The Cauchy problem for the DMKP equation

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Abstract

In this work, we study the dissipation-modified Kadomtsev-Petviashvili equation in two space-dimensional case. We establish that the Cauchy problem for this equation is locally well-posed in anisotropic Sobolev spaces. We show in some sense that our result is sharp. We also prove the global well-posedness for this equation under suitable conditions.

Keywords: DMKP equation, Well-posedness, Anisotropic Sobolev spaces.
Mathematical subject classification: 35B30, 35Q55, 35Q72.

1 Introduction

In this work, we consider the initial value problem for the dissipation-modified Kadomtsev-Petviashvili (DMKP) equation

\begin{equation}
\begin{aligned}
&\left\{
\begin{array}{l}
(ut + uxxx + uu_x + \alpha (uxx + uxxxx) + \beta (u^2)_{xx})_x + \varepsilon u_{yy} = 0, \\
u(x, y, 0) = \varphi(x, y),
\end{array}
\right.
\end{aligned}
\end{equation}

where \( \alpha > 0 \) and \( \beta \) are real constants and \( \varepsilon = \pm 1 \). The DMKP equation (1.1) arises in studying spontaneous generation of long waves in the presence of a conservation law in isotropic systems (e.g., Bénard-Marangoni waves), near the instability threshold [1, 8, 18]. In [6, 7], the author has also investigated another version of (1.1).

Equation (1.1) is also a natural two-dimensional version of the KdV-Kuramoto-Sivashinsky (KdV-KS) equation

\begin{equation}
\begin{aligned}
&u_t + u_{xxx} + uu_x + \alpha (uxx + uxxxx) = 0,
\end{aligned}
\end{equation}

which arises in interesting physical situations, for example as a model for long waves on a viscous fluid flowing down an inclined plane [20] and to derive drift waves in a plasma [5].

The DMKP equation, when \( \beta = 0 \), is a dissipative version of the Kadomtsev-Petviashvili (KP) equation

\begin{equation}
\begin{aligned}
&(ut + u_{xxx} + uu_x)x + \varepsilon u_{yy} = 0,
\end{aligned}
\end{equation}

which is universal model for nearly one directional weakly nonlinear dispersive waves with weak transverse effects. The KP equation, in turn, is a two-dimensional extension of the KdV equation

\begin{equation}
\begin{aligned}
&u_t + u_{xxx} + uu_x = 0.
\end{aligned}
\end{equation}

Our principal aim here is to study the local well-posedness for the initial value problem associated to the DMKP equation in the anisotropic Sobolev spaces \( H^{s_1, s_2}(\mathbb{R}^2) \), \( s_1, s_2 \in \mathbb{R} \).

In the past decades, Bourgain developed a new method, clarified by Ginibre in [9], for the study of the Cauchy problem for nonlinear dispersive equations. This method was further successfully applied

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to Schrödinger, KdV and KPII equations (cf. [2, 3, 4, 11, 12]). The original Bourgain method makes extensive use of the Strichartz inequalities in order to derive the bilinear estimates corresponding to the nonlinearity. On the other hand, Kenig et al. [11, 12] simplified Bourgain’s proof and improved the bilinear estimates using only elementary techniques, such as Cauchy-Schwartz inequality and simple calculus inequalities (see also [10, 19]).

It was also shown by Molinet and Ribaud [14, 15, 16] that the Bourgain spaces can be used to study the Cauchy problems associated to semi-linear equations with a linear part containing both dispersive and dissipative terms; and consequently this applies to the KdV-Burgers (KdVB) equation

$$u_t + u_{xxx} + uu_x = u_{xx}$$

(1.5)

and the Kadomtsev-Petviashvili-Burgers (KPB) equation

$$(u_t + u_{xxx} + uu_x - u_{xx})_x + \varepsilon u_{yy} = 0.$$  

(1.6)

By introducing a Bourgain space associated to the usual KP equation, related only to the dispersive part of the linear symbol of (1.6), Molinet and Ribaud [15] proved global existence for the Cauchy problem associated to the KPB equation (1.6), by using Strichartz-type estimates for the KP equation injected into the framework of Bourgain spaces. More precisely, authors in [15] showed the KPB-I equation ($\varepsilon = -1$) is locally well-posed in $H^{s_1,s_2}(\mathbb{R}^2)$ if $s_1 > 0$ and $s_2 \geq 0$; and the KPB-II equation ($\varepsilon = 1$) is locally well-posed in $H^s(\mathbb{R}^2)$ if $s \geq 0$. The global well-posedness is followed by means of a priori estimates.

Recently Kojok in [13] obtained a sharp result by proving that the KPB-II equation is globally well-posed in $H^{s_1,s_2}(\mathbb{R}^2)$ for $s_1 > -1/2$ and $s_2 \geq 0$.

In this paper, we will apply the ideas of [14, 15, 16] and introduce a Bourgain-type space associated to the KP equation. This space is in fact the intersection of the space introduced in [4] and of a Sobolev space. The advantage of this space is that it contains both the dissipative and dispersive parts of the linear symbol of (1.1). Next we establish the local existence for (1.1) with initial value $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$ when $s_1 > -1/2$ and $s_2 \geq 0$; and we also show that the Cauchy problem (1.1) is globally well-posed in $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$ if $\beta = 0, s_1 > -1$ and $s_2 \geq 0$. We prove also that our local existence theorem is optimal by constructing a counterexample showing that the application $\varphi \rightarrow u$ from $H^{s_1,s_2}(\mathbb{R}^2)$ to $C(\mathbb{R} \times [0,T] ; H^{s_1,s_2}(\mathbb{R}^2))$ cannot be regular for $s_1 < -1/2$ and $s_2 = 0$.

This existence result, in some sense, is quite surprising. There is no difference in the existence result for $\varepsilon = \pm 1$ in (1.1). However, despising the dissipation terms in (1.1), we obtain the KP equation (1.3), where the KP-I and KP-II models are quite distinct.

This paper is organized as follows. In Section 2, we introduce some notations and our main results. In Section 3, we derive linear estimates and some smoothing properties for the operator arising from (1.1) in the Bourgain spaces (Lemma 3.2). Section 4 is devoted to establish bilinear estimates by using Strichartz-type estimates for the KP equation. In Section 5, using bilinear estimates, a standard fixed point argument and some smoothing properties, we prove uniqueness and local existence of the solution of (1.1) in anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -1/2$ and $s_2 \geq 0$; and global existence of the solution of (1.1), with $\beta = 0$, in $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -1$ and $s_2 \geq 0$. Finally in Section 6 we show that our results are sharp in the sense that the flow map of the DMKP equation fail to be $C^2$ in $H^{s_1,0}(\mathbb{R}^2))$ for $s_1 < -1/2$.

2 Notations and Main Results

For the simplicity, throughout the paper we assume that $\beta = 1$ (if $\beta \neq 0$) and $\alpha = 1$. Before stating our main result, we introduce our notations that are used in this paper.

We denote $(\cdot) = 1 + |\cdot|$. The notation $A \lesssim B$ means that there exists the constant $C > 0$ such that $A \leq CB$. Similarly, we will write $A \sim B$ to mean $A \lesssim B$ and $A \gtrsim B$. 
For $n \in \mathbb{N}$, we denote by $\hat{f}$ the Fourier transform of $f$, defined as

$$\hat{f}(\omega) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot \omega} \, dx.$$ 

For $b, s_1, s_2 \in \mathbb{R}$, we denote $H^b = H^b(\mathbb{R})$, $\dot{H}^b = \dot{H}^b(\mathbb{R})$ and $H^{s_1, s_2} = H^{s_1, s_2}(\mathbb{R}^2)$ as the nonhomogeneous Sobolev, the homogeneous Sobolev and the anisotropic Sobolev spaces, respectively, defined by

$$H^b = \left\{ f \in \mathscr{S}'(\mathbb{R}) : \|f\|_{H^b} = \|\langle \tau \rangle^b \hat{f}(\tau)\|_{L^2} < \infty \right\},$$

$$\dot{H}^b = \left\{ f \in \mathscr{S}'(\mathbb{R}) : \|f\|_{\dot{H}^b} = \|\langle \tau \rangle^b \hat{f}(\tau)\|_{L^2} < \infty \right\},$$

$$H^{s_1, s_2} = \left\{ f \in \mathscr{S}'(\mathbb{R}^2) : \|f\|_{H^{s_1, s_2}} = \|\langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{f}(\xi, \eta)\|_{L^2_{\xi, \eta}} < \infty \right\}.$$ 

Let $U(\cdot)$ be the unitary group in $H^{s_1, s_2}$, $s_1, s_2 \in \mathbb{R}$, defining the free evolution of the KP equation (1.3), which is given by

$$U(t) = \exp(itP(D_x, D_y)),$$

where $P(D_x, D_y)$ is the Fourier multiplier with symbol $P(\zeta) = P(\xi, \eta) = \xi^3 - \varepsilon \eta^2 / \xi$, with $\varepsilon = \pm 1$.

We introduce a Bourgain space which is in relation with both the dissipative and dispersive parts of (1.1) at the same time, we define this space by

$$X^{b, s_1, s_2} = \left\{ f \in \mathscr{S}'(\mathbb{R}^3) : \|f\|_{X^{b, s_1, s_2}} < \infty \right\},$$

equipped with the norm

$$\|f\|_{X^{b, s_1, s_2}} = \left\| (\sigma + g(\xi))^{b} \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{f}(\tau, \zeta) \right\|_{L^2(\mathbb{R}^3)},$$

where $\sigma = \tau - P(\zeta)$ and $g(\xi) = \xi^4 - \xi^2$.

