DIRICHLET SERIES OF INTEGERS WITH MISSING DIGITS

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Abstract. For certain sequences $A$ of positive integers with missing $g$-adic digits, the Dirichlet series $F_A(s) = \sum_{a \in A} a^{-s}$ has abscissa of convergence $\sigma_c < 1$. The number $\sigma_c$ is computed. This generalizes and strengthens a classical theorem of Kempner on the convergence of the sum of the reciprocals of a sequence of integers with missing decimal digits.

Let $A_{10,9}$ be the set of positive integers whose decimal representation contains no 9. A classical theorem of Kempner [2] states that the harmonic series $\sum_{a \in A_{10,9}} 1/a$ converges. There is a straightforward generalization of this result. Let $g$ be an integer such that $g \geq 2$. Every integer $n$ in the interval $[g^{m-1}, g^m - 1]$ has a unique $g$-adic representation

$$n = \sum_{i=0}^{m-1} c_i g^i$$

with

$$c_i \in \{0, 1, 2, \ldots, g-1\} \quad \text{for all } i \in \{0, 1, \ldots, m-1\}$$

and

$$c_{m-1} \neq 0.$$ 

Let $u \in [0, g-1]$ and let $A_{g,u}$ be the set of positive integers whose $g$-adic representation contains no digit $c_i = u$. The series $\sum_{a \in A} 1/a$ converges. This is Theorem 144 in Hardy and Wright [1].

It is natural to ask if Kempner’s convergence theorem can be strengthened. Does there exist a real number $\sigma < 1$ such that the infinite series $\sum_{a \in A_{g,u}} 1/a^\sigma$ converges? A sharper question is: Compute the abscissa of convergence of the Dirichlet series

$$F_{A_{g,u}}(s) = \sum_{a \in A_{g,u}} a^{-s}.$$ 

We shall prove that this series has abscissa of convergence

$$\sigma_c = \frac{\log(g-1)}{\log g}$$

and that $F_{A_{g,u}}(\sigma_c)$ diverges. This is a corollary of Theorem 2 below.

Let $N = \{1, 2, 3, \ldots\}$ be the set of positive integers and $N_0 = \{0, 1, 2, 3, \ldots\}$ the set of nonnegative integers. For $x, y \in N_0$, define the interval of integers $[x, y] = \{n \in N_0 : x \leq n \leq y\}$.

Fix an integer $g \geq 2$. For all $i \in N_0$, let $U_i$ be a proper subset of $[0, g-1]$, and let $U = (U_i)_{i=0}^\infty$. Let $A_{g,U}$ be the set of positive integers $n$ with $g$-adic representation
Let \( M \) be the set of digits \( c \) choices for the digit \( g \).

Consider the Dirichlet series

\[
F_{A_{g,U}}(s) = \sum_{a \in A_{g,U}} \frac{1}{a^s}.
\]

This series converges if \( \sigma = \Re(s) > 1 \). We have

\[
A_{g,U} \cap \{g^m - 1, g^{m-1} \} \neq \emptyset
\]

if and only if \( U_{m-1} \neq [1, g-1] \). Let

\[
\mathcal{M} = \{ m \in \mathbb{N} : U_{m-1} \neq [1, g-1] \}.
\]

The set \( A_{g,U} \) is infinite if and only if the set \( \mathcal{M} \) is infinite. If \( \mathcal{M} \) is finite, then the series \( F_A(s) \) is a Dirichlet polynomial, which is an entire function. If \( \mathcal{M} \) is infinite, then \( F_A(0) \) diverges, and so the Dirichlet series \( F_{A_{g,U}}(s) \) has abscissa of convergence \( \sigma_c \) with \( 0 \leq \sigma_c \leq 1 \). We shall compute \( \sigma_c \) for a large class of sets of integers with missing \( g \)-adic digits.

