A change in a stochastic system has three representations: Probabilistic, statistical, and informational: (i) is based on random variable \( u(\omega) \rightarrow \tilde{u}(\omega) \); this induces (ii) the probability distributions \( F_u(x) \rightarrow F_{\tilde{u}}(x), x \in \mathbb{R}^n \); and (iii) a change in the probability measure \( \mathbb{P} \rightarrow \tilde{\mathbb{P}} \) under the same observable \( u(\omega) \). In the informational representation a change is quantified by the Radon-Nikodym derivative \( \ln \left( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) \right) = -\ln \left( \frac{dF_{\tilde{u}}}{dF_u}(\omega) \right) \) when \( x = u(\omega) \). Substituting a random variable into its own density function creates a fluctuating entropy whose expectation has been given by Shannon. Informational representation of a deterministic transformation on \( \mathbb{R}^n \) reveals entropic and energetic terms, and the notions of configurational entropy of Boltzmann and Gibbs, and potential of mean force of Kirkwood. Mutual information arises for correlated \( u(\omega) \) and \( \tilde{u}(\omega) \); and a nonequilibrium thermodynamic entropy balance equation is identified.

### I. INTRODUCTION

A change according to classical physics is simple: if one measures \( x_1 \) and \( x_2 \) which are traits, in real numbers, of a “same” type, then \( \Delta x = x_2 - x_1 \) is the mathematical representation of the change; \( \Delta x \in \mathbb{R}^n \). How to characterize a change in a complex world? To represent a complex, stochastic world \([1]\), the theory of probability developed by A. N. Kolmogorov envisions an abstract space \((\Omega, \mathcal{F}, \mathbb{P})\) called a probability space. Similar to the Hilbert space underlying quantum mechanics \([2]\), one does not see or touch the objects \( u(\omega) \), called random variables which map \( \Omega \rightarrow \mathbb{R}^n \). Rather, one observes the probability space through functions, say \( u(\omega) \), called random variables which map \( \Omega \rightarrow \mathbb{R}^n \). The same function maps the probability measure \( \mathbb{P} \) to a cumulative probability distribution function (cdf) \( F_u(x), x \in \mathbb{R}^n \).

Now a change occurs; and based on observation(s) the \( F_u(x) \) is changed to \( F_{\tilde{u}}(x) \). In the current statistical data science, one simply works with the two functions \( F_u(x) \) and \( F_{\tilde{u}}(x) \). In fact, the more complete description in the statistical representation is a joint probability distribution \( F_{u\tilde{u}}(x_1, x_2) \) whose marginal distributions are \( F_u(x_1) = F_{u\tilde{u}}(x_1, \infty) \) and \( F_{\tilde{u}}(x_2) = F_{u\tilde{u}}(\infty, x_2) \).

If, however, one explores a little more on the “origin” of the change, one realizes that there are two possible sources: A change in the \( \mathbb{P} \), or a change in the \( u(\omega) \). If the \( \mathbb{P} \rightarrow \tilde{\mathbb{P}} \) while the \( u(\omega) \) is fixed, then according to the measure theory, one can characterize this “change of measure” in terms of a Radon-Nikodym (RN) derivative \( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) \). In the rest of this paper, we will assume that all the measures under consideration are absolutely continuous with respect to each other and that all measures on \( \mathbb{R}^n \) are absolutely continuous with respect to the Lebesgue measure. This ensures that all RN derivatives are well-defined. Note, this is a mathematical object that is defined on the invisible probability space. It actually is itself a random variable, with expectation, variance, and statistics.

