Tensor distributions on signature-changing space-times

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Abstract

Irregularities in the metric tensor of a signature-changing space-time suggest that field equations on such space-times might be regarded as distributional. We review the formalism of tensor distributions on differentiable manifolds, and examine to what extent rigorous meaning can be given to field equations in the presence of signature-change, in particular those involving covariant derivatives. We find that, for both continuous and discontinuous signature-change, covariant differentiation can be defined on a class of tensor distributions wide enough to be physically interesting.

1 Introduction

A classical signature-changing space-time $M$ has a metric tensor $g$ whose signature is Lorentzian in some regions and Euclidean in others. Whenever two regions of different signature exist on a connected component of $M$, there must be some surface at which the metric tensor is either degenerate (implying a singularity in the inverse metric) or discontinuous. In order to give meaning to field equations involving the metric on such space-times, either severe restrictions must be placed on the allowed
class of fields, such as assuming they be class $C^2$ (see also [1]), or a distributional point of view must be adopted [8] (see also [4, 2]).

In this note, we investigate to what extent the formalism of tensor distributions can be applied to signature-changing space-times. Our main conclusion is that rigorous meaning can be given to field equations for a class of fields wide enough to be physically interesting.

Of course, tensor distributions on manifolds are by no means new: admirable treatments of the topic can be found in many places (eg [10, 13, 3]). Indeed, the formalism is quite standard in studying shock waves in general relativity [12, 2]. However, in such applications, it is usual to assume the metric is continuous and non-degenerate, with irregularities first appearing at the level of the connection. Since either of these conditions on the metric can be violated under signature-change, the standard formalism should be re-examined in this new context.

The difficulty is not defining tensor distributions in the presence of signature-change, since the space-time metric plays no part in establishing the topological spaces of test functions and test tensors on a manifold. Rather, problems arise when tensor distributions are associated with locally integrable tensors, since the volume element of $g$ is usually used for this purpose. Under signature-change, this may be degenerate, so an alternative volume element must be introduced. Similarly, the volume element and connection are often used to define differentiation of distributions, so care is needed here too. More problematic is the definition of covariant differentiation, since both the tensor fields and the connection components are distributional in nature.

We begin in sections 2 and 3 with a brief review of the standard treatment of distributions on manifolds, highlighting the choices appropriate to signature-change. Distributions related to hypersurfaces are discussed in section 4. Section 5 then contains applications to signature-changing space-times, including some simple examples. We discuss our conclusions in section 6. A more detailed review of tensor distributions on degenerate space-times can be found elsewhere [6].

## 2 Distributions

In this section, we review the standard formalism of tensor distributions on differentiable manifolds, partly to establish our notation. For missing details, we refer to standard works [10, 13, 3]. Here, as elsewhere, we assume all functions and tensors are smooth (ie $C^\infty$) unless specifically mentioned. Similar definitions hold for lower degrees of differentiability.

Let $M$ be a paracompact $n$-dimensional differentiable manifold. Test functions on $M$ are (smooth) functions with compact support. Scalar distributions on $M$ are real or complex-valued continuous linear functionals on the space of test functions, which is equipped with a suitable topology. We denote the action of a distribution $D$ on a test function $f$ by $D : f \mapsto D[f]$.

In a similar fashion, test tensors on $M$ are tensors with compact support. The space of test tensors can be equipped with a suitable topology by introducing an auxiliary Riemannian metric and connection. However, the space-time metric is not required in this definition. In fact, an equivalent topology can be defined without using any metric at all [2]. Tensor distributions are then real or complex-valued continuous linear functionals on the topological space of test tensors, and we write $T : U \mapsto T[U]$. As a linear functional on test tensors, $T$ can be assigned a tensor type, which must be dual to that of the test tensor $U$.

