Nonlinear dynamical systems and classical orthogonal polynomials

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It is demonstrated that nonlinear dynamical systems with analytic nonlinearities can be brought down to the abstract Schrödinger equation in Hilbert space with boson Hamiltonian. The Fourier coefficients of the expansion of solutions to the Schrödinger equation in the particular occupation number representation are expressed by means of the classical orthogonal polynomials. The introduced formalism amounts a generalization of the classical methods for linearization of nonlinear differential equations such as the Carleman embedding technique and Koopman approach.

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I. INTRODUCTION

In 1931 Koopman [1] showed that one can associate with Hamiltonian systems of ordinary differential equations the Schrödinger equation in Hilbert space of square integrable functions. The observations of Koopman have become an important tool in the spectral theory of dynamical systems [2]. In 1932 Carleman [3] following ideas of Poincaré and Fredholm demonstrated that nonlinear systems of ordinary differential equations with polynomial nonlinearities can be reduced to an infinite system of linear differential equations. This approach is nowadays referred to as the Carleman linearization or Carleman embedding. The Carleman approach has been successfully applied to the solution of numerous nonlinear problems (see [4] and references therein). For example it was used for calculating Lyapunov exponents [5] and finding first integrals for the Lorenz system [6]. In his book [7] Varadarajan extended the Koopman linearization to the very general case of the phase space replaced with a $G$-space. When $G=\mathbb{R}$ and action of the group is given by the flow the Varadarajan observations can be regarded as a generalization of the Koopman approach to the case with non-Hamiltonian systems. Recently, such a generalization was rediscovered by Alanson [8] who reported the possibility of reformulation of dynamical systems in the Hilbert space of square integrable functions. In 1982 Steeb [9] demonstrated that the Carleman embedding matrix can be expressed with the help of Bose creation and annihilation operators. Inspired by this observation the author introduced in 1987 the Hilbert space approach to nonlinear dynamical systems [10]. The formalism is based on the reduction of nonlinear dynamical systems with analytic nonlinearities to the abstract, linear Schrödinger-like equation in Hilbert space with non-Hermitian boson Hamiltonian. The treatment amounts a far-reaching generalization of the Carleman linearization technique which corresponds to the particular case with the occupation number representation for the Schrödinger-like equation. On the other hand, it works also in the case with partial differential equations. The approach has been developed in a series of papers (see monograph [11] and references therein) and it has been shown therein to be an effective tool in the study of both ordinary and partial differential equations.

As remarked by Carleman [3] (see also [12]) the simplest polynomial ansatz utilized in his linearization scheme can be replaced by an orthogonal polynomial. Indeed, the author showed in [13] that the Koopman linearization is nothing but a version of the Carleman technique with the polynomial linearization ansatz coinciding with a multidimensional generalization of the Hermite polynomials. It is worthwhile to note that Carleman who discussed the Koopman formalism in [3] and wrote down the linearization transformation coinciding with that given by Hermite polynomials did not recognize such interpretation of the Koopman observations. So in both cases of the Carleman and Koopman approach the system is linearized via a polynomial ansatz. We remark that the polynomials utilized by Carleman are orthogonal ones only in the complex domain. Now, using the Hilbert space formalism introduced by the author one can connect the Carleman
polynomial linearization ansatz with the Schrödinger-like equation in Hilbert space. The reason of the nonhermicity of the corresponding Hamiltonian is the fact that we deal with complex orthogonal polynomials. In summary, the common feature of the Carleman and Koopman approaches is an orthogonal polynomial ansatz which allows to reduce the nonlinear dynamical systems to the Schrödinger or Schrödinger-like equation in Hilbert space. The aim of this work is to introduce a general method for linearization of nonlinear dynamical systems with analytic nonlinearities including the Carleman technique and Koopman approach as a special case. Namely, we show that the linearization ansatz enabling reduction of dynamical systems to the Schrödinger equation is generated by an arbitrary classical orthogonal polynomial.

The paper is organized as follows. In section 2 we briefly recall the Carleman technique and its Hilbert space generalization. Section 3 is devoted to a short exposition of the Koopman linearization. In section 4 we describe the method for linearization of nonlinear dynamical systems in the general case of an arbitrary classical orthogonal polynomial. The explicit relations for the concrete polynomials are discussed in section 5.

II. CARLEMAN LINEARIZATION AND THE HILBERT SPACE APPROACH TO NONLINEAR DYNAMICAL SYSTEMS

We first briefly recall the Carleman technique [3]. Consider the real analytic system

$$\dot{x} = F(x),$$

(2.1)

where $F : \mathbb{R}^k \to \mathbb{R}^k$ and $F$ is analytic in $x$. Setting

$$\phi_n = \prod_{i=1}^{k} x_i^{n_i},$$

(2.2)

where $x = (x_1, \ldots, x_k)$ satisfies (2.1) and $n_i \in \mathbb{Z}_+$ (the set of nonnegative integers), we arrive at the linear differential-difference equation such that

$$\dot{\phi}_n = \sum_{m \in \mathbb{Z}_+^k} C_{nm} \phi_m.$$

(2.3)

Since we can introduce an order in the set $\mathbb{Z}_+^k$, therefore (2.3) is equivalent to the infinite linear system of ordinary differential equations. In view of (2.2) the finite system (2.1) is embedded into the infinite system implied by (2.3). Indeed, it follows from (2.2) that the solution $x$ to (2.1) is linked to the solution $\phi_n$ of (2.3) by

$$x_i = \phi_{e_i}, \quad i = 1, \ldots, k,$$

(2.4)

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector of $\mathbb{R}^k$. Therefore, the Carleman linearization is also referred to as the Carleman embedding technique.

We now outline the Hilbert space approach to nonlinear dynamical systems [10]. Consider the system (complex or real)

$$\dot{z} = F(z),$$

(2.5)

where $F : \mathbb{C}^k \to \mathbb{C}^k$ and $F$ is analytic in $z$. Let us introduce the vectors in Hilbert space of the form

$$|z\rangle' = \exp\left(\frac{1}{2} \sum_{i=1}^{k} |z_i|^2\right) |z\rangle,$$

(2.6)

where $z$ fulfil (2.5) and $|z\rangle$ are normalized coherent states (see appendix B). On differentiating both sides of (2.6) with respect to time we arrive at the following linear, abstract Schrödinger-like equation in Hilbert space satisfied by the vectors $|z\rangle'$.
\[
\frac{d}{dt}|z\rangle' = M'|z\rangle',
\]
\[\text{(2.7)}\]

where \(M'\) is the boson operator such that
\[
M' = \sum_{i=1}^{k} a_i^\dagger F_i(a).
\]
\[\text{(2.8)}\]

Here \(a_i^\dagger, a_j, i = 1, \ldots, k,\) are the standard Bose creation and annihilation operators, respectively (see appendix A). Evidently, the solutions to the nonlinear system (2.5) are linked to the solutions to the linear equation (2.7) by
\[
a|z\rangle' = z|z\rangle'.
\]
\[\text{(2.9)}\]

It thus appears that the integration of the nonlinear dynamical system (2.5) can be brought down to the solution of the linear Schrödinger-like equation in Hilbert space (2.7).

