On the Existence of Non-Supersymmetric Black Hole Attractors for Two-Parameter Calabi-Yau’s and Attractor Equations

Payal Kaura\(^{(a)}\)\(^1\) and Aalok Misra\(^{(a), (b)}\)\(^2\)

\(^{(a)}\) Indian Institute of Technology Roorkee, Roorkee - 247 667, Uttaranchal, India
\(^{(b)}\) Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Abstract

We look for possible nonsupersymmetric black hole attractor solutions for type II compactification on (the mirror of) \(CY_3(2, 128)\) expressed as a degree-12 hypersurface in \(WCP^1[1, 1, 2, 2, 6]\). In the process, (a) for points away from the conifold locus, we show that the existence of a non-supersymmetric attractor along with a consistent choice of fluxes and extremum values of the complex structure moduli, could be connected to the existence of an elliptic curve fibered over \(C^8\) which may also be “arithmetic” (in some cases, it is possible to interpret the extremization conditions for the black-hole superpotential as an endomorphism involving complex multiplication of an arithmetic elliptic curve), and (b) for points near the conifold locus, we show that existence of non-supersymmetric black-hole attractors corresponds to a version of \(A_1\)-singularity in the space \(\text{Image}(\mathbb{Z}^6 \to \mathbb{R}^2(\hookrightarrow \mathbb{R}^3))\) fibered over the complex structure moduli space. The (derivatives of the) effective black hole potential can be thought of as a real (integer) projection in a suitable coordinate patch of the Veronese map: \(\mathbb{C}P^5 \to \mathbb{C}P^{20}\) fibered over the complex structure moduli space. We also discuss application of Kallosh’s attractor equations (which are equivalent to the extremization of the effective black-hole potential) for nonsupersymmetric attractors and show that (a) for points away from the conifold locus, the attractor equations demand that the attractor solutions be independent of one of the two complex structure moduli, and (b) for points near the conifold locus, the attractor equations imply switching off of one of the six components of the fluxes. Both these features are more obvious using the attractor equations than the extremization of the black hole potential.

---

\(^1\)email: pa123dph@iitr.ernet.in
\(^2\)e-mail: aalokfph@iitr.ernet.in
1 Introduction

It has been shown that extremal black holes exhibit an interesting phenomenon - attractor mechanism [1] - the moduli are “attracted” to some fixed values determined by the charges of the black hole, independent of the asymptotic values of the moduli. Supersymmetric black holes at the attractor point, correspond to minimizing the central charge and the effective black hole potential, whereas nonsupersymmetric attractors [2], which have recently been (re)discussed [3], at the attractor point, correspond to minimizing only the potential and not the central charge. Recently, attractor equations for (non) supersymmetric black holes and flux vacua were given by Kallosh [4] (For an earlier derivation, see [5])

and some examples verifying the same were studied in [7] including IIB compactified on one-parameter Calabi-Yau’s - the attractor equations, however, are equivalent to extremizing the effective black hole potential (See [6] and references therein). In this paper, we discuss the existence of possible nonsupersymmetric attractor solutions to type IIB compactified on a two-parameter Calabi-Yau, both, from the equivalent points of view of extremizing an effective black-hole potential and also by using the attractor equations. We get some interesting connections between arithmetic and geometry and nonsupersymmetric black-hole attractors. We emphasize that we stress more on the forms of the various equations rather than their numerical content.

The plan of the paper is as follows. Section 2 consists of the bulk of the calculations and results as regards the non-supersymmetric black hole attractors from minimizing the effective black-hole potential. It is divided into two parts - 2.1 deals with points in the moduli space away from the singular conifold locus, and 2.2 deals with points near the same - 2.1 is further subdivided into two parts: 2.1.1 deals with positive eigenvalues of the mass matrix and 2.1.2 deals with null eigenvalues of the mass matrix. Section 3 has a discussion on the use of the new attractor equations of [4] to get non-supersymmetric attractors; it is divided into two (short) parts - 3.1 is for points in the moduli space away from the singular conifold locus and 3.2 is for points close to the same. There are three appendices relevant to the calculations in sections 2 and 3. Section 4 has the conclusions and discussion on future directions.

2 The Black Hole Potential Extremization, the Mass Matrix and Attractor Solutions

In this section we work out possible attractor solutions obtained by extremizing the effective black-hole potential for points in the moduli space, both away and near the conifold locus of the mirror to a two-parameter Calabi-Yau with \( h^{1,1} = 2, h^{2,1} = 128 \), expressed as a degree-12 hypersurface in \( \text{WCP}^4[1,1,2,2,6] \).

2.1 Away from the Singular Conifold Locus

The defining hypersurface for the mirror to the aforementioned Calabi-Yau is:

\[
x_0^2 + x_1^{12} + x_2^{12} + x_3^6 + x_4^6 - 12\psi x_0 x_1 x_2 x_3 x_4 - 2\phi x_1^6 x_2^6 = 0,
\]

with \( h^{1,1} = 128 \) and \( h^{2,1} = 2 \). Under the symplectic decomposition of the holomorphic three-form \( \Omega \) canonical homology \( (A_a, B^a, a = 1, 2, 3) \) and cohomology bases \( (\alpha_a, \beta^a) \), defining the periods as \( \int_{A_a} \Omega = z^a, \int_{B^a} \Omega = F_a \).
such that $\Omega = z^a \alpha_a - F_a \beta^a$. Then, the Kähler potential $K$ is given by: $-\ln(-i(\tau - \tau) - \ln(-i f_{CY} \Omega \wedge \overline{\Omega}) = ln(-i(\tau - \tau) - ln(-i \Pi \Pi \Sigma \Pi)$, $\Pi$ being the six-component period vector and $\Sigma = \left( \begin{array}{cc} 0 & 1_3 \\ -1_3 & 0 \end{array} \right)$.

Expanding about a point in the moduli space away from the conifold locus, such as $\phi = 2$ (or equivalently $z = 0$) and $\psi = 0$ (See [8, 9, 10]), one gets the following period vector:

\[ \Pi = \left\{ \left( \begin{array}{c} \rho_1 \\ \rho_2 \end{array} \right) \right\} \left( -576 \ 2F_1(1/2, 7/12, 1, 1/4) + 48 z \ 2F_1(1/2, 7/12, 1, 1/4) + 7 z \ 2F_1(13/12, 19/12, 2, 1/4) \right) \]

\[ = \left\{ \frac{-i}{72} \left( \begin{array}{c} 3 + 2i \end{array} \right) \pi \frac{2}{3} \left( -576 \ 2F_1(1/2, 7/12, 1, 1/4) + 48 z \ 2F_1(1/2, 7/12, 1, 1/4) + 7 z \ 2F_1(13/12, 19/12, 2, 1/4) \right) \right\} \]

\[ \frac{-i}{36} \left( \begin{array}{c} 2 \pi \frac{2}{3} \left( -576 \ 2F_1(1/2, 7/12, 1, 1/4) + 48 z \ 2F_1(1/2, 7/12, 1, 1/4) + 7 z \ 2F_1(13/12, 19/12, 2, 1/4) \right) \Gamma(\frac{5}{6})^3 \right\} \]

\[ \left( \begin{array}{c} -108 \psi^2 32 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \end{array} \right) \]

\[ \frac{\psi^2 \left( -128 \psi^2 \left( -128 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \right) \right) \Gamma(\frac{5}{6})^3 \right\} \]

\[ \frac{\psi^2 \left( -128 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \right) \Gamma(\frac{5}{6})^3 \right\} \]

\[ \left( \begin{array}{c} -108 \psi^2 32 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \end{array} \right) \]

where the complete elliptic integral of the first kind $EllipticK(\nu) \equiv \int_0^\nu \frac{d\phi}{\sqrt{1 - \nu \sin^2 \phi}}$. One then constructs the superpotential:

\[ W = f^T \Pi = \frac{1}{72} \left( -2i 2 \pi \left( -1 + (-1) \frac{2}{3} \right) f_i \pi \frac{2}{3} \left( -576 \ 2F_1(1/2, 7/12, 1, 1/4) + 48 z \ 2F_1(1/2, 7/12, 1, 1/4) + 7 z \ 2F_1(13/12, 19/12, 2, 1/4) \right) \right) \]

\[ + \frac{(2 - 2i) \pi F_3 \pi \frac{2}{3} \left( -576 \ 2F_1(1/2, 7/12, 1, 1/4) + 48 z \ 2F_1(1/2, 7/12, 1, 1/4) + 7 z \ 2F_1(13/12, 19/12, 2, 1/4) \right) \Gamma(\frac{5}{6})^3 \right\} \]

\[ \left( \begin{array}{c} -108 \psi^2 32 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \end{array} \right) \]

\[ \frac{\psi^2 \left( -128 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \right) \Gamma(\frac{5}{6})^3 \right\} \]

\[ \left( \begin{array}{c} -108 \psi^2 32 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \end{array} \right) \]

\[ \frac{\psi^2 \left( -128 \sqrt{6} EllipticK(\frac{2}{3}) + 9 \pi \ 2F_1(5/4, 7/4, 2, 1/4) \right) \Gamma(\frac{5}{6})^3 \right\} \]
\[-108 \psi^2 (-128 \sqrt{6} \text{Elliptic} K (\frac{2}{3}) + 32 \sqrt{6} z \text{Elliptic} K (\frac{2}{3}) + 9 \pi z \ 2F_1 (\frac{5}{4}, \frac{7}{4}, \frac{2}{4}))\]

\[-i f_1 \left( \frac{2^3 (3 + 2 i + 4 (-1)^{\frac{1}{2}} + (-1)^{\frac{1}{2}}) \pi^2 (-576 \ 2F_1 (\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{\pi}) + 48 z \ 2F_1 (\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{\pi}) + 7 z \ 2F_1 (\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{\pi}))}{\Gamma (\frac{5}{4})^3} \right)\]

\[-108 \psi^2 (-128 \sqrt{6} \text{Elliptic} K (\frac{2}{3}) + 32 \sqrt{6} z \text{Elliptic} K (\frac{2}{3}) + 9 \pi z \ 2F_1 (\frac{5}{4}, \frac{7}{4}, \frac{2}{4}))\]

\[-2 i f_4 \left( \frac{2^3 \pi^2 (-576 \ 2F_1 (\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{\pi}) + 48 z \ 2F_1 (\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{\pi}) + 7 z \ 2F_1 (\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{\pi}))}{\Gamma (\frac{5}{4})^3} \right)\]

\[-54 \psi^2 (-128 \sqrt{6} \text{Elliptic} K (\frac{2}{3}) + 32 \sqrt{6} z \text{Elliptic} K (\frac{2}{3}) + 9 \pi z \ 2F_1 (\frac{5}{4}, \frac{7}{4}, \frac{2}{4}))\]

\[+ i F_5 \left( \frac{2^3 \pi^2 (-576 \ 2F_1 (\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{\pi}) + 48 z \ 2F_1 (\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{\pi}) + 7 z \ 2F_1 (\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{\pi}))}{\Gamma (\frac{5}{4})^3} \right)\]

\[-54 \psi^2 (-128 \sqrt{6} \text{Elliptic} K (\frac{2}{3}) + 32 \sqrt{6} z \text{Elliptic} K (\frac{2}{3}) + 9 \pi z \ 2F_1 (\frac{5}{4}, \frac{7}{4}, \frac{2}{4}))\]

The Kähler potential is given by:

\[
K = -\log \left( a + b \psi^2 + c z + d \psi^2 z + g z \bar{\psi}^2 + \bar{h} \bar{z} \bar{\psi}^2 + c \bar{z} + h \bar{\psi} \bar{z} + j \psi^2 \bar{z} + \bar{\psi} \bar{z} \bar{\psi} \bar{z} + z + i k \psi \bar{\psi} (\bar{z} + \bar{z}) \right),
\]

from which one calculates the metric:

\[
g_{ij} = \begin{pmatrix} g_{z \bar{z}} & g_{z \psi} \\ g_{\psi \bar{z}} & g_{\psi \psi} \end{pmatrix}, \tag{2}
\]

where

\[g_{z \bar{z}} = \frac{\bar{c}^2 - a h + h^2 \bar{z}}{(a + c z + c \bar{z})^2},\]

\[g_{\psi \bar{z}} = \frac{4 |\psi|^2 (g z + \bar{b} + \bar{d} \bar{z}) (b + d z + \bar{g} \bar{z})}{(a + c z + c \bar{z})^2},\]

\[g_{z \psi} = \frac{2 \bar{\psi} ((c + h \bar{z}) (g z + \bar{b} + \bar{d} \bar{z}) - (a + c z + c \bar{z}) (g + j \bar{z}))}{(a + c z + c \bar{z})^2}.
\]

The effective black hole potential in type II theories is given by:

\[
V = e^K (g^{ij} D_i W D_j \bar{W} + |W|^2), \tag{3}
\]

\(W\) being the superpotential, \(K\) the Kähler potential and the covariant derivative \(D_i W \equiv \partial_i W + \partial_i K W\).

