Manifold and metric in numerical solution of the quasi-static electromagnetic boundary value problems

Pasi Raumonen*, Saku Suuriniemi, Timo Tarhasaari, Lauri Kettunen
Tampere University of Technology, Institute of Electromagnetics,
P.O. Box 692, FIN-33101 Tampere, Finland.

Abstract

Classical vector analysis is the predominant formalism used by engineers of computational electromagnetism, despite the fact that manifold as a theoretical concept has existed for a century. This paper discusses the benefits of manifolds over the traditional approach in practical problems of modelling. With a structural approach, it outlines the role and interdependence of coordinate systems, metric, constitutive equations, and fields, and relates them to practical problems of quasi-static computational electromagnetics: mesh generation, open-boundary problems, and electromagnetic-mechanical coupled problems involving motion and deformation. The proposed procedures also imply improvements to the flexibility of the modelling software.

1 Introduction

The objective of solver software systems for electromagnetic boundary value problems (BVP) is to solve an arbitrary problem from a specific problem class. However, this is not possible in practice: One of the restricting factors is the difficulty of the mesh generation. This is related to the fact that to pose a BVP, the user needs to define a geometry of the BVP domain. The geometry is described with a coordinate system and certain choices of

*This work was supported by Academy of Finland, project number 5211066.
coordinate systems may cause too much round-off error with floating point arithmetic.

Often the problem geometries lend themselves to a particular coordinate system, and therefore the software systems usually offer a list of coordinate systems to choose from. To offer a user interface with a list of coordinate systems, many structures in the software are fixed: For example, the software often has one intrinsic coordinate system, usually Cartesian, which it uses in all calculations. Furthermore, the basic material parameters, such as the permittivity of a vacuum, as well as the metric (inner product) in the intrinsic coordinate system are pre-decided. When the user gives the model’s geometry, for example, in the spherical coordinates, the software maps them to the intrinsic Cartesian system.

Frequently, the floating point problems leave the modeler without any mesh at all and prevent the solution of the BVP: Practical computations with the intrinsic coordinate system requires that the coordinate numbers are represented with floating point numbers. This makes mesh generation particularly challenging for BVPs with details of the domain varying significantly in size. For example, consider the power line on top of figure 1. The thickness of the cables and the structures in the supports are much smaller in scale than the distance between the two supports. Meshes are difficult to generate because the floating point representation gives us labels to only a finite, although large, number of points. For example, as a floating point number grows in magnitude, the distance between points separable by numeric labels increases, and, at some point, round-off errors become big enough to make labels of distinct points coincide unintentionally. Generally, mesh generation is based on geometric predicates [1], some of which are difficult to calculate with floating point numbers. Mesh generation becomes hard or even impossible with double precision floating point numbers when the scale variation, the ratio of the largest and the smallest dimensions, is in the order of $10^4$ or higher. This estimate for scale variation is based on the experiences with the BVP presented in the section 7.

The mesh generation problems arise from the way the BVP was described in a particular coordinate system. Is it possible to alleviate these problems? With the help of modern mathematics, manifolds and differential geometry, the role of mathematical structures on the description of a BVP can be outlined. In this paper we strongly promote a structural approach. Particularly, we will show that for a given BVP, the elements of the triplet \{coordinate system, metric, material parameters\} are interdependent, and the triplet constitutes an entity. In fact, there is a whole class of these triplets for each BVP: Each triplet describes a geometry of the BVP domain and the constitutive equations, which are needed to define an electromagnetic BVP. Specifically,
the numerical values of material parameters are respect to given metric. For example, the numerical value of the permittivity of vacuum is different with respect to meters than with respect to inches. Although the metric appears in the triplets, the electromagnetic theory essentially concerns the works related to displacements of particles and the total energy of systems. The role of metric is central only in the classical view—vector analysis—whereas the modern view of differential geometry reveals the independence of the electromagnetic theory from a particular metric. This has been known for long \[2\] \[3\] \[4\] \[5\] \[6\], but not that well recognized in engineering. Hence, the modeller may select the metric at will. This is much greater a freedom in the context of manifold than a redefinition of the inner product in the single codomain vector space of a vector field in the classical view. We use the modern view to show how to formulate electromagnetic problems with the classical vector formalism in different metrics.

From the software design point of view, there is no need to fix the whole triplet in a software system. Just like the software system is not intended to solve a particular problem, but any problem from a class of problems, the solver software should provide the possibility to use any triplet from the triplet class. It should not be even restricted to a single triplet, because manifold promotes local coordinate systems that provide further flexibility. This will help overcome floating point problems in the calculations of the geometric predicates and thus will enlarge the class of practically solvable problems: Our approach suggests giving up the metric and the coordinate system strongly suggested by human intuition. For example, if the mesh generation fails with the coordinate system in the top of figure 1, one could use the coordinate system in the bottom of the same figure, when the calculations of the predicates are easier. This time it is more likely that one gets...
a mesh and thus a numerical solution. However, there is usually a price to pay: numerical error can be large because the mesh quality may be poor.

The problems of mesh generation are just one example of practical challenges in numerics that can be engaged with tools provided by manifold. The problems with motion and deformations, such as the deformation effects of magnetostriction, can be modelled with a single mesh: motions and deformations can be modelled as changes in the metric. This can speed up the solution process substantially. Finally, the solutions for open boundary problems, that are solved by finite domains with so-called infinite elements in the boundary region, are an example of an application where a convenient triplet is chosen for solution.