We now discuss the connection of the Hilbert space formalism with the Carleman linearization. Writing the abstract equation (2.7) in the occupation number representation (see appendix A) we obtain
\[
\dot{z}_n = \sum_{m \in \mathbb{Z}_+^k} M'_{nm} z_m,
\]
\[\text{(2.10)}\]

where \(z_n = \langle n|z\rangle'\) and \(M'_{nm} = \langle n|M'|m\rangle\). Using (2.6) and (B.6) we get
\[
\dot{z}_n = \prod_{i=1}^{k} \frac{\dot{z}_i}{\sqrt{n_i!}},
\]
\[\text{(2.11)}\]

where \(z = (z_1, \ldots, z_k)\) obeys (2.5). Thus it turns out that the Carleman embedding technique corresponds to the particular occupation number representation within the Hilbert space approach. We note that the polynomials (2.11) form the orthonormal complete set in the Fock-Bargmann space specified by the inner product (B.7) (see (B.9)). In other words, (2.11) is simply the normalized version of (2.2). It should be noted that in the case with monomials like (2.11) with \(k = 1\), the passage to the complex domain is necessary for obtaining the complete orthonormal set. Indeed, it can be easily demonstrated that there is no Hilbert space spanned by real monomials \(p_n(x) = c_n x^n\). However, it might be observed that (2.7) and (2.9) hold in the real domain as well.

### III. THE KOOPMAN LINEARIZATION

This section is devoted to a brief exposition of the Koopman linearization, more precisely its generalization mentioned in the introduction, with the help of the Hilbert space approach described in [11]. Consider the real analytic system
\[
\dot{x} = F(x).
\]
\[\text{(3.1)}\]

Let \(|x\rangle\), where \(x\) fulfills (3.1), be the normalized eigenvectors of the position operators (see appendix C). Using (C.5) and (C.2) we find that these vectors satisfy the linear Schrödinger-like equation in Hilbert space of the form
\[
\frac{d}{dt}|x\rangle = M|x\rangle,
\]
\[\text{(3.2)}\]

where the operator \(M\) is given by
\[
M = -i \sum_{i=1}^{k} \hat{p}_i F_i(\hat{q}),
\]
\[\text{(3.3)}\]
where $\hat{p}_i, \hat{q}_j, i = 1, \ldots, k$, are the momentum and position operators, respectively. Clearly, the solutions to the nonlinear system (3.1) are related to the solutions of the linear equation (3.2) by

$$\hat{q}|x\rangle = x|x\rangle.$$  \hfill (3.4)

We have thus shown that the nonlinear dynamical system (3.1) can be cast into the linear Schrödinger-like equation (3.2). Moreover, an easy calculation based on (3.2) and (C.3) shows that the vectors defined by

$$|x\rangle := e^{\frac{1}{2}\int_0^t \text{div}\mathbf{F}(x)dt} |x\rangle,$$ \hfill (3.5)

obey the Schrödinger equation of the form

$$i\frac{d}{dt}|x\rangle = H|x\rangle,$$ \hfill (3.6)

where the Hamiltonian is

$$H = \frac{1}{2} \sum_{i=1}^{k} (\hat{p}_i F_i(\hat{q}) + F_i(\hat{q}) \hat{p}_i).$$ \hfill (3.7)

In view of (3.5) the solutions to (3.1) and (3.6) are related by

$$\hat{q}|\tilde{x}\rangle = x|\tilde{x}\rangle.$$ \hfill (3.8)

Thus it turns out that the nonlinear dynamical system (3.1) can be brought down to the abstract Schrödinger equation (3.6). Observe that the Hamiltonian $H$ is nothing but the Hermitian part of the operator $iM$. One may ask if we could analogously symmetrize the operator $M'$ given by (2.8). The negative answer follows from observations of Kano [13] who showed that whenever Hamiltonian is not linear in the Bose creation operators then the coherent states become unstable, that is the relation (2.9) is violated during the time evolution. We finally remark that the $L^2$ version of the relations (3.5) and (3.6) was originally introduced by Alanson [8].

**IV. LINEARIZATION IN HILBERT SPACE AND CLASSICAL ORTHOGONAL POLYNOMIALS**

**A. Classical orthogonal polynomials**

As we promised in the introduction we now describe a general method for reduction of nonlinear dynamical systems to the Schrödinger-like or Schrödinger equation based on the linearization ansatz generated by an arbitrary classical orthogonal polynomial. We begin by recalling the basic properties of classical orthogonal polynomials. Let $p_n(x)$ be a normalized classical orthogonal polynomial, that is we have

$$\int_a^b dx w(x)p_n(x)p_m(x) = \delta_{nm},$$ \hfill (4.1)

where $w(x)$ is the weight function and $n, m \in \mathbb{Z}_+$. The orthogonal polynomials satisfy the recurrence formula

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \quad n = 0, 1, 2, \ldots,$$ \hfill (4.2)

where $p_{-1}(x) = 0$ and

$$A_n = \frac{q_{n+1}}{q_n}, \quad B_n = A_n (r_{n+1} - r_n), \quad C_n = \frac{A_n}{A_{n-1}}.$$ \hfill (4.3)
The coefficients \( q_n \) and \( r_n \) in (4.3) are given by

\[
p_n(x) = q_n x^n + q'_n x^{n-1} + \ldots ,
\]
\[
r_n = \frac{q'_n}{q_n}
\]

(4.4)

We note that the last formula of (4.3) on \( C_n \) holds only for the normalized orthogonal polynomials.

We now specialize to the case with normalized classical orthogonal polynomials. We then have a generalized Rodrigues formula \[14\]

\[
p_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n} [w(x)X^n],
\]

(4.5)

where \( K_n \) are constant and \( X \) is a polynomial with coefficients independent of \( n \), and the differentiation formula \[14\]

\[
X \frac{dp_n(x)}{dx} = (\alpha_n + \frac{n}{2} X''x)p_n(x) + \beta_n p_{n-1}(x),
\]

(4.6)

where

\[
\alpha_n = n X'(0) - \frac{1}{2} X'' r_n, \quad A_n \beta_n = -C_n [q_1 K_1 + (n - \frac{1}{2}) X''].
\]

(4.7)

### B. Reduction to the evolution equation in Hilbert space

We now come to the discussion of the linearization of nonlinear dynamical systems in Hilbert space. As it is well-known the orthogonal polynomials form the complete set in \( L^2_w \) \[14\]. Based on this observation we introduce the Hermitian operator \( \hat{x} \) with the complete set of eigenvectors, such that

\[
\hat{x} |x\rangle = x |x\rangle,
\]

(4.8)

and

\[

\langle n|x\rangle = w(x)^{\frac{1}{2}} p_n(x),
\]

(4.9)

where \( |n\), \( n \in \mathbb{Z}_+ \), are the basis vectors of the occupation number representation (see appendix A) and \( p_n(x) \) is a normalized classical orthogonal polynomial. Evidently, the resolution of the identity for the states \( |x\rangle \) can be written as

\[
\int_a^b dx \langle x|\langle x| = I.
\]

(4.10)

On writing the eigenvalue equation (4.8) in the occupation number representation and using (4.9), (4.2), (4.3) and (A.10) we arrive at the following boson realization of the operator \( \hat{x} \):

\[
\hat{x} = a - \frac{1}{A_{N-1} \sqrt{N}} + \frac{1}{A_{N-1} \sqrt{N}} a^\dagger - \frac{B_N}{A_N} - \frac{1}{A_{N-1} \sqrt{N}} a^\dagger + r_N - r_{N+1},
\]

(4.11a)

where \( N \) is the number operator. Consider now the differential equation

\[
\dot{x} = F(x),
\]

