Sparse reconstruction of log-conductivity in current density impedance tomography

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Abstract

A new non-linear optimization approach is proposed for the sparse reconstruction of log-conductivities in current density impedance imaging. This framework comprises of minimizing an objective functional involving a least squares fit of the interior electric field data corresponding to two boundary voltage measurements, where the conductivity and the electric potential are related through an elliptic PDE arising in electrical impedance tomography. Further, the objective functional consists of a $L^1$ regularization term that promotes sparsity patterns in the conductivity and a Perona-Malik anisotropic diffusion term that enhances the edges to facilitate high contrast and resolution. This framework is motivated by a similar recent approach to solve an inverse problem in acousto-electric tomography. Several numerical experiments and comparison with an existing method demonstrate the effectiveness of the proposed method for superior image reconstructions of a wide-variety of log-conductivity patterns.

Keywords: Inverse problems, PDE-constrained optimization, proximal methods, edge-enhancement, sparsity patterns, current density impedance imaging.

MSC: 35R30, 49J20, 49K20, 65M08, 82C31

1 Introduction

Electrical impedance tomography (EIT) is an imaging modality, where one attempts to recover the conductivity of a body from the boundary measurement of current and voltage [8]. The underlying inverse problem is highly ill-posed and non-linear yet very important due to its wide range applications in the fields such as medical imaging [44] and engineering [20, 43]. The mathematical formulation of the EIT inverse problem is given by the following conductivity equation

$$-\nabla \cdot (\sigma(x)\nabla u(x)) = 0 \quad x \in \Omega,$$

$$\sigma(x)\frac{\partial u}{\partial v}(x) = f(x) x \in \Gamma,$$ (1)

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where $\Omega \subset \mathbb{R}^n$ is a convex and bounded domain with Lipschitz boundary and $\Gamma$ is the boundary of $\Omega$. In this model, $\sigma$ is the electrical conductivity, $u$ represents the electric potential and $f$ is the current applied to the boundary.

The reconstructions obtained through the EIT setup usually have high contrast but limited spatial resolution \cite{38}. On the other hand, reconstructions obtained through ultrasound imaging has very high resolution but limited contrast \cite{3,32}. In recent years, attempts have been made to combine multiple imaging modalities to obtain image reconstructions with both high contrast and high resolution. This led to the emergence of hybrid imaging methods that belong to class of coupled-physics imaging modalities to generate images of superior quality. One of such imaging methods, known as current density impedance imaging (CDII) combines the classical EIT setup with magnetic resonance (MR) scanning \cite{12,27}. It is alternatively known as magnetic resonance EIT (MREIT). Current or voltage is applied through the electrodes, which give rise to an interior electric field and the corresponding generated magnetic field, represented as $B = (B_x, B_y, B_z)$, is measured by the MR scanner. The corresponding inverse problem is to solve for the conductivity $\sigma$ from $B_z$ using the well-known iterative Harmonic $B_z$-algorithm \cite{10,22}. Convergence of the harmonic $B_z$ algorithm has been well-studied \cite{10,22,23}. In particular, it has been shown that for small contrast values of the target conductivity, the harmonic $B_z$-algorithm is stable and convergent, provided we have a good initial guess \cite{22}. Thus, it is not clear that one can recover good quality images for high contrast objects through Harmonic $B_z$-algorithm.

An alternate approach to solve the CDII inverse problem is to use the knowledge of interior electric field, which is obtained from the magnetic field. Correspondingly, the magnitude of the interior electric field is also determined \cite{37,39}, which is given by

$$H(\sigma(x)) = \sigma(x)|\nabla u(x)|, \quad x \in \Omega. \quad (2)$$

The formulation of reconstruction problem is as follows: Given the boundary data $f$ for, possibly, several choices of boundary patterns and the corresponding interior measurement data $H$, find the conductivity distribution $\sigma$. In this framework, we use the internal function $H(\sigma)$ to replace $\sigma$ in the EIT equation (1) to get the following nonlinear equation

$$\nabla \cdot \left( \frac{H}{|\nabla u|} \nabla u \right) = 0 \text{ in } \Omega,$$

$$\frac{H}{|\nabla u|} \frac{\partial u}{\partial \nu} = f \text{ on } \Gamma. \quad (3)$$

For the CDII inverse problem, the solution to the boundary value problem \cite{3} is crucial but it is difficult to use it in practice because of its highly nonlinear behaviour and also because the data represented by the measured values of $H$ enter as a coefficient of the differential model \cite{38}. Even with the additional measurements, analysis and application of the 1-Laplacian relies on an iterative localized algorithm, wherein one considers an approximation of the CDII problem. This subsequently led to several computational approaches in solving the CDII inverse problem. In \cite{16}, it was proved that the linearized problem is elliptic and hence solvable, if there are at least $n$ set of measurements $\{H_i(\sigma)\}_{i=1}^n$ and corresponding to $n$ boundary data $\{f_i\}_{i=1}^n$ such that $\nabla u_i$ and $\nabla u_j$ are nowhere collinear for $i \neq j$. It has been shown in \cite{19} that the solution of the above 1-Laplacian equation with the Neumann boundary condition is non-existent unless additional measurements with different boundary current patterns are used. Recovery of isotropic conductivity in regions
where the magnetic field is transversal using two internal current distributions was done using an explicit local formula \[21\]. Moreover, using the information of two internal current distributions, the authors in \[19\] uniquely determine the singular support of the conductivity function. In \[25\], the authors showed that the conductivity in the planar domain can be recovered from a single voltage-current on a part of boundary and the magnitude of one interior current density. In the same article, they also provide sufficient conditions on Dirichlet boundary data to guarantee unique recovery of conductivity. In \[26\], the recovery of Hölder continuous conductivities have been established for domains with connected boundary from the interior measurement of the magnitude of one current density. Determination of isotropic conductivity variations from measurements of two current density vector fields was studied in \[12\]. In \[41\], authors showed the recovery of planar conductivities by solving the 1-Laplace equation with partial boundary data.

The well-known numerical reconstruction algorithm using the internal current distribution is an iterative \(J\)-substitution algorithm which was first introduced by \[17\] and subsequently considered in other works, see for e.g., in \[14, 18, 26, 27\]. It has been shown that the \(J\)-substitution algorithm is able to reconstruct the conductivity with high resolution \[18, 19\]. Another numerical reconstruction iterative method is the regularized D-bar method \[15\] that provides images with high resolution. In \[24\], the authors use an alternating split Bregman algorithm for solving a minimization problem related to the energy functional corresponding to the 1-Laplacian equation \[3\]. Also, in \[13\], Picard and Newton type algorithms are implemented to solve the 1-Laplacian problem. But there is not enough evidence to suggest that these existing algorithms (linearized or localized iterative methods) can provide high contrast images, specially for objects with holes or inclusions, which are inherent to CDII reconstructions.

