Radiation reaction as a non-conservative force

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Abstract
We study a system of a finite size charged particle interacting with a radiation field by exploiting Hamilton’s principle for a non-conservative system recently introduced by Galley [1]. This formulation leads to the equation of motion of the charged particle that turns out to be the same as that obtained by Jackson [2]. We show that the radiation reaction stems from the non-conservative part of the effective action for a charged particle. We notice that a charge interacting with a radiation field modeled as a heat bath affords a way to justify that the radiation reaction is a non-conservative force. The topic is suitable for graduate courses on advanced electrodynamics and classical theory of fields.

Keywords: radiation reaction, Hamilton’s principle for non-conservative systems, Abraham–Lorentz equation

1. Introduction
An accelerating charge radiates and therefore loses energy [2–4]. Larmor power accounts for the rate of loss of energy for a non-relativistic accelerated point charge. Loss of energy could be conceived to be caused by a damping force operating on the charge produced by the radiation field. This damping force is known as the radiation reaction [2].

An accelerating charge is thus a non-conservative system which experiences the radiation reaction—a non-conservative force. However, an accelerating charge together with its own field as a system conserves energy and is therefore conservative.

Although the radiation reaction has been studied extensively in the past [5–10] with several notable attempts to rectify the issues of causality violation and runaway solutions [11–17], it continues to be the subject of several recent studies [18–21].
We perform a systematic study of the radiation reaction as a non-conservative force using Hamilton’s principle for non-conservative systems to obtain the equation of motion of a charge and establish its non-conservative nature.

The radiation reaction, studied in an ad hoc manner in the past in light of Hamilton’s principle [22–24] for the construction of the relevant Lagrangian, could be attributed essentially to the prescribed Lagrangian in Galley’s formulation of Hamilton’s principle. Polonyi [25] has studied the radiation reaction by working out the effective dynamics of a single point charge having recast Schwinger’s closed time path formalism. Polonyi’s effective action approach to the radiation reaction appears similar to that of the derived effective action of Galley’s formulation; however, his method does not have an immediate scope to construct a prescribed Lagrangian like Galley’s formulation.

Galley [1] has formulated Hamilton’s principle compatible with initial data which leads to the Euler–Lagrange equations of motion for both conservative and non-conservative systems. In the present work, we obtain the equation of motion for an accelerating charge by exploiting Galley’s formulation of Hamilton’s principle for a non-conservative system. The equation of motion so obtained for a rigid, spherically symmetric charge instantaneously at rest turns out to be the same as that obtained by Jackson [2]. The radiation reaction is shown to stem from the non-conservative part of the effective action for a charged particle. The Abraham–Lorentz equation [26] is derived using the effective non-conservative Lagrangian for a point charge. We notice, on the basis of a correspondence between a charge interacting with radiation field and that of a particle interacting with infinite bath oscillators, that the radiation reaction could be realized as a non-conservative force.

In section 2, we briefly review Hamilton’s principle for a non-conservative system due to Galley. In section 3, we observe distinctly why the usual derivation of the Euler–Lagrange equation from the effective action provides an incomplete equation of motion for an accelerating charge. In sub-sections 3.1 and 3.2, we derive the correct equation of motion for a finite size charge and a point charge respectively using Galley’s formulation of Hamilton’s principle for a non-conservative system.

2. Preliminaries

We understand that Hamilton’s principle is formulated as a boundary value problem in time to account for conservative systems. In this section, we shall briefly discuss the formulation of Hamilton’s principle for a non-conservative system due to Galley [1].

2.1. Hamilton’s action principle for a non-conservative system

Galley’s formulation of Hamilton’s principle, which is compatible with initial value problems, is endowed with a systematic algorithm as to how to obtain the Euler–Lagrange equation for a non-conservative system. The essential ingredients required to obtain the Euler–Lagrange equation for a non-conservative system are:

1. a dynamical system and
2. coordinates $q_i$ and velocities $\dot{q}_i$.

Consider a dynamical system with coordinates $q_i$ and velocities $\dot{q}_i$. The action, a functional of the dynamical variables $q_i$, is defined by
Under an arbitrary variation of the paths, \( q_i(t) \rightarrow q_i(t) + \epsilon \eta_i(t) \), with the end points held fixed, i.e., \( \eta(t_i) = 0 = \eta(t_f) \), Hamilton’s principle states that for the actual paths the first-order variation of \( S \) vanishes

\[
\left. \frac{\delta S}{\delta \epsilon} \right|_{\epsilon = 0} = \left. \frac{\delta}{\delta \epsilon} \int_{t_i}^{t_f} L(q_i(t), \dot{q}_i(t)) \, dt \right|_{\epsilon = 0} = 0,
\]

which in turn leads to the Euler–Lagrange equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i(t)} - \frac{\partial L}{\partial q_i(t)} = 0.
\]

Thus, Hamilton’s principle is formulated not as an initial value problem but as a boundary value problem in time since its formulation requires knowledge of the initial and final values of \( q_i(t) \). However, the solution to the derived equation of motion requires initial data, i.e., the values of \( q_i(t) \) and \( \dot{q}_i(t) \) at the initial time. It is this disparity between the two that lies at the heart of the incompatibility of Hamilton’s principle with dissipative (time-asymmetric) systems. To this end, we consider the following example. Consider the action for a harmonic oscillator

\[
S = \int_{t_i}^{t_f} \, dt \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right)
\]

which leads to the equation of motion

\[
\left( m \frac{d^2}{dt^2} + k \right) x(t) = 0.
\]

