Real Space Effective Interaction and Phase Transition in the Lowest Landau Level

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The transition between the stripe state and the liquid state in a high magnetic field is studied by the density-matrix renormalization-group (DMRG) method. Systematic analysis on the ground state of two-dimensional electrons in the lowest Landau level shows that the transition from the stripe state to the liquid state at $\nu \sim 3/8$ is caused by a reduction of repulsive interaction around $r \sim 3\ell$. The same reduction of the interaction also stabilizes the incompressible liquid states at $\nu = 1/3$ and $2/5$, which shows a similarity between the two liquid states at $\nu \sim 3/8$ and $1/3$. It is also shown that the strong short-range interaction around $r \sim \ell$ in the lowest Landau level makes qualitatively different stripe correlations compared with that in higher Landau levels.

KEYWORDS: stripe, liquid, transition, fractional quantum Hall effect, two dimension, density matrix, renormalization group

1. Introduction

In two-dimensional systems under a high perpendicular magnetic field, the kinetic energy of electrons is completely quenched, and macroscopic degeneracy appears in each Landau level. The macroscopic degeneracy is lifted by Coulomb interaction and various interesting ground states, including incompressible liquids,\(^1\)\\(^2\) compressible liquids,\(^3\)\\(^4\) CDW states\(^5\)\\(^6\) called stripes, bubbles and Wigner crystal are realized depending on the filling $\nu$ of Landau levels.

In the lowest Landau level, the incompressible liquid states are realized at various fractional fillings $\nu = n/(2n \pm 1)$, which are known as fractional quantum Hall states.\(^7\)\\(^8\) The Wigner crystal is realized at low fillings and its formation has been observed as a transition to an insulating state below $\nu \sim 1/5$, which is thought to be caused by the pinning of Wigner crystal by impurity potentials.\(^9\)\\(^10\)\\(^11\)

Concerning the ground state between the incompressible liquid states at $\nu = n/(2n \pm 1)$ and $1/5$, there still exist many questions. Even though recent numerical calculations based on the density-matrix renormalization-group (DMRG) method show the existence of weak stripe states,\(^12\) clear experimental evidence of the stripe formation has not yet been obtained. The stripe state obtained in the DMRG has significantly small amplitude of the stripes and the short-range correlations are similar to those of the Wigner crystal.\(^12\) It has also been shown that a transition from the stripe state to a liquid state occurs as the short-range repulsion between the electrons is reduced,\(^13\) which means that the stripe state in the lowest Landau level is realized only in narrow quantum wells.

In this paper we investigate the ground state at $\nu = 3/8$, where the stripe state is realized in ideal two-dimensional systems,\(^12\)\\(^13\) and show why the stripe state in the lowest Landau level is realized only in narrow quantum wells. From the analysis on the change in the pair correlation function through the transition to a liquid state, we show that the stripe state is stabilized by the strong repulsive interaction around $r \sim 3\ell$. We also show that the same repulsive interaction around $r \sim 3\ell$ reduces the excitation gap of the incompressible liquid states at $\nu = 1/3$ and $2/5$, which suggests a similarity between the incompressible liquid state at $\nu = 1/3$ and the liquid state at $\nu = 3/8$. We finally explain the origin of the difference in the stripe correlation between the lowest and higher Landau levels by comparing the Coulomb interaction projected onto each Landau level.

2. Model and Method

We use the Hamiltonian of two-dimensional electrons in a perpendicular magnetic field. Since the kinetic energy of the electrons is completely quenched, we can omit this energy. The Hamiltonian of the electrons in the lowest Landau level is then written by

$$H = \sum_{i<j} \sum_{\mathbf{q}} e^{-q^2\ell^2/2}V(q) e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)}$$

(1)

where $R_i$ is the guiding center coordinate of the $i$th electron, which satisfies the commutation relation, $[R^x_i, R^y_j] = i\ell^2 \delta_{ij}$ and $V(q)$ is the Fourier transform of the Coulomb interaction, which is given by $2\pi e^2/\varepsilon \ell$ in ideal two-dimensional systems. We set the magnetic length $\ell$ equal to unity, and use $e^2/\varepsilon \ell$ as units of energy. We assume that the magnetic field is strong enough to polarize the spins and suppress the Landau level mixing.

