Some exact results for the velocity of cracks propagating in non-linear elastic models

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We analyze a piece-wise linear elastic model for the propagation of a crack in a stripe geometry under mode III conditions, in the absence of dissipation. The model is continuous in the propagation direction and discrete in the perpendicular direction. The velocity of the crack is a function of the value of the applied strain. We find analytically the value of the propagation velocity close to the Griffith threshold, and close to the strain of uniform breakdown. Contrary to the case of perfectly harmonic behavior up to the fracture point, in the piece-wise linear elastic model the crack velocity is lower than the sound velocity, reaching this limiting value at the strain of uniform breakdown. We complement the analytical results with numerical simulations and find excellent agreement.

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I. INTRODUCTION

The velocity of a crack propagating in a brittle material is known to be related to the sound velocity in the material. This general statement can be qualitatively justified by noticing that a crack is a sort of elastic disturbance, although of course of extreme non-linear nature. Thus it is not surprising that its velocity is related to the velocity of propagation of small amplitude elastic deformations. However, when we want to be more precise about the relation between crack velocity and sound velocity, difficulties appear. In text book treatments of linear elastic fracture mechanics, it is suggested that the maximum crack velocity is given by the Raleigh velocity \( v_R \). This limit is expected to be achieved at large driving forces (i.e., large applied strain), since for low driving forces the discrete (atomic) nature of the material may reduce the velocity drastically (this is called the lattice trapping effect). Experimentally, an increase of the velocity with the applied strain is observed in general, however the limiting Raleigh velocity is almost never achieved. Microscopical observation of crack paths in different kinds of samples reveal one source of this discrepancy: at velocities roughly close to \( v_R/3 \) a straight crack path destabilizes, becoming wandering, and generating side branches at larger velocities. Some people have claimed that if this effect is taken into account (i.e., the true microscopic crack path length is larger than the apparent macroscopic length) then the classical prediction is verified. But this cannot be claimed to be always the case. Even restricting to cracks propagating is a stationary fashion along a perfectly linear path, a careful analysis reveals that crack propagation velocity cannot be determined independently of the microscopic details close to the crack tip. This means that a purely macroscopic analysis using continuous approximations for the material leaves the velocity of the crack undetermined. This is the reason why detailed models of the breaking phenomena at the atomic scale are necessary in determining crack velocities.

A class of fully consistent models on which crack velocities can be calculated (albeit numerically) are lattice spring models where the material is represented by a set of point masses joined by springs. These springs can break when some threshold deformation is reached giving rise to cracks in the form of connected sets of broken springs.

It has been recently established that the propagation velocity in this kind of model crucially depends on the presence of anharmonicities of the springs. These anharmonicities are also called hyperelastic effects. The most spectacular case is that to hyperelastic stiffening (i.e., springs becoming stiffer at large deformation), that can produce supersonic crack propagation, something that had been considered impossible in classical treatments of fracture. However, the case of hyperelastic softening is by far the expected most common case, since most decohesion potentials typically interpolate smoothly between the weakly deformed material and the broken material, in which the elastic constants are formally zero. In this case, and even in the absence of other effects such as crack velocity oscillation or crack branching, hyperelastic softening produces a noticeable reduction of the crack velocity.

Even in the relatively simple class of lattice spring models, quantitative predictions of crack velocity is elusive, since, as already stated, breaking of the material is a form of non-linear behavior, and it is typically very difficult to find exact results for non-linear models. The situation is even worse in the presence of hyperelasticity, which is an additional source of non-linear behavior.

In this paper we show that taking a continuous approximation in the propagation direction in a class of lattice spring models, some exact results can be obtained for the crack velocity even in the presence of hyperelastic softening. These results shed light on the effect of hyperelasticity on crack propagation, and serve as a starting point for other (most likely numerical) studies in more realistic models.

We have also implemented the model numerically and compared the simulated results with the analytical ones, finding excellent agreement.
by the equation:

\[
\rho \frac{d^2 u_j(x, t)}{dt^2} = \frac{d}{dx} \left[ k_0 \eta \left( \frac{du_j}{dx} \right) \frac{du_j(x, t)}{dx} \right] + \nu(u_{j+1} - u_j) + \nu(u_j - u_{j-1}),
\]

with the functions \( \eta \) and \( \nu \) defined as

\[
\eta(y) = 1 \quad \text{if} \quad |y| < u_{nl}, \\
\eta(y) = \gamma \quad \text{if} \quad |y| > u_{nl}, \\
\nu(y) = y \quad \text{if} \quad |y| < u_{bk}, \\
\nu(y) = 0 \quad \text{if} \quad |y| > u_{bk}
\]

and \( \rho \) being the density of each chain.

