Static deformation of a heavy spring
due to gravity and centrifugal force

Hanno Essén and Arne Nordmark

Department of Mechanics, KTH, SE-100 44 Stockholm, Sweden
E-mail: hanno@mech.kth.se

Received 15 January 2010, in final form 10 February 2010
Published 7 April 2010
Online at stacks.iop.org/EJP/31/603

Abstract
The static equilibrium deformation of a heavy spring due to its own weight is calculated for two cases: first for a spring hanging in a constant gravitational field, and then for a spring which is at rest in a rotating system where it is stretched by the centrifugal force. Two different models are considered: first a discrete model assuming a finite number of point masses connected by springs of negligible weight, and then the continuum limit of this model. In the second case, the differential equation for the deformation is obtained by demanding that the potential energy is minimized. In this way a simple application of the variational calculus is obtained.

1. Introduction
The ideal linear spring, which is quite well approximated by a coil spring, is an important concept in the education of a physicist. Simple problems in mechanics often involve a spring of stiffness $k$ and negligible mass, with a weight of mass $m$ hanging at the free end. In the static case, the extension of the spring is then $\Delta \ell = mg/k$. A real spring has mass which is evenly distributed along the unloaded spring. It is then natural to ask how such a spring deforms under its own weight in a force field. Here we investigate this problem for two different force fields: first in constant gravity, and then in the centrifugal force field due to rotation of the reference frame with constant angular velocity. We also investigate two different models for the finite mass springs: first a discrete model where the spring is assumed to consist of point masses connected by weightless springs, and then the continuum limit of this model is considered. It is pointed out that the deformation in this limit should be given by the function that minimizes the potential energy. Results are then easily obtained from the variational principle.

Various authors have considered similar problems involving springs of finite mass. The statics of a slinky has been studied by Mak [1]. Several authors have been interested in the effects on dynamics of the finite spring mass [2–6]. Rotating springs with [7] and without

$1$ $k$ is also called the force constant or spring constant.
finite mass have also attracted attention. The approach and most results presented here are, however, either new or, at least, hard to find in the literature. The discrete model could be taught at the elementary level. The continuum model should be useful in the teaching of variational principles as simple examples of their use.

2. Extension of a light spring by weight

Here we introduce notation for later use and make elementary derivations for comparison with results obtained below. When a particle of mass \( m \) hangs in a light (negligible mass) spring of natural length \( \ell \) and stiffness \( k \), the equation for static equilibrium is

\[
0 = mg - k\Delta\ell.
\]

(1)

where \( \Delta\ell \) is the extension of the spring, so the length of the loaded spring is

\[
\ell + \Delta\ell = \ell(1 + \Delta_g),
\]

(2)

Here \( \Delta_g \) is the dimensionless relative extension of the spring.

If the same light spring rotates freely with one end fixed on the rotation axis a particle of mass \( m \) at the other end remains at a fixed distance if

\[
-m\omega^2(\ell + \Delta\ell) = -k\Delta\ell.
\]

(4)

Here \( \omega \) is the constant angular velocity of the rotation and \( \Delta\ell \) is the extension of the spring. The dimensionless relative extension \( \Delta_\omega \), defined analogously to \( \Delta_g \) of equation (2), becomes

\[
\Delta_\omega = \frac{m\omega^2}{k}\frac{1}{1 - \frac{m\omega^2}{k}}.
\]

(5)

To compare this with the gravitational result we chose the angular velocity, \( \omega^2 = g/\ell \), so that the centripetal acceleration at the end of the unloaded spring is \( g \). One then finds that

\[
\Delta_\omega > \Delta_g,
\]

(6)

which is natural since the centrifugal force grows with distance from the rotation axis. Below we compare these results for a light spring with corresponding results for a heavy spring.

3. Gravity, discrete case

Consider a chain of \( N \) particles, each of mass \( \mu \), connected by \( N \) light springs of natural (neutral, unloaded, uncompressed) length \( a \) and stiffness (spring constant) \( \kappa \) (see figure 1). We must first establish the properties of the resulting finite mass spring of natural length \( \ell = Na \) and total mass \( m = N\mu \). The stiffness or spring constant \( k \) must be determined by the formula for the effective spring constant of springs in series. This means that

\[
k = \frac{1}{\sum_{i=1}^{N} \frac{1}{\kappa}} = \frac{\kappa}{N}
\]

