}^{BW} = \frac{R^{(i)}}{s - s_{R_i}}, \text{ but with } -\infty_{II} < s < \infty$$

Unlike the conventional Breit-Wigner for which s is taken over \(s_0 \leq s < \infty\), the Breit-Wigner in (5.26) is an idealized or exact Breit-Wigner whose domain extends to \(-\infty_{II}\) in the second (unphysical) sheet.

By (5.25) we have associated each resonance pole at \(s_{R_i}\) to an exact Breit-Wigner (5.26) which we obtain by omitting the integral over the well behaved functions \((-s|\psi^-)(+s|\phi^+) \in \hat{S} \cap H^2\) from (5.25). By (5.24) we have associated each resonance pole at \(s_{R_i}\) with vectors \(|s_{R_i}\rangle = |\hat{p}j_3[s_{R_i},j]^{-}\rangle\) which we call in analogy to (2.18) the relativistic Gamow vector or Gamow ket \([19]\\)

We obtain a representation of the Gamow vector by using the equality (5.24)= (5.25) and omitting the arbitrary \(\psi^- \in \Phi_+\) (which represents the decay products defined by the detector). Thus the defining relation of this ket (functional over \(\Phi^+\)) is

$$|s_{R_i}^{-}\rangle = \frac{i}{2\pi} \int_{-\infty_{II}}^{+\infty} |s^-\rangle \frac{1}{s - s_{R_i}} \langle +s|\phi^+ \rangle.$$  (5.27)

But various other “normalizations” will also be used, e.g., the one with the factor \(\sqrt{2\pi \Gamma}\) in (2.18). Reverting the short form notation (5.13) and using the notation that includes the degeneracy quantum numbers, the relativistic Gamow kets are usually defined as:

$$|\hat{p}j_3[s_{R_i},j]^{-}\rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds |\hat{p}j_3[s_j]\eta^-\rangle \frac{1}{s - s_{R_i}}.$$  (5.28)

The Gamow kets (5.28) are a superposition of the exact –not asymptotically free – “out-plane waves” \(|\hat{p}j_3[s_j]\eta^-\rangle\). The degeneracy quantum numbers \(\eta\) of the Gamow kets \(|\hat{p}j_3[s_{R_i},j]^{-}\rangle\) are the same as the ones chosen for the Lippmann-Schwinger out plane wave kets \(|\hat{p}j_3[s_j]\eta^-\rangle\). However, whereas for the Lippmann-Schwinger kets one can choose generalized eigenvectors of any
complete set of commuting observables, e.g., one could choose momentum eigenkets \(|p_j \eta\rangle\), one does not have the same freedom for the Gamow kets. Since in the contour deformations that one uses to get from (5.14) to (5.17), and ultimately to (5.22)–(5.25), one makes an analytic extension in the variable \(s\) to complex values. If one chooses the momentum to label the basis vectors then because of \(p_{\mu}p^{\mu} = s\), the \(p_{\mu}\) also change and become complex when \(s\) is extended to the complex plane. Thus, \(p_{\mu}\) could not be kept at one and the same value during this analytic continuation and the Gamow vector on the l.h.s. of (5.28) would be a complicated (continuous) superposition (integral) over different values of \(p\) and not just a superposition over different values of \(s\). For this reason, the momentum \(p\) is not a good choice as a label for the basis vectors. In contrast, the space components of the 4-velocity \(\hat{p} = p/\sqrt{s}\) is a good choice because then we can impose the condition that \(p\) will become complex in the analytic continuation to complex \(s\) in such a way that \(\hat{p}^{\mu} = p^{\mu}/\sqrt{s}\) will always be real. This condition restricts the arbitrariness of analytic continuation and makes the momentum only “minimally complex”. As we shall discuss later, minimally complex momentum keeps the representations of the Lorentz subgroup of the Poincaré group unitary. In the analytic continuation in \(s\) under the restriction that \(\hat{p}\) be real, only the representations of the space-time translations turn into (causal) semigroup representations. The homogeneous Lorentz transformations are the same as in Wigner’s representations. We will call this subclass of semigroup representations of \(\mathcal{P}\) minimally complex. They will be the subject of a subsequent paper [31].

