Scalar products of the elliptic Felderhof model and elliptic Cauchy formula

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Abstract

We analyze the scalar products of the elliptic Felderhof model introduced by Foda-Wheeler-Zuparic as an elliptic extension of the trigonometric face-type Felderhof model by Deguchi-Akutsu. We derive the determinant formula for the scalar products by applying the Izergin-Korepin technique developed by Wheeler to investigate the scalar products of integrable lattice models. By combining the determinant formula for the scalar products with the recently-developed Izergin-Korepin technique to analyze the wavefunctions, we derive a Cauchy formula for elliptic Schur functions.

Elliptic integrable models are classes of integrable models described by elliptic functions. Investigations of elliptic integrable models lead to new discoveries of mathematical structures. An instance is the notion of elliptic quantum groups [1, 2, 3, 4] which are extensions of the quantum groups [5, 6, 7], introduced through the analysis of the eight-vertex model, eight-vertex solid-on-solid model and their generalizations [8, 9, 10, 11, 12]. Recently, there are also progresses on partition functions of the eight-vertex solid-on-solid models [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26] from the viewpoint of the quantum inverse scattering method [27, 28], vertex operator method [29] and so on.

In this paper, we investigate another class of elliptic integrable model. We analyze the scalar products of the elliptic Felderhof model introduced by Foda-Wheeler-Zuparic [30] as an extension of the face-type Felderhof model [31] by Deguchi-Akutsu [32]. The elliptic Felderhof model (Foda-Wheeler-Zuparic model), and its closely related elliptic Perk-Schultz model (Okado-Deguchi-Fujii-Martin model) constructed by Okado [33], Deguchi-Fujii [34] and Deguchi-Martin [35] as an elliptic extension of the Perk-Schultz model [36], are interesting models to be investigated, since the corresponding trigonometric models were discovered by number theorists recently to be related with automorphic representation theory and deformations of Weyl character formulas (Tokuyama formulas) for symmetric functions. Bump-Brubaker-Friedberg [37] constructed free-fermion models by themselves and showed that the wavefunctions are given as a product of a deformed Vandermonde determinant and

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Schur functions. One of the consequences of their results is the natural construction of the
Tokuyama formula [38] as wavefunctions of integrable models, which is a one-parameter de-
formation of the Weyl character formula for the Schur functions. Their result is the one of the
main motivations to study the elliptic Felderhof model of Foda-Wheeler-Zuparic and the el-
lipic Perk-Schultz model of Okado, Deguchi-Fujii and Deguchi-Martin, since these models can
be regarded as elliptic analogues of the free-fermion model which Bump-Brubaker-Friedberg
introduced and analyzed (the quantum group structure of the trigonometric models can be
found in [32, 39, 40] for example). There are not so much studies on the partition functions
of these elliptic models. Foda-Wheeler-Zuparic showed the factorization of the domain wall
boundary partition functions of these models [30] by applying the Izergin-Korepin technique
[41, 42], which is a classical method to analyze the domain wall boundary partition functions
of integrable models. Recently, we extended the Izergin-Korepin technique to be able to
analyze the wavefunctions [43, 44, 45], and showed that the wavefunctions of these elliptic
models are given as a deformed elliptic Vandermonde determinant and elliptic symmetric
functions which can be viewed as elliptic Schur functions (see Schlosser [46] or Noumi [47, 48]
for other types of elliptic Schur functions introduced from the viewpoint of combinatorics,
special functions and classical integrable systems). The results can be viewed as elliptic
analogues of the one by Bump-Brubaker-Friedberg.

In this paper, we investigate another special class of partition functions called the scalar
products. One of the motivations to study this class of partition functions comes from the
recent active line of researches on the application of the correspondence between symmetric
functions and wavefunctions of integrable models to derivations of various algebraic identities.
For the free-fermionic models, see [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62,
63, 64] for examples on integrability approach to symmetric functions, as well as closely
related non-intersecting lattice paths approach. There are also investigations on the six-
vertex models and face models related to the XXX, XXZ and XYZ quantum integrable spin
chains and q-boson models, where the Schur, Grothendieck, Hall-Littlewood polynomials and
their generalizations appear as the wavefunctions. See [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75,
76, 77, 78] for examples on various studies of these models. Among these active studies, it was
realized that the analysis on the scalar products lead us to Cauchy formulas for symmetric
functions. Directly evaluating the scalar products to get determinant formulas in one way,
and comparing the expressions with another way of evaluation by inserting completeness
relation and express it as the sum of products of the wavefunctions whose explicit forms
are given by symmetric functions, one can get Cauchy formulas for symmetric functions.
This quantum integrability approach often enables us to derive algebraic identities which
are almost impossible to find by any other means. In this paper, we apply this idea to the
elliptic Felderhof model, and derive the Cauchy formula for the elliptic Schur functions. The
main part of this paper is the direct evaluation of the scalar products. We apply the Izergin-
Korepin technique developed by Wheeler [79] to derive the scalar products of integrable
models. In his paper, Wheeler showed that his technique can be applied to the $U_q(sl_2)$ six-
vertex model to derive the Slavnov’s determinant formula [80] for the XXZ spin chain for
example, by introducing and listing the properties which uniquely defines the intermediate
scalar products, and showing the explicit determinant forms satisfying all the properties. We
apply his technique to the elliptic Felderhof model and obtain the determinant formula for the
scalar products. Together with our results on the correspondence between the wavefunctions
and the elliptic Schur functions [44, 45] obtained by the Izergin-Korepin analysis on the
wavefunctions, we derive the Cauchy formula for the elliptic Schur functions.

This paper is organized as follows. In Section 2, we recall the properties of theta functions and the Foda-Wheeler-Zuparic (elliptic Felderhof) model. In Section 3, we introduce the scalar products, and derive the determinant formula by applying the Izergin-Korepin technique developed by Wheeler. In Section 4, by combining with another evaluation of the scalar products using the correspondence between the wavefunctions and the elliptic Schur functions, we derive the Cauchy formula for the elliptic Schur functions. Section 5 is devoted to the conclusion of this paper.

1 Foda-Wheeler-Zuparic (elliptic Felderhof) model

In this section, we first introduce elliptic functions and list their properties, and introduce the Foda-Wheeler-Zuparic model which is an elliptic analogue of the Felderhof model. The theta functions

\[ H(u) = 2 \sinh u \prod_{n=1}^{\infty} (1 - 2 q^{2n} \cosh(2u) + q^{4n})(1 - q^{2n}), \]  

where \( q \) is the elliptic nome (\( 0 < q < 1 \)). For the description of the matrix elements of the dynamical R-matrix of the elliptic Felderhof model, we introduce the following notation

\[ [u] = H(\pi i u). \]  

The theta function \([u]\) is an odd function \([−u] = −[u]\) and satisfies the quasi-periodicities

\[ [u + 1] = −[u], \]  

\[ [u - i \log(q)/\pi] = −q^{-1} \exp(-2\pi i u)[u]. \]

We use the following property about the elliptic polynomials \([13, 81]\) presented below.

A character is a group homomorphism \( \chi \) from multiplicative groups \( \Gamma = \mathbb{Z} + \tau \mathbb{Z} \) to \( \mathbb{C}^\times \). An \( N \)-dimensional space \( \Theta_N(\chi) \) is defined for each character \( \chi \) and positive integer \( N \), which consists of holomorphic functions \( \phi(y) \) on \( \mathbb{C} \) satisfying the quasi-periodicities

\[ \phi(y + 1) = \chi(1) \phi(y), \]  

\[ \phi(y + \tau) = \chi(\tau) e^{-2\pi i N y - \pi i N \tau} \phi(y). \]

The elements of the space \( \Theta_N(\chi) \) are called elliptic polynomials. The space \( \Theta_N(\chi) \) is \( N \)-dimensional \([13, 81]\) and the following fact holds for the elliptic polynomials.

