INFINITE SYSTOLIC GROUPS ARE NOT TORSION

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Abstract. We study $k$-systolic complexes introduced by T. Januszkiewicz and J. Świątkowski, which are simply connected simplicial complexes of simplicial nonpositive curvature. Using techniques of filling diagrams we prove that for $k \geq 7$ the 1-skeleton of a $k$-systolic complex is Gromov hyperbolic. We give an elementary proof of the so-called Projection Lemma, which implies contractibility of 6-systolic complexes. We also present a new proof of the fact that an infinite group acting geometrically on a 6-systolic complex is not torsion.

Introduction. Simply connected nonpositively curved metric spaces, called CAT(0) spaces, have been intensively studied for over 50 years, and they are one of the major parts of geometric group theory [2]. Because of a theorem by Gromov, a special place among CAT(0) spaces is occupied by CAT(0) cube complexes: a cube complex is CAT(0) if and only if it is simply connected and satisfies an easily checked, local combinatorial condition (links are flag).

Therefore the question naturally arises if there exists a similar characterization for simplicial complexes. A partial answer is the notion of systolic complexes. These are simply connected simplicial complexes whose links satisfy a certain combinatorial condition called 6-largeness. This makes them good analogues of CAT(0) cube complexes. This condition may be treated as an upper bound for simplicial curvature around the vertex, and hence complexes with $k$-large links for $k \geq 6$ are also called complexes of simplicial nonpositive curvature (SNPC). The $k$-largeness condition is geometrically motivated by the 2-dimensional case, where systolic complexes actually are CAT(0). In the general case, SNPC neither implies nor is implied by metric nonpositive curvature.

Systolic complexes were introduced by V. Chepoi [4] under the name of bridged complexes, although their 1-skeleta, the bridged graphs, were studied earlier by V. Chepoi, R. E. Jamison, M. Farber and V. P. Soltan [5 9 3].

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In the context of group theory, systolic complexes were rediscovered independently by F. Haglund \cite{11} and by J. Świątkowski and T. Januszkiewicz \cite{13}, and they were used to construct word-hyperbolic groups of large cohomological dimension that do not come from isometries of the hyperbolic $n$-space. However, systolic complexes turned out to be of interest on their own. It has been shown that in many situations systolic complexes behave like CAT(0) spaces, and share many of their properties \cite{13,15,7,8}.

In the current work we mainly focus on analogies between systolic complexes and CAT(0) spaces. We present new and often more direct proofs of various results of \cite{13}, mostly using techniques of filling diagrams. Since our article is self-contained, it may also be treated as an introduction to the theory of systolic complexes. Another introductory source is \cite{18}.

Let us briefly describe the content of the article. In Section 1 we give preliminary definitions and fix the notation. In Section 2 we define $k$-large and $k$-systolic complexes, and we introduce combinatorial tools to study these complexes. In Section 3 we prove that the 1-skeleton of a $k$-systolic complex is $\delta$-hyperbolic for $k \geq 7$. The proofs in these sections are usually simplified versions of ones in \cite{13}, however, we include them for completeness. From Section 4 on, our research focuses on the case $k = 6$. We give an elementary proof of the simplicial version of the Cartan–Hadamard Theorem, which states that systolic complexes are contractible. Finally we turn to group theory. In Section 6 we discuss properties of directed geodesics, and in Section 7 we use them to give a direct (i.e. not invoking biautomaticity) proof that an infinite group acting geometrically on a systolic complex is not torsion.

1. Notation, basic definitions and combinatorial Gauss–Bonnet.

Let $X$ be a simplicial complex. We do not assume that $X$ is finite-dimensional or that it is locally finite. However, when $X$ is finite-dimensional, we define its dimension $\dim X$ to be the largest number $n$ such that $X$ contains an $n$-simplex. We equip $X$ with a CW-complex topology, and always consider $X$ as a topological space, rather than an abstract simplicial complex. Most of the time, though, we are only interested in the combinatorial structure of $X$.

Given a subset $\{v_1, \ldots, v_n\}$ of vertices of $X$ let $\text{span}\{v_1, \ldots, v_n\}$ denote the largest subcomplex of $X$ that has $\{v_1, \ldots, v_n\}$ as its vertex set. We will call it the subcomplex spanned by $\{v_1, \ldots, v_n\}$. If this subcomplex is a simplex, we denote it by $[v_0, \ldots, v_n]$. The subcomplex spanned by a collection $\{X_1, \ldots, X_n\}$ of subcomplexes is defined as the subcomplex spanned by all the vertices of all $X_i$’s.

We say that $X$ is flag if every set of vertices pairwise connected by edges spans a simplex of $X$. By a cycle in $X$ we understand a subcomplex
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If \( \gamma \) is a cycle, we denote by \( |\gamma| \) the number of edges in \( \gamma \) and we call it the length of \( \gamma \). A diagonal in \( \gamma \) is an edge connecting two nonconsecutive vertices of \( \gamma \). The link of a vertex \( v \), denoted by \( X_v \), is the subcomplex of \( X \) that consists of all simplices \( \sigma \subset X \) that do not contain \( v \), but that together with \( v \) span a simplex of \( X \).

A simplicial map \( f : X \to Y \) between simplicial complexes \( X \) and \( Y \) is a map which sends vertices to vertices, and if vertices \( v_0, \ldots, v_k \in X \) span a simplex \( \sigma \) of \( X \) then their images span a simplex \( \tau \) of \( Y \) and we have \( f(\sigma) = \tau \). Therefore a simplicial map is determined by its values on the vertex set of \( X \). A simplicial map is called nondegenerate if it is injective on each simplex.

We now introduce the notion of combinatorial curvature and the combinatorial version of the Gauss–Bonnet Theorem.

**Definition 1.1.** Let \( S \) be a triangulated surface, possibly with boundary. For a vertex \( v \in S \), we denote by \( \angle_S(v) \) the number of triangles (2-simplices) of \( S \) containing \( v \). (If \( S \) is clear from the context we may write simply \( \angle(v) \).) Let \( \text{int} S \) denote the set of interior vertices of the triangulation of \( S \), and \( \partial S \) the set of boundary vertices. Then we define

- for \( v \in \text{int} S \), the curvature of \( v \) to be \( \kappa(v) = 6 - \angle_S(v) \),
- for \( v \in \partial S \), the boundary curvature of \( v \) to be \( \kappa_\partial(v) = 3 - \angle_S(v) \).

**Theorem 1.2 (Combinatorial Gauss–Bonnet).** Let \( S \) be a compact triangulated surface. Let \( \chi(S) \) denote the Euler characteristic of \( S \). Then

\[
6 \chi(S) = \sum_{v \in \partial S} \kappa_\partial(v) + \sum_{v \in \text{int} S} \kappa(v).
\]

**Proof.** We give a proof only for \( S \) homeomorphic to the 2-disc. The proof of the general version can be found in [14, Theorem V.3.1].

We proceed by induction on the number of triangles in \( S \). Since \( S \) is contractible, its Euler characteristic is equal to 1. If \( S \) is a single triangle \( [v_1, v_2, v_3] \), then each \( v_i \) is a boundary vertex and we have \( \kappa_\partial(v_i) = 2 \), hence \( \sum_{i=1}^3 \kappa_\partial(v_i) = 6 = 6 \chi(S) \).

Now assume that \( S \) consists of more than one triangle and choose a triangle \( T = [v_1, v_2, v_3] \) in \( S \) such that at least one of its edges, say \( [v_1, v_2] \), is a boundary edge. We have the following three cases to consider. If \( v_3 \) is an interior vertex, then we are in situation (b) of Figure 1. If \( v_3 \) is a boundary vertex, then either both \( [v_1, v_3] \) and \( [v_2, v_3] \) are interior edges, or exactly one of them is a boundary edge. In the former case we are in situation (c) of Figure 1 and in the latter case we are in situation (a).

We first consider situations (a) and (b). Remove \( T \) from \( S \) and call the resulting surface \( S' \). By the inductive assumption \( S' \) satisfies the Gauss–Bonnet formula. We will show that after adding \( T \) back the formula still holds. In
situation (a), after adding $T$, the boundary curvature of both $v_1$ and $v_2$ decreases by 1. The new vertex $v_3$ contributes to the Gauss–Bonnet sum its boundary curvature, which is 2, hence the whole sum remains unchanged.

In situation (b) the vertex $v_3$ becomes an interior vertex, so its curvature increases by 3, but now it is contained in one more triangle, hence altogether its curvature increases by 2. The boundary curvatures of both $v_1$ and $v_2$ again decrease by 1, so the sum remains unchanged.

In situation (c) we proceed as follows. Consider two subsurfaces $S'_1$ and $S'_2 \cup T$ of $S$ (i.e. we cut $S$ along the edge $[v_1, v_3]$). By the assumption $S'_1$ and $S'_2 \cup T$ each have fewer triangles than $S$ has, and hence each of them satisfies the Gauss–Bonnet formula.

Let us compare the sum of the Gauss–Bonnet sums of $S'_1$ and $S'_2 \cup T$ with the Gauss–Bonnet sum of $S$. These two sums differ only in summands corresponding to $v_1$ and $v_3$. In the first sum we have four summands

$$(3 - \angle_{S'_1(v_1)}) + (3 - \angle_{S'_2 \cup T(v_1)}) + (3 - \angle_{S'_1(v_3)}) + (3 - \angle_{S'_2 \cup T(v_3)}).$$

In the second sum we have two summands

$$(3 - \angle_S(v_1)) + (3 - \angle_S(v_3)).$$

Since clearly

$$\angle_{S'_1(v_i)} + \angle_{S'_2 \cup T(v_i)} = \angle_S(v_i)$$

for $i \in \{1, 3\}$, the first sum is 6 greater than the second. This finishes the proof since by the inductive assumption the first sum is 12. $\blacksquare$

2. $k$-largeness condition. In this section we introduce two conditions for simplicial complexes called $k$-largeness and local $k$-largeness. The definition is purely combinatorial, but still geometrically inspired by the 2-dimensional case. In terms of these conditions we define $k$-systolic complexes, which are the main objects of our discussion. We then discuss relations between $k$-large and $k$-systolic complexes, and finally we give some examples and non-examples of those.
DEFINITION 2.1. Given a natural number $k \geq 4$, a simplicial complex $X$ is $k$-large if it is flag and if every cycle of length less than $k$ has a diagonal. We say that $X$ is locally $k$-large if the links of all vertices of $X$ are $k$-large.