With (5.26) and (5.28), we have obtained for each resonance defined by the pole of the \(j\)-th partial \(S\)-matrix at \(s = s_R\) an “exact” Breit-Wigner (5.26) and associated to it a set of “exact” Gamow kets (5.28). These Gamow kets span, like the Dirac kets \(|p_j \eta\rangle\) (4.1), (4.11) over the Schwartz space \(\Phi\), the space of an irreducible representation \([s_R,j]\) of Poincaré transformations. But, unlike the space spanned by the ordinary Dirac kets, the representation space spanned by the kets of (5.28) is not a representation space of a unitary group transformations.
We have the correspondence

\[ a_{j}^{BW}(s) = \frac{R_{i}}{s - s_{R}} \iff \langle [s_{R}j]f^{-} \rangle = \sum_{j_{3}} \int \frac{d^{3}\hat{p}}{2p_{0}} [\hat{p}j_{3}[s_{R}j]^{-}] f_{j_{3}}(\hat{p}) \] (5.29)

for \(-\infty < s < \infty\) for all functions \(f_{j_{3}}(\hat{p}) \in S(\mathbb{R}^{3}), -j \leq j_{3} \leq j\.

The Gamow vectors \(|\hat{p}[s_{R}j]f^{-}\rangle\) have, according to (5.26) and (5.28), also the representation

\[ \langle [s_{R}j]f^{-} \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds [s_{j}]f^{-} \frac{1}{s - s_{R}} \] (5.30)

They are functionals on the Hardy space \(\Phi_{+}\), i.e., \(|[s_{R}j]f^{-}\rangle \in \Phi_{+}^{*}\).

Equation (5.30) is reminiscent of the continuous basis vector expansion (4.26a) of \(\psi^{-} \in \Phi_{+} \subset H\) with respect to the generalized eigenvectors \(|[s_{j}]f^{-}\rangle\) of \(P_{\mu}P^{\mu}\) with eigenvalue \(s\). However, in (4.26a) \(s\) extends over \(s_{0} \leq s < \infty\), whereas in (5.30) \(s\) extends over \(-\infty < s < +\infty\) and the “wave function” \(\psi^{G}(s) \equiv \frac{i}{2\pi s - s_{R}}\) is not a well behaved Hardy function like \(\psi^{-}(s) \equiv \langle -s|\psi^{-}\rangle \in S \cap H^{2}_{+}\) of (4.26a). Also, in the exact Breit-Wigner “wave function” \(\psi^{G}(s) = \frac{i}{2\pi s - s_{R}}\) in (5.30), the variable \(s\) extends over \(-\infty < s < \infty\). Thus the continuous linear superpositions (5.30), which define the relativistic Gamow vectors, are entirely different mathematical entities than the \(\psi^{-}\) of (4.26).

The Gamow vectors \(|[s_{R}j]f^{-}\rangle\) and also the Gamow kets \(|\hat{p}j_{3}[s_{R}j]\eta^{-}\rangle\) of (5.28) are functionals over the space \(\Phi_{+}\). The equations (5.30) and (5.28) are thus functional equations over the space \(\Phi_{+}\), and (5.28) can be stated in terms of the smooth Hardy class functions \(\bar{\psi}^{-}(s) \equiv \langle -[s_{j}]|\bar{\psi}^{-}\rangle = \langle \psi^{-} | \hat{p}j_{3}[s_{j}]\eta^{-}\rangle \in \bar{S} \cap H^{2}_{+}\) as

\[ \langle \psi^{-} | \hat{p}j_{3}[s_{R}j]\eta^{-}\rangle \equiv -\frac{i}{2\pi} \oint ds \langle \psi^{-} | \hat{p}j_{3}[s_{j}]\eta^{-}\rangle > \frac{1}{s - s_{R}} \] (5.31)

\[ \langle \psi^{-} | \hat{p}j_{3}[s_{R}j]\eta^{-}\rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \langle \psi^{-} | \hat{p}j_{3}[s_{j}]\eta^{-}\rangle \frac{1}{s - s_{R}} \] (5.32)

for all \(\psi^{-} \in \Phi_{+}\). This is just another form of the definition (5.30) in terms of the well-behaved functions \(\langle \psi^{-} | \hat{p}j_{3}[s_{j}]\eta^{-}\rangle\) rather than the singular kets \(|\hat{p}j_{3}[s_{j}]\eta^{-}\rangle\); (5.32) is the Titchmarsh theorem for the Hardy function \(\langle \psi^{-} | \hat{p}j_{3}[s_{j}]\eta^{-}\rangle\).

