High Rayleigh number variational multiscale large eddy simulations of Rayleigh-Bénard Convection

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Abstract

The variational multiscale (VMS) formulation is used to develop residual-based VMS large eddy simulation (LES) models for Rayleigh-Bénard convection. The resulting model is a mixed model that incorporates the VMS model and an eddy viscosity model. The Wall-Adapting Local Eddy-viscosity (WALE) model is used as the eddy viscosity model in this work. The new LES models were implemented in the finite element code Drekar. Simulations are performed using continuous, piecewise linear finite elements. The simulations ranged from \( Ra = 10^6 \) to \( Ra = 10^{14} \) and were conducted at \( Pr = 1 \) and \( Pr = 7 \). Two domains were considered: a two-dimensional domain of aspect ratio 2 with a fluid confined between two parallel plates and a three-dimensional cylinder of aspect ratio 1/4. The Nusselt number from the VMS results is compared against three dimensional direct numerical simulations and experiments. In all cases, the VMS results are in good agreement with existing literature.

Keywords: Rayleigh-Bénard convection, large eddy simulation, variational multiscale formulation

1. Introduction

Rayleigh-Bénard convection is the buoyancy-driven flow of a fluid confined between two parallel, horizontal plates where the bottom plate is at a higher temperature than the top plate. An initially quiescent fluid will be set into motion for a sufficiently large temperature difference between the two plates. This deceptively simple configuration provides for an exceptionally rich variety of fluid motion. Indeed, Rayleigh-Bénard convection has been used as a proxy for the phenomenon of thermal convection, which is responsible for a dizzying array of fluid phenomena from the geophysical through the astrophysical. Rayleigh-Bénard convection is also known for being one of the original flow fields studied in the field of hydrodynamic stability theory \[1\][2]. In his pioneering work, Rayleigh used linear stability theory to show precisely when an initially quiescent fluid will bifurcate from the quiescent, conduction state to the first convection state \[3\]. The primary control parameter governing this bifurcation is now known as the Rayleigh number \( Ra \), which is a measure of the ratio of buoyancy-driven inertial forces to viscous forces. Over the years, Rayleigh-Bénard convection has been studied well beyond the theory of fluid stability.

One major research thrust has been the focus on quantifying how the heat transport through the fluid layer depends on \( Ra \) \[4\][5]. The primary diagnostic quantity in Rayleigh-Bénard convection is the dimensionless heat transport expressed as the ratio of total heat transport to conduction heat transport and called the Nusselt number (\( Nu \)). Significant theoretical, computational, and experimental resources have been devoted to determining the relationship between \( Nu \) and \( Ra \). A major question focuses on the exponent in the power law relationship \( Nu \propto Ra^{\beta} \). Theoretical arguments have been used to show that \( \beta \approx 1/3 \) \[6\][7] while other rigorous mathematical arguments have established bounds that show \( \beta \leq 1/2 \) \[8\][9]. Classical work based on turbulence mixing length models has predicted that \( \beta \) transitions to 1/2 for very large \( Ra \) with logarithmic corrections in \( Ra \) \[10\]. Other recent work has proposed models of \( Nu \) that are not pure power laws in \( Ra \) \[11\]. There has been much discussion on recent experimental results at high \( Ra \) that observe a transition to \( \beta = 1/2 \) \[12\][13] or not at very high \( Ra \) \[14\][15]. Numerical calculations have shown that \( \beta \approx 0.28 \pm 0.3 \) up to the largest \( Ra \) currently achievable \[16\][17]. Very recently, two-dimensional numerical simulations up to \( Ra = 10^{14} \) observed a transition to \( \beta = 1/2 \) \[18\][19], while three-dimensional simulations up to \( Ra = 10^{15} \) have not observed this transition \[17\]. At high \( Ra \), it becomes prohibitively expensive to perform fully-resolved direct numerical simulations of Rayleigh-Bénard convection. Hence, there is great interest in the development of turbulence models that will permit accurate simulations at high \( Ra \).

