Symmetric Vacua in Heterotic M–Theory

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Abstract

Symmetric vacua of heterotic M–theory, characterized by vanishing cohomology classes of individual sources in the three–form Bianchi identity, are analyzed on smooth Calabi–Yau three–folds. We show that such vacua do not exist for elliptically fibered Calabi–Yau spaces. However, explicit examples are found for Calabi–Yau three–folds arising as intersections in both unweighted and weighted projective space. We show that such symmetric vacua can be combined with attractive phenomenological features such as three generations of quarks and leptons. Properties of the low energy effective actions associated with symmetric vacua are discussed. In particular, the gauge kinetic functions receive no perturbative threshold corrections, there are no corrections to the matter field Kähler metric and the associated five–dimensional effective theory admits flat space as its vacuum.
1 Introduction:

A general property of theories containing branes is that bulk fields are excited by sources located on those branes. This effect is particularly pronounced in heterotic M–theory [1, 2, 3, 4] where $\mathcal{N} = 1$ supersymmetric $E_8$ gauge multiplets are localized on two ten–dimensional orbifold planes in eleven–dimensional space–time. These boundary sources lead, roughly, to a linear variation of the bulk fields along the single transverse direction and, hence, to strong effects that grow with the size of this dimension.

When constructing vacua of heterotic M–theory preserving four supercharges, these effects cause a deformation of the zeroth order Calabi–Yau background [3]. This deformation can be determined using a strong coupling expansion and is, at present, known only in the linearized approximation [3, 4]. Requiring the validity of this approximation places a bound on the size of the eleventh dimension [3] and, hence, certain regions of the moduli space are unavailable. In the four–dimensional effective action, these deformations induce threshold corrections to the gauge kinetic functions [3, 5, 6, 7] which differ on the observable and hidden sector branes and can lead to a strong gauge coupling in one of the sectors. Furthermore, they induce corrections to the matter field Kähler potential [5].

Such deformations of the Calabi–Yau background occur quite generically in heterotic M–theory. They are caused by source terms in the Bianchi identity of the M–theory three–form field. These source terms have support on each of the orbifold planes and, in general, are non–vanishing cohomologically. As an example, the standard embedding of the spin connection into one of the $E_8$ gauge groups leads to cohomologically non–trivial sources on each of the orbifold planes. Essentially, this happens because the gravitational contribution to the Bianchi identity is split equally between the two orbifold planes. Generically, the same property is shared by non–standard embedding vacua [9, 10, 11, 12] as well. Typically, the sources turn out to be non–trivial in cohomology and the Calabi–Yau background receives corrections. However, we will show that not all non–standard embedding vacua have this property.

Specifically, in this paper, we would like to study the possibility of setting the sources in the three–form Bianchi identity to zero individually, on each orbifold plane, at least in cohomology. Vacua with this property are called “symmetric vacua”. We want to emphasize that the concepts of symmetric vacua and standard embeddings are different, the latter always having non–vanishing individual sources in the three–form Bianchi identity. For heterotic M–theory on K3, symmetric vacua have been studied in ref. [13]. In ref. [5], it has been shown that a class of heterotic orbifold models with vanishing threshold corrections exists. In these models, the vanishing corrections are due to a combination of topological properties and special properties at the orbifold point.

Here, we would like to construct symmetric vacua of heterotic M–theory based on smooth
Calabi–Yau three–folds. From what we have said, this requires non–standard embedding vacua and, hence, the analysis of certain classes of semi–stable holomorphic vector bundles $V$ over Calabi–Yau three–folds $X$. As we will see, the relevant vector bundles needed to construct symmetric vacua are those with the property that $c_2(V) = \frac{1}{2}c_2(TX)$, where $c_2(V)$ and $c_2(TX)$ are the second Chern classes of $V$ and the tangent bundle of $X$ respectively. We will call such vector bundles symmetric as well. Symmetric vacua have interesting properties, such as the absence of the threshold and Kähler potential corrections to the low energy effective action mentioned above. They also admit flat space as the vacuum solution to the associated five–dimensional effective theory. It is also possible that they constitute valid solutions even in the region of moduli space with large orbifold radius, which is usually not accessible.

In section 2, we start by reviewing the general context and by presenting some of the essential formulae. Section 3 is devoted to the study of elliptically fibered Calabi–Yau spaces with sections and semi–stable holomorphic bundles of the Friedman–Morgan–Witten type \[24\]. In section 4, we move on to discuss Calabi–Yau spaces defined as intersections in unweighted and weighted projective spaces and holomorphic vector bundles of the monad type. Finally, in section 5, we discuss the special properties of symmetric vacua and their associated low energy effective actions.

Our results can be summarized as follows. We show that, within the class of elliptically fibered Calabi–Yau three–folds and Friedman–Morgan–Witten vector bundles, no symmetric vector bundles and, hence, no symmetric vacua exist. This confirms the expectation that symmetric vacua are, in fact, rare and that generically Calabi–Yau vacua do receive corrections. For Calabi–Yau three–folds defined as intersections in projective space, we first prove several general properties of symmetric bundles (of the monad type) such as a lower bound on the number of generations. Then, we construct explicit examples of symmetric bundles on various Calabi–Yau spaces within this class. Again, such bundles turn out to be relatively rare. For example, for the five Calabi–Yau three–folds defined as intersections in an unweighted projective space, there exist exactly four symmetric bundles (of the monad type), three for the quintic polynomial and one for the intersection of two cubic polynomials in $\mathbf{CP}^5$. Furthermore, using two of our examples, one in unweighted and the other in weighted projective space, we show that symmetric vacua can be combined with phenomenologically interesting properties such as three generations of quarks and leptons. To first non–trivial order, symmetric vacua do not receive corrections at the level of the Calabi–Yau zero modes. However, massive first order corrections are generically present. We point out that, due to the vanishing massless vacuum corrections, all first order strong coupling corrections to the associated four– and five–dimensional low energy effective actions vanish for symmetric vacua. In particular, the threshold corrections to the gauge kinetic functions and the correction to the matter field Kähler metric vanish. We also speculate that symmetric vacua might be valid in the region...
of moduli space with large orbifold size where the strong coupling expansion breaks down in the non–symmetric case.

2 General Framework:

In this section, we would like to present the general framework for the discussion. Our starting point is the effective action for the strongly coupled heterotic string \[^1\] given by M–theory on the orbifold $S^1/Z_2$. As usual, the orbifold coordinate $x^{11}$ is taken to be in the range $x^{11} \in [-\pi\rho, \pi\rho]$ with the $Z_2$ symmetry acting as $x^{11} \rightarrow -x^{11}$. This leads to two fixed ten-planes at $x^{11} = 0$ and $\pi\rho$, each of which carries an $\mathcal{N}=1$ supersymmetric $E_8$ gauge multiplet.

