Abstract. There exists a $C^2$-open and $C^1$-dense subset of vector fields exhibiting singular-hyperbolic attracting sets (with codimension-two stable bundle), in any $d$-dimensional compact manifold ($d \geq 3$), which mix exponentially with respect to any physical/SRB invariant probability measure.

More precisely, we show that given any connected singular-hyperbolic attracting set for a $C^2$-vector field $X$, there exists a $C^1$-close multiple of $X$ of class $C^2$, generating a topologically equivalent flow, which is robustly exponentially mixing with respect to any physical measure for all vector fields in a $C^2$ neighborhood. That is, every singular-hyperbolic attracting set mixes exponentially with respect to its physical measures modulo an arbitrarily small change in the speed of the flow.

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1. Introduction

A central concept in Smooth Ergodic Theory is that of physical (or Sinai-Ruelle-Bowen) measure for a flow or a transformation. For a flow $X_t$ induced by a vector field $X$ on a compact manifold, such measure is an invariant probability measure $\mu$ for which the family of points $z$ satisfying

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \psi(X_s(z)) \, ds = \int \psi \, d\mu,$$

for all continuous observables (functions) $\psi$, is a positive Lebesgue (volume) measure subset $B(\mu)$ (the ergodic basin) of the ambient space. So these time averages are considered a priori physically observable.

These measures were first rigorously obtained for (uniformly) hyperbolic diffeomorphisms by Sinai, Ruelle and Bowen [35, 34, 17]. For non-uniformly hyperbolic transformations and flows these measures were studied more recently: we mention only [12, 25, 13] on the existence of physical measures for singular-hyperbolic attractors. Statistical properties of such measures are an active field of study: see e.g. [2, 23, 15, 6, 11, 10, 14]. The general motivation is that the family $\{\psi \circ X_t\}_{t \geq 0}$ should behave asymptotically as a family of independent and identically distributed random variables.

One of the features is the speed of convergence of the time average to the space average: regarding $\varphi$ and $\psi \circ X_t : M \to \mathbb{R}$ as random variables with law $\mu$, mixing means that the random variables $\varphi$ and $\psi \circ X_t$ are asymptotically independent: $\mathbb{E}(\varphi \cdot (\psi \circ X_t))$ converges to $\mathbb{E}(\varphi) \cdot \mathbb{E}(\psi)$ when $t \to +\infty$. More precisely, the correlation function

$$C^\mu(t, \varphi, \psi) = \mathbb{E}(\psi \cdot (\varphi \circ X_t)) - \mathbb{E}(\varphi) \cdot \mathbb{E}(\psi) = \int \psi \cdot (\varphi \circ X_t) \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu$$

satisfies $|C^\mu(t, \varphi, \psi)| \to 0$ for all integrable observables. Exponential mixing means that there exist $C, \gamma > 0$ so that

$$|C^\mu(t, \varphi, \psi)| \leq Ce^{-\gamma t} \|\psi\| \|\varphi\|, \quad t > 0,$$
on a Banach space of usually more regular observables than just integrable ones (mostly Hölder continuous, some times differentiable).

The speed of mixing is a subtle issue for flows. For hyperbolic diffeomorphisms, Sinai, Ruelle and Bowen [35, 34, 17] in the 70’s obtained exponential mixing. Only much later, a significant breakthrough was obtained by Dolgopyat [20]. Here, for the first time, exponential mixing was proved for Anosov flows with respect to physical measures, under rather strong assumptions on smoothness of stable and unstable foliations, plus their uniform non-integrability, which are not robust – these properties can be destroyed by small perturbations.

Later superpolynomial mixing was obtained for $C^2$-open and $C^r$-dense families (hence robust) of $C^r$ hyperbolic vector fields, $r \geq 2$, by Field, Melbourne and Torok [21] refining Dolgopyat techniques, but achieving only a speed of mixing for $C^r_t(\psi, \varphi)$ faster than any polynomial in $t$.

Singular-hyperbolicity is a non-trivial extension of uniform hyperbolicity encompassing continuous times systems like the Lorenz attractor in a unified theory, founded on the work of Morales, Pacífico and Pujals [28]. This allows to rigorously frame Lorenz-like attractors after the the work of Tucker [38].

The existence of physical measures for singular-hyperbolic attractors and some of their properties were obtained in full generality for the first time in [12]. Robust exponential mixing for physical measures was first proved in [15] for a $C^2$-open subset of vector fields with a geometric Lorenz attractor. For the original Lorenz attractor exponentially mixing was proved in [11, 9]. Similar techniques to [11, 9] allowed to obtain robust exponential mixing for Axiom A attractors [6] whose stable bundle has codimension two [17], and have been extended to prove robust exponential mixing for Anosov flows [18] and, recently, to achieve robust exponential mixing for singular-hyperbolic attracting sets with any finite number of hyperbolic singularities [14].

With different techniques superpolynomial mixing for a $C^2$ open and dense subset of singular-hyperbolic attracting sets was obtained in [10]. More recently, following a different strategy, Tsujii and Zhang [37] claim exponential mixing for all equilibrium states (of which physical measure are but an example) with respect to Hölder continuous potentials for an open and dense subset of topologically mixing $C^\infty$ Anosov flows on 3-manifolds.

In this work we extend the result of existence of a $C^2$ open subset of vector fields admitting robust exponential mixing for singular-hyperbolic attracting sets, to a $C^2$-open and dense subset of vector fields on compact $d$-manifold ($d \geq 3$) which admits a singular hyperbolic attracting set, with or without singularities. This includes hyperbolic attractors and Anosov flows whose stable bundle has codimension two, which improves on the results of Tsujii-Zhang.

1.1. Preliminary definitions. We fix $M$ a compact boundaryless $d$-dimensional manifold. Given an integer $k \geq 1$, we denote by $\mathcal{X}^k(M)$ the set of $C^k$ vector fields on $M$ endowed with the $C^k$ topology. We fix a smooth Riemannian structure on $M$,.
denote induced distance by \( \text{dist} \) and the volume measure by \( \text{Leb} \), which we assume are normalized: the diameter \( \text{diam}(M) \) of \( M \) and its volume \( \text{Leb}(M) \) are both equal to 1.

For \( X \in \mathcal{X}^k(M) \) we denote by \( X_t : M \to M, t \in \mathbb{R} \), the flow induced by \( X \). For each \( x \in M \) and each interval \( I \subset \mathbb{R} \) we define \( X_I(x) := \{ X_t(x) : t \in I \} \). The orbit of \( x \) by the flow of \( X \) is the set \( O_X(x) = X_{\mathbb{R}}(x) \).

We say that \( x \in M \) is regular for the vector field \( X \) if \( X(x) \neq 0 \). Otherwise we say that \( x \) is an equilibrium or singularity of \( X \). We also say that the corresponding orbit is regular or singular, respectively. The orbit of \( p \in M \) is periodic for \( X \), if the set \( \{ t \in \mathbb{R}^+ : X_t(p) = p \} \) is nonempty and the number \( T := \inf \{ t \in \mathbb{R}^+ : X_t(p) = p \} \) is positive, which is the period of \( p \).

We say that a set \( \Lambda \subset M \) is invariant by \( X \) if \( X_t(\Lambda) = \Lambda \) for all \( t \in \mathbb{R} \). A compact invariant set \( \Lambda \) is isolated if we can find an open neighborhood \( U \supset \Lambda \) so that \( \Lambda = \bigcap_{t \in \mathbb{R}} X_t(U) \). If \( U \) also satisfies Closure \( X_t(U) \subset U \) for all \( t > 0 \), then we \( \Lambda \) is an attracting set and \( U \) is a trapping region for \( \Lambda \). In this case we get \( \Lambda = \cap_{t>0} X_t(U) \).

We say that the attracting set is proper if it is not the whole ambient manifold \( M \). The topological basin of an attracting set \( \Lambda \) is given by

\[
W^s(\Lambda) = \left\{ x \in M : \lim_{t \to +\infty} \text{dist}(X_t(x), \Lambda) = 0 \right\}.
\]

Given \( x \in M \) the \( \omega \)-limit set of \( x \) by the flow \( X_t \) is given by the set

\[
\omega(x) = \omega_X(x) = \left\{ y \in M : \exists t_k \nearrow +\infty \text{ such that } \lim_{k \to +\infty} \text{dist}(X_{t_k}(x), y) = 0 \right\}.
\]

An invariant set \( \Lambda \) is transitive for \( X \) if there exists a regular point \( x \in M \) such that \( \Lambda = \omega_X(x) \). It is non-trivial if it is neither a finite set of periodic orbits nor a finite set of equilibria. Otherwise we say that \( \Lambda \) is trivial. A compact invariant set \( \Lambda \subset M \) is an attractor for a vector field \( X \) if it is a transitive attracting set for \( X \).

1.1.1. Singular-hyperbolic attracting sets. A compact invariant set \( \Lambda \) is partially hyperbolic if the tangent bundle over \( \Lambda \) can be written as a continuous \( DX_t \)-invariant sum \( T_{\Lambda}M = E^s \oplus E^{cu} \), where \( d_s = \dim E^s_x \geq 1 \) and \( d_{cu} = \dim E^{cu}_x \geq 2 \) for \( x \in \Lambda \), and there exist constants \( C > 0 \), \( \lambda \in (0, 1) \) such that for all \( x \in \Lambda \), \( t \geq 0 \), we have both uniform contraction along \( E^s \): \( \| DX_t |E^s_x \| \leq C \lambda^t \); and domination of the splitting: \( \| DX_t |E^s_x \| \cdot \| DX_{-t} |E^{cu}_x \| \leq C \lambda^t \).

Then \( E^s \) is the stable bundle and \( E^{cu} \) the center-unstable bundle. A partially hyperbolic attracting set is a partially hyperbolic set that is also an attracting set.

The center-unstable bundle \( E^{cu} \) is volume expanding if there exists \( K, \theta > 0 \) such that \( | \det(DX_t|E^{cu}_x) | \geq K e^{\theta t} \) for all \( x \in \Lambda \), \( t \geq 0 \).

The compact invariant set \( \Lambda \) is hyperbolic if it is partially hyperbolic and the center-unstable bundle admits a continuous \( DX_t \)-invariant splitting \( E^{cu} = E^X \oplus E^u \), where \( E^X = \mathbb{R} \cdot X \) is the flow direction, and for all \( x \in \Lambda, t \geq 0 \) we have
uniform expansion along $E^u$: $\| (DX_t | E^u_0)^{-1} \| \leq C \lambda^t$.

A periodic orbit is a hyperbolic if $O_X(p)$ is a (trivial) hyperbolic set. A singularity $\sigma$ is hyperbolic if $\{ \sigma \}$ is a hyperbolic set with $E^X_{\sigma} = \{ 0 \}$, i.e., $T_\sigma M = E^s_{\sigma} \oplus E^u_{\sigma}$.

**Definition 1.1.** Let $\Lambda$ be a compact invariant set for $X \in \mathcal{X}(M)$. We say that $\Lambda$ is a *singular-hyperbolic set* if all equilibria in $\Lambda$ are hyperbolic, and $\Lambda$ is partially hyperbolic with volume expanding two-dimensional center-unstable bundle ($d_{cu} = 2$). A singular-hyperbolic set which is also an attracting set is called a *singular-hyperbolic attracting set*.

**Remark 1.2.** A singular-hyperbolic attracting set contains no isolated periodic orbits. For such a periodic orbit would have to be a periodic sink, violating volume expansion. Analogously, there are no isolated singularities. Hence, all singular-hyperbolic attracting sets are non-trivial.

Clearly, all hyperbolic sets are singular-hyperbolic and singular-hyperbolicity is a natural extension of hyperbolicity because of the following.