We should note that $X^{b, s_1, s_2}$ is the intersection of the Bourgain space associated with the dispersive part of equation (1.1) and Sobolev space. Indeed, one can easily see that

$$\|f\|_{X^{b, s_1, s_2}} \approx \|U(-t) f\|_{H^{s_1}_{b, 0} H^{s_2}_{b, 0}} + \|f\|_{L^2_{\tau} H_{x, y}^{s_1 + 2b, s_2}}.$$ 

For $T > 0$, we define the restricted spaces $X^{b, s_1, s_2}_T$ by the norm

$$\|f\|_{X^{b, s_1, s_2}_T} = \inf_{f \in X^{b, s_1, s_2}} \left\{ \|g\|_{X^{b, s_1, s_2}} : g(t) = f(t) \text{ on } [0, T] \right\}.$$ 

We denote by $W(\cdot)$ the semi-group associated with the free evolution of (1.1),

$$(W(t)f)^{\wedge_z}(\zeta) = \exp(itP(\zeta) - t\varrho(\xi)), \quad f \in \mathscr{S}', \quad z = (x, y), \quad t \geq 0.$$ 

Also, we can extend $W$ to a linear operator defined on the whole real axis by setting

$$(W(t)f)^{\wedge_z}(\zeta) = \exp(itP(\zeta) - |t|\varrho(\xi)), \quad f \in \mathscr{S}', \quad t \in \mathbb{R}.$$ 

By the Duhamel integral formulation, equation (1.1) can be written

$$u(t) = W(t) \varphi - \int_0^t W(t - t') \Lambda(u^2(t')) \, dt', \quad t \geq 0, \quad (2.1)$$

where $\Lambda = \frac{1}{2}\partial_x + \partial_x^2$. 

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To prove the local existence result, we will apply a fixed point argument to a truncated version of (2.1) which is defined on all the real axis by
\[ u(t) = \theta(t)W(t)\varphi - \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t W(t-t')\Lambda \left( \theta_T^2(t')u^2(t') \right) \, dt', \quad t \geq 0, \] (2.2)
where \( t \in \mathbb{R} \) and \( \theta \) indicates a time cutoff function:
\[ \theta \in C_0^\infty(\mathbb{R}), \quad \text{supp}(\theta) \subset [-2, 2], \quad \theta \equiv 1 \quad \text{on} \quad [-1, 1], \]
and \( \theta_T(\cdot) = \theta(\cdot/T) \).

We note that that if \( u \) solves (2.2) then \( u \) is a solution of (2.1) on \([0, T]\), \( T \leq 1 \). Thus it is sufficient to solve (2.2) for a small time (\( T \leq 1 \) is enough).

Let us now state our results.

**THEOREM 2.1** Let \( s_1 > -1/2, s_2 \geq 0, s'_1 \in (-1/2, \min\{0, s_1\}] \) and \( \varphi \in H^{s_1, s_2} \). Then there exists a time \( T = T(\|\varphi\|_{H^{s_1, 0}}) > 0 \) and a unique solution \( u \) of (1.1) in
\[ \mathcal{Y}_T = C([0, T], H^{s_1, s_2}) \cap X^{1/2, s_1, s_2}_T. \] (2.3)
Moreover, \( u \) belongs \( C([0, T]; H^{s_1, s_2}) \cap C((0, T]; H^{\infty, s_2}) \) and the map \( \varphi \mapsto u \) is analytic from \( H^{s_1, s_2} \) to \( \mathcal{Y}_T \).

**THEOREM 2.2** Let \( s < -1/2 \). Then it does not exist a time \( T > 0 \) such that equation (1.1) admits a unique solution in \( C([0, T]; H^{s,0}) \) for any initial data in some ball of \( H^{s,0} \) centered at the origin and such that the map \( \varphi \mapsto u \) is \( C^2 \)-differentiable at the origin from \( H^{s,0} \) to \( C([0, T], H^{s,0}) \).

**THEOREM 2.3** Let \( \beta = 0 \). Then Theorem 2.1 holds for \( s_1 > -1, s_2 \geq 0 \) and \( \varphi \in H^{s_1, s_2} \); and the corresponding local solution \( u \) of the Cauchy problem (1.1) extends globally in time.

### 3 Linear Estimates

In this section we are going to obtain some appropriate linear estimates for (2.2). The proofs of the linear estimates follow closely the proofs given in [14, 15, 16]. In this section we study the linear operator \( \theta V \).

**LEMMA 3.1** Let \( s_1, s_2 \in \mathbb{R} \), then for all \( \varphi \in H^{s_1, s_2} \), we have
\[ \|\theta(t)W(t)\varphi\|_{X^{1/2, s_1, s_2}} \lesssim \|\varphi\|_{H^{s_1, s_2}}. \] (3.1)

**Proof.** By definition of \( W(\cdot) \) and \( X^{1/2, s_1, s_2} \), and by performing the change of variable \( \tau \mapsto \sigma := \tau - P(\zeta) \), we have
\[ \|\theta(t)W(t)\varphi\|_{X^{1/2, s_1, s_2}} = \left\| (|\sigma + \varrho(\xi)|^{1/2} \langle |\varphi| \rangle^{s_1} \langle |\eta| \rangle^{s_2} \left( \theta(t)e^{-|t|\varrho(\xi)} \hat{\varphi}(\xi) \right)^{\wedge} \right\|_{L^2} \] (3.2)
\[ \lesssim I + II, \]
where
\[ I = \left\| (|\xi|^{s_1} \langle |\eta| \rangle^{s_2} \varrho(\xi))^{1/2} \hat{\varphi}(\xi) \|g_\xi(t)\|_{L^2} \right\|_{L^2}. \]
\[
II = \left\| |(\xi)^{s_1} (\eta)^{s_2} \hat{\varphi}(\zeta)| g_\xi(t) \right\|_{L^2_t},
\]
and
\[
g_\xi(t) = \theta(t) e^{-|\xi| |t|}.
\]

**Contribution of I.** When \(|\xi| \geq \sqrt{2}\), we have \(g(\xi) \geq 2\), then we can obtain
\[
\|g_\xi\|_{L^2_t} \leq \|e^{-|\xi| |t|}\|_{L^2_t} \sim |\theta(\xi)|^{-1/2} \lesssim \frac{1}{(|\theta(\xi)|)^{1/2}}.
\]
When \(|\xi| \leq \sqrt{2}\), then \(-1/4 \leq \theta(\xi) \leq 2\) implies that
\[
\|g_\xi\|_{L^2_t} \leq \|e^{\theta(t/2)}\|_{L^2_{t[-2,2]}} \lesssim 1 \lesssim (\theta(\xi))^{-1/2}.
\]
Then we deduce that
\[
I \lesssim \|\varphi\|_{H^{s_1-s_2}}.
\]

**Contribution of II.** When \(|\xi| \geq \sqrt{2}\), we use the Young inequality to see that
\[
\|g_\xi\|_{H^{1/2}} = \|\theta^{1/2} \ast (e^{-|t| |\xi|}) \hat{\theta}(\tau)\|_{L^2}\]
\[
\lesssim \|\theta^{1/2} \hat{\theta}(\tau)\|_{L^1_t} \|e^{-|t| |\xi|}\|_{L^2_t} + \|\hat{\theta}\|_{L^1_t} \|e^{-|t| |\xi|}\|_{H^{1/2}}\]
\[
\lesssim \frac{1}{(|\theta(\xi)|)^{1/2}} \lesssim 1.
\]
When \(|\xi| \leq \sqrt{2}\), since \(|\theta(\xi)| \leq 2\), we have
\[
\|g_\xi\|_{H^{1/2}} \leq \sum_{j \geq 0} \frac{2^j}{j!} \|t^j \theta(t)\|_{H^{1/2}} \lesssim 1.
\]
Since \(||t|^j \theta(t)||_{H^{1/2}} \leq |||t|^j \theta(t)||_{H^1} \lesssim j\), for \(j \geq 1\), therefore we deduce that
\[
II \lesssim \|\varphi\|_{H^{s_1-s_2}}.
\]

**Lemma 3.2** Let \(0 < \delta \leq 1/2\) and \(s_1, s_2 \in \mathbb{R}\), there exists \(C = C_{\delta} > 0\) such that for all \(w \in X^{-1/2+\delta, s_1-4\delta, s_2}\), we have
\[
\left\| \theta(t) \chi_{\mathbb{R}^+} (t) \int_0^t w(t') w(t') dt' \right\|_{X^{1/2, s_1, s_2}} \leq C \|w\|_{X^{-1/2+\delta, s_1-4\delta, s_2}}.
\]

**Proof.** Let \(b \in \mathbb{R}\). For \(\zeta \in \mathbb{R}^2\) fixed, we define the following time-Sobolev space
\[
\mathcal{W}_\zeta^b = \{w \in \mathcal{S}'(\mathbb{R}^3); \|w\|_{\mathcal{W}_\zeta^b} = \|(\iota \tau + \phi(\xi))^{b} \hat{w}(\tau, \zeta)\|_{L^2} < \infty\}.
\]
First we shall show that for \(\zeta \in \mathbb{R}^2\), \(0 < \delta \leq 1/2\) and \(w \in \mathcal{S}'(\mathbb{R}^3)\), the following estimate holds:
\[
\|K_\zeta(t)\|_{\mathcal{W}_\zeta^{1/2}} \lesssim \langle \zeta \rangle^{-4\delta} \|w\|_{\mathcal{W}_\zeta^{-1/2+\delta}},
\]
where
\[
K_\zeta(t) = \theta(t) \int_0^t e^{-|t-t'| |\phi(t)|} w(t', \zeta) dt'.
\]
By a simple calculation, similar to [15], one can easily show that

\[ K_\zeta(t) = \theta(t) \int_{\mathbb{R}} \frac{e^{it\zeta} - e^{-|t|\phi(\xi)}}{i\tau + \phi(\xi)} \tilde{w}(\tau, \zeta) d\tau. \]

We split \( K_\zeta \) into \( K_\zeta = K_{1,0} + K_{1,\infty} + K_{2,0} + K_{2,\infty} \), where

\[ K_{1,0} = \theta(t) \int_{|\tau| \leq 1} \frac{e^{it\tau} - 1}{i\tau + \phi(\xi)} \tilde{w}(\tau, \zeta) d\tau, \quad K_{1,\infty} = \theta(t) \int_{|\tau| \geq 1} \frac{e^{it\tau}}{i\tau + \phi(\xi)} \tilde{w}(\tau, \zeta) d\tau, \]

\[ K_{2,0} = \theta(t) \int_{|\tau| \leq 1} \frac{1 - e^{-|t|\phi(\xi)}}{i\tau + \phi(\xi)} \tilde{w}(\tau, \zeta) d\tau, \quad K_{2,\infty} = \theta(t) \int_{|\tau| \geq 1} \frac{e^{-|t|\phi(\xi)}}{i\tau + \phi(\xi)} \tilde{w}(\tau, \zeta) d\tau; \]

and then we examine each \( K_\cdot \) in (3.7).