Multiplication of tensor distributions by functions, and tensor products and contractions with ordinary tensors present no difficulty: for a tensor distribution $T$,
function $f$, vector $X$ and tensor $S$,

\[
\begin{align*}
(fT)[U] &= T[fU] \\
(S \otimes T)[W \otimes U] &= T[(S,W)U] \\
T_X[U] &= T[X \otimes U],
\end{align*}
\] (1)

where $\langle S, W \rangle$ represents the total contraction of $S$ and $W$ (which has tensor type dual to that of $S$). Tensor products of tensor distributions are not defined in general (just as for scalar products of scalar distributions).

Given a local frame $\{X_a\}$ and dual coframe $\{\theta^a\}$ on an open set $N$ in $M$, the *components* of $T$ are scalar distributions defined in an obvious way. For example, if $\alpha$ is a covector distribution, then the component distributions $\alpha_a$ are given by

\[
\alpha_a[p] = \alpha[p X_a] \tag{2}
\]

for any test function $p$ with $\text{supp } p \subset N$. This leads to a local expression for any tensor distribution $T$ in terms of its components and frame and coframe elements, completely analogous to local expressions of ordinary tensors. For example, on test vectors $V$ with $\text{supp } V \subset N$,

\[
\alpha[V] = \alpha_a \theta^a [V]. \tag{3}
\]

The components transform in the usual way under changes of frame, and the tensorial operations defined above have the usual expressions in terms of components. In other words, tensor distributions can equally be regarded as distribution-valued tensors.

Up to this point, no special structures (metric etc) have been assumed on $M$. Now let $\omega$ be a volume element (i.e. a nowhere vanishing $n$-form) on $M$. To every locally integrable tensor $S$ on $M$ we can associate a tensor distribution denoted $\hat{S}$ by

\[
\hat{S}[U] = \int_M \langle S, U \rangle \omega. \tag{4}
\]

It is possible to avoid the introduction of a volume element by using de Rham currents [10, 1], and replacing test functions by test $n$-forms. Tensor distributions can be defined in terms of their components, and a distribution $\hat{f}$ associated with a locally integrable function $f$ by $\hat{f}[\phi] = \int_M f \phi$ for any test $n$-form $\phi$. Here we shall follow the more conventional approach.

## 3 Differentiation

In order to define differentiation of distributions, we make further use of the volume form. For an ordinary vector $X$ and a scalar distribution $D$, we set

\[
(XD)[p] = -D[\text{div}(pX)] \tag{5}
\]

for all test functions $p$, where the divergence is defined (cf. [14]) by

\[
\text{div}(X)\omega = \mathcal{L}_X \omega = d_i X \omega. \tag{6}
\]

The definition (5) is compatible with the action of $X$ on an ordinary function: $X \hat{f} = \hat{Xf}$. In a coordinate chart $\{x^a\}$ with $\omega = k dx^1 \wedge ... \wedge dx^n$ (where $k$ is a function in general), we have the partial derivatives (in agreement with (3))

\[
\partial_a D[p] = -D[k^{-1} \partial_a (kp)]. \tag{7}
\]
It is interesting to note that if a non-smooth volume element \( \omega \) were used then distributions would be only finitely differentiable, in contrast with the usual presentation of distributions on \( \mathbb{R}^n \).

Other differential operators on tensor distributions are built from this basic definition in the same fashion as for ordinary tensors. For example, the exterior derivative \( d \) is uniquely prescribed by the requirements that it be an anti-derivation of degree 1 on exterior form distributions, that \( d^2 = 0 \), and that \( dD \) for a scalar distribution \( D \) satisfy

\[
\frac{\partial}{\partial x^i} dD[p] = -D[\text{div}(pX)]
\]

or, for any test vector \( V \),

\[
dD[V] = -D[\text{div}V]. \tag{8}
\]