Instead of directly resolving all scales of motion, large eddy simulation (LES) coarse grains the fields and equations and simulates only the largest scales of motion. The price to pay is that this coarse graining procedure introduces correlations between resolved and unresolved terms that cannot be neglected and must be modeled. The goal of LES turbulence modeling is...
to develop models that account for the effect of the unresolved scales on the resolved scales. Fluid simulations with the finite element method have been challenging due to the need to satisfy (or circumvent) the inf-sup condition, satisfy the incompressibility constraint, and stabilize spurious oscillations for highly convective flows. Stabilized finite element methods were developed to overcome these challenges [20, 21] and were eventually shown to derive from the variational multiscale (VMS) method [22]. Since its original development, the VMS method has been used to develop LES models for a variety of fluid flows [23, 24, 25, 26, 27, 28]. Recently, researchers developed a VMS-based LES model for Rayleigh-Bénard convection with application to heating systems [29]. In the current work, we propose a mixed VMS method for Rayleigh-Bénard convection at high \( Ra \). We perform simulations up to \( Ra = 10^{14} \) for two different Prandtl numbers \( (Pr = 1 \) and \( Pr = 7 \) ) in both two and three dimensions for rectangular and cylindrical geometries.

The remainder of the paper is organized as follows. In section 2 we provide the governing equations, the VMS formulation for Rayleigh-Bénard convection, and a description of the code used to perform the simulations. Following this, section 3 presents the results of the simulations. Section 4 summarizes the work and discusses ongoing and future work.

2. Background

2.1. Rayleigh-Bénard Convection

Rayleigh-Bénard convection is concerned with the buoyancy-driven flow of a fluid confined between two parallel, horizontal plates separated by a distance \( H \). The two plates are maintained at constant temperatures such that the temperature difference between the top and bottom plates is \( \Delta T = T_{\text{top}} - T_{\text{bot}} > 0 \). Within the Oberbeck-Boussinesq approximation, density variations are assumed to be important only in the buoyancy term and these variations are taken to depend linearly on the temperature. The fluid is otherwise incompressible. The velocity field \( u(x,t) = (u,v,w) \) evolves according to the Oberbeck-Boussinesq equations,

\[
\begin{align*}
\rho_0 \frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) &= -\nabla p + \mu \nabla^2 u + \alpha_v(T - T_0)\vec{Y} \\
\nabla \cdot u &= 0
\end{align*}
\] (1)

where \( \rho_0 = \rho(T_0) \) is the reference density evaluated at a reference temperature \( T_0 \), \( p = P(x,t) \) is the kinematic pressure of the fluid, \( \mu \) is the kinematic viscosity, \( \alpha_v \) is the coefficient of volume expansion of the fluid, \( g \) is the acceleration due to gravity, and \( \vec{Y} \) is the unit vector in the vertical direction. The temperature field \( T = T(x,t) \) evolves according to an advection-diffusion equation,

\[
\rho_0 C_p \left( \frac{\partial T}{\partial t} + \nabla \cdot (uT) \right) = k \nabla^2 T
\] (2)

where \( C_p \) and \( k \) are the specific heat and thermal conductivity of the fluid, respectively. The velocity field uses no-slip boundary conditions on all solid surfaces. The temperature is held at a uniform constant temperature on the top and bottom surfaces such that the bottom surface is hotter than the top surface. In the present work, we consider two different geometries and therefore the boundary conditions on the “sides” are specified differently depending on which geometry is being considered. In two-dimensional Rayleigh-Bénard convection between two infinite parallel planes the velocity and temperature fields have periodic boundary conditions in the \( x \) direction. In three-dimensional Rayleigh-Bénard convection in a right circular cylinder, the surface of the cylinder is insulated and the temperature field uses homogeneous Neumann boundary conditions on the side-walls of the cylinder.

