HOMOTOPY G-ALGEBRAS AND MODULI SPACE OPERAD

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Abstract. This paper emphasizes the ubiquitous role of moduli spaces of algebraic curves in associative algebra and algebraic topology. The main results are: (1) the space of an operad with multiplication is a homotopy G- (i.e., homotopy graded Poisson) algebra; (2) the singular cochain complex is naturally an operad; (3) the operad of decorated moduli spaces acts naturally on the de Rham complex $\Omega^\bullet X$ of a Kähler manifold $X$, thereby yielding the most general type of homotopy G-algebra structure on $\Omega^\bullet X$.

Recently, shortly after enunciation of a conjecture of Deligne [4], the structure of a homotopy G- (i.e., homotopy graded Poisson) algebra on the Hochschild cochain space $V = C^\bullet(A, A)$ of an associative algebra was discovered [8, 6]. It has also been pointed out in [8] that a similar structure exists for the singular cochain complex $V = C^\bullet X$ of a topological space, due to Baues [2].

In this paper, we find a very general pattern which works for these two examples: in both cases, $V$ has a natural structure of an operad. Together with a multiplication, it yields the structure of a homotopy G-algebra on $V$, see Sections 1 and 2.

The rest of the paper is dedicated to the geometry of the conjecture, which, in fact, assumed something more than mere algebraic structure:

Conjecture (Deligne). The Hochschild cochain complex has a natural structure of an algebra over a chain operad of the little squares operad.

In Section 4.2, we use the construction of Gromov-Witten invariants to propose a way of proving a result analogous to the conjecture in the case of a singular cochain complex: the singular chain operad of decorated moduli spaces of genus zero algebraic curves acts naturally on the de Rham complex $\Omega^\bullet X$ of a Kähler manifold $X$, thereby yielding the most general type of homotopy G-algebra structure on $\Omega^\bullet X$.

Acknowledgment. We thank J. Stasheff for helpful comments on an early version of the manuscript. The second author would like to thank

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Research of the first author was supported in part by an NSA grant.
Research of the second author was supported in part by an NSF grant.
1. Homotopy G-algebra structure on an operad

1.1. The brace structure on an operad. Let \( \{O(n) \mid n \geq 1\} \) be an operad of vector spaces. Such an object relates to the collection of spaces \( \text{Hom}(V^\otimes n, V) \), \( n \geq 1 \), of all \( n \)-ary operations on a vector space \( V \) as much as an abstract group relates to the group \( \text{GL}(V) \) of linear transformations. The structure of an operad on a collection of spaces \( O(n) \) consists in a number of operations \( \gamma : O(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \to O(n_1 + \cdots + n_k) \) satisfying certain natural associativity axioms. These axioms can be read off from the endomorphism operad \( \text{End}_V(n) := \text{Hom}(V^\otimes n, V) \), where \( \gamma \) is the substitution of the values of \( k \) operations in a \( k \)-nary operation as inputs. The precise definition can also be read off from many recent papers on operads, for example, [8, 9, 11]. In addition, one usually assumes an action of the symmetric group \( S_n \) on each \( O(n) \) and a unit element \( \text{id} \in O(1) \) modeled after the action of \( S_n \) by permuting the inputs in \( \text{Hom}(V^\otimes n, V) \) and the identity operator in \( \text{Hom}(V, V) \), respectively. We will not assume an \( S_n \) action here, i.e., work with non-\( \Sigma \) operads.

Consider the graded vector space \( O = \bigoplus_n O(n) \), the sum of all components of the operad, now regarded as the graded components of \( O \). Denote by \( \deg x \) the degree of an element \( x \in O \) and by \( |x| \) the degree in the desuspension \( O[1] \) of the graded vector space \( O \), i.e., \( \deg x = n, |x| = n - 1 \), whenever \( x \in O(n) \). Define the following collection of multilinear operations, braces on \( O \):

\[
x\{x_1, \ldots, x_n\} := \sum (-1)^\varepsilon \gamma(x; \text{id}, \ldots, \text{id}, x_1, \text{id}, \ldots, x_n, \text{id}, \ldots, \text{id})
\]

for \( x, x_1, \ldots, x_n \in O \), where the summation runs over all possible substitutions of \( x_1, \ldots, x_n \) into \( x \) in the prescribed order and \( \varepsilon := \sum_{p=1}^n |x_p| i_p \) being the total number of inputs of \( \gamma(x; \text{id}, \ldots, \text{id}, x_1, \ldots, \text{id}, x_n, \ldots, \text{id}) \) in front of \( x_p \). The braces \( x\{x_1, \ldots, x_n\} \) are homogeneous of degree \(-n\), i.e.,

\[
\deg x\{x_1, \ldots, x_n\} = \deg x + \deg x_1 + \cdots + \deg x_n - n.
\]

