By a direct resummation of perturbation theory in the limit of very high energy and small transferred momentum (the so-called “eikonal” limit), we derive expressions for the truncated–connected quark, antiquark and gluon propagators in an external gluon field, both for scalar and fermion gauge theories. These are the basic ingredients to derive “soft” high–energy parton–parton scattering amplitudes, using the LSZ reduction formulae and a functional integral approach.

1 Introduction

In 1991 Nachtmann developed a nonperturbative analysis, based on QCD, of the elastic scattering processes at very high squared energies $s$ in the center of mass and small squared transferred momentum $t$ ($s \to \infty$, $t \ll s$). He derived formal expressions for the quark–quark scattering amplitudes in the above–mentioned limit, by using a functional integral approach and an eikonal approximation to the solution of the Dirac equation in the presence of an external non–Abelian gauge field.

In a previous paper we proposed an alternative approach to high–energy quark–quark scattering based on a first–quantized path–integral description of quantum–field theory developed by Fradkin in the early 1960s. In this approach one obtains convenient expressions for the truncated–connected scalar propagators in an external (gravitational, electromagnetic, etc.) field, and the eikonal approximation can be easily recovered in the relevant limit. Knowing the truncated–connected propagators, one can then extract, in the manner of Lehmann, Symanzik and Zimmermann (LSZ), the scattering matrix elements in the framework of a functional integral approach.

The same problem can be addressed in an even more immediate way: by a direct resummation of perturbation theory in the high–energy limit that we are considering, we can derive expressions for the truncated–connected quark, antiquark and gluon propagators in an external gluon field, both for scalar and fermion gauge theories. In the following, we shall briefly outline this procedure and the main results that one obtains: we refer the reader to Ref. for more details.
2 The scattering of partons in an external gluon field

We begin, for simplicity, with the case of scalar QCD, i.e., the case of a spin–0 quark, described by the scalar field \( \phi_i \) \((i = 1, \ldots, N_c)\) coupled to a non–Abelian gauge field \( A^\mu \equiv A^\mu_a T_a \), \( T_a \) being the generators of the Lie algebra of the colour group \( SU(N_c) \) in the fundamental representation. We limit ourselves to the case of one single flavour. The unrenormalized Lagrangian is:

\[
L(\phi, \phi^\dagger, A) = [D^\mu \phi]^\dagger D^\mu \phi - m_0^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu \nu}^a F^a_{\mu \nu},
\]

(1)

where \( D^\mu = \partial^\mu + igA^\mu \) is the covariant derivative.

Let us define the “physical” quark mass \( m \), taken to be the pole mass, and the residue \( Z_W \) at the pole of the unrenormalized quark propagator by the following two equations:

\[
\left[ \tilde{S}(p) \right]^{-1} |_{p^2=m^2} = 0, \quad \tilde{S}_{ij}(p) \underset{p^2=m^2}{\approx} \frac{iZ_W \delta_{ij}}{p^2 - m^2 + i\epsilon},
\]

(2)

where \( \tilde{S}_{ij}(p) \) is the unrenormalized propagator in the momentum space. In order to derive the scattering matrix elements following the LSZ approach, we need to know the on–shell truncated–connected Green functions, which are obtained from the connected Green functions by removing the external legs calculated on–shell. We first consider the scattering of a quark in a given external gluon field \( A^\mu \):

\[
\phi(p, j) \rightarrow \phi(p', i),
\]

(3)

where \( i, j \) are colour indices \((i, j = 1, \ldots, N_c)\). We define the truncated–connected propagator in the external gluon field \( A^\mu \), in the momentum space as:

\[
\tilde{S}_{ij}^{(tc)}(p, p'| A) \equiv \lim_{p^2, p'^2 \rightarrow m^2} \frac{p^2 - m^2}{i} \tilde{S}_{ij}(p, p'| A) \frac{p'^2 - m^2}{i},
\]

(4)

where \( m \) is the physical mass defined above and \( \tilde{S}_{ij}(p, p'| A) \) is the Fourier transform of \( S_{ij}(x, y | A) \), the scalar propagator in an external gluon field, in the coordinate representation.