The two classical non-dimensional parameters emerging from the system in (1)–(3) are the Rayleigh and Prandtl numbers. The Prandtl number is a fluid property and is given by,

\[
Pr = \frac{\nu}{\kappa}
\] (4)

where \( \nu = \mu/\rho_0 \) and \( \kappa = k/\rho_0 C_p \). The Rayleigh number is,

\[
Ra = \frac{g\alpha_v \Delta T H^3}{\nu \kappa}
\] (5)

and is interpreted as a measure of the strength of buoyancy-driven inertial forces. In the conduction state, the heat transport is \( \mathcal{H}_{\text{cond}} = \kappa \Delta T / H \), independent of \( Ra \) and \( Pr \). After convection sets in, the heat transport is quantified by the Nusselt number \( Nu \) as the ratio of total heat transfer to conduction heat transfer. The Nusselt number is,

\[
Nu = -\frac{H}{\Delta T} \frac{dT}{dy} \bigg|_{y=\pm \infty}
\] (6)

where \( \bar{\cdot} \) represents an average over the plane orthogonal to the wall-normal coordinate. In statistically steady state, after integrating across the width of the fluid layer, the Nusselt number can be written as,

\[
Nu = 1 + \frac{\langle vT \rangle}{\mathcal{H}_{\text{cond}}}
\] (7)

where \( \langle \cdot \rangle \) is a space-time average. Once a statistically stationary state has been reached, the time-average of (6) is equal to (7). In the present work, the Nusselt number was calculated using (6) and (7) with identical results.

2.2. Variational Multiscale Formulation for Rayleigh-Bénard Convection

The variational statement of the equations governing Rayleigh-Bénard convection is: Find \( U \in \mathcal{V} \) s.t. \( \forall W \in \mathcal{V} \)

\[
\mathcal{R}(W, U) = (W, F)
\] (8)

where \( U = [u, p, T]^T \) is a vector of solutions, \( W = [w, q, s]^T \) is a vector of weighting functions, and \( F = \{ f^l, 0, f^l \}^T \) is a vector of forcing functions. Note that in the current work this forcing is zero. As per usual convection, the notation \( \langle \cdot, \cdot \rangle \) denotes
an $L_2$ inner product of two functions over the domain $\Omega$. The semilinear form is,

$$ \mathcal{A}(W, U) = A^V(W, U) + (q, \nabla \cdot u) + A^T(W, U) $$  \hspace{1cm} (9)

where

$$ A^V(W, U) = \left( w, \rho_0 \frac{\partial u}{\partial t} \right) - \left( \nabla w, \rho_0 u \otimes u \right) - \left( \nabla \cdot w, P \right) + \left( \nabla w, \mu \nabla u \right) - \left( w, \alpha \nabla g(T - T_0) \right) $$  \hspace{1cm} (10)

and

$$ A^T(W, U) = \left( s, \rho_0 C_p \frac{\partial T}{\partial t} \right) - \left( \nabla s, \rho_0 C_p uT \right) + \left( \nabla s, k \nabla T \right). $$  \hspace{1cm} (11)

No boundary terms appear in the variational formulation due to the periodic, Dirichlet, and homogeneous Neumann boundary conditions in the problems considered in this work.

We consider a finite element method in which the discretized solutions $U^h \in \mathcal{V}^h \subset \mathcal{V}$ are linear combinations of bilinear quadrilateral or hexahedral basis functions. The straightforward discretization leading to the Galerkin statement: Find $U^h \in \mathcal{V}^h$ s.t. \forall $W^h \in \mathcal{V}^h$, $\mathcal{A}(W^h, U^h) = (W^h, F)$ is not sufficient due to the instabilities inherent in the Galerkin method for highly convective flows. To overcome this limitation, we develop a mixed variational multiscale formulation for Rayleigh-Bénard convection. The VMS method induces a sum-decomposition of the solution field $U$ into resolved $U^h$ and unresolved $U'$ components so that $U = U^h + U'$. The resulting VMS formulation for our problem with linear finite elements is:

$$ \mathcal{A}(W^h, U^h) - \left( \nabla w^h, \rho_0 u^h \otimes u^h \right) - \left( \nabla w^h, \rho_0 u' \otimes u^h \right) \quad $$

$$ \text{SUPG} \text{ cross stresses} $$

$$ - \left( \nabla w^h, \rho_0 u^h \otimes u' \right) - \left( \nabla \cdot w^h, P \right) - \left( w^h, \alpha \nabla g(T - T_0) \right) \quad $$

