New Higher Anomalies, SU(N) Yang-Mills Gauge Theory and $\mathbb{C}P^{N-1}$ Sigma Model

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We hypothesize a new and more complete set of anomalies of certain quantum field theories (QFTs) and then give an eclectic verification. First, we propose a set of 't Hooft higher anomalies of 4d time-reversal symmetric pure SU(N)-Yang-Mills (YM) gauge theory with a second-Chern-class topological term at $\theta = \pi$, via 5d cobordism invariants (higher symmetry-protected topological states), with $N = 2, 3, 4$ and others. Second, we propose a set of 't Hooft anomalies of 2d $\mathbb{C}P^{N-1}$-sigma models with a first-Chern-class topological term at $\theta = \pi$, by enlisting all possible 3d cobordism invariants and selecting the matched terms. Based on algebraic/geometric topology, QFT analysis, manifold generator correspondence, condensed matter inputs such as stacking SU(N)-generalized Haldane quantum spin chains, and additional physics criteria, we derive a correspondence between 5d and 3d new invariants. Thus we broadly prove a potentially complete anomaly-matching between 4d SU(N) YM and 2d $\mathbb{C}P^{N-1}$ models at $N = 2$, and suggest new (but maybe incomplete) anomalies at $N = 4$. We formulate a higher-symmetry analog of "Lieb-Schultz-Mattis theorem" to constrain the low-energy dynamics.

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Determining the dynamics and phase structures of strongly coupled quantum field theories (QFTs) is a challenging but important problem. For example, one of the Millennium Problems is partly on showing the existence of quantum Yang-Mills (YM) gauge theory [1] and the mass gap. The fate of a pure YM theory with an SU(N) gauge group (i.e. we simply denote it as an SU(N)-YM), without additional matter fields, without topological term ($\theta = 0$), is confined and trivially gapped in an Euclidean spacetime $\mathbb{R}^4$ [2]. A powerful tool to constrain the dynamics of QFTs is based on non-perturbative methods such as the 't Hooft anomaly-matching [3]. Although anomaly-matching may not uniquely determine the quantum dynamics, it can rule out some impossible quantum phases with mismatched anomalies, thus guiding us to focus only on favorable anomaly-matched phases for low energy phase structures of QFTs. The importance of dynamics and anomalies is not merely for a formal QFT side, but also on a more practical application to high-energy ultraviolet (UV) completion of QFTs, such as on a lattice regularization or condensed matter systems. (See, for instance [4] and references therein, a recent application of the anomalies, topological terms and dynamical constraints of SU(N)-YM gauge theories on UV-regulated condensed matter systems, obtained from dynamically gauging the SU(N)-symmetric interacting generalized topological superconductors/insulators [5, 6], or more generally Symmetry-Protected Topological state (SPTs) [7–9]).

In this work, we attempt to identify the potentially complete 't Hooft anomalies of 4d pure SU(N)-YM gauge theory with a $\theta = \pi$ topological term (a second-Chern-class topological term) and 2d CP$^{N-1}$-sigma model with a $\theta = \pi$ topological term (a first-Chern-class topological term) in an Euclidean spacetime. Here $4d$ denotes a $d$-dimensional spacetime. For the convenience of readers, our main result is summarized in Fig. 3 and Fig. 4.

By completing 't Hooft anomalies of QFTs, we need to first identify the relevant (if not all of) global symmetry $G$ of QFTs. Then we couple the QFTs to classical background-symmetric gauge field of $G$, and try to detect the possible obstructions of such coupling [3]. Such obstructions, known as the obstruction of gauging the global symmetry, are named “'t Hooft anomalies.” In the literature, when people refer to “anomalies,” however, they can means different things. To fix our terminology, we refer “anomalies” to one of the followings:

1. Classical global symmetry is violated at the quantum theory, such that the classical global symmetry fails to be a quantum global symmetry, e.g. the original Adler-Bell-Jackiw anomaly [10, 11].

2. Quantum global symmetry is well-defined and preserved. (Global symmetry is sensible, not only at a classical theory [if there is any classical description], but also for a quantum theory.) However, there is an obstruction to gauge the global symmetry. Specifically, we can detect a certain obstruction to even weakly gauge the symmetry or couple the symmetry to a non-dynamical background probed gauge field. (We will refer this as a back-
ground field, abbreviated as “bgd.field.”) This is known as “‘t Hooft anomaly,” or sometimes regarded as a “weakly gauged anomaly” in condensed matter. Namely, the partition function $Z$ does not sum over background gauge connections, but only fix a background gauge connection and only depend on the background gauge connection as a classical field (as a classical coupling constant).

3. Quantum global symmetry is well-defined and preserved. However, once we promote the global symmetry to a gauge symmetry of the dynamical gauge theory, then the gauge theory becomes ill-defined. Some people call this as a “dynamical gauge anomaly” which makes a quantum theory ill-defined. Namely, the partition function $Z$ after summing over dynamical gauge connections becomes ill-defined.

Now “‘t Hooft anomalies” (for simplicity, from now on, we may abbreviate them as “anomalies”) have at least three intertwined interpretations:

Interpretation (1): In condensed matter physics, “‘t Hooft anomalies” are known as the obstruction to lattice-regularize the global symmetry’s quantum operator in a local on-site manner at UV due to symmetry-twists. (See [12–14] for QFT-oriented discussion and references therein.) This “non-onsite symmetry” viewpoint is generically applicable to both, perturbative anomalies, and non-perturbative anomalies:

- **perturbative anomalies** — Computable from perturbative Feynman diagram calculations.
- **non-perturbative or global anomalies** — Examples of global anomalies include the old and the new SU(2) anomalies [15, 16] (a caveat: here we mean their ‘t Hooft anomaly analogs if we view the SU(2) gauge field as a non-dynamical classical background, instead of dynamical field) and the global gravitational anomalies [17].

The occurrence of these anomalies are sensitive to the underlying UV-completion not only of fermionic systems, but also of bosonic systems [13, 18–20]. We call the anomalies of QFT whose UV-completion requires only the bosonic degrees of freedom as bosonic anomalies [18]; while those must require fermionic degrees of freedom as fermionic anomalies.

Interpretation (2): In QFTs, the obstruction is on the impossibility of adding any counter term in its own dimension $(d-d)$ in order to absorb a one-higher-dimensional counter term (e.g. $(d+1)d$ topological term) due to background $G$-field [21]. This is named the “anomaly-inflow [22].” The $(d+1)d$ topological term is known as the $(d+1)d$ SPTs in condensed matter physics [7, 8].

Interpretation (3): In math, the $dd$ anomalies can be systematically captured by $(d+1)d$ topological invariants [15] known as cobordism invariants [23–26].

There is a long history of relating these two particular $4d$ SU(N)-YM and $2d$ $\mathbb{C}P^{N-1}$ theories, since the work of Atiyah [27], Donaldson [28] and others, in the interplay of QFTs in physics and mathematics. Recently three key progresses shed new lights on their relations further:

(i) Higher symmetries and higher anomalies: The familiar 0-form global symmetry has a charged object of 0d measured by the charge operator of $(d-1)d$. The generalized $q$-form global symmetry, introduced by [29], demands a charged object of $qd$ measured by the charge operator of $(d-q-1)d$ (i.e. codimension-$(q+1)$). This concept turns out to be powerful to detect new anomalies, e.g. the pure SU(N)-YM at $\theta = \pi$ (See eq. (6)) has a mixed anomaly between 0-form time-reversal symmetry $Z^T_0$ and 1-form center symmetry $Z_{N,[1]}$ at an even integer $N$, firstly discovered in a remarkable work [30]. We review this result in Sec. IV, then we will introduce new anomalies (to our best understanding, these have not yet been identified in the previous literature) in later sections (Fig. 3 and Fig. 4).

(ii) Relate (higher)-SPTs to (higher)-topological invariants: Follow the condensed matter literature, based on the earlier discussion on the symmetry twist, it has been recognized that the classical background-field partition function under the symmetry twist, called $Z_{\text{sym.twist}}$ in $(d+1)d$ can be regarded as the partition function of $(d+1)d$ SPTs $Z_{\text{SPTs}}$. These descriptions are applicable to both low-energy infrared (IR) field theory, but also to the UV-regularized SPTs on a lattice, see [12, 13, and 24] and Refs. therein for a systematic set-up. Schematically, we follow the framework of [13],

$$Z_{(d+1)d}^{\text{sym.twist}} = Z_{\text{SPTs}}^{(d+1)d} = Z_{\text{topo.inv}}^{(d+1)d} = Z_{\text{cobordism.inv}}^{(d+1)d}$$

$$\longleftrightarrow d(d)-\text{(higher) ‘t Hooft anomaly. (1)}$$

In general, the partition function $Z_{\text{sym.twist}} = Z_{\text{SPTs}}[A_1, B_2, w_1, \ldots]$ is a functional containing background gauge fields of 1-form $A_1$, 2-form $B_2$ or higher forms; and can contain characteristic classes [31] such as the $i$-th Stiefel-Whitney class $(w_i)$ and other geometric probes such as gravitational background fields, e.g. a gravitational Chern-Simons 3-form $CS_3(\Gamma)$ involving the Levi-Civita connection or the spin connection $\Gamma$. For convenience, we use the capital letters $(A, B, \ldots)$ to denote non-dynamical background gauge fields (which, however, later they may or may not be dynamically gauged), while the little letters $(a, b, \ldots)$ to denote dynamical gauge fields.

More generally,

- For the ordinary 0-form symmetry, we can couple the charged 0d point operator to 1-form background gauge field (so the symmetry-twist occurs in the Poincaré dual codimension-1 sub-space time $[dd]$ of SPTs).
- For the 1-form symmetry, we can couple the charged 1d line operator to 2-form background gauge field

- Following the condensed matter literature, based on the earlier discussion in physics and mathematics.
(so the symmetry-twist occurs in the Poincaré dual codimension-2 sub-space-time \([d-1]d\) of SPTs).

- For the \(q\)-form symmetry, we can couple the charged \(q\)d extended operator to \((q+1)\)-form background gauge field. The charged \(q\)d extended operator can be measured by another charge operator of codimension-\((q+1)\) \([i.e. \,(d-q)d]\). So the symmetry-twist can be interpreted as the occurrence of the codimension-\((q+1)\) charge operator. Namely, the symmetry-twist happens at a Poincaré dual codimension-\((q+1)\) sub-space-time \([(d-q)d]\) of SPTs. We can view the measurement of a charged \(q\)d extended object, happening at any \(q\)-dimensional intersection between the \((q+1)d\) form background gauge field and the codimension-\((q+1)\) symmetry-twist or charge operator of this SPT vacua.

For SPTs protected by higher symmetries (for generic \(q\), especially for any SPTs with at least a symmetry of \(q > 0\), we refer them as higher-SPTs). So our principle above is applicable to higher-SPTs [32–34]. In the following of this article, thanks to eq. (1), we can interchange the usages and interpretations of "higher SPTs \(Z_{\text{SPTs}}^{(d+1)d}\)" "higher topological terms due to symmetry-twist \(Z_{\text{sym.twist}}^{(d+1)d}\)" "higher topological invariants \(Z_{\text{topo.inv}}^{(d+1)d}\) or "cobordism invariants \(Z_{\text{Cobordism.inv}}^{(d+1)d}\)" in \((d+1)d\). They are all physically equivalent, and can uniquely determine a \(dd\) higher anomaly, when we study the anomaly of any boundary theory of the \((d+1)d\) higher SPTs living on a manifold with \(dd\) boundary. Thus, we regard all of them as physically tightly-related given by eq. (1). In short, by turning on the classical background probed field (denoted as "bgd.field" in eq. (2)) coupled to \(dd\) QFT, under the symmetry transformation (i.e. symmetry twist), its partition function \(Z_{\text{QFT}}^{dd}\) can be shifted

\[
Z_{\text{QFT}}^{dd}|_{\text{bgd.field}=0} \rightarrow Z_{\text{QFT}}^{dd}|_{\text{bgd.field} \neq 0} \cdot Z_{\text{SPTs}}^{(d+1)d}(\text{bgd.field}),
\]

to detect the underlying \((d+1)d\) toplogical terms/counter term/SPTs, namely the \((d+1)d\) partition function \(Z_{\text{SPTs}}^{(d+1)d}\). To check whether the underlying \((d+1)d\) SPTs really specifies a true \(dd\) ‘t Hooft anomaly unremovable from \(dd\) counter term, it means that \(Z_{\text{SPTs}}^{(d+1)d}(\text{bgd.field})\) cannot be absorbed by a lower-dimensional SPTs \(Z_{\text{SPTs}}^{dd}(\text{bgd.field})\), namely

\[
Z_{\text{QFT}}^{dd}|_{\text{bgd.field}} \cdot Z_{\text{SPTs}}^{(d+1)d}(\text{bgd.field}) \neq Z_{\text{QFT}}^{dd}|_{\text{bgd.field}} \cdot Z_{\text{SPTs}}^{dd}(\text{bgd.field}).
\]

(iii) Dimensional reduction: A very recent progress shows that a certain anomaly of 4d SU(N)-YM theory can be matched with another anomaly of 2d \(\mathbb{CP}^{N-1}\) model under a 2-torus \(T^2\) reduction in [35], built upon previous investigations [36, 37]. This development, together with the mathematical rigorous constraint from 4d and 2d instantons [27, 28], provides the evidence that the complete set of (higher) anomalies of 4d YM should be fully matched with 2d \(\mathbb{CP}^{N-1}\) model under a \(T^2\) reduction.\(^1\)

In this work, we draw a wide range of knowledges, tools, comprehensions, and intuitions from:

- Condensed matter physics and lattice regularizations. Simplicial-complex regularized triangulable manifolds and smooth manifolds. This approach is related to our earlier Interpretation (1), and the progress (ii).
- QFT (continuum) methods: Path integral, higher symmetries associated with extended operators, etc. This is related to our earlier Interpretation (2), and the progress (i), (ii) and (iii).
- Mathematics: Algebraic topology methods include cobordism, cohomology and group cohomology theory. Geometric topology methods include the embedding of manifolds, and Poincaré duality, etc. This is related to our earlier Interpretation (3), and the progress (ii) and (iii).

Built upon previous results, we are able to derive a consistent story, which identifies, previously missing, thus, new higher anomalies in YM theory and in \(\mathbb{CP}^{N-1}\) model. A sublimed version of our result may count as an eclectic proof between the anomaly-matching between two theories under a \(2\)-torus \(T^2\) reduction from the 4d theory reduced to a 2d theory.

Earlier we stated that our aim is to provide potentially complete \(\text{t} \) 't Hooft anomalies of 4d pure SU(N)-YM gauge theory with a \(\theta = \pi\) topological term and 2d \(\mathbb{CP}^{N-1}\)-sigma model with a \(\theta = \pi\) topological term. It turns out that our recent work Ref. [38] suggests there are indeed different versions of 4d pure SU(N)-YM gauge theory with a \(\theta = \pi\) topological term. What happened is that Ref. [38] founds the different versions of YM theories can be characterized at least partially by the different quantum numbers associated with the extended operator (Wilson line) of SU(N) YM. In simple words, Wilson line of YM can have:

1. time-reversal \(\mathcal{T}\) quantum number, say labeled by \(K_1 \in \{0, 1\} = \mathbb{Z}_2 [38]\), under \(\mathcal{T}\)-symmetry tranformation, as:
   - Kramers singlet \((\mathcal{T}^2 = +1)\)
   - Kramers doublet \((\mathcal{T}^2 = -1)\)

---

\(^{1}\) The complex projective space \(\mathbb{CP}^{N-1}\) is obtained from the moduli space of flat connections of SU(N) YM theory. (See [36] and Fig. 2.) This moduli space of flat connections do not have a canonical Fubini-Study metric and may have singularities. However, this subtle issue, between the \(\mathbb{CP}^{N-1}\) target and the moduli space of flat connections, only affects the geometry issue, and should not affect the topological issue concerning non-perturbative global discrete anomalies that we focus on in this work.
2. spin-statistics quantum number, say labeled by $K_2 \in \{0,1\} = \mathbb{Z}_2$ [38], as:

- bosonic (integer spin-statistics)
- fermionic (half-integer spin-statistics).

More physically intuitively, imagine in the ultraviolet lattice cut-off energy scale, the closed Wilson line can be opened up as an open Wilson line with two open ends. Such that each open end can host very highly-energetically massive 0D particle. This 0D particle can be Kramers singlet $T^2 = +1$ or Kramers doublet $T^2 = -1$ under time-reversal. Under self-spinning by $2\pi$, this 0D particle can also be bosonic (getting a $+1$ sign) or fermionic (getting a $-1$ sign). Ref. [38] focuses on SU(2) YM and gives mathematical interpretations of the $K_1, K_2$ term, based on the gauge bundle constraint,

$$w_2(V_{PSU(2)}) = w_2(V_{SO(3)}) = B + K_1w_1(TM^5)^2 + K_2w_2(TM^5).$$

Thus $K_1$ and $K_2$ are the choices of the gauge bundle constraint, with $K_1, K_2 \in \mathbb{Z}_2$. The $w_j(TM)$ is $j$-th Stiefel-Whitney (SW) classes of tangent bundle $TM$. Ref. [38] shows that putting different siblings of 4d YM on unorientable manifolds and turning on background $B$ fields, give us the access to different versions of ’t Hooft anomalies. Ref. [38] suggests the Wilson line quantum numbers are related to the $(K_1, K_2)$ via:

$$\begin{align*}
(K_1, K_2) &= (0, 0), \text{ Kramers singlet bosonic,} \\
(K_1, K_2) &= (1, 0), \text{ Kramers doublet bosonic,} \\
(K_1, K_2) &= (0, 1), \text{ Kramers doublet fermionic,} \\
(K_1, K_2) &= (1, 1), \text{ Kramers singlet fermionic.}
\end{align*}$$

In this article, we do not use the approach of Ref. [38]. Instead, we like to relate different versions (four siblings of Ref. [38]) of 4d YM simply based on possible 4d ’t Hooft anomalies (5d topological terms) satisfying physical constraints (given in Sec. V). Amusingly, we can find out at least two versions of YM with two different ’t Hooft anomalies. We also relate different versions of 4d SU(N) YM to different versions of 2d $\mathbb{C}P^{N-1}$-sigma model with a $\theta = \pi$ topological term, via a 2-torus dimensional reduction. We also consider a slight generalization of the above gauge-bundle constraint when $N \geq 2$, e.g. see Sec. VIC. The details of a further generalized gauge-bundle constraint including the charge conjugation $C$ quantum number for different siblings of YM theories is reported in an upcoming future work [39].

The outline of our article goes as follows.

In Sec. II, we comment and review on QFTs (relevant to YM theory and $\mathbb{C}P^{N-1}$ model), their global symmetries, anomalies and topological invariants. This section can serve as an invitation for condensed matter colleagues, while we also review the relevant new concepts and notations to high energy/QFT theorists and mathematicians.

In Sec. III, we provide the concrete explicit results on the cobordisms, SPTs/topological terms, and manifold generators. This is relevant to our classification of all possible higher ’t Hooft anomalies. Also it is relevant to our later eclectic proof on the anomalies of YM theory and $\mathbb{C}P^{N-1}$ model.

In Sec. IV, we review the known anomalies in 4d YM theory and 2d $\mathbb{C}P^{N-1}$ model, and explain their physical meanings, or re-derive them, in terms of mathematically precise cobordism invariants.

In Sec. V, Sec. VI and Fig. 2, we should cautiously remark that how 4d SU(N)-YM theory is related to 2d $\mathbb{C}P^{N-1}$ model.

In Sec. VI, in particular, we give our rules to constrain the anomalies for 4d YM theory and 2d $\mathbb{C}P^{N-1}$ model, and for 5d and 3d invariants.

In Sec. VI, we present mathematical formulations of dimensional reduction, from 5d to 4d of cobordism/SPTs/topological term, and from 4d to 2d of anomaly reduction.

In Sec. VII, we present new higher anomalies for 4d SU(N) YM theory.

In Sec. VIII, we present new anomalies for 2d $\mathbb{C}P^{N-1}$ model.

In Sec. IX, with the list of potentially complete ’t Hooft anomalies of the above 4d SU(N)-YM and 2d $\mathbb{C}P^{N-1}$-model at $\theta = \pi$, we constrain their low-energy dynamics further, based on the anomaly-matching. We discuss the higher-symmetry analog Lieb-Schultz-Mattis theorem. In particular, we check whether the ’t Hooft anomalies of the above 4d SU(N)-YM and 2d $\mathbb{C}P^{N-1}$-model can be saturated by a symmetry-extended TQFT of their own dimensions, by the (higher-)symmetry-extension method generalized from the method of Ref. [14]. (See also our companion work [40]) We also discuss their dynamical fates which become spontaneously symmetry-breaking (SSB) phases.

In Sec. X, we summarize our main results of a more complete set of ’t Hooft anomalies of the 4d SU(N)-YM and 2d $\mathbb{C}P^{N-1}$-model and their dimensional reduction in Sec. XA for $N = 2$ and Sec. XB for $N = 4$.

We conclude in Sec. XI.

II. COMMENTS ON QFTS: GLOBAL SYMMETRIES AND TOPOLOGICAL INVARIANTS

A. 4d Yang-Mills Gauge Theory

Now we consider a 4d pure SU(N)-Yang-Mills gauge theory with $\theta$-term, with a positive integer $N \geq 2$, for a Euclidean partition function (such as an $\mathbb{R}^4$ spacetime)

The path integral (or partition function) $Z_{YM}^{4d}$ is formally written as
\[
Z_{\text{YM}}^{\text{4d}} \equiv \int [D\alpha] \exp \left( -S_{\text{YM}+\theta}[\alpha] \right) = 
\int [D\alpha] \exp \left( - S_{\text{YM}}[\alpha] \right) \exp \left( - S_{\theta}[\alpha] \right) \equiv \int [D\alpha] \exp \left( \left( - \int_{M^4} \left( \frac{1}{g^2} \text{Tr} F_{\alpha} \wedge \star F_{\alpha} \right) + \int_{M^4} \left( \frac{i}{8\pi^2} \text{Tr} (F_{\alpha} \wedge F_{\alpha}) \right) \right) \right). 
\]

- \(a\) is the 1-form SU(N)-gauge field connection obtained from parallel transporting the principal-SU(N) bundle over the spacetime manifold \(M^4\). The \( a = a_{\mu} dx^\mu = a_{\mu}^a T^a dx^\mu\); here \(T^a\) is the generator of Lie algebra \(g\) for the gauge group \((\text{SU}(N))\), with the commutator \([T^a, T^b] = i f^{a\beta\gamma} T^\beta\), where \(f^{a\beta\gamma}\) is a fully antisymmetric structure constant. Locally \(dx^\mu\) is a differential 1-form, the \(\mu\) runs through the indices of coordinate of \(M^4\). Then \(a_{\mu} = a_{\mu}^a T^a\) is the Lie algebra valued gauge field, which is in the adjoint representation of the Lie algebra. (In physics, \(a_{\mu}\) is the gluon vector field for quantum chromodynamics.) The \([D\alpha]\) is the path integral measure, for a certain configuration of the gauge field \(\alpha\). All allowed gauge inequivalent configurations are integrated over within the path integral measures \(\int [D\alpha]\), where gauge redundancy is removed or mod out. The integration is under a weight factor \(\exp \left( i S_{\text{YM}+\theta}[\alpha] \right)\).

- The \(F_{\alpha} = da - i a \wedge a\) is the SU(N) field strength, while \(d\) is the exterior derivative and \(\wedge\) is the wedge product; the \(\star F_{\alpha}\) is \(F_{\alpha}\)’s Hodge dual. The \(g\) is YM coupling constant.

- The \(\text{Tr}(F_{\alpha} \wedge \star F_{\alpha})\) is the Yang-Mills Lagrangian \([1]\) (a non-abelian generalization of Maxwell Lagrangian of U(1) gauge theory). The \(\text{Tr}\) denotes the trace as an invariant quadratic form of the Lie algebra of gauge group (here SU(N)). Note that \(\text{Tr}[F_{\alpha}] = \text{Tr}[da - i a \wedge a] = 0\) is traceless for a SU(N) field strength. Under the variational principle, YM theory’s classical equation of motion (EOM), in contrast to the linearity of U(1) Maxwell theory, is non-linear.

- The \(\left( \frac{\theta}{8\pi^2} \text{Tr} F_{\alpha} \wedge F_{\alpha} \right)\) term is named the \(\theta\)-topological term, which does not contribute to the classical EOM.

- This path integral is physically sensibly well-defined, but not precisely mathematically well-defined, because the gauge field can be freely chosen due to the gauge freedom. This problem occurs already for quantum U(1) Maxwell theory, but now becomes more troublesome due to the YM’s non-abelian gauge group. One way to deal with the path integral and the quantization is the method by Faddeev-Popov \([41]\) and De Witt \([42]\). However, in this work, we actually do not need to worry about of the subtlety of the gauging fixing and the details of the running coupling \(g\) for the full quantum theory part of this path integral. The reason is that we only aim to capture the 5d classical background field partition function \(Z_{\text{sym.twist}}^{(d+1)\text{4d}} = Z_{\text{SPTs}}^{(d+1)\text{4d}}\) in eq. (1) that 4d YM theory must couple with in order to match the ‘t Hooft anomaly. Schematically, by coupling YM to background field, under the symmetry transformation, we expect that

\[
Z_{\text{YM}}^{\text{4d}}|_{\text{bgd.field=}0} \rightarrow Z_{\text{SPTs}}^{\text{5d}}(\text{bgd.field}) \cdot Z_{\text{YM}}^{\text{4d}}|_{\text{bgd.field}\neq 0}.
\]

For example, when the bkgd.field is \(B\),

\[
Z_{\text{YM}}^{\text{4d}}(B = 0) \rightarrow Z_{\text{SPTs}}^{\text{5d}}(B \neq 0) \cdot Z_{\text{YM}}^{\text{4d}}(B \neq 0).
\]

Our goal will be identifying the 5d topological term (5d SPTs) eq. (7) under coupling to background fields. We will focus on the Euclidean path integral of eq. (6).

\section{SU(N)-YM theory: Mixed higher-anomalies}

Below we warm up by re-deriving the result on the mixed higher-anomaly of time-reversal \(Z_{\mathbb{Z}^2_2}\) and 1-form center \(Z_{\mathbb{Z}_2}\)-symmetry of SU(N)-YM theory, firstly obtained in \([30]\), from scratch. Our derivation will be as self-contained as possible, meanwhile we introduce useful notations.

\subsection{Global symmetry and preliminary}

For 4d SU(N)-Yang-Mills (YM) theory at \(\theta = 0\) and \(\pi\) mod \(2\pi\), on an Euclidean \(\mathbb{R}^4\) spacetime, we can identify its global symmetries: the 0-form time-reversal \(\mathbb{Z}_{\mathbb{Z}_2}^\theta\) symmetry with time reversal \(\mathcal{T}\) (see more details in Sec. II B 4), and 0-form charge conjugation \(\mathbb{Z}_2^C\) with symmetry transformation \(\mathcal{C}\) (see more details in Sec. II B 6). Since the parity \(\mathcal{P}\) is guaranteed to be a symmetry due to the \(\mathcal{CPT}\) theorem (see more details in Sec. II B 6, or a version for Euclidean \([43]\)), we can denote the full 0-form symmetry as \(G_{[0]} = \mathbb{Z}_{\mathbb{Z}_2}^\theta \times \mathbb{Z}_2^C\). We also have the 1-form electric \(G_{[1]} = \mathbb{Z}_{\mathbb{Z}_2^N}[1]\) center symmetry \([29]\).

So we find that the full global symmetry group “schematically” as

\[
G = \mathbb{Z}_{\mathbb{Z}_2^N}[1] \rtimes (\mathbb{Z}_{\mathbb{Z}_2}^\theta \times \mathbb{Z}_2^C),
\]

here we focus on the discrete part, but we intentionally omit the continuous part (shown later in Sec. III) of the spacetime symmetry group.\(^2\)

\(^2\) One may wonder the role of parity \(\mathcal{P}\) (details in Sec. II B 6), and a potential larger symmetry group \((\mathbb{Z}_{\mathbb{Z}_2}^\theta \times \mathbb{Z}_2^C \times \mathbb{Z}_2^C)\) for \(G_{[0]}\). As
For \( N = 2 \), we actually have the semi-direct product “\( \times \)“ reduced to a direct product “\( \times \)”, so we write
\[
G = Z^e_2 \times Z^f_2,
\]
where also do not have the \( Z^c_2 \) charge conjugation global symmetry, due to that now becomes part of the SU(2) gauge group of YM theory. The non-commutative nature (the semi-direct product “\( \times \)”) of eq. (9) between 0-form and 1-form symmetries will be explained in the end of Sec. II B 6, after we first derive some preliminary knowledge below:

- The 0-form \( Z^f_2 \) symmetry can be probed by “background symmetry-twist” if placing the system on non-orientable manifolds. The details of time-reversal symmetry transformation will be discussed in Sec. II B 4.
- The 1-form electric \( Z^e_N,1 \) symmetry (or simply 1-form \( Z^e_N \) symmetry) can be coupled to 2-form background field \( B_2 \). The charged object of the 1-form \( Z^e_N,1 \) symmetry is the gauge-invariant Wilson line
\[
W_e = \text{Tr}_R(P \exp(i \int a)).
\]

The Wilson line \( W_e \) has the \( a \) viewed as a connection over a principal Lie group bundle (here SU(\( N \))), which is parallel transported around the integrated closed loop resulting an element of the Lie group. \( P \) is the path ordering. The Tr is again the trace in the Lie algebra valued, over the irreducible representation \( R \) of the Lie group (here SU(\( N \))). The spectrum of Wilson line \( W_e \) includes all representations of the given Lie group (here SU(\( N \))). Specifying the local Lie algebra \( g \) is not enough, we need to also specify the gauge Lie group (here SU(\( N \))) and other data, such as the set of extended operators and the topological terms, in order to learn the global structure and non-perturbative physics of gauge theory (See [44], and [4] for many examples).

For the SU(\( N \)) gauge theory we concern, the spectrum of purely electric Wilson line \( W_e \) includes the fundamental representation with a \( \mathbb{Z}_N \) class, which can be regarded as the \( \mathbb{Z}_N \) charge subgroup of 1-form \( Z^e_N,1 \)-symmetry.

The 2-surface charge operator that measures the 1-form \( Z^e_N,1 \)-symmetry of the charged Wilson line is the electric 2-surface operator that we denoted as \( U_e \). The higher \( q \)-form symmetry (\( q > 0 \)) needs to be abelian [29], thus the 1-form electric symmetry is associated with the \( \mathbb{Z}_N \) center subgroup part of SU(\( N \)), known as the 1-form \( Z^e_N,1 \)-symmetry.

If we place the Wilson line along the \( S^1 \) circle of the time or thermal circle, it is known as the Polyakov loop, which nonzero expectation value (i.e. breaking of the 0-form center dimensionally-reduced from 1-form center symmetry) serves as the order parameter of confinement-deconfinement transition.

Below we illuminate our understanding in details for the SU(2) YM theory (so we set \( N = 2 \)), which the discussion can be generalized to SU(\( N \)) YM.

