Different Asymptotic Behavior versus Same Dynamical Complexity

Xueting Tian*

School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China

E-mail: xuetingtian@fudan.edu.cn; tianxt@amss.ac.cn

Abstract

For shifts of finite type or uniformly hyperbolic systems (expanding maps, etc.), we distinguish several periodic-like recurrent sets and find that they all carry full topological entropy and so do their various curious complementary sets. Moreover, we cooperate periodic-like recurrent sets with irregular sets and obtain lots of multi-fractal analysis for all continuous observable functions.

Roughly speaking, we combine various different “eyes” (i.e., observable functions and periodic-like recurrences) to observe the dynamical complexity and obtain a Refined Dynamical Structure for Recurrence Theory and Multi-fractal Analysis.
1 Introduction

In the theory of dynamical systems, i.e., the study of the asymptotic behavior of orbits \( \{T^n(x)\}_{n \in \mathbb{N}} \) when \( T : X \to X \) is a continuous map of a compact metric space \( X \), one may say that two fundamental problems are to understand how to partition different asymptotic behavior and how the points with same asymptotic behavior control or determine the complexity of system \( T \).

1.1 Entropy

Topological entropy is a classical concept to describe the dynamical complexity.

Let \( T : X \to X \) be a continuous map of a compact metric space \( X \). Let \( M(T, X) \), \( M_{\text{erg}}(T, X) \) denote the set of all \( T \)-invariant measures and \( T \)-ergodic measures respectively. Let \( x \in X \). The dynamical ball \( B_n(x, \varepsilon) \) is the set

\[
B_n(x, \varepsilon) := \{ y \in X | \max\{d(T^j(x), T^j(y)) | 0 \leq j \leq n-1 \} \leq \varepsilon \}.
\]

Let \( E \subseteq X \), and \( \mathcal{F}_n(E, \varepsilon) \) be the collection of all finite or countable covers of \( E \) by sets of the form \( B_m(x, \varepsilon) \) with \( m \geq n \). We set

\[
C(E; t, n, \varepsilon, T) := \inf \left\{ \sum_{B_m(x, \varepsilon) \in C} 2^{-tm} : C \in \mathcal{F}_n(E, \varepsilon) \right\},
\]

and

\[
C(E; t, \varepsilon, T) := \lim_{n \to \infty} C(E; t, n, \varepsilon, T).
\]

Then

\[
h_{\text{top}}(E, \varepsilon, T) := \inf \{ t : C(E; t, \varepsilon, T) = 0 \} = \sup \{ t : C(E; t, \varepsilon, T) = \infty \}
\]

and the topological entropy of \( E \) is defined as

\[
h_{\text{top}}(T, E) := \lim_{\varepsilon \to 0} h_{\text{top}}(E, \varepsilon, T).
\]

In particular, if \( E = X \), we also denote \( h_{\text{top}}(T, X) \) by \( h_{\text{top}}(T) \). It is known that if \( E \) is an invariant compact subset, then the topological entropy \( h_{\text{top}}(T, E) \) is same as the classical definition.

Let \( \xi = \{ V_i | i = 1, 2, \cdots, k \} \), be a finite partition of measurable sets of \( X \). The entropy of \( \nu \in M(T, X) \) with respect to \( \xi \) is

\[
H(\nu, \xi) := -\sum_{V_i \in \xi} \nu(V_i) \log \nu(V_i).
\]

We write \( T^{\wedge n} \xi := \wedge_{k \in \Lambda} T^{-k} \xi \). The entropy of \( \nu \in \mathcal{M}_T(X) \) with respect to \( \xi \) is

\[
h(T, \nu, \xi) := \lim_{n \to \infty} \frac{1}{n} H(\nu, T^{\wedge n} \xi),
\]

where \( H(\nu, T^{\wedge n} \xi) \) is the entropy of \( \nu \) with respect to the partition \( T^{\wedge n} \xi \).
and the *metric entropy* of $\nu$ is

$$h_\nu(T) := \sup_\xi h(T, \nu, \xi).$$

It is known that every set with totally full measure (i.e., being full measure for all invariant measures) has full topological entropy, deduced from classical Variational Principle. More precisely, let $Y \subseteq X$ be a set with totally full measure. From \cite{9} for any subsets $Y_1 \subseteq Y_2 \subseteq X$,

$$h_{\text{top}}(T, Y_1) \leq h_{\text{top}}(T, Y_2) \tag{1.1}$$

and for any ergodic measure $\mu$, if $\mu(Y) = 1$, then

$$h_\mu(T) \leq h_{\text{top}}(T, Y). \tag{1.2}$$

Thus by Variational Principle,

$$h_{\text{top}}(T, Y) \leq h_{\text{top}}(T, X) = h_{\text{top}}(T) = \sup_{\mu \in M(T, X)} h_\mu(T)$$

$$= \sup_{\mu \in M_{\text{erg}}(T, X)} h_\mu(T) \leq h_{\text{top}}(T, Y). \tag{1.3}$$

### 1.2 Periodic and Periodic-like Recurrence

One important way to partition points with different asymptotic behavior is according to the recurrence property.

In the classical study of general dynamical systems, an important concept is non-wandering points. A point $x \in X$ is called *wandering*, if there is a neighborhood $U$ of $x$ such that the sets $T^{-n} U$, $n \geq 0$, are mutually disjoint. Otherwise, $x$ is called non-wandering. Let $\Omega(T)$ denote the set of all non-wandering points, called *non-wandering set*. For shifts of finite type or general transitive maps, the non-wandering set is the whole space. The interesting action of $T$ takes place in $\Omega(T)$ and recall that from \cite{36} $\Omega(T)$ is always invariant, compact, carries totally full measure and owns the whole complexity of the system

$$h_{\text{top}}(T) = h_{\text{top}}(T, \Omega(T))$$

so that one can always consider the subsystem $T : \Omega(T) \to \Omega(T)$ to replace the original system $T : X \to X$. For transitive maps, it is the same. However, it is an interesting to ask for general dynamical systems, how about the dynamical complexity of $X \setminus \Omega(T)$? Unfortunately, in this paper the systems studied are mainly transitive so here we still face the case $X = \Omega(T)$.

The set $\Omega(T)$ consists of those points with a weak recurrence property. Now let us recall the concept of recurrent point. It is known that recurrent points play important roles in the ergodic theory of dynamical systems. Given $x \in X$, let $\omega_T(x)$ denote the $\omega$-limit set. We call $x \in X$ to be *recurrent*, if

$$x \in \omega_T(x).$$
Let $\text{Rec}(T)$ denote the set of all recurrent points. It is known that recurrent set has totally full measure so that by (1.3)

**Theorem.** *For any continuous dynamical system $T : X \to X$ of a compact metric space $X$, the recurrent set $\text{Rec}(T)$ has full topological entropy.*

In the study of (smooth or topological) dynamical systems, many people pay attention to refine recurrent set according to the ‘recurrent frequency’. A standard and important kind of recurrent point with same asymptotic behavior is periodic point, which returns itself through finite iterates. Let $\text{Per}(T)$ denote the set of periodic points. Then

$$\text{Per}(T) \subseteq \text{Rec}(T).$$

A fundamental question in dynamical systems is to search the existence of periodic points. For shifts of finite type(or uniformly hyperbolic systems), it is well known that the set of periodic points is dense in the whole space and moreover, it is countable and thus has zero entropy. However, a classical result states that the topological entropy can be characterized by the exponential growth of periodic points with same period. More precisely,

$$h_{\text{top}}(T) = \lim_{n \to \infty} \frac{1}{n} \log \# P_n(T)$$

where $P_n(T) = \{ x \mid T^n(x) = x \}$ and $\#A$ denotes the cardinality of the set $A$. Roughly speaking, periodic points have same complexity as the system itself from the viewpoint of two different ways to interpret complexity. This result holds for Axiom A diffeomorphisms in any dimension(see [8]), but this is a somewhat special situation. It is well known that Axiom A diffeomorphisms are not dense in $\text{Diff}^1(M)$ (the space of all diffeomorphisms on a compact Riemannian manifold $M$). In 2004 Kaloshin [17] showed that in general $\#P_n(T)$ can grow much faster than entropy. Moreover, it is well known that for $C^1$ generic diffeomorphisms, all periodic points are hyperbolic so that countable and they form a dense subset of the non-wandering set(from classical Kupka-Smale theorem and Mañé’s Ergodic Closing lemma of the smooth ergodic theory). For nonuniformly hyperbolic systems, many people studied the existence of periodic point by showing closing lemma, for example, see [18]. In particular, an interesting result from [18] is that any surface diffeomorphism with positive entropy always carries a lot of periodic points. All in all, periodic points have been studied more and more in the research of modern dynamical systems.

In general, it is well known that there are lots of topological dynamical systems without periodic points. The standard example is minimal systems with positive entropy(for example, minimal homeomorphisms on torus with positive entropy [28] and minimal subshifts with positive entropy in [13][14]). So some generalizations of periodic points are introduced. One such kind ‘periodic-like’ point is almost periodic point. A point $x \in X$ is *almost periodic*, if for every open neighborhood $U$ of $x$, there exists $N > 0$ such that $f^k(x) \in U$ for some $k \in [n, n + N]$ and every integer $n \geq 1$. Let $A(T)$ denote the set of all almost periodic points. It is well known that a point $x$ is almost periodic if and only if $x$ is minimal, i.e., $x \in \omega_T(x)$ and $\omega_T(x)$ is minimal. Here an invariant set $E \subseteq X$ is called minimal, if for every point $y \in E$, $\omega_T(y) = E$. 

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Remark that for any uniquely ergodic system, the support of the invariant measure must be minimal. However, there are minimal invariant sets which are not uniquely ergodic\cite{23}. Note that \(E \subseteq X\) is a minimal invariant subset if and only if the support of any invariant measure supported on \(E\) coincides with \(E\). By Zorn’s lemma, one can show that any dynamical system contains at least one minimal invariant subset. So different from periodic points, almost periodic points naturally exist and thus played important roles in the study all topological dynamical systems. Moreover, constructions of minimal examples are studied a lot by many researchers. For homeomorphisms, there are many examples of subshifts which are strictly ergodic and has positive entropy \cite{13, 14}. For systems on manifolds, from \cite{28} there are minimal homeomorphisms on 2-torus with positive entropy and it was proved in \cite{5} that any compact manifold of dimension \(d \geq 2\) which carries a minimal uniquely ergodic homeomorphism also carries a minimal uniquely ergodic homeomorphism with positive topological entropy. M. Herman asked whether, for diffeomorphisms, positive topological entropy was compatible with minimality or strict ergodicity. It was constructed in \cite{16} a 4-dimensional example of a minimal (but not strictly ergodic) diffeomorphism with positive topological entropy. For any \(C^{1+\alpha}\) surface diffeomorphism with positive entropy, by classical Pesin theory\cite{18} there is some Smale horseshoe which implies existence of periodic points so that the system is not minimal.

For any system (for example, hyperbolic systems, shifts of finite type etc.) with an ergodic measure \(\mu\) whose support is not minimal, the almost periodic set \(A(T)\) is of \(\mu\)-zero measure so that in general \(A(T)\) is not a periodic-like set with totally full measure. More precisely, by contradiction, \(\mu(A(T)) > 0\). By ergodicity and invariance of \(A(T)\), \(\mu(A(T)) = 1\). By ergodicity, there is \(z \in A(T)\) such that the orbit of \(z\) is dense in the support of \(\mu\), denoted by \(S_\mu\). This implies \(S_\mu\) is consisted of just the minimal subset \(\omega_T(z)\), which contradicts the original assumption. Thus, if one want to find some kind of periodic-like points with totally full measure, we need to generalize almost periodic point to be more general. Zhou etc. (see \cite{38, 39, 40}) introduced such two more general concepts of ‘periodic-like’ points which have totally full measure for all dynamical systems.

One is called weakly almost periodic and another is more weaker called quasi-weakly almost periodic. Different ‘recurrent frequency’ determines different asymptotic behavior. If \(E \subseteq X\) is nonempty and \(x \in X\), define

\[
P_x(E) := \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)) \quad \text{and} \quad \overline{P}_x(E) := \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)).
\]

In other words,

\[
\overline{P}_x(E) = \liminf_{n \to \infty} \Upsilon_n(x)(E), \quad P_x(E) = \limsup_{n \to \infty} \Upsilon_n(x)(E),
\]

where \(\Upsilon_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}\) and \(\delta_y\) is the Dirac probability measure supported at \(y \in M\). If \(P_x(E) = \overline{P}_x(E)\), we denote by \(P_x(E)\), which means the probability of the orbit of \(x\) enters in \(E\). Let \(V_\varepsilon(x)\) denote \(\varepsilon\)-neighborhood of \(x\), i.e., \(V_\varepsilon(x) = \{y \in M \mid d(x, y) < \varepsilon\}\).
Definition 1.1. ((quasi-)weakly almost periodic) We call $x$ to be weakly almost periodic point, if for any $\varepsilon > 0$, 
$$\mathcal{P}_x(V_\varepsilon(x)) > 0.$$ 
$x$ is called to be quasi-weakly almost periodic point, if for any $\varepsilon > 0$, 
$$\overline{\mathcal{P}_x}(V_\varepsilon(x)) > 0.$$ 

Let $QW(T)$ and $W(T)$ denote the set of weakly almost periodic point and quasi-weakly almost periodic point respectively. Remark that 
$$\Omega(T) \supseteq \text{Rec}(T) \supseteq QW(T) \supseteq W(T) \supseteq A(T) \supseteq \text{Per}(T) \quad (1.4)$$ 
and the first three sets are of full measure for any invariant measure. In fact, for any ergodic measure $\mu$, let $G(\mu)$ denote the set of all points satisfying that $\Upsilon_n(x)$ converges to $\mu$ in weak* topology. By Birkhoff ergodic theorem $G(\mu)$ is of $\mu$ full measure. Let $S_\mu$ denote the support of $\mu$, meaning that $S_\mu$ is the minimal compact invariant set with $\mu$ full measure. So $S_\mu \cap G(\mu)$ is of $\mu$ full measure and every $x \in S_\mu \cap G(\mu)$ satisfies $\Upsilon_n(x) \to \mu$ in weak* topology which implies that 
$$\mathcal{P}_x(V_\varepsilon(x)) = \liminf_{n \to \infty} \Upsilon_n(x)(V_\varepsilon(x)) \geq \mu(V_\varepsilon(x)) > 0.$$ 
That is, for any ergodic measure $\mu$, $x$ a.e. $x$ belongs to $W(T)$. Thus by Ergodic Decomposition theorem, $W(T)$ has full measure for any invariant measure and so does $QW(T)$. Recall that $A(T)$ maybe not have totally full measure(for example, shifts of finite type and uniformly hyperbolic systems), thus the concepts of weakly and quasi-weakly almost periodic is more general than almost periodic from the probabilistic perspective. By (1.3) and (1.4) 

**Theorem.** For any continuous dynamical system $T : X \to X$ of a compact metric space $X$, each one of $\text{Rec}(T)$, $QW(T)$ and $W(T)$ carries full topological entropy.

In particular, note that the sets $QW(T) \setminus W(T)$ and $\text{Rec}(T) \setminus QW(T)$ have zero measure for any invariant measure. Many people pay attention to which system has nonempty gap between two periodic-like level-sets(see [10, 21, 15, 37]). They are mainly interested in which system satisfies $QW(T) \setminus W(T) \neq \emptyset$. For generic diffeomorphisms and systems with Bowen’s specification such as shifts and hyperbolic systems, $QW(T) \setminus W(T)$ always contains a dense $G_\delta$ subset(see Theorem 5.8 and 5.10 below). This is a characterization from topological or geometric perspective so that the concept of quasi-weakly almost periodic is more general than weakly almost periodic point. In this paper we are mainly concerned with one deeper question: how about the dynamical complexity of such periodic-like gap-sets:

- $\text{Rec}(T) \setminus QW(T)$: it is the complementary of quasi-weakly almost periodic set restricted on recurrent set $\text{Rec}(T)$. It is always zero measure for any invariant measure.
In present paper for a series of dynamical systems, we will give a positive answer to the complementary set of $QW(T)$, $X \setminus QW(T)$. It carries full topological entropy which can be deduced from Theorem 2.1 (C). If one consider disjoint level-sets $\{Y_i\}_{i=1}^n$, from [6] we know that the topological entropy satisfies

$$h_{top}(T, \bigcup_{i=1}^n Y_i) = \sup_{1 \leq i \leq n} h_{top}(T, Y_i).$$

(1.5)

So, either $Rec(T) \setminus QW(T)$ carries full topological entropy or $X \setminus Rec(T)$ carries full topological entropy. This can be as a partial answer to $Rec(T) \setminus QW(T)$. However, it should be right for shifts of finite type, because specification property implies topological mixing so that in this case the main used idea of present paper([26] by Pfister and Sullivan, see Lemma 4.9 below) should be also right restricted on the set of transitive points. This needs to adapt the proof of [26] to obtain a refined result restricted on the set of transitive points. This is a straightforward argument but complex work and $Rec(T) \setminus QW(T)$ is just one case of our paper so that here we omits the details.

