Noncommutative Generalized NS and Super Matrix KdV Systems from a Noncommutative Version of (Anti-)Self-Dual Yang-Mills Equations

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Abstract

A noncommutative version of the (anti-) self-dual Yang-Mills equations is shown to be related via dimensional reductions to noncommutative formulations of the generalized $(SO(3)/SO(2))$ nonlinear Schrödinger (NS) equations, of the super- Korteweg - de Vries (super-KdV) as well as of the matrix KdV equations. Noncommutative extensions of their linear systems and bicomplexes associated to conserved quantities are discussed.

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1 Introduction

Noncommutative geometry has recently been involved in a noncommutative version of gauge theory [1] related to strings and has been stimulated by different works on field theories defined over noncommutative spaces (for example [2, 3, 4] and references therein). Naturally, classical integrable models have also been generalized to noncommutative spaces ([5, 6, 7] and references therein), and, for instance, noncommutative versions of the Toda, nonlinear Schrödinger (NS) and Korteweg - de Vries (KdV) equations have been formulated. Some properties of these “deformed” versions have also been shown. Using bicomplexes, an infinite set of conserved quantities has been found, which as well suggest the complete integrability of these modified systems. In view of the results on the deformation of the ADHM construction [8] and its twistor interpretation [9], a formulation of (anti-)self-dual Yang-Mills (abbreviated below (A-) SDYM) equations on noncommutative spaces has been presented [10]. Many twistor and integrability properties were shown to be preserved in this setting. The “deformation” equations are simply obtained by substituting the product of fields with a Moyal product in the classical form of the (A-) SDYM equations. Dimensional reductions to the (noncommutative) principal chiral field model and Hitchin equations are discussed in [10], and integrability properties inherited from the (A-) SDYM equations.

In this short article, dimensional reductions of a noncommutative version of the (A-) SDYM equations on noncommutative Euclidean (E^4) and pseudo-Euclidean (E^{2,2}) spaces are studied with also the help of conditions on the gauge fields. First, generalized nonlinear Schrödinger (NS) equations and matrix KdV equations on noncommutative spaces are derived with associated linear systems (or Lax pairs). The integrability of these systems is suggested
from the (anti-) self-dual Yang-Mills equations, the presence of an infinite set of conserved quantities, and bicomplexes, which are themselves linked to the reduced linear systems. A noncommutative version of a supersymmetric KdV system is also derived in a similar manner. Some of these results are an adaptation of certain methods used in ref. [11] and references therein for closely related problems, i.e. for the commutative version of the systems presented below. It relates to the approaches of refs [6, 7] and [10]. However, it does not appear that general commutative reductions could be extended to noncommutative versions in the same manner.

Section 2 introduces the notation and different elements needed for the description of systems on noncommutative space, such as Lax pairs or linear systems on noncommutative Euclidean and signature zero pseudo-Euclidean spaces. Then in section 3, the linear systems, bicomplexes, and generalized nonlinear Schrödinger equations are obtained as dimensional reductions accompanied with specific gauge fields forms (or ansatzes) of the corresponding (A-) SDYM equations and their linear system on an identical noncommutative space. Section 4 deals with the matrix KdV equations, and also presents a noncommutative version of the supersymmetric (matrix) KdV equations, again as reductions of the SDYM equations and their linear system(s) on the appropriate noncommutative space. Finally, section 5 suggests developments and comments on the previous sections.
2 Bicomplexes, Linear Systems, and Noncommutative Formulations

Definitions and applications of bicomplexes can be found for example in the following refs [12, 13, 14, 15]. For our purposes, let us use the following definition below (see for example [15]). A bicomplex corresponds to a linear space over $\mathbb{R}$ or $\mathbb{C}$, here denoted $V$, endowed with a grading over the non-negative integers, i.e.

$$V = \bigoplus_{i \geq 0} V^i,$$

and two (linear) maps (operators) $d$ and $\delta$ between successive spaces $V^i$ and $V^{i+1}$ in other words, $d : V^i \to V^{i+1}$, and $\delta : V^i \to V^{i+1}$, such that :

$$d^2 \cdot = 0, \quad \delta^2 \cdot = 0, \quad (\delta d + d\delta)\cdot = 0,$$

(2.1)

where $\cdot$ stands for an element of $V$.