**Contribution of** \( K_{2,\infty} \). In this case, since \(|\tau| \geq 1\), note that

\[ \| (i\tau + \phi(\xi))^{1/2} K_{2,\infty} \|_{L^2_x} \leq \| (i\tau + \phi(\xi))^{1/2} (g_\zeta(t))^{1/1} (\tau) \|_{L^2_x} \left( \int_{|\tau| \geq 1} \frac{\tilde{w}(\tau, \zeta)}{(i\tau + \phi(\xi))^{1+2\delta}} d\tau \right), \]

where \( g_\zeta \) is defined in (3.3). Exactly the same computations as in Lemma 3.1 lead to

\[ \left\| (i\tau + \phi(\xi))^{1/2} (g_\zeta(t))^{1/1} (\tau) \right\|_{L^2_x} \lesssim 1. \]

Therefore, by the Cauchy-Schwarz inequality, we obtain

\[ \| K_{2,\infty} \|_{\phi^{1/2}} \lesssim \left( \int_{\mathbb{R}} \frac{\tilde{w}(\tau, \zeta)^2}{(i\tau + \phi(\xi))^{1-2\delta}} d\tau \right)^{1/2} \left( \int_{|\tau| \geq 1} \frac{d\tau}{(i\tau + \phi(\xi))^{1+2\delta}} \right)^{1/2}. \]

When \(|\xi| \geq \sqrt{2}\), a change of variable gives

\[ \| K_{2,\infty} \|_{\phi^{1/2}} \lesssim \langle \xi \rangle^{-\delta} \| w \|_{\phi^{1/2}}. \]  

(3.8)

When \(|\xi| \leq \sqrt{2}\), it follows \( \langle \xi \rangle^{-\delta} \sim 1 \); so that (3.8) holds.

**Contribution of** \( K_{1,\infty} \). In this case, by using the Young inequality, we see that

\[ \| K_{1,\infty} \|_{\phi^{1/2}} = \left\| (i\tau + \phi(\xi))^{1/2} \left[ \hat{\theta}(\tau') \ast \left( \frac{\tilde{w}(\tau', \zeta)}{(i\tau' + \phi(\xi))^{1+2\delta}} \right) \right] \right\|_{L^2_x} \]

\[ \lesssim \left\| \langle \tau' \rangle^{1/2} \hat{\theta}(\tau') \left( \frac{\tilde{w}(\tau', \zeta)}{(i\tau' + \phi(\xi))^{1+2\delta}} \right) \right\|_{L^2_x} + \left\| \langle \tau' \rangle^{1/2} \hat{\theta}(\tau') \right\|_{L^2_x} \left\| \frac{\tilde{w}(\tau, \zeta)}{(i\tau + \phi(\xi))^{1+2\delta}} \right\|_{L^2_x}. \]

\[ \lesssim \left\| \frac{\tilde{w}(\tau, \zeta)}{(i\tau + \phi(\xi))^{1+2\delta}} \right\|_{L^2_x} \lesssim \langle \xi \rangle^{-\delta} \| w \|_{\phi^{1/2}}. \]

**Contribution of** \( K_{2,0} \). First we notice that

\[ \| K_{2,0} \|_{\phi^{1/2}} \leq \left( \int_{|\tau| \leq 1} \frac{\tilde{w}(\tau, \zeta)}{(i\tau + \phi(\xi))} d\tau \right) \left\| (i\tau + \phi(\xi))^{1/2} \left( \theta(t) \left( 1 - e^{-|t|\phi(\xi)} \right) \right) \right\|_{L^2_x}. \]  

(3.9)
Now, as in the proof of Lemma 3.1, we consider two cases. When $|\xi| \geq \sqrt{2}$, we have $\varrho(\xi) \geq 2$, so that
\begin{equation}
\| \langle i\sigma + \varrho(\xi) \rangle^{1/2} \left( \theta(t) \left( 1 - e^{-|t|\varrho(\xi)} \right) \right)^{\lambda^t} \|_{L^2_{t}} \lesssim \| \theta \|_{H^{1/2}_{t}} + \langle \varrho(\xi) \rangle^{1/2} \| \theta \|_{L^2_{t}} + \| g_{\xi} \|_{H^{1/2}_{t}} + \langle \varrho(\xi) \rangle^{1/2} \| g_{\xi} \|_{L^2_{t}} \lesssim \| \varrho(\xi) \|^{1/2}.
\end{equation}

On the other hand, we have
\begin{align}
\int_{|\tau| \leq 1} \frac{|\tau + \varrho(\xi)|}{|\tau + \varrho(\xi)|^2} d\tau \approx & \int_{|\tau| \leq 1} \frac{d\tau}{|\tau + \varrho(\xi)|^2} + \int_{|\tau| \leq 1} \frac{d\tau}{|\tau + \varrho(\xi)|} \\
\lesssim & \int_{0}^{1} \frac{d\tau}{\tau^2 + \varrho^2(\xi)} + \frac{1}{|\varrho(\xi)|} \\
\lesssim & \frac{1}{\varrho(\xi)} \int_{0}^{1} \frac{d\tau}{1 + \left( \frac{\tau}{|\varrho(\xi)|} \right)^2} \left( \frac{\tau}{|\varrho(\xi)|} \right) + \frac{1}{|\varrho(\xi)|} \lesssim \frac{1}{|\varrho(\xi)|}.
\end{align}

From (3.9)-(3.11), we deduce that
\begin{align}
\| K_{2,0} \|_{\varphi^{-1/2}} \lesssim & \| \varrho(\xi) \|^{1/2} \left( \int_{|\tau| \leq 1} \frac{|\hat{\omega}(\tau, \xi)|^2}{|\tau + \varrho(\xi)|} d\tau \right)^{1/2} \left( \int_{|\tau| \leq 1} \frac{|\tau + \varrho(\xi)|}{|\tau + \varrho(\xi)|^2} d\tau \right)^{1/2} \\
\lesssim & \left( \int_{|\tau| \leq 1} \frac{|\hat{\omega}(\tau, \xi)|^2}{|\tau + \varrho(\xi)|} d\tau \right)^{1/2} \lesssim \langle \xi \rangle^{-4\delta} \| w \|_{\varphi^{-1/2+\delta}}.
\end{align}

When $|\xi| \leq \sqrt{2}$, then $|\varrho(\xi)| \leq 2$ and we have
\begin{equation}
\| \langle i\sigma + \varrho(\xi) \rangle^{1/2} \left( \theta(t) \left( 1 - e^{-|t|\varrho(\xi)} \right) \right)^{\lambda^t} \|_{L^2_{t}} \lesssim \| \theta(t) \left( 1 - e^{-|t|\varrho(\xi)} \right) \|_{H^{1/2}_{t}}.
\end{equation}

Then arguing again as in Lemma 3.1, we obtain that
\begin{equation}
\| \theta(t) \left( 1 - e^{-|t|\varrho(\xi)} \right) \|_{H^{1/2}_{t}} \leq \sum_{j \geq 0} \frac{|\varrho(\xi)|^j}{j!} \| t^j \theta(t) \|_{H^{1/2}_{t}} \lesssim |\varrho(\xi)| \sum_{j \geq 0} \frac{|\varrho(\xi)|^j}{j!} \lesssim |\varrho(\xi)|.
\end{equation}

From (3.9) and (3.11)-(13.13), we get
\begin{equation}
\| K_{2,0} \|_{\varphi^{-1/2}} \lesssim \langle \xi \rangle^{-4\delta} \| w \|_{\varphi^{-1/2+\delta}}.
\end{equation}

**Contribution of $K_{1,0}$.** Since $K_{1,0}$ can be written as
\[ K_{1,0} = \theta(t) \sum_{j \geq 1} \int_{|\tau| \leq 1} \frac{\langle i\tau \rangle^j}{j!} \frac{\hat{\omega}(\tau, \xi)}{|\tau + \varrho(\xi)|} d\tau, \]
we deduce from the Cauchy-Schwarz inequality that
\begin{align}
\| \langle i\tau + \varrho(\xi) \rangle^{1/2} K_{1,0}(\tau) \|_{L^2_{t}} \lesssim & \sum_{j \geq 1} \frac{1}{j!} \left( \| t^j \theta(t) \|_{H^{1/2}_{t}} + \langle \varrho(\xi) \rangle^{1/2} \| t^j \theta(t) \|_{L^2_{t}} \right) \int_{|\tau| \leq 1} \frac{|\tau| \| \hat{\omega}(\tau, \xi) \|}{|\tau + \varrho(\xi)|} d\tau \\
\lesssim & \langle \varrho(\xi) \rangle^{1/2} \left( \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau, \xi)|^2}{|\tau + \varrho(\xi)|} d\tau \right)^{1/2} \left( \int_{|\tau| \leq 1} \frac{|\tau|^2 \| i\tau + \varrho(\xi) \|}{|\tau + \varrho(\xi)|^2} d\tau \right)^{1/2} \\
\lesssim & \left( \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau, \xi)|^2}{|\tau + \varrho(\xi)|} d\tau \right)^{1/2}.
\end{align}
Finally, since for $\langle i\tau + \varrho(\xi) \rangle^{1/2} \geq \langle i\tau + \varrho(\xi) \rangle^{1 - 2\delta} \langle \varrho(\xi) \rangle^{2\delta}$, we get
\[
\|K_{1,0}\|_{\varphi_{\xi}^{1/2}} \lesssim \langle \xi \rangle^{-4\delta} \|w\|_{\varphi_{\xi}^{-1/2+\delta}};
\]
which completes the proof of (3.7).

