Positivity of the Density for Rough Differential Equations

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Abstract
Due to recent developments of Malliavin calculus for rough differential equations, it is now known that, under natural assumptions, the law of a unique solution at a fixed time has a smooth density function. Therefore, it is quite natural to ask whether or when the density is strictly positive. In this paper we study this problem from the viewpoint of Aida–Kusuoka–Stroock’s general theory.

Keywords Gaussian rough path · Malliavin calculus · Positivity of probability density function

Mathematics Subject Classification (2020) 60L20 · 60H07 · 60G15

1 Introduction
In this paper, we study the following rough differential equation (RDE) driven by a Gaussian rough path \( w \):

\[
\mathrm{d} y_t = \sum_{i=1}^{d} V_i(y_t) \mathrm{d} w^i_t + V_0(y_t) \mathrm{d} t \quad \text{with} \quad y_0 = a \in \mathbb{R}^e. \tag{1.1}
\]

Here, \( V_i \) (\( 0 \leq i \leq d \)) are nice vector fields on \( \mathbb{R}^e \). The main example of Gaussian rough paths we have in mind is a fractional Brownian rough path (fBRP) with Hurst parameter \( H \in (1/4, 1/2] \). We are interested in the law of \( y_t \) for a fixed time \( t \). In
particular, we take up the problem of positivity of the density of the law (when the density with respect to the Lebesgue measure exists).

Let us review results for a usual stochastic differential equation (which coincides with (1.1) driven by Brownian rough path). Thanks to Malliavin calculus, the law of \( y_t \) has a smooth density if \( V_i \) (\( 0 \leq i \leq d \)) satisfy Hörmander’s condition at the starting point \( a \). Positivity of the density was first studied by Ben Arous and Léandre [3]. Later Aida, Kusuoka and Stroock [1] generalized the positivity theorem for very general Wiener functionals. (The key theorem is [1, Theorem 2.8]. Fortunately, its proof is not long.)

Malliavin calculus for RDE (1.1) was established by [5,10]. For a class of Gaussian rough paths including fBRP with Hurst parameter \( H \in (1/4, 1/2] \), it is now known that the law of \( y_t \) has a smooth density under Hörmander’s condition at the starting point \( a \). Therefore, it is quite natural to ask whether or when the density is strictly positive. Concerning this problem, there are two results [2,7], in both of which \( w \) is fBRP with Hurst parameter \( H \in (1/4, 1/2] \). In these works, they proved everywhere-positivity of the density under the uniform ellipticity or the uniform Hörmander condition along the lines of [3].

The aim of this paper is to look at this problem from the viewpoint of Aida–Kusuoka–Stroock’s general theory. It is true that these two existing results are quite nice, but it is also true that we can still improve them by using this theory.

The organization of this paper is as follows. In Sect. 2, we review Aida–Kusuoka–Stroock’s theory on an abstract Wiener space. In Sect. 3, we collect some known deterministic results from rough path theory, which will be used in the proof of our main result. Following [6, Section 15], we recall basic results on Gaussian rough paths in Sect. 4. Section 5 is the core of this paper. We prove that Lyons–Itô map, i.e., the solution map for RDE (1.1) is twice \( \mathcal{K} \)-differentiable in the sense of [1]. (See Proposition 5.2. This is one of the key points in proving the positivity.)

In Sect. 6, we present our main result in Theorem 6.1 and compare it with the preceding results in Remark 6.3.

Throughout this paper the following notation will be used. We work on the time interval \([0, T]\), where \( T \in (0, \infty) \) is arbitrary but fixed. For a Banach space \( \mathcal{X} \), the set of \( \mathcal{X} \)-valued continuous paths over \([0, T]\) is denoted by \( C(\mathcal{X}) \). For \( a \in \mathcal{X} \), the subset of continuous paths that start at \( a \) is \( C_a(\mathcal{X}) \). For \( p \geq 1 \), set of \( \mathcal{X} \)-valued continuous paths over \([0, T]\) with finite \( p \)-variation is denoted by \( C_p^{p-\text{var}}(\mathcal{X}) \). In a similar way, \( C_a^{p-\text{var}}(\mathcal{X}) \) is defined. Let \( f : U \to \mathcal{Y} \), where \( U \) is an open subset of \( \mathcal{X} \) and \( \mathcal{Y} \) is another Banach space. For \( 0 \leq k < \infty \), \( f \) is said to be of \( C^k_b \) on \( U \) if \( f \) is a bounded \( C^k \)-map from \( U \) to \( \mathcal{Y} \) whose derivatives up to order \( k \) are all bounded. The set of such \( f \) is denoted by \( C^k_b(U, \mathcal{Y}) \). The \( C^k_b \)-norm of \( f \) is given by \( \sum_{i=0}^k \sup_{x \in U} \| D^i f(x) \|_{\mathcal{Y}} \). If \( f \) is of \( C^k_b \) for all \( 0 \leq k < \infty \), \( f \) is said to be of \( C^\infty_b \) on \( U \).

### 2 Review of Abstract Wiener Space and \( \mathcal{K} \)-Regularity

In this section, we recall Aida–Kusuoka–Stroock’s result on the positivity of the density for non-degenerate Wiener functionals (see [1]).
In this section, \((\mathcal{W}, \mathcal{H}, \mu)\) is an abstract Wiener space in the sense of [9]. That is, \((\mathcal{W}, \| \cdot \|_{\mathcal{W}})\) is a separable Banach space, \((\mathcal{H}, \| \cdot \|_{\mathcal{H}})\) is a separable Hilbert space, \(\mathcal{H}\) is a dense subspace of \(\mathcal{W}\) and the inclusion map is continuous, and \(\mu\) is the (necessarily unique) probability measure on \((\mathcal{W}, \mathcal{B}_{\mathcal{W}})\) with the property that

\[
\int_{\mathcal{W}} \exp\left(\sqrt{-1} \lambda^* (\lambda, w)_{\mathcal{W}}\right) \mu(dw) = \exp\left(-\frac{1}{2} \|\lambda\|_{\mathcal{H}}^2\right), \quad \lambda \in \mathcal{W}^* \subset \mathcal{H}^*,
\]

where we have used the fact that \(\mathcal{W}^*\) becomes a dense subspace of \(\mathcal{H}\) when we make the natural identification between \(\mathcal{H}^*\) and \(\mathcal{H}\) itself. Hence, \(\mathcal{W}^* \hookrightarrow \mathcal{H}^* = \mathcal{H} \hookrightarrow \mathcal{W}\) and both inclusions are continuous and dense. We denote by \(\{\langle k, \bullet \rangle \colon k \in \mathcal{H}\}\) the family of centered Gaussian random variable defined on \(\mathcal{W}\) indexed by \(\mathcal{H}\) (i.e. the homogeneous Wiener chaos of order 1). If \(\langle k, \bullet \rangle_{\mathcal{H}} \in \mathcal{H}^*\) extends to an element of \(\mathcal{W}^*\), then the extension coincides with the random variable \(\langle k, \bullet \rangle\). We also denote by \(\tau_k \colon \mathcal{W} \to \mathcal{W}\) the translation \(\tau_k(w) = w + k\).

