DYNAMICS OF COMPOSITE SYMPLECTIC DEHN TWISTS

WENMIN GONG, ZHIJING WENDY WANG, JINXIN XUE

Abstract. This paper appears as the confluence of hyperbolic dynamics, symplectic topology and low dimensional topology, etc. We show that composite symplectic Dehn twists have certain form of nonuniform hyperbolicity: it has positive topological entropy as well as two families of local stable and unstable Lagrangian manifolds, which are analogous to signatures of pseudo Anosov mapping classes. Moreover, we show that the rank of the Floer cohomology group of these compositions grows exponentially under iterations, which partially answers a question of Smith concerning the classification of symplectic mapping class group in higher dimensions. Finally, we propose a conjecture on the positive metric entropy of our model and point out its relationship with the standard map.

1. Introduction

In this paper, we study the dynamical behaviors of elements in the symplectic mapping class group of a symplectic manifold.

When the symplectic manifold is a surface, the classical Nielsen-Thurston theory provides valuable insights into the dynamics of mapping classes: every automorphism $f$ of $S$ is homotopic to a homeomorphism $\phi$ that satisfies one of the following, periodic ($\phi^k = \text{id}$ for some $k \in \mathbb{N}$), reducible (there exists a closed loop $\gamma \subset S$ preserved by $\phi$), and pseudo-Anosov. The latter is characterized by two significant properties, among others:

(a) Existence of two singular transversal foliations invariant under a representative $\phi$;
(b) Expansion/contraction along the leaves, measured by positive topological entropy.

Pseudo-Anosov mapping classes are of particular interest due to their close connection to Anosov systems, which are central objects studied in the field of hyperbolic dynamical systems. Furthermore, pseudo-Anosov mapping classes constitute the majority of classes within the mapping class group.

To gain a clearer understanding of these concepts, let’s consider the classification on the two dimensional torus. We examine the linear automorphisms in $\text{PSL}_2\mathbb{Z} = \text{SL}_2\mathbb{Z}/\{\pm \text{id}\}$ that act on the torus $\mathbb{R}^2/\mathbb{Z}^2$. An automorphism $f$ in $\text{PSL}_2\mathbb{Z}$ can be classified into one of the following three types:

1. $\text{tr}(f) < 2$: In this case, $f$ is periodic, and there exists some $k \in \mathbb{N}$ such that $f^k = \text{id}$.
2. $\text{tr}(f) = 2$: Here, $f$ is reducible, it fixes a closed curve and exhibits linear growth of intersection numbers.
(3) \(\text{tr}(f) > 2\): This type corresponds to Anosov automorphisms. In this case, 
\(f\) expands along one eigen-direction while contracting along the other eigen-
direction, which form the stable and unstable foliations of the automorphism.
It also exhibits exponential growth of intersection numbers.

In 4-dimensional case, for the symplectic mapping class group of \(A^2_m\), where the
symplectic manifold \(A^n_m\) is
\[A^n_m = \{x_1^2 + x_2^2 + \ldots + x_n^2 = x_{n+1}^{m+1} + \frac{1}{2}\} \subset (\mathbb{C}^{n+1}, \omega_0),\]
given in example 2.2.1, our first main result gives a dynamical classification in terms
of Lagrangian Floer cohomology, which provides the first example answering Smith’s
question in [37, Section 3.2.2].

**Definition 1.1.** A Lagrangian isotopy on a symplectic manifold \((M, \omega)\) is a smooth
family of Lagrangian embedding \(\Phi : L \times [0,1] \rightarrow M\).

**Theorem 1.1.** For each symplectic mapping class \([f] \in \pi_0(\text{Symp}_c(A^2_m, \omega))\), any
representative \(f\) satisfies one of the following:

- **f is reducible:** some power \(f^k\) preserves the Lagrangian isotopy class of a La-
grangian sphere \(S\), i.e. \(f^k(S)\) is Lagrangian isotopic to \(S\);
- **f is periodic:** for any Lagrangian spheres \(\alpha, \beta \subseteq A^2_m\), the sequence rank \(HF^\ast(\alpha, f^n \beta)\)
is periodic in \(n\);
- **f is hyperbolic:** for any Lagrangian spheres \(\alpha, \beta \subseteq A^2_m\) the sequence rank \(HF^\ast(\alpha, f^n \beta)\)
grows exponentially in \(n\).

A classical example of an Anosov automorphism is the Arnold’s cat map:
\[f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.\]

We observe that it can be expressed as a composition of upper and lower triangular matrices:
\[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\] and \[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\]
which are instances of Dehn twists on the torus. Dehn twists involve cutting the surface
along a closed curve, twisting a tubular neighborhood of one side of the curve by \(2\pi\),
and then reassembling the curve.

In general, the composition of Dehn twists is a common method for constructing
pseudo-Anosov elements. This algorithm can be found in the works of Thurston [38],
Penner [30], Fathi [16], and others.

It was noted by Arnold [3] that “the natural generalization of SL(2) is, from many
points of view, not SL(n) but rather the linear symplectomorphisms group \(\text{Sp}(n)\) of
2n-space, and correspondingly the group \(\text{Symp}(M)\).” Therefore, it is natural to try
and extend the picture to 2n-dimensional symplectic manifolds, where \(n > 1\). Arnold
[3] and Seidel [34] introduced a generalization of Dehn twists as a symplectomorphism
on \(T^\ast\mathbb{S}^n\). If \(L\) is a Lagrangian sphere on a symplectic manifold \(M\), the Dehn twist \(\tau_L\)
with respect to \(L\) roughly twists a tubular neighborhood of \(L\) in a way that maps the
points on $L$ to its antipodal points and transition to identity outside the neighborhood. For $n = 2$, the square of a symplectic Dehn twist is smoothly but not symplectically isotopic to the identity [34]. Keating and Randal-Williams further discuss this in different dimensions, see [25].

We anticipate that by composing symplectic Dehn twists, one can generate symplectic mapping classes that exhibit characteristics of pseudo-Anosov mapping classes. We aim to uncover the hyperbolicity of such elements and establish properties similar to pseudo-Anosov elements mentioned above (see Theorems 1.2, 1.3, 1.4 below). Additionally, we will investigate the exponential growth of ranks in Floer cohomology groups associated with these elements (see Theorems 1.5 and 1.6 below). This also leads to a classification of the symplectic mapping class group of the $A_m$ manifold in terms of the growth of Floer cohomology. Moreover, we will explore the complexity of their dynamics by comparing them to the standard map.

The study of composite symplectic Dehn twists has gained attention in recent years from the perspective of categorical dynamics. Interested readers can refer to [6, 14, 27] and the references therein for more information.

1.1. **Positive topological entropy.** We begin by demonstrating that specific natural representatives of compositions of symplectic Dehn twists exhibit positive topological entropy.

**Theorem 1.2.** Let $M$ be a symplectic manifold containing an $A_2$-configuration of Lagrangian spheres $S_1$ and $S_2$ that intersect transversely at a single point. Then there exist symplectic Dehn twists $\tau_1$ and $\tau_2$ of $S_1$ and $S_2$, respectively, such that for any $k, \ell \in \mathbb{Z}$ where $k\ell \neq 0, 1, 2, 3, 4$, the composition $\tau = \tau_1^k \tau_2^\ell$ of symplectic Dehn twists has a positive topological entropy, i.e., $h_{\text{top}}(\tau_1^k \tau_2^\ell) > 0$.

In cases where there are more than two Lagrangian spheres, a natural setting to consider is the $A_m$ configuration, which has been extensively studied in the literature. It is known that in four real dimension, the symplectic mapping class group of the $A_m$ configuration is generated by symplectic Dehn twists [13, 39]. In higher dimensions, the natural braid group action is faithful [26]. We will describe this configuration in Example 2.2.1, following [34].

**Theorem 1.3.** Let $L_i, 1 \leq i \leq m$, be the $m$ embedded Lagrangian spheres in $A_m^n$ as described in Example 2.2.1. Then, there exist Dehn twists $\tau_i$ on $S_i$ for each $1 \leq i \leq m$ such that for any $k_i \in \mathbb{Z}$ satisfying $k_ik_{i+1} < 0$ for all $i = 1, 2, \ldots, m - 1$, the topological entropy of $\tau = \tau_1^{k_1} \cdots \tau_m^{k_m}$ is positive, i.e.,

$$h_{\text{top}}(\tau_1^{k_1} \cdots \tau_m^{k_m}) > 0.$$  

The proof of this theorem involves identifying an invariant submanifold on which the restrictions of the symplectic Dehn twists correspond to the usual Dehn twists on surfaces. The restricted dynamics is a generalization of the linked twist map and related to the egg-beater map in the work of [32]. Positive topological entropy is only one of the many consequences of the presence of the linked twist map. Additionally, we show that the dynamics on the subsystem exhibit hyperbolic behavior, characterized by positive Lyapunov exponent on a full Lebesgue measure set of the subsystem. Using
Pesin’s stable manifold theory (Theorem A.5), we further demonstrate the existence of two families of Lagrangian submanifolds invariant under the composite Dehn twist.

**Theorem 1.4.** In the setting of the Theorem 1.2 with $k \ell < 0$ and Theorem 1.3, there exist two families of Lagrangian submanifolds $\mathcal{F}^s, \mathcal{F}^u$ on $M$ in a neighbourhood $N$ of $\cup S_i$ which are invariant under $\tau$. Furthermore, there is an invariant subset $P$ of $N$ admitting an $SO(n)$ action that commutes with $\tau$ and preserves each leaf in $\mathcal{F}^s$ and $\mathcal{F}^u$, such that

1. $P/\text{SO}(n)$ is a two dimensional manifold with boundary,
2. either of $\mathcal{F}^s/\text{SO}(n)$ and $\mathcal{F}^u/\text{SO}(n)$ has full Lebesgue measure on $P/\text{SO}(n)$
3. for a.e. $x \in P/\text{SO}(n)$, the leaf $f^s(x)$ (resp. $f^u(x)$) of $\mathcal{F}^s/\text{SO}(n)$ (resp. $\mathcal{F}^u/\text{SO}(n)$) passing through $x$ is contracted exponentially under iterations of $\tau$ (resp. $\tau^{-1}$).

The question of existence of stable/unstable Lagrangian foliations for pseudo-Anosov elements generalizing Nielsen-Thurston theory was initially posed as an open problem in [12], and was studied in [28] recently. Our Theorem 1.4 presented in this paper offers a non-trivial example of existence of stable and unstable lamination for symplectic diffeomorphisms. Furthermore, the presence of nonuniformly hyperbolic dynamics yields various standard consequences. For instance, it leads to the existence of horseshoes on $P/\text{SO}(n)$ that are conjugate to Bernoulli shifts. As a result, exponential growth of periodic orbits and other related phenomena such as certain form of ergodicity can be observed. For further exploration of these topics, interested readers are referred to Chapter 5.5 of [23].

### 1.2. Exponential growth of the ranks of Floer cohomology groups.

The aforementioned results rely on a specific choice of representative in the symplectic mapping classes, analogous to the Nielsen-Thurston theory. As a consequence, Theorems 1.2–1.4 may not generally remain robust under perturbations. However, due to the symplectic nature of the problem, it is natural and necessary to describe the complexity of the dynamics using symplectic invariants that are insensitive to the choice of representative in the symplectic mapping classes.

In this paper, we consider the rank of Floer cohomology as the invariant that characterize the dynamics. We primarily consider Lagrangian Floer cohomology group $HF(S_1, \tau^n S_2)$, where $\tau$ represents the composite symplectic Dehn twists as mentioned earlier. Throughout the paper, unless otherwise specified, we work with Lagrangian Floer cohomology using $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Let’s define the relative symplectic growth rate as follows:

**Definition 1.2** (Relative symplectic growth rate). Let $(M, \omega)$ be a symplectic manifold, and let $\phi$ be a symplectomorphism of $M$. Consider a pair of connected compact Lagrangian submanifolds $(L_1, L_2)$ in $M$. Assume that the Lagrangian Floer cohomology groups $HF(L_1, \phi^n(L_2))$ are well defined for all $n \in \mathbb{N}$. Then the symplectic growth rate of $\phi$ relative to the pair of Lagrangians $(L_1, L_2)$ is defined as:

$$\Gamma(\phi, L_1, L_2) = \liminf_{n \to \infty} \frac{\log \text{rank} \ HF(L_1, \phi^n(L_2))}{n}.$$
From the works of Arnold [4] and Seidel [36], the relative symplectic growth rate (when defined) has a uniform bound that only depends on $M$ and $\phi$.

We now proceed to demonstrate that exponential growth can be achieved by composite symplectic Dehn twists, in the symplectic manifolds $A^n_m$. This is summarized in the following theorem:

**Theorem 1.5.** Let $(M^{2n}, \omega_0)$ be the $A^n_m$-manifold with Lagrangian spheres $(S_1, S_2)$. Consider the symplectic Dehn twists $\tau_1$ and $\tau_2$ along the spheres $S_1$ and $S_2$, respectively. The following statements hold:

1. For any $k, \ell \in \mathbb{Z}$ with $k\ell \neq 0, 1, 2, 3, 4$, we have $\Gamma(\tau_1^k \tau_2^\ell, S_1, S_2) > 0$.
2. For any $k, \ell \in \mathbb{Z}$ with $k\ell \in \{0, 1, 2, 3, 4\}$, we have $\Gamma(\tau_1^k \tau_2^\ell, S_1, S_2) = 0$.
3. For $i, j \in 1, 2$ with $i \neq j$, we have $\Gamma(\tau_i^q, S_j, S_j) = 0$.

While the symplectic growth rate is zero in case (3) of the theorem, it can be shown that the rank of $\text{HF}(S_j, \tau_i^{2n}(S_j))$ grows with a polynomial rate (see Section 6.1). A similar computation of the rank of Lagrangian Floer homology reveals that the “slow volume growth behavior” of the symplectic Dehn twist on the cotangent bundle $T^*\mathbb{S}^d$ of the $d$-sphere is positive (see [20]). Additionally, in [8], the authors develop a categorical machinery that yields similar but distinct results regarding exponential growth. Based on Theorem 1.5, we will confirm that the volume entropy and barcode entropy (which is introduced in [11]) are also positive for these compositions in a forthcoming paper [21].

We extend the results to the more general $A^n_m$-type configuration as follows:

**Theorem 1.6.** Consider $m$ embedded Lagrangian spheres $S_i, 1 \leq i \leq m$, in an $A^n_m$-configuration of Lagrangian $n$-spheres as described in Example 2.2.1. Let $\tau_i, i = 1, \ldots, m$, be the symplectic Dehn twists along the $n$-spheres $S_i$ respectively. For $\tau = \tau_1^{k_1} \cdots \tau_m^{k_m}$, where each $k_i \in \mathbb{Z} \setminus \{0\}$ and $k_i \cdot k_{i+1} < 0$ for all $i = 1, \ldots, m-1$, we have $\Gamma(\tau, S_k, S_k) > 0$ for all $k, \ell \in 1, \ldots, m$ with $k \neq \ell$.

These results are proved by combining the techniques developed in [26] and the classical theory of mapping class groups of surfaces. The last theorem can be considered as an analogue of Theorem 3.1 in [30].

Let $P_\psi(S_1, S_2, \ldots, S_m)$ and $P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m)$ be the plumbing spaces, and let $\gamma_i, i = 1, \ldots, m$ be the generating curves for $S_i$, see Section 5.2 for the notations. One technical result in the proof of Theorems 1.5 and 1.6 that may be of independent interest is the following.

**Lemma 1.7.** If the composition $\tau = \tau_1^{k_1} \cdots \tau_m^{k_m}$ of symplectic Dehn twists along spheres $S_i$ satisfies that $\tau^n(\gamma_j)$ is not isotopic to $\gamma_i$, then

$$\text{rank } \text{HF}(S_i, \tau^n(S_j)) = i([\gamma_i], [\tau^n(\gamma_j)]).$$

where $i([\alpha], [\beta])$ represents the geometric intersection number of two closed curves $\alpha, \beta$.

**Remark 1.8.** In the case of dimension two, Dimitrov, Haiden, Katzarkov, and Kontsevich [12] showed that for two non-homotopic curves $\alpha$ and $\beta$ on a closed oriented surface $\Sigma_g$ of genus $g$, we have $\text{rank } \text{HF}(\alpha, \tau^n(\beta)) = i([\alpha], [\tau^n(\beta)])$. This result provides a connection between the Thurston classification and a classification of symplectic mapping classes based on the growth rate of Floer cohomology.
However, in higher dimensions, finding analogues of this result has remained a longstanding challenge, as mentioned in [37, Section 3.2.2]. Our Lemma 1.7 presents a non-trivial case in this direction, making the first attempt towards understanding the classification of symplectic mapping classes and their growth rates in higher dimensions.

1.3. Speculations on the complexity of the dynamics of composite Dehn twists. It is important to note that the study presented so far does not uncover the full complexity of the dynamics of composite symplectic Dehn twists. In this section, we aim to discuss the richness of the dynamics of composite symplectic Dehn twists.

In the field of hyperbolic dynamics, a significant indicator of the complexity of a differentiable dynamical system is the presence of positive metric entropy with respect to the volume measure. This is equivalent to having a positive Lyapunov exponent on a set of positive measure, where the Lyapunov exponent is defined as $\lambda(x) := \limsup_{n \to \infty} \frac{\log |Df^n(x)|}{n}$ for a map $f : M \to M$ and $x \in M$. The existence of positive metric entropy allows for strong conclusions regarding ergodicity. However, establishing positive metric entropy for a given system is generally a highly challenging task.