We would like to consider vacua of this theory associated with $\mathcal{N}=1$ supergravity theories in four dimensions. To lowest order, this implies the space–time structure

$$M_{11} = M_4 \times S^1/Z_2 \times X$$

where $M_4$ is four–dimensional Minkowski space and $X$ is a Calabi–Yau three–fold. As is well–known \[^2\] , the above direct product structure is valid only to lowest order in an expansion in the eleven–dimensional Newton constant $\kappa$. More precisely, the first corrections appear at order $\kappa^{2/3}$ and produce a “deformation” of the above space in both the orbifold and Calabi–Yau components. The key ingredient in the theory that leads to these deformations is the Bianchi identity

$$(dG)_{11IJKL} = 4\sqrt{2}\pi \left(\frac{\kappa}{4\pi}\right)^{2/3} \left( J^{(0)}\delta(x^{11}) + J^{(N+1)}\delta(x^{11} + \pi\rho) + \sum_{i=1}^N \frac{1}{2} J^{(i)}(\delta(x^{11} - x_n) + \delta(x^{11} + x_n)) \right)_{IJKL}. \quad (2)$$

for the field strength $G = 6dC$ of the M-theory three–form field $C$. Here $J^{(0)}$ and $J^{(N+1)}$ are sources supported on the orbifold planes and defined in terms of the two $E_8$ field strengths $F^{(1)}$, $F^{(2)}$ and the curvature $R$ by

$$J^{(0)} = -\frac{1}{16\pi^2} \left( \text{tr} F^{(1)} \wedge F^{(1)} - \frac{1}{2} \text{tr} R \wedge R \right)\big|_{x^{11}=0} ,$$

$$J^{(N+1)} = -\frac{1}{16\pi^2} \left( \text{tr} F^{(2)} \wedge F^{(2)} - \frac{1}{2} \text{tr} R \wedge R \right)\big|_{x^{11}=\pi\rho} . \quad (3)$$

For the vacua under consideration, the gauge fields $F^{(1)}$ and $F^{(2)}$ are associated with holomorphic vector bundles $V_1$ and $V_2$ on the Calabi–Yau space $X$ while the curvature $R$ is associated with the tangent bundle $TX$. Note that the gravitational contribution $\text{tr}(R \wedge R)$ to the Bianchi identity has been split equally between the two orbifold planes. This can be seen from the factors $1/2$ in

\[^1\]Indices $\bar{I}, \bar{J}, \bar{K}, \cdots = 0, \ldots 9$ specify the 10–dimensional space orthogonal to the orbifold.
front of $\text{tr}(R \wedge R)$ that appear in eq. (3). For generality, we have also allowed for additional sources $J^{(i)}$, where $i = 1, \ldots, N$, in the Bianchi identity which could arise from five–branes in the vacuum. Such vacua with five–branes have been constructed and analyzed in ref. [12, 7, 13]. Here, we will only need some elementary properties. The $N$ five–branes are oriented transversely to the orbifold and located at $x^{11} = x_1, \ldots, x_N$. To preserve four–dimensional Poincaré invariance and $\mathcal{N} = 1$ supersymmetry, they stretch across the uncompactified space $M_4$ and wrap on holomorphic curves $C_2^{(i)}$ within the Calabi–Yau space $X$. This orientation implies for their sources that

$$J^{(i)} = \delta(C_2^{(i)}), \quad i = 1, \ldots, N,$$

where $\delta(C_2^{(i)})$ is the Poincaré dual four–form to the holomorphic curve $C_2^{(i)}$.

Integrating Bianchi identity (2) over an arbitrary four–cycle in the Calabi–Yau space times the orbifold cycle, one finds

$$c_2(V_1) + c_2(V_2) + [W] = c_2(TX)$$

where

$$c_2(V_i) = -\frac{1}{16\pi^2} \left[ \text{tr} F^{(i)} \wedge F^{(i)} \right], \quad c_2(TX) = -\frac{1}{16\pi^2} \left[ \text{tr} R \wedge R \right]$$

are the second Chern classes for the vector bundle $V_i$ and the tangent bundle $TX$ of $X$ respectively and

$$[W] = \sum_{i=1}^N \left[ \delta(C_2^{(i)}) \right]$$

is the four–form class of the five–branes. The brackets $[\ldots]$ indicate the cohomology class of the associated four–form. This topological condition guarantees anomaly–freedom of the vacuum and it is necessary (and sufficient) for the Bianchi identity to be soluble. It states that the total right hand side of the Bianchi identity is topologically trivial, as it should be. However, this is not the case for the cohomology classes of the individual source terms $[J^{(n)}]$, for $n = 0, \ldots, N + 1$, which are generally non–vanishing. Hence, the sources themselves generally do not vanish, that is, $J^{(n)} \neq 0$. This leads to a non–vanishing field strength $G$ which, in turn, causes a deformation of the space–time (1).

As a familiar example, let us consider the standard embedding. The anomaly constraint (3) can be satisfied if one takes

$$c_2(V_1) = c_2(TX), \quad c_2(V_2) = 0$$

Footnote 2: We will consider vector bundles $V$ with $c_1(V) = 0$ throughout the paper.
and

$$[W] = 0.$$  \hspace{1cm} (9)

The last statement asserts that the standard embedding does not allow five–branes in the vacuum, as expected. Therefore, $N = 0$. Note from eq. (8) and (9) that the cohomology classes of the remaining sources are given by

$$[J^{(0)}] = -\frac{1}{32\pi^2} [\text{tr}(R \wedge R)] , \quad [J^{(1)}] = \frac{1}{32\pi^2} [\text{tr}(R \wedge R)].$$  \hspace{1cm} (10)

Since $c_2(TX) \neq 0$, the cohomology class of the source on each of the orbifold planes is non–vanishing. This is due to the aforementioned fact that the gravitational part of the Bianchi identity has been equally distributed onto each of the two orbifold planes. Solution (8), (9) is an attractive choice, since it can be explicitly realized at the level of fields by embedding the spin connection into the first $E_8$ gauge group and choosing the second $E_8$ gauge vacuum to be trivial. That is

$$\text{tr}(F^{(1)} \wedge F^{(1)}) = \text{tr}(R \wedge R), \quad \text{tr}(F^{(2)} \wedge F^{(2)}) = 0.$$  \hspace{1cm} (11)

It follows from eq. (3) that the sources are explicitly given by

$$J^{(0)} = -\frac{1}{32\pi^2} \text{tr}(R \wedge R), \quad J^{(1)} = \frac{1}{32\pi^2} \text{tr}(R \wedge R).$$  \hspace{1cm} (12)

Since the source cohomology classes are non-vanishing, it follows that the sources themselves cannot be set to zero. Therefore, the standard embedding leads to a non–vanishing field strength $G$ and, hence, to a deformation of the Calabi–Yau vacuum [3].