1. We write the SU(2)-YM theory with a background \( B \) field (more precisely the \( B_2 \) 2-cochain field) coupling to 1-form \( Z^e_N,1 \)-symmetry as:
\[
Z^e_{SU(2)YM}[B] = \int [D\lambda] Z^e_{YM} \exp(i \pi \int \lambda \cup (w_2(E) - B_2) + \frac{i}{2} \mathcal{P}(B_2)),
\]
where \( w_2(E) \) is the Stiefel-Whitney (SW) class of gauge bundle \( E \), and \( B_2 \) is a \( Z^2 \)-valued 2-cochain, both are non-dynamical probes. We see that integrating out \( \lambda \), set \( (w_2(E) - B_2) = 0 \) mod 2, thus \( B_2 = w_2(E) \) is related. For \( B_2 = 0 \), there is no symmetry twist \( w_2(E) = 0 \).

For \( B_2 = w_2(E) \neq 0 \), there is a twisted bundle or a so called symmetry twist. So we have an additional \( \frac{i}{2} \mathcal{P}(B_2) \) depending on \( p \in \mathbb{Z}_4 \). The Pontryagin square term \( \mathcal{P}_2 : \mathbb{H}^2(-, \mathbb{Z}_{2p}) \to \mathbb{H}^4(-, \mathbb{Z}_{2p+1}) \), here is given by
\[
\mathcal{P}_2(B_2) = B_2 \cup B_2 = B_2 \cup \delta B_2 = B_2 \cup B_2 + B_2 \cup 2\text{Sq}^1 B_2,
\]
see Sec. II B 3. With \( \cup \) is a normal cup product and \( \cup \) is a higher cup product. For readers who are not familiar with the mathematical details, see the introduction to mathematical background in [34]. The physical interpretation of adding \( \frac{i}{2} \mathcal{P}(B_2) \) with \( p \in \mathbb{Z}_4 \), is related to the fact of the YM vacua can be shifting by a higher-SPTs protected by 1-form symmetry, see Sec. II B 3.

2. The electric Wilson line \( W_e \) in the fundamental representation is dynamical and a genuine line operator. Wilson line \( W_e \) can live on the boundary of a magnetic 2-surface \( U_m = \exp(i \pi \int w_2(E)) \). However, we can set \( B_2 = 0 \) since it is a probed field. So \( W_e \) is a genuine line operator, i.e. without the need to be at the boundary of 2-surface [29].

3. The magnetic ‘t Hooft line \( T_m \) is on the boundary of an electric 2-surface \( U_e = \exp(i \pi \int \lambda) \). Since \( \lambda \) is dynamical, ‘t Hooft line is not genuine thus not in the line spectrum.

4. The electric 2-surface \( U_e = \exp(i \pi \int \lambda) \) measures 1-form \( e \)-symmetry, and it is dynamical. This can be seen from the fact that the 2-surface \( w_2(E) \) is
defined as a 2-surface defect (where each small 1-loop of ‘t Hooft line linked with this $w_2(E)$ getting a nontrivial $\pi$-phase $e^{i\pi}$). The $w_2(E)$ has its boundary with Wilson loop $W_{\alpha}$ such that $U_{\alpha}U_m \sim \exp(i\pi \lambda w_2(E))$ specifies that when a 2-surface $\lambda$ links with (i.e. wraps around) a 1-Wilson loop $W_{\alpha}$, there is a nontrivial statistical $\pi$-phase $e^{i\pi} = -1$. This type of a link of 2-surface and 1-loop in a 4d spacetime is widely known as the generalized Aharonov-Bohm type of linking, captured by a topological link invariant, see e.g. [45, 46] and references therein.

2. YM theory coupled to background fields

First we make a 2-form $Z_N$ field out of 2-form and 1-form $U(1)$ fields. The 1-form global symmetry $G_{[1]}$ can be coupled to a 2-form background $Z_N$-gauge field $B_2$. In the continuum field theory, consider firstly a 2-form $U(1)$-gauge field $B_2$ and 1-form $U(1)$-gauge field $C_1$ such that

\begin{align}
B_2 & \text{ as a 2-form } U(1) \text{ gauge field,} \\
C_1 & \text{ as a 1-form } U(1) \text{ gauge field,} \\
NB_2 & = dC_1, \quad B_2 \text{ as a 2-form } Z_N \text{ gauge field.} \quad (16)
\end{align}

that satisfactorily makes the continuum formulation of $B_2$ field as a 2-form $Z_N$-gauge field when we constrain an enclosed surface integral

\begin{equation}
\int B_2 = \frac{1}{N} \oint dC_1 \in \frac{1}{N} 2\pi Z. \quad (17)
\end{equation}

Now based on the relation $PSU(N) = \frac{SU(N)}{Z_N} = \frac{U(N)}{U(1)}$, we aim to have an $SU(N)$ gauge theory coupled to a background 2-form $Z_N$ field. Here

$a$ as an $SU(N)$ 1-form gauge field,

$F_a = da - ia \wedge a$, as an $SU(N)$ field strength,

$\text{Tr}[F_a] = \text{Tr}[da - ia \wedge a] = 0$ traceless for $SU(N). \quad (18)$

We then promote the $U(N)$ gauge theory with 1-form $U(N)$ gauge field $a'$, such that its normal subgroup $U(1)$ is coupled to the background 1-form probed field $C_1$. Here we can identify the $U(N)$ gauge field to a combination of $SU(N)$ and $U(1)$ gauge fields via, up to details of gauge transformations [30],

\begin{align}
a' & \text{ as an } U(N) \text{ 1-form gauge field,} \\
a' & \simeq a + \frac{1}{N} C_1, \\
\text{Tra'} & \simeq \text{Tr} + C_1 = C_1. \quad (19)
\end{align}

$F_{a'} = da' - ia' \wedge a'$, as a $U(N)$ field strength,

$\text{Tr}[F_{a'}] = \text{Tr}[da' - ia' \wedge a'] = \text{Tr}[da'] = dC_1$, its trace is a $U(1)$ field strength. \quad (20)

To associate the $U(1)$ field strength $\text{Tr}[F_{a'}] = \text{Tr}[da'] = dC_1$ to the background $U(1)$ field strength, we can impose a Lagrange multiplier 2-form $u$,

\begin{align}
\int [Du] \exp \left( i \int_{M^4} \frac{1}{2\pi} u \wedge (\text{Tr}F_{a'} - dC_1) \right) \\
= \int [Du] \exp \left( i \int_{M^4} \frac{1}{2\pi} u \wedge d(\text{Tra'} - C_1) \right). \quad (21)
\end{align}

We also have $NB_2 = dC_1$, so we can impose another Lagrange multiplier 2-form $u'$,

\begin{equation}
\int [Du'] \exp \left( i \int_{M^4} \frac{1}{2\pi} u' \wedge (NB_2 - dC_1) \right) \quad (22)
\end{equation}

From now we will make the YM kinetic term implicit, we focus on the $\theta$-topological term associated with the symmetry transformation. The YM kinetic term does not contribute to the anomaly (in QFT language) and is not affected under the symmetry twist (in condensed matter language [13]). Overall, with only a pair $(B_2, C_1)$ as background fields (or sometimes simply written as $(B, C)$), we have,

\begin{align}
\int [Du][Du'][Du'] \exp (i \int_{M^4} (\frac{\theta}{8\pi^2} \text{Tr} F_a \wedge F_a) + i \int_{M^4} \frac{1}{2\pi} u \wedge \text{d}(\text{Tra'} - C_1) + i \int_{M^4} \frac{1}{2\pi} u' \wedge (NB_2 - dC_1))

= \int [Da][Du] \exp \left( i \int_{M^4} \left( \frac{\theta}{8\pi^2} \text{Tr} F_a \wedge F_a \right) \right) \exp \left( i \int_{M^4} \frac{1}{2\pi} u \wedge d(\text{Tra'} - C_1) \right) \bigg|_{NB_2 = dC_1}

= \int [Da] \exp \left( i \int_{M^4} \left( \frac{\theta}{8\pi^2} \text{Tr} F_a \wedge F_a \right) \right) \bigg|_{\text{Tr}(F_{a'}) = \text{Tr} da' = dC_1 = NB_2 = B_2(\text{Tr} 1)}. \quad (23)
\end{align}
here I is a rank-N identity matrix, thus (Tr I) = N.

Next we rewrite the above path integral in terms of U(N) gauge field, again up to details of gauge transformations [30],

\[
a' \simeq a + \frac{1}{N} C_1, \\
F_{a'} = \text{d}a' - \frac{1}{N} F_a \\
= \left( \text{d}a + \frac{1}{N} \text{d}C_1 \right) - i(a + \frac{1}{N} C_1) \wedge (a + \frac{1}{N} C_1) \\
= (\text{d}a + B_2) - i(a + \frac{1}{N} C_1) \wedge (a + \frac{1}{N} C_1) = F_a + B_2 \tag{24}
\]

Now, to fill in the details of gauge transformations,

\[
B_2 \rightarrow B_2 + \text{d}\lambda, \\
C_1 \rightarrow C_1 + \text{d}\eta + N\lambda, \\
a' \rightarrow a' - \frac{1}{N} \lambda + \text{d}\eta_a, \\
a \rightarrow a + \text{d}\eta_a, 
\]

The infinitesimal and finite gauge transformations are:

\[
a'_\mu \rightarrow a'_\mu + \partial_\mu \eta_a^\alpha + f^{\alpha\beta\gamma}_\mu \eta_a^\beta \eta_a^\gamma, \\
a \rightarrow V(a + i \text{d}) V^\dagger \equiv e^{i \eta_a^\alpha T^\alpha} (a + i \text{d}) e^{-i \eta_a^\alpha T^\alpha},
\]

where we denote 1-form $\lambda$ and 0-form $\eta, \eta_a$ for gauge transformation parameters. Here $\eta_a$ with subindex $a$ is merely an internal label for the gauge field $a$’s transformation $\eta_a$. Here $\alpha \beta \gamma$ are the color indices in physics, and also the indices for the adjoint representation of Lie algebra in math, which runs from 1, 2, \ldots, $d(G_{\text{gauge}})$ with the dimension $d(G_{\text{gauge}})$ of Lie group $G_{\text{gauge}}$ (YM gauge group), especially here $d(G_{\text{gauge}}) = d(\text{SU}(N)) = N^2 - 1$. By coupling $Z_{\text{YM}}^4$ to 2-form background field $B$, we obtain a modified partition function

\[
Z_{\text{YM}}^4[B] = \int [Da] \exp \left( i \int_{M^4} \left( \frac{\theta}{8\pi^2} \text{Tr} (F_{a'\mu} - B_{2\mu}) \wedge (F_{a'\mu} - B_{2\mu}) \right) \right) |_{\text{Tr}(F_{a'}) = \text{Tr}(\text{d}a') = \text{d}C_1 = N B_2 = B_2 (\mathrm{Tr} I)} \tag{31}
\]

\[
= \int [Da] \exp \left( i \int_{M^4} \left( \frac{\theta}{8\pi^2} \text{Tr} (F_{a'\mu} \wedge F_{a'\mu}) \right) - \frac{2\theta \eta_a^\mu B_2 \wedge B_2}{8\pi^2} + \frac{\theta \eta_a^\mu B_2 \wedge B_2}{8\pi^2} \right) = \int [Da] \exp \left( i \int_{M^4} \left( \frac{\theta}{8\pi^2} \text{Tr} (F_{a'\mu} \wedge F_{a'\mu}) \right) - \frac{N \eta_a^\mu B_2 \wedge B_2}{8\pi^2} \right). 
\]

3.  $\theta$-periodicity and the vacua-shifting of higher SPTs

Normally, people say $\theta$ has the 2$\pi$-periodicity,

\[
\theta \simeq \theta + 2\pi. \tag{32}
\]

However, this identification is imprecise. Even though the dynamics of the vacua $\theta$ and $\theta + 2\pi$ is the same, the $\theta$ and $\theta + 2\pi$ can be differed by a short-ranged entangled gapped phase of SPTs of condensed matter physics. In [30]’s language, the vacua of $\theta$ and $\theta + 2\pi$ are differed by a counter term (which is the 4d higher-SPTs in condensed matter physics language). We can see the two vacua are differed by $\exp(-i \int_{M^4} \frac{\Delta \eta_a^\mu}{8\pi^2} B_2 \wedge B_2) \big|_{\Delta \phi = 2\pi}$, which is

\[
\exp(i \int_{M^4} \frac{-\theta \eta_a^\mu B_2 \wedge B_2}{8\pi^2}) = \exp(i \int_{M^4} \frac{-\pi}{N} B_2 \wedge B_2), \tag{33}
\]

where on the right-hand-side (rhs), we switch the notation from the wedge product ($\wedge$) of differential forms to the cup product ($\cup$) of cochain field, such that $B_2 \rightarrow \frac{2\pi}{N} B_2$ and $\wedge \rightarrow \cup$. More precisely, when $N = 2^k$ as a power of 2, the vacua is differed by

\[
\exp(i \int_{M^4} \frac{-\pi}{N} \mathcal{P}_2(B_2)), \tag{34}
\]

where a Pontryagin square term $\mathcal{P}_2 : \mathbb{H}^2(-, \mathbb{Z}_{2^k}) \rightarrow \mathbb{H}^4(-, \mathbb{Z}_{2^{k+1}})$ is given by eq. (13) $\mathcal{P}_2(B_2) = B_2 \cup B_2 \cup 2\mathbb{S}^1 \cdot B_2$. This term is related to the generator of group cohomology $\mathbb{H}^4(B^2 \mathbb{Z}_2, U(1)) = \mathbb{Z}_4$ when $N=2$, and $\mathbb{H}^4(B^2 \mathbb{Z}_N, U(1))$ for general $N$. This term is also related to the generator of cobordism group $\Omega^4_{\text{SO}}(B^2 \mathbb{Z}_2, U(1)) \equiv \text{Tor} \Omega^4_{\text{SO}}(B^2 \mathbb{Z}_2) = \mathbb{Z}_4$ when $N=2$, and $\Omega^4_{\text{SO}}(B^2 \mathbb{Z}_N, U(1)) \equiv \text{Tor} \Omega^4_{\text{SO}}(B^2 \mathbb{Z}_N)$ for general $N$. For the even integer $N = 2^k$, we have $\Omega^4_{\text{SO}}(B^2 \mathbb{Z}_N = 2^k, U(1)) = \mathbb{Z}_{2N = 2^{k+1}}$, via $\mathbb{Z}_{2^k}$-valued 2-cochain in 2d to $\mathbb{Z}_{2^{k+1}}$ in 4d. For our concern (e.g. $N = 2, 4, \text{etc.}$), we have $\Omega^4_{\text{SO}}(B^2 \mathbb{Z}_N, U(1)) = \mathbb{Z}_{2N}$, and the Pontryagin square is well-defined. For the odd integer $N$ that we concern (e.g. $N = 3$ or say $N = p$ an odd prime), Pontryagin square still can be defined, but it is $\mathbb{H}^2(-, \mathbb{Z}_{p^k}) \rightarrow \mathbb{H}^4(-, \mathbb{Z}_{p^{k+1}})$. So we do not have Pontryagin square at $N = 3$ in 4d. See more details on the introduction to mathematical background in [34]. Since we know that the probe-field topological term characterizes SPTs [13], which are classified by group cohomology [7, 9] or cobordism theory [24–26]; we had identified the precise SPTs (eq. (33), eq. (34)) differed between the vacua of $\theta$ and $\theta + 2\pi$.  

\[^3\] We use the same notation $B_2$ for the differential form (which is 2$\pi$ periodic) and the cocycle (which is N periodic).
4. Time reversal $\mathcal{T}$ transformation

As mentioned in eq. (9), the global symmetry of YM theory (at $\theta = 0$ and $\theta = \pi$) contains a time-reversal symmetry $\mathcal{T}$. We denote the spacetime coordinates $\mu$ for 2-form $B = B_2$ and 1-form $C_1$ gauge fields as $B_{2,\mu\nu}$ and $C_{1,\mu}$ respectively. We denote $a_\mu = a_\mu^\alpha T^\alpha$. Then, time reversal acts as:

$$\mathcal{T} : a_0 \rightarrow -a_0, \quad a_i \rightarrow +a_i, \quad (t, x_i) \rightarrow (-t, x_i).$$

$$C_{1,0} \rightarrow -C_{1,0}, \quad C_{1,i} \rightarrow +C_{1,i},$$

$$B_{2,0i} \rightarrow -B_{2,0i}, \quad B_{2,ij} \rightarrow +B_{2,ij}.$$  \hspace{1cm} (35)

Thus the path integral transforms under time reversal, schematically, becomes $Z_{YM}^{4d}[\mathcal{T}B]$. By $\mathcal{T}B$, we also mean $\mathcal{T}BT^{-1}$ in the quantum operator form of $B$ (if we canonically quantize the theory). More precisely,

\[
\begin{align*}
Z_{YM}^{4d}[B] &= \int [Da] \exp \left( i \int_{M^4} \left( \frac{\theta}{8\pi^2} \text{Tr} (F_{a'} \land F_{a'}) - \frac{\theta N}{8\pi^2} B_2 \land B_2 \right) \right) \\
Z_{YM}^{4d}[\mathcal{T}B] &= \int [Da] \exp \left( i \int_{M^4} \left( -\frac{\theta}{8\pi^2} \text{Tr} (F_{a'} \land F_{a'}) - \frac{\theta N}{8\pi^2} B_2 \land B_2 \right) \right) = Z_{YM}^{4d}[B] \cdot \int [Da] \exp \left( i \int_{M^4} \left( -\frac{2\theta}{8\pi^2} \text{Tr} (F_{a'} \land F_{a'}) - \frac{2\theta N}{8\pi^2} B_2 \land B_2 \right) \right).
\end{align*}
\]  \hspace{1cm} (36)

- When $\theta = 0$, this remains the same $Z_{YM}^{4d}[\mathcal{T}B] = Z_{YM}^{4d}[B]$.
- When $\theta = \pi$, this term transforms to

\[
\begin{align*}
Z_{YM}^{4d}[B] &\cdot \int [Da] \exp \left( i(-2\pi) \int_{M^4} \left( \frac{1}{8\pi^2} \text{Tr} (F_{a'} \land F_{a'}) - \frac{N}{8\pi^2} B_2 \land B_2 \right) \right) \\
&= Z_{YM}^{4d}[B] \cdot \exp \left( i(-2\pi)(-c_2) + \frac{(-2\pi)i}{8\pi^2} \int_{M^4} \left( \text{Tr} F_{a'} \land \text{Tr} F_{a'} - NB_2 \land B_2 \right) \right) \\
&= Z_{YM}^{4d}[B] \cdot \exp \left( i2\pi c_2 + \frac{(-2\pi)i}{8\pi^2} \int_{M^4} (N(N-1)B_2 \land B_2) \right) = Z_{YM}^{4d}[B] \cdot \exp \left( -\frac{iN(N-1)}{4\pi} \int_{M^4} (B_2 \land B_2) \right). \hspace{1cm} (37)
\end{align*}
\]

where we apply the 2nd Chern number $c_2$ identity:

\[
\frac{1}{8\pi^2} \int_{M^4} \left( \text{Tr} F_{a'} \land \text{Tr} F_{a'} - \text{Tr} (F_{a'} \land F_{a'}) \right) = c_2 \in \mathbb{Z}. \hspace{1cm} (38)
\]

We can add a 4d SPT state (of a higher form symmetry) as a counter term. Consider again a 1-form $Z_{N}$-symmetry ($Z_{N,[1]}$) protected higher-SPTs, classified by a cobordism group $\Omega^{4}_{SO}(B^2Z_{N}, U(1))$,

\[
\exp(i \int_{M^4} \frac{pN}{N} \mathcal{P}_2(B_2)) \sim \exp \left( i \frac{pN}{4\pi} \int_{M^4} B_2 \land B_2 \right) \hspace{1cm} (39)
\]

here we again convert the 2-cochain field $B_2$ to 2-form field $B_2$ (to recall, see Sec. II B 3). For any 4-manifold, according to $[29, 46],$

\[
\frac{Np}{2} \in \mathbb{Z} \quad \text{(For even N, } p \in \mathbb{Z}. \text{ For odd N, } p \in 2\mathbb{Z}). \hspace{1cm} (40)
\]

\[
p \simeq p + 2N. \hspace{1cm} (41)
\]

For even $N$, there are 2N classes of 4d higher SPTs for $p \in \mathbb{Z}$. For odd $N$, there are $N$ classes of 4d higher SPTs for $p \in 2\mathbb{Z}$.

For spin 4-manifolds (when $p$ and $N$ are odd):

\[
p \in \mathbb{Z}. \hspace{1cm} (42)
\]

\[
p \simeq p + N. \hspace{1cm} (43)
\]

In this case, there are $N$ classes on the spin manifold. This 4d higher SPTs (counter term) under TR sym changes to: $\frac{pN}{4\pi} \int B_2 \land B_2 \rightarrow -\frac{pN}{4\pi} \int B_2 \land B_2$, or more precisely,

\[
\int \frac{pN}{N} \mathcal{P}_2(B_2) \rightarrow -\int \frac{pN}{N} \mathcal{P}_2(B_2). \hspace{1cm} (44)
\]
5. **Mixed time-reversal and 1-form-symmetry anomaly**

Now we discuss the mixed time-reversal $\mathcal{T}$ and 1-form $\mathbb{Z}_N$ symmetry anomaly of \([30]\) in details. We re-derive based on our language in \([13]\). The charge conjugation

$$
\mathbb{Z}_{\text{YM}}^{\text{4d}}(B) \cdot \exp \left( i \left( -\frac{N(N-1)}{4 \pi} + \frac{2pN}{4 \pi} \right) \int_{M^4} \mathcal{B}^2 \wedge \mathcal{B}_2 \right) = \mathbb{Z}_{\text{YM}}^{\text{4d}}(B) \cdot \exp \left( -\frac{iN(N-1+2p)}{4 \pi} \int_{M^4} \mathcal{B}^2 \wedge \mathcal{B}_2 \right). 
$$

(45)

1. For even $N$, and $\theta = \pi$, here the 4d higher SPTs (counter term) labeled by $p$ becomes labeled by $-(N-1-p)$. To check whether there is a mixed anomaly or not, which asks for the identification of two 4d SPTs before and after time-reversal transformation. Namely $(N-1+2p) = 0$ (mod out the classification of 4d higher SPTs given below eq. \((34)\)) cannot be satisfied for any $p \in \mathbb{Z}$ (actually $p \in \mathbb{Z}_{2k+1}$, via the Pontryagin square, which sends a $\mathbb{Z}_{2k}$-valued 2-form in 2d to $\mathbb{Z}_{2k+1}$-class of 4d higher SPTs. For $N = 2$, we have $p \in \mathbb{Z}_4$.)

So this indicates that for any $p$ (with or without 4d higher SPTs/counter term) in the YM vacua, we detect the mixed time-reversal $\mathcal{T}$ and 1-form $\mathbb{Z}_N$ symmetry anomaly, which requires a 5d higher SPTs to cancel the anomaly. We will write down this 5d higher SPTs/counter term in Sec. \III.

2. For even $N$, and $\theta = 0$, we have $\mathbb{Z}_{\text{YM}}^{\text{4d}}(TB) = \mathbb{Z}_{\text{YM}}^{\text{4d}}(B)$ without 4d SPTs. With 4d SPTs, the only shift is eq. \((44)\), so to check the anomaly-free condition, we need $p = -p$, or $2p = 0$, mod out the classification of 4d higher SPTs given below eq. \((34)\). This anomaly-free condition can be satisfied for $p = 0$. For $N = 2$, we can also have $2p = 0$ mod 4, which is true for $p = 0, 2$, even with the $p = 2$-class of 4d SPTs. In that case, there is no mixed higher anomaly of $\mathcal{T}$ and $\mathbb{Z}_{N,[1]}$ symmetry,

3. For odd $N$, and $\theta = \pi$, the $(N-1+2p) = 0$ (mod out the classification of 4d higher SPTs given below eq. \((34)\)) can be satisfied for some $p = \frac{1-N}{2} \in \mathbb{Z}$, but $p$ needs to be even $p \in 2\mathbb{Z}$ on a non-spin manifold. If $p = \frac{1-N}{2} \in 2\mathbb{Z}$, the 4d SPTs can be defined on a non-spin manifold. If $p = \frac{1-N}{2} \in \mathbb{Z}$, the 4d SPTs can only be defined on a spin manifold. So, for an odd $N$, there can be no mixed anomaly at $\theta = \pi$, a 4d higher SPTs/counter term of $p = \frac{1-N}{2}$ preserves the $\mathcal{T}$-symmetry and 1-form $\mathbb{Z}_N$-symmetry (such that two symmetries can be regulated locally onsite \([12–14]\)).

6. **Charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, reflection $\mathcal{R}$, CT, CP transformations, $\mathbb{Z}_{\text{CT}}^2 \times (\mathbb{Z}_{N,[1]}^2 \times \mathbb{Z}_{N,2}^2)$ and $\mathbb{Z}_{\text{CT}}^2 \times \mathbb{Z}_{N,2}^2$, and their higher mixed anomalies**

Follow Sec. \II B 4 and the discrete $\mathcal{T}$ transformation in eq. \((35)\), and we denote $a_{\mu} = a_{\mu}^\alpha T^\alpha$, below we list down additional discrete transformations including charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, CT, CP:

$$
\mathcal{C} : a_{\mu}^\alpha(t, x) \rightarrow a_{\mu}^\alpha(t, x), \quad (t, x_i) \rightarrow (t, x_i). 
$$

(46)

$$
T^\alpha \rightarrow -T^\alpha, \quad a_{\mu} \rightarrow -a_{\mu}. 
$$

(47)

$$
\mathcal{P} : a_0 \rightarrow a_0, \quad a_{i} \rightarrow -a_{i}, \quad (t, x_i) \rightarrow (t, -x_i). 
$$

(48)

$$
\mathcal{CT} : a_0 \rightarrow +a_0, \quad a_{i} \rightarrow -a_{i}, \quad (t, x_i) \rightarrow (-t, x_i). 
$$

(49)

$$
\mathcal{CP} : a_{\mu} \rightarrow +a_{\mu}, \quad (t, x_i) \rightarrow (-t, -x_i). 
$$

(50)

The * means the complex conjugation. In Euclidean spacetime, we can regard the former $\mathcal{T}$ in eq. \((35)\) (or $\mathcal{CT}$ eq. \((48)\)) as a reflection $\mathcal{R}$ transformation \([43]\), which we choose to flip any of the Euclidean coordinate. See further discussions of the crucial role of discrete symmetries in YM gauge theories in \([4]\).

We can ask whether there is any higher mixed anomalies between the above discrete symmetries and the 1-form center symmetry. We can easily check that whether $\frac{2\pi}{\pi} \text{Tr} (F_a \wedge F_a)$ term flips sign to $-\frac{2\pi}{\pi} \text{Tr} (F_a \wedge F_a)$ under any of the discrete symmetries. Among the $\mathcal{C}, \mathcal{P}, \mathcal{T}$, and $\mathcal{CT}$, only the $\mathcal{C}$ does not flip the $\theta$ term and $\mathcal{C}$ is a good global symmetry for all $\theta$ values. So the answer is that each of the $\mathcal{T}, \mathcal{P}, \mathcal{CT}$ and $\mathcal{CP}$

have itself mixed anomalies with the 1-form center symmetry. Only

$$
\mathcal{C}, \mathcal{PT}, \text{ and } \mathcal{CPT}, 
$$

(52)
do not have mixed anomalies with the 1-form center symmetry.

Now, we come back to explain the non-commutative nature (the semi-direct product ‘$\times$’) of eq. \((9)\) between 0-form and 1-form symmetries

$$
\mathbb{Z}_{N,[1]}^2 \times (\mathbb{Z}_{N}^2 \times \mathbb{Z}_{N}^2). 
$$
Obviously $(\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ is due to that $C$ and $T$ commute, and the combined diagonal group $\text{diag}(\mathbb{Z}_2^T \times \mathbb{Z}_2^C) = \mathbb{Z}_2^{CT}$ has the group generator $CT$.

We note that to physically understand some of the following statements, it may be helpful to view the symmetry transformation in the Minkowski/Lorentz signature instead of the Euclidean signature.\footnote{In the Minkowski case, we also need to regard the time-reversal}

- The non-commutative nature $\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^C$ is due to that the $\mathbb{Z}_2^T$ keeps 1-Wilson loop $W_e = \text{Tr}_R(P \exp(i(\hat{a} a))) \rightarrow \text{Tr}_R(P \exp((-1)(-\hat{a} a))) = W_e$ invariant, while $\mathbb{Z}_2^C$ flips the 2-surface $U_e \rightarrow U_e^1 = U_e^{-1}$ due to the orientation of $U_e$ and its boundary ’t Hooft line is flipped. Thus, the 1-form $\mathbb{Z}_{N,[1]}^e$-symmetry charge of $W_e$, measured by the topological number of linking between $W_e$ and $U_e$, now flips from $n \in \mathbb{Z}_N$ to $-n = N - n \in \mathbb{Z}_N$. Since the charge operator of $\mathbb{Z}_{N,[1]}^e$ symmetry, $U_e$, is flipped thus does not commute under the $\mathbb{Z}_2^C$ symmetry, this effectively defines the semi-direct product in a dihedral group like structure of $\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^T$.

- The commutative nature $\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^{CT}$ is due to that the $\mathbb{Z}_2^{CT}$ flips 1-Wilson loop $W_e = \text{Tr}_R(P \exp(i(\hat{a} a))) \rightarrow W_e = W_e$ invariant, while $\mathbb{Z}_2^{CT}$ keeps the 2-surface $U_e \rightarrow U_e$ invariant. We can see that the $\mathbb{Z}_2^{CT}$ and $\mathbb{Z}_2^T$ flips the 1-loop and 2-surface oppositely. Thus, the 1-form $\mathbb{Z}_{N,[1]}^e$ symmetry charge of $W_e$, measured by the topological number of linking between $W_e$ and $U_e$, again flips from $n \in \mathbb{Z}_N$ to $-n = N - n \in \mathbb{Z}_N$. But the charge operator of $\mathbb{Z}_{N,[1]}^e$ symmetry, $U_e$, is invariant thus does commute under the $\mathbb{Z}_2^{CT}$ symmetry, this effectively defines the direct product in a group structure of $\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^{CT}$.

- The non-commutative nature $\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^C$ is due to that the $\mathbb{Z}_2^C$ in eq. (46) flips $W_e = \text{Tr}_R(P \exp(i(\hat{a} a))) \rightarrow \text{Tr}_R(P \exp(i(-\hat{a} a))) = \text{Tr}_R(P \exp(i(\hat{a} a)))^* = \text{Tr}_R(P \exp(i(\hat{a} a)))^! = W_e^! = W_e^{-1}$, while $\mathbb{Z}_2^C$ also flips the 2-surface $U_e \rightarrow U_e^1 = U_e^{-1}$ for the same reason. Thus, the 1-form $\mathbb{Z}_{N,[1]}^e$-symmetry charge of $W_e$, measured by the topological number of linking between $W_e$ and $U_e$, is invariant under $\mathbb{Z}_2^C$. But the charge operator of $\mathbb{Z}_{N,[1]}^e$ symmetry, $U_e$, is flipped thus does not commute under the $\mathbb{Z}_2^C$ symmetry, this effectively defines the semi-direct product in a dihedral group like structure of $\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^C$.

Note that the potentially related dihedral group structure of Yang-Mills theory under a dimensional reduction to $\mathbb{R}^3 \times S^1$ is recently explored in [30, 47].

When $N = 2$, it is obvious that we simply have the direct product $\mathbb{Z}_{e,[1]}^N \times \mathbb{Z}_2^T$ as eq. (10).