**Theorem 1.** For all \( i \in \mathbb{N}_0 \), let \( U_i \) be a proper subset of \([0, g-1)\) such that (i) the set \( \mathcal{M} = \{ m \in \mathbb{N} : U_{m-1} \neq [1, g-1] \} \) is infinite, and (ii) there exist nonnegative real numbers \( \alpha_0, \alpha_1, \ldots, \alpha_{g-1} \) such that, for all \( m \in \mathbb{N} \) and \( k \in [0, g-1] \),

\[
\text{(1)} \quad \text{card}\{i \in [0, m-1] : |U_i| = k\} = \alpha_k m + \varepsilon_k(m)
\]

and

\[
\text{(2)} \quad \lim_{m \to \infty} \frac{\varepsilon_k(m)}{m} = 0.
\]

Let \( U = (U_i)_{i=0}^\infty \). Let \( A_{g,U} \) be the set of positive integers \( n \) with \( g \)-adic representation \( n = \sum_{i=0}^{m-1} c_i g^i \) such that \( c_{m-1} \neq 0 \) and \( c_i \notin U_i \) for all \( i \in [0, m-1] \). The Dirichlet series

\[
F_{A_{g,U}}(s) = \sum_{a \in A_{g,U}} \frac{1}{a^s}
\]

has abscissa of convergence

\[
\sigma_c = \frac{1}{\log g} \sum_{k=0}^{g-1} \alpha_k \log(g - k).
\]

**Proof.** Let \( A = A_{g,U} \). For all \( m \in \mathbb{N} \), let

\[
I_m = [g^{m-1}, g^m - 1]
\]

and let \( n \in A \cap I_m \) have the \( g \)-adic representation \( n = \sum_{i=0}^{m-1} c_i g^i \). For \( i \in [0, m-2] \) there are \( g - |U_{m-1}| \) choices for the digit \( c_i \). If \( 0 \in U_{m-1} \), there are \( g - |U_{m-1}| \) choices for the digit \( c_{m-1} \). If \( 0 \notin U_{m-1} \), there are \( g - |U_{m-1}| - 1 \) choices for the digit \( c_{m-1} \). It follows that

\[
\text{(3)} \quad |A \cap I_m| = \prod_{i=0}^{m-1} (g - |U_i|) \quad \text{if } 0 \in U_{m-1}
\]

and

\[
\text{(4)} \quad |A \cap I_m| = \left( \frac{g - |U_{m-1}| - 1}{g - |U_{m-1}|} \right) \prod_{i=0}^{m-1} (g - |U_i|) \quad \text{if } 0 \notin U_{m-1}.
\]
Let
\[ \sigma > \sigma_c = \frac{1}{\log g} \sum_{k=0}^{g-1} \alpha_k \log(g - k). \]

We shall prove that the infinite series \( F_A(\sigma) \) converges.

Choose a real number \( \delta \) such that
\[ 0 < \delta < \sigma - \sigma_c = \sigma - \frac{1}{\log g} \sum_{k=0}^{g-1} \alpha_k \log(g - k). \]

Choose \( m_0 = m_0(\delta) \) such that
\[ |\varepsilon_k(m)| < \left(\frac{\delta}{g}\right)^m \quad \text{for all} \quad k \in [0, g-1] \quad \text{and} \quad m \geq m_0. \]

Equations 1, 3, and 4 imply that, for \( m \geq m_0 \), we have
\[ |A \cap I_m| \leq \prod_{i=0}^{m-1} (g - |U_i|) = \prod_{k=0}^{g-1} (g - k)^{\alpha_k m + \varepsilon_k(m)} \]
\[ \leq g^{\sum_{k=0}^{g-1} |\varepsilon_k(m)|} \prod_{k=0}^{g-1} (g - k)^{\alpha_k m} \]
\[ < g^{\delta m} \prod_{k=0}^{g-1} (g - k)^{\alpha_k m} \]
\[ = \left( g^{\delta} \prod_{k=0}^{g-1} (g - k)^{\alpha_k} \right)^m. \]

It follows that
\[ F_A(\sigma) = \sum_{a \in A} \frac{1}{a^\sigma} = \sum_{a \in A} \frac{1}{a^\sigma} + \sum_{a \in A} \frac{1}{a^\sigma} \]
\[ = \sum_{a \in A} \frac{1}{a^\sigma} + \sum_{m=m_0}^{\infty} \frac{1}{a^\sigma} \]
\[ \leq \sum_{a \in A} \frac{1}{a^\sigma} + \sum_{m=m_0}^{\infty} \frac{|A \cap I_m|}{g^{(m-1)\sigma}} \]
\[ \leq \sum_{a \in A} \frac{1}{a^\sigma} + \sum_{m=m_0}^{\infty} \frac{g^\delta \prod_{k=0}^{g-1} (g - k)^{\alpha_k}}{g^{(m-1)\sigma}} \]
\[ = \sum_{a \in A} \frac{1}{a^\sigma} + g^\sigma \sum_{m=m_0}^{\infty} \left( g^\delta \prod_{k=0}^{g-1} (g - k)^{\alpha_k} \right)^m. \]