**Theorem 1.3.** [27, Lemma 3] Every compact invariant set without singularities of a singular-hyperbolic set is hyperbolic.

**Definition 1.4.** A *singular-hyperbolic attractor* is a transitive singular-hyperbolic attracting set. A hyperbolic attractor is a transitive hyperbolic attracting set.

**Remark 1.5.** A non-trivial hyperbolic attractor\(^1\) can be seen as an attracting basic piece of a Axiom A vector field of Smale [36] and, after Theorem 1.3 together with Remark 1.2, also as a singular-hyperbolic attractor without equilibria.

1.1.2. **Physical measures.** The existence of a unique invariant and ergodic physical measure for hyperbolic attractors is well-known [34]. For singular-hyperbolic attractors this was first proved for 3-dimensional manifolds in [12] and extended in [13]. In fact, singular-hyperbolic attracting sets have finitely many ergodic physical measures which are equilibrium states for the central-unstable Jacobian, just like Axiom A attracting sets.

**Theorem 1.6.** [13, Theorem 1.7] Let $\Lambda$ be a singular-hyperbolic attracting set for a $C^2$ vector field $X$ with the open subset $U$ as trapping region. Then

1. there are finitely many ergodic physical/SRB measures $\mu_1, \ldots, \mu_k$ supported in $\Lambda$ such that the union of their ergodic basins covers $U$ Lebesgue almost everywhere: $\text{Leb} \left( U \setminus \left( \bigcup_{i=1}^{k} B(\mu_i) \right) \right) = 0$.

2. Moreover, for each $X$-invariant ergodic probability measure $\mu$ supported in $\Lambda$ the following are equivalent
   
   (a) $h_\mu(X_1) = \int \log \left| \det DX_1 \right| d\mu > 0$;

---

\(^1\)We are excluding periodic sinks and fixed sinks.
(b) $\mu$ is a SRB measure, that is, admits an absolutely continuous disintegration along unstable manifolds;
(c) $\mu$ is a physical measure, i.e., its basin $B(\mu)$ has positive Lebesgue measure.

(3) The family $E$ of all $X$-invariant probability measures which satisfy item (2a) above is the convex hull $E = \left\{ \sum_{i=1}^{k} t_i \mu_i : \sum_i t_i = 1; 0 \leq t_i \leq 1, i = 1, \ldots, k \right\}$.

We note that there are many examples of singular-hyperbolic attracting sets, non-transitive and containing non-Lorenz-like singularities; see e.g. [14, Section 2.4].

1.2. Statement of results. We can now state our main results. In what follows, we write $C^{k+\eta}(M)$, where $\eta \in (0, 1]$ is a real number and $k \geq 0$ is a non-negative integer, for the set of functions $\varphi : M \to \mathbb{R}$ which are of class $C^k$ and the $k$th derivative $D^k \varphi$ is $\eta$-Hölder. This is a Banach space with norm given by
\[
\|\varphi\|_{k+\eta} := \sum_{i=0}^{k} |D^i \varphi|_{\infty} + |D^k \varphi|_{\eta},
\]
where for any function $\psi : M \to \mathbb{R}$ we set $|\psi|_{\infty} := \sup_{x \in M} |\psi(x)|$ and $|\psi|_{\eta} := \sup_{x \neq y} |\psi(x) - \psi(y)| / \text{dist}(x, y)^{\eta}$.

**Theorem A** (Open and dense exponential mixing among singular-hyperbolic attracting sets). Let $U \subset \mathfrak{X}^2(M)$ be the $C^2$ open subset of vector fields so that every $X \in U$ admits a non-trivial connected singular-hyperbolic attracting set. For each $X \in U$ and each $\epsilon > 0$ there exists a $C^\infty$ function $\varphi : M \to (1/2, 3/2)$ such that
- $Y = \varphi X$ satisfies $\|Y - X\|_{C^1} < \epsilon$; and
- there exists a $C^2$-neighborhood $\mathcal{V}$ of $Y$ so that for all $Z \in \mathcal{V}$, for each connected singular-hyperbolic attracting set $\Lambda$ for the flow of $Z$, and for a physical measure $\mu$ supported in $\Lambda$, there exist constants $c, C > 0$ so that for any $\eta \in (0, 1]$ we have $|C^\mu_t (\varphi, \psi)| \leq C e^{-ct} \|\varphi\|_{\eta} \|\psi\|_{\eta}$, for all $\varphi, \psi \in C^n(M)$ and $t > 0$.

The previous statement implies that exponential mixing is $C^2$-open and $C^1$-dense among vector fields admitting singular-hyperbolic attracting sets. In addition, given any connected singular-hyperbolic attracting set $\Lambda$ for a $C^2$-vector field $X$, there exists a $C^1$-close $C^2$-multiple of $X$, generating a flow with the exact same trajectories, which mixes exponentially with respect to any physical measure. That is, “every singular-hyperbolic attracting set mixes exponentially with respect to its physical measures modulo an arbitrarily small change in the speed of the flow”. On 3-dimensional manifolds we obtain the following.

**Corollary B.** Among all $C^2$-vector fields of a 3-dimensional compact manifold $M$ having a singular-hyperbolic attracting set $\Lambda$, a $C^2$-open and $C^1$-dense subset of vector fields mix exponentially with respect to any physical measure supported on the attractor.
In particular, given any geometrical Lorenz attractor $\Lambda$ of a vector field $X$ of class $C^2$, there exists a $C^1$-close $C^2$-multiple $Y$ of $X$ so that $Y$ mixes exponentially with respect to the unique physical measure supported on the same attractor $\Lambda$.

The same strategy provides similar results for Axiom A attractors and transitive Anosov flows with one-dimensional unstable direction, as presented in [4], whose main technical result we use in our proof.

It is known that fast decay of correlations for a dynamical system implies many other statistical properties. The base map of a hyperbolic skew-product semiflow is known to satisfy exponential mixing for Hölder observables with respect to its physical measures; see e.g. [8]. This in turn automatically implies certain statistical properties for the induced measure on the suspension flow: the Central Limit Theorem (CLT), the Law of the Iterated Logarithm (LIL) and the Almost Sure Invariance Principle (ASIP); see e.g. [20].

For mixing and the speed of mixing, the properties of the base map do not extend to the suspension flow in general: the suspension flow does not even have to be mixing. More precisely, see [32], rates of mixing of suspension flows can be arbitrarily slow even if the base map is exponentially mixing.

Having a flow which mixes exponentially fast implies more subtle statistical properties, including CLT and ASIP for the time-1 map of the flow. The ASIP implies the CLT and also the functional CLT (weak invariance principle), and the law of the iterated logarithm together with its functional version, as well as numerous other results; see e.g. [31] for a comprehensive list.

Therefore, the flow near any connected singular-hyperbolic attracting set is topologically equivalent to $C^1$-close flow of class $C^2$ for which all the above properties hold.

1.3. Overall strategy. As in previous works on robust exponential mixing for geometric Lorenz attractors [15, 11, 9, 14] and hyperbolic attractors [6] and also Anosov flows [18], the proof relies on finding a convenient conjugation between the flow in a neighborhood of the attracting set $\Lambda$ and a skew-product semiflow satisfying strong dynamical and ergodic properties. We present the detailed definitions in Section 2.

In all the previous works, including [16], it was assumed that the suspension semiflows are given by a roof function which is constant along stable leaves of the flow, that is, every point of each stable leaf returns to the base of the suspension in the same instant.

This assumption is used in the proofs of [16, Lemma 8.2], [9, Theorem 3.3] and [14, Theorem 2.4 & Corollary 5.12], to pass from exponential mixing from the expanding semiflow, to exponential mixing for the hyperbolic skew-product semiflow. The uniform contraction of the stable leaves by the skew-product base map is essential in this argument; see e.g. [18, Subsection 2.1] for a a short explanation.
However, the reduction of the flow dynamics near a singular-hyperbolic attracting set $\Lambda$ to a suspension semiflow on a global Poincaré map, always admits a $C^{1+}$ smooth piecewise expanding map of the interval as a quotient over the stable leaves; see e.g. [10] and the detailed overview in Section 3. The suspension of the global Poincaré map, absent a smoothness property on the stable foliation extended to an open neighborhood of $\Lambda$, cannot in general be constructed with a $C^1$ smooth roof function which is constant along stable leaves. But this roof function is naturally at least Lipschitz along each stable leaf, and its Lipschitz constant depends only on the vector field.

Hence, the missing step to be able to conclude exponential mixing, is to pass the mixing estimates from the $C^{1+}$ expanding semiflow, to the hyperbolic skew-product semiflow with a slowly varying roof function along the stable leaves.

We show, in Subsection 2.2, that such a suspension semiflow can be naturally conjugated to a hyperbolic skew-product semiflow whose roof function is constant on stable leaves. Moreover, this conjugation can be made in such a manner that the space of smooth observables needed to obtain exponential mixing in [9, Theorem 3.3], or [14, Theorem 2.4 & Corollary 5.12], is preserved by the conjugation.

In this way, with the extra step outlined in the previous paragraph, we reduce the study of the speed of mixing on a neighborhood of a singular-hyperbolic attracting set to a hyperbolic skew-product semiflow, just as in [14].

The procedure is completed by a perturbation of the norm of the vector field, i.e. the speed of the flow, along a certain periodic orbit, to obtain the “Uniform Non-Integrability” (UNI) condition, just like the one described in [14]. Since this is the only step of the previous procedure where we need to potentially change the vector field which we started with, this explains why in this way we obtain an open and dense subset of vector fields satisfying all the necessary conditions (because the UNI condition is naturally an open condition).

The same arguments can be applied to a hyperbolic attractor whose stable bundle is of codimension two, since these attractors are singular-hyperbolic. In particular, the support of an ergodic physical probability measure, without singularities, inside a singular-hyperbolic attracting set is such an hyperbolic attractor, to which we may apply the previous procedure; see Subsection 3.3.

We conjecture, naturally, that the dimensional restriction can be overcome and a similar exponential mixing result, valid for open and dense subsets of vector field with sectional-hyperbolic attracting sets, can also be obtained, complementing the existence of physical measures recently obtained [25, 3].

In addition, a refinement of these techniques should be applied to obtain open and dense exponential mixing for all equilibrium states with respect to Hölder potentials supported on singular-hyperbolic (sectional-hyperbolic) attractors, extending and adapting the recent results from Daltro and Varandas [19].

1.4. Organization of the text. The remainder of the paper consists of two sections.
Section 2 details the main results on suspension semiflows that are crucial in our arguments: Subsection 2.1 describes the “standard” hyperbolic skew-product flows used in the previous works in the subject; and Subsection 2.2 details the “generalized” hyperbolic skew-product flows which are the focus of this text and we prove the existence of a natural conjugation between the “standard” and the “generalized” versions of these semiflows.

In Section 3, we state the main result on representation of a singular-hyperbolic flow around an attracting set as a hyperbolic skew-product suspension semiflow. In the remaining of this section we provide a detailed overview of the reduction of the study of the speed of mixing for singular-hyperbolic attractors to a generalized hyperbolic skew-product semiflow, to which we apply the results of Section 2 to conclude the proof of the main theorem.

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2. Exponential mixing for generalized hyperbolic skew-product semiflows

The main strategy of this work is to take a flow admitting a connected singular-hyperbolic attracting set, or a codimension two hyperbolic attractor, and reduce its dynamics to the setting that we present in this section. After obtaining the results for hyperbolic skew product semiflows, we explain how to take them to the original flow.

We present this model semiflow step by step in Subsections 2.1 and 2.2 and prove the existence of a natural conjugation between the “standard” and the “generalized” versions of these semiflows.

2.1. Hyperbolic skew product semiflow. Here we describe the “standard” hyperbolic skew-product flows used in the previous works to obtain exponential mixing for the physical measure of certain classes of flows.