2 Structural Background

We need many mathematical structures to pose and describe an electromagnetic BVP. The starting point of this description is a point set that is the BVP domain. The points of this set correspond to observations on distance measurements. Then we will add one structure at a time to this point set, such that finally we have all that is needed for the description.

2.1 Metric Space

At first we need to specify a point set for the BVP domain. The points in the domain correspond to the points observed in reality, in the following sense: The observed points are considered distinct if and only if a nonzero distance can be measured between them. We denote this abstract point set by $M$. By the way the set $M$ was constructed, it is naturally endowed with the structure of metric space [7]. The distance between a given pair of points refers to an observation of lengths in multiples of some reference object called the rigid body. (Observe here that numerical values of distances depend on the chosen rigid body but distances can be discussed without a specified reference body.) Furthermore, the metric induces the structure of topological space for $M$ [7]. The dimension of $M$, which is two or three, must be the same at every point of space $M$.

2.2 Manifolds

For computational analysis, we need to characterize displacements arithmetically and thus must parameterize the point set. In the classical view this is done with coordinate systems, which assign unique coordinates—that is, a tuple of real numbers—to each point of $M$. Formally this means a mapping
of type $M \to \mathbb{R}^n$. Furthermore, $\mathbb{R}^n$ is endowed with the norm topology and the mapping is homeomorphic, i.e. continuous bijection with continuous inverse. However, the abstract point set $M$ is often ignored completely and the points of $M$ are identified with the coordinate numbers. Furthermore, often a metric connotation is given to the coordinate numbers. Thus the domain of the BVP is taken to be some particular subset of $\mathbb{R}^n$. Yet the coordinates are just labels for the points of $M$, such that they allow arithmetic and need not carry metric information. The coordinates can be changed with diffeomorphic mappings of type $\mathbb{R}^n \to \mathbb{R}^n$.

Instead of a particular coordinate system, the modern view uses manifolds as domains for BVPs. In manifolds, the abstract point set $M$ is the primary object, which is not identified with any particular set of coordinate numbers. The coordinate systems, or charts, are considered secondary, which is natural because we can choose the charts in many ways, and the real numbers assigned to each point are not canonical. In particular, the physics—work and energy—does not depend at all on the chosen chart. Furthermore, $M$ is not expected to be coverable by a single chart, as was the case for coordinate systems: Charts are homeomorphic mappings from the open connected subsets of $M$ to the open subsets of $\mathbb{R}^n$. Moreover, every point of $M$ is required to be in the domain of some chart. Any topological space, that can be covered with charts, is called a topological manifold [8]. Thus the structure of topological manifold emphasizes the existence of charts, rather than a particular chart.

Topological manifold is not enough for physics, because the differentiability of functions needs an extra structure. Let us require a collection of charts, that are bound together by their transition maps, or change-of-chart maps: The differentiability of a real-valued function $h$ defined on $M$ is defined indirectly through the differentiability of the functions $h \circ f^{-1}$, where $f$ is a chart. The chart can be changed with the transition map $f \circ g^{-1}$ as follows: $(h \circ f^{-1}) \circ (f \circ g^{-1}) = h \circ g^{-1}$. We do not want the $n$-times differentiability of $h$ to depend on the choice of chart, and therefore require that the transition map be (at least) $n$-times differentiable. If this holds, the charts are considered equivalent. This relation constitutes an equivalence class, a differentiable structure, of charts. The set $M$ together with a differentiable structure is a differentiable manifold or here just manifold for short [8] [9] [10].

2.3 Standard Parameterization

The rigid body can be used as a pair of dividers to specify spheres in $M$. If the image of every sphere in $M$ under a chart is a sphere also in $\mathbb{R}^n$ when measured with the Euclidean 2-norm, we call the chart a standard parameterization. For example, the chart $f$ in figure 2 is a standard parameterization. All standard parameterizations are identical up to scaling,
Figure 2: Standard parameterization: The line drawing on top refers to a real coaxial cable. The rigid body we chose gave us a pair of dividers and enabled us to specify spheres. Chart \( f \) from \( M \) to \( \mathbb{R}^n \) is a standard parameterization, for the images of the spheres in \( M \) are also spheres in \( \mathbb{R}^n \) in the sense of Euclidean 2-norm. Consequently, chart \( g \) is not a standard parameterization.

Figure 3: Tangent space and tangent vector. A tangent vector \( \mathbf{v} \) of the tangent space of the point \( p \) in manifold \( M \).
rotation, and translation (plus reflection). The standard parameterizations are important because in literature the numerical values of the material parameters are given in a specific class of standard parameterizations. This is natural because the numerical values of the material parameters are meaningful only with respect to a given metric, and in practice, the given metric is always based on a length unit system in which some rigid body is chosen for reference.

2.4 Tangent Spaces

Charts enable us to talk about displacements in small neighborhoods, i.e. virtual translations: the possibility to parameterize $M$ implies a local $n$-dimensional vector space at each point of $M$. The vector space is called the tangent space, and the elements of this space are tangent vectors. Virtual translations can be recognized as tangent vectors, such as $v$ at point $p \in M$ in figure 3. Note here that whereas in the classical vector analysis the fields are mappings from the domain to a single vector space, the points of a manifold all have their distinct tangent spaces.