We can rewrite the action as

\[
\int_{t_i}^{t_f} \left( \frac{1}{2} m \dot{x}^2(t) - \frac{1}{2} k x^2(t) \right) \, dt = \int_{t_i}^{t_f} \left( \frac{1}{2} m \frac{d}{dt}(\dot{x} x) - \frac{1}{2} m x \dot{x} - \frac{1}{2} k x^2(t) \right) \, dt
\]

\[
= -\int_{t_i}^{t_f} \frac{1}{2} \dot{x}(t) \left( m \frac{d^2}{dt^2} + k \right) x(t) \, dt.
\]

The total derivative term \( \frac{1}{2} m \frac{d}{dt}(\dot{x} x) \) vanishes as the end points are held fixed. Note that the integrand in (4) is time-symmetric. Now, we consider the equation of motion for a damped harmonic oscillator

\[
\left( m \frac{d^2}{dt^2} + b \frac{d}{dt} + k \right) x(t) = 0.
\]

In order to derive the Euler–Lagrange equation for the damped harmonic oscillator using Hamilton’s principle, getting inspired from (4), one might be tempted to define the action as follows

\[
S_d[x(t)] = \int_{t_i}^{t_f} \left( \frac{1}{2} m \frac{d^2}{dt^2} + b \frac{d}{dt} + k \right) x(t) \, dt.
\]

However, under the arbitrary variation of the path \( x(t) \rightarrow x(t) + \epsilon \eta(t) \) with the endpoints held fixed, \( \eta(t_i) = 0 = \eta(t_f) \), the vanishing of the first-order variation of \( S_d \)
leads to the incorrect equation of motion

$$m\frac{d^2 x}{dt^2} + kx = 0$$

for the damped harmonic oscillator because the velocity term in (5) appears as a total derivative and does not contribute.

Galley has developed a consistent formulation of Hamilton’s principle that is compatible with initial value problems. The formulation of Hamilton’s principle for a generic system is accomplished through the following steps.

- Double the dynamical variables \([27–29]\) \(q_i \rightarrow (q_{i1}, q_{i2})\). This is done to facilitate the nonconservativity.
- Define action, a functional of the coordinates \(q_{i1}\) and \(q_{i2}\), as

$$S[q_{i1}, q_{i2}] = \int_{t_0}^{t_f} dt \Lambda(q_{i1}, q_{i2})$$

where the new Lagrangian reads

$$\Lambda(q_{i1}, q_{i2}, \dot{q}_{i1}, \dot{q}_{i2}) = L(q_{i1}, \dot{q}_{i1}) - L(q_{i2}, \dot{q}_{i2}) + K(q_{i1}, q_{i2}, \dot{q}_{i1}, \dot{q}_{i2}, t)$$

(7)

\(K\) depends on both variables and encodes the important aspects of the formulation. When

\[K = 0,\] two paths \(q_{i1}\) and \(q_{i2}\) are decoupled and \(\Lambda\) describes a conservative system,

\[\neq 0,\] the two paths \(q_{i1}\) and \(q_{i2}\) get coupled with each other and \(\Lambda\) describes a non-conservative system.

It is thus the non-zero \(K\) that determines the non-conservative nature of a dynamical system.

- Subject an arbitrary variation of the paths: \(q_{i1,\epsilon}(t) \rightarrow q_{i1,\epsilon}(t, 0) + \epsilon \eta_{i1,\epsilon}(t)\) with the following pair of boundary conditions
  i. \(\eta_{i1}(t_i) = \eta_{i2}(t_i) = 0\) at the initial time as well as
  ii. \(q_{i1}(t, \epsilon) = q_{i2}(t, \epsilon)\) and \(\dot{q}_{i1}(t, \epsilon) = \dot{q}_{i2}(t, \epsilon)\) (equality conditions) at the final timeso that Hamilton’s principle for the corresponding change in the action becomes

$$\left. \frac{\delta S[q_{i1}, q_{i2}]}{\delta \epsilon} \right|_{\epsilon = 0} \bigg|_{q_{i1}=q_{i2}=\eta_i} = 0,$$

where \(q_{i1} = q_{i2} = q\) is called the physical limit. Hamilton’s principle leads to the following equation of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \left( \frac{\partial}{\partial q_{i2}} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_{i2}} \right) K(q_{i1}, q_{i2}, \dot{q}_{i1}, \dot{q}_{i2}, t) \bigg|_{q_{i1}=q_{i2}=\eta_i}.$$

It is noteworthy that the equality conditions imply that the data \(q_{i1,\epsilon}(t_i)\) and \(\dot{q}_{i1,\epsilon}(t_i)\) are not held fixed (see figure 1).
Change the variables \( q_{1,2} \) (for convenience’s sake only) to
\[
q_+ = \frac{q_1 + q_2}{2} \quad \text{and} \quad q_- = q_1 - q_2
\]
such that \( q_- \to 0 \) and \( q_+ \to q_i \) for \( q_1 = q_2 \) (physical limit (PL)). Hamilton’s principle in terms of these new variables
\[
\left[ \frac{\delta S}{\delta \varepsilon} \right]_{\varepsilon=0, q_-, q_+ = q_i}^{q_-=0} = 0
\]
leads to the Euler–Lagrange equation for a generic system as
\[
\frac{d\pi_i}{dr} = \left[ \frac{\partial \Lambda}{\partial q_i} \right]_{PL} - \left[ \frac{\partial L}{\partial q_i} \right]_{PL} + \left[ \frac{\partial K}{\partial q_i} \right]_{PL}
\]
where the conjugate momenta \( \pi_i \) are given by
\[
\pi_i = \left[ \frac{\partial \Lambda}{\partial q_i} \right]_{PL} - \left[ \frac{\partial L}{\partial q_i} \right]_{PL} + \left[ \frac{\partial K}{\partial q_i} \right]_{PL}
\]

\( K \) is zero for conservative systems whereas open systems, which can exchange energy with the environment, possess non-zero \( K \). A non-zero \( K \) gives rise to the non-conservative force as we shall see in section 3. Non-zero \( K \) could be either

- \textit{prescribed} when the underlying variables that cause non-conservative (e.g. dissipative) forces are neither given nor modeled or
- \textit{derived} when all the degrees of freedom of a closed system are given or modeled.

