Stochastic quantum field dynamics in the proper time

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Abstract
We consider a quantization of relativistic wave equations which allows to treat quantum fields together with interacting particles at a finite time. We discuss also a dissipative interaction with the environment. We introduce a stochastic wave function whose dynamics is determined by a non-linear Schrödinger-type evolution equation in an additional time parameter. The correct classical limit requires the proper time interpretation of the time parameter. An average over the proper time leads to the conventional quantum field theory of particles which are free at an infinite space separation. We consider models with scalar and vector fields on a pseudo-Riemannian manifold. A quantization of the Einstein gravity in this approach is briefly discussed.

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I Introduction
The classical non-relativistic mechanics describes system’s history $x_t$ as a function of time (a succession of events). In a classical relativistic theory time and space should be treated on an equal footing. If so then we should consider world lines (histories) $x_{\mu}(\cdot)$ rather than events. In quantum mechanics the wave function $\psi(x)$ gives an amplitude of probability of detecting a particle at the position $x$. Only in the classical limit $|\psi(x)|^2 \approx |\psi(x_t)|^2$ we regain the time-dependent trajectory. By an analogy to the non-relativistic quantum mechanics in an explicitly relativistic treatment we would require that in the classical limit the probability of an occurrence of the history $x_{\mu}(\cdot)$ in a state $\psi$ should be $|\psi(x_{\mu}(\cdot))|^2$. This probability amplitude can vary depending on a certain universal time $\tau$ which should be a Lorentz scalar. In classical relativistic physics there is a good candidate for this universal time (the proper time) which describes e.g. the periods of oscillations of a monochromatic light emitted by an atom or the frequencies of crystal vibrations in their rest frames.

The relativistic quantum mechanics is usually rejected in favor of the quantum field theory (QFT). There is at least one good reason for that: the need
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to describe the processes of particle creation and annihilation. The QFT successfully and consistently treats the scattering processes with particle creation and annihilation. However, in order to describe bound states (e.g. the Lamb shift in the hydrogen atom) we resign of a relativistic description and combine quantum field theory with relativistic quantum mechanics. It seems that the conventional QFT is unable to treat any space-time description of relativistic particles. In particular, the limit $\hbar \to 0$ of quantum field theory exists \[\] but it describes the classical field theory rather than the classical mechanics. Nevertheless, we would still be interested e.g. in the probability density of finding a relativistic electron close to the nucleus and a classical limit of this probability.

In this paper we consider a non-linear relativistic quantum wave mechanics which is inseparably bound with the quantum field theory. The quantum effects are achieved by an addition of a noise to the relativistic field wave equation. It is shown that the expectation values of the random part of the field after an average over the proper time coincide with the time-ordered vacuum expectation values of the conventional QFT.

II Relativistic wave equation

We assume that the evolution of the wave function $\psi_\tau$ is determined by its initial value $\psi$. In a classical limit we can imagine $\psi$ as a wave packet $\psi(x)$ concentrated on a certain space-time point. We consider a relativistic particle in an electromagnetic field $A$. We suggest the following analogue of the Schrödinger equation (see earlier papers on such wave equations \[\] )

$$
i\hbar \partial_\tau \psi = \frac{1}{2M} g^{\mu\nu} (-i\hbar \partial_\mu + \frac{1}{c} A_\mu)(-i\hbar \partial_\nu + \frac{1}{c} A_\nu)\psi \equiv -\frac{\hbar^2}{2M} \Box A \psi \tag{1}$$

where $c$ is the velocity of light and $g^{\mu\nu} = (1, 1, 1, -1)$ is the Minkowski metric (we shall also use the notation $\mathbb{R}^d$ for the Minkowski space suggesting an arbitrary dimension $d$ of this space). If we write (where $W_\tau$ may be complex)

$$
\psi_\tau = e^{\frac{i}{\hbar} W_\tau} \tag{2}
$$

Then, from eq. (1) it follows

$$
\partial_\tau W_\tau + \frac{1}{2M} (\partial_\mu W_\tau + \frac{1}{c} A_\mu)(\partial^\mu W_\tau + \frac{1}{c} A^\mu) - \frac{i\hbar}{2M} (\Box W_\tau + \frac{1}{c} \partial_\mu A^\mu) = 0 \tag{3}
$$

In a formal limit $\hbar \to 0$ we obtain the Hamilton-Jacobi equation

$$
\partial_\tau W_\tau + \frac{1}{2M} (\partial_\mu W_\tau + \frac{1}{c} A_\mu)(\partial^\mu W_\tau + \frac{1}{c} A^\mu) = 0 \tag{4}
$$

Eq. (4) does not coincide with the conventional Hamilton-Jacobi equation in classical relativistic dynamics \[\] which has no $\partial_\tau W$ term. Nevertheless, eq. (4)
can be derived from classical mechanics. Let us recall that if the relativistic Lagrangian is chosen in the form invariant under the reparametrization $x(\gamma) \to x(f(\gamma))$ then the canonical Hamiltonian

$$H = \frac{1}{2M} (p_\mu + \frac{1}{c} A_\mu)(p^\mu + \frac{1}{c} A^\mu)$$  \hspace{1cm} (5)$$

is identically equal to zero. This constraint $H=0$ generates correct equations of motion if the time parameter is interpreted as the proper time. If from the beginning we choose $\gamma$ as the proper time then $H \neq 0$. In such a case we may pose the problem of a canonical change of coordinates (determined by the generating function $W$) such that in the new coordinates $H \to H + \partial_\tau W = 0$. The generating function $W$ (hence also the classical dynamics) is defined by the solution of the Hamilton-Jacobi equation (4). Eq.(1) can be considered as a quantization of eq.(4). In the standard quantization scheme of constrained systems one argues that the quantum theory should be invariant under the choice of the parameter $\gamma$ (reparametrization invariance). Hence, the dependence on this parameter is gauged away and what remains is the Klein-Gordon equation $H\psi = 0$. However, in our interpretation the proper time has a physical meaning. Hence, we make this preferred choice of $\gamma$ in the Lagrangian. In such a case the canonical Hamiltonian (5) is different from zero. It again generates the correct equations of motion. Then, the conventional quantization scheme leads to eq.(1) rather than to the Klein-Gordon equation. We show below that conversely the classical dynamics (4) is determined as a limit $\hbar \to 0$ of the quantum dynamics (1) if $\tau$ is identified with the classical proper time.

For further purposes we write the metric tensor in eq.(1) in terms of complex vierbeins $v$

$$i g^{\mu\nu} = v^\mu_a v^\nu_a$$  \hspace{1cm} (6)$$

We can take as a solution of eq.(6)

$$v^k_a = \lambda \delta^k_a$$

if $k=1,2,3$ and

$$v^0_a = \lambda \delta^0_a$$

where

$$\lambda = \sqrt{i} = \frac{1}{\sqrt{2}} (1 + i)$$

The Feynman integral supplies a simple intuitive way of proving the classical limit. We assume that all functions we deal with are analytic. Then, we can express the solution of eq.(1) by the following rigorous form of the Feynman integral ($\tau \geq 0$, see refs. [10] for a more precise formulation and all assumptions)

$$\psi_\tau(x) = E \left[ \exp \left( \frac{i}{\hbar} \int_0^\tau ds A_\mu (x + \sigma v_b) \sigma v^\mu_a d\theta^a \right) \psi (x + \sigma v_b) \right]$$  \hspace{1cm} (7)$$
where $b$ is the Brownian motion i.e. the real Gaussian process with values in $\mathbb{R}^d$ and the covariance

$$E[b^a(s)b^c(\tau)] = \delta^{ac}\min(s, \tau)$$

In eq.(7)

$$\sigma = \sqrt{\frac{\hbar}{M}}$$

We write eq.(1) and eq.(7) still in another (equivalent) form. Let $W_\tau$ be the solution of the Hamilton-Jacobi equation (4) with the initial condition $W_0$. We express the wave function $\psi$ as a product

$$\psi_\tau = \exp\left(\frac{i \bar{\hbar}}{\hbar} W_\tau \right) \Phi_\tau$$

(8)

If $\psi_\tau$ is a solution of eq.(1) then $\Phi_\tau$ fulfills the equation (assuming the Lorentz gauge $\partial_\mu A^\mu = 0$)

$$\partial_\tau \Phi_\tau = -\frac{i}{\hbar} \tilde{H} \Phi_\tau = \frac{\hbar}{2M} \Box \Phi_\tau - \frac{1}{M} (\partial^\mu W_\tau + \frac{1}{c} A^\mu(q(s))) \partial_\mu \Phi_\tau - \frac{1}{2M} \Box W_\tau \Phi_\tau$$

(9)

where

$$\Box = g^{\mu\nu} \partial_\mu \partial_\nu$$

The equivalent form of the Feynman formula follows from eq.(9). So, if $q_\mu(s)$ is the solution of the equation (for $0 \leq s \leq \tau$)

$$dq_\mu = -\frac{1}{M} \left( \partial_\mu W(\tau - s, q(s)) + \frac{1}{c} A_\mu(q(s)) \right) ds + \sigma v_\mu db^\mu(s)$$

(10)

where $W(\tau)$ is the solution of the Hamilton-Jacobi equation (4), then the solution of eq.(1) with the initial condition $\psi = \exp(\frac{i \bar{\hbar}}{\hbar} W) \Phi$ reads

$$\psi_\tau(x) = \exp\left(\frac{i \bar{\hbar}}{\hbar} W(\tau, x) \right) E \left[ \exp\left( -\int_0^\tau \frac{1}{2M} \Box W(\tau - s, q(s)) ds \right) \Phi(q(\tau)) \right]$$