\(^{20}\) The integral \(\oint\) is defined to be counter-clockwise whereas the integration around \(C_{i}\) in (5.23) is clockwise, cf. Figure 2. This explains the sign difference.
The first equality \((5.31)\) is again the well known Cauchy formula for the analytic function \(\psi^-(s) = \langle \psi^-|s^-\rangle\). The second equality \((5.32)\) is the Titchmarsh theorem (B1. of Appendix B) for the Hardy class function \(\psi^-(s)\) in the lower half plane of the second sheet. The integration path extends as in \((5.30)\) along the real axis in the second sheet, which agrees for physical values \(s_0 \leq s < \infty\) only with the integration along the real axis on the first sheet.

With \((5.29)\) we have associated a space of vectors \((5.30)\) with the Breit-Wigner partial wave amplitude \((5.26)\). The relativistic Breit-Wigner \((5.26)\) is the pole term of the relativistic \(j\)-th partial \(S\)-matrix element \((5.21)\). The vectors \((5.30)\) are spanned by the basis vectors \(|p_j[s_R]^3\rangle\) defined by \((5.28)\). This space of vectors \((5.29)\) is labeled by the complex generalized eigenvalue \(s_R\) (\(S\)-matrix pole position) and the angular momentum \(j\) of the partial wave in which the resonance occurs. This means the space of superpositions \((5.29)\) is very similar to Wigner’s unitary representation spaces of the group of Poincaré transformations for stable relativistic particles, the only difference being that Wigner’s representation spaces are characterized by real square mass \(m^2\) and by spin \(j [m^2j]\), whereas the spaces of \((5.29)\) are labeled by the complex number \(s_R\) and by \(j\), the total angular momentum of the scattering system of the \(j\)-th partial wave \(a_j(s)\).

The association \((5.29)\) between representation spaces \([sj]\) and partial wave amplitude \(a_j^{BW}(s)\) required a very specific form for the partial wave amplitude, namely the one given by the Cauchy kernel \((5.20)\). Similar associations of vectors to a partial wave amplitude will not be possible if the partial wave amplitude has not the special form of \((5.26)\). Even for the Breit-Wigner \((5.26)\) we had to extend the values of \(s\) from the phenomenologically testable values \(s_0 \leq s < \infty\) to the negative axis and introduce the idealization of an “exact” Breit-Wigner \((5.26)\) for which \(s\) extends over \(-\infty_{II} < s < +\infty\). Only for the exact Breit-Wigner \((5.26)\) could we use the Titchmarsh theorem in \((5.32)\) and associate to the amplitude \(a_j^{BW}(s)\) a vector which is defined by this exact Breit-Wigner amplitude. And in order to apply the Titchmarsh theorem we had to restrict the admissible wave functions \(\overline{\psi}(s) = \langle \psi^-|s^-\rangle\) and \(\phi^+(s) = \langle +s|\phi^+\rangle\) to be Hardy class in the lower half plane \((4.29)\). That means we had to specify the in-state vector \(\phi^+\) and the out-observable vector

\(^{21}\)For instance, it would not be possible for the most popular form of a relativistic Breit-Wigner with an energy dependent width given by the on-the-mass-shell renormalization scheme \([5, 20]\).
that can appear in the $S$-matrix element to be in the spaces $\Phi_-$ and $\Phi_+$, respectively. Only then could we define the Gamow kets $| [s_R] \eta^- \rangle$ in terms of the Dirac-Lippmann-Schwinger kets $| [s] \eta^- \rangle$ by (5.30) or (5.32), as functionals over the Hardy space $\Phi_+$. The Gamow vectors cannot be defined as functionals over the Schwartz space $\Phi$ like the usual Dirac kets. Thus the Hardy spaces $\Phi_-$ and $\Phi_+$, and therewith the new hypothesis (4.29), (4.36) had to be introduced (as in the non-relativistic theory [2]) in order to be able to construct vectors (5.28), (5.30) with a Breit-Wigner energy distribution.