2.1.1. Uniformly expanding maps. Let \( \alpha \in (0, 1] \) and \( \Delta \) be a compact interval of \( \mathbb{R} \). Without loss of generality we assume that \( \Delta = [0, 1] \) in this section. Let \( \mathcal{P} = \{(c_m, d_m) : m \in \mathbb{N}\} \) be a countable partition (Leb mod0) of \( \Delta \). Let \( F : \Delta \to \Delta \) be \( C^{1+\alpha} \) on each element \( J \) of the partition \( \mathcal{P} \) with \( F(J) = \Delta \) and such that \( F \) extends to a homeomorphism from \( J \) to \( \Delta \), for every \( J \in \mathcal{P} \).

Given \( J \in \mathcal{P} \), we say that a map \( h : \Delta \to \overline{J} \) is an inverse branch of \( F \) if \( F \circ h = \text{id} \). We denote by \( \mathcal{H} \) and \( \mathcal{H}_n \) the set of all inverse branches of \( F \) and \( F^n \), respectively, for all \( n \geq 1 \). Given a function \( \psi : \Delta \to \mathbb{R} \) we write

\[
|\psi|_\infty := \sup_{x \in \Delta} |\psi(x)| \quad \text{and} \quad |\psi|_\alpha := \sup_{x \neq y} |\psi(x) - \psi(y)|/|x - y|^\alpha.
\]
We say that $F$ is a $C^{1+\alpha}$ uniformly expanding map if there exist constants $C > 0$ and $\rho \in (0, 1)$ such that

1. $|h'|_{\infty} \leq C \rho^n$ for all $h \in H_n$,
2. $|\log |h'||_\alpha \leq C$ for all $h \in H$.

**Remark 2.1.** It follows from (1) and (2) that $\sum_{h \in H_n} |h'|_{\infty} < \infty$.

It is standard that $C^{1+\alpha}$ uniformly expanding maps have a unique absolutely continuous $F$-invariant ergodic measure with $\alpha$-Hölder positive density function bounded from above and below away from zero. We denote this measure by $\mu_F$.

### 2.1.2. $C^{1+}$ expanding semiflows.

Consider a function $r : \Delta \to (0, +\infty)$ which is $C^1$ on each element of the partition $\mathcal{P}$. We assume the following conditions on $r$:

3. bounded derivative: $| (r \circ h)' |_{\infty} \leq C$ for all $h \in H$.
4. $r$ has exponential tail: there exists $\varepsilon > 0$ such that $\sum_{h \in H} e^{\varepsilon |r \circ h|_{\infty}} |h'|_{\infty} < \infty$;
5. uniform non-integrability (UNI): it is not possible to write $r = \psi + \varphi \circ F - \varphi$ with $\psi : \Delta \to \mathbb{R}$ constant in elements of the partition $\mathcal{P}$ and $\varphi : \Delta \to \mathbb{R}$ a $C^1$ function.

Let $\Delta' = \{(x, u) \in \Delta \times \mathbb{R} : 0 \leq u \leq r(x)\}/\sim$ be a quotient space, where $(x, r(x)) \sim (F(x), 0)$, and define the suspension semiflow $F_t : \Delta' \to \Delta'$ with roof function $r$ by $F_t(x, u) = (x, u + t)$, for all $t \geq 0$, computed modulo the given identification. The semiflow $F_t$ has an ergodic invariant probability measure $\mu_F' = (\mu_F \times \text{Leb}) / \int_{\Delta} r \, d\mu_F$. If conditions (1)-(4) hold, then we say that $F_t$ is a $C^{1+\alpha}$ expanding semiflow.

### 2.1.3. Decay of correlations for $C^{1+}$ expanding semiflows.

We define $C^{\alpha}_{\text{loc}}(\Delta')$ to consist of $L^\infty$ functions $\psi : \Delta' \to \mathbb{R}$ such that $\|\psi\|_\alpha = |\psi|_{\infty} + |\psi|_{\alpha, \text{loc}} < \infty$, where

$$
|\psi|_{\alpha, \text{loc}} = \sup_{h \in H} \sup_{(x_1, u) \neq (x_2, u)} \frac{|\psi(hx_1, u) - \psi(hx_2, u)|}{|x_1 - x_2|^{\alpha}}.
$$

Given an integer $k \geq 1$, define $C^{\alpha, k}_{\text{loc}}(\Delta')$ to consist of $C^{\alpha}_{\text{loc}}(\Delta')$ functions $\psi$ with $\|\psi\|_{\alpha, k} = \sum_{j=0}^k \|\partial^j_t \psi\|_\alpha < \infty$, where $\partial_t$ denotes the differentiation along the semiflow direction.

**Theorem 2.2.** (Exponential mixing for expanding semiflows) [14] Theorem 2.2 If conditions (1)-(5) hold, then there are constants $c, C > 0$ so that we have $|C^{\alpha}_{\text{loc}}(\varphi, \psi)| = |\int (\varphi \circ F_t) \psi \, d\mu^F_t - \int \varphi \, d\mu^F_t \int \psi \, d\mu^F_t| \leq Ce^{-ct} |\varphi|_{\infty} \|\psi\|_{\alpha, 2}$ for all $\varphi \in L^\infty(\Delta')$, $\psi \in C^{\alpha, 2}_{\text{loc}}(\Delta')$ and $t > 0$.

Theorem [22.2] is a generalization of [9] Theorem 2.1] from $\alpha$-Hölder observables to the more general class of observables presented above.
2.1.4. Hyperbolic skew products. Now we introduce stable directions on the model. Let \( F : \Delta \to \Delta \) be a \( C^{1+\alpha} \) expanding map, as in Subsection [2.1.1] and \( \Omega \) the closure of a small open ball around the origin in \( \mathbb{R}^N \), for some integer \( N \geq 1 \). Let \( \hat{\Delta} = \Delta \times \Omega \) be a direct product endowed with the distance given by \(|(x_1, y_1) - (x_2, y_2)| = |x_1 - x_2| + |y_1 - y_2| \). Consider also \( G : \hat{\Delta} \to \Omega \) a \( C^{1+\alpha} \) map and define \( \hat{F} : \hat{\Delta} \to \hat{\Delta} \) by \( \hat{F}(x, y) = (F(x), G(x, y)) \). We say that \( \hat{F} \) is a uniformly hyperbolic skew product if it satisfies

\[
\text{(uniform contraction along } \Omega) \text{ there exist constants } C > 0 \text{ and } \gamma \in (0, 1) \text{ such that } |\hat{F}^n(x_1, y_1) - \hat{F}^n(x_2, y_2)| \leq C\gamma^n|y_1 - y_2|, \text{ for all } x \in \Delta \text{ and } y_1, y_2 \in \Omega.
\]

For each integer \( n \geq 1 \), we denote the iterates of \( \hat{F} \) by \( \hat{F}^n(x, y) = (F^n(x), G_n(x, y)) \) for all \((x, y) \in \hat{\Delta}\). Hence, item (6) above becomes \( |G_n(x, y_1) - G_n(x, y_2)| \leq C\gamma^n|y_1 - y_2| \), for all \((x, y_1) \in \hat{\Delta}, i = 1, 2\).

Fix \( y_0 \in \Omega \). Let \( \pi : \hat{\Delta} \to \Delta \times \{y_0\} \) be the projection \( \pi(x, y) = (x, y_0) \), for all \((x, y) \in \hat{\Delta}\). Note that \( \pi \circ \hat{F} = F \circ \pi \) where \( F_0(x, y_0) = (Fx, y_0) \) is naturally identified with \( F \), that is, \( \pi \) is a semiconjugacy between \( \hat{F} \) and \( F_0 \). Moreover, property (4) says that the leaf \( \pi^{-1}(x) \) is exponentially contracted by the skew product \( \hat{F} \), for all \( x \in \Delta \).

2.1.5. Invariant probability measure for the hyperbolic skew product. In the following proposition we recall how to obtain a \( \hat{F} \)-invariant probability measure using the (absolutely continuous) invariant probability measure \( \mu_F \) for the map \( F \).

**Proposition 2.3**. [12] Section 6] Let \( \varphi : \hat{\Delta} \to \mathbb{R} \) be a continuous function and define \( \varphi_\pm : \Delta \to \mathbb{R} \) by \( \varphi_+(x) = \sup_{y \in \Omega} \varphi(x, y) \) and \( \varphi_-(x) = \inf_{y \in \Omega} \varphi(x, y) \). Then the limits \( \lim_{n \to +\infty} \int_\Delta (\varphi \circ \hat{F}^n)_+ \, d\mu_F \) and \( \lim_{n \to +\infty} \int_\Delta (\varphi \circ \hat{F}^n)_- \, d\mu_F \) exist, are equal, and define a \( \hat{F} \)-invariant probability measure \( \mu_{\hat{F}} \) such that \( \pi_* \mu_{\hat{F}} = \mu_F \).

2.1.6. Hyperbolic skew product semiflow (with constant roof function on stable leaves). Let \( F : \Delta \to \Delta \) be a \( C^{1+\alpha} \) uniformly expanding map with partition \( \mathcal{P} \); \( \hat{F} : \hat{\Delta} \to \hat{\Delta} \) a \( C^{1+\alpha} \) hyperbolic skew product with \( \pi \circ \hat{F} = F_0 \circ \pi \) as in the previous Subsections [2.1.2] and [2.1.4] and \( r : \Delta \to (0, +\infty) \) be \( C^1 \) on elements of the partition \( \mathcal{P} \) with \( \inf r > 0 \).

We extend the definition of \( r \) to \( \hat{\Delta} \) by setting \( r(x, y) = r(x) \) for all \((x, y) \in \hat{\Delta}\). We note that here we are assuming that the return time to the base of the semiflow is constant on stable leaves.

Considering the quotient space \( \hat{\Delta}^r = \{(z, u) \in \hat{\Delta} \times \mathbb{R} : 0 \leq u \leq r(z)\}/\sim \), where \((z, r(z)) \sim (\hat{F}(z), 0)\), we define the suspension semiflow \( \hat{F}_t \) with roof function \( r \) by \( \hat{F}_t(z, u) = (z, u + t) \), for all \( t \geq 0 \), computed modulo the given identification. This semiflow has an ergodic invariant probability measure \( \mu^r_{\hat{F}} = \mu_{\hat{F}} \times \text{Leb} / \int_\Delta r \, d\mu_{\hat{F}} \). If \( r \) satisfies the conditions (3) and (4), then we say that the \( \hat{F}_t \) is a \( C^{1+\alpha} \) hyperbolic skew product semiflow.
2.1.7. **Exponential mixing for hyperbolic skew product semiflows.** Let \( C^\alpha_{\text{loc}}(\hat{\Delta}^r) \) denote the subset of \( L^\infty \) functions \( \psi: \hat{\Delta}^r \to \mathbb{R} \) such that \( \|\psi\|_{\alpha} = |\psi|_\infty + |\psi|_{\alpha,\text{loc}} \), where

\[
|\psi|_{\alpha,\text{loc}} = \sup_{h \in \mathcal{H}} \sup_{(x_1,y_1,u) \neq (x_2,y_2,u)} \left| \frac{\psi(hx_1,y_1,u) - \psi(hx_2,y_2,u)}{|x_1 - x_2|^{\alpha} + |y_1 - y_2|} \right|
\]

and let \( C^{\alpha,k}_{\text{loc}}(\hat{\Delta}^r) \) be the subset of \( C^\alpha_{\text{loc}}(\hat{\Delta}^r) \) given by those functions \( \varphi: \hat{\Delta}^r \to \mathbb{R} \) such that \( \|\varphi\|_{\alpha,k} := \sum_{j=0}^{k} |\partial^j_i \varphi|_{\alpha} < \infty \), where \( \partial_i \) denotes the differentiation along the semiflow direction and \( k \geq 1 \) is a given integer. The following is a extension of [9, Theorem 3.3] to this more general class of observables.