In the field of CDII imaging, we present a new optimization framework that uses tools from PDE control models and anisotropic diffusion theory to potentially obtain reconstructions with high resolution and contrast. Such a framework was first used in \[1, 31\] to reconstruct log-conductivity in acousto-electric tomography (AET). The results obtained demonstrated that such a framework was robust and accurate for imaging modalities arising through a partial differential equation (PDE). In this paper, we consider a similar optimization framework developed in \[31\] for reconstructing the log-conductivity in CDII. We formulate a minimization problem, where given interior electric field intensity data, we aim at determining the variation in conductivity from a known background conductivity. We, further, assume that this variation demonstrates a sparsity pattern. This is incorporated in our model through a \(L^2 - L^1\) regularization term in our objective functional. To obtain sharp edges and, thus, improve spatial resolution of the reconstructed images, we use a Perona-Malik anisotropic diffusion filtering term in our functional. The resulting optimality system gives rise to an elliptic adjoint equation with a \(L^2\) source term. Classical cell-nodal finite difference schemes are not applicable for solving such equations. We, thus, use a averaged cell-nodal scheme to solve such equations. Finally, we solve the optimization problem using a variable inertial proximal scheme that efficiently handles the non-differentiable terms in the objective functional. We demonstrate through several examples that our method can be used to obtain superior quality reconstructions for objects with holes and inclusions.

The article is organized as follows: In the Section 2 we formulate the minimization problem for the CDII. In the Section 3 we present some theoretical results about our optimization problem. We also characterize the optimality system. The variable inertial proximal scheme and the averaged cell-nodal schemes to solve the optimization problem are discussed in Section 4. In the Section 5 we present simulation results of our CDII framework and compare them with the reconstructions...
obtained using the Picard scheme proposed in [13], which validate our framework for CDII and demonstrate the effectiveness of our method to reconstruct wide variety of objects with corners, holes and inclusions. A section on conclusions completes our work.

2 A minimization problem

We consider the conductivity equation in \( \mathbb{R}^2 \) arising in EIT

\[
-\nabla \cdot (e^{\sigma(x,y)} \nabla u(x,y)) = 0 \quad \text{in} \quad \Omega,
\]

\[
u(x,y)|_{\Gamma} = f_D(x,y),
\]

where \( \Omega \subset \mathbb{R}^2 \) is bounded, \( \Gamma \) is the boundary of \( \Omega \), \( e^\sigma \) is the conductivity coefficient and \( u \in H^1_{[\Omega]}(\Omega) = \{u \in H^1(\Omega) : u = f_D \text{ on } \Gamma \} \) is the electric potential.

We assume that \( \sigma \) is a sparse conductivity coefficient which we want to recover, given the fact that the conductivity of the background is 1. The conductivity equation (4) can also be written as

\[
\mathcal{L}(u, \sigma, f_D) = 0,
\]

where \( \sigma(x,y) \in L_{ad} = \{\sigma \in H^1_0(\Omega) : \sigma_l \leq \sigma(x,y) \leq \sigma_u, \forall (x,y) \in \Omega\} \), \( \sigma_u > 0 \) and \( \sigma_l = -\frac{1}{2} \sigma_t \).

We consider an optimization-based approach for reconstructing \( \sigma \) given \( H_1(\sigma), H_2(\sigma) \), where

\[
H(\sigma) = e^{\sigma|\nabla u|}
\]

is the interior electric field corresponding to the voltage potential \( u \). We consider the following cost functional

\[
J(\sigma, u_1, u_2) = \sum_{j=1}^{2} \frac{1}{2} \int_{\Omega} (H_j(x,y) - H^j_j(x,y))^2 \, dx \, dy + \frac{\beta}{2} \|\sigma\|^2_{L^2(\Omega)}
\]

\[
+ \gamma \|\sigma\|_{L^1(\Omega)} + \frac{\delta}{2} \int_{\Omega} \log(1 + |\nabla \sigma(x,y)|^2) \, dx \, dy
\]

where \( u_1, u_2 \) satisfy (4) with boundary data \( f_D^1, f_D^2 \). We now consider the following minimization problem

\[
\begin{align*}
\min_{\sigma} & \quad J(\sigma, u_1, u_2), \\
\text{s.t.} & \quad \mathcal{L}(u_1, \sigma, f_D^1) = 0, \\
& \quad \mathcal{L}(u_2, \sigma, f_D^2) = 0.
\end{align*}
\]

The term \( \gamma \|\sigma\|_{L^1(\Omega)}, \gamma > 0 \) in the functional, defined in (5), implements a \( L^1 \) regularization of the minimization problem that promotes sparsity patterns in the reconstruction of conductivity. Such a regularization method mirrors the well known compressed-sensing technique; see [5]. In recent past, optimal control with \( L^1 \) cost functionals has become a topic of major interest [40], because one obtains sparse controls through this procedure, which finds numerous applications. The motivation for sparse log-conductivity patterns is based on the assumption that the background conductivity is known to be 1 in a substantial part of the domain \( \Omega \) after normalization and
varies considerably from this value in correspondence to different kind of objects present within the domain.

The combined $L^2$-$L^1$ regularization allows for the reconstruction of conductivity, and thus the imaging of, possibly, irregular objects inside $\Omega$. This does not serve the ultimate goal of reconstructing objects like tissues in medical imaging, which are more regular, save for the edges that eventually define them. We infuse this additional apriori knowledge into our model through the last term in our functional \ref{functional} that, commonly, appears in the field of anisotropic diffusion. Such a term plays an important role in dampening image noise while keeping significant parts of the image content such as edges and other anatomical details that are of utmost importance in the interpretation of the image. Anisotropic diffusion means non-uniform diffusion in different directions. The regions where $|\nabla \sigma|$ is very small corresponds to noise and thus, the process of smoothening occurs. At the edges or singularities of an object, where the value of $|\nabla \sigma|$ is large, there is a small amount of smoothening and this preserves the edges. A standard technique to implement anisotropic diffusion, in order to obtain a good contrast, is to use a total variation (TV) regularization \cite{7, 33}. But, this regularization method gives rise to a non-differentiable term in the functional \ref{functional}, thus requiring more sophisticated optimization algorithms. On the other hand, anisotropic diffusion is inherent to Perona-Malik (PM) filtering \cite{28}. It is well-known that the diffusion process governed by the PM equation leads to a decrease in the total variation during its evolution \cite{34}. We, thus, choose the energy functional of the Perona-Malik equation for anisotropic diffusion \cite{28}. One can note that the PM regularization term is differentiable and, thus, easier to handle than the TV regularization term.