For a finite-dimensional subspace \(K\) of \(\mathcal{H}\), \(P_K : \mathcal{H} \to K\) stands for the orthogonal projection and we write \(P_K^\perp = \text{Id}_{\mathcal{H}} - P_K\). This projection naturally extends to \(\bar{P}_K : \mathcal{W} \to K\) as follows:

\[
\bar{P}_K(w) = \sum_{i=1}^{\dim K} \langle e_i, w \rangle e_i,
\]

where \(\{e_i\}_{i=1}^{\dim K}\) is an orthonormal basis of \(K\). (This right-hand side is independent of the choice of \(\{e_i\}\).) We set \(\bar{P}_K^\perp = \text{Id}_{\mathcal{W}} - \bar{P}_K\).

Now we recall the definitions of \(\mathcal{K}\)-continuity, \(\mathcal{K}\)-regularity, uniformly \(\mathcal{K}\)-regularity, and \(l\)-times \(\mathcal{K}\)-regular differentiability, which were first introduced by [1]. Note that in these definitions, functions and maps on \(\mathcal{W}\) are everywhere-defined ones (not equivalence classes with respect to \(\mu\)).

Assume that \(\mathcal{K} = \{K_n\}_{n=1}^\infty\) is a non-decreasing, countable exhaustion of \(\mathcal{H}\) by finite dimensional subspaces, that is, \(K_n \subset K_{n+1}\) for all \(n\) and \(\bigcup_{n=1}^\infty K_n\) is dense in \(\mathcal{H}\). Set \(P_n = P_{K_n}\), define \(\tilde{P}_n, \tilde{P}_n^\perp\) accordingly. We say that a map \(F\) from \(\mathcal{W}\) into a Polish space \((E, \rho_E)\) is \(\mathcal{K}\)-continuous if it is measurable and, for each \(n \in \mathbb{N} := \{1, 2, \ldots\}\), there is a measurable map \(F_n : \mathcal{W} \times K_n \longleftrightarrow E\) with the properties that \(F \circ \tau_k = F_n(\cdot, k)\) (a.s. \(\mu\)) for each \(k \in K_n\) and \(F_n(w, k) \in E\) is continuous for each \(w \in \mathcal{W}\).

Given a \(\mathcal{K}\)-continuous map \(F\), we set

\[
F_n^\perp(w, k) = F_n(w, -\tilde{P}_n(w) + k) \quad \text{for } n \in \mathbb{N} \text{ and } k \in K_n. \quad (2.2)
\]

Given a measurable map \(F : \mathcal{W} \to E\), we will say that \(F\) is \(\mathcal{K}\)-regular if \(F\) is \(\mathcal{K}\)-continuous, and there is a continuous map \(\tilde{F} : \mathcal{H} \to E\) such that

\[
\lim_{n \to \infty} \mu\left(\left\{w : \rho_E(\tilde{F} \circ \tilde{P}_n(w), F(w)) \vee \rho_E(\tilde{F}(h), F_n^\perp(w, P_n(h))) \geq \epsilon\right\}\right) = 0 \quad (2.3)
\]
holds for every $\epsilon > 0$ and $h \in \mathcal{H}$. In this case, $\tilde{F}$ will be called a $\mathcal{K}$-regularization of $F$.

If $F$ is a map from $\mathcal{W}$ into a Polish space $E$, we will say that it is uniformly $\mathcal{K}$-regular if it is $\mathcal{K}$-regular and (2.3) can be replaced by the condition that

$$\lim_{n \to \infty} \mu \left( \left\{ w : \sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \rho_E (\tilde{F}(\bar{P}_n(w) + k), F_{m\vee n}(w, k)) \vee \rho_E (\tilde{F}(h + k), F_{m\vee n}(w, h + k)) \geq \epsilon \right\} \right) = 0$$

for every $m \in \mathbb{N}, r > 0, \epsilon > 0$ and $h \in \mathcal{H}$.

Let $E$ be a separable Banach space and $F$ be a map from $\mathcal{W}$ into $E$. Given $l \in \mathbb{N}$ we will say that $F$ is $l$-times $\mathcal{K}$-regularly differentiable if $F$ is uniformly $\mathcal{K}$-regular, $F_n(w, \cdot)$ is $l$-times continuously Fréchet differentiable on $K_n$ for each $n \in \mathbb{N}$ and $w \in \mathcal{W}$, $\tilde{F}$ is $l$-times continuously Fréchet differentiable on $\mathcal{H}$, and (2.4) can be replaced by the condition that

$$\lim_{n \to \infty} \mu \left( \left\{ w : \| \tilde{F}(\bar{P}_n(w) + \cdot) - F_{m\vee n}(w, \cdot) \|_{C^l_b(K_m(0,r),E)} \vee \| \tilde{F}(h + \cdot) - F_{m\vee n}(w, h + \cdot) \|_{C^l_b(K_m(0,r),E)} \geq \epsilon \right\} \right) = 0$$

for every $m \in \mathbb{N}, r > 0, \epsilon > 0$ and $h \in \mathcal{H}$. Here, $B_{K_m}(0, r) = \{ k \in K_m : \|k\|_{\mathcal{H}} < r \}$.

The following theorem is [1, Theorem 2.8] and plays a key role in this paper. It is a quite general result on the positivity of the density function of the law of a non-degenerate Wiener functional. (In this theorem, any choice of $\tilde{F}$ and $\tilde{G}$ will do. If $G \equiv 1$, then $f$ is the density function of the law of $F$ on $\mathbb{R}^e$.)

**Theorem 2.1** Let $F : \mathcal{W} \to \mathbb{R}^e, e \in \mathbb{N},$ and $G : \mathcal{W} \to [0, +\infty)$ be functions which are infinitely differentiable in the sense of the Malliavin calculus for $(\mathcal{W}, \mathcal{H}, \mu)$, and assume that the associated Malliavin covariance matrix $DF \cdot DF^T = \{ \langle DF^i(w), DF^j(w) \rangle_{\mathcal{H}} \}_{1 \leq i, j \leq e}$ is non-degenerate in the Malliavin sense, namely,

$$\left( \det (DF \cdot DF^T) \right)^{-1} \in \bigcap_{p \in [1, +\infty)} L^p(\mathcal{W}; \mu).$$

Here, $D$ stands for the $\mathcal{H}$-derivative. Then, there exists a unique nonnegative function $f \in C^\infty(\mathbb{R}^e, \mathbb{R})$ with the properties that $f$ has rapidly decreasing derivatives of all orders and

$$\int_{\mathcal{W}} (\phi \circ F)(w) G(w) \mu(dw) = \int_{\mathbb{R}^e} \phi(x) f(x) dx, \quad \phi \in C_b(\mathbb{R}^e, \mathbb{R}).$$

Assume further that $F$ is twice $\mathcal{K}$-regularly differentiable and $G$ is $\mathcal{K}$-regular with their $\mathcal{K}$-regularizations $\tilde{F}$ and $\tilde{G}$, respectively. Then, for $y \in \mathbb{R}^e$, the following are equivalent:

\[ Springer \]
\begin{itemize}
  \item $f(y) > 0$.
  \item There exists $h \in \mathcal{H}$ such that $D\tilde{F}(h) : \mathcal{H} \rightarrow \mathbb{R}^e$ has rank $e$, $\tilde{F}(h) = y$ and $\tilde{G}(h) > 0$.
\end{itemize}

3 Some Deterministic Results from Rough Path Theory

In this and the next sections we recall basic results on rough paths and RDEs. In this section, we summarize deterministic facts which will be used later.