We can rewrite eq. (9)’s 0-form and 1-form symmetries

$$\mathbb{Z}_2^{CT} \times (\mathbb{Z}_{N,[1]}^e \times \mathbb{Z}_2^C).$$

We can rewrite eq. (10) as

$$\mathbb{Z}_2^{CT} \times \mathbb{Z}_2^C.$$\footnote{In the Minkowski case, we also need to regard the time-reversal}

It is related to the fact that for SU(2) YM theory, the charge conjugation $\mathbb{Z}_2^C$ is inside the gauge group, because there is no outer automorphism of SU(2) but only an inner automorphism ($\mathbb{Z}_2$) of SU(2). For $N=2$, the charge conjugation matrix $\mathcal{C}_{SU(2)} = e^{i \frac{\pi}{2} \sigma^2}$ is a matrix that provides an isomorphism map between fundamental representations of SU(2) and its complex conjugate. We have $\mathcal{C}_{SU(2)}^\alpha \mathcal{C}_{SU(2)}^{-1} = -(\sigma^\alpha)^*$. Let $\mathcal{U}_{SU(2)}$ be the unitary SU(2) transformation on the SU(2)-fundamentals, so $\mathcal{C}_{SU(2)}^\alpha \mathcal{U}_{SU(2)} \mathcal{C}_{SU(2)}^{-1} = \exp(-i \frac{\pi}{2}(\sigma^\alpha)^*) = \mathcal{U}_{SU(2)}^*$, which is a $\mathbb{Z}_2$ inner automorphism of SU(2).

We propose that the structure of eq. (9), eq. (10), eq. (53) and eq. (54) can be regarded as an analogous 2-group. It can be helpful to further organize this 2-group like data into the context of [48].

C. 2d $\mathbb{C}P^{N-1}$-sigma model

Here we consider 2d $\mathbb{C}P^{N-1}$-model [49], which is a 2d sigma model with a target space $\mathbb{C}P^{N-1}$. The $\mathbb{C}P^{N-1}$ model is a 2d toy model which mimics some similar behaviors of 4d YM theory: dynamically-generated energy gap and asymptotic freedom, etc. We will focus on 2d $\mathbb{C}P^{N-1}$-model at $\theta = \pi$.

1. Related Models

The path integral of 2d $\mathbb{C}P^{N-1}$-model is

\[
\mathcal{Z}_{\mathbb{C}P^{N-1}}^{2d} \equiv \int [Dz][D\bar{z}][Da'] \exp \left( - S_{\mathbb{C}P^{N-1} + \theta}[z, \bar{z}, a'] \right) \equiv \int [Dz_j][D\bar{z}_j][Da'] \exp \left( - S_{\mathbb{C}P^{N-1}}[z_j, \bar{z}_j, a'] \right) \exp \left( - S_{\theta}[a'] \right)
\]

\[
\equiv \int [Dz][D\bar{z}][Da'] \delta(|z|^2 - \nu^2) \exp \left( - \int_{M^2} \frac{d^2 \bar{z}}{g^2} |Da_{\nu}^*|^2 + \int_{M^2} \frac{1}{2\pi} \theta F_{a^*} \right). \tag{55}
\]
The $z_j \in \mathbb{C}$ is a complex-valued field variable, with an index $j = 1, \ldots, N$. (In math, the $z_j \sim c z_j$, identified by any complex number $c \in \mathbb{C}^\times$ excluding the origin, is known as the homogeneous coordinates of the target space $\mathbb{C}P^{N-1}$.) The delta function imposes a constraint: $|z|^2 \equiv \sum_{j=1}^N |z_j|^2 = r^2$, here $r \in \mathbb{R}$ specifies the size of $\mathbb{C}P^{N-1}$. The delta function $\delta(|z|^2 - r^2)$ may be also replaced by a potential term, such as the $\frac{1}{4}(|z|^2 - r^2)^2$ potential, at large $\lambda$ coupling energetically constraining $|z|^2 = r^2$. Here $|D a'_\mu z|^2 \equiv (D a'_\mu z)(D a'_\mu z)$. Here $F a' = d a'$ is the U(1) field strength of $a'$.

For 2d CP$^1$ (2d CP$^{N-1}$ at $N = 2$), we can rewrite the model as the O(3) nonlinear sigma model (NLSM). The O(3) NLSM is parametrized by an O(3) = SO(3)$\times Z_2$ Néel vector $\vec{n} = (n_1, n_2, n_3)$, which obeys

$$\vec{n} = \frac{1}{r} z_i^* \vec{\sigma}_j z_j$$

(56)

with $|\vec{n}|^2 = 1$ and Pauli matrix $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. It is called Néel vector because the 2d CP$^1$ or O(3) NLSM describes the Heisenberg anti-ferromagnet phase of quantum spin system [50, 51]. To convert eq. (55) to eq. (60), notice that we do not introduce the kinetic Maxwell term $|F a|^2$ for the U(1) photon field $a'$, thus $a'$ is an auxiliary field, that can be integrated out and eq. (55) is constrained by the EOM:

$$a'_\mu = -\frac{1}{r^2} \sum_{j=1}^2 \bar{z}_j \partial_{\mu} z_j = \frac{i}{2r^2} \sum_{j=1}^2 (\bar{z}_j \partial_{\mu} z_j - \bar{z}_j \partial_{\mu} z_j),$$

(57)

and we can derive:

$$|D a'_\mu z|^2 = \sum_{j=1}^2 |D a'_\mu z_j|^2 = \left(\frac{1}{2}\right)^2 |\partial_{\mu} \vec{n} \cdot \partial^{\mu} \vec{n}|,$$

(58)

$$\frac{i}{2\pi} \epsilon^{\mu \nu \lambda} \partial_{\nu} a'_\mu = \left(\frac{i}{8\pi}\right) \epsilon^{\mu \nu \lambda} |(\partial_{\nu} \vec{n} \times \partial_{\mu} \vec{n})|.$$  

(59)

Then we rewrite $Z_{\text{CP}^1}^{2d}$ as $Z_{O(3)}^{2d}$ of the O(3) NLSM path integral:

\[
Z_{O(3)}^{2d} = \int [D\vec{n}] \delta(|\vec{n}|^2 - 1) \exp \left( - S_{O(3)+U(1)}[\vec{n}] \right)
\]

\[
= \int [D\vec{n}] \delta(|\vec{n}|^2 - 1) \exp \left( - \int d^2 x \left( \frac{1}{g^2} \vec{n} \cdot \partial^{\mu} \vec{n} + \frac{i}{8\pi} \epsilon^{\mu \nu \lambda} |(\partial_{\nu} \vec{n} \times \partial_{\mu} \vec{n})| \right) \right),
\]

(60)

Note that $\epsilon^{\mu \nu \lambda} \partial_{\nu} \vec{n} \cdot \partial^{\mu} \vec{n} = \left(\frac{i}{8\pi}\right) \epsilon^{\mu \nu \lambda} |(\partial_{\nu} \vec{n} \times \partial_{\mu} \vec{n})|$.

The O(3) NLSM coupling $g''$ in $\left(\frac{1}{g''} \vec{n} \cdot \partial^{\mu} \vec{n} \right) = \left(\frac{1}{g''} \right)^2 |\partial_{\mu} \vec{n} \cdot \partial^{\mu} \vec{n}|$ is related to the CP$^1$ model via $g'' = (2g'/r)$, which is inverse proportional to the radius size of the 2-sphere CP$^1 = S^2$.

\[
Z_{\text{SU(N)}_k}^{\text{WZW}} = \int [DU] [DU^\dagger] \exp \left( - \frac{k}{8\pi} \int_{M^2} d^2 x \text{Tr} \left( \partial_{\mu} U \partial^{\mu} U \right) + \frac{i k}{12\pi} \int_{M^3} \text{Tr} \left( (U^\dagger dU)^3 \right) \right),
\]

(61)

with $M^2 = \partial(M^3)$. At N = 2, the UV theory of $Z_{\text{CP}^1}^{2d}$ flows to this 2d CFT called the SU(2)$_1$-WZW CFT at IR. The global symmetry can be preserved at IR.

For the general 2d CP$^{N-1}$-model, its global symmetry can also be embedded into another SU(N)$_1$-WZW model at UV; although unlike N = 2 case, CP$^{N-1}$-models for N > 2 conventionally and generically do not flow to an

symmetries ($T$ and $CT$) as anti-unitary symmetry, instead of the unitary symmetry (as the Euclidean case).

In fact, the UV high energy theory of $Z_{\text{CP}^1}^{2d} = Z_{O(3)}^{2d}$ is known to be, in Renormalization Group (RG) flow, flowing to the same IR conformal field theory CFT from another UV model via the SU(2)$_1$-WZW model (Wess-Zumino-Witten model [52–54]). The SU(N)$_k$-WZW model at the level $k$ is IR CFT. For N > 2, there exist UV-symmetry preserving relevant deformations driving the RG flow away from an IR CFT. The global symmetry may be spontaneously broken, and the vacua can be gapped and/or degenerated. See for example [55, 56] and references therein.

\section{Global symmetry: $Z_{\text{CP}^1}^{2d} \times \text{PSU}(2) \times Z_2^C$ and $Z_{\text{CP}^1}^{2d} \times (\text{PSU}(N) \times Z_2^C)$}

Let us check the global symmetry of 2d CP$^{N-1}$-model.
Continuous global symmetry: In eq. (55), it is easy to see the continuous global SU(N) transformation rotating between the SU(N) fundamental scalar multiplet $z_j$ via $z \to V_z z_j = (e^{i \theta P_T}) z_j$ which has its $Z_N$-center subgroup being gauged away by the U(1) gauge field $\alpha'$. So we have the net continuous global symmetry

$$\text{PSU}(N) = \text{SU}(N)/Z_N = \text{U}(N)/\text{U}(1).$$ (62)

which acts on gauge invariant object faithfully (e.g. the PSU(2) = SO(3) symmetry can act on the gauge-invariant $\vec{n}$ vector in the 2d CP$^1$-model or O(3) NLSM faithfully).

Now we explore 2d CP$^{N-1}$-model’s discrete global symmetries as finite groups.

Discrete global symmetry for $N = 2$:

- **$Z_T^2$-time-reversal symmetry**, there is a $T$-symmetry allowed for any $\theta$, acting on fields and coordinates of eq. (55) and eq. (60), whose transformations become.

$$Z_T^2 : z_i \to \epsilon_{ij} z_j, \quad \bar{n} \to -\bar{n},$$

$$a'_i, a'_j \to (a'_i, -a'_j), \quad (t, x) \to (-t, x).$$ (63)

Here a Pauli matrix $\sigma_{ij}^2$ gives $\epsilon_{ij} = i \sigma_{ij}^2$.

- **$Z_2^x$-translation symmetry (as $Z_2^x$ as an effective charge conjugation symmetry $Z_T^x$) allowed for $\theta = 0, \pi$, acts as.

$$Z_2^x (\equiv Z_2^C) : z_i \to \epsilon_{ij} z_j, \quad \bar{n} \to -\bar{n},$$

$$a'_i, a'_j \to (-a'_i, a'_j), \quad (t, x) \to (t, x).$$ (64)

It is easy to understand the role of $Z_2^x$-translation on the UV-lattice model of Heisenberg anti-ferromagnet (AFM) phase of quantum spin system $[50, 51]$. Its AFM Hamiltonian operator is

$$\hat{H} = \sum_{\langle i, j \rangle} |J| \hat{S}_i \cdot \hat{S}_j + \ldots$$ (65)

where $\langle i, j \rangle$ is for the nearest-neighbor lattice sites $i$ and $j$’s AFM interaction between spin operators $\hat{S}_i$ and $|J| > 0$ is the AFM coupling. So $Z_2^x$-translation flips the spin orientation, also flips the AFM’s Néel vector $\bar{n} \to -\bar{n}$.

- **$Z_2^P$-parity symmetry allowed for $\theta = 0, \pi$ acts as.

$$Z_2^P : z_i \to z_i, \quad \bar{n} \to +\bar{n},$$

$$a'_i, a'_j \to (a'_i, -a'_j), \quad (t, x) \to (-t, -x).$$ (66)

- **CPT-symmetry** (as $Z_2^{CPT} = \text{diag}(Z_2^C, Z_2^P, Z_T^x)$), the diagonal symmetry generator of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ allowed for any $\theta$ acts as.

$$Z_2^{CPT} : z_i \to z_i, \quad \bar{n} \to +\bar{n},$$

$$a'_i, a'_j \to (-a'_i, a'_j), \quad (t, x) \to (-t, -x).$$ (67)

- **$Z_2^C$-another charge conjugation symmetry of CP$^{N-1}$-model allowed for $\theta = 0, \pi$ acts as.

$$Z_2^C : z_i \to \bar{z}_i, \quad (n_1, n_2, n_3) \to (-n_1, -n_2, n_3),$$

$$a'_i, a'_j \to (-a'_i, a'_j), \quad (t, x) \to (t, x).$$ (68)

- **$Z_2^{CT}$-symmetry allowed for $\theta = 0, \pi$ acts as.

$$Z_2^{CT} : z_i \to \epsilon_{ij} z_j, \quad (n_1, n_2, n_3) \to (-n_1, n_2, -n_3),$$

$$(a'_i, a'_j) \to (-a'_i, a'_j), \quad (t, x) \to (-t, x).$$ (69)

- **$Z_2^T$-symmetry (as $Z_2^T$ as another choice of time-reversal allowed for $\theta = 0, \pi$, acts as.

$$Z_2^T (\equiv Z_2^{CT}) : z_i \to z_i, \quad \bar{n} \to \bar{n},$$

$$(a'_i, a'_j) \to (-a'_i, a'_j), \quad (t, x) \to (-t, x).$$ (70)

Next we check the commutative relation between the above continuous PSU(N) and the discrete symmetries

For $N = 2$, we see that $Z_T^2$ commutes with PSU(2), because $TV_z = i \sigma_z^y V_z^* = i \sigma_z^y V_z = V_T^2$. Similarly, $Z_2^x$ commutes with PSU(2). So, $Z_2^{CT}$ commutes with PSU(2). We see that $Z_T^2$ does not commute with PSU(2), because $CV_z = (V_z)^* = V_T^* z$ while $V_{C^*} z = V_T^2$. Therefore, $Z_2^{CT}$ is another choice of time-reversal and it does not commute with PSU(2).

Similar to eq. (51) for 4d YM theory, here for 2d CP$^{N-1}$ model, we can check that whether the $\frac{i}{2} \sigma^y_{ij}(F_{ij})$ term flips sign to $-\frac{i}{2} \sigma^y_{ij}(F_{ij})$ under any of the discrete symmetries. Among the $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ for 2d CP$^{N-1}$ model, only the $\mathcal{T}$ does not flip the $\theta$ term and $\mathcal{T}$ is a good global symmetry for all $\theta$ values. So each of the $\mathcal{C}$, $\mathcal{P}$, $\mathcal{CT}$, and $\mathcal{PT}$, (71)

plays the similar role for the 2d anomalies at $\theta = \pi$. Only $\mathcal{T}$, $\mathcal{CP}$, and $\mathcal{CTP}$, (72)

are good symmetries for all $\theta$.

Global symmetry for $N = 2$:

Overall, for 2d CP$^1$ model at $\theta = 0, \pi$, we can combine the above to get the full 0-form global symmetries

$$Z_T^2 \times \text{PSU}(2) \times Z_2^x = Z_T^2 \times \text{O}(3)$$ (73)

which is the same as

$$Z_T^2 \times \text{PSU}(2) \times Z_2^C$$

with a semi-direct product “×” since PSU(2) and $Z_2^C$ do not commute.

It is very natural to regard $Z_T^2$-symmetry as the new $Z_T^{CT}$-symmetry, because it flips the time coordinates $t \to -t$, but it does not complex conjugate the $z$. So we may
define\textsuperscript{5}

\[ Z^\text{CT}_2 \equiv Z^x_2. \quad (74) \]

Similarly, we may regard the \( Z^x_2 \)-translation as a new charge conjugation symmetry \( Z^x_2 \equiv Z^x_2 \).

Therefore, 0-form global symmetries eq. (73) can also be

\[
\begin{align*}
Z^\text{CT}_2 \times \text{PSU}(2) \times Z^x_2 & \equiv Z^\text{CT}_2 \times \text{PSU}(2) \times Z^x_2 \\
& \equiv Z^\text{CT}_2 \times \text{O}(3) \\
\end{align*}
\]

Global symmetry for \( N \geq 2 \):
For 2d \( \mathbb{C}P^{N-1} \) model eq. (55) at \( \theta = 0, \pi, N > 2 \), we follow the above discussion and the footnote \( 5 \), we again can define a natural definition of \( Z^\text{CT}_2 \) (without involving the complex conjugation of \( z \) fields). Then we have instead the full 0-form global symmetries:

\[
Z^\text{CT}_2 \times (\text{PSU}(N) \times Z^x_2),
\]

where again \( Z^x_2 \) acts on \( z_i \rightarrow \overline{z_i}, a'_\mu \rightarrow -a'_\mu \) and \( (t, x) \rightarrow (t, x) \) as eq. (68).

We remark that the SU(2) (or \( N = 2 \) for \( \mathbb{C}P^1 \) model) is special because its order-2 automorphism is an inner automorphism. The SU(2) fundamental representation is equivalent to its conjugate. This is related to the fact that both \( Z^\text{CT}_2 \) and \( Z^x_2 \) can commute with the SU(2) or PSU(2), also the remark we made in the footnote \( 5 \).

For SU(N) with \( N > 2 \), we gain an order-2 automorphism as an outer automorphism, which is the \( Z_2 \) symmetry of Dynkin diagram \( A_{N-1} \), swapping fundamental with anti-fundamental representations. Although we have \( Z^\text{CT}_2 \times \text{PSU}(N) \) in eq. (76), we would have \( Z^x_2 \times \text{PSU}(N) \) for \( N > 2 \). See related and other detailed discussions in [4].

The above we have considered the global symmetry (focusing on the internal symmetry, and the discrete sector of the spacetime symmetry) without precisely writing down their continuous spacetime symmetry part. In Sec. III, we like to write down the “full” global symmetry including the spacetime symmetry group.

\textsuperscript{5} Above we discuss \( Z^\text{CT}_2 \equiv Z^x_2 \) and \( Z^x_2 \) both commute with the \( \text{PSU}(2) \) (also \( \text{SU}(2) \)) for bosonic systems (bosonic QFTs). Indeed, the \( Z^\text{CT}_2 \) and \( Z^x_2 \) reminisce the discussion of [4] (e.g. Sec. II), for the case including the fermions (with the fermion parity symmetry \( Z^x_2 \) acted by \((-1)^F\)), we have the natural \( Z^\text{CT}_2 \)-time-reversal symmetry, without taking complex conjugation on the matter fields, which gives rise to the full symmetry \( \text{Pin}^+ \times \text{SU}(2) \); while the other \( Z^x_2 \)-time-reversal symmetry, involving complex conjugation on the matter fields, gives rise to \( \text{Pin}^- \times \text{SU}(2) \).

III. COBORDISMS, TOPOLOGICAL TERMS, AND MANIFOLD GENERATORS:
CLASSIFICATION OF ALL POSSIBLE HIGHER \textquoteleft \textquoteleft T HOOFT ANOMALIES

A. Mathematical preliminary and co/bordism groups

Since we have obtained the full global symmetry \( G \) (including the 0-form and higher symmetries) of 4d YM and 2d \( \mathbb{C}P^{N-1} \) model, we can now use the knowledge that their \textquoteleft \textquoteleft T Hooft anomalies are classified by 5d and 3d cobordism invariants of the same global symmetry \[26\]. Namely, we can classify the \textquoteleft \textquoteleft T Hooft anomalies by enlisting the complete set of all possible cobordism invariants from their corresponding 5d and 3d bordism groups, whose 5d and 3d manifold generators endorsed with the \( G \) structure.

To begin with, we should rewrite the global symmetries in previous sections (e.g. (eq. (9)/eq. (53)), (eq. (10)/eq. (54))) into the form of

\[ G \equiv \left(G_{\text{spacetime}} \rtimes \mathcal{G}_{\text{internal}}\right)_{N_{\text{shared}}}, \]

where the \( G_{\text{spacetime}} \) is the spacetime symmetry, the \( \mathcal{G}_{\text{internal}} \) the internal symmetry,\textsuperscript{6} the \( \rtimes \) is a semi-direct product specifying a certain “twisted” operation (e.g. due to the symmetry extension from \( \mathcal{G}_{\text{internal}} \) by \( G_{\text{spacetime}} \)) and the \( N_{\text{shared}} \) is the shared common normal subgroup symmetry between the two numerator groups.

The theories and their \textquoteleft \textquoteleft T Hooft anomalies that we concern are in \( d \)d QFTs (4d YM and 2d \( \mathbb{C}P^{N-1} \) model), but the topological/cobordism invariants are defined in the \( Dd = (d + 1)d \) manifolds. The manifold generators for the bordism groups are actually the closed \( Dd = (d + 1)d \) manifolds. We should clarify that although there can be \textquoteleft \textquoteleft T Hooft anomalies for \( d \)d QFTs so \( \mathcal{G}_{\text{internal}} \) may not be gauge-able, the SPTs/topological invariants defined in the closed \( Dd = (d + 1)d \) manifolds actually have \( \mathcal{G}_{\text{internal}} \) always gauge-able in that \( Dd = (d + 1)d \).\textsuperscript{7} This is related to the fact that in condensed matter physics, we say that the \textquoteleft \textquoteleft bulk \textquoteleft \textquoteleft \( Dd = (d + 1)d \) SPTs has an onsite local internal \( \mathcal{G}_{\text{internal}} \)-symmetry, thus this \( \mathcal{G}_{\text{internal}} \) must be gauge-able.

The \textit{new} ingredient in our present work slightly going beyond the cobordism theory of \[26\] is that the \( \mathcal{G}_{\text{internal}} \)-symmetry may not only be an ordinary 0-form global

\textsuperscript{6} Later we denote the probed background spacetime \( M \) connection over the spacetime tangent bundle \( TM \), e.g. as \( w_j(TM) \) where \( w_j \) is \( j \)-th Stiefel-Whitney (SW) class \[31\]. We may also denote the probed background internal-symmetry/gauge connection over the principal bundle \( E \), e.g. as \( w_j(E) = w_j(V_{\mathcal{G}_{\text{internal}}}) \) where \( w_j \) is also \( j \)-th SW class. In some cases, we may alternatively denote the latter as \( w_j'(E) = w_j'(V_{\mathcal{G}_{\text{internal}}}) \).

\textsuperscript{7} This idea has been pursued to study the vacua of YM theories, for example, in [4] and references therein. See more explanations in Sec. XI’s eq. (202).
symmetry, but also include higher global symmetries. The details of our calculation for such “higher-symmetry-
group cobordism theory” are provided in [34].

Based on a theorem of Freed-Hopkin [26] and an extended
generalization that we propose [34], there exists a
1-to-1 correspondence between “the invertible topologi-
cal quantum field theories (iTQFTs) with symmetry (in-
cluding higher symmetries)” and “a cobordism group.”
In condensed matter physics, this means that there is a
1-to-1 correspondence between the symmetric invertible
topological order with symmetry (including higher sym-
metries)’ that can be regularized on a lattice in its own
dimensions’ and “a cobordism group,” at least at lower
dimensions, at least at lower
dimensions. More precisely, it is a 1-to-1 correspon-
dence (isomorphism “≈”) between the following two well-
defined “mathematical objects” (these “objects” turn out
to be abelian groups):
\[
\begin{aligned}
\text{Deformation classes of reflection positive } & D\text{-dimensional extended } \\
& \text{topological field theories (iTQFT) with}
\end{aligned}
\]
\[\begin{array}{c}
\text{symmetry group } G_{\text{spacetime}} \ltimes G_{\text{internal}} \\
\text{symmetry group } N_{\text{shared}}
\end{array}\]
\[\cong \left[MT\left(G_{\text{spacetime}} \ltimes G_{\text{internal}}\right), \Sigma^{D+1}IZ\right]_{\text{tors}}.
\]

Let us explain the notation above: \(MTG\) is the Madsen-
Tillmann spectrum [58] of the group \(G\), \(\Sigma\) is the sus-
pension, \(IZ\) is the Anderson dual spectrum, and \(\tau\)s means taking only the finite group sector (i.e. the torsion

group).

Namely, we classify the deformation classes of sym-
metric iTQFTs and also symmetric invertible topological
orders (iToSs), via this particular cobordism group
\[
\begin{aligned}
\Omega_D^G & \cong \Omega_D^{G_{\text{spacetime}} \ltimes G_{\text{internal}}} \\
& \equiv \text{TP}_D(G) \equiv [MTG, \Sigma^{D+1}IZ].
\end{aligned}
\]

by classifying the cobordant relations of smooth, dif-
ferentiable and triangulable manifolds with a stable
\(G\)-structure, via associating them to the homotopy groups
of Thom-Madsen-Tillmann spectra [58, 59], given by a
theorem in Ref. [26]. Here \(\text{TP}\) means the abbreviation
of “Topological Phases” classifying the above symmetric
iTQFT, where our notations follow [26] and [34]. (For an
introduction of the mathematical background for physi-
cists, the readers can consult the Appendix A of [4].)

Moreover, there are only the discrete/finite \(\mathbb{Z}_n\)-classes
of the non-perturbative global ’t Hooft anomalies for YM
and \(\mathbb{C}_p^{N-1}\) model (so-called the torsion group for \(\mathbb{Z}_n\-
class); there is no \(\mathbb{Z}\)-class perturbative anomaly (so-called
the free class) for our QFTs. So, we concern only the
torsion group part of data in eqn. (78), this is equivalent
for us to simply look at the bordism group:
\[
\Omega_D^G \equiv \Omega_D^{G_{\text{spacetime}} \ltimes G_{\text{internal}}} / N_{\text{shared}},
\]
in order to classify all the ’t Hooft anomalies for YM and
\(\mathbb{C}_p^{N-1}\) model.

Therefore, below we focus on the unoriented bordism
groups and also some oriented bordism groups, replacing
the orthogonal \(O\) group to a special orthogonal \(SO\) group:

Let \(X\) be a fixed topological space, define the unori-
ented cobordism group
\[
\Omega_D^G(X) = \{(M, f) \mid M \text{ is a closed } D\text{-manifold,}
\]
\[
f : M \to X \text{ is a map}\}/ \sim
\]
\[
(\sigma, \varphi) \sim (\sigma', \varphi') \text{ if there exists a compact } (D + 1)
\]
manifold \(N\) and a map \(h : N \to X\) such that \(\partial N = M \sqcup M'\), and \(h |_M = f, h |_{M'} = f'\).

Let \(X\) be a fixed topological space, define the oriented
cobordism group
\[
\Omega_D^{SO}(X) = \{(M, f) \mid M \text{ is a closed oriented } D\text{-manifold,}
\]
\[
f : M \to X \text{ is a map}\}/ \sim
\]
\[
(\sigma, \varphi) \sim (\sigma', \varphi') \text{ if there exists a compact oriented } (D + 1)
\]
manifold \(N\) and a map \(h : N \to X\) such that \(\partial N = M \sqcup M'\), the orientations of \(M\) and \(M'\) are induced from
that of \(N\), and \(h |_M = f, h |_{M'} = f'\).

Here \(\sqcup\) is the disjoint union.

In particular, when \(X = B^2\mathbb{Z}_n\), \(f : M \to B^2\mathbb{Z}_n\) is a
cohomology class in \(H^2(M, \mathbb{Z}_n)\). When \(X = BG\), with \(G\) is
a Lie group or a finite group (viewed as a Lie group
with discrete topology), then \(f : M \to BG\) is a principal
\(G\)-bundle over \(M\). To explain our notation, here \(BG\) is
a classifying space of \(G\), and \(B^2\mathbb{Z}_n\) is a higher classifying
space (Eilenberg-MacLane space \(K(\mathbb{Z}_n, 2)\)) of \(\mathbb{Z}_n\).

We have the following well-known facts:

• Unoriented cobordism groups are always \(\mathbb{Z}_2\)-vector
spaces.
• \(\Omega_D^{SO}(X)\) is a subgroup of \(\Omega_D^G(X)\) for \(D \neq 0 \mod 4\).

Our conventions in the following subsections are:

• A map between topological spaces is always assumed
to be continuous.

---

8 We have used a mathematical fact that all smooth and dif-
erentiable manifolds are triangulable manifolds, based on Morse
theory. On the contrary, triangulable manifolds are smooth mani-

folds at least for dimensions up to \(D = 4\) (i.e. the “if and only
if” statement is true below \(D \leq 4\)). The concept of piecewise
linear (PL) and smooth structures are equivalent in dimensions
\(D \leq 6\). Thus all symmetric iTQFT classified by the cobordant
properties of smooth manifolds have a triangulation (thus a lat-
tice regularization on a simplicial complex (thus a UV com-
petition on a lattice). This implies a correspondence between “the
symmetric iTQFTs (on smooth manifolds)” and “the symmetric
invertible topological orders (on triangulable manifolds)” for
\(D \leq 4\). See a recent application of this mathematical fact on
the lattice regularization of symmetric iTQFTs and symmetric
invertible topological orders in [57] for various Standard Models
of particle physics.
• For a top degree cohomology class with coefficients \(\mathbb{Z}_2\), we often suppress an explicit integration over the manifold (i.e. pairing with the fundamental class \([M]\) with coefficients \(\mathbb{Z}_2\)), for example: 
\[ w_3(TM)w_3(TM) \equiv \int_M w_3(TM)w_3(TM) \] 
where \(M\) is a 5-manifold.

• The group operation in cobordism group is induced from the disjoint union of manifolds.

• If \(\Omega^D_{\phi}(X) = \mathbb{Z}_2\), then the group homomorphisms \(\phi_i : \Omega^D_{\phi}(X) \to \mathbb{Z}_2\) for \(1 \leq i \leq r\) form a complete set of cobordism invariants of \(\Omega^D_{\phi}(X)\) if \(\phi = (\phi_1, \ldots, \phi_r) : \Omega^D_{\phi}(X) \to \mathbb{Z}_2^r\) is a group isomorphism.

• The elements of \(\Omega^D_{\phi}(X)\) are manifold generators if their images in \(\mathbb{Z}_2^r\) under \(\phi\) generate \(\mathbb{Z}_2^r\).

In the following subsections, we consider the potential cobordism invariants/topological terms (5d and 3d [higher] SPTs for 4d YM and 2d \(\mathbb{C}P^{N-1}\) model), and their manifold generators for bordism groups, as the complete classification of all of their possible candidate higher ‘t Hooft anomalies.

First, we can convert the time reversal \(\mathbb{Z}_2^T\) or \(\mathbb{Z}_2^{CT}\) to the orthogonal \(O(D)\)-symmetry group for such an underlying UV-completion of bosonic system (all gauge-invariant operators are bosons), where the \(O(D)\) is an extended symmetry group from \(SO(D)\) via a short extension:

\[ 1 \to SO(D) \to O(D) \to \mathbb{Z}_2^T \to 1. \] (83)

The \(SO(D)\) is the spacetime Euclidean rotational symmetry group for \(Dd\) bosonic systems. \(^9\)

Then we can easily list their converted full symmetry group \(G\) and their relevant bordism groups, for \(SU(2)\) YM (eq. (10)/eq. (54)), \(SU(N)\) YM (eq. (9)/eq. (53)), \(\mathbb{C}P^1\) model (eq. (73)/eq. (78)), and \(\mathbb{C}P^{N-1}\) model (eq. (76)), into the eq. (77)’s form:

\begin{enumerate}
  \item \(\Omega^5_O(B^2\mathbb{Z}_2) \equiv \Omega^{(O\times B^2\mathbb{Z}_2)}_5\): This is the bordism group for \(\mathbb{Z}_2^{CT} \times \mathbb{Z}_2^{CT}_{[1]}\) in eq. (54) without \(\mathbb{Z}_2^{CT}\), which we will study in Sec. III B, here eq. (77)’s \(G = O(D) \times B\mathbb{Z}_2\) or \(G = O(D) \times \mathbb{Z}_2^{CT}_{[1]}\).
  
