Lie symmetry analysis and invariant solutions of 3D Euler equations for axisymmetric, incompressible, and inviscid flow in the cylindrical coordinates

R. Sadat, Praveen Agarwal, R. Saleh and Mohamed R. Ali

Abstract
Through the Lie symmetry analysis method, the axisymmetric, incompressible, and inviscid fluid is studied. The governing equations that describe the flow are the Euler equations. Under intensive observation, these equations do not have a certain solution localized in all directions \((r, t, z)\) due to the presence of the term \(1/r\), which leads to the singularity cases. The researchers avoid this problem by truncating this term or solving the equations in the Cartesian plane. However, the Euler equations have an infinite number of Lie infinitesimals; we utilize the commutative product between these Lie vectors. The specialization process procures a nonlinear system of ODEs. Manual calculations have been done to solve this system. The investigated Lie vectors have been used to generate new solutions for the Euler equations. Some solutions are selected and plotted as two-dimensional plots.

Keywords: Euler equations; Axisymmetric flow; Lie point symmetries; Analytical solutions

1 Introduction
Suppose that the Euler equations have the form [1–4]

\[
\begin{align*}
\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial r} + u \frac{\partial w}{\partial z} + \frac{v^2}{r} + \frac{\partial p}{\partial r} &= 0, \\
\frac{\partial v}{\partial t} + w \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} - \frac{vw}{r} &= 0, \\
\frac{\partial u}{\partial t} + w \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} + \frac{\partial p}{\partial z} &= 0, \\
\frac{\partial w}{\partial r} + w \frac{\partial u}{\partial r} + \frac{\partial u}{\partial z} &= 0.
\end{align*}
\]

That describes the dynamics of incompressible, axisymmetric flow with swirl [3], where \(w(r, t, z), u(r, t, z),\) and \(v(r, t, z)\) are the components of the velocity in the cylindrical coordinates \((r, \phi,\) and \(z)\), and \(p(r, t, z)\) is the pressure. The flow is called axisymmetric flow if
the velocity component and the pressure are independent of \( \phi \). Navier–Stokes and Euler equations in the cylindrical coordinates can describe any pipe fluid flow that has more applications, especially in the medical field. For example, blood flow in stenoses narrow artery [5–8]. System (1) had been solved using numerical methods in [1, 2, 9]. Manipulation of the results in most applications needs explicit solutions. The Lie symmetry analysis is one of the most important and powerful methods for obtaining closed-form solutions [10, 11]. The method proves its dependence in the fluid mechanics, turbulence field, and turbulent plane jet model [12–18]. Other researchers apply the method to other applications [19–25]. In (2007), Oberlack et al. [3] deduced five Lie point symmetries for Euler equations. Here, we use the commutative product to explore new Lie infinitesimals for system (1), then we use the investigated Lie vectors to reduce system (1) to the system of ODEs. By solving these ODEs, we explore new analytical solutions for Euler equations.

2 Investigation of Lie infinitesimals for Euler equations

System (1) possesses Lie infinitesimals as follows:

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t} + f_1(t) \frac{\partial}{\partial r} + f_3(t) \frac{\partial}{\partial \theta} + \left(-f''_1(t)z + f_2(t)\right) \frac{\partial}{\partial p}, \\
X_2 &= f_3(t) \frac{\partial}{\partial \theta} + f_2(t) \frac{\partial}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} + \left(\frac{1}{r^2} - f''_5(t)z + f_6(t)\right) \frac{\partial}{\partial p}, \\
X_3 &= r \frac{\partial}{\partial r} + f_3(t) \frac{\partial}{\partial \theta} + \left(f_2(t) - u\right) \frac{\partial}{\partial t} - w \frac{\partial}{\partial \theta} - v \frac{\partial}{\partial r} + \left(-2p - f''_7(t)z + f_8(t)\right) \frac{\partial}{\partial p}, \\
X_4 &= r \frac{\partial}{\partial r} + \left(z + f_7(t)\right) \frac{\partial}{\partial \theta} + \left(u + f_8(t)\right) \frac{\partial}{\partial t} + w \frac{\partial}{\partial \theta} + v \frac{\partial}{\partial r} + \left(2p - f''_7(t)z + f_8(t)\right) \frac{\partial}{\partial p}. 
\end{align*}
\]

