Higher Secant Varieties of Segre-Veronese varieties.

M.V. Catalisano, A.V. Geramita, A. Gimigliano.

0. Introduction.

The problem of determining the dimensions of the higher secant varieties of the classically studied projective varieties (and to describe the defective ones) is a problem with a long and interesting history.

In the case of the Segre varieties there is much interest in this question, and not only among geometers. In fact, this particular problem is strongly connected to questions in representation theory, coding theory, algebraic complexity theory (see our paper [CCG2] for some recent results as well as a summary of known results, and also [BCS]) and, surprisingly enough, also in algebraic statistics (e.g. see [GHKM] and [GSS]).

We address a slight generalization of this problem here; more precisely we will study the higher secant varieties of

$$X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_t} = \mathbb{P}^N, \ n = (n_1, \ldots, n_t)$$

embedded in the projective space \( \mathbb{P}^N \) (\( N = \Pi (n_i + n) - 1 \)) by the morphism \( \nu_{n,a} \) given by \( O_{\mathbb{P}^N}(a) \), where \( a = (a_1, \ldots, a_t) \) (a_i positive integers). We denote the embedded variety \( \nu_{n,a}(X) \) by \( V_{n,a} \), and call it a Segre-Veronese variety and the embedding a Segre-Veronese embedding (see [BuM]).

In Section 1 we recall some classical results by Terracini regarding secant varieties and we also introduce one of the fundamental observations ([Theorem 1.1]) which allows us to convert certain questions about ideals of varieties in multiprojective space to questions about ideals in standard polynomial rings.

In Section 2 we concentrate on \( t = 2, 3 \) and let \( a \) be arbitrary. In [Theorem 2.1] we give the dimensions of all the higher secant varieties for \( V_{n,a} \) (where \( n = (1, 1) \), and \( a \) is arbitrary). For \( n = (k, n), \ a = (1, k + 1) \) we find that \( V_{n,a} \) has no deficient higher secant varieties ([Proposition 2.3]) and this gives an interesting conclusion about the Grassmann defectivity for the \( (k + 1) \)-Veronese embedding of \( \mathbb{P}^n \) (\( k \geq 2, \ n \geq 2 \)). We also state our theorem (the proof will appear elsewhere) which gives the dimensions of all the higher secant varieties to \( V_{(1,1,1),(a,b,c)} \) for any positive integers \( a, b \) and \( c \) ([Theorem 2.5]).

Section 3 is dedicated to results on regularity of secant varieties of Segre Veronese varieties which can be deduced from results for the Segre varieties and by studying the multigraded Hilbert function of a scheme of 2-fat points in \( \mathbb{P}^n \). Also we give several examples of defective and Grassmann defective Segre-Veronese varieties.

Finally, in Section 4 we describe a way of thinking about the points of Segre-Veronese varieties as (partially symmetric) tensors.

Our method is essentially this (see §1): we use Terracini’s Lemma (as in [CGG2], [CGG4]) to translate the problem of determining the dimensions of higher secant varieties into that of calculating the value, at \( (a_1, \ldots, a_t) \), of the Hilbert function of generic sets of 2-fat points in \( \mathbb{P}^n \). Then we show, by passing to an affine chart in \( \mathbb{P}^n \) and then homogenizing in order to pass to \( \mathbb{P}^n, n = n_1 + \ldots + n_t \), that this last calculation amounts to computing...
the Hilbert function of a very particular subscheme of $\mathbb{P}^n$. Finally, we study the postulation of these special subschemes of $\mathbb{P}^n$.

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1. Preliminaries, the Multiprojective-Affine-Projective Method.

Let us recall the notion of higher secant varieties.

**Definition 1:** Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety of dimension $n$. The $s$th higher secant variety of $X$, denoted $X^s$, is the closure of the union of all linear spaces spanned by $s$ independent points of $X$.

Recall that, for $X$ as above, there is an inequality involving the dimension of $X^s$. Namely,

$$\dim X^s \leq \min\{N, sn + s - 1\}$$  (1)

and one “expects” the inequality should, in general, be an equality.

When $X^s$ does not have the expected dimension, $X$ is said to be $(s-1)$-defective, and the positive integer

$$\delta_{s-1}(X) := \min\{N, sn + s - 1\} - \dim X^s$$

is called the $(s-1)$-defect of $X$. Probably the most well known defective variety is the Veronese surface, $X$ in $\mathbb{P}^5$ for which $\delta_1(X) = 1$.

As a generalization of the higher secant varieties of a variety, one can also consider the following varieties.

**Definition 2:** Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety of dimension $n$. The $(k,s-1)$-Grassmann secant variety, denoted $\text{Sec}_{k,s-1}(X)$, is the Zariski closure (in the Grassmaniann of $k$-dimensional linear subspaces of $\mathbb{P}^N$) of the set

$$\{l \in G(k, N) \mid l \text{ lies in the span of } s \text{ independent points of } X\}.$$

In case $k = 0$ we get $X^s = \text{Sec}_{0,s-1}(X)$.

As a generalization of the analogous result for the higher secant varieties, one always has

$$\dim \text{Sec}_{k,s-1}(X) \leq \min\{sn + (k + 1)(s - k - 1), (k + 1)(N - k)\},$$

with equality being what is generally “expected”.

When $\text{Sec}_{k,s-1}(X)$ does not have the expected dimension then we say that $X$ is $(k,s-1)$-defective and in this case we define the $(k,s-1)$-defect of $X$ as the number:

$$\delta_{k,s-1}(X) = \min\{sn + (k + 1)(s - k - 1), (k + 1)(N - k)\} - \dim \text{Sec}_{k,s-1}(X).$$

(For general information on these defectivities see [ChCo] and [DF].)

In this note we study the defectivities of $V_{n,a}$. For convenience, and when no doubts can arise about the variety we are considering, we will just write $V$ for $V_{n,a}$.