In what follows, we will always assume $2 \leq p < 4$, $1 \leq q < 2$. The integer part of $p$ is denoted by $\lfloor p \rfloor$.

The geometric rough path space with $p$-variation topology over $\mathbb{R}^d$ is denoted by $G\Omega_p(\mathbb{R}^d)$. An element of $G\Omega_p(\mathbb{R}^d)$ is denoted by $x = (x^1, \ldots, x^{[p]}) = (x^{1}_{t}, \ldots, x^{[p]}_{t})_{0 \leq s \leq t \leq T}$.

Recall that the $p$-variation topology is induced by the following variation norms:

$$
\|x^i\|_{p/i-\text{var}} := \sup_{0=t_0 < \cdots < t_K=T} \left( \sum_{k=1}^{K} |x^i_{t_{k-1}, t_k}|^{p/i} \right)^{i/p}, \quad 1 \leq i \leq [p].
$$

Here, $\{0 = t_0 < \cdots < t_K = T\}$ runs over all finite partition of $[0, T]$. For more details, see [6] for example.

Now we introduce an RDE. Recall that an RDE itself is deterministic. In this work, we only treat the first level path of the solution of an RDE and therefore we simply call it a solution of the RDE.

Let $V_t : \mathbb{R}^e \rightarrow \mathbb{R}^e$ be a vector field on $\mathbb{R}^e$ with sufficient regularity ($0 \leq i \leq d$) and let $G\Omega_p(\mathbb{R}^d)$ be the geometric rough path space over $\mathbb{R}^d$ with $p$-variation topology ($2 \leq p < 4$).

We consider the following RDE driven by $x \in G\Omega_p(\mathbb{R}^d)$:

$$
\text{dy}_t = \sum_{i=1}^{d} V_i(y_t) \text{dx}_t^i + V_0(y_t) \text{dt} \quad \text{with} \quad y_0 = a \in \mathbb{R}^e. \quad (3.1)
$$

If $V_i$’s are of $C_{b}^{[p]+1}$, then a unique solution $y = y(x)$ exists, which is denoted by $\Phi(x)$. Moreover, $\Phi : G\Omega_p(\mathbb{R}^d) \rightarrow C^p_{a-\text{var}}(\mathbb{R}^e)$ is locally Lipschitz continuous, that is, Lipschitz continuous on every bounded subset of $G\Omega_p(\mathbb{R}^d)$. This map is called Lyons–Itô map (associated with $V_i$’s). We remark that $\Phi$ actually takes values in $C^p_{a-\text{var}}(\mathbb{R}^e) \cap C^0_{\leq p-\text{var}}(\mathbb{R}^e)$. Here, $C^0_{\leq p-\text{var}}(\mathbb{R}^e)$ is a separable Banach subspace of $C^p_{\leq p-\text{var}}(\mathbb{R}^e)$ defined as the closure of the set of all $\mathbb{R}^e$-valued $C^1$-path.

Let $1 \leq q < 2$. For $h \in C^q_{\leq p-\text{var}}(\mathbb{R}^d)$, the ordinary differential equation in the sense of Young integral (Young ODE) that corresponds to (3.1) is given as follows:

$$
\text{dy}_t = \sum_{i=1}^{d} V_i(y_t) \text{dh}_t^i + V_0(y_t) \text{dt} \quad \text{with} \quad y_0 = a \in \mathbb{R}^e. \quad (3.2)
$$
If $V_i$’s are of $C^2_b$, then a unique solution $y$ exists, which is denoted by $\Psi(h)$. Under the same condition, $\Psi: C_0^{q\text{-var}}(\mathbb{R}^d) \to C_0^{d\text{-var}}(\mathbb{R}^e)$ is locally Lipschitz continuous. Using Young integration, we define $h \in G\Omega_p(\mathbb{R}^d)$ by

$$h_{s,t}^i = \int_{s \leq u_1 \leq \cdots \leq u_i \leq t} dh_{u_1} \otimes \cdots \otimes dh_{u_i}, \quad 0 \leq s \leq t \leq T, \quad 1 \leq i \leq [p].$$

We often write $h = \mathcal{L}(h)$, too. The lift map $\mathcal{L}: C_0^{q\text{-var}}(\mathbb{R}^d) \to G\Omega_p(\mathbb{R}^d)$ is also locally Lipschitz continuous, injective and its image is dense. As one can easily guess, $\Psi(h) = \Phi(h)$ holds. In this sense, RDE (3.1) generalizes Young ODE (3.2).

Under the condition that $1/p + 1/q > 1$, there exists a translation on the geometric rough path space which is compatible with the usual translation on the usual path space.

It is called Young translation $T: G\Omega_p(\mathbb{R}^d) \times C_0^{q\text{-var}}(\mathbb{R}^d) \to G\Omega_p(\mathbb{R}^d)$ and is characterized as a unique continuous map that satisfies $T_h(k) = \mathcal{L}(h + k)$ for every $h, k \in C_0^{q\text{-var}}(\mathbb{R}^d)$. $T_h(x) = T_hx$ should be viewed as the translation of $x$ by $h$. (See [6, Subsection 9.4] for example.)

Next, let us see how derivatives of $\Psi$ look like. For brevity, we write $\sigma = [V_1, \ldots, V_d]$ and $b = V_0$ and view them as an $e \times d$ matrix-valued and an $\mathbb{R}^e$-valued function, respectively. Then, (3.2) simply reads $dy_t = \sigma(y_t)dh_t + b(y_t)dt$. By formal differentiation of (3.2) in the direction of $l \in C_0^{q\text{-var}}(\mathbb{R}^d)$, $D_l y_t$ should satisfy the following Young ODE (if it exists):

$$d\xi_t^{[1]} = \nabla \sigma(y_t)\langle \xi_t^{[1]} \rangle dh_t + \nabla b(y_t)\langle \xi_t^{[1]} \rangle dt + \sigma(y_t)dl_t \quad \text{with} \quad \xi_0^{[1]} = 0 \in \mathbb{R}^e. \quad (3.3)$$

$\langle \cdot \rangle$ denotes the average operator. Here, $D_l$ stands for the directional derivative on $C_0^{q\text{-var}}(\mathbb{R}^d)$ in the direction $l$ and $\nabla$ stands for the standard gradient on $\mathbb{R}^e$.

