Dissipation power homogenization of anisotropic linear viscoelastic composites

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The current work presents a thermodynamic consistent derivation of the volume-specific dissipation power and energy function based on an anisotropic generalized Maxwell model. Further the equivalence to a physically meaningful effective dissipation power and energy density of a heterogeneous microstructure, exhibiting macroscopic direction-dependent linear viscoelastic material behavior, is shown.

KEYWORDS
anisotropic viscoelasticity, heterogeneous material, homogenization, power dissipation

1 INTRODUCTION

A suitable method for the determination of effective material properties of a micro-heterogeneous media is the mechanical homogenization. Beside particular analytical procedures, like [1] or [2], numerical treatments of general homogenization problems have become indispensable. Based on the mean field theory (MFT), see e.g. [3], the finite element method (FEM) enables the calculation of volume-averaged characteristics for almost any microscopically heterogeneous structure. Within such a scheme, the considered structure is modelled on the microscopic scale as a representative volume element (RVE) on which convenient boundary conditions (BCs) are imposed to generate a given homogeneous stress or strain state. The unknown corresponding macroscopic stress or strain state is achieved by averaging the related field size on the microscale, which comes from the numerical solution of the generated boundary value problem (BVP). Finally, the linkage of the given and the determined effective state variables with a homogeneous substitute material (HSM) allows a corresponding parameter identification of a convenient constitutive law. One requirement for this procedure is the Hill-Mandel condition,[4] which postulates the equivalence of the volume-averaged strain energy density of the RVE and the strain energy density of the HSM. Within the setting of small deformations, an identical relationship for the stress power density may be derived simply.[5] To satisfy this energy or power equivalence, certain BCs may be prescribed.[6,7] Regardless, in the case of a linear viscoelastic microstructure the volume-averaged stress power density consists of a reversible and an irreversible part (elastic stress power density, dissipation power density), whereas this separation is also mandatory for a corresponding homogeneous viscoelastic material model. However, the Hill-Mandel condition does not ensure the equivalence of these two stress power components on the micro- and macroscale.

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For that reason, an effective material model might be eligible for the reproduction of homogeneous stress-strain relations and therefore also for the homogeneous stress power but not for the elastic and dissipative part of it.

To overcome this issue for anisotropic linear viscoelastic materials, Section 3.1 presents the generalized Maxwell model (GMM) with an appropriate allocation to achieve a well known stress-strain convolution integral formulation for direction-dependent material behavior. Within Section 3.2 analytical expressions for the irreversible volume-specific stress power and energy are derived. Section 3.3 deduces a physically meaningful determination for the effective dissipation power and energy density of heterogeneous viscoelastic materials and proves the equivalence to the corresponding expression derived from the GMM. Practical usages for the given considerations are the adequate calculation of the homogeneous dissipation power and energy for arbitrary anisotropic linear viscoelastic materials described by the GMM or a parameter identification of such materials based on the homogenized dissipation power or energy. Especially, the latter possibility enables a nonlinear fitting of the scalar relaxation times for anisotropic materials by one scalar energy function, whereas their adequately determined values tremendously simplify the stress-strain based identification of the direction-dependent stiffness parameters of the GMM. Material examples might be given with viscoelastic polymers reinforced with any kind of elastic or viscoelastic fibres (e.g. woven or unidirectional fabrics). The current work is based on infinitesimal strains, linear viscoelastic material behavior, constant temperatures and microstructures with ideal adherence beneath the phases. Moreover, solely strain-driven processes are taken into account whereas similar considerations for stress-driven processes lead to analogous insights. Finally, the reader is referenced to appendix “Nomenclature/Notation” for a better understanding of the applied notation.

2 | THEORETICAL FRAMEWORK

2.1 | Anisotropic and isotropic viscoelastic stress-strain relations

According to [8,9], the relation between the Cauchy stress tensor \( \sigma(t) \) and the time derivation of the linear strain tensor \( \dot{\varepsilon}(t) \) for a linear viscoelastic material can be written by means of a convolution integral

\[
\sigma(t) = \int_0^t R(t - \varphi) \cdot \dot{\varepsilon}(\varphi) d\varphi,
\]

with the relaxation function \( R(t) \) as a positive-definite fourth-order tensor with time-dependent coefficients. The total number of independent coefficients is restricted to 21 due to equivalent symmetry properties as an elastic stiffness tensor. For the case of general anisotropy[10] proves the following choice for \( R(t) \) as viable

\[
R(t) = R_\infty + \sum_{j=1}^N R_j e^{-\frac{t}{\tau_j}},
\]

what represents Maxwell-type relaxation behavior as a special kind of the general relaxation function \( R(t) \) within Equation (1).

