Multiplicity fluctuation from hydrodynamic noise

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Abstract

We discuss multiplicity fluctuation caused by noises during hydrodynamic evolution of the quark-gluon fluid created in high-energy nuclear collisions.

Keywords: Quark gluon plasma, Relativistic fluctuating hydrodynamics, Fluctuation theorem

1. Introduction

Event-by-event analysis of high-energy nuclear collisions has been performed to understand transport properties of the quark gluon plasma (QGP). Higher order harmonics of collective flow is intimately related with initial state fluctuation in nucleus-nucleus (A+A) collisions \cite{1}. In addition to this, the system created in high multiplicity proton-nucleus (p+A) events also exhibits collective behaviour \cite{2,3}. These can be described by conventional hydrodynamic simulations \cite{4}. However, the apparent success of hydrodynamics in such small systems has been questioned since the hydrodynamics poses on a condition of gradually changing thermodynamic variables in both time and space direction for its validity.

Motivated by these facts, we extended the conventional hydrodynamic framework by introducing causal hydrodynamic fluctuations \cite{5}. Within framework of the relativistic fluctuating hydrodynamics, we discuss multiplicity fluctuation caused by noises during hydrodynamic evolution of the quark-gluon fluid created in high-energy nuclear collisions. We first discuss the stochastic equations for dissipative currents coupled with the temporal evolution equation for the energy density in one-dimensionally expanding coordinate system \cite{6} to demonstrate that the final entropy fluctuates from event to event for a given initial condition. We next discuss the fluctuation theorem in non-equilibrium statistical mechanics \cite{7,8}. We finally discuss consequences of hydrodynamic fluctuations on final multiplicity.

2. Fluctuating hydrodynamics in Bjorken coordinates

To demonstrate the time evolution of the QGP in high-energy nuclear collisions under hydrodynamic fluctuations, we introduce an one-dimensionally expanding coordinate system and assume boost invariant scaling ansatz \cite{9}. In Bjorken coordinates ($\tau = \sqrt{t^2 - z^2}$, $x, y, \eta_s = \frac{1}{2} \ln[(t + z)/(t - z)]$), relativistic hydrodynamic equations reduce to \cite{9}

\begin{align}
\frac{de}{d\tau} &= -\epsilon + P_s \left( \frac{1}{sT} \right), \\
\pi &= \pi'' - \pi''', \tag{1}
\end{align}

\begin{align}
\pi &= \frac{1}{sT} \left( 1 - \frac{\pi}{sT} + \Pi \right), \tag{2}
\end{align}
where \(e\) is energy density, \(P_s\) is hydrostatic pressure, \(T\) is temperature, \(s = (e + P_s)/T\) is entropy density, \(\Pi\) is bulk pressure and \(\pi^\mu_\nu\) is shear stress tensor. In the case of perfect fluids, \(\pi^\mu_\nu = 0\), the entropy \(S = s\Delta t\Delta x\Delta y\) in a fluid element is conserved. In the case of dissipative fluids, it is easy to show

\[
\frac{1}{\Delta t\Delta x\Delta y} \frac{dS}{d\tau} = s + \tau \frac{ds}{d\tau} + \tau \frac{de}{T} \frac{d\tau}{d\tau} = s + \tau \left(\frac{-e + P_s}{T}\right) \left(1 - \frac{\pi}{sT} + \frac{\Pi}{sT}\right)
\]

\[
= \frac{\pi}{T} - \frac{\Pi}{T}.
\]

Thus the production rate of the entropy in one fluid element becomes

\[
\sigma = \frac{dS}{d\tau} = \left(\frac{\pi}{T} - \frac{\Pi}{T}\right)\Delta t\Delta x\Delta y
\]

In relativistic fluctuating hydrodynamics [5], the constitutive equations in the differential form are

\[
\tau_\pi \frac{d\pi}{d\tau} + \pi = \frac{4\eta}{3\tau} + \xi_\pi
\]

\[
\tau_\Pi \frac{d\Pi}{d\tau} + \Pi = -\frac{\zeta}{\tau} + \xi_\Pi
\]

Here \(\tau_\pi\) and \(\tau_\Pi\) are relaxation times, \(\eta\) and \(\zeta\) are shear and bulk viscosities and \(\xi_\pi\) and \(\xi_\Pi\) are Gaussian white noises for shear and bulk pressure, respectively. It is worthwhile mentioning that the right hand side of Eq. (4) is not positive definite due to the existence of noises in Eqs. (5) and (6). In the conventional hydrodynamic framework, constitutive equations are so designed to obey the second law of thermodynamics. On the other hand, stochastic constitutive equations are obtained in fluctuating hydrodynamic framework so that solutions of these equations obey the fluctuation-dissipation relation by taking an ensemble average.

3. Fluctuation theorem

After the linear response theory was established [10], a variety of progress has been made in non-equilibrium statistical mechanics. Among them, the fluctuation theorem [7, 8] has become a milestone in that field. Since the fluctuation theorem contains the Green-Kubo formula at long-time limit, it is believed to capture some important properties of non-equilibrium processes away from equilibrium. As seen in the previous section, entropy production can be negative in a certain sample event. Interestingly, the probability for the system having negative entropy production can be quantified through the fluctuation theorem shown below.