Inequality 5 implies
\[ 0 < \frac{g^\delta \prod_{k=0}^{g-1} (g - k)^{\alpha_k}}{g^\sigma} < 1. \]

and so the infinite series \( F_A(\sigma) \) converges if \( \sigma > \sigma_c. \)
Let $\sigma < \sigma_c$. We shall prove that the infinite series $F_A(\sigma)$ diverges. Choose a real number $\delta$ such that

\begin{equation}
0 < \delta < \sigma - \sigma_c = \frac{1}{\log g} \sum_{k=0}^{g-1} \alpha_k \log(g - k) - \sigma.
\end{equation}

(7)

Let $m \geq m_0$. If $0 \leq |U_{m-1}| \leq g - 2$, then

$$
\frac{g - |U_{m-1}| - 1}{g - |U_{m-1}|} \geq \frac{1}{2}.
$$

From $[1]$, $[3]$, $[4]$, and $[5]$, we obtain

$$
|A \cap I_m| \geq \left(\frac{g - |U_i| - 1}{g - |U_i|}\right)^{m-1} \prod_{i=0}^{m-1} (g - |U_i|)
$$

\begin{align*}
&\geq \frac{1}{2} \prod_{i=0}^{m-1} (g - |U_i|) \\
&= \frac{1}{2} \prod_{k=0}^{g-1} (g - k)^{\alpha_k + \varepsilon_k(m)} \\
&\geq \frac{1}{2} g^{-\sum_{k=0}^{g-1} \varepsilon_k(m)} \prod_{k=0}^{g-1} (g - k)^{\alpha_k m} \\
&> \frac{1}{2} g^{-\delta m} \prod_{k=0}^{g-1} (g - k)^{\alpha_k m} \\
&= \frac{1}{2} \left(g^{-\delta} \prod_{k=0}^{g-1} (g - k)^{\alpha_k}\right)^m.
\end{align*}

If $|U_{m-1}| = g - 1$ and $m \in \mathcal{M}$, then $0 \in U_{m-1}$ and $g - |U_{m-1}| = 1$. It follows that

$$
|A \cap I_m| = \prod_{i=0}^{m-1} (g - |U_i|) = \prod_{k=0}^{g-1} (g - k)^{\alpha_k m + \varepsilon_k(m)}
$$

\begin{align*}
&\geq g^{-\sum_{k=0}^{g-1} |\varepsilon_k(m)|} \prod_{k=0}^{g-1} (g - k)^{\alpha_k m} \\
&> g^{-\delta m} \prod_{k=0}^{g-1} (g - k)^{\alpha_k m} \\
&= \left(g^{-\delta} \prod_{k=0}^{g-1} (g - k)^{\alpha_k}\right)^m.
\end{align*}
If \( n \in \mathbb{N}_0 \setminus \mathcal{M} \), then \( U_{m-1} = [1, g-1] \) and \( A \cap I_m = \emptyset \). We have

\[
F_A(\sigma) = \sum_{a \in A} \frac{1}{a^\sigma} = \sum_{a \in A} \frac{1}{a^\sigma} \sum_{m=m_1}^{\infty} \sum_{a \in A \cap I_m} \frac{1}{a^\sigma} \geq \sum_{m=m_1}^{\infty} \frac{|A \cap I_m|}{g^{m\sigma}} \geq \frac{1}{2} \sum_{m=m_1}^{\infty} \frac{(g^{-\delta} \prod_{k=0}^{g-2}(g-k)^{\alpha_k})^m}{g^{m\sigma}} = \frac{1}{2} \sum_{m=m_1}^{\infty} \frac{(g^{-\delta} \prod_{k=0}^{g-2}(g-k)^{\alpha_k})^m}{g^{m\sigma}}.
\]

Inequality (7) implies

\[
\frac{g^{-\delta} \prod_{k=0}^{g-2}(g-k)^{\alpha_k}}{g^{m\sigma}} > 1
\]

and so the infinite series \( F_A(\sigma) \) diverges if \( \sigma < \sigma_c \). This completes the proof. \( \square \)

**Corollary 1.** Let \( u_i \in [0, g-1] \) and \( U_i = \{u_i\} \) for all \( i \in \mathbb{N}_0 \). Let \( A = A_{g, \mathcal{U}} \) be the set of positive integers whose \( g \)-adic representation contains no digit \( c_i = u_i \). The Dirichlet series \( F_A(s) = \sum_{a \in A} a^{-s} \) has abscissa of convergence

\[
\sigma_c = \frac{\log(g-1)}{\log g}.
\]

In particular, Kempner’s series \( F_{A_{10,0}}(s) \) has abscissa of convergence \( \log 9/\log 10 \).

