Probability Bifurcations of Lévy Bridges

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A Lévy bridge—a stable Lévy stochastic process conditioned to arrive at some state at some later time—can exhibit behavior differing dramatically from the more widely studied case of conditioned Brownian (Gaussian) processes. This difference stems from a structural change in the conditioned probability density at intermediate times as the arrival position varies. This structural shift gives rise to a distinction between “short” and “long” jumps. We explore the consequences of this idea for the statistics of Lévy vs. Brownian bridges, with applications to the analysis of the boundary-crossing problem and a computationally useful representation of Lévy bridges that does not carry over directly from the Gaussian case.

Lévy processes generalize Gaussian stochastic processes—they exhibit characteristic large jumps (Fig. 1) unexplained by ordinary diffusion. Furthermore, the stable Lévy processes are universal for random walks generated by heavy-tailed distributions, in the same sense that Gaussian processes are universal for finite-variance steps, due to a generalized central limit theorem. The Lévy flights exhibited by the stable processes play an important role in understanding a wide range of phenomena [1, 2], including ecology [3], finance [4], fluid flows [5], chaotic transport [6], and optimal stochastic searches under certain conditions [7, 8]. They have been of particular interest in the context of laser-cooled atoms [9–16]. The stable processes also produce strikingly counterintuitive behavior. For example, intriguing work has shown that the image method fails to predict first-passage times [17, 18]; this is related to the leapover phenomenon [19], where Lévy walkers are first absorbed inside a perfectly absorbing region, rather than on the boundary as in the Gaussian case. As discussed in Ref. [20], related issues have even led to erroneous results in the literature.

A Brownian bridge is a continuous-time Gaussian stochastic process conditioned to arrive at some final location (state). The properties and statistics of Brownian bridges have been thoroughly studied [21]. They are important in diverse areas, occurring in financial mathematics (such as in modeling bond prices [22] and accelerating convergence of Monte Carlo simulations [23]), models of animal movements [24], and Monte Carlo path-integral methods in quantum mechanics [25, 26].

Thus, it comes as something of a surprise that similarly conditioned Lévy processes have so far received relatively little attention in the literature. They have been formalized and applied to the pricing of financial assets even when the latter are “close to” Gaussian.

Definitions. The continuous-time \( \alpha \)-stable Lévy processes are specified in terms of the characteristic function \( \langle e^{ikx(t)} \rangle = e^{-t|\alpha| \langle k \rangle} \) at time \( t \), provided \( x(0) = 0 \) [31]; the Fourier transform yields the probability density \( f_a(x; t) \) for \( x(t) \), thus being “stable” under iterated convolutions. For simplicity we will only consider symmetric...
Lévy bridges. In a Lévy bridge, the arrival point is specified as \( x(T) = L \) for some arrival time \( T > 0 \). Then the intermediate position \( x_{1/2} := x(T/2) \) has the conditional density (“midpoint density”)

\[
f_\alpha(x_{1/2}; T/2|x; L; T) = \frac{f_\alpha(x_{1/2}; T/2) f_\alpha(L-x_{1/2}; T/2)}{f_\alpha(L; T)}
\]

in terms of the unconditioned density \( f_\alpha(x; t) \). Once \( x(T/2) \) is sampled, the bridge is effectively bisected into two bridges, and the midpoint-sampling process may be iterated to sample the Lévy bridge to any desired time resolution. For \( \alpha = 2 \) the midpoint density retains the same Gaussian form as the unconditioned density, but the conditioned and unconditioned forms differ for any \( \alpha < 2 \).

The Cauchy \((\alpha = 1)\) case is a good example of what happens for \( \alpha < 2 \). For \( L < \sigma T \), this distribution has a single peak at \( x_{1/2} = L/2 \), which has a seemingly intuitive interpretation: if a particle travels from \( x = 0 \) to \( L \) in time \( T \), the most probable intermediate position at \( T/2 \) is \( L/2 \). However, this intuition breaks down at the special arrival point \( L_0 = \sigma T \), beyond which the midpoint density becomes bimodal, and the single maximum bifurcates into a pair at \( x_{1/2} = [L \pm (L^2 - \sigma^2 T^2)^{1/2}]/2 \) (Fig. 2).