In a similar way, $D_{l,l} y_t$ should satisfy the following Young ODE (if it exists):

$$d\xi_t^{[2]} = \nabla \sigma(y_t)\langle \xi_t^{[2]} \rangle dh + \nabla b(y_t)\langle \xi_t^{[2]} \rangle dt + \nabla^2 \sigma(y_t)\langle \xi_t^{[1]} \rangle, \xi_t^{[1]}, dh_t \rangle + 2\nabla \sigma(y_t)\langle \xi_t^{[1]} \rangle dl_t + \nabla^2 b(y_t)\langle \xi_t^{[1]} \rangle, \xi_t^{[1]}, dl_t \rangle \quad \text{with} \quad \xi_0^{[2]} = 0 \in \mathbb{R}^e. \quad (3.4)$$

If $V_i$’s are of $C^4$, the following facts are known to hold (see [11,12] for example). The system of Young ODEs (3.2), (3.3) and (3.4) has a unique global solution for every $h$ and $l$. (We write $\xi^{[1]}_t(h, l)$ and $\xi^{[2]}_t(h, l)$ when necessary.) The mapping

$$C_0^{q\text{-var}}(\mathbb{R}^d \oplus \mathbb{R}^d) \ni (h, l) \mapsto (y(h), \xi^{[1]}_t(h, l), \xi^{[2]}_t(h, l)) \in C_0^{q\text{-var}}((\mathbb{R}^e)^{\oplus 3})$$

is locally Lipschitz continuous. Moreover, $\Psi$ is of Fréchet-$C^2$ and $D_l \Psi(h)$ and $D_{l,l} \Psi(h)$ coincide with $\xi^{[1]}_t(h, l)$ and $\xi^{[2]}_t(h, l)$, respectively.

Now we go back to RDEs. We assume that $V_i$’s are of $C^{|p|+3}_b$. The RDEs driven by $x \in G\Omega_p(\mathbb{R}^d)$ and $l \in C_0^{q\text{-var}}(\mathbb{R}^d)$ which correspond to (3.3)–(3.4) are given as
It is known that the system of RDEs (3.1), (3.5) and (3.6) has a unique global solution for every \((x, l)\). (See [10] or [6, Section 10.7], for example.) When we specify the driver, we will write \(\xi^{[j]}(x, l)\), \(j = 1, 2\). For \(h \in C^0_0 \text{-var}(\mathbb{R}^d)\), we have \(\xi^{[j]}(h, l) = \xi^{[j]}(\mathcal{L}(h), l)\), \(j = 1, 2\). Since explosion never happens, Lyons’ continuity theorem still holds for this system of RDEs, namely, the following map is locally Lipschitz continuous:

\[
G_{\rho}(\mathbb{R}^d) \times C^0_0 \text{-var}(\mathbb{R}^d) \ni (x, l) \mapsto (y(x), \xi^{[1]}(x, l), \xi^{[2]}(x, l)) \in C^{0, \rho} \text{-var}((\mathbb{R}^e)^{\oplus 3}).
\]

This property plays a key role in Sect. 5.

### 4 Gaussian Rough Path

We introduce a stochastic process. Let \((w_t)_{0 \leq t \leq T} = (w^1_t, \ldots, w^d_t)_{0 \leq t \leq T}\) be a centered, continuous, \(d\)-dimensional Gaussian process with i.i.d. components which start at 0. We denote its law by \(\mu\) and its Cameron–Martin space by \(\mathcal{H}\). Then, \((\mathcal{W}, \mathcal{H}, \mu)\) becomes an abstract Wiener space, where \(\mathcal{W}\) is the closure of \(\mathcal{H}\) with respect to the usual sup-norm in \(C_0(\mathbb{R}^d) := \{x : [0, T] \to \mathbb{R}^d : \text{continuous and } x_0 = 0\}\). When \(d = 1\), we write \((\mathcal{W}_1, \mathcal{H}_1, \mu_1)\). Obviously, \(\mathcal{W} = (\mathcal{W}_1)^{\oplus d}\) (the direct sum), \(\mathcal{H} = (\mathcal{H}_1)^{\oplus d}\) (the orthogonal sum), \(\mu = (\mu_1)^{\oplus d}\) (the product measure).

Let \(R(s, t) = \mathbb{E}[w^1_rw^1_s]\) be the covariance function.

For the rest of this paper, we will assume the following condition:

\[
R(s, t) \text{ is of finite 2D } \rho\text{-variation for some } \rho \in [1, 2), \tag{4.1}
\]

(For the definition of 2D \(\rho\)-variation, see [6, Section 15.1].) Under (4.1) many facts were shown. Some of them are as follows. First, \(\mathcal{H}\) is continuously embedded in \(C^0_0 \text{-var}(\mathbb{R}^d)\) (see [6, Proposition 15.7]). This implies that we can use Young integration for \(h \in \mathcal{H}\) since \(\rho < 2\). Second, the natural lift \(w\) of \(w\) exist as a \(G_{\rho}(\mathbb{R}^d)\)-valued random variable for \(p \in (2\rho, 4)\). All known reasonable rough path lift of \(w\) coincides with \(w\). Those include the limit of piecewise linear, mollifier, and Karhunen–Loève approximations of \(w\) (see [6, Theorem 15.33, Definition 15.34]). In this work, we will only use Karhunen–Loève approximations. We will recall it in the next paragraph.

Let \(\{h^i_i\}_{i=1}^{\infty}\) be an orthonormal basis of \(\mathcal{H}_1\) such that \(\langle h_i, \bullet \rangle \in (\mathcal{W}_1)^*\) and \(\{e^d_j\}_{j=1}^{d}\) be the canonical orthonormal basis of \(\mathbb{R}^d\). Set \(h_{i, j} = h_i e_j \in \mathcal{H}\), then \(\{h_{i, j}\}_{1 \leq i < \infty, 1 \leq j \leq d}\) forms an orthonormal basis of \(\mathcal{H}\) and \(\langle h_{i, j}, \bullet \rangle \in \mathcal{W}_1^*\). As is well-known, \(\{\langle h_{i, j}, \bullet \rangle\}_{1 \leq i < \infty, 1 \leq j \leq d}\) is an i.i.d. sequence defined on \((\mathcal{W}, \mathcal{H}, \mu)\) with the law of \(\langle h_{i, j}, \bullet \rangle\) being the standard normal distribution.
Define Gaussian processes $w^{[N]}$ and $w^{*N}$ for $w \in \mathcal{W}$ and $N \geq 1$ by

$$w^{[N]} = \sum_{i=1}^{N} \sum_{j=1}^{d} (h_{i,j}, w) h_{i,j}, \quad w^{*N} = w - w^{[N]}.$$ 

Note that these are everywhere-defined random variables defined on $\mathcal{W}$ taking values in $H$ and $W$, respectively. Since the former is $H$-valued, the lift $w^{[N]} := L(w^{[N]})$ exists for every $w$. The covariance of $w^{*N}$ also satisfies (4.1) with the same $\rho$ (see [6, p. 438]). Therefore, the natural lift of $w^{*N}$ of $w^{*N}$ exists, too.

Now we recall Karhunen–Loéve approximations for Gaussian rough paths. The following is (a special case of) [6, Theorem 15.47].