Non-zero \textit{derived} \( K \) could be obtained from the effective action (such as the Fokker action) that describes some interacting system, for instance, a system of a charge interacting with the radiation field. It is worth noting that the interim variables \( (q_{1,2}) \) need not be associated with any physical meaning until after the physical limit is applied. The novelty of the formulation lies in its generality in the sense that

- it is compatible with both conservative and non-conservative systems and

\[\text{Figure 1. The solid lines represent the paths } q_{1i}(t) \text{ and } q_{2i}(t) \text{ and the dashed lines stand for their infinitesimal } (\varepsilon \ll 1) \text{ departures. At the initial time } t_i \text{ both variables are fixed, i.e., } q_{1i}(t_i) = q_{2i}(t_i) = q_0 \text{ and at the final time } t_f \text{ their values coincide, i.e., } q_{1i}(t_f, \varepsilon) = q_{2i}(t_f, \varepsilon), \text{ but have an arbitrary variation.} \]
• in this formulation it is possible to have a consistent Lagrangian consisting of distinct conservative as well as non-conservative part with the freedom to have either prescribed or derived $K$ for the non conservative part of Lagrangian.

3. Effective action for an accelerating charge

Consider a system of a finite size charged particle of charge $e$ coupled with its own electromagnetic field that is subjected to an external potential $V(\tilde{x})$. Such a system of a charged particle plus electromagnetic field is described by the Lagrangian [30]

$$ L = L_0 + L_{em} + L_{int} $$  \hspace{1cm} (12)

where

$$ L_0 = \frac{1}{2}m\dot{x}_i^2 - V(\tilde{x}) $$  \hspace{1cm} (13)

$$ L_{em} = \int d^3x \left( \frac{1}{2} \varepsilon_0 (-\nabla\phi - \dot{A}_i)^2 - \frac{1}{2\mu_0} (\nabla \times A_i)^2 \right) $$

$$ = \int d^3x \left( \frac{1}{2} \varepsilon_0 (\partial_i \phi + \dot{A}_i)^2 - \frac{1}{2\mu_0} (\partial_j A_j \partial_i A_j - \partial_i A_j \partial_j A_i) \right) $$  \hspace{1cm} (14)

$$ L_{int} = \int d^3x (-\rho \phi + J_i A_i) $$  \hspace{1cm} (15)

$\rho$ and $J_i$ are charge and current densities of the charged particle respectively; $\phi$ and $A_i$ are the potentials associated to the self field of the charged particle. The action associated with the Lagrangian (12) is given by

$$ S[x_i, \phi, A_i] = \int dt L(x_i, \dot{x}_i, \phi, \dot{\phi}, A_i, \dot{A}_i, \partial_j A_i) $$

$$ = \int dt \left( \frac{1}{2}m\dot{x}_i^2 - V(\tilde{x}) \right) $$

$$ + \int dtd^3x \left( \frac{1}{2} \varepsilon_0 (\partial_i \phi + \dot{A}_i)^2 - \frac{1}{2\mu_0} (\partial_j A_j \partial_i A_j - \partial_i A_j \partial_j A_i) \right) $$

$$ + \int dtd^3x (-\rho \phi + J_i A_i). $$  \hspace{1cm} (16)

Variations of the action (16) with respect to potentials $\phi$ and $A_i$ in the Lorentz gauge

$$ \nabla \cdot \hat{A} + \varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t} = 0 $$

lead to the following equations of motion for $\phi$ and $A_i$

$$ \Box \phi = -\frac{1}{\varepsilon_0} \rho $$  \hspace{1cm} (17)

$$ \Box A_i = -\mu_0 J_i. $$  \hspace{1cm} (18)
The appropriate solutions to equations (17) and (18) for radiation reaction can be given by

\[ \phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\tilde{x}'d'G_{\text{ret}}(\vec{x} - \vec{x}', t - t')\rho(\vec{x}', t') \] (19)

\[ A_i(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3\tilde{x}'d'G_{\text{ret}}(\vec{x} - \vec{x}', t - t')J_i(\vec{x}', t') \] (20)

where the function \( G_{\text{ret}}(\vec{x} - \vec{x}', t - t') \) stands for the retarded Green function and has nonzero support at \( t - t' = \frac{|\vec{x} - \vec{x}'|}{c} \). We obtain the effective action by substituting (19) and (20) in the action (16) as

\[ S_{\text{eff}} = \int dt \left[ \frac{1}{2} m\dddot{x}^2 + \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int d^3x \int d^3x'd' \rho(\vec{x}, t) \rho(\vec{x}', t') G_r(\vec{x} - \vec{x}', t - t') \right. \\
- \rho(\vec{x}, t) \rho(\vec{x}', t') G_{\text{ret}}(\vec{x} - \vec{x}', t - t')] \].

\[ (21) \]

The products \( J(\vec{x}, t)J(\vec{x}', t') \) and \( \rho(\vec{x}, t)\rho(\vec{x}', t') \) are symmetric under the exchange of variables \( x_i \leftrightarrow x_i' \) and \( t \leftrightarrow t' \) and couple only to the symmetric part of the retarded Green function i.e.