We will also assume the following conventions:

\[
x\{} := x, \quad x \circ y := x\{y\}.
\]

Remark. The desuspension grading and sign are motivated by the example \( \text{End}_V[1] \) of the endomorphism operad of the desuspension \( V[1] \),
where the vector space $V$ is regarded as a graded vector space concentrated in degree 0. In this case, the sign $(-1)^\varepsilon$ is picked up by rearranging the sequence of letters $x\{x_1, \ldots, x_n\}(v_1, \ldots, v_m)$, where $v_1, \ldots, v_m \in V[1]$, $m$ is such that $x\{x_1, \ldots, x_n\} \in \text{Hom}(V[1] \otimes m, V[1])$, into the sequence
\[
\gamma(x; v_1, \ldots, v_1, x_1(v_1+1), \ldots, v_i, x_n(v_n+1), \ldots, v_m)
\]
in accordance with the usual sign convention.

One can immediately check the following identities:

(2) \[ x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} = \sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} (-1)^\varepsilon x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_m}, x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n\}, \]
where $\varepsilon := \sum_{p=1}^m |x_p| \sum_{q=1}^{i_p} |y_q|$, i.e., the sign is picked up by the $x_i$’s passing through the $y_j$’s in the shuffle.

Remark. The identity for $m = n = 1$ implies that the degree $-1$ bracket
\[ [x, y] := x \circ y - (-1)^{|x||y|} y \circ x \]
defines the structure of a graded Lie algebra on $O$.

Definition 1. A brace algebra is a graded vector space with a collection of braces $x\{x_1, \ldots, x_n\}$ of degree $-n$ satisfying the identities (2).

Thus we have made the following observation.

Proposition 1. For every operad $O$ of vector spaces, the braces (1) define the natural structure of a brace algebra on the underlying graded vector space $O$.

1.2. Homotopy G-algebras. A multiplication on an operad $O$ is an element $m \in O(2)$ such that $m \circ m = 0$.

Proposition 2. (1) A multiplication on an operad is equivalent to a morphism $\mathcal{A}s \rightarrow O$ of operads, where $\mathcal{A}s$ is the associative operad, cf. [9].

(2) If $V$ is an algebra over an operad with multiplication, then $V$ is naturally an associative algebra.

(3) The product
\[ x \cdot y := (-1)^{\deg x} m\{x, y\} \]
of degree 0 and the differential
(5) \[ dx := [m, x] = m \circ x - (-1)^{|x|} x \circ m, \quad d^2 = 0, \quad \text{deg } d = 1, \]
define the structure of a differential graded (DG) associative algebra on \( O \).

Analogously, a multiplication on a brace algebra \( V = \bigoplus_n V^n \) is an element \( m \in V^2 \) such that \( m \circ m = 0 \). It also provides \( V \) with a DG algebra structure.

Amazingly, a multiplication generates a much richer algebraic structure on an operad, as the following theorem implies. Before making the statement, we should define the algebraic structure in question.

**Definition 2.** A homotopy G-algebra is a brace algebra \( V = \bigoplus_n V^n \) provided with a differential \( d \) of degree one and a dot product \( xy \) of degree 0 making \( V \) into a DG associative algebra. The dot product must satisfy the following compatibility identities:

\[
(x_1 \cdot x_2) \{ y_1, \ldots, y_n \} = \sum_{k=0}^{n} (-1)^{\varepsilon} x_1 \{ y_1, \ldots, y_k \} \cdot x_2 \{ y_{k+1}, \ldots, y_n \},
\]

where \( \varepsilon = |x_2| \sum_{p=1}^{k} |y_p| \), and

\[
\begin{align*}
d(x_1, \ldots, x_{n+1}) - (dx)\{x_1, \ldots, x_{n+1}\} & \\
& - (-1)^{|x|} \sum_{i=1}^{n+1} (-1)^{|x_1|+\cdots+|x_{i-1}|} x_1 \{ x_1, \ldots, dx_i, \ldots, x_{n+1} \} \\
& = (-1)^{|x||x_1|+1} x_1 \cdot x_2, \ldots, x_{n+1} \\
& + (-1)^{|x|} \sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_{i-1}|} x_1 \{ x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_{n+1} \} \\
& - x_2 \{ x_1, \ldots, x_n \} \cdot x_{n+1}
\end{align*}
\]