The electric field is an object that yields the virtual electromotive force $dU$ (virtual work up to charge) for every virtual translation $v$. In other words, the electric field is a functional field over the tangent spaces. Correspondingly, the electric flux is an object that yields the virtual flux for every pair of virtual translations, which define a virtual surface. That is, the electric flux is also a functional field over the tangent spaces. The solutions of quasi-static BVPs are such functional fields.

2.5 Inner Product

In the classical view we use a vector field to express the electric field. This possibility is due to Riesz representation theorem, that enables us to represent the electric field as a vector field, once an inner product is available. In the classical view the inner product is defined in the codomain vector space, but in the modern view each tangent space of $M$ has its own inner product. The pointwise representation of the electric field functional by a vector $E$ requires an inner product into each of them. At each point, with the inner product denoted by $(\cdot, \cdot)$, the virtual emf $dU(v)$ corresponding to the virtual translation vector $v$ can be represented as $dU(v) = (E, v)$. This representation makes the inner product an inherent part of the classical view: the vector $E$ depends on the chosen inner product whereas $dU$ does not. When the tangent spaces of a manifold are equipped with an inner product which varies smoothly from point to point, the manifold is called a Riemannian manifold. The inner products on the tangent spaces of
Figure 4: Correspondences of tangent vectors under different charts. A tangent vector $v$ at $p \in M$ corresponding to vectors $\mathbf{dr}_f$ at $f(p)$ and $\mathbf{dr}_g$ at $g(p)$.

$M$ are usually called metric tensors. They induce a metric\footnote{The metric tensors on the tangent spaces of $M$ are selected such that the induced metric on $M$ is the same as the metric $M$ already has: the metric given by the measurements with some rigid body.} on $M$ and also bring about the notion of angles between the tangent vectors\footnote{For practical computations with tangent vectors, such as evaluations of metric tensors, we need the counterparts of the tangent vectors and the metric tensors on $\mathbb{R}^n$. For each tangent vector $v$ at $p$, a unique counterpart vector (the so-called push forward\footnote{For each metric tensor $(\cdot, \cdot)_M$ given for $M$, there is a unique equivalent metric tensor $(\cdot, \cdot)_f$ for $\mathbb{R}^n$, that satisfies}}.

For practical computations with tangent vectors, such as evaluations of metric tensors, we need the counterparts of the tangent vectors and the metric tensors on $\mathbb{R}^n$. For each tangent vector $v$ at $p$, a unique counterpart vector (the so-called push forward\footnote{We can use these counterpart vectors $\mathbf{dr}_f$ to define metric tensors on $\mathbb{R}^n$ such that the tensors uniquely correspond to the ones on the tangent spaces of $M$: For each metric tensor $(\cdot, \cdot)_M$ given for $M$, there is a unique equivalent metric tensor $(\cdot, \cdot)_f$ for $\mathbb{R}^n$, that satisfies}) $\mathbf{dr}_f$ exists in $\mathbb{R}^n$, induced by the chart $f$, see figure $4$. We can use these counterpart vectors $\mathbf{dr}_f$ to define metric tensors on $\mathbb{R}^n$ such that the tensors uniquely correspond to the ones on the tangent spaces of $M$: For each metric tensor $(\cdot, \cdot)_M$ given for $M$, there is a unique equivalent metric tensor $(\cdot, \cdot)_f$ for $\mathbb{R}^n$, that satisfies\footnote{Equivalent metric tensors on $\mathbb{R}^n$ and $M$ make $f$ an isometry. However, the metric tensor chosen for $\mathbb{R}^n$ need not be equivalent with the one on $M$, i.e. the charts need not be isometries.}

$$
(\mathbf{dr}_f, \mathbf{dr}'_f)_f = (v, v')_M
$$

at all points and for all pairs $\mathbf{dr}_f, \mathbf{dr}'_f$ of $\mathbb{R}^n$ and all pairs of corresponding tangent vectors $v$, $v'$. Equivalent metric tensors on $\mathbb{R}^n$ and $M$ make $f$ an isometry. However, the metric tensor chosen for $\mathbb{R}^n$ need not be equivalent with the one on $M$, i.e. the charts need not be isometries.
3 Equivalent Descriptions of BVP

The theory of electromagnetics is independent of the chosen metric tensor and the chart, as stated in the introduction. In fact, Maxwell’s equations do not need a metric tensor at all. They are meaningful on a differentiable manifold (not necessarily a Riemannian manifold) [14][15][16]: the virtual works related to the displacements of a point charge are independent of the metric tensors and chosen charts. Naturally, the energy stored in a system does not depend on the chosen charts either. However, the constitutive equations need to be described to define the energy and there are at least two ways to proceed: First, a metric tensor is selected in order to represent the fields as vector fields, and constitutive equations are defined couple the two vector fields. Second way to proceed is to couple the functional fields directly, in which case the constitutive equations couple different types of fields. Although a metric tensor is not needed to construct the constitutive equations in this case, constitutive equations induce metric tensors on the manifold [16]. Yet in neither strategy does the energy itself depend on the selected or the induced metric tensor. Thus an electromagnetic BVP is independent of particular chart or metric tensor, and it is uniquely specified with manifold $M$, Maxwell’s equations, the constitutive equations, and the boundary values.

