Supersymmetric Many-particle Quantum Systems with Inverse-square Interactions

Pijush K. Ghosh

Department of Physics, Siksha-Bhavana, Visva-Bharati University, Santiniketan, PIN 731 235, India.

Abstract

The development in the study of supersymmetric many-particle quantum systems with inverse-square interactions is reviewed. The main emphasis is on quantum systems with dynamical $OSp(2|2)$ supersymmetry. Several results related to exactly solved supersymmetric rational Calogero model, including shape invariance, equivalence to a system of free super-oscillators and non-uniqueness in the construction of the Hamiltonian, are presented in some detail. This review also includes a formulation of pseudo-hermitian supersymmetric quantum systems with a special emphasis on rational Calogero model. There are quite a few number of many-particle quantum systems with inverse-square interactions which are not exactly solved for a complete set of states in spite of the construction of infinitely many exact eigen functions and eigenvalues. The Calogero-Marchioro model with dynamical $SU(1,1|2)$ supersymmetry and a quantum system related to short-range Dyson model belong to this class and certain aspects of these models are reviewed. Several other related and important developments are briefly summarized.

Contents

1 Introduction 3

2 Supersymmetry: General Formalism 7

1.1 Clifford Algebra 8

2.2 Realization of $\mathcal{N} = 2$ Super-algebra 10

*email: pijushkanti.ghosh@visva-bharati.ac.in
3 Exactly Solved Systems in One Dimension
  3.1 Rational $A_{N+1}$ Calogero Model ......................... 14
  3.2 Shape Invariance and Exact Solvability .................... 16

4 $OSp(2|2)$ Supersymmetric Systems .......................... 18
  4.1 Non-uniqueness of the Construction ....................... 20
  4.2 Mapping to Free Super-oscillators ....................... 22
    4.2.1 Supersymmetry-preserving Phase ..................... 23
    4.2.2 Supersymmetry-breaking Phase ....................... 25
    4.2.3 Generalization ....................................... 26

5 Pseudo-hermitian Supersymmetric Systems ................... 27
  5.1 Isospectral Deformation .................................. 29
    5.1.1 Deformation involving bosonic coordinates .......... 29
    5.1.2 Pseudo-hermitian Realization of Clifford Algebra ... 31
  5.2 Many-particle non-Dirac-hermitian Supersymmetry ........ 33
  5.3 Examples: Rational Calogero Models ...................... 35

6 Quantum Systems in Higher Dimensions ...................... 36
  6.1 Supersymmetric Calogero-Marchioro Model ................. 37
  6.2 Extended Superconformal Symmetry ....................... 39

7 Systems with Internal Degrees of Freedom .................. 40
  7.1 Supersymmetry & XY model ................................ 42
  7.2 Systems related to short-range Dyson models ............ 42

8 Omitted topics, Open Arena and Summary .................... 45
  8.1 Omitted Topics ........................................... 45
  8.2 Open Arena ............................................... 46
  8.3 Summary .................................................. 48

9 Acknowledgment .............................................. 50

10 Appendices .................................................. 50
  10.1 Appendix-I: Matrix representation of the real Clifford algebra 50
  10.2 Appendix-II: $OSp(2|2)$ superalgebra .................... 51
  10.3 Appendix-III: $SU(1,1|2)$ superalgebra .................. 52
  10.4 Appendix-IV: Rational $BC_{N+1}$ Calogero Model ........ 53
    10.4.1 Shape Invariance & Exact Solvability .............. 54
    10.4.2 Mapping to free super-oscillators ................. 56
1 Introduction

Supersymmetry plays an important role in understanding unification of different fundamental forces in nature. In the earlier days of the development of the subject, supersymmetric quantum mechanics was introduced to study different aspects of supersymmetric quantum field theory within a much simpler and tractable framework\textsuperscript{[1, 2]}. However, the importance of developing supersymmetric quantum mechanics as an independent subject was soon apparent to physicists as well as mathematicians working in this area. Several aspects of supersymmetric quantum mechanics have been investigated over the last three decades and many of these important developments are summarized in a few review articles and books, for example, in Refs. \textsuperscript{[3, 4, 5]}. A study on supersymmetric quantum systems with many degrees of freedom was initiated in Ref. \textsuperscript{[6]}, just within two years of the development of the subject. However, unlike systems with one bosonic and one fermionic degrees of freedom, not many exactly solved supersymmetric many-particle quantum systems are known. This is because of the following reasons. Supersymmetric systems with one bosonic and one fermionic degrees of freedom are described in terms of two Hamiltonians, known as supersymmetric partner Hamiltonians. The eigen functions and eigenvalues of one of these Hamiltonians can be obtained from the other one and the vice versa, by using the intertwining relations arising from the underlying superalgebra\textsuperscript{[3]}. The complete eigen spectra can be obtained algebraically provided the partner Hamiltonians satisfy the condition of shape invariance and examples of such quantum systems are abundant in the literature\textsuperscript{[3, 4, 5]}. In the case of systems with $N$ bosonic and $N$ fermionic degrees of freedom, the supersymmetric Hamiltonian can be expressed in the fermionic basis as a $2^N \times 2^N$ block-diagonal matrix with $N + 1$ components corresponding to $N + 1$ fermionic sectors on the diagonal\textsuperscript{[7]}. Each sector is characterized by the total fermion number. The zero-fermion and the $N$-fermion sectors are described by two different Hamiltonians, while matrix-operators of different dimensions appear in rest of the $N - 1$ sectors\textsuperscript{[7]}. The underlying super algebra allows an one-to-one correspondence between the eigenfunctions and the eigenvalues of
these two Hamiltonians for only systems with one bosonic and one fermionic degrees of freedom, i.e. \( N = 1 \). It is true that part of the spectrum of a block-operator corresponding to a fixed fermionic sector coincide with the spectra of neighbouring blocks for systems with \( N > 1 \). However, a successful implementation of the condition of shape invariance, which is essential for showing exact solvability, meets with difficulty for arbitrary \( N \). In general, the diagonalization of the block-diagonal matrix operator is also a highly non-trivial task. These technical limitations appear as the main stumbling block for finding physically relevant exactly solved many-particle supersymmetric quantum systems. Nevertheless, a class of exactly solved many-particle supersymmetric quantum systems exists for which the many-body interaction varies inverse-squarely.

Inverse-square interactions appear naturally in physical systems in two or higher space dimensions as the centrifugal barrier. There are several low dimensional condensed matter systems governed by inverse-square interactions[8]. Systems with purely inverse-square interaction are also example of conformal mechanics[9,10] which are relevant in diverse branches of physics. The Calogero model[11] is an exactly solved system of \( N \) particles interacting on a line through pair-wise inverse-square and harmonic interactions. The model is exactly solved even if the pair-wise harmonic interaction is replaced by a common confining harmonic potential or a periodic version of the inverse-square interaction without the confining harmonic term is considered[12]. Several other generalizations of the original Calogero model have been considered over a period of more than four decades and this class of exactly solved many-particle systems is generically known as Calogero-Moser-Sutherland systems. There are excellent reviews on the topic, for example, in Refs. [13,14,15,16,17]. The study of Calogero-Moser-Sutherland systems have produced many interesting results which are relevant in the context of a diverse branches of physics, including exclusion statistics[18,19], quantum Hall effect[20], Tomonaga-Luttinger liquid[21], quantum chaos[22], electric transport in mesoscopic systems[23], novel correlations[24], spin-chains[25,26,27,28,29,30,31,32,33] etc. These developments are also important in the context of mathematical physics, for example, algebraic and integrable structure[34,35,36], mapping of rational model to Calogero model with Coulomb-like potential[37], self-adjoint extensions[38,39], equivalence to a system of free oscillators[40,41], collective field formulation of many-particle systems[42] etc.

A supersymmetric version of the rational Calogero Model was introduced by Freedman and Mende[43] in 1990. Various aspects of this class of supermodels have been studied over the last two decades, enriching a general understanding of the integrable structure of many-particle supersymmetric quantum
Apart from being examples of exactly solved quantum systems, the importance of these models is due to their relevance in the study of black-holes, Seiberg-Witten theory, matrix models and string theory, collective field theory, many-particle superconformal quantum mechanics, superpolynomials, pseudo-hermitian supersymmetric Calogero models, etc.

The main emphasis of this topical review article is on systems with dynamical supersymmetry. The rational $A_{N+1}$ and $BC_{N+1}$ Calogero models belong to this class and an algebraic construction of spectra of these two models is discussed. The eigenvalue problem of the Hamiltonians appearing in the zero and $N$ fermion sectors of the supersymmetric Hamiltonian is solved by using the Dunkl operators and the idea of shape invariance. The supersymmetric Hamiltonian can also be mapped to a system of free superoscillators through a non-unitary similarity transformation, facilitating a construction of the complete set of states from the free superoscillator basis. However, only those free superoscillator states are acceptable which are invariant under the discrete symmetries of the original many-body Hamiltonian. The discrete symmetries of the rational $A_{N+1}$ and $BC_{N+1}$-type Calogero models are different. Thus, although both of these Hamiltonians can be mapped to the same free superoscillator Hamiltonian, the eigen spectra are not identical.

The mathematical aspects of dynamical supersymmetry is described in Refs. It is known that $OSp(2|2)$ admits ‘typical’ as well as ‘atypical’ representations. The quadratic and the cubic Casimir operators are necessarily zero in the ‘atypical’ representation. The supersymmetric Calogero model introduced by Freedman and Mende corresponds to ‘typical’ representation of $OSp(2|2)$ group. A supersymmetric version of the rational Calogero model corresponding to the ‘atypical’ representation of the $OSp(2|2)$ group is presented in this article.

The study of $\mathcal{PT}$ symmetric non-hermitian quantum system has received considerable attention in the literature over the last decade. Several pseudo-hermitian quantum systems with an exact description of the norm in the Hilbert space have been considered. The rational Calogero model and its variants have also been considered in the literature within the same context. A general construction of pseudo-hermitian supersymmetric quantum systems and rational Calogero model is presented.

There are many higher dimensional generalizations of the rational Calogero model. Although infinitely many exact eigenstates can be ob-
tained analytically for these models, not a single Hamiltonian belonging to this class is known to be exactly solved for a complete set of states. The Calogero-Marchioro model [100] is one such example of a ‘partially solved’ system, which has interesting connections with complex random matrix theory [103, 104, 105], two dimensional Bose systems [104], quantum Hall effect [106], quantum dot [107], collective field theory [108]. The construction of supersymmetric Calogero-Marchioro model in arbitrary $D$ space dimensions with $OSp(2|2)$ symmetry is presented. It is also shown that the same model in two space dimensions has an extended $SU(1,1|2)$ superconformal symmetry for arbitrary number of particles and the generic value of the coupling constant [64].

There are generalizations of rational Calogero model where each particle interacts only with the nearest-neighbour and the next-nearest-neighbour particles through an inverse-square interaction [109, 110]. Infinitely many exact eigenstates can be obtained analytically for this system for open or periodic boundary condition. However, these states do not form a complete set of states. This model has a close connection with random banded matrix theory describing short-range Dyson model [111] and spin chains [112, 113, 114, 115]. A supersymmetric version of this model with dynamical $OSp(2|2)$ symmetry is presented in this article [112]. The supersymmetric Hamiltonian can also be equivalently described as an interacting system of $N$ particles with spin degrees of freedom, where each particle interacts with only its nearest-neighbour and the next-nearest-neighbour through inverse-square interaction. Such an interpretation is admissible by expressing fermionic operators in terms of Pauli matrices via Jordan-Wigner transformation. In an appropriate limit, the spin degrees of freedom can be completely decoupled from the bosonic degrees of freedom and models of nearest-neighbour $XY$ model in an external magnetic field on a non-uniform lattice can be obtained.

The plan of presenting the results in this review are the following. The basic formalism for discussing many-particle supersymmetric quantum systems is discussed in the next section. The supersymmetric rational $A_{N+1}$ Calogero model is studied in section 3. The eigen spectra of the Hamiltonians corresponding to the zero and the $N$ fermion sectors are obtained with the help of Dunkl operators and shape invariance. The section 4 is an exploration of systems with $OSp(2|2)$ symmetry. The non-uniqueness in constructing model Hamiltonians with $OSp(2|2)$ symmetry is pointed out. It is also shown that the bosonic $O(2,1) \times U(1)$ sub-algebra of $OSp(2|2)$ can be exploited to show an equivalence between the many-particle Hamiltonian and a system of free superoscillator. In section 5, a general construction of pseudo-hermitian su-
persymmetric quantum system with rational Calogero model as an example is
given. The supersymmetric version of the Calogero-Marchioro models is studied
in section 6, while models related to short-range Dyson model is presented in
section 7. Finally, topics not included in this review are briefly summarized
in section 8. This section also includes discussions on open problems and a
summary of the results obtained. Several appendices containing mainly mathemati-
cal prerequisites (except section 9.4) are included in section 9. The results
for supersymmetric rational $BC_{N+1}$ Calogero model is included in section 9.4.

A few points regarding the unit and convention used in this article. The
velocity of light, the Planck’s constant and mass of identical particles are taken
to be unity. The angular frequency of the common harmonic confining term is
denoted as $\omega$, which is assumed to be equal to unity, unless mentioned otherwise
explicitly. All the supersymmetric Hamiltonians considered in this review article
correspond to ‘typical’ representation of $OSp(2|2)$, except for Hamiltonians in
section 4.1. The supersymmetric phase is characterized by conditions on the
parameters arising from the criteria that single particle momentum operators
are self-adjoint for the normalizable zero-energy groundstate wavefunction.

2 Supersymmetry: General Formalism

The super-algebra that is relevant in the description of a supersymmetric quan-
tum system has the general form[1],

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H, \quad [H, Q_\alpha] = 0, \alpha, \beta = 1, 2, \ldots N, \quad (1)$$

where $Q_\alpha$ are $N$ real supercharges and $H$ is the supersymmetric Hamiltonian.
The super-algebra in Eq. (1) can be shown to be a sub-algebra of the relativistic
two dimensional $N$ extended super-algebra[6]. Higher symmetry may
persists for specific quantum systems. The main focus in this review article is on
$OSp(2|2)$ supersymmetry with $N = 2$ and on $SU(1,1|2)$ supersymmetry with
$N = 4$. The relevant superalgebra are given in appendix-II and Appendix-III,
respectively.

The supersymmetric Hamiltonian $H$ depends on the bosonic as well as
fermionic co-ordinates. The positions and the momenta operators correspond to
the bosonic operators, while fermionic co-ordinates generally describe internal
degrees of freedom which may be identified as spin for some specific cases. A
minimal realization of the superalgebra [1] for $N = 2$ describes a single parti-
cle suspersymmetric quantum system, where the fermionic degrees of freedom
are realized in terms of the Pauli matrices. Elements of the Clifford algebra
are used to realize the superalgebra for a many-particle quantum system. A
general discussion in this regard on Clifford algebra and realization of $\mathcal{N} = 2$
super-algebra are discussed in the next two sections.

2.1 Clifford Algebra

The real Clifford algebra of $2N$ entities $\xi_p$ is described by the relations[116],

$$\{\xi_p, \xi_q\} = 2\delta_{pq}, \ p, q = 1, 2, \ldots 2N. \quad (2)$$

An idempotent operator $\xi_{2N+1}$ may be introduced in terms of these elements
as,

$$\xi_{2N+1} = (-i)^N \xi_1 \xi_2 \ldots \xi_{2N-1} \xi_{2N}, \quad (3)$$

which anti-commutes with all the $\xi_p$’s. The operator $\xi_{2N+1}$ facilitates the con-
struction of the projection operators $\xi^{\pm}_{2N+1}$,

$$\xi^{\pm}_{2N+1} = \frac{1}{2}(1 \pm \xi_{2N+1}), \quad (\xi^{\pm}_{2N+1})^2 = \xi^{\pm}_{2N+1}, \quad \xi^{\pm}_{2N+1} \xi^{\mp}_{2N+1} = 0. \quad (4)$$

A particular realization of the generators of the $O(2N)$ group is in terms of the
elements of the Clifford algebra,

$$J_{pq} = \frac{i}{4} [\xi_p, \xi_q]. \quad (5)$$

The generators of the group $O(2N + 1)$ are realized by including $\xi_{2N+1}$ and
allowing $p, q = 1, 2, \ldots 2N + 1$ in Eq. (5). A matrix representation of the
elements $\xi_p$ is given in Appendix-I.

A set of fermionic variables $\psi_i$ and their conjugates $\psi^\dagger_i$ may be introduced
in terms of $\xi_i$’s as,

$$\psi_i = \frac{1}{2} (\xi_i - i\xi_{N+i}), \quad \psi^\dagger_i = \frac{1}{2} (\xi_i + i\xi_{N+i}), \quad i, j = 1, 2, \ldots N, \quad (6)$$

which satisfy the complex Clifford algebra,

$$\{\psi_i, \psi_j\} = \{\psi^\dagger_i, \psi^\dagger_j\} = 0, \quad \{\psi_i, \psi^\dagger_j\} = \delta_{ij}. \quad (7)$$

The fermionic permutation operator is defined[44] as,

$$K_{ij} := \frac{1}{2} \left[\psi^\dagger_i - \psi^\dagger_j, \psi_i - \psi_j\right] = 1 - (\psi_i - \psi_j) \left(\psi^\dagger_i - \psi^\dagger_j\right), \quad (8)$$

which acts on fermionic operators and satisfies the following properties:

$$K_{ij} \psi_{ij}^\dagger = \psi_{ij}^\dagger K_{ij}, \quad K_{ij} \psi_{ik}^\dagger = \psi_{ik}^\dagger K_{ij} \text{ for } k \neq i, j, \quad K^2_{ij} = 1. \quad (9)$$
The first two relations in Eq. (9) may be re-written in a compact form as [51],

\[
K_{ij}\psi_k^{\dagger} = \sum_l T_{(ij)lk}\psi_l^{\dagger}K_{ij}
\]

\[
T_{(ij)lk} = \delta_{lk} - \delta_{li}\delta_{ki} - \delta_{lj}\delta_{kj} + \delta_{li}\delta_{kj}.
\]

(10)

The action of \(K_{ij}\) on any bosonic operator or variable leaves it unchanged.

The fermionic vacuum \(|0\rangle\) and its conjugate \(|\bar{0}\rangle\) in the \(2^N\) dimensional fermionic Fock space are defined as, \(\psi_i|0\rangle = 0, \psi_i^{\dagger}|\bar{0}\rangle = 0 \forall i\). The fermion number operator \(n_i = \psi_i^{\dagger}\psi_i\) corresponding to the \(i^{th}\) fermion has the eigenvalue 0 or 1. The total fermion number operator is denoted as, \(N_f = \sum_i n_i\), with \(N_f = 0\) and \(N_f = N\) corresponding to the fermionic and the conjugate vacuum, respectively.

The action of \(\psi_i, \psi_i^{\dagger}\) on an arbitrary eigenstate \(|n_1, \ldots, n_i, \ldots, n_N\rangle\) of \(N_f\) is the following:

\[
\psi_i|n_1, \ldots, n_i, \ldots, n_N\rangle = 0, \text{ if } n_i = 0
\]

\[
= |n_1, \ldots, 0, \ldots, n_N\rangle, \text{ if } n_i = 1
\]

\[
\psi_i^{\dagger}|n_1, \ldots, n_i, \ldots, n_N\rangle = 0, \text{ if } n_i = 1
\]

\[
= |n_1, \ldots, 1, \ldots, n_N\rangle, \text{ if } n_i = 0.
\]

(11)

It may be noted that the eigenstate \(|n_1, \ldots, n_i, \ldots, n_N\rangle\) of \(N_f\) can be constructed in the fermionic Fock space by taking linear superposition of \(N_C_{N_f}\) number of base states:

\[
|i_1, i_2, \ldots, i_{N_f}\rangle := \psi_{i_1}^{\dagger}\psi_{i_2}^{\dagger}\ldots\psi_{i_{N_f}}^{\dagger}|0\rangle, \text{ } i_1 < i_2 < \ldots i_{N_f}.
\]

(12)

The action of the permutation operator on the base states is the following [51],

\[
K_{ij}|i_1, \ldots, i, \ldots, j, \ldots, i_{N_f}\rangle = |i_1, \ldots, j, \ldots, i, \ldots, i_{N_f}\rangle
\]

\[
K_{ij}|i_1, \ldots, i, \ldots, i_{N_f}\rangle = |i_1, \ldots, j, \ldots, i_{N_f}\rangle \text{ } j \neq i_1, i_2, \ldots i_{N_f}
\]

\[
K_{ij}|i_1, \ldots, i_{N_f}\rangle = |i_1, \ldots, i_{N_f}\rangle \text{ } \text{ } i, j \neq i_1, i_2, \ldots i_{N_f}.
\]

(13)

The permutation operator \(K_{ij}\) leaves the vacuum of fermionic Fock space invariant. The operator \(K_{ij}\) realizes [51] a tensor representation of rank \(N_f\) of the symmetric group \(S_N\) of permutations of fermionic operators \(\psi_i^{\dagger}\) on the base states [12] with fixed fermion number \(N_f\).

An equivalent expression of \(\xi_{2N+1}\) in terms of \(n_i\) can be written as,

\[
\xi_{2N+1} = (-1)^N \prod_{i=1}^{N} (2n_i - 1).
\]

(14)
The action of $\xi_{2N+1}$ on a state $|N_f\rangle$ with fermion number $N_f$ is,

$$
\xi_{2N+1}|N_f\rangle = (-1)^N_f|N_f\rangle, \quad 0 \leq N_f \leq N.
$$

(15)

Note that $\xi_{2N+1}$ leaves the fermionic vacuum invariant, i.e., $\xi_{2N+1}|0\rangle = |0\rangle$. On the other hand, $\xi_{2N+1}^*|\bar{0}\rangle = (-1)^N|\bar{0}\rangle$, implying that the conjugate vacuum is invariant for even $N$ and changes sign for odd $N$.

2.2 Realization of $\mathcal{N} = 2$ Super-algebra

The $\mathcal{N} = 2$ superalgebra is realized in terms of the real supercharges $Q_1$ and $Q_2$ as follows,

$$
Q_1 = \frac{1}{\sqrt{2}} \sum_{i=1}^{N} \left[ \xi_i p_i + \xi_{N+i} W_i \right],
$$

$$
Q_2 = -\frac{1}{\sqrt{2}} \sum_{i=1}^{N} \left[ \xi_{N+i} p_i - \xi_i W_i \right], \quad W_i \equiv \frac{\partial W}{\partial x_i}.
$$

(16)

where $x_i, p_i = -i \frac{\partial}{\partial x_i}$ are the position and the momentum operators, respectively. The function $W(x_1, x_2, \ldots, x_N)$ is identified as the superpotential. The Hamiltonian for the above choices of the supercharges is expressed as,

$$
H = \frac{1}{2} \sum_{i=1}^{N} (p_i^2 + W_i^2) - \frac{i}{2} \sum_{i,j=1}^{N} \xi_i \xi_{N+j} W_{ij}, \quad W_{ij} \equiv \frac{\partial^2 W}{\partial x_i \partial x_j}.
$$

(17)

The matrix representation of $\xi_p$’s are taken to be hermitian. Consequently, the Hamiltonian is hermitian for any real superpotential. It may be noted that $W_{ij}$ is symmetric with respect to its indexes, i.e. $W_{ij} = W_{ji}$. Further, the identity $\sum_{i=1}^{N} W_i = 0$ holds for any transnational invariant superpotential.