**Remark.** 1. Note that every homotopy G-algebra is in particular a DG Lie algebra with respect to the commutator \([3]\), which is a graded derivation of the dot product up to null-homotopy:

\[
[x, yz] - [x, y]z - (-1)^{|x||y|+1} y[x, z]
= (-1)^{|x|+|y|+1} (d(x \{ y, z \}) - (dx)\{ y, z \})
- (-1)^{|x|} x \{ dy, z \} - (-1)^{|x||y|} x \{ y, dz \}).
\]

Moreover, the multiplication is always homotopy graded commutative:

\[
(9) \quad xy - (-1)^{(|x|+1)(|y|+1)} yx = (-1)^{|x|}(d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy).
\]
2. Since a brace algebra $V$ with multiplication is a DG algebra, one can define its Hochschild cochain complex $C^\bullet(V,V)$ as usual, with the differential

$$Df := d \circ f - (-1)^{|f|} f \circ d + m \circ f - (-1)^{|f|} f \circ m$$

and the cup product

$$f_1 \cup f_2 := (-1)^{|f_1|+1} m\{f_1, f_2\},$$

the degree $|f|$ meaning the desuspended total degree. (Usually, the cup product has a different sign.) Then the relations (6) and (7) mean that the correspondence

$$V \to C^\bullet(V,V),$$

$$x \mapsto \sum_{n=0}^{\infty} x\{x_1, \ldots , x_n\},$$

the summation being in fact finite, is a morphism of DG algebras.

**Theorem 3.** A multiplication on an operad $\mathcal{O}(n)$ defines the structure of a homotopy $G$-algebra on $\mathcal{O} = \oplus \mathcal{O}(n)$. A multiplication on a brace algebra is equivalent to the structure of a homotopy $G$-algebra on it.

**Proof.** The differential, the dot product and the braces have already been defined. What remains is to check the compatibility identities. In view of (4) and (5), they both are particular cases of (2). $\square$

### 2. Applications: Hochschild cochains and singular cochains

#### 2.1. Hochschild complex

Applying this theorem to the Hochschild cochain complex $C^\bullet(A,A)$, which is at the same time the total space $\mathcal{E}nd_A$ of the endomorphism operad $\mathcal{E}nd_A(n) = C^n(A,A) = \text{Hom}(A^\otimes n, A)$ as in Section 1.1 and whose multiplication cocycle $m(a,b) := ab$, $m \in C^2(A,A)$, is a multiplication on this operad, we obtain the following result conjectured by Deligne [4] and proved in [6, 8].

**Corollary 4.** The Hochschild complex $C^\bullet(A,A)$ of an associative algebra $A$ has a natural structure of homotopy $G$-algebra.

It is clear that the product obtained this way is the usual cup product

$$ (x \cup y)(a_1, \ldots , a_{k+l}) = x(a_1, \ldots , a_k)y(a_{k+1}, \ldots , a_{k+l}) $$

altered by the sign $(-1)^{|x|+1}|y|+1)$. The bracket $[z,z]$ plays the role of a primary obstruction in deformation theory. It was introduced by Gerstenhaber in [5]. The higher braces for the Hochschild complex have been introduced in Kadeishvili and Getzler’s works [10, 7], where the
brackets are used to define the Hochschild cohomology of a homotopy associative algebra.

2.2. $G$-algebras. The structure inherited by the Hochschild cohomology was introduced in [5] and has been discovered in a number of places in mathematics and physics since then. A $G$-algebra is a graded vector space $H$ with a dot product $xy$ defining the structure of a graded commutative algebra and with a bracket $[x,y]$ of degree $-1$ defining the structure of a graded Lie algebra, such that the bracket with an element is a derivation of the dot product:

$$[x, yz] = [x, y]z + (-1)^{|x||y|+1} y[x, z].$$

In other words, a $G$-algebra is a specific graded version of a Poisson algebra.

