ON HOMOGENEOUS POLYNOMIALS DETERMINED BY THEIR PARTIAL DERIVATIVES

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Abstract. We prove that a generic homogeneous polynomial of degree $d$ is determined, up to a nonzero constant multiplicative factor, by its partial derivatives of order $k$ whenever $k \leq \frac{d}{2} - 1$.

1. Introduction

We investigate in this note the reconstructibility of a homogeneous polynomial from its partial derivatives. The study can date back to J. Carlson and Ph. Griffiths, who in [1] showed that a generic homogeneous polynomial could be reconstructed, up to a nonzero constant multiple, from its Jacobian ideal, or equivalently, from its first order partial derivatives; in that paper, they used this result to study variation of Hodge structures and proved the global Torelli theorem for hypersurfaces. For further developments of the determination of a homogeneous polynomial by its Jacobian ideal, see [3] and references therein.

In the classical theory of variation of Hodge structures for smooth hypersurfaces, as in [1], only first order derivatives of the defining homogeneous polynomials are involved. We can also construct higher order versions of this classical theory; see for instance [2]. In this higher order analogous theory, a problem arises concerning the reconstructibility of a homogeneous polynomial from its higher order partial derivatives. In this paper, we will solve this problem and prove that a generic homogeneous polynomial has the desired property.

Let $S = \mathbb{C}[x_0, x_1, \cdots, x_n]$ be the graded polynomial ring in $n + 1$ variables with coefficients in $\mathbb{C}$

$$S = \bigoplus_{d=0}^{\infty} S_{n,d},$$

where $S_{n,d}$ is the vector space of homogeneous polynomials of degree $d$. Given $f \in S_{n,d}$ and a natural number $k \geq 0$. Denote by $J_k(f)$ the graded ideal of $S$ generated by all partial derivatives of $f$ of order $k$ and by $E_{d-k}(f)$ the degree $d-k$ homogeneous component of $J_k(f)$, that is, the vector space spanned by all $k$-th order partial derivatives of $f$. We will prove the following theorem.

Theorem 1.1. Given $n \geq 1$ and $d \geq 3$, and $k \geq 1$ a natural number such that $k \leq \frac{d}{2} - 1$. Suppose $f$ is a generic homogeneous polynomial in $S_{n,d}$.

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Let \( g \) be another homogeneous polynomial in \( S_{n,d} \) such that \( E_k(f) = E_k(g) \), then \( g \in C^*f \).

The underlying idea in the proof is very simple, so we give an outline here. We will show that \( E_{k-1}(g) = E_{k-1}(f) \), then apply induction on \( k \) to obtain \( E_r(g) = E_r(f) \) for all \( 0 \leq r \leq k \). Since \( E_0(f) \) is essentially nothing but \( Cf \), the conclusion follows immediately.

Note that we already have a more precise result, Theorem 1.1 in [3], for the case \( k = 1 \). But we do not need to use it to prove Theorem 1.1; instead, we will use induction on \( k \) until the case \( k = 0 \) is reached. In addition, the restriction \( k \leq \frac{d}{2} - 1 \) is given in order to ensure that \( \dim E_{k+1}(f) = \dim S_{n,k+1} \) for a generic \( f \), see Lemma 2.5 below.

We almost proved the above theorem when we wrote the paper [3], but we did not write down the complete proof due to the lack of adequate knowledge of its applications. We would like to thank Professor A. Dimca for his kindly pointing out the applications to the study of higher Jacobians, associated polar maps, and so on. We thank the referee for valuable comments. We also thank Yau Mathematical Sciences Center for their financial support and wonderful working atmosphere.

**Notations**

As in the introduction, \( S_{n,d} \) denotes the vector space of homogeneous polynomials of degree \( d \).

