Abstract: Fractional photon-assisted tunneling is investigated both numerically and analytically in a double-well lattice. While integer photon-assisted tunneling is a single-particle effect, fractional photon-assisted tunneling is an interaction-induced many-body effect. Double-well lattices with few particles in each double well are ideal to study this effect far from the mean-field effects. It is predicted that the 1/4-resonance is observable in such systems. Fractional photon-assisted tunneling provides a physically relevant model, for which $N$-th order time-dependent perturbation theory can be large although all previous orders are small. All predicted effects will be observable with an existing experimental setup [1].

$\Delta E \approx \hbar \omega = 4 \Delta E$

Sketch of 1/4-“photon” resonance: one photon has enough energy to make four particles tunnel ($\hbar \omega = 4 \Delta E$). The “photons” are time-periodic potential modulations in the kilohertz regime.

Fractional photon-assisted tunneling of ultra-cold atoms in periodically shaken double-well lattices

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1. Introduction

Optical lattices are an important system for research with ultra-cold atoms [2–4]. Experimental developments enable the creation of lattices of controllable double-well potentials [5, 6] that can be engineered such that the tunneling between neighboring double wells can be discarded [7], allowing treatment of the system as a single double well. Loading the lattice from a Mott-insulator state allows deterministic population of fewer than six atoms in each well [1, 8]. Combined with the ability to count the atoms in each well, this makes the double well system ideal for investigating fractional photon-assisted tunneling via periodic shaking of the lattice that is typically a small effect in other systems [9–11]. The “photons” are time-dependent potential modulations in the kilohertz regime; a sketch of the 1/4-photon resonance can be seen in the title figure.

Research on periodic shaking has focused on effects ranging from destruction of tunneling [12–14] and dynamic localization [15] over tunneling control [16–18] to field-induced barrier transparency [19], two-dimensional solitons [20], super Bloch oscillations [21–23], phase-jumps [24, 25] and dynamics of bound pairs in optical lattices [26, 27], or NOON-states [28].

Complementary studies of Bose-Einstein condensates (BECs) loaded into double well lattices include investigations into controlled transport of BECs between two
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H = \{1, 45, 46\} body Hamiltonian with a two-mode approximation as follows \[H = \frac{\hbar \Omega}{2} \left( \hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1 \right) + \hbar \kappa \left( \hat{c}_1^\dagger \hat{c}_1 \hat{c}_1^\dagger \hat{c}_1 + \hat{c}_2^\dagger \hat{c}_2 \hat{c}_2^\dagger \hat{c}_2 \right) + \hbar \mu \left( \mu_0 + \mu_1 \sin(\omega t) \right) \left( \hat{c}_2 \hat{c}_2 - \hat{c}_1 \hat{c}_1 \right).\]

The operators \(\hat{c}_j / \hat{c}^\dagger_j\) annihilate/create a boson in well \(j\); \(\hbar \Omega\) is the tunneling splitting, \(\hbar \mu\) denotes the tilt between well 1 and well 2 and \(\hbar \mu_1\) is the driving amplitude. The on-site pair interaction is denoted by \(2 \hbar \kappa\). For calculations beyond this model see, e.g., [47–50]. Here, we use the two-mode approximation as the experiment [33] demonstrates that this approximation describes the physics of the 1/2-photon resonance. Our focus lies in identifying and understanding interesting signatures of photon-assisted tunneling rather than quantitative predictions; calculations including effects of higher energy levels (cf. [33]) will subsequently depend on precise experimental details like the depth of the lattice.

In order to characterize the photon-assisted tunneling, we use the experimentally measurable time-averaged particle transfer probability,

\[
\langle P_{\text{trans}} \rangle_T = \frac{1}{NT} \int_0^T \langle \Psi(t) | \hat{c}_1^\dagger \hat{c}_2 | \Psi(t) \rangle dt.
\]