A set of bicomplexes can be related to linear systems of the (A-)SDYM equations. For instance, the (A-)SDYM equations on 4-dimensional Euclidean space ($\mathbb{E}^4$) have the following linear systems [16]:

$$[D_1 + iD_2 - \lambda(D_3 \pm iD_4)]\Psi(x, \lambda, \bar{\lambda}) = 0$$

$$[D_3 \mp iD_4 + \lambda(D_1 - iD_2)]\Psi(x, \lambda, \bar{\lambda}) = 0,$$

(2.2)

$$\partial_\lambda \Psi(x, \lambda, \bar{\lambda}) = 0,$$

where the lower sign in the above equations correspond to the A-SDYM formulation, $\lambda \in \mathbb{C}P^1$, $D_\mu = \partial_\mu + A_\mu$, and $\Psi$ is a $\mathbb{C}$-valued column vector.

On the 4-dimensional pseudo-Euclidean space of signature 0 ($\mathbb{E}^{(2,2)}$) with diagonal metric $(+,+,\mp,-,\mp)$, one finds the following set of linear equations
\[ D_1 + iD_2 + \lambda(D_3 \mp iD_4)]\Psi(x, \lambda, \bar{\lambda}) = 0 \]
\[ (D_3 \pm iD_4 + \lambda(D_1 - iD_2)]\Psi(x, \lambda, \bar{\lambda}) = 0, \tag{2.3} \]
\[ \partial_{\bar{\lambda}}\Psi(x, \lambda, \bar{\lambda}) = 0, \]

where here too, the lower sign applies to the A-SDYM equations, and \( \lambda \in \) (a sheet of hyperboloid \( \mathbb{H}^2 \)).

The compatibility equations of the linear systems (2.2) and (2.3) are, respectively, the (A-)SDYM equations on \( E^4 \) or \( E^{(2,2)} \).

Note that, for simplicity in later calculations, the A-SDYM equations on \( E^4 \) could be transformed to [10, 19]:

\[ F_{z_1 z_2} = 0, \quad F_{\bar{z}_1 \bar{z}_2} = 0, \quad F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \tag{2.4} \]

where \( F_{z_1 z_2} = \partial_{z_1}A_{z_2} - \partial_{z_2}A_{z_1} + [A_{z_1}, A_{z_2}] \), and \( F_{z_1 \bar{z}_2} = \partial_{z_1}A_{\bar{z}_2} - \partial_{\bar{z}_2}A_{z_1} + [A_{z_1}, A_{\bar{z}_2}] \), using the following change to null variables: \( z_1 = x_3 + ix_4, \bar{z}_2 = x_1 + ix_2 = x_1 - ix_2, \) with \( i, j = 1, 2 \).

Accordingly, the associated linear system becomes:

\[ [(\partial_{z_1} - \lambda\partial_{z_2}) + (A_{z_1} - \lambda A_{z_2})]\Psi(z_i, \bar{z}_j, \lambda, \bar{\lambda}) = 0, \]
\[ [(\partial_{\bar{z}_2} + \lambda\partial_{\bar{z}_1}) + (A_{\bar{z}_2} + \lambda A_{\bar{z}_1})]\Psi(z_i, \bar{z}_j, \lambda, \bar{\lambda}) = 0, \tag{2.5} \]
\[ \partial_{\bar{\lambda}}\Psi(z_i, \bar{z}_j, \lambda, \bar{\lambda}) = 0. \]