**Corollary 5.** The dot product and the bracket

$$[x, y] := x \circ y - (-1)^{|x||y|} y \circ x$$

define the structure of a $G$-algebra on the Hochschild cohomology $H^\bullet(A,A)$ of an associative algebra $A$.

**Proof.** A simple computation shows that the identity (1) yields the Jacobi identity for the bracket. Equation (9) implies that the differential is a derivation of the bracket:

$$d[x, y] - [dx, y] - (-1)^{|x|} [x, dy] = 0.$$

Therefore, even before passing to cohomology, the Hochschild complex forms a DG Lie algebra with respect to the bracket and a DG associative algebra with respect to the dot product.

Thus, we will be through if we see that

(i) the two operations take cocycles into cocycles and are independent of the choice of representatives of cohomology classes, and up to coboundaries,

(ii) the dot product is graded commutative and

(iii) the bracket is a derivation of the dot product.

It easy to observe Fact (i) from the definition of the dot product and the differential in any operad with multiplication (see [4] and [5]) and from the fact that $d$ is a derivation of the bracket (11). Facts (ii) and (iii) have already been mentioned — see Equations (8) and (9). □

2.3. Singular cochain complex. Let $C^\bullet X$ be the singular cochain complex of a topological space (or a simplicial set) $X$. For an $n$-simplex $\sigma : \Delta(n) \to X$ with $\Delta(n)$ being the standard $n$-simplex, let $\sigma(n_0,\ldots,n_k)$ denote its face spanned by the vertices $n_0,\ldots,n_k$, where
\(i \mapsto n_i\) is an injective monotone function \(\Delta(k) \to \Delta(n)\). The singular cochain complex \(C^\bullet X\) has a natural operad structure, \(S(n) = C^n X\), defined by the compositions:

\[
\gamma : S(k) \otimes S(n_1) \otimes \ldots \otimes S(n_k) \to S(n_1 + \cdots + n_k),
\]

\[
\gamma(\varphi; \varphi_1, \ldots, \varphi_k)(\sigma) := \varphi(\sigma(0, n_1 + n_2, \ldots, n_1 + \cdots + n_k))
\]

\[
\varphi_1(\sigma(0, 1, \ldots, n_1))\varphi_2(\sigma(n_1, \ldots, n_1 + n_2)) \ldots \\
\varphi_k(\sigma(n_1 + \cdots + n_{k-1}, \ldots, n_1 + \cdots + n_k)).
\]

This automatically yields the structure of a brace algebra on \(C^\bullet X\), according to Proposition 1. Define the multiplication \(m \in C^2 X\) as

\[
m(\sigma) := 1 \quad \text{for any 2-simplex } \sigma.
\]

Then Theorem 3 immediately recovers the following statement, which is an interpretation of a result of Baues [2].

**Corollary 6.** The singular cochain complex \(C^\bullet X\) of a topological space \(X\) has a natural structure of homotopy \(G\)-algebra.

The dot product determined by the multiplication \(m\) as in (4) is nothing but the familiar cup product, up to the sign \((-1)^{|\varphi|+1}(|\psi|+1)\),

\[
(\varphi \cup \psi)(\sigma) = \sum_{k=0}^{n} \varphi(\sigma(0, \ldots, k))\psi(\sigma(k, \ldots, n))
\]

and the differential determined by (5) is merely the familiar coboundary operator, up to the sign \((-1)^{|\varphi|+1}\),

\[
(d\varphi)(\sigma) = \sum_{k=0}^{n+1} (-1)^k \varphi(\sigma(0, 1, \ldots, \hat{k}, \ldots, n + 1)).
\]

Moreover, the brace \(\varphi\{\psi\}\) is the Steenrod operation \(\varphi \cup_1 \psi\) and higher braces are multilinear generalizations of it.