However, the material parameters, or rather the numeric values of permittivity, permeability, and conductivity depend on the chosen metric. Furthermore, to describe an abstract manifold concretely, we need to do that by charts, even though the manifold itself is independent of particular charts. Therefore, to pose the BVP, we need charts and metric tensors. When a chart and a metric tensor field are chosen, the numerical values of the material parameters can be specified. Then the material parameters are real- or matrix-valued fields defined on the chart. Thus we have a triplet {chart, metric tensor, material parameters} that describes a geometry of the BVP domain and the constitutive equation. Because we can choose any chart from the differentiable structure, and define any metric tensor on the chart, there are infinitely many geometries that we can use to pose the BVP. In fact, there is an equivalence class of these triplets {chart, metric tensor, material parameters}, that describe the same unique BVP. We call the equivalence relation producing these classes the material equivalence.

We can now express a BVP on any chart: we need a triplet {chart, metric tensor, material parameters}, Maxwell’s equations, and the boundary conditions. To express the BVP on any other chart, we only need to select another triplet, because Maxwell’s equations are independent of the metric and of particular chart, as are the boundary conditions. Notice, that we can also keep the same chart and change the metric tensor and the material
parameters. These descriptions of the BVP have equivalent solutions; that is, in any description, the electric vector field corresponds to the same virtual emf functional field. We obtain a new view of BVPs from the material equivalence: a BVP can be seen as a material equivalence class together with Maxwell’s equations and boundary conditions. This view suggests an approach to construct solver software systems:

Solver for BVPs should allow the user to select any triplet from the material equivalence class.

This way the user can use the triplet that is the most suitable for numerical solution. The benefit of choosing any triplet is that we can solve numerically a larger class of problems. Let us derive the material parameters for given chart and metric tensor, such that the material equivalence holds with another triplet that is known.

4 Material Parameters for Given Chart And Metric Tensor

To determine the material parameters for given chart and metric tensor, we require that the energy stored in a system and the virtual works related to the displacements of a point charge be invariant of our choice of chart and metric tensor. For simplicity, let us now focus on electrostatics, because other quasi-static cases can be treated conceptually in a similar fashion.

Following the standard convention, we use $\mathbf{dr} \cdot \mathbf{dr}'$ to denote $(\mathbf{dr}, \mathbf{dr}')$ in $\mathbb{R}^n$. Also, for each metric tensor field $\cdot$ in $\mathbb{R}^n$, there is a unique symmetric positive definite matrix field $S$ such that $\mathbf{dr} \cdot \mathbf{dr}' = \mathbf{dr}^T S \mathbf{dr}'$ holds pointwise for all $\mathbf{dr}$ and $\mathbf{dr}'$. Let us assume we know a triplet $\{f_i, \cdot_i, \epsilon_i\}$, consisting of chart $f_i$ of $M$, metric tensor $\cdot_i$ on $\mathbb{R}^n$ and the material parameters $\epsilon_i$.

Next, we choose chart $f_j$ from the differentiable structure and use metric tensor $\cdot_j$ on $\mathbb{R}^n$ and then determine the material parameters $\epsilon_j$ for the triplet $\{f_j, \cdot_j, \epsilon_j\}$ such that the two triplets are equivalent.

4.1 Invariance of Virtual Work

The counterparts or the push forwards of a tangent vector $\mathbf{v}$ at $p \in M$ are $\mathbf{dr}_i$ and $\mathbf{dr}_j$ under charts $f_i$ and $f_j$, respectively. The virtual emf $dU(\mathbf{v})$ corresponding to the virtual translation $\mathbf{v}$ is $\mathbf{E}_i \cdot_i \mathbf{dr}_i$, or $\mathbf{E}_j \cdot_j \mathbf{dr}_j$, where $\cdot_i$ and $\cdot_j$ are metric tensors in $\mathbb{R}^n$. These expressions must equal pointwise:

$$\mathbf{E}_j \cdot_j \mathbf{dr}_j = \mathbf{E}_i \cdot_i \mathbf{dr}_i. \quad (2)$$
Let \( f_j \circ f_i^{-1} : \text{ran}(f_i) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the transition map from the range of \( f_i \) to the codomain of \( f_j \). Denoting by \( J \) the Jacobian matrix \( \text{Jacobian matrix} \) of \( f_j \circ f_i^{-1} \), we have
\[
\text{dr}_j = J \text{dr}_i. \quad (3)
\]
Substituting (3) for \( \text{dr}_j \) in (2), and relying on the invariance of the virtual work, we get
\[
\mathbf{E}_j^T S_j J \text{dr}_i = \mathbf{E}_i^T S_i \text{dr}_i, \quad \forall \text{dr}_i, \quad (4)
\]
where \( S_i \) and \( S_j \) are the matrix presentations of the metric tensors \( \cdot_i \) and \( \cdot_j \). Because (4) holds for every point and for every virtual translation \( \text{dr}_i \), we can express \( \mathbf{E}_i \) in terms of \( \mathbf{E}_j \) and matrices \( S_i, S_j, \) and \( J \) as follows:
\[
\mathbf{E}_i = S_i^{-1} J^T S_j \mathbf{E}_j, \quad (5)
\]
If we know either \( \mathbf{E}_i \) or \( \mathbf{E}_j \), we know \( dU \) for every virtual translation, hence the solution of the BVP.