Within Equation (2), \( R_\infty \) represents a long-term stiffness tensor and \( R_j \) is a stiffness tensor by means of a relaxation strength related to a scalar relaxation time \( \tau_j \). [11–13] and [14], among others, give an insight into the application of Equations (1) and (2) for the description of viscoelastic fibre-reinforced plastics (FRP). For modeling an isotropic linear viscoelastic material, Equation (2) can be rewritten as follows

\[
R^{iso}(t) = \left\{ k_\infty + \sum_{j=1}^N k_j e^{-\frac{t}{\tau_j}} \right\} \frac{1}{1} + 2 \left\{ G_\infty + \sum_{j=1}^N G_j e^{-\frac{t}{\tau_j}} \right\} \frac{S_D}{1},
\]

with \( k_\infty \) and \( G_\infty \) being the bulk and shear modulus for the long-term answer. Moreover, \( k_j \) and \( G_j \) are the volumetric and deviatoric relaxation strengths conjugated with the relaxation times \( \tau_k^j \) and \( \tau_G^j \), respectively. A more detailed discussion of Equation (3) and its application to polymers can be found in [15].

2.2 | Mean-field homogenization of linear viscoelastic microstructures

In the following, a brief overview of the mechanical homogenization based on the mean-field theory referring to [3] as well as [16] is given. For that reason, the RVE of a microscopically heterogeneous structure, described as an area \( \mathcal{C} \) with the volume \( V \)
and the boundary $\partial \mathcal{G}$, is considered (see Figure 1). $\mathcal{G}$ consists of several phases $(i)$, each being homogeneous on the microscale and exhibiting a volume fraction $\epsilon^{(i)} = \frac{V^{(i)}}{V}$. Moreover the volume $V$ needs to be infinitesimal small on the macroscale and large enough to represent the structure on the microscale. Within a mean-field homogenization of a viscoelastic composite, the following initial boundary value problem (IBVP) for $\mathcal{G}$ needs to be solved with given constitutive equations on microscale for each phase

\begin{equation}
\nabla \cdot \sigma(t, r) = 0 \quad \forall \ r \in \mathcal{G}, \tag{4a}
\end{equation}

\begin{equation}
u(t, r^+) - u(t, r^-) = \varepsilon(t) \cdot (r^+ - r^-) \quad \forall \ r \in \partial \mathcal{G}. \tag{4b}
\end{equation}

Herein $r^+$ and $r^-$ are microscopic local vectors on opposite surface points, $u(t, r^+)$ and $u(t, r^-)$ corresponding microscopic displacement vectors and $\varepsilon(t)$ expresses the homogeneous strain tensor. The solution of the problem yields the time-dependent microscopic stress and strain tensor field, which may be homogenized to an effective stress and strain tensor by the volume averaging $A(t) = \langle A(t, r) \rangle = \frac{1}{V} \int_{\mathcal{G}} A(t, r) \, dV$ as follows

\begin{equation}
\sigma(t) = \langle \sigma(t, r) \rangle = \sum_i c^{(i)} \langle \sigma^{(i)}(t, r) \rangle, \tag{5a}
\end{equation}

\begin{equation}
\varepsilon(t) = \langle \varepsilon(t, r) \rangle = \sum_i c^{(i)} \langle \varepsilon^{(i)}(t, r) \rangle. \tag{5b}
\end{equation}

Furthermore, Equation (4b) represents periodic displacement boundary conditions (PDBCs), which enable a defined prescription of $\varepsilon(t)$ for non periodic and periodic microstructures. The stated coupling between the micro- and macroscale (Equations (5a) and (5b)) was introduced by [4] for elastic materials. To integrate a physical plausibility into this homogenization scheme, [4] also introduced the Hill-Mandel lemma, which originally presumed that the volume averaging of the microscopic strain energy density is identical to the strain energy density arising from the effective stress and strain tensor. However, within the setting of small deformations the generalized stress power density formulation of the Hill-Mandel lemma is equivalent to its original formulation and given by

\begin{equation}
\langle \sigma(t, r) \cdot \dot{\varepsilon}(t, r) \rangle = \langle \sigma(t, r) \rangle \cdot \langle \dot{\varepsilon}(t, r) \rangle = \sigma(t) \cdot \dot{\varepsilon}(t). \tag{6}
\end{equation}