Mathematically, one can consider the PM filtering as the gradient flow generated by the non-convex and lower semi-continuous functional given by

$$J_{PM}(\sigma) = \int_{\Omega} \log(1 + |\nabla \sigma(x,y)|^2) \, dxdy.$$ 

We refer to \cite{6, 34} for a general introduction to anisotropic diffusion and a detailed discussion on the PM functional. Further, in \cite{31}, the PM model was used in the reconstruction of log-conductivities in AET and it was observed that the reconstructions obtained demonstrated superior contrast and resolution. Thus, for the current setup in CDII, we use a similar PM anisotropic diffusion filter to facilitate high contrast and high resolution images.

3 Theory of the minimization problem

In this section, we discuss the existence of solutions of the minimization problem \ref{problem} and its characterization through a first-order optimality system. We refer to this minimization problem as the CDII sparse reconstruction problem (CDII-SR). Our analysis of this problem begins with the discussion concerning the existence and uniqueness of weak solutions of $L(u, \sigma, f_D) = 0$, which can be proved by standard arguments of Riesz representation theorem \cite[Chapter 8]{11}.

Proposition 1. Let $\sigma \in L_{ad}$ and $f_D \in H^{1/2}(\partial \Omega)$. Then the problem \ref{problem} has a unique solution in $H_{f_D}^1(\Omega)$.

The solvability of the CDII inversion problem depends on the type of Dirichlet boundary data $f_D^j$, $j = 1, 2$. In this context, we have the following lemma from \cite{2}.
Lemma 3.1 (Boundary data). Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, whose boundary $\Gamma$ is a simple closed curve. Let $f = (f^1, f^2)$ be a mapping $\Gamma \to \mathbb{R}^2$ which is a homeomorphism of $\Gamma$ onto a convex closed curve $C$, and let $D$ denote the bounded convex domain bounded by $C$. Let $\sigma \in L^\infty(\Omega)$, and let $U = (u_1, u_2)$ be the $\sigma$-harmonic mapping whose components $u_1$ and $u_2$ are solutions to the Dirichlet problem $[4]$ with $f_D^1 = f_D^1$ and $f_D^2 = f_D^2$, respectively, and $f_D^1, f_D^2 \in H^1(\Omega) \cap C(\Omega)$, $J = 1, 2$. Then $U$ is a homeomorphism of $\Omega$ onto $D$. In particular, for all $\omega \subset \subset \Omega$ we have either $\det(\nabla u_1, \nabla u_1) > 0$ or $\det(\nabla u_1, \nabla u_1) < 0$ almost everywhere in $\omega$.

In $[10]$, the authors have shown that in 2D, the boundary condition pair $f_D^1 = x$ and $f_D^2 = y$ satisfies the conditions of Lemma 3.1 and, thus, the corresponding solutions to $[4]$ $u_1$ and $u_2$ have no critical points and $\nabla u_1, \nabla u_2$ are not collinear in $\Omega$. We will use these boundary conditions for our numerical experiments in Section 5.

Next, we consider the Fréchet differentiability of the mapping $u(\sigma)$.

Lemma 3.2. The map $u(\sigma)$ defined by $[4]$ is Fréchet differentiable as a mapping from $L_{ad}$ to $H^1_{f_D}(\Omega)$.

For the proof of this Lemma, we refer to [10]. Using Lemma 3.2, we introduce the reduced cost functional

$$\tilde{J}(\sigma) = J(\sigma, u_1(\sigma), u_2(\sigma)), \quad (6)$$

where $u_i(\sigma), i = 1, 2$ denotes the unique solution of $[4]$ given $\sigma$ and $f_D^i, i = 1, 2$. The constrained optimization problem $[\mathbb{P}]$ can be formulated as an unconstrained one as follows

$$\min_{\sigma \in L_{ad}} \tilde{J}(\sigma). \quad (7)$$

We next investigate the existence of a minimizer to the CDII-SR problem $[\mathbb{P}]$. We first consider the case when $\delta = 0$, i.e., the Perona-Malik term in the functional $J$ is absent.

Proposition 2. Let $f_D^1, f_D^2 \in H^{1/2}(\Omega)$ such that $|\nabla u_1| > 0, |\nabla u_2| > 0$ and let $\delta = 0$. Then there exists a triplet $(\sigma^*, u_1^*, u_2^*) \in L_{ad} \times H^1_{f_D^1}(\Omega) \times H^1_{f_D^2}(\Omega)$ such that $u_i^*, i = 1, 2$ are solutions to $L(\sigma, u_i, f_D^i) = 0, i = 1, 2$ and $\sigma^*$ minimizes $\tilde{J}$ in $L_{ad}$.

Proof. Boundedness from below of $\tilde{J}$ guarantees the existence of a minimizing sequence $(\sigma^m)$. Since $L_{ad}$ is reflexive and $\tilde{J}$ is sequentially weakly lower semi-continuous, this sequence is bounded. Therefore it contains a weakly convergent subsequence $(\sigma^{m_i})$ in $L_{ad}$, $\sigma^{m_i} \rightharpoonup \sigma^*$. Correspondingly, the sequence $(u_1^{m_i}, u_2^{m_i})$, where $u_i^{m_i} = u_i(\sigma^{m_i})$, is bounded in $H^1_{f_D^1}(\Omega) \times H^1_{f_D^2}(\Omega)$. Therefore the sequence converges weakly to $(u_1^*, u_2^*)$. Now, using the Rellich Kondrachew compactness theorem in $\mathbb{R}^2$, we have that $L_{ad}$ is compactly embedded in $L^2(\Omega)$. This results in a strong convergence of the subsequence $\sigma^{m_i}$ in $L^2(\Omega)$ to $\sigma^*$. We, now, consider the weak formulation of the solutions of the elliptic problem $[4]$ and, thus, focus on $\langle \nabla \cdot (\sigma^{m_i} \nabla u_i^{m_i}), \psi \rangle_{L^2(\Omega)}$ for any $\psi \in H^1_0(\Omega)$. Using integration by parts, we have $\langle \nabla \cdot (\sigma^{m_i} \nabla u_i^{m_i}), \psi \rangle_{L^2(\Omega)} = -\langle \sigma^{m_i} \nabla u_i^{m_i}, \nabla \psi \rangle_{L^2(\Omega)}$. From the above discussion, the sequence of products $\sigma^{m_i} \nabla u_i^{m_i}$ is weakly convergent in $L^2(\Omega)$, that is, $\langle \sigma^{m_i} \nabla u_i^{m_i}, \nabla \psi \rangle_{L^2(\Omega)} \to \langle \sigma^* \nabla u_i^*, \nabla \psi \rangle_{L^2(\Omega)}$. This preparation and using the continuity of the maps $u_i(\sigma)$, it follows that $(u_1^*, u_2^*) = (u_1(\sigma^*), u_2(\sigma^*))$, and the triplet $(\sigma^*, u_1^*, u_2^*)$ minimizes the objective $J$. \qed
In the case $\delta \neq 0$, we first note that the function $\log(1 + z^2)$ is not convex. Therefore the PM functional, and, hence, the functional $\hat{J}$ in (3) is not weakly lower semi-continuous on $W^{1,p}(\Omega)$ for any $1 < p < \infty$. Nevertheless, $\hat{J}$ is a bounded below, lower semi-continuous Lipschitz functional, for which a minimizer exists, provided that $L_{ad}$ is compact.