It is sometimes convenient to study supersymmetric quantum mechanics in terms of complex supercharges,

$$
Q := \frac{1}{\sqrt{2}} (Q_1 - iQ_2) = \sum_{i=1}^{N} \psi_i A_i, \quad A_i := p_i - iW_i,
$$

$$
Q^\dagger := \frac{1}{\sqrt{2}} (Q_1 + iQ_2) = \sum_{i=1}^{N} \psi_i A_i^\dagger, \quad A_i^\dagger := p_i + iW_i.
$$

(18)

The operators $A_i$ and $A_i^\dagger$ satisfy the following relations:

$$
[A_i, A_j] = 0 = \left[ A_i^\dagger, A_j^\dagger \right], \quad \left[ A_i, A_j^\dagger \right] = \left[ A_j, A_i^\dagger \right] = 2W_{ij}.
$$

(19)
The superalgebra (1) is re-written in terms of $Q$ and its adjoint $Q^\dagger$ as,

$$H = \frac{1}{2} \{Q, Q^\dagger\}, \quad Q^2 = 0 = (Q^\dagger)^2, \quad [H, Q] = 0 = [H, Q^\dagger].$$  \hspace{1cm} (20)

The supersymmetric Hamiltonian $H$ in Eq. (17) has the following expression in terms of the fermionic operators:

$$H = \frac{1}{4} \sum_{i=1}^{N} \{A_i, A_i^\dagger\} + \frac{1}{4} \sum_{i,j=1}^{N} \left[ A_i, A_j^\dagger \right] \left[ \psi_i^\dagger, \psi_j \right]$$

$$= \frac{1}{2} \sum_{i=1}^{N} \left( p_i^2 + W_i^2 - W_{ii} \right) + \sum_{i,j=1}^{N} W_{ij} \psi_i^\dagger \psi_j.$$

The supersymmetry-preserving phase is characterized by the existence of state(s) with ground state energy $E^s_0 = 0$, while that of supersymmetry-breaking phase as $E^s_0 > 0$. The wave-function should be well-behaved in both the cases. The first equation of (20) implies that the ground state of $H$ with $E^s_0 = 0$ is annihilated by both $Q$ and $Q^\dagger$. The ground states $\Phi_0, \Phi_N$ are determined from the defining relations of the supercharge in Eq. (18),

$$\Phi_0 = e^{-W}|0>, \quad \Phi_N = e^W|\bar{0} >.$$

Both $\phi_0$ and $\phi_N$ will not be normalizable simultaneously for the type of super-potential that will be considered in this article. The supersymmetric phase is characterized by normalizable $\Phi_0$ or/and $\Phi_N$ for which each $p_i$ is self-adjoint.

The total fermion number operator $N_f$ commutes with the Hamiltonian and simultaneous eigenstates of $H$ and $N_f$ can be constructed. The $2^N$ dimensional fermionic Fock space is decomposed into $N + 1$ fermionic sectors with $0 \leq N_f \leq N$. The projection of the Hamiltonian $H$ to a fixed fermionic sector with the fermion number $N_f$ produces $N_{CN_f} \times N_{CN_f}$ matrix-Hamiltonian $H^{(N_f)}$ and the identity $\sum_{N_f=0}^{N} N_{CN_f} = 2^N$ holds trivially. The projected Hamiltonian $H^{(N_f)}$ is obtained by evaluating $H$ in the basis given in Eq. (12):

$$H^{(N_f)} = \langle i_{N_f}, \ldots, i_1 | H | i_1, \ldots, i_{N_f} \rangle.$$  \hspace{1cm} (22)

In the same basis, the Hamiltonian $H$ has a block-diagonal structure, $H = \text{diag}\{H^{(N)}, H^{(N-1)}, \ldots, H^{(N_f)}, \ldots, H^{(1)}, H^{(0)}\}$. The zero-fermion sector and the $N$-fermion sector of $H$ define the Hamiltonians,

$$H^{(0)} = \frac{1}{2} \sum_{i=1}^{N} \left( p_i^2 + W_i^2 - W_{ii} \right),$$

$$H^{(N)} = \frac{1}{2} \sum_{i=1}^{N} \left( p_i^2 + W_i^2 + W_{ii} \right).$$  \hspace{1cm} (23)
If the superpotential depends on an overall multiplicative parameter \( \lambda \), i.e. \( W \sim \lambda W \), then the bosonic potentials of \( H^{(0)} \) and \( H^{(N)} \) are shape-invariant. In particular, \( H^{(N)}(\lambda) = H^{(0)}(-\lambda) \) for any choice of the superpotential. The bosonic potentials of these two Hamiltonians may be shape invariant for specific choices of \( W \). However, as Eq. (28) implies, the eigenspectra of \( H^{(0)} \) and \( H^{(N)} \) are not in one-to-one correspondence. Thus, the shape invariance condition cannot be implemented as in the case of system with \( N = 1 \). The \( N \times N \) matrix Hamiltonian corresponding to \( N_f = 1 \) has the form,

\[
[H^{(1)}]_{ij} = \delta_{ij} \left( \frac{1}{2} \sum_{k=1}^{N} (p_k^2 + W_k^2 - W_{kk}) \right) + W_{ij}. \tag{24}
\]

The matrix-Hamiltonians \( H^{(N_f)} \) for higher \( N_f < N \) can be obtained in a similar way. However, diagonalizing the Hamiltonian \( H^{(N_f)} \) for arbitrary \( N_f \) or implementing the shape invariance condition successfully is a daunting task.

For quantum systems with only one degree of freedom, i.e. \( N = 1 \), the fermionic Fock space is two dimensional. The Hamiltonian \( H \) has a block-diagonal structure \( H = \text{diag}(H^{(1)}, H^{(0)}) \), where \( H^{(0)} \) and \( H^{(1)} \) are identified as partner Hamiltonians\[3\]. In the same fermionic basis\[12\], the supercharge \( Q \) and \( Q^\dagger \) can be expressed as \( 2 \times 2 \) matrix operators:

\[
Q = \begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A_1^\dagger & 0 \end{pmatrix}. \tag{25}
\]

The superalgebra relates the partner Hamiltonians through the intertwining relations,

\[
[H, Q] = 0 \Rightarrow H^{(1)} A_1 = A_1 H^{(0)},
\]

\[
[H, Q^\dagger] = 0 \Rightarrow H^{(0)} A_1^\dagger = A_1^\dagger H^{(1)}. \tag{26}
\]

These relations allow an algebraic derivation of the complete spectra and the associated eigen states of the partner Hamiltonians for shape invariant potentials\[3\].

The scenario for many-particle systems with \( N \geq 2 \) is quite different from the one described above. The supercharge \( Q \) (\( Q^\dagger \)) changes the fermion number from \( N_f \) to \( N_f + 1(N_f - 1) \) and has the following over-diagonal(under-diagonal) structure\[7\] in the fermionic basis\[12\],

\[
[Q]_{pq} = \delta_{p, p+1} Q_{p-1, p}, \quad [Q^\dagger]_{pq} = \delta_{p-1, p} Q^\dagger_{p, p-1}, \quad p, q = 1, 2, \ldots, N_f, \tag{27}
\]

where \( \delta_{pq} \) is the Kronecker delta, \( Q_{p-1, p} := \langle i_p, \ldots, i_1|Q|i_1, \ldots, i_{p-1} \rangle \) is a matrix of dimension \( N C_{p-1} \times N C_p \) depending on the operators \( A_i \), and \( Q^\dagger_{p, p-1} := \langle i_{p-1}, \ldots, i_1|Q^\dagger|i_1, \ldots, i_{p} \rangle \) is a matrix of dimension \( N C_p \times N C_{p-1} \) depending on
the operators $A_i$. The underlying superalgebra leads to the following intertwining relations:

$$
H^{(N-i)}Q_{i,i+1} = Q_{i,i+1}H^{(N-i)},
$$

$$
H^{(N-i-1)}Q_{i+1,i} = Q_{i+1,i}H^{(N-i)}, \quad i = 0, 1, \ldots, N - 1.
$$

It follows that a part of the spectrum of $H^{(N_f)}$ coincide with the spectra of neighbouring block-Hamiltonians $H^{(N_f-1)}$ and $H^{(N_f+1)}$. However, a successful scheme of implementing the condition of shape invariance is still beyond the reach for arbitrary $N$. The diagonalization of $H^{(N_f)}$ involves solving a set of $N_C N_f$ coupled second order partial differential equations which is in general a difficult task. These technical limitations restrict the number of physically relevant exactly solved many-particle supersymmetric quantum systems.

### 3 Exactly Solved Systems in One Dimension

Several supersymmetric many-particle quantum systems may be obtained by suitably choosing the superpotential $W$. Different choices of superpotential lead to different interaction terms in the many-particle Hamiltonian $H$. The main focus of this review article is on many-body systems with inverse-square interactions. The bosonic potential in $H$ scales inverse-squarely if the superpotential is chosen as,

$$
W = W_0 \equiv -\ln G(x_1, \ldots, x_N), \quad \sum_{i=1}^{N} x_i \frac{\partial G}{\partial x_i} = d,
$$

where $d$ is the degree of the homogeneous function $G$. In general, Hamiltonians with purely scale-invariant bosonic potentials do not admit bound states. In the present article, harmonic confining potential will be added for the description of bound states for which the superpotential is of the form,

$$
W = W_0 + \frac{\omega}{2} \sum_{i=1}^{N} x_i^2.
$$

The bosonic potential in $H$ due to the superpotential $W$ in Eq. (30) contains harmonic confining potential and inverse-square many body interactions. It may be noted that the cross-term arising from $\sum_{i=1}^{N} W_i^2$ produces an additive constant in $H$ that is equal to the degree of the homogeneous function $G$.

The supersymmetric Hamiltonian $H_0$ corresponding to the superpotential $W_0$ has $O(2,1)$ symmetry. The Dilatation operator $D$ and the conformal oper-
ator $K$,

$$D = -\frac{1}{4} \sum_{i=1}^{N} (x_i p_i + p_i x_i), \quad K = \frac{r^2}{2}, \quad r^2 \equiv \sum_{i=1}^{N} x_i^2,$$

(31)

along with $H_0$ satisfy the $O(2, 1)$ algebra (154). The Casimir operator $C$ of the $O(2, 1)$, as given in Eq. (155), has the following expression,

$$C = \frac{1}{4} \left[ \sum_{i<j=1}^{N} L_{ij}^2 + r^2 V + \frac{1}{4} N(N-4) \right],$$

$$V = \sum_{i=1}^{N} \left[ \left( \frac{\partial W_0}{\partial x_i} \right)^2 - \frac{\partial^2 W_0}{\partial x_i^2} \right] + 2 \sum_{i,j=1}^{N} \frac{\partial^2 W_0}{\partial x_i \partial x_j} \psi_i^\dagger \psi_j,$$

(32)

where $L_{ij} \equiv x_i p_j - x_j p_i$ are defined as the angular momentum operators. The potential $V$ scales inverse-squarely. Consequently, in the $N$-dimensional hyperspherical coordinate system, $r^2 V$ contains only angular variables. The Hamiltonian $H = H_0 + \omega^2 K$ thus always can be separated into an angular and a radial part in the $N$-dimensional hyperspherical co-ordinate. The Hamiltonian $H$ contains a harmonic plus an inverse-square interaction,

$$H = \frac{p^2}{2} + \frac{\omega^2}{2} r^2 + \frac{C'}{2 r^2}, \quad C' := 4C - \frac{1}{4} N(N-4),$$

(33)

with the co-efficient of the inverse-square term determined in terms of the eigenvalues of the Casimir. The Hamiltonian is integrable for a fixed eigenvalue of $C$. Infinitely many exact eigenstates and the corresponding eigenspectra may be obtained analytically for a fixed eigen value of $C$. However, the eigen value equation for $C$ can be solved completely for specific superpotentials only and the corresponding potentials belong to exactly solved systems.

3.1 Rational $A_{N+1}$ Calogero Model

There are very few many-particle quantum systems for which the complete eigen spectra and the associated eigen states can be obtained analytically, the rational Calogero model being one of them [11]. The Hamiltonian for this model describes $N$ particles interacting with each other on a line through pair-wise inverse-square plus harmonic interactions. The supersymmetric version of this Hamiltonian was first considered in Ref. [43] and was shown to be exactly solvable. The supersymmetric generalizations of Calogero-Sutherland models based on all the root systems with rational, trigonometric and hyperbolic potentials have also been considered [45, 46, 49, 50, 79]. The main emphasis of this topical review being systems with $OSp(2|2)$ supersymmetry, the discussion is restricted to
rational models corresponding to different root systems. The rational \( A_{N+1} \) Calogero model is described below and rational \( BC_{N+1} \) model is included in Appendix-IV in section 9.4.

The superpotential for the \( A_{N+1} \)-type rational Calogero model is given by,

\[
W = -\lambda \ln \prod_{i<j} x_{ij} + \frac{\omega}{2} \sum_{i} x_{i}^2, \quad x_{ij} = x_i - x_j.
\]  

(34)

The Hamiltonian (21), with the above choice of \( W \), has the following form,

\[
H = -\frac{1}{2} \sum_{i} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda (\lambda - 1) \sum_{i\neq j} x_{ij}^{-2} + \frac{\omega^2}{2} \sum_{i} x_i^2 - \frac{\omega}{2} N [1 + \lambda (N - 1)]
\]

\[
+ \omega \sum_{i} \psi_i^\dagger \psi_i + \lambda \sum_{i\neq j} x_{ij}^{-2} \left( \psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j \right).
\]  

(35)

The Hamiltonian is invariant under the permutation symmetry:

\[
x_i \leftrightarrow x_j, \psi_i \leftrightarrow \psi_j, \psi_i^\dagger \leftrightarrow \psi_j^\dagger.
\]  

(36)

The last term in (35) can be expressed in terms of the fermionic exchange operator \( K_{ij} \) defined in Eq. (8) as,

\[
\sum_{i\neq j} x_{ij}^{-2} K_{ij}.
\]  

(37)

The configuration space of the system is divided into \( N! \) different sectors characterized by a definite ordering of the coordinates of the particles, \( x_1 < x_2 < \ldots < x_N \) and its all possible permutations. The inverse-square interaction is singular at the coinciding points \( x_i = x_j \). The many-body wave-functions and the associated probability currents are taken to be vanishing at these points which allows a smooth continuation of the wave-function from a given sector in the configuration space to all other sectors.

The ground state wave-function of the super-Hamiltonian with \( E_0 = 0 \) reads,

\[
\Phi_0 = \phi_0 \mid 0 \rangle, \quad \phi_0 \equiv \prod_{i<j=1}^{N} x_{ij}^{\lambda e^{-\frac{\omega}{2} \sum_{i=1}^{N} x_i^2}},
\]  

(38)

which is normalizable for \( \lambda > -\frac{1}{2} \). However, a negative value for the parameter in the range \( -\frac{1}{2} < \lambda < 0 \) necessarily leads to singularities in \( \Phi \) at the coinciding points \( x_i = x_j \). A stronger criteria that each momentum operator \( p_i \) is self-adjoint for the wave-functions of the form \( \Phi_0 \) requires \( \lambda > 0 \). The supersymmetry is preserved for \( \lambda > 0 \), while it is broken for \( \lambda < 0 \).
3.2 Shape Invariance and Exact Solvability

The Hamiltonian in the zero-fermion sector reduces to the rational $A_{N+1}$ Calogero model:

$$H^{(0)}(\lambda, \omega) = \mathcal{H}^{A_{N+1}} - E_0^{A_{N+1}}$$

$$\mathcal{H}^{A_{N+1}} := -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda (\lambda - 1) \sum_{i \neq j} x_{ij}^{-2} + \frac{\omega^2}{2} \sum_i x_i^2,$$

$$E_0^{A_{N+1}} \equiv \frac{\omega}{2} N [1 + \lambda (N - 1)]. \quad (39)$$

The complete eigenvalues and the eigenstates of $H^{(0)}$ and $\mathcal{H}^{A_{N+1}}$ can be obtained using the ideas of supersymmetry and shape invariance[49]. It may be noted that many-body potentials of the Hamiltonians $H^{(0)}$ and $H^{(N)}$ are shape-invariant,

$$H^{(N)}(\lambda, \omega) = H^{(0)}(\lambda + 1, \omega) + \frac{\omega}{2} N (N + 1)$$

$$= -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda (\lambda + 1) \sum_{i \neq j} x_{ij}^{-2} + \frac{\omega^2}{2} \sum_i x_i^2$$

$$+ \frac{\omega N}{2} [1 - \lambda (N - 1)]. \quad (40)$$

However, no direct relation between the eigen-spectra of $H^{(0)}$ and $H^{(N)}$ can be shown, as in the case of supersymmetric quantum mechanics with one bosonic and one fermionic degrees of freedom. Nevertheless, the shape invariance can be used to obtain the spectrum of $H^{(0)}$ or $H^{(N)}$ with the introduction of permutation and Dunkl operators[49].

The Dunkl operator for the rational $A_{N+1}$ Calogero model is defined as,

$$\pi_i = p_i + i \lambda \sum_{j \neq i} x_{ij}^{-1} M_{ij}, \quad (41)$$

where the exchange operator $M_{ij}$ satisfies the following properties[14, 15],

$$M_{ij} = M_{ij}^{-1} = M_{ij}^\dagger = M_{ji}, \quad M_{ij} \phi^\pm = \pm \phi^\pm,$$

$$M_{ij} O_i = O_j M_{ij}, \quad M_{ij} O_k = O_k M_{ij} \quad if \; i, j, k \; distinct,$$

$$M_{ijk} := M_{ij} M_{jk}, \quad M_{ijk} = M_{jik} = M_{kij}. \quad (42)$$

The function $\phi^+(\phi^-)$ appearing above is a(an) symmetric(anti-symmetric) function of the $N$ bosonic co-ordinates and $O_i$ is a single-particle bosonic operator in the phase space. The exchange operator $M_{ij}$ acting on any fermionic operator leaves it unchanged. The Dunkl operators $\pi_i$ commute with each other, i.e.
\[ [\pi_i, \pi_j] = 0 \]. A set of operators \( a_i, a_i^\dagger \) are introduced,

\[
a_i := \pi_i - i\omega x_i, \quad a_i^\dagger := \pi_i + i\omega x_i,
\]

which satisfy an extended version of the Heisenberg algebra involving the permutation operators \( M_{ij} \):

\[
\left[ a_i, a_j^\dagger \right] = 2\omega \delta_{ij} \left( 1 + \lambda \sum_{k \neq i} M_{ik} \right) - 2 (1 - \delta_{ij}) \lambda \omega M_{ij}.
\]

(43)

All other commutators involving \( a_i \) and their adjoints vanish identically. The Hamiltonian \( \mathcal{H} \) and its partner Hamiltonian \( \tilde{\mathcal{H}} \) are introduced as follows:

\[
\mathcal{H} = \frac{1}{2} \sum_{i=1}^{N} a_i^\dagger a_i, \quad \tilde{\mathcal{H}} = \frac{1}{2} \sum_{i=1}^{N} a_i a_i^\dagger.
\]

(44)

The Hamiltonian \( \mathcal{H} \) reduces to \( H^{(0)} \) if \( M_{ij} \) acts on symmetric functions only, whereas \( \tilde{\mathcal{H}} \) becomes \( H^{(N)} \) if \( M_{ij} \) acts on antisymmetric functions only. No choice between the symmetric and the antisymmetric functions will be made in the discussions below and the results obtained are valid for either cases. The convention is that an upper(lower) sign in the expressions below corresponds to the case that \( M_{ij} \) only acts on symmetric(antisymmetric) functions.

The extended Heisenberg algebra can be used to show one to one correspondence between the non-zero energy eigen values of \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \). In particular, if \( \phi \) is the eigenstate of \( \mathcal{H} \) with eigenvalue \( E > 0 \), then \( A_1 \phi \) is the eigenstate of \( \tilde{\mathcal{H}} \) with eigenvalue \( E + \delta_1 \) i.e.

\[
\tilde{\mathcal{H}}(A_1 \phi) = [E + \delta_1](A_1 \phi),
\]

(45)

where the operator \( A_1 \) is defined as,

\[
A_1 := \sum_{i} a_i, \quad \delta_1 = [(N - 1) \pm \lambda N (N - 1)] \omega.
\]

(46)

Similarly, if \( \tilde{\phi} \) is the eigen state of \( \tilde{\mathcal{H}} \) with eigenvalue \( \tilde{E} \), then \( A_1^\dagger \tilde{\phi} \) is the eigenfunction of \( \mathcal{H} \) with eigenvalue \( \tilde{E} - \delta_1 \) i.e.

\[
\mathcal{H}(A_1^\dagger \tilde{\phi}) = \left( \tilde{E} - \delta_1 \right)(A_1^\dagger \tilde{\psi}).
\]

(47)

The energy eigenvalues and eigenfunctions of the two partner Hamiltonians \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) are thus related through the following relations:

\[
\tilde{E}_n = E_{n+1} + \delta_1, \quad E_0 = 0, \quad n = 0, 1, 2, \ldots
\]

\[
\tilde{\phi}_n = \frac{A_1 \phi_{n+1}}{\sqrt{E_{n+1} + \delta_1}}, \quad \phi_{n+1} = \frac{A_1^\dagger \tilde{\phi}_n}{\sqrt{E_{n+1}}}.
\]

(48)
The standard results of supersymmetric quantum systems with one bosonic and one fermionic degrees of freedom are reproduced for $N = 1$. The energy levels of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are non-degenerate for $\delta \neq 0$.

The shape invariance condition for the partner Hamiltonians $\mathcal{H}$ and $\tilde{\mathcal{H}}$ reads,

$$\tilde{\mathcal{H}}(\lambda) = \mathcal{H}(\lambda) + R(\lambda),$$

$$R(\lambda) = [N \pm \lambda N (N - 1)]\omega = \omega + \delta_1 .$$  \hspace{1cm} (50)

Following the standard formalism of supersymmetric quantum mechanics\[3] and using the first equation of (49), the spectrum of $\mathcal{H}$ is determined\[49]:

$$E_n = n(R(\lambda) - \delta_1) = n\omega .$$  \hspace{1cm} (51)

The eigenvalues of $\mathcal{H}^{A_{N+1}}$ are thus equal to the eigenvalues of $N$ free harmonic oscillators shifted by a constant,

$$E^{A_{N+1}}_n = E_n + E^{A_{N+1}}_0 = \omega n + \frac{\omega}{2} N \left[1 + \lambda (N - 1)\right].$$  \hspace{1cm} (52)

The $n^{th}$ eigenstate is obtained as, $\phi_n = \left(A_1^\dagger\right)^n \phi_0$, since $A_1$ and $A_1^\dagger$ can be identified as the annihilation and creation operators, respectively. In general, the following identities hold true,

$$A_n := \sum_{i=1}^{N} a_i^n, \quad A_n^\dagger := \sum_{i=1}^{N} (a_i^\dagger)^n, \quad n \leq N,$$

$$[\mathcal{H}, A_n] = -n A_n, \quad [\mathcal{H}, A_n^\dagger] = n A_n^\dagger .$$  \hspace{1cm} (53)

Further, the operators $A_n, A_n^\dagger, \mathcal{H}, \tilde{\mathcal{H}}$ satisfy relations which are analogous to those given by Eqs. (46) and (48). This facilitates a construction of all the degenerate states $\phi_{\{n_i\}}$,

$$\phi_{\{n_i\}} = \prod_{i=1}^{N} \left(A_i^\dagger\right)^{n_i} \phi_0, \quad k = \sum_{i=1}^{N} n_i ,$$  \hspace{1cm} (54)

corresponding to a particular value of $k$. All the corresponding states of $\tilde{\mathcal{H}}$ can be obtained by applying the same $A_1$ on $\phi_{\{n_i\}}$.