(4.12)
where $F$ is analytic in $x$. Our aim is to study the dynamics of the time-dependent vectors $|x\rangle$, where $x$ satisfies (4.12). These states will be seen to be stable with respect to the time evolution given by (4.12), that is (4.8) holds at any time. Expanding the time-dependent vector $|x\rangle$ in the basis of the vectors $|n\rangle$, differentiating with respect to time and using (4.9), (4.6), (4.2), (4.5) and (A.10) we find that the vectors $|x\rangle$ satisfy the Schrödinger-like equation in Hilbert space of the form

$$\frac{d}{dt}|x\rangle = M|x\rangle,$$  \hspace{1cm} (4.13)

where the operator $M$ is

$$M = \hat{\pi} \frac{F(\hat{x})}{X}.$$  \hspace{1cm} (4.14)

Here

$$\hat{\pi} = \frac{1}{2} \left\{ a \frac{1}{A_{N-1}\sqrt{N}} [X''(N-1) + q_1K_1] - \frac{1}{A_{N-1}\sqrt{N}} q_1^\dagger + [X''(N-1) + q_1K_1]r_N - [X''N + q_1K_1]r_{N+1} + \frac{B_0K_1}{K_0} - X' + 2NX'(0) \right\}. \hspace{1cm} (4.15)$$

In formulas (4.14) and (4.15) we have not designated for brevity the dependence of $X$ and $X'$ on $\hat{x}$. It should also be noted that $X''$ is constant for arbitrary classical orthogonal polynomial. Therefore, $X''$ is a $c$-number in (4.15). Now, it can be verified that the following commutation relation holds for arbitrary (normalized) classical orthogonal polynomial

$$[\hat{x}, \hat{\pi}] = X.$$ \hspace{1cm} (4.16)

Therefore, in view of (4.14) we have

$$[\hat{x}, M] = F(\hat{x}).$$ \hspace{1cm} (4.17)

Hence, with the use of (4.13) we find that (4.16) is precisely the condition for stability of the time-dependent states $|x\rangle$, i.e. (4.8) where $x$ fulfills (4.12), is valid at any time. In other words, the solution to (4.12) and the solution of (4.13) are related by (4.8). It thus appears that the solution of the nonlinear differential equation (4.12) can be brought down to the solution of the linear, evolution, Schrödinger-like equation in Hilbert space (4.13). We remark that (4.16) is equivalent to the following identities satisfied by an arbitrary normalized classical orthogonal polynomial:

$$[X''(n - 2) + q_1K_1]r_{n-1} - 2[X''(n - 1) + q_1K_1]r_n + [X''n + q_1K_1]r_{n+1} - 2X'(0) = 0,$$ \hspace{1cm} (4.18)

$$\frac{1}{A_{n-1}^2}[X''(n - \frac{3}{2}) + q_1K_1] - \frac{1}{A_n^2}[X''(n + \frac{1}{2}) + q_1K_1] = X(r_n - r_{n+1}) = X \left( -\frac{B_n}{A_n} \right),$$ \hspace{1cm} (4.19)

where $X(r_n - r_{n+1}) \equiv X(y)|_{y=r_n-r_{n+1}}$. We finally note that the formula (4.18) is a direct consequence of the more general relation such that

$$[X''(n - 1) + q_1K_1]r_n - [X''n + q_1K_1]r_{n+1} + 2nX'(0) + \frac{B_0K_1}{K_0} = 0.$$ \hspace{1cm} (4.20)

**C. Reduction to the Schrödinger equation**

We now demonstrate that (4.12) can be furthermore reduced to the Schrödinger equation. First observe that (4.20) is equivalent to the operator formula

$$\hat{\pi} + \hat{\pi}^\dagger = -X'.$$ \hspace{1cm} (4.21)
Further, owing to (4.16) the Hermitian operator $\hat{k}$ defined as
\begin{equation}
\hat{k} = \frac{i}{2}(\hat{x} - \hat{\pi}^\dagger),
\end{equation}
satisfies the commutation relation
\begin{equation}
[\hat{x}, \hat{k}] = iX.
\end{equation}
Using (4.23), (4.22) and (4.21) we find that the Hermitian operator $H$ such that
\begin{equation}
H = \frac{1}{2} \left( \hat{k} \frac{F(\hat{\pi})}{X} + \frac{F(\hat{x})}{X} \hat{k} \right),
\end{equation}
is related to the operator $M$ of the form (4.14) by
\begin{equation}
H = i \left( M + \frac{1}{2} \frac{dF}{dx} \right).
\end{equation}
An easy inspection based on (4.13), (4.8) and (4.25) shows that the vectors defined by
\begin{equation}
|\tilde{x}\rangle := e^{\frac{i}{2} \int_0^t \frac{dF}{dx} d\tau} |x\rangle,
\end{equation}
where $|x\rangle$ fulfills (4.13), satisfy the Schrödinger equation of the form
\begin{equation}
i\frac{d}{dt} |\tilde{x}\rangle = H |\tilde{x}\rangle.
\end{equation}
By virtue of (4.8) and (4.26) we have
\begin{equation}
\hat{x} |\tilde{x}\rangle = x |\tilde{x}\rangle,
\end{equation}
where $x$ obey (4.12). Thus it turns out that the solution of the nonlinear equation (4.12) can be cast into the solution of the Schrödinger equation (4.27). We remark that in view of (4.25) the Hamiltonian $H$ coincides with the symmetrization of the operator $iM$.

D. Polynomial linearization ansatz

We note that (4.9) can be regarded as a linearization ansatz for (4.12). Indeed, on writing (4.13) in the occupation number representation we arrive at the linear differential-difference equation satisfied by $\langle n|x \rangle$. In order to show that the actual treatment can be regarded as a generalization of the Carleman linearization technique we should reformulate it to deal with polynomial linearization ansatz instead of the nonpolynomial one given by (4.9). Consider the vectors
\begin{equation}
|x\rangle' := w(x)^{-\frac{1}{4}} |x\rangle,
\end{equation}
where $|x\rangle$ fulfills (4.13). Taking into account (4.13) we find that these vectors satisfy the following evolution equation in Hilbert space:
\begin{equation}
\frac{d}{dt} |x\rangle' = M' |x\rangle',
\end{equation}
where $M'$ is given by
\begin{equation}
M' = \hat{\pi}' \frac{F(\hat{\pi})}{X},
\end{equation}
where the operator $\hat{\pi}'$ is
We remark that in view of (4.32a) \[ [\hat{x}, \hat{\pi}'] = X, \] (4.33) which leads to \[ [\hat{x}, M'] = F(\hat{x}). \] (4.34) Clearly, the solution to (4.12) is linked to the solution of (4.30) by \[ \hat{x}|x\rangle' = x|x\rangle'. \] (4.35) We have thus shown that the solution to the nonlinear equation (4.12) can be reduced to the solution of the linear Schrödinger-like equation (4.30). On writing the abstract equation (4.30) in the occupation number representation we arrive at the following equation:

\[ \dot{x}_n = \sum_{m \in \mathbb{Z}_+} M'_{nm} x_m, \] (4.36)

where \( x_n = \langle n|x\rangle' \) and \( M'_{nm} = \langle n|M'|m \rangle \). Eqs. (4.29) and (4.9) taken together yield

\[ x_n = p_n(x). \] (4.37)

Thus it turns out that in the particular occupation number representation the presented formalism describes linearization of (4.12) by means of the polynomial ansatz (4.37) given by an arbitrary classical orthogonal polynomial.