In his paper [Te2], Terracini gives a link between these two kinds of defectivity for a variety $X$ as above (see [DF] for a modern proof):
**Proposition 1.0:** (Terracini) Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension $n$. Let $\sigma : X \times \mathbb{P}^k \to \mathbb{P}^{(k+1)(N+1)-1}$ be the (usual) Segre embedding. Then $X$ is $(k, s - 1)$-defective with defect $\delta_{k,s-1}(X) = \delta$ if and only if $\sigma(X \times \mathbb{P}^k)$ is $(s - 1)$-defective with $(s - 1)$-defect $\delta_{s-1}(X \times \mathbb{P}^k) = \delta$.

A classical result about higher secant varieties is Terracini’s Lemma (see [Te], [CGG2]):

**Terracini’s Lemma:** Let $(X, \mathcal{L})$ be a polarized, integral, non-singular scheme; if $\mathcal{L}$ embeds $X$ into $\mathbb{P}^N$, then:

$$T_P(X^s) = \langle T_{P_1}(X), ..., T_{P_s}(X) \rangle,$$

where $P_1, ..., P_s$ are $s$ generic points on $X$, and $P$ is a generic point of $\langle P_1, ..., P_s \rangle$ (the linear span of $P_1, \ldots, P_s$); here $T_P(X)$ is the projectivized tangent space of $X$ in $\mathbb{P}^N$.

Let $Z \subset X$ be a scheme of $s$ generic 2-fat points, that is a scheme defined by the ideal sheaf $\mathcal{I}_Z = \mathcal{I}_{P_1} \cap \cdots \cap \mathcal{I}_{P_s} \subset \mathcal{O}_X$, where $P_1, ..., P_s$ are $s$ generic points. Since there is a bijection between hyperplanes of the space $\mathbb{P}^N$ containing the subspace $\langle T_{P_1}(X), ..., T_{P_s}(X) \rangle$ and the elements of $H^0(X, \mathcal{I}_Z)\langle \mathcal{L} \rangle$, we have:

**Corollary:** Let $X, \mathcal{L}, Z$, be as above; then

$$\dim X^s = \dim \langle T_{P_1}(X), ..., T_{P_s}(X) \rangle = N - \dim H^0(X, \mathcal{I}_Z)\langle \mathcal{L} \rangle.$$

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ and let $V_{n,a} = V \subset \mathbb{P}^N$ be the embedding of $X$ given by $\mathcal{L} = \mathcal{O}_X(a_1, \ldots, a_t)$. By applying the corollary above to our case (i.e. $V = V_{n,a}$), we get

$$\dim V^s = H(Z, a) - 1,$$

where $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is a set of $s$ generic 2-fat points, and where $\forall j \in \mathbb{N}^t$, $H(Z, j)$ is the Hilbert function of $Z$ i.e.

$$H(Z, j) = \dim R_j - \dim H^0(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}, \mathcal{I}_Z(j)),$$

where $R = k[x_{0,1}, ..., x_{n_1,1}, ..., x_{0,t}, ..., x_{n_t,t}]$ is the multi-graded homogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$.

Now let $n = n_1 + \cdots + n_t$ and consider the birational map

$$g : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \dashrightarrow \mathbb{A}^n,$$

where:

$$((x_{0,1}, ..., x_{n_1,1}), ..., (x_{0,t}, ..., x_{n_t,t})) \mapsto \left(\frac{x_{1,1}}{x_{0,1}}, \frac{x_{2,1}}{x_{0,1}}, ..., \frac{x_{n_1,1}}{x_{0,1}}, \frac{x_{1,2}}{x_{0,2}}, ..., \frac{x_{n_1,2}}{x_{0,2}}, ..., \frac{x_{1,t}}{x_{0,t}}, ..., \frac{x_{n_t,t}}{x_{0,t}}\right).$$

This map is defined in the open subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ given by $\{x_{0,1}x_{0,2}...x_{0,t} \neq 0\}$.

Let $S = k[z_{0}, z_{1,1}, ..., z_{n_1,1}, z_{1,2}, ..., z_{n_2,2}, ..., z_{1,t}, ..., z_{n_t,t}]$ be the coordinate ring of $\mathbb{P}^n$ and consider the embedding $\mathbb{A}^n \to \mathbb{P}^n$, whose image is the chart $\mathbb{A}^n_0 = \{z_0 = 1\}$. By composing the two maps above we get:

$$f : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \dashrightarrow \mathbb{P}^n,$$

with

$$((x_{0,1}, ..., x_{n_1,1}), ..., (x_{0,t}, ..., x_{n_t,t})) \mapsto \left(\frac{1}{x_{0,1}}, \frac{x_{1,1}}{x_{0,1}}, ..., \frac{x_{1,2}}{x_{0,2}}, ..., \frac{x_{1,t}}{x_{0,t}}, ..., \frac{x_{n_1,1}}{x_{0,1}}, ..., \frac{x_{n_t,t}}{x_{0,t}}\right).$$
\[(x_{1,1}x_{2,1} \cdots x_{0,t}, x_{1,1}x_{2,1} \cdots x_{0,t}, x_{1,1}x_{2,1} \cdots x_{0,t}, \ldots, x_{0,1} \cdots x_{0,t} - x_{n,t},)\]

Let \(Z \subset \mathbb{P}^n \times \cdots \times \mathbb{P}^n\) be a zero-dimensional scheme which is contained in the affine chart \(\{x_{0,1}x_{0,2} \cdots x_{0,t} \neq 0\}\) and let \(Z' = f(Z)\). We want to construct a scheme \(W \subset \mathbb{P}^n\) such that \(\dim(I_W)_a = \dim(I_Z)_{(a_1, \ldots, a_t)}\), where \(a = a_1 + \ldots + a_t\).

