Uncertainty Relation on Wigner-Yanase-Dyson Skew Information

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Abstract. We give a trace inequality related to the uncertainty relation of Wigner-Yanase-Dyson skew information. This inequality corresponds to a generalization of the uncertainty relation derived by S.Luo [7] for the quantum uncertainty quantity excluding the classical mixture.

Key Words: Uncertainty relation, Wigner-Yanase-Dyson skew information

1 Introduction

Wigner-Yanase skew information

$$I_\rho(H) = \frac{1}{2} Tr \left[ (i [\rho^{1/2}, H])^2 \right]$$
$$= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H]$$

was defined in [9]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state $\rho$ and an observable $H$. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$I_{\rho,\alpha}(H) = \frac{1}{2} Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$
$$= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0, 1]$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to $\rho$ was successfully proven by E.H.Lieb in [6]. From the physical point of view, an observable $H$ is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we

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consider $H \in B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [8]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [4, 10]. In our paper [10], we defined a generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we discuss various properties of the Wigner-Yanase-Dyson skew information. Finally in section 3, we give our main result and its proof.

2 Trace inequalities of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable $H$ in a quantum state $\rho$ is expressed by $\text{Tr}[\rho H]$. It is natural that the variance for a quantum state $\rho$ and an observable $H$ is defined by $V_\rho(H) = \text{Tr}[\rho(H - \text{Tr}[\rho H]I)^2] = \text{Tr}[\rho H^2] - \text{Tr}[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2$$

for a quantum state $\rho$ and two observables $A$ and $B$. The further strong results was given by Robertson and Schrödinger

$$V_\rho(A)V_\rho(B) - |\text{Cov}_\rho(A, B)|^2 \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2,$$

where the covariance is defined by $\text{Cov}_\rho(A, B) = \text{Tr}[\rho(A - \text{Tr}[\rho A]I)(B - \text{Tr}[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [8, 4, 10])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2},$$

then he derived the uncertainty relation on $U_\rho(H)$ in [7]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2.$$
Note that we have the following relation

\[ 0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (2.4) \]

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In this section, we study one-parameter extended inequality for the inequality (2.3).

**Definition 2.1** For \( 0 \leq \alpha \leq 1 \), a quantum state \( \rho \) and an observable \( H \), we define the Wigner-Yanase-Dyson skew information

\[
I_{\rho,\alpha}(H) = \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \quad (2.5)
\]

and we also define

\[
J_{\rho,\alpha}(H) = \frac{1}{2} \text{Tr}[[\rho^\alpha, H_0] \{\rho^{1-\alpha}, H_0\}] = \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \quad (2.6)
\]

where \( H_0 = H - \text{Tr}[\rho H]I \) and we denote the anti-commutator by \( \{X, Y\} = XY + YX \).

Note that we have

\[
\frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]
\]

but we have

\[
\frac{1}{2} \text{Tr}[[\rho^\alpha, H_0] \{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2} \text{Tr}[[\rho^\alpha, H] \{\rho^{1-\alpha}, H\}].
\]

Then we have the following inequalities:

\[
I_{\rho,\alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho,\alpha}(H), \quad (2.7)
\]

since we have \( \text{Tr}[\rho^{1/2} H \rho^{1/2}] \leq \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H] \). (See [1, 2] for example.) If we define

\[
U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2}, \quad (2.8)
\]

as a direct generalization of Eq.(2.2), then we have

\[
0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H) \quad (2.9)
\]

due to the first inequality of (2.7). We also have

\[
U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}. \]
From the inequalities (2.4),(2.8),(2.9), our situation is that we have
\[ 0 \leq I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq U_{\rho}(H) \]
and
\[ 0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H). \]
Our concern is to show an uncertainty relation with respect to \( U_{\rho,\alpha}(H) \) as a direct generalization of the inequality (2.3) such that
\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4} |Tr[\rho[A,B]]|^2 \tag{2.10} \]

On the other hand, we introduced a generalized Wigner-Yanase skew information which is a generalization of the inequality (2.10), but different from the Wigner-Yanase-Dyson skew information defined in (2.5) and gave the following theorem in [11].