\[ G_+ \equiv \frac{G_{\text{ret}}(\vec{x} - \vec{x}', t - t') + G_{\text{ret}}(\vec{x}' - \vec{x}, t' - t)}{2} \] (22)

\[ = \frac{G_{\text{ret}}(\vec{x} - \vec{x}', t - t') + G_{\text{adv}}(\vec{x} - \vec{x}', t - t')}{2} \]

since \( G_{\text{ret}}(\vec{x}' - \vec{x}, t' - t) = G_{\text{adv}}(\vec{x} - \vec{x}', t - t') \) which has nonzero support at \( t' - t = \frac{|\vec{x} - \vec{x}'|}{c} \). The effective action thus becomes

\[ S_{\text{eff}} = \int dt \left[ \frac{1}{2} m\dddot{x}^2 + \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int d^3x \int d^3x'd' \rho(\vec{x}, t) \rho(\vec{x}', t') G_+ \right. \\
- \rho(\vec{x}, t) \rho(\vec{x}', t') G_{\text{ret}}(\vec{x} - \vec{x}', t - t')] \].

\[ (23) \]

The equation of motion for the finite size charged particle (rigid) could be obtained by the variation of an effective action (23) with respect to \( x_i \). We have the Euler–Lagrange equation as

\[ m\dddot{x} + \int d^3x \int d^3x'd' \rho(\vec{x}, t) \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{c^2} \frac{\partial G_+}{\partial t} \tilde{f}(\vec{x}', t') + \vec{\nabla} G_+ \rho(\vec{x}', t') \right] = 0 \]

\[ (24) \]

where we have used for a rigid object \( J(\vec{x}, t) = \rho(\vec{x}, t)\dot{s}(t) \). Equation (24) would exhibit non-causal behavior because the presence of \( G_{\text{adv}} \) in \( G_+ \) would render the evolution of equation of motion acausal. The second term of (24)

\[ \int d^3x \int d^3x'd' \rho(\vec{x}, t) \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{c^2} \frac{\partial G_+}{\partial t} \tilde{f}(\vec{x}', t') + \vec{\nabla} G_+ \rho(\vec{x}', t') \right] \]

subject to small \( \frac{g}{c} \), rigid, spherically symmetric and instantaneously at rest charge distribution, after performing a Taylor series expansion [2], takes the following form

\[ \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{1 + (-1)^n}{n! c^{n+2}} \right) \frac{\partial^{n+1}}{\partial t^{n+1}} \tilde{f}(t) \int d^3x \int d^3x' \rho(\vec{x}) R^{n-1} \rho(\vec{x}') \]
where $R = |\vec{x} - \vec{x}'|$. Equation (24) now becomes

$$m\ddot{\vec{x}} + \frac{1}{2} \frac{1}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} 2 \left( \frac{1}{n! c^{n+2}} \right) \frac{\partial^{n+1}}{\partial t^{n+1}} \ddot{\vec{x}}(t) \int d^3x' \int d^3\rho(x') R^{n-1} \rho(x') = 0$$

(25)

which in fact does not represent the correct equation of motion for an accelerating charge. Owing to the merely temporally symmetric contribution to (25), the second term in (25) contains terms proportional to only an even order time derivative like $\ddot{\vec{x}}(t)$, $\dddot{\vec{x}}(t)$ etc and hence does not incorporate a radiation reaction term which is proportional to $\dot{\vec{x}}(t)$. It is the retarded Green function $G_{\text{ret}} = G_+ + G_-$ that can provide the correct equation of motion

$$m\ddot{\vec{x}} + \frac{1}{2} \frac{1}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} 2 \left( \frac{1}{n! c^{n+2}} \right) \frac{\partial^{n+1}}{\partial t^{n+1}} \ddot{\vec{x}}(t) \int d^3x' \int d^3\rho(x') R^{n-1} \rho(x') = 0.$$  

(26)

In the following section, we shall apply Hamilton’s principle for a non-conservative system due to Galley for an accelerating charge in order to obtain its correct equation of motion.

### 3.1. Equation of motion for an accelerated charge using Hamilton’s principle for a non-conservative system

The Lagrangian that describes a system of an accelerating charge plus its electromagnetic field reads

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) + \int d^3x \left[ \frac{\varepsilon_0}{2} ((\partial_j \phi)^2 + \dot{A}_i^2 + 2\partial_j \phi \dot{A}_i) - \frac{1}{2\mu_0} ((\partial_j A_i)^2 - \partial_j A_i \partial_j A_i) \right]$$

$$- \rho \phi + J_i A_i].$$

(27)

We begin with doubling the variables as follows

$$x_i \rightarrow x_{i1}, x_{i2}$$

$$A_i \rightarrow A_{i1}, A_{i2}$$

$$\phi \rightarrow \phi_1, \phi_2.$$  

(28)

Next, we subject the change of variables as given below

$$A_{i+} = \frac{A_{i1} + A_{i2}}{2}, \phi_+ = \frac{\phi_1 + \phi_2}{2}, x_{i+} = \frac{x_{i1} + x_{i2}}{2}$$

$$A_{i-} = \frac{A_{i1} - A_{i2}}{2}, \phi_- = \frac{\phi_1 - \phi_2}{2}, x_{i-} = \frac{x_{i1} - x_{i2}}{2}$$

(29)

so that the action becomes

$$S[x_{i+}, x_{i-}, \phi_+, \phi_-, A_{i+}, A_{i-}]$$

$$= \int dt \Lambda$$

$$= \int dt \left[ m \dot{x}_{i+} \dot{x}_{i-} - V + \int d^3x (\varepsilon_0 \partial_i \phi_+ \partial_i \phi_+ + \varepsilon_0 \dot{A}_{i+} \dot{A}_{i-} - \frac{1}{\mu_0} \partial_i A_{i-} \partial_i A_{i+} + \frac{1}{\mu_0} \partial_i A_{i+} \partial_i A_{i-} + \frac{1}{\mu_0} \partial_i A_{i+} \partial_i A_{i-} + \frac{1}{2\mu_0} \partial_i A_{i+} \partial_i A_{i-}) \right]$$

