Comparing \textit{EQP} and \textit{MOD}_{p^k}P using Polynomial Degree Lower Bounds

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Abstract

We show that an oracle \( A \) that contains either \( \frac{1}{4} \) or \( \frac{3}{4} \) of all strings of length \( n \) can be used to separate \textit{EQP} from the counting classes \textit{MOD}_{p^k}P, where \( p \) is a prime. Our proof makes use of the degree of a representing polynomial over \( \mathbb{Z}_{p^k} \). We show a linear lower bound on the degree of this polynomial. We also show an upper bound of \( O(n^{1/\log_p m}) \) on the degree over the ring \( \mathbb{Z}_m \), whenever \( m \) is a squarefree composite with largest prime factor \( p \).

1 Introduction

One of the central goals of complexity theory is to understand the various relationships between complexity classes. In particular, with the introduction of quantum complexity theory, an exciting new challenge has arisen in understanding the relationship between classical and quantum classes. In particular, one asks about the strength of \textit{BQP}, the class of all problems that can be efficiently solved using a quantum computer with bounded error, and \textit{EQP}, the class of problems that can be efficiently solved using a quantum computer which always gives the right answer, compared to classical complexity classes. Unfortunately, questions in this direction are notoriously hard to settle.

A more feasible task however, is to show that relative to some oracle, a certain relationship between two complexity classes holds. Early results include a relativized separation of \textit{BQP} from \textit{BPP} by Bernstein and Vazirani [BV97], and a relativized separation of \textit{EQP} from \textit{NP} \( \cup \text{coNP} \) by Berthiaume and Brassard [BB94]. Green and Pruim [GP01] improved upon the latter result by exhibiting an oracle relative to which \( \text{EQP} \not\subseteq \text{P}^{\text{NP}} \).

In this paper we ask whether \textit{EQP} can be separated from \textit{MOD}_{m}P by an oracle. Note that due the linear lower bound on the degree of a polynomial representing the parity function over the reals (Beals et al. [BBC\textsuperscript{+}98]), the other direction, separating \textit{MOD}_{m}P from \textit{EQP}, is easy. Recall that \textit{MOD}_{m}P is the class of languages decided by non-deterministic polynomial time machines that accept iff the number of accepting computation paths is nonzero modulo \( m \). In particular, we ask whether an oracle that is promised to hold either \( \frac{1}{4} \) or \( \frac{3}{4} \) of all the strings of each length can be used to separate \textit{EQP} from \textit{MOD}_{m}P. This leads us to investigate the degree of a polynomial

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\( q : \mathbb{Z}_m^n \rightarrow \mathbb{Z}_m \) that for \( x \in \{0,1\}^n \) has \( q(x) \neq 0 \) if \( |x| = n/4 \), and \( q(x) = 0 \) if \( |x| = 3n/4 \), where \( |x| \) denotes the number of 1's in a binary string. When \( m \) is a prime power, we show a linear lower bound on the degree of any such polynomial. This implies that for prime \( p \), EQP can be separated from MOD\(_p^p\)P (and specifically \( \oplus \)P). We then focus our attention on composite \( m \). If \( m \) is a squarefree composite, we show an upper bound on the degree of \( O(n^{1/\log p m}) \), where \( p \) is the largest prime factor of \( m \). As is the case with the OR function (Barrington, Beigel, and Rudich [BBR92]), this gives another example of a Boolean function whose representing degree drops significantly if we go from prime power moduli to composite moduli.

2 Preliminaries

2.1 Complexity Theory

We assume familiarity with the basics of classical and quantum complexity theory. For the former [Pap94] provides an excellent introduction, for the latter we recommend [NC00]. In particular we are interested in the complexity classes MOD\(_m^p\)P and EQP, definitions of which are provided here for completeness’ sake. Let \( M \) be a non-deterministic Turing machine. By \( \#M(x) \) we denote the number of accepting computations of \( M \) on input \( x \).

