Universality driven analytic structure of QCD crossover: radius of convergence and QCD critical point

Andrew Connelly\textsuperscript{a}, Gregory Johnson\textsuperscript{a}, Swagato Mukherjee\textsuperscript{b}, Vladimir Skokov\textsuperscript{a,c}

\textsuperscript{a}North Carolina State University, Raleigh, NC 27695, USA
\textsuperscript{b}Brookhaven National Laboratory, Upton, NY 11973, USA
\textsuperscript{c}Riken-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract
Recent lattice QCD calculations show strong indications that the crossover of QCD at zero baryon chemical potential ($\mu_B$) is a remnant of the second order chiral phase transition. The non-universal parameters needed to map temperature $T$ and $\mu_B$ to the universal properties of the second order chiral phase transition were determined by lattice QCD calculations. Motivated by these advances, first, we discuss the analytic structure of the partition function – the so-called Yang-Lee edge singularity – in the QCD crossover regime, solely based on universal properties. Then, utilizing the lattice calculated non-universal parameters, we map this singularity to the real $T$ and complex $\mu_B$ plane, in order to find the radius of convergence for a Taylor series expansion of QCD partition function around $\mu_B = 0$ in the QCD crossover regime. Our most important findings are: (i) An universality-based estimate of the radius of convergence around $\mu_B = 0$; (ii) Universality and lattice QCD based constraints on the location of the QCD critical point in the $T - \mu_B$ plane.

Keywords: QCD, phase diagram, critical point

1. Introduction

Lattice QCD calculations conclusively showed that the approximate chiral symmetry gets nearly restored at a pseudo-critical temperature $T_{pc} = 156.5 \pm 1.5$ MeV \cite{ref} via a smooth crossover \cite{ref}. The same is true also for small-to-moderate values of $\mu_B$ modulo the dependence of the transition temperature on the baryon chemical potential $T_{pc}(\mu_B)$ \cite{ref}. Beyond this first-principle knowledge, it is conjectured that at some values of $\mu_B$ the chiral restoration in QCD takes place via a first order transition (for an alternative, see Ref. \cite{ref}); the first order phase transition line terminates and turns to crossover at the QCD critical point.

Due to the fermion sign problem, lattice QCD calculations provide limited guidance on the existence and location of the QCD critical point in the $T - \mu_B$ phase diagram. The present lattice calculations can be extended to non-zero $\mu_B$ either by carrying out Taylor expansions around $\mu_B = 0$ \cite{ref} or through analytic continuation from purely imaginary values of $\mu_B$ \cite{ref, ref, ref}. These approaches rely on the assumption that the QCD partition function is an analytic function of $\mu_B$ within a radius of convergence. To what extent these lattice QCD results are trustworthy, and how far in $\mu_B$ these methods might be extended can be answered...
only if the radius of convergence of the QCD partition around $\mu_B = 0$ is known. In this talk, we use non-universal input from lattice QCD calculations to map O(4) universality class scaling functions and its singularity structure to the QCD $T$ and $\mu$ plane and extract the radius of convergence.

2. Results and Discussions

Lattice QCD provides a compelling evidence that for massless $u$ and $d$ quarks, the chiral symmetry restoration in QCD takes place via true second order phase transition belonging to the O(4) universality class\footnote{It is not completely ruled out that the phase transition is of a first order in the chiral limit; it turns into a second order phase transition of the three-dimensional $Z(2)$ universality class at some small but non-zero light quark mass. If it is indeed the case, the mapping in Eq. (1) has to be modified, but the main idea still holds.}. The “external field” explicitly breaking O(4) symmetry $h$ is related to the mass of the light $u$ and $d$ quarks, conventionally $h = m_t/m_s^{\text{phys}}$. Then, for a non-zero $h$, but within the O(4) scaling regime the thermodynamics of QCD is driven by the free energy scaling function $f_s = h^{+1/\beta} f(z)$ and the magnetic equation of state for the order parameter $M \equiv \delta f_s / \delta h = h^{1/\beta} g(z)$. Here $z$ is the scaling variable $z = h^{-1/\beta} t$ and $t$ is the reduced temperature; $z$ can be related to physical $T$ and $\mu_B$ via:

$$z = z_0 \left(\frac{m_t}{m_s^{\text{phys}}}\right)^{-\frac{1}{\beta}} \left[ T - T_{c0} \right] + \kappa_2^{B} \left(\frac{\mu_B}{T_{c0}}\right)^{2} + \kappa_4^{B} \left(\frac{\mu_B}{T_{c0}}\right)^{4} + \ldots \right].$$