Let us recall that the coordinate ring of \(\mathbb{P}^n\) is \(S = k[z_0, z_{1,1}, \ldots, z_{n,1}, \ldots, z_{1,2}, \ldots, z_{n,2}, \ldots, \ldots, z_{1,t}, \ldots, z_{n,t}]\), and let \(Q_0, Q_{1,1}, \ldots, Q_{n,1}, Q_{1,2}, \ldots, Q_{n,2}, \ldots, Q_{n,t}\) be the coordinate points of \(\mathbb{P}^n\). Consider the linear subspace \(\Pi_i \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n\), where \(\Pi_i = \langle Q_{1,i}, \ldots, Q_{n,i} \rangle\). The defining ideal of \(\Pi_i\) is:

\[I_{\Pi_i} = (z_0, z_{1,1}, \ldots, z_{n,1}; \ldots, \hat{z}_{1,i}, \ldots, \hat{z}_{n,i}; \ldots; z_{1,t}, \ldots, z_{n,t}).\]

Let \(W_i\) be the subscheme of \(\mathbb{P}^n\) denoted by \((a - a_i)\Pi_i\), i.e. the scheme defined by the ideal \(I_{\Pi_i}^{a-a_i}\). Notice that \(W_i \cap W_j = \emptyset\) for \(i \neq j\).

**Theorem 1.1:** Let \(Z, Z'\) be as above and let \(W = Z' + W_1 + \ldots + W_t \subset \mathbb{P}^n\). Then we have:

\[\dim(I_W)_a = \dim(I_Z)_{(a_1, \ldots, a_t)}\]

where \(a = a_1 + \ldots + a_t\).

**Proof:** First note that

\[R_{(a_1, \ldots, a_t)} = \langle (x_{0,1}z_{1,1} - s_{1,1}N_1)(x_{0,2}z_{2,1} - s_{2,1}N_2) \cdots (x_{0,t}z_{n,1} - s_{n,1}N_t)\rangle\]

where every \(N_i\) varies among all monomials in \((x_{1,i}, \ldots, x_{n,i})_{s_i}\), for all \(s_i \leq a_i\).

By dehomogenizing (via \(f\) above) and then substituting \(z_{i,j}\) for \((x_{i,j}/x_{0,j})\), and finally homogenizing with respect to \(z_0\), we see that

\[R_{(a_1, \ldots, a_t)} \cong \langle z_0^{a-s_{1,1}} \cdots z_0^{a-s_{n,1}}M_1M_2 \cdots M_t \rangle\]

where every \(M_i\) varies among all monomials in \((z_{1,i}, \ldots, z_{n,i})_{s_i}\).

**Claim:**

\[(I_{W_1} + \ldots + I_{W_t})_a = (I_{W_1} \cap \ldots \cap I_{W_t})_a = \langle z_0^{a-s_{1,1}} \cdots z_0^{a-s_{n,1}}M_1M_2 \cdots M_t \rangle\]

where every \(M_i\) varies among all monomials in \((z_{1,i}, \ldots, z_{n,i})_{s_i}\), for all \(s_i \leq a_i\).

**Proof of Claim:** \(\subseteq\): Since both vector spaces are generated by monomials, it is enough to show that the monomials of the left hand side of the equality are contained in the right hand side of the equality.

Consider \(M = z_0^{a-s_{1,1}} \cdots z_0^{a-s_{n,1}}M_1M_2 \cdots M_t\) (as above). We now show that this monomial is in \(I_{W_i}\) (for each \(i\)). Notice that \(M_j \in I_{\Pi_i}^{s_{j,1}}\) (for \(j \neq i\)) and that \(z_0^{a-s_{1,1}} \cdots z_0^{a-s_{n,1}} \in I_{\Pi_i}^{(a-s_{1,1}) + \ldots + (a-s_{n,1}) + s_{j,1} + \ldots + s_{j,1}} = I_{\Pi_i}^{a-s_{j,1}}\). Since \(s_i \leq a_i\) we have \(a - a_i \leq a - s_i\) and so \(M \in I_{\Pi_i}^{a-a_i}\) as well, and that is what we wanted to show.

\(\supseteq\): To prove this inclusion, consider an arbitrary monomial \(M \in S_a\). Such an \(M\) can be written \(M = z_0^{a_1}M_1 \cdots M_t\) where \(M_i \in (z_{1,i}, \ldots, z_{n,i})\) is a monomial of degree \(a_i\).
Now, $M \in (I_{W_1} + \ldots + I_{W_t})_a$ means $M \in (I_{W_i})_a$ for each $i$, hence
\[
\alpha_0 + \alpha_1 + \cdots + \alpha_t + a_i \geq a - a_i
\]
for $i = 1, \ldots, t$. Since $\alpha_0 + \alpha_1 + \cdots + \alpha_t = a$, then $a - a_i \geq a - a_i$ for each $i$, and so $\alpha_i \leq a_i$ for each $i$. That finishes the proof of the Claim.

Now, since $Z$ and $Z'$ are isomorphic ($f$ is an isomorphism between the two affine charts $\{z_0 \neq q0\}$ and $\{x_0, 1x_{0, 2} \ldots x_{0, t} \neq 0\}$, it immediately follows (via the two different dehomogeneizations) that $(I_Z)_{(a_1, \ldots, a_t)} \equiv (I_W)_a$.

\[\square\]

When $Z$ is given by $s$ generic 2-fat points, we have the obvious corollary:

**Corollary 1.2:** Let $Z \subset \mathbb{P}^n \times \ldots \times \mathbb{P}^n$ be a generic set of $s$ 2-fat points, let $W \subset \mathbb{P}^n$ be as in Theorem 1.1, then we have:
\[
\dim V^s = H(Z, (a_1, \ldots, a_t)) - 1 = N - \dim(I_W)_a.
\]

