Compatibility and Binary Correlations of Fibonacci Partial Words

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Abstract. The sequences of Fibonacci words play an important role in formal language theory and combinatorics on words. Fibonacci partial words, arrays and their combinatorial properties such as palindromes and primitivity was established. In this paper, we extend some fundamental results about Fibonacci words to Fibonacci partial words such as compatibility, periodicity and also binary correlations which turns out to be an effective technique of portraying sets of periods of Fibonacci partial words briefly with comprehensive scope.

1. Introduction

The study of sequence of Fibonacci word is a great interest in some aspects of formal language theory. In theory of combinatorics, the sequence of Fibonacci words which is uniformly recurrent play a vital role due to their remarkable properties, some of which have been studied by Knuth [13] in relation with string matching problems and by Duval [10] in the study of “periodicity” of words. Aldo De Luca [9] discussed primitivity and palindromic properties of Fibonacci words and later on S.S.Yu and Yu-Kuang Zhao [16] discussed d-primitive and palindrome properties of Fibonacci words and have also shown that Fibonacci languages are regular free. The urge for the introduction of partial words came from the molecular biology of nucleic acids. There, among other things, the properties of the DNA sequences experienced in the genome of organisms are determined. These are considered as strings over the alphabet {A, C, G, T} of four bases respectively. In DNA sequencing, some part of information may be absent or unseen. This can be revealed by positions denoting missing symbols in a word. Thus, instead of complete words, partial words are considered. Fischer and Paterson [11] introduced partial words as strings containing don’t care letters. Berstel and Boasson [3] initiated the study of combinatorics on partial words and was later pursued by Blanchet-Sadri et al. [4, 5, 6, 7]. The critical factorization theorem and Fine and Wilf theorem [6, 8] are the two basic results on periodicity which is one of the most used and known results on words, has extensions to partial words. In [14] the notion of Fibonacci partial words, arrays and their combinatorial properties such as palindromes and primitivity are established. Guibas et al. [12] represented their results using binary vectors called correlations which
turned out to be a convenient way of describing periods sets briefly with comprehensive scope. The motive for recent studies on combinatorics of words is the study of molecules such as DNA that play a major part in molecular biology. Therefore several basic results on words have been extended to partial words [1, 2, 4, 7, 9, 12, 14, 15]. In this paper, we extend some basic results about Fibonacci words to Fibonacci partial words. An outline of this paper follows. Some basic definitions are recollected in section 2. In section 3, correlations and compatibility of Fibonacci partial words are discussed. Finally section 4 concludes this paper.

2. Preliminaries

Here we recall fundamental notions of partial words, correlations of partial words and Fibonacci words. Let $\Sigma$ be a non-empty finite set of letters. These letters are also termed as symbols and the set is termed as an alphabet. Any string over $\Sigma$ is called a word or a total word. If a word $u$ is of the form $xy$, where $x$ is empty then $u_R = yx$ is called a rotation of $u$ of degree $|u|$. The correlation of a word $u$ over a word $v$ is defined to be binary vector of the same length as $u$, composed as follows. The $i$th bit (from the left) of the correlation is determined by placing $v$ under $u$ so that the leftmost character of $v$ is under the $i$th character of $u$ (from the left). Then, if all pairs of characters that are directly over each other match, the $i$th bit of the correlation is 1, else it is 0. The same process is applicable for binary correlation of a word over itself. The sequence or word that contains a number of “do not know” symbols or “holes” denoted as $\diamond$ is termed as a partial word. The symbol $\diamond$ does not belong to the alphabet $\Sigma$. A partial word over $\Sigma_\diamond$ with length $n$ is a partial function $r_\diamond : \{0, 1, 2 \cdots, n - 1\} \to \Sigma_\diamond = \Sigma \cup \{\diamond\}$ defined by $r_\diamond(i) = r(i)$ if $i \in D(r), \diamond$ if $i \in H(r)$, where $D(r)$ and $H(r)$ are the domain set and hole set of $r$ respectively. A partial word $r$ over $\Sigma_\diamond$ is $p$- periodic if a non-negative integer $p$ exists such that $i \equiv j \bmod p$ whenever $r(i) = r(j)$ for every $i, j \in D(r)$. If two partial words say $x$ and $y$ are of equal length and if all the elements in domain of $x$ are also in domain of $y$ with $x(i) = y(i)$ for every $i \in D(x)$, then $x$ is contained in $y$ and is denoted by $x \subset y$. Two partial words $x$ and $y$ are compatible, denoted by $x \uparrow y$ if $x(i) = y(i)$ for all $i \in D(x) \cap D(y)$. The least upper bound of $x$ and $y$ is denoted by $x \lor y$ where $D(x \lor y) = D(x) \cup D(y)$. Consider an alphabet $\Sigma$ with $|\Sigma| \geq 2$. The Fibonacci sequence of words $\{f_n\}$, $n \geq 1$ over the alphabet $\Sigma$ is defined inductively as $f_n = f_{n-1}f_{n-2}, f_1 = a, f_2 = b$ where $a \neq b$. Length of the Fibonacci sequence of words $|f_n|$ is denoted by $F(n)$. Also $|f_1| = |f_2| = 1$ and $|f_n| = |f_{n-1}| + |f_{n-2}|$ for all $n \geq 2$.