  \item \(\Omega^3_O(BO(3))\): This is the bordism group for \(\mathbb{Z}_2^{CT} \times O(3)\) in eq. (75), which we will study in Sec. III C, here eq. (77)’s \(G = O(D) \times O(3)\).
  
  \item \(\Omega^3_O(B^2\mathbb{Z}_4)\): This is the bordism group for \(\mathbb{Z}_2^{CT} \times (\mathbb{Z}_4^{CT} \times \mathbb{Z}_2^{CT})\) in eq. (53) at \(N = 4\) without \(\mathbb{Z}_2^{CT}\), which we will study in Sec. III D, here eq. (77)’s \(G = O(D) \times B\mathbb{Z}_4\) or \(G = O(D) \times \mathbb{Z}_4^{CT}_{[1]}\).
\end{enumerate}

(4iv) \(\Omega^5_O(B^2\mathbb{Z}_2 \times B^2\mathbb{Z}_4) \equiv \Omega^{(O \times (B^2\mathbb{Z}_2 \times B^2\mathbb{Z}_4))}_5\) and \(\Omega^3_O(B^2\mathbb{Z}_2 \times B^2\mathbb{Z}_4) \equiv \Omega^{(O \times (B^2\mathbb{Z}_2 \times B^2\mathbb{Z}_4))}_3\).

The first is the bordism group with a CT-time reversal, for \(\mathbb{Z}_2^{CT} \times (\mathbb{Z}_4^{CT} \times \mathbb{Z}_2^{CT})\) in eq. (53) at \(N = 4\), which we will study in Sec. III E, here eq. (77)’s \(G = O(D) \times (\mathbb{Z}_2 \times B\mathbb{Z}_4)\) or \(G = O(D) \times (\mathbb{Z}_2^{CT} \times \mathbb{Z}_2^{CT})\).

The second is actually the re-written bordism group with a T-time reversal, for \(\mathbb{Z}_2^{CT} \times (\mathbb{Z}_2^{CT} \times \mathbb{Z}_2^{CT})\) at \(N = 4\), here eq. (77)’s \(G = (O(D) \times \mathbb{Z}_2^{CT} \times \mathbb{Z}_2^{CT})\) or \(G = (O(D) \times \mathbb{Z}_2^{CT} \times \mathbb{Z}_2^{CT}) \times (\mathbb{Z}_4^{CT} \times \mathbb{Z}_4^{CT})\). But we will not study this, since it is simply a more complicated re-writing of the same result of Sec. III E.

(5v) \(\Omega^3_O(B\mathbb{Z}_2 \times \text{PSU}(4))\): This is the bordism group for \(\mathbb{Z}_2^{CT} \times (\text{PSU}(N) \times \mathbb{Z}_2^{CT})\) in eq. (76) at \(N = 4\), which we will study in Sec. III F, here eq. (77)’s \(G = O(D) \times (\text{PSU}(N) \times \mathbb{Z}_2^{CT})\).

Based on the relation between bordism groups and their \(Dd = (d+1)\) bordism invariants to the \(d\) anomalies of QFTs, below we may simply abbreviate “5d cobordism invariants for characterizing 4d YM theory’s anomaly” as

“5d (Yang-Mills) terms.”

We may simply abbreviate “3d cobordism invariants for characterizing 2d \(\mathbb{C}P^{N-1}\) model’s anomaly” as

“3d (\(\mathbb{C}P^{N-1}\)) terms.”

B. \(\Omega^3_O(B^2\mathbb{Z}_2)\)

Follow Sec. III A, now we enlist all possible ‘t Hooft anomalies of 4d pure \(SU(2)\) YM at \(\theta = \pi\) (but when the \(\mathbb{Z}_2^{CT}\)-background field is turned off) by obtaining the 5d cobordism invariants from bordism groups of (eq. (10)/eq. (54)).

We are given a 5-manifold \(M^5\) and a map \(f : M^5 \to B^2\mathbb{Z}_2\). Here the map \(f : M^5 \to B^2\mathbb{Z}_2\) is the 2-form \(B = B_2\) gauge field in the YM gauge theory eq. (12) (and eqn. (31) at \(N = 2\)). We like to obtain the bordism invariants of \(\Omega^3_O(B^2\mathbb{Z}_2)\). We find the bordism group \([34]\):

\(\Omega^3_O(B^2\mathbb{Z}_2) = \mathbb{Z}_4^4,\)

(84)

whose cobordism invariants are generated by

\[
\begin{align*}
B_2 & \cup \text{Sq}^1B_2, \\
\text{Sq}^2\text{Sq}^1B_2, \\
w_1(TM^5)^2\text{Sq}^1B_2, \\
w_2(TM^5)^2\text{Sq}^1B_2, \\
w_2(TM^5)^2w_3(TM^5).
\end{align*}
\]

(85)

\(^9\) For the case of time-reversal symmetry, where there must be an underlying UV-completion of fermionic system (some gauge-invariant operators are fermions), the more subtle time-reversal extension scenario is discussed in [26] and [4].

\(^{10}\) Interestingly, the oriented version of the bordism group \(\Omega^3_O(B^2\mathbb{Z}_2)\) has also been studied recently in a different context in [60]. Here we study instead the unoriented bordism group \(\Omega^3_O(B^2\mathbb{Z}_2)\) new to the literature [34].
Here $TM^5$ means the spacetime tangent bundle over $M^5$, see footnote 6. See Ref. [34], note that we derive on a 5d closed manifold,

$$Sq^2 Sq^1 B_2 = (w_2(TM^5) + w_1(TM^5)^2) Sq^1 B_2 = (w_2(TM^5) + w_1(TM^5)^3) B_2,$$
$$w_1(TM^5)^2 Sq^1 B_2 = w_1(TM^5)^3 B_2. \quad (86)$$

We have a group isomorphism

$$\Phi_1 : \Omega^5_3(B^2Z_2) \to Z^4 \ \ (M^5, B^2) \to (B_2 \cup Sq^1 B_2, Sq^2 Sq^1 B_2, w_1(TM^5)^2 \cup Sq^1 B_2, w_2(TM^5)w_3(TM^5)). \quad (88)$$

1. Consider $(M^5, B) = (\mathbb{R}P^2 \times \mathbb{R}P^3, \alpha \cup \beta)$. Here $\alpha$ is the generator of $H^1(\mathbb{R}P^2, \mathbb{Z}_2)$, and $\beta$ is the generator of $H^4(\mathbb{R}P^3, \mathbb{Z}_2)$.

Since

$$\begin{cases} Sq^1(\alpha \cup \beta) = \alpha^2 \cup \beta + \alpha \cup \beta^2, \\
w_1(T(\mathbb{R}P^2 \times \mathbb{R}P^3)) = \alpha, \\
w_2(T(\mathbb{R}P^2 \times \mathbb{R}P^3)) = \alpha^2, \\
and w_3(T(\mathbb{R}P^2 \times \mathbb{R}P^3)) = 0, \end{cases} \quad (89)$$

thus the $\Phi_1$ maps $(\mathbb{R}P^2 \times \mathbb{R}P^3, \alpha \cup \beta)$ to $(1, 0, 0, 0)$.

2. Consider $(M^5, B) = (S^1 \times \mathbb{R}P^4, \gamma \cup \zeta)$. Here $\gamma$ is the generator of $H^1(S^1, \mathbb{Z}_2)$, and $\zeta$ is the generator of $H^4(\mathbb{R}P^4, \mathbb{Z}_2)$.

Since

$$\begin{cases} Sq^1(\gamma \cup \zeta) = \gamma \cup \zeta^2, \\
w_1(T(S^1 \times \mathbb{R}P^4)) = \zeta, \\
and w_2(T(S^1 \times \mathbb{R}P^4)) = 0, \end{cases} \quad (90)$$

thus the $\Phi_1$ maps $(S^1 \times \mathbb{R}P^4, \gamma \cup \zeta)$ to $(0, 1, 1, 0)$.

3. Consider $(M^5, B) = (S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2, \gamma \alpha_1)$. Here $\gamma$ is the generator of $H^1(S^1, \mathbb{Z}_2)$, and $\alpha_1$ is the generator of $H^4(\mathbb{R}P^2, \mathbb{Z}_2)$ of the $i$-th factor $\mathbb{R}P^2$.

Since

$$\begin{cases} Sq^1(\gamma \alpha_1) = \gamma \alpha_2, \\
w_1(T(S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2)) = \alpha_1 + \alpha_2, \\
and w_2(T(S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2)) = \alpha_1^2 + \alpha_2^2 + \alpha_1 \alpha_2, \end{cases} \quad (91)$$

thus the $\Phi_1$ maps $(S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2, \gamma \alpha_1)$ to $(0, 0, 1, 0)$.

4. Let $W$ be the Wu manifold $SU(3)/SO(3)$.

Since $Sq^1 w_2(TW) = w_3(TW)$, thus the $\Phi_1$ maps $(W, w_2(TW))$ to $(1, 1, 0, 1)$, and $\Phi_1$ maps $(W, 0)$ to $(0, 0, 0, 1).

So we conclude that a generating set of manifold generators for $\Omega^5_3(B^2Z_2)$ is

$$\{(M^5, B) = ((\mathbb{R}P^2 \times \mathbb{R}P^3, \alpha \cup \beta), (S^1 \times \mathbb{R}P^4, \gamma \cup \zeta), (S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2, \gamma \alpha_1), (W, 0))\}. \quad (92)$$

Note that $(W, 0)$ can be replaced by $(W, w_2(TW))$.

The 4d Yang-Mills theory at $\theta = \pi$ has no 4d 't Hooft anomaly once the $T$ symmetry is not preserved. This means that all 5d higher SPTs/cobordism invariant for 4d YM theory must vanish at $\Omega^5_3(B^2Z_2)$ when $T$ (or $CT$) is removed. Compare with the data of $\Omega^5_3(B^2Z_2)$ given in Ref. [34], thus we find that the 5d terms (5d higher SPTs) for this 4d $SU(2)$ YM are chosen among:

$$\begin{bmatrix} B_2 \cup Sq^1 B_2 + Sq^2 Sq^1 B_2, \\
w_1(TM^5)^2 Sq^1 B_2, \end{bmatrix} \quad (93)$$

This information will be used later to match the $SU(2)$ YM anomalies at $\theta = \pi$.

C. $\Omega^5_3(BO(3))$

Follow Sec. III A, we enlist all possible 't Hooft anomalies of 2d $\mathbb{C}P^1$ model, or equivalently $O(3)$ NSLM, at $\theta = \pi$, by obtaining the 3d cobordism invariants from bordism groups of (eq. (73)/eq. (75)). From physics side, we will interpret the unoriented $O(D)$ spacetime symmetry with the time reversal from $CT$ instead of $T$.

We are given a 3-manifold $M^3$ and a map $f : M^3 \to BO(3)$. Here the map $f : M^3 \to BO(3)$ is a principal $O(3)$ bundle whose associated vector bundle is a rank 3 real vector bundle $E$ over $M^3$.

We like to obtain the bordism invariants of $\Omega^3_3(BO(3))$. We compute the bordism group [34]:

$$\begin{align*} 
\Omega^5_3(BO(3)) &= \mathbb{Z}_4^4, \\
\text{whose cobordism invariants are generated by} & \begin{cases} 
w_1(E)^3, \\
w_1(E)w_2(E), \\
w_3(E), \\
w_1(E)w_1(TM^3)^2. \end{cases} \quad (95) 
\end{align*}$$

We have a group isomorphism

$$\Phi_2 : \Omega^5_3(BO(3)) \to \mathbb{Z}_4^4, \quad (M^3, E) \mapsto (w_1(E)^3, w_1(E)w_2(E), w_3(E), w_1(E)w_1(TM^3)^2). \quad (96)$$

Let $l_{\mathbb{R}P^n}$ denote the tautological line bundle over $\mathbb{R}P^n$ ($\mathbb{R}P^1 = S^1$). If $x_n \in H^1(\mathbb{R}P^n, \mathbb{Z}_2)$ denotes the generator, then $w(l_{\mathbb{R}P^n}) = 1 + x_n, w(T(\mathbb{R}P^n)) = (1 + x_n)^{n+1}$.

Let $n$ denote the trivial real vector bundle of rank $n$, and let $+ \text{ denote the direct sum.}$

By the Whitney sum formula, $w(E \oplus F) = w(E)w(F)$. Here $w(E) = 1 + w_1(E) + w_2(E) + \cdots$ is the total Stiefel-Whitney class of $E$. Then we find:

1. Since $w(3l_{\mathbb{R}P^3}) = (1 + x_3)^3 = 1 + x_3 + x_3^2 + x_3^3$, and $w_1(T(\mathbb{R}P^1)) = 0$, thus the $\Phi_2$ maps $(\mathbb{R}P^3, 3l_{\mathbb{R}P^3})$ to $(1, 1, 1, 0)$.

2. Since $w(l_{\mathbb{R}P^3} + 2) = 1 + x_3$, thus the $\Phi_2$ maps $(\mathbb{R}P^3, l_{\mathbb{R}P^3} + 2)$ to $(1, 0, 0, 0)$. 


3. Since \(w(l_{s1} + 2) = 1 + x_1\), and \(w_1(T(S_1^1 \times \mathbb{RP}^2)) = x_2\), thus the \(Φ_2\) maps \((S_1^1 \times \mathbb{RP}^2, l_{s1} + 2)\) to \((0, 0, 0, 1)\).

4. Since \(w(l_{s1} + l_{\mathbb{RP}^2} + 1) = (1 + x_1)(1 + x_2) = 1 + x_1 + x_2 + x_1 x_2\), thus the \(Φ_2\) maps \((S_1^1 \times \mathbb{RP}^2, l_{s1} + l_{\mathbb{RP}^2} + 1)\) to \((1, 1, 0, 1)\).

So a generating set of manifold generators for \(Ω^0_5(BO(3))\) is

\[\{(M^3, E)\} = \{(\mathbb{RP}^3, 3\mathbb{RP}^3), (\mathbb{RP}^3, l_{\mathbb{RP}^3} + 2), (S_1^1 \times \mathbb{RP}^2, l_{s1} + 2), (S_1^1 \times \mathbb{RP}^2, l_{s1} + l_{\mathbb{RP}^2} + 1)\}\].

Note that \((S_1^1 \times \mathbb{RP}^2, l_{s1} + 2l_{\mathbb{RP}^2})\) is also a manifold generator. Note \(w(l_{s1} + 2l_{\mathbb{RP}^2}) = (1 + x_1)(1 + x_2)^2 = 1 + x_1 + x_2^2 + x_1 x_2^2\), therefore \(Φ_2\) maps \((S_1^1 \times \mathbb{RP}^2, l_{s1} + 2l_{\mathbb{RP}^2})\) to \((0, 1, 1, 1)\).

D. \(Ω^0_5(B^2 Z_4)\)

Follow Sec. IIIA, now we enlist all possible 't Hooft anomalies of 4d pure SU(4) YM at \(θ = π\) (but when the \(Z_2^7\)-background field is turned off) by obtaining the 5d cobordism invariants from bordism groups of \(eq. (9)/eq. (53)\).

We are given a 5-manifold \(M^5\) and a map \(f : M^5 \to B^2 Z_4\). Here the map \(f : M^5 \to B^2 Z_4\) is the 2-form \(B = B_2\) gauge field in the YM gauge theory eq. (12) (and eqn. (31) at \(N = 4\)).

We compute the bordism invariants of \(Ω^0_5(B^2 Z_4)\), we find the bordism group [34]:

\[Ω^0_5(B^2 Z_4) = Z_2^4\],

(98)

whose cobordism invariants are generated by

\[
\begin{align*}
B_2 & \cup β_{(2,4)} B_2, \\
S_0^2 & β_{(2,4)} B_2, \\
w_1(T M^5) & β_{(2,4)} B_2, \\
w_2(T M^5) & w_3(T M^5).
\end{align*}
\]

where \(β_{(2,4)} : H^*(M^5, Z_4) \to H^{*+1}(M^5, Z_2)\) is the Bockstein homomorphism associated with the extension \(Z_2 \to Z_8 \to Z_4\) (see Appendix A).

We have a group isomorphism

\[
Φ_3 : Ω^0_5(B^2 Z_4) \to Z_2^4
\]

\[
(M^5, B_2) \to (B_2 \cup β_{(2,4)}, B_2, S_0^2 β_{(2,4)}, B_2, w_1(T M^5) β_{(2,4)} B_2, w_2(T M^5) w_3(T M^5)).
\]

(100)

Let \(K\) be the Klein bottle.

1. Let \(α'\) be the generator of \(H^1(S_1^1, Z_4)\), \(β'\) be the generator of the \(Z_4\) factor of \(H^1(K, Z_4) = Z_4 \times Z_3\) (see Appendix C), and \(γ'\) be the generator of \(H^3(S^2, Z_4)\). Note that \(β_{(2,4)} β' = σ\) where \(σ\) is the generator of \(H^3(K, Z_2) = Z_2\) (see Appendix C).

Since \(β_{(2,4)} (α' \cup β' + γ') = α' \cup σ\) and \(w_2(T(S_1^1 \times K \times S^2)) = w_1(T(S_1^1 \times K \times S^3))^2 = 0\), we find that \(Φ_3\) maps \((S_1^1 \times K \times S^2, α' \cup β' + γ')\) to \((1, 0, 0, 0)\).

2. Following the notation of [61], \(X_2\) is a simply-connected 5-manifold which is orientable but non-spin. Let \(θ'\) and \(η'\) be two generators of \(H^0(X_2, Z_4) = Z_2^4\), then \(β_{(2,4)} θ'\) is one of the two generators of \(H^3(X_2, Z_2) = Z_2^2\). Since \(w_2(T X_2) = (θ' + η') \mod 2\), \(w_1(T X_2) = 0\) and \(w_3(T X_2) = 0\), we find that \(Φ_3\) maps \((X_2, θ', η')\) to \((1, 1, 0, 0)\).

3. Since \(w_1(T(S_1^1 \times K \times \mathbb{RP}^2))^2 = w_2(T(S_1^1 \times K \times \mathbb{RP}^2)) = α^2\) where \(α\) is the generator of \(H^4(\mathbb{RP}^2, Z_2)\), we find that \(Φ_3\) maps \((S_1^1 \times K \times \mathbb{RP}^2, α' \cup β')\) to \((0, 1, 1, 0)\).

4. \(W\) is the Wu manifold, while \(Φ_3\) maps \((W, 0)\) to \((0, 0, 0, 1)\).

So a generating set of manifold generators for \(Ω^0_5(B^2 Z_4)\) is

\[\{(M^5, B)\} = \{(S_1^1 \times K \times S^2, α' \cup β' + γ'), (X_2, θ', η'), (S_1^1 \times K \times \mathbb{RP}^2, α' \cup β'), (W, 0)\}\].

(101)

Note that

1. \((S_1^1 \times K \times T^2, α' \cup β' + γ')\) is also a generator where \(γ'\) is the generator of \(H^3(T^2, Z_4)\). Since \(β_{(2,4)} (α' \cup β' + γ') = α' \cup σ\) and \(w_2(T(S_1^1 \times K \times T^2)) = w_1(T(S_1^1 \times K \times T^3))^2 = 0\), we find \(Φ_3\) maps \((S_1^1 \times K \times T^2, α' \cup β' + γ')\) to \((1, 0, 0, 0)\).

2. \((K \times S^3/Z_4, β' \cup ε' + φ')\) is also a generator where \(S^3/Z_4\) is the Lens space \(L(4, 1)\), \(ε'\) is the generator of \(H^1(S^3/Z_4, Z_4)\), \(φ'\) is the generator of \(H^2(S^3/Z_4, Z_4)\). Since \(β_{(2,4)} (β' \cup ε' + φ') = σ \cup ε' + β' \cup φ\) where \(φ\) is the generator of \(H^2(S^3/Z_4, Z_2)\), and \(w_2(T(K \times S^3/Z_4)) = w_1(T(K \times S^3/Z_4))^2 = 0\), we find that \(Φ_3\) maps \((K \times S^3/Z_4, β' \cup ε' + φ')\) to \((1, 0, 0, 0)\).

E. \(Ω^0_5(BZ_2 \times B^2 Z_4) \equiv Ω^0_5(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)\)

Follow Sec. IIIA, now we enlist all possible 't Hooft anomalies of 4d pure SU(4) YM at \(θ = π\) (when the \(Z_2^7\)-background field can be turned on) by obtaining the 5d cobordism invariants from bordism groups of \(eq. (9)/eq. (53)\).

Note that again from physics side, we will interpret the unoriented \(O(D)\) spacetime symmetry with the time reversal from \(CT\) instead of \(T\). So we choose the former \(Ω^0_5(BZ_2 \times B^2 Z_4) \equiv Ω^0_5(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)\) for \(T\), rather than the more complicated latter \(Ω^0_5(BZ_2 \times B^2 Z_4) \equiv Ω^0_5(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)\) for \(T\).

Before we dive into \(Ω^0_5(BZ_2 \times B^2 Z_4) \equiv Ω^0_5(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)\), we first study the simplified “untwisted” bordism group \(Ω^0_5(BZ_2 \times B^2 Z_4)\).

We are given a 5-manifold \(M^5\) and a 1-form field \(A : M \to BZ_2\) and a 2-form \(B = B_2 : M^5 \to B^2 Z_4\) gauge field in the YM gauge theory eq. (12) (and eqn. (31) at \(N = \))
We compute the bordism invariants of $\Omega^O_5(BZ_2 \times \mathbb{B}^2 Z_4)$, we find the bordism group $[34]$

$$\Omega^O_5(BZ_2 \times \mathbb{B}^2 Z_4) = \mathbb{Z}_2^2,$$  \hspace{1cm} (102)

whose cobordism invariants are generated by

$$\begin{align*}
B_2 \cup \beta_{(2,4)} B_2, \\
\text{Sq}^2 \beta_{(2,4)} B_2, \\
w_1(TM^5)^2 \beta_{(2,4)} B_2, \\
w_2(TM^5) w_9(TM^5), \\
A^5, A^2 \beta_{(2,4)} B_2, \\
A^3 B_2, A^2 w_1(TM^5)^2, \\
AB_2, A w_1(TM^5)^4, \\
AB_2 w_1(TM^5)^2, A w_2(TM^5)^2.
\end{align*}$$  \hspace{1cm} (103)

We also compute the bordism invariants of $\Omega^O_5(BZ_2 \times \mathbb{B}^2 Z_4)$, we find $[34]$

$$\Omega^O_5(BZ_2 \times \mathbb{B}^2 Z_4) = \mathbb{Z}_2^2,$$  \hspace{1cm} (104)

whose cobordism invariants are generated by

$$\begin{align*}
\text{Sq}^2 \beta_{(2,4)} B_2, \\
w_2(TM^5) w_9(TM^5), \\
A^5, A^3 B_2, \\
AB_2, A w_2(TM^5)^2.
\end{align*}$$  \hspace{1cm} (105)

The 4d Yang-Mills theory at $\theta = \pi$ has no 4d ’t Hooft anomaly once the $\mathcal{C}T$ (or $\mathcal{T}$) symmetry is not preserved (as we discussed before that $\mathcal{C}$-symmetry is a good symmetry for any $\theta$ which has no anomaly directly from mixing with $\mathcal{C}$ by its own). This means that all 5d higher SPTs/cobordism invariants for 4d YM theory must vanish at $\Omega^O_5(BZ_2 \times \mathbb{B}^2 Z_4)$ when $\mathcal{C}T$ (or $\mathcal{T}$) is removed. So the 5d SPTs for this 4d YM are chosen among:

$$\begin{align*}
\{ B_2 \cup \beta_{(2,4)} B_2, \\
w_1(TM^5)^2 \beta_{(2,4)} B_2, \\
A^3 \beta_{(2,4)} B_2, A^2 w_1(TM^5)^2, \\
A w_1(TM^5)^4, AB w_1(TM^5)^2. \}
\end{align*}$$  \hspace{1cm} (106)

Follow Sec. III A, we enlist all possible ’t Hooft anomalies of 2d $\mathbb{CP}^N-1$ model at $N = 4$, at $\theta = \pi$, by obtaining the 3d cobordism invariants from bordism groups of (eq. (76)). From physics side, we will interpret the unoriented O(D) spacetime symmetry with the time reversal from $\mathcal{C}T$ instead of $\mathcal{T}$.

We are given a 3-manifold $M^3$ and a map $f : M^3 \to B(Z_2 \ltimes \text{PSU}(4))$ which corresponds to a principal $Z_2 \ltimes \text{PSU}(4)$ bundle $E$ over $M^3$.

We compute the bordism invariants of $\Omega^O_5(BO(3))$, we find the bordism group $[34]$

$$\Omega^O_5(B(Z_2 \ltimes \text{PSU}(4))) = \mathbb{Z}_2^4,$$  \hspace{1cm} (110)

whose cobordism invariants are generated by

$$\begin{align*}
w_1(E)^3, \\
w_1(E) w_1(TM^5)^2, \\
\beta_{(2,4)} w_2(E), \\
w_1(E) (w_2(E) \mod 2).
\end{align*}$$  \hspace{1cm} (111)

where $E$ is a principal $Z_2 \ltimes \text{PSU}(4)$ bundle over $M^3$ which is a pair $(w_1(E), w_2(E)) \in H^1(M^3, Z_2) \times H^2(M^3, Z_4, w_1(E))$ where $H^2(M^3, Z_4, w_1(E))$ is the twisted cohomology, $w_1(E)$ can be viewed as a group homomorphism $\pi_1(M^3) \to \text{Aut}(Z_4) = Z_2$.

In the following discussion, we use the ordinary cohomology instead of the twisted cohomology. If $w_1(E) = 0$
or \( w_2(E) = 0 \), then this simplification has no effect, while for the term \( w_1(E) / (w_2(E) \mod 2) \), though both \( w_1(E) \) and \( w_2(E) \) may be nonzero, but we will see that for certain 3-manifold \( M^3 \), for example, \( M^3 = S^1 \times T^2 \), if the action of \( \pi_1(M^3) = \mathbb{Z}^3 \) on \( Z_4 \) is nontrivial on only one factor \( Z \), namely, \( w_1(E) = \gamma \), then \( w_2(E) \) may not be twisted by \( w_1(E) \), for example, \( w_2(E) = \zeta' \), then the ordinary cohomology is sufficient for our discussion.

We have a group isomorphism

\[
\Phi_4 : \Omega^3_\mathbb{Z}(B(Z_2 \rtimes \text{PSU}(4))) \to Z_4^4
\]

\[
(M^3, w_1(E), w_2(E)) \mapsto (w_1(E)^3, w_1(E)(w_2(E) \mod 2), \beta_{(2,4)}w_2(E), w_1(1)w_1(TM^3)^2).
\]

1. Recall that \( \beta \) is the generator of \( H^1(\mathbb{R}P^3, Z_2) \). Since \( w_1(T\mathbb{R}P^3) = 0 \), \( \Phi_4 \) maps (\( \mathbb{R}P^3, \beta, 0 \)) to \((0, 1, 0, 0)\).
2. Recall that \( \gamma \) is the generator of \( H^1(S^1, Z_2) \). \( \gamma' \) is the generator of \( H^2(S^2, Z_4) \). \( \Phi_4 \) maps \((S^1 \times S^2, \gamma, \gamma')\) to \((0, 1, 0, 0)\).
3. \( K \) is the Klein bottle. Recall that \( \alpha' \) is the generator of \( H^1(S^1, Z_2) \). \( \alpha' \) is the generator of \( H^2(K, Z_2) \). \( \Phi_4 \) maps \((S^1 \times K, 0, \alpha' \cup \beta')\) to \((0, 0, 0, 0)\).
4. Recall that \( \gamma \) is the generator of \( H^1(S^1, Z_2) \). Since \( w_1(T(S^1 \times \mathbb{R}P^2)) \equiv \alpha \) where \( \alpha \) is the generator of \( H^1(\mathbb{R}P^2, Z_2) \), \( \Phi_4 \) maps \((S^1 \times \mathbb{R}P^2, \gamma, 0)\) to \((0, 0, 0, 1)\).

So a generating set of manifold generators for \( \Omega^3_\mathbb{Z}(B(Z_2 \rtimes \text{PSU}(4))) \) is

\[
\{(M^3, w_1(E), w_2(E)) \mapsto (\mathbb{R}P^3, \beta, 0), (S^1 \times S^2, \gamma, \gamma'), (S^1 \times K, 0, \alpha' \cup \beta'), (S^1 \times \mathbb{R}P^2, \gamma, 0)\}.
\]

Note that \((S^1 \times T^2, \gamma, \gamma')\) is also a manifold generator, where \( \gamma \) is the generator of \( H^1(S^1, Z_2) \), \( \gamma' \) is the generator of \( H^2(T^2, Z_4) \). \( \Phi_4 \) maps \((S^1 \times T^2, \gamma, \gamma')\) to \((0, 1, 0, 0)\).

IV. REVIEW AND SUMMARY OF KNOWN ANOMALIES VIA COBORDISM INVARIANTS

Follow Sec. III, we have obtained the co/bordism groups relevant from the given full \( G \)-symmetry of 4d YM and 2d \( \mathbb{C}P^{N-1} \) models. Therefore, based on the correspondence between \( d \) \( ' \)t Hooft anomalies and \( Dd=(d+1)d \) topological terms/cobordism/SPTs invariants, we have obtained the classification of all possible higher \( ' \)t Hooft anomalies for these 4d YM and 2d \( \mathbb{C}P^{N-1} \) models.

Below we first match our result to the known anomalies found in the literature, and we shall put these known anomalies into a more mathematical precise thus a more general framework, under the cobordism theory. We will write down the precise \( d \) \( ' \)t Hooft anomalies and \( Dd=(d+1)d \) cobordism/SPTs invariants for them. We will also clarify the physical interpretations (e.g. from condensed matter inputs) of anomalies.

A. Mixed higher-anomaly of time-reversal \( Z_2^{CT} \) and 1-form center \( Z_2 \)-symmetry of SU(N)-YM theory

First recall in Sec. IIB5, we re-derives the mixed higher-anomaly of time-reversal \( Z_2^{CT} \) and 1-form center \( Z_2 \)-symmetry of 4d SU(N)-YM, at even \( N \), discovered in [30]. By turning on 2-form \( Z_2 \)-background field \( B = B_2 \) coupling to YM theory, the \( Z_2^{CT} \)-symmetry shifts the 4d YM with an additional 5d higher SPTs term eq. (IIB5). We also learned that the same mixed higher-anomaly occurs in replacing \( Z_2^{CT} \) to eq. (51),

\[
Z_2^{CT}, Z_2^P, \text{ and } Z_2^{CP},
\]

For our preference, we focus on \( CT \) instead of \( T \). This type of anomaly has the linear dependence on \( CT \) (thus linear also \( T \)) and quadratic dependence on \( B_2 \). Compare with our eq. (85), we find that the precise form for 5d cobordism invariant/ 4d higher \( ' \)t Hooft anomaly is:

\[
B_2 Sq^1 B_2.
\]

More precisely, we need to consider instead eq. (169), \( B_2 Sq^1 B_2 + Sq^2 Sq^1 B_2 = \frac{1}{2} w_1(TM) P_2(B_2)_2 \), see Sec. VII A for details and derivations.