**Proposition 1.1.** \([13, 81]\) Suppose there are two elliptic polynomials \( P(y) \) and \( Q(y) \) in \( \Theta_N(\chi) \), where \( \chi(1) = (-1)^N \), \( \chi(\tau) = (-1)^N e^{\alpha} \). If those two polynomials are equal \( P(y_j) = Q(y_j) \) at \( N \) points \( y_j, j = 1, \ldots, N \) satisfying \( y_j - y_k \notin \Gamma, \sum_{k=1}^{N} y_k - \alpha \notin \Gamma \), then the two polynomials are exactly the same \( P(y) = Q(y) \).

This property ensure the uniqueness of the Izergin-Korepin analysis on the wavefunctions of elliptic integrable models. For example, it is used in \([13, 14, 15]\) on the analysis on the domain wall boundary partition functions of the eight-vertex solid-on-solid model \([10]\).
that the property above is an elliptic analogue of the following fact for ordinary polynomials: if \( P(y) \) and \( Q(y) \) are polynomials of degree \( N - 1 \) in \( y \), and if these polynomials match at \( N \) distinct points, then the two polynomials are exactly the same.

The trigonometric face-type Felderhof model was first introduced by Deguchi-Akutsu \[32\], and its elliptic extension was constructed by Foda-Wheeler-Zuparic \[30\]. The dynamical \( R \)-matrix of the elliptic Felderhof model is given by \[30\]

\[
R_{ab}(u, v|p, q|h) = \begin{pmatrix}
[u - v + p + q] & 0 & 0 & 0 \\
0 & \frac{|h|^{1/2}[h + 2p + 2q]^{1/2}[u - v + q + p + h]}{[h + 2p]^{1/2}[h + 2q]^{1/2}} & 0 & \frac{|2p|^{1/2}[2q]^{1/2}[-u + v + q + p + h]}{[h + 2p]^{1/2}[h + 2q]^{1/2}} \\
0 & 0 & \frac{|h|^{1/2}[h + 2p + 2q]^{1/2}[u - v + q + p]}{[h + 2p]^{1/2}[h + 2q]^{1/2}} & 0 \\
0 & \frac{|2p|^{1/2}[2q]^{1/2}[u - v + q + p + h]}{[h + 2p]^{1/2}[h + 2q]^{1/2}} & 0 & \frac{|-u + v + p + q|}{[h + 2p]^{1/2}[h + 2q]^{1/2}} \\
\end{pmatrix} \tag{1.7}
\]

acting on the tensor product \( W_a \otimes W_b \) of the complex two-dimensional space \( W_a \). The parameters \( u \) and \( v \) are spectral parameters, and \( p \) and \( q \) are complex parameters. \( h \) is called as the height or dynamical variable. One can think that the space \( W_a \) carries the parameters \( u \) and \( p \), while the parameters \( v \) and \( q \) are associated with the space \( W_b \). See Figure (1) for the graphical representations of the dynamical \( R \)-matrix \[1.7\].

![Diagram of the matrix elements of the elliptic Felderhof model](image)

**Figure 1:** The matrix elements of the elliptic Felderhof model \[1.7\]. The states \(|0\rangle, \langle 0|\) are represented as \(\oplus\), while the states \(|1\rangle, \langle 1|\) are represented as \(\ominus\). This kind of graphical representation for the case of trigonometric vertex models can be found in Bump-Brubaker-Friedberg \[37\] and Bump-McNamara-Nakasuji \[55\] for example.

We denote the orthonormal basis of \( W_a \) and its dual as \( \{|0\rangle_a, |1\rangle_a\} \) and \( \{\langle 0|_a, \langle 1|_a\} \), and the matrix elements of the dynamical \( R \)-matrix as \( R_{ab}(u, v|p, q|h)_{\langle \alpha|_a|\beta\rangle_b} \).
\[ R(u, v|p, q|h)^{\alpha\beta}_{\gamma\delta}. \] The matrix elements of the dynamical \( R \)-matrix are explicitly given as

\[
\begin{align*}
\alpha(0) |b, 0 R_{ab}(u, v|p, q|h) 0, a| b, 0 &= [u - v + p + q], \\
\alpha(1) |b, 1 R_{ab}(u, v|p, q|h) 0, a| b, 1 &= \frac{[h]^{1/2}[h + 2p + 2q]^{1/2}[u - v + q - p]}{[h + 2p]^{1/2}[h + 2q]^{1/2}}, \\
\alpha(0) |b, 1 R_{ab}(u, v|p, q|h) 1, a| b, 0 &= \frac{[2p]^{1/2}[2q]^{1/2}[-u + v + q + p + h]}{[h + 2p]^{1/2}[h + 2q]^{1/2}}, \\
\alpha(1) |b, 0 R_{ab}(u, v|p, q|h) 1, a| b, 1 &= \frac{[2p]^{1/2}[2q]^{1/2}[u - v + q + p]}{[h + 2p]^{1/2}[h + 2q]^{1/2}}.
\end{align*}
\]

(1.8) (1.9) (1.10) (1.11) (1.12) (1.13)

In statistical physics, \( |0 \rangle \) or its dual \( \langle 0 | \) can be regarded as a hole state, while \( |1 \rangle \) or its dual \( \langle 1 | \) can be interpreted as a particle state. We thus sometimes use the terms hole states and particle states to describe states constructed from \( |0 \rangle, \langle 0 |, |1 \rangle \) and \( \langle 1 | \) since they are convenient for the description of the states.

For later convenience, we also define the following Pauli spin operators \( \sigma^+ \) and \( \sigma^- \) as operators acting on the (dual) orthonormal basis as

\[
\begin{align*}
\sigma^+ |1 \rangle &= |0 \rangle, & \langle 0 | \sigma^+ &= \langle 1 |, & \langle 1 | \sigma^+ &= 0, \\
\sigma^- |0 \rangle &= |1 \rangle, & \langle 1 | \sigma^- &= 0, & \langle 0 | \sigma^- &= 0.
\end{align*}
\]

(1.14) (1.15)

The dynamical \( R \)-matrix (1.7) satisfies the dynamical Yang-Baxter (face-type Yang-Baxter, star-triangle) relation (Figure 2):

\[
R_{ab}(u, v|p, q|h) R_{ac}(u, w|p, r|h + 2q) R_{bc}(v, w|q, r|h) = R_{bc}(v, w|q, r|h + 2p) R_{ac}(u, w|p, r|h) R_{ab}(u, v|p, q|h + 2r),
\]

(1.16)

acting on \( W_a \otimes W_b \otimes W_c \).

To construct partition functions of integrable lattice models, we identify one of the complex two-dimensional spaces \( W_b \) of the tensor product space \( W_a \otimes W_c \) with the quantum space. Let us denote the quantum space by \( \mathcal{F}_j \), and define the \( L \)-operator \( L_{aj}(u, v|p, q|h) \) acting on \( W_a \otimes \mathcal{F}_j \) as

\[
L_{aj}(u, v|p, q|h) = R_{aj}(u, v|p, q|h).
\]

(1.17)

The next step is to define the monodromy matrix from the \( L \)-operators. For convenience, one denotes the sum of complex numbers \( q_1, q_2, \ldots, q_j \) as \( \overline{q_j} \)

\[
\overline{q_j} := \sum_{k=1}^{j} q_k.
\]

(1.18)

The monodromy matrix \( T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h) \) is the product of \( L \)-operators

\[
T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h) = L_{a1}(u, v_1|p, q_1|h) L_{a2}(u, v_2|p, q_2|h + 2\overline{q_1}) \cdots L_{aM}(u, v_M|p, q_M|h + 2\overline{q_{M-1}}),
\]

(1.19)
Figure 2: The dynamical Yang-Baxter relation (1.16). The left and right figure represents $R_{ab}(u, v|p, q|h)R_{ac}(u, w|p, r|h + 2q)R_{bc}(v, w|q, r|h)$ and $R_{bc}(v, w|q, r|h)R_{ab}(u, v|p, q|h + 2r)$, respectively.

acting on $W_a \otimes F_1 \otimes \cdots \otimes F_M$.