4 *OSp*(2|2) Supersymmetric Systems

The supersymmetric rational $A_{N+1}$ Calogero model has a dynamical *OSp*(2|2) symmetry\[43]. The celebrated Calogero model is obtained by projecting the
supersymmetric Hamiltonian $H$ in the zero fermion sector, i.e. $N_f = 0$. There exists a possibility of constructing a new supersymmetric many-particle quantum system with dynamical $OSp(2|2)$ supersymmetry that is different from $H$, yet it reduces to the celebrated Calogero model in appropriate limit. In particular, the Hamiltonian $H$ constructed by Freedman and Mende$^\text{43}$ corresponds to ‘typical’ representation of $OSp(2|2)$. The ‘atypical’ representation of $OSp(2|2)$ with the same superpotential $W$ in $\text{(34)}$ produces a many-particle supersymmetric quantum system that is different from $H$ $\text{(47)}$. Thus, the construction of $OSp(2|2)$ supersymmetric Calogero model is not unique.

The structure equations of $OSp(2|2)$ symmetry are given in Appendix-II. In this section, the superpotential $W$ is replaced by $W_0$ as defined in Eq. $\text{(29)}$ and the corresponding Hamiltonian $H$ in Eq. $\text{(17)}$ or in Eq. $\text{(21)}$ is denoted as $H_0$. A coordinate realization of the dilatation operator $D$ and the conformal generator $K$ is given in Eq. $\text{(31)}$. The real supercharges $Q_1, Q_2$ are described in Eq. $\text{(16)}$. The remaining three generators $S_1, S_2$ and $Y$ corresponding to the ‘typical’ representation of $OSp(2|2)$ are realized in the following way:

$$S_1 = \frac{1}{\sqrt{2}} \sum_{i=1}^{N} \xi_i x_i, \quad S_2 = -\frac{1}{\sqrt{2}} \sum_{i=1}^{N} \xi_{N+i} x_i,$$

$$Y = -\frac{1}{4} \left( i \sum_{i=1}^{N} \xi_{N+i} \xi_i + 2d \right).$$

These operators can be expressed in terms of fermionic operators,

$$S := \frac{1}{\sqrt{2}} (S_1 - iS_2) = \sum_{i=1}^{N} \psi_i^\dagger x_i, \quad S^\dagger := \frac{1}{\sqrt{2}} (S_1 + iS_2) = \sum_{i=1}^{N} \psi_i x_i,$$

$$Y = -\frac{N_f}{2} + \frac{N}{4} - \frac{d}{2}. \quad (56)$$

The hypercharge $Y$ is related to the total fermion number operator $N_f$ and $S$ factorizes the conformal generator $K$.

The Hamiltonian $H_0$ is scale invariant and its ground state wave-function,

$$\phi_0 = G(x_1, \ldots, x_N) \ket{0}, \quad (57)$$

is not even plane-wave normalizable. The time-evolution of such scale invariant supersymmetric systems are generally studied in terms of the operators $\text{9, 10, 47}$,

$$\mathcal{H}_\pm = R \pm \omega Y, \quad R := \frac{1}{2} \left( H_0 + \omega^2 K \right). \quad (58)$$

The generator of the compact rotation $R$ of $O(2,1)$ and the hypercharge $Y$ are simultaneously diagonal in the Cartan basis. The operator $\mathcal{H}_+$ is related to $\mathcal{H}_-$.
and the vice versa through an automorphism of the $OSp(2|2)$ algebra. Thus, either $H_+$ or $H_-$ may be identified as the supersymmetric Hamiltonian and the choice in this article is the Hamiltonian $H_-$. The Hamiltonian $H_-$ can be expressed in terms of supercharges $F$ and $F^\dagger$ as follows,

$$ F := \frac{1}{\sqrt{2}} (Q - i\omega S), \quad F^\dagger := \frac{1}{\sqrt{2}} (Q^\dagger + i\omega S^\dagger), \quad H_- = \frac{1}{2} \{ F, F^\dagger \}. \quad (59) $$

The Hamiltonian $H$ in Eqs. (17, 21) with the superpotential as given in Eq. (30) is identical with $2H_-$. The bosonic generators $H_0, D, K, Y$ and the fermionic generators $Q_1, Q_2, S_1, S_2$ with the co-ordinate realization described above satisfy the structure equations of $OSp(2|2)$. The quadratic and the cubic Casimir operators of $OSp(2|2)$ are non-vanishing and hence, this particular co-ordinate realization of generators corresponds to the ‘typical’ representation of $OSp(2|2)$. The arguments in favour of non-vanishing $C_2$ and $C_3$ are as follows. The terms appearing in the expressions of $C_2$ and $C_3$ in Eq. (156) have the following co-ordinate realization:

$$ i[Q_1, S_1] = \frac{N}{2} - \frac{i}{2} \sum_{i,j=1}^{N} \xi_i \xi_j L_{ij} + i \sum_{i,j=1}^{N} \xi_{N+i} \xi_j x_j W_i, $$

$$ i[Q_2, S_2] = \frac{N}{2} - \frac{i}{2} \sum_{i,j=1}^{N} \xi_{N+i} \xi_{N+j} L_{ij} - i \sum_{i,j=1}^{N} \xi_{N+i} \xi_j x_i W_j, $$

$$ Y^2 = \frac{1}{16} \left( 4d^2 + N + 4d \sum_i \xi_{N+i} \xi_i + \frac{1}{2} \sum_{i,j \neq j} [\xi_{N+i}, \xi_{N+j}] \xi_i \xi_j \right). \quad (60) $$

The operator $i[Q_1, S_1] + i[Q_2, S_2] - Y^2$ appearing in $C_2$ does not contain any term proportional to $L_{ij}^2$. However, the Casimir operator $C$ in Eq. (32) contains a term proportional to $L_{ij}^2$. This implies that $C_2$ in (156) can not vanish identically. The cubic Casimir operator $C_3$ is also non-vanishing, since it contains a term of the form $L_{ij}^2 Y$ which can not be canceled from rest of the terms appearing in its definition in Eq. (156).

### 4.1 Non-uniqueness of the Construction

The operators $H_0, D, K, Q_1$ and $S_1$ have identical coordinate realizations in ‘typical’ as well as in ‘atypical’ representations of $OSp(2|2)$. The operators $Q_2, S_2$ and $Y$ have different coordinate realizations corresponding to two different representations of the group. The operators $Q_2, S_2$ and $Y$ in the ‘atypical’ representation of $OSp(2|2)$ are realized in the following way,

$$ \hat{Q}_2 = -i\xi_{2N+1} Q_1, \quad \hat{S}_2 = -i\xi_{2N+1} S_1 $$
\[
\hat{Y} = \frac{\xi_{2N+1}}{2} \left[ -\frac{i}{2} \sum_{i,j=1}^{N} \xi_i \xi_j L_{ij} + i \sum_{i,j=1}^{N} \xi_{N+i} \xi_j x_j + \frac{N}{2} \right]. \tag{61}
\]

The structure equations of \( OSp(2|2) \) are now satisfied by the bosonic generators \( H_0, D, K, \hat{Y} \) and the fermionic generators \( Q_1, \hat{Q}_2, S_1, \hat{S}_2 \). The 'atypical' realization of the \( OSp(2|2) \) superalgebra is possible only for \( N \geq 2 \). No independent coordinate realization of \( Q_2, S_2 \) and \( Y \), other than the one given in Eqs. (16,55), is admissible for systems with one bosonic and fermionic degrees of freedom.

The cubic Casimir operator can be expressed in terms of the quadratic Casimir operator in the 'atypical' representation:

\[
C_3 = \left( \hat{Y} - \frac{\xi_{2N+1}}{4} \right) C_2. \tag{62}
\]

Further, the Casimir operators \( C_s, \bar{C}_s \) defined in Eqs. (158) and (160) are identical in this representation. Using the second equation of (161), definition of \( C_s \) and the identity,

\[
\hat{Y}^2 = \frac{1}{4} \left( C_s^2 + C_s + \frac{1}{4} \right), \tag{63}
\]

it follows that the quadratic Casimir operator \( C_2 \) vanishes identically. Consequently, the cubic Casimir \( C_3 \) is identically equal to zero for this particular representation. The spectrum is not completely specified by the eigenvalues of Casimir operators and hence, the representation is 'atypical'. The Casimir \( C_s \) can be used to determine the spectrum, since it commutes with the bosonic generators and anti-commutes with the fermionic generators.

A new supersymmetric Hamiltonian preserving the \( OSp(2|2) \) symmetry may be introduced in the 'atypical' representation:

\[
\hat{H}_\pm = \frac{1}{2} \left( H_0 + \omega^2 K \right) \pm \omega \hat{Y}
= \frac{1}{4} \sum_i \left( p_i^2 + W_i \right) - W_{ii} + \omega^2 x_i^2 + \frac{1}{4} \sum_{i,j} \psi_i^\dag \psi_j W_{ij} \pm \frac{\omega \xi_{2N+1}}{4} \left[ N + 2d \right]
- i \sum_{i,j} \left( \psi_i^\dag \psi_j \right) e^W L_{ij} e^{-W} - i \sum_{i,j} \left( \psi_i \psi_j - \psi_j^\dag \psi_i \right) e^{-W} L_{ij} e^W
- \sum_{i,j} \left( \psi_i^\dag \psi_j + \psi_j^\dag \psi_i \right) \left( x_i W_j + x_j W_i \right), \tag{64}
\]

where \( W \) is identified with \( W_0 \) defined in Eq. (23). The Hamiltonian \( \hat{H}_\pm \) can also be casted in a manifestly supersymmetric form,

\[
\hat{\mathcal{H}}_\pm = \frac{1}{2} \{ \hat{\mathcal{F}}, \hat{\mathcal{F}}^\dag \}
\]

\[
\hat{\mathcal{F}} := \xi_{2N+1} (Q_1 - i \omega S_1), \quad \hat{\mathcal{F}}^\dag := \xi_{2N+1}^* (Q_1 + i \omega S_1). \tag{65}
\]
The difference between $\mathcal{H}_\pm$ and $\hat{\mathcal{H}}_\pm$ lies in the expressions of the hyper-charges $Y$ and $\hat{Y}$, respectively. Unlike in the case of $\mathcal{H}_\pm$, the total fermion number operator $N_f$ does not commute with $\hat{\mathcal{H}}_\pm$. Consequently, the eigenstates of $\hat{\mathcal{H}}_\pm$ can not be constructed as simultaneous eigenstates of $N_f$. However, $\hat{\mathcal{H}}_\pm$ commutes with $\hat{Y}$ and simultaneous eigenstates of these two operators can be constructed. If the eigenvalue of $\hat{Y}$ is chosen as $Y^+ = \frac{N}{4} (N + 2d)$ and $R$ is projected in the fermionic vacuum $|0\rangle$, then $2\hat{\mathcal{H}}_+$ reduces to the purely bosonic Hamiltonian $H^{(0)}$. Similarly, $2\hat{\mathcal{H}}_-$ reduces to the purely bosonic Hamiltonian $H^{(N)}$ for the choice of the eigenvalue $Y^-$ as $Y^- = \frac{N}{4} (N - 2d)$ and projection of $R$ in the conjugate fermionic vacuum $|\bar{0}\rangle$. Thus, the same Hamiltonians $H^{(0)}$ (or $H^{(N)}$) may be obtained from two different supersymmetric Hamiltonians with $OSp(2|2)$ symmetry.

The Hamiltonian for the rational $A_{N+1}$ Calogero model is given by,

$$
\hat{\mathcal{H}}_\pm = \frac{1}{4} \sum_i \left( p_i^2 + \omega^2 x_i^2 + \sum_{j \neq i} \frac{\lambda(\lambda - 1)}{(x_i - x_j)^2} \right) + \frac{\lambda}{2} \sum_{i \neq j} \left( \psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j \right) \left( x_i - x_j \right)^2
$$

$$
\pm \frac{\omega \xi_{2N+1}}{4} \left[ N + \lambda N(N - 1) - i \sum_{i,j} \left( \psi_i^\dagger \psi_j^\dagger + \psi_i^\dagger \psi_j \right) \left( GL_{ij} G^{-1} \right) - i \sum_{i,j} \left( \psi_i \psi_j - \psi_j^\dagger \psi_i^\dagger \right) \left( G^{-1} L_{ij} G \right) - \lambda \sum_{i,j} \left( \psi_i^\dagger \psi_j^\dagger + \psi_i^\dagger \psi_j \right) \left( \sum_{k \neq j} \frac{x_i}{x_i - x_k} + \sum_{k \neq i} \frac{x_j}{x_i - x_k} \right) \right], \quad (66)
$$

where $G = \prod_{i<j} x_{ij}^\lambda$. It is not known whether $\hat{\mathcal{H}}_\pm$ is integrable or not. Many exact eigenstates of $\hat{\mathcal{H}}_\pm$ can be constructed explicitly using the underlying $OSp(2|2)$ supersymmetry. The quadratic and the cubic Casimir operators being zero in the ‘atypical’ representation, these operators can not be used to characterize the spectra. Further, as in the case of ‘typical’ Calogero model, $OSp(2|2)$ is not the full spectrum generating algebra of $\hat{\mathcal{H}}_\pm$. Alternative methods are required to establish (non-)integrability of $\hat{\mathcal{H}}_\pm$.

4.2 Mapping to Free Super-oscillators

The Hamiltonian $H$ in Eq. (35) is exactly solvable in both supersymmetry-preserving and supersymmetry-breaking phases\textsuperscript{[43, 44, 45]}. The spectrum in the supersymmetry-preserving phase is identical to that of the $N$ free super-oscillators, while in the supersymmetry-breaking phase, it has no counter-part in the super-oscillator model\textsuperscript{[43]}. This is primarily because of the fact that the
supersymmetry is always preserved in the super-oscillator model, once the convention for choosing the ground state in either zero or $N_f$ fermion sector has been made. This is a good indication that the rational $A_{N+1}$ Calogero model in the supersymmetry-preserving phase may be mapped to a set of free superoscillators through a similarity transformation, much akin to its non-supersymmetric version[40]. The superpotential of the rational $A_{N+1}$ Calogero model is given in Eq. [44] and without any loss of generality $\omega$ is taken to be unity in this section.

4.2.1 Supersymmetry-preserving Phase

The mapping of $H$ to a system of free super-oscillators is achieved as follows:

$$H_1 = e^W H e^{-W}$$

$$= \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right) - S,$$

$$S := \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \lambda \sum_{i \neq j} x_{ij}^{-1} \frac{\partial}{\partial x_i} - \lambda \sum_{i \neq j} x_{ij}^{-2} \left( \psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j \right).$$

It may be noted that $N_f$ commutes with both $H_1$ and $S$. Making use of the following identity,

$$\left[ \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right), S \right] = -2S,$$

the operator $H_1$ is mapped to the operator $H_2$,

$$H_2 = e^{\frac{\theta}{2}} H_1 e^{-\frac{\theta}{2}}$$

$$= THT^{-1}$$

$$= \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right), \quad T := e^{\frac{\theta}{2}} e^W,$$

which may be considered as a supersymmetric generalization of the Euler operator. The familiar form of the super-oscillator Hamiltonian may be obtained in the following way,

$$H_{sho} = e^{-\frac{1}{2} \sum_i x_i^2} e^{-\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2}} H_2 e^{\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2}} e^{\frac{1}{2} \sum_i x_i^2}$$

$$= \frac{1}{2} \sum_i \left( -\frac{\partial^2}{\partial x_i^2} + x_i^2 \right) + \sum_i \psi_i^\dagger \psi_i - \frac{N}{2}.$$
complete spectrum of $H$ is obtained from that of $H_2$ or $H_{sho}$ by using the similarity operator $T$. Thus, the similarity operator $T$ should not take the original Hamiltonian out of its Hilbert space and the domain of $T$, $H$ and $H_2(H_{sho})$ should be identical.

If $P_{n,k}$ is an eigen-function of $H_2$ with the eigen-value $E_{n,k}$, then, $H$ has the same eigen-value $E_{n,k}$ with the eigen-function given by,

$$\chi = T^{-1}P_{n,k}|0>.$$  \hfill (72)

The eigenfunctions $P_{n,k}$, which are invariant under the combined exchange of the bosonic and the fermionic coordinates, i.e. $(x_i, \psi_i) \leftrightarrow (x_j, \psi_j)$, produce physically acceptable $\chi$. Any departure from this prescription for choosing $P_{n,k}$ produces essential singularity in $\chi$ and is not physically acceptable. The complete eigenstates and eigen spectrum of $H$ can be reproduced using the similarity transformation and a complete set of $P_{n,k}$ that is constructed using the prescription described above. This shows the equivalence between $H$ and $H_{sho}$. An example of $P_{n,k}$ that corresponds to $N_f = 1$ solution of $H_2$ with energy eigenvalue $E_{n,k} = 2n + k$ is given as,

$$P_{n,k} = i^{2n} \sum_{i=1}^{N} x_i^{k-1} \psi_i^\dagger, \quad n = 0, 1, \ldots, k = 1, 2, \ldots.$$  \hfill (73)

The action of $S^m$, $m \geq 1$ on $P_{n,k}$ does not produce any singularity. In general, $P_{n,k}$ for arbitrary $N_f$ may be chosen as,

$$P_{n,k} = \frac{1}{N_f!} i^{2n} \sum_{i_1,i_2,\ldots,i_{N_f}} f_{i_1,i_2\ldots,i_{N_f}}(x_1,x_2,\ldots,x_N) \psi_{i_1}^\dagger \psi_{i_2}^\dagger \ldots \psi_{i_{N_f}}^\dagger,$$  \hfill (74)

where $f_{i_1,i_2\ldots,i_{N_f}}$ is anti-symmetric under the exchange of any two indices and is a homogeneous function of degree $k - N_f$. The anti-symmetric nature of $f$ ensures that $P_{n,k}$ is permutation invariant under the combined exchange of bosonic and fermionic coordinates. No closed form expression of $f_{i_1,i_2\ldots,i_{N_f}}$ for arbitrary $N_f$ is known.

An algebraic construction of the eigen spectrum of $H$ is allowed through the introduction of the operators,$\[16$,

$$b_i^- = ip_i, \quad b_i^+ = 2x_i, \quad \psi_i^\dagger := \psi_i^\dagger, \quad \psi_i^- := \psi_i,$$

$$B_n^\pm = T^{-1} \left( \sum_{i=1}^{N} b_i^{\pm n} \right) T, \quad F_n^\pm = T^{-1} \left( \sum_{i=1}^{N} \psi_i^\mp b_i^{\pm n-1} \right) T,$$

$$q_n^\pm = T^{-1} \left( \sum_{i=1}^{N} \psi_i^\mp \left( b_i^\pm \right)^n \right) T.$$  \hfill (75)
The operators $B_n^\pm$ and $F_n^\pm$ satisfy the algebra of $N$ independent superoscillators with frequencies $1, 2, \ldots, N$, namely,

$$[H, B_n^\pm] = \pm n B_n^\pm, \quad [H, F_n^\pm] = \pm n F_n^\pm,$$

$$\{q_i^+, F_n^\pm\} = B_n^+, \quad [q_i^+, B_n^\pm] = 2 n F_n^\pm,$$  \hspace{1cm} (76)

and so on. Thus,

$$\chi_{n_1 \ldots n_N \nu_1 \ldots \nu_N} = \prod_{k=1}^N B_k^{\nu_k} F_k^{\nu_k} \Phi_0,$$  \hspace{1cm} (77)

is the eigenfunction of $H$ with the eigen-value,

$$E = \sum_{k=1}^N k(n_k + \nu_k), \quad n_k = 0, 1, \ldots; \nu_k = 0, 1.$$  \hspace{1cm} (78)

The spectrum of $H$ is identical, including degeneracies at each level, to that of $N$ superoscillators with frequencies $1, 2, \ldots, N$.

### 4.2.2 Supersymmetry-breaking Phase

The eigen-spectrum of the rational $A_{N+1}$ Calogero model in the supersymmetry-breaking phase can also be constructed from the known super-oscillator basis by making use of a duality property of the model[46]. In particular, a new super-Hamiltonian $H_d$ may be constructed[43] from $H$ by using the transformation:

$$\lambda \rightarrow -\lambda \text{ and } \psi_i \leftrightarrow \psi_i^\dagger,$$

$$H_d = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda(\lambda - 1) \sum_{i \neq j} x_{ij}^{-2} + \frac{1}{2} \sum_i x_i^2 + \frac{1}{2} N [1 + \lambda(N - 1)]$$

$$- \sum_i \psi_i^\dagger \psi_i + \lambda \sum_{i \neq j} x_{ij}^{-2} \left( \psi_i^\dagger \psi_i^\dagger - \psi_i^\dagger \psi_j^\dagger \right),$$

$$H = H_d + 2N f - N [1 + \lambda(N - 1)].$$  \hspace{1cm} (79)

The ground state of $H_d$ is in the $N$ fermion sector,

$$\Phi = e^{-\hat{W}} |\bar{0}\rangle = \prod_{i<j} x_{ij}^{-\lambda} e^{-\frac{1}{2} \sum_i x_i^2} |\bar{0}\rangle, \quad \lambda < 0.$$  \hspace{1cm} (80)

The supersymmetric phase of $H_d$ is described by $\lambda < 0$. This condition on $\lambda$ ensures that each momentum operator $p_i$ is self-adjoint for wave-functions of the form $\Phi$. The Hamiltonian $H_d$ differs from the original Hamiltonian $H$ by the fermionic number operator and a constant. This implies that any eigen-function of this dual model is also a valid eigen-function of the rational $A_{N+1}$ Calogero model. Of course, the corresponding energy eigen-values are different from each
other. For example, the wave-function $\tilde{\Phi}$ is also an eigen-state of $H$ with positive energy. This is, in fact, the ground state of $H$ in the supersymmetry-breaking phase\[43\]. The complete spectrum of $H$ in this phase can be obtained from $H_d$ by making use of the second equation of (79).