(30)

where $V. \equiv V(\vec{x}_1) - V(\vec{x}_2)$. Equations of motion for $A_{i\pm}$ and $\phi_\pm$ that follow from the variation of action (30) are given by
\[ A_{i\pm} = -\mu_0 J_{i\pm} \]
\[ \phi_{i\pm} = -\frac{1}{\epsilon_0} \rho_{i\pm} \]

where we have identified
\[ \rho_+ = \frac{\rho_1 + \rho_2}{2} = \frac{1}{2} \left[ \rho \left( \bar{x}_+ + \frac{1}{2} \bar{y}_- \right) + \rho \left( \bar{x}_+ - \frac{1}{2} \bar{y}_- \right) \right] \]
\[ \rho_- = \rho_1 - \rho_2 = \rho \left( \bar{x}_+ + \frac{1}{2} \bar{y}_+ \right) - \rho \left( \bar{x}_+ - \frac{1}{2} \bar{y}_+ \right) \]

and
\[ J_{i+} = \frac{J_1 + J_2}{2} = \dot{x}_{i+} \rho_+ + \frac{1}{4} \dot{x}_{i-} \rho_- \]
\[ J_{i-} = J_1 - J_2 = \dot{x}_{i+} \rho_- + \dot{x}_{i-} \rho_+ \]

Now the solutions to (31) and (32) in the Lorentz gauge are given by
\[ A_{i+}(\bar{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' d't' G_{\text{ret}}(\bar{x} - \bar{x}', t - t') J_{i+}(\bar{x}', t') \]
\[ \phi_{i+}(\bar{x}, t) = \frac{1}{4\pi \epsilon_0} \int d^3x' d't' G_{\text{ret}}(\bar{x} - \bar{x}', t - t') \rho_+(\bar{x}', t') \]

up to solutions of the homogeneous equations, which for the radiation reaction could be appropriately set to zero and
\[ A_{i-}(\bar{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' d't' G_{\text{adv}}(\bar{x} - \bar{x}', t - t') J_{i-}(\bar{x}', t') \]
\[ \phi_{i-}(\bar{x}, t) = \frac{1}{4\pi \epsilon_0} \int d^3x' d't' G_{\text{adv}}(\bar{x} - \bar{x}', t - t') \rho_-(\bar{x}', t') \]

We can obtain the effective action by substituting (36), (37), (38) and (39) in the action (30).

The effective action turns out to be
\[ S_{\text{eff}} = \int dt \left[ m \dot{x}_{i+} \dot{x}_{i-} - V_- \right] + \frac{\mu_0}{4\pi} \int d^3x J_{i-}(\bar{x}_\pm, t) \int d^3x' d't' G_{\text{ret}}(\bar{x} - \bar{x}', t - t') J_{i+}(\bar{x}', t') \]
\[ - \frac{1}{4\pi \epsilon_0} \int d^3x \rho_-(\bar{x}_\pm, t) \int d^3x' d't' G_{\text{ret}}(\bar{x} - \bar{x}', t - t') \rho_+(\bar{x}', t') \]

The effective Lagrangian could be read from the effective action (40) as
\[ \Lambda = m \dot{x}_{i+} \dot{x}_{i-} - V_- \]
\[ + \frac{\mu_0}{4\pi} \int d^3x d^3x' dt' \left[ \frac{\mu_0}{4\pi} J_{i-}(\bar{x}_\pm, t) G_{\text{ret}}(\bar{x} - \bar{x}', t - t') J_{i+}(\bar{x}', t') \right] \]
\[ - \frac{1}{4\pi \epsilon_0} \rho_-(\bar{x}_\pm, t) G_{\text{ret}}(\bar{x} - \bar{x}', t - t') \rho_+(\bar{x}', t') \]

Thus, the effective Lagrangian \( \Lambda \) consists of conservative as well as non-conservative parts. The non-conservative part \( K \) of \( \Lambda \) is given by
\[ K = \int d^3x \frac{\mu_0}{4\pi} \left[ \frac{1}{4\pi\varepsilon_0} \rho_\omega (\vec{x}_\omega, t) G_{\text{ret}}(\vec{x}, \vec{x}', t - t') J_+(\vec{x}_\omega, t') \right] \]

while the conservative part of \( \Lambda \) (in the physical limit) reads

\[ L = \frac{1}{2} m \dot{x}_i^2 - V(\vec{x}). \]

The Euler–Lagrange equation for \( x_i(t) \)

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial x_i} = \left[ \frac{\partial K}{\partial \dot{x}_i} + \frac{d}{dt} \frac{\partial K}{\partial t} \right]_{\text{PL}} \tag{44} \]

follows from the variation of action \( S_{\text{eff}} \) which in turn yields

\[ m \ddot{x}_i + \int d^3x \int d^3x' \dot{\rho}(\vec{x}, t) \frac{1}{4\pi\varepsilon_0} \left[ \frac{1}{c^2} \frac{\partial}{\partial t} [G_{\text{ret}} J_i(\vec{x}', t')] \right] \]

\[ + \partial_t [G_{\text{ret}} \dot{\rho}(\vec{x}', t')] = F_{i,\text{ext}}. \tag{45} \]

where

\[ \left[ \frac{\partial K}{\partial \dot{x}_i} \right]_{\text{PL}} = \frac{\mu_0}{4\pi} \int d^3x \frac{1}{4\pi\varepsilon_0} \rho(\vec{x}, t) G_{\text{ret}}(\vec{x} - \vec{x}', t - t') J_i(\vec{x}', t') \tag{46} \]

\[ \left[ \frac{\partial K}{\partial x_i} \right]_{\text{PL}} = -\frac{1}{4\pi\varepsilon_0} \int d^3x \int d^3x' \rho(\vec{x}, t) \partial_t [G_{\text{ret}}(\vec{x} - \vec{x}', t - t') \rho(\vec{x}', t')] \tag{47} \]

\[ -\frac{\partial V}{\partial x_i} = F_{i,\text{ext}}. \tag{48} \]