2. **On Segre-Veronese with two or three factors.**

First we consider the case $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $t = 2$, $n_1 = n_2 = 1$, for all $a = (a_1, a_2) \in \mathbb{N}^2$. We get that the $\Pi_i$’s are points; let $\Pi_1 = A_1 = (0, 1, 0)$, $\Pi_2 = A = A - 2 = (0, 0, 1)$, in $\mathbb{P}^2$, and let $Z = 2R_1 + \ldots + 2R_s \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a set of $s$ generic 2-fat points. We may assume that $R_i = ((1, \alpha), (1, \beta))$, so that $f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ is such that:
\[
f(R_i) = P_i = (1, \alpha_i, \beta_i) \in \mathbb{P}^2,
\]
and $A_1, A_2, P_1, \ldots, P_s$ will be generic points of $\mathbb{P}^2$. Let
\[
W = a_2A_1 + a_1A_2 + 2P_1 + \ldots + 2P_s \subset \mathbb{P}^2
\]
be the scheme defined by the ideal sheaf $I_W = \mathcal{I}_{A_1} \cap \mathcal{I}_{A_2} \cap \mathcal{I}_{P_1} \cap \ldots \cap \mathcal{I}_{P_s}$. By Theorem 1.1, the data of the Hilbert function of $Z$ in bidegree $(a_1, a_2)$ is equivalent to the data of the Hilbert function of $W$ in degree $(a_1 + a_2)$, in fact, from $(I_W)_{a_1 + a_2} \equiv (I_Z)_{(a_1, a_2)}$, we easily get:
\[
H(W, (a_1 + a_2)) = H(Z, (a_1, a_2)) + \deg(a_2A_1 + a_1A_2) = H(Z, (a_1, a_2)) + \left(\frac{a_1 + 1}{2}\right) + \left(\frac{a_2 + 1}{2}\right)
\]
since
\[
\dim(I_Z)_{(a_1, a_2)} = (a_1 + 1)(a_2 + 1) - H(Z, (a_1, a_2))
\]
\[
\dim(I_W)_{(a_1 + a_2)} = \left(\frac{a_1 + a_2 + 2}{2}\right) - H(W, (a_1, a_2)).
\]
Now let $V = V_{1,a} = \nu_{1,a}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{a_1a_2 + a_1 + a_2}$ be the Segre-Veronese embedding; by Corollary 1.3 we get:
\[
\dim V^s = H(Z, (a_1, a_2)) - 1 = a_1a_2 + a_1 + a_2 - \dim(I_W)_{(a_1 + a_2)} =
\]
\[
= H(W, (a_1 + a_2)) - 1 - \left(\frac{a_1 + 1}{2}\right) - \left(\frac{a_2 + 1}{2}\right)
\]
Hence, in order to compute \( \dim V^s \), we should study a scheme of generic fat points \( W \) in \( \mathbb{P}^2 \); without loss of generality, we may suppose \( a_1 \geq a_2 \).

**Theorem 2.1:** Let \( V = V_{1,a} = \nu_{1,a}(\mathbb{P}^1 \times \mathbb{P}^1) \), then \( V^s \) has the expected dimension, except for

\[
    a_1 = 2d, \quad a_2 = 2, \quad d \geq 1, \quad \text{and} \quad s = a_2 + 1
\]

\((\dagger)\)

In this case \( V^s \) is defective, and \( \dim V^s = 3s - 2 \) (its defectiveness is 1).

This theorem could be proved by methods similar to those used in \([\text{CGG}3],[\text{CGG}4] \), but we omit the lengthy and tedious proof here. Notice that in \([\text{ChCi}]\) a total classification of all the surfaces with some defective secant variety can be found and in \([\text{La}]\) the case of rational scroll is treated, which covers also the \( \mathbb{P}^1 \times \mathbb{P}^1 \) case.

We wish to thank Monica Idà for showing us a direct proof of Theorem 2.1 (\([\text{Id}]\)) which uses the "differential Horace" method.

Theorem 2.1 yields (by interpreting things via Theorem 1.1) that for fat points in \( \mathbb{P}^2 \), we have:

**Remark 2.2:** Let \( a_1, a_2, s \) be positive integers, with \( a_1 \geq a_2 \). Let \( W = a_2 A_1 + a_1 A_2 + 2P_1 + \ldots + 2P_s \subset \mathbb{P}^2 \). Then

\[
    H(W, a_1 + a_2) = \min \{ (a_1 + a_2 + 2)/2, (a_1 + 1)/2 + (a_2 + 1)/2 + 3s \},
\]

\( (*) \)

except when \( a_1 = 2d, a_2 = 2 \) and \( s = 2d + 1 \); in this case \( H(W, a_1 + a_2) \) is 1 less than expected.

Notice that in the exceptional case it is easy to check what is the geometrical situation: there is an unique (rational) curve \( C \) through \( dA_1 + A_2 + P_1 + \ldots + P_{2d+1} \), and \( 2C \) gives an “unexpected element” of \( (I_W)_{2d+2} \).

Now let us consider another case; namely the products \( \mathbb{P}^r \times \mathbb{P}^k \); we have the following:

**Proposition 2.3:** Let \( r, k \geq 1 \), and \( V \subset \mathbb{P}^N \) be the \((k + 1, 1)\) Segre-Veronese embedding of \( \mathbb{P}^r \times \mathbb{P}^k \). Then for any \( s \geq 1 \), \( V^s \) has the expected dimension.

**Proof:** Notice that, since \( N = \binom{r+k+1}{r}(k+1) - 1 \), for \( s = \binom{r+k}{k} \) we get that the expected dimension of \( V^s \) is exactly \( N \), hence the statement will follow if we prove that \( V^{\binom{r+k+1}{r}} = \mathbb{P}^N \).

Consider the scheme \( W = \Pi_1 + (k+1)\Pi_2 + 2P_1 + \ldots + 2P_s \subset \mathbb{P}^{r+k} \), where \( \Pi_1 \cong \mathbb{P}^{r-1} \) and \( \Pi_2 \cong \mathbb{P}^{k-1} \) are linear spaces and \( s = \binom{r+k}{k} \); then, by Corollary 1.2, we get that

\[
    \dim V^s = N - h^0(\mathbb{P}^{r+k}, I_W(k + 2)).
\]

From what we have seen before, we will be done if \( \dim(I_W)_{k+2} = 0 \).