B. Mixed anomaly of \( Z_2^C = Z_2^T \)- and time-reversal \( Z_2^{CT} \) or SO(3)-symmetry of \( \mathbb{C}P^1 \)-model

Now we move on to 2d \( \mathbb{C}P^1 \) or O(3) NLSM model at \( \theta = \pi \), we get the full 0-form global symmetries eq. (75), \( Z_2^{CT} \times \text{PSU}(2) \times Z_2^2 = Z_2^{CT} \times \text{PSU}(2) \times Z_2^2 = Z_2^{CT} \times O(3) \).

It has been known that there is a non-perturbative global discrete anomaly from the \( Z_2^C \) (a discrete translational \( Z_2 \) symmetry) since the work of Gepner-Witten [63]. More recently, this non-perturbative global discrete anomaly has been revisited by [64, 65] to understand the nature of symmetry-protected gapless critical phases.
obtained from group cohomology data $H^2(BSO(3),U(1)) = \mathbb{Z}_2$ and $H^3(BZ^T_2,U(1)) = \mathbb{Z}_2$ \cite{7}. If the time-reversal or $SO(3)$ symmetry is preserved, the boundary has 2-fold degenerate spin-1/2 modes on each 1d edge. The layer stacking of such spin-1/2 modes to a 2d boundary (encircled by the dashed-line rectangle in Fig. 1) can actually give rise to gapless 2d $\mathbb{CP}^1$ / $O(3)$ NLSM / $SU(2)_1$-WZW model. Part of its anomaly is captured by the $\mathbb{Z}_2^T$-translational anomaly of the same system, and detects the anomaly $w_3(E) = w_3(V_{O(3)})$, we can convert it to

\[ w_3(E) = w_3(V_{O(3)}) = w_1(V_{O(3)})^3 + w_1(V_{SO(3)})^2 + w_3(V_{SO(3)})^3 \]

\[ = w_1(\mathbb{Z}_2^T)^3 + w_1(\mathbb{Z}_2^T)w_2(V_{SO(3)}) + w_3(V_{SO(3)})^3 \]

\[ = w_1(E)^3 + w_1(E)w_2(V_{SO(3)}) + w_3(V_{SO(3)}) \]

\[ = w_1(E)w_2(V_{SO(3)}) + w_1(TM)w_2(E) \]

\[ = w_1(E)^3 + w_1(E)w_2(V_{SO(3)}) + w_1(TM)w_2(V_{SO(3)}) \]
We also note that
\[ w_1(E)w_2(E) = w_1(V_{O(3)})w_2(V_{O(3)}) \]
\[ = w_1(V_{O(3)})^3 + w_1(V_{O(3)})w_2(V_{SO(3)}) \]
\[ = w_1(Z_2^x)^3 + w_1(Z_2^x)w_2(V_{SO(3)}) \]
\[ = w_1(E)^3 + w_1(E)w_2(V_{SO(3)}). \] (118)

Similar equality and anomaly are discussed in [68] in a different topic on Chern-Simons matter theories.

To summarize, we note that:
The \( w_1(E)^3 \) is \((0, 1, 0, 0)\) in the basis of eq. (96).
The \( w_1(E)w_2(E) \) is \((0, 1, 0, 0)\) in the basis of eq. (96).
The \( w_1(E)w_2(V_{SO(3)}) \) is \((1, 1, 0, 0)\) in the basis of eq. (96).
The \( w_3(V_{SO(3)}) \) is \((1, 1, 0, 0)\) in the basis of eq. (96).
The \( w_3(E)w_2(E) \) is \((0, 1, 1, 0)\) in the basis of eq. (96).
The \( w_3(E)w_2(V_{SO(3)}) \) is \((0, 0, 1, 0)\) in the basis of our eq. (96).

Therefore, Ref. [67]'s anomaly eq. (117) given by \( w_3(E) = w_1(E)^3 + w_1(E)w_2(V_{SO(3)}) + w_1(TM)w_2(E) \) coincides with one of the cobordism invariant as \((0, 0, 1, 0)\) in the basis of our eq. (96). We had explained the physical meaning of \( w_1(E)w_2(V_{SO(3)}) \) term in eq. (115). We will explain the meaning of \( w_1(E)^3 \) in Sec. IV D and the meaning of \( w_1(TM)w_2(E) \) in Sec. IV E.

D. Cubic anomaly of \( Z_2^C \) of \( \mathbb{CP}^1 \)-model

Now we like to capture the physical meaning of a cubic anomaly of \( Z_2^C = Z_2^C \)-symmetry in eq. (117):
\[ w_1(E)^3 \equiv w_1(Z_2^C)^3 \equiv (A_x)^3 \] (119)
which is a sensible cobordism invariant as the \((0, 0, 0, 0)\) in the basis of eq. (96). Ref. [62] also points out this \( w_1(E)^3 \) or the \( A_x^3 \)-anomaly, where \( A_x \) is regarded as the \( Z_2^C \)-translational background gauge field. We know that the 2d boundary physics we look at in Fig. 1 (encircled by the dashed-line rectangle) describes the gapless CFT theory of \( SU(2)_1 \) WZW model at the level \( k = 1 \). The \( SU(2)_1 \) WZW model at \( k = 1 \) is equivalent to a \( c = 1 \) compact non-chiral boson theory (the left and right chiral central charge \( c_L = c_R = 1 \), but the chiral central charge \( c_- = c_L - c_R = 0 \)) at the self-dual radius [69]. Although properly we could use non-Abelian bosonization method [54], here focusing on the abelian \( Z_2^C \)-symmetry and its anomaly, we can simply use the Abelian bosonization.

Since the \( SU(2)_1 \) WZW model at \( k = 1 \) is equivalent to a \( c = 1 \) compact non-chiral boson theory at the self-dual radius, we consider an action
\[ S_{2d} = \frac{1}{2\pi \alpha'} \int d\sigma d\bar{\sigma} \left( \partial_\sigma \Phi \right) \left( \partial_{\bar{\sigma}} \Phi \right) + \ldots, \] (120)
\[ S_{2d} = \frac{1}{4\pi} \int dt dx \left( K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J \right) + \ldots \]
requiring a rank-2 symmetric bilinear form \( K \)-matrix,
\[ K_{IJ} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) + \ldots; \quad V_{IJ} = \left( \begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon \end{array} \right) + \ldots \] (121)

The first form of the action is familiar in string theory and a \( c = 1 \) compact non-chiral boson theory at the self-dual radius. (In string theory, we are looking at \( R = \sqrt{\alpha'} = \sqrt{2} \).)

The second form of the action is the familiar 2d boundary of 3d bosonic SPTs. This second description is also known as Tomonaga-Luttinger liquid theory [70–72] in condensed matter physics. It is a \( K \)-matrix multiplet generalization of the usual chiral boson theory of Floreeni and Jackiw [73]. The reason we write \ldots in eq. (112) is that there could be additional 3d SPTs sectors for 2d \( \mathbb{CP}^1 \)-model (e.g. eq. (179)), more than what we focus on in this subsection. Here we trade the boson scalar \( \Phi \) to \( \phi_1 \), while \( \phi_2 \) is the dual boson field. We can determine the bosonic anomalies [18] by looking at the anomalous symmetry transformation on the 2d theory, living on the boundary of which 3d SPTs. We use the mode expansion for a multiplet scalar boson field theory [18], with zero modes \( \phi_{01} \) and winding modes \( P_{\phi, i} \):
\[ \phi_1(x) = \phi_{01} + K_{IJ}^{-1} P_{\phi, I} \frac{2\pi}{L} x + i \sum_{n \neq 0} \lambda_{I,n} e^{-i n x \frac{2\pi}{L}}, \]
which satisfy the commutator \([\phi_{01}, P_{\phi, I}] = i\delta_{I,J} \). The Fourier modes satisfy a generalized Kac-Moody algebra: \([\lambda_{I,n}, \lambda_{J,m}] = n K_{IJ}^{-1} \delta_{n,-m} \). For a modern but self-contained pedagogical treatment on a canonical quantization of \( K \)-matrix multiplet (non-)chiral boson theory, the readers can consult Appendix B of [74].

Follow [62], based on the identification of spin observables of Hamiltonian model eq. (65) and the abelian bosonized theory, we can map the symmetry transformation to the continuum description on the boson multiplet \( \phi_I(x) = (\phi_1(x), \phi_2(x)) \). The commutation relation is \([\phi_I(x_1), K_{IJ'} \partial_x \phi_J(x_2)] = 2\pi i \delta_{IJ'} \delta(x_1 - x_2) \). The continuum limit of 2d anomalous symmetry transformation is [75] [18]:
\[ S^{(p)}_N = \exp \left[ \frac{i}{N} \int_0^L dx \partial_x \phi_2 + p \int_0^L dx \partial_x \phi_1 \right], \] (122)
\[ S^{(p)}_N \left( \phi_1(x), \phi_2(x) \right) = \left( \phi_1(x), \phi_2(x) \right) + \frac{2\pi}{N} \left( \begin{array}{c} 1 \\ p \end{array} \right). \]

Here \( L \) is the compact spatial \( S^1 \) circle size of the 2d theory. For 2d \( \mathbb{CP}^1 \)-model, we have \( N = 2 \) and \( p = 1 \), this is indeed known as the Type I bosonic anomaly in [18], which also recovers one anomaly found in [62] and in [67]'s eq. (117).

E. Mixed anomaly of time-reversal \( Z_2^T \) and 0-form flavor \( Z_N \)-center symmetry of \( \mathbb{CP}^1 \)-model

Ref. [36, 37] point out another anomaly of \( \mathbb{CP}^1 \)-model, which mixes between time-reversal (which we have chosen to be \( CT \)) and the \( PSU(2) \) symmetry (which is viewed as the twisted flavor symmetry in [36, 37]). Compare
with eq. (95), we can interpret the above 2d anomalies are captured by a 3d cobordism invariant for $N = 2$ case:

$$w_1(TM)w_2(V_{SO(3)}) = w_1(TM)w_2(E) = w_3(V_{SO(3)}).$$

(123)

This also coincides with the last anomaly term in eq. (117)’s $w_3(E)$. We derive the above first equality in eq. (123) based on $Sq^1(w_1(E))^2 = 2w_1(E)Sq^1w_1(E) = 0$ and combine Wu formula, $Sq^1(w_1(E)^2) = w_1(TM)(w_1(E))^2 = 0$. Thus,

$$w_1(TM)w_2(V_{SO(3)}) = w_1(TM)(w_2(E) + w_1(E)^2) = w_1(TM)w_2(E) = w_1(TM)w_2(V_{SO(3)}).$$

(124)

The last equality in eq. (123) is due to $w_1(TM)w_2(V_{SO(3)}) = Sq^1w_2(V_{SO(3)}) = w_3(V_{SO(3)})$.

See Ref. [34], we can combine the Steenrod-Wu formula for $i < j$:

$$Sq^i(w_j) = w_iw_j + \sum_{k=1}^{i} \binom{j - i - 1 + k}{k} w_{i-k}w_{j+k},$$

(125)

and Wu formula

$$Sq^{d-j}(x_j) = u_{d-j}x_j, \text{ for any } x_j \in H^j(M^d; \mathbb{Z}_2)$$

(126)

to obtain:

$$w_1(E)w_2(E) + w_3(E) = Sq^1(w_2(E)) = w_1(TM)w_2(E),$$

$$\Rightarrow w_3(E) = (w_1(E) + w_1(TM))w_2(E)$$

$$\Rightarrow w_1(TM)w_2(E) = w_3(E) + w_1(E)w_2(E).$$

(127)

so we derive $w_1(TM)w_2(E)$ is $(0, 1, 1, 0)$ in the basis of eq. (96). The physical meaning of the 2d anomaly eq. (123) will be explored later in Sec. V, Sec. VI and in Fig. 2, which can be understood as the dimensional reduction of 4d anomaly of YM theory compactified on a 2-torus with twisted boundary conditions [36] [35].

In Sec. IVD, We had checked some of the 2d bosonic anomaly by dimensional reducing from 4d to 2d, can be captured by abelian bosonization method as Type I bosonic anomaly in [18]. Some of the anomalies in the above may be also related to other (Type II or Type III) bosonic discrete anomalies, when we break down the global symmetry to certain subgroups.

V. RULES OF THE GAME FOR ANOMALY MATCHING CONSTRAINTS

With all the QFT and global symmetries information given in Sec. II, and all the possible anomalies enumerated by the cobordism theory computed in Sec. III, and all the known anomalies in the literature derived and re-written in terms of cobordism invariants organized in Sec. IV, now we are ready to set up the rules of the game to determine the full anomaly constraints for these QFTs (4d SU(N) YM theory and 2d CP$^{N-1}$ model at $\theta = \pi$).

Below we simply abbreviate the “5d invariant” as the 5d cobordism/(higher) SPTs invariants which captures the anomaly of 4d SU(N) YM at $\theta = \pi$ at even N, and “3d invariant” as the 3d cobordism/SPTs invariants which captures the anomaly of 2d CP$^{N-1}$ at $\theta = \pi$ at even N. Our convention chooses the natural time-reversal symmetry transformation as $C\mathbb{T}$.

Rules:

Rule 1. For 5d invariant, for 4d SU(N) YM at $\theta = \pi$ of an even integer N must have analogous anomaly captured by 5d cobordism term of $\sim u_1(TM)(B_2)^2$ (up to some properly defined normalization and quantization). (It will become transparent later in eq. (169) that the precise term needs to be $\frac{1}{2}w_1(TM)[P_2(B_2)] = B_2Sq^1B_2 + Sq^2B_2$.)

Rule 2. The chosen 5d invariants may be non-vanished in O-bordism group, but they are vanished in SO-bordism group.

Rule 3. The 3d invariant for 2d CP$^1$ model must include the 3d cobordism invariants discussed in Sec. IV, in particular, eq. (179).

Rule 4. The 3d invariant for other 2d CP$^{N-1}$ for even N (e.g. 2d CP$^3$) model must include some of familiar terms generalizing that of 2d CP$^1$ model.

Rule 5. Due to the physical meanings of $C\mathbb{T}$ and $\mathbb{T}$ (and other orientation-reversal symmetries), we must impose a swapping symmetry for 5d invariants.

Rule 6. Relating the 5d and 3d invariants: There is a dimensional reductional constraint and physical meanings between the 5d and 3d invariants, for example, by the twist-compactification on 2-torus $T^2$.

Rule 7. The 5d invariants for a 4d pure YM theory must involve the nontrivial 2-form $B_2$ field. The 5d terms that involve no $B_2$ dependence should be discarded.
4d SU(N)$_{\theta=\pi}$ Yang-Mills gauge theory/ boundary of 5d higher-SPTs on a 5-manifold $M^5$

\[ S_y^1 \times S_z^1 \times S_x^1 \times \mathbb{R} \]
moduli space of flat connections of YM becomes
gauge bundle $E$ /
target space $\mathbb{C}P^{N-1}$

\[ 2d \text{ spacetime of } \mathbb{C}P^{N-1}\text{-model}/ \]
boundary of 3d SPTs on a 3-manifold $N^3$

't Hooft twisted boundary condition

FIG. 2. Follow the setup of the twisted boundary condition induced 't Hooft boundary condition [76] along the 2-torus $T^2_{yz} \equiv S_y^1 \times S_z^1$, and the twisted compactification [36] [35], we examine that the higher anomaly of 4d SU(N) YM theory at $\theta = \pi$ induces the anomaly of 2d $\mathbb{C}P^{N-1}$ model at $\theta = \pi$. The 4d YM on $S_y^1 \times S_y^1 \times S_z^1 \times \mathbb{R}$ is compactified along the small size of $T^2_{yz} \equiv S_y^1 \times S_z^1$, whose moduli space of flat connections becomes the target space $\mathbb{C}P^{N-1}$ [77, 78], while the remained $S_x^1 \times \mathbb{R}$ becomes the 2d spacetime of 2d $\mathbb{C}P^{N-1}$ model. Our goal, in Sec. V, VII and VIII is to identify the underlying 't Hooft anomalies of 4d SU(N) YM and 2d $\mathbb{C}P^{N-1}$ model, namely identifying the theories living on the boundary ($\equiv$ bdry) of 5d and 3d (higher) SPTs when all the (higher) global symmetries needed to be regularized strictly on site and local (e.g. [12–14]). (Higher global symmetries can be regularized, instead on the 0-simplex, on the higher-simplices: 1-simplex, 2-simplex, etc.) The twisted boundary condition of 4d YM for 1-form $Z_N$-center symmetry (as a higher symmetry twist of [13]) can be dimensionally reduced to the 0-form $Z_N$-flavor symmetry twisted [79] in the 2d $\mathbb{C}P^{N-1}$ model. In fact, the twisted boundary conditions can be designed numerically, and the twisted boundary conditions can result in a fractional instanton number in a gauge theory, see, for example, recent numerical attempts and Reference therein [80].

Here are the explanations for our rules.

Rule 1 is based on Sec. II B, for 4d SU(N) YM at $\theta = \pi$ of an even integer $N$ must have analogous anomaly captured by 5d cobordism term of $\sim w_1(TM)B^2_2$ (up to some properly defined normalization and quantization), where we choose the linear time-reversal symmetry transformation from $CT$ and a quadratic term of 2-form fields $B_2$ coupling to 1-form center symmetry.

Rule 2’s physical reasoning is that the time-reversal symmetry transformation from $CT$ plays an important role for the anomaly. We can see from Sec. II B 6 that only when time-reversal or orientation reversal is involved ($T$, $P$, $CT$ and $CP$), we have the mixed higher anomalies for YM theory; while for the others ($C$, $PT$ and $CPT$), we do not gain mixed anomalies (e.g. with the 1-form center symmetry).

Rule 3 is dictated by the known physics derivations in Sec. IV and in the literature.

Rule 4 will become clear in Sec. VIII.

Rule 5, the swapping symmetry for 5d invariants between $CT$ and $T$ (and other orientation-reversal symmetries), we will interpret the unoriented $O(D)$ spacetime symmetry with the time reversal from $CT$ or from $T$ can be swapped. This means that we can choose the 5d topological invariant from the former $\Omega^5_5(BZ_2 \ltimes B^2 Z_4) \equiv \Omega^5_5(O \ltimes (Z_2 \times BZ_4))$ for $CT$, rather than the more complicated latter $\Omega^5_5(BZ_2 \ltimes B^2 Z_4) \equiv \Omega^5_5(O \times Z_2) \times BZ_4$ for $T$. We focus on the 5d terms involving $CT$-symmetry.

Rule 6 about the dimensional reduction from 5d...
to 3d (or 4d to 2d) is explained in Fig. 2 and the main text, such as in Sec. VI. We should also find the mathematical meanings behind this constraint in Sec. VI.

Rule 7 is based on the physical input that there should be no obstruction to regularize a pure YM theory by imposing only ordinary 0-form symmetry along on-site. The obstruction only comes from regularizing a pure YM theory with the involvement of restricting both the higher 1-form center symmetry to be on-link and local, and the ordinary 0-form symmetry to be on-site and local. Here on-link means that the symmetry acts locally on the 1-simplex, and on-site means that the symmetry acts locally on the 0-simplex or a point.

Thus, it is necessary to turn on the 2-form background field B_2 in order to detect the t Hooft anomaly of YM theory. Namely, the 5d cobordism invariants of the form \( w_1(TM)^t \cup A^t \) with \( t + a = 5 \) should be discarded out of the candidate list of 5d term for 4d YM anomalies.

Therefore, we can refine the set eqn. (106) to a smaller subset satisfying Rule 7:

\[
\begin{align*}
B_2 \cup \beta_{(2,4)} B_2, \\
w_1(TM^5)^2 \beta_{(2,4)} B_2, \\
A^2 \beta_{(2,4)} B_2, \\
AB w_1(TM^5)^2.
\end{align*}
\] (128)

VI. RULES OF DIMENSIONAL REDUCTION: 5D TO 3D

Now we aim to utilize the Rule 6 in Sec. V and the new anomaly of 2d CP^{N-1}-model found in Sec. VIII, to deduce the new higher anomaly of 4d YM theory — which later will be organized in Sec. VII.

From the physics side, follow [36], see Fig. 2, we choose the 4d YM living on \( S^1_y \times S^1_y \times S^1_y \), such that the size \( L_y, L_z \) of \( S^1_y \times S^1_y \) is taken to be much smaller than the size \( L_x \) of \( S^1_z \), namely \( L_y, L_z \ll L_x \). Then, below the energy gap scale

\[ \Delta_E \ll L_y^{-1} \text{ and } L_z^{-1}, \]

the resulting 2d theory on \( S^1_x \times \mathbb{R} \) is given by a sigma model with a target space of \( \text{CP}^{N-1} \). There are several indications that the low energy theory is a 2d CP^{N-1}-model:

- The 4d and 2d instanton matchings in [27, 28] and other mathematical works. The \( \theta = \pi \)-topological term of SU(N) YM is mapped to the \( \theta = \pi \)-topological term of 2d \( \text{CP}^{N-1} \)-model.

- The moduli space of flat connections on the 2-torus \( T^2 = S^1_y \times S^1_z \) of 4d YM theory is the projective space \( \text{CP}^{N-1} \) [77, 78] (up to the geometry details of no canonical Fubini-Study metric and singularities mentioned in [36] and footnote 1). See Fig. 2.

- The 1-form \( Z_N \)-center symmetry of 4d YM is dimensionally reduced, in addition to 1-form symmetry itself, also to a 0-form \( Z_N \)-flavor of 2d \( \text{CP}^{N-1} \) model. The twisted boundary condition of 4d YM for 1-form \( Z_N \)-center symmetry (e.g., [37] as a higher symmetry twist of [13]) can be dimensionally reduced to the 0-form \( Z_N \)-flavor symmetry twisted [78] in the 2d \( \text{CP}^{N-1} \) model.

- Ref. [35] derives that the physical meaning of the 2d anomaly eq. (123) is directly descended from the 4d anomaly eq. (114) of YM theory by twisted \( T^2 \) compactification.

Encouraged by the above physical and mathematical evidences, in this section, we formalize the 4d and 2d anomaly matching under the twisted \( T^2 \) compactification, into a mathematical precise problem of the 5d and 3d cobordism invariants (SPTs/topological terms) matching, under a 2-torus \( T^2 \) dimensional reduction.

Below we follow our notations of the bordism groups in Sec. III, and their \( \mathcal{D} = \mathcal{D} \mathcal{C} \) of QFTs. We may simply abbreviate,

"5d cobordism invariants for 4d YM theory’s anomaly"

\[ \equiv 5d \text{ (Yang-Mills) terms.} \]

We may simply abbreviate,

"3d cobordism invariants for 2d \( \text{CP}^{N-1} \) model’s anomaly"

\[ \equiv 3d \text{ (CP}^{N-1} \text{) terms.} \]

A. Mathematical Set-Up

We aim to find the corresponding 3d \( \text{CP}^{N-1} \) terms which are obtained from certain 5d Yang-Mills terms under a "\( T^2 \) reduction".

Recall that we have the following group isomorphisms in Sec. III B, Sec. III C, Sec. III E, and Sec. III F:

\[ \Phi_1 : \Omega^5_{BO(3)} \rightarrow \mathbb{Z}_4, \]
\[ (M^5, B) \mapsto (B \cup S_1 B, S_1^2 S_1 B_2, w_1(TM^5)^2 \cup S_1 B, w_2(TM^5) w_3(TM^5)). \]

\[ \Phi_2 : \Omega^5_{BO(3)} \rightarrow \mathbb{Z}_4, \]
\[ (M^5, E) \mapsto (w_1(E)^2, w_1(E) w_2(E), w_3(E), w_1(E) w_1(TM^4)^2). \]

\[ \Phi_3 : \Omega^5_{BO(3)} \rightarrow \mathbb{Z}_4, \]
\[ (M^5, A, B_2) \mapsto (B_2 \cup \beta_{(2,4)} B_2, AB w_1(TM^5)^2, w_1(TM^5)^2 \beta_{(2,4)} B_2, A^2 \beta_{(2,4)} B_2, A^2 w_1(TM^5)^2, w_2(TM^5) w_3(TM^5), S_1^2 \beta_{(2,4)} B_2, w_2(TM^5) w_3(TM^5), A^2, AB_2^2, w_2(TM^5)^2). \]
and
\[ \Phi_4 : \Omega^3(B(Z_2 \ltimes \text{PSU}(4))) \to \mathbb{Z}_2, \]
\[ (M^3, w_1(E), w_2(E)) \mapsto (w_1(E))^3, w_1(E)(w_2(E) \text{ mod 2}), \]
\[ \beta_{(2,4)}w_2(E), w_1(E)w_1(TM^3)^2. \]  
(132)

We have the group isomorphisms
\[ \Phi'_1 : \Omega^3_5(B^2\mathbb{Z}_2)/\Omega^\text{SO}_5(B^2\mathbb{Z}_2) \to \mathbb{Z}_2^2, \]
\[ [(M^5, B_2)] \mapsto (B_2 \cup Sq^1 B_2 + Sq^2 Sq^1 B_2, \]
\[ w_1(TM^5)^2 \cup Sq^1 B_2). \]
(133)

and
\[ \Phi'_3 : \Omega^3_5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4)/\Omega^\text{SO}_5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4) \to \mathbb{Z}_2^6 \]
\[ [(M^5, A, B_2)] \mapsto (B_2 \cup \beta_{(2,4)} B_2, AB_2w_1(TM^5)^2, \]
\[ w_1(TM^5)^2\beta_{(2,4)}B_2, A^2\beta_{(2,4)} B_2, \]
\[ A^3 w_1(TM^5)^2, Aw_1(TM^5)^4. \]
(134)

1. General Set-Up

In general, if V and W are two vector spaces, \( \rho : V \to W \) is a linear map, and we choose the bases of V and W to be \((e_1, \ldots, e_n)\) and \((f_1, \ldots, f_m)\) respectively. Suppose the matrix of \( \rho \) with respect to the bases chosen above is an \( m \times n \) matrix A, namely, we have
\[ \rho(e_1, \ldots, e_n) = (f_1, \ldots, f_m)A. \]

Let \( \rho^* : W^* \to V^* \) be the dual linear map of \( \rho \), and \((e_1^*, \ldots, e_n^*)\) and \((f_1^*, \ldots, f_m^*)\) be the dual bases of \( V^* \) and \( W^* \) respectively. Suppose \( A = (a_{ij}) \), then we have
\[ \rho(e_i) = \sum_{j=1}^m a_{ij} f_j, \]
and
\[ \rho^*(f_j^*) = \sum_{i=1}^n a_{ji} e_i^*. \]

Namely, the matrix of \( \rho^* \) with respect to the dual bases chosen above is \( A^T \), the transpose of A.

2. Mathematical Set-Up for \( N = 2 \)

Below we elaborate on the \( N = 2 \) case first.

Let \( W \) in this subsection be the quotient group \( \Omega^3_5(B^2\mathbb{Z}_2)/\Omega^\text{SO}_5(B^2\mathbb{Z}_2) \).

We choose a basis of \( W \) to be \((f_1, f_2)\) where
\[ \begin{cases} 
 f_1 = \Phi_1^{-1}(1, 0), \\
 f_2 = \Phi_1^{-1}(0, 1).
\end{cases} \]
(135)

The dual basis of \( W^* \) is \((f_1^*, f_2^*)\) where
\[ \begin{cases} 
 f_1^* = BSq^1 B + Sq^2 Sq^1 B, \\
 f_2^* = w_1(TM^5)^2 Sq^1 B.
\end{cases} \]
(136)

Let \( V = \Omega^6_3(BO(3)) \).

We choose a basis of \( V \) to be \((e_1, e_2, e_3, e_4)\) where
\[ \begin{cases} 
 e_1 = \Phi_2^{-1}(1, 0, 0, 0), \\
 e_2 = \Phi_2^{-1}(0, 1, 0, 0), \\
 e_3 = \Phi_2^{-1}(0, 0, 1, 0), \\
 e_4 = \Phi_2^{-1}(0, 0, 0, 1).
\end{cases} \]
(137)

The dual basis of \( V^* \) is \((e_1^*, e_2^*, e_3^*, e_4^*)\) where
\[ \begin{cases} 
 e_1^* = w_1(E)^3, \\
 e_2^* = w_1(E)w_2(E), \\
 e_3^* = w_3(E), \\
 e_4^* = w_4(E)w_1(TN^3)^2.
\end{cases} \]
(138)

We aim to construct a linear map \( F : V \to W \); then we can find the image of \( f_i^* \) under \( F^* \), which is the desired 3d term reduced from the 5d term \( f_i^* \).

3. Mathematical Set-Up for \( N = 4 \)

Below we elaborate on the \( N = 4 \) case.

Let \( W \) be the subgroup of the quotient group \( \Omega^3_5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4)/\Omega^\text{SO}_5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4) \) defined by
\[ W = \{(\{M^5, A, B\}) \in \Omega^3_5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4)/\Omega^\text{SO}_5(B\mathbb{Z}_2 \ltimes B^2\mathbb{Z}_4) \mid \text{the 3-rd to 6-th components of } \Phi''_3(M^5, A, B) \text{ are 0}. \}
(139)

First, the 5-th to 6-th components of \( \Phi''_3(M^5, A, B) \) are 0 because we derive that from eqn. (128), for eqn. (134), only \( B_2 \beta_{(2,4)} B_2, w_1(TM^5)^2 \beta_{(2,4)} B_2, A^2 \beta_{(2,4)} B_2, \) and \( AB_2w_1(TM^5)^2 \) are possible 5d terms in eqn. (134) for YM theory, while other terms \( A^3 w_1(TM^5)^2 \) and \( Aw_1(TM^5)^4 \) must not appear in the 4d anomaly of SU(2) YM at \( N = 4 \). Moreover, we will discuss the possibility of \( w_1(TM^5)^2 \beta_{(2,4)} B_2 \) and \( A^2 \beta_{(2,4)} B_2 \) separately on an upcoming work. Here we assume that \( w_1(TM^5)^2 \beta_{(2,4)} B_2 \) and \( A^2 \beta_{(2,4)} B_2 \) are unlikely for a 4d SU(4) YM theory, thus we will set the 3-rd to 6-th components of \( \Phi''_3(M^5, A, B) \) to be 0.

We choose a basis of \( W \) to be \((f_1, f_2)\) where
\[ \begin{cases} 
 f_1 = \Phi''_1^{-1}(1, 0, 0, \ldots, 0), \\
 f_2 = \Phi''_1^{-1}(0, 1, 0, \ldots, 0).
\end{cases} \]
(140)

The dual basis of \( W^* \) is \((f_1^*, f_2^*)\) where
\[ \begin{cases} 
 f_1^* = B \beta_{(2,4)} B, \\
 f_2^* = Aw_1(TM^5)^2 B.
\end{cases} \]
(141)

Let \( V = \Omega^6_3(B(Z_2 \ltimes \text{PSU}(4))) \).
We choose a basis of $V$ to be $(e_1, e_2, e_3, e_4)$ where
\[
\begin{aligned}
e_1 &= \Phi_4^{-1}(1, 0, 0, 0), \\
e_2 &= \Phi_4^{-1}(0, 1, 0, 0), \\
e_3 &= \Phi_4^{-1}(0, 0, 1, 0), \\
e_4 &= \Phi_4^{-1}(0, 0, 0, 1).
\end{aligned}
\]  
(142)

The dual basis of $V^*$ is $(e_1^*, e_2^*, e_3^*, e_4^*)$ where
\[
\begin{aligned}
e_1^* &= w_1(E)^3, \\
e_2^* &= w_1(E)(w_2(E) \text{ mod } 2), \\
e_3^* &= \beta(2, 1)w_2(E), \\
e_4^* &= w_1(E)w_3(TN^3)^2.
\end{aligned}
\]  
(143)

We aim to construct a linear map $G : V \to W$, then we can find the image of $f_i^*$ under $G^*$, which is the desired 3d term reduced from the 5d term $f_i^*$.