Independent of the material properties of ideally adhering phases $(i)$ and the size of a RVE containing them, Equation (6) is fulfilled under homogeneous displacement boundary conditions (HDBC) as well as PDBCs Equation (4b). The latter ones reduce the influence of the RVE’s extent on the convergence behavior of the effective stress for non periodic microstructures and eliminate this influence for periodic ones. Detailed discussions of BCs and convergence issues of effective properties within a homogenization may be found in [5–7,17–19].
Within the present case, effective properties of linear viscoelastic microstructures have to be determined, which are represented by the relaxation modulus $R(t)$ within the constitutive law in Equation (1). For that reason, the IBVP, defined by the Equations (4a) and (4b), needs to be solved within a time interval of interest with a suitable time-dependent effective strain tensor applied via PDBCs to determine the time-dependent stress tensor according to Equation (5a). As a result, the parameter of a mathematical choice for $R(t)$, e.g. Equation (2), may be identified with an appropriate optimization algorithm for a maximum of 21 coefficients in the case of anisotropic behavior. Since $R(t)$ has to be extracted from a convolution integral, this last step requires special techniques which may include the viscoelastic correspondence principle. Examples for the homogenization of anisotropic viscoelastic FRP are given with [11,12,14].

3 | A PHYSICALLY MEANINGFUL DISSIPATION POWER DENSITY FUNCTION FOR ANISOTROPIC LINEAR VISCOELASTIC MATERIALS

3.1 | Model of anisotropic linear viscoelasticity

Based on [8,9,22–24], the GMM according to Figure 2 is taken into account. $\sigma^\infty$, $\sigma^\text{in}_j$, and $\sigma^\text{el}_j$ are the equilibrium, viscous (inelastic) and elastic stress tensor, respectively. Accordingly, the total, the viscous and the elastic strain tensors are symbolized by $\varepsilon$, $\varepsilon^\text{in}_j$, and $\varepsilon^\text{el}_j$, respectively. Furthermore, each of the $j = 1 \ldots N$ parallel connected submodels will be termed Maxwell model or Maxwell branch $j$. The necessary connection relations of the GMM are

$$\varepsilon = \varepsilon^\text{in}_j + \varepsilon^\text{el}_j, \quad (7a)$$
$$\sigma_j = \sigma^\text{in}_j = \sigma^\text{el}_j, \quad (7b)$$
$$\sigma = \sigma^\infty + \sum_{j=1}^{N} \sigma_j, \quad (7c)$$

whereas Equation (7a) solely holds within the infinitesimal strain theory. The elementary elastic and inelastic constitutive laws for a single Maxwell model $j$ are introduced according to the projection method proposed by

$$\sigma^\text{el}_j = R_j \varepsilon^\text{el}_j, \quad (8a)$$
$$\sigma^\text{in}_j = D_j \dot{\varepsilon}^\text{in}_j. \quad (8b)$$

Here $R_j$ and $D_j$ are anisotropic fourth-order stiffness and viscosity tensors, for which identical eigenspaces have to be assumed. At this point, this is restricted to

$$D_j = \tau_j R_j, \quad (9)$$
According to that, \( R_j \) and \( D_j \) undergo a proportional relation with the scalar relaxation time \( \tau_j \) as constant of proportionality. Note that this assumption does not restrict the anisotropy of the stress-strain behavior within a Maxwell branch \( j \) and, thus, leads to a straightforward and appropriate material law (see Equations (17) to (20)). With \( R_\infty \) being the anisotropic equilibrium stiffness tensor, the material law for the corresponding part of the GMM reads

\[
\sigma_\infty = R_\infty \cdot \varepsilon.
\]  

### 3.2 Phenomenological derivation of reversible and irreversible stress power and strain energy density

To derive a thermodynamic consistent stress-strain relation for the GMM within the current setting, the following presentation of the Clausius-Planck inequality (CPI) is required

\[
\sigma \cdot \dot{\varepsilon} - \rho \dot{\psi} \left( \varepsilon, \varepsilon^{el} \right) \geq 0.
\]  

