\textbf{{\textit{p}}-adic Rankin product \textit{L}-functions}

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\textbf{Abstract}

We describe Panchishkin’s construction of the \(p\)-adic Rankin product \(L\)-function.

Let \(p\) be an odd prime. In this article we give a construction of the \(p\)-adic Rankin product \(L\)-function which interpolates \(p\)-adically the special values of the convolution of two cusp forms on the complex upper half plane. The argument given here closely follows Panchishkin’s original argument \cite{Pan88} where the \(S\)-adic non-archimedean \(L\)-function associated to the Rankin product of two modular forms was constructed, for \(S\) any set of finite primes including \(p\). In this exposition we will specialize the argument given in \cite{Pan88} to the case \(S = \{p\}\). We also provide some background details and correct a sign error along the way which does not seem to have been noticed in the subsequent literature.

1 Introduction

1.1 Rankin product \textit{L}-functions

Let \(N\) be an arbitrary natural number. We consider a cusp form \(f\) of weight \(k \geq 2\) for the congruence subgroup \(\Gamma_0(N)\) and nebentypus \(\psi\). We suppose that \(f\) is a primitive cusp form, i.e., it is a normalized newform of some level \(C_f\) dividing \(N\); \(C_f\) is called the conductor of \(f\). Let \(g\) be another primitive cusp form of conductor \(C_g\) and weight \(2 \leq l < k\) for \(\Gamma_0(N)\) and nebentypus \(\omega\). We set \(e(z) = e^{2\pi iz}\) and let

\[ f(z) = \sum_{n=1}^{\infty} a(n)e(nz), \quad g(z) = \sum_{n=1}^{\infty} b(n)e(nz) \]  

be the Fourier expansions of \(f\) and \(g\). The Rankin convolution of the modular forms \(f\) and \(g\) is defined by means of the equality

\[ D(s, f, g) := L_N(2s + 2 - k - l, \psi \omega) L(s, f, g), \]  

where

\[ L(s, f, g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s}, \]

and \(L_N(2s + 2 - k - l, \psi \omega)\) denotes the Dirichlet \(L\)-series with character \(\psi \omega\), and the subscript \(N\) indicates that the factors corresponding to the prime divisors of \(N\) are omitted from the Euler product. A classical method of Rankin and Selberg \cite{Ran39} enables one to construct an analytic continuation of the function \(D(s, f, g)\) to the whole complex plane and prove that it satisfies a functional equation. Let

\[ f^p(z) := \sum_{n=1}^{\infty} \overline{a(n)}e(nz), \quad g^p(z) := \sum_{n=1}^{\infty} \overline{b(n)}e(nz). \]
Further, define
\[ \Psi(s, f, g) = \gamma(s)\mathcal{D}(s, f, g), \] (1.3)
where \( \gamma(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - l + 1) \) consists of \( \Gamma \)-functions. Though we do not use it here, \( \Psi(s, f, g) \) has a well-known functional equation. For instance, if \( \psi, \omega \) and \( \psi^{-1}\omega \) all have conductor \( N \) and \( C_f = C_g = N \), then the functional equation is (see [Hid93, §9.5, Theorem 1]):
\[ \Psi(s, f, g) = W(f^\rho, g)N^{3(-s+(k+l-1)/2)}\Psi(k + l - 1 - s, f, g^\rho), \] (1.4)
where
\[ W(f^\rho, g) = (-1)^l\Lambda(f^\rho)\Lambda(g)\frac{G(\psi^{-1}\omega)}{|G(\psi^{-1}\omega)|}, \]
\( G(\psi^{-1}\omega) \) is the Gauss sum associated to \( \psi^{-1}\omega \) and \( \Lambda(f^\rho), \Lambda(g) \) are the root numbers associated to \( f^\rho, g \) respectively (defined in §2). Shimura [Shi77] established the following algebraicity result for the special values of \( \mathcal{D}(s, f, g) \) (see [Hid93, §10.2, Theorem 1]): the numbers
\[ \Psi(s, f, g)(\pi^{1-l}(f, f)_{C_f})^{-1} \in \overline{\mathbb{Q}}, \] (1.5)
for all integers \( l \leq s \leq k - 1 \). Here \( (f, f)_{C_f} \) is the Petersson inner product defined by
\[ (f, f)_{C_f} := \int_{\mathcal{H}/\Gamma_0(C_f)} |f(z)|^2 y^{k-2} dxdy, \quad z = x + iy, \]
where \( \mathcal{H}/\Gamma_0(C_f) \) is a fundamental domain for the upper half plane \( \mathcal{H} \) modulo the action of \( \Gamma_0(C_f) \). The integers \( s = l, \ldots, k-1 \) in (1.5) are “critical” in the sense of Deligne [Del79]. They are precisely the values of \( s \) for which neither of the functions \( \gamma(s) \) and \( \gamma(k + l - 1 - s) \) in the functional equation have poles.

1.2 Main theorem

Let \( \mathbb{C}_p = \overline{\mathbb{Q}}_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \). Let \( |\cdot|_p \) be the norm on \( \mathbb{C}_p \), normalized so that \( |p|_p = 1/p \). For any topological group \( G \), let \( X(G) \) denote the group of continuous homomorphisms from \( G \) to \( \mathbb{C}_p^\times \). The domain of definition of \( p \)-adic \( L \)-functions is the \( \mathbb{C}_p \)-analytic Lie group \( X_p = X(\mathbb{Z}_p^\times) \), where \( \mathbb{Z}_p^\times \) is the group of units of \( \mathbb{Z}_p \). We put \( X_p^{\text{tors}} = \{ \chi \in X_p \mid \chi \text{ has finite order} \} \). Let \( x_p \) denote the embedding \( \mathbb{Z}_p \hookrightarrow \mathbb{C}_p \). For a precise statement of the results we introduce the notation
\[ g(\chi) := \sum_{n=1}^{\infty} \chi(n)b(n)e(nz), \]
for the cusp form \( g \) twisted by the Dirichlet character \( \chi \). We fix an embedding of \( \overline{\mathbb{Q}} \) into \( \mathbb{C} \) and an embedding \( i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \). Then every Dirichlet character \( \chi \) whose conductor \( C_\chi \) is a power of \( p \) can be identified with an element of \( X_p^{\text{tors}} \) and vice versa. By Theorem 2.6 with \( g(\chi) \) replaced by \( g^\rho(\overline{\chi}) \), the numbers
\[ \frac{\Psi(l+r, f, g^\rho(\overline{\chi}))}{\pi^{1-l}(f, f)_{C_f}} \in \overline{\mathbb{Q}}, \]
for \( r = 0, 1, \ldots, k - l - 1 \). In this article we construct a \( \mathbb{C}_p \)-analytic function on \( X_p \) which interpolates the numbers
\[ i_p \left( \frac{\Psi(l+r, f, g^\rho(\overline{\chi}))}{\pi^{1-l}(f, f)_{C_f}} \right), \]
for all $\chi \in X_p^{\text{tors}}$ and $r = 0, 1, \ldots, k - l - 1$. We work under the assumption that $f$ is a $p$-ordinary form, i.e., $a(p)$ is a unit in $\mathbb{C}_p$. In other words
\[
|\iota_p(a(p))|_p = 1. \tag{1.6}
\]
In addition, we suppose that
\[
(C_f, C_g) = 1, \quad (p, C_f) = (p, C_g) = 1, \tag{1.7}
\]
and we set $C = C_fC_g$. Let $\alpha(p)$ denote the root of the Hecke polynomial $X^2 - a(p)X + \psi(p)p^{k-1}$, for which $|\iota_p(\alpha(p))|_p = 1$ and let $\alpha'(p)$ be the other root. For every prime $q \nmid N$, let
\[
X^2 - a(q)X + \psi(q)q^{k-1} = (X - \alpha(q))(X - \alpha'(q)), \quad
X^2 - b(q)X + \omega(q)q^{l-1} = (X - \beta(q))(X - \beta'(q)). \tag{1.8}
\]
We extend the definition of $\alpha(n)$ to all natural numbers of the form $p^r$ by setting $\alpha(p^r) := \alpha(p)^r$.

**Theorem 1.1. (Main theorem)** Under the assumptions (1.6) and (1.7), there exists a unique measure $\mu$ on $\mathbb{Z}_p^\times$ satisfying the following interpolation property: for all characters $\chi \in X_p^{\text{tors}}$ and all integers $r$ with $0 \leq r \leq k - l - 1$, the value of the function $x^r_p\chi$ under the measure $\mu$
\[
\int_{\mathbb{Z}_p^\times} x^r_p\chi \, d\mu
\]
is given by the image under $\iota_p$ of the following algebraic number
\[
(-1)^r \omega(C_{\chi}) \frac{G(\chi)^2C_{\chi}^{2r-1}}{\alpha(C_{\chi}^2)} \cdot \frac{\Psi(l + r, f, g^r(\chi))}{\pi^{1-l}(f, f)C_f}.
\]

This is exactly [Pan88] Thm. 1.4], noting that the extra Euler factors $A(r, \chi)$ there do not appear here because $S = \{p\}$. Finally we remark that if $\mu$ is a $\mathbb{C}_p$-valued measure on $\mathbb{Z}_p^\times$, as in the main theorem above, then the function $L_\mu$ (the $p$-adic $L$-function attached to $\mu$) defined by
\[
L_\mu(\chi) = \mu(\chi) = \int_{\mathbb{Z}_p^\times} \chi \, d\mu, \quad \forall \chi \in X_p, \tag{1.9}
\]
always turns out to be a $\mathbb{C}_p$-analytic function $L_\mu : X_p \rightarrow \mathbb{C}_p$.

To make sense of the last statement we briefly recall the $\mathbb{C}_p$-analytic structure on $X_p = X(\mathbb{Z}_p^\times)$. We set
\[
U = \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \mod p\},
\]
units of $\mathbb{Z}_p$ congruent to 1 mod p. Then we have the following decomposition
\[
X_p = X((\mathbb{Z}/p\mathbb{Z})^\times) \times X(U).
\]
Therefore every $\chi \in X_p$ can be written as $\chi_0\chi_1$ with $\chi_0 \in X((\mathbb{Z}/p\mathbb{Z})^\times)$ and $\chi_1 \in X(U)$. The characters $\chi_0$ and $\chi_1$ are called the tame part and the wild part of the character $\chi$ respectively.

We claim the function $\varphi$ defined by $\varphi(\chi) := \chi(1 + p)$, where $1 + p$ is a topological generator of the group $U$, induces an isomorphism of groups
\[
\varphi : X(U) \xrightarrow{\sim} T := \{t \in \mathbb{C}_p^\times \mid |t - 1|_p < 1\}.
\]
This isomorphism defines an analytic structure on $X(U)$, which can easily be checked to be independent of the choice of generator $1 + p$. We first check that $\varphi$ is well defined, i.e., $\varphi$ takes
\footnote{Except that we have added the sign $(-1)^r$ which we feel is necessary (see subsequent footnotes).}
values in $T$. Let $\chi \in X(U)$. Since $(1 + p)^{p^n} \rightarrow 1$ as $n \rightarrow \infty$, by the continuity of $\chi$ we have $(\chi(1 + p))^{p^n} \rightarrow 1$. Hence, $|\chi(1 + p)|_p = 1$ and $|\chi(1 + p) - 1|_p \leq \max\{|\chi(1 + p)|_p, 1\} \leq 1$. We now claim that $|\chi(1 + p) - 1|_p < 1$. Suppose not, then $|\chi(1 + p) - 1|_p = 1$. Therefore

$$1 = |(\chi(1 + p) - 1)^{p^n}|_p$$

$$= \left| \sum_{k=1}^{p^n} \left( \frac{p^n}{k} \right) (\chi(1 + p) - 1)^k \right|_p$$

$$= |(\chi(1 + p) - 1 + 1)^{p^n} - 1|_p$$

$$= |(\chi(1 + p))^{p^n} - 1|_p.$$ 

But, this contradicts $(\chi(1 + p))^{p^n} \rightarrow 1$. Hence, $|\chi(1 + p) - 1|_p < 1$. We now show that $\varphi$ is an isomorphism. Every character $\chi \in X_p$ is uniquely determined by $\chi(1 + p)$, since $1 + p$ is a topological generator of $U$, hence $\varphi$ is injective. For $t \in T$, define $\chi_t((1 + p)^n) = t^n$, for all $n \in \mathbb{Z}$. Extending $\chi_t$ to all of $1 + p\mathbb{Z}_p$ by continuity we get an element of $X(U)$ which maps to $t$ under $\varphi$. Hence $\varphi$ is also surjective. A function $F : T \rightarrow \mathbb{C}_p$ is said to be analytic if $F(t)$ can be expressed as a power series, i.e., $F(t) = \sum_{i=0}^{\infty} a_i(t - 1)^i$, $a_i \in \mathbb{C}_p$, which converges absolutely for all $t \in T$. The isomorphism $\varphi : X(U) \simeq T$ allows us to define an analytic structure on $X(U)$. Finally the notion of analyticity can be extended to all of $X_p$ by translation.

In closing this introduction, we remark that Hida [Hid88] subsequently constructed a more general measure interpolating the critical Rankin product $L$-values of two cusp forms which themselves vary in $p$-adic families.\footnote{The sign mentioned in the previous footnote is consistent with the sign in [Hid88 Theorem 1].} and in a different direction, Vienney [Vie00] has generalized Panchishkins argument to cases where $a(p)$ is not a $p$-adic unit.\footnote{Again, the author adds a sign of $(-1)^s$.}

## 1.3 Outline of the paper

We recall notation and results from the theory of modular forms in §2. In §3 we recall generalities about distributions and measures and state a criterion for a distribution whose values are known on a specific set of functions to be a measure in terms of the abstract Kummer congruences. The measure in Theorem 1.1 is obtained from certain complex-value distributions $\Psi_s$, which we construct in §4 using the definition of the convolution (1.2). The distributions $\Psi_s$ take values in $\overline{\mathbb{Q}}$ on $X^\text{tors}_p$ for integers $l \leq s \leq k - 1$. In §5 we obtain an integral representation for these distribution values using the Rankin-Selberg method and holomorphic projection. In §6 we prove that the $\mathbb{C}_p$-valued distributions $i_p(\Psi_s)$ satisfy the abstract Kummer congruences to finish the proof of Theorem 1.1.

## 2 Background on Modular forms

In this section we recall a few results from the theory of classical modular forms. Most of the material covered here is well known. In this section $f$ and $g$ are arbitrary functions which need not satisfy the assumptions of §1 unless otherwise stated. Also, $\chi$, $\psi$, $\omega$ denote Dirichlet characters. Let $M_2(\mathbb{Z})$ denote the set of $2 \times 2$ matrices with entries in $\mathbb{Z}$. Let $SL_2(\mathbb{Z})$ denote the set of matrices with determinant 1 in $M_2(\mathbb{Z})$. Let $\mathbb{C}$ denote the complex plane. We write an element $z \in \mathbb{C}$ as $x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$. For $z = x + iy \in \mathbb{C}$, sometimes we denote $x$ and $y$ by $\text{Re}(z)$ and $\text{Im}(z)$ respectively.