Observe that if $k \leq m$ then $m$-largeness implies $k$-largeness. Now we will show that $k$-largeness implies local $k$-largeness. Later we will show that under some additional assumptions the converse also holds.

LEMMA 2.2. If a simplicial complex $X$ is $k$-large, then it is locally $k$-large.

Proof. Let $v_0$ be any vertex of $X$, and let $\gamma$ be a cycle of length $m < k$ in the link $X_{v_0}$. Then $\gamma$ is also a cycle in $X$, so it must have a diagonal, say $[v_1, v_2]$. Since $X$ is flag, there is a simplex $[v_0, v_1, v_2] \subset X$. This means that the edge $[v_1, v_2]$ belongs to $X_{v_0}$ and thus it is a required diagonal for $\gamma$. ■

Here we state a converse to Lemma 2.2.

THEOREM 2.3. For $k \geq 6$, let $X$ be simply connected and locally $k$-large simplicial complex. Then $X$ is $k$-large.

To prove this theorem, we need to introduce the notion of a filling diagram. Filling diagrams together with the Gauss–Bonnet Theorem will be our fundamental tools in this article. First we prove the following lemma.

LEMMA 2.4. Let $X$ be a $k$-large simplicial complex, and let $S^1_m$ denote the triangulation of the circle that consists of $m$ edges, where $m < k$. Then for any simplicial map $f_0: S^1_m \to X$ there exists a simplicial map $f: D \to X$, where $D$ is a triangulated 2-disc, such that $\partial D = S^1_m$, $f|_{\partial D} = f_0$ and $D$ has no interior vertices.

Proof. We proceed by induction on $m$. If $m = 3$ then we get the required extension, since $X$ is flag. Assume that $m > 3$ and label vertices of the triangulation of $S^1_m$ by $(v_1, \ldots, v_m)$. Then since $X$ is $k$-large and $m < k$, there are two nonconsecutive vertices, say $v_i, v_j \in S^1_m$ with $i < j$, whose images under $f_0$ either coincide, or are connected by an edge. Add to $S^1_m$ the edge $[v_i, v_j]$. Then $S^1_m \cup [v_i, v_j]$ is the union of two cycles $S^1_A = (v_1, \ldots, v_i, v_j, \ldots, v_m)$ and $S^1_B = (v_i, \ldots, v_j)$. Note that by the choice of $v_i$ and $v_j$, the restrictions of $f_0$ to $S^1_A$ and $S^1_B$, denoted by $f^A_0$ and $f^B_0$ respectively, are well defined simplicial maps. Since $v_i$ and $v_j$ are nonconsecutive, both $S^1_A$, $S^1_B$ are shorter than $m$, and hence by inductive hypothesis we have maps $f^A: D_A \to X$ and $f^B: D_B \to X$ extending $f^A_0$ and $f^B_0$ respectively such that neither $D_A$ nor $D_B$ has an interior vertex. Gluing $D_B$ and $D_A$ along the edge $[v_i, v_j]$ and setting $f = f_A \cup f_B$ gives the required extension of $f_0$ since $D = D_A \cup D_B$ has no interior vertices. ■

DEFINITION 2.5. Let $\gamma$ be a cycle in a simplicial complex $X$. A filling diagram for $\gamma$ is a simplicial map $f: D \to X$, where $D$ is a triangulated
2-disc, and $f|_{\partial D}$ maps $\partial D$ isomorphically onto $\gamma$. A filling diagram $f: D \to X$ for $\gamma$ is called

- **minimal area** (or **minimal**) if $D$ consists of the least possible number of triangles (2-simplices) among all filling diagrams for $\gamma$,
- **locally $k$-large** if $D$ is a locally $k$-large simplicial complex,
- **nondegenerate** if $f$ is a nondegenerate map.

**Remark 2.6.** We do not assume that $D$ is a simplicial complex, i.e. it may have multiple edges and loops. Consequently, the attaching maps of 2-cells of $D$ can be loops or (not necessarily embedded) cycles of length 2 or 3. We will still call this cell structure a "triangulation".

A **simplicial map** for a nonsimplicial $D$ is defined in an analogous way. In particular, a loop is always collapsed to a vertex, a pair of double edges is mapped to a single vertex or edge, and any 2-cell whose boundary is not an embedded triangle is collapsed to a vertex or an edge.

Observe that a diagram can be locally $k$-large only if $D$ is simplicial, as required in the definition. In fact nonsimplicial diagrams appear only in the proof of Theorem 2.7.

Finally, note that for a simplicial 2-disc, being locally $k$-large is equivalent to every interior vertex being contained in at least $k$ triangles.

**Theorem 2.7.** Let $\gamma$ be a homotopically trivial cycle in a locally $k$-large complex $X$. Then

(1) there exists a filling diagram for $\gamma$,
(2) any minimal filling diagram for $\gamma$ is simplicial, locally $k$-large and nondegenerate.

**Proof.** (1) Triangulate $S^1$ with $|\gamma|$ edges, and define $f_0: S^1 \to X$ as a simplicial isomorphism from $S^1$ to $\gamma$. Since $\gamma$ is homotopically trivial, $f_0$ extends to a map $f: D \to X$, where $D$ is a 2-disc. Using the relative Simplicial Approximation Theorem [16, Section 3.4] we deduce that $D$ can be given a triangulation which extends the triangulation of $S^1 = \partial D$, and we get a simplicial map $f_1: D \to X$ which agrees with $f_0$ on $S^1$. This proves the existence.

(2) Let $f: D \to X$ be a minimal filling diagram for $\gamma$. We will show that $D$ is simplicial, locally $k$-large and nondegenerate.

First we prove that $D$ is simplicial. We have to show that there are no double edges and no loops in $D$. We proceed by contradiction. First suppose that we have two edges $e$ and $e'$ joining vertices $v_1$ and $v_2$. Then we can remove the subdisc bounded by $e \cup e'$, and glue $e$ to $e'$, which gives a triangulation of $D$ with fewer triangles, together with a simplicial map induced by $f$. Since $X$ is simplicial, the images of $e$ and $e'$ under $f$ coincide, hence the induced map is well defined. This contradicts the minimality of $D$. 
Now assume that we have a loop in $D$. Then there exists a maximal loop, i.e. one which is not properly contained in the disc bounded by any other loop. Let $e$ be any such loop. Let $D_e$ denote the disc bounded by $e$, and let $T$ be the triangle adjacent to $e$ outside $D_e$. Then we have two possibilities: either the other two edges of $T$ are embedded, or they are both loops. Both situations are shown in Figure 2.

![Fig. 2. Two possibilities for the loop $e$](image)

In situation (a) two other sides of $T$ have the same endpoints, which contradicts the fact that there are no double edges. In situation (b) the disc bounded by one of the other two sides of $T$ contains $e$, which contradicts the maximality of $e$.

Now we show that $D$ is locally $k$-large. We do this by showing that there are no interior vertices of $D$ with $\angle(v) < k$. First observe that in $D$ there are no vertices with $\angle(v) \in \{1, 2\}$. Indeed, a vertex $v$ with $\angle(v) = 1$ would force $D$ to have a loop, and a vertex $v$ with $\angle(v) = 2$ would give a loop or two edges with the same endpoints in $D$.

Assume then that we have a vertex $v \in \text{int} D$ with $3 \leq \angle(v) < k$. Then the link $D_v$ is a cycle of length $\angle(v)$. The restriction $f|_{D_v} : D_v \to X_{f(v)}$ is a simplicial map from a cycle of length $\angle(v) < k$ to the $k$-large complex $X_{f(v)}$, hence by Lemma 2.4 we get a triangulation of a subdisc $D_0$ bounded by $D_v$ with no interior vertices, and a simplicial map $f_0 : D_0 \to X_{f(v)}$ that agrees with $f$ on $D_v$. Thus we can replace the triangulation of $D_0$ with the one given by Lemma 2.4 and we define a map $f_1$ to be $f_0$ on $D_0$ and $f$ elsewhere. This yields the filling diagram $f_1$ for $\gamma$ with fewer simplices than $D$ has, which is a contradiction. Hence $D$ is locally $k$-large.

To prove that $D$ is nondegenerate, assume the contrary. Then there is an edge $e$ mapped by $f$ to a vertex. Take two triangles containing $e$, delete the interior of their union and glue the remaining four edges pairwise. Since $X$ is simplicial, the values of $f$ on the glued edges coincide, hence the map induced by the gluing is well defined. This results in a smaller diagram for $\gamma$. 

We are now ready to prove Theorem 2.3.
Proof of Theorem 2.3. First we show that $X$ satisfies the $k$-largeness condition, and then we show the flagness of $X$. Let $\gamma$ be a cycle in $X$ with no diagonal. We need to show that $|\gamma| \geq k$. Since $X$ is simply connected, by Theorem 2.7 there exists a locally $k$-large diagram $f : D \rightarrow X$ for $\gamma$. Then

- $D$ has at least one interior vertex, because $\gamma$ has no diagonal,
- interior vertices of $D$ are each contained in at least $k$ triangles, because $D$ is locally $k$-large,
- boundary vertices are each contained in at least two triangles, because there is no diagonal in $\gamma$.