In the following we shall compute the truncated–connected propagator \( \tilde{S}_{ij}^{(tc)}(p, p'| A) \) in the so–called eikonal approximation, which is valid in the case of scattering particles with very high energy \((E \equiv p^0 \simeq |\vec{p}| \gg m)\) and small transferred momentum \( q \equiv p' - p \) (i.e., \( \sqrt{t} \ll E \), where \( t = q^2 \)). For example, if \( p'^\mu \simeq p'^\mu \simeq (E, E, 0, 0) \), one has that

\[
p_+ \simeq p'_+ \simeq 2E, \quad p_- \simeq p'_- \simeq 0,
\]

(5)
where \( V_+ \equiv V^0 + V^1 \), \( V_- \equiv V^0 - V^1 \). We shall call \( V_+ \) and \( V_- \) the "longitudinal" components of the four–vector \( V^\mu \), while \( \tilde{V}_\perp \equiv (V^2, V^3) \) is the component of \( V^\mu \) in the "transverse" plane \((y, z)\).

Our strategy consists in evaluating the truncated–connected propagator \( \tilde{S}_{ij}^{(tc)}(p, p'/A) \) in each order in perturbation theory considering \( L_\phi \equiv [D^\mu \phi]^\dagger D_\mu \phi - m_\phi^2 \phi^3 = L_0 + L_{int} \), where

\[
L_0 = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^3 \tag{6}
\]

is the "free" (i.e., unperturbed) quark Lagrangian, which defines the "free" quark propagator \( i/(p^2 - m^2 + i\varepsilon) \), with the physical mass \( m \), and \( L_{int} \) is the "interaction" Lagrangian, i.e., the "perturbation".

Let us start, therefore, by evaluating the \( n \)-th order term \((n \geq 1)\) in the perturbative expansion of the truncated–connected scalar propagator in an external gluon field \( A^\mu \), in the eikonal approximation. This contribution, that we shall indicate as \( [\tilde{S}_{ij}^{(tc)}(p, p'/A)]_{(n)} \), is schematically represented in Fig. 1a: only the quark–quark–gluon vertex contributes to the propagator in the eikonal limit that we are considering. The key–point is that, in the high–energy limit we are considering, \( p \approx p' \) and quarks retain their large longitudinal momenta during their scattering process. In the eikonal limit, one finds that:

\[
[\tilde{S}_{ij}^{(tc)}(p, p'/A)]_{(n)} \approx 2E \int [d^3 b] e^{ibq} \int d\tau_1 \ldots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) 
\times \{[-igp_{\mu n} A_{\mu n}(b + p\tau_n)] \ldots [-igp_{\mu 1} A_{\mu 1}(b + p\tau_1)]\}_{ij}, \tag{7}
\]

where we have used the notation:

\[
[d^3 b] \equiv d^2 \vec{b}_\perp db_- . \tag{8}
\]

As a general rule, in \("[d^3 b]"\) one must not include the longitudinal component of \( b^\mu \) which is parallel to \( p^\mu \). In other words, if \( p^\mu \approx p'^\mu \approx (E, E, 0, 0) \) (i.e., \( p_+ \approx 2E, p_- \approx 0 \)), one has that \( [d^3 b] \equiv d^2 \vec{b}_\perp db_- \), while, if \( p^\mu \approx p'^\mu \approx (E, -E, 0, 0) \) (i.e., \( p_+ \approx 0, p_- \approx 2E \)), then \( [d^3 b] \equiv d^2 \vec{b}_\perp db_+ \).