In order to obtain the ground-state wave function, we use the DMRG method, which was originally developed for one-dimensional quantum systems.\(^14\) We apply this method to two-dimensional systems divided into unit cells $L_x \times L_y$ with the periodic boundary conditions for both $x$- and $y$-directions.\(^15\)\\(^16\)\\(^17\) This method enables us to obtain the essentially exact ground state of large systems extending the limitation of the exact diagonalization method with controlled accuracy. The truncation error in the ground-state wave function is typically $10^{-4}$ with keeping 200 eigenvectors of the density matrix.

We calculate the ground-state energy and the wave function for various size of systems with up to 18 elec-
tions in the unit cell, and analyze the pair correlation function $g(r)$ defined by

$$g(r) \equiv \frac{L_x L_y}{N_e(N_e-1)} \langle \Psi | \sum_{i \neq j} \delta(r + R_i - R_j) | \Psi \rangle$$  \hspace{1cm} (2)$$

where $| \Psi \rangle$ is the ground state with $N_e$ being the number of electrons in the unit cell. Since the period of the stripes is artificially modified by changing the aspect ratio $L_x/L_y$, we need to find the unit cell that has the energy minimum with respect to $L_x/L_y$. The correlation functions in such unit cell are expected to have the correct period of the stripes realized in the thermodynamic limit.

3. **Pseudopotentials and Phase Transition**

In ideal two-dimensional systems, the stripe ground state in the lowest Landau level is realized around $\nu = 0.42, 0.37$ and between 0.32 and 0.15. However, experimental evidence of the stripe formation has not been obtained. Instead, the transport experiments on ultra high mobility wide quantum wells suggest the formation of liquid states even at the fillings where the stripe state is numerically obtained. Since the finite width of the two-dimensional system reduces the short-range repulsion between the electrons, we here study the stability of the stripe state against the reduction of short-range interaction.

We first consider the effect of the decrease in short-range components of Haldane’s pseudopotentials, $V_1$ and $V_3$. Since the change in $V_1$ does not affect the relative energy difference between the stripe and liquid states, we consider the case of $\delta V_1 = \delta V_3$, where $\delta V_m = V_m - (V_m)_{2D}$ with $(V_m)_{2D}$ being the pseudopotentials of the pure Coulomb interaction in ideal two-dimensional systems. We calculate the ground state for wide range of $\delta V_3$ to confirm the transition to the liquid state. The ground-state energies of the stripe state and the liquid state at $\nu = 3/8$ are shown in Fig. 1. We find relative decrease in the energy of the liquid state with the decrease in $\delta V_3$. For small $V_3$ below $\delta V_3 \sim -0.032$, the liquid state has lower energy independent of $L_x/L_y$, which clearly shows the existence of a transition to the liquid state.

The correlation functions for the liquid ground state at $\delta V_1 = \delta V_3 = -0.024$ and the stripe ground state at $\delta V_1 = \delta V_3 = 0$ are presented in Figs. 2 (a) and (b), respectively. The size dependence of the stripe correlation has been analyzed for systems with up to 24 electrons in the unit cell, and the stripe correlations are shown to be almost the same even for large systems. Since the correlation function and the total momentum of the stripe and liquid states are both different, we expect a first order transition even in the thermodynamic limit. The critical value of $\delta V_3$ in the case of $\delta V_m = 0$ for $m \geq 5$ is expected to be around $-0.02$, below which the energy of the liquid state has clear minima at $L_x/L_y \sim 1.3$ and 2.2, which continuously connect to the corresponding minima of the liquid state below $\delta V_3 \sim -0.032$.