Now by definition of $X^{1/2,s_1,s_2}$, we see that
\[
\left\| \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t W(t-t')w(t')dt' \right\|_{X^{1/2,s_1,s_2}}
= \left\| \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} (i\sigma + \varrho(\xi))^{1/2} \left( \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t W(t-t')w(t')dt' \right)^{\wedge_t} (\tau, \zeta) \right\|_{L^2(\mathbb{R}^3)}.
\]
We also note that
\[
\left( \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t W(t-t')w(t')dt' \right)^{\wedge_t} (\tau, \zeta)
= \left( \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t e^{-|t-t'|\varrho(\xi)} e^{i\varrho(\xi)\vartheta((t-t')\vartheta'(t', \zeta)dt') \right)^{\wedge_t} (\tau, \zeta)
= \left( \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t e^{-|t-t'|\varrho(\xi)} e^{-i\varrho(\xi)\vartheta'(t', \zeta)dt'} (U(t)w)^{\wedge_t} (t', \zeta)dt' \right)^{\wedge_t} (\tau, \zeta)
= \left( \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t e^{-|t-t'|\varrho(\xi)} e^{-i\varrho(\xi)\vartheta'(t', \zeta)dt'} (w)^{\wedge_t} (t', \zeta)dt' \right)^{\wedge_t} (\tau - P(\zeta), \zeta);
\]
and hence
\[
\left\| \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t W(t-t')w(t')dt' \right\|_{X^{1/2,s_1,s_2}}
= \left\| \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \left( \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t e^{-|t-t'|\varrho(\xi)} (U(-t)w)^{\wedge_t} (t', \zeta)dt' \right)^{\wedge_t} (t) \right\|_{L^2(\varphi_{\xi}^{1/2})}.
\]
Now define $v(t, \zeta) = (U(-t)w)^{\wedge_t} (t, \zeta) \in \mathcal{F}(\mathbb{R}^3)$. Then by applying (3.7), we obtain
\[
\left\| \theta(t)\chi_{\mathbb{R}^+}(t) \int_0^t W(t-t')w(t')dt' \right\|_{X^{1/2,s_1,s_2}}
\lesssim \left\| \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \|v\|_{\varphi_{\xi}^{-1/2+\delta}} \langle \xi \rangle^{-4\delta} \right\|_{L^2_{\zeta}}
\lesssim \left\| \langle \xi \rangle^{s_1-4\delta} \langle \eta \rangle^{s_2} (i\sigma + \varrho(\xi))^{-1/2+\delta} \vartheta(\tau) \right\|_{L^2_{\zeta}}
\lesssim \left\| \langle \xi \rangle^{s_1-4\delta} \langle \eta \rangle^{s_2} (i\sigma + \varrho(\xi))^{-1/2+\delta} (U(-t)w)^{\wedge_t} (t, \zeta) \right\|_{L^2(\mathbb{R}^3)}
\lesssim \left\| \langle \xi \rangle^{s_1-4\delta} \langle \eta \rangle^{s_2} (i\sigma + \varrho(\xi))^{-1/2+\delta} \vartheta(\tau + P(\zeta), \zeta) \right\|_{L^2(\mathbb{R}^3)}.
\]
Finally, by performing a change of variable, we deduce (3.6); and the proof of Lemma 3.2 is complete. \qed

**Lemma 3.3** Let $s_1, s_2 \in \mathbb{R}$ and $0 < \delta \leq 1/2$. Then for all $f \in X^{-1/2+\delta,s_1-4\delta,s_2}$, we have
\[
N : t \mapsto \int_0^t W(t-t')f(t')dt' \in C(\mathbb{R}^+; H^{s_1,s_2}).
\]
Moreover, if \( \{f_n\} \) is a sequence with \( f_n \to 0 \) in \( X^{-1/2+\delta,s_1,-4\delta,s_2} \) as \( n \to \infty \), then

\[
\left\| \int_0^t W(t-t')f_n(t')dt' \right\|_{L^\infty(\mathbb{R}^+;H^{s_1,-2})} \to 0. \tag{3.15}
\]

**Proof.** By Fubini theorem, and by the definition of \( W(\cdot) \) we have

\[
N(t) = \int_0^t \left( e^{-|t-t'|}e^{i(t-t')P(\xi)}(f(t'))^\vee \right) v \, dt' = U(-t) \left( \int_0^t e^{-|t-t'|}e^{i(t-t')P(\xi)}(g(t',\cdot))^\vee \, dt' \right) v, \tag{3.16}
\]

where \( g(t,z) = U(-t)f(t,\cdot)(z) \). Since \( U \) is a strongly continuous unitary group in \( L^2(\mathbb{R}^2) \), it is enough to prove that

\[
F(\cdot,\xi) : t \in \mathbb{R}^+ \mapsto \langle \xi \rangle^{s_1}\langle \eta \rangle^{s_2} \int_0^t e^{-|t-t'|}e^{i(t-t')P(\xi)}(g(t',\cdot))^\vee \, dt'
\]

is continuous from \( \mathbb{R}^+ \) in \( L^2_\delta(\mathbb{R}^2) \) for \( f \in X^{-1/2+\delta,s_1,-4\delta,s_2}, 0 < \delta \leq 1/2 \). We note that by Fubini theorem we have

\[
F(t,\xi) = \langle \xi \rangle^{s_1}\langle \eta \rangle^{s_2} \int_{\mathbb{R}} \hat{g}(\tau,\xi) \frac{e^{it\tau} - e^{-t\phi(\xi)}}{i\tau + \phi(\xi)} \, d\tau.
\]

Fix \( t_0 \in \mathbb{R}^+ \) and define for all \( t \in \mathbb{R} \),

\[
H(t,\xi) = F(t,\xi) - F(t_0,\xi)
\]

\[
= \langle \xi \rangle^{s_1}\langle \eta \rangle^{s_2} \int_{\mathbb{R}} \hat{g}(\tau,\xi) \left[ \frac{e^{it\tau} - e^{-t\phi(\xi)}}{i\tau + \phi(\xi]} \right] \, d\tau.
\]

We will use the Lebesgue dominated convergence theorem to show that

\[
\lim_{t \to t_0} \| H(t,\cdot) \|_{L^2(\mathbb{R}^2)} = 0. \tag{3.17}
\]

First we note that

\[
\lim_{t \to t_0} h(t,\tau,\xi) = 0, \quad \text{a.e. } (\tau,\xi) \in \mathbb{R}^3, \tag{3.18}
\]

where

\[
h(t,\tau,\xi) = \frac{\hat{g}(\tau,\xi)}{i\tau + \phi(\xi)} \left[ \frac{e^{it\tau} - e^{-t\phi(\xi)} + e^{-t_0\phi(\xi)}}{i\tau + \phi(\xi)} \right]. \tag{3.19}
\]

Moreover, since \( t \to t_0 \), we can suppose that \( 0 \leq t \leq T \), and then,

\[
|h(t,\tau,\xi)| \leq (2 + e^{t/4} + e^{t_0/4}) \frac{|\hat{g}(\tau,\xi)|}{|i\tau + \phi(\xi)|} \lesssim \frac{|\hat{g}(\tau,\xi)|}{|i\tau + \phi(\xi)|} \tag{3.20}
\]

We deduce from the Cauchy-Schwarz inequality that

\[
\int_{\mathbb{R}} \frac{|\hat{g}(\tau,\xi)|}{|i\tau + \phi(\xi)|} \, d\tau \lesssim \left\| \frac{|i\tau + \phi(\xi)|^{1/2-\delta}}{|i\tau + \phi(\xi)|} \right\|_{L^2_\delta} \left\| \frac{\hat{g}(\tau,\xi)}{|i\tau + \phi(\xi)|^{1/2-\delta}} \right\|_{L^2_\delta}.
\]

By the hypotheses on \( g \), we deduce

\[
\int_{\mathbb{R}} \frac{|\hat{g}(\tau,\xi)|}{|i\tau + \phi(\xi)|} \, d\tau \lesssim \left\| \frac{\hat{g}(\tau,\xi)}{|i\tau + \phi(\xi)|^{1/2-\delta}} \right\|_{L^2_\delta}, \tag{3.21}
\]

for almost every \( \xi \in \mathbb{R}^2 \). We use (3.18)-(3.21) and the Lebesgue dominated convergence theorem to conclude that