**Proposition 4.1** Let the notation be as above and assume (4.1) and $p \in (2\rho, 4)$. Then, we have the following:

- There exists a positive constant $\eta$ independent of $N$ such that

$$\sup_{N \geq 1} \mathbb{E} \left[ \exp \left( \eta \sum_{i=1}^{[p]} \| (w^{[N]})^i \|_{p/i-\text{var}}^{2/i} \right) \right] < \infty.$$ 

- For every $r \in [1, \infty)$ and $1 \leq i \leq [p],$

$$\| (w^{[N]})^i - w^i \|_{p/i-\text{var}} \to 0 \quad \text{in } L^r(\mu) \text{ as } n \to \infty.$$ 

- For every $r \in [1, \infty)$ and $1 \leq i \leq [p],$

$$\| (w^{*N})^i \|_{p/i-\text{var}} \to 0 \quad \text{in } L^r(\mu) \text{ as } n \to \infty.$$ 

We further assume the following condition, which is called complementary Young regularity in [6, Condition 15.56]. When we work under this condition, we pick (any) $p$ and $q$ as in the statement of this condition.

**(CYR)** In addition to (4.1), there exist $p \in (2\rho, 4)$ and $q \in [1, 2)$ such that (1) $1/p + 1/q > 1$ and (2) $\mathcal{H}$ is continuously embedded in $C^q_{0-\text{var}}(\mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued continuous paths of finite $q$-variation that start at 0.

**Remark 4.2** Assumption (CYR) holds if one of the following conditions holds:

(i) $R$ is of finite 2D $\rho$-variation for some $\rho \in [1, 3/2]$.

(ii) $w$ is fractional Brownian motion (fBM) with Hurst parameter $H \in (1/4, 1/2]$.

**Lemma 4.3** Under (CYR), we have $w^{*N} = T_{-w^{[N]}}(w)$, almost surely. Here, $T : G\Omega_p(\mathbb{R}^d) \times C^q_{0-\text{var}}(\mathbb{R}^d) \to G\Omega_p(\mathbb{R}^d)$ stands for the Young translation.

**Proof** For $w \in \mathcal{W}$ and $k \geq 1$, we denote by $w(k)$ the dyadic piecewise linear approximation associated with $|l2^{-k}T: 0 \leq l \leq 2^k|$. By the piecewise linear approximation theorem for Gaussian rough path [6, Theorem 15.42], we have $L(w(k)) \to w$ a.s. and
\( \mathcal{L}(w^*N(k)) = T_{-w|N_1(k)} \mathcal{L}(w(k)) \to w^*N \) a.s. as \( k \to \infty \), taking a subsequence if necessary. Take \( \delta > 0 \) so small that \( 1/p + 1/(q + \delta) > 1 \). By [6, Theorem 5.23], we have \( h_{i,j}(k) \to h_{i,j} \) in \( C_0^{q+\delta-\var}(\mathbb{R}^d) \) as \( k \to \infty \). Hence, for all \( w \in \mathcal{W} \), \( w^{|N_1}(k) \to w^{|N} \) in \( C_0^{q+\delta-\var}(\mathbb{R}^d) \) as \( k \to \infty \). Using the continuity of \( T \), we finish the proof. \( \square \)

By Proposition 4.1, we can find a subsequence \( \{N_n\}_{n=1}^\infty \) such that \( w^{|N_n} \to w \) a.s. and \( w^*N_n \to 0 \) a.s. in \( G\Omega_p(\mathbb{R}^d) \) as \( n \to \infty \). Here, 0 stands for the zero rough path. (In some literature, it is denoted by 1.) We take such \( \{N_n\}_{n=1}^\infty \) and set

\[ \tilde{A} := \{ w \in \mathcal{W} : \{w^{|N_n}\}_{n=1}^\infty \text{ converges in } G\Omega_p(\mathbb{R}^d) \}. \]

As a version of \( w \), we choose the following everywhere-defined Borel-measurable map \( L : \mathcal{W} \to G\Omega_p(\mathbb{R}^d) \):

\[ L(w) = \begin{cases} \lim_{n \to \infty} w^{|N_n} & \text{if } w \in \tilde{A} \\ 0 & \text{if otherwise.} \end{cases} \]

Since \( h^{|N} \to h \) in \( \mathcal{H} \) as \( N \to \infty \) and we have continuous injections \( \mathcal{H} \hookrightarrow C_0^{q-\var}(\mathbb{R}^d) \hookrightarrow G\Omega_p(\mathbb{R}^d) \), we can easily see that \( h \in \tilde{A} \) and \( L(h) = L(h) \) for every \( h \in \mathcal{H} \).

Note that if \( \{w^{|N_n}\} = \mathcal{L}(w^{|N_1}) \) is convergent then so is \( \mathcal{L}((w + h)^{|N_n}) \) for every \( h \in \mathcal{H} \) since

\[ \mathcal{L}((w + h)^{|N_1}) = \mathcal{L}(w^{|N_1}) + h^{|N_1}) = T_h|N_1 \mathcal{L}(w^{|N_1}) \to T_hL(w) \text{ as } n \to \infty. \]

This implies that \( \tilde{A} \) is invariant under \( \tau_h \) for every \( h \in \mathcal{H} \), where \( \tau_h(w) = w + h \), and that \( L(w + h) = T_hL(w) \) for every \( w \in \tilde{A} \) and \( h \in \mathcal{H} \).

By Lemma 4.3, we can choose \( T_{-w|N_1}L(w) \) as an everywhere-defined Borel-measurable version of \( w^*N \) and will fix this version from now on. Then, we can find a subsequence of \( \{N_n\} \), which will be denoted by the same symbol again, such that the following subset

\[ A := \{ w \in \tilde{A} : \lim_{n \to \infty} w^*N_n = 0 \text{ in } G\Omega_p(\mathbb{R}^d) \}. \]

is of full measure with respect to \( \mu \). Then, \( A \) is also invariant under \( \tau_h \) for every \( h \in \mathcal{H} \). Indeed, if \( w \in A \), then

\[ T_{-(w + h)|N_1}L(w + h) = T_{h}|N_1 T_{-w|N_1}L(w) \to T_00 = 0 \text{ as } n \to \infty, \]

which implies \( w + h \in A \). In particular, \( \mathcal{H} \subset A \) since \( 0 \in A \).

Choose a subsequence \( \{N_n\} \) as above and fix it in what follows. Let \( K_n \subset \mathcal{H} \) be the finite-dimensional subspace spanned by \( \{h_{i,j}\}_{1 \leq i \leq N_n, 1 \leq j \leq d} \) and denote by \( P_n : \mathcal{H} \to K_n \) the orthogonal projection onto this subspace. As is well-known, \( P_n \) naturally extends to the bounded projection \( \tilde{P}_n : \mathcal{W} \to K_n \) given explicitly by \( \tilde{P}_n(w) = w^{|N_n} \).

We set \( P_n^\perp(h) = h - P_n(h) \) for \( h \in \mathcal{H} \) and \( \tilde{P}_n^\perp(w) = w - \tilde{P}_n(w) \) for \( w \in \mathcal{W} \).