### 3.1 Characterization of local minima

To characterize the solution of our optimization problem through first-order optimality conditions, we write the reduced functional $\hat{J}$ as

$$\hat{J} = \hat{J}_1 + \hat{J}_2, \quad \hat{J}_i : L_{ad} \to \mathbb{R}^+, \quad i = 1, 2,$$

where

$$\hat{J}_1(\sigma) = \frac{\alpha_1}{2} \| e^\sigma |\nabla u_1| - g_1^\delta \|_{L^2(\Omega)}^2 + \frac{\alpha_2}{2} \| e^\sigma |\nabla u_2| - g_2^\delta \|_{L^2(\Omega)}^2 + \frac{\beta}{2} \| \sigma \|_{L^2(\Omega)}^2,$$

$$\hat{J}_2(\sigma) = \gamma \| \sigma \|_{L^1(\Omega)}.$$

(8)

**Remark 3.1.** The functional $\hat{J}_1$ is smooth and possibly non-convex, while $\hat{J}_2$ is non-smooth and convex.

We next state some properties of the reduced functional $\hat{J}_1(\sigma)$ which can be proved using the arguments in [16, Lemma 3.1].

**Proposition 3.** The reduced functional $\hat{J}_1(\sigma)$ is weakly lower semi-continuous, bounded below and Fréchet differentiable.

We now define the subdifferential of a non-smooth functional.

**Definition 3.1** (Subdifferential). If $\hat{J}$ is finite at a point $\sigma$, the Fréchet subdifferential of $\hat{J}$ at $\sigma$ is defined as follows [3]

$$\partial \hat{J}(\bar{\sigma}) := \left\{ \phi \in L_{ad}^* : \liminf_{\sigma \to \bar{\sigma}} \frac{\hat{J}(\sigma) - \hat{J}(\bar{\sigma}) - \langle \phi, \sigma - \bar{\sigma} \rangle}{\| \sigma - \bar{\sigma} \|_2} \geq 0 \right\},$$

(9)

where $L_{ad}^*$ is the dual space of $L_{ad}$. An element $\phi \in \partial \hat{J}(\sigma)$ is called a subdifferential of $\hat{J}$ at $\sigma$.

In our setting, we have the following

$$\partial \hat{J}(\sigma) = \nabla \hat{J}_1(\sigma) + \partial \hat{J}_2(\sigma),$$

since $\hat{J}_1$ is Fréchet differentiable by Prop. [3]. Moreover, for each $\alpha > 0$, it holds that

$$\partial (\alpha \hat{J}) = \alpha \partial \hat{J}.$$

The following proposition gives a necessary condition for a local minimum of $\hat{J}$ (see [31]).
Proposition 4 (Necessary condition). If \( \hat{J} = \hat{J}_1 + \hat{J}_2 \), with \( \hat{J}_1, \hat{J}_2 \) given by \( \text{[8]} \), attains a local minimum at \( \sigma^* \in L_{ad} \), then

\[
0 \in \partial \hat{J}(\sigma^*),
\]
or equivalently

\[
-\nabla \hat{J}_1(\sigma^*) \in \partial \hat{J}_2(\sigma^*).
\]

The following variational inequality holds for each \( \lambda \in \partial \hat{J}_2(\sigma^*) \) (see \( \text{[40]} \)).

\[
\langle \nabla \hat{J}_1(\sigma^*) + \lambda, \sigma - \sigma^* \rangle \geq 0, \quad \forall \sigma \in L_{ad}.
\] (10)

Using the definition of \( \hat{J}_2 \) in \( \text{[8]} \) and the fact that \( L_{ad} \) is reflexive, the inclusion \( \lambda \in \partial \hat{J}_2(\sigma^*) \) gives the following characterization of space of \( \lambda \)

\[
\lambda \in \Lambda_{ad} := \{ \lambda \in L^2(\Omega) : 0 \leq \lambda \leq \gamma, \text{ a.e. in } \Omega \}.
\]

A pointwise analysis of the variational inequality \( \text{[10]} \) leads to the existence of a non-negative functions \( \lambda^{*\sigma}_1, \lambda^{*\sigma}_u \in L^2(\Omega) \) that correspond to Lagrange multipliers for the inequality constraints in \( L_{ad} \). We, thus, have the following first-order optimality system.

Proposition 5 (First-order necessary conditions). The optimal solution of the minimization problem \( \text{[7]} \) can be characterized by the existence of \( (\lambda^*, \lambda^{*\sigma}_1, \lambda^{*\sigma}_u) \in \Lambda_{ad} \times L^2(\Omega) \times L^2(\Omega) \) such that

\[
\nabla_{\sigma} \hat{J}_1(\sigma^*) + \lambda^* + \lambda^{*\sigma}_u - \lambda^{*\sigma}_1 = 0,
\] (11)

\[
\lambda^{*\sigma}_u \geq 0, \quad \sigma_u - \sigma^* \geq 0, \quad \langle \lambda^{*\sigma}_u, \sigma_u - \sigma^* \rangle = 0,
\] (12)

\[
\lambda^{*\sigma}_1 \geq 0, \quad \sigma^* - \sigma_1 \geq 0, \quad \langle \lambda^{*\sigma}_1, \sigma^* - \sigma_1 \rangle = 0,
\] (13)

\[
\lambda^* = \gamma \text{ a.e. on } \{ x \in \Omega : \sigma^*(x) > 0 \},
\] (14)

\[
0 \leq \lambda^* \leq \gamma \text{ a.e. on } \{ x \in \Omega : \sigma^*(x) = 0 \}.
\] (15)

The conditions \( \text{[12]}-\text{[15]} \) are known as the complementarity conditions for \( (\sigma^*, \lambda^*) \).