(7)

is the stiffness of the full chain. The first spring has a free end and we attach this end to a fixed point and let the chain hang vertically from this point. Introduce a downward-directed
x-axis along the chain with origin at the point of suspension. If we introduce \( \delta \equiv \mu g / \kappa \), the equilibrium equations for the \( N \) particles can be written as

\[
0 = -(x_1 - a) + (x_2 - x_1 - a) + \delta \\
0 = -(x_2 - x_1 - a) + (x_3 - x_2 - a) + \delta \\
\ldots \\
0 = -(x_{N-1} - x_{N-2} - a) + (x_N - x_{N-1} - a) + \delta \\
0 = -(x_N - x_{N-1} - a) + \delta. 
\]

This system is most easily solved by noting that the \( i \)th spring is extended by the weight of the particles below it: \( \delta_i = (N + 1 - i) \delta \). The \( x \)-coordinate of the \( n \)th particle is thus

\[
x_n = na + \sum_{i=1}^{n} \delta_i = na + \frac{n(2N + 1 - n)}{2} \delta. 
\]

Putting \( n = N \) here gives the length of the hanging massive chain

\[
x_N = Na + \frac{N(N + 1)}{2} \delta. 
\]

Here we can use \( \ell = Na \), \( \delta = \mu g / \kappa \), \( \mu = m / N \) and \( \kappa = N k \) to get

\[
x_N = \ell + \left( 1 + \frac{1}{N} \right) \frac{mg}{k} = \ell \left[ 1 + \left( 1 + \frac{1}{N} \right) \frac{\Delta g}{2} \right]. 
\]

For a real coil spring with continuously distributed mass, we must let \( N \to \infty \). Comparing to (3) we then find that a heavy spring of mass \( m \) and stiffness \( k \) is extended half as much \( (\Delta g / 2) \) as the corresponding light spring with a weight of mass \( m \) at the end.

### 4. Passing to the continuum

To find how gravity deforms a spring with continuously distributed mass we will now use that potential energy minimization determines the static equilibrium. To pass to the continuum one can follow a procedure by Goldstein [9] (it can also be found in Sakurai [10]). He derives the wave equation for longitudinal waves along a one-dimensional elastic continuum by first considering the discrete case of a horizontal chain of masses and springs. In the limit of infinitely many particles the sum which is the Lagrangian for the discrete case is replaced by an integral of a Lagrangian density. The Euler–Lagrange equation of the corresponding action is the wave equation. Here we adapt to the static hanging case by skipping the kinetic energy and instead including the gravitational potential energy.

The potential energy of the discrete system of the previous section is given by

\[
V(x_n) = \sum_{i=1}^{N} \frac{K}{2} (x_i - x_{i-1} - a)^2 - \mu g x_i, 
\]

Figure 1. The unloaded (\( g = 0 \)) discrete model system, for \( N = 8 \), treated in this paragraph.
where we assume $x_0 = 0$. We rewrite this in the form
\begin{equation}
V(x_n) = \sum_{i=1}^{N} \kappa \left( \frac{x_i - x_{i-1}}{a} - 1 \right)^2 - \frac{m\ell}{\ell N} g x_i,
\end{equation}
(13)
put $\lambda = m/\ell$, $Y = \kappa a$, and note that $\ell/N = a$, to get
\begin{equation}
V(x_n) = \sum_{i=1}^{N} a \left[ \frac{Y}{2} \left( \frac{x_i - x_{i-1}}{a} - 1 \right)^2 - \lambda g x_i \right].
\end{equation}
(14)
Here $Y$ is Young’s modulus$^2$ and $\lambda$ the linear mass density of the undeformed spring. We can now pass to the continuum by the identifications
\begin{equation}
x_i \to \xi(x), \quad a \to dx, \quad \frac{x_i - x_{i-1}}{a} \to \frac{d\xi}{dx}.
\end{equation}
(15)
Here $0 \leq x \leq \ell$ represents the $x$-coordinate of points on the undeformed spring, and $\xi(x)$ is the $x$-coordinate of the same point on the spring deformed by gravity under its own weight.

For the continuum case the potential energy is now a functional of the function $\xi(x)$ which is the limit of the sum (14) as $N \to \infty$ and $a = dx \to 0$. The potential energy functional is thus
\begin{equation}
V[\xi(x)] = \int_{0}^{\ell} dx \, v(\xi, \xi') = \int_{0}^{\ell} dx \left[ \frac{Y}{2} \left( \frac{d\xi}{dx} - 1 \right)^2 - \lambda g \xi(x) \right],
\end{equation}
(16)
where it is assumed that $\xi(0) = 0$. Here, $v(\xi, \xi') = (Y/2)(\xi' - 1)^2 - \lambda g \xi$ is a potential energy per unit length.