$$ \text{Reynolds stresses} $$

$$ - \left( \nabla s^h, \rho_0 C_p u'T \right) - \left( \nabla s^h, \rho_0 C_p u'T^h \right) \quad $$

$$ \text{T SUPG} \text{ VMS cross stresses} $$

$$ - \left( \nabla s^h, \rho_0 C_p u' \right) \quad $$

$$ \text{T Reynolds stresses} $$

$$ = \left( W^h, F \right) \quad $$

The formulation in (12) neglects terms involving time derivatives of unresolved fields as well as inner products of gradients of resolved and unresolved fields. Although not pursued here, approaches exist to model the transient effects of the unresolved scales [35]. In residual-based VMS formulations, the unresolved fields are proportional to the residual of the partial differential equations (PDEs),

$$ U' \approx -\tau R(U^h) $$  \hspace{1cm} (13)

where

$$ R(U^h) = \left[ \rho_0 \frac{\partial u^h}{\partial t} + \rho_0 \nabla \cdot (u^h \otimes u^h) + \nabla p^h - \mu \nabla^2 u^h - \alpha \nabla g(T^h - T_0) \hat{y} \right] $$  \hspace{1cm} (14)

and $\tau$ is the stabilization matrix. We use a diagonal stabilization matrix $\tau = \text{diag} (\tau^V, \tau^C, \tau^T)$ where

$$ \tau^V = \left[ \left( \frac{2C_1 \rho_0}{\Delta t} \right)^2 + \rho_0 u^h \cdot G u^h + (C_1 \mu)^2 \|G\|^2 + \rho_0^2 \alpha \|g(T^h - T_0)\| \right]^{-1/2}, \quad i = 1, \ldots, n_{sd} $$  \hspace{1cm} (15)

$$ \tau^C = \left( C_1 \text{trace} (G) \tau^V \right)^{-1} $$  \hspace{1cm} (16)

$$ \tau^T = \left[ \left( \frac{2C_1 \rho_0 C_p}{\Delta t} \right)^2 + (\rho_0 C_p)^2 \|u^h \cdot G u^h + (C_1 k)^2 \|G\|^2 \right]^{-1/2}, $$  \hspace{1cm} (17)

$G$ is the metric tensor, $n_{sd}$ the number of spatial dimensions, $\Delta t$ is the time-step, and $C_1 = C_t = 1$. The components of the metric tensor $G$ are given by,

$$ G_{ij} = \frac{\partial \xi_i}{\partial x_j} \frac{\partial \xi_k}{\partial x_j} $$  \hspace{1cm} (18)

where $\xi$ are the coordinates in the parametric (finite element) space. The first-order approximation to the unresolved scales [35] has been shown to be insufficient to model correlations of unresolved scales (the Reynolds stresses, $u' \otimes u'$ and $u'T'$) for highly turbulent flows [31]. We expect the Reynolds
Bénard convection is: find $\mathbf{U} \in \mathcal{V}$ s.t. $\mathbf{u} \in \mathcal{V}^T$

$$\mathcal{A} \left( \mathbf{W}^h, \mathbf{U}^h \right) - C_{\text{VMS}} \left( \nabla \mathbf{w}^h, \rho_0 \mathbf{u} \otimes \mathbf{u} \right) - C_{\text{SUPG}} \left( \nabla \mathbf{w}^h, \rho_0 \mathbf{u} \right) - C_P \left( \nabla \mathbf{w}^h, \mathbf{P} \right) + C_{\text{SUPG}} \left( \mathbf{v}^h, \mathbf{u} \right) - C_{\text{SUPG}} \left( \mathbf{v}^h, \mathbf{u} T^h \right) \right) - C^T_{\text{SUPG}} \left( \nabla s^h, \mathbf{u} T^h \right)$$

$$+ C^T_{\text{VMS}} \left( \mathbf{v}^h, \mathbf{v} \right) + C^T_{\text{VMS}} \left( \mathbf{v}^h, \rho_0 C_p k_T \nabla T^h \right) = \left( \mathbf{W}^h, \mathbf{F} \right)$$

where $\nabla \mathbf{u} = \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) / 2$. In [19], the first term on the first line represents the Galerkin discretization, the second term represents a VMS cross-stress term, and the third term represents the SUPG stabilization. The first term on the second line is the pressure stabilization term, the second term represents temperature fluctuations in the buoyancy term, and the third term is used to overcome the inf-sup condition for finite element discretizations of the incompressible Navier-Stokes equations. The first term on the third line is an upwinding stabilization term for the temperature advection-diffusion equation and the second term is an additional cross-stress term from the VMS formulation.

