The aim of the article is to clarify the status of Shapiro plane wave solutions of the Schrödinger’s equation in the frames of the well-known general method of separation of variables. To solve this task, we use the well-known cylindrical coordinates in Riemann and Lobachevsky spaces, naturally related with Euler angle-parameters. Conclusion may be drawn: the general method of separation of variables embraces all plane wave solutions; the plane waves in Lobachevsky and Riemann space consist of a small part of the whole set of basis wave functions of Schrödinger equation.

In space of constant positive curvature $S^3$, a complex analog of horospherical coordinates of Lobachevsky space $H^3$ is introduced. To parameterize real space $S^3$, two complex coordinates $(r, z)$ must obey additional restriction in the form of the equation $r^2 = e^{z - z^*} - e^{2z}$. The metrical tensor of space $S^3$ is expressed in terms of $(r, z)$ with additional constraint, or through pairs of conjugate variables $(r, r^*)$ or $(z, z^*)$; correspondingly exist three different representations for Schrödinger Hamiltonian. Shapiro plane waves are determined and explored as solutions of Schrödinger equation in complex horispherical coordinates of $S^3$. In particular, two oppositely directed plane waves may be presented as exponentials in conjugated coordinates. $\Psi_- = e^{-\alpha z}$ and $\Psi_+ = e^{-\alpha z^*}$. Solutions constructed are single-valued, finite, and continuous functions in spherical space and correspond to discrete energy levels.

Published in:

E.M. Bychkovskaya, N.G. Tokarevskaya, V.M. Red’kov. Shapiro’s plane waves in spaces of constant curvature and separation of variables in real and complex coordinates. Nonlinear Phenomena in Complex Systems. 12, 1–15 (2009).

---

1or E.M. Bychkovskaya
2E-mail: redkov@dragon.bas-net.by
1. Introduction

It is well known that in field theory of elementary particles the most used elementary solutions are plane waves. However, in any curved space-time such simple plane wave solutions do not exist. Instead, only very symmetrical space-time models may have some analogues of such waves. In particular, there are known Shapiro’s plane waves[2], in the Lobachevsky space $H_3$, they were introduced in the context of expansion of scattering amplitudes in relativistic spherical functions. In the context of quantum mechanics for Schrödinger particle these wave functions are eigenfunctions of projection of generator of displacements on Lobachevsky space on arbitrary direction described by 3-vector (about the notation see below):

$$\Psi = e^{-i\epsilon t} (u_0 + \mathbf{n} \mathbf{u})^\alpha , \quad (\mathbf{P}_n) \Psi = \alpha \Psi , \quad \alpha = 1 \pm i \sqrt{2 \epsilon - 1} . \quad (1)$$

Solutions of that type can be constructed explicitly in arbitrary coordinate system of the space $H_3$ in [5], the plane Shapiro waves were considered on the base of generalized cylindrical coordinates. Recently, that extended plane waves were used in solving the scattering problem for Schrödinger particle on Coulomb center in Lobachevsky space [4]. Besides, solutions of that type were constructed for Maxwell equations in Lobachevsky space [5], [6], they were specified in two coordinate systems: cylindrical and horospheric. In should be noted that the scalar wave equation in these coordinates in Lobachevsky model was extensively investigated many years ago by Vilenkin and Smorodinsky [3].

The status of Shapiro plane waves for both models $H_3$ and $S_3$ from the viewpoint of the well-known general method of separation of variables was explored on the base of corresponding cylindrical coordinates. Conclusion was been drawn: the general method of separation of variables embraces the all plane wave solutions; the plane waves in Lobachevsky and Riemann space consist of a small part of the whole set of basis wave functions of Schrödinger equation. In contrast to cylindric systems for $H_3$ and $S_3$, the orispherical system exists only in Lobachevsky space, the full list of appropriate coordinate system relevant to method of separation of variables was given by Olevsky [7]: in the model $H_3$ we have 34 systems, in the model $S_3$ we have only 6 ones. However, in [8], it was pointed out one method to have some analogues for all 34 coordinate systems of the space $H_3$ in the space model $S_3$ as well: it suffices to permit the use of complex coordinates in real space $S_3$. The aim of the present paper is to develop this approach for complex orispherical coordinates in real spherical space $S_3$ on this base to examine the plane waves Shapiro’s type in spherical model.

It should be stressed that special interest may have the task of extending the general method of separation of variables to complex curvilinear coordinates.

2. Shapiro’s plane waves in Lobachevsky space and cylindric coordinates

In Olevsky paper [7], among 34 coordinate systems we see the following one

$$dS^2 = dt^2 - dr^2 - sh^2 r \, d\phi^2 - ch^2 r \, dz^2 , \quad r \in [0, +\infty) , \quad z \in (-\infty, +\infty) , \quad \phi \in [0, 2\pi) ,$$

$$u_1 = sh \, r \, \cos \phi , \quad u_2 = sh \, r \, \sin \phi , \quad u_3 = ch \, r \, sh \, z , \quad u_0 = ch \, r \, ch \, z . \quad (2)$$

all coordinates are dimensionless. In the limit of vanishing curvature, relations (2) will define the ordinary cylindric coordinates.
In quasi-Cartesian coordinates \( u_a \) of \( H_3 \)-model, Shapiro’s solutions are determined by

\[
    u_0^2 - u^2 = 1, \quad \Psi = e^{-iEt} (u_0 + n u)^\alpha, \tag{3}
\]

where \( n \) stands for any unit 3-vector. Taking orientation vector \( n \) along third axis we get

\[
    n = (0, 0, +1), \quad \Psi = e^{-iEt/h} (u_0 + u_3)^\alpha = e^{-iEt/h} (\chi r e^z)^\alpha. \tag{4}
\]

It is the matter of simple calculation to show that eq. (4) defines an exact solution of Schrödinger equation in coordinates (2):

\[
    i \frac{\hbar}{\sqrt{-\gamma}} \frac{\partial}{\partial t} \Psi = H \Psi, \quad H = \frac{1}{2M\rho^2} \left[ \frac{i\hbar}{\sqrt{-\gamma}} \frac{\partial}{\partial x} \left( -g^{kl}(x) i \frac{\hbar}{\sqrt{-\gamma}} \right) \right] =
\]

\[
    = -\frac{\hbar^2}{2M\rho^2} \left[ \frac{1}{\sh r} \frac{\partial}{\partial r} \sh r \frac{\partial}{\partial r} \sh r + \frac{1}{\sh^2 r} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sh^2 r} \frac{\partial^2}{\partial z^2} \right]; \tag{5}
\]

indeed, eq. (5) reads

\[
    \frac{1}{\sh r} \frac{\partial}{\partial r} \sh r \frac{\partial}{\partial r} \sh r + \alpha^2 \sh r \frac{\partial}{\partial r} + \alpha^2 \sh r + 2\epsilon \sh r = 0, \quad \epsilon = \frac{E}{\hbar^2/M\rho^2};
\]

which becomes identity when parameter \( \alpha \) obeys

\[
    \alpha^2 + 2\alpha + 2\epsilon = 0.
\]

Thus, Schrödinger equation has the following solutions

\[
    \Psi = e^{-iEt/h} \ch\alpha r \ch\alpha z, \quad \alpha = \alpha_\pm = -1 \pm i \sqrt{2\epsilon - 1}. \tag{6}
\]

where \((2\epsilon - 1) \geq 0\) because in space \( H_3 \) the energy spectrum of a free particle starts with a definite minimal value \( \epsilon > \frac{1}{2} \), or \( E > \hbar^2/2M\rho^2 \). The limiting procedure to flat space is realized according to

\[
    e^{\alpha Z} = \exp \left[ \frac{1}{\rho} (-1 \pm \sqrt{1 - \frac{2EM\rho^2}{\hbar^2}}) \rho z \right], \quad \exp \left[ \pm i \sqrt{\frac{2EM}{\hbar^2}} z \right] = e^{\pm ipz/h},
\]

