ZEROS OF POLYNOMIALS WITH FOUR-TERM RECURRENCE

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Abstract. For any real numbers $b, c \in \mathbb{R}$, we form the sequence of polynomials \( \{H_m(z)\}_{m=0}^{\infty} \) satisfying the four-term recurrence
\[
H_m(z) + cH_{m-1}(z) + bH_{m-2}(z) + zH_{m-3}(z) = 0, \quad m \geq 3,
\]
with the initial conditions $H_0(z) = 1$, $H_1(z) = -c$, and $H_2(z) = -b + c^2$. We find necessary and sufficient conditions on $b$ and $c$ under which the zeros of $H_m(z)$ are real for all $m$, and provide an explicit real interval on which $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense where $\mathcal{Z}(H_m)$ is the set of zeros of $H_m(z)$.

1. Introduction

Consider the sequence of polynomials \( \{H_m(z)\}_{m=0}^{\infty} \) satisfying a finite recurrence
\[
\sum_{k=0}^{n} a_k(z)H_{m-k}(z) = 0, \quad m \geq n,
\]
where $a_k(z)$, $1 \leq k \leq n$, are complex polynomials. With certain initial conditions, one may ask for the locations of the zeros of $H_m(z)$ on the complex plane. There are two common approaches: asymptotic location and exact location. The first direction asks for a curve where the zeros of $H_m(z)$ approach as $m \to \infty$, and the second direction aims at a curve where the zeros of $H_m(z)$ lie exactly on for all $m$ (or at least for all large $m$). Recent works in the first direction include [1, 2, 3, 5, 6]. Results in this first direction are useful to prove the necessary condition for the reality of zeros of $H_m(z)$ that we will see in Section 3.

It is not an easy problem to find an explicit curve (if such exists) where the zeros of all $H_m(z)$ lie for a general recurrence in (1.1). For three-term recurrences with degree two and appropriate initial conditions, the curve containing zeros is given in [10]. The corresponding curve for a three-term recurrence with degree $n$ is given in [11]. Among all possible curves containing the zeros of the $H_m(z)$s, the real line plays an important role. We say that a polynomial is hyperbolic if all of its zeros are real. There are a lot of recent works on hyperbolic polynomials and on linear operators preserving hyperbolicity of polynomials, see for example [4, 7]. In some cases, we could find the curve on the plane containing the zeros of a sequence of polynomials by claiming certain polynomials are hyperbolic, see for example [8].

The main result of this paper is the identification of necessary and sufficient conditions on $b, c \in \mathbb{R}$ under which the zeros of the sequence of polynomials $H_m(z)$ satisfying the recurrence
\[
H_m(z) + cH_{m-1}(z) + bH_{m-2}(z) + zH_{m-3}(z) = 0, \quad m \geq 3,
\]
(1.2)
\[
H_0(z) \equiv 1, \quad H_1(z) \equiv -c, \quad H_2(z) \equiv -b + c^2,
\]
are real. We use the convention that the zeros of the constant zero polynomial are real. Let \( Z(H_m) \) denote the set of zeros of \( H_m(z) \).

**Theorem 1.** Suppose \( b, c \in \mathbb{R} \), and let \( \{H_m(z)\}_{m=0}^{\infty} \) be defined as in (1.2). The zeros of \( H_m(z) \) are real for all \( m \) if and only if one of the two conditions below holds

(i) \( c = 0 \) and \( b \geq 0 \)
(ii) \( c \neq 0 \) and \( -1 \leq b/c^2 \leq 1/3 \).

Moreover, in the first case with \( b > 0 \), \( \bigcup_{m=0}^{\infty} Z(H_m) \) is dense on \( (-\infty, \infty) \). In the second case, \( \bigcup_{m=0}^{\infty} Z(H_m) \) is dense on the interval

\[
\left( c^3, -\frac{2 + 9b/c^2 - 2\sqrt{(1 - 3b/c^2)^3}}{27} \right].
\]

Our paper is organized as follows. In Section 2, we prove the sufficient condition for the reality of the zeros of all \( H_m(z) \) in the case \( c \neq 0 \). The case \( c = 0 \) follows from similar arguments whose key differences will be mentioned in Section 3. Finally, in Section 4, we prove the necessary condition for the reality of the zeros of \( H_m(z) \).

**2. The case \( c \neq 0 \) and \( -1 \leq b/c^2 \leq 1/3 \)**

We write the sequence \( H_m(z) \) in (1.2) using its generating function

\[
\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + ct + bt^2 + zt^3}.
\]

From (2.1), if we make the substitutions \( t \to t/c \), \( b/c^2 \to a \), and \( z/c^3 \to z \), it suffices to prove the following form of the theorem.

**Theorem 2.** Consider the sequence of polynomials \( \{H_m(z)\}_{m=0}^{\infty} \) generated by

\[
\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + t + at^2 + zt^3}
\]

where \( a \in \mathbb{R} \). If \( -1 \leq a \leq 1/3 \) then the zeros of \( H_m(z) \) lie on the real interval

\[
I_a = \left( -\infty, -\frac{2 + 9a - 2\sqrt{(1 - 3a)^3}}{27} \right]
\]

and \( \bigcup_{m=0}^{\infty} Z(H_m) \) is dense on \( I_a \).

We will see later that the density of the union of zeros on \( I_a \) follows naturally from the proof that \( Z(H_m) \subset I_a \) and thus we focus on proving this claim. We note that each value of \( a \in [-1, 1/3] \) generates a sequence of polynomials \( H_m(z, a) \). The lemma below asserts that it suffices to prove that \( Z(H_m(z, a)) \subset I_a \) for all \( a \) in a dense subset of \([-1, 1/3]\).

**Lemma 3.** Let \( S \) be a dense subset of \([-1, 1/3]\), and let \( m \in \mathbb{N} \) be fixed. If \( Z(H_m(z, a)) \in I_a \)
for all $a \in S$ then

$$Z(H_m(z, a^*)) \in I_a^*$$

for all $a^* \in [-1, 1/3]$.

