Emergence of magnetic field due to spin-polarized baryon matter in neutron stars

M. Kutschera

Astrophysics Division, H. Niewodniczański Institute of Nuclear Physics, ul. Radzikowskiego 152, 31-342 Kraków, Poland

Institute of Physics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland

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ABSTRACT

A model of the ferromagnetic origin of magnetic fields of neutron stars is considered. In this model, the magnetic phase transition occurs inside the core of neutron stars soon after formation. However, owing to the high electrical conductivity the core magnetic field is initially fully screened. We study how this magnetic field emerges for an outside observer. After some time, the induced field that screens the ferromagnetic field decays enough to uncover a detectable fraction of the ferromagnetic field. We calculate the time-scale of decay of the screening field and study how it depends on the size of the ferromagnetic core. We find that the same fractional decay of the screening field occurs earlier for larger cores. We conjecture that weak fields of millisecond pulsars, $B \sim 10^8 - 10^9 G$, could be identified with ferromagnetic fields of unshielded fraction $\epsilon \sim 10^{-4} - 10^{-3}$ resulting from the decay of screening fields by a factor $1 - \epsilon$ in $\sim 10^8$ yr since their birth.

Key words: dense matter - stars: magnetic fields - stars: neutron - pulsars: general

1 INTRODUCTION

The physical origin of the magnetic field of neutron stars remains an open problem. Some researches adopt a working hypothesis that the field is inherited from the progenitor star. There also exist more specific models of generation of the magnetic field during the early cooling of a young neutron star. Generally, this kind of approach links the magnetic field of neutron stars to macroscopic currents generated either by the motion of the ionized material or by heat transport (Chanmugam 1992).

For neutron stars, unlike for any other astrophysical object, there exists a different physical possibility, namely that the magnetic field is of microscopic origin and reflects spin-polarization properties of the neutron star matter itself. The ground state of dense baryonic matter is determined by nuclear interactions that possess very strong spin components. It seems likely that these interactions lead to spin ordering in the neutron star matter making the ground state of the system a permanently magnetized phase, at least in some range of densities.

Mechanisms of spin ordering in dense matter have been studied by a number of authors (Kutschera & Wójcik 1989, 1990, 1993, 1996, 1997; Kutschera, Broniowski & Kotlorz 1990a,b; Niembro et al. 1990; Marcos et al. 1991; Kutschera 1994a,b). Kutschera & Wójcik (1989) have shown that neutron star matter with a low proton fraction of a few per cent is particularly susceptible to ferromagnetic spin ordering. We have shown (Kutschera & Wójcik 1990, 1993, 1995; Kutschera 1994a,b) that protons in neutron star matter with such a small proton admixture behave as polarons that can localize at higher densities. Neutron star matter with localized protons displays a ferromagnetic instability (Kutschera & Wójcik 1989, 1996,1997) and the spin-ordered system develops a permanent magnetization, $M \sim n_p \mu_{eff}$, where $n_p$ is the proton number density and $\mu_{eff}$ is the effective magnetic moment of the localized proton. Typical values of magnetization are in the range $10^{13} - 10^{14} G$ (Kutschera & Wójcik 1996).

There exist also other model mechanisms that result in a spontaneous magnetization of hadronic matter inside neutron stars. The presence of the neutral pion condensate in the nucleon matter can lead to ordering of nucleon spins (Dautry & Nyman 1979). The same mechanism can operate in quark matter with chiral condensate (Kutschera et al. 1990a,b). The magnetization can be higher in this case, up to $\sim 10^{15} G$.

Uncertainties in the physics of dense hadronic matter prevent a firm conclusion concerning the nature of the ground state of baryon matter in neutron stars. However, ubiquity of spin interactions between hadrons with strong couplings makes the spin-polarized ground state of dense baryon matter a possibility that deserves serious consideration.