(11)

In the limit $\hbar \to 0$ of the stochastic process we obtain the flow (here $0 \leq s \leq \tau$)

$$\frac{d\xi_\mu}{ds} = -\frac{1}{M} \left( \partial_\mu W(\tau - s, \xi(s)) + \frac{1}{c} A_\mu(\xi(s)) \right)$$

(12)

Till $O(\hbar)$ terms we have in the semi-classical approximation

$$\psi_\tau(x)^{cl} = \exp\left(\frac{i \bar{\hbar}}{\hbar} W(\tau, x) \right) \Phi(\xi(\tau, x)) \exp\left( -\int_0^\tau \frac{1}{2M} \Box W(\tau - s, \xi(s)) ds \right)$$

(13)
If we differentiate eq.(12) once more over \( s \) and make use of the Hamilton-Jacobi equation (4) then we conclude that \( \xi \) fulfills the equation

\[
M \frac{d^2 \xi_\mu}{ds^2} = F_{\mu\nu}(\xi) \frac{d\xi^\nu}{ds}
\]  

(14)

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

with the boundary conditions

\[
\xi(s)|_{s=0} = x
\]

\[
\frac{d\xi_\mu}{ds}|_{s=\tau} = -\frac{1}{M}(\partial_\mu W(x) + \frac{1}{c} A_\mu(x))
\]  

(15)

We could also obtain eq.(14) directly from the Feynman formula (7). For this purpose we make a shift of variables \( b^a \rightarrow b^a + f^a \). Then, for any functional \( \chi \) of the Brownian motion \( (b(s), s \leq \tau) \) we have (the Cameron Martin formula [12])

\[
E[\chi(b)] = E\left[\exp\left(-\int_0^\tau f^a db^a - \frac{1}{2} \int_0^\tau \frac{df^a}{ds} \frac{df^a}{ds} ds\right) \chi(b+f)\right]
\]  

(16)

We apply the shift (16) to the Feynman formula (7) with \( \psi = \exp(\frac{1}{\hbar} W)\Phi \) and \( \sigma v f = \xi - x \), where \( \xi \) is the solution of the equation (14) with the boundary conditions (15). Then, expanding in \( \sigma \) we obtain the formula (13) till \( O(\sigma) \).

Summarizing the result (14) derived either from the Feynman formulas (7) and (11) or directly from the Schrödinger equation in the form (9) we can say that the limit \( \hbar \rightarrow 0 \) exists if and only if there exists \( \xi \) such that the equations of motion (14) are satisfied. As an immediate but important consequence of the equations of motion (14) we obtain

\[
\frac{d^2 \xi_\mu}{ds^2} \frac{d\xi_\mu}{ds} = 0
\]

Hence, the square of the covariant velocity is time-independent. From the boundary condition (15) we obtain that

\[
\frac{d\xi_\mu}{ds} \frac{d\xi_\mu}{ds} = \frac{1}{M^2}(\partial_\mu W(x) + \frac{1}{c} A_\mu(x))(\partial^\mu W(x) + \frac{1}{c} A^\mu(x))
\]

If in the Hamilton-Jacobi equation (4)

\[
\partial_\tau W_\tau = \frac{Mv^2}{2}
\]  

(17)

Then

\[
\frac{d\xi_\mu}{ds} \frac{d\xi_\mu}{ds} = -c^2
\]  

(18)
Eq. (18) means that $s$ (hence also $\tau$ in the Schrödinger equation) is the proper time. Moreover, this interpretation must be preserved in higher orders of $\hbar$ because the subsequent quasiclassical expansion is determined by the first order. It is well-known from classical relativistic dynamics that the equations of motion (14) with a time parameter $s$ have a correct form only if $s$ coincides with the proper time. If the dependence of $W_\tau$ on $\tau$ in eq. (8) is determined by eq. (17) then $\exp(\frac{i}{\hbar}W_\tau(x))$ also solves the Klein-Gordon equation (in such a case we regain the conventional relativistic wave equation). On the other hand $\psi_\tau \approx \exp(\frac{i}{\hbar}W_\tau(x))$ means that the wave function oscillates rapidly with a very small period of oscillations. It is known from classical mechanics that such additional rapid oscillations have little effect on the mean behavior in a slowly varying potential. We suggest that even if the condition (17) is not satisfied then the evolution in the proper time describes rapid oscillations which have little effect on the evolution in the Minkowski time for most physically relevant potentials $A_\mu$. Hence, after an averaging over the proper time we obtain the conventional quantum theory.

For an understanding of eq. (1) it is important to see its non-relativistic limit (the problem is discussed in a different way in ref. [13]). The non-relativistic energy $\epsilon$ is related to the relativistic energy $p_0$, mass $M$ and the momentum $p$ by the formula

$$\epsilon = c(p^2 + M^2c^2)^{\frac{1}{2}} - Mc^2 = \frac{p^2}{2M} + o(\frac{1}{c})$$

Let

$$\psi_\tau = \exp(\frac{-iMc^2(\tau - 2t)}{2\hbar})\tilde{\psi}_\tau$$

Then, in the limit $c \to \infty$ we obtain (we write $A = (A, V)$ i.e. $A_0 = V$)

$$i\hbar\partial_\tau \tilde{\psi}_\tau = -i\hbar\partial_t \tilde{\psi}_\tau + \frac{1}{2M}(-i\hbar \nabla + \frac{1}{c}A)^2 \tilde{\psi}_\tau + V(x, t)\tilde{\psi}_\tau \equiv K\tilde{\psi}_\tau$$ (19)

Strictly speaking we should have omitted the $\frac{1}{c}A$ term in the limit $c \to \infty$ but we keep it in order to be in agreement with the conventional procedure. If the potentials $A$ and $V$ are $t$-independent then we can express eq. (19) as the Schrödinger equation with a new time $\hat{t} = t + \tau$. In such a case $\tau$ is just a global shift of time in the non-relativistic quantum mechanics.

If the potentials are time-dependent then we obtain Howland’s description [14] of the time evolution in time-dependent potentials. In such a case $t$ is treated as a coordinate on an equal footing with $x$ (the eigenvalues of $K$ are called quasienergies and constitute a standard tool in an investigation of time-dependent systems).

The relation between the $\tau$-evolution and the conventional Schrödinger evolution $U_{sch}$ is well-known [14]

$$\exp(-\frac{i\tau}{\hbar}K)\tilde{\psi}(t, x) = U_{sch}(t, t - \tau)\tilde{\psi}(t - \tau, x)$$ (20)
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In particular, it follows that the scattering theories in terms of $\exp(-i\tau K)$ and $U_{\text{sch}}$ are equivalent.

III Quantum free fields and free particles

Let us consider a time evolution of the wave function with no interaction

$$i\hbar \partial_\tau \phi_\tau = -\frac{\hbar^2}{2M} \Box \phi_\tau$$

(21)

An easy computation shows that the wave packet

$$\phi(x) = \int dp \tilde{\phi}(p) \exp\left(\frac{i}{\hbar} p_\mu x^\mu\right)$$

evolves into

$$\phi_\tau(x) = \int dp \tilde{\phi}(p) \exp\left(-\frac{i p^2 \tau}{2M\hbar}\right) \exp\left(\frac{i}{\hbar} p_\mu x^\mu\right)$$

(22)

If $\tilde{\phi}(p)$ is regular in $\hbar$ and its support is concentrated at $p_c$ e.g.

$$\tilde{\phi}(p) = \exp\left(-\frac{1}{2\mu} (p - p_c)^2 - \frac{1}{2\mu} (p_0 + p_{0c})^2\right)$$

then we can conclude that

$$\phi_\tau(x) \approx \phi(x - \frac{p_c \tau}{M})$$

(23)

Eq.(23) correctly describes the evolution of a relativistic wave packet if $\tau$ is interpreted as the proper time.

We can describe a wave packet of $k$-particles by a generalization of eq.(22)

$$\phi(x(1),...,x(k)) = \int dp(1)....dp(k) \tilde{\phi}(p(1),...,p(k)) \exp\left(\sum_{j=1}^k \frac{i}{\hbar} p_\mu(j) x^\mu(j)\right)$$

Its time evolution is determined by a generalization of eq.(21)

$$i\hbar \partial_\tau \phi(x(1),...,x(k)) = -\frac{\hbar^2}{2M} \sum_{j=1}^k \Box_j \phi(x(1),...,x(k))$$

(24)

The solution of eq.(24) shows an independent free evolution of each particle

$$x_\tau(j) = x(j) + \frac{p_{0c}(j) \tau}{M}$$

By a free quantum field we understand an object which can describe any number of free particles as excitations of the field at any point of the space-time. By its
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physical meaning the quantum field of a large intensity is supposed to behave as its classical counterpart (keep in mind the example of the electromagnetic field). It should also describe fluctuations corresponding to the Heisenberg uncertainty principle. We represent these fluctuations by the Gaussian field $B$ with the covariance

$$E[B(\tau, x)B(s, y)] = \min(\tau, s)\delta(x - y)$$  \hfill (25)

We quantize the wave equation (21) by adding the noise (when $M \to \infty$ then $\phi$ is the Brownian motion)

$$d\phi(\tau, x) = \frac{i\hbar}{2M}\Box\phi(\tau, x)d\tau + \sqrt{2\hbar}dB(\tau, x)$$  \hfill (26)

The solution of eq. (26) is a sum of two pieces (see the subsequent section): the first part is the wave function (21) and the second one is a noise. We describe the quantum field by means of the random part relating the correlation functions of the random field to time-ordered products of non-commutative operators.