2.3. Numerical Methodology

All simulations were conducted using the Drekar finite element code [37, 38] using linear quadrilateral or hexahedral finite elements. The 2D simulations used an SDIRK22 (single diagonally implicit Runge-Kutta 2nd order, 2 stage) time-integration method while the 3D simulations used a second order BDF method. The two-dimensional computations were run on up to 2000 cores while the three-dimensional simulations were run on up to 8000 cores. All simulations were performed on a capacity cluster with an Intel Haswell based CPU. The following section provides a broad overview of the time integration and solver formulations used to perform the simulations.

2.3.1. Fully-implicit Time Integration and Strongly-coupled Newton-Krylov-AMG Solver

Fully-implicit formulations, coupled with effective robust nonlinear iterative solution methods, have the potential to provide stable, higher-order time-integration of multiphysics systems when long dynamical time-scales are of interest. These methods can follow the desired dynamical time-scales as opposed to time-scales determined by either numerical stability or by temporal order-of-accuracy reduction [39, 40, 41, 42, 43, 47, 38]. For time-integration of the governing equations in (1) - (3), the L-stable SDIRK22 method is used [37, 38] as it provides a high-order time integration with damping of the highest unresolved wavenumbers [44].

A finite element discretization of the VMS equations [12] gives rise to a system of coupled, nonlinear, nonsymmetric algebraic equations, the numerical solution of which can be very challenging. These equations are linearized using an inexact form of Newton’s method. A formal block matrix representation of these discrete linearized equations is given by

$$\begin{bmatrix}
\mathbf{D}_n & \mathbf{B}^T & \mathbf{Q} \\
\mathbf{B} & \mathbf{L}_p & 0 \\
\mathbf{C} & 0 & \mathbf{D}_T
\end{bmatrix} \begin{bmatrix}
\delta \hat{\mathbf{u}} \\
\delta \hat{\mathbf{P}} \\
\delta \hat{T}
\end{bmatrix} = - \begin{bmatrix}
\mathbf{F}_u \\
\mathbf{F}_P \\
\mathbf{F}_T
\end{bmatrix}.$$
sure Laplacian”. The matrix $Q$ corresponds to the coupling of velocity and the temperature field (buoyancy term), and the matrix $C$ the coupling of temperature gradient to the velocity field. The right hand side vectors contain the residuals for Newton’s method. The existence of the nonzero matrix $L_F$ in the stabilized finite element discretization, is in contrast to Galerkin methods for incompressible flow using mixed interpolation that produce a zero block on the total mass continuity diagonal. The existence of this block matrix helps to enable the solution of the linear systems with a number of algebraic, domain decomposition (DD), and algebraic multilevel (AMG) type preconditioners/smoothers that rely on non-pivoting ILU type factorization, or in some cases methods such as Jacobi or Gauss-Seidel as sub-domain solvers [37][38]. Although the above formal block matrix representation provides insight into the system, the actual linear algebra implementation in the application employs an ordering by finite element mesh node with each degree of freedom (dof) ordered consecutively. The Jacobian is evaluated analytically using automatic differentiation [45].

Fully-coupled Newton-Krylov techniques [46] where a Krylov solver is used to solve the linear system generated by a Newton’s method are robust. However, efficient solution of the large sparse linear system that must be solved for each nonlinear iteration is challenging [47] [48]. The performance and scalability of the preconditioner is critical [47]. It is well known in the literature that Schwarz DD preconditioners do not scale due to lack of global coupling [49]. Multigrid methods are one of the most efficient techniques for solving large linear systems [50]. A Newton-Krylov preconditioned by AMG solution method has been described in our previous work in detail [48][51][37][38][52] and we will therefore only provide a very brief description here.