We want to obtain the solution to this equation when the external strain \( \delta \) is in between two limiting values. The lowest possible value for propagation corresponds to the Griffith’s threshold \( \delta_G \), at which the elastic energy available in the system ahead of the crack equals that stored in the broken springs behind the crack. An easy calculation shows that \( \delta_G = u_{bk}/\sqrt{2N + 1} \). On the other hand the maximum external strain that can be supported by the system is the one that would break the system even in the absence of any pre-existing crack. Clearly this stress for uniform breaking \( \delta_U \) is given by \( \delta_U = u_{bk} \).

A few remarks are in order. Our model is obviously anisotropic, as there are continuous chains along the \( x \) direction, whereas the system is discrete in the \( y \) direction. Another source of anisotropy lies in the fact that hyperelastic softening is introduced only inside the chains, but not in the inter-chain springs. We have previously indicated that in fact it is the hyperelasticity in the propagation direction that drives the non-trivial evolution of the system. There is no point in introducing hyperelasticity in the interchain springs, as this has no important effect in the dynamics and complicates greatly the analytical treatment. Note also that chains are not allowed to break, it is only the vertical inter-chain springs that break. This forces the crack to remain in the center of the stripe and avoids effects such as crack branching.

The wave velocity inside each chain is given by \( V_w = \sqrt{k_0/\rho} \). In the highly stretched case, the spring constant changes in a factor of \( \gamma \), so \( \gamma \) we can define the stretched wave velocity \( V_w' \) as \( V_w' = \sqrt{k_0/\rho} \). We will solve the model under the assumption that there is a stable propagation of a crack in the middle of the stripe, with a velocity \( V \). As we will see, this velocity –if not zero– will never be larger than \( V_w \), nor lower than \( V_w' \).

III. SCALING PROPERTIES OF THE SOLUTION

Before presenting the analytical results we have derived, it is interesting to indicate some constraints on the solution that can be obtained using scaling arguments only. Let us suppose we have obtained the solution \( u_j(x, t) \) corresponding to a given set of parameters \( u_{nl}, \)

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**FIG. 1:** A sketch of the model studied: a set of \( 2N \) continuous non-linear elastic chains (represented by the continuous lines) are coupled through perfectly harmonic interactions if the vertical distance between chains is lower than a threshold value \( u_{bk} \). If this value is exceeded (this may occur only between chains \( u_1 \) and \( -u_1 \)) the two chains decouple defining the crack. In the figure, intact springs are shadowed. The crack advances to the right as the system evolves. Boundary conditions are fixed displacements imposed at the top and bottom of the figure.
stationary solution \( \tilde{u} \), and some applied stress \( \delta \), and that this solution has a velocity \( V \). It is then immediate to verify that a rescaled solution \( \alpha u \) is also a solution of the problem for an applied strain \( \alpha \delta \), with the same velocity \( V \) if the parameters \( \alpha \tilde{u} \) and \( \alpha u_n \) are rescaled to \( \alpha \tilde{u} \) and \( \alpha u_n \). This means that the velocity can be written as a function of the combinations \( \tilde{u} / u_n \), and \( \delta / u_n \).

A less trivial scaling can be obtained by changing a solution of the form \( u(x,t) \equiv u(Ax,Bt) \), and finding \( A \) and \( B \) and new coefficients of the model for \( u(x,t) \) to be a solution. The calculation is straightforward, and we present only the result, that can be stated as the fact that the velocity of the crack should be of the form

\[
\sqrt{1 - \left( \frac{V}{V_w} \right)^2} = \sqrt{1 - \gamma f \left( N, \frac{\delta}{u_n \sqrt{1 - \gamma}}, \frac{\tilde{u}}{u_n \sqrt{1 - \gamma}} \right)} \tag{3}
\]

where \( f \) is an undetermined function of the arguments and where we also made use of the previously obtained scaling.