• $QW(T) \setminus W(T)$: it is called ‘truly’ quasi-weakly almost periodic set and its element is called truly quasi-weakly almost periodic point. It always has zero measure for any invariant measure. In present paper for a series of dynamical systems, we will give a positive answer to show that $QW(T) \setminus W(T)$ carries full topological entropy, see Theorem 2.1 (B).

• $W(T) \setminus Per(T)$: every element is weakly almost periodic but ‘truly’ non-periodic. This set naturally has full topological entropy:

**Theorem.** For any continuous dynamical system $T : X \to X$ of a compact metric space $X$ and a set $Y$ with totally full measure,

$$Y \setminus Per(T)$$

(1.6)

In particular, $W(T) \setminus Per(T)$ carries full topological entropy and by (1.4) so do $QW(T) \setminus Per(T)$ and $Rec(T) \setminus Per(T)$.

In fact, it is trivial for systems with zero topological entropy so that we only consider the systems with positive topological entropy. Note that periodic measures always carry zero topological entropy. By classical Variational Principle, we only need to consider ergodic measures which are not periodic. Then for any ergodic measure which is not periodic, $W(T) \setminus Per(T)$ always has full measure. Otherwise, there is an ergodic measure $\mu$ which is not periodic such that $\mu(Per(T)) > 0$. By ergodicity and invariance of $Per(T)$, $\mu(Per(T)) = 1$. By ergodicity, there is $z \in Per(T)$ such that the orbit of $z$ is dense in the support of $\mu$, $S_\mu$. This implies $S_\mu$ is consisted of just the periodic orbit of $z$, which contradicts the original assumption. Similar as the discussion of (1.3),

$$h_{top}(T, W(T) \setminus Per(T)) \leq h_{top}(T, X) = h_{top}(T) = \sup_{\mu \in M(T,X)} h_{\mu}(T)$$

$$= \sup_{\mu \in M_{erg}(T,X)} h_{\mu}(T) \leq \sup_{\mu \in M_{erg}(T,X) \setminus M_{per}(X,T)} h_{\mu}(T) \leq h_{top}(T, W(T) \setminus Per(T)).$$
Here $M_p(T, X)$ denote the set of all $T$-periodic measures. Recall that $W(T)$ and $QW(T)$ have full measure for any invariant measure. Then $W(T) \setminus Per(T)$ carries full topological entropy and by \([13]\) so do $QW(T) \setminus Per(T)$ and $Rec(T) \setminus Per(T)$.

Note that

$$W(T) \setminus Per(T) = (W(T) \setminus A(T)) \sqcup (A(T) \setminus Per(T)).$$

by the above discussion of $W(T) \setminus Per(T)$ and \([15]\).

**Theorem.** For any continuous dynamical system $T : X \to X$ of a compact metric space $X$, at least one of $W(T) \setminus A(T)$ and $A(T) \setminus Per(T)$ carries full topological entropy.

- $W(T) \setminus A(T)$: For shifts of finite type (or uniformly hyperbolic systems etc.), it carries full topological entropy. This can be deduced from Theorem 3.1. In fact, transitive points in such systems are not minimal. So

$$W(T) \setminus A(T) \supseteq W(T) \cap D(T),$$

where $D(T)$ denotes the set of all transitive points.

- $A(T) \setminus Per(T)$: For every minimal dynamical system $T : X \to X$, $A(T) = X$ and thus from \([16]\) $A(T) \setminus Per(T)$ has full topological entropy. Moreover, we have

**Theorem.** For shifts of finite type (or uniformly hyperbolic systems etc.), $A(T) \setminus Per(T)$ carries full topological entropy. In particular, the almost periodic set $A(T)$ carries full topological entropy.

It is not difficult to prove. In fact, denote the set of all ergodic measures of all uniquely ergodic minimal subshifts with positive entropy by $M_{\ast \text{erg}}^{\ast}(T, X)$. Then

$$\bigcup_{\mu \in M_{\ast \text{erg}}^{\ast}(T, X)} S_{\mu} \subseteq A(T) \setminus Per(T).$$

Recall a result from \([13, 14]\) that for any shifts of finite type, there exist uniquely ergodic minimal subshifts with any given entropy. So

$$h_{\text{top}}(T) = \sup_{\mu \in M_{\ast \text{erg}}^{\ast}(T, X)} h(T, S_{\mu}) = \sup_{\mu \in M_{\ast \text{erg}}^{\ast}(T, X)} h_{\mu}(T).$$

Thus similar as the discussion of \([13]\),

$$h_{\text{top}}(T, A(T) \setminus Per(T)) \leq h_{\text{top}}(T, X) = h_{\text{top}}(T) = \sup_{\mu \in M_{\ast \text{erg}}^{\ast}(T, X)} h(T, S_{\mu}) \leq h_{\text{top}}(T, A(T) \setminus Per(T)). \quad (1.7)$$

All in all, for shifts of finite type, all the gap-sets of different periodic-like points carry full topological entropy so that each one of generalized periodic-like points is non-trivial and essentially important in the role to determine the dynamical complexity.
### 1.3 Regularity and Irregularity

Another important way to observe points is from Birkhorff Ergodic theorem, called regular points. Let \( \phi : X \to \mathbb{R} \) be a continuous observable function. A point \( x \in X \) is called to be \( \phi \)-regular, if the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x))
\]

exists. Define the \( \phi \)-regular set to be the set of all \( \phi \)-regular points, that is,

\[
R_{\phi}(T) := \{ x \in X | x \text{ is } \phi \text{ - regular} \}.
\]

It is known from Birkhoff ergodic theorem, \( R_{\phi}(T) \) has totally full measure so that by (1.3) any \( \phi \)-regular points essentially determine the dynamical complexity of any system \( T : X \to X \).

**Theorem.** For any continuous dynamical system \( T : X \to X \) of a compact metric space \( X \) and any continuous function \( \phi : X \to \mathbb{R} \), \( R_{\phi}(T) \) carries full topological entropy.

In contrast to the \( \phi \)-regular points, its complementary set, called irregular points and denoted by \( I_{\phi}(T) \), can be considered as carrying same asymptotic behavior. Their Birkhoff average does not exist under the observation of \( \phi \). By Birkhoff’s ergodic theorem, the irregular set is always of zero measure for any invariant measure. However, in recent several years many people have focused on studying the dynamical complexity of irregular set from different sights, for example, in the sense of dimension theory and topological entropy(or pressure) etc. Pesin and Pitskel [24] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. Barreira, Schmeling, etc. studied the irregular set in the setting of shifts of finite type and beyond, see [4, 2, 33] etc. Recently, Thompson shows in [34, 35] that the every \( \phi \)-irregular set \( I_{\phi}(T) \) either is empty or carries full topological entropy (or pressure) when the system satisfies (almost) specification, which is inspired from [26] by Pfister and Sullivan and [33] by Takens and Verbitskiy.

**Theorem.** Let \( T \) be a continuous map of a compact metric space \( X \) with (almost) specification. Then for any continuous function \( \phi : X \to \mathbb{R} \), the \( \phi \)-irregular set \( I(\phi, T) \) either is empty or carries full topological entropy.

A point \( x \in X \) is called (essentially) regular, if it is \( \phi \)-regular for any continuous function \( \phi : X \to \mathbb{R} \). Denote the set of all regular points by \( R(T) \), called regular set, then

\[
R(T) = \bigcap_{\phi \in C^0(X)} R_{\phi}(T),
\]

where \( C^0(X) \) denotes the space of all continuous functions on \( X \). By Birkhoff ergodic theorem, \( R(T) \) still has totally full measure so that by (1.3) regular points essentially determine the dynamical complexity of any system \( T : X \to X \).
Theorem. For any continuous dynamical system \( T : X \to X \) of a compact metric space \( X \), \( R(T) \) carries full topological entropy.

Let \( I(T) \) denotes the complementary set of regular set, called irregular set, then

\[
I(T) = \bigcup_{\phi \in C^0(X)} I_\phi(T).
\]

\( I(T) \) contains any \( \phi \)-irregular set so that it has similar results as \([1, 2, 34, 35]\) etc.

Theorem. Let \( T \) be a continuous map of a compact metric space \( X \) with (almost) specification. Then the irregular set \( I(T) \) either is empty or carries full topological entropy.

Moreover, if further assume that the system is not uniquely ergodic, then \( I(T) \) is not empty. In fact, by assumption, there are two different invariant measures \( \mu_1, \mu_2 \). By weak* topology, there is a continuous function \( \phi : X \to \mathbb{R} \) such that

\[
\int \phi d\mu_1 \neq \int \phi d\mu_2.
\]

Then

\[
\inf_{\mu \in M(T, X)} \int \phi(x) d\mu < \sup_{\mu \in M(T, X)} \int \phi(x) d\mu.
\]

Under the assumption of (almost) specification, from \([35, 34]\) this implies \( I_\phi(T) \neq \emptyset \) and so does \( I(T) \) (see the paragraph behind of Lemma 2.1 in \([35]\), as a corollary of Lemma 2.1 and Theorem 4.1 there, P. 5397, and see Lemma 1.6 of \([34]\) for the case of specification). Thus we have

Theorem. Let \( T \) be a continuous map of a compact metric space \( X \) with (almost) specification. If \( T \) is not uniquely ergodic, then the irregular set \( I(T) \) either carries full topological entropy.

1.4 Overlap of Regularity and Periodic-like Recurrence

Regularity and recurrence are two different “eyes” to study asymptotic behavior. Inspired from above analysis, a natural idea is to consider the recurrence and (ir)regularity simultaneously. Roughly speaking, under the observation of two “eyes”, we will obtain more deeper results.

Firstly, we try to cooperate periodic-like points with regular set. Remark that

\[
R(T) \cap Rec(T) \cap QW(T) \cap W(T) = R(T) \cap W(T)
\]

has totally full measure so that for any dynamical systems, it always carries full topological entropy. In general, \( R(T) \cap Rec(T) \subseteq W(T) \) (see Corollary 4.2) and obviously \( Per(f) \subseteq \)
From (1.6) for any dynamical system, \( R(T) \setminus \text{Per}(T) \) always carries full topological entropy. So we are just to cooperate \( R(T) \) with \( A(T) \). Remark that

\[
\bigcup_{\mu \in \mathcal{M}_{erg}(T,X)} S_\mu \subseteq R(T) \cap (A(T) \setminus \text{Per}(T)).
\]

So similar as the discussion of \( A(T) \setminus \text{Per}(T) \) (see (1.7)), we have

**Theorem.** For shifts of finite type (or uniformly hyperbolic systems etc.), \( R(T) \cap (A(T) \setminus \text{Per}(T)) \) carries full topological entropy and so does \( R(T) \cap A(T) \).  

Recall that the maximal entropy measure \( \mu \) for shifts of finite type is ergodic and has full support. Let \( G(\mu) \) denote the set of all points satisfying that \( \Upsilon_n(x) \) converges to \( \mu \) in weak* topology. By Birkhoff ergodic theorem \( G(\mu) \) is of \( \mu \) full measure. By ergodicity, it is well known that the set of transitive points, denoted by \( D(T) \), has \( \mu \) full measure. So the intersection \( G(\mu) \cap D(T) \) still has \( \mu \) full measure. By (1.1) and (1.2),

\[
h_{\text{top}}(T, G(\mu) \cap D(T)) \leq h_{\text{top}}(T, X) = h_{\text{top}}(T) = h_\mu(T) \leq h_{\text{top}}(T, G(\mu) \cap D(T)).
\]

Remark that \( G(\mu) \cap D(T) \subseteq R(T) \cap D(T) \subseteq R(T) \setminus A(T) \). Then by (1.1)

**Theorem.** For shifts of finite type (or uniformly hyperbolic systems etc.), each one of \( R(T) \setminus A(T) \) and \( R(T) \cap D(T) \) carries full topological entropy.

In this section we mainly consider periodic-like points and regular or irregular points. For cooperating these points with transitive set, we will deal with independently in subsection 3.1.

It is still unknown that for shifts of finite type, whether \( A(T) \setminus R(T) \) (or \( A(T) \cap I(T) \)) carries full topological entropy? If there is a result different from [13, 14] that for any shift of finite type, there exist minimal but not uniquely ergodic subshifts with any given entropy, then the idea of present paper works. However, due to my best knowledge, it is still an open problem.

Secondly, let us to cooperate other periodic-like points with irregular set. Using weakly almost periodic and quasi-weakly almost periodic to classify irregular points or using irregular points to classify different periodic-like recurrent points, we study the dynamical complexity on three types of irregular point, called irregular level-sets, which exhibit separate asymptotic behavior:

**I.** We will use different periodic-like recurrence to partition irregular set \( I(T) \):

- \( W(T) \setminus R(T) \): it is the set of all points which are weakly almost periodic but not regular. For a series of dynamical systems, we will give a positive answer to show that it carries full topological entropy, see Theorem 2.1 (A).

- \( QW(T) \setminus W(T) \): it is same as above subsection, see Theorem 2.1 (B). In particular, it is contained in the irregular set \( I(T) \), see Corollary 4.2.
• $I(T) \setminus QW(T)$: it is the set of all points which are irregular but not quasi-weakly almost periodic. In other words, it is the complementary of quasi-weakly almost periodic set restricted on irregular set $I(T)$. For a series of dynamical systems, we will give a positive answer to show that it carries full topological entropy, see Theorem 2.1 (C).

At the beginning to study such recurrent and irregular gap-sets, from (1.5) we expect that one irregular level-set has same complexity as the dynamical system itself but others not. In this case, the expected irregular level-set should be the genuine core of irregular points, which essentially determines the dynamical complexity of whole system. But surprisingly we find that every irregular level-set is expected. That is, all irregular level-sets have same complexity as the dynamical system itself. In other words, each one of them can determine the dynamical complexity of the whole system.

Now let us to cooperate periodic-like points with $\phi$—regular set under the observation of a continuous function.

(II). We will use different recurrence to partition $\phi$—regular set:

- $R_\phi(T) \cap W(T) \setminus R(T)$: Theorem 2.2 (A’).
- $R_\phi(T) \cap QW(T) \setminus W(T)$: Theorem 2.2 (B’).
- $R_\phi(T) \cap I(T) \setminus QW(T)$: Theorem 2.2 (C’).

For a series of dynamical systems, under any observable continuous function, each one of them carries full topological entropy simultaneously. Remark that $R(T) \subseteq R_\phi(T)$ implies

$$R(T) \cap (A(T) \setminus Per(T)) \subseteq R_\phi(T) \cap (A(T) \setminus Per(T))$$

and

$$R(T) \setminus A(T) \subseteq R_\phi(T) \setminus A(T).$$

Thus

**Theorem.** For shifts of finite type (or uniformly hyperbolic systems etc.) and any continuous function $\phi$, each one of $R_\phi(T) \cap (A(T) \setminus Per(T))$ and $R_\phi(T) \setminus A(T)$ carries full topological entropy.

Now let us to cooperate periodic-like points with $\phi$—irregular set under the observation of a continuous function.

(III). We will use different recurrence to partition $\phi$—irregular set,

- $I_\phi(T) \cap W(T) \setminus R(T)$: Theorem 2.2 (A”).
- $I_\phi(T) \cap QW(T) \setminus W(T)$: Theorem 2.2 (B”).
- $I_\phi(T) \cap I(T) \setminus QW(T)$: Theorem 2.2 (C”).

For a series of dynamical systems, under any observable continuous function, either all of them are empty sets or each one of them carries full topological entropy simultaneously.
In general the fundamental idea is to construct lots of irregular points such that you can use ‘so many points’ to prove full topological entropy. However, in the new requirements, we need to construct lots of irregular points which are irregular and satisfy additional conditions. Inspired from [26] by Pfister and Sullivan, we can use variational principle to solve the problem directly.