We will proceed by double induction on \( k \) and \( r \).

Let us consider the case \( k = 1 \) (any \( r \)) first. When \( k = 1 \), \( W \) is the schematic union: \( W = \Pi_1 + 2\Pi_2 + 2P_1 + \ldots + 2P_s \subset \mathbb{P}^{r+1} \); where \( \Pi_2 \) is a point. It is enough, in order to prove the case \( k = 1 \), to show that \( (I_W)_3 = \{0\} \) (here \( s = r + 1 \)).

Let us work by induction on \( r \); for \( r = 1 \) we trivially have \( h^0(\mathbb{P}^2, I_W(3)) = 0 \) (since \( W \) is made of three 2-fat points and one simple point). When \( r > 1 \), consider the exact sequence:

\[
    0 \to I_W(2) \to I_W(3) \to I_{W \cap H}(3) \to 0
\]
where \( H \subset \mathbb{P}^{r+1} \) is the hyperplane \( H = \langle P_1, ..., P_{r+1} \rangle \) and \( W' = \Pi_1 + 2\Pi_2 + P_3 + ... + P_{r+1} \). We get \( h^0(H, I_{W \cap H, H}(3)) = 0 \) by induction, and \( h^0(\mathbb{P}^{r+1}, I_W(2)) = 0 \) since any of its element should give a quadric cone with vertex in \( \Pi_2 \), but, since \( \Pi_1 \cong \mathbb{P}^{r-1} \), the cone should split into the hyperplane \( \langle \Pi_1, \Pi_2 \rangle \), and another hyperplane containing \( \Pi_2, P_3, ..., P_{r+1} \), which is impossible by their genericity.

Hence the case \( k = 1 \) is done.

Now let us consider the case \( r = 1 \); here we have \( \mathbb{P}^1 \times \mathbb{P}^k \to \mathbb{P}^N \), and \( W = \Pi_1 + (k+1)\Pi_2 + 2P_3 + ... + 2P_{r+1} \subset \mathbb{P}^{k+1} \), where \( \Pi_1 \) is a point and we must show that \( (I_W)_{k+2} = \{0\} \).

The hyperplanes \( H_i = \langle P_i, \Pi_2 \rangle, i = 1, ..., k \), are fixed components for the hypersurfaces given by the forms in \( (I_W)_{k+2} \). Hence (by removing such fixed components), we get

\[
\dim(I_W)_{k+2} = \dim(I_{\Pi_1 + P_3 + ... + P_{r+1}}) = 0.
\]

So case \( r = 1 \) is done.

Now we can consider \( r, k \geq 2 \) and work by double induction on them.

Let \( H \subset \mathbb{P}^{r+k} \) be a hyperplane such that \( \Pi_2 \subset H \) and \( \Pi_1 \) is not contained in \( H \). Let \( \Pi_1' = H \cap \Pi_1 \cong \mathbb{P}^{r-2} \), specialize \( P_1, ..., P_{s'}, s' = \binom{r+k-1}{k} \) on \( H \) and consider the exact sequence:

\[
0 \to I_Z(k+1) \to I_W(k+2) \to I_{W \cap H, H}(k+2) \to 0,
\]

where \( Z = \Pi_1 + k\Pi_2 + P_3 + ... + P_{s'} + 2P_{s'+1} + ... + 2P_s \subset \mathbb{P}^{r+k} \) and \( W \cap H = \Pi_1' + (k+1)\Pi_2 + 2P_3 + ... + 2P_{s'}|_H \).

We have \( h^0(I_{W \cap H, H}(k+2)) = 0 \) by induction on \( r \); so we will be done if \( h^0(I_Z(k+1)) = 0 \), since the above sequence would yield \( h^0(I_W(k+2)) = 0 \).

Let us consider now a hyperplane \( H' \subset \mathbb{P}^{r+k} \) with \( \Pi_1 \subset H' \) and \( \Pi_2 \) not contained in \( H' \); let \( \Pi_1'' = H' \cap \Pi_2 \cong \mathbb{P}^{k-2} \), then specialize \( P_{s'+1}, ..., P_s \) on \( H' \) and consider the exact sequence:

\[
0 \to I_{Z'}(k) \to I_Z(k+1) \to I_{Z' \cap H', H'}(k+1) \to 0,
\]

where \( Z' = k\Pi_2 + P_3 + ... + P_s \subset \mathbb{P}^{r+k} \) and \( Z' \cap H' = \Pi_1' + k\Pi_2' + 2P_{s+1} + ... + 2P_s|_{H'} \). Notice that \( s - s' = \binom{r+k}{k} - \binom{r-1+k}{k-1} = \binom{r+k-1}{k-1} \), so that \( h^0(I_{Z' \cap H', H'}(k+1)) = 0 \) by induction on \( r \) and \( k \).