**Definition 1** Let \( L \subseteq \{0,1\}^* \). We say that \( L \in \text{MOD}_{m^p}^P \) iff there exists a polynomial time non-deterministic Turing machine \( M \), such that

1. \( x \in L \Rightarrow \#M(x) \mod m \neq 0 \)
2. \( x \notin L \Rightarrow \#M(x) \mod m = 0 \)

**Definition 2** Let \( L \subseteq \{0,1\}^* \). We say that \( L \in \text{EQP} \) iff there exists a polynomial time quantum Turing machine \( M \), such that

1. \( x \in L \Rightarrow \Pr[M \text{ accepts } x] = 1 \)
2. \( x \notin L \Rightarrow \Pr[M \text{ accepts } x] = 0 \)

We define relativized versions of these complexity classes in the usual way.

2.2 Combinatorics

For natural numbers \( n \) and \( k \), we denote by \((n)_k \) the \( k\)-ary representation of \( n \), i.e. the string \( \ldots a_2 a_1 a_0 \), with \( 0 \leq a_i < k \), such that \( n = \sum_i a_i k^i \). Note that the first (from the right) nonzero digit of \((n)_k \) is given by the least \( i \) such that \( k^i \nmid n \), an observation to which we shall frequently refer.

In 1878 Lucas [Luc78] gave a method to easily determine the value of \( \binom{n}{k} \mod p \), for prime \( p \), and the following theorem is now known as Lucas’ Theorem. It is one of the main ingredients in the proofs of our results. By \( x[i] \) we denote the symbol at the \( i \)th position of string \( x \).

**Theorem 1 (Lucas)** Let \( p \) be a prime number, and \( n, k \) positive integers, then

\[
\binom{n}{k} \mod p = \prod_{i=1}^{m} \left( \binom{n}{k} \mod p \right) \mod p,
\]

where \( m \) is the maximal index \( i \) such that \( \binom{n}{k} \mod p \neq 0 \) or \( \binom{k}{i} \mod p \neq 0 \), and we use the convention that \( \binom{0}{0} = 1 \) whenever \( x > 0 \).
Another theorem that we shall make use of in this paper is the Chinese Remainder Theorem. We state it here for completeness’ sake.

**Theorem 2 (Chinese Remainder Theorem)** Let \( r_1, \ldots, r_\ell \) be pairwise relatively prime, and \( m = \prod_{i=1}^\ell r_i \). Then

\[
\mathbb{Z}_m \cong \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_\ell},
\]

where the isomorphism is given by \( \psi(x \bmod m) \mapsto (x \bmod r_1, \ldots, x \bmod r_\ell) \).

**2.3 Representation of Boolean Functions over \( \mathbb{Z}_m \)**

We now define what it means for a polynomial over \( \mathbb{Z}_m \) to represent a Boolean function. We should note that there are different opinions on what would be the most natural definition of representing a Boolean function by a polynomial over \( \mathbb{Z}_m \), see for instance the discussion in Tardos and Barrington [TB95]. The definition we use here, is what is sometimes called one-sided representation.

**Definition 3** Let \( g : \{0,1\}^n \to \{0,1\} \) be a Boolean function, and \( p : \mathbb{Z}_m^n \to \mathbb{Z}_m \) a polynomial. We say that \( p \) represents \( g \) over \( \mathbb{Z}_m \) iff for all \( x \in \{0,1\}^n \), \( p(x) = 0 \iff g(x) = 0 \). By the degree \( \deg(p) \) of a polynomial \( p : \mathbb{Z}_m^n \to \mathbb{Z}_m \), we mean the size of its largest monomial. The degree of a Boolean function \( g : \{0,1\}^n \to \{0,1\} \) over \( \mathbb{Z}_m \) is then defined as \( \deg(g,m) = \min \{ \deg(p) \mid p \text{ represents } g \text{ over } \mathbb{Z}_m \} \).

Note that since for all \( x \in \{0,1\} \) and \( \ell > 0 \), we have that \( x^\ell = x \), we can restrict ourselves to multilinear polynomials.