We believe these phenomenons are adaptable to a large class of dynamical systems such as (non-)uniformly hyperbolic systems and topological dynamical systems with (almost) specification etc. But for consideration of making sure the paper clear, concise and accurate, here in this paper we mainly focus on one-sided shifts of finite type (or two-sided shifts of finite type, systems conjugated to shifts of finite type such as hyperbolic elementary sets of Axiom A systems and expanding maps, etc.). That is, in this paper we always assume that

Let $T : X \rightarrow X$ be the one-sided shift on $X = \prod_{n=0}^{+\infty} Y$ where $Y = \{0, 1, \cdots, k - 1\}$ and $k$ is a positive integer $\geq 2$.

2 Results

Before stating our main results we need to recall some concepts.

A set $E$ is said to be a center of attraction of $T$ with respect to $X_0 \subseteq X$, if $E$ is a closed invariant set, and for any $x \in X_0$ and $\varepsilon > 0$, the limit

$$P_x(V(E, \varepsilon)) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x))$$

exists and equals to 1, where $V(E, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $E$, i.e., $V(E, \varepsilon) = \{y \in X | d(E, y) < \varepsilon\}$. A set $E$ is called minimal center of attraction relative to $X_0$ if $E$ is a center of attraction relative to $X_0$ but no proper subset of $E$ also is. Denote by $C(X_0)$ the minimal center of attraction relative to $X_0$. In particular, let

$$C_x := C(\{x\}), x \in X.$$

Let $M_x(T)$ be the set of all limits of $\Upsilon_n(x)$ in weak* topology. Recall that $S_\mu$ denotes the support of a measure $\mu$.

2.1 Partition of irregular set

Remark that

$$I(T) = (W(T) \setminus R(T)) \sqcup (QW(T) \setminus W(T)) \sqcup (I(T) \setminus QW(T)),$$

see Figure [I]. For these three kinds of irregular subset, all of them are of zero measure for any invariant measure. From the viewpoint of irregular point, they have same asymptotic behavior but from the viewpoint of weakly almost periodic and quasi-weakly almost
periodic they have three different kinds of asymptotic behavior and each one behaves its own asymptotic feature. We will show that these three subsets of irregular set have same topological entropy as the system itself. These observations will help us more better to understand the dynamical complexity on various irregular level-sets with different asymptotic behavior.

\[
\begin{array}{|c|c|c|}
\hline
W(T) \setminus R(T) & QW(T) \setminus W(T) & I(T) \setminus QW(T) \\
\hline
\end{array}
\]

\[ I(T) \]

Figure 1: Classification of Irregular set.

**Theorem 2.1. (Main Theorem I)** All following irregular level-sets, 
\[ W(T) \setminus R(T), QW(T) \setminus W(T), I(T) \setminus QW(T), \]

have full topological entropy. That is,
\[
(A) \quad h_{\text{top}}(T, W(T) \setminus R(T)) = h_{\text{top}}(T); \\
(B) \quad h_{\text{top}}(T, QW(T) \setminus W(T)) = h_{\text{top}}(T); \\
(C) \quad h_{\text{top}}(T, I(T) \setminus QW(T)) = h_{\text{top}}(T). 
\]

Moreover, \( QW(T) \setminus W(T) \) can be divided into two disjoint subsets in which each one has full topological entropy. More precisely,
\[
(B_1) \quad h_{\text{top}}(T, \{x \in QW(T) \setminus W(T) | \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x \}) = h_{\text{top}}(T); \\
(B_2) \quad h_{\text{top}}(T, \{x \in QW(T) \setminus W(T) | \forall \mu \in M_x(T) \text{ s.t. } S_\mu \neq C_x \}) = h_{\text{top}}(T). 
\]

**2.2 A question by Zhou and Feng**

There is an open problem in \[40\] by Zhou and Feng that whether the set
\[
\{QW(T) \setminus W(T) | \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x \} \neq \emptyset ?
\]
Let us denote this set by $V$ for convenience. It has been solved positively by constructing examples, see \cite{21, 15, 37}. From our Theorem 2.1, for a series of dynamical systems such as shifts of finite type (or uniformly hyperbolic system etc.), $V$ is not only nonempty but also has full topological entropy (and so does its complementary set in $QW(T) \setminus W(T)$). In other words, $V$ has very strong dynamical complexity which reaches the complexity of dynamical system itself. In particular, we know that positive topological entropy implies $V$ has uncountable elements. So our Theorem 2.1 can be as a strong answer for Zhou and Feng’s open problem.

### 2.3 Partition of irregular set under an observable function

In this subsection we study the irregular level-sets under a fixed observable function.

Recall that for any $\phi \in C^0(X)$, $R_\phi(T) \supseteq R(T)$ and $R_\phi(T)$ has same complexity as the system itself. So it is natural to ask how about $R_\phi(T) \setminus R(T)$. Now we consider $\phi$-irregular set $I_\phi(T)$ and its complementary set in $I(T)$, i.e., $R_\phi(T) \setminus R(T)$ ($= I(T) \setminus I_\phi(T)$). That is, for any $\phi \in C^0(X)$,

$$I(T) = I_\phi(T) \sqcup (R_\phi(T) \setminus R(T)).$$

Similar as Theorem 2.1 we can divide $I_\phi(T)$ and $R_\phi(T) \setminus R(T)$ into three kinds of subset respectively, see Figure 2. The following theorem studies the dynamical complexity on these irregular level-sets, which implies Theorem 2.1 and thus can be as a generalization.

![Figure 2: Classification of Irregular points under an observable function $\phi$.](image)

**Theorem 2.2. (Main Theorem II)** For any $\phi \in C^0(X)$,

1. all following irregular level-sets,

$$R_\phi(T) \cap W(T) \setminus R(T), \ R_\phi(T) \cap QW(T) \setminus W(T), \ R_\phi(T) \cap I(T) \setminus QW(T),$$

have full topological entropy. That is, (A')

$$h_{\text{top}}(T, R_\phi(T) \cap W(T) \setminus R(T)) = h_{\text{top}}(T);$$

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In particular, \( h_{\text{top}}(T, R_\phi(T) \cap QW(T) \setminus W(T)) = h_{\text{top}}(T) \);

Moreover, \( h_{\text{top}}(T, R_\phi(T) \cap I(T) \setminus QW(T)) = h_{\text{top}}(T) \).

(B') 
\[
h_{\text{top}}(T, R_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x \}) = h_{\text{top}}(T); \]

(C') 
\[
h_{\text{top}}(T, R_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \forall \mu \in M_x(T) \text{ s.t. } S_\mu \neq C_x \}) = h_{\text{top}}(T). \]

In particular,

(D) 
\[
h_{\text{top}}(T, R_\phi(T) \setminus R(T)) = h_{\text{top}}(T, I(T) \setminus I_\phi(T)) = h_{\text{top}}(T). \]

Moreover, \( R_\phi(T) \cap QW(T) \setminus W(T) \) can be divided into two disjoint subsets in which each one has full topological entropy. More precisely, 

(B1') 
\[
h_{\text{top}}(T, R_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x \}) = h_{\text{top}}(T); \]

(B2') 
\[
h_{\text{top}}(T, R_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \forall \mu \in M_x(T) \text{ s.t. } S_\mu \neq C_x \}) = h_{\text{top}}(T). \]

(2) if \( I_\phi(T) \neq \emptyset \), then all following irregular level-sets, 

\( I_\phi(T) \cap W(T) \setminus R(T), \ I_\phi(T) \cap QW(T) \setminus W(T), \ I_\phi(T) \cap I(T) \setminus QW(T), \)

have full topological entropy. That is, 

(A'') \[
h_{\text{top}}(T, I_\phi(T) \cap W(T) \setminus R(T)) = h_{\text{top}}(T); \]

(B'') \[
h_{\text{top}}(T, I_\phi(T) \cap QW(T) \setminus W(T)) = h_{\text{top}}(T); \]

(C'') \[
h_{\text{top}}(T, I_\phi(T) \cap I(T) \setminus QW(T)) = h_{\text{top}}(T). \]

In particular,

\[
h_{\text{top}}(T, I_\phi(T)) = h_{\text{top}}(T). \]

Moreover, \( I_\phi(T) \cap QW(T) \setminus W(T) \) can be divided into two disjoint subsets in which each one has full topological entropy. More precisely, 

(B1'') 
\[
h_{\text{top}}(T, I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x \}) = h_{\text{top}}(T); \]

(B2'') 
\[
h_{\text{top}}(T, I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \forall \mu \in M_x(T) \text{ s.t. } S_\mu \neq C_x \}) = h_{\text{top}}(T). \]

Remark 2.3. Remark that in [34] \( \phi \)-irregular set \( I_\phi(T) \) either is empty or has full topological entropy. \( I_\phi(T) \) is just a subset of irregular set \( I(T) \). Here this theorem shows that its complementary set in \( I(T) \) (i.e., \( I(T) \setminus I_\phi(T) \)) also has full topological entropy.
2.4 Partition of level-sets of irregular set under an observable function

Some classical results are known on multi-fractal analysis of Birkhoff averages, for example see [26, 33, 22]. More precisely, firstly let’s recall $R_{\phi,a}(T)$ which denote the set of points whose Birkhoff average by $\phi$ equal to $a$, that is,

$$R_{\phi,a}(T) := \{x \in X | \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x)) = a\}.$$  

Remark that

$$R_{\phi}(T) = \bigsqcup_{a \in \mathbb{R}} R_{\phi,a}(T).$$

For $\phi \in C^0(X)$ and $a \in \mathbb{R}$, by the continuity of $\phi$ and weak* topology,

$$x \in R_{\phi,a}(T) \iff \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = a \iff M_x(T) \subseteq \{\rho | \int \phi d\rho = a\}.$$  \hspace{1cm} (2.8)

So by (2.8)

$$x \in R_{\phi}(T) \iff \exists \ a \in \mathbb{R}, \ M_x(T) \subseteq \{\rho | \int \phi d\rho = a\}.$$  \hspace{1cm} (2.9)

Thus one has

$$x \in I_{\phi}(T) \iff \exists \ \mu_1, \mu_2 \in M_x(T) \text{ such that } \int \phi d\mu_1 \neq \int \phi d\mu_2.$$  \hspace{1cm} (2.10)

In particular,

$$I_{\phi}(T) \neq \emptyset \Rightarrow \inf\{\int \phi d\mu | \mu \in M(T, X)\} < \sup\{\int \phi d\mu | \mu \in M(T, X)\}. $$  \hspace{1cm} (2.11)

For any dynamical system $T : X \to X$ with (almost) specification and any $a \in \mathbb{R}$,

$$h_{top}(T, R_{\phi,a}(T)) = \sup\{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\}.$$  

If $a$ is not in the closed interval

$$L_{\phi} := [\inf\{\int \phi d\mu | \mu \in M(T, X)\}, \sup\{\int \phi d\mu | \mu \in M(T, X)\}],$$  

the result is trivial because two sides are all restricted on empty sets. In next theorem we will show that level-sets of Birkhoff average restricted on the above three types of irregular set also have same dynamical complexity in the sense of topological entropy. Let $\phi : X \to \mathbb{R}$ be a continuous function. If $I_{\phi}(T) = \emptyset$, then there is only one level-set of Birkhoff average for $\phi$, $R_{\phi,a}$ (for some $a \in \mathbb{R}$), which equals to $R_{\phi}(T) = X$. In this case

$$R_{\phi,a}(T) \cap W(T) \setminus R(T) = W(T) \setminus R(T), \ R_{\phi,a}(T) \cap QW(T) \setminus W(T) = QW(T) \setminus W(T),$$
\[ R_{\phi,a}(T) \cap I(T) \setminus QW(T) = I(T) \setminus QW(T), \]

and thus by Theorem 2.1 all of them have full topological entropy. So we only need to consider continuous functions with \( I_\phi(T) \neq \emptyset \). Let \( \text{Int}(L_\phi) \) denote the interior of \( L_\phi \). That is,

\[ \text{Int}(L_\phi) = (\inf \{ \int \phi d\mu | \mu \in M(T, X) \}, \sup \{ \int \phi d\mu | \mu \in M(T, X) \}). \]

Remark that if \( I_\phi(T) \neq \emptyset \), by (2.10) \( \text{Int}(L_\phi) \) is a nonempty and open interval.

**Theorem 2.4. (Main Theorem III)** Let \( \phi \in C^0(X) \) satisfy \( I_\phi(T) \neq \emptyset \) and let \( a \in \text{Int}(L_\phi) \). Then

(A*)

\[ h_{\text{top}}(T, R_{\phi,a}(T) \cap W(T) \setminus R(T)) = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0; \]

(B*)

\[ h_{\text{top}}(T, R_{\phi,a}(T) \cap QW(T) \setminus W(T)) = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0; \]

(C*)

\[ h_{\text{top}}(T, R_{\phi,a}(T) \cap I(T) \setminus QW(T)) = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0. \]

In particular,

(D*)

\[ h_{\text{top}}(T, R_{\phi,a}(T) \setminus R(T)) \quad (\text{or} \quad h_{\text{top}}(T, R_{\phi,a}(T) \cap I(T))) \]

\[ = h_{\text{top}}(T, R_{\phi,a}(T)) = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0. \]

Moreover, \( R_{\phi,a}(T) \cap QW(T) \setminus W(T) \) can be divided into two disjoint subsets with same positive topological entropy. More precisely,

(B₁)

\[ h_{\text{top}}(T, R_{\phi,a}(T) \cap \{ x \in QW(T) \setminus W(T) | \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \}) \]

\[ = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0; \]

(B₂)

\[ h_{\text{top}}(T, R_{\phi,a}(T) \cap \{ x \in QW(T) \setminus W(T) | \forall m \in M_x(T) \text{ s.t. } S_m \neq C_x \}) \]

\[ = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0. \]
Remark that from [26] for any system $T : X \to X$ with (almost) specification,

$$h_{\text{top}}(T, R_{\phi,a}(T)) = \sup\{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\}.$$ 

From its proof in [26], in fact one can have

$$h_{\text{top}}(T, R_{\phi,a}(T) \cap R(T)) = \sup\{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\}.$$ 

From Theorem 2.4 the set $R_{\phi,a}(T) \setminus R(T)$, being the complementary set of $R_{\phi,a}(T) \cap R(T)$, has same complexity as $R_{\phi,a}(T)$. Moreover, the sets of

$$W(T) \setminus R(T), \ QW(T) \setminus W(T), \ I(T) \setminus QW(T)$$

restrict on $R_{\phi,a}(T) \setminus R(T)$ have same complexity as $R_{\phi,a}(T)$. So Theorem 2.4 are some new observations of [26, 33, 22] on multi-fractal analysis of Birkhoff averages.

**Remark 2.5.** In Theorem 2.4 we require $a$ to satisfy

$$\inf\{\int \phi d\mu | \mu \in M(T, X)\} < a < \sup\{\int \phi d\mu | \mu \in M(T, X)\}.$$ 

That is, $a$ is not allowed to choose the extreme points. For example, let $\phi$ be a continuous function which restricts on two periodic orbits with value 0 and 1 respectively, and the values of other points are in the open interval of $(0, 1)$. Then if $a = 0$, or 1,

$$h_{\text{top}}(T, R_{\phi,a}(T) \setminus R(T)) = h_{\text{top}}(T, R_{\phi,a}(T) \cap I(T))$$

$$\leq h_{\text{top}}(T, R_{\phi,a}(T)) = \sup\{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\} = 0.$$ 

That is, in this case $R_{\phi,a}(T) \setminus R(T)$ has zero topological entropy.

## 3 Other Related Discussion

### 3.1 Another Recurrence: Transitive Points

Let $D(T)$ denote the set of all points whose orbits are dense in the whole space $X$. That is,

$$D(T) = \{x \in X \mid \overline{\text{Orbit}(x)} = X\}$$

where $\text{Orbit}(x) = \{T^n(x)\}_{n \geq 0}$. It is also a subset of recurrent set $\text{Rec}(T)$. Remark that for any minimal dynamical system, $D(T) = A(T)$ and for any dynamical system which is not minimal, $D(T)$ does not contain any periodic points and almost periodic points.