4. Construction of maps: $F$ for $N = 2$ and $G$ for $N = 4$

To construct the map $F$ and $G$, we first discuss the background fields for the global symmetries in both 4d/5d and 2d/3d.

1. For $N = 2$, because our bordism calculations are only for the cases where time reversal commutes with other symmetries, the time reversal symmetry in 5d is identified as $CT$, and the background field is $w_1(TM^5)$. However, for $N = 2$, the charge conjugation symmetry is trivial. In 3d, because both $T, C$ and $CT$ commute with all other symmetries, we identify time reversal symmetry as $T$ whose background is $w_1(TN^3)$, and the charge conjugation symmetry in 3d is identified as $C$ whose background is $w_1(E)$.

2. For $N = 4$, because our bordism calculations are only for the cases where time reversal commutes with other symmetries, in 5d, we identify $CT$ as time reversal symmetry whose background field is $w_1(TM^5)$, and the charge conjugation is denoted as $C$ whose background field is $A$. In 3d, for the same reason as above, we identify $CT$ as time reversal symmetry whose background field is $w_1(TN^3)$, and the charge conjugation is denoted as $C$ whose background field is $w_1(E)$.

We proceed to discuss the reduction rules for the symmetry background fields. We first focus on $N = 2$.

1. $CT$ in 5d reduces to $CT$ in 3d. Correspondingly the background field for time reversal in 5d $w_1(TM^5)$ reduces to $w_1(TN^3) + w_1(E)$. This is because restricting $w_1(TM^5)$ to a submanifold should be of the form $w_1(TN^3) + (\ldots)$. To determine $(\ldots)$, we notice that the 2d $\mathbb{CP}^1$ model with a theta term at $\theta = \pi \text{ mod } 2\pi$ (but not other $\theta$ except $\theta = 0 \text{ mod } 2\pi$), respects the charge conjugation symmetry. On the other hand, the 4d YM with a theta term at $\theta = \pi \text{ mod } 2\pi$ (but not other $\theta$ except $\theta = 0 \text{ mod } 2\pi$), respects the $CT$ symmetry. We thus demand $w_1(TM^5)$ reduces to $w_1(TN^3) + w_1(E)$.

Formally, we need to find an embedding $\iota : N^3 \hookrightarrow M^5$ such that
\[
w_1(TM^5) |_{N^3} = w_1(TN^3) + w_1(E).
\]  
(144)

In summary, we find the symmetry reduction
\[
5d : CT \to 3d : CT.
\]  
(145)

2. Following Ref. [36], the twisted boundary condition by the center symmetry (which is $B$) for the Yang-Mills is reduced to the twisted boundary condition by the $Z_N$ global symmetry (which is $w_2(E) + \bar{K}_1w_1(E)^2$ for $N = 2$) of $\mathbb{CP}^{N-1}$. Here we find two possibilities of the reduction, labeled by $K_1 \in Z_2$. Formally, we need to find $B \in H^2(M^5, Z_2)$ such that
\[
B |_{N^3} = w_2(E) + \bar{K}_1w_1(E)^2
\]  
(146)

We will determine $\bar{K}_1$ at the end of Sec. VI.B.

After choosing an embedding $\iota : N^3 \hookrightarrow M^5$, we have the Poincaré dual PD : $H^2(M^5, \mathbb{Z}) \sim H_3(M^5, \mathbb{Z})$, we denote $B' = \text{PD}^{-1}(\iota_*([N^3]))$. We require that $B'$ is the cup product of two different degree-1 cohomology classes. We impose this condition because, as discussed at the beginning of Sec. VI, the 3d submanifold $N^3$ is reduced from the 5d manifold $M^5$ by a 2-torus. Suppose the normal bundle of the embedding $N^3 \hookrightarrow M^5$ is $\nu$, then $\nu$ is the direct sum of two different line bundles, and the condition $B' |_{N^3} = w_2(\nu)$ is satisfied. We also require that $\text{Sq}^1 B = \text{Sq}^1 B'$ to ensure that the map $F$ in Lemma VI.1 is well-defined.

Now we construct the map $F$, which we organize as the following lemma.

**Lemma VI.1.** The map $F$ is defined as
\[
F : \Omega^3_0(BO(3)) \to \Omega^3_0(B^2\mathbb{Z}_2) / \Omega^3_0(BO(2)^2\mathbb{Z}_2) / \Omega^3_0(BO(3)) \to \Omega^3_0(BO(3)) / \Omega^3_0(B^2\mathbb{Z}_2) / \Omega^3_0(B^2\mathbb{Z}_2).
\]  
(147)

sending $(N^3, E) \in \Omega^3_0(BO(3))$ to $[(M^5, B)] \in \Omega^3_0(B^2\mathbb{Z}_2) / \Omega^3_0(B^2\mathbb{Z}_2)$. $F$ is well defined.

**Proof.** For the map $F$ to be well-defined, the trivial element in $\Omega^3_0(BO(3))$ must be mapped to the trivial element in $\Omega^3_0(B^2\mathbb{Z}_2) / \Omega^3_0(B^2\mathbb{Z}_2)$. Hence we need to prove that
\[
\begin{aligned}
\int_{N^3} w_1(E)^3 &= 0, \\
\int_{N^3} w_1(E)w_2(E) &= 0, \\
\int_{N^3} w_1(E) &= 0, \\
\int_{N^3} w_3(TN^3)^2 &= 0.
\end{aligned}
\]  
(148)

implies
\[
\begin{aligned}
\int_{M^5} \text{Sq}^1 B + \text{Sq}^2 \text{Sq}^1 B &= 0, \\
\int_{M^5} w_1(TM^5)^2 \text{Sq}^1 B &= 0
\end{aligned}
\]  
(149)
Since \( TM^5 \mid_{N^3} = TN^3 \oplus \nu \) where \( \nu \) is the normal bundle, by the Whitney sum formula for the total Stiefel-Whitney class, we have \( w(TM^5) \mid_{N^3} = w(TM^3)w(\nu) \). So \( w_1(TM^5) \mid_{N^3} = w_1(TM^3) + w_1(\nu) \) implies \( w_1(E) = w_1(\nu) \), and \( w_2(TM^5) \mid_{N^3} = w_2(TM^3) + w_2(TM^3)w_1(\nu) + w_1(TM^3)w_2(\nu) \), if \( w_1(E)w_1(TM^3) = 0 \), then \( w_2(TM^3)w_1(\nu) = 0 \) since \( w_2(TM^3) = w_1(TM^3)^2 \). Also since \( w_3(TM^3) = 0 \), so \( w_3(TM^5) \mid_{N^3} = w_1(TM^3)w_2(\nu) \).

Consider \( E \oplus \nu \), by splitting principle, the total Stiefel-Whitney class \( w(E \oplus \nu) \) is the product of linear factors (of the form \( 1 + x \) where \( x \) is a degree-1 cohomology class). Since \( w_1(E \oplus \nu) = w_1(E) + w_1(\nu) = 0 \), so \( w_2(E \oplus \nu) \) is a sum of squares.\(^\text{11}\) Since \( w_2(E \oplus \nu) = w_2(E) + w_2(\nu) + w_1(E)w_1(\nu) = w_2(E) + w_1(\nu)^2 \), we have \( S^1(E) = S^1(\nu) \). By Wu formula, we have \( w_1(TM^3)w_2(\nu) = w_1(TM^3)w_2(E) \).

By Wu formula, we have \( S^1(E) = S^0(E) \). We have \( \int_M w_2(TM^5)S^1B' = \int_M w_5(TM^5)^2S^1B' = \int_M w_5(TM^5)S^1B' = \int_M w_5(TM^3)S^1B' = 0 \).

By Wu formula, we have \( S^1(E) = S^0(E) \). We have \( \int_M w_2(TM^5)S^1B' = \int_M w_3(TM^5)S^1B' = \int_M w_3(TM^5)S^1B' = \int_M w_3(TM^5)S^1B' = 0 \).

By Wu formula, we have \( S^1(E) = S^0(E) \). We have \( \int_M w_2(TM^5)S^1B' = \int_M w_3(TM^5)S^1B' = \int_M w_3(TM^5)S^1B' = \int_M w_3(TM^5)S^1B' = 0 \).

Since we impose the condition \( S^1(B) = S^0(B) \), we have proved the statement. \( \square \)

We further discuss the reduction rules for the symmetry background fields in the case \( N = 4 \).

1. \( \mathcal{C}T \) in 5d reduces to \( \mathcal{C}T \) in 3d. Correspondingly the background field for time reversal in 5d \( w_1(TM^3) \) reduces to \( w_1(TM^3) \). Moreover, the charge conjugation \( \mathcal{C} \) in 4d YM is a symmetry for any \( \theta \), while \( T \) in 2d CP\(^1\) model is also a symmetry for any \( \theta \). Thus we demand that \( A \) in 5d reduces to \( w_1(TM^3) + w_1(E) \) (which is the background for \( \mathcal{C} \)) in 3d. Formally, we need to find an embedding \( : N^3 \hookrightarrow M^5 \) and \( A \in H^1(M^5, \mathbb{Z}_2) \) such that \( w_1(TM^5) \mid_{N^3} = w_1(TM^3) \).

\[ A \mid_{N^3} = w_1(TM^3) + w_1(E). \] \hspace{1cm} (151)

In summary, we find the symmetry reduction

\[ 5d : (\mathcal{C}T, C) \rightarrow 3d : (\mathcal{C}T, T). \] \hspace{1cm} (152)

2. Following Ref. [36], the twisted boundary condition by the center symmetry (which is \( B \)) for the Yang-Mills is reduced to the twisted boundary condition by the \( \mathbb{Z}_N \) global symmetry (which is \( w_2(E) \)) for \( N = 4 \) of CP\(^{N-1}\). Formally, we need to find \( B \in H^2(M^5, \mathbb{Z}_4) \) such that \( B \mid_{N^3} = w_2(E). \) \hspace{1cm} (153)

After choosing an embedding \( : N^3 \hookrightarrow M^5 \), we have the Poincaré dual PD : \( H^2(M^5, \mathbb{Z}_2) \rightarrow H_3(M^5, \mathbb{Z}_2) \), we denote \( B' = PD^{-1}(\iota_*[N^3]) \). We require that \( B' \) is the cup product of two different degree-1 cohomology classes. We impose this condition because, as discussed at the beginning of Sec. VI, the 3d submanifold \( N^3 \) is reduced from the 5d manifold \( M^5 \) by a 2-torus. We also require that \( \tilde{\beta}(B') = 0 \), and \( w_1(TM^5) \mid_{N^3} = w_1(TM^3) \) implies \( w_2(\nu) = 0 \). We claim that \( \nu \) is a trivial bundle, thus \( M^5 = N^3 \times T^2 \), the 3rd to 6th components of \( \Phi^\gamma_4([M^5, A, B]) \) are zero and \( ([M^5, A, B]) \in W \).

Now we construct the map \( G \), which we organize as the following lemma.

**Lemma VI.2.** The map \( G \) is defined as \( G : \Omega^0_3(B(Z_2 \times PSU(4))) \rightarrow W \subset \Omega^0_3(B(Z_2 \times B^2Z_2, \Omega^0_3(B(Z_2 \times B^2Z_4)) \text{ sending } (N^3, w_1(E), w_2(E)) \in \Omega^0_3(B(Z_2 \times PSU(4))) \text{ to } ([M^5, A, B]) \in W \). \( G \) is well defined.

**Proof.** For the map \( G \) to be well-defined, the trivial element in \( \Omega^0_3(B(Z_2 \times PSU(4))) \) must be mapped to the trivial element in \( W \). Hence we need to prove that

\[ \int_M w_1(E) = 0, \]

\[ \int_M w_2(E) = 0, \]

\[ \int_M \beta(2,4)w_2(E) = 0, \]

\[ \int_M w_1(E)w_1(TM^3)^2 = 0. \] \hspace{1cm} (154)

implies

\[ \int_M B \beta(2,4) = 0, \]

\[ \int_M w_1(TM^3)^2B = 0 \] \hspace{1cm} (155)

\(^{11}\) Suppose the total Stiefel-Whitney class \( w(E \oplus \nu) = (1 + x_1)^n(1 + x_2)^m(1 + x_3)^n, \) then \( w_1(E \oplus \nu) = n_1x_1 + n_2x_2 + n_3x_3 = 0 \) implies \( n_1 = n_2 = n_3 = 0 \) mod 2, so \( n_i = 2k_i \), and \( w_2(E \oplus \nu) = k_1x_1^2 + k_2x_2^2 + k_3x_3^2 \).

\(^{12}\) Every orientable real line bundle is a trivial bundle. Since \( B' \) is the cup product of two different degree-1 cohomology classes, \( \nu \) is the direct sum of two different line bundles, each of them is orientable, thus a trivial bundle.
We have \( \int_{M^4} B' \beta_{(2,4)} B = \int_{N^3} \beta_{(2,4)} w_2(E) = 0 \) and \( \int_{M^4} A w_1(TM^5) \beta' B' = \int_{N^3}(w_1(TN^3) + w_1(E))w_1(TN^3)^2 = 0 \).

Since we impose the conditions \((B + B') \beta_{(2,4)} B = 0\) and \(A w_1(TM^5) (B + B') = 0\), we have proved the statement.

\[ \square \]

### B. From \( \Omega^0_\Sigma(BO(3)) \) to \( \Omega^0_\Sigma(B^2Z_2)/\Omega^0_{\Sigma}(B^2Z_2) \)

In this subsection, we use the linear dual of the map \( F \) defined in Lemma VI.1 to reduce the bordism invariants of \( \Omega^0_\Sigma(B^2Z_2)/\Omega^0_{\Sigma}(B^2Z_2) \) to the bordism invariants of \( \Omega^0_\Sigma(BO(3)) \).

We first consider the 5d cobordism invariants that characterize the 4d SU(2) YM theory’s anomaly. We may also name these 5d invariants as “5d Yang-Mills terms,” “5d terms,” “Yang-Mills terms,” “5d anomaly polynomial of Yang-Mills” or “5d iTQFTs whose boundary can live 4d Yang-Mills.”

Based on the discussions around eqn. (93) in Sec. III B and the Rule 1 in Sec. V, we can safely propose that the 5d Yang-Mills term for \( N = 2 \) is at most

\[ BSq^1 B + Sq^2 Sq^1 B + K_1 w_1(TM^5)^2 Sq^1 B \]

where \( K_1 \in Z_2 = \{0, 1\} \).

Amusingly, Ref. [38] actually derive eqn. (156) based on putting 4d YM on unorientable manifolds, and then turning on background fields. Ref. [38] also gives mathematical and physical interpretations of the \( K_1 \) term, based on the gauge bundle constraint,

\[ w_2(V_{SO(3)}) = B + K_1 w_1(TM^5)^2 + K_2 w_2(TM^5) \]

We should emphasize that our approach in this work to derive this possible term eqn. (156) is sharply distinct from Ref. [38], although we obtain the same result! Although the starting definitions of the \( K_1 \) in eqn. (156) and the \( K_1 \) in eqn. (157) (and in Ref. [38]) are distinct, below we should also derive that the two \( K_1 \) are actually equivalent. Thus, we use the same label for both \( K_1 \).

We show that, using the linear dual of the map \( F \), the 5d Yang-Mills term in eqn. (156) reduces to the anomaly polynomial in 3d in theorem VI.3.

### Theorem VI.3. The 5d anomaly polynomial for the SU(2) YM theory

\[ BSq^1 B + Sq^2 Sq^1 B + K_1 w_1(TM^5)^2 Sq^1 B \]

reduces to the anomaly polynomial of 2d \( \mathbb{C}P^1 \) theory

\[
(\bar{K}_1 + 1)w_1(E)^3 + w_3(E) + K_1(w_1(E)^3 + w_1(E)w_1(TN^3)^2)
\]

\[ = \bar{K}_1 w_1(E)^3 + w_1(TN^3)w_2(V_{SO(3)}) + w_1(E)w_2(V_{SO(3)}) + K_1(w_1(E)^3 + w_1(E)w_1(TN^3)^2) \]

---

**Proof.** We first define the following notations to simplify the proof.

1. \( \alpha \) is the generator of the cohomology \( H^1(\mathbb{R}P^2, Z_2) \).
2. \( \beta \) is the generator of \( H^1(\mathbb{R}P^3, Z_2) \).
3. \( \gamma \) is the generator of \( H^1(S^1, Z_2) \).
4. \( \zeta \) is the generator of \( H^1(\mathbb{R}P^4, Z_2) \).

Recall that the manifold generators of \( \Omega^0_\Sigma(BO(3)) \) are

\[
(N^3, E) = \begin{cases} 
(\mathbb{R}P^3, l_{\mathbb{R}P^3} + 2) & \text{(1)}, \\
(\mathbb{R}P^3, 3l_{\mathbb{R}P^3}) & \text{(2)}, \\
(S^1 \times \mathbb{R}P^2, l_{S^1}, l_{\mathbb{R}P^2} + 1) & \text{(3)}, \\
(S^1 \times \mathbb{R}P^2, l_{S^1} + 2) & \text{(4)}.
\end{cases}
\]

Using the definitions in Sec. VIA, we find \( \Phi_2(\text{(1)}) = (1, 0, 0, 0), \phi_2(\text{(2)}) = (1, 1, 1, 0), \phi_2(\text{(3)}) = (1, 1, 0, 1), \phi_2(\text{(4)}) = (0, 0, 0, 1). \) Thus \( e_1 = \text{(1)} \), \( e_2 = \text{(1)} + \text{(3)} + \text{(4)} \), \( e_3 = \text{(2)} + \text{(3)} + \text{(4)} \), \( e_4 = \text{(4)} \).

---

\[ \square \]

Following the construction of the map \( F \) in Lemma VI.1, we have:\footnote{Note that \( (N^3, E) = (S^1 \times \mathbb{R}P^2, l_{S^1} + 2l_{\mathbb{R}P^2}) \) is also a manifold generator of \( \Omega^1_\Sigma(BO(3)) \), actually \( e_2 = \epsilon_2 + \epsilon_3 + \epsilon_4 \). Since \( w_1(TN^3) = \alpha, w_1(E) = \gamma, w_2(E) = \alpha^2, w_2(E) + K_1 w_1(E)^2 = \alpha^3, w_1(TN^3) + w_1(E) = \gamma + \alpha, \) we have \( M^5 = S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2 \), \( B = \alpha \gamma + \alpha^2 \) where \( B = \gamma \alpha_{1} + \alpha_{2} \) where \( \alpha_{1} = \gamma \alpha + \alpha \). So \( F(\text{(3)}) \) is well-defined.}

1. \( (N^3, E) = (\mathbb{R}P^3, l_{\mathbb{R}P^3} + 2) \)
   
   Since \( w_1(TN^3) = 0, w_1(E) = \beta, w_2(E) = 0, w_2(E) + K_1 w_1(E)^2 = K_1 \beta^2, w_1(TN^3) + w_1(E) = \beta, \) we have \( M^5 = S^1 \times \mathbb{R}P^4 \), \( w_1(TM^5) = \gamma, B = \gamma \zeta + \bar{K}_1 \zeta^2 \) where \( B' = \gamma \zeta \).

2. \( (N^3, E) = (\mathbb{R}P^3, 3l_{\mathbb{R}P^3}) \)
   
   Since \( w_1(TN^3) = 0, w_1(E) = \beta, w_2(E) = \beta^2, w_2(E) + K_1 w_1(E)^2 = (K_1 + 1) \beta^2, w_1(TN^3) + w_1(E) = \beta, \) we have \( M^5 = S^1 \times \mathbb{R}P^4 \), \( w_1(TM^5) = \gamma, B = \gamma \zeta + (\bar{K}_1 + 1) \zeta^2 \) where \( B' = \gamma \zeta \).

3. \( (N^3, E) = (S^1 \times \mathbb{R}P^2, l_{S^1} + l_{\mathbb{R}P^2} + 1) \)
   
   Since \( w_1(TN^3) = \alpha, w_1(E) = \gamma + \alpha, w_2(E) = \gamma \alpha, w_2(E) + K_1 w_1(E)^2 = \gamma \alpha + K_1 \alpha^2, w_1(TN^3) + w_1(E) = \gamma, \)
we have $M^5 = \mathbb{RP}^2 \times \mathbb{RP}^3$, $w_1(TM^5) = \alpha + K_1\beta^2$ where $B' = \alpha \beta$.

3. $(N^3, E) = (S^1 \times \mathbb{RP}^2, f|_{S^1})$;

Since $w_1(TN^3) = \alpha, w_2(T) = \gamma, w_2(E) = 0, w_2(E) + K_1w_1(E)^2 = 0, w_1(TN^3) + w_1(E) = \gamma + \alpha$, we have $M^5 = S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^3, w_1(TM^5) = \alpha_1 + \alpha_2, B = \gamma \alpha_1$

where $B' = \gamma \alpha_1$.

Recall that in Sec. VI A $f_1 = \Phi_1^{-1}(1, 0), f_2 = \Phi_1^{-1}(0, 1)$, we have $F(1) = (K_1 + 1)f_1 + f_2, F(2) = K_1f_1 + f_2, F(3) = (K_1 + 1)f_1, F(4) = f_2$.

So $F(e_1) = (K_1 + 1)f_1 + f_2, F(e_2) = 0, F(e_3) = f_1, F(e_4) = f_2$, and $F^*(f_1) = (K_1 + 1)e_1 + e_3, F^*(f_2) = e_1 + e_4$.

\[ \square \]

Compared with the 5d anomaly polynomial eqn. (158), the 3d anomaly polynomial eqn. (159) contains an extra parameter $K_1$. In the following, we argue that $K_1 = 1$ by comparing with the results in the literature.

1. In Ref. [62], the authors computed the anomaly for the $\mathbb{CP}^4$ model with the $Z^X_2 \equiv Z^X_4$-translation symmetry and the SO(3) symmetry, on an oriented manifold. Denote $T_x$ as the generator of $Z^X_2$. Because in Ref. [62] $T_x$ acts trivially on all physical observables, in their notation they only considered the case $K_1 = 0$. The $Z^X_2$ symmetry background field is $w_1(E)$, and the SO(3) background field is $w_2(V_{SO(3)})$. Ref. [62] found the following anomaly polynomial

\[ w_1(E)^3 + w_1(E)w_2(V_{SO(3)}) \]  

By setting $w_1(TN^3) = 0$ (because Ref. [62] only discussed oriented manifold) and $K_1 = 0$ in eqn. (159) and comparing with eqn. (161), we find $K_1 = 1$.

2. In Ref. [67], the authors computed the anomaly for the $\mathbb{CP}^1$ model with the $Z^X_2 \equiv Z^X_4$-translation symmetry, the SO(3) symmetry, and time reversal symmetry on an unorientable manifold. Using the same notation as above and furthermore we denote the time reversal background as $w_1(TN^3)$, Ref. [67] found the anomaly polynomial

\[ w_1(E)^3 + w_1(E)w_2(V_{SO(3)}) + w_1(TN^3)w_2(V_{SO(3)}) \]  

By setting $K_1 = 0$ in eqn. (159), this again requires $K_1 = 1$.

C. From $\Omega^5_0(B(\mathbb{Z}_2 \ltimes SU(4)))$ to $W \subseteq \Omega^5_0(B(\mathbb{Z}_2 \ltimes SU(4)))/\Omega^5_0(B(\mathbb{Z}_2 \ltimes SU(4)))$

In this subsection, we use the linear dual of the map $G$ defined in Lemma VI.2 to reduce the bordism invariants of $W$ to the bordism invariants of $\Omega^5_0(B(\mathbb{Z}_2 \ltimes SU(4)))$.

We first consider the 5d cobordism invariants that characterize the 4d SU(4) YM theory’s anomaly (abbreviate them as “5d Yang-Mills terms”). Based on the discussions around eqn. (106) in Sec. III E and the Rule 1 and Rule 7 in Sec. V, we can safely propose that the 5d Yang-Mills term for $N = 4$ is at most

\[ B\beta_{(2,4)}B + K'_1w_1(TM^5)^2\beta_{(2,4)}B + K'CA^2\beta_{(2,4)}B \]
\[ + K'CAw_1(TM^5)^2B, \]

where $K'_1, K'_C, K'_{C,1} \in Z_2 = \{0, 1\}, K'_1, K'_C, K'_{C,1}$ are distinct couplings different from the gauge bundle constraint couplings $K_1, K, K_{C,1}$ later in (165).

In Sec. VI A 3, we also mentioned the appearances of $w_1(TM^5)^2\beta_{(2,4)}B$ and $A^2\beta_{(2,4)}B$ for the anomaly of 4d SU(4) YM are unlikely and the full discussion is left for the future work [39]. Following the derivation in Ref. [38], we find that the 5d Yang-Mills term for $N = 4$ is

\[ B\beta_{(2,4)}B + KCAw_1(TM^5)^2B \]

where $K_{C,1}$ is from the gauge bundle constraint

\[ w_2(V_{PSU(4)}) = B + 2K_1w_1(TM^5)^2 + K_2w_2(TM^5) \]
\[ + KCA^2 + KCAw_1(TM^5) \text{ mod 4.} \]  

The detailed derivation will be left for the future work [39].

In the following, we show that, using the linear dual of the map $G$, the 5d Yang-Mills term in eqn. (164) reduces to the anomaly polynomial in 3d in theorem VI.4.

---

**Theorem VI.4.** The 5d anomaly polynomial for the SU(4) YM theory

\[ B\beta_{(2,4)}B + KCAw_1(TM^5)^2B \]

reduces to the anomaly polynomial of 2d CP^3 theory

\[ \beta_{(2,4)}w_2(E) + K_{C,1}w_1(E)w_1(TM^3)^2 = \beta_{(2,4)}w_2(V_{PSU(4)}) + K_{C,1}w_1(E)w_1(TM^3)^2. \]  

\[ \square \]

**Proof.** For simplicity, we define the following notations:

1. $K$ is the Klein bottle.

2. $\alpha'$ is the generator of $H^1(S^1, \mathbb{Z}_4) = \mathbb{Z}_4$.

3. $\beta'$ is the generator of the $\mathbb{Z}_4$ factor of $H^1(K, \mathbb{Z}_4) =$
\[ Z_4 \times Z_2 \] (see Appendix C). Note that \((\beta' \mod 2)^2 = 2\beta(2,4)\beta' = 0\).

4. \(\alpha\) is the generator of \(H^1(\mathbb{R}P^2, Z_2) = Z_2\).

5. \(\gamma\) is the generator of \(H^1(S^1, Z_2) = Z_2\).

Using the definitions in Sec. VI A, recall that the manifold generators of \(\Omega^2_5(B(Z_2 \times PSU(4)))\) are

\[
(N^3, w_1(E), w_2(E)) = \begin{cases}
(\mathbb{R}P^3, \beta, 0) = e_1, \\
(T^3, \gamma_1, \alpha_2^2 \alpha_3^3) = e_2, \\
(S^1 \times K, 0, \alpha' \beta') = e_3, \\
(S^1 \times \mathbb{R}P^3, \gamma, 0) = e_4.
\end{cases}
\] (168)

Following the construction of the map \(G\) in Lemma VI 2, we have:

1. \((N^3, w_1(E), w_2(E)) = (\mathbb{R}P^3, \beta, 0), w_1(TN^3) = 0,\) we have \(M^5 = T^5 \times \mathbb{R}P^3,\) since both \(B\beta(2,4)\beta\) and \(Aw_1(TM^5)\beta\) must vanish on \(M^5,\) so \(G(e_1) = 0.\)

2. \((N^3, w_1(E), w_2(E)) = (T^3, \gamma_1, \alpha_2^2 \alpha_3^3), w_1(TN^3) = 0,\) we have \(M^5 = T^5,\) since both \(B\beta(2,4)\beta\) and \(Aw_1(TM^5)\beta\) must vanish on \(M^5,\) so \(G(e_2) = 0.\)

3. \((N^3, w_1(E), w_2(E)) = (S^1 \times K, 0, \alpha' \beta'),\) \(w_1(TN^3) = \beta' \mod 2,\) we have \(M^5 = K \times T^5,\) \(w_1(TM^5) = \beta' \mod 2,\) \(A = \beta' \mod 2,\) \(B = \alpha_2^2 \beta + \alpha_2^2 \alpha_3^3,\) where \(B' = \alpha_2^2 \alpha_3^3 \mod 2.\) So \(G(e_3) = f_1.\)

4. \((N^3, w_1(E), w_2(E)) = (S^1 \times \mathbb{R}P^2, \gamma, 0), w_1(TN^3) = \alpha,\) we have \(M^5 = T^3 \times \mathbb{R}P^2, w_1(TM^5) = \alpha,\) \(A = \gamma_1 + \alpha, B = \alpha_2 \alpha_3^3,\) where \(B' = \alpha_2 \alpha_3^3 \mod 2.\) So \(G(e_4) = f_2.\)

So \(G^*(f_1^*) = e_3^2, G^*(f_2^*) = e_4^2.\)

Next we can elaborate the new higher anomaly of 4d YM theory in Sec. VII.

VIIL NEW HIGHER ANOMALIES OF 4D SU(N)-YM THEORY

We provide more details on the anomaly of 4d YM theory. We deduce the new higher anomaly of 4d YM theory written in terms of invariants given in Sec. III, and satisfying Rules in Sec. V and following the physical/mathematical 5d to 3d reduction scheme in Sec. VI.

A. SU(N)-YM at \(N = 2\)

Let us formulate the potentially complete \(t\) Hooft anomaly for 4d SU(N)-YM for \(N = 2\) at \(\theta = \pi,\) written in terms of a 5d cobordism invariant in Sec. III.

Based on Rule 3 and Rule 6 in Sec. V, we deduce that 4d anomaly must match 2d \(\mathbb{C}P^1\)-model anomaly's eq. (179) via the sum of following two terms (5d SPTs).