When the modulus is a prime, we have the following two interesting lemmas. Both are usually stated as being folklore results. See [Bei93] for an overview of these and other similar results.

**Lemma 3** Let \( p \) be a prime, and \( g : \mathbb{Z}_p^n \to \mathbb{Z}_p \) be a polynomial of degree \( d \), then there is a polynomial \( h : \mathbb{Z}_p^n \to \mathbb{Z}_p \) of degree \((p-1)d\), such that for all \( x \in \{0,1\}^n \), \( h(x) \in \{0,1\} \), and \( h(x) = 0 \iff g(x) = 0 \).

**Proof** Take \( h = g^{p-1} \). By Fermat’s little theorem, \( h(x) \equiv 1 \bmod p \) iff \( g(x) \neq 0 \). \( \square \)

We should note that Theorem 19, item (ii) in [Bei93] contains an erroneous proof. We have learned about a correct result via Richard Beigel (personal communication, October 2002). It is stated in the next lemma.

**Lemma 4** Let \( k \) be a positive integer, and \( p \) a prime. If \( g : \mathbb{Z}_p^n \to \mathbb{Z}_p \) is a polynomial of degree \( d \), then there exists a degree \( d(2p^{k-1} - 1) \) polynomial \( h : \mathbb{Z}_p^n \to \mathbb{Z}_p \), such that for all \( x \in \{0,1\}^n \), \( h(x) = 0 \iff g(x) = 0 \).

**Proof** By Theorem 1, we have that for every prime \( p \), and positive integer \( m \)

\[
m \equiv 0 \bmod p^k \iff \forall i < k \left[ \binom{m}{p^i} \equiv 0 \bmod p \right]. \tag{1}
\]

Define the \( i \)th elementary symmetric function of the \( n \) variables \( y_1, \ldots, y_n \), \( i \leq n \), as

\[
\sum_{1 \leq \ell_1 < \cdots < \ell_i \leq n} \prod_{j=1}^i y_{\ell_j}.
\]
Note that if each \( y_i \in \{0,1\} \), and exactly \(|y|\) of them are 1, then the value of the above expression is \( (|y| \choose i) \). Now write \( g \) as a sum of monomials of coefficient 1, i.e., replace for example \( 3x_1x_2 \) by \( x_1x_2 + x_1x_2 + x_1x_2 \). Let \( (g(x) \choose i) \) be the \( i \)-th elementary symmetric function of the monomials in \( g \). Define \( h(x) \) as

\[
    h(x) = \sum_{i=0}^{k-1} \left( g(x) \choose p^i \right) \prod_{j=0}^{i-1} \left( 1 - \left( g(x) \choose p^j \right)^{p-1} \right).
\]

The degree of \( h(x) \) is \( d(2p^{k-1} - 1) \). If \( g(x) \equiv 0 \mod p \), then by Equation 1, \( (g(x) \choose p^i) \equiv 0 \mod p \) for all \( 0 \leq i < k \), hence \( h(x) \equiv 0 \mod p \). On the other hand, if \( g(x) \not\equiv 0 \mod p \), then using Equation 1, let \( r \) be the least value such that \( (g(x) \choose p^r) \not\equiv 0 \mod p \). Note that the \( r \)th term in \( h(x) \) is nonzero modulo \( p \), but all the others are zero modulo \( p \), since all terms after the \( r \)th contain the factor \( (1 - (g(x) \choose p^r)^{p-1}) = 0 \), and hence \( h(x) \not\equiv 0 \mod p \).

3 Linear Lower Bound for Prime Power Moduli

In this section we restrict ourselves to the field \( \mathbb{Z}_{p^k} \), where \( p \) is a prime. For a binary string \( x \), let \( |x| \) denote its Hamming weight (number of 1’s). We consider any Boolean function \( g : \{0,1\}^n \to \{0,1\} \), that has \( g(x) = 1 \) if \( |x| = n/4 \) and \( g(x) = 0 \) if \( |x| = 3n/4 \), and prove that it has \( \deg(g, p^k) = \Omega(n) \).