### 2.1 Classical modular forms

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ denote the complex upper half plane, on which the group $\text{GL}_2^+ (\mathbb{R})$ of real $2 \times 2$ matrices with positive determinant acts by fractional linear transformations. For
any natural number \( k \), we have a weight \( k \) action of \( GL_2^+(\mathbb{R}) \) on functions \( f : \mathcal{H} \to \mathbb{C} \) given by:

\[
(f|k\gamma)(z) = (\det\gamma)^{k/2}(cz + d)^{-k}f\left(\frac{az + b}{cz + d}\right), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).
\]

For any natural number \( N \), we have the following subgroups:

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\},
\]

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv d \equiv 1 \mod N \right\},
\]

\[
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \mid b \equiv 0 \mod N \right\}.
\]

**Definition 2.1.** A subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \) is called a congruence subgroup if \( \Gamma(N) \subset \Gamma \) for some \( N > 0 \). The smallest \( N \) satisfying this condition is called the level of the congruence subgroup.

If \( \Gamma \) is a congruence group, then \( M_k(\Gamma) \) denotes the complex vector space of modular forms of weight \( k \) for \( \Gamma \). These consist of holomorphic functions \( f : \mathcal{H} \to \mathbb{C} \) which satisfy \( f|k\gamma = f \), for all \( \gamma \in \Gamma \), and a holomorphicity condition at the cusps of \( \Gamma \). Let \( S_k(\Gamma) \) denote the subspace of cusp forms consisting of those \( f \) which in addition vanish at the cusps.

**Notation.** Throughout the article we use the following notation:

(i) For every integer \( M \), let \( S(M) \) denote the set of primes dividing \( M \).

(ii) For every \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{M}_2(\mathbb{Z}) \), and Dirichlet character \( \psi \) we put \( \psi(\gamma) = \psi(d) \).

(iii) Let \( \chi_0 \) denote the principal character on \( \mathbb{Z} \). It is given by \( \chi_0(n) = 1, \forall n \in \mathbb{Z} \).

If \( \psi \) is a Dirichlet character mod \( N \), we set

\[
M_k(N, \psi) = \{ f \in M_k(\Gamma_1(N)) \mid f|k\gamma = \psi(\gamma)f, \forall \gamma \in \Gamma_0(N) \},
\]

\[
S_k(N, \psi) = S_k(\Gamma_1(N)) \cap M_k(N, \psi).
\]

For an arbitrary modular form \( h \in M_k(N, \psi) \) with \( k \geq 1 \) and a cusp form \( f \in S_k(N, \psi) \) the Petersson inner product is defined by the integral

\[
\langle f, h \rangle_N = \int_{\mathcal{H}/\Gamma_0(N)} f(z)h(z)y^{k-2} \, dx \, dy,
\]

where \( \mathcal{H}/\Gamma_0(N) \) is a fundamental domain for the upper half plane \( \mathcal{H} \) modulo the action of \( \Gamma_0(N) \). Observe that if \( \gamma \in GL_2^+(\mathbb{R}) \) normalizes \( \Gamma_0(N) \) and \( \gamma^2 \) is a scalar matrix, then (see [Miy89, Theorem 2.8.2])

\[
\langle f|k\gamma, h \rangle_N = \langle f, h|k\gamma \rangle_N.
\]

For the rest of this subsection assume that \( M, N \) are positive integers such that \( S(NM) = S(N) \). Since \( S(NM) = S(N) \) it can be checked that \( [\Gamma_0(N) : \Gamma_0(NM)] = M \) and

\[
\{ b_u = \begin{pmatrix} 1 \\ uN \end{pmatrix} \mid u = 1, \ldots, M \}
\]

is a set of coset representatives for \( \Gamma_0(NM)/\Gamma_0(N) \). Therefore, for every \( \gamma \in \Gamma_0(N) \) and \( b_u \), there exists unique \( \gamma_u \in \Gamma_0(NM) \) and \( b_{u'} \) such that \( \gamma \equiv \gamma_u b_u \). Since \( b_u, b_{u'} \equiv 1_2 \mod N \) we have

\[
\gamma \equiv \gamma_u b_u \equiv \gamma_u \mod N.
\]
For a Dirichlet character \( \psi \) modulo \( N \) and \( h \in M_k(NM, \psi) \) we have

\[
\left( \sum_{u=1}^{M} h|_{\kappa} \beta_u \right)|_{\kappa} = \sum_{u=1}^{M} h|_{\kappa} \gamma_u \beta_u' = \sum_{u=1}^{M} \psi(\gamma_u) \cdot h|_{\kappa} \beta_u' = \psi(\gamma) \cdot \sum_{u=1}^{M} h|_{\kappa} \beta_u.
\]

Therefore \( \sum_{u=1}^{M} h|_{\kappa} \beta_u \in M_k(N, \psi) \). For \( M, N \) positive integers such that \( S(NM) = S(N) \) and a Dirichlet character \( \psi \) modulo \( N \), define the trace operator \( \text{Tr}^{NM}_N : M_k(NM, \psi) \rightarrow M_k(N, \psi) \) by the equality

\[
\text{Tr}^{NM}_N(h) = \sum_{u=1}^{M} h|_{\kappa} \beta_u = \sum_{u=1}^{M} h|_{k} \left( \frac{1}{uN} \right).
\]

(2.3)

**Remark 2.2.** The definition of the trace above depends on the choice of coset representatives \( \{\beta_1, \ldots, \beta_M\} \) of \( \Gamma_0(NM) \backslash \Gamma_0(N) \). In the computations below, we always use this choice.

**Lemma 2.3.** Let \( \psi \) be a Dirichlet character modulo \( N \). Let \( f \in S_k(N, \psi) \) and \( h \in M_k(NM, \psi) \). If \( S(M) \subset S(N) \), then \( \langle f, h \rangle_{NM} = \langle f, \text{Tr}^{NM}_N(h) \rangle_N \).

**Proof.** Let \( \{\beta_1, \ldots, \beta_M\} \) be as above. If \( D \) is a fundamental domain for \( \Gamma_0(N) \), then \( \bigcup_{u=1}^{M} \beta_u D \) is a fundamental domain for \( \Gamma_0(NM) \). Therefore

\[
\langle f, h \rangle_{NM} = \int_{H/\Gamma_0(NM)} \overline{f(z)} h(z) y^{k-2} \, dx \, dy = \sum_{u=1}^{M} \int_{\beta_u D} \overline{f(z)} h(z) y^{k-2} \, dx \, dy = \sum_{u=1}^{M} \int_{D} \overline{(f|_{k} \beta_u)(z)} (h|_{k} \beta_u)(z) y^{k-2} \, dx \, dy = \sum_{u=1}^{M} \int_{D} \overline{(f(z))} (h|_{k} \beta_u)(z) y^{k-2} \, dx \, dy = \langle f, \text{Tr}^{NM}_N(h) \rangle_N.
\]

For any integer \( k \), complex number \( s \) and Dirichlet characters \( \chi, \psi \) modulo \( L, M \) respectively, we define (see [Miy89] Chapter 7) the non-holomorphic Eisenstein series of weight \( k \) by

\[
E_k(z,s; \chi, \psi) = y^s \sum_{c,d = -\infty}^{\infty} \chi(c) \psi(d) (cz + d)^{-k|cz + d|^{-2s}}, \forall z \in \mathcal{H},
\]

(2.4)

where the prime means that the sum is over all \((c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\} \). The series converges for \( \text{Re}(k + 2s) > 2 \) and can be continued meromorphically to the whole complex plane as a function of \( s \). Further, if \( k \geq 3 \), then \( E_k(z,0; \chi, \psi_0) \in M_k(L, \chi) \) [Miy89] Lemma 7.1.4, Lemma 7.1.5].

### 2.2 Operators acting on modular forms

Let \( f \in S_k(N, \psi) \) be a cusp form with the Fourier expansion

\[
f(z) = \sum_{n=1}^{\infty} a(n)e(nz).
\]
If \( d \) is a natural number, then define
\[
\begin{align*}
  f|U_d &= \sum_{n=1}^{\infty} a(dn)e(nz) = d^{k/2-1} \sum_{u \mod d} f|_k \left( \begin{array}{c} 1 \\ 0 \\ u/d \end{array} \right), \\
  f|V_d &= f(dz) = d^{-k/2} f|_k \left( \begin{array}{c} d \\ 0 \\ 0 \end{array} \right) \in S_k(Nd, \psi), \\
  f^0(z) &= f(-z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_k(N, \overline{\psi}), \\
  f|w_d &= (\sqrt{d}z)^{-k} f \left( \begin{array}{c} -1 \\ d \end{array} \right) = f|_k \left( \begin{array}{c} 0 \\ d \\ -1 \end{array} \right), \ f|w_N \in S_k(N, \overline{\psi}).
\end{align*}
\]

Also, the Hecke operators \( T_n : M_k(N, \psi) \rightarrow M_k(N, \psi) \) are defined by \((T_n f)(z) = \sum_{m=0}^{\infty} a(m, T_n f)e(mz)\), where \( a(m, T_n f) = \sum_{0<d|(m,n)} \psi(d)d^{k-1}a(mn/d^2) \).
When \( S(NM) = S(N) \), we have the following identity, which will be used for explicit computations:
\[
T_n^{NM}(f) = (-1)^k M^{-k/2} f|_{w_NM} U_M w_N, \ \forall \ f \in S_k(N, \psi). \tag{2.5}
\]
The above identity follows from the definitions and the matrix identity:
\[
\begin{pmatrix} 1 & 0 \\ uN & 1 \end{pmatrix} = -(NM)^{-1} \begin{pmatrix} 0 & -1 \\ NM & 0 \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & M \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.
\]

**Lemma 2.4.** Let \( f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_k(N, \psi) \) and \( U_d, V_d, T_n \) be as above.

1. If \( d^2 | N \) and \( \psi \) is a Dirichlet character mod \( N/d \), then \( f|U_d \in S_k(N/d, \psi) \).
2. For \( n \geq 1 \), we have \( T_n f = \sum_{ad=n} \psi(d)d^{k-1} f|_{U_d} \). Hence, \( T_p = U_p \) if \( p | N \) is a prime.

**Proof.** Let \( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma_0(N/d) \) and \( 0 \leq u, u' < d \). Then
\[
\begin{pmatrix} 1 & u \\ 0 & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} (x + uz) + yu' - (x + uz)u' \\ dz \end{pmatrix} \begin{pmatrix} z & w \end{pmatrix} = \begin{pmatrix} x + uz \\ dz \end{pmatrix} \frac{y + uw - (x + uz)u'}{w - zu'} \in \Gamma_0(N).
\]

We observe that if \( d^2 | N \), then \( d | z \) and \( (x, d) = 1 \). So \( x + uz \) is a unit in \( \mathbb{Z}/d\mathbb{Z} \). Hence, for every \( 0 \leq u < d \), there exists unique \( u' \mod d \) such that \( d | (y + uw - (x + uz)u') \). This implies that for every \( u \mod d \) there exist unique \( u' \mod d \) such that
\[
\begin{pmatrix} x_u & y_u \\ z_u & w_u \end{pmatrix} \in \Gamma_0(N).
\]

Therefore
\[
(f|U_d)_{kU_d} = \sum_{u \mod d} f|_k \begin{pmatrix} x_u & y_u \\ z_u & w_u \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & d \end{pmatrix} = \sum_{u' \mod d} \psi(w_u) f|_k \begin{pmatrix} 1 & u' \\ 0 & d \end{pmatrix} = \psi(w) \sum_{u' \mod d} f|_k \begin{pmatrix} 1 & u' \\ 0 & d \end{pmatrix} (\because w_u \equiv w \mod (N/d)) = \psi(w)(f|U_d).
\]
Hence (1) follows. For the second statement we compare the Fourier expansion of both sides. From the definition of \( U_a, V_d \) it follows that
\[
\sum_{ad=n} \psi(d) d^{k-1} f|U_aV_d(z) = \sum_{d|n} \psi(d) d^{k-1} (f|U_{n/d})(dz) \quad \text{(substituting } a = n/d) \\
= \sum_{d|n} \psi(d) d^{k-1} \sum_{m=1}^{\infty} a(mn/d) e(mdz) \\
= \sum_{d[n,d|m]} \psi(d) d^{k-1} \sum_{m=1}^{\infty} a(mn/d^2) e(mz) \\
= \sum_{m=1}^{\infty} \left( \sum_{d(n,m)} \psi(d) d^{k-1} a(mn/d^2) \right) e(mz) \\
= (T_n f)(z). \ 
\]
\[
\square
\]

**Definition 2.5.** We call an element \( f \in S_k(N,\psi) \) a primitive cusp form of conductor \( N \) if the following conditions are satisfied:

1. \( f \) is an eigenform, i.e., \( f(z) \) is an eigenvector for the Hecke operators \( T_n \), for all \( n \in \mathbb{N} \),
2. \( a(1) = 1 \), where \( f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \),
3. \( f \) is a newform, i.e., it is orthogonal to all (old)forms lying in the images of the maps \( V_d : S_k(N/d,\psi) \to S_k(N,\psi) \), for \( d | N, C_\psi | (N/d) \), under \( \langle , \rangle_N \).

If \( f \in S_k(N,\psi) \) is a primitive cusp form, then \( T_q(f) = a(q) f \) and \( f|U_{q'} = T_{q'}(f) = a(q') f \) for all \( q \nmid N \) and \( q' \mid N \) respectively. Hence, \( f \) is uniquely determined by the eigenvalues of the Hecke operators \( T_n \). Further, we also have the following:

**Euler Product** \( L(s,f) = \sum_{n=1}^{\infty} a(n) e(nz) = \prod_q (1 - a(q)q^{-s} + \psi(q)q^{k-1-2s}) \).

**Functional Equation** \( \Lambda_N(s;f) = i^k \Lambda_N(k-s;f|w_N) \) where \( \Lambda_N(s;f) = (2\pi/\sqrt{N})^{-s}\Gamma(s)L(s,f) \).

From the theory of newforms (see [Miy89, Theorem 4.6.15]) it follows that if \( f \in S_k(N,\psi) \) is a primitive cusp form of conductor \( C_f \), then
\[
f|w_{C_f} = \Lambda(f) f^p, \tag{2.6}
\]
where \( \Lambda(f) \) is called the root number associated to \( f \).