**Proof.** Let $a^* \in [-1, 1/3]$ be given. By the density of $S$ in $[-1, 1/3]$, we can find a sequence $\{a_n\}$ in $S$ such that $a_n \to a^*$. For any $z^* \notin I_{a^*}$, we will show that $H_m(z^*, a^*) \neq 0$. We note that the zeros of $H_m(z, a_n)$ lie on the interval $I_{a_n}$ whose right endpoint approaches the right endpoint of $I_{a^*}$ as $n \to \infty$. If we let $z_k^{(n)}, 1 \leq k \leq \deg H_m(z, a_n)$, be the zeros of $H_m(z, a_n)$ then

$$|H_m(z^*, a_n)| = \gamma^{(n)} \prod_{k=1}^{\deg H_m(z, a_n)} |z^* - z_k^{(n)}|$$

where $\gamma^{(n)}$ is the lead coefficient of $H_m(z, a_n)$. We will see in the next lemma that $\deg H_m(z, a_n)$ is at most $\lfloor m/3 \rfloor$. From this finite product and the assumption that $z^* \notin I_a$, we conclude that there is a fixed (independent of $n$) $\delta > 0$ so that $|H_m(z^*, a_n)| > \delta$, for all large $n$. Since $H_m(z^*, a)$ is a polynomial in $a$ for any fixed $z^*$, we conclude that

$$H_m(z^*, a^*) = \lim_{n \to \infty} H_m(z^*, a_n) \neq 0$$

and the lemma follows. \qed

Lemma 3 allows us to ignore some special values of $a$. In particular, we may assume $a \neq 0$. In our main approach, we count the number of zeros of $H_m(z)$ on the interval $I_a$ in (2.3) and show that this number of zeros is at least the degree of $H_m(z)$. To count the number of zeros of $H_m(z)$ on $I_a$, we write $z = z(\theta)$ as a strictly increasing function of a variable $\theta$ on the interval $(2\pi/3, \pi)$. Then we construct a function $g_m(\theta)$ on $(2\pi/3, \pi)$ with the property that $\theta$ is a zero of $g_m(\theta)$ on $(2\pi/3, \pi)$ if and only if $z(\theta)$ is a zero of $H_m(z)$ on $I_a$. From this construction, we count the number of zeros of $g_m(\theta)$ on $(2\pi/3, \pi)$ which will be the same as the number of zeros of $H_m(z)$ on $I_a$ by the monotonicity of the function $z(\theta)$. We first obtain an upper bound for the degree of $H_m(z)$ and provide heuristic arguments for the formulas of $z(\theta)$ and $g_m(\theta)$.

**Lemma 4.** The degree of the polynomial $H_m(z)$ defined by (2.2) is at most $\lfloor m/3 \rfloor$.

**Proof.** We rewrite (2.2) as

$$(1 + t + at^2 + zt^3) \sum_{m=0}^{\infty} H_m(z) t_m = 1.$$

By equating the coefficients in $t$ of both sides, we see that the sequence $\{H_m(z)\}_{m=0}^{\infty}$ satisfies the recurrence

$$H_{m+3}(z) + H_{m+2}(z) + aH_{m+1}(z) + zH_m(z) = 0$$

and the initial condition $H_0(z) \equiv 1$, $H_1(z) \equiv -1$, and $H_2(z) \equiv 1 - a$. The lemma follows from induction. \qed

2.1. **Heuristic arguments.** We now provide heuristic arguments to motivate the formulas for two special functions $z(\theta)$ and $g_m(\theta)$ on $(2\pi/3, \pi)$. Let $t_0 = t_0(z)$, $t_1 = t_1(z)$, and $t_2 = t_2(z)$ be the three zeros of the denominator $1 + t + at^2 + zt^3$. We will show rigorously in Section 2.2 that $t_0, t_1, t_2$ are nonzero and distinct with $t_0 = t_1$. We let $q = t_1/t_0 = e^{2i\theta}, \theta \neq 0, \pi$. We have

$$\sum_{m=0}^{\infty} H_m(z) t_m = \frac{1}{1 + t + at^2 + zt^3} = \frac{1}{z(t - t_0)(t - t_1)(t - t_2)}.$$
We apply partial fractions to rewrite the generating functions $1/(z - t_0)(t - t_1)(t - t_2)$ as

$$
\frac{1}{z(t - t_0)(t_0 - t_1)(t_0 - t_2)} + \frac{1}{z(t - t_1)(t_1 - t_0)(t_1 - t_2)} + \frac{1}{z(t - t_2)(t_2 - t_0)(t_2 - t_1)}
$$

which can be expanded as a series in $t$ below

$$(2.4) \quad - \sum_{m=0}^{\infty} \frac{1}{z} \left( \frac{1}{(t_0 - t_1)(t_0 - t_2)t_0^{m+1}} + \frac{1}{(t_1 - t_0)(t_1 - t_2)t_1^{m+1}} + \frac{1}{(t_2 - t_0)(t_2 - t_1)t_2^{m+1}} \right) t^m.
$$

From this expression, we deduce that $z$ is a zero of $H_m(z)$ if and only if

$$(2.5) \quad \frac{1}{(1 - t_1/t_0)(1 - t_2/t_0)} + \frac{1}{(t_1/t_0 - 1)(t_1/t_0 - t_2/t_0)(t_1/t_0)^{m+1}} + \frac{1}{(t_2/t_0 - 1)(t_2/t_0 - t_1/t_0)(t_2/t_0)^{m+1}} = 0.
$$

After multiplying the left side of (2.5) by $t_0^{m+3}$ we obtain the equality

$$
\frac{1}{(1 - t_1/t_0)(1 - t_2/t_0)} + \frac{1}{(t_1/t_0 - 1)(t_1/t_0 - t_2/t_0)(t_1/t_0)^{m+1}} + \frac{1}{(t_2/t_0 - 1)(t_2/t_0 - t_1/t_0)(t_2/t_0)^{m+1}} = 0.
$$

Setting $\zeta = t_2/t_0 e^{i\theta}$, we rewrite the left side as

$$
\frac{1}{(1 - e^{2i\theta})(1 - e^{i\theta})} + \frac{1}{(e^{2i\theta} - 1)(e^{2i\theta} - \zeta e^{i\theta})(e^{2i\theta})^{m+1}} + \frac{1}{(\zeta e^{i\theta} - 1)(\zeta e^{i\theta} - e^{2i\theta})(\zeta e^{i\theta})^{m+1}}.
$$

or equivalently

$$
\frac{1}{e^{2i\theta}(-2i \sin \theta)(e^{-i\theta} - \zeta)} + \frac{1}{(2i \sin \theta)(e^{i\theta} - \zeta)(e^{2i\theta})^{m+2}} + \frac{1}{(\zeta - e^{-i\theta})(\zeta - e^{i\theta})(\zeta)^{m+1}(e^{i\theta})^{m+3}}.
$$