A prominent example in this context is the “standard map” $f : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $(x, y) \mapsto (x + y + \epsilon \sin(x), y + \epsilon \sin(x))$. It is conjectured that this map exhibits positive metric entropy for the Lebesgue measure, for all $\epsilon > 0$. To help topologists appreciate this conjecture, we can reframe it as follows: For a map with an Anosov mapping class in $\text{SL}_2(\mathbb{Z})$ such as $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, as well as its small perturbations, it is relatively straightforward to establish positive metric entropy. However, for a reducible element such as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the metric entropy is zero. The standard map corresponds to an element with the mapping class $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and we would like to show that the complicated dynamical behaviors appear naturally by perturbing $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. However, proving this conjecture has proven to be an extremely difficult task. From the same perspective, it is known that for $\dim(M) = 4$ or 12 the compositions of Dehn twists $\tau_1^k \tau_2^\ell$, where $k, l \in 2\mathbb{Z}$, are isotopic to the identity but symplectically nontrivial [34]. This implies that we obtain diffeomorphisms with trivial mapping class but intricate dynamics.

Motivated by the analogy with the standard map conjecture, we put forward the following conjecture: the measure-theoretic entropy of the composition $\tau_1^k \tau_2^\ell$ as described in Theorem 1.2 is positive.

**Conjecture 1.9.** Let $S_1$ and $S_2$ be two Lagrangian $n$-spheres in an $A_n^2$-configuration, intersecting transversely at a single point. Then, there exist symplectic Dehn twists $\tau_1$ and $\tau_2$ of $S_1$ and $S_2$ respectively, such that for any $k, \ell \in \mathbb{Z}$ with $k\ell < 0$, the measure-theoretic entropy of the composition $\tau_1^k \tau_2^\ell$ of symplectic Dehn twists with respect to the Lebesgue measure $\text{vol}$ is positive, i.e., $h_{\text{vol}}(\tau_1^k \tau_2^\ell) > 0$.

Conjecture 1.9 represents a significantly stronger statement than Theorem 1.2. If this conjecture were true, it would provide an intriguing example of a dynamical system
with positive entropy that is homotopic to the identity. However, we acknowledge that this conjecture poses a considerable challenge, as we will discuss in Section 9.

1.4. Organization of the paper. In Section 2, we provide a review of the definitions of symplectic Dehn twists and plumbing spaces.

In Section 3, we find a subsystem of $A_n^m$ equivalent to Dehn twists on surfaces and prove Theorems 1.2 and 1.3, which establish positive topological entropy for composite symplectic Dehn twists. In Section 4, we establish the hyperbolicity of the subsystem, demonstrating the presence of positive Lyapunov exponents on a positive measure set. In Section 5, we prove Theorem 1.4, establishing the existence of local stable and unstable Lagrangian manifolds within the dynamics of composite symplectic Dehn twists.

In Section 6, we provide the proof for Theorems 1.5, 1.6 and 1.1, which focus on the growth of Floer cohomology groups. In Section 8, we establish a connection between the rank of Floer cohomology groups and intersection numbers, completing the proof of several technical lemmas from Section 6. In Section 9, we discuss the conjecture on metric entropy and address the challenges associated with it.

Finally, we include three appendices. Appendix A provides preliminaries on topological entropy, metric entropy, and Pesin theory for non-uniformly hyperbolic dynamics. Appendix B presents the preliminaries on Lagrangian Floer cohomology. Additionally, in Appendix C, we offer a proof of Lemma 6.3 and its generalization, which are necessary for verifying the conditions (C1) and (C2) in Theorem 6.1 within our specific setting.

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2. Preliminaries on symplectic Dehn twist

In this section we shall introduce the preliminary definitions of symplectic Dehn twist and plumbing space.

**Definition 2.1** (Symplectic Dehn twist). We define the model Dehn twist $\tau$ on $T^* S^n$ to be

$$(2.1) \quad \tau(x, v) = \begin{cases} \sigma(e^{ir(|v|)})(x, v) & |v| > 0 \\ (A(x), 0) & |v| = 0 \end{cases}$$

for $x \in S^n, v \in T_x S^n$, where $\sigma : S^1 \times T^* S^n \to T^* S^n$ is the circle action on $T^* S^n$ given by the moment map $\mu(x, v) = |v|$, $A(x)$ is the antipodal map on $S^n$ and $r(t) \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is a smooth function such that $r(0) = \pi, r(t) = 0$ for $t \geq \epsilon$, and $\frac{dr}{dt}(0) = 0, \forall k \geq 1$. Here $|v|$ is taken under the standard metric.
In this paper, we consider the compositions of symplectic Dehn twists which are supported in small neighborhoods of some Lagrangian spheres that intersect transversely. A model example of the configuration of such Lagrangian spheres is the plumbing space.

**Definition 2.2 (Plumbing).** Given two $n$-spheres $S_1$ and $S_2$ with points $p_1 \in S_1$ and $p_2 \in S_2$, we take local neighborhoods $p_1 \in U_1 \subset S_1$ and $p_2 \in U_2 \subset S_2$, with local coordinates $\psi_1 : U_1 \to \mathbb{R}^n$ and $\psi_2 : U_2 \to \mathbb{R}^n$. They induce symplectomorphisms $T^*\psi_1 : T^*U_1 \to T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ and $T^*\psi_2 : T^*U_2 \to T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. Take a standard linear symplectomorphism $J : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ given by $J(x_i) = dx_i, J(dx_i) = -x_i, i = 1, 2, ..., n$. We have a symplectomorphism $T^*U_1 \to T^*U_2$ given by $\psi = (T^*\psi_2)^{-1}JT^*\psi_1$.

The plumbing space of $S_1$ and $S_2$ is defined as

$$P_\psi(S_1, S_2) = T^*S_1 \cup T^*S_2 = T^*S_1 \cup T^*S_2 / \sim$$

where $p \sim q$ if and only if $p \in U_1, q \in U_2, q = \psi(p)$.

We see by the definition that the symplectic structure of the plumbing space does not depend on the maps $\psi_1$ and $\psi_2$ (in the sense of symplectic diffeomorphisms between two symplectic manifolds).

We can also plumb multiple $n$-spheres together, by plumbing the spheres one by one. More precisely, taking $\psi_i$ to be the local identification plumbing $T^*S_i$ and $T^*S_j$ together, we take the plumbing space

$$P_\psi(S_1, S_2, ..., S_m) = T^*S_1 \cup \psi_1 T^*S_2 \cup \psi_2 ... \cup \psi_{m-1} T^*S_m = \cup_{j=1}^m T^*S_j / \sim,$$

where $x \sim \psi_i(x), \forall i$.

We note that the symplectic structure of $P_\psi(S_1, ..., S_m)$ depends only on the configuration of the plumbing (in other word, it only depends on how the spheres intersect with each other), but not on the points, the symplectomorphism $J$, or the identifications of neighbourhoods of $S_i$ with $\mathbb{R}^n$.

The following example illustrates that transversely intersecting Lagrangian 2-spheres, as given in the plumbing space, occurs naturally on algebraic surfaces.

**Example 2.2.1 ($A_m$-configuration, see Section 8 of [34]).** We consider the symplectic manifold $A_m^n$ given as an affine hypersurface

$$A_m^n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid x_1^2 + x_2^2 + \ldots + x_n^2 = x_{n+1}^2 + \frac{1}{2} \right\} \subset \mathbb{C}^{n+1}$$

in $\mathbb{C}^{n+1}$ equipped with the standard symplectic form $\omega_{\mathbb{C}^{n+1}} = \frac{i}{2\pi} \sum_{j=1}^{n+1} dz_j \wedge d\bar{z}_j$.

For $n = 2$, we briefly describe these Lagrangian spheres. Following Seidel [34] we consider the projection $\pi : A_2^m \to \mathbb{C}^2 : (x, y, z) \to (x, y)$, then $\pi$ is a $(m + 1)$-fold covering branched along $C = \{ x^2 + y^2 = \frac{1}{2} \}$ and the covering group is generated by $\sigma : (x, y, z) \to (x, y, e^{\frac{2\pi i}{m+1}} z)$.

Take the figure-8 map

$$f : \mathbb{R}^3 \supset S^2 \to \mathbb{C}^2 \setminus C : (t_1, t_2, t_3) \mapsto (t_2(1 + it_1), t_3(1 + it_2)),$$
then the map is an immersion with one double point 0 at which the north and south pole of $S^2$ meets transversely. Lifting the immersion to $A_m^2 \subset \mathbb{C}^3$, the map becomes an embedding $\tilde{f}$ with $\tilde{f}(-1,0,0) = \sigma(1,0,0)$. Then we see that

$$S_1 = \text{Im}(\tilde{f}), \quad S_2 = \sigma(S_1), \ldots, \quad S_m = \sigma^{m-1}S_1$$

are a family of smoothly embedded 2-spheres whose only intersections are $S_i \cap S_{i+1}, 1 \leq i \leq m-1$. We can further show that these embedded 2-spheres can be perturbed into transversely intersecting Lagrangian spheres $L_i$ that such that

$$|L_i \cap L_j| = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases}.$$

For $A_m^n$ with dimension $n \geq 3$ similar construction also gives us $m$ Lagrangian $n$-spheres with the same relation.

The symplectic manifold $A_m^n$ is symplectomorphic to the plumbing space of $m$ $n$-spheres at antipodal points, see [2].

3. Topological entropy of composite Dehn twists

In this section, we give the proofs of Theorems 1.2 and 1.3 on positive topological entropy of composite Dehn twists. The key observation is that we can find an invariant subset of dimension 2 on which the system is reduced to the usual Dehn twists on surfaces. We refer readers to Appendix A for the definition of topological entropy.

First, it is not hard to show that the topological entropy of a single Dehn twist is always zero, which we leave as an exercise.

**Proposition 3.1.** The Dehn twist $\tau$ on the symplectic manifold $T^*\mathbb{S}^n$ has zero topological entropy, i.e. $h_{\text{top}}(\tau) = 0$.

In the $A_m^n$ configuration $A_m^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid x_1^2 + x_2^2 + \ldots + x_n^2 = x_{n+1}^2 + \frac{1}{2}\} \subset \mathbb{C}^{n+1}$, we can define a map

$$\iota : A_m^n \to A_m^n : (x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, -x_2, \ldots, -x_n, x_{n+1}).$$

(3.1)

We see that $\iota$ is an involution on $A_m^n$ preserving the symplectic form. We denote by $X$ the set of fixed point of $A_m^n$ under the involution $\iota$. Then we see that the fixed set $X = \{(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid x_1^2 = x_{n+1}^2 + \frac{1}{2}, x_2 = \ldots = x_n = 0\} \subset \mathbb{C}^2 \subset \mathbb{C}^{n+1}$ is simply the real 2-dimensional $A_m^1$ configuration. By the construction of the Lagrangian spheres $L_i \subset A_m^n$, they restrict to Lagrangian circles $\gamma_i \subset X$.

It is shown in (6.3) that we may choose the Dehn twists on $A_m^n$ to be commutative with $\iota$ such that the restriction of a single symplectic Dehn twist $\tau$ on $X$ is exactly some Dehn twist of $\gamma_i$ on $A_m^1$.

Thus, in the case $m = 2$, the symplectic Dehn twists restricted to $X$ is simply the Dehn twists on a torus.

**Lemma 3.2.** $X$ is diffeomorphic to the punctured torus $\mathbb{T}^2 \setminus \{\ast\}$ and the restriction of $\tau_1$ and $\tau_2$ on $X$ is conjugate to the maps $T_1$ and $T_2$ on $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ of the form

$$T_1(x, y) = (x + r(y), y); \quad T_2(x, y) = (x, y - r(x))$$

where $r(t)$ is the smooth function $r \in C^\infty(\mathbb{R}, \mathbb{R})$ in Definition 2.1.
With this lemma, we prove the positive topological entropy of the composite Dehn twists using Yomdin’s inequality (Theorem A.2).

**Proof of Theorem 1.2.** Let $\mathcal{T} = T_1^k T_2^\ell$ on $\mathbb{T}^2$, then under appropriate basis in $H^1(X, \mathbb{R})$, the induced linear map $\mathcal{T}^* : H^1(X, \mathbb{R}) \to H^1(X, \mathbb{R})$ is exactly $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^\ell$. Thus $\mathcal{T}^*$ has spectral radius $\rho(\mathcal{T}^*) > 1$ if $k \ell \neq 0, 1, 2, 3, 4$.

Thus by Yomdin’s inequality (Theorem A.2), we have $h_{\text{top}}(\mathcal{T}) \geq \log \rho(\mathcal{T}^*) > 0$. Combined with Lemma 3.2, we have

$$h_{\text{top}}(\tau_1^k \tau_2^\ell, M) \geq h_{\text{top}}(\tau_1^k \tau_2^\ell, X) = h_{\text{top}}(\mathcal{T}, \mathbb{T}^2 \setminus \{\ast\}) = h_{\text{top}}(\mathcal{T}, \mathbb{T}^2) > 0.$$ 

□

For $m > 2$, the manifold $A_m^1$ is a surface $S_{m/2}$ of genus $[m/2]$ with one or two punctured points. If we view the surface $S_{m/2}$ as a translation surface on $\mathbb{R}^2$ with parallel sides identified, the Lagrangians $\gamma_i$ are the curves presented as the median lines in the $\delta$-neighborhoods $Y_\delta^i$, $i = 1, 2$, see Figure 1.

**Lemma 3.3.** The manifold $X \subset A_m^n$ fixed by the involution $\iota$ is diffeomorphic to the punctured translation surface $S_{[m+1]/2} \{\ast\}$ which we view as a subset of $\mathbb{R}^2$ with sides identified, and the Dehn twists $\tau_i$ on $X$ are conjugate to Dehn twists $T_i$ on $S_{m+1}/2$ of the form $T_i(x, y) = (x + r(y - i\pi), y)$ for $i$ odd and $T_j(x, y) = (x, y - r(x - (j + 1)\pi))$ for $j$ even.

Given this lemma, by Penner’s construction of pseudo-Anosov diffeomorphisms (Theorem 6.8) and the exponential growth of spectral radius of a pseudo-Anosov diffeomorphism, we see that $h_{\text{top}}(\mathcal{T}, S) \geq \log \rho(\mathcal{T}^*) > 0$. So we have

$$h_{\text{top}}(\tau, A_m^n) \geq h_{\text{top}}(\tau, X) = h_{\text{top}}(\mathcal{T}, S_{m+1}/2 \{\ast\}) = h_{\text{top}}(\mathcal{T}, S_{m+1}/2) > 0$$

which proves Theorem 1.3.

We shall also present a different proof in Section 4.1 using the hyperbolicity of the subsystem $\mathcal{T}$ on $S_{[m+1]/2}$.

### 4. Hyperbolicity of the subsystem

In this section, we shall prove the hyperbolicity of the composition $T_1^{k_1} \ldots T_m^{k_m}$, which by Pesin theory gives not only positive topological entropy, but also stable and unstable manifolds. We can apply this result for Theorem 1.4 on the existence of stable/unstable laminations of Dehn twists and for an alternative proof of Theorems 1.2 and 1.3. For a preliminary on Pesin theory, we refer readers to Appendix A. In the following we set $\mathcal{T} = T_1^{k_1} \ldots T_m^{k_m}$.

#### 4.1. Statement and Proof of Hyperbolicity

**Theorem 4.1.** Let $Y$ be the punctured surface and let $T_i$ be the Dehn twists as in Lemma 3.3, then for $\mathcal{T} = T_1^{k_1} \ldots T_m^{k_m}$, $k_i k_{i+1} < 0$, the Lyapunov exponent $\chi^+(x)$ (see Definition A.3) of $\mathcal{T}$ is positive for almost every $x \in Y$. 
Remark 4.2. The subsystem for \( m = 2 \), i.e., \((\tau_1^{k_1}\tau_2^{k_2}, X_\epsilon) \simeq (T_1^{k_1}T_2^{k_2}, \mathbb{T}^2)\), is similar to the linked twist map on the torus studied in literature [9]. However, there is a difference between them. Indeed, the linked twist map considered there is defined by compositing \( T_1, T_2 \) with \( r'(t) > 0, -\epsilon < t < \epsilon \) for all \( \epsilon > 0 \). However, in our case, we have \( r'(0) = 0 \) and \( r'(t) > 0, \forall \ 0 < |t| < \epsilon \). When \( k\ell < 0 \), the corresponding map is called co-rotating linked twist map in [9] and \( k\ell > 0 \) counter-rotating. Technically, the latter case is much harder, so we focus only on the co-rotating case in this paper.

In what follows we shall extend the ideas to \( m > 2 \) and study the multi-linked twist map \((Y, T)\).