E. Linearization of multidimensional nonlinear dynamical systems

We now generalize the actual treatment to the case with multidimensional nonlinear dynamical systems. Having in mind the relation (2.2) which is crucial for the Carleman linearization, it is natural to postulate the following multidimensional generalization of classical orthogonal polynomials:

\[ p_n(x) = \prod_{i=1}^{k} p_n(x_i), \] (4.38)

where \( p_n(x_i) \) is a (normalized) classical orthogonal polynomial and \( n \in \mathbb{Z}^k_+ \). Evidently, \( p_n(x) \) form the orthonormal, complete set in the \( k \)-fold tensor product of \( L^2_w \). We can thus introduce the Hermitian operators \( \hat{x}_i, i = 1, \ldots, k, \) with complete set of eigenvectors \( |x\rangle, x \in \mathbb{R}^k \), satisfying

\[ \hat{x}_i \langle x| = x_i \langle x|, \] (4.39)

\[ \langle n|x\rangle = \left( \prod_{i=1}^{k} w(x_i)^{\frac{1}{2}} \right) p_n(x), \] (4.40)

where \( |n\rangle, n \in \mathbb{Z}^k_+ \), span the occupation number representation. Clearly, the resolution of the identity for the states \( |x\rangle \) is
\[
\int_{(a,b)^k} d^k x |x\rangle\langle x| = I. \tag{4.42}
\]

Furthermore, it can be easily checked (see (A.1) and (A.2)) that the \(k\)-dimensional generalization of (4.11) is

\[
\hat{x}_i = a_i \frac{1}{A_{N_i-1}\sqrt{N_i}} + \frac{1}{A_{N_i-1}\sqrt{N_i}} a_i^\dagger - B_{N_i} \frac{1}{A_{N_i}} \tag{4.43a}
\]

\[
= a_i \frac{1}{A_{N_i-1}\sqrt{N_i}} + \frac{1}{A_{N_i-1}\sqrt{N_i}} a_i^\dagger + r_{N_i} - r_{N_i+1}, \quad i = 1, \ldots, k. \tag{4.43b}
\]

Consider now the nonlinear dynamical system

\[
\dot{x} = F(x), \tag{4.44}
\]

where \(F : \mathbb{R}^k \to \mathbb{R}^k\) and \(F\) is analytic in \(x\). Proceeding as with (4.12) we find that the time-dependent vectors \(|x\rangle\), where \(x\) fulfills (4.44), obey

\[
\frac{d}{dt} |x\rangle = M|x\rangle. \tag{4.45}
\]

The boson operator \(M\) is given by

\[
M = \sum_{i=1}^{k} \hat{\pi}_i F_i(\hat{x}) \frac{X(\hat{x}_i)}{X(\hat{x})}, \tag{4.46}
\]

where

\[
\hat{\pi}_i = \frac{1}{2} \left\{ a_i \frac{1}{A_{N_i-1}\sqrt{N_i}} [X''(N_i - 1) + q_1 K_1] - [X''(N_i - 1) + q_1 K_1] \frac{1}{A_{N_i-1}\sqrt{N_i}} a_i^\dagger \right. \\
\left. + [X''(N_i - 1) + q_1 K_1] r_{N_i} - [X''N_i + q_1 K_1] r_{N_i+1} + \frac{B_0 K_1}{K_0} - X'(\hat{x}_i) + 2N_iX'(0) \right\}, \tag{4.47}
\]

\(i = 1, \ldots, k\).

From (4.43), (4.47) and (4.16) (see also (A.1) and (A.2)) it follows that

\[
[\hat{x}_i, \hat{\pi}_j] = \delta_{ij} X(\hat{x}_j), \quad i, j = 1, \ldots, k. \tag{4.48}
\]

Hence

\[
[\hat{x}, M] = F(\hat{x}). \tag{4.49}
\]

As with (4.16) we find that (4.48) is the condition for the stability of the time-dependent states \(|x\rangle\). Therefore, the eigenvalue equation (4.40) is valid at any time. In other words, the nonlinear dynamical system (4.44) can be brought down to the solution of the linear evolution equation in Hilbert space (4.45).

It is easy to show using observations of section C that (4.44) can be cast into the Schrödinger equation. Namely, we introduce the Hermitian operators

\[
\hat{k}_i = \frac{i}{2} (\hat{\pi}_i - \hat{\pi}_i^\dagger), \quad i = 1, \ldots, k, \tag{4.50}
\]

which in view of (4.48) obey

\[
[\hat{x}_r, \hat{k}_s] = i\delta_{rs} X(\hat{x}_s), \quad r, s = 1, \ldots, k. \tag{4.51}
\]

Taking into account (4.51), (4.46) and the generalization of (4.21) such that
\[ \hat{\pi}_i + \hat{\pi}_i^\dagger = -\frac{dX(\hat{x}_i)}{d\hat{x}_i}, \quad i = 1, \ldots, k, \]  

(4.52)

we find that the Hermitian operator \( H \) defined as

\[ H = \frac{1}{2} \sum_{i=1}^{k} \left( \hat{k}_i F_i(\hat{x}) \frac{X(\hat{x}_i)}{X(\hat{x}_i)} + \frac{F_i(\hat{x})}{X(\hat{x}_i)} \hat{k}_i \right), \]  

(4.53)

is linked to the operator \( M \) by

\[ H = i(M + \frac{1}{2}\text{div}F). \]  

(4.54)

Therefore, the vectors

\[ |\mathbf{x}\rangle_{\tilde{\rangle}} := e^{\frac{i}{2} \int_0^t \text{div}F \, d\tau} |\mathbf{x}\rangle, \]  

(4.55)

where \( |\mathbf{x}\rangle \) obey (4.45), satisfy the Schrödinger equation

\[ i\frac{d}{dt} |\mathbf{x}\rangle_{\tilde{\rangle}} = H |\mathbf{x}\rangle_{\tilde{\rangle}}. \]  

(4.56)

Obviously, the following relation holds:

\[ \hat{x}_i |\mathbf{x}\rangle_{\tilde{\rangle}} = x_i |\mathbf{x}\rangle_{\tilde{\rangle}}, \]  

(4.57)

where \( x \) fulfil (4.44). It thus appears that the nonlinear dynamical system (4.44) can be reduced to the abstract Schrödinger equation (4.56).

We finally discuss the multidimensional generalization of the actual treatment in the case with the polynomial linearization ansatz (4.37). The resulting formalism generalizes the approach taken up by Carleman. Let us introduce the vectors of the form

\[ |\mathbf{x}\rangle' = \left( \prod_{i=1}^{k} w((x_i) - \frac{1}{2}) \right) |\mathbf{x}\rangle, \]  

(4.58)

where \( |\mathbf{x}\rangle \) obey (4.45). Proceeding as in the case with (4.29) we arrive at the following evolution equation in Hilbert space satisfied by the vectors (4.58):

\[ \frac{d}{dt} |\mathbf{x}\rangle' = M' |\mathbf{x}\rangle', \]  

(4.59)

where \( M' \) is

\[ M' = \sum_{i=1}^{k} \hat{\pi}_i' F_i(\hat{x}) \frac{X(\hat{x}_i)}{X(\hat{x}_i)}, \]  

(4.60)

Here the operators \( \hat{\pi}_i' \), \( i = 1, \ldots, k \), are given by

\[ \hat{\pi}_i' = \hat{\pi}_i - \frac{1}{2} \frac{w'(\hat{x}_i)}{w(\hat{x}_i)} X(\hat{x}_i) \]  