Asymptotic behavior of the solution is given by

\[
    n = (0, 0, +1),
\]

\[
    (\ch\alpha r)_{r \to 0} \to 1, \quad (\ch\alpha r)_{r \to +\infty} \to \infty, \quad z \to +\infty, \quad \sh r \to 0, \quad z \to -\infty, \quad \sh r \to \infty. \tag{7}
\]

When starting with opposite orientation

\[
    n = (0, 0, -1), \quad \Psi = e^{-iEt/h} (u_0 - u_3)^\alpha =
\]

\[
    = e^{-iEt/h} (\ch r \sh r + \ch r \ch r z)^\alpha = e^{-iEt/h} \ch\alpha r e^{-\alpha z}, \tag{8}
\]

we arrive at the other asymptotic behavior in \( z \)-variable:

\[
    z \to +\infty, \quad e^{-\alpha z} \to \infty, \quad z \to -\infty, \quad e^{-\alpha z} \to 0. \tag{9}
\]

It may be easily shown that Shapiro plane waves satisfy the eigenvalue equation:

\[
    (\mathbf{P} n) \Psi = \alpha \Psi, \quad \Psi = e^{-iEt} (u_0 + n u)^\alpha, \tag{10}
\]
where \( \mathbf{P} \) plays a role of operator momentum in \( H_3 \). In accordance with symmetry of spaces of constant curvature, the operators \( \vec{P}, \vec{L} \) obeys Lie algebra of \( SO(3,1) \), respectively: and \( SO(4) \):

\[
\mathbf{P} = -i(1 + \mathbf{q} \cdot \mathbf{q}) \frac{\partial}{\partial \mathbf{q}}, \quad \vec{L} = [\mathbf{q} \cdot \mathbf{P}] = -i[\mathbf{q}, \frac{\partial}{\partial \mathbf{q}}],
\]

(11)

the sign \(-\) corresponds to \( H_3 \)-model, the sign \(+\) is referred to \( S_3 \)-model; vector variable is defined by

\[
\mathbf{q} = \frac{\mathbf{u}}{u_0}, \quad u_0^2 - u^2 = 1;
\]

the quantities \( \vec{L} \) and \( \vec{P} \) are measured in \( \bar{\hbar} \) and \( \bar{\hbar}/R \); respectively. To verify eq. (10), let us translate the above solutions to \( \mathbf{q} \)-variables:

\[
\Psi = e^{-i\epsilon t}(1 - q^2)^{-\alpha/2}(1 + \mathbf{n} \cdot \mathbf{q})^{\alpha}.
\]

(12)

Taking into account relation

\[
(\mathbf{P} \cdot \mathbf{n}) \Psi = -i(n_i \frac{\partial}{\partial q_i} - n_j q_j \frac{\partial}{\partial q_i}) \left[ (1 - q^2)^{-\alpha/2}(1 + \mathbf{n} \cdot \mathbf{q})^{\alpha} \right] =
\]

\[
= -i(1 - q^2)^{-\alpha/2}(1 + \mathbf{n} \cdot \mathbf{q})^{\alpha} \left[ \frac{\alpha}{1 - q^2} + \frac{\alpha}{1 + \mathbf{n} \cdot \mathbf{q}} - \frac{(n_j q_j)(q_i n_i)}{1 - q^2} \right],
\]

(13)

after simple calculation we arrive at

\[
(\mathbf{P} \cdot \mathbf{n}) \Psi = -i\alpha \Psi.
\]

Relating the wave function with Schrödinger solution, we can establish connection between the eigenvalue \(-i\alpha\) and the energy value \( \epsilon \) (see (6)):

\[
- i\alpha = -i\alpha_{\pm} = +i \sqrt{2\epsilon - 1},
\]

(14)

in usual units this relationships exhibits correct behavior when vanishing the curvature:

\[
- i\alpha \frac{\hbar}{\rho} = +i \frac{\hbar}{\rho} \pm \sqrt{2EM - \frac{\hbar^2}{\rho^2}} \quad \rightarrow \quad \pm\sqrt{2EM} \quad \rho \rightarrow \infty.
\]

Taking the vector \( \mathbf{n} \) in three different ways, one can construct eigenfunctions of \( P_1, P_2, P_3 \) respectively:

\[
\mathbf{n} = (1, 0, 0) , \quad \hat{P}_1 \Psi_1 = -i\alpha \Psi_1 , \quad \Psi_1 = e^{-i\epsilon t}(1 - q^2)^{-\alpha/2}(1 + q_1)^{\alpha}.
\]

\[
\mathbf{n} = (0, 1, 0) , \quad \hat{P}_2 \Psi_2 = -i\alpha \Psi_2 , \quad \Psi_2 = e^{-i\epsilon t}(1 - q^2)^{-\alpha/2}(1 + q_2)^{\alpha}.
\]

\[
\mathbf{n} = (0, 0, 1) , \quad \hat{P}_3 \Psi_3 = -i\alpha \Psi_3 , \quad \Psi_3 = e^{-i\epsilon t}(1 - q^2)^{-\alpha/2}(1 + q_3)^{\alpha}.
\]

(15)

For flat space model, one can multiply three elementary functions

\[
\Psi_1^0 = e^{ik_1 x}, \quad \Psi_2^0 = e^{ik_2 y}, \quad \Psi_3^0 = e^{ik_3 z},
\]

4
and produce an eigenfunction of the vector-operator $\hat{P}$. In the curved space, such method does nod work because of no-commutativity components of the momentum – see (11).

Let us find expression for $P_3$ in coordinates $x^i = (r, \phi, z)$. Starting from

$$q_1 = \frac{\text{th } r}{\text{ch } z} \cos \phi, \quad q_2 = \frac{\text{th } r}{\text{ch } z} \sin \phi, \quad q_3 = \text{th } z;$$

$$\text{th } z = q_3, \quad \text{ch } z = \frac{1}{\sqrt{1 - q_3^2}}, \quad \text{th } r = \sqrt{q_1^2 + q_2^2}, \quad \text{ch } r = \sqrt{1 - q_3^2},$$

$$\cos \phi = \frac{q_1}{\sqrt{q_1^2 + q_2^2}}, \quad \sin \phi = \frac{q_2}{\sqrt{q_1^2 + q_2^2}}. \quad \text{(16)}$$

for $P_3$ we get

$$P_3 = -i \left[ \frac{\partial x^i}{\partial q_3} \frac{\partial}{\partial x^i} - q_3 \left( q_1 \frac{\partial x^i}{\partial q_1} + q_2 \frac{\partial x^i}{\partial q_2} + q_3 \frac{\partial x^i}{\partial q_3} \right) \frac{\partial}{\partial x^i} \right]. \quad \text{(17)}$$

Taking into account identities

$$\frac{\partial r}{\partial q_1} = \text{ch}^2 r \text{ ch } z \cos \phi, \quad \frac{\partial r}{\partial q_2} = \text{ch}^2 r \text{ ch } z \sin \phi, \quad \frac{\partial r}{\partial q_3} = \text{sh } r \text{ ch } r \text{ sh } z \text{ ch } z, \quad \frac{\partial \phi}{\partial q_1} = -\frac{\text{ch } z}{\text{th } r} \sin \phi, \quad \frac{\partial \phi}{\partial q_2} = +\frac{\text{ch } z}{\text{th } r} \cos \phi, \quad \frac{\partial \phi}{\partial q_3} = 0, \quad \frac{\partial z}{\partial q_1} = 0, \quad \frac{\partial z}{\partial q_2} = 0, \quad \frac{\partial z}{\partial q_3} = \text{ch}^2 z. \quad \text{(18)}$$

after simple calculation we arrive at the formula

$$P_3 = -\frac{\partial}{\partial z}. \quad \text{(18)}$$

Immediately, one produces the identity expected:

$$-i \frac{\partial}{\partial z} \left[ e^{-iEt/\hbar} \text{ ch}^\alpha r e^{\alpha z} \right] = -i \alpha \left[ e^{-iEt/\hbar} \text{ ch}^\alpha r e^{\alpha z} \right].$$