What is the relationship between this informational representation of the change and the observed \( F_u \) and \( F_{\tilde{u}} \)? For \( u(\omega), \tilde{u}(\omega) \in \mathbb{R} \), the answer is \([5, 2]\):

\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) = \left( \frac{dF_{\tilde{u}}}{dF_u}(\omega) \right)^{-1}.
\]

On the rhs of \([1] \), \( \frac{dF_{\tilde{u}}}{dF_u}(x) \) is like a probability density function, which is only defined on \( \mathbb{R} \). However, substituting the random variable \( u(\omega) \) into the probability density function, one obtains the lhs of \([1] \). Putting a random variable back into the logarithm of its own density function to create a new random variable is the fundamental idea of fluctuating entropy in stochastic thermodynamics \([6, 7]\), and the notion of self-information \([8–10]\). Its expected value then becomes the Shannon information entropy or intimately related relative entropy \([11]\). The result in \([1] \) can be generalized to \( u, \tilde{u} \in \mathbb{R}^n \).

In this case,

\[
\frac{dF_{\tilde{u}}}{dF_u}(x_1, \cdots, x_n) = \frac{\partial^n F_{\tilde{u}}}{\partial x_1 \cdots \partial x_n} \frac{\partial^n F_u}{\partial x_1 \cdots \partial x_n}.
\]

In the rest of the paper, we shall consider the \( u(\omega) \in \mathbb{R} \). But the results are generally valid for multidimensional \( u(\omega) \).

In this paper, we present key results based on this informational representation of stochastic change. We show all types of entropy are unified under the single theory. The discussions are restricted on very simple cases; we only touch upon the stochastic change with a pair of correlated \( u(\omega) \rightarrow \tilde{u}(\omega) \), which have respective generated \( \sigma \)-algebras that are non-identical in general. The notion of “thermodynamic work” will appear then \([8]\).

The informational and probabilistic representations of stochastic changes echo the Schrödinger and Heisenberg pictures in quantum dynamics \([12]\): in terms of wave functions in the abstract, invisible Hilbert space and in terms of self-adjoint operators as observables.

### II. INFORMATIONAL REPRESENTATION OF STOCHASTIC CHANGE

Statistics and information: Push-forward and pull-back. Consider a sequence of real-valued random variables of a
same physical origin and their individual cdfs:
\[ F_{u_1}(x), F_{u_2}(x), \ldots, F_{u_T}(x). \]  

(3)

According to the axiomatic theory of probability built on \((\Omega, \mathcal{F}, \mathbb{P})\), there is a sequence of random variables
\[ u_1(\omega), u_2(\omega), \ldots, u_T(\omega), \]

(4)

in which each \(u_k(\omega)\) maps the \(\mathbb{P}\) to the push-forward measure \(F_k(x), x \in \mathbb{R}\). Eq. 5 illustrate this with \(u\) and \(\tilde{u}\) stand for any \(u_i\) and \(u_j\),

\[ \frac{d\tilde{P}}{dP}(\omega) = \left[ \frac{dF_{u_1}}{dF_{u_k}}(u(\omega)) \right]^{-1} d\mathbb{P}(\omega). \]

(5)

Informational representation, thus, considers the \(u\) as the push-forward of a sequence of measures \(\mathbb{P}_1 = \mathbb{P}, \mathbb{P}_2, \ldots, \mathbb{P}_T\), under a single observable, say \(u_1(\omega)\). This sequence of \(\mathbb{P}_k\) can be represented through the fluctuating entropy inside the \([\cdots]\) below:

\[ d\mathbb{P}_k(\omega) = \left[ \frac{dF_{u_1}}{dF_{u_k}}(u_1(\omega)) \right]^{-1} d\mathbb{P}(\omega). \]

(6)

Narratives based on information representation have rich varieties. We have discussed above the information representation cast with a single, common random variable \(u(\omega); \mathbb{P}_k(\omega) \xrightarrow{u_k} F_k(x)\). Alternatively, one can cast the information representation with a single, given \(F^*(x)\) on \(\mathbb{R}: \mathbb{P}_k(\omega) \xrightarrow{u_k} F^*(x)\) for any sequence of \(u_k(\omega)\). Actually there is a corresponding sequence of measures \(P_k\), whose push-forward are all \(F^*(x)\), on the real line independent of \(k\). Then parallel to Eq. 5, we have a schematic:

\[ \frac{d\tilde{P}}{dP}(\omega) = \left[ \frac{dF_{u_1}}{dF_{F_k}}(u_1(\omega)) \right]^{-1} \frac{d\mathbb{P}_k(\omega)}{d\mathbb{P}(\omega)}. \]