The main point of this paper is to look for solutions of the anomaly constraint (5) which possibly allow for the vanishing of the right hand side of the Bianchi identity. In other words, we are interested in vacua where each source term $J^{(n)}$, for $n = 0, \ldots, N+1$, on the right hand side of the Bianchi identity potentially vanishes. For such vacua, the four–form field strength $G$ can be set to zero and, hence, the space–time (3) remains uncorrected, at least to first non–trivial order in the $\kappa$ expansion. The task of finding such vacua can be broken into two steps. First, if each of the sources is to vanish, it necessarily must vanish in cohomology, $[J^{(n)}] = 0$. This is easily achieved for the five–brane sources. We simply consider vacua without five–branes, that is, we take $N = 0$. In this case, the five–brane sources $J^{(i)}$ and their cohomology classes simply do not exist. In the following, we concentrate exclusively on vacua without five–branes. The situation is more complicated for the sources on the orbifold planes. For those sources to be cohomologically trivial, that is $[J^{(0)}] = [J^{(1)}] = 0$, it follows from eq. (3) that we need to set

$$\left[\text{tr}(F^{(1)} \wedge F^{(1)})\right] = \left[\text{tr}(F^{(2)} \wedge F^{(2)})\right] = \frac{1}{2} [\text{tr}(R \wedge R)].$$  \hspace{1cm} (13)
Consequently, we have
\[ c_2(V_1) = c_2(V_2) = \frac{1}{2} c_2(TX). \] (14)
which satisfies the topological constraint (5) if we take
\[ [W] = 0, \] (15)
consistent with the above assumption. We will refer to vacua with property (14) and no five–branes as symmetric vacua. Second, having chosen a Calabi–Yau three–fold \( X \) and vector bundles \( V_1 \) and \( V_2 \) realizing such a symmetric vacuum, one has removed the topological obstruction of setting the orbifold sources to zero. To actually set the sources to zero, however, one needs to explicitly choose a spin connection on \( TX \) and specific gauge connections on \( V_1 \) and \( V_2 \) so that \( J(0) = J(1) = 0 \). This would imply that
\[ \text{tr}(F^{(1)} \wedge F^{(1)}) = \text{tr}(F^{(2)} \wedge F^{(2)}) = \frac{1}{2} \text{tr}(R \wedge R). \] (16)
Unlike in the case of the standard embedding where the relations (11) are satisfied at the level of fields, it is not clear that vacuum solutions satisfying constraint (13) can be achieved. Therefore, for the purposes of this paper, we restrict ourselves to solving the topological part of the problem specified by eq. (13). However, if we cannot be sure that sources \( J(0) \) and \( J(1) \) vanish, then it is conceivable that the field strength \( G \) is again non–vanishing and, hence, spacetime (1) is deformed. What simplification, then, has been achieved by using symmetric vacua? The answer is, considerable simplification. To see this, note that we have shown elsewhere \([5, 16]\) that the corrections, due to the deformation of space–time (1), to the the effective action of the zero modes are all proportional to charges \( \beta_i \) defined by
\[ \beta_i(*\omega^i) = c_2(V_1) - \frac{1}{2} c_2(TX) = -c_2(V_2) + \frac{1}{2} c_2(TX) \] (17)
Here \( \{\omega_i\}_{i=1,\ldots,h^{1,1}} \) is a basis of \( H^{1,1}(X) \). The quantities on the right hand side of this equation are exactly those that are set to zero for symmetric vacua. We conclude that all charges \( \beta_i \) vanish for symmetric vacua. Therefore, for symmetric vacua, even if there is a deformation of space–time (1) due to non–vanishing sources \( J(0) \) and \( J(1) \), to first non–trivial order, this deformation does not affect the low energy zero mode action as a consequence of the vanishing of the cohomology classes \([J(0)] = [J(1)] = 0\). This motivates us to search for symmetric vacua in this paper. Finally, note that eq. (14) is much stronger than the anomaly cancellation condition (5) so that we are, in fact, looking for a small sub–class of all non–anomalous vacua.

The problem we are going to address, then, is to find, for a given Calabi–Yau three–fold \( X \), holomorphic vector bundles \( V \) with the property that
\[ c_2(V) = \frac{1}{2} c_2(TX). \] (18)
We will call such vector bundles symmetric bundles of the Calabi–Yau space $X$. It is understood, in what follows, that a symmetric embedding can then be constructed by choosing both relevant bundles $V_1$ and $V_2$ of the symmetric type. Note again that the standard embedding implies that $c_2(V) = c_2(TX)$, which differs from the condition (18) by a crucial factor of $1/2$. Hence, the standard embedding does not provide us with symmetric bundles.

For phenomenological reasons, we will be interested in the number of chiral generations of quarks and leptons associated with a (symmetric) bundle $V$. It is given by

$$N_{\text{gen}} = \frac{1}{2} \left| \int_X c_3(V) \right|.$$  \hspace{1cm} (19)

Furthermore, we will focus on holomorphic vector bundles with structure group $G = SU(n)$ only. This is phenomenologically motivated, since the choices $n = 3, 4$ and 5 lead to the low–energy groups (the commutants of $G$ in $E_8$) $E_6$, $SO(10)$ and $SU(5)$, that is, to attractive grand–unified groups.

### 3 Elliptically Fibered Calabi–Yau Three–Folds:

We will now consider the possibility of symmetric vacua for elliptically fibered Calabi–Yau three–folds. Let us first review the properties of these spaces that are essential to our discussion.

The relevant Calabi–Yau three–folds $X$ are given as an elliptic fibration over a two–fold base $B$ with section $\sigma : B \to X$. It can be shown \cite{23} that the base space $B$ is restricted to del Pezzo, Enriques, or Hirzebruch surfaces or certain blow–ups of the latter. The second Chern class of such a Calabi–Yau space can be expressed in terms of the base $B$ and the section $\sigma$ as

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B),$$  \hspace{1cm} (20)

where $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of the base respectively. Next, we have to specify the class of holomorphic vector bundles over elliptically fibered Calabi–Yau spaces on which we intend to focus. We concentrate on the vector bundles with structure group $SU(n)$ constructed by Friedman, Morgan and Witten \cite{24}. These bundles are parameterized by a class $\eta \in H_2(B)$ and a rational number $\lambda$ subject to the constraints

$$n \text{ is odd, } \lambda = m + \frac{1}{2} \hspace{1cm} (21)$$

$$n \text{ is even, } \lambda = m, \hspace{0.5cm} \eta = c_1(B) \text{ mod } 2 \hspace{1cm} (22)$$

where $m$ is an integer. The second Chern class of these bundles $V$ can be expressed in terms of $\eta$, $\lambda$, $n$ and properties of the fibration. One finds \cite{24} that

$$c_2(V) = \eta \sigma - \frac{1}{24} c_1(B)^2 \left( n^3 - n \right) + \frac{1}{2} \left( \lambda^2 - \frac{1}{4} \right) n \eta \left( \eta - nc_1(B) \right).$$  \hspace{1cm} (23)
Let us now try to find symmetric vector bundles in this class. The obstruction to having such a symmetric bundle is given by

$$U = c_2(V) - \frac{1}{2} c_2(TX).$$  \hspace{1cm} (24)

In order to solve the equation $U = 0$, we split $U$ into a base component and a fiber component as

$$U = U_B + aF. \hspace{1cm} (25)$$

Here $U_B$ is a second homology class in the base while $a$ is an integer that counts the number of fibers $F$. Inserting the above Chern classes, one finds

$$U_B = \sigma(\eta - 6c_1(B)) \hspace{1cm} \text{(26)}$$

$$a = -\frac{1}{24} c_1(B)^2 (n^3 - n) + \frac{1}{2} \left( \lambda^2 - \frac{1}{4} \right) n \eta (\eta - n c_1(B)) - \frac{11}{2} c_1(B)^2 - \frac{1}{2} c_2(B). \hspace{1cm} (27)$$