The dual Hamiltonian can be shown to be equivalent to a free super-oscillator Hamiltonian through a similarity transformation\[46\]. The eigen-spectrum thus obtained from the super-oscillator model via the dual Hamiltonian indeed correctly describes the supersymmetry-breaking phase of the rational $A_{N+1}$ Calogero model. An algebraic construction of the complete set of eigenstates is admissible with the introduction of the bosonic creation operator $\hat{B}^+_n$ and the fermionic creation operator $\hat{F}^+_n$:

$$\hat{B}^+_n = \sum_i \hat{T}^{-1} b_i^{n+} \hat{T}, \quad \hat{F}^+_n = \hat{T}^{-1} \left( \sum_i \psi_i b_i^{n-} \right) \hat{T},$$

$$\hat{T} := e^{\tilde{S} \tilde{W}}, \quad \tilde{W} := \lambda \ln \prod_{i<j} x_{ij} + \frac{1}{2} \sum_i x_i^2.$$  \hspace{3cm} (81)

The eigenstates and the associated eigen values are,

$$\tilde{\Phi}_{n_1,...,n_N,\nu_1,...,\nu_N} = \prod_{k=1}^N \hat{B}^{+n_k}_k \hat{F}^{+\nu_k}_k \tilde{\Phi},$$

$$E = N [1 - \lambda (N - 1)] + \sum_{k=1}^N [kn_k + (k - 2)n_k].$$  \hspace{3cm} (82)

The bosonic quantum numbers $n_k$’s are non-negative integers, while the fermionic quantum numbers $\nu_k$’s are either 0 or 1.

4.2.3 Generalization

The mapping of $H$ in Eq. (21) to a set of free superoscillators through a similarity transformation is valid for a class of superpotential of the form given in Eq. (30)\[46\]:

$$H_2 = \tilde{T} H \tilde{T}^{-1}, \quad \tilde{T} := e^{\tilde{S} \tilde{W}}$$

$$\tilde{S} := \sum_i \left( \frac{1}{2} \frac{\partial^2}{\partial x_i^2} + W_i \frac{\partial}{\partial x_i} \right) + \sum_i W_{ii} \psi_i^\dagger \psi_i + \sum_{i \neq j} W_{ij} \psi_i^\dagger \psi_j.$$  \hspace{3cm} (83)

The homogeneity condition on $G$ ensures that the resulting Hamiltonian has a dynamical $OSp(2|2)$ supersymmetry with the bosonic sub-algebra $O(2,1) \times U(1)$. The presence of the symmetry algebra $O(2,1) \times U(1)$ in a Hamiltonian is enough to show its equivalence to free super-oscillators. The supersymmetry of the
Hamiltonian does not play any role. The mapping to free super-oscillator is achieved even if the $OSp(2|2)$ symmetry of a Hamiltonian is lost, but, has only $O(2, 1) \times U(1)$ symmetry. The mapping can be considered as a necessary condition, while the construction of the complete spectrum and associated well-behaved eigen-functions of the original Hamiltonian from the super-oscillator basis is sufficient to claim the equivalence between these two Hamiltonian.

The rational $BC_{N+1}$ Calogero model with $OSp(2|2)$ supersymmetry can be mapped to free superoscillators on the half-line\cite{46}. The complete eigen spectra, including the degeneracy at each level, can be obtained from the free super-oscillator basis. Relevant results in this regard are described in section 9.4. The mapping is also applicable to a class of $OSp(2|2)$ supersymmetric models\cite{112} related to short-range Dyson models\cite{109} which are discussed in section 7. Although infinitely many exact eigenstates can be constructed from the superoscillator basis, the exact solvability is not known for this class of models. Thus, an equivalence between such models with super-oscillator cannot be claimed. A careful analysis is required to check whether the similarity operator is taking the original Hamiltonian out of its Hilbert space or not. A similar study on the domain of the similarity operator and the Hamiltonian is also desirable.

A comment is in order before the end of this section. A mapping of purely rational Calogero model (i.e. $H$ without the harmonic confinement and the fermion number operator terms) to a system of free particles has been introduced later in Ref. \cite{53} by using a unitary operator. This has been achieved at the cost of making the special conformal generator of the underlying $O(2, 1)$ symmetry non-local. An attempt to generalize this result to the case of mapping the Calogero model with harmonic confinement term to that of a system of free oscillators essentially leads to non-unitary similarity operator\cite{53}. The non-unitary nature of the similarity operator is in conformity with the existing results\cite{46}. However, the explicit form of the similarity operator is quite different for these two cases and the readers are referred to the relevant references\cite{46, 53} for details.

5 Pseudo-hermitian Supersymmetric Systems

The definition of a hermitian operator crucially depends on the choice of the inner-product (/norm/metric) in the Hilbert space. The metric in the Hilbert space is always chosen as an identity operator in the standard treatment of quantum mechanics and a hermitian operator is defined to be equal to its own
complex-conjugate transpose. Operators not satisfying the above criteria are termed non-hermitian and have been used extensively to simulate dissipative quantum processes. Hermitian operators in a Hilbert space that is endowed with an identity operator as the metric is known as Dirac-hermitian operator in the current literature.

The question of necessity of Dirac-hermitian operators in formulating quantum physics is as old as the subject itself. A renewed interest has been generated over the last decade in addressing the same question in a systematic manner. The current understanding is that a quantum system with unbroken combined Parity($P$) and Time-reversal($T$) symmetry admits entirely real spectra even though the system may be non-Dirac-hermitian. It has been further shown that quantum system with unbroken $\mathcal{PT}$-symmetry also admits a symmetry which is identified as a charge-conjugation ($C$) symmetry. A consistent quantum description including reality of the entire spectra and unitary time-evolution of the non-Dirac-hermitian system is possible with the choice of a new inner-product involving the $\mathcal{CPT}$-symmetry.

An alternative description of $\mathcal{PT}$-symmetric theories is in terms of pseudo-hermitian operator. An operator $\hat{O}$ that is related to its hermitian-conjugate $\hat{O}^\dagger$ through a similarity transformation is defined as a pseudo-hermitian operator,

$$\hat{O}^\dagger = \eta \hat{O} \eta^{-1}.$$  \hfill (84)

The hermitian conjugation of $\hat{O}$ is taken in the Hilbert space $H_D$ that is endowed with the inner product $\langle \cdot | \cdot \rangle$. The Hilbert space that is endowed with the inner product $\langle \langle \cdot | \cdot \rangle \rangle_\eta_+ := \langle \cdot | \eta_+ \cdot \rangle$ is denoted as $H_{\eta_+}$. In general, the similarity operator $\eta$ is not unique. However, if a positive definite similarity operator $\eta_+$ exists, the operator $\hat{O}$ can be shown to be hermitian in the Hilbert space $H_{\eta_+}$. Further, $\hat{O}$ can be mapped to a hermitian operator $\hat{O}$ through a similarity transformation, i.e. $\hat{O} = \rho \hat{O} \rho^{-1}$, where $\rho := \sqrt{\eta_+}$. Consequently, a consistent quantum description including reality of the entire spectra and unitary time evolution is possible for the operator $\hat{O}$ which is non-Dirac-hermitian, but, hermitian in a Hilbert space with the metric $\eta_+$.

Several non-hermitian quantum systems admitting an entirely real spectra and unitary time-evolution have been constructed in $H_D$, for example, in Refs. $[89, 90, 93, 94, 95]$. The $\mathcal{PT}$ symmetric extensions of Calogero models have been studied in Refs. $[84, 96, 97, 98, 99]$. A general construction of many-particle pseudo-hermitian supersymmetric systems is presented below which is valid for any superpotential. The pseudo-hermitian supersymmetric rational Calogero model is included as an example. The discussions below are based on Ref. $[84]$. 

28
5.1 Isospectral Deformation

Pseudo-hermitian quantum systems can be constructed by isospectral deformation of known Dirac-hermitian quantum systems[93]. The general method involves a realization of the basic canonical commutation relations defining the quantum system in terms of non-Dirac-hermitian operators, which are hermitian with respect to a pre-determined positive-definite metric in the Hilbert space. Appropriate combinations of these operators produce a large number of pseudo-hermitian quantum system.

5.1.1 Deformation involving bosonic coordinates

The metric $\eta^b_{+}$ in the Hilbert space $H_{\eta^b_{+}}$ is chosen as,

$$\eta^b_{+} := e^{-2(\delta B + Re(W_-))}, \quad W_{\pm} = \frac{1}{2} (W_1 \pm W_2) \quad \delta \in R,$$

(85)

where $W_{1,2}$ are two complex functions of the $N$ bosonic co-ordinates. The functions $W_+(W_-)$ can be made real even for complex $W_1,W_2$. For example, $W_1,W_2$ may be decomposed as,

$$W_1 = W + \chi + i\theta_1, \quad W_2 = W - \chi + i\theta_2,$$

(86)

where $W,\chi,\theta_1,\theta_2$ are four real functions of the $N$ bosonic co-ordinates. A real $W_+ = W$ is obtained for $\theta_1 = -\theta_2 \equiv \theta$. The operator $\hat{B}$ acts on the bosonic co-ordinates only. The bosonic co-ordinates $x_i$ and the momenta $p_i$ are assumed to be non-hermitian in $H_{\eta^b_{+}}$ for the type of operator $\hat{B}$ that will be considered in this article. A set of hermitian co-ordinates $X_i$ and momenta $P_i$ in $H_{\eta^b_{+}}$ may be introduced as follows,

$$X_i = \rho^{-1} x_i \rho, \quad P_i = \rho^{-1} p_i \rho, \quad \rho := \sqrt{\eta^b_{+}}.$$

(87)

The non-Dirac-hermitian operators $X_i, P_i$ trivially satisfy the basic canonical commutation relations $[X_i, P_j] = \delta_{ij}$. It may be noted that the similarity transformation (87) keeps the length in the momentum space as well as in the coordinate space invariant.

There are several choices for the operator $\hat{B}$ appearing in the metric $\eta^b_{+}$ in Eq. (85). For example, $\hat{B}$ may be chosen as a linear combination of the angular momentum operators,

$$\hat{B} := \sum_{i,j=1}^{N} c_{ij} \mathcal{L}_{ij}, \quad \mathcal{L}_{ij} := x_i p_j - x_j p_i, \quad c_{ij} = -c_{ji} \in R,$$

(88)
with restrictions on the co-efficients $c_{ij}$ such that all the eigenvalues of $\hat{B}$ are real. The reality condition on the eigenvalues of $\hat{B}$ ensures the positivity of $\eta^b_{\mathcal{D}}$.

A simple form of $\hat{B}$ is chosen in this article:

$$\hat{B} := L_{12} = x_1 p_2 - x_2 p_1.$$  \hspace{1cm} (89)

The co-ordinates $x_1, x_2$ and the momenta $p_1, p_2$ are not hermitian in $\mathcal{H}_{\eta^b_{\mathcal{D}}}$. It follows from the relation (87) that hermitian canonical conjugate operators in the Hilbert space $\mathcal{H}_{\eta^b_{\mathcal{D}}}$ have the following expressions:

$$X_1 = x_1 \cosh\delta + i x_2 \sinh\delta,$$
$$X_2 = -ix_1 \sinh\delta + x_2 \cosh\delta, \quad X_i = x_i \text{ for } i > 2,$$
$$P_1 = p_1 \cosh\delta + ip_2 \sinh\delta,$$
$$P_2 = -ip_1 \sinh\delta + p_2 \cosh\delta, \quad P_i = p_i \text{ for } i > 2.$$  \hspace{1cm} (90)

The operator $L_{12} = X_1 P_2 - X_2 P_1 = L_{12}$ is hermitian both in $\mathcal{H}_{\mathcal{D}}$ and $\mathcal{H}_{\eta^b_{\mathcal{D}}}$, ensuring a positive-definite $\eta^b_{\mathcal{D}}$.

A pseudo-hermitian Hamiltonian may be introduced as follows:

$$\mathcal{H} = \sum_{i=1}^{N} \Pi_i^2 + V(X), \quad \Pi_i := P_i + iW_{-i}, \quad W_{-i} = \frac{\partial W}{\partial X_i},$$  \hspace{1cm} (91)

where $V(X)$ is a function of the co-ordinates $X_1, \ldots, X_N$ and is hermitian in $\mathcal{H}_{\eta^b_{\mathcal{D}}}$. The real function $\theta$ appearing in $\Pi_i$ via $W_{-i}$ can always be rotated away by using the unitary operator $U := e^{-i\theta}$. Without loss of any generality, $\theta$ is chosen as zero and the generalized momentum operators $\Pi_i$ now read,

$$\Pi_i = P_i + i\chi_i, \quad \chi_i := \frac{\partial \chi}{\partial X_i}.$$  \hspace{1cm} (92)

The operators $\Pi_i$ contain imaginary gauge potentials $\chi_i$. Physical systems with imaginary gauge potentials have a wide range of applicability. The Hamiltonian $\mathcal{H}$ is non-hermitian in $\mathcal{H}_{\mathcal{D}}$ and hermitian in $\mathcal{H}_{\eta^b_{\mathcal{D}}}$. An anti-linear $\mathcal{PT}$ transformation for the bosonic coordinates may be introduced as follows:

$$\mathcal{P} : x_1 \leftrightarrow x_2, \quad p_1 \leftrightarrow p_2, \quad (x_i, p_i) \rightarrow (x_i, p_i) \forall i \geq 2;$$
$$\mathcal{T} : i \rightarrow -i, \quad x_i \rightarrow x_i, \quad p_i \rightarrow -p_i;$$
$$\mathcal{PT} : X_1 \leftrightarrow X_2, \quad P_1 \leftrightarrow -P_2, \quad (X_i, P_i) \rightarrow (X_i, -P_i) \forall i \geq 2;$$
$$\mathcal{PT} : \Pi_1 \leftrightarrow -\Pi_2, \quad \Pi_i \rightarrow -\Pi_i \forall i \geq 2.$$  \hspace{1cm} (93)

where the real function $\chi$ is assumed to be invariant under the discrete transformation $\mathcal{P}$. The operators $P_1^2 (\Pi_1^2)$ or $P_2^2 (\Pi_2^2)$ are not $\mathcal{PT}$-symmetric individually.
However, the combinations $P_1^2 + P_2^2$ and $\Pi_1^2 + \Pi_2^2$ are always $\mathcal{PT}$-symmetric. The Hamiltonian $\mathcal{H}$ is invariant under $\mathcal{PT}$ transformation provided the real potential $V$ remains invariant under the transformation $X_1 \leftrightarrow X_2$.

5.1.2 Pseudo-hermitian Realization of Clifford Algebra

The generators (5) of $O(2N)$ may be used to obtain a multi-parameter dependent pseudo-hermitian realization of the Clifford algebra.\(^8\) The metric $\eta_+^f$ in the Hilbert space $\mathcal{H}_{\eta_+^f}$ is chosen as,

$$\eta_+^f := \prod_{i=1}^{N} e^{2\gamma_i J_{N+i} \gamma_i} = \prod_{i=1}^{N} e^{-2\gamma_i \psi_i}.$$  

$$\rho_+^f := \sqrt{\eta_+^f} = \prod_{i=1}^{N} e^{\gamma_i J_{N+i} \gamma_i} = \prod_{i=1}^{N} e^{-\gamma_i \psi_i} \quad \gamma_i \in \mathbb{R} \forall i. \quad (94)$$

The ordering of the generators $J_{N+i}$ is not required in Eq.(94), since the commutators $[J_{N+i}, J_{N+j}] = 0$ for any $i$ and $j$. A set of elements $\Gamma_p$ of the real Clifford algebra is introduced as follows,

$$\Gamma_p := (\rho_+^f)^{-1} \xi_p \rho_+^f, \quad (95)$$

implying the following expressions:

$$\Gamma_i = \xi_i \cosh\gamma_i + i\xi_{N+i} \sinh\gamma_i,$$

$$\Gamma_{N+i} = -i\xi_i \sinh\gamma_i + \xi_{N+i} \cosh\gamma_i, \quad (96)$$

which are hermitian in $\mathcal{H}_{\eta_+^f}$. The analog of $\xi_{2N+1}$ in Eq. is denoted as $\Gamma_{2N+1}$,

$$\Gamma_{2N+1} := (-i)^N \Gamma_1 \Gamma_2 \ldots \Gamma_{2N-1} \Gamma_{2N} = \xi_{2N+1}, \quad (97)$$

which anti-commutes with all the $\Gamma_p/\xi_p$’s and squares to unity. The element $\Gamma_{2N+1}$ facilitates a pseudo-hermitian realization of the generators of the group $O(2N+1)$. The readers are referred to Ref. for a detail discussion on other aspects of pseudo-hermitian realization of Clifford algebra.

The fermionic operators $\Psi_i$’s and their adjoints $\Psi^\dagger_i$ in $\mathcal{H}_{\eta_+^f}$,

$$\Psi_i := \frac{1}{2} (\Gamma_i - i\Gamma_{N+i}) = e^{-\gamma_i} \psi_i,$$

$$\Psi^\dagger_i := \frac{1}{2} (\Gamma_i + i\Gamma_{N+i}) = e^{\gamma_i} \psi^\dagger_i, \quad (98)$$

satisfy the basic canonical anti-commutation relations.

$$\{\Psi_i, \Psi_j\} = 0 = \{\Psi^\dagger_i, \Psi^\dagger_j\}, \quad \{\Psi_i, \Psi^\dagger_j\} = \delta_{ij}. \quad (99)$$

31
The total fermion number operator $N_f$ has identical expressions,
\[
N_f = \sum_{i=1}^{N} N \psi_i^\dagger \psi_i = \sum_{i=1}^{N} \Psi_i^\dagger \Psi_i,
\]
(100)
in $\mathcal{H}_D$ as well as in $\mathcal{H}_{\eta f}$. The relation between an eigenstate $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D}$ of $N_f$ in $\mathcal{H}_D$, to the corresponding state $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}}$ in the Hilbert space $\mathcal{H}_{\eta f}$ is determined as,
\[
|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}} = \prod_{k=1}^{N} e^{\gamma_k f_k} |n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D}, \quad n_i = 0, 1 \forall i.
\]
(101)
The $2^N$ states $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}}$ form a complete set of orthonormal states in $\mathcal{H}_{\eta f}$, while $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D}$ constitute a complete set of orthonormal states in $\mathcal{H}_D$. The action of $\Psi_i (\Psi_i^\dagger)$ on $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}}$ is identical to that of $\psi_i (\psi_i^\dagger)$ on $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D}$:
\[
\Psi_i |n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}} = 0, \quad \text{if } n_i = 0,
\]
\[
\quad = |n_1, \ldots, 0, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}}, \quad \text{if } n_i = 1,
\]
\[
\Psi_i^\dagger |n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}} = 0, \quad \text{if } n_i = 1,
\]
\[
\quad = |n_1, \ldots, 1, \ldots, n_N\rangle_{\mathcal{H}_{\eta f}}, \quad \text{if } n_i = 0.
\]
(102)
The pseudo-hermitian odd elements $\Gamma_i$ or $\Psi_i, \Psi_i^\dagger$ is used to construct pseudo-hermitian supersymmetric quantum systems. The fermionic permutation operator $\tilde{K}_{ij}$,
\[
\tilde{K}_{ij} := \frac{1}{2} \left[ \Psi_i^\dagger - \Psi_i, \Psi_j - \Psi_j^\dagger \right] = 1 - \left( \Psi_i - \Psi_j \right) \left( \Psi_i^\dagger - \Psi_j^\dagger \right) = K_{ij},
\]
(103)
is hermitian in $\mathcal{H}_D$ as well as in $\mathcal{H}_{\eta f}$. The metric $\eta_{\eta f}^f$ is invariant under the action of $\tilde{K}_{ij}$. As in the case of bosonic coordinates in $\mathcal{H}_{\eta b}$, an anti-linear $\mathcal{P}\mathcal{T}$ transformation for the elements $\xi_p$ may be introduced as follows:
\[
\mathcal{T} : \quad i \rightarrow -i, \quad \xi_p \rightarrow \xi_p \forall p;
\]
\[
\mathcal{P} : \quad \xi_i \rightarrow \tilde{\xi}_i = \xi_i \cos \beta + \xi_{N+i} \sin \beta,
\]
\[
\xi_{N+i} \rightarrow \tilde{\xi}_{N+i} = \xi_i \sin \beta - \xi_{N+i} \cos \beta, \quad 0 \leq \beta \leq 2\pi,
\]
(104)
where $\beta$ appears as a phase which may be fixed at some specific value depending on the physical requirements. The action of the $\mathcal{P}\mathcal{T}$ transformation on the
fermionic variables $\psi_i$ is as follows,

$$
P : \psi_i \rightarrow e^{-i\beta} \psi_i, \quad \psi_i^\dagger \rightarrow e^{i\beta} \psi_i^\dagger,
$$

$$
T : \psi_i \rightarrow \psi_i^\dagger, \quad \psi_i^\dagger \rightarrow \psi_i,
$$

$$
PT : \psi_i \rightarrow e^{i\beta} \psi_i, \quad \psi_i^\dagger \rightarrow e^{-i\beta} \psi_i^\dagger.
$$

(105)

The supersymmetric Hamiltonian in Eq. (21) contains bi-linear terms of the form $\psi_i^\dagger \psi_j$ which are $\mathcal{PT}$-invariant for any $\beta$. The Hamiltonian (64) in the ‘atypical’ representation of $OSp(2|2)$ contains bi-linear terms of the form $\psi_i \psi_j, \psi_i^\dagger \psi_j^\dagger$ which are $\mathcal{PT}$-invariant only for $\beta = 0$. However, $\Gamma_{2N+1}$ appearing in the same Hamiltonian is $\mathcal{PT}$-invariant for any $\beta$. This can be checked by expressing $\Gamma_{2N+1}$ in terms of fermionic variables as,

$$
\Gamma_{2N+1} = (-1)^N \prod_{i=1}^{N} \left(2\psi_i^\dagger \psi_i - 1\right).
$$

(106)

Similarly, the fermionic exchange operator $\tilde{K}_{ij} = K_{ij}$ is invariant under $\mathcal{PT}$ for any $\beta$. This provides a framework for constructing $\mathcal{PT}$ symmetric non-Dirac-hermitian Hamiltonian.