From the proofs below we will find that Theorem 2.1, 2.2 and 2.4 are still true for $W(T) \setminus R(T), QW(T) \setminus W(T)$ and its two divided subsets if one restricts them on $D(T)$, because in these proofs the chosen points are transitive.
Theorem 3.1. $(W(T) \setminus R(T)) \cap D(T)$ and $(QW(T) \setminus W(T)) \cap D(T)$ both carry full topological entropy. In particular, transitive set $D(T)$ and the intersection of weakly almost periodic set and transitive set, $W(T) \cap D(T)$, carries full topological entropy.

In particular, this implies the transitive irregular set $D(T) \cap I(T) = D(T) \setminus R(T)$ has full topological entropy. Recall from the introduction that $D(T) \setminus R(T)$ also carries full topological entropy. Now let us consider $R(T) \setminus D(T)$ and $I(T) \setminus D(T)$.

Recall that $M^*_\mathrm{erg}(T, X)$ denotes the set of all ergodic measures of all uniquely ergodic minimal subshifts with positive entropy and from [13, 14] we know that for any shifts of finite type, there exist uniquely ergodic minimal subshifts with any given entropy. So

$$\bigcup_{\mu \in M^*_\mathrm{erg}(T, X)} S_\mu \subseteq R(T) \setminus D(T).$$

So similar as the discussion of $A(T) \setminus P\mathrm{er}(T)$ (see (1.7)), $R(T) \setminus D(T)$ carries full topological entropy.

Recall a classical result in §7.3 of [36] that any shift of finite type (for example, $k$ symbols) has truth sub-shifts with topological entropy equal to any given positive real number less than the topological entropy of the shift itself (see Lemma 4.10 below). The constructed sub-shift is the well known $\beta-$shift ($\beta > 1$) and it is known that any $\beta-$shift is not minimal (which contains a fixed point) so that it is not uniquely ergodic. For convenience to state, denote the subshift by $T_\beta$ and the subspace by $\Sigma_\beta \subseteq X$. From [35, 26], every $\beta-$shift satisfies almost specification. Recall that from the introduction we know that for non-uniquely ergodic systems with almost specification, $I(T_\beta)$ is nonempty and carries full topological entropy of $\log \beta$. Notice that

$$I(T_\beta) \subseteq I(T) \cap \Sigma_\beta \subseteq I(T) \cap (X \setminus D(T)) = I(T) \setminus D(T)$$

and thus $I(T) \setminus D(T)$ has topological entropy of at least $\log \beta$. Let $\beta \uparrow k$, then $I(T) \setminus D(T)$ carries full topological entropy of $\log k$.

As said in the introduction, $R\mathrm{ec}(T) \setminus QW(T)$ and its subset $D(T) \setminus QW(T)$ should also carry full topological entropy, whose proof needs to adapt the idea of present paper (Lemma 4.9 below). In particular, this implies $(D(T) \setminus QW(T)) \cap I(T)$ and $(R\mathrm{ec}(T) \setminus QW(T)) \cap I(T)$ both carry full topological entropy. Moreover, remark that

$$D(T) \setminus QW(T) \subseteq R\mathrm{ec}(T) \setminus QW(T) \subseteq I(T),$$

because

$$R\mathrm{ec}(T) \cap R(T) \subseteq W(T) \subseteq QW(T),$$

(see Corollary 4.2).

All in all, with the help to various periodic-like recurrence and regularity of points, we obtain a refined classification of transitive points and each one carries full topological entropy.
3.2 Two-sided shifts, Hyperbolic systems and Lyapunov exponents

Similarly, one has same results as Theorem 2.1, 2.2 and 2.4 for two-sided shifts. From the classical uniform hyperbolicity theory, uniformly hyperbolic systems can be conjugated to shifts. So the above result is also valid to uniformly hyperbolic or expanding systems etc. (for example, Smale’s horseshoe and \( T : S^1 \to S^1, x \mapsto kx \mod 1 \) for an integer \( k \geq 2 \)).

In particular, we can use Lyapunov exponents to observe the periodic-like recurrence. Let \( f : M \to M \) be a \( C^1 \) diffeomorphism of a compact Riemannian manifold \( M \). Let \( E \subset TM \) be a \( Df \)-invariant subbundle. Define the (maximal) Lyapunov exponent of \( E \) at a point \( x \in M \) by

\[
\lim_{n \to \infty} \frac{1}{n} \log \| Df^n|_{E_x} \|
\]

if the limit exists. Such points are called Lyapunov-regular. Otherwise, the points are called Lyapunov-irregular. Denote the sets of all Lyapunov-regular and Lyapunov-irregular points by \( R_{Ly}(f) \) and \( I_{Ly}(f) \) respectively. The subbundle \( E \) is called conformal, if for any \( x \in M, n \geq 1 \)

\[
\| Df^n|_{E_x} \| = \prod_{j=0}^{n-1} \| Df|_{E_{f_j(x)}} \|
\]

Let \( \phi(x) = \| Df|_{E_x} \| \), if \( E \) is a continuous subbundle then it is a continuous function. Thus, using this \( \phi \) in Theorem 2.2 and 2.4 we have

Theorem 3.2. Let \( f : M \to M \) be a \( C^1 \) diffeomorphism of a compact Riemannian manifold \( M \) and let \( E \subset TM \) be a continuous conformal \( Df \)-invariant subbundle. Then Theorem 2.2 and 2.4 hold, replacing \( R_{\phi}(f) \) and \( I_{\phi}(f) \) by \( R_{Ly}(f) \) and \( I_{Ly}(f) \) respectively.

In other words, we can use the ‘eye’ of Lyapunov exponents to distinguish different periodic-like recurrence.

3.3 Hausdorff Dimension

It is known that for shifts, full topological entropy and full Hausdorff dimension appear simultaneously. In fact, it is known that every subset \( Z \subset X \) satisfying

\[
Dim_H(Z) = \frac{1}{h_{top}(T)} h_{top}(T, Z).
\]

So

Theorem 3.3. For dynamical systems of one-sided shifts and two-sided shifts, all the level-sets in Theorem 2.1, 2.3 have full Hausdorff dimension.
3.4 Surface diffeomorphisms and Non-uniformly hyperbolic diffeomorphisms

Using Pesin theory, the above results hold for all surface diffeomorphisms.

**Theorem 3.4.** Let \( f : M \to M \) be a \( C^{1+\alpha} \) diffeomorphism on a compact surface \( M \). Then \( f \) satisfies all results of Theorem 2.1, 2.2 and 2.4.

The proof is simple. If \( h_{\text{top}}(f) = 0 \), then it is trivial. If \( h_{\text{top}}(f) > 0 \), then for any \( \frac{h_{\text{top}}(f)}{2} > \varepsilon > 0 \) by Variational Principle, there is an ergodic measure \( \mu \) such that

\[
h_{\mu}(f) > h_{\text{top}}(f) - \varepsilon > 0.
\]

From Ruelle’s inequality[29] for \( f \) and \( f^{-1} \), the two Lyapunov exponents of \( \mu \) is non-zero. One is positive and another is negative. So \( \mu \) is an ergodic hyperbolic measure. By classical Pesin theory[3, 18], there is a Smale’s horseshoe \( \Lambda \subseteq M \) such that

\[
h_{\text{top}}(f, \Lambda) > h_{\mu}(f) - \varepsilon > h_{\text{top}}(f) - 2\varepsilon > 0.
\]

By Subsection 3.3 all results of Theorem 2.1, 2.2 and 2.4 hold for the horseshoe \( \Lambda \). Then one can use this horseshoe to get Theorem 3.4. For example,

\[
\begin{align*}
  h_{\text{top}}(f, W(f) \setminus R(f)) &\geq h_{\text{top}}(f, \Lambda \cap W(f) \setminus R(f)) \\
  &= h_{\text{top}}(f, \Lambda) > h_{\text{top}}(f) - 2\varepsilon.
\end{align*}
\]

By the arbitrariness of \( \varepsilon \), we complete the proof.

For any hyperbolic ergodic measure \( \mu \) (meaning that all Lyapunov exponents are non-zero) with positive entropy, by classical Pesin theory[3, 18] for any \( \varepsilon > 0 \) there is a Smale’s horseshoe \( \Lambda \subseteq M \) such that

\[
h_{\text{top}}(f, \Lambda) > h_{\mu}(f) - \varepsilon.
\]

Thus similar as above discussion, one has

**Theorem 3.5.** Let \( f : M \to M \) be a \( C^{1+\alpha} \) diffeomorphism on a compact Riemannian manifold \( M \). Then one can replace “\( = h_{\text{top}}(T) \)” in the statements of Theorem 2.4 by

“\( \geq \sup \{ h_{\mu}(f) | \mu \in M_{\text{erg}}(f) \text{ is hyperbolic } \} \)”.

For example,

\[
\begin{align*}
  h_{\text{top}}(f, W(f) \setminus R(f)) &\geq \sup \{ h_{\mu}(f) | \mu \in M_{\text{erg}}(f) \text{ is hyperbolic } \}.
\end{align*}
\]

Similarly one can change the statements of Theorem 2.2 and 2.4.
In particular, recall an example of [9] that there is a $C^{1+\alpha}$ non-uniformly hyperbolic system $f$ such that the Lyapunov exponents of any invariant measure are all non-zero but $f$ is not uniformly hyperbolic. For convenience, we call such systems to be completely non-uniformly hyperbolic. In this case

$$\sup\{h_\mu(f) | \mu \in M_{\text{erg}}(f) \text{ is hyperbolic}\} = \sup\{h_\mu(f) | \mu \in M_{\text{erg}}(f)\} = h_{\text{top}}(f).$$

Thus, we have

**Theorem 3.6.** Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemannian manifold $M$. If $f$ is completely non-uniformly hyperbolic, then one have the results of Theorem 2.1, 2.2 and 2.4.

4 Some definitions and known facts

In this section we recall some definitions and introduce some known facts which are useful to prove our main theorems.

4.1 Equivalent statement of regular and irregular points

For an invariant measure $\mu$, recall that $G(\mu)$ denotes the set of all points satisfying that $\Upsilon_n(x)$ converges to $\mu$ in weak$^*$ topology. Remark that by definition and weak$^*$ topology,

$$x \in R(T) \iff \exists \mu \in M(T, X), x \in G(\mu) \iff M_x(T) \text{ is a singleton}.$$

More precisely, on one hand, if $x$ belongs to some $G(\mu)$, then by weak$^*$ topology the Birkhoff average $\frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x))$ is always convergent to the integral of $\int \phi d\mu$. On the other hand, if $x$ is $\phi$-regular for any continuous function $\phi : X \to \mathbb{R}$ but $\Upsilon_n(x)$ do not converge to a unique measure, then this sequence has at least two different limit points $\mu$ and $\nu$. By weak$^*$ topology for any continuous function $\phi : X \to \mathbb{R}$, the limit of

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x))$$

equals to $\int \phi(x) d\mu$ and $\int \phi(x) d\nu$. In other words,

$$\int \phi(x) d\mu = \int \phi(x) d\nu$$

holds for any continuous functions. This implies $\mu = \nu$, which is a contradiction. So

$$x \in I(T) \iff M_x(T) \text{ is not a singleton}. \quad (4.12)$$
4.2 Equivalent statements of weakly almost periodic points and quasi-weakly almost periodic points

Let’s recall some equivalent statements of weakly almost periodic points and quasi-weakly almost periodic points. From [38, 39, 40] we know some basic facts:

**Lemma 4.1.**

\[ C_x = \bigcup_{m \in M_x(T)} \overline{S_m}; \]  
\[ \forall x \in X, C_x \subseteq \omega_T(x); \]  
\[ \text{For any recurrent point } x \in \text{Rec}(T), \]
\[ x \in W(T) \iff C_x = S_\mu, \forall \mu \in M_x(T) \iff \omega_T(x) = S_\mu, \forall \mu \in M_x(T); \]  
\[ x \in QW(T) \iff x \in C_x \iff \omega_T(x) = C_x \iff \omega_T(x) = \bigcup_{\mu \in M_x(T)} \overline{S_\mu}. \]

**Corollary 4.2.**

\[ \text{Rec}(T) \cap R(T) \subseteq W(T), \ QW(T) \setminus W(T) \subseteq I(T). \]

**Proof.** On one hand, if \( x \in \text{Rec}(T) \cap R(T) \), then there is \( \mu \in M(T, X) \) such that \( x \in G(\mu) \) (the set of all regular points of \( \mu \)). So \( M_x(T) = \{ \mu \} \) and by (4.13) \( C_x = \bigcup_{m \in M_x(T)} \overline{S_m} = S_\mu \). Then by (4.15) \( x \in W(T) \).

On the other hand, let \( x \in QW(T) \setminus W(T) \). Then by (1.4) \( x \in \text{Rec}(T) \). If \( x \in R(T) \), by the first statement \( x \in W(T) \), which is a contradiction to \( x \in QW(T) \setminus W(T) \).

If \( C_x = X \), then by (4.14) \( \omega_T(x) = X \ni x \) so that \( x \in \text{Rec}(T) \cap C_x \). By (4.16), \( x \in QW(T) \). That is, we have following corollary.

**Corollary 4.3.**

\[ C_x = X \Rightarrow x \in QW(T). \]

4.3 Bowen’s specification property

Now we recall the definition of Bowen’s *specification property* and some classical properties, see [10, 31, 7, 8].
Definition 4.4. Let $T : X \to X$ be a continuous map on a compact metric space $X$. We say that the dynamical system $T$ satisfies specification property, if the following holds: for any $\epsilon > 0$ there exists an integer $M(\epsilon)$ such that for any $k \geq 2$, any $k$ points $x_1, \ldots, x_k$, any integers

$$a_1 \leq b_1 < a_2 \leq b_2 \cdots < a_k \leq b_k$$

with $a_{i+1} - b_i \geq M(\epsilon)$ ($2 \leq i \leq k$), and for any integer $p \geq M(\epsilon) + b_k - a_1$, there exists a point $x \in X$ with $T^p(x) = x$ such that

$$d(T^j(x), T^j(x_i)) < \epsilon, \quad \text{for} \quad a_i \leq j \leq b_i, \quad 1 \leq i \leq k.$$

Following are three results from [10] when the system satisfies specification property. Since any one-sided shift of finite type always has specification property, then we have following properties.

Lemma 4.5. (1) The set of periodic points is dense in the whole space $X$.
(2) The set of periodic measures is dense in the set of $T$-invariant measures.

Lemma 4.6. There is a dense $G_\delta$ subset $R$ of $T$-invariant measures such that for any $\mu \in R$, $\mu$ is ergodic and $S_\mu = X$ (in this case we say $\mu$ has full support).

Lemma 4.7. For any compact connected nonempty set $K \subseteq M(X, T)$,

$$G_K := \{x \in X | M_x(T) = K\}$$

(called saturated set of $K$) is nonempty and dense in $X$. In particular,

$$G_{\text{max}} := \{x \in X | M_x(T) = M(X, T)\}$$

is nonempty and contains a dense $G_\delta$ subset of $X$.

By weak* topology and the continuity of $\phi \in C^0(X)$, it is easy to see that if $R$ is a dense subset of $M(T, X)$, then

$$\left\{ \int \phi(x) d\nu | \nu \in R \right\}$$

is dense in $L_\phi$. This implies that

Lemma 4.8. Let $\phi \in C^0(X)$ with $I_\phi(T) \neq \emptyset$. Then for any $\nu \in M(T, X)$, there is $\rho \in R$ such that $\int \phi d\rho \neq \int \phi d\nu$. Moreover, for any $n \in \mathbb{Z}^+$ and $a \in \text{Int}(L_\phi)$, there is $\mu_1, \mu_2, \ldots, \mu_n \in R$ and $\nu_1, \nu_2, \cdots, \nu_n \in R$ such that

$$\int \phi(x) d\mu_n < \cdots < \int \phi(x) d\mu_1 < a < \int \phi(x) d\nu_1 < \int \phi(x) d\nu_2 < \cdots < \int \phi(x) d\nu_n.$$

By Lemma 4.5 and 4.6, $R$ can be as the set of periodic measures or the set of ergodic measures with full support.
Now we state a result from [26].

**Lemma 4.9. (Variational Principle)** For any compact connected nonempty set \( K \subseteq M(X, T) \),
\[
h_{\text{top}}(T, G_K) = \inf \{ h_{\mu}(T) \mid \mu \in K \},
\]
where \( G_K = \{ x \in X \mid M_x(T) = K \} \).