Already from this scaling relation a very important result can be obtained. This corresponds to the case in which there is no hyperelastic softening, i.e., in which the spring constants within the chains remain always equal to \( k_0 \). This can be formally obtained by letting \( u_n \) go to infinity. Then the r.h.s. of Eq. (3) becomes \( \sqrt{1 - \gamma f(N,0,0)} \). Since obviously in this limit the crack velocity cannot depend on \( \gamma \), the only possibility is that \( f(N,0,0) = 0 \), which implies \( V = V_w \). This means that in the absence of hyperelastic effects the crack velocity is equal to the wave velocity for any \( \delta > \delta_G \).

**IV. EXACT SOLUTION FOR \( N = 1 \)**

We present here the exact analytic solution of the previous non-linear model in the case \( N = 1 \). This will be a reference result for the further discussion of the more interesting cases with \( N > 1 \).

Assuming a stationary propagation, we introduce the stationary solution \( \tilde{u}(x) \equiv u_1(x - Vt, 0) \), where \( V \) is the propagation velocity to be determined self-consistently.

We choose the new reference system in such a way that the crack tip is located at \( x = 0 \). Thus, we must search for solutions of the piece-wise defined equation (we eliminate the tilde for simplicity):

\[
1 - \left( \frac{V}{V_w} \right)^2 \frac{d^2u}{dx^2} = u - \frac{3\delta}{2}; \quad x < 0, \quad \left| \frac{du}{dx} \right| < u_n \quad \tag{4}
\]

\[
1 - \left( \frac{V}{V_w} \right)^2 \frac{d^2u}{dx^2} = 3u - \frac{3\delta}{2}; \quad x > 0, \quad \left| \frac{du}{dx} \right| < u_n \quad \tag{5}
\]

\[
\gamma - \left( \frac{V}{V_w} \right)^2 \frac{d^2u}{dx^2} = u - \frac{3\delta}{2}; \quad x < 0, \quad \left| \frac{du}{dx} \right| > u_n \quad \tag{6}
\]

\[
\left[ \gamma - \left( \frac{V}{V_w} \right)^2 \right] \frac{d^2u}{dx^2} = 3u - \frac{3\delta}{2}; \quad x > 0, \quad \left| \frac{du}{dx} \right| > u_n \quad \tag{7}
\]

with the additional condition to be satisfied at the crack tip: \( u(0) = \tilde{u}_{kk} / 2 \).

Non-trivial solutions of this non-linear equation of motion can be obtained by matching the solution in the different sectors. It can be shown in general that the crack tip (located at \( x = 0 \)) must also be the point that separates low and high stretching regions, i.e., \( |du/dx| < u_n \) for \( x > 0 \), and \( |du/dx| > u_n \) for \( x < 0 \). In fact, according to the differential equation, when the non-linear threshold is reached, there is a change of sign in the pre-factor of \( d^2u/dx^2 \), that passes from \( 1 - (V/V_w)^2 \) to \( \gamma - (V/V_w)^2 \). But \( d^2u/dx^2 \) cannot change sign, otherwise the first derivative \( du/dx \) would be an extreme at that point, and this is inconsistent since we assumed \( |du/dx| < u_n \) to the right and \( |du/dx| > u_n \) to the left.

Then the change of sign of the pre-factor of \( d^2u/dx^2 \) must be compensated by a change of sign on the right hand side of the equation, and this is only possible at the point where the system is breaking and the right hand side changes from \( 3u - 3\delta/2 \) to \( u - 3\delta/2 \). This justifies our statement that exactly at the crack tip, the value of \( |du/dx| = u_n \).

For \( x > 0 \) the solution of the differential equation has the form

\[
u(x) = \frac{\delta}{2} - \left( \frac{\delta}{2} - \frac{\tilde{u}_{kk}}{2} \right) \exp \left[ -\frac{x\sqrt{2}}{\sqrt{1 - (V/V_w)^2}} \right], \quad \tag{8}
\]

where we have already used the constraint \( u(0) = \tilde{u}_{kk} / 2 \), and from the requirement \( |du/dx|_{x=0} = u_n \) we get the velocity as

\[
V(\delta) = V_w \sqrt{1 - \frac{3}{4} \left( \frac{\tilde{u}_{kk} - \delta}{u_n} \right)^2} \quad \tag{9}
\]

Note that the velocity is independent of the value of \( \gamma \), and that it is consistent with Eq. (3), as it must be. We also see explicitly that \( V = V_w \) if \( u_n \to \infty \). To check that this is a consistent solution, we must verify that there is a reasonable form of \( u(x) \) for \( x < 0 \).