We multiply this expression by $(\zeta - e^{-i\theta})(\zeta - e^{i\theta})e^{i(m+3)\theta}$ and set the summation to 0 and rewrite (2.5) as

$$
0 = \frac{(\zeta - e^{i\theta})e^{i(m+1)\theta}}{2i \sin \theta} + \frac{e^{-i\theta} - \zeta}{(2i \sin \theta)e^{i(m+1)\theta}} + \frac{1}{\zeta^{m+1}}
$$

$$
= \frac{(\zeta - e^{i\theta})e^{i(m+1)\theta} - (\zeta - e^{-i\theta})e^{-i(m+1)\theta}}{2i \sin \theta} + \frac{1}{\zeta^{m+1}}
$$

$$
= \frac{\zeta(e^{i(m+1)\theta} - e^{-i(m+1)\theta}) + e^{-i(m+2)\theta} - e^{i(m+2)\theta}}{2i \sin \theta} + \frac{1}{\zeta^{m+1}}
$$

$$
= \frac{\zeta(2i \sin(m + 1)\theta - 2i \sin(m + 2)\theta)}{2i \sin \theta} + \frac{1}{\zeta^{m+1}}
$$

$$
= \frac{2i \zeta \sin(m + 1)\theta - 2i \sin(m + 1)\theta \cos - 2i \cos(m + 1)\theta \sin \theta}{2i \sin \theta} + \frac{1}{\zeta^{m+1}}
$$

$$
= \frac{(\zeta - \cos \theta) \sin(m + 1)\theta - \cos(m + 1)\theta}{\sin \theta} + \frac{1}{\zeta^{m+1}}.
$$

(2.6)

The last expression will serve as the definition of $g_m(\theta)$. 
We next provide a motivation for the specific form of $z(\theta)$. Since $t_0$, $t_1$, and $t_2$ are the zeros of $D(t, z) = 1 + at^2 + zt^3$, they satisfy the three identities

\[ t_0 + t_1 + t_2 = -\frac{a}{z}, \]
\[ t_0t_1 + t_0t_2 + t_1t_2 = \frac{1}{z}, \quad \text{and} \]
\[ t_0t_1t_2 = -\frac{1}{z}. \]

If we divide the first equation by $t_0$, the second by $t_2^2$, and the third by $t_3^3$ then these identities become

\[ 1 + e^{2i\theta} + \zeta e^{i\theta} = -\frac{a}{zt_0}, \quad (2.7) \]
\[ e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta} = \frac{1}{zt_0^2}, \quad \text{and} \]
\[ \zeta e^{3i\theta} = -\frac{1}{zt_0^3}. \quad (2.8) \]

We next divide the first identity by second, and the second by the third to obtain

\[ \frac{1 + e^{2i\theta} + \zeta e^{i\theta}}{e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta}} = -at_0, \quad \text{and} \]
\[ \frac{e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta}}{\zeta e^{3i\theta}} = -t_0, \]

from which we deduce that

\[ (1 + e^{2i\theta} + \zeta e^{i\theta})\zeta e^{3i\theta} = a(e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta})^2. \]

This equation is equivalent to

\[ (e^{-i\theta} + e^{i\theta} + \zeta)e^{4i\theta} = ae^{4i\theta}(1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^2, \]

or simply

\[ (2 \cos \theta + \zeta)\zeta = a(1 + 2\zeta \cos \theta)^2. \]

**Lemma 5.** For any $a \in [-1, 1/3]$ and $\theta \in (2\pi/3, \pi)$, the zeros in $\zeta$ of the polynomial

\[ (2 \cos \theta + \zeta)\zeta = a(1 + 2\zeta \cos \theta)^2 \]

are real and distinct.

**Proof.** We consider the discriminant of the above polynomial in $\zeta$:

\[ \Delta = (1 - 4a) \cos^2 \theta + a. \]

There are three possible cases depending on the value of $a$. If $1/4 \leq a \leq 1/3$, the inequality $\Delta > 0$ comes directly from $a \geq 4a - 1 > (1 - 4a) \cos^2 \theta$. If $0 \leq a < 1/4$, the claim $\Delta > 0$ is trivial since $1 - 4a > 0$. Finally, if $a < 0$, we have $\Delta \geq (1 - 4a)/4 + a = 1/4$. It follows that the zeros of (2.10) are real and distinct for any $a \in [-1, 1/3]$ and $\theta \in (2\pi/3, \pi)$.

To obtain the formula $z(\theta)$, we multiply both sides of (2.7) and (2.8)

\[ (1 + e^{2i\theta} + \zeta e^{i\theta})(e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta}) = -\frac{a}{zt_0^3}. \]
and divide by (2.13) to arrive at the resulting equation

\[
\frac{ae^{3i\theta}\zeta}{(1 + e^{2i\theta} + \zeta e^{i\theta})(e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta})} = \frac{ae^{3i\theta}\zeta}{e^{3i\theta}(e^{-i\theta} + e^{i\theta} + \zeta)(1 + \zeta e^{-i\theta} + \zeta e^{i\theta})} = \frac{a\zeta}{(2 \cos \theta + \zeta)(1 + 2 \zeta \cos \theta)}.
\]

(2.11)

2.2. Rigorous proof. Motivated by Section 2.1, we will rigorously prove Theorem (2) in this Section 2.2. We start our proof of Theorem (2) by defining the function \( \zeta(\theta) \) according to (2.10):

\[
\zeta = \zeta(\theta) = \frac{(2a - 1) \cos \theta + \sqrt{(1 - 4a) \cos^2 \theta + a}}{1 - 4a \cos^2 \theta}
\]

which, from Lemma (3), is a real function on \((2\pi/3, \pi)\) with a possible vertical asymptote at

\[
\theta = \cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right)
\]

when \(1/4 < a \leq 1/3\). However we note that the function \(1/\zeta(\theta)\) is a real continuous function on \((2\pi/3, \pi)\).

Lemma 6. Let \( \zeta(\theta) \) be defined as in (2.12). Then \(|\zeta(\theta)| > 1\) for every \(a \in (-1, 1/3)\) and every \(\theta \in (2\pi/3, \pi)\) with \(1 - 4a \cos^2 \theta \neq 0\).