(7)

If the invariant \(F^*(x)\) is the uniform distribution, e.g., when one chooses the Lebesgue measure on \(\mathbb{R}\) with \(F^*(x) = x\), then one has

\[ - \mathbb{E}^P \left[ \ln \left( \frac{dF_{u_k}}{dx} (u_k(\omega)) \right) \right] = - \int f_{u_k}(x) \ln f_{u_k}(x) \, dx. \]

This is precisely the Shannon entropy! More generally with a fixed \(F^*(x)\), one has

\[ \mathbb{E}^P \left[ \ln \left( \frac{dF_{u_k}}{dF^*} (u_k(\omega)) \right) \right] = \int f_{u_k}(x) \ln \frac{f_{u_k}(x)}{f^*(x)} \, dx = - \int f_{u_k}(x) \ln f^*(x) \, dx + \int f_{u_k}(x) \ln f_{u_k}(x) \, dx. \]

(8)

This is exactly the relative entropy w.r.t. the stationary \(f^*(x)\). In statistical thermodynamics, this is called free energy. \(-\beta^{-1} \ln f^*(x)\) on the rhs of (8) is called internal energy, where \(\beta\) stands for the physical unit of energy. The integral is the mean internal energy.

**Essential facts on information.** Several key mathematical facts concerning the information, as a random variable defined in 11, are worth stating.

First, even though the \(u(\omega)\) appears in the rhs of the equation, the resulting lhs is independent of the \(u(\omega)\): It is a random variable created from \(\mathbb{P}, \tilde{\mathbb{P}}\), and random variable \(\tilde{u}(\omega)\), as clearly shown in Eq. 5. This should be compared with a well-known result in elementary probability: For a real-valued random variable \(\eta(\omega)\) and its cdf \(F^*\), the constructed random variable \(\tilde{F}^*(\eta(\omega))\) is a uniform distribution on \([0, 1]\) independent of the nature of \(\eta(\omega)\).

Second, if we denote the logarithm of 11 as \(\xi(\omega), \xi(\omega) = \ln \frac{d\tilde{P}}{dP}\), then one has a result based on the change of measure for integration:

\[ \begin{align*}
\mathbb{E}^P \left[ \eta(\omega) \right] &= \int \eta(\omega) \, d\mathbb{P}(\omega) = \int \eta(\omega) \left( \frac{d\tilde{P}}{d\mathbb{P}} (\omega) \right) \, d\tilde{P}(\omega) \\
&= \int \eta(\omega) e^{-\xi(\omega)} \, d\tilde{P}(\omega) \\
&= \mathbb{E}^{\tilde{P}} \left[ \eta(\omega) e^{-\xi(\omega)} \right].
\end{align*} \]

(9a)

And conversely,

\[ \mathbb{E}^{\tilde{P}} \left[ \eta(\omega) \right] = \mathbb{E}^P \left[ \eta(\omega) e^{\xi(\omega)} \right]. \]

(9b)

In particular, when the \(\eta = 1\), the log-mean-exponential of fluctuating \(\xi\) is zero. The incarnations of this equality have been discovered numerous times in thermodynamics, such as Zwanzig’s free energy perturbation method [13], the Jarzynski-Crooks equality [13][36], and the Hatano-Sasa equality [16].

Third, one has an inequality,

\[ \ln \mathbb{E}^{\tilde{P}} \left[ \xi(\omega) \right] \leq \ln \mathbb{E}^{\tilde{P}} e^{\xi(\omega)} = 0. \]

(10)

As we have discussed in 15, this inequality is the mathematical origin of almost all inequalities in connection to entropy in thermodynamics and information theory.