To determine the gradient \( \nabla_{\sigma} \hat{J}_1 \), we use the adjoint approach (see for e.g., \( \text{[29, 30]} \)). This gives the following reduced gradient of \( \hat{J}_1 \)

\[
\nabla_{\sigma} \hat{J}_1(\sigma^*) = (e^{\sigma^*}|\nabla u_1| - g_1^\delta)|\nabla u_1| + (e^{\sigma^*}|\nabla u_2| - g_2^\delta)|\nabla u_2| + \nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2 + \beta \sigma^*,
\] (16)

where \( u_1, u_2 \) satisfy the forward equations \( L(u_1, \sigma^*, f^1_D) = 0, \quad L(u_2, \sigma^*, f^2_D) = 0 \), respectively, and \( v_1, v_2 \) satisfy the adjoint equations

\[
-\nabla \cdot (e^{\sigma^*} \nabla v_1) = \nabla \cdot \left[ e^{\sigma^*} (e^{\sigma^*}|\nabla u_1| - g_1^\delta) \text{ sign}(\nabla u_1) \right] \text{ in } \Omega,
\] (17)

\[
v_1|_{\Gamma} = 0,
\]

\[
-\nabla \cdot (e^{\sigma^*} \nabla v_2) = \nabla \cdot \left[ e^{\sigma^*} (e^{\sigma^*}|\nabla u_2| - g_2^\delta) \text{ sign}(\nabla u_2) \right] \text{ in } \Omega,
\] (18)

\[
v_2|_{\Gamma} = 0.
\]
The complementarity conditions (12)-(15) can be rewritten in a compact form as follows. Define

$$\mu^* = \lambda^* + \lambda^*_{\sigma_l} - \lambda^*_{\sigma_u}. \quad (19)$$

Then the triplet \((\lambda^*, \lambda^*_{\sigma_l}, \lambda^*_{\sigma_u})\) is obtained by solving the following equations

$$\begin{align*}
\lambda^* &= \min(\gamma, \max(0, \mu^*)) , \\
\lambda^*_{\sigma_l} &= -\min(0, \mu^* + \gamma) , \\
\lambda^*_{\sigma_u} &= \max(0, \mu^* - \gamma) ,
\end{align*} \quad (20)$$

(see [40]). For each \(k \in \mathbb{R}^+\), define the following quantity

$$E(\sigma^*, \mu^*) = \sigma^* - \max\{0, \sigma^* + k(\mu^* - \gamma)\} + \max\{0, \sigma^* - \sigma_u + k(\mu^* - \gamma)\} - \min\{0, \sigma^* + k(\mu^* + \gamma)\} + \min\{0, \sigma^* - \sigma_l + k(\mu^* + \gamma)\}. $$

The following lemma determines the complementarity conditions (12)-(15) in terms of \(E\) (see [40, Lemma 2.2]).

**Lemma 3.3.** The complementarity conditions (12)-(15) are equivalent to the following

$$E(\sigma^*, \mu^*) = 0, \quad (21)$$

where \(\mu\) is defined in (19).

Using the gradients in (16) and Lemma 3.3, the optimality conditions (11)-(15) for the CDII-SR problem can be rewritten as follows

**Proposition 6.** A local minimizer \((u_1, u_2, \sigma^*)\) of the problem (P) can be characterized by the existence of \((v_1, v_2, \mu^*) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L_{ad}\), such that the following system is satisfied

$$\begin{align*}
-\nabla \cdot (e^{\sigma^*} \nabla u_1) &= 0 \text{ in } \Omega, \\
|u_1|_{\Gamma} &= f_{D_1}, \\
-\nabla \cdot (e^{\sigma^*} \nabla v_1) &= \nabla \cdot \left[ e^{\sigma^*} (e^{\sigma^*} |\nabla u_1| - g_{D_1}^\delta) \sign(\nabla u_1) \right] \text{ in } \Omega, \\
v_1|_{\Gamma} &= 0, \\
-\nabla \cdot (e^{\sigma^*} \nabla u_2) &= 0 \text{ in } \Omega, \\
|u_2|_{\Gamma} &= f_{D_2}, \\
-\nabla \cdot (e^{\sigma^*} \nabla v_2) &= \nabla \cdot \left[ e^{\sigma^*} (e^{\sigma^*} |\nabla u_2| - g_{D_2}^\delta) \sign(\nabla u_2) \right] \text{ in } \Omega, \\
v_2|_{\Gamma} &= 0, \\
(e^{\sigma^*} |\nabla u_1| - g_{D_1}^\delta)e^{\sigma^*} |\nabla v_1| + (e^{\sigma^*} |\nabla u_2| - g_{D_2}^\delta)e^{\sigma^*} |\nabla v_2| + \nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2 + \beta \sigma^* + \mu^* &= 0, \text{ a.e. in } \Omega, \\
E(\sigma^*, \mu^*) &= 0, \quad \text{a.e. in } \Omega.
\end{align*} \quad (22)$$
4 Numerical solution of the CDII-SR problem

In this section, we discuss numerical optimization and approximation schemes to solve the CDII-SR problem. In this context, we first discuss proximal methods that consists of identifying a smooth and a non-smooth part in the reduced objective $\hat{J}(\sigma)$. Thus, we consider the following optimization problem

$$\min_{\sigma \in L_{ad}} \hat{J}(\sigma) := \hat{J}_1(\sigma) + \hat{J}_2(\sigma). \quad (23)$$

We assume that $\nabla_\sigma \hat{J}_1(\sigma)$, given in (16) is Lipschitz continuous and the upper bound for the Lipschitz constant is obtained using a backtracking search scheme, which will be discussed later. Also, from (8), we have that $\hat{J}_2(\sigma)$ is a continuous, convex, and nondifferentiable functional. The formulation of proximal methods depends, essentially, on the following lemma [31]

Lemma 4.1. Let $\hat{J}_1$ be differentiable with a Lipschitz continuous gradient with Lipschitz constant $L(\hat{J}_1)$. Then the following holds

$$\hat{J}_1(\sigma) \leq \hat{J}_1(\tilde{\sigma}) + \left\langle \nabla \hat{J}_1(\tilde{\sigma}), \sigma - \tilde{\sigma} \right\rangle + \frac{L}{2} \| \sigma - \tilde{\sigma} \|^2, \quad \forall \sigma, \tilde{\sigma} \in L_{ad}, \quad (24)$$

for all $L \geq L(\hat{J}_1) > 0$.

We note that $L := L(\hat{J}_1)$ represents the smallest value of $L$ such that (24) holds true.