\[
\lim_{t \to t_0} H(\xi,t) = 0, \quad \text{a.e. } \xi \in \mathbb{R}^2. \tag{3.22}
\]
Next we show that there exists $G \in L^2(\mathbb{R}^2)$ such that
\[
|H(t, \zeta)| \leq |G(\zeta)|, \quad (3.23)
\]
for all $\zeta \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$. When $|\xi| \geq \sqrt{2}$, we get from the Cauchy-Schwarz inequality and (3.20) that
\[
|H(t, \zeta)| \lesssim \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \left\| \frac{\langle i\tau + \varphi(\xi) \rangle^{1/2 - \delta}}{|i\tau + \varphi(\xi)|} \right\|_{L^2_t} \left\| \frac{\widehat{g}(\tau, \xi)}{|i\tau + \varphi(\xi)|^{1/2 - \delta}} \right\|_{L^2_t}.
\]
Since $\varphi(\xi) \geq 2$, we have
\[
\left\| \frac{\langle i\tau + \varphi(\xi) \rangle^{1/2 - \delta}}{|i\tau + \varphi(\xi)|} \right\|_{L^2_t} \lesssim \left( \int_{\mathbb{R}} \frac{1}{|i\tau + \varphi(\xi)|^{1+2\delta}} d\tau \right)^{1/2} \lesssim \langle \xi \rangle^{-4\delta},
\]
then using the hypotheses on $g$, we conclude that for all $t \in \mathbb{R}^+$,
\[
|H(t, \zeta)| \lesssim \langle \xi \rangle^{s_1 - 4\delta} \langle \eta \rangle^{s_2} \left\| \frac{\widehat{g}(\tau, \xi)}{|i\tau + \varphi(\xi)|^{1/2 - \delta}} \right\|_{L^2_t} \in L^2(\mathbb{R}^2),
\]
which proves (3.23) in this case. When $|\xi| \leq \sqrt{2}$, then we have $|\varphi(\xi)| \leq 2$, so that
\[
|H(t, \zeta)| \lesssim \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \int_{\mathbb{R}} \left| \frac{\widehat{g}(\tau, \xi)}{|i\tau + \varphi(\xi)|} \right| e^{-t\varphi(\xi)} - e^{-t_0\varphi(\xi)} d\tau + \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \int_{\mathbb{R}} \left| \frac{\widehat{g}(\tau, \xi)}{|i\tau + \varphi(\xi)|} \right| e^{i\tau} - e^{i\tau_0} d\tau = I + II.
\]
We first evaluate $II$. Using the Cauchy-Schwarz inequality
\[
II \leq |t - t_0| \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \int_{|\tau| \leq 1} \left| \frac{\langle i\tau + \varphi(\xi) \rangle^{1/2 - \delta}}{|i\tau + \varphi(\xi)|} \right| d\tau + 2 \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \int_{|\tau| \geq 1} \left| \frac{\widehat{g}(\tau, \xi)}{|i\tau + \varphi(\xi)|} \right| d\tau
\]
\[
\lesssim \langle \xi \rangle^{s_1 - 4\delta} \langle \eta \rangle^{s_2} \left( \int_{|\tau| \leq 1} \frac{\langle \widehat{g}(\tau, \xi) \rangle^2}{|i\tau + \varphi(\xi)|^{1+2\delta}} d\tau \right)^{1/2} \left[ \left( \int_{|\tau| \leq 1} |\tau|^{1-2\delta} d\tau \right)^{1/2} + \left( \int_{|\tau| \geq 1} |\tau|^{-1-2\delta} d\tau \right)^{1/2} \right]
\]
\[
\lesssim \langle \xi \rangle^{s_1 - 4\delta} \langle \eta \rangle^{s_2} \left( \frac{\langle \widehat{g}(\tau, \xi) \rangle^2}{|i\tau + \varphi(\xi)|^{1+2\delta}} d\tau \right)^{1/2} \in L^2(\mathbb{R}^2).
\]
We next turn to $I$ and again use the Cauchy-Schwarz inequality to see that
\[
I \leq |t - t_0| \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \left( \int_{\mathbb{R}} \frac{\langle \widehat{g}(\tau, \xi) \rangle^2}{|i\tau + \varphi(\xi)|^{1-2\delta}} d\tau \right)^{1/2} \langle \varphi(\xi) \rangle \left( \int_{\mathbb{R}} \frac{|i\tau + \varphi(\xi)|^{1-2\delta}}{|i\tau + \varphi(\xi)|^2} d\tau \right)^{1/2},
\]
and we compute
\[
\left( \int_{\mathbb{R}} \frac{|i\tau + \varphi(\xi)|^{1-2\delta}}{|i\tau + \varphi(\xi)|^2} d\tau \right)^{1/2} \lesssim \left( \int_{\mathbb{R}} \frac{1}{|i\tau + \varphi(\xi)|^{1+2\delta}} d\tau \right)^{1/2} + \left( \int_{\mathbb{R}} \frac{1}{|i\tau + \varphi(\xi)|^{1+2\delta}} d\tau \right)^{1/2}
\]
\[
\lesssim \frac{1}{\sqrt{|\varphi(\xi)|}} + \frac{1}{|\varphi(\xi)|^{\delta}}.
\]
Then, since $|\varphi(\xi)| \leq 2$, we conclude that
\[
I \lesssim \langle \xi \rangle^{s_1 - 4\delta} \langle \eta \rangle^{s_2} \left( \int_{\mathbb{R}} \frac{\langle \widehat{g}(\tau, \xi) \rangle^2}{|i\tau + \varphi(\xi)|^{1-2\delta}} d\tau \right)^{1/2} \in L^2(\mathbb{R}^2).
\]
Thus (3.23) still remains true in this case. We use (3.22), (3.23) and the dominated convergence theorem to prove (3.17).

To show (3.15) it suffices to notice that one has

$$\lim_{t \to 0} \sup_{t < T} \|F_n(t)\|_{L^2(R^3)} \leq Cn^{-1/2} \|f_n\|_{X^{1/2, 
1/2, R^3}} \leq \|f_n\|_{X^{1/2 + \delta, R^3}}$$

where $F_n$ is defined as $F$ with $g_n(t, z) = U(t)f_n(t, \cdot)(z)$ instead of $g$. This completes the proof. \qed

4. Bilinear Estimates

In this section, we are going to obtain suitable estimates for the nonlinear terms (1.1). Before stating this result, we will give certain multilinear estimates which are necessary to treat the nonlinear term $A(u^2)$ in $X^{b, a_1, a_2}$.

**Lemma 4.1 ([13, 15])** Let $u, v, w \in L^2(R^3)$ with compact support in $\{ (x, y) \in R^3 : |t| \leq T \}$. Then for $b > 0$ and $c > 0$ small enough there exists $\mu > 0$ such that

$$\int_{R^6} \frac{\|\hat{u}(r, \zeta)\| \|\hat{v}(r_1, \zeta_1)\| \|\hat{w}(r_2, \zeta_2)\| {d} r \hat{r} \hat{d} r_1 \hat{d} \zeta_1 \leq C T^\mu \|u\|_{L^2(R^3)} \|v\|_{L^2(R^3)} \|w\|_{L^2(R^3)}.$$  \hspace{1cm} (4.1)

where

$$\zeta = (\xi, \eta), \quad \zeta_1 = (\xi_1, \eta_1), \quad \zeta_2 = \zeta - \zeta_1$$  \hspace{1cm} (4.2)

and

$$\sigma = \tau - P(\zeta), \quad \sigma_1 = \tau_1 - P(\zeta_1), \quad \sigma_2 = \tau_2 - P(\zeta_2).$$

**Lemma 4.2 ([13, 15])** Let $u, v, w \in L^2(R^3)$ with compact support in $\{ (x, y) \in R^3 : |t| \leq T \}$. Then for $b > 0$ and $c > 0$ small enough there exists $\mu > 0$ such that

$$\int_{R^6} \frac{\|\hat{u}(r, \zeta)\| \|\hat{v}(r_1, \zeta_1)\| \|\hat{w}(r_2, \zeta_2)\| {d} r \hat{r} \hat{d} r_1 \hat{d} \zeta_1 \leq C T^\mu \|u\|_{L^2(R^3)} \|v\|_{L^2(R^3)} \|w\|_{L^2(R^3)}.$$  \hspace{1cm} (4.3)

**Theorem 4.3** Let $\delta > 0$ small enough, $s_2 \geq 0$ and $s_1 > -1/2$. For all $u, v \in X^{1/2, a_1, a_2}$ with compact support in time and included in the subset $\{ t, x, y, t \in [-T, T] \}$, there exists $\mu > 0$ such that the following bilinear estimate holds

$$\|A(u^2)v\|_{X^{-1/2 + \delta, a_1, a_2}} \leq C T^\mu \|u\|_{X^{1/2, a_1, a_2}} \|v\|_{X^{1/2, a_1, a_2}}.$$  \hspace{1cm} (4.4)

**Proof.** We proceed by duality. It is equivalent to show that for $\delta > 0$ small enough and for all $w \in X^{1/2, a_1, a_2},$

$$\langle A(u^2)w, \rangle \leq C T^\mu \|u\|_{X^{1/2, a_1, a_2}} \|v\|_{X^{1/2, a_1, a_2}} \|w\|_{X^{1/2 - \delta, a_1, a_2}}.$$  \hspace{1cm} (4.5)

Let $f, g$ and $h$ respectively defined by

$$\hat{f}(r, \zeta) = (i(r - P(\zeta)) + \phi(\zeta))^{1/2} \langle \zeta \rangle^{a_1} \hat{u}(r, \zeta),$$  \hspace{1cm} (4.6)

$$\hat{g}(r, \zeta) = (i(r - P(\zeta)) + \phi(\zeta))^{1/2} \langle \zeta \rangle^{a_1} \hat{v}(r, \zeta),$$  \hspace{1cm} (4.7)

$$\hat{h}(r, \zeta) = (i(r - P(\zeta)) + \phi(\zeta))^{-1/2 + \delta} \langle \zeta \rangle^{-a_1 + \delta} \hat{w}(r, \zeta).$$  \hspace{1cm} (4.8)

It is clear that

$$\|u\|_{X^{1/2, a_1, a_2}} = \|f\|_{L^2(R^3)}, \quad \|v\|_{X^{1/2, a_1, a_2}} = \|g\|_{L^2(R^3)} \quad \text{and} \quad \|w\|_{X^{-1/2 + \delta, a_1, a_2}} = \|h\|_{L^2(R^3)}.$$
Thus by Plancherel theorem, inequality (4.5) is equivalent to

\[
\int_{\mathbb{R}^3} \frac{|q(\xi)| \hat{g}^2(\tau_1, \xi_1) \hat{h}(\tau_2, \xi_2) |\hat{A}(\tau, \xi)|^{s_1/4} |\hat{G}(\eta_1)|^{s_2/4}}{(i\sigma + \hat{Q}(\xi))^{1/2} (i\sigma_1 + \hat{Q}(\xi_1))^{1/2} (i\sigma_2 + \hat{Q}(\xi_2))^{1/2}} \, d\tau_1 d\xi_1 d\eta_1 \leq CT^u ||u||_{L^2_{\mu}} ||v||_{L^2_{\nu}} ||w||_{L^2_{T}},
\]

where \( q(\xi) = |\xi| + \xi^2 \). We can assume that \( s_2 = 0 \) and \( s_1 \leq 0, \) since in the case \( s_1, s_2 \geq 0, \) we have

\[
\frac{(|\eta|)^{s_2}}{(\xi_1)^{s_1} (\eta_2)^{s_2}} \leq 1 \quad \text{and} \quad \frac{(|\xi|)^{s_1}}{(\xi_1)^{s_1} (\eta_2)^{s_2}} \leq 1,
\]

for all \( \xi_1, \xi, \eta_1, \eta \in \mathbb{R} \). We note that it suffices to prove (4.9) for \( q(\xi) = \xi^2 \).