Let \( g \in S_k(N,\omega) \) be a primitive cusp form of conductor \( C_g \). If the conductor \( C_\chi \) of the primitive Dirichlet character \( \chi \) is coprime to \( C_g \), then the twisted cusp form \( g(\chi) \in S_k(C_g C_\chi^2,\omega \chi^2) \) [Miy89, Lemma 4.3.10 (2)] is primitive, and
\[
\Lambda(g(\chi)) = \omega(C_\chi) \omega(C_g) G(\chi)^2/C_\chi \Lambda(g), \tag{2.7}
\]
where
\[
G(\chi) = \sum_{u \bmod C_\chi} \chi(u) e^{2\pi i u/C_\chi},
\]
is the Gauss sum [Miy89, Theorem 4.3.11].
2.3 Rankin-Selberg convolution

The proof of Theorem 2.4 makes constant use of the classical Rankin-Selberg method (see [Ran39, Ran52]). For the sake of completeness, we recall a few consequences of the Rankin-Selberg method in this section. Let \( f \in S_k(N, \psi) \), \( g \in S_l(N, \omega) \) be primitive cusp forms. Let \( \alpha(q), \alpha'(q), \beta(q), \beta'(q) \) be as in [Hid93, §10.2] for all \( q \mid N \). Put \( \alpha(q) = a(q) \) and \( \beta(q) = b(q) \) and \( \alpha'(q) = \beta'(q) = 0 \) for all \( q \mid N \). Then the \( L \)-function associated to \( f \) and \( g \) has the Euler product

\[
L(s, f) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_q [(1 - \alpha(q)q^{-s})(1 - \alpha'(q)q^{-s})]^{-1},
\]

\[
L(s, g) = \sum_{n=1}^{\infty} b(n)n^{-s} = \prod_q [(1 - \beta(q)q^{-s})(1 - \beta'(q)q^{-s})]^{-1}.
\]

Before we state the result we introduce another class of Eisenstein series which are different from \([2.4]\). For every Dirichlet character \( \psi \mod N \), we set

\[
E_{k,N}(z, s, \psi) = y^s \sum_{c,d=\infty}^{\infty} \psi(d)(cNz + d)^{-k}|cNz + d|^{-2s}.
\]

Let \( f \in S_k(N, \psi) \) and \( g \in S_l(N, \omega) \) be primitive cusp forms as in the Introduction (so \( l < k \)) and let \( \mathcal{D}(s, f, g) \) be as in \([2.2]\). Then the Rankin-Selberg method states that

1. The Rankin product \( L \)-function has the Euler product

\[
\mathcal{D}(s, f, g) = \prod_q [(1 - \alpha(q)\beta(q)q^{-s})(1 - \alpha(q)\beta'(q)q^{-s})]^{-1}
\]

\[
\times (1 - \alpha'(q)\beta(q)q^{-s})(1 - \alpha'(q)\beta'(q)q^{-s})]^{-1}.
\]

2. For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 + \frac{k+l}{2} \), the Rankin product \( L \)-function \( \mathcal{D}(s, f, g) \) has the integral representation given by

\[
2(4\pi)^{-s}\Gamma(s)\mathcal{D}(s, f, g) = \langle f^p, gE_{k-l,N}(z, s - k + 1, \psi \omega) \rangle_N.
\]

We now state an algebraic result for the Rankin product \( L \)-function which is crucial for the construction of the \( p \)-adic Rankin product \( L \)-function, due to Shimura.

**Theorem 2.6.** ([Shim77, Theorem 4], [Hid93 §10.2, Corollary 1]) Let \( f \in S_k(N, \psi) \) and \( g \in S_l(N, \omega) \) be primitive cusp forms of conductor \( C_f \) and \( C_g \) respectively. Then for every Dirichlet character \( \chi \) and for all integers \( s \) with \( l \leq s \leq k - 1 \), we have

\[
\frac{\Psi(s, f, g(\chi))}{\pi^{1-l}(f, f)_{C_f}} \in \overline{\mathbb{Q}}.
\]

2.4 Nearly holomorphic modular forms

In this section we recall some facts about nearly holomorphic modular forms due to Shimura (see [Hid93 §10.1]).

The Maass-Shimura differential operator of weight \( k \in \mathbb{C} \) on \( C^\infty \)-functions on \( \mathcal{H} \) is the operator:

\[
\delta_k = \frac{1}{2\pi i} \left( \frac{k}{2iy} + \frac{\partial}{\partial z} \right), \quad \text{where} \quad z = x + iy, \quad \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).
\]

For every positive integer \( r \), we define \( \delta_k^r := \delta_{k+2r-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k \) and \( \delta_k^0 f = f \). Let \( d := \frac{1}{2\pi i} \frac{\partial}{\partial z} \). The Maass-Shimura differential operator satisfies the following properties:
(1) \( \delta_{k+s}(fg) = (\delta_k f)g + f(\delta_s g) + (\delta_s f)g + f(\delta_k g) \), \( \forall \ s, k \in \mathbb{C} \),
(2) \( \delta_k(f) = y^{-k}d(y^k f) \), \( \forall \ k \in \mathbb{C} \),
(3) \( \delta_k^r = \delta_{k+2}^r \circ \delta_k \),
(4) \( \delta_k^r(f) = \sum_{j=0}^{r} \binom{r}{j} \Gamma(r+k) (-4\pi y)^{-r} \partial^j f \), \( \forall \ k \in \mathbb{C} \), \( r \in \mathbb{N} \).

**Definition 2.7.** Let \( k, r \) be non-negative integers. A function \( f : \mathcal{H} \to \mathbb{C} \) is said to be a nearly holomorphic modular form of weight \( k \) and depth less than or equal to \( r \) for the congruence subgroup \( \Gamma \), if the following hold:

(1) \( f \) is smooth as a function of \( x \) and \( y \),
(2) \( f|_{k\gamma} = f \), for all \( \gamma \in \Gamma \),
(3) there exist holomorphic functions \( f_0, \ldots, f_r \) on \( \mathcal{H} \) such that \( f(z) = \sum_{j=0}^{r} (4\pi y)^{-j} f_j(z) \),
(4) \( f \) is slowly increasing, i.e., for every \( \alpha \in \text{SL}_2(\mathbb{Z}) \), there exists positive real numbers \( A \) and \( B \) such that \( |(f|_{k\alpha \gamma})(z)| \leq A(1+y^B) \) as \( y \to \infty \).

The space of nearly holomorphic modular forms of weight \( k \) and depth less than or equal to \( r \) for the congruence subgroup \( \Gamma \) is denoted by \( \mathcal{N}^r_k(\Gamma) \). It is clear that for \( r = 0 \) we obtain the space of (holomorphic) modular forms \( \mathcal{M}_k(\Gamma) \). Let \( \mathcal{N}_k(\Gamma) = \bigcup_{r=0}^{\infty} \mathcal{N}^r_k(\Gamma) \). Let \( \mathcal{N}_k^+(\Gamma) \) be a graded \( \mathbb{C} \)-algebra. Further, let \( \mathcal{N}_k^+(N, \chi) = \{ f \in \mathcal{N}_k^+(\Gamma_1(N)) \mid f(\gamma z) = \chi(\gamma) f(z), \forall \gamma \in \Gamma_0(\Gamma_0(N)) \}. \]

We say a function \( h \in \mathcal{N}_k^+(\Gamma) \) is rapidly decreasing if for every \( B \in \mathbb{R} \) and \( \alpha \in \text{SL}_2(\mathbb{Z}) \), there exists a positive constant \( A \) such that \( |(h|_{k\alpha \gamma})(z)| \leq A(1+y^B) \) as \( y \to \infty \). We denote the subspace of rapidly decreasing functions in \( \mathcal{N}_k^+(\Gamma) \), \( \mathcal{N}_k^+(N, \chi) \) and \( \mathcal{N}_k(\Gamma) \) by \( \mathcal{NS}_k^+(\Gamma) \), \( \mathcal{NS}_k^+(N, \chi) \) and \( \mathcal{NS}_k(\Gamma) \) respectively (cf. Lemma 2.15).

**Lemma 2.8.** If \( h : \mathcal{H} \to \mathbb{C} \) is a \( C^\infty \)-function, then \( (\delta_k^r h)|_{k+2r} = \delta_k^r(h|_{k\gamma}) \), for all \( \gamma \in \text{GL}_2^+(\mathbb{R}) \).

**Proof.** Observe that by induction on \( r \), it is enough to prove the lemma for \( r = 1 \) and for all \( k \in \mathbb{C} \). So, it is enough to show

\[
(\delta_k^r h)|_{k+2r} = \delta_k^r(h|_{k\gamma}).
\]

For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \), the left hand side is given by

\[
((\delta_k h)|_{k+2\gamma})(z) = \frac{1}{2\pi i} \left( \left( \frac{kh}{2i\text{Im}(z)} + \frac{\partial h}{\partial z} \right) \bigg|_{k+2} \gamma \right)(z) = \frac{1}{2\pi i} \left( (cz + d)^{-k-2}k \frac{\partial (\gamma z)}{\partial \gamma} + (cz + d)^{-k-2} \frac{\partial h}{\partial \gamma}(\gamma z) \right) = \frac{1}{2\pi i} \left( (cz + d)^{-k-2} |cz + d|^2 \frac{k}{2iy} h(\gamma z) + (cz + d)^{-k-2} \frac{\partial h}{\partial \gamma}(\gamma z) \right).
\]

The right hand side is given by

\[
\delta_k(h|_{k\gamma})(z) = \delta_k((cz + d)^{-k}h(\gamma z)) = \frac{1}{2\pi i} \left( \frac{k}{2iy} + \frac{\partial}{\partial z} \right) ((cz + d)^{-k}h(\gamma z)) = \frac{1}{2\pi i} \left( \frac{k}{2iy} ((cz + d)^{-k}h(\gamma z) - ck(cz + d)^{-k-1}h(\gamma z) + (cz + d)^{-k-2} \frac{\partial h}{\partial z}(\gamma z)) \right) = \frac{1}{2\pi i} \left( \frac{k}{2iy} (cz + d)^{-k-1}h(\gamma z)(cz + ciy + d - 2ciy) + (cz + d)^{-k-2} \frac{\partial h}{\partial z}(\gamma z) \right) = \frac{1}{2\pi i} \left( \frac{k}{2iy} (cz + d)^{-k-2} |cz + d|^2 h(\gamma z) + (cz + d)^{-k-2} \frac{\partial h}{\partial z}(\gamma z) \right),
\]

10
which proves that \( (\delta_k h)|_{k+2} = \delta_k(h|_{k}) \) and completes the proof.

Let \( h : \mathcal{H} \to \mathbb{C} \) be a holomorphic function such that \( h(z) = \sum_{n=0}^\infty a(n)e(nz/N) \). Then \( e(-z/N)(h(z) - a_0) \) is holomorphic on \( \mathcal{H} \cup \{\infty\} \). Thus, there exists a positive real number \( C \) such that

\[
|h(z)| \leq |h(\infty)| + Ce^{-2\pi y/N} \quad \text{as} \quad y \to \infty.
\]

(2.14)

**Proposition 2.9.** For \( k, r \in \mathbb{N} \), the operator \( \delta_k^r \) induces a linear map of \( \mathbb{C} \)-vector spaces \( \delta_k^r : M_k(\Gamma) \to \mathcal{N}_{k+2r}(\Gamma) \).

**Proof.** Clearly \( \delta_k^r \) is \( \mathbb{C} \)-linear. So it is enough to show \( \delta_k^r(f) \in \mathcal{N}_{k+2r}(\Gamma) \), for every \( f \in M_k(\Gamma) \). Let \( f \in M_k(\Gamma) \). Recall that

\[
\delta_k^r(f) = \sum_{j=0}^r \binom{r}{j} \frac{\Gamma(r+k)}{\Gamma(j+k)} (-4\pi y)^{j-r} d^j f.
\]

Clearly \( d^j f \) is holomorphic and \( y^{j-r} \) is smooth. Hence, \( \delta_k^r(f) \) satisfies (1) and (3) of Definition 2.7. From Lemma 2.8, it follows that

\[
(\delta_k^r f)|_{k+2r} = \delta_k^r(f|_\Gamma) = \delta_k^r(f), \quad \text{for all} \quad \gamma \in \Gamma,
\]

(2.15)

hence (2) also holds. It remains to check that \( \delta_k^r f \) is slowly increasing. If \( \alpha \in \text{SL}_2(\mathbb{Z}) \), then \( f|_\Gamma \) is also \( C^\infty \), so

\[
(\delta_k^r f)|_{k+2r} = \delta_k^r(f|_\Gamma) = \sum_{j=0}^r \binom{r}{j} \frac{\Gamma(r+k)}{\Gamma(j+k)} (-4\pi y)^{j-r} d^j (f|_\Gamma).
\]

Note that the \( (-4\pi y)^{j-r} \) are bounded as \( y \to \infty \) and the \( d^j (f|_\Gamma) \) are holomorphic. It follows from (2.11) that, for every \( 0 \leq j \leq r \), there exists positive numbers \( A_j, B_j \) such that \( |(-4\pi y)^{j-r} d^j (f|_\Gamma)| \leq A_j(1 + e^{-2\pi y/B_j}) \) as \( y \to \infty \). Since \( e^{-y} \) decays faster than \( y^{-n} \) for any \( n \geq 0 \) as \( y \to \infty \), we have \( |(\delta_k^r f)|_{k+2r}| \leq A_\alpha(1 + y^{-B_\alpha}) \) as \( y \to \infty \) for some positive numbers \( A_\alpha, B_\alpha \).

Now we will show that \( E_k(z; s; \chi, \chi_0) \) is a nearly holomorphic modular form if \( \chi \) is a Dirichlet character modulo \( N \) and \( s \leq 0 \) is an integer such that \( k + 2s > 2 \). To prove this, we need to consider the action of the Maass-Shimura operator on Eisenstein series. Observe that for \( k, r \) positive integers and \( s \leq 0 \) an integer such that \( k + 2s > 2 \), we have

\[
\delta_k^r(y^s) = \sum_{j=0}^r \binom{r}{j} \frac{\Gamma(r+k+1)}{\Gamma(k+j)} (-4\pi y)^{j-r} d^j y^s
\]

\[
= \sum_{j=0}^r \binom{r}{j} \frac{\Gamma(r+k+1)}{\Gamma(k+j)} (-4\pi y)^{j-r} \left( \frac{-1}{4\pi} \right)^j \frac{\Gamma(s+1)}{\Gamma(s-j+1)} y^{s-j}
\]

\[
= (-4\pi)^{-r} y^{s-r} \sum_{j=0}^r \binom{r}{j} \frac{\Gamma(r+k+1)}{\Gamma(k+j)} \frac{\Gamma(s+1)}{\Gamma(s-j+1)} y^{s-j}
\]

\[
= (-4\pi)^{-r} \frac{\Gamma(s+k+r)}{\Gamma(s+k)} y^{s-r},
\]

(2.16)

where the last equality follows by comparing the coefficient of \( X^r \) in \((1 + X)^s(1 + X)^{k+r-1}\) and \((1 + X)^{s+k+r-1}\). For \((c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\), let \( \gamma = \left( \begin{array}{cc} c & -d \\ d & c \end{array} \right) \). Since \( y^s|_{k\gamma} = (cz + d)^{-k}|cz + d|^{-2s}y^s \),
we have
\[ \delta_k^r((cz + d)^{-k}|cz + d|^{-2s}y^s) = \delta_k^r(y^s|k\gamma) = \delta_k^r(y^s)|_{k+2r}\gamma \quad \text{(By Lemma 2.10)} \]
\[ = (-4\pi)^{-r} \frac{\Gamma(s + k + r)}{\Gamma(s + k)} (y^{s-r})|_{k+2r}\gamma \]
\[ = (-4\pi)^{-r} \frac{\Gamma(s + k + r)}{\Gamma(s + k)} (cz + d)^{-k-2r}|cz + d|^{-2(s-r)}y^{s-r}. \]

Let \( \chi \) be a Dirichlet character modulo \( N \). Multiplying both sides of the equation above by \( \chi(c) \) and then taking the sum over all \((c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}\) (ignoring convergence issues) we get
\[ (-4\pi)^{-r} \frac{\Gamma(s + k)}{\Gamma(s + k + r)} \delta_k^r(E_k(z,s;\chi,\chi_0)) = E_{k+2r}(z,-r;\chi,\chi_0). \tag{2.17} \]

From [Miy89, Chapter 7] we know that if \( k \geq 3 \), \( E_k(z,0;\chi,\chi_0) = \sum_{c,d} \chi(c)(cz + d)^{-k} \) is a usual holomorphic modular form in \( M_k(N,\chi) \). It follows from (2.17) and Proposition 2.9 that for all integers \( r \geq 0 \) and \( k \geq 3 \):
\[ E_{k+2r}(z,-r;\chi,\chi_0) \in \mathcal{N}^r_{k+2r}(N,\chi). \]

A similar argument for the Eisenstein series \( E_{k,N}(z,0,\overline{\chi}) \in M_k(N,\chi), k \geq 3 \) (cf. (2.9)), gives:

**Proposition 2.10.** Let \( k, r \) be integers such that \( 0 \leq r < k/2 - 1 \). If \( \chi \) is a Dirichlet character mod \( N \), then \( E_k(z,-r;\chi,\chi_0), E_{k,N}(z,-r,\overline{\chi}) \in \mathcal{N}^r_k(N,\chi) \).