The first derivative of the potential is given by (See [11]):

\[
\partial_i V = e^K (g^{ij} D_i D_j W D_k \bar{W} + \partial_i g^{ij} D_j W D_k \bar{W} + 2 D_i W \bar{W}). \tag{4}
\]
the expression \( \partial_z \) space valued \( f \) six flux components (imaginary parts) of the effective black-hole potential about the extremum, is given by:

\[
g^{\bar{y}y} D_2 D_\psi D_3 \tilde{W}, \partial_z g^{\bar{y}y} D_1 \tilde{W} D_3 \tilde{W} \sim \sum_i e^{i\alpha_i \arg(y)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),
\]

\[
D_2 \tilde{W} \tilde{W} \sim \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}},
\]

(5)

and

\[
g^{\bar{y}y} D_2 D_\psi D_3 \tilde{W}, \partial_z g^{\bar{y}y} D_1 \tilde{W} D_3 \tilde{W} \sim \frac{1}{|\psi|} \sum_i e^{i\alpha_i \arg(y)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),
\]

\[
D_\psi \tilde{W} \tilde{W} \sim |\psi| \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right).
\]

(6)

where \( \alpha_i = 2, -2, 0 \). This implies that

\[
\partial_z V \sim \sum_i e^{i\alpha_i \arg(y)} \left( \frac{A_i + B_i z + C_i \bar{z}}{A_i' + B_i' z + C_i' \bar{z}} \right),
\]

\[
\partial_\psi V \sim \frac{1}{|\psi|} \sum_i e^{i\alpha_i \arg(y)} \left( \frac{A_i + B_i z + C_i \bar{z}}{A_i' + B_i' z + C_i' \bar{z}} \right).
\]

(7)

If one complexifies and projectivizes the \( f_i \)s, then the effective potential extremization conditions \( \partial_z V = \partial_\psi V = 0 \) could correspond to real integer projections of intersection of quadrics in a suitable patch of \( \mathbb{CP}^5(f_1 : \ldots : f_6) \) fibered over \( \mathbb{C}(z) \times R(\arg(y)) \), which correspond to four real non-linear constraints on the six flux components \( f_i \)s and the two complex complex structure moduli \( z, \psi \). It is interesting to note that the expression \( \partial_z V \), for a given extremum values of the complex structure moduli (for complex projective space valued \( f_i \)s) would correspond to the Veronese map: \( \mathbb{CP}^5(f_1 : \ldots : f_6) \rightarrow \mathbb{CP}^{20}(f_1^2 : f_1 f_2 : \ldots : f_6^2) \sim \mathbb{CP}^6_z \) where the \( \mathbb{Z}_2 \) flips the signs of all the \( f_i \)s). Veronese surfaces and maps have been shown to have connection with moduli spaces relevant to MSSM (See [12]).

The mass matrix corresponding to fluctuations (assumed to have been separated into their real and imaginary parts) of the effective black-hole potential about the extremum, is given by:

\[
M = \begin{pmatrix}
2 [\text{Re}(\partial_i \partial_3 V)] + [\text{Re}(\partial_i \partial_1 V)] & -2 [\text{Im}(\partial_i \partial_3 V) + \text{Im}(\partial_i \partial_2 V)] \\
-2 [\text{Im}(\partial_i \partial_2 V) + \text{Im}(\partial_i \partial_3 V)] & 2 [\text{Re}(\partial_i \partial_2 V) - \text{Re}(\partial_i \partial_1 V)]
\end{pmatrix}.
\]

(8)

The second derivatives of the black hole potential are given as (See [11]):

\[
\partial_i \partial_1 V = e^K (g^{kl} D_k D_i D_j W D_i \tilde{W} + \partial_a g^{kl} D_k D_j W D_i \tilde{W} + \partial_3 g^{kl} D_k D_i W D_i \tilde{W}) + 3 D_2 D_3 \tilde{W} \tilde{W} + \partial_i \partial_3 g^{kl} D_k W D_i \tilde{W} - g^{kl} \partial_a g_{kl} D_k W \tilde{W}) ,
\]

\[
\partial_2 \partial_3 V = e^K (g^{kl} D_k D_i D_j W D_i \tilde{W} + [2 |\tilde{W}|^2 + g^{kl} D_k W D_i \tilde{W} |g_{ij} + \partial_i g^{kl} D_k W D_i \tilde{W} + \partial_2 g^{kl} D_k D_i D_j W D_i \tilde{W} + 3 D_2 D_3 \tilde{W} + \partial_2 \partial_3 g^{kl} D_k W D_i \tilde{W}) .
\]

(9)
Using the results of the appendix A, one can show that up to $\mathcal{O}(\text{second order in } z \text{ and/or } \psi \text{ and their complex conjugates})$ in the numerators and the denominators:

$$g^{ij} D_i D_j W D_j \bar{W}, \partial_z g^{ij} D_j W D_j \bar{W}, \partial_z \partial_z g^{ij} D_i W D_j \bar{W} \sim \sum_i e^{i \alpha_i \text{arg}(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),$$

$$D_z D_j \bar{W} \sim \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}},$$

$$g^{ij} \partial_z g_{ij} D_i W \bar{W} \sim \sum_i e^{i \alpha_i \text{arg}(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),$$

where $\alpha_i = -2, 0, 2$.

Therefore,

$$\partial_z \partial_z V \sim \sum_i e^{i \alpha_i \text{arg}(\psi)} \left( \frac{A_i + B_i z + C_i \bar{z}}{A_i' + B_i' z + C_i' \bar{z}} \right).$$

Again, using the results of the appendix A, one sees that:

$$\partial_{\psi} g^{ij} D_i D_j \psi W D_j \bar{W} \sim \frac{e^{-2 \text{arg}(\psi)}}{|\psi|^2} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),$$

$$g^{ij} D_i D_j \psi W D_j \bar{W} \sim \frac{e^{-2 \text{arg}(\psi)}}{|\psi|} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),$$

$$D_{\psi} D_j \psi W \bar{W} \sim \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),$$

$$\partial_{\psi} \partial_{\psi} g^{ij} D_i D_j \psi W D_j \bar{W} \sim \frac{1}{|\psi|^2} \sum_i e^{i \beta_i \text{arg}(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),$$

$$g^{ij} \partial_{\psi} g_{ij} D_i \psi W \bar{W} \sim \frac{1}{|\psi|} \sum_i e^{i \gamma_i \text{arg}(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),$$

where $\beta_i = -2, -4$; $\gamma_i = 1, 3$. This yields:

$$\partial_{\psi} \partial_{\psi} V \sim \frac{1}{|\psi|^2} \sum_i e^{i \beta_i \text{arg}(\psi)} \left( \frac{\tilde{A}_i + \tilde{B}_i z + \tilde{C}_i \bar{z}}{\tilde{A}_i' + \tilde{B}_i' z + \tilde{C}_i' \bar{z}} \right).$$

Similarly, using the results of the appendix A, one sees that:

$$g^{ij} D_i D_j D_i \psi W D_j \bar{W} = \frac{1}{|\psi|^2} \sum_i e^{i \alpha_i \text{arg}(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),$$

$$\partial_z g^{ij} D_i D_j \psi W D_j \bar{W} + D_{\psi} g^{ij} D_j D_i W D_j \bar{W} \sim \frac{1}{|\psi|^2} \sum_i e^{i \beta_i \text{arg}(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a_i' + b_i' z + c_i' \bar{z}} \right),$$

$$\partial_z \partial_{\psi} g^{ij} D_i D_j \psi W \bar{W} \sim \frac{e^{-3i \text{arg}(\psi)}}{|\psi|} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),$$

$$g^{ij} \partial_{\psi} g_{ij} D_i \psi W \bar{W} \sim e^{i \text{arg}(\psi)} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),$$

$$D_z D_{\psi} W \bar{W} \sim \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),$$

(14)
where \( \alpha_i = \pm 1, \beta_i = \pm 1, -3 \). Therefore, one obtains:

\[
\partial_\bar{z} \partial_\psi V \sim \frac{1}{|\psi|} \sum_i e^{i \alpha_i \arg(\psi)} \left( \frac{A_i + B_i z + C_i \bar{z}}{A'_i + B'_i z + C'_i \bar{z}} \right),
\]

(15)

We now come to the evaluation of \( \partial_\psi \partial_\bar{z} V \) - the other ingredient necessary for the evaluation of the mass matrix (8). Referring again to the appendix A, one sees:

\[
g^{ij} D_i D_j W D_j D_i \bar{W}, \ g_{ij} g^{ij} D_i W D_j \bar{W} \sim \sum_i e^{i \alpha_i \arg(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a'_i + b'_i z + c'_i \bar{z}} \right),
\]

\[
\partial_\bar{z} g^{ij} D_i W D_j \bar{W} \sim \frac{e^{-3i \arg(\psi)}}{|\psi|} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

\[
\partial_\psi \partial_\bar{z} g^{ij} D_i W D_j \bar{W} \sim e^{2i \arg(\psi)} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

\[
\partial_\psi g^{ij} D_i D_j \bar{W} \sim \sum_i e^{i \alpha_i \arg(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a'_i + b'_i z + c'_i \bar{z}} \right),
\]

\[
g_{z \bar{z}} |W|^2, \ D_z W D_z \bar{W} \sim \left( \frac{a + b z + \bar{b} \bar{z}}{a' + b' z + b' \bar{z}} \right).
\]

(16)

One therefore finally gets:

\[
\partial_\psi \partial_\bar{z} V \sim \frac{e^{-3i \arg(\psi)}}{|\psi|} \left( \frac{a + b z + \bar{b} \bar{z}}{a' + b' z + b' \bar{z}} \right).
\]

(17)

Similarly, using the results from the appendix A, one arrives at:

\[
g^{ij} D_i \bar{D}_j W D_j \bar{D}_i \bar{W} \sim \frac{1}{|\psi|^2} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

\[
\partial_\psi g^{ij} D_i \bar{D}_j W \bar{D}_j \bar{D}_i \bar{W} + \partial_\bar{z} g^{ij} D_i \bar{D}_j W D_j \bar{D}_i \bar{W} \sim \frac{1}{|\psi|^2} \sum_i e^{i \alpha_i \arg(\psi)} \left( \frac{a_i + b_i z + c_i \bar{z}}{a'_i + b'_i z + c'_i \bar{z}} \right),
\]

\[
\partial_\psi \partial_\bar{z} g^{ij} D_i \bar{D}_j \bar{W} \sim \frac{1}{|\psi|^2} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