Herein \( \sigma \cdot \dot{\varepsilon} \) expresses the volume-specific stress power and \( \rho \) the constant reference mass density at a homogenized material point, which shall replace a certain area \( G \) of a microstructure (see Section 2.2). \( \psi(\varepsilon, \varepsilon^{el}) \) stands for the mass-specific free energy (Helmholtz free energy), which is merely reversible and positive by definition. At this point, \( \psi(\varepsilon, \varepsilon^{el}) \) is modelled by the elastic contributions of the GMM. The free energy for one Maxwell model \( j \) reads as

\[
\psi_j \left( \varepsilon^{el}_j \right) = \frac{1}{2} \rho \varepsilon^{el}_j \cdot R_j \cdot \varepsilon^{el}_j,
\]  

which reveals the requirement of positive definite stiffness tensors \( R_j \) and leads to the CPI for the branch \( j \)

\[
\sigma_j \cdot \dot{\varepsilon} - \left( R_j \cdot \varepsilon^{el}_j \right) \cdot \dot{\varepsilon}^{el}_j \geq 0.
\]  

Note that the total strain tensor \( \varepsilon_j \) of any branch \( j \) corresponds to the total strain tensor \( \varepsilon \) of the GMM. With respect to the Equations (7a), (7b) and (8a) it follows

\[
\left( \sigma - R_j \cdot \left( \varepsilon - \varepsilon^{in}_j \right) \right) \cdot \dot{\varepsilon} + \sigma^{in} \cdot \dot{\varepsilon}^{in} \geq 0.
\]  

A suitable restriction of the present considerations is the separate evaluation of each term within inequality (14). To achieve a constitutive law of practical use, the first term on the left-hand side of Equation (14) is set to zero, what leads to

\[
\sigma = R_j \cdot \left( \varepsilon - \varepsilon^{in}_j \right) + \sigma^z \text{ with } \sigma^z \cdot \dot{\varepsilon} = 0.
\]  

Equation (15) represents a material law for one Maxwell branch \( j \) with the viscous strain tensor \( \varepsilon^{in}_j \) and the additional stress tensor \( \sigma^z \), which covers modes without a power contribution, like a hydrostatic stress applied to an incompressible material. For the current case, \( \sigma^z \) is not necessary and will be set to 0. The evaluation of the second term within inequality (14) reveals (see Equation (8b))

\[
p^{\text{diss}}_j = \sigma^{in}_j \cdot \dot{\varepsilon}^{in}_j = \varepsilon^{in}_j \cdot D_j \cdot \dot{\varepsilon}^{in} \geq 0.
\]  

The irreversible stress power \( p^{\text{diss}}_j \) of a Maxwell model \( j \) needs to be equal to or greater than zero. For that reason, \( D_j \) has to be positive definite what leads, in conjunction with Equation (9), to the necessity of positive relaxation times \( \tau_j \). To eliminate the inelastic strain tensor \( \varepsilon^{in}_j \) within Equation (15), Equations (8a) and (8b) are inserted in Equation (7b) and the remaining elastic strain tensor \( \varepsilon^{el}_j \) is substituted by Equation (7a) leading to

\[
\dot{\varepsilon}^{in}_j = D^{-1} \cdot R_j \cdot \left( \varepsilon - \varepsilon^{in}_j \right).
\]
With equation (9) we obtain
\[ \dot{\varepsilon}_{\text{in}}^j = \frac{1}{\tau_j} \left( \varepsilon - \varepsilon_{\text{in}}^j \right). \]  

Equation (18)

The resulting evolution equation can be resolved analytically as follows
\[ \varepsilon_{\text{in}}^j(t) = \varepsilon(t) - \int_0^t e^{-\frac{t-\varphi}{\tau_j}} \dot{\varepsilon}(\varphi) d\varphi. \]  

Equation (19)

With Equation (19), Equation (15) changes over to
\[ \sigma_j(t) = R_j \cdot \left( \int_0^t e^{-\frac{t-\varphi}{\tau_j}} \dot{\varepsilon}(\varphi) d\varphi \right). \]  

Equation (20)

Note that without the proportional relation given by Equation (9), Equation (17) needs to be solved by a numerically evolution of the inelastic strain, what inhibits the simple calculation of the stress-tensor according to Equation (20) and implies difficulties in terms of a parameter identification or finite element implementation (see [26]) of the current material model. Moreover, the volume-specific dissipation power within inequality (16) can be modified with the time derivative of Equation (19) and Equation (9) as follows
\[ p_j^{\text{diss}}(t) = \frac{1}{\tau_j} \left( \int_0^t e^{-\frac{t-\varphi}{\tau_j}} \dot{\varepsilon}(\varphi) d\varphi \right) \cdot R_j \cdot \left( \int_0^t e^{-\frac{t-\varphi}{\tau_j}} \dot{\varepsilon}(\varphi) d\varphi \right). \]  