**Theorem 2.11.** [Hid93, §10.1, Theorem 1] Suppose that \( r \geq 0 \) and \( k \geq 1 \). If \( f \in \mathcal{N}^r_{k+2r}(N,\chi) \), then
\[ f = \sum_{j=0}^{r} \delta_{k+2r-2j}^r h_j, \text{ where } h_j \in M_{k+2r-2j}(N,\chi). \tag{2.18} \]

More precisely,
\[ \mathcal{N}_{k+2r}(N,\chi) \cong \bigoplus_{j=0}^{r} M_{k+2r-2j}(N,\chi), \]
\[ \mathcal{N}^r_{k+2r}(N,\chi) \cong \bigoplus_{j=0}^{r} S_{k+2r-2j}(N,\chi), \]
and the isomorphisms are equivariant under the \( \Gamma_k \) action of \( \text{GL}^+_2(\mathbb{R}) \).

The projection \( f \mapsto h_0 \) induces a map
\[ \mathcal{H} : \mathcal{N}_{k+2r}(N,\chi) \to M_{k+2r}(N,\chi), \tag{2.19} \]
which is called the holomorphic projection.

**Lemma 2.12.** Let \( f \in S_k(N,\psi) \) and let \( g : \mathcal{H} \to \mathbb{C} \) be a smooth function which is slowly increasing such that \( g|_{k\gamma} = \psi(\gamma)g \) for every \( \gamma \in \Gamma_0(N) \). Then \( \langle f,g \rangle_N := \int_{\mathcal{H}/\Gamma_0(N)} f(z)g(z) y^{k-2}dxdy \) converges.

**Proof.** This follows from Lemma 2.15 (1) below and [Hid93, §9.3, (6)]. \( \square \)

**Lemma 2.13.** Suppose \( f \in S_k(N,\chi) \) and \( g \in \mathcal{N}^r_k(N,\chi) \). If \( r < k/2 \), then \( \langle f,g \rangle_N = \langle f,\mathcal{H}(g) \rangle_N \).

Further, if \( g \in \mathcal{N}^r_k(N,\chi) \), then \( \mathcal{H}(g) \) is the unique cusp form with the property \( \langle f,g \rangle_N = \langle f,\mathcal{H}(g) \rangle_N, \forall f \in S_k(N,\chi) \).
Proof. The first part follows from [Hd93, §10.1, Corollary 1]. By Theorem 2.11, we have \( \text{Hol}(g) \in S_k(N, \chi) \). Now the first part shows that \( \text{Hol}(g) \) satisfies the required property. From Lemma 2.12, we have \( f \mapsto \langle f, g \rangle_N \) defines an anti-linear functional on \( S_k(N, \chi) \). The uniqueness statement follows from the fact that Petersson inner product induces a non-degenerate pairing \( \langle \ , \ \rangle_N : S_k(N, \chi) \times S_k(N, \chi) \to \mathbb{C} \). \( \square \)

**Lemma 2.14. (Holomorphic Projection lemma)** [Zag92] Appendix C] Let \( \Phi : \mathcal{H} \to \mathbb{C} \) be a smooth function satisfying:

1. \( \Phi(\gamma(z)) = (cz + d)^k \Phi(z) \), \( \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) and \( z \in \mathcal{H} \),
2. \( \Phi(z) = c_0 + O(y^{-\epsilon}) \) as \( y = \text{Im}(z) \to \infty \),

for some integer \( k > 2 \) and numbers \( c_0 \in \mathbb{C} \) and \( \epsilon > 0 \). If \( \Phi(z) = \sum_{n=0}^{\infty} c_n(y)e(nx) \), then the function \( \phi(z) := \sum_{n=0}^{\infty} c_n(y)e(nz) \) with

\[
c_n = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^\infty c_n(y)e^{-2\pi ny}y^{k-2}dy
\]

for \( n > 0 \) belongs to \( M_k(\Gamma) \) and satisfies \( \langle f, \phi \rangle_\Gamma = \langle f, \Phi \rangle_\Gamma \), \( \forall f \in S_k(\Gamma) \).

Any rapidly decreasing function \( \Phi : \mathcal{H} \to \mathbb{C} \) which satisfies hypothesis (1) of Lemma 2.14 automatically satisfies hypothesis (2) with \( c_0 = 0 \). For such \( \Phi \), we set \( \text{Hol}(\Phi) := \phi \),

where \( \phi \) is as defined in Lemma 2.14. It is easy to see that \( \phi \) is a cusp form. Recall that the elements of \( NS_k^2(N, \chi) \) are rapidly decreasing. The definition of \( \text{Hol} \) given just above in fact extends the definition of the holomorphic projection \( \text{Hol} \) given in (2.19), by the uniqueness part of Lemma 2.13.

We now state a result which will enable one to apply the lemma above.

**Lemma 2.15.** Let \( k, N \) be a positive integers and \( \chi \) a Dirichlet character mod \( N \). Then

1. If \( h \in S_k(N, \chi) \), then \( |(h|\gamma)(z)| = O(y^{-B}) \), for all positive real numbers \( B \) and all \( \gamma \in \text{SL}_2(\mathbb{Z}) \), as \( y \to \infty \). In particular \( h \) is rapidly decreasing.
2. For any compact set \( T \subset \mathbb{R} \) and \( \gamma \in \text{SL}_2(\mathbb{Z}) \), there exists positive real numbers \( A \) and \( B \) such that if \( \chi \neq \chi_0 \)

\[
|E_k(z, s; \chi, \chi_0)| \leq A(1 + y^{-B}) \text{ as } y \to \infty \text{ as long as } \text{Re}(z) \in T.
\]

**Proof.** Observe that if \( h \in S_k(N, \chi) \), then \( h \) vanishes at the cusps. Now, the first part of the lemma follows from (2.14). For the second part see [Hd93, §9.3, Lemma 3]. \( \square \)

It follows from Lemma 2.15 that if \( h \) is a (holomorphic) cusp form of weight \( 2 \leq l < k \) (in our application below \( h \) will be the slash of a twist of \( g \) from the Introduction), then \( h(z)E_{k-l}(z, s; \chi, \chi_0) \) has weight \( k > 2 \) and satisfies the hypotheses of Lemma 2.14 with \( c_0 = 0 \). So \( \text{Hol}(h(z)E_{k-l}(z, s; \chi, \chi_0)) \) is defined, and we can calculate its Fourier expansion using Lemma 2.14 if we know the Fourier expansion of \( h(z)E_{k-l}(z, s; \chi, \chi_0) \).

### 3 Distributions and Measures

In this section, we define distributions and measures following [Pan88]. Most of the material covered in this section can also be found in [Was97], [MSD74]. Finally, we state the abstract Kummer congruences which is the key tool used in the construction of the \( p \)-adic \( L \)-function.
3.1 Distributions

Let $Y$ be a compact, Hausdorff and totally disconnected topological space. Then $Y$ is a projective limit of finite discrete spaces $Y_i$,

$$Y = \lim_{\rightarrow} Y_i,$$

(3.1)

with respect to transition maps $\pi_{ij} : Y_i \to Y_j$, for $i \geq j$, $i$, $j$ in some directed set $I$. We assume that the $\pi_{ij}$ are surjections, so the canonical maps $\pi_i : Y \to Y_i$ are projections. Let $R$ be a commutative ring and let $\text{Step}(Y, R)$ be the set of $R$-valued locally constant functions on $Y$.

**Definition 3.1.** A distribution on $Y$ with values in an $R$-module $A$ is a homomorphism of $R$-modules

$$\mu : \text{Step}(Y, R) \to A.$$

We use the notation

$$\mu(\varphi) = \int_Y \varphi \ d\mu = \int_Y \varphi(y) \ d\mu(y),$$

for $\varphi \in \text{Step}(Y, R)$. Any distribution $\mu$ can be given by a system of functions $\{\mu^{(i)} : Y_i \to A\}$, satisfying the following finite additivity condition:

$$\mu^{(j)}(y) = \sum_{x \in \pi_i^{-1}(y)} \mu^{(i)}(x), \forall \ y \in Y_j, \ x \in Y_i, \ i \geq j.$$  

(3.2)

Indeed, given such a system of functions $\{\mu^{(i)} : Y_i \to A \mid i \in I\}$, if $\delta_{i,x}$ is the characteristic function of the inverse image $\pi_i^{-1}(x) \subset Y$, for $x \in Y_i$, define

$$\mu(\delta_{i,x}) = \mu^{(i)}(x)$$

and extend the definition of $\mu$ to all of $\text{Step}(Y, R)$ by linearity. Conversely, given a distribution $\mu$, in order to construct such a system, set $\mu^{(i)}(x) = \mu(\delta_{i,x}) \in A$, $\forall \ x \in Y_i$.

It can be checked that a system of functions $\{\mu^{(i)} : Y_i \to A\}$ satisfies (3.2) if and only if for all $j \in I$ and all $\varphi_j : Y_j \to R$,

the sum

$$\sum_{x \in Y_i} \varphi_i(x) \mu^{(i)}(x) \text{ does not depend on } i, \ \forall \ i \geq j,$$

(3.3)

where $\varphi_i := \varphi_j \circ \pi_{ij} : Y_i \to R$. If $\mu$ is the corresponding distribution and $\varphi = \varphi_j \circ \pi_j \in \text{Step}(Y, R)$, then $\mu(\varphi)$ is just the sum above. If $Y = G = \lim_{\rightarrow} G_i$ is a profinite abelian group and $R$ is an integral domain containing all roots of unity of order dividing the cardinality of $G$ (perhaps a transfinite cardinal, in which case $R$ contains all roots of unity), then one needs to verify (3.3) only for all characters of finite order $\chi : G \to R^\times$, since the orthogonality relations imply that their linear span over $R \otimes \mathbb{Q}$ coincides with $\text{Step}(Y, R \otimes \mathbb{Q})$ (see [MSD74]).

**Example 3.2.** Let $p$ be an odd prime. Then $\mathbb{Z}_p^\times = \lim_{\leftarrow} (\mathbb{Z}/p^n\mathbb{Z})^\times$. We consider

$$X_p = X(\mathbb{Z}_p^\times) = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times), \ B = \{\chi \in X(\mathbb{Z}_p^\times) \mid \chi \text{ has finite order}\}.$$

We claim that $B$ is a basis for $\text{Step}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ as a $\mathbb{C}_p$-vector space. For every $x \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, let $\delta_{n,x}$ be the characteristic function of the basic open set $\{a \in \mathbb{Z}_p^\times \mid a \equiv x \mod p^n\}$. Then, by the orthogonality relations, we have

$$\delta_{n,x} = \frac{1}{\varphi(p^n)} \sum_{\chi \in X((\mathbb{Z}/p^n\mathbb{Z})^\times)} \bar{\chi}(x) \chi.$$  

(3.4)

Since every locally constant function $\mathbb{Z}_p^\times \to \mathbb{C}_p$ is a $\mathbb{C}_p$-linear combination of characteristic functions, we see that $B$ spans $\text{Step}(\mathbb{Z}_p^\times, \mathbb{C}_p)$. For linear independence, let $\chi_1, \ldots, \chi_n \in B$ and suppose $\sum_{i=1}^{n} a_i \chi_i = 0$, with $a_i \in \mathbb{C}_p$. By choosing $m$ sufficiently large we may assume $\chi_i \in X((\mathbb{Z}/p^m\mathbb{Z})^\times)$, for all $i$. By linear independence of characters, we have $a_i = 0$, for all $i$.  

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3.2 Measures

Let \( R \) be a topological ring with topology induced by a norm. Let \( C(Y, R) \) denote the \( R \)-module of continuous \( R \)-valued functions on \( Y \) and equip \( C(Y, R) \) with the corresponding sup norm topology. In this article we will take \( R = \mathbb{C} \) (or) \( \mathbb{C}_p \) (or) \( \mathcal{O}_p := \{ x \in \mathbb{C}_p \mid |x|_p \leq 1 \} \).

Definition 3.3. A measure on \( Y \) with values in a topological \( R \)-module \( A \) is a continuous homomorphism of \( R \)-modules \( \mu : C(Y, R) \to A \).

The restriction of a measure \( \mu \) to the \( R \)-submodule \( \text{Step}(Y, R) \subset C(Y, R) \) is a distribution, which we denote by the same symbol. Since \( Y \) is compact, we have \( \text{Step}(Y, R) \) is dense in \( C(Y, R) \). So every measure is uniquely determined by its values on \( \text{Step}(Y, R) \). We take for \( R \) a closed subring of \( \mathbb{C}_p \), and let \( A \) be a complete \( R \)-module with topology induced by a norm \( |\cdot|_A \) on \( A \). We further assume that \( |\cdot|_A \) is compatible with \( |\cdot|_R \), i.e., \( |ra|_A = |r|_p|a|_A \) for all \( r \in R \) and \( a \in A \). Then the condition that a distribution \( \{ \mu(i) : Y \to A \} \) gives rise to an \( A \)-valued measure on \( Y \) is equivalent to the condition that the \( \mu(i) \) are bounded, i.e., there is a uniform constant \( B > 0 \) such that for all \( i \in I \) and all \( x \in Y_i \), we have \( |\mu(i)(x)|_A < B \). The proof of this fact is easy using the non-archimedean property and completeness of the norm \( |\cdot|_A \) (see [Was97, Proposition 12.1]). In particular, if \( A = R = \mathcal{O}_p = \{ x \in \mathbb{C}_p \mid |x|_p \leq 1 \} \) is the ring of integers of \( \mathbb{C}_p \), then distributions are the same as measures. The most important tool in the construction of the \( p \)-adic \( L \)-function is the following criterion for the existence of a measure with prescribed properties.