To further investigate the nature of the transition, we next compare the correlation functions between the two states. The difference between the two correlation functions in Figs. 2 (a) and (b) is presented in Fig. 2 (c). We find a clear peak at $x \sim y \sim 2$ and a dip around $r \sim 4$, which means that the electrons in the liquid state are more likely to approach each other. The existence of the clear peak at $r \sim 3$ $(x \sim y \sim 2)$ also suggests that the energy difference between the two states is sensitive to the change in the interaction at $r \sim 3$.

Figure 3 shows the Haldane’s pseudopotentials for relative momentum $m$ represented in real space which is
defined by the following equation,
\[
\mathcal{V}_m(r) = \frac{1}{2\pi} \int d^2 q \, L_m(q^2) e^{-q^2/2} \exp(-i\mathbf{q} \cdot \mathbf{r})
\tag{3}
\]
where \(L_m(x)\) are Laguerre polynomials. The effective interaction \(\mathcal{V}_{\text{eff}}(\mathbf{r})\) between the two-dimensional electrons in guiding center coordinates is then given by
\[
\mathcal{V}_{\text{eff}}(\mathbf{r}) = \sum_m 2V_m \mathcal{V}_m(\mathbf{r}).
\tag{4}
\]
We find that the pseudopotential \(\mathcal{V}_m(r)\) for \(m = 3\) has a maximum at \(r \sim 3\), which means the decrease in \(V_3\) stabilizes the liquid state. This is consistent with the results in Fig. 1.

In contrast to the pseudopotential for \(m = 3\), the pseudopotential for \(m = 5\) has a minimum around \(r \sim 3\), which suggests a relative increase in the energy of the liquid state with the decrease in \(V_5\). This is actually shown in Fig. 4, where we find clear energy minimum for the stripe state, which shows the stability of the stripe state even in the case of \(\delta V_1 = \delta V_3 = 0.016\).

With increasing the relative momentum \(m\), the pseudopotential \(\mathcal{V}_m(r)\) oscillates rapidly as shown in Fig. 3. To see the effect of large \(m\), we next decrease \(V_m\) up to \(m = 11\). The results for the case of \(\delta V_5 = \delta V_7 = \delta V_9 = \delta V_{11} = \delta V_{13}/4\) is presented in Fig. 5. We find the energy of the liquid state relatively decreases with the decrease in \(V_m\). This means the effect of \(V_5\) is canceled by the decrease in \(V_m\) of higher \(m\). Indeed we find a peak of pseudopotential at \(r \sim 3\) for \(m = 7\). The tendency of the stabilization of the liquid state does not change even for larger systems of 18 electrons in the unit cell as shown in Figs. 5 (c) and (d).

**4. Similarity to the Incompressible Liquids**

To clarify the similarity and the difference between the two liquid states at \(\nu = 3/8\) and \(\nu = n/(2n + 1)\), we here consider the role of short-range repulsion in the incompressible states at \(\nu = n/(2n + 1)\). At \(\nu = 1/3\), it has been shown that the Laughlin state is an exact ground state in the limit of \(V_1 \to \infty\). This means the incompressible liquid state is stabilized by the increase in \(V_1\). Figure 6 actually shows that the excitation gap at \(\nu = 1/3\) and \(2/5\) monotonically decreases with the decrease in \(V_1\) and it seems to vanish at \(\delta V_1 \sim -0.1\) whose value is comparable to the size of the gap at \(\nu = 1/3\) at \(\delta V_m = 0\). This is contrasted to the fact that \(V_1\) does not make clear energy difference between the liquid and stripe states at \(\nu \sim 3/8\).