The multi-index set
\[
\mathbb{N}^{n+1} = \{(i_0, i_1, \ldots, i_n) : i_j \geq 0 \text{ for } j = 0, 1, \ldots, n\}.
\]

We denote by \( I \) an element of \( \mathbb{N}^{n+1} \). We shall see \( \mathbb{N}^{n+1} \) as a subset of the vector space \( \mathbb{R}^{n+1} \); among operations on \( \mathbb{N}^{n+1} \) are addition, subtraction and multiplication by a positive integer:
\[
I \pm I' = (i_0 \pm i'_0, i_1 \pm i'_1, \ldots, i_n \pm i'_n)
\]
and
\[
mI = (mi_0, \ldots, mi_n)
\]
for \( I = (i_0, \ldots, i_n), I' = (i'_0, \ldots, i'_n) \), and \( m \in \mathbb{N} \).

Denote by \( e_j, j = 0, \ldots, n \) the canonical basis of \( \mathbb{R}^{n+1} \):
\[
e_j = (0, \ldots, 0, 1, 0, \ldots, 0),
\]
where 1 lies in the \( j \)-th entry. Using this basis, we may write \( I = (i_0, \ldots, i_n) \) as \( I = \sum_{j=0}^{n} i_j e_j \).

Moreover, there is an obvious partial ordering “\( \geq \)” on \( \mathbb{N}^{n+1} \), with
\[
I = (i_0, i_1, \ldots, i_n) \geq I' = (i'_0, \ldots, i'_n) \iff i_j \geq i'_j, \quad j = 0, \ldots, n,
\]
or more concisely,
\[
I \geq I' \iff I - I' \in \mathbb{N}^{n+1}.
\]
The order of $I = (i_0, \cdots, i_n)$:

$$|I| = i_0 + \cdots + i_n.$$ 

For $f \in S_{n,d}$, the partial derivative of $f$ of type $I$ is

$$D_I f = \frac{\partial^{|I|} f}{\partial x_0^{i_0} \partial x_1^{i_1} \cdots \partial x_n^{i_n}}.$$ 

By definition, $E_k(f)$ is the vector subspace of $S_{n,d-k}$ spanned by $D_I f$, $|I| = k$; thus we have

$$E_k(f) = \langle D_I f : |I| = k \rangle.$$ 

2. Polynomials determined by higher order derivatives

In this section, we will give the proof of Theorem 1.1. We begin our proof with the following lemma.

**Lemma 2.1.** Let $f \in S_{n,d}$. If $k \geq 1$ and $\dim E_k(f) = \dim S_{n,k}$, then $\dim E_{k-1}(f) = \dim S_{n,k-1}$.

**Proof.** The proof is almost obvious: if we are given a linear relation

$$\sum_{|I|=k-1} a_I D_I f = 0,$$

by taking differentiation with respect to the variable $x_0$, it follows that

$$\sum_{|I|=k-1} a_I D_{I+e_0} f = 0.$$ 

On the other hand, the assumption of $E_k(f)$ implies that $\{D_{I+e_0} f : |I| = k-1\}$ are linearly independent, so $a_I = 0$ for all $I$. □

An induction on $k$ gives the following corollary.

**Corollary 2.2.** Let $f \in S_{n,d}$. If $k \geq 1$ and $\dim E_k(f) = \dim S_{n,k}$, then $\dim E_r(f) = \dim S_{n,r}$ for all $0 \leq r \leq k$.

As a second step to the proof of Theorem 1.1, we show the following proposition.

**Proposition 2.3.** Given $n \geq 1$, $d \geq 3$, and $k \geq 1$. Let $f, g \in S_{n,d}$ be such that $E_k(g) = E_k(f)$ and $\dim E_{k+1}(f) = \dim S_{n,k+1}$, then $E_{k-1}(g) = E_{k-1}(f)$.

**Proof.** We will show $E_{k-1}(g) \subseteq E_{k-1}(f)$. This is sufficient for our purpose because the two vector spaces have the same dimension by Corollary 2.2.

From $E_k(g) = E_k(f)$, we have a system of linear relations as follows: for all $I \in \mathbb{N}^{n+1}$ such that $|I| = k$, we have

$$D_I g = \sum_{|I'|=k} a_{I,I'} D_{I'} f.$$

for some $a_{I,I'} \in \mathbb{C}$.