Now, let us relate the above plane wave solutions (6) and (8) with all solutions of Schrödinger equation constricted within general method of separation of variables in cylindric coordinates:

$$\Psi = e^{-iEt/\hbar} e^{im\phi} e^{\alpha z} G(r),$$

$$\left[ \frac{d^2}{dr^2} + \left( \frac{\text{ch } r}{\text{sh } r} + \frac{\text{sh } r}{\text{ch } r} \right) \frac{d}{dr} - \frac{m^2}{\text{ch}^2 r} + \frac{\alpha^2}{\text{ch}^2 r} + 2\epsilon \right] G(r) = 0. \quad \text{(19)}$$

In new variable $\text{ch}^2 r = y$, $y \in [1, +\infty)$ eq. (19) takes the form

$$\left[ 4y(y-1) \frac{d^2}{dy^2} + 4(2y-1) \frac{d}{dy} - \frac{m^2}{y-1} + \frac{\alpha^2}{y} + 2\epsilon \right] G = 0. \quad \text{(20)}$$

With the use of the following substitution

$$G(y) = (y-1)^a y^b Y(y).$$
we get
\[\begin{align*}
4(y - 1)y Y'' &+ 4[2(a + b + 1)y - (2b + 1)] Y' + \\
+ \{ &4a(a - 1) + 8ab + 4b(b - 1) + 4a + 4b + 4a + 4b + 2e] + \\
+ [-4b(b - 1) - 4b + \alpha^2] \frac{1}{y} + [4a(a - 1) + 4a - m^2] \frac{1}{y - 1} \} Y = 0. \tag{21}
\end{align*}\]

Let us require
\[\begin{align*}
-4b(b - 1) - 4b + \alpha^2 &= 0, \quad b = \pm \frac{\alpha}{2}; \\
4a(a - 1) + 4a - m^2 &= 0, \quad a = \pm \frac{m}{2}. \tag{22}
\end{align*}\]
then eq. (21) reduces to
\[\begin{align*}
4(y - 1)y Y'' &+ 4[2(a + b + 1)y - (2b + 1)] Y' + \\
+ [4a(a + 1) + 8ab + 4b(b + 1) + 2e] Y = 0; \tag{23}
\end{align*}\]
that is of the hypergeometric type
\[\begin{align*}
(y - 1)y Y'' &+ \left( (A + B + 1)y - C \right) Y' + AB Y = 0,
\end{align*}\]
if
\[\begin{align*}
C &= 2b + 1, \quad A + B = 2a + 2b + 1, \quad AB = a(a + 1) + 2ab + b(b + 1) + e/2,
\end{align*}\]
or
\[\begin{align*}
AB &= (a + b + 1/2)^2 - \frac{(1 - 2e)}{4}, \quad A + B = 2(a + b + 1/2).
\end{align*}\]
Thus, the problem of wave functions is solved:
\[\begin{align*}
G(y) &= (y - 1)^a y^b F(A, B, C; y), \quad y = \text{ch}^2 r, \\
a &= \pm \frac{|m|}{2}, \quad b = \pm \frac{\alpha}{2}, \quad C = 2b + 1, \\
A &= a + b + \frac{1}{2} \pm i \frac{\sqrt{2e - 1}}{2}, \quad B = a + b + \frac{1}{2} \mp i \frac{\sqrt{2e - 1}}{2}. \tag{24}
\end{align*}\]
Let us restrict ourselves and consider a part of solutions (24):
\[\begin{align*}
a &= \pm m/2 = 0, \quad b = +\alpha/2 \\
\Psi(r, z) &= e^{\alpha z} (\text{ch} r)^\alpha F(A, B, C; \text{ch}^2 r), \\
2A &= \alpha + 1 \pm i \sqrt{2e - 1}, \quad 2B = \alpha + 1 \mp i \sqrt{2e - 1}. \tag{25}
\end{align*}\]
Requiring \(A = 0\) or \(B = 0\), we arrive at
\[\begin{align*}
A &= 0, \quad \alpha = -1 \mp i \sqrt{2e - 1}, \\
F(0, B, C; \text{ch}^2 r) &= 1, \quad \Psi(r, z) = e^{\alpha z} (\text{ch} r)^\alpha;
\end{align*}\]
\[ B = 0, \quad \alpha = -1 \mp i \sqrt{2\epsilon - 1}, \]
\[ F(A, 0, C; ch^2 r) = 1, \quad \Psi(r, z) = e^{\alpha z} (ch r)^\alpha; \]  
(26)

which coincides with (6).

There exists another symmetric variant
\[ a = \pm m/2 = 0, \quad 2b = -\alpha : \]
\[ \Psi(r, z) = e^{\alpha z} (ch r)^{-\alpha} F(A, B, C; ch^2 r), \]
\[ 2A = -\alpha + 1 \pm i \sqrt{2\epsilon - 1}, \quad 2B = -\alpha + 1 \mp i \sqrt{2\epsilon - 1}. \]  
(27)

which leads to
\[ A = 0, \quad \alpha = +1 \mp i \sqrt{2\epsilon - 1}, \quad \Psi(r, z) = e^{\alpha z} (ch r)^{-\alpha}, \]
\[ B = 0, \quad \alpha = +1 \mp i \sqrt{2\epsilon - 1}, \quad \Psi(r, z) = e^{\alpha z} (ch r)^{-\alpha}; \]  
(28)

this solution coincides with (8).

Conclusion may be drawn: the general method of separation of variables embraces the all plane wave solutions; the plane waves in Lobachevsky space consists of a small part of the whole set of basis wave functions of Schrödinger equation.

3. Plane wave in spherical space and separation of variables

Let us extend results of previous section to the case of space of positive constant curvature, spherical space \( S_3 \). In this space there exist similar cylindric coordinates
\[ dt = \left[ d\rho^2 + \sin^2 \rho d\phi^2 + \cos^2 \rho dz^2 \right], \quad \rho \in [0, \pi/2], \quad \phi, z \in [-\pi, \pi]; \]
\[ u_0 = \cos \rho \cos z, \quad u_3 = \cos \rho \sin z, \quad u_1 = \sin \rho \cos \phi, \quad u_2 = \sin \rho \sin \phi. \]  
(29)

Shapiro’s wave in this space model is given by
\[ u_0^2 + u_3^2 = 1, \quad \Psi = e^{-iEt/\hbar} \left( u_0 + i \mathbf{n} \cdot \mathbf{u} \right)^\alpha. \]  
(30)

Taking the vector \( \mathbf{n} \) along third axis, we get
\[ \mathbf{n} = (0, 0, +1), \quad \Psi = e^{-iEt/\hbar} \left( u_0 + i u_3 \right)^\alpha = e^{-iEt/\hbar} \cos^\alpha \rho \ e^{i\alpha z}. \]  
(31)

This function satisfies Schrödinger equation in \( S_3 \):
\[ \frac{1}{\sin \rho \cos \rho} \frac{\partial}{\partial \rho} \sin \rho \cos \rho \frac{\partial}{\partial \rho} \cos^\alpha \rho - \alpha^2 \cos^{\alpha-2} \rho + 2\epsilon \cos^\alpha \rho = 0, \]
which is an identity if
\[ \alpha^2 + 2\alpha - 2\epsilon = 0, \quad \alpha = \alpha_\pm = -1 \pm \sqrt{2\epsilon + 1}. \]  
(32)

Thus, the plane wave solutions to Schrödinger equation are
\[ \Psi_\pm^\alpha = e^{-iEt/\hbar} \cos^\alpha \rho \ e^{i\alpha z}, \quad \alpha_\pm = -1 \pm \sqrt{2\epsilon + 1}. \]  
(33)
Let us require the wave functions to be finite, continuous and single-valued function of point in space $S_3$. From demand of periodicity in variable $z$ we get

$$\sqrt{2\epsilon + 1} = n = +1, +2, \ldots \implies \epsilon = \frac{n^2 - 1}{2},$$

$$\alpha_- = -1 - n = -2, -3, \ldots, \quad \alpha_+ = -1 + n = 0, +1, +2, \ldots \quad (34)$$