As for the SDYM equations on \( E^{(2,2)} \), a change to null variables [11, 17, 18] :

\[ t = \frac{1}{\sqrt{2}}(x^2 - x^4), \quad y = \frac{1}{\sqrt{2}}(x^1 - x^3), \]
\[ u = \frac{1}{\sqrt{2}}(x^2 + x^4), \quad z = \frac{1}{\sqrt{2}}(x^1 + x^3), \tag{2.6} \]
leads to the following corresponding linear system on $\mathbb{E}^{(2,2)}$:

\[
(D_z + \omega D_u)\Psi(x, \omega, \bar{\omega}) = 0,
\]
\[
(D_t - \omega D_y)\Psi(x, \omega, \bar{\omega}) = 0,
\]
\[
\partial_{\bar{\omega}}\Psi(x, \omega, \bar{\omega}) = 0,
\]

(2.7)

where the parameter $\omega = i\frac{1 - \lambda}{(1 + \lambda)}$, and $D_t = \frac{1}{\sqrt{2}}(D_2 - D_4)$, $D_u = \frac{1}{\sqrt{2}}(D_2 + D_4)$, $D_y = \frac{1}{\sqrt{2}}(D_1 - D_3)$, $D_z = \frac{1}{\sqrt{2}}(D_1 + D_3)$.

However, if one builds a bicomplex based on the previous type of linear systems (2.2, 2.3, 2.5) with parameter $\lambda$

\[
D_1\Psi = [O_1 + \lambda O_1^\lambda] = 0,
\]
\[
D_2\Psi = [O_2 + \lambda O_2^\lambda] = 0,
\]

(2.8)

as the set of two operators $d$ and $\delta$ on $\Psi : \mathbb{R}^4 \to \mathbb{C}^n, \in V^0 \mathbb{R}, \mathbb{C}$:

\[
d\Psi = O_1\Psi_1 + O_2\Psi_2,
\]
\[
\delta\Psi = O_1^\lambda\Psi_1 + O_2^\lambda\Psi_2,
\]

(2.9)

or, in short, $d\Psi + \lambda\delta\Psi = 0$, which resembles the “linear equation” formulation of [3, 4], where $\xi_1, \xi_2 \in \Lambda^1$, (which can be simply extended to $V$), then the conditions for these operators to form a bicomplex: $d^2 = 0, \delta^2 = 0, d\delta + \delta d = 0$, correspond exactly to the compatibility or integrability conditions of the linear system (2.8), and provide the (A-)SDYM equations on the respective space for the linear systems (2.2), (2.3), (2.5). One notes that the system (2.9) is as well invariant under gauge transformations.

Now, it would of interest to introduce a noncommutative version of the four dimensional pseudo-Euclidean space of signature 0 ($\mathbb{E}^{(2,2)}$). Let us as-
sume for the coordinates $x^\mu, \mu, \nu = 1, 2, 3, 4$, the following commutation relations:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (2.10)$$

where the quantities $\theta^{\mu\nu}$ are real constants. The associative product on the space of functions over $\mathbb{E}^{(2,2)}$ is then substituted here by the (associative, but noncommutative) Moyal product $\star$ denoted $\ast$, of two functions $f$ and $g$ on $\mathbb{E}^{(2,2)}$:

$$(f \ast g)(x) = \exp\left[ \sum_{\mu, \nu=1}^{4} \frac{i}{2} \theta^{\mu\nu} \partial_x \partial_{\tilde{x}} \right] f(x) g(\tilde{x}) \big|_{x^\mu = \tilde{x}^\mu}. \quad (2.11)$$

A simple noncommutative version of the (A-)SDYM equations on $\mathbb{E}^4$ or $\mathbb{E}^{(2,2)}$ can then be obtained by using the above $\ast$-product instead of the usual commutative product of two functions. Thus:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \ast A_\nu - A_\nu \ast A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (2.12)$$

stands for the field strength of the gauge field components $A_\mu$. The (A-)SDYM equations are then invariant under the gauge transformations:

$$A'_\mu = g^{-1} \ast A_\mu \ast g + g^{-1} \ast \partial_\mu g, \quad (2.13)$$

with $g^{-1}$ the inverse of $g$, such that $g^{-1} \ast g = 1$. Note that $\partial_\mu$ is still a "derivation" on the noncommutative spaces $\mathbb{E}^4$ or $\mathbb{E}^{(2,2)}$, and that the gauge group has not been specified yet, but such gauge theories have been explored for $U(n)$ as gauge group $[11]$. Also, it is noticed that this version differs from versions of noncommutative theories found for instance in refs $[2]$ and $[3]$. 