Independently of the mechanism of polarization, one can study the phenomenological consequences of the existence of spin-polarized baryon matter in neutron stars. If the ground state of dense matter is permanently magnetized...
at baryon number densities exceeding some critical density \( n_f \), then there could exist a class of neutron stars possessing magnetic cores provided \( n_f < n_{f,\text{max}} \), where \( n_{f,\text{max}} \) is the central baryon density of the maximum mass neutron star. Only sufficiently massive stars, with central densities \( n_c > n_f \) would belong to this class. The critical density, \( n_f \), is estimated to exceed twice the nuclear saturation density. The magnetized core is a source of the magnetic field. However, the phase transition producing the magnetized core occurs in the inner part of the star, which has very high electrical conductivity. Thus, any magnetic field due to the sudden appearance of the spontaneous polarization of baryonic matter becomes fully screened in such a medium. It takes some time for the magnetized core to become detectable by an outside observer. The aim of this paper is to investigate this problem.

The fact that any magnetic moment spontaneously created in the neutron star core is initially fully screened has important consequences for the spin structure of the magnetic core, which can form a single magnetic domain. Such a domain is the most favourable configuration as far as the hadronic energy gain resulting from spin ordering is concerned. Because of screening, electromagnetic interactions do not force the single domain core to fragment into smaller magnetic domains.

In the next section we describe the model of the magnetized core. In Section 3 we study the screening of the ferromagnetic field. Then in Section 4 the decay of the screening currents is considered. Section 5 contains astrophysical implications of the model. Some useful formulae are given in Appendix A.

2 THE MAGNETIZED CORE INSIDE A NEUTRON STAR

A single domain structure of the magnetic core is favoured by the dynamics of the magnetic phase transition. As the neutron star matter cools, the conditions for the spontaneous spin ordering to occur are met for the first time at the centre of the star. This is because the energy gain resulting from spin ordering in the ferromagnetic phase increases with density (Kutschera & Wójcik 1990, 1993, 1995, 1996). Thus, additionally, the critical temperature increases with density, and, for a roughly isothermal core, the first small bubble of the spin-ordered phase is nucleated at the centre of the neutron star. This first small domain grows as the temperature drops with nucleon spins in the next layers of matter being polarized in the same direction. A single domain is the minimum hadronic energy configuration of the spin-polarized phase of neutron star matter. So far we have neglected electromagnetic interactions in the polarized phase. One can worry that, when magnetostatic energy is included, such a single domain may fragment into smaller randomly oriented domains. We show at the end of Section 3 that this is not the case.

To model the magnetized core we assume that the spontaneously polarized matter forms a single domain with magnetization in the z-direction: \( \mathbf{M} = M(r) \mathbf{z} \). For this study the only relevant quantity is the magnetization as a function of baryon number density, \( M \equiv M(n_b) \). Calculations reported by Kutschera & Wójcik (1996) show that the magnetization depends only weakly on density, \( M \approx M_0 \approx 10^{13} - 10^{14} G \) in the whole range of relevant densities. In the case of polarized quark matter with chiral condensate, the magnetization is essentially density independent (Kutschera et al. 1990a,b).

We can thus parametrize the magnetization distribution inside neutron stars as

\[
M(r) = M_0 f(r),
\]

where the function \( f(r) \) accounts for the radial variation of magnetization. In the following we adopt a simple form:

\[
f(r) = \begin{cases} 1 & \text{for } r < r_a \\ 0 & \text{for } r > r_b \end{cases}
\]

This function corresponds to a uniformly magnetized sphere of radius \( r_a \) with a surface layer of thickness \( h = r_b - r_a \) where the magnetization drops linearly to zero at \( r = r_b \) with \( f(r) = (r - r_b)/(r_a - r_b) \) (Fig.1). This simple function is flexible enough to account for the main features of the magnetization distribution in neutron stars.

The magnetic moment of the core is

\[
d_{\text{core}} = \int_0^{r_b} M(r) d^3r,
\]

with \( M(r) \) given by equation (1). The size of the core is determined by the critical density for spontaneous polarization, \( n_f \). Thus the nonzero magnetization, \( M(r) \neq 0 \), exists only in the inner core of stars of density \( n_f(r) > n_f \). The core radius \( r_b \) corresponds to the critical density \( n_f \): \( n_f(r_b) = n_f \).