Equation (21)

To obtain a thermodynamic consistent stress-strain relation for the GMM, the entire free energy \( \psi \) needs to be formulated as the superimposition of the elementary contributions (small strain theory). With \( \psi_j \) according to Equation (12) as well as an additional contribution resulting from the equilibrium spring, \( \psi \) appears as
\[ \psi \left( \varepsilon, \varepsilon_{\text{in}}^j \right) = \frac{1}{2\rho_e} \varepsilon \cdot \varepsilon + \sum_{j=1}^N \psi_j. \]  

Equation (22)

The insertion of the time derivative of Equation (22) in inequality (11) and the consideration of Equations (12), (7a), (7b) and (8b) yields
\[ \left( \sigma - R_{\infty} \cdot \varepsilon - \sum_{j=1}^N R_j \cdot (\varepsilon - \varepsilon_{\text{in}}^j) \right) \cdot \ddot{\varepsilon} + \sum_{j=1}^N D_j \cdot \ddot{\varepsilon}_{\text{in}}^j \geq 0. \]  

Equation (23)

To eliminate the viscous strain tensors \( \varepsilon_{\text{in}}^j \), Equation (19) is utilized what leads with inequality (16) and Equation (2) to
\[ \left( \sigma - \int_0^t R(t-\varphi) \cdot \ddot{\varepsilon}(\varphi) d\varphi \right) \cdot \ddot{\varepsilon} + \sum_{j=1}^N p_j^{\text{diss}}(t) \geq 0, \]  

Equation (24)

whereas \( p_j^{\text{diss}}(t) \) can be calculated according Equation (21). Analogous to the evaluation of inequality (14), the resulting constitutive law of the GMM reads as already given with Equation (1) containing the relaxation function associated with Equation (2). In other words, the assumptions represented by Equations (8a), (8b), (9) and (10) for the GMM lead to the well-known stress-strain relation for anisotropic viscoelastic materials. Moreover, the right term on the left-hand side in inequality (24), which is the volume-specific dissipation power of the GMM, has to be equal to or greater than zero, which is fulfilled with positive definite \( D_j \). An expression for the entire dissipation power density is given as
\[ p^{\text{diss}}(t) = \sum_{j=1}^N p_j^{\text{diss}}(t) = \sum_{j=1}^N \frac{1}{\tau_j} \left( \int_0^t e^{-\frac{t-\varphi}{\tau_j}} \ddot{\varepsilon}(\varphi) d\varphi \right) \cdot R_j \cdot \left( \int_0^t e^{-\frac{t-\varphi}{\tau_j}} \ddot{\varepsilon}(\varphi) d\varphi \right). \]  

Equation (25)
3.3 | Homogenization of the reversible and irreversible stress power and strain energy density

In this section, it will be shown that the dissipation power given by Equation (25) and the dissipation energy are always identical to their corresponding homogenized functions for certain requirements. To this end, a microstructure under the IBVP according to the Equations (4a) and (4b) is taken into account. For illustration purposes only, the following considerations are related to a FRP with an elastic fibre-reinforcement and a viscoelastic polymer matrix and the dependence on time is further omitted within each equation. The indexes \(i\) for separate phases of a microstructure turn into \((f)\) and \((m)\) for fibre and matrix, respectively. The latter one is modelled as GMM according to Figure 2. Thus, the required relations for stresses and strains of the matrix material are according to the Equations (7a) to (7c). Formulated for the microscale, they read

\[
\begin{align*}
\varepsilon^{(m)}(r) &= \varepsilon^{(m)}(r) + \varepsilon^{(f)}(r), \\
\sigma^{(m)}(r) &= \sigma^{(m)}(r) + \sigma^{(f)}(r), \\
\varepsilon^{(m)}(r) &= \varepsilon^{(m)}(r) + \sum_{j=1}^{N} \varepsilon^{(m)}(r).
\end{align*}
\]

with

\[
\begin{align*}
\sigma^{(m)}(r) &= R^{(m)} \varepsilon^{(m)}(r), \\
\sigma^{(f)}(r) &= R^{(f)} \varepsilon^{(f)}(r).
\end{align*}
\]