For a symmetric bundle, we must demand that $U_B = 0$. This fixes the bundle parameter $\eta$ to be

$$\eta = 6c_1(B). \hspace{1cm} (28)$$

From eqs. (21) and (22), we see that this choice of $\eta$ is always acceptable for odd $n$. For even $n$, it may or may not be compatible with the constraint (22). Keeping this restriction in mind, we try to satisfy the remaining condition for a symmetric bundle, namely that $a = 0$. Inserting $\eta$ in eq. (28) into (27) leads to the relation

$$c_2(B) = N(\lambda, n) c_1(B)^2 \hspace{1cm} (29)$$

between the first and second Chern class of the base. The numbers $N(\lambda, n)$ are given by

$$N(\lambda, n) = \left( 6\lambda^2 - \frac{3}{2} \right) n (6 - n) - \frac{1}{12} (n^3 - n) - 11. \hspace{1cm} (30)$$

What remains to be checked is whether the relation (29) can be satisfied for certain base spaces $B$ and numbers $\lambda, n$. To do this, let us look at the explicit values of $c_1(B)^2$ and $c_2(B)$ for the various allowed base spaces $B$ as given in Table 1. For the Enriques surface $E$ and the del Pezzo surface $dP_9$ we have

$$c_1(B)^2 = 0, \quad c_2(B) \neq 0. \hspace{1cm} (31)$$

This is incompatible with the relation (29) and, hence, no symmetric bundles exist for these base spaces. From Table 1, we have for all other base spaces that

$$\frac{1}{2} \leq \frac{c_2(B)}{c_1(B)^2} \leq 11. \hspace{1cm} (32)$$

On the other hand, it is easy to show that the numbers $N(\lambda, n)$ satisfy either $N(\lambda, n) < 0$ or $N(\lambda, n) \geq 20$ for all allowed values of $n$ and $\lambda$. Hence the relation (29) can never be satisfied.

We conclude that for the base spaces listed in Table 1 no symmetric vector bundles of the Friedman–Morgan–Witten type exist. This result shows clearly that the property of being a symmetric bundle is quite restrictive.
Table 1: Base spaces and their Chern classes

| Name       | $B$       | $c_1(B)^2$ | $c_2(B)$ |
|------------|-----------|------------|----------|
| del Pezzo  | $dP_r$, $r = 1, \ldots, 9$ | $9 - r$    | $3 + r$  |
| Hirzebruch | $F_r$     | 8          | 4        |
| Enriques   | $E$       | 0          | 12       |

4 Intersections in Weighted Projective Space:

The condition (18) for a symmetric vector bundle is an equation in the vector space $H^{2,2}(X)$ and, therefore, provides $h^{1,1}$ constraints on the bundle data. In the previous section we have, unsuccessfully, looked for symmetric bundles over elliptically fibered Calabi–Yau three–folds. For those Calabi–Yau space, we always have $h^{1,1} \geq 2$. Therefore, it seems worthwhile to minimize the number of constraints and consider Calabi–Yau spaces with $h^{1,1} = 1$.

A large class of Calabi–Yau three–folds with this property is provided by intersections in both unweighted and weighted projective space. We will focus our attention on such spaces in this section \[.\] Partially, this is motivated by earlier approaches to heterotic model building within this class, particularly in ref. \[19, 20\]. Let us review the relevant properties of these spaces following ref. \[21, 22\]. The starting point is the weighted projective space $\mathbb{CP}_w^{N+3}$ with homogeneous coordinates $(X^\nu)_{\nu = 0, \ldots, N+3}$ and weights $w = (w_\nu)_{\nu = 0, \ldots, N+3}$. The space $X \subset \mathbb{CP}_w^{N+3}$ is defined as the zero locus of $N$ polynomials $P_1(X), \ldots, P_N(X)$ with degrees $p = (p_\alpha)_{\alpha = 1, \ldots, N}$, where $p_\alpha = \deg(P_\alpha)$.

The first Chern class of $X$ is given by

$$c_1(TX) = \left( \sum_{\nu=0}^{N+3} w_\nu - \sum_{\alpha=1}^{N} p_\alpha \right) J$$

Here $J = c_1(O(1))$ is the first Chern class of the hyperplane bundle $O(1)$. For $X$ to be a Calabi–Yau space, we want that $c_1(TX) = 0$ and, hence, that

$$\sum_{\nu=0}^{N+3} w_\nu = \sum_{\alpha=1}^{N} p_\alpha .$$

Provided this condition is satisfied, one finds for the second Chern class

$$c_2(TX) = \frac{1}{2} \left( \sum_{\alpha=1}^{N} p_\alpha^2 - \sum_{\nu=0}^{N+3} w_\nu^2 \right) J^2 .$$

\[3\] Weighted projective space is singular. If these singularities intersect the Calabi–Yau space they have to be blown up in order to arrive at a smooth manifold. This creates new classes and, hence, a manifold with $h^{1,1} > 1$. For the purpose of this paper, we concentrate on cases where this does not happen, so that indeed $h^{1,1} = 1$. 
Another useful relation which we need below in order to evaluate the number of generations is

\[ \int_X J^3 = \frac{\prod_{\alpha=1}^{N} P_\alpha}{\prod_{\nu=0}^{N^2} w_\nu}. \]  

(36)

Next, we have to specify a class of holomorphic vector bundles. In the following, we are going to use the so called monads defined by the short exact sequence \([19, 23, 20]\)

\[ 0 \rightarrow V \rightarrow \bigoplus_{a=1}^{n+M} \mathcal{O}(n_a) \otimes Q_a^i(X) \rightarrow \bigoplus_{i=1}^{M} \mathcal{O}(m_i) \rightarrow 0. \]  

(37)

Here, the map between the two sums of line bundles is defined by the polynomials \(Q_a^i(X)\) with degree \(\text{deg}(Q_a^i) = m_i - n_a\). Consequently, one requires that

\[ m_i > n_a > 0 \]  

(38)

for all \(i = 1, \ldots, M\) and all \(a = 1, \ldots, n + M\). The vector bundle \(V\) is then specified by the two sets of integers \((m_i)_{i=1,\ldots,M}\) and \((n_a)_{a=1,\ldots,n+M}\). It is denoted by

\[ V = V(m_1, \ldots, m_M; n_1, \ldots, n_{n+M}). \]  

(39)

For the first Chern classes of such a bundle \(V\), one finds that

\[ c_1(V) = \left(\sum_{a=1}^{n+M} n_a - \sum_{i=1}^{M} m_i\right) J \]  

(40)

The condition that \(V\) be a semi–stable bundle and the fact that \(h^{1,1} = 1\) imply \(c_1(V) = 0\). As a result, \(V\) is a bundle with structure group \(SU(n)\). Therefore, we require

\[ \sum_{a=1}^{n+M} n_a = \sum_{i=1}^{M} m_i. \]  