In this parametrization, we have a few non-universal parameters: (i) $T_{c0}$ is the critical temperature of the chiral phase transition in the limit $m_t = 0$; (ii) $\kappa_2^{B}$ defines the curvature of the transition line at non-zero chemical potential; (iii) $z_0$ is a non-universal constant fixed by universal behaviour of the order parameter as a function of the light quark mass. These non-universal parameters were determined in LQCD calculations: $T_{c0} = 132^{+3}_{-6}$ MeV \cite{2, 10}, $\kappa_2^{B} = 0.012(2)$ \cite{1}, $z_0 = 1 - 2 \kappa_2^{B}$. LQCD found that $\kappa_2^{B}$ is consistent with 0. The ratio of light to heavy quark masses is $m_t/m_s^{\text{phys}} = 1/27$ at the physical point. We approximate $z_0$ as a constant. As a non-universal parameter, $z_0$ can, in principal, have a residual dependence on quark mass, temperature and chemical potential.

It is very well known that for an O($N$) universality class, the functions $g(z)$ and $f(z)$ have a singularity in the complex $z$ plane, see Fig. [1]. This is the so called Yang-Lee edge singularity – the remnant of the
second order phase transition. The singularity can be treated as an ordinary critical point; it belongs to Z(2)
universality class of $\phi^4$ theory. The symmetry of the partition function with respect to $h \to -h$ and $h \to h^*$
dictates that $z_c = |z_c| \exp \frac{i\pi}{3}$ with O(4) critical exponents. The magnitude $|z_c|$, the main ingredient in defining
the radius of convergence, was unknown before our studies except for the mean-field approximation or in
the large $N$ limit. For pedagogical demonstration let's consider the former. The Landau mean-field model
for the phase transition is $\Omega = \frac{1}{4} tr^2 + \frac{1}{4} tr^4 - h r$, where without any loss of generality we set the quartic
coefficient to 1/4 (this fixes $z_0$ to 1). The equation of motion for the $\sigma$ field is thus
\[ \frac{\partial \Omega}{\partial \sigma} = t \sigma + \sigma^3 - h = 0. \] (2)
Introducing $M = h^{1/3} g$ (that is $\delta$ in mean-field approximation is 3) and $z = t/h^{2/3}$ (which also suggests that
$\beta = 1/2$) this equation can be rewritten in a canonical form
\[ g(z)[z + g^2(z)] = 1. \] (3)
The position of the singularity is given by the zero of the derivative of the inverse function $z'(g_c) = 0$. In
combination with Eq. (3), this gives $z_c = \frac{1}{\sqrt{3}} e^{i\pi/4}$.

For this talk, we performed Functional Renormalization Group (FRG) to establish the value of $z_c$ beyond the
mean-field approximation. FRG is based on a functional differential equation for a scale dependent
effective action which begins with a bare classical action and iteratively incorporates quantum fluctuations
by the momentum type scale resulting in the fully renormalized action. A standard basic approach to solving
the FRG flow equations for the effective action is to take the lowest order derivative expansion of the effective
action together with a truncated Taylor series for the $\Omega$ action. If this is done for $\Omega$ up to and beyond
$z_{\beta}$, the main ingredient in defining the radius of convergence in the $\mu_T$ plane for different
values of $m_T^0$, using $z_0 = 2$, $O(4)$ critical exponents, and other lattice QCD-determined non-universal
parameters described above. Note that, in the chiral limit, QCD free energy is singular at $T = T^0_c$, $\mu_B = 0$
and, therefore, the radius of convergence at this point is zero.

In Figure 2, right panel, provides a more realistic estimate for the radius of convergence in $\mu_B$ in the $T - \mu_B$
plane for $m_T^{\text{phys}}$ by varying $|z_c|$ around its FRG value; as to account the systematics related to the FRG
truncation used in our calculations. While the variation of $|z_c|$ leads to a limited uncertainty of the radius of
convergence, more precise lattice QCD results for $z_0$ are needed to improve this estimate.

3. Conclusions

Relying only on the universal behavior of QCD in the chiral crossover region we investigated the analytic
behavior of the free energy. We argued that if the chiral behavior of QCD is well-described by the universal
scaling, as borne out in recent the lattice QCD calculations, then analytical structure of the free energy will
be completely governed by the corresponding universal scaling function in the complex scaling variable.
We extracted the position of the relevant singularity of the scaling function by performing FRG calculations.
We showed how this can be translated to the singularity in the complex-$\mu_B$ plane to determine the radius of
convergence in $\mu_B$, solely based on the universal critical exponents and well-determined non-universal
parameters from lattice QCD calculations. Figure 2 summarizes our universality- and QCD-based estimate
for the radius of convergence in $\mu_B$ for temperatures in the vicinity of the QCD chiral crossover.
This shows that the radius of convergence is larger than $|\mu_B| > 400$ MeV, implying that the present lattice QCD
calculations based on Taylor expansions in $\mu_B$ and analytic continuations from imaginary values of $\mu_B$ can
be reliable below this region, as suggested also by recent lattice QCD calculations [1, 5, 11].
Andrew Connelly, Gregory Johnson, Swagato Mukherjee, and Vladimir Skokov / Nuclear Physics A 00 (2020) 1–4