2. Model

2.1. Hamiltonian

For both the case of a few ultra-cold atoms, or a small BEC loaded into the double-well potential modulated at frequency \(\omega\), the system can be described using a many-body Hamiltonian with a two-mode approximation as follows \([1, 45, 46]\)

\[
\hat{H} = \frac{\hbar \Omega}{2} \left( \hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1 \right) + \hbar \kappa \left( \hat{c}_1^\dagger \hat{c}_1 \hat{c}_1^\dagger \hat{c}_1 + \hat{c}_2^\dagger \hat{c}_2 \hat{c}_2^\dagger \hat{c}_2 \right) + \hbar \mu \left( \mu_0 + \mu_1 \sin(\omega t) \right) \left( \hat{c}_2 \hat{c}_2 - \hat{c}_1 \hat{c}_1 \right)
\]

which is just one of the tools used to predict new phenomena related to a 1/2-photon resonance. This has subsequently been predicted to be a small effect \([9–11]\). Reference [43] introduced and thoroughly tested a simplified model to understand the tunneling process. In preliminary investigations, this was applied to predicting co-tunneling of two particles being a large effect for parameters which relate to a 1/2-photon resonance. This has subsequently been confirmed experimentally [33]. A related experiment on photon-assisted tunneling of strongly correlated atoms can be found in [44]. In the present letter, the method of [43], combined with time-dependent perturbation theory, is just one of the tools used to predict new phenomena for fractional photon-assisted tunneling like co-tunneling of more than two particles.

The letter is organized as follows: Sec. 2 introduces the model describing atoms in the periodically shaken lattice, whilst Sec. 3 explores the effect for both few and hundreds of atoms per well. In Sec. 4 we demonstrate that for four particles, both the 1/4- and the 1/2-resonance will provide clear experimental signatures. Sec. 5 explains a feature in the photon-assisted tunneling plot reminiscent of avoided crossings. In Sec. 6 we show that for some parameters, even though the first few orders of time-dependent perturbation theory might be small, higher order perturbation theory can still correctly predict the fact that fractional photon-assisted tunneling is a large effect.

\[i \hbar \hat{a}_\nu(t) = \langle \nu | H_1 | \nu + 1 \rangle a_{\nu + 1}(t) + \langle \nu | H_1 | \nu - 1 \rangle a_{\nu - 1}(t), \quad \nu = 0, 1, \ldots, N,\]

which uses the notation \(a_{\nu - 1} \equiv a_{N + 1} \equiv 0\); the phase factors read:

\[
h_{\nu}(t) = \exp \left\{ i \left[ 2(N - 1 - 2\nu) \kappa t - 2\mu_0 t + \frac{2\mu_1 \cos(\omega t)}{\omega} \right] \right\}
\]
2.2. Time-dependent perturbation theory

In the following, we always use the experimentally realistic initial condition that all the atoms are in the lower well at $t=0$, i.e. $a_0(0)=1$ [1, 8]. Then at a later time $t$, zeroth-order time-dependent perturbation theory gives the Fock-state amplitudes as:

$$a_0^{(0)}(t) \equiv 1,$$  \hspace{1cm} (7)

$$a_j^{(0)}(t) \equiv 0, \quad j > 0.$$  \hspace{1cm} (8)

The first non-zero order of $a_j^{(k)}$ is obtained for $k = j$:

$$a_j^{(j)}(t) \equiv \sqrt{\frac{N-j+1}{2}} \Omega \times$$

$$\times \int_0^t d\tau h_{j-1}(\tau)^* a_{j-1}^{(j-1)}(\tau), \quad j \geq 1,$$  \hspace{1cm} (9)

where $^*$ denotes the complex conjugate.

2.3. Tunneling dynamics

To understand the tunneling dynamics, we expand the oscillatory term at frequency $\omega$ in the phase factors of Eq. (9) in terms of Bessel functions [51]

$$e^{i \omega t} = \sum_{k=-\infty}^{\infty} J_k(z) e^{i k \omega t}.$$  \hspace{1cm} (10)