The rough idea behind our proof is the following. Using Lemmas 3 and 4 we restrict ourselves to polynomials over \( \mathbb{Z}_p \) that are always 0/1 valued on inputs from the domain \( \{0,1\}^n \). This only increases the degree by a multiplicative constant. Now assume there exists a low degree polynomial \( q \) that represents \( g \). We then use the property that for all \( x \in \{0,1\}^n \), \( q(x) \not\equiv 0 \) if \( |x| = n/4 \) and \( q(x) = 0 \) if \( |x| = 3n/4 \) to set up a system of linear equations over the coefficients of the monomials in \( q \). Using Theorem 1 we then show that this system is unsolvable, and conclude that no such low degree polynomial \( q \) exists.

**Theorem 5** Let \( p \) be a prime, \( n = 4p^r \), and \( g : \{0,1\}^n \to \{0,1\} \) be such that \( g(x) = 1 \) if \( |x| = n/4 \), and \( g(x) = 0 \) if \( |x| = 3n/4 \). Then

\[
    \deg(g, p^k) \geq \frac{n}{4(2p^{k-1} - 1)(p-1)}.
\]

**Proof** We will prove the lemma for primes \( p > 3 \). The case where \( p \in \{2,3\} \) has an identical proof, and we leave this to the reader.

Consider any degree \( d < \frac{n}{4(2p^{k-1} - 1)(p-1)} \) multilinear polynomial \( p \) over \( \mathbb{Z}_{p^k} \) that represents \( g \). Using Lemmas 3 and 4, transform \( p \) into a polynomial \( q \) that represents \( g \) over \( \mathbb{Z}_p \), and that has \( q(x) \in \{0,1\} \) for all \( x \in \{0,1\}^n \). This will only increase the degree of \( q \) by a multiplicative factor \( (p-1)(2p^{k-1} - 1) \). We now prove a lower bound of \( n/4 \) on the degree of \( q \). Write \( q \) as

\[
    q(x_1, \ldots, x_n) = \sum_{S \subseteq [n], |S| < n/4} c_S \cdot \text{mon}(S),
\]

where \( \text{mon}(S) = \prod_{i \in S} x_i \), \( |S| \) denotes the size of \( S \), and each \( c_S \in \mathbb{Z}_p \). On input \( x \in \{0,1\}^n \), with \( x_1 = \ldots = x_{3n/4} = 1 \) and \( x_{3n/4+1} = \ldots = x_n = 0 \), we have that

\[
    \sum_{S \subseteq [3n/4], |S| < n/4} c_S \equiv 0 \mod p.
\]
However, every input $x \in \{0,1\}^n$ with exactly $n/4$ out of the first $3n/4$ variables set to 1 gives a constraint

$$\sum_{S \subseteq T} c_S \equiv 1 \mod p,$$

where $T \subseteq [3n/4]$, is the set of $n/4$ indices of variables that are set to 1 in the input $x$. Note that the total number of such constraints modulo $p$ is

$$\binom{3n/4}{n/4} \mod p = \binom{3p^r}{p^r} \mod p = 3,$$

by Theorem 1. Also, note that every monomial $\text{mon}(S)$ with $S \subseteq [3n/4]$, of degree $0 < \ell < n/4 = p^r$ occurs in exactly

$$\binom{3n/4 - \ell}{n/4 - \ell} \mod p = \binom{3p^r - \ell}{p^r - \ell} \mod p = 1,$$

constraints modulo $p$, which follows again from Theorem 1. To see this, note that the first $r$ digits in the $p$-ary representation of $3p^r - \ell$ and $p^r - \ell$ for $0 < \ell < p^r$ are all equal, but the $(r+1)$st digit of $(p^r - \ell)_p$ is 0, and that of $(3p^r - \ell)_p$ is 2.