**Theorem 2.4** (Exponential mixing for hyperbolic skew-product semiflows). Suppose that \( \tilde{F}_t: \hat{\Delta}^r \to \hat{\Delta}^r \) is a \( C^{1+\alpha} \) hyperbolic skew product with roof function \( r \) satisfying the UNI condition (5). Then there exist constants \( c, C > 0 \) such that

\[
|\tilde{F}_t(\varphi \circ \hat{F}_t) - \int \varphi \, \mu_{\hat{F}}^t| \leq Ce^{-ct}\|\varphi\|_{\alpha} \|\psi\|_{\alpha,2}, \text{ for all } \varphi \in C^\alpha_{\text{loc}}(\hat{\Delta}^r), \psi \in C^{\alpha,2}_{\text{loc}}(\hat{\Delta}^r)
\]

and \( t > 0 \).

**Proof.** This is obtained in [14, Theorem 2.4] – see [18, Section 2.1] for an overview of the argument. \( \square \)

2.2. **Generalized hyperbolic skew product semiflow.** Here we present the new technical result at the heart of the main statements, detailing the “generalized” hyperbolic skew-product flows and proving the existence of a natural conjugation between the “standard” and the “generalized” versions of these semiflows, under certain mild conditions.

Consider the same base maps \( F: \Delta \to \Delta \) and \( \hat{F}: \hat{\Delta} \to \hat{\Delta} \) as in the previous Subsection 2.1.6 with the extra

**domination condition:** there exists \( c \in (0,1) \) such that for all \( (x,y) \in \hat{\Delta} \) we have

\[
|DG_1(x,y)| \leq c \cdot |DF(x)|.
\] (2.1)

Moreover, we consider an extension of \( r \) to \( \hat{\Delta} \) which satisfies the following

**Lipschitz condition:** there exists a constant \( C > 0 \) so that

\[
|\tilde{r}(x,y) - \tilde{r}(x,y')| < C \operatorname{dist}(y,y') \text{ for all } x \in \Delta \setminus \mathcal{P}, y,y' \in \Omega.
\] (2.2)

We stress that here we are NOT assuming that the return time to the base of the semiflow is constant on stable leaves.

Considering the quotient space \( \hat{\Delta}^r = \{(z,u) \in \hat{\Delta} \times \mathbb{R} : 0 \leq u \leq \tilde{r}(z)\}/\sim \), where \( (z,\tilde{r}(z)) \sim (\tilde{F}(z),0) \), we define the suspension semiflow \( \tilde{F}_t \) with roof function \( \tilde{r} \) as before: \( \tilde{F}_t(z,u) = (z,u+t) \), for all \( t \geq 0 \), computed modulo the identification. This semiflow also has an ergodic invariant probability measure \( \mu_{\tilde{F}}^t = \mu_{\hat{F}} \times \text{Leb} / \int_{\hat{\Delta}} \tilde{r} \, d\mu_{\hat{F}} \).

We assume that there exists a point \( 0 \in \Omega \) such that

\[
(\tilde{r} \circ \pi)(x,y) = \tilde{r}(x,0) = r(x), \quad (x,y) \in \hat{\Delta}.
\]
In what follows, to simplify notation, we denote $\tilde{r}$ by $r$ and write $r \circ \pi$ to denote a roof function over $\hat{\Delta}$, constant on stable leaves. In this way, $r$ is an extension of the restriction $r|_\Delta$ satisfying the domination and Lipschitz conditions.

If, in addition, $r \circ \pi$ satisfies the conditions (3) and (4), then we say that the suspension semiflow $\hat{F}_t$ is a $C^{1+\alpha}$ generalized hyperbolic skew product semiflow.

2.2.1. Exponential mixing for generalized hyperbolic skew-product semiflow. We extend the previous exponential mixing result to this generalized setting. Since $\Delta$ can be identified with a subset $\Delta \times \{0\}$ of $\hat{\Delta}$, then the UNI condition (5) still makes sense in this generalized setting.

**Theorem 2.5.** Suppose that $\hat{F}_t: \hat{\Delta}^r \rightarrow \hat{\Delta}^r$ is a $C^{1+\alpha}$ generalized hyperbolic skew product with roof function $r$ so that $r \circ \pi$ satisfies the UNI condition (5). Then there are constants $c, C > 0$ such that $|\int (\varphi \circ \hat{F}_t) \cdot \psi \, d\mu_{\hat{F}} - \int \varphi \, d\mu_{\hat{F}} \int \psi \, d\mu_{\hat{F}}| \leq Ce^{-ct}\|\varphi\|_0\|\psi\|_{\alpha,2}$, for all $\varphi \in C^0_{\text{loc}}(\hat{\Delta}^r)$, $\psi \in C^{\alpha, 2}_{\text{loc}}(\hat{\Delta}^r)$ and $t > 0$.

**Proof.** The proof of this result is based on a reduction to the setting of Theorem 2.4.

A map $q: \hat{\Delta}^r \rightarrow \hat{\Delta}^{r \circ \pi}((w, t) \mapsto (w, t + \chi(w)))$ conjugates the two suspension flows $\hat{F}_t : \hat{\Delta}^r \circ \text{ and } \hat{F}_t : \hat{\Delta}^{r \circ \pi}$, that is, $q \circ \hat{F}_t = \hat{F}_t \circ q$ if $\chi : \hat{\Delta} \rightarrow \mathbb{R}$ satisfies the cocycle relation

$$\chi \circ \hat{F}_t - \chi = r - r \circ \pi. \tag{2.3}$$

Let $\Xi = \hat{\Delta} \setminus \bigcup \cup l \hat{\Delta}(l)$ be the family of stable leaves where $\hat{F}$ is not defined and let $\Xi_0 = \hat{\Delta} \setminus \bigcup_{n \geq 1} \hat{F}^{-n} \Xi$ be the subset of points of $\hat{\Delta}$ which never visit $\Xi$ in all future iterates.

**Lemma 2.6.** The following defines a function on $\Xi_0$

$$\chi(w) = \sum_{m \geq 0} [r(\hat{F}^m \pi w) - r(\hat{F}^m w)], \quad w \in \Xi_0.$$  

Moreover, the relation (2.3) holds and there exists $C > 0$ so that $\chi |_{\pi^{-1}(x)}$ is $C(1 - \gamma)^{-1}$-Lipschitz for every $x \in \Xi_0$.

To be able to convert the relevant observables on $\hat{\Delta}^{r \circ \pi}$ to good observables on $\hat{\Delta}^r$ we need more regularity of $\chi$ along the unstable direction. Given $0 < \theta \leq 1$, we say that a function $\phi : \hat{\Delta} \rightarrow \mathbb{R}$ belongs to the space $C^\theta_{\text{loc}}(\Delta)$ if $\phi$ is measurable and

$$|\phi|_{\alpha, \text{loc}} = \sup_{h \in H(x_1, y_1) \neq (x_2, y_2)} \frac{|\phi(hx_1, y_1) - \phi(hx_2, y_2)|}{|x_1 - x_2|^{\theta} + |y_1 - y_2|} < \infty.$$  

Next we use the following version of Livsic’s Theorem.

**Lemma 2.7.** Let $\chi : \Xi_0 \rightarrow \mathbb{R}$ be a function satisfying $\chi \circ F - \chi = \phi$ where $\phi \in C^\theta_{\text{loc}}(\hat{\Delta})$ for some $\theta \in (0, 1]$. Given any $\hat{F}$-invariant and transitive set $\Gamma \subset \hat{\Delta}$, then $\chi$ admits a $\theta$-Hölder extension to the closure $\Gamma$ of $\Gamma$.  

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We use the previous lemma with \( \phi = r - r \circ \pi \) and \( \theta = 1 \), noting that \( \phi \in C^{1}_{\text{loc}}(\hat{\Delta}) \).

We consider the compact \( \hat{F} \)-transitive set \( \Gamma = \text{supp} \mu_{\hat{F}} \) for each \( \hat{F} \)-invariant ergodic and physical measure \( \mu_{\hat{F}} \), and write \( \Gamma_{r} \) and \( \Gamma_{r0} \) for the sets \( \Gamma \times \mathbb{R} \) quotiented over the spaces \( \hat{\Delta}^{r} \) and \( \hat{\Delta}^{r0} \), respectively, and note that \( \text{Leb}_{\hat{\Delta}}(\Gamma) > 0 \).

Lemma 2.7 ensures that we may extend \( \chi \) to a Lipschitz function first on \( \Gamma \), then on \( \hat{\Delta} \), satisfying the cocycle relation (2.3) on the invariant set \( \Gamma \). The map \( \hat{\Delta}^{r} \) becomes an ergodic equivalence between \( (\hat{\Delta}^{r}, \hat{F}_{t}, \mu_{\hat{F}}^{r}) \) and \( (\hat{\Delta}^{r0}, \hat{F}_{t}, \mu_{\hat{F}}^{r0}) \), since supp \( \mu_{\hat{F}}^{r} = \Gamma_{r} \) and supp \( \mu_{\hat{F}}^{r0} = \Gamma_{r0} \).

Moreover, if \( \varphi \in C^{\alpha,2}_{\text{loc}}(\hat{\Delta}^{r0}) \), then \( \varphi \circ q \in C^{\alpha,2}_{\text{loc}}(\hat{\Delta}^{r} \cap \Gamma_{r}) \), since \( \partial_{i}^{2}(\varphi \circ q)(w,t) = (\partial_{i}^{2} \varphi)(q(w,t)), i = 1, 2 \) by the expression of \( q \). Similarly, if \( \varphi \in C^{\alpha,2}_{\text{loc}}(\hat{\Delta}^{r}) \), then \( \varphi \circ q^{-1} \in C^{\alpha,2}_{\text{loc}}(\hat{\Delta}^{r0} \cap \Gamma_{r0}) \), where \( q^{-1}(w,t) = (w,t - \chi(w)) \). In addition, it is easy to check

\[
\| \varphi \circ q^{\pm 1} \|_{\alpha,2} \leq \| \varphi \|_{\alpha,2}(1 + \| \chi \|_{\alpha,\text{loc}}) \quad \text{and} \quad \| \chi \|_{\alpha,\text{loc}} \leq \| \chi \|_{\alpha,2}.
\]

Since the integrals in the statement of Theorem 2.4 are with respect to the measures \( \mu_{\hat{F}}^{r0} \), and \( q_{\ast}^{-1} \mu_{\hat{F}}^{r0} = \mu_{\hat{F}}^{r} \) then, for each \( \varphi \in C^{\alpha}_{\text{loc}}(\hat{\Delta}^{r}) \) and \( \psi \in C^{\alpha,2}_{\text{loc}}(\hat{\Delta}^{r}) \), we can write

\[
\left| \int (\varphi \circ q^{-1} \circ \check{F}_{t}) \cdot \psi \circ q^{-1} \, d\mu_{\hat{F}}^{r0} - \int \varphi \circ q^{-1} \, d\mu_{\hat{F}}^{r0} \int \psi \circ q^{-1} \, d\mu_{\hat{F}}^{r0} \right| \\
= \left| \int (\varphi \circ \check{F}_{t}) \cdot \psi \, d\mu_{\hat{F}}^{r} - \int \varphi \, d\mu_{\hat{F}}^{r} \int \psi \, d\mu_{\hat{F}}^{r} \right| \\
\leq C(1 + \| \chi \|_{\alpha,2})e^{-ct}\| \varphi \|_{\alpha,2}\| \psi \|_{\alpha,2},
\]

using Theorem 2.4 applied to \( \mu_{\hat{F}}^{r0} \) together with the previous estimates. This completes the proof of Theorem 2.3 except for the proof of Lemmas 2.6 & 2.7.