In the notation of the Gauss–Bonnet Theorem, the above conditions give the following inequalities:

(1) for $v \in \partial D$ we have $\angle(v) \geq 2$, so $\kappa_\partial(v) \leq 1$, and since $|\partial D| = |\gamma|$, we obtain

$$\sum_{v \in \partial D} \kappa_\partial(v) \leq \sum_{v \in \partial D} 1 = |\gamma|,$$

(2) for $v \in \text{int } D$ we have $\angle(v) \geq k$, so $\kappa(v) \leq 6 - k$, and since $k \geq 6$, all the terms $\kappa(v)$ are nonpositive. Because $\text{int } D$ contains at least one vertex, we get

$$\sum_{v \in \text{int } D} \kappa(v) \leq \sum_{v \in \text{int } D} (6 - k) \leq 6 - k.$$

Inserting (1) and (2) into the Gauss–Bonnet formula gives

$$6 = 6\chi(D) = \sum_{v \in \partial D} \kappa_\partial(v) + \sum_{v \in \text{int } D} \kappa(v) \leq |\gamma| + (6 - k).$$

Hence $|\gamma| \geq k$.

Now we show the flagness of $X$. Note that since all links of $X$ are flag by definition, it suffices to show that the 2-skeleton $X^{(2)}$ is flag. Indeed, assume that the 1-skeleton of an $n$-simplex $[v_1, \ldots, v_n]$ lies in $X$. Then if $X^{(2)}$ is flag, for any vertex $v_i$ we have the 1-skeleton of an $(n - 1)$-simplex in $X_{v_i}$, which by flagness of the link gives an $n$-simplex $[v_1, \ldots, v_n]$ in $X$.

To show that $X^{(2)}$ is flag, let $(v_1, v_2, v_3)$ be a cycle of length 3 in $X$. We need to show that this cycle bounds a single 2-simplex of $X$. Since $X$ is simply connected, $(v_1, v_2, v_3)$ is homotopically trivial. Thus by Theorem 2.7 it has a locally $k$-large filling diagram $f : D \rightarrow X$. If the disc $D$ is a single triangle then we are done, so assume it consists of at least two triangles. Then $v_1, v_2, v_3$ are each contained in at least two triangles, hence $\kappa_\partial(v_i) \leq 1$ for $i \in \{1, 2, 3\}$. For a vertex $v$ in the interior of $D$, by $k$-largeness we have $\kappa(v) \leq 0$. Hence applying the Gauss–Bonnet Theorem we get

$$6 = 6\chi(D) = \kappa_\partial(v_1) + \kappa_\partial(v_2) + \kappa_\partial(v_3) + \sum_{v \in \text{int } \Delta} \kappa(v) \leq 3,$$
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a contradiction. Therefore $D$ must consist of precisely one triangle, which means that $[v_1, v_2, v_3] \subset X$.

In this set-up we introduce systolic complexes, the main object of our further discussion.

**Definition 2.8.** Let $k \geq 4$. A simplicial complex $X$ is $k$-systolic if it is locally $k$-large, connected and simply connected. For $k = 6$ we abbreviate 6-systolic to systolic.

Note that if $k \geq 6$ then Theorem 2.3 implies that a $k$-systolic complex is $k$-large, so in particular it is flag. Hence any $k$-systolic complex is determined by its 1-skeleton. Here we give some basic examples of $k$-large and $k$-systolic complexes:

- If $X$ is a graph, then $X$ is $k$-large if and only if $X$ has no cycle of length less than $k$, and $X$ is $k$-systolic if and only if it is a tree.
- Equilaterally triangulated Euclidean plane is systolic.
- Equilaterally triangulated hyperbolic plane, by triangles with angles $2\pi/k$, is $k$-systolic.
- For any $k \geq 6$, no triangulation of the sphere $S^2$ is $k$-large, hence it is not systolic (by Gauss–Bonnet).
- Here is a locally 6-large triangulation of the torus:

![Fig. 3. The locally 6-large torus. Links are 6-cycles.](image)

- In a similar way, for any $g \geq 2$, a closed oriented genus $g$ surface admits a locally 6-large triangulation.

**3. 7-systolic complexes are Gromov hyperbolic.** The aim of this section is to prove that if $X$ is a 7-systolic complex then $X^{(1)}$, the 1-skeleton of $X$, with the standard metric is Gromov hyperbolic. The standard metric on $X^{(1)}$ is defined as follows. First declare each edge to be isometric to the Euclidean unit interval. Then for any two points $x_1, x_2 \in X^{(1)}$ define their distance to be the length of a shortest path between them, and denote it by $|x_1, x_2|$. Similarly, the length of a path $\gamma$ is denoted by $|\gamma|$. Although this metric is defined for all points in $X^{(1)}$, most of the time we are interested in distances between vertices, and the distance between any two vertices is
just the number of edges in the shortest path between them. Note that the
length of a cycle defined in Section 1 is exactly its length in the $X^{(1)}$-metric.
Also note that any simplicial map is 1-Lipschitz with respect to this metric.

**Definition 3.1.** Let $(X, d)$ be a metric space and $x_1, x_2 \in X$.

- A **geodesic** joining $x_1$ to $x_2$ is an isometry $\gamma: [0, d(x_1, x_2)] \to X$ such that
  $\gamma(0) = x_1$ and $\gamma(d(x_1, x_2)) = x_2$.
- A space $(X, d)$ is called **geodesic** if any two points of $X$ can be joined by
  a geodesic.
- A **geodesic triangle** in $X$ consists of three points and three geodesics joining
  these points. The images of these geodesics are called the **sides** of this triangle.

**Definition 3.2.** A geodesic metric space $(X, d)$ is called **$\delta$-hyperbolic** (or
**Gromov hyperbolic**) if there exists $\delta > 0$ such that for every geodesic triangle
in $X$ each side is contained in the union of the $\delta$-neighborhoods of the other
two sides.

**Convention 3.3.** Note that $X^{(1)}$ is a geodesic metric space, where the
geodesics are precisely the shortest paths between points.

From now on, by *geodesics* we always mean $X^{(1)}$-geodesics (even if we
work in $X$, not $X^{(1)}$), and whenever we refer to a metric, we mean the
$X^{(1)}$-metric. Moreover, since geodesics are injective maps, we may identify
a geodesic with its image.

Finally, observe that any geodesic $\gamma$ whose endpoints are vertices is
uniquely determined by the sequence of vertices $(\gamma(0), \ldots, \gamma(n))$.

To prove the main theorem of this section we need a lemma which gives
an estimate on the boundary curvature of a geodesic path.

**Lemma 3.4.** Let $D$ be a simplicial 2-disc, and let $\alpha$ be a geodesic in $D$
contained in $\partial D$. Then

$$\sum_{v \in \text{int} \alpha} \kappa_{\partial}(v) \leq 1,$$

where for a path $\alpha$, we denote by $\text{int} \alpha$ all the vertices of $\alpha$ except for its
endpoints.

**Proof.** Since $\alpha$ is a geodesic, every vertex of $\text{int} \alpha$ is contained in at least
two triangles. If not, then part (a) of Figure 4 shows that there exists a short-
cut. Moreover, there cannot be two consecutive vertices each contained in
two triangles: if that happens, again there is a shorter path between vertices
adjacent to these two, as shown in part (b) of Figure 4. And whenever there
are two vertices, each contained in two triangles, between them there has to
be a vertex contained in at least four triangles as in part (c) of Figure 4; otherwise we would be able to find another shortcut. Hence the number of
vertices contained in two triangles each is at most 1 greater than the number of vertices contained in four triangles each. Since any other vertex is contained in three or more triangles, the claim follows. □

(a) \hspace{2cm} (b) \hspace{2cm} (c)

| a part of $\alpha$ |
|---------------------|
| a possible shortcut |

Fig. 4. There are shortcuts in (a) and (b).

**Theorem 3.5.** Let $X$ be a 7-systolic complex. Then $X^{(1)}$ is Gromov hyperbolic.

**Proof.** The proof is due to Januszkiewicz and Świątkowski [13, Theorem 2.1]. We need to show that geodesic triangles in $X^{(1)}$ are $\delta$-thin for some $\delta > 0$. First we show that this condition is satisfied by geodesic triangles with vertices in $X^{(0)}$. Note that a geodesic triangle in $X^{(1)}$ can have self-intersections, so it splits into finitely many embedded geodesic bigons and an embedded geodesic triangle, joined by their endpoints or joined by some geodesics between their endpoints.

So to show $\delta$-thinness of any geodesic triangle with vertices in $X^{(0)}$, it is enough to prove that embedded geodesic bigons and embedded geodesic triangles are $\delta$-thin. Let $\alpha_1 \cup \alpha_2$ be a bigon formed by two geodesics $\alpha_1$ and $\alpha_2$ with common endpoints. Since $X$ is simply connected, by Theorem 2.7 there exists a minimal area filling diagram for $\alpha_1 \cup \alpha_2$. Denote this diagram by $f: D \to X$. Since $f$ is an isomorphism on $\partial D$, we keep denoting the preimages $f^{-1}(\alpha_i)$ by $\alpha_i$. Now we work in the disc $D$. By Lemma 3.4 we have $\sum_{v \in \text{int} \alpha_i} \kappa_\partial(v) \leq 1$ for both $\alpha_1$ and $\alpha_2$. For $v_0$ and $v_1$, the endpoints of these geodesics, we have $\angle(v_i) \geq 1$ so $\kappa_\partial(v_i) \leq 2$ for $i \in \{0,1\}$. Finally by Theorem 2.7 the disc $D$ is locally 7-large, so for the interior vertices we have $\angle(v) \geq 7$, hence $\kappa(v) \leq -1$. Thus by the Gauss–Bonnet formula we get

$$6 = 6\chi(D) = \kappa_\partial(v_0) + \kappa_\partial(v_1) + \sum_{v \in \text{int} \alpha_1} \kappa_\partial(v) + \sum_{v \in \text{int} \alpha_2} \kappa_\partial(v) + \sum_{v \in \text{int} D} \kappa(v) \leq 2 + 2 + 1 + 1 + \sum_{v \in \text{int} D} (-1) = 6 - |\text{int} D|.$$  