Similarly to the Gamow vectors (5.28) and (5.30), we can define another kind of Gamow ket $| [s] \ast R [j] \eta^+ \rangle \in \Phi_-$ in terms of the Dirac-Lippmann-Schwinger kets $| [s] \eta^+ \rangle$ for the $S$-matrix pole at $s_\ast R = (M_R + i \Gamma/2)^2$ in the upper half plane of the second sheet. We do not want to discuss this here.

From the relativistic Gamow vectors we can now calculate consequences without any further mathematical assumption. This means they are just consequences of the hypothesis (4.29) (4.36) for the relativistic Gamow vectors defined as elements of $\Phi_\times$ from the $S$-matrix pole.

The relativistic Gamow vector $| \hat{p}_j [s_R] \rangle$ is a generalized eigenvector of $P^\mu$ with a complex eigenvalue. To see this, we use the Titchmarsh theorem (5.32) for the vector $\psi^- \equiv P^\mu \psi^- \in \Phi_+$:

$$
\langle P^\mu \psi^- | \hat{p}_j [s_R] \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \frac{\langle P^\mu \psi^- | \hat{p}_j [s] \rangle}{s - s_R} = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \frac{\sqrt{s} \hat{p}_\mu \langle \psi^- | \hat{p}_j [s] \rangle}{s - s_R} = \sqrt{s_R \hat{p}_\mu} \langle \psi^- | \hat{p}_j [s_R] \rangle .
$$

In (5.33), we used (4.33a) to write $\langle P^\mu \psi^- | \hat{p}_j [s] \rangle = \sqrt{s} \hat{p}_\mu \langle \psi^- | \hat{p}_j [s] \rangle$ and (1.12) to assert that $\sqrt{s} \hat{p}_\mu \langle \psi^- | \hat{p}_j [s] \rangle$ is again a Hardy function from below, so that Titchmarsh theorem (B.1), Appendix B can be applied to this Hardy function to obtain the last equality. Similarly, for the square mass operator $M^2 = P^\mu P^\mu$ we calculate

$$
\langle M^2 \psi^- | \hat{p}_j [s_R] \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \frac{\langle M^2 \psi^- | \hat{p}_j [s] \rangle}{s - s_R} = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \frac{s \langle \psi^- | \hat{p}_j [s] \rangle}{s - s_R} = s_R \langle \psi^- | \hat{p}_j [s_R] \rangle .
$$

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Equation (5.34), valid for all $\psi^{-} \in \Phi_{+}$, is the mathematical expression that $|\hat{p}\bar{J}_{3}[s_{R}j]\rangle$ is a generalized eigenvector of the square mass operator $M^{2}$ with the complex eigenvalue $s_{R}$. This is written equivalently as

$$M^{2} |\hat{p}\bar{J}_{3}[s_{R}j]\rangle = s_{R} |\hat{p}\bar{J}_{3}[s_{R}j]\rangle. \quad (5.35)$$

In the same way one can calculate the eigenvalue of the spin operator

$$\hat{W} = -\hat{w}_{\mu} \hat{w}^{\mu}, \quad \text{with} \quad \hat{w}^{\mu} = \epsilon^{\mu\nu\rho\sigma} \hat{P}_{\nu} J_{\rho\sigma}$$

where $\hat{P}_{\mu} = P_{\mu} M^{-1}$. One obtains, just in the same way as for the basis vectors of Wigner’s unitary representations

$$\hat{W} |\hat{p}\bar{J}_{3}[s_{R}j]\rangle = j(j + 1) |\hat{p}\bar{J}_{3}[s_{R}j]\rangle \quad (5.36a)$$