The contribution of the magnetic moment (2) to the magnetic field at the pole is

\[
B_{\text{pol}}^{\text{fer}} = \frac{2d_{\text{core}}}{R_{\text{NS}}^3},
\]

where \( R_{\text{NS}} \) is the radius of the neutron star. This field is \( B_{\text{pol}} \sim 10^{12} - 10^{13} G \) for the above typical values of the magnetization.

3 SCREENING OF THE FERROMAGNETIC FIELD

Calculations in various models suggest the energy per baryon in the polarized phase could be below that for the normal phase by at least \( \sim 1 MeV \). The phase transition from normal matter, with no spin ordering, to magnetized matter is thus expected to occur very soon after the formation of the neutron star. The phase transition is completed quickly. For any practical purposes one can assume that the ferromagnetic field is switched on instantaneously. This is because the time-scale for magnetic field diffusion, determined by the electrical conductivity, is much longer than the duration of the phase transition.

The neutron star matter outside the magnetic core is a medium of very high electrical conductivity \( \sigma \). Realistic calculations show that conductivities corresponding to the neutron star crust are, typically, \( \sigma_{\text{crust}} \sim 10^{23} s^{-1} \) (Channugum 1992). These values are lower than those corresponding to the core, \( \sigma_{\text{core}} \sim 10^{29} s^{-1} \) (Channugum 1992). It is rather obvious that sudden switching-on of the magnetic field of the magnetized core will result in the induction of the screening field, which will fully shield the ferromagnetic field. This happens because for such a high conductivity the magnetic flux through any loop is conserved on the time-scale of the phase transition. Flux conservation then requires
that $\partial \mathbf{B}/\partial t = 0$. We assume that there is no magnetic field before the phase transition occurs, $\mathbf{B} = 0$. Thus, at the instant $t = 0$ the switched-on ferromagnetic field, $\mathbf{B}_{fe}$, and the induced field, $\mathbf{B}_{ind}(t)$, cancel one another exactly,

$$\mathbf{B}_{fe} + \mathbf{B}_{ind}(0) = 0. \quad (4)$$

The ferromagnetic field of the core is easily calculated. The vector potential has only the $\phi$-component

$$A_\phi = \frac{R(r)}{r} \sin \theta, \quad (5)$$

where the function $R(r)$ reads

$$R(r) = d(r)/r. \quad (6)$$

Here $d(r)$ is the magnetic moment inside the sphere of radius $r$,

$$d(r) = \int_0^r M(r')dr'. \quad (7)$$

Components of the magnetic field are

$$B_{e}^{fe} = \frac{2R(r)}{r^2} \cos \theta, \quad (8)$$

$$B_{\theta}^{fe} = -\frac{1}{r} \frac{\partial R(r)}{\partial r} \sin \theta. \quad (9)$$

The flux conservation condition, equation (4), gives the components of the induced field that screens the ferromagnetic field,

$$B_{e}^{ind}(0) = -B_{e}^{fe}, \quad (10)$$

$$B_{\theta}^{ind}(0) = -B_{\theta}^{fe}. \quad (11)$$

These formulae show that the induced field components, $B_{e}^{ind}$ and $B_{\theta}^{ind}$, satisfy equations (8) and (9) with the function $R_{ind}(r,t)$, which, however, depends also on time. At $t = 0$ the condition (4) implies that

$$R_{ind}(r,0) = -R(r). \quad (12)$$

One can also obtain the space structure of the current sustaining the induced field. Generally, this current changes in time, $\mathbf{J}_{ind} \equiv J_{ind}(r,\theta,t)$. From the formula $\nabla \times \mathbf{B}_{ind} = 4\pi/\mathbf{J}_{ind}$ it follows that

$$\frac{4\pi}{c} J_{\phi}^{ind} = \frac{\sin \theta}{r} \left( \frac{\partial^2 R_{ind}}{\partial \phi^2} - \frac{2R_{ind}}{r^2} \right). \quad (13)$$