A Newton-Krylov (NK) method is an implementation of Newton’s method in which a Krylov iterative solution technique is used to approximately solve the linear systems, $J_k s_{k+1} = -F_k$, that are generated at each step of Newton’s method. For efficiency, an inexact Newton method [53][54][55] is employed, whereby one approximately solves the linear systems generated in the Newton method by choosing a forcing term $\eta_k$ and stopping the Krylov iteration when the inexact Newton condition, $\|F_k + J_k s_{k+1}\| \leq \eta_k \|F_k\|$ is satisfied. The particular Krylov method that is used in this study is a robust non-restarted GMRES method that is capable of iteratively converging to the solution of very large non-symmetric linear systems provided a sufficiently robust and scalable preconditioning method is available [48][51][37][38][52]. Two nonlinear convergence criteria are used to ensure that the numerical solution error is below discretization error. The first is a sufficient reduction in the relative nonlinear residual norm, $\|F_1\|/\|F_0\| < 10^{-2}$. In general, and specifically in the results presented in this paper, this requirement is easily satisfied. The second convergence criterion is based on a sufficient decrease of a weighted norm of the Newton update vector. This latter criterion requires that the correction, $\Delta \chi_i^k$, for any variable, $\chi_i$, is small compared to its magnitude, $|\chi_i^k|$, and is given by

$$\sqrt{\frac{1}{N_u} \sum_{i=1}^{N_u} \left( \frac{|\Delta \chi_i^k|}{\epsilon_r |\chi_i^k| + \epsilon_a} \right)^2} < 1,$$

where $N_u$ is the total number of unknowns, $\epsilon_r$ is the relative error tolerance between the variable correction and its magnitude, and $\epsilon_a$ is the absolute error tolerance of the variable correction. Essentially $\epsilon_a$ sets the magnitude of components that are to be considered to be numerically zero. In the numerical results that are presented in this paper the relative-error, and absolute-error tolerance are set to $10^{-3}$ and $10^{-6}$ respectively for all of the test cases. In general, each linear system in Newton’s method is solved to a moderate level of accuracy (e.g. $\eta = 10^{-3}$) since the outer nonlinear Newton iteration controls convergence at each time step.

A scalable preconditioner for the iterative linear solver is necessary to achieve solutions efficiently. For this reason, a fully-coupled algebraic multigrid method is employed [56]. In general AMG methods are significantly easier to implement and integrate with complex unstructured mesh discretizations than geometric multigrid methods [57][58][59]. Our fully-coupled AMG preconditioner employs a nonsmoothed aggregation approach with uncoupled aggregation. For systems of partial differential equations (PDEs), aggregation is performed on the graph where all the PDEs per mesh node are represented by a single vertex, as in our VMS discretization of the governing equations with each dof ordered at each finite element node consecutively. The discrete equations are projected to the coarser level employing a Galerkin projection with a triple matrix product, $A_{\ell+1} = R_{\ell} A_{\ell} P_{\ell}$, where $R_{\ell}$ restricts the residual from level $\ell$ to level $\ell+1$, $A_{\ell}$ is the discretization matrix on level $\ell$ and $P_{\ell}$ prolongates the correction from level $\ell+1$ to $\ell$. We typically employ both pre- and post-smoothing on each level of the multigrid V-cycle except the coarsest level where a serial sparse direct solve is performed. In the computations carried out in this study the AMG preconditioner employs a DD-ILU(k) smoother with one level of overlap and a moderate level of fill-in (e.g. $k = 1$) A detailed discussion of the scalability of the solvers and comparisons of differing preconditioning techniques is out of scope of this study. However these fully-coupled Newton-Krylov-AMG solver have been studied extensively, with results demonstrating scaling on up to 1M+ cores for challenging resistive MHD type problems [38][52].