(18)

which finally yields:

\[
\partial_\psi \partial_\bar{z} V \sim \frac{1}{|\psi|^2} \left( \frac{A(\arg \psi) + B(\arg \psi)z + \bar{B}(\arg \psi)\bar{z}}{A' + B' z + B' \bar{z}} \right).
\]

(19)

Finally, using again the results from the appendix A, one sees that:

\[
g^{ij} D_i D_j W \bar{D}_j \bar{D}_i \bar{W} \sim \frac{1}{|\psi|^2} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

\[
\partial_\bar{z} g^{ij} D_i W \bar{D}_j \bar{D}_j \bar{W} + \partial_\psi g^{ij} D_i W D_j \bar{D}_j \bar{W} \sim \frac{e^{-2i \arg(\psi)}}{|\psi|} \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

\[
g_{z \bar{z}} |W|^2, \ D_z W \bar{D}_z \bar{W} \sim \psi \left( \frac{a + b z + c \bar{z}}{a' + b' z + c' \bar{z}} \right),
\]

\[
\partial_\psi \partial_\bar{z} g^{ij} D_i W \bar{D}_j \bar{W} \sim \frac{1}{|\psi|} \sum_i \left( \frac{a_i + b_i z + c_i \bar{z}}{a'_i + b'_i z + c'_i \bar{z}} \right),
\]

(20)
which gives:

$$
\partial_z \partial_\psi V \sim \frac{1}{|\psi|^2} \left( \frac{a + bz + cz}{a' + b'z + c'z} \right).
$$

(21)

Hence, the mass matrix can be written as:

$$
M \sim \begin{pmatrix}
\frac{1}{|\psi|^2} \left( \frac{\xi_1 + \xi_2^z + \xi_2^z}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\xi_1 + \xi_2^z + \xi_2^z}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) \\
\frac{1}{|\psi|^2} \left( \frac{\xi_1 + \xi_2^z + \xi_2^z}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\xi_1 + \xi_2^z + \xi_2^z}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) \\
\frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\lambda_1 + \lambda_2 + \lambda_2^z}{\omega_1 + \omega_2 + \omega_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\lambda_1 + \lambda_2 + \lambda_2^z}{\omega_1 + \omega_2 + \omega_2^z} \right) \\
\frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\chi_1 - i(\xi_2z - \xi_2^z)}{\eta_1 + \eta_2 + \eta_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\lambda_1 + \lambda_2 + \lambda_2^z}{\omega_1 + \omega_2 + \omega_2^z} \right) & \frac{1}{|\psi|^2} \left( \frac{\lambda_1 + \lambda_2 + \lambda_2^z}{\omega_1 + \omega_2 + \omega_2^z} \right)
\end{pmatrix}
$$

(22)

The $A_i$s, $B_i$s and $\bar{B}_i$s are quadratic in the fluxes $f_j$s.

### 2.1.1 Non-zero Positive Eigenvalues of the Mass Matrix and (Arithmetic) Elliptic Curves

If the eigenvalues of the mass matrix are positive then one gets an attractor solution - for negative eigenvalues, the interpretation is not very clear (See section 4). The eigenvalues of $M$ are given by:

$$
\frac{1}{|\psi|^2} \left( (1) \pm \sqrt{2} \right) \pm \sqrt{3},
$$

where

1. $A_3 + A_5 + (B_3z + B_5z + c.c.) \in \mathbb{R}$,
2. $A_4 + \frac{1}{2} A_3 - \frac{1}{2} A_5 + \frac{(B_3 + B_5)z}{2} + c.c. \in \mathbb{R}$,
3. $A + Bz + \bar{B}z \in \mathbb{R}$.

(23)

We will now impose the following real non-linear (in the fluxes) constraint:

$$
(3) \equiv A + Bz + \bar{B}z = 0.
$$

(24)

Now, the following is part of the expression “(3)”: 

$$
-2 A_1^2 + 2 A_2^2 - A_4^2 + A_3 A_5 - (4 A_1 B_1 z + 4 A_2 B_2 z + A_5 B_3 z - 2 A_4 B_4 z + A_3 B_5 z + c.c.)(25)
$$

If (25) is set to zero, then one can recast (24) in the following form:

$$
A_3^3 + A_3^2 \alpha_2 (A_5, B_3, \bar{B}_3; z, \bar{z}) + A_3 \alpha_4 (A_1, A_5, B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3, B_5, \bar{B}_5; z, \bar{z}) \\
= A_4^2 + A_4 \alpha_3 (B_4, \bar{B}_4) + \alpha_6 (A_1, A_2, A_5, B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3, B_5, \bar{B}_5; z, \bar{z}),
$$

(26)
which, is an elliptic curve fibered over \(\mathbb{C}^8(A_1 + i\text{arg}(y), A_2 + iA_5, B_1, B_2, B_3, B_4, B_5, z)\). One can compare (26) with the following elliptic curve over any field:

\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,
\]

(27)

for which the \(j\)-invariant is defined as: \(j = \frac{(a_1^2 + 4a_2)^2 - 24(a_1a_3 + a_2)}{\Delta} \) where the discriminant \(\Delta \equiv -(a_1^2 + 4a_2)^2(a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_6^2) + 9(a_2^2 + 4a_2)(a_1a_3 + 2a_4)(a_2^2 + 4a_6) - 8(a_1a_3 + 2a_4)^3 - 27(a_2^2 + 4a_6)^2\).

Interestingly, the equations (7) can be rewritten as:

\[
\begin{pmatrix}
A_1 & -\frac{C_1B_1z}{C_2} \\
A_2 & -\frac{C_1B_2z}{C_2}
\end{pmatrix}
\begin{pmatrix}
1 \\
C_1z
\end{pmatrix}
= -C_1 \bar{z}
\begin{pmatrix}
1 \\
C_1\bar{z}
\end{pmatrix}.
\]

(28)

If the \(2 \times 2\) matrix in (28) is \(SL(2, \mathbb{Z})\)-valued, then (28) can be compared with following endomorphism \(E \rightarrow E\) requiring \(\lambda(\mathbb{Z} + \tau\mathbb{Z}) \subset \mathbb{Z} + \tau\mathbb{Z}, \lambda \in \mathbb{C}\), for an elliptic curve \(E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\):

\[
\begin{pmatrix}
N & A \\
-C & M
\end{pmatrix}
= \lambda
\begin{pmatrix}
1 \\
\tau
\end{pmatrix},
\]

(29)

implying a complex multiplication \(\mathbb{Z} + \omega\mathbb{Z}\) represented as: \(m_11 + m_2 \left(\frac{1}{2}(d + b) \begin{pmatrix} a \\ -c \end{pmatrix}, \frac{1}{2}(D - b) \right)\), where \((A, N - M, C) = l(a, b, c)\) (\(l\) being the greater common factor) and \(D \equiv b^2 - 4ac\) (See [14]). The modular parameter \(\tau\), which is supposed to satisfy: \(a\tau^2 + b\tau + c = 0\), gets identified with \(-\frac{C}{C_1}\). It would be interesting to see if one could further impose the condition that this value of \(\tau\) satisfies the above definition of the \(j\)-invariant function where it is understood that \(j = j(\tau = -\frac{C}{C_1}; \{A_i\}, \{B_i\}, \{\bar{B}_i\})\). Such an elliptic curve is what is referred to as an “arithmetic elliptic curve” (See [14])\(^4\).

To ensure that the eigenvalues are real, we now impose the following additional real and again non-linear(in the fluxes) constraint:

\[-A_4A_5 - A_3A_4 + z(A_4B_5 - A_4B_3 + c.c.) = 0.\]

(30)

Thus one is guaranteed to have two, doubly degenerate, real eigenvalues of \(M \frac{1}{|\psi|}(1 \pm \sqrt{2})\). One thus sees the possibility of getting attractor as well as repeller (see section 4) solutions depending on whether \((1) > \sqrt{2}\) or \((1) < \sqrt{2}\).

To summarize, from (7), one gets two complex, or four real constraints and then three additional real constraints from (24), (30) and (25) on the six integer-valued fluxes \(f_i\)'s, the complex structure moduli \(z, \psi\).

\(^4\)Related to complex multiplication, one can choose a Weierstrass model for \(E\) given by(See [15]):

\[
\begin{align*}
y^2 &= 4x^3 - c(x + 1), \quad c = \frac{27j}{j - (12)^3}, \quad j \neq 0, (12)^3, \\
y^2 &= x^3 = 1, \quad j = 0, \\
y^2 &= x^3 + x, \quad j = (12)^3.
\end{align*}
\]

If \(\text{gcd}(a, b, c) = 1\) and \(D\) is the fundamental discriminant (which means a discriminant of a quadratic imaginary field \(K_D \equiv \mathbb{Q}[i\sqrt{|D|}] = \{a + ib\sqrt{|D|} : a, b \in \mathbb{Q}\}\)), then \(j(\tau)\) is an algebraic integer of order equal to the number of equivalence classes of integral binary forms \(\begin{pmatrix} a & b \\ b & c \end{pmatrix}\) using \(SL(2, \mathbb{Z})\)-valued matrices for similarity transformations. Also, \(K_D(j(\tau))\) is Galois over \(K_D\) and independent of \(\tau\), where each \(\tau_i\) corresponds to the distinct ideal classes in the order \(\mathcal{O}(K_D)\).
2.1.2 Zero Eigenvalues of the Mass Matrix

We assume that one or more of the four eigenvalues of the mass matrix $M$, vanish. Now, if one wishes to ensure that one still gets an attractor solution for the eigenvalue(s) zero of $M$, then one needs to show that the effective potential when expanded about the extremum, has no cubic terms and that the quartic terms are positive [11]. Abbreviating $\frac{A_i + B_i z + B_{iz} \bar{z}}{|\phi|^2}$ as $\Omega_i$, the mass term can be written as:

$$
\begin{pmatrix}
\delta Re(z) \\
\delta Re(\psi) \\
\delta Im(z) \\
\delta Im(\psi)
\end{pmatrix}
\begin{pmatrix}
0 & \Omega_1 & 0 & \Omega_3 \\
\Omega_1 & \Omega_2 & -\Omega_3 & \Omega_4 \\
0 & \Omega_3 & 0 & \Omega_1 \\
-\Omega_3 & \Omega_4 & \Omega_1 & -\Omega_5
\end{pmatrix}
\begin{pmatrix}
\delta Re(z) \\
\delta Re(\psi) \\
\delta Im(z) \\
\delta Im(\psi)
\end{pmatrix}.
$$

(31)

A null eigenvalue would therefore satisfy:

$$
\begin{pmatrix}
0 & \Omega_1 & 0 & \Omega_3 \\
\Omega_1 & \Omega_2 & -\Omega_3 & \Omega_4 \\
0 & \Omega_3 & 0 & \Omega_1 \\
-\Omega_3 & \Omega_4 & \Omega_1 & -\Omega_5
\end{pmatrix}
\begin{pmatrix}
\delta Re(z) \\
\delta Re(\psi) \\
\delta Im(z) \\
\delta Im(\psi)
\end{pmatrix} = 0.
$$

One sees that the cubic terms can be made to vanish by imposing:

$$
\Omega_1 = \Omega_3 = 0.
$$

(32)

One can show that the extremum effective potential can be written as:

$$
V_{eff} \sim \left( \frac{a(\arg(\psi_0)) + b(\arg(\psi_0))z_0}{A(\arg(\psi_0)) + B(\arg(\psi_0))z_0} \right),
$$

(33)

which for $z_0 \to z_0 + \delta z_0$, setting $\delta \psi_0 = 0$, when expanded in powers of $\delta z_0$, can be shown to be as given in Tables 1 and 2 below:

| Type of Term | Coefficient |
|--------------|-------------|
| $\delta \bar{z}^3$ | $B^2(-Ab + aB)$ |
| $\delta z \delta \bar{z}^2$ | $B(2Abb + B(\bar{A}b - 3aB))$ |
| $(\delta z_0)^2 \delta z_0$ | $B(A\bar{B}b + 2Abb - 3a|B|^2)$ |
| $(\delta z_0)^3$ | $B^2(\bar{A}b - aB)$ |

One sees that the cubic terms can be made to vanish by imposing:

$$
Im(b) = Im(B) = 0, \quad Ab = aB,
$$

(34)

and that the quartic term, given by $a|B|^4 > 0$ if $a > 0$. One therefore gets ten constraints ((7), (32),(34) and $a > 0$) on the ten parameters: $f, s, z, \psi$. This indicates the possibility of the existence of attractor solutions for two-parameter Calabi-Yau’s away from the singular loci in the moduli space of the same.