Fourth, let us again consider a real-valued random variable \(\eta(\omega)\), with probability density function \(f_{\eta}(x), x \in \mathbb{R}\), and its information, e.g., fluctuating entropy \(\xi(\omega) = -\ln f_{\eta}(\eta(\omega))\).

Then one has a new measure \(\tilde{\mathbb{P}}\) whose \(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) = e^{\xi(\omega)}\), and

\[ \begin{align*}
\int_{x_1 < \eta(\omega) \leq x_2} d\tilde{\mathbb{P}}(\omega) &= \int_{x_1 < \eta(\omega) \leq x_2} e^{\xi(\omega)} \, d\tilde{\mathbb{P}}(\omega) \\
&= \int_{x_1 < \eta(\omega) \leq x_2} \left[ f_{\eta}(\eta(\omega)) \right]^{-1} \, d\tilde{\mathbb{P}}(\omega) \\
&= \int_{x_1}^{x_2} \left( f_{\eta}(y) \right)^{-1} f_{\eta}(y) \, dy \\
&= x_2 - x_1.
\end{align*} \]

(11)
This means that under the new measure $\hat{P}$, the random variable $\eta(\omega)$ has an unbiased uniform distribution on $\mathbb{R}$. Note that the measure $\hat{P}(\omega)$ is non-normalizable if $\eta(\omega)$ is not a bounded function.

Entropy is the greatest “equalizer” of random variables, as physical observables inevitably biased!

The foregoing discussion leaves no doubt that entropy (or negative free energy) $\xi(\omega)$ is a quantity to be used in the form of $e^{\xi(\omega)}$. This clearly points to the origin of partition function computation in statistical mechanics. In fact, it is fitting to call the tangent space in the affine structure of the space of measures its “space of entropies” [5]; which represents the change of information.

III. CONFIGURATIONAL ENTROPY IN CLASSICAL DYNAMICS

By classical dynamics, we mean the representation of dynamical change in terms of a deterministic mathematical description, with either discrete space-time or continuous space-time. The notion of “configurational entropy” arose in this context in the theories of statistical mechanics, developed by L. Boltzmann and J. W. Gibbs, either as Boltzmann’s transformations [17] as a non-normalizable density function, or its cumulative distribution. Boltzmann’s entropy emerges from the conserved quantities in a microscopic deterministic map in the space of observables, dynamics.

In our present approach, the classical dynamics is a deterministic map in the space of observables, $\mathbb{R}^n$. The informational representation demands a description of the change via a change of measures, and we now show the notion of configurational entropy arises.

Information in deterministic change. Let us now consider a one-to-one deterministic transformation $\mathbb{R} \rightarrow \mathbb{R}$, which maps the random variable $u(\omega)$ to $v(\omega) = g^{-1}(u(\omega))$, $g'(x) > 0$. Then

$$\frac{dF_u(x)}{dF_u(x)} = \left( \frac{f_u(g(x))}{f(x)} \right) \cdot \frac{g'(x)}{g'(x)}. \tag{12}$$

Applying the result in [1] and [5], the corresponding RN derivative

$$- \ln f_u(g(x)) - \ln f_u(x) + \ln g' \bigg|_{x=g^{-1}(\omega)} \tag{13a}$$

$$= - \ln f_u(v(\omega)) + \ln \left( \frac{dg}{dx}(v(\omega)) \right) \tag{13b}$$

$$- \left( - \ln f_u(u(\omega)) \right) \tag{13c}$$

in which the information of $v = g^{-1}(u)$, under observable $u(\omega)$, has two distinctly different contributions: energetic part and entropic part.

The energetic part represents the “changing location”, which characterizes movement in the classical dynamic sense: A point to a point. It experiences an “energy” change where the internal energy is defined as $-\beta^{-1} \ln f_u(x) = \varphi(x)$.