The $B$-operator and the $C$-operator are matrix elements of the monodromy matrix (1.19) with respect to the auxiliary space $W_a$

$$B(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h) = a\langle 0|T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|1\rangle_a, \quad (1.20)$$

$$C(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h) = a\langle 1|T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|0\rangle_a, \quad (1.21)$$

which acts on $F_1 \otimes \cdots \otimes F_M$ and its dual $F_1^* \otimes \cdots \otimes F_M^*$ (Figure 3).

The following domain wall boundary partition functions (Figure 4) is one of the most well-investigated classes of partition functions [41]

$$Z_N(u_1, \ldots, u_N|v_1, \ldots, v_N|h) = N\langle \Omega|B(u_N|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N - 1)p) \times \cdots \times B(u_2|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2p)B(u_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|\Omega\rangle_N. \quad (1.22)$$

Here, $|\Omega\rangle_N := |0\rangle_1 \otimes \cdots \otimes |0\rangle_N \in F_1 \otimes \cdots \otimes F_N$ and $N\langle \Omega| := 1(1 \otimes \cdots \otimes N(1) \in F_1^* \otimes \cdots \otimes F_N^*$ are the vacuum state and the dual particle-occupied state in the tensor product of quantum spaces.

In the paper in which the elliptic Felderhof model was introduced, Foda-Wheeler-Zuparic showed the following factorized expression for the domain wall boundary partition functions [30, 63, 64].

**Theorem 1.2.** (Foda-Wheeler-Zuparic [30]) The domain wall boundary partition functions
of the elliptic Felderhof model $Z_N(u_1, \ldots, u_N|v_1, \ldots, v_N|h)$ has the following factorized form

$$Z_N(u_1, \ldots, u_N|v_1, \ldots, v_N|h) = \left[ \frac{h + \sum_{j=1}^{N} v_j - \sum_{j=1}^{N} u_j + Np + \sum_{j=1}^{N} q_j}{h + 2\sum_{j=1}^{N} q_j} \right]^{1/2} \frac{1}{h + 2Np} \prod_{j=1}^{N} \left[ u_j - u_k + 2p \right] \left[ v_j - v_k + q_j + q_k \right].$$

We use (1.23) for the analysis on the scalar products of the elliptic Felderhof model in the next section.

## 2 Scalar Products

In this section, we introduce the scalar products of the Foda-Wheeler-Zuparic model, and prove the determinant formula. The scalar products are defined as the following partition functions (Figure 5)

$$Q_{M,N}(u_1, \ldots, u_N|w_1, \ldots, w_N|v_1, \ldots, v_M|h) = \left\langle M|\Omega \right| C(w_N|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2Np) \times \cdots \times B(w_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2Np) \times \left\langle v_1, \ldots, v_M|p|q_1, \ldots, q_M|h\right| \Omega \right\rangle_M,$$
Theorem 2.1. We have the following determinant formula for the scalar products of the elliptic Felderhof model

\[
Q_{M,N}(u_1, \ldots, u_N|w_1, \ldots, w_N|v_1, \ldots, v_M|h) = [2p]^N \frac{[h + 2Np]^{1/2}[h + 2Np + 2qM]^{1/2}}{[h + 4Np]^{1/2}[h + 2qM]^{1/2}} \prod_{1 \leq j < k \leq N} \frac{[u_j - u_k + 2p][w_j - w_k - 2p]}{[u_j - u_k][w_j - w_k]} \times \det_N \left( \frac{[w_j + h + 2Np - u_k]}{[h + 2Np]} a(w_j) d(u_k) - \frac{[w_j + h + 2Np - u_k + 2qM]}{[h + 2Np + 2qM]} a(u_k) d(w_j) \right),
\]

(2.2)

where \( \overline{q} = \sum_{k=1}^J q_k \), and \( a(u) \) and \( d(u) \) are given by

\[
a(u) = \prod_{\ell=1}^M [u - v_\ell + p + q_\ell], \quad d(u) = \prod_{\ell=1}^M [u - v_\ell + p - q_\ell].
\]

(2.3)

We apply Wheeler’s method \[79\] which extends the Izergin-Korpein technique \[42, 41\] from the domain wall boundary partition functions to the scalar products. To this end, we introduce the following intermediate scalar products \[79\] which is an intermediate object between the scalar products and the domain wall boundary partition functions (Figure 6)

\[
Q_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_N|v_1, \ldots, v_M|h)
\]

\[
= \langle 0^{M-N+n} 1^{N-n} | C(w_n|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N + n - 1)p) \times \cdots \times C(w_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2Np) \times B(u_N|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N - 1)p) \times \cdots \times B(u_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|\Omega \rangle_M.
\]

(2.4)
The scalar products $Q_{M,N}(u_1,\ldots,u_N|w_1,\ldots,w_N|v_1,\ldots,v_M|h)$ (2.1).

where

$$\langle 0^{M-N+n}1^{N-n}| = 1\langle 0| \otimes \cdots \otimes (M-N+n)\langle 0| \otimes (M-N+n+1)\langle 1| \otimes \cdots \otimes M\langle 1|.$$ (2.5)

The special case $n = N$ of the intermediate scalar products corresponds to the scalar products $Q_{M,N}(u_1,\ldots,u_N|w_1,\ldots,w_N|v_1,\ldots,v_M|h)$, while the case $n = 0$ is essentially the domain wall boundary partition functions.

The first thing to do is to list the properties of the intermediate scalar products which uniquely characterize it, which is given below.

**Proposition 2.2.** The intermediate scalar products of the elliptic Felderhof model $Q_{M,N,n}(u_1,\ldots,u_N|w_1,\ldots,w_n|v_1,\ldots,v_M|h)$ satisfies the following properties.

1. $$\prod_{j=M-N+n+1}^{M} [w_n - v_j + q_j - p]^{-1} Q_{M,N,n}(u_1,\ldots,u_N|w_1,\ldots,w_n|v_1,\ldots,v_M|h)$$ is an elliptic polynomial of $w_n$ in $\Theta_{M-N+n}(\chi)$.

2. The intermediate scalar products $Q_{M,N,n}(u_1,\ldots,u_N|w_1,\ldots,w_n|v_1,\ldots,v_M|h)$ is invariant under the simultaneous exchange of $v_j$, $q_j$ and $v_k$, $q_k$ for $1 \leq j < k \leq M - N + n$. 

Figure 5: The scalar products $Q_{M,N}(u_1,\ldots,u_N|w_1,\ldots,w_N|v_1,\ldots,v_M|h)$ (2.1).
One notes that the dynamical $R$-matrices in the right part of the bottom row are already frozen due to the ice-rule of the dynamical $R$-matrix.

(3) The following recursive relations between the intermediate scalar products hold:

$$Q_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) = \prod_{j=1}^{M-N+n-1} \left[ v_{M-N+n} - q_{M-N+n} - v_j - q_j - 2p \right]$$

$$\times Q_{M,N,n-1}(u_1, \ldots, u_N|w_1, \ldots, w_{n-1}|v_1, \ldots, v_M|h).$$

(2.6)

(4) The following evaluation holds for the case $n = 0$

$$Q_{M,N,0}(u_1, \ldots, u_N|v_1, \ldots, v_M|h) = \frac{[h + 2Np + 2q_M]}{[h + 2q_M]^{1/2}[h + 2q_{M-N} + 2Np]^{1/2}}$$

$$\times \prod_{1 \leq j < k \leq N} \left[ \frac{u_j - u_k + 2p}{u_j - u_k} \right] \prod_{M-N+1 \leq j < k \leq M} \left[ \frac{v_k - v_j + q_j + q_k}{v_k - v_j + q_j - q_k} \right] \prod_{j=1}^{M} \prod_{k=1}^{M} [u_j - v_k + p + q_k]$$

$$\times \prod_{j=1}^{N} [2p]^{1/2} \prod_{j=M-N+1}^{M} [2q_j]^{1/2} \mathrm{det}_N \left( \frac{[h + (2N - 1)p + 2q_M - q_{M-N+k} + v_{M-N+k} - u_j]}{[h + 2Np + 2q_M] [u_j + p - v_{M-N+k} + q_{M-N+k}]} \right).$$

(2.7)
Figure 7: The intermediate scalar products $Q_{M,N,n}(u_1, \ldots, u_N|v_1, \ldots, w_n|v_1, \ldots, v_M|h)$ (2.1) evaluated at $w_n = v_{M-N+n} - p - q_{M-N+n}$, which gives (2.6) since the dynamical $R$-matrices at the bottom row are frozen and the remaining part is $Q_{M,N,n-1}(u_1, \ldots, u_N|w_1, \ldots, w_{n-1}|v_1, \ldots, v_M|h)$.