Hence $|\text{int} D| \leq 0$. Thus there are no interior vertices in $D$, so every vertex of $\alpha_1$ is connected by an edge to some vertex of $\alpha_2$. This implies that every point from $\alpha_1$ is at distance at most $\frac{3}{2}$ from some vertex of $\alpha_2$ and vice versa, thus $\partial D$ is $\frac{3}{2}$-thin. Then since $f$ is 1-Lipschitz, it cannot increase distances, hence a bigon $\alpha_1 \cup \alpha_2 \subset X^{(1)}$ is also $\frac{3}{2}$-thin.
Now we pass to geodesic triangles. Let $\alpha_1 \cup \alpha_2 \cup \alpha_3$ be such a triangle. Pick a minimal diagram $f : D \to X$ for $\alpha_1 \cup \alpha_2 \cup \alpha_3$. Now $\partial D$ consists of three geodesics: $\alpha_1$, $\alpha_2$ and $\alpha_3$. Using the same estimates as above we get

$$6 = 6\chi(D) = \kappa_\partial(v_0) + \kappa_\partial(v_1) + \kappa_\partial(v_2) + \sum_{v \in \text{int } \alpha_1} \kappa_\partial(v) + \sum_{v \in \text{int } \alpha_2} \kappa_\partial(v) + \sum_{v \in \text{int } \alpha_3} \kappa_\partial(v) + \sum_{v \in \text{int } D} \kappa(v)$$

$$\leq 2 + 2 + 2 + 1 + 1 + 1 + \sum_{v \in \text{int } D} (-1) = 9 - |\text{int } D|.$$ 

Hence $|\text{int } D| \leq 3$. As there are at most three interior vertices in $D$, every point of $\alpha_i$ is at distance at most $4\frac{1}{2}$ from some vertex of $\alpha_j \cup \alpha_k$ where \{i, j, k\} = \{1, 2, 3\}. So $\partial D$ seen as a geodesic triangle is $4\frac{1}{2}$-thin. Since $f$ is 1-Lipschitz, we deduce that $\alpha_1 \cup \alpha_2 \cup \alpha_3 \subset X^{(1)}$ is also $4\frac{1}{2}$-thin. Thus we have proved that geodesic triangles with vertices in $X^{(0)}$ are $4\frac{1}{2}$-thin.

It can be shown that this fact implies that any geodesic triangle in $X^{(1)}$ is $\delta'$-thin, for some other $\delta' > 0$. Indeed, given any geodesic triangle $[x_1, x_2, x_3]$ in $X^{(1)}$, one turns it into a geodesic 6-gon with vertices in $X^{(0)}$ by taking edges containing $x_i$’s to be the new geodesics of length 1. Then one can show that it is $\delta'$-thin by splitting it into four geodesic triangles with vertices in $X^{(0)}$. A detailed proof can be found in [13, Theorem 2.1].

4. Projection Lemma for systolic complexes. From now on, we focus on 6-systolic complexes, so we just write systolic. An important result which we prove in this section is the Projection Lemma. This theorem, first proved by V. Chepoi [3], is an extremely useful tool in working with systolic complexes. Both its statement and proof are purely combinatorial: our proof relies on filling diagrams and the Gauss–Bonnet Theorem. First we need some preparation.

**Definition 4.1.** Let $X$ be a simplicial complex and let $\sigma$ be a simplex of $X$. We define the residue of $\sigma$ in $X$, denoted by $\text{Res}(\sigma, X)$, to be the subcomplex of $X$ spanned by all simplices that contain $\sigma$.

**Definition 4.2.** Given a simplicial complex $X$ and a vertex $v_0 \in X$, we define the *combinatorial ball* of radius $n$ centered at $v_0$ to be

$$B_n(v_0, X) = \text{span}\{v \in X \mid |v, v_0| \leq n\},$$

and the *combinatorial sphere* to be

$$S_n(v_0, X) = \text{span}\{v \in X \mid |v, v_0| = n\}.$$ 

The idea of the Projection Lemma is to enable projecting simplices contained in $S_n(v_0, x)$ onto $S_{n-1}(v_0, X)$ in a nice way.
Theorem 4.3 (Projection Lemma). Let $X$ be a systolic complex and let $v_0$ be a vertex of $X$. Pick $n > 0$. Then for any simplex $\sigma \subset S_n(v_0, X)$, the intersection $\text{Res}(\sigma, X) \cap S_{n-1}(v_0, X)$ is a nonempty simplex. We call it the projection of $\sigma$ onto $S_{n-1}(v_0, X)$ and denote it by $\pi_{v_0}(\sigma)$.

Remark 4.4. If it is clear in what sphere $\sigma$ is contained, we may also call $\pi_{v_0}(\sigma)$ the projection of $\sigma$ in the direction of $v_0$.

Note that since $X$ is flag, the subcomplex $\text{Res}(\sigma, X) \cap S_{n-1}(v_0, X)$ is spanned by the set of vertices in $S_{n-1}(v_0, X)$ which are connected by an edge to every vertex of $\sigma$. So to prove that $\text{Res}(\sigma, X) \cap S_{n-1}(v_0, X)$ is a simplex, it is enough to show that any two vertices in this set are connected by an edge. The proof is quite long, so we divide it into a few steps. First we prove two lemmas about filling diagrams in systolic complexes, then we show that the projection of $\sigma$ is a simplex, and finally that it is nonempty (which is the hardest part).

Lemma 4.5. Let $\gamma_1 \cup \gamma_2$ be a geodesic bigon between vertices $v_0$ and $v$ in a systolic complex $X$, and let $\gamma_1(0) = \gamma_2(0) = v$. Then $|\gamma_1(1), \gamma_2(1)| \leq 1$, i.e. we have one of the situations shown in Figure 5.

\begin{align*}
(a) & \ |\gamma_1(1), \gamma_2(1)| = 0 & (b) & \ |\gamma_1(1), \gamma_2(1)| = 1 \\
\end{align*}

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) (v0) {$v_0$};
\node at (0,2) (v) {$v$};
\node at (1,-1) (gamma1) {$\gamma_1(1)$};
\node at (2,-1) (gamma2) {$\gamma_2(1)$};
\node at (1,1) (gamma12) {$\gamma_1 \cup \gamma_2$};
\draw (v0) -- (v); \draw (gamma1) -- (gamma2); \draw (gamma1) -- (gamma12) -- (gamma2);
\end{tikzpicture}
\caption{The distance $|\gamma_1(1), \gamma_2(1)|$ is at most 1.}
\end{figure}

Proof. If $\gamma_1(1) = \gamma_2(1)$ then we are in situation (a) of Figure 5. So assume $\gamma_1(1) \neq \gamma_2(1)$. We will show that in this case we are in situation (b). We can assume that $\gamma_1$ and $\gamma_2$ do not intersect except at their endpoints, i.e. the bigon $\gamma_1 \cup \gamma_2$ is homeomorphic to $S^1$. If this is not the case, instead of $\gamma_1 \cup \gamma_2$ consider the bigon $\gamma_1|[0,m] \cup \gamma_2|[0,m]$, where $m$ is the smallest number such that $\gamma_1(m) = \gamma_2(m)$. By the choice of $m$, the new bigon is homeomorphic to $S^1$.

Since $\gamma_1 \cup \gamma_2$ is homeomorphic to $S^1$, as in the proof of Theorem 3.5, by Theorem 2.7 we have a minimal filling diagram $f: D \rightarrow X$ for $\gamma_1 \cup \gamma_2$. Since $f|\partial D: \partial D \rightarrow \gamma_1 \cup \gamma_2$ is an isomorphism, we keep the same notation for the
preimages of $\gamma_1$ and $\gamma_2$ under $f$, and we apply the Gauss–Bonnet Theorem to $D$:

$$6 = 6\chi(D) = \kappa_0(v_0) + \kappa_\partial(v) + \sum_{v \in \text{int} \gamma_1} \kappa_0(v) + \sum_{v \in \text{int} \gamma_2} \kappa_\partial(v) + \sum_{v \in \text{int} D} \kappa(v).$$

By Theorem 2.7 the disc $D$ is locally 6-large, so $\angle(v) \geq 6$ and thus $\kappa(v) \leq 0$ for $v \in \text{int} D$. By Lemma 3.4 we get $\sum_{v \in \text{int} \gamma_1} \kappa_\partial(v) \leq 1$, hence altogether

$$6 = 6\chi(D) \leq 1 + 1 + 3 - \angle(v_0) + 3 - \angle(v),$$

so $\angle(v_0) + \angle(v) \leq 2$. Since $\gamma_1 \cup \gamma_2$ is homeomorphic to $S^1$, both $\angle(v_0)$ and $\angle(v)$ are at least 1. Thus $\angle(v_0) = \angle(v) = 1$, which means that $|\gamma_1(1), \gamma_2(1)| = 1$ in $D$. Since $f$ cannot increase distances, we have $|\gamma_1(1), \gamma_2(1)| = 1$ in $X$ as well.

The following lemma is very similar to Lemma 4.5:

**Lemma 4.6.** Let $v_0, v_1$ and $v_2$ be vertices in a systolic complex $X$ such that $v_1, v_2 \in S_n(v_0, X)$ for some $n > 0$, and $|v_1, v_2| = 1$. For $i \in \{1, 2\}$, let $\gamma_i$ be a geodesic joining $v_0$ to $v_i$ and $\gamma_i(0) = v_i$. Then either $|\gamma_1(1), \gamma_2(1)| \leq 1$, or $|\gamma_1(1), \gamma_2(1)| = 2$ with the middle vertex $v$ of a geodesic realizing this distance contained in $S_{n-1}(v_0, X)$, and joined by edges to both $v_1$ and $v_2$ (see Figure 6).

(a) $|\gamma_1(1), \gamma_2(1)| = 0$  
(b) $|\gamma_1(1), \gamma_2(1)| = 1$  
(c) $|\gamma_1(1), \gamma_2(1)| = 2$

Fig. 6. The distance $|\gamma_1(1), \gamma_2(1)|$ is at most 2.