5The most general eigenvector would be:

$$
\begin{pmatrix}
\frac{1}{\Omega_1} (\Omega_1 - \Omega_2) \delta Re(y) \\
\frac{1}{\Omega_2} (2 - \Omega_2) \delta Re(y) \\
\frac{1}{\Omega_3} (\Omega_3 - \Omega_4) \delta Re(y) \\
\frac{1}{\Omega_4} \delta Re(y)
\end{pmatrix},
$$

where $\Omega_1 \neq 0, \Omega_3 \neq 0$. The calculations are more involved but the main idea remains the same.
Table 2: Terms quartic in fluctuations

| Type of Term | Coefficient |
|--------------|-------------|
| $(\delta z_0)^4$ | $B^3(-Ab + aB)$ |
| $(\delta z_0)^3 \delta z_0$ | $B^2(3BbA + Abb - 4a|B|^2)$ |
| $|\delta z_0|^4$ | $|B|^2(BbA + BAb)$ |
| $\delta z_0 (5z_0)^2$ | $B^2(AbB + 3AbB)$ |
| $(\delta z_0)^4$ | $B^3(-Ab + aB)$ |

2.2 Near the Singular Conifold Locus

For points near the singular conifold locus: $\phi = 1 - 864\psi^6$, the period vector $\Pi$, in the symplectic basis, is
given by:

$$\Pi = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix},$$

(35)

where $w_i$'s, the components in the Picard-Fuchs basis, are given as (See [10]):

$$w_i = \frac{c_i}{2\pi i} \left( 2\pi i \frac{1 - 864\psi^6 - \phi}{(1 - \phi)^2} \right) \ln(1 - 864\psi^6 - \phi) + f_i(\phi, \psi),$$

(36)

where $f_i(\phi, \psi)$ are analytic functions of $\phi$ and $\psi$, $c_i = (1, 1, -1, -2, 2, 1)$. Defining $y \equiv 1 - 864\psi^6 - \phi$, the $w_i$s, about $\phi = 0, y = 0 : \frac{\phi}{y} \to 0$, are given as:

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \frac{1}{14.8\pi} \begin{pmatrix} -11.6 - 0.5i & 2.811 + 1.626i & 1.9 + 1.2i \\ -13.3 - 1.4i & 1.896 - 6.649i & 1.5 - 6.2i \\ -20.5 - 3.5i & 10.53 - 2.842i & 12.1 - 24.4i \\ -34.2 - 25.9i & 7.079 - 0.264i & 8.25 - 7.7i \\ -71.2 - 82.5i & 73.904 + 144.422i & 58.7 + 138i \\ 81.6 - 50.2i & 156.6 + 107.911i & 156.6 + 126.2i \end{pmatrix} \begin{pmatrix} 1 \\ \phi \\ y \end{pmatrix},$$

(37)

Near $\phi = y = 0$, the Kähler potential is given as:

$$K = -\ln \left( A + B\phi + \bar{B}\phi + C y + \bar{C}\bar{Y} + D|y|^2\ln|y|^2 \right),$$

(38)

which gives the following metric:

$$g_{ij} = \begin{pmatrix} B \bar{B} & \frac{B (\bar{C} + D y (1 + \log(|y|^2)))}{(A + C y + B z + \bar{C} y + \bar{B} z + D |y|^2 \log(|y|^2))^2} \\ \frac{(A + C y + B z + \bar{C} y + \bar{B} z + D |y|^2 \log(|y|^2))^2}{(A + C y + B z + \bar{C} y + \bar{B} z + D |y|^2 \log(|y|^2))^2} & \frac{B (\bar{C} + D y (1 + \log(|y|^2)))}{(A + C y + B z + \bar{C} y + \bar{B} z + D |y|^2 \log(|y|^2))^2} \end{pmatrix},$$

(39)
Using the results of the appendices B and C, one can see that for \(|\phi| < < 1, |y| < < 1\),
\[
g^{ij} D_\phi D_i W D_j W, \partial_\phi g^{ij} D_i W D_j W, \\
\sim \frac{|\ln(y)|^2}{\ln(|y|^2)} \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right), \\
D_\phi W W \sim \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right), 
\]
(40)

and
\[
g^{ij} D_y D_i W D_j W, \partial_\phi g^{ij} D_i W D_j W, D_\phi W W \\
\sim \left( \frac{\ln(y)}{|y \ln(|y|^2)|}, \frac{|\ln(y)|^2}{y (|y|^2)^2}, \ln(y) \right) \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right). 
\]
(41)

In equations (40), (41) and other similar equations below, it is assumed that only the forms and not the details of the different terms, apart from the \((\ln|y|)\alpha\) pieces, are the same. This implies that
\[
\partial_\phi V \sim \ln|y| \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right), \\
\partial_\phi V \sim \left( \frac{1}{|y|} \right) \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right), 
\]
(42)

For the purpose of constructing the mass matrix, one needs to evaluate second derivatives of the black hole potential.

Using the results of the appendices B and C, one can show that up to \(O\)(second order terms in \(z\) and/or \(y\) and their complex conjugates) in the numerators and denominators:
\[
g^{ij} D_\phi D_i D_j W, \partial_\phi g^{ij} D_j D_\phi W, \partial_\phi \partial_\phi g^{ij} D_i D_j W, D_\phi D_\phi W W, g^{ij} \partial_\phi g^{ij} D_i W W \\
\sim \left( 1 \text{ or } \frac{\ln|y|}{|\ln|y||}, \frac{\ln|y|}{\ln|y|^2}, \frac{1}{|\ln|y|^2|}, \frac{1}{|y|}, \frac{1}{\ln|y|^2}, \frac{1}{\ln|y|^2 y^2} \right) \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right). 
\]
(43)

Therefore,
\[
\partial_\phi \partial_\phi V \sim \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right). 
\]
(44)

Again, using the results of the appendices B and C, one sees that:
\[
\partial_y g^{ij} D_i D_y W D_j W, D_y D_y W W, g^{ij} \partial_y g^{ij} D_i W W, g^{ij} D_y D_i D_y W D_j W, \partial_\phi \partial_\phi g^{ij} D_i W D_j W \\
\sim \left( \frac{|\ln(y)|^2}{y (|\ln|y|^2|)^2}, \frac{1}{|y|^2 \ln(y)}, \frac{1}{|y|^2 \ln|y|^2}, \frac{1}{|y|^2 \ln|y|^2 |y|^2} \right) \left( \frac{a + b\phi + c\phi + f y + g y + h y \ln(y) + k y \ln(y) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b'\phi + c'\phi + f' y + g' y + n' |y|^2 \ln(|y|^2)} \right). 
\]
(45)
This yields:

\[
\partial_y \partial_y V \sim \frac{1}{|y|^2 (\ln |y|)^2} \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right). 
\]  

(46)

Similarly, using the results of the appendix B, one sees that:

\[
g^{ij} D_\phi D_\phi W D_j \bar{W}, \ \partial_\phi g^{ij} D_\phi W D_j \bar{W} + D_\phi g \bar{\phi} D_\phi W D_j \bar{W}, \ \partial_\phi \partial_\phi g^{ij} D_\phi W D_j \bar{W}, \ g^{ij} \partial_\phi \partial_\phi g^{ij} D_\phi W \bar{W}, \ D_\phi W D_\phi \bar{W} \\
\sim \left( \frac{\ln(y)}{y \ln |y|^2}, \frac{1}{|y|}, \frac{\ln(y)}{y (\ln |y|^2)^2}, \frac{\ln(y)}{|y|^2} \right) \\
\times \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right). 
\]  

(47)

Therefore, one obtains:

\[
\partial_\phi \partial_\phi V \sim \frac{1}{|y|} \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right). 
\]  

(48)

We now come to the evaluation of \( \partial_\phi \partial_\phi V - \) the other ingredient necessary for the evaluation of the mass matrix (8). Referring again to the appendix B, one sees:

\[
g^{ij} D_\phi D_\phi W D_j \bar{W}, \ g^{ij} g \bar{\phi} D_\phi W D_j \bar{W}, \ g^{ij} \partial_\phi \partial_\phi g^{ij} D_\phi W \bar{W}, \ g^{ij} g \bar{\phi} \partial_\phi \partial_\phi g^{ij} D_\phi W \bar{W}, \ |W|^2 g_\phi \phi \bar{\phi}, \ D_\phi W D_\phi \bar{W} \\
\sim \left( \ln |y|, \ln |y|, \ln |y|, 1, \ln |y|, 1, 1 \right) \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right). 
\]  

(49)

One therefore finally gets:

\[
\partial_\phi \partial_\phi V \sim \ln |y| \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right). 
\]  

(50)

Similarly, using the results from the appendix B, one arrives at:

\[
g^{ij} D_\phi D_\phi W D_j \bar{W}, \ g^{ij} g \bar{\phi} D_\phi W D_j \bar{W}, \ g^{ij} \partial_\phi \partial_\phi g^{ij} D_\phi W \bar{W}, \ g^{ij} g \bar{\phi} \partial_\phi \partial_\phi g^{ij} D_\phi W \bar{W}, \ |W|^2 g_\phi \phi \bar{\phi}, \ D_\phi W D_\phi \bar{W} \\
\sim \left( \frac{1}{|y|^2 (\ln |y|^2)^2}, \frac{1}{|y|^2 \ln |y|^2}, \frac{1}{|y|^2 \ln |y|^2}, \frac{\ln(y)^2}{|y|^2} \right) \\
\times \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right) 
\]  

(51)

which finally yields:

\[
\partial_\phi \partial_\phi V \sim \frac{1}{|y|^2 \ln |y|} \left( \frac{a + b \phi + c \bar{\phi} + f y + g \bar{y} + h y \ln(y) + k \bar{y} \ln(\bar{y}) + l \bar{y} \ln(y) + m y \ln(y) + n |y|^2 \ln(|y|^2)}{a' + b' \phi + c' \bar{\phi} + f' y + g' \bar{y} + n' |y|^2 \ln(|y|^2)} \right). 
\]  