The entire volume-specific free energy (see Section 3.2) for the matrix material model at any point \(r \in \mathcal{O}^{(m)}\) is given by

\[
\psi^{(m)}(r) = \frac{1}{2\rho^{(m)}} \left[ \sigma^{(m)}(r) \cdot \varepsilon^{(m)}(r) + \sum_{j=1}^{N} \sigma^{(m)}(r) \cdot \varepsilon^{(m)}(r) \right].
\]

For the pure elastic fibre-reinforcement it follows at any point \(r \in \mathcal{O}^{(f)}\)

\[
\psi^{(f)}(r) = \frac{1}{2\rho^{(f)}} \sigma^{(f)}(r) \cdot \varepsilon^{(f)}(r).
\]

Moreover, the homogenized CPI (see inequality (11)) without respect to a special point on the microscale reads as

\[
\left\langle \sigma \cdot \dot{\varepsilon} - \rho \psi \left( \varepsilon^{(m)}(r), \varepsilon^{(m)}(r), \dot{\varepsilon}^{(f)}(r) \right) \right\rangle \geq 0.
\]

Since we can split an integral of a sum into the separate parts, inequality (30) turns into

\[
\sigma \cdot \dot{\varepsilon} - \left\langle \rho \psi \left( \varepsilon^{(m)}(r), \varepsilon^{(m)}(r), \dot{\varepsilon}^{(f)}(r) \right) \right\rangle \geq 0.
\]

Here, \(\langle \sigma \cdot \dot{\varepsilon} \rangle\) has already been replaced by \(\sigma \cdot \dot{\varepsilon}\), since the Hill-Mandel condition Equation (6) holds for the current case. The substitution of the time derivative of Equations (28) and (29) yields

\[
\sigma \cdot \dot{\varepsilon} - c^{(f)} \left\langle \dot{\varepsilon}^{(f)} \right\rangle - c^{(m)} \left\langle \dot{\varepsilon}^{(m)} \right\rangle \geq 0.
\]

The elastic strain rate tensor \(\dot{\varepsilon}^{(m)}\) is replaced by substituting Equation (26a) into inequality (32), which reveals

\[
\sigma \cdot \dot{\varepsilon} - c^{(f)} \left\langle \dot{\varepsilon}^{(f)} \right\rangle - c^{(m)} \left\langle \dot{\varepsilon}^{(m)} \right\rangle \geq 0.
\]
Moreover, the middle term can be replaced by Equation (26c), which turns inequality (33) into

\[ \sigma \cdot \dot{\varepsilon} - c(f) \langle \sigma(f) \rangle \cdot \dot{\varepsilon} + c(m) \langle \sigma(m) \rangle \cdot \dot{\varepsilon} + c(m) \langle \sum_{j=1}^{N} \sigma_j(m)(r) \cdot \dot{\varepsilon}_j(in(m)) \rangle \geq 0. \]  (34)

For the evaluation of Equation (34) the following identity is required

\[ \langle \sigma(r) \cdot \dot{\varepsilon}(r) \rangle = c(f) \langle \sigma(f)(r) \cdot \dot{\varepsilon}(r) \rangle + c(m) \langle \sigma(m)(r) \cdot \dot{\varepsilon}(r) \rangle. \]  (35)

This can be easily found by splitting the integral \( \frac{1}{V} \int \sigma(r) \cdot \dot{\varepsilon}(r) \, dV = \langle \sigma(r) \cdot \dot{\varepsilon}(r) \rangle \) into separate parts for fibre and matrix material. With the premise of Equation (6) and Equations (35) as well as (3a), Equation (34) finally turns into

\[ (\sigma - c(f) \langle \sigma(f)(r) \rangle) \cdot \dot{\varepsilon} + c(m) \langle \sum_{j=1}^{N} \sigma_j(m)(r) \cdot \dot{\varepsilon}_j(in(m)) \rangle \geq 0. \]  (36)

The evaluation of inequality (36) then yields

\[ \sigma = c(f) \langle \sigma(f)(r) \rangle + c(m) \langle \sigma(m)(r) \rangle, \]  (37)

which corresponds to Equation (5a) in the current framework. Moreover, the microstructure’s irreversible effective dissipation power density can be identified as the right term on the left-hand side within inequality (36) as follows

\[ p^{diss} = c(m) \langle p^{diss(m)} \rangle = c(m) \langle \sum_{j=1}^{N} \sigma_j(m)(r) \cdot \dot{\varepsilon}_j(in(m)) \rangle. \]  (38)