Proof. Properties (1), (2) and (3) can be shown by using standard arguments. Property (1) can be shown as follows. We insert the completeness relation into the intermediate scalar products

$Q_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) = \langle 0^{M-N+n}1^{N-n}|C(w_n|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N + n - 1)p) \times \cdots \times C(w_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N - 1)p) \times \cdots \times B(u_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)\Omega\rangle_M$ \hspace{1cm} (2.8)

We calculate the matrix elements of $C(w_n|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N + n - 1)p)$ based

\[
\begin{align*}
\frac{w_n}{h + 2(N + n)p} &= v_{M-N+n} - p - q_{M-N+n} \\
\frac{h}{u_N} &= h + 2Np \\
\frac{h}{u_1} &= h + 2(N + n)p \\
\frac{h}{v_m} &= h + 2q_M \\
\frac{h}{v_M} &= h + 2(q_M + 1)Np \\
\frac{h}{v_M} &= h + 2(N + n)p
\end{align*}
\]
Figure 8: The intermediate scalar products $Q_{M,N,0}(u_1,\ldots,u_N||v_1,\ldots,v_M|h)$. One can see that the inner states of the left part of the lattice models are frozen which make contribution $\prod_{j=1}^N \prod_{k=1}^{M-N} |u_j - v_k + p + q_k|$ to the intermediate scalar products, and the right unfrozen part are the domain wall boundary partition functions $Z_N(u_1,\ldots,u_N|v_{M-N+1},\ldots,v_M|h + 2q_{M-N})$.

on its definition to get

$$
\langle 0^{M-N+n} | C(u_1,\ldots,u_N|v_1,\ldots,v_M|h + 2(N+n)p) | 0^{k-1}10^{M-N+n-k}1^{N-n} \rangle = \prod_{j=1}^{k-1} \frac{[h + 2(N + n - 1)p + 2q_{j-1}]^{1/2}[h + 2(N + n)p + 2q_j]^{1/2}[w_n - v_j - q_j + p]}{[h + 2(N + n)p + 2q_{j-1}]^{1/2}[h + 2(N + n - 1)p + 2q_j]^{1/2}} \prod_{j=k+1}^{M-n+N} [w_n - v_j + p + q_j] \times \prod_{j=M-N+n+1}^{M} \frac{[h + 2(N + n - 1)p + 2q_{j-1}]^{1/2}[h + 2(N + n)p + 2q_j]^{1/2}[w_n - v_j + q_j - p]}{[h + 2(N + n)p + 2q_{j-1}]^{1/2}[h + 2(N + n - 1)p + 2q_j]^{1/2}}.
$$

(2.9)

One can see from (2.9) that $Q_{M,N,n}(u_1,\ldots,u_N|v_1,\ldots,v_M|h)$ has $\prod_{j=M-N+n+1}^{M} [w_n - v_j + q_j - p]$ as an overall factor (one can also show this by the graphical representation of the intermediate scalar products in Figure 8 in which one can show that the dynamical $R$-matrices of the right part of the bottom row are already frozen due to the ice-rule of the dynamical $R$-matrix, and the matrix elements of the $R$-matrices of the frozen parts contain the factor $\prod_{j=M-N+n+1}^{M} [w_n - v_j + q_j - p]$). Let us denote the intermediate scalar products
divided by this factor as $\tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h)$:

$$\tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) = \prod_{j=M-N+n+1}^{M} [w_n - v_j + q_j - p]^{-1} Q_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h).$$  \hspace{1cm} (2.10)$$

By calculating the quasi-periodicities of $(0^{M-N+n+1}N^{-n})C(w_n|v_1, \ldots, v_M|h)p_1, \ldots, q_M|h+2(N+n-1)p)(0^{k-1}10^{M-N+n-k}1^{N-n})$ with respect to $w_n$, one finds those of the elliptic functions $\tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h)$ are given by

$$\tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n+1|v_1, \ldots, v_M|h)$$

$$= (-1)^{M-N+n} \tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h),$$  \hspace{1cm} (2.11)$$

$$= (-q^{-1})^{M-N+n} \exp \left(-2\pi i \left((M-N+n)w_n + h + (M+N+3n-2)p \right) - \sum_{j=1}^{M-N+n} q_j \right) \tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h),$$  \hspace{1cm} (2.12)$$

which shows that $\tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h)$ is an elliptic polynomial of $w_n$ in $\Theta_{M-N+n}(\chi)$ of periods 1 and $\tau = -i\log(q)/\pi$ with the characters $\chi(1) = (-1)^{M-N+n}$ and $\chi(\tau) = \exp \left(-2\pi i \left(h + (M+N+3n-2)p - \sum_{j=1}^{M-N+n} q_j \right) \right)$. This shows Property (1).

Property (2) can be shown as a consequence of the commutation relation between the vertical monodromy matrices, which is a standard procedure using the dynamical Yang-Baxter relation, thus we omit the details.

Property (3) can be shown by substituting $w_n = v_{M-N+n} - p - q_{M-N+n}$ into (2.13), after which only one of the summands $k = M-N+n$ survives. Equivalently, this property can also be shown by the graphical description of the intermediate scalar products (Figure 8).

Let us show Property (4). From its graphical description (Figure 8), one easily finds that for the case $n = 0$, the intermediate scalar products $Q_{M,N,0}(u_1, \ldots, u_N|v_1, \ldots, v_M|h)$ is just a concatenation of frozen parts and the domain wall boundary partition functions

$$Q_{M,N,0}(u_1, \ldots, u_N|v_1, \ldots, v_M|h) = \prod_{j=1}^{N} \prod_{k=1}^{M-N} [u_j - v_k + p + q_k] Z_N(u_1, \ldots, u_N|v_{M-N+1}, \ldots, v_M|h + 2q_{M-N}).$$  \hspace{1cm} (2.13)$$

We insert the factorization formula for the domain wall boundary partition functions by
Foda-Wheeler-Zuparic (1.23) in Theorem 1.2 into the right hand side of (2.13) to get

\[
Q_{M,N,0}(u_1,\ldots,u_N|v_1,\ldots,v_M|h) = \prod_{j=1}^{N} \prod_{k=1}^{M-N} (u_j - v_k + p + q_k) \\
\times \left[ \frac{h + 2qM-N + \sum_{j=M-N+1}^{M} v_j - \sum_{j=1}^{N} u_j + Np + \sum_{j=M-N+1}^{M} q_j}{[h + 2qM-N]^{1/2}[h + 2qM-N + 2Np]^{1/2}} \right] \\
\times \prod_{j=1}^{N} [2p]^{1/2} \prod_{j=M-N+1}^{M} [2q_j]^{1/2} \prod_{1\leq j<k\leq N} [u_j - u_k + 2p] \prod_{M-N+1\leq j<k\leq M} [v_k - v_j + q_j + q_k].
\] (2.14)

(2.14) is already an explicit form corresponding to the initial condition of the Izergin-Korepin recursion process between the intermediate scalar products, and is a very compact expression since it is a factorized form. However, we further rewrite it in a determinant form. Going back to a complicated expression is because in the next proposition, we present the explicit determinant form of the intermediate scalar products, and we have to check that it satisfies the case \(n = 0\), which is immediate to see if one rewrites in the determinant form (2.7).