2.3. Proof of lemmata. For completeness, we present the proof of Lemmas 2.6 & 2.7.

Proof of Lemma 2.6. Because \( w, \pi w \) belong to the same stable leaf, the skew-product structure ensures that \( \hat{F}^{m}w, \hat{F}^{m} \pi w \) also lie in one stable leaf whenever the image is defined. Hence, for \( w \in \hat{\Delta}_{0} \) and \( m \geq 0 \) we get from the contraction of stable leaves given by condition (6):

\[
|r(\hat{F}^{m} \pi w) - r(\hat{F}^{m}w)| \leq C(1 + \gamma |w - \pi w| \leq C \gamma^{m} |w - \pi w| \leq C \gamma^{m},
\]

and so the series defining \( \chi(w) \) is absolutely convergent. Moreover, the relation (2.3) is a trivial consequence of the definition of \( \chi \) and, if \( \pi w' = \pi w \), then

\[
|\chi(w) - \chi(w')| \leq \sum_{m \geq 0} |r(\hat{F}^{m} w') - r(\hat{F}^{m} w)| \leq \sum_{m \geq 0} C \gamma^{m} |w' - w| = \frac{C |w - w'|}{1 - \gamma}
\]

which completes the proof of Lemma 2.6.

Proof of Lemma 2.7. We follow the usual path of Hyperbolic Dynamics, obtaining first the following version of the Closing Lemma for a hyperbolic skew-product.
Lemma 2.8. Given \( w \in \Xi_0, \varepsilon > 0 \) and \( m > 0 \) such that \( |w - \hat{F}^m w| < \varepsilon \), then there exists a \( m \)-periodic point \( p \in \Xi_0 \) so that, for \( \varphi = \max\{\gamma, \rho\} \) and a constant \( K > 0 \) depending only on \( \hat{F} \), we get

\[
|\hat{F}^i p - \hat{F}^i w| \leq K \varphi^\min\{i, m-i\}|w - \hat{F}^m w| \leq K \varphi^\min\{i, m-i\}, \quad i = 0, \ldots, m.
\]

Now the proof of our version of Livsic’s Theorem is straightforward. Let \( \chi, \phi \) and \( \Gamma \) be as in the statement of Lemma 2.7. Since, by assumption, periodic orbits \( p \) of \( \hat{F} \) belong to \( \Xi_0 \), then \( S_k \phi(p) = \sum_{i=0}^{k-1} \phi(\hat{F}^i p) = 0 \) for any period \( k \) of \( p \).

Let \( w \in \Gamma \) be such that \( \text{Closure}\{w_i : i \geq 0\} \supset \Gamma \) where \( w_i = (x_i, y_i) = \hat{F}^i w, i \geq 0 \). Using the cocycle relation (2.3) we get \( \chi(w_i) = \chi(w) + S_i \phi(w), i \geq 0 \).

Thus, whenever we have \( \varepsilon = |w - w_i| \) small for some \( i > 0 \), we have a periodic point \( p = (z, \hat{z}) \) of period \( i \) satisfying the Closing Lemma 2.8. We write \( p_k = (z_k, \hat{z}_k) = \hat{F}^k p \) and let \( h_k \) be the inverse branch of \( F \mid \mathcal{P}(z_k), k = 0, \ldots, i-1 \). Then because \( \phi \in \mathcal{C}_{\alpha, \text{loc}}^\theta(\Delta) \)

\[
|\chi(w_i) - \chi(w)| = |S_i \phi(w)| = |S_i \phi(w) - S_i \phi(p)| \leq \sum_{k=0}^{i-1} |\phi(w_k) - \phi(p_k)|
\]

\[
\leq \sum_{k=0}^{i-1} |\phi(h_k F x_{k+1}, y_k) - \phi(h_k F z_{k+1}, \hat{z}_k)| \leq \sum_{k=0}^{i-1} \|\phi\|_{\alpha, \text{loc}}(|z_{k+1} - x_{k+1}|^\theta + |y_k - \hat{z}_k|)
\]

\[
\leq \|\phi\|_{\alpha, \text{loc}} \left( \sum_{k=1}^{i} K \varphi^\min\{k, i-k\}|w - w_i| + \sum_{k=0}^{i-1} K \varphi^\min\{i, i-k\}|w - w_i| \right)
\]

\[
\leq \frac{\max\{K, K \varphi\} \|\phi\|_{\alpha, \text{loc}}}{1 - \varphi^\theta}|w - w_i|^{\theta}.
\]

Hence, \( \chi \) is \( \theta \)-Hölder on a dense subset of \( \Gamma \). Thus, \( \chi \) admits a unique \( \theta \)-Hölder extension, with the same Hölder constant, to the closure of \( \Gamma \). This completes the proof of Lemma 2.7. \( \square \)

3. The main reduction result

We present the reduction result at the core of Theorem A. We construct a \( C^2 \)-open and \( C^1 \)-dense subset of vector fields that are semiconjugated to a \( C^{1+\alpha} \) generalized hyperbolic skew product semiflow and have the necessary properties that allow us to transfer the decay of correlations obtained in Theorem 2.5 to the original flow.

Theorem 3.1. Let \( \mathcal{U} \) be the \( C^2 \) open subset of vector fields so that every \( X \in \mathfrak{X}^2(M) \) admits a non-trivial connected singular-hyperbolic attracting set \( \Lambda \) with a neighborhood \( U \) as trapping region. For any \( X \in \mathcal{U} \) there exists \( \alpha \in (0, 1) \) so that, for all small enough \( \varepsilon > 0 \) the following holds.

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We can find a \( C^\infty \) function \( \varphi : M \to (1/2, 3/2) \), which is \( \varepsilon C^2 \)-close to 1, and such that \( Y = \varphi \cdot X \) admits a \( C^2 \)-neighborhood \( \mathcal{V} \) so that for each ergodic physical measure \( \mu \in \mathcal{V} \cap \mathcal{U} \) supported in \( U \), there exists a \( C^{1+\alpha} \) generalized hyperbolic skew product semiflow \( \tilde{F}_t : \hat{\Delta}^r \to \hat{\Delta}^r \) with roof function \( r \) satisfying the UNI condition and a map \( p : \hat{\Delta}^r \to U \) satisfying:

(i) \( Z_t \circ p = p \circ \tilde{F}_t \), for all \( t > 0 \) and \( p_* \mu^r_{\tilde{F}} = \mu \);
(ii) there exists a constant \( C > 0 \) such that \( \| \varphi \circ p \|_\alpha \leq C |\varphi|_{C^1} \) for all \( \varphi \in C^1(U) \) and \( \| \psi \circ p \|_{\alpha,2} \leq C |\psi|_{C^3} \) for all \( \psi \in C^3(U) \).

This ensures that the flow \( Z_t : U \to U \) also mixes exponentially for sufficiently smooth observables and for each physical/SRB probability measure. If \( \mu \) is an ergodic physical/SRB probability measure supported on the singular-hyperbolic attracting set \( \Lambda \), and \( \phi, \psi : U \to \mathbb{R} \) are \( C^r \) for some \( r > 1 \) and \( \phi \circ p, \psi \circ p \) are also \( C^{\alpha,2}_{\text{loc}}(\hat{\Delta}_0^r) \) observables on \( \hat{\Delta}_0^r \), where \( p \) is the conjugacy provided by the previous theorem, then

\[
\int \phi \cdot (\psi \circ Z_t) \, d\mu = \int (\phi \circ p) \cdot (\psi \circ Z_t \circ p) \, d\mu^r_{\tilde{F}} = \int (\phi \circ p) \cdot (\psi \circ p) \circ \tilde{F}_t \, d\mu^r_{\tilde{F}}
\]

tends to zero exponentially fast. A standard approximation argument [20, Proof of Corollary 1] enables us to deduce exponential mixing for Hölder observables.

To obtain the same result for any physical/SRB probability measure \( \mu \) supported on the singular-hyperbolic attracting set \( \Lambda \), use Theorem 1.6 to write \( \mu = \sum_{i=1}^{k} t_i \mu_i \) as a linear convex combination of finitely many ergodic physical/SRB measures supported on \( \Lambda \), and then in the same setting as in the previous paragraph

\[
\int \phi \cdot (\psi \circ Z_t) \, d\mu = \sum_{i=1}^{k} \int \phi \cdot (\psi \circ Z_t) \, d\mu_i
\]

converges to zero exponentially fast, proving the statement of Theorem A.

To present a detailed overview of the proof of Theorem 3.1, we first synthesize results already published on the type of equilibria and the construction and properties of the global Poincaré return map for singular-hyperbolic attracting sets, in Subsections 3.1 through 3.4. Then we show how to use these properties to represent the flow on a singular-hyperbolic attracting set as a generalized \( C^{1+} \) hyperbolic skew-product semiflow, in Subsections 3.5 through 3.7. Finally, we describe how to obtain the UNI condition on an open and dense subset of vector fields with singular-hyperbolic attracting sets, in Subsection 3.8.

3.1. Equilibria for singular-hyperbolic attracting sets.

Proposition 3.2. [10; Proposition 2.6] Suppose that \( \Lambda \) is a singular-hyperbolic attractor and let \( \sigma \in \Lambda \) be an equilibrium. Then \( \sigma \) is Lorenz-like. That is, \( DG(\sigma)|E^c_{\sigma} \) has real eigenvalues \( \lambda^s, \lambda^u \) satisfying \(-\lambda^u < \lambda^s < 0 < \lambda^u\).

Remark 3.3. Some consequences of singular-hyperbolicity follow.
(1) Partial hyperbolicity of Λ implies that the direction $X(x)$ of the flow is contained in the center-unstable bundle $E_{x}^{cu}$ at every point $x$ of Λ (see [5] Lemma 5.1).

(2) The index of a singularity $σ$ in a singular-hyperbolic set Λ equals either $\dim E_{σ}^{s}$ or $1 + \dim E_{σ}^{s}$. That is, $σ$ is either a hyperbolic saddle with $\dim M - \dim E_{σ}^{s} = 2$ (that is, the codimension of $E_{σ}^{s}$ equals 2) or a Lorenz-like singularity.

(3) If a singularity $σ$ in a singular-hyperbolic set Λ is not Lorenz-like, then there is no regular orbit of Λ that accumulates $σ$ in the positive time direction. In other words, there is no $x ∈ Λ$ regular such that $σ ∈ ω(x)$ (see [13] Remark 1.5))

### 3.2. Construction of the global Poincaré return map.

In [12] the construction of a global Poincaré map for any singular-hyperbolic attractor is carried out based on the existence of “adapted cross-sections” and $C^{1+α}$ stable holonomies on these cross-sections. With the results just presented this construction can be performed for any singular-hyperbolic attracting set. This construction was presented in [10], Sections 3 and 4, so from there we obtain:

- a finite collection $Σ_1, \ldots, Σ_m$ of (pairwise disjoint) cross-sections for $X$ so that
  - each $Σ_i$ is diffeomorphically identified with $(-1, 1) × B^{d_s}$;
  - the stable boundary $∂^s Σ_i ≃ \{±1\} × B^{d_s}$ consists of two curves contained in stable leaves; and
  - each $Σ_i$ is foliated by $W_{x}^{s}(Σ_i) = \bigcup_{|t|<\varepsilon_0} X_{t}(W_{x}^{s}) ∩ Σ_i$ for a small fixed $\varepsilon_0 > 0$. We denote this foliation by $W_{x}^{s}(Σ_i), i = 1, \ldots, m$;

- a Poincaré map on the disjoint union $Ξ = Σ_1 + \cdots + Σ_m$ given by $P : Ξ \setminus Γ → Ξ, P(x) = X_{τ(x)}(x)$ with $τ : Ξ \setminus Γ → [T, +∞)$ the associated return time for a big enough $T > 0$, which is $C^2$ smooth in $Σ_i \setminus Γ, i = 1, \ldots, m$, and preserves the foliation $W_{x}^{s}(Ξ)$, where $Γ = Γ_0 ∪ Γ_1$ is a finite family of stable disks $W_{x}^{s}(Ξ)$ defined as follows
  - $Γ_0 = \{x ∈ Ξ : X_{T+1}(x) ∈ \bigcup_{σ ∈ S} γ_{σ}^{0}\}$ for $S = S(X, Λ) = \{σ ∈ Λ : X(σ) = 0\}$ and $γ_{σ}^{0}$ is the local stable manifold of $σ$ in a small fixed neighborhood of $σ ∈ S$;
  - $Γ_1 = \{x ∈ Ξ : P(x) ∈ ∂^s Ξ = \bigcup_{i=1}^m ∂^s Σ_i\}$;

- and open neighborhoods $V_{σ}$ for each $σ ∈ S$ so that defining $V_0 = ∪_{σ ∈ S} V_{σ}$ we have that every orbit of a regular point $z ∈ U_0 \setminus V_0$ eventually hits $Ξ$ or else $z ∈ Γ$.