At $t = 0$ using equation (4), we find

$$J_{ind}(r,\theta,0) = -c \nabla \times \mathbf{M}. \quad (14)$$

The above discussion allows us now to address the question of the stability of a large single domain against fragmentation into smaller randomly oriented domains. In the case of terrestrial ferromagnets, large domains are not stable. Their magnetostatic energy can be lowered by producing a number of smaller domains with apparently randomly oriented magnetic moments. Let us stress that reduction of the energy is due to dipole magnetic interactions between these magnetic moments. The magnetic core in neutron stars forming a single domain with screened magnetic field cannot lower its magnetostatic energy by fragmenting into smaller domains. This is because the magnetic moment of any new domain would also be fully screened and thus, unlike in terrestrial ferromagnets, there would be no magnetic dipole interactions between neighbouring domains. The core is thus stable against fragmentation into small domains.

### 4 DECAY OF SCREENING FIELDS

The induced current, $\mathbf{J}_{ind}$, will suffer ohmic decay as the electrical conductivity, $\sigma$, though very high, is finite. The nonzero net field, $\mathbf{B} = \mathbf{B}_{fe} + \mathbf{B}_{ind}(t) \neq 0$, will eventually emerge. Let us note that ohmic decay is the only relevant mechanism of magnetic field decay as long as the net magnetic field is low. Ambipolar diffusion and Hall drift, which play a crucial role in the decay of strong fields (Goldreich & Reisenegger 1992), can be safely neglected.

To calculate the time behaviour of the screening field we apply a standard analysis of decay modes (Wendell, Van Horn & Sargent 1987). To avoid unnecessary complications in our exploratory study we neglect spatial and temporal dependence of the conductivity and assume that $\sigma = \text{const}$. The time dependence of the induced field can be found from the expansion of the function $R_{ind}(r,t)$ into eigenfunctions

$$X_n(x) = \sqrt{2} n \pi x j_1(n \pi x), \quad (15)$$

where $j_1$ is the Bessel function and $x = r/R_*$ is the normalized radial variable with a suitably chosen radius $R_*$. The expansion reads

$$R_{ind}(x,t) = \sum_n C_n X_n(x) \exp(-t/\tau_n), \quad (16)$$

where $\tau_n = 4\pi R_0^2 \sigma/(c \pi n)^2$ is the decay time of the $n$th mode.

The expansion coefficients, $C_n$, are obtained using the function $R_{ind}(x,0) = -R(x)$ as the initial condition:

$$C_n = \int_0^1 R_{ind}(x,0) X_n(x)dx. \quad (17)$$

With our choice of the magnetization profile, $f(r)$, the function $R(x)$ reads

$$R(x) = \frac{4}{3} \pi R_0^2 g(x), \quad (18)$$

where the function $g(x)$ is determined entirely by the magnetization profile function $f(r)$. The function $g(x)$, calculated by integrating the magnetization in equation (7), is given in Appendix A.

The form (18) of the function $R(x)$ indicates that the time behaviour of the screening field is sensitive to the magnetization profile. To study this behaviour we have calculated the coefficients $C_n$ of the expansion, equation (17), which are also given in Appendix A. The unshielded fraction, $\epsilon(x,t)$, of the ferromagnetic field emerging after time $t$
is

$$\epsilon(x, t) = \frac{|B_{fer} + B_{ind}(t)|}{|B_{fer}|} = 1 - \frac{R_{ind}(x, t)}{R_{ind}(x, 0)}.$$  \hspace{1cm} (19)

We are interested mostly in the value of $\epsilon(x, t)$ at the surface of the star, $x = 1$, which we denote as $\epsilon(t) \equiv \epsilon(1, t)$.