For \( L \gg L_0 \) the peaks are well separated, with maxima approaching asymptotes \( x_{1/2} \sim 0, L \). In this case, the interpretation of the midpoint changes: the long jump \( L \) tends to correspond to one large step of order \( L \) and one small step, rather than two steps roughly equal to \( L/2 \). Thus, a bridge with sufficiently large overall transition length \( L \) will tend to maintain this as a single jump discontinuity.

Similar structural changes in the midpoint density occur for all \( \alpha < 2 \). Fig. 3 shows typical possibilities of how the bifurcation occurs as \( L \) increases. For \( \alpha = 1.5 \) there is a pitchfork bifurcation, as in the Cauchy case, where two maxima and a minimum are created from a single maximum. However, closer to the Gaussian limit \((\alpha = 1.99999)\), the structure is more complicated: first, a pair of side peaks is born via tangent bifurcations, and then the two associated minima collide with the central maximum to form a minimum in a reverse-pitchfork bifurcation. For small and large \( L \), the end results are the same in either case: a unimodal density transforms into a bimodal density with well separated peaks.

**Bifurcation length:** variation with \( \alpha \). An obvious characterization of the bifurcation length \( L_b \) is the value of \( L \) for which the curvature of the midpoint density \( f_\alpha \) at \( x_{1/2} = L/2 \) changes sign (Fig. 4). However, for \( \alpha \) above a critical value \( \alpha_c \), as we have seen, the midpoint density does not exhibit a simple bifurcation to a bimodal density; rather, there are three distinct transitions. (The critical value \( \alpha_c \approx 1.799933 \) occurs when both the second and fourth derivatives of the midpoint density vanish at \( x_{1/2} = L/2 \).) All three bifurcation lengths are shown in Fig. 4 for \( \alpha > \alpha_c \). They all usefully characterize the structural changes of the distribution, though in practice the particular choice of \( L_b \) is not too important—as we will see, the transition between “short” and “long” jumps is not sharp. (We use the curvature-change criterion except where noted.)

Fig. 4 also shows the transition away from power-law tails in the limit \( \alpha \rightarrow 2 \). The bifurcation length diverges in this limit, so that for the Gaussian (\( \alpha = 2 \)) case, any final step \( L \) corresponds to a “short step.” The nature of this divergence may be analyzed using using the asymptotic density, \( f_\alpha(x; t) \sim f_2(x; 1) + \delta|x|^{3-\alpha} \), valid for large \( |x| \) and small \( \delta := 2 - \alpha \) [32]. One can show that \( L_b \) (defined by the curvature-sign-change criterion) diverges as \( L_b \sim [-4\sigma^2 T \log(\pi \delta^2/2)]^{1/2} \). Numerically, \( L_b \) seems to diverge similarly according to the other criteria as well. Thus, even very close to the Gaussian limit \( \alpha = 2 \), \( L_b \) remains relatively small (cf. Fig. 3, third panel).

**Conditioned sampling.** As noted above, when sampling the intermediate state of a Lévy bridge for \( L > L_b \), a jump of order \( L \) tends to persist. Upon further recursive sampling of the bridge’s intermediate states, this behavior locks in: \( L_0 \) is effectively smaller when sampling the sub-bridges on progressively smaller time intervals, so that the jump length \( L \) tends to exceed \( L_b \) by an ever increasing margin, making it progressively less likely to be split into smaller jumps. Fig. 5 illustrates this: for \( L = 1.5L_b \) there is typically a single long jump that

![Fig. 2. (Color online.) Bifurcation diagram showing maxima (solid/red) and minima (dashed/blue) of the conditioned density (1) for \( \alpha = 1 \). Inset: conditioned density before and after the bifurcation.](image-url)
FIG. 3. (Color online.) Variation of the midstep density (1) with Lévy index $\alpha$ and arrival point $L$. Curves highlighting maxima (solid/red) and minima (dashed/blue) are superimposed.