Having this, the same arguments from [12] (see [10] Proposition 4.1 and Theorem 4.3 and [3] Section 2.3]) show that $DP$ contracts $TW_{x}^{s}(Ξ)$ and expands vectors on the unstable cones $\{C_{x}^{u}(Ξ) := C_{x}^{u}(a) ∩ T_{x}Ξ\}_{x ∈ Ξ \setminus Γ}$. The stable holonomies for $P$ enable us to reduce its dynamics to a one-dimensional map, as follows.

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2We write $A + B$ the union of the disjoint subsets $A$ and $B$. 

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Let $\Sigma$ be a cross-section in $\Xi$. A smooth curve $\gamma : [0,1] \to \Sigma$ is called a $u$-curve if $\gamma'(t) \in C^u_{\gamma(t)}(\Xi)$ for all $t \in [0,1]$. We also use the term curve to denote the image of the curve. We say that the $u$-curve $\gamma$ crosses $\Sigma$ if each leaf $W^s_\gamma(\Sigma)$ of $\Sigma$ intersects $\gamma$ in a unique point.

Let $\gamma_i \subset \Sigma_i$ be $u$-curves that cross $\Sigma_i$, $i = 1,2,\ldots,m$. The (sectional) stable holonomy $\pi_\gamma : \Xi \to \gamma = \sum_{i=1}^m \gamma_i$ is defined by setting $\pi_\gamma(x)$ to be the intersection point of $W^s_\gamma(\Sigma_i)$ with $\gamma_i$, for $x \in \Sigma$ and $i = 1,2,\ldots,m$.

**Lemma 3.4.** [10 Lemma 7.1] The stable holonomy $\pi_\gamma$ is $C^{1+\alpha}$ for some $\alpha > 0$.

**Remark 3.5.** The result of Lemma 3.4 is crucial in our argument and is the reason for the restriction on the codimension of the stable bundle of $\Lambda$. If $E^s$ has codimension higher than two, the $C^{1+}$-smoothness of the stable holonomy $\pi$ is expected to fail in general; see e.g. [33] and [24].

Following the same arguments in [12] (see also [10] Section 7]) we obtain a one-dimensional piecewise $C^{1+\alpha}$ quotient map over the stable leaves $f_\gamma : \gamma \setminus \Gamma \to \gamma$ for some $0 < \alpha < 1$ so that $\pi_\gamma(P(x)) = f_\gamma(x)$ and $|f_\gamma'(x)| > 2$, for all $x \in \gamma \setminus \Gamma$.

Let $\gamma_i : I_i \to \Sigma_i$ be a smooth parametrization of a $u$-curve in $\Sigma_i$ for each $i = 1,2,\ldots,m$. We assume that $\{I_i : i = 1,2,\ldots,m\}$ is a family of disjoint intervals, define $I = I_1 + \cdots + I_m$ and set a parametrization of $\gamma$ as $\gamma : I \to \Xi$ by $\gamma(t) = \gamma_i(t)$ if $t \in I_i$. Using the last parametrization we can identify $f_\gamma$ with the one-dimensional map $f : I \setminus \mathcal{D} \to I$ by $f(x) = \gamma^{-1}(f_\gamma(\gamma(x)))$, where $\mathcal{D} = \gamma^{-1}(\pi_\gamma(\Gamma))$ is the critical set for $f$. Moreover, defining the singular set $\mathcal{S} = \gamma^{-1}(\pi_\gamma(\Gamma_0))$ we get, as shown in [10], that $f'|_{I \setminus \mathcal{D}}$ behaves like a power of the distance near $\mathcal{S}$ in the following sense: there exist constants $\eta \in (0,1)$ and $C,q > 0$ such that

1. $C^{-1} \text{dist}(x,\mathcal{S})^q \leq |f'(x)| \leq C \text{dist}(x,\mathcal{S})^{-q}$, for all $x \in I \setminus \mathcal{S}$;
2. $|\log |f'(x)| - \log |f'(y)|| \leq C|x-y|^q(|f'(x)|^{-q}+|f'(x)|^q)$, for all $x,y \in I \setminus \mathcal{S}$, with $|x-y| < \text{dist}(x,\mathcal{S})/2$;

where we let dist denote the Euclidean distance in the interval $I$.

**Remark 3.6.** With the identifications above and in order to simplify notations, we make no distinction between $I$ and $\gamma$, $f$ and $f_\gamma$, $\mathcal{D}$ and $\pi_\gamma(\Gamma)$ and $\mathcal{S}$ and $\pi_\gamma(\Gamma_0)$. We assume in what follows that $I = [0,1]$.

Indeed, quotient maps are conjugated:

1. For $j = 1,2$ let $\gamma^j = \sum \gamma_i^j$, where $\gamma_i^j$ is a $u$-curve in $\Sigma_i$. If $f_j : \gamma^j \setminus \Gamma \to \gamma^j$ are two quotients along stable leaves (as explained above), then they are $C^{1+\alpha}$ conjugated. Indeed, let $\pi_j$ be the stable holonomy with respect to $\gamma^j$. Defining $g : \gamma^1 \to \gamma^2$ by $g = \pi_2 |_{\gamma^1}$ it follows that $g$ is a $C^{1+\alpha}$ diffeomorphism. We claim that $g$ is a conjugacy between $f_1$ and $f_2$. By the invariance of the stable leaves under the Poincaré map, we have that $P(g(x)) \in W^s_{P(x)}(\Xi)$ for all $x \in \gamma^1$. Hence $g(f_1(x)) = g(\pi_1(P(x))) = \pi_2(P(g(x))) = f_2(g(x))$ for all $x \in \gamma^1$.
(b) Moreover, it follows from (a) that there exists a constant $C > 0$, depending only on the holonomy map $g$, such that

$$C^{-1}|f_2(g(x_1)) - f_2(g(x_2))| \leq |f_1(x_1) - f_1(x_2)| \leq C|f_2(g(x_1)) - f_2(g(x_2))|, \quad (3.1)$$

for all $x_1, x_2 \in \gamma_i^1, i = 1, 2, \ldots, m$.

For $0 < \delta < 1$, we set $\delta$-truncated distance of $y$ to $D$ on $I$ by

$$\text{dist}_\delta(y, D) = \text{dist}(y, D) \cdot 1_{B(D, \delta)}(y) + 1_{I \setminus B(D, \delta)}(y), \quad y \in I,$$

where $1_A$ is the characteristic function of the subset $A$. Given $\delta > 0$, let $B(\Gamma, \delta) = \{x \in \Xi : \text{dist}(x, \Gamma) < \delta\}$. We say that a function $\varphi : \Xi_0 = \Xi \setminus \Gamma \to \mathbb{R}$ has logarithmic growth near $\Gamma$ if there is a constant $C = C(\varphi) > 0$ such that for every small $\delta$ it holds $|\varphi(x)| \cdot 1_{B(\Gamma, \delta)}(x) \leq C|\text{log dist}_\delta(\pi_\gamma(x), D)|$, for all $x \in \Xi_0$.

The construction outlined above can be summarized as in [13, Theorem 2.8], as follows.

**Theorem 3.7.** [13, Theorem 2.8] Let $X \in \mathcal{X}^2(M)$ be a vector field admitting a non-trivial connected singular-hyperbolic attracting set $\Lambda$. Then there exists $\alpha > 0$, a finite family $\Xi$ of cross-sections and a global Poincaré map $P : \Xi_0 \to \Xi$, $P(x) = X_{\tau(x)}(x)$ such that

1. the domain $\Xi_0 = \Xi \setminus \Gamma$ is the entire cross-sections with a family $\Gamma$ of finitely many smooth arcs removed and
   (a) $\tau : \Xi_0 \to [\tau_0, +\infty)$ is a smooth function with logarithmic growth near $\Gamma$ and bounded away from zero by some uniform constant $\tau_0 > 0$;
   (b) there exists a constant $C > 0$ so that $|\tau(x) - \tau(y)| < C\text{dist}(x, y)$ for all points $y \in W^s_x(\Xi)$;
2. We can choose coordinates on $\Xi$ so that the map $P$ can be written as $F : \tilde{Q} \to Q$, $F(x, y) = (f(x), g(x, y))$, where $Q = I \times B^d$, $I = [0, 1]$ and $\tilde{Q} = Q \setminus \tilde{\Gamma}$ with $\tilde{\Gamma} = D \times B^d$ and $D = \{c_1, \ldots, c_n\} \subset I$ a finite set of points.
3. The map $f : I \setminus D \to I$ is a piecewise $C^{1+\alpha}$ map with finitely many branches, defined on the connected components of $I \setminus D$, with finitely many ergodic absolutely continuous invariant probability measures $\mu^i_f, i = 1, \ldots, k$, whose ergodic basins cover $I$ Lebesgue modulo zero. Also $\inf\{|f'| : x \in I \setminus D\} > 2$.
4. The map $g : \tilde{Q} \to B^d$ preserves and uniformly contracts the vertical foliation $\mathcal{F} = \{x \times B^d\}_{x \in I}$ of $Q$: there is $\lambda \in (0, 1)$ so that $\text{dist}(g(x, y_1), g(x, y_2)) \leq \lambda \cdot |y_1 - y_2|$ for each $x \in I \setminus D$ and $y_1, y_2 \in B^d$. Moreover, $g$ is dominated by $f$: there exists $0 < c < 1$ so that $\|D_1g(x, y)\| \leq c\|DF(x)\|$ for all $(x, y) \in \tilde{Q}$.
5. The map $F$ admits a finite family of physical ergodic probability measures $\mu^i_F$, which are induced by $\mu^i_f$ in a standard way; see Proposition 2.8 and [12] Section 6.1] where it is shown how to get $(\pi_\gamma)_*\mu^i_F = \mu^i_f$. Moreover, the Poincaré time $\tau$ is integrable both with respect to each $\mu^i_f$ and with respect to the two-dimensional Lebesgue area measure of $Q$. 

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Remark 3.8. If \( \Lambda \) is a singular-hyperbolic attractor, then the number \( k \) of \( f \)-invariant and \( F \)-invariant measures in the above statement is \( k = 1 \); see e.g. [12]. In particular, we get \( k = 1 \) if \( \Lambda \) is a uniformly hyperbolic attractor, or a basic piece of the “Spectral Decomposition” of an Axiom A vector field, with codimension two stable direction.

We recall that each of the measures \( \mu^i_f \) and \( \mu^i_F \) induce an ergodic physical measure \( \mu^i_X \) for the flow of \( X \) supported in \( \Lambda \); see e.g. [12, Section 6].