When \( \delta \) is large enough, the solution for \( x < 0 \) consists of a concatenation of similar pieces of the form

\[
u(x) = \frac{3\delta}{2} + \frac{u_{kk} - 3\delta}{2} \cos \left( \frac{x}{\sqrt{(V/V_w)^2 - \gamma}} \right) - u_n \sqrt{(V/V_w)^2 - \gamma} \sin \left( \frac{x}{\sqrt{(V/V_w)^2 - \gamma}} \right) \quad \tag{10}
\]

This kind of solution is sketched in Fig. 2. When two pieces of this form are matched together, the derivative of \( u \) has a jump. A momentum conservation condition must be satisfied at those points. In fact, the integral of the force on an infinitesimal piece of chain during the time in which it passes through the singular point, must
The singular part is an oscillation that has an amplitude which is composed by a smooth part and a singular part. The solution has the previous form given in Eq. (10). For that in this case the solution for \( u(x) \) is singular in our macroscopic system we should study the case of a large number of chains. When the number of chains extends to \( N \to \infty \), the oscillation for \( x < 0 \) diverges. In fact, if according to Eq. (9) the velocity would be lower than \( V_c \), then neither the solutions given by Eq. (10) nor Eq. (12) exist. It can be shown that in this case the velocity of crack propagation is actually \( V = V_c \). This regime is not particularly interesting to us, and from now on we will always assume to have chosen values of \( \gamma \) such that \( V > V_c \).

Note that according to Eq. (9) the crack velocity at the Griffith’s threshold is finite if \( \frac{w_{nk}}{u_{nl}} < \frac{\sqrt{3}}{2} \), and is given by

\[
V(\delta_G) = V_c \left( 1 - \frac{\frac{w_{nk}}{u_{nl}}}{\frac{\sqrt{3}}{2}} \right) \frac{\left( \sqrt{3} - 1 \right)^2}{4} \tag{14}
\]

This is in contrast to cases in which the system is discrete in the direction of crack propagation. In those cases, due to lattice trapping effects the crack cannot propagate if the Griffith’s threshold is not overpassed by a finite amount.

The values in Eq. (13) for the velocity as a function of \( \delta \) must be compared with the result that is obtained in the absence of hyperelastic effect, namely \( V(\delta) = V_w \). The non-trivial result contained in Eq. (9) is a consequence of hyperelasticity in the system. We will see that the same qualitative effects exist in the more interesting cases with \( N > 1 \).

V. EXACT RESULTS FOR \( N > 1 \)

The previous case \( N = 1 \) is a good starting point in which the analytical solution can be worked out in full detail. But obviously, if we are interested in modeling a macroscopic system we should study the case of a large number of chains.

For \( N > 1 \) the exact value of the velocity for arbitrary \( \delta \) cannot be obtained, in general. However, we can provide exact results in some neighborhood of \( \delta_G \) and \( \delta_V \).
Consider first the case $\delta \sim \delta_G$. Let us concentrate on one half (the upper one) of the system, since the other is symmetric. Sufficiently close to the Griffith’s threshold, only one chain (the one adjacent to the crack) will enter the hyperelastic region. In this regime the problem can be separated in three sectors as shown in Fig. 4. We match the solutions, requiring continuity of the function and derivative of $u_j(x)$, except for the derivative of $u_1(x)$, in which (as in the $N = 1$ case) a discontinuity of the derivative of the form $\delta x_0$ exists between regions II and III. The solution obtained will be valid as long as no chain other than the first enters the non-linear regime and $u_1(x) < u_2(x)$. The width of zone II is determined as part of the solution. The problem stands as a system of $4N + 1$ nonlinear algebraic equations, which we solve to any desired accuracy through an iterative method. The results for the velocity are plotted in Fig. 4 as a function of $\delta/\delta_G$, and in Fig. 5 as a function of $\delta/\delta_U$. We plot data in the full range in which the method is reliable and gives the exact value of the velocity. As it can be observed, the velocity is only weakly dependent on $\gamma$ (always assuming $V_w^2 < V$).

It is interesting to observe from Fig. 4 that even in the limit $N \to \infty$ our method provides the solution in a finite range of $\delta$. This means that in all this interval, for a system of infinite chains, there is a single one that explores the hyperelastic regime, and is responsible for the full reduction of the velocity from $V_w$ to the actual value.