This means that the numbers $[s_{R}j]$ that label the spaces of Gamow vectors are indeed the eigenvalues of square mass and spin, only that this generalized eigenvalue $\sqrt{s}$ of the mass operator for the relativistic Gamow vectors is now a complex number. The representation spaces $[s_{R}j]$ for resonances is an eigenspace (of generalized eigenvectors) of the mass and spin operators

$$(P^{\mu} P_{\mu})^{\times} [s_{R}j] f^{-} = s_{R} [s_{R}j] f^{-} \quad (5.36a)$$

$$\hat{W} [s_{R}j] f^{-} = j(j + 1) [s_{R}j] f^{-} \quad (5.36b)$$

in complete analogy to the Wigner representation spaces $[m^{2}j]$ for stable particles.

In the subsequent paper we shall derive physically important properties of these representation spaces characterized by $[s_{R}j]$ and spanned by the Gamow kets (5.28). We shall show that, in contrast to Wigner’s unitary representations $[m^{2}j]$ for stable particles, the representations $[s,j]$ are not irreducible representation of the Poincaré group but representation spaces of the subsemigroup into the forward light cone. This will lead to causal propagation of Born probabilities.

In the subsequent paper, we shall also discuss the definition of resonance mass and width of a relativistic resonance. That the complex number $\sqrt{s_{R}}$, defined by the $S$-matrix pole position, characterizes a relativistic resonance does not yet tell us how one should parameterize this complex number in
terms of two real numbers which could be conveniently called resonance mass and width. One has an infinite number of possibilities to express the complex number $\sqrt{s_R}$ in terms of resonance mass and width of which some popular suggestions are \[5, 20, 37\]

$$\sqrt{s_R} = (M_R - i\Gamma_R/2) = \sqrt{M_Z^2 - iM_Z\Gamma_Z} = \sqrt{\frac{m_1^2 - im_1\Gamma_1}{1 + \frac{\Gamma_1}{m_1}}}$$ (5.37)

The lineshape of the resonance will not be sufficient to discriminate between them. The transformation property for the Gamow vectors, considered as state vectors of the resonance will allow us to choose precisely the values $M_R$ and $\Gamma_R$ as the mass and width of a relativistic resonance.

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Appendices

A Overview of Rigged Hilbert Space Concepts

A Rigged Hilbert Space \[30\] is the result of the completion of a scalar product space with respect to three different topologies. The completion of a vector space with respect to some topology $\tau$ amounts to including in this space the limit points of all $\tau$-Cauchy sequences. If one starts with a scalar-product space $\Psi$ and completes it with respect to the norm induced by the scalar product

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle},$$

a Hilbert space $\mathcal{H}$ is obtained. On the other hand, if $\Psi$ is completed with respect to a topology $\tau_\Phi$ defined by a countable number of norms with some qualifications, a countably normed space $\Phi$ is obtained. This countably normed topology $\tau_\Phi$ is finer than the Hilbert space topology $\tau_\mathcal{H}$ so that there are more $\tau_\Phi$-neighborhoods than $\tau_\mathcal{H}$-neighborhoods. Hence:

$$\Psi \subset \Phi \subset \mathcal{H}.$$
A third space of interest is the space of antilinear functionals on $\Phi$, denoted by $\Phi^\times$. Since $\Phi \subset \mathcal{H}$, it follows that $\mathcal{H}^\times \subset \Phi^\times$. But, from Hilbert space theory, $\mathcal{H} = \mathcal{H}^\times$. Hence

$$\Phi \subset \mathcal{H} \subset \Phi^\times. \quad \text{(A.1)}$$

The triplet (A.1) is called a Rigged Hilbert Space when $\Phi$ is nuclear and dense in $\mathcal{H}$ (with respect to $\tau_\mathcal{H}$). The fundamental physical axiom of the Rigged Hilbert Space formulation of quantum physics is that the set of states of the physical system do not inhabit the entire Hilbert space $\mathcal{H}$ but an appropriately defined dense subspace $\Phi$ of $\mathcal{H}$. The countably normed topology of $\Phi$ is constructed so as to yield the algebra of relevant physical observables continuous as mappings on $\Phi$. It is this feature of Rigged Hilbert Space theory that is made use of in Section 2 in making the distinction between the set of prepared states $\Phi^-$ and registered observables $\Phi^+$ by taking $\Phi^\pm$ as dense subspaces of the same Hilbert space $\mathcal{H}$ as in (2.15) (2.16). This distinction is what allows semigroup time evolution to be incorporated into the quantum mechanical theory.