FIG. 4. (Color online.) Variation of boundaries between “small” and “large” steps with $\alpha$. The curves indicate the bifurcation length $L_b$ for which the center of the midstep density has vanishing curvature (red/solid), the half-step distribution develops side peaks (blue/dashed), and the side peaks are equal in height to the center peak (green/dot-dashed). Inset: magnified view for $\alpha > \alpha_c$.

FIG. 5. (Color online.) Typical sample paths of Lévy bridges for $\alpha = 1.9$, illustrating the qualitative transition with $L$. Each path was generated through 10 iterations of recursive subsampling from the midstep distribution (1).

This behavior under conditioned subsampling shows that the bifurcation length $L_b$ yields a natural boundary between “short” and “long” jumps of an $\alpha$-stable process. Specifically, an observed final displacement $|x(T)| \gg L_b$ most likely corresponds to a single, similarly large jump discontinuity, even if the detailed evolution up to the final time $T$ is not known. Meanwhile, a smaller final displacement $|x(T)| \sim L_b$ is much more likely to be a composite event comprising multiple smaller jumps. This is a powerful qualitative inference based only on the endpoints of the process; it could be useful in problems of interpolation of a stochastic process between observations (e.g., animal movement [24] and kriging [33]), if the underlying process is heavy-tailed. Additionally, this provides a means for inferring whether a rare, significant event occurred between observations. Criteria like this are important for the analysis of statistical extremes [34] and for specific problems like detecting market crashes [35].

This distinction between short and long jumps is perhaps surprising in light of the Lévy–Khintchine representation [36] of the stable processes, which states that every step is a “long” jump discontinuity compared to a Gaussian step. However, we have seen that it is useful to distinguish between the short and long steps of stable processes beyond whether increments correspond to discontinuities, and that this distinction captures how, visually and intuitively, the large-scale structure of stable Lévy processes seem similar to Gaussian processes punctuated by discrete, long jumps (Fig. 1).

**Application: stretched Lévy bridges.** In the Gaussian case, one important representation of the Brownian bridge is [37]

$$W(t) = B(t) + \frac{t}{T}[W(T) - B(T)],$$

(2)
FIG. 6. (Color online) Simulated probability for Lévy bridges generated via Eq. (2) to cross a boundary at $d = \sigma T^{1/\alpha}$, as the rejection threshold $L_{\text{thresh}}$ varies.

where $W(t)$ is a Wiener process (unconditioned Lévy process with $\alpha = 2$, $\sigma = 1/\sqrt{2}$), and $B(t)$ is a Brownian bridge [Wiener process conditioned to have a fixed arrival $B(T)$]. Intuitively, in the “standard bridge” case $B(T) = 0$, the second term is the ballistic trajectory from 0 to $W(T)$, while $B(t)$ comprises the random fluctuations. This representation provides a simple way to simulate Brownian bridges using any Wiener-process algorithm. It is natural to wonder if this representation carries over to $\alpha < 2$ stable processes, in particular as part of a numerical algorithm to use ordinary Lévy increments [38] to simulate bridges (e.g., for efficiency on graphics hardware). Naively, it seems like this representation should be valid: Dividing the evolution into time steps $\Delta t$, the increments of the stable process and bridge are of order $\Delta t^{1/\alpha}$, while the ballistic correction is of order $\Delta t$. The ballistic component is thus of order $\Delta t^{1-1/\alpha}$ relative to the Lévy-process steps, and thus should be negligible as $\Delta t \rightarrow 0$ provided $\alpha > 1$. In the Gaussian case this heuristic argument is correct, and the representation (2) is valid—any ballistic “stretch” does not affect the Gaussian statistics in the continuum limit. It fails, however, for $\alpha < 2$: if $x(T)$ corresponds to a sufficiently large jump, then the stretch is excessive, and the resulting “bridges” produce erroneous results in simulations. (Ref. [29] noted this inequivalence between stretched and conditioned bridges [39].)