Likewise, covariant differentiation of tensor distributions is completely determined by equation (5) and the usual properties of covariant differentiation upon fixing the connection map \( \nabla \) from vector distributions to type (1,1) tensor distributions, satisfying the usual requirements

\[
\nabla(Y + Z) = \nabla Y + \nabla Z \tag{9}
\]

\[
\nabla(DX) = dD \otimes X + D\nabla X \tag{10}
\]

for all vector distributions \( Y, Z \), scalar distributions \( D \) and ordinary vectors \( X \). Explicit formulae for covariant derivatives may be written in terms of local components. For example, if \( Z \) is a vector distribution with local coordinate components \( Z^a \), then

\[
(\nabla Z)^a_b \equiv \partial_b Z^a + \Gamma^a_{bc} Z^c. \tag{11}
\]

Implicit in condition (10) is the requirement that the product \( D\nabla X \) be well-defined. Typically, the connection map on distributions arises from a smooth connection, also denoted \( \nabla \), on \( M \) (ie \( \nabla X \) is smooth for all smooth vectors \( X \)). Setting \( \nabla X = \nabla X \) for all smooth vectors \( X \) consistently determines the distributional connection map, since \( D\nabla X \) is then well-defined for all scalar distributions \( D \).

Given a smooth connection \( \nabla \) on \( M \), it is straightforward to show that the divergence defined in equation (6) and the alternative definition \( \nabla \cdot X = \text{Tr}(\nabla X) \) satisfy

\[
\text{div} X = \nabla \cdot X \tag{12}
\]

for all vectors \( X \) if, and only if

\[
\nabla \omega + \text{Tr}(\tau) \otimes \omega = 0, \tag{13}
\]

where \( \tau \) is the type (1,2) torsion tensor of \( \nabla \). If these conditions are met, then

\[
(\nabla D)[p] = -D[\nabla \cdot (pX)] \tag{14}
\]

can be used as an equivalent starting point for the differentiation of distributions, leading to an elegant expression for the absolute derivative:

\[
\nabla T[U] = -T[\nabla \cdot U], \tag{15}
\]

in which the first argument of \( U \) must be a covector, and \( \nabla \cdot U \) is the trace over the first two arguments of \( \nabla U \). However, we are interested in the possibility that \( \nabla \) not be smooth (see below), let alone satisfy condition (13), so we forego (14) and (15), and adhere to our choice of definition (8).

If \( \nabla \) is not smooth, by which we mean that it is not associated with a smooth connection on \( M \), then the covariant derivative may not be defined for all tensor distributions. In particular, products such as \( D\nabla X \) from property (10) will not be defined for all \( D \) and \( X \). The same holds for terms like \( \Gamma^a_{bc} Z^c \) from the component version (11). However, by restricting the class of tensor distribution \( T \) being differentiated with respect to a given \( \nabla \), it may still be possible to give meaning to \( \nabla T \).
4 Hypersurface distributions

Since we are interested in discontinuities and singularities at a change of signature, it is worth examining distributions with support on and between regions of different signature. For our purposes, it suffices to suppose $M$ is orientable and divided into two disjoint open regions $M^+$ and $M^-$ by an $(n-1)$-dimensional submanifold $\Sigma$, which in turn is defined by the equation $\lambda = 0$ for some function $\lambda$ on $M$ satisfying $d\lambda \neq 0$ in a neighbourhood of $\Sigma$. We fix an orientation by taking $\Sigma = \partial M^-$. 

The Heaviside scalar distributions $\Theta^\pm$ are defined by

$$\Theta^\pm[p] = \int_{M^\pm} p\omega$$

(16)

for any test function $p$, while the Dirac 1-form distribution $\delta$ is

$$\delta[V] = \int_\Sigma i_V \omega$$

(17)

where $V$ is any test vector.