If we use this notation, \( w^*N_n = L(w - w^{|N_1}) = L(P_n^\perp(w)) \) for all \( w \in A \).
Remark 4.4 Consider the case $\mathcal{W} = C_0(\mathbb{R}^d)$ (e.g. the case that $\omega$ is fBM with $1/4 < H \leq 1/2$). Then, the situation described above can be summarized by the following commutative diagram:

Here, all the maps except $L$ are continuous. Recall also that $\Psi = \Phi \circ \mathcal{L}$. (If $\mathcal{W} \neq C_0(\mathbb{R}^d)$, we just need to replace $C_0^{q-\text{var}}(\mathbb{R}^d)$ in this diagram by the closure of $\mathcal{H} \cap C_0^{q-\text{var}}(\mathbb{R}^d)$ with respect to the $q$-variation norm.)

5 Twice $\mathcal{K}$-Differentiability of Lyons–Itô Map

For $\mathcal{K} = \{K_n\}_{n=1}^\infty$ as in the previous section, the rough path lift map is $\mathcal{K}$-regular and so is the solution of an RDE driven a Gaussian rough path. Note that $\Phi \circ L$ equals a.s. to the solution of RDE (3.1) with $x$ being replaced by $w$.

Proposition 5.1 Let the notation be as above and assume (CYR). Then, we have the following:

1. The measurable map $L: \mathcal{W} \to G_{\Omega_R}(\mathbb{R}^d)$ is uniformly $\mathcal{K}$-regular.
2. If, in addition, $V_i$ is of $C_b^{[p]+1}$ for all $0 \leq i \leq d$, then $\Phi \circ L: \mathcal{W} \to C_0^{0,p-\text{var}}(\mathbb{R}^e)$ is uniformly $\mathcal{K}$-regular.

Proof First we show (1). We set $E' = G_{\Omega_R}(\mathbb{R}^d)$, $G = L$ and $\tilde{G} = \mathcal{L} \upharpoonright \mathcal{H}: \mathcal{H} \to E'$. Set also $G_n: \mathcal{W} \times K_n \mapsto E'$ by $G_n(w,k) = T_k L(w)$ if $w \in A$ and $G_n(w,k) = 0$ if $w \not\in A$. Then, for all $w \in A$ and $k \in \mathcal{H}$, we have

$$G \circ \tau_k(w) = L(w + k) = T_k L(w) = G_n(w,k),$$
$$G_n^{-1}(w,k) = G_n(w,-\tilde{P}_n(w) + k) = L(w - \tilde{P}_n(w) + k) = L(\tilde{P}_n^{-1}(w) + k) = T_k w^{w N_n}.$$ 

Thanks to these explicit expressions, we may and will view $G_n$ and $G_n^{-1}$ as maps from $\mathcal{W} \times \mathcal{H}$ to $E'$.

Let us check (2.4). Take $w \in A$. Note that $\tilde{G}(\tilde{P}_n(w) + k) = \mathcal{L}(\tilde{P}_n(w) + k) = T_k w^{w N_n}$ and $G_{m \vee n}(w,k) = T_k L(w)$. Since $w^{N_n} \to L(w)$ as $n \to \infty$, $\{w^{N_n}\}_n$ is bounded in $E'$. Since $T: E' \times \mathcal{H} \to E'$ is locally Lipschitz continuous, we see that

$$\sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(\tilde{G}(\tilde{P}_n(w) + k), G_{m \vee n}(w,k)) = \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(T_k w^{[N_n]}, T_k L(w)) \leq C_r, w \rho_{E'}(w^{[N_n]}, L(w)) \to 0 \text{ as } n \to \infty.$$ 

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Here, $C_{r,w}$ is a positive constant which depends only on $r > 0$ and $w \in \mathcal{A}$ (and may vary from line to line). In a similar way, since $w^{*N_n} \to 0$ as $n \to \infty$,

$$
\sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E}(\tilde{G}(h + k), G_n^*(w, P_n(h) + k)) = \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E}(T_{k+h}0, T_k + P_n(h)w^{*N_n}) \leq C_{r,w,h} \rho_{E}(w^{*N_n}, 0) + \|P_n(h) - h\|_{\mathcal{H}} \to 0
$$

as $n \to \infty$. Here, $C_{r,w,h}$ is a positive constant which depends only on $r > 0$, $h \in \mathcal{H}$ and $w \in \mathcal{A}$. Thus, we have shown (1).

Next we show (2). We set $E = C^{0,p-\text{var}}(\mathbb{R}^e)$, $\Phi = \Phi \circ L$ and $\tilde{\Phi} = \Phi \circ L |_{\mathcal{H}} : \mathcal{H} \to E$. Note that $\tilde{F} = \Psi |_{\mathcal{H}}$. Set also $F_n : \mathcal{W} \times K_n \mapsto E$ by $F_n(w, k) = \Phi(G_n(w, k))$. Take any $w \in \mathcal{A}$ and $k \in \mathcal{H}$. It is clear that $F \circ \tau_k(w) = F_n(w, k)$. We also have $F_{n\perp}(w, k) = \Phi(T_kw^{*N_n})$. Again, we may and will view $F_n$ and $F_{n\perp}$ as maps from $\mathcal{W} \times \mathcal{H}$ to $E$.

Since Lyons–Itô map $\Phi$ is locally Lipschitz continuous and both $\tilde{G}(\tilde{P}_n(w) + k)$ and $G_{m\vee n}(w, k)$ stay bounded when $n \geq 1$ and $k \in \mathcal{H}$ (with $\|k\|_{\mathcal{H}} \leq r$) vary, we have

$$
\sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E}(\tilde{F}(\tilde{P}_n(w) + k), F_{m\vee n}(w, k)) \leq C_{r,w} \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E}(\tilde{G}(\tilde{P}_n(w) + k), G_{m\vee n}(w, k)).
$$

As we have seen, the right hand tends to zero as $n \to \infty$. In same way, we can show

$$
\sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E}(\tilde{F}(h + k), F_{n\perp}(w, P_n(h) + k)) \to 0 \quad \text{as } n \to \infty.
$$

These imply (2.4) for $F$. Thus, we have shown (2). \hfill \Box

The following is the key result of this paper. Note that twice $\mathcal{K}$-regular differentiability is shown at the level of the path space unlike in [1, Theorem 3.41]. Of course, it immediately implies that $y_t = y_t(w)$ is twice $\mathcal{K}$-regularly differentiable for every $t \in [0, T]$, where $y = (y_t(w))_{0 \leq t \leq T}$ the unique solution of RDE (3.1) with $x$ being replaced by the Gaussian rough path $w$.

**Proposition 5.2** Assume (CYR). If $V_i$ is of $C^{|p|+3}_b$ for all $0 \leq i \leq d$, then $\Phi \circ L : \mathcal{W} \to C^{0,p-\text{var}}(\mathbb{R}^e)$ is twice $\mathcal{K}$-regularly differentiable.