**Proof.** Consider a cycle $\gamma_1 \cup \gamma_2 \cup [v_1, v_2]$. If $\gamma_1(1) = \gamma_2(1)$ then we are in situation (a) of Figure 6. Now assume that $|\gamma_1(1), \gamma_2(1)| \geq 1$. As in the proof of Lemma 4.5 we can assume that the cycle $\gamma_1 \cup \gamma_2 \cup [v_1, v_2]$ is homeomorphic to $S^1$. By Theorem 2.7 we can pick a minimal filling diagram $f: D \to X$ for $\gamma_1 \cup \gamma_2 \cup [v_1, v_2]$. Using the same notation and estimates as in the proof of Lemma 4.5 we find that $\kappa(v) \leq 0$ for all $v$ in $\text{int} D$, and $\sum_{v \in \text{int} \gamma_i} \kappa_\partial(v) \leq 1$.
for both $\gamma_1$ and $\gamma_2$. Therefore the Gauss–Bonnet Theorem gives

$$6 = \sum_{v \in \gamma_1} \kappa_\partial(v) + \sum_{v \in \gamma_2} \kappa_\partial(v) + \sum_{v \in \text{int } D} \kappa(v) \leq 1 + 1 + 3 - \angle(v_0) + 3 - \angle(v_1) + 3 - \angle(v_2),$$

so

$$\angle(v_0) + \angle(v_1) + \angle(v_2) \leq 5.$$  

We have $\angle(v_i) \geq 1$ for $i \in \{0, 1, 2\}$, because $\gamma_1 \cup \gamma_2 \cup [v_1, v_2]$ is embedded, hence solving the last displayed inequality we see that either both $\angle(v_1)$ and $\angle(v_2)$ are 2, or at least one of them is 1. Let us consider these two cases:

- $\angle(v_1) = 1$ (or $\angle(v_2) = 1$). This means that $\gamma_1(1)$ is connected to $v_2$ by an edge. Then applying Lemma 4.5 to a geodesic bigon formed by $\gamma_2$ and $[v_2, \gamma_1(1)] \cup \gamma_1[1, n]$ we conclude that $|\gamma_1(1), \gamma_2(1)| = 1$ and we are in situation (b) of Figure 6.

- $\angle(v_1) = \angle(v_2) = 2$. Since $v_1$ and $v_2$ are each contained in two triangles, we easily see that there exists a vertex $v$ connected by edges to $v_1$, $v_2$, $\gamma_1(1)$ and $\gamma_2(1)$. Thus, it remains to check that $|v, v_0| = n - 1$. Clearly, $|\gamma_1(1), v_0| = n - 1$ and $|v, \gamma_1(1)| = 1$, so $|v_0, v|$ is at most $n$. On the other hand, $|v_1, v_0| = n$ and $|v_1, v| = 1$, so $|v_0, v| \geq n - 1$. Therefore we only need to show that $|v_0, v|$ cannot be $n$. But if $|v_0, v| = n$ then again by Lemma 4.5 applied to the geodesics $[v, \gamma_1(1)] \cup \gamma_1[1, n]$ and $[v, \gamma_2(1)] \cup \gamma_2[1, n]$ we conclude that $\gamma_1(1)$ and $\gamma_2(1)$ are connected by an edge. Hence we are in situation (b) of Figure 6. Otherwise $|v, v_0| = n - 1$ and we are in situation (c).

This proves that for the cycle $\gamma_1 \cup \gamma_2 \cup [v_1, v_2] \subset D$ we have one of the possibilities shown in Figure 6. Since $f$ is simplicial, it cannot increase distances, and therefore we have the same possibilities for $\gamma_1 \cup \gamma_2 \cup [v_1, v_2]$ in $X$. □

Having these two lemmas, we are ready to prove the Projection Lemma.

**Proof of Theorem 4.3.** First we prove that for a given $\sigma \subset S_n(v_0, X)$, the projection $\pi_{v_0}(\sigma) \subset S_{n-1}(v_0, X)$ is a simplex. Since $X$ is flag, it is enough to show that any two vertices of the projection are connected by an edge. Let $w_1$ and $w_2$ be such vertices. They are in $\pi_{v_0}(\sigma)$, so each is connected by an edge to every vertex of $\sigma$. Pick any vertex $v \in \sigma$ and consider the geodesic bigon formed by two geodesics: $[v, w_1] \cup \gamma_1$ and $[v, w_2] \cup \gamma_2$, where $\gamma_i$ is a geodesic from $w_i$ to $v_0$. By Lemma 4.5 we have $|w_1, w_2| \leq 1$, so either $w_1 = w_2$ or they are connected by an edge. This proves that $\pi_{v_0}(\sigma)$ is a simplex.

Now we need to show that $\pi_{v_0}(\sigma)$ is nonempty. We proceed by induction on the dimension of $\sigma$. First assume $\dim \sigma = 0$, i.e. $\sigma$ is a single vertex
$v \in S_n(v_0, X)$. Since $X$ is connected, there is a geodesic $\gamma$ from $v$ to $v_0$, i.e. $\gamma(0) = v$. Then $\gamma(1) \in S_{n-1}(v_0, X)$ and $\gamma(1) \in \pi_{v_0}(\sigma)$ by the definition of the projection.

Now assume that $\dim \sigma = 1$ and let $\sigma = [v_1, v_2]$ where $v_1, v_2 \in S_n(v_0, X)$. Choose geodesics $\gamma_1$ and $\gamma_2$ joining respectively $v_1$ and $v_2$ to $v_0$. Applying Lemma 4.6 to the cycle $[v_1, v_2] \cup \gamma_1 \cup \gamma_2$ shows that $\pi_{v_0}([v_1, v_2])$ is nonempty.

For the inductive step let $\dim \sigma = k - 1$ for some $k > 2$ and let $\sigma = [v_1, \ldots, v_k]$. Consider the faces $\tau_1 = [v_1, \ldots, v_{k-1}]$ and $\tau_2 = [v_2, \ldots, v_k]$. By inductive hypothesis, the projections $\pi_{v_0}(\tau_1)$ and $\pi_{v_0}(\tau_2)$ are nonempty. Choose vertices $w_1 \in \pi_{v_0}(\tau_1)$ and $w_2 \in \pi_{v_0}(\tau_2)$. Since $k > 2$, there is a vertex $v_i \in \sigma$ such that $i \notin \{1, k\}$. By construction, $v_i$ is connected by edges to both $w_1$ and $w_2$. Choose geodesics $\gamma_1$ and $\gamma_2$ that join respectively $w_1$ and $w_2$ to $v_0$, and consider the geodesic bigon $[v_i, w_1] \cup \gamma_1 \cup [v_i, w_2] \cup \gamma_2$ (see Figure 7). By Lemma 4.5 we have $|w_1, w_2| \leq 1$.

If $w_1 = w_2$, then $w_1$ is connected by edges to all $v_i$ for $i \in \{1, \ldots, k\}$, and hence belongs to $\pi_{v_0}(\sigma)$.

If $|w_1, w_2| = 1$, then we consider the cycle $(w_1, w_2, v_k, v_1)$. Since it has length 4, it has a diagonal. If this diagonal is $[w_1, v_k]$ then $w_1$ is connected by edges to all $v_i$’s, and hence $w_1 \in \pi_{v_0}(\sigma)$. If not, then it must be $[w_2, v_1]$, and thus $w_2 \in \pi_{v_0}(\sigma)$. ■

Fig. 7. Inductive step

5. Contractibility of systolic complexes. Contractibility of systolic complexes follows almost directly from the Projection Lemma. Intuitively, the Projection Lemma enables us to collapse the complex, by projecting simplices onto smaller and smaller balls centered at a chosen vertex. In this section we turn this intuition into a proof. Contractibility of systolic complexes has been essentially proved in [1], where the stronger property of dis-
mantlability is established for bridged graphs (which are 1-skeleta of systolic complexes).

**Theorem 5.1.** Every finite-dimensional systolic complex is contractible.

Note that since locally 6-large complexes are supposed to be simplicial analogues of locally CAT(0) spaces, Theorem 5.1 is a simplicial version of the Cartan–Hadamard Theorem, which states that the universal cover of a locally CAT(0) space is contractible [2, Theorem II.4.1]. In our situation the universal cover of a locally k-large complex is systolic, hence contractible.

The proof of Theorem 5.1 is based on the Projection Lemma and the following result.

**Lemma 5.2.** Let $X$ be a simplicial complex, and let $\sigma \subset X$ be a simplex which is properly contained in exactly one maximal (with respect to inclusion) simplex of $X$. Then $X - \text{Res}(\sigma, X)$ is a strong deformation retract of $X$. (Here $X - \text{Res}(\sigma, X)$ denotes the subcomplex of $X$ obtained by removing all simplices that contain $\sigma$.)

**Proof.** Let $\tau$ be the unique maximal simplex containing $\sigma$. If $\sigma$ is a codimension 1 face of $\tau$, the required retraction is called the elementary collapse [6, Section I.2]. To prove the general case, we construct the retraction as the composition of finitely many elementary collapses.

Let $\tau' \subset \tau$ be a codimension 1 face of $\tau$ such that $\sigma \subset \tau'$. Let $S = \{ \rho \mid \sigma \subset \rho \subset \tau' \}$. This set is partially ordered by inclusion, hence we can extend this order to a linear order and we get $S = \{ \sigma = \rho_0 < \cdots < \rho_n = \tau' \}$. Since $\tau' \subset \tau$ is a codimension 1 face, there is a vertex $v \in \tau$ such that $\tau = \tau' * v$. Then we can perform a sequence of elementary collapses for pairs $\rho_i \subset \rho_i * v$ starting from $i = n$ with the pair $\tau' \subset \tau$, terminating at $i = 0$ with the pair $\sigma = \rho_0 \subset \rho_0 * v$. Indeed, at each step, $\rho_i$ is a face of exactly one simplex $\rho_i * v$, because we have already removed all other subsimplices of $\tau$ which might properly contain $\rho_i$.