Theorem 3.4. (The abstract Kummer congruences) ([Kat78, Proposition 4.0.6], [CP04])

Let \( \{ f_i \} \) be a system of continuous \( \mathcal{O}_p \)-valued functions on \( Y \) such that the \( \mathbb{C}_p \)-linear span of \( \{ f_i \} \) is dense in \( C(Y, \mathbb{C}_p) \). Let \( \{ a_i \} \) be any system of elements with \( a_i \in \mathcal{O}_p \). Then the existence of an \( \mathcal{O}_p \)-valued measure \( \mu \) on \( Y \) (i.e., \( \mu(C(Y, \mathcal{O}_p)) \subset \mathcal{O}_p \)) with the property that

\[
\int_Y f_i \, d\mu = a_i
\]

is equivalent to the following: for an arbitrary choice of elements \( b_i \in \mathbb{C}_p \), almost all of which vanish, and any \( n \geq 0 \), we have the following implication of congruences:

\[
\sum_i b_i f_i(y) \in p^n \mathcal{O}_p, \quad \forall \ y \in Y \implies \sum_i b_i a_i \in p^n \mathcal{O}_p.
\]

(3.5)

Proof. The necessity is obvious. Indeed if \( \sum_i b_i f_i(y) \in p^n \mathcal{O}_p \), then

\[
\sum_i b_i a_i = \sum_i \int_Y b_i f_i \, d\mu
= p^n \int_Y \left( p^{-n} \sum_i b_i f_i \right) \, d\mu \in p^n \mathcal{O}_p.
\]

In order to prove the sufficiency we need to construct a measure \( \mu \) from the numbers \( a_i \). For a function \( f \in C(Y, \mathcal{O}_p) \) and a positive integer \( n \), there exists \( b_i \in \mathbb{C}_p \) such that \( b_i = 0 \) for almost all \( i \), and

\[
f - \sum_i b_i f_i \in C(Y, p^n \mathcal{O}_p)
\]

by the density of the \( \mathbb{C}_p \)-span of the \( \{ f_i \} \) in \( C(Y, \mathbb{C}_p) \). Now, we claim that the value \( \sum_i b_i a_i \) belongs to \( \mathcal{O}_p \) and is well defined modulo \( p^n \), i.e., it doesn’t depend on the choice of \( b_i \). Since \( f \in C(Y, \mathcal{O}_p) \), clearly \( \sum_i b_i f_i \in C(Y, \mathcal{O}_p) \). Therefore, by (3.5), we have \( \sum_i b_i a_i \in \mathcal{O}_p \). Let \( c_i \in \mathbb{C}_p \)
be another set of numbers with $c_i \neq 0$ only for finitely many $i$ such that $f - \sum_i c_i f_i \in \mathcal{C}(Y, p^n\mathcal{O}_p)$. Then
\[ \sum_i (c_i - b_i) f_i = (f - \sum_i b_i f_i) - (f - \sum_i c_i f_i) \in \mathcal{C}(Y, p^n\mathcal{O}_p). \]

By (3.5), we have $\sum_i c_i a_i \equiv \sum_i b_i a_i \mod p^n$. Therefore, $\sum_i b_i a_i$ is well defined modulo $p^n$. We denote this value by
\[ \left( \sum_i b_i f_i \right) \quad \left( \sum_i c_i f_i \right) \equiv (\int_Y f d\mu \mod p^n) \mod p^n\mathcal{O}_p. \]

So, we may define $\mu$ on $\mathcal{C}(Y, \mathcal{O}_p)$ via
\[ \int f \ d\mu = \left\{ \left( \int f \ d\mu \mod p^n \right) \right\}_{n \geq 1} \in \lim_{n \to \infty} \mathcal{O}_p/p^n\mathcal{O}_p = \mathcal{O}_p. \]

Since every element of $\mathcal{C}(Y, \mathcal{C}_p)$ is bounded, by rescaling, the above definition of $\mu$ extends to all of $\mathcal{C}(Y, \mathcal{C}_p)$. A check shows $\mu : \mathcal{C}(Y, \mathcal{C}_p) \to \mathcal{C}_p$ is continuous, so $\mu$ is an $\mathcal{O}_p$-valued measure. Clearly $\int_Y f d\mu = a_i$. \hfill \Box

Recall that $X_p = \text{Hom}_{\text{cont}}(\mathbb{Z}^\times_p, \mathbb{C}^\times_p)$ has an analytic structure described in the Introduction. If $\mu : \mathcal{C}(\mathbb{Z}^\times_p, \mathcal{C}_p) \to \mathcal{C}_p$ is a measure, the non-archimedean Mellin transform of $\mu$, defined by
\[ L_\mu(\chi) = \mu(\chi) = \int_{\mathbb{Z}^\times_p} \chi\ d\mu, \ \forall \chi \in X_p, \quad \text{(3.6)} \]
gives a bounded $\mathcal{C}_p$-analytic function $L_\mu : X_p \to \mathcal{C}_p$ (see [MSD74 § 7.4], [Man73 Theorem 8.7]). Here ‘analytic’ means that the integral (3.6) depends analytically on the parameter $\chi \in X_p$. The converse is also true: any bounded $\mathcal{C}_p$-analytic function on $X_p$ is the Mellin transform of some measure $\mu$. These measures with the convolution operation form an algebra, which essentially coincides with the Iwasawa algebra (see [CP04 §(1.4.3), §(1.5.2)]).

4 Construction of Complex-valued Distributions

From now on, let $f$ and $g$ be the primitive cusp forms as in the Introduction. In this section we define two complex-valued distributions associated to $f$ and $g$ and compare them.

Let $p$ be a prime as in the Introduction. The $p$-stabilization of $f$ is defined by
\[ f_0(z) = f(z) - \alpha'(p)f(pz) = f(z) - \alpha'(p)(f|_{V_p})(z), \quad \text{(4.1)} \]
where as before $f(z) = \sum_{n=1}^\infty a(n, f)e(nz) \in S_k(C_f, \psi)$. Let $f_0(z) = \sum_{n=1}^\infty a(n, f_0)e(nz)$ be the Fourier expansion of $f_0$. Comparing the Fourier coefficients in (4.1), we get $a(n, f_0) = a(n, f) - \alpha'(p)a(n/p, f)$. Hence, we have the following identity for the corresponding Dirichlet series:
\[ L(s, f_0) = \sum_{n=1}^\infty a(n, f_0)n^{-s} = (1 - \alpha'(p)p^{-s})\left( \sum_{n=1}^\infty a(n, f)n^{-s} \right) = (1 - \alpha'(p)p^{-s})L(s, f). \quad \text{(4.2)} \]

From (4.1), it follows that $f_0 \in S_k(pC_f, \psi)$ and from (2.8) and (4.2), we have
\[ L(s, f_0) = (1 - \alpha'(p)p^{-s})\left( \prod_{q \text{ prime}} (1 - \alpha(q)q^{-s})^{-1}(1 - \alpha'(q)q^{-s})^{-1} \right) \]
\[ = (1 - \alpha(p)p^{-s})^{-1}\left( \sum_{n=1}^\infty a(n, f)n^{-s} \right). \quad \text{(4.3)} \]
Thus we have the following multiplicative relation
\begin{equation}
    a(p^ r n, f_0) = \alpha(p^ r) a(n, f_0), \quad \forall \, r, n \geq 0.
\end{equation}
Hence, \( f_0 \) is a \( U_p \)-eigenvector with eigenvalue \( \alpha(p) \), i.e., \( f_0 | U_p = \alpha(p) f_0 \).

Recall \( g \in S_l(N, \omega) \). From the definition of the operators \( w_d \) and \( V_d \) given in Section 2.2, one checks that
\begin{equation}
    g|w_{AB} = A^{1/2} g|B V_A,
\end{equation}
where \( A, B \) are positive integers.

Recall that complex valued Dirichlet characters \( \chi \) of \( p \)-power conductor are the same as finite order characters \( \chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times \). As in Example 3.2, we have \( B = \{ \chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times \text{ of finite order} \} \) forms a basis of \( \text{Step}(\mathbb{Z}_p^\times, \mathbb{C}) \). Therefore every complex-valued function on \( B \) extends to a complex-valued distribution on \( \mathbb{Z}_p^\times \).

Let \( \chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times \) be a Dirichlet character of conductor \( C_\chi \). Then \( g(\chi) = \sum_{n=1}^{\infty} \chi(n) b(n) e(nz) \) lies in \( S_l(C_\chi^2, \omega \chi^2) \), where here and below we use the convention that \( \chi(n) = 0 \) if \( n \notin p\mathbb{Z} \).

For every \( s \in \mathbb{C} \), define a quantity \( \Psi^M_s(\chi) \) as follows:
\begin{equation}
    \Psi^M_s(\chi) = \frac{(p M')^{s-1/2} C_\chi^2}{\Lambda(g) \alpha(p M')} \cdot \Psi(s, f_0 | V_{C_f}, g(\chi) | w_{C_0 M'}) \cdot \pi^{1-1}(f, f | C_f),
\end{equation}
where \( C_0, M' \) are natural numbers satisfying:
\begin{equation}
    C_0 = p C = p C_f C_g, \quad p^2 C_\chi^2 | M' \quad \text{and} \quad S(M') = \{ p \}.
\end{equation}

A priori, the definition of \( \Psi^M_s(\chi) \) depends on \( M' \), though we show below that it does not, whence \( \Psi^M_s(\chi) \) extends to a (well-defined) complex-valued distribution on \( \mathbb{Z}_p^\times \). To do this, for each \( s \in \mathbb{C} \), consider the complex-valued distribution \( \Psi_s \) on \( \mathbb{Z}_p^\times \) whose value on the Dirichlet character \( \chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times \) is given by:
\begin{equation}
    \Psi_s(\chi) := \frac{\omega(C_\chi) G(\chi)^2 C_\chi^{2s-l-1}}{\alpha(C_\chi)^2} \cdot \Psi(s, f, g^p(\chi)) \cdot \pi^{1-1}(f, f | C_f).
\end{equation}

**Proposition 4.1.** Let \( p \) be an odd prime for which \( f \) is a \( p \)-ordinary form. Then for every Dirichlet character \( \chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times \) and positive integer \( M' \) such that \( p^2 C_\chi^2 | M' \) and \( S(M') = \{ p \} \), we have
\begin{equation}
    \Psi^M_s(\chi) = \Psi_s(\chi).
\end{equation}

In particular, \( \Psi^M_s(\chi) \) does not depend on \( M' \).

**Proof.** First we simplify the right side of \( 4.6 \). From \( 1.2 \) and \( 1.3 \) it follows that
\begin{equation}
    \Psi(s, f_0 | V_{C_f}, g(\chi) | w_{C_0 M'}) = (2\pi)^{-2s} \Gamma(s) \Gamma(s-l+1) L_{pc}(2s+2-k-l, \psi \omega \chi^2) \times L(s, f_0 | V_{C_f}, g(\chi) | w_{C_0 M'}),
\end{equation}
noting that \( S(p C) = S(C_0 M') \), for the joint level \( C_0 M' \) of the forms \( f_0 | V_{C_f} \) and \( g(\chi) | w_{C_0 M'} \).

We define \( A(n) \) and \( B(n) \) to be the coefficients in the Dirichlet series
\begin{equation}
    L(s, f_0) = \sum_{n=1}^{\infty} A(n) n^{-s},
\end{equation}
\begin{equation}
    L(s, g(\chi) | w_{\rho^2 C_g C_\chi^2}) = \sum_{n=1}^{\infty} B(n) n^{-s}.
\end{equation}
Then, by the multiplicative property (4.4), we have
\[ A(Mn) = \alpha(M)A(n), \text{ for all } M \text{ such that } S(M) = \{p\}. \] (4.11)

Let \( M' = pC_\chi^2M_1 \). Applying (4.5) with \( A = M_1C_f \) and \( B = p^2C_\chi^2 \), we get
\[ g(\chi)|w_{C_\chi M'} = g(\chi)|w_{M_1C_f}p^2C_\chi^2 = (M_1C_f)^{1/2}g(\chi)|w_{p^2C_\chi^2}V_{M_1C_f} \]
\[ = (M_1C_f)^{1/2} \sum_{n=1}^{\infty} B(n)e(M_1C_fnz). \] (4.12)

We transform the last \( L \)-function in (4.9) as follows:
\[ L(s, f_0|V_{C_f}, g(\chi)|w_{C_\chi M'}) \]
\[ = \quad (M_1C_f)^{1/2} \sum_{n=1}^{\infty} A(nC_f^{-1})B(nM_1^{-1}C_f^{-1})n^{-s} \]
\[ = \quad (M_1C_f)^{1/2} \sum_{n=1}^{\infty} A(nM_1)B(n)(nM_1C_f)^{-s} \]
\[ \equiv \quad (M_1C_f)^{1/2-s} \alpha(M_1) \sum_{n=1}^{\infty} A(n)B(n)n^{-s} \]
\[ = \quad (M_1C_f)^{1/2-s} \alpha(M_1)L(s, f_0, g(\chi)|w_{p^2C_\chi^2}) \]
\[ = \quad \frac{\alpha(M')}{\alpha(pC_\chi^2)} \left( \frac{M'C_f}{pC_\chi^2} \right)^{1/2-s} L(s, f_0, g(\chi)|w_{p^2C_\chi^2}). \] (4.13)

If we substitute (4.13) in (4.10), we see that (4.10) does not depend on \( M' \). In order to obtain the more precise expression given by (4.8), it is enough to establish the following equality:

\[ \Psi(s, f_0, g(\chi)|w_{p^2C_\chi^2}) = \alpha(p)^2p^{l-2s}\Lambda(g(\chi))\Psi(s, f, g^\theta(\chi)) \] (4.14)

where \( \Lambda(g(\chi)) \) is the root number associated to \( g(\chi) \), i.e., \( g(\chi)|w_{C_\chi^2} = \Lambda(g(\chi))g^\theta(\chi) \), since by (2.7) we have \( \Lambda(g(\chi)) = \omega(C_\chi)\chi(C_\chi)G(\chi)^2C_\chi^{-1}\Lambda(g) \).

To derive (4.14) we find an appropriate expression for \( g(\chi)|w_{p^2C_\chi^2} \). Applying (4.5) once more with \( A = p^2 \) and \( B = C_\chi C_f^2 \), we get
\[ g(\chi)|w_{p^2C_\chi^2} = p^l g(\chi)|w_{C_\chi^2}V_{p^2} = p^l \Lambda(g(\chi))g^\theta(\chi)|V_{p^2}, \]
so that
\[ L(s, f_0, g(\chi)|w_{p^2C_\chi^2}) = p^l \Lambda(g(\chi))L(s, f_0, g^\theta(\chi)|V_{p^2}). \]

A computation similar to that of (4.13) shows that
\[ L(s, f_0, g^\theta(\chi)|V_{p^2}) = p^{-2s}L(s, f_0|U_{p^2}, g^\theta(\chi)) \]
\[ = \alpha(p)^2p^{-2s}L(s, f_0, g^\theta(\chi)), \]
where we used \( f_0|U_p = \alpha(p)f_0 \) in the last step. Therefore
\[ \Psi(s, f_0, g(\chi)|w_{p^2C_\chi^2}) = \alpha(p)^2\Lambda(g(\chi))p^{l-2s}\Psi(s, f_0, g^\theta(\chi)). \]

Substituting this in (4.14) we are reduced to proving
\[ \Psi(s, f_0, g^\theta(\chi)) = \Psi(s, f, g^\theta(\chi)). \]
From (4.1), it follows that
\[
L(s, f_0, g^\theta(\chi)) = L(s, f, g^\theta(\chi)) - \alpha'(p)L(s, f|V_p, g^\theta(\chi)) = L(s, f, g^\theta(\chi)) \quad (\because \chi(p) = 0).
\]

Further, for every character \(\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times\) we have \(S(pC_f C_g C_\chi^2) = S(CC_\chi^2)\) (except if \(\chi\) is the trivial character) so that
\[
L_{pC_f C_g C_\chi^2}(2s + 2 - k - l, \psi \omega \chi^2) = L_{CC_\chi^2}(2s + 2 - k - l, \psi \omega \chi^2)
\]
in all cases (since if \(\chi\) is the trivial character, \(\chi(p) = 0\)). From (4.15) and (4.16), it follows that \(\Psi(s, f, g^\theta(\chi)) = \Psi(s, f, g^\theta(\chi))\). Thus we obtain (4.14).