(52)
Finally, using again the results from the appendices B and C, one sees that:
\[ g^{ij} D_i W D_j D_y \dot{W}, \partial \partial \partial g^{ij} D_i W D_j D_y \dot{W}, \partial \partial \partial g^{ij} D_i W D_j D_y \dot{W}, |W|^2 g_{\phi \phi}, D_\phi W D_\phi \dot{W} \]
\[ \sim \left( \frac{\ln(y)}{|y|^{3/2}}, \left( \frac{1}{|y|^{1/2}} \ln|y|^2, \ln(y) \right) \right) \times \left( \frac{a + b\phi + c\phi^2 + fy + g\bar{y} + hy \ln(y) + k\bar{y} \ln(y) + l\bar{y} \ln(y) + my \ln(\bar{y}) + n|y|^2 \ln(|y|^2)}{a' + b' \phi + c' \phi^2 + f' \bar{y} + g' \bar{y} + n'|y|^2 \ln(|y|^2)} \right), \]
which gives:
\[ \partial \partial \partial \phi \partial \gamma V = \frac{\ln(|y|)}{|y|} \left( \frac{a + b\phi + c\phi^2 + fy + g\bar{y} + hy \ln(y) + k\bar{y} \ln(y) + l\bar{y} \ln(y) + m \ln(\bar{y}) + n|y|^2 \ln(|y|^2)}{a' + b' \phi + c' \phi^2 + f' \bar{y} + g' \bar{y} + n'|y|^2 \ln(|y|^2)} \right). \]

One thus sees that the mass matrix of (8) is given by (retaining again only the most dominant terms):
\[ M \sim \frac{1}{|y|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & \Lambda_2 \\ 0 & 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 & \Lambda_3 \end{pmatrix}, \]
where \( \Lambda_i \equiv \frac{a_i + b_i \phi + c_i \phi^2 + f_i \bar{y} + g_i \bar{y} \ln(y) + h_i \bar{y} \ln(y) + i_i \bar{y} \ln(y) + j_i \bar{y} \ln(y) + k_i |y|^2 \ln(|y|^2)}{a_i' + b_i' \phi + c_i' \phi^2 + f_i' \bar{y} + g_i' \bar{y} + n_i |y|^2 \ln(|y|^2)}. \)

Hence, \( M \) will have at least one doubly degenerate null eigenvalue. One corresponding eigenvector of fluctuations in \( \phi \) and \( y \) will be given by:
\[ \begin{pmatrix} \delta \Re(\phi) \\ \delta \Im(\phi) \\ 0 \end{pmatrix}, \]
alongwith the constraint:
\[ \Lambda_2^2 = \Lambda_1 \Lambda_3. \]

Thus, from equations (34), (42), (56) and “\( a > 0 \)” one gets nine constraints on the six fluxes \( f_i \)-s and the complex structure moduli \( \phi, y \).

One has to remember that the \( \Lambda_i \)-s are real-valued quantities constructed from the square of the fluxes and the complex structure moduli at the extremum of the effective black-hole potential. This is very interesting - \( \Lambda_i \in \mathbb{R} \), which implies that one gets, for null eigenvalues of the mass matrix, for points in the moduli space near the singular conifold locus, a version of an \( A_1 \)-singularity wherein one gets the embedding: \( \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \), which is the real projection of the familiar \( T^*(S^2) \) for \( \mathbb{C}^2 \)/\( \mathbb{Z}_2 \) - in short, the singular conifold locus in the moduli space of the two-parameter Calabi-Yau, corresponds to some version of \( A_1 \)-singularity in the space \( \text{Image}(\mathbb{Z}^6 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3) \) fibered over \( \mathbb{C}^2(\phi, y) \), when looking for nonsupersymmetric black-hole attractor solutions.

\section{Attractor equations for non-supersymmetric Attractors}

In this section, we now discuss getting non-supersymmetric attractor solutions using the “new attractor” equations of Kallosh [4], which are as follows:
\[ \Sigma, f = 2e^K \text{Im} \left( W \bar{\Pi} - g^{ij} D_i W D_j \bar{\Pi} \right). \]
3.1 Away from the conifold locus

Using the results of appendix A, one can show that the RHS, up to terms linear in the complex structure moduli, \( z, \psi \) is independent of \( \psi \), and the attractor equations can be written as:

\[
\begin{pmatrix}
    f_4 \\
    f_5 \\
    f_6 \\
    -f_1 \\
    -f_2 \\
    -f_3
\end{pmatrix}
= \begin{pmatrix}
    a_1 + b_1 z + c_1 \bar{z} \\
    a_2 + b_2 z + c_2 \bar{z} \\
    a_3 + b_3 z + c_3 \bar{z} \\
    a_4 + b_4 z + c_4 \bar{z} \\
    a_5 + b_5 z + c_5 \bar{z} \\
    a_6 + b_6 z + c_6 \bar{z}
\end{pmatrix}
\]

where \( a_i, b_i, c_i \) depend on the fluxes \( f_i \). This is not in contradiction with the analysis of section 2, where it is shown that the results depend, at best, on the phase of \( \psi \) and not its modulus - the attractor equations go one step further in showing that the attractors are also independent of the phase. The attractor equations (58) bring out a feature, which would become apparent in the analysis of section 2 involving extremization of the effective black-hole potential only after a complete numerical calculation, namely that for points away from the conifold locus, the nonsupersymmetric attractors are independent of one of the complex structure moduli (\( \psi \)).

3.2 Near the conifold locus

Using results of appendices B and C, one sees that

\[
Im(W\Pi) \sim \begin{pmatrix}
    \tilde{\Sigma}_1 \\
    \tilde{\Sigma}_2 \\
    \tilde{\Sigma}_3 \\
    \tilde{\Sigma}_4 \\
    \tilde{\Sigma}_5 \\
    \tilde{\Sigma}_6
\end{pmatrix}
\]

and

\[
Im(g^{ij} D_i W D_j \Pi \sim g^{\bar{y}y} D_y W D_{\bar{y}} \Pi) \sim \frac{|\ln(y)|^2}{ln|y|^2} \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    \Sigma_4
\end{pmatrix},
\]

where

\[
\Sigma_4, \tilde{\Sigma}_i \equiv \left( \frac{a_i + b_i \phi + \tilde{b}_i \phi + f_i y + \tilde{f}_i \bar{y} + h_i y \ln(y) + \tilde{h}_i \ln(y) + l_i \bar{y} \ln(y) + \tilde{l}_i y \ln(y) + n_i |y|^{2} \ln(|y|^2)}{a_i' + b_i' \phi + \tilde{b}_i' \phi + f_i' y + \tilde{f}_i' \bar{y} + n_i' |y|^{2} \ln(|y|^2)} \right).
\]

The only way to satisfy the attractor equations (57) is to impose

\[
f_1 = \Sigma_4 = 0.
\]

(59)
Thus, the attractor equations show that the attractor solutions of section 2 (obtained by extremization of the effective black hole potential and analysis of the eigenvalues of the mass matrix) must include switching off of one of the six components of the fluxes - this would become apparent only after a complete numerical analysis of section 2.

4 Conclusion

We looked at an example of (the mirror to) a two-parameter Calabi-Yau (expressed as a hypersurface in a weighted complex projective space) and looked at possible non-supersymmetric black-hole attractor solutions by extremization of an effective potential, for points away and close to the singular conifold locus. For the former, we showed a connection between non-supersymmetric black hole attractors and an elliptic curve and found a system of seven (for positive eigenvalues of the mass matrix for points in the moduli space away from the conifold locus) or nine (for null eigenvalues of the mass matrix for points in the moduli space near the conifold locus) or ten (for null eigenvalues of the mass matrix for points in the moduli space away from the conifold locus) constraints on the six integer fluxes and the two complex structure moduli. It might be possible to interpret the black-hole extremization as an endomorphism involving complex multiplication of a possibly arithmetic elliptic curve. For points close to the conifold locus, we found a connection between non-supersymmetric black hole attractors and an $A_1$ singularity. From the point of view of the attractor equations of [4], we saw that for the former case, the nonsupersymmetric attractor solutions are independent of one of the two complex structure moduli. For the latter, the attractor equations of [4] imply switching off of one of the six components of the fluxes. Both would become manifest only after a detailed numerical computation involving extremization of the effective potential and analysis of mass matrix eigenvalues and therefore serve as good checks on the numerics involved in the analysis of section 2 - one must however make note of the fact that the black hole potential extremization analysis, even without doing any detailed numerical analysis, already tells us that the nonsupersymmetric attractors for points in the moduli space away from the singular conifold locus, can have, at best, only a phase-factor dependence on $\psi$ and are independent of $|\psi|$, and the attractor equations analysis says that even the phase factor dependence is absent. The mass matrix can take negative eigenvalues, in addition to positive and null - the eigenmodes for the negative eigenvalues could perhaps be interpreted as non-supersymmetric repellers⁶, or might be interpretable as a flop transition in the extended Kähler cone [13].

Using tools from computational algebraic geometry, one could hope to do a better job in actually doing the numerical computations related to the present work on supersymmetric black-hole attractors (and also

⁶This was suggested by R.Kallosh to one of us(AM).
flux vacua ([16]) attractors)\(^7\). Attractor basins ([17]) and area codes, is another aspect which could be looked into. Further, it would be nice to see whether the particular Calabi-Yau considered in this work is an “arithmetic attractor” (See [14])\(^8\).

**Acknowledgements**

One of us (AM) acknowledges the support from the Abdus Salam ICTP under the junior associateship scheme and the Enrico Fermi Institute, University of Chicago for its hospitality and financial support, where part of this work was completed. He also thanks the Department of Atomic Energy, India for a research grant related to the Department of Atomic Energy Young Scientist Award scheme. AM would also like to thank R.Kallosh for a useful correspondence. We gratefully acknowledge extremely useful correspondences with S.Ferrara which resulted in revision of the interpretations of the results in section 3 after the first version was submitted to the archive.

\(^7\)The basic idea is to use the “splitting principle” in which for some positive integer \(l\), the algebraic variety \(L\) corresponding to the radical ideal \(\sqrt{I}\) is expressed as: \(L(\sqrt{I}) = L((I : f^\infty)) \cup L((I : f^\infty))\) for some polynomial \(f\) and the ideal \(I = (f_1, ..., f_n)\), where the first term on the right hand side is the algebraic variety corresponding to the radical of “saturating” of the ideal \(I\), implying a subvariety for which \(f \neq 0\). For the purposes of finding (non)supersymmetric attractors and/or flux vacua one chooses \(f_i\)s to be the numerators of \(D_iW_i\)s and \(I\) to be \(\langle \partial V \rangle\). Then (See [16])

\[
L((\partial V)) = L((\partial V, D_1W, ..., D_nW)) \cup L((V, D_1W, ..., D_{i-1}W, D_{i+1}W, ..., D_nW : D_iW^\infty)) \\
\cup \cup_{j} L(((\partial V, D_1W, ..., D_{i-1}W, D_{i+1}W, ..., D_{j-1}W, D_{j+1}W, ..., D_nW)) : D_iW^\infty)) : D_jW^\infty) \\
\cup L((\partial V : D_iW^\infty) : ...D_{i-1}W^\infty) : D_nW^\infty),
\]

implying that one gets a SUSY vacuum from the first term, and non-SUSY vacua for the rest with, e.g., the second term implying violation of one of the \(n\) F-flatness conditions and the last implying violation of all \(n\) F-flatness conditions. Stable isolated vacua are associated with the real roots of the zero-dimensional primary decomposition.