### 4.2 Invariance of Energy

The energy \( W \) stored in the electric field in \( f_i(M) \) can be expressed as
\[
W = \int_{f_i(M)} \mathbf{E}_i \cdot \epsilon_i \mathbf{E}_i \, dv_i = \int_{f_i(M)} \mathbf{E}_i^T S_i \epsilon_i \mathbf{E}_i \, dv_i, \quad (6)
\]
where the matrix \( \epsilon_i \) contains the material parameters in the chart \( f_i \). Equation (6) induces an inner product
\[
(\mathbf{E}_k, \mathbf{E}_l) = \int_{f_i(M)} \mathbf{E}_k^T S_i \epsilon_i \mathbf{E}_l \, dv_i \quad (7)
\]
for the fields. Here \( \mathbf{E}_k \) and \( \mathbf{E}_l \) are any two vector fields. Even though there is a metric tensor in the expression of the inner product of the fields, the energy is independent of chosen metric tensor: the product \( S_i \epsilon_i \) describes the constitutive equation, such that the value of energy is the same regardless of the choice of \( S_i \). This is what is meant by the claim that electromagnetic theory is independent of the chosen metric. Note that the inner product defined in (7) allows us to formulate the Galerkin method.

For any electric field, the energy stored in the field should be invariant under the choice of chart, or
\[
\int_{f_i(M)} \mathbf{E}_i^T S_i \epsilon_i \mathbf{E}_i \, dv_i = \int_{f_j(M)} \mathbf{E}_j^T S_j \epsilon_j \mathbf{E}_j \, dv_j \quad (8)
\]
must hold for any equivalent fields $E_i$ and $E_j$. Using the change of variables theorem [17], we can write the left hand side of (8) in the codomain of $f_j$ as

$$
\int_{f_j(M)} E_i^T S_i \epsilon_i E_i \, dv_i = \int_{f_j(M)} E_i^T S_i \epsilon_i E_i \mid J^{-1} \mid \, dv_j,
$$

(9)

where $|J^{-1}|$ is the determinant of the inverse of the Jacobian matrix $J$. Substituting the expression given in equation (5) for $E_i$ in (9), we get

$$
\int_{f_j(M)} E_j^T S_j J S_i^{-1} \epsilon_i S_i^{-1} J^T S_j E_j \mid J^{-1} \mid \, dv_i = \int_{f_j(M)} E_j^T S_j \epsilon_j E_j \, dv_j.
$$

(10)

Now, because this holds for any vector field $E_j$ defined on $f_j(M)$, the matrix $\epsilon_j$ can be written in terms of matrices $\epsilon_i, S_i, S_j, J$, as follows:

$$
\epsilon_j = J \epsilon_i S_i^{-1} J^T S_j \mid J^{-1} \mid.
$$

(11)

5 Equivalence of Numerical Solutions

The solutions for equivalent triplets are equivalent from a mathematical point of view, and so are the numerical solutions: Next we define equivalent meshes and show that under equivalent meshes, the system matrices in FEM are identical up to the round-off errors of the floating point representation.

First, assume a mesh in $M$. The mesh points in $M$ have corresponding points in every codomain of the chart of every triplet; consequently, we get an equivalent mesh for every chart (an example of equivalent meshes in figure 5). Assuming real number arithmetics, equation (8) implies that the system matrices related to them are identical by construction. Thus the FEM solutions are also equivalent. However, in practice the system matrices are assembled and the numerical solutions are calculated with floating point numbers, and this leads to round-off errors. Thus the system matrices are identical only up to the round-off errors.

Although the solutions are equivalent, in practice every BVP has triplets—sometimes including the standard parameterizations—that do not allow mesh generation, and thus no numerical solution. In these cases, mesh generation may well be possible with some other triplets. However, if the mesh quality criteria are not changed according to the chosen triplet, then there is usually

---

2 Notice that if the edges of mesh elements are straight in some chart, in an equivalent mesh of some other chart the edges can be curved.

3 Elements that are long in one direction only can give good solutions if the field has little variations in that direction.
Figure 5: Example of equivalent meshes. Two equivalent meshes for the codomains of the charts \( f \) and \( g \) of figure 2. The mesh in the codomain of the standard parameterization \( f \) has long elements in the middle, which may imply a poor quality. However, it is not always possible to generate any reasonable mesh for the standard parameterizations.

...
of this possibility in practice. First, we show how a chart can be changed when the metric tensor in $\mathbb{R}^n$ is fixed, which is usually the case. Second, we suggest a scheme to use multiple charts in mesh generation and problem solution. Third application is open boundary problems which are solved by selecting a triplet in which the problem domain in the chart is bounded. Finally, we discuss how to model motion with only one mesh by changing the metric.

6.1 Reparameterization Under Fixed Chart Metric

Because the numerical values of the material parameters are always given in literature in some length unit system, it is easiest to describe the constitutive relations with isometric standard parameterizations. Consequently, the programmer of a finite element software system often assumes that the user applies only isometric standard parameterizations; that is, the user can control only the unit of length, and rotate and translate the model. Furthermore, due to the isometry assumption, the metric tensor is implicitly fixed such that it induces the Euclidean 2-norm. Therefore the matrix representing the metric tensor in the intrinsic coordinate system of the solver is implicitly hardwired to be the identity matrix.

Let us assume that we know how to express a BVP in some isometric standard parameterization $f$, that is, we know the triplet $\{f, S_f, \epsilon_f\}$. However, instead of using $f$ directly, we would like to reparameterize with chart $g$, which is not a standard parameterization, to avoid floating point problems in numerical modeling. Because of the same hardwired metric tensor in the codomains of $f$ and $g$, the distances between any two coordinates (tuples of real numbers) do not change when changing chart from $f$ to $g$. However, a change of chart makes the coordinates correspond with different points in the manifold $M$, and thus the change of chart in this case implies a change of metric, hence the chart $g$ is not isometry.