\(\sigma(t)\) is supposed to be the production rate of entropy and

\[
\bar{\sigma}(t) = \frac{1}{t} \int_0^t d\tau' \sigma(t')
\]

is its average over time duration \(t\). A relation between a probability of \(\bar{\sigma} = \alpha\) and the one of \(\bar{\sigma} = -\alpha\) holds as

\[
P(\bar{\sigma} = \alpha) = \exp(\alpha t),
\]

\[
\lim_{t \to \infty} \frac{1}{t} \ln \frac{P(\bar{\sigma} = \alpha)}{P(\bar{\sigma} = -\alpha)} = \alpha.
\]

Equations (8) and (9) are called “transient fluctuation theorem” and “steady state fluctuation theorem”, respectively. For details, see Refs. [7, 8].

Suppose entropy production rate obeys Gaussian

\[
P(\bar{\sigma}) = \frac{1}{\sqrt{2\pi\alpha^2}} \exp \left[-\frac{(\bar{\sigma} - \langle \bar{\sigma} \rangle)^2}{2\alpha^2}\right].
\]
the transient fluctuation theorem\cite{8} leads to $2\langle \dot{\mathcal{S}} \rangle /a^2 = t$. Here $\langle \cdots \rangle$ denotes an ensemble average. Thus entropy distribution at time $t$ is

$$P(S) \propto \exp \left[ -\frac{(S - \langle S \rangle)^2}{2t^2a^2} \right].$$

(11)

So far, the above results have been obtained in general under the fluctuation theorem.

4. Entropy fluctuation

Now we focus on the Bjorken expansion case discussed in Sec.\cite{2} Relative width of entropy distribution at time $\tau$ (being sufficiently larger than the relaxation time $\tau_{\eta}\Pi$) becomes

$$\Delta S / \langle S \rangle (\tau) = \frac{\sqrt{2\langle \Delta (\tau s) \rangle}}{\tau_0 s_0 + \langle \Delta (\tau s) \rangle} \frac{1}{\sqrt{\Delta \eta_s \Delta x \Delta y}}.$$  

(12)

Here $\tau_0$ is initial time, $s_0$ is initial entropy density and $\langle \Delta (\tau s) \rangle$ is average entropy production per volume of local thermal system $\Delta V = \tau \Delta \eta_s \Delta x \Delta y$. In heavy ion collisions, the number of independent local thermal system in the transverse plane can be estimated as $N = A(b) / \Delta x \Delta y$. Here $A(b)$ is the effective transverse area of collision geometry for a given impact parameter $b$. Finally relative fluctuation for total entropy per space-time rapidity window $\Delta \eta_s$ becomes

$$\frac{\Delta S_{\text{tot}}}{\langle S_{\text{tot}} \rangle} = \frac{1}{\sqrt{N}} \frac{\Delta S}{\langle S \rangle} = \frac{\sqrt{2\langle \Delta (\tau s) \rangle}}{\tau_0 s_0 + \langle \Delta (\tau s) \rangle} \frac{1}{\sqrt{\Delta \eta_s A(b)}} \frac{1}{\sqrt{2X_{\text{ini}}}}.$$  

(13)

From the first line to the second line in Eq. (13), we utilise an inequality $\sqrt{x^2 + c^2} \leq 1 / \sqrt{2} c$ for $x = \sqrt{\langle \Delta (\tau s) \rangle}$ and $c = \sqrt{\tau_0 s_0}$. This result suggests fluctuation of total entropy is bounded by a value evaluated solely from the initial entropy.

It turns out that entropy fluctuation per rapidity window $\Delta \eta_s$ depends on a factor $1 / \sqrt{\Delta \eta_s A}$. Figure 1 shows $1 / \sqrt{A}$ as a function of impact parameter $b$ by assuming $A$ corresponds to overlap area of collisions of two nuclei with a radius of 6 fm. The factor $1 / \sqrt{A}$ gradually increases with $b$ but rapidly enhances in peripheral collisions $b \sim 10$-$12$. 

Figure 1. Impact parameter dependence of a factor $1 / \sqrt{A}$ which controls the entropy fluctuation.
Identifying inelastic cross section of p+p collisions at RHIC and the LHC with the above transverse area $A$, $1/\sqrt{\sigma_{\text{in}}} \approx 0.4$–0.5 fm$^{-1}$. These values are comparable with the ones in peripheral A+A collisions. As expected, the effects of entropy fluctuation manifest in small system.

5. Summary

Entropy and, in turn, multiplicity fluctuate from event to event due to hydrodynamic fluctuation of dissipative currents such as shear stress tensor and bulk pressure even if the initial state is the same in a macroscopic sense. Although event-averaged entropy has to increase with time so that the system obeys the second law of thermodynamics, entropy in a certain event can, however, decrease with time temporarily and locally due to the hydrodynamic fluctuations. The probability of decreasing entropy during hydrodynamic evolution is tiny in general. Nevertheless, the probability is quantified by the fluctuation theorem as known in the non-equilibrium statistical mechanics. The multiplicity fluctuation caused by hydrodynamic noise must play a crucial role in small system such as in peripheral A+A and/or in high multiplicity p+A and p+p events.

Acknowledgement

The work was supported by JSPS KAKENHI Grant Numbers 25400269 (T.H.) and 12J08554 (K.M.). The work of R.K. and K.M. was supported by an Advanced Leading Graduate Course for Photon Science grant in the University of Tokyo. The work of K.M. was supported by a JSPS Research Fellowship for Young Scientists.

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