The solution of eq. (26) depends on the proper time parameter $\tau$. We suggest that correlations which are observed in experiments result from an average over the rapid oscillations in the proper time. We shall show that time-ordered vacuum expectation values of the conventional QFT of massless particles coincide with an average over the proper time

$$<\phi(x_1)....\phi(x_k)> = \lim_{T \to \infty}(T - \tau_0)^{-1}\int_{\tau_0}^{T} d\tau E[\phi(\tau, x_1)....\phi(\tau, x_k)]$$  \hfill (27)

For massive particles with mass $M$ we should add the term $\frac{\hbar}{2M}M\phi$ on the r.h.s. of eq.(26) i.e. $\frac{1}{2\hbar}\Box \to \frac{1}{2\hbar}(\Box - M^2)$ (in eq.(1) $M$ fulfills the role of the mass only in the non-relativistic limit).

There is an alternative to the stochastic quantization (26) of the wave equation (21). We could simply treat eq.(21) as an operator equation in the Fock space. Then, the conventional quantum free field is a particular $\tau$-independent solution of eq.(21). We could add an interaction and continue the proper time quantization in the operator formalism. We outline such an approach in the Appendix.

IV Mathematical aspects of the linear stochastic equation

We consider a general form of the linear stochastic differential equation

$$d\phi_{\tau} = -iA\phi_{\tau}d\tau + \sqrt{2\hbar}dB_{\tau}$$  \hfill (28)
Eq. (28) is treated as an equation in the Gelfand triple \[ S(R^d) \subset L^2(R^d) \subset S'(R^d) \]
where \( S'(R^d) \) is the Schwartz space of tempered distributions. \( \mathcal{A} \) is a real self-adjoint operator in \( L^2(R^d) \). Hence, there exists the unitary group \( U \)
\[ U(\tau) = \exp(-i\tau \mathcal{A}) = \cos(\tau \mathcal{A}) - i \sin(\tau \mathcal{A}) \] (29)
The solution of eq. (28) with the initial condition \( \phi \) at \( \tau_0 \) reads
\[ \phi_\tau = U(\tau - \tau_0)\phi + \sqrt{2\hbar} \int_{\tau_0}^{\tau} U(\tau - s)dB_s \] (30)
It is common to work with real processes. For this purpose let us decompose the complex field \( \phi = \phi_1 + i\phi_2 \) into its real and imaginary parts
\[ \phi_1(\tau) = \cos(\mathcal{A}(\tau - \tau_0))\phi_1 + \sin(\mathcal{A}(\tau - \tau_0))\phi_2 + \sqrt{2\hbar} \int_{\tau_0}^{\tau} \cos(\mathcal{A}(\tau - s))dB_s \] (31)
and
\[ \phi_2(\tau) = \cos(\mathcal{A}(\tau - \tau_0))\phi_2 - \sin(\mathcal{A}(\tau - \tau_0))\phi_1 - \sqrt{2\hbar} \int_{\tau_0}^{\tau} \sin(\mathcal{A}(\tau - s))dB_s \] (32)
The solutions \( \phi_1(\tau) \) and \( \phi_2(\tau) \) determine the transition function (from the initial point \( (\phi_1, \phi_2) \) to a set \( \Gamma \subset S' \)) of the stochastic process
\[ P(\tau_0, \phi_1, \phi_2; \tau, \Gamma) = P((\phi_1(\tau), \phi_2(\tau)) \in \Gamma) \] (33)
The stochastic process \( \phi_\tau \) determines a solution of the differential equation
\[ \partial_\tau \Phi_\tau = \hbar^2 \text{Tr}(D^2\Phi_\tau) - i(\mathcal{A}\phi, D\Phi_\tau) \] (34)
where
\[ (D\Phi(\phi), f) = \lim_{\epsilon \to 0} \epsilon^{-1} \left( \Phi(\phi + \epsilon f) - \Phi(\phi) \right) \]
is the Frechet derivative; the second order derivative \( D^2\Phi \) is an operator whose trace gives the Laplacian in an infinite number of dimensions (for a diffusion in infinite dimensional spaces see [14]). In physicists’ notation
\[ (\mathcal{A}\phi, D\Phi) = \int dx (A\phi)(x) \frac{\delta \Phi}{\delta \phi(x)} \]
and
\[ \text{Tr}(D^2\Phi) = \int dx \frac{\delta^2 \Phi}{\delta \phi(x) \delta \phi(x)} \]
The solution of the differential equation (34) with the initial condition $\Phi$ reads

$$\Phi_\tau(\phi) = E\left[ \Phi_\tau(\phi) \right]$$  \hspace{1cm} (35)

where $\phi_\tau(\phi)$ is the solution of eq.(28) with the real initial condition $\phi$ at $\tau_0$.

We can also express the solution by means of the transition probability (33)

$$\Phi_\tau(\phi) = \int P(\tau_0, \phi, 0; \tau, d\phi'_1, d\phi'_2) \Phi(\phi'_1 + i\phi'_2)$$  \hspace{1cm} (36)

$P(\tau, \phi, \Gamma)$ also solves eq.(34) with a $\delta$-type initial condition. The transition probability $P$ is defined by the Gaussian measure which is uniquely determined by its mean and covariance. The mean and the covariance can be calculated explicitly from eqs.(31)-(32). It can be seen from these equations that the average (27) over $\tau$ does not exist separately for $\phi_1$ and $\phi_2$. We show however that the average (27) does exist for the complex field $\phi = \phi_1 + i\phi_2$. First of all, concerning the mean value

$$\int_{\tau_0}^{T} d\tau (f, \exp(-iA(\tau - \tau_0))\phi) = i((\exp(-i(T - \tau_0)A) - 1)A^{-1}f, \phi)$$  \hspace{1cm} (37)

Hence, the limit (27) of the mean value is equal to zero if $A$ is invertible on $f$ i.e. if $f \in \text{Range}(A)$. On the other hand if we consider $A = -\hbar^2 \Box$ and the initial condition $\phi$ satisfies the free wave equation

$$\Box \phi = 0$$  \hspace{1cm} (38)

then such an initial wave function gives a non-trivial contribution to the average (27). We shall discuss this term later. Now, consider the covariance

$$E[(\phi_\tau, f) - E[(\phi_\tau, f)])(\phi_{\tau'}, f') - E[(\phi_{\tau'}, f')]] = \frac{\hbar^2}{2i} (f, A^{-1}\left(\exp(-iA|\tau - \tau'|) - \exp(-i(\tau + \tau' - 2\tau_0)A)\right)f')$$  \hspace{1cm} (39)

The average (27) is

$$\lim_{T \to \infty}(T - \tau_0)^{-1} \int_{\tau_0}^{T} d\tau E[(\phi_\tau, f) - E[(\phi_\tau, f)])(\phi_\tau, f') - E[(\phi_\tau, f')]] = \frac{\hbar^2}{2i} (f, A^{-1}f') - \frac{\hbar^2}{4} \lim_{T \to \infty}(T - \tau_0)^{-1}(A^{-1}f, \left(\exp(-2i(T - \tau_0)A) - 1\right)A^{-1}f')$$  \hspace{1cm} (40)

The limit of the second term on the r.h.s. is equal to zero under the assumption that $A^{-1}f$ and $A^{-1}f'$ exist. So, we define $A^{-1}$ first on a restricted set of functions ( with no support on the mass shell). However, it is not sufficient to define quantum fields only on test functions $f$ with no support on the mass-shell ($p^2 = 0$ or more general $p^2 = -M^2$ for a massive particle when $A = -\hbar(\Box - M^2)$). We have to specify the correlation functions also on the mass shell. In the
distribution theory, this means an extension of a linear functional to the whole of $S$. Defining the limit $T \to \infty$ is equivalent to a particular extension. There is a natural definition of this extension resulting from the solution (30) of eq.(28) and related to the treatment of indefinite integrals and especially oscillatory integrals. We make the replacement ($\epsilon > 0$)

$$A \to A - i\epsilon$$

(41)

Then, the distribution

$$\Box^{-1}(x,y) = \lim_{\epsilon \to 0} (2\pi)^{-d} \int dp \exp(-ip(x-y))(-p^2-i\epsilon)^{-1} = \Delta_F(x-y)$$

(42)

coincides with Feynman’s causal function which is equal to the time-ordered vacuum expectation value of the real scalar free field

$$<0|T(\phi(x)\phi(y))|0> = i\hbar \Delta_F(x-y)$$

(43)

Note that if $(x-y)^2 \neq 0$ in eq.(42) then $\Delta_F$ is a regular function. This property remains true for the kernel $K_T(x-y)$ of the operator

$$K_T = \left(\exp(-2i(T-T_0)A) - 1\right)A^{-2}$$

( eq.(40) but now with $A = -\frac{\hbar}{2M} \Box$). Hence, if $(x-y)^2 \neq 0$ then the limit $T \to \infty$ in eq.(40) holds true not only in a distributional sense but also for the operator kernels (in particular $K_T(x-y)$ tends to zero for every $x$ and $y$ if $(x-y)^2 \neq 0$). This stronger mode of convergence of the $\tau$-averages may be important for a convergence in a model with an interaction when there is a non-trivial renormalization and the distributional convergence is not sufficient.