For \( 0 \leq \delta < \epsilon < 1 \), we take
\[
Y_1^\delta = \{(x, y) \in Y \mid \exists j \in \mathbb{N}, \ s.t. \ \delta < |x - j\pi| < \epsilon - \delta; \ \delta < |y - (j - 1)\pi| < \epsilon - \delta \}
\]
to be the intersections of \( \delta \)-interior of \( \bigcup_{i=1}^{m+1} A_i \cap B_i \) minus the \( \delta \)-neighborhood of the median lines. We also take
\[
Y_2^\delta = \{(x, y) \in Y \mid \exists j \in \mathbb{N}, \ s.t. \ \delta < |x - j\pi| < \epsilon - \delta; \ \delta < |y - j\pi| < \epsilon - \delta \}
\]
to be the intersections of \( \delta \)-interior of \( \bigcup_{i=1}^{m+1} B_i \cap A_{i+1} \) minus the \( \delta \)-neighborhood of the median lines as shown in Figure 1.

![Figure 1. Y_1^\delta and Y_2^\delta in Y](image)

We denote the set of points returning sufficiently often to \( Y_1^\delta \) by
\[
A_\delta = \{p \in Y \mid \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} 1_{Y_1^\delta}(T^k p) \geq \delta \},
\]
and the set of points returning sufficiently often to \( Y_2^\delta \) by
\[
B_\delta = \{p \in Y \mid \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} 1_{Y_2^\delta}(T^k p) \geq \delta \}.
\]

We first show that \( T \) is hyperbolic on \( A_\delta \) and \( B_\delta \).
Lemma 4.3. Given \( \delta > 0 \), there exists \( c_\delta \) such that for any \( p \in A_\delta \cup B_\delta \), the Lyapunov exponent \( \chi^+(p) \geq c_\delta \).

Lemma 4.4. Let \( \mu \) be the Lebesgue probability measure on \( Y \), we have \( \mu(\cup_{\delta>0}(A_\delta \cup B_\delta)) = 1 \).

We postpone the proofs to Section 4.2.

Theorem 4.1 comes as a direct corollary of the above lemmata. Furthermore, these lemmata give an alternative proof of Theorem 1.3.

Alternative proof of Theorem 1.3. By Lemma 4.4 we have \( \mu(\cup_{\delta>0}(A_\delta \cup B_\delta)) = 1 \), thus there exists \( \delta_0 > 0 \) for which \( \mu(A_{\delta_0} \cup B_{\delta_0}) > 0 \). By Lemma 4.3, there exists \( c > 0 \) such that \( \chi^+(x) > c \) for any \( x \in A_{\delta_0} \cup B_{\delta_0} \). Thus by Pesin entropy formula we have \( h_\mu(\mathcal{T}) \geq c \mu(A_{\delta_0} \cup B_{\delta_0}) > 0 \), so by the variational principle we have

\[
h_{\text{top}}(\tau) \geq h_{\text{top}}(\tau | X) = h_{\text{top}}(\mathcal{T}) \geq h_\mu(\mathcal{T}) > 0.
\]

4.2. Proofs of Lemmata 4.3–4.4. In this section, we give different proofs of Lemmata 4.3–4.4 based on the above hyperbolicity result.

Proof of Lemma 4.3. In this proof we take \( \prod \) to be the ordered product which is read from right to left, and we denote by \( 1_N \) the indicator function that takes value 1 on \( N \) and 0 elsewhere. For any \( n \in \mathbb{N} \) and any \( p \in A_\delta \cup B_\delta \), we have

\[
D\mathcal{T}^n(p) = \prod_{i=1}^{n} \left( \begin{array}{c} 1 \ t_i^1 \\ 0 \ 1 \end{array} \right) \left( \begin{array}{c} 1 \ 0 \\ t_i^j \ 1 \end{array} \right) \ldots \left( \begin{array}{c} 1 \ 0 \\ t_i^{m} \ 1 \end{array} \right) 1_N((m+1)/2) \cdot t_i^m
\]

where

\[
t_i^j = \begin{cases} k_j r(y(\prod_{k=1}^{j+1} T_{i+1}^k \mathcal{T}^{i-1} p) - j\pi) & \text{if } j \text{ odd} \\ k_j r(x(\prod_{k=1}^{j+1} T_{i+1}^k \mathcal{T}^{i-1} p) - (j+1)\pi) & \text{if } j \text{ even} \end{cases}
\]

Thus we have \( t_i^j \geq 0 \) for any \( 1 \leq j \leq m, 1 \leq i \leq n \).

For \( p \in A_\delta \cup B_\delta \), assume without loss of generality that \( p \in A_\delta \), then for any \( n \) sufficiently large, we have \#\{\{\mathcal{T}^i(p) \mid 0 \leq i \leq n\} \cap A_\delta\} \geq \delta n/2 \). We number the iterations when \( p \) hits \( A_\delta \) by \( \{n_k\}_{k \in \mathbb{N}} = \{n \in \mathbb{N} \mid \mathcal{T}^n(p) \in A_\delta\} \) where \( n_i < n_{i+1} \). Then for any \( k \in \mathbb{N} \), \( \mathcal{T}^{n_k}(p) \) and \( \mathcal{T}^{n_{k+1}}(p) \) are both in \( A_\delta \). Suppose \( \mathcal{T}^{n_k}(p) \in A_{j(n_k)} \cap B_{j(n_k)} \), \( \mathcal{T}^{n_{k+1}}(p) \in A_{j(n_{k+1})} \cap B_{j(n_{k+1})} \) then we claim that

\[
t_{2j(n_k)}^{n_k} > c_1 \text{ and } t_{2j(n_{k+1})}^{n_{k+1}-1} > c_1,
\]

where the constant \( c_1 > 0 \) is taken so that \( r'(t) > c_1, \forall \delta \leq |t| \leq \epsilon - \delta \) and \( c_1 \) depends only on \( r \) and \( \delta \).

To prove the claim we just note that if \( q = \mathcal{T}^{n_k}(p) \) starts from \( A_\delta \cap A_{j(n_k)} \cap B_{j(n_k)} \), \( T_j \) acts by identity on \( q \) for \( j > 2j(n_k) \) and so \( t_{2j(n_k)}^{n_k} > k_{2j(n_k)} c_1 \). Similarly, whenever \( q = \mathcal{T}^{n_{k+1}}(p) \) lands in \( A_\delta \cap A_{j(n_{k+1})} \cap B_{j(n_{k+1})} \), we see that \( T_{j-1}^{-1}(q) = q \) for \( j < 2j(n_{k+1}) - 1 \) which ensures that \( t_{2j(n_{k+1})}^{n_{k+1}-1} > k_{2j(n_{k+1})-1} c_1 \).

Thus we have that for \( p \in A_\delta \), it holds that \#\{1 \leq i \leq n \mid t_i^j > c_1 \text{ for some } j\} \geq \delta n/2 \) for \( n \) sufficiently large. The case \( p \in B_\delta \) can be proved in the same way.
We next introduce a partial ordering $\leq$ on $\mathbb{R}^2$ given by $u = (u_1, u_2) \leq v = (v_1, v_2)$ if $u_1 \leq v_1, u_2 \leq v_2$. We note that $T_j^{k_i} u \leq T_j^{k_j} v$ if $u \leq v$ and $T_j^{k_i} u \geq u$ if $u \geq 0$. Also, for $u \geq 0, t_j^i > c_1$, we have

$$
\left( \begin{array}{c}
1 \\
1_N(j/2) \cdot t_j^i
\end{array} \right) u \geq \left( \begin{array}{c}
1 \\
1_N((j + 1)/2) \cdot t_j^i
\end{array} \right) u.
$$

Thus by our claim above, for $u$ in the first quadrant, we have

$$DT^n(u)[u] \geq \left( \begin{array}{c}
1 \\
1_N(j/2)
\end{array} \right)^{[\delta n/2]} u = \left( \begin{array}{c}
1 \\
1_N((j + 1)/2)
\end{array} \right)^{[\delta n/2]} u.
$$

This implies that

$$\|DT^n(u)[u]\| \geq \left\| \left( \begin{array}{c}
1 \\
1_N(1 + c_1^2)
\end{array} \right)^{[\delta n/2]} u \right\| \geq c_2 \lambda^{[\delta n/2]} \|u\|
$$

for a constant $c_2$ depending on $u$ and $c_1$, where $\lambda$ is the larger eigenvalue of $\left( \begin{array}{cc}
1 & c_1 \\
1 & 1 + c_1^2
\end{array} \right)$. Thus we have $\chi^+(p, u) \geq \lambda^{\delta/2} = c_3$ for any $u \geq 0$. So we get our desired result $\chi^+(p) \geq c_3 > 0$. □

The following lemma by Burton and Easton shows that almost every point that hits a subset of $Y$ must hit the subset sufficiently often (which can be viewed as a generalization of Poincaré recurrence theorem).

**Lemma 4.5** (Burton and Easton, [9]). Let $(F, X, \mu)$ be a measure preserving dynamical system, and $V$ be a subset of $X$. Suppose

$$P(V) = \{ p \in X \mid \exists N \in \mathbb{N}, \text{ s.t. } F^N(p) \in V \}$$

is the set of points that hits $V$,

$$Z(V) = \{ p \in X \mid \liminf_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} 1_V(F^N(p)) = 0 \}$$

is the set of points that do not hit $V$ at a positive average rate, then we have $\mu(P(V) \cap Z(V)) = 0$.

**Proof.** By definition $Z(V)$ is an $F$-invariant set of $X$. Thus we have

$$\int_{Z(V)} 1_V(x) d\mu(x) = \int_{Z(V)} 1_V(F^N(x)) d\mu(x) = \frac{1}{N} \sum_{k=0}^{N} 1_V(F^N(x)) d\mu(x) = 0.$$

So we get $\mu(V \cap Z(V)) = 0$. Furthermore, we have $\mathcal{T}^{-k}(V \cap Z(V)) = \mathcal{T}^{-k}(V) \cap Z(V)$ for any $k \in \mathbb{N}$. This implies that $\mu(P(V) \cap Z(V)) = \mu(\cup_{k \in \mathbb{N}}(\mathcal{T}^{-k}(V) \cap Z(V))) = 0$. □

We next show that indeed almost very point in $Y$ hits $Y_0^1 \cup Y_0^2$ eventually.
Lemma 4.6. For Lebesgue almost every point $p$ in $Y$, there exists $N \in \mathbb{N}$ such that $T^N(p) \in Y_0^1 \cup Y_0^2$.

Proof. We notice that

$$Y_0^1 \cup Y_0^2 = \bigcup_{i \neq j} (Y_i \cap Y_j) - \{(x, y) \mid x \in \mathbb{N} \text{ or } y \in \mathbb{N}\}.$$ 

First we note that the set of points ever hitting $\{(x, y) \mid x \in \mathbb{N} \text{ or } y \in \mathbb{N}\}$ has zero Lebesgue measure since $\operatorname{vol}(\{(x, y) \mid x \in \mathbb{N} \text{ or } y \in \mathbb{N}\}) = 0$. Thus we only need to consider points that never hits $\bigcup_{i \neq j} (Y_i \cap Y_j)$.

If $T^n(p) \notin \bigcup_{i \neq j} (Y_i \cap Y_j)$, $\forall n \in \mathbb{N}$, then suppose $p \in Y_i$, then any iteration $T^n(p)$ is in $Y_i \setminus \bigcup_{j \neq i} Y_j$. $T$ restricted to $Y_i \setminus \bigcup_{j \neq i} Y_j$ is simply $T_{k_i}$ which is a single Dehn twist. Thus for $T^n(p) \in Y_i \setminus \bigcup_{j \neq i} Y_j$, $\forall n$, the point $p$ has to be periodic with $|T(p) - p| > \epsilon$. Thus we see that such points must lie on finitely many 1-dimensional segments, so they make up a measure zero subset of $Y$. \hfill $\square$

Now we finish the proof the the lemmas.

Proof of Lemma 4.4. We only need to prove that $\mu(\bigcup_{n \in \mathbb{N}} (A_{\frac{1}{n}} \cup B_{\frac{1}{n}})) = 1$.

We follow the notations in Lemma 4.6. By Lemma 4.6, since $\bigcup_{\delta > 0} Y_0^i = Y_0^i$, $i = 1, 2$, for almost every $p \in Y$, there exists $n > 0$, $N \in \mathbb{N}$ such that $T^N(p) \in Y_0^1 \cup Y_0^2$, i.e.

$$\mu\left(\bigcup_{n \in \mathbb{N}} (P(Y^1_{\frac{1}{n}}) \cup P(Y^2_{\frac{1}{n}}))\right) = 1.$$

By Lemma 4.5, we have $\mu(P(Y^i_{\frac{1}{n}}) \cap Z(Y^i_{\frac{1}{n}})) = 0, \forall \delta$ which implies

$$\mu\left(\bigcup_{m \in \mathbb{N}} \left(P(Y^i_{\frac{1}{n}}) \cap \left\{p \in Y \mid \liminf_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} 1_{Y^1_{\frac{1}{n}}} (F^k(p)) > \frac{1}{m}\right\}\right)\right) = \mu(P(Y^i_{\frac{1}{n}})),$$

for $i = 1, 2$. Now since $Y^i_{\frac{1}{n}} \subset Y^i_0$, for $n < m$, the last equation implies that

$$\mu(P(Y^1_{\frac{1}{n}}) \cap (\bigcup_{m \in \mathbb{N}} A_{\frac{1}{m}})) = \mu(P(Y^1_{\frac{1}{n}})), \quad \mu(P(Y^2_{\frac{1}{n}}) \cap (\bigcup_{m \in \mathbb{N}} B_{\frac{1}{m}})) = \mu(P(Y^2_{\frac{1}{n}}))$$

for any $n \in \mathbb{N}$. Combining this with $P(Y^i_{\frac{1}{n}}) \subset P(Y^i_{\frac{1}{m}})$ for any $n < m$ and $\mu(\bigcup_{n \in \mathbb{N}} (P(Y^1_{\frac{1}{n}}) \cup P(Y^2_{\frac{1}{n}}))) = 1$, we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} (A_{\frac{1}{n}} \cup B_{\frac{1}{n}})\right) = \mu\left(\bigcup_{n \in \mathbb{N}} (P(Y^1_{\frac{1}{n}}) \cup P(Y^2_{\frac{1}{n}}))\right) \geq \mu\left(\bigcup_{n \in \mathbb{N}} (P(Y^1_{\frac{1}{n}}) \cap (\bigcup_{m \in \mathbb{N}} A_{\frac{1}{m}})) \cup (\bigcup_{n \in \mathbb{N}} (P(Y^2_{\frac{1}{n}}) \cap (\bigcup_{m \in \mathbb{N}} B_{\frac{1}{m}}))\right) = \mu\left(\bigcup_{n \in \mathbb{N}} (P(Y^1_{\frac{1}{n}}) \cup P(Y^2_{\frac{1}{n}}))\right) = 1.$$

This completes the proof. \hfill $\square$
5. Rotation map and the local stable/unstable Lagrangian manifolds

As a corollary of the non-uniform hyperbolicity of composite Dehn twists on surfaces established in Lemma 4.3 and Lemma 4.4, in this section, we also prove the existence of local stable/unstable Lagrangian manifolds using Pesin’s stable manifold theorem and a rotation construction. We start with the definition of rotation map that brings curves from $X = \mathbb{A}_m^n$ back to Lagrangian submanifolds in $\mathbb{A}_m^n$. In this section, we use the standard identification of $\mathbb{A}_m^n$ with the plumbing space of $m$ cotangent spaces of $n$-dimensional spheres $P_\psi(S_1, S_2, \ldots, S_n)$ and perform the rotation map on the plumbing space.

5.1. Rotation map on the standard sphere. First we define the rotation map on the standard sphere $\mathbb{S}^n$.

**Definition 5.1.** Let $\gamma$ be the great circle $\{(y_1, 0, \ldots, 0, y_{n+1})\}$ on $\mathbb{S}^n$ passing through the two points $n = (0, \ldots, 0, 1)$ and $-n = (0, \ldots, 0, -1)$ called north pole and south pole respectively. Let $\mathbb{S}^{n-1} = \{ (\theta, 0) \in \mathbb{R}^n \times \mathbb{R} \mid |\theta|^2 = 1 \}$ be the equator of $\mathbb{S}^n$. We suppose that $\gamma$ has angle coordinate $\theta = (1, 0, \ldots, 0) \in \mathbb{S}^{n-1}$ and $\gamma_\theta$ is a great circle with angle $\theta \in \mathbb{S}^{n-1}$. We associate to $\theta$ the rotation map $R_\theta$, an element in $\text{SO}(n)$ sending $T^*\gamma$ to $T^*\gamma_\theta$ isometrically. More precisely, let $x = (\theta, 0)$ with $\theta \in \mathbb{S}^{n-1}$ be a point on the equator. We define

$$R_\theta : T^*\gamma = \{ (y_1, 0, \ldots, 0, y_{n+1}, v_1, 0, \ldots, 0, v_{n+1}) \in T^*\mathbb{S}^n \} \rightarrow T^*\mathbb{S}^n$$

$$(y_1, 0, \ldots, 0, y_{n+1}, v_1, 0, \ldots, 0, v_{n+1}) \mapsto (y_{n+1} \cdot n + y_1 \cdot x, v_{n+1} \cdot n + v_1 \cdot x).$$

Hereafter we call $R_\theta$ the rotation map.

Note that for all points $\theta \in \mathbb{S}^{n-1}$ we have $R_\theta(\pm n, 0) = (\pm n, 0)$, and if $p \notin \{ (\pm n, 0) \}$ in $T^*\gamma$ the union of images $\cup_{\theta \in \mathbb{S}^{n-1}} R_\theta(p)$ is an embedded $(n-1)$-sphere in $T^*\mathbb{S}^n$.