Hence summing all these constraints gives

$$2c_0 + \sum_{S \subseteq [3n/4], |S| < n/4} c_S \equiv 3 \mod p,$$

where the term $2c_0$ is due to the fact that the free term $c_0$ of $q$ occurs in 3 constraints modulo $p$. Since $c_0 \in \{0,1\}$ (because $q(0^p) \in \{0,1\}$), we thus have a contradiction with Equation 2. Hence $q$ must have degree $\geq n/4$. \qed

As a consequence of Theorem 5 we have a relativized separation of $\text{EQP}$ from $\text{MOD}_{p^r}\text{P}$.

**Corollary 6** There exists an oracle $A$, such that

$$\text{EQP}^A \not\subseteq \text{MOD}_{p^r}\text{P}^A.$$

**Proof** For fixed $r$, define $r^* = \lfloor \log_2 4p^r \rfloor$, and for each $A \subseteq \{0,1\}^*$, define $A^{r^*}$ to be the restriction of $A$ to the lexicographically first $4p^r$ strings of length $r^*$. Consider oracles $A$ with the property that $|A^{r^*}| \in \{p^r, 3p^r\}$ for all $r$. For such $A$, define

$$L_A = \{0^r \mid |A^{r^*}| = p^r\}.$$

Grover’s algorithm [Gro96] has the property that if either a 1/4 or a 3/4 fraction of the total search space is a solution, then we can find out which of the two is the case with certainty using just one query. This observation was first made by Boyer et al. [BBHT98], and later generalized by Brassard et al. [BHMT00]. Using this observation, it is not hard to see that for all appropriate $A$, $L_A \in \text{EQP}^A$.

We now show the existence of an $A$ such that $L_A \not\subseteq \text{MOD}_{p^r}\text{P}^A$. The construction of $A$ will be in stages. Let $M_1, M_2, \ldots$ be an enumeration of $\text{MOD}_{p^r}\text{P}$ oracle machines. In stage $i$, run $M_i$ on input $0^{r_i}$, where $r_i$ is chosen large enough as not to interfere with any previous stages. Note that we may assume that $M_i$ only makes queries to the lexicographically first $4p^{r_i}$ strings of length $r_i^*$. Call these strings $y_1, y_2, \ldots, y_{4p^{r_i}}$. Take an arbitrary computation path of $M_i$ on input $0^{r_i}$, and let $y_{i_1}, y_{i_2}, \ldots, y_{i_t}$ be the queries made along this path. Note that $\ell$ is upper bounded by a polynomial
in $r_i$. Now for each possible appropriate setting of $A$ on length $r_i^*$, see if this path accepts. If it does, create a monomial which is the product of all the variables $y_i$ (if $y_i \in A$) or $(1-y_i)$ (if $y_i \notin A$), for $1 \leq i \leq \ell$. Repeat this procedure for all other computation paths. The sum of all monomials thus obtained is a polynomial $q : \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$, that for $x \in \{0,1\}^{4p^r}$ has $q(x) \neq 0$ if $|x| = p^r$, and $q(x) = 0$ if $|x| = 3p^r + 1$. Furthermore, the degree of $q$ is bounded by a polynomial in $r_i$. But Theorem 5 states that such a polynomial does not exist. Hence there must exist a setting of $A$ on length $r_i^*$ such that $M_i$ is incorrect on input $0^{r_i}$. Set $A$ in this way on length $r_i^*$, this ensures that $M_i$ can not decide $L_A$. Continue with stage $i + 1$.

In the other direction, an oracle separation of MOD$_2$P from EQP is easy to achieve. For instance, to separate MOD$_2$P (= ⊕P) from EQP, we can use the following construction. Let $B \subseteq \{0,1\}^*$, and define

$$L_B = \{0^r \mid \text{the parity of the number of strings in } B \text{ of length } r \text{ is odd} \}.$$  

Clearly, $L_B \in$ MOD$_2$P$^B$. However, using the fact that the degree over the reals of the representing polynomial for the parity function on $n$ variables is $n$ (Beals et al. [BBC+98]), we can show that there exists a $B$ such that $L_B \notin$ EQP$^B$.