Including all these terms in analytic calculations does, in principle, lead to analytic results for the tunneling. However, evaluating these analytic formulae is numerically much more intensive than solving the time-dependent Schrödinger equation [10]. Combining the rotating-wave-approximation based approach, which includes only the slowly oscillating terms of $h_j$ [43] with the above time-dependent perturbation theory leads to a simpler form for the non-zero perturbations of Eq. (9):

$$a_j^{(j)}(t) \equiv \begin{cases} \frac{1}{2} \sqrt{\frac{N-j+1}{2}} \Omega J_k(z) \left(\frac{2 \mu_j}{\omega}\right) \times \right. \\
\int_0^t d\tau \exp \{i \eta_k^{(j)} \} a_{j-1}^{(j-1)}(\tau), \quad j \geq 1,
\end{cases}$$  \hspace{1cm} (11)

where the integer $k_j$ is chosen such that $|\eta_k|$, with

$$\eta_k^{(j)} \equiv -k \omega + 2 \mu_0 - 2 \left|N - (2j - 1)\right| \kappa,$$  \hspace{1cm} (12)

is minimized at $k = k_j$. As in [43], it might sometimes be preferable to minimize $| \sum_j |\eta_k^{(j)}| \right|$ rather than each $\eta_k^{(j)}$ separately. For other cases (e.g. near a zero of one of the Bessel functions) more than one term will have to be included in the above sums. We define the number of “photons” involved in the tunneling process,

$$\#_{\text{photons}} = \sum_{j=1}^{N} k_j,$$  \hspace{1cm} (13)

such that it corresponds to the total “energy” transferred (rather than taking the sum of the moduli).

3. BEC or few particles per double well?

In order to demonstrate why it is necessary to use a few atoms per double well as opposed to a BEC to observe fractional resonances, we compare the results obtained for $N = 100$ particles with $N = 4$. Fig. 1 shows the time averaged particle transfer probability for $N = 100$ atoms initially in the lower well. While there are no resonances visible at higher frequencies, the one-photon-resonance which starts at $\omega = 3 \Omega$ for small interactions is clearly visible. For a BEC initially in the upper well, the one-photon resonance would move toward higher frequencies for increasing interaction.

Fig. 2 shows the time-averaged transfer probability for $N = 4$ calculated using the same parameters as Fig. 1 with...
many-particle tunneling. For larger particle numbers, similar features will only be visible on even larger time-scales (cf. Sec. 6). In Fig. 3, both the 1/2-photon and the 1/4-photon resonance are visible for a broad range of interaction strengths.

Integer-photon resonances essentially are single-particle effects which survive interactions. Fractional photon assisted tunneling, however, is a true many-particle quantum effect. For four particles, one expects [11] to observe the 1/2- and the 1/4-resonance, both of which are clearly visible in Fig. 3. Odd fractions like the 1/3-photon resonance (cf. Sec. 6) only occur for odd particle numbers \( N \geq 3 \) (cf. [11]). Fractional resonances also appear for the case of \( N = 100 \) (cf. [9]). However, for the experimentally motivated comparatively short timescales used both here and in [9], fractional photon-assisted tunneling is a very small effect even for small BECs. We therefore focus on the case of few-atoms per well for the remainder of the paper.

### 4. The 1/2- and 1/4- resonance

For fractional resonances in a double well lattice with few atoms per double well, most of the physics of the tunneling process can be understood by taking the approach of Sec. 2.3 to a level beyond perturbation theory. Within the approximation motivated by the rotating wave-approximation, this leads to time-dependent \((N+1) \times (N+1)\) matrices, which can, in some cases, even be solved analytically (cf. [43]).

For \( N = 4 \), the analytic calculations for the 1/2-photon resonance of [43] can be extended by calculating all the eigenvalues and eigenvectors of the resulting 5×5-matrix. After projecting the time-dependent solution of the simplified Schrödinger equation to

\[
\langle \psi(t) \mid i \rangle = |a_i(t)|^2,
\]

the time-dependent transfer \( P_{\text{trans}}(t) \) is expressed in terms of the amplitudes \( a_i(t) \) as

\[
P_{\text{trans}}(t) = \frac{1}{4} \left[ 4|a_4(t)|^2 + 3|a_3(t)|^2 + 2|a_2(t)|^2 + |a_1(t)|^2 \right].
\]

Fig. 4 shows that the 1/2-photon resonance will be clearly visible for four particles. As the full width at half maximum is an order of magnitude larger than 1% (the typical error [52], with which the interaction can be fixed in experiments as [1]), the 1/2-photon resonance could be observed with the existing experimental setup of [1, 33].