**Theorem 2.1** For \( 0 \leq \alpha \leq 1 \), a quantum state \( \rho \) and an observable \( H \), we define a generalized Wigner-Yanase skew information by
\[ K_{\rho,\alpha}(H) = \frac{1}{2} Tr \left[ \left( i \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right) \right]^2 \]
and we also define
\[ L_{\rho,\alpha}(H) = \frac{1}{2} Tr \left[ \left( i \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right) \right]^2, \]
and
\[ W_{\rho,\alpha}(H) = \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)}. \]
Then for a quantum state \( \rho \) and observables \( A, B \) and \( \alpha \in [0,1] \), we have
\[ W_{\rho,\alpha}(A)W_{\rho,\alpha}(B) \geq \frac{1}{4} \left| Tr \left[ \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [A,B] \right] \right|^2. \]

### 3 Main Theorem

We give the main theorem as follows;

**Theorem 3.1** For a quantum state \( \rho \) and observables \( A, B \) and \( 0 \leq \alpha \leq 1 \), we have
\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1 - \alpha)|Tr[\rho[A,B]]|^2. \tag{3.1} \]
We use the several lemmas to prove the theorem 3.1. By spectral decomposition, there exists an orthonormal basis \( \{ \phi_1, \phi_2, \ldots \} \) consisting of eigenvectors of \( \rho \). Let \( \lambda_1, \lambda_2, \ldots \) be the corresponding eigenvalues, where \( \sum_{i=1}^{\infty} \lambda_i = 1 \) and \( \lambda_i \geq 0 \). Thus, \( \rho \) has a spectral representation

\[
\rho = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle \langle \phi_i|.
\] (3.2)

**Lemma 3.1**

\[
I_{\rho, \alpha}(H) = \sum_{i<j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2.
\]

**Proof of Lemma 3.1.** By (3.2),

\[
\rho H_0^2 = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle \langle \phi_i| H_0^2.
\]

Then

\[
Tr[\rho H_0^2] = \sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0^2 | \phi_i \rangle| = \sum_{i=1}^{\infty} \lambda_i \| H_0 | \phi_i \|^2.
\] (3.3)

Since

\[
\rho^\alpha H_0 = \sum_{i=1}^{\infty} \lambda_i^\alpha |\phi_i\rangle \langle \phi_i| H_0
\]

and

\[
\rho^{1-\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{1-\alpha} |\phi_i\rangle \langle \phi_i| H_0,
\]

we have

\[
\rho^\alpha H_0 \rho^{1-\alpha} H_0 = \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle \langle \phi_j | H_0 | \phi_i \rangle|.
\]

Thus

\[
Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] = \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle \langle \phi_j | H_0 | \phi_i \rangle| = \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle|^2.
\] (3.4)
From (2.5), (3.3), (3.4),

$$I_{\rho,\alpha}(H) = \sum_{i=1}^{\infty} \lambda_i \|H_0|\phi_i\|^2 - \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle|^2$$

$$= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2$$

$$= \sum_{i<j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2.$$  

\[ \square \]

**Lemma 3.2**

$$J_{\rho,\alpha}(H) \geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2.$$  

**Proof of Lemma 3.2.** By (2.6), (3.3), (3.4), we have

$$J_{\rho,\alpha}(H) = \sum_{i=1}^{\infty} \lambda_i \|H_0|\phi_i\|^2 + \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle|^2$$

$$= \sum_{i,j=1}^{\infty} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2$$

$$= 2 \sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0 | \phi_i \rangle|^2 + \sum_{i \neq j} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2$$

$$= 2 \sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0 | \phi_i \rangle|^2 + \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2$$

$$\geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2.$$  

\[ \square \]