The first term is:

\[
B_2S\Psi B_2 + S\Psi^2 S\Psi B_2 = \frac{1}{2} \bar{w}_1(TM)P_2(B_2).
\] (169)

which is dictated by Rule 1 in Sec. V. (Note that \(S\Psi^2 S\Psi = (B_2 \cup B_2) \cup (B_2 \cup B_2)\). Here \(\bar{w}_1(TM) \in H^1(M, Z_{w_1})\) is the mod 4 reduction of the twisted first Stiefel-Whitney class of the tangent bundle \(TM\) of a 5-manifold \(M\) which is the pullback of \(w_1\) under the classifying map \(M \to BO(5).\) Here \(Z_{w_1}\) denotes the orientation local system, the twisted first Stiefel-Whitney class \(\bar{w}_1 \in H^1(BO(n), Z_{w_1})\) is the pullback of the nonzero element of \(H^1(BO(1), Z_{w_1}) = Z_2\) under the determinant map \(B det : BO(n) \to BO(1).\) Since \(2\bar{w}_1(TM) = 0\) mod 4, \(\bar{w}_1(TM)P_2(B_2)\) is even, so it makes sense to divide it by 2. If \(w_1(TM) = 0,\) then \(Z_{w_1} = \bar{Z}\) and \(H^1(BO(1), Z_{w_1}) = H^1(BO(1), \bar{Z}) = 0,\) so \(\bar{w}_1 = 0.\) Namely, \(\frac{1}{2} \bar{w}_1(TM)P_2(B_2)\) vanishes when \(w_1(TM) = 0.\)

We can derive the last equality of eq. (169) by proving that both LHS and RHS are bordism invariants of \(\Omega^5_2(B^2Z_2)\) and they coincide on manifold generators of \(\Omega^5_2(B^2Z_2).\)

We can also prove that

\[
\beta_{(2,4)}P_2(B_2) = \frac{1}{4} \delta P_2(B_2) \mod 2
\]

\[
= \frac{1}{4} \delta(B_2 \cup B_2 \cup B_2 \cup \delta B_2)
\]

\[
= \frac{1}{4} (\delta B_2 \cup B_2 \cup B_2 \cup \delta B_2 + \delta(B_2 \cup \delta B_2))
\]

\[
= \frac{1}{4} (2B_2 \cup \delta B_2 + B_2 \cup B_2 \cup \delta B_2)
\]

\[
= B_2 \cup (\frac{1}{2} \delta B_2 + \frac{1}{2} \delta B_2) \cup (\frac{1}{2} \delta B_2)
\]

\[
= B_2S\Psi B_2 + S\Psi B_2 + S\Psi S\Psi B_2
\]

\[
= B_2S\Psi B_2 + S\Psi^2 S\Psi B_2.
\] (170)

Here we have used \(\beta_{(2,4)} = \frac{1}{4} \delta \mod 2,\)

\[
\delta (u \cup v) = u \cup v - u \cup u + \delta u \cup v + u \cup \delta v
\] (171)

for 2-cocoin \(u\) and 3-cocoin \(v\) \([81],\) \(S\Psi^1 = \beta_{(2,2)} = \frac{1}{2} \delta \mod 2,\) and \(S\Psi^k \zeta_n = \zeta_n + \zeta_n.\) The first term contains two appear together in order to satisfy Rule 2.

The other term is:

\[
w_1(TM)^2S\Psi B_2.
\] (172)

We also check that the sum of two terms satisfy the Rule 5 in Sec. V. Besides, Rule 7 restricts us to focus on the bordism group \(\Omega^5_2(B^2Z_2)\) and discards other terms involving \(\Omega^5_2(B^2Z_2 \times B^2Z_2).\) Our final answer of 4d anomaly and 5d cobordism/SPTs invariant is combined and given in eq. (206):

\[
B_2S\Psi B_2 + S\Psi^2 S\Psi B_2 + K_1w_1(TM)^2S\Psi B_2.
\] (173)
To our understanding, the whole expression indicates a new higher anomaly for this YM theory, which turns out to be new to the literature.

B. SU(N)-YM at \( N = 4 \)

Let us propose some 't Hooft anomaly for 4d SU(N)-YM at \( N = 4 \) at \( \theta = \pi \), written in terms of a 5d constraint invariant in Sec. III. Here we do not claim to have a complete set of 't Hooft anomaly. As at \( N = 4 \) we need to specify:

1. the gauge bundle constraint (165).

2. the appropriate charge conjugation symmetry background field coupling to YM.

However, we only have a potentially complete gauge bundle constraint (165), but we do not yet know whether we have captured all possible charge conjugation symmetry background field coupling to YM. The second issue will be left in the future work.

Based on Rule 4 in Sec. V, we deduce the 2d \( \mathbb{C}P^3 \)-model anomaly’s eq. (183) generalizing the eq. (179). Based on Rule 3 and Rule 6, we deduce that 4d anomaly must match 2d \( \mathbb{C}P^3 \)-model anomaly’s eq. (183) via the sum of following two terms (5d SPTs). The first term is:

\[
B_2 \beta_{(2,4)} B_2 = \frac{1}{4} \tilde{w}_1(TM) P_2(B_2),
\]

(which is dictated by Rule 1 in Sec. V). Here \( \tilde{w}_1(TM) \in H^1(M, \mathbb{Z}_{8,w_1}) \) is the mod 8 reduction of the twisted first Stiefel-Whitney class of the tangent bundle \( TM \) of a 5-manifold \( M \) which is the pullback of \( \tilde{w}_1 \) under the classifying map \( M \to BO(5) \). Here \( \mathbb{Z}_{w_1} \) denotes the orientation local system, the twisted first Stiefel-Whitney class \( \tilde{w}_1 \in H^1(BO(n), \mathbb{Z}_{w_1}) \) is the pullback of the nonzero element of \( H^1(BO(1), \mathbb{Z}_{w_1}) = \mathbb{Z} \) under the determinant map \( B det : BO(n) \to BO(1) \). Since \( 2\tilde{w}_1 = 0 \), \( \tilde{w}_1(TM) P_2(B_2) \) is divided by 4, so it makes sense to divide it by 4. If \( w_1(TM) = 0 \), then \( \mathbb{Z}_{w_1} = \mathbb{Z} \) and \( H^1(BO(1), \mathbb{Z}_{w_1}) = H^1(BO(1), \mathbb{Z}) = 0 \), so \( \tilde{w}_1 = 0 \). Namely, \( \frac{1}{4} \tilde{w}_1(TM) P_2(B_2) \) vanishes when \( w_1(TM) = 0 \).

We can derive the last equality by proving that both LHS and RHS are bordism invariants of \( \Omega_8^5(B^2\mathbb{Z}_4) \) and they coincide on manifold generators of \( \Omega_8^5(B^2\mathbb{Z}_4) \).

We can also prove that

\[
\beta_{(2,8)} P_2(B_2) = \frac{1}{8} \delta P_2(B_2) \mod 2
\]

\[
= \frac{1}{8} (\delta B_2 \cup B_2 + B_2 \cup \delta B_2)
\]

\[
= \frac{1}{8} (\delta B_2 \cup B_2 + B_2 \cup \delta B_2 + \delta (B_2 \cup \delta B_2))
\]

\[
= \frac{1}{8} (2B_2 \cup \delta B_2 + B_2 \cup \delta B_2)
\]

\[
= B_2 \cup (\frac{1}{4} \delta B_2) + 2(\frac{1}{4} \delta B_2) \cup (\frac{1}{4} \delta B_2)
\]

\[
= B_2 \beta_{(2,4)} B_2 + 2\delta_{(2,4)} B_2 \cup \beta_{(2,4)} B_2
\]

\[
= B_2 \beta_{(2,4)} B_2 + 2 \delta_{(2,4)} B_2 \cup \beta_{(2,4)} B_2
\]

\[
= B_2 \beta_{(2,4)} B_2
\]

which is dictated by Rule 1 in Sec. V. (Note that \( \tilde{B}_2 = B_2 \mod 2 \).) Here we have used \( \beta_{(2,8)} = \frac{1}{8} \delta \mod 2 \).

\[
\delta (u \cup v) = u \cup v - v \cup u + \delta u \cup v + u \cup \delta v
\]

for 2-cochain \( u \) and 3-cochain \( v \). \( \beta_{(2,4)} = \frac{1}{4} \delta \mod 2 \), and \( Sq^2 z_n = z_{n-k} \cup z_n \).

The other term is:

\[
Aw_1(TM)^2 B_2.
\]

We also check that the sum of two terms satisfy the Rule 2 and Rule 5 in Sec. V. By imposing Rule 7, we can rule out thus discard many other 5d terms in the bordism group \( \Omega_8^5(B\mathbb{Z}_2 \times B^2\mathbb{Z}_4) \). In summary, our final answer of 4d anomaly and 5d cobordism/SPTs invariant is combined and given in eq. (207):

\[
B_2 \beta_{(2,4)} B_2 + K_{C,1} Aw_1(TM)^2 B_2.
\]

To our understanding, the whole expression indicates a new higher anomaly for this YM theory, new to the literature.

VIII. NEW ANOMALIES OF 2D \( \mathbb{C}P^{N-1} \)-MODEL

In this section, we provide more details and summarize the anomaly for the \( \mathbb{C}P^{N-1} \)-model. For 2d \( \mathbb{C}P^1 \)-model at \( \theta = \pi \), in theorem VI.3, we find that the 't Hooft anomaly is the combination of the cobordism invariants eq. (115), eq. (117), eq. (119) and eq. (123), which we repeat for readers’ convenience:

\[
2d \mathbb{C}P^1 \text{-model anomaly : } \left[ w_1(E)^3 + w_1(TM) w_2(V_{SO(3)}) + w_1(E) w_2(V_{SO(3)}) + K_1 (w_1(E)^3 + w_1(E) w_1(TM)^2) \right]
\]

\[
= w_3(E) + K_1 (w_1(E)^3 + w_1(E) w_1(TM)^2)^2.
\]

Recall that, under the basis \( \langle w_1(E)^3, w_1(E) w_2(E), w_3(E), w_1(E) w_1(TM)^2 \rangle \), we can express the following cobordism
invariants using eq. (96)

\[
\begin{align*}
\begin{aligned}
  w_1(E) &= (1, 0, 0, 0), \\
  w_1(E)w_2(\mathbb{V}_{SO(3)}) &= (1, 1, 0, 0), \\
  w_1(TN^2)w_2(\mathbb{V}_{SO(3)}) &= (0, 1, 1, 0), \\
  w_1(E)w_1(TN^3) &= (0, 0, 0, 1), \\
  w_3(E) &= (0, 0, 1, 0).
\end{aligned}
\]

To summarize, the overall anomaly of 2d $\mathbb{CP}^1$-model can be expressed as a 3d cobordism invariant/topological term eq. (179), which is $\left(K_1, 0, 1, K_1\right)$ under the basis $(w_1(E)^3, w_1(E)w_2(E), w_3(E), w_1(E)w_1(TN^3)^2)$ of eq. (96).

For 2d $\mathbb{CP}^{N-1}$-model at $\theta = \pi$, at even N, Ref. [67] proposes an important quantity (called $u_3$ in Ref. [67]), which is an element $u_3 \in H^3(BPSU(N) \times Z_2^C, \mathbb{Z}^C)$ as an anomaly for that 2d theory. First we notice that one needs to generalize the second SW class from $w_2 \in H^2(BPSU(2), \mathbb{Z}_2) = \mathbb{Z}_2$ to $\tilde{w}_2 \in H^2(BPSU(N), \mathbb{Z}_N) = \mathbb{Z}_N$. Moreover, there is an additional $\mathbb{Z}_C^N$ twist modifying the PSU(2)-bundle to PSU(N)$\times \mathbb{Z}_C^N$-bundle. In the definition of $u_3 \in H^3(BPSU(N) \times Z_2^C, \mathbb{Z}_C^N)$, $\mathbb{Z}_C^N$, $C$ specifies the symmetry as a charge conjugation $Z_2^C$. This means that $d_{u_3} \neq 0$, but $d_A u_3 = 0$, where $d_A$ is a twisted differential. The construction of these classes is a Bockstein operator for the extension applied to $u_2 \in H^2(BPSU(N) \times Z_2^C, \mathbb{Z}_C^N)$. Eventually, the 3d invariant for the 2d anomaly term of Ref. [67] is $u_3 \in H^3(BPSU(N) \times Z_2^C, \mathbb{Z}_C^N)$. In our setup, we consider $\tilde{w}_3(E) \equiv \tilde{w}_3(V_{PSU(N) \times \mathbb{Z}_2}) \in H^3(BPSU(N) \times Z_2^C, \mathbb{Z}_2) = \mathbb{Z}_2$ here $E$ is the background gauged bundle of $PSU(N) \times \mathbb{Z}_2$.

For $N = 2$, we derive that $\tilde{w}_3(E) = w_3(E) = w_1(E)w_2(E) + w_1(TN)w_2(E)$ in eq. (117). We emphasize that $w_1(TN)$ and $w_0(E)$ are the symmetry background fields for $T$ and $C$ respectively.

For $N = 4$, in theorem VI.4, we find that the 3d anomaly polynomial is

\[
\beta_{(2,4)}w_2(E) + K_{C,1}w_1(E)w_1(TN^3)^2 = \beta_{(2,4)}w_2(V_{PSU(4)}) + K_{C,1}w_1(E)w_1(TN^3)^2. \tag{180}
\]

Let us remind the notations explained in Sec. III F. When $N = 4$, we have $E$ is the principal $Z_2 \times PSU(4)$ bundle, while $w_2(E) \in H^2(M, \mathbb{Z}_4)$ is a $\mathbb{Z}_4$-valued second twisted cohomology class, $w_1(E) \in H^1(M, \mathbb{Z}_2)$ is a group homomorphism $\pi_1(M) \to \text{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$.

We emphasize that when $K_{C,1} = 0$, our anomaly polynomial

\[
\beta_{(2,4)}w_2(E) = \beta_{(2,4)}w_2(V_{PSU(4)}) \tag{181}
\]

is consistent with the result in Ref. [67]. They derived that the anomaly polynomial is

\[
\tilde{w}_3(E) = \frac{1}{2}w_1(E)w_2(E) + \beta_{(2,4)}w_2(E) = \frac{1}{2}w_1(E)w_2(E) + \frac{1}{2}\tilde{w}_1(TN^3)w_2(E). \tag{182}
\]

Compared with eqn. (181) there is an additional $\frac{1}{2}w_1(E)w_2(E)$ in eqn. (182). This superficial mismatch is because $w_1(TN^3)$ is identified as the background field for $CT$ in our work, while $T$ in Ref. [67]. If we replace $w_1(TN^3)$ in eqn. (181) by $w_1(TN^3) + w_0(E)$, we correctly obtain eqn. (182).

Based on Rule 4 in Sec. V, we propose that 3d invariant for the anomaly of 2d $\mathbb{CP}^3$-model is:

\[
2d \mathbb{CP}^3\text{-model anomaly : } \beta_{(2,4)}w_2(E) + K_{C,1}w_1(E)w_1(TN^3)^2
\]

\[
= \frac{1}{2}\tilde{w}_1(TN^3)w_2(E) + K_{C,1}w_1(E)w_1(TN^3)^2 \tag{183}
\]

\[
= \frac{1}{2}w_1(TN^3)w_2(V_{PSU(4)}) + K_{C,1}w_1(E)w_1(TN^3)^2.
\]

We should mention our anomaly term contains the previous anomaly found in the literature for more generic even $N$ [65, 67, 82].

IX. SYMMETRIC TQFT, SYMMETRY-EXTENSION AND HIGHER-SYMMETRY ANALOG OF LIEB-SCHULTZ-MATTIS THEOREM

Since we know the potentially complete 't Hooft anomalies of the above 4d SU(N)-YM and 2d $\mathbb{CP}^{N-1}$-model at $\theta = \pi$, we wish to constrain their low-energy dynamics further, based on the anomaly-matching. This thinking can be regarded as a formulation of a higher-symmetry analog of “Lieb-Schultz-Mattis theorem” [83] [84].” For example, the consequences of low-energy dynamics, under the anomaly saturation can be:
1. Symmetry-breaking:
   - (say CT- or T-symmetry or other discrete or continuous $G$-symmetry breaking).

2. Symmetry-preserving:
   - Gapless, conformal field theory (CFT),
   - Intrinsic topological orders.
   - Degenerate ground states.

3. Symmetry-extension [14]: Symmetry-extension is another exotic possibility, which does not occur naturally without fine-tuning or artificial designed, explained in [14]. However, symmetry-extension is a useful intermediate step, to obtain another earlier scenario: symmetry-preserving TQFT, via gauging the extended-symmetry.

Recently Lieb-Schultz-Mattis theorem has been applied to higher-form symmetries acting on extended objects, see [85] and references therein.

In this section, we like to ask, whether it is possible to have a fully symmetry-preserving TQFT to saturate the higher anomaly we discussed earlier, for 4d SU(N)-YM and 2d CP^{N-1}-model? We use the systematic approach of symmetry-extension method developed in Ref. [14]. We will consider its generalization to higher-symmetry-extension method, also developed in our parallel work Ref. [40].

We will trivialize the 4d and 2d 't Hooft anomaly of 4d YM and 2d CP^{N-1} models (again we abbreviate them as 5d Yang-Mills and 3d CP^{N-1} terms) by pullback the global symmetry to the extended symmetry. If the pullback trivialization is possible, then it means that we can use the “symmetry-extension” method of [14] to construct a fully symmetry-preserving TQFT, at least as an exact solvable model.

In below, when we write an induced fiber sequence:

$$[BK] \to BG' \to BG,$$  \hspace{1cm} (184)

we mean that $[BK]$ is the extension from a finite group $K$ with the classifying space $BK$, while $BG$ is the classifying space of the original full symmetry $G$ (including the higher symmetry). Moreover, the bracket in $[BK]$ means that the full-anomaly-free $K$ can be dynamically gauged to obtain a dynamical $K$ gauge theory as a symmetry-$G$ preserving TQFT, see [14].

However, as noticed in [14, 38, 40], there are a few possibilities of dynamical fates for the attempt to construct a theory via the symmetry-extension eqn. (184):

I. **No $G'$-symmetry extended gapped phase:**

$G'$-symmetry extended gapped phase is impossible to construct via eqn. (184). Namely, a $G$-anomaly cannot be trivialized by pulling back to be $G'$-anomaly free. Although we cannot prove that the symmetry-preserving gapped phase is impossible in general (say, beyond the symmetry-extension method of [14]), the recent works [40, 90, 91] suggest a strong correspondence between “the impossibility of symmetric gapped phase” and “the non-existence of such $G'$.”

II. **$G'$-symmetry extended gapped phase:**

$G'$-symmetry extended gapped phase can be constructed via eqn. (184). Namely, there exist certain $G'$, such that a $G$-anomaly can be trivialized by pulling back to be $G'$-anomaly free. However, there are at least two possible fates after dynamically gauging $K$:

(a) $G$-spontaneously symmetry-breaking (SSB) phase:

After dynamically gauging $K$, the $G$-symmetry would be spontaneously broken.

(b) **Anomalous $G$-symmetry-preserving $K$-gauge phase:**

After dynamically gauging $K$, the $G$-symmetry would not be broken, thus we obtain a $G$-symmetry-preserving and dynamical $K$-gauge TQFT.

In this work, we will mainly focus on determining whether the phases can be $G$-symmetry extended gapped phase (namely the phase II) or not (namely the phase I). If the $G'$-symmetry extended gapped phase is possible, we will comment briefly about the dynamics after gauging the extended $K$: Whether it will be spontaneously symmetry-breaking (namely the phase (a)) or symmetry-preserving (namely the phase (b)). Further detail discussions about the fate of 4d SU(N)_{\theta=\pi} YM dynamics of these phases are pursuit recently in Ref. [38] and [90, 91].

The new ingredient and generalization here we need to go beyond the symmetry-extension method of [14] are:

1. **Higher-symmetry extension:** We consider a higher group $G$ or higher classifying space $BG$.
2. **Co/Bordism group and group cohomology of higher group $G$ or higher classifying space $BG$**

Another companion work of ours [40] also implements this method, and explore the constraints on the low energy dynamics for adjoint quantum chromodynamics theory in 4d (adjoint QCD).

We first summarize the mathematical checks, and then we will explain their physical implications in the end of this section and in Sec. XI.
In the following subsections, we will not directly present quantum Hamiltonian models involving these higher-group cohomology cocycles and Stiefel-Whitney classes. Nonetheless, we believe that it is fairly straightforward to generalize the quantum Hamiltonian models of \cite{93-95} to obtain lattice Hamiltonian models for our Sec. IX A, Sec. IX B, Sec. IX C and Sec. IX D below. A sketch of the design of the lattice Hamiltonian models can be found in Ref. \cite{38}.

A. $\Omega_2^G (B^2 \mathbb{Z}_2)$: $Z_{4,[1]}$-symmetry-extended but $Z_{2,[1]}$-spontaneously symmetry breaking

We consider $B_2 S(q^1 B_2 + S^2 S(q^1 B_2 + K_1 w_1(TM)^2) S^1 B_2$ of eq. (173) and eq. (206) for 4d SU(N)$_{g=-}$-YM’s anomaly at $N = 2$.

Since $S^2 S(q^1 B_2 = (w_2(TM) + w_2(TM)) S^1 B_2$ and $S^1 B_2$ can be trivialized by $B^2 Z_4 \to B^2 Z_2$ since when $B_2 = B_2 \mod 2$, $B_2 : M \to B^2 Z_4$, and $S^1 B_2 = 2B_{[2,4]} B_2 = 0$ (see Appendix A).

So $B_2 S(q^1 B_2 + S^2 S(q^1 B_2 + K_1 w_1(TM)^2) S^1 B_2$ can be trivialized via

$$[B^2 Z_{2,[1]}] \to BO(d) \times B^2 Z_{4,[1]} \to BO(d) \times B^2 Z_{2,[1]},$$

(185)

which we shorthand the above induced fibration as

$$[B^2 Z_{2,[1]}] \to BG' \to BG.$$  

(186)

Given

$$\omega^G = B_2 S(q^1 B_2 + S^2 S(q^1 B_2 + K_1 w_1(TM)^2) S^1 B_2 = (B_2 + (1 + K_1) w_1(TM)^2) S^1 B_2,$$

(187)

we find that pulling back $G$ to $G'$, we need to solve that

$$\omega^G = \delta \beta_4'$$

(188)

with the split cohomology solution

$$\beta_4' = (B_2 + (1 + K_1) w_1(TM)^2 + w_2(TM)) \cup \gamma_2'$$

(189)

Here we define that $\gamma_2'$ satisfies

$$S_1 B_2' = \delta \gamma_2'$$

(190)

with a solution

$$\gamma_2' (g') = g'^2 - g' \mod 2, \quad \gamma_2 \in C^2(B^2 Z_4, Z_2).$$

(191)

with $g' \in Z_4$. For a 2-simplex/2-plaquett $ijk$, let $g' = d_{ijk}$, so $\gamma_2(g') = (g'^2 - g')/2 \mod 2$, where $g' \in Z_4$ assigned on a 2-simplex, while $\gamma_2(g')$ maps the input $g' \in Z_4$ to the output $Z_2$-valued cochain in $C^2(B^2 Z_4, Z_2)$. This boils down to simply show eqn. (190) $S_1 B_2 = B_2 \cup B_2$ on a 3-simplex say with vertices 0-1-2-3 can be split into 2-cochains $\gamma_2$ in the following way:

$$\delta(\gamma_2)(g') = -\gamma_2(g_01, 2) + \gamma_2(g_01, 3) - \gamma_2(g_02, 3) + \gamma_2(g_12, 3) = -\gamma_2(g_0) + \gamma_2(g_0) - \gamma_2(g_0) + \gamma_2(g_0 - g_0 + g_0) = (g_0 + g_0)(g_0 + g_0) \mod 2 = S_1 B_2 (r(g')).$$

(192)

Therefore, we can also show eqn. (189) that $\beta_4' = (B_2 + (1 + K_1) w_1(TM)^2 + w_2(TM)) \cup \gamma_2'$ via the above given $\gamma_2$.

Using the data, $\omega^G$, we can construct a $G'$-symmetry extended gapped phase (namely the phase II). Using the pair of the above data, $\omega^G$ and $\beta_4'$, we also hope to construct the 4d fully symmetry-preserving TQFT with an emergent 2-form $Z_2$ gauge field (given by $\beta_4'$ and via the gauging the 1-form $Z_{2,[1]}$-symmetry) living on the boundary of 5d SPT (given by $\omega_2^G$). However, it turns out that gauging $K$ results in $G$-spontaneously symmetry breaking (SSB in 1-form $Z_2$, namely the phase (a)). The SSB phase agrees with the analysis in Sec. 8 of Ref. \cite{38} and \cite{90, 91}.

B. $\Omega_3^G (BO(3))$: $Z_2$-symmetry-extended but $Z_2$-spontaneously symmetry breaking

We consider $w_1(E)^3 + w_1(TM)^2 w_2(V_{SO(3)}) + w_1(E) w_2(V_{SO(3)}) + K_1 w_1(E)^3 + w_1(E) w_1(TM)^2$ of eq. (179) and eq. (203) for 2d CP$^{N-1}$ model’s anomaly at $N = 2$.

Since $w_2(V_{SO(3)})$ can be trivialized in SU(2) = Spin(3). Also $w_1(E)^3$ can be trivialized by

$$Z_2 \to Z_2^C,$$

and since $S^2 w_1(E) = (w_2(TM)^2 + w_2(TM)^2) w_1(E) = 0, w_1(E) w_1(TM)^2 = w_1(E) w_2(TM)^3$ can be trivialized by

$$\text{Pin}^+ (d) \to O(d).$$

In summary, $w_1(E)^3 + w_1(TM)^2 w_2(V_{SO(3)}) + w_1(E) w_2(V_{SO(3)}) + K_1 w_1(E)^3 + w_1(E) w_1(TM)^2$ can be trivialized via an induced fiber sequence:

$$[B(Z_2)^3] \to B \text{Pin}^+ (d) \times BSU(2) \times \mathbb{Z}_4^C \to \text{BO}(d) \times B \text{PSU}(2) \times \mathbb{Z}_4^C.$$

(193)

The above shows that the anomaly can be trivialized in $G'$ = $\text{Pin}^+ \times SU(2) \times Z_2^C$, we can construct a $G'$-symmetry extended gapped phase (namely the phase II). However, it turns out that gauging $K$ results in $G$-spontaneously symmetry breaking (SSB in 0-form symmetry here, namely the phase (a)). The SSB phase agrees with the analysis in Appendix A.2.4 of Ref. \cite{14} and \cite{90, 91}.

C. $\Omega_3^G (BZ_2 \times BZ_4)$: $Z_4$-symmetry-extended but $Z_4$-spontaneously symmetry breaking

We consider $B_2 S(2,4) B_2 + K C_{1} A w_1(TM)^2 B_2$ of eq. (178) and eq. (207) for 4d SU(N)$_{g=-}$-YM’s anomaly at $N = 4$. 

Notice $\beta_{(2,4)}B_2$ can be trivialized by $B^2Z_8 \to B^2Z_4$, and notice that $B_2 = B_2'$ mod 4, $B_2' : M \to B^2Z_8$, $\beta_{(2,4)}B_2 = 2\beta_{(2,8)}B_2' = 0$ (see Appendix A). Since $w_1(TM^3)^2$ is trivialized in the group $E(d) \subset O(d) \times Z_4$ defined in [26] which consists of the pairs $(A, j)$ with $\det A = j^2$.

So $B_2\beta_{(2,4)}B_2 + K_{C,1}Aw_1(TM)^2B_2$ can be trivialized via an induced fiber sequence:

$$[BZ_2 \times B^2Z_{2,[1]}] \to BE(d) \times BZ^C_2 \times B^2Z^C_{2,[1]} \to BO(d) \times BZ^C_2 \times B^2Z^C_{2,[1]}.$$  (194)

The above shows that the anomaly can be trivialized in $\mathcal{G}$, we can construct a $\mathcal{G}'$-symmetry extended gapped phase (namely the phase II). However, it turns out that gauging $K$ results in $G$-spontaneously symmetry breaking (SSB in 1-form $Z^C_{2,[1]}$ symmetry here, namely the phase (a)). The SSB phase agrees with the analysis in Appendix A.2.4 of Ref. [14] and [90, 91]. It also agrees with the fact found in Ref. [14] that the 1+1D symmetry-preserving bosonic TQFT is not robust against local perturbation, thus this TQFT flows to the SSB phase.

D. \(\Omega^0_B(B(Z_2 \times PSU(4)))\): $Z^T_4 \times PSU(4)$-symmetry-extended but $Z^S_4 \times PSU(4)$-spontaneously symmetry breaking

We consider the 3d term eq. (183) and eq. (204) for 2d $\mathbb{CP}^{N-1}_{\theta = \pi}$-model’s anomaly at $N = 4$: $\beta_{(2,4)}w_2(E) + K_{C,1}w_1(E)w_1(TN^3)^2 = \frac{1}{2}w_1(TN^3)^2w_2(V_{PSU(4)}) + K_{C,1}w_1(E)w_1(TN^3)^2$.

Since there is a short exact sequence of groups: $1 \to Z_4 \to Z^C_2 \times SU(4) \to Z^S_4 \times PSU(4) \to 1$, we have an induced fiber sequence: $BZ_4 \to B(Z^C_2 \times SU(4)) \to B(Z^S_4 \times PSU(4)) \cong BZ^S_4$, so $w_2(V_{PSU(4)})$ can be trivialized by

$$B(Z^S_4 \times PSU(4)) \to B(Z^C_2 \times PSU(4)).$$

Also since $Sq^2w_1(E) = 0$, $w_1(E)w_1(TN^3)^2 = w_1(E)w_2(TN^3)$ can be trivialized by

$$\text{Pin}^+(d) \to O(d).$$

So $\beta_{(2,4)}w_2(E) + K_{C,1}w_1(E)w_1(TN^3)^2$ can be trivialized via an induced fiber sequence:

$$[BZ_2 \times BZ_4] \to \text{BPin}^+(d) \times B(Z^C_2 \times SU(4)) \to BO(d) \times B(Z^C_2 \times PSU(4)).$$  (195)

The above shows that the anomaly can be trivialized in $G' = \text{Pin}^+ \times (Z^C_2 \times SU(4))$, we can construct a $G'$-symmetry extended gapped phase (namely the phase II). However, it turns out that gauging $K$ results in $G$-spontaneously symmetry breaking (SSB in 0-form symmetry here, namely the phase (a)). The SSB phase agrees with the analysis in Appendix A.2.4 of Ref. [14] and [90, 91]. It also agrees with the fact found in Ref. [14] that the 1+1D symmetry-preserving bosonic TQFT is not robust against local perturbation, thus this TQFT flows to the SSB phase.

E. Summary on the fate of dynamics

In summary, in this section, for all examples Sec. IX A, Sec. IX B, Sec. IX C and Sec. IX D, we have found that there exists such a finite $K$ extension such that the $G$-anomaly becomes $G'$-anomaly free, via the pull back procedure of eqn. (184).

Namely, we can obtain various symmetry-$G$ extended TQFTs (namely the phase II) to saturate (higher) ’t Hooft anomalies of YM theories and $\mathbb{CP}^{N-1}_{\theta = \pi}$-model, via the $[BK]$ extension to a higher-symmetry $G'$ or a higher-classifying space $BG'$.

However, when $K$ is dynamically gauged to obtain a dynamical $K$ gauge topologically ordered TQFT, thanks to a caveat in footnote 15, we find that the above particular examples of 0-form-symmetric 2d TQFT and 1-form-symmetric 4d TQFT become $G$-spontaneously symmetry breaking (SSB in 0-form symmetry for 2d and SSB in 1-form symmetry for 4d). Namely, dynamically gauging $K$ result in the symmetry-breaking phase (a) in our examples. We do not obtain symmetry-preserving gapped phase (b) in the end.

X. MAIN RESULTS SUMMARIZED IN FIGURES

A. SU(N)-YM and $\mathbb{CP}^{N-1}_{\theta = \pi}$-model at $N = 2$

In Fig. 3, we organize the $N = 2$ case of 4d anomalies and 5d topological terms of 4d SU(2)$_{\theta = \pi}$ YM theory (these 5d terms are abbreviated as “5d YM terms”), as well as the 2d anomalies and 3d topological terms of 2d $\mathbb{CP}^1_{\theta = \pi}$-model (these 3d topological terms are abbreviated as “3d $\mathbb{CP}^1_{\theta = \pi}$ terms”).