Our discussions in the sequel will be divided into two steps.
**Step 1: Differentiating equations.** Fix $I$ and $0 \leq p \leq n$ such that $I \geq e_p$. For any $0 \leq q \leq n$, we will apply the equality $D_{e_q}D_I g = D_{e_q}D_{(I-e_p)+e_q}g$ to equation (1); to this end, we obtain first

$$D_{e_q}D_I g = D_{e_q}\left(\sum_{|I'|=k} a_{I,I'} D_{I'} f\right) = \sum_{|I'|=k} a_{I,I'} D_{I'+e_q} f;$$

second,

$$D_{e_q}D_{(I-e_p)+e_q} g = D_{e_q}\left(\sum_{|I'|=k} a_{(I-e_p)+e_q,I'} D_{I'} f\right) = \sum_{|I'|=k} a_{I-e_p+e_q,I'} D_{I'+e_p} f.$$

From our assumption $\dim E_{k+1}(f) = \dim S_n,k+1$, it follows that \{ $D_J f : |J| = k+1$ \} are linearly independent. Therefore, using $D_{e_q}D_I g = D_{e_q}D_{(I-e_p)+e_q}g$ and comparing the coefficients of each term $D_J f$, we obtain that

$$a_{I,J-e_q} = a_{I-e_p+e_q,J-e_p}$$

for all $|J| = k + 1$. Here we used the convention that $a_{I,J-e_q} = 0$ if $J \not\geq e_q$.

Since the above conclusion holds for all $I,J,p,q$ satisfying $I \geq e_p$, it follows that for all $I,I',p,q$ such that $|I| = |I'| = k$ and $I \geq e_p$,

$$a_{I,I'} = a_{I-e_p+e_q,I'-e_p+e_q}.$$

**Step 2: Considering $(k-1)$-th order partial derivatives.** Let $K \in \mathbb{N}^{n+1}$ be such that $|K| = k - 1$, then the Euler formula for $D_K g$ gives

$$(d - k + 1)D_K g = \sum_{p=0}^{n} x_p D_{K+e_p} g.$$

Substituting (1) into (3), we have

$$(d - k + 1)D_K g = \sum_{p=0}^{n} x_p a_{K+e_p,I'} D_{I'} f.$$ 

By (2), we deduce first of all that $a_{K+e_p,I'} = 0$ if $I' \not\geq e_p$, and thus

$$(d - k + 1)D_K g = \sum_{p=0}^{n} \sum_{I' \geq e_p} x_p a_{K+e_p,I'} D_{I'} f = \sum_{p=0}^{n} \sum_{I' \geq e_p} x_p a_{K+e_p,(I'-e_p)+e_p} D_{I'} f,$$
or written in a more convenient way,

\[(d - k + 1)D_K g = \sum_{p=0}^{n} \sum_{|K'|=k-1} x_p a_{K'+e_p,K'+e_p} D_{K'+e_p} f \]

\[= \sum_{|K'|=k-1} \left( \sum_{p=0}^{n} x_p a_{K'+e_p,K'+e_p} D_{K'+e_p} f \right).\]

Now the relations (2) imply that \(a_{K'+e_p,K'+e_p} = a_{K'+e_q,K'+e_q}\) for any \(p, q = 0, \cdots, n\); therefore, we obtain

\[(d - k + 1)D_K g = \sum_{|K'|=k-1} a_{K'+e_0,K'+e_0} \left( \sum_{p=0}^{n} x_p D_{K'+e_p} f \right).\]

By the Euler formula for \(D_{K'} f\), we have that

\[\sum_{p=0}^{n} x_p D_{K'} f = (d - k + 1)D_K f,\]

so

\[D_K g = \sum_{|K'|=k-1} a_{K'+e_0,K'+e_0} D_{K'} f.\]

Since this holds for all \(K\) satisfying \(|K'| = k - 1\), it follows that \(E_{k-1}(g) \subseteq E_{k-1}(f)\). \(\square\)

2.4. **Linear independence of partial derivatives.** As a final step to our proof of Theorem 1.1, we need the following lemma, which is interesting in its own right; see also [2], Proposition 3.4.