The entropic part represents the resolution for measuring information. This is distinctly a feature of dynamics in a continuous space, it is related to the Jacobian matrix in a deterministic transformation: According to the concept of Markov partition developed by Kolmogorov and Sinai for chaotic dynamics [18], there is the possibility of continuous entropy production in dynamics with increasing “state space fineness”. This term is ultimately related to Kolmogorov-Sinai entropy and Ruelle’s folding entropy [19].

Entropy and Jacobian matrix. We note that the

$$- \ln f_u(v(\omega)) + \ln \left( \frac{dg}{dx}(v(\omega)) \right) \neq - \ln f_u(v(\omega)). \tag{14}$$

This difference precisely reflects the effect of “pull-back” from $\mathbb{R}$ to the $\Omega$ space, there is a breaking symmetry between $u(\omega)$ and $v(\omega)$ in (13b), when setting $x = v(\omega)$. Actually, the full “information entropy change” associated with the deterministic map

$$- \ln f_u(v(\omega)) + \ln f_u(u(\omega)) = - \ln \left( \frac{dg}{dx}(v(\omega)) \right), \tag{15}$$

as expected. The rhs is called configurational entropy. Note this is an equation between three random variables, i.e., fluctuating entropies, that is valid for all $\omega$. For a one-to-one map in $\mathbb{R}^n$, the above $|dg(x)/dx|$ becomes the absolute value of the determinant of $n \times n$ Jacobian matrix, which has a paramount importance in the theory of functions, integrations, and deterministic transformations. The matrix is associated with an invertible local coordinate transform, $y_i = g_i(x)$, $1 \leq i \leq n, x \in \mathbb{R}^n$:

$$\frac{\mathcal{D}[y_1, \ldots , y_n]}{\mathcal{D}[x_1, \ldots , x_n]} = \left[ \begin{array}{cccc} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_n}{\partial x_n} \end{array} \right], \tag{16}$$

whose determinant is the local “density change”. Classical Hamiltonian dynamics preserves the Lebesgue volume.

Measure-preserving transformation. A change with information preservation means the lhs of (13) being zero for all $\omega \in \Omega$. This implies the function $g(x)$ necessarily satisfies

$$- \ln f_u(x) + \ln \left( \frac{dg(x)}{dx} \right) = - \ln f_u(g(x)). \tag{17}$$

Eq. [17] is actually the condition for $g$ preserving the measure $F_u(x)$, with density $f_u(x)$, on $\mathbb{R}$ [18]:

$$f_u(x) = f_u(g(x)) \left( \frac{dg(x)}{dx} \right) = f_v(x), \tag{18a}$$

$$F_u(x) = \int_{-\infty}^{x} f_u(z)dz = F_v(g(x)). \tag{18b}$$
In this case, the inequality in (14) becomes an equality.
In terms of the internal energy $\varphi(x)$, then one has

$$
\beta = \frac{1}{\varphi'(g(x))} \ln \left( \frac{dg(x)}{dx} \right),
$$

(19)

which should be heuristically understood as the ratio $\Delta S / \Delta F$ where $\Delta S = \ln dg(x) - \ln dx$ and $\Delta E = \varphi(g(x)) - \varphi(x)$. For an infinitesimal change $g(x) = x + \varepsilon(x)$, we have

$$
\beta = \frac{\varepsilon'(x)}{\varphi'(x)\varepsilon(x)}.
$$

(20)

**Entropy balance equation.** The expected value of the lhs of (13) according to measure $\mathbb{P}$ is non-negative. In fact, consider the Shannon entropy change associated with $F_u(x) \rightarrow F_u(x)$:

$$
\int_R f_u(x) \ln f_u(x) dx - \int_R f_u(x) \ln f_u(x) dx = \int_R f_u(x) \ln \left( \frac{f_u(x)}{f_u(x)} \right) dx + \int_R \left[ f_u(x) - f_u(x) \right] \ln f_u(x) dx
$$