**Proof** We use the same notation as in the proof of Proposition 5.1, (2). An unimportant positive constant which depends only on the parameter $\star$ is denoted by $C_{\star}$, which may vary from line to line. As is well-known, Fréchet-$C^n$ and Gâteaux-$C^n$ are equivalent (see [4, Theorem (2.1.27)] for example). Hence, we only consider Gâteaux derivatives.

We will prove (2.5) for $l = 2$ by showing that

$$
\|\tilde{F}(\tilde{P}_n(w) + \bullet) - F_{m\vee n}(w, \bullet)\|_{C^2_B(\mathcal{H}(0,r), E)} \vee \|\tilde{F}(h + \bullet) - F_{n\perp}(w, P_n(h) + \bullet)\|_{C^2_B(\mathcal{H}(0,r), E)} \to 0 \quad \text{as } n \to \infty \quad (5.1)
$$

for every $w \in \mathcal{A}$, $r > 0$ and $h \in \mathcal{H}$.
Convergence of the zeroth order in (5.1) was already shown in the proof of Proposition 5.1, (2).

Now we calculate the first order derivative. For the rest of the proof, let $r, r' > 0$, $w \in \mathcal{A}, k, l, h \in \mathcal{H}$. Since $\tilde{F} = \Psi$, we have

$$D_l \tilde{F}(\tilde{P}_n(w) + \bullet)|_{\bullet = k} = \tilde{\xi}^{[1]}(\tilde{P}_n(w) + k, l) = \tilde{\xi}^{[1]}(T_k w^{[N_n]}, l)$$

Due to the local Lipschitz continuity of $\tilde{\xi}^{[1]}$ mentioned in (3.7), we have that, if $\|k\|_{\mathcal{H}} \leq r$ and $\|l\|_{\mathcal{H}} \leq r'$, then

$$\|D_l \tilde{F}(\tilde{P}_n(w) + \bullet)|_{\bullet = k} - \xi^{[1]}(T_k w, l)\|_E = \|\tilde{\xi}^{[1]}(T_k w^{[N_n]}, l) - \tilde{\xi}^{[1]}(T_k L(w), l)\|_E$$

$$\leq C_{r,r',w} \rho_{E'}(w^{[N_n]}, L(w)) \to 0 \quad \text{as} \quad n \to \infty.$$  

In particular, this uniform convergence in $(k, l)$ implies that

$$D_l F_{m\vee n}(w, \bullet)|_{\bullet = k} = D_l \Phi(T_m L(w))|_{\bullet = k} = \tilde{\xi}^{[1]}(T_k L(w), l)$$

and $k \mapsto D_l F_{m\vee n}(w, \bullet)|_{\bullet = k}$ is continuous and

$$\sup_{\|k\|_{\mathcal{H}} \leq r} \|D_l \tilde{F}(\tilde{P}_n(w) + \bullet)|_{\bullet = k} - D_l F_{m\vee n}(w, \bullet)|_{\bullet = k}\|_{\mathcal{H} \to E}$$

$$\leq \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \leq l} \|\tilde{\xi}^{[1]}(T_k w^{[N_n]}, l) - \tilde{\xi}^{[1]}(T_k L(w), l)\|_E \to 0 \quad \text{as} \quad n \to \infty, \quad (5.2)$$

where $\| \cdot \|_{\mathcal{H} \to E}$ stands for the operator norm for bounded operators from $\mathcal{H}$ to $E$.

Since $F_n^{\perp}(w, k) = \Phi(T_k w^{[N_n]}) = \Phi(T_{k-w^{[N_n]}} L(w))$, we can show in the same way as above that $D_l F_n^{\perp}(w, \bullet)|_{\bullet = k} = \tilde{\xi}^{[1]}(T_{k-w^{[N_n]}} L(w), l) = \tilde{\xi}^{[1]}(T_k w^{[N_n]}, l)$. Hence, if $\|k\|_{\mathcal{H}} \leq r$ and $\|l\|_{\mathcal{H}} \leq r'$, then

$$\|D_l \tilde{F}(h + \bullet)|_{\bullet = k} - D_l F_n^{\perp}(w, P_n(h) + \bullet)|_{\bullet = k}\|_E \leq \|\tilde{\xi}^{[1]}(T_h + P_n(h) w^{[N_n]}, 0, l) - \tilde{\xi}^{[1]}(T_k w^{[N_n]}, l)\|_E$$

$$\leq C_{r,r',w,h} \rho_{E'}(w^{[N_n]}, 0) \to 0 \quad \text{as} \quad n \to \infty.$$  

It is also easy to see from this that $k \mapsto D_l F_n^{\perp}(w, P_n(h) + \bullet)|_{\bullet = k}$ is continuous and

$$\sup_{\|k\|_{\mathcal{H}} \leq r} \|D_l \tilde{F}(h + \bullet)|_{\bullet = k} - D_l F_n^{\perp}(w, P_n(h) + \bullet)|_{\bullet = k}\|_{\mathcal{H} \to E} \to 0 \quad \text{as} \quad n \to \infty. \quad (5.3)$$

Convergence of the first-order derivatives in (5.1) follows immediately from (5.2) and (5.3).

Next we calculate the second-order derivatives. We have

$$D_{l,l}^2 \tilde{F}(\tilde{P}_n(w) + \bullet)|_{\bullet = k} = \tilde{\xi}^{[2]}(\tilde{P}_n(w) + k, l) = \tilde{\xi}^{[2]}(T_k w^{[N_n]}, l).$$
Due to the Lipschitz continuity of $\xi^{[2]}$ we mentioned in (3.7), we have that, if $\|k\|_{\mathcal{H}} \leq r$ and $\|l\|_{\mathcal{H}} \leq r'$, then

$$
\|D^2_{l,l} \tilde{F}(\tilde{P}_n(w) + \bullet)\|_{\mathcal{H}} = \|\xi^{[2]}(T_k w^{[N_n]}, l) - \xi^{[2]}(L(w), l)\|_{E} \\
\leq C_{r,r',w} \rho_{E}(w^{[N_n]}, L(w)) \to 0 \quad \text{as } n \to \infty.
$$

(5.4)

By polarization, we can easily see that, for $\|k\|_{\mathcal{H}} \leq r$ and $\|l\|_{\mathcal{H}} \vee \|\hat{l}\|_{\mathcal{H}} \leq r'$,

$$
\|4D^2_{l,l} \tilde{F}(\tilde{P}_n(w) + \bullet)\|_{\mathcal{H}} = \frac{1}{4}\|\xi^{[2]}(T_k L(w), l + \hat{l}) + \xi^{[2]}(T_k L(w), l - \hat{l})\|_{E} \\
\leq C_{r,r',w} \rho_{E}(w^{[N_n]}, L(w)) \to 0 \quad \text{as } n \to \infty.
$$

This uniform convergence in $(k, l)$ implies that

$$
D_{l,l} F_{m \vee n}(w, \bullet)\|_{\mathcal{H}} = D^2_{l,l} \Phi(T_k L(w))\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty.
$$

(5.5)

where $\| \cdot \|_{\mathcal{H}} \to E$ stands for the standard norm for bounded bilinear maps from $\mathcal{H} \times \mathcal{H}$ to $E$.