### 3. Generalities

#### 3.1. \(A_\infty\) version.** Here we would like to consider \(A_\infty\) generalizations of our results in the spirit of [8]. An \(A_\infty\)-multiplication on an operad \(O\) (a brace algebra \(V\)) is a formal sum \(m = m_1 + m_2 + m_3 + \ldots\) with \(m_n \in \mathcal{O}(n)\) (or \(V^n\), such that \(m \circ m = 0\). In this case (5) is also a differential, but not homogeneous, \(d = d_0 + d_1 + \ldots\). As above, an \(A_\infty\)-multiplication on an operad \(O\) defines

1. a morphism \(A_\infty \to \mathcal{O}\) of operads, where \(A_\infty\) is the \(A_\infty\) (homotopy associative) operad, see [9],
(2) a natural structure of \( A_\infty \)-algebra on each algebra over the operad \( \mathcal{O} \),

(3) the structure of an \( A_\infty \)-algebra on \( \mathcal{O} \) itself with higher products \( M_n \) defined by the formula:

\[
M_n(x_1, \ldots, x_n) := m\{x_1, \ldots, x_n\}, \quad \text{for } n > 1,
\]

\[
M_1(x) := dx := m \circ x - (-1)^{|x|} x \circ m.
\]

For any \( A_\infty \)-algebra \( V \) the same formulas define the structure of an \( A_\infty \)-algebra on the Hochschild complex \( C^\bullet(V, V) \), see Getzler [7].

**Theorem 7.** An \( A_\infty \)-multiplication on an operad \( \mathcal{O}(n) \) defines the following \( A_\infty \) version of a homotopy G-algebra on \( V = \mathcal{O} = \oplus \mathcal{O}(n) \). It is a brace algebra \( V \) and an \( A_\infty \)-algebra at the same time, such that the correspondence

\[
V \to C^\bullet(V, V),
\]

\[
x \mapsto \sum_{n=0}^\infty x\{\ldots\},
\]

is a morphism of \( A_\infty \)-algebras.

### 3.2. Bar interpretation.

In this section we want to make a translation of the algebraic notions introduced above into the dual language of bar constructions, following ideas of Getzler-Jones' work [8]. For a graded vector space \( V = \bigoplus_n V_n \), let \( V[-1] = sV \), \( V[-1]_n := V_{n-1} \), be its suspension and

\[
BV = \bigoplus_{n=0}^\infty (V[-1])^\otimes n,
\]

the bar coalgebra with the usual coproduct

\[
\Delta[x_1 \ldots | x_n] = \sum_{i=0}^n [x_1 \ldots | x_i] \otimes [x_{i+1} \ldots | x_n],
\]

\([x_1 \ldots | x_k]\) denoting an element of \( V[-1]^\otimes k \subset BV \).

We call a product \( BV \otimes BV \overset{\cup}{\to} BV \) left nonincreasing if \( \deg(x \cup y) \geq \deg x \), where by definition \( \deg[x_1 \ldots | x_n] = n \).

**Lemma 8.** The structure of a brace algebra on a graded vector space \( V \) is equivalent to the structure of a bialgebra on the bar coalgebra \( BV \) defined by a left nonincreasing product.
Proof. A product $\cup$ on $BV$ compatible with the coproduct $\Delta$ and satisfying $\deg(x \cup y) \geq \deg x$ determines the braces uniquely by the formula
\[
[x_1 \ldots x_m] \cup [y_1 \ldots y_n] = \sum (-1)^e [y_1 \ldots y_i | x_1 \{y_{i+1}, \ldots \}] \ldots [y_m | x_m \{y_{m+1}, \ldots \}] \ldots [y_n],
\]
where the sign is the same as in (2). The associativity of the product is then equivalent to the relations (2).

Let $V$ be a brace algebra, $BV$ the corresponding bar bialgebra. A DG-bialgebra is a bialgebra with a degree $-1$ differential which is simultaneously a derivation and a coderivation.

**Lemma 9.** An $A_\infty$-multiplication on a brace algebra $V$ is equivalent to the structure of a DG-bialgebra on the bar bialgebra $BV$.

**Proof.** An $A_\infty$-multiplication $m$ on $V$ is equivalent to a codifferential $\delta$ on $BV$, as has been well-known since Stasheff [15]. That $\delta$ is a derivation is equivalent to the compatibility condition of Theorem 7. \qed