**Proof** This is Theorem 1.1 of [26] which was stated for systems with \( g \)-almost product property and uniform separation property (see definitions in [26] for details). It is known that specification property is stronger than \( g \)-almost product property and expansiveness is stronger than the uniform separation property, and every shift always satisfies specification property and expansiveness. Thus Theorem 1.1 of [26] can be stated for the shift \( T : X \to X \).

\[ \square \]

### 4.4 Completeness of sub-shifts

Recall a classical result in §7.3 of [36] that any shift has truth sub-shifts with topological entropy equal to any given positive real number less than the topological entropy of the shift itself. That is,

**Lemma 4.10.** For any \( \beta \in [0, h_{\text{top}}(T)) \), there is a \( T \)-invariant compact truth subset \( X_\beta \subseteq X \) such that \( h_{\text{top}}(T, X_\beta) = \log \beta \).

Remark that \( X_\beta \subseteq X \) can be deduced from \( h_{\text{top}}(T, X_\beta) = \log \beta < h_{\text{top}}(T) \). Thus by Variational Principle we have following corollary.

**Corollary 4.11.** For any \( \beta \in [0, h_{\text{top}}(T)) \), there is a \( \mu \in M_{\text{erg}}(T, X) \) such that \( S_{\mu} \subseteq X \) and
\[
h_{\mu}(T) = \beta.
\]
In particular, for any \( \varepsilon > 0 \), there is a \( \mu \in M_{\text{erg}}(T, X) \) such that \( S_{\mu} \subseteq X \) and
\[
h_{\mu}(T) > h_{\text{top}}(T) - \varepsilon.
\]

**Proof** Suppose \( I_\beta(T) \neq \emptyset \) and fix \( \varepsilon > 0 \). By Lemma 4.10 take a \( T \)-invariant compact truth subset \( X_\ast \subseteq X \) such that \( h_{\text{top}}(T|_{X_\ast}) = \beta \). By Variational Principle on the subsystem \( T|_{X_\ast} \) and expansiveness of \( T \), we can choose \( \nu \in M_{\text{erg}}(T, X_\ast) \) (called measure with maximal entropy) such that
\[
h_{\nu}(T|_{X_\ast}) = \sup_{\rho \in M_{\text{erg}}(T, X_\ast)} h_{\rho}(T|_{X_\ast}) = h_{\text{top}}(T|_{X_\ast}) = \beta.
\]
Remark that \( S_{\nu} \subseteq X_\ast \subseteq X \). Define
\[
\mu(B) := \nu(B \cap X_\ast)
\]
for any Borel set \( B \subseteq X \). Then \( \mu \in M(T, X) \) and by the definition of metric entropy,
\[
h_{\mu}(T) = h_{\nu}(T|_{X_\ast}) = \beta.
\]
Moreover, \( S_{\mu} = S_{\nu} \subseteq X \).
5 Partition of \( \phi \)-irregular set

In this section we divide Theorem 2.2 (2) into several propositions to prove and then Theorem 2.2 implies Theorem 2.1.

**Proposition 5.1. (Theorem 2.2 (A"))** For any \( \phi \in C^0(X) \), either \( I_\phi(T) = \emptyset \) or
\[
h_{\text{top}}(T, I_\phi(T) \cap W(T) \setminus R(T)) = h_{\text{top}}(T).
\]

**Proof.** Suppose \( I_\phi(T) \neq \emptyset \) and fix \( \varepsilon > 0 \). By variational principle, we can take \( \mu \in M(T, X) \) such that
\[
h_{\mu}(T) > h_{\text{top}}(T) - \varepsilon.
\]
By Lemma 4.8 we can take \( \nu \in M(T, X) \) such that \( S_\nu = X \) and \( \int \phi d\nu \neq \int \phi d\mu \). Then we can choose two different numbers \( 0 < \theta_1 < \theta_2 < 1 \) close to 1 enough such that for \( \omega_i = \theta_i \mu + (1 - \theta_i) \nu, i = 1, 2 \), one has
\[
h_{\omega_i}(T) = \theta_i h_{\mu}(T) + (1 - \theta_i) h_{\nu}(T) \geq \theta_i h_{\mu}(T) > h_{\text{top}}(T) - \varepsilon, i = 1, 2.
\]
Remark that \( \theta_1 \neq \theta_2 \) and \( \int \phi d\mu \neq \int \phi d\nu \) imply
\[
\int \phi d\omega_1 \neq \int \phi d\omega_2;
\]
and \( S_\nu = X \) implies
\[
S_{\omega_i} = S_\mu \cup S_\nu = X, i = 1, 2.
\]
Let
\[
K = \{ \tau \omega_1 + (1 - \tau) \omega_2 | \tau \in [0, 1] \}.
\]
Then by (5.20) and (5.18) for any \( m = \tau \omega_1 + (1 - \tau) \omega_2 \in K \),
\[
S_m = X, \ h_m(T) \geq \min\{h_{\omega_1}(T), h_{\omega_2}(T)\} > h_{\text{top}}(T) - \varepsilon.
\]
By Lemma 4.13
\[
h_{\text{top}}(T, G_K) = \inf\{h_m(T) | m \in K\} \geq h_{\text{top}}(T) - \varepsilon,
\]
where \( G_K = \{ x \in X | M_x(T) = K \} \).
To complete the proof, we only need to prove \( G_K \subseteq I_\phi(T) \cap W(T) \). On one hand, for any \( x \in G_K \), notice that \( \omega_1, \omega_2 \in K = M_x(T) \) and thus by (2.10), (5.19) implies \( x \in I_\phi(T) \). On the other hand, by (4.13) the equality
\[
C_x = \bigcup_{\kappa \in M_x(T)} S_\kappa = \bigcup_{\kappa \in K} S_\kappa = X = S_m
\]
hold for all \( m \in K = M_x(T) \). By (4.14) \( x \in X = C_x \subseteq \omega_T(x) \) and thus \( x \in \text{Rec}(T) \). So from (4.15) one has \( x \in W(T) \). \( \square \)
Proposition 5.2. (Theorem 2.2 (B′″)) For any φ ∈ C^0(X), either I_φ(T) = ∅ or
\[ h_{top}(T, A) = h_{top}(T), \]
where \( A = I_φ(T) \cap \{ x ∈ QW(T) \setminus W(T) | \exists \omega ∈ M_x(T) \text{ s.t. } S_ω = C_x \} \).

**Proof.** Suppose \( I_φ(T) ≠ ∅ \) and fix \( ε > 0 \). By Corollary 4.11 we can choose \( μ ∈ M_{erg}(T, X) \) such that \( S_μ \subsetneq X \) and
\[ h_μ(T) > h_{top}(T) - ε. \]

By Lemma 4.8 \( I_φ(T) ≠ ∅ \) implies that we can take \( ν ∈ M(T, X) \) such that \( S_ν = X \) and
\[ \int φdν ≠ \int φdμ. \]
Then we can take \( θ ∈ (0, 1) \) close to 1 such that \( h_ω(T) ≥ θh_μ(T) > h_{top}(T) - ε \) where \( ω = θμ + (1 - θ)ν \). Remark that \( S_ω = S_μ \cup S_ν = X \) and
\[ \int φdω ≠ \int φdμ. \tag{5.21} \]

Let
\[ K = \{ τω + (1 - τ)μ | τ ∈ [0, 1] \}. \]

Then for any \( m = τω + (1 - τ)μ ∈ K \setminus \{ μ \}, S_m = X \) and for any \( m = τω + (1 - τ)μ ∈ K \)
\[ h_m(T) ≥ \min \{ h_ω(T), h_μ(T) \} > h_{top}(T) - ε. \]

By Lemma 4.13,
\[ h_{top}(T, G_K) = \inf \{ h_m(T) | m ∈ K \} > h_{top}(T) - ε, \]
where \( G_K = \{ x ∈ X | M_x(T) = K \}. \)

To complete the proof, we only need to prove \( G_K ⊆ A \). In fact, fix \( x ∈ G_K \). On one hand, notice that \( ω, μ ∈ K = M_x(T) \) and thus by (2.10), (5.21) implies \( x ∈ I_φ(T) \). On the other hand, by (4.13) one has the following equality
\[ C_x = \bigcup_{m ∈ M_x(T)} S_m = \bigcup_{m ∈ K} S_m = X. \]

By (4.17) \( x ∈ QW(T) \). Thus by (4.14) \( x ∈ Rec(T) \). Notice that \( μ ∈ M_x(T) \) and \( C_x = X ≠ S_μ \)
so that from (4.15) \( x ∈ X \setminus W(T) \). Recall that \( ω ∈ K = M_x(T) \) and \( S_ω = X \). So \( x ∈ A \). □

Proposition 5.3. (Theorem 2.2 (B″)) For any φ ∈ C^0(X), either I_φ(T) = ∅ or
\[ h_{top}(T, A) = h_{top}(T), \]
where \( A = I_φ(T) \cap \{ x ∈ QW(T) \setminus W(T) | ∀ m ∈ M_x(T) \text{ s.t. } S_m ≠ C_x \} \).

**Proof.** Suppose \( I_φ(T) ≠ ∅ \) and fix \( ε > 0 \). By Corollary 4.11 we can choose \( μ ∈ M_{erg}(T, X) \) such that \( S_μ ⊆ X \) and
\[ h_μ(T) > h_{top}(T) - ε. \]
From the expansiveness of shifts, periodic points with same period is finite so that the set of all periodic points are countable, denoted by \( \{x_i\}_{i=1}^\infty \). Denote the \( T \)-invariant measure supported on the periodic orbit of \( x_i \) by \( m_i \). By Lemma 4.5

\[
\{x_i\}_{i=1}^\infty = X, \quad \{m_i\}_{i=1}^\infty = M(T, X).
\]

Take a strictly increasing sequence of \( \{\theta_i | \theta_i \in (0, 1)\}_{i=1}^\infty \) such that

\[
\lim_{i \to +\infty} \theta_i = 1
\]

and

\[
h_{\nu_i}(T) \geq \theta_i h_{\mu}(T) > h_{\text{top}}(T) - \varepsilon
\]

where \( \nu_i = \theta_i \mu + (1 - \theta_i)m_i, \ i = 1, 2, 3, \ldots \). Remark that for any \( i, S_{\nu_i} = S_{\mu} \cup S_{m_i} \).

By Lemma 4.8 if let \( \mathcal{R} = \{m_i\}_{i=1}^\infty \), then \( I_\phi(T) \neq \emptyset \) implies that we can take one periodic measure \( m_{i_0} \) such that \( \int \phi dm_{i_0} \neq \int \phi d\mu \). Without loss of generality, we can assume \( m_1 = m_{i_0} \). Then

\[
\int \phi dm_{i_1} \neq \int \phi d\mu. \tag{5.22}
\]

Now we consider

\[
K = \bigcup_{\tau \geq 1} \{\tau \nu_i + (1 - \tau)\nu_{i+1} | \tau \in [0, 1]\} \cup \{\mu\}.
\]

Remark that \( K \) is a nonempty connected compact subset of \( M(T, X) \) because \( \nu_i \to \mu \) in weak* topology. So we can use Lemma 4.9 to get that

\[
h_{\text{top}}(T, G_K) = \inf \{h_m(T) | m \in K\} = \min \{\inf \{h_{\nu_i}(T), h_\mu(T)\} \geq h_{\text{top}}(T) - \varepsilon, \]

where \( G_K = \{x \in X | M_x(T) = K\} \).

To complete the proof, we only need to prove \( G_K \subseteq A \). Fix \( x \in G_K \). Recall that \( \nu_1, \mu \in K = M_x(T) \) and thus by (2.10), (5.22) implies \( x \in I_\phi(T) \). Clearly by (1.13)

\[
C_x = \bigcup_{m \in M_x(T)} S_m = \bigcup_{m \in K} S_m \supseteq \bigcup_{i \geq 1} S_{\nu_i} = \bigcup_{i \geq 1} (S_{\nu_i} \cup S_{\mu}) \supseteq \bigcup_{i \geq 1} \{x_i\} = X.
\]

So \( C_x = X \). By (1.17) \( x \in QW(T) \). By (1.4) \( x \in \text{Rec}(T) \). Notice that \( C_x = X \neq \mu \) and \( \mu \in M_x(T) \) so that from (1.15) \( x \in X \setminus W(T) \). For any \( m \in M_x(T) = K \), by definition of \( K \) there is some \( i \) such that \( S_m \subseteq S_\mu \cup S_{m_i} \cup S_{m_{i+1}} \). Note that \( X \setminus S_\mu \supseteq X \setminus K \) is a nonempty open set so that \( X \setminus S_\mu \) is not a finite set. But \( S_{m_i} \cup S_{m_{i+1}} \) is just composed of two periodic orbits and thus is a finite set. Hence \( S_m \not\subseteq X = C_x \). I.e., \( S_m \neq C_x \).

Remark 5.4. If we do not use expansiveness, one can take \( \{x_i\}_{i=1}^\infty \) to be a countable subset of periodic points such that \( \{x_i\}_{i=1}^\infty = X, \ \{m_i\}_{i=1}^\infty = M(T, X) \). More precisely, by compactness of \( X \) and \( M(T, X) \) from Lemma 4.5 we can take a countable subset \( \mathcal{B} \) of periodic points and a countable subset \( \mathcal{B} \) of periodic measures such that \( \overline{B} = X \) and \( \overline{B} = M(T, X) \).
For any $x \in B$, denote the $T$-invariant measure supported on the periodic orbit of $x$ by $m_x$. Then

$$B \cup \bigcup_{m \in B} S_m$$

is countable and dense in $X$, and

$$\bigcup_{x \in B} m_x \cup B$$

is countable and dense in $M(T,X)$. In this case one take $\{x_i\}_{i=1}^\infty = B \cup \bigcup_{m \in B} S_m$ and denote the $T$-invariant measure supported on the periodic orbit of $x_i$ by $m_i$. Then we have $\{x_i\}_{i=1}^\infty = X$, $\{m_i\}_{i=1}^\infty = M(T,X)$.

**Proposition 5.5. (Theorem 2.2 (C'))** For any $\phi \in C^0(X)$, either $I_\phi(T) = \emptyset$ or

$$h_{top}(T, I_\phi(T) \cap I(T) \setminus QW(T)) = h_{top}(T).$$

**Proof.** Suppose $I_\phi(T) \neq \emptyset$ and fix $\varepsilon > 0$. By Corollary 4.11 we can choose $\mu \in M_{erg}(T, X)$ such that $S_\mu \subseteq X$ and

$$h_\mu(T) > h_{top}(T) - \varepsilon.$$ 

Since $X \setminus S_\mu$ is nonempty and open, by density of periodic points (Lemma 4.5)

$$B := \{\nu \in M_\mu(T, X) \mid S_\nu \setminus S_\mu \neq \emptyset\} \neq \emptyset.$$ 

Now we will construct an invariant measure $\kappa$ such that the set $S_\kappa \setminus S_\mu$ is composed of one periodic orbit and

$$\int \phi d\kappa \neq \int \phi d\mu.$$ 

More precisely, if there is a periodic measure $\nu \in B$ such that $\int \phi d\nu \neq \int \phi d\mu$, then take $\kappa = \nu$ and remark that $S_\kappa \setminus S_\mu = S_\nu$. Otherwise, for any $\nu \in B$, $\int \phi d\nu = \int \phi d\mu$. Take such a measure $\nu$. By Lemma 4.9, $I_\phi(T) \neq \emptyset$ implies

$$Y := \{\tau \mid \int \phi d\tau \neq \int \phi d\mu, \tau \in M_\mu(T, X)\} \neq \emptyset.$$ 

Then we can take $\nu' \in Y$ such that $\int \phi d\nu' \neq \int \phi d\mu$. Remark that in this case $Y \cap B = \emptyset$ so that $S_{\nu'} \setminus S_\mu = \emptyset$. Then $S_{\nu'} \subseteq S_\mu$. So if we take $\kappa = \frac{1}{2}(\nu + \nu')$, then $\int \phi d\kappa \neq \int \phi d\mu$. Note that $S_\kappa = S_\nu \cup S_{\nu'}$ and $S_\kappa \setminus S_\mu = S_{\nu'}$.