(41)

For vanishing first Chern class, the second and third Chern classes are given by

\[ c_2(V) = -\frac{1}{2} \left(\sum_{a=1}^{n+M} n_a^2 - \sum_{i=1}^{M} m_i^2\right) J^2 \]  

(42)

\[ c_3(V) = \frac{1}{3} \left(\sum_{a=1}^{n+M} n_a^3 - \sum_{i=1}^{M} m_i^3\right) J^3. \]  

(43)

Putting together eq. \([19], (36)\) and \([13]\) one obtains for the number of generations

\[ N_{\text{gen}} = \frac{1}{6} \left|\sum_{a=1}^{n+M} n_a^3 - \sum_{i=1}^{M} m_i^3\right| \frac{\prod_{\alpha=1}^{N} P_\alpha}{\prod_{\nu=0}^{N^2} w_\nu}. \]  

(44)
Before we discuss explicit examples, it is useful to prove some general properties of symmetric bundles in the setting described above. Let us focus on a specific Calabi–Yau three–fold $X$ represented by weights $(w_\nu)$ and polynomials of degree $(p_\alpha)$. For this space $X$, define the quantity

$$Q = \frac{1}{2} \left( \sum_{\alpha=1}^{N} p_\alpha^2 - \sum_{\nu=0}^{N+3} w_\nu^2 \right).$$

(45)

Then we are interested in vector bundles $V = V(n_a;m_i)$ on $X$ with the properties

$$\sum_{i=1}^{M} m_i = \sum_{a=1}^{n+M} n_a \equiv S$$

(46)

$$\sum_{i=1}^{M} m_i^2 = \sum_{a=1}^{n+M} n_a^2 + Q.$$ 

(47)

The first condition is just the statement that $c_1(V) = 0$. The second one states that $c_2(V) = \frac{1}{2} c_2(TX)$ and, hence, that $V$ is a symmetric bundle. Furthermore, we define the quantity $C$ by

$$\sum_{i=1}^{M} m_i^3 = \sum_{a=1}^{n+M} n_a^3 + C.$$ 

(48)

Using the bound (48), it is then easy to prove the inequalities

$$C > 2Q \geq 2S$$

(49)

which must hold for any symmetric monad vector bundle on $X$. Recall here that $Q$ depends on Calabi–Yau data only while $C$ and $S$ depend on the vector bundle. The first part of this inequality can be used, together with eq. (44), to find a lower bound on the number of generations associated with symmetric vector bundles. This bound is given by

$$N_{\text{gen}} > \frac{Q}{3} \frac{\prod_{\alpha=1}^{N} p_\alpha}{\prod_{\nu=0}^{N+3} w_\nu}.$$ 

(50)

Note that the right hand side depends on data of the Calabi–Yau three–fold only. Therefore, the above bound must be satisfied for any symmetric (monad) vector bundle on the given Calabi–Yau space $X$.

So far the integers $(m_i)$ and $(n_a)$ defining the vector bundle are not bounded from above. Similarly, the number $M$ of line bundles is not bounded. It turns out, however, that symmetric bundles are possible only if those numbers do not exceed certain maximal values. This is summarized in the following two statements.

**Statement 1** If $n_a > n_{\text{max}}$ for any $a$ or $m_i > m_{\text{max}}$ for any $i$, where $n_{\text{max}} = Q + 1 - n - M$ and $m_{\text{max}} = Q + 2 - 2M$, then $V = V(m_i;n_a)$ is not a symmetric bundle.
Statement 2 If $M > M_{\text{max}}$ where $M_{\text{max}} = Q - n$, then $V = V(m_i; n_a)$ is not a symmetric bundle.

These two statements can easily be proven using the inequality (49). They are useful because, for a given Calabi–Yau three–fold, they only leave a finite set of monad vector bundles as candidates for symmetric bundles. Scanning this finite set, we can then find all symmetric monad bundles for a given Calabi–Yau space.

We would now like to apply the above results to a number of explicit examples, thereby showing that symmetric vector bundles indeed exist. As already mentioned, for phenomenological reasons, we are mainly interested in $SU(n)$ bundles with $n = 3, 4, 5$. In the following examples, we will focus on these three cases. We start with the five Calabi–Yau spaces that can be defined as intersections in a single unweighted projective space.

Example 1: Quintic polynomial in $\mathbb{CP}^4$

Using the above notation, the quintic is specified by $N = 1$, $w = (1, 1, 1, 1, 1)$ and $p = (5)$. This leads to

\[ Q = 10, \quad \int_X J^3 = 5. \]

The maximal integers for which symmetric bundles are possible are then given by

\[ n_{\text{max}} = 11 - n - M, \quad m_{\text{max}} = 12 - 2M, \quad M_{\text{max}} = 10 - n. \]  

Scanning the region $n_a \leq n_{\text{max}}$, $m_i \leq m_{\text{max}}$, $M \leq M_{\text{max}}$ one finds three symmetric monad bundles. They are given in Table 2. We conclude that, for the quintic, there is exactly one symmetric monad bundle for each rank $n = 3, 4, 5$.

Example 2: Intersection of two cubic polynomials in $\mathbb{CP}^5$

This space is defined by $N = 2$, $w = (1, 1, 1, 1, 1, 1)$ and $p = (3, 3)$. One finds

\[ Q = 6, \quad \int_X J^3 = 9 \]  

| $(n, M)$ | $V = V(m_i; n_a)$ | $N_{\text{gen}}$ |
|---------|------------------|-----------------|
| (3, 2)  | (3, 3; 2, 1, 1, 1, 1) | 35 |
| (4, 3)  | (3, 2; 1, 1, 1, 1, 1, 1) | 30 |
| (5, 5)  | (2, 2, 2, 2; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) | 25 |

Table 2: Symmetric bundles for quintic in $\mathbb{CP}^4$
Table 3: Symmetric bundles for the intersection of two cubics in $\mathbb{CP}^5$

| $(n,M)$ | $V = V(m_i;n_a)$ | $N_{\text{gen}}$ |
|---------|------------------|------------------|
| (3,3)   | (2,2,2;1,1,1,1,1,1) | 27               |

and

$$n_{\text{max}} = 7 - n - M, \quad m_{\text{max}} = 8 - 2M, \quad M_{\text{max}} = 6 - n.$$  \hspace{1cm} (54)

Using these maximal numbers, one can show that there exists a unique rank 3 symmetric vector bundle for this space. It is given in Table 3.

**Example 3:** Other intersections in unweighted projective space

There are three more Calabi–Yau spaces that can be defined as intersections in a single unweighted projective space, namely the intersection of a quadric and a quartic in $\mathbb{CP}^5$, the intersection of two quadrics and a cubic in $\mathbb{CP}^6$ and the intersection of four quadrics in $\mathbb{CP}^7$. Using the method described above, one can show that no symmetric monad bundles of rank $n = 3, 4, 5$ exist for these three spaces.