## 4 Sublinear Upper Bound for Squarefree Composite Moduli

We now focus our attention on representing $g$ over the ring $\mathbb{Z}_m$ of integers modulo $m$, where $m$ is a squarefree composite. We prove the following theorem.

**Theorem 7** Let $m$ be a squarefree composite with largest prime factor $p$, $4 \mid n$, and $g : \{0,1\}^n \to \{0,1\}$ be such that $g(x) = 1$ if $|x| = n/4$, and $g(x) = 0$ if $|x| = 3n/4$. Then $\deg(g,m) = O(n^{1/\log_p m})$.

We will prove this result using two separate lemmas. Note that $g$ is only well-defined on lengths $n$ such that $4 \mid n$. We split these possible lengths in 2 different categories, namely those such that $n/4 \not\equiv 0 \bmod m$, and those such that $n/4 \equiv 0 \bmod m$. For each of these two lengths, we separately prove an upper bound. A key insight that we shall need is provided by the following lemma.

**Lemma 8** If $m = p_1p_2 \cdots p_r$, then for all $1 \leq i \leq r$,

$$\left( \frac{am^b}{p_i^\ell} \right) \bmod m \neq \left( \frac{3am^b}{p_i^\ell} \right) \bmod m$$

if and only if the $(\ell + 1)$st digit of the $p_i$-ary representations of $am^b$ and $3am^b$ differ.

**Proof** Let $1 \leq j \leq r$ with $j \neq i$. We have that $p_j \nmid p_i^\ell$, but $p_j \mid a \cdot m^b$. Hence the first digit of $(am^b)_{p_j}$ is 0 and the first digit of $(p_i^\ell)_{p_j}$ is nonzero. By Theorem 1, it follows that

$$\left( \frac{am^b}{p_i^\ell} \right) \bmod p_j = 0,$$

for $1 \leq j \leq r$, and $j \neq i$. Note that again by Theorem 1, the value of $(am^b)_{p_i^\ell}$ mod $p_i$ is determined by the $(\ell + 1)$st digit of $(am^b)_{p_i}$, since the only nonzero digit of $(p_i^\ell)_{p_i}$ is the $(\ell + 1)$st and has value 1. Likewise, the value of $(3am^b)_{p_i^\ell}$ is determined by the $(\ell + 1)$st digit of $(3am^b)_{p_i}$. Now apply the Chinese Remainder Theorem. □
The use of Lemma 8 stems from the following fact. Assume we have a polynomial \( p : \mathbb{Z}_m^n \rightarrow \mathbb{Z}_m \), where all the monomials of degree \( d \) have coefficient 1, and all other monomials have coefficient 0. Then for \( x \in \{0,1\}^n \), \( p(x) \) has the value \( \binom{n}{d} x^d \) mod \( m \). Specifically, if \( \binom{n}{d} \) mod \( m \neq \binom{3n}{d} \) mod \( m \), then \( p(x) - \binom{3n}{d} x^d \) is a representing polynomial for \( g \) of degree \( d \).

We shall first prove that if \( n/4 = am^b \) for some \( b > 0 \), and \( 0 < a < m \), then there exists a degree at most \( p^{b+1} = O(n/\log_p m) \) representing polynomial for \( g \) over \( \mathbb{Z}_m \), where \( p \) is the prime factor of \( m \) not occurring in \( a \).

**Lemma 9** Let \( m = p_1 p_2 \cdots p_r \) be a squarefree composite with \( p_i < p_{i+1} \), \( 0 < a < m \), \( b > 0 \), and \( p \) the least prime factor of \( m \) not occurring in \( a \). Then the following hold.