The action of an element $F \in \Phi^\times$ on $\phi \in \Phi$, $F(\phi)$, is denoted—in the Dirac bra-ket notation—by

$$F(\phi) = \langle \phi | F \rangle.$$ 

Since $\mathcal{H} \subset \Phi^\times$, it follows that the Dirac bra-ket $\langle \phi | F \rangle$ is an extension of the Hilbert space scalar product in the sense that

$$\langle \phi | F \rangle = (\phi, F) \quad \text{for} \quad F \in \mathcal{H}.$$ 

The topology on $\Phi^\times$, denoted by $\tau_{\Phi^\times}$, is the weak*-topology induced by $\Phi$ on $\Phi^\times$. This means that convergence in $\Phi^\times$ is defined by

$$F_i \xrightarrow{\tau_{\Phi^\times}} F \iff \langle \phi | F_i \rangle \rightarrow \langle \phi | F \rangle, \quad \text{for all} \ \phi \in \Phi. \quad \text{(A.2)}$$

To every $\tau_\Phi$-continuous operator $A$ on $\Phi$, there corresponds a $\tau_{\Phi^\times}$-continuous operator $A^\times$ defined on $\Phi^\times$ by

$$\langle \phi | A^\times F \rangle \equiv \langle A\phi | F \rangle, \quad \text{for all} \ \phi \in \Phi, \ F \in \Phi^\times. \quad \text{(A.3)}$$

The operator $A^\times$ is called the conjugate operator of $A$. It is an extension of the Hilbert space adjoint operator $A^\dagger$, since for $F \in \mathcal{H}$ we have

$$\langle \phi | A^\times F \rangle = (A\phi, F) = (\phi, A^\dagger F) \quad \text{for} \ F \in \mathcal{H}. \quad \text{(A.4)}$$
Hence,

\[ A|\Phi \rangle \subset A^\dagger \subset A^\times. \] (A.5)

It should be stressed that the conjugate operator \(A^\times\) can be defined as a \(\tau_\Phi\)-continuous operator only when \(A\) is a continuous linear operator on \(\Phi\). In quantum mechanics, it is impossible (empirically) to restrict oneself to continuous (and therefore bounded) operators \(A\) in \(\mathcal{H}\). However, one can restrict oneself to algebras of observables \(\{A, B, \cdots\}\) described by continuous operators in \(\Phi\), if the topology of \(\Phi\) is suitably chosen. Then, \(A^\times, B^\times, \cdots\) are defined and continuous in \(\Phi^\times\).

A generalized eigenvector \(|F\rangle\) of a \(\tau_\Phi\)-continuous operator \(A\) with a generalized eigenvalue \(\omega \in \mathbb{C}\) is defined by the relation

\[ \langle A\phi|F\rangle = \langle \phi|A^\times F\rangle = \omega \langle \phi|F\rangle, \quad \text{for all } \phi \in \Phi. \] (A.6)

Since the vector \(\phi\) in (A.6) is arbitrary, (A.6) can be formally expressed as

\[ A^\times|F\rangle = \omega|F\rangle. \] (A.7)

In the Dirac notation the \(\times\) in (A.7) is suppressed so that (A.7) reads

\[ A|F\rangle = \omega|F\rangle. \] (A.8)

If \(A\) is a self-adjoint operator, suppressing the \(\times\) as in (A.8) does not lead to confusion since \(A = A^\dagger \subset \Phi^\times\). However, if \(A\) is not self-adjoint, a clear distinction between the operator and its conjugate should be made. The concept of generalized eigenvectors (A.7) in Rigged Hilbert Space mathematics allows the description of “eigenstates” which do not exist in the Hilbert space. For instance, the Dirac scattering kets are generalized eigenvectors with eigenvalues belonging to the continuous spectrum, and they are not Hilbert space elements. The Gamow vectors, which are used to describe decaying states, are also generalized eigenvectors which are not in \(\mathcal{H}\), but, unlike in the case of scattering states, their complex eigenvalues do not belong to the Hilbert space spectrum of the Hamiltonian.