5.2 Many-particle non-Dirac-hermitian Supersymmetry

The metric $\eta_+$ in the Hilbert space $\mathcal{H}_{\eta_+} := \mathcal{H}_{\eta_+^b} \otimes \mathcal{H}_{\eta_+^f}$ of the supersymmetric Hamiltonian is chosen as,

$$
\eta_+ := \eta_+^b \otimes \eta_+^f,
$$

(107)

where $\eta_+^b$ and $\eta_+^f$ are given by Eqs. (35) and (11), respectively. The supercharges are introduced as follows,

$$
\tilde{Q}_1 = \sum_{i=1}^{N} e^{-\gamma_i} \psi_i (P_i + iW_{1,i})
$$

$$
\tilde{Q}_2 = \sum_{i=1}^{N} e^{\gamma_i} \psi_i^\dagger (P_i - iW_{2,i}),
$$

$$
W_{1,i} = \frac{\partial W_1}{\partial X_i}, \quad W_{2,i} = \frac{\partial W_2}{\partial X_i},
$$

(108)

leading to the the supersymmetric Hamiltonian,

$$
\tilde{H} := \frac{1}{2} \{ \tilde{Q}_1, \tilde{Q}_2 \}
$$

$$
= \frac{1}{2} \sum_{i=1}^{N} \left[ \Pi_i^2 + (W_{+,i})^2 \right] - W_{+,ii} + \sum_{i,j=1}^{N} e^{\gamma_i - \gamma_j} W_{+,ij} \psi_i^\dagger \psi_j,
$$

$$
W_{\pm,i} = \frac{1}{2} (W_{1,i} \pm W_{2,i}), \quad W_{+,ij} = \frac{\partial^2 W_+}{\partial X_i \partial X_j}.
$$

(109)
The last term in $\tilde{H}$ is manifestly non-hermitian in $H_D$. The non-hermiticity of the generalized momentum operators in $H_D$ appears due to the presence of imaginary gauge potentials $\chi_i$. It is worth re-emphasizing that imaginary gauge potentials appear in the study of diverse branches of physics including metal-insulator transitions or depinning of flux lines from extended defects in type-II superconductors and unzipping of DNA. The purely bosonic potentials containing in the second and the third terms are functions of the coordinates $X_i$ and are in general non-hermitian in $H_D$. The Hamiltonian $\tilde{H}$ is hermitian in $H_{\eta+}$, provided the complex functions $W_1, W_2$ are taken to be of the form (86) with $\theta_1 = -\theta_2 \equiv \theta$. Finally, $\theta$ can gauged away from the Hamiltonian through a unitary transformation. The hermiticity of $\tilde{H}$ in $H_{\eta+}$ may be checked by re-expressing it as,

$$\tilde{H} = \frac{1}{2} \sum_{i=1}^{N} \left[ \Pi_i^2 + (W_i)^2 - W_{ii} \right] + \sum_{i,j=1}^{N} W_{ij} \Psi_i^\dagger \Psi_j, \quad (110)$$

It may be noted that the non-Dirac-hermitian operators $\Pi_i = p_i + i\chi_i$ and non-Dirac-hermitian functions $W_i, W_{ij}$ are hermitian in $H_{\eta+}$. The Hamiltonian $\tilde{H}$ is isospectral with the Dirac-hermitian Hamiltonian $H$ in Eq. (21),

$$H = (U\rho) \tilde{H} (U\rho^{-1}). \quad (111)$$

An exactly solved non-Dirac-hermitian quantum system $\tilde{H}$ may be constructed for the choice of $W$ for which exactly solvable many-particle supersymmetric quantum systems $H$ is known. A set of orthonormal eigenfunctions $\chi_n$ of $\tilde{H}$ in $H_{\eta+}$ may be constructed from the orthonormal eigenfunctions $\Phi_n$ of $H$ in $H_D$ by using the relation, $\psi_n = (U\rho)^{-1}\Phi_n$.

The main focus of this article is on systems with inverse-square interactions. The superpotential $W$ is chosen as,

$$W(X_1, X_2, \ldots, X_N) \equiv W_0 = -\ln G(X_1, X_2, \ldots, X_N) \quad (112)$$

where $G$ is a homogeneous function of degree $d$,

$$\sum_{i=1}^{N} X_i \frac{\partial G(X_1, \ldots, X_N)}{\partial X_i} = dG(X_1, \ldots, X_N). \quad (113)$$

The Hamiltonian $\tilde{H}_0$ corresponding to $W_0$, along with the dilatation operator $\tilde{D}$ and the conformal operator $\tilde{K}$,

$$\tilde{D} = \frac{1}{4} \sum_{i=1}^{N} (X_i \Pi_i + \Pi_i X_i), \quad \tilde{K} = \frac{1}{2} \sum_{i=1}^{N} X_i^2 = K, \quad (114)$$

The last term in $\tilde{H}$ is manifestly non-hermitian in $H_D$. The non-hermiticity of the generalized momentum operators in $H_D$ appears due to the presence of imaginary gauge potentials $\chi_i$. It is worth re-emphasizing that imaginary gauge potentials appear in the study of diverse branches of physics including metal-insulator transitions or depinning of flux lines from extended defects in type-II superconductors and unzipping of DNA. The purely bosonic potentials containing in the second and the third terms are functions of the coordinates $X_i$ and are in general non-hermitian in $H_D$. The Hamiltonian $\tilde{H}$ is hermitian in $H_{\eta+}$, provided the complex functions $W_1, W_2$ are taken to be of the form (86) with $\theta_1 = -\theta_2 \equiv \theta$. Finally, $\theta$ can gauged away from the Hamiltonian through a unitary transformation. The hermiticity of $\tilde{H}$ in $H_{\eta+}$ may be checked by re-expressing it as,

$$\tilde{H} = \frac{1}{2} \sum_{i=1}^{N} \left[ \Pi_i^2 + (W_i)^2 - W_{ii} \right] + \sum_{i,j=1}^{N} W_{ij} \Psi_i^\dagger \Psi_j, \quad (110)$$

It may be noted that the non-Dirac-hermitian operators $\Pi_i = p_i + i\chi_i$ and non-Dirac-hermitian functions $W_i, W_{ij}$ are hermitian in $H_{\eta+}$. The Hamiltonian $\tilde{H}$ is isospectral with the Dirac-hermitian Hamiltonian $H$ in Eq. (21),

$$H = (U\rho) \tilde{H} (U\rho^{-1}). \quad (111)$$

An exactly solved non-Dirac-hermitian quantum system $\tilde{H}$ may be constructed for the choice of $W$ for which exactly solvable many-particle supersymmetric quantum systems $H$ is known. A set of orthonormal eigenfunctions $\chi_n$ of $\tilde{H}$ in $H_{\eta+}$ may be constructed from the orthonormal eigenfunctions $\Phi_n$ of $H$ in $H_D$ by using the relation, $\psi_n = (U\rho)^{-1}\Phi_n$.

The main focus of this article is on systems with inverse-square interactions. The superpotential $W$ is chosen as,

$$W(X_1, X_2, \ldots, X_N) \equiv W_0 = -\ln G(X_1, X_2, \ldots, X_N) \quad (112)$$

where $G$ is a homogeneous function of degree $d$,

$$\sum_{i=1}^{N} X_i \frac{\partial G(X_1, \ldots, X_N)}{\partial X_i} = dG(X_1, \ldots, X_N). \quad (113)$$

The Hamiltonian $\tilde{H}_0$ corresponding to $W_0$, along with the dilatation operator $\tilde{D}$ and the conformal operator $\tilde{K}$,

$$\tilde{D} = \frac{1}{4} \sum_{i=1}^{N} (X_i \Pi_i + \Pi_i X_i), \quad \tilde{K} = \frac{1}{2} \sum_{i=1}^{N} X_i^2 = K, \quad (114)$$

The last term in $\tilde{H}$ is manifestly non-hermitian in $H_D$. The non-hermiticity of the generalized momentum operators in $H_D$ appears due to the presence of imaginary gauge potentials $\chi_i$. It is worth re-emphasizing that imaginary gauge potentials appear in the study of diverse branches of physics including metal-insulator transitions or depinning of flux lines from extended defects in type-II superconductors and unzipping of DNA. The purely bosonic potentials containing in the second and the third terms are functions of the coordinates $X_i$ and are in general non-hermitian in $H_D$. The Hamiltonian $\tilde{H}$ is hermitian in $H_{\eta+}$, provided the complex functions $W_1, W_2$ are taken to be of the form (86) with $\theta_1 = -\theta_2 \equiv \theta$. Finally, $\theta$ can gauged away from the Hamiltonian through a unitary transformation. The hermiticity of $\tilde{H}$ in $H_{\eta+}$ may be checked by re-expressing it as,

$$\tilde{H} = \frac{1}{2} \sum_{i=1}^{N} \left[ \Pi_i^2 + (W_i)^2 - W_{ii} \right] + \sum_{i,j=1}^{N} W_{ij} \Psi_i^\dagger \Psi_j, \quad (110)$$

It may be noted that the non-Dirac-hermitian operators $\Pi_i = p_i + i\chi_i$ and non-Dirac-hermitian functions $W_i, W_{ij}$ are hermitian in $H_{\eta+}$. The Hamiltonian $\tilde{H}$ is isospectral with the Dirac-hermitian Hamiltonian $H$ in Eq. (21),

$$H = (U\rho) \tilde{H} (U\rho^{-1}). \quad (111)$$

An exactly solved non-Dirac-hermitian quantum system $\tilde{H}$ may be constructed for the choice of $W$ for which exactly solvable many-particle supersymmetric quantum systems $H$ is known. A set of orthonormal eigenfunctions $\chi_n$ of $\tilde{H}$ in $H_{\eta+}$ may be constructed from the orthonormal eigenfunctions $\Phi_n$ of $H$ in $H_D$ by using the relation, $\psi_n = (U\rho)^{-1}\Phi_n$.

The main focus of this article is on systems with inverse-square interactions. The superpotential $W$ is chosen as,

$$W(X_1, X_2, \ldots, X_N) \equiv W_0 = -\ln G(X_1, X_2, \ldots, X_N) \quad (112)$$

where $G$ is a homogeneous function of degree $d$,

$$\sum_{i=1}^{N} X_i \frac{\partial G(X_1, \ldots, X_N)}{\partial X_i} = dG(X_1, \ldots, X_N). \quad (113)$$

The Hamiltonian $\tilde{H}_0$ corresponding to $W_0$, along with the dilatation operator $\tilde{D}$ and the conformal operator $\tilde{K}$,

$$\tilde{D} = \frac{1}{4} \sum_{i=1}^{N} (X_i \Pi_i + \Pi_i X_i), \quad \tilde{K} = \frac{1}{2} \sum_{i=1}^{N} X_i^2 = K, \quad (114)$$
satisfy the $O(2,1)$ algebra which appears as a bosonic sub-algebra of the $OSp(2|2)$ super-group. The ‘typical’ representation of the $OSp(2|2)$ is realized with the following definition of the operators:

$$
\tilde{S} = \frac{1}{2} \sum_{i=1}^{N} e^{-\gamma_i} \psi_i X_i, \quad \tilde{S}^\dagger = \frac{1}{2} \sum_{i=1}^{N} e^{\gamma_i} \psi_i^\dagger X_i, \quad \hat{Y} = Y. \quad (115)
$$

The generators $\tilde{H}_0$, $\tilde{D}$, $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{S}$ and $\tilde{S}^\dagger$ of $OSp(2|2)$ are hermitian in $H_{\eta}$. The Dirac-hermitian generators $K$ and $Y$ are also hermitian in $H_{\eta}$. Similarly, the ‘atypical’ representation of $OSp(2|2)$ in $H_{\eta}$ is obtained trivially by replacing $x_i \rightarrow X_i, p_i \rightarrow \Pi_i$ and $\xi \rightarrow \Gamma_i$ in the corresponding expressions of the generators in $H_D$. It may be recalled in this regard that $L_{12} = \hat{L}_{12}$ and $\xi_{2N+1} = \Gamma_{2N+1}$.

5.3 Examples: Rational Calogero Models

The pseudo-hermitian supersymmetric rational Calogero model is presented as an example for which the superpotential is chosen as,

$$
W = -\lambda \sum_{i<j=1}^{N} \ln (X_i - X_j) + \frac{1}{2} \sum_{i=1}^{N} X_i^2, \quad (116)
$$

and the corresponding non-Dirac-hermitian Hamiltonian reads,

$$
\tilde{H} = \frac{1}{2} \sum_{i=1}^{N} \Pi_i^2 + \frac{1}{2} \lambda (\lambda - 1) \sum_{i\neq j=1}^{N} X_{ij}^{-2} + \frac{1}{2} \sum_{i=1}^{N} x_i^2
$$

$$
+ \lambda \sum_{i\neq j=1}^{N} X_{ij}^{-2} \left( \psi_i^\dagger \psi_j - e^{\gamma_i - \gamma_j} \psi_i^\dagger \psi_j \right) + \sum_{i=1}^{N} \psi_i^\dagger \psi_i - \frac{N}{2} - \frac{\lambda}{2} N (N - 1),
$$

$$
X_{12} = (x_1 - x_2) \cosh \delta + i (x_1 + x_2) \sinh \delta, \quad X_{1j} = x_1 \cosh \delta + i x_2 \sinh \delta - x_j, \quad j > 2,
$$

$$
X_{2j} = -i x_1 \sinh \delta + x_2 \cosh \delta - x_j, \quad j > 2,
$$

$$
X_{ij} = x_i - x_j, \quad (i,j) > 2. \quad (117)
$$

The following differences between the system governed by $\tilde{H}$ and the standard rational Calogero Hamiltonian in Eq. (35) are to be noted. The many-body inverse-square interaction term in $\tilde{H}$ is neither invariant under translation nor singular for $x_1 = x_i, i > 1$ and $x_2 = x_i, i > 2$. The permutation symmetry (36) of $H$ is no longer a symmetry of $\tilde{H}$. However, the Hamiltonian $\tilde{H}$ is invariant under a combined $PT$ operation as defined in Eqs. (105) and (93). The Hamiltonian $\tilde{H}$ and the Dirac-hermitian rational $A_{N+1}$ Calogero model are related to
each other through a non-unitary similarity transformation. This implies that
these models are isospectral provided identical boundary conditions have been
used. However, there are no compelling reasons to solve these systems under
identical boundary conditions. The Hamiltonian $\hat{H}$ has $(N-3)(N-2)$ number
of less singular points compared to the standard Calogero model\cite{11} due to the
non-singular points at $x_1 = x_i, i > 1$ and $x_2 = x_i, i > 2$. Thus, the configura-
tion spaces of these two Hamiltonians are different allowing different boundary
conditions. It is to be seen whether a consistent description of $\hat{H}$, including an
entirely real spectrum, is allowed for any modified boundary condition or not.

6 Quantum Systems in Higher Dimensions

There are many higher dimensional generalizations of the rational Calogero
model for which infinitely many exact eigen states and eigen values can be ob-
tained analytically\cite{100,101,102}. The exact eigenstates owe their existence to
mainly the underlying $O(2,1)$ symmetry. However, not a single model belong-
ing to this class is exactly solved for a complete set of states. Nevertheless,
the study of these systems gives a better understanding of quantum systems
in higher dimensions. The Calogero-Marchioro model\cite{100} is one such example
which describes a $D > 1$ dimensional quantum system of particles interacting
with each other through two-body and three-body inverse-square interaction
terms. The importance of $D = 2$ dimensional Calogero-Marchioro model lies
in its relevance in the study of a host of different subjects, like normal ma-
trix model\cite{103,104,105,42}, two dimensional Bose system\cite{104}, quantum Hall
effect\cite{106}, quantum dot\cite{107}, extended superconformal symmetry\cite{64}. It is
known that the two dimensional Calogero-Marchioro model at some specific
value of the coupling constant describes the dynamics of a Gaussian ensemble
of normal matrices in the large $N$ limit\cite{103,51,105,108}. The low energy limit of
2+1 dimensional Yang-Mills theory, dimensionally reduced to 0+1 dimensions,
is described by the Gaussian action of normal matrices\cite{64}. Thus, the Calogero-
Marchioro model is also indirectly related to 2+1 dimensional Yang-Mills theory.
The supersymmetric version of the $D$ dimensional Calogero-Marchioro model is
presented in this section.

It has been suggested\cite{54} that the rational Calogero model with extended
$N = 4$ $SU(1,1|2)$ superconformal symmetry may describe the motion of a
test super-particle in the near-horizon geometry of 3+1 dimensional extremal
black holes. The first initiative to construct such a model was not completely
successful\cite{65}. In particular, the resulting Hamiltonian is $SU(1,1|2)$ supercon-
formal only for specific values of the strength of the inverse-square interaction. Several attempts have been made thereafter to construct models with $N = 4$ superconformal symmetry \[66, 67, 71, 72, 73, 74, 75, 76, 77\]. The development in this regard has been described in the review article \[78\], which also contains a few examples of generalized Calogero-type models in diverse dimensions. An example of $N = 4$ superconformal model has been constructed in Ref. \[77\], whose bosonic sector is not described by the standard rational Calogero model. In fact, the construction of a $N = 4$ superconformal Hamiltonian for arbitrary number of particles and the generic values of the coupling constant, which reduces to the standard Calogero or Calogero-Marchioro model in the purely bosonic sector, is still beyond the reach in $D \neq 2$ dimensions. It is shown that the $D = 2$ dimensional Calogero-Marchioro model can naturally be embedded into an extended $N = 4$ $SU(1, 1|2)$ superconformal Hamiltonian\[64\]. The construction of rational Calogero-Marchioro model with $SU(1, 1|2)$ superconformal symmetry is discussed in some detail.

### 6.1 Supersymmetric Calogero-Marchioro Model

The supercharge $Q_1$ and its conjugate $Q_1^\dagger$ are defined as,

$$Q_1 = \sum_{i,\mu} \psi_{i,\mu}^\dagger A_{i,\mu}, \quad Q_1^\dagger = \sum_{i,\mu} \psi_{i,\mu} A_{i,\mu}^\dagger,$$

$$A_{i,\mu} := p_{i,\mu} - i W_{i,\mu}, \quad A_{i,\mu}^\dagger := p_{i,\mu} + i W_{i,\mu},$$

$$p_{i,\mu} = -i \frac{\partial}{\partial x_{i,\mu}}, \quad W_{i,\mu} = \frac{\partial W}{\partial x_{i,\mu}}, \quad i = 1, \ldots, N, \quad \mu, \nu = 1, \ldots, D, \quad (118)$$

where $W$ is the superpotential and the $ND$ fermionic variables $\psi_{i,\mu}$’s satisfy the Clifford algebra,

$$\{\psi_{i,\mu}, \psi_{j,\nu}\} = 0 = \{\psi_{i,\mu}^\dagger, \psi_{j,\nu}^\dagger\}, \quad \{\psi_{i,\mu}, \psi_{j,\nu}^\dagger\} = \delta_{ij} \delta_{\mu,\nu}. \quad (119)$$

The superpotential is chosen as,

$$W \equiv W_0 = -g \sum_{i<j} \ln |\vec{r}_{ij}|, \quad \vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j, \quad (120)$$

which results in the following supersymmetric Hamiltonian,

$$H_0 = \frac{1}{2} \sum_{i,\mu} p_{i,\mu}^2 + \frac{g}{2} (g + D - 2) \sum_{i\neq j} \vec{r}_{ij}^{-2} + \frac{g^2}{2} \sum_{i\neq j\neq k} (\vec{r}_{ij} \cdot \vec{r}_{ik}) \vec{r}_{ij}^{-2} \vec{r}_{ik}^{-2}$$

$$+ g \sum_{i\neq j} \left( 2 (x_{i,\mu} - x_{j,\mu})^2 \vec{r}_{ij}^{-2} - 1 \right) \vec{r}_{ij}^{-2} \left( \psi_{i,\mu}^\dagger \psi_{i,\mu} - \psi_{i,\mu}^\dagger \psi_{j,\mu} \right)$$

$$+ 2g \sum_{i\neq j, \mu \neq \nu} (x_{i,\mu} - x_{j,\mu}) (x_{i,\nu} - x_{j,\nu}) \vec{r}_{ij}^{-4} \left( \psi_{i,\mu}^\dagger \psi_{i,\nu} - \psi_{i,\mu}^\dagger \psi_{j,\nu} \right) \quad (121)$$

37
The Hamiltonian $H_0$ along with the Dilatation operator $D$ and the conformal generator $K$,

$$D = -\frac{1}{4} \sum_{i,\mu} \{x_{i,\mu}, p_{i,\mu}\}, \quad K = \frac{1}{2} \sum_{i,\mu} x_{i,\mu}^2, \quad (122)$$

satisfy the $O(2,1)$ algebra given in Eq. (154).

The super-Hamiltonian $H_0$ does not have a normalizable ground-state. The quantum evolution of the system can instead be studied in terms of the operator $R$ or $H$:

$$H = R + B - T, \quad R = H_0 + K, \quad B = \frac{1}{2} \sum_{i,\mu} \left[ \psi_{i,\mu}^\dagger, \psi_{i,\mu} \right], \quad T = \frac{g}{2} N(N-1). \quad (123)$$

The zero-fermion sector of the supersymmetric Hamiltonian $H$ describes $D$ dimensional Calogero-Marchioro Hamiltonian. The supersymmetric rational $A_{N+1}$ Calogero model is obtained from (123) for $D = 1$. The supersymmetric ground state of $H$ is obtained in the region $g > 0$,

$$\Phi_0 = \phi_0 |0\rangle, \quad \phi_0 \equiv \prod_{i < J} |r_{ij}|^g e^{-\frac{1}{2} \sum_i r_i^2}, \quad (124)$$

where $|0\rangle$ is now the fermionic vacuum in $2^{ND}$ dimensional Fock space. A set of exact eigenstates is constructed below in the supersymmetric phase. The analysis in the supersymmetry-breaking phase ($g < 0$) is similar to the case of one dimensional supersymmetric rational Calogero model and is given in Ref. [64].

The complete $OSp(2|2)$ algebra is realized by the introduction of the following operators,

$$S_1 := \sum_{i,\mu} \psi_{i,\mu}^\dagger x_{i,\mu}, \quad S_1^\dagger := \sum_{i,\mu} \psi_{i,\mu} x_{i,\mu},$$

$$\mathcal{F}_1 = Q_1 - i S_1, \quad \mathcal{F}_2 = Q_1^\dagger - i S_1^\dagger,$$

$$\mathcal{F}_1^\dagger = Q_1^\dagger + i S_1^\dagger, \quad \mathcal{F}_2^\dagger = Q_1 + i S_1, \quad (125)$$

The supersymmetric Hamiltonian $H$ in Eq. (123) is re-expressed in terms of the operators $\mathcal{F}_1, \mathcal{F}_1^\dagger$ as, $H = \frac{1}{2} \{ \mathcal{F}_1, \mathcal{F}_1^\dagger \}$. The following algebra,

$$\mathcal{B}_2^\dagger := -\frac{1}{4} (\mathcal{F}_1^\dagger, \mathcal{F}_2^\dagger),$$

$$[H, \mathcal{B}_2^\dagger] = 2 \mathcal{B}_2^\dagger, \quad [H, \mathcal{F}_2^\dagger] = 2 \mathcal{F}_2^\dagger, \quad (126)$$

allows a construction of the excited states,

$$\Phi_{n,\nu} = \mathcal{B}_2^\nu \mathcal{F}_2^\nu \Phi_0, \quad (127)$$
with the energy eigen values $E_{n,\nu} = 2(n + \nu)$. The bosonic quantum number $n$ can take any non-negative integer values, while the fermionic quantum number $\nu = 0, 1$. The spectrum does not reduce to that of $D$ dimensional $N$ free superoscillators in the limit $g \to 0$. The set of exact eigenstates \[141\] is thus incomplete and the complete spectrum is not known.