Given the fact that the 1/2-photon resonance has already been observed experimentally [33], it would be even more interesting to investigate the 1/4-photon resonance as \( N = 4 \) is the lowest number of particles for which it can be observed. Just because 4 particles produce a large effect in

Figure 2 (online color at www.lphys.org) Time-averaged transfer probability to well 2 for \( N = 4 \) calculated using the same parameters as in Fig. 1 but with an averaging time of \( \Omega T = 10 \). Experimentally, this could be realized in the double-well lattice of [1, 8]

Figure 3 (online color at www.lphys.org) Time-averaged transfer calculated for the same parameters as Fig. 2 except for \( \Omega T = 100 \). While the 1/2-photon resonance is clearly visible near \( \omega = 6\Omega \), the 1/4-resonance can also be found for some interactions near \( \omega = 12\Omega \)

an averaging time of \( \Omega T = 10 \). For this shorter averaging time-scale, Fig. 2 displays many features also seen in Fig. 1. The visible lines are either tunneling resonances which can be understood on the single-particle level like the 1-photon resonance near \( \kappa \approx 0 \) and \( \omega \approx 3\Omega \) or the horizontal lines. These correspond to, e.g., the energy of all particles being in the lower well and one particle having tunnelled being equal. The straight lines with non-zero gradient correspond to adding one or several photons to those horizontal lines.

More interesting features, including fractional photon resonances, emerge for larger averaging times (Fig. 3). While the short-time effects visible in Fig. 2 can be explained by simply looking at tunneling of a single particle, Fig. 3 displays many features, which are due to
Figure 4 (online color at www.lphys.org) 1/2-photon resonance for four particles; displayed is the transfer to the second well at time $t^* = 13.115$ as a function of the scaled interaction parameter $\kappa/\Omega$. Dotted/magenta: model on which Eq. (14) is based, dashed/blue: full numerics. The parameters are: $\omega = 6\Omega$, $\mu_0 = 1.5\Omega$, $2\mu_1/\omega = 4.567$. As the full width at half maximum is an order of magnitude larger than the typical experimental accuracy for $\kappa/\Omega$ [52], the 1/2-photon resonance could be observed with existing experimental setups. Choosing the correct time will also be feasible: for $\kappa = 1.5\Omega$, numerics shows that missing the time $t^*$ by 10% only leads to deviations of $P_{\text{trans}}(t^*)$ from $P_{\text{trans}}(t^*) \approx 0.92$ by less than 3%.

the numerics does, however, not automatically imply that it is a true 4-particle effect: many features which are already visible for the short averaging times of Fig. 2 would also occur at least similarly for lower particles. Thus, even if we restrict our search for a 1/4-resonance to parameters near $\omega = 12\Omega$, this could coincide with large tunneling for lower particle numbers. Due to the experimental way to load double-well lattices via a Mott-insulator [33], the harmonic confinement will prevent experiments with all double-wells being filled with exactly 4 particles. However, as can be clearly seen from Fig. 5, only the wells initially loaded with 4 atoms can contribute to the observable signature of the 1/4-photon resonance.

5. Avoided-crossing-type features

Fig. 3 shows several features which resemble avoided crossings. The smallest particle number for which they can occur is $N = 2$, for which the feature is particularly strong at the 1/2-photon frequency near zeros of the Bessel function responsible for the tunneling of the first particle. Such a situation is depicted in Fig. 6. For parameters near the center of this figure, i.e. $\mu_0 = 1.5\Omega$, $\omega = 6\Omega$ and $\kappa = 1.5\Omega$, the tunneling of the first particle would normally be described by a 0-photon process while the tunneling of the second particle would be a 1-photon process. Only the average number of photons per tunneling process justifies the word “1/2-photon resonance”. If, however, $J_0(2\mu_1/\omega) = 0$ (as chosen for Fig. 6), we have an entirely different situation. Now the tunneling of the first particle consists of two competing processes: a 1-photon process and a $-1$-photon process, making the overall tunneling a superposi-
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This shows in the numerics of Fig. 6.