4. Topological and mirror reflections

4.1. Moduli spaces and little squares. As predicted by Deligne [4], the structure of a homotopy G-algebra on the Hochschild complex arises from an action of a chain complex of the little squares operad. The following combinatorial version of this statement was proved by Getzler and Jones. Consider Fox-Neuwirth’s cellular partition of the configuration spaces $F(n, \mathbb{R}^2)$ of $n$ distinct points in $\mathbb{R}^2$: cells are labeled by ordered partitions of the set $\{1, \ldots, n\}$ into subsets with orderings within each subset. This reflects grouping points lying on common vertical lines on the plane and ordering the points lexicographically. For each $n$, take the quotient cell complex $K_n \mathcal{M}(n)$ by the action of translations $\mathbb{R}^2$ and dilations $\mathbb{R}^*_+$. These quotient spaces do not form an operad, but one can glue lower $K_n \mathcal{M}(n)$’s to the boundaries of higher $K_n \mathcal{M}(n)$’s to form a cellular operad $K_\bullet \mathcal{M}(n)$. The resulting space $\overline{\mathcal{M}}(n)$ is a circle bundle over the real compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli space $\mathcal{M}_{0,n}$ of $n$-punctured curves of genus zero, see [1, 8, 11, 12]. The space $\overline{\mathcal{M}}(n)$ can be also interpreted as a “decorated” moduli space, see next section. Cells in this cellular operad $K_\bullet \overline{\mathcal{M}}(n)$ are enumerated by pairs $(T, p)$, where $T$ is a tree with $n$ initial vertices and one terminal vertex, labeling a component of the boundary of $\overline{\mathcal{M}}(n)$, and $p$ is a partition, as above, of the set $\text{in}(v)$ of incoming vertices for each vertex $v$ of the tree $T$. 

In [8], it is shown that a complex $V$ is an $A_{\infty}$ homotopy G-algebra, iff it is an algebra over the operad $K_{*}M(n)$ satisfying the following condition. The structure mappings
\[ K_{*}M(n) \to \text{Hom}(V^{\otimes n}, V), \]
of the algebra $V$ over the operad $K_{*}M(n)$ send all cells in $K_{*}M(n)$ to zero, except cells of two kinds:

1. $(\delta_n, ((i_1), (i_2, \ldots, i_n)))$, where $\delta_n$ is the corolla, the tree with one root and $n$ edges, connecting it to the remaining $n$ vertices, corresponding to the configuration where the points $i_2, \ldots, i_n$ sit on a vertical line, the $i_k$th point being below the $i_{k+1}$st, and the $i_1$st point is in the half-plane to the left of the line;
2. $(\delta_n, ((i_1, i_2, \ldots, i_n)))$, corresponding to the configuration where all the points sit on a single vertical line, the $i_k$th point being below the $i_{k+1}$st.

Cells of the first kind map to the braces $x_{i_1} \{ x_{i_2}, \ldots, x_{i_n} \}$, $n \geq 1$, and cells of the second kind map to the $A_{\infty}$ products $M_n(x_{i_1}, \ldots, x_{i_n})$, $n > 1$.

Thus, the conditions of Theorem 7 and, in particular, when $M_n = 0$ for $n > 2$ the relations (6) and (7), follow from the combinatorial structure of the cell complex.

This construction of Getzler and Jones is essentially combinatorial and the question of a natural topological construction, perhaps, similar to those which come from quantum field theory (cf. [11]), where algebraic operations are obtained by integration over cycles, remains a mystery. At least in known examples, we anticipate that other cells give rise to nonzero multilinear operations.

**Wish 10.** In all the examples, where a homotopy G-algebra structure occurs, e.g., the Hochschild complex or the singular cochain complex, the structure extends nontrivially to a natural structure of an algebra over the operad $K_{*}M(n)$.

This operad is closer to the homotopy G-operad in the sense of Ginzburg-Kapranov [9]. The latter operad is the operad cobar construction for the Koszul dual to the Gerstenhaber operad, which is the homology operad $H_{*}M(n)$, see [3, 8].

**4.2. The construction.** Here we are going to sketch how to make Wish [10] come true in case of the homotopy G-algebra $C^{*}X$. Our construction is a real version of Kontsevich’s construction of Gromov-Witten invariants in [13-14]. The difference is that we replace the moduli spaces $\overline{M}_{0,n+1}$, which are compact complex manifolds, with the
spaces $\mathcal{M}(n)$, which are circle bundles over the real compactifications $\mathcal{M}_{0,n+1}$, which are compact real manifolds with corners.

Let $(X, \omega)$ be a compact manifold with a sufficiently positive Kähler form $\omega$. We will replace singular cochains $C^\bullet X$ with smooth forms $V = \Omega^\bullet X$.

We want to define the natural structure of an algebra over the singular chain operad $C^\bullet \mathcal{M}(n)$ on $V$. In particular, this will yield the structure of an algebra over the cellular chain operad $K^\bullet \mathcal{M}(n)$ on $V$, giving a solution of Wish [10].