Therefore setting \( s = -s_1 \geq 0, \) it is enough to estimate

\[
I = \int_{\mathbb{R}^3} \frac{|\hat{g}(\tau_1, \xi_1) \hat{g}(\tau_2, \xi_2) |\hat{h}(\tau, \xi)|^{s_2} |\hat{A}(\tau, \xi)|^{s_1}}{(i\sigma + \hat{Q}(\xi))^{1/2} (i\sigma_1 + \hat{Q}(\xi_1))^{1/2} (i\sigma_2 + \hat{Q}(\xi_2))^{1/2}} \, d\tau_1 d\xi_1 d\eta_1.
\]

By a symmetry argument we can restrict ourselves to the set

\[
A = \{ (\tau_1, \xi_1, \tau, \xi) \in \mathbb{R}^6 : |\sigma_2| \leq |\sigma_1| \}.
\]

Let \( \mathcal{K} \gg 4 \). We divide \( A \) into the following subregions:

\[
\begin{align*}
A_1 &= \{ (\tau_1, \xi_1, \tau, \xi) \in A : |\xi| \leq \mathcal{K}, |\xi_1| \leq 2\mathcal{K} \}, \\
A_2 &= \{ (\tau_1, \xi_1, \tau, \xi) \in A : |\xi| \leq \mathcal{K}, |\xi_1| \geq 2\mathcal{K} \}, \\
A_3 &= \{ (\tau_1, \xi_1, \tau, \xi) \in A : |\xi| \geq \mathcal{K}, \min\{ |\xi_1|, |\xi_2| \} \leq 2 \}, \\
A_4 &= \{ (\tau_1, \xi_1, \tau, \xi) \in A : |\xi| \geq \mathcal{K}, \min\{ |\xi_1|, |\xi_2| \} \geq 2 \}.
\end{align*}
\]

**Case 1.** Contribution of \( A_1 \) to \( I \). In this case we have \( |\xi_2| \lesssim 1 \) and we see that

\[
\frac{\xi^2 (\xi_2)^{s_2} (\xi_1)^{s_1}}{\xi^{s_1+4\delta}} \lesssim 1;
\]

and hence,

\[
I \lesssim \int_{\mathbb{R}^3} \frac{|\hat{g}(\tau_1, \xi_1) |\hat{g}(\tau_2, \xi_2) |\hat{h}(\tau, \xi)|^{s_2}}{(i\sigma + \hat{Q}(\xi))^{1/2} (i\sigma_1 + \hat{Q}(\xi_1))^{1/2} (i\sigma_2 + \hat{Q}(\xi_2))^{1/2}} \, d\tau_1 d\xi_1 d\eta_1 \leq \int_{\mathbb{R}^3} \frac{|\hat{g}(\tau_1, \xi_1) |\hat{g}(\tau_2, \xi_2) |\hat{h}(\tau, \xi)|}{(\sigma)^{1/2} (\sigma_1)^{1/2} (\sigma_2)^{1/2}} \, d\tau_1 d\xi_1 d\eta_1.
\]

By applying Lemma 4.2, we deduce that

\[
I \lesssim T^u ||u||_{L^2_{\mu}} ||v||_{L^2_{\nu}} ||w||_{L^2_{T}}.
\]

**Case 2.** Contribution of \( A_2 \) to \( I \). Since we have, in this case, \( |\xi| \leq |\xi_1|/2 \), it follows that \( |\xi_1| \sim |\xi - \xi_1| \). Therefore

\[
I \lesssim \int_{\mathbb{R}^3} \frac{|\hat{g}(\tau_1, \xi_1) |\hat{g}(\tau_2, \xi_2) |\hat{h}(\tau, \xi)|^{s_2} (\xi_1)^{s_2} (\xi_2)^{s_2}}{(i\sigma + \hat{Q}(\xi))^{1/2} (i\sigma_1 + \hat{Q}(\xi_1))^{1/2} (i\sigma_2 + \hat{Q}(\xi_2))^{1/2}} \, d\tau_1 d\xi_1 d\eta_1 \leq \int_{\mathbb{R}^3} \frac{|\hat{g}(\tau_1, \xi_1) |\hat{g}(\tau_2, \xi_2) |\hat{h}(\tau, \xi)|}{(\sigma)^{1/2} (\sigma_1)^{1/2} (\sigma_2)^{1/2}} \, d\tau_1 d\xi_1 d\eta_1.
\]

By applying again Lemma 4.2, for \( s < 1 - 2\delta \), we obtain that

\[
I \lesssim T^u ||u||_{L^2_{\mu}} ||v||_{L^2_{\nu}} ||w||_{L^2_{T}}.
\]
Case 3. Contribution of $A_3$ to $I$. We first assume that $\min\{\|\xi_1\|, \|\xi_2\|\} = \|\xi_1\|$ and thus $2 \leq \|\xi_2\|$ and $\|\xi_1\| + \|\xi\| \leq (2 + C)\|\xi\|$, for $C > 0$, and therefore $|\xi| \sim |\xi_2|$. It follows that

$$\frac{\xi^2(\xi_1)^* \xi_2^s}{(\xi)^{s+4\delta}} \lesssim |\xi|^{2-4\delta}.$$ 

Since $(\sigma + \rho(\xi))^{1/2 - \delta} \gtrsim |\xi|^{2-4\delta}$, it results that

$$I \lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi \lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi,$$

for any $\ell \in (0, 1/2)$. The estimate $I \lesssim T^s \|u\|_{L_{t,x}^2}^2 \|\nu\|_{L_{t,x}^2}^2 \|w\|_{L_{t,x}^2}$ is now derived from Lemma 4.1. The other case where $\min\{\|\xi_1\|, \|\xi_2\|\} = \|\xi_2\|$, follows exactly in the same manner.

Case 4. Contribution of $A_4$ to $I$. In this case we need to divide $A_4$ in two regions defined by

$$A_4^1 = \{(\tau_1, \zeta_1, \tau, \zeta) \in A_4 : |\xi| \geq \hat{K} \min\{\|\xi_1\|, \|\xi_2\|\}\},$$

$$A_4^2 = \{(\tau_1, \zeta_1, \tau, \zeta) \in A_4 : |\xi| \leq \hat{K} \min\{\|\xi_1\|, \|\xi_2\|\}\}.$$

Case 4.1. Contribution of $A_4^1$ to $I$. By a symmetry argument we can assume that $\min\{\|\xi_1\|, \|\xi_2\|\} = \|\xi_1\|$. It follows $|\xi| \sim |\xi_2|$. Thusly $|\xi_2| \leq |\xi| + |\xi| \lesssim |\xi|$ and $|\xi| \leq |\xi_1| + |\xi_2| \leq |\xi|/\hat{K} + |\xi_2|$ and consequently, $|\xi| \sim |\xi_2|$. It results that

$$\frac{\xi^2(\xi_1)^s \xi_2^s}{(\xi)^{s+4\delta}} \lesssim |\xi|^{2-4\delta}|\xi_1|^s.$$ 

Hence $(\sigma + \rho(\xi_2))^{1/2} \gtrsim (\sigma_2)\|1/2^{-s/4-\ell}|\xi_2|^{4\ell} \gtrsim (\sigma_2)^{1/2-s/4-\ell}|\xi_1|^{4\ell}$ gives

$$I \lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi \lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi,$$

for any $3s/4 < \ell < 1/2 - s/4$. Therefore a use of Lemma 4.1 provides a good bound for $I$ in this case.

Case 4.2. Contribution of $A_4^2$ to $I$. To estimate $I$ in this case we need to split $A_4^1$ into the following two subregions:

$$A_4^{21} = \{(\tau_1, \zeta_1, \tau, \zeta) \in A_4^1 : |\sigma_1| \geq |\sigma|\},$$

$$A_4^{22} = \{(\tau_1, \zeta_1, \tau, \zeta) \in A_4^1 : |\sigma| \geq |\sigma_1|\}.$$ 

Case 4.21. Contribution of $A_4^{21}$ to $I$. In this case, by a symmetry argument we assume that $\min\{\|\xi_1\|, \|\xi_2\|\} = \|\xi_1\|$. We have $|\xi| \lesssim |\xi_1|$, $|\xi| \lesssim |\xi_2|$ and $(\sigma_1 + \rho(\xi)) \gtrsim (\sigma_1 + \rho(\xi_1))$, and therefore we obtain

$$I \lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi \lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi,$$

$$\lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi,$$

$$\lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi,$$

$$\lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi,$$

$$\lesssim \int_{\mathbb{R}^6} \left| \frac{\hat{f}(\tau_1, \zeta_1) |\hat{g}(\tau_2, \zeta_2)||\hat{b}(\tau, \zeta)}{\langle \sigma_1 \rangle^{1/2} (\sigma_2 + \rho(\xi))^{1/2}} \right|^2 d\tau d\sigma d\zeta d\xi.$$

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for any $3s/4 < \ell < 1/2 - s/4$, and thus we can apply Lemma 4.1 to estimate $I$ in this case.