We can generalize this result to an average of an arbitrary number of fields

$$\langle ((\phi_\tau,f_1) - E[(\phi_\tau,f_1)])((\phi_\tau,f_2) - E[(\phi_\tau,f_2)])...((\phi_\tau,f_{2n}) - E[(\phi_\tau,f_{2n})])\rangle = \hbar^2 \sum_{\text{pairs}} \prod_{(j,k)} \frac{1}{2}(f_j, A^{-1} f_k)$$

(44)

where the sum is over the product of all pairs $(j,k)$ in agreement with the Gaussian integral combinatorics (the expectation value (44) is equal to zero if the number of fields is odd). With $A = -\hbar(\Box - M^2)$ eq.(44) can be proved by means of explicit computations (as in eqs.(39)-(40)) through an application of the Fourier transform.

The limit (27) exists on test functions such that $A^{-1} f$ is well-defined i.e. $f \in \text{Range}(A)$. We could instead of eq.(28) consider the equation for $A\phi$

$$d(A\phi) = -iA(A\phi)d\tau + \sqrt{2}\hbar dB$$

Then, we would not have the problem of an inverse in eq.(44) (we just let $f_j \to Af_j$ in eq.(44)). We would have obtained the correlation functions of
A\phi = \chi. The ambiguity in the inversion problem arises if we wish to express \phi by \chi (this problem can be considered as another linear stochastic equation). Such an equation in \mathcal{S}' poses the problem of an extension of a linear functional defined on a subspace of \mathcal{S} to the whole of \mathcal{S}.

In this section we have discussed only the random field corresponding to the quantum real scalar field. If the quantum field has more components then we proportionally increase the number of components of the random field (as well as the number of Brownian motions in eq.(26)). So, for example we treat the complex scalar field as a real doublet \( (\phi_1, \phi_2) \). The stochastic equation for a free electromagnetic field depends on whether we add the gauge fixing terms to the Lagrangian or not. Without any gauge fixing terms it reads

\[ \partial_\tau A_\mu = \hbar \partial^\nu F_{\mu\nu} d\tau + \sqrt{2}\hbar dB_\mu \]  

(45)

If an external current \( J_\mu \) is added to the Lagrangian then the stochastic equation takes the form

\[ \partial_\tau A_\mu = i\hbar \partial^\nu F_{\mu\nu} d\tau - i\hbar J_\mu d\tau + \sqrt{2}\hbar dB_\mu \]  

(46)

We would not obtain a finite limit \( T \to \infty \) of the correlation functions (27) of \( A \) without any gauge fixing. However, we may restrict ourselves to gauge invariant observables e.g. to \( F_{\mu\nu} \). Then, we can easily rewrite eq.(46) in a gauge invariant form as an equation for \( F_{\mu\nu} \)

\[ \partial_\tau F_{\alpha\mu} = i\hbar \partial^\nu (\partial_\alpha F_{\mu\nu} - \partial_\mu F_{\alpha\nu}) d\tau - i\hbar (\partial_\alpha J_\mu - \partial_\mu J_\alpha) d\tau + \sqrt{2}\hbar (\partial_\alpha dB_\mu - \partial_\mu dB_\alpha) \]

We can still simplify the problem of solving this equation if we take a divergence of both sides

\[ \partial_\tau \partial^\alpha F_{\alpha\mu} = -i\hbar \Box F_{\alpha\mu} d\tau - i\hbar \partial^\alpha (\partial_\alpha J_\mu - \partial_\mu J_\alpha) d\tau + \sqrt{2}\hbar \partial^\alpha (\partial_\alpha dB_\mu - \partial_\mu dB_\alpha) \]  

(47)

Now, we can apply the formulas (29)-(30) where \( \mathcal{A} = \hbar \Box \). The exponential of this operator and its kernel are well-known. We obtain a formula expressing the \( \tau \)-averages of \( \partial^\nu F_{\alpha\nu} \) by a Gaussian field \( \chi_\alpha(x) \) with known correlation functions. The gauge problem appears as the non-uniqueness of the potential \( A \) solving an equation \( \partial^\nu F_{\alpha\nu} = \chi_\alpha \). We discuss here these elementary aspects of linear stochastic equations because they will appear in a more complex form in quantum gravity discussed briefly at the end of this paper.

Eq.(45) without the noise is treated as a wave equation for the photon. Not surprisingly the solution is complex and

\[ F_{jk}(\tau, x)F_{jk}(\tau, x) + F_{0k}(\tau, x)F_{0k}(\tau, x) \]  

(48)

can be interpreted as the probability density (unnormalized) of finding a photon in the space-time point \( x \) measured at the proper time \( \tau \). Note that eq.(48) gives a generalization of the conventional statistical interpretation of the electromagnetic field when we consider \( \tau \)-independent (real) solutions (then eq.(48) defines the electromagnetic energy density).
V General Lagrangians

We consider now an interaction among relativistic fields $\phi_a$ described by a general Lagrangian $L(\phi_a)$. Its action integral is denoted $L(\phi_a)$. The $\tau$-independent (no noise) solutions of the wave equations for $\phi_a$ should coincide with the classical non-linear waves. A proper generalization of stochastic equations of sec.4 reads

$$d\phi_a(\tau, x) = i\hbar \frac{\delta L}{\delta \phi_a(\tau, x)} d\tau + \sqrt{2\hbar} dB(\tau, x)$$

where

$$E[B_{a}(\tau, x)B_{c}(s, y)] = min(\tau, s)\delta_{ac}\delta(x - y)$$

We write eq.(49) in a symbolic form

$$d\phi = -iA\phi d\tau - igG(\phi)d\tau + \sqrt{2\hbar} dB$$

Let (as in eq.(35)) $\Phi_\tau(\phi) = E[\Phi(\phi_\tau(\phi))]$ then it follows from a general theory of stochastic equations [16] that $\Phi_\tau(\phi)$ is a solution of the equation

$$\partial_\tau \Phi_\tau \equiv G\Phi_\tau = \hbar^2 Tr(D^2 \Phi_\tau) - i(\Phi_\tau(\phi) + gG(\phi), D\Phi_\tau)$$

We can solve eq.(52) by means of an expansion in $g$.

$$\Phi_\tau = \sum_{n=0}^{\infty} g^n \Phi_\tau^{(n)}$$

Then

$$\partial_\tau \Phi_\tau^{(n)} = \hbar^2 Tr(D^2 \Phi_\tau^{(n)}) - i(\Phi_\tau^{(n)} + gG(\phi), D\Phi_\tau^{(n-1)})$$

The solution of this equation can be expressed by means of the transition function (33)

$$\Phi_\tau^{(n)}(\phi_1 + i\phi_2) = -i \int P(\tau_0, \phi_1, \phi_2; \tau, d\phi_1', d\phi_2')(G(\phi'), D\Phi_\tau^{(n-1)})$$

where $\phi' = \phi_1' + i\phi_2'$. In this way we obtain a perturbative solution of eq.(52) (the convergence of the series for interactions $G(\phi)$ of physical interest remains an open problem).

In the Appendix we outline a proper time quantization of interacting quantum fields in the Fock space. We show in the lowest order of the perturbation theory that such a quantization leads to the same results as the stochastic quantization (49). We supply also some non-perturbative arguments to this conjecture. However, we think that the stochastic approach although equivalent to the operator one avoids many difficulties related to non-commutativity. Moreover, it is useful for computations; in particular, for numerical simulations (see refs. [15] [13] for numerical simulations of complex Langevin equations of the form (49) in the context of the stochastic quantization scheme of refs. [19] [20]).
VI An average over the proper time

When we average over the proper time $\tau$ (as in eq.(27)) then we obtain a linear functional $f$ on a set of functions of fields

$$f(\Phi) = \langle \Phi \rangle$$ (56)

We are interested in obtaining equations which could determine the averaged values. For real processes and real diffusion equations the averaged value $f(\Phi)$ can be expressed by a measure called an invariant measure. Let us recall this definition [12]. The solution $\Phi(\phi) = E[\Phi(\phi_\tau(\phi))]$ defines a semigroup $\Phi_\tau(\phi) \equiv (P_\tau\Phi)(\phi)$. We say that $\nu$ is an invariant measure if for a dense set of functions

$$\int d\nu(\phi)(P_\tau\Phi)(\phi) = \int d\nu(\phi)\Phi(\phi)$$ (57)

So, we could say that computing the expectation values with respect to the noise $B$ is equivalent to a functional integral with respect to $\nu$.

Differentiating eq.(57) over time at $\tau = 0$ we obtain

$$\int d\nu(\phi)\mathcal{G}\Phi(\phi) \equiv \int d(\mathcal{G}^*\nu)\Phi = 0$$ (58)

where the adjoint operator $\mathcal{G}^*$ is defined by the duality in the space of linear functionals.

We shall show that the averaged value (27) is formally expressed by the Feynman integral. In fact, the averaging over the proper time could have been considered as a rigorous definition of the Feynman integral. We have already calculated the average values as the limit $T \to \infty$ of the correlations of the free fields (44). On the other hand a formal Feynman integral gives the same result

$$\int d\phi \exp\left(\frac{i}{\hbar} L_{\text{free}}(\phi)\right) \phi(x_1) \ldots \phi(x_{2n}) = \langle \phi(x_1) \ldots \phi(x_{2n}) \rangle_{\text{free}}$$

here

$$L_{\text{free}}(\phi) = -\frac{1}{2} \int \phi \Box \phi$$

We shall show that in each order of the perturbation expansion the correlation functions defined by an average over the proper time fulfil the same differential equations as the ones resulting from a perturbative calculation of the Feynman integral (the statement has been shown by means of a formal argument concerning distributional limits of oscillatory integrals in refs. [21][22]; the method applied here has been introduced first in the context of the stochastic quantization in ref. [23]). We neglect problems of the ultraviolet divergencies. In order to avoid the singularities we should regularize the noise $B$. Then, we prove that the $\tau$-averaged stochastic correlation functions coincide with the regularized correlation functions of the conventional QFT. In two-dimensional QFT we need no
ultraviolet regularization, hence the proof applies directly (but because of the infrared problems we add the mass $\Box \to \Box - M^2$).