We conclude this section by making an observation on the algebraicity of \(\Psi^{(M')}\), which will be used in later sections.

**Corollary 4.2.** Let \(\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times\) be a finite order character and \(M'\) as in Proposition 4.1. Then for every integer \(s\) with \(l \leq s \leq k - 1\), we have \(\Psi^{(M')}_s(\chi) \in \overline{\mathbb{Q}}\).

**Proof.** From Theorem 2.6 we have \(\Psi_s(\chi)\) is algebraic for every integer \(s\) with \(l \leq s \leq k - 1\). Hence, by the previous proposition, we have \(\Psi^{(M')}_s\) is also algebraic for every integer \(s\) in the interval \([l, k - 1]\).

Dirichlet characters actually take values in \(\overline{\mathbb{Q}} \subset \mathbb{C}\). Via our fixed embedding \(i_p : \overline{\mathbb{Q}} \to \mathbb{C}_p\), we may think of them as \(\mathbb{C}_p\)-valued. Moreover, by the corollary above we may similarly think of \(\Psi^{(M')}_s\) as \(\mathbb{C}_p\)-valued for \(s \in [l, k - 1]\). Thus, for such \(s\), all the measures in this section can (and later will be) thought of as \(p\)-adic entities.

## 5 Integral representation for Distributions

In this section we obtain an integral expression for the distribution \(\Psi^{(M')}\) given by (4.6) involving the Petersson inner product of certain cusp forms. We also compute the Fourier expansion of one of these cusp forms. This will be needed in the last section in order to explicitly verify the Kummer congruences.

Recall the following classical integral formula of Rankin (cf. (2.11)). For \(F \in S_k(N, \psi)\) and \(G \in M_l(N, \omega)\), we have
\[
\Psi(s, F, G) = 2^{-1} \Gamma(s - l + 1) \pi^{-s} \langle F^\rho, GE(s - k + 1) \rangle_N,
\]
where
\[
F^\rho(z) = \frac{F(-\overline{z})}{\psi}, \quad E(z, s) = E_{k-l,N}(z, s, \psi \omega) = \sum_{c,d=\infty} \psi \omega(d) \epsilon_N(z + d)^{-s-l}|\epsilon_N(z + d)|^{-2s}.
\]

Let \(\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times\) be a finite order character. Let \(M'\) be as in (4.7), i.e., \(p^2 C_\chi^2 | M'\) and \(S(M') = \{p\}\). We apply (5.1) with
\[
N = C_0 C_f M',
F = f_0|V_{C_f} \in S_k(pC_f^2, \psi) \subset S_k(C_0 C_f M', \psi),
G = g(\chi)|w_{C_0 M'} \in S_l(C_0 M', \omega \chi^2) \subset S_l(C_0 C_f M', \omega \chi^2).
\]
For every integer \( s \) such that \( l \leq s \leq k - 1 \), we transform the definition of the distribution \((4.6)\) into the following integral:

\[
\Psi(s, f_0|V_{C_f}, g(\chi)|w_{C_0 M'}) = 2^{-1} \Gamma(s - l + 1) \pi^{-s} (f_0^p|V_{C_f}, G E(s - k + 1))_{C_0 M'},
\]

where \( E(z, s - k + 1) = E_{k-l,C_0 M'}(z, s - k + 1, \psi \omega \chi^2) \). If we set

\[
K(s) = G \cdot E(z, s),
\]

then the formula for the values of the distribution \((4.6)\) takes the form

\[
\Psi^{(M')}_{s}(\chi) = (p M')^{s-l/2} C_{f}^{s-l/2} \chi(C_{g}) \Lambda(g)^{-1} \alpha(p M')^{-1} \times 2^{-1} \Gamma(s - l + 1) \pi^{-s} \frac{(f_0^p|V_{C_f}, K(s - k + 1))_{C_0 M'}}{\pi^{-l} (f, f)_{C_f}}.
\]

By Lemma 2.2 (with \( N = C_0 C_f, M' = M', f = f_0^p|V_{C_f} \) and \( g = K(s) \)), we obtain

\[
\langle f_0^p|V_{C_f}, K(s) \rangle_{C_0 M'} = \langle f_0^p|V_{C_f}, T^{C_0 M'}_{F C_f} (K(s)) \rangle_{C_0 C_f} = (-1)^k M'^{n-k/2} \langle f_0^p|V_{C_f}, K'(s) \rangle_{U M' w_{C_0 C_f}} {\rangle}_{C_0 C_f},
\]

where \( K'(s) = K(s)|w_{C_0 C_f} M' \). Hence,

\[
\Psi^{(M')}_{s}(\chi) = (-1)^k C_{f}^{l/2} g(\chi)|V_{C_f} \times 2^{-1} \Gamma(s - l + 1) \pi^{-s} \frac{(f_0^p|V_{C_f}, K'(s - k + 1))_{U M' w_{C_0 C_f}} {\rangle}_{C_0 C_f}}{\pi^{-l} (f, f)_{C_f}}.
\]

Now we compute the Fourier coefficients of \( K'(s) \) for special values of \( s \) (more precisely, for \( l - k + 1 \leq s \leq 0, s \in \mathbb{Z} \)). We rewrite \( K'(s) \) as

\[
K'(s) = g' \cdot E'(z, s),
\]

where

\[
g' = g(\chi)|w_{C_0 M'} w_{C_0 C_f} M' \quad \text{and} \quad E'(z, s) = E(z, s)|w_{C_0 C_f} M'.
\]

It follows from the definition of \( w_{C_0 M'}, w_{C_0 C_f} M' \) that

\[
g' = (-1)^k C_{f}^{l/2} g(\chi)|V_{C_f}.
\]

The Fourier expansion of the Eisenstein series \( E'(z, s) \) will be computed in the next section, from which we will obtain the Fourier expansion of \( K'(s) \).

### 5.1 Fourier expansion of Eisenstein series

Here we follow [Miy89 §7.2] to compute the Fourier expansion of \( E'(z, s) \). The procedure given in [Miy89] describes the Fourier expansion of more general Eisenstein series. Let \( \mathcal{H}' = \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \) denote the right half plane. For \( \alpha \in \mathbb{C} \) and \( \beta, z \in \mathcal{H}' \), the Whittaker function \( W(z; \alpha, \beta) \) is defined by the following integral:

\[
W(z; \alpha, \beta) = \Gamma(\beta)^{-1} \int_{0}^{\infty} (u + 1)^{\beta - 1} u e^{-zu} \, du.
\]

The convergence of the above integral follows from [Miy89 Lemma 7.2.1 (2)].
Lemma 5.1. The function $W(z; \alpha, \beta)$ can be continued analytically to a holomorphic function on $\mathcal{H}' \times \mathbb{C} \times \mathbb{C}$ satisfying:

1. $W(z; \alpha, \beta) = z^{1-\alpha-\beta}W(z;1-\beta,1-\alpha)$, $\forall (z, \alpha, \beta) \in \mathcal{H}' \times \mathbb{C} \times \mathbb{C}$.
2. $W(z; \alpha, 0) = 1$, $\forall (z, \alpha) \in \mathcal{H}' \times \mathbb{C}$.
3. $W(y; \alpha, \beta) = \sum_{i=0}^{r} (-1)^i \frac{r}{i} \Gamma(\alpha) \Gamma(\alpha-i) y^{r-i} W(y; \alpha-i, \beta+r)$, $\forall r \geq 0$, $y \in \mathbb{R}^+$, $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$.

Proof. Note that $\omega(z; \alpha, \beta)$ defined by [Miy89] (7.2.31) equals to $z^\beta W(z; \alpha, \beta)$ for all $(z, \alpha, \beta) \in \mathcal{H}' \times \mathbb{C} \times \mathcal{H}'$. The lemma now follows from [Miy89] Theorem 7.2.4 (1), [Miy89] Lemma 7.2.6, and [Miy89] (7.2.40).

By part (3) of Lemma 5.1, with $\beta = -r$, and by part (2), we obtain for all $y > 0$ that

$$W(y; \alpha, -r) = \sum_{i=0}^{r} (-1)^i \frac{r}{i} \Gamma(\alpha) \Gamma(\alpha-i) y^{r-i} W(y; \alpha-i, 0),$$

(5.6)

Recall that the Eisenstein series $E_k(z; s; \theta, \varphi)$ for $\theta$ and $\varphi$ Dirichlet characters mod $L$ and $M$ respectively is defined by (cf. (2.21))

$$E_k(z; s; \theta, \varphi) = y^s \sum_{c,d \in -\infty} \theta(c) \varphi(d)(cz+d)^{-k} |cz+d|^{-2s}.$$

We now state a result about the Fourier expansion of Eisenstein series.

Theorem 5.2. Let $\theta$ and $\varphi$ be Dirichlet characters mod $L$ and $M$, respectively, satisfying $\theta(-1)\varphi(-1) = (-1)^k$. Then for any integer $k$, the Eisenstein series $E_k(z; s; \theta, \varphi)$ can be analytically continued to a meromorphic function on the whole $s$-plane and has the Fourier expansion

$$E_k(z; s; \theta, \varphi) = C(s)y^s + D(s)y^{1-k-s} + A(s)y^s \sum_{n=1}^{\infty} a_n(s)(4\pi/M)^s e(nz/M) W(4\pi y/M; k + s, s)$$

$$+ B(s)y^s \sum_{n=1}^{\infty} a_n(s)(4\pi/M)^{s+k} e(-nz/M) W(4\pi y/M; s, k + s),$$

where

$$C(s) = \begin{cases} 2LM(2s + k, \varphi), & \text{if } \theta = \chi_0, \\ 0, & \text{otherwise,} \end{cases}$$

$$D(s) = \begin{cases} 2\sqrt{\pi} i^{-k} \prod_{p \mid M} (1 - p^{-1}) \Gamma(s)^{-1} \Gamma(s+k)^{-1} \\ \times \Gamma\left(\frac{2s+k-1}{2}\right) \Gamma\left(\frac{2s+k}{2}\right) L_L(2s + k - 1, \theta), & \text{if } \varphi \text{ is the trivial character mod } M, \\ 0, & \text{otherwise}, \end{cases}$$

$$A(s) = 2^{k+1} G(\varphi^0)(\pi/M)^{s+k} \Gamma(s+k)^{-1},$$

$$B(s) = 2^{1-k} G(\varphi(-1)) \Gamma(\pi/M)^{s+k} \Gamma(s)^{-1},$$

$$a_n(s) = \sum_{0 < c \mid n} \theta(n/c) e^{k+2s-1} \sum_{0 < d \mid (l,c)} d\mu(l/d) \varphi(0)(l/d) \varphi^0(c/d).$$

Here $\varphi^0$ denotes the primitive character associated with $\varphi$ of conductor $m_\varphi = M/l$ and $\mu$ is the Möbius function.
Proof. This is [Miy89 Theorem 7.2.9], noting that $E_k(z, s; \theta, \varphi)$ differs from the one defined in [Miy89 (7.2.1)] by a factor of $y^s$ and $\omega(y; \alpha, \beta)$ equals $y^\beta W(y; \alpha, \beta)$, $\forall (\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$. \hfill $\square$

We apply the above theorem to compute the Fourier expansion of $E'(z, s)$. Recall

$$E'(z, s) = E(z, s)|_{wC_0C_f' M'} = E_{k-l, C_0C_f' M'}(z, s, \bar{\psi} \omega^{s})|_{wC_0C_f' M'}$$

$$= (C_0C_f' M')^{-(k-l+2s)/2} E_{k-l, z, s, \bar{\psi} \omega^{s}}(\chi_0) \quad \text{(by direct computation).} \quad (5.7)$$

For convenience we introduce the normalized Eisenstein series

$$G^*(z, s) = \frac{(C_0C_f' M')^{(k-l+2s)/2} \Gamma(k-l+s)}{(-2\pi i)^{k-l-s}} E'(z, s) \quad (5.8)$$

$$\Gamma(k-l+s) E_{k-l, z, s, \bar{\psi} \omega^{s}}(\chi_0). \quad (5.9)$$

If $s$ is an integer such that $s \leq 0$ and $k - l + s > 0$, then from (5.5) and Theorem 5.2, we have

$$G^*(z, s) = \varepsilon(k-l, y, s, \bar{\psi} \omega^{s}) + 2(4\pi y)^s \sum_{n=1 \atop 1 < c|n} \sum_{d|n} \bar{\psi}(n/c)\epsilon^{k-l+2s-1}W(4\pi yn; k - l + s, s)e(nz)$$

$$= \varepsilon(k-l, y, s, \bar{\psi} \omega^{s}) + 2(4\pi y)^s \sum_{n=1 \atop d'd''=n} \bar{\psi}(d')\epsilon^{k-l+2s-1}W(4\pi yn; k - l + s, s)e(nz), \quad (5.10)$$

where

$$\varepsilon(k-l, y, s, \bar{\psi} \omega^{s}) = \frac{\Gamma(k-l+s)}{(-2\pi i)^{k-l-s}} (C(s)y^s + D(s)y^{1-k+l-s}),$$

with $C(s)$, $D(s)$ denoting the same constants as in Theorem 5.2 (corresponding to $\theta = \bar{\psi} \omega^{s}$, $\varphi = \chi_0$). The term with $\bar{\tau}$ doesn’t appear as for such $s$ we have $B(s) = 0$ because the Gamma function $\Gamma(s)$ in the denominator of $B(s)$ has a pole at $s \leq 0$ and the function $a_n(s)W(4\pi yn/M, k + s)$ is holomorphic in $s$.

### 5.2 Integral representation via holomorphic projection

Taking $s$ equal to $s - k + 1 \leq 0$ in (5.8), we get

$$E'(z, s - k + 1) = (C_0C_f' M')^{-(2s+2-k-l)/2}(-1)^{k-l}k^{-k-l+1} \times \Gamma(s - l + 1)^{-1} G^*(z, s - k + 1). \quad (5.11)$$

Substituting (5.11) and (5.4) into (5.3), and substituting $C_0 = pC = pC_f C_g$, we get

$$\Psi_\alpha^{(M')}(\chi) = \frac{2k-l+1-p^{k+2-k-l}}{\alpha(pM')\Lambda(g)C^{(2s+2-k-l)/2}} \cdot \frac{\langle f_0^\alpha |_{wC_f' G} \rangle C_f G^*(z, s - k + 1)}{\langle f, f \rangle_{C_f}}$$

$$\times U_{M'} w_{C_0C_f' C_0C_f' G} \cdot G^*(z, s - k + 1). \quad (5.12)$$

in which we have set

$$\gamma(M') = 2^{k-l-1-k-l} p^{k/2-1} C_f^{-1} C_g^{(l-k)/2} \alpha(pM')^{-1} \Lambda(g)^{-1},$$

$$K^*(s) = C_f^{-s} C_g^{-s} \bar{\chi}(C_g) g(\chi) |_{C_f} G^*(z, s). \quad (5.13)$$

This formula differs from [Pan88 (4.22)] by $(-1)^{s-k+1}$ and is the source of the sign discrepancy in Theorem 5.1 mentioned in the first footnote. Without the sign in (5.11), it becomes difficult to verify the abstract Kummer congruences in the proof of Proposition 5.1 (2) later.
Observe that $\gamma(M')$ is an algebraic number. Moreover, $i_p(\gamma(M'))$ is $p$-integral if $i_p(A(g))$ is a $p$-adic unit. One can check this last fact using explicit formulas for the root number in terms of Gauss sums when the automorphic representation attached to $g$ has no supercuspidal local factors; it is apparently also true in general [Hod88, (5.4a), (5.4b)]. In any case $i_p(\gamma(M'))$ is bounded independent of $M'$, which is all we shall need later.