So we are only left to prove \( h^0(I_Z(k)) = 0 \). The sections of \( I_Z(k) \) correspond to degree \( k \) hypersurfaces in \( \mathbb{P}^{r+k} \) which, in order to contain \( k\Pi_2 \) have to be cones with \( \Pi_2 \) as vertex. Let \( H'' \cong \mathbb{P}^r \) be a generic \( r \)-dimensional linear subspace of \( \mathbb{P}^{r+k} \); then \( h^0(I_{Z'}(k)) = h^0(I_{Z'' \cap H'', H''}(k)) \), where \( Z'' \subset H'' \) is the projection of \( Z' \) into \( H'' \) from \( \Pi_2 \). We have \( Z'' = Q_1 + ... + Q_{s'} + Q_{s+1} + ... + Q_s \), where \( Q_{s+1}, ..., Q_s \) are generic in \( H'', \) while \( Q_1, ..., Q_{s'} \) are contained in the linear space \( H \cap H'' \cong \mathbb{P}^{r-1} \) (where they are generic). A hypersurface \( F \) of degree \( k \) in \( H'' \) cannot contain \( Q_1, ..., Q_{s'} \) without containing all \( H \cap H'' \) (\( F \) intersects \( H \cap H'' \) in something of degree \( k \), but the \( Q_i, i = 1, ..., s' = \binom{r+k-1}{k-1} \) are generic in \( H \cap H'' \), so they are not contained in a hypersurface of degree \( k \) of \( \mathbb{P}^{r-1} = H \cap H'' \)). So, if \( F \) vanishes on \( Q_1, ..., Q_s \), we have that \( F \) is the union of \( H \cap H'' \) and a hypersurface \( F' \) of degree \( k-1 \) vanishing on \( Q_{s+1}, ..., Q_s \), but again this cannot happen because they are \( \binom{r+k-1}{k-1} \) generic points in \( \mathbb{P}^r \), so no form of degree \( k-1 \) vanishes at them. Thus \( h^0(I_{Z'' \cap H'', H''}(k)) = 0 \) and we are done.

\[\square\]

From this result we get immediately the following:

**Corollary 2.4:** Let \( r, k \geq 2 \), and \( V \) be the \((k+1)\)-ple (Veronese) embedding of \( \mathbb{P}^r \). Then \( V \) is not (Grassman) \((k, s-1)\)-defective, for any \( s \).
Proof: By Proposition 1.0, this statement is equivalent to Proposition 2.3.

In the case $t = 3$, i.e. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{P}^1)^3$, the situation for all Segre-Veronese embeddings can be analyzed with the same methods used above; if we consider the embedding $V_a$ of $(\mathbb{P}^1)^3$ given by the forms of tridegree $a = (a_1, a_2, a_3)$, in order to compute the dimension of $V_a^s$ we will have to study (by Theorem 1.1) the scheme of fat points

$$W_s = (a_2 + a_3)A_1 + (a_1 + a_3)A_2 + (a_1 + a_2)A_3 + 2P_1 + ... + 2P_s \subseteq \mathbb{P}^3,$$

where $A_1, A_2, A_3$ are coordinate points.

A complete description of what happens is given by the following result:

**Theorem 2.5:** Let $a_1 \geq a_2 \geq a_3 \geq 1$, $\alpha \in \mathbb{N}$ and $V = V_a$ be a Segre-Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $V_s$ has the expected dimension, except for:

$$(a_1, a_2, a_3) = (2, 2, 2), \quad \text{and} \quad s = 7;$$

$$(a_1, a_2, a_3) = (2\alpha, 1, 1), \quad \text{and} \quad s = 2\alpha + 1.$$

In these cases $V_s$ is defective, and its defectivity is 2 in the first case and 1 in the second.

The proof of the Theorem uses the same kind of procedures as Proposition 2.3, but it also makes use of the *Horace differential method* (see [AH]), and its proof is quite long since a lot of different cases have to be considered. A complete proof can be found in [CGG3].

**Corollary 2.6:** Let $a_1, a_2 \geq 1$, and $V = V_{a_1, a_2}$ be the Segre-Veronese $(a_1, a_2)$-embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $V$ is not Grassman defective, except when $(a_1, a_2) = (2\alpha, 1)$, and in this case $V$ is $(1, 2\alpha)$-defective.

Proof: By Proposition 1.0, this statement is equivalent to Theorem 2.5.

\[\square\]

3. Other results.

The correspondence between the dimension of $V_{n,a}$ and the Hilbert function of a scheme $Z$ made of $s$ generic 2-fat points in $\mathbb{P}^n$ (see Corollary 1.2) allows us to deduce results on $\dim V_{n,a}^s$ from previous results on Segre Varieties and from properties of Hilbert functions. Namely we have (notations as in §1):

**Proposition 3.1:** Let $V_{n,a}$ be a Segre-Veronese variety and $s \geq 2$ be such that $\dim V_{n,a}^s = ns + s - 1$ (the expected dimension). Then also $V_{n,b}^s$ has the (same) expected dimension for any $b$ such that $b_i \geq a_i \ \forall i = 1, ..., t$.

Proof: This is an immediate consequence of Corollary 1.2, since our hypothesis on $V_{n,a}^s$ amounts to say that for a generic scheme $Z \subseteq \mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_t}$ made of $s$ 2-fat points, $H(Z, a) = s(n + 1) = \text{length } Z$ and this trivially implies that $H(Z, b) = s(n + 1) = \text{length } Z$ for all $b$ such that $b_i \geq a_i \ \forall i = 1, ..., t$.

From this follows:
Proposition 3.2: Let $1 \leq n_1 \leq n_2 \leq \ldots \leq n_t$ and $V = V_{n,a}$ be the Segre-Veronese embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_t}$. If we are not in the case $t = 2$, $a = (1, 1)$, then $\dim V^s = s(n_1 + \ldots + n_t + 1) - 1$ for all $s \leq n_1 + 1$.

Proof: If $t \geq 3$, from [CGG2], Proposition 2.3, we have that $H(Z, 1) = s(n_1 + \ldots + n_t + 1) = length Z$, so we can conclude by Proposition 3.1.

When $t = 2$, we know that for $a = (1, 1)$, $V^s$ is defective for all $2 \leq s \leq n_1$ (e.g. see [CGG2], Proposition 2.3 again). We will be done if we show that (again by Proposition 3.1), $V_{1,2}^{n_1+1}$ and $V_{2,1}^{n_1+1}$ are not defective.

Without loss of generality, we can consider $Z$ such that $\text{Supp } Z = \{P_0, \ldots, P_{n_1}\} \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, where $P_i$ is the coordinate point associated to the bihomogeneous ideal $p_i = (x_0, \ldots, x_{n_1}, y_0, \ldots, y_{n_2})$ in the ring $k[x_0, \ldots, x_{n_1}; y_0, \ldots, y_{n_2}]$. Let $I = p_0^2 \cap \ldots \cap p_{n_1}^2$ be the ideal associated to $Z$; since $I$ is a monomial ideal (for the $P_i$'s are coordinate points), we only need to show that the monomials not in $I_{1,2}$ and those not in $I_{2,1}$ are $(n_1 + 1)(n_1 + n_2 + 1)$ in number.