6.2 Extended Superconformal Symmetry

The coefficients of the bosonic two-body and the thee-body interaction terms are identical for $D = 2$. Consequently, the supersymmetric Hamiltonian $H$ can be embedded into an extended $N = 4$ superconformal symmetry in $D = 2$ space dimensions. The general form of the superpotential in $D = 2$ that gives rise to Hamiltonian with extended superconformal symmetry may be expressed as \[64\],

$$W_0 = -\ln G, \quad G = f(z_1, z_2, \ldots, z_N) \, g(z^*_1, z^*_2, \ldots, z^*_N),$$

where $G$ is a homogeneous function. The homogeneity condition on $G$ implies that the (anti-)holomorphic function $(g) f$ should also be homogeneous. The superpotentials of the Calogero-Marchioro model and a nearest-neighbor variant of this model\[109\] in $D = 2$ satisfy the above criteria. The rest of the discussions is based on two dimensional superconformal Calogero-Marchioro model\[64\].

The Hamiltonian $H_0$ has an internal $SU(2)$ symmetry. The generators of the $SU(2)$ are the even operator $B$ in Eq. (123) and the operators,

$$Y = \frac{1}{2} \sum_i \epsilon_{\mu\nu} \psi_{i,\mu} \psi_{i,\nu}, \quad Y^\dagger = -\frac{1}{2} \sum_i \epsilon_{\mu\nu} \psi_{i,\mu}^\dagger \psi_{i,\nu}^\dagger,$$

satisfying the algebra

$$[Y, Y^\dagger] = -B, \quad [B, Y] = -2Y, \quad [B, Y^\dagger] = 2Y^\dagger.$$

The repeated indices of the Levi-Civita pseudo-tensor $\epsilon_{\mu\nu}$ are always summed over. The $SU(1, 1, |2)$ requires the introduction of the following odd operators:

$$Q_2 = \sum_i \epsilon_{\mu,\nu,\mu,\nu} \psi_{i,\mu} \psi_{i,\nu}, \quad Q_2^\dagger = \sum_i \epsilon_{\mu,\nu,\mu,\nu} \psi_{i,\mu}^\dagger \psi_{i,\nu}^\dagger,$$

$$S_2 = \sum_i \epsilon_{\mu,\nu,\mu,\nu} x_{i,\mu}, \quad S_2^\dagger = \sum_i \epsilon_{\mu,\nu,\mu,\nu} x_{i,\mu}^\dagger.$$

Defining two new odd operators in terms of $Q_2, S_2$ and their adjoints,

$$\tilde{F}_1 := Q_2 - iS_2, \quad \tilde{F}_2 := Q_2^\dagger - iS_2^\dagger,$$
the following algebra holds,

\[
\frac{1}{2}\{\tilde{F}_2, \tilde{F}_2^\dagger\} = H + 2T, \quad \frac{1}{2}\{F_1, \tilde{F}_2^\dagger\} = -\frac{1}{2}\{F_1^\dagger, \tilde{F}_2\} = -i\tilde{J}, \quad \frac{1}{2}\{F_2, \tilde{F}_2^\dagger\} = \tilde{H},
\]

\[
\frac{1}{2}\{\tilde{F}_1, \tilde{F}_1^\dagger\} = \tilde{H} - 2T, \quad \frac{1}{2}\{F_2, \tilde{F}_1^\dagger\} = -\frac{1}{2}\{F_1^\dagger, \tilde{F}_1\} = -i\tilde{J}, \quad \tilde{H} := H - B + 2T
\]

(133)

The operator angular momentum operator \(\tilde{J}\) appearing above is defined as,

\[
\tilde{J} = \sum_i \epsilon_{\mu\nu} \left( x_{i,\nu} p_{i,\mu} + i\psi_{i,\mu}^\dagger\psi_{i,\nu} \right).
\]

(134)

All other non-vanishing anti-commutators are,

\[
-\frac{1}{2}\{F_1, \tilde{F}_1^\dagger\} = \frac{1}{2}\{\tilde{F}_2, F_1^\dagger\} = 2Y^\dagger,
\]

\[
-\frac{1}{2}\{\tilde{F}_1, F_1^\dagger\} = \frac{1}{2}\{F_2, \tilde{F}_1^\dagger\} = 2Y,
\]

\[
\frac{1}{4}\{F_1, F_2\} = \frac{1}{4}\{\tilde{F}_1, \tilde{F}_2\} = -B_2.
\]

(135)

The evolution can be described either in terms of the Hamiltonian \(H\) or its dual \(\tilde{H}\). The dual Hamiltonian \(\tilde{H}\) is used to study the spectrum in supersymmetry-breaking phase \(g < 0\). The algebra in Eq. (133) is not in a diagonal form because of the presence of \(\tilde{J}\). However, a diagonal form of the algebra can be obtained by introducing two new supercharges as a linear combination of \(F_1\) and \(\tilde{F}_2\).

7 Systems with Internal Degrees of Freedom

The Calogero models with internal degrees of freedom have been studied previously in the literature. Such models naturally appear in the reductions of various matrix models to many-particle quantum systems. A supersymmetric Calogero-type model with internal \(U(2)\) degrees of freedom has been obtained via the reductions of certain gauged matrix models. The relevant discussions in this regard have been included in the review article. Calogero models with \(k\) numbers of internal degrees of freedom may also be introduced directly in terms of permutation operators constructed out of the generators of \(SU(k)\). A generalization of the Calogero-Sutherland models, where the indices corresponding to the particles internal degrees of freedom form a representation of the \(gl(n|m)\) graded Lie algebra, has also been considered in the literature. These models are integrable and have many interesting
properties. These models are also termed ‘supersymmetric’[26], because of the presence of the ‘graded permutation operator’ in the Hamiltonian. However, the full superalgebra (1) is realized for the corresponding quantum Hamiltonians only for the simplest case of $gl(1|1)$. Thus, discussions of this class of models with $n \neq 1, m \neq 1$ are beyond the scope of this paper.

The supersymmetric Hamiltonian $H$ in Eq. (21) has a block-diagonal structure in the fermionic representation, thereby, giving internal structures to the bosonic particles, which may not be always interpreted in terms of spin degrees of freedom. It may be recalled that the fermionic exchange operator appearing in the supersymmetric rational Calogero Hamiltonian realizes a tensor representation of symmetric group $S_N$ of fermionic operators[51]. A class of supersymmetric many-particle Hamiltonians is constructed in this section which can be interpreted as $N$ bosonic particles with internal spin degrees of freedom.

Several spin chain Hamiltonians, including the celebrated Haldane-Shastry model[25, 26], may be obtained from Calogero models with internal degrees of freedom in the strong interaction limit, known as ‘freezing limit’[14]. The supersymmetric Hamiltonians considered in this section reduce to $XY$ model on a non-uniform lattice in the ‘freezing limit’. The ‘freezing limit’ for a supersymmetric system may be taken in the following way. The supersymmetric Hamiltonian (21) can be re-written as,

$$H = \frac{1}{2} \sum_i (\psi_i^\dagger \psi_i + W_i^2) + \frac{1}{2} \sum_{i,j} W_{ij} \left( \psi_i^\dagger \psi_j - \psi_i \psi_j^\dagger \right). \quad (136)$$

The bosonic and the fermionic parts of the supersymmetric Hamiltonian decouple from each other for a superpotential satisfying $W_{ij} = constant \ \forall \ i, j$. This implies that the superpotential is a quadratic form of the bosonic co-ordinates. The Hamiltonian of superoscillators is one such example. In general, the bosonic and the fermionic degrees of freedom can not be decoupled for any other choices of $W$. However, The fermionic part can be decoupled from the parent Hamiltonian in the ‘freezing limit’[14]. In particular, the superpotential $W$ is taken to be proportional to an overall coupling constant $\lambda$. The coefficient of the bosonic potential term $W_i^2$ becomes $\lambda^2$, while it is $\lambda$ for the fermionic part of the Hamiltonian. An effective Hamiltonian in the strong interaction limit $\lambda \to \infty$ may be obtained by first multiplying $H$ with $\lambda^{-2}$ and then taking $\lambda \to \infty$. The leading relevant term in this limit is,

$$H \equiv \frac{1}{2\lambda} \sum_{i,j} W_{ij} \left( \psi_i^\dagger \psi_j - \psi_i \psi_j^\dagger \right) + O(\lambda^{-2}), \quad (137)$$

where the bosonic coordinates in $W_{ij}$ take the value of their classical minimum equilibrium configurations, $W_i = 0$. The non-dynamical term $\sum_i W_i^2$ vanishes.
identically for the classical minimum equilibrium configurations. The Hamiltonian $H$ in Eq. (137) contains only bilinear terms in the fermionic operators and is always diagonalizable.

### 7.1 Supersymmetry & XY model

The superpotential $W$ is chosen such that,

$$W_{ij} = \delta_{ij} g_i(x_1, x_2, \ldots, x_N) + \delta_{i,j+1} h_i(x_1, x_2, \ldots, x_N) + \delta_{i,j-1} h_i(x_1, x_2, \ldots, x_N),$$

where $h_i$’s and $g_i$’s are arbitrary functions of the bosonic coordinates. The supersymmetric Hamiltonian (21) now reads,

$$H = \frac{1}{2} \sum_i \left( p_i^2 + W_i^2 \right) + \frac{1}{2} \sum_i \left[ g_i (2n_i - 1) + 2h_i \left( \psi_i^\dagger \psi_{i+1} - \psi_i \psi_{i+1}^\dagger \right) \right].$$

The Hamiltonian $H$ can be re-written in terms of spin degrees freedom by using the Jordan-Wigner transformation (See Appendix V) with the periodic boundary conditions:

$$H = \frac{1}{2} \sum_i \left( p_i^2 + W_i^2 \right) + \frac{1}{2} \sum_i \left[ g_i \sigma_i^z + h_i \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) \right].$$

The use of the Jordan-Wigner transformation and hence, the interpretation of $H$ as a system of spin-$\frac{1}{2}$ particles interacting with each other through inverse-square interactions fails, if the fermion-fermion interaction is beyond the next-neighbour. The situation is saved for the particular choice of the superpotential in Eq. (138). The supersymmetric Hamiltonian $H$, in general, describes an $N$ particle system with both kinematic and internal spin degrees of freedom. Note that both $g_i$’s and $h_i$’s depend on the bosonic coordinates. In the FL, as described above, it is possible to decouple the spin degrees of freedom from the coordinate degrees of freedom. For such cases, solving the supersymmetric Hamiltonian $H$, one would in fact also be able to solve the corresponding spin chain problem.

### 7.2 Systems related to short-range Dyson models

A quantum Hamiltonian with nearest-neighbour and next-nearest-neighbour inverse-square interactions among the particles was introduced and studied in [109]. The model has relevance in the context of random banded matrix theory.
describing short-range Dyson model\textsuperscript{[111]}. In particular, the norm of the ground-state of this many-body system can be identified with the joint-probability distribution function of the Gaussian random banded matrix theory. Consequently, different correlation functions of this Hamiltonian can be calculated exactly from the known results of random banded matrix theory. The model has also relevance in the study of nearest-neighbour spin chain models\textsuperscript{[112, 113, 114, 115]}.

A supersymmetric version of this nearest-neighbour analog of rational $A_{N+1}$ Calogero Hamiltonian is constructed in this section. The readers are referred to Ref. \textsuperscript{[112]} for discussions on a nearest-neighbour analog of rational $BC_{N+1}$ Calogero Hamiltonian.

The superpotential is chosen as,

$$W = -\lambda \sum_{i=1}^{N} \ln(x_i - x_{i+1}) + \frac{\omega}{2} \sum_{i=1}^{N} x_i^2, \quad x_{N+1} = x_1.$$  \hfill (141)

leading to the following expressions for $g_i$ and $h_i$,

$$h_i = -\lambda(x_i - x_{i+1})^{-2}, \quad g_i = \omega - (h_i + h_{i-1}).$$  \hfill (142)

The Hamiltonian $H$ for the superpotential now reads,

$$H = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\lambda^2}{2} \sum_{i=1}^{N} \left[ 2(x_i - x_{i+1})^{-2} - (x_{i-1} - x_i)^{-1}(x_i - x_{i+1})^{-1} \right]$$

$$+ \lambda \sum_{i=1}^{N} \left[ (x_i - x_{i+1})^{-2} \left( n_i + n_{i+1} - 1 - \psi_i \psi_{i+1} + \psi_i \psi_{i+1}^\dagger \right) \right]$$

$$+ \frac{1}{2} \omega^2 \sum_{i=1}^{N} x_i^2 + \omega \sum_{i=1}^{N} n_i - \frac{N \omega}{2} - \lambda \omega N,$$  \hfill (143)

with the periodic boundary conditions on the fermionic variables: $\psi_{N+i} = \psi_i$. Both nearest-neighbour and next-nearest-neighbour interaction terms are present in the bosonic many-body potential. However, only nearest-neighbour interaction terms are present for the fermions. This allows a mapping of the bilinear terms involving fermionic operators in terms of spin-spin interaction terms. The third term with the coefficient $\lambda$ contains the XY Hamiltonian in terms of fermionic operators. Thus, particles in this model are also having internal spin degrees of freedom.

The Hamiltonian $H$ reduces to the supersymmetric rational Calogero model for $N = 3$ due to the periodic boundary conditions imposed on the bosonic and the fermionic coordinates. The system is not exactly solved for a complete set of states for $N > 3$. However, infinitely many exact eigenstates can be constructed
analytically. An algebraic construction of these states in the supersymmetric phase is described here. The ground state of $H$ in the supersymmetric phase ($\lambda > 0$) is given by,

$$\Phi_0 = \phi_0 |0\rangle, \quad \phi_0 = \prod_i (x_i - x_{i+1})^\lambda e^{-\frac{\omega}{2} \sum_i x_i^2}.$$ (144)

The Hamiltonian $H$ can be mapped to a system of free superoscillators satisfying the Eq. (70) with the operator $S$ having the following expression:

$$S = \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \lambda \sum_i (x_i - x_{i+1})^{-1} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)$$

$$- \lambda \sum_i (x_i - x_{i+1})^{-2} \left[ \left( \psi_i^\dagger \psi_i + \psi_{i+1}^\dagger \psi_{i+1} \right) - \left( \psi_i^\dagger \psi_{i+1} + \psi_i \psi_{i+1}^\dagger \right) \right].$$ (145)

The construction of a partial set of eigen states from the free superoscillator basis is given in Ref. [112]. Based on this analysis, the following set of operators are introduced,

$$b_i^- = ip_i = \frac{\partial}{\partial x_i}, \quad b_i^+ = 2\omega x_i$$

$$B_n^- = \sum_{i=1}^N T^{-n} b_i^- T, \quad B_n^+ = \sum_{i=1}^N T^{-n} b_i^+ T, \quad n = 1, 2, 3,$$

$$F_m^- = \sum_i T^{-m} \psi_i b_i^- T, \quad F_m^+ = \sum_i T^{-m} \psi_i^\dagger b_i^+ T, \quad m = 1, 2, 3, 4,$$

$$q_n^- = T^{-1} \sum_i \psi_i^\dagger b_i^- T, \quad q_n^+ = T^{-1} \sum_i \psi_i b_i^+ T, \quad n = 1, 2, 3$$ (146)

which satisfy an algebra that reduces to (76) for $\omega = 1$. Unlike the case of rational Calogero model, there are only three bosonic annihilation(creation) operators $B_n^-$ ($B_n^+$) and four fermionic annihilation(creation) operators $F_m^- (F_m^+)$. The operators $B_n^+ (F_m^+)$ lead to singular wave-function for $n \geq 4$ ($m \geq 5$) and are not acceptable. The eigenstates can now be created in an algebraic manner by using the above relations. In particular,

$$\chi_{n_1...n_3 \nu_1...\nu_4} = \prod_{k=1}^3 B_{k}^{+n_k} F_{k}^{+\nu_k} F_{4}^{+\nu_4} \Phi_0,$$ (147)

is the eigenfunction with the eigen-value,

$$E = \omega \left( \sum_{k=1}^3 k(n_k + \nu_k) + 4\nu_4 \right).$$ (148)
The bosonic quantum numbers \( n_k \)'s are non-negative integers, while the fermionic quantum numbers \( \nu_k \)'s are either 0 or 1. It appears that the eigen spectrum is not complete.

The Hamiltonian contains two independent parameters \( \lambda \) and \( \omega \). The ‘freezing limit’ is obtained by first scaling \( \omega \) as \( \lambda \omega \) in \( H \) and then taking the limit \( \lambda \to \infty \) of the operator \( \lambda^2 H \). The spin degrees of freedom decouple completely from the kinematic ones in this strong interaction limit with all the particles frozen at their classical equilibrium configurations,

\[
W_i = \lambda \omega x_i + \lambda \left[ (x_{i-1} - x_i)^{-1} - (x_i - x_{i+1})^{-1} \right] = 0.
\] (149)

The XY Hamiltonian on a non-uniform lattice appears as a leading term in this limit,

\[
H_{XY} = \sum_i \left[ (\bar{x}_i - \bar{x}_{i+1})^{-2} \left( \frac{1}{2} (n_i + n_{i+1}) - \psi_i^\dagger \psi_{i+1} + \psi_i \psi_{i+1}^\dagger \right) + \frac{\omega}{2} n_i \right],
\] (150)

where \( \bar{x}_i \)'s are determined from (149). The general solution of (149) is not known. It is argued in Ref. [112] that the equilibrium configurations of the particles necessarily constitute a non-uniform lattice.

### 8 Omitted topics, Open Arena and Summary

#### 8.1 Omitted Topics

(i) \( SU(1,1|2) \) Supersymmetric Models in One Dimension:

The motion of a test super-particle in the near-horizon limit of \( 3+1 \) dimensional extremal Reissner-Nordström black hole is suggested to be described by the rational Calogero model with \( \mathcal{N} = 4 \) \( SU(1,1|2) \) superconformal symmetry [54]. A \( 2+1 \) dimensional many-particle system with extended \( SU(1,1|2) \) superconformal symmetry is given in Ref. [64] and described in Section 6 of this article. However, such a construction in space dimensions \( D \neq 2 \) meets with difficulty [65, 66, 67, 71, 72, 73, 74, 75, 76]. The rational \( A_{N+1} \) Calogero model with \( SU(1,1|2) \) superconformal symmetry is presented in Ref. [65] for specific values of the strength of the inverse-square interaction. The construction is based on \( N \) bosonic and \( 4N \) fermionic degrees of freedom and within this approach, the \( SU(1,1|2) \) conformal symmetry is absent for the generic values of the strength of the inverse-square interaction [65]. Several other attempts have been made to construct \( \mathcal{N} = 4 \) superconformal models [66, 67, 71, 72, 73, 74, 77]. An
important result in this context is described in Ref. [77] and an account of the whole development in this regard is given in the review article [78]. However, an explicit construction of the Hamiltonian for arbitrary number of particles and for the generic values of the coupling constant(s), which contains the standard Calogero model in its purely bosonic sector, is still beyond the reach.

(ii) **Matrix Models, Collective Field theory etc.**:

An appropriate reduction of hermitian matrix model leads to rational $A_{N+1}$ Calogero model [14]. Similarly, a supersymmetric version of the Marinari-Parisi model can be reduced to the supersymmetric rational $A_{N+1}$ Calogero model [56]. Matrix model description of Calogero model appears in many other context, for example, in Refs. [57, 58, 59, 60]. The matrix models related to Calogero Hamiltonian have also been studied in the large $N$ limit through continuum collective theory techniques [61, 62, 63]. Developments in this regard are described in the review article [63].

(iii) **Superpolynomial**:

The zero fermion sector of the operator $H_1$ in Eq. (68) gives a realization of the generalized Hermite polynomials. The orthogonal eigenfunctions of the operator $H_1$ for any $N_f$ is known as generalized hermitian superpolynomial [81]. An algebraic construction of superpolynomials related to supersymmetric Calogero-Sutherland systems are described in Refs. [79, 80, 81, 82, 83]. The integrable structure of these supersymmetric systems are also explored.

8.2 **Open Arena**

(i) **Self-adjoint Extensions**:

The self-adjoint extension of rational Calogero model has been studied in Refs. [38, 39]. The scale invariance of the rational Calogero model with purely inverse-square interaction gets broken at the quantum level due to the imposition of modified boundary conditions and the Hamiltonian admits bound states. Similarly, this new quantization scheme for the rational Calogero model with the harmonic confinement term leads to non-equispaced energy levels with a negative energy bound state [38]. The supersymmetric Hamiltonian [35] reduces to the Calogero model in the zero fermion sector. The results stated above are thus equally valid in the zero fermion sector. A systematic study on self-adjoint
extensions of the supersymmetric rational Calogero model for arbitrary $N_f$ is desirable.

(ii) ‘Atypical’ Calogero Model:

The construction of the rational Calogero Hamiltonian with $OSp(2|2)$ supersymmetry is not unique. The standard Hamiltonian \[ \text{(35)} \text{ or } \text{(164)} \] corresponds to the ‘typical’ representation. A new Hamiltonian corresponding to the ‘atypical’ representation has been introduced in Eq. \[ \text{(64)}. \] The complete spectrum and the integrable structure of this Hamiltonian is not known. A set of eigenvalues and the eigenfunctions of this Hamiltonian corresponding to the underlying $OSp(2|2)$ symmetry may be obtained analytically. The quadratic and the cubic Casimir operators being zero in the ‘atypical’ representation, these operators can not be used to characterize the spectra. Further, as in the case of ‘typical’ Calogero model, the spectrum generating algebra is expected to be larger than $OSp(2|2)$. This expectation stems from the fact that superoscillator Hamiltonian in the ‘atypical’ representation of $OSp(2|2)$ contains a spin-orbit interaction term and its spectrum is different from the superoscillator model corresponding to ‘typical’ representation. It may be noted that a non-trivial mixing of angular momentum operators $L_{ij}$ and $J_{ij}$ also appears in the ‘atypical’ Calogero Hamiltonian \[ \text{(35)}. \] A study on the exact solvability of the ‘atypical’ model may reveal some of its hidden surprises.

(iii) Generalized Calogero-type Models:

There are many supersymmetric systems with inverse-square interactions for which only a part of the complete spectrum can be obtained analytically. Two such physically relevant systems are presented in this review article in sections 6 and 7. Any supersymmetric Hamiltonian corresponding to the ‘typical’ representation of $OSp(2|2)$ can be mapped to a system of free superoscillators through a similarity transformation. However, in general, the spectrum of the original many-body Hamiltonian is not identical with that of superoscillators. It appears that the similarity operator takes the many-body Hamiltonian out of its Hilbert space. The choice of the free superoscillator basis respecting the discrete symmetries of the rational $A_{N+1}(B_{N+1})$ Calogero model gives the complete eigen spectrum of the model. Further investigations on hidden symmetries, if any, of the generalized Calogero-type models are desirable. In general, the nature of the similarity operator and its action on free superoscillator basis needs further
investigation for a better understanding of generalized Calogero-type models.

(iv) **Pseudo-hermitian supersymmetric Calogero Models:**

The deformation of the rational $A_{N+1}$ Calogero model without the confining term lead to a pseudo-hermitian supersymmetric system with broken translational invariance. The configuration space of the original Hamiltonian is different from the deformed Hamiltonian and the eigen spectra of these models become identical, only when solved for identical boundary conditions. The deformed model with allowed modified boundary condition(s) is expected to have different spectra and need further investigations.