Take $\theta \in (0,1)$ close to 1 such that $\omega = \theta \mu + (1 - \theta)\kappa$ satisfies $h_\omega(T) \geq \theta h_\mu(T) > h_{top}(T) - \varepsilon$. Then $\omega$ also satisfies that $\int \phi d\omega \neq \int \phi d\mu$. Remark that $S_\omega = S_\mu \cup S_\nu \cup S_{\nu'} = S_\mu \cup S_{\nu'}$. Let $K = \{\tau \mu + (1 - \tau)\omega \mid \tau \in [0,1]\}$. Then one can use Lemma 4.9 to get

$$h_{top}(T, GK) = \inf \{h_m(T) \mid m \in K\} \geq \min \{h_\mu(T), h_\omega(T)\} > h_{top}(T) - \varepsilon,$$

where $G_K = \{x \in X \mid M_x(T) = K\}$. 

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To complete the proof, we only need to prove
\[ G_K \subseteq I_\phi(T) \setminus QW(T). \]
For any \( x \in G_K \), recall that \( \omega, \mu \in K = M_k(T) \) and \( \int \phi d\omega \neq \int \phi d\mu \). Thus by (2.10) \( x \in I_\phi(T) \). If \( x \in QW(T) \), then by (1.4) \( x \in \text{Rec}(T) \) so that by (1.16) \( x \in C = \omega_T(x) \).
Then by (4.13) \( x \in \omega_T(x) = C = \bigcup_{m \in M_k(T)} S_m = \bigcup_{m \in K} S_m = S_\mu \cup S_\nu \), which implies \( x \in S_\mu \) or \( S_\nu \). So by invariance of \( S_\mu \) and \( S_\nu \), one has \( \text{Orbit}(x) \subseteq S_\mu \) or \( \text{Orbit}(x) \subseteq S_\nu \).

**Remark 5.6.** Remark that in fact there essentially exists a periodic measure \( \nu \) satisfying \( \int \phi d\nu \neq \int \phi d\mu \) and simultaneously \( S_\nu \subseteq X \setminus S_\mu \), even though it is not a trouble in our above proof. This is because we can modify the proof of [10] that for any open set \( U \subseteq X \), by specification (Definition 4.4) one can choose the shadowing periodic point beginning from \( U \) so that the set of periodic measures whose support cross \( U \) is dense in the set of \( T \)-invariant measures. Here \( X \setminus S_\mu \) is an open and invariant set so that the set of periodic measures whose support is contained in \( X \setminus S_\mu \) is dense in the set of \( T \)-invariant measures. So by Lemma 4.8 there is a periodic measure \( \nu \) satisfying \( \int \phi d\nu \neq \int \phi d\mu \) and simultaneously \( S_\nu \subseteq X \setminus S_\mu \).

**Proof of Theorem 2.1** Let \( \mu_i \) be two periodic measures which are supported on two different periodic orbits \( \text{Orbit}(p_i), i = 1, 2 \). If
\[ K := \{ \tau \mu_1 + (1 - \tau)\mu_2 \mid \tau \in [0, 1] \}, \]
by Lemma 4.7 \( G_K \) is dense in \( X \). Since \( \text{Orbit}(p_i), i = 1, 2 \) are different closed finite sets, we can choose a continuous function \( \phi \) such that
\[ \phi|_{\text{Orbit}(p_i)} \equiv i, \quad i = 1, 2. \]
Then \( \int \phi d\mu_1 \neq \int \phi d\mu_2 \). Thus for every point \( x \in G_K \), the Birkhoff average
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) \]
does not exist. Then \( I_\phi(T) \supseteq G_K \neq \emptyset \) so that the above Propositions imply Theorem 2.1 because \( I_\phi(T) \subseteq I(T) \).

From the proofs of above four propositions, all the considered sets above contains some \( G_K \) which is dense in the whole space by Lemma 4.7. That is,
Proposition 5.7. Let $\phi \in C^0(X)$. If $I_\phi(T) \neq \emptyset$ then the following sets of

(I) \[ I_\phi(T) \cap W(T) \setminus R(T); \]

(II) \[ I_\phi(T) \cap I(T) \setminus QW(T); \]

(III) \[ I_\phi(T) \cap \{ x \in QW(T) \setminus W(T) | \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \} \]

(IV) \[ I_\phi(T) \cap \{ x \in QW(T) \setminus W(T) | \forall m \in M_x(T) \text{ s.t. } S_m \neq C_x \} \]

are all dense subsets of $X$.

In particular, we have a better characterization for the set of

\[ I_\phi(T) \cap \{ x \in QW(T) \setminus W(T) | \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \}, \]

which is to answer the open problem in Subsection 2.2 from different sight.

Proposition 5.8. Let

\[ IC^0(X) := \{ \phi \in C^0(X) | I_\phi(T) \neq \emptyset \}. \]

Then the set

\[ \bigcap_{\phi \in IC^0(X)} I_\phi(T) \cap \{ x \in QW(T) \setminus W(T) | \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \} \]

contains a dense $G_\delta$ subset of $X$ (called residual in $X$).

**Proof** By Lemma 4.7, $G_{\text{max}} = \{ x \in X | M_x(T) = M(X, T) \}$ is residual in $X$. So we only need to prove for any $\phi \in IC^0(X)$,

\[ I_\phi(T) \subseteq \bigcap_{\phi \in IC^0(X)} I_\phi(T) \cap \{ x \in QW(T) \setminus W(T) | \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \}. \]

Given $x \in G_{\text{max}}$, by (2.11) we can take $\omega, \mu \in M(X, T) = M_x(T)$ such that they have different integrals for $\phi$, thus by weak* topology $x \in I_\phi(T)$. By density of periodic points (Lemma 4.5),

\[ C_x = \bigcup_{m \in M_x(T)} S_m = \bigcup_{m \in M(T, X)} S_m \supseteq \bigcup_{m \in M(T, X) \text{ is periodic measure}} S_m = X \]

(Remark that $C_x = X$ can be also obtained from the existence of invariant measure with full support by Lemma 4.6. By (1.17) $X = C_x$ implies $x \in QW(T)$. By (1.4) $x \in Rec(T)$. By Lemma 4.5 one can take a periodic measure $\nu \in M(T, X) = M_x(T)$ whose support $S_\nu \subseteq X = C_x$, then by (1.15) $x \in X \setminus W(T)$. Take an invariant measure $\mu$ with full support $X$ by Lemma 4.6 then $\mu \in M(T, X) = M_x(T)$ and $S_\mu = X = C_x$. We complete the proof.

Remark that $G_{\text{max}}$ has zero topological entropy from Lemma 4.9 because $M(T, X)$ contains periodic measures which have zero entropy. Recall Bowen’s result in [6] that
Lemma 5.9. Let $f : X \to X$ be a continuous map on a compact metric space $X$. Then for any sequence of subsets $Y_i \subseteq X$ ($i \in \mathbb{N}$),

$$h_{\text{top}}(f, \bigcup_i Y_i) = \sup_i h_{\text{top}}(f, Y_i).$$

From this lemma we know that the complementary set of $G_{\text{max}}$,

$$\{ x \in I_{\phi}(T) \cap QW(T) \setminus W(T) | \exists \omega \in M_x(T) \ s.t. \ S_\omega = C_x \} \setminus G_{\text{max}},$$

has full topological entropy. This complementary set is just dense but not residual in $X$ (density can be seen from Lemma 4.7 and the choose of $G_K$ in the proof of Proposition 5.2).

At the end of this section, we show that Proposition 5.8 holds for a larger class of systems, such as $C^1$ generic (volume-preserving) diffeomorphisms, which means the open problem of [10] by Zhou and Feng is solved for generic systems. Let $M$ be a compact Riemannian manifold and $m$ be a volume measure on $M$. Let $\text{Diff}^1(M)$ and $\text{Diff}^1_m(M)$ denote the space of all $C^1$ diffeomorphisms on $M$ and all volume-preserving $C^1$ diffeomorphisms on $M$ respectively. If $f : M \to M$ and $\Lambda$ is a compact subset of $M$, let

$$IC_0^0(f)(\Lambda) := \{ \phi \in C^0(\Lambda) | I_\phi(f) \neq \emptyset \}.$$

Theorem 5.10. (1) Let $\Lambda$ be an isolated non-trivial transitive set of a $C^1$ generic diffeomorphism $f \in \text{Diff}^1(M)$. Then the set

$$\bigcap_{\phi \in IC_0^0(\Lambda)} I_\phi(f) \cap \{ x \in \Lambda \cap QW(f) \setminus W(f) | \exists \omega \in M_x(f) \ s.t. \ S_\omega = C_x \}$$

contains a dense $G_\delta$ subset of $\Lambda$ (called residual in $\Lambda$).

(2) Let $f \in \text{Diff}^1_m(M)$ be a $C^1$ generic volume-preserving diffeomorphism. Then the set

$$\bigcap_{\phi \in IC_0^0(f)(M)} I_\phi(f) \cap \{ x \in QW(f) \setminus W(f) | \exists \omega \in M_x(f) \ s.t. \ S_\omega = C_x \}$$

contains a dense $G_\delta$ subset of $M$.

Proof. For the first case, by the main result of [32]

$$G_{\text{max}} = \{ x \in \Lambda | M_x(f) = M(\Lambda, f) \}$$

is residual in $X$. Recall Theorem 3.5 of [1] that generic invariant measures have full support. Thus, forward the proof of Proposition 5.8 one can replace Lemma 4.6 can be replaced by Theorem 3.5 of [1] to prove.

For the second case, it is known that generic $f \in \text{Diff}^1_m(M)$ is transitive so that we can take $\Lambda = M$. Notice that Theorem 3.5 of [1] and the main result of [32] also can be stated as the volume-preserving case. Then the proof is similar as above. Here we omit the details. \[\square\]
6 Partition of $\phi$-regular set and its level-sets

In last section we divide irregular set of an observable function into several kinds of subsets, for which the points in a same subset have same asymptotic behavior. By contrast, in this section we will use same way to divide regular set of an observable function. Similar as Section 5 we divide Theorem 2.4 into four propositions as follows. Moreover, Theorem 2.2 (1) can be deduced from Theorem 2.4. Before that we need to recall some known results which are necessary when we study the level-sets of Birkhorff average.

Now let's recall the entropy-dense property in [25], which is stated for systems with approximate (L-)product property which is weaker than specification (also see [12]). So we can state this result for the shift of finite type. Roughly speaking, any invariant probability measure $\mu$ is the limit of a sequence of ergodic measures $\{\mu_n\}_{n=1}^\infty$ in weak* topology such that the entropy of $\mu$ is the limit of the entropies of $\mu_n$. Let $M(X)$ denote the space of all Borel probability measures.

**Lemma 6.1.** $T$ has entropy-dense property, that is, for any $\nu \in M(T, X)$, any neighborhood $G \subseteq M(X)$ of $\mu$ and any $h_\nu < h_\nu(T)$, there exists an ergodic measure $\mu \in G \cap M(T, X)$ such that $S_\mu \neq X$ and $h_\mu(T) > h_\nu$.

**Proof** Here we add $S_\mu \neq X$ in this lemma. In fact, note that $M(T, X)$ is not a singleton and thus we can take an open subset $G' \subseteq \overline{G} \subseteq G$ such that $\nu \in G'$ and $M(T, X) \setminus \overline{G'} \neq \emptyset$. Note that $M(T, X) \setminus \overline{G'}$ is open in $M(T, X)$ and for any $m \in M(T, X) \setminus \overline{G'}$ there is a periodic measure $\omega \in M(T, X) \setminus \overline{G'}$. Let $\omega \in M_{\text{erg}}(T, X)$.

From the proof of Proposition 2.3 (1) of [25], one construct a closed invariant set $Y$ and there exists $n_{G'} \in \mathbb{N}$ such that $h_{\text{top}}(T, Y) > h_\nu$ and for any $y \in Y$ and any $n \geq n_{G'}$, $\Upsilon_n(y) \in G'$. So for any $m \in M_{\text{erg}}(T, Y)$, by Birkhorff ergodic theorem there is $y \in Y$ such that $\Upsilon_n(y)$ converge to $m$ in weak* topology and thus $m \in \overline{G'}$. In other words, $M_{\text{erg}}(T, Y) \subseteq \overline{G'}$. If $Y = X$, then $\omega \in M_{\text{erg}}(T, X) = M_{\text{erg}}(T, Y) \subseteq \overline{G'}$, which contradicts $\omega \in M(T, X) \setminus \overline{G'}$. By Variational Principle, for $0 < \varepsilon < h_{\text{top}}(T, Y) - h_\nu$, take a $\mu \in M(T, Y)$ such that $h_\mu(T) > h_{\text{top}}(T, Y) - \varepsilon > h_\nu$. Then $\mu$ is the measure we need. For more details, see [25].

Remark that Lemma 6.1 implies the particular case of Corollary 4.11. However, in order to prove Theorem 2.4 it not only needs the approximation of entropy but also need the approximation of measures in weak* topology simultaneously. In other words, Corollary 4.11 is not enough to prove Theorem 2.4 so that here we state Lemma 6.1.

Now we divide Theorem 2.4 into several propositions and start to prove.

**Proposition 6.2.** (Theorem 2.4 (D*))

Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\phi)$,

$$h_{\text{top}}(T, R_{\phi, a}(T) \setminus R(T)) = h_{\text{top}}(T, R_{\phi, a}(T))$$

$$= \sup \{h_\nu(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\} > 0.$$
Fix \( a \in \text{Int}(L_\phi) \). The last equality is essentially from the proof of Proposition 7.1 in [20]. Since \( h_{\text{top}}(T, R_{\phi,a}(T) \setminus R(T)) \leq h_{\text{top}}(T, R_{\phi,a}(T)) \), then we only need to prove

\[
h_{\text{top}}(T, R_{\phi,a}(T) \setminus R(T)) \geq \sup \{ h_\rho(T) \mid \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0.
\]

Take a \( \omega \in M(T, X) \) with positive entropy. If \( \int \phi d\omega = a \), then

\[
\sup \{ h_\rho(T) \mid \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} > 0.
\]

Otherwise, without loss of generality, we assume \( \int \phi d\omega > a \). By Lemma [4.8] take a periodic measure \( \sigma \) such that \( \int \phi d\sigma < a \). Then we can choose suitable \( \xi \in (0, 1) \) such that \( \nu = \xi \omega + (1 - \xi)\sigma \) satisfies \( \int \phi d\nu = a \). Remark that \( h_\nu(T) \geq \xi \omega(T) > 0 \). So

\[
\sup \{ h_\rho(T) \mid \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} \geq h_\nu(T) > 0.
\]

By Lemma [4.8] we can take two different periodic measures of \( \mu_i (i = 1, 2) \) such that

\[
\int \phi d\mu_i < a < \int \phi d\mu_2.
\]

Then we can choose \( \theta \in (0, 1) \) such that \( \mu = \theta \mu_1 + (1 - \theta)\mu_2 \) has integral by \( \phi \) equal to \( a \). Clearly the entropy of \( \mu \) is zero.

Let \( F_a = \{ \rho \mid \int \phi d\rho = a \} \). If \( G^{F_a} = \{ x \in X \mid M_x(T) \subseteq F_a \} \), then by [2.8]

\[
G^{F_a} = R_{\phi,a}(T).
\]

For any \( \rho \in F_a \) with positive entropy and \( \tau \in (0, 1) \), define

\[
K_\tau = \{ t\rho + (1-t)(\tau \rho + (1-\tau)\mu) \mid t \in [0, 1] \}.
\]

Remark that \( h_\rho(T) > 0 = h_\nu(T) \) implies \( \rho \neq \mu \). So \( K_\tau \) is not a singleton and thus \( G_{K_\tau} \subseteq I(T) \).

Note that \( \int \phi d\mu = a = \int \phi d\rho \) implies for any \( \tau \in (0, 1) \), \( K_\tau \subseteq F_a \) so that \( G_{K_\tau} \subseteq G^{F_a} \), where \( G_{K_\tau} = \{ x \in X \mid M_x(T) = K_\tau \} \). Then \( G_{K_\tau} \subseteq G^{F_a} \cap I(T) \). Hence,

\[
h_{\tau \rho + (1-\tau)\mu}(T) = \inf_{m \in K_\tau} h_m(T) = h_{\text{top}}(T, G_{K_\tau}) \leq h_{\text{top}}(T, G^{F_a} \cap I(T)).
\]

Letting \( \tau \to 1 \), this implies \( h_\rho(T) \leq h_{\text{top}}(T, G^{F_a} \cap I(T)) \) so that

\[
\sup \{ h_\rho(T) \mid \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} \leq h_{\text{top}}(T, R_{\phi,a}(T) \setminus R(T)).
\]

**Proposition 6.3. (Theorem 2.4 (A*))** Let \( \phi : X \to \mathbb{R} \) be a continuous function with \( I_\phi(T) \neq \emptyset \). Then for any real number \( a \in \text{Int}(L_\phi) \),

\[
h_{\text{top}}(T, R_{\phi,a}(T) \cap W(T) \setminus R(T)) = \sup \{ h_\rho(T) \mid \rho \in M(T, X) \text{ and } \int \phi d\rho = a \}.
\]
Proof. Fix $a \in \text{Int}(L_\phi)$ and let $t = \sup\{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\}$. Fix $\varepsilon > 0$. We need to construct two measures as follows, which are also useful to prove other propositions.