How to rewrite (2.14) in the determinant form goes as follows. We set \(\lambda = -h - 2Np - 2qM\), \(z_j = u_j + p\), \(w_k = v_{M+N+k} - q_{M-N+k}\) into the Frobenius determinant formula

\[
\det_N \left( \frac{\lambda + \sum_{j=1}^{N} (z_j - w_j)}{[\lambda]} \prod_{1\leq j<k\leq N} [z_j - w_k] \right) = \frac{[\lambda]}{[\lambda]} \prod_{1\leq j,k\leq N} [z_j - w_k],
\] (2.15)

to get the following identity

\[
\left[ h + 2qM-N + \sum_{j=M-N+1}^{M} v_j - \sum_{j=1}^{N} u_j + Np + \sum_{j=M-N+1}^{M} q_j \right] \\
\times \prod_{1\leq j<k\leq N} [u_j - u_k] \prod_{M-N+1\leq j<k\leq M} [v_k - v_j - q_k + q_j] \\
\times \det_N \left( \frac{[h + (2N-1)p + 2qM - qM-N+k + v_{M-N+k} - u_j]}{[h + 2Np + 2qM][u_j + p - v_{M-N+k} + q_{M-N+k}]} \right). (2.16)
\]
Substituting (2.16) into the right hand side of (2.14), we get

\[
Q_{M,N,0}(u_1, \ldots, u_N|v_1, \ldots, v_M|h)
\]

\[
= \frac{1}{[h + 2Np + 2qM]} \prod_{j=1}^{N} \prod_{k=M-N+1}^{M} [u_j - v_k + p + q_k]
\]

\[
\times \prod_{1 \leq j < k \leq N} [u_j - u_k] \prod_{M-N+1 \leq j < k \leq M} [v_k - v_j - q_j + q_j]
\]

\[
\times \det_N \left( \frac{[h + (2N - 1)p + 2qM - q_{M-N+k} + u_j]}{[h + 2Np + 2qM]} \right)
\]

\[
\times \prod_{j=1}^{N} \prod_{j=M-N+1}^{M} [2q_j]^{1/2} \prod_{1 \leq j < k \leq N} [u_j - u_k + 2p] \prod_{M-N+1 \leq j < k \leq M} [v_k - v_j + q_j + q_k]
\]

\[
= \frac{1}{[h + 2Np + 2qM]} \prod_{1 \leq j < k \leq N} [u_j - u_k + 2p] \prod_{M-N+1 \leq j < k \leq M} [v_k - v_j + q_j + q_k]
\]

\[
\times \prod_{j=1}^{N} \prod_{j=M-N+1}^{M} [2q_j]^{1/2} \det_N \left( \frac{[h + (2N - 1)p + 2qM - q_{M-N+k} + u_j]}{[h + 2Np + 2qM]} \right),
\]

and Property (4) is proved.

\[\square\]

The next thing to do is to find the explicit forms of the intermediate scalar products satisfying all the properties in Proposition 2.2. One can show the following determinant representation.

**Theorem 2.3.** The intermediate scalar products \(Q_{M,N,n}(u_1, \ldots, u_N|v_1, \ldots, v_M|h)\) have the following determinant form:

\[
Q_{M,N,n}(u_1, \ldots, u_N|v_1, \ldots, v_M|h)
\]

\[
= D_{M,N,n} \prod_{1 \leq j < k \leq N} [u_j - u_k + 2p] \prod_{M-N+n+1 \leq j < k \leq M} [v_k - v_j + q_j + q_k]
\]

\[
\times \prod_{1 \leq j < k \leq N} [w_j - w_k - 2p] \det_N \left( P_{M,N,n}(u_1, \ldots, u_N|v_1, \ldots, v_M|h) \right),
\]  

(2.18)
where \( D_{M,N,n} \) is given by

\[
D_{M,N,n} = [2p]^{(N+n)/2} \prod_{j=n+1}^{N} [2qM - N + j]^{1/2} \times \frac{[h + 2Np]^{1/2}[h + 2Np + 2M]^{1/2}[h + 2(N + n)p + 2M]^{1/2}}{[h + 2M]^{1/2}[h + 2(N + n)p]^{1/2}[h + 2(N + n)p + 2M]^{1/2}},
\]

and \( P_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \) is an \( N \times N \) matrix whose matrix elements are given by

\[
P_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h)_{jk} = \begin{cases} 
\prod_{\ell=M-N+n+1}^{M} [w_j - v_\ell - p + q_\ell]^{-1} \frac{[w_j + h + 2Np - u_k]}{[h + 2Np]} a(w_j)d(u_k) - \frac{[w_j + h + 2Np - u_k + 2M]}{[h + 2Np + 2M]} a(u_k)d(w_j), & (1 \leq j \leq n) \\
\prod_{\ell \neq M-N+j}^{M} [u_k - v_\ell + p + q_\ell], & (n + 1 \leq j \leq N) 
\end{cases}
\]

Proof. One can check directly that the right hand side of (2.18) satisfies Properties (1), (2), (3) and (4) in Proposition 2.2. We give some comments. Let us denote the right hand side of (2.18) as \( \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \) and set

\[
\tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) = \prod_{j=M-N+n+1}^{M} [w_n - v_j + q_j - p]^{-1} \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h).
\]

Property (1) can be shown by calculating the quasiperiodicites of the function \( \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \) with respect to \( w_n \). Expanding the determinant in the right hand side of (2.18), recalling that \( a(u) \) and \( d(u) \) are defined as

\[
a(u) = \prod_{\ell=1}^{M} [u - v_\ell + p + q_\ell], \quad d(u) = \prod_{\ell=1}^{M} [u - v_\ell + p - q_\ell],
\]

concentrating on the factors depending on \( w_n \), one finds

\[
\tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n + 1|v_1, \ldots, v_M|h) = (-1)^{M-N+n} \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h),
\]

\[
\tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n - i\log(q)/\pi|v_1, \ldots, v_M|h) = (-q^{-1})^{M-N+n} \exp \left( -2\pi i \left( (M - N + n)w_n + h + (M + N + 3n - 2)p \right) - \sum_{j=1}^{M-N+n} v_j + \sum_{j=1}^{M-N+n} q_j \right) \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h).
\]
which are exactly the same with those for \( \tilde{Q}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n+1|v_1, \ldots, v_M|h) \) \((2.11)\) and \((2.12)\).

One can also show that \( \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n+1|v_1, \ldots, v_M|h) \) has apparent singularities at \( w_n = v_j, \ j = 1, \ldots, n \) coming from the zeroes of the denominators of the matrix elements \((2.20)\), which cancel with the corresponding zeroes of the numerators. There are also apparent singularities at \( w_n = v_j, \ j = 1, \ldots, n-1 \) and \( w_n = v_\ell - p - q_\ell, \ \ell = M - N + n + 1, \ldots, M \). Again, one finds that in these cases two rows of the matrix \( \tilde{F}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \) become proportional when taking the limits \( w_n \to w_j, \ j = 1, \ldots, n-1 \) or \( w_n \to v_\ell - p - q_\ell, \ \ell = M - N + n + 1, \ldots, M \), hence there are no singularities and \( \tilde{R}_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n+1|v_1, \ldots, v_M|h) \) is an elliptic polynomial as a function of \( w_n \).