Since $F_{n}^{\perp}(w, k) = \Phi(T_{k-w[N_n]} L(w))$, we see that $D_{l} F_{n}^{\perp}(w, \bullet)\|_{\mathcal{H}} = \xi^{[2]}(T_k w^{*N_n}, l)$. Hence, if $\|k\|_{\mathcal{H}} \leq r$ and $\|l\|_{\mathcal{H}} \vee \|\hat{l}\|_{\mathcal{H}} \leq r'$, then

$$
\|D^2_{l,l} \tilde{F}(h + \bullet)\|_{\mathcal{H}} = \|D^2_{l,l} F_{n}^{\perp}(w, P_n(h) + \bullet)\|_{\mathcal{H}} \\
\leq \|\xi^{[2]}(T_{h+k} 0, l + \hat{l}) - \xi^{[2]}(T_{k} w^{*N_n}, l + \hat{l})\|_{E} \\
+ \|\xi^{[2]}(T_{h+k} 0, l - \hat{l}) - \xi^{[2]}(T_{k} w^{*N_n}, l - \hat{l})\|_{E} \\
\leq C_{r,r',w} \rho_{E}(w^{*N_n}, 0) + \|h - P_n(h)\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty.
$$

(5.6)

It is also easy to see from this that $k \mapsto D^2 F_{n}^{\perp}(w, P_n(h) + \bullet)\|_{\mathcal{H}}$ is continuous and

$$
\sup_{\|k\|_{\mathcal{H}}} \|D^2 \tilde{F}(h + \bullet)\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty.
$$

(5.6)

Convergence of the second order derivatives in (5.1) follows immediately from (5.5) and (5.6). This completes the proof. \qed

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6 Summary of Our Result and Comparison with Preceding Ones

Finally, we summarize our main result on positivity of the law of the solution of RDEs in the next theorem.

As in Propositions 5.1 and 5.2, we denote by $y = (y_t(w))_{0 \leq t \leq T}$ the unique solution of RDE (3.1) with $x$ being replaced by the Gaussian rough path $w$. Also, $\mathcal{K} = \{K_n\}_{n=1}^\infty$ is the same non-decreasing, countable exhaustion of $\mathcal{H}$ as in these propositions.

**Theorem 6.1** Assume (CYR) and that $V_i$ is of $C^\infty_b$ for all $0 \leq i \leq d$. Let $t \in (0, T]$ and assume further that $y_t = y_t(w)$ is non-degenerate in the Malliavin sense (2.6). Let $G: \mathcal{W} \to [0, +\infty)$ be functions which is infinitely differentiable in the sense of the Malliavin calculus. Then, there exists a unique nonnegative function $f_t \in C^\infty(\mathbb{R}^e, \mathbb{R})$ with the properties that $f_t$ has rapidly decreasing derivatives of all orders and

$$\int_{\mathcal{W}} \phi(y_t(w))G(w)\mu(dw) = \int_{\mathbb{R}^e} \phi(z)f_t(z)dz, \quad \phi \in C_b(\mathbb{R}^e, \mathbb{R}).$$

Assume in addition that $G$ is $\mathcal{K}$-regular with its $\mathcal{K}$-regularization $\tilde{G}$. Then, for $z \in \mathbb{R}^e$, the following are equivalent:

- $f_t(z) > 0$.
- There exists $h \in \mathcal{H}$ such that $D\Psi(h)_t: \mathcal{H} \to \mathbb{R}^e$ has rank $e$, $\Psi(h)_t = z$ and $\tilde{G}(h) > 0$.

Here, $\Psi$ is the solution map for the corresponding Young ODE (3.2).

**Proof** It is shown in [10] that if we assume (CYR) and that $V_i$ is of $C^\infty_b$, then $y_t(w)$ is infinitely differentiable in the sense of the Malliavin calculus for every $t$. The rest of the proof follows immediately from Proposition 5.2 and Theorem 2.1. □

**Remark 6.2** We make a remark on the non-degeneracy of $y_t(w)$, which is the key assumption of the above theorem. First, we recall Hörmander’s bracket generating condition on $V_i$’s. Set $V_1 = \{V_1, \ldots, V_d\}$ and $V_k = \{V_i, W: 0 \leq i \leq d, W \in V_{k-1}\}$ for $k \geq 2$. We say $\{V_0, V_1, \ldots, V_d\}$ satisfies the Hörmander condition at the starting point $a \in \mathbb{R}^e$ if $\{W(a): W \in \bigcup_{k=1}^\infty V_k\}$ spans $\mathbb{R}^e$.

Non-degeneracy of $y_t(w)$ under this condition at the starting point $a$ was proved in [5,8] for a class of Gaussian processes, wherein the assumption on $w$ is stronger than (CYR), but examples still include fBM with $H \in (1/4, 1/2]$.

**Remark 6.3** Let us compare Theorem 6.1 above with two preceding results [2,7], in which $w$ is fBM with $H \in (1/4, 1/2]$. In this case, we have Remark 6.2 above for the non-degeneracy assumption in Theorem 6.1.

1. Suppose that $\{V_i: 1 \leq i \leq d\}$ satisfies the ellipticity condition everywhere, i.e., $\{V_i(z): 1 \leq i \leq d\}$ spans $\mathbb{R}^e$ at every $z \in \mathbb{R}^e$. In this case, $D\Psi(h)_t: \mathcal{H} \to \mathbb{R}^e$ has rank $e$ for every $h \in \mathcal{H}$. By standard argument, one can easily find for every $z$ a $C^1$-path $h$ such that $\Psi(h)_t = z$. Since a $C^1$-path belongs to Cameron-Martin space for fBM, we can use Theorem 6.1 to conclude that $f_t(z) > 0$ for all $z \in \mathbb{R}^e$ and $t > 0$ when $G \equiv 1$. Therefore, Theorem 6.1 implies the positivity result in [2, Theorem 1.4].
(2) We explain that Theorem 6.1 implies the positivity result in [7], in which $V_0 \equiv 0$ and \{$V_i : 1 \leq i \leq d$\} is assumed to satisfy the uniform Hörmander condition on $\mathbb{R}^e$. By using Sard’s theorem and Chow–Radevski’s theorem, they proved in [7] that, for every $z \in \mathbb{R}^e$, there exists $h \in \mathcal{H}$ such that $D\Psi(h)_t : \mathcal{H} \to \mathbb{R}^e$ has rank $e$, $\Psi(h)_t = z$. (This part seems non-trivial.) So, we can use Theorem 6.1 to recover the positivity result in [7], namely, $f_t(z) > 0$ for all $z \in \mathbb{R}^e$ and $t > 0$ when $G \equiv 1$.

(3) Other reasons why Theorem 6.1 are more general than the results in [2,7] are as follows:

(a) In [2,7], only the case $G \equiv 1$ is considered.
(b) In [2,7], the Gaussian process $w$ is restricted to fBM with $H \in (1/4, 1/2]$ only.
(c) The results in [2,7] prove everywhere-positivity for the density. The case $f_t(z) = 0$ was not studied there unlike in Theorem 6.1 above.

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