**Case 4.22.** Contribution of $A_{3}^{2}$ to $I$. In this case, by a symmetry argument we again assume that

\[
\min\{|\xi|, |\zeta|\} = |\xi|.
\]

We have $|\zeta| \lesssim |\xi|$ and $|\xi| \lesssim |\zeta|$, and therefore it follows that

\[
I \lesssim \int_{\mathbb{R}^{6}} |\tilde{f}(\tau_{1}, \xi_{1})| |\tilde{g}(\tau_{2}, \xi_{2})| |\tilde{h}(\tau, \zeta)| |(\xi)|^{2-s/4} |(\xi)| \rmd \tau_{1} \rmd\zeta_{1}
\]

\[
\lesssim \int_{\mathbb{R}^{6}} |\tilde{f}(\tau_{1}, \xi_{1})| |\tilde{g}(\tau_{2}, \xi_{2})| |\tilde{h}(\tau, \zeta)| |(\xi)|^{2-s/4} |(\xi)| \rmd \tau_{1} \rmd\zeta_{1}
\]

\[
\lesssim \int_{\mathbb{R}^{6}} |\tilde{f}(\tau_{1}, \xi_{1})| |\tilde{g}(\tau_{2}, \xi_{2})| |\tilde{h}(\tau, \zeta)| |(\xi)|^{2-s/4} |(\xi)| \rmd \tau_{1} \rmd\zeta_{1}
\]

\[
\lesssim \int_{\mathbb{R}^{6}} |\tilde{f}(\tau_{1}, \xi_{1})| |\tilde{g}(\tau_{2}, \xi_{2})| |\tilde{h}(\tau, \zeta)| |(\xi)|^{2-s/4} |(\xi)| \rmd \tau_{1} \rmd\zeta_{1},
\]

for any $\ell \in (3s/4, 1/2 - s/4)$. Finally by applying Lemma 4.1, we estimate $I$ in this case.

This completes the proof of Theorem 4.3.

\[\square\]

**Corollary 4.4** Let $\delta >$ sufficiently small, $s_{1} \geq s_{1}^{'}, s_{2} \geq 0$. Then for $u, v \in X^{1/2, s_{1}, s_{2}}$, compact supported in time in $\{(t, x, y) \in \mathbb{R}^{3} : t \in [-T, T]\}$, there exists $\mu > 0$ such that

\[
\|A(uv)\|_{X^{-1/2+\delta, s_{1}-4\delta, s_{2}}} \lesssim T^{\mu} \left( \|u\|_{X^{1/2, s_{1}, s_{2}}} \|v\|_{X^{1/2, s_{1}, s_{2}}} + \|u\|_{X^{1/2, s_{1}, s_{2}}} \|v\|_{X^{1/2, s_{1}, s_{2}}} \right).
\]

**Proof.** The proof is a direct consequence of Theorem 4.3 together with the following inequalities:

\[
\langle \xi \rangle^{s_{1}} \leq \langle \xi \rangle^{s_{1}^{'}} \langle \xi \rangle^{s_{1}-s_{1}^{'}} + \langle \xi \rangle^{s_{1}^{'}} \langle \xi - \xi \rangle^{s_{1}-s_{1}^{'}}, \quad s_{1} \geq s_{1}^{'},
\]

\[
\langle \eta \rangle^{s_{2}} \leq \langle \eta \rangle^{s_{2}} + \langle \eta - \eta \rangle^{s_{2}}.
\]

\[\square\]

**REMARK 4.5** It is noteworthy that by an argument similar to Theorem 4.3, one can show that the bilinear estimate of Theorem 4.3 holds for $s_{1} > -1$ and $s_{2} \geq 0$, if $\beta = 0$ in (1.1).

### 5 Existence

Now we are ready to prove Theorem 2.1 and Theorem 2.3.

**Proof of Theorem 2.1.** Let $\varphi \in H^{s_{1}, s_{2}}$ with $s_{1} > -1/2, s_{2} \geq 0$ and $s_{1}^{'}, s_{2}^{'}, s_{2}^{'}, s_{2}^{(1)} \in (-1/2, \min\{0, s_{1}\}]$. We suppose that $T \leq 1$, if $u$ is a solution of the integral equation $u = \Phi(u)$ with

\[
\Phi(u) = \theta(t) \left( W(t)\varphi - \chi_{\mathbb{R}^{+}}(t) \int_{0}^{t} W(t - t')\Lambda \left( \theta_{2}^{2}(t')u^{2}(t') \right) \rmd t' \right),
\]

then $u$ solve the DMKP equation on $[0, T/2]$. We introduce the Bourgain spaces defined by

\[
\mathcal{Z}_{1} = \left\{ u \in X^{1/2, s_{1}, s_{2}} : \|u\|_{\mathcal{Z}_{1}} = \|u\|_{X^{1/2, s_{1}, s_{2}}} + \kappa_{1}\|u\|_{X^{1/2, s_{1}, s_{2}} < \infty} \right\},
\]

\[
\mathcal{Z}_{2} = \left\{ u \in X^{1/2, s_{1}, 0} : \|u\|_{\mathcal{Z}_{2}} = \|u\|_{X^{1/2, s_{1}, 0}} + \kappa_{2}\|u\|_{X^{1/2, s_{1}, 0} < \infty} \right\},
\]

or...
where \( \kappa_1 = \frac{\| \varphi \|_{H^s}}{\| \varphi \|_{H^{s+2+\varepsilon}}} \), \( \kappa_2 = \frac{\| \varphi \|_{H^s}}{\| \varphi \|_{H^{s+2+\varepsilon}}} \).

We show that there exist \( T_1 = T_1(H^{s+1,0}) \) and a solution \( u \) of (5.1) in a ball of \( Z_1 \), and then we solve (5.1) in \( Z_2 \) in order to check that the time of existence \( T = T(H^{s+1,0}) \).

First, by Lemmas 3.1 and 3.2, we have

\[
\| \Phi(u) \|_{X^{1/2,s+1,s+2}} \lesssim \| \varphi \|_{H^{s+1,0}} + T^{2\mu} \| \Lambda(\theta_T^2 u^2) \|_{X^{-1/2+\delta_1-4\delta_2,0}},
\]

\[
\| \Phi(u) \|_{X^{1/2,s+1,s+2}} \lesssim \| \varphi \|_{H^{s+1,0}} + T^{2\mu} \| \Lambda(\theta_T^2 u^2) \|_{X^{-1/2+\delta_1-4\delta_2,0}}.
\]

By Theorem 4.3, Corollary 4.4, Leibniz rule for fractional derivative and Sobolev inequalities in time, we can deduce

\[
\| \Phi(u) \|_{X^{1/2,s+1,s+2}} \lesssim \| \varphi \|_{H^{s+1,0}} + T^{2\mu} \| u \|_{X^{1/2,s+1,s+2}}^2
\]

and consequently we obtain

\[
\| \Phi(u) \|_{Z_1} \leq C (\| \varphi \|_{H^{s+1,0}} + \kappa_1 \| \varphi \|_{H^{s+1,0}}) + CT^{2\mu} \| u \|_{Z_1}^2.
\]

(5.4)

Analogously, we can get

\[
\| \Phi(u) - \Phi(v) \|_{Z_1} \leq C T^{2\mu} \| u - v \|_{Z_1} + \| u + v \|_{Z_1}.
\]

(5.5)

Hence by setting

\[
T_1 = \left[ 4C^2 (\| \varphi \|_{H^{s+1,0}} + \kappa_1 \| \varphi \|_{H^{s+1,0}}) \right]^{-2/\mu} = \left[ 8C^2 \| \varphi \|_{H^{s+1,0}} \right]^{-2\mu},
\]

(5.6)

we can deduce from (5.4) and (5.5) that \( \Phi \) is strictly contractive on the ball of radius

\[ 2C (\| \varphi \|_{H^{s+1,0}} + \kappa_1 \| \varphi \|_{H^{s+1,0}}) \]

in \( Z_1 \). This proves the existence of a unique solution \( u_1 \) to (5.1) in \( X^{1/2,s_1,s_2} \) with \( T_1 \) defined above.

On the other hand, Since \( \varphi \in H^{s_1,s_2} \), it follows that \( \theta(\cdot)W(\cdot)\varphi \in C([0,T_1], H^{s_1,s_2}) \), moreover since \( u_1 \in X^{1/2,s_1,s_2} \), we can deduce from Theorem 4.3 that \( \Lambda(u_1^2) \in X^{-1/2+\delta_1-4\delta_2,0} \) and from Lemma 3.3, we obtain that

\[
t \mapsto \int_0^t W(t-t')\Lambda(u_1^2)dt' \in C([0,T_1], H^{s_1,s_2}).
\]

(5.7)

Thus \( u_1 \) belongs \( C([0,T_1], H^{s_1,s_2}). \)

An argument as above in \( Z_2 \) shows that \( \Phi \) is also strictly contractive on the ball of radius

\[ 2C (\| \varphi \|_{H^{s+1,0}} + \kappa_2 \| \varphi \|_{H^{s+1,0}}) \]

in \( Z_2 \) with

\[ T_2 = \left[ 4C^2 (\| \varphi \|_{H^{s+1,0}} + \kappa_2 \| \varphi \|_{H^{s+1,0}}) \right]^{-1/\mu}. \]

Therefore by definition of \( \kappa_2 \), it follows that \( T_2 = T_2(\| \varphi \|_{H^{s+1,0}}) \); which it follows that there exists a unique solution \( u_1 \) of (5.1) in \( C([0,T_2]; H^{s_1,0}) \cap X^{1/2,s_1,0} \). If we indicate by \( T^* = T_{\text{max}} \) the maximum time of the existence in \( Z_1 \) then by uniqueness, we have \( u_1 = u_2 \) on \( [0, \min\{T_2, T^*\}) \) and this gives that \( T^* \geq T_2(\| \varphi \|_{H^{s+1,0}}) \).

The continuity of map \( \varphi \mapsto u \) from \( H^{s_1,s_2} \) to \( X^{1/2,s_1,s_2} \) follows from classical argument, while the continuity from \( H^{s_1,s_2} \) to \( C([0,T_1], H^{s_1,s_2}) \) follows again from Lemma 3.3. The analyticity of the flow-map is a direct consequence of the implicit function theorem.
The uniqueness of the solution to the truncated integral equation (5.1) is consequence of the contraction argument. We deduce the uniqueness of the solution to the integral equation (2.1) by using the ideas of [16].