Property (2) can be easily checked by recalling that \( \eta_M \) and \( \eta_{M-N+n} \) which appear in \( R_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \) are defined as \( \eta_M = \sum_{j=1}^M q_j \) and \( \eta_{M-N+n} = \sum_{j=1}^{M-N+n} q_j \) from which one can see that they are invariant under the exchange of \( q_j \) and \( q_k \) for \( 1 \leq j < k \leq M - N + n \). Property (3) can be checked by a long and tedious but straightforward computation. We remark that expanding the determinant in the right hand side of \((2.18)\) \( \det_N \left( P_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \right) \) based on its definition, and rewriting the prefactor \( D_{M,N,n} \) in the function \( R_{M,N,n}(u_1, \ldots, u_N|w_1, \ldots, w_n|v_1, \ldots, v_M|h) \) as \( D_{M,N,n} \)

\[
D_{M,N,n} = \frac{[h + 2Np + 2\eta_M]}{[h + 2\eta_M]^{1/2}[h + 2\eta_{M-N} + 2Np]^{1/2}} \prod_{j=1}^N [2p]^{1/2} \prod_{j=1}^n [2\eta_{M-N+j}]^{1/2} \\
\times \prod_{j=1}^n \frac{[h + 2(N + j - 1)p]^{1/2} [h + 2(N + j)p + 2\eta_{M-N+j-1}]^{1/2} [h + 2(N + j)p + 2\eta_{M-N+j}]^{1/2}}{[h + 2(N + j)p]^{1/2} [h + 2(N + j)p + 2\eta_{M-N+j}]^{1/2}},
\]

makes things easier to check Property (3).

Property (4) can be checked immediately by setting \( n = 0 \) in the right hand side of \((2.18)\).

As a consequence of Theorem 2.3 one gets the determinant formula for the scalar products \((2.2)\) by specializing \((2.18)\) to \( n = N \), hence we have proved Theorem 2.4.

3 Elliptic Cauchy formula

We derive the Cauchy formula for elliptic symmetric functions by combining the determinant formula for the scalar products proved in the last section with another evaluation based on the correspondence between the wavefunctions and the symmetric functions. Let us first recall the correspondence [45]. A detailed proof of the correspondence can also be found for the closely related Okado-Deguchi-Fujii-Martin [33, 34, 35] (elliptic Perk-Schultz) model [44].
We introduce a class of partition functions $W_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)$ defined as the matrix elements of the product of the $B$-operators \[(1.20)\] as follows:

$$W_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) = \langle x_1 \cdots x_N | B(u_N|v_1, \ldots, v_M)pq_1, \ldots, q_M|h + 2(N-1)p \rangle \times \cdots \times B(u_2|v_1, \ldots, v_M)pq_1, \ldots, q_M|h + 2p \rangle B(u_1|v_1, \ldots, v_M)pq_1, \ldots, q_M|h |\Omega\rangle_M,$$  \[(3.1)\]

where $\langle x_1 \cdots x_N |$ are the dual $N$-particle states.

$$\langle x_1 \cdots x_N | = (1|0) \otimes \cdots \otimes_M (0) \prod_{j=1}^{N} \sigma_{x_j}^{+} \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M,$$ \[(3.2)\]

which are states labelling the configurations of particles $1 \leq x_1 < x_2 < \cdots < x_N \leq M$. We call this class of partition functions as wavefunctions in this paper since it is an analogue of wavefunctions of integrable vertex models.

We also define another class of wavefunctions $V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)$ as matrix elements of the $C$-operators \[(1.21)\] as

$$V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) = \langle x_1 \cdots x_N | C(u_N|v_1, \ldots, v_M)pq_1, \ldots, q_M|h + 2(N-1)p \rangle \times \cdots \times C(u_2|v_1, \ldots, v_M)pq_1, \ldots, q_M|h + 2p \rangle C(u_1|v_1, \ldots, v_M)pq_1, \ldots, q_M|h |x_1 \cdots x_N \rangle,$$ \[(3.3)\]

where $|x_1 \cdots x_N \rangle$ are the $N$-particle states.

$$|x_1 \cdots x_N \rangle = \prod_{j=1}^{N} \sigma_{x_j}^{-} \langle (0)|_1 \otimes \cdots \otimes_M (0)|_M \rangle \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M.$$ \[(3.4)\]

See Figures 9 and 10 for graphical representations of the wavefunctions (3.1), (3.3).

The wavefunctions (3.1), (3.3) can be explicitly expressed using deformed elliptic Vandermonde determinants and elliptic symmetric functions defined below.

**Definition 3.1.** We define the following elliptic Schur function $S_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)$ which depends on the symmetric variables $u_1, \ldots, u_N$, two sets of complex parameters $v_1, \ldots, v_M$ and $q_1, \ldots, q_M$, two complex parameters $h, p$ and integers $x_1, \ldots, x_N$ satisfying $1 \leq x_1 < \cdots < x_N \leq M$,

$$S_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) = \sum_{\sigma \in \mathcal{S}_N} \prod_{1 \leq j < k \leq N} \frac{1}{[u_{\sigma(j)} - u_{\sigma(k)}]} \prod_{j=1}^{N} \prod_{k=x_{j+1}}^{N} [u_{\sigma(j)} - v_k - q_k + p] \times \prod_{j=1}^{N} \frac{[h + 2jp + 2q_M]^{1/2}[2p]^{1/2} [2q_{x_j}^{-1}]^{1/2}}{[h + 2j - 1]p + 2q_M^{1/2} [h + 2Np + 2q_{x_j}^{-1}]^{1/2} [h + 2Np + 2q_{x_j}^{-1}]^{1/2}} \times \prod_{j=1}^{N} \frac{-u_{\sigma(j)} + v_{x_j} + h + (2N - 1)p + q_{x_j} + 2q_{x_j}^{-1}}{-u_{\sigma(j)} + v_{x_j} + h + (2N - 1)p + q_{x_j} + 2q_{x_j}^{-1}} \prod_{j=1}^{N} \prod_{k=x_j-1}^{N} [u_{\sigma(j)} - v_k + p + q_k],$$ \[(3.5)\]

$$= \prod_{1 \leq j < k \leq N} \frac{1}{[u_j - u_k]} \det_N (f_{x_j}(u_k|v_1, \ldots, v_M)),$$ \[(3.6)\]

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Figure 9: The wavefunctions $W_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)$ (3.1). The figure illustrates the case $M = 7$, $N = 4$, $x_1 = 2$, $x_2 = 5$, $x_3 = 6$, $x_4 = 7$.

\[
f_{x_j}(u|v_1, \ldots, v_M) = \left[ h + 2jp + 2q_{x_j} \right]^{1/2} \left[ 2p \right]^{1/2} \left[ 2q_{x_j} \right]^{1/2} \\
\times [-u + v_{x_j} + h + (2N - 1)p + q_{x_j} + 2q_{x_j - 1}] \prod_{k=1}^{x_j - 1} \left[ u - v_k + p + q_k \right] \prod_{k=x_j + 1}^{M} \left[ u - v_k + p - q_k \right].
\]

(3.7)

Recall that $q_j$ is defined as $q_j = \sum_{k=1}^{j} q_k$.

We also define another elliptic Schur function $T_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)$ which depends on the symmetric variables $u_1, \ldots, u_N$, two sets of complex parameters $v_1, \ldots, v_M$ and $q_1, \ldots, q_M$, two complex parameters $h$, $p$ and integers $x_1, \ldots, x_N$ satisfying $1 \leq x_1 < \cdots < x_N \leq M$,

\[
T_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) = \sum_{\sigma \in S_N} \prod_{1 \leq j < k \leq N} \left[ u_{\sigma(j)} - u_{\sigma(k)} \right] \prod_{j=1}^{N} \prod_{k=x_j + 1}^{M} \left[ u_{\sigma(j)} - v_k + q_k + p \right] \\
\times \prod_{j=1}^{N} \left[ h + 2(j - 1)p \right]^{1/2} \left[ 2p \right]^{1/2} \left[ 2q_{x_j} \right]^{1/2} \\
\times \prod_{j=1}^{N} \left[ u_{\sigma(j)} - v_{x_j} + h + p + q_{x_j} + 2q_{x_j - 1} \right] \prod_{j=1}^{x_j - 1} \prod_{k=1}^{N} \left[ u_{\sigma(j)} - v_k + p - q_k \right],
\]

(3.8)

\[
= \prod_{1 \leq j < k \leq N} \frac{1}{u_j - u_k} \det_N(h_{x_j}(u_k|v_1, \ldots, v_M)),
\]

(3.9)
The wavefunctions $V_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$ (3.3). The figure illustrates the case $M = 7$, $N = 4$, $x_1 = 1$, $x_2 = 3$, $x_3 = 5$, $x_4 = 6$.