**Lemma 2.5.** Given \(n \geq 1\) and \(d \geq 3\). Suppose \(0 \leq k \leq \frac{d}{2}\). Then for a generic \(f \in S_{n,d}\), we have

\[\dim E_k(f) = \dim S_{n,k}.\]

**Proof.** Suppose given a linear relation

\[(4) \quad \sum_{|I|=k} a_I D_I f = 0.\]

We first show that our proof can be reduced to the case where \(d = 2k\). Indeed, the cases \(k = 0, 1\) are rather trivial. The remaining proof will be divided into two steps. **Step 1: Reduction.** If \(k > 1\) and \(d > 2k\), take derivative \(D_{(d-2k)e_0}\) in (4), we obtain

\[\sum_{|I|=k} a_I D_I (D_{(d-2k)e_0} f) = 0.\]

Note that \(D_{(d-2k)e_0} : S_{n,d} \to S_{n,2k}\) is a linear surjective morphism, hence for a generic \(f \in S_{n,d}\), the polynomial \(D_{(d-2k)e_0} f \in S_{n,2k}\) is also generic.
Step 2: The case $d = 2k$. From the linear relation (4), we obtain for any $I' \in \mathbb{N}^{n+1}$ with $|I'| = k$,

$$\sum_{|I|=k} a_I D_{I+I'} f = 0.$$ 

Hence $\{a_I : |I| = k\}$ satisfies a system of linear equations with coefficients given by $\{D_{I+I'} f : |I| = k, |I'| = k\}$. Note that $D_{I+I'} f$ is a constant since $|I + I'| = 2k = \deg f$.

Recall that the lexicographic order on the set $\{I = (i_0, \cdots, i_n) : |I| = k\}$ is given as follows:

$$I = (i_0, \cdots, i_n) \prec I' = (i'_0, \cdots, i'_n)$$

if and only if there exists $0 \leq j \leq n$ such that

$$i_0 = i'_0, \cdots, i_j = i'_j, i_j < i'_j.$$ 

One can use this order to write the sequence $\{D_{I+I'} f : |I| = k, |I'| = k\}$ into a square matrix, denote by $S(f)$, whose rows and columns are both indexed by the set $\{I \in \mathbb{N}^{n+1} : |I| = k\}$ and whose $(I, I')$-entry is given by $D_{I+I'} f$.

To finish the proof of Lemma 2.5, we need to show the matrix $S(f)$ is nonsingular for a generic $f$. To this end, it suffices to find one $f$ for which $S(f)$ is nonsingular, because the subset of $f \in S_{n,2k}$ with nonsingular $S(f)$ is clearly a Zariski open subset of $S_{n,2k}$. Just take the polynomial $f = \sum_{|I| = k} x^{2I}$; then the matrix $S(f)$ is a diagonal matrix whose $(I, I)$-entry is the nonzero number $(2I)!$, hence it is nonsingular. □

Remark 2.6. In view of the obvious bound for $\dim E_k(f)$ given by

$$\dim E_k(f) \leq \min\{\dim S_{n,k}, \dim S_{n,d-k}\},$$

the condition on $k$ in Lemma 2.5 is optimal.

2.7. Proof of Theorem 1.1. Let $f$ be a generic polynomial in $S_{n,d}$ and $E_k(g) = E_k(f)$. Under the assumption $k \leq \frac{d}{2} - 1$, it follows that $k + 1 \leq \frac{d}{2}$, hence by Lemma 2.5, we have $\dim E_{k+1}(f) = \dim S_{n,k+1}$; therefore the requirements in Proposition 2.3 are satisfied. By Proposition 2.3, it follows that $E_{k-1}(g) = E_{k-1}(f)$. Note that by Corollary 2.2, we have $\dim E_k(f) = \dim S_{n,k}$, so the requirements in Proposition 2.3 are satisfied with $k$ replaced by $k - 1$ and we obtain $E_{k-2}(g) = E_{k-2}(f)$. These arguments can be repeated until we obtain $E_0(g) = E_0(f)$. By definition, we have $E_0(g) = Cg$ and $E_0(f) = Cf$, therefore $g$ is a constant multiple of $f$.