One should take special attention to the value $\alpha_+ = 0$, when we obtain a very special wave function at $\epsilon = 0$:

$$\Psi^{\alpha=0} = (\cos \rho e^{-i\phi})^0 = 1, \quad (35)$$

which may be associated with uniform distribution of the particle over the whole space $S_3$. To test the functions constructed we must recall peculiarities of parametrization of $S_3$ by cylindric coordinates: The first peculiarity is

$$\rho = 0, \quad (0, \phi, z) \iff \cos^2 z + \sin^2 z = 1;$$

on this closed the function (33) behaves correctly

$$\Psi(\rho \to +0, \phi, z) = e^{-iEt/\hbar} (\cos \rho \to +1)^{\alpha_+} e^{i\alpha z +} = e^{i\alpha z +}, \quad \alpha_+ \in +1, +2, \ldots$$

$$\Psi(\rho \to +0, \phi, z) = e^{-iEt/\hbar} (\cos \rho \to +1)^{\alpha_-} e^{i\alpha z -} = e^{i\alpha z -}, \quad \alpha_- \in -1, -2, \ldots$$

The second peculiarity is

$$\rho = \pi/2, \quad (\pi/2, \phi, z) \iff \cos^2 \phi + \sin^2 \phi = 1.\quad$$

the solutions constructed behave themselves as follows

$$\Psi(\rho \to +\pi/2, \phi, z) = e^{-iEt/\hbar} (\cos \rho \to +1)^{\alpha_+} e^{i\alpha z +} = 0, \quad \alpha_+ \in +1, +2, \ldots$$

$$\Psi(\rho \to +\pi/2, \phi, z) = e^{-iEt/\hbar} (\cos \rho \to +1)^{\alpha_-} e^{i\alpha z -} = \infty e^{i\alpha z -}, \quad \alpha_- \in -1, -2, \ldots$$

Therefore, all solutions with negative $\alpha$ must be rejected as being discontinuous on the whole closed line on the sphere $S_3$ specified by $\rho = \pi/2$. Thus, physical solutions to Schrödinger equations on the sphere are

$$\Psi^{\alpha_+} = (\cos \rho e^{+i\phi})^{\alpha_+}. \quad (36)$$

It should be stressed that taking the orientation vector in the opposite direction

$$\mathbf{n} = (0, 0, -1), \quad \Psi = e^{-iEt/\hbar} (u_0 - iu_3)^{\alpha} = e^{-iEt/\hbar} \cos^{\alpha} \rho e^{-i\alpha z}, \quad (37)$$

we arrive at

$$\Psi^{\alpha_\pm} = e^{-iEt/\hbar} \cos^{\alpha_\pm} \rho e^{-i\alpha z}, \quad \alpha_\pm = -1 \pm \sqrt{2\epsilon + 1}, \quad \sqrt{2\epsilon + 1} = 0, 1, 2, \ldots,$$

from which physical ones are

$$\alpha_+ \in +1, +2, \ldots \quad \Psi^{\alpha_+}(\rho, \phi, z). \quad (38)$$
Let us show that the plane wave solution may be considered as an eigenfunction in the problem

\[ (P n) \Psi = \alpha \Psi, \quad \Psi = e^{-i\epsilon t} (u_0 + i n \mathbf{u})^\alpha, \quad (39) \]

\[ P = -i(1 + \mathbf{q} \cdot \mathbf{q}) \frac{\partial}{\partial \mathbf{q}}, \quad \mathbf{q} = \frac{\mathbf{u}}{u_0}, \quad u_0^2 + \mathbf{u}^2 = 1. \quad (40) \]

It should be noted that the variable \( q_i \) parameterizes correctly only elliptic model, because it does not distinguish between \((+u_0, +\mathbf{u})\) and \((-u_0, -\mathbf{u})\); for simplicity below we perform calculation for this case. In variable \( q_i \):

\[
\begin{align*}
    u_0 &= \frac{1}{\sqrt{1 + q^2}}, \quad u = \frac{q}{\sqrt{1 + q^2}}, \\
    \Psi &= e^{-i\epsilon t} (1 + q^2)^{-\alpha/2} (1 + i n \mathbf{q})^\alpha, \quad (41)
\end{align*}
\]

after simple calculation we get

\[
(P n) \Psi = \alpha \Psi. \quad (42)
\]

Substituting the function \( \Psi \) into Schrödinger equation, we relate \( \alpha \) with energy \( \epsilon \):

\[
\alpha = \alpha_\pm = -1 \pm \sqrt{2\epsilon + 1}. \quad (43)
\]

In usual units one can easily perform the limiting procedure

\[
\alpha \frac{\hbar}{\rho} = -\frac{\hbar}{\rho} \pm \frac{\hbar}{\rho} \sqrt{\frac{2EM\rho^2}{\hbar^2} + 1} = -\frac{\hbar}{\rho} \pm \sqrt{\frac{2EM + \hbar^2}{\rho^2}} \rightarrow \pm\sqrt{2EM} \quad \rho \rightarrow \infty.
\]

Now we want to relate the plane waves on the sphere \( S_3 \) with solutions arising in the frames of general method of separation of variables. Let start with Schrödinger equation for cylindric waves

\[
\Phi(\rho, \phi, z) = e^{im\phi} e^{i\alpha z} R(\rho),
\]

\[
\begin{align*}
    \left[ \frac{d^2}{d\rho^2} + \left( \frac{\cos \rho}{\sin \rho} - \frac{\sin \rho}{\cos \rho} \right) \frac{d}{d\rho} + 2\epsilon - \frac{m^2}{\sin^2 \rho} - \frac{K^2}{\cos^2 \rho} \right] R(\rho) &= 0.
\end{align*}
\]

The functions \( R(\rho) \) are expressed in hypergeometric functions (for more detail see [9]):

\[
R(\rho) = \sin^a \rho \cos^b \rho \ F(A, B, C; \cos^2 \rho),
\]

\[
\begin{align*}
    a &= \pm m, \quad b = \pm \alpha, \quad C = b + 1, \\
    A &= (a + b + 1 - \sqrt{2\epsilon + 1})/2, \quad B = (a + b + 1 + \sqrt{2\epsilon + 1})/2.
\end{align*}
\]

(44)
To have wave functions $\Phi(\rho, \phi, z)$, single-valued and finite on $S_3$, we must require
\[ M = \{0, \pm 1, \pm 2, \ldots\}, \quad N = \{0, \pm 1, \pm 2, \ldots\}, \quad a = + |M|, \quad b = |K| . \tag{45} \]
and make the hypergeometric series a polynomials:
\[ A = -n, \quad n = 0, 1, 2, 3, \ldots, \quad \epsilon_N = \frac{1}{2} (N^2 - 1), \quad N = a + b + 1 + 2n ; \tag{46} \]
When $\rho = 0$ we have
\[ n_0 = \cos z, \quad n_1 = 0, \quad n_2 = 0, \quad n_3 = \sin z ; \tag{47} \]
so the construct $(0, \phi, z)$ represents points $n_a = (\cos z, 0, 0, \sin z)$; correspondingly, the wave function $\Phi_{\epsilon m \alpha}$ behaves as follows
\[ m \neq 0, \quad \Phi_{\epsilon m \alpha} = 0, \quad m = 0, \quad \Phi_{\epsilon 0 \alpha} = e^{i \epsilon z} F(A, B, C; 1) = \Phi(n) ; \tag{48} \]
being a single valued and continuous on $S_3$. In the same manner, when $\rho = \pi/2$ we have
\[ n_0 = 0, \quad n_1 = \cos \phi, \quad n_2 = \sin \phi, \quad n_3 = 0; \quad \alpha \neq 0, \quad \Phi_{\epsilon m \alpha} = 0, \quad K = 0, \quad \Phi_{\epsilon 0 \alpha} = e^{i m \phi} F(A, B, C; 0) . \tag{49} \]
To obtain the Shapiro plane waves, one must separate in (44) a subset with $m = 0$, and then restrict oneself by $A = 0$ (what corresponds to $n = 0$), as result we arrive at
\[ \Phi(\rho, \phi, z) = e^{i \epsilon z} \cos^{|\alpha|} \rho F(A, B, C; \cos^2 \rho) = e^{i \epsilon z} \cos^{|\alpha|} \rho , \quad b = |\alpha |, \quad \alpha = 0, \pm 1, \pm 2, \ldots, \quad C = b + 1, \quad B = \frac{b + 1 + \sqrt{2 \epsilon + 1}}{2} , \quad A = 0, \quad \epsilon = \frac{(b + 1)^2 - 1}{2} . \tag{50} \]
The previous conclusion may be repeated: the general method of separation of variables embraces the all plane wave solutions; the plane waves in spherical space consist of a small part of the whole set of basis wave functions of Schrödinger equation. We gave special attention to plane wave solutions in spherical model $S_3$, expecting in the future to extend results of [4] on the scattering problem on Coulomb center in Lobachevsky space to spherical model.