As a first step in the proof we note that if the average over the proper time exists then from eq.(52)

$$0 = \lim_{T \to \infty} (T - \tau_0)^{-1} \int_{\tau_0}^T d\tau \partial_{\tau} E[\Phi(\phi_\tau(\phi))] = \lim_{T \to \infty} (T - \tau_0)^{-1} \int_{\tau_0}^T d\tau G_\phi E[\Phi(\phi_\tau(\phi))] = G_\phi < \Phi(\phi_\tau(\phi)) >$$

If the average in the last line of eq.(59) is expressed by a (complex) measure $\nu$ then eq.(57) defines an equation for this measure

$$G^* \nu(\phi) = 0$$

or

$$D\left(h^2 D + iA\phi + igG(\phi)\right) \nu = 0$$

If $A = -\Box$ and

$$G(\phi) = hDV(\phi)$$

then the solution of eq.(60) as a formal complex Feynman measure reads

$$d\nu(\phi) = d\phi \exp\left(\frac{i}{h} L(\phi)\right)$$

where

$$L(\phi) = \int dx L(\phi(x)) = - \int dx \left(\frac{1}{2} \Box \phi + gV(\phi(x))\right)$$

**VII Scalar quantum electrodynamics**

We discuss in more detail the scalar electrodynamics with $(\phi^2)^2$ interaction (this interaction is needed for stability in less than four dimensions and additionally for renormalizability in four dimensions). We treat a complex scalar field as a doublet $\phi_a (a = 1, 2)$. Then eqs.(49) read (in the Feynman gauge; $e$ denotes the electric charge)

$$d\phi_a = \hbar \Box \phi_a d\tau + i\hbar(\phi_a^2 + \phi_b^2)\phi_a d\tau - i\epsilon_{ab} A_\mu \partial_\mu \phi_a d\tau - \frac{i}{\hbar} e^2 A_\mu A^\mu d\tau + \sqrt{2} h dB_a$$

$$A_\mu = \hbar \Box A_\mu d\tau + i\hbar \epsilon_{ab} \phi_b \partial_\mu \phi_a d\tau + \sqrt{2} h dB_\mu$$

We write eq.(64) in an integral form specifying the initial condition

$$A_\mu(\tau, x) = \exp(i\hbar(\tau - \tau_0) \Box) A_\mu + i\hbar \int_{\tau_0}^\tau \exp(i\hbar(\tau - s) \Box) \epsilon_{ab} \phi_b \partial_\mu \phi_a ds + \sqrt{2} h \int_{\tau_0}^\tau \exp(i\hbar(\tau - s) \Box) dB_\mu(s) \equiv A_\mu^0 + A_\mu^Q$$
where we denoted by $A_{\mu}^{cl}$ the noise-independent part of $A_{\mu}$. Then, eq.(63) is rewritten in an integral form

$$
\phi(\tau, x) = \exp(i\bar{h}(\tau - \tau_0) \Box A)\phi + i\bar{h} \int_{\tau_0}^{\tau} \exp(i\bar{h} \Box A(\tau - s))\left(\phi_1^2 + \phi_2^2\right)\phi(s)ds + \sqrt{2\bar{h}} \int_{\tau_0}^{\tau} \exp\left(i\bar{h} \Box A(\tau - s)\right)dB(s)
$$

(65)

It is understood that $\phi$ in this equation is a two-dimensional vector and $A_{\mu}$ is a matrix with matrix elements $(A_{\mu})_{ab} = \epsilon_{ab} A_{\mu}$. If we insert 0 as the initial conditions in eqs.(63)-(64) then as follows from secs.6 and 8 after averaging over $\tau$ we obtain the standard scalar QED. In particular, the formula (74) of the subsequent section can be applied for a computation of the scattering amplitudes. If the initial conditions $\phi$ in eq.(63) and $A^{cl}$ in eq.(64) are different from zero then our formalism goes beyond the conventional QFT. We restrict ourselves here to a discussion of the approximation with $e = 0$ in eq.(64) whereas in eq.(63) we neglect the noise and the terms non-linear in $\phi$. Then, eq.(63) is just the relativistic wave equation (1) with the electromagnetic field $A$ which is the sum $A^{cl} + A^Q$. In such a case the solution (65) of eq.(63) is again expressed by eq.(7) and eq.(11). We can compute the contribution of the quantum electromagnetic fluctuations upon the meson scattering processes as well as the effect of these fluctuations on the energy levels of mesonic atoms (the Lamb shift).

In this approximation to QFT we have a well-defined limit $\hbar \to 0$ of QFT when the first term on the r.h.s. of eq.(65) describes a localized trajectory of eq.(13). The method of calculations in the framework of the proper time stochastic field theory may be considered as a rigorous version of the argument of Welton. We consider the approximation (10)-(13). The solution of the Hamilton-Jacobi equation (4) can be expressed by the solution of the Newton equation (14)

$$
W(\tau, x) = W(\xi(\tau, x)) + \int_0^{\tau} \left(\frac{M}{2} \frac{d\xi_{\mu}}{ds} \frac{d\xi^\mu}{ds} + e_{c} A_{\mu} \frac{d\xi_{\mu}}{ds}\right)ds
$$

We compute perturbatively the effect of $A^Q$ upon the trajectory (we performed similar computations in ref.[26]). We write

$$
\xi = \xi^{(cl)} + \xi^{(Q)} = \xi^{(cl)} + \hbar \xi^{(1)} + \hbar^2 \xi^{(2)}
$$

where

$$
M \frac{d^2 \xi^{(cl)}_{\mu}}{ds^2} = F_{\mu\nu}^{(cl)}(\xi^{(cl)}) \frac{d\xi^{(cl)\nu}}{ds}
$$

and

$$
M \frac{d^2 \xi^{(Q)}_{\mu}}{ds^2} = (F_{\mu\nu}(\xi^{(cl)} + \xi^{(Q)}) - F^{(cl)}_{\mu\nu}(\xi^{(cl)})) \frac{d\xi^{(cl)\nu}}{ds} + F_{\mu\nu}(\xi^{(cl)} + \xi^{(Q)}) \frac{d\xi^{(Q)\nu}}{ds}
$$

(66)

We can compute from eq.(66) perturbatively $\xi^{(Q)}$ in powers of $A^Q$ (with $A^{cl} = (0, 0, 0, -e/\bar{h})$ we obtain Bethe’s and Welton’s $\delta(x)$ correction to the
Stochastic quantum field

potential). Then, the evolution of the relativistic wave function in the electromagnetic field $A^{cl} + A^Q$ is computed from the formula

$$\phi_\tau = \langle\langle \exp(i\hbar(\tau - \tau_0)\Box_A)\phi \rangle\rangle$$

where $\langle\langle .. \rangle\rangle$ means the average over the electromagnetic field in the approximation $e = 0$ in eq.(64).

VIII Large time behavior and scattering

If the interaction is weak at large distances then we expect that the wave function $\phi_\tau$ or the corresponding stochastic field behaves at large time as a solution of the equation with no interaction (eq.(28) with $A \to A_0$, a differential operator with constant coefficients). Without noise such a behavior means

$$\phi_\tau \approx \exp(-i\tau A_0)\phi_{in}$$  \hspace{1cm} (67)

when $\tau \to -\infty$ (on a formal level such a behavior can be imposed if we rewrite eq.(49) in an integral form) and

$$\phi_\tau \approx \exp(-i\tau A_0)\phi_{out}$$

when $\tau \to +\infty$.

In non-relativistic quantum mechanics if the Schrödinger time evolution is determined by $H$ then a local distortion of the behavior (scattering) is described by the operator

$$S = \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) = \lim_{t \to \infty, t_0 \to -\infty} \exp(\frac{i}{\hbar} H_0 t) \exp(-\frac{i}{\hbar} H(t - t_0)) \exp(-\frac{i}{\hbar} H_0 t_0)$$  \hspace{1cm} (68)

If $H - H_0 = V$ then $U$ satisfies the equation (useful for perturbative calculations)

$$i\hbar \partial_t U = V(t)U$$  \hspace{1cm} (69)

where

$$V(t) = \exp(\frac{i}{\hbar} H_0 t) V \exp(-\frac{i}{\hbar} H_0 t)$$

Similar formulas hold true for a relativistic quantum mechanics described by the Hamiltonian $A$

$$S_{rel} = \lim_{\tau \to -\infty, \tau_0 \to -\infty} U_{rel}(\tau, \tau_0) = \lim_{\tau \to -\infty, \tau_0 \to -\infty} \exp(iA_0 \tau) \exp(-iA(\tau - \tau_0)) \exp(-iA_0 \tau_0)$$  \hspace{1cm} (70)

If $A - A_0 = V$ then $U_{rel}$ satisfies the equation

$$\partial_\tau U_{rel} = V(\tau)U_{rel}$$  \hspace{1cm} (71)
where

\[ \mathcal{V}(\tau) = \exp(iA_0\tau)\mathcal{V}\exp(-iA_0\tau) \]

When \( c \to \infty \) then (as we have shown at the end of sec. 2) the evolution operator of eq. (1) tends to the evolution operator (19) of Howland. Then, it follows from the formula (20) (see also [14]) that the scattering operator (70) of the evolution equation (20) in the limit \( c \to \infty \) coincides with \( S \) of eq. (68).