The composition of these elementary collapses is the required deformation retraction: we have removed $\text{Res}(\sigma, X)$ because we have removed all the simplices $\rho_i$ and $\rho_i * v$ such that $\sigma \subset \rho_i$. □

**Proof of Theorem 5.1.** Pick any vertex $v_0$ of $X$. By Whitehead’s Theorem [12, Theorem 4.5], in order to show that $X$ is contractible, it is enough to prove that for any $k \geq 1$, any map $f : (S^k, s_0) \to (X, v_0)$ is homotopic to the constant map. The image of $S^k$ is compact, hence it intersects only finitely many simplices of $X$. Thus it is contained in $B_n(v_0, X)$ for $n$ sufficiently large. We prove that for any $n$ the ball $B_n(v_0, X)$ is contractible. We do so by showing how to homotopy retract $B_n(v_0, X)$ onto $B_{n-1}(v_0, X)$; then the result follows by induction since $B_0(v_0, X) = v_0$. 
Let $\sigma$ be a maximal simplex in $S_n(v_0, X) \subset B_n(v_0, X)$ and let $m = \dim \sigma$. Then $\sigma * \pi_{v_0}(\sigma)$ is a simplex of $B_n(v_0, X)$ containing $\sigma$. We claim that $\sigma * \pi_{v_0}(\sigma)$ is the unique maximal simplex containing $\sigma$. To see that, let $\tau$ be a simplex of $B_n(v_0, X)$ with $\sigma \subset \tau$. Let $v$ be a vertex of $\tau$ which does not belong to $\sigma$. Since $\sigma$ is a maximal simplex of $S_n(v_0, X)$, the vertex $v$ must lie in $S_{n-1}(v_0, X)$. But $v$ is connected by edges to all vertices of $\sigma$, so by definition it lies in the projection of $\sigma$. Repeating this argument for any such vertex, we finally see that every vertex of $\tau$ belongs to either $\sigma$ or $\pi_{v_0}(\sigma)$, hence $\tau$ is a subsimplex of $\sigma * \pi_{v_0}(\sigma)$. This proves the claim.

Thus $\sigma \subset B_n(v_0, X)$ meets the assumptions of Lemma 6.2, i.e. it is contained in exactly one maximal simplex of $B_n(v_0, X)$. By Lemma 6.2 the ball $B_n(v_0, X)$ homotopy retracts onto $B_n(v_0, X) - \text{Res}(\sigma, B_n(v_0, X))$. We can apply Lemma 6.2 to all maximal simplices in $S_n(v_0, X)$ and obtain a complex which is a deformation retract of $B_n(v_0, X)$, where deformation retraction is obtained by performing all retractions from Lemma 6.2 simultaneously. Call this complex $B_n(v_0, X)'$, and let $S_n(v_0, X)'$ denote $S_n(v_0, B_n(v_0, X))'$.

By the above procedure we have removed all maximal simplices of $S_n(v_0, X)$ (removing the residue of a simplex removes the simplex itself), thus in $S_n(v_0, X)'$ simplices of dimension $m - 1$ are maximal. Hence in the same way as above, we show that any such simplex is contained in a unique maximal simplex of $B_n(v_0, X)'$, and we can again apply Lemma 6.2. Applying Lemma 6.2 to all maximal simplices in $S_n(v_0, X)'$ we obtain a complex $B_n(v_0, X)''$, with maximal simplices in $S_n(v_0, X)''$ having dimension $m - 2$. Continuing this procedure, we finally obtain a complex $B_n(v_0, X)^{(m+1)}$ with no simplices in $S_n(v_0, X)^{(m+1)}$, hence $B_n(v_0, X)^{(m+1)} = B_{n-1}(v_0, X)$.

Since every map used in the above procedure restricts to the identity on $B_{n-1}(v_0, X)$, we also conclude that $B_{n-1}(v_0, X) = B_n(v_0, X)^{(m+1)}$ is a deformation retract of $B_n(v_0, X)$, where deformation retraction is the composition of $m + 1$ ‘simultaneous retractions’ arising in the procedure. Now the claim follows by induction.

### 6. Directed geodesics.

In this section we study directed geodesics. These are sequences of simplices satisfying certain local conditions, which make them behave better than ordinary geodesics in many situations. Originally they were introduced in [13, Definition 9.1]. In our case the main purpose of discussing them is to prove Theorem 7.4. The properties of directed geodesics which we show in this section, namely Theorem 6.3 and Lemma 6.4, were proved in [13, Fact 8.3.2, Lemma 9.3, Lemma 9.6]. However, our proofs are different and in some way more elementary.

**Definition 6.1.** A sequence $(\sigma_0, \ldots, \sigma_n)$ of simplices in a systolic complex $X$ is called a directed geodesic if it satisfies the following two conditions:
(i) for any $0 \leq i \leq n - 1$ the simplices $\sigma_i$ and $\sigma_{i+1}$ are disjoint and together they span a simplex of $X$,
(ii) $\text{Res}(\sigma_{i+2}, X) \cap B_1(\sigma_i, X) = \sigma_{i+1}$ for any $0 \leq i \leq n - 2$.

Here for a simplex $\sigma$ we define the ball $B_1(\sigma, X)$ to be the simplicial span of the set $\{v \in X \mid |v, \sigma| \leq 1\}$.

**Remark 6.2.** The above definition is slightly different from [13, Definition 9.1], where (ii) is stated as $\text{Res}(\sigma_i, X) \cap B_1(\sigma_{i+2}, X) = \sigma_{i+1}$. Our modification is needed to avoid notational problems in the construction of infinite directed geodesics.

The following theorem justifies the term ‘geodesic’ in Definition 6.1 and is the crucial step in the proof of Theorem 7.4.

**Theorem 6.3.** Let $(\sigma_0, \ldots, \sigma_n)$ be a directed geodesic in a systolic complex $X$. Then any sequence $(v_0, \ldots, v_n)$ of vertices such that $v_i \in \sigma_i$ is a geodesic in $X^{(1)}$.

**Proof.** First we will show that this property is satisfied locally, i.e. any triple $(v_i, v_{i+1}, v_{i+2})$ is a geodesic. Indeed, assume otherwise, i.e. $|v_i, v_{i+2}| = 1$. This means that $v_{i+2} \in B_1(v_i, X) \subset B_1(\sigma_i, X)$, and since $v_{i+2} \in \text{Res}(\sigma_{i+2}, X)$, we get $v_{i+2} \in B_1(\sigma_i, X) \cap \text{Res}(\sigma_{i+2}, X)$. By definition of a directed geodesic, $B_1(\sigma_i, X) \cap \text{Res}(\sigma_{i+2}, X) = \sigma_{i+1}$, hence $v_{i+2} \in \sigma_{i+1}$, which contradicts the fact that $\sigma_{i+1}$ and $\sigma_{i+2}$ are disjoint. Thus $(v_i, v_{i+1}, v_{i+2})$ is a geodesic. This fact will be used several times.

Now observe that in order to prove the theorem it is enough to prove that $|v_0, v_n| = n$. Indeed, then any sequence of vertices $(v_0, v_1, \ldots, v_n)$ where $v_i \in \sigma_i$ gives a path from $v_0$ to $v_n$ of length $n$, hence a geodesic path.

Also note that if a geodesic $\gamma$ from $v_0$ to $v_n$ satisfies $\gamma(i) \in \sigma_i$ for some $1 \leq i \leq n - 1$, then we are done by induction: assume that the assertion of the theorem holds for any shorter sequence of vertices; then since $\gamma(i) \in \sigma_i$, by inductive hypothesis we have $|v_0, \gamma(i)| = i$ and $|\gamma(i), v_n| = n - i$, hence $|v_0, v_n| = n$.

Thus it suffices to show that there is no geodesic from $v_0$ to $v_n$ that is disjoint from the sequence $\sigma_1, \ldots, \sigma_{n-1}$. Assume conversely that there is such a geodesic.

Consider the set $C = \{(v_0, v_1, \ldots, v_n) \cup \gamma\}$ of cycles where $v_i \in \sigma_i$ and $\gamma$ is a geodesic from $v_0$ to $v_n$ disjoint from $\sigma_i$ for $1 \leq i \leq n - 1$. By our assumption $C$ is nonempty. Any cycle in $C$ is homotopically trivial, hence by Theorem 2.7 it has a locally 6-large filling diagram. Thus we can pick a cycle $(v_0, v_1, \ldots, v_n) \cup \gamma$ whose filling diagram $f : D \to X$ is minimal among all diagrams for cycles in $C$. 

We would like to apply the Gauss–Bonnet Theorem to the diagram $D$, so we need to find certain curvature estimates. We keep the notation $(v_0, v_1, \ldots, v_{n-1}, v_n) \cup \gamma$ for the boundary of $D$.

- $\sum_{w \in \text{int} \gamma} \kappa_\partial(w) \leq 0$.

We show that there are no vertices in $\text{int} \gamma$ with positive boundary curvature. Since $\gamma$ is a geodesic, for any interior vertex $w$ we have $\kappa_\partial(w) \leq 1$ (cf. Lemma 3.4). Suppose there is a vertex $w_i \in \text{int} \gamma$ with $\kappa_\partial(w_i) = 1$. This means that $w_i$ is contained in two triangles, call them $[w_{i-1}, w_i, w]$ and $[w_{i+1}, w_i, w]$. Remove these two triangles from $D$ and replace the edges $[w_{i-1}, w_i]$ and $[w_i, w_{i+1}]$ by $[w_{i-1}, w]$ and $[w, w_i]$. This gives a new geodesic $\gamma'$ and a new diagram $D'$ with smaller area, which contradicts the minimality of $D$. Figure 8 shows the replacement procedure.

- $\sum_{i=1}^{n-1} \kappa_\partial(v_i) \leq 1$.

First note that there cannot be any vertex $v_i$ with $\kappa_\partial(v_i) = 2$, since then $v_{i-1}$ is connected by an edge to $v_{i+1}$, which contradicts $(v_{i-1}, v_i, v_{i+1})$ being a geodesic.