There are an infinite number of possibilities for these vectors as the presence of arbitrary functions \( f_i(t) \), \( i = 1 \ldots 8 \). Using the commutative product between these infinitesimals listed in Table 1 authorizes us to specialize these vectors through the same procedure as in [10, 26]. Firstly, we generate the commutator table as follows in Table 1, where

\[
\begin{align*}
a_1 &= -zf''_3 + f'_3 - f_1f''_3 + df_1''', \\
a_2 &= f'_3 - tf_1, \\
a_3 &= f'_5 - f'_3 - tf'_1, \\
a_4 &= -zf''_3 + f'_3 - f_1f''_5 + df_1'' + 2zf_1''' - 2f_2 + tfz'''' - tf''', \\
a_5 &= f''_7 + f_1, \\
a_6 &= f''_5 + f_1, \\
a_7 &= -zf''_3 + f'_3 - f_1f''_3 - zf''_1 + 2f_2 + f_1f''''', \\
a_8 &= tf''_3 - f'_3, \\
a_9 &= -f_3f''_3 + 2zf_3''' - 2f_4 + \frac{2}{r^2} + tfz''''' - tf''', \\
a_{10} &= -\frac{4}{r^2} + f''_3 - f_1f''_3 - zf_3''' + 2f_4, \\
a_{11} &= tf''_5 + f_5, \\
a_{12} &= tf''_5 + f_5, \\
a_{13} &= -tzf'''' + tf''_8 + f_3f'''' - f_5f'''' - zf_5''' + 2f_6 + 2f_6 - 2zf_5'''.
\end{align*}
\]
We use these vectors (6) to reproduce the commutator table (Table 2). Through manual calculations this system has been solved, and the results are

\[ f_1 = \frac{1}{t}, \quad f_2 = \frac{1}{t^2}, \quad f_3 = f_4 = 0, \]
\[ f_5 = 1, \quad f_6 = \frac{1}{t^2}, \quad f_7 = -\ln(t), \quad f_8 = \frac{-\ln(t)}{t^2}. \] (5)

Substituting from (5) into (2), we obtain

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t} + \frac{\partial}{\partial X} - \frac{1}{t^2} \frac{\partial}{\partial t} + \frac{1}{t^2} \frac{\partial}{\partial t^2}, \\
X_2 &= \frac{\partial}{\partial t^2} + \frac{1}{t} \frac{\partial}{\partial t} + \frac{1}{t^2} \frac{\partial}{\partial t^2}, \\
X_3 &= t \frac{\partial}{\partial t} + \frac{\partial}{\partial X} - u \frac{\partial}{\partial u} - w \frac{\partial}{\partial w} - v \frac{\partial}{\partial v} + (-2p + \frac{1}{t^2}) \frac{\partial}{\partial p}, \\
X_4 &= r \frac{\partial}{\partial t} + (z - \ln(t)) \frac{\partial}{\partial X} + (u - \frac{1}{t}) \frac{\partial}{\partial u} + w \frac{\partial}{\partial w} + v \frac{\partial}{\partial v} + 2p \frac{\partial}{\partial p} - \frac{\ln(t)}{t^2} \frac{\partial}{\partial p}.
\end{align*}
\] (6)

We use these vectors (6) to reproduce the commutator table (Table 2).
3 Reduction of the independent variables in Euler equations

3.1 Using Lie vector $X_1$

To snaffle the similarity variables, we solve the associated Lagrange system

$$\frac{dt}{T} = \frac{dz}{z} = -\frac{du}{t} = \frac{dp}{\left(-\frac{1}{3}z + \frac{1}{t}\right)}.$$  \hspace{1cm} (7)

The similarity variables of system (1) are

$$u(r, t, z) = R(y, x) + \frac{1}{t}, \quad w(r, t, z) = F(y, x), \quad v(r, t, z) = G(y, x),$$

$$p(r, t, z) = H(y, x) + \frac{z}{t^2},$$ \hspace{1cm} (8)

where, $y = r, x = z - \ln(t)$.