In a proximal scheme, one usually minimizes an upper bound of the objective functional at each iteration, instead of minimizing the functional directly. From Lemma 4.1, we obtain the following

$$\min_{\sigma \in L_{ad}} \left\{ \hat{J}_1(\sigma) + \hat{J}_2(\sigma) \right\} \leq \min_{\sigma \in L_{ad}} \left\{ \hat{J}_1(\tilde{\sigma}) + \left\langle \nabla \hat{J}_1(\tilde{\sigma}), \sigma - \tilde{\sigma} \right\rangle + \frac{L}{2} \| \sigma - \tilde{\sigma} \|^2 + \hat{J}_2(\sigma) \right\},$$

where equality holds if $\sigma = \tilde{\sigma}$. Furthermore, we have the following equation

$$\arg \min_{\sigma \in L_{ad}} \left\{ \hat{J}_1(\tilde{\sigma}) + \left\langle \nabla \hat{J}_1(\tilde{\sigma}), \sigma - \tilde{\sigma} \right\rangle + \frac{L}{2} \| \sigma - \tilde{\sigma} \|^2 + \hat{J}_2(\sigma) \right\} = \arg \min_{\sigma \in L_{ad}} \left\{ \frac{L}{2} \left\| \sigma - \left( \tilde{\sigma} - \frac{1}{L} \nabla \hat{J}_1(\tilde{\sigma}) \right) \right\|^2 + \hat{J}_2(\sigma) \right\}. \quad (25)$$

Using the definition of $\hat{J}_2(\sigma) = \gamma \| \sigma \|_{L^1(\Omega)}$, we have the following lemma from [35] that helps in characterizing the solution of (25).

Lemma 4.2. The following equation holds

$$\arg \min_{\sigma \in L_{ad}} \left\{ \tau \| \sigma \|_{L^1} + \frac{1}{2} \| \sigma - \tilde{\sigma} \|^2 \right\} = S^{L_{ad}}_{r}(\tilde{\sigma}) \quad \text{for any} \quad \tilde{\sigma} \in L^2(\Omega),$$

where the left-hand side represents the proximal function and the projected soft thresholding function on the right-hand side is defined as follows

$${S}^{L_{ad}}_{r}(\tilde{\sigma}) := \begin{cases} \min\{\tilde{\sigma} - \tau, \sigma_u\} & \text{on } \{(x,y) \in \Omega : \tilde{\sigma}(x,y) > \tau\} \\ 0 & \text{on } \{(x,y) \in \Omega : |\tilde{\sigma}(x,y)| \leq \tau\} \\ \max\{\tilde{\sigma} + \tau, \sigma_l\} & \text{on } \{(x,y) \in \Omega : \tilde{\sigma}(x,y) < -\tau\} \end{cases}. \quad (26)$$
Using this lemma, the solution to (25) is given by
\[
\arg\min_{\sigma \in L_{ad}} \left\{ \tilde{J}_2(\sigma) + \frac{L}{2} \left\| \sigma - \left( \tilde{\sigma} - \frac{1}{L} \nabla \tilde{J}_1(\tilde{\sigma}) \right) \right\|_2^2 \right\} = S_{L_{ad}}^L \left( \tilde{\sigma} - \frac{1}{L} \nabla \tilde{J}_1(\tilde{\sigma}) \right).
\]
This gives rise to the following iterative scheme
\[
\sigma_{k+1} \leftarrow S_{L_{ad}}^L \left( \sigma_k - \frac{1}{L} \nabla \tilde{J}_1(\sigma_k) \right),
\]
starting from a given \( \sigma_0 \) and is known as the iterative shrinkage-thresholding algorithm (ISTA) scheme [35]. We note that the argument of \( S_{L_{ad}}^L \) represents a gradient update in a steepest descent scheme with a fixed step size \( s = 1/L \) in conjunction with a regularized PM filter [31]. Further, to accelerate the ISTA scheme described above, one can consider a sequence \( \{t_k, v_k\} \) [35, 36] such that
\[
t_0 = 1, \quad t_k := 1 + \sqrt{1 + 4t_{k-1}^2}/2, \quad (27)
\]
and
\[
v_0 := \sigma_0, \quad v_k := \sigma_k + \frac{(t_{k-1} - 1)}{t_k}(\sigma_k - \sigma_{k-1}). \quad (28)
\]
This gives us the following update for the optimization variable \( \sigma_k \)
\[
\sigma_{k+1} \leftarrow S_{L_{ad}}^L \left( v_k - \frac{1}{L} \nabla \tilde{J}_1(v_k) \right). \quad (29)
\]
Replacing \( v_k \) in (29) with (28), and assuming that \( \nabla \tilde{J}_1(\sigma_k) \approx \nabla \tilde{J}_1(v_k) \), we obtain the following iterative scheme [36]
\[
\sigma_{k+1} \leftarrow S_{L_{ad}}^L \left( \sigma_k - s_k \nabla \tilde{J}_1(\sigma_k) + \theta_k (\sigma_k - \sigma_{k-1}) \right), \quad (30)
\]
where \( \sigma_{-1} = \sigma_0 \).

The above discussion is valid for any \( L \geq L(\tilde{J}_1) \). However, since the quantity \( s = 1/L \) represents the step size in a gradient update, we use a backtracking line search algorithm to determine the optimal step size in each iteration. This leads to the computation of an upper bound \( L_k \) that satisfies \( L_k \geq L(\tilde{J}_1) \) at each iteration step. Thus, we define our variable step size as \( s_k = 1/L_k \) and substitute \( \tau \) in (26) with \( \gamma s_k \). The variable step size causes the factor \((t_{k-1} - 1)/t_k\) in (28) to be non-optimal and we replace it by the fixed inertial parameter \( \theta \). This leads to a variable inertial proximal (VIP) scheme, which is described in Algorithm 4.1.

With our VIP scheme, we aim at determining an optimal \( \sigma \in L_{ad} \subset H^0(\Omega) \). But in the update step of the algorithm, we have the argument of the thresholding function \( S_{L_{ad}}^L \) as \( \sigma_k - s_k \nabla \sigma \tilde{J}_1(\sigma_k) \). The term \( \nabla \sigma \tilde{J}_1(\sigma_k) \) is only in \( L^2(\Omega) \) and the resulting update gives us the argument of \( S_{L_{ad}}^L \) in \( L^2(\Omega) \), which is not desired. We, thus, use the \( H^1 \) gradient instead of the \( L^2 \) gradient, which are related by the equation \((\nabla \sigma \tilde{J}_1(\sigma), v)_{H^1(\Omega)} = (\nabla \sigma \tilde{J}_1, v)_{L^2(\Omega)} \) for all \( v \in H^1(\Omega) \). But such a \( H^1 \) gradient results in a highly diffused \( \sigma \) with blurred edges. We, instead, consider a weighted \( H^1 \) product that represents a suitable denoising of the \( \nabla \sigma \tilde{J}_1(\sigma) \). We apply the denoising operator
\[ R(c) = (I - c\Delta)^{-1} \] with a small denoising parameter \( c \) (we take \( c = 10^{-3} \)) and define \( (\nabla_{\sigma} \hat{J}_1)_{H^1} = R(c) \nabla_{\sigma} \hat{J}_1 \). Note that a higher value of \( c \) results in a greater blurring of the edges along with noise removal. On the other hand, since the PM term in the functional \( J \) promotes better resolution with edge-enhancement, we choose the value of \( c \) in proportion to the weight \( \eta \) of the PM functional term (we choose \( \eta = 10^{-2} \)).