(4.61a)

\[ \quad = \frac{1}{2} \left\{ \frac{1}{A_{N_i} \sqrt{N_i}} X''(N_i - 1) - \frac{X''(N_i - 1)}{X''(N_i - 1)} + \frac{1}{A_{N_i} \sqrt{N_i}} a_i' \right. \]  

\[ - q_1 K_1 (r_{N_i} - r_{N_i+1}) - \frac{B_0 K_1}{K_0} \left\} \right. \]  

(4.61b)

Notice that
\[ [\hat{x}_i, \hat{p}_j] = \delta_{ij} X(\hat{x}_j), \quad i, j = 1, \ldots, k, \quad \text{(4.62)} \]
\[ [\hat{x}, M'] = \mathbf{F(\hat{x})}. \quad \text{(4.63)} \]

Clearly,
\[ \hat{x}|x\rangle' = x|x\rangle'. \quad \text{(4.64)} \]

Thus it turns out that the nonlinear dynamical system (4.44) can be brought down to the linear Schrödinger-like equation in Hilbert space (4.59). Writing (4.59) in the occupation number representation we get
\[ \dot{x}_n = \sum_{m \in Z^k_+} M'_{nm} x_m, \quad \text{(4.65)} \]
where \( x_n = \langle n|x\rangle' \) and \( M'_{nm} = \langle n|M'|m\rangle \). Taking into account (4.58) and (4.41) we find
\[ x_n = p_n(x). \quad \text{(4.66)} \]

In view of the form of the relations (2.2), (2.3), (4.66), (4.38) and (4.65) it is plausible to treat the actual formalism as a generalization of the Carleman embedding technique to the case with the linearization ansatz given by an arbitrary classical orthogonal polynomial. We recall that the orthogonal polynomials (2.13) are complex. Nevertheless, as we demonstrate in the next section the introduced approach works also in such the case.

V. EXAMPLES

A. Complex orthogonal polynomials

In this section we illustrate the general formalism described above by the examples of the concrete classical orthogonal polynomials. We begin with the simplest case of the complex polynomials (2.13) for \( k = 1 \). As a matter of fact, these polynomials are not considered as classical ones. Nevertheless, they satisfy the most important relations characteristic for classical orthogonal polynomials such as the recurrence, Rodrigues and differentiation formulas. No wonder that the introduced approach covers the case of polynomials (2.13). To see this consider the complex polynomials
\[ p_n(z) = \frac{z^n}{\sqrt{n!}}. \quad \text{(5.1)} \]
Evidently, for these polynomials (see (4.1), (4.2), (4.4) and (2.14))
\[ w(z, z^*) = e^{-zz^*}, \quad A_n = \frac{1}{\sqrt{n+1}}, \quad B_n = 0, \quad C_n = 0, \quad \text{(5.2)} \]
\[ q_n = \frac{1}{\sqrt{n!}}, \quad r_n = 0. \quad \text{(5.3)} \]

We remark that the formula (4.3) on \( C_n \) does not hold in the case with complex orthogonal polynomials. Furthermore, the Rodrigues formula can be written as
\[ p_n(z) = \frac{1}{K_n w(z, z^*)} \frac{d^n}{dz^{*n}} [w(z, z^*) X^n], \quad \text{(5.4)} \]
where
\[ X = 1, \quad K_n = (-1)^n n! \sqrt{n!}. \quad \text{(5.5)} \]
It can be easily checked with the use of (5.2), (5.3) and (5.5) that the differentiation formula (4.6) is also valid by the polynomials (5.1). Now we can write the complex counterpart of relations (4.8) and (4.9) of the form

$$\hat{z}|z\rangle = z|z\rangle,$$  \hfill (5.6)

$$\langle n|z\rangle = w(z, z^*) \hat{p}_n(z).$$  \hfill (5.7)

Proceeding as with (4.8) and (4.9) we find

$$\hat{z} = a,$$  \hfill (5.8)

that is the states $|z\rangle$ satisfying (5.6) are nothing but the standard coherent states (see appendix B). Referring back to the earlier remark concerning the coefficient $C_n$ in the recurrence formula (4.2) we note that the form of the operator $\hat{z}$ is implied by (5.2) and

$$\hat{z} = a \frac{1}{A_{N-1} \sqrt{N}} + \frac{C_N}{A_N \sqrt{N}} a^\dagger - B \frac{A_N}{A_N}.$$

(5.9)

This formula is a generalization of (4.11) in the case with complex polynomials when the last relation of (4.3) does not take place.

Consider the differential equation

$$\dot{z} = F(z),$$  \hfill (5.10)

where $F$ is analytic in $z$. In opposite to the case of classical orthogonal polynomials the time-dependent vectors $|z\rangle$, where $z$ fulfil (5.10) do not satisfy the linear evolution equation in Hilbert space. Indeed, applying the algorithm described in section 4 we get

$$\frac{d}{dt}|z\rangle = [\text{Re} F(z) + a^\dagger F(a)]|z\rangle.$$  \hfill (5.11)

On the other hand, the vectors defined as

$$|z\rangle' = w(z, z^*) \frac{1}{2} |z\rangle = e^{\frac{1}{4}zz^*} |z\rangle,$$  \hfill (5.12)

which are the complex counterparts of the vectors (4.29) are easily seen to obey

$$\frac{d}{dt}|z\rangle' = M'|z\rangle',$$  \hfill (5.13)

where the boson operator $M'$ is

$$M' = a^\dagger F(a).$$  \hfill (5.14)

Notice that since $X = 1$ (see (5.5)), the operator (5.14) has the structure analogous to (4.31). Moreover, identifying $\hat{z}'$ with $a^\dagger$ we see that (4.33) is valid as well. Furthermore, as an immediate consequence of (5.14) and (A.1) we find

$$[a, M'] = F(a).$$  \hfill (5.15)

As with (4.17) we conclude that the eigenvalue equation

$$a|z\rangle' = z|z\rangle',$$  \hfill (5.16)

where $z$ satisfies (5.10), holds at any time, that is the nonlinear equation (5.10) reduces to the linear Schrödinger-like equation in Hilbert space (5.13).

The generalization of the linearization algorithm in the case with the nonlinear dynamical system (2.5) is straightforward. Evidently, the counterparts of (4.38), (4.39), (4.40), (4.41) and (4.58) can be written as
\[ p_n(z) = \prod_{i=1}^{k} p_n(z_i) = \prod_{i=1}^{k} \frac{z_i}{\sqrt{n_i!}}, \quad (5.17a) \]
\[ [a_i, a_j] = 0, \quad i, j = 1, \ldots, k, \quad (5.17b) \]
\[ a(z) = z \langle z \rangle, \quad (5.17c) \]
\[ \langle n | z \rangle = \exp \left( -\frac{1}{2} \sum_{i=1}^{k} |z_i|^2 \right) p_n(z), \quad (5.17d) \]
\[ |z\rangle' = \left( \prod_{i=1}^{k} w(z_i, z_i^*) \right)^{-\frac{1}{2}} |z\rangle = \exp \left( \frac{1}{2} \sum_{i=1}^{k} |z_i|^2 \right) |z\rangle. \quad (5.17e) \]

Proceeding analogously as in the case of (4.58) we arrive at eq. (2.7). Furthermore, identifying \( \hat{a_i}' \) with \( a_i^\dagger \) we find that the formula (4.60) takes place also in the case with complex orthogonal polynomials. Using (2.8) and (A.1) we get the following counterpart of the relation (4.63):
\[ [a, M'] = F(a). \quad (5.18) \]

Clearly, (5.18) leads to the formula (2.9). Finally, the formula (2.11) is a direct consequence of (5.17c). We have thus shown that the particular case with the complex orthogonal polynomials (5.17a) corresponds to the Hilbert space approach discussed in section 2.