**Lemma 6.4.** There are two different measures $\omega, \omega' \in M(T, X)(\omega \neq \omega')$ such that

$$h_\omega(T) > t - \varepsilon, h_{\omega'}(T) > t - \varepsilon$$

and $\int \phi d\omega = \int \phi d\omega' = a, S_\omega = S_{\omega'} = X$.

More precisely, by Lemma 4.8 we can take three different ergodic measures of $\mu_i (i = 0, 1, 2)$ with support $X$ such that

$$\int \phi d\mu_0 < a < \int \phi d\mu_1 < \int \phi d\mu_2.$$ 

Then we can choose suitable $\theta_i \in (0, 1)(i = 1, 2)$ such that $\nu_i = \theta_i \mu_0 + (1 - \theta_i) \mu_i$ satisfy

$$\int \phi d\nu_i = a, i = 1, 2.$$ 

Remark that by ergodicity of $\mu_i, \nu_1 \neq \nu_2$ and $S_{\nu_i} = S_{\mu_0} \cup S_{\mu_i} = X, i = 1, 2$.

By Proposition 6.2 we can take $\mu \in M(T, X)$ such that $\int \phi d\mu = a$ and $h_\mu(T) > t - \varepsilon$.

Then we can choose $0 < \theta < 1$ close to 1 such that $\omega = \theta \mu + (1 - \theta) \nu_1, \omega' = \theta \mu + (1 - \theta) \nu_2$ satisfy

$$h_\omega(T) = \theta h_\mu(T) + (1 - \theta) h_{\nu_1}(T) \geq \theta h_\mu(T) > t - \varepsilon,$$

$$h_{\omega'}(T) = \theta h_\mu(T) + (1 - \theta) h_{\nu_2}(T) \geq \theta h_\mu(T) > t - \varepsilon.$$ 

Remark that $\int \phi d\omega = \int \phi d\omega' = a, S_\omega = S_{\omega'} = X$ and $\nu_1 \neq \nu_2$ implies $\omega \neq \omega'$.

Now we continue the proof. Let $K = \{\tau \omega + (1 - \tau) \omega' | \tau \in [0, 1]\}$, then for any $m = \tau \omega + (1 - \tau) \omega' \in K$,

$$S_m = X, \quad h_m(T) \geq \min\{h_\omega(T), h_{\omega'}(T)\} > t - \varepsilon, \quad \int \phi dm = a.$$ 

By Lemma 4.3

$$h_{top}(T, G_K) = \inf\{h_m(T) | m \in K\} > t - \varepsilon,$$

where $G_K = \{x | x \in X \text{ and } M_x(T) = K\}$. We only need to prove $G_K \subseteq R_{\phi, a}(T) \cap W(T) \setminus R(T)$.

Fix $x \in G_K$. Then $M_x(T) = K$ so that for any $m \in M_x(T), \int \phi dm = a$. Thus $M_x(T) \subseteq \{\rho | \int \phi d\rho = a\}$ and so by (2.8) $x \in R_{\phi, a}(T)$. Notice that for any $m \in M_x(T), \int \phi dm = a$ by (4.13)

$$C_x = \bigcup_{m \in M_x(T)} S_m = \bigcup_{m \in K} S_m = X = S_m.$$ 

By (4.14) $x \in X = C_x \subseteq \omega_T(x)$ and thus $x \in \text{Rec}(T)$. So from (4.15) one has $x \in W(T)$. Since $M_x(T) = K$ is not a singleton, by (4.12) $x \in I(T)$. □
Proposition 6.5. (Theorem 2.4 (B')) Let \( \phi : X \to \mathbb{R} \) be a continuous function with \( I_\phi(T) \neq \emptyset \). Then for any real number \( a \in \text{Int}(L_\phi) \),
\[
\begin{align*}
h_{\text{top}}(T, R_{\phi, a}(T) \cap \{ x \in QW(T) \setminus W(T) | \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \})
&= \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \}.
\end{align*}
\]

Proof. Fix \( a \in L_\phi(T) \) and let \( t = \sup \{ h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} \). Fix \( \varepsilon \in (0, t) \). We need to construct a measure as follows, which is also useful to prove other propositions.

Lemma 6.6. There is a measure \( \omega \in M(T, X) \) such that
\[
h_\omega(T) > t - \varepsilon \text{ and } \int \phi d\omega = a, \quad S_\omega \subsetneq X.
\]

More precisely, take a \( \nu \in M(T, X) \) such that \( h_\nu(f) > t - \frac{\varepsilon}{3} \) and \( \int \phi d\nu = a \). By Lemma 4.8, take two periodic measures \( \nu_i(i = 1, 2) \) such that \( b_1 := \int \phi d\nu_i > a > \int \phi d\nu_2 := b_2 \). Let \( \delta > 0 \) small enough such that
\[
\min \left\{ \frac{b_1 - a}{b_1 - a + \delta}, \frac{a - b_2}{a - b_2 + \delta} \right\} > \frac{t - \varepsilon}{t - \frac{2\varepsilon}{3}}.
\]
Then by entropy-dense property of Lemma 6.1, we can take one ergodic measure \( \mu \) close to \( \nu \) enough (in weak* topology) such that \( S_\mu \subsetneq X \) and
\[
| \int \phi d\mu - a | = | \int \phi d\mu - \int \phi d\nu | < \delta, \quad h_\mu(f) > t - \frac{2\varepsilon}{3}.
\]
If \( \int \phi d\mu = a \), then take \( \omega = \mu \). Otherwise, \( \int \phi d\mu \neq a \). Without loss of generality, we assume \( \int \phi d\mu < a \). Take
\[
\omega = \frac{b_1 - a}{b_1 - \int \phi d\mu} \mu + (1 - \frac{b_1 - a}{b_1 - \int \phi d\mu}) \nu_1.
\]
Then \( \int \phi d\omega = a, \ h_\omega(f) \geq \frac{b_1 - a}{b_1 - \int \phi d\mu} h_\mu(f) + \frac{b_1 - a}{b_1 - \int \phi d\mu} h_\nu(f) > t - \varepsilon \). Recall that \( S_{\nu_1} \) is a finite closed set but \( X \setminus S_\mu \) is nonempty, open and thus uncountable. So \( S_\omega = S_\mu \cup S_{\nu_1} \neq X \).

Now we continue the proof. One can construct \( \omega' \) same as in Lemma 6.3 such that
\[
h_{\omega'}(T) > t - \varepsilon, \int \phi d\omega' = a \text{ and } S_{\omega'} = X.
\]
Clearly \( S_\omega \neq S_{\omega'} \) and thus \( \omega \neq \omega' \).

Let \( K = \{ \tau \omega + (1 - \tau)\omega' | \tau \in [0, 1] \} \), then for any \( m = \tau \omega + (1 - \tau)\omega' \in K \setminus \{ \omega \} \), \( S_m = X \) and for any \( m = \tau \omega + (1 - \tau)\omega' \in K \),
\[
h_m(T) \geq \min \{ h_\omega(T), h_{\omega'}(T) \} > t - \varepsilon, \int \phi dm = a.
\]
By Lemma 4.10,
\[ h_{\text{top}}(T, G_K) = \inf \{ h_m(T) \mid m \in K \} > t - \varepsilon, \]
where \( G_K = \{ x \in X \mid M_x(T) = K \} \). We only need to prove
\[ G_K \subseteq \{ x \in R_{\phi,a}(T) \cap QW(T) \setminus W(T) \mid \exists m \in M_x(T) \text{ s.t. } S_m = C_x \}. \]

Fix \( x \in G_K \). Then \( M_x(T) = K \) so that for any \( m \in M_x(T), \int \phi dm = a \). Thus
\[ M_x(T) \subseteq \{ \rho \mid \int \phi d\rho = a \} \]
and so by (2.8) \( x \in R_{\phi,a}(T) \). Notice that by (4.13)
\[ C_x = \bigcup_{m \in M_x(T)} S_m = \bigcup_{m \in K} S_m = X. \]
So by (4.17) \( C_x = X \) implies \( x \in QW(T) \). By (1.4) \( x \in \text{Rec}(T) \). Then from (4.15) \( C_x = X \neq S_{\omega} \) and \( \omega \in M_x(T) \) imply \( x \in X \setminus W(T) \). Recall \( S_{\omega'} = X \) and \( \omega' \in K = M_x(T) \). So
\[ x \in R_{\phi,a}(T) \cap \{ x \in QW(T) \setminus W(T) \mid \exists m \in M_x(T) \text{ s.t. } S_m = C_x \}. \]

**Proposition 6.7. (Theorem 2.4 (B))** Let \( \phi : X \to \mathbb{R} \) be a continuous function with \( I_{\phi}(T) \neq \emptyset \). Then for any real number \( a \in \text{Int}(I_{\phi}), \)
\[ h_{\text{top}}(T, R_{\phi,a}(T) \cap \{ x \in QW(T) \setminus W(T) \mid \forall m \in M_x(T) \text{ s.t. } S_m \neq C_x \}) \]
\[ = \sup \{ h_\rho(T) \mid \rho \in M(T,X) \text{ and } \int \phi d\rho = a \}. \]

**Proof.** Fix \( a \in L_{\phi}(T) \) and let \( t = \sup \{ h_\rho(T) \mid \rho \in M(T,X) \text{ and } \int \phi d\rho = a \}. \) Fix \( \varepsilon > 0 \). Take same \( \omega \) as in Lemma 6.6 such that \( \int \phi d\omega > a, h_\omega(f) > t - \varepsilon \) and \( S_\omega \neq X \).

Let \( \{ x_i \}_{i=1}^\infty \) and \( \{ m_i \}_{i=1}^\infty \) be same as in the proof Proposition 5.3 or Remark 5.4. They are composed of periodic points and periodic measures and \( \{ x_i \}_{i=1}^\infty = X, \{ m_i \}_{i=1}^\infty = M(T,X), \bigcup_{i \geq 1} S_{m_i} = \{ x_i \}_{i=1}^\infty \).

Let
\[ K_1 := \{ m \mid \int \phi dm > a, m \in \{ m_i \}_{i=1}^\infty \}, \]
\[ K_2 := \{ m \mid \int \phi dm < a, m \in \{ m_i \}_{i=1}^\infty \} \]
and
\[ K_3 := \{ m \mid \int \phi dm = a, m \in \{ m_i \}_{i=1}^\infty \}. \]

By Lemma 4.8 if let \( \mathcal{R} = \{ m_i \}_{i=1}^\infty \), then it is easy to see that \( K_1 \) and \( K_2 \) are countable. Remark that \( K_3 \) may be empty, finite or countable. Without loss of generality, we can
assume $K_i = \{m_j^{(i)}\}_{j=1}^\infty, i = 1, 2, 3..$ Then we can choose suitable $\theta_{j,k} \in (0, 1)$ such that $m_{j,k} = \theta_{j,k}m_j^{(1)} + (1 - \theta_{j,k})m_k^{(2)}$ satisfies $\int \phi dm_{j,k} = a$. For any $n \geq 1$, let

$$l_n = \frac{\sum_{j+k=n} m_{j,k} + m_n^{(3)}}{n},$$

then $\int \phi dl_n = a$. Remark that every $S_{l_n}$ is composed of finite periodic orbits and $\bigcup_{n \geq 1} S_{l_n} = \{x_i\}_{n=1}^\infty$ is dense in $X$.

Take an increasing sequence of $\{\theta_i | \theta_i \in (0, 1)\}_{i=1}^\infty$ convergent to $1$ such that $h_{\omega_i}(T) > t - \varepsilon$ where $\omega_i = \theta_i \omega + (1 - \theta_i)l_i$. Remark that $S_{\omega_i} = S_\omega \cup S_{l_i}$. In particular, for all $i$, $\int \phi d\omega_i = \int \phi d\omega = a$.

Now we consider

$$K = \{\omega\} \cup \bigcup_{i \geq 1} \{\tau \omega_i + (1 - \tau) \omega_{i+1} | \tau \in [0, 1]\}.$$

Then $K$ is nonempty connected compact subset of $M(T, X)$ because $\omega_i \to \omega$ in weak* topology. So we can use Lemma 4.9 to get that

$$h_{top}(T, G_K) = \inf \{h_\nu(T) | \nu \in K\} = \min \{\inf_{i \geq 1} \{h_{\omega_i}(T)\}, h_\omega(T)\} \geq t - \varepsilon,$$

where $G_K = \{x \in X | M_x(T) = K\}$. We only need to prove

$$G_K \subseteq \{x \in R_{\phi,a}(T) \cap QW(T) \setminus W(T) | \forall m \in M_x(T) \text{ s.t. } S_m \neq C_x\}.$$

Fix $x \in G_K$. Note that for any $m \in M_x(T) = K$, $\int \phi dm = a$. Thus $M_x(T) \subseteq \{\rho | \int \phi d\rho = a\}$ and so by (2.3) $x \in R_{\phi,a}(T)$. Notice that by (4.13)

$$C_x = \bigcup_{m \in M_x(T)} S_m = \bigcup_{m \in K} S_m \supseteq \bigcup_{i \geq 1} S_{\omega_i} = \bigcup_{i \geq 1} (S_{l_i} \cup S_\omega) \supseteq \bigcup_{i \geq 1} \{x_i\} = X.$$

So $X = C_x$. From (4.17) one has $x \in QW(T)$. By (1.3) $x \in Rec(T)$. Notice that from (4.15) $C_x = X \neq S_\omega$ and $\omega \in M_x(T)$ imply $x \in X \setminus W(T)$. For any $m \in M_x(T) = K$, there is some $i$ such that $S_m \subseteq S_{l_i} \cup S_{l_{i+1}} \subseteq X = C_x$. This is because $S_{l_i} \cup S_{l_{i+1}}$ is a finite set but $X \setminus S_\omega$ is nonempty, open and thus uncountable. Hence $S_m \neq C_x$.

Proposition 6.8. (Theorem 2.4 (C*)) Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,

$$h_{top}(T, R_{\phi,a}(T) \cap I(T) \setminus QW(T)) = \sup \{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\}.$$  

Proof. Fix $a \in L_\phi(T)$ and let $t = \sup \{h_\rho(T) | \rho \in M(T, X) \text{ and } \int \phi d\rho = a\}$. Fix $\varepsilon > 0$. Take same $\omega$ as in Lemma 6.6 such that $\int \phi d\omega = a$, $h_\omega(f) > t - \varepsilon$ and $S_\omega \neq X$.

Let $B := \{m | S_m \setminus S_\omega \neq \emptyset\}$. Since $X \setminus S_\omega$ is nonempty, open and invariant, by density of periodic points(Lemma 4.15) $B \neq \emptyset$. If there is $m \in B$ such that $\int \phi dm = a$, take $\mu = m$. Otherwise, for any $m \in B$, $\int \phi dm \neq a$. Take one $\mu_1 \in B$. Then $\int \phi dm_{\mu_1} \neq a$. Without loss
of generality, we assume $\int \phi d\mu_1 < a$. By Lemma 4.9 we can take a periodic measure $\mu_2$ such that $\int \phi d\mu_2 > a$. Then we can choose suitable $\theta \in (0, 1)$ such that $\mu = \theta \mu_1 + (1 - \theta)\mu_2$ satisfies $\int \phi d\mu = a$. Remark that $S_\mu = S_{\mu_1} \cup S_{\mu_2}$ and $S_\mu \setminus S_\omega$ is composed of one periodic orbit or two periodic orbits containing $S_{\mu_1}$.

Take $\theta' \in (0, 1)$ close to 1 such that $\omega' = \theta' \omega + (1 - \theta')\mu$ satisfies $h_{\omega'}(T) > t - \varepsilon$. Remark that $\int \phi d\omega' = a$ and $S_{\omega'} \setminus S_\omega = S_{\mu_1} \setminus S_\omega$ is composed of one periodic orbit or two periodic orbits. So $S_{\omega'} \setminus S_\omega$ is nonempty, finite, invariant and compact and thus $\omega \neq \omega'$.