Summing all orders \((n \geq 1)\), we finally obtain (see also Ref. [3]):

\[
\tilde{S}_{ij}^{(tc)}(p, p'/A) \approx 2E \int [d^3 b] e^{ibq} \left[T \exp \left(-ig \int_{-\infty}^{+\infty} A_\mu(b + p\tau)p^\mu d\tau\right) - 1\right]_{ij} 
= 2E \int [d^3 b] e^{ibq}[W_p(b) - 1]_{ij}, \tag{9}
\]
Figure 1. a) The Feynman diagram corresponding to the $n$-th order term ($n \geq 1$) in the perturbative expansion of the truncated-connected quark propagator in an external gluon field $A_a^\mu$, in the eikonal approximation. b) The Feynman diagram which defines the $n$-th order perturbative term ($n \geq 1$) of the gluon matrix element (17). Crosses represent insertions of the external gluon field $A_a^\mu$. The four-momenta $q_1, \ldots, q_n$ are taken to be flowing into the diagram.

where $W_p(b) = T \exp(\ldots)$ is the time-ordered exponential, defined as:

$$W_p(b) \equiv T \exp \left( -ig \int_{-\infty}^{+\infty} A_\mu(b + p \tau) p^\mu d\tau \right)$$

$$\equiv \sum_{n=0}^{\infty} \int d\tau_1 \ldots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1)$$

$$\times \left[ -ig p^\mu A_\mu (b + p \tau_n) \right] \ldots \left[ -ig p^\mu A_\mu (b + p \tau_1) \right].$$

(10)

Eq. (10) gives the expression for the truncated-connected scalar propagator in an external gluon field, in the eikonal approximation.

We want now to extend these results to the case of “real” fermion QCD: that is, a spin-1/2 quark coupled to a non-Abelian gauge field. As before,
we limit ourselves to the case of one single flavour. The unrenormalized QCD Lagrangian (for a “real” spin–1/2 quark) is:

\[ L(\psi, \psi^\dagger, A) = \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \]

where \( D^\mu = \partial^\mu + igA^\mu \) is the covariant derivative and \( \psi \) stands for a vector \( \psi_i \) \((i = 1, \ldots, N_c)\) in the colour vector space of the fundamental representation.

Let us define the “physical” quark mass \( m \), taken to be the pole mass, and the residue \( Z_W \) at the pole of the unrenormalized quark propagator by the following two equations:

\[
\left[ \tilde{G}(p) \right]^{-1} \bigg|_{p^2=m^2} = 0, \quad \tilde{G}_{ij}(p) \underset{p^2 \to m^2}{\sim} \frac{iZ_W \delta_{ij}}{\hat{p} - m + i\epsilon},
\]

where \( \tilde{G}_{ij}(p) \) is the unrenormalized propagator in the momentum space. We have used the notation: \( \hat{a} \equiv \gamma^\mu a_\mu \).

As in the scalar case, we consider the scattering of a quark in a given external gluon field \( A^\mu \). The truncated–connected fermion propagator in the momentum space is defined as:

\[
\tilde{G}^{(tc)}_{ij}(p, p'|A) \equiv \lim_{p^2, p'^2 \to m^2} \frac{\hat{p}' - m}{i} \tilde{G}_{ij}(p, p'|A) \frac{\hat{p} - m}{i},
\]

where \( m \) is the physical quark mass defined above and \( \tilde{G}_{ij}(p, p'|A) \) is the Fourier transform of \( G_{ij}(x, y|A) \), the truncated–connected fermion propagator in the coordinate representation. The matrix element for the scattering of a quark in a given external gluon field \( A^\mu \),

\[ \psi(p, j, \beta) \rightarrow \psi(p', i, \alpha), \]

where \( i, j \) are colour indices and \( \alpha, \beta \) are spin indices, is given by \( \overline{\psi}_a(p') \tilde{G}^{(tc)}_{ij}(p, p'|A) u_\beta(p) \), where \( u_\alpha(p) \) are the “positive–energy” spinors with the usual relativistic normalization: \( \overline{\psi}_a(p) u_\alpha(p) = 2m\delta_{\alpha\alpha} \). In the high–energy limit we are considering, one finds the following result, for each order \( n \) in the perturbative expansion:

\[ \left[ \overline{\psi}_a(p') \tilde{G}^{(tc)}_{ij}(p, p'|A) u_\beta(p) \right]_{(n)} \simeq \delta_{\alpha\beta} \cdot \left[ \tilde{S}^{(tc)}_{ij}(p, p'|A) \right]_{(n)}, \]

where \( \tilde{S}^{(tc)}_{ij}(p, p'|A) \) is the truncated–connected propagator for a scalar (i.e., spin–0) quark in the external gluon field \( A^\mu \), which was discussed in the first part of this section.

The \( \delta \) function in front simply reflects the fact that fermions retain their helicities during the scattering process in the high–energy limit. Therefore, using the result (3) derived in the first part of this section, we find the
following expression in the eikonal approximation (see also Ref. 3):

\[
\overline{u}_\alpha(p') \tilde{G}_{ij}^{(tc)}(p, p'|A) u_\beta(p) \simeq \delta_{\alpha,\beta} \cdot \tilde{S}_{ij}^{(tc)}(p, p'|A)
\]

\[
\simeq \delta_{\alpha,\beta} \cdot 2E \int [d^3b] \, e^{i\eta b} \left[ T \exp \left( -i g \int_{-\infty}^{+\infty} A_\mu(b + pr)p^\mu d\tau \right) - 1 \right]_{ij}
\]

\[
= \delta_{\alpha,\beta} \cdot 2E \int [d^3b] \, e^{i\eta b}[W_p(b) - 1]_{ij} \ ,
\]

(16)

where \(W_p(b)\) is the time–ordered Wilson string along the path \(x(\tau) = b + pr\), \(\tau \in [-\infty, +\infty]\).

Proceeding exactly in the same way, we can also derive the expression for the scattering matrix element of an antiquark in an external gluon field \(A^\mu\), \(\overline{\psi}(p, j, \beta) \rightarrow \overline{\psi}(p', i, \alpha)\), in the eikonal approximation: as expected, the scattering amplitude of a quark in the charge–conjugated (C–transformed) gluon field \(A'_\mu = -A^\mu = -A^\mu_\mu\). (In other words, going from quarks to antiquarks corresponds just to the change from the fundamental representation \(T_a \in SU(N_c)\) to the complex conjugate representation \(T'_a = -T^*_a\).)

By using the same techniques developed for the previous cases, we can evaluate the “truncated–connected gluon propagator” in a given external gluon field \(A^\mu_v\), in the eikonal approximation. We call this quantity \(\langle_D^{(tc)}(k, k'|A)\): its \(n\)–th order perturbative term is defined schematically in Fig. 1b, where the external legs are supposed to be truncated on–shell \((k^2, k'^2 \rightarrow 0)\). More precisely, we shall evaluate the following quantity:

\[
\varepsilon^{\mu, \ast}_\lambda(k') \tilde{D}^{(tc)}_{\mu', \lambda'}(k, k'|A) \varepsilon^\mu_\lambda(k) \ ,
\]

(17)

where \(\varepsilon^{\mu}_\lambda(k)\) are the polarization four–vectors: \(\varepsilon_\lambda(k) \cdot \varepsilon^{\ast}_\lambda(k) = -\delta_{\lambda\lambda'}\), \(k \cdot \varepsilon_\lambda(k)|_{k^2=0} = 0\), with \(\lambda, \lambda' \in \{1, 2\}\). This quantity should describe (under certain approximations) the scattering matrix element of a gluon in a given external gluon field \(A^\mu_v\):

\[
g(k, a, \lambda) \rightarrow g(k', a', \lambda') \ ,
\]

(18)