We next see the effect of \(V_3\). Figure 6 also shows the gap obtained in the case of \(\delta V_3 = \delta V_1\). We find the gap is always larger than that for \(\delta V_3 = 0\), which means the reduction in \(V_3\) enhances the excitation gap. This is similar to the stabilization of the liquid state at \(\nu \sim 3/8\) with the decrease in \(V_3\). To further confirm the similarity to the liquid state at \(\nu \sim 3/8\), we next decrease \(V_3\). The gap at \(\delta V_5 = \delta V_1\) is presented in Fig. 6 with the dotted line. In this case the gap is smaller than that obtained at \(\delta V_5 = 0\), which shows the decrease in \(V_5\) destabilizes the liquid state. These results suggest that the liquid state at \(\nu = 3/8\) has a similar character to the liquid states at \(\nu = n/(2n + 1)\) although the liquid states at \(\nu = n/(2n + 1)\).
Fig. 6. \( V_1, V_3 \) and \( V_5 \) dependences of the excitation gap \( \Delta \) in the incompressible liquid states at \( \nu = 1/3 \) and 2/5. \( N_e = 12 \).

Fig. 7. The effective interaction between the electrons in the \( N \)th Landau level in guiding center coordinates. \( R_c \) is the classical cyclotron radius.

have large excitation gap, which is controlled by \( V_1 \).

5. Effective Potential in Higher Landau Levels

We finally consider the origin of the difference in the stripe correlation between the lowest and higher Landau levels. The stripe state in the lowest Landau level is qualitatively different from the stripe state in higher Landau levels as shown in Figs. 2 (b) and (d). The stripes in higher Landau levels have large amplitude and the pair correlation function has a shoulder structure around \( r \sim 2 \) along the perpendicular direction to the stripes. These results show that the electrons are likely to form clusters in higher Landau levels in guiding center coordinates.

The ground state in each Landau level is determined by the Coulomb interaction projected onto each Landau level. This interaction is represented by a set of Haldane's pseudopotentials \( V_m \). In Fig. 7 we show the real space representation of the pseudopotentials defined in eq. (4). In the lowest Landau level we find monotonic decrease in the pseudopotential, but in higher Landau levels we find non-monotonic decrease. The non-monotonic structure appears below twice the classical cyclotron radius \( 2R_c = 2\sqrt{2N + 1} \), which is shown by the arrows in Fig. 7.

When we fix the guiding centers of the two electrons, the other electrons feel the potential made by the two fixed electrons, which is shown in Fig. 8 by the solid lines, where \( \Delta x \) is the distance between the two fixed electrons in guiding center coordinates. In the lowest Landau level, we always find the potential minimum at the center of the two fixed electrons. This is due to the monotonic decrease in the effective interaction \( V_{eff} \). Thus the electrons always keep away from the other electrons, and this is the reason why the short-range correlations in the type-II stripe state are similar to those of Wigner crystal. In higher Landau levels, however, the effective potential has potential minima near the two fixed electrons, which are made by the non-monotonic decrease in \( V_{eff} \). These potentials suggest that the electrons in higher Landau levels tend to form clusters when the mean distance between the electrons is not so long compared with the classical cyclotron radius \( R_c \) as is discussed in the existing paper.\(^{20}\) This is the origin of the formation of the stripes and bubbles in higher Landau levels. Thus the difference in the stripe correlation between the lowest and higher Landau levels is naturally explained from the qualitative difference in the effective interaction between the electrons.

6. Summary

In the present study we have calculated the ground state of two-dimensional electrons for various pseudopotentials. The obtained results show that the liquid state at \( \nu \sim 3/8 \) is stabilized by the decrease in the effective repulsion around \( r \sim 3 \). Similar stabilization of the liquid state at \( \nu = 1/3 \) and 2/5 suggests that the liquid states at \( \nu \sim 3/8 \) and \( n/(2n + 1) \) have a similar char-
acter although the liquid states at $\nu = n/(2n + 1)$ have large excitation gap which is enhanced by the increase in $V_1$. We have also investigated the reason of the difference in the stripe correlations between the lowest and higher Landau levels and shown that the monotonic real-space profile of the effective interaction in the lowest Landau level is the origin of the difference.

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