For an arbitrary composite with viscoelastic phases \( i \) modelled with the GMM on the microscale, an analogous consideration leads to

\[ p^{diss} = \sum_{i} c^{(i)} \langle p^{diss(i)} \rangle = \sum_{i} c^{(i)} \langle \sum_{j=1}^{N} \sigma_j^{(i)}(r) \cdot \dot{\varepsilon}_j(in(i)) \rangle. \]  (39)

Note that the homogeneous dissipation power given by Equation (39) and (25) are derived by splitting a homogeneous expression of a density-weighted time derivative of the free energy (see deduction of inequality (24) and (36)). As a consequence of this, the corresponding CPI contain expressions of the entire stress power, which are identical if the Hill-Mandel lemma is fulfilled. For that case, the split expressions of the volume-averaged and homogeneous dissipation power according to Equation (39) and (25) are always identical, if the the left-hand term on the left-hand side within inequality (24) and (36) is zero. This is the case, if Equation (37) and (1) coincide for arbitrary strain-driven processes. In other words, if the constitutive law according to Equation (1) including a suitably adjusted relaxation function (2) reproduces the effective stress of a considered microstructure according to Equation (37) for arbitrary strains, the corresponding GMM is a suitable material model to reproduce the microstructure’s effective dissipation power as well as energy in a physically meaningful manner. Alternatively, the viscous strain tensors within the GMM can be interpreted as macroscopic internal variables with an analytical solution of their evolution according to Equation (19) and the viscous stress tensors coincide with macroscopic viscous driving forces given by Equation (20) or (8b).[27] In contrast to the more general remarks within [27], they fulfill the principle of the equivalence of the macroscopic and volume-averaged dissipation power automatically for the current case and the stated requirements. It should be noted, that the current considerations are based on the linear viscoelasticity and small deformations within the time domain as a special case, whereas there are some possibilities for other special cases or generalisations. For instance, a consideration in the frequency (steady state) domain would lead to similar results, whereas the
consideration within the Laplace-domain would open the possibility for the application of all methods from the anisotropic linear elasticity. Furthermore, analogous examinations for large deformations considering different strain measurements would be worthwhile.

4 | CONCLUSION

Within the current work, a suitable choice for elementary constitutive laws for the generalized Maxwell model is presented, which leads to a well-known and thermodynamically consistent stress-strain relation for anisotropic viscoelastic materials. Moreover, an analytical expression for the dissipation power density Equation (25) as well as energy (integral over time of Equation (25)) is obtained. For this, the equivalence to the effective dissipation power density, and therefore also for the dissipation energy density, of a corresponding microstructure under the fulfillment of the Hill-Mandel lemma is demonstrated. It follows that the (effective) reversible counterparts of the microstructure and the material model coincide, too. The given considerations, including the employed GMM for anisotropic linear viscoelastic materials as well as microstructures exhibiting the same material behavior, provide a general methodology to proof and not to postulate the identity of a given dissipation energy function and a homogenized equivalence. To the knowledge of the authors such a derivation has not been carried out before and enables a physically meaningful description of the dissipation power with the generalized Maxwell model. Another application of the current work is given by an energy based parameter identification of the generalized Maxwell model. The latter two examples will be demonstrated in detail within a further publication including several numerical examples for anisotropic viscoelastic fibre-reinforced plastics exhibiting a strong direction dependency in their relaxation behavior.

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**APPENDIX: NOMENCLATURE/NOTATION**

The present work deals with viscoelastic and thus time-dependent materials on the micro- as well as macro-scale. For that reason, power- or energy-based as well as dynamic and kinematic variables are generally a function of time, what will only be specified with “(t)” if necessary for comprehension. Moreover time-dependent material laws are always denoted with “(t)” to distinguish them from time-independent material parameters. To separate micro- and macroscopic variables, the first type is always denoted with “(r)” as the microscopic local vector. The order of a tensor is represented by the number of underlines below the tensor symbol. Thus, a vector reads \( \mathbf{r} \) and a second-order tensor is denoted by \( \mathbf{X} \). The tensor’s coefficients are related to orthonormal base vectors, e.g. \( \mathbf{X} = X_{ab} \mathbf{e} \otimes \mathbf{e}_b \). The base vectors \( \{ \mathbf{e}_a \} = \{ \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \} \) refer to a Cartesian coordinate system. The treatment of the indices corresponds to the Einstein summation convention. The tensor product is symbolized by ‘\( \otimes \)’ and the operator ‘\( \cdot \)’ describes a single contraction between base vectors. Multiple dots mean multiple contractions such that