The formula (19) shows that $\epsilon(t)$ does not depend on the magnetization $M_0$. This means that the relative rate of emergence of the ferromagnetic field depends only on the geometry of the magnetized core. For the case of uniform conductivity we consider here this behaviour is even more universal. We can express the time behaviour of the emerging magnetic field in terms of the dimensionless variable $y = t/\tau_1$, where

$$\tau_1 = \frac{4R^2\sigma}{\epsilon^2\pi}$$  \hspace{1cm} (20)

is the decay time of the longest-living (fundamental) decay mode. The formula (19) becomes

$$\epsilon(y) = 1 - \frac{\sum_a C_n X_n(1) \exp(-yn^2)}{\sum_a C_n X_n(1)}.$$  \hspace{1cm} (21)

The main results of our analysis are presented in Fig.2 where we show $\epsilon(y)$ for indicated values of the parameters $z_a \equiv r_a/R_\ast$ and $z_b \equiv r_b/R_\ast$ governing the spatial distribution of magnetization in our simple model. As one can notice, the shape of the curves is determined essentially by the value of $z_b$ with much smaller influence of $z_a$.

The general tendency is that the higher is $z_b$ the higher is $\epsilon$. The curves for the same $z_b$ and various $z_a$ are very close to each other. We see that the time dependence of the unshielded fraction $\epsilon$ is most sensitive to the total radial extension of the ferromagnetic core. The radial distribution of magnetization, which is our model is controlled by $z_a$, has a much smaller influence on $\epsilon$. This means that the ferromagnetic field for cores of larger extension becomes uncovered earlier than for smaller ones. The curves shown in Fig.2 prove that the unshielded fraction $\epsilon$ is very sensitive to $z_b$ and is much less sensitive to $z_a$. For each value of $z_b$ we show curves corresponding to two extreme values of $z_a$: $z_a = 0.001$ (dashed curves) and $z_a = 0.9z_b$ (solid curves).

In our discussion above we have made a number of simplifying assumptions regarding the neutron star structure. The conductivity is assumed to be uniform and constant in time, which is equivalent to constant temperature. Also, Euclidean geometry is used. These simplifications are not expected to affect our qualitative conclusions as far as the decay time-scale of the screening field is concerned.

**5 CONCLUSIONS AND IMPLICATIONS**

The main conclusion from the above analysis is that the emergence time of the ferromagnetic field, at low values of the unshielded fraction $\epsilon$, is very sensitive to the size of the polarized core. For large cores, comprising $\sim 90$ per cent of the radius, a fraction $\epsilon \sim 10^{-4}$ of the core field is visible after $t \sim 10^{10}\pi$. For small cores, $z_b \sim 0.4$, the same fractional field is visible after $t \sim 10^{-10}\pi$, i.e. a factor of $\sim 10^7$ later.

To explore the astrophysical consequences of this model we must specify the relevant values of the decay time $\tau_1$, or equivalently the electrical conductivity $\sigma$. For magnetized cores in neutron stars, the relevant conductivity is that of the liquid core matter, $\sigma_{core} \sim 10^{29}s^{-1}$. The corresponding decay time is $\tau_1 \sim 10^{12}$ yr. One should notice that the choice of the $\sigma_{core}$ implies that the presence of the neutron star crust becomes somewhat irrelevant for our considerations here. This is because the crust conductivity is lower by a factor $\sim 10^{-6}$. An important consequence is that the value of the radius $R_\ast$ we use in the definition of the variable $x$ should be identified with the radius of the liquid core, $R_\ast = R_{lc}$, rather than the total neutron star radius (which includes the crust). Correspondingly, $z_a$ and $z_b$ are the fractions of the liquid core radius, $z_a = r_a/R_{lc}$ and $z_b = r_b/R_{lc}$. With this observation the values $z_b \geq 0.9$ do not seem unrealistic, especially for soft equations of state.

Applying our model to neutron stars we find that the magnetic field of the magnetized core formed soon after the birth of the star emerges in $\sim 10^6$ yr at a level of $\sim 10^{-3}B_{fer}^{\ast} \sim 10^8G$ provided the ferromagnetic core is large enough. Both the time-scale of $10^6$ yr and the magnetic field of $10^8G$ are typical for millisecond pulsars. One can thus conjecture that the magnetic fields of millisecond pulsars are due to spin-polarized matter inside neutron stars.