Based on the discussions around eq. (93) in Sec. III B and the Rule 1 in Sec. V, we proposed that the 5d Yang-Mills
5d higher-SPT/cobordism invariant capturing the 4d higher 't Hooft anomaly of 4d SU(2) YM:

\( (BSq^1 B + Sq^2 Sq^1 B) + K_1 w_1(TM^5)^2 Sq^1 B = \frac{1}{2} \tilde{w}_1(TM) P_2(B_2) + K_1 w_1(TM^5)^2 Sq^1 B, \)

(196)

where \( K_1 \in \mathbb{Z}_2 = \{0, 1\}. \)

Amusingly, recently Ref. [38] derived this precise anomaly based on a different method: putting 4d YM on unorientable manifolds, and then turning on background \( B \) fields. Ref. [38] also gives mathematical and physical interpretations of the \( K_1 \) term, based on the gauge bundle constraint,

\( w_2(V_{PSU(2)}) = w_2(V_{SO(3)}) = B + K_1 w_1(TM^5)^2 + K_2 w_2(TM^5). \)

(197)

Following Ref. [38], \( K_1 \) and \( K_2 \) in eqn. (197) are the choices of the gauge bundle constraint, with \( K_1 \in \{0, 1\} = \mathbb{Z}_2 \) and \( K_2 \in \{0, 1\} = \mathbb{Z}_2 \). The \( K_1 \) is associated with Kramers singlet (\( T^2 = +1 \)) or Kramers doublet (\( T^2 = -1 \)) of Wilson line under time-reversal symmetry. The \( K_2 \) is related to bosonic or fermionic properties of Wilson line under quantum statistics.

FIG. 3. A main result of our work obtains the higher 't Hooft anomalies of 4d SU(N)\( _{\theta=\pi} \) YM and 't Hooft anomalies of 2d \( CP^{N-1}_{\theta=\pi} \) at \( N = 2 \). We express the 4d and 2d anomalies in terms of 5d and 3d SPT/cobordism invariants respectively. We related the 4d and 2d anomalies by compactifying a 2-torus with twisted boundary conditions. Some useful formulas are proved in Sec. VII, VIII and Ref. [34]. — Note that on a closed 5-manifold, we rewrite: \( B_2 Sq^1 B_2 + Sq^2 Sq^1 B_2 = \frac{1}{2} \delta(P_2(B_2)) = \beta(2A)P_2(B_2) = \frac{1}{2} \tilde{w}_1(TM)P_2(B_2), \) \( Sq^2 Sq^1 B_2 = w_2(TM) + w_1(TM)^2 Sq^1 B_2 = (w_3(TM) + w_1(TM)^2)B_2 \), and \( B_2 Sq^1 B_2 = (\frac{1}{2} \tilde{w}_1(TM)P_2(B_2) - (w_3(TM) + w_1(TM)^2)B_2). \)

B. SU(N)-YM and \( CP^{N-1}_{\theta=\pi} \)-model at \( N = 4 \)

In Fig. 4, we organize the \( N = 4 \) case of 4d anomalies and 5d topological terms of 4d SU(4)\( _{\theta=\pi} \) YM theory (these 5d terms are abbreviated as “5d YM terms”):

\( B\beta(2A)B + K'_{1} w_1(TM^5)^2 \beta(2A)B + K' C A^2 \beta(2A)B + K' C A w_1(TM^5)^2 B, \)

(198)

where \( K'_{1}, K'_C, K'_{C,1} \in \mathbb{Z}_2 = \{0, 1\}. \) \( K'_{1}, K'_C, K'_{C,1} \) are distinct couplings different from the gauge bundle constraint couplings \( K_1, K_C, K_{C,1} \) in (165). In Sec. VI A 3, we also mentioned the appearances of \( w_1(TM^5)^2 \beta(2A)B \) and \( A^2 \beta(2A)B \) for the anomaly of 4d SU(4) YM are unlikely and the full discussion is left for the future work [39]. Thus we focus on:

\( B\beta(2A)B + K' C A w_1(TM^5)^2 B. \)

(199)

Amusingly, similar to the discussion in Ref. [38], we find that the 5d Yang-Mills term for \( N = 4 \) is

\( B\beta(2A)B + K' C A w_1(TM^5)^2 B, \)

(200)

where \( K'_{C,1} \) is from the gauge bundle constraint similar to the generalization in Ref. [38],

\( w_2(V_{PSU(4)}) = B + 2(K_1 w_1(TM^5)^2 + K_2 w_2(TM^5) + K_C A^2 + K'_{C,1} A w_1(TM^5)) \mod 4. \)

(201)

The full discussion will be left in a future work [39]. We also organize the \( N = 4 \) case of 2d anomalies and 3d topological terms of 2d \( CP^3_{\theta=\pi} \)-model (these 3d topological terms are abbreviated as “3d \( CP^3_{\theta=\pi} \) terms”) in the bottom part of Fig. 4.
5d higher-SPT/cobordism invariant capturing the 4d higher 't Hooft anomaly of 4d SU(4)$_{\theta=\pi}$ YM:

$$B\beta_{(2,4)}B + K_{C,1} Aw_1(TM^5)^2 B$$

$$= \beta_{(2,4)}w_2(E) + K_{C,1} w_1(E)w_1(TN^3)^2$$

3d SPT/cobordism invariant capturing the 2d 't Hooft anomaly of 2d $CP^3_{\theta=\pi}$-model

FIG. 4. A main result of our work obtains the higher 't Hooft anomalies of 4d SU(N)$_{\theta=\pi}$ YM and 't Hooft anomalies of 2d $CP^{N-1}_{\theta=\pi}$ at $N = 4$. We express the 4d and 2d anomalies in terms of 5d and 3d SPT/cobordism invariants respectively. We related the 4d and 2d anomalies by compactifying a 2-torus with twisted boundary conditions. Some useful formulas are proved in Sec. VII, VIII and Ref. [34] — Note that on a closed 5-manifold, we rewrite: $B_2\beta_{(2,4)} B_2 = \frac{1}{4} \tilde{w}(TM) P_1(B_2)$.

XI. CONCLUSION AND MORE COMMENTS: ANOMALIES FOR THE GENERAL N

In this work, we propose a new and more complete set of 't Hooft anomalies of certain quantum field theories (QFTs): time-reversal symmetric 4d SU(N)-Yang-Mills (YM) and 2d $CP^{N-1}$ models with a topological term $\theta = \pi$, and then give an eclectic “proof” of the existence of these full anomalies (of ordinary 0-form global symmetries or higher symmetries) to match these QFTs. Our “proof” is formed by a set of analyses and arguments, combining algebraic/geometric topology, QFT analysis, condensed matter inputs and additional physical criteria.

We mainly focus on $N = 2$ and $N = 4$ cases. As known in the literature, we actually know that $N = 3$ case is absent from the strict 't Hooft anomaly. The absence of obvious 't Hooft anomalies also apply to the more general odd integer $N$ case (although one needs to be careful about the global consistency or global inconsistency, see [30]). For a general even $N$ integer, it has not been clear about the global consistency or global inconsistency, see [30]). For a general even $N$ integer, it has not been clear about the global consistency or global inconsistency, see [30]).

Physically we follow the idea that coupling the global symmetry of $dd$ QFTs to background fields, we can detect the higher dimensional ($d + 1d$) SPTs/counter term as eq. (2):

$$Z_{\text{QFT}}^{dd}|_{\text{bgd.field}=0} \rightarrow Z_{\text{SPTs}}^{(d+1)d}(\text{bgd.field}) \cdot Z_{\text{QFT}}^{dd}|_{\text{bgd.field} \neq 0},$$

that cannot be absorbed by $dd$ SPTs. (Here, for condensed matter oriented terminology, we follow the conventions of [13].) This underlying $d+1d$ SPTs means that the $dd$ QFTs have an obstruction to be regularized with all the relevant (higher) global symmetries strictly local or onsite. Thus this indicates the obstruction of gauging, which indicates the $dd$ 't Hooft anomalies (See [12–14] for QFT-oriented discussion and references therein).

We comment that the above idea eq. (2) is distinct from another idea also relating to coupling QFTs to SPTs, for example used in [4]: There one couples $dd$ QFTs to $dd$ SPTs/topological terms,

$$Z_{\text{QFT}}^{dd}(A_1, B_2, \ldots) \rightarrow \int [DA_1][DB_2] \ldots Z_{\text{QFT}}^{dd}(A_1, B_2, \ldots) \cdot Z_{\text{SPTs}}^{dd}(A_1, B_2, \ldots),$$

(202)

with the allowed global symmetries, and then dynamically gauging some of global symmetries. A similar framework outlining the above two ideas, on coupling QFTs to SPTs and gauging, is also explored in [21].

Follow the idea of eq. (2) and the QFT and global symmetries information given in Sec. II, we classify all the possible anomalies enumerated by the cobordism theory computed in Sec. III. Then constrained by the known anomalies in the literature Sec. IV, we follow the rules for the anomaly constraint we set in Sec. V and a dimensional reduction method in Sec. VI, we deduce the new anomalies of $2d$-$CP^{N-1}$ models in Sec. VIII and of $4d$ SU(N)-Yang-Mills (YM) in Sec. VII.

A. Anomaly of 2d $CP^1$ model

To summarize the $dd$ anomalies and the $(d+1)$ cobordism/SPTs invariants of the above QFTs,
we propose that a general anomaly formula (3d cobordism/SPT invariant) for 2d $\mathbb{C}P^{N-1}_{g=\pi}$ model at $N = 2$ as:

$$Z_{\mathbb{C}P^{1}_{g=\pi}}^{2d}(w_j(TM), w_j(E), \ldots)Z_{\mathrm{SPTs}}^{3d}$$

$$\equiv Z_{\mathbb{C}P^{1}_{g=\pi}}^{2d}(w_j(TM), w_j(E), \ldots) \exp(i\pi \int_{M^3} (w_1(E)^3 + w_1(TN^3)w_2(V_{\mathrm{SO}(3)}) + w_1(E)w_2(V_{\mathrm{SO}(3)})$$

$$+ K_1(w_1(E)^3 + w_1(E)w_1(TN^3)^2)))$$

$$= Z_{\mathbb{C}P^{1}_{g=\pi}}^{2d}(w_j(TM), w_j(E), \ldots) \exp(i\pi \int_{M^3} (w_3(E) + K_1(w_1(E)^3 + w_1(E)w_1(TN^3)^2))) \ (203)$$

Note that we used and derived that $w_1(E)^3 + w_1(E)w_2(V_{\mathrm{SO}(3)}) = w_1(E)w_2(E)$, $w_1(TN^3)w_2(V_{\mathrm{SO}(3)}) = w_1(TN^3)w_2(E)$

Schematically, the 2d anomaly of $\mathbb{C}P^{1}_{g=\pi}$, written as a qualitative expression for its 3d SPT term, behaves as:

$$\sim (A_x)^3 + T w_2(V_{\mathrm{SO}(3)}) + A_x w_2(V_{\mathrm{SO}(3)}) + K_1((A_x)^3 + A_x T^2) \mod 2.$$  

Here $T$ means the dependence on the time-reversal background field $w_1(TN^3)$. Here $w_2(V_{\mathrm{SO}(3)})$ behaves as a topological term for the $SO(3)$-symmetric 1+1D Haldane chain. From eqn. (119), the $w_1(E)$ behaves as a $\mathbb{Z}_2$-translantion background gauge field written as $w_1(Z_2^3)$ or $A_x$. The $w_1(TN^3)^2$ behaves as a topological term for the $Z_2^3$-symmetric 1+1D Haldane chain.

### B. Anomaly of 2d $\mathbb{C}P^3$ model

We propose that a general anomaly formula (3d cobordism/SPT invariant) for 2d $\mathbb{C}P^{N-1}_{g=\pi}$ model at $N = 4$ as:

$$Z_{\mathbb{C}P^{3}_{g=\pi}}^{2d}(w_j(TM), \bar{w}_j(E), \ldots)Z_{\mathrm{SPTs}}^{3d}$$

$$\equiv Z_{\mathbb{C}P^{3}_{g=\pi}}^{2d}(w_j(TM), \bar{w}_j(E), \ldots) \exp(i\pi \int_{M^3} (\frac{1}{2} \bar{w}_1(TN^3)w_2(V_{\mathrm{PSU}(4)}) + K_{C,1}w_1(E)w_1(TN^3)^2)))$$

$$= Z_{\mathbb{C}P_{g=\pi}}^{2d}(w_j(TM), \bar{w}_j(E), \ldots) \exp(i\pi \int_{M^3} (\frac{1}{2} \bar{w}_1(TN^3)w_2(E) + K_{C,1}w_1(E)w_1(TN^3)^2))) \ (204)$$

Using the fact that $w_1(TN^3)$ is the background field for $CT$, $w_1(TN^3)$ can be schematically written as $T + A_x$. Hence the 2d anomaly of $\mathbb{C}P^{3}_{g=\pi}$, written as a qualitative expression up to a normalization factor for its 3d SPT term, behaves as:

$$\sim (T + A_x)w_2(V_{\mathrm{PSU}(4)}) + K_{C,1}A_x(T + A_x)^2 \mod 2$$

$$= T w_2(V_{\mathrm{PSU}(4)}) + A_x w_2(V_{\mathrm{PSU}(4)}) + K_{C,1}(A_x^3 + A_x T^2) \mod 2 \ (205)$$

Here $T$ means the dependence on the background field for time-reversal symmetry $Z_2^T$. Here $w_2(V_{\mathrm{PSU}(4)})$ behaves as a topological term for a $PSU(4)$-symmetric 1+1D generalized spin chain. From eqn. (119), the $w_1(E)$ behaves as a $\mathbb{Z}_2$-translantion background gauge field written as $w_1(Z_2^3)$ or $A_x$. The $w_1(TN^3)^2$ behaves as a topological term for the $Z_2^T$-symmetric 1+1D Haldane chain.
C. Higher Anomaly of 4d SU(2) Yang-Mills theory

We propose that a general anomaly formula (5d cobordism/higher SPT invariant) for 4d SU(N)_{\theta=\pi}-YM theory at N = 2 as:

\[
Z_{\text{SU}(2)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) Z_{\text{higher-SPTs}}^{4}\equiv \begin{vmatrix} Z_{\text{SU}(2)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) & \exp(i\pi \int_{M^5} (B_2 S q^1 B_2 + S q^2 S q^1 B_2 + K_1 w_1(TM)^2 S q^1 B_2) ) \end{vmatrix}
\]

(206)

Schematically, the 4d anomaly of SU(2)_{\theta=\pi}-YM theory, written as a qualitative expression for its 5d SPT term, behaves as:

\[
\sim (T B B + K_1 T^3 B) \mod 2.
\]

Here T means the dependence on the time-reversal background field w_1(TM). Here B means the dependence on the \( Z_{2,1}^{4} \) background field B.

D. Higher Anomaly of 4d SU(4) Yang-Mills theory

We propose that a general anomaly formula (5d cobordism/higher SPT invariant) for 4d SU(N)_{\theta=\pi}-YM theory at N = 4 as:

\[
Z_{\text{SU}(4)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) Z_{\text{higher-SPTs}}^{4}\equiv \begin{vmatrix} Z_{\text{SU}(4)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) & \exp(i\pi \int_{M^5} (B_2 \beta_{(2,4)} B_2 + K_{C,1} A w_1(TM)^2 B_2) ) \end{vmatrix}
\]

(207)

Schematically the 4d anomaly of SU(2)_{\theta=\pi}-YM theory, written as a qualitative expression for its 5d SPT term, behaves as:

\[
\sim (T B B + K_{C,1} A T^2 B) \mod 2
\]

Here T means the dependence on the time-reversal background field w_1(TM). Here B means the dependence on the \( Z_{4,1}^{4} \) background field B. Here A \equiv A_C means the dependence on the charge conjugation C background field. We will leave the discussions for possible additional anomalies at N = 4 in an upcoming work [39].

E. Higher Anomaly of 4d SU(N) Yang-Mills theory

When N is an even number, we propose that a partial list of the 4d anomaly formula (5d cobordism/higher SPT invariant) the 4d SU(N)_{\theta=\pi}-YM theory

\[
Z_{\text{SU}(N)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) Z_{\text{higher-SPTs}}^{4}\equiv \begin{vmatrix} Z_{\text{SU}(N)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) & \exp(i\pi \int_{M^5} (B_2 \beta_{(2,N)} B_2 + \frac{N}{2} S q^2 \beta_{(2,N)} B_2 + \ldots) \end{vmatrix}
\]

\[
= Z_{\text{SU}(N)\text{YM}_{\theta=\pi}}^{4d}(w_j(TM), A, B_2, \ldots) \exp(i\pi \int_{M^5} (\frac{1}{N} \tilde{w}_1(TM) P_2(B_2) + \ldots))
\]

(208)

Note that we can derive \( B_2 \beta_{(2,N=2^N)} B_2 + \frac{N}{2} S q^2 \beta_{(2,N)} B_2 = \frac{1}{N} \tilde{w}_1(TM) P_2(B_2) \), where Pontryagin square \( P_2 \) : \( H^2(-, \mathbb{Z}_{2^N}) \to H^4(-, \mathbb{Z}_{2^{N+1}}) \). However, when charge conjugation C is an additional \( Z_2 \)-discrete symmetry for SU(N) YM with N > 2, we foresee the additional new anomalies can happen, such as \( A w_1(TM)^2 B_2 \) where \( A \equiv A_C \) is the charge conjugation C background field. We will leave this additional anomalies in an upcoming work [39].

We have commented about the higher symmetry analogy of “Lieb-Schultz-Mattis theorem” in Sec. IX, for ex-
ample, the consequences of low-energy dynamics due to the anomalies. (For the early-history and the recent explorations on the emergent dynamical gauge fields and anomalous higher symmetries in quantum mechanical and in condensed matter systems, see for example, [96] and [97] respectively, and references therein.) In all examples of 4d SU(N)θ=π-YM and 2d CF^{N−1} model in Sec. IX, we find the symmetry-extension method [14] or higher-symmetry-extension method [40] can construct their symmetry-extended gapped phases. However, the dynamical fates of the gauged topologically ordered gapped phases suggest them to flow to become spontaneously symmetry breaking instead of symmetry preserving [38]. The fact that symmetry-preserving gapped phase is not allowed is consistent with Ref. [90 and 91]. We hope to address more about the dynamics in future work.

XII. ACKNOWLEDGMENTS

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Appendix A: Bockstein Homomorphism

In general, given a chain complex C_· and a short exact sequence of abelian groups:

\[ 0 \to A' \to A \to A'' \to 0, \quad (A1) \]

we have a short exact sequence of cochain complexes:

\[ 0 \to \text{Hom}(C_·,A') \to \text{Hom}(C_·,A) \to \text{Hom}(C_·,A'') \to 0. \quad (A2) \]

Hence we obtain a long exact sequence of cohomology groups:

\[ \cdots \to H^n(C_·,A') \to H^n(C_·,A) \to H^n(C_·,A'') \to \cdots, \quad (A3) \]

the connecting homomorphism \( \partial \) is called Bockstein homomorphism.

For example, \( \beta_{(n,m)} : H^*(-,Z_m) \to H^{*+1}(-,Z_n) \) is the Bockstein homomorphism associated with the extension \( Z_n \to Z_{nm} \to Z_m \) where \( m \) is the group homomorphism given by multiplication by \( m \). In particular, \( \beta_{(2,2^n)} = \frac{1}{2^n}\delta \mod 2 \).

Since there is a commutative diagram

\begin{equation}
\begin{array}{ccc}
Z_n & \overset{m}{\longrightarrow} & Z_{nm} \mod m \overset{k}{\longrightarrow} Z_m \\
\downarrow & & \downarrow \\
Z_n & \overset{km}{\longrightarrow} & Z_{km} \mod km \overset{k}{\longrightarrow} Z_{km},
\end{array}
\end{equation}

by the naturality of connecting homomorphism, we have the following commutative diagram:

\[ H^*(-,Z_m) \overset{\beta_{(n,m)}}{\longrightarrow} H^{*+1}(-,Z_n) \]

Hence we prove that

\[ \beta_{(n,m)} = \beta_{(n,km)} \cdot k. \quad (A6) \]

In particular, since \( \text{Sq}^1 = \beta_{(2,2)} \), we have \( \text{Sq}^1 = \beta_{(2,4)} \cdot 2 \). This formula is used in Sec. IX.

Appendix B: Poincaré Duality

An orientable manifold is R-orientable for any ring R, while a non-orientable manifold is R-orientable if and only if R contains a unit of order 2, which is equivalent to having \( 2 = 0 \) in R. Thus every manifold is Z_2-orientable.

**Poincaré Duality:** Let M be a closed connected n-dimensional manifold, R is a ring, if M is R-orientable, let \([M] \in H_n(M,R)\) be the fundamental class for M with coefficients in R, then the map \( PD : H^k(M,R) \to H_{n-k}(M,R)\) defined by \( PD(\alpha) = [M] \cap \alpha \) is an isomorphism for all k.

Appendix C: Cohomology of Klein bottle with coefficients Z_4

In this Appendix, we derive the relation of \( \beta_{(2,4)}x = z \), where x is the generator of the \( Z_4 \) factor of \( H^1(K,\mathbb{Z}_4) = \mathbb{Z}_4 \times \mathbb{Z}_2 \) and z is the generator of \( H^2(K,\mathbb{Z}_2) = \mathbb{Z}_2 \).

One \( \Delta \)-complex structure of Klein bottle is shown in Fig. 5. Let \( \alpha_i \) denote the dual cochain of the 1-simplex \( a_i \).
with coefficients $Z_4$, $\lambda_1$ the dual cochain of the 2-simplex $u_1$ with coefficients $Z_4$, let $\ast$ denote its mod 2 reduction and let $\{ \}$ denote the cohomology class.

**FIG. 5.** One $\Delta$-complex structure of Klein bottle

The 2-simplices and 1-simplices are related by the boundary differential $\partial$ of chains, namely $\partial u_1 = 2a_1 + a_3$, $\partial u_2 = 2a_2 - a_3$, so we deduce that the boundary differential $\delta$ of cochains have the following relation: $\delta_1 = 2\lambda_1$, $\delta_2 = 2\lambda_2$, $\delta_3 = \lambda_1 - \lambda_2$. So we deduce that the cohomology classes $\{\lambda_1\} = \{\lambda_2\}$ are the same.

Since $\delta(a_1 - a_2 - 2a_3) = 0$, $\delta(2a_1) = 0$, $H^1(K, Z_4) = Z_4 \times Z_2$. Let $x = \{a_1 - a_2 - 2a_3\}$, $y = \{2a_1\}$, then $x$ generates $Z_4$, $y$ generates $Z_2$, $x \mod 2 = \{\delta_1 + \delta_2\}$, $y \mod 2 = 0$.

By the definition of cup product, $\alpha_2(u_1) = \alpha_1(a_1) \cdot \alpha_1(a_1) = 1$, $\alpha_2(u_2) = \alpha_1(a_2) \cdot \alpha_1(a_2) = 0$, so $\alpha_2 = \lambda_2$, similarly $\alpha_2 = \lambda_2$.

$\{\delta_1 + \delta_2\}^2 = \{\delta_1\}^2 + \{\delta_2\}^2 = 2z = 0$ where $z = \{\lambda_1\} = \{\lambda_2\}$ is the generator of $H^2(K, Z_2) = Z_2$, so $\beta_2(x) = z$.

**Appendix D: Twisted cohomology $H^n(BZ_2, Z_4, \rho)$**

Here $\rho : Z_2 \to Aut(Z_4)$ is a nontrivial homomorphism. Since $BZ_2 = S^\infty/Z_2$ whose universal covering space is $S^\infty$, by the definition of twisted cohomology [100], $H^n(BZ_2, Z_4, \rho)$ is the $n$-th cohomology group of the cochain complex $\operatorname{Hom}(\pi_2 Z_2(C_\bullet(S^\infty), Z_4))$ where both $C_\bullet(S^\infty)$ and $Z_4$ are left $ZZ_2$-modules, $C_\bullet(S^\infty)$ is the cellular chain complex of $S^\infty$ with two cells in each dimension. Denote the two cells in dimension $n$ by $e^n_1$ and $e^n_2$, and denote the action of $Z_2$ on $C_\bullet(S^\infty)$ by $\rho$.

Then $\rho'(e^n_1) = (-1)^n e^n_2$, and $\rho'(e^n_2) = (-1)^n e^n_1$.

(D1)

We have

$$\partial_n(e^n_1) = \partial_n(e^n_2) = e^{n-1}_1 - e^{n-1}_2. \quad (D2)$$

$$f_n \in \operatorname{Hom}(ZZ_2(C_n(S^\infty), Z_4))$$ satisfies

$$f_n(\rho'(e^n_1)) = \rho(f_n(e^n_1)) = (f_n(e^n_1))^{-1}. \quad (D3)$$

By (D2), if $f_n \in \operatorname{Ker}\partial_n$, then $f_n(e^n_1) = f_n(e^n_2)$. While by (D2), if $f_n \in \operatorname{Im}\partial_{n-1}$, then there exists $f_{n-1}$

$\operatorname{Hom}(Z_{2\ast-1}(S^\infty), Z_4)$ such that $f_n(e^n_1) = f_n(e^n_2) = f_{n-1}(e^{n-1}_1) / f_{n-1}(e^{n-1}_2)$.

If $n$ is odd, then by (D1) and (D3), for any $f_n \in \operatorname{Hom}(Z_{2\ast-1}(C_n(S^\infty), Z_4))$, we have $f_n(e^n_1) = f_n(e^n_2)$.

If $n$ is even, then by (D1) and (D3), for any $f_n \in \operatorname{Hom}(Z_{2\ast-1}(C_n(S^\infty), Z_4))$, we have $f_n(e^n_1) = (f_n(e^n_2))^{-1}$.

So if $n$ is odd, then $\operatorname{Ker}\partial_n = Z_4$, and $\operatorname{Im}\partial_{n-1} = 2Z_4$, so $H^n(BZ_2, Z_4, \rho) = Z_4/2Z_4 = Z_2$. While if $n$ is even, then $\operatorname{Ker}\partial_n = 2Z_4$, and $\operatorname{Im}\partial_{n-1} = 0$, so $H^n(BZ_2, Z_4, \rho) = 2Z_4 = Z_2$.

**Appendix E: Cohomology of $BZ_2 \times B^2 Z_4$**

The reference for this appendix is the appendix of [101].

In order to compute $H^n(BZ_2 \times B^2 Z_4)$, we need the data of $H^n(BZ_2 \times B^2 Z_4, Z_2)$ for $n \leq 5$.

Let $G$ be a 2-group with $BG = BZ_2 \times B^2 Z_4$. By the Universal Coefficient Theorem,

$$H^n(BG, Z_2) = H^n(BG, Z) \otimes Z_2 \oplus \operatorname{Tor}(H^{n+1}(BG, Z), Z_2). \quad (E1)$$

So we need only compute $H^n(BZ_2 \times B^2 Z_4, Z)$ for $n \leq 6$. $H^n(B^2 Z_4, Z)$ is computed in Appendix C of [102].

For the 2-group $G$ defined by the nontrivial action $\rho$ of $Z_2$ on $Z_4$ and nontrivial fibration

$$BZ_4 \longrightarrow BG \longrightarrow BZ_2 \quad (E3)$$

classified by the nonzero Postnikov class $\pi \in H^3(BZ_2, Z_4)$. Here we consider the fiber sequence $B^2 Z_4[i] \rightarrow BG \rightarrow BZ_2 \rightarrow B^3 Z_4[i] \rightarrow \ldots$ induced from a short exact sequence $1 \rightarrow Z_4[i] \rightarrow G \rightarrow Z_2 \rightarrow 1$. We have the Serre spectral sequence

$$H^p(BZ_2, H^q(B^2 Z_4, Z)) \Rightarrow H^{p+q}(BG, Z), \quad (E4)$$

equipped with the E₂ page of the Serre spectral sequence is the $p$-equivariant cohomology $H^p(BZ_2, H^q(B^2 Z_4, Z))$. The shape of the relevant piece is shown in Fig. 6.

Note that $p$ labels the columns and $q$ labels the rows.

The bottom row is $H^0(BZ_2, Z)$. The universal coefficient theorem tells us that $H^3(B^2 Z_4, Z) = H^2(B^2 Z_4, \mathbb{R}/Z) = \operatorname{Hom}(H_2(B^2 Z_4, \mathbb{R}/Z) = \operatorname{Hom}(\pi_2(B^2 Z_4), \mathbb{R}/Z) =$
More precisely, the value of $\langle \pi, - \rangle_q$ on the simplex $(v_0, \ldots, v_3)$ is $\langle \pi(v_0, \ldots, v_3), - \rangle_q$ which is in $\hat{Z}_4$.

The next possibly non-zero differentials are on the $E_4$ page:

$$H^j(BZ_2, \hat{Z}_4) \to H^{j+3}(BZ_2, R/Z) \to H^{j+4}(BZ_2, Z). \quad (E6)$$

The first map is contraction with $\pi$.

The last relevant possibly non-zero differential is on the $E_6$ page:

$$H^0(BZ_2, H^5(B^2Z_4, Z)) \to H^6(BZ_2, Z). \quad (E7)$$

Following the appendix of [101], this differential is actually zero.

So the only possible differentials in Fig. 6 below degree 5 are $d_3$ from $(0,5)$ to $(3,3)$ and $d_4$ from the third row to the zeroth row.

Since $q(k) = e^{2\pi k^2}$, if $\pi(v_0, \ldots, v_3) = k$, then

$$\langle \pi(v_0, \ldots, v_3), \pi(v_0, \ldots, v_3) \rangle_q = q(\pi(v_0, \ldots, v_3))^2 = e^{2\pi k^2} = 1$$

if $k = 0 \mod 2$, so $\text{Ker} d_3(0,5) = 2\mathbb{Z}_6 = \mathbb{Z}_4$.

If we identify $\hat{Z}_4$ with $\mathbb{Z}_4$, then the nonzero element in the image of $q \to \langle \pi, - \rangle_q$ is just $\pi$. So the differential $d_3(0,5)$ is nontrivial.

The differential $d_3^{(0,3)} : H^0(BZ_2, \hat{Z}_4) \to H^3(BZ_2, R/Z)$ is defined by

$$d_3^{(2,3)}(\lambda)(v_0, \ldots, v_3) = \lambda(\pi(v_0, \ldots, v_3))$$

which is actually zero since $\pi(v_0, \ldots, v_3) \in 2\mathbb{Z}_4$, if we identify $\hat{Z}_4$ with $\mathbb{Z}_4$, then this is just the cup product of $\lambda$ and $\pi$, and $\lambda \in 2\mathbb{Z}_4$.

The differential $d_4^{(2,3)} : H^2(BZ_2, \hat{Z}_4) \to H^5(BZ_2, R/Z)$ is defined by

$$d_4^{(2,3)}(\chi)(v_0, \ldots, v_5) = (\chi(v_0, \ldots, v_2))(\pi(v_2, \ldots, v_5))$$

which is also actually zero since $\pi(v_2, \ldots, v_5) \in 2\mathbb{Z}_4$, if we identify $\hat{Z}_4$ with $\mathbb{Z}_4$, then this is just the cup product of $\chi$ and $\pi$, and $\pi(v_0, \ldots, v_2) \in 2\mathbb{Z}_4$.

The position $(3,3)$ corresponds to the term $A^3B_2$ where $A$ and $B_2$ are explained in Sec. III.E. So only the $A^3B_2$ vanishes in $H^6(BZ_2 \times B^2Z_4, Z_2)$, hence in $\Omega^6_0(BZ_2 \times B^2Z_4)$. 

\[\text{FIG. 6. Serre spectral sequence for } (BZ_2, B^2Z_4)\]