Moreover, for a rigid, spherically symmetric charged particle instantaneously at rest, equation of motion \( (45) \) takes the form

\[ m \ddot{x}_i + \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{2}{3} \frac{(-1)^n}{n!c^{n+2}} \frac{\partial^{n+1} \dot{x}(t)}{\partial x_i^{n+1}} \int d^3x \int d^3x' \rho(\vec{x}) R^{n-1}(\vec{x}') = \vec{F}_{i,\text{ext}}. \tag{49} \]

This is in fact the same equation as obtained by Jackson [2]. We observe that the radiation reaction

\[ \vec{F}_{\text{Rad}} = \frac{\mu_0 e^2}{6\pi c} \vec{\dot{x}} \]

comes from the expansion of the second term for \( n = 1 \) in the LHS of equation \( (49) \). Thus the radiation reaction stems from \( K \). This is important to note that it is now possible to obtain distinctly the non-conservative force from the non-conservative part \( K \) of the effective Lagrangian \( \Lambda \). Moreover, it is now feasible to predict almost effortlessly a prescribed Lagrangian

\[ \Lambda = m \ddot{x}_i - \vec{\dot{x}}_i + \vec{x}_i \cdot \vec{F}_{\text{ext}} - \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{2}{3} \frac{(-1)^n}{n!c^{n+2}} \frac{\partial^{n+1} \dot{x}_i}{\partial x_i^{n+1}} \vec{\dot{x}}_i + I_n. \tag{50} \]
that would result the same equation of motion as (49), where
\[ I_n \equiv \int d^3x \int d^3x' \rho(x) R^{n-1}\rho(x'). \] (51)

We note that the prescribed Lagrangian (50) is seemingly different from the derived one (41); however, both of them lead to the same equation of motion in the physical limit. We have an advantage of this formalism that we can have the freedom to choose \( K \) (the last term in (50)). For example, \( K = \frac{1}{\pi n!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \epsilon^{n-2}} R^n \frac{\partial^\mu}{\partial \mu}, I_n \) is an equally acceptable choice.

### 3.2. Equation of motion for a point charge from the effective Lagrangian

We can obtain the effective Lagrangian for an accelerating point charge from \( \Lambda \) in (41), using the following form of Green’s function
\[ G_{\text{eff}}(\vec{x} - \vec{x}', t - t') = \frac{1}{R} \delta(t' - t + \frac{R}{c}) \] (52)
and performing integration over the variable \( t' \), as follows
\[ \Lambda = m_\gamma \dot{z}_\gamma - V_\gamma + \frac{\mu_0}{4\pi} \int d^3x d^3x' J_\gamma(x, \vec{z}_\gamma, t) \frac{1}{R} \dot{J}_\gamma(x', \vec{z}_\gamma, t) \]
\[ - \frac{1}{4\pi \epsilon_c} \int d^3x d^3x' \rho_\gamma(x, \vec{z}_\gamma, t) \frac{1}{R} \dot{\rho}_\gamma(x', \vec{z}_\gamma, t) \] (53)
where \( t_r = t - R/c \) and \( V_\gamma \) is given by
\[ V_\gamma = V \left( \vec{z}_\gamma + \frac{1}{2} \frac{x}{c} \right) - V \left( \vec{z}_\gamma - \frac{1}{2} \frac{x}{c} \right). \] (54)

If the velocity of the charge is small compared to that of light, then the change in charge and current densities could be considered negligible \([31]\) over the time interval \( \frac{R}{c} \). Hence, in the point charge limit \((R \to 0)\), we can neglect the higher order terms in the expansion of \( \rho_\gamma(t_r) \) and \( \dot{J}_\gamma(t) \) about \( t \)
\[ J_{\gamma+}(t_r) \approx J_{\gamma+}(t) = \frac{R}{c} \dot{J}_{\gamma+}(t) \]
\[ \rho_{\gamma+}(t_r) \approx \rho_{\gamma+}(t) = \frac{R}{c} \dot{\rho}_{\gamma+}(t) + \frac{R^2}{2c^2} \ddot{\rho}_{\gamma+}(t) - \frac{R^3}{6c^3} \dddot{\rho}_{\gamma+}(t). \] (55)

Using the above expansions, the effective Lagrangian \( \Lambda \) can now be expressed as
\[ \Lambda = m_\gamma \dot{z}_\gamma - V_\gamma \]
\[ + \int d^3x d^3x' \left[ \frac{\mu_0}{4\pi} J_{\gamma+}(x, \vec{z}_\gamma, t) \frac{1}{R} \dot{J}_{\gamma+}(x', \vec{z}_\gamma, t) - \frac{1}{4\pi \epsilon_c} \frac{1}{2c^2} \rho_{\gamma+}(x, \vec{z}_\gamma, t) R \dot{\rho}_{\gamma+}(x', \vec{z}_\gamma, t) \right] \]
\[ - \frac{\mu_0}{4\pi c} J_{\gamma-}(x, \vec{z}_\gamma, t) \dot{J}_{\gamma+}(x', \vec{z}_\gamma, t) + \frac{1}{4\pi \epsilon_c} \frac{1}{6c^3} \rho_{\gamma+}(x, \vec{z}_\gamma, t) R \ddot{\rho}_{\gamma+}(x', \vec{z}_\gamma, t) \]
\[ - \frac{1}{4\pi \epsilon_c} \rho_{\gamma+}(x, \vec{z}_\gamma, t) \frac{1}{R} \dot{\rho}_{\gamma+}(x', \vec{z}_\gamma, t) - \frac{1}{4\pi \epsilon_c} \rho_{\gamma+}(x, \vec{z}_\gamma, t) \dddot{\rho}_{\gamma+}(x', \vec{z}_\gamma, t) \] (57)
The equation of motion

\[ \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{z}_{i\text{PL}}} \right] = \frac{\partial L}{\partial z_{i\text{PL}}} \]  