Proof. From (2.10), we note that \(\zeta_+ := \zeta(\theta)\) and \(\zeta_- := \frac{(2a - 1) \cos \theta - \sqrt{(1 - 4a) \cos^2 \theta + a}}{1 - 4a \cos^2 \theta}\) are the zeros of

\[
f(\zeta) := (2 \cos \theta + \zeta)\zeta - a(1 + 2\zeta \cos \theta)^2.
\]

Note that

\[
f(-1)f(1) = (-1 + 2 \cos \theta)(1 + 2 \cos \theta)(4a^2 \cos^2 \theta - (a - 1)^2).
\]

If \(\theta \in (2\pi/3, \pi)\) and \(a \in (-1, 1/3)\), this product is negative since

\[
4a^2 \cos^2 \theta - (a - 1)^2 \leq 4a^2 - (a - 1)^2 = (a + 1)(3a - 1) < 0.
\]

Thus exactly one of the zeros of the quadratic function \(f(\zeta)\) lies outside the interval \([-1, 1]\). The claim follows from the fact that |\(\zeta_+\)| > |\(\zeta_-\)|. \(\square\)

Although one can prove Lemma (3) for the extreme value \(a = -1\) or \(a = 1/3\), that will not be necessary by Lemma (3). Next we define the real function \(z(\theta)\) by

\[
z = z(\theta) := \frac{a\zeta}{(2 \cos \theta + \zeta)(1 + 2 \zeta \cos \theta)}
\]

on \((2\pi/3, \pi)\). From Lemma (3) \(1 + 2\zeta \cos \theta \neq 0\) and so does \(2 \cos \theta + \zeta\) by (2.10). Dividing the numerator and the denominator by \(\zeta^2(\theta)\) and combining with the fact that \(1/\zeta(\theta)\) is continuous on \((2\pi/3, \pi)\), we conclude that the possible discontinuity of \(z(\theta)\) in (2.13) is removable. Finally, motivated by (2.6), we define the function \(g_m(\theta)\) by

\[
g_m(\theta) := \frac{(\zeta - \cos \theta) \sin (m + 1)\theta}{\sin \theta} - \cos (m + 1)\theta + \frac{1}{\zeta^{m+1}}
\]
which has the same vertical asymptote as that of $\zeta(\theta)$ in \((2.13)\) when $1/4 < a \leq 1/3$.

From Lemma 6, we see that the sign of the function $g_m(\theta)$ alternates at values of $\theta$ where $\cos(m + 1)\theta = \pm 1$. Thus by the Intermediate Value Theorem the function $g_m(\theta)$ has at least one root on each subinterval whose endpoints are the solution of $\cos(m + 1) = \pm 1$. However, in the case $1/4 \leq a \leq 1/3$, one of the subintervals will contain a vertical asymptote given in \((2.13)\). The lemma below counts the number of zeros of $g_m(\theta)$ on such a subinterval.

**Lemma 7.** Let $g_m(\theta)$ be defined as in \((2.15)\). Suppose $1/4 < a \leq 1/3$ and $m \geq 6$. Then there exists $h \in \mathbb{N}$ such that

$$\theta_{h-1} := \frac{h - 1}{m + 1} \pi < \cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \leq \frac{h}{m + 1} \pi =: \theta_h$$

where $\lfloor 2(m + 1)/3 \rfloor + 1 \leq h - 1 < h \leq m + 1$. Furthermore, as long as

$$\cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \neq \frac{h}{m + 1} \pi,$$

the function $g(\theta)$ has at least two zeros on the interval

$$\theta \in \left(\frac{h - 1}{m + 1} \pi, \frac{h}{m + 1} \pi\right) := J_h$$

whenever $h$ is at most $m$, and at least one zero when $h$ is $m + 1$.

**Proof.** Suppose $a \in [1/4, 1/3]$. Since the function $\cos^{-1}\left(-1/2\sqrt{a}\right)$ is decreasing on the interval $[1/4, 1/3]$, we conclude that

$$\cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \geq \frac{5\pi}{6}.$$

Then the existence of $h$ comes directly from the inequality

$$\frac{\lfloor 2(m + 1)/3 \rfloor + 1}{m + 1} \pi < \frac{5\pi}{6},$$

when $m \geq 6$.

The vertical asymptote at $\cos^{-1}\left(-1/2\sqrt{a}\right)$ of $g_m(\theta)$ divides the interval $J_h$ in \((2.17)\) into two subintervals. We will show that each subinterval contains at least a zero of $g_m(\theta)$ if $h \leq m$. In the case $h = m + 1$, only the subinterval on the left contains at least zero of $g_m(\theta)$. We analyze these two subintervals in the two cases below.

We consider the first case when $\theta \in I_h$ and $\theta < \cos^{-1}\left(-1/2\sqrt{a}\right)$. By Lemma 6 and \((2.15)\) we see that the sign of $g_m(\theta_{h-1})$ is $(-1)^{h-1}$. We now show that the sign of $g_m(\theta)$ is $(-1)^{h-1}$ when $\theta \rightarrow \cos^{-1}\left(-1/2\sqrt{a}\right)$.

From \((2.12)\), we observe that $\zeta(\theta) \rightarrow +\infty$ as $\theta \rightarrow \cos^{-1}\left(-1/2\sqrt{a}\right)$. Since $\theta \in I_h$, the sign of $\sin(m + 1)\theta$ is $(-1)^{h-1}$ and consequently the sign of $g_m(\theta)$ is $(-1)^{h-1}$ when $\theta \rightarrow \cos^{-1}\left(-1/2\sqrt{a}\right)$ by \((2.15)\). By the Intermediate Value Theorem, we obtain at least one zero of $g_m(\theta)$ in this case.

Next we consider the case when $\theta \in I_h$ and $\theta > \cos^{-1}\left(-1/2\sqrt{a}\right)$. In this case the sign of $g_m(\theta_h)$ is $(-1)^{h-1}$ if $h \leq m$ by Lemma 6. Since $\zeta(\theta) \rightarrow -\infty$ as $\theta \rightarrow \cos^{-1}\left(-1/2\sqrt{a}\right)$ and the sign of $\sin(m + 1)\theta$ is $(-1)^{h-1}$, the sign of $g_m(\theta)$ is $(-1)^h$ as $\theta \rightarrow \cos^{-1}\left(-1/2\sqrt{a}\right)$. By the Intermediate Value Theorem, we obtain at least one zero of $g_m(\theta)$ in this case and $h \leq m$.

Note that we may assume \((2.15)\) by Lemma 6. From the fact that the sign of $g_m(\theta)$ in \((2.15)\) alternates when $\cos(m + 1)\theta = \pm 1$, we can find a lower bound for the number of zeros of $g_m(\theta)$ on $\left(2\pi/3, \pi\right)$ by the Intermediate Value Theorem. We will relate the zeros of $g_m(\theta)$ to the zeros of
Proof. We first note that \( t_0 = \frac{e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta}}{\zeta e^{3i\theta}} \), \( t_1 = t_0 e^{2i\theta} \), \( t_2 = t_0 e^{2i\theta} \).

where \( \zeta := \zeta(\theta) \) is given in (2.12).

Lemma 8. Let \( \theta \in (0, \pi) \) such that \( \theta \neq \cos^{-1}( -1/2\sqrt{a} ) \) whenever \( a > 1/4 \). The zeros in \( t \) of \( 1 + t + at^2 + z(\theta)t^3 \) are

\[
t_0 = -\frac{e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta}}{\zeta e^{3i\theta}}, \quad t_1 = t_0 e^{2i\theta}, \quad t_2 = t_0 e^{2i\theta}
\]

where \( \zeta := \zeta(\theta) \) is given in (2.12).