To summarize our results so far, we have shown that for the five Calabi–Yau spaces defined in a single unweighted projective space, there exist exactly four symmetric monad bundles of rank $n = 3, 4, 5$, three for the quintic in $\mathbb{CP}^4$ and one for the intersection of two cubics in $\mathbb{CP}^5$. Let us now turn to two further examples in weighted projective space.

**Example 4:** Degree 6 polynomial in $\mathbb{CP}^4_{1,1,1,1,2}$

This space is characterized by $N = 1$, $w = (1,1,1,1,2)$ and $p = (6)$. We find

| $(n,M)$ | $V = V(m_i;n_a)$ | $N_{\text{gen}}$ |
|---------|------------------|------------------|
| (3,2)   | (4,3;2,2,1,1,1)  | 36               |
| (3,4)   | (3,3,3;2,2,2,2,1,1) | 33               |
| (4,2)   | (4,2;1,1,1,1,1,1) | 33               |
| (4,3)   | (3,3,3;2,2,1,1,1,1) | 30               |
| (5,3)   | (3,3,2;1,1,1,1,1,1) | 27               |

Table 4: Symmetric bundles for a Calabi–Yau space defined by a degree 6 polynomial in $\mathbb{CP}^4_{1,1,1,1,2}$

$$Q = 14, \quad \int_X J^3 = 3$$ \hspace{1cm} (55)
Given those bounds, we find exactly the five symmetric bundles listed in Table 4.

**Example 5:** Two degree 6 polynomials in $\mathbb{CP}^{5}_{1,1,2,2,3,3}$

This space is defined by $N = 2$, $w = (1, 1, 2, 2, 3, 3)$ and $p = (6, 6)$. It has also been used in ref. 20 to construct a (non–symmetric) three–family model with the standard model gauge group. We find

$$Q = 22, \quad \int_X J^3 = 1$$

and

$$n_{\text{max}} = 23 - n - M, \quad m_{\text{max}} = 24 - 2M, \quad M_{\text{max}} = 22 - n.$$  \hspace{0.5cm} (58)

Using those bounds, we find a total of 15 symmetric bundles for this space. They are listed in Table 5.

This concludes our list of explicit examples. We have seen that symmetric vector bundles on Calabi–Yau three–folds in both unweighted and weighted projective spaces exist. The five Calabi–Yau spaces in unweighted projective space allow for exactly four symmetric (monad) bundles of rank $n = 3, 4, 5$. Our final two examples showed that it is somewhat easier to find symmetric bundles on Calabi–Yau three–folds in weighted projective space. Still, it is clear that such symmetric bundles are relatively rare objects.

Based on the above experience, we would now like to ask whether symmetric vacua can have interesting phenomenological properties, such as three chiral families of quarks and leptons in the observable sector. The above list of examples does not contain a single case with three generations. Given the lower bound (50) on the number of generations for symmetric (monad) bundles, this is not surprising. However, as usual, we are not necessarily interested in getting three generations on the original Calabi–Yau three–fold $X$. Instead, in order to be able to break the grand unified group by Wilson lines, we would like to consider non–simply connected Calabi–Yau three–folds defined by $Y = X/D$, where $D$ is a freely acting discrete automorphism group on $X$. Assuming that we are able to lift the automorphism $D$ to the vector bundle $V$ (so it defines a bundle $V_Y$ on $Y$), the “new” number of generations on $Y$ is given by

$$N_{\text{gen}}(Y) = N_{\text{gen}}/|D|$$

where $N_{\text{gen}}$ is the number of generations on $X$ and $|D|$ is the order of the group $D$. It is for this new number of generations that we require

$$N_{\text{gen}}(Y) = 3.$$  \hspace{0.5cm} (60)
Table 5: Symmetric bundles for a Calabi–Yau space defined by two degree 6 polynomials in \( \mathbb{CP}^5 \)

Note that if \( q : X \to Y \) is the covering map, then

\[
c_2(V_Y) = \frac{1}{|D|} q_* c_2(V), \quad c_2(TY) = \frac{1}{|D|} q_* c_2(TX)
\]

and symmetry property (14) continues to hold on \( Y \). Hence, the quotient vacuum is a symmetric vacuum. In order to construct three–family quotient manifolds \( Y \), the interesting symmetric vacua on \( X \) are those with a generation number that is a multiple of three. Indeed, there are a few such examples contained in the above tables.

There is one more constraint that has to be satisfied in order for the quotient symmetric vacuum to be consistent. This is the level matching condition of ref. 27. Let us briefly summarize this constraint. Consider a Calabi–Yau three–fold defined as the intersection of polynomials in projective space with coordinates \( X^\nu \). Let \( D = \mathbb{Z}_N \) be a discrete group with generator \( g \) which acts on the coordinates as

\[
g : X^\nu \to \alpha^{k_\nu} X^\nu,
\]

where \( \alpha = \exp(2\pi i/N) \) and \( k_\nu \) are integer charges and assume that \( D \) is an automorphism of this Calabi–Yau space. Also assume that this automorphism lifts to a vector bundle \( V \) over the Calabi–
Yau three-fold. This will be the case if one chooses the $\mathbb{Z}_N$ charges $\tilde{k}_a$ of the coordinates $\zeta_a$ of the line bundles $\mathcal{O}(n_a)$ that appear in the exact sequence (37) in a specific way [23, 20], to be discussed below. For the two vector bundles $V_1$ and $V_2$ on the orbifold planes there are two sets of charges $\tilde{k}_{1a}$ and $\tilde{k}_{2a}$, respectively. Then, the level matching condition of ref. [27] states that these charges should satisfy

$$\sum_{\nu} k_{\nu}^2 = \sum_a \tilde{k}_{1a}^2 + \sum_a \tilde{k}_{2a}^2 \mod 2N \quad (63)$$

$$\sum_{\nu} k_{\nu} = \sum_a \tilde{k}_{1a} = \sum_a \tilde{k}_{2a} = 0 \mod 2 \quad (64)$$

for $N$ even. For $N$ odd we only have the first constraint with $2N$ replaced by $N$.

In order to make the above line of thought explicit, we consider the Calabi–Yau three–fold of Example 2 defined by the intersection of two cubic polynomials in $\mathbb{CP}^5$. As we have seen, this space has a unique rank 3 symmetric vector bundle

$$V = V(2, 2; 1, 1, 1, 1, 1, 1) \quad (65)$$

with $N_{\text{gen}} = 27$. Let us choose the two cubic polynomials

$$P_1(X) = \sum_{\nu=0}^5 a_{\nu}(X^\nu)^3, \quad P_2(X) = \sum_{\nu=0}^5 b_{\nu}(X^\nu)^3. \quad (66)$$

to define the Calabi–Yau three–fold $X$. Then, for generic choices of the coefficients $a_{\nu}$ and $b_{\nu}$, the manifold $X$ is non–singular. Furthermore, $X$ admits an automorphism $D = \mathbb{Z}_3 \times \mathbb{Z}_3$ with generators $g_1$ and $g_2$ acting as

$$g_1 : X^\nu \to \alpha^{k_{\nu}} X^\nu, \quad g_2 : X^\nu \to \alpha^{l_{\nu}} X^\nu \quad (67)$$