(21a)

$$
= \mathbb{E} \left[ \ln \left( \frac{d\mathbb{P}}{d\mathbb{P}}(\omega) \right) \right] + \mathbb{E} \left[ \ln \left( \frac{dF_u}{dx}(\omega) \right) - \ln \left( \frac{dF_u}{dx}(u(\omega)) \right) \right].
$$

(21b)

This equation in fact has the form of the fundamental equation of nonequilibrium thermodynamics [20, 21]: $\Delta S = \Delta S^{\text{tr}} + \Delta S^{(e)}$. The entropy production $\Delta S^{\text{tr}}$ on the rhs is never negative, the entropy exchange $\Delta S^{(e)}$ has no definitive sign. If $f_u(x) = \frac{dF_u(x)}{dx} = C$ is a uniform distribution, then $\Delta S^{(e)} = 0$ and entropy change is the same as entropy production.

**Contracted description, endomorphism, and matrix volume.** We have discussed above the $\mathbb{R}^n \rightarrow \mathbb{R}^n$ deterministic invertible transformation $y = g(x)$ and shown that the determinant of its Jacobian matrix is indeed the information entropy change. The informational representation, in fact naturally, allows us to also consider a non-invertible transformation in the space of observable $u(\omega) \in \mathbb{R}^m$ through a much lower dimensional $g(x) \in \mathbb{R}^m$, $m \ll n$. In this case, the original $\mathbb{R}^n$ is organized by the $m$-dimensional observables, called “macroscopic” (thermodynamic) variables in classical physics, or “principle components” in current data science and model reduction [22].

The term *endomorphism* means that a deterministic map $u \rightarrow v = g^{-1}(u)$ in (12) is many-to-one. In this case, a simple approach is to divide the domain of $u$ into invertible parts. Actually, in terms of the probability distributions of $g(x) = (g_1, \cdots, g_m)(x)$, the non-normalizable density function

$$
W(y_1, \cdots, y_m) = \frac{\partial^m}{\partial y_1 \cdots \partial y_m} \int_{y_1(\omega) \leq y_1, \cdots, y_m(\omega) \leq y_m} dx,
$$

(22)

now plays a crucial role in the information representation. In fact, the $W$ in (22) is the reciprocal of the matrix volume, e.g., the absolute value of the “determinant” of the rectangular matrix [23]

$$
\det \left( \mathcal{D}_{y_1, \cdots, y_m} \mathcal{D}_{x_1, \cdots, x_n} \right),
$$

(23)

which can be computed from the product of the singular values of the non-square matrix. Boltzmann’s *Wahrscheinlichkeit*, his thermodynamic probability, is when $m = 1$. In that case,

$$
W(y) = \frac{d}{dy} \int_{g(x) \leq y} dx = \int_{g(x) = y} d\omega \| \nabla g(x) \|^1
$$

(24)

in which the integral on rhs is the surface integral on the level surface of $g(x) = y$.

One has a further, clear physical interpretation of $-\ln W(y_1, \cdots, y_m)$ as a potential of entropic force [24], with force:

$$
\frac{\partial \ln W}{\partial y_\ell} = \frac{1}{W} \frac{\partial W}{\partial y_\ell} (1 \leq \ell \leq m).
$$

(25)

In the polymer theory of rubber elasticity, a Gaussian density function emerges due to central limit theorem, and Eq. 23 yields a three-dimensional Hookean linear spring [25].

**IV. CONDITIONAL PROBABILITY, MUTUAL INFORMATION AND FLUCTUATION THEOREM**

In addition to describing change by $\Delta x = x_2 - x_1$, another more in-depth characterization of a pair of observables $(x_1, x_2)$ is by their functional dependency $x_2 = g(x_1)$, if any. In connection to stochastic change, this leads to the powerful notion of conditional probability, which we now discuss in terms of the informational representation.