Proof. We first note that

\[
P(t_0) = 1 + t_0 + at_0^2 + zt_0^3
\]

\[
= -\frac{1}{\zeta e^{i\theta}} - e^{2i\theta} + \frac{a}{\zeta^2 e^{2i\theta}} (1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^2 - \frac{z}{\zeta^3 e^{3i\theta}} (1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^3.
\]

where \( \zeta \) is a root of the quadratic equation \( (2 \cos \theta + \zeta) - a(1 + 2 \cos \theta)^2 = 0 \). We apply the following identities

\[
(1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^2 = (1 + 2 \zeta \cos \theta)^2 = \frac{1}{a} (2 \cos \theta + \zeta)\zeta = \frac{1}{a} (e^{-i\theta} + e^{i\theta} + \zeta)\zeta
\]

and

\[
z = \frac{a \zeta}{(2 \cos \theta + \zeta)(1 + 2 \zeta \cos \theta)} = \frac{\zeta^2}{(1 + 2 \zeta \cos \theta)^3} = \frac{\zeta^2}{(1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^3}
\]

to conclude that \( P(t_0) = 0 \). Similarly, we have

\[
P(t_1) = 1 + t_0 e^{2i\theta} + at_0^2 e^{4i\theta} + zt_0 e^{6i\theta}
\]

\[
= -\frac{e^{i\theta}}{\zeta} - e^{2i\theta} + \frac{ae^{2i\theta}}{\zeta^2} (1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^2 - \frac{ze^{3i\theta}}{\zeta^3} (1 + \zeta e^{-i\theta} + \zeta e^{i\theta})^3
\]

\[
= \frac{e^{i\theta}}{\zeta} - e^{2i\theta} + \frac{ae^{2i\theta}}{\zeta^2} (e^{-i\theta} + e^{i\theta} + \zeta)\zeta - \frac{ze^{3i\theta}}{\zeta^3} e^{2i\theta} = 0.
\]

Finally

\[
P(t_2) = P(\zeta t_0 e^{i\theta})
\]

\[
= -\zeta e^{-i\theta} - \zeta e^{i\theta} + a \left(1 + \zeta e^{-i\theta} + \zeta e^{i\theta}\right)^2 - z \left(1 + \zeta e^{-i\theta} + \zeta e^{i\theta}\right)^3
\]

\[
= -\zeta e^{-i\theta} - \zeta e^{i\theta} + a \left(e^{-i\theta} + e^{i\theta} + \zeta\right) \zeta - \zeta^2 = 0.
\]

As a consequence of Lemma [8] if \( \theta \in (2\pi/3, \pi) \), then the zeros of \( 1 + t + at^2 + z(\theta)t^3 \) will be distinct and \( t_1 = \overline{t_0} \) since \( \zeta \in \Re \) by Lemma [5]. Thus we can apply partial fractions given in the beginning of Section 2.1. From this partial fraction decomposition, we conclude that if \( \theta \) is a zero of \( g_m(\theta) \), then \( z(\theta) \) will be a zero of \( H_m(z) \). In fact, we claim that each distinct zero of \( g_m(\theta) \) on \((2\pi/3, \pi)\) produces a distinct zero of \( H_m(z) \) on \( I_\alpha \). This is the content of the following two lemmas.
Lemma 9. Let \( \zeta(\theta) \) be defined as in \( \text{(2.12)} \). The function \( z(\theta) \) defined as in \( \text{(2.14)} \) is increasing on \( \theta \in (2\pi/3, \pi) \).

Proof. Lemma 8 gives

\[
-\frac{z}{t_0^3} = 1 + t_0 + at_0^2 = 1 + t_1 + at_1^2.
\]

We differentiate the three terms and obtain

\[
(2.19) \quad \frac{dz}{d\theta} = \frac{3 + 2t_0 + at_0^2}{t_0^4} dt_0 = \frac{3 + 2t_1 + at_1^2}{t_1^4} dt_1,
\]

where

\[
dt_1 = d(t_0e^{2i\theta}) = e^{2i\theta} dt_0 + 2it_0e^{2i\theta} d\theta.
\]

If we set

\[
f(t_0) = \frac{3 + 2t_0 + at_0^2}{t_0^4}, \quad f(t_1) = \frac{3 + 2t_1 + at_1^2}{t_1^4}
\]

then \( f(t_0) = f(t_1) \), and consequently \( f(t_0)f(t_1) \geq 0 \). Thus \( \text{(2.19)} \) implies that

\[
f(t_0)dt_0 = f(t_1)(e^{2i\theta} dt_0 + 2it_0e^{2i\theta} d\theta).
\]

After solving this equation for \( dt_0 \) and substituting it into \( \text{(2.19)} \), we obtain

\[
(2.20) \quad \frac{dz}{d\theta} = \frac{2f(t_0)f(t_1)t_0e^{2i\theta}}{f(t_0) - f(t_1)e^{2i\theta}}.
\]

With \( t_0 = \tau e^{-i\theta} \), \( \tau \in \mathbb{R} \), we have

\[
\frac{f(t_0) - f(t_1)e^{2i\theta}}{2it_0e^{2i\theta}} = \frac{f(t_0)e^{-i\theta} - f(t_1)e^{i\theta}}{2it_0e^{i\theta}}
= \frac{3}{\tau} \left( f(t_0)e^{-i\theta} \right)
= \frac{1}{\tau} \left( \frac{3 + 2t_0 + at_0^2}{t_0^4} e^{-i\theta} \right).
\]

We now substitute \( 3 = -3t_0 - 3at_0^2 - 3zt_0^3 \) and have

\[
\frac{f(t_0) - f(t_1)e^{2i\theta}}{2it_0e^{2i\theta}} = \frac{1}{\tau} \left( \frac{-t_0 - 2at_0^2 - 3zt_0^3}{t_0^4} \right) e^{-i\theta}
= \frac{1}{\tau^4} \left( -e^{2i\theta} - 2a\tau e^{i\theta} - 3\tau^2 \right)
= \frac{1}{\tau^4} \left( -\sin 2\theta - 2a\tau \sin \theta \right)
= \frac{2\sin \theta}{\tau^4} \left( -\cos \theta - a\tau \right).
\]