Introducing a Leray form $\sigma$ in a neighbourhood of $\Sigma$, defined by

$$\omega = d\lambda \wedge \sigma$$

(18)

allows us to define the usual scalar Dirac distribution $\delta(\lambda)$ as

$$\delta(\lambda)[p] = \int_\Sigma p\sigma.$$  

(19)

The right-hand side is independent of the choice of $\sigma$ satisfying (18), but depends on the choice of the function $\lambda$ used to describe $\Sigma$. The scalar $\delta(\lambda)$ and ($\lambda$-independent) 1-form $\delta$ are related by

$$\delta[V] = \int_\Sigma ((i_V d\lambda)\sigma - d\lambda \wedge i_V \sigma)$$

$$= \delta(\lambda)[(d\lambda, V)]$$

(20)

since $d\lambda$ vanishes under pullback to $\Sigma$. Thus

$$\delta = \delta(\lambda)d\lambda.$$  

(21)

The usual scaling law $\delta(a\lambda) = \frac{1}{a}\delta(\lambda)$ is simply an expression of the fact that $\delta(\lambda)$ transforms as the component of a 1-form.

Applying the exterior derivative (8) to the Heaviside distribution $\Theta^+$ we have, for any test vector $V$,

$$d\Theta^+[V] = -\Theta^+[\text{div } V] = -\int_{M^+} \text{div } \omega = -\int_{\partial M^+} i_V \omega = \delta[V]$$

(22)

since $\partial M^+ = -\Sigma$. A similar calculation can be done for $\Theta^-$, so we finally have the satisfying result

$$d\Theta^\pm = \pm \delta.$$  

(23)

A function $f$ on $M$ is regularly $C^k$ discontinuous at $\Sigma$ if $f$ and its first $k$ derivatives are continuous on $M^\pm$ and converge uniformly to limits $f^\pm_\Sigma$ etc at $\Sigma$. We will consider only regularly $C^\infty$ discontinuous functions and suppress the degree of differentiability. A regularly discontinuous tensor $S$ is one whose components in any given chart intersecting $\Sigma$ are regularly discontinuous functions. The discontinuity $[S]$ of $S$ is an ordinary continuous tensor over $\Sigma \subset M$ defined by

$$[S] = S^+_\Sigma - S^-_\Sigma.$$  

(24)
Let \( \hat{f} \) be the distribution associated with a regularly discontinuous function \( f \). Then \( \hat{f} \) can be written
\[
\hat{f} = \Theta^+ f^+ + \Theta^- f^-
\] (25)
where \( f^\pm \) are arbitrary smooth extensions of \( f|_{M^\pm} \) to \( M \). In particular, \( f^\pm|_{\Sigma} = f^\pm_{\Sigma} \).

It follows that
\[
d\hat{f} = \Theta^+ df^+ + \Theta^- df^- + \|f\| \delta,
\] (26)
where the last term is well-defined because \( \|f\| \) is continuous on \( \text{supp} \delta = \Sigma \). Expression (25) extends in the obvious way to regularly discontinuous tensors, and we have, for smooth \( \nabla \) and regularly discontinuous \( S \)
\[
\nabla \hat{S} = \Theta^+ \nabla S^+ + \Theta^- \nabla S^- + \delta \otimes [\nabla S].
\] (27)

Secondly, we consider the possibility of defining a distributional connection which is not only discontinuous, but contains a Dirac \( \delta \) part. In this case, it will not be possible to make sense of condition (10) even when \( D \) is merely discontinuous. Accordingly, we restrict our attention to those distributions associated with smooth tensors. As described earlier, fixing the connection map on vector distributions consistently determines the covariant derivative of other tensor distributions.

For a vector \( X \) we postulate a connection map of the form
\[
\nabla \hat{X} = \nabla \bar{X} + \delta \otimes k(X),
\] (29)
where \( \nabla \) is some regularly discontinuous connection and \( k \) is a fixed \( T(M) \)-valued function, whose properties are to be determined. The Leibniz rule (11) applied to both \( \nabla \) and \( \nabla \) implies that, for any function \( f \),
\[
\delta \otimes k(fX) = f\delta \otimes k(X),
\] (30)
which is satisfied if \( k \) is a type (1,1) tensor. With this condition on \( k \), the connection map given by (29) defines a connection \( \nabla \) on tensor distributions associated with smooth tensors. Extension of definition (29) to \( \nabla \bar{Y} \) with \( Y \) regularly discontinuous is ruled out by the same argument: \( Y \) is a sum of terms \( fX \) where \( f \) is regularly discontinuous and \( X \) is smooth, so condition (30) still applies. However, the right-hand side is undefined for general \( f \) (eg \( f = \Theta^\pm \)) unless \( \delta \otimes k(X) = 0 \) for \( X \) smooth, which reduces definition (29) to \( \nabla \bar{Y} = \nabla \bar{Y} \).