In the limit $\delta \to \delta_G$, we have $x_0 \to 0$, and region II shrinks to zero. In this limit we obtain the exact values of the velocity and its derivative with respect to $\delta$ by solving a linear system of $2N$ algebraic equations. Both $V(\delta_G)$ and $\frac{dV}{d\delta_G}$ turn out to be independent of $\gamma$. From this independence and the scaling form Eq. 3, we can conclude that

$$V(\delta_G, N) = V_w \sqrt{1 - \left(\frac{u_{bk}}{u_{nl}} F_1(N)\right)^2} \quad (15)$$

The values of the functions $F_1$ and $F_2$ for different $N$ are shown in Fig. 5. The limiting value $F_1(N \to \infty) = 1/2$ can be obtained analytically through an appropriate analysis of the equations in this limit. Extrapolation of the finite $N$ exact values of $F_2(N)$ suggests also that $\lim_{N \to \infty} \left(\frac{F_2(N)}{\sqrt{N/2}}\right) = 1$, but we have not verified it analytically.

We can then write:

$$\sqrt{1 - \left(\frac{V}{V_w}\right)^2} = \frac{u_{bk}}{2u_{nl}} \quad (17)$$

$$d\sqrt{1 - \left(\frac{V}{V_w}\right)^2} \bigg|_{\delta_G, N \to \infty} = \frac{u_{bk}}{2u_{nl}} \quad (18)$$

We emphasize again the reduction of the velocity from the value $V_w$ due to the hyperelastic effect. This effect disappears if $u_{nl} \to \infty$.

A second limiting case can be solved analytically, and that is the asymptotic form of the velocity very close to
\[ \sqrt{N/2 - F_2(N)} \]

\( F_3(N) \) is another \( N \)-dependent dimensionless function. When \( \delta \to \delta_U \) (and \( V/V_w \to 1 \)), it can be seen that the solution of the equations for sector II are \( N - 1 \) exponential modes with a diverging decaying constant, and a trigonometric mode with finite frequency. Taking into account that the width of region II (namely \( |x_0| \)) remains finite even for \( V \to V_w \), we can neglect exponential modes that grow toward negative \( x \). This allows us to obtain the velocity in this limit by solving a system of 2\( N \) linear equations, matching the solutions between regions I and II only.

The result we obtain is that to lowest order in \( \delta - \delta_U \), the velocity can be written as

\[ \sqrt{1 - \left( \frac{V}{V_w} \right)^2} \bigg|_{\delta \to \delta_U} = \frac{\delta - \delta_U}{u_{nl}} F_3(N) \]  

where \( F_3 \) is another \( N \)-dependent dimensionless function. Note that this result is again independent of \( \gamma \), and consistent with the general expression in Eq. (3). Values of \( F_3(N) \) are plotted in Fig. 8. By analyzing in more detail the \( N \to \infty \) limit, it can be shown that \( F_3 \) goes to zero as \( 1/\ln(N) \).

VI. COMPARISON WITH NUMERICAL RESULTS

Although the main results of our work are the analytical findings of the previous section, we include here some results of numerical simulations for two reasons: first of all, some of the results of the previous section are not fully intuitive, and then we think it is clarifying to check them against a numerical simulation. Secondly, numerical simulations can be done in the full range between \( \delta = \delta_G \) and \( \delta = \delta_U \), filling the gap between the two analytical limits.

In our numerical simulations we are forced to consider a system that is discrete also in the \( x \) direction. We do this by introducing softer springs in the \( x \) direction than in the \( y \) direction. The ratio between horizontal and vertical spring constants will be noted \( \kappa \), and it is a measure of the degree of anisotropy of the lattice. For \( \kappa = 1 \) the lattice is isotropic, whereas for \( \kappa \to 0 \) we recover the continuous limit of the analytical treatments. As we will see, keeping a finite but small \( \kappa \) is also an appropriate form of regularizing the singular results that may appear in the continuum case.

In Figs. 9 and 10 we present results for \( N = 1 \). They compare very well with the analytical results of the previous Section. Note in particular in Fig. 10(b) the oscillation of \( du/dx \) for \( x < x_0 \). This represents the singular
FIG. 9: a) Analytical (line) and numerical results (points) for $N = 1$, $\delta/u_{nl} = 1.23$ and $u_{bk}/u_{nl} = 2$ ($\kappa = 1/1600$ in the numerics). For clarity, only one every four points of the simulated system is shown. b) The plot of $du/dx$, from the simulations. We calculate the derivative in the discrete system for chain $j$ as $du/dx \equiv \sqrt{\kappa \left[ u_{i+1,j} - u_{i,j} \right]}$, where the subindex $i$ is the discretization along the $x$ axis. Note that the only piece of chain in the non-linear regime is that with $x_0 < x < 0$. The oscillation for $x < x_0$ carries (in the form of kinetic energy) the excess of elastic energy that is present in the system.