In the formula of \( t_0 \) in Lemma 8, we substitute \( \tau = -1/\zeta - 2\cos \theta \) and obtain

\[
\frac{f(t_0) - f(t_1)e^{2i\theta}}{2it_0e^{2i\theta}} = \frac{2\sin \theta}{\tau^4} \left( -\cos \theta + a/\zeta + 2a\cos \theta \right).
\]
We finish this lemma by showing that \(-\cos \theta + a/\zeta + 2a \cos \theta > 0\), which implies \(f(t_0) = f(t_1) \neq 0\) and the lemma will follow from (2.20). We expand and divide both sides of (2.10) by \(\zeta\) to get
\[
\zeta(1 - 4a \cos^2 \theta) + 2 \cos \theta(1 - 2a) - a/\zeta = 0,
\]
or equivalently,
\[
\zeta(1 - 4a \cos^2 \theta) + \cos \theta(1 - 2a) = -\cos \theta + 2a \cos \theta + a/\zeta.
\]
Finally, using the definition of \(\zeta\) in (2.12) and Lemma 5, we note that
\[
\zeta(1 - 4a \cos^2 \theta) + \cos \theta(1 - 2a) = \sqrt{(1 - 4a) \cos^2 \theta + a} > 0.
\]
The proof is complete. \(\square\)

Lemma 10. The function \(z(\theta)\) as defined in (2.14) maps the interval \((2\pi/3, \pi)\) onto the interior of \(I_a\).

Proof. Since \(z(\theta)\) is a continuous increasing function on \((2\pi/3, \pi)\), we only need to evaluate the limits of \(z(\theta)\) at the endpoints. Since \(|\zeta| > 1\) by Lemma 6, the formula of \(\zeta(\theta)\) in (2.12) implies that \(\zeta(\theta) \to 1^+\) as \(\theta \to (2\pi/3)^+\). Consequently, (2.18) gives
\[
\lim_{\theta \to (2\pi/3)^+} z(\theta) = -\infty.
\]
Finally, from the fact that
\[
\lim_{\theta \to \pi} \zeta(\theta) = \frac{1 - 2a + \sqrt{1 - 3a}}{1 - 4a},
\]
and (2.13), we have
\[
\lim_{\theta \to \pi} z(\theta) = \frac{a(1 - 2a + \sqrt{1 - 3a})(1 - 4a)}{(-1 + 6a + \sqrt{1 - 3a})(-1 - 2\sqrt{1 - 3a})}
\]
\[
= \frac{a(-1 + 4a)^2(-2 + 9a) + 2a(-1 + 3a)(-1 + 4a)^2\sqrt{1 - 3a}}{27(1 - 4a)^2a}
\]
\[
= \frac{-2 + 9a - 2\sqrt{1 - 3a}^3}{27},
\]
where we obtain (2.21) by multiply and divide the fraction by \((-1 + 6a - \sqrt{1 - 3a})(-1 + 2\sqrt{1 - 3a})\).

Before making final arguments to connect all results in this section, we check the sign of \(g_m(\theta)\) at one of the endpoints.

Lemma 11. If \(-1 \leq a < 1/4\) then the sign of \(g_m(\pi^-)\) is \((-1)^m\).

Proof. Since \(-1 \leq a < 1/4\), one can check that
\[
\lim_{\theta \to \pi^-} \zeta(\theta) = \frac{1 - 2a + \sqrt{1 - 3a}}{1 - 4a} \geq 1.
\]
The result then follows directly from (2.15) and the fact that
\[
\lim_{\theta \to \pi^-} \frac{\sin(m + 1)\theta}{\sin(\theta)} = (m + 1)(-1)^m.
\]
\(\square\)
With all the lemmas at our disposal, we produce the final arguments to finish the proof of Theorem 2. We consider the function \( g_m(\theta) \) at the points \( \theta_h = \frac{h \pi}{m+1} \in \left( \frac{2 \pi}{3}, \pi \right) \), \( \quad \left\lfloor \frac{2(m+1)}{3} \right\rfloor + 1 \leq h \leq m \).

We note that the number of such values of \( h \) is
\[
\frac{m}{3} - \left\lfloor \frac{2(m+1)}{3} \right\rfloor = \left\lfloor \frac{m}{3} \right\rfloor,
\]
where the equality can be checked by considering \( m \) congruent to 0, 1, or 2 modulus 3. From the formula of \( g_m(\theta) \) in (2.15) and Lemma 6, the sign of \( g_m(\theta_h) \) is \((-1)^{h-1}\). By the Intermediate Value Theorem and Lemma 7, there are at least \( \frac{m}{3} - 1 \) zeros of \( g_m(\theta) \) on \((2\pi/3, \pi)\). In the case \(-1 \leq a < 1/4\), we obtain one more zero of \( g_m(\theta) \) from Lemma 11. On the other hand, if \( 1/4 < a \leq 1/3 \), then we obtain another zero of \( g_m(\theta) \) by Lemma 7. From Lemmas 9 and 10, we obtain at least \( \left\lfloor \frac{m}{3} \right\rfloor \) zeros of \( H_m(z) \) on \( I_a \). Since the degree of \( H_m(z) \) is at most \( \frac{m}{3} \) by Lemma 4, all the zeros of \( H_m(z) \) lie on \( I_a \). Recall that we can ignore the case \( a = 1/4 \) by Lemma 3. The density of \( \bigcup_{m=0}^{\infty} \mathcal{Z}(H_m(z)) \) on \( I_a \) comes from the density of \( \bigcup_{m=0}^{\infty} \mathcal{Z}(g_m(\theta)) \) on \((2\pi/3, \pi)\) and from \( z(\theta) \) being a continuous map.

3. The case \( c = 0 \) and \( b \geq 0 \)

It is trivial that if \( c = 0 \) and \( b = 0 \) then the zeros of \( H_m(z) \) are real under the convention that the constant zero polynomial is hyperbolic. In the case \( b > 0 \), we make a substitution \( t \rightarrow t/\sqrt{b} \) and reduce the problem to the following theorem.

**Theorem 12.** The zeros of the sequence of polynomials \( H_m(z) \) generated by
\[
(3.1) \quad \sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + t^2 + zt^3}
\]
are real, and the set \( \bigcup_{m=0}^{\infty} \mathcal{Z}(H_m) \) is dense on \((−\infty, \infty)\).

The proof of Theorem 12 follows from a similar procedure seen in Section 2. We will point out some key differences. The following lemma comes directly from the recurrence relation
\[
H_m(z) + H_{m-2}(z) + zH_{m-3}(z) = 0
\]
and induction.

**Lemma 13.** The degree of the polynomial \( H_m(z) \) generated by (3.1) is at most
\[
\begin{cases} \frac{m}{3} & \text{if } m \equiv 0 \pmod{3} \\ \frac{m-4}{3} & \text{if } m \equiv 1 \pmod{3} \\ \frac{m-2}{3} & \text{if } m \equiv 2 \pmod{3}. \end{cases}
\]
We define the functions
\[
\zeta(\theta) = -\frac{1}{2\cos\theta},
\]
\[
g_m(\theta) = -\frac{\sin(m+1)\theta}{2\cos\theta \sin\theta} (2 + \cos 2\theta) - \cos(m+1)\theta + (-2\cos\theta)^{m+1},
\]
\[
z(\theta) = \frac{2\cos\theta}{\sqrt{(1-4\cos^2\theta)^3}}
\]
on the interval \((\pi/3, \pi/2)\).