5 Signature-change

We now consider \( M \) to be a signature-changing space-time, with \( \Sigma \) the surface of signature-change in the metric \( g \). Two types of signature-change are considered
here: *discontinuous*, in which \( g \) is regularly discontinuous at \( \Sigma \) with \( \det g^\pm \neq 0 \), and *continuous*, in which \( g \) is smooth, but \( \det g \) vanishes on \( \Sigma \). We further demand that the induced metrics on \( \Sigma \) from each side agree and are non-degenerate. With these assumptions, \( g \) can be written

\[
g = N \mathrm{d}\lambda \otimes \mathrm{d}\lambda + h, \tag{31}
\]

where \( N \) changes sign at \( \Sigma \), being regularly discontinuous with \( N^\Sigma \neq 0 \) in one case, and smooth with \( N|_{\Sigma} = 0 \) in the other.

The Levi-Civita connection for the restrictions of this metric to the open sets \( M^\pm \) can be computed as usual. In particular, in adapted coordinates such that \( \partial_\lambda \) is normal to the surfaces of constant \( \lambda \) near \( \Sigma \), we have

\[
d\lambda(\nabla_{\partial_\lambda} \partial_\lambda)|_{M^\pm} = \frac{\partial_\lambda N}{2N} \tag{32}
\]

For discontinuous signature-change, this allows us to construct a regularly discontinuous connection \( \nabla^\ast \) based on arbitrary smooth extensions \( \nabla^\pm \) to \( M \) of the Levi-Civita connections on \( M^\pm \). From section 4 we know that such a connection can be applied to those distributions associated with locally integrable tensors, including regularly discontinuous tensors. Hence, it is possible to discuss first order field equations for this class of fields, or second order field equations for smooth tensor fields. The metric tensor is included in the class of covariantly differentiable tensor distributions, so the metric compatibility of \( \nabla \) can be established from equation (28) as

\[
\nabla^\ast g = \Theta^+ \nabla^+ g^+ + \Theta^- \nabla^- g^- + \delta \otimes [g] \tag{33}
\]

Thus the (metric-compatible) Levi-Civita connections on \( M^\pm \) give rise to a non-metric compatible connection \( \nabla^\ast \) for distributions on \( M^\ast \). To rectify this, we might be tempted to add a Dirac \( \delta \) term to \( \nabla \), as discussed in section 4, but this would make \( \nabla^\ast g \) undefined, since \( g \) is not smooth.

As a simple concrete example of discontinuous signature-change, consider the metric tensor

\[
g = \text{sign}(t) \mathrm{d}t \otimes \mathrm{d}t + a(t)^2 \mathrm{d}x \otimes \mathrm{d}x \tag{34}
\]
on \( M = \mathbb{R}^2 \), where \( a > 0 \). In this particular case, the metric volume forms on \( M^\pm \) extend to give a smooth volume form \( \omega = \mathrm{d}t \wedge \mathrm{d}x \), so this is a natural choice. The non-zero Levi-Civita connection components \( \Gamma^a_{bc} = \mathrm{d}x^a(\nabla_{\partial_c} \partial_b) \) \((a, b, c = t, x)\) for \( g \) restricted to \( M^\pm \) are

\[
\Gamma^t_{xx}|_{M^\pm} = \mp a \dot{a} \quad \Gamma^x_{xt}|_{M^\pm} = \frac{\dot{a}}{a} \tag{35}
\]

and these expressions extend smoothly to the whole of \( M \), determining a regularly discontinuous connection \( \nabla \). As anticipated in equation (33), the connection is not metric compatible, since (using \( \Gamma^\pm_{tt} = 0 \))

\[
\nabla_{tt} g_{tt} = \partial_t g_{tt} = 2\delta(t). \tag{36}
\]