**Lemma 5.1.** Let $c : [0, 1] \rightarrow T^*\gamma$ be an embedded curve. Then the set

$$L_c = \bigcup_{\theta \in \mathbb{S}^{n-1}} R_\theta(c([0, 1]))$$

is an immersed Lagrangian submanifold (possibly with boundary) of $T^*\mathbb{S}^n$ which possibly has non-smooth points at $(\pm n, 0)$.

**Proof.** Let $c(t) = (y_1(t), 0, \ldots, 0, y_n(t), v_1(t), 0, \ldots, 0, v_n(t))$. By definition we have

$$L_c = \{ (y_1(t)\theta, y_n(t)); v_1(t)\theta, v_n(t)) \mid t \in [0, 1], \theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \}.$$ 

Thus it is a smoothly immersed submanifold outside the south and north pole.

To finish the proof we notice that $\omega|_{L_c} = 0$ if and only if $\sum_{i=1}^{n+1} dy_i \wedge dv_i|_{L_c} = 0$. To verify the latter, we compute

$$\sum_i d(y_1(t)\theta_i) \wedge d(v_1(t)\theta_i) + dy_{n+1}(t) \wedge dv_{n+1}(t)$$

$$= (y_1(t)v_1'(t) - v_1(t)y_1'(t)) \sum_i \theta_i d\theta_i = \frac{(y_1(t)v_1'(t) - v_1(t)y_1'(t))}{2} d\left( \sum_i \theta_i^2 \right) = 0,$$
where in the last inequality we have used $\sum \theta_i^2 = 1$. Thus we see that $L_c$ is locally Lagrangian. \hfill \Box

By the definition of the symplectic Dehn twist $\tau$ on $T^*\mathbb{S}^n$, we see that it is commutative with the rotation map.

**Lemma 5.2.** For the symplectic Dehn twists $\tau$ on $T^*\mathbb{S}^n$ we have
\begin{equation}
\mathcal{R}_\theta \circ \tau|_{T^*\gamma} = \tau \circ \mathcal{R}_\theta|_{T^*\gamma},
\end{equation}
and
\begin{equation}
\tau(L_c) = L_{\tau(c)}
\end{equation}
for any smooth curve $c : [0,1] \to T^*\gamma$.

### 5.2. Rotation map on the plumbing space.

Let $\gamma_i \subset S_i$, $i = 1, \ldots, m$ be a geodesic circle which contains a plumbing point and its antipodal point (denoted by $p_i$ and $-p_i$) of $S_i$ as given in Section 2. Then for each $i = 1, \ldots, m-1$, the plumbing map $\psi_i$ sends an open subset of $p_i$ in $T^*\gamma_i$ onto that of $-p_{i+1}$ in $T^*\gamma_{i+1}$. In other words, the restriction of $\psi_i$ to $T^*\gamma_i$ gives rise to the plumbing map from $T^*\gamma_i$ to $T^*\gamma_{i+1}$, from which we get a plumbing space denoted by $P_{\psi}(\gamma_1, \gamma_2, \ldots, \gamma_m)$.

We identify each $S_i$ with $\mathbb{S}^n$ through a map $\phi_i : S_i \to \mathbb{S}^n$ such that the image of the great circle $\gamma$ is exactly $\gamma_i$. Then for each $\theta \in \mathbb{S}^{n-1}$, the rotation maps defined above give rise to a symplectic map $\mathcal{R}_\theta^i : T^*\gamma_i \to T^*S_i$ by
\begin{equation}
\mathcal{R}_\theta^i = (T^*\phi_i)^{-1} \circ \mathcal{R}_\theta \circ T^*\phi_i|_{T^*\gamma_i},
\end{equation}
which we call the rotation map on $S_i$.

To extend the Lagrangian submanifolds $L_i$ from a single sphere to the full space $P_{\psi}(S_1, S_2, \ldots, S_m)$, we need to know the effect of the rotation maps $\mathcal{R}_\theta^i$ under the local identification maps $\psi_i : T^*U_i^+ \to T^*S_{i+1}$, where the open subsets $U_i^\pm \subset S_i$ are the preimages of the open subsets
\begin{equation}
U_{\pm} = \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid |x|^2 = 1, \pm x_{n+1} > 0\} \subset \mathbb{S}^n
\end{equation}
der under the map $\phi_i : S_i \to \mathbb{S}^n$.

**Lemma 5.3.** We may choose local identification maps $\psi_i$ of $P_{\psi}(S_1, \ldots, S_n)$ so that for any $\theta \in \mathbb{S}^{n-1}$ and any geodesic circle $\gamma$ of $S_i$ containing the plumbing point $p$ of $S_i$ with $S_{i+1}$, we have
\begin{equation}
\mathcal{R}_\theta^{i+1} \circ \psi_i = \psi_i \circ \mathcal{R}_\theta^i \quad \text{on } T^*\gamma \cap U_i^+,
\end{equation}
where $U_i^+$ is the plumbing region of $p_i$ in $T^*S_i$.

**Proof.** We take the plumbing map $\psi_i$ to be given by $(d\varphi_{i+1}^-)^{-1}Jd\varphi_i^+$ where $\varphi_{i+1}^\pm = \phi_{i+1}
|_{U_i^\pm} \varphi_{j} : U_j^\pm \to \mathbb{R}^n$ with
\begin{equation}
\varphi_{j} : (x_1, \ldots, x_{n+1}) \mapsto \frac{\arccos(\pm x_{n+1})}{\sqrt{1 - x_{n+1}^2}}(x_1, \ldots, x_{n}).
\end{equation}
Then by the definition of $\mathcal{R}_\theta$ we calculate that the conjugation of $\mathcal{R}_\theta^j$ via $d\varphi^j_\pm$ acting on $(d\varphi^j_\pm)^{-1}(T^*\gamma_j) \subset T^*\mathbb{R}^n$ is simply

$$d\varphi^j_\pm \mathcal{R}_\theta^j (d\varphi^j_\pm)^{-1} = d\varphi_\pm \mathcal{R}_\theta d\varphi_\pm^{-1}: (t, 0, \ldots, 0; s, 0, \ldots, 0) \mapsto (t\theta, s\theta) \in T^*\mathbb{R}^n$$

for any $j$.

Therefore, the rotation map rotates both the base coordinates $x = (t, 0, \ldots, 0)$ and the fiber coordinate $v = (s, 0, \ldots, 0)$ to the direction of $\theta$ while the plumbing map from $S_i$ to $S_{i+1}$ switches the roles of base and fiber via the complex map $J$. Thus we see that they are commutative.

Since we have $\mathcal{R}_{\{1,0,\ldots,0\}}(T^*\gamma) = T^*\gamma$ and $\mathcal{R}_{\{1,0,\ldots,0\}}^2 = Id$, i.e. $\mathcal{R}_{\{1,0,\ldots,0\}}$ acts as a reflection on $T^*\gamma$, by Lemma 5.3 one can define a reflection map

$$\mathcal{R}: P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m) \longrightarrow P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m)$$

$$(5.5) \mathcal{R}(z) = \mathcal{R}_{\{1,0,\ldots,0\}}(z) \quad \text{if } z \in T^*\gamma_i.$$ 

**Definition 5.2.** We call a smooth embedded curve $c \subset P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m)$ admissible if $\mathcal{R}(c) = c$. The set of admissible curves in $P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m)$ is denoted by $\mathcal{C}_{ad}$.

**Lemma 5.4.** Let $c$ be a smooth embedded admissible curve satisfying

$$c([0, 1]) \in P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

We denote by $L_c = \bigcup_{i=1}^m \bigcup_{\theta \in S^n} \mathcal{R}_\theta(c)$ the image of $c$ under rotations, then $L_c$ is a smoothly embedded Lagrangian submanifold in $P_\psi(S_1, \ldots, S_m)$.

**Proof.** By Lemma 5.1 and Lemma 5.3, $L_c$ is an immersed Lagrangian submanifold. Furthermore, the condition that $c$ is admissible ensures that $L_c$ does not have any self-intersection. Therefore, it is a smoothly embedded Lagrangian submanifold.

**Definition 5.3.** Let $c \in \mathcal{C}_{ad}$ be an admissible curve. We call $c$ the generating curve of the Lagrangian submanifold $L_c$. We also say that $L_c$ is generated by $c$.

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** The group $SO(n)$ acts on the plumbing space via its standard action on the sphere $S^{n-1}$, more precisely, it is given by $SO(n) \times P_\psi(S_1, \ldots, S_n) \to P_\psi(S_1, \ldots, S_n) : (g, p) \mapsto \mathcal{R}_g(1,0,\ldots,0)p$. We take $N$ to be the $\epsilon$–neighbourhood of $\cup S_i$ in the plumbing space.

By Theorem 4.1, we see from Pesin’s stable manifold theorem (see Theorem A.5) that $X_t$ admits stable and unstable curves $W^s(p)$ and $W^u(p)$ for almost every $p \in X_t$. Let $\Lambda$ be the set of points $p \in X_t$ that is hyperbolic under $\tau$. Since the stable curves are given by the Dehn twists which is commutative with the rotation map $\mathcal{R}_\theta$, we see that $\Lambda$ is invariant under the reflection map $\mathcal{R}$ and the family of stable curves $W^s(p)$ is admissible under $\mathcal{R}$, i.e. $\{W^s(p) \mid p \in \Lambda\} = \{\mathcal{R}(W^s(p)) \mid p \in \Lambda\}$. Thus by Lemma 5.4 we define a family of invariant Lagrangian submanifolds $\mathcal{F}^s = \{L_{W^s(p)} \mid p \in \Lambda\}$. Similarly, the family of unstable curves $W^u(p)$ in $X_t$ given by the inverse map also gives us a family of invariant Lagrangian submanifolds $\mathcal{F}^u$. \qed
6. Growth of Floer cohomology groups

In this section, we explore the symplectic aspect of the composite Dehn twists. As we have explained in the introduction, we are interested in the growth of Floer cohomology group. For a brief introduction to Lagrangian Floer theory, we refer readers to Appendix B.

In order to estimate the growth of Lagrangian Floer cohomology groups, we shall employ a result due to Khovanov and Seidel [26].

Let \((M, \omega)\) be an exact symplectic manifold with \(\omega = d\lambda\) which is equal to a symplectization near infinity. We assume in addition that \(M\) admits an involution, i.e., \(\iota : M \to M\) with \(\iota \circ \iota = \text{Id}\) and \(\iota^* \omega = \omega\) and \(\iota^* \lambda = \lambda\). Clearly, the fixed point set \(M^\iota\) is a symplectic submanifold of \(M\). Moreover, when \(L\) is a Lagrangian submanifold of \(M\), its fixed part \(L^\iota = L \cap M^\iota\) is again a Lagrangian submanifold of \(M^\iota\).

**Theorem 6.1** (Khovanov and Seidel [26, Proposition 5.15]). Let \((S_1, S_2)\) be a pair of closed \(\lambda\)-exact Lagrangian submanifolds of \((M, \omega)\) with \(\iota(S_i) = S_i, i = 1, 2\). Suppose that

\[
\begin{align*}
\text{(C1)} \quad & \text{the intersection } C = S_1 \cap S_2 \text{ is clean}, \\
\text{(C2)} \quad & \text{there is no continuous map } u : [0, 1] \times [0, 1] \to M^\iota \text{ such that } u(0, t) = x \text{ and } u(1, t) = y \text{ for all } t, \text{ and } u(s, 0) \in S_1 \text{ and } u(s, 1) \in S_2 \text{ for all } s, \text{ where } x \text{ and } y \text{ are two different points of } S_1 \cap S_2.
\end{align*}
\]

Then \(\text{rank } \text{HF}(S_1, S_2) = \text{rank } H^*(C, \mathbb{Z}/2) = |C^\iota|\).

The proof of this theorem is based on equivariant transversality and a symmetry argument, for which we refer readers to [26] for details. It is important to note that the involution plays the role of reducing the dimension of the symplectic manifold and Lagrangian submanifold for the purpose of calculating the rank of Lagrangian Floer cohomology, similar to what we did in the last section when proving positive topological entropy.

Now we consider an explicit involution on the plumbing space \(P_y(S_1, S_2, \ldots, S_m)\) which corresponds to the involution on \(A^n_\mathbb{R}\) given in Equation (3.1). This would be useful to compute the rank of Lagrangian Floer cohomology.

To do this, we first consider a canonical involution on the cotangent bundle of the standard \(n\)-sphere. As before, we write

\[T^*\mathbb{S}^n = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} | |x| = 1, \langle x, v \rangle = 0\}.\]

Set \(x = (x_2, \ldots, x_n)\) and \(v = (v_2, \ldots, v_n)\). We can define a symplectic involution map on \(T^*\mathbb{S}^n\) by

\[\iota_0(x_1, x, x_{n+1}, v, v_{n+1}) = (x_1, -x, x_{n+1}, v_1, -v, v_{n+1}).\]

Then the fixed point set of \(\iota_0\) is precisely the cotangent bundle of a great circle \(\gamma \in \mathbb{S}^n\)

\[(T^*\mathbb{S}^n)^{\iota_0} = T^*\gamma = \{(x_1, 0, x_{n+1}, v_1, 0, v_{n+1}) \in T^*\mathbb{S}^n\}.\]

\(^1\)Here “clean intersection” means that \(C = S_1 \cap S_2\) is a smooth manifold and \(TC = TS_1|_C \cap TS_2|_C\).
Using the involution $\iota_0$ we can now define an involution on $P_\psi(S_1, S_2, \ldots, S_m)$ as follows. For each sphere $S_i$ we pick the geodesic circle $\gamma_i$, which contains every plumbing point of $S_i$ and a symplectomorphism $\psi_i : T^*S_i \to T^*S^n$ such that $\psi_i$ maps the zero section $S_i$ of $T^*S_i$ to $S^n$ of $T^*S^n$, and $\gamma_i$ to $\gamma$. Then we define the involution $\iota_{\gamma_i}$ on $T^*S_i$ by $\iota_{\gamma_i} = (\psi_i)^{-1} \circ \iota_0 \circ \psi_i$.

Furthermore, by the definition of $P_\psi(S_1, S_2, \ldots, S_m)$ we can assume that $\iota_{\gamma_i}(p) = \iota_{\gamma_{i+1}}(p)$ for all points $p \in T^*S_i \cap T^*S_{i+1}$, $i = 1, \ldots, m - 1$. Then the involution $\iota$ on $P_\psi(S_1, S_2, \ldots, S_m)$ is defined by

$$\iota(p) = \iota_{\gamma_i}(p) \quad \text{if} \quad p \in T^*S_i.$$  

By (6.1), for each sphere $S_i$ we have

$$(T^*S_i)^{\iota} = T^*\gamma_i,$$

which implies that the fixed point set of $\iota$ is symplectomorphic to the plumbing space $P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m)$. Also, each zero section $S_i$ of $P_\psi(S_1, S_2, \ldots, S_m)$ is invariant under the map $\iota$. Moreover, a direct calculation shows that $\iota$ and the symplectic Dehn twist $\tau_i$ along $S_i$ have the following relations:

$$\iota \circ \tau_i = \tau_i \circ \iota, \quad i = 1, \ldots, m.$$  

Furthermore, we note that the involution and the rotation map on $S^n$ satisfy

$$\iota \circ R_\theta = R_\theta \circ \iota$$

which gives us the following lemma.

**Lemma 6.2.** Let $\gamma_i$ be circles chosen as above and $c$ be an embedded curve in $T^*\gamma_i$. Then Lagrangian submanifold $L_c$ is invariant under the involution $\iota$, that is,

$$\iota(L_c) = L_c.$$  

6.1. **Proof of Theorem 1.5.** In this section, we give the proof of Theorem 1.5. In order to apply Theorem 6.1, we need to verify the conditions (C1) and (C2). In the following, for any $k, \ell \in \mathbb{Z}$ we denote $\tau = \tau^k_1 \tau^\ell_2$ the composition of symplectic Dehn twists.

**Lemma 6.3.** If $\gamma_1$ and $\tau^n(\gamma_2)$ intersect transversely, then $S_1$ and $\tau^n(S_2)$ satisfy condition (C1) of Theorem 6.1.

**Remark 6.4.** The above lemma may be well-known to symplectic topologists by using the equivariant Lefschetz fibration (ELF). But we do not want to introduce this notion here because in general the condition (C2) in Theorem 6.1 does not hold and we do not know how to use the ELF setting to obtain Lemma 1.7, which makes it possible to use Penner’s result (Theorem 6.8) and Fathi’s result (Theorem 6.9) to give a positive relative symplectic growth rate of the composition of some symplectic Dehn twits in high dimensions. Instead, we shall give a self-contained proof of this lemma in Appendix C.

**The advantage of Lemma C.1 and C.2 is that they allow us to perturb $S_i$ in a smooth exact Lagrangian isotopic way (hence in a Hamiltonian isotopic one) via perturbing the generating curves so that the intersections of the resulting closed curves $\gamma_i^k$ and $\tau^n(\gamma_j)$ become minimal, see Section 8.**
To achieve the condition (C2) in Theorem 6.1, we use the following concept which can be found in the book [15].