In the following sections, slight modifications of known reductions of the (commutative) (A-)SDYM equations to (classical) integrable systems will be
used to derive (noncommutative) versions of the same integrable models, via translational reductions of the (A-)SDYM equations. Corresponding reduced noncommutative linear systems will also be obtained, and their integrability associated to the compatibility of the original (A-)SDYM equations.

3 Generalized NS Equations

As presented in refs \[19\] and \[21\], the NS equations can be derived from the SDYM equations on \(\mathbb{E}^{(2,2)}\) using translational invariance along \(z_2\) and \(z_1 - \bar{z}_1\), with the “ansatz”:

\[
A_{z_1} = 0 \quad A_{\bar{z}_1} = \begin{bmatrix} 0 & -q \\ r & 0 \end{bmatrix} \]

\[
A_{z_2} = -\kappa \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_{\bar{z}_2} = -\frac{1}{2\kappa} \begin{bmatrix} q \ast r & q_x \\ r_x & -r \ast q \end{bmatrix}. \tag{3.1}
\]

The residual linear system (2.5) takes the form:

\[
\begin{align*}
[\partial_x + (A_{\bar{z}_1} - \lambda A_{z_2})] \ast \Psi &= 0, \\
[\partial_t + \lambda \partial_x + A_{\bar{z}_2}] \ast \Psi &= 0, \\
\partial_\lambda \Psi &= 0,
\end{align*} \tag{3.2}
\]

and then have the noncommutative generalized NS equations given below as compatibility equations:

\[
2\kappa q_t = q_{xx} + 2 q \ast r \ast q,
\]

\[
2\kappa r_t = -(r_{xx} + 2 r \ast q \ast r), \tag{3.3}
\]

where: \(x = z_1 + \bar{z}_1\), \(t = \bar{z}_2\), and \(\kappa\) is a constant.
The equations (3.3) coincide with the equations obtained in ref. [5] from an almost similar bicomplex, with \( q = \bar{r} \) and \( \kappa = \frac{i}{2} \). Let us add that conserved quantities for this noncommutative system would be derivable in a manner similar to the approach found in ref. [5].

4 Super Matrix KdV Equations

In a follow up to the work of refs ([21]) and ([22]) on the reduction of the (A-)SDYM equations to the (commutative) KdV equation, the (commutative) matrix KdV equations were deduced from the same original (A-)SDYM equations via translational symmetries ([17], [18]), and then using Lie superalgebra valued gauge fields, a supersymmetric version of the matrix KdV model was exhibited ([11]).

The symmetry reductions applied in refs ([19]) and ([21]) have not allowed us to derive a noncommutative form of these equations via the same procedure used on noncommutative (A-)SDYM equations. Instead, the formulations of refs ([11], [17], [18], and [22]) have been found more suitable for this purpose. Indeed, starting from the linear system (2.7) and imposing translational symmetries along the coordinates \( u \) and \( y - z \), one finds the residual linear equations:

\[
\frac{\partial}{\partial t} + A_t + \omega(A_z - A_y) + \omega^2 A_u \right] \Psi = 0,
\]

\[
\text{[} \frac{\partial}{\partial x} + A_z + \omega A_u \text{]} \right] \Psi = 0,
\]

(4.1)