To set up our BVP correctly under $g$, we must impose the material equivalence on $g$. For this we can use only the material parameters $\epsilon_g$, because $g$ is chosen by the modeller, and the metric tensor is hardwired by the programmer. Equation (11) shows that the material parameters $\epsilon_g$ require information on $g$ relative to $f$. Yet everything related to $f$, including $f$ itself, exists only in the modeller’s mind. We are solving the BVP using a triplet $\{g, S_g, \epsilon_g\}$, and we interpret this description of the BVP under $g$ as being equivalent to that under $f$. However, the software programmer who assumes only standard parameterizations and has no information about $f$, might think of $g$ as a standard parameterization and very likely interpret the problem geometry very differently. In this sense, we would be “misusing” the software.
To impose the material equivalence, we need the Jacobian between the charts and material parameters $\epsilon_f$. After substituting $I$ for $S_f$ and $S_g$ in (11), we get the following equivalent material parameters:

$$\epsilon_g = \frac{1}{|J|} J \epsilon_f J^T,$$

where the identity $|J| |J^{-1}| = 1$ is used (Henrotte et al. [18] arrived at a similar expression but one of their motivations was open boundary problems, see section 6.3). It is easy to see from (12) that the material parameters in $g$ may not be a scalar multiple of the identity matrix, even if they were in the chart $f$, a point expressed in [16] from the opposite point of view.

The solution of the problem are fields $E_g$ and $D_g$. However, to present the fields in the standard parameterization $f$, where they may be more intuitive, we must use (5) and $D_f = \epsilon_f E_f$ to obtain the equivalent fields.

**Remark 1** $D_g$ cannot be transformed into $D_f$ with the same transformation as $E_g$ is transformed into $E_f$ in [5]. That is, the electric field intensity $E$ and the electric flux density $D$ transform differently between charts. To get the $D_f$, one can use (5) to transform $E_g$ to $E_f$ and then use the constitutive law $D_f = \epsilon_f E_f$.

The above method is fully realizable with any mesh generator and solver software that allows for pointwise definition of material parameters as a matrix. However, if the solver does not allow for pointwise definition, only charts $g$ with piecewise affine transition maps from the standard parameterization $f$ can be used.

### 6.2 Modelling With Multiple Charts

Previously we had two charts that both cover the whole domain. However, manifold can be covered with multiple charts, such that only a part of the manifold is covered by each chart. All that is required is that every point of the manifold belongs to the domain of some chart and that all the charts belong to the same differentiable structure. The use of multiple charts can benefit numerical analysis, for example in the case of figure 6. First of all, the mesh generation gets easier, because the detailed regions could be covered separately with their own charts, with the origins located such that there are as many floating point numbers available as possible. Second, for the same reason, the calculations of the inner products for the system matrix are more accurate.
Figure 6: Multiple charts. Top: Standard parameterization, with three rectangular regions. Bottom: Three charts, that each cover one of the rectangular regions, but the origins are moved and scales are changed.

Even though multiple charts are used, we still assume here that the manifold *can* be covered with one $\mathbb{R}^n$ chart. This makes it possible to partition the manifold into separate domains such that the domains only overlap on their boundary. Furthermore, this guarantees that the manifold can be embedded into the $n$-dimensional Euclidean space. Next, based on these assumptions, we suggest an implementation of multiple charts into software systems.

First, the user implicitly decides on one isometric standard parameterization, which covers the whole domain. Let us call it the universal chart, because it works as the reference to the other charts, and the material parameters are given on it. However, the universal chart is not used in the calculation or in the mesh generation, but these are done in separate charts that cover only parts of the manifold: the user partitions the universal chart into multiple regions, such that the regions only overlap on their boundary (see figure 6). Then the user gives new charts for these regions and gives the transition maps to the universal chart. In practice the user only gives the separate charts and the transition maps to the software system, and then the universal chart is constructed from these by the software. It is also possible that the universal chart is not constructed at all in the software system but exists only implicitly. The metric tensors for these separate charts could be fixed or given by the user. The material parameters for the charts are given as if they are standard parameterizations, and the software calculates automatically the material parameters to be used in the calculations. This is possible because the metric tensors are known, as are the transition maps, whose Jacobians can be automatically calculated.

Finally, in order to attain compatibility of meshes in different charts, the meshes have to agree at the boundaries of the regions. The mesh generation could proceed as follows: First a mesh is generated on one of the regions with
the chart that covers it. Then the coordinates of the nodes on the common boundary with a second region are mapped with the transition map to the chart that covers the second region, and the rest of the mesh in the second region can be generated. The assembly of the system matrix can be done as usually.

6.3 Open Boundary Problems

The open boundary problems are often solved by truncating the domain and setting the fields zero at an artificial distant boundary. This usually gives a good enough approximation for the fields near the sources, because the fields tend to zero quite fast as the distance from the source increases. The distance of the artificial boundary from the sources and regions of importance is decided by the modeller. However, the error from this truncation is hard to estimate and the number of elements needed to cover all that empty space can be large.

To overcome these problems, many methods have been proposed over the years: The so called "ballooning method" [19], in which the true distance of the boundary is put very far away with a thin layer of special elements.
Another method uses infinite elements [20], which are special decaying basis functions used in FEM. Yet another method couples the FEM with analytical solutions as in [21]. Finally, the transformation (or shell transformation) methods presented for example in [18][22] put the infinite boundary to a finite distance with the help of a suitable change of coordinates. However, many of these methods are hard to implement into most production solvers, because they require modification of the solver codes: For example, the modeller needs to define new basis functions or needs to give pointwise Jacobians of the change of coordinates.