**Definition 6.1** (Geometric intersection number). Let \( \alpha \) and \( \beta \) be two free homotopy classes of simple closed curves in a surface \( S \). We call the minimal number of intersection points between a representative curve \( a \) in the class \( \alpha \) and a representative curve \( b \) in the class \( \beta \) the geometric intersection number between \( \alpha \) and \( \beta \), which we denote by \( i(\alpha, \beta) \).

**Lemma 6.5.** If \( \gamma_1 \) and \( \tau^n(\gamma_2) \) are not isotopic, we have the following equality
\[
(6.5) \quad \text{rank} \, \HF(S_1, \tau^n(S_2)) = i([\gamma_1], [\tau^n(\gamma_2)]).
\]

We postpone the proof of the lemma to the next section and complete the proof of Theorem 1.5.

**Proof of Theorem 1.5.** Since symplectic Dehn twists \( \tau_1 \) and \( \tau_2 \) are supported in a compact neighborhood of the zero section of \( P_\psi(S_1, S_2) \), as in Section 3 we identify the action of \( \tau = \tau_2^k \tau_1^\ell \) on an invariant submanifold \( X_\psi \subseteq P_\psi(\gamma_1, \gamma_2) \) with the action of \( T = T_2^k T_1^\ell \) on an open subset \( Y \) of 2-torus \( \mathbb{T}^2 \). And the curves \( \gamma_1 \) and \( \gamma_2 \) are identified with \( \beta_1 = \{(s, 0) \in \mathbb{T}^2\} \) and \( \beta_2 = \{(0, t) \in \mathbb{T}^2\} \) respectively. Then we get
\[
(6.6) \quad i([\gamma_1], [\tau^n(\gamma_2)]) = i_Y([\beta_1], [T^n(\beta_2)]) \geq i_{T \beta}([\beta_1], [T^n(\beta_2)]).
\]

So by (6.5) to prove Theorem 1.5 we need to estimate the growth rate of the geometric intersection number \( i_{T \beta}([\beta_1], [T^n(\beta_2)]) \) as \( n \to \infty \).

It is well-known that the mapping class group \( \text{Mod}(\mathbb{T}^2) \) of \( \mathbb{T}^2 \) is \( \text{PSL}_2 \mathbb{Z} \), and the nontrivial free homotopy classes of oriented simple closed curves in \( \mathbb{T}^2 \) are in bijective correspondence with primitive elements of \( \mathbb{Z}^2 \). Moreover, the geometric intersection number of two such homotopy classes \( (a, b), (c, d) \in \mathbb{Z}^2 \) can be computed as
\[
(6.7) \quad i_{T \beta}((a, b), (c, d)) = |ad - bc|,
\]
see, for instance, [15, Section 1.2.3]. From Lemma 3.2 we see that \( T_1 \) and \( T_2 \), representing an element of \( \text{Mod}(\mathbb{T}^2) \), have the forms
\[
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}
\]
respectively. For \( k, \ell \in \mathbb{N} \), the mapping class of \( T = T_2^k T_1^\ell \) is given by
\[
\begin{pmatrix} 1 & k \ell & 1 \\ -k & 1 \end{pmatrix}.
\]
If \( k \ell \neq 0, 1, 2, 3, 4 \), then this matrix has a similar diagonalizable matrix
\[
\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda \end{pmatrix}
\]
with the eigenvalue \( |\lambda| > 1 \). In this case, using (6.7) an easy calculation shows that
\[
(6.8) \quad \lim_{n \to \infty} \sqrt[n]{i_{T \beta}([\beta_1], [T^n(\beta_2)])} = |\lambda|.
\]

Clearly, for sufficiently large \( n \), \( \beta_1 \) and \( T^n(\beta_2) \) are not isotopic, and so for \( \gamma_1 \) and \( \tau^n(\gamma_2) \). Then it follows from (6.5), (6.6) and (6.8) that
\[
\Gamma(\tau, S_1, S_2) = \liminf_{n \to \infty} \frac{\log \text{rank} \, \HF(S_1, \phi^n(S_2))}{n} \geq \log |\lambda| > 0.
\]

The proof of the case that \( \tau = \tau_1^k \tau_2^\ell \) is similar, and so we omit it here. This completes the proof of statement (1).
Proof of statement (2). When $k\ell = 0, 1, 2, 3, 4$, then we have that the matrix $\begin{pmatrix} 1 - k\ell \ell & -k \\ -k & 1 \end{pmatrix}$ is a periodic or reducible mapping class, then the growth of Floer cohomology group is zero.

Proof of statement (3). The claim that $\Gamma(\tau^2, S_j, S_j) = 0$, $i, j \in \{1, 2\}$ with $i \neq j$ follows from the following well-known result immediately.

**Lemma 6.6** ([22, Proposition 4.7]). Let $(M^{2n}, \omega)$ be a connected Liouville domain $2c_1(M, \omega) = 0$. If $M$ contains an $A_2$-configuration of Lagrangian spheres $(S_1, S_2)$, then for any $k \in \mathbb{N} \cup \{0\}$, it holds that
$$\text{rank } HF(S_i, \tau_j^{2k}(S_i)) = 2k, \quad i, j \in \{1, 2\} \text{ with } i \neq j.$$ The proof of this lemma is essentially due to Seidel’s long exact sequence for Floer cohomology of Dehn twists [34, Theorem 1], see also Keating [24, Section 6]. The plumbing space $P(S_1, S_2)$ is a symplectization of Liouville domain with contact-type boundary which satisfies the conditions of the above lemma, see, for instance, [1, Section 2.3].

6.2. Proof of Theorem 1.6. The proof of Theorem 1.6 is similar to that of Theorem 1.5. Let $P(\psi(S_1, \ldots, S_m))$ and $P(\gamma_1, \gamma_2, \ldots, \gamma_m)$ be the plumbing spaces as given in Section 5.2. We need to verify the conditions (C1) and (C2) of Theorem 6.1. Similar to Lemma 6.5, we have the following lemma (see Lemma 1.7).

**Lemma 6.7.** If the composition $\tau = \tau_1^{k_1} \cdots \tau_m^{k_m}$ of symplectic Dehn twists along spheres $S_i$ satisfies that $\tau^n(\gamma_j)$ is not isotopic to $\gamma_i$, then We have the equality
$$\text{rank } HF(S_i, \tau^n(S_j)) = i([\gamma_i], [\tau^n(\gamma_j)]).$$

To complete the proof of Theorem 1.6 the remaining task is to show that $i([\gamma_i], [\tau^n(\gamma_j)])$ grows exponentially as $n \to \infty$. To do this, we need some knowledge from the subject of mapping class groups, which we will present below.

By definition, a collection of isotopy classes of simple closed curves in a closed surface $S$ with genus $g \geq 2$ fills $S$ if the complement in $S$ of the representatives in the surface is a collection of topological disks. (Equivalently, any simple closed curve in $S$ has positive geometric intersection with some isotopy class in the collection). A multicurve in $S$ is the union of a finite collection of disjoint simple closed curves in $S$.

**Theorem 6.8** (Penner [30, Theorem 3.1]). Let $A = \{c_1, \ldots, c_n\}$ and $B = \{d_1, \ldots, d_m\}$ be two multicurves in a surface $S$ which fill $S$. Then any composition of positive powers of Dehn twists $T_{c_i}$ along $c_i$ and negative powers of Dehn twists $T_{d_j}$ along $d_j$ is pseudo-Anosov, where each $c_i$ and $d_j$ appears at least once.

**Theorem 6.9** ([17]). Let $f$ be a pseudo-Anosov mapping class of a closed surface $S$ of genus $g \geq 2$ with stretch factor $\lambda > 1$. Then for any two isotopy classes of curves $\alpha$ and $\beta$ we have $\lim_{n \to \infty} \sqrt[n]{i(\alpha, f^n(\beta))} = \lambda$.

Proof of Theorem 1.6. Notice that each symplectic Dehn twist $\tau_i$ is supported in a compact neighborhood of the zero section of $S_i$. One can identify the action of $\tau =
Figure 2. Curves filling a surface of genus \( g = k \)

\[ \tau_1^{k_1} \tau_2^{k_2} \ldots \tau_m^{k_m} \]

on an invariant submanifold \( X \subseteq P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m) \) with the action of \( T = T_1^{k_1} T_2^{k_2} \ldots T_m^{k_m} \) on an open subset \( Y \) of a closed surface of genus \( g \). More precisely, for \( m = 2k, 2k+1 \) one can embed curves \( \gamma_1, \ldots, \gamma_m \) into a surface \( S \) of genus \( g = k \) with images \( \beta_1, \ldots, \beta_m \) as illustrated in Picture 2 (when \( m = 2k \), we do not need \( \beta_{2k+1} \)), and \( X \) is identified with the union \( Y \) of open neighborhoods of simple closed curves \( \beta_1, \ldots, \beta_m \) in \( S \), and each \( \tau_i \) corresponds to the Dehn twist \( T_i \) along \( \beta_i \). So we have

(6.10) \[ i([\gamma_i], [\tau^n(\gamma_j)]) = i_Y([\beta_i], [T^n(\beta_j)]) \geq i_S([\beta_i], [T^n(\beta_j)]). \]

Notice that by our construction the collection of curves \( \beta_1, \ldots, \beta_m \) fills \( S \), and that \( \beta_i \) and \( \beta_{i+2} \), \( i = 1, \ldots, m - 2 \) are disjoint. By Theorem 6.8, if either \( k_i > 0 \) for \( i \) odd and \( k_i < 0 \) for \( i \) even, or \( k_i < 0 \) for \( i \) even and \( k_i > 0 \) for \( k \) odd, \( T \) is pseudo-Anosov. Then by Theorem 6.9 we have \( \lim_{n \to \infty} \sqrt{i_Y([\beta_i], [T^n(\beta_j)])} = \lambda \). This, together with (6.9) and (6.10), implies the desired result.

\[ \square \]

7. Classification of symplectic mapping class group

In this section, we prove Theorem 1.1 which is an analogue of Nielsen-Thurston classification for symplectic mapping class group in terms of growth of Floer cohomology in four dimension.

7.1. Proof of Theorem 1.1. We use the following results of Weiwei Wu on the classification of Lagrangian spheres in an \( A^2_m \)-surface.

Theorem 7.1 ([39, Theorem 1.1]). Lagrangian spheres in \( A^2_m \)-surface are Hamiltonian isotopic to the standard spheres up to symplectic Dehn twists along them.

Theorem 7.2 ([39, Theorem 1.3]). Every compactly supported symplectomorphism of \( A^2_m \)-surface is Hamiltonian isotopic to a composition of symplectic Dehn twists along the standard spheres.

In the following we identify \( P_\psi(\gamma_1, \ldots, \gamma_m) \) with a translation surface embedded in \( S_g \) as in the proof of Theorem 1.6. We shall use the following lemma to establish the existence of invariant Lagrangian spheres for reducible symplectic mapping classes. Recall that a curve is called admissible if it is fixed by the rotation map \( \bar{R} \). We need to rotate an admissible curve to recover a Lagrangian sphere in the symplectic manifold. 
Proposition 7.3. For any free homotopy class $[\gamma]$ in $P_\psi(\gamma_1, \ldots, \gamma_m)$, there exists an admissible representative $\gamma'$ which generates a smoothly embedded Lagrangian sphere $L_{\gamma'} = \bigcup_{i \in \mathbb{Z}} \mathcal{R}_0^i(\gamma')$ in $P_\psi(S_1, \ldots, S_m)$.

We postpone the proof to Section 7.2.

Proof of Theorem 1.1. Let $f$ be a compactly supported symplectomorphism of $A^2_m$ surface. It follows from Theorem 7.2 that $f = \phi_0 \circ \tau_0$ for some compactly supported Hamiltonian diffeomorphism $\phi_0$ of $A^2_m$ and a composition $\tau_0$ of symplectic Dehn twists $\tau_{S_i}, i = 1, \ldots, m$ along spheres $S_i$, i.e., $\tau_0 = \tau_{S_{j_1}}^{k_1} \circ \cdots \circ \tau_{S_{j_m}}^{k_m}$ for some integers $k_i \in \mathbb{N}$, where $j_1, \ldots, j_m$ is a permutation from 1 to $n$. For any Lagrangian spheres $\alpha, \beta \subseteq A^2_m$, by Theorem 7.1, we have

$$\alpha = \phi_1 \circ \tau_1(S_i), \quad \beta = \phi_2 \circ \tau_2(S_j),$$

for some $i, j \in \{1, \ldots, m\}$, where $\phi_1, \phi_2$ are compactly supported Hamiltonian diffeomorphisms of $A^2_m$ and $\tau_1, \tau_2$ are compositions of symplectic Dehn twists. We have that $f^n \beta$ is Lagrangian isotopic to $\tau_0^n \tau_2(S_j)$, since we can isotope $\phi_0$ and $\phi_2$ to identity through a path of Hamiltonian diffeomorphisms. Therefore Hamiltonian isotopic to $\tau_0^n \tau_2(S_j)$ due to $H^1(S^2; \mathbb{R}) = 0$. Since Lagrangian Floer cohomology is invariant with respect to Hamiltonian isotopies and symplectic Dehn twists are symplectomorphisms, we get

$$HF(\alpha, f^n \beta) = HF(\tau_1(S_i), \tau_0^n \tau_2(S_j)) = HF(S_i, \tau_1^{-1} \tau_0^n \tau_2(S_j)).$$

Now we discuss in two cases:

1. $\alpha$ is Lagrangian isotopic to $f^n \beta$;
2. $\alpha$ is not Lagrangian isotopic to $f^n \beta$.

In case (1), the map $f$ is automatically reducible. We next focus on case (2), then $\tau_1(S_i)$ is not Lagrangian isotopic to $\tau_0^n \tau_2(S_j)$, and hence, by Lemma 8.1, $\gamma_i$ is not isotopic to $\tau_1^{-1} \tau_0^n \tau_2(\gamma_j)$ in the sense of Definition 8.1. Then by Theorem 1.7,

$$\text{rank } HF(\alpha, f^n \beta) = \iota(\gamma_i, \tau_1^{-1} \tau_0^n \tau_2(\gamma_j)) = \iota(\tau_1(\gamma_i), \tau_0^n \tau_2(\gamma_j)).$$

Notice that for $m = 2g, 2g + 1$, the plumbing space $P_\psi(\gamma_1, \ldots, \gamma_m)$ is homeomorphic to an oriented surface $\hat{S}_g$ of genus $g$ with one or two punctures. By the Nielsen-Thurston’s classification [15, Theorem 13.2], the mapping class $[\tau_0] \in \text{Mod}(\hat{S}_g)$ has a representative $\phi$ of one of the three types:

I. Reducible: $\phi$ leaves invariant a finite collection of pairwise disjoint simple closed curves in $\hat{S}_g$;
II. Periodic: $\phi^m = Id$ for some positive integer $m$;
III. Pseudo-Anosov: there are transverse measured foliations $(\mathcal{F}^s, \mu_s)$ and $(\mathcal{F}^u, \mu_u)$ on $\hat{S}_g$ and a real number $\lambda > 1$ so that

$$\phi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda \mu_s) \quad \text{and} \quad \phi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda^{-1} \mu_u)$$

Therefore, the geometric intersection number $\iota(\tau_1(\gamma_i), \tau_0^n \tau_2(\gamma_j))$ is periodic in $n$ for type II, and grows exponentially in $n$ for type III, see for instance [15, Theorem 14.24]; and so does $\text{rank } HF(\alpha, f^n \beta)$ for these two types.
In the case that \( \phi \) is reducible, for some positive integer \( k \), \( \phi^k \) preserves a closed curve in \( P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m) \). By Proposition 7.3, we may choose an admissible representative \( \gamma \) (in the sense of Definition 5.2) such that it has exactly two fixed points under the reflection \( \bar{R} \). Hence this curve generates a Lagrangian sphere in \( P_\psi(S_1, S_2, \ldots, S_m) \).

Therefore, by Proposition 8.1, \( \tau_0^k \) preserves the Lagrangian sphere generated by \( \gamma \) up to Lagrangian isotopy, and so does \( f^k \). \( \square \)

7.2. Proof of Lemma 7.3 (Existence of admissible representatives). In what follows, we identify \( P_\psi(\gamma_1, \ldots, \gamma_m) \) with a translation surface as illustrated in Figure 1 and equip it with the Euclidean metric. Denote by \( d \) and \( d_H \) the distance between two points and the Hausdorff distance between two sets induced by this metric respectively. By abuse of notations, for a simple closed curve \( \gamma \) lying in \( P_\psi(\gamma_1, \ldots, \gamma_m) \), we also use \( \gamma \) to denote its image in it. We shall proceed in two steps:

- First we show that every free homotopy class of simple closed curve has an admissible representative \( \gamma' \). Thus by Lemma 5.4, the submanifold \( L_{\gamma'} \) generated by \( \gamma' \) under the rotation maps is a smoothly embedded Lagrangian submanifold of \( A^2_m \).
- Next we show that such a representative must have exactly two fixed points under \( \bar{R} \). Therefore, the submanifold \( L_{\gamma'} \) must be a topological sphere, hence a smoothly embedded Lagrangian sphere in \( A^2_m \).

Now we prove these two claims.

Lemma 7.4. Any free homotopy class of simple closed curve in \( P_\psi(\gamma_1, \ldots, \gamma_m) \) has a representative of an admissible simple closed curve.