It follows from (5.10) that for integers $l - k < s \leq 0$ we have

$$K^*(s) = \sum_{n=1}^{\infty} \sum_{C_f n_1 + n_2 = n} d(n_1, n_2; y, s) e(nz),$$

(5.14)

where for $p \mid n$, the Fourier coefficients are given by

$$d(n_1, n_2; y, s) = C_f^{-s} C_g^{-s} (C_f) \chi(n_1)b(n_1) \times 2(4\pi y)^s \sum_{n_2 = \dd} \psi \omega \chi_x(d') d^{2s+k-l-1} W(4\pi n_2 y, s - l + k, s).$$

(5.15)

Here we used that if $p \mid n$ there is no contribution to the coefficient of $e(nz)$ in $K^*(s)$ from the constant ($n_2 = 0$) term of Eisenstein series $G^*(\gamma, s)$ because the coefficient of $e(C_f n_1 z)$ in $g(\gamma)|V_{C_f}$ is zero for $p \mid n_1$ since $\chi(n_1) = 0$.

The expression (5.12) for $\Psi^*_f(\gamma)(\chi)$ involves $K^*(s - k + 1)$ whose Fourier coefficients contain Whittaker functions which are difficult to handle. To get rid of the Whittaker functions we consider its holomorphic projection. We first check that $\mathcal{H}\mathcal{O}(K^*(s - k + 1))$ is defined. From Proposition 2.10 it follows that if $(k + l)/2 < s \leq k - 1$, then $E_{k-l}(z, s-k+1; \psi \omega \chi_x, \chi_0)$ belongs to $\mathcal{N}^{s+k-1}_{k-l}(C_0 C_f M', \psi \omega \chi^2)$, hence so does $G^*(z, s-k+1)$, by (5.9). Thus $K^*(s - k + 1) \in \mathcal{N}^{s+k-1}_{k-l}(C_0 C_f M', \psi)$ if $s > (k+l)/2$. So for such $s$ one can define the holomorphic projection $\mathcal{H}\mathcal{O}(K^*(s - k + 1))$ of $K^*(s - k + 1)$ in the sense of Theorem 2.11. However, for $l \leq s \leq (k+l)/2$ it is not clear (to us) that $K^*(s - k + 1)$ is a nearly holomorphic form. So we cannot use Theorem 2.11 to define the holomorphic projection of $K^*(s - k + 1)$ for $l \leq s \leq (k+l)/2$. Nevertheless, by the discussion at the end of §2, we know that $K^*(s)$ is rapidly decreasing and satisfies the hypotheses of Lemma 2.14 with $c_0 = 0$. Thus one can define the holomorphic projection of $K^*(s - k + 1)$ for any integer $l \leq s \leq -k - 1$.

We now study:

$$\tilde{K}_{M'}(s) := \mathcal{H}\mathcal{O}(K^*(s))|U_{M'},$$

for integers $l - k + 1 \leq s \leq 0$. We begin by computing the level and nebentypus of $\tilde{K}_{M'}(s)$. Since $K^*(s)|_{\gamma} = \psi(\gamma) K^*(s)$ for all $\gamma \in \Gamma_0(C_0 C_f M')$, we have $\mathcal{H}\mathcal{O}(K^*(s)) \in S_k(C_0 C_f M', \psi)$, by the remarks after Lemma 2.14. As $p^2 \mid M'$ we have $\mathcal{H}\mathcal{O}(K^*(s))|U_p \in S_k(C_0 C_f M'/p, \psi)$, by Lemma 2.1 (1). Repeatedly applying Lemma 2.1 (1) we get $\mathcal{H}\mathcal{O}(K^*(s))|U_{M'} \in S_k(C_0 C_f, \psi)$.

We now state the main result of this section.

**Proposition 5.3.** Let the notation be as above. For $s \in \mathbb{Z}$ with $l \leq s \leq -k - 1$ one has following equality

$$\Psi^*_f(\gamma)(\chi) = \gamma(M')(f, f)_C^{-1} (f_0|^{|} V_{C_f}, \tilde{K}_{M'}(s - k + 1)|_{w_{C_0 C_f}}) C_0 C_f.$$

(5.16)

Moreover, for $s \in \mathbb{Z}$ with $l - k + 1 \leq s \leq 0$ we have

$$\tilde{K}_{M'}(s) = \sum_{n=1}^{\infty} \sum_{C_f n_1 + n_2 = M' n} d(n_1, n_2; s, \chi) e(nz) \in S_k(C_0 C_f, \psi)$$

(5.17)

is a cusp form with algebraic Fourier coefficients given by

$$d(n_1, n_2; s, \chi) = 2C_f^{-s} C_g^{-s} (C_f) \chi(n_1)b(n_1) \sum_{n_2 = \dd} \psi \omega \chi_x(d') d^{2s+k-l-1} P_s(n_2, M' n).$$

(5.18)

---

5 The formula differs from [Pan88 (4.27)] by $(-1)^s$ due to the sign error mentioned in the previous footnote.

6 The formula differs from [Pan88 (4.29)] by the same sign as in the previous footnote.
and

\[ P_s(x, y) = \sum_{i=0}^{s} (-1)^i \binom{-s}{i} \Gamma(s + k - l) \Gamma(k - i - 1) \frac{y^{s-i}}{\Gamma(k-1)} x^{-s-i} y^i \]

\[ = x^{-s} \frac{y}{\Gamma(k-1)} Q_s(x, y), \text{ where } s \leq 0 \text{ and } Q_s(x, y) \in \mathbb{Z}[x, y]. \]  

**Proof.** The proof of the lemma is an application of the holomorphic projection lemma (Lemma 2.14). We first note that \( \text{Hol} \) commutes with the action of the \( w_N \)-operator. Indeed, by Lemma 2.14 and (2.2), we have

\[ \langle h, \text{Hol}(\Phi|w_N) \rangle_N = \langle h, \Phi|w_N \rangle_N = \langle h|w_N, \Phi \rangle_N = \langle h, \text{Hol}(\Phi) \rangle_N = \langle h, \text{Hol}(\Phi)|w_N \rangle_N, \]

for all modular rapidly decreasing \( \Phi \) and all cusp forms \( h \) of weight \( k \) and level \( N \geq 1 \), whence \( \text{Hol}(\Phi|w_N) = \text{Hol}(\Phi)|w_N \). A similar argument shows that \( \text{Hol} \) commutes with the \( U_p \)-operator. Thus, by Lemma 2.13 and Lemma 2.14 we have

\[ \langle f_0', V_{C_f}, K^*(s-k+1)|U_{M'}w_{C_{f'}} \rangle_{C_{f'}} = \langle f_0'|V_{C_f}, \text{Hol}(K^*(s-k+1)|U_{M'}w_{C_{f'}}) \rangle_{C_{f'}} \]

\[ = \langle f_0'|V_{C_f}, \text{Hol}(K^*(s-k+1)|U_{M'}w_{C_{f'}}) \rangle_{C_{f'}} \]

Substituting the above expression in (5.12), we obtain (5.16). It follows from (5.14), (5.15) that

\[ K^*(s)|U_{M'} = M'^{k/2-1} \sum_{u \mod \text{M'}} K^*(s)| \frac{1}{u} \begin{pmatrix} M' \\ 0 \end{pmatrix} \]

\[ = M'^{-1} \sum_{u \mod \text{M'}} \sum_{n=1}^{\infty} \sum_{n_1+n_2=n} d(n_1, n_2; y/M', s) e(n(z+u)/M') \]

\[ = \sum_{n=1}^{\infty} \sum_{n_1+n_2=n} d(n_1, n_2; y/M', s) e(nz/M') M'^{-1} \sum_{u \mod \text{M'}} e(un/M') \]

\[ = \sum_{n=1}^{\infty} \sum_{n_1+n_2=M'n} d(n_1, n_2; y/M', s) e(nz). \]

(5.20)

Now we use Lemma 2.14 to compute the Fourier coefficients of \( \tilde{K}_{M'}(s-k+1) = \text{Hol}(K^*(s-k+1)|U_{M'}) \) for \( l \leq s \leq k-1 \). Let \( s' = s-k+1 \) then \( l-k+1 \leq s' \leq 0 \). From (5.20) and Lemma 2.14 it follows that

\[ \tilde{K}_{M'}(s') = \sum_{n=1}^{\infty} \sum_{n_1+n_2=M'n} \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \left( \int_0^{\infty} d(n_1, n_2; y/M', s') e^{-2\pi ny} e^{-2\pi ny k-2} dy \right) e(nz). \]

(5.21)

Note that if \( C_{f_1} + n_2 = M'n \), the quantity \( d(n_1, n_2; y/M', s) \) is as in (5.15), with \( y \) replaced by \( y/M' \), because \( p \mid M' \), since \( p \mid M' \). We get

\[ d(n_1, n_2; s', \chi) := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \int_0^{\infty} d(n_1, n_2; y/M', s') e^{-4\pi ny} y^{k-2} dy, \]

\[ = 2(C_f C_g)^{-s'} \chi(C_g)(n_1) b(n_1) \sum_{n_2=dd'} \psi(n) x^2(d') d^{2s'+k-l-1} \]

\[ \times \left( \frac{4\pi y}{M'} \right)^{s'} W \left( \frac{4\pi n_2 y}{M'}, s' + k - l, s' \right) e^{-4\pi y k-2} dy. \]

(5.22)
Since \( l - k + 1 \leq s' \leq 0 \), we can use (5.10) to compute \( W(\frac{4\pi n_2 y}{M'}, s' + k - l, s') \). We obtain

\[
\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \int_0^\infty \left( \frac{4\pi y}{M'} \right)^{s'} W\left( \frac{4\pi n_2 y}{M'}, s' + k - l, s' \right) e^{-4\pi ny} y^{k-2} dy
\]

\[= \sum_{i=0}^{-s'} (-1)^i \left( \frac{-s'}{i} \right) \frac{\Gamma(s' + k - l)}{\Gamma(s' + k - l - i) \Gamma(k-1)} \int_0^\infty \left( \frac{4\pi n}{M} \right)^{k-1} \left( \frac{4\pi n_2 y}{M'} \right)^{-s'-i} e^{-4\pi ny} y^{k-2} dy
\]

\[= \sum_{i=0}^{-s'} (-1)^i \left( \frac{-s'}{i} \right) \frac{\Gamma(s' + k - l)}{\Gamma(s' + k - l - i) \Gamma(k-1)} n_2^{-s'-i} M^i \int_0^\infty (4\pi ny)^{k-1} (4\pi y)^{-i} e^{-4\pi ny} dy \]

\[= \sum_{i=0}^{-s'} (-1)^i \left( \frac{-s'}{i} \right) \frac{\Gamma(s' + k - l)}{\Gamma(s' + k - l - i) \Gamma(k-1)} n_2^{-s'-i} (M'n)^i \int_0^\infty y^{-k-1} y^{-i} e^{-y} dy \]

\[= \sum_{i=0}^{-s'} (-1)^i \left( \frac{-s'}{i} \right) \frac{\Gamma(s' + k - l)}{\Gamma(s' + k - l - i) \Gamma(k-1)} n_2^{-s'-i} (M'n)^i = P_\gamma(n_2, M'n).
\]

Therefore, for every \( n_1, n_2 \) such that \( C_f n_1 + n_2 = M'n \), (5.22) becomes

\[d(n_1, n_2; s', \chi) = 2(C_f C_g)^{-s'} \tilde{\chi}(C_g) \chi(n_1) b(n_1) \sum_{n_2 \equiv \text{odd}} \psi \omega \chi^2(d') \text{d}^{2s'+k-l-1} P_\gamma(n_2, M'n).
\]

Substituting the above expression in (5.21) finishes the proof. \( \square \)

## 6 Kummer congruences for the distributions

In this section, we show that the distributions in (4.10) for \( s = l + r \), where \( 0 \leq r \leq k - l - 1 \) patch together into a measure, by verifying the abstract Kummer congruences.

By Proposition 5.3 with \( s = l + r \), where \( 0 \leq r \leq k - l - 1 \), we have

\[\Psi_{1+r}^{(M')}(\chi) = \gamma(M')(f, f)_{C_f}^{-1}(f_0^0|V_{C_f}, K_{M'}(r - k + l + 1)|w_{C_0 C_f})_{C_0 C_f} \]

\[= \gamma(M')(f, f)_{C_f}^{-1}(f_0^0|V_{C_f}, w_{C_0 C_f}, K_{M'}(r - k + l + 1))_{C_0 C_f} \]  

(6.1)

By Corollary 4.2 and (5.13), we have \( \Psi_{1+r}^{(M')}(\chi) \) and \( \gamma(M') \) are algebraic numbers. Hence,

\[\gamma(M')(f, f)_{C_f}^{-1}(f_0^0|V_{C_f}, K_{M'}(r - k + l + 1)|w_{C_0 C_f})_{C_0 C_f} \in \overline{\mathbb{Q}}. \]  

(6.2)

Further, note that the cusp form \( \tilde{K}_{M'}(r - k + l + 1) \) has algebraic Fourier coefficients. Let

\[S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}) = \{ h \in S_k(C_0 C_f, \psi) \mid \text{h has algebraic Fourier coefficients} \}.
\]

We now claim that \( f_0^0|V_{C_f} w_{C_0 C_f} \in S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}) \). Clearly \( f_0^0|V_{C_f} w_{C_0 C_f} \) belongs to \( S_k(C_0 C_f, \psi) \). So it is enough to show that the Fourier coefficients of \( f_0^0|V_{C_f} w_{C_0 C_f} \) are algebraic. Observe that

\[f_0^0|V_{C_f} w_{C_0 C_f} = f_0^0|V_{C_f} w_{C_0 C_f} - \alpha'(p) f_0^0|V_p V_{C_f} w_{C_0 C_f} - C_f^{-k/2} f_0^0|w_{C_0} - \alpha'(p)(p C_0)^{-k/2} f_0^0|w_{C_f} C_0 \]  

(2.6)

\[= (p C_f C_0^{-1})^{k/2} f_0^0|w_{C_f} V_p C_0 - \alpha'(p)(p C_f C_0^{-1})^{k/2} f_0^0|w_{C_f} V_{C_f} \]  

(2.7)

\[= (p C_f C_0^{-1})^{k/2} \Lambda(f_0^0) f_0^0|V_p C_0 - \alpha'(p)(p C_f C_0^{-1})^{k/2} \Lambda(f_0^0) f_0^0|V_{C_f}. \]  

(2.8)
Since \( f \) is primitive, it follows that \( f^0|_{V_C} w_{C_h C_f} \) has algebraic Fourier coefficients. Define the linear functional \( \mathcal{L} : S_k(C_0 C_f, \psi) \rightarrow \mathbb{C} \), by

\[
\mathcal{L}(K) = \frac{\langle f^0|_{V_C} w_{C_h C_f}, K \rangle_{C_h C_f}}{\langle f, f \rangle_{C_f}}. \tag{6.3}
\]