### 5. Finding the minimum potential energy

Variation of the functional (16) in the usual way$^{12}$ gives
\begin{equation}
\delta V = \int_{0}^{\ell} \left( \frac{\partial v}{\partial \xi} \delta \xi + \frac{\partial v}{\partial \xi'} \delta \xi' \right) dx = 0,
\end{equation}
(17)
assuming that the functional is stationary. Then one notes that $\delta \xi' = d\delta \xi/dx$ is a derivative and takes advantage of integration by parts to get rid of this derivative. After this step one has
\begin{equation}
\delta V = \int_{0}^{\ell} \left( \frac{\partial v}{\partial \xi} - \frac{d}{dx} \frac{\partial v}{\partial \xi'} \right) \delta \xi dx + \left[ \frac{\partial v}{\partial \xi'} \delta \xi \right]_{0}^{\ell} = 0
\end{equation}
(18)
for the variation of $V$. At this point one usually invokes the requirement that $\delta \xi(0) = \delta \xi(\ell) = 0$ and thus arrive at the usual Euler–Lagrange equation. Here we must have $\delta \xi(0) = 0$ since the top end of the spring is fixed. At the lower end, however, this is not obvious.

The variation $\delta V = 0$ therefore results in the equation
\begin{equation}
\int_{0}^{\ell} \left( \frac{\partial v}{\partial \xi} - \frac{d}{dx} \frac{\partial v}{\partial \xi'} \right) \delta \xi(x) dx + \left( \frac{\partial v}{\partial \xi'} \right)_{x = \ell} \delta \xi(\ell) = 0.
\end{equation}
(19)
Since both the function $\delta \xi(x)$ and its value at $x = \ell$ are arbitrary, apart from the condition $\delta \xi(0) = 0$, the only way to satisfy this equation is to have both the Euler–Lagrange equation
\begin{equation}
\frac{d}{dx} \frac{\partial v}{\partial \xi'} - \frac{\partial v}{\partial \xi} = 0
\end{equation}
(20)

$^2$ This terminology is used by Goldstein [9] and is copied by Sakurai [10]. It is, however, not really the same quantity, usually denoted by $E$, that goes under this name in the theory of elasticity [11].
and the boundary condition
\[ \left( \frac{\partial v}{\partial \xi'} \right)_{\xi=\ell} = 0 \]  \hspace{1cm} (21)
satisfied.

Using the explicit form for \( v(\xi, \xi') \) of equation (16) we get the simple equation
\[ \frac{d^2 \xi}{dx^2} = -\lambda g Y \]  \hspace{1cm} (22)
from the Euler–Lagrange equation. Here the constant on the right-hand side can also be written as \(-1/\ell^2)(mg/k)\) using the definitions above (equation (14)). The boundary condition gives
\[ \left( \frac{d\xi}{dx} \right)_{\xi=\ell} \equiv \xi'(\ell) = 1. \]  \hspace{1cm} (23)

Since \( \xi(x) = x \) corresponds to an undeformed spring this result has the simple physical interpretation that the spring is not stretched (deformed) at its lower end where there is no mass below that pulls on it.

The solution of (22) is trivial to find once the boundary conditions \( \xi(0) = 0 \) and \( \xi'(\ell) = 1 \) are taken into account. One finds that the new \( x \)-coordinates of the particles of the spring are given in terms of the unloaded \( x \)-coordinates by
\[ \xi(x) = x + \frac{mg}{k} \left( 1 - \frac{1}{2} \frac{x}{\ell} \right). \]  \hspace{1cm} (24)

The total length of the hanging spring is then given by
\[ \xi(\ell) = \ell + \frac{1}{2} \frac{mg}{k} = \ell \left( 1 + \frac{\Delta^2}{2} \right) \]  \hspace{1cm} (25)
in agreement with the discrete result (11) when \( N \to \infty \).

6. Spring stretched by centrifugal force

We first consider the discrete case. The first spring of the chain is attached on a fixed rotation axis. The spring rotates freely with angular velocity \( \omega \). We choose a rotating \( z \)-axis parallel to the spring with the origin on the axis. The centrifugal force on a particle of the spring is then
\[ F_n = \mu \omega^2 z_n. \]

The equilibrium equations corresponding to equation (8) in the gravitational case are then
\[ 0 = -(z_1 - a) + (z_2 - z_1 - a) + \eta z_1 \]
\[ 0 = -(z_2 - z_1 - a) + (z_3 - z_2 - a) + \eta z_2 \]
\[ \ldots \ldots \]
\[ 0 = -(z_{N-1} - z_{N-2} - a) + (z_N - z_{N-1} - a) + \eta z_{N-1} \]
\[ 0 = -(z_N - z_{N-1} - a) + \eta z_{N}, \]  \hspace{1cm} (26)
where we have introduced \( \eta = \mu \omega^2 / k \). The trick used in the gravitational case to find \( x_n \) does not work here. Instead one might note that the typical equation of this sequence is the recursive relation
\[ z_{n+1} - (2 - \eta)z_n + z_{n-1} = 0. \]  \hspace{1cm} (27)
This is a so-called difference equation that can be treated by standard methods [13], but here we chose to proceed directly to the continuum case.
Recalling that the potential energy of the centrifugal force is $-\mu \omega^2 z^2/2$, we can take the result (16) for the gravitational case and get