The proof of the lemma below is similar to that of Lemma 8. We leave the detailed computations to the reader.

**Lemma 14.** Suppose \(\theta \in (\pi/3, \pi/2)\), \(\zeta = \zeta(\theta)\), and \(z = z(\theta)\) defined by (3.2). The three zeros of \(1 + t^2 + z(\theta)t^3\) are
\[
t_0 = -\frac{e^{-i\theta}}{z(2\cos\theta + \zeta)},
\]
\[
t_1 = t_0 e^{2i\theta},
\]
\[
t_2/t_0 = \zeta e^{i\theta}.
\]

Looking at \(z'(\theta)\), one can check that \(z(\theta)\) is strictly decreasing on the interval \((\pi/3, \pi/2)\). By the partial fraction decomposition of
\[
\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + t^2 + zt^3} = \frac{1}{z(t-t_0)(t-t_1)(t-t_2)}
\]
similar to the previous section, we conclude that for each zero of \(g_m(\theta)\) on the interval \((\pi/3, \pi/2)\) we obtain two zeros \(\pm z(\theta)\) of \(H_m(z)\). We can also check by induction that \(z = 0\) is a simple zero of \(H_m(z)\) if \(m\) is odd, and \(z = 0\) is not a zero of \(H_m(z)\) when \(m\) is even. The formula of \(g_m(\theta)\) implies that the sign of this function alternates when \(\cos(m+1)\theta = \pm 1\), that is,
\[
(m+1)\theta = k\pi, \quad \frac{m+1}{3} < k < \frac{m+1}{2}.
\]
Since \(g_m(\theta)\) is continuous on \((\pi/3, \pi/2)\), we may apply the Intermediate Value Theorem to compute the number of zeros of \(g_m(\theta)\) on \((\pi/3, \pi/2)\) and correspond to the number of zeros of \(H_m(z)\) on \((-\infty, \infty)\). We will see that this number is equal to the degree of \(H_m(z)\) and Theorem 12 follows.

We quickly summarize in the six cases below where \(\theta^*\) denotes the smallest solution \((m+1)\theta = k\pi\) on the interval \((\pi/3, \pi/2)\).

1. \(m \equiv 1 \pmod{3}\) and \(m\) is even: there are
\[
\frac{m}{2} - \frac{m+2}{3} = \frac{m-4}{6}
\]
zeros of \(g_m(\theta)\) on \((\pi/3, \pi/2)\), which gives \((m-4)/3\) zeros of \(H_m(z)\) on \((-\infty, \infty)\).

2. \(m \equiv 1 \pmod{3}\) and \(m\) is odd: there are
\[
\frac{m-1}{2} - \frac{m+2}{3} = \frac{m-7}{6}
\]
zeros of \(g_m(\theta)\) on \((\pi/3, \pi/2)\) which gives \((m-7)/3\) nonzero roots of \(H_m(z)\). We add a simple zero \(z = 0\) and obtain \((m-4)/3\) zeros of \(H_m(z)\) on \((-\infty, \infty)\).
(3) \( m \equiv 0 \pmod{3} \) and \( m \) is even: with the observation that \( \lim_{\theta \to \pi/3} g_m(\theta) = -3 < 0 \) and \( g_m(\theta^*) > 0 \) we obtain
\[
\frac{m}{2} - \left( \frac{m}{3} + 1 \right) + 1 = \frac{m}{6}
\]
zeros of \( g_m(\theta) \) on \((\pi/3, \pi/2)\), which gives \( m/3 \) zeros of \( H_m(z) \) on \((-\infty, \infty)\).

(4) \( m \equiv 0 \pmod{3} \) and \( m \) is odd: with the observation that \( \lim_{\theta \to \pi/3} g_m(\theta) = 3 > 0 \) and \( g_m(\theta^*) < 0 \) we obtain
\[
\frac{m - 1}{2} - \left( \frac{m}{3} + 1 \right) + 1 = \frac{m - 3}{6}
\]
zeros of \( g_m(\theta) \) on \((\pi/3, \pi/2)\), which gives \((m - 3)/3\) nonzero roots of \( H_m(z) \). We add a simple zero \( z = 0 \) and obtain \( m/3 \) zeros of \( H_m(z) \) on \((-\infty, \infty)\).

(5) \( m \equiv 2 \pmod{3} \) and \( m \) is even: with the observation that \( g_m(\pi/3) = 0, g_m'(\pi/3) > 0 \), and \( g_m(\theta^*) < 0 \) we obtain
\[
\frac{m}{2} - \left( \frac{m + 1}{3} + 1 \right) + 1 = \frac{m - 2}{6}
\]
zeros of \( g_m(\theta) \) on \((\pi/3, \pi/2)\), which gives \((m - 2)/3\) zeros of \( H_m(z) \) on \((-\infty, \infty)\).

(6) \( m \equiv 2 \pmod{3} \) and \( m \) is odd: with the observation that \( g_m(\pi/3) = 0, g_m'(\pi/3) < 0 \), and \( g_m(\theta^*) > 0 \) we obtain
\[
\frac{m - 1}{2} - \left( \frac{m + 1}{3} + 1 \right) + 1 = \frac{m - 5}{6}
\]
zeros of \( g_m(\theta) \) on \((\pi/3, \pi/2)\), which gives \((m - 5)/3\) nonzero roots of \( H_m(z) \). We plus add simple zero \( z = 0 \) and obtain \((m - 2)/3\) zeros of \( H_m(z) \) on \((-\infty, \infty)\).

In all cases above the number of zeros of \( H_m(z) \) on \((-\infty, \infty)\) corresponds to the degree of \( H_m(z) \) and Theorem 12 follows.

4. NECESSARY CONDITION FOR THE REALITY OF ZEROS

To prove the necessary condition of Theorem 12 we first show that if \( c = 0 \) and \( b < 0 \) then not all polynomials \( H_m(z) \) are hyperbolic. In fact, with the substitution \( t \) by \( it \), we conclude that all the zeros of \( H_m(z) \) will be purely imaginary by Theorem 12.