Fig. 2. **Left panel:** Radius of convergence $\mu_B$ as a function for different values of the light up/down quark masses. The minimum of the curves shifts to higher temperatures by the amount $\Delta T = Re \left( \frac{m_l}{m_s} \right) \beta \delta$. **Right panel:** Radius of convergence in $\mu_B$ for physical quark masses. The orange band is for $z_0 = 2$ and incorporates a 15% uncertainty on the value of $|z_c|$. The yellow region depicts LQCD disfavoured region for the location of critical end point. The blue dots show freeze-out $T$ and $\mu_B$ for a given collisions energy in GeV.

The present state-of-the-art lattice QCD calculations do not find any evidence for an additional singularity for $\mu_B \lesssim 400$ MeV [1, 5, 11]. Our result on the radius of convergence $|\mu_B| \gtrsim 400$ MeV, coupled with these lattice QCD results, suggest that QCD critical point, if one exists, will most likely be located at $\mu_B \gtrsim 400$ MeV. Such conclusion will potentially have an important impact on the on-going beam energy scan experiments at RHIC and SPS, as well as on the future experiments, such as at FAIR and NICA. Since the critical point is located somewhere along the chiral crossover boundary, naturally, the corresponding singularity is continuously connected to the crossover Yang-Lee edge singularity. In fact, a critical point is located where the Yang-Lee edge singularity and its complex conjugate pinch the real chemical potential axis [12]. Thus, the curves in Figure 2 help understand how to map critical Ising direction, $t$, to QCD, which is of relevance not only for static [13] but also dynamic properties [14] near a possible critical point.

This work was supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics: (2) Through the Contracts No. [de-sc0012704] and [de-sc0020081], (iii) Within the framework of the Beam Energy Scan Theory (BEST) Topical Collaboration.

References

[1] A. Bazavov, et al., Chiral crossover in QCD at zero and non-zero chemical potentials, Phys. Lett. B795 (2019) 15–21.
[2] T. Bhattacharya, et al., QCD Phase Transition with Chiral Quarks and Physical Quark Masses, Phys. Rev. Lett. 113 (8) (2014) 082001.
[3] A. Bazavov, et al., The chiral and deconfinement aspects of the QCD transition, Phys. Rev. D85 (2012) 054503.
[4] R. D. Pisarski, V. V. Skokov, A. M. Tsvelik, Fluctuations in cool quark matter and the phase diagram of Quantum Chromodynamics, Phys. Rev. D99 (7) (2019) 074025.
[5] A. Bazavov, et al., The QCD Equation of State to $\mathcal{O}(\mu_B^6)$ from Lattice QCD, Phys. Rev. D95 (5) (2017) 054504.
[6] C. Bonati, M. D’Elia, M. Mariti, M. Mesiti, F. Negro, F. Sanfilippo, Curvature of the chiral pseudocritical line in QCD: Continuum extrapolated results, Phys. Rev. D92 (5) (2015) 054503.
[7] R. Bellwied, S. Borsanyi, Z. Fodor, J. Genter, S. D. Katz, C. Ratti, K. K. Szabo, The QCD phase diagram from analytic continuation, Phys. Lett. B751 (2015) 559–564.
[8] S. Borsanyi, Z. Fodor, J. N. Guenther, S. K. Katz, K. K. Szabo, A. Pasztor, I. Portillo, C. Ratti, Higher order fluctuations and correlations of conserved charges from lattice QCD, JHEP 10 (2018) 205.
[9] H. T. Ding, et al., Chiral Phase Transition Temperature in (2+1)-Flavor QCD, Phys. Rev. Lett. 123 (6) (2019) 062002.
[10] H. T. Ding, P. Hegde, F. Karsch, A. Lahiri, S. T. Li, S. Mukherjee, P. Petreczky, Chiral phase transition of (2+1)-flavor QCD, Nucl. Phys. A982 (2019) 211–214.
[11] S. Borsanyi, Z. Fodor, J. N. Guenther, S. K. Katz, A. Pasztor, I. Portillo, C. Ratti, K. K. Szab, Towards the equation of state at finite density from the lattice, Nucl. Phys. A982 (2019) 223–226.
[12] M. A. Stephanov, QCD critical point and complex chemical potential singularities, Phys. Rev. D73 (2006) 094508.
[13] M. S. Pradeep, M. Stephanov, Universality of the critical point mapping between Ising model and QCD at small quark mass, Phys. Rev. D100 (5) (2019) 056003.
[14] M. Martinez, T. Schfer, V. Skokov, Critical behavior of the bulk viscosity in QCD, Phys. Rev. D100 (7) (2019) 074017.