$$V[\xi(z)] = \int_0^\ell dz\, v(\xi, \xi') = \int_0^\ell dz\left[ \frac{Y}{2} \left( \frac{d\xi}{dz} - 1 \right)^2 - \frac{1}{2} \lambda \omega^2 \xi^2(z) \right]$$

for the centrifugal case. The Euler–Lagrange differential equation is now

$$\frac{d^2 \xi}{dz^2} = -\frac{1}{\ell^2} \frac{m \omega^2}{k} \xi.$$  \hfill (29)

If we introduce the notation $\omega_0 = \sqrt{k/m}$, the solution obeying the boundary conditions $\xi(0) = 0$, $\xi'(\ell) = 1$ is easily found to be

$$\xi(z) = \frac{\ell}{\omega_0} \cos \left( \frac{\omega z}{\omega_0 \ell} \right).$$  \hfill (30)

This function and the corresponding one for gravity (24) are plotted and compared in figure 2.

The length of the spring in the centrifugal force field is

$$\xi(\ell) = \ell \frac{\omega_0}{\omega} \tan \left( \frac{\omega}{\omega_0} \right) \approx \ell \left[ 1 + \frac{1}{3} \left( \frac{\omega}{\omega_0} \right)^2 + \frac{2}{15} \left( \frac{\omega}{\omega_0} \right)^4 + \cdots \right].$$  \hfill (31)

For small angular velocity $\omega$ we get the approximate result

$$\xi(\ell) \approx \ell \left( 1 + \frac{1}{3} \frac{m \omega^2}{k} \right).$$  \hfill (32)

A light rotating spring with mass $m$ at the end becomes longer by the amount given in equation (5), which gives

$$\Delta_\omega \approx \frac{m \omega^2}{k} = \frac{\omega^2}{\omega_0^2},$$  \hfill (33)
for small angular velocity $\omega$. Equation (32) shows that if the mass instead is distributed along the spring the extra relative length is reduced to $\Delta L/3$, to the first order in $\omega$. This concludes our study of the rotating heavy spring.

7. Conclusions

The study of the deformation of springs under their own weight presented above provides nice illustrations of some general principles of statics. The most important of these is the fact that static equilibrium often is given simply by minimizing potential energy. The most well-known non-trivial example of this principle found in the literature is the Catenary, which usually is presented, together with the Brachistochrone, as a basic application of the variational calculus [12]. Unfortunately these classical examples yield quite difficult differential equations whose solution requires considerable mathematical skill. The examples here, giving the deformation of heavy springs, provide simple but interesting, non-trivial and easily understood results.

References

[1] Mak S Y 1987 The static effectiveness mass of a Slinky Am. J. Phys. 55 994–7
[2] Cushing J T 1984 The spring–mass system revisited Am. J. Phys. 52 925–33
[3] Cushing J T 1984 The method of characteristics applied to the massive spring problem Am. J. Phys. 52 933–7
[4] Ruby L 2000 Equivalent mass of a coil spring Phys. Teach. 38 140–1
[5] Christensen J 2004 An improved calculation of the mass for the resonant spring pendulum Am. J. Phys. 72 818–28
[6] Santos F C, Coutinho Y A, Ribeiro-Pinto L and Tort A C 2006 The motion of two masses coupled to a finite mass spring Eur. J. Phys. 27 1037–51
[7] Wildey R L 1989 A correction for spring mass in the ubiquitous centripetal force experiment of freshman physics Am. J. Phys. 57 1098–102
[8] Kenyon K E 2001 Exciting a rotating mass on a spring without change to its rotation rate Eur. J. Phys. 22 471–5
[9] Goldstein H 1980 Classical Mechanics 2nd edn (Reading, MA: Addison-Wesley)
[10] Sakurai J J 1967 Advanced Quantum Mechanics (Reading, MA: Addison-Wesley)
[11] Landau L D and Lifshitz E M 1986 Theory of Elasticity 3rd edn (Oxford: Butterworth-Heinemann)
[12] Fox C 1987 An Introduction to the Calculus of Variations (New York: Dover)
[13] Fox C A and Korn T M 1968 Mathematical Handbook for Scientists and Engineers—Definitions, Theorems and Formulas for Reference and Review (New York: McGraw-Hill)