3. Applications

As pointed out in the introduction, the most remarkable application of the results in this paper lies in the study of higher order analogue of variation of Hodge structures for hypersurfaces; see [2]. In this section, we give some other applications in the study of deformations of homogeneous polynomials.

For $k \geq 0$, denote by $U_{n,d}(k)$ the following set

$$U_{n,d}(k) = \{ f \in S_{n,d} : \dim E_k(f) = \dim S_{n,k}\}.$$
From semi-continuity of $\dim E_k(f)$ with respect to $f$, we see that $\mathcal{U}_{n,d}(k)$ is a Zariski open subset of $S_{n,d}$. Obviously, we have $\mathcal{U}_{n,d}(k) = \emptyset$ if $k > \frac{d}{2}$. From Lemma 2.5, we have the following result.

**Corollary 3.1.** Given $n \geq 1$ and $d \geq 3$. For $k \leq \frac{d}{2}$, the set $\mathcal{U}_{n,d}(k)$ is a Zariski open dense subset of $S_{n,d}$.

In addition, for any $f \in \mathcal{U}_{n,d}(k)$, we have by definition that $\dim E_k(f) = \dim S_{n,k}$; by Lemma 2.1, we deduce that $\dim E_{k-1}(f) = \dim S_{n,k-1}$, that is $f \in \mathcal{U}_{n,d}(k-1)$. In other words, for fixed $n$ and $d$, the sequence of sets $\{\mathcal{U}_{n,d}(k)\}$ satisfies the following relations

$$\mathcal{U}_{n,d}(0) \supseteq \mathcal{U}_{n,d}(1) \supseteq \cdots \supseteq \mathcal{U}_{n,d}(k) \supseteq \mathcal{U}_{n,d}(k+1) \supseteq \cdots.$$  

Note that $\mathcal{U}_{n,d}(k)$ is a cone in $S_{n,d}$, hence we can consider its projectivization, denoted by $\mathbb{P}(\mathcal{U}_{n,d}(k))$, in $\mathbb{P}(S_{n,d})$. Similar to the construction in [3], the assignment $[f] \mapsto \mathbb{P}(E_k(f))$ gives a well-defined map, denoted by $\varphi_k$, from $\mathbb{P}(\mathcal{U}_{n,d}(k))$ to an obvious Grassmannian for $k \leq \frac{d}{2}$.

Using Proposition 2.3 and Lemma 2.1, we prove the following result which gives an extension of Corollary 7.7 in [3].

**Corollary 3.2.** For $k \leq \frac{d}{2} - 1$, the map $\varphi_k : \mathbb{P}(\mathcal{U}_{n,d}(k)) \ni [f] \mapsto \mathbb{P}(E_k(f))$ is injective when restricted to $\mathbb{P}(\mathcal{U}_{n,d}(k+1))$. In particular, it is generically injective.

**Proof.** To begin the proof, suppose $[f]$ and $[g]$ are two elements of $\mathbb{P}(\mathcal{U}_{n,d}(k+1))$ such that $\varphi_k([f]) = \varphi_k([g])$. By the definition of $\varphi_k$, this means that $E_k(f) = E_k(g)$. Now the assumption $[f] \in \mathbb{P}(\mathcal{U}_{n,d}(k+1))$ implies that $\dim E_{k+1}(f) = \dim S_{n,k+1}$, hence by Proposition 2.3, we obtain $E_{k+1}(f) = E_{k+1}(g)$. An induction argument on $k$ gives $[f] = [g]$, which goes exactly the same as the proof of Theorem 1.1 where only the properties $\dim E_{k+1}(f) = \dim S_{n,k+1}$ and $E_k(f) = E_k(g)$ are essentially used. Thus, $\varphi_k$ is injective on $\mathbb{P}(\mathcal{U}_{n,d}(k+1))$. □

**Remark 3.3.** We do not know whether $\varphi_k$ is injective on $\mathbb{P}(\mathcal{U}_{n,d}(k))$ or not, except the case $k = \frac{d}{2}$ where $\varphi_{\frac{d}{2}}$ is a constant map, because in this case $E_k(f) = S_{n,k}$ for any $f \in \mathcal{U}_{n,d}(k)$.

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