We summarize the variable inertial proximal (VIP) scheme for our CDII-SR setup in Algorithm 4.1 below.

**Algorithm 4.1** (Variable inertial proximal (VIP) method).

1. **Input:** \( \beta, \hat{J}_1, \sigma_0 = \sigma_{-1}, L_{ad}, TOL, n > 1, L_0 > 0 \)

   **Initialize:** \( E_0 = 1, k = 0 \), choose \( \theta \in (0, 1) \) and \( c_1 < 2 \) and \( c_2 > 0 \);

2. **While** \( \|E_{k-1}\| > TOL \) **do**

3. **Compute** \( \nabla_{\sigma} \hat{J}_1(\sigma_k) \)

4. **Backtracking:** Find the smallest nonnegative integer \( i \) such that with

   \[
   \tilde{L} = n^i L_{k-1}
   \]

   \[
   \hat{J}_1(\tilde{\sigma}) \leq \hat{J}_1(\sigma_k) + \left\langle \nabla_{\sigma} \hat{J}_1(\sigma_k), \tilde{\sigma} - \sigma_k \rightangle + \frac{\tilde{L}}{2} \|\tilde{\sigma} - \sigma_k\|^2
   \]

   where \( \tilde{\sigma} = \mathbb{S}^{L_{ad}}_{\sigma_k} \left( \sigma_k - s (\nabla_{\sigma} \hat{J}_1)_{H^1}(\sigma_k) + \theta(\sigma_k - \sigma_{k-1}) \right), s = c_1(1 - \theta)/(\tilde{L} + 2c_2), \)

5. **Set** \( L_k = \tilde{L} \) and \( s_k = c_1(1 - \theta)/(L_k + 2c_2) \).

6. \( \sigma_{k+1} = \mathbb{S}^{L_{ad}}_{s_k} \left( \sigma_k - s_k (\nabla_{\sigma} \hat{J}_1)_{H^1}(\sigma_k) + \theta(\sigma_k - \sigma_{k-1}) \right) \)

7. \( \mu_k = -\alpha \sigma_k - (\nabla_{\sigma} \hat{J}_1)_{H^1}(\sigma_k) \)

8. \( E_k = E(\sigma_k, \mu_k) \)

9. \( k = k + 1 \)

10. **end**

In the VIP algorithm, we need to compute the reduced gradient \( \nabla_{\sigma} \hat{J}_1 \). This, in turn, requires an accurate numerical solution of the forward and the corresponding adjoint EIT problems as given in Proposition 6. For the forward EIT equation (4), we use the cell-nodal finite-difference approximation. We consider a sequence of uniform grids \( \{\Omega_h\}_{h>0} \) given by

\[
\Omega_h = \{(x_i, y_j) \in \mathbb{R}^2 : (x_i, y_j) = (a + ih, a + jh), (i, j) \in \{0, \ldots, N\}^2 \} \cap \Omega,
\]

where \( N \) represents the number of cells in each direction and \( h = \frac{(b - a)}{N} \) is the mesh size. The corresponding cell-nodal scheme for (4), at the grid point \((x_i, y_j)\), is given as follows.
\[
\frac{1}{h^2} \left\{ (e^{\sigma_{i+1/2,j}} + e^{\sigma_{i-1/2,j}} + e^{\sigma_{i,j+1/2}} + e^{\sigma_{i,j-1/2}}) u_{i,j} - e^{\sigma_{i+1/2,j}} u_{i+1,j} - e^{\sigma_{i-1/2,j}} u_{i-1,j} - e^{\sigma_{i,j+1/2}} u_{i,j+1} - e^{\sigma_{i,j-1/2}} u_{i,j-1} \right\} = 0, \quad 1 \leq i, j \leq N - 1,
\]

where \( \sigma_{i\pm1,j} = \sigma(x_i \pm h, y_j) \), \( \sigma_{i,j\pm1} = \sigma(x_i, y_j \pm h) \). The required intermediate values of \( \sigma \) are computed as follows

\[
\sigma_{i\pm1/2,j} = \frac{1}{2} \left( \sigma_{i\pm1,j} + \sigma_{i,j} \right) \quad \text{and} \quad \sigma_{i,j\pm1/2} = \frac{1}{2} \left( \sigma_{i,j\pm1} + \sigma_{i,j} \right).
\]

The Dirichlet boundary data \( f_D \) is included in the usual way in the right-hand side of the algebraic equation.

For the adjoint equations (17) and (18), we first note that the cell nodal finite difference scheme is not applicable to the right-hand side term in both the equations as they are of the form \( G = \nabla \cdot F \), where \( F \) is in \( L^2(\Omega) \). We modify the cell nodal scheme by replacing the nodal value of \( G \) at \((x_i, y_j)\) with a cell average of \( G \) given as follows

\[
G_a = \frac{1}{h^2} \int_{C_{ij}} G(x, y) \, dxdy,
\]

where the cell \( C_{ij} \) is defined by

\[
C_{ij} := (x_i - \frac{h}{2}, x_i + \frac{h}{2}) \times (y_j - \frac{h}{2}, y_j + \frac{h}{2}), \quad 1 \leq i, j \leq N - 1.
\]

Since \( G = \nabla \cdot F \), using the divergence theorem we have

\[
G_a = \frac{1}{h^2} \int_{C_{ij}} \nabla \cdot F(x, y) \, dxdy = \frac{1}{h^2} \int_{\partial C_{ij}} F(x, y).n \, ds
\]

The above integral can be approximated with a midpoint quadrature rule along each edge of \( C_{ij} \). This results in the following approximation

\[
G_a = \frac{F_{i+1/2,j}^1 - F_{i-1/2,j}^1}{h} + \frac{F_{i,j+1/2}^2 - F_{i,j-1/2}^2}{h},
\]

where \( F = (F^1, F^2) \).

### 5 Numerical experiments

In this section, we validate our CDII-SR framework using different experiments that validate the choice of different features in our formulation and demonstrate its effectiveness in reconstructing a wide variety of objects. We choose the two boundary conditions as \( f_D^1 = x \), \( f_D^2 = y \) on \( \Gamma \), which is the boundary of \( \Omega = (-1, 1) \times (-1, 1) \). The weights in the functional (5) are chosen as follows:
\( \alpha_1 = \alpha_2 = 1.0, \beta = 0.3, \gamma = 0.01, \delta = 0.01. \) The value of the denoising parameter is \( c = 0.001. \) The parameters of our VIP scheme are chose as \( \theta = 0.5, \, c_1 = 1.9, \, c_2 = 0.001, \, TOL = 10^{-4} \) with the maximum number of iterations as 20. Even though there is a specified tolerance for the termination of the algorithm, due to the high non-linearity of the problem, our VIP scheme terminates due to the maximum number of iterations.