Then, the additional expressions of the gauge components with values in a real form of \( sl(n, \mathbb{C}) \otimes \mathcal{A} \), where \( \mathcal{A} \) identifies the set of functions on noncom-
mutative \( \mathbb{E}^{(2,2)} \), are introduced:

\[
A_u(t, x) = \begin{bmatrix}
0_n & 0_n \\
-1_n & 0_n
\end{bmatrix}, \quad A_z(t, x) = \begin{bmatrix}
0_n & 0_n \\
U_n(t, x) & 0_n
\end{bmatrix},
\]

\[
A_{z-y}(t, x) \equiv A_z(t, x) - A_y(t, x) = \begin{bmatrix}
0_n & 0_n \\
0_n & 0_n
\end{bmatrix}, \quad (4.2)
\]

\[
A_t(t, x) = \begin{bmatrix}
0_n & 0_n \\
3U_n * U_n + U_{n,xx} & 0_n
\end{bmatrix},
\]

where the subindex \( n \) indicates the dimension of the matrix involved, i.e. \( n \times n \).

One can mention that the formulation of ref. \[22\] can also provide the same noncommutative version of the KdV equations, which arise as the compatibility of the above linear system (4.1) with components (4.2):

\[
U_{n,t} = 3(U_{n,x} * U_n + U_n * U_{n,x}) + U_{n,xxx}, \quad (4.3)
\]

which provides when \( n = 1 \):

\[
U_t = 3(U_x * U + U * U_z) + U_{xxx}, \quad (4.4)
\]

originally presented in ref \[7\] as a noncommutative version of the KdV equation, but with a different path. An infinite set of conserved densities can be derived using a noncommutative version of the transformation presented in \[22\] : \( U = W + \lambda W_x + \lambda^2 W * W \) \[7\].

On the other hand, a noncommutative version of a supersymmetric KdV equation can also be produced from the linear systems (4.1) by inserting the following ansatz for the gauge field components into the compatibility
The reduced equations SDYM equations using the above fields with values in the Lie superalgebra $gl(2n/n)$, where $\theta$ is an odd Grassmann variable, and with $U$ and $\phi$ being respectively even and odd degree variables depending on $x$ and $t$, have the form:

\[ U_t = 3U_x * U + U_{xx} - \frac{3}{2} \phi * \phi_x + \frac{3}{2} \phi_x * \phi - \frac{3}{2} \phi_x * \phi_x, \]
\[ \phi_t = \phi_{xxx} + \frac{3}{2} \phi_x * U + \frac{3}{2} \phi * U_x + \frac{3}{2} U_x * \phi + \frac{3}{2} U * \phi_x \]  

It can be verified that these noncommutative equations are left invariant under the following supersymmetry transformations, induced by the odd Grassmann parameter $\epsilon$:

\[ \delta_\epsilon U = \epsilon \phi_x, \quad \text{and} \quad \delta_\epsilon \phi = \epsilon U. \]  

A derivation of a noncommutative formulation of supersymmetric matrix KdV equations is similar and the resulting equations can be written by adding a subscript $n$ to the variables $U$ and $\phi$ in the equation (4.6) above.
5 Summary/Conclusion

This paper has shown a relation via the procedure of reduction \([24, 25, 26]\) for translations between a noncommutative version of (anti-) self-dual Yang-Mills equations and noncommutative formulations of diverse integrable systems, the generalized NS and matrix KdV equations, as well as a supersymmetric integrable model, the super matrix KdV system. It could be seen as an extension of results published by ref. [21] and ref. [11] in the direction of noncommutative theories. For each of these noncommutative versions of integrable models, a corresponding noncommutative linear system has been exhibited, and a link to bicomplexes provided. Conserved densities would be obtainable in a similar fashion to the cases presented by refs [6, 7].

Many directions can then be followed as future developments. One may want to explore the set of solutions of these noncommutative models, and more explicitly, look at the possibilities for solitons, or related solutions in the 0 limit of \(\theta\) (see [7]). The integrability, Hamiltonian, and reduced twistor interpretations could also be probed, as well as further reductions to other integrable equations such as (2+1) systems [27], using varied constraints and symmetries. Moreover, other formulations of noncommutative gauge theories could be examined in similar manners through a reduction process.

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