Substituting from (8) into (1), we get the following system with two independent variables:

$$y\frac{\partial F}{\partial y} + y\frac{\partial R}{\partial x} + F = 0,$$

$$F\frac{\partial G}{\partial y} + R\frac{\partial G}{\partial x} + FG = 0,$$

$$-F\frac{\partial F}{\partial y} - R\frac{\partial F}{\partial x} + G^2 - y\frac{\partial H}{\partial y} = 0,$$

$$F\frac{\partial R}{\partial y} + R\frac{\partial R}{\partial x} + \frac{\partial H}{\partial x} = 0.$$ \hspace{1cm} (9)

System (9) has five Lie vectors as follows:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial H}, \quad V_3 = y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x},$$

$$V_4 = \frac{1}{y^2G}\frac{\partial}{\partial G} - \frac{1}{y^2}\frac{\partial}{\partial x}, \quad V_5 = F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G} + 2H\frac{\partial}{\partial H} + R\frac{\partial}{\partial R}.$$ \hspace{1cm} (10)

3.1.1 Using vector $V_3$

This Lie vector will reduce system (9) to

$$-\eta T d\theta d\eta + \theta d\theta d\eta + \frac{d\beta}{d\eta} = 0,$$

$$-\eta T dE d\eta + \theta dE d\eta + ET = 0,$$

$$\eta dT d\eta - T - \frac{d\theta}{d\eta} = 0,$$

$$\eta dT d\eta - \theta dT d\eta + E^2 + \eta \frac{d\beta}{d\eta} = 0,$$ \hspace{1cm} (11)
where the new dependent variables have been obtained from solving the characteristic equation that the \( V_3 \) was generated.

\[
E(\eta) = G(y, x), \quad T(\eta) = F(y, x), \quad \beta(\eta) = H(y, x),
\]

\[
\theta(\eta) = R(y, x), \quad \eta = \frac{x}{y}.
\]

The solutions for system (11) are as follows:

\[
T(\eta) = c_3 \eta + c_4 \sqrt{1 + \eta^2},
\]

\[
\theta(\eta) = -c_4 \sinh^{-1}(\eta),
\]

\[
E(\eta) = \mp \sqrt{-c_3(c_4 \eta^3 + c_3 \eta^2 \sqrt{1 + \eta^2} + c_4 \eta + c_4 \sinh^{-1}(\eta) \sqrt{1 + \eta^2} - c_2 \sqrt{1 + \eta^2})},
\]

\[
\beta(\eta) = -\frac{1}{2} \left(c_4 \sinh^{-1}(\eta)\right)^2
\]

\[-c_4 \left(c_3 \left(\frac{1}{2} \eta \sqrt{1 + \eta^2} - \frac{1}{2} \sinh^{-1}(\eta)\right) - c_2 \sinh^{-1}(\eta) + \frac{1}{2} c_4 \eta\right) + c_1.
\]

Back substitution to the original variables using similarity variables in (8) and (12) leads to

\[
w(r, t, z) = c_3 \frac{(z - \ln(t))}{r} + c_4 \sqrt{1 + \left(\frac{z - \ln(t)}{r}\right)^2},
\]

\[
u(r, t, z) = -c_4 \sinh^{-1}\left(\frac{(z - \ln(t))}{r}\right),
\]

\[
p(r, t, z) = -\frac{1}{2} \left(c_4 \sinh^{-1}(\delta)\right)^2
\]

\[-c_4 \left(c_3 \left(\frac{1}{2} \delta \sqrt{1 + \delta^2} - \frac{1}{2} \sinh^{-1}(\delta)\right) - c_2 \sinh^{-1}(\delta) + \frac{1}{2} c_4 (\delta)\right) + c_1,
\]

where \( \delta = \frac{(z - \ln(t))}{r} \).