Hence, \( S_{\text{rel}} \to S \).

We expect that in the scattering of particles of low energy the quantum field theory should give the same results as the quantum mechanics. We could establish this property in scalar QED of sec. 7 when the iterative solution \( A \) of eq. (64) starts from \( A^{cl} \) such that \( \Box A^{cl} = 0 \). Then, \( \phi_\tau = \exp(-i\bar{\hbar}\tau\Box A^{cl})\phi \) plus the noise and higher order terms. If the noise and terms of higher order in the coupling constant are neglected then we obtain the time evolution and scattering of relativistic quantum mechanics.

The quantum field theory is represented as a non-linear stochastic wave mechanics. The asymptotic free behavior (67) makes well sense in such a non-linear wave mechanics. The S-matrix can be defined as

\[ \phi_{\text{out}} = S\phi_{\text{in}} \] (72)

Its matrix elements are products of the in and out states. So, if the in-state \( f_q(x) \) satisfies \( A_0 f_q = 0 \) (in eq. (51) we let \( A = A_0 = -\hbar(\Box - M^2) \)) then the matrix element is

\[ (f_q, \phi_{\text{out}}) = \lim_{T \to \infty} (f_q, \exp(-iT A_0)\phi_{\text{out}}) \approx \lim_{T \to \infty} \int_0^T (f_q, \partial_\tau \phi_\tau) d\tau \]

We treat the multiparticle scattering as a scattering of independent particles connected only by the common noise. Then, a suggested generalization of the one particle matrix element to the multiparticle one could be

\[ \langle p_1, ..., p_n | S | q_1, ..., q_m \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_j \prod_k (\partial_\tau f_{p_j})(f_{q_k}, \partial_\tau f_{p_k}) d\tau \]

If we replace \( \partial_\tau \phi_\tau \) by \(-iA_0 \phi_\tau \) (which follows by a formal differentiation of eq. (67)), average over \( \tau \) first (before the differentiation contained in \( A_0 = \Box - M^2 \)) then from the stochastic field theory we obtain the standard LSZ formula for the scattering matrix [24]

\[ \langle p_1, ..., p_n | S | q_1, ..., q_m \rangle = \left\langle dx_1 ... dx_n dy_1 ... dy_m \int_{p_1} ... \int_{p_n} (x_1) ... (x_n) f_{q_1}(y_1) ... f_{q_m}(y_m) \right\rangle \]

\[ \sum_j \sum_k (\partial_\tau f_{p_j})(f_{q_k}, \partial_\tau f_{p_k}) \]

\[ (73) \]

\[ (74) \]
Dissipative relativistic quantum mechanics

Quantum theory of microscopic phenomena cannot be separated from a description of the macroscopic world of measuring devices. An interaction of a quantum particle with a macroscopic environment is often described by a random wave function or (equivalently) by a density matrix

$$\rho(x, y) = E[\phi(x)\phi(y)]$$  \hspace{1cm} (75)

If we require that the trace and positivity of $\rho$ are preserved by the time evolution then we obtain the Lindblad equation \[28\] (Lindblad equation for the relativistic quantum mechanics has been discussed earlier in ref.\[29\])

$$\partial_\tau \rho = -i[A, \rho] - \frac{1}{2} \sum_k R_k^+ R_k \rho - \frac{1}{2} \sum_k \rho R_k^+ R_k + \sum_k R_k \rho R_k^+$$  \hspace{1cm} (76)

where $R$ describes a (phenomenological) dissipation.

We can equivalently represent the dissipative dynamics by the Ito stochastic wave equation (i.e. a random perturbation of eq.(1)) which resembles eq.(26)

$$d\phi = -iA\phi d\tau + i\sum_k R_k \phi dB_k - \frac{1}{2} \sum_k R_k^+ R_k \phi d\tau$$  \hspace{1cm} (77)

where $B_k$ are independent complex Brownian motions

$$E[B_k B_l] = 0$$

$$E[B_k(s, x) B_k(\tau, y)] = \delta_{kl} \min(\tau, s) \delta(x - y)$$

If the operators $R$ are Hermitian then eq.(77) can be expressed in a more compact form

$$d\phi = -iA\phi d\tau + i\sum_k R_k \phi \circ dB_k$$  \hspace{1cm} (78)

where the circle denotes the Stratonovitch differential \[12\] and $\hat{B}_k$ are independent real Brownian motions. When we solve eq.(77) and calculate the expectation value (75) then we can see that the solution of eq.(77) determines a solution of the Lindblad equation (76). If the wave function $\phi$ has more components then these components add discrete indices to the density matrix. As an example, the random wave equation for the electromagnetic field in the gauge $A_0 = 0$ reads

$$dA_j = i\hbar \partial_\tau F_{j0} d\tau - i\hbar \partial_k F_{jk} d\tau + i R_{ljk} A_k dB_l - \frac{1}{2} \bar{R}_{lkj} R_{lmk} A_m d\tau$$  \hspace{1cm} (79)

We define

$$\rho_{jk}(x, y) = E[A_j(x)\overline{A_k(y)}]$$
Then the Lindblad equation takes the form

$$\partial_\tau \rho_{jk} = i[A, \rho]_{jk} - \frac{1}{2} \bar{R}_{lrj} R_{lrm} \rho_{mk} - \frac{1}{2} \bar{R}_{lrm} R_{lrk} \rho_{jm} + \bar{R}_{jkm} R_{ljr} \rho_{rm} \tag{80}$$

where

$$A_{jk,lm} = \hbar \delta_{jl} \delta_{km} \Box - \hbar \delta_{km} \partial_j \partial_l$$

Eq.(80) can describe a time evolution of an ensemble of photons (of various polarizations corresponding to different indices of $\rho$) which undergoes a measurement $R$. In particular, a continuous observation of photon’s position corresponds to $R_{ijk} = \delta_{jk} x_i a_l$ (where the vector $a$ selects an orientation in space).

Let us note that if we add a dissipation in the form (78) to the field equation (28)

$$d\phi = -iA\phi d\tau + iR\phi \circ dB + \sqrt{2}\hbar dB \tag{81}$$

and first calculate the expectation value over $B$

$$\rho(x, y) = E[E_B[\phi(x)]E_B[\phi(y)]]$$

then we obtain again the density matrix $\rho$ satisfying eq.(76). However, we interpret eq.(81) in a different way. We consider eq.(81) as an equation for a quantum field interacting in a dissipative way with an environment. Then, both noises $B$ and $\tilde{B}$ are treated on an equal footing i.e. we average over the noise at the end. We generalize eq.(81) to the non-linear quantum field theory (51) allowing $R$ to be a real non-linear function of $\phi$

$$d\phi = -iA\phi d\tau - igG(\phi) d\tau + i\gamma R(\phi) \circ dB + \sqrt{2}\hbar dB \tag{82}$$

Let (as in eq.(35)) $\Phi_\tau(\phi) = E[\Phi(\phi (\tau))]$ then it follows from eq.(82) that $\Phi_\tau(\phi)$ is a solution of the equation

$$\partial_\tau \Phi_\tau = \tilde{G}\Phi_\tau = \hbar^2 Tr(D^2 \Phi_\tau) - \frac{\gamma^2}{2} Tr(R(\phi) DR(\phi) D\Phi_\tau) - i(A\phi + gG(\phi), D\Phi_\tau) \tag{83}$$

We average again over $\tau$ and look for a measure $\nu$ such that an average over $\nu$ is equivalent to the $\tau$-average. We obtain a condition which is an analog of eq.(58)

$$\int d\nu(\phi) \tilde{G}\Phi(\phi) \equiv \int d(G^* \nu) \Phi = 0 \tag{84}$$

We rewrite eq.(84) as a differential equation for $\nu$

$$D\left(\hbar^2 D - \frac{\gamma^2}{2} R(\phi) R(\phi) D + \frac{\gamma^2}{2} R(\phi) DR(\phi) + iA\phi + igG(\phi)\right) \nu = 0 \tag{85}$$

Let as in eq.(60) $A = -\hbar \Box$ and

$$G(\phi) = \hbar DV(\phi) \tag{86}$$
We express the solution of eq.(85) as a formal complex Feynman measure

\[ d\nu(\phi) = d\phi \exp\left(\frac{i}{\hbar} L_\gamma(\phi)\right) \]

where

\[ L_\gamma(\phi) = \int dx \mathcal{L}(\phi(x)) = \int dx \left( -\frac{1}{2} \phi \Box \phi - gV(\phi(x)) + \frac{\gamma^2}{2} F(\phi) \right) \]  

(87)

Then, eq.(85) for \( \nu \) can be rewritten as a linear equation for \( \mathcal{F} \)

\[ \hbar^2 D\mathcal{F} = R(\phi) R(\phi) DL_\gamma + i\hbar R(\phi) DR(\phi) \]  

(88)

Its solution is a complex function of \( \phi \). The imaginary part of \( \mathcal{F} \) describes a dissipation. In the lowest order in \( \gamma \) we obtain

\[ \text{Im}\mathcal{F} = \frac{1}{2\hbar} \int dx R(\phi(x))^2 \]

The Feynman integral representing the averaged correlation functions (27) is an oscillatory integral describing the interference phenomena in quantum mechanics. The addition of a dissipation damps the interference leading to decoherence \cite{30} and a smooth equilibrium limit \cite{31} represented by the complex Gibbs measure (87).