Let $K = \{\tau\omega + (1 - \tau)\omega' | \tau \in [0, 1]\}$, then for any $m \in K$,

$$h_m(T) \geq \min\{h_\omega(T), h_{\omega'}(T)\} > t - \varepsilon, \int \phi dm = a.$$  

By Lemma 4.9

$$h_{\text{top}}(T, G_K) = \inf\{h_m(T) | m \in K\} > t - \varepsilon,$$

where $G_K = \{x \in X | M_x(T) = K\}$. We only need to prove $G_K \subseteq R_{\phi, a}(T) \cap I(T) \setminus QW(T)$.

Fix $x \in G_K$. Since $K$ is not singleton, by (4.12) $x \in I(T)$. Note that for any $m \in M_x(T) = K$, $\int \phi dm = a$. Thus $M_x(T) \subseteq \{\rho | \int \phi d\rho = a\}$ and so by (2.9) $x \in R_{\phi, a}(T)$. If $x \in QW(T)$, by (1.4) $x \in \text{Rec}(T)$. By (4.10) $x \in C_x = \omega_T(x)$. Then by (4.13)

$$x \in \omega_T(x) = C_x = S_{\omega'} = S_{\omega} \setminus (S_{\omega'} \setminus S_\omega).$$

So $x \in S_{\omega}$ or $S_{\omega'} \setminus S_\omega$. Recall that $S_{\omega'} \setminus S_\omega$ is compact and invariant and so does $S_{\omega}$. One has $\text{Orbit}(x) \subseteq S_\omega$ or $\text{Orbit}(x) \subseteq S_{\omega'} \setminus S_\omega$ and thus $\omega_T(x) \subseteq S_\omega \subseteq S_{\omega} \setminus S_\omega = C_x = \omega_T(x)$ or $\omega_T(x) \subseteq (S_{\omega'} \setminus S_\omega) \subseteq S_\omega \setminus (S_{\omega'} \setminus S_\omega) = C_x = \omega_T(x)$. That $\omega_T(x) \subsetneq \omega_T(x)$ is a contradiction. Hence, $x$ is not in $QW(T)$.

Remark 6.9. Here we remark that the measure $\omega'$ can be also chosen as follows. By Remark 5.6 that periodic measures in $T$-invariant open set $X \setminus S_\omega$ are dense in all $T$-invariant measures, by Lemma 4.8 we can take two periodic measures $\mu_i(i = 1, 2)$ such that the support of every $\mu_i$ is in $X \setminus S_\omega$ and $\int \phi d\mu_1 < a < \int \phi d\mu_2$. Then take suitable $\theta \in (0, 1)$ such that $\int \phi d\mu = a$ where $\mu = \theta \mu_1 + (1 - \theta)\mu_2$. Take $\theta'$ close to 1 such that $\omega' = \theta' \omega + (1 - \theta')\mu$ satisfies $h_{\omega'}(T) > t - \varepsilon$. Remark that $\int \phi d\omega' = a$ and $S_{\omega'} \setminus S_\omega$ is composed of two periodic orbits.

Proof of Theorem 2.2 (1)

If $I_{\phi}(T)$ is empty, then $R_{\phi}(T) = X$. By Theorem 2.1 it is trivial. Now we consider $I_{\phi}(T) \neq \emptyset$.

Fix $\varepsilon > 0$. We can take a number $a \in \text{Int}(L_{\phi})$ such that $h_{\text{top}}(T, R_{\phi, a}(T)) > h_{\text{top}}(T) - \varepsilon$. In fact, by Variational Principle, we can take one $\mu \in M(T, X)$ such that

$$h_\mu(T) > h_{\text{top}}(T) - \varepsilon.$$  

Then by (2.11) there is another $\nu \in M(T, X)$ such that

$$\int \phi d\mu \neq \int \phi d\nu.$$  

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Take $\theta \in (0, 1)$ close to 1 such that
\[ h_\omega(T) > h_{\text{top}}(T) - \varepsilon, \]
where $\omega = \theta \mu + (1 - \theta) \nu$. Remark that the value of $\int \phi d\omega$ is between $\int \phi d\mu$ and $\int \phi d\nu$ so that if $a = \int \phi d\omega$, then $a \in \text{Int}(L_\phi)$ and by Theorem 6.2
\[ h_{\text{top}}(T, R_{\phi,a}(T)) = \sup \{ h_\rho(T) \mid \rho \in M(T, X) \text{ and } \int \phi d\rho = a \} \geq h_\omega(T) > h_{\text{top}}(T) - \varepsilon. \]

Recall that $R_{\phi}(T) = \bigsqcup_{b \in \mathbb{R}} R_{\phi,b}(T)$. So by Theorem 2.4,
\[ h_{\text{top}}(T, R_{\phi}(T) \cap \xi) \geq h_{\text{top}}(T, R_{\phi,a}(T) \cap \xi) = h_{\text{top}}(T, R_{\phi,a}(T)) > h_{\text{top}}(T) - \varepsilon, \]
where
\[ \xi = W(T) \setminus R(T), \]
\[ QW(T) \setminus W(T), \]
\[ I(T) \setminus QW(T), \]
\[ \{ x \in QW(T) \setminus W(T) \mid \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x \}, \]
\[ \{ x \in QW(T) \setminus W(T) \mid \forall \mu \in M_x(T) \text{ s.t. } S_\mu \neq C_x \}. \]

By arbitrariness of $\varepsilon$ we complete the proof. \qed

From the proofs of above propositions, for shifts, all the considered sets above contains some $G_K$ which is dense in the whole space by Lemma 4.7

**Proposition 6.10.** Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\phi)$, each one of following subsets is dense in $X$:

(I) 
\[ R_{\phi,a}(T) \cap W(T) \setminus R(T); \]

(II) 
\[ R_{\phi,a}(T) \setminus QW(T) = R_{\phi,a}(T) \cap I(T) \setminus QW(T); \]

(III) 
\[ R_{\phi,a}(T) \cap \{ x \in QW(T) \setminus W(T) \mid \exists \omega \in M_x(T) \text{ s.t. } S_\omega = C_x \}; \]

(IV) 
\[ R_{\phi,a}(T) \cap \{ x \in QW(T) \setminus W(T) \mid \forall m \in M_x(T) \text{ s.t. } S_m \neq C_x \}. \]

Before ending this section we point out a result on the continuity of entropy function when the level-sets change.
Proposition 6.11. Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then every entropy map $\Psi_\xi : \text{Int}(L_\phi) \to \mathbb{R}$:

$$a \mapsto h_{\text{top}}(f|_{\xi \cap R_\phi,a(T)})$$

is a positive concave function and thus continuous, where

$$\xi = W(T) \setminus R(T),$$

$$QW(T) \setminus W(T),$$

$$I(T) \setminus QW(T),$$

$$\{x \in QW(T) \setminus W(T) | \exists \mu \in M_x(T) \text{ s.t. } S_\mu = C_x\},$$

$$\{x \in QW(T) \setminus W(T) | \forall \mu \in M_x(T) \text{ s.t. } S_\mu \neq C_x\}.$$

Proof. For any $a_1, a_2 \in L_\phi, \theta \in [0, 1]$,

$$\theta \sup \{h_\mu(f) : \int \phi(x)d\mu = a_1\} + (1 - \theta) \sup \{h_\mu(f) : \int \phi(x)d\mu = a_2\}$$

$$= \sup \{\theta h_{\mu_1}(f) + (1 - \theta) h_{\mu_2}(f) : \int \phi(x)d\mu_i = a_i, i = 1, 2\}$$

$$= \sup \{h_{\theta \mu_1 + (1 - \theta) \mu_2}(f) : \int \phi(x)d\mu_i = a_i, i = 1, 2\}$$

$$\leq \sup \{h_\mu(f) : \int \phi(x)d\mu = \theta a_1 + (1 - \theta)a_2\}$$

By classical convex analysis theory, convex function is always (locally Lipshitz) continuous over interior subset of the domain. By the propositions in this section we complete the proof. \hfill \Box

However, it is unknown the higher smoothness of the concave function $\Psi_\xi$.

7 Comments and questions

As said in the introduction, we believe the above results are adaptable to more systems. Here we consider time-1 map of hyperbolic flows and $\beta-$shifts.

Firstly we recall the role of Corollary 4.11 in the above proof. It is used to choose an ergodic measure with ‘large’ entropy and non-full support. So one can use Lemma 6.1 to replace the role. However, we prefer to state Corollary 4.11 because it is familiar with more people. Note that from Lemma 6.1 the ergodic measures with non-full support are entropy-dense. In particular ergodic measures are dense in $M(X, T)$. Recall that the set of ergodic measures is naturally a $G_\delta$ set, for example, see [1].

Secondly, one needs that the periodic points are dense in the whole space $X$ and and the periodic measures are dense in $M(X, T)$.

Thirdly, recall that the main technique of Lemma 4.7 was originally stated for any system with $g-$almost product property and uniform separation property. And recall that Lemma 6.1 holds for any system with $g-$almost product property.
From Proposition 21.11 of [10] we know that the set of invariant measures with full support is either empty or dense $G_δ$ in $M(X, T)$. So, if there is an invariant measure $\rho$ with full support, then the set of invariant measures with full support is dense $G_δ$ in $M(X, T)$. Recall that the intersection of two dense $G_δ$ sets is still dense $G_δ$ and from Lemma 6.1 the set of ergodic measures is dense $G_δ$. Thus, in this case, the set of ergodic measures with full support is dense $G_δ$ in $M(X, T)$. Moreover, we claim the set of invariant measures with full support is entropy-dense in $M(X, T)$.

Recall that the intersection of two dense $G_δ$ sets is still dense $G_δ$ and from Lemma 6.1 the set of ergodic measures is dense $G_δ$. Thus, in this case, the set of ergodic measures with full support is dense $G_δ$ in $M(X, T)$. Then the proofs of Theorem 2.1, 2.2 and 2.4 can be adaptable to more systems.

**Theorem 7.1.** Let $f$ be a continuous map of a compact metric space $X$ with $g$–almost product property and uniform separation property. If the periodic points are dense in $X$ and the periodic measures are dense in $M(X, T)$, then the results of Theorem 2.1 2.2 and 2.4 hold.

In particular, we have

**Theorem 7.2.** Let $f$ be a continuous map of a compact metric space $X$ with $g$–almost product property and uniform separation property. Then the results of Theorem 2.1 2.2 and 2.4 hold.

If we do not have any information of periodic points and measures, based on above analysis, the proofs of (A), (A'), (A''), (A*), (B_1), (B''_1) still work. That is,

**Theorem 7.3.** Let $f$ be a continuous map of a compact metric space $X$ with Bowen’s specification and uniform separation property. If there is an invariant measure with full support, then the results of

$$(A), (A'), (A''), (A*), (B_1), (B''_1)$$

in Theorem 2.1 2.2 and 2.4 hold. In particular, (B), (B'') hold.

### 7.1 Time-1 map of hyperbolic flows

Let $f : X → X$ be the time-1 map of a transitive Anosov flow of a compact Riemannian manifold $X$. In this case, $f$ is partially hyperbolic with one-dimension central bundle. The system $f$ of (C) is partially hyperbolic with one dimensional central bundle and thus $f$ is far from tangency so that $f$ is entropy-expansive from [19] (or see [11, 27]). Recall that from [20] entropy-expansive implies asymptotically $h$–expansive and from [20] (Theorem 3.1) any expansive or asymptotically $h$–expansive system satisfies uniform separation property. Recall that the maximal entropy measure of the flow has full support and note that the invariant measure of the flow is also invariant measure for the time-1 map. Thus, $f$ has the results of Theorem 7.3.
7.2 $\beta$–shift

Let us recall the definition of $\beta$–shift ($\beta > 1$) $(\Sigma_\beta, \sigma_\beta)$ in [36] (Chapter 7.3). We only need to consider that $\beta$ is not an integer. Consider the expansion of 1 in powers of $\beta^{-1}$, i.e.

$$1 = \sum_{n=-\infty}^{+\infty} a_n \beta^{-n}$$

where $a_1 = [\beta]$ and $a_n = \beta^n - \sum_{i=-\infty}^{n-1} a_i \beta^{n-i}$. Here $[t]$ denotes the integral part of $t \in \mathbb{R}$. Let $k = [\beta] + 1$. Then $0 \leq a_n \leq k-1$ for all $n$ so we can consider $a = \{a_n\}_1^\infty$ as a point in the space $X = \prod_{n=1}^{+\infty} Y$ where $Y = \{0, 1, \ldots, k-1\}$. Consider the lexicographical ordering on $X$, i.e. $x = \{x_n\}_1^\infty < y = \{y_n\}_1^\infty$ if $x_j < y_j$ for the smallest $j$ with $x_j \neq y_j$. Let $f : X \to X$ denote the one-sided shift transformation. Note that $f^n a \leq a$ for all $n \geq 0$. Let

$$\Sigma_\beta := \{x = \{x_n\}_1^\infty \mid x \in X \text{ and } f^n(x) \leq a \text{ for all } n \geq 0\}.$$ Then $\Sigma_\beta$ is a closed subset of $X$ and $f(\Sigma_\beta) = \Sigma_\beta$. Let $\sigma_\beta := f|_{\Sigma_\beta}$. Then $(\Sigma_\beta, \sigma_\beta)$ is one-sided $\beta$–shift. One can obtain the two-sided $\beta$–shift by letting

$$\hat{\Sigma}_\beta := \{x = \{x_n\}_{-\infty}^\infty \mid x \in \prod_{n=-\infty}^{+\infty} Y \text{ and } (x_i, x_{i+1}, \ldots) \in \Sigma_\beta \text{ for all } i \in \mathbb{Z}\}.$$ Then $\hat{\Sigma}_\beta$ is a closed subspace of $\prod_{n=-\infty}^{+\infty} Y$ invariant under the two-sided shift

$$\hat{f} : \prod_{n=-\infty}^{+\infty} Y \to \prod_{n=-\infty}^{+\infty} Y.$$ The topological entropy of $\beta$–shift ($\beta > 1$) is $\log \beta$. Remark that by Variational Principle, there is an ergodic measure with positive entropy. Note that the Dirac measure supported on the fixed point $x = \{0\}_1^\infty \in \Sigma_\beta$ has zero entropy. So every $\beta$–shift is not uniquely ergodic.

It is known that every $\beta$–shift ($\beta > 1$) $(\Sigma_\beta, \sigma_\beta)$ is expansive (as a subshift of finite type), its maximal entropy measure has full support and satisfies $g$–almost product property from [26] (see the Example on P.934). So every $\beta$–shift has the results of Theorem 7.3.

7.3 (Almost) specification

As Thompson [35] proved that for any system with almost specification, every $\phi$–irregular set either is empty or carries full topological entropy, we hope to generalize the above results to any system with (almost) specification. The problem is

**Question 7.4.** Let $f$ be a continuous map of a compact metric space $X$ with (almost) specification. Whether all the results in Theorem 2.1, 2.2 and 2.4 hold? If not, which results hold, which results do not hold and how to construct such a counterexample?

By the idea of Thompson [35], one needs to take two needed ergodic measures and then use these two measures to construct a set $F \subseteq I_\beta(T)$ such that the topological entropy of $F$ is larger than $h_{top}(f) - \epsilon$. In this process, Entropy Distribution Principle plays an important role. One can see [34, 35] for more details. The constructed measures are required ergodic.
However, the constructed measures in present paper are not all ergodic so that we are
not sure the idea of [34, 35] works. So it is still an interesting job to find more general
adaptable topological or smooth dynamical systems. In particular, for diffeomorphisms,
recall that uniformly hyperbolic systems are not dense in the set of diffeomorphisms so it is
not known that the results of present paper can be adaptable to generic diffeomorphisms.
Inspired by Theorem 5.10, it is possible to ask

**Question 7.5.** Let \( \Lambda \) be an isolated non-trivial transitive set of a \( C^1 \) generic diffeomor-
phism \( f \in \text{Diff}^1(M) \) or let \( f \in \text{Diff}^1_m(M) \) be a \( C^1 \) generic volume-preserving diffeomorphism.
Then whether all the results in Theorem 2.1, 2.2 and 2.4 hold?

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