The wavefunctions of the elliptic Felderhof model can be expressed as products of one-parameter deformations of the elliptic Vandermonde determinant and the elliptic Schur functions $S_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$ and $T_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$ defined above. We present the correspondence below.

**Theorem 3.2.** The wavefunctions of the elliptic Felderhof model

$W_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$ is explicitly expressed as the product of a one-parameter deformation of an elliptic Vandermonde determinant $\prod_{1 \leq j < k \leq N} [u_j - u_k + 2p]$ and the elliptic Schur functions $S_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$

$$W_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h) = \prod_{1 \leq j < k \leq N} [u_j - u_k + 2p]S_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h). \quad (3.11)$$

The wavefunctions $V_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$ is explicitly expressed as the product of a one-parameter deformation of an elliptic Vandermonde determinant $\prod_{1 \leq j < k \leq N} [u_j - u_k + 2p]$ and the elliptic Schur functions $S_{M,N}(u_1,\ldots,u_N|v_1,\ldots,v_M|x_1,\ldots,x_N|h)$

$$h_{x_j}(u|v_1,\ldots,v_M) = \frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{x_j}]^{1/2}}{[h+2jp]^{1/2}[h+2q_{x_j-1}]^{1/2}[2q_{x_j}]^{1/2}} \times [u - v_{x_j} + h + p + q_{x_j} + 2q_{x_j-1}] \prod_{k=1}^{x_j-1} [u - v_k + p - q_k] \prod_{k=x_j+1}^{M} [u - v_k + p + q_k]. \quad (3.10)$$
and the elliptic Schur functions \( T_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) \)

\[
V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) = \prod_{1 \leq j < k \leq N} [u_j - u_k - 2p] T_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h).
\]

(3.12)

The relation (3.11) is shown in [45]. In the Appendix, we give a proof of (3.12) in some detail.

Now, combining the determinant formula for the scalar products (Theorem 2.1) and the correspondence between the wavefunctions and the elliptic Schur functions (Theorem 3.2), one can derive the Cauchy formula for the elliptic Schur functions.

**Theorem 3.3.** We have the following Cauchy formula for the elliptic Schur functions

\[
\sum_{1 \leq x_1 < x_2 < \cdots < x_N \leq M} S_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)
\times T_{M,N}(w_1, \ldots, w_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h + 2Np)
= [2p]^N \frac{[h + 2Np]^{1/2}[h + 2Np + 2qM]^{1/2}}{[h + 2Np + 2qM]^{1/2}} \prod_{1 \leq j < k \leq N} [u_j - u_k][w_j - w_k]
\times \det_N \left( \begin{array}{c} [w_j + h + 2Np - u_k]a(w_j)d(u_k) - [w_j + h + 2Np - u_k + 2qM]a(u_k)d(w_j) \\ [w_j + h + 2Np] \end{array} \right).
\]

(3.13)

**Proof.** The scalar products (2.1) can be evaluated by inserting the completeness relation

\[
\sum_{1 \leq x_1 < x_2 < \cdots < x_N \leq M} |x_1 \cdots x_N \rangle \langle x_1 \cdots x_N| = \text{Id},
\]

(3.14)

and using the correspondence between the wavefunctions and the elliptic Schur functions
Figure 11: A graphical representation of a summand corresponding to $x_1 = 2$, $x_2 = 5$, $x_3 = 7$, $x_4 = 11$ in the decomposition of the scalar products into the sum over products of the wavefunctions (3.15).

(3.15) See Figure 11 for a graphical description of the decomposition (3.15). The Cauchy formula
for the elliptic Schur functions (3.13) follows from comparing (3.15) with the direct evaluation which gives the determinant formula for the scalar products (2.2) (Theorem 2.1).

4 Conclusion

In this paper, we examined the scalar products of the elliptic Felderhof model introduced by Foda-Wheeler-Zuparic [30], which is an elliptic extension of the face-type Felderhof model [31] of Deguchi-Akutsu [32]. By applying the Izergin-Korepin technique developed by Wheeler [79] to analyze the scalar products, we derived the determinant formula for the scalar products of the elliptic Felderhof model by constructing the explicit determinant forms of the intermediate scalar products, which connects the scalar products and the domain wall boundary partition functions.

The scalar products can be evaluated in another way as a sum over products of the wavefunctions. For the case of the elliptic Felderhof models, the wavefunctions are expressed as a deformed elliptic Vandermonde determinant and elliptic Schur functions, which can be shown for example by the recently-developed Izergin-Korepin technique to analyze the wavefunctions [43, 44, 45]. By combining this way of evaluation with the direct evaluation which gives the determinant formula, we obtained the Cauchy formula for the elliptic Schur functions.

For the case of trigonometric integrable models, other boundary conditions such as the reflecting boundaries, half-turn boundaries are investigated [52, 53, 54], where in some cases symplectic Schur functions emerge as the wavefunctions. There are also progresses on the introduction of a higher rank model called as the metaplectic ice [58], where connections with metaplectic Whittaker functions are established. It seems valuable to lift those works to the elliptic setting, which may lead to new developments in number theory as well as integrable models.

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Appendix

In this Appendix, we give a proof of (3.12) in some detail, based on the idea initiated by Korepin [41], listing the properties of the domain wall boundary partition functions which uniquely characterize it. This characterization lead Izergin [42] to found the determinant representation (Izergin-Korepin determinant) of the domain wall boundary partition functions of the $U_q(sl_2)$ six-vertex model. We recently extended the Izergin-Korepin technique to be able to analyze the wavefunctions [44, 45, 43]. We apply this technique here. As usual, we first list the properties which uniquely characterize the wavefunctions.
Proposition 4.1. The wavefunctions $V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h)$ satisfies the following properties.

(1) If $x_N = M$, the wavefunctions $V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h)$ is an elliptic polynomial of $v_M$ in $\Theta_N(x)$.

(2) The wavefunctions $V_{M,N}(u_\sigma(1), \ldots, u_\sigma(N) | v_1, \ldots, v_M | x_1, \ldots, x_N | h)$ with the ordering of the spectral parameters permuted $u_\sigma(1), \ldots, u_\sigma(N)$, $\sigma \in S_N$ are related to the wavefunctions $V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h)$ by the following relation

$$\prod_{1 \leq j < k \leq N \atop \sigma(j) > \sigma(k)} [u_\sigma(j) - u_\sigma(k) - 2p] V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h) = \prod_{1 \leq j < k \leq N \atop \sigma(j) > \sigma(k)} [u_\sigma(k) - u_\sigma(j) - 2p] V_{M,N}(u_\sigma(1), \ldots, u_\sigma(N) | v_1, \ldots, v_M | x_1, \ldots, x_N | h).$$

(3) If $x_N = M$, the following recursive relations between the wavefunctions hold:

$$V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h)|_{v_M = u_N + p + q_M} = [2p]^{1/2} [2q_M]^{1/2} [h + 2q_{M-1}]^{1/2} [h + 2(N-1)p]^{1/2} \prod_{j=1}^{N-1} [u_j - u_N - 2p] \prod_{j=1}^{M-1} [u_N - v_j + p - q_j] \times V_{M-1,N-1}(u_1, \ldots, u_{N-1} | v_1, \ldots, v_{M-1} | x_1, \ldots, x_{N-1} | h).$$

If $x_N \neq M$, the following factorizations hold for the wavefunctions:

$$V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h) = \prod_{j=1}^{N} [u_j - v_M + q_M + p] V_{M-1,N}(u_1, \ldots, u_N | v_1, \ldots, v_{M-1} | x_1, \ldots, x_N | h).$$

(4) The following evaluation holds for the case $N = 1$, $x_1 = M$

$$V_{M,1}(u | v_1, \ldots, v_M | h) = \frac{[h]^{1/2} [2p]^{1/2} [u - v_M + h + p + q_M + 2q_{M-1}]}{[h + 2p]^{1/2} [h + 2q_{M-1}]^{1/2} [h + 2q_M]^{1/2}} \prod_{k=1}^{M-1} [u - v_k + p - q_k].$$

Proposition 4.1 can be proved in a similar way with the Proposition 2.2 for the scalar products. The next step is to find the explicit forms of the functions which satisfy all the properties in Proposition 4.1. This is given in the next proposition. The proof of the proposition also concludes the proof of (3.12).