A Coxter-invariant superpotential was constructed in Ref. [50] leading to a unified description of supersymmetric Calogero-Moser-Sutherland models based on all the root systems with the rational, trigonometric and hyperbolic potentials. Such a universal description of pseudo-hermitian supersymmetric Calogero models presented in this review article is desirable. Further, pseudo-hermitian non-supersymmetric Calogero models have been constructed by considering complex root spaces that are invariant under anti-linear involutions related to all Coxter groups [99]. The deformations considered in Ref. [99] involve discrete transformations in the root space, while only continuous deformation was considered in section 5. A construction of supersymmetric version of the models considered in Ref. [99] is desirable.

### 8.3 Summary

The main results presented in this article are based on previously published works [46, 47, 49, 64, 84, 112] and may be summarized as follows.

- The condition of shape invariance can be used in conjunction with Dunkl operator to obtain the complete eigen spectrum of the rational Calogero model, i.e. the Hamiltonian appearing in the $N_f = 0$ sector of the supersymmetric Hamiltonian.

- The rational Calogero model in the supersymmetry-preserving phase can be mapped to a set of free superoscillators through a similarity transformation. The complete eigen spectrum of the Calogero model can be constructed from those eigenstates of the free superoscillator Hamiltonian which are invariant under the discrete symmetries of the many-body parent Hamiltonian.
• The supersymmetry-breaking phase of the rational Calogero model can also be studied by mapping a dual Hamiltonian to free superoscillator Hamiltonian. All the eigenstates in the supersymmetry-breaking phase can be constructed from permutation-symmetric superoscillator basis via the dual Hamiltonian.

• A ‘necessary condition’ for the equivalence of a many-body supersymmetric Hamiltonian with $OSp(2|2)$ symmetry to a system of free superoscillators is that the Hamiltonian should commute with the total fermion number operator $N_f$. The proof of ‘sufficient condition’ is model dependent and nontrivial. The equivalence can be proved for rational Calogero models.

• The super-extension of rational Calogero model with $OSp(2|2)$ supersymmetry is not unique. A new Hamiltonian corresponding to the ‘atypical’ representation of $OSp(2|2)$ has been constructed. The quadratic and the cubic Casimir operators vanish identically and cannot be used to characterize the spectrum. It is not known whether the ‘atypical’ Calogero model is integrable or not.

• A construction of pseudo-hermitian supersymmetric Calogero model has been given, which is isospectral with the standard Calogero model. This construction is valid for rational, trigonometric or hyperbolic versions of Calogero models and also for any root system.

• A $2 + 1$ dimensional many-body system with extended $N = 4$ $SU(1,1|2)$ superconformal symmetry has been presented for the generic values of the coupling constant as well as for arbitrary number of particles.

• A supersymmetric Hamiltonian describing $N$ spinless particles with nearest-neighbour and next-nearest-neighbour inverse-square interaction has been shown to be equivalent to a system of interacting bosonic particles with internal spin degrees of freedom. Models of XX spin chains may be obtained from this supersymmetric Hamiltonian in appropriate limits.

The findings of this study are relevant in enriching a general understanding of the integrable structure of many-particle supersymmetric quantum systems. Further studies may reveal new avenues on the applicability of the mathematical techniques related to shape-invariance and supersymmetry to generic quantum systems with more than one bosonic and one fermionic degrees of freedom. Further, the conformal symmetry is a recurrent theme in many areas like blackholes, matrix models, string theory, strongly correlated system etc. and offers
universal description of many apparently diverse physical systems. The super-conformal systems studied in this review thus have potential applications in diverse subjects, including a possible futuristic realization of many-body quantum systems with both spatial and internal degrees of freedom in the laboratory.

9 Acknowledgment

The Author would like to thank B. Basu-Mallick, G. Date, T. Deguchi, Kumar S. Gupta, A. Khare, S. P. Khastgir, M. V. N. Murthy, R. Sasaki and M. Sivakumar for discussions on the topic at various points of time and contributing to his understanding of the subject.

10 Appendices

A few mathematical results which are relevant in the discussions of the main text are discussed in sections 9.1, 9.2, 9.3 and 9.5. The rational $BC_{N+1}$ Calogero model has not been included in the main text. The relevant discussions in this regard are included in Appendix-IV in section 9.4.

10.1 Appendix-I: Matrix representation of the real Clifford algebra

A $2^N \times 2^N$ matrix representation of the elements of the Clifford algebra [2] may be given in terms of the Pauli matrices $\sigma^a, a = 1, 2, 3$ and the $2 \times 2$ identity matrix $I$ as follows,

\begin{align*}
\xi_1 &= \sigma^1 \otimes I \otimes I \otimes \ldots \otimes I, \quad \xi_{N+1} = \sigma^2 \otimes I \otimes I \otimes \ldots \otimes I, \\
\xi_2 &= \sigma^3 \otimes \sigma^1 \otimes I \otimes \ldots \otimes I, \quad \xi_{N+2} = \sigma^3 \otimes \sigma^2 \otimes I \otimes \ldots \otimes I, \\
&\vdots \\
\xi_i &= \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^1 \otimes I, \quad \xi_{N+i} = \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^2 \otimes I \otimes \ldots \otimes I, \\
&\vdots \\
\xi_{N-1} &= \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^1 \otimes I, \quad \xi_{2N-1} = \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^2 \otimes I, \\
\xi_N &= \sigma^3 \otimes \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^1, \quad \xi_{2N} = \sigma^3 \otimes \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^2. \quad (151)
\end{align*}

The matrices $\xi_p$ are hermitian. The matrices $\xi_i$ are symmetric, while $\xi_{N+i}$ are anti-symmetric for any $i$. The fermionic operators $\psi_i$ and $\psi_i^\dagger$ have the matrix
representation,
\[
\psi_i = \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^- \otimes I \otimes \ldots \otimes I \\
\psi^\dagger_i = \sigma^3 \otimes \ldots \otimes \sigma^3 \otimes \sigma^+ \otimes I \otimes \ldots \otimes I,
\]
(152)
where \( \sigma_{\pm} = \frac{1}{2} (\sigma^1 \pm i \sigma^2) \) are in the \( i^{th} \) position, preceded by the tensor-product of \( i - 1 \) numbers of \( \sigma^3 \) and followed by the tensor-product of \( N - i \) numbers of the matrix \( I \).

10.2 Appendix-II: \( OSp(2|2) \) superalgebra

The structure-equations of the \( OSp(2|2) \) superalgebra are described in terms of a set of fermionic generators \( f \equiv \{ Q_1, Q_2, S_1, S_2 \} \) and a set of bosonic generators \( b \equiv \{ H, D, K, Y \} \) as follows \[85, 86\]:

\[
\{ Q_\alpha, Q_\beta \} = 2 \delta_{\alpha \beta} H, \quad \{ S_\alpha, S_\beta \} = 2 \delta_{\alpha \beta} K, \quad \{ Q_\alpha, S_\beta \} = -2 \delta_{\alpha \beta} D + 2 \epsilon_{\alpha \beta} Y, \\
[H, Q_\alpha] = 0, \quad [H, S_\alpha] = -i Q_\alpha, \quad [K, Q_\alpha] = i S_\alpha, \quad [K, S_\alpha] = 0, \\
[D, Q_\alpha] = -\frac{i}{2} Q_\alpha, \quad [D, S_\alpha] = \frac{i}{2} S_\alpha, \quad [Y, Q_\alpha] = \frac{i}{2} \epsilon_{\alpha \beta} Q_\beta, \quad [Y, S_\alpha] = \frac{i}{2} \epsilon_{\alpha \beta} S_\beta, \\
[Y, H] = [Y, D] = [Y, K] = 0, \quad \alpha, \beta = 1,2.
\]
(153)

The bosonic operators \( H, D \) and \( K \) generate the \( O(2,1) \) sub-algebra of \( OSp(2|2) \),
\[
[H, D] = iH, \quad [H, K] = 2iD, \quad [D, K] = iK,
\]
(154)
with its Casimir operator having the form,
\[
C = \frac{1}{2} (HK + KH) - D^2.
\]
(155)

The quadratic and the cubic Casimir operators of \( OSp(2|2) \) are given by \[85, 86\],
\[
C_2 = C + \frac{i}{4} [Q_1, S_1] + \frac{i}{4} [Q_2, S_2] - Y^2, \\
C_3 = C_2 Y - \frac{Y}{2} + \frac{i}{8} \left([Q_1, S_1] Y + [Q_2, S_2] Y + [S_1, Q_2] D - [S_2, Q_1] D + [Q_1, Q_2] K + [S_1, S_2] H \right).
\]
(156)

Both ‘typical’ and ‘atypical’ representations of the supergroup \( OSp(2|2) \) are allowed. The quadratic and the cubic Casimir operators vanish identically for the ‘atypical’ representation of the group.

The subgroup \( OSp(1|1) \) of \( OSp(2|2) \) is described either by the set of generators \( A_1 \equiv \{ H, D, K, Q_1, S_1 \} \) or \( A_2 \equiv \{ H, D, K, Q_2, S_2 \} \). The Casimir of the \( OSp(1|1) \) corresponding to the set \( A_1 \) is given by,
\[
C_1 = C + \frac{i}{4} [Q_1, S_1] + \frac{1}{16}.
\]
(157)
An even operator $C_s$, known as Scasimir, may be defined as follows,

$$C_s = i[Q_1, S_1] - \frac{1}{2}, \quad (158)$$

which has the property that it commutes with all the bosonic generators and anti-commutes with all the fermionic generators of the set $A_1$. The Scasimir $C_s$ is related to the Casimir operators $C$ and $C_1$ through the relations,

$$C_1 = \frac{1}{4} C_s^2, \quad C = \frac{1}{4} C_s (C_s - 1) - \frac{3}{16}. \quad (159)$$

The Casimir and the Scasimir for the set $A_2$ are given by,

$$\bar{C}_1 = C + \frac{i}{4} [Q_2, S_2] + \frac{1}{16}, \quad \bar{C}_s = i[Q_2, S_2] - \frac{1}{2}, \quad (160)$$

which satisfy the identities

$$\bar{C}_1 = \frac{1}{4} \bar{C}_s^2, \quad C = \frac{1}{4} \bar{C}_s (\bar{C}_s - 1) - \frac{3}{16}. \quad (161)$$

This implies that, in general, the Casimir $C$ can be factorized in two different ways, either in terms of $C_s$ or $\bar{C}_s$.

### 10.3 Appendix-III: $SU(1,1|2)$ superalgebra

The structure equations of $SU(1,1|2)$ superalgebra are described in terms of a set of fermionic generators ($Q_p, S_p, Q_p^\dagger, S_p^\dagger$), a set of bosonic generators ($J_a, H, D, K$) and a central element $T$ as follows:

\begin{align*}
\{ Q_p, Q_r^\dagger \} &= 2\delta_p^r H, & \{ S_p, S_r^\dagger \} &= 2\delta_p^r K, & \{ Q_p, Q_r \} &= 0, & \{ S_p, S_r \} &= 0, \\
\{ Q_p, S_r^\dagger \} &= 2i (\sigma^a)^r_p J_a - 2\delta_p^r D - i\delta_p^r T, & [J_a, Q_p] &= -\frac{1}{2} (\sigma^a)^r_p Q_r, \\
\{ S_p, Q_r^\dagger \} &= -2i (\sigma^a)^r_p J_a - 2\delta_p^r D + i\delta_p^r T, & [J_a, S_p] &= -\frac{1}{2} (\sigma^a)^r_p S_r, \\
[Q_p, D] &= \frac{i}{2} Q_p, & [Q_p^\dagger, D] &= \frac{i}{2} Q_p^\dagger, & [S_p, D] &= -\frac{i}{2} S_p, & [S_p^\dagger, D] &= -\frac{i}{2} S_p^\dagger, \\
[K, Q_p] &= i S_p, & [K, Q_p^\dagger] &= i S_p^\dagger, & [H, S_p] &= -i Q_p, & [H, S_p^\dagger] &= -i Q_p^\dagger, \\
[J_a, J_b] &= i \epsilon_{abc} J_c, & p, r &= 1, 2, & a, b, c &= 1, 2, 3. \quad (162)
\end{align*}

The bosonic operators $H, D$ and $K$ generate the $O(2,1)$ algebra given in Eq. (154). A summation over the repeated indices is implied in the above expressions.
10.4 Appendix-IV: Rational $BC_{N+1}$ Calogero Model

The superpotential for the rational $BC_{N+1}$-type Calogero model has the following expression,

$$W(\lambda, \lambda_1, \lambda_2) = -\lambda \sum_{i<j=1}^{N} \ln \left( x_i^2 - x_j^2 \right) - \sum_{i=1}^{N} [\lambda_1 \ln x_i + \lambda_2 \ln(2x_i)]$$

$$+ \frac{\omega}{2} \sum_{i=1}^{N} x_i^2,$$  \hspace{1cm} (163)

where $\lambda$, $\lambda_1$ and $\lambda_2$ are arbitrary parameters. The $D_{N+1}$-type model is described by $\lambda_1 = \lambda_2 = 0$, while $\lambda_1 = 0 (\lambda_2 = 0)$ describes $C_{N+1} (B_{N+1})$-type Hamiltonian. The discussion in this article is restricted to the $B_{N+1}$-type Calogero model for which the Hamiltonian is given by,

$$H_{B_{N+1}} = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2} \sum_i x_i^2 + \frac{1}{2} \lambda(\lambda - 1) \sum_{i \neq j} \left[ x_{ij}^{-2} + (x_i + x_j)^{-2} \right]$$

$$+ \frac{1}{2} \lambda_1(\lambda_1 - 1) \sum_i x_i^{-2} + \omega \sum_i \psi_i^\dagger \psi_i + \lambda_1 \sum_i \psi_i^\dagger \psi_i x_i^{-2}$$

$$+ \lambda \sum_{i \neq j} \left[ x_{ij}^{-2} \left( \psi_i^\dagger \psi_j - \psi_i \psi_j^\dagger \right) + (x_i + x_j)^{-2} \left( \psi_i^\dagger \psi_j + \psi_i \psi_j^\dagger \right) \right]$$

$$- \frac{\omega}{2} N \left[ 1 + 2\lambda(N - 1) + \lambda_1 \right].$$  \hspace{1cm} (164)

It may be noted that the many-body inverse-square interaction is not transnational invariant. Apart from the transnational invariant mutual inverse-square interaction between any pair of particles, each particle also interacts with the images of all other particles and also with itself. This kind of Hamiltonians are suitable for describing systems with boundaries. The many-body wave-functions are taken to be vanishing at the singular points $x_i = 0, x_i = \pm x_j \forall i, j$ and the eigen-value problem is solved in the $0 < x_1 < x_2 < \ldots < x_N$ sector of the configuration space. The Hamiltonian is invariant under the permutation symmetry $\mathcal{P}$ and the reflection symmetry $\mathcal{R}$:

$$\mathcal{P} : \quad x_i \to x_j, \quad \psi_i \to \psi_j, \quad \psi_i^\dagger \to \psi_j^\dagger$$

$$\mathcal{R} : \quad x_i \to -x_i, \quad \psi_i \to -\psi_i, \quad \psi_i^\dagger \to -\psi_i^\dagger.$$  \hspace{1cm} (165)

These two symmetries allow a smooth continuation of the wave-functions from a given sector of the configuration space to all other sectors. The reflection symmetry also has an interesting consequence on the spectrum. The ground-state of the Hamiltonian in the supersymmetric phase is given by,

$$\Phi_0 = \phi_0 \vert 0 \rangle, \quad \phi_0 = \prod_{i<j} \left( x_i^2 - x_j^2 \right)^{\lambda} \prod_k x_k^{\lambda} e^{-\frac{1}{2} \sum_i x_i^2}, \quad \lambda, \lambda_1 > 0,$$  \hspace{1cm} (166)
which is normalizable for \( \lambda, \lambda_1 > -\frac{1}{2} \). However, the positivity of \( \lambda \) and \( \lambda_1 \) is imposed so that each momentum operator \( p_i \) is self-adjoint for the wave-function of the form \( \Phi_0 \).

10.4.1 Shape Invariance & Exact Solvability

The \( N_f = 0 \) sector of the the supersymmetric Hamiltonian is described by the bosonic Hamiltonians \( \mathcal{H}^{(0)} \):

\[
\mathcal{H}^{(0)}(\lambda, \lambda_1, \omega) = \mathcal{H}^{B_{N+1}} - E^{B_{N+1}}_0 = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \lambda(\lambda - 1) \sum_{i \neq j} \left[ (x_i - x_j)^{-2} + (x_i + x_j)^{-2} \right] + \frac{1}{2} \lambda_1(\lambda_1 - 1) \sum_{i=1}^N x_i^{-2} + \frac{\omega}{2} \sum_i x_i^2
\]

\[
E^{B_{N+1}}_0 = \frac{\omega}{2} N \left[ 1 + 2 \lambda(N - 1) + \lambda_1 \right].
\]

The bosonic Hamiltonian \( \mathcal{H}^{(N)} \) corresponding to the \( N_f = N \) sector of the supersymmetric Hamiltonian is related to \( \mathcal{H}^{(0)} \) through the shape-invariance condition,

\[
\mathcal{H}^{(N)}(\lambda, \lambda_1, \omega) = \mathcal{H}^{(0)}(\lambda + 1, \lambda_1 + 1, \omega) + \frac{N \omega}{2} (2N + 1). \tag{168}
\]

However, the complete spectra of \( \mathcal{H}^{(0)} \) and \( \mathcal{H}^{(N)} \) cannot be obtained by using this shape-invariance condition. This is because of the absence of intertwining relations between these two Hamiltonians which appear in the description of supersymmetric system with one bosonic and one fermion degrees of freedom. The present situation is circumvented by the use of a modified shape-invariance condition and exchange operator formalism as applied to the system with \( B_{N+1} \)-type Hamiltonian.

The Dunkl operator is introduced in terms of the exchange operator \( M_{ij} \) and the reflection operator \( t_i \):

\[
\mathcal{D}_i = -i \partial_i + i \lambda \sum_{j \neq i} \left[ (x_i - x_j)^{-1} M_{ij} + (x_i + x_j)^{-1} \tilde{M}_{ij} \right] + i \lambda_1 x_i^{-1}, \quad \tilde{M}_{ij} := t_i t_j M_{ij}.
\]

The reflection operator \( t_i \) satisfies the following relations,

\[
t_i x_j = x_j t_i \text{ for } i \neq j, \quad t_i x_i + x_i t_i = 0 \quad \forall \ i, \quad t_i^2 = 1 \quad \forall \ i,
\]

\[
M_{ij} t_i = t_j M_{ij}, \quad \tilde{M}_{ij} = M_{ij}, \quad t_i \mathcal{D}_i = -\mathcal{D}_i t_i, \quad t_i \mathcal{D}_j = \mathcal{D}_i t_i \text{ for } j \neq i,
\]

\[
\tilde{M}_{ij} t_i = -\mathcal{D}_i M_{ij}, \quad t_i \phi(x_1, \ldots, x_i, \ldots, x_N) = \phi(x_1, \ldots, -x_i, \ldots, x_N). \tag{170}
\]

54
The Dunkl operators $D_i$ commute among themselves and the following identity holds:

$$ b_i := D_i - i\omega x_i, \quad b_i^\dagger := D_i + i\omega x_i, $$

$$ [b_i, b_j^\dagger] = 2\omega\delta_{ij} \left( 1 + \lambda \sum_{k(\neq i)} (M_{ik} + \tilde{M}_{ik}) + 2\lambda_1 t_i \right) $$

$$ -2(1 - \delta_{ij})\lambda\omega(M_{ij} - \tilde{M}_{ij}). \quad (171) $$

All other commutators involving $b_i$ and their adjoints vanish identically. The supersymmetric partner Hamiltonians $H$ and $\tilde{H}$ for the $B_{N+1}$ case may be defined in a similar way to that of rational $A_{N+1}$ Calogero model,

$$ H = \frac{1}{2} \sum_{i=1}^N b_i^\dagger b_i, \quad \tilde{H} = \frac{1}{2} \sum_{i=1}^N b_i b_i^\dagger. \quad (172) $$

The Hamiltonian $H$ reduces to $H^{(0)}$ in Eq. (167) if $M_{ij}$ acts on symmetric functions, whereas it reduces to $H^{(N)}$ if $M_{ij}$ is restricted to the subspace of antisymmetric functions. A set of annihilation and creation operators, similar to $A_n$, $A_n^\dagger$ in Eq. (47), are introduced as follows:

$$ \hat{B}_n = \sum_{i=1}^N b_i^2, \quad \hat{B}_n^\dagger = \sum_{i=1}^N (b_i^\dagger)^2 \quad n \leq N. \quad (173) $$

It can be shown that if $\phi (\tilde{\phi})$ is the eigenfunction of $H(\tilde{H})$ with eigenvalue $E (\tilde{E})$ then

$$ H(\hat{B}_2^\dagger \phi) = (\tilde{E} - \hat{\delta}_2)(\hat{B}_2^\dagger \phi), \quad \tilde{H}(\tilde{B}_2 \phi) = (E + \hat{\delta}_2)(\hat{B}_2 \phi). \quad (174) $$

where,

$$ \hat{\delta}_2 = [N - 2 \pm 2\lambda N(N - 1) + 2\lambda_1 N]\omega. \quad (175) $$

It may be noted that he operator which brings in a correspondence between the eigenstates $\phi$ and $\tilde{\phi}$ is $B_2$, not $B_1$. This is because the reflection symmetry of the $BC_{N+1}$ Hamiltonian is also a symmetry of the wave-functions provided the operator $B_2$ is used instead of $B_1$. The shape invariance condition for $H$ and $\tilde{H}$ of the $B_N$ model reads,

$$ \tilde{H}(\lambda, \lambda_1, \omega) = H(\lambda, \lambda_1, \omega) + R_2(\lambda, \lambda_1, \omega) $$

$$ R_2(\lambda, \lambda_1, \omega) \equiv [N \pm 2\lambda N(N - 1) + 2\lambda_1 N]\omega. \quad (176) $$

Following the standard treatment, the spectrum of $H^{(0)}$ is determined as,

$$ E_n = n(R_2 - \hat{\delta}_2) = 2n\omega. \quad (177) $$
The reflection symmetry is manifested in the spectrum which now depends on $2n\omega$, instead of $n\omega$ as in the case of rational $A_{N+1}$ Calogero model. The eigenfunctions are obtained by acting the operators $B_n$ on the ground state $\phi_0$ of the $B_{N+1}$ model, as in Eq. (54) for $A_{N+1}$ model.

10.4.2 Mapping to free super-oscillators

The supersymmetric rational $B_{N+1}$-type Calogero Hamiltonian in Eq. (164) can be mapped to a system of $N$ free superoscillators by using Eq. (83) with $W(\lambda, \lambda_1, \lambda_2 = 0)$ given by Eq.(163). The supersymmetric and the supersymmetry-breaking phases are discussed separately.