The triplet approach of BVPs suggest a strategy to solve open boundary problems: Just select a suitable triplet, such that in the chart the infinity boundary is at a finite distance, such as the chart $g$ in the figure 7. This is effectively identical to many of the above methods. For example, in the transformations method the Jacobian of the change of coordinates is implemented in the software code whereas in the triplet approach the Jacobian is included in the material parameters given by the user.

The triplet should be selected such that a small region including the source regions and other regions of interest is modelled as usually, but the complement of this region is mapped to a small bounded outer region as in the chart $g$ in the figure 7. The elements nearest the infinity boundary of the domain are such that the points at the boundary are mapped to the infinity by the transition map form $g$ to the corresponding standard parameterization. A suitable transition mapping could be the following: The region outside the unit disc is mapped to the ring with exterior radius two, such that the ring surrounds the unit disc and the boundary of the disc is the inner boundary of the ring, see figure 8. A point outside the unit disc at the distance $r$ from the center of the unit disc is mapped to a point at the distance $R$ from the center, such that the distances have the following relation:

$$R = 2 - \frac{1}{r}.$$  

This method can be used with almost any solver: Only the material parameters have to be changed accordingly, which is a responsibility of the modeller. However, the procedure is easy to automate: the user only specifies the regions where the outer boundary is mapped to the infinity by the transition map from $g$ to the standard parameterization, and the software system would take care of the rest. The system could have built-in such a mapping or mappings which adapt according to the coordinates of the infinity boundary and the boundary of the infinite regions. If necessary, the user gives the correspondence between these boundaries and then the system modifies the user-given material parameters according to the adapted mapping.
6.4 Modelling Motion With Single Chart

When modelling motion, we have to solve a sequence of BVPs whose only
difference is that some object changes its position relative to the other ob-
jects. For an example, consider the force calculations of an electromagnet,
see figure 9: The magnet attracts the load, which then moves upwards. One
solution is to make whole model for each position of the moving object. For
each position, we have to define a new geometry and generate a new mesh
for it, and furthermore assembly the whole system matrix again. This is very
time-consuming.

The triplet approach can speed up the process: Keep the chart and the
material parameters fixed and change the metric tensor. This changes the
geometry and takes into account the motion: When a rigid object moves in a
fluid (air) such that the topology does not change, i.e. the moving object does
not touch any other material than the fluid, then the underlying differentiable
manifold of the domain does not change (this underlying manifold is often
referred to as material manifold). However, the Riemannian manifold will
change, because the metric tensor changes. This means that the distances
between points of the manifold are changed. Thus we may select a single
chart \( g \) from the differentiable structure, to cover all the different Riemannian
manifolds.

The metric tensor for the selected chart \( g \) again depends on the material
equivalence. This requires that some standard parameterization \( f \) for each
Riemannian manifold is known, when the corresponding Jacobians are also
known. Then the matrix $S_g$ representing the metric tensor on $g$ can be solved from equation (11): Assume the material parameter $\epsilon_f$ to be a scalar and require the material parameters to be equal, that is $\epsilon_g = \epsilon_f$. Then $S_g$ in terms of the Jacobian can be solved:

$$S_g = \left| J \right| J^{-T} J^{-1}. \quad (13)$$

Now, for the example in the case of figure 9, the metric tensor only changes in the air between the magnet and the load according to the equation (13). Because the mesh remains the same, only the elements in the system matrix corresponding to the mesh elements in this air region will change. Thus only a partial re-assembly of system matrix is needed for each new step. Furthermore, in the case of iterative solvers, a preconditioner used for one step can be used effectively for many successive steps, because the changes in the system matrix can be very small. Also the previous solution may be a good initial guess. Thus, all these things—only one mesh, partial re-assembly of system matrix, the preconditioner, and the initial guess—can speed up the solution process substantially.

Many solvers do not offer the possibility to change the metric tensor, yet the modelling of motion with a single mesh is still possible: we can equivalently change the material parameters, see section 6.1. One needs to know a standard parameterization for each Riemannian manifold and then use the material parameters given by the equation (12). Finally, all things said about modelling motion apply also for modelling the geometric effects of deformations, such as the effects of magnetostriction.

Figure 9: Modelling motion. Electromagnet: coil with current $I$, ferromagnetic core and load, which is attracted upwards.
7 Numerical Example

We demonstrate a combination of proposed applications: an open boundary problem (section 6.3) with reparameterization under fixed chart metric (section 6.1). The problem to solve is the electric potential of a three-phase
high-voltage line, a 3D-Laplace’s problem. Figure 10 shows a standard parameterization of the domain. The domain is the half-space above the ground apart from the line itself. The infinity boundary is mapped to a finite distance from the line, as explained in section 6.3. There is also great variation in the scale of the details of the domain: the length of the lines is order of hundreds of meters, whereas the smallest dimensions are order of centimeters. Based on experience, we can say that the mesh generation with the standard parameterization is difficult or even impossible. To avoid mesh generation problems, the lines are scaled down, as is the height of the pillars. The model used in the calculations is shown in figure 11. The result is shown in figure 12. The figures and the calculations are produced with GetDP \cite{GetDP} and Gmsh \cite{Gmsh}.