The monomials not in $I_{1,2}$ are of two types:

$x_i y_j$, with $i = 0, \ldots, n_1$, $j = 0, \ldots, n_2 \ (n_1 + 1)(n_2 + 1)$ of them);

$x_i y_j^2$, with $i \neq j$, both in $\{0, \ldots, n_1\}$ $(n_1 + 1)n_1$ of them).

For a total number of $(n_1 + 1)(n_1 + n_2 + 1)$, as requested.

The monomials not in $I_{2,1}$ are of two types:

$x_i y_k$, with $i \neq j$, both in $\{0, \ldots, n_1\}$ and $k = i$ or $k = j \ (n_1 + 1)n_1$ of them);

$x_i^2 y_j$, with $i = 0, \ldots, n_1$, $j = 0, \ldots, n_2 \ (n_1 + 1)(n_2 + 1)$ of them).

For a total number of $(n_1 + 1)(n_1 + n_2 + 1)$, as requested.

□

From this result we get immediately the following:

Corollary 3.3: Let $r, k \geq 1$, $d \geq 2$, and $V$ be the $d$-ple (Veronese) embedding of $\mathbb{P}^r$. Then $V$ is not (Grassman) $(k, s - 1)$-defective, for $s \leq \min\{r + 1, k + 1\}$.

Proof: By Proposition 1.0, this statement is equivalent to say that $\mathbb{P}^r \times \mathbb{P}^k$ in the Segre-Veronese embedding of bidegree $(d, 1)$ is not $(s - 1)$-defective, hence the results follows from Proposition 3.2.

□

Up to this point in this section we only proved results about Segre-Veronese varieties which are NOT defective. What follows is a list of examples of defective varieties which are defective. The way one can check the defectivity in all this examples is the same: we should have $V_{n,a}^s = \mathbb{P}^N$, but instead it is easy to find a way to split $a = b + c = (b_1, \ldots, b_t) + (c_1, \ldots, c_t)$ in such a way that there is a form $f_1$ of multidegree $(b_1, \ldots, b_t)$ and an $f_2$ of multidegree $(c_1, \ldots, c_t)$ passing through $s$ generic (simple) points, hence there is at least a form of degree $a$ (namely, $f_1f_2$) through $s$ generic 2-fat points which was not supposed to exist.

In the following list we always have $m \geq 1$, and we give values $s, n, a$ for which $V_{n,a}^s$ is defective:

$\mathbb{P}^1 \times \mathbb{P}^m$, $a = (2k, 2)$, $b = c = (k, 1)$, $k \geq 1$, $s = \lceil \frac{(2k+1)(m+1)}{2} \rceil$;

$\mathbb{P}^2 \times \mathbb{P}^2$, $a = (2, 2)$, $c = (1, 1)$, $s = 8$;

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m$, $a = (1, 1, 2)$, $b = (1, 0, 1)$, $c = (0, 1, 1)$, $k \geq 1$, $s = 2m + 1$;

$\mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^m$, $a = (2k, 1, 1)$, $b = (k, 1, 0)$, $c = (k, 0, 1)$, $k \geq 1$, $s = km + k + m$;

$\mathbb{P}^1 \times \mathbb{P}^r \times \mathbb{P}^m$, $a = (r + m, 1, 1)$, $b = (r, 1, 0)$, $c = (m, 0, 1)$, $s = rm + r + m$;
\(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m, \ a = (2, 2, 2), \ b = c = (1, 1, 1), \ m \leq 3, \ s = 4m + 3;\)
\(\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^m, \ a = (2, 1, 1), \ b = (1, 1, 0), \ c = (1, 0, 1), \ s = 3m + 2;\)
\(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^5, \ a = (2, 1, 1, 1), \ b = (1, 1, 1, 0), \ c = (1, 0, 0, 1), \ s = 11;\)
\(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{2m-1}, \ a = (m, 1, 1, 1), \ b = (m-1, 1, 1, 0), \ c = (1, 0, 0, 1), \ m > 1, \ s = 4m - 1;\)
\(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^{2m}, \ a = (m, 1, 1, 1), \ b = (m-1, 1, 1, 0), \ c = (1, 0, 0, 1), \ m \geq 4, \ s = 4m - 1.\)

Notice that the penultimate case for \(m = 1\) is defective too, but it is not of the same kind of all the others (\(V^s\) has dimension 13 and not 14); for this example see [CGG 4], Example 2.2.

Of course from these examples we can derive examples of Grassmann defectivity, again by using Proposition 1.0; we will just notice what we get from the last two cases:

**Corollary 3.4:** Let \(V\) be the \((m, 1, 1)\)-ple (Segre-Veronese) embedding of \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\). Then \(V\) is (Grassman) \((2m - 1, 4m - 2)\)-defective, for \(m \geq 1\) and \((2m, 4m - 2)\)-defective, for \(m \geq 4\).

The Corollary shows that the Segre Veronese varieties given by a \((2\alpha, 1, 1)\)-embedding of \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) are both defective (see Theorem 2.5) and Grassmonn defective.