where $\alpha = \exp(2\pi i/3)$ and $k_{\nu}, l_{\nu}$ are integer charges. Pick, for example

$$k = (0, 0, 1, 1, 2, 2), \quad l = (0, 1, 2, 0, 1, 2). \quad (68)$$

The set of fixed points under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ transformations (67) can be shown to have complex dimension one. Generically, a three–fold $X$ in $\mathbb{CP}^5$ does not intersect a curve. Hence, generically, $D$ is freely acting on $X$ and we can define the quotient Calabi–Yau three–fold $Y = X/D$. We should also lift $D$ to an automorphism of the bundle $V$. As a first step, we have to choose the $\mathbb{Z}_3 \times \mathbb{Z}_3$ charges $(\tilde{k}_a, \tilde{l}_a)$ for the coordinates $\zeta_a$ of the line bundles $\mathcal{O}(n_a)$ that appear in the exact sequence (37). Furthermore, we should pick explicit polynomials $Q_a^i(X)$ in this exact sequence. These polynomials each inherit a $\mathbb{Z}_3 \times \mathbb{Z}_3$ charge from the charges on $X^\nu$. Then $D$ lifts to a symmetry of the vector bundle $V$ if the above charges are chosen in such a way that $\zeta_a Q_a^i(X)$ is
invariant under $D$ for all $a$ and $i$. For the case at hand, this can indeed be done. For example, we can choose the charges of $\zeta_a$ as

$$\tilde{k} = (0, 1, 1, 2, 2, 2), \quad \tilde{l} = (0, 1, 2, 1, 2, 2).$$  

(69)

It follows from the structure of the vector bundle $V$ (65), that all polynomials $Q_i^a(X)$ should be linear in $X^\nu$. It is not hard to show that these linear polynomials can be chosen so that they inherit the $\mathbb{Z}_3 \times \mathbb{Z}_3$ charges $(-\tilde{k}_a, -\tilde{l}_a)$ from the coordinates $X^\nu$. It is then clear that $\zeta_a Q_i^a(X)$ is indeed invariant for all $a$ and $i$. Hence, $D$ lifts to an automorphism of the vector bundle $V$. As a consequence, $V$ defines a symmetric vector bundle on the quotient space $Y$ with $N_{\text{gen}}(Y) = 3$.

A symmetric vacuum can be constructed by taking, for example, $V_1 = V_2 = V$. Finally, we must check the level matching constraint. Having chosen $V_1 = V_2 = V$ then, for the first $\mathbb{Z}_3$ symmetry, we set $\tilde{k}_{1a} = \tilde{k}_{2a} = \tilde{k}_a$. It is easy to check that the level matching condition is satisfied. Similarly, this can be verified for the second $\mathbb{Z}_3$.

To summarize, we have found a symmetric rank 3 vector bundle $V$ with three generations. Choosing $V_1 = V_2 = V$, we obtain a symmetric vacuum with low energy gauge group $E_6 \times E_6$ and three generations in the observable as well as in the hidden sector. Furthermore, since the space $Y$ is not simply–connected, we can introduce Wilson lines to break the observable sector gauge group $E_6$ spontaneously to $SU(3) \times SU(2) \times U(1)^3$.

Let us consider another example with similar properties, this time in weighted projective space. We start with the intersection of two polynomials of degree 6 in $\mathbb{CP}^{5}_{1,1,2,2,3,3}$, as in Example 5 above. From Table [3], we use the second to last bundle

$$V = V(4, 3, 3, 3; 2, 2, 2, 1, 1, 1, 1, 1).$$  

(70)

This is a rank 5 bundle with $N_{\text{gen}} = 18$. We consider the symmetry $D = \mathbb{Z}_6$ generated by

$$g : X^\nu \to \alpha^{k_\nu} X^\nu$$  

(71)

with $\alpha = \exp(2\pi i/6)$ and the charges $k_\nu$ given by

$$k = (0, 1, 1, 3, 3, 4).$$  

(72)

One can choose two degree 6 polynomials that admit this symmetry and define a non–singular manifold. The set of fixed points under the $\mathbb{Z}_6$ transformations (71) can be shown to be at most a complex curve. Again, generically, a three–fold $X$ in $\mathbb{CP}^{5}_{1,1,2,2,3,3}$ does not intersect a curve and, therefore, the symmetry is freely acting. It follows that we can define the quotient Calabi–Yau three–fold $Y = X/D$. We must now lift $D$ to an automorphism of the bundle $V$. To do this, we have to apply the same procedure as in the previous example. We choose $\mathbb{Z}_6$ charges $\tilde{k}_a$ for the
coordinates \( \zeta_a \) and pick explicit polynomials \( Q^i_a(X) \). Then the combinations \( \zeta_a Q^i_a(X) \) should be invariant under \( D \) for all \( a \) and \( i \). For the case at hand, let us choose

\[
\tilde{k} = (1, 1, 0, 0, 4, 4, 5, 5, 0).
\]

(73)

It is then easy to show that one can pick polynomials \( Q^i_a(X) \) with the correct properties. Hence \( D \) lifts to an automorphism of the bundle \( V \). Furthermore, \( V \) specifies a bundle on \( Y \) with \( N_{\text{gen}} = 3 \). A symmetric vacuum is obtained by choosing, for example, \( V_1 = V_2 = V \). Setting \( \tilde{k}_{1a} = \tilde{k}_{2a} = \tilde{k}_a \) in eq. (63) and (64), we can verify that the level matching constraints are satisfied.

In summary, we have found a symmetric rank 5 bundle with three generations. The low energy theory has a gauge group \( SU(5) \times SU(5) \) with three generations in both the observable and the hidden sector. Again, we can introduce Wilson lines on \( Y \) to break the observable sector gauge group \( SU(5) \) spontaneously to \( SU(3) \times SU(2) \times U(1) \).

5 Properties of symmetric vacua

We would now like to discuss some of the properties of symmetric vacua and their associated four- and five-dimensional effective actions.

Let us begin with the implications for the four-dimensional effective action. Generically, this action has two types of strong coupling corrections at first non-trivial order. First, there is the well-known threshold correction to the gauge kinetic functions [7, 5, 8, 16]

\[
f^{(1,2)} = S \pm \epsilon_S \beta_i T^i
\]

(74)

where \( f^{(1)} \) and \( f^{(2)} \) correspond to the observable and hidden sector. Secondly, there are corrections to the matter field Kähler metric [8, 16] which has the form

\[
Z_{IJ} = e^{-K_T/3} \left[ K_{BIJ} - \frac{\epsilon_S \beta_i}{S + S} \tilde{\Gamma}^i_{BIJ} \right].
\]

(75)

Here \( K_T \) is the Kähler potential of the T moduli, \( K_B \) is a bundle Kähler metric and \( \tilde{\Gamma} \) some associated connection. The indices \( I, J, \ldots \) run over different generations. These quantities have been defined in ref. [25] but need not concern us in detail here. The corrections are of linear order in the strong coupling expansion parameter \( \epsilon_S \) given by

\[
\epsilon_S = \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{2\pi \rho}{v^{2/3}}.
\]