\section{Quantum fields on a general curved manifold}

The proper time formalism is especially useful if fields are to be defined on a pseudoriemannian manifold \( \mathcal{M} \). In such a case there is no candidate for a time as an evolution parameter (the coordinate \( x_0 \) is only locally defined). Moreover, the classical proper time (intrinsic time) is a function of the metric. Hence, in quantum gravity, when the metric becomes a dynamical variable, then the proper time acquires the same dynamical content as the coordinate \( x \) in non-relativistic quantum mechanics. We can easily generalize eq.(1) to a general pseudoriemannian manifold \( \mathcal{M} \) (we set \( c = 1 \) in this section)

\[ i\hbar \partial_\tau \psi = \frac{1}{2M} g^{\mu\nu} (-i\hbar \partial_\mu + A_\mu + \hbar g^{\alpha\beta} \Gamma_{\mu\alpha\beta})(-i\hbar \partial_\nu + A_\nu) \psi \]  

(89)

where \( g^{\mu\nu} \) is the Riemannian metric and \( \Gamma \) is the Christoffel symbol of the Levi-Civita connection. Assume that \( W_\tau \) is a solution of the Hamilton-Jacobi equation

\[ \partial_\tau W_\tau + \frac{1}{2M} g^{\mu\nu}(\partial_\mu W_\tau + A_\mu)(\partial_\nu W_\tau + A_\nu) = 0 \]  

(90)
with the initial condition \( W \). Let us consider eq.(89) with the initial condition \( \phi = \exp(\frac{i}{\hbar}W)\Phi \). Then, \( \phi_\tau \) is a solution of eq.(89) if and only if \( \Phi_\tau \) is the solution of the equation (the Lorentz gauge for \( A_\mu \) is assumed)

\[
i\hbar \partial_\tau \Phi = -\frac{\hbar^2}{2M} \Box g \Phi - \frac{i\hbar}{M} g^{\mu\nu} (\partial_\mu W_\tau + A_\mu) \partial_\nu \Phi_\tau - \frac{i\hbar}{2M} \Box g \Phi_\tau \quad (91)
\]

where \( \Box_g \) is the wave operator on the pseudoriemannian manifold \( M \)

\[
\Box_g = g^{\mu\nu} \partial_\mu \partial_\nu + \frac{1}{2} g^{\nu\rho} \Gamma^\mu_{\nu\rho} \partial_\mu \partial_\nu
\]

In the formal limit \( \hbar \to 0 \) of eq.(91) we obtain \( \phi_\tau(x) \approx \Phi(\xi(\tau)) \) where \( \xi \) is the solution of the equation (0 \( \leq s \leq \tau \)

\[
\frac{d\xi^\mu}{ds} = -\frac{1}{M} g^{\mu\nu}(\xi(s)) \left( \partial_\nu W(\tau - s, \xi(s)) + A_\nu(\xi(s)) \right) \quad (92)
\]

Differentiating eq.(92) once more and using the Hamilton-Jacobi equation (90) we obtain the equation

\[
\frac{d^2 \xi^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{d\xi^\nu}{ds} \frac{d\xi^\rho}{ds} = \frac{1}{M} F^{\mu\nu}(\xi(s)) \frac{d\xi^\nu}{ds} \quad (93)
\]

Eq.(93) shows that the correct classical limit (as a geodesic equation) of a motion of the quantum particle on a general pseudoriemannian manifold results if and only if \( \tau \) is the proper time. The interpretation of \( \tau \) as the classical proper time remains true not only in the leading order in \( \hbar \) but also in all subsequent terms because the leading order determines the subsequent interpretation of \( \tau \).

We can express the quantum corrections in terms of the Brownian motion. Let us define the complex vierbeins \( v_{\mu a} \)

\[
v_{\mu a} v_{\nu a} = i g_{\mu\nu}
\]

Then, we consider a complex Markov process \( q \) as a solution of the set of covariant equations (for the case of the Riemannian manifold and imaginary time see [12] or [37]; the real time is discussed in [11], it can be considered as a complexification of the diffusion)

\[
dq^\mu = -\frac{1}{M} g^{\mu\nu}(q_s) \left( \partial_\nu W(\tau - s, q_s) + A_\nu(q_s) \right) ds + \sigma v^\mu_a(q_s) \circ db^a \quad (95)
\]

\[
dv^\mu_a + \Gamma^\mu_{\nu\rho} v^\nu_a \circ dq^\rho = 0 \quad (96)
\]

Then, the solution of eq.(91) with the initial condition \( \psi = \exp(\frac{i}{\hbar}W)\Phi \) reads

\[
\psi_\tau(x) = \exp(\frac{i}{\hbar}W_\tau(x)) E\left[ \exp\left( -\frac{1}{2M} \int_0^\tau \Box_g W(\tau - s, q(s)) ds \right) \Phi(q(\tau)) \right] \quad (97)
\]

Eq.(97) can be applied for a semiclassical expansion in powers of \( \hbar \).
We can continue now (as in the earlier sections) with the quantization of the wave equation (89). So, for the free field the stochastic equation coincides with eq.(28) where
\[ \mathcal{A} = -\hbar \Box_g \] (98)
In the computation of an average over the proper time (27) we need to define \( \Box^{-1}_g \). At finite T we obtain a linear functional on a subset of test-functions \( S(\mathcal{M}) \). Taking the limit \( T \to \infty \) is equivalent to a definition of an extension of this linear functional. The definition of the inverse is not unique but it is restricted by the requirements that the quantum field theory resulting from \( \Box^{-1}_g \) is causal and that the quantum fields should be defined in a Hilbert space (with a non-negative scalar product). It remains unclear whether such a definition of \( \Box^{-1}_g \) exists on an arbitrary manifold. This can be achieved on a globally hyperbolic manifold \([32][33][34][35]\). In such a case there exists a complete set of solutions of the wave equation
\[ \Box_g u_j = 0 \] (99)
Then, we can define
\[ \Delta_g^{(+)}(x,y) = \sum_j \pi_j(x)u_j(y) \] (100)
Eq.(100) determines a non-negative bilinear form. We can define by means of the standard Fock space methods the quantum annihilation \( \phi^{(+)}(x) \) (as \( \phi^{(+)} = \sum a_ju_j \)) and creation \( \phi^{(-)}(x) \) parts of the field operator \( \phi = \phi^{(+)} + \phi^{(-)} \).
Now, on a globally hyperbolic manifold there exists a choice of the \( x_0 \)-coordinate \([34]\) such that
\[ \Box^{-1}_g(x,y) \equiv \Delta_F(x,y) = \theta(x_0 - y_0)\Delta_g^{(+)}(x,y) + \theta(y_0 - x_0)\Delta_g^{(+)}(y,x) \] (101)
is independent of the choice of coordinates. Eq.(101) defines the Feynman propagator on a globally hyperbolic manifold. The definition (101) coincides with the one resulting from the \( \epsilon \)-prescription \([30] \) (\( \epsilon > 0 \))
\[ \Box^{-1}_g = \lim_{\epsilon \to 0} \int_0^{\infty} id\varepsilon \exp(-is\Box_g - \epsilon s) \] (102)
The interacting field is defined by the stochastic equation (49) with \( \mathcal{A} = -\hbar \Box_g \).
After the propagator is defined there is no further difficulty (up to the ultraviolet problems) in defining perturbatively an interacting field by means of an iterative solution of the stochastic equation (49).
A really ambitious program concerns a quantization of the metric \( g \). The proper time approach supplies a useful method at the problem where other quantization methods encounter insurmountable difficulties. The Feynman integral is a formal tool. It can be defined either in the imaginary time or for complex coordinates \([10]\). However, these methods fail for Einstein gravity which has the
action unbounded from below. The method of Euclidean stochastic quantiza-
tion \cite{39,40} works well. However, it remains unclear whether the model can be
continued back to the real time (pseudoriemannian manifold). An application
of the stochastic quantization \cite{19} with a complex action and real time to the
Einstein gravity has been suggested by Rumpf \cite{38}. We interprete the fictitious
time of Parisi and Wu as the proper time with a physical meaning. Such an
interpretation should have experimental consequences for a time evolution of
the graviton wave function (in ref. \cite{41} an experiment showing the interference
in time is suggested).