Let $u, v \in X^{1/2, s_1, s_2}$ be two solutions of the integral equation (2.1) on the time interval $[0, T]$ and let $\bar{u} - \bar{v}$ be an extension of $u - v$ in $X^{1/2, s_1, s_2}$ such that

$$\| \bar{u} - \bar{v} \|_{X^{1/2, s_1, s_2}} \leq \| u - v \|_{X^{1/2, s_1, s_2}}$$

with $0 \leq \kappa \leq T/2$. It results by Lemmas 3.1 and 3.2 that

$$\| u - v \|_{X^{1/2, s_1, s_2}} \leq \left\| \theta(t) \chi_{[0, T]}(t) \int_0^t W(t - t') \Lambda(\theta^2(\kappa)(u^2 - v^2))dt' \right\|_{X^{1/2, s_1, s_2}}$$

$$\leq \| \Lambda(\theta^2(\kappa)(u^2 - v^2))(t') \|_{X^{-1/2 + \delta, s_1 - \delta, s_2}}$$

$$\leq C\kappa^{\mu/2} \| u + v \|_{X^{1/2, s_1, s_2}} \| \bar{u} - \bar{v} \|_{X^{1/2, s_1, s_2}}$$

$$\leq 2C\kappa^{\mu/2} (\| u \|_{X^{1/2, s_1, s_2}} + \| v \|_{X^{1/2, s_1, s_2}}) \| u - v \|_{X^{1/2, s_1, s_2}}.$$

for some $\mu > 0$. By considering $\kappa \leq \left[ 4C(\| u \|_{X^{1/2, s_1, s_2}} + \| v \|_{X^{1/2, s_1, s_2}}) \right]^{-\mu/2}$, it follows that $u \equiv v$ on $[0, \kappa]$. Iterating this argument, one extends the uniqueness result on the whole time interval $[0, T]$. □

A proof of Theorem 2.3 is now in sight.

**Proof of Theorem 2.3.** The local existence is obtained by an argument similar to Theorem 2.1 and Remark 4.5. To show the global existence when $\beta = 0$, we note that $\partial_x(u^2) \in X^{-1/2 + \delta, s_1 - \delta, s_2}$. Therefore by Lemma 3.3, we obtain that

$$t \mapsto \int_0^t W(t - t') \partial_x(u^2(t'))dt' \in C([0, T]; H^{s_1 + \epsilon, s_2}).$$

(5.8)

Note that

$$W(\cdot, \varphi) \in C([0, +\infty; H^{s_1, s_2}) \cap C((0, +\infty); H^{\infty, s_2});$$

and consequently

$$u \in C([0, T]; H^{s_1 + \epsilon, s_2}) \cap C((0, T); H^{s_1 + \epsilon, s_2}).$$

Noting that $T = T(\| \varphi \|_{H^{s_1, 0}})$ with $s'_1 > -1$ and using the uniqueness result, we deduce by induction that $u \in C([0, T]; H^{\infty, s_2})$. This allows us to take the $L^2$-scalar product of the DMKP equation with $u$, which shows that $t \mapsto \| u(t) \|_{L^2}$ is nonincreasing on $[0, T]$. Since the time of local existence $T$ only depends on $\| \varphi \|_{H^{s_1, 0}}$, this clearly gives that the solution is global in time. □

6 Ill-Poseness

In this section, we prove the ill-posedness result for the DMKP equation stated in Theorem 2.2. We start by constructing a sequence of initial data $\{ \varphi_n \}_n$ which will ensure the nonregularity of the map $\varphi \mapsto u$ from $H^{s_0}$ to $C([0, T]; H^{s_0})$ for $s < -1/2$.

**Proof of Theorem 2.2.** Let $u$ be a solution of (1.1). Then we have

$$u(x, y, t, \varphi) = W(t)\varphi(x, y) - \int_0^t W(t - t')\Lambda(u^2(x, y, t', \varphi))dt'.$$
We will argue by contradiction and suppose that the map \( \varphi \to u \) is \( C^2 \). Since \( u(x,y,0,\varphi) = 0 \), it is straightforward to verify that

\[
 u_1(x,y,t) = \frac{\partial u}{\partial \varphi}(x,y,t,0)[h] = W(t)h,
\]

\[
 u_2(x,y,t) = \frac{\partial^2 u}{\partial \varphi^2}(x,y,t,0)[h, h] = -\int_0^t W(t-t')\Lambda \left( (W(t')h)^2 \right) dt'.
\]

The assumption of \( C^2 \)-regularity of the map solution implies that

\[
 \|u_j(\cdot, \cdot, t)\|_{H^{s,0}} \lesssim \|h\|_{H^{s,0}}^2, \quad j = 1, 2, \quad \forall h \in H^{s,0}.
\] (6.1)

First recall the definitions of \( \zeta_1, \zeta \) and \( \zeta_2 \) in (4.2). A straightforward calculation reveals that

\[
 (u_2(\cdot, \cdot, t))^\zeta (\zeta) = i(\xi + \eta^2)e^{itP(\zeta)} \int_{\mathbb{R}^2} \hat{\varphi}(\zeta_1)\hat{\varphi}(\zeta_2) \frac{e^{-t(\varphi(\zeta_1) + \varphi(\zeta_2))e^{it\mathcal{R}(\zeta, \zeta_1)} - e^{-t\varphi(\zeta)}}}{\mathcal{M}(\xi, \zeta_1) + i\mathcal{R}(\zeta, \zeta_1)} d\zeta_1,
\]

where \( \mathcal{R}(\zeta, \zeta_1) = P(\zeta_1) + P(\zeta_2) - P(\zeta) \) and \( \mathcal{M}(\xi, \zeta_1) = \varphi(\zeta_1) + \varphi(\zeta_2) - \varphi(\zeta) \). Note that from definitions of \( P(\zeta) \) and \( \varphi(\zeta) \), it is readily seen that

\[
 \mathcal{R}(\zeta, \zeta_1) = \mathcal{R}(\xi, \eta, \xi_1, \eta_1) = 3\xi_1\xi_2 + \varepsilon \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi_1\xi_2}
\]

and

\[
 \mathcal{M}(\xi, \zeta_1) = -2\xi_1\xi_2(\xi_1^3 - \xi_1^2 + 2\xi^2 - 1).
\]

We choose now a sequence of initial data \( \{ \varphi_N \}_N \), \( N > 0 \), defined through its Fourier transform by

\[
 \hat{\varphi}_N(\xi, \eta) = N^{-3/2 - s}(\chi_{A_N}(\xi, \eta) + \chi_{B_N}(\xi, \eta))
\]

where \( N \) is a positive parameter such that \( N \gg 1 \), and \( A_N, B_N \) are the rectangles in \( \mathbb{R}^2 \) defined by

\[
 A_N = \left[ N/2, N \right] \times \left[ -6N^2, 6N^2 \right], \quad B_N = \left[ N, 2N \right] \times \left[ 2N^2, 3N^2 \right]
\]

Note first that \( \|\varphi_N\|_{H^{s,0}} \sim 1 \). Let us denote by \( u_{2,N} \) the sequence of the second iteration \( u_2 \) associated with \( \varphi_N \). Hence it is readily seen that

\[
 \|u_{2,N}\|_{H^{s,0}}^2 \gtrsim N^{-4 s - 6} \int_{\mathbb{R}^2} (|\xi| + \xi^2)^2 (1 + |\xi|^2)^s \left| \int_{k_\zeta} \mathcal{K}(\zeta, \zeta_1, t) d\zeta \right|^2 d\zeta,
\] (6.2)

where

\[
 \mathcal{K}(\zeta, \zeta_1, t) = \frac{e^{\varphi(\zeta_1) + \varphi(\zeta_2)}e^{it\mathcal{R}(\zeta, \zeta_1)} - e^{-t\varphi(\zeta)}}{\mathcal{M}(\xi, \zeta_1) + i\mathcal{R}(\zeta, \zeta_1)}
\]

and

\[
 k_\zeta = \{ \zeta_1 : \zeta_1 \in B_N, \zeta_2 \in A_N \} \cup \{ \zeta_1 : \zeta_1 \in A_N, \zeta_2 \in B_N \}.
\]

Now the definition of \( \mathcal{M} \) shows that \( |\mathcal{M}(\xi, \zeta_1)| \lesssim N^4 \). On the other hand, by Lemma 7.1 in [13], we deduce from the inequality

\[
 |\mathcal{R}(\zeta, \zeta_1)| \leq 3(1 + \varepsilon)|\xi_1\xi_2| + \left| 3\xi_1\xi_2 - \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi_1\xi_2} \right|
\]

that \( |\mathcal{R}| \lesssim N^3 \); so that \( |\mathcal{M}(\xi, \zeta_1) + i\mathcal{R}(\zeta, \zeta_1)| \lesssim N^4 \). Note that for any \( \zeta = (\xi, \eta) \in \mathbb{R}^2 \) with \( \xi \in [3N/2, 3N] \) and \( \eta \in [-4N^2, 9N^2] \), we have \( |k_\zeta| \gtrsim N^3 \).
Now, for $0 < \epsilon \ll 1$ fixed, we choose a sequence of times $\{t_N\}_N$ defined by $t_N = N^{-4-\epsilon}$. For $N \gg 1$, it can be easily seen that $e^{-t_N |\xi|} > C > 0$. Hence

$$|e^{t_N |\xi|} \mathcal{X}(\xi, \zeta_1, t)| = \frac{1}{N^{4+\epsilon}} + O\left(\frac{1}{N^{4+2\epsilon}}\right).$$

This implies that

$$\left|\int_{k_\zeta} e^{t_N |\xi|} \mathcal{X}(\xi, \zeta_1, t) d\zeta_1\right| \gtrsim |k_\zeta| N^{-4-\epsilon} \gtrsim N^{-1-\epsilon}.$$

Therefore it follows from (6.2) that

$$1 \gtrsim \|u_{2,N}\|_{L^{1,0}_{x,t}}^2 \gtrsim N^{-4s-6} N^{-2-2\epsilon} \int_{3N/2}^{3N} \int_{-4N^2}^{9N^2} (|\xi| + \xi^2)^2 (1 + \xi^2)^s d\zeta \gtrsim N^{-2s-1-2\epsilon}.$$

This leads to a contradiction for $N \gg 1$, since we have $s < -1/2 - 2\epsilon$; and the proof of Theorem 2.2 is complete. □

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