B  Hardy Class Functions on a Half-plane

Definition B.1 (\(\mathcal{H}_p^\pm 1 \leq p < \infty\)). Appendix. A complex function \(f(x+iy)\) analytic in the open lower half complex plane \((y < 0)\) is said to be a
Hardy class function from below of order $p$, $\mathcal{H}_p^-$, if $f(x + iy)$ is $L^p$-integrable as a function of $x$ for any $y < 0$ and

$$\sup_{y < 0} \int_{-\infty}^{\infty} dx \ |f(x + iy)|^p < \infty.$$  \hspace{1cm} (B.1a)

Similarly, a complex function $f(x + iy)$ analytic in the open upper half complex plane ($y > 0$) is said to be a Hardy class function from above of order $p$, $\mathcal{H}_p^+$, if $f(x + iy)$ is $L^p$-integrable as a function of $x$ for all $y > 0$, and

$$\sup_{y > 0} \int_{-\infty}^{\infty} dx \ |f(x + iy)|^p < \infty.$$  \hspace{1cm} (B.1b)

A property of $\mathcal{H}_p^\pm$ functions is that their boundary values on the real axis exist almost everywhere and define an $L^p$-integrable function, i.e., if $f \in \mathcal{H}_p^\pm$, then its boundary values $f(x) \in L^p(\mathbb{R})$. Conversely, the values of any $\mathcal{H}_p^\pm$ function on the upper/lower half-plane are determined from its boundary values on the real axis. This result is provided by a theorem of Titchmarsh:

**Theorem B.1 (Titchmarsh theorem).** If $f \in \mathcal{H}_p^-$, then

$$f(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt , \quad \text{for} \ \text{Im} \ z < 0,$$

and

$$\int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt = 0 , \quad \text{for} \ \text{Im} \ z > 0.$$

Similarly, if $f \in \mathcal{H}_p^+$, then

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt , \quad \text{for} \ \text{Im} \ z > 0,$$

and

$$\int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt = 0 , \quad \text{for} \ \text{Im} \ z < 0.$$

This one-to-one correspondence between the $\mathcal{H}_p^\pm$ functions and their boundary values on $\mathbb{R}$ allows the identification of $f(z)$ with $f(x)$ for $f \in \mathcal{H}_p^\pm$.

The following results are related to the decay properties of the Hardy class functions. They are straightforward generalizations of the corresponding results of [24] and are needed for the construction of the relativistic Gamow vectors.
Proposition B.1. Let $C_\infty$ be the infinite semi-circle in the lower half complex plane. If $f \in \mathcal{H}_p^-$, then
\[ \int_{C_\infty} \frac{|f(z)|}{z} \, dz = 0. \]

Proof. Let $C_r$ be the arc with radius $r$ shown in Figure 3. Then
\[ \left| \int_{C_r} \frac{f(z)}{z} \, dz \right| \leq \int_{C_r} |f(re^{i\theta})| \, d\theta = \int_{1/r}^{\pi - 1/r} |f(-re^{i\theta})| \, d\theta. \]

Since $f \in \mathcal{H}_p^-$, then there exists $C$ such that
\[ |f(-re^{i\theta})| \leq \frac{C}{(r \sin \theta)^{1/p}}, \quad \text{(cf. [10] page 149).} \]

Thus
\[ \int_{C_r} \frac{|f(z)|}{z} \, dz \leq \frac{2C}{r^{1/p}} \int_{1/r}^{\pi/2} \frac{1}{(\sin \theta)^{1/p}} \, d\theta. \] (B.2)