In all the experiments, the domain \( \Omega \) is uniformly discretized into \( N = 150 \) subintervals in both the \( x \) and \( y \) directions with \( h = 0.013. \) The generation of the synthetic interior electric field data \( H^\delta \) is done as follows: we first solve for \( u \) in (4) with given value of \( \sigma \) on a finer mesh with \( N = 400 \) using the finite difference method outlined in Section 4. Then, we compute \( \nabla u \) with one-sided finite differences to obtain \( H^\delta \) on the finer mesh. In the final step, we restrict the obtained \( H^\delta \) onto the coarser mesh with \( N = 150 \) and choose this as our given data to which we also add noise in some of the experiments.

In Test Case 1, we consider a disk phantom for \( \sigma \) that is represented by a disk centered at \((0.25, 0.25)\) with radius 0.25. The value of \( \sigma \) inside the disk is 1 with the background value chosen as 0. The plots of the actual \( \sigma \) and the reconstructed \( \sigma \) are shown in Figure 1.
Figure 1: Test Case 1- The actual and reconstructed disk with different choices of the values of the regularization weights.

Figure [1b] shows the reconstruction of $\sigma \in L_{ad}$ without any regularization terms, i.e. $\beta = \gamma = \delta = 0$ and no denoising, i.e., $c = 0$. Presence of strong artifacts can be observed in this case, which
is inherent to the inverse problem and not the algorithm. A study of the pattern of such artifacts are very challenging and is out of the scope of the paper.

Figure 1c shows the result of CDII-SR reconstruction without the denoising and the Perona-Malik regularization term, \( c = \delta = 0 \), but with the \( L^2 - L^1 \) regularization. We observe that the artifacts are reduced to some extent, but are still present. Figure 1d shows the reconstruction with the \( L^2 - L^1 \) regularization and the \( H^1 \) denoising but without the PM filter. In this case, we observe that the artifacts diminish by a huge amount, but the edges are more blunt and the value of \( \sigma \) is lowered, leading to a loss of resolution and contrast. We correct this loss using the PM regularization term as can be observed in Figure 1e where the edges are fairly well seen and the recovered parameter values are very close to the true ones. We also compare our results with the reconstruction in Figure 1f obtained with the Picard algorithm proposed in [13]. We observe a lot of artifacts and a significant loss of contrast in Figure 1f in comparison to the reconstruction shown in Figure 1e which suggests that our CDII-SR scheme outperforms the Picard scheme.

In Test Case 2, we consider the heart and lung phantom for the true \( \sigma \). It consists of two ellipses representing lungs with the value is 1 and a circular region representing the heart with the value is 0.5. The background value of \( \sigma = 0 \). The plots of the actual \( \sigma \) and the reconstructed \( \sigma \) are shown in Figure 2.
Figure 2: Test Case 2- The actual and reconstructed heart and lung phantom with and without noisy data.

We see from Figure 2c that our CDII-SR algorithm results in reconstruction of $\sigma$ with high resolution. Moreover, the values of the reconstructed $\sigma$ are very close to the true values, with the background value exactly equals to 0, which implies a high contrast reconstruction. We further compare our results to the Picard algorithm proposed in [13]. The corresponding reconstruction is shown in Figure 2b. It can be seen that the CDII-SR provides a better contrast image, yet maintaining the same resolution as that of the Picard scheme. Also, there are far more artifacts through the Picard reconstruction method whereas the sparsity assumption and the $H^1$ denoising in CDII-SR scheme results in an image with very less artifacts. Further, to test the robustness of our method, we introduce 10% multiplicative Gaussian noise in the interior data $H^\delta$, which is fed as input to our CDII-SR algorithm. The corresponding reconstruction is shown in Figure 2d, which again possesses high contrast and high resolution, demonstrating that the CDII-SR algorithm is robust even in the presence of noisy data.

In Test Case 3, we consider a more generalized form of the heart and lung phantom. The heart is represented by a cardioid with the value of $\sigma = 0.5$. The two lungs are represented by “boomerangs” with the value of $\sigma = 1.0$. The plots of the actual and the reconstructed $\sigma$ are
shown in Figure 3.

Figure 3: Test Case 3- The actual and reconstructed modified heart and lung phantom.

We again note good quality reconstructions, specially at the corners of the phantom, which demonstrates the effectiveness of the CDII-SR scheme for irregular shaped phantoms.

In Test Case 4, we consider a combination of phantoms, where one is supported on a square annulus \( S_a = \{(x,y) \in \mathbb{R}^2 : -0.8 < x < -0.7, -0.2 < x < -0.1, -0.8 < y < -0.7, -0.2 < y < -0.1\} \) with \( \sigma = 3.0 \); the other one consists of 2 disks centered at (0.7,0.7) with radius 0.2 and \( \sigma = 1.0 \) and at (0.55,0.55) with radius 0.15 and \( \sigma = 2.0 \). The value of \( \sigma \) inside the square annulus has a value -2.0. The plots of the actual and the reconstructed \( \sigma \) is shown in Figure 4.
Figure 4: Test Case 4- The actual and reconstructed mixed phantom with and without noisy data.

Figure 4c shows the reconstruction with the CDII-SR algorithm. We compare the results, as shown in Figure 4b, obtained with the Picard algorithm in [13] and note that the CDII-SR scheme results in a superior contrast image while preserving the same resolution. It should also be noted that the value inside the square annulus is recovered to be close to the actual negative value and the background is obtained to be exactly 0 due to the sparsity assumption, which suggests that the CDII-SR algorithm is robust and more effective in reconstructing objects with holes and inclusions. We observe similar features in the reconstruction even in the presence of 10% Gaussian noise in the interior data as shown in Figure 4d.

6 Conclusions

In this paper, we propose a new framework to facilitate high contrast and high resolution reconstructions in CDII. Our framework is based on formulating the CDII inverse problem as a PDE-constrained optimization problem, where we minimize an objective functional comprising of least square interior data fitting terms corresponding to two boundary voltage measurements, a $L^1$ functional as follows:
penalization term of the log-conductivity that helps promotes sparsity patterns and a PM filtering term to sharpen the edges.

We characterized the solution of the optimization problem through an optimality system that was solved with using a proximal scheme, coupled with $H^1$ denoising to remove artifacts in the reconstructions. We then demonstrated the effectiveness of our proposed scheme through several numerical experiments and compared our results with an existing scheme. Our scheme facilitated reconstructions of a wide variety of conductivity patterns with good contrast and resolution.

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