Proof. Firstly, we notice that for any simple closed curve \( c \) lying in \( P_\psi(\gamma_1, \ldots, \gamma_m) \), \( c \) and \( \bar{R} \circ c \) are homotopic since \( \bar{R} \) is isotopic to the identity map. Let \( \gamma \) be a smooth simple closed curve in the class \([\gamma]\). Set \( T := [0,1]/\{0,1\} \). The distance between any two points \([s],[t]\) \( \in T \) is given by

\[
\rho([s],[t]) = \inf_{i \in \{0,1\}} |s - t + i|.
\]

If \( \gamma = \bar{R}(\gamma) \), nothing needs to be proved. Otherwise, one can choose a homotopy

\[
F : [0,1] \times T \to P_\psi(\gamma_1, \ldots, \gamma_m), \quad F(0,\cdot) = \gamma, \quad F(1,\cdot) = \bar{R} \circ \gamma
\]

such that the image of \( F \), denoted by \( \mathcal{D} \) (a compact subset of \( P_\psi(\gamma_1, \ldots, \gamma_m) \)), is invariant under \( \bar{R} \).

For any closed curve \( \sigma : T \to P_\psi(\gamma_1, \ldots, \gamma_m) \), we denote

\[
\text{Lip}(\sigma) := \sup_{[s],[t] \in T} \frac{d(\sigma(s),\sigma(t))}{\rho([s],[t])}.
\]

Given \( C > \text{Lip}(\gamma) \), we pick \( \epsilon > 0 \) small enough \( (C\epsilon \ll 1) \) and consider the set of curves

\[
\mathcal{C}(\epsilon,C) = \{ \sigma \in C^\infty(T,\mathcal{D}) | [\sigma] = [\gamma], \quad \text{Lip}(\sigma) \leq C, \quad d(\sigma(s),\sigma(t)) \geq \epsilon \text{ if } \rho([s],[t]) > C\epsilon \}.
\]

Clearly, \( \gamma, \bar{R}(\gamma) \in \mathcal{C}(\epsilon,C) \) for \( \epsilon \) small enough.
For any \( \sigma \in \mathcal{C}(\epsilon, C) \), we denote by \( A_\sigma = \cap \mathcal{R}[0, 1] \times \mathbb{T} \) where \( \mathcal{R} : [0, 1] \times \mathbb{T} \rightarrow \mathcal{D} \) is taken over all possible homotopies from \( \sigma \) to \( \mathcal{R}(\sigma) \) inside \( \mathcal{D} \).

We observe that \( A_\sigma \) is a compact subset of \( \mathcal{D} \) that is bounded by \( \sigma \) and \( \mathcal{R}(\sigma) \). We claim that the region \( A_\sigma \) must be invariant under \( \mathcal{R} \). If not, then since \( \mathcal{R} \circ \mathcal{R} = \mathcal{I} \), \( \mathcal{R}(A_\sigma) \) is also a region bounded by \( \mathcal{R}(\sigma) \) and \( \mathcal{R}(\sigma) \). If \( \sigma \) and \( \mathcal{R}(\sigma) \) intersect, then the set of intersection points is \( \mathcal{R} \)-invariant. And we see that \( A_\sigma \) is a union of bigons (possibly including some line segments) whose boundaries are \( \mathcal{R} \)-invariant, which implies that \( A_\sigma \) is \( \mathcal{R} \)-invariant. If \( \sigma \) and \( \mathcal{R}(\sigma) \) do not intersect, then the two regions \( A_\sigma \) and \( \mathcal{R}(A_\sigma) \) are either the same, or their union must be the whole surface \( P_\psi(\gamma_1, \ldots, \gamma_m) \). The latter is impossible since \( \mathcal{R} \) fixes the plumbing points and their antipodal points.

We now consider the map \( \mathcal{F}_\mathcal{R} : \mathcal{C}(\epsilon, C) \rightarrow \mathbb{R}_{\geq 0} \),

\[
\mathcal{F}_\mathcal{R}(\sigma) = \text{Area}(A_\sigma).
\]

Here, we take the unsigned (non-negative) area of a region given by the metric on the surface.

Since \( \mathcal{C}(\epsilon, C) \) is compact in the space of Lipschitz continuous curves from \( \mathbb{T} \) to \( \mathcal{D} \) with respect to the Lipschitz norm, one can take \( \gamma' \in \mathcal{C}(\epsilon, C) \) to achieve the minimum of \( \mathcal{F}_\mathcal{R} \). We claim that \( \gamma' = \mathcal{R}(\gamma') \). Indeed, if \( \gamma' \neq \mathcal{R}(\gamma') \), then for any curve \( \sigma \) in the homotopy class lying in \( A_{\gamma'} \), the area \( \mathcal{F}_\mathcal{R}(\sigma) \) is smaller than \( \mathcal{F}_\mathcal{R}(\gamma') \). This contradiction implies that \( \gamma' = \mathcal{R}(\gamma') \).

Moreover, the curve \( \gamma' \) is an admissible Lipschitz curve in the homotopy class of \( \gamma \) whose only self intersection points are those \( \gamma'(s) = \gamma'(t) \) with \( \rho([s], [t]) \leq C\epsilon \). Hence, for a small enough \( \epsilon \), any self-intersection of \( \gamma' \) must give rise to a small disk and thus can be removed by deforming the curve in the homotopy class. Since \( \gamma' \) is admissible, the resulting simple closed curve is also admissible.

**Lemma 7.5.** Any smooth admissible simple closed curve in \( P_\psi(\gamma_1, \ldots, \gamma_m) \) has exactly two fixed points under \( \mathcal{R} \).

**Proof.** For an admissible curve \( \gamma \subseteq P_\psi(\gamma_1, \ldots, \gamma_m) \), we assume with out loss of generality that \( \gamma(0) \) is not a fixed point and \( \gamma \) is unit speed under the Euclidean metric on the translation surface. Under this assumption, there exists a unique \( t \in [0, 1] \) such that \( \gamma(t) = \mathcal{R}(\gamma(0)) \). We claim that \( \gamma \) has at least two fixed point by \( \mathcal{R} \). Indeed, since \( \gamma \) is admissible, simple closed, and unit speed, we have \( \gamma(s) = \mathcal{R} \circ \gamma(t-s) \) and \( \gamma(t+s) = \mathcal{R} \circ \gamma(1-s) \); or conversely \( \gamma(s) = \mathcal{R} \circ \gamma(1-s) \) and \( \gamma(t+s) = \mathcal{R} \circ \gamma(t-s) \), for any \( s \in [0, t] \). This shows that either \( \gamma(t/2) \) and \( \gamma(1+t/2) \) are fixed points of \( \mathcal{R} \), or \( \gamma(0) \) and \( \gamma(1/2) \) are fixed points of \( \mathcal{R} \).

We next show that an admissible curve has at most two fixed points. We assume without loss of generality that \( \gamma \) is unit speed and \( \gamma(0) \) is a fixed point by \( \mathcal{R} \), then by the same reasoning as above, \( \gamma(1/2) \) must be a fixed point of \( \mathcal{R} \).

Let \( t = \min \{ s \in (0, 1] \mid \mathcal{R} \circ \gamma(t) = \gamma(t) \} \). Then we have \( \gamma(t+s) = \mathcal{R} \circ \gamma(t-s) \) for any \( 0 \leq s \leq t \). So we have \( \gamma(0) = \gamma(s) \), but since \( \gamma \) is a simple closed curve, we have \( s = 1 \). Therefore, \( t = 1/2 \), and \( \gamma \) only has two fixed points. \(\square\)
8. Floer cohomology groups and geometric intersection numbers

In this section, we relate Lagrangian Floer cohomology groups to the geometric intersection numbers in the reduced space \( P_\psi(\gamma_1, \ldots, \gamma_n) \) and give the proof of Lemma 6.5.

8.1. Rotation maps and Lagrangian isotopies. The next lemma is important for constructing Lagrangian isotopies through isotopic curves in \( W := P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m) \). Denote by \( \Sigma_\psi \) the set consisting of plumbing points and their antipodal points in \( W \). Note that if \( c \in C_{ad} \) is an admissible curve containing exactly two points of \( \Sigma_\psi \), then \( L_c \) is a Lagrangian sphere in \( P_\psi(S_1, S_2, \ldots, S_m) \). Let \( \text{Diff}_c(W, \Sigma_\psi) \) be the group of compactly supported diffeomorphisms \( f: W \to W \) with \( f(\Sigma_\psi) = \Sigma_\psi \).

Definition 8.1. Two curves \( \gamma_0, \gamma_1 \in C_{ad} \) are called isotopic if there is an isotopy \( (f_s)_{0 \leq s \leq 1} \) in \( \text{Diff}_c(W, \Sigma_\psi) \) such that \( \gamma_s = f_s(\gamma_0) \) is a family of admissible curves connecting \( \gamma_0 \) to \( \gamma_1 \).

Definition 8.2. A Lagrangian isotopy on a symplectic manifold \( (M, \omega) \) is a smooth family of Lagrangian embedding \( \Phi: L \times [0, 1] \to M \). Since \( \Phi^*_s \omega \) vanishes on the fibers \( L \times \{ \text{point} \} \), we have \( \Phi^*_s \omega = \theta_s \wedge ds \). We call a Lagrangian isotopy \( \Phi \) exact if \( \theta_s \in \Lambda^1(L) \) is exact for all \( s \).

The following lemma follows immediately from Lemma 5.1.

Lemma 8.1. If \( c_1, c_2 \in C_{ad} \) are isotopic in \( P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m) \), then \( L_{c_1} \) and \( L_{c_2} \) are Lagrangian isotopic.

By (6.4), we know that \( L_c \) is invariant under the involution \( \iota \) associated to the circles \( \gamma_1, \gamma_2, \ldots, \gamma_m \), i.e.,

\[ \iota(L_c) = L_c. \]  

8.2. Lagrangian Floer cohomology and geometric intersection number. The geometric intersection number between \( \alpha \) and \( \beta \) is a homotopic invariant. If two simple closed curves \( a \) and \( b \) realize the minimal intersection in their homotopy classes \( \alpha = [a] \) and \( \beta = [b] \), i.e., \( i(\alpha, \beta) = \sharp(a \cap b) \), then we say that \( a \) and \( b \) in minimal position. The following bigon criterion from [15, Proposition 1.7] is useful for verifying the minimal position.

Lemma 8.2 (The bigon criterion). Two transverse simple closed curves \( a \) and \( b \) in a surface \( S \) are in minimal position if and only if they do not form a bigon. Here a bigon refers to an embedded disk in \( S \) whose boundary is the union of an arc of \( a \) and an arc of \( b \) intersecting in exactly two points.

Thus, in order to prove Lemma 6.5, it is enough to show how to eliminate bigons.

Lemma 8.3. Let \( \gamma_1 \) and \( \gamma_2 \) be the big circles that we have chosen for the plumbing space \( P_\psi(S_1, S_2) \). Then any bigon formed by \( \gamma_1 \) and \( \tau^n(\gamma_2) \) on \( P_\psi(\gamma_1, \gamma_2) \) can be eliminated by homotoping \( \gamma_1 \) provided that \( \gamma_1 \) and \( \tau^n(\gamma_2) \) are not isotopic.
Proof. We parameterize $T^*\gamma_2$ by $T^*\gamma_2 \cong S^1 \times \mathbb{R} = \{ (\theta, t) \in [0, 2\pi) \times \mathbb{R} \}$. Then we can define two subsets $T^*_\pm \gamma_2 \subset T^*\gamma_2$ by

$$T^+_\gamma \cong \{ (\theta, t) \in [0, 2\pi) \times [0, \infty) \}, \quad T^-_\gamma \cong \{ (\theta, t) \in [0, 2\pi) \times (-\infty, 0] \}.$$ 

Let $\gamma^+_1$ (resp. $\gamma^-_1$) be one of half geodesic circles of $\gamma$ connecting the plumbing point $p$ to its antipodal point $q_1$ such that $T^*_\gamma^+_1 \subset T^*_\gamma_1$ (resp. $T^*_\gamma^-_1 \subset T^*_\gamma_1$) is identified with $T^+_\gamma \sim (\theta, t) \in [0, 2\pi) \times [0, \infty)$ near $p$ under the plumbing map $\psi$, as illustrated in Figure 3.

![Figure 3](image.png)

**Figure 3.** $P^+_\psi(\gamma_1, \gamma_2)$ and $P^-_\psi(\gamma_1, \gamma_2)$ in the plumbing space of $T^*\gamma_1$ and $T^*\gamma_2$

Clearly, we have $T^*\gamma_1 = T^*_\gamma^+_1 \cup T^*_\gamma^-_1$ and $T^*\gamma_2 = T^+_\gamma \cup T^-_\gamma$. Denote

$$P^+_\psi(\gamma_1, \gamma_2) = T^*_\gamma^+_1 \cup \psi T^+_\gamma, \quad P^-_\psi(\gamma_1, \gamma_2) = T^*_\gamma^-_1 \cup \psi T^-_\gamma.$$ 

Then $P_\psi(\gamma_1, \gamma_2) = P^+_\psi(\gamma_1, \gamma_2) \cup P^-_\psi(\gamma_1, \gamma_2)$ and $\gamma_2 = P^+_\psi(\gamma_1, \gamma_2) \cap P^-_\psi(\gamma_1, \gamma_2)$.

We define the rotation map $\mathcal{R} : P^+_\psi(\gamma_1, \gamma_2) \to P_\psi(S_1, S_2)$ by (see (5.5) for $\mathcal{R}$)

$$\mathcal{R}(z) = \begin{cases} \mathcal{R}^1(z) & \text{if } z \in T^*_\gamma^+_1; \\ \mathcal{R}^2(z) & \text{if } z \in T^+_\gamma. \end{cases}$$

The image of $\mathcal{R}$ is precisely $P^-_\psi(\gamma_1, \gamma_2)$. By Lemma C.1,

$$\mathcal{R}(\tau^n(\gamma_2) \cap P^+_\psi(\gamma_1, \gamma_2)) = \tau^n(\gamma_2) \cap P^-_\psi(\gamma_1, \gamma_2),$$

and hence, $\tau^n(\gamma_2)$ is invariant under the map $\mathcal{R}$. Note that $\gamma_1$ is also invariant under $\mathcal{R}$. So both $\gamma_1$ and $\tau^n(\gamma_2)$ belong to $\mathcal{C}_ad$, and all bigons of $\gamma_1$ and $\tau^n(\gamma_2)$ appear in pairs related by $\mathcal{R}$. Let $q_1$ and $q_2$ be the antipodal points of the plumbing point $p$ in $S_1$ and $S_2$ respectively. Denote $\Sigma = \{ p, q_1, q_2 \}$. As a map on $P_\psi(\gamma_1, \gamma_2)$, each symplectic Dehn
twist \( \tau_i \) satisfies \( \tau_i(\Sigma) = \Sigma \), hence the curve \( \tau^n(\gamma_2) \) contains precisely two elements of \( \Sigma \). Then all bigons \( \Omega \) of \( \gamma_1 \) and \( \tau^n(\gamma_2) \) are divided into three types:

I. \( \Omega \cap \Sigma = \{p, q_1\} \);
II. \( \Omega \cap \Sigma = \{p\} \) or \( \{q_1\} \);
III. \( \Omega \cap \Sigma = \emptyset \).

Let \( D_+ \) and \( D_- = R(D_+) \) be a pair of such bigons. If \( \gamma_1 \) and \( \tau^n(\gamma_2) \) has a bigon of type I, then \( \tau^n(\gamma_2) \) contains a simple closed curve homotopic to \( \gamma_1 \) and thus \( \tau^n(\gamma_2) \) itself is homotopic to \( \gamma_1 \). If \( D_+ \) and \( D_- \) are of type II or type III, one can eliminate these two bigons by deforming \( \gamma_1 \) to an isotopic curve \( \gamma'_1 \in \mathcal{C}_{ad} \) with \( p \) and \( q_1 \) being fixed as illustrated in the Figure 4.

---

**Figure 4.** Deforming \( \gamma_1 \) to eliminate the bigons

The last lemma enables us to complete the proof of Lemma 6.5.

**Proof of Lemma 6.5.** For the two simple closed curves \( \tau^n(\gamma_2), \gamma_1 : \mathbb{R}/\mathbb{Z} \rightarrow P_\psi(\gamma_1, \gamma_2) \), whenever \( \tau^n(\gamma_2) \) and \( \gamma_1 \) are not isotopic, by successively killing the unnecessary intersection points we can find an isotopy

\[
h : [0, 1] \times \mathbb{R}/\mathbb{Z} \rightarrow P_\psi(\gamma_1, \gamma_2), \quad h(0, t) = \gamma_1(t), \ h(1, t) = \gamma'_1(t)
\]
satisfying the following properties: during the isotopy each $h_s = h(s, \cdot)$ is a simple closed curve containing $\{p, q_1\}$, and $\gamma'_1$ and $\tau^n(\gamma_2)$ are transverse and in minimal position in $P_\psi(\gamma_1, \gamma_2)$, and the image of each $h_s$ is invariant under $R$.