(58)
of the point charge gives

\[ m \ddot{z}_i + \frac{\mu_0}{4\pi} \int d^3 x d^3 y' \rho (\vec{x} - \vec{z}, t) \frac{1}{R} \frac{\partial}{\partial t} \left[ J_i (\vec{x}', t) + \frac{\partial R}{2R^2} \dot{\rho} (\vec{x}' - \vec{z}, t) \right] \]

\[- \frac{1}{4\pi \varepsilon_0} \frac{1}{c^2} \int d^3 x d^3 y' \rho (\vec{x} - \vec{z}, t) \frac{\partial^2}{\partial t^2} \left[ J_i (\vec{x}', t) + \frac{\partial R^2}{6} \dot{\rho} (\vec{x}' - \vec{z}, t) \right] = F_{\text{ext}} \]  

(59)

where

\[ \frac{\partial L}{\partial \dot{z}_{i\text{PL}}} = m \ddot{z}_i + \int d^3 x d^3 y' \rho (\vec{x} - \vec{z}, t) \frac{\mu_0}{4\pi} \left[ \frac{1}{R} J_i (\vec{x}', \vec{z}, t) - \frac{1}{c} J_i (\vec{x}', \vec{z}, t) \right] \]

\[ \frac{\partial L}{\partial \dot{z}_{i\text{PL}}} = - \frac{\partial V}{\partial x_i} - \int d^3 x d^3 y' \left[ \frac{1}{4\pi \varepsilon_0} \frac{1}{2c^2} \rho (\vec{x} - \vec{z}, t) \partial_x R^2 \rho (\vec{x}' - \vec{z}, t) \right. \]

\[- \frac{1}{4\pi \varepsilon_0} \frac{1}{6c} \left. \partial_x \rho (\vec{x} - \vec{z}, t) \partial_x \dot{R} \rho (\vec{x}' - \vec{z}, t) \right) \]

\[- \frac{1}{4\pi \varepsilon_0} \frac{1}{c} \partial_x (\vec{x} - \vec{z}, t) \partial_x \dot{R} \rho (\vec{x}' - \vec{z}, t) \]

\[ 0 = \int d^3 x d^3 y' \rho (\vec{x} - \vec{z}, t) \partial_x \left[ \frac{1}{R} \rho (\vec{x}' - \vec{z}, t) \right] \]

\[ 0 = \partial_t \int d^3 x d^3 y' \rho (\vec{x} - \vec{z}, t) \dot{\rho} (\vec{x}' - \vec{z}, t) \]

For a spherically symmetric charge distribution (see pp 752–3 of [2])

\[ \vec{J} (\vec{x}', t) + \frac{\partial}{\partial t} (\rho (\vec{x}' - \vec{z}, t) + \frac{\partial R^{n+1}}{(n+1)(n+2)} = \frac{2}{3} \dot{z}_i \rho (\vec{x}' - \vec{z}, t). \]  

(60)

By making use of (60) and substituting \( \rho (\vec{x} - \vec{z}) = e \delta (\vec{x} - \vec{z}) \) and \( J_i (\vec{x}, t) = e \dot{z}_i \delta (\vec{x} - \vec{z}) \) in (59), followed by integration over \( \vec{x} \) and \( \vec{x}' \), we obtain the equation of motion of a point charge as

\[ m \ddot{z} + \lim_{R \to 0} \frac{\mu_0 e^2}{6\pi R} \ddot{z} - \frac{\mu_0 e^2}{6\pi c} \ddot{z} = F_{\text{ext}}. \]  

(61)

The term \( \frac{\mu_0 e^2}{6\pi R} \) which is divergent in the limit \( R \to 0 \) possesses the dimension of mass and couples with the bare mass \( m \) to give the renormalized mass \( M \),

\[ M = m + \lim_{R \to 0} \frac{\mu_0 e^2}{6\pi R} \]  

(62)

so that the equation of motion can now be expressed as the famous Abraham–Lorentz equation,

\[ M (\ddot{z} - \tau \dot{z}) = F_{\text{ext}} \]  

(63)

where \( \tau = \frac{\mu_0 e^2}{6\pi Mc} \). We note that the Abraham–Lorentz equation can also be obtained directly from (49) by substituting \( \rho (\vec{x}) = e \delta (\vec{x}) \) and taking the limit \( R \to 0 \). Moreover, the details of
the current section are important in their own right as they enable us to obtain the relevant effective Lagrangian which describes the effective dynamics of a point charge.

We know that the Abraham–Lorentz equation suffers from the problems of causality-violation and runaway solutions. It has been shown by Rohrlich \[16, 32\] that if the external force varies slowly over the size of a charged particle (of the order \(c^7\)) such that

\[
\left| \frac{\tau}{dt} \vec{F}_{\text{ext}} \right| \ll |\vec{F}_{\text{ext}}(t)|
\]

then the (finite) charge distribution can be approximated as a point charge and the effect of radiation reaction on the motion of charge is negligible. This allows us to write

\[ M \ddot{\vec{z}} \approx \vec{F}_{\text{ext}} \]  

or

\[ \ddot{z} \approx \frac{\vec{F}_{\text{ext}}}{M}. \]

The equation of motion obtained by substituting for \(\ddot{z}\) in (63) is free of the above mentioned problems—causality-violation and runaway solutions—and is given by

\[ M \ddot{z} = \vec{F}_{\text{ext}} + \tau \vec{F}_{\text{ext}}. \]  

A Lagrangian may now be prescribed for an accelerating point charge in light of equation (67) as follows

\[ \Lambda = M \dot{z}_+ \cdot \dot{z}_- + \dot{z}_- \cdot \vec{F}_{\text{ext}} - \tau \dot{z}_- \cdot \vec{F}_{\text{ext}} \]

with the non-conservative part of the Lagrangian given by

\[ K = \dot{z}_- \cdot \vec{F}_{\text{ext}} - \tau \dot{z}_- \cdot \vec{F}_{\text{ext}}. \]

The prescribed Lagrangian (68) produces the same equation of motion as (67).