4. Plane waves in orispherical coordinates of Lobachevsky space
As was shown in [4], for Shapiro waves the horispherical coordinate play a special role. In [7] this coordinate system has been listed under the number XIV,
\[ u_1 = r e^{-z} \cos \phi , \quad u_2 = r e^{-z} \sin \phi , \quad u_3 = \text{sh} \ z + \frac{1}{2} r^2 e^{-z} = \frac{1}{2} \left[ e^{+z} + (r^2 - 1) e^{-z} \right] , \quad u_0 = \text{ch} \ z + \frac{1}{2} r^2 e^{-z} = \frac{1}{2} \left[ e^{+z} + (r^2 + 1) e^{-z} \right] ; \quad dS^2 = dt^2 - e^{-2z} dr^2 - e^{-2z} r^2 d\phi^2 - dz^2 . \tag{51} \]

To avoid misunderstanding it should be noted that in [3], [6] instead of $(r, \phi, z)$ were used variables $(r, \pi/2 - \phi, -a,)$
In the limit of flat space they reduce to ordinary cylindric coordinates.

Arbitrary Shapiro’s wave is determined by
\[ u_0^2 - u^2 = 1, \quad \Psi = e^{-iEt} (u_0 + n \mathbf{u})^\alpha, \]
taking the orientation vector \( \mathbf{n} \) according to \( \mathbf{n} = (0, 0, -1) \) we get
\[ \Psi = e^{-iEt/\hbar} (u_0 - u_3)^\alpha = e^{-iEt/\hbar} e^{-\alpha z}, \] (52)
It may be recognize as a solution of Schrödinger equation in \( H_3 \)
\[ i \partial_t \Psi = -\frac{\hbar^2}{2M\rho^2} \left[ \frac{e^{2z}}{r} \frac{\partial}{\partial r} + \frac{e^{2z}}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{e^{-2z}} \frac{\partial}{\partial z} e^{-2z} \frac{\partial}{\partial z} \right] \Psi ; \]
which with the substitution (52) gives
\[ 2e e^{-\alpha z} = (\frac{\partial^2}{\partial z^2} - 2 \frac{\partial}{\partial z}) e^{-\alpha z}, \quad \alpha^2 - 2\alpha + 2e = 0, \]
\[ \alpha = 1 \mp i \sqrt{2e - 1}; \quad \Psi = e^{-iEt/\hbar} e^{(-1 \pm i \sqrt{2e-1})z}. \] (53)

Let us find expression for momentum operator \( P_3 \) in these coordinates \( x^i = (r, \phi, z) \). Starting with
\[ q_1 = \frac{2r \cos \phi}{e^{2z} + r^2 + 1}, \quad q_2 = \frac{2r \sin \phi}{e^{2z} + r^2 + 1}, \quad q_3 = \frac{e^{2z} + r^2 - 1}{e^{2z} + r^2 + 1}, \] (54)
and inverse ones
\[ r = \frac{\sqrt{q_1^2 + q_2^2}}{1 - q_3}, \quad e^{2z} = \frac{1 - q_3^2}{(1 - q_3)^2}, \]
\[ \cos \phi = \frac{q_1}{\sqrt{q_1^2 + q_2^2}}, \quad \sin \phi = \frac{q_2}{\sqrt{q_1^2 + q_2^2}}. \] (55)
Further, with the help of formulas
\[ \frac{\partial r}{\partial q_1} = \frac{1}{2} (e^{2z} + r^2 + 1) \cos \phi, \quad \frac{\partial r}{\partial q_2} = \frac{1}{2} (e^{2z} + r^2 + 1) \sin \phi, \quad \frac{\partial r}{\partial q_3} = \frac{1}{2} r (e^{2z} + r^2 + 1), \]
\[ \frac{\partial \phi}{\partial q_1} = -\frac{(e^{2z} + r^2 + 1)}{2r} \sin \phi, \quad \frac{\partial \phi}{\partial q_2} = \frac{(e^{2z} + r^2 + 1)}{2r} \cos \phi, \quad \frac{\partial \phi}{\partial q_3} = 0, \]
\[ \frac{\partial z}{\partial q_1} = -\frac{1}{2} e^{-2z} r (e^{2z} + r^2 + 1) \cos \phi, \quad \frac{\partial z}{\partial q_2} = -\frac{1}{2} e^{-2z} r (e^{2z} + r^2 + 1) \sin \phi, \quad \frac{\partial z}{\partial q_3} = \frac{1}{4} e^{-2z} [(e^{2z} + 1)^2 - r^4]. \]

after simple calculation we get
\[ P_3 = -i \left[ \frac{\partial x^i}{\partial q_3} \frac{\partial}{\partial x^i} - q_3 (q_1 \frac{\partial x^i}{\partial q_1} + q_2 \frac{\partial x^i}{\partial q_2} + q_3 \frac{\partial x^i}{\partial q_3}) \right] = -i (r \frac{\partial}{\partial r} + \frac{\partial}{\partial z}). \] (56)

Therefore, the plane wave is an eigenfunction of \( P_3 \):
\[ \mathbf{n} = (0, 0, -1), \quad \Psi_- = e^{-iEt/\hbar} (u_0 - u_3)^\alpha = e^{-iEt/\hbar} e^{-\alpha z}, \quad P_3 \Psi_- = +i\alpha \Psi_- . \] (57)
It should be noted a plane wave with opposite direction:
\[ \mathbf{n} = (0, 0, +1) \, , \, \Psi_+ = e^{-iEt/\hbar} (u_0 + u_3) = e^{-iEt/\hbar} (e^z + r^2 e^{-z})^\alpha, \quad P_3 \Psi_+ = -i\alpha \Psi_+ ; \] (58)

Therefore, both \( \Psi_{\pm} \) should be considered as representing plane waves. Also, we may verify by simple calculation that \( \Psi_+ \) satisfies the Schrödinger equation as well.

\[
\begin{align*}
    i \partial_t \Psi_+ &= -\frac{\hbar^2}{2M \rho^2} \left[ \frac{e^{2z}}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{e^{2z}}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{e^{-2z}} \frac{\partial}{\partial z} e^{-2z} \frac{\partial}{\partial z} \right] \Psi_+ ; \\
    \Psi_+ &= e^{-iEt/\hbar} (e^z + r^2 e^{-z})^\alpha, \quad \alpha = 1 \mp i \sqrt{2\epsilon - 1} .
\end{align*}
\]

5. Complex orispherical coordinates in \( S_3 \)

To introduce complex horispherical coordinates in spherical space \( S_3 \), let us start with real horispherical coordinates (29) in Lobachevsky space \( H_3 \)

\[
\begin{align*}
    u_1 &= r e^{-z} \cos \phi , \quad u_2 = r e^{-z} \sin \phi , \\
    u_3 &= \frac{1}{2} \left[ e^{+z} + (r^2 - 1)e^{-z} \right] , \quad u_0 = \frac{1}{2} \left[ e^{+z} + (r^2 + 1)e^{-z} \right] .
\end{align*}
\] (59)

Inverse formulas are

\[
\begin{align*}
    \cos \phi &= \frac{u_1}{\sqrt{u_1^2 + u_2^2}} , \quad \sin \phi = \frac{u_2}{\sqrt{u_1^2 + u_2^2}} , \\
    e^z &= \frac{1}{u_0 - u_3} , \quad r^2 = \frac{u_0 + u_3}{u_0 - u_3} - \frac{1}{(u_0 - u_3)^2} .
\end{align*}
\] (60)