Let $\mathcal{M}(n)(X, \beta)$ be the moduli space of stable holomorphic maps $(C; p_1, \ldots, p_{n+1}; \tau_1, \ldots, \tau_m, \tau_\infty; \phi)$ from a (degenerated) curve $C$ of genus 0 to $X$:

$$\phi : C \to X$$

mapping the fundamental class of $C$ to a given homology class $\beta \in H_2(X, \mathbb{Z})$. Here the curve $C$ has $n + 1$ punctures $p_1, \ldots, p_{n+1}$ and all the singularities of $C$ must be $m$ double points. For each $i$, $1 \leq i \leq m$, $\tau_i$ is the choice of a tangent direction at the $i$th double point to the irreducible component that is farther away from the “root”, i.e., from the component of $C$ containing the puncture $\infty := p_{n+1}$. $\tau_\infty$ is a tangent direction at $\infty$. The stability of a map is understood in the sense of Kontsevich [13, 14]: each irreducible component of $C$ contracted to a point by $\phi$ must be stable, i.e., admit no infinitesimal automorphisms. Because of a finite group of automorphisms, the moduli space $\mathcal{M}(n)(X, \beta)$ is only a compact stack. In some cases, e.g., when $X$ is a homogeneous space, it is a smooth stack with corners. Let us assume it is one.

Notice that the configuration space $\mathcal{M}(n)$ considered in Section [11] is the same as the moduli space of data $(C; p_1, \ldots, p_{n+1}; \tau_1, \ldots, \tau_m, \tau_\infty)$ as above, except that all components of $C$ must be stable, cf. [8, 11]. The operad composition is given by attaching the $\infty$ punctures on curves to the other punctures on another curve, remembering the tangent direction at each new double point. Let

$$\pi : \mathcal{M}(n)(X, \beta) \to \mathcal{M}(n)$$

denote the forgetful map of the space $\mathcal{M}(n)(X, \beta)$ of maps to the space $\mathcal{M}(n)$ of curves.

There is a universal (evaluation) map

$$\Phi : C \to X$$

from the universal curve over $\mathcal{M}(n)(X, \beta)$ to the manifold $X$. The natural projection $C \to \mathcal{M}(n)(X, \beta)$ admits $n + 1$ canonical sections.
s_1, \ldots, s_{n+1}, sending a point of the moduli to the i-th puncture on the universal curve.

Now we are ready to define the structure of an algebra over the operad \( C_{\ast}M(n) \) on \( V = \Omega^\ast X \). Let \((\varphi, \psi) := \int_X \varphi \wedge \psi\) be the Poincaré pairing on \( V \). Leaving aside problems with pairings and duals for infinite dimensional vector spaces and replacing singular cochains with differential forms once again, it suffices to construct mappings

\[
(12) \quad f_n : V^\otimes n+1 \to \Omega^\ast M(n)
\]

which will define the structure of an algebra over the operad

\[
C_{\ast}M(n) \to \text{Hom}(V^\otimes n, V)
\]
after dualizing \( V \) with the help of the Poincaré pairing. (Honestly speaking, we would have to replace singular chains with currents then). We define the mapping \((12)\) by the formula

\[
f_n(\varphi_1, \ldots, \varphi_{n+1}) := \sum_{\beta \in H_2(X, \mathbb{Z})} \exp(-\int_\beta \omega) \pi_* (s_1^\ast \Phi^* \varphi_1 \wedge \cdots \wedge s_{n+1}^\ast \Phi^* \varphi_{n+1}),
\]

where \( \Phi^* \) and \( s_i^\ast \)'s denote pullbacks and \( \pi_* \) a push-forward (fiberwise integration). In fact, because of the summation over the lattice \( H_2(X, \mathbb{Z}) \), we have to replace the ground field \( \mathbb{C} \) with formal power series in \( \beta \in H_2(X, \mathbb{Z}) \).

Claim 11. The maps \((12)\) define a morphism of operads, that is, the structure of an algebra over the chain operad \( C_{\ast}M(n) \) on the de Rham complex \( V = \Omega^\ast X \).

As soon as some hard problems with the construction, such as the smoothness of the stack of stable maps, are solved, the verification of the operad properties of this claim is automatic. So, we postpone the proof of it until better times.

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