We note from (6.1) and (6.3) that, for every finite order character \( \chi : \mathbb{Z}^\times \rightarrow \mathbb{C}^\times \),

\[
\Psi^{(M')}_{l+r}(\chi) = \gamma(M') \mathcal{L}(\tilde{K}_M(r - k + l + 1)). \tag{6.4}
\]

**Lemma 6.1.** Let \( \mathcal{L} \) be defined as above. Then

1. \( \mathcal{L} \) is defined over \( \overline{\mathbb{Q}} \), i.e., \( \mathcal{L}(S_k(C_0 C_f, \psi; \overline{\mathbb{Q}})) \subset \overline{\mathbb{Q}} \).
2. Let \( K(z) = \sum_{n=1}^{\infty} a(n, K) e(nz) \) be an element of \( S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}) \). Then there exists \( m \in \mathbb{N} \) and \( \xi_1, \ldots, \xi_m \in \overline{\mathbb{Q}} \) such that

\[
\mathcal{L}(K) = \sum_{n=1}^{m} \xi_n a(n, K). \tag{6.5}
\]

**Proof.** Choose an orthogonal basis \( f_1, \ldots, f_d \) of \( S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}) \) such that \( f_1 = f^0|_{V_C} w_{C_h C_f} \). By Proposition 5.3, we know that \( \tilde{K}_M(r - k + l + 1) \in S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}) \) for all integers \( 0 \leq r \leq k - l - 1 \). Let \( \tilde{K}_M(r - k + l + 1) = \sum_{i=1}^{d} c_i f_i \), for some \( c_i \in \overline{\mathbb{Q}} \). It follows from (6.2) and orthogonality that

\[
\mathcal{L}(\tilde{K}_M(r - k + l + 1)) = c_1 \mathcal{L}(f^0|_{V_C} w_{C_h C_f}) \in \overline{\mathbb{Q}}.
\]

Choose \( r, \chi \) such that \( \Psi^{(M')}_{l+r}(\chi) = \gamma(M') \mathcal{L}(\tilde{K}_M(r - k + l + 1)) \neq 0 \). Such a choice exists, otherwise all the twisted \( L \)-values of the Rankin product \( L \)-function vanish by (4.8) and Proposition 4.1 so the \( p \)-adic Rankin product \( L \)-function, or more precisely the measure \( \mu \) in Theorem 1.1, can be taken to be identically zero. Hence, \( c_1 \neq 0 \) and \( \mathcal{L}(f^0|_{V_C} w_{C_h C_f}) \in \overline{\mathbb{Q}} \). Therefore \( \mathcal{L}(S_k(C_0 C_f, \psi; \overline{\mathbb{Q}})) = \overline{\mathbb{Q}} \mathcal{L}(f^0|_{V_C} w_{C_h C_f}) = \overline{\mathbb{Q}} \). This finishes the proof of the first part.

Let \( T_k(C_0 C_f, \psi) \) denote the \( \overline{\mathbb{Q}} \)-subalgebra of \( \text{End}_{\mathbb{C}}(M_k(C_0 C_f, \psi)) \) generated by the Hecke operators \( T_n \), for all \( n \in \mathbb{N} \). Clearly \( T_k(C_0 C_f, \psi) \) is a finite dimensional \( \overline{\mathbb{Q}} \)-vector space. By [Miy89], Theorem 4.5.13 and [Miy89] (4.5.27) we obtain that \( \{T_n\}_{n \in \mathbb{N}} \) spans \( T_k(C_0 C_f, \psi) \) as a \( \overline{\mathbb{Q}} \)-vector space. Hence, by finite dimensionality, there exists \( m \) such that \( T_1, \ldots, T_m \) span \( T_k(C_0 C_f, \psi) \) as a \( \overline{\mathbb{Q}} \)-vector space. There is an isomorphism of \( \overline{\mathbb{Q}} \)-vector spaces given by (see [Gha02] Lemma 2)

\[
T_k(C_0 C_f, \psi) \rightarrow \text{Hom}_{\overline{\mathbb{Q}}}(S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}), \overline{\mathbb{Q}})
\]

\[
T \rightarrow a(1, T f).
\]

By the first part of the lemma we know that \( \mathcal{L} \in \text{Hom}_{\overline{\mathbb{Q}}}(S_k(C_0 C_f, \psi; \overline{\mathbb{Q}}), \overline{\mathbb{Q}}) \). Therefore, \( \mathcal{L}(K) = a(1, T f) \), for some \( T \in T_k(C_0 C_f, \psi) \). Since, \( T_1, \ldots, T_m \) span \( T_k(C_0 C_f, \psi) \) as \( \overline{\mathbb{Q}} \)-vector space, there exists \( \xi_1, \ldots, \xi_m \in \overline{\mathbb{Q}} \) such that

\[
T = \sum_{n=1}^{m} \xi_n T_n. \quad \text{So,} \quad \mathcal{L}(K) = \sum_{n=1}^{m} \xi_n a(n, K, \psi) \in \overline{\mathbb{Q}}.
\]

As mentioned earlier, every complex-valued Dirichlet character \( \chi \) on \( \mathbb{Z}_p^\times \) takes values in \( \overline{\mathbb{Q}} \subset \mathbb{C} \). From now on we think of such character as taking values in \( \mathbb{C}_p \) via our fixed embedding \( i_p : \overline{\mathbb{Q}} \rightarrow \mathbb{C}_p \). Since \( \Psi_{l+r}(\chi) \in \overline{\mathbb{Q}} \), for \( 0 \leq r \leq k - l - 1 \), by Corollary 4.2 we have \( i_p(\Psi_{l+r}(\chi)) \in \mathbb{C}_p \). Thus we may think of the complex distribution \( \Psi_{l+r} \), as a \( \mathbb{C}_p \)-valued distribution. We shall denote these distributions by \( i_p(\Psi_{l+r}) \), for \( 0 \leq r \leq k - l - 1 \). We now define a candidate for the measure in Theorem 1.1 namely we take

\[
\mu := i_p(\Psi_{l}). \tag{6.6}
\]
By Proposition [4.1] and [4.2], we have

\[ \Psi_{t+r}(\chi) = \gamma(M')L(\overline{K}_M'(r - k + l + 1)) \]

\[ = \gamma(M')\sum_{n=1}^{m} \xi_n a(n, \overline{K}_M'(r - k + l + 1)) \text{ (by Lemma 6.1 (2))} \]

\[ = \gamma(M')\sum_{n=1}^{m} \xi_n \sum_{C/m_1+n_2=M'} d(n_1, n_2; r - k + l + 1, \chi), \quad (6.7) \]

by Proposition [4.3] where \( M' \) is a sufficiently large power of \( p \) chosen depending on \( \chi \), and \( \gamma(M') \) is as defined in [5.13]. As remarked earlier, \( \gamma(M') \) is \( p \)-integral in many cases (apparently in all, but in any case has bounded denominator, coming from \( \Lambda(g) \), since \( \alpha(pM') \) is a \( p \)-adic unit. Similarly, the \( \xi_n \in \overline{Q} \) have bounded denominators. Finally the \( d(n_1, n_2; s, \chi) \) also have denominators at worst \( \Gamma(k - 1) \) by [5.18], [5.19]. Hence multiplying \( i_p(\Psi_{t+r}) \) by a suitable \( \text{(fixed) power of } p \) we may and do assume that \( i_p(\Psi_{t+r}(\chi)) \) lies in \( \mathcal{O}_p \) for all \( \chi \). Proving that this rescaled distribution is an \( \mathcal{O}_p \)-valued measure will imply that \( i_p(\Psi_{t+r}) \) is a (not necessarily \( \mathcal{O}_p \)-valued) measure.

**Proposition 6.2.** For all integers \( 0 \leq r \leq k - l - 1 \), we have

1. The \( \mathbb{C}_p \)-valued distributions \( i_p(\Psi_{t+r}) \) are bounded. Hence, \( i_p(\Psi_{t+r}) \) are measures on \( \mathbb{Z}_p^\times \).

2. Moreover, with \( \mu \) as in [6.6], the following equality holds \( ^\dagger \)

\[ \int_{\mathbb{Z}_p^\times} \chi x_p^r \, d\mu = (-1)^r \int_{\mathbb{Z}_p^\times} \chi d_i(\Psi_{t+r}). \quad (6.8) \]

**Proof.** Fix an integer \( 0 \leq r \leq k - l - 1 \). Recall that the linear span of \( \mathcal{B} = \{ \chi \mid \chi : \mathbb{Z}_p^\times \to \mathbb{C}_p \text{ has finite order} \} \) is dense in \( \mathcal{C}(\mathbb{Z}_p^\times, \mathbb{C}_p) \). We claim that the distribution \( i_p(\Psi_{t+r}) \) satisfies the abstract Kummer congruences [5.5] with \( \mathcal{B} \) as the system of functions. We need to prove that for every finite set of characters \( \chi_1, \ldots, \chi_t \in \mathcal{B} \), constants \( c_1, \ldots, c_t \in \mathbb{C}_p \) and \( m \geq 0 \),

\[ \text{if } \sum_{i=1}^{t} c_i \chi_i \equiv 0 \mod{p^m}, \text{ then } \sum_{i=1}^{t} c_i i_p(\Psi_{t+r}(\chi_i)) \equiv 0 \mod{p^m}. \]

Choose \( M' \) sufficiently large so that (6.7) holds for each of the \( \chi_i \). By (6.7), this is equivalent to proving

\[ \text{if } \sum_{i=1}^{t} c_i \chi_i \equiv 0 \mod{p^m}, \text{ then } \sum_{i=1}^{t} c_i d(n_1, n_2; r - k + l + 1, \chi) \equiv 0 \mod{p^m}. \quad (6.9) \]

for each \( n \), and each \( n_1, n_2 \) satisfying \( C_f n_1 + n_2 = M' n \).

If \( p \mid n_1 \), then \( d(n_1, n_2; r - k + l + 1, \chi_i) \equiv 0 \), by [5.18], since \( \chi_i(n_1) = 0 \). So, the relation (6.9) is trivially true. Hereafter, we assume \( p \nmid n_1 \). Since \( p \nmid C_f \), \( M' \) and \( C_f n_1 + n_2 = M' n \), we have \( p \nmid n_1 \) if and only if \( p \nmid n_2 \). So we have \( p \nmid n_2 \) and \( d/d' \) is a \( p \)-adic unit, for \( dd' = n_2 \). Let \( s = r - k + l + 1 \). We may also assume \( M' \) has been chosen large enough so that \( p^m \mid M'/\Gamma(k - 1) \).

From [5.19] and the equality \( C_f n_1 + n_2 = M' n \), it follows that

\[ P_s(n_2, M') \equiv n_2^{k-l-1-r} \equiv (dd')^{k-l-1-r} \mod{p^m}, \]

\[ \chi(n_1) = \overline{\chi}(-C_f)\chi(n_2) = \overline{\chi}(-C_f)\chi(dd'). \quad (6.10) \]

\(^\dagger\)The formula (6.8) differs from [Pan88] (5.6) by the factor \((-1)^r\). This factor is forced on us in view of the sign corrections mentioned in the previous footnotes. Moreover, this sign has theoretical significance: (6.8) matches with a general expectation about measures attached to \( L \)-functions of motives [CP80] (4.16)].
By (6.12), we have the congruence
\[ d(n_1, n_2; r - k + l + 1, \chi_i) \equiv 2\chi(-C)C^{k-l-r-1}b(n_1) \sum_{n_2 = dd'} \psi(d')(d')^{k-l-r-1}d' \quad (\text{mod } p^m). \]

Therefore,
\[ (-1)^{r} \sum_{i=1}^{t} c_i d(n_1, n_2; r - k + l + 1, \chi_i) \equiv 2b(n_1) \sum_{n_2 = dd'} \psi(d')(d')^{k-l-1} \sum_{i=1}^{t} c_i \chi_i \left( -\frac{d}{d'C} \right) \left( -\frac{d}{d'C} \right)^r \quad (\text{mod } p^m). \]

By assumption, \( \sum_{i} c_i \chi_i \equiv 0 \pmod{p^m} \), so \( \sum_{i} c_i \chi_i(-d/d'C) \equiv 0 \pmod{p^m} \). Since each \(-d/d'C\) is a \( p \)-adic unit, we obtain \( \sum_{i=1}^{t} c_i d(n_1, n_2; r - k + l + 1, \chi_i) \equiv 0 \pmod{p^m} \). Thus (6.9) holds and this finishes the proof of (1).

For (2), we claim there exists a \( \mathbb{C}_p \)-valued measure \( \nu \) such that
\[ \int_{\mathbb{Z}_p^\times} \chi x_r^d \, d\nu = (-1)^r \int_{\mathbb{Z}_p^\times} \chi \, d\mu(\Psi_{l+r}), \quad \forall \ 0 \leq r \leq k - l - 1. \]

Let \( \mathcal{B}' = \{ \chi x_r^d \mid \chi \in \mathcal{B} \text{ and } 0 \leq r \leq k - l - 1 \} \). To prove the existence of this measure, it is enough to verify the abstract Kummer congruences hold for \( \mathcal{B}' \) as the system of functions. As in (1), we need to prove for every finite set of characters \( \chi_i \in \mathcal{B}' \) and \( c_{i,r} \in \mathbb{C}_p \),

if \( \sum_{i,r} c_{i,r} \chi_i x_r^d = 0 \pmod{p^m} \), then \( \sum_{i,r} (-1)^r c_{i,r} d(n_1, n_2; r - k + l + 1, \chi_i) = 0 \pmod{p^m} \). (6.12)

As observed above, if \( p \nmid n_1 \), then \( d(n_1, n_2; r - k + l + 1, \chi_i) = 0 \), so (6.12) holds. For \( p \nmid n_1 \), it follows from (6.11) that
\[ \sum_{i,r} (-1)^r c_{i,r} d(n_1, n_2; r - k + l + 1, \chi_i) \equiv \sum_{n_2 = dd'} 2b(n_1) \psi(d')(d')^{k-l-1} \]
\[ \times \left( \sum_{i,r} c_{i,r} \chi_i \left( -\frac{d}{d'C} \right) \left( -\frac{d}{d'C} \right)^r \right) \quad (\text{mod } p^m). \]

By the assumption in (6.12), the inner sum is congruent to 0 (mod \( p^m \)). Thus
\[ \sum_{i,r} (-1)^r c_{i,r} d(n_1, n_2; r - k + l + 1, \chi_i) \equiv 0 \pmod{p^m}, \]
so again (6.12) holds. This proves that \( \nu \) as claimed above exists. Further \( \mu \) and \( \nu \) agree on \( \mathcal{B} \) (take \( r = 0 \)) which spans Step(\( \mathbb{Z}_p^\times, \mathbb{C}_p \)). Hence, \( \mu = \nu \). This completes the proof of (2). \hspace{1cm} \Box

Let \( \mu \) be the distribution in (6.6). By Proposition 6.2 (1) with \( r = 0 \), we see that \( \mu \) is a measure. By (6.8) and (4.8) with \( s = l+r \), we see that \( \mu \) satisfies the interpolation property \(^8\) of Theorem 1.1. This completes the proof of Theorem 1.1.

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\(^8\) The sign \((-1)^r\) in (6.8) directly contributes to the corrected sign \((-1)^r\) in Theorem 1.1.
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