### B. Hermite polynomials

We now examine the case of the Hermite polynomials within the actual treatment. Let \( H_n(x) \) be the Hermite polynomials. For these polynomials we have (see (4.1))
\[ a = -\infty, \quad b = \infty, \quad w(x) = e^{-x^2}. \quad (5.19) \]

Now let \( p_n(x) \) be normalized Hermite polynomials, i.e.
\[ p_n(x) = \pi^{-\frac{1}{4}} 2^{-\frac{n}{2}} \frac{1}{\sqrt{n!}} H_n(x). \quad (5.20) \]

The polynomials \( p_n \) satisfy the recurrence formula (4.2), where
\[ A_n = \sqrt{\frac{2}{n+1}}, \quad B_n = 0, \quad C_n = \sqrt{\frac{n}{n+1}}. \quad (5.21) \]

These formulas can be recovered from (4.3) with the help of the relations
\[ q_n = \pi^{-\frac{1}{4}} 2^{-\frac{n}{2}} \frac{1}{\sqrt{n!}}, \quad r_n = 0. \quad (5.22) \]

The Rodrigues formula for polynomials \( p_n \) is given by (4.5), where
\[ X = 1, \quad K_n = \pi^{\frac{1}{2}} (-1)^n 2^{\frac{n}{2}} \sqrt{n!}. \quad (5.23) \]

Using (5.21) we find that the operator (4.11) is
\[ \hat{x} = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad (5.24) \]
that is
\[ \hat{x} = \hat{q}, \quad (5.25) \]
where \( \hat{q} \) is the position operator (see appendix C). Furthermore, taking into account (5.21), (5.22) and (5.23) we obtain the following formula on the operator (4.15):
\[
\hat{\pi} = \frac{1}{\sqrt{2}}(a^\dagger - a).
\] (5.26)

A look at (5.26) is enough to conclude that
\[
\hat{\pi} = -i\hat{p},
\] (5.27)
where \( \hat{p} \) is the momentum operator (see appendix C). Therefore, the operator given by (4.22) coincides with the momentum operator, i.e.
\[
\hat{k} = \hat{p}.
\] (5.28)

We now return to the Schrödinger-like equation (4.45). Eqs. (4.46), (4.43), (4.47), (5.23), (5.25) and (5.27) taken together lead to the operator \( M \) given by (3.3). Thus, the nonlinear system (4.44) reduces to the linear Schrödinger-like equation (4.45), where \( M \) is expressed by (3.3). Moreover, by virtue of (4.53) and (5.28) the system can be furthermore brought down to the Schrödinger equation (4.56), where the Hamiltonian \( H \) is given by (3.7). It thus appears that the particular case of the Hermite polynomials within the presented approach refers to the Koopman linearization.

C. Jacobi polynomials

Our purpose now is to discuss the case of the Jacobi polynomials within the introduced formalism. The Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) are specified by
\[
a = -1, \quad b = 1, \quad w(x) = (1-x)^\alpha(1+x)^\beta,
\] (5.29)
where \( \alpha > -1 \) and \( \beta > -1 \). Let \( p_n \) be the normalized Jacobi polynomials, so
\[
p_n(x) = \left[ \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right]^{1/2} P_n^{(\alpha,\beta)}(x).
\] (5.30)

The coefficients in the recurrence formula (4.2) are of the form
\[
A_n = \frac{2n+\alpha+\beta+2}{2} \sqrt{\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}},
\]
\[
B_n = \frac{\alpha^2-\beta^2}{2(2n+\alpha+\beta)} \sqrt{\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}},
\]
\[
C_n = \frac{2n+\alpha+\beta+2}{2n+\alpha+\beta} \sqrt{\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)(2n+\alpha+\beta+3)}{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)(2n+\alpha+\beta-1)}}.
\] (5.31)

Accordingly, the coefficients \( q_n \) and \( r_n \) (see (4.4)) are
\[
q_n = \left[ \frac{2n+\alpha+\beta+1}{2^{n+\alpha+\beta+1}\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)} \right]^{1/2} \Gamma(2n+\alpha+\beta+1),
\]
\[
r_n = \frac{(\alpha-\beta)n}{2n+\alpha+\beta}.
\] (5.32)

The expressions for the polynomial \( X \) and the coefficient \( K_n \) in the Rodrigues formula (4.5) are
\[X = 1 - x^2, \quad K_n = (-1)^n \left[ \frac{2^{2n\alpha+\beta+1} \Gamma(n+1) \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1)} \right]^{\frac{1}{2n}}. \quad (5.33)\]

Now taking into account (5.31) we find that the formula (4.11) takes the form
\[
\dot{x} = a \frac{2}{2N + \alpha + \beta} \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} \\
+ \frac{2}{2N + \alpha + \beta} \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} a^\dagger + \frac{\beta^2 - \alpha^2}{(2N + \alpha + \beta)(2N + \alpha + \beta + 1)}. \quad (5.34)
\]

Furthermore, making use of (4.15), (5.31), (5.32) and (5.33) we get
\[
\dot{\pi} = -a \frac{2N + \alpha + \beta - 2}{2N + \alpha + \beta} \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} \\
+ \frac{2N + \alpha + \beta + 2}{2N + \alpha + \beta} \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} a^\dagger + \frac{\beta^2 - \alpha^2}{(2N + \alpha + \beta)(2N + \alpha + \beta + 2)}. \quad (5.35)
\]

Finally, by virtue of (4.22) and (5.32) we have
\[
\dot{k} = i \left(-a \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} + \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} a^\dagger \right). \quad (5.36)
\]

Consider now the system (4.44). Using (4.46) and (5.33) we find that it can be reduced to the linear evolution equation in Hilbert space of the form (4.45), with
\[
M = \sum_{i=1}^{k} \pi_i \frac{F_i(\dot{x})}{1 - \dot{x}_i^2}, \quad (5.37)
\]
where in view of (4.43) and (4.47) \( \dot{x}_i \) and \( \pi_i \) can be obtained immediately from (5.34) and (5.35) by formal replacement of \( a \) and \( N \) by \( a_i \) and \( N_i \), respectively. That is \( \dot{x}_i = \dot{x}(a = a_i, N = N_i) \) and \( \pi_i = \pi(a = a_i, N = N_i) \).

The system (4.44) can be furthermore brought down to the Schrödinger equation (4.56). On taking into account (5.36) we arrive at the following form of the corresponding Hamiltonian (4.53):
\[
H = \frac{1}{2} \sum_{i=1}^{k} \left( \dot{k}_i \frac{F_i(\dot{x})}{1 - \dot{x}_i^2} + \frac{F_i(\dot{x})}{1 - \dot{x}_i^2} \dot{k}_i \right), \quad (5.38)
\]
where \( \dot{k}_i = \dot{k}(a = a_i, N = N_i) \).