Then by Lemma 8.1 the Lagrangian spheres $S'_1 = L_{\gamma'_1}$ and $S_1$ are Lagrangian isotopic via $L_{h_s}$, and hence exact Lagrangian isotopic because of $H^1(S^n, \mathbb{R}) = 0$ with $n \geq 2$. As before, the condition (C1) of Theorem 6.1 for $S'_1$ and $\tau^n(S_2)$ are still satisfied. Since $S'_1 \cap M^r = \gamma'_1$ and $\tau^n(S_2) \cap M^r = \tau^n(\gamma_2)$, by Lemma 8.2 the condition (C2) of Theorem 6.1 for $S'_1$ and $\tau^n(S_2)$ holds. Then by Theorem 6.1 we have

$$\text{rank}(\text{HF}(S_1, \tau^n(S_2)) = \text{rank}(\text{HF}(S'_1, \tau^n(S_2))) = \text{rank}(\text{HF}(C, \mathbb{Z}/2), \tag{8.2}$$

where $C = S'_1 \cap \tau^n(S_2)$ and $C^n = \gamma'_1 \cap \tau^n(\gamma_2)$. Replacing $\gamma_1$ by $\gamma'_1$ in (C.2), we find that if $z \in C^n \setminus \{p, q_1\}$ then the corresponding connected component of $C$ is a sphere and thus its contribution to $\text{rank}(\text{HF}(C, \mathbb{Z}/2)$ is 2, and if $z \in \{p, q_1\}$ then its contribution is 1. Since $\gamma'_1$ and $\tau^n(\gamma_2)$ are invariant under $R$, the intersection points of $\gamma'_1$ and $\tau^n(\gamma_2)$ appear in pairs except for $\{p, q_1\} \subset C^n$. So we get $\text{rank}(\text{HF}(C, \mathbb{Z}/2) = i([\gamma'_1], [\tau^n(\gamma_2)])]$. This, together with (8.2), implies the statement. \qed

9. Speculations on Measure theoretic entropy

Since Dehn twists are symplectomorphisms on the manifold $M$, they also preserve the Lebesgue measure $\mu$ on $M$. We conjecture that the measure theoretic entropy of $\tau = \tau^k_1 \tau^l_2$, $kl < 0$ on $P_\psi(S_1, S_2)$ with respect to $\mu$ is also positive. However, there are certain substantial obstacles, similar to the ones we see for the standard map.

Let us first point out the difficulty in the standard map for the purpose of comparison with the current setting. In [10] the authors adopted a special form of a class of 2 dimensional maps including the standard map: $F(x, y) = (y + L\psi(x), -x)$ defined on the torus, where $L$ is a parameter and $\psi$ is a generic smooth function on $\mathbb{T}^1$. The derivative matrix has the form $DF(x, y) = \begin{bmatrix} L\psi'(x) & 1 \\ -1 & 0 \end{bmatrix}$. For large $L$ in most part of the domain ($\{|\psi'(x)| > L^{-1/2}\}$), the matrix has a large eigenvalue with almost horizontal eigenvector. However, when $L\psi'(x)$ is close to zero, the matrix $DF$ is almost a rotation by $\pi/2$, which can mix expanding and contracting directions.

We shall see below that similar problem appears in our composite Dehn twists in a disguised form.

9.1. Periodic points. We consider the action of $\tau = \tau_1^{-1} \tau_2$ on $P_\psi(S_1, S_2)$ for $\dim(S_1) = \dim(S_2) = 2$ and for any monotone $r(t)$. Before trying to calculate Lyapunov exponents on the whole manifold, we first consider the Lyapunov exponents at the periodic points, which are given by the eigenvalues of the differential of iterations.

Proposition 9.1. There exist finitely many 1-dimensional periodic circles on $P_\psi(S_1, S_2)$ on which the Lyapunov exponent of $\tau_1^{-1} \tau_2$ is zero.

Proof. We notice that $(q, v) \in T^*S_1$ is a periodic point of $\tau_1$ if and only if $r(|v|) \in \mathbb{Q}\pi$. Thus if a point $p_0 = (q_2, s_2) = \psi(q_1, s_1) \in P_\psi(S_1, S_2), (q_1, s_1) \in T^*S_1, (q_2, s_2) \in T^*S_2$ satisfies $r(|s_i|) \in \mathbb{Q}\pi, r(|s_i|) > 2\epsilon, i = 1, 2$, then $p_0$ is a periodic point under $\tau$. For
such periodic points, we explicitly calculate the differentials of the Dehn twist in local coordinates.

We consider the plumbing space $P_\psi(S_1, S_2)$ plumbed at the point $p = (0, 0, 1) \in S_1 \subset \mathbb{R}^3$, given by local coordinates $\phi_1: S_1 \to \mathbb{R}^2$ given by $\phi_1(x_1, x_2, x_3) = (x_1, x_2)$, and $\nabla^* \phi_1(x_1, x_2, x_3, y_1, y_2, y_3) = (x_1, x_2, y_1 - \frac{x_1y_3}{x_3}, y_2 - \frac{x_2y_3}{x_3})$. We also suppose that $r' \leq \frac{C}{\epsilon}$ for some constant $C$.

So if $\tau_1^k(p) \in U_1$ for $p \in U_1$, take $\phi_1(p) = (u, v) \in \mathbb{R}^4$, $u, v, \in \mathbb{R}^2$, then we have, in local coordinates

$$\tilde{\tau}_1^k(u, v) := \phi_1 \tau_1^k \phi_1^{-1}(u, v) = \left( f_1(u, v)u + \frac{\sin(kr'(\rho))}{\rho}v, f_2(u, v) - \frac{f_2}{f_1}u + \cos(kr(\rho))v \right),$$

where $\rho(u, v) = \sqrt{|v|^2 - \frac{(u, v)^2}{1 - |u|^2} + \frac{(u, v)^2}{1 - |u|^2}}$ and

$$f_k(u, v) = \frac{\sin(kr(\rho))}{1 - |u|^2} \frac{u}{\rho} f_2(u, v) = -\sin(kr(\rho) - \cos(kr(\rho)) \frac{\langle u, v \rangle}{1 - |u|^2}.$$

Suppose $\tau_1^k(\phi_1(u, v)) \in U_1$, then treating $u, v \in \mathbb{R}^2$ as column vectors, when $(u, v) = \phi_1^{-1}(p_0)$ with $p_0$ a periodic point of $\tau_1$ with period $k$, we have

$$d\tilde{\tau}_1^k(u, v) = \left( I \begin{bmatrix} 0 & \rho \frac{\partial u}{\partial \rho} \\ \rho \frac{\partial u}{\partial \rho} & 0 \end{bmatrix} - \frac{k r'(\rho)}{1 - |u|^2} \begin{bmatrix} \frac{\partial u}{\partial v} & \frac{\partial u}{\partial v} \\ \frac{\partial u}{\partial v} & \frac{\partial u}{\partial v} \end{bmatrix} \right) \begin{bmatrix} du \\ dv \end{bmatrix}.$$

Furthermore, if we suppose $\phi(p_0) = (u, v)$ with $\langle u, v \rangle = 0$, then we have the exact equality

$$d\tilde{\tau}_1^k(u, v) = \left( I \begin{bmatrix} kr'(|u|) \hat{u} \hat{v} \\ 0 \end{bmatrix} \right) \begin{bmatrix} du \\ dv \end{bmatrix}$$

where we write $\hat{u} = \frac{u}{|u|}$.

Next we consider the action of the second Dehn twist $\tau_2$, suppose $p_0$ is a periodic point of $\tau_2$ of period $l$. Since the coordinates $(u, v)$ on $T^*S_1$ is identified to coordinates $(u', v')$ on $T^*S_2$ through $J: (u, v) \mapsto (u', v') = (-v, u)$, we have

$$d(J^{-1}\phi_2 \tau_2 \phi_2^{-1}J^k)(u', v') = \left( I \begin{bmatrix} 0 & \rho \frac{\partial u}{\partial \rho} \\ \rho \frac{\partial u}{\partial \rho} & 0 \end{bmatrix} - \frac{l r'(|u|)}{1 - |u'|^2} \begin{bmatrix} \frac{\partial u}{\partial v} & \frac{\partial u}{\partial v} \\ \frac{\partial u}{\partial v} & \frac{\partial u}{\partial v} \end{bmatrix} \right) \begin{bmatrix} du \\ dv \end{bmatrix}.$$

Now we calculate the differential of an iteration of $\tau$ at $p_0 \in T^*U_1 \cap T^*U_2 \subset P_\psi(S_1, S_2)$. Suppose $p_0 = (q_2, s_2) = \psi(q_1, s_1) \in P_\psi(S_1, S_2), (q_1, s_1) \in T^*S_1, (q_2, s_2) \in T^*S_2$ with $r(|s_1|) = \frac{1}{k} \geq 2\epsilon, r(|s_2|) = \frac{1}{l} \geq 2\epsilon$, then $p_0$ is also a periodic point of $\tau_1\tau_2^{-1}$, of period $k + |\ell| - 1$. By the calculations above, we see that for $p_0 = \phi^{-1}(u, v)$ with $\langle u, v \rangle = 0$, we have

$$d(\tau_1^{k-\ell-1}(u, v)) = d(\tau_1^k) d(\tau_2^{-k})(u, v) = \left( I \begin{bmatrix} \frac{kr'(|v|)}{\hat{u} \hat{v}} \\ 0 \end{bmatrix} \right) \begin{bmatrix} du \\ dv \end{bmatrix}.$$
Thus \(d(\tau_1^{-1})^{k+\ell-1}(u, v)\) has eigenvalues 1, so we have \(\chi^+(p_0) = 0\) for finitely many \(p_0\).

### 9.2. Hyperbolic cones near the elliptic points

Now if we remove the condition \(\langle u, v \rangle = 0\) for \(p_0\) and keep the conditions that \(p_0\) is periodic under both \(\tau_1\) and \(\tau_2\) we would get a partially hyperbolic periodic point of \(\tau\). However, we shall see that in a neighborhood of the elliptic periodic point, the eigenvectors of the hyperbolic periodic points “flip”, so the arguments we used in Section 3 to prove positive measure theoretic entropy on the subsystem cannot be applied to prove positive entropy on the whole system with respect to the Lebesgue measure.

Now for \(p_0 = (q_2, s_2) = \psi(q_1, s_1) \in P_\psi(S_1, S_2), (q_1, s_1) \in T^*S_1, (q_2, s_2) \in T^*S_2\) with \(r(|s_i|) \in \mathbb{Q}, r(|s_i|) > 2\epsilon, i = 1, 2\), we see from the second condition that \(|u|, |v| < \epsilon\), which gives us

\[
d\hat{\tau}_1^k(u, v) = \left( \begin{pmatrix} I & kr'(|v|)\hat{v}\hat{v}^t \\ 0 & I \end{pmatrix} + O(\langle u, v \rangle) \right) \left( \frac{du}{dv} \right).
\]

Applying the same argument for \(\tau_2\), we have

\[
d\hat{\tau}_2^\ell(u, v) = \left( \begin{pmatrix} I & 0 \\ -\ell r'(|u|)\hat{u}\hat{u}^t & I \end{pmatrix} + O(\langle u, v \rangle) \right) \left( \frac{du}{dv} \right).
\]

Thus \(p_0\) as also a periodic point of \(\tau_1\tau_2^{-1}\) of period \(k + |\ell| - 1\) has differential

\[
d(\tau_1\tau_2^{-1})^{k+\ell-1}(u, v) = \left( \begin{pmatrix} \ell + k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle\hat{v}\hat{u}^t & kr'(|v|)\hat{v}\hat{v}^t \\ \ell r'(|u|)\hat{u}\hat{u}^t & I \end{pmatrix} + O(\langle u, v \rangle/\epsilon) \right) \left( \frac{du}{dv} \right).
\]

Thus for \(p_0 = \phi_1(u, v)\) with \(\langle u, v \rangle \neq 0\), we have \(tr(d(\tau_1\tau_2^{-1})^{k+\ell-1}) > 4\) which shows that \(p_0\) has positive Lyapunov exponent. The eigenvalues of \(d(\tau_1\tau_2^{-1})^{k+\ell-1}\) are approximately

\[
\lambda_1 = 1 + \ell r'(|u|)\langle \hat{u}, \hat{v} \rangle + \frac{k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle + \sqrt{(k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle)^2 + 4(kr'(|v|))^2}}{2},
\]

\[
\lambda_2 = \lambda_3 = 1,
\]

\[
\lambda_4 = 1 + \ell r'(|u|)\langle \hat{u}, \hat{v} \rangle - \frac{k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle - \sqrt{(k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle)^2 + 4(kr'(|v|))^2}}{2}
\]

with eigenvectors

\[
w_1 = \left( \begin{pmatrix} k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle + \sqrt{(k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle)^2 + 4(kr'(|v|))^2} \hat{v} \\ \hat{u} \end{pmatrix} \right)
\]

and

\[
w_4 = \left( \begin{pmatrix} -k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle + \sqrt{(k\ell r'(|u|)r'(|v|)\langle \hat{u}, \hat{v} \rangle)^2 + 4(kr'(|v|))^2} \hat{v} \\ \hat{u} \end{pmatrix} \right).
\]

So we see that if \(\langle u, v \rangle > 0\) we have \(\lambda_1 > 1\) and \(\lambda_4 < 1\), and when \(\langle u, v \rangle < 0\) we have \(\lambda_1 < 1\) and \(\lambda_4 > 1\). However the corresponding eigenvectors \(w_1\) and \(w_4\) is continuous with respect to \(u, v\). Thus if a tangent vector returns to the neighborhood of an elliptic periodic point, it can start shrinking even if it was originally expanding.
Appendix A. Topological entropy and non-uniformly hyperbolic dynamics

Here we introduce the basic definitions of topological and measure theoretic entropy and its relation to hyperbolicity of the system. The contents of this subsection is classical and can be found in e.g. Chapters 3, 4 and Supplement of [23].

Definition A.1 (Topological entropy). The topological entropy of a map \( f : X \to X \) on a compact metric space \((X, d)\) is given by

\[
h_d(f) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_d(f, \delta, n)
\]

where \( S_d(f, \delta, n) \) is the minimal cardinality of set of balls \( B(x, \delta, d^f_n) \) that covers the space \( X \). Here the balls \( B(x, \delta, d^f_n) = \{ y \in X : d^f_n(x, y) < \delta \} \) are taken with respect to the metric \( d^f_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)) \).

The topological entropy has the following properties.

Proposition A.1. (1) The topological entropy does not depend on the metric \( d \). If \( d' \) is a metric on \( X \) such that \((X, d)\) and \((X, d')\) has the same topology then \( h_d(f) = h_{d'}(f) \). Thus we define \( h_{top}(f) = h_d(f) \) for any metric \( d \) compatible with the topology.

(2) The topological entropy is invariant under conjugacy, i.e. \( h_{top}(f) = h_{top}(g^{-1}fg) \) for any homeomorphism \( g : X \to Y \).

(3) If \( \Lambda \) is a closed \( f \)-invariant subset of \( X \), then \( h_{top}(f |_\Lambda) \leq h_{top}(f) \).

(4) \( h_{top}(f^m) = mh_{top}(f) \).

Theorem A.2 (Yomdin’s inequality [40]). Let \( X \) be a compact smooth manifold and \( f : X \to X \) be a \( C^\infty \) diffeomorphism. Then we have

\[
h_{top}(f) \geq \log \rho(f^* : H^*(X) \to H^*(X)).
\]

Where \( \rho(\cdot) \) denotes the largest absolute value of eigenvalues of a linear map.

Definition A.2 (Measure theoretic entropy). A measurable partition of a probability space \((X, B, \mu)\) is a collection of measurable subsets \( \xi = \{ C_\alpha \in B : \alpha \in I \} \) such that \( \mu(\cup C_\alpha) = 1 \) and \( \mu(C_\alpha \cap C_\beta) = 0 \) or \( \mu(C_\alpha \Delta C_\beta) = 0, \forall \alpha \neq \beta \).

The entropy of a measurable partition \( \xi = \{ C_\alpha \in B : \alpha \in I \} \) is given by

\[
H(\xi) = -\sum_{\alpha \in I} \mu(C_\alpha) \log(\mu(C_\alpha)).
\]

The entropy of a measure-preserving transformation \( T \) on \((X, \mu)\) is defined as

\[
h_{\mu}(T) = \sup_{\xi} \lim_{n \to \infty} \frac{1}{n} H(\{ \cap_{i=1}^n T^{-i+1}C_i \ : \ C_i \in \xi \})
\]

where \( \xi \) is taken over any measurable partition with \( H(\xi) < \infty \).

The measure theoretic entropy and the topological entropy of are related by the variational principle.
Theorem A.3 (Variational Principle). Let $X$ be a compact metric space, $f : X \to X$ a homeomorphism. Set $M(f)$ to be the set of Borel probability measures that are $f$-invariant. Then we have

$$h_{\text{top}}(f) = \sup_{\mu \in M(f)} h_{\mu}(f).$$

One way of calculating measure theoretic entropy is by examining the “hyperbolicity” of the map $f$. As we shall see in Pesin entropy formula, a diffeomorphism $f$ has positive entropy if it possesses some hyperbolicity. A diffeomorphism $f : X \to X$ on a Riemannian manifold is said to be hyperbolic if a every point $p$, $Df$ expands a subspace $E^+(p) \subset T_pX$ at rate $\lambda > 1$ and contracts a subspace $E^-(p) \subset T_pX$ at rate $1/\lambda < 1$ such that $T_pX = E^+(p) \oplus E^-(p)$ and $E^+, E^-$ are invariant under $f$.

A more precise description of hyperbolicity at a generic point is given by the Lyapunov exponent which describes the expansion rate of $f$ on a tangent vector $(p, u)$.