We need now the Brownian motion $B$ which is a symmetric matrix whose
entries constitute independent Brownian motions

$$E[B_{ac}(s,x)B_{fd}(\tau,y)] = (\delta_{af}\delta_{cd} + \delta_{ad}\delta_{ef})\min(\tau,s)\delta(x-y)$$

Then, we introduce the infinite dimensional pseudoriemannian manifold $G(M)$
of metric tensors $g$ on $M$ with DeWitt’s supermetric \cite{12} on this manifold

$$G^{\alpha\beta;\mu\nu} = (\det(-g))^{-\frac{1}{2}}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})$$  (103)

We need a basis of complex frames on $G(M)$ (tetrads) fulfilling the relation

$$G^{\mu\nu;\alpha\beta} = \xi^{\mu\nu;ac}\xi^{\alpha\beta;ac}$$

If $L$ is the action for the metric and the matter fields then the stoch astic
equation for $g$ according to the prescriptions (49) and (95)-(96) reads (see refs.
\cite{37,39,40} for the case of infinite dimensional Riemannian manifolds)

$$dg^{\mu\nu} = i\hbar\frac{\delta L}{\delta g^{\mu\nu}}d\tau + \hbar\xi^{\mu\nu;ac}\circ dB_{ac}$$  (104)

$$d\xi^{\mu\nu;ac} + \Gamma^{(\mu\nu)}_{(\alpha\beta)(\gamma\rho)}\xi^{\gamma\rho;ac}\circ dg^{\alpha\beta} = 0$$  (105)

Eqs.(104)-(105) need a regularization of the Brownian motion $B$ for a proper
interpretation of the Christoffel symbol appearing in eq.(105) (see \cite{39}). Note
that if $L$ is a sum of the Einstein action and the action for the matter fields
then eq.(104) reads

$$dg^{\mu\nu} = i\hbar(\det(-g))^{\frac{1}{2}}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)d\tau - i\hbar\kappa(\det(-g))^{\frac{1}{2}}T^{\mu\nu}d\tau + \hbar\xi^{\mu\nu;ac}\circ dB_{ac}$$  (106)

where $\kappa$ is proportional to the gravitational constant and $T_{\mu\nu}$ is the energy-
momentum tensor for the matter fields. If we start from a linearized form of
eq.(106) then the first difficulty which we encounter involves the zero modes of
the linear part which cause a trouble with the large time limit. We can deal
with the problem of zero modes either by means of the stochastic gauge fixing
\cite{13} or restricting ourselves to gauge invariant observables. We would like to
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Outline here the method introduced for the electromagnetic field at eq. (47) (the Einstein tensor $G_{\mu\nu} = (\det(-g))^{1/2} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$ is an analog of $F_{\mu\nu}$, it has a gauge invariance resembling the one of non-Abelian gauge fields [42]). So, for pure gravity ($T_{\mu\nu} = 0$) we have

$$d((\det(-g))^{1/2} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)) = \frac{\delta L}{\delta g_{\alpha\beta}} \circ dg_{\alpha\beta}$$

(107)

or more explicitly

$$d((\det(-g))^{1/2} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)) = i\bar{\hbar} \frac{\delta L}{\delta g_{\mu\nu}} \left( (\det(-g))^{1/2} (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) d\tau - iE_{\alpha\beta;ac} \circ dB_{ac} \right)$$

(108)

Let $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ where $g_{\mu\nu}^{(0)}$ is the constant metric tensor of the Minkowski space-time. Denote by $G(h)$ the part of the Einstein tensor linear in $h$. Then eq. (108) reads

$$dG^{\mu\nu}(h) = 2i\bar{\hbar} \Box G^{\mu\nu}(h) d\tau + p(G(h))^{\mu\nu} d\tau + C^{\mu\nu;ac}(G(h)) \circ dB_{ac}$$

(109)

where $p$ and $C$ are non-linear in $G(h)$. In eq.(109) we have inverted the linear relation $h \rightarrow G(h)$ i.e. we have chosen a particular solution $h_{\mu\nu}$ of the equation $G_{\mu\nu}(h) = G_{\mu\nu}$. We can solve eq.(109) iteratively (we assume a coordinate invariant regularization of the ultraviolet divergencies [39]). Then, the proper time average (27) of $G_{\mu\nu}(h)$ exists in a perturbation expansion with $\Box^{-1}$ defined by $\triangle_{\Box}(x,y)$. We compute in this way the time-averaged correlation functions of $G(h)$. These correlation functions determine a random variable $\chi_{\mu\nu}$. Then, we can obtain the metric tensor as a particular solution of the linear equation $G_{\mu\nu}(h) = \chi_{\mu\nu}$. Subsequently, we can express $g_{\mu\nu}$ in terms of $\chi_{\mu\nu}$. We expect that in spite of the arbitrariness (gauge dependence) of the relation between $G(h)$ and $h$ the coordinate independent variables e.g. $R_{\mu\nu\rho\tau}(g)R_{\beta\alpha\gamma\delta}(g)$ (where $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ and $R$ is the Riemann curvature) will not depend on the choice of the gauge (as in the case of the stochastic gauge fixing). In general, we could start from a background metric $g_{\mu\nu}^{(0)}$ instead of the Minkowski one. In such a case in addition to the processes of graviton creation and annihilation (and mutual scattering) we could describe the time evolution and localization of a single graviton in interaction with other fields.

## XI Summary

We have discussed a method of quantization which can be considered as a version of the Feynman integral. The method is related to the stochastic quantization of Parisi and Wu [13]. We considered the Bose fields only. If we are interested in correlation functions of Fermi fields then we need stochastic equations with
Grassmann variables. Our main emphasis was on the proper time interpretation of the "fictitious time parameter". The proper time interpretation supplies a relativistically covariant quantization. It can give a meaning to the probability of detecting a particle inside a bounded region (sec.7). It is rather difficult to formulate such questions in the framework of the conventional QFT. Moreover, a proper time interpretation can have experimental consequences for time measurements [41]. For quantum field theory on a manifold and quantum gravity it may be valuable to treat the background metric and gravitons at the same footing. The conventional Euclidean quantization of Einstein gravity is impossible because the action is unbounded from below. The stochastic Euclidean quantization of gravity is feasible [39][40]. However, in such an approach it remains unclear how to continue to the real physical time. The direct quantization in the real time avoids these problems. We have clarified some mathematical aspects of the proper time stochastic quantization scheme. We have shown the convergence of the averaged values. The convergence rate of the average over the time interval \([0,T]\) is usually proportional to \(T^{−1}\). With purely Hamiltonian dynamics it cannot be as fast as in the Euclidean version of the stochastic quantization where the convergence can be exponential. Nevertheless, in view of the difficulties of the conventional oscillatory Feynman integral the stochastic version can be a useful tool for numerical calculations (stochastic simulations of complex stochastic equations are discussed in [17][18][11]). In the dissipative quantum field theory of sec.9 the convergence of the averaged correlation functions can again be exponential as in the Euclidean framework.

### XII Appendix

We outline in this section an operator quantization in the proper time. The equation of motion reads

\[
\partial_\tau \phi(\tau, x) = i\hbar \frac{\delta L}{\delta \phi(\tau, x)} \tag{A.1}
\]

We are looking for a solution of this equation with an initial condition at \(-\infty\)

\[
\lim_{\tau \to -\infty} \phi_\tau(x) = \phi_{\text{in}}(x) \tag{A.2}
\]

where \((\Box - M^2)\phi_{\text{in}}=0\) is the quantum scalar free field of mass \(M\). We expect that as \(\tau \to \infty\)

\[
\phi_\tau(x) \to \phi_{\text{out}}(x) \tag{A.3}
\]

If this asymptotic behavior holds true then we can define

\[
\phi_{\text{out}} = S^{-1} \phi_{\text{in}} S \tag{A.4}
\]
We can develop a perturbative theory now. Assume the Lagrangian (62), then we can rewrite eq.(A.1) as an integral equation
\[ \phi_{\tau}(x) = \phi_{in}(x) - i \hbar \int_{-\infty}^{\tau} \exp\left(-i\hbar(\Box - M^2)(\tau - s)\right)V(\phi_{s})ds \] (A.5)

We are interested in computing the time-ordered products of quantum fields. We first solve eq.(A.5) perturbatively (fixing the initial condition at \( \tau_0 \rightarrow -\infty \))
\[ \phi_{\tau} = \phi_{in} - i \hbar \int_{\tau_0}^{\tau} \exp\left(-i\hbar(\Box - M^2)(\tau - s)\right)V(\phi_{in})ds \]
\[ -i(\hbar)^2 \int_{\tau_0}^{\tau} \exp\left(-i\hbar(\tau - s)(\Box - M^2)\right)V(\phi_{in})ds \]
\[ \int_{\tau_0}^{s'} \exp\left(-i\hbar(\Box - M^2)(s - s')\right)V(\phi_{in})ds' + \ldots \] (A.6)

We assume that V is defined by the Wick normal product. Then, the Wick theorem allows us to calculate explicitly \(< O|T(\phi_{\tau}(x)\phi_{\tau}(x')))|0 >\) where the T-ordering is understood in \( x_0 \) rather than in \( \tau \). Subsequently, the average over the proper time can be calculated. We obtain a closed formula in terms of the Green's functions of \( \exp(-is(\Box - M^2)) \) and \((\Box - M^2)^{-1}\). We must define \( \Box^{-1}(x,y) = \Delta_F(x,y) \) if the time-ordered correlation functions are to agree with the conventional ones. We have checked the agreement with the conventional QFT only at the lowest order of \( g \). We have obtained the conventional QFT because of the asymptotic condition (A.2). Eq.(A.1) at \( g = 0 \) has more solutions which in general are \( \tau \)-dependent. Starting with such solutions we would go beyond the conventional QFT.

Another (implicit) argument concerning the relation to the conventional QFT at each order of \( g \) comes from the identity
\[ 0 = \lim_{R \to \infty} \frac{1}{\pi} \int_{0}^{R} d\tau \left< \partial_{\tau}(\phi_{\tau}(x_1)\ldots\ldots\phi_{\tau}(x_n)) \right> = \lim_{R \to \infty} \frac{1}{\pi} \int_{0}^{R} d\tau \sum_{k=1}^{n} \left< \phi_{\tau}(x_1)\ldots\ldots((\Box - M^2)\phi_{\tau}(x_k) - gV'(\phi_{\tau}(x_k))\ldots\phi_{\tau}(x_n)) \right> \] (A.7)

We can use this equation in order to prove that if at \( g = 0 \) the time-ordered correlation functions coincide with the conventional free ones then at each order of \( g \) the proper time average coincides with the one resulting from the solution of the operator equation
\[ (\Box - M^2)\phi(x) = gV'(\phi(x)) \]

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