Proposition 4.2. The product of a deformed elliptic Vandermonde determinant and the elliptic Schur functions $\prod_{1 \leq j < k \leq N} [v_j - u_k - 2p] T_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h)$ satisfy all the properties listed in Proposition 4.1, which the wavefunctions of the elliptic Felderhoff model $V_{M,N}(u_1, \ldots, u_N | v_1, \ldots, v_M | x_1, \ldots, x_N | h)$ must satisfy.
Proof. This can be proved by showing that

\[
H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) := \prod_{1 \leq j < k \leq N} [u_j - u_k - 2p]T_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)
\]

\[
= \prod_{1 \leq j < k \leq N} [u_j - u_k - 2p] \sum_{\sigma \in S_N} \prod_{1 \leq j < k \leq N} \frac{1}{|u_{\sigma(j)} - u_{\sigma(k)}|} \prod_{j=1}^{N} \prod_{k=1}^{M} [u_{\sigma(j)} - v_k + q_k + p]
\]

\[
\times \prod_{j=1}^{N} [h + 2(j-1)p]^{1/2}[2p]^{1/2}[2q_{x_j}]^{1/2}
\times \prod_{j=1}^{N} [h + 2j p]^{1/2}[h + 2q_{x_j-1}]^{1/2}[h + 2q_{x_j}]^{1/2},
\]

\[
\times \prod_{j=1}^{N} [u_{\sigma(j)} - v_{x_j} + h + p + q_{x_j} + 2q_{x_j-1}] \prod_{j=1}^{N} \prod_{k=1}^{M} [u_{\sigma(j)} - v_k + p - q_k],
\]

(4.5)

satisfies Properties (1), (2), (3) and (4) in Proposition 4.1. For example, Property (1) can be checked by computing the quasi-periodicities of \(H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_{N-1}, M|h)\) with respect to \(v_M\), which are given by

\[
H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M + 1|x_1, \ldots, x_{N-1}, M|h) = (-1)^N H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_{N-1}, M|h),
\]

(4.6)

\[
H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M - \frac{i\log(q)/\pi}{x_1, \ldots, x_{N-1}, M|h}) = (-q^{-1})^N \exp\left(-2\pi i \left(Nv_M - h - 2q_{M-1} - Nq_M - Np - \sum_{j=1}^{N} u_j \right)\right)
\times H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_{N-1}, M|h).
\]

(4.7)

These explicit quasi-periodicities show that \(H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_{N-1}, M|h)\) is an elliptic polynomial of degree \(N\) in \(v_M\). Also, the factors \((-1)^N\) and \((-q^{-1})^N\exp\left(-2\pi i \left(Nv_M - h - 2q_{M-1} - Nq_M - Np - \sum_{j=1}^{N} u_j \right)\right)\) in (4.6) and (4.7) are the same with the ones which appear when we examine the quasi-periodicities of the wavefunctions \(V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_{N-1}, M|h)\) as a function of \(v_M\).

Property (3) for the case \(x_N = M\) can be shown as follows. The factors \(\prod_{j=1}^{N-1} [u_{\sigma(j)} - v_M + q_M + p]\) in each summand in \(H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)\) means that we only have to deal with the summands satisfying \(\sigma(N) = N\) in (4.5) which survive after the substitution \(v_M = u_N + p + q_M\). Then we find that \(H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)\) evaluated at \(v_M = u_N + p + q_M\) can be expressed using the symmetric group \(S_{N-1}\) where
every $\sigma' \in S_{N-1}$ satisfies $\{\sigma'(1), \ldots, \sigma'(N-1)\} = \{1, \ldots, N-1\}$ as follows:

$$H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)|_{v_M = u_N + p + q_M}$$

$$= \prod_{1 \leq j < k \leq N-1} [u_j - u_k - 2p] \prod_{j=1}^{N-1} [u_j - u_N - 2p]$$

$$\times \sum_{\sigma' \in S_{N-1}} \prod_{1 \leq j < k \leq N-1} \frac{1}{(u_{\sigma'(j)} - u_{\sigma'(k)})} \prod_{j=1}^{N-1} \frac{1}{(u_{\sigma'(j)} - u_N)}$$

$$\times \prod_{j=1}^{N-1} \prod_{k=x_j+1}^{M-1} [u_{\sigma'(j)} - u_k + q_k + p] \prod_{j=1}^{N-1} [u_{\sigma'(j)} - u_N]$$

$$\times \prod_{j=1}^{N-1} \prod_{k=x_j+1}^{M-1} [u_{\sigma'(j)} - v_{x_j} + h + p + q_{x_j} + 2q_{x_j - 1}]$$

$$\times \prod_{j=1}^{N-1} \prod_{k=x_j+1}^{M-1} [u_{\sigma'(j)} - v_{x_j} + p - q_k] \prod_{k=1}^{M-1} [u_N - v_k + p - q_k].$$

(4.8)

After appropriately cancelling and rearranging the expression above, we find that (4.8) can be rewritten as

$$H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)|_{v_M = u_N + p + q_M}$$

$$= \frac{[2p]^{1/2}[2q_M]^{1/2}[h + 2q_{M-1}]^{1/2}[h + 2(N-1)p]^{1/2}}{[h + 2q_M]^{1/2}[h + 2Np]^{1/2}} \prod_{j=1}^{N-1} [u_j - u_N - 2p] \prod_{j=1}^{M-1} [u_N - v_j + p - q_j]$$

$$\times \prod_{1 \leq j < k \leq N-1} [u_j - u_k - 2p] \sum_{\sigma' \in S_{N-1}} \prod_{1 \leq j < k \leq N-1} \frac{1}{(u_{\sigma'(j)} - u_{\sigma'(k)})}$$

$$\times \prod_{j=1}^{N-1} \prod_{k=x_j+1}^{M-1} [u_{\sigma'(j)} - u_k + q_k + p] \prod_{j=1}^{N-1} [h + 2(j-1)p]^{1/2}[2p]^{1/2}[2q_{x_j}]^{1/2}$$

$$\times \prod_{j=1}^{N-1} \prod_{k=x_j+1}^{M-1} [u_{\sigma'(j)} - v_{x_j} + h + p + q_{x_j} + 2q_{x_j - 1}] \prod_{j=1}^{N-1} \prod_{k=x_j+1}^{M-1} [u_{\sigma'(j)} - v_k + p - q_k]$$

$$= \frac{[2p]^{1/2}[2q_M]^{1/2}[h + 2q_{M-1}]^{1/2}[h + 2(N-1)p]^{1/2}}{[h + 2q_M]^{1/2}[h + 2Np]^{1/2}} \prod_{j=1}^{N-1} [u_j - u_N - 2p] \prod_{j=1}^{M-1} [u_N - v_j + p - q_j]$$

$$\times H_{M-1,N-1}(u_1, \ldots, u_{N-1}|v_1, \ldots, v_{M-1}|x_1, \ldots, x_{N-1}|h),$$

(4.9)

which is exactly the same with the one (4.2) for the wavefunctions of the elliptic Felderhof model $V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h).$
\( H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) \) is nothing but the explicit form of the wavefunctions \( V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) \)

\[
V_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) = H_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h),
\]

\[(4.10)\]

hence (3.12) in Theorem 3.2 is proved.

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