3.3. The support of a non-singular ergodic physical measure is a hyperbolic attractor. The following enables us to interpret our results in the uniformly hyperbolic setting, as a consequence of Theorem 1.3 together with the absolutely continuous disintegration of physical/SRB measures.

Proposition 3.9. Let \( \mu \) be a physical/SRB ergodic probability measure whose support \( \text{supp} \mu \) is contained in a sectional-hyperbolic attracting set \( \Lambda = \Lambda_G(U) \). If \( \text{supp} \mu \) does not contain Lorenz-like singularities, then \( \text{supp} \mu \) is a hyperbolic attractor.

Remark 3.10. Since each hyperbolic attractor for a \( C^2 \) vector field supports a unique physical/SRB measure \([22]\), and is singular-hyperbolic if the stable bundle has codimension two, then we can apply our results to this class of hyperbolic attractors.

Proof of Proposition 3.9. Let \( \mu \) be a physical ergodic probability measure supported in \( \Lambda \), and let us assume that \( A = \text{supp} \mu \) contains no Lorenz-like singularities. Since \( A \) is transitive, then by Remark 3.3 there can be no other (hyperbolic) singularities in \( A \). Then the compact invariant subset \( A \) is uniformly hyperbolic, by Theorem 1.3.

The SRB property can be geometrically described as follows; see e.g. [30]. For \( \mu \)-a.e. \( x \) there exists a neighborhood \( V_x \) where \( \mu \) admits a disintegration \( \{ \mu^\gamma \}_{\gamma \in \mathcal{F}(V_x)} \) over the strong-unstable leaves \( \mathcal{F}(V_x) \) that cross \( V_x \) which is absolutely continuous with respect to the induced volume measure on the leaves. More precisely, we have

- \( \mu(\varphi) = \int \mu^\gamma(\varphi) \, d\hat{\mu}(\gamma) \) for each bounded measurable observable \( \varphi : M \to \mathbb{R} \), where \( \hat{\mu} \) is the quotient measure on the leaf space induced by \( \mu \) and, in addition

- \( \mu^\gamma = \psi_\gamma \text{Leb}_\gamma \) for \( \hat{\mu} \)-a.e. \( \gamma \), where \( \text{Leb}_\gamma \) denotes the measure induced on the leaf \( \gamma \in \mathcal{F}(V_x) \) (which is a submanifold of \( M \)) by the Lebesgue volume measure; and \( \psi_\gamma : \gamma \to [0, +\infty) \) is a strictly positive measurable and \( \text{Leb}_\gamma \)-integrable density function.

In particular, \( \text{Leb}_\gamma \)-a.e. point of \( \hat{\mu} \)-a.e. leaf in \( \mathcal{F}(V_x) \) belongs to \( A \). Indeed, given any full measure subset \( A_1 \) of \( A \), we have that \( \mu^\gamma(A_1) = 1 \) for \( \hat{\mu} \)-a.e. \( \gamma \) and hence \( \text{Leb}_\gamma(A_1) = 1 \) also. Since a full Lebesgue measure subset is dense and \( A \) is closed, we see that the unstable leaf \( \gamma \) is contained in \( A \) for a \( \mu \)-positive measure subset of \( V_x \).

In addition, unstable leaves of inner radius \( \epsilon > 0 \) are defined on all points of \( A \) by uniform hyperbolicity and the map \( A \ni x \mapsto W^u_\epsilon(x) \) is continuous in the \( C^1 \) topology of disk embeddings; see e.g. [22, Chapter 6].

Then, since the previous property holds for a full \( \mu \)-measure subset of points \( x \), which is dense in \( A \), we see that \( A \) contains the unstable leaves through a dense
subset of its points. By continuity of the unstable foliation in a hyperbolic set, we conclude that \( A \) contains the unstable manifold through each of its points. This ensures that \( A \) is a hyperbolic attractor:

- the support of the ergodic measure \( \mu \) is topologically transitive;
- the union the center stable leaves \( W^s(y) \), through each point \( y \) of the local strong-unstable leaf \( W^\text{uu}_t(x) \), is a open subset contained in the topological basin of \( A \);
- the flow is at least \( \mathcal{C}^2 \), hence we can apply [17, Proposition 5.4 & Theorem 5.6].

The proof is complete. \( \square \)

3.4. Properties of the global Poincaré return time. In what follows we state the linearization result of [29] in the particular case of a saddle singularity for a 2-dimensional flow.

**Lemma 3.11.** [29, Theorem 1.5] Let \( M \) be a surface and \( Z \in \mathcal{X}^{1+\alpha}(M) \), with \( 0 < \alpha < 1 \). If \( \sigma \in M \) is a singularity of saddle type for \( Z \) and \( L = DZ(\sigma) \), then there are a neighborhood \( V \subset M \) of \( \sigma \), a real number \( \beta \in (0, \alpha) \) and a \( \mathcal{C}^{1+\beta} \) diffeomorphism \( h \) from \( V \) onto its image such that \( h(\sigma) = 0 \) and \( h(Z_t(x)) = L_t(h(x)) \), for all \( t \in \mathbb{R} \) such that \( V \cap Z_{-t}(V) \neq \emptyset \) and all \( x \in V \cap Z_{-t}(V) \).

Every Lorenz-like singularity \( \sigma \) admits local central-unstable invariant manifolds \( W = W_\sigma \) in a neighborhood of \( \sigma \), as smooth as the vector field \( X \), such that \( T_\sigma W = E^\text{cu}_\sigma = E^u_\sigma \oplus E^s_\sigma \), where \( E^u_\sigma \) and \( E^s_\sigma \) are the eigenspaces of \( DX(\sigma) \) corresponding to the positive and least negative eigenvalues of \( DX(\sigma) \).

**Remark 3.12.** In fact, given any cross-section \( \Sigma_\sigma \) transverse to the local weak stable manifold of \( \sigma \), as smooth as the vector field \( X \), such that \( T_\sigma W = E^\text{cu}_\sigma = E^u_\sigma \oplus E^s_\sigma \), where \( E^u_\sigma \) and \( E^s_\sigma \) are the eigenspaces of \( DX(\sigma) \) corresponding to the positive and least negative eigenvalues of \( DX(\sigma) \).

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By Remark 3.6(2) and the previous observation, the following estimate on the Poincaré return time function \( \tau \) (cf. Subsection 3.2), obtained using a central-unstable leaf, holds for all \( cu \)-curves \( \gamma \) if we choose coordinates given by projections along the stable foliation.

**Lemma 3.13.** [14, Lemma 3.6] Let \( x, y \in I \setminus \mathcal{D} \) such that there is no element of \( \mathcal{D} \) between \( x \) and \( y \). Then there exist constants \( \alpha, C > 0 \) so that

\[
|\tau(x) - \tau(y)| \leq C \left( \frac{|x - y|}{\min\{\text{dist}(x, \Gamma), \text{dist}(y, \Gamma)\}} + |x - y|^\alpha \right) \quad \& \quad |\tau'(x)| \leq \frac{C}{\text{dist}(x, \Gamma)}.
\]
Remark 3.14 (Horizontal lines are $u$-curves). We choose an identification $L : I \times B^d \to \Xi$ such that for all $c \in B^d$, the curve $\gamma_{i,c} : I_i \to \Sigma_i$ defined by $\gamma_{i,c}(t) = L(t, c)$ is a $u$-curve in $\Sigma_i$ for all $i = 1, 2, \ldots, m$. In other words, with the identification given by $L$ we may assume that each horizontal line $\{(t, c) : t \in I_i\}$ is a $u$-curve for all $c \in B^d$.

3.5. Induced piecewise expanding Markov map for the one-dimensional quotient transformation. For each $f$-invariant ergodic absolutely continuous probability measure $\nu$, we fix a point $p \in \text{int}(\text{supp}\ \nu)$ and an integer $N \geq 1$ such that $\cup_{j=0}^{N} f^{-j}(p)$ is $\delta_1/3$-dense in $\text{supp} \ \nu$ and does not contain any element of $\mathcal{D}$. The following is [14, Theorem 4.10 & Remark 3.9].

Theorem 3.15. [1, 15, 23] There exists a neighborhood $\Delta \subset \text{int}(\text{supp} \ \nu) \setminus \mathcal{D}$, a countable Lebesgue modulo zero partition $\mathcal{P}$ of $\Delta$ into sub-intervals; a function $R : \Delta \to \mathbb{N}$ defined almost everywhere, constant on elements of the partition $\mathcal{P}$; and constants $C > 0$, $0 < \rho < 1$ such that, for all $J \in \mathcal{P}$ and $R = R(J)$, the map $F := f^R : J \to \Delta$ is a $C^{1+\alpha}$ diffeomorphism, satisfies the bounded distortion property and is uniformly expanding: for each $x, y \in J$

$$|F'(x)/F'(y) - 1| \leq C|F(x) - F(y)|^\alpha \quad \text{and} \quad |F(x) - F(y)| > \rho^{-1}|x - y|.$$

3.6. The $C^{1+\alpha}$ expanding semiflow. Now we check the conditions (3) and (4) from Subsection 2.1 to obtain an expanding semiflow associated to each ergodic absolutely continuous $f$-invariant probability measure.

3.6.1. The induced roof function. Let $r : \Delta \to \mathbb{R}$ be defined as $r(x) = \sum_{j=0}^{R(x)-1} \tau(f^j(x))$, for all $x \in \Delta$. Then $r$ satisfies conditions (3) and (4) of Subsection 2.1.2.

Proposition 3.16. [14] Lemma 3.9, Propositions 3.10 & 3.11] The function $r$ has exponential tail and, for all $h \in \mathcal{H}$, we have that $|(r \circ h)'|_\infty < +\infty$.

3.7. Generalized $C^{1+\alpha}$ skew product semiflow. Now we note that from Theorem 3.7 we already have all that is needed to obtain a generalized $C^{1+\alpha}$ skew product semiflow, with the exception of the UNI condition (5), which we focus on the Subsection 3.8.

Indeed, let $\hat{\Delta} = \cup_{x \in \Delta} W_x^s$ and define $\hat{F} : \hat{\Delta} \to \hat{\Delta}$ by $\hat{F}(x) = P^{R(px)}(x)$ and the suspension semiflow $\hat{F}_t : \hat{\Delta} \to \hat{\Delta}$ with base map $\hat{F}$ and roof function $\hat{r}(x) = \sum_{j=0}^{R(px)-1} \tau(\hat{F}^j x)$.

Through the identification of $\hat{\Delta}$ with $\Delta \times B^d$, it follows from item 4 of Theorem 3.7 that $\hat{F}_t$ satisfies condition (4) of a hyperbolic skew product semiflow, from Subsection 2.1.

3.8. The $C^2$-robust UNI condition after small $C^1$-perturbations. We denote by $\mathcal{X}_{sh}^s(M)$ the subset of $\mathcal{X}^s(M)$ that admits a singular-hyperbolic attracting set, for some $s \geq 2$.

We construct a $C^2$-open and dense subset $\mathcal{V}$ of $\mathcal{X}_{sh}^2(M)$ where all vector fields in $\mathcal{V}$ have a roof function that satisfies the UNI condition. In particular, we construct a
family of suspension semiflows, one for each ergodic physical measure of the attracting set, which satisfy the conditions of Theorem 2.5 and we deduce exponentially mixing for them.

Recall that there exists a one-to-one correspondence between the periodic points of the Poincaré map and its quotient along the stable leaves.

**Lemma 3.17.** A point \( x \in I \) is periodic for \( f \) if and only if there exists a periodic point \( w \in \pi^{-1}(x) \) for the Poincaré map \( P \).

**Remark 3.18.** Using Lemma 3.17, from now on we make no distinction between a periodic point of the Poincaré map and its quotient along stable leaves.