We now discuss the Hilbert space counterpart (4.59) of the system (4.44). Using (4.32) we get
\[
\dot{\pi}' = -a \frac{2(N - 1)}{2N + \alpha + \beta} \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} \\
+ \frac{2(N + \alpha + \beta + 1)}{2N + \alpha + \beta} \sqrt{\frac{(N + \alpha)(N + \beta)(N + \alpha + \beta)}{(2N + \alpha + \beta - 1)(2N + \alpha + \beta + 1)}} a^\dagger + \frac{2(\alpha - \beta)N(N + \alpha + \beta + 1)}{(2N + \alpha + \beta)(2N + \alpha + \beta + 2)}. \quad (5.39)
\]

Therefore, the operator \( M' \) given by (4.60) takes the form

15
\[ M' = \sum_{i=1}^{k} \frac{\hat{\pi}'_i M_i(x)}{1 - \hat{x}_i}, \]  

where \( \hat{\pi}' = \hat{\pi}'(a = a_i, N = N_i) \). We finally recall that Gegenbauer polynomials, Legendre polynomials and Chebyshev polynomials are special cases of Jacobi polynomials. We also point out that the actual treatment cannot be applied in the case with Chebyshev polynomials of first kind \( T_n(x) \). Indeed, we have \( T_{-1}(x) = x \neq 0 \), which implies violating of the recurrence formula (4.2) for \( n = 0 \).

D. Laguerre polynomials

We finally study the case of (generalized) Laguerre polynomials \( L_\alpha^n(x) \) within the actual approach. These polynomials correspond to

\[ a = 0, \quad b = \infty, \quad w(x) = x^\alpha e^{-x}, \]  

where \( \alpha > -1 \). Let \( p_n \) be the normalized Laguerre polynomials, that is

\[ p_n(x) = \left[ \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right]^{\frac{1}{2}} L_\alpha^n(x). \]  

The coefficients in the recurrence formula (4.2) are

\[ A_n = -\frac{1}{\sqrt{(n+1)(n+\alpha+1)}}, \quad B_n = \frac{2n+\alpha+1}{\sqrt{(n+1)(n+\alpha+1)}}, \quad C_n = \frac{\sqrt{n(n+\alpha)}}{(n+1)(n+\alpha+1)}. \]  

The constants \( q_n \) and \( r_n \) in (4.3) are of the form

\[ q_n = \frac{(-1)^n}{\sqrt{\Gamma(n+1)\Gamma(n+\alpha+1)}}, \quad r_n = -n(n+\alpha). \]  

The expressions for the polynomial \( X \) and the constant \( K_n \) in the Rodrigues formula are

\[ X = x, \quad K_n = \sqrt{\Gamma(n+1)\Gamma(n+\alpha+1)}. \]  

Now, owing to (4.11) and (5.43) we get

\[ \hat{x} = -a\sqrt{N+n+\alpha} - \sqrt{N+n+\alpha} a^\dagger + 2N+\alpha+1. \]  

Furthermore, making use of (4.15) one obtains

\[ \hat{\pi} = \frac{1}{2}(a\sqrt{N+n+\alpha} - \sqrt{N+n+\alpha} a^\dagger - 1). \]  

Hence, in view of (4.22)

\[ \hat{k} = \frac{i}{2}(a\sqrt{N+n+\alpha} - \sqrt{N+n+\alpha} a^\dagger). \]  

We are now in a position to write down eqs. (4.46) and (4.53) in the case with Laguerre polynomials. We have

\[ M = \sum_{i=1}^{k} \frac{\hat{\pi}_i}{\hat{x}_i} F_i(\hat{x}) \]
and
\[
H = \frac{1}{2} \sum_{i=1}^{k} \left( \tilde{k}_i \frac{F_i(\hat{x})}{\hat{x}_i} + \frac{F_i(\hat{x})}{\hat{x}_i} \tilde{k}_i \right),
\]  
(5.50)

where \( \hat{x}_i = \hat{x}(a = a_i, N = N_i) \), \( \tilde{\pi}_i = \tilde{\pi}(a = a_i, N = N_i) \) and \( \tilde{k}_i = \tilde{k}(a = a_i, N = N_i) \). We now return to (4.30). Using (4.32) we find
\[
\hat{\pi}' = -\sqrt{N + a^t} + N.
\]  
(5.51)

Therefore, the operator (4.60) is
\[
M' = \sum_{i=1}^{k} \tilde{\pi}'_i \frac{F_i(\hat{x})}{\hat{x}_i},
\]  
(5.52)

where \( \tilde{\pi}'_i = \tilde{\pi}'(a = a_i, N = N_i) \).

VI. CONCLUSION

We have introduced in this work the approach generalizing the classical methods for linearization of nonlinear dynamical systems such as the Carleman embedding technique and Koopman formalism. Moreover, in the light of the observations of section 5 the Hilbert space approach developed by the author can be also regarded as a special case of the treatment introduced herein. The role played by classical orthogonal polynomials in the presented formalism is remarkable. On the one hand, the classical orthogonal polynomials have been shown to be the most natural tool for the Hilbert space linearization of nonlinear dynamical systems. On the other hand, the stability of classical orthogonal polynomials with respect to the nonlinear time-evolution has been demonstrated to be one of the properties which actually determine these polynomials via “canonical” algebraic relations like (4.16) and (4.21). As a consequence of the general algorithm the new methods have been found in this work for linearization of nonlinear dynamical systems connected with an arbitrary classical orthogonal polynomial, excluding the case with Hermite polynomials which has been shown herein to correspond to the classical Koopman approach. We note that in opposite to the existing approaches mentioned above, these methods cover the case of systems with the phase space different from the whole \( \mathbb{R}^k \). For example, in the case when the Hilbert space counterpart of the system (4.44) is (4.45) with \( M \) given by (5.37), we have the restrictive condition \( x_i \neq 1 \), \( i = 1, \ldots, k \), where \( x \) satisfies (4.44). Analogously, (5.49) leads to the requirement that \( x_i \neq 0 \), \( i = 1, \ldots, k \), where \( x \) fulfills (4.44). It is suggested that the phase space of the linearized nonlinear dynamical system (4.44) should coincide with a subset of \( (a, b)^k \) (see (4.1) and (4.38)), where \( (a, b) \) is the interval associated with the corresponding classical orthogonal polynomial. Therefore, the case of (5.37) refers to the system (4.44) such that the solution remains in a \( k \)-dimensional cube \((-1, 1)^k\). Clearly, an arbitrary system performing finite motion can be reduced to such one by appropriate rescaling. On the other hand, (5.49) corresponds to systems (4.44) with \( x \in \mathbb{R}^k_+ \), where \( \mathbb{R}^k_+ \) is the set of positive real numbers. The well-known example of such systems are rate equations of chemical kinetics. The experience with the Koopman linearization, the Carleman embedding technique and the Hilbert space approach indicates that the formalism introduced herein would be a useful tool in the theory of nonlinear dynamical systems. Last but not least, we point out that results of this paper would be of importance in the theory of orthogonal polynomials. Indeed, it might be observed that the formulas (4.8) and (4.9) as well as the relations (4.16) and (4.21) mentioned above seem to provide a new algebraic approach to the theory of classical orthogonal polynomials. We also point out that relations (4.51) amount a generalization of the Heisenberg algebra (C.1). On the other hand, we have shown earlier (see section 4) that this algebra can be interpreted as a condition for stability of solutions to (3.2).
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APPENDIX A: BOSE OPERATORS AND OCCUPATION NUMBER REPRESENTATION