As \( J_{-1}(x) = -J_1(x) \), we have:

\[
a_1^{(1)} = \frac{\Omega}{\sqrt{2}} J_1 \left[ \frac{2\mu_1}{\omega} \right] \left[ \exp \left( \frac{i\eta_1^{(1)}}{\omega} t \right) + \exp \left( \frac{-i\eta_1^{(1)}}{\omega} t \right) \right] - \frac{\Omega}{\sqrt{2}} J_1 \left( \frac{2\mu_1}{\omega} \right) \left[ \frac{1}{i\eta_1^{(1)}} + \frac{1}{-i\eta_1^{(1)}} \right].
\]

This equation already explains why there is no transfer visible at \( \mu_0 = 1.5\Omega, \omega = 6\Omega, \) and \( \kappa = 1.5\omega \), as

\[
\eta_1^{(1)} = -\eta_1^{(1)}.
\]

For these parameters, in the vicinity of this point, the time-independent part of Eq. (15) will be responsible for \( a_2 \) eventually becoming larger, in particular, for parameters for which \( \eta_1^{(2)} = 0 \):

\[
a_2^{(2)} \sim \frac{\Omega^2}{2} J_2 \left( \frac{2\mu_1}{\omega} \right) \left[ \frac{1}{i\eta_1^{(1)}} + \frac{1}{-i\eta_1^{(1)}} \right] t.
\]

Combined with the fact that \( \eta_1^{(2)} = 0 \) corresponds to the line

\[
\kappa = \omega - 3\Omega,
\]

this explains why transfer becomes large when following this line away from the point \( (\omega = 6\Omega, \kappa = 1.5\Omega) \); for the line perpendicular to this line through the same point such increased transfer is neither to be expected nor does it show in the numerics of Fig. 6.

6. 1/\(N\)-resonance: large effect only in \(N\)th order perturbation theory

The system investigated here offers the unique possibility to construct physically relevant examples for which the perturbation theory is small up to \((N-1)\)th order while the \(N\)th order produces results that dominate the dynamics.

If

\[
\eta_{k_j}^{(1)} \neq 0
\]

for all \( j = 1, \ldots, N \) and furthermore\(^1\)

\[
\sum_{\mu = \ell}^j \eta_{j,\mu}^{(e)} \neq 0 : \ell < j < N, \quad 0 : \ell = 1, j = N,
\]

perturbation theory will give only (small) oscillatory terms up to \((N-1)\)th order. However, in \(N\)th order we obtain a term linear in time proportional to:

\[
\left[ a_N^{(N)}(t) \right]_{\text{leading}} \propto \left[ \prod_{j=1}^N J_{k_j} \left( \frac{2\mu_1}{\omega} \right) \right] t.
\]

However, just because Eq. (21) becomes large this does not automatically imply that this can be observed in the numerics: as soon as \( a_N^{(N)}(t) \) becomes large, \( a_{N-1}(t) \) changes which in turn influences \( a_N(t) \). Nevertheless, there are examples for which Eq. (21) well describes the tunneling.

For \( N = 3 \), the above criteria are fulfilled, e.g., for \( \mu_0 = 1.5\Omega, \kappa = 2.1\Omega, \omega = 9\Omega, \) and \( 2\mu_1/\omega = 0.75 \):

\[
\eta_0^{(1)} = -5.4\Omega, \quad \eta_0^{(2)} = 3\Omega, \quad \eta_0^{(3)} = 2.4\Omega.
\]

The tunneling can thus be understood to be a 1-photon-process, the photon being responsible for the tunneling of the third particle with no photons being involved in the tunneling of the other two. Thus:

\[
\left[ a_3^{(3)}(t) \right]_{\text{leading}} = \frac{3\Omega^3}{4\eta_0^{(1)} \left( \eta_0^{(1)} + \eta_0^{(2)} \right)} J^2 \left( \frac{2\mu_1}{\omega} \right) J_1 \left( \frac{2\mu_1}{\omega} \right) t.
\]

Fig. 7 shows that Eq. (22) well describes the timescale, on which the probability for all particles having tunneled to the other well becomes large. The figure furthermore shows that the probability to find either one or two