3. Compatibility and Binary correlations of Fibonacci partial words

Here we define sequence of Fibonacci partial words and discuss their compatibility and binary correlations. Binary correlations of Fibonacci partial words are the binary vectors specifying the periods of Fibonacci partial words.

**Definition 3.1** [14] Consider an alphabet $\Sigma_\diamond$ with $|\Sigma_\diamond| \geq 2$. The sequence of Fibonacci partial words $\{f_n^\diamond\}$, $n \geq 0$ over the alphabet $\Sigma_\diamond$ is defined as $f_{n+2}^\diamond = f_{n+1}^\diamond f_n^\diamond$. Length of Fibonacci partial word $|f_n^\diamond + 2|$ is denoted by $F^\diamond(n + 2)$ such that $|f_n^\diamond + 2| = |f_{n+1}^\diamond| + |f_n^\diamond|$ for all $n \geq 0$ and $f_0^\diamond, f_1^\diamond$ are initial Fibonacci partial words with $|f_0^\diamond|, |f_1^\diamond| \geq 2$.

**Example 3.2** Consider $f_0^\diamond = \diamond b$ and $f_1^\diamond = aa\diamond$ as initial Fibonacci partial words where $\Sigma_\diamond = \{a, b\} \cup \{\diamond\}$. Here $|f_0^\diamond| = 2$ and $|f_1^\diamond| = 3$. Then the sequence of Fibonacci partial words are as
follows:

\[ f_0^\diamond = \diamond b \]
\[ f_1^\diamond = aa\diamond \]
\[ f_2^\diamond = aa\diamond\diamond b \]
\[ f_3^\diamond = aa\diamond ba\diamond \]
\[ f_4^\diamond = aa\diamond ba\diamond aa\diamond b \]
\[ f_5^\diamond = aa\diamond ba\diamond aa\diamond ba\diamond ba\diamond ba\diamond ba\diamond ba\diamond. \]

Length of \( f_5^\diamond \) is determined by

\[ |f_{n+2}^\diamond| = |f_{n+1}^\diamond| + |f_n^\diamond| \]
\[ |f_5^\diamond| = |f_3^\diamond| + |f_2^\diamond| = 21. \]

**Definition 3.3** A Fibonacci partial word \( f_n^\diamond \) in the sequence \( \{f_n^\diamond\} \) over \( \Sigma_\diamond \) is \( p \)-periodic if \( f_n^\diamond(i) = f_n^\diamond(j) \) whenever \( i, j \) belongs to domain of \( f_n^\diamond \) and \( i \equiv j \mod p \). Here \( p \) is a positive integer termed as a strong period.

**Definition 3.4** Two Fibonacci partial words \( f_n^\diamond \) and \( g_n^\diamond \) of equal length in the sequences \( \{f_n^\diamond\} \) and \( \{g_n^\diamond\} \) are compatible, denoted by \( f_n^\diamond \uparrow g_n^\diamond \), if \( f_n^\diamond(i) = g_n^\diamond(i) \) for all \( i \in D(f_n^\diamond) \cap D(g_n^\diamond) \).

**Property 3.1** Let \( f_m^\diamond, f_n^\diamond \) and \( f_p^\diamond \) be Fibonacci partial words of equal length in the sequences \( \{f_m^\diamond\} \), \( \{f_n^\diamond\} \) and \( \{f_p^\diamond\} \) over \( \Sigma_\diamond \). The following three properties proves that compatibility on Fibonacci partial words is an equivalence relation.

Reflexive property: Trivial

Symmetric property: If \( \{f_m^\diamond\} \uparrow \{f_n^\diamond\} \) then \( \{f_n^\diamond\} \uparrow \{f_m^\diamond\} \) since \( f_m^\diamond(i) = f_n^\diamond(i) \) for all \( i \in D(f_m^\diamond) \cap D(f_n^\diamond) \).