We will now show that no two consecutive vertices have boundary curvature 1. Assume conversely that vertices $v_i, v_{i+1}$ are each contained in two triangles (one triangle is common for $v_i$ and $v_{i+1}$), say $[v_{i-1}, v_i, x], [v_i, v_{i+1}, x]$ and $[v_{i+1}, v_{i+2}, x]$. We have $x \notin \sigma_i$, for otherwise $(x, v_{i+1}, v_{i+2})$ would not be geodesic since $x$ and $v_{i+2}$ are connected by an edge. This, together with the definition of a directed geodesic and the fact that $x \in B_1(v_{i-1}, X)$, implies that $x \notin \text{Res}(\sigma_{i+1}, X)$. Hence there exists a vertex $w_{i+1} \in \sigma_{i+1}$ which is not connected by an edge to $x$. Since $w_{i+1}$ belongs to $\sigma_{i+1}$, by definition of a directed geodesic it is connected to all vertices of both $\sigma_i$ and $\sigma_{i+2}$, so in particular it is connected by an edge to $v_{i+2}$ and to $v_i$. Hence we have a cycle $(v_i, x, v_{i+2}, w_{i+1})$ of length 4. By Theorem 2.3 the complex $X$ is 6-large, so this cycle has a diagonal (see Figure 9). By the choice of $w_{i+1}$ it
cannot be \([w_{i+1}, x]\), so it has to be \([v_i, v_{i+2}]\), contrary to \((v_i, v_{i+1}, v_{i+2})\) being a geodesic.

Fig. 9. Possible diagonals of the cycle \((v_i, x, v_{i+2}, w_{i+1})\)

Next, take any vertex \(v_i\) such that \(\kappa_\partial(v_i) = 1\) and suppose that \(\kappa_\partial(v_{i+1}) = 0\). We then have four triangles: \([v_{i-1}, x, v_i]\), \([v_i, x, v_{i+1}]\), \([x, y, v_{i+1}]\) and \([v_{i+1}, y, v_{i+2}]\). Again, \(x \not\in \sigma_i\), so there is a vertex \(w_{i+1}\) of \(\sigma_{i+1}\) which is not connected by an edge to \(x\), but is connected to both \(v_i\) and \(v_{i+2}\). Thus now we have a cycle \((v_i, x, y, v_{i+2}, w_{i+1})\) of length 5, which must have a diagonal (see Figure 10). It can be neither \([w_{i+1}, x]\) nor \([v_i, v_{i+2}]\). Also it cannot be \([x, v_{i+2}]\) because in that case we would have a cycle of length 4 and the first two edges are the only possible diagonals for that cycle. So the only possibility is that we have diagonals \([v_i, y]\) and \([w_{i+1}, y]\). Then we can do the following: replace the vertex \(v_{i+1}\) by \(w_{i+1}\) and the triangles \([v_i, x, v_{i+1}]\), \([x, y, v_{i+1}]\) and \([v_{i+1}, y, v_{i+2}]\) by \([v_i, x, y]\), \([v_i, y, w_{i+1}]\) and \([w_{i+1}, y, v_{i+2}]\), and call the resulting diagram \(D'\). The diagram \(D'\) is still minimal, but now we have \(\kappa_\partial(v_i) = 0\) and \(\kappa_\partial(w_{i+1}) = 1\).

Fig. 10. Pushing upstairs the positive boundary curvature
So whenever we have a vertex with boundary curvature equal to 1, we can ‘push it upstairs’ along vertices with curvature equal to 0, till we arrive at a vertex with negative curvature or at $v_n$. Hence $\sum_{i=1}^{n-1} \kappa_\partial(v_i) \leq 1$.

- $\sum_{v \in \text{int} D} \kappa_\partial(v) \leq 0$.

Indeed, since $X$ is locally 6-large, for any $v \in \text{int} D$ we have $\kappa_\partial(v) \leq 0$.

- $\kappa_\partial(v_0) \leq 2$, $\kappa_\partial(v_n) \leq 2$.

This follows from the fact that $(v_0, \ldots, v_n) \cup \gamma$ belongs to $C$.

To finish the argument we put all these inequalities into the Gauss–Bonnet formula:

$$6 = \kappa_\partial(v_0) + \kappa_\partial(v_n) + \sum_{v \in \text{int} D} \kappa_\partial(v) + \sum_{i=1}^{n-1} \kappa_\partial(v_i) + \sum_{v \in \text{int} \gamma} \kappa_\partial(v)$$

$$\leq 2 + 2 + 0 + 1 + 0 = 5,$$

a contradiction. ■

Note that the second condition in the definition of a directed geodesic is somehow similar to the condition appearing in the Projection Lemma. Using Theorem 6.3 we can make this relation explicit.

**Lemma 6.4.** Let $(v_0 = \sigma_0, \ldots, \sigma_n)$ be a a sequence of simplices in a systolic complex $X$, starting at a vertex $v_0$. Then $(v_0 = \sigma_0, \ldots, \sigma_n)$ is a directed geodesic if and only if for all $i \in \{0, \ldots, n-1\}$ the simplex $\sigma_i \subset S_i(v_0, X)$ is the projection of $\sigma_{i+1}$ in the direction of $v_0$.

**Proof.** First assume that $(v_0, \ldots, \sigma_n)$ is a directed geodesic. Recall the definition of the projection of $\sigma_i$ in the direction $v_0$. By Lemma 6.3 for any $i$ we have $\sigma_i \subset B_i(v_0, X)$, so in particular the projection makes sense:

$$\pi_{v_0}(\sigma_{i+2}) = \text{Res}(\sigma_{i+2}, X) \cap B_{i+1}(v_0, X).$$

And by the definition of a directed geodesic,

$$\sigma_{i+1} = \text{Res}(\sigma_{i+2}, X) \cap B_1(\sigma_i, X).$$

We have $\sigma_i \subset S_i(v_0, X)$, so $B_1(\sigma_i, X) \subset B_{i+1}(v_0, X)$, hence $\sigma_{i+1} \subset \pi_{v_0}(\sigma_{i+2})$. We need to show the other inclusion. Assume inductively that for all $0 \leq k \leq i$ we have $\sigma_k = \pi_{v_0}(\sigma_{k+1})$. The projection of a simplex is contained in the projection of any of its faces (follows directly from the definition), so since $\sigma_{i+1} \subset \pi_{v_0}(\sigma_{i+2})$, we get $\pi_{v_0}(\pi_{v_0}(\sigma_{i+2})) \subset \pi_{v_0}(\sigma_{i+1})$ and the latter equals $\sigma_i$ by the inductive assumption. Hence $\pi_{v_0}(\sigma_{i+2}) \subset B_1(\sigma_i, X)$ and therefore $\pi_{v_0}(\sigma_{i+2}) \subset \sigma_{i+1}$ by the definition of a directed geodesic. Thus $\pi_{v_0}(\sigma_{i+2}) = \sigma_{i+1}$. Clearly, for $i = 0$ we have $v_0 = \pi_{v_0}(\sigma_1)$, so the claim follows by induction.
To prove the other direction assume \( \sigma_k = \pi_{v_0}(\sigma_{k+1}) \) for all \( 0 \leq k \leq n \). Condition (i) of Definition 6.1 is satisfied by the definition of the projection of a simplex, thus we need to prove that condition (ii) holds. We have \( \sigma_i = \text{Res}(\sigma_{i+1}, X) \cap B_i(v_0, X) \), and we need to show that this is equal to \( \text{Res}(\sigma_{i+1}, X) \cap B_1(\sigma_{i-1}, X) \).

By the assumption \( \sigma_{i-1} \subset S_i(v_0, X) \), so \( B_1(\sigma_{i-1}, X) \subset B_i(v_0, X) \), and hence \( \text{Res}(\sigma_{i+1}, X) \cap B_1(\sigma_{i-1}, X) \subset \text{Res}(\sigma_{i+1}, X) \cap B_i(v_0, X) = \sigma_i \). Conversely, \( \sigma_{i-1} = \pi_{v_0}(\sigma_i) \), so \( \sigma_i \subset B_1(\sigma_{i-1}, X) \), and thus \( \sigma_i = \text{Res}(\sigma_{i+1}, X) \cap B_i(v_0, X) \subset \text{Res}(\sigma_{i+1}, X) \cap B_1(\sigma_{i-1}, X) \). So \( \text{Res}(\sigma_{i+1}, X) \cap B_1(\sigma_{i-1}, X) = \sigma_i \), and therefore the triple \( (\sigma_{i-1}, \sigma_i, \sigma_{i+1}) \) satisfies condition (ii) of Definition 6.1. This argument works for any \( 0 \leq i \leq n - 2 \), hence the claim follows.

Remark 6.5. There is a more general version of the Projection Lemma, which allows one to project in the direction of an arbitrary simplex (see [13, Lemma 7.7]). Lemma 6.4 is then also true if we let \( \sigma_0 \) be any simplex, not necessarily a vertex.

7. Infinite systolic groups are not torsion. Here we turn to studying systolic groups, i.e. groups acting geometrically on systolic complexes. The aim of this section is to prove that an infinite systolic group contains an element of infinite order. This result is true for hyperbolic groups [2, Proposition III.Γ.2.22] and for CAT(0) groups [17, Theorem 11]. As pointed out by the referee, this statement for systolic groups follows from their biautomaticity [13, Theorem 13.1] and the fact that infinite biautomatic groups are not torsion [10].

However, our proof is direct and more elementary. It is based on the approach used in the CAT(0) case, but instead of the usual geodesics we use the directed geodesics discussed in the previous section. First we show the existence of infinite directed geodesics in infinite systolic complexes.

Lemma 7.1. Let \( X \) be a locally finite, infinite systolic complex. Then \( X \) contains an infinite directed geodesic.

Proof. Pick a vertex \( v_0 \in X \). We will construct the required geodesic by defining a sequence of projections onto \( v_0 \). Since \( X \) is infinite and locally finite, it contains an infinite \( X^{(1)} \)-geodesic. We can take this geodesic to be issuing from \( v_0 \), so in particular for every \( n \), the sphere \( S_n(v_0, X) \) is nonempty.