(76)

Here, we recall that \( \kappa \) is the 11-dimensional Newton constant and \( \rho \) is the radius of the orbifold while \( v \) is the Calabi–Yau volume. Furthermore, these corrections are proportional to the topological
charges $\beta_i$. This observation is crucial in our context. As already mentioned, in terms of the underlying bundles these charges are specified by

$$\beta_i(\ast \omega^i) = c_2(V_1) - \frac{1}{2} c_2(TX) = -c_2(V_2) + \frac{1}{2} c_2(TX) \ .$$

(77)

where $\{\omega_i\}_{i=1,\ldots,h^{1,1}}$ is a basis of $H^{1,1}(X)$. The quantities on the right hand side of this equation are exactly those that are set to zero for symmetric vacua. Hence we conclude that all charges $\beta_i$ vanish for symmetric vacua. Most importantly, this implies the vanishing of the strong coupling corrections in the gauge kinetic functions (74) and the matter field Kähler metric (75) above. From the above argument, those corrections vanish for topological reasons and, hence, irrespectively of the specific values of moduli. Furthermore, the gauge kinetic functions are not expected to receive corrections at higher loop order. Therefore, they are perturbatively uncorrected for symmetric vacua and are simply given by

$$f^{(1,2)} = S \ .$$

(78)

This equation holds in the weakly coupled limit as well. However, there it is valid only approximately since $|\epsilon_S \beta_i T^i| \ll |S|$ in this region of moduli space. For symmetric vacua, the important difference is that the threshold correction vanishes exactly throughout all of (large radius) moduli space. Similarly, for symmetric vacua the Kähler metric does not receive strong coupling corrections to this order and is given by

$$Z_{IJ} = e^{-K_T/3} K_{BIJ} \ .$$

(79)

Let us next discuss the five–dimensional effective action [14, 15, 16] of heterotic M–theory. This theory is a five–dimensional $\mathcal{N} = 1$ supergravity theory coupled to two four–dimensional $\mathcal{N} = 1$ theories on the orbifold planes. The strong coupling corrections manifest themselves in a gauging of the bulk supergravity. Specifically, a certain $U(1)$ isometry associated with the three–form axion in the universal hypermultiplet coset space $SU(2,1)/U(2)$ is gauged. The gauge connection is the fixed linear combination $\beta_i A^i$, where the sum runs over the graviphoton in the supergravity multiplet and the $h^{(1,1)} - 1$ vector fields in the vector supermultiplets. The $\beta_i$ coefficients are the above topological charges. As usual, this gauging implies the existence of potential energy terms in the bulk supergravity theory proportional to the coefficients $\beta_i \beta_j$. This potential obstructs flat space from being a solution of the equations of motion. Instead, the “ground state” of the five–dimensional theory turns out to be a non–trivial BPS double three–brane given by

$$ds^2_5 = a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ • 19
where

\begin{align}
a &= k_1 V_1^{1/6} \\
b &= k_2 V_2^{1/3} \\
V &= \left( \frac{1}{6} d_{ijk} f^i f^j f^k \right)^2 \tag{81}
\end{align}

and

\[ d_{ijk} f^i f^j = H_i, \quad H_i = 2\sqrt{2}k_3 \beta_i |y| + c_i \tag{82} \]

Here V is the dilaton field, \( f^i \) are functions of \( y \), \( d_{ijk} \) are the intersection numbers of the Calabi–Yau three–fold, \( k_a \) and \( c_i \) are constants and the \( \beta_i \) are the topological charges. To obtain the four–dimensional effective theory one has to reduce on this non–trivial BPS domain wall. Now for symmetric vacua, as we have seen above, \( \beta_i = 0 \). Hence the gauging, the gauge connection and the associated potential terms are absent in five–dimensional effective theories based on such vacua. In this case, the functions \( H_i \) in (82) become constants, as do the functions \( f^i \). Hence, \( a, b \) and \( V \) in (81) are constants and the BPS three–brane degenerates to a solution with flat space–time \( S^1/Z_2 \times M_4 \) and a constant dilaton.

Finally, we would like to discuss some properties of symmetric vacua. Let us first review the general situation for (not necessarily symmetric) vacua. The eleven–dimensional solution describing a vacuum is determined as an expansion around a pure Calabi–Yau background \[3\]. Only the first non–trivial terms in this expansion are known and they have been determined in ref. \[5, 12\]. The size of these first order corrections is controlled by \( \epsilon_S \), defined above, and

\[ \epsilon_R = \frac{\nu^{1/6}}{\pi \rho} \tag{83} \]

More precisely, the first order corrections are given as an expansion in harmonics on the Calabi–Yau three–fold. Hence, they have a massless part corresponding to zero eigenvalue harmonics (zero modes of the Calabi–Yau space) and a massive part corresponding to the non–zero eigenvalue harmonics. These massless and massive parts are of order \( \epsilon_S \) and \( \epsilon_S \epsilon_R \), respectively.

It is clear that, generically, this linearized solution is sensible only as long as \( \epsilon_S \ll 1 \) and \( \epsilon_S \epsilon_R \ll 1 \). If these constraints are violated, higher order terms in the equations of motion (for example quadratic terms in the eleven–dimensional Einstein equation) become important and the linear approximation breaks down. Also, beyond linear order one expects (partially unknown) corrections of order \( \kappa^{1/3} \) to the eleven–dimensional action to become important. At any rate, if the parameters (76) and (83) approach unity the linearized supersymmetric background is invalidated and, at present, there is no “all order” version that could replace it. This remains true even if
one had arranged both gauge couplings on the orbifold planes to be perturbative. As a result, supersymmetric vacua are known only in a restricted portion of the moduli space.

Let us now discuss what happens for symmetric vacua. As we have already mentioned, the corrections to the Calabi–Yau background are caused by the non–vanishing source terms in the Bianchi identity \( \text{(2)} \). For symmetric vacua, each source term in this Bianchi identity vanishes in cohomology. As a consequence, the massless part of the corrections vanishes. Indeed, the massless part is proportional to \( \epsilon_S \beta_i \) with the charges \( \beta_i \) defined above. For a symmetric embedding, the source terms in this Bianchi identity, although zero in cohomology, might not, in general, vanish identically. Correspondingly, the massive part of the corrections to the vacuum does not necessarily vanish for symmetric vacua.

Symmetric vacua remove the first obvious obstruction to making \( \epsilon_S \) large. To see this let us assume that \( \epsilon_S \epsilon_R \sim \kappa^{2/3} / v^{1/2} \) still remains small so that we do not need to worry about the massive part of the solution. Then there are no sizeable linear corrections to the Calabi–Yau background (since the massless part vanishes) and, at the same time, higher order terms in the equations of motion remain small. We still have to worry about unknown correction of order \( \kappa^{4/3} \) to the action that we have not taken into account. Those may reintroduce large corrections at the quadratic order \( \text{(4)} \). We do not know whether or not this happens but we may at least speculate that symmetric vacua are special enough to prevent such higher order corrections to occur. Then such vacua would allow one to access the part of the moduli space with \( \epsilon_S \geq 1 \).

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