Transitive property: Let \( \{f_m^\diamond\} \uparrow \{f_n^\diamond\} \) and \( \{f_n^\diamond\} \uparrow \{f_p^\diamond\} \). Consider two partial words \( u \) and \( v \) such that \( \{f_m^\diamond\} \subset w, \{f_n^\diamond\} \subset v \) and \( \{f_p^\diamond\} \subset v \). Let \( w \) represent \( u \vee w \). Then \( \{f_m^\diamond\} \subset w \), \( \{f_n^\diamond\} \subset w \) and \( \{f_p^\diamond\} \subset w \). Hence \( \{f_m^\diamond\} \uparrow \{f_p^\diamond\} \).

**Theorem 3.1** If two initial Fibonacci partial words of two different sequences of Fibonacci partial words are compatible then both the sequences of Fibonacci partial words are compatible.

**Proof.** Let \( \{f_m^\diamond\} \) and \( \{g_n^\diamond\} \) be two sequence of Fibonacci partial words over \( \Sigma_\diamond \) with \( |\Sigma_\diamond| \geq 2 \). The above statement is provable if the initial partial words of both the sequences are of equal length say \( |f_0^\diamond| = |g_0^\diamond| \) and \( |f_1^\diamond| = |g_1^\diamond| \).

We claim that \( 1 \leq r \leq 2 \), if \( f_r^\diamond \uparrow g_r^\diamond \) then \( f_r^\diamond \uparrow g_r^\diamond \vee r \leq n \) and \( \{f_n^\diamond\} \uparrow \{g_n^\diamond\} \). Let us prove by contradiction. Let \( f_2^\diamond \not\uparrow g_2^\diamond \forall 1 \leq r \leq 2 \), and let \( f_2^\diamond \uparrow g_2^\diamond \forall 1 \leq r \leq 2 \).

From the notion of sequence of Fibonacci partial words,

\[ f_r^\diamond = f_{r-1}^\diamond f_{r-2}^\diamond \]
\[ g_r^\diamond = g_{r-1}^\diamond g_{r-2}^\diamond \]

Then \( f_{(r-1)}^\diamond \not\uparrow g_{(r-1)}^\diamond \) and \( f_{(r-2)}^\diamond \not\uparrow g_{(r-2)}^\diamond \) \( \forall r \leq n \). Consider \( r = 3 \). Then \( f_3^\diamond \not\uparrow g_3^\diamond \). But this contradicts the fact that \( \forall 1 \leq r \leq 2 \), \( f_r^\diamond \uparrow g_r^\diamond \) since \( f_3^\diamond = f_2^\diamond f_1^\diamond \) and \( g_3^\diamond = g_2^\diamond g_1^\diamond \). Therefore \( \forall 1 \leq r \leq 2 \), if \( f_r^\diamond \uparrow g_r^\diamond \) then \( f_r^\diamond \uparrow g_r^\diamond \forall r \leq n \) and \( \{f_n^\diamond\} \uparrow \{g_n^\diamond\} \).

**Theorem 3.2** Consider the Fibonacci partial words \( f_x^\diamond, f_y^\diamond, f_m^\diamond \) and \( f_n^\diamond \) of equal length in the sequences \( \{f_x^\diamond\}, \{f_y^\diamond\}, \{f_m^\diamond\} \) and \( \{f_n^\diamond\} \). Then \( f_m^\diamond \uparrow f_n^\diamond \) and \( f_x^\diamond \uparrow f_y^\diamond \) if \( f_m^\diamond f_x^\diamond \uparrow f_n^\diamond f_y^\diamond \).
Proof. If \( f_m^o \uparrow f_y^o \) then \( f_m^o f_y^o \subseteq f_m^o f_x^o \lor f_y^o f_y^o \), and \( f_m^o f_y^o \subseteq f_m^o f_x^o \lor f_y^o f_y^o \), where \( \lor \) denotes the least upper bound of \( f_m^o f_y^o \) and \( f_y^o f_y^o \). Then the domain of \( f_m^o f_x^o \) and \( f_y^o f_y^o \) is exactly equal to the union of domains of \( f_m^o f_x^o, f_y^o \), and \( f_y^o f_y^o \).  

\[
D(f_m^o f_x^o \lor f_y^o f_y^o) = D(f_m^o) \cup D(f_y^o) \cup D(f_m^o) \cup D(f_y^o)
\]

\( D(f_m^o) \cup D(f_y^o) = D(f_m^o \lor f_y^o) \) implies that \( f_m^o \uparrow f_y^o \) and \( f_m^o \subseteq f_m^o \lor f_y^o \). Similarly we can show that \( f_x^o \uparrow f_y^o \) which completes the proof.