1. if \( p > 2 \), then \( \binom{am^b}{p^b} \) mod \( m \neq \binom{3am^b}{p^b} \) mod \( m \)
2. if \( p = 2 \), then \( \binom{am^b}{p^{b+1}} \) mod \( m \neq \binom{3am^b}{p^{b+1}} \) mod \( m \)

**Proof** To prove item 1, we will show that for all \( b \), the \((b+1)\)st digit of \( (am^b)_p \) and \( (3am^b)_p \) are different. The result then follows by Lemma 8. We distinguish between the case where \( p = 3 \), and \( p > 3 \).

If \( p = 3 \), then since for all \( i \leq b \), \( 3^i \mid am^b \), but \( 3^{b+1} \nmid am^b \), the \((b+1)\)st digit of \( (am^b)_3 \) is the first nonzero digit. However, since \( 3^i \mid 3am^b \) for \( i \leq b + 1 \), the \((b+1)\)st digit of \( (3am^b)_3 \) is 0.

If \( p > 3 \), then for all \( 0 < i < b \), we have that \( p^i \mid am^b \), and \( p^i \nmid 3am^b \), but \( p^{b+1} \mid am^b \) and \( p^{b+1} \nmid 3am^b \). Hence, we have that the \((b+1)\)st digit of both \( (am^b)_p \) and \( (3am^b)_p \) is the first nonzero digit. We now claim that \( (am^b \mod p^{b+1})/p^b \neq (3am^b \mod p^{b+1})/p^b \), i.e., \( (am^b)_p \) and \( (3am^b)_p \) differ in their \((b+1)\)st digit. To prove this, note that \( am^b = c \cdot p^{b+1} + r \cdot p^b \), for some integer \( c \) and \( 0 < r < p \). Hence, \( 3am^b = c' \cdot p^{b+1} + 3r \cdot p^b \). But \( 3r \mod p \neq r \), for all \( 0 < r < p \), if \( p > 3 \) and \( p \) is a prime.

To prove item 2, we show that the \((b+2)\)nd digit of \( (am^b)_2 \) and \( (3am^b)_2 \) differ, for all \( b \). Note that for both \( (am^b)_2 \) and \( (3am^b)_2 \), the first nonzero bit is the \((b+1)\)st, so both \( (am^b)_2 \) and \( (3am^b)_2 \) are of the form \( \ldots 10^b \). Now if \( (am^b)_2 \) has a 0 as its \((b+2)\)nd bit, i.e., \( (am^b)_2 \) is of the form \( \ldots 010^b \), then \( (3am^b)_2 \) is of the form \( \ldots 110^b \). On the other hand, if \( (am^b)_2 \) has a 1 as its \((b+2)\)nd bit, i.e. \( (am^b)_2 \) is of the form \( \ldots 110^b \), then \( (3am^b)_2 \) is of the form \( \ldots 010^b \). Hence the \((b+2)\)nd bit of \( (am^b)_2 \) and \( (3am^b)_2 \) are different.

If \( n \) is such that \( n/4 \neq 0 \) mod \( m \), then we can prove that constant degree suffices.

**Lemma 10** Let \( m = p_1 p_2 \cdots p_r \) be a squarefree composite with \( p_i < p_{i+1} \). Then the following hold.

1. if \( c \neq m/2 \), then \( \binom{am^c}{1} \) mod \( m \neq \binom{3am^c}{1} \) mod \( m \), for \( 0 < c < m \)
2. if \( c = m/2 \), then \( \binom{am^{m/2}}{2} \) mod \( m \neq \binom{3am^{m/2}}{2} \) mod \( m \)

**Proof** We first prove item 1. If \( c \neq m/2 \), then \( c \neq 3c \) mod \( m \). Hence, \( (am + c) \) mod \( m \neq (3am + 3c) \) mod \( m \).