Transition to spherical model can be realized for formal change:

\[
\begin{align*}
    u_0 = V_0 , \quad u = i \, V , \quad V_0^2 + V^2 = 1 ;
\end{align*}
\]

therefore, relations defining complex orispherical coordinates in \( S_3 \) are

\[
\begin{align*}
    \cos \phi &= \frac{V_1}{\sqrt{V_1^2 + V_2^2}} , \quad \sin \phi = \frac{V_2}{\sqrt{V_1^2 + V_2^2}} , \\
    e^z &= \frac{1}{V_0 - iV_3} , \quad r^2 = \frac{V_0 + iV_3}{V_0 - iV_3} - \frac{1}{(V_0 - iV_3)^2} .
\end{align*}
\] (61)

To parameterize real space \( S_3 \), one must impose restrictions on complex coordinate. Evidently, \( \phi \) is real-valued coordinate; besides, allowing for identitie \( V_0 - iV_3 = e^{-z} \, , \, V_0 + iV_3 = e^{-z^*} \), we arrive at the following relationship for \( r, z \):

\[
    r^2 = e^{z-z^*} - e^{2z} .
\] (62)

Let us introduce notation for complex variables:

\[
\begin{align*}
    r = A + iB , \quad z = a + ib \, ,
\end{align*}
\]

Eqs. (62) take the real form

\[
\begin{align*}
    A^2 - B^2 &= - \cos 2b (e^{2a} - 1) , \quad 2AB = - \sin 2b (e^{2a} - 1) .
\end{align*}
\] (63)
from whence we get \(^4\):

\[
+ (e^{2a} - 1) = A^2 + B^2, \quad a \in [0, +\infty),
\]

\[
\cos 2b = - \frac{A^2 - B^2}{A^2 + B^2}, \quad \sin 2b = - \frac{2AB}{A^2 + B^2}.
\]

(64)

In turn, eqs. (64) give

\[
r^2 = (A + iB)^2 = -(e^a - 1)(\cos 2b + i \sin 2b) = -rr^*e^{z-z^*},
\]

therefore eq. (62) may be transformed into

\[
e^{2z} = - \frac{r}{r^*} - r^2, \quad \Rightarrow \quad (e^{2z} + r^2)(e^{2z} + r^2)^* = 1. \quad (65)
\]

Eqs. (63) can be easily solved with respect to variables \(A, B\)^5:

\[
A = - \sqrt{e^{2a} - 1} \sin b, \quad B = + \sqrt{e^{2a} - 1} \cos b. \quad (66)
\]

Below, we will use \((a, b)\) as two real independent parameters:

\[
z = a + ib, \quad r = i \sqrt{e^{2a} - 1} e^b \quad (67)
\]

Inverse to (61) are (compare with (59)):

\[
iV_1 = re^{-z} \cos \phi, \quad iV_2 = re^{-z} \sin \phi,
\]

\[
iV_3 = \frac{1}{2} [ e^{+z} + (r^2 - 1)e^{-z} ], \quad V_0 = \frac{1}{2} [ e^{+z} + (r^2 + 1)e^{-z} ]. \quad (68)
\]

The rule to perform a limiting procedure in (68) to the flat space model is given by

\[
\rho V_1 \rightarrow x, \quad \rho V_2 \rightarrow y, \quad \rho V_3 \rightarrow z, \quad (-i\rho r) \rightarrow r, \quad \phi \rightarrow \phi, \quad (-iz\rho) \rightarrow z
\]

\[
x = r \cos \phi, \quad y = r \sin \phi, \quad z = z. \quad (69)
\]

From (68) it follows

\[
V_1 = \sqrt{e^{2a} - 1} e^{-a} \cos \phi, \quad V_2 = \sqrt{e^{2a} - 1} e^{-a} \sin \phi,
\]

\[
V_3 = e^{-a} \sin b, \quad V_0 = e^{-a} \cos b. \quad (70)
\]

in turn, eqs. (70) give

\[
V_1^2 + V_2^2 = 1 - e^{-2a}, \quad \frac{V_2}{V_1} = \tan \phi, \quad V_0^2 + V_3^2 = e^{-2a}, \quad \frac{V_3}{V_9} = \tan b. \quad (71)
\]

\(^4\)Second solution \(A^2 + B^2 = -(e^{2a} - 1)\) must be rejected.

\(^5\)In general, \((A, B)\) can be found up to \(\pm\) sign.
We may note two peculiarities in coordinates \((a, b, \phi)\):

\[
a \in [0, +\infty), \quad b \in [0, 2\pi], \quad \phi \in [0, 2\pi],
\]

\[
a \to 0, \quad V_1 = 0, \quad V_2 = 0, \quad V_0^2 + V_3^2 = 1;
\]

\[
a \to +\infty, \quad V_0 = \cos b, \quad V_3 = \sin b, \quad \phi - \text{mute variable}
\]

\[
a \to 0, \quad V_0 = 0, \quad V_3 = 0, \quad V_1^2 + V_2^2 = 1;
\]

\[
a \to +\infty, \quad V_1 = \cos \phi, \quad V_2 = \sin \phi, \quad b - \text{mute variable}.
\] (72)

The identity (62) permits us to express all four real coordinates \((V_0, \mathbf{V})\) given by (68) in terms of variables \((z, z^*, \phi)\):

\[
V_1 = \sqrt{1 - e^{-z-z^*}} \cos \phi, \quad V_2 = \sqrt{1 - e^{-z-z^*}} \sin \phi,
\]

\[
V_3 = \frac{e^{-z} - e^{-z^*}}{2i}, \quad V_0 = \frac{e^{-z^*} + e^{-z}}{2}.
\] (73)

Alternatively, with the help of identities (see (65))

\[
e^z = i \sqrt{1 + rr^*} \frac{r}{r^*}, \quad e^{-z} = -i \sqrt{\frac{1 + rr^*}{rr^*}} \frac{r^*}{r},
\]

all four coordinates \((V_0, \mathbf{V})\) (68) can be expressed in terms of variables \((r, r^*, \phi)\):

\[
V_1 = -\sqrt{rr^*} \cos \phi, \quad V_2 = -\sqrt{rr^*} \sin \phi,
\]

\[
V_3 = \frac{r + r^*}{2} \sqrt{\frac{1}{1 + rr^*}} \frac{1}{rr^*}, \quad V_0 = \frac{r^* - r}{2i} \sqrt{\frac{1}{1 + rr^*}} \frac{1}{rr^*}.
\] (74)

6. Plane waves in complex coordinates of \(S_3\)

Let us specify Shapiro plane waves (for shortness we will omit the factor \(e^{-iEt/\hbar}\)):

\[
V_0^2 + \mathbf{V}^2 = 1, \quad \Psi = (V_0 + i \mathbf{n} \mathbf{V})^\alpha.
\]

in complex orispherical coordinates. Taking two opposite orientations we get

\[
\mathbf{n} = (0, 0, -1), \quad \Psi_- = (V_0 - i V_3)^\alpha = e^{-\alpha z} = e^{-\alpha(a+ib)},
\]

\[
\mathbf{n} = (0, 0, +1), \quad \Psi_+ = (V_0 + i V_3)^\alpha = e^{-\alpha z^*} = (e^z + r^2 e^{-z})^\alpha = e^{-\alpha(a-ib)}. \tag{75}
\]

These solutions are eigenfunctions of \(P_3\) in \(S_3\):

\[
P_3 = -i \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial z} \right), \quad P_3 \Psi_- = +i\alpha \Psi_- , \quad P_3 \Psi_+ = -i\alpha \Psi_+ . \tag{76}
\]

we should note the expression for \(\psi_+\) in (75) as a function of conjugate variable \(z^*\), which agrees with the use of variables \((z, z^*, \phi)\) or \((r, r^*, \phi)\) as independent ones. We might expect other symmetric variant in variables \(r, r^*\); indeed,

\[
e^{-z} = \sqrt{-\frac{r^*}{r} \frac{1}{1 + rr^*}}, \quad \Psi_- = \left[ \sqrt{-\frac{r^*}{r} \frac{1}{1 + rr^*}} \right]^\alpha; \tag{77}
\]
For the eave with different orientation we have an equation constructed in the frame of general method of separation of variables.