**Theorem 3.3** If the Fibonacci partial words \( f_m^o \) and \( f_y^o \) of equal length in the sequences \( \{f_m^o\} \) and \( \{f_y^o\} \) are compatible then \( f_m^o f_y^o \lor f_y^o f_m^o \subseteq (f_m^o \lor f_y^o)^2 \).

Proof. Since \( f_m^o \) and \( f_y^o \) are compatible, \( f_m^o \) and \( f_y^o \) are contained in \( f_m^o \lor f_y^o \). This implies that \( f_m^o f_y^o \) and \( f_y^o f_m^o \) are contained in \( (f_m^o \lor f_y^o)^2 \). For instance let \( f_3^o = a \circ ab \circ a \circ b \circ a \) with initial words \( b \circ a \) and \( g_3^o = a \circ a \circ b \circ \circ a \) with initial words \( b \circ a \) and \( \circ a \). Since \( f_3^o \uparrow g_3^o \) implies that \( f_3^o \subseteq f_3^o \lor g_3^o \), \( g_3^o \subseteq f_3^o \lor g_3^o \) and also \( f_3^o g_3^o \) and \( g_3^o f_3^o \) is \( (f_3^o \lor g_3^o)^2 \). Then by the definition of least upper bound,

\[
f_3^o g_3^o \lor g_3^o f_3^o \subseteq (f_3^o \lor g_3^o)^2.
\]

Therefore \( f_m^o f_y^o \lor f_y^o f_m^o \subseteq (f_m^o \lor f_y^o)^2 \).

**Definition 3.5** Consider a Fibonacci partial word \( f_3^o \) of length \( |f_3^o| \) in the sequence of Fibonacci partial words \( \{f_3^o\} \) over \( \Sigma_0 \) with \( |\Sigma_0| \geq 2 \). The binary vector \( x \) with length \( |f_3^o| \) is the binary correlation of \( f_3^o \) if \( x_i = 0 \) if \( i \) belongs to the set of all periods of \( f_3^o \) which is denoted as \( \phi(f_3^o) \) and \( 0 \) otherwise.

**Example 3.6** Consider \( f_0^o = b \circ b \) and \( f_1^o = \circ a \) as initial Fibonacci partial words over \( \Sigma_0 = \{a, b\} \cup \{\circ\} \). Here \( |f_0^o| = 3 \) and \( |f_3^o| = 2 \). Then

\[
f_3^o = \circ a \ b \circ b \circ a
\]

\[
\rightarrow \circ a \ b \circ b \circ a \ldots \ldots 1
\]

\[
\rightarrow \ast \circ a \ b \circ b \circ a \ldots \ldots 0
\]

\[
\rightarrow \ast \ast \circ a \ b \circ b \ldots \ldots 0
\]

\[
\rightarrow \ast \ast \ast \circ a \ b \circ b \ldots \ldots 0
\]

\[
\rightarrow \ast \ast \ast \ast \circ a \ b \ldots \ldots 0
\]

\[
\rightarrow \ast \ast \ast \ast \circ a \ b \ldots \ldots 1
\]

\[
\rightarrow \ast \ast \ast \ast \ast \circ a \ b \ldots \ldots 0
\]

The Fibonacci partial word \( f_3^o \) has correlation 1000010 with period 5 and 7.

**Property 3.2** Two compatible Fibonacci partial words of two different sequence of Fibonacci partial words need not have same binary correlation.

**Example 3.7** Consider the initial Fibonacci partial words \( f_0^o = b \circ b \) and \( f_1^o = a \circ a \) in the sequence of Fibonacci partial words \( \{f_n^o\} \) and \( g_0^o = b \circ b \) and \( f_1^o = a \circ a \) in the sequence of Fibonacci partial words.
\{g_n^\circ\} over \Sigma_\circ with |\Sigma_\circ| \geq 2. Consider \(f_2^\circ\) and \(g_2^\circ\) where \(f_2^\circ \uparrow g_2^\circ\).

\[
f_2^\circ = \quad a \quad a \quad b \\
\to \quad a \quad a \quad b \ldots \ldots 1 \\
\to \quad \ast \quad a \quad \ast \ast \ast \ldots \ldots 1 \\
\to \quad \ast\ast \quad a \quad \ast \ldots \ldots 0 \\
\to \quad \ast\ast\ast \quad a \quad \ast \ldots \ldots 0 \\
\to \quad \ast\ast\ast\ast \quad \ast\ldots \ldots 0 \\
\]

\[
g_2^\circ = \quad \ast \ast \ast \ast \quad a \quad b \\
\to \quad \ast \ast \ast \ast \quad a \quad b \ldots \ldots 1 \\
\to \quad \ast \ast \ast \ast \quad a \quad \ast \ldots \ldots 0 \\
\to \quad \ast \ast \ast \ast \quad a \quad \ast \ldots \ldots 0 \\
\to \quad \ast \ast \ast \ast \quad \ast \ldots \ldots 0 \\
\]

Thus \(f_2^\circ\) and \(g_2^\circ\) are compatible Fibonacci partial words of two different sequence of Fibonacci partial words with different binary correlation.