For continuous signature-change, we can take \( N = \lambda \) near \( \Sigma \) if we further assume \( dN \neq 0 \) on \( \Sigma \). Then the right-hand side of equation (22) simplifies to give

\[
d\lambda(\nabla_{\partial_\lambda} \partial_\lambda)|_{M^\pm} = \frac{1}{2\lambda} \tag{37}
\]

which is singular on \( \Sigma \). Hence we cannot construct a regularly discontinuous connection from the Levi-Civita connections on \( M^\pm \). Nonetheless, the connection can
be given a meaning directly, and for all distributions. This is most easily seen from the component expression (11) and the fact that $\frac{1}{\lambda} D$ can be defined for any distribution $D$ (see eq 3). However, this can be done in many ways, as the distribution $E$ satisfying $\lambda E = D$ is determined only up to addition of a term proportional to $\delta(\lambda)$. Up to such ambiguities, it is possible to discuss field equations of any order for any tensor distributions within continuous signature-change. The question of metric compatibility can again be posed, and here, the natural connection may be metric compatible.

A simple example of continuous signature-change, close to (34) above for the discontinuous case, is given by the metric tensor

$$g = t dt \otimes dt + a(t)^2 dx \otimes dx$$

(38)
on $M = \mathbb{R}^2$, where again $a > 0$. The metric volume forms on $M^\pm$ vanish on $\Sigma$, so an alternative must be introduced: $\omega = adt \wedge dx$ is an obvious choice in these coordinates. The non-zero Levi-Civita connection components for $g$ restricted to $M^\pm$ are

$$\Gamma^t_{tt} |_{M^\pm} = \frac{1}{2t} \quad \Gamma^t_{x|} |_{M^\pm} = -\frac{a\dot{a}}{t} \quad \Gamma^x_{xt} |_{M^\pm} = \frac{\dot{a}}{a},$$

(39)
giving rise to singularities at $t = 0$. As discussed above, the connection map determined by these expressions is defined for general vector distributions, fixed up to addition of terms proportional to $\delta(t)$. In $\nabla \hat{g}$, these terms are annulled by the smooth function $g_{tt} = t$, since $t \delta(t) = 0$. A simple calculation verifies that the connection is metric compatible.

6 Discussion

We began with a review of the standard theory of tensor distributions on manifolds, emphasising that the space-time metric need not be used in establishing the formalism. We chose definitions for ordinary and covariant differentiation which depend on an auxiliary volume element rather than on a full metric tensor, and which separate the role of this auxiliary volume element from that of the connection components. This avoids introducing unnecessary irregularities when the metric tensor and connection are not smooth.

We then considered distributions associated with a hypersurface, detailing how the Dirac $\delta$ distribution may conveniently be regarded as an exact 1-form, $\delta = d\Theta$, where $\Theta$ is a Heaviside distribution, and discussing the important class of distributions arising from regularly discontinuous tensors. Extensions of the covariant derivative to cases where the connection components are not smooth were described, and shown to be well-defined, at least on a restricted class of distributions.

Finally, we applied the formalism to signature-changing metrics, both continuous and discontinuous, and showed to what extent we can make rigorous sense of covariant differentiation, and thus of tensor field equations. A surprising result for discontinuous signature-change is that the Levi-Civita connections on the Lorentzian and Euclidean regions extend to include the hypersurface in such a way that the overall connection is no longer metric-compatible! Einstein’s equations in this setting have been discussed from a variational approach in [5].

The main obstruction to extending covariant differentiation with respect to a connection based on the metric to all distributions is the occurrence of undefined products of distributions. This deficiency might be remedied by resorting to the Colombeau algebra of distributions, as has been recently proposed for cosmic strings [4] and for other distributional sources [1].
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