\(a, a' \in \{1, \ldots, N_c^2 - 1\}\) are colour indices and \(\lambda, \lambda' \in \{1, 2\}\) are spin indices. In the eikonal approximation the dominant interaction between the incident gluon and the external gluon field is represented by the three–gluon vertex, which is linear in the four–momentum of the gluon (while the four–gluon vertex is not dependent on the momentum). In the eikonal approximation, summing all orders \((n \geq 1)\), one finally obtains:

\[
\varepsilon^{\mu, \ast}_\lambda(k') \tilde{D}^{(tc)}_{\mu', \lambda'}(k, k'|A) \varepsilon^\mu_\lambda(k)
\]
\[ \simeq \delta_{\lambda'\lambda} \cdot 2E \int [d^3b] \ e^{iqb} [\mathcal{V}_k(b) - 1]_{a'a} , \quad (19) \]

where \( q \equiv k' - k \) is the transferred momentum and \( \mathcal{V}_k(b) \) is the Wilson string along the path \( x(\tau) = b + k\tau \ (\tau \in [-\infty, +\infty]) \), in the adjoint representation, defined as:

\[
\mathcal{V}_k(b) \equiv T \exp \left( -ig \int_{-\infty}^{+\infty} A_\mu(b + k\tau) k^\mu d\tau \right) \
\equiv \sum_{n=0}^{\infty} \int d\tau_1 \ldots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) \\
\times [-igk^{\mu_n} A_\mu_n(b + p\tau_n)] \ldots [-igk^{\mu_1} A_\mu_1(b + k\tau_1)] . \quad (20)\]

Eq. (19) gives the expression for the scattering matrix element of a gluon in a given external gluon field, in the eikonal approximation.

### 3 Conclusions and outlook

In the previous section, we have derived expressions for the truncated–connected quark, antiquark and gluon propagators in a given external gluon field \( A_\mu \), by a direct resummation of perturbation theory in the limit of very high energy and small transferred momentum.

The truncated–connected propagators in an external gluon field are the basic ingredients to derive high–energy parton–parton scattering amplitudes, using the LSZ reduction formulae and a functional integral approach. This was done in Ref. and it has been also quickly reviewed in Ref. for the convenience of the reader. As a result, the high–energy parton–parton scattering amplitude, in the center–of–mass reference system, turns out to be related to the functional average \( \langle \ldots \rangle_{A_\mu} \), with respect to the gluon field \( A_\mu \), of two Wilson lines \( W_1 \) and \( W_2 \) taken along the unperturbed classical trajectories of the two colliding partons. Therefore, using this procedure, one derives the same results already found in Refs. in a different and even more immediate way, i.e., by a direct resummation of perturbation theory, in a background gluon field, in the limit of very high energy and small transferred momentum.

Once we have found these nonperturbative expressions for the high–energy scattering amplitudes, the natural question which arises is: How can we evaluate them directly? The answer to this question is highly nontrivial and it is also strictly connected with the renormalization properties of Wilson–line operators. For a possible approach to the problem, we refer the reader to
Refs. 8, 9, 10, where it was found that the v.e.v. of two Wilson lines in the Euclidean theory, forming a certain Euclidean angle $\theta$ in the longitudinal plane, and the v.e.v. of two Wilson lines in the Minkowskian theory, forming a certain hyperbolic angle $\chi$ in the longitudinal plane, are connected by the analytic continuation $\theta \to -i\chi$ in the angular variables. (See also Ref. 11, where a similar analytic continuation from Minkowskian to Euclidean theory was proposed to study the small-$x_{Bj}$ behaviour of the structure functions of deep inelastic lepton–nucleon scattering.) The analytic continuation proposed in Refs. 8, 9, 10 has opened the possibility of studying the high–energy scattering amplitude using the Euclidean formulation of the theory: it has been recently adopted in Ref. 12, in order to study the high–energy scattering in strongly coupled gauge theories using the AdS/CFT correspondence, and also in Ref. 13, in order to investigate instanton–induced effects in QCD high–energy scattering. In our opinion, a considerable progress could be achieved by a direct investigation of the high–energy scattering problem on the lattice along this line in the near future.

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