The solutions have been plotted for different values of time as depicted in Figs. 1–4.
Figure 1  Velocity component $w(r,t,z)$ at $z = 2$, $c_3 = 1$, and $c_4 = 1$

Figure 2  Positive case of velocity component $v(r,t,z)$ at $z = 2$ and $c_4 = -1$
Figure 3  Velocity component $u(r, t, z)$ at $z = 5$, $c_2 = 1$, and $c_4 = -1$

Figure 4  The pressure $p(r, t, z)$ at $z = 5$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, and $c_4 = 1$
3.1.2 Using $V = V_1 + V_4$

This vector produces a system of nonlinear ODEs as follows:

\[
\begin{align*}
\eta \frac{dT}{d\eta} + T &= 0, \\
\eta^2 T \frac{d\theta}{d\eta} - 1 &= 0, \\
- \eta^2 \frac{dT}{d\eta} + E - \eta \frac{d\beta}{d\eta} &= 0, \\
\eta^2 T \frac{dE}{d\eta} + 2\eta TE + 2\theta &= 0,
\end{align*}
\]

where the new dependent variables are

\[
\begin{align*}
E(\eta) &= \frac{-2x + y^2 G(y,x)^2}{y^2}, \\
T(\eta) &= F(y,x), \\
\beta(\eta) &= H(y,x) + \frac{x}{y^2}, \\
\theta(\eta) &= R(y,x) \quad \text{where } \eta = y.
\end{align*}
\]

By solving system (15), new solutions for Euler equations have been produced:

\[
\begin{align*}
T(\eta) &= \frac{c_4}{\eta}, \\
\theta(\eta) &= \frac{\ln(\eta)}{c_4} + c_3, \\
E(\eta) &= \frac{-\eta^2 \ln(\eta) + 0.5\eta^2 - c_1 c_4 \eta^2 + c_2 c_4^2}{(c_4 \eta)^2}, \\
\beta(\eta) &= -0.5 \left( \frac{c_4^2}{\eta^2} + \frac{(\ln(\eta))^2}{c_4^2} - \frac{\ln(\eta)}{c_4} + 2 \frac{c_1 \ln(\eta)}{c_4} + \frac{c_2}{\eta^2} - 2c_1 \right).
\end{align*}
\]

Using the similarity variables in (8) and (16) leads to back substitution to the original variables:

\[
\begin{align*}
w(r,t,z) &= \frac{c_4}{r}, \\
u(r,t,z) &= \frac{\ln(r)}{c_4} + c_3 + t^{-1}, \\
v(r,t,z) &= \sqrt{-r^2 \ln(r) + 0.5r^2 - c_1 c_4 r^2 + c_2 c_4^2 + 2 \left( \frac{(z - \ln(t))}{r^2} \right)}, \\
p(r,t,z) &= -0.5 \left( \frac{c_4^2}{r^2} + \frac{(\ln(r))^2}{c_4^2} - \frac{\ln(r)}{c_4} + 2 \frac{c_1 \ln(r)}{c_4} + \frac{c_2}{r^2} - 2c_1 \right) - \left( \frac{(z - \ln(t))}{r^2} \right) + zt^{-2}.
\end{align*}
\]
3.1.3 Using Lie vector \( \mathbf{V} = \mathbf{V}_1 + \mathbf{V}_5 \)

Through the same previous procedure system (9) has been reduced to

\[
\begin{align*}
T \frac{d\theta}{d\eta} + \theta^2 + 2\beta &= 0, \\
\eta \frac{dT}{d\eta} + T + \eta \theta &= 0, \\
\eta T \frac{dE}{d\eta} + \eta \theta E + ET &= 0, \\
-\eta T \frac{dT}{d\eta} + E^2 - \eta T \theta - \eta \frac{d\beta}{d\eta} &= 0,
\end{align*}
\]