Now consider the set \( P(1, 2) \) of all simplices in \( S_1(v_0, X) \) which are projections in the direction of \( v_0 \) of some simplices of \( S_2(v_0, X) \). This set is nonempty, because \( S_2(v_0, X) \) is nonempty. Similarly, for any \( n \) we define \( P(1, n) \) to be the set of all simplices of \( S_1(v_0, X) \) which come from iterating
a projection of some simplex of $S_n(v_0, X)$:

$$P(1, n) = \{ ((\pi_{v_0})^{(n-1)}(\sigma) \mid \sigma \subset S_n(v_0, X) \}.$$ 

Again, any such set is nonempty. Moreover, directly from the definition we have $P(1, n) \subset P(1, n-1)$. So we have a descending family of sets $\{P(1, n)\}_{n>1} \subset S_1(v_0, X)$. Since $X$ is locally finite, the sphere $S_1(v_0, X)$ is finite, so for $N$ sufficiently large this family stabilizes, i.e. $P(1, n)=P(1, N)$ for any $n \geq N$, and obviously $P(1, N)$ is nonempty. Pick any simplex $\sigma_1$ of $P(1, N)$.

Now define $P(m, n)$ to be the subset of $S_m(v_0, X)$ consisting of simplices which project onto $\sigma_{m-1}$ and come from the iterated projection of some simplex of $S_n(v_0, X)$ ($n > m$). Again, for any $m$, the family $\{P(m, n)\}_{n>m}$ has the same properties as for $m = 1$, i.e. it is nonempty, descending and stabilizes for $n$ sufficiently large. Hence, we can define $\sigma_m$ to be any simplex of $P(m, N_m)$ where $N_m$ is taken such that $P(m, n) = P(m, N_m)$ for all $n \geq N_m$.

This procedure gives an infinite sequence of simplices $(v_0, \sigma_1, \ldots)$, which is a directed geodesic by Lemma 6.4 because by construction $\sigma_i$ is the projection of $\sigma_{i+1}$ in the direction of $v_0$.

**Definition 7.2.** Let $G$ be a finitely generated group which acts on a topological space $X$. We say that the action is

- **proper** if for every compact subset $K \subset X$ the set $\{ g \in G \mid gK \cap K \neq \emptyset \}$ is finite,
- **cocompact** if there exists a compact subset $K \subset X$ such that $X = GK$.

**Definition 7.3.** A finitely generated group $G$ is called **systolic** if it acts properly and cocompactly by simplicial automorphisms on a systolic complex $X$.

**Theorem 7.4.** Let $G$ be an infinite systolic group. Then $G$ contains an element of infinite order.

**Proof.** We follow the idea of Swenson’s proof of an analogous result for CAT(0) groups [17, Theorem 11]. Let $X$ denote the systolic complex on which $G$ acts. As $G$ is infinite and the action is proper and cocompact, the complex $X$ has to be locally finite and infinite. Thus, by Lemma 7.1 we can pick a vertex $v_0 \in X$ and an infinite directed geodesic $\gamma = (v_0, \sigma_1, \ldots)$. We will use $\gamma$ to construct an element of infinite order.

Pick any sequence $(\sigma_k)$ of disjoint simplices of $\gamma$. We will be modifying this sequence by passing to a subsequence many times, which we will view as restricting the indices to some set $I \subset \mathbb{N}$. First note that since the action is cocompact, any simplex of $(\sigma_k)$ is in a translate of a compact set $K$. As $K$ contains finitely many simplices, there are only finitely many possible values
of \( \dim \sigma_k \). Thus one particular dimension \( d_0 \) appears in \( (\sigma_k) \) infinitely many times, and we take a subsequence \( (\sigma_k)_{k \in I} \) such that \( \dim \sigma_k = d_0 \). Again by cocompactness, for every \( \sigma_k \) there exists \( g_k \in G \) such that \( g_k(\sigma_k) \subset K \). Since \( K \) is compact, there is a subsequence of \( g_k(\sigma_k) \) which converges to some simplex \( \tau_0 \subset K \). Because \( G \) acts by simplicial automorphisms, for \( k \) sufficiently large we have \( g_k(\sigma_k) = \tau_0 \).

Next, for every \( \sigma_k \) we look at the simplices \( \sigma_{k-1} \) and \( \sigma_{k+1} \). Again we have only finitely many possibilities for their dimensions, hence we can pass to a subsequence \( (\sigma_k)_{k \in I} \) such that for every \( k \) we have \( \dim \sigma_{k-1} = d_{-1} \) and \( \dim \sigma_{k+1} = d_1 \). We have \( \{\sigma_{k-1}, \sigma_{k+1}\} \subset B_1(\sigma_k, X) \), so also \( \{g_k(\sigma_{k-1}), g_k(\sigma_{k+1})\} \subset B_1(\tau_0, X) \). Since \( B_1(\tau_0, X) \) is compact, passing to a subsequence we get \( g_k(\sigma_{k-1}) \rightarrow \tau_{-1} \) and \( g_k(\sigma_{k+1}) \rightarrow \tau_1 \) for some simplices \( \tau_{-1} \) and \( \tau_1 \) contained in \( B_1(\tau_0, X) \). So taking \( k \) sufficiently large we obtain \( g_k(\sigma_{k-1}, \sigma_{k+1}) = (\tau_{-1}, \tau_0, \tau_1) \).

Having \( (\sigma_k) \) thus modified, we are ready to construct the desired element. Take \( l > k \) such that \( g_k \) and \( g_l \) both satisfy the above condition, i.e. \( g_k(\sigma_{k-1}, \sigma_k, \sigma_{k+1}) = g_l(\sigma_{l-1}, \sigma_l, \sigma_{l+1}) = (\tau_{-1}, \tau_0, \tau_1) \). We claim that \( h = g_l^{-1} g_k \) is of infinite order.

Indeed, \( (\sigma_k, \ldots, \sigma_l) \subset \gamma \) is a directed geodesic, and so is \( (h(\sigma_k), \ldots, h(\sigma_l)) \), because \( h \) is an \( X^{(1)} \)-isometry. Now consider their concatenation. By construction of \( h \) we have \( (h(\sigma_{k-1}), h(\sigma_k), h(\sigma_{k+1})) = (\sigma_{l-1}, \sigma_l, \sigma_{l+1}) \), hence \( (\sigma_{k-1}, \ldots, \sigma_{l+1}) \) and \( (h(\sigma_{k-1}), \ldots, h(\sigma_{l+1})) \) intersect in three simplices. This is enough for \( (\sigma_k, \ldots, \sigma_l = h(\sigma_k), \ldots, h(\sigma_l)) \) to be a directed geodesic, because the condition defining a directed geodesic involves checking triples of consecutive simplices, and every triple in the concatenation is contained in either \( (\sigma_{k-1}, \ldots, \sigma_{l+1}) \) or \( (h(\sigma_{k-1}), \ldots, h(\sigma_{l+1})) \). Iterating this procedure shows that for any \( n > 0 \) the concatenation of \( (h^n(\sigma_k), \ldots, h^n(\sigma_l)) \) for \( m = 1 \) to \( n \) is a directed geodesic.

To finish the argument, assume that \( h^n = e \) for some \( n > 0 \). In particular this means \( h^n(\sigma_k) = \sigma_k \), which in a view of Theorem 6.3 contradicts the fact that \( (\sigma_k, \ldots, h^n(\sigma_k)) \) is a directed geodesic.

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REFERENCES

[1] R. P. Anstee and M. Farber, On bridged graphs and cop-win graphs, J. Combin. Theory Ser. B 44 (1988), 22–28.
[2] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. 319, Springer, Berlin, 1999.

[3] V. Chepoi, *Bridged graphs are cop-win graphs: an algorithmic proof*, J. Combin. Theory Ser. B 69 (1997), 97–100.

[4] V. Chepoi, *Graphs of some CAT(0) complexes*, Adv. Appl. Math. 24 (2000), 125–179.

[5] V. Chepoi and V. P. Soltan, *Conditions for invariance of set diameters under d-convexification in a graph*, Kibernetika (Kiev) 1983, no. 6, 14–18 (in Russian).

[6] M. M. Cohen, *A Course in Simple-Homotopy Theory*, Grad. Texts in Math. 10, Springer, New York, 1973.

[7] T. Elsner, *Flats and the flat torus theorem in systolic spaces*, Geom. Topol. 13 (2009), 661–698.

[8] T. Elsner, *Isometries of systolic spaces*, Fund. Math. 204 (2009), 39–55.

[9] M. Farber and R. E. Jamison, *On local convexity in graphs*, Discrete Math. 66 (1987), 231–247.

[10] R. H. Gilman, *Groups with a rational cross-section*, in: Combinatorial Group Theory and Topology (Alta, UT, 1984), Ann. of Math. Stud. 111, Princeton Univ. Press, Princeton, NJ, 1987, 175–183.

[11] F. Haglund, *Complexes simpliciaux hyperboliques de grande dimension*, Prépublication Orsay 71 (2003).

[12] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, 2002.

[13] T. Januszkiewicz and J. Świątkowski, *Simplicial nonpositive curvature*, Publ. Math. Inst. Hautes Études Sci. 104 (2006), 1–85.

[14] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Classics Math. (reprint of the 1977 ed.), Springer, Berlin, 2001.

[15] P. Przytycki, *Systolic groups acting on complexes with no flats are word-hyperbolic*, Fund. Math. 193 (2007), 277–283.

[16] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[17] E. L. Swenson, *A cut point theorem for CAT(0) groups*, J. Differential Geom. 53 (1999), 327–358.

[18] J. Świątkowski, *Notes from the mini-course on Simplicial Nonpositive Curvature*, Montreal, 2006; [http://www.math.uni.wroc.pl/~swiatkow/montreal/notes.pdf](http://www.math.uni.wroc.pl/~swiatkow/montreal/notes.pdf)

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