\[
e^z + r^2 e^{-z} = \sqrt{-\frac{r^*}{r} \frac{1}{1 + rr^*}} \left( -\frac{r}{r^*} \right) = \sqrt{-\frac{r^*}{r} \frac{1}{1 + rr^*}}, \quad \Rightarrow \Psi_+ = \left[ \sqrt{-\frac{r^*}{r} \frac{1}{1 + rr^*}} \right]^\alpha.
\]

(78)

Now, let us turn to Schrödinger equation in complex coordinates. We are to get the metric of space in these coordinates, staring from

\[
dS^2 = dt^2 - dl^2, \quad dl^2 = dV_0^2 + dV_1^2 + dV_2^2 + dV_3^2
\]

and allowing for identities

\[
dV_1 = -i d(e^{-z} \cos \phi) = -i e^{-z} \left[ \cos \phi (dr - r dz) - r \sin \phi d\phi \right],
\]

\[
dV_2 = -i d(e^{-z} \sin \phi) = -i e^{-z} \left[ \sin \phi (dr - r dz) + r \cos \phi d\phi \right],
\]

\[
dV_3 = -i \frac{1}{2} \left[ (e^{+z} dz + e^{-z} 2R dr - r^2 e^{-z} dz) + e^{-z} dz \right],
\]

\[
dV_0 = \frac{1}{2} \left[ (e^{+z} dz + e^{-z} 2R dr - r^2 e^{-z} dz) - e^{-z} dz \right],
\]

we get

\[
dS^2 = dt^2 + e^{-2z} dr^2 + e^{-2z} d\phi^2 + dz^2, \quad \sqrt{-g} = \sqrt{-e^{4z} r^2} = i r e^{-2z};
\]

(79)

take notice on four signs + + ++ in metrical tensor. Correspondingly, Schrödinger Hamiltonian is given by

\[
H = \frac{\hbar^2}{2M^2 \rho^2} \frac{1}{\sqrt{-g}} \partial_i \sqrt{-gg^{ij}} \partial_j = \frac{\hbar^2}{2M^2 \rho^2} \left[ e^{2z} \frac{\partial}{r} \frac{\partial}{dr} + e^{2z} \frac{\partial^2}{r^2} \frac{\partial}{\phi^2} + \frac{1}{e^{-2z} \partial_z} \frac{\partial}{\partial z} \right].
\]

(80)

One may easily verify that above constructed plane waves satisfy the Schrödinger equation (80). It is so for the wave \( \Psi_\pm \):

\[
\Psi_- = e^{-Et/\hbar} e^{-\alpha z}, \quad 2e e^{-\alpha z} = \left( \frac{d^2}{dz^2} - 2 \frac{d}{dz} \right) e^{-\alpha z}, \quad \alpha^2 + 2\alpha - 2\epsilon = 0, \quad \alpha = -1 \pm \sqrt{2\epsilon + 1}.
\]

(81)

For the eave with different orientation we have an equation

\[
2\epsilon(e^z + r^2 e^{-z})^\alpha = \left( e^{2z} \frac{\partial^2}{\partial r^2} + e^{2z} \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial \phi^2} - 2 \frac{\partial}{\partial z} \right)(e^z + r^2 e^{-z})^\alpha.
\]

(82)

which is identity if

\[
\alpha^2 + 2\alpha = 2\epsilon, \quad \alpha = -1 \pm \sqrt{2\epsilon + 1}.
\]

The plane wave (82), as well as and its counterpart in \( H_3 \), can hardly be considered as constructed in the frame of general method of separation of variables.

Now we should examine the plane waves constructed with respect to their continuity properties in the space \( S_3 \):

\[
\Psi_- = e^{-\alpha z} = e^{-\alpha (a+ib)}, \quad \Psi_+ = e^{-\alpha z^*} = e^{-\alpha (a-ib)}, \quad \alpha = -1 \pm \sqrt{2\epsilon + 1};
\]

\[
V_1 = \sqrt{e^{2a} - 1} e^{-a} \cos \phi, \quad V_2 = \sqrt{e^{2a} - 1} e^{-a} \sin \phi, \quad V_3 = e^{-a} \sin b, \quad V_0 = e^{-a} \cos b.
\]
Evidently, one must require $\Psi_\pm$ to be $2\pi$-periodic in variable $b$:

$$
\alpha = -1 + \sqrt{2\epsilon + 1} = n = 0, +1, +2, \ldots ; \quad \epsilon = \frac{(n + 1)^2 - 1}{2};
$$

$$
\alpha = -1 - \sqrt{2\epsilon + 1} = n = -2, -3, \ldots ; \quad \epsilon = \frac{(n + 1)^2 - 1}{2};
$$

(83)

There arise four types of solutions:

$$
\begin{align*}
\Psi_{\alpha \geq 0}^- &= e^{(1-\sqrt{2\epsilon+1})a} e^{+i(1-\sqrt{2\epsilon+1})b}, \\
\Psi_{\alpha \leq 0}^- &= e^{(1+\sqrt{2\epsilon+1})a} e^{+i(1+\sqrt{2\epsilon+1})b}, \\
\Psi_{\alpha \geq 0}^+ &= e^{(1-\sqrt{2\epsilon+1})a} e^{-i(1+\sqrt{2\epsilon+1})b}, \\
\Psi_{\alpha \leq 0}^+ &= e^{(1+\sqrt{2\epsilon+1})a} e^{-i(1+\sqrt{2\epsilon+1})b}.
\end{align*}
$$

(84)

Having remembered the peculiarities of these coordinates – see (?) – we must conclude that second and fourth solutions are to be rejected. Thus, physical solutions are

$$
\Psi_{\alpha > 0}^- = e^{(1-\sqrt{2\epsilon+1})a} e^{+i(1-\sqrt{2\epsilon+1})b},
\Psi_{\alpha > 0}^+ = e^{(1-\sqrt{2\epsilon+1})a} e^{-i(1-\sqrt{2\epsilon+1})b};
$$

(85)

they are related by complex conjugation. In a particular case, $\epsilon = 0$, $\alpha = 0$, we have very specific solution

$$
\epsilon = 0, \alpha = 0, \quad \Psi_{\epsilon = 0}^\pm = (V_0 \pm iV_3)^0 = 1.
$$

(86)

which represents a quantum state with uniform probability distribution in spherical space $S_3$.

Having in mind the possibility to express the plane waves in variable $z, z^*$ or $(r, r^*)$, or $(a, b)$ let us consider the task of translating the metric tensor to those coordinates.

First, with the help of relation $r^2 = e^{z-z^*} - e^{2z}$ one can exclude the variable $r$, it results in

$$
dS^2 = dt^2 - \frac{e^{z+z^*} - 1}{e^{z+z^*}} d\phi^2 - \frac{1}{4e^{z+z^*}(e^{z+z^*} - 1)} [dz^2 + dz^{*2} + 2(2e^{z+z^*} - 1)dz^*d\phi^*].
$$

(87)

Allowing for $z = a + ib$, we produce

$$
dS^2 = dt^2 - \frac{e^{2a} - 1}{e^{2a}} d\phi^2 - \frac{da^2}{e^{2a} - 1} - \frac{db^2}{e^{2a}}, \quad \sqrt{-g} = \frac{1}{e^{2a}}.
$$

(88)

Corresponding Schrödinger equation is

$$
-2\epsilon \Psi = (\frac{e^{2a}}{e^{2a} - 1} \partial_b^2 + \frac{e^{2a}}{e^{2a}} \partial_a \frac{e^{2a} - 1}{e^{2a}} \partial_a + e^{2a} \partial_a^2) \Psi.
$$

(89)