Since we have a distinction between short and long jumps, it is possible to deal with the excessive stretches. The fix is to define a threshold $L_{\text{thresh}}$, and an unconditioned Lévy sample path is only stretched as in Eq. (2) if its final point $L = x(T)$ is within $L_{\text{thresh}}$ of the bridge’s arrival point. Otherwise, it is rejected and other paths attempted until a bridge is successfully generated. The $\alpha$-dependent bifurcation length $L_b$ from Fig. 4 marks a scale $L_{\text{thresh}}$ below which the stretching algorithm should yield an accurate set of Lévy bridges. A test of this algorithm for simulations, computing the probability $P_{\text{cross}}$ for Lévy bridges (with $L = 0$) to cross a boundary at $d = \sigma T^{1/\alpha}$ before time $T$ illustrates this transition (Fig. 6) [40]. In particular, the simulated $P_{\text{cross}}$ rapidly becomes accurate when $L_{\text{thresh}}$ decreases below $L_b$ (the bridge construction is exact in the limit $L_{\text{thresh}} \rightarrow 0$). As a practical bridge-generation method, this is much more efficient than using $\sigma \Delta t^{1/\alpha}$ (the smallest natural length scale) for $L_{\text{thresh}}$.

A particularly interesting feature in Fig. 6 is that $P_{\text{cross}} = 0.9\%$ is so small for the case $\alpha = 0.5$. (By contrast, $P_{\text{cross}} = 31.5\%$ in the unconditioned case.) Intuitively, conditioning on $L = 0$ also conditions away the tendency to have large jumps (and thus to easily cross the boundary), especially for small $\alpha$.

**Application: conditioned first passage.** First passage times, defined here as the first time a process $x(t)$ exceeds a boundary $d$, are of broad importance [41]. They are especially interesting for Lévy processes due to the universal Sparre Andersen scaling [17, 18, 42], where the tail of the first-passage-time distribution is $\alpha$-independent. However, as we have seen, conditioned Lévy bridges have a particularly sensitive transition as $\alpha \rightarrow 2$, a pattern that continues for first-passage times.

An intuitive picture of the conditioned first-passage time follows from the qualitative appearance of the sample paths for $L = 1.5L_c$ in Fig. 5. A long jump is consistently present among the paths, but not at any particular time. This can be regarded as an outcome of recursively sampling the midpoint density (1). For $L \gg L_b$, a large jump likely persists under sampling iterations, but due to the symmetry of the midstep distribution, the jump is equally likely to be associated with any time subinterval. Since the first-passage time is likely due to the long jump, the first-passage time should be uniformly distributed. Fig. 7 confirms this intuition with simulations of the first

FIG. 7. (Color online) Simulated conditioned first passage time distributions for $\alpha = 1.99999$ and $d = L/2$ are shown for $L = 0.1L_b$ (blue/squares) and $L = 2L_b$ (red/triangles). Exact densities for $\alpha = 2$ [21] for the same $L$ values are shown for comparison in each case (blue/solid and red/dashed, respectively).
passage density [43]. For \( L = 2L_b \), the first passage density is indeed uniform. A small change from \( \alpha = 1.99999 \) to the Gaussian case yields a remarkably different distribution: approximately Gaussian, centered at \( t \approx T/2 \). The Gaussian result follows intuitively from the bridge representation (2), since the most likely bridges in this regime are concentrated around the ballistic path to the endpoint.

For a smaller overall jump (\( L = 0.1L_b \)), there is a closer match in the first passage time density between the \( \alpha = 1.99999 \) and Gaussian cases. This is consistent with the observation that for \( L \ll L_b \), the conditioned Lévy bridges are qualitatively similar to Brownian bridges. Nevertheless, the rare but important long jumps generate remarkably non-Gaussian behavior, even very close to the Gaussian limit.

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