It remains to consider the sequence \( H_m(z) \) generated by
\[
\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + t + at^2 + zt^3}
\]
and show that if \( \alpha \notin [-1, 1/3] \) then there is an \( m \) such that not all the zeros of \( H_m(z) \) are real. In fact, we will show if \( \alpha \notin [-1, 1/3] \), \( H_m(z) \) is not hyperbolic for all large \( m \). To prove this, let us introduce some definitions (discussed in [9]) related to the root distribution of a sequence of functions
\[
f_m(z) = \sum_{k=1}^{n} \alpha_k(z)\beta_k(z)^m
\]
where \( \alpha_k(z) \) and \( \beta_k(z) \) are analytic in a domain \( D \). We say an index \( k \) dominant at \( z \) if \( |\beta_k(z)| \geq |\beta_l(z)| \) for all \( l \), \( 1 \leq l \leq n \). Let \( D_k = \{ z \in D : k \text{ is dominant at } z \} \).
Let \( \lim \inf Z(f_m) \) be the set of all \( z \in D \) such that every neighborhood \( U \) of \( z \) has a non-empty intersection with all but finitely many of the sets \( Z(f_m) \). Let \( \lim \sup Z(f_m) \) be the set of all \( z \in D \) such that every neighborhood \( U \) of \( z \) has a non-empty intersection infinitely many of the sets \( Z(f_m) \). We will need the following theorem from Sokal ([13 Theorem 1.5]).

**Theorem 15.** Let \( D \) be a domain in \( \mathbb{C} \), and let \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n (n \geq 2) \) be analytic function on \( D \), none of which is identically zero. Let us further assume a 'no-degenerate-dominance' condition: there do not exist indices \( k \neq k' \) such that \( \beta_k \equiv \omega \beta_{k'} \) for some constant \( \omega \) with \( |\omega| = 1 \) and such that \( D_k (= D_{k'}) \) has nonempty interior. For each integer \( m \geq 0 \), define \( f_m \) by

\[
f_m(z) = \sum_{k=1}^{n} \alpha_k(z) \beta_k(z)^m.
\]

Then \( \lim \inf Z(f_m) = \lim \sup Z(f_m) \), and a point \( z \) lies in this set if and only if either

(i) there is a unique dominant index \( k \) at \( z \), and \( \alpha_k(z) = 0 \), or
(ii) there are two or more dominant indices at \( z \).

If \( z^* \in \mathbb{C} \) such that the zeros in \( t \) of \( 1 + t + at^2 + z^*t^3 \) are distinct then by partial fractions given in (2.4) and Theorem 15, \( z^* \) will belong to \( \lim \inf Z(H_m) \) when the two smallest (in modulus) roots of \( 1 + t + at^2 + z^*t^3 \) have the same modulus. We also note that \( t_0(z), t_1(z), \) and \( t_2(z) \) are analytic in a neighborhood of \( z^* \) by the Implicit Function Theorem. If we let \( \omega = e^{i\theta} \) then the 'no-degenerate-dominance' condition in Theorem 15 comes directly from equations (2.14) and (2.12) since \( \theta \) is a fixed constant (and thus \( z \) is a fixed point which has empty interior).

Suppose \( a \notin [-1,1/3] \). With the arguments in the previous paragraph, our main goal is to find a \( z^* \notin \mathbb{R} \) so that the zeros of \( 1 + t + at^2 + z^*t^3 \) are distinct and the two smallest (in modulus) zeros of this polynomial have the same modulus. If we can find such a point, then \( z^* \in \lim \inf Z(H_m) = \lim \sup Z(H_m) \). This implies that on a small neighborhood of \( z^* \) which does not intersect the real line, there is a non-real zero of \( H_m(z) \) for all large \( m \) by the definition of \( \lim \inf Z(H_m) \). Our choice of \( z^* = z(\theta^*) \) comes from (2.14) for a special \( \theta^* \). Unlike in Section 2, \( \theta^* \) will not belong to \((2\pi/3, \pi)\) to ensure that \( z^* \notin \mathbb{R} \). In particular, we consider the two cases \( a < -1 \) and \( a > 3 \).

**Case** \( a < -1 \). We will pick \( \theta^* \neq \pi/2 \) but sufficiently close to \( \pi/2 \) on the left. Since from (2.12)

\[
\lim_{\theta \to \pi/2} \zeta(\theta) = i\sqrt{|a|},
\]
we can pick \( 0 < \theta^* < \pi/2 \) sufficiently close to \( \pi/2 \) so that \( \zeta := \zeta(\theta^*) \in \mathbb{C} \setminus \mathbb{R} \) and \( |\zeta(\theta^*)| > 1 \). By Lemma 8, we have \( t_2 = \zeta t_0 e^{i\theta} \) and \( t_1 = t_0 e^{i\theta} \). The fact that \( |\zeta| > 1 \) and \( \theta^* \neq 0, \pi/2 \) implies that the polynomial \( 1 + t + at^2 + z(\theta^*)t^3 \) have distinct zeros and not all its zeros are real. We will show that \( z(\theta^*) \notin \mathbb{R} \) by contradiction. In deed, if \( z(\theta^*) \in \mathbb{R} \) then the zeros of the polynomial \( 1 + t + at^2 + z(\theta^*)t^3 \in \mathbb{R}[t] \) satisfy \( t_0 = t_1 \) and

\[
t_2 = t_0 \zeta e^{i\theta} \in \mathbb{R}.
\]

This gives a contradiction because the first equation implies that \( t_0 e^{i\theta} \in \mathbb{R} \), while the second equation implies that \( t_0 e^{i\theta} \notin \mathbb{R} \) since \( \zeta \notin \mathbb{R} \).

**Case** \( a > 1/3 \). We will pick \( \theta^* \) so that \( \cos \theta^* = \beta^+ \) where \( \beta = \sqrt{a/(4a-1)} < 1 \). The definition of \( \zeta(\theta) \) in (2.12) gives

\[
\lim_{\cos \theta \to \beta} \zeta(\theta) = \begin{cases} \frac{\sqrt{4a^2 - a}}{1-2a} & \text{if } a \neq 1/2 \\ \infty & \text{if } a = 1/2 \end{cases}
\]
where we can easily check that
\[
\frac{\sqrt{4a^2-a}}{1-2a} > 1, \quad a > 1/3.
\]

Thus if \( \cos \theta^* = \beta^+ \) then \( 0 < \theta^* < \pi/2, |\zeta(\theta^*)| > 1, \) and \( |\zeta(\theta^*)| \notin \mathbb{R} \) where the last statement comes from (2.12) and the inequality
\[
(1-4a) \cos^2 \theta^* + a < 0.
\]

With \( 0 < \theta^* < \pi/2, |\zeta(\theta^*)| > 1, \) and \( |\zeta(\theta^*)| \notin \mathbb{R}, \) we apply the same arguments given in the previous case to complete this section.

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