4. Conclusion and discussion

We have shown that Hamilton’s principle for a non-conservative system such as a finite size charged particle interacting with a radiation field furnishes the same equation of motion as that obtained by Jackson [2]. Moreover, in the limit of a point charge, we obtain the Abraham–Lorentz equation. We find that the radiation reaction is derivable from the non-conservative part of the effective Lagrangian. A systematic study of the radiation reaction, based on the interaction of a finite size charged particle with a radiation field modeled as a heat bath, has been made in the past \[13, 33\]. This model affords a way to justify that an accelerating charge interacting with a radiation field is a non-conservative system. Since if, for a particle interacting with a heat bath modeled as a large number of independent harmonic oscillators, the number of the bath oscillators is about 20 or larger, the Poincaré recurrence time (the amount of time after which a system returns to a state very close to the initial state) turns out to be infinite \[34, 35\], which renders the dynamics of the interacting particle non-conservative.

References

[1] Galley C R 2013 Classical mechanics of non-conservative systems Phys. Rev. Lett. 110 174301
[2] Jackson J D 1999 Classical Electrodynamics 3rd edn (New York: Wiley) ch 16
[3] Barut A O 1980 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Dover) ch 5
[4] Schwinger J, DeRaad L L Jr., Milton K, Tsai W and Norton J 1998 *Classical Electrodynamics* (Boulder, CO: Westview) ch 32
[5] Boyer T H 1972 Mass renormalization and radiation damping for a charged particle in uniform circular motion *Am. J. Phys.* 40 1843
[6] Lenie H, Moniz E J and Sharp D H 1977 Motion of extended charges in classical electrodynamics *Am. J. Phys.* 45 75
[7] Griffiths D J and Szeto E W 1978 Dumbbell model for the classical radiation reaction *Am. J. Phys.* 46 244
[8] Templin J D 1998 An approximate method for the direct calculation of radiation reaction *Am. J. Phys.* 66 403
[9] Templin J D 1999 Radiation reaction and runaway solutions in acoustics *Am. J. Phys.* 67 407
[10] Rohrlich F 2000 The self-force and radiation reaction *Am. J. Phys.* 68 1109
[11] Cook R J 1984 Radiation reaction revisited *Am. J. Phys.* 52 894
[12] Valentini A 1988 Resolution of causality violation in the classical radiation reaction *Phys. Rev. Lett.* 61 17
[13] Ford G W and O’Connell R F 1991 Radiation reaction in electrodynamics and the elimination of runaway solutions *Phys. Lett. A* 157 4 5
[14] Ford G W and O’Connell R F 1993 Relativistic form of radiation reaction *Phys. Lett. A* 174 182
[15] Spohn H 2000 The critical manifold of the Lorentz–Dirac equation *Europhys. Lett.* 50 287
[16] Rohrlich F 2008 Dynamics of a charged particle *Phys. Rev. E* 77 046609
[17] Griffiths D J, Proctor T C and Schroeter D F 2010 Abraham–Lorentz versus Landau–Lifshitz *Am. J. Phys.* 78 391
[18] Haque A 2014 A simple derivation of Lorentz self-force *Eur. J. Phys.* 35 5
[19] Aashish S and Haque A 2015 Average Lorentz self-force from electric field lines *Eur. J. Phys.* 36 055012
[20] Steane A M 2015 Reduced-order Abraham–Lorentz–Dirac equation and the consistency of classical electromagnetism *Am. J. Phys.* 83 256
[21] Steane A M 2015 Tracking the radiation reaction energy when charged bodies accelerate *Am. J. Phys.* 83 703
[22] Bateman H 1931 On dissipative systems and related variational principles *Phys. Rev.* 38 815
[23] Barone P M V B and Mendes A C R 2007 Lagrangian description of the radiation damping *Phys. Lett. A* 364 438
[24] Bender C M and Gianfreda M 2015 PT-symmetric interpretation of the electromagnetic self-force *J. Phys. A: Math. Theor.* 48 34FT01
[25] Polonyi J 2014 Effective dynamics of a classical point charge *Ann. Phys.* 342 239
[26] Griffiths D J 2012 *Introduction to Electrodynamics* 4th edn (Harlow: Pearson) sections 11.2.2 and 11.2.3
[27] Staruszkiewicz A 1970 An example of a consistent relativistic mechanics of point particles *Ann. Phys. (NY)* 480 362
[28] Jaranowski P and Schäfer G 1997 Radiative 3.5 post-Newtonian ADM Hamiltonian for many-body point-mass systems *Phys. Rev. D* 55 4712
[29] Königsdörfer C, Faye G and Schäfer G 2003 Binary black-hole dynamics at the third-and-a-half post-Newtonian order in the ADM formalism *Phys. Rev. D* 68 044004
[30] Zangwill A 2013 *Modern Electrodynamics* 1st edn (Cambridge: Cambridge University Press) section 24.3
[31] Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (Course of Theoretical Physics Series) vol 2 4th edn (Amsterdam: Elsevier) section 65.75
[32] Rohrlich F 2002 Dynamics of classical quasi-point charge *Phys. Lett. A* 303 307–10
[33] O’Connell R F 2012 Radiation reaction: general approach and applications, especially to electrodynamics *Contemp. Phys.* 53 4
[34] Weiss U 2008 *Quantum Dissipative Systems* 3rd edn (Singapore: World Scientific)
[35] Griener W 2003 *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics* (New York: Springer)