First, for two random variables $(u_1, u_2)(\omega)$, the conditional probability distribution on $\mathbb{R} \times \mathbb{R}$:

$$
F_{u_1|u_2}(x; y) = \mathbb{P} \left\{ y < u_2(\omega) \leq y + dy, u_1(\omega) \leq x \right\}.
$$

(26)

Then it generates an “informational” random variable

$$
\xi_{12}(\omega) = \ln \left[ \frac{\partial F_{u_1|u_2}(x; y)/\partial x}{\partial F_{u_1}(x)/\partial x} \right]_{x=u_1(\omega), y=u_2(\omega)},
$$

(27)

which is a conditional information, whose expected value is widely know in information theory as *mutual information* [26]:

$$
\mathbb{E} [\xi_{12}(\omega)] = \int_\mathbb{R} \int f_{u_1,u_2}(x, y) \ln \left[ \frac{\partial F_{u_1|u_2}(x; y)/\partial x}{\partial F_{u_1}(x)/\partial x} \right] dx dy
$$

$$
= \int_\mathbb{R} \int f_{u_1,u_2}(x, y) \ln \left[ \frac{\partial F_{u_1|u_2}(x; y)/\partial x}{\partial F_{u_1}(x)/\partial x} \right] dx dy
$$

$$
= \int_\mathbb{R} \int f_{u_1,u_2}(x, y) \ln \left[ \frac{\partial F_{u_1|u_2}(x; y)/\partial x}{\partial F_{u_1}(x)/\partial x} \right] dx dy
$$

$$
= \text{MI}(u_1, u_2).
$$

(28)
We note that $\xi_{12}(\omega)$ is actually symmetric w.r.t. $u_1$ and $u_2$. The term inside the $[\cdot \cdot \cdot ]$ in (27)
\[
\frac{f_{u_1|u_2}(x; y)}{f_{u_1}(x)} = \frac{f_{u_1|u_2}(x, y)}{f_{u_1}(x)} = \frac{f_{u_2|u_1}(x; y)}{f_{u_2}(x)} = \frac{f_{u_2|u_1}(x, y)}{f_{u_2}(x)}. \tag{29}
\]
In fact the equality $\xi_{12}(\omega) = \xi_{21}(\omega)$ is the Bayes’ rule. Furthermore, $\mathrm{MI}(u_1, u_2) \geq 0$. It has been transformed into a distance function that satisfies triangle inequality (27, 28).

In statistics, “distance” is not only measured by their dissimilarity, but also statistical dependence. This realization gives rise to the key notion of independent and identically distributed (i.i.d.) random variables. In the informational representation, this means $u_1$ and $u_2$ have same amount of information on the probability space, and they have zero mutual information.

Finally, but not the least, for $F_{u_1|u_2}(x, y)$, one can introduce a $F_{u_1\to u_2}(x, y) = F_{u_2\to u_1}(y, x)$. Then entropy production,
\[
\xi(\omega) = \ln \frac{\partial F_{u_1\to u_2}(x, y)}{\partial F_{u_1\to u_2}(x, y)} \bigg|_{x=u_1(\omega), y=u_2(\omega)} \tag{30a}
\]
\[
= \ln \frac{\partial^2 F_{u_1\to u_2}(x, y)}{\partial x \partial y} \bigg|_{x=u_1(\omega), y=u_2(\omega)} + \ln \frac{\partial^2 F_{u_2\to u_1}(y, x)}{\partial y \partial x} \bigg|_{x=u_1(\omega), y=u_2(\omega)}, \tag{30b}
\]
satisfies the fluctuation theorem (35, 37):
\[
\mathbb{P}\{a < \xi(\omega) \leq a + da\} = e^a \mathbb{P}\{-a - da < \xi(\omega) \leq -a\} \tag{31}
\]
for any $a \in \mathbb{R}$. Eq. (31) characterizes the statistical asymmetry between $u_1(\omega)$ and $u_2(\omega)$, not independent in general. Being identical and independent is symmetric, so is a stationary Markov process with reversibility (35).