\tag{19}

where the similarity variables are

\[
\begin{align*}
E(\eta) &= G(y,x), \quad e^{-x}, \\
T(\eta) &= F(y,x), \quad e^{-x}, \\
\beta(\eta) &= H(y,x), \quad e^{-2x}, \\
\theta(\eta) &= R(y,x), \quad e^{-x}, \quad \eta = y.
\end{align*}
\]

System (19) has closed form solutions as follows:

\[
\begin{align*}
T(\eta) &= -c_3 e^{0.5\eta^2 - \frac{4}{c_1}} + c_3 e^{0.5\eta^2 - \frac{4}{c_1}} + c_4 e^{0.5\eta^2 - \frac{4}{c_1}}, \\
\theta(\eta) &= -I(c_3 e^{-\frac{0.5\eta^2}{c_1}} + c_3 e^{-\frac{0.5\eta^2}{c_1}} - c_4 e^{-\frac{0.5\eta^2}{c_1}}), \\
E(\eta) &= \pm \sqrt{2c_3^2 + 2c_3 c_4 e^{-\frac{1}{c_1}} - 2c_3 c_4 - c_3^2 e^{-\frac{1}{c_1}} - c_4^2 e^{-\frac{1}{c_1}} - c_2^2 e^{-\frac{1}{c_1}}}, \\
\beta(\eta) &= 2c_3(c_3 - c_4) e^{-\frac{1}{c_1}}.
\end{align*}
\]

\tag{20}

Back substitution using the similarity variables in (20) and (8) is as follows:

\[
\begin{align*}
w(r,t,z) &= -c_3 e^{0.5\rho^2 - \frac{4}{c_1}} + c_3 e^{0.5\rho^2 - \frac{4}{c_1}} + c_4 e^{0.5\rho^2 - \frac{4}{c_1}} e^{(z-\ln(t))}, \\
u(r,t,z) &= -I(c_3 e^{0.5\rho^2 - \frac{4}{c_1}} + c_3 e^{0.5\rho^2 - \frac{4}{c_1}} - c_4 e^{0.5\rho^2 - \frac{4}{c_1}}) e^{(z-\ln(t))} + t^{-1}, \\
v(r,t,z) &= \pm \sqrt{2c_3^2 + 2c_3 c_4 e^{-\frac{1}{c_1}} - 2c_3 c_4 - c_3^2 e^{-\frac{1}{c_1}} - c_4^2 e^{-\frac{1}{c_1}} - c_2^2 e^{-\frac{1}{c_1}}} e^{(z-\ln(t))}, \\
p(r,t,z) &= 2c_3(c_3 - c_4) e^{(z-\ln(t))} + z t^{-2}.
\end{align*}
\]

\tag{22}

The solutions have been plotted in Figs. 5–8.
Figure 5  Velocity component $w(r,t,z)$ at $z = 2$, $c_1 = 1$, $c_3 = 1$, and $c_4 = 2$

Figure 6  Positive case velocity component $v(r,t,z)$ at $z = 2$, $c_1 = 1$, $c_3 = l$, and $c_4 = 2l$
Figure 7  Velocity component $u(r, t, z)$ at $z = 2$, $c_1 = 1$, $c_3 = 1$, and $c_4 = 2$

Figure 8  The pressure $p(r, t, z)$ at $c_1 = 1$, $c_3 = 1$, and $c_4 = 2$
3.2 Using Lie vector $X = X_3 + X_4$

By solving the subsidiary equation, we explore the similarity variables

$$u(r, t, z) = R(y, x) + \frac{1}{t}, \quad w(r, t, z) = F(y, x), \quad v(r, t, z) = G(y, x),$$

$$p(r, t, z) = H(y, x) + \frac{z}{t^2},$$

where $y = \frac{r}{t}, x = \frac{z - \ln(t)}{r},$ (23)

which reduce system (1) to

$$-\frac{\partial G}{\partial y} + x F \frac{\partial G}{\partial x} + y R \frac{\partial G}{\partial x} - R \frac{\partial G}{\partial x} - FG = 0,$$