### 3. Results

VMS-based large eddy simulations of Rayleigh-Bénard convection were run in two- and three-dimensional domains. In both cases, the no-slip boundary condition was used for the velocity field on all surfaces while the temperature field was prescribed at the top and bottom surfaces. Statistics such as the Nusselt number were obtained after the simulations reached a statistically steady state by averaging over a number of free-fall times, $t_f = H/U_f$ where the free-fall velocity is given by,

$$U_f = \sqrt{\rho \nu \Delta T H}.$$  (20)
3.1. VMS Simulations of Two-dimensional Rayleigh-Bénard Convection

Two-dimensional simulations were run at $Pr = 1$. The aspect ratio for the two-dimensional problem was equal to 2 and periodic boundary conditions were used in the streamwise ($x$) direction. Two types of meshes were used for the 2D simulations. The first mesh used a uniform discretization in both coordinate directions. The second type of mesh was uniform in the $x$- direction but stretched in the $y$- direction. The stretching was accomplished using,

$$y_j^s = \frac{H}{2}\left(1 - \cos\left(\frac{(y_j - y_B)\pi}{2}\right)\right) + y_B, \quad j = 0, \ldots, N_y$$  \hspace{1cm} (21)

where $y_j = j\Delta y + y_B$, $\Delta y$ is a uniform mesh spacing, $N_y$ is the number of elements in the wall-normal direction, and $y_B$ and $y_T$ are the bottom and top coordinates of the plates, respectively. For all of the two-dimensional simulations, $y_T = 1$ and $y_B = -1$ leading to $H = 2$. We note that, especially for the high $Ra$ cases, the simulations may be under-resolved in the boundary layer. This may have significant implications for the $Nu$ results. Even so, the residual-based LES models do provide some measure of robustness and adaptivity in the near-wall region. For $Ra = 10^{13}$, we found $y^s < 1$, which is an indication that the mesh is resolved. However, a more rigorous mesh convergence study should be performed. All statistics were measured after the simulations reached a statistically steady state. A representative temporal evolution of $Nu$ is presented in Figure 1 for $Ra = 10^{13}$. The lightly-shaded region represents the transient portion and transition to a statistically steady state while the darkly-shaded region depicts the period within which statistics were collected. A representative snapshot of the temperature field from the statistically steady portion of the simulation is presented in Figure 2. The visualization uses a Schlieren-type coloring to bring out the features of the flow. Table 2 presents a summary of the simulation parameters and results for the 2D VMS runs.

The $Nu - Ra$ scaling for the 2D VMS simulations is compared to recent 2D DNS results in Figure 3. The 2D VMS and DNS results are in excellent agreement up to $Ra \approx 10^{13}$ at which point the DNS results indicate an increase in transport. Recent work [18, 19] has reported observations of a possible transition to the ultimate regime in direct numerical simulations of two-dimensional Rayleigh-Bénard convection. Although the present VMS results do not indicate such a transition, we emphasize that the VMS simulations may be under-resolved in the boundary layer. Additional research is needed to determine the effectiveness of the VMS simulations for an under-resolved boundary layer. In this particular study, the 2D VMS simulations show $Nu = 0.151Ra^{0.278}$ for $10^{10} \leq Ra \leq 10^{14}$.

3.2. VMS-WALE Model Simulations of Three-dimensional Rayleigh-Bénard Convection

The three-dimensional simulations were performed at $Pr = 7$ in a circular cylinder of aspect ratio 1/4. The height of the cylinder was $H = 100$ for each case. Homogeneous Neumann boundary conditions were used for the temperature on the surface of the cylinder. No-slip velocity boundary conditions were used on all surfaces of the cylinder. Most of the 3D simulations used the classical WALE model [13, 60] as an eddy viscosity model in the VMS formulation with a turbulent Prandtl number equal to unity. This corresponds to the WALE-VMS model in Table [1]. Table 3 presents a summary of the simulation parameters and results for the three-dimensional simulations with the WALE-VMS model. The results from the WALE-VMS model were compared to SUPG and VMS results using a sequence of

![Figure 1: Nusselt evolution showing development to the statistically steady state. Statistics were collected in the statistically stationary region, indicated here by the darkly-shaded region.](image)