The ferromagnetic origin of the magnetic fields of millisecond pulsars could explain the discrepancy between the birth rate of low mass X-ray binaries (LMXB), which are supposed to be the progenitors of millisecond pulsars, and the number of millisecond pulsars in the Galaxy (Bhattacharya 1995). In this case, single neutron stars born with short rotation periods and low magnetic fields (much less than $10^7G$) could become millisecond pulsars in $\sim 10^9$ yr. This would also help to understand the existence of single millisecond pulsars without invoking the companion evaporation scenario. Also, this model would shed some new light on the old question of the decay time of the inherited magnetic field of normal radio pulsars. Currently, a popular view is that only magnetic fields of neutron stars that accreted matter in binary evolution decay significantly. In the ferromagnetic model, the inherited magnetic field of an isolated neutron star could decay in $\sim 10^7$ yr, as was found in early studies of the subject (Ostriker & Gunn 1969). There would be no need for direct connection between the amount of matter accreted by the neutron star in LMXBs and the amount of decay of its magnetic field. Recent analysis by Wijers (1997) provides some evidence against field decay being proportional to the mass accreted by a neutron star during the LMXB evolution, supporting our conjecture.

Let us stress finally that looking for the presence (or absence) of a ferromagnetic component of the magnetic field of neutron stars is of great importance for the physics of hadronic matter. Physical conditions prevailing in neutron stars (high density, low temperature and $\beta = equilibrium$) are not accessible to any laboratory experiment. Magnetic
field could serve as a direct probe of dense hadronic matter in the neutron star interior. Evidence in favour of the presence of spin-polarized matter inside neutron stars would introduce new qualitative features of the ground state of dense hadronic matter. An opposite conclusion would mainly constrain the critical density \( n_f \). The fact that a magnetic field resulting from the ferromagnetic core grows with time, while the inherited field decays in time, could potentially allow one to distinguish the two components.

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**APPENDIX A:**

For our choice of the magnetization profile, \( f(r) \), the function \( g(x) \) is a smooth function, which is given analytically below:

\[
g(x) = x^2, \quad x < z_a, \quad \text{(A1)}
\]

\[
g(x) = \frac{A}{x} + Bx^2 + Cx^3, \quad z_a < x < z_b, \quad \text{(A2)}
\]

where

\[
A = z^3_a + \frac{0.75 z^4_a - z_b z^3_a}{z_b - z_a}, \quad \text{(A3)}
\]

\[
B = \frac{z_b}{z_b - z_a}, \quad \text{(A4)}
\]

and,

\[
C = -\frac{0.75}{z_b - z_a}, \quad \text{(A5)}
\]

and, finally,

\[
g(x) = \frac{D}{x}, \quad z_b < x < 1, \quad \text{(A6)}
\]

where

\[
D = z^3_a + \frac{1}{z_b - z_a} (\frac{3}{4} (z_a^4 - z_b^4) + z_b (z_b^3 - z_a^3)). \quad \text{(A7)}
\]

The expansion coefficients \( C_n \) in equation (19) read:

\[
C_n = -\frac{4}{3} \pi M_0 R_c^2 c_n, \quad \text{(A8)}
\]

where

\[
c_n = 2 [a_n(z_a) + A(b_n(z_a) - b_n(z_b)) + B(a_n(z_b) - a_n(z_a)) + \frac{C}{(n \pi)^3} (d_n(z_b) - d_n(z_a))] + D b_n(z_b). \quad \text{(A9)}
\]

Here

\[
a_n(z) = -\frac{3n \pi z \cos(n \pi z) + 3 \sin(n \pi z)}{(n \pi)^3} \quad \text{(A10)}
\]

\[
b_n(z) = \frac{\sin(n \pi z)}{n \pi z} \quad \text{(A11)}
\]

and

\[
d_n(z) = 8 \cos(n \pi z) - 4 (n \pi z)^2 \cos(n \pi z) + 8n \pi z \sin(n \pi z) - (n \pi z)^3 \sin(n \pi z). \quad \text{(A12)}
\]

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