\(^1\) The above conditions are necessary to construct examples for which \(N\)th order perturbation theory is the first to be large. It is, however, possible to find parameters for which the 1/\(N\)-resonances do not meet all those conditions (cf. Fig. 6). Furthermore, \( |a_N^{(N)}| \) being large is, in general, neither a necessary nor a sufficient condition for \( |a_N| \) to be large.
Figure 7 (online color at www.lphys.org) Probability to find \( n \) particles in the upper well if initially all \( N = 3 \) particles are in the lower well for \( \mu_0 = 1.5 \Omega, \kappa = 2.1 \Omega, \omega = 9 \Omega, \) and \( 2\mu_1/\omega = 0.75 \). (a) – black solid line: numerics using the model (1); magenta/grey dotted line: as predicted by numerically evaluated perturbation theory by adding \( |a_1^{(1)}|^2 + |a_2^{(2)}|^2 \) as defined in Eq. (9). The sum is much smaller than the probability to transfer all 3 particles which is displayed in the lower panel. (b) – solid black line: numerics using the model (1), dotted magenta/grey line: \( |a_3^{(3)}|^2 \) as defined in Eq. (9), and dashed blue/black line: analytic estimate of Eq. (22).

Figure 8 (online color at www.lphys.org) Probability to find \( n \) particles in the upper well if initially all \( N = 4 \) particles are in the lower well for \( \mu_0 = 1.5 \Omega, \omega = 12 \Omega, \kappa = 1.95 \Omega, \) and \( 2\mu_1/\omega = 0.5 \). (a) – black line: numerics using the model (1); magenta/grey line: as predicted by numerically evaluated perturbation theory by adding \( |a_1^{(1)}|^2 + |a_2^{(2)}|^2 + |a_3^{(3)}|^2 \) as defined in Eq. (9). The sum is much smaller than the probability to transfer all 3 particles which is displayed in the lower panel. (b) – from bottom to top: numerics using the model (1), analytic estimate of Eq. (23), numeric evaluation of \( |a_4^{(4)}|^2 \) as defined in Eq. (9). The upper two curves lie very close together.

changing the driving amplitude might change which is the dominating contribution.

For \( N = 4 \) similar parameters can be found. Using \( \mu_0 = 1.5 \Omega, \omega = 12 \Omega, \kappa = 1.95 \Omega, \) and \( 2\mu_1/\omega = 0.5 \), one has as the leading order contribution:

\[
\begin{align*}
\eta_0^{(1)} &= -8.7 \Omega, & \eta_0^{(2)} &= -0.9 \Omega, \\
\eta_0^{(3)} &= 6.9 \Omega, & \eta_1^{(4)} &= 2.7 \Omega,
\end{align*}
\]
with
\[
\left| \begin{align*}
(a^{(4)}_4(t))_{\text{leading}} &= \left( \frac{3\Omega^4 J^3_0 \left( \frac{2\mu u}{\omega} \right) J_1 \left( \frac{2\mu u}{\omega} \right)}{2} \right) \left( \eta_0^{(1)} + \eta_0^{(2)} \right) \left( \eta_0^{(1)} + \eta_0^{(2)} + \eta_0^{(3)} \right) \right|^t.
\end{align*} \right.
\]

This analytic function correctly predicts that the tunneling of all 4 particles at once takes place at a much longer time-scale than for three particles (Fig. 8). As for the previous figure, the tunneling can be labeled a co-tunneling process, now of four particles.

7. Conclusion

We demonstrate that ultra-cold atoms loaded in a double-well lattice are preferable to small BECs in a double well for the observation of fractional photon-assisted tunneling. After preparing identical initial states in all double wells of the superlattice, state-of-the-art experiments enable the average atom number in one of the wells to be determined [1,33]. As all of our results show strong signatures in the average number of atoms per lattice site, our predictions can be tested using existing experiment setups, for example the 1/2-photon resonance has already been observed [1]. Our main results can be summarized as follows:

Firstly, we predict that both the 1/3-photon resonance and the 1/4-photon resonance should be observable experimentally. For those predictions we use both numeric simulations of the many-particle Hamiltonian in two-mode approximation and an analytic model.

Secondly, by using second-order perturbation theory, we give a physical explanation of the avoided-crossing type features observable for photon-assisted tunneling.

Thirdly, we predict that co-tunneling of both 3 and 4 particles is observable in double-well lattices. For \( N \) particles, co-tunneling implies that although all orders perturbation theory below the \( N \)th order are small, the \( N \)th order predicts a large effect. For even larger particle numbers than investigated here, both the time-scales for the transfer to the upper well becomes large and the model (1) becomes less valid. However, co-tunneling will still be observable experimentally for larger particle numbers than investigated so far [33].

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