For a vector field \( X \in \mathcal{X}_{sh}^2(M) \) we can repeat the constructions of the previous subsections. We need to perform the constructions for more than one vector field so, where necessary, we make the dependence on the vector field explicit in what follows.

For instance, \( P_X : \Xi \setminus \Gamma_X \to \Xi \) denotes the Poincaré map of \( X \) with Poincaré time given by \( \tau_X \); and \( f_X : I \setminus \mathcal{D} \to I \) is the corresponding one-dimensional quotient map.

**Definition 3.19.** Let \( \Lambda \) be a singular-hyperbolic attracting set for \( X \in \mathcal{X}_{sh}^2(M) \). Let \( P : \Xi \setminus \Gamma \to \Xi \) and \( f : I \setminus \mathcal{D} \to I \) be the global Poincaré map and its quotient along stable leaves, respectively, for the flow of \( X \) as in Subsection 3.2.

We say that \( X \) satisfies the UNI condition if, for each ergodic physical measure \( \mu \) of \( X \) corresponding to an ergodic physical measure \( \hat{\nu} \) of \( P \) given by an ergodic \( f \)-invariant absolutely continuous probability measure \( \nu \), there exists an open interval \( \Delta \subset \text{int}(\text{supp } \nu) \subset I \) and an inducing time function \( R : \Delta \to \mathbb{N} \) (as in Theorem 3.15) such that the induced roof function \( r : \Delta \to \mathbb{R} \) given by \( r(x) = \sum_{j=0}^{R(x)-1} \tau(f^j(x)) \) satisfies the UNI condition.

We note that in this definition we are calculating \( \tau \) at the central-unstable curve \( \gamma \) used to define the map \( f = f_\gamma \); see Subsection 3.2. We need to consider also \( \hat{r} : \hat{\Lambda} \to \mathbb{R} \) given by \( \hat{r}(w) = \sum_{j=0}^{R(\pi w)-1} \tau(P^j(w)) \) the induced roof function for the Poincaré map, associated to \( \hat{F} \).

Let us fix an ergodic physical measure \( \hat{\nu} \) for \( P \) and \( \nu \) for \( f \), and let \( F = f^R : \Delta \to \Delta \) be an induced full branch Markov map constructed for \( X \in \mathcal{X}_{sh}^2(M) \) as in Theorem 3.15. For a function \( \varphi : \Delta \to \mathbb{R} \) and \( n \geq 1 \) we denote \( S^F_n \varphi = \sum_{j=0}^{n-1} \varphi \circ F^j \).

Now we describe the subset \( \mathcal{V} \) where Theorem 3.1 holds. We define \( \mathcal{V} \) to be the subset of vector fields in \( \mathcal{X}_{sh}^2(M) \) such that

- for all \( X \in \mathcal{V} \), each physical measure \( \nu \) of \( f \) and each corresponding induced Markov map \( F : \Delta \to \Delta \), there exist two distinct periodic points \( x_1, x_2 \in \Delta \subset \text{int}(\text{supp } \nu) \) for the induced Markov map \( F \) with the same period \( p \) and satisfying:
  1. the orbits are distinct; and they visit the interior of the same elements of the partition \( \mathcal{P} \) the same number of times as the other, but necessarily in some different order to each other; and
(ii) \( S^F_p r(x_1) \neq S^F_p r(x_2) \).

Since \textit{not} satisfying the UNI condition implies that \( r|_\Delta \) is \( C^1 \)-cohomologous to a function which is constant on each element of the partition \( \mathcal{P} \) of the uniformly expanding map \( F \), then we obtain the following.

**Lemma 3.20.** [14, Lemma 5.4] The vector fields in \( \mathcal{V} \) satisfy the UNI condition.

The proofs of the next two propositions follow the same steps presented in [14], because there exists a conjugation between \( \hat{\Delta}^r \) and \( \hat{\Delta}^{r \circ \pi} \) thanks to the existence of \( \chi \) satisfying the relation (2.3). In particular, this ensures that

\[(S^F_p \pi)(w_i) = (S^F_p r)(x_i), \quad i = 1, 2;\]

where \( w_i \) is the periodic orbit of \( F \) corresponding to the periodic orbit \( x_i \) of \( F \), that is, \( x_i = \pi w_i \).

The next proposition shows that, if we start with a vector field \( X \in \mathfrak{X}_{sh}^2(M) \) which does not satisfy the UNI condition and change slightly the velocity of a well chosen periodic orbit, then the new vector field satisfies the UNI condition and is arbitrarily \( C^1 \)-close to the initial vector field \( X \). In particular, we get that the subset of vector fields in \( \mathfrak{X}_{sh}^2(M) \) that satisfy the UNI condition is dense in the \( C^1 \) topology.

**Proposition 3.21.** The set \( \mathcal{V} \) is \( C^1 \)-dense in \( \mathfrak{X}_{sh}^2(M) \): for each \( X \in \mathfrak{X}_{sh}^2(M) \) there exists \( \delta > 0 \) and a \( \delta \)-\( C^1 \)-close vector field \( Y \in \mathcal{V} \) which is a multiple of \( X \).

In other words, any \( X \in \mathfrak{X}_{sh}^2(M) \) admits a \( C^1 \) close time reparametrization which lies in \( \mathcal{V} \).

**Proof.** We follow the proof of [14, Proposition 5.5]. Let \( X \in \mathfrak{X}_{sh}^2(M) \) and assume that \( X \) does not satisfy the UNI condition. Hence, there exist an ergodic \( f_X \)-invariant absolutely continuous probability measure \( \nu \), an induced map \( F_X : \Delta_X \to \Delta_X \) with \( \Delta_X \subset \text{int}(\text{supp} \nu) \), a \( C^1 \) function \( \varphi : \Delta \to \mathbb{R} \), and a function \( \psi : \Delta_X \to \mathbb{R} \) constant on elements of the partition \( \mathcal{P}_X \) together with \( \varphi : \Delta_X \to \mathbb{R} \) of class \( C^1 \) such that \( r_X = \psi + \varphi \circ F_X - \varphi \).

Let \( x_1, x_2 \in \Delta_X \) be two periodic points with the same period \( p \) for \( F_X \). Since \( F_X \) is a full branch Gibbs-Markov map, we may assume that the orbits are distinct and visit the same elements of the partition \( \mathcal{P}_X \) the same number of times but in a different order. If \( J_1, J_2 \) are two disjoint elements of the partition \( \mathcal{P}_X \), we can choose \( p = 4 \), \( x_1 \) and \( x_2 \) so that

\[ x_1, F_X(x_1) \in J_1, F_X^2(x_1), F_X^3(x_1) \in J_2 \quad \text{and} \quad x_2, F_X^2(x_2) \in J_1, F_X(x_2), F_X^3(x_2) \in J_2. \]

By construction we get \( S^F_X r(x_1) = S^F_X \psi(x_1) = S^F_X \psi(x_2) = S^F_X r(x_2) \). If \( w_i \) is the unique \( \hat{F}_X \)-periodic orbit such that \( x_i = \pi w_i \), then \( t_0 = S^F_X \pi (w_i) = S^F_X r(x_i) \) is a multiple of the period of \( x_i \) with respect to the flow of \( X \), and does not depend on the functions \( \varphi \) and \( \psi \). We have \( X_{t_0}(w_i) = w_i, i = 1, 2 \).
We change the roof function $\tau : \hat{\Delta} \to \mathbb{R}^+$ around $w_2$ keeping the orbit of $w_1$ to ensure $(S_p^F x)(w_1) > S_p^F x (w_2)$ and so obtain $S_p^F x (x_1) > S_p^F x (x_2)$. Let $V_0$ and $V_1$ be open small neighborhoods of $w_2$ which do not intersect the orbit of $w_1$, so that $\overline{V_0} \subseteq V_1$ and take a $C^\infty$ bump function $\xi : M \to [0, 1]$ such that $\xi|_{V_0} \equiv 1$ and $\xi|_{M \setminus V_1} \equiv 0$. For any small $\delta > 0$, let $X_\delta = X + \delta X$ be a vector field which is $\delta$-close to $X$ in the $C^1$-topology. Inside $V_0$ we have $Y := X_\delta$ equal to $(1 + \delta)X$ and outside of $V_1$ it equals $X$. Hence, $w_2$ is still a periodic orbit for $Y$ but with a smaller period than before. Thus, we get $S_p^F x (x_1) > S_p^F x (x_2)$ as desired.

From Lemma 3.20 the field $Y$ satisfies the UNI condition. Moreover, we have $\|X - Y\|_1 = \delta \|X\|_1$ and so we can make the perturbation arbitrarily close to the original vector field in $C^1$ topology.

**Remark 3.22.** If we start with a $C^s$ vector field, for some $s \geq 2$, then the same argument gives a $\delta$-$C^s$-close vector field $X_\delta \in \mathcal{V}$.

In the following, we show that the inequality that we obtained in the previous proposition remains valid for $C^2$-close vector fields in $\mathcal{V}$.

**Proposition 3.23.** The set $\mathcal{V}$ is $C^2$-open in $\mathcal{X}_{sh}^2(M)$.

**Proof.** The same as the proof of [13, Proposition 5.7].

### 3.9. Exponential mixing for singular-hyperbolic attracting sets.

In order to prove Theorem 3.1, we show that the original flow is semiconjugated to a generalized suspension semiflow $\hat{F}_t$ and that, given observables with certain amount of regularity for the original flow, we get observables in the right space for the generalized suspension semiflow, and the measure in the original flow is the pushforward of the measure for the generalized suspension semiflow. This provides what is needed to transfer the results about decay of correlations from the generalized suspension semiflow to the original flow.

Let $X \in \mathcal{V}$ and, for each ergodic physical measure supported on the attracting set, let $F : \Delta \to \Delta$ be the induced Markov map for $X$ with inducing function given by $R$, as before. Again we consider also $\hat{\Delta} = \cup_{x \in \Delta} W_x^s$ and $\hat{F} : \hat{\Delta} \circ \hat{\sigma}$ defined by $\hat{F}(w) = P^{R(w)}(w)$ together with the generalized suspension semiflow $\tilde{F}_t : \hat{\Delta}^r \circ \hat{\sigma}$ with roof function $r(x) = \sum_{j=0}^{R(x)-1} R(p^j(x))$.

Using the identification of $\hat{\Delta}$ with $\Delta \times B^{d_1}$, it follows that $\hat{F}_t$ and $\mu_{\hat{F}}^x$ satisfy Theorem 2.5 that is, we have exponentially fast decay of correlations in the function spaces $C_{loc}^s(\hat{\Delta}^r)$ and $C_{loc}^{s,2}(\hat{\Delta}^r)$ for the generalized skew product semiflow associated to each physical measure of the global Poincaré return map.

#### 3.9.1. From the generalized suspension flow to the original flow.

The harder part of Theorem 3.1 is item (ii), which needs the next result, enabling us to pass from the suspension semiflow to the ambient manifold $M$ using the map $p : \hat{\Delta}^r \to M$ given by $p(x, y, u) = X_u(x, y)$. 

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Theorem 3.24. There exists $C > 0$ so that for all $(x_1, y_1, u_1), (x_2, y_2, u_2) \in \hat{\Delta}^r$, we have $|p(x_1, y_1, u_1) - p(x_2, y_2, u_2)| \leq C(|F(x_1) - F(x_2)|^\alpha + |y_1 - y_2| + |u_1 - u_2|)$.

Proof. This result follows using Remark 3.12 and the same arguments in the proof of [14, Theorem 5.8].

Given a fixed vector field $X \in \mathcal{V}$ and a fixed ergodic physical measure and its corresponding generalized skew product semiflow, the proof of Theorem 3.1 follows using the previous results as in [14, Theorem 2.5].

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