Using
\[ \sin \theta \geq \theta - \theta^3/6 \geq \theta(1 - \pi^2/24), \quad \text{for } 1/r \leq \theta \leq \pi/2, \]
we obtain for (B.2)
\[ \int_{C_r} \frac{|f(z)|}{z} \, dz \leq \frac{2C}{r^{1/p}(1 - \pi^2/24)^{1/p}} \int_{1/r}^{\pi/2} \frac{d\theta}{\theta^{1/p}} \]
\[ = \frac{2C}{(1 - \pi^2/24)^{1/p} r^{1/p}} \left\{ \log \left( \frac{\pi}{2} \right) \right\} \begin{cases} \frac{1}{p} & 1 < p < \infty \end{cases} \]
\[ \left( \frac{\pi}{2} \right)^{1-1/p} - \left( \frac{1}{r} \right)^{1-1/p} \]
(B.3)

Therefore, as $r \to \infty$, we obtain
\[ \int_{C_\infty} \frac{|f(z)|}{z} \, dz = 0. \]

\[ \square \]

Corollary B.1. Let $f \in S \cap \mathcal{H}_2^-$, $g \in S \cap \mathcal{H}_2^-$, then
\[ \int_{C_\infty} |f(z)g(z)| \, dz = 0. \]
Proof. Since \( f \in S \cap H^2 \), then \( xf(x) \in S \cap H^2 \) [24]. A straightforward application of Hölder’s inequality shows that \( xf(x)g(x) \in H^1_\cdot \). Then, from the above lemma

\[
\int_{C_\infty} |f(z)g(z)dz| = \int_{C_\infty} \left| \frac{zf(z)g(z)}{z} \right| dz = 0.
\]

\[\square\]

A remarkable property of Hardy class functions that is used in [24] is that they are uniquely determined from their boundary values on a semi-axis on the real line. This result is provided by a theorem of van Winter [41]. Before stating the van Winter’s theorem below, we define first the Mellin transform

**Definition B.2 (Mellin transform).** Let \( f(x) \) be a function on \( \mathbb{R}_+ \). Its Mellin transform is a function defined almost everywhere on \( \mathbb{R} \) as

\[
H(s) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} f(x)x^{is-1/2}dx,
\]

provided that the integral exists for almost all \( s \in \mathbb{R} \).

**Theorem B.2 (van Winter).** A function \( f(x) \in L^2(\mathbb{R}^+) \) can be extended to \( \mathbb{R}^- = (-\infty, 0] \) to become a function in \( H^2_\cdot \) if and only if its Mellin transform satisfies

\[
\int_{-\infty}^{\infty} (1 + e^{2\pi s})|H(s)|^2 ds < \infty.
\]

This extension is unique. The values of \( f(z) \) for \( z = \rho e^{i\theta} \) for \( 0 \leq \theta \leq \pi \), \( \rho > 0 \) are given by

\[
f(\rho e^{i\theta}) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H(s)(\rho e^{i\theta})^{-is-1/2}ds.
\]

In particular for negative values, the function is given by

\[
f(-x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H(s)(xe^{i\pi})^{-is-1/2}ds.
\]

A similar result can be obtained for \( H^2_\cdot \).
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Figure 1a: Preparation of the controlled in-state.

Figure 1b: The uncontrolled out-state: $\phi_{out} = S \phi_{in}$. 
Figure 1c: Registration of the detector defined out-observable.

Figure 1d: Combining the preparation of the state and the registration of the observable in a scattering experiment.

Figure 1: The subdivision of a quantum mechanical scattering experiment into a registration part and a preparation part.
Figure 2: Contour of integration of (5.15) after contour deformation. The figure shows the second sheet of the $S$-matrix $S(s)$ with the cut from $s_0$ to $\infty$, and two resonance poles at $s_{R_1}$ and $s_{R_2}$ in the second sheet. The upper half is the first sheet of $S(s)$ which one reaches when one goes across the real axis between $s_0$ and $\infty$. The original contour of integration of (5.14) was along the cut. This contour is then deformed into the contour shown: $C_-$, $C_\infty$, $C_1$, $C_2$, $C_\infty$ which leads to the integrals in (5.15).

Figure 3: (Proof of Proposition B.1 of Appendix B). Arc $C_r$. 

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