\[
(\mathbf{e}_a \otimes \mathbf{e}_b) \cdot (\mathbf{e}_c \otimes \mathbf{e}_d) = \delta_{bc} \mathbf{e}_a \otimes \mathbf{e}_d, \tag{A1a}
\]

\[
(\mathbf{e}_a \otimes \mathbf{e}_b) \cdot \cdot (\mathbf{e}_c \otimes \mathbf{e}_d) = \delta_{bc} \delta_{ad}, \tag{A1b}
\]

where \( \delta \) is the Kronecker symbol. Within the infinitesimal strain theory, the material time derivative \( \dot{\mathbf{X}} \) is defined by

\[
\dot{\mathbf{X}} = \dot{X}_{ab} \mathbf{e}_a \otimes \mathbf{e}_b, \tag{A2}
\]

with fixed base vectors. Moreover, two isotropic fourth-order tensors are introduced as follows

\[
\frac{1}{3} \mathbb{I} \cdot \cdot \mathbf{X} = I_1(\mathbf{X}) \mathbb{I}, \tag{A3a}
\]

\[
\frac{SD}{3} \mathbb{I} \cdot \cdot \mathbf{X} = \frac{1}{2} (\mathbb{I} \cdot \mathbf{X} + \mathbf{X}^T) - \frac{1}{3} I_1(\mathbf{X}) \mathbb{I}, \tag{A3b}
\]

\( \mathbb{I} \) is the second-order identity tensor and \( I_1(\mathbf{X}) = \mathbb{I} \cdot \cdot \mathbf{X} \) the first main invariant of \( \mathbf{X} \). Any part (elastic, inelastic) of the volume-specific stress power may be defined as

\[
p = \sigma \cdot \cdot \varepsilon. \tag{A4}
\]

The corresponding volume-specific stress energy is then given by \( e = \int p \, dt \). The expressions “density of power or energy” as well as “power- or energy density” always refer to this kind of volume-specific measures. Within the present work, the inelastic parts of these two expressions solely arises from viscous components and are also denoted with “dissipation power density” or “dissipation energy density”, respectively. Finally, Table A1 gives an overview of further used symbols and indices.
| Symbol | Description |
|--------|-------------|
| $\mathcal{G}$ | of a heterogeneous microstructure occupied volume |
| $A, V$ | area, volume |
| $c$ | volume fraction |
| $e, p$ | volume-specific stress energy, volume-specific stress power |
| $G, k$ | shear modulus, bulk modulus |
| $\delta_{ab}$ | Kronecker delta |
| $\delta_j$ | parameter of dissipation energy density |
| $\rho$ | mass density |
| $\tau$ | relaxation time |
| $\varphi$ | variable of integration |
| $\psi$ | free energy density |
| $\tau$ | microscopic local vector |
| $\psi$ | displacement vector |
| $\sigma, \varepsilon$ | Cauchy stress tensor, linear strain tensor as the symmetric part of the displacement gradient tensor |
| $R, D$ | fourth-order stiffness and viscosity tensor |
| $R(t)$ | time-dependent fourth-order tensor with the meaning of a relaxation function |
| $(\cdot)^+, (\cdot)^-$ | refer to two opposite points at the boundary of a microstructure |
| $(\cdot)i$ | belonging to a time increment $i$ |
| $(\cdot)j$ | belonging to the Maxwell model $j$ of the Generalized Maxwell model |
| $0, \infty$ | belonging to the equilibrium stiffness tensor of the Generalized Maxwell model |
| $(\cdot)^{\text{diss}}$ | belonging to a dissipative part |
| $(\cdot)^{\text{elas}}, (\cdot)^{\text{elas}}$ | belonging to a deviatoric or volumetric part |
| $(\cdot)^{\text{elas}}, (\cdot)^{\text{elas}}$ | belonging to an inelastic or elastic part |
| $(\cdot)^{(i)}$ | belonging to the phase $(i)$ of a microstructure with $(m)$ for matrix and $(f)$ for fibre as a special case |