\(^8\)In fact, as shown in [14], the two-parameter Calabi-Yau expressed as a degree-eight hypersurface in \(\text{WCP}^4[1, 1, 2, 2, 2]\):

\[
x_1^6 + x_2^4 + x_3^4 + x_4^4 = x_5^8 - 8\psi \prod_{i=1}^{5} x_i - 2\phi x_1 x_2 = 0
\]

is an arithmetic attractor for \(\psi = 0\). The ratio of the the periods is related to a Schwarz triangle functions \(s_k(z) = \frac{\phi_0^{(k)}(z)}{\phi_k^{(k)}(z)}, k = 0, 1, \infty\) where corresponding to a given \(2F_1(a, b; c; z)\),

\[
\begin{align*}
\phi_0^{(0)}(z) &= \frac{2F_1(a, b; c; z)}{z^{1-c} 2F_1(a+1-c, b+1-c; 2-c; z)}, \\
\phi_0^{(1)}(z) &= \frac{2F_1(a, b; 1-c+a+b; 1-z)}{(1-z)^{a-b} 2F_1(c-a, c-b; 1+c-a-b; 1-z)}, \\
\phi_1^{(0)}(z) &= \frac{z^{-a} 2F_1(a, a+1-c; 1+a-b; \frac{1-z}{z})}{z^{-b} 2F_1(b, b+1-c; 1-a+b; \frac{1-z}{z})}
\end{align*}
\]

for the triangle arithmetic group (corresponding to reflections in the sides of a (curved) triangle with angles \(\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}\): \(\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} = \pi > 0 < 1\) for Euclidian or sperical or hyperbolic triangles respectively, \(l, m, n\) being positive integers greater than or equal to two \((2, 4, 4)\)
A Covariant derivatives relevant to the calculations

In this appendix, we give analytic expressions for (almost) all covariant derivatives of the period vector and the superpotential for points in the moduli space away from the conifold locus. It will be understood that one has dropped terms quadratic in (complex conjugates of) \( z, \bar{z} \) and their products in the numerators and denominators of all expressions in this appendix - this is indicated by “\(^{\sim}\)”.

A.1 Covariant derivatives of \( \Pi \)

For the purpose of discussing the generalized attractor equations of [4] for non-supersymmetric attractors, one would need expressions for \( D_{z} \Pi \) which we give below:

\[
(i) \quad D_{z} \Pi \sim \left\{ \frac{1}{2^{7} \Gamma(\frac{5}{6})^{3}} \frac{i}{18} \pi^{\frac{7}{2}} (-1 + \text{Conjugate}((-1) \bar{\Pi})) \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right) \right) \right.
\]

\[
\left. \quad - \left(\frac{c}{a} + z + \bar{c} \bar{z}\right) \left(576 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + \bar{c} \bar{z} \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right)\right) \right\},
\]

\[
\left\{ \frac{1}{2^{7} \Gamma(\frac{5}{6})^{3}} \frac{i}{36} \pi^{\frac{7}{2}} (3 - 2i + 4 \text{Conjugate}((-1) \bar{\Pi}) + \text{Conjugate}((-1) \bar{\Pi})) \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right) \right. \]

\[
\left. \quad - \left(\frac{c}{a} + z + \bar{c} \bar{z}\right) \left(576 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + \bar{c} \bar{z} \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right)\right) \right\},
\]

\[
\left\{ \frac{1}{2^{7} \Gamma(\frac{5}{6})^{3}} \frac{i}{18} \pi^{\frac{7}{2}} (576 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + \bar{c} \bar{z} \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right)),
\]

\[
\left. \quad + 576 c \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 a \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right) + 7 c z \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right) \right\},
\]

\[
\left\{ \frac{1}{2^{7} \Gamma(\frac{5}{6})^{3}} \frac{i}{36} \pi^{\frac{7}{2}} (48 a \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 a \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right) + 7 c z \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right))\right. \]

\[
\left. \quad + 576 c \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 a \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right) + 7 c z \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right) \right\},
\]

\[
\left\{ \frac{1}{2^{7} \Gamma(\frac{5}{6})^{3}} \frac{i}{36} \pi^{\frac{7}{2}} (1 + \text{Conjugate}((-1) \bar{\Pi})) \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right) \right. \]

\[
\left. \quad - \left(\frac{c}{a} + z + \bar{c} \bar{z}\right) \left(576 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + \bar{c} \bar{z} \left(48 \ _2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}\right) + 7 \ _2F_1\left(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}\right)\right)\right) \right\},
\]
\[D_{\psi} \Pi \sim \left\{ \frac{1}{2\pi^2 (a + cz + cz)} \right\}^{\frac{1}{3}} \left\{ -\frac{i}{9} \pi^2 \bar{\psi} \left( -1 + \text{Conjugate}((-1)\bar{z}) \right) \bar{\psi} (g z + \bar{b} + \bar{d} + z \text{Conjugate}(j)) \bar{z} \right\} \]

\[\times (g z + \bar{b} + (\bar{d} + z \text{Conjugate}(j)) \bar{z}) \left( -576 \ 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \right) + \bar{z} (48 \ 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \right) + 7 \ 2F_1 \left( \frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4} \right)) \right) \]

\[\left\{ -\frac{i}{36} \pi \bar{\psi} \left( \frac{1}{(a + cz + cz)} \Gamma \left( \frac{5}{9} \right) \right)^{\frac{1}{3}} \left[ 2^\frac{2}{3} \pi^2 \bar{\psi} (g z + \bar{b} + (\bar{d} + z \text{Conjugate}(j)) \bar{z}) \right] \left( -576 \ 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \right) + \bar{z} (48 \ 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \right) + 7 \ 2F_1 \left( \frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4} \right)) \right) \right\} \]

\[\times (g z + \bar{b} + (\bar{d} + z \text{Conjugate}(j)) \bar{z}) \left( -576 \ 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \right) + \bar{z} (48 \ 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \right) + 7 \ 2F_1 \left( \frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4} \right)) \right) + 54 (-128 \sqrt{6} \text{EllipticK} \left( \frac{2}{3} \right) + \bar{z} (32 \sqrt{6} \text{EllipticK} \left( \frac{2}{3} \right) + 9 \pi \ 2F_1 \left( \frac{5}{4}, \frac{7}{4}, 2, \frac{1}{4} \right)) \right) \}

A.2 Covariant derivatives of W

In this subsection, we list the covariant derivatives of the superpotential. It is understood that all expressions below are expressed as complex rational functions in the complex structure moduli \( z, \psi \) retaining terms only linear in the same in the numerators and denominators of the expressions.
\[ \Delta \nabla \}\] extremizing the superpotential (equations (4) - (7)) and for studying the mass matrix (equations (8) - (22)).

We give below expressions for the double covariant derivatives of the superpotential that will be relevant to extremizing the effective potential via equations (4) - (7), and also for studying the generalized attractor equations for non-supersymmetric attractors.

\[
(i) D_{\psi} W \sim \frac{1}{2^4 \left( \frac{1}{2} \right)^{5/6}} \Gamma \left( \frac{5}{6} \right)^3 \left[ -\frac{i}{36} \left( 2 \left( -1 + (-1)^{\frac{7}{12}} \right) f_1 + \left( 3 + 2 i + 4 \left( -1 \right)^{\frac{7}{12}} + (-1)^{\frac{7}{12}} \right) f_1 \right) \\
+ \left( 2 + 2 i \right) f_3 + 2 f_4 - f_5 + F6 + (-1)^{\frac{7}{12}} F6 \right) \pi^{\frac{7}{12}} \left( 48 a \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) + 576 c \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) \right) \\
= \left( 7 a \ 2 F_1 \left( \frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4} \right) + \bar{z} \left( 48 c \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) + 576 h \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) + 7 c \ 2 F_1 \left( \frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4} \right) \right) \right],
\]

\[
(ii) D_{\psi} W \sim \frac{i}{36} \pi \psi \left( \frac{1}{a + c z + c \bar{z}} \right) \Gamma \left( \frac{5}{6} \right)^3 \left[ \left( 2 \left( -1 + (-1)^{\frac{7}{12}} \right) f_1 + \left( 3 + 2 i + 4 \left( -1 \right)^{\frac{7}{12}} + (-1)^{\frac{7}{12}} \right) f_1 \right) f_2 + (2 + 2 i) f_3 \\
+ 2 f_4 - f_5 + f_6 + (-1)^{\frac{7}{12}} f_6 \right) \pi^{\frac{7}{12}} \left( b + d z + \left( j + g \bar{z} \right) \right) \left( -576 \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) + 48 z \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) \right) \\
= \left( 108 f_1 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 32 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 9 \pi z \ 2 F_1 \left( \frac{5}{4}, \frac{7}{4}, 2, \frac{1}{4} \right) \right) \right) \\
+ \left( 108 f_4 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 32 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) - 54 f_5 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) \right) \right) \\
+ \left( 108 f_6 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 32 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 9 \pi z \ 2 F_1 \left( \frac{5}{4}, \frac{7}{4}, 2, \frac{1}{4} \right) \right) \right]
\]

\[ \Delta \nabla \]

A.2.1 $D_i W$

We give below expressions for the double covariant derivatives of the superpotential that will be relevant to extremizing the effective potential (equations (4) - (7)) and for studying the mass matrix (equations (8) - (22)).

\[
(i) D_{\psi} D_{\psi} W = \frac{i}{36} \pi \psi \left( \frac{1}{a + c z + c \bar{z}} \right) \Gamma \left( \frac{5}{6} \right)^3 \left[ \left( 2 \left( -1 + (-1)^{\frac{7}{12}} \right) f_1 + \left( 3 + 2 i + 4 \left( -1 \right)^{\frac{7}{12}} + (-1)^{\frac{7}{12}} \right) f_1 \right) f_2 + (2 + 2 i) f_3 \\
+ 2 f_4 - f_5 + f_6 + (-1)^{\frac{7}{12}} f_6 \right) \pi^{\frac{7}{12}} \left( \left( -576 \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) + 48 z \ 2 F_1 \left( \frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4} \right) \right) \\
= \left( 108 f_1 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 32 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 9 \pi z \ 2 F_1 \left( \frac{5}{4}, \frac{7}{4}, 2, \frac{1}{4} \right) \right) \right) \\
+ \left( 108 f_4 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 32 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) - 54 f_5 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) \right) \right) \\
+ \left( 108 f_6 \left( -128 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 32 \sqrt{6} \text{ EllipticK} \left( \frac{2}{3} \right) + 9 \pi z \ 2 F_1 \left( \frac{5}{4}, \frac{7}{4}, 2, \frac{1}{4} \right) \right) \right]
\]

A.2.2 $D_{\psi} D_{\psi} W$

We give below expressions for the double covariant derivatives of the superpotential that will be relevant to extremizing the effective potential (equations (4) - (7)) and for studying the mass matrix (equations (8) - (22)).
We give below expressions for the triple covariant derivatives of the superpotential which will be relevant to the calculation of the mass matrix (via equations (8) - (22)). For triple covariant derivatives of the superpotential, an example of a short expression is:

(i) \( D_\psi D_\psi W \sim \frac{1}{2^4 (a^3 + 3a^2(c z + c \bar{z})) \Gamma(\frac{5}{6})^3} \left[ \frac{i \pi}{6} (2 (-1 + (-1)^{\frac{1}{3}}) f_1 + (3 + 2 i + 4 (-1)^{\frac{1}{3}} + (-1)^{\frac{5}{3}}) f_2 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 + (-1)^{\frac{5}{6}} f_6) \pi \frac{(c + h \bar{z})^2 (48 a - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 576 c - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 7 a - 2 F_1(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}) + \bar{z} (48 c - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 76 h - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 7 c - 2 F_1(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}))}{\Gamma(\frac{5}{6})^3} \right] \),

and an example of a long expression is:

(ii) \( D_\psi D_\psi W \sim \frac{1}{(a^3 + 3a^2(c z + c \bar{z})) \Gamma(\frac{5}{6})^3} \frac{i \pi}{36} \left[ 2^5 \left( 2 (-1 + (-1)^{\frac{1}{3}}) f_1 + (3 + 2 i + 4 (-1)^{\frac{1}{3}} + (-1)^{\frac{5}{3}}) f_2 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 + (-1)^{\frac{5}{6}} f_6) \pi \frac{(c + h \bar{z})^2 (48 a - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 576 c - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 7 a - 2 F_1(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}) + \bar{z} (48 c - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 76 h - 2 F_1(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{4}) + 7 c - 2 F_1(\frac{13}{12}, \frac{19}{12}, 2, \frac{1}{4}))}{\Gamma(\frac{5}{6})^3} \right] \).
\[-\frac{1}{\Gamma\left(\frac{3}{6}\right)} \left[ 2 \frac{\sqrt{2}}{\pi} (2 (-1 + (-1)^{1/2}) f_1 + (3 + 2 i + 4 (-1)^{1/2} + (-1)^{1/2}) f_1 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 \right] \]

\[+ \frac{1}{\Gamma\left(\frac{3}{6}\right)} \left[ 2 \frac{\sqrt{2}}{\pi} (2 (-1 + (-1)^{1/2}) f_1 + (3 + 2 i + 4 (-1)^{1/2} + (-1)^{1/2}) f_1 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 \right] \]

\[+ (1)^{1/2} f_6) \pi^{1/2} \left( a^2 c + 2a c (z + \bar{z}) + a^2 h \bar{z} \right) \left( \frac{1}{(a + cz + c \bar{z})} \Gamma\left(\frac{5}{6}\right)^3 \right) \left[ 2 \frac{\sqrt{2}}{\pi} (2 (-1 + (-1)^{1/2}) f_1 + (3 + 2 i + 4 (-1)^{1/2} + (-1)^{1/2}) f_1 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 \right] \]

\[- \frac{1}{(a + cz + c \bar{z})^2} \Gamma\left(\frac{5}{6}\right)^3 \left[ 2 \frac{\sqrt{2}}{\pi} (2 (-1 + (-1)^{1/2}) f_1 + (3 + 2 i + 4 (-1)^{1/2} + (-1)^{1/2}) f_1 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 \right] \]

\[+ (1)^{1/2} f_6) \pi^{1/2} \left( b + d z + \bar{g} \bar{z} + h \bar{z} \right) \left( \frac{1}{(a + cz + c \bar{z})} \Gamma\left(\frac{5}{6}\right)^3 \right) \left[ 2 \frac{\sqrt{2}}{\pi} (2 (-1 + (-1)^{1/2}) f_1 + (3 + 2 i + 4 (-1)^{1/2} + (-1)^{1/2}) f_1 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 \right] \]

\[+ (1)^{1/2} f_6) \pi^{1/2} \left( b + d z + \bar{g} \bar{z} + h \bar{z} \right) \left( \frac{1}{(a + cz + c \bar{z})} \Gamma\left(\frac{5}{6}\right)^3 \right) \left[ 2 \frac{\sqrt{2}}{\pi} (2 (-1 + (-1)^{1/2}) f_1 + (3 + 2 i + 4 (-1)^{1/2} + (-1)^{1/2}) f_1 + (2 + 2 i) f_3 + 2 f_4 - f_5 + f_6 \right] \]
\[ (+1) \frac{\bar{z}}{f_0} \pi \frac{\bar{z}}{f_0} (b+d z+\bar{g} \bar{z}) (48 \ 2 F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{1}{4} \right) + 7 \ 2 F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{1}{4} \right)) + \frac{1}{(a+c z+c \bar{z}) \Gamma \left( \frac{5}{6} \right)^3} \left[ \frac{2 \pi (2 (1-1) \pi f_1 + (3+2 i +4 (-1) \pi f_6 + (-1) \pi f_6) \pi \frac{\bar{z}}{f_0} (d+j \bar{z}) (-576 \ 2 F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{1}{4} \right)) + \frac{1}{(a^2 +2ac(z+\bar{z}) \Gamma \left( \frac{5}{6} \right)^3} \left[ \frac{2 \pi (2 (1-1) \pi f_1 + (3+2 i +4 (-1) \pi f_6 + (-1) \pi f_6) \pi \frac{\bar{z}}{f_0} (bc+cd z+c \bar{g} \bar{z} + bh \bar{z}) (-576 \ 2 F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{1}{4} \right)) + 108 f_1 (32 \sqrt{6} \text{EllipticK} \left( \frac{2}{3} \right) + 9 \pi \ 2 F_1 \left( \frac{5}{4}, \frac{5}{4}, \frac{3}{4} \right)) + 108 f_1 (32 \sqrt{6} \text{EllipticK} \left( \frac{2}{3} \right)) \right], \right. \]

Because of the length of the expressions involved, we do not give the explicit forms of \( D_z D_\psi D_z W, D_\psi D_z D_z W, D_\psi D_z D_\psi W, D_\psi D_\psi D_z W, D_\psi D_\psi D_\psi W, D_\psi D_\psi D_z W, D_\psi D_\psi D_\psi W \).

**B  Covariant derivatives relevant to the calculations near the conifold locus**

We first write down the expressions for the period vector in the symplectic basis:

\[ \Pi = \begin{pmatrix} b_0 y + c_0 \phi \\ a_1 + b_1 y + c_1 \phi \\ a_2 + b_2 y + c_2 \phi \\ a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(1-\phi)^2} \\ a_4 + b_4 y + c_4 \phi \\ a_5 + b_5 y + c_5 \phi \end{pmatrix}, \]

and then the superpotential:

\[ W = f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(1-\phi)^2}). \]
Now, we give expressions for the covariant derivatives of the superpotential relevant to the calculations in this paper. In all the following expressions, analogous to the results in appendix A, one retains terms linear in \( \phi, y \) as well terms of \( \mathcal{O}(|y|\ln|y|, |y|^2\ln|y|, \ln|y|^2) \) in the numerators and denominators.

### B.1 \( D_{\phi}W \) and \( D_{\bar{\phi}}\bar{\Pi} \)

We write out expressions for the first derivatives of the superpotential and the complex conjugate of the period that would be relevant, e.g., to the attractor equations of section 3.2:

(i) \( D_{\phi}W \sim c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 \frac{2 f_4 y \ln(y)}{(-1 + \phi)^3} \frac{B}{A + C y + B \phi + B \phi + D |y|^2 \ln|y|^2} (f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) + f_4 (a_3 + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2})) \)

(ii) \( D_y W \sim b_0 f_1 + b_1 f_2 + b_2 f_3 + b_4 f_5 + b_5 f_6 + \frac{f_4 (f + b_3 (-1 + \phi)^2 + f \ln(y))}{(-1 + \phi)^2} \frac{1}{A + C y + B \phi + C y + B \phi + D |y|^2 \ln|y|^2} (f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) + f_4 (a_3 + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2})) (C + D \bar{y} (1 + \ln|y|^2)) \)

(iii) \( D_{\bar{\phi}}\bar{\Pi} \sim \begin{pmatrix} \bar{c}_0 - \frac{B (b_0 \bar{y} + c_0 \phi)}{A + C y + B \phi + C \bar{y} + B \phi + D |y|^2 \ln|y|^2} \\ \bar{c}_1 - \frac{B (a_1 + b_1 \bar{y} + c_1 \phi)}{A + C y + B \phi + C \bar{y} + B \phi + D |y|^2 \ln|y|^2} \\ \bar{c}_2 - \frac{B (a_2 + b_2 \bar{y} + c_2 \phi)}{A + C y + B \phi + C \bar{y} + B \phi + D |y|^2 \ln|y|^2} \\ \bar{c}_3 - \frac{2 f \bar{y} \ln(|\bar{y}|)}{(-1 + \phi)^4} - \frac{B (a_3 + b_3 \bar{y} + c_3 \phi + \frac{f y \ln(y)}{(-1 + \phi)^2})}{A + C y + B \phi + C \bar{y} + B \phi + D |y|^2 \ln|y|^2} \\ \bar{c}_4 - \frac{B (a_4 + b_4 \bar{y} + c_4 \phi)}{A + C y + B \phi + C \bar{y} + B \phi + D |y|^2 \ln|y|^2} \\ \bar{c}_5 - \frac{B (a_5 + b_5 \bar{y} + c_5 \phi)}{A + C y + B \phi + C \bar{y} + B \phi + D |y|^2 \ln|y|^2} \end{pmatrix} \)

(iv) \( D_y \bar{\Pi} \sim \begin{pmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \tilde{b}_2 \\ \frac{\tilde{b}_3 (-1 + \phi)^2 + \bar{f} (1 + \ln|\bar{y}|)}{(-1 + \phi)^2} \\ \tilde{b}_4 \\ \tilde{b}_5 \end{pmatrix} \)
B.2 \( D_1D_2W \)

We list the second derivatives of the superpotential which would be relevant to the evaluation of the mass matrix in (55):

\[
(i) \quad D_\phi D_\phi W \sim \frac{6 f_4 y \ln(y)}{(-1 + \phi)^3} - \frac{B (c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 - \frac{2 f_4 y \ln(y)}{(-1 + \phi)^3})}{A + C y + B \phi + C \tilde{y} + B \phi} \\
+ \frac{B^2}{(A + C y + B \phi + C \tilde{y} + B \phi)^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \\
- \frac{B}{A + C y + B \phi + C \tilde{y} + B \phi} \left( c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 - \frac{2 f_4 y \ln(y)}{(-1 + \phi)^3} \right) \\
- \frac{B}{A + C y + B \phi + C \tilde{y} + B \phi} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) \right] \\
\]

\[
(ii) \quad D_y D_\phi W \sim \frac{-2 f_4}{(-1 + \phi)^3} \frac{2 f_4 y \ln(y)}{(-1 + \phi)^3} - \frac{B}{A + C y + B \phi + C \tilde{y} + B \phi + D |y|^2 \ln|y|^2} (b_0 f_1 + b_1 f_2 + b_2 f_3 + b_4 f_5 + b_5 f_6 \\
+ f_4 (f + b_3 (-1 + \phi)^2 + f \ln(y))) + \frac{B}{(A + C y + B \phi + C \tilde{y} + B \phi + D |y|^2 \ln|y|^2)^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) \\
+ f_3 (a_2 + b_2 y + c_2 \phi) + f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) (C + D \tilde{y} (1 + \ln|y|^2)) \\
- \frac{1}{A + C y + B \phi + C \tilde{y} + B \phi + D |y|^2 \ln|y|^2} (C + D \tilde{y} (1 + \ln|y|^2)) \left( c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 \\
- \frac{2 f_4 y \ln(y)}{(-1 + \phi)^3} - \frac{B}{A + C y + B \phi + C \tilde{y} + B \phi + D |y|^2 \ln|y|^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) \right] \\
\]

\[
(iii) \quad D_\phi D_y W \sim \frac{2 b_3 f_4}{-1 + \phi} - \frac{2 f_4 (f + b_3 (-1 + \phi)^2 + f \ln(y))}{(-1 + \phi)^3} \\
+ \frac{B}{(A + C y + B \phi + C \tilde{y} + B \phi + D |y|^2 \ln|y|^2)^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) (C + D \tilde{y} (1 + \ln|y|^2)) \\
\]
\[
\begin{align*}
- \frac{1}{(A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} & \left( c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 + \frac{2 f f_4 y \ln(y)}{(-1 + \phi)^3} \right) (C + D \bar{y} (1 + \ln|y|^2)) \\
- \frac{B}{A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} & \left( b_0 f_1 + b_1 f_2 + b_2 f_3 + b_4 f_5 + b_5 f_6 + \frac{f_4 (f + b_3 (-1 + \phi)^2 + f \ln(y))}{(-1 + \phi)^2} \right) \\
- \frac{1}{A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} & \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) (C + D \bar{y} (1 + \ln|y|^2)) \right)
\end{align*}
\]