### 4. Partially symmetric tensors.

Now we want to describe how to interpret our Segre-Veronese embeddings from the point of view of tensors (e.g. see [Ge] or [IK] for the Veronese case and [CGG2] for the Segre case). In order to look at the Veronese variety (or \(t\)-uple embedding of \(\mathbb{P}^r\)) \(\nu_t(\mathbb{P}^n) \subset \mathbb{P}^N, \ N = \binom{t+r}{r}\) one can consider the Segre Variety \(\nu(\mathbb{P}^r \times \ldots \times \mathbb{P}^r)\), with \(t\) factors, in \(\mathbb{P}^M, \ M = (r + 1)^t - 1\), and then consider the action of the symmetric group \(S_t\) on \(\mathbb{P}^M\) where, if the variables in \(\mathbb{P}^M\) are \(\{z_{(1,0,...,0),(1,0,...,0)}, \ldots, z_{(0,...,0,1),(0,...,0,1)}\}\), the action of an element \(\sigma \in S_t\) is defined by

\[\sigma(z_{(1,0,...,0),(1,0,...,0)}, \ldots, z_{(0,...,0,1),(0,...,0,1)}) = (z_{\sigma((1,0,...,0),(1,0,...,0))), \..., z_{\sigma((0,...,0,1),(0,...,0,1))}}.\]

The invariant subspace of \(\mathbb{P}^M\) with respect to this action is actually a linear space \(\cong \mathbb{P}^N\), and the linear equations which define it give the required symmetries for the tensors parameterized by the points of \(\mathbb{P}^M\). We can view \(\mathbb{P}^M\) as the parameterizing space of all the \((r + 1)^t\) tensors, and \(\mathbb{P}^N\) inside it as the subspace of symmetric ones: then the Segre Variety and the Veronese parameterize the rank one (decomposable) tensors.

Notice that the symmetric tensors of rank one correspond to forms that can be written as powers of linear forms. Notice also that when we say that a symmetric tensor has rank one, i.e. that it is decomposable, we mean that it is decomposable as an element of the Tensor Algebra \(V \otimes \ldots \otimes V = V^\otimes t\) (where \(P^r = P(V)\)), not of the symmetric algebra \(\text{Sym}_t(V)\).

Consider for example a rational normal curve \(C_t \subset \mathbb{P}^4\); we are used to view its ideal as generated by the \(2 \times 2\) minors of a \(2 \times t\) catalecticant matrix of indeterminates (or also by the \(2 \times 2\) minors of a different catalecticant matrix, see e.g. [Pu]). From the point of view above we should look at the ideal of the Segre embedding \(V_t: (\mathbb{P}^1)^t \to \mathbb{P}^{2^t-1}\), which is generated by the \(2 \times 2\) minors of a \(2 \times 2 \times \ldots \times 2\) (\(t\) times) tensor (e.g. see [Gr] and [Ha]); the ideal of \(C_t\) comes from the ideal of \(V_t\) modulo the symmetry relations (given by the action of the symmetric group \(S_t\) on \(\mathbb{P}^{2^t-1}\)) which define a linear space \(\mathbb{P}^t\) in \(\mathbb{P}^{2^t-1}\).

This can be thought as a more “complete” way to view those ideals, with respect to the usual way (as given by minors of catalecticant matrices) since the tensor represents “more faithfully” their symmetries.

Now consider e.g. the case \(t = 3\); we can think of “stopping halfway” between the Segre variety \(V_3\) (parameterizing \(2 \times 2 \times 2\) decomposable tensors in \(\mathbb{P}^7\)) and the rational normal curve \(C_t\) (which parameterizes decomposable
2 × 2 × 2 symmetric tensors) by considering the Segre-Veronese embedding \( V_{(2,1)} \) of \( \mathbb{P}^1 \times \mathbb{P}^1 \) of degree \((2,1)\) into \( \mathbb{P}^5 \).

We can consider the action of the symmetric group \( S_2 \) on \( \mathbb{P}^7 \) which symmetrizes its variables \( x_{ijk}, i,j,k \in 0,1 \) only with respect to \( i \) and \( j \). The invariant space for this action is a linear space \( \mathbb{P}^5 \subset \mathbb{P}^7 \), and it cuts \( V_3 \) exactly in \( V_{(2,1)} \). Hence the variety \( V_{(2,1)} \) parameterizes \( 2 \times 2 \times 2 \) “partially symmetric tensors”, i.e. tensors whose entries are symmetric only with respect to the first two indices.

In general, consider \((\mathbb{P}^r)^t = \mathbb{P}^r \times \cdots \times \mathbb{P}^r\), \( t \) times, and its Segre-Veronese embedding \( V_{r,a} = (r, \ldots, r) \) and \( a = (a_1, \ldots, a_t) \), into the space \( \mathbb{P}^M \). Let \( a = a_1 + \cdots + a_t \), and consider the Segre embedding of \((\mathbb{P}^r)^a\) into \( \mathbb{P}^M \), where \( M = (r + 1)^a - 1 \). We can view \( \mathbb{P}^N \) inside \( \mathbb{P}^M \) as the space of tensors which are invariant with respect to the actions of \( S_{a_1}, \ldots, S_{a_t} \) on the variables of relative indeces. So those are “partially symmetric” tensors (for \( t = 1 \) we get symmetric tensors and \( V_{r,a} \) is the Veronese variety, while for \( a_1 = \ldots = a_t = 1 \) they are generic tensors and \( V_{r,a} \) is the Segre embedding of \((\mathbb{P}^r)^t\)).

So the Segre-Veronese variety \( V_{r,a} \) will parameterize the partially symmetric tensors (with respect to the actions of \( S_{a_1}, \ldots, S_{a_t} \)) in \( \mathbb{P}^M \) which are decomposable. Since those are the tensors of tensor rank 1 (e.g. see [CGG2]), the secant varieties of \( V_{r,a} \) give the stratification by tensor rank of those partially symmetric tensors.

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M.V.Catalisano, Dip. Matematica, Univ. di Genova, Italy.
e-mail: catalisa@dima.unige.it

A.V.Geramita, Dept. Math. and Stats. Queens’ Univ. Kingston, Canada and Dip. di Matematica, Univ. di Genova. Italy.
e-mail: geramita@dima.unige.it ; tony@mast.queensu.ca

A. Gimigliano, Dip. di Matematica and C.I.R.A.M., Univ. di Bologna, Italy.
e-mail: gimiglia@dm.unibo.it