We recall the basic properties of the standard Bose operators and the occupation number representation. The Bose creation ($a_i^\dagger$) and annihilation ($a_i$) operators, where $a = (a_1, \ldots, a_k)$ and $\mathbf{a} = (a_1^\dagger, \ldots, a_k^\dagger)$, satisfy the Heisenberg-Weyl algebra

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_j^\dagger, a_i^\dagger] = 0, \quad i, j = 1, \ldots, k.$$  \hspace{1cm} (A.1)

The Hermitian operators $N_i = a_i^\dagger a_i$, $i = 1, \ldots, k$, are called the number operators. These operators obey

$$[N_i, N_j] = 0, \quad [N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \quad i, j = 1, \ldots, k.$$  \hspace{1cm} (A.2)

Let us assume that there exists in the Hilbert space of states $H$, a unique, normalized vector $|0\rangle$ (vacuum vector) such that

$$a|0\rangle = 0.$$  \hspace{1cm} (A.3)

We also assume that there is no nontrivial closed subspace of $H$ which is invariant under the action of the Bose operators. The state vectors $|n\rangle$, $n \in \mathbb{Z}^k_+$, defined as

$$|n\rangle = \left( \prod_{i=1}^k \frac{a_i^\dagger}{\sqrt{n_i!}} \right) |0\rangle,$$  \hspace{1cm} (A.4)

are the common eigenvectors of the number operators, i.e.

$$N|n\rangle = n|n\rangle.$$  \hspace{1cm} (A.5)

These vectors form the orthonormal basis of $H$, that is

$$\langle n|m\rangle = \prod_{i=1}^k \delta_{n_i,m_i}, \quad \sum_{n \in \mathbb{Z}^k_+} |n\rangle\langle n| = I.$$  \hspace{1cm} (A.6)

The action of the Bose operators on the vectors $|n\rangle$ has the following form:

$$a_i|n\rangle = \sqrt{n_i} |n - e_i\rangle, \quad a_i^\dagger|n\rangle = \sqrt{n_i + 1} |n + e_i\rangle, \quad i = 1, \ldots, k.$$  \hspace{1cm} (A.8)

We finally write down the following formulas corresponding to the case with $k = 1$, which were frequently used throughout this work:

$$f(N)a = af(N - 1), \quad a^\dagger f(N) = f(N - 1)a^\dagger,$$  \hspace{1cm} (A.9)

where $N = a^\dagger a$ is the number operator,

$$a|n\rangle = \sqrt{n} |n - 1\rangle, \quad a^\dagger|n\rangle = \sqrt{n + 1} |n + 1\rangle.$$  \hspace{1cm} (A.10)
We now outline the main facts about the standard coherent states. Consider the coherent states $|z\rangle$, where $z \in \mathbb{C}^k$, that is the eigenvectors of the Bose annihilation operators

$$a(z) = z|z\rangle.$$  \hfill (B.1)

The normalized coherent states can be defined as

$$|z\rangle = \exp\left(-\frac{1}{2} \sum_{i=1}^{k} |z_i|^2\right) \exp\left(\sum_{i=1}^{k} z_i a_i^\dagger\right) |0\rangle,$$  \hfill (B.2)

where $|0\rangle$ is the vacuum vector. The coherent states are not orthogonal. We have

$$\langle z|w\rangle = \exp\left[-\frac{1}{2} \sum_{i=1}^{k} (|z_i|^2 + |w_i|^2 - 2 z_i^* w_i)\right].$$  \hfill (B.3)

These states form the complete (overcomplete) set. Namely,

$$\int_{\mathbb{R}^2} d\mu(z) |z\rangle\langle z| = I,$$  \hfill (B.4)

where

$$d\mu(z) = \prod_{i=1}^{k} \frac{1}{\pi} d(\text{Re} z_i) d(\text{Im} z_i).$$  \hfill (B.5)

The passage from the occupation number representation to the coherent state representation is given by the kernel

$$\langle n|z\rangle = \left(\prod_{i=1}^{k} \frac{z_i^n}{\sqrt{n_i!}}\right) \exp\left(-\frac{1}{2} \sum_{i=1}^{k} |z_i|^2\right).$$  \hfill (B.6)

On taking into account (B.4), (A.7) and (B.6) we arrive at the Fock-Bargmann space of analytic (entire) functions specified by the inner product

$$\langle \phi|\psi\rangle = \int_{\mathbb{R}^{2k}} d\mu(z) \exp\left(-\sum_{i=1}^{k} |z_i|^2\right) (\tilde{\phi}(z^*))^* \tilde{\psi}(z^*),$$  \hfill (B.7)

where $\tilde{\phi}(z^*) = \langle z|\phi\rangle \exp\left(\frac{1}{2} \sum_{i=1}^{k} |z_i|^2\right)$ and $z^* = (z_1^*, \ldots, z_k^*)$. We remark that in view of (B.4) and (B.6) the polynomials

$$p_n(z) = \prod_{i=1}^{k} \frac{z_i^n}{\sqrt{n_i!}}$$  \hfill (B.8)

form the orthonormal complete set in the Fock-Bargmann space that is, we have

$$\int_{\mathbb{R}^2} d\mu(z) \exp\left(-\sum_{i=1}^{k} |z_i|^2\right) (p_n(z^*))^* p_m(z^*) = \delta_{nm},$$  \hfill (B.9)

where $\delta_{nm} = \prod_{i=1}^{k} \delta_{n,m_i}$.
APPENDIX C: POSITION AND MOMENTUM OPERATORS

We finally collect some basic properties of position and momentum operators. The position ($\hat{q}$) and momentum ($\hat{p}$) operators, where $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_k)$ and $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_k)$ satisfy the Heisenberg algebra

\[
[\hat{q}_r, \hat{p}_s] = i\delta_{rs}, \quad [\hat{q}_r, \hat{q}_s] = [\hat{p}_r, \hat{p}_s] = 0, \quad r, s = 1, \ldots, k.
\]  

(C.1)

These operators are related to the Bose operators by

\[
\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a),
\]

\[
a = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad a^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}).
\]  

(C.2)

An immediate consequence of (C.1) is

\[
[\hat{p}, f(\hat{q})] = -i\frac{\partial f(\hat{q})}{\partial \hat{q}}.
\]  

(C.3)

Consider now the eigenvectors $|q\rangle$, $q \in \mathbb{R}^k$, of the position operators

\[
|q\rangle = q|q\rangle.
\]  

(C.4)

The normalized eigenvectors can be expressed by

\[
|q\rangle = \pi^{-\frac{k}{4}} \exp\left(\frac{1}{2}q^2\right) \exp\left[-\frac{1}{2}(a^\dagger - \sqrt{2}q)^2\right]|0\rangle,
\]  

(C.5)

where $|0\rangle$ is the vacuum vector. The states $|q\rangle$ form the orthogonal and complete set, namely

\[
\langle q | q' \rangle = \delta(q - q'),
\]  

\[
\int d^kq \langle q | q \rangle = I.
\]  

(C.6)\hspace{1cm} (C.7)

The passage from the coordinate representation spanned by the vectors $|q\rangle$ to the occupation number representation is given by the kernel

\[
\langle n | q \rangle = \left(\prod_{i=1}^k p_{n_i}(q_i)\right) \exp\left(-\frac{1}{2}q^2\right),
\]  

(C.8)

where $p_n(q)$ are normalized Hermite polynomials, that is

\[
p_n(q) = \pi^{-\frac{k}{2}} 2^{-\frac{k}{4}} \frac{1}{\sqrt{n!}} H_n(q).
\]  

(C.9)
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