We now prove item 2. Since \( m \) is an even squarefree number, \( m/2 \) is odd. Hence the first bit of \( (am + m/2)_2 \) is 1. We thus have that \( (am + m/2)_2 \) has the form \( \ldots 11 \) or \( \ldots 01 \). In the first case, \( (3am + 3m/2)_2 \) then has the form \( \ldots 01 \), in the second case, \( (3am + 3m/2)_2 \) has the form \( \ldots 11 \). In other words, \( (am + m/2)_2 \) and \( (3am + 3m/2)_2 \) differ in their 2nd bit. Using Theorem 1 and the Chinese Remainder Theorem, the result follows.

□
Together, Lemmas 9 and 10 imply Theorem 7:

**Proof (of Theorem 7)** Let $m$ be a squarefree composite with largest prime factor $p$, $4 \mid n$, and $g : \{0, 1\}^n \to \{0, 1\}$ be such that $g(x) = 1$ if $|x| = n/4$, and $g(x) = 0$ if $|x| = 3n/4$. We will exhibit a representing polynomial of degree $O(n^{1/\log_p m})$ for $g$ on each length $n$. We distinguish two different cases for $n$:

1. $n/4 \equiv 0 \mod m$, i.e., $n/4 = am^b$ with $0 < a < m$, and $b > 0$. In this case, Lemma 9 tells us that if $p$ is the least prime factor of $m$ not in $a$, then either $(am^b) \mod m \neq (3am^b) \mod m$ (if $p > 2$), or $(am^b) \mod m \neq (3am^b) \mod m$ (if $p = 2$). In the former case, the polynomial
   \[
   \left(\frac{3am^b}{p^b}\right) - \sum_{S \subseteq [n], |S| = p^b} \text{mon}(S),
   \]
   where $\text{mon}(S) = \prod_{i \in S} x_i$, represents $g$. This polynomial has degree $p^b$. In the latter case
   \[
   \left(\frac{3am^b}{p^{b+1}}\right) - \sum_{S \subseteq [n], |S| = p^{b+1}} \text{mon}(S)
   \]
   is a representing polynomial of degree $p^{b+1}$.

2. $n/4 \not\equiv 0 \mod m$, i.e., $n/4 = am + c$. In this case Lemma 10 tells us that either $am + c \mod m \neq 3am + 3c \mod m$ (if $c \neq m/2$), or $(am+c) \mod m \neq (3am+3c) \mod m$ (if $c = m/2$). In the former case, the polynomial
   \[
   3am + 3c - \sum_{i=1}^{n} x_i
   \]
   is a representing polynomial for $g$ of degree 1. In the latter case,
   \[
   \left(\frac{3am + 3c}{2}\right) - \sum_{S \subseteq [n], |S| = 2} \text{mon}(S)
   \]
   is a degree 2 representing polynomial for $g$.

\[\square\]

5 **Discussion and Open Problems**

We studied the degree of a polynomial $q : \mathbb{Z}_m^n \to \mathbb{Z}_m$, that for all $x \in \{0, 1\}^n$ has $q(x) \neq 0$ if $|x| = n/4$, and $q(x) = 0$ if $|x| = 3n/4$. We have proven a linear lower bound when $m$ is a prime power, and an upper bound of $O(n^{1/\log_p m})$, if $m$ is a squarefree composite with largest prime factor $p$. The former result implies a relativized separation of $\text{EQP}$ from $\text{MOD}_p \text{P}$.

A number of open questions are left by this research. First of all, can we prove that the upper bound of $O(n^{1/\log_p m})$ is tight? And second, what can we say about general composite $m$, instead of squarefree composite $m$? Establishing a good lower bound in the latter case would show a relativized separation of $\text{EQP}$ from $\text{MOD}_m \text{P}$ for all $m$.

Another interesting direction is to investigate whether one can exhibit an oracle relative to
which EQP is not contained in $\Sigma^A_2$ or higher levels of $PH$. This will require, however, a different and presumably more complex oracle construction than the one we have used here, since the language that separates $EQP^A$ from $P^{NP^A}$ and $\text{MOD}_{\rho^A}P^A$ is in $BPP^A$, and hence in $\Sigma^A_2$.

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