(i) \[D_y D_y W \sim \frac{f f_4}{y (1 + \phi)^2} - \frac{D}{y (A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} \left[ f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right] (C + D \bar{y} (1 + \ln|y|^2)) \]

\[+ \frac{1}{(A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} \left[ f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right] (C + D \bar{y} (1 + \ln|y|^2)) \]

\[+ \frac{1}{A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} \left( C + D \bar{y} (1 + \ln|y|^2) \right) \left( b_0 f_1 + b_1 f_2 + b_2 f_3 + b_4 f_5 + b_5 f_6 + \frac{f_4 (f + b_3 (-1 + \phi)^2 + f \ln(y))}{(-1 + \phi)^2} \right) \]

\[+ \frac{1}{A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) (C + D \bar{y} (1 + \ln|y|^2)) \]

\[+ \frac{1}{A + C y + B \phi + C \bar{y} + B \bar{\phi} + D |y|^2 \ln|y|^2)^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \\
+ f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) (C + D \bar{y} (1 + \ln|y|^2)) \]

\[B.3 \quad D_i D_j D_k W \]

Because of the length of the expressions involved, we give below one example of a triple covariant derivative of the superpotential - triple derivatives are relevant to the evaluation of the mass matrix in (55):

\[D_y D_y D_y W \sim \frac{6 f f_4}{(1 + \phi)^4} + \frac{6 f f_4 \ln(y)}{(1 + \phi)^4} + \frac{2 B f f_4 (1 + \ln(y))}{(1 + \phi)^3} (A + C y + B \phi + C \bar{y} + B \bar{\phi}) \]

\[+ \frac{B \phi (c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 - \frac{2 f f_4 y \ln(y)}{(1 + \phi)^3})}{(A + C y + B \phi + C \bar{y} + B \bar{\phi})^2} \]

\[+ \frac{2 B f f_4 (1 + \ln(y))}{(A + C y + B \phi + C \bar{y} + B \bar{\phi})} \left( b_0 f_1 + b_1 f_2 + b_2 f_3 + b_4 f_5 + b_5 f_6 + \frac{f_4 (f + b_3 (-1 + \phi)^2 + f \ln(y))}{(-1 + \phi)^2} \right) \]

\[+ \frac{2 B f f_4 (1 + \ln(y))}{(A + C y + B \phi + C \bar{y} + B \bar{\phi})^3} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \right) \]
\[ + f_3 (a_2 + b_2 y + c_2 \phi) + f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \]

\[ \frac{B}{A + C y + B \phi + C \bar{y} + \bar{B} \phi} \left( -2 f f_4 \frac{2 f f_1 \ln(y)}{(-1 + \phi)^3} \right) \frac{B (b_0 f_1 + b_1 f_2 + b_2 f_3 + b_4 f_5 + b_5 f_6 + f_4 (f + b_4 (-1 + \phi)^2 + f \ln(y)))}{A + C y + B \phi + C \bar{y} + \bar{B} \phi} \]

\[ + \frac{BC}{(A + C y + B \phi + C \bar{y} + \bar{B} \phi)^2} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) \right. \]

\[ + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \]

\[ \left. + f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) \]

\[ + \frac{(C + D \bar{y} (1 + \ln|y|^2))}{A + C y + B \phi + C \bar{y} + \bar{B} \phi} \left[ \frac{BC (c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6}{A + C y + B \phi + C \bar{y} + \bar{B} \phi} \left( f_1 (b_0 y + c_0 \phi) + f_2 (a_1 + b_1 y + c_1 \phi) \right. \right. \]

\[ + f_3 (a_2 + b_2 y + c_2 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \]

\[ \left. + f_5 (a_4 + b_4 y + c_4 \phi) + f_6 (a_5 + b_5 y + c_5 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \right) \]

\[ \left[ - \frac{6 f f_4 y \ln(y)}{2 f f_4 y \ln(y)} \frac{B (c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 - 2 f f_4 y \ln(y))}{A + C y + B \phi + C \bar{y} + \bar{B} \phi} \right. \]

\[ + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \]

\[ \left. - (B (c_0 f_1 + c_1 f_2 + c_2 f_3 + c_4 f_5 + c_5 f_6 - 2 f f_4 y \ln(y)) \frac{B}{A + C y + B \phi + C \bar{y} + \bar{B} \phi} \left( f_1 (b_0 y + c_0 \phi) \right. \right. \]

\[ + f_2 (a_1 + b_1 y + c_1 \phi) + f_3 (a_2 + b_2 y + c_2 \phi) + f_4 (a_3 + c_3 \phi + b_3 y + \frac{f y \ln(y)}{(-1 + \phi)^2}) \] \]

C The Complex Structure Moduli Space Metric (Inverse) and Its Derivatives Near the Conifold Locus

We summarize below the forms of the complex structure moduli space metric inverse, and its various relevant (anti)holomorphic derivatives - \( \mathcal{G}_i \equiv a_i + b_i \phi + c_i \phi + f_i y + h_i y \ln(y) + h_i \bar{y} \ln(\bar{y}) + l_i \bar{y} \ln(y) + n_i |y|^2 \ln(|y|^2) \) below:

\[ g^{\bar{i} \bar{j}} \sim \left( \frac{\text{Constant}}{\hat{\mathcal{G}}_1 \ln|y|^2} \right), \]

\[ \partial_{\phi} g^{\bar{i} \bar{j}} \sim \left( \frac{\text{Constant}}{\ln|y|^2} \right), \]

\[ \partial_{y} g^{\bar{i} \bar{j}} \sim \left( \frac{\text{Constant}}{\ln|y|^2} \right), \]
\[ \partial_y g^{ij} \sim \left( \frac{y \ln |y|}{y^2 \ln |y|^2} \frac{G_2}{y (\ln |y|)^2} \right), \]
\[ \partial_y g_{ij} \sim \left( \frac{G_4}{G_6} \frac{G_5}{\ln |y|^2 G_7} \right), \]
\[ \partial_y g_{ij} \sim \left( \frac{\ln |y|^2 G_7}{y G_9} \frac{\ln |y|^2 G_8}{y G_{10}} \right), \]
\[ \partial_\phi \partial_y g^{ij} \sim \left( \text{Constant} \begin{array}{c} 0 \\ 0 \end{array} \right), \]
\[ \partial_\phi \partial_y g^{ij} \sim \left( \text{Constant} \begin{array}{c} \frac{\ln |y|^2}{y^2 (\ln |y|^2)^2} \left( \text{Constant} \right) \\ \frac{\ln |y|^2}{y^2 (\ln |y|^2)^2} \left( \text{Constant} \right) \end{array} \right), \]
\[ \partial_y \partial_\phi g^{ij} \sim \left( \text{Constant} \begin{array}{c} \frac{\ln |y|^2}{y^2 (\ln |y|^2)^3} \left( \text{Constant} \right) \\ \frac{\ln |y|^2}{y^2 (\ln |y|^2)^3} \left( \text{Constant} \right) \end{array} \right), \]
\[ \partial_y \partial_\phi g^{ij} \sim \left( \text{Constant} \begin{array}{c} \frac{\ln |y|^2}{y^2 (\ln |y|^2)^3} \left( \text{Constant} \right) \\ \frac{\ln |y|^2}{y^2 (\ln |y|^2)^3} \left( \text{Constant} \right) \end{array} \right). \]

References

[1] S. Ferrara, R. Kallosh and A. Strominger, N=2 extremal black holes, Phys. Rev. D 52, 5412 (1995)[arXiv:hep-th/9508072]; A. Strominger, Macroscopic Entropy of N = 2 Extremal Black Holes, Phys. Lett. B 383, 39 (1996)[arXiv:hep-th/9602111]; H. Ooguri, A. Strominger and C. Vafa, Black hole attractors and the topological string, Phys. Rev. D 70, 106007 (2004)[arXiv:hep-th/0405146].

[2] S. Ferrara, G. W. Gibbons and R. Kallosh, Black holes and critical points in moduli space, Nucl. Phys. B 500, 75 (1997)[arXiv:hep-th/9702103].

[3] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, Non-supersymmetric attractors, Phys. Rev. D 72, 124021 (2005)[arXiv:hep-th/0507096]; A. Sen, Black hole entropy function and the attractor mechanism in higher derivative gravity, JHEP 0509, 038 (2005)[arXiv:hep-th/0506177]; B. Sahoo and A. Sen, Higher derivative corrections to non-supersymmetric extremal black holes in N = 2 supergravity,[arXiv:hep-th/0603149].

[4] R. Kallosh, New attractors, JHEP 0512, 022 (2005)[arXiv:hep-th/0510024].

[5] S. Ferrara, M. Bodner and A. C. Cadavid, Calabi-Yau supermoduli space, field strength duality and mirror manifolds, Phys. Lett. B 247, 25 (1990).
[6] M. Alishahiha and H. Ebrahim, *New attractor, entropy function and black hole partition function*, arXiv:hep-th/0605279.

[7] A. Giryavets, *New attractors and area codes*, JHEP 0603, 020 (2006)[arXiv:hep-th/0511215].

[8] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair Of Calabi-Yau Manifolds As An Exactly Soluble Superconformal Theory*, Nucl. Phys. B 359, 21 (1991); P. Candelas, X. De La Ossa, A. Font, S. Katz and D. R. Morrison, *Mirror symmetry for two parameter models. I*, Nucl. Phys. B 416, 481 (1994)[arXiv:hep-th/9308083].

[9] A. Giryavets, S. Kachru, P. K. Tripathy and S. P. Trivedi, *Flux compactifications on Calabi-Yau three-folds*, JHEP 0404, 003 (2004)[arXiv:hep-th/0312104].

[10] A. Misra and A. Nanda, *Flux vacua statistics for two-parameter Calabi-Yau’s*, Fortsch. Phys. 53, 246 (2005) [arXiv:hep-th/0407252].

[11] P. K. Tripathy and S. P. Trivedi, *Non-supersymmetric attractors in string theory*, JHEP 0603, 022 (2006) [arXiv:hep-th/0511117].

[12] J. Gray, Y. H. He, V. Jejjala and B. D. Nelson, *Exploring the vacuum geometry of N = 1 gauge theories*, [arXiv:hep-th/0604208].

[13] A. Chou, R. Kallosh, J. Rahmfeld, S. J. Rey, M. Shmakova and W. K. Wong, *Critical points and phase transitions in 5d compactifications of M-theory*, Nucl. Phys. B 508 (1997) 147 [arXiv:hep-th/9704142].

[14] G. W. Moore, *Arithmetic and attractors*, arXiv:hep-th/9807087.

[15] G. W. Moore, *Les Houches lectures on strings and arithmetic*, arXiv:hep-th/0401049.

[16] J. Gray, Y. H. He and A. Lukas, *Algorithmic algebraic geometry and flux vacua*, arXiv:hep-th/0606122.