8 Conclusion

Most solver software systems for quasi-static electromagnetic boundary value problems are based on the formulation of electromagnetics with classical vector analysis. This has many restrictions for numerical solution, because many
mathematical structures have limited freedom or they are completely fixed. For example, the fields are presented as mappings from some global coordinate system to a single vector space and the metrics for the coordinate systems and the vector space are fixed. The restrictions limit the possible coordinate systems that can be used.

The formulation of electromagnetics with manifolds and differential geometry is less restrictive and thus can help in a numerical solution. This view exposes the independence of the electromagnetic theory on the chosen coordinates or metric. Based on this it is shown that there is an equivalence class of triplets \{coordinate system, metric, material parameters\} for each boundary value problem. A triplet describes a geometry of the problem domain and the constitutive equations which are needed to pose the problem in the coordinate system. We propose that the solvers should allow to use any triplet, that is, the modeller should be able to choose any coordinate system and metric at will. These possibilities come built-in in differential geometry, whereas they are not self-evident in classical vector analysis. We discuss how this can be done. It is not hard to apply some of the triplet strategies in most production solvers, because not all of them require modifications to the code. We show that despite the freedom in the choice of the triplet, the FEM system matrices of a given BVP, assembled for a mesh represented under different triplets, are identical.

We show how the triplet approach can be exploited in applications: Potentially prohibitive difficulties of mesh generation in problem domains that have large scale variation can be alleviated by a suitable choice of coordinate system. This extends the class of solvable problems, although possibly at the expense of mesh quality. Open boundary problems can be solved by choosing a convenient triplet for the solution. Solution of problems involving motion and deformations can be accelerated when a single mesh is used on every time step and only the metric is altered.

References

[1] J. R. Shewchuk, Adaptive Precision Floating-Point Arithmetic and Fast Robust Geometric Predicates. in Discrete and Computational Geometry, 18, 305-363, 1997.

[2] F. Kottler, Maxwell’sche Gleichungen und Metrik (Maxwell’s equations and metric). In Sitzungsber. Akad. Wien IIa, 131 (1922) 119-146.

[3] E. Cartan, Sur les variétés à connection affine et la théorie de la relativité généralisée. In Annales de l’école normale superieure, 40 (1923) 325-412, 41 (1924) 1-25.
[4] D. Van Dantzig, The fundamental equations of electromagnetism, independent of metrical geometry. In Proc. Cambridge Phil. Soc., 30 (1934) 421-427.

[5] E. Schrödinger, Space-Time Structure. Cambridge University Press: Cambridge, 1950.

[6] E. J. Post, Formal structure of electromagnetics. North-Holland: Amsterdam, 1962.

[7] C. Nash, S. Sen, Topology and geometry for physicists. Academic Press, London, 1983.

[8] W. M. Boothby, An introduction to differentiable manifolds and Riemannian geometry (Revised 2nd ed.). Academic Press: London, 2003.

[9] K. Jänich, Vector analysis. Springer-Verlag: New York, Berlin, Heidelberg, 2001.

[10] F. W. Warner, Foundations of differentiable manifolds and Lie groups. Springer-Verlag: New York, Berlin, Heidelberg, 1983.

[11] W. L. Burke, Applied differential geometry. Cambridge University Press: Cambridge, 1985.

[12] A. Bossavit, Computational electromagnetism. Academic Press: San Diego CA., Chestnut Hill, MA., 1998.

[13] K. Yosida, Functional analysis. Springer-Verlag: New York, Berlin, Heidelberg, 1980.

[14] A. Di Carlo, The geometry of linear heat conduction. In Trends and applications of mathematics to mechanics, (New York), Longman, 1991. (F. Ziegler, W. Schneider, H. Troger, eds.).

[15] A. Bossavit, On the geometry of electromagnetism. (4): Maxwell’s house. In J. Japan Soc. Appl. Electromagn. & Mech., 6, pp. 318–326, 1998.

[16] A. Bossavit, On the notion of anisotropy of constitutive laws: Some implications of the 'Hodge implies metric' result. In COMPEL, 20(1), pp. 233–239, 2001.

[17] K. R. Stromberg, An introduction to classical real analysis. Wadsworth and Brooks/Cole Advanced Books and Software: Belmont, Ca. 1981.
[18] F. Henrotte, B. Meys, H. Hedia, P. Dular, W. Legros, Finite element modelling with transformation techniques. In *IEEE Trans. Magn.*, 35(3), pp. 1434–1437, 1999.

[19] P. P. Silvester, D. A. Lowther, C. J. Carpenter, E. A. Wyatt, Exterior finite elements for 2-dimensional field problems with open boundaries. In *PROC. IEE*, 124(12), pp. 1267-1270, 1977.

[20] P. Bettess, Finite element modelling of exterior electromagnetic problems. In *IEEE Trans. Magn.*, 24(1), pp. 238-243, 1988.

[21] P. P. Silvester, M. S. Hsieh, Finite element solution of two dimensional exterior field problems. In *PROC. IEE*, 118, pp. 1743-1747, 1971.

[22] J. F. Imhoff, G. Meunier, J. C. Sabonnadiere, Finite element modeling of open boundary problems. In *IEEE Trans. Magn.*, 26(2), pp. 588-591, 1990.

[23] P. Dular, C. Geuzaine, *GetDP: a General Environment for the Treatment of Discrete Problems*. [http://www.geuz.org/getdp/](http://www.geuz.org/getdp/)

[24] C. Geuzaine, J.-F. Remacle, *Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities*. [http://www.geuz.org/gmsh/](http://www.geuz.org/gmsh/)