FIG. 10: a) Analytical (line) and numerical results (points) for $N = 1$ with $\delta/u_{nl} = 1.53$ and $u_{bk}/u_{nl} = 2$ ($\kappa = 1/1600$ in the numerics). For clarity, only one every five points of the simulated system is shown. The solution behaves as described in Eq. (10). The numerical results differ at the left border of the system, because the system is finite in the numerical simulation.

FIG. 11: a) Analytical (lines) and numerical results (points) for $N = 20$, $\delta/u_{nl} = 0.33$ and $u_{bk}/u_{nl} = 2$ ($\kappa = 1/625$ in the numerics). For clarity, only one every four points of the simulated system is shown. At this strain value only the first chain enters the non-linear regime. b) Numerical values of $du/dx$, where the singular oscillation behind the crack is observable.

VII. THE CONTINUOUS LIMIT: $N \to \infty$

From the finite $N$ results of the previous sections we can try to obtain the behavior of a ‘macroscopic’ material by studying the limit $N \to \infty$. We must keep in mind however that the results we are about to discuss still depend strongly on the microscopic details of the model, other microscopic realization giving rise probably to different macroscopic behavior. In fact, we have already emphasized that the fracture of a macroscopic object cannot
We also know that eventually \( V \) decay with \( N \) value of \( N \) close to \( \infty \). We present here a non-rigorous argument, which we think reproduces the right tendency. First of all note that this reduction of the velocity does not exist when \( \delta/\delta_G \) is a numerical constant. This is in fact an extremely slow convergence to this limit, from a practical point of view we can say that hyperelastic softening produces an appreciable reduction of the limiting velocity. In particular, we see that this reduction of the velocity does not exist when hyperelasticity is absent (namely, for \( u_{nl} \rightarrow \infty \)).

**VIII. DISCUSSION AND CONCLUSIONS**

We have analyzed the effect of hyperelastic softening in a model of crack propagation in a stripe geometry under mode III fixed displacement boundary conditions. The model is continuous in the propagation direction and has a finite number of chains in the perpendicular direction. The two central chains of the stripe can decouple when they separate more than a critical distance \( u_{kk} \), generating a crack in the model.

In the case in which the chains are perfectly harmonic the velocity of crack propagation is equal to the wave velocity \( V_w \) in the full range of external strain \( \delta \) between the Griffith’s threshold \( \delta_G \) and the strain of uniform breakdown \( \delta_{bj} \).

We have studied how this result is affected by the inclusion of hyperelastic softening in the chains, namely, a softening of the spring constant of the chains when the stretching is greater than a threshold value. We have provided analytical results in some cases, and complemented
them with numerical simulations.

For the case of a single chain the full analytical solution has been worked out. It is clearly seen already in this simple case that hyperelastic softening reduces the velocity from the harmonic case. Now the velocity has a non-trivial dependence on $\delta$, and becomes equal to $V_w$ only at $\delta_U$.

We have given the analytical solution for the velocity in neighborhoods of $\delta = \delta_G$ and $\delta = \delta_U$. The main results in this case are the following. The crack velocity at $\delta_G$ is strictly lower than $V_w$. It decreases as a function of the number of chains but may well be finite in the $N \to \infty$ limit for some range of the parameters of the model. There is a finite range of $\delta/\delta_G$ in which only the chain adjacent to the crack enters the hyperelastic regime. This range remains finite for large $N$. This means that our analytical treatment provides the exact value of the velocity in a finite range around $\delta/\delta_G = 1$ even in the $N \to \infty$ case.

The crack velocity tends always to $V_w$ when $\delta \to \delta_U$. In the large $N$ limit this can be stated as the fact that $V$ tends to $V_w$ for $\delta/\delta_G \to \infty$. However, ours estimations show that this convergence is very slow, namely like $\sim 1/\ln^2(\delta/\delta_G)$.

IX. ACKNOWLEDGMENTS

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