The plane waves satisfy this equation, indeed

$$
\Psi = \Psi_{\pm} = e^{-\alpha(a \pm ib)} ,
$$

$$
-2\epsilon \Psi = e^{-\alpha(a \pm ib)} = [ (e^{2a} - 1) \partial_a^2 + 2 \partial_a + e^{2a} \partial_a^2 ] e^{-\alpha(a \pm ib)},
$$

or

$$
-2\epsilon e^{-\alpha(a \pm ib)} = [ (e^{2a} - 1)\alpha^2 - 2 \alpha - e^{2a} \alpha^2 ] e^{-\alpha(a \pm ib)};
$$
and further
\[ \alpha^2 + 2\alpha - 2\epsilon = 0, \quad \alpha = -1 \pm \sqrt{2\epsilon + 1}. \]

In the same manner let us specify the Schrödinger equation in variables \((z, z^*, \phi)\). With notation \((z, z^*) = (z, W)\), the metric \((86)\) reads
\[
f = e^{z+W}, \quad dS^2 = dt^2 - \frac{f-1}{f} d\phi^2 - \frac{1}{4f(f-1)} \left[ dz^2 + dW^2 + 2(2f-1) dzdW \right]. \tag{90}
\]

Allowing for relations
\[
g_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4f(f-1)} & -\frac{2f-1}{4f(f-1)} & 0 \\ 0 & -\frac{2f-1}{4f(f-1)} & \frac{1}{4f(f-1)} & 0 \\ 0 & 0 & 0 & -\frac{1}{f-1} \end{vmatrix}, \quad \det (g_{\alpha\beta}) = \frac{1}{4f^2}. \tag{91}
\]

the Schrödinger equation
\[
2\epsilon \Psi = \left[ \frac{1}{\sqrt{g}} \partial_\phi \sqrt{g} g^{\phi\phi} \partial_\phi + \frac{1}{\sqrt{g}} \partial_z \sqrt{g} g^{W\phi} \partial_z + \frac{1}{\sqrt{g}} \partial_W \sqrt{g} g^{W\phi} \partial_W + \frac{1}{\sqrt{g}} \partial_z \sqrt{g} g^{Wz} \partial_W + \frac{1}{\sqrt{g}} \partial_W \sqrt{g} g^{Wz} \partial_z \right] \Psi
\]

reduces to the form
\[
2\epsilon \Psi = \left[ -\frac{f}{f-1} \partial_\phi^2 + f \partial_z \frac{1}{f} \partial_z + f \partial_W \frac{1}{f} \partial_W - 2f \partial_z \frac{1}{2f} (2f-1) \partial_W - 2f \partial_W \frac{1}{2f} (2f-1) \partial_z \right] \Psi.
\]

Further, taking into account identities
\[
f = e^{z+W}, \quad \frac{1}{f} = e^{-z-W}, \quad f \partial_z \frac{1}{f} = -1, \quad f \partial_W \frac{1}{f} = -1, \quad -2f \partial_z \frac{1}{2f} (2f-1) = -1, \quad -2f \partial_W \frac{1}{2f} (2f-1) = -1.
\]

we translate the above equation to the form
\[
2\epsilon \Psi = \left[ -\frac{f}{f-1} \partial_\phi^2 + \partial_z^2 - \partial_z + \partial_W^2 - \partial_W + -(2f-1) \partial_z \partial_W - \partial_W - (2f-1) \partial_W \partial_z - \partial_z \right] \Psi; \tag{92}
\]
remembering on \( W = z^* \). Two plane wave satisfy eq. (92):

\[
\Psi^- = e^{-\alpha z}, \quad 2\epsilon e^{-\alpha z} = \left[ \partial_z^2 - 2\partial_z \right] e^{-\alpha z},
\]

\[
\Psi^+ = e^{-\alpha W}, \quad 2\epsilon e^{-\alpha W} = \left[ \partial_W^2 - 2\partial_W \right] e^{-\alpha W},
\]

\[
2\epsilon = \alpha^2 + 2\alpha, \quad \alpha = -1 \pm \sqrt{2\epsilon + 1}.
\]

(93)

At last, one can readily translate metrical tensor to the variables \((r, r^*, \phi)\):

\[
dS^2 = dt^2 + \frac{1}{4(1 + rr^*)^2} \frac{dr^2}{r^2} + \frac{1}{4(1 + rr^*)^2} \frac{dr^*2}{r^{*2}} - 2 \frac{(2rr^* + 1)}{4(1 + rr^*)^2} \frac{1}{rr^*} dr dr^* - \frac{rr^*}{1 + rr^*} d\phi^2,
\]

(94)

One could produce corresponding form of Schrödinger Hamiltonian and then verify that expressed in variables \( r, r^* \) plane wave are exact solution of quantum mechanical equation in these variables.

Let us summarize results.

The general method of separation of variables embraces the all plane wave solutions; the plane waves in Lobachevsky and Riemann space consist of a small part of the whole set of basis wave functions of Schrödinger equation.

In space of constant positive curvature \( S_3 \), a complex analog of orispherical coordinates of Lobachevsky space \( H_3 \) is introduced. To parameterize real space \( S_3 \), two complex coordinates \((r, z)\) must obey additional restriction in the form of the equation \( r^2 = e^z - z^* - e^{2z} \). The metrical tensor of space \( S_3 \) is expressed in terms of \((r, z)\) with additional constraint, or through pairs of conjugate variables \((r, r^*)\) or \((z, z^*)\); correspondingly exist three different representations for Schrödinger Hamiltonian. Shapiro plane waves are determined and explored as solutions of Schrödinger equation in complex horispherical coordinates of \( S_3 \). In particular, two oppositely directed plane waves may be presented as exponentials in conjugated coordinates. \( \Psi^- = e^{-\alpha z} \) and \( \Psi^+ = e^{-\alpha z^*} \). Solutions constructed are single-valued, finite, and continuous functions in spherical space and correspond to discrete energy levels.

References

[1] I.S. Shapiro. Dokl. Akad. Nauk SSSR. 1956. Vol 106. P. 647.

[2] I.S. Shapiro. expansion of the scattering amplitude in relativistic spherical functions. Phys. Lett. 1962. Vol.1. no 7. P. 253-255 .

[3] N.Ya. Vilenkin, Ya.A. Smorodinsky. Invariant expansions of relativistic amplitudes. 1964. JETF, Vol. 46. P. 1793-1808.

[4] A.A. Bogush, Yu.A. Kurochkin, V.S. Otechik. Scattering by Coulomb field in the Lobachevsky space. Doklady of the National Academy of Sciences of Belarus. 2003. Vol. 47. No 5. P. 54–57

[5] E.M. Bychkovskaya. About solutions of the Maxwell equations in the three-dimensional Lobachevsky space. Proc. of the National Academy of Sciences of Belarus. Series of Physical-Mathematical Sciences. 2006. No 5. P. 45-48.
[6] A.A. Bogush, Yu.A. Kurochkin, V.S. Otchik, E.M. Bychkovskaya. Analog of the plane electromagnetic waves in the Lobachevsky space. in: Proc. of the 5-th International Conference Bolyai-Gauss-Lobachevsky (BGL-5). Non-Euclidean Geometry in Modern Physics. Minsk, October 10 - 13, 2006. P. 111-115.

[7] M.N. Olevsky. Three-orthogonal coordinate systems in spaces of constant curvature, in which equation $\Delta_2 U + \lambda U = 0$ permits the full separation of variables. Mathematical collection. 1950. Vol. 27. P. 379-426 (in Russian).

[8] A.A. Bogush, V.C. Otchik, V.M. Red’kov. The Runge-Lenz vector for quantum Kepler problem in the space of positive constant curvature and complex parabolic coordinates. Pages 135-144 in: Proc. of 5th International Conference Bolyai-Gauss-Lobachevsky: Non-Euclidean Geometry In Modern Physics (BGL-5). 10-13 Oct 2006, Minsk, Belarus; arxiv:hep-th/0612178

[9] V.M. Red’kov. On Solutions of Schrodinger and Dirac Equations in Einstein Stationary Space-Time, Spherical, and Elliptical Models. NPCS. 2007. Vol. 10. No 4. P. 312-334.