![Figure 2: Temperature snapshot of the $Ra = 10^{13}$ simulation using a Schlieren-type visualization.](image)

![Table 2: Simulation parameters and results for the two-dimensional runs. $N_y$ and $N_x$ are the number of linear finite elements used for spatial discretization. $n_f$ represents the number of free-fall times over which statistics were computed. The last column indicates if a uniform or stretched mesh was used in the wall-normal direction.](image)
finer meshes at $Ra = 10^{10}$. Table 4 compares the Nusselt number between these three models at the finest mesh. At this mesh resolution, all three models are in good agreement. Figure 4 shows a snapshot of the temperature field at $Ra = 10^{10}$ taken at a plane in the boundary layer ($y = 0.001$) for the simulations in the cylinder. Similarly to the 2D runs, the $y^+$ value was computed and determined to be less than 1 for most runs. Figure 5 presents temperature contours colored by vertical velocity at $Ra = 10^{10}$.

The Nusselt-Rayleigh scaling is presented in Figure 6. The results are presented over several decades of $Ra$ for a number of studies, including the present results. The results from the VMS models compare favorably to direct numerical simulations as well as experiments and the classical $Nu \propto Ra^{1/3}$ scaling emerges over several decades in $Ra$. In particular, the VMS-WALE model shows a scaling of $Nu = 0.104Ra^{0.310}$ for $10^{10} \leq Ra \leq 10^{14}$, while very recent DNS results show $Nu = 0.0525Ra^{0.331}$ for $10^{10} \leq Ra \leq 10^{15}$ [17].

| $Ra$ | Mesh Elements | VMS | VMS-WALE | SUPG |
|------|---------------|-----|----------|------|
| $10^{10}$ | 2,467,584 | 131.1 | 131.8 | 131.7 |

Table 4: Comparison of $Nu$ at $Ra = 10^{10}$ between simulations in an aspect ratio $1/4$ cylinder using the VMS, VMS-WALE, and SUPG models.

4. Conclusions

A residual-based VMS formulation was derived for Rayleigh-Bénard convection and augmented with eddy viscosity models to account for the Reynolds stresses. This new mixed model was implemented in the finite element code Drekar. In the current work, the WALE model was used for the eddy viscosity model. A number of two and three dimensional simulations were performed to compute the heat transport scaling in each system up to $Ra = 10^{14}$. The new VMS simulations are in good agreement with previous direct numerical simulations of two- and three- dimensional Rayleigh-Bénard convection re-
sults. When compared to recent 2D DNS simulations, the VMS simulations begin to show a difference in $Nu \sim Ra$ scaling at around $Ra = 10^{13}$. Before claiming this as evidence for the absence of a transition to the ultimate regime, we suggest several avenues for additional research regarding the VMS models.

Resolution of the boundary layer in Rayleigh-Bénard convection is critical [65]. Residual-based VMS formulations have an ability to automatically adapt to regions of the flow that are under-resolved, but this may not be sufficient to capture the underlying heat release from the boundary layer. Moreover, when using linear finite elements, the viscous terms in the residual vanish identically. The impact of this incomplete residual on the $Nu \sim Ra$ scaling should be assessed. Previous work has introduced techniques for reconstructing the diffusive flux for linear finite elements [36]. We have implemented this diffusive flux reconstruction into the Drekar code and are currently testing its impact on Rayleigh-Bénard convection. In addition to the diffusive flux reconstruction, a more thorough mesh convergence study should be performed along with a rigorous assessment of the near-wall behavior of the models. Comparisons to simulations that use higher-order elements would also provide a useful perspective.

Beyond Rayleigh-Bénard convection, we will implement and use VMS models on rotating Rayleigh-Bénard convection. Early results from Drekar compare favorably with experiments in this regime. Additional future work will include magnetoh-convection with applications to geophysical and astrophysical problems.

Acknowledgments

The authors would like to thank Professor Jon A. Aurnou (UCLA) for sharing data from their experimental system. The work of John N. Shadid, Roger P. Pawlowski, Thomas Smith, and Sidafa Conde was partially supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research. Sandia National Laboratories is a multi-mission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-NA0003525. This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

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