**Property 3.3** For all \(n \geq 1\), \(f_n^\circ\) is unequal to any rotation of \(f_n^\circ\) other than itself but the binary correlation of \(f_n^\circ\) can be equal to one or more binary correlation of any rotation of \(f_n^\circ\). 

**Property 3.4** If \(f_n^\circ\) and \(g_n^\circ\), \(n \geq 2\) are Fibonacci partial words in the sequence of Fibonacci partial words \(\{f_n^\circ\}\) and \(\{g_n^\circ\}\) over \(\Sigma_\circ\), then

\[
\varphi(f_n^\circ \cup \varphi(g_n^\circ) \subset \varphi(f_n^\circ \cap g_n^\circ) \quad (1)
\]

\[
\varphi(f_n^\circ \cap \varphi(g_n^\circ) \subset \varphi(f_n^\circ \cup g_n^\circ) \quad (2)
\]

where \((f_n^\circ \cap g_n^\circ)\) is the greatest lower bound of \(f_n^\circ\) and \(g_n^\circ\) and \((f_n^\circ \cup g_n^\circ)\) is the least upper bound of \(f_n^\circ\) and \(g_n^\circ\).

**Theorem 3.4** Consider a Fibonacci partial word \(f_1^\circ\) in the sequence of Fibonacci partial words \(\{f_n^\circ\}\) over \(\Sigma_\circ\). Let \(r\) be a set of full words over \(\Sigma\) with period \(p\). Then \(p\) is a period of \(f_1^\circ\) if \(r\) is compatible with \(f_1^\circ\).

**Proof.** Consider \(r = \{r_1, r_2, \ldots, r_n\}\) as a set of total words over \(\Sigma\) with period \(p\). Then

\[
\varphi(r) = \varphi(r_1) \cup \varphi(r_2) \cup \ldots \cup \varphi(r_n)
\]

where \(\varphi(r)\) indicates the set of all periods (period set) of \(r\). Consider \(r \uparrow f_1^\circ\). Then by the definition of strong period, \(i \equiv j \mod p\) whenever \(r(i) = r(j) = f_1^\circ(i) = f_1^\circ(j)\) for all \(i, j\) belongs to domain of \(f_1^\circ\). This shows that \(p\) is a period of \(f_1^\circ\). For instance let \(r = \{r_1, r_2, r_3\}\) be a set of full words over \(\Sigma\) with period \(p\) such that \(r_1 = aabbaa, r_2 = bababa\) and \(r_3 = aababa\) Then

\[
\varphi(r) = \varphi(r_1) \cup \varphi(r_2) \cup \varphi(r_3).
\]

Consider \(f_1^\circ = b^\circ\) and \(f_1^\circ = a^\circ\) as initial Fibonacci partial words in the sequence of Fibonacci partial words \(\{f_n^\circ\}\) over \(\Sigma_\circ\). We get \(f_2^\circ = a^\circ b^\circ \quad \circ a\cdot Then

\[
\varphi(f_2^\circ) = \varphi(r_1) \cup \varphi(r_2) \cup \varphi(r_3) \quad = \{2, 4, 5, 6\}
\]
are the words \( r \) compatible with \( f_n^\ominus \). Thus the set of all periods of any Fibonacci partial word \( f_n^\ominus \) over \( \Sigma_\ominus \) is equal to the union of the set of all periods of all total words \( r \) over \( \Sigma \) compatible with \( f_n^\ominus \). Hence the proof.

**Corollary 3.1** The set of all periods of any Fibonacci partial word \( f_n^\ominus \) in the sequence of Fibonacci partial words \( \{ f_n^\ominus \} \) over \( \Sigma_\ominus \) is equal to every union of the set of all periods of all total words \( x \) over \( \Sigma \) compatible with \( f_n^\ominus \).

### 4. Conclusion

In this paper, binary correlations and compatibility of Fibonacci partial words are discussed with illustrations. In future, few more properties such as weak periodicity, ternary correlations etc can be studied for Fibonacci partial words.

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