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} - F + \frac{\partial R}{\partial x} = 0,$$

$$-\frac{\partial R}{\partial y} + x F \frac{\partial R}{\partial x} + y R \frac{\partial R}{\partial x} - R \frac{\partial R}{\partial x} - \frac{\partial H}{\partial x} = 0,$$

$$-\frac{\partial F}{\partial y} + x F \frac{\partial F}{\partial x} + y F \frac{\partial F}{\partial y} - R \frac{\partial F}{\partial x} + G^2 + \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} y = 0.$$ (24)

This system possesses three Lie vectors as follows:

$$V_1 = \frac{\partial}{\partial H}, \quad V_2 = y \frac{\partial}{\partial x} + \frac{\partial}{\partial R}, \quad V_3 = y \frac{\partial}{\partial y} - F \frac{\partial}{\partial y} - G \frac{\partial}{\partial G} - 2H \frac{\partial}{\partial H} - R \frac{\partial}{\partial R}. \quad (25)$$

- Using $V = V_1 + V_2$

Following the same procedure system (24) will be reduced to

$$-\frac{dE}{d\eta} + \eta T \frac{dE}{d\eta} - ET = 0,$$

$$\frac{dT}{d\eta} + \eta T \frac{dT}{d\eta} + E^2 + \eta \frac{d\beta}{d\eta} = 0,$$

$$-\eta \frac{d\theta}{d\eta} - \theta + \eta \frac{d\theta}{d\eta} - 1 = 0,$$

$$\eta^2 \frac{d\theta}{d\eta} - \eta \theta - 1 = 0.$$ (26)

with new variables

$$E(\eta) = G(y, x), \quad T(\eta) = F(y, x), \quad \beta(\eta) = -H(y, x) + \frac{x}{y},$$

$$\theta(\eta) = R(y, x) - \frac{x}{y}, \quad \eta = y.$$ (27)
By solving system (26), we have

\[ T(\eta) = \frac{-1}{2\eta}, \]
\[ \theta(\eta) = -1 + \frac{c_3}{\eta^{2/3}}, \]
\[ E(\eta) = c_2\eta^{2/3}, \]
\[ \beta(\eta) = \frac{-3c_2^2}{2} \eta^{3/3} - \frac{3}{8\eta^2} + c_1. \]  

(28)

Using the similarity variables in (23) and (27) authorizes us to back substitution to the original variables

\[ w(r, t, z) = -\frac{r}{2t}, \]
\[ u(r, t, z) = -1 + \frac{c_3 (\frac{t}{r})^{4}}{4} - \frac{z - \ln(t)}{t} + t^{-1}, \]
\[ v(r, t, z) = c_2 \left(\frac{t}{r}\right)^{2/3}, \]
\[ p(r, t, z) = \frac{-3c_2^2}{2} \left(\frac{t}{r}\right)^{2/3} - \frac{3}{8\left(\frac{t}{r}\right)^2} \frac{z - \ln(t)}{t} + c_1 + zt^{-2}. \]

(29)

The results have been plotted as shown in Figs. 9–12.
4 Conclusions

We deduce an infinite number of Lie infinitesimals, and through commutative product properties, we minimize these vectors to four Lie vectors. Through some combinations between these vectors, we explore exact solutions for Euler equations. The results illustrate that the velocity components decrease with increasing the spatial or temporal coordinates. The pressure may be appearing as a negative value, and this is reasonable according to the human pressure in the case of the tapered artery [6].
Acknowledgements
The authors thank the reviewers.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
1Department of Mathematics, Zagazig Faculty of Engineering, Zagazig University, Zagazig, Egypt. 2Department of Mathematics, Anand International College of Engineering, Jaipur, 302012, India. 3Department of Basic Science, Faculty of Engineering at Benha, Benha University, Benha, 13512, Egypt.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 August 2021 Accepted: 19 October 2021 Published online: 06 November 2021

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