Definition A.3. Let $X$ be a Riemannian manifold with $f : X \to X$ a diffeomorphism of $X$. The Lyapunov exponent of $f$ at a point $(p, u) \in TX$ is defined as

$$\chi^{+}(p, u) = \lim_{m \to \infty} \frac{1}{m} \log \|df_p^m(u)\|$$

if the limit exists.

Although a priori the Lyapunov exponent of a tangent vector may not exist, Oseledeets [23] proved that for almost every point on the manifold $X$ the Lyapunov exponent at any vector is well defined, and we can split the tangent space into subspaces on which $f$ has uniform hyperbolicity.

Theorem A.4 (Oseledeets Multiplicative Ergodic Theorem). Let $X$ be a Riemannian manifold with $f : X \to X$ a diffeomorphism, let $\mu$ be a $f$-invariant Borel measure on $X$ such that $\log^+ \|Df(\cdot)\| \in L^1(X, \mu)$, then there exists a set $Y \subset X$ with $\mu(X \setminus Y) = 0$ such that for each $p \in Y$, there exists a decomposition $T_pX = \bigoplus_{i=1}^{k(p)} H_i(p)$ that is invariant under $Df$ and a set of $f$-invariant functions $\chi_1(p) < \chi_2(p) < \ldots < \chi_{k(p)}(p)$ (called the Lyapunov exponents of $p$) such that the Lyapunov exponent exists for any $u \in T_pX$ and $\chi^+(p, u) = \pm \chi_i(p), \forall u \in H_i(p)$.

Furthermore, we can use the splitting of the tangent spaces to generate stable and unstable local submanifolds of a non-uniformly hyperbolic map. Here, to avoid technical details, we only state the result for Riemannian surfaces which we shall use in Section 1.3. More details on this subject can be found in Chapter 7 and 8 [7].

Theorem A.5 (Pesin’s stable manifold theorem[7]). Let $X$ be a Riemannian surface and $f : X \to X$ a diffeomorphism such that the Lyapunov exponent $\chi_1(p) < 0$ for almost every $p \in X$, then there exists Borel functions $C : X \to \mathbb{R}$ and $\lambda : X \to (0, 1)$, such that for almost every $p \in X$, there exists a stable manifold (curve) $W^s_f(p)$ such that $x \in W^s_f(p), T_pW^s(p) = H_1(p)$, and for any $q \in W^s_f(p)$ we have

$$\text{dist}(f^n(p), f^n(q)) \leq C(x) \lambda^n(x)\text{dist}(x, y).$$

Furthermore, the stable manifolds satisfy $W^s(q) = W^s(p)$ for any $q \in W^s(p)$ and $W^s(q) \cap W^s(p) = \emptyset$ if $q \notin W^s(p)$. 
Similarly, if we consider local stable manifolds $W^s_{f^{-1}}(p)$ of $f^{-1}$, then the action of $f$ expands the distance of points on the manifold exponentially. We call them the local unstable manifolds of $f$ and denote by $W^u_f(p) = W^s_{f^{-1}}(p)$.

The Lyapunov exponents is connected with the entropy of a map by the Pesin entropy formula.

**Theorem A.6** (Pesin entropy formula). Let $X$ be a compact Riemannian manifold with $f : X \to X$ a diffeomorphism, suppose that $f$ preserves a smooth Borel measure $\mu$ on $X$, then we have

$$h_\mu(f) = \int \chi^+d\mu$$

where $\chi^+(p) = \sum_{\chi_i(p) > 0} \chi_i(p)$ with $\chi_i$ the Lyapunov exponents of $p$ as given in the Oseledec Multiplicative Ergodic Theorem.

**Appendix B. Floer cohomology**

In the section we give a brief review of Lagrangian Floer cohomology. The definition of Floer cohomology here is essentially Floer’s original one [18]. For our purpose in the present paper, we work with ungraded groups.

Let $(\tilde{M}, d\tilde{\lambda})$ be a $2n$-dimensional exact symplectic manifold with contact-type boundary, and let $(M, d\lambda)$ be its symplectization. Let $(S_1, S_2)$ be a pair of connected compact exact Lagrangian submanifolds of $M$ with $\lambda|_{S_i} = df_i, i = 1, 2$, where $f_i \in C^\infty(S_i, \mathbb{R})$. The action functional on the path space

$$\mathcal{P}(S_1, S_2) = \{\gamma : [0, 1] \to M | \gamma(0) \in S_1, \gamma(1) \in S_2\}$$

is defined as

$$\mathcal{A}_{S_1, S_2}(\gamma) = \int \gamma^*\lambda + f_1(\gamma(0)) - f_2(\gamma(1)).$$

Clearly, the critical points of $\mathcal{A}_{S_1, S_2}$ are constant paths $\gamma_x$ at the intersection points $x$ of $S_1$ and $S_2$. If $S_1$ and $S_2$ intersect transversely, the Floer cohomology $HF(S_1, S_2)$ is well defined by the standard argument of transversality and gluing, see [29]. In general, to define Floer cohomology one needs to consider Hamiltonian perturbations to achieve transversality. Let us give a quick review of this construction. Let $H \in \mathcal{H} = C^\infty([0, 1] \times M, \mathbb{R})$. For every $\gamma \in \mathcal{P}(S_1, S_2)$, the Hamiltonian action of $\gamma$ is

$$\mathcal{A}^H_{S_1, S_2}(\gamma) = \int \gamma^*\lambda - \int_0^1 H(t, \gamma(t))dt + f_1(\gamma(0)) - f_2(\gamma(1)).$$

Denote $\mathcal{P}^H(S_1, S_2) \subset \mathcal{P}(S_1, S_2)$ the critical points of this functional, which are the flow lines of the Hamiltonian flow $\phi^H_t$, i.e., $\gamma : [0, 1] \to M$ such that $\dot{\gamma}(t) = X_H(t, \gamma(t))$, $\gamma(0) \in S_1$ and $\gamma(1) \in S_2$. So there is a one-to-one correspondence between $\mathcal{P}^H(S_1, S_2)$ and $\phi^H_1(S_1) \cap S_2$. For generic $H \in \mathcal{H}^{reg} \subseteq \mathcal{H}$, $\phi^H_1(S_1)$ and $S_2$ intersect transversely. The Floer cochain $CF(S_1, S_2)$ is the vector space over $\mathbb{Z}/2$ with a base given by these intersection points. Denote by $\mathcal{F}$ the set of one-parameter families $(J_t)_{0 \leq t \leq 1}$ of almost complex structures on $M$ such that each $J_t$ is $d\lambda$-compatible, and is independent of $t$.
and tame at infinity (for the definition of “tame” see [5]). For \( J \in \mathcal{J} \), consider the maps \( u : \mathbb{R} \times [0, 1] \to M \) which solves the perturbed Cauchy-Riemannian equation

\[
\partial_s u + J(t, u)(\partial_t u - X_H(t, u)) = 0
\]

subject to the boundary conditions

\[
u(s, 0) \in S_1 \text{ and } u(s, 1) \in S_2, \quad s \in \mathbb{R}
\]

\[
\lim_{s \to +\infty} u(s, t) = x, \quad \lim_{s \to +\infty} u(s, t) = y
\]

and the finite energy condition

\[
E(u) = \frac{1}{2} \int_0^1 \int_0^1 |\partial_s u|^2 + |\partial_t u - X_H(u)|^2 ds dt < \infty.
\]

Denote by \( \widehat{\mathcal{M}}(x, y; H, J) \) the space of the above maps \( u \). There is a natural \( \mathbb{R} \)-translation on \( \mathcal{M}(x, y; H, J) \) in \( s \)-direction, and its quotient space is denoted by \( \mathcal{M}(x, y; H, J) \). Solutions of (B.1) can be thought as negative gradient flow lines for \( \mathcal{A}^H_{S_1, S_2} \) in an \( L^2 \)-metric on \( \mathcal{P}(S_1, S_2) \). For each \( u \in \widehat{\mathcal{M}}(x, y; H, J) \) we linearize (B.1) and obtain a Fredholm operator \( D_{H,J,u} \) in suitable Sobolev spaces. Under the assumptions that \( \langle [\omega], \pi_2(M, S_i) \rangle = 0, i = 1, 2 \), there is a dense subspace \( \mathcal{J}^{reg} \subset \mathcal{J} \) of almost complex structures such that \( D_{H,J,u} \) are onto for all \( u \in \widehat{\mathcal{M}}(x, y; H, J) \), see [18]. Hence the spaces \( \widehat{\mathcal{M}}(x, y; H, J) \), as well as \( \mathcal{M}(x, y; H, J) \), are smooth manifolds.

In the setting of our present paper, these topological assumptions are met, and then we can define the Floer differential \( d_{H,J} : \text{CF}(S_1, S_2) \to \text{CF}(S_1, S_2) \) by counting isolated points in \( \mathcal{M}(x, y; H, J) \) mod 2, i.e.,

\[
d_{H,J}(y) = \sum_x \#_2 \mathcal{M}(x, y; H, J) x.
\]

This map has square zero, i.e., \( d_{H,J}^2 = 0 \), and hence \( \text{CF}(S_1, S_2) \) is a complex over the coefficient \( \mathbb{Z}/2 \). \( \text{HF}(S_1, S_2) \) is defined to be its cohomology \( \text{Ker}(d_{H,J})/\text{Im}(d_{H,J}) \). It can be shown that Floer cohomology \( \text{HF}(S_1, S_2) \) is independent of the pair \( (H, J) \) up to canonical isomorphism, see [33, 35].

**Appendix C. Rotation maps and Dehn twists**

Set \( M = P_\psi(S_1, S_2) \) and \( \tau = \tau_2^k \tau_1^\ell \). As before, we identify two open neighborhoods \( U_1 \) and \( U_2 \) of the plumbing point \( p \) in \( T^*S_1 \) and \( T^*S_2 \) by a symplectomorphism \( \psi \). Pick a geodesic circle \( \gamma_1 \) of \( S_1 \) containing \( p \), and let \( \gamma_2 \) be the unique geodesic circle of \( S_2 \) passing through \( p \) such that two open subsets of \( T^*\gamma_1 \) and \( T^*\gamma_2 \) are identified near \( p \) under the map \( \psi \). Let \( \mathcal{R}_\theta^i : T^*\gamma_i \to M, i = 1, 2 \) be the rotation maps associated to \( S_i \) as described in Section 5.2.

**Lemma C.1.** Let \( c_1 \subseteq T^*\gamma_1 \) and \( c_2 \subseteq T^*\gamma_2 \) be two smooth curves. Then for any \( k \in \mathbb{Z} \) and any \( \theta \in \mathbb{S}^{n-1} \) we have

\[
\tau_1^k(\mathcal{R}_\theta^2(c_2)) = \mathcal{R}_\theta^2(\tau_1^k(c_2) \cap T^*\gamma_2) \cup_\psi \mathcal{R}_\theta^1(\tau_1^k(c_2) \cap T^*\gamma_1),
\]

\[
\tau_2^k(\mathcal{R}_\theta^1(c_1)) = \mathcal{R}_\theta^1(\tau_2^k(c_1) \cap T^*\gamma_1) \cup_\psi \mathcal{R}_\theta^2(\tau_2^k(c_1) \cap T^*\gamma_2).
\]
Proof. \((C.1)\) follows from \((5.2)\) and \((5.4)\) and the fact that every symplectic Dehn twist along a sphere preserves all cotangent bundles of geodesic circles of this sphere. We only prove the former. Let \(c \subset T^*\gamma_2\) be a smooth curve. Then we have

\[
\begin{align*}
\tau_1(\mathcal{R}_\theta^2(c)) &= \tau_1(\mathcal{R}_\theta^2(c) \cap T^*\gamma_1) \cup \tau_1(\mathcal{R}_\theta^2(c) \setminus T^*\gamma_1) \\
&= \tau_1(\mathcal{R}_\theta^2(c \cap T^*\gamma_1)) \cup \tau_1(\mathcal{R}_\theta^2(c \setminus T^*\gamma_1)) \\
&= \mathcal{R}_\theta^1(\tau_1(c \cap T^*\gamma_1)) \cup \tau_1(\mathcal{R}_\theta^2(c \setminus T^*\gamma_1)) \\
&= \mathcal{R}_\theta^2(\tau_1(c \cap T^*\gamma_1) \cup T^*\gamma_2) \cup \tau_1(\tau_1(c \cap T^*\gamma_1) \setminus T^*\gamma_2) \cup (\mathcal{R}_\theta^2(c \cap T^*\gamma_1)) \\
&= \mathcal{R}_\theta^2(\tau_1(c \cap T^*\gamma_1) \cup T^*\gamma_2) \cup \tau_1(\tau_1(c \cap T^*\gamma_1) \setminus T^*\gamma_2) \\
&= \mathcal{R}_\theta^2(\tau_1(c \cap T^*\gamma_1) \cup T^*\gamma_2) \cup \tau_1(\tau_1(c \cap T^*\gamma_1)).
\end{align*}
\]

Now we are ready to give the proof of Lemma 6.3.

**Proof of Lemma 6.3.** Since \(S_i = \bigcup_{\theta \in S^{n-1}} \mathcal{R}_\theta^i(\gamma_i), i = 1, 2,\) we have

\[
S_1 \cap \tau^n(S_2) = \bigcup_{\theta \in S^{n-1}} (\mathcal{R}_\theta(\gamma_1) \cap \tau^n(S_2)) = \bigcup_{\theta \in S^{n-1}} \bigcup_{\theta' \in S^{n-1}} (\mathcal{R}_\theta^1(\gamma_1) \cap \tau^n(\mathcal{R}_{\theta'}^2(\gamma_2)))
\]

\[\text{(C.2)}\]

where in the last equality we have used Lemma C.1 and the fact that each cotangent bundle of geodesic circle \(\mathcal{R}_\theta^1(T^*\gamma_1) \subset T^*S_1\) is identified locally with an unique one \(\mathcal{R}_\theta^2(T^*\gamma_2) \subset T^*S_2\) near \(p\) under the plumbing map \(\psi\). Here we note that both \(\gamma_1\) and \(\tau^n(\gamma_2)\) locate in the plumbing space \(P_{\psi}(\gamma_1, \gamma_2)\). From \((C.2)\) we can see that the intersection of \(S_1\) and \(\tau^n(S_2)\) is the union of \((n - 1)\)-spheres and/or the plumbing point \(\{p\}\) and/or its antipodal point \(\{-p\}\). Furthermore, when \(\gamma_1\) and \(\tau^n(\gamma_2)\) intersect transversely in \(P_{\psi}(\gamma_1, \gamma_2)\), the Lagrangian \(n\)-spheres \(S_1\) and \(\tau^n(S_2)\) intersect cleanly. Since \(L_{\gamma_1} = S_1\) and \(L_{\tau^n(\gamma_2)} = \tau^n(S_2)\), it follows from \((8.1)\) that

\[
\iota(S_1) = S_1, \quad \iota(\tau^n(S_2)) = \tau^n(S_2).
\]

Set \(N = S_1 \cap \tau^n(S_2)\). Since \(M^t = P_{\psi}(\gamma_1, \gamma_2)\), we have that \(N^t = \gamma_1 \cap \tau^n(\gamma_2)\). Clearly, for each sphere \(\alpha\) belonging to \(N\), one can pick an \(t\)-invariant Morse function \(f\) on \(\alpha\) such that the critical points of \(f\) are precisely the unique maximal and minimal points belonging to \(\alpha \cap N^t\).

\[\square\]

**Lemma C.2.** Let \(c_i \subset T^*\gamma_i\) be a smooth curve. Then for any \(k \in \mathbb{Z}\) we have

\[
\tau_j^k(\mathcal{R}_\theta^j(c_i)) = \mathcal{R}_\theta^j(\tau_j^k(c_i) \cap T^*\gamma_1) \cup \psi_1 \cdots \cup \psi_{m-1} \mathcal{R}_\theta^m(\tau_j^k(c_i) \cap T^*\gamma_m), \quad j = 1, \ldots, m.
\]

Put \(\tau = \tau_1^{k_1} \tau_2^{k_2} \cdots \tau_m^{k_m}\). With the last lemma, similar to \((C.2)\), we have

\[
S_i \cap \tau^n(S_j) = \bigcup_{\theta \in S^{n-1}} \mathcal{R}_\theta^i(\gamma_i \cap \tau^n(\gamma_j)).
\]

Moreover, \(\gamma_i\) and \(\tau^n(\gamma_j)\) are invariant under the map \(\mathcal{R}\) as in \((5.5)\). If \(\gamma_i\) and \(\tau^n(\gamma_j)\) are not homotopic, similar to the proof of Lemma 8.3, one can find a curve \(\gamma'_i \in \mathcal{C}_{ad}\)
which is isotopic to $\gamma_i$ in $P_\psi(\gamma_1, \gamma_2, \ldots, \gamma_m)$ such that $\gamma'_i$ and $\tau^n(\gamma_j)$ are transverse and in minimal position.

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School of Mathematical Sciences, Beijing Normal University, Beijing, China
Email address: wmgong@bnu.edu.cn

Yau Mathematical Sciences Centre, Tsinghua University, Beijing, China
Email address: zj-wang19@mails.tsinghua.edu.cn

Department of Mathematics, Tsinghua University, Beijing, China
Email address: jxue@tsinghua.edu.cn