V. CONCLUSION AND DISCUSSION

The theory of information (26) as a discipline that exists outside the field of probability and statistics owes to its singular emphasis on the notion of entropy as a quantitative measure of information. Our theory shows that it is indeed a unique representation of stochastic change that is distinctly different from, but complementary to, the traditional mathematical theory of probability. In statistical physics, there is a growing awareness of a deep relation between the notion of thermodynamic free energy and information (29). The present work clearly shows that it is possible to narrate the statistical thermodynamics as a subject of theoretical physics in terms of the measure-theoretic information, as suggested by some scholars who studied deeply thermodynamics and the concept of entropy (30). Just as differential calculus and Hilbert space providing the necessary language for classical and quantum mechanics, respectively, the Kolmogorovian probability, including informational representation, provides many known results in statistical physics and chemistry with a deeper understanding, such as phase transition and symmetry breaking (31). Gibbsian ensemble theory (32), nonequilibrium thermodynamics (33, 38), and the unification of the theories of chemical kinetics and chemical thermodynamics (39). In fact, symmetry principle (40) and emergent probability distribution via limit laws (41) can both be understood as providing legitimate measures a priori, $\mathbb{P}(\omega)$, for the physical world or the biological world. And stochastic kinematics dictates entropic force and its potential function, the free energy (42, 53).

For a long time the field of information theory and the study of large deviations (LD) in probability were not integrated: A. Ya Khinchin’s book was published the same year as the Sanov theorem (43, 44). Researchers now are agreed upon that Boltzmann’s original approach to the canonical equilibrium energy distribution (45), which was based on a maximum entropy argument, is a part of the contraction of Sanov theorem in the LD theory (46). In probability, the contraction principle emphasizes the LD rate function for the sample mean of a random variable (54). In statistics and data science, the same mathematics has been used to justify the maximum entropy principle which emphasizes a bias, as a conditional probability, introduced by observing a sample mean (47). In all these work, Shannon’s entropy and its variant relative entropy, as a single numerical characteristic of a probability distribution, has a natural and logical role. Many approaches have been further advanced in applications, e.g., surprisal analysis and maximum caliber principle (48, 49).

The idea that information can be itself a stochastic quantity originated in the work of Tribus, Kolmogorov (8, 9), and probably many other mathematically minded researchers (50). In physics, fluctuating entropy and entropy production arose in the theory of nonequilibrium stochastic thermodynamics. This development has significantly deepened the concept of entropy, both to physics and as the theory of information. The present work further illustrates that the notion of information, together with fluctuating entropy, actually originates from a perspective that is rather different from that of strict Kolmogorovian; with complementarity and contradistinctions. In pure mathematics, the notion of change of measures goes back at least to 1940s (51), if not earlier.

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\[ \bar{F}_u(x) = \int_{u(\omega) \leq x} \bar{d}P(\omega) = \left[ \int_{u(\omega) \leq x} \frac{dF_u}{d\bar{F}_u}(u(\omega)) \right] d\bar{P}(\omega) \]

\[ = \int_{-\infty}^{x} \left[ \frac{dF_u}{d\bar{F}_u}(u) \right] f_u(u) du = \int_{-\infty}^{x} \left[ \frac{dF_u}{d\bar{F}_u}(u) \right] du = F_u(x). \]

The equality in (1) is restricted on all the \( B \in \sigma(u^{-1}(B)) \subseteq \mathcal{F}. \)

[52] Even though the mathematical steps in Boltzmann’s work on ideal gas are precisely these in the LD contraction computation, their interpretations are quite different: Total mechanical energy conservation figured prominently in Boltzmann’s work, which is replaced by the conditioning on a given sample mean in the LD theory. The subtle relation between conservation law and conditional probability was discussed in [32].

[53] There is a remarkable logic consistency between fundamental physics pursuing symmetries in appropriate spaces under appropriate group actions and the notion of invariant measures in a theory of stochastic dynamics as the legitimate prior probability [33, 34].