**Case I: Supersymmetric phase: $\lambda, \lambda_1 > 0$:**

The construction of the eigen functions requires the following choice of the free superoscillator basis:

$$\hat{P}_{n,k} = \frac{1}{N_f!} \epsilon^{2n} \sum_{i_1, i_2, \ldots, i_{N_f}} f_{i_1 i_2 \ldots i_{N_f}} (x_{i_1} \psi^\dagger_{i_1}) (x_{i_2} \psi^\dagger_{i_2}) \cdots (x_{i_{N_f}} \psi^\dagger_{i_{N_f}}),$$  \hspace{1cm} (178)

where $f$ is anti-symmetric under the exchange of any two indices and a homogeneous function of degree $2k$ of the bosonic coordinates. The functions $\hat{P}_{n,k}$ are invariant under the discrete symmetry (165). The action of the similarity operator $\tilde{T}$ on functions without the discrete symmetry (165) produces essential singularity in the wave-functions which are not physically acceptable. Thus, the complete spectrum of $H_{B_{N+1}}$ is described by a subset of the spectrum of super-oscillators,$$

E_{B_{N+1}} = 2(n + k + N_f), \quad E_{sho} = 2n + k + N_f. \hspace{1cm} (179)

The rational $A_{N+1}$ Calogero model is equivalent to a set of $N$ free super-oscillators, whereas the rational $B_{N+1}$ Calogero model is equivalent to a set of $N$ free ‘super-half-oscillators’[46].

The eigen-spectrum can be constructed in an algebraic way by defining the creation and annihilation operators as,

$$\hat{B}^+_n = \tilde{T}^{-1} \sum_i b_i^{+2n} \tilde{T}, \quad \hat{F}^+_n = \tilde{T}^{-1} \sum_i \psi_i^\dagger b_i^{+2n-1} \tilde{T},$$

which are invariant under the discrete symmetry (165). Thus, the eigen-functions obtained by operating these operators on the ground-state $\Phi_0$ in Eq. (166) are
also invariant under the same discrete symmetry. The eigen-states are obtained as,
\[ \chi_{n_1...n_N \nu_1...\nu_N} = \prod_{k=1}^{N} \tilde{B}_k^{n_k} \tilde{F}_k^{\nu_k} \Phi_0. \]  
(181)

with the energy \( \mathcal{E} = \sum_{k=1}^{N} 2k(n_k + \nu_k) \). The bosonic quantum numbers are non-negative integers, while the fermionic quantum numbers are 0 or 1.

Case II: Supersymmetry-breaking phase:

There are three regions in the parameter space for which the supersymmetry is broken: (i) \( \lambda < 0, \lambda_1 < 0 \), (ii) \( \lambda < 0, \lambda_1 > 0 \) and (iii) \( \lambda > 0, \lambda_1 < 0 \). The eigen-spectrum of the Hamiltonian in the region (i) can be obtained in a similar way as in the case of supersymmetry-breaking phase of the rational \( A_{N+1} \) Calogero model. However, modified treatments are required for obtaining the spectra in the regions (ii) and (iii). The readers are referred to Ref. [46] for details. The eigen spectra in these three regions are given below:

\( \lambda < 0, \lambda_1 < 0 : \quad E = N \left[ 1 - 2\lambda(N - 1) - \lambda_1 \right] + E_{n_k,\nu_k} \)  
(182)

\( \lambda < 0, \lambda_1 > 0 : \quad E = N \left[ \frac{3}{2} - 2\lambda(N - 1) \right] + E_{n_k,\nu_k} \)  
(183)

\( \lambda > 0, \lambda_1 < 0 : \quad E = N \left[ \frac{3}{2} - 2\lambda_1(N - 1) \right] + E_{n_k,\nu_k} \)  
(184)

where \( E_{n_k,\nu_k} = \sum_{k=1}^{N} 2 [kn_k + (k - 1)\nu_k] \). The bosonic quantum numbers \( n_k \)'s are non-negative integers, while the fermionic quantum numbers \( \nu_k \) are 0 or 1.

10.4.3 Nearest-neighbour variant of rational \( BC_{N+1} \) Calogero model

The Hamiltonian \( H \) in terms of the variables \( q_i = x_i - x_{i+1} \), \( \bar{q}_i = x_i + x_{i+1} \) reads,
\[
H = \frac{1}{2} \sum_i p_i^2 + \lambda^2 \sum_i \left[ q_i^{-2} + \bar{q}_i^{-2} - (q_{i-1}^{-1} - q_{i-1}^{-1})(q_i^{-1} + q_i^{-1}) \right] \\
- \lambda \omega \left( 2 + \frac{\lambda_1}{\lambda} \right) N + \frac{\lambda}{2} \sum_i q_i^{-2} \left[ (n_i + n_{i+1}) - 2 \left( \psi_i^\dagger \psi_{i+1} - \psi_i \psi_{i+1}^\dagger \right) \right] \\
+ \lambda \frac{\omega}{2} q_i^{-2} \left[ (n_i + n_{i+1}) + 2 \left( \psi_i^\dagger \psi_{i+1} - \psi_i \psi_{i+1}^\dagger \right) \right] \\
+ \frac{1}{2} \sum_i \left[ \left( \omega + \frac{\lambda_1}{x_i^2} \right) n_i + \omega^2 x_i^2 + \frac{\lambda_1}{x_i^2} \right]. 
\]  
(185)
The spectrum \[E = 2\omega(n_1 + \nu_1) + 4\omega\nu_2, \quad (186)\] corresponding to the exactly solved states is that of a superoscillator with the frequency \(2\omega\) and a fermionic oscillator with the frequency \(4\omega\). The complete spectrum is not known.

10.5 Appendix V: Jordan-Wigner transformation

The spin operators \(S^a_i, a = 1, 2, 3\) for the \(i\)th spin-\(\frac{1}{2}\) particle is realized in terms of the Pauli matrices \(\sigma^a_i\) as, \(S^a_i := \frac{1}{2}\sigma^a_i\). The Jordan-Wigner transformation is defined as,

\[
\psi_j = e^{i\pi\sum_{k=1}^{j-1}\sigma_k^+ \sigma_k^-} \psi_j, \quad \psi_j^\dagger = e^{-i\pi\sum_{k=1}^{j-1}\sigma_k^+ \sigma_k^-} \psi_j^\dagger, \quad \sigma_i^\pm := \frac{1}{2}(\sigma_i^1 \pm i\sigma_i^2), \quad (187)
\]

which relates \(\sigma^a_i\)'s or the spin operators to the fermionic variables \(\psi_i\)'s. The inverse transformation of Eq. (187) is given by,

\[
\sigma_j^- = e^{i\pi\sum_{k=1}^{j-1}\psi_k^\dagger \psi_k} \psi_j, \quad \sigma_j^+ = e^{-i\pi\sum_{k=1}^{j-1}\psi_k^\dagger \psi_k} \psi_j^\dagger. \quad (188)
\]

The Jordan-Wigner transformation implies the following identities:

\[
\sigma_i^+ \sigma_{i+1}^- = \psi_i^\dagger \psi_{i+1}, \quad \sigma_i^- \sigma_{i+1}^+ = -\psi_i \psi_{i+1}^\dagger, \quad \psi_i^\dagger \psi_i = \sigma_i^+ \sigma_i^- = \frac{1}{2}(1 + \sigma_i^z), \quad (189)
\]

which can be used to map a XX spin-chain system with nearest-neighbour interaction to a system of free fermions.

References

[1] E. Witten, Nucl. Phys B188, 513(1981); Nucl. Phys. B202, 253(1982).
[2] F. Cooper and B. Freedman, Ann. Phys. 146, 262(1983); C. Bender, F. Cooper and A. Das, Phys. Rev. D28, 1473 (1983).
[3] F. Cooper, A. Khare and U. Sukhatme, Phys. Rept. 251, 267(1995); F. Cooper, A. Khare and U. Sukhatme, Supersymmetry in Quantum Mechanics, World Scientific(2001).
[4] B. Bagchi, Supersymmetry in Classical and Quantum Mechanics, Chapman & Hall/CRC(2001).
[5] A. Ganogopadhyaya, J. V. Mallow and C. Rasinariu, Supersymmetric Quantum Mechanics: An Introduction, World Scientific(2011).
[6] M. de Crombrugghe and V. Rittenberg, Annals Phys. 151, 99 (1983).

[7] A. A. Andrianov, N. V. Borisov, M. I. Eides and M. V. Ioffe, Phys. Lett. A109, 143 (1985).

[8] Y. Kuramoto and Y. Kato, Dynamics of one-dimensional quantum systems: inverse-square interaction models, Cambridge University Press, 2009.

[9] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cimento A34, 569 (1976).

[10] S. Fubini and E. Rabinovici, Nucl. Phys. B245, 17 (1984).

[11] F. Calogero, J. Math. Phys. (N.Y.) 10, 2191 (1969); 10, 2197 (1969).

[12] B. Sutherland, J. Math. Phys. (N.Y.) 12, 246 (1971); 12, 251 (1971); Phys. Rev. A 4, 2019 (1971).

[13] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71, 314 (1981); 94, 6 (1983).

[14] A. Polychronakos, Les Houches Lectures 1998, hep-th/9902157.

[15] A. P. Polychronakos, J. Phys. A39, 12793 (2006).

[16] V. Pasquier, hep-th/9405104.

[17] M. Milekovic, S. Meljanac and A. Samsarov, SIGMA 2, 035 (2006).

[18] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. 73, 3331 (1994).

[19] Z. N. Ha, Quantum many-body systems in one dimension, Series on Advances in Statistical Mechanics, Vol. 12 (World-Scientific, 1996).

[20] H. Azuma and S. Iso, Phys. Lett. B331, 107 (1994).

[21] N. Kawakami and S.-K. Yang, Phys. Rev. Lett. 67, 2493 (1991).

[22] B. D. Simons, P. A. Lee and B. Altshuler, Phys. Rev. Lett. 72, 64 (1994); S. Jain, Mod. Phys. Lett. A11, 1201 (1996).

[23] C. W. J. Beenakker and B. Rejaei, Phys. Rev. B49, 7499 (1994); M. Caselle, Phys. Rev. Lett. 74, 2776 (1995).

[24] G. Date, P. K. Ghosh and M. V. N. Murthy, Phys. Rev. Lett. 81, 3051 (1998).

[25] F. D. M. Haldane, Phys. Rev. Lett. 60, 635 (1988); B. S. Shastry, Phys. Rev. Lett. 60, 639 (1988).
[26] F. D. M. Haldane, Proceedings of the 16th Taniguchi Symposium on Condensed Matter Physics, edited by A. Okiji and N. Kawakami, Springer-Verlag, Berlin, 1994.

[27] J. A. Minahan and A. P. Polychronakos, Phys. Lett. B302, 265(1993).

[28] C. Ahn and W. M. Koo, Phys. Lett. B365, 105(1996).

[29] B. Basu-Mallick, Nucl. Phys. B482, 713(1996).

[30] K. Hikami and B. Basu-Mallick, Nucl. Phys. B566, 511 (2000).

[31] B. Basu-Mallick, F. Finkel and A. Gonzalez-Lopez, Nucl. Phys. B812, 402 (2009).

[32] B. Basu-Mallick, N. Bondyopadhaya and Diptiman Sen, Nucl. Phys. B795, 596(2008).

[33] J.C. Barba, F. Finkel, A. Gonzalez-Lopez and M.A. Rodriguez, Phys. Rev. B77, 214422 (2008).

[34] K. Hikami and M. Wadati, Phys. Rev. Lett. 73, 1191(1994); H. Ujino and M. Wadati, J. Phys. Soc. Jap. 63, 3585(1994).

[35] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, Nucl. Phys. B449, 347(1995).

[36] A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 102, 499 (1999), hep-th/9905011; S. P. Khastgir, A. J. Pocklington and R. Sasaki, J. Phys. A: Math. Gen. 33, 9033(2000), hep-th/0005277.

[37] P. K. Ghosh and A. Khare, J. Phys. A: Math. Gen. 32, 2129(1999).

[38] B. Basu-Mallick, P. K. Ghosh and Kumar S. Gupta, Phys. Lett A311, 87(2003), hep-th/0208132; B. Basu-Mallick, P. K. Ghosh and Kumar S. Gupta, Nucl. Phys. B659, 437 (2003), hep-th/0207040; B. Basu-Mallick, P. K. Ghosh and Kumar S. Gupta, Pramana-J. Phys. 62, 691 (2004); B. Basu-Mallick and Kumar S. Gupta, Phys. Lett. A292, 36 (2001), hep-th/0109022.

[39] N. Yonezawa and I. Tsutsui, J. Math. Phys. 47, 012104 (2006); L. Feher, I. Tsutsui and T. Fulop, Nucl. Phys. B715, 713 (2005).

[40] N. Gurappa and P. K. Panigrahi, Phys. Rev. B59, R2490 (1999), cond-mat/9710035, quant-ph/9710019; N. Gurappa, A. Khare and P. K. Panigrahi, Phys. Lett. A 224, 467(1998), cond-mat/9804207.
[41] T. Brezinski, C. Gonera and P. Maslanka, Phys. Lett. A254, 185(1999).
[42] V. Bardek, J. Feinberg, S. Meljanac, JHEP 08, 018(2010); V. Bardek, J. Feinberg, S. Meljanac, Annals of Physics 325, 691 (2010).
[43] D. Z. Freedman and P. F. Mende, Nucl. Phys. B344, 317 (1990).
[44] L. Brink, T. H. Hansson, S. Konstein and M. A. Vasiliev, Nucl. Phys. B401, 591 (1993), hep-th/9302023.
[45] L. Brink, A. Turbiner and N. Wyllard, J. Math. Phys. 39, 1285 (1998), hep-th/9705219.
[46] P. K. Ghosh, Nucl. Phys. B595, 519(2001).
[47] P. K. Ghosh, Nucl. Phys. B681, 359(2004).
[48] B. S. Shastry and B. Sutherland, Phys. Rev. Lett. 70, 4029 (1993).
[49] P. K. Ghosh, A. Khare and M. Sivakumar, Phys. Rev. A58, 821 (1998), cond-mat/9710206. C. Efthimiou and D. Spector, Phys. Rev. A56, 208 (1997), quant-ph/9702017.
[50] A. J. Bordner, N. S. Manton and R. Sasaki, Prog. Theor. Phys. 103, 463 (2000), hep-th/9910033.
[51] M. V. Ioffe and A. I. Neelov, J. Phys. A33, 1581 (2000), quant-ph/0010063.
[52] M. V. Ioffe and A. I. Neelov, J. Phys. A35, 7613(2002).
[53] A. Galajinsky, O. Lechtenfeld and K. Polovnikov, Phys. Lett. B643, 221(2006) A. Galajinsky, I. Masterov, Phys. Lett. B675, 116(2009).
[54] G. W. Gibbons and P. K. Townsend, Phys. Lett. B 454, 187 (1999), hep-th/9812034.
[55] E. D’Hoker and D. H. Phong, hep-th/9912271. A. Gorsky and A. Mironov, hep-th/0011197.
[56] A. Dabholkar, Nucl. Phys. B368, 293(1992).
[57] S. James Gates Jr., A. Jellal, E. L. Hassan Saidi and M. Schreiber, JHEP 0411, 075(2004).
[58] H. L. Verlinde, e-Print: hep-th/0403024
[59] J. McGreevy, S. Murthy and H. L. Verlinde, JHEP 0404, 015(2004).
[60] A. Agarwal and A. P. Polychronakos, JHEP 0608, 034(2006).

[61] J. P. Rodrigues and A. J. van Tonder, Int. J. Mod. Phys. A8, 2517(1993), hep-th/9204061; A. Jevicki and J. P. Rodrigues, Phys. Lett. B268, 53(1991).

[62] R. de Mello Koch and J. P. Rodrigues, Phys.Rev. D51, 5847(1995).

[63] I. Aniceto and A. Jevicki, J.Phys. A39, 12765(2006).

[64] P. K. Ghosh, J. Phys. A34, 5583 (2001), hep-th/0009055.

[65] N. Wyllard, J. Math.Phys. 41, 2826(2000), hep-th/9910160.

[66] S. Bellucci, A. Galajinsky and Sergey Krivonos, Phys. Rev. D68, 064010 (2003); A. V. Galajinsky, Mod. Phys. Lett. A18, 1493 (2003).

[67] S. Bellucci, S. Krivonos and A. Sutulin, Nucl. Phys. B805, 24(2008).

[68] Wen-Yu Wen, e-Print: arXiv:0807.0633.

[69] M. Kojima and N. Ohta, Nucl. Phys. B473, 455 (1996); N. Ohta, J. Phys. Soc. Japan 65, 3769 (1996).

[70] S. Fedoruk, E. Ivanov and O. Lechtenfeld, Phys. Rev. D79, 105015(2009); JHEP 1004, 129(2010).

[71] S. Fedoruk, E. Ivanov, O. Lechtenfeld, JHEP 0908, 081(2009).

[72] O. Lechtenfeld, K. Schwerdtfeger, J. Thuerigen, SIGMA 7, 023(2011).

[73] A. Galajinsky, O. Lechtenfeld, K. Polovnikov, JHEP 0711, 008(2007).

[74] A. V. Galajinsky, Mod.Phys.Lett. A18, 1493(2003).

[75] A. Galajinsky, Nuclear Physics B832, 586 (2010).

[76] A. Galajinsky, O. Lechtenfeld and K. Polovnikov, JHEP 0903, 113 (2009); A. Galajinsky, O. Lechtenfeld, Phys. Rev. D 80, 065012 (2009).

[77] S. Kirivonos and O. Lechtenfeld, JHEP 1102, 042(2011).

[78] S. Fedoruk, E. Ivanov and O. Lechtenfeld, arXiv:1112.1947.

[79] P. Desrosiers, L. Lapointe and P. Mathieu, Nucl. Phys. B606, 547(2001), hep-th/0103178.
[80] P. Desrosiers, L. Lapointe and P. Mathieu, J. Phys. A37, 1251(2004); e-Print: math/0412306
[81] P. Desrosiers, L. Lapointe and P. Mathieu, Nucl. Phys. B674, 615(2003).
[82] P. Desrosiers, L. Lapointe and P. Mathieu, Comm. Math. Phys. 242, 331(2003);
[83] P. Desrosiers, L. Lapointe and P. Mathieu, Commun.Math.Phys. 233, 383(2003).
[84] P. K. Ghosh, J. Phys. A: Math. Theor. 44, 215307 (2011).
[85] W. Nahm and M. Scheunert, J. Math. Phys. 17, 868 (1976); M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. 18, 146(1977); 18, 155 (1977).
[86] L. Frappat, P. Sorba and A. Sciarrino, hep-th/9607161
[87] E. D’Hoker and L. Vinet, Comm. Math. Phys. 97, 391 (1985).
[88] D. Arnaudon and M. Bauer, Lett. Math. Phys. 40, 307 (1997), hep-th/9605020; D. Arnaudon, M. Bauer and L. Frappat, Comm. Math. Phys. 187, 429 (1997), hep-th/9605021.
[89] C. M. Bender, Contemp. Phys. 46, 277(2005); C.M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243(1998).
[90] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7, 1191(2010), arXiv:0810.5643; A. Mostafazadeh, J. Math Phys. 43, 205(2002); 43, 2814(2002); 43, 3944(2002).
[91] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. 213, 74 (1992).
[92] P. Dorey, C. Dunning and R. Tateo, J. Phys. A34, 5679(2001); J. Phys. A40, R205(2007).
[93] P. K. Ghosh, J. Phys. A:Math. Theor. 43, 125203(2010); P. K. Ghosh, Phys. Lett. A375, 3250(2011); P. K. Ghosh, A note on topological insulator phase in non-hermitian quantum system, arXiv:1109.1697.
[94] P. K. Ghosh, J. Phys. A38, 7313 (2005); T. Deguchi and P. K. Ghosh, Phys. Rev. E 80, 021107(2009); T. Deguchi, P. K. Ghosh and K. Kudo, Phys. Rev. E 80, 026213 (2009).
[95] T. Deguchi and P. K. Ghosh, J. Phys. A42, 475208(2009).
B. Basu-Mallick and A. Kundu, Phys. Rev. B62, 9927(2000); B. Basu-Mallick, T. Bhattacharya and B. P. Mandal, Mod. Phys. Lett. A20, 543(2005); B. Basu-Mallick and B. P. Mandal, Phys. Lett. A284, 231(2001).

P. K. Ghosh, Int. J. Theo. Phys. 50, 1143(2011); Pijush K. Ghosh and Kumar S. Gupta, Physics Letters A323, 29 (2004).

M. Znojil and M. Tater, J. Phys. A34, 1793(2001); A. fringe, Mod. Phys. Lett. A21, 691 (2006); A. Fring and M. Znojil, J. Phys. A41, 194010(2008); P. E. G. Assis and A. Fring, J. Phys. A: Math. Theor. 42, 425206 (2009).

A. Fring and M. Smith, Non-Hermitian multi-particle systems from complex root spaces, arXiv:1108.1719. J. Phys. A: Math. Theor. 43, 325201 (2010).

F. Calogero and C. Marchioro, J. Math. Phys. 14, 182(1973).

M. V. N. Murthy, R. K. Bhadury and D. Sen, Phys. Rev. Lett. 76, 4103 (1996), cond-mat/9603155. R. K. Bhadury, A. Khare, J. Law, M. V. N. Murthy and D. Sen, J. Phys. A 30, 2557 (1997). cond-mat/9609012

P. K. Ghosh, Phys. Lett. A229, 203(1997), cond-mat/9610024

A. Khare and K. Ray, Phys. Lett. A230, 139(1997).

M. V. Feigel' man and M. A. Skvortsov, Nucl. Phys. B506[FS], 665 (1997), cond-mat/9703215

G. Oas, Phys. Rev. E55, 205 (1997), cond-mat/9610073

C. Kane, S. Kivelson, D.-H. Lee and S. C. Zhang, Phys. Rev. B43, 3255(1991).

G. Date, M. V. N. Murthy and R. Vathsan, Jr. of Phys. : Condensed Matter 10, 5876(1998), cond-mat/9802034

J. Feinberg, Nucl. Phys. B705, 403(2005).

S. R. Jain and A. Khare, Phys. Lett. A 262, 35(1999); G. Auberson, S. R. Jain and A. Khare, Phys. Lett. A267, 293 (2000).

G. Auberson, S. R. Jain and A. Khare, J. Phys. A34, 695(2001).
[111] B. Grem’and and S. R. Jain, J. Phys. A 31, L637 (1998); E. Bogomol’ny, U. Gerland and C. Scimt, Phys. Rev. E 59, R1315 (1999); H. D. Parab and S. R. Jain, J. Phys. A 29, 3903 (1996).

[112] T. Deguchi and P. K. Ghosh, J. Phys. Soc. Jap. 70, 3225 (2001), hep-th/0012058.

[113] A. Enciso, F. Finkel, A. Gonzalez-Lopez and M.A. Rodriguez, Phys. Lett. B 605, 214 (2005).

[114] A. Enciso, F. Finkel, A. Gonzalez-Lopez, M.A. Rodriguez, J. Phys. A 40, 1857 (2007).

[115] A. Enciso, F. Finkel, A. Gonzalez-Lopez and M.A. Rodriguez, SIGMA 2, 073 (2006).

[116] R. Coquereaux, Phys. Lett. B 115 (1982) 389.