**Proof.** Apply Theorem 1 with \( U_i = \{u_i\} \) for all \( i \in \mathbb{N}_0 \). We have \( \alpha_0 = 0, \alpha_1 = 1, \alpha_k = 0 \) for all \( k \in [2, g-1] \), and \( \varepsilon_k(m) = 0 \) for all \( k \in [0, g-1] \).

For Kempner’s series, let \( g = 10 \) and \( u_i = 9 \) for all \( i \in \mathbb{N}_0 \). \( \square \)

**Theorem 2.** For all \( i \in \mathbb{N}_0 \), let \( U_i \) be a proper subset of \([0, g-1]\) such that (i) the set \( \mathcal{M} = \{m \in \mathbb{N} : U_{m-1} \neq [1, g-1]\} \) is infinite, and (ii) there exist nonnegative real numbers \( \alpha_0, \alpha_1, \ldots, \alpha_{g-1} \), and \( \beta \) such that, for all \( m \in \mathbb{N} \) and \( k \in [0, g-1] \),

\[
\text{card}\{i \in [0, m-1] : |U_i| = k\} = \alpha_k m + \varepsilon_k(m)
\]

and

\[
|\varepsilon_k(m)| < \beta.
\]

Let \( \mathcal{U} = (U_i)_{i=0}^{\infty} \). Let \( A_{g, \mathcal{U}} \) be the set of positive integers \( n \) with \( g \)-adic representation \( n = \sum_{i=0}^{m-1} c_i g^i \) such that \( c_{m-1} \neq 0 \) and \( c_i \notin U_i \) for all \( i \in [0, m-1] \). The Dirichlet series

\[
F_{A_{g, \mathcal{U}}}(s) = \sum_{a \in A_{g, \mathcal{U}}} \frac{1}{a^s}
\]

has abscissa of convergence

\[
\sigma_c = \frac{1}{\log g} \sum_{k=0}^{g-1} \alpha_k \log(g-k).
\]

Moreover, \( F_{A_{g, \mathcal{U}}}(\sigma_c) \) diverges.
Proof. The only difference between Theorem 1 and Theorem 2 is that condition (2) has been replaced by the more restrictive condition (9). Thus, $F_{A_{g,\mathcal{U}}}(s)$ has abscissa of convergence $\sigma_c$. We shall prove that $F_{A_{g,\mathcal{U}}}(\sigma_c)$ diverges. Note that (10) is equivalent to

$\prod_{k=0}^{g-1}(g-k)^{\alpha_k} = 1$

and that (9) implies

$$\sum_{k=0}^{g-1} |\varepsilon_k(m)| < g^3$$

for all $m \geq 1$. Following the proof of Theorem 1, we see that

$$|A_{g,\mathcal{U}} \cap I_m| \geq \frac{1}{2} \prod_{i=0}^{m-1} (g - |U_i|)$$

$$= \frac{1}{2} \prod_{k=0}^{g-1} (g-k)^{\alpha_k m + \varepsilon_k(m)}$$

$$\geq \frac{1}{2} g^{-\sum_{k=0}^{g-1} |\varepsilon_k(m)|} \prod_{k=0}^{g-1} (g-k)^{\alpha_k m}$$

$$> \frac{1}{2g^3g^3} \left( \prod_{k=0}^{g-1} (g-k)^{\alpha_k} \right)^m.$$
Dirichlet series $F_{A_g,t}(s) = \sum_{a \in A} a^{-s}$ has abscissa of convergence

$$\sigma_c = \frac{\log(g-1)}{\log g}$$

and $F_{A_g,t}(\sigma_c)$ diverges. In particular, Kempner’s series $F_{A_{10,9}}(s)$ has abscissa of convergence $\log 9/\log 10$, and $F_{A_{10,9}}(\log 9/\log 10)$ diverges.

References

[1] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 6th ed., Oxford University Press, Oxford, 2008.

[2] A. J. Kempner, A curious convergent series, Amer. Math. Monthly 21 (1914), no. 2, 48–50.