**Lemma 3.3** For any $t > 0$ and $0 \leq \alpha \leq 1$, the following inequality holds;

$$(1 - 2\alpha)^2 (t - 1)^2 - (t^\alpha - t^{1-\alpha})^2 \geq 0.$$  

(3.5)
Proof of Lemma 3.3. If \( \alpha = 0 \) or \( \frac{1}{2} \) or 1, then it is clear that (3.5) is satisfied. Now we put
\[
F(t) = (1 - 2\alpha)^2(t - 1)^2 - (t^\alpha - t^{1-\alpha})^2.
\]
We have
\[
F'(t) = 2(1 - 2\alpha)^2t - 2\alpha t^{2\alpha-1} - 2(1 - \alpha) t^{1-2\alpha} + 8\alpha(1 - \alpha).
\]
And we also have
\[
F''(t) = 2(1 - 2\alpha)^2 - 2\alpha(2\alpha - 1) t^{2\alpha-2} - 2(1 - \alpha)(1 - 2\alpha) t^{-2\alpha}
\]
and
\[
F'''(t) = 4\alpha(1 - 2\alpha)(1 - \alpha) t^{-2\alpha-1} - 4\alpha(1 - 2\alpha)(1 - \alpha) t^{2\alpha-3}
= 4\alpha(1 - 2\alpha)(1 - \alpha) \left( \frac{1}{t^{1+2\alpha}} - \frac{1}{t^{3-2\alpha}} \right).
\]
If \( \frac{1}{2} < \alpha < 1 \), then \( 1 + 2\alpha > 3 - 2\alpha \). Then it is easy to show that \( F'''(t) < 0 \) for \( t < 1 \) and \( F'''(t) > 0 \) for \( t > 1 \). On the other hand if \( 0 < \alpha < \frac{1}{2} \), then \( 1 + 2\alpha < 3 - 2\alpha \). Then it is easy to show that \( F'''(1) = 0 \), we can get \( F'''(t) > 0 \). Since \( F'(1) = 0 \), we also have \( F'(t) < 0 \) for \( t < 1 \) and \( F'(t) > 0 \) for \( t > 1 \). Since \( F(1) = 0 \), we finally get \( F(t) \geq 0 \) for all \( t > 0 \). Therefore we have (3.5).

Proof of Theorem 3.1. We put \( t = \frac{\lambda_i}{\lambda_j} \) in (3.5). Then we have
\[
(1 - 2\alpha)^2 \left( \frac{\lambda_i}{\lambda_j} - 1 \right)^2 - \left( \left( \frac{\lambda_i}{\lambda_j} \right)^\alpha - \left( \frac{\lambda_i}{\lambda_j} \right)^{1-a} \right)^2 \geq 0.
\]
And we get
\[
(1 - 2\alpha)^2(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-a} - \lambda_i^{1-a} \lambda_j^a)^2 \geq 0
\]
and
\[
(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-a} - \lambda_i^{1-a} \lambda_j^a)^2 \geq 4\alpha(1 - \alpha)(\lambda_i - \lambda_j)^2
\]
and
\[
(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-a} + \lambda_i^{1-a} \lambda_j^a)^2 \geq 4\alpha(1 - \alpha)(\lambda_i + \lambda_j)^2.
\]
Since
\[
Tr[\rho[A,B]] = Tr[\rho[A_0,B_0]]
\]
and
\[
\lambda_1 \approx \lambda_2 \approx \lambda_3 _{\approx} \lambda_4 \approx \lambda_5 \approx \lambda_6 \approx \lambda_7 \approx \lambda_8 \approx \lambda_9.
\]
We remark that (2.3) is derived by putting 

Hence we have the final result (3.1).

By (3.6) and Schwarz inequality,

Then we have

\[ |\text{Tr}[\rho|A, B]|^2 \leq 4\sum_{i<j} (\lambda_i - \lambda_j)^2 |\text{Im}\langle \phi_i|A_0|\phi_j\rangle \langle \phi_j|B_0|\phi_i\rangle|^2. \]

By (3.6) and Schwarz inequality,

\[
\alpha(1 - \alpha)|\text{Tr}[\rho|A, B]|^2 \\
\leq 4\alpha(1 - \alpha)\left\{ \sum_{i<j} |\lambda_i - \lambda_j||\text{Im}\langle \phi_i|A_0|\phi_j\rangle \langle \phi_j|B_0|\phi_i\rangle| \right\}^2 \\
= \left\{ \sum_{i<j} 2\sqrt{\alpha(1 - \alpha)}|\lambda_i - \lambda_j||\text{Im}\langle \phi_i|A_0|\phi_j\rangle \langle \phi_j|B_0|\phi_i\rangle| \right\}^2 \\
\leq \left\{ \sum_{i<j} 2\sqrt{\alpha(1 - \alpha)}|\lambda_i - \lambda_j||\langle \phi_i|A_0|\phi_j\rangle|\langle \phi_j|B_0|\phi_i\rangle| \right\}^2 \\
\leq \left\{ \sum_{i<j} |(\lambda_i + \lambda_j)^2 - (\lambda_i^a\lambda_j^{1-a} + \lambda_i^{1-a}\lambda_j^a)^2|^{1/2}|\langle \phi_i|A_0|\phi_j\rangle|\langle \phi_j|B_0|\phi_i\rangle| \right\}^2 \\
\leq \left\{ \sum_{i<j} (\lambda_i + \lambda_j - \lambda_i^a\lambda_j^{1-a} - \lambda_i^{1-a}\lambda_j^a)|\langle \phi_i|A_0|\phi_j\rangle|^2 \right. \\
\times \left. \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^a\lambda_j^{1-a} + \lambda_i^{1-a}\lambda_j^a)|\langle \phi_i|B_0|\phi_j\rangle|^2. \right\}
\]

Then we have

\[ I_{\rho, \alpha}(A)J_{\rho, \alpha}(B) \geq \alpha(1 - \alpha)|\text{Tr}[\rho|A, B]|^2. \]

We also have

\[ I_{\rho, \alpha}(B)J_{\rho, \alpha}(A) \geq \alpha(1 - \alpha)|\text{Tr}[\rho|A, B]|^2. \]

Hence we have the final result (3.1).

\[ \square \]

**Remark 3.1** We remark that (2.3) is derived by putting \( \alpha = 1/2 \) in (3.1). Then Theorem 3.1 is a generalization of the result of Luo [7].
Remark 3.2 We remark that Conjecture 2.3 in [11] does not hold in general. The Conjecture is (2.10). A counterexample is given as follows. Let

\[ \rho = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \frac{1}{3}. \]

We have

\[ I_{\rho,\alpha}(A)I_{\rho,\alpha}(B) = I_{\rho,\alpha}(B)I_{\rho,\alpha}(A) = 0.22457296 \ldots \]

and \(|Tr[\rho[A, B]]|^2 = 1\). These imply

\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.22457296 \ldots < \frac{1}{4}|Tr[\rho[A, B]]|^2 = 0.25. \]

On the other hand we have

\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) > \alpha(1 - \alpha)|Tr[\rho[A, B]]|^2 = 0.2222222 \ldots. \]

We also give a counterexample for Conjecture 2.10 in [11]. The inequality

\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[(\rho^\alpha + \rho^{1-\alpha})^2[A, B]]|^2 \]

is not correct in general, because LHS = 0.22457296 \ldots, RHS = 0.23828105995 \ldots.

Remark 3.3 In the recent literature another generalization for inequality (2.3) has been proved in [5] as follows; for any \( \rho, A, B \) and \( 0 \leq \alpha \leq 1 \)

\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[(\rho - \rho^{2\alpha - 1})[A, B]]|^2. \]

However we gave the counter example for this inequality. Let

\[ \rho = \begin{pmatrix} \frac{1}{51} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{59}{64} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \frac{3}{4}. \]

Then we have

\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.00170898 \ldots, \]

\[ \frac{1}{4}|Tr[(\rho - \